The Ring of Twisted Differential Operators of Reflection Group $I_2(5)$

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Summary

The concept of a reflection and, hence, of a group generated by reflections is one of the major objects of modern geometry. It has been studied for several centuries, beginning with the pioneering works by Klein, advancing due to the work of Coxeter, Humphreys, and Hiller, and culminating in recent developments and applications to Schubert calculus by Kostant and Kumar. The aim of this research project was to generalize the Kostant-Kumar construction of an algebra of differential operators twisted by reflections of the group $I_2(5)$ of symmetries of a regular pentagon.

Golden Ratio

The **Golden Ratio** ϕ is a famous constant. It is used to describe the patterns on flower petals, and the ancient Egyptians may have used it to construct pyramids.

Integers a and b are said to be in the Golden Ratio if

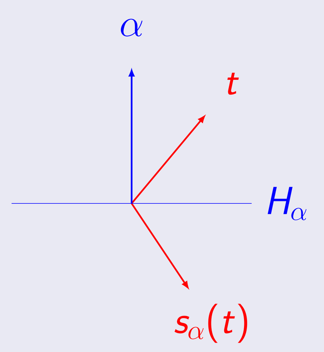
$$\frac{a+b}{a} = \frac{a}{b} = \phi$$

Reflections

Choose a vector α in a vector space V. A **reflection** with respect to α is a linear operator that we will denote s_α. It acts on $t \in V$ in the following way:

$$s_\alpha t = t - 2\frac{\langle t, \alpha \rangle}{\langle \alpha, \alpha \rangle}\alpha.$$

Here is an visualization of a reflection, where H_α is the plane whose normal vector is α:



A group that is generated by reflections is called a **Reflection Group**.

Root System

A **Root System** Φ in a vector space V is a non-empty subset of vectors that satisfies two properties. For all $a \in \Phi$:

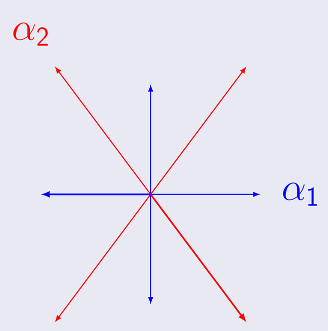
$$\Phi \cap \mathbf{c}a = \{a, -a\} \text{ for all } \mathbf{c} \in \mathbf{R} \quad \text{and} \quad \mathbf{s_a \Phi = \Phi}.$$

The subset of **Simple Roots** $\Delta = \{\beta_1, \dots, \beta_n\}$ of Φ satisfies the following condition. Let $\alpha_i \in \Phi$. Then

$$\alpha_i = b_1\beta_1 + \dots + b_n\beta_n$$

where the coefficients $b_1, \dots, b_n$ are either all non-negative or all non-positive.

Here is an example of a root system with simple roots $\{\alpha_1, \alpha_2\}$:



Every root system Φ is associated with a reflection group W.

The reflections that are associated with simple roots are called **Simple Reflections**. An important fact is that

every reflection group is generated by its simple reflections.

Coxeter Groups

Let ϕ be a root system with reflection group W and simple system $\Delta = \{\beta_1, \dots, \beta_n\}$. Consider the following group

$$S = \{s_{\beta_i} | (s_{\beta_i}s_{\beta_j})^{m(\beta_i,\beta_j)} = 1\}$$

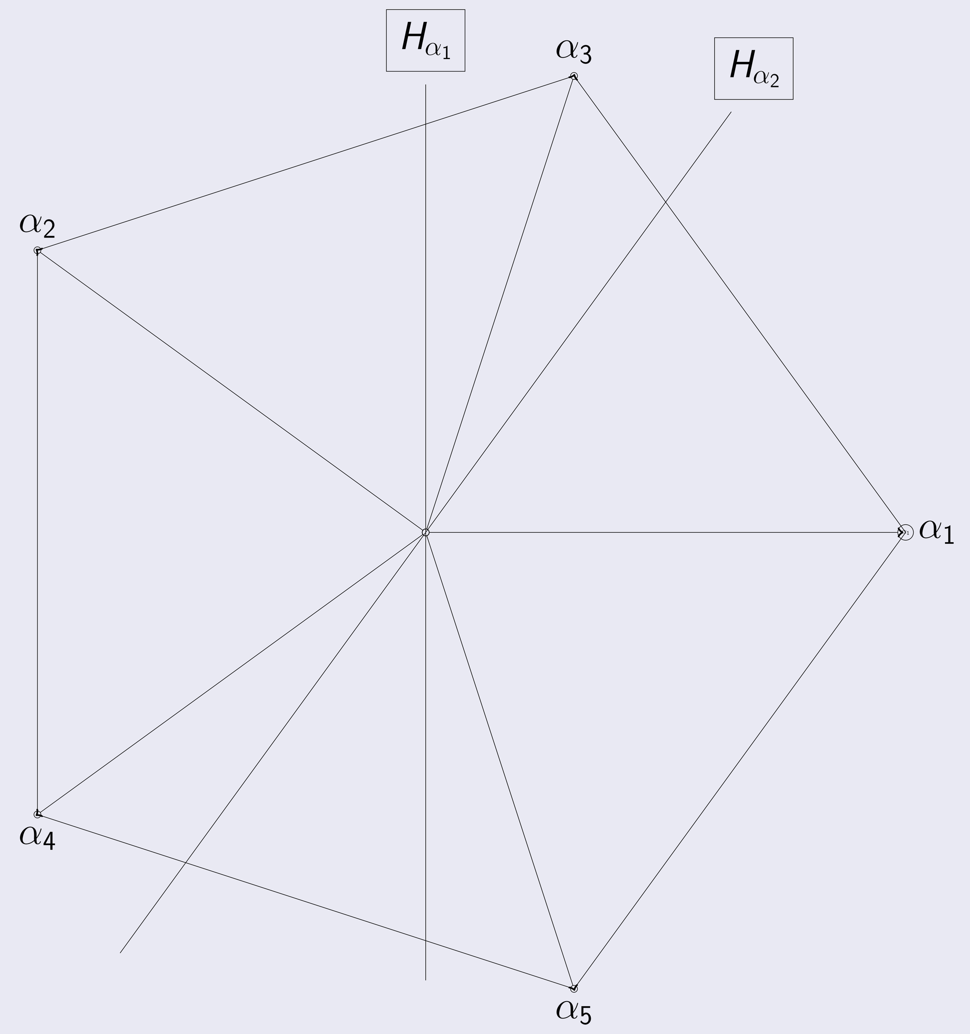
where $m(\beta_i, \beta_j)$ denotes the order of $s_{\beta_i}s_{\beta_j}$ in W.

S is called a **Coxeter group** of W. An important fact is that

every reflection group is generated by a Coxeter group.

The Golden Pentagon

The five vectors representing the vertices of a regular pentagon on the complex plane, along with their negatives, form a root system, whose associated reflection group W is called $I_2(5)$. We will denote these vectors $\pm\alpha_1, \dots, \pm\alpha_5$, and, interestingly, they are all related to the Golden Ratio ϕ. We will choose α_1 and α_2 as our simple roots. Below is an illustration of this root system, where H_{α_1} and H_{α_2} are planes whose normal vectors are α_1 and α_2 respectively.



Let $s_1 = s_{\alpha_1}$ and $s_2 = s_{\alpha_2}$. Then the group $W = I_2(5)$ consists of the following composites of reflections

$$\{1, s_1, s_2, s_1s_2, s_2s_1, s_1s_2s_1, s_2s_1s_2, s_1s_2s_1s_2, s_2s_1s_2s_1, s_1s_2s_1s_2s_1\}$$

This presentation of the group $I_2(5)$ was obtained from its Coxeter group generated by s_1 and s_2.

Root System of $I_2(5)$

Let $\mathbf{R}$ denote the ring $\{a + b\phi \mid a \text{ and } b \text{ are integers}\}$.

The roots $\{\alpha_1, \dots, \alpha_5\}$ of the root system associated with $I_2(5)$ can be written as linear combinations of α_1 and α_2 over the ring $\mathbf{R}$ with stricly non-negative or non-positive coefficients.

$$\alpha_1 = (1, 0)$$
$$\alpha_2 = (-\tfrac{\phi}{2}, \tfrac{\sqrt{3-\phi}}{2})$$
$$\alpha_3 = \phi(\alpha_1 + \alpha_2)$$
$$\alpha_4 = -(\phi\alpha_1 + \alpha_2)$$
$$\alpha_5 = -(\alpha_1 + \phi\alpha_2)$$

The result of applying a reflection of $I_2(5)$ to a root of this root system is summarized by the following table.

1	α_1	$-\alpha_1$	α_2	$-\alpha_2$	α_3	$-\alpha_3$	α_4	$-\alpha_4$	α_5	$-\alpha_5$
s_1	$-\alpha_1$	α_1	-α_4	α_4	$-\alpha_5$	α_5	$-\alpha_2$	α_2	$-\alpha_3$	α_3
s_2	$-\alpha_5$	α_5	$-\alpha_2$	α_2	-α_4	α_4	$-\alpha_3$	α_3	$-\alpha_1$	α_1
s_1s_2	α_3	$-\alpha_3$	α_4	$-\alpha_4$	α_2	$-\alpha_2$	α_5	$-\alpha_5$	α_1	$-\alpha_1$
s_2s_1	α_5	$-\alpha_5$	α_3	$-\alpha_3$	α_1	$-\alpha_1$	α_2	$-\alpha_2$	α_4	$-\alpha_4$
$s_1s_2s_1$	$-\alpha_3$	α_3	$-\alpha_5$	α_5	$-\alpha_1$	α_1	$-\alpha_4$	α_4	$-\alpha_2$	α_2
$s_2s_1s_2$	$-\alpha_4$	α_4	$-\alpha_3$	α_3	$-\alpha_2$	α_2	$-\alpha_1$	α_1	$-\alpha_5$	α_5
$s_1s_2s_1s_2$	α_2	$-\alpha_2$	α_5	$-\alpha_5$	α_4	$-\alpha_4$	α_1	$-\alpha_1$	α_3	$-\alpha_3$
$s_2s_1s_2s_1$	α_4	$-\alpha_4$	α_1	$-\alpha_1$	α_5	$-\alpha_5$	α_3	$-\alpha_3$	α_2	$-\alpha_2$
$s_1s_2s_1s_2s_1$	$-\alpha_2$	α_2	$-\alpha_1$	α_1	$-\alpha_3$	α_3	$-\alpha_5$	α_5	$-\alpha_4$	α_4

The Twisted Differential Operator

Let $\mathbf{Q}$ denote the ring of **Laurent polynomials** $\mathbb{Z}[\alpha_1^{\pm 1}, \dots, \alpha_5^{\pm 1}]$ in variables $\alpha_1, \dots, \alpha_5$ with integer coefficients.

A **Twisted Differential Operator** of $I_2(5)$, denoted X_i, $i = 1, \dots, 5$, is an operator that acts on elements $q \in \mathbf{Q}$ in the following way:

$$X_i(q) = \frac{q - s_i(q)}{\alpha_i} = \frac{1}{\alpha_i}(\mathbf{1} - s_i)(q), \text{ where } s_i = s_{\alpha_i},$$

or simply

$$X_i = \frac{1}{\alpha_i}(\mathbf{1} - s_i).$$

An important fact about twisted differential operators is that they reduce the order of their argument by one.

Here is an example of X_1 acting on an element of $\mathbf{Q}$:

$$X_1(\alpha_1\alpha_2) = \frac{\alpha_1\alpha_2 - s_1(\alpha_1\alpha_2)}{\alpha_1} = \frac{\alpha_1\alpha_2 - (-\alpha_1)(-\alpha_4)}{\alpha_1} = \alpha_2 - \alpha_4$$

The Twisted Group Ring

Let $\tilde{\mathbf{Q}}[I_2(5)]$ denote the ring of functions $I_2(5) \to \mathbf{Q}$ with a point-wise multiplication and addition.

If δ_w denotes a function that is supported at $w \in I_2(5)$, then the elements of $\tilde{\mathbf{Q}}[I_2(5)]$ can be described as finite linear combinations of elements $q\delta_w$, where $q \in \mathbf{Q}$.

Elements of $\tilde{\mathbf{Q}}[I_2(5)]$ are multiplied together using the following commuting rule (this is why this group ring is "twisted"):

$$(q_1\delta_{s_1})(q_2\delta_{s_2}) = (q_1 s_1(q_2))(\delta_{s_1}\delta_{s_2}).$$

The ring $\tilde{\mathbf{Q}}[I_2(5)]$ is called the **Twisted Group Ring** of $I_2(5)$.

Our twisted differential operators X_i can be viewed as elements of $\tilde{\mathbf{Q}}[I_2(5)]$. The following example shows us how two differential operators are composed (multiplied):

$$X_1X_2 = (\frac{1}{\alpha_1}(\mathbf{1} - \delta_{s_1}))(\frac{1}{\alpha_2}(\mathbf{1} - \delta_{s_2})) = \frac{1}{\alpha_1\alpha_2}(\mathbf{1} - \delta_{s_2}) - \frac{1}{\alpha_1}s_1(\frac{1}{\alpha_2})(\delta_{s_1} - \delta_{s_1s_2}).$$

The Ring of Twisted Differential Operators

The subring of $\tilde{\mathbf{Q}}[I_2(5)]$ generated by the elements $X_1, \dots, X_5$ is called the ring of **Twisted Differential Operators** of the group $I_2(5)$. We will denote this ring by $D_{I_2(5)}$.

We define two rings $\mathbf{T}$ and $\mathbf{T}'$ as follows:

$$\mathbf{T} = \{a + b\phi + c\alpha_2 + d\phi\alpha_2 + e\alpha_1 + f\phi\alpha_1 \mid a, d, c, d, e, f \in \mathbb{Z}\}$$

and $\mathbf{T}'$ is the ring of Laurent polynomials in variables of $\mathbf{T}$ with integer coefficients.

To shorten the length of the expressions, we use the following notation:

(i) $\alpha_i = i$ for $i = 1, \dots, 5$
(ii) $x = [x]$ for all $x \in \mathbb{Z}$
(iii) $X_iX_j = X_{ij}$ for $i, j = 1, \dots, 5$

Generating $D_{I_2(5)}$ - The Main Result

The ring $D_{I_2(5)}$ is generated by X_1 and X_2 over the coefficient ring $\mathbf{T}'$, modulo the following relations:

(1) $$X_3 = \tfrac{[1]}{\phi(1+2)}(\mathbf{1} - (\mathbf{1} - 1X_1)(\mathbf{1} - 2X_2)(\mathbf{1} - 1X_1)(\mathbf{1} - 2X_2)(\mathbf{1} - 1X_1))$$

(2) $$X_4 = \tfrac{-[1]}{\phi+2}(\mathbf{1} - (\mathbf{1} - 1X_1)(\mathbf{1} - 2X_2)(\mathbf{1} - 1X_1))$$

(3) $$X_5 = \tfrac{-[1]}{1+\phi^2}(\mathbf{1} - (\mathbf{1} - 2X_2)(\mathbf{1} - 1X_1)(\mathbf{1} - 2X_2))$$

(4) $$X_{12121} = X_{21212}$$

(5) $$X_i^2 = 0, \ i = 1, \dots, 5$$

Conclusion

These relations help to make computations in $D_{I2(5)}$, e.g. to simplify otherwise complicated equations between twisted differential operators. This is emphasized by the following example:

$$X_{2121212} = X_{21}(X_{21212}) = X_{21}(X_{12121}) = X_2(X_1^2)X_{2121} = 0.$$

In other words, any product of more than five twisted differential operators X_1 and X_2 equals zero. The results of this research project may become useful in representation theory and in the theory of finite reflection groups.

References

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