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APPLICATION OF A VARIATIONAL METHOD TO FLOW OVER
A FLAT PLATE IN THE ENTRANCE REGION WITH
VARIABLE PHYSICAL PROPERTIES

By

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Submitted to the Faculty of Science and
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partial fulfillment of the requirements for the
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APPLICATION OF A VARIATIONAL METHOD TO FLOW OVER
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Advisor

U

Candidate

V

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ABSTRACT

A variational method has been used to solve the flow over a flat plate in the entrance region at constant wall temperature. The physical properties, i.e. thermal conductivity and viscosity, were assumed to be linear functions of temperature in the study. Two coupled equations were derived from the variational formulation and then solved by the analog/hybrid computer. Consequently, momentum boundary layer thickness, thermal boundary layer thickness, local Nusselt number and local friction factor were found for the flow. A comparison of the constant properties case was made between the results from this study and those obtained from the exact solution.

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NOMENCLATURE

A	viscosity coefficient
Amp	amplifier
a	analog input signal.
B	conductivity coefficient.
b	analog output signal.
C	comparator.
C_p	specific heat at constant pressure.
C_v	specific heat at constant volume.
D/A	direct/analog (switch).
$d_e s$	entropy change due to reaction of the system with its surrounding.
$d_i s$	internal production of entropy in an irreversible reaction.
e	internal energy per unit mass.
e_o	output analog signal.
e_1, e_2	input analog signal.
E	local potential.
E_p	entropy production per unit time.
f	local friction factor.
H.G.	high gain.
h	heat transfer coefficient.
IC	initial condition.
J_i	generalized fluxes.
K	amplification.
k	thermal conductivity.
L	logic signal.
L_{ij}	phenomenological coefficients.
l	characteristic length of the flat plate.

Nu_x	local Nusselt number.
n_i	unit vector normal to the surface.
P	potentiometers, set by servomotor.
p	pressure
Pr	Prandtl number.
q	heat flux.
Re	Reynolds number.
s	surface.
S1, S2	switch 1, switch 2.
T	temperature.
t	time.
u	velocity in the x-direction.
v	velocity in the y-direction.
V	volume.
X_i	generalized forces.
x_0	unheated starting length.
x, y	cartesian coordinates.
Y	(Δ_t/Δ) ratio between thermal and momentum boundary layer thickness.

Greek Symbols

α	thermal diffusivity.
Δ	momentum boundary layer thickness.
Δ_t	thermal boundary layer thickness.
δ	variation notation.
θ	dimensionless temperature variable.
μ	dynamic viscosity.
ν	kinematic viscosity.

- ρ density.
 σ entropy production per unit time and volume.

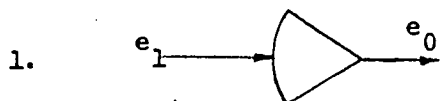
Subscripts

- i, j tensorial indices.
 x, y derivative with respect to x (y).
 w wall property.
 ∞ free stream property.

Superscripts

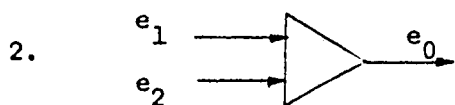
- * dimensionless quantity.
o stationary state.

PROGRAMMER SYMBOLS



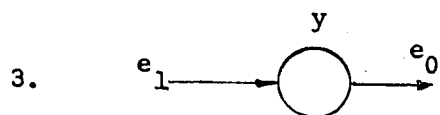
High gain dc amplifier

$$e_0 = -K e_1 \text{ (K large, normally greater than } 10^8 \text{)}$$



Summer-inverter

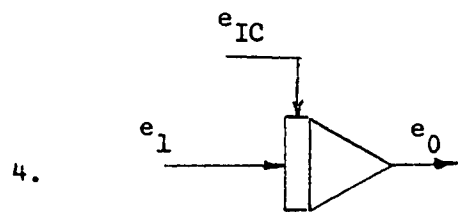
$$e_0 = - (e_1 + e_2)$$



Grounded potentiometer

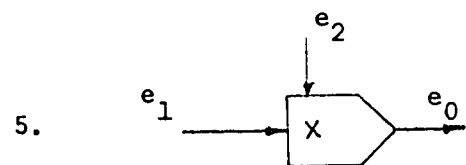
$$e_0 = y e_1$$

$$0 \leq y \leq 1$$



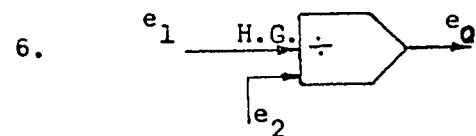
Integrator

$$e_0 = - \int e_1 \cdot dt - e_{IC}$$



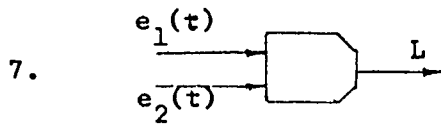
Multiplier

$$e_0 = -e_1 e_2$$



Divider

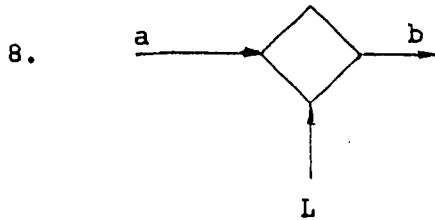
$$e_0 = - \frac{e_1}{e_2}$$



Comparator

$e_1(t) + e_2(t) > 0$, L is TRUE

$e_1(t) + e_2(t) < 0$, L is FALSE



D/A SWITCH

a and b are connected, L is TRUE

b is grounded, L is FALSE

CHAPTER I

INTRODUCTION

The application of the variational method in the theory of elasticity and other related fields has been well established. An important formulation of the variational principle in thermoscience was derived by Glansdroff et al.⁽¹⁾ in 1962 based on minimum entropy production. Its application is shown in the literature^(2, 3). However, the formulation is only applicable to those systems in which

- (1) the phenomenological coefficients are constant or expressed in specific forms
- (2) the Onsager reciprocal relations are valid
- (3) the convective terms are negligible.

These represent a rather restricted class of systems. Later, Glansdroff and Prigogine⁽⁴⁾ removed the above restrictions by modifying the formulation using the concept of local potential (generalized entropy production). The application of this formulation is shown in the literature^(5, 6, 7). The formulation of the variational principle in thermoscience is discussed in detail in the next chapter.

The purpose of this study is to apply the variational method based on the local potential theory in solving the flow over a flat plate in the entrance region at constant wall temperature with variable physical properties. This involves the formulation of two coupled momentum and energy boundary equations and their solution by

analog/hybrid computer. The quantities to be determined include momentum and thermal boundary layer thickness, local Nusselt number and local friction factor. In certain cases a comparison will be made between the results from this work and those available from the literature, i.e., constant properties case.

The combination of the variational method and analog/hybrid computer in solving this class of engineering problem is a rather new approach. As a result of this study some degree of confidence in this combined solution approach is established. It is hoped that the approach can be effectively utilized in obtaining solution to other problems in this area.

CHAPTER II

THEORETICAL DEVELOPMENT

In general, the behavior of a physical system can be described by differential equations, subject to certain prescribed boundary and initial conditions. For certain cases or for certain types of problems, it is very difficult or impossible to obtain solutions by conventional methods. In such cases another approach, the so called variational method, may be useful. The variational method is based on the techniques of the calculus of variations. In this approach, a functional (function of functions), is first constructed according to

- (1) the differential equations and associated boundary conditions of the system, and
- (2) some physical laws.

Different values of a functional are calculated for different expressions of the functions in the functional. Only for certain expressions of functions, does the functional have a stationary value. These functions are the solutions of the problem. The solution functions in general can be expressed in polynomial forms and their coefficients can be calculated by Ritz method ⁽⁸⁾.

In order to find a functional in thermoscience, one starts from the theorem of minimum entropy production (see Glansdorff et al. ⁽⁹⁾ and DeGroot et al. ⁽¹⁰⁾). The entropy change may be split into two parts

$$dS = d_e S + d_i S$$

Here $d_e s$ is the entropy change due to the reaction of the system with its surrounding and $d_i s$ is the term which represents the internal production of entropy in an irreversible reaction.

From the second law,

$$d_i s \geq 0$$

Glansdorff et al. have stated that "This expression indeed provides us with a general evolution criterion in the form of a non-total differential".⁽⁴⁾

The entropy production per unit time may be expressed as

$$E_p = \frac{d_i S}{dt} = \int_V \sigma dV \geq 0 \quad (1)$$

Here σ denotes the entropy production per unit time and volume. It is well known that under certain restrictive conditions σ is a bilinear form which consists of a sum of terms, each containing the product of a generalized flux J_i and force X_i .

i.e.,
$$\sigma = \sum_{\lambda} J_{\lambda} X_{\lambda} \geq 0 \quad (2)$$

These restrictive conditions are

- (1) the validity of the linear phenomenological laws

$$J_{\lambda} = \sum_j L_{\lambda j} X_j \quad (3)$$

- (2) the validity of Onsager's reciprocity relations

$$L_{\lambda j} = L_{j\lambda} \quad (4)$$

(3) the phenomenological coefficients are constant

$$L_{ij} = \text{constant} \quad (5)$$

From Eq. (1) through Eq. (5), Glansdorff et al.⁽⁹⁾ showed that the entropy production can only decrease with respect to time for time independent boundary conditions

i.e.,
$$\frac{\partial E_P}{\partial t} \leq 0. \quad (6)$$

where equality refers to a stationary state, or

$$E_P = \text{minimum} \quad (7)$$

This is the so-called theorem of minimum entropy production.

Differentiating Eq. (2) with respect to time and substituting the result into Eq. (1), one obtains

$$\begin{aligned} \frac{\partial E_P}{\partial t} &= \int_V \sum_i J_i \frac{\partial X_i}{\partial t} dV + \int_V \sum_i X_i \frac{\partial J_i}{\partial t} dV \\ &= \int_V \frac{\partial_x \sigma}{\partial t} dV + \int_V \frac{\partial_J \sigma}{\partial t} dV \\ &= \frac{\partial_x E_P}{\partial t} + \frac{\partial_J E_P}{\partial t} \end{aligned} \quad (8)$$

where
$$\frac{\partial_x \sigma}{\partial t} = J_i \frac{\partial X_i}{\partial t}, \quad \frac{\partial_J \sigma}{\partial t} = X_i \frac{\partial J_i}{\partial t}$$

$$\frac{\partial_x E_P}{\partial t} = \int_V \frac{\partial_x \sigma}{\partial t} dV \quad \text{and} \quad \frac{\partial_J E_P}{\partial t} = \int_V \frac{\partial_J \sigma}{\partial t} dV$$

Under certain general conditions^(9, 11)

$$\frac{\partial_x E_P}{\partial t} = \int_V \sum_i J_i \frac{\partial X_i}{\partial t} dV \leq 0 \quad (9)$$

holds during the evolution of the system, without any reference to the phenomenological relations between the fluxes J_i and the forces

X_i . Note that Eq.(9) is valid for nonlinear processes. Still, Eq.(9) only considers the evolution under the influence of dissipative forces, it does not include the effects of the inertia forces.

In order to develop a general criterion describing the stationary state, Glansdorff et al. ⁽⁴⁾ formulated a universal evolution criterion and definition of a local potential. That is, there exists a quantity $\frac{\partial E}{\partial t}$ of the form

$$\frac{\partial E}{\partial t} = \int_V \sum_{\lambda} J_{\lambda}' \frac{\partial X_{\lambda}'}{\partial t} dV \leq 0 \quad (10)$$

This is similar to Eq.(9), but the forces X_i' and the fluxes J_i' here include both inertia terms and dissipative terms. Similarly, $\frac{\partial E}{\partial t}$ is non-positive for time independent boundary conditions.

For the sake of clearness, it is best to demonstrate these ideas by presenting a simple example ⁽⁴⁾. The example here is to determine the temperature distribution in a solid body.

In the absence of convection, the equation of conservation of energy is

$$\rho \frac{\partial e}{\partial t} = - \frac{\partial q_{\lambda}}{\partial x_{\lambda}} \quad (11)$$

Multiplying both sides of this expression by $\frac{\partial}{\partial t} \left(\frac{1}{T} \right)$, one obtains

$$\rho \frac{\partial}{\partial t} \left(\frac{1}{T} \right) \frac{\partial e}{\partial t} = - \frac{\partial}{\partial x_{\lambda}} \left[q_{\lambda} \frac{\partial}{\partial t} \left(\frac{1}{T} \right) \right] + q_{\lambda} \frac{\partial}{\partial x_{\lambda}} \left(\frac{\partial}{\partial t} \frac{1}{T} \right) \quad (12)$$

Since $\frac{\partial e}{\partial t} = c_v \frac{\partial T}{\partial t}$ (c_v is a constant volume specific heat)

therefore, a function Ψ can be defined so that

$$\Psi = \rho C_v \frac{\partial T}{\partial t} \frac{\partial}{\partial t} \frac{1}{T} = - \frac{\rho C_v}{T^2} \left(\frac{\partial T}{\partial t} \right)^2 \leq 0. \quad (13)$$

From Eqs.(12), and (13), one obtains

$$\Psi = - \frac{\partial}{\partial x_i} \left[q_i \frac{\partial}{\partial t} \left(\frac{1}{T} \right) \right] + q_i \frac{\partial}{\partial x_i} \left(\frac{\partial}{\partial t} \frac{1}{T} \right) \leq 0 \quad (14)$$

where ϕ is obtained by integrating Ψ over the volume

$$\phi = \int_V \Psi dV = \int_V \left\{ - \frac{\partial}{\partial x_i} \left[q_i \frac{\partial}{\partial t} \left(\frac{1}{T} \right) \right] + q_i \frac{\partial}{\partial x_i} \left(\frac{\partial}{\partial t} \frac{1}{T} \right) \right\} dV \leq 0. \quad (15)$$

The first term of the integrand is equivalent to a certain surface integral. According to Gauss' theorem Eq. (15) becomes

$$\phi = - \int_S \left(q_i \frac{\partial}{\partial t} \frac{1}{T} \right) n_i dS + \int_V q_i \frac{\partial}{\partial x_i} \left(\frac{\partial}{\partial t} \frac{1}{T} \right) dV \leq 0 \quad (16)$$

where S is the surface enclosing V and n_i is a unit vector normal to the surface. For time independent boundary conditions, the surface integral vanishes, Eq.(16) now gives

$$\phi = \int_V q_i \frac{\partial}{\partial t} \left(\frac{\partial}{\partial x_i} \frac{1}{T} \right) dV \leq 0 \quad (17)$$

If one introduces Fourier's law $q_i = -k(T) \frac{\partial T}{\partial x_i}$ into Eq.(17)

then

$$\phi = \int_V k(T) T^2 \frac{\partial T^{-1}}{\partial x_i} \cdot \frac{\partial}{\partial t} \left(\frac{\partial T^{-1}}{\partial x_i} \right) dV \leq 0$$

or

$$\phi = \int_V \frac{1}{2} k(T) T^2 \frac{\partial}{\partial t} \left(\frac{\partial}{\partial x_i} \frac{1}{T} \right)^2 dV \leq 0 \quad (18)$$

Suppose that the stationary state corresponds to the temperature distribution $T^0(x_i)$, then in the neighbourhood of this stationary

state $k(T)T^2$ in Eq.(18) may be replaced by $k(T^0)T^{02}$ and let $k^0 = k(T^0)$. It follows that

$$\dot{\phi} = \dot{\phi}(T, T^0) = \frac{\partial}{\partial t} \left[\int_V \frac{1}{2} K^0 T^{02} \left(\frac{\partial T^{-1}}{\partial x_{\lambda}} \right)^2 dV \right] \leq 0 \quad (19)$$

This form corresponds to Eq.(10). Thus the volume integral

$$E = \frac{1}{2} \int_V K^0 T^{02} \left(\frac{\partial T^{-1}}{\partial x_{\lambda}} \right)^2 dV \quad (20)$$

can only decrease with time and E is a minimum at the stationary state. Here E can be identified as the rate of entropy production at the stationary state (as E_p in Eq.(1)) and it is called local potential.

Taking variation of Eq.(20) with respect to T, it becomes

$$\left\{ \frac{\delta E(T, T^0)}{\delta T} \right\}_{T^0} = 0$$

$$\text{or} \quad \frac{\partial}{\partial x_{\lambda}} \left(K^0 T^{02} \frac{\partial T^{-1}}{\partial x_{\lambda}} \right) = 0 \quad (21)$$

Using the subsidiary condition

$$T^0 = T$$

one obtains

$$\frac{\partial \varrho_{\lambda}}{\partial x_{\lambda}} = 0 \quad (22)$$

Eq.(22) corresponds to the Euler-Lagrange equation of the calculus of variations (8).

From Eq.(22), it is clear that the minimum condition for Eq.(20) is equivalent to the vanishing of the divergence of the heat flow

for $T = T^0$ (stationary state). Since Eq. (22) agrees with the steady state condition on Eq.(11), one can conclude that Eq. (20) is the functional of the problem.

The construction of a variational formulation for the problem starting from the governing equations and following the concept developed in this chapter will be shown in the next chapter.

CHAPTER III

VARIATIONAL FORMULATION OF THE PROBLEM

The conservation equations of mass, momentum and energy for two-dimensional, incompressible boundary layer flow ⁽¹²⁾ can be expressed

as
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (23)$$

$$\rho \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = - \frac{\partial p}{\partial x} + \frac{\partial}{\partial y} \left(\mu \frac{\partial u}{\partial y} \right) \quad (24)$$

$$\rho c_p \left(\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right) = \frac{\partial}{\partial y} \left(k \frac{\partial T}{\partial y} \right) + \mu \left(\frac{\partial u}{\partial y} \right)^2 \quad (25)$$

Although the problem under the study is a steady state case, one needs however to retain the time-dependent character of the equations when forming the local potential.

The pressure gradient is zero for flow over a flat plate. In addition, the heat dissipation may be neglected for flow at low velocity and low Prandtl number ⁽¹³⁾. Thus Eqs. (24) and (25) become

$$\rho \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = \frac{\partial}{\partial y} \left(\mu \frac{\partial u}{\partial y} \right) \quad (26)$$

$$\rho c_p \left(\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right) = \frac{\partial}{\partial y} \left(k \frac{\partial T}{\partial y} \right) \quad (27)$$

Here μ and k are functions of T .

The closed form solution of simultaneous Eqs. (23), (26) and (27) is usually very difficult to obtain even for the simplest geometry. This is primarily because Eqs. (26) and (27) are nonlinear. In this study an attempt will be made to solve Eqs. (23), (26) and (27) by the variational technique.

In order to construct a local potential for the problem for use in the variational method, a technique used by Glansdorff and Prigogine⁽⁴⁾ is followed. Upon multiplying Eq.(23) by $-\left(\frac{\rho}{2}\right)\left(\frac{\partial v^2}{\partial t}\right)$, Eq.(26) by $\frac{\partial u}{\partial t}$ and Eq.(27) by $\frac{\partial T}{\partial t}$, summing the results and rearranging the terms, one obtains

$$\begin{aligned} \psi &= -\rho \left(\frac{\partial u}{\partial t}\right)^2 - \rho c_p \left(\frac{\partial T}{\partial t}\right)^2 \\ &= \rho u \frac{\partial u}{\partial t} \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial t} \frac{\partial u}{\partial y} - \frac{\partial u}{\partial t} \left[\frac{\partial}{\partial y} (\mu \frac{\partial u}{\partial y}) \right] - \frac{\rho}{2} \frac{\partial v^2}{\partial t} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \\ &\quad + \rho c_p u \frac{\partial T}{\partial x} \frac{\partial T}{\partial t} + \rho c_p v \frac{\partial T}{\partial y} \frac{\partial T}{\partial t} - \frac{\partial}{\partial y} (\kappa \frac{\partial T}{\partial t} \frac{\partial T}{\partial y}) + \frac{\kappa}{2} \frac{\partial}{\partial t} \left(\frac{\partial T}{\partial y} \right)^2 \leq 0 \end{aligned} \quad (28)$$

Since Eqs.(23),(26) and (27) describe a two-dimensional flow. ϕ is defined in this case by integrating over the area.

$$\phi = \iint_S \psi \, dx \, dy \leq 0 \quad (29)$$

where S is an area of interest in the x-y plane which is bounded by curve C.

Applying integration by parts on integrand ψ , it yields

$$\begin{aligned} \psi &= \frac{\partial}{\partial x} \left(\rho u^2 \frac{\partial u}{\partial t} \right) + \frac{\partial}{\partial y} \left(\rho u v \frac{\partial u}{\partial t} \right) - \mu \frac{\partial}{\partial y} \left(\frac{\partial u}{\partial y} \frac{\partial u}{\partial t} \right) - \rho u \frac{\partial u}{\partial t} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \\ &\quad - \rho u^2 \frac{\partial}{\partial t} \left(\frac{\partial u}{\partial x} \right) - \rho u v \frac{\partial}{\partial t} \left(\frac{\partial u}{\partial y} \right) + \frac{\mu}{2} \frac{\partial}{\partial t} \left(\frac{\partial u}{\partial y} \right)^2 - \frac{\rho}{2} \frac{\partial v^2}{\partial t} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \\ &\quad + \rho c_p u \frac{\partial T}{\partial x} \frac{\partial T}{\partial t} + \rho c_p v \frac{\partial T}{\partial y} \frac{\partial T}{\partial t} - \frac{\partial}{\partial y} (\kappa \frac{\partial T}{\partial t} \frac{\partial T}{\partial y}) + \frac{\kappa}{2} \frac{\partial}{\partial t} \left(\frac{\partial T}{\partial y} \right)^2 \leq 0 \end{aligned} \quad (30)$$

Combining Eqs.(29),(30) and using Gauss' theorem, one obtains

$$\begin{aligned} \phi = & \iint_S \left[-\rho u^2 \frac{\partial}{\partial t} \left(\frac{\partial u}{\partial x} \right) - \rho u v \frac{\partial}{\partial t} \left(\frac{\partial u}{\partial y} \right) + \frac{\mu}{2} \frac{\partial}{\partial t} \left(\frac{\partial u}{\partial y} \right)^2 - \frac{\rho}{2} \frac{\partial u^2}{\partial t} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \right. \\ & \left. - \frac{\rho}{2} \frac{\partial v^2}{\partial t} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + \rho c_p u \frac{\partial T}{\partial x} \frac{\partial T}{\partial t} + \rho c_p v \frac{\partial T}{\partial y} \frac{\partial T}{\partial t} + \frac{\kappa}{2} \frac{\partial}{\partial t} \left(\frac{\partial T}{\partial y} \right)^2 \right] dx dy \\ & + \int_c \left(\rho u^2 \frac{\partial u}{\partial t} dy + \rho u v \frac{\partial u}{\partial t} dx - \mu \frac{\partial u}{\partial y} \frac{\partial u}{\partial t} dx - \kappa \frac{\partial T}{\partial y} \frac{\partial T}{\partial t} dx \right) \leq 0 \quad (31) \end{aligned}$$

Near the stationary state, ϕ yields

$$\begin{aligned} \phi = & \frac{\partial}{\partial t} \iint_S \left[-\rho u^{\circ 2} \frac{\partial u}{\partial x} - \rho u^{\circ} v^{\circ} \frac{\partial u}{\partial y} + \frac{\mu^{\circ}}{2} \left(\frac{\partial u}{\partial y} \right)^2 - \frac{\rho u^{\circ 2}}{2} \left(\frac{\partial u^{\circ}}{\partial x} + \frac{\partial v^{\circ}}{\partial y} \right) \right. \\ & \left. - \frac{\rho v^{\circ 2}}{2} \left(\frac{\partial u^{\circ}}{\partial x} + \frac{\partial v^{\circ}}{\partial y} \right) + \rho c_p u^{\circ} T \frac{\partial T^{\circ}}{\partial x} + \rho c_p v^{\circ} T \frac{\partial T^{\circ}}{\partial y} + \frac{\kappa^{\circ}}{2} \left(\frac{\partial T}{\partial y} \right)^2 \right] dx dy \\ & + \frac{\partial}{\partial t} \int_c \left(\rho u^{\circ 2} u dy + \rho u^{\circ} v^{\circ} u dx - \mu^{\circ} \frac{\partial u^{\circ}}{\partial y} u dx - \kappa^{\circ} \frac{\partial T^{\circ}}{\partial y} T dx \right) \leq 0 \quad (32) \end{aligned}$$

Therefore from Eq.(10), the local potential is

$$\begin{aligned} E = & \iint_S \left[-\rho u^{\circ 2} \frac{\partial u}{\partial x} - \rho u^{\circ} v^{\circ} \frac{\partial u}{\partial y} + \frac{\mu^{\circ}}{2} \left(\frac{\partial u}{\partial y} \right)^2 - \rho \left(\frac{u^{\circ 2}}{2} + \frac{v^{\circ 2}}{2} \right) \left(\frac{\partial u^{\circ}}{\partial x} + \frac{\partial v^{\circ}}{\partial y} \right) \right. \\ & \left. + \rho c_p u^{\circ} T \frac{\partial T^{\circ}}{\partial x} + \rho c_p v^{\circ} T \frac{\partial T^{\circ}}{\partial y} + \frac{\kappa^{\circ}}{2} \left(\frac{\partial T}{\partial y} \right)^2 \right] dx dy \\ & + \int_c \left(\rho u^{\circ 2} u dy + \rho u^{\circ} v^{\circ} u dx - \mu^{\circ} \frac{\partial u^{\circ}}{\partial y} u dx - \kappa^{\circ} \frac{\partial T^{\circ}}{\partial y} T dx \right) \quad (33) \end{aligned}$$

with the subsidiary conditions

$$u^{\circ} = u$$

$$v^{\circ} = v$$

$$T^{\circ} = T$$

The line integral of Eq.(33) can be simplified by using the boundary conditions. In this study the area of interest is a semi-infinite strip bounded by the lines $x = 0$, $x = l$ and $y = 0$. Note that the boundary conditions for the problem are

$$u = 0 \quad \text{at} \quad y = 0$$

$$v = 0 \quad \text{at} \quad y = 0$$

$$u = u_{\infty} \quad \text{at} \quad y = \Delta$$

$$u = u_{\infty} \quad \text{at} \quad x = 0$$

and

$$T = T_w \quad \text{at} \quad y = 0$$

$$T = T_{\infty} \quad \text{at} \quad y = \Delta_t$$

$$T = T_{\infty} \quad \text{at} \quad x = 0$$

Therefore the contribution from the line integral is

$$\begin{aligned} E_{line} &= \int_C (\rho u^2 u dy + \rho u^2 v^2 u dx - \mu^{\circ} \frac{\partial u^{\circ}}{\partial y} u dx - \kappa^{\circ} \frac{\partial T^{\circ}}{\partial y} T dx) \\ &= \int_0^{\Delta} (\rho u^2 u|_{x=l} - \rho u^2 u|_{x=0}) dy - \int_0^l [\rho u^2 v^2 u|_{y=\Delta} \\ &\quad - \rho u^2 v^2 u|_{y=0} - \mu^{\circ} \frac{\partial u^{\circ}}{\partial y} u|_{y=\Delta} + \mu^{\circ} \frac{\partial u^{\circ}}{\partial y} u|_{y=0} - \kappa^{\circ} \frac{\partial T^{\circ}}{\partial y} T|_{y=\Delta_t} \\ &\quad + \kappa^{\circ} \frac{\partial T^{\circ}}{\partial y} T|_{y=0}] dx. \end{aligned}$$

Imposing the boundary conditions, one obtains

$$E_{line} = \int_0^{\Delta} (\rho u^2 u|_{x=l} - \rho u_{\infty}^3) dy - \int_0^l \rho u_{\infty}^2 v^2|_{y=\Delta} dx \quad (34)$$

By using Eqs. (23) and (34), Eq. (33) can be further reduced to

$$E = \iint \left[-\rho u^{\circ 2} \frac{\partial u}{\partial x} - \rho u^{\circ} v^{\circ} \frac{\partial u}{\partial y} + \frac{u^{\circ}}{2} \left(\frac{\partial u}{\partial y} \right)^2 + \rho C_p u^{\circ} T \frac{\partial T^{\circ}}{\partial x} + \rho C_p v^{\circ} T \frac{\partial T^{\circ}}{\partial y} + \frac{K^{\circ}}{2} \left(\frac{\partial T^{\circ}}{\partial y} \right)^2 \right] dx dy + \int_0^{\Delta} (\rho u^{\circ 2} u|_{x=l} - \rho u_{\infty}^3) dy - \int_0^l \rho u_{\infty}^2 v^{\circ}|_{y=\Delta} dx \quad (35)$$

or
$$E = \iint F(x, y, u, u_x, u_y, T, T_x, T_y) dx dy + \int_0^{\Delta} (\rho u^{\circ 2} u|_{x=l} - \rho u_{\infty}^3) dy - \int_0^l \rho u_{\infty}^2 v^{\circ}|_{y=\Delta} dx \quad (35-a)$$

where
$$F(x, y, u, u_x, u_y, T, T_x, T_y) = -\rho u^{\circ 2} \frac{\partial u}{\partial x} - \rho u^{\circ} v^{\circ} \frac{\partial u}{\partial y} + \frac{u^{\circ}}{2} \left(\frac{\partial u}{\partial y} \right)^2 + \rho C_p u^{\circ} T \frac{\partial T^{\circ}}{\partial x} + \rho C_p v^{\circ} T \frac{\partial T^{\circ}}{\partial y} + \frac{K^{\circ}}{2} \left(\frac{\partial T^{\circ}}{\partial y} \right)^2 \quad (35-b)$$

Again it can be seen that the local potential as defined for two-dimensional boundary layer is composed of two parts. One part is an area integral and other is a line integral. The line integral enters because the velocity u is not specified at $x=l$. Hence the variation in velocity does not vanish at $x=l$ as it does for $x=0$, $y=0$ and $y=\Delta$. This problem corresponds to the one-free end boundary condition in the calculus of variations^(14, 15). Therefore the boundary condition at $x=l$ must be a natural boundary condition.

In order to prove Eq. (35) is the local potential of the problem the following operations must be performed. Taking variation of local potential E (Eq. (35-a)) with respect to u and T , one obtains

$$\left(\frac{\delta E}{\delta u} \right)_{u^{\circ}} = 0 \quad (36)$$

$$\left(\frac{\delta E}{\delta T} \right)_{T^{\circ}} = 0 \quad (37)$$

Eq. (36) can be written as

$$\frac{\partial F}{\partial u} - \frac{\partial}{\partial x} \left(\frac{\partial F}{\partial u_x} \right) - \frac{\partial}{\partial y} \left(\frac{\partial F}{\partial u_y} \right) = 0 \quad (38)$$

Substituting Eq. (35-b) into Eq. (38), it follows that

$$\frac{\partial}{\partial x}(-\rho u^{\circ 2}) - \frac{\partial}{\partial y}(-\rho u^{\circ} v^{\circ}) - \frac{\partial}{\partial y}(\mu^{\circ} \frac{\partial u}{\partial y}) = 0 \quad (39)$$

Using the subsidiary conditions

$$u^{\circ} = u$$

$$v^{\circ} = v$$

Eq.(39) becomes

$$\frac{\partial}{\partial x}(\rho u^2) + \frac{\partial}{\partial y}(\rho u v) - \frac{\partial}{\partial y}(\mu \frac{\partial u}{\partial y}) = 0$$

or

$$2\rho u \frac{\partial u}{\partial x} + \rho u \frac{\partial v}{\partial y} + \rho v \frac{\partial u}{\partial y} - \frac{\partial}{\partial y}(\mu \frac{\partial u}{\partial y}) = 0$$

Rearranging the terms, the above equation gives

$$\rho u (\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) + \rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} - \frac{\partial}{\partial y}(\mu \frac{\partial u}{\partial y}) = 0$$

or

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(\mu \frac{\partial u}{\partial y})$$

This is the momentum boundary layer equation in the x-direction for constant pressure. Similarly, from Eq.(37) it can be shown that

$$\rho c_p u \frac{\partial T}{\partial x} + \rho c_p v \frac{\partial T}{\partial y} = \frac{\partial}{\partial y}(\kappa \frac{\partial T}{\partial y})$$

This is the energy boundary layer equation. Thus it has been shown that Eq.(35) is the local potential of the problem.

In order to proceed, the following velocity and temperature profiles are assumed:

$$\frac{u}{u_{\infty}} = \frac{3}{2} \frac{y}{\Delta} - \frac{1}{2} \left(\frac{y}{\Delta} \right)^3 \quad 0 \leq y \leq \Delta \quad (40)$$

$$\frac{T - T_w}{T_{\infty} - T_w} = \frac{3}{2} \left(\frac{y}{\Delta_t} \right) - \frac{1}{2} \left(\frac{y}{\Delta_t} \right)^3 \quad 0 \leq y \leq \Delta_t \quad (41)$$

These satisfy the boundary conditions

$$\left. \begin{array}{l} u = 0 \\ \frac{\partial^2 u}{\partial y^2} = 0 \\ T = T_w \\ \frac{\partial^2 T}{\partial y^2} = 0 \end{array} \right\} \text{ at } y = 0$$

$$\left. \begin{array}{l} u = u_{\infty} \\ \frac{\partial u}{\partial y} = 0 \end{array} \right\} \text{ at } y = \Delta$$

$$\left. \begin{array}{l} T = T_{\infty} \\ \frac{\partial T}{\partial y} = 0 \end{array} \right\} \text{ at } y = \Delta_t$$

In this study, only the case where $\Delta > \Delta_t$ is considered, where Δ, Δ_t are functions of x to be determined*.

For simplicity, the viscosity and thermal conductivity are chosen as linear functions of temperature

$$\text{i.e., } \frac{\mu}{\mu_\infty} = 1 + A\theta \quad (42)$$

$$\frac{K}{K_\infty} = 1 + B\theta \quad (43)$$

$$\text{where } \theta = \frac{T - T_\infty}{T_w - T_\infty}$$

In the above expressions, a positive A and a negative B indicate cooling of the fluid, while a negative A and a positive B indicate heating of the fluid.

These expressions for velocity, temperature, viscosity and conductivity are substituted into Eq.(35). By imposing certain variational arguments, the following two coupled equations are obtained. (The rather tedious calculation involved in making this step are given in Appendix I).

$$2\gamma^{*2} \Delta^{*2} \gamma^{*'} = a_5 - \gamma^{*3} \Delta^{*} \Delta^{*'} \quad (44)$$

$$-a_1 \Delta^{*} \Delta^{*'} = -a_2 - a_3 \gamma^{*} + a_4 \gamma^{*3} \quad (45)$$

where

$$a_1 = \frac{21}{320} Re_\infty$$

$$a_2 = \frac{3}{5}$$

* The approach here is similar to that of the Ritz Method.

$$a_3 = \frac{9}{4} A \cdot \frac{3}{8}$$

$$a_4 = \frac{9}{4} A \cdot \frac{1}{6}$$

$$a_5 = \frac{128}{9 \cdot Pr_\infty \cdot Re_\infty} \left(\frac{3}{5} + \frac{177}{320} B \right)$$

Here $Y = \Delta_t / \Delta$ and dimensionless quantities x^* , Δ^* and Δ_t^* are defined as

$$x^* = x/l$$

$$\Delta^* = \Delta/l$$

$$\Delta_t^* = \Delta_t/l$$

The analog/hybrid computer solution of Eqs.(44) and (45) is discussed in the next chapter.

CHAPTER IV

ANALOG/HYBRID COMPUTER SOLUTION TO THE PROBLEM

Note first that computer solution of the problem of simultaneous development of momentum and thermal boundary layer is intractable. This is because the leading edge of the flat plate is a singular point, i.e., at this point $Y = 0/0$ and hence is undefined. In order to circumvent this difficulty, an unheated starting length is assumed. This length is denoted by x_0 (see Fig. 1) and its dimensionless counterpart is $x_0^* = x_0/l$

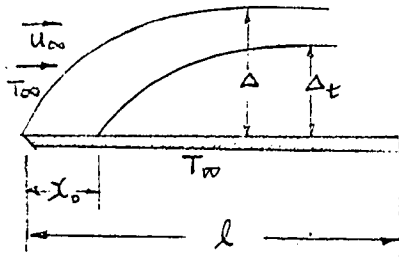


Fig. 1 Schematic Diagram of the Problem

In the initial phases of the study a value of 0.1 was used for x_0^* . The effects of smaller values were later investigated.

A magnitude scaled analog computer circuit suitable for generating the solution to Eq.(44) and (45) is given in Fig. 2. As noted on this circuit, the relation $x^* = 0.1 t$, is assumed between the problem independent variable x^* and the computer independent variable t . As a consequence of this assumption, the solution time on the computer is 10 seconds (i.e., in 10 seconds x^* covers its range of 0 to 1). All computational work for this problem was carried out on the

EAI-680 analog/hybrid computer facility of the Analysis Laboratory,
National Research Council, Ottawa, Canada,

As indicated earlier, the particular problem considered is associated with an unheated starting length given by $x_0^* = 0.1$. The implementation of this requires that the output of Amp 45 (Fig. 2) remain at zero for x^* in the interval $(0, x_0^*)$. This is achieved by generating a logic signal L whose sense changes from FALSE to TRUE at $x^* = x_0^*$ (integrator Amp 20 and comparator C 19 of Fig. 3). This logic signal in turn is used to control switch S1 (Fig. 3) and thereby disconnect the input to Amp 45 when $0 \leq x^* \leq x_0^*$. This then achieves the desired effect.

The purpose of the additional switch S2 in the circuit of Fig. 3 is to disable the division operation in the interval $0 \leq x^* \leq x_0^*$. The disabling of this operation is necessary to avoid overloading the division unit in this interval where the variable Y is held at zero.

The ranges of the parameters Re_∞ , Pr_∞ , A and B used in this study are shown in Table I. Viscosity coefficients A and conductivity coefficients B have been selected based on two criteria.

- (1) physical consideration, and
- (2) the value of $Y(\Delta_t/\Delta)$ remaining less than 1 over the whole solution.

In general, for incompressible fluids, the thermal conductivity is slightly dependent on temperature. In this study, conductivity is assumed slightly increasing with temperature. On the other hand,

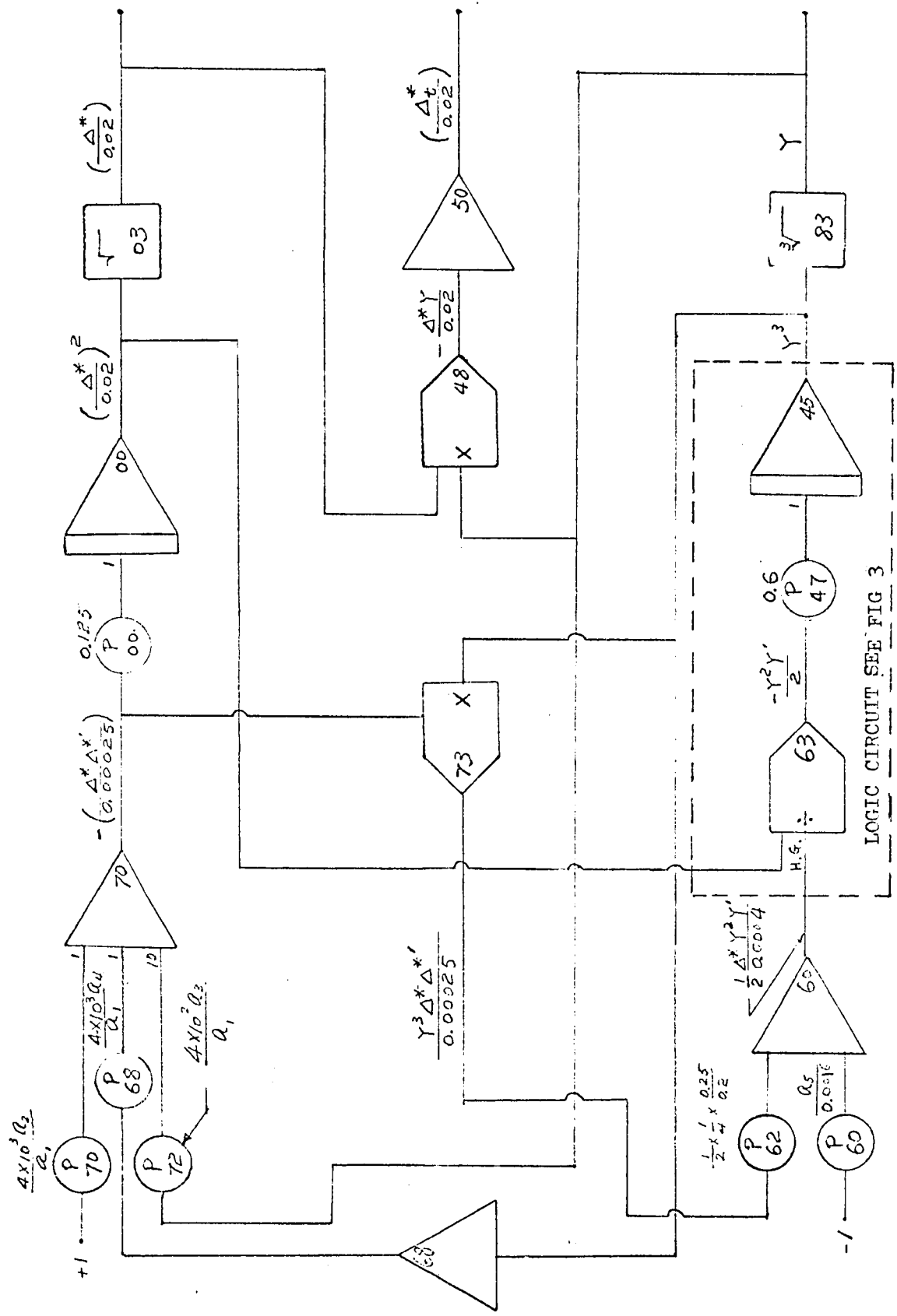


Fig. 2 Magnitude Scaled Analog Computer Circuit Diagram of the Problem

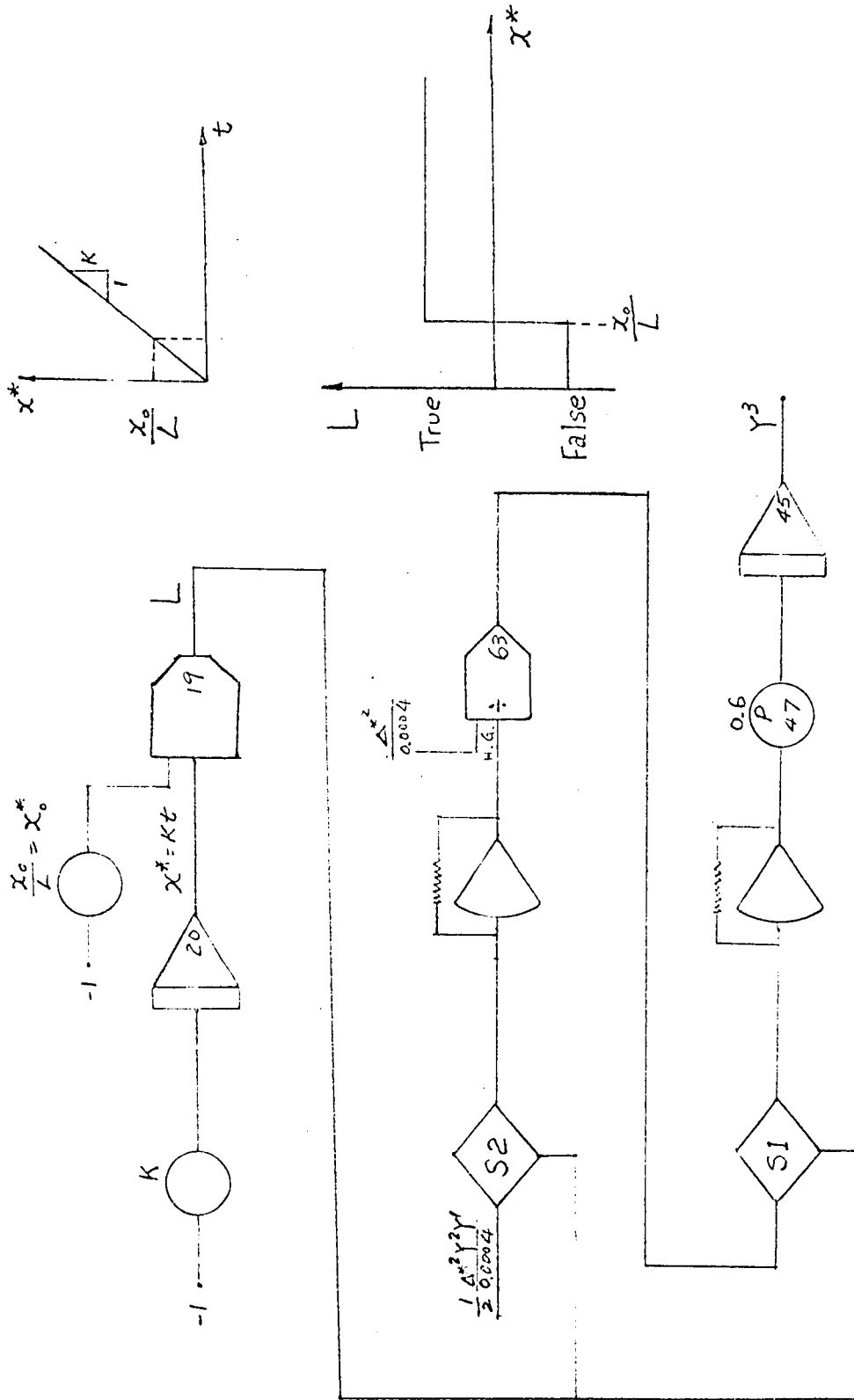


Fig. 3 The Logic Circuit of the Problem

Table I. : Parameters for Run Number 1 to Run number 29

with $x_o^*=0.1$

Re_∞ No.	Pr_∞ No.	A	B	Run No.
80,000	2.0	1.0	0.0	1
		0.7	0.0	2
		0.3	0.0	3
		0.0	0.0	4
		-0.5	0.0	5
		-0.5	0.1	6
	5.0	1.5	-0.1	7
		0.5	-0.2	8
	10.0	1.5	-0.1	9
		1.0	-0.2	10
		0.5	-0.1	11
200,000	2.0	1.0	-0.2	12
		0.1	-0.1	13
		0.0	0.0	14
		-0.5	0.0	15
		-0.5	0.2	16
	5.0	1.5	-0.2	17
		1.5	-0.1	18
		0.5	-0.1	19
	10.0	2.0	-0.2	20
		0.8	-0.2	21
0.3		-0.1	22	
320,000	2.0	1.0	-0.2	23
		0.0	0.0	24
	5.0	2.0	-0.2	25
		1.5	-0.1	26
		0.5	-0.1	27
	10.0	1.5	-0.2	28
		0.5	-0.1	29

the viscosity is always rapidly decreasing with temperature.

For $x_o^* = 0.1$, 29 runs were performed for different combination of parameters as shown in Table I. The results are shown in Fig. 4-22. For values of $x_o^* < 0.1$ the scale factor associated with the quantity $Y'Y^2$ was increased from 2 to accommodate the larger values of $Y'Y^2$. Four more runs were obtained, one for $x_o^* = 0.00286$, one for $x_o^* = 0.000572$ and two for $x_o^* = 0.01$ as shown in Table II. The results are shown in Fig. 23 to Fig. 26. The runs with x_o^* set to 0.000572 closely approximate the case of simultaneous development of momentum and thermal boundary layers. Notice that Δ^* and Δ_t^* are direct outputs of the computer. Local Nusselt number, Nu_x , and local friction factor, f , were calculated according to the following relations i.e. Eqs. (50) and (51).

From Fourier's law

$$\begin{aligned} q &= -K \left(\frac{\partial \theta}{\partial y} \right)_{y=0} \\ &= -\frac{3K}{2} \frac{1}{\Delta_t} (T_\infty - T_w) \end{aligned} \tag{46}$$

$$\text{also, } q = h (T_w - T_\infty) \tag{47}$$

From Eqs. (46) and (47), it follows

$$h = \frac{3K}{2} \frac{1}{\Delta_t} \tag{48}$$

Rearranging the terms in Eq. (48), one obtains

$$Nu_x = \frac{hx}{K} = \frac{3}{2} \frac{x}{\Delta_t} \tag{49}$$

$$\text{or } Nu_{x^*} = \frac{3}{2} \frac{x^*}{\Delta_t^*} \tag{50}$$

Table II: Parameters for Run Number 30 to Run Number 33
with $Re_{\infty} = 80,000$

Pr_{∞} No.	x_o^*	A	B	Run No.
2.0	0.01	0.0	0.0	30
	0.00286	1.0	-0.2	31
10.0	0.01	1.0	-0.2	32
	0.000572	1.0	-0.2	33

By definition

$$\begin{aligned} f &= \frac{\mu_w \left(\frac{\partial u}{\partial y} \right)_{y=0}}{\frac{\rho u_\infty^2}{2}} \\ &= \frac{3 \nu_\infty (1+A)}{u_\infty \cdot \Delta} \end{aligned}$$

or $f = \frac{3(1+A)}{Re_\infty \cdot \Delta^*}$ (51)

The results are presented and discussed in the next chapter.

CHAPTER V

RESULTS AND DISCUSSIONS

For the case of $x_o^* = 0.1$, 29 runs were obtained. The results are shown from Fig. 4 to Fig. 22. These include momentum and thermal boundary layer thickness, local Nusselt number and local friction factor. The results Fig 4-10, Fig. 11-16, Fig. 17-22 correspond respectively to values of Re_∞ of 80,000, 200,000, and 320,000. Each group has three subgroups corresponding to parameter values of $Pr_\infty = 2, 5$ and 10. Figs. 23 and 24 show the case of $x_o^* = 0.01$ at $Re_\infty = 80,000$. Figs. 25 and 26 show the case of $x_o^* = 0.00572$ at $Re_\infty = 80,000$. A comparison has been made for the case of constant properties (i.e. $A=0.0, B=0.0$) between the results from this study and Blasius' exact solution as shown in Fig. 27-30.

For constant properties case Eqs. (44) and (45) become

$$2Y^* \Delta^{*2} Y^{*'} = \frac{128}{7 \cdot Pr_\infty \cdot Re_\infty} \left(\frac{3}{5}\right) - Y^{*3} \Delta^* \Delta^{*'} \quad (52)$$

$$\frac{21}{320} Re_\infty \Delta^* \Delta^{*'} = \frac{3}{5} \quad (53)$$

Eq. (53) can be solved as

$$\Delta = 4.26 \sqrt{\frac{Ux}{u_\infty}} \quad (54)$$

or

$$\frac{\Delta}{x} = \frac{4.26}{\sqrt{Re_x}} \quad (55)$$

Substituting Eq. (53) into Eq. (52), one obtains

$$\frac{15}{14} \frac{d}{dx} \left(Y^3 + 4xY^2 \frac{dY}{dx} \right) = 1 \quad (56)$$

The solution of Eq. (56) ⁽¹⁶⁾ is

$$Y = \frac{1}{1.024 \sqrt[3]{Pr}} \sqrt[3]{1 - \left(\frac{x_0}{x}\right)^{\frac{3}{4}}} \quad (57)$$

Now
$$\Delta_t = \Delta \cdot Y = \frac{4.16x}{\sqrt{Re_x} \sqrt[3]{Pr}} \sqrt[3]{1 - \left(\frac{x_0}{x}\right)^{\frac{3}{4}}} \quad (58)$$

Comparing Eq. (55) with Blasius' exact solution ⁽¹⁶⁾, the momentum boundary layer thickness is in error by 14.8 percents (see also Fig. 27) Because of the similarity between Y as given in Eq. (57) and the exact solution, it follows that the error in the thermal boundary layer thickness is also in the order of 14.8 percent.

From Eq. (51)

$$f = \frac{3}{Re_\infty \cdot \Delta^*} \quad (A=0.0, \text{ for constant properties}) \quad (59)$$

Substituting Eq. (55) into Eq. (59), one obtains

$$f = 0.704 \frac{1}{\sqrt{Re_x}} \quad (60)$$

The result is in error by 6 percent relative to the exact solution. (see also Fig. 28-30)

Substituting Eq. (58) into Eq. (49), one obtains

$$Nu_x = 0.36 \sqrt[3]{Pr} \sqrt{Re_x} \frac{1}{\sqrt[3]{1 - \left(\frac{x_0}{x}\right)^{\frac{3}{4}}}} \quad (61)$$

The result for the Nusselt number is in error by 8.34 percent relative to the exact solution. (see also Fig. 28-30)

For the case of flow over a flat plate with variable properties, there are no results available from the literature. Consequently, only some qualitative remarks can be made regarding the effects of

the parameters on the flow properties. Some of these qualitatively agree with results from the literature ⁽¹⁷⁾ for the flow through a pipe with variable properties.

(1) For constant conductivity coefficients B, Nusselt number increases with decreasing viscosity coefficients A for all x^* . But friction factor increases with increasing A for all x^* . This is due to the fact that positive A indicates cooling of liquid in the flow, thus corresponding increase in viscosity near the wall slows down the flow, results in a lower rate of heat transfer, relative to the constant properties case.

(2) The curves in Fig. 6 and Fig. 12 show that for constant viscosity coefficients A, Nusselt number decreases with increasing B (conductivity coefficient) for all x^* , while friction factor remains unchanged.

(3) For constant Reynolds number, Nusselt number increases with Prandtl number for all x^* .

(4) The curves in Fig. 26 show that for constant Reynolds number and constant Prandtl number, Nusselt number decreases with decreasing unheated starting length x_0^* for all x^* , while friction factor decreases slightly with decreasing unheated starting length x_0^* for all x^* .

It is believed that the results from this study can be improved by assuming a higher order velocity profile and a higher order temperature profile in the calculation process. Similarly, more realistic expressions of viscosity and thermal conductivity would be useful.

Nevertheless, the variational method combined with an analog/hybrid computer solution technique has been found to be very fruitful in the solution of this complex problem. It is felt that the great potential of such a combined technique still remains to be fully exploited to other engineering problems.

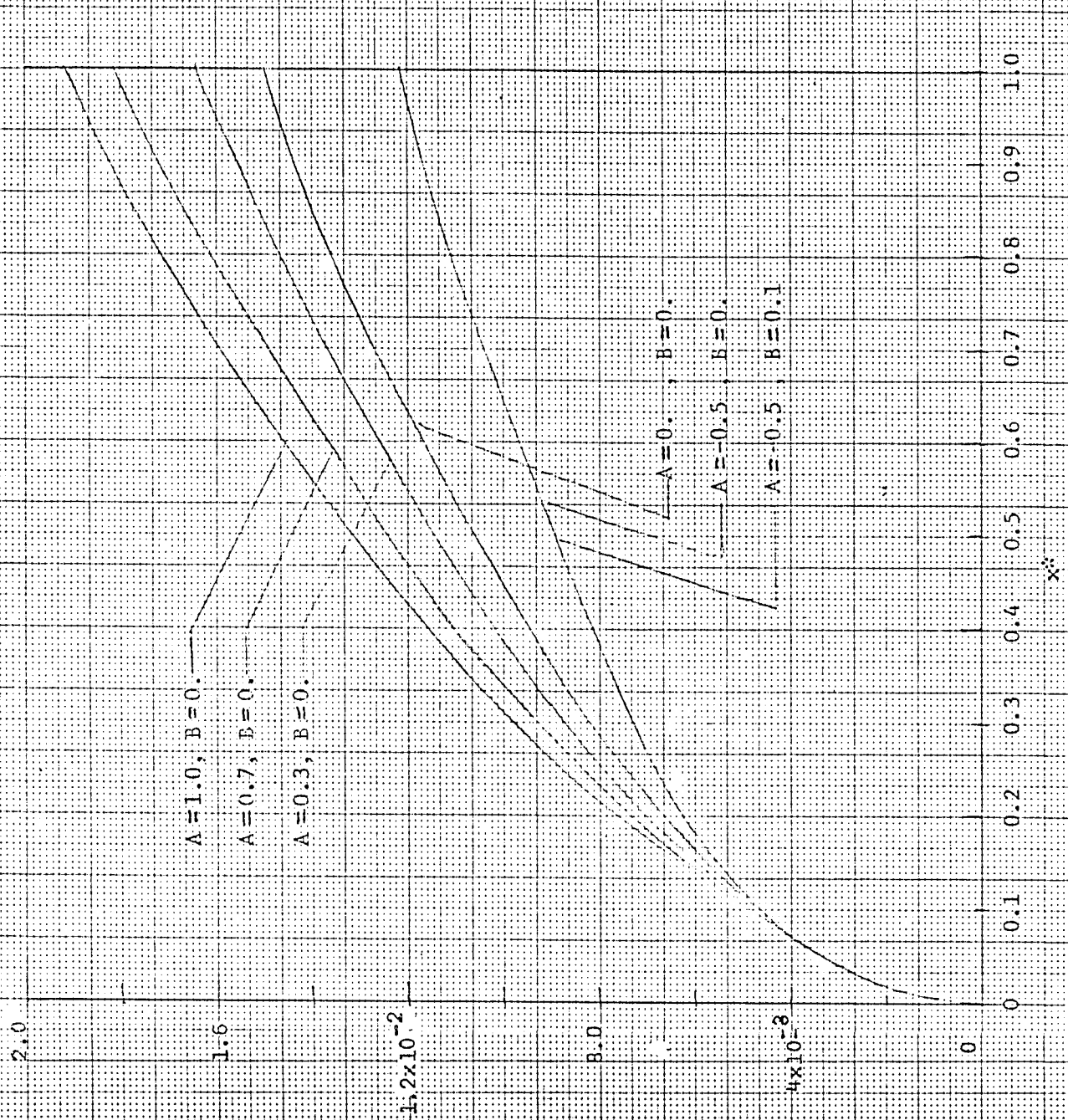


Fig. 4 Momentum Boundary layer Thickness for $Re_{\infty}=80,000, Pr_{\infty}=2$

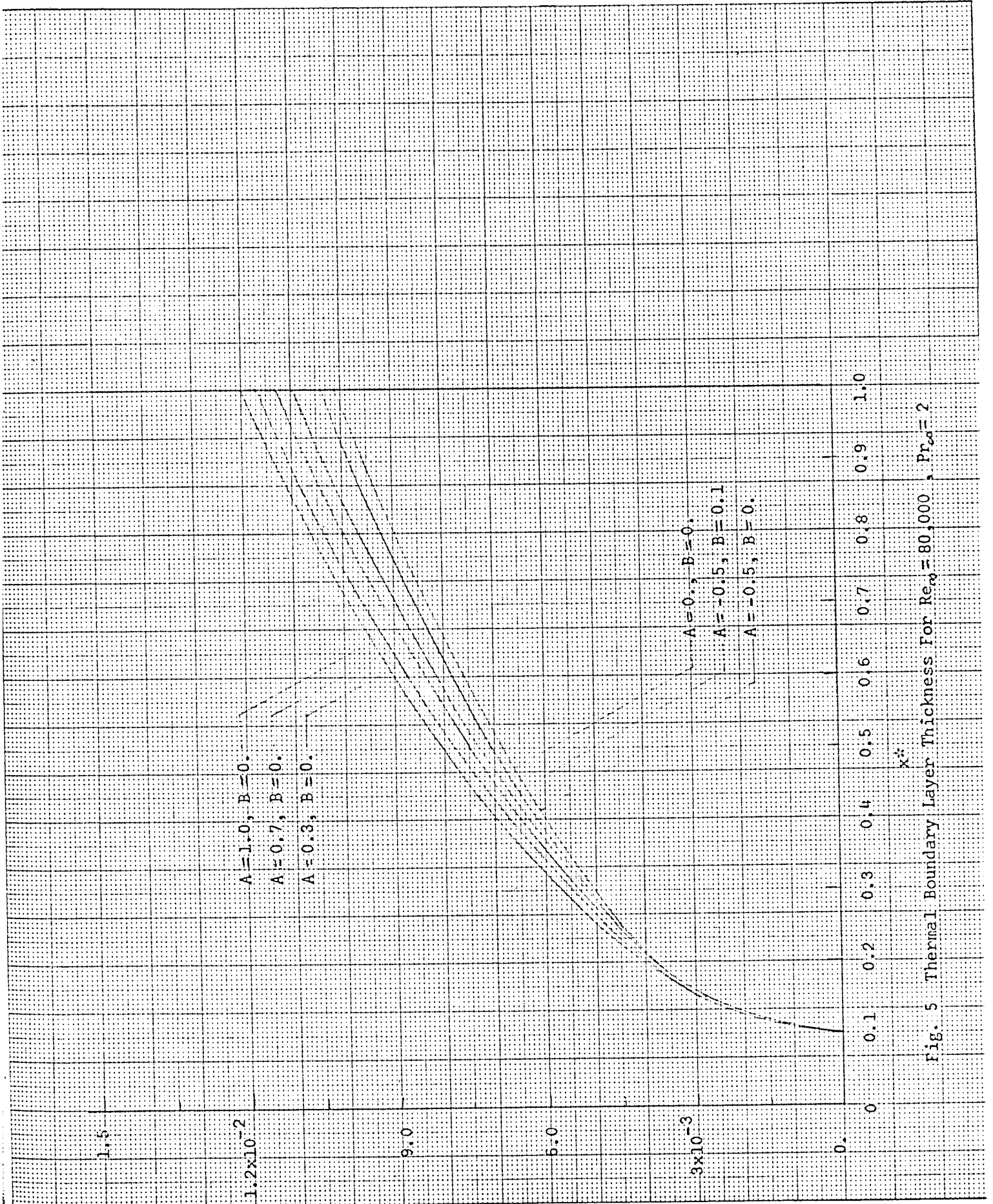


Fig. 5 Thermal Boundary Layer Thickness For $Re_{\infty} = 80,000, Pr_{\infty} = 2$

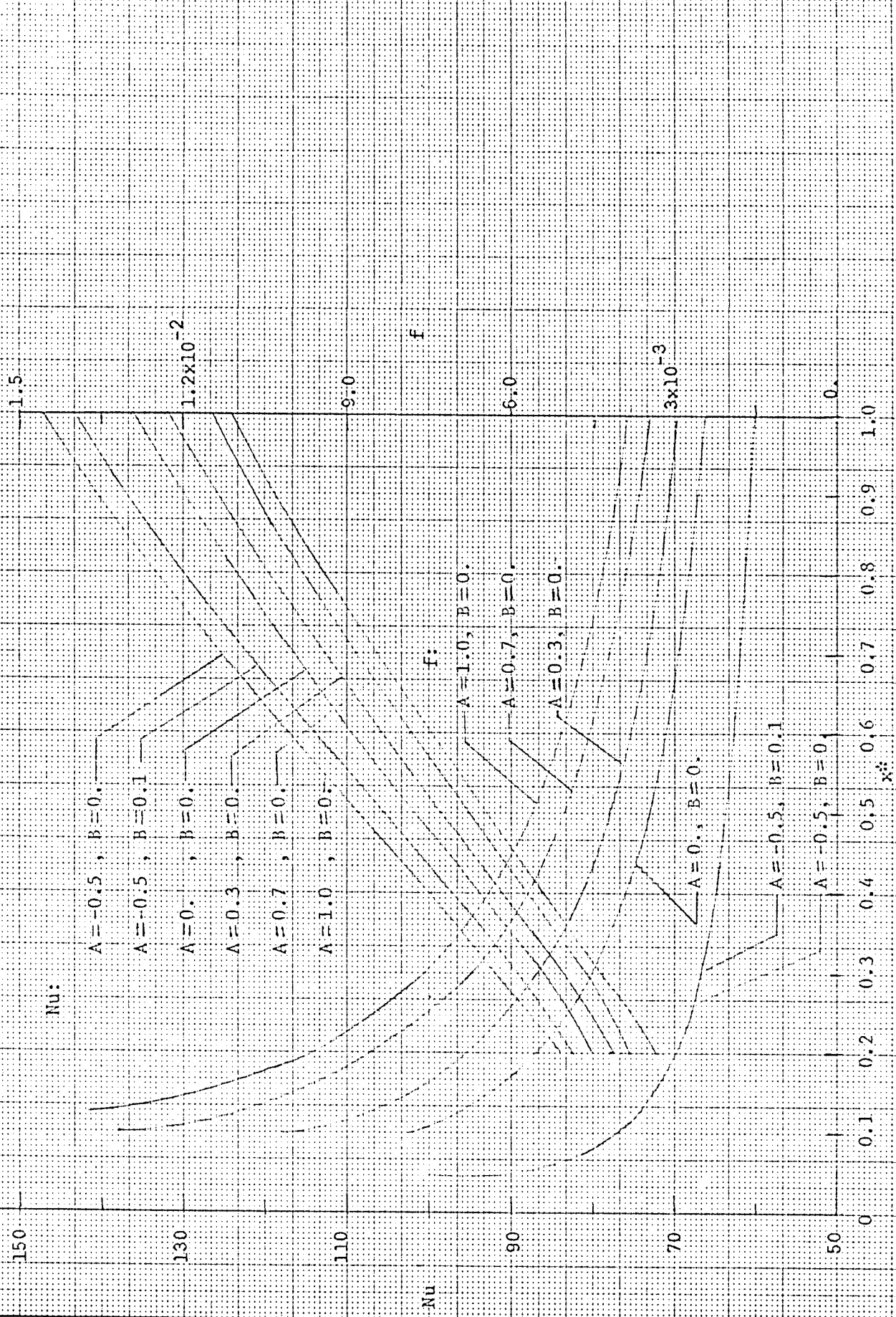


Fig. 6 Local Nusselt Number and Local Friction Factor For $Re_{\infty} = 80,000$, $Pr_{\infty} = 2$

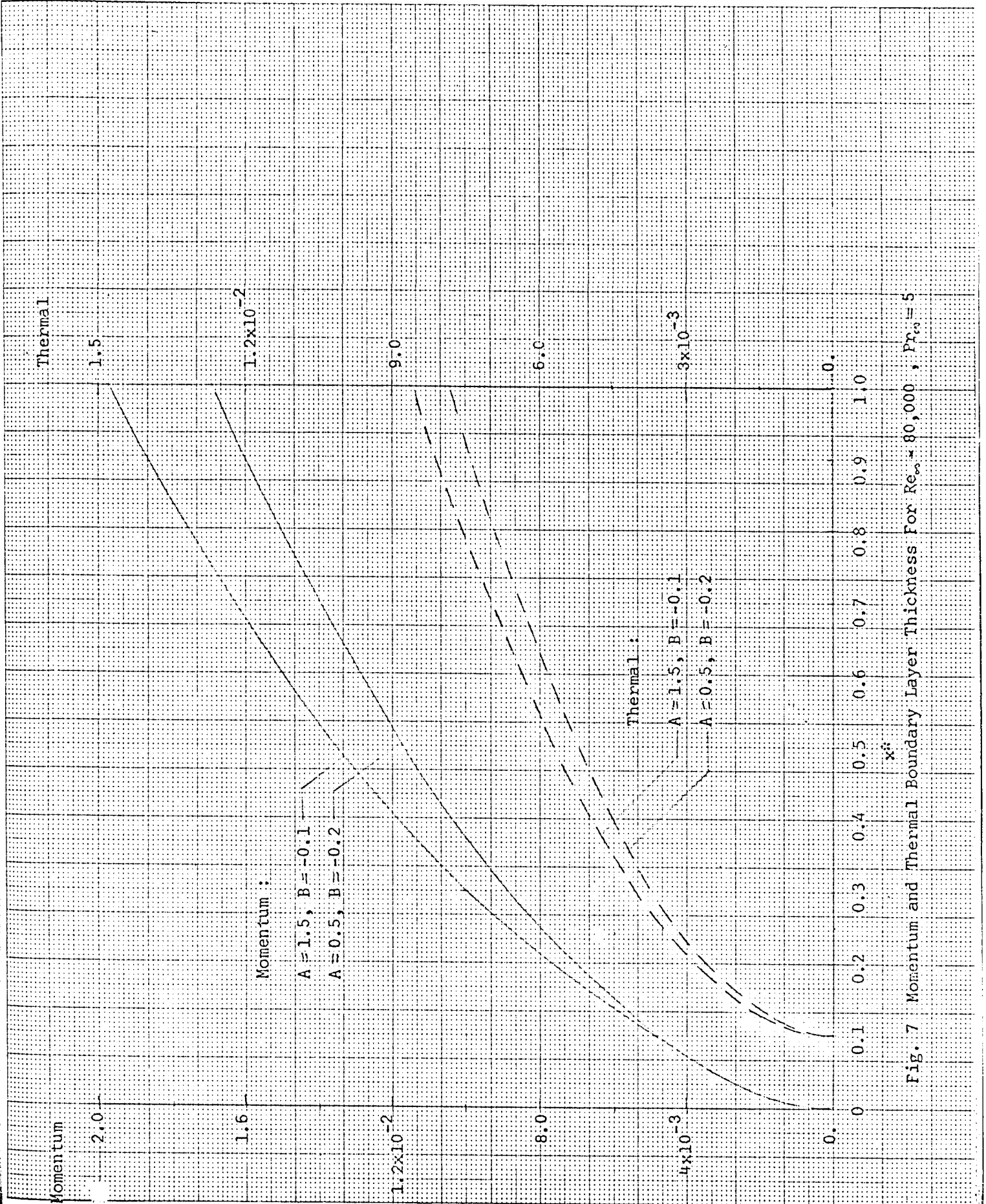


Fig. 7 Momentum and Thermal Boundary Layer Thickness For $Re_\infty = 80,000$, $Pr_{e3} = 5$

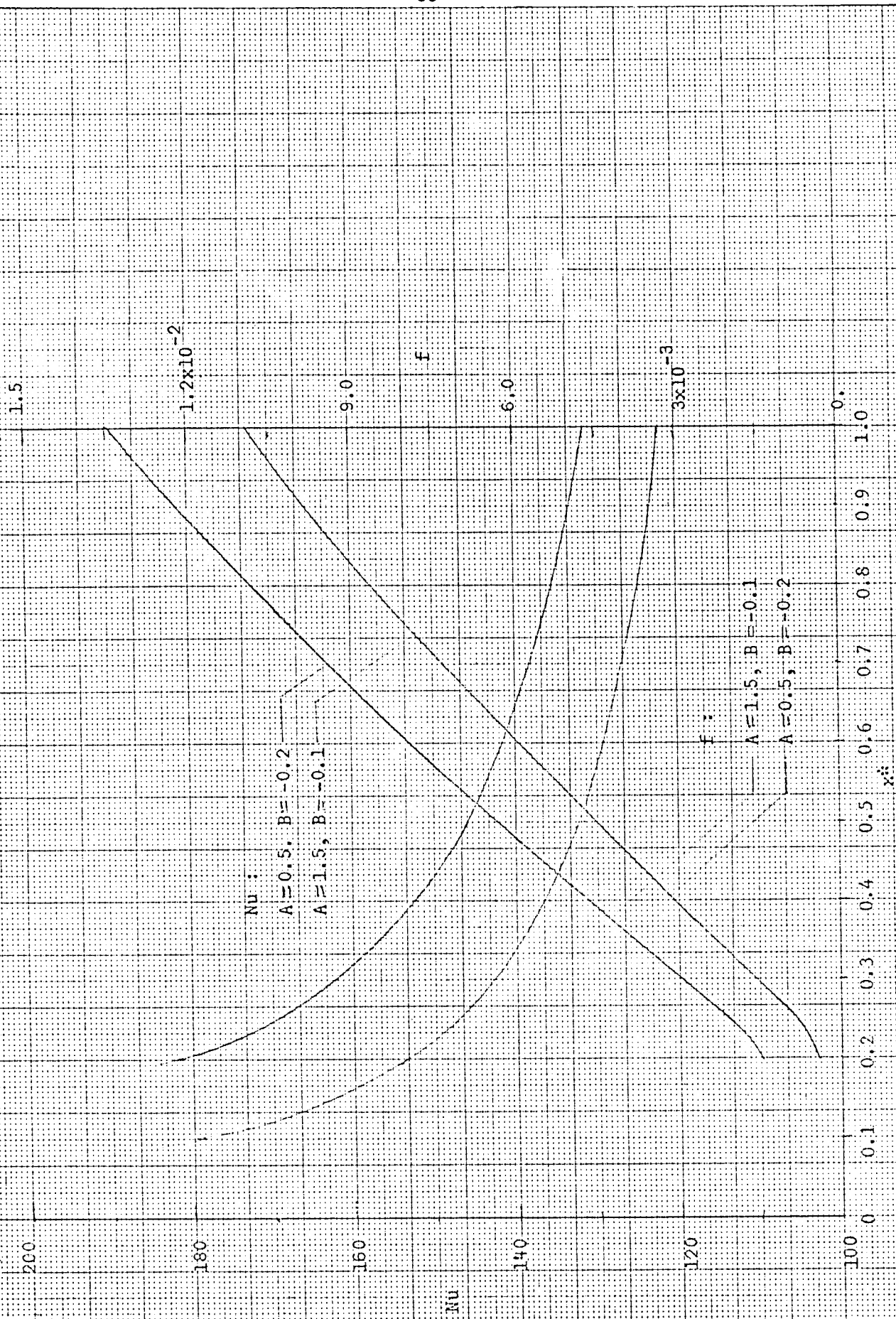


Fig. 8 Local Nusselt Number and Local Friction Factor For $Re_{\delta} = 80,000$, $Pr_{\delta} = 5$

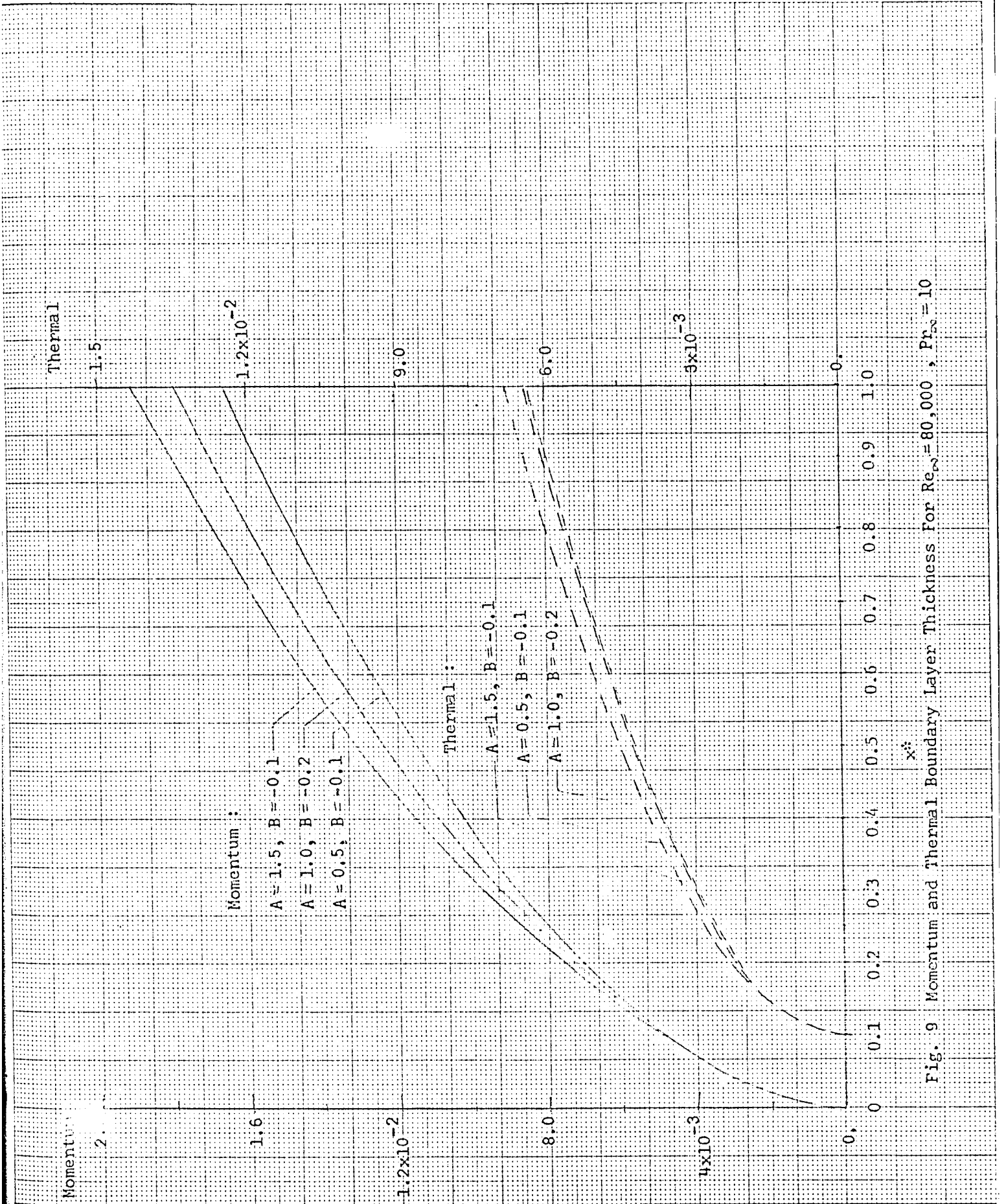


Fig. 9 Momentum and Thermal Boundary Layer Thickness For $Re_{\infty} = 80,000$, $Pr_{\infty} = 10$

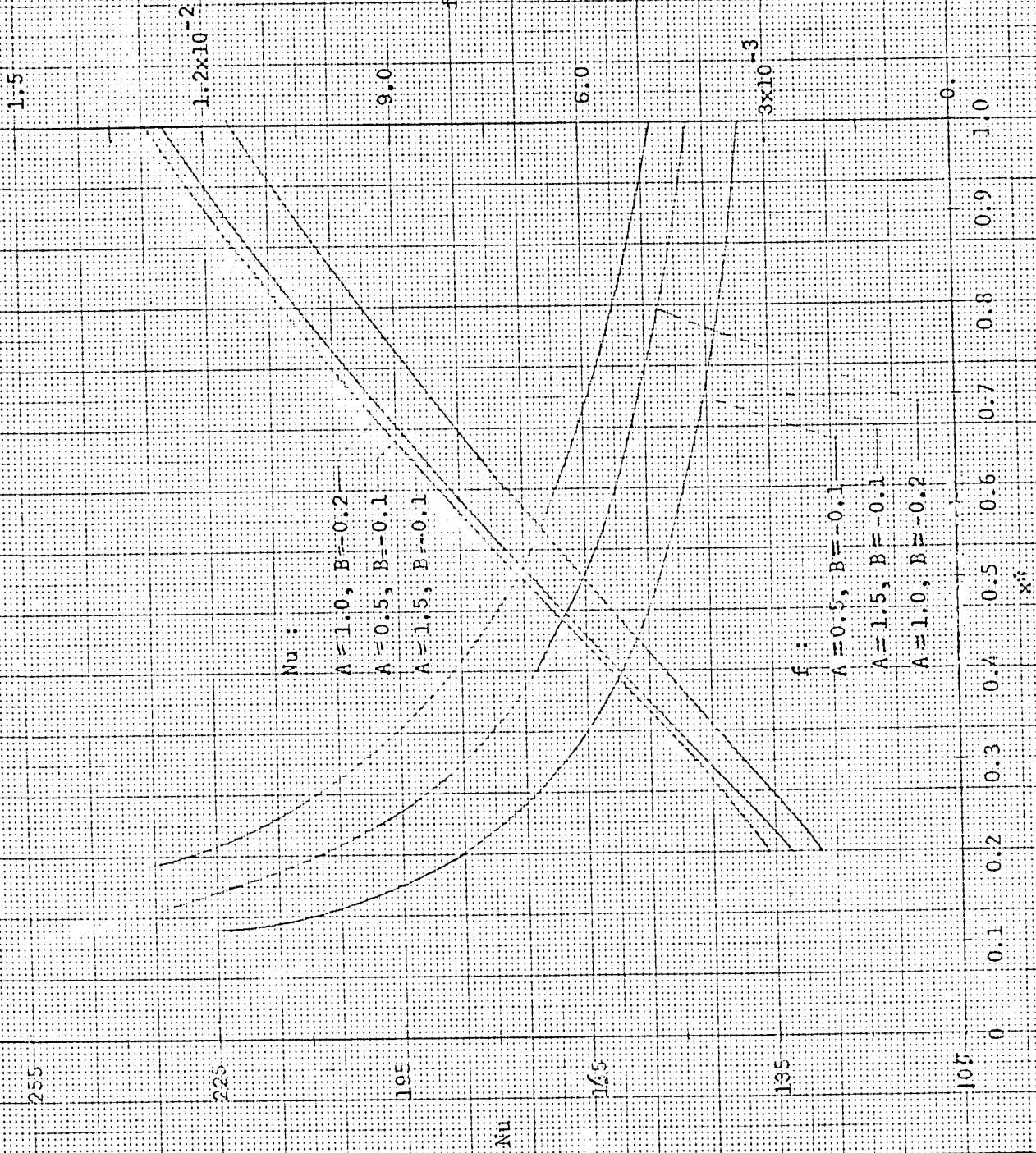


Fig. 10 Local Nusselt Number and Local Friction Factor For $Re_b = 80,000$, $Pr_b = 10$

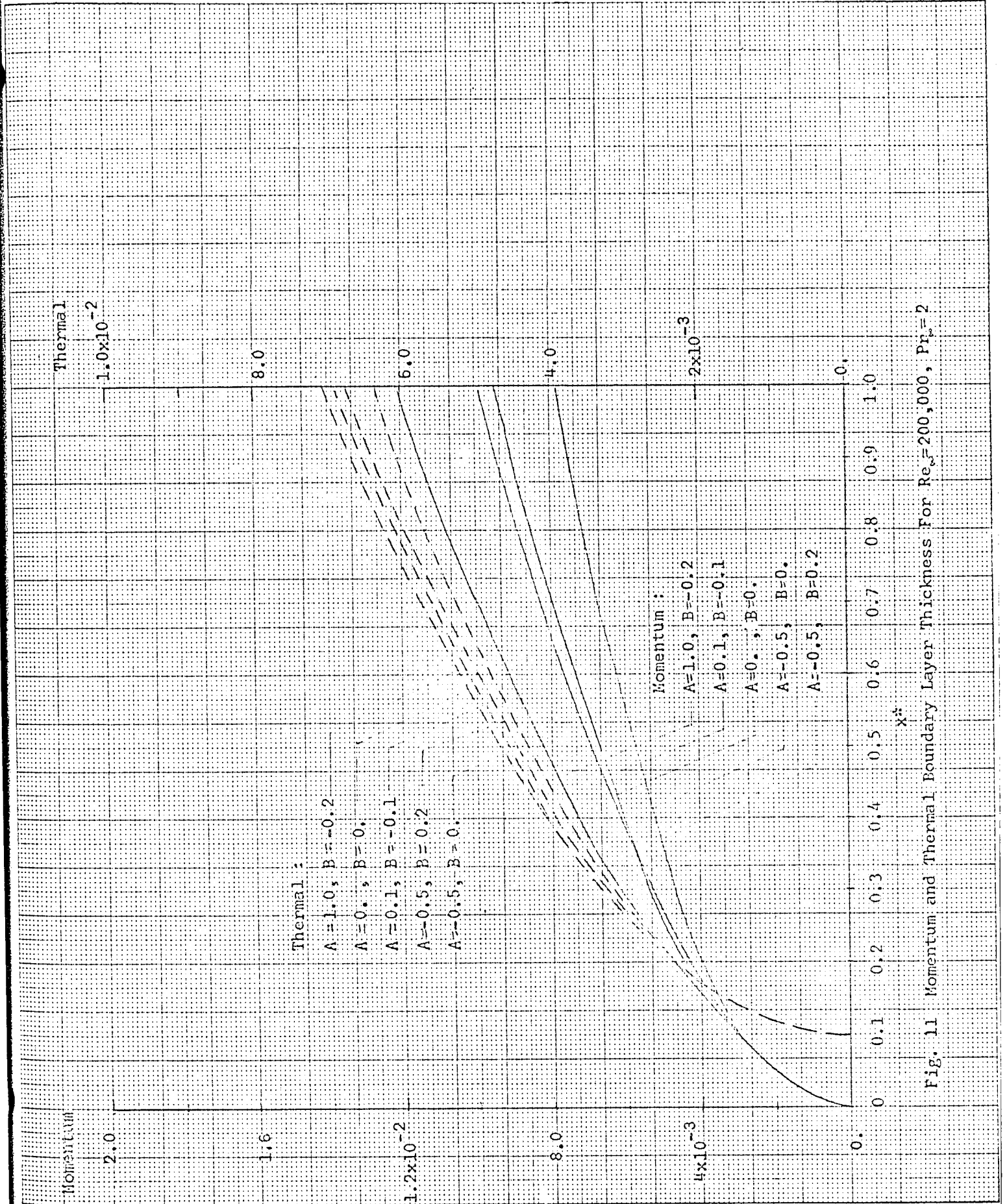


Fig. 11 Momentum and Thermal Boundary Layer Thickness For $Re_c=200,000, Pr=2$

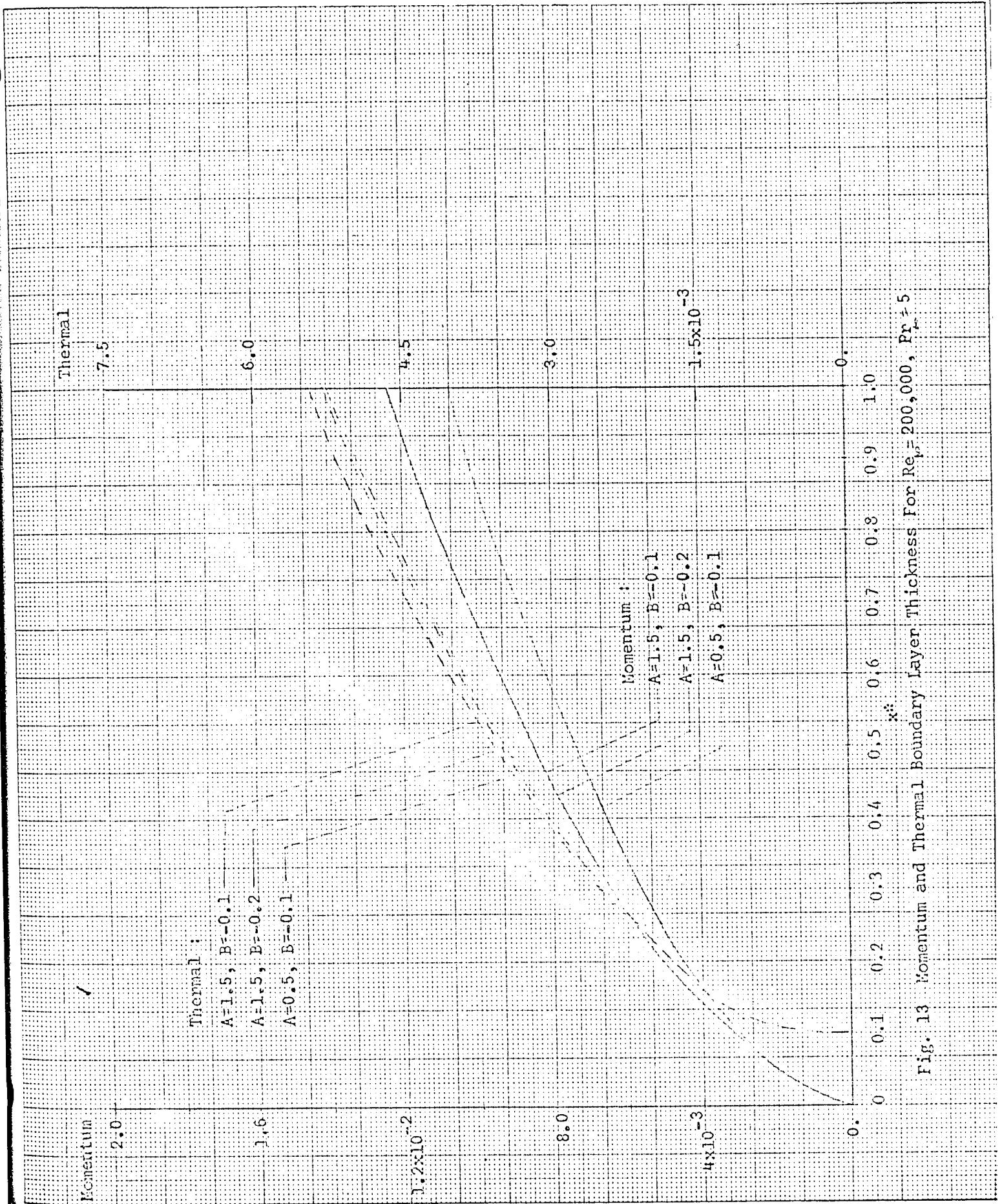


Fig. 13 Momentum and Thermal Boundary Layer Thickness For $Re_b = 200,000$, $Pr = 5$

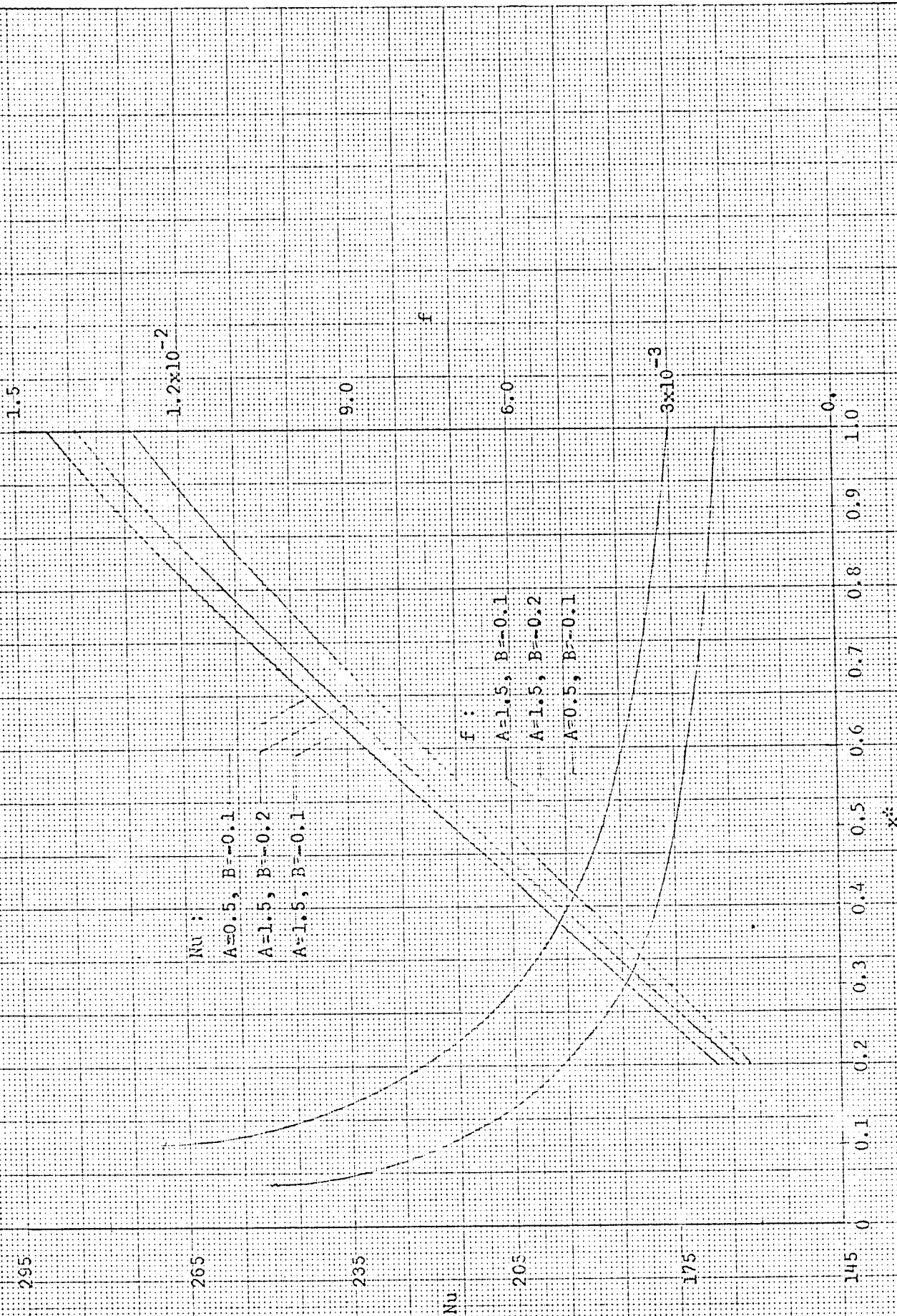


Fig. 14 Local Nusselt Number and Local Friction Factor For $Re = 200,000$, Pr. = 5

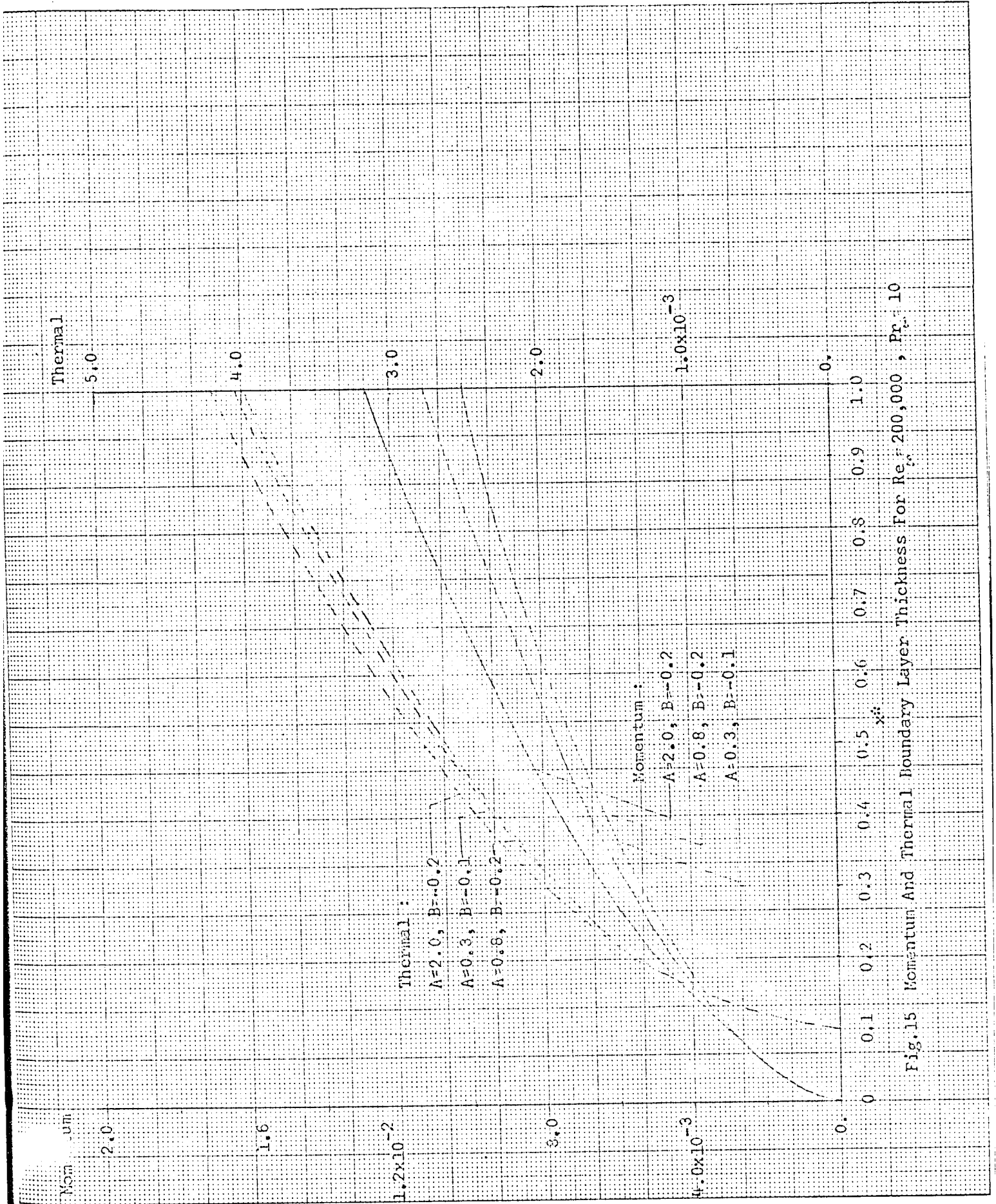


Fig. 15 Momentum And Thermal Boundary Layer Thickness for $Re = 200,000$, $Pr = 10$

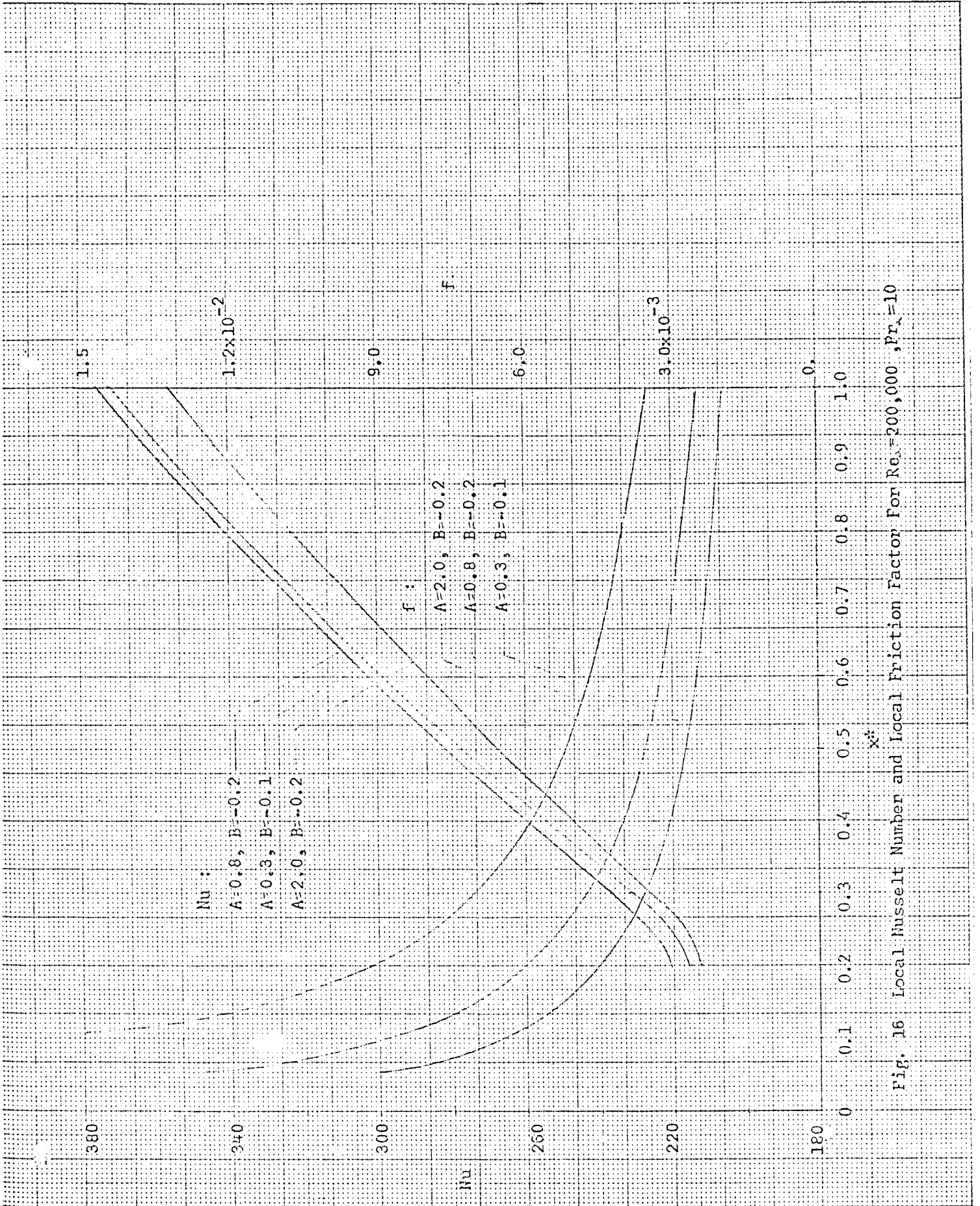


Fig. 16 Local Nusselt Number and Local Friction Factor For $Re_{\delta} = 200,000, Pr_{\delta} = 10$

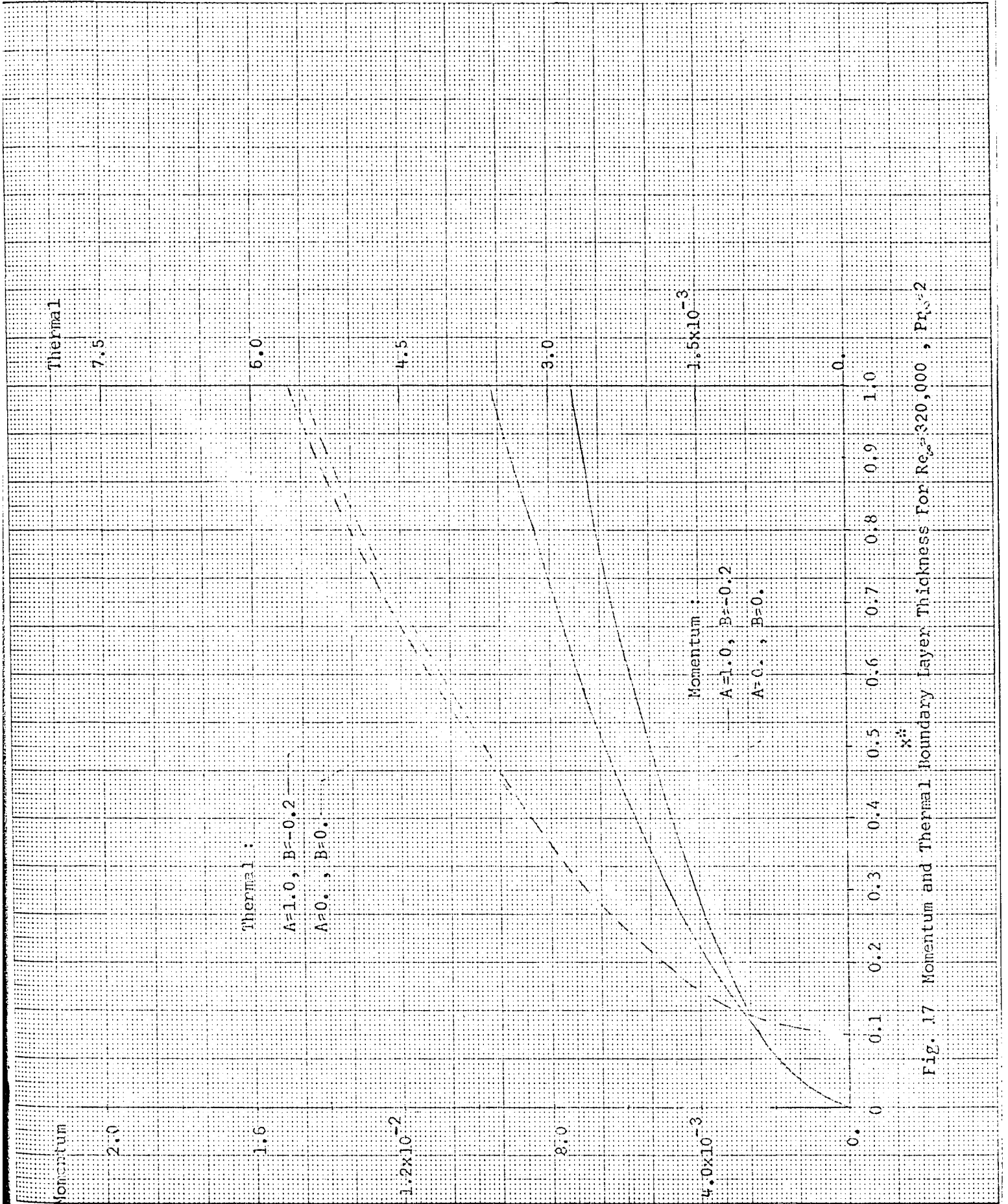


Fig. 17 Momentum and Thermal Boundary layer Thickness For $Re_\infty=320,000$, $Pr_\infty=2$

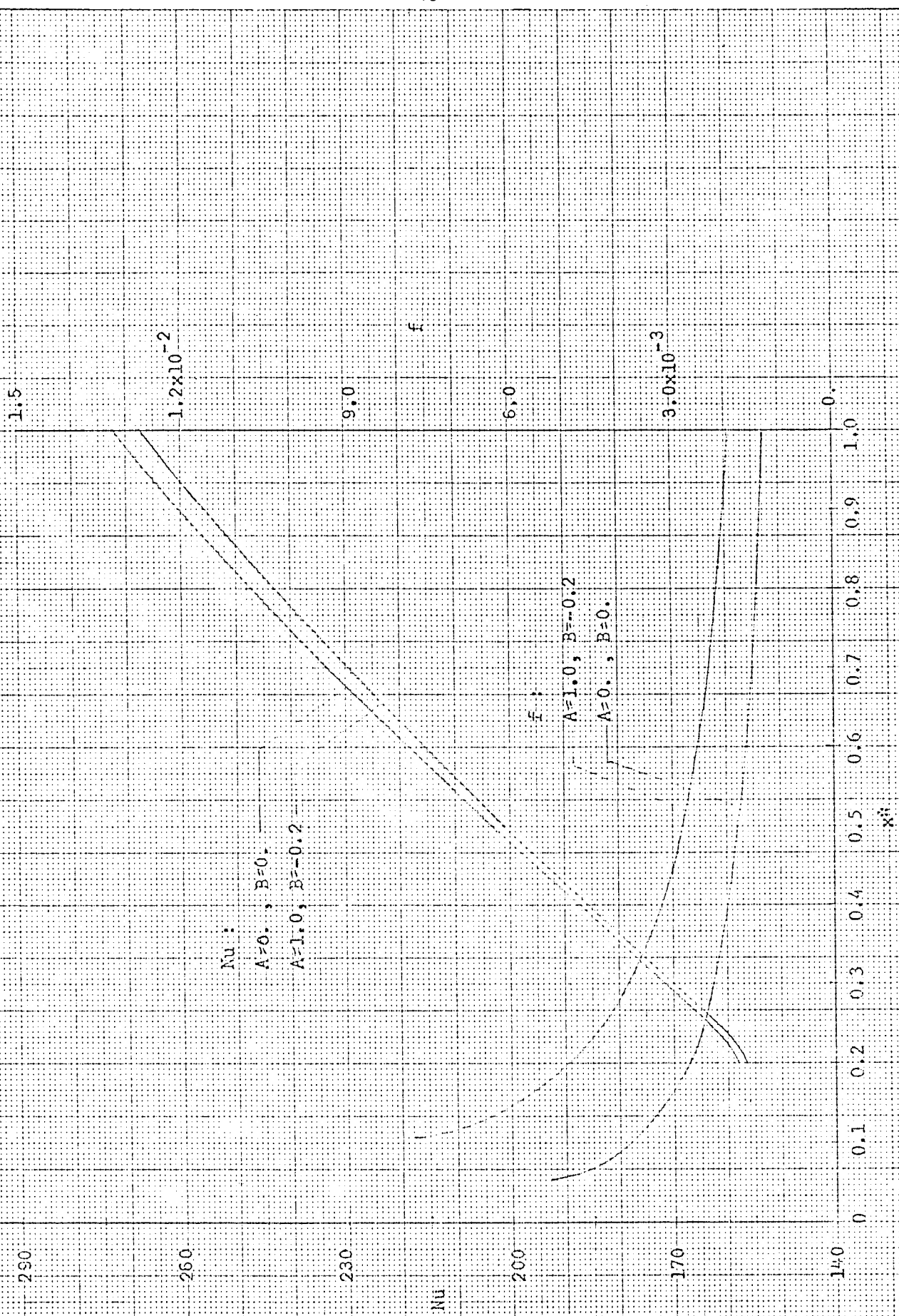


Fig. 18 Local Nusselt Number and Local Friction Factor For $Re_b = 320,000$, $Pr = 2$

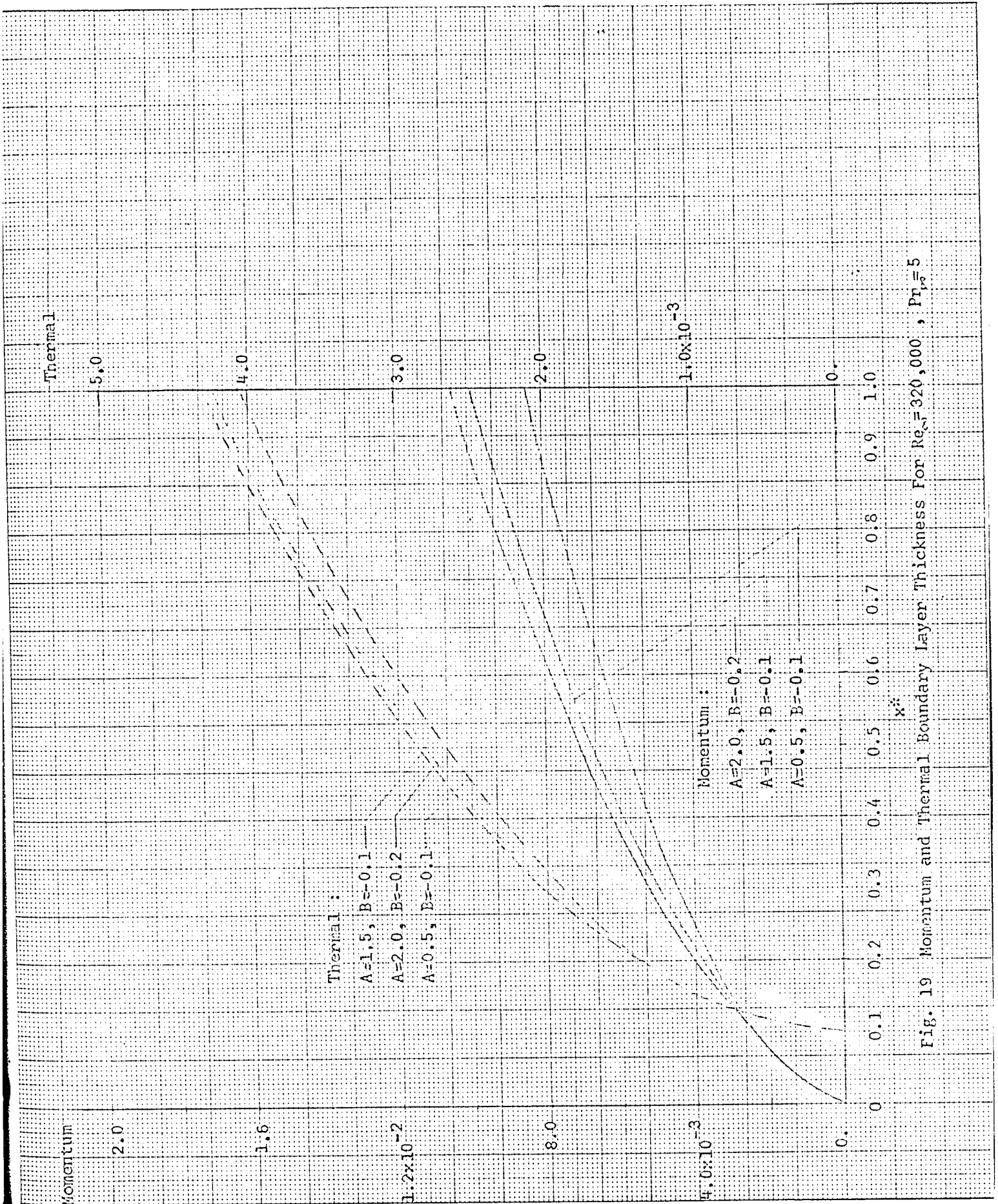


Fig. 19 Momentum and Thermal Boundary Layer Thickness For $Re_{\infty} = 320,000$, $Pr_{\infty} = 5$

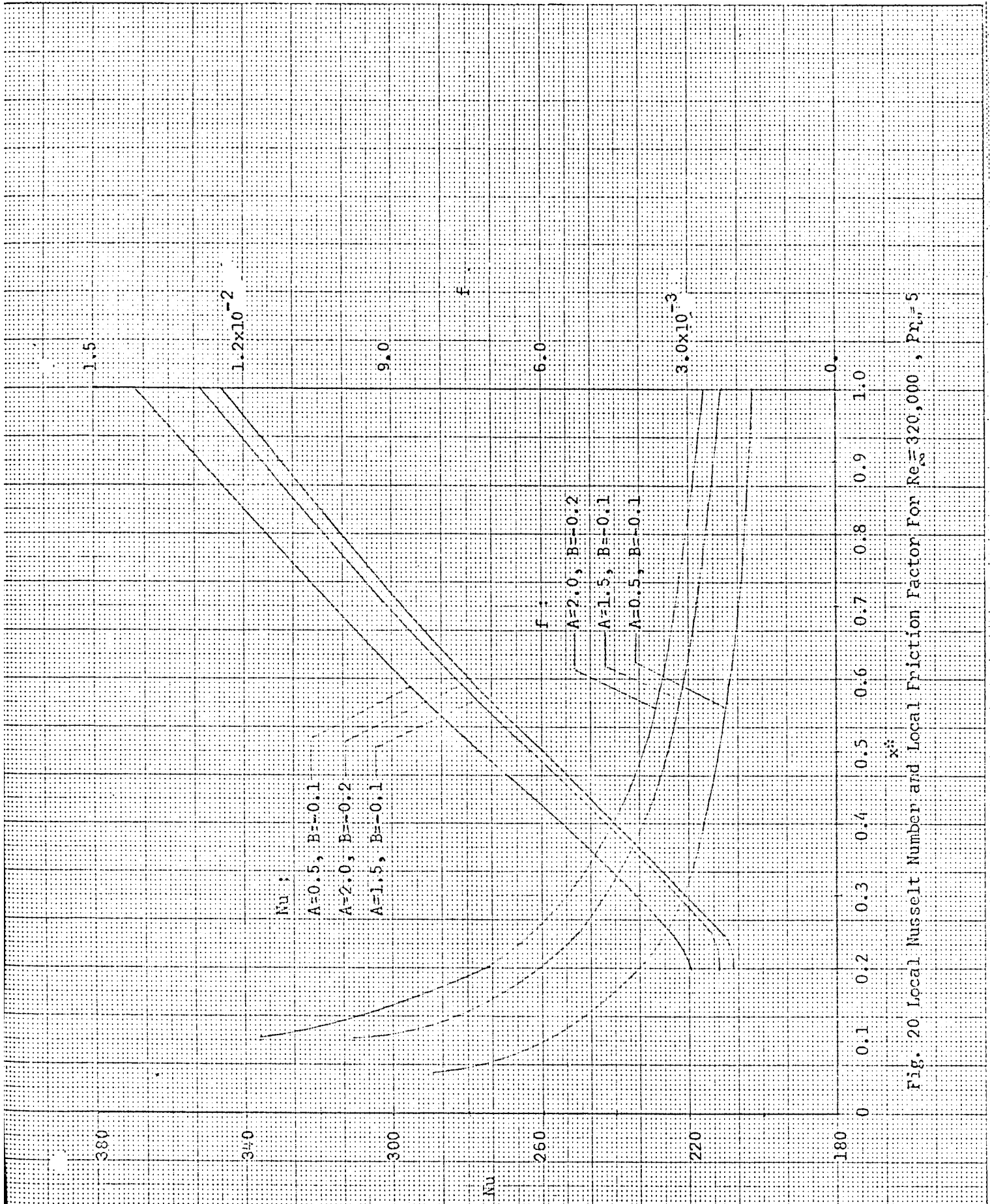


Fig. 20 Local Nusselt Number and Local Friction Factor For $Re = 320,000$, $Pr = 5$

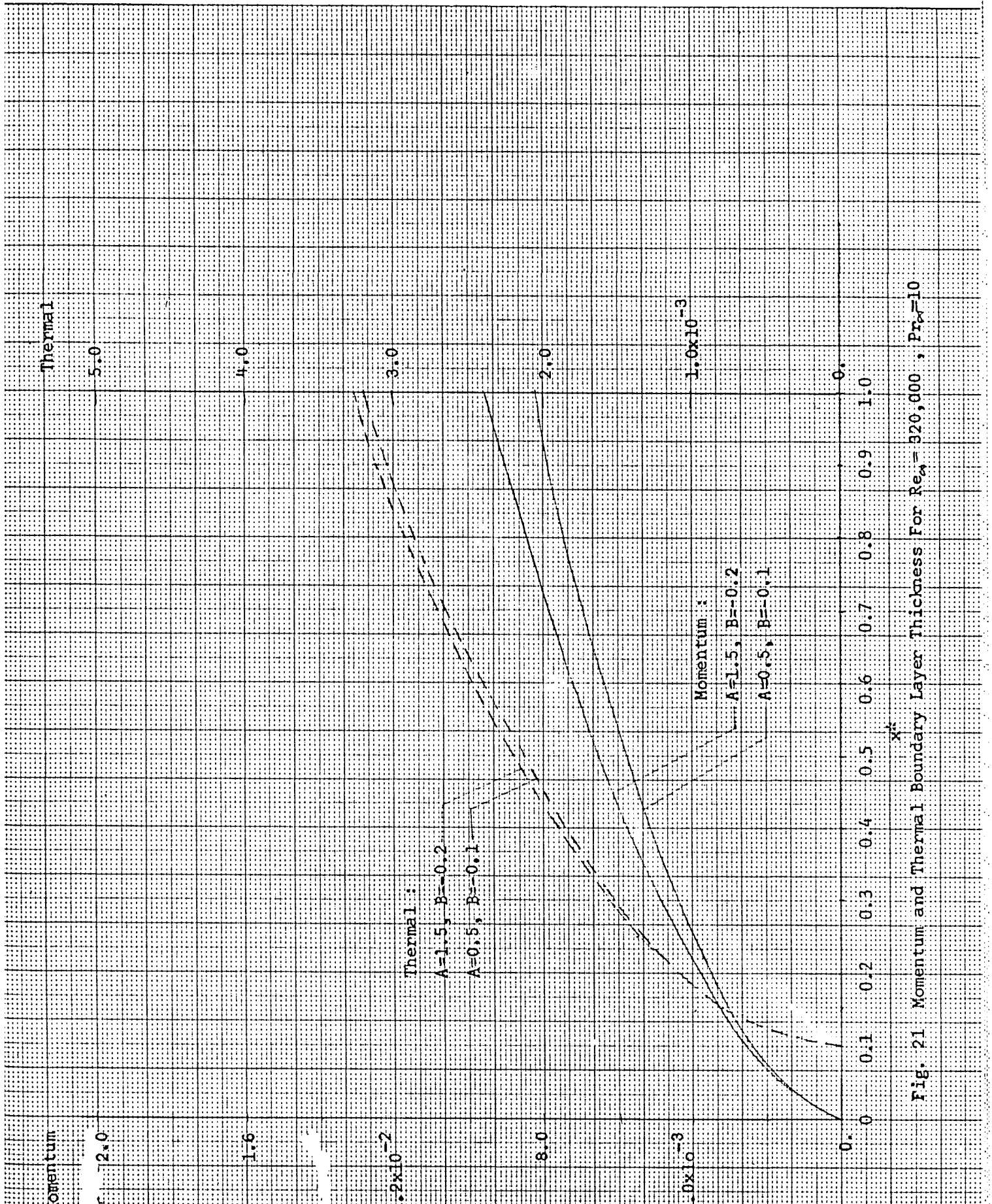


Fig. 21 Momentum and Thermal Boundary Layer Thickness For $Re_x = 320,000$, $Pr = 10$

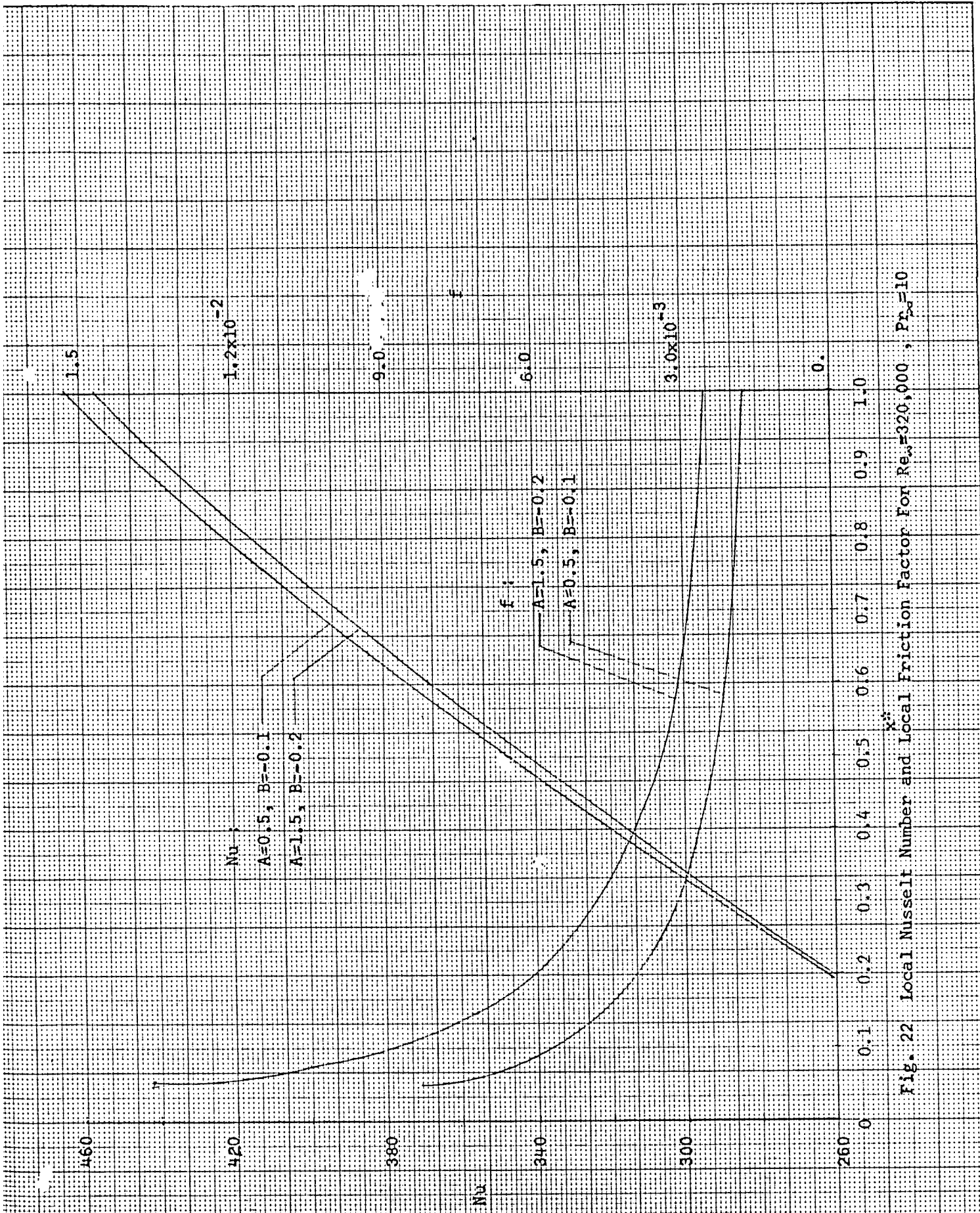


Fig. 22 Local Nusselt Number and Local Friction Factor For $Re_s = 320,000$, $Pr_s = 10$

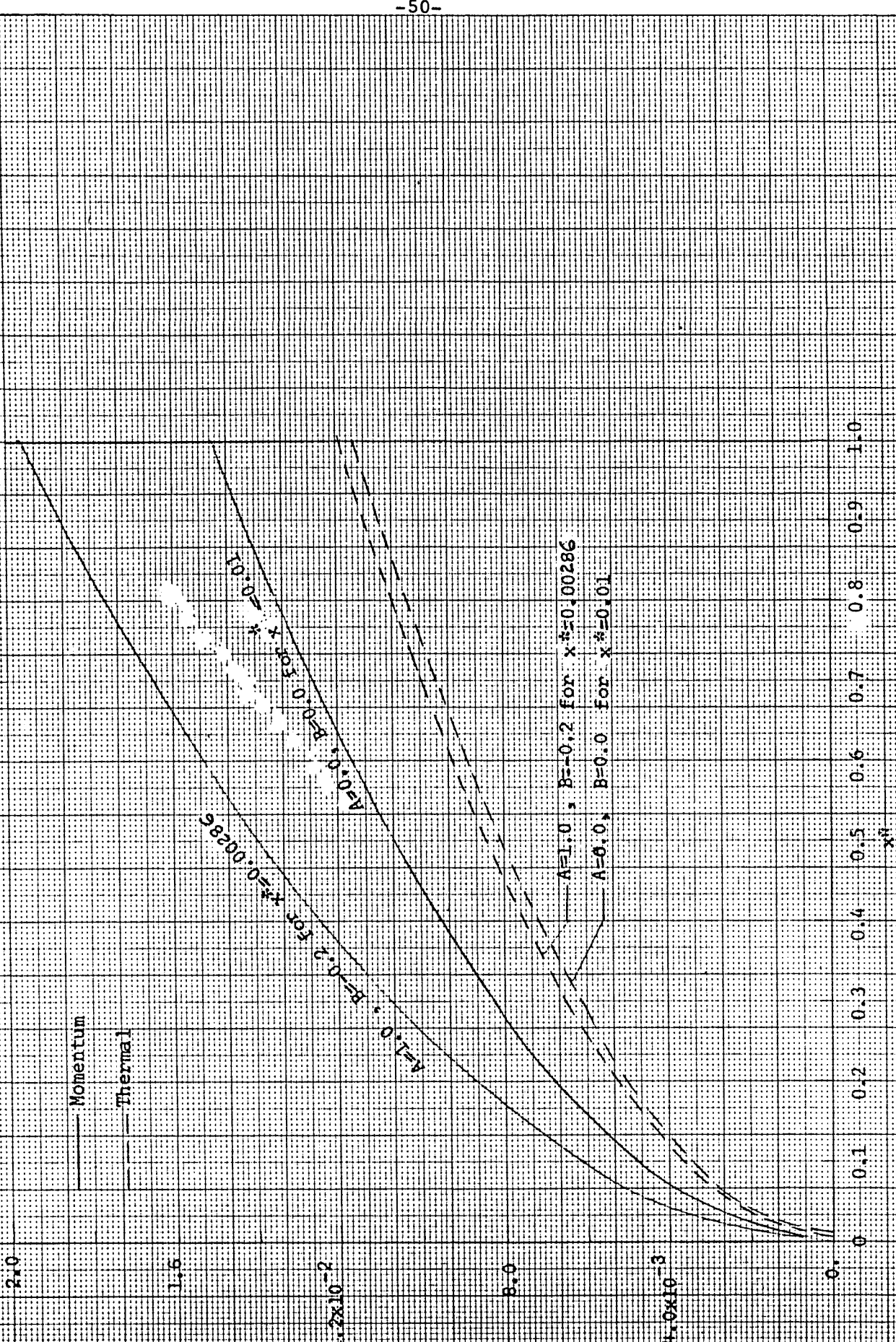


Fig. 23 Momentum and Thermal Boundary layer Thickness for $Re_x = 80,000$, $Pr = 2$
When x_0^* Approaching Zero.

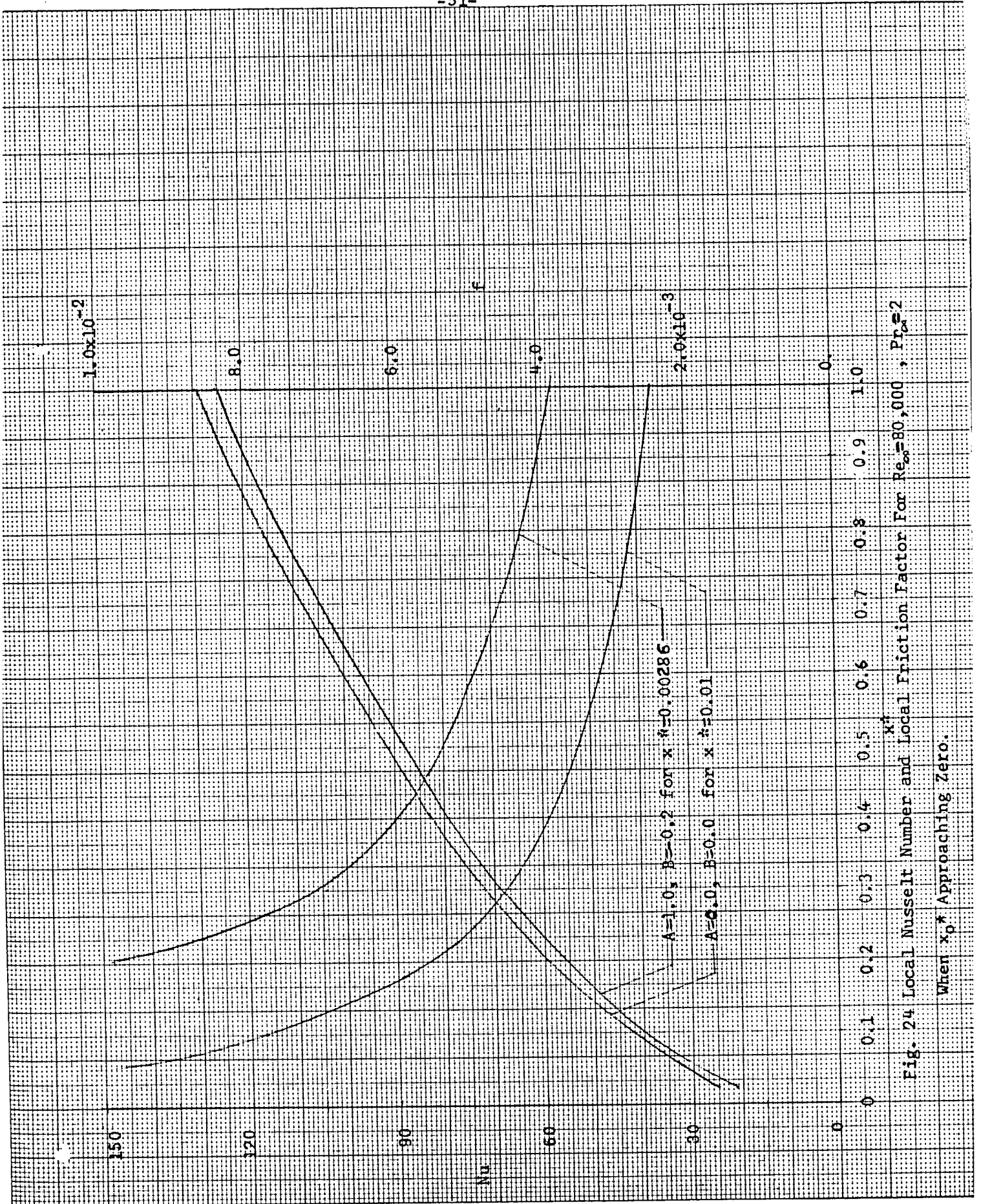


Fig. 24 Local Nusselt Number and Local Friction Factor For $Re_{\infty} = 80,000, Pr_{\infty} = 2$ When x_p^* Approaching Zero.

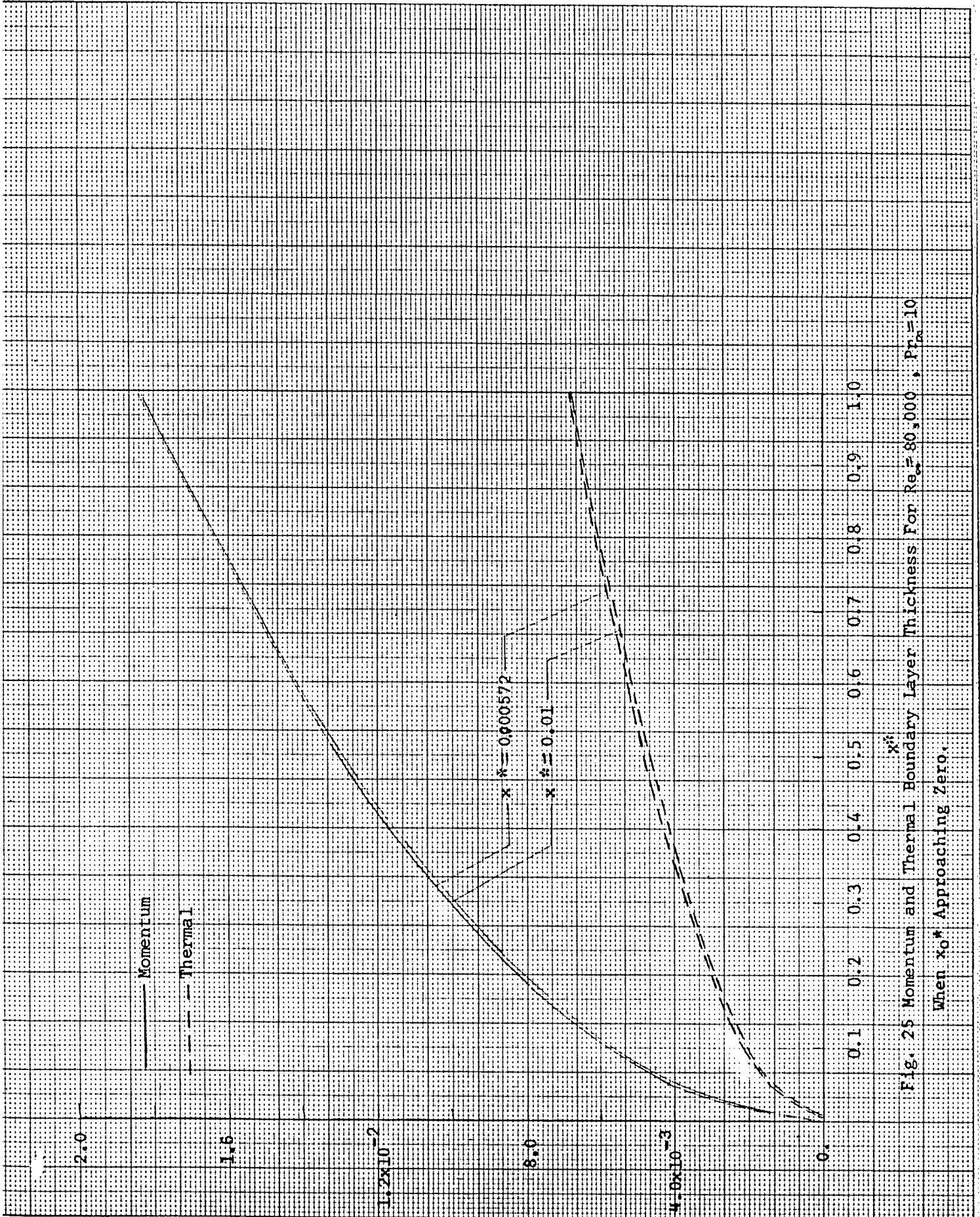


Fig. 25 Momentum and Thermal Boundary Layer Thickness For $Re_{\delta} = 60,000$, $Pr = 10$ When x_0^* Approaching Zero.

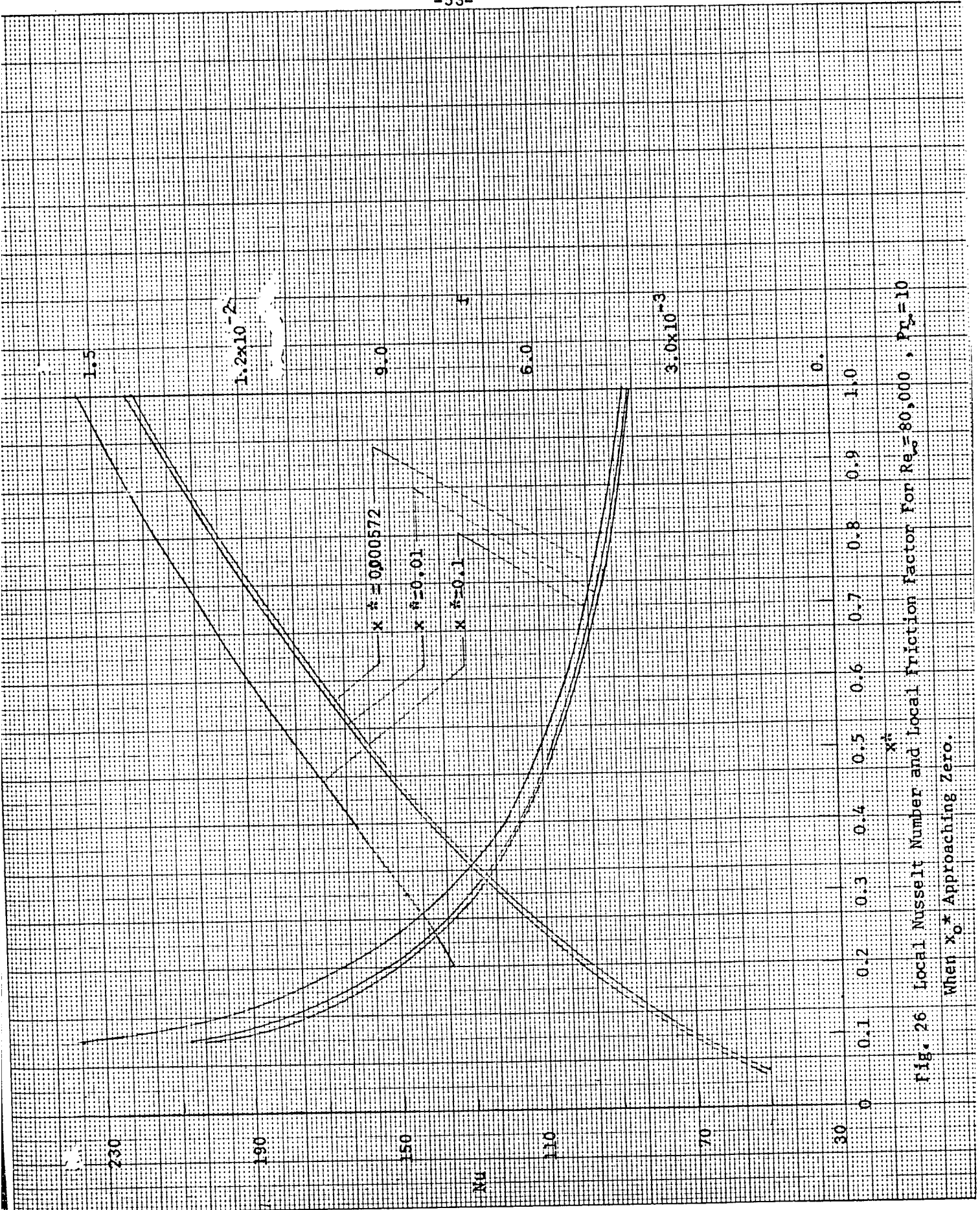


Fig. 26 Local Nusselt Number and local friction factor for $Re_\infty = 80,000$, $Pr_\infty = 10$ When x_0^* Approaching Zero.

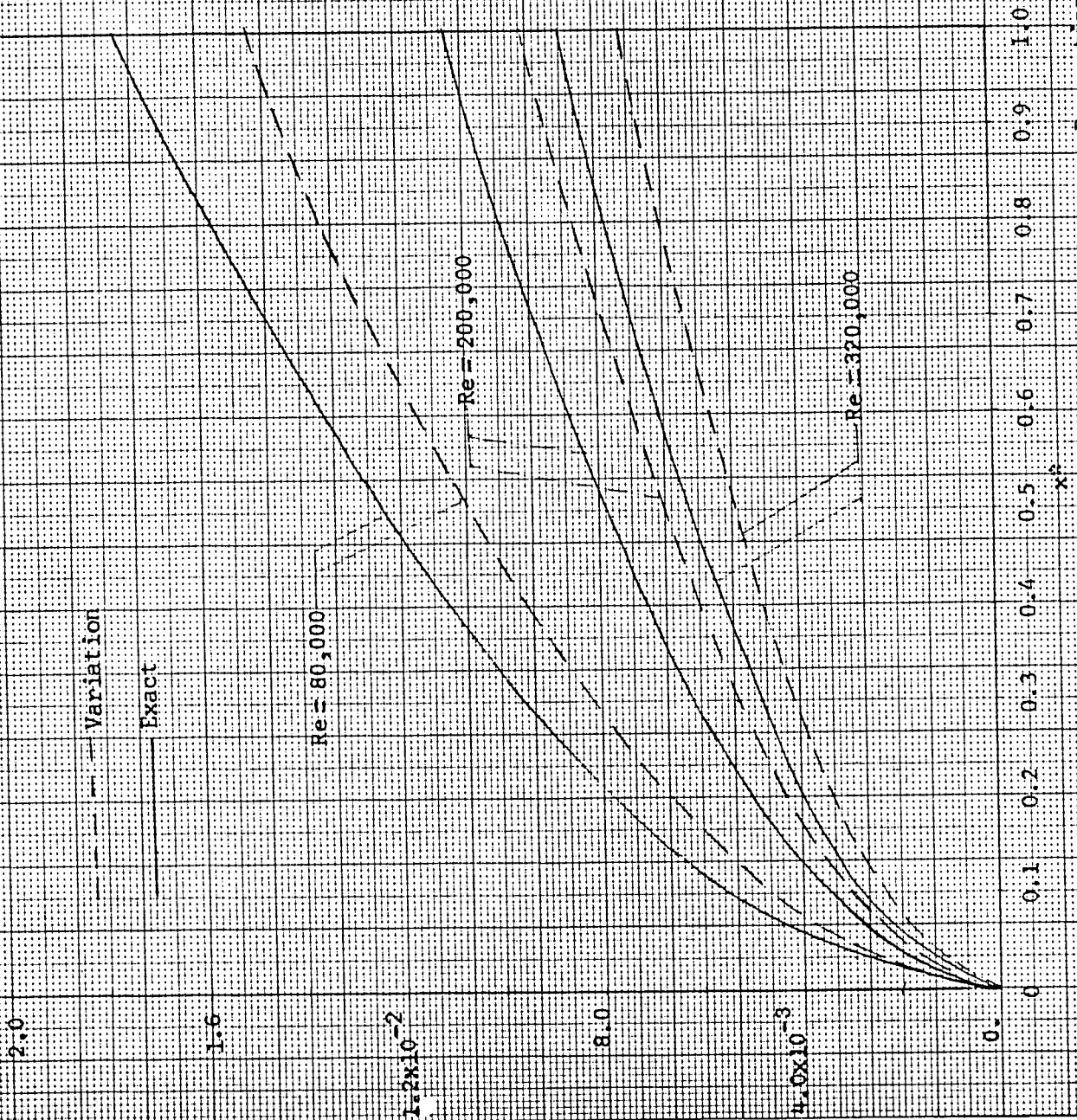


Fig. 27 Comparison of Momentum Boundary Layer Thickness For Constant Properties case

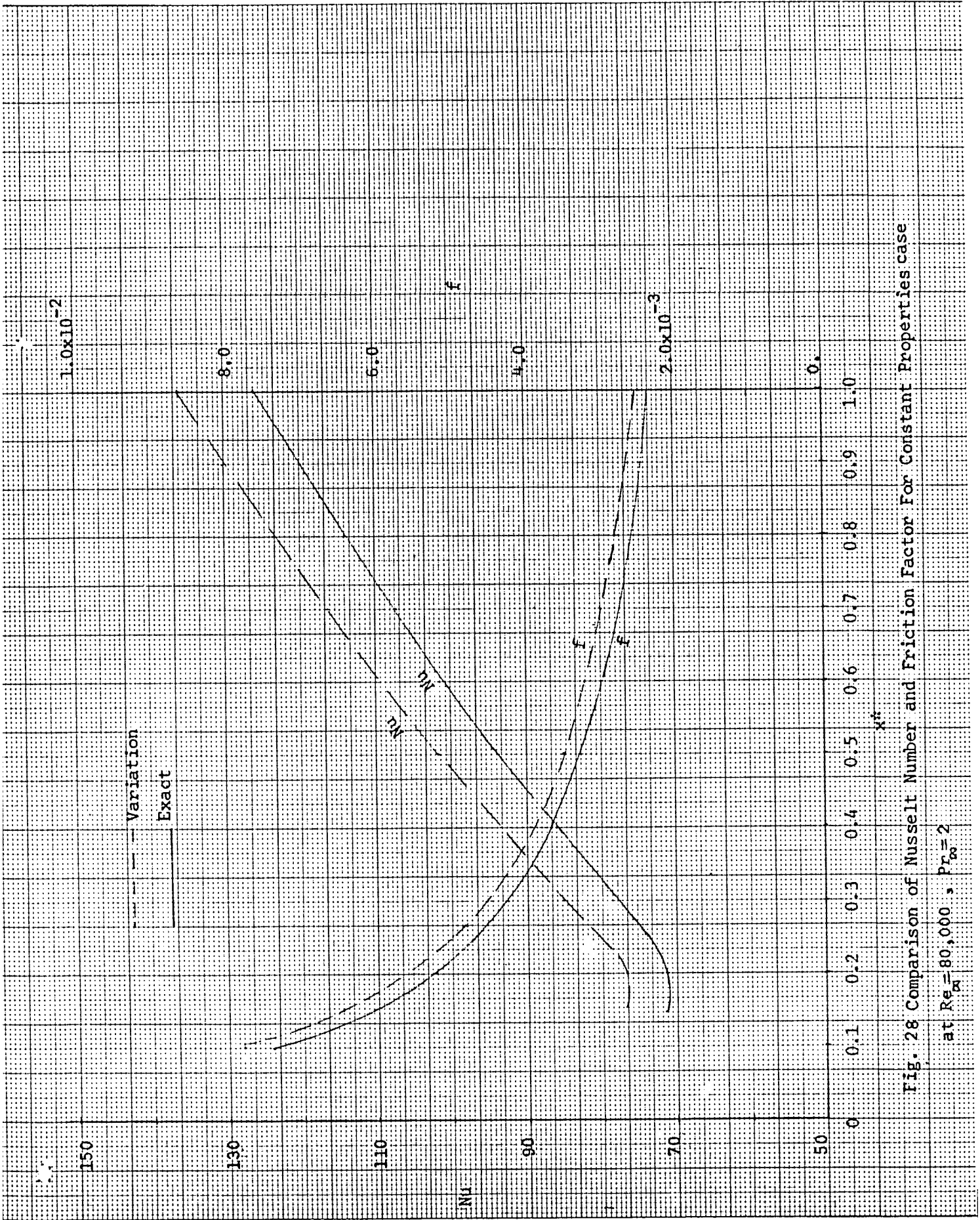


Fig. 28 Comparison of Nusselt Number and Friction Factor For Constant Properties case
 at $Re_{\rho} = 80,000$, $Pr_{\rho} = 2$

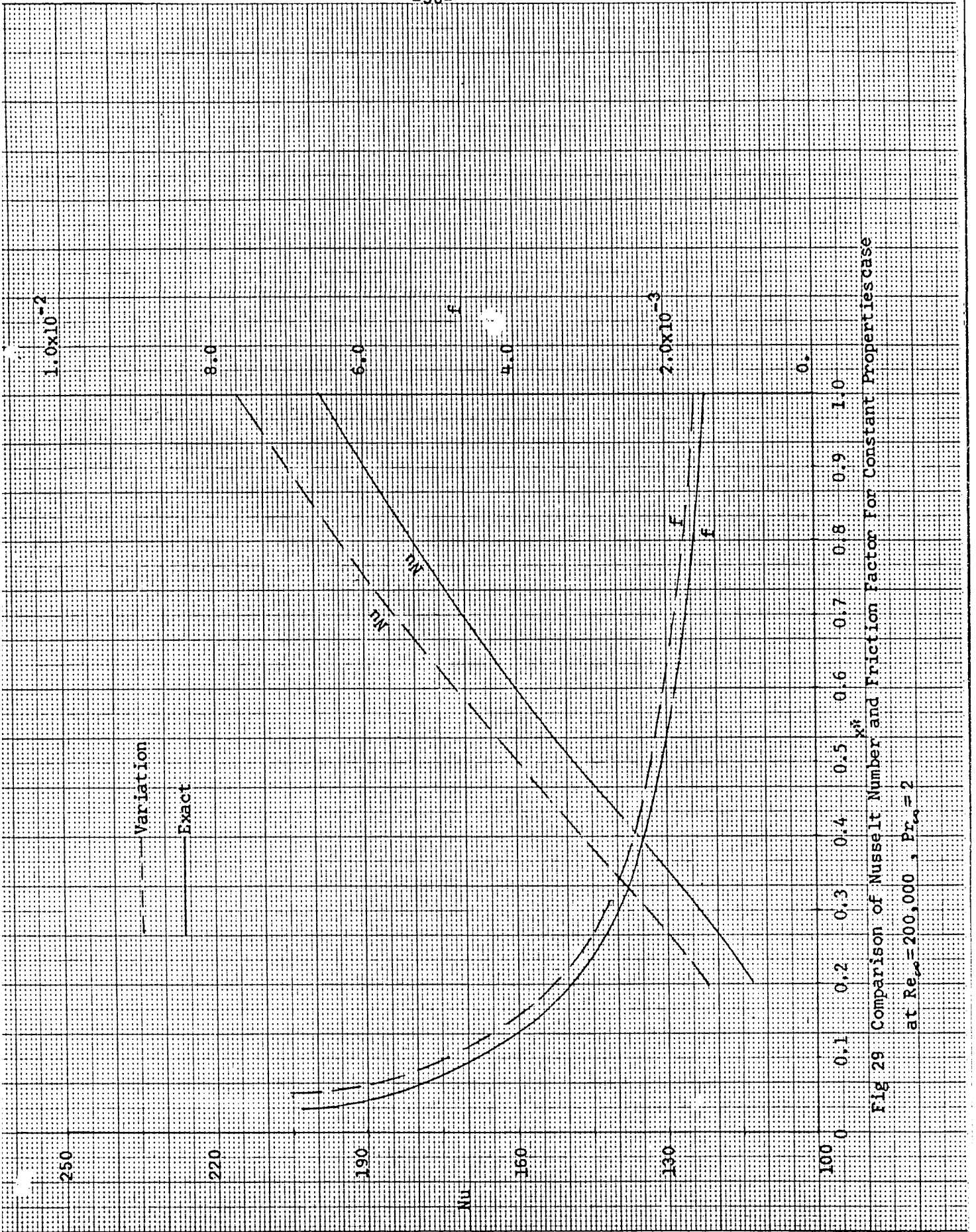


Fig 29 Comparison of Nusselt Number and Friction Factor for Constant Properties case at $Re_\rho = 200,000$, $Pr_\rho = 2$

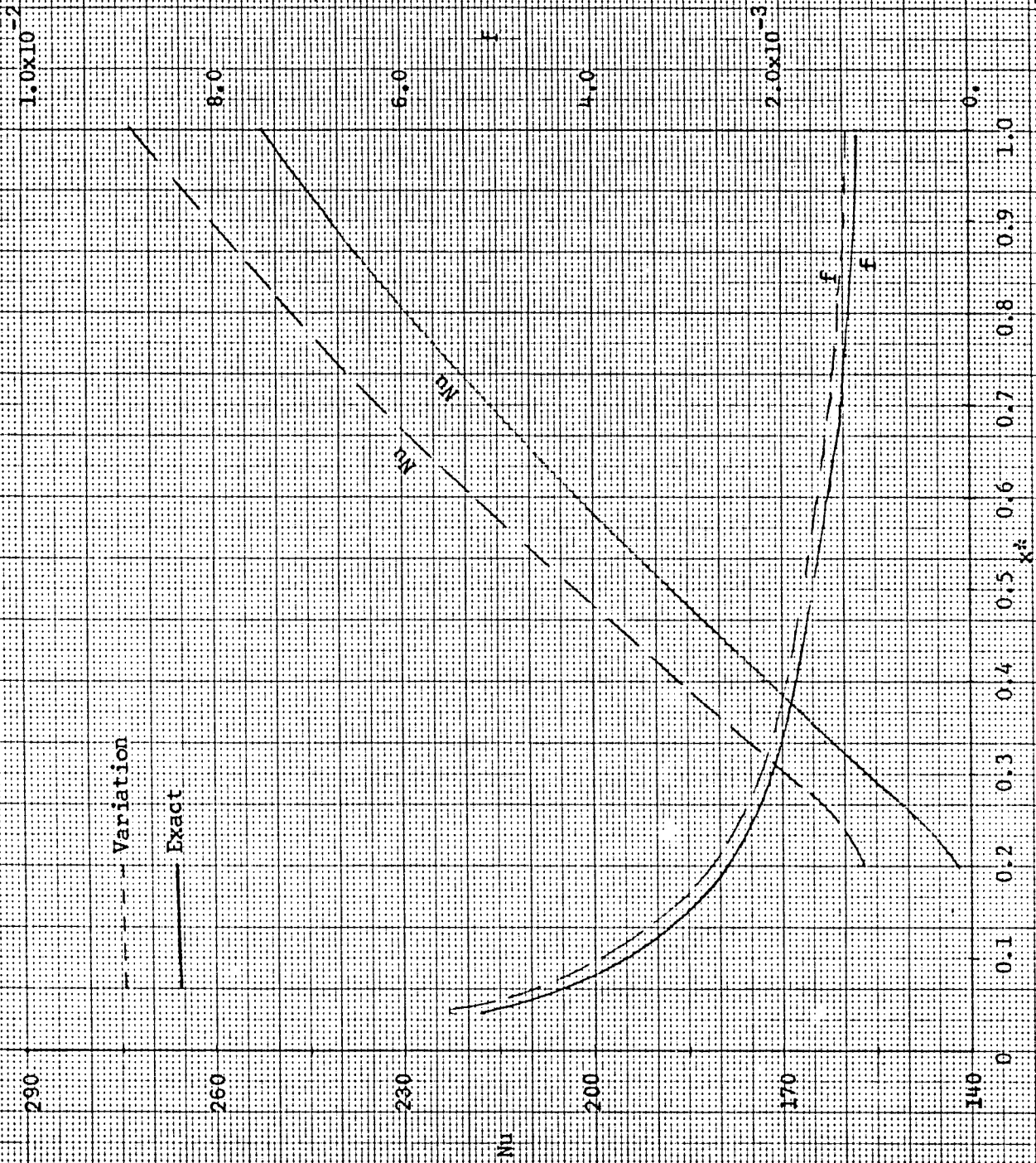


Fig 30 Comparison of Nusselt Number and Friction Factor for Constant Properties case at $Re_{\infty} = 320,000$, $Pr_{\infty} = 2$

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APPENDIX I

DERIVATION OF THE COUPLED EQUATIONS FROM THE VARIATIONAL
FORMULATION OF THE PROBLEM

First, we re-write Eq. (35)

$$E = \iint \left\{ \frac{\rho^0}{2} \left(\frac{\partial T}{\partial y} \right)^2 + \frac{\mu^0}{2} \left(\frac{\partial u}{\partial y} \right)^2 + \rho^0 c_p^0 u^0 T \frac{\partial T}{\partial x} + \rho^0 c_p^0 v^0 T \frac{\partial T}{\partial y} \right. \\ \left. - \rho^0 u^0 \frac{\partial u}{\partial x} - \rho^0 v^0 \frac{\partial u}{\partial y} \right\} dx dy + \int_0^{\Delta} (\rho^0 u^0 |_{x=l} - \rho^0 u_{\infty}^0) dy \\ - \int_0^l \rho^0 u_{\infty}^0 v^0 |_{y=\Delta} dx \quad (A-1)$$

Since

$$\frac{u}{u_{\infty}} = \frac{3}{2} \frac{y}{\Delta} - \frac{1}{2} \left(\frac{y}{\Delta} \right)^3$$

it follows that

$$u = u_{\infty} \left\{ \frac{3}{2} \frac{y}{\Delta} - \frac{1}{2} \left(\frac{y}{\Delta} \right)^3 \right\}, \\ \frac{\partial u}{\partial x} = u_{\infty} \left\{ -\frac{3}{2} \frac{y \Delta'}{\Delta^2} + \frac{3}{2} \frac{y^3 \Delta'}{\Delta^4} \right\}$$

and

$$\frac{\partial u}{\partial y} = u_{\infty} \left\{ -\frac{3}{2} \frac{1}{\Delta} - \frac{3}{2\Delta} \frac{y^2}{\Delta^2} \right\}$$

From the continuity equation

$$\frac{\partial v}{\partial y} = - \frac{\partial u}{\partial x}$$

therefore

$$v = \int_0^y - \frac{\partial u}{\partial x} dy \\ = u_{\infty} \int_0^y \left\{ \frac{3}{2} \left(\frac{y \Delta'}{\Delta^2} \right) - \frac{3}{2} \left(\frac{y^3 \Delta'}{\Delta^4} \right) \right\} dy \\ = u_{\infty} \left\{ \frac{3}{4} \left(\frac{y \Delta'}{\Delta^2} \right) - \frac{3}{8} \left(\frac{y^4 \Delta'}{\Delta^4} \right) \right\}$$

Also, it is assumed that

$$\frac{T - T_w}{T_\infty - T_w} = \frac{3}{2} \left(\frac{y}{\Delta_t} \right) - \frac{1}{2} \left(\frac{y}{\Delta_t} \right)^3$$

and so

$$T = (T_\infty - T_w) \left\{ \frac{3}{2} \left(\frac{y}{\Delta_t} \right) - \frac{1}{2} \left(\frac{y}{\Delta_t} \right)^3 \right\} + T_w,$$

$$\frac{\partial T}{\partial y} = (T_\infty - T_w) \left\{ \frac{3}{2\Delta_t} - \frac{3}{2\Delta_t} \left(\frac{y}{\Delta_t} \right)^2 \right\}$$

and

$$\frac{\partial T}{\partial x} = (T_\infty - T_w) \left\{ -\frac{3}{2} \left(\frac{y \Delta_t'}{\Delta_t^2} \right) + \frac{3}{2} \left(\frac{y^3 \Delta_t'}{\Delta_t^4} \right) \right\}$$

Also

$$\mu = \mu_\infty (1 + A\theta)$$

and

$$k = k_\infty (1 + B\theta)$$

Substituting all the above expressions into Eq.(A-1) gives

$$\begin{aligned} E = & \iiint \left\{ \frac{k_\infty - Bk_\infty \left(\frac{3y}{2\Delta_t^2} - \frac{y^3}{2\Delta_t^3} - 1 \right)}{2} (T_\infty - T_w)^2 \left(\frac{3}{2\Delta_t} - \frac{3y^2}{2\Delta_t^3} \right)^2 \right. \\ & + \frac{\mu_\infty - A\mu_\infty \left(\frac{3y}{2\Delta_t^2} - \frac{y^3}{2\Delta_t^3} - 1 \right)}{2} (u_\infty)^2 \left(\frac{3}{2\Delta_t} - \frac{3y^2}{2\Delta_t^3} \right)^2 \\ & + \rho C_p^0 u_\infty \left(\frac{3y}{2\Delta_t^2} - \frac{y^3}{2\Delta_t^3} \right) \left\{ (T_\infty - T_w) \left(\frac{3y}{2\Delta_t} - \frac{y^3}{2\Delta_t^3} \right) + T_w \right\} (T_\infty - T_w) \left(\frac{-3y\Delta_t'}{2\Delta_t^2} + \frac{3y^3\Delta_t'}{2\Delta_t^4} \right) \\ & + \rho C_p^0 u_\infty \left(\frac{3y^2\Delta_t'}{4\Delta_t^2} - \frac{3y^4\Delta_t'}{8\Delta_t^4} \right) \left\{ (T_\infty - T_w) \left(\frac{3y}{2\Delta_t} - \frac{y^3}{2\Delta_t^3} \right) + T_w \right\} (T_\infty - T_w) \left(\frac{3}{2\Delta_t} - \frac{3y^2}{2\Delta_t^3} \right) \\ & - \rho u_\infty^3 \left(\frac{3}{2} \frac{y}{\Delta_t^2} - \frac{y^3}{2\Delta_t^3} \right)^2 \left(-\frac{3y\Delta_t'}{2\Delta_t^2} + \frac{3y^3\Delta_t'}{2\Delta_t^4} \right) \\ & - \rho u_\infty^3 \left(\frac{3}{2} \frac{y}{\Delta_t^2} - \frac{y^3}{2\Delta_t^3} \right) \left(\frac{3y^2\Delta_t'}{4\Delta_t^2} - \frac{3y^4\Delta_t'}{8\Delta_t^4} \right) \left(\frac{3}{2\Delta_t} - \frac{3y^2}{2\Delta_t^3} \right) \left. \right\} dx dy \\ & + \int_0^\Delta (\rho u_\infty^2 u|_{x=\Delta} - \rho u_\infty^3) dy - \int_0^\Delta \rho u_\infty^2 v^0|_{y=\Delta} dx \end{aligned} \quad (A-2)$$

Taking the variation of E (Eq.(A-2)) with respect to Δ_t , we obtain

$$\begin{aligned} \delta E = \int_0^l \int_0^{\Delta_t} \left\{ \left[k_\infty - B k_\infty \left(\frac{3}{2} \frac{y}{\Delta_t} - \frac{y^3}{2\Delta_t^3} - 1 \right) \right] (T_\infty - T_w)^2 \left(\frac{3}{2\Delta_t} - \frac{3y^2}{2\Delta_t^3} \right) \left(\frac{-3}{2\Delta_t^2} + \frac{9y^2}{2\Delta_t^4} \right) \right. \\ + \int C_p^0 u_\infty \left(\frac{3}{2} \frac{y}{\Delta_t} - \frac{y^3}{2\Delta_t^3} \right) (T_\infty - T_w)^2 \left(\frac{-3y}{2\Delta_t^2} + \frac{3y^3}{2\Delta_t^4} \right) \left(-\frac{3y\Delta_t'}{2\Delta_t^3} + \frac{3y^3\Delta_t'}{2\Delta_t^4} \right) \\ \left. + \int C_p^0 u_\infty \left(\frac{3y^2\Delta_t'}{4\Delta_t^3} - \frac{3y^4\Delta_t'}{8\Delta_t^5} \right) (T_\infty - T_w)^2 \left(\frac{-3y}{2\Delta_t^2} + \frac{3y^3}{2\Delta_t^4} \right) \left(\frac{3}{2\Delta_t} - \frac{3y^2}{2\Delta_t^3} \right) \right\} \delta \Delta_t dx dy \end{aligned} \quad (A-3)$$

There is no longer any need to distinguish between the varied (Δ_t) and unvaried (Δ_t^0) versions of thermal boundary layer thickness. Dropping the superscript and integrating y from 0 to Δ_t , we find

$$\begin{aligned} \delta E = \int_0^l \left\{ -\frac{3}{5} (T_\infty - T_w)^2 \frac{k_\infty}{\Delta_t^2} - \frac{177}{320} B (T_\infty - T_w)^2 \frac{k_\infty}{\Delta_t^2} \right. \\ \left. + \int C_p u_\infty (T_\infty - T_w)^2 \left(\frac{9}{64} \frac{\Delta_t^2 \Delta_t'}{\Delta} - \frac{3}{160} \frac{\Delta_t^4 \Delta_t'}{\Delta^3} - \frac{9}{128} \frac{\Delta_t^3 \Delta_t'}{\Delta^2} + \frac{9}{640} \frac{\Delta_t^5 \Delta_t'}{\Delta^4} \right) \right\} \delta \Delta_t dx \end{aligned} \quad (A-4)$$

Since δE must vanish for all $\delta \Delta_t$, therefore

$$-\frac{3}{5} \frac{k_\infty}{\Delta_t^2} - \frac{177}{320} B \frac{k_\infty}{\Delta_t^2} + \int C_p u_\infty \left(\frac{9}{64} \frac{\Delta_t^2 \Delta_t'}{\Delta} - \frac{3}{160} \frac{\Delta_t^4 \Delta_t'}{\Delta^3} - \frac{9}{128} \frac{\Delta_t^3 \Delta_t'}{\Delta^2} + \frac{9}{640} \frac{\Delta_t^5 \Delta_t'}{\Delta^4} \right) = 0 \quad (A-5)$$

Introducing the ratio $Y = \frac{\Delta_t}{\Delta}$, Eq.(A-5) becomes

$$-\frac{3}{5} \alpha - \frac{177}{320} \alpha B + u_\infty \left\{ \frac{9}{128} (Y^3 \Delta \Delta' + 2Y^2 \Delta^2 Y') - \frac{1}{128} (4Y^4 \Delta^2 Y' + Y^5 \Delta \Delta') \right\} = 0 \quad (A-6)$$

Similarly, we take the variation of E (Eq.(A-2)) with respect to Δ ,

$$\begin{aligned} \delta E = \int_0^l \int_0^\Delta \left\{ \left[u_\infty - A u_\infty \left(\frac{3}{2} \frac{y}{\Delta_t} - \frac{y^3}{2\Delta_t^3} - 1 \right) \right] (u_\infty)^2 \left(\frac{3}{2\Delta} - \frac{3y^2}{2\Delta^3} \right) \left(-\frac{3}{2\Delta^2} + \frac{9y^2}{2\Delta^4} \right) \delta \Delta \right. \\ - \int u_\infty^3 \left(\frac{3y}{2\Delta^0} - \frac{y^3}{2\Delta^3} \right) \left(\frac{3y^2\Delta_t'}{4\Delta_t^3} - \frac{3y^4\Delta_t'}{8\Delta_t^5} \right) \left(-\frac{3}{2\Delta^2} + \frac{9y^2}{2\Delta^4} \right) \delta \Delta \\ - \int u_\infty^3 \left(\frac{3y}{2\Delta^0} - \frac{y^3}{2\Delta^3} \right)^2 \left[\left(\frac{3y\Delta_t'}{\Delta^3} - \frac{6y^3\Delta_t'}{\Delta^5} \right) \delta \Delta + \left(-\frac{3}{2} \frac{y}{\Delta^2} + \frac{3}{2} \frac{y^3}{\Delta^4} \right) \delta \Delta' \right] \right\} dx dy \\ + \int_0^\Delta \left\{ \int u_\infty^3 \left(\frac{3y}{2\Delta^0} - \frac{1}{2} \frac{y^3}{\Delta^3} \right)^2 \left(-\frac{3y}{2\Delta^2} + \frac{3y^3}{2\Delta^4} \right) \delta \Delta \right\}_{x=l} dy \quad (A-7) \end{aligned}$$

As before, there is no longer any need to distinguish between the varied (Δ) and unvaried (Δ^0) versions of the momentum boundary layer thickness. By using the fact that $\delta\Delta' = \frac{d(\delta\Delta)}{dx}$, the term involving $\delta\Delta'$ on the right-hand side of Eq.(A-7) can be written as

$$\begin{aligned} I &= -\int_0^l \int_0^\Delta \rho u_\infty^3 \left(\frac{3y}{2\Delta} - \frac{y^3}{2\Delta^3} \right)^2 \left(-\frac{3y}{2\Delta^2} + \frac{3y^3}{2\Delta^4} \right) \delta\Delta' dx dy \\ &= -\int_0^l \int_0^\Delta \frac{3}{8} \rho u_\infty^3 \left(\frac{3y}{\Delta} - \frac{y^3}{\Delta^3} \right)^2 \left(-\frac{y}{\Delta^2} + \frac{y^3}{\Delta^4} \right) \frac{d(\delta\Delta)}{dx} dx dy \quad (A-8) \end{aligned}$$

Integration by parts yields,

$$\begin{aligned} I &= -\int_0^\Delta \frac{3}{8} \rho u_\infty^3 \left(\frac{3y}{\Delta} - \frac{y^3}{\Delta^3} \right)^2 \left(-\frac{y}{\Delta^2} + \frac{y^3}{\Delta^4} \right) \delta\Delta \Big|_{x=l} dy \\ &\quad + \int_0^l \int_0^\Delta \frac{d}{dx} \left\{ \frac{3}{8} \rho u_\infty^3 \left(\frac{3y}{\Delta} - \frac{y^3}{\Delta^3} \right)^2 \left(-\frac{y}{\Delta^2} + \frac{y^3}{\Delta^4} \right) \right\} \delta\Delta dx dy \quad (A-9) \end{aligned}$$

Finally, integrating y from 0 to Δ of Eq. (A-7), we obtain

$$\begin{aligned} \delta E &= \int_0^l \left\{ -\frac{3}{5} \mu_\infty u_\infty^2 \frac{1}{\Delta^2} - \frac{9}{4} A \mu_\infty u_\infty^2 \left(\frac{3}{8} \frac{\Delta_t}{\Delta^3} - \frac{\Delta_t^3}{6\Delta^5} + \frac{3}{80} \frac{\Delta_t^5}{\Delta^7} \right) \right. \\ &\quad \left. + \frac{21}{320} \rho u_\infty^3 \frac{\Delta'}{\Delta} \right\} \delta\Delta dx \quad (A-10) \end{aligned}$$

Since δE must vanish for all $\delta\Delta$, therefore

$$-\frac{3}{5} \mu_\infty u_\infty^2 \frac{1}{\Delta^2} - \frac{9}{4} A \mu_\infty u_\infty^2 \left(\frac{3}{8} \frac{\Delta_t}{\Delta^3} - \frac{1}{6} \frac{\Delta_t^3}{\Delta^5} + \frac{3}{80} \frac{\Delta_t^5}{\Delta^7} \right) + \frac{21}{320} \rho u_\infty^3 \frac{\Delta'}{\Delta} = 0 \quad (A-11)$$

Introducing the ratio $Y = \frac{\Delta_t}{\Delta}$, Eq. (A-11) becomes

$$-\frac{3}{5} \mu_\infty - \frac{9}{4} A \mu_\infty \left(\frac{3}{8} Y - \frac{1}{6} Y^3 + \frac{3}{80} Y^5 \right) + \frac{21}{320} \rho u_\infty \Delta \Delta' = 0 \quad (A-12)$$

Since Δ_t is assumed to be smaller than Δ , it follows that $Y < 1$, hence Eqs. (A-6) and (A-12) can be simplified by neglecting the higher order terms Y^4 and Y^5

$$-\frac{3}{5}\alpha - \frac{177}{320}\alpha B + u_\infty \left\{ \frac{9}{128} (\gamma^3 \Delta \Delta' + 2\gamma^2 \Delta^2 \gamma') \right\} = 0 \quad (\text{A-13})$$

$$-\frac{3}{5}u_\infty - \frac{9}{4}A u_\infty \left(\frac{3}{8}\gamma - \frac{1}{6}\gamma^3 \right) + \frac{21}{320} \rho u_\infty \Delta \Delta' = 0 \quad (\text{A-14})$$

Introducing the dimensionless quantities

$$x^* = \frac{x}{l}$$

$$\Delta^* = \frac{\Delta}{l}$$

$$\Delta_t^* = \frac{\Delta t}{l}$$

we obtain

$$-\frac{3}{5} - \frac{177}{320} B + \frac{9 \cdot Pr_\infty \cdot Re_\infty}{128} (\gamma^{*3} \Delta^* \Delta^{*'} + 2\gamma^{*2} \Delta^{*2} \gamma^{*'}) = 0 \quad (\text{A-15})$$

$$-\frac{3}{5} - \frac{9}{4} A \left(\frac{3}{8} \gamma^* - \frac{1}{6} \gamma^{*3} \right) + \frac{21}{320} Re_\infty \Delta^* \Delta^{*'} = 0 \quad (\text{A-16})$$

Finally we rewrite Eq. (A-15) and (A-16) as

$$2\gamma^{*2} \Delta^{*2} \gamma^{*'} = a_5 - \gamma^{*3} \Delta^* \Delta^{*'} \quad (\text{A-17})$$

$$-a_1 \Delta^* \Delta^{*'} = -a_2 - a_3 \gamma^* + a_4 \gamma^{*3} \quad (\text{A-18})$$

where

$$a_1 = \frac{21}{320} Re_\infty$$

$$a_2 = \frac{3}{5}$$

$$a_3 = \frac{9}{4} A \cdot \frac{3}{8}$$

$$a_4 = \frac{9}{4} A \cdot \frac{1}{6}$$

$$a_5 = \frac{128}{9 \cdot Pr_\infty \cdot Re_\infty} \left(\frac{3}{5} + \frac{177}{320} B \right)$$

Eqs (A-17) and (A-18) are same as Eqs.(44) and (45) in chapter III.