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MATHEMATICAL MODELS FOR DIRECT AND INVERSE SOLUTIONS OF THE ELECTRICAL ACTIVITY OF HEART

by

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ABSTRACT

Relationship between the electrical activity of heart and the electrocardiograms on the body surface has been investigated for many decades to help in diagnosing the condition of heart and to develop some understanding of the physical process.

The purpose of the research described in this work was to explore the available techniques and to find an improved method for relating electrical activity of the heart to the resulting electrocardiograms on the body surface, such that the parameters derived from this relationship could be used to determine equivalent electrical activity of the heart when only body surface electrocardiograms were available.

A brief description of heart-human thorax relationship is given in section 1.2. Figure 1.1 was drawn from actual measurements on a subject since these measurements were to be used later. Sequence of ventricular depolarization is described in section 1.3.

In Chapter 2, some of the proposed mathematical models are discussed followed by fundamental representation of distributed charge double layer by a discrete number of dipoles as developed by the author.

In Chapter 3, a method is developed for relating electrical activity of the heart represented by a number of equivalent dipole sources to resulting unipolar electrocardiograms at a number of points on the body surface. The method is based on multivariate regression analysis. Although regression technique is extensively used in statistics, the material developed in Chapter 3, is new so far as

the systematic evaluation and analysis of parameters is concerned.

Results of computations with the actual electrocardiograms are described in Chapter 4, with detailed analysis of the parameters obtained. These parameters were used to obtain the equivalent dipole sources in the case of normal cardiac activity. Comments on the results obtained, limitations, and application of the method are discussed at the end of Chapter 4.

The digital computer programs in appendices III, VII, VIII, X, XI and XII were developed and written by the author. The material presented in appendices II, IX, XIII and XIV was also developed by the author.

TABLE OF CONTENTS

CHAPTER 1

INTRODUCTION

1.1 General

The contraction of cardiac muscle is immediately preceded by electrical events. An essential part is the creation of potential differences in the heart (muscle) which in turn give rise to electrical currents. The heart muscle is surrounded by body fluids, which by virtue of their electrolyte content act as electrical conductors. In this manner the cardiac currents flow in all directions. The body can, therefore, be regarded as volume conductor.

The nature of this electrical activity has always intrigued investigators. As early as 1913, Einthoven already speculated that cardiac electrical activity can be represented by an equivalent dipole.

Since the introduction of this concept, the problem of finding an equivalent system for the "cardiac generator" has been approached in several ways.

The models proposed were intended to serve some of the following objectives:

(1) to understand more clearly electrical activity of the heart;

(2) to relate this electrical activity to the electrocardiograms appearing on the surface of the body.

(3) to investigate the dielectric properties of the medium surrounding the heart i.e. the human thorax.

(4) to be able to predict electrical activity inside the heart from the electrocardiograms on the surface of the body.

At this stage, no attempt will be made to go into details of the work carried out by other investigators. The interested reader may find the details in references [1 - 15].

1.2 The Heart

The heart is a hollow, muscular organ of a somewhat conical form; it lies between the lungs and is enclosed in the pericardium. It is placed obliquely in the chest behind the body of the sternum and adjoining parts of the rib cartilages and projects farther into the left than in the right half of the thoracic cavity, so that about one-third of it is situated on the right of the median plane and two-thirds on the left [16]. The size of the adult heart is about 12 cms from base to apex, 8 - 9 cms transversely at the broadest part and 6 cms antero-posteriorly.

The position of the normal heart within the human thorax varies considerably in different subjects. The heart may be vertical in tall individuals and horizontal in those of short stature, obese and pregnant women.

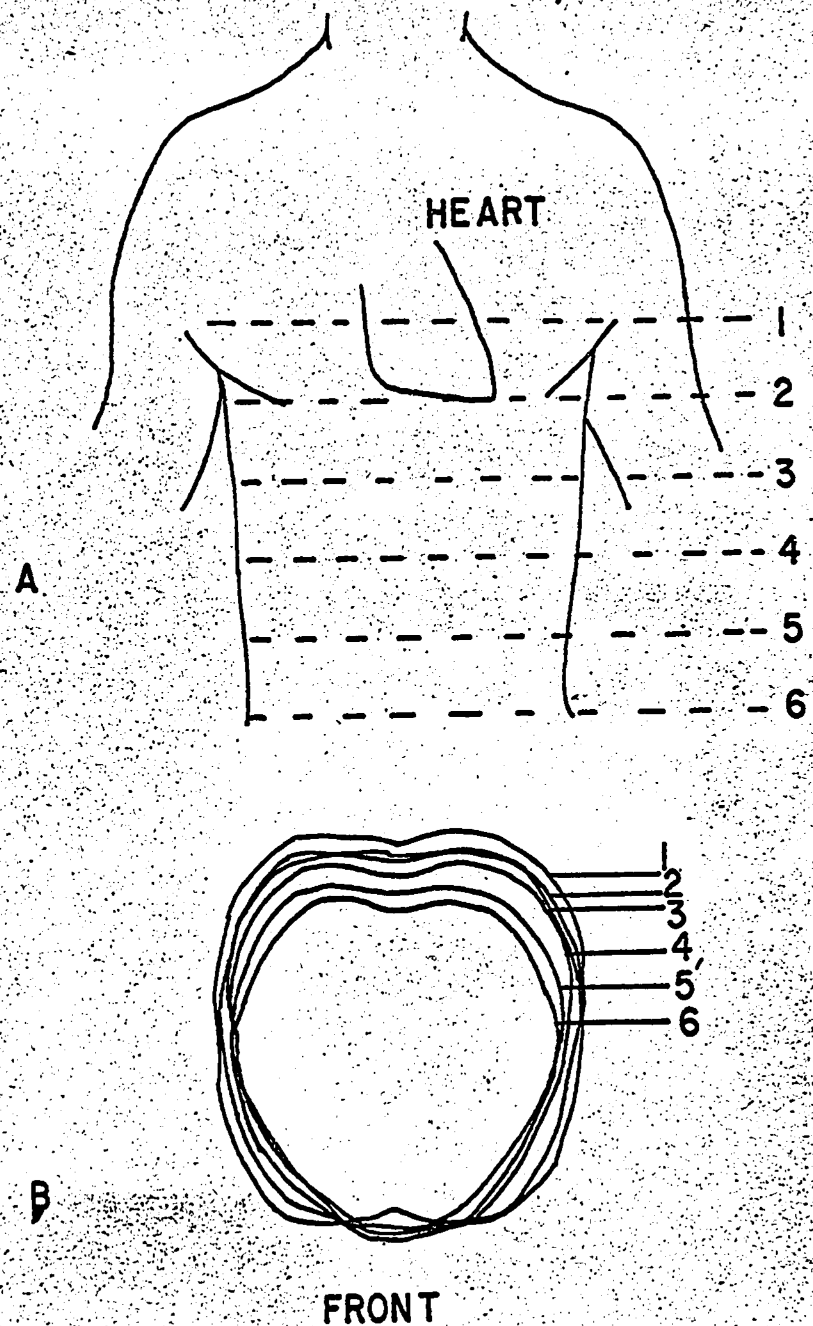


Fig. 1.1 Position of heart and sectional view of outerwalls of thorax in subject SIAL1966339 as described in the text.

Figure 1.1A shows the sketch of the outer walls of the thorax from the measurements on a subject, who weighed 130 lbs. and measured 5'6" in height. The position of heart shown is the most common position in individuals and was drawn from the description in reference [16].

The dotted lines in the figure show the levels at which measurements were taken.

Figure 1.1B gives the sectional view of the outer walls of the thorax at the six levels.

1.3 Electrical Activity of the Heart

The initiation and the sequence of excitation of a normal human heart can be described fairly accurately. Although most of this information has been derived from experiments on dogs, the extrapolation is justified by two facts: (i) human and canine hearts are anatomically similar, both grossly and histologically, and (ii) electrocardiograms of similar shape can be recorded from both hearts [17, 18].

The cardiac tissue has the ability to beat rythmically without external stimuli. The cells with most inherent rythm are called "pacemaker cells". Pacemaker activity is normally confined to sinuatrial (SA) node and atrioventricular (AV) node, with SA node being the dominant pacemaker (Fig. 1.2). The exact process by which the SA and AV nodes generate impulses is not known [19].

The spread of activity in the atrium commences in the SA node, spreads around and reaches the borders of atria and interatrial septum. The duration of atrial excitation is about 50 msec. This is followed by an apparently quiet period i.e. no current flows from one region to another.

The period from the beginning of atrial activity to the beginning of ventricular activity is about 160 msec. The pathway of ventricular activity has been studied in detail [20, 21, 18, 23, 24], and generally there is a good agreement between the findings of different investigations carried out independently.

Activity starts, usually a few milliseconds earlier, on the left, beginning in two separate areas which are supplied by the anterior and posterior terminations of the left bundle, respectively. The activity on the right starts at the septal terminations of the right bundle in the region of the anterior papillary muscle of the right ventricle. The early activity on the left is directed from left to right and the activity on

Fig. 1.2 A close-up of sectional view of the heart.

1. Pacemaker (region)
2. Interatrial septum
3. Conduction system
4. Right ventricle (RV)
5. Interventricular septum
6. Left ventricle (LV)

the right is directed from right to left. Since the size of activated tissue on the left is larger than on the right, the resultant activity is directed to the right.

The regions near the septal termination of the bundles are soon activated, and the activity spreads very rapidly over a large portion of the endocardium near the apex of the heart and in the central region on both sides. Within a short period most of the central and apical endocardium on both sides is activated. The activity in central and apical endocardium then proceeds from inside out whereas the net activity in the septal region has right to left direction. The activated regions coelesce and at about 25 msec from the start of ventricular excitation, most of the ventricular surface is activated. The last area to be excited lies in the basal septum bordering the atrium.

The sequence of activation of interventricular septum and free ventricular walls is partly shown in Fig. 1.3.

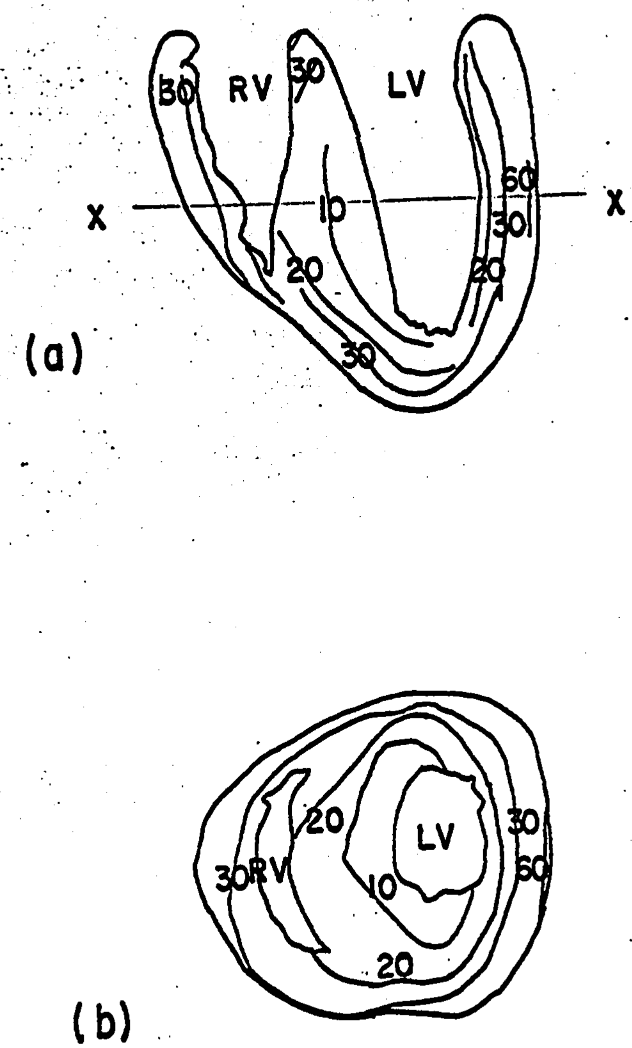


Fig. 1.3 Activation wavefront at different points on interventricular septum and of the free ventricular walls. The numbers show the time of arrival of activation wavefront in msec.

Figure (a) gives vertical view and figure (b) the sectional view, the section drawn at level X-X.

CHAPTER 2

ELECTRICAL ACTIVITY OF THE HEART

MATHEMATICAL APPROACH

2.1 Mathematical Interpretation of the Electrical Activity of the Heart

In Chapter 1, a description was given of the initiation and sequence of excitation of the heart muscle as determined from experiments carried out in the years 1950 - 1960.

Electrical activity of the heart gives rise to potentials on the surface of the body. For more than half a century these potentials have been recorded and used by physicians to make diagnoses of conditions of the heart. The earliest system, for recording potential differences between different points on the body, as proposed by Einthoven is shown in Fig. 2.1. The records were given the name of "electrocardiograms" or ECGs (also known as electrokardiogramme or EKG).

To understand these electrocardiograms relative to one another and in relation to the electrical activity of the heart, Einthoven in 1913 introduced the "triangle hypothesis" [25] based on the following assumptions:

(1) the body is a spherical homogeneous volume conductor of infinite size.

(2) the extremities of three limbs used in the standard leads (I, II, and III) form the apices of an equilateral triangle lying in the frontal plane of the body. Therefore,

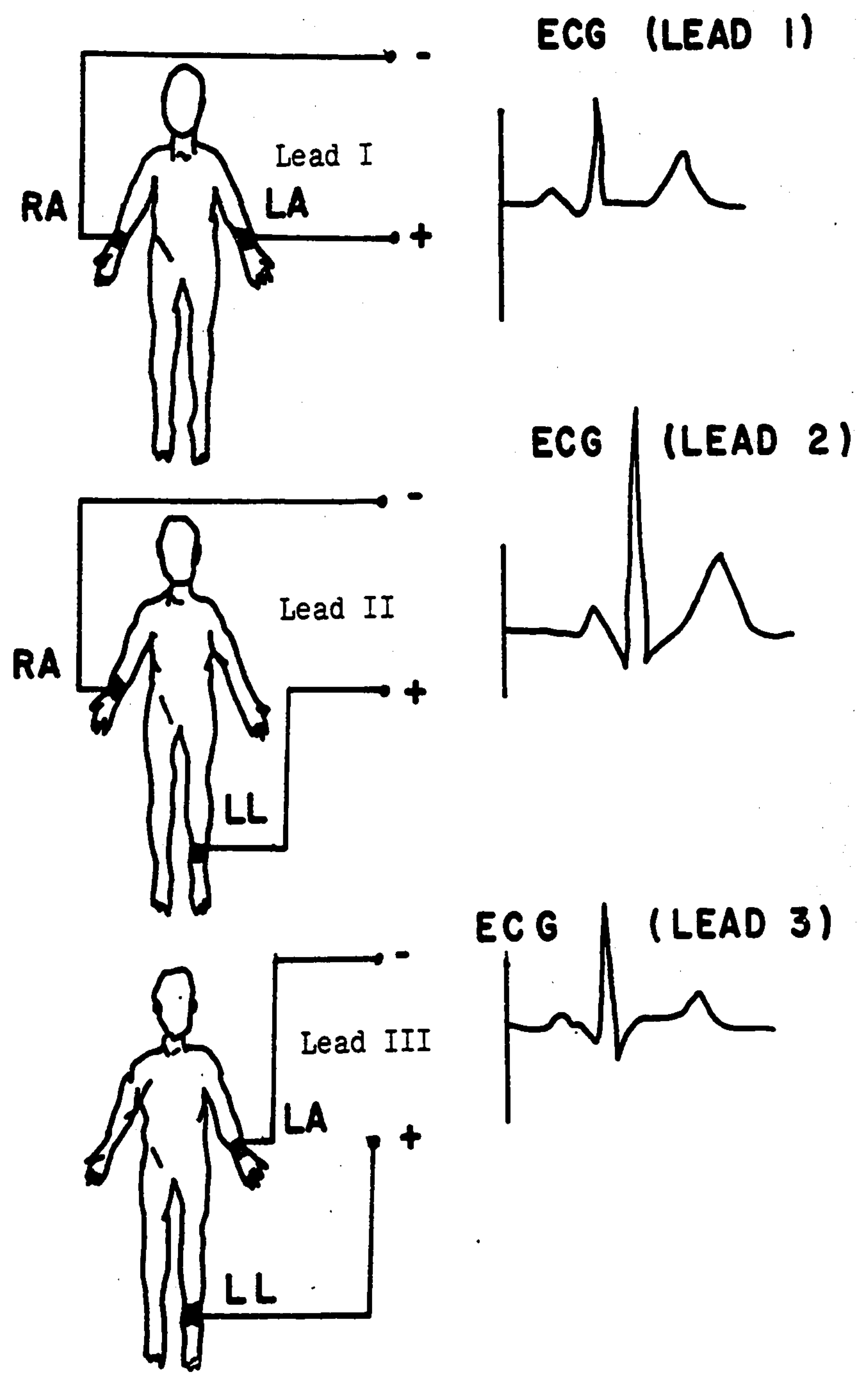


Fig. 2.1 The standard Lead connections, as introduced by Einthoven and corresponding typical normal ECG's from these bipolar leads.

the electrodes are considered equidistant from one another and also from the centre of the heart.

(3) the electrical activity of the heart can be represented by an equivalent "dipole" at the centre of this equilateral triangle.

Einthoven's "triangle hypothesis" marked the beginning of a new era in electrocardiography. It was the first step in the evolution of mathematico-physical models of electrical activity of the heart.

Einthoven's model, though extensively used in the interpretation of the electrocardiograms, is not accurate because of the following facts:

(1) the body is an irregularly shaped cylinder with inhomogeneities especially in the areas around the heart [26 - 31]. Moreover, the size of the body is not infinite compared to the electrically activated portion of the heart.

(2) the right arm (RA), left arm (LA) and left foot (LF) as shown in Fig. 2.1 do not represent three equidistant points from the so called electrical centre of the heart.

(3) the electrical activity of the heart cannot be represented by a single dipole as the equivalent source [32].

Einthoven's "triangle hypothesis", although not very accurate, has been widely used in clinical electrocardiography because of its simplicity. Simultaneously search has continued for a more accurate system.

Essentially the problem can be divided into two parts, i.e.:

(i) <u>representation of electrical activity of the heart by an equivalent source or cardiac generator.</u>

Putting this source inside an assumed geometrical shape, we can determine potentials appearing on the surface due to this source. If the geometrical shape chosen can be made realistically comparable to the human thorax, then the theoretical results obtained on the model can be compared to actual electrocardiographic measurements. In the earlier models a dipole source was assumed to be immersed in homogeneous conducting medium bounded by a sphere, cylinder, circular lamina or a prolate spheroid [1 - 5]. In the later models, the equivalent source was represented by multipole expansions [7, 9 - 12]. We will return to the discussion of the equivalent source in the next section.

(ii) <u>determining a system of lead connections for recording electrocardiograms.</u>

Modification of Einthoven's leads were necessary to account for the fact that (1) the electrical activity of the heart could not be replaced by a single equivalent dipole and (2) the three extremities, i.e. left arm, right arm and left foot did not form apices of an equilateral triangle. Studies were carried out on models resulting into a variety of systems [33 - 42], in general making some improvement over the previous one.

2.2 Mathematical Representation for Electrical Activity of the Heart

2.2.1 Discussion

As we have described in section 2.1, it was realized by many investigators that a single dipole representation did not sufficiently account for electrical activity of the heart. Gabor and Nelson [43] presented a method for deriving equivalent source consisting of one, two or more dipoles, from the potential measurements on the surface of the body. However, complexities involved in applying the method for the case of more than one dipole were such that the method could rarely be applied in practice. Yeh and Martinek [45, 7] proposed a multipole source. A multipole can be defined as a set of sources and sinks (Fig. 2.2) infinitesimally apart (as compared to the distance of the point of observation). A multipole expansion contains dipole terms, quadrupole terms, octupole terms etc., depending on the order of expansion. These expansions have been described in reference [7]. The concept of multipole source has been preferred over the multiple-dipole source mainly because of the fact that

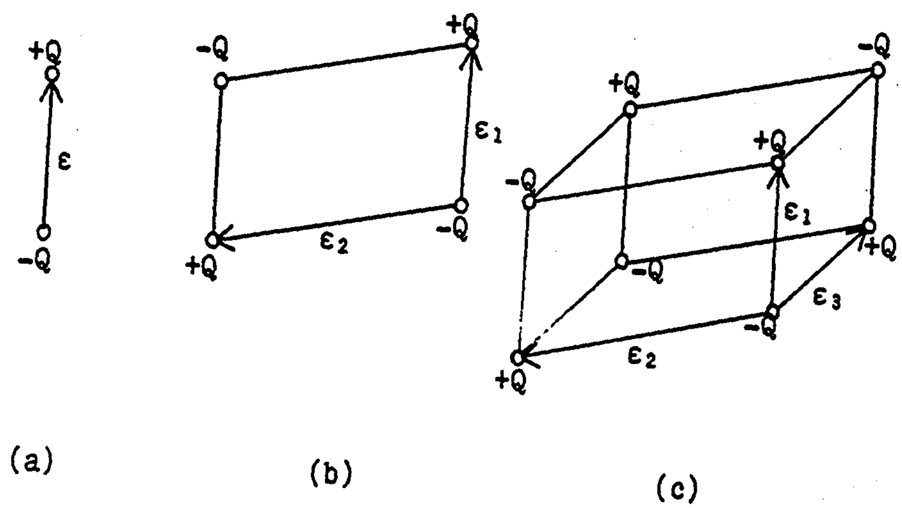


Fig. 2.2

(a) Two charges +Q and -Q a small distance ε apart represent a dipole.

(b) Four charges of alternating sign placed at the vertices of a parallelogram, represent a quadrupole.

(c) Eight charges of alternating sign placed at the vertices of a regular parallelopiped, represent an octupole the former is amenable to more detailed mathematical analysis [9]. It has been suggested that electrical activity of the heart can be represented accurately by an equivalent multipole source [11, 12].

The techniques for application of a multipole source are still too involved and complex to be useful in practice. Moreever, it could be seen from the description of electrical activity of the heart in Chapter 1, that representing this "activity" by a multipole source is a gross departure from actual situation. A multipole source might give a better "fit" for the variations of the potentials on the surface of the body, but it will fail to relate, physically, the changes in the pathway of depolarization wavefront in the heart to the changes in the potentials recorded on the body surface. Thus, to a large extent, a multipole source is theoretical in nature.

2.2.2 Formulation

In this section we will explore techniques for representing electrical activity of the heart in terms of multiple dipoles.

Consider a number of dipoles located on the circumference of a circular ring; all the dipole moments directed radially outward (Fig. 2.3). Since our purpose will be, mainly, to establish a principle, we will also assume that (1) the medium is unrestricted and (2) the probe will lie in the same plane as that of the ring.

From Appendix II, the expressions for the potentials are:

(1) Let V_p = Potential at P, a point outside the ring

i.e. $F > 1$

then $$V_p = \left(\frac{1}{4\pi\varepsilon}\right) \frac{MR}{AP^3} \left(F \cos(\theta - \emptyset) - 1 \right) \qquad (2.2.1)$$

where

ε = permittivity of the medium

M = dipole moment (magnitude)

θ = angle that the dipole moment vector makes with axis ox

$\emptyset$ = angle of the radius vector from centre of dipole to probe P

R = radius of ring

AP = length of radius 'vector AP'

where $AP = R\sqrt{F^2 + 1 - 2F \cos(\theta - \emptyset)}$

$F = OP/_R$

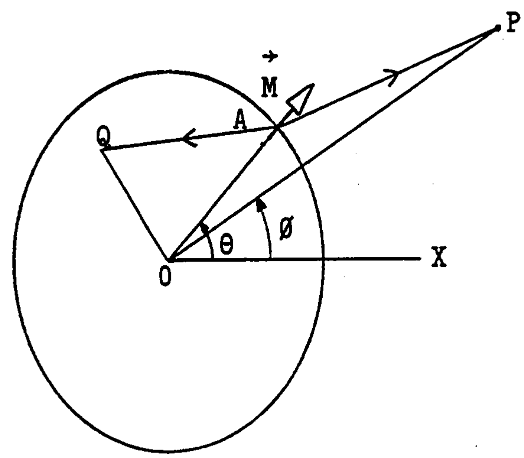


Fig. 2.3 Dipoles located on the ring with their moments directed radially outward. P and Q are the given positions of probes outside and inside the ring respectively.

(2) Let V_Q = potential at Q, a point inside the ring

i.e. $E<1$

then $V_Q = (\frac{1}{4\pi\varepsilon}) \frac{MR}{AQ^3} (E \cos (\theta - \phi) - 1)$ (2.2.2)

where ε, M, R, θ are the same as for (1)

ϕ = angle of the radius vector from centre of dipole to probe Q

AQ = length of radius "vector AQ"

where $AQ = R \sqrt{E^2 + 1 - 2E \cos (\theta - \phi)}$

$E = \frac{OQ}{R}$

It should be noted that equations 2.2.1 and 2.2.2 are essentially the same. These equations are valid only if the distance of the probe from the dipole is large as compared to the separation between positive and negative charges of the dipole itself.

If the number of dipoles is N, the potential at a given position, from equations (2.2.1) and (2.2.2) is as follows:

$$Vp = (\frac{1}{4\pi\varepsilon}) R \sum_{i=1}^{N} \frac{M_i (F \cos (\theta_i - \phi) - 1)}{AP_i^3} \qquad (2.2.3)$$

where

M_i = moment of i-th dipole

θ_i = angle that ith dipole moment vector makes with axis ox

$$AP_i = R \sqrt{F^2 + 1 - 2F \operatorname{Cos} (\theta_i - \phi)} \qquad (2.2.4)$$

and

$$V_Q = \left(\frac{1}{4\pi\varepsilon}\right) R \sum_{i=1}^{N} \frac{M_i \left(E \operatorname{Cos} (\theta_i - \phi) - 1 \right)}{AQ_i^3} \qquad (2.2.5)$$

where

$$AQ_i = R \sqrt{E^2 + 1 - 2E (\theta_i - \phi)} \qquad (2.2.6)$$

In the following subsections we will make use of these equations to study some of the special features.

2.2.2.1 Distributed and Discrete Charges

From equation (2.2.3)

$$Vp = \left(\frac{1}{4\pi\varepsilon}\right) R \sum_{i=1}^{N} \frac{M_i \left(F \operatorname{Cos} (\theta_i - \phi) - 1 \right)}{AP_i^3}$$

Let

$$M_1 = M_2 = - - - - - = M_i = M = K/N$$

where K is a constant

TABLE 2.2.1 Potential (function) due to dipoles located on the circumference of a ring with their moments directed radially outward.

N.	F = 1.1	F = 1.2	F = 1.4	F = 1.7	F = 2.0	F = 2.5	F = 3.0
3	33.1130	8.17659	1.95856	.586922	.261335	.0988817	.0477814
6	16.3904	3.95730	.899096	.255272	.112242	.0439531	.0225211
9	10.8219	2.57967	.592929	.185287	.090310	.0394137	.0212702
12	8.05887	1.93025	.478447	.168871	.0870531	.0390753	.0212207
15	6.42765	1.57814	.433230	.165114	.0866084	.0390572	.0212214
18	5.36814	1.37845	.415329	.164297	.0865584	.0390595	
24	4.11763	1.18149	.405677	.1641119	.0865654		
30	3.45136	1.11132	.404334	.164128			
36	3.07525	1.08617	.404218				
40	2.91774	1.07941	.404245				
45	2.78548	1.07551					
60	2.61310	1.07346					
72	2.57466	1.07364					
90	2.55993						
120	2.55903						

Potential is given in arbitrary units

N = Number of dipoles

R = radius of ring = 1 (unit)

F = ratio of distances i.e. OP/R (Fig. 2.3)

(Dipole moment) x (No. of dipoles) = constant

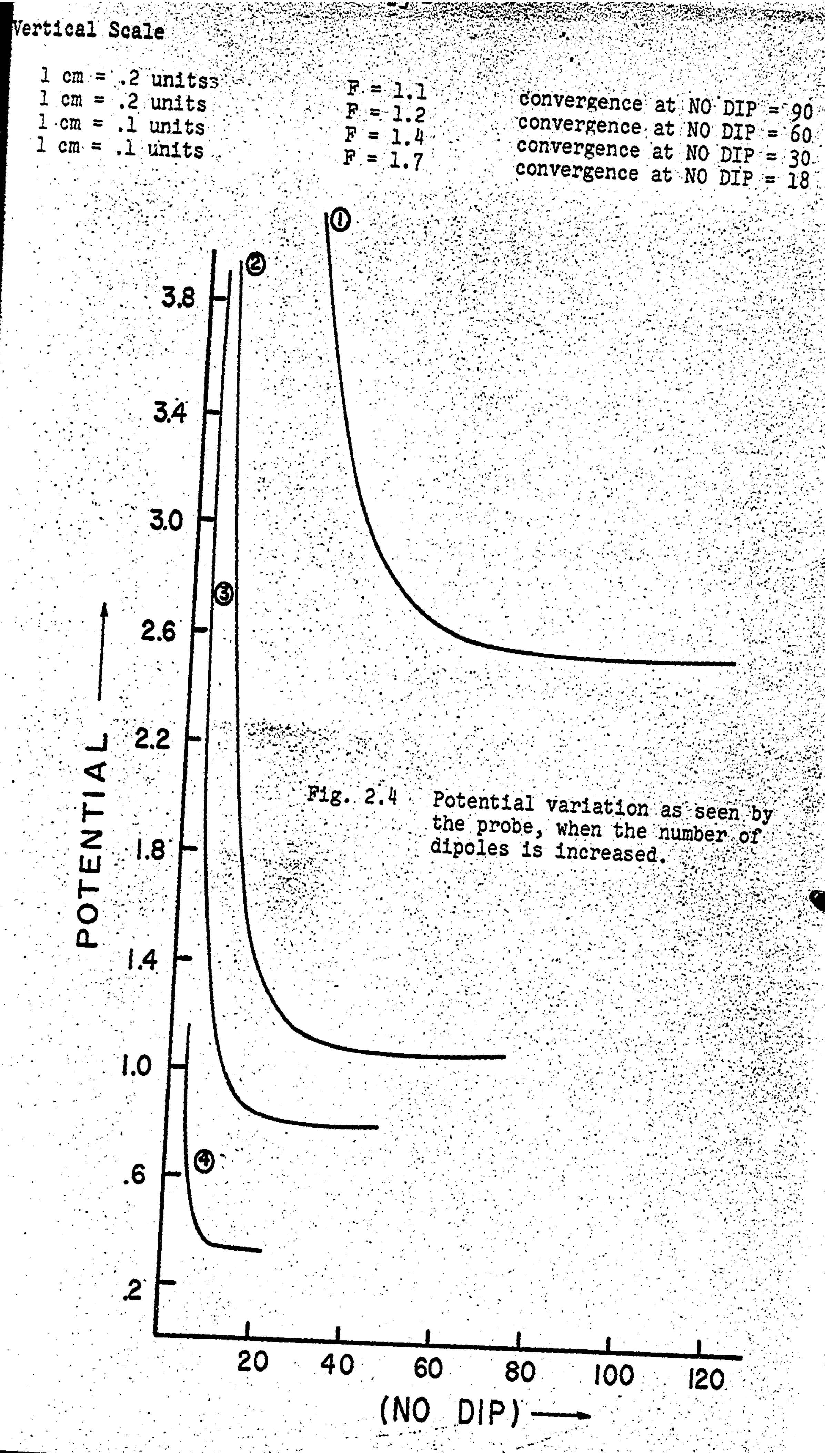


Fig. 2.4 Potential variation as seen by the probe, when the number of dipoles is increased.

Potentials were computed for various positions of the probe. As the number of dipoles was increased the charges became more and more uniformly distributed over the ring, the total charge remaining the same. The purpose of computing potentials as shown in Table 2.2.1, was to determine the number of dipoles necessary to simulate uniformly distributed charge around the ring, for a number of different positions of the probe.

Some typical cases are plotted in Fig. 2.4. We can see that when the probe is sufficiently far from the region of activity, a very small number of dipoles can represent a uniformly distributed layer of charges also known as a "double layer".

The ring shaped wavefront of depolarization (or double layer), as described above, is generally not encountered in practice (see section 1.3).

The irregularly shaped wavefront of depolarization can be considered to consist of a number of segments, where it is possible to assign a direction in which the activity is proceeding. In figure 2.5 we have considered one such hypothetical segment. Potentials were computed using equation (2.2.3). A simple generalized program was written to compute most of the results discussed in this section (Appendix III).

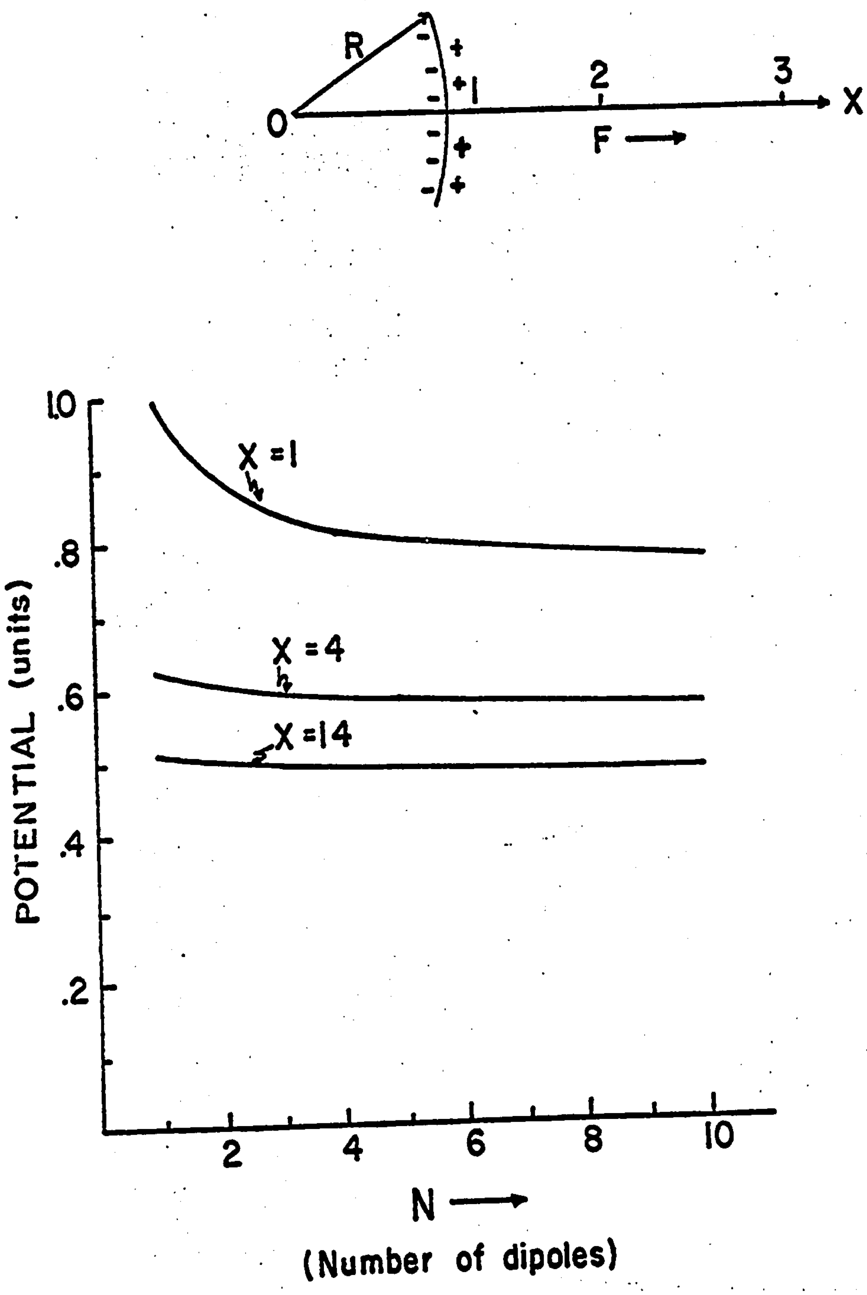


Fig. 2.5 Potential due to dipoles distributed over the segment (symmetrical to axis OX). F indicates the position of probe along the axis OX.

x = OP - R (see Fig. 2.3)

TABLE 2.2.2 Potentials due to dipoles distributed over the segment as shown in Fig. 2.5.

F	Number of dipoles (N)										
	1	2	3	4	5	6	7	8	9	10	
2	1.	.8846	.8410	.8181	.8040	.7944	.7875	.7822	.7781	.7748	10^{0}
3	.2500	.2351	.2286	.2251	.2229	.2215	.2204	.2196	.2189	.2184	10^{0}
4	.1111	.1063	.1041	.1029	.1022	.1016	.1013	.1010	.1008	.1006	10^{0}
5	.6250	.6024	.5920	.5863	.5828	.5804	.5786	.5772	.5761	.5653	10^{-1}
6	.4000	.3872	.3813	.3781	.3760	.3746	.3736	.3728	.3722	.3717	10^{-1}
7	.2777	.2696	.2659	.2638	.2625	.2616	.2609	.2604	.2600	.2597	10^{-1}
8	.2041	.1985	.1959	.1944	.1935	.1929	.1925	.1921	.1918	.1916	10^{-1}
9	.1563	.1522	.1503	.1492	.1486	.1481	.1478	.1475	.1473	.1472	10^{-1}
10	.1235	.1204	.1189	.1181	.1175	.1173	.1170	.1168	.1167	.1166	10^{-1}
11	1.	.9759	.9646	.9583	.9543	.9516	.9497	.9482	.9470	.9460	10^{-2}
12	.8265	.8071	.7980	.7929	.7897	.7876	.7860	.7848	.7838	.7830	10^{-2}
13	.6944	.6786	.6711	.6669	.6643	.6625	.6612	.6602	.6594	.6588	10^{-2}
14	.5917	.5785	.5722	.5687	.5666	.5651	.5640	.5631	.5625	.5619	10^{-2}
15	.5102	.4990	.4937	.4907	.4889	.4876	.4867	.4860	.4854	.4850	10^{-2}

Potential is given in arbitrary units

N = Number of dipoles

F = OP/R (Fig. 2.3)

Last column is the factor by which the entries in the corresponding rows should be multiplied to obtain the value of potential.

As will be seen from the Fig. 24 and table 2.2.2, when the probe is close to the wavefront of activity we need a large number of dipoles to simulate the action of the wavefront. However, for the ratio F≥4, the wavefront can be represented, approximately, by a single dipole placed along the axis ox, at a point where the segment crosses this axis.

From Table 2.2.2, it will be seen that there is always some change in potential when the number of dipoles is increased, however, part of this variation can be attributed to errors due to multiple computations.

2.2.2.2. Potential Variation as a Function of Distance

We will consider here both cases that were discussed in section 2.2.1. For the first case it was shown that when the number of dipoles around the ring is larger than 90, it is equivalent to a double layer as seen by the probe lying as close as F = 1.1. For F>1.1, the action can be simulated by a much smaller number of dipoles. Table 2.2.3 gives the values of potential for a number of values of F.

In Fig. 2.6, we have plotted the values of potential versus distance. As will be seen from this figure, the potential decreases very sharply as the probe moves away from F = 1.1 to about F = 1.7, where the rate of decrease of potential starts

TABLE 2.2.3 Potential variation as a function of distance.

F	Potential	F	Potential	F	Potential
	10^{0}		10^{-2}		10^{-3}
1.1	2.55993	2.0	8.65710	4.0	8.43966
1.2	1.07363	2.1	7.22838	5.0	4.21036
1.3	.61631	2.2	6.10913	6.0	2.40383
1.4	.40430	2.3	5.21735	7.0	1.50206
1.5	.28617	2.4	4.49647	8.0	1.00148
1.6	.21290	2.5	3.90634	9.0	.70129
1.7	.16414	2.6	3.41785	10.0	.51026
1.8	.13001	2.7	3.00958	11.0	.38289
1.9	.10518	2.8	2.66533	12.0	.29471
		2.9	2.37279	13.0	.23170
		3.0	2.12239	14.0	.18548
				15.0	.15081

F = OP/R (Fig. 2.3)

where OP = distance of the probe from the centre of ring

R = radius of ring

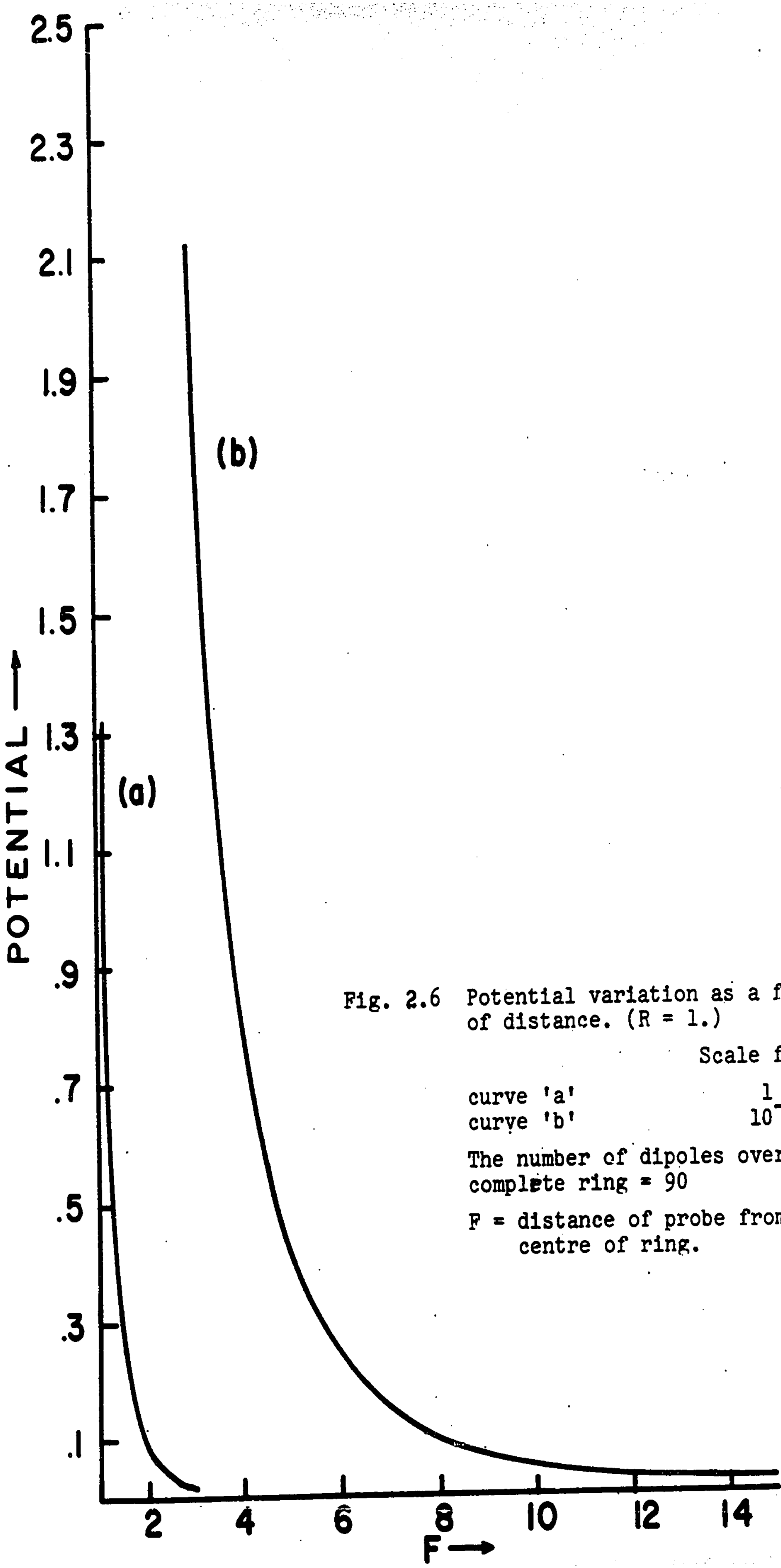


Fig. 2.6 Potential variation as a function of distance. (R = 1.)

	Scale factor
curve 'a'	1
curve 'b'	10^{-2}

The number of dipoles over the complete ring = 90

F = distance of probe from the centre of ring.

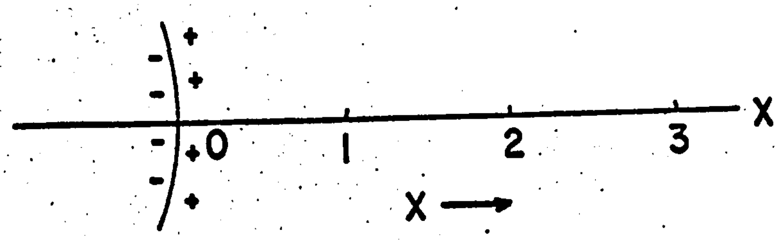
to decrease. For values of $F>3$, the potential varies inversely to approximately cube power of distance.

However, it should be pointed out that the distance considered here is the distance of the probe from the centre of the ring.

For the second case we had considered a segment instead of a complete ring. Using Table 2.2.2, we have drawn curves for two cases i.e., (i) for N = 1 and (ii) N = 5 (Fig. 2.7).

For N = 1, the dipole lies along the axis ox, and the potential varies inversely to the square power of x, as seen by the probe moving along the axis ox.

Consider that we can represent the potential, V with a function of the type $V = Ax^{-B}$ where A is a constant and B may or may not be a constant. Then, for the case of a single dipole just discussed above, B = 2. In case there is more than one dipole, e.g. N = 5, the value of parameter B is found to be <2, particularly for smaller values of x. However, for large values of x, e.g. for x = 13, parameter $B \simeq 2$. Typically $1.8 < B \leq 2$.



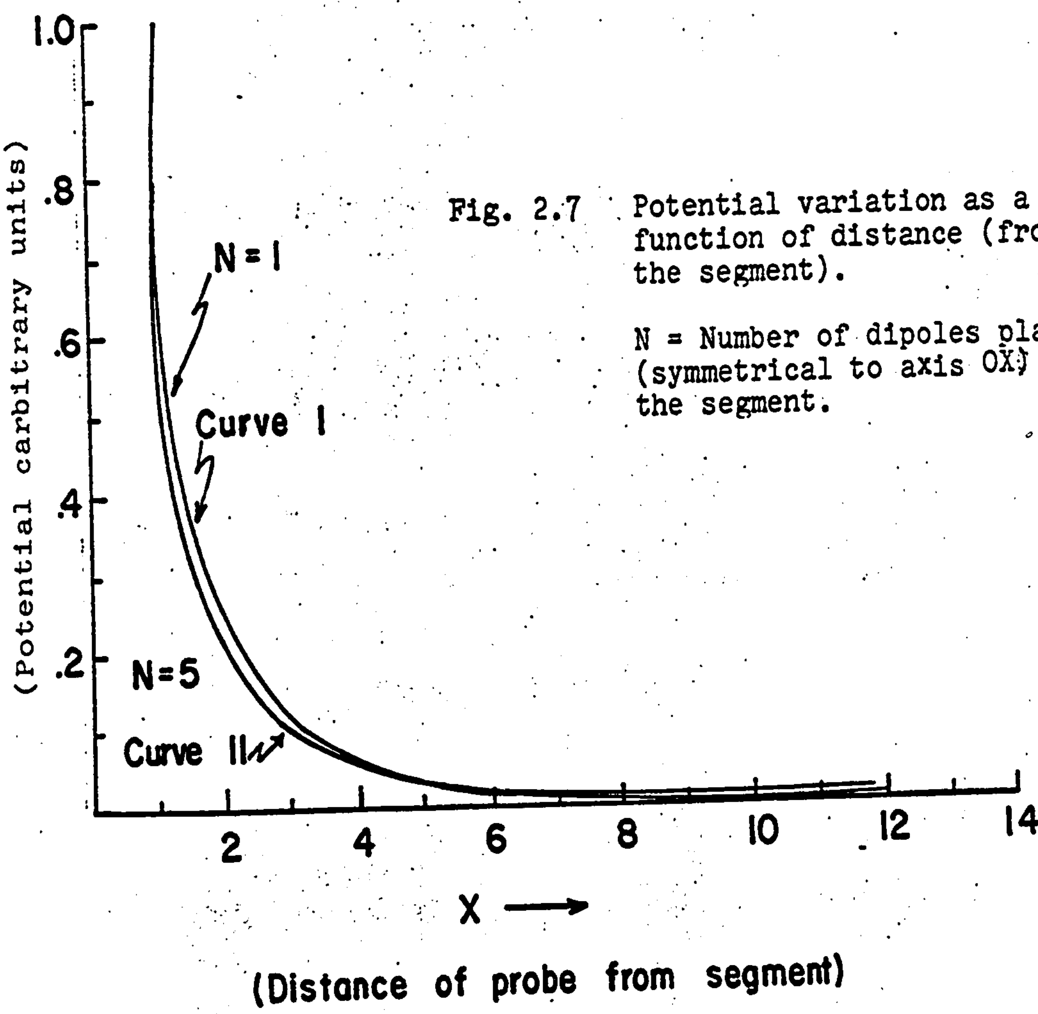


Fig. 2.7 Potential variation as a function of distance (from the segment).

N = Number of dipoles placed (symmetrical to axis OX) on the segment.

CHAPTER 3

MULTIPLE DIPOLE MODEL

3.1 Introduction

In section 1.3, it was shown that the sequence of ventricular depolarization could be described in terms of a number of fairly well defined wavefronts (Fig. 1.3). From our discussion in section 2.2, we have seen that any wavefront of electrical activity can be represented, equivalently, by a number of dipoles.

In calculating potentials at a point due to a number of dipoles (section 2.2), we had limited ourselves to unrestricted homogeneously conducting medium. When the medium is restricted equations such as (2.2.1) and (2.2.2) have to be modified to account for the boundary conditions. If the medium is bound by a regular shape, such as a sphere, a cylinder, a disc or a prolate spheroid, equations can be derived for a single dipole placed anywhere inside these regular bodies [1 - 5]. For a large number of dipoles these methods become impractical. Moreover the shape of the human thorax has no regular geometrical form.

Gelernter and Swihart [13] have shown that it is possible to satisfy boundary conditions by an iterative procedure. In their method potentials are calculated due to a given configuration of dipoles considering them to be in an infinite medium. Then boundary conditions at a finite number of area elements (m) can be satisfied by writing "m" simultaneous equations. The resulting square matrix is inverted by an iterative procedure. The solution may or may not converge for a small number of iterations. Results obtained compared favourably with those derived through direct techniques.

A somewhat similar technique has been suggested by Barr et al [45]. Both of these methods rely heavily on the convergence properties of the iterative procedure. Although essentially the technique is the same in both methods, it has not been possible to compare the results [46].

In section 1.3, we discussed the pathway of ventricular depolarization of the heart. As a result of the experimental work described there, it has been possible to represent ventricular depolarization process of the heart by a suitable number of dipoles located inside the ventricles [47]. The scheme envisages the ventricular mass to be divided into a number of segments, such that the electrical activity in each segment can be represented by a single dipole source of current. Twenty such dipoles have been proposed (Fig. 3.1). The size and the time history of dipole moments was obtained via an extrapolation of Scher's work [21]. The twenty dipole sources were, each, associated with dipole moment functions occurring at a particular

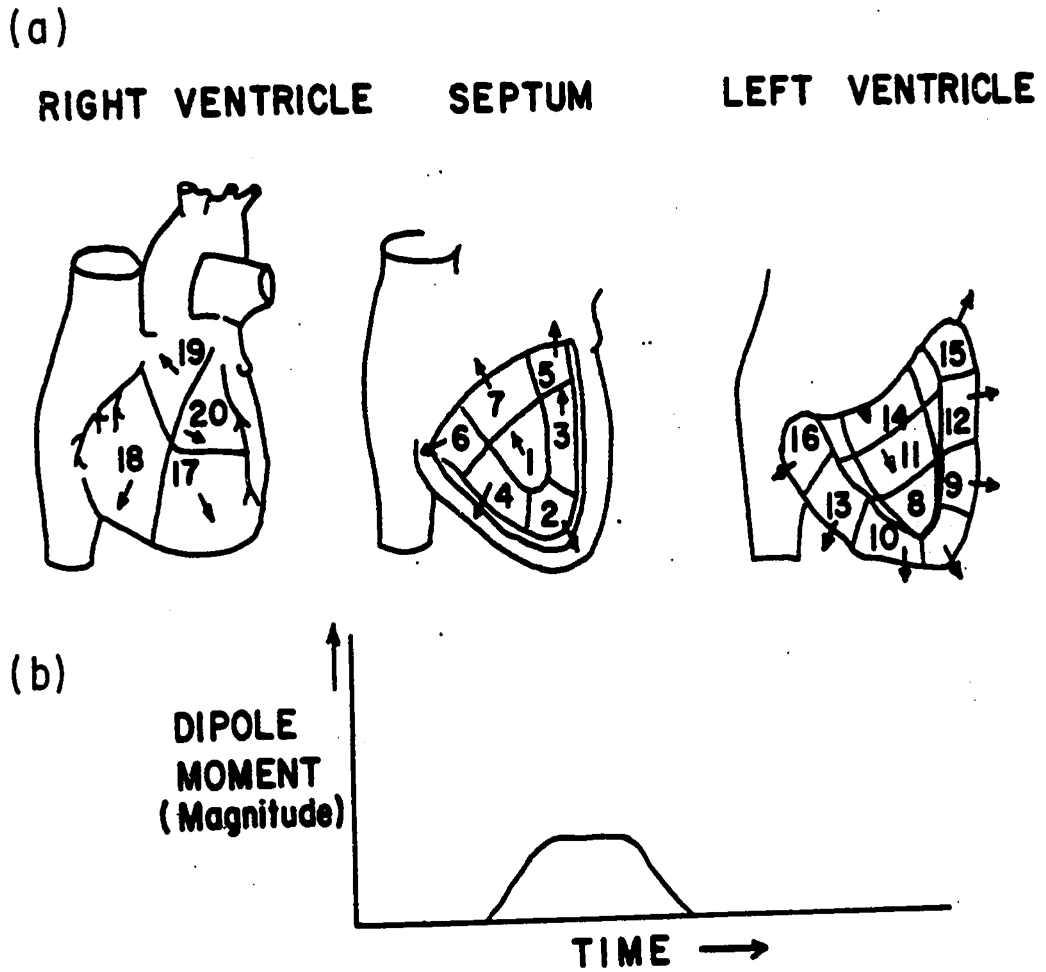


Fig. 3.1 (a) Proposed number of segments with the equivalent dipole sources.

(b) A typical dipole moment function.

For details of time history of all the sources see Appendix IV.

instant corresponding to the time of arrival of the activation wavefront at the segment.

The model as proposed above, for the ventricular depolarization of the heart is a significant step toward multiple-dipole representation. It has been shown that these equivalent sources can be used to produce potentials on the surface of the body in the case of a normal heart and some special cases of heart abnormalities [47, 48].

However, much work has yet to be done to establish the accuracy of the model in respect of (i) dipole moment function associated with each source e.g. its magnitude, time of maxima and the duration and (ii) the optimum number of equivalent sources. This could be achieved either by direct techniques through further experimental exploration of depolarization process or by mathematical studies.

A significant contribution in this regard has come from Bellman et al [48]. In their work, the dipole moment functions such as the one shown in figure 3.1 (b) has been expressed mathematically as a solution of the ordinary differential equation:

$$\frac{dF_i}{dt} = K_i \, (t_i - t) \, F_i \qquad 0 \leq t \leq 80 \text{ msec}, \quad i = 1, 2, ----, M \tag{3.1}$$

and $F_i(0) = h_i$

where F_i - is the dipole moment function for the i-th equivalent source.

K_i - is related to the broadness of the curve.

t_i - is time at which the maximum of the moment occurs.

h_i - maximum value of the moment.

M - total number of equivalent sources.

It should be pointed out that this mathematical representation produces the equivalent sources which are somewhat different from the ones proposed in the original model. Assuming that the sources can be represented by equation (3.1), and under certain conditions, iterative techniques can be used to determine a set of parameters $\{K_i, t_i, h_i\}$ from the surface potentials such that there is "best" agreement between the calculated and the observed values of the surface potentials. The details of the technique are available in literature [49, 50, 51].

In the next section we will present our method for relating a set of equivalent sources to surface electrocardiograms and vice versa.

3.2 Formulation of the Problem

The multiple dipole model shown in Fig. 3.1, was proposed by Selvester et. al. [47]. They used this model to simulate vectorcardiograms. However, in their studies, it has not been shown, whether each of the assumed equivalent sources contributes significantly to the resulting vectorcardiogram. We have already discussed, in section 3.1, some of the other methods for calculating potential resulting from assumed distribution of sources. A second and more important aspect of electrocardiography is to be able to predict the changes in the electrical activity of the heart from the electrocardiograms appearing on the body.

In this chapter we will develop a method to achieve the following main objectives:

(1) to determine the significance of each of the proposed equivalent sources in explaining the electrocardiograms on the surface of the body. The method will be used, mainly, to study the equivalent sources shown in Fig. 3.1. However, any other equivalent sources, whose time history is known, can be used. The solution will be valid provided it satisfies certain conditions described in sections 3.3 - 3.5. Once the significant sources are determined, the transfer (coefficients) matrix relating these sources to the surface electrocardiograms will be evaluated. This will be called direct solution.

(2) to determine whether the transfer matrix found in the direct solution can be used to predict the equivalent sources inside the heart from the electrocardiograms on the body. This is known as the inverse problem [14].

3.2.1 Method

The time history of potential at point "k" on the body due to a number of equivalent dipole sources can be expressed as:

$$Y_k(t) = C_k + \sum_{i=1}^{M} A_{ik} X_i(t) \qquad (3.2)$$

where $X_i(t)$ = i-th equivalent source.

A_{ik} = coefficient relating i-th equivalent source to point 'k' on the body.

C_k = constant term indicating contribution which is not part of equivalent sources.

M = total number of equivalent sources.

It should be noted that the contribution of one equivalent source $X_i(t)$ to potential $Y_k(t)$ is:

$$Y_{ik} = A_{ik}X_i(t) \qquad (3.3)$$

But the potential at point 'k' due to a dipole in an infinite homogeneous (conducting) medium is:

$$V_{ik} = G \frac{\vec{P_i} \cdot \vec{r_{ik}}}{r_{ik}^3} \qquad (3.4)$$

where G = constant of the medium.

$\vec{P_i}$ = dipole moment vector.

$\vec{r_{ik}}$ = radius vector from centre of dipole to point k

Equation (3.4) can be rewritten as:

$$V_{ik} = G \frac{P_i \text{ Cos } \theta_{ik}}{r_{ik}^2} \tag{3.5}$$

where θ_{ik} is the angle between dipole moment vector $\vec{P_i}$ and radius vector $\vec{r}_{ik}$. If θ_{ik} is held constant, equation (3.5) reduces to

$$V_{ik} = B_{ik}P_i \tag{3.6}$$

This, in fact, is identical in form to equation (3.3). Thus coefficients A_{ik} have the same meaning as B_{ik} if $X_i(t)$ represents the moment of an equivalent dipole whose direction is fixed. The basic assumption in arriving at equation (3.3) or (3.6) is that the medium is infinite. This assumption is justified, if each dipole (equivalent) source represents only a small segment of the heart [14, 47].

Let us return now to equation (3.2). We can write

$$\hat{Y}_k(t) = c_k + \sum_{i=1}^{M} a_{ik}X_i(t) \tag{3.7}$$

where $\hat{Y}(t)$ = estimated value of $Y_k(t)$, c_k and a_{ik} are the estimates of parameters C_k and A_{ik} in equation (3.2)

In order to study the significance of each of the equivalent sources in explaining body electrocardiograms, solutions such as (3.7) should be determined for a large number of points covering the entire surface area of the body surrounding the heart. Thus equation (3.7) transforms into a matrix form.

$$[\hat{Y}] = [c] + [a]\,[x] \tag{3.8}$$

where $[\hat{Y}]$ is L x 1 matrix for number 'L' of points on the body .

$[a]$ is L x M matrix characterizing the system.

$[X]$ is M x 1 matrix

$[c]$ is L x 1 matrix

The details of the method for finding coefficients a_{ik} and terms c_k in equation (3.7), and the associated tests are developed in sections 3.3 - 3.6.

3.3 Least Squares Method and Regression Analysis - Introduction [52, 53]

We will begin with a simple example. Let us consider that we are observing a quantity Y, which is linearly related to a variable X i.e.

$$Y_i = \alpha + \beta X_i \tag{3.9}$$

where α and β are the unknown parameters. Given a set of measurements on Y and X, our purpose is to establish the relationship

$$\hat{Y}_i = A + BX_i \tag{3.10}$$

where A and B are the values of the parameters α and β such that the estimated values $\hat{Y}_i$ are as close as possible to the given values Y_i. The straight line in equation (3.9) is called the line of regression of Y on X.

Now $\quad Y_i$ = i-th observed value of Y

$\hat{Y}_i$ = estimate for the i-th observed value of Y

and $\quad d_i$ = deviation between Y_i and $\hat{Y}_i$

$$= Y_i - \hat{Y}_i = Y - A - BX_i \tag{3.11}$$

We will assume that the error terms d_i are independently distributed (according to Gaussian distribution). A discussion concerning this assumption is justified and the associated test will follow in the next section. Under the above assumption, the probability of obtaining a measurement within an interval dY of Y_i is

$$P_i = \frac{dY}{\sigma\sqrt{2\pi}} \exp\left[-(Y_i - \hat{Y}_i)^2/2\sigma^2\right] \tag{3.12}$$

where σ characterizes parent distribution from which Y_i have been obtained. For N measurements

$$P = \prod_{i=1}^{N} P_i$$

$\therefore$ from equation (3.12)

$$P = \left(\frac{dY}{\sigma\sqrt{2\pi}}\right)^N \exp\left[-\frac{1}{2\sigma^2}\sum_{i=1}^{N}(Y_i - \hat{Y}_i)^2\right] \qquad (3.13)$$

Assuming that the set of measurements available to us is the most probable set of measurements, the proper value of Y is the one which maximizes P in equation (3.13). P is maximum for

$$\frac{1}{2\sigma^2}\sum_{i=1}^{N}(Y_i - \hat{Y}_i)^2 \text{ minimum}$$

thus, the most probable estimated values $\hat{Y}$ are obtained by minimizing

$$\sum_{i=1}^{N}(Y_i - \hat{Y}_i)^2 = \sum_{i=1}^{N} d_i^2 = E \qquad (3.14)$$

but

$$E = \sum_{i=1}^{N}(Y_i - A - BX_i)^2 \qquad (3.15)$$

The quantity $E = \sum_{i=1}^{N} d_i^2$ is known as the sum squares deviation, and is minimized by assuming

$$\partial E/_{\partial B} = 0 \quad \text{and } \partial E/_{\partial A} = 0 \qquad (3.16)$$

Applying conditions (3.16) to equation (3.15), we get

$$AN + B\,\Sigma X_i = \Sigma Y_i$$
$$A\Sigma X_i + B\Sigma X_i^2 = \Sigma X_i Y_i \qquad (3.17)$$

Equations (3.17) are known as the normal equations where the number of equations is equal to the number of unknowns. The values of B and A are (from the above equations):

$$B = \frac{N\Sigma Y_i X_i - \Sigma X_i \Sigma Y_i}{N\Sigma X_i^2 - (\Sigma X_i)^2}$$

$$= \frac{\Sigma(Y_i - \overline{Y})(X_i - \overline{X})}{\Sigma(X_i - \overline{X})^2}$$

and

$$A = \overline{Y} - B\overline{X} \tag{3.18}$$

where

$$\overline{X} = \frac{1}{N}\sum_{i=1}^{N} X_i \quad \text{and} \quad \overline{Y} = \frac{1}{N}\sum_{i=1}^{N} Y_i$$

3.4 Statistics of Regression Analysis

3.4.1 Definition of terms and their significance [54, 55]

(1) Variance and standard deviation

Variance or mean square deviation of a set of observations is defined as

$$\sigma_X^2 = \frac{1}{N-1}\sum_{i=1}^{N}(X_i - \overline{X})^2 \tag{3.19}$$

where N = No. of observations

$N-1$ = Available degrees of freedom

X_i = i-th observed value of the variable

$\overline{X}$ = average value of the variable

Square root of the variance, σ_x is the standard deviation. This is also known as the root-mean-square deviation. Equation (3.19) can also be written as

$$\sigma_x^2 = \frac{1}{N-1} \left[\Sigma X_i^2 - N(\overline{X})^2\right] \tag{3.20}$$

(2) Correlation coefficient

In section 3.2, we determined the regression line $\hat{Y}_i = A + BX_i$, giving a linear relationship between X and Y

where B = slope of the line

A = intercept of the line on Y-axis

thus $B = o$ indicates that there is no correlation between X and Y

Consider, now, that we attempt to find the regression line

$$\hat{X}_i = A' + B' Y_i \tag{3.21}$$

Proceeding, as in section 3.2, it can be shown that

$$B' = \frac{\Sigma(Y_i - \overline{Y})(X_i - \overline{X})}{\Sigma(Y_i - \overline{Y})^2} \tag{3.22}$$

and rewriting equation (3.21)

$$Y_i = -\frac{A'}{B'} + \frac{1}{B'}\hat{X}_i \tag{3.23}$$

where $1/_{B'}$ = slope of regression line

$-A'/_{B'}$ = intercept of regression line on Y-axis

If equation (3.10) is a regression line passing through all the given points, then equations (3.10) and (3.23) should be identical and

$B = 1/_{B'}$ i.e. $BB' = 1$, representing perfect correlation between X and Y. In general $BB' \neq 1$, in statistical work, the correlation coefficient is then defined as

$$r = \sqrt{BB'} \tag{3.24}$$

using equations (3.18) and (3.22)

$$r = \frac{\Sigma(Y_i - \overline{Y})\ (X_i - \overline{X})}{\sqrt{(Y_i - \overline{Y})^2}\ \sqrt{\Sigma(X_i - \overline{X})^2}} \tag{3.25}$$

and using equation (3.20)

$$r = \frac{\Sigma(Y_i - \overline{Y})\ (X_i - \overline{X})}{(N - 1)\ \sigma_y\ \sigma_x}$$

$$= \frac{\Sigma YX - N(\overline{Y})(\overline{X})}{(N-1)\ \sigma_y\ \sigma_x} \tag{3.26}$$

Once we calculate the value of r, we can compare it with the probability of obtaining a value of r as large as this from the observations on two variables which are not really correlated (Appendix V). The correlation coefficient is thus an index of linear relationship.

(3) Regression coefficient

Slope of the regression line can now be redefined

$$B = \frac{\Sigma(Y_i - \bar{Y})\,(X_i - \bar{X})}{\Sigma(X_i - \bar{X})^2}$$

$$= r\,\frac{\sigma_y}{\sigma_x} \qquad (3.27)$$

(4) Coefficient of determination

let S_y^2 = sum squares deviation of Y

then $\sigma_y^2 = \frac{S_y^2}{N-1}$

and $S_{y/x}^2$ = sum squares deviation explainable by the regression of Y on X

$= r^2 S_y^2$

$\sigma_{y/x}^2$ = estimated variance of Y given X

$$= \frac{S_y^2 - S_{y/x}^2}{N-2} \qquad (3.28)$$

then the coefficient of determination is defined as

$$R^2 = 1 - \frac{\sigma^2_{y/x}}{\sigma^2_y} \tag{3.29}$$

where $0 \leq R^2 \leq 1$

If the regression line is such that all the given points lie very close to it, then R^2 is very large. However, as the scatter of points becomes larger, R^2 becomes smaller. Thus R^2 is a measure of the strength of the relationship in the regression of Y on X. This is also, known as the measure of goodness of fit. We will denote coefficient of determination by the symbol "CD".

(5) Coefficient of variation

$$C_v = \frac{\sigma_{y/x}}{\bar{Y}} \tag{3.30}$$

This indicates the ratio of dispersion of the given points from regression line relative to the average value of the dependent variable.

(6) Standard error of coefficient (or slope) and t-test

Standard error of the coefficient is defined as

$$\sigma_B = \frac{\sigma_{y/x}}{S_x}$$

$$\text{where} \quad S_x = \sqrt{\Sigma(X_i - \bar{X})^2} \tag{3.31}$$

The application of the numerical technique of regression analysis will always give some value of the coefficient $B_{y/x}$ (defined as the value of the coefficient found by regression of Y on X) even if no relation exists between Y and X. Therefore the value of the coefficient has to be tested for the "hypothesis" that the value obtained is non-chance deviation from zero. Since estimates of slope are assumed to have Student's "t" distribution, the test is defined as

$$t = \frac{B_{y/x}}{\sigma_B} \qquad (3.32)$$

This calculated t-value is compared with the values given in the table at a given significance level (Appendix VI).

Let $T(n,S)$ be the value of "t" determined from the table

where n = number of degrees of freedom

S = % significance level for null hypothesis

Therefore:

If t calculated $< T(n,S)$

Parameter $B_{y/x}$ is rejected

If t calculated $\geq T(n,S)$

Parameter $B_{y/x}$ is accepted.

3.4.2 Independence of Regression Disturbances

A basic assumption in the application of least squares method is that the error terms in the regression model are independent. However this may not be true in all the cases. Thus it is necessary to test for the independence of error terms. The first major contribution in this regard came from Durbin and Watson [57, 58].

They have outlined a procedure to test for the independence of error terms based on von Neumann ratio of the least squares estimated disturbances, Q.

where

$$Q = \frac{\Sigma\{d_i - d_{i-1}\}^2}{\Sigma d_i^2} \quad (3.33)$$

where d_i = least square estimate of disturbance in the regression equation for the i-th point of observation.

A significant modification of Durbin and Watson's test came from Theil and Nagar [59]. We will limit ourselves to the discussion of the latter method. Assuming that under the null hypothesis Q is stochastically independent of its denominator (i.e. any moment of Q is equal to the ratio of corresponding moments of numerator and denominator) and by fitting a beta distribution to the true distribution of Q, the test values of Q can be calculated as follows:

$$Q(s) = \frac{J^2 - 1}{N^2} + (4 - \frac{J^2 + 2}{N^2})X \quad (3.34)$$

where Q(s) = desired value of von Neumann ratio at the given significance level

J = total number of variates in the regression solution (=M+1)

X = value of beta variate at the desired significance level.

Equation (3.34) is straight forward except for the value of the beta variate "X". There is no direct method available for computing "X" for a given combination of N and J at a desired significance level. Tables of the Incomplete Beta Function are available [60], and the values not given there may be found by extrapolation. We have developed a program for comptuing values of beta variate for any combination of N,J and S (Appendix VII).

Thus the procedure for evaluating Q(s) is as follows:

(1) Compute von Neumann ratio, Q, from equation (3.33). This is generally included in the program developed for regression analysis.

(2) Determine the values of beta-variate for the given N and J at the desired significance level.

(3) Compute Q(s) (Appendix VII)

then if $Q < Q(s)$, the error terms are not independently distributed (Null hypothesis is rejected).

if $Q \geq Q(s)$, the error terms are independently distributed and the least squares estimates are valid. (Null hypothesis is not rejected.)

Theil and Nagar [59] have also suggested a method for re-computing the values of regression coefficients in case the error terms are not independently distributed. However, in our work, we have found that their method cannot be applied in a variety of cases.

The correction factor defined by them is

$$F = \left(\frac{N^2 + J^2 - 1}{N^2 - J^2 - .5}\right) - \left(\frac{.5N^2}{N^2 - J^2 - .5}\right) Q \qquad (3.35)$$

In figure 3.2, we have plotted the behavior of this factor for various values of Q.

Corresponding values of Q(s) were found from equation (3.34) according to the procedure outlined. The values are also shown on the figure for the four cases described there.

In cases (1) and (2), we see that this factor is zero at Q = 2.86 and Q = 2.56 respectively, although in both cases

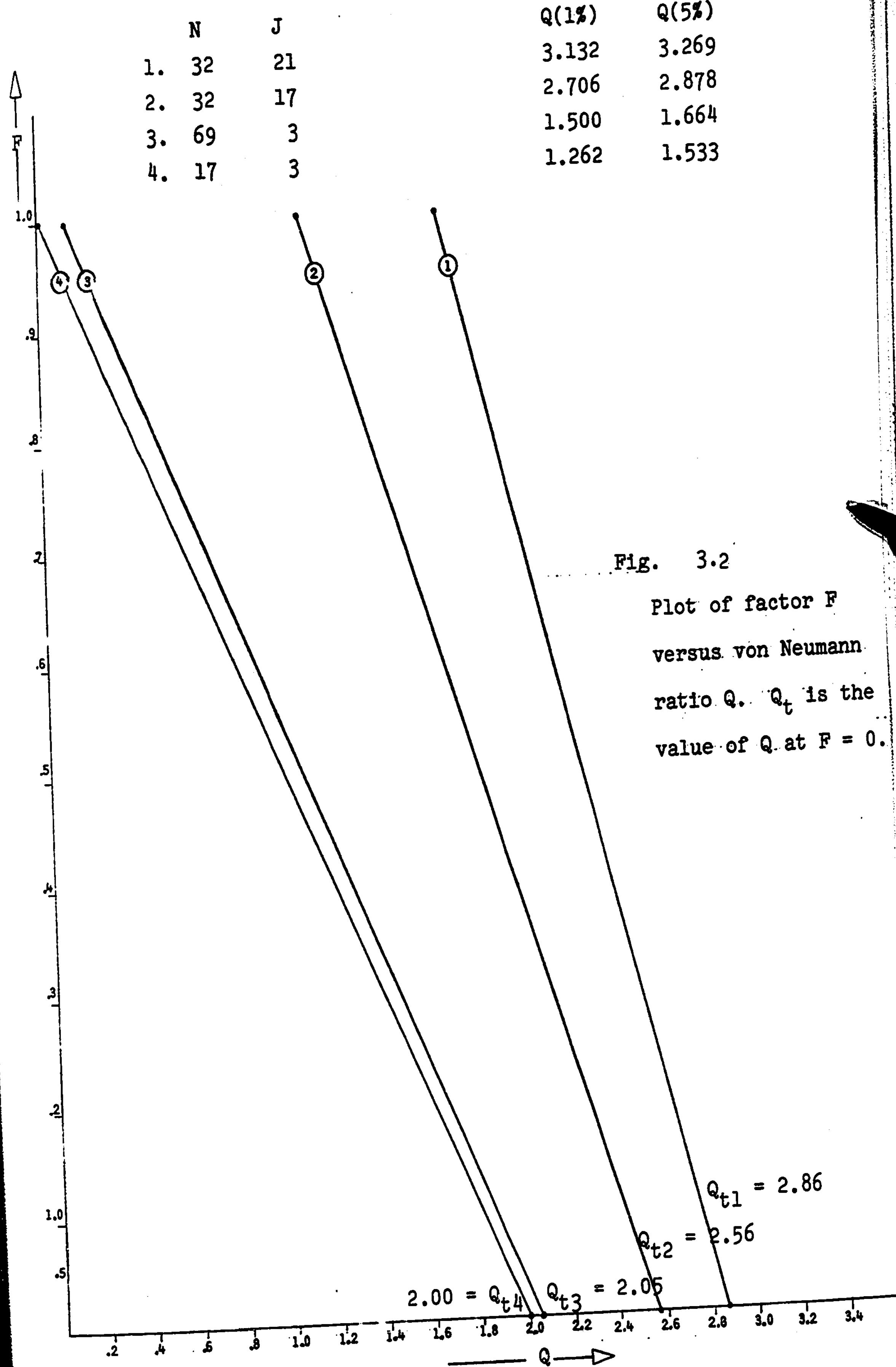


Fig. 3.2

Plot of factor F versus von Neumann ratio Q. Q_t is the value of Q at F = 0.

$Q<Q(s)$ at the given significant levels.

On the other hand for cases (3) and (4), factor F goes to zero at much higher values than the desired level Q(s).

Thus for cases where the number of observations is not very large as compared to the number of variates in the regression, use of F results in under compensation. When the number of observations is very large as compared to the number of variates, use of F results in over compensation.

From equation (3.35), let Q_t be the value of Q at which $F = 0$

then
$$Q_t = \frac{N^2 + J^2 - 1}{.5N^2} \tag{3.36}$$

Substituting $Q(s) = Q_t$ in equation (3.28), we can find the corresponding value of beta variate, which is $X = 0.5$. Thus factor F passes through zero at a fixed value of beta variate resulting in shifting of significance level for different values of N and J.

We are suggesting, a modification of equation (3.35)

i.e.
$$F = \frac{.5N^2}{N^2 + J^2 - 1} (Q(s) - Q) \tag{3.37}$$

This will result in achieving a desired significance level whenever a correction is needed. For some pertinent remarks attention is drawn to footnotes in reference [59].

3.5 Simple Regression Model

In this section a typical regression solution for the equation (3.2) with numerical evaluation of the statistics discussed in section 3.2, will be given. In Table 3.1, values are given for a set of observations on the independent variable X and the dependent variable Y.

In appendix VIII, a digital computer program is given which will compute all the relevant statistics for this simple regression model.

The results are shown in Table 3.2, and a plot of the regression line is given in Fig. 3.3. Brief analysis of the results is as follows:

Correlation coefficient is greater than .75, showing a significant correlation between Y and X.

TEE = 4.1256 > T(13,05) = 2.16, thus the slope estimate is reliable.

Q = 2.37 > Q(s) = 1.36, thus error terms are independently distributed.

Coefficient of determination is small (=.533), showing that the fit obtained is not very good. This indicates that there are some other explanatory variables missing, which should be included in the regression.

TABLE 3.1 Input Data for Regression Model
(Single Explanatory Variable)

No. of observation	Dependent variable	Independent Variable
N	X	Y
1	54.	15.
2	19.	16.
3	30.	14.
4	64.	22.
5	60.	24.
6	53.	19.
7	29.	13.
8	55.	15.
9	62.	23.
10	33.	12.
11	68.	25.
12	42.	17.
13	45.	18.
14	39.	19.
15	39.	18.

Note: This example will be continued in the next section for a multivariate case.

TABLE 3.2 Computer output for the data given in Table 3.1

Standard Deviation of X	= 14.6672
Standard Deviation of Y	= 4.03578
Correlation Coefficient	= 0.752974
Regression Coefficient	= 0.207185
Constant	= 8.44185
Standard Error of Coefficient	= 0.0502187
*Tee Value	= 4.12565
Coefficient of Variation	= .153110
Sum Squares Deviation Explained	= 129.283
Sum Squares Deviation of Y	= 228.025
Coefficient of Determination	= .533660

*Test value for null hypothesis at given significance level (see section 3.4.1 (vi)

$T(13,05) = 2.16$ [Appendix VI]

Y-GIVEN (Y_i)	Y-CALC ($\hat{Y}$)	ERROR (E_i)
15.	19.6298	-4.62986
15.	12.3783	3.62162
14.	14.6574	-0.657409
22.	21.7017	.298294
24.	20.8729	3.12703
19.	19.4226	- .422668
13.	14.4502	-1.45022
15.	19.8370	-4.83704
23.	21.2873	1.71266
12.	15.2789	-3.27896
25.	22.5304	2.46955
17.	17.1436	- .143631
18.	17.7652	.234809
19.	16.5220	2.47792
18.	16.5220	1.47792

† von Neumann ratio of the least squares estimated disturbances = 2.37356

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† Test value for null hypothesis at given significance level (see section 3.4.2)

Q(05) = 1.36

[Appendix VII]

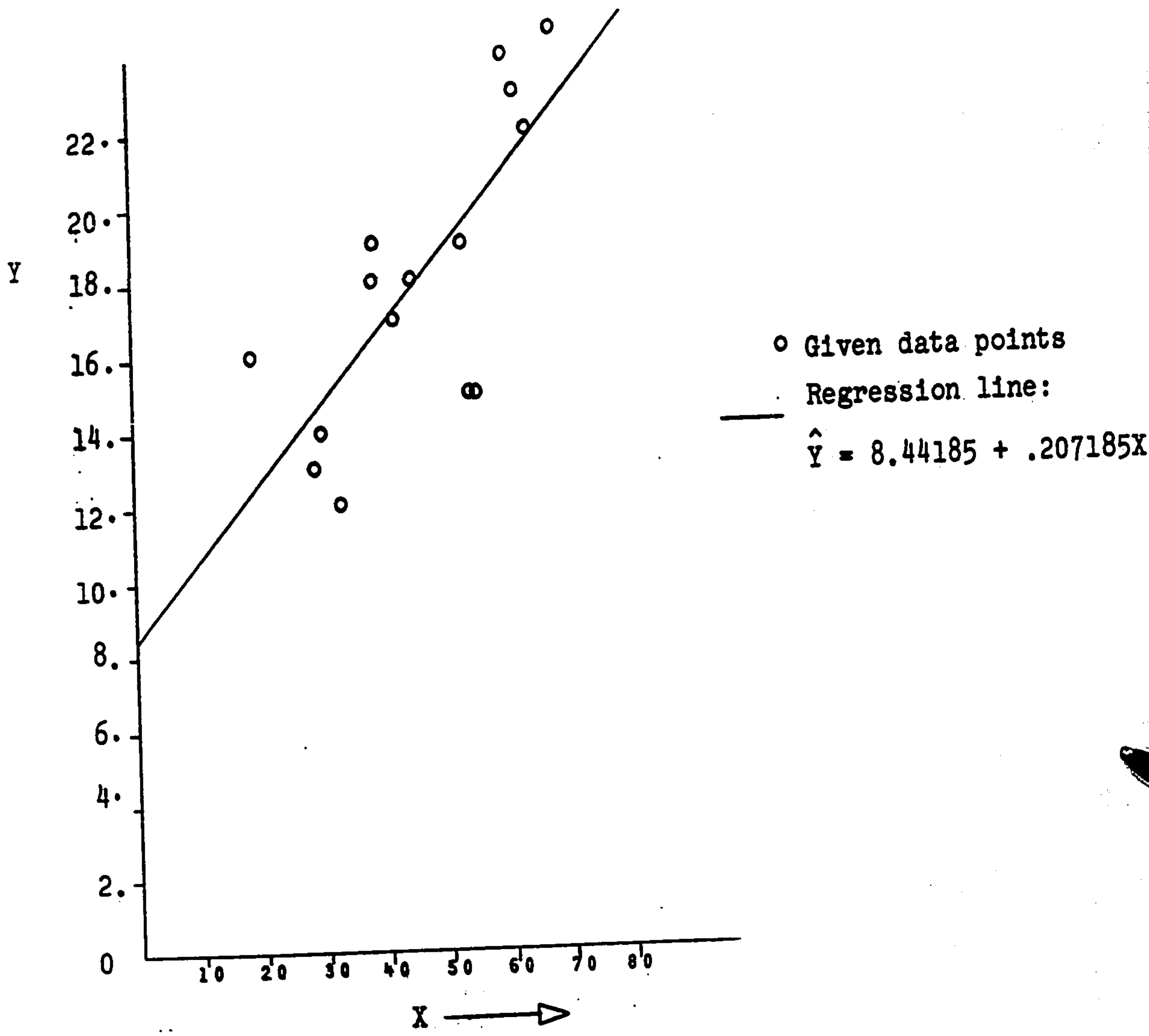


Fig. 3.3 Regression of Y on X

It has been suggested [59, 61] that in the case of single explanatory variable, data should be arranged in the order of increasing values of the explanatory variable before computing Q. Following this procedure Q = 1.62547 in the above example. However this is still larger than Q(s).

3.6 Multivariate Regression Model

We will now discuss the model proposed in section 3.2 i.e.

$$\hat{Y}(t) = c_k + \sum_{i=1}^{M} a_{ik} X_i(t)$$

Let

$$y_j = Y(t_j) - \bar{Y} \tag{3.38}$$

$$x_{ij} = X_i(t_j) - \bar{X} \tag{3.39}$$

where $Y(t_j)$ and $X_i(t_j)$ are the observed values at $t = t_j$, then from the criterion developed in section 3.3, in order to find best possible values of parameters a_{ik}, we should minimize

$$E = \sum_{j=1}^{N} [\, y_j - \sum_{i=1}^{M} a_{ik} x_{ij}]^2 \tag{3.40}$$

and the residual constant term would, then, be determined from the relation

$$c_k = \bar{Y} - \sum_{i=1}^{M} a_{ik} \bar{X}_i \tag{3.41}$$

To minimize E, in equation (3.40), we must equate to zero each of its partial derivatives

$$\partial E/\partial a_{1k}, \ \partial E/\partial a_{2k}, \ ----- \ \partial E/\partial a_{mk}$$

which for $\partial E/\partial a_{1k}$, for example, gives the equation

$$a_{1k}\sum_{j=1}^{M} x_{1j}x_{1j} + a_{2k}\sum_{j=1}^{N} x_{1j}x_{2j} + ----- a_{mk}\sum_{j=1}^{N} x_{1j}x_{mj}$$
$$= \sum_{j=1}^{N} x_{1j}y_j \qquad (3.42)$$

We will, thus, obtain a set of M equations in M unknowns, known as the normal equation. In practice these equations can be written down by looking at the right hand side of equation 3.40 [62].

Let

$$s_{mn} = \sum_{j=1}^{N} x_{mj}x_{nj}$$

and

$$g_m = \sum_{j=1}^{N} x_{mj}y_j \qquad (3.43)$$

then the normal equations are

$$a_{1k}s_{11} + a_{2k}s_{12} + ------- a_{mk}s_{1m} = g_1$$
$$a_{1k}s_{21} + a_{2k}s_{22} + ------- a_{mk}s_{2m} = g_2$$
$$\vdots \qquad (3.44)$$
$$a_{1k}s_{m1} + a_{2k}s_{m2} + ------- a_{mk}s_{mm} = g_m$$

In matrix notations equations (3.44) are written as:

$$[s]\,[a_k] = [g] \tag{3.45}$$

where

$[s]$ is a square matrix of order M; it is a symmetric matrix since $s_{mn} = s_{nm}$

$[a_k]$ and $[g]$ are column vectors.

From equation (3.45)

$$[a_k] = [s]^{-1}\,[g] \tag{3.46}$$

In our work [s] will be, in general, a high order matrix.

von Neumann and Goldstine [63, 64] have given a detailed analysis of problems involved in inverting a high order matrix. A complete survey of the methods for solving equations such as (3.45) appears in a paper by Forsythe [65]. The development of digital computer field has resulted in the development of methods which minimize the errors that are intrinsic to the computation [67].

Two of the well known methods [67, 68, 69] for solving systems of linear equations on digital computer are (i) Gauss-Jordan Elimination Method (ii) Gauss-Seidel Iterative Method. In the following subsections we will, first, discuss briefly these methods and then give details of the procedure particularly suited for our system.

3.6.1 Gauss-Jordan Method

In this method, the coefficient matrix is reduced to a unity matrix. The reduced column vector [g] (see equation 3.45) is, then the desired solution.

Consider a general system represented by

$$
\begin{aligned}
s_{11}x_1 + s_{12}x_2 + \text{- - - - - - -}\ s_{1m}x_m &= g_1 \\
s_{21}x_1 + s_{22}x_2 + \text{- - - - - - -}\ s_{2m}x_m &= g_2 \\
&\vdots \\
s_{mn}x_1 + s_{m2}x_2 + \text{- - - - - - -}\ s_{mm}x_m &= g_m
\end{aligned}
\qquad (3.47)
$$

The augmented coefficient matrix for this system is

$$
\begin{bmatrix}
s_{11} & s_{12} & s_{13} & \text{- - - - - - - -} & s_{1m} & s_{1n} \\
s_{21} & s_{22} & s_{23} & \text{- - - - - - - -} & s_{2m} & s_{2n} \\
\vdots & & & & & \\
s_{m1} & s_{m2} & s_{m3} & \text{- - - - - - - -} & s_{mm} & s_{mn}
\end{bmatrix}
$$

where $n = m + 1$, and $s_{1n} = g_1$, $s_{2n} = g_2$ etc.....

The algorithm for eliminating the off diagonal elements and obtaining unity digonal elements is as follows:

At the i-th stage in the elimination process, the new elements in matrix (3.48) are

$$s_{ij} = \frac{s_{ij}}{s_{ii}} \quad \text{for } j = 1, 2, ----, n$$

$$s_{kj} = s_{kj} - s_{ij}\frac{s_{ki}}{s_{ii}} \quad \text{for } k = 1, 2, ----, m \text{ except } k = i \text{ and for } j = 1, 2, ----, n \qquad (3.49)$$

The elimination process is completed when i = m. Gauss-Jordan method has an advantage in accuracy provided the matrix is positive definite [69]. Moreover, in the case of symmetric coefficients, the labor can be cut in half.

3.6.2 Gauss-Seidel Method

Gauss-Seidel method is based on iterative technique. Consider the system represented in equation (3.47). We can write

$$x_1 = \frac{1}{s_{11}} (g_1 - s_{12}x_2 - s_{13}x_3 ------ s_{1n}x_m)$$

$$x_2 = \frac{1}{s_{22}} (g_2 - s_{21}x_1 - s_{23}x_3 ------ s_{2n}x_m) \qquad (3.50)$$

$$\vdots$$

$$x_m = \frac{1}{s_{nn}} (g_m - s_{m1}x_1 - s_{m2} ------ s_{m,\, m-1}x_{m-1})$$

An initial approximation to the solution is taken as

$$\begin{aligned} x_1 &= g_1/s_{11} \\ x_2 &= g_2/s_{22} \\ &\vdots \\ x_m &= g_m/s_{mm} \end{aligned} \qquad (3.51)$$

For the next approximation values from equation (3.51) are substituted into the right hand side of equation (3.50). The process is continued until successive approximations for all unknowns are within some preset tolerance.

Gauss-Seidel method is useful for equations which can be so arranged that diagonal elements of coefficient matrix dominate the other elements [70]. A sufficient condition for the convergence is

$$\begin{aligned} |s_{ii}| > |s_{i1}| + |s_{i2}| + - - - - |s_{i,\, i-1}| \\ + |s_{i,\, i+1}| + |s_{im}| \end{aligned} \qquad (3.52)$$

3.6.3 Stepwise Regression solution using Gauss-Jordan method

From the discussion in section 3.2, it will be seen that we are not only interested in the final solution obtainable by methods described in sections 3.6.1 and 3.6.2, but some intermediate results are also required. For this purpose the method particularly suited is stepwise regression solution [71]. In this method solutions are obtained by including one explanatory variable at a time. The variable included is the one which contributes most to the sum squares deviation of the dependent variable. This process is continued and at each step one more explanatory variable is included. This is important in studying the influence of each explanatory variable on the solution obtained.

An outline of the computational scheme is as follows: From equations (3.43) and (3.45), [s] is a square matrix whose elements are

$$s_{mn} = \sum_{j=1}^{N} x_{mj} x_{nj}$$

A unity diagonal element can be obtained in the matrix by transforming the elements s_{mn} into simple correlation coefficients between the variables x_m and x_n.

Thus, from equation (3.26) and (3.43)

$$r_{mn} = \frac{s_{mn}}{(N-1)\sigma_m \sigma_n} \tag{3.53}$$

and

$$g_{1m} = \frac{g_m}{(N-1)\sigma_m \sigma_y} \tag{3.54}$$

Let

$$z = \frac{\sum_{j=1}^{N} y_j^2}{(N-1)\sigma_m \sigma_y} \tag{3.55}$$

Therefore, we can represent R, G and Z as the normalized form of matrices whose elements are given by equations (3.53 - 3.55).

where R is an mxm matrix

G is lxm matrix

and Z is a scalar

We can apply linear transformations to the partitioned matrix:

$$\begin{bmatrix} R & G' & I \\ G & Z & D \\ -I & B & C \end{bmatrix} \tag{3.56}$$

Matrix (3.56) is a square matrix of order 2m + 1,

where

$$(G')_{ml} = (G)_{lm}$$
$$(B) = (C) = (D) = 0 \qquad \text{(initially)} \tag{3.57}$$
$$I_{mn} = \delta_{mn}$$

also C and I are both mxm matrices.

Although matrix (3.56) is of order (2m + 1), it can be handled in the computer as (m + 1) x (m + 1) square matrix, since all other terms are zero or unity and can be generated as needed.

At every step, in successive row elimination of R matrix, B matrix contains the regression coefficients and C contains inverse of the partitioned part of R matrix corresponding to the variables in the regression at this step.

In appendix IX, we have carried out, step by step, computation in the case of regression with three explanatory variables. The example was chosen from Efroymson [71] for comparison of results. The statistics of regression model were calculated according to our discussion in the earlier sections of this chapter.

The digital computer program for carrying out regression analysis was written in Fortran II, for use on IBM System 1620. The results reported in section 4.2 were obtained on this computer. The program was modified when the computing centre at Ottawa University acquired IBM System 360/Model 40. This program is included in Appendix X with a sample of printout.

In some cases, if it is desired to obtain only the final solutions, then the methods described in sections 3.6.1 and 3.6.2 can be used directly. For this purpose three subroutines were developed:

(i) SUBROUTINE CORRE

this subroutine reduces the system given by equation (3.7) into normalized matrices R and G as described by equations (3.53), (3.54) and (3.56).

(ii) SUBROUTINE SIMULT

this subroutine solves the system of equations represented by matrix [R G] by Gauss-Jordan elimination method.

(iii) SUBROUTINE SOLVGS

this subroutine solves the system of equations represented by matrix [R G] by Gauss-Seidel iterative method.

Any two of these subroutines may be used with a main program to obtain the desired solution. The digital computer output using (i) & (ii) and (i) & (iii) is shown in Appendices XI and XII.

CHAPTER 4

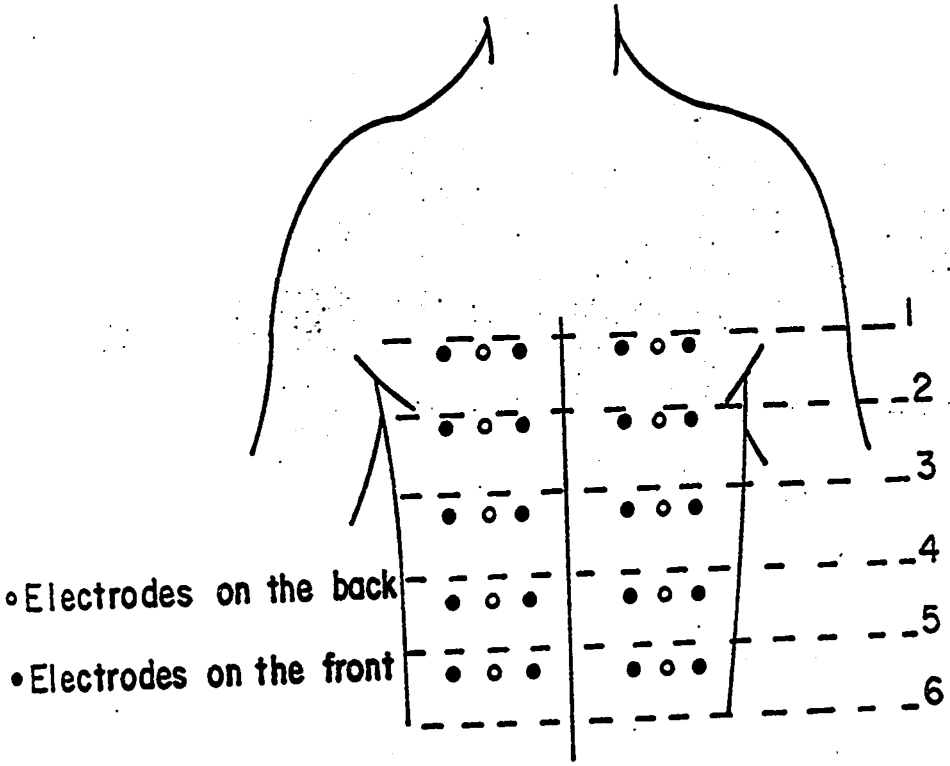
EVALUATION AND ANALYSIS

OF SYSTEM PARAMETERS

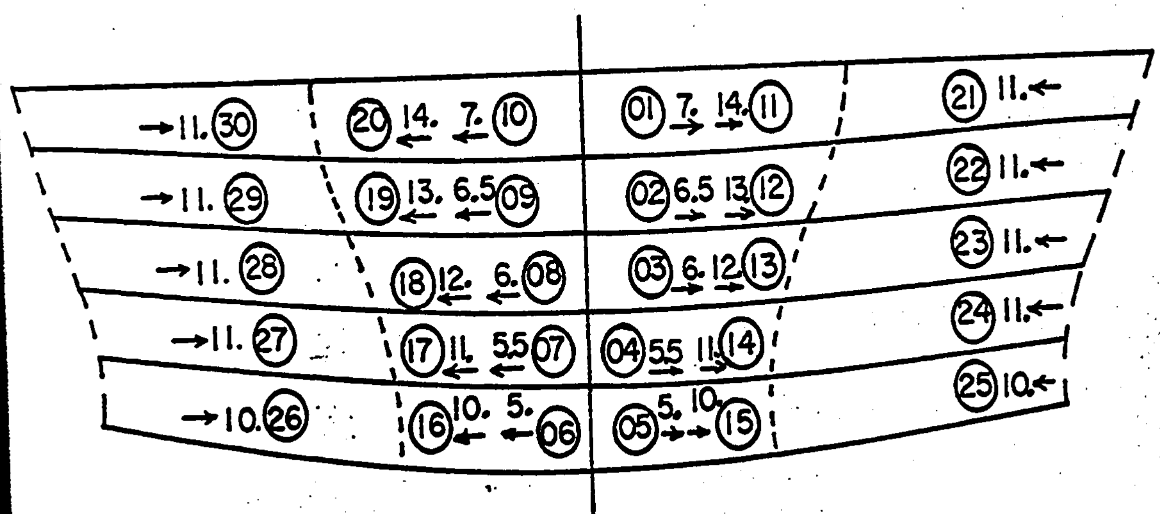
4.1 Preparation of Input Data

In order to solve the model proposed in section 3.1, it is required to record a number of electrocardiograms on the surface of the body. Moreover the points at which these electrocardiograms are recorded should be distributed in such a manner that we will be able to observe the influence of each individual equivalent source inside the heart.

For present studies, we developed the configuration shown in figure 4.1. A total of thirty points were marked to record the unipolar ECGs. Twenty of the points were in the front and ten on the back. The points were marked 01, 02, - - - -, 30 and these are the values of the subscript "K" in equation (3.1). For recording the ECGs, facilities were made available to us by the Ottawa General Hospital. Conventional lead connections were used. Each unipolar electrocardiogram was visually monitored along with a lead II reference electrocardiogram. At least three sets of ECGs were photographed for each point. Each set of records contained a unipolar and a lead II ECG with time markings 40 msec apart. Sensitivity of the recording unit was calibrated by a standard pulse and kept constant throughout. After ECGs had been taken at all the points, the photographic paper was developed and the records were examined. QRS portion of these ECGs is reproduced



(a)



(b)

Fig. 4.1 (a) General layout of the positions of electrodes for recording unipolar electrocardiograms.

(b) The portion between levels 1-6 in figure 'a' has been unfolded to show the position of each electrode. Distance (in cms) is shown under each position (with reference to the middle of front and back).

(Subject: Weight 130 lbs.; Height 5'6")

in figure 4.2. It may be pointed out that the quality of the ECGs obtained by us may not be the best possible and could be improved through the use of better techniques.

Thirty two samples, each 2.5 msec apart, were taken on each ECG for use on digital computer. These samples are shown in appendix XIV. Fig. 4.3, shows general computational procedure used in carrying out regression analysis of the direct solution.

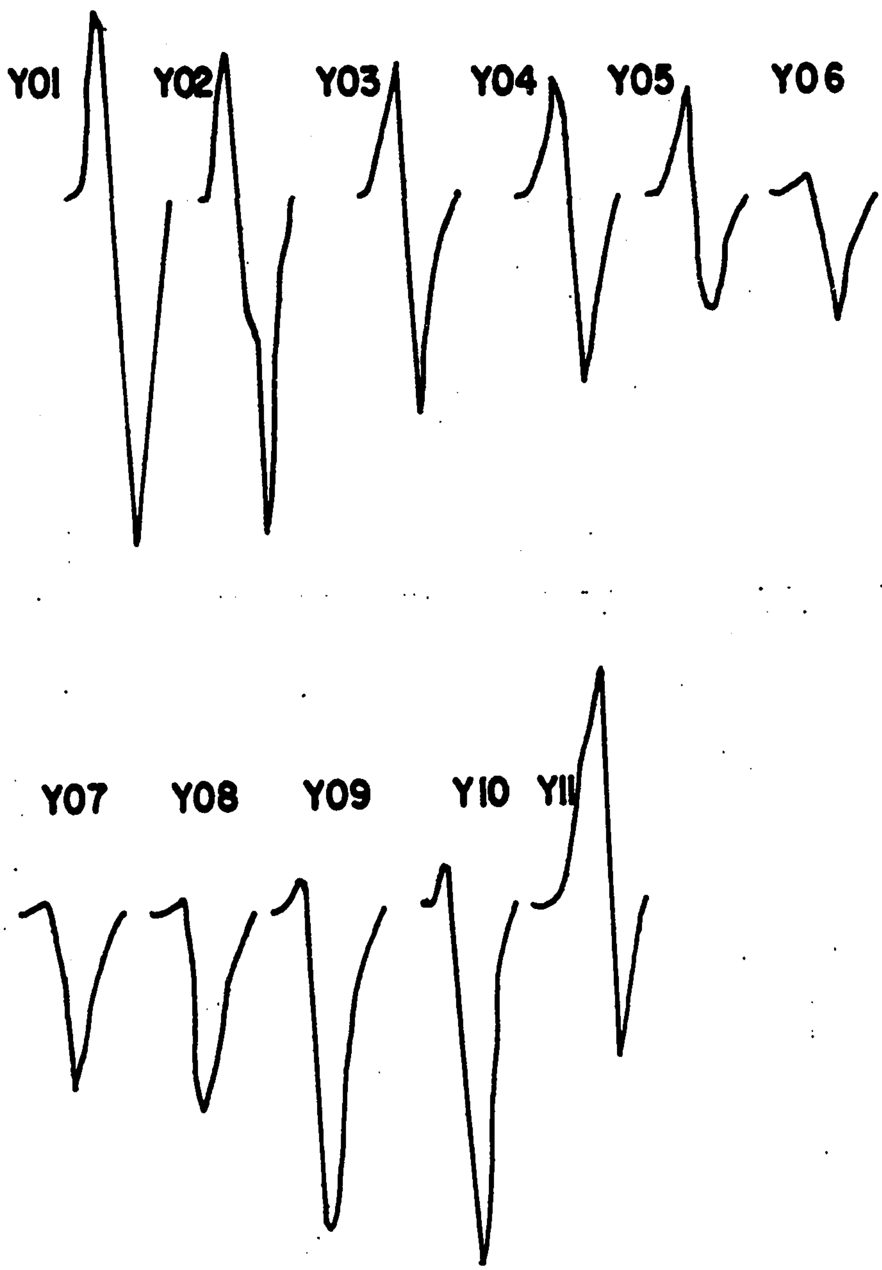
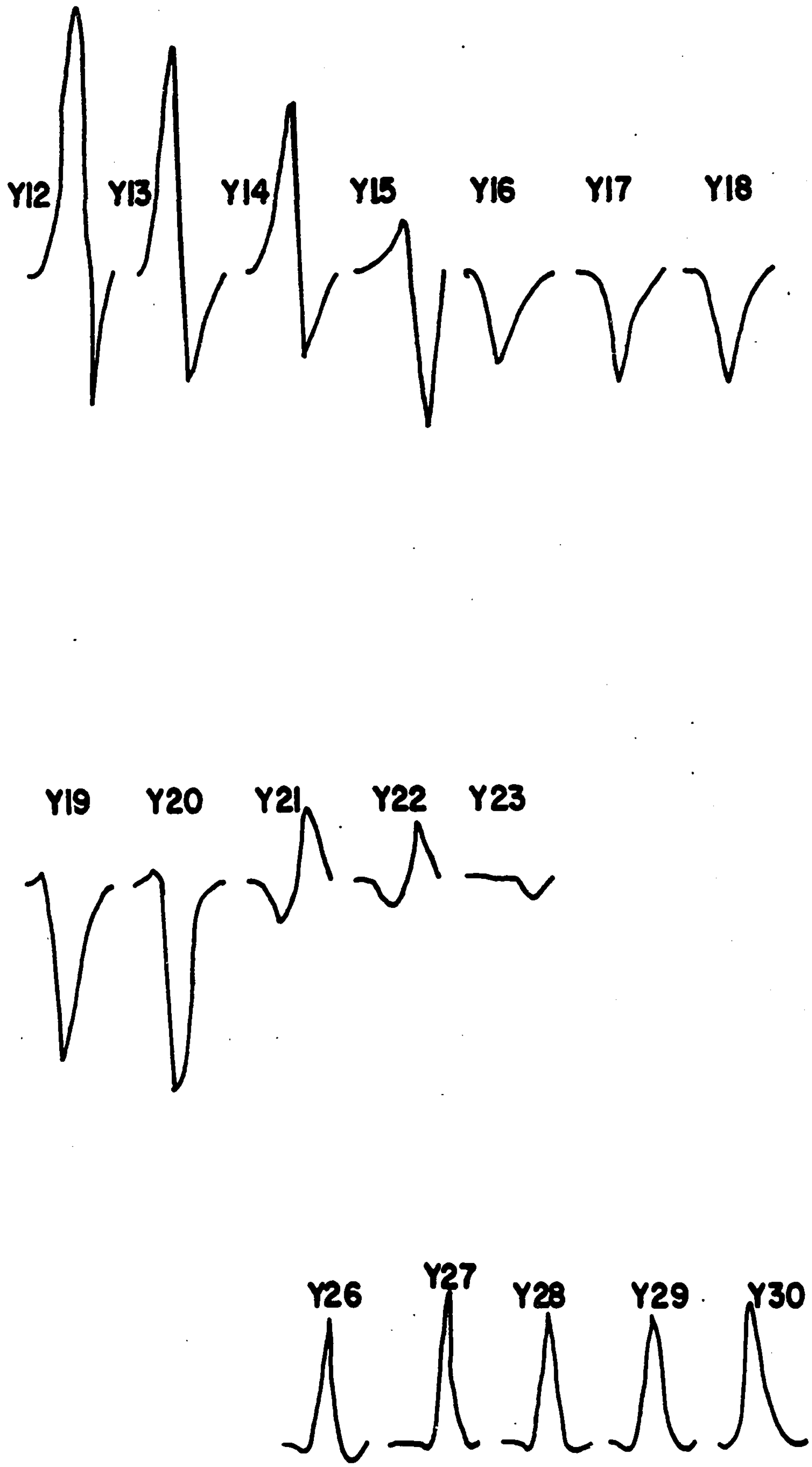


Fig. 4.2 QRS portion of unipolar ECGs, taken at position as shown in Fig. 4.1 Duration of each wavefront is 80 msec. Y24 and Y25 have not been shown because their patterns were not discernible.

Fig. 4.2 (continued)



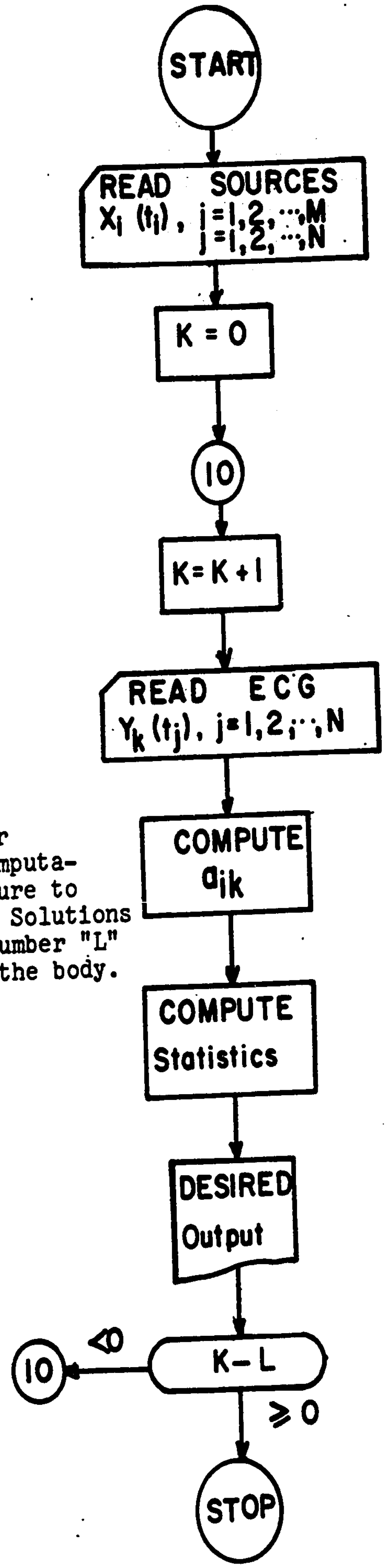


Fig. 4.3 Flow Chart for Outline of computational Procedure to obtain Direct Solutions for ECGs at number "L" of points on the body.

4.2 First computation

The sample values of the electrocardiograms Y01, Y02, - - - - -, Y30 (Fig. 4.2), were punched on cards, one set of cards for each electrocardiogram. Similarly 20 sets of cards were prepared for the equivalent sources (Appendix IV and XIII). Thus for each electrocardiogram, multiple regression solutions were obtained by taking the 20 equivalent sources inside the heart as explanatory variables. The digital computer program is given in appendix X. The program utilizes stepwise procedure as described in section 3.6. In this procedure, at each step, only one explanatory variable is included; the variable included is the one which contributes most in explaining* the given electrocardiogram. Thus explanatory variables (equivalent sources) are entered into the solution in order of their significance in explaining the given ECG. The solution can be stopped at a predetermined level i.e., when new variables entering, do not improve the preceding solution.

Fig. 4.4, shows a typical solutions obtained as the number of equivalent sources included in the solution is increased. Table 4.1(a) - (g) gives the values of the coefficient of determination at each step of regression solution i.e., when a new explanatory variable is included. We have chosen this to

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* The term "explaining" stands for "explaining the sum squares deviation of".

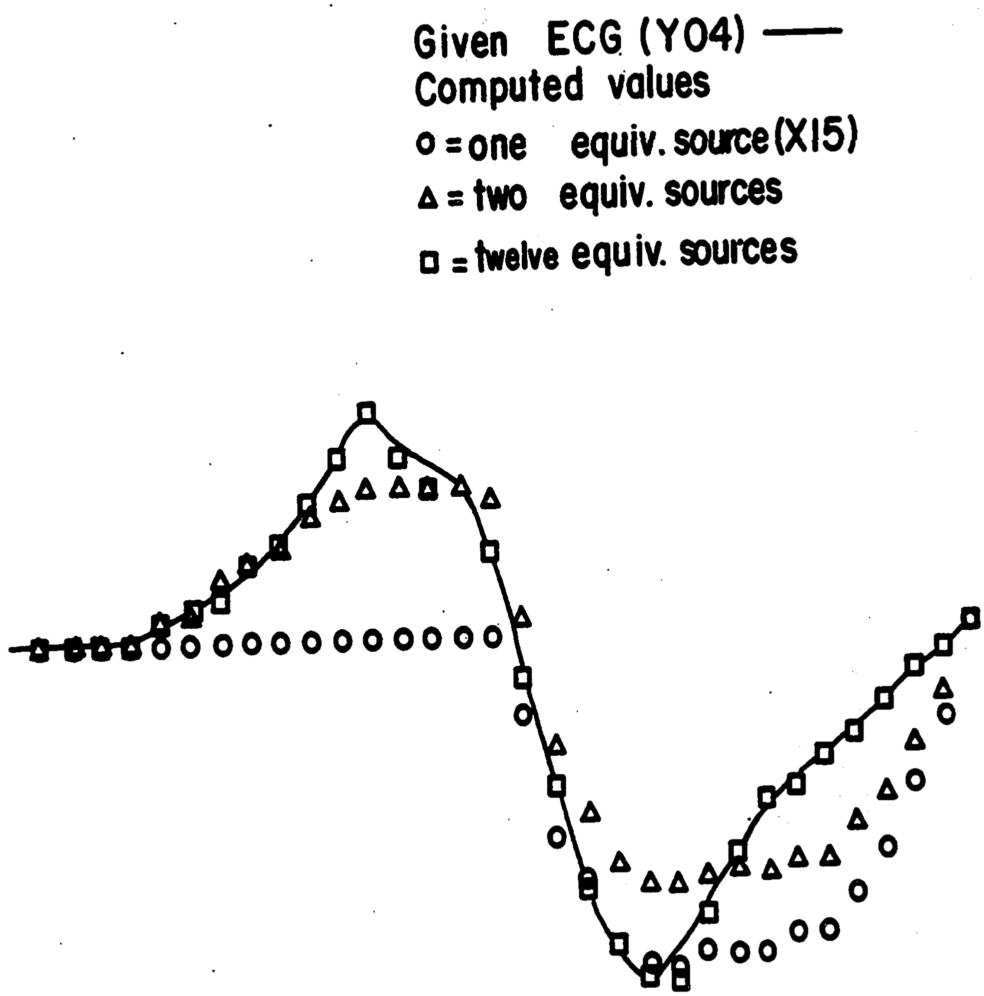


Fig. 4.4 Typical solutions obtained for a given electrocardiogram when the number of equivalent sources included in the solution is increased. The equivalent sources included being in order of their significance in improving the preceding solution.

study the behavior of solution obtained. Consider, e.g., the regression for electrocardiogram Y01 (Table 4.1a). In the beginning, as the number of explanatory variables increases, the solution improves. However, after a certain number of explanatory variables have been included, admission of a new variable does not improve the solution. This is due to either of the following reasons:

(1) The new explanatory variable contributes very little in explaining Y01.

(2) The new explanatory variable is almost identical to the one already included in explaining Y01.

Studied as an isolated case, the solution should be stopped at this stage. In fact such a provision is embodied in the computer program. However, in the integrated systems, like the one we are studying, we will postpone our decision to exclude certain variables until all the solutions Y01, Y02, - - - -, Y30 have been obtained.

Table 4.1 also demonstrates some other features of interest. The coefficient of determination obtained in cases Y05, Y11, Y12, Y14, Y20, Y26, Y27, Y28 is not very high and therefore these cases do not represent very good fit.

Y09, Y17, Y18, Y19, Y20, Y28, Y29, Y30 show similarity in that regression in these cases starts with explanatory variable X09. Similarly in the case of Y02, Y03, Y05, Y06, Y10,

TABLE 4.1 Sequence of inclusion of explanatory variables and the corresponding changes in the goodness of fit.

(a)

Y01		Y02		Y03		Y04	
X	CD	X	CD	X	CD	X	CD
X07	.84026	X20	.79961	X20	.69354	X15	.78748
X02	.90596	X03	.92222	X08	.90507	X08	.87860
X05	.93451	X12	.96745	X10	.97299	X05	.97670
X08	.96400	X17	.98229	X14	.97873	X04	.98339
X01	.96845	X05	.98436	X13	.98415	X14	.99096
X14	.97160	X18	.98504	X18	.98606	X11	.99308
X18	.97771	X07	.98539	X03	.98884	X09	.99460
X06	.98029	X10	.98555	X04	.98863	X03	.99551
X10	.98908	X09	.98770	X05	.98845	X19	.99653
X04	.99133	X01	.98807	X16	.98822	X16	.99711
X03	.99354	X04	.98763	X09	.98801	X17	.99745
X17	.99490	X13	.98719	X11	.98781	X10	.99746
X16	.99519	X19	.98681	X06	.98750	X13	.99744
X15	.99536	X11	.98719	X19	.98728	X18	.99736
X13	.99613	X15	.98669	X07	.98701	X02	.99755
X12	.99648	X14	.98609	X17	.98633	X07	.99749
X19	.99656	X06	.98573	X01	.98554	X06	.99733
X11	.99694	X08	.98481	X02	.98475	X12	.99714
X09	.99681	X02	.98357	X12	.98395	X01	.99691
X20	.9966-	X06	.98208	X15	.98280	X20	.99664

YK = Electrocardiogram at point K of the body where K = 01, 02, - - - -, 30 as shown in figure 4.1 and 4.2.

X = Explanatory variables as given in appendix IV

CD = Coefficient of determination as a measure of goodness of fit described in section 3.3.1

(b)

Y05		Y06		Y07		Y08	
X	CD	X	CD	X	CD	X	CD
X20	.70735	X20	.88797	X18	.83411	X18	.85810
X19	.89374	X14	.91841	X14	.91058	X06	.90622
X02	.94162	X15	.94316	X07	.95464	X19	.95491
X01	.94640	X17	.94924	X11	.97156	X13	.96266
X18	.94745	X04	.97232	X19	.98207	X12	.96513
X13	.96243	X05	.97879	X20	.98387	X03	.97062
X08	.97166	X08	.98022	X09	.98447	X02	.97944
X06	.97708	X07	.98615	X02	.98626	X08	.98074
X16	.97856	X18	.98852	X15	.98817	X07	.98258
X05	.97826	X12	.98930	X06	.98895	X15	.98581
X12	.97805	X06	.98919	X13	.99141	X17	.98819
X07	.97713	X11	.98896	X16	.99158	X16	.98966
X14	.97794	X10	.98916	X03	.99139	X09	.99054
X17	.97861	X16	.98910	X04	.99155	X05	.99076
X09	.97769	X01	.98860	X12	.99143	X20	.99154
X15	.97668	X13	.98802	X17	.99130	X11	.99199
X10	.97534	X09	.98739	X05	.99083	X14	.99172
X11	.94408	X03	.98661	X08	.99056	X01	.99119
X03	.97222	X02	.98557	X01	.99033	X04	.99063
X04	.96977	X19	.98429	X10	.98981	X10	.98979

(c)

Y09		Y10		Y11		Y12	
X	CD	X	CD	X	CD	X	CD
X09	.84718	X20	.91083	X10	.73867	X10	.69887
X08	.87008	X07	.95052	X06	.79900	X02	.73751
X01	.94244	X14	.95624	X02	.89591	X14	.86207
X04	.95995	X18	.96204	X01	.90543	X09	.88716
X19	.96963	X19	.98426	X05	.91651	X15	.90002
X20	.97766	X08	.98776	X08	.92756	X05	.90998
X03	.97783	X06	.98922	X04	.94193	X19	.91980
X10	.98146	X11	.99187	X20	.94914	X06	.92585
X11	.98138	X12	.99259	X18	.95095	X01	.92432
X17	.98143	X16	.99283	X03	.95133	X04	.92235
X18	.98127	X02	.99297	X17	.96089	X20	.92024
X12	.98280	X15	.99402	X07	.96138	X07	.91917
X02	.98454	X04	.99436	X11	.96139	X08	.92602
X13	.98703	X03	.99635	X14	.96032	X03	.92298
X16	.98745	X05	.99642	X09	.95812	X17	.92214
X14	.98740	X13	.99672	X13	.95547	X18	.91865
X07	.98891	X17	.99670	X12	.95234	X11	.91710
X05	.98816	X10	.99651	X19	.97872	X13	.91114
X15	.98726	X01	.99622	X15	.94449	X16	.90389
X06	.98621	X09	.99588	X16	.93945	X12	.89515

(d)

Y13		Y14		Y15		Y16	
X	CD	X	CD	X	CD	X	CD
X02	.81329	X08	.74529	X14	.54870	X17	.84322
X20	.85368	X02	.81443	X05	.82816	X02	.87908
X19	.95118	X14	.92216	X06	.89067	X09	.96781
X11	.96370	X05	.93200	X12	.93715	X01	.97460
X14	.97434	X20	.94611	X16	.95134	X06	.97655
X08	.97737	X04	.95107	X10	.96246	X18	.98083
X05	.98281	X15	.95851	X08	.97675	X05	.98405
X18	.98542	X13	.95891	X04	.98196	X08	.98530
X13	.98671	X11	.95931	X15	.98566	X10	.98749
X09	.98859	X01	.95878	X11	.98603	X19	.98839
X03	.99043	X07	.95801	X09	.98800	X16	.98909
X17	.99061	X10	.95723	X13	.98783	X04	.98911
X15	.99024	X09	.95633	X17	.98731	X07	.98865
X10	.98982	X19	.95534	X07	.98669	X15	.99085
X06	.98929	X18	.95368	X20	.98615	X14	.99137
X01	.98869	X03	.95114	X01	.98536	X12	.99133
X12	.98791	X17	.94849	X18	.98438	X13	.99096
X16	.98700	X12	.94530	X19	.98331	X20	.99032
X04	.98593	X06	.94303	X02	.9820	X11	.98962
X07	.98466	X16	.93974	X03	.98038	X03	.98867

(e)

Y17		Y18		Y19		Y20	
X	CD	X	CD	X	CD	X	CD
X09	.69288	X09	.83846	X09	.84799	X09	.79337
X18	.81439	X19	.86510	X08	.86846	X01	.81372
X08	.87836	X15	.93970	X02	.92546	X08	.88081
X02	.96002	X07	.96746	X04	.94143	X02	.93462
X03	.97419	X11	.97467	X19	.95208	X10	.93802
X12	.97619	X14	.98193	X11	.96828	X05	.94100
X04	.97618	X08	.98193	X03	.97353	X13	.95705
X06	.97655	X02	.98456	X10	.97925	X06	.96579
X19	.98147	X13	.98593	X18	.98527	X19	.97057
X10	.98239	X18	.99058	X13	.98680	X20	.97410
X07	.98303	X03	.99128	X16	.98689	X03	.97599
X15	.98667	X10	.99312	X14	.98773	X07	.97581
X16	.99038	X06	.99413	X15	.98921	X14	.97668
X05	.99388	X16	.99434	X12	.98936	X18	.97606
X13	.99379	X04	.99436	X05	.98904	X12	.97499
X01	.99364	X17	.99435	X17	.98853	X15	.97361
X20	.99327	X12	.99444	X06	.98774	X04	.97178
X14	.99276	X20	.99438	X20	.98685	X17	.96964
X11	.99217	X01	.99400	X01	.98578	X16	.96717
X17	.99146	X05	.99348	X07	.98450	X11	.96418

(f)

Y21		Y22		Y23		Y26	
X	CD	X	CD	X	CD	X	CD
X14	.78432	X14	.78803	X14	.91712	X10	.54802
X08	.90711	X03	.90638	X15	.96075	X02	.74161
X01	.92044	X05	.94578	X06	.96125	X03	.92465
X05	.93323	X01	.97789	X09	.97201	X15	.94860
X09	.97314	X09	.98591	X08	.97808	X19	.95120
X15	.97656	X07	.98795	X03	.98217	X04	.95356
X04	.98211	X20	.99017	X19	.98685	X14	.95678
X03	.98401	X15	.99185	X05	.98915	X07	.95897
X13	.98504	X04	.99255	X11	.98938	X06	.96122
X06	.98763	X02	.99285	X07	.99064	X20	.96565
X19	.98999	X18	.99304	X16	.99203	X17	.96509
X18	.99211	X13	.99432	X13	.99196	X08	.96458
X07	.99275	X06	.99431	X10	.99192	X13	.96311
X20	.99492	X11	.99491	X01	.99167	X12	.96212
X12	.99529	X12	.99518	X17	.99145	X05	.96081
X02	.99526	X17	.99529	X04	.99105	X09	.95898
X11	.99498	X08	.99515	X18	.99064	X11	.95664
X17	.99468	X10	.99489	X20	.98993	X16	.95389
X10	.99423	X19	.99450	X02	.98910	X18	.95037
X16	.99371	X16	.99400	X12	.98811	X01	.94594

(g)

Y27		Y28		Y29		Y30	
X	CD	X	CD	X	CD	X	CD
X20	.67174	X09	.56082	X09	.68497	X02	.84776
X19	.70516	X02	.75857	X08	.72099	X10	.95424
X12	.6955	X03	.93588	X02	.95436	X08	.96111
X15	.82681	X15	.94797	X14	.96660	X01	.96494
X18	.87564	X18	.95599	X15	.97455	X18	.96882
X11	.91217	X14	.95571	X19	.97830	X06	.97425
X09	.92874	X07	.96279	X04	.98036	X19	.98222
X07	.94303	X05	.96390	X06	.98054	X05	.98532
X13	.96194	X06	.96638	X03	.98104	X04	.98586
X14	.96199	X19	.96862	X10	.98388	X13	.98609
X05	.96433	X11	.97056	X07	.98389	X15	.98639
X06	.96847	X16	.97131	X16	.98427	X07	.98717
X04	.96897	X12	.97088	X17	.98384	X16	.98802
X03	.97551	X20	.96973	X12	.98313	X20	.98822
X02	.97502	X08	.96812	X05	.98270	X12	.98809
X17	.97624	X04	.96641	X11	.98218	X09	.98831
X16	.97484	X17	.96429	X13	.98104	X03	.98777
X01	.97303	X01	.96185	X01	.97987	X17	.98814
X08	.97096	X10	.95894	X20	.97836	X11	.98727
X10	.96833	X13	.95524	X18	.97646	X14	.98633

Y27, regression starts with X20; in case of Y15, Y21, Y22, Y23, regression starts with X14.

In table 4.2, we have given partial summary of the statistical evaluation of regression solutions. One important parameter to note, in addition to the coefficient of determination discussed already, is the von Neumann ratio of the least squares estimated disturbances, Q. The significance level Q(1%) = 3.132 (see section 3.2.2 & Fig. 3.2). Thus in all cases, the solutions are less than satisfactory except Y17. However for cases where $2.5 < Q < 3.132$, the solutions do not represent a poor case.

Table 4.3 shows the coefficients determined from regression solutions. Since we have already pointed out that the regression solutions obtained are not very satisfactory, the values of the coefficients are not of immediate importance to us, except for the t-test that will be described shortly. These values, can, however, be used for comparison when a modified solution is presented.

<u>t-test</u>

$$t_{ik} = \frac{a_{ik}}{\text{Standard error of } a_{ik}}$$ (From equation 3.32, section 3.2.1 (vi))

These computed values were compared with

T(11, 10) = 1.796 from the table in appendix VI.

TABLE 4.2 Partial statistical evaluation of regression solutions

	CVR	SDR	MSDR	CD	von Neumann Q
Y01	-.12479	1.12835	1.27318	.99660	2.95488
Y02	-.30971	2.13704	4.56696	.98208	2.49734
Y03	-.80093	1.52678	2.33106	.98280	1.97020
Y04	-.24159	.63870	.40794	.99664	3.14992
Y05	-.85347	1.40822	1.98310	.96977	2.36541
Y06	-.17821	.71005	.50417	.98429	2.32486
Y07	-.11996	.79440	.63108	.98981	2.96752
Y08	-.12077	.92578	.85707	.98979	3.08807
Y09	-.13909	1.73039	2.99426	.98621	2.75232
Y10	-.08568	1.07935	1.16500	.99588	3.00213
Y11	-.65616	3.48590	1.21515	.93945	2.25399
Y12	.72461	5.69726	32.45878	.89515	2.39600
Y13	.52249	1.74055	3.02952	.98466	2.58024
Y14	.88525	2.81620	7.93100	.93974	2.43067
Y15	-.54729	1.25023	1.56309	.98038	2.81852
Y16	-.10618	.50170	.25170	.98867	2.83360
Y17	-.10294	.47931	.22974	.99146	3.22005
Y18	-.08423	.44565	.19860	.99348	2.97444
Y19	-.14443	1.19065	1.41766	.98450	2.74928
Y20	-.26019	2.1047	4.42980	.96418	2.70682
Y21	.43779	.39401	.15524	.99371	2.55216

TABLE 4.2 (continued)

	CVR	SDR	MSDR	CD	von Neumann Q
Y22	.53175	.28581	.08169	.99400	2.61008
Y23	-.15182	.13427	.01802	.98811	3.18576
Y26	.56873	1.40674	1.97891	.94594	2.76703
Y27	.31629	1.22763	1.50709	.96833	2.60228
Y28	.40883	1.29116	1.66840	.95524	2.58303
Y29	.27742	1.03601	1.07331	.97646	2.53741
Y30	.16346	.81886	.67054	.98633	2.69363

CVR = coefficient of variation

SDR = standard deviation from regression

MSDR = mean squares deviation from regression

CD = coefficient of determination

Q = von Neumann ratio of the least squares estimated disturbances (see section 3.2.2)

In table 4.4, results of this test are shown. A "T" entry shows that the t-test indicated the corresponding explanatory variable to be "true" or significantly different from zero and an "F" entry shows that the t-test indicated the explanatory variable to be "false" or not significantly different from zero. Thus e.g., for Y01, a "T" is shown under X03, implying that there is a significant contribution from X03 in explaining Y01. A vertical line has been drawn when none of the variables are significant in explaining the given ECG.

We will now state the principle upon which certain variables will be excluded from the system.

Exclusion Principle:

If a particular equivalent source (explanatory variable) does not contribute to any of the electrocardiograms recorded on body surface surrounding the heart, then that source is not significant for the system and can be excluded.

Invoking this principle, we find from table 4.4, that sources X01, X02, X11, X12 are to be excluded. It is important to point out that these sources have been excluded on the basis of regression analysis of the ECGs Y01, Y02, - - - -, Y30.

The particular equivalent sources that are found to be significant, depend only on the sequence of ventricular

TABLE 4.3 Coefficients a_{1k} obtained from regression solution with twenty explanatory variables

No. (Source)	Y01	Y02	Y03	Y04	Y05
01	-0.40642	0.33456	-0.61717	-0.06201	-0.38092
02	2.81940	-0.58259	1.92254	-0.11454	-0.83696
03	16.48798	12.70022	-4.90001	7.06951	-1.26007
04	-15.11219	2.31744	-3.30334	-10.80438	-0.86458
05	-3.36120	-7.75191	-5.50902	-1.20591	-2.71471
06	0.30609	-2.30722	-6.26175	-0.85691	-4.58234
07	5.27339	3.43707	-5.42972	1.67056	-5.06084
08	1.80677	-0.54432	3.85215	2.64211	0.55114
09	1.22880	-2.19185	1.35378	1.07840	1.69170
10	1.29535	4.08661	-1.72877	-1.08516	-0.65514
11	2.04616	2.22447	2.99234	-1.04618	1.08366
12	2.51175	0.18900	1.21318	0.24008	0.35673
13	3.39124	2.55612	3.52779	0.85238	5.00647
14	-0.14905	3.47842	1.25303	-2.94236	3.52048
15	-10.23590	-8.32723	3.22051	0.94232	-2.48004
16	-1.00764	-0.26388	-1.06150	2.25207	1.36429
17	-9.36436	-4.93098	0.32907	-1.98687	5.44484
18	-18.62627	-8.29636	-3.46237	-2.47001	-9.20755
19	-4.40345	-3.85004	-3.86477	1.33405	0.46172
20	-1.42257	-6.73604	-5.34767	-0.24922	-5.82085
C_k (constant)	0.56164	-0.15361	0.76148	0.15839	0.72115

TABLE 4.3 (continued)

No. (Source)	Y06	Y07	Y08	Y09	Y10
01	-0.19458	-0.32039	-0.22581	1.09547	-0.06367
02	0.34231	1.69497	1.04150	0.26295	-0.85779
03	-0.83079	-4.11666	-2.18741	-9.67981	8.74780
04	-3.23709	0.41678	-1.19860	9.17735	-11.10252
05	-0.36617	-2.28236	-3.22292	2.09622	-4.55793
06	1.93948	2.79402	7.16142	1.59532	4.54061
07	3.99180	-16.93741	-22.06186	-8.13302	-4.22658
08	0.89944	0.64094	1.80731	-2.54545	2.52201
09	0.49709	-1.97862	-1.50420	-6.12437	-0.14463
10	-0.89989	0.73560	-0.16324	3.32598	-0.25166
11	-0.68291	-0.15500	-0.91395	1.51613	-1.51871
12	0.28999	0.42228	1.48467	-3.21439	-0.12965
13	0.33924	-0.88489	-0.37123	-3.24839	1.45947
14	-0.32133	0.70364	-1.03703	4.73137	-5.29553
15	1.06922	10.34620	12.97452	-2.47836	9.29180
16	1.00909	2.47433	5.29829	-4.76850	4.63215
17	1.77344	0.70295	-1.53902	7.67536	-2.95825
18	-2.95180	2.73492	-1.25219	10.59560	-9.44492
19	-0.22857	-2.77275	-5.55905	-3.81483	0.12219
20	1.21176	0.26997	3.25794	1.78245	-0.20695
C_k	-0.17821	0.27725	0.08814	-0.77063	0.33675

TABLE 4.3 (continued)

No. (Source)	Y11	Y12	Y13	Y14	Y15
01	0.72654	1.01390	-0.29109	-0.33043	0.26272
02	0.35921	-0.38075	2.38282	-0.84964	-0.48671
03	16.13997	15.75147	-6.06316	5.64341	-0.23293
04	3.80087	14.23772	-0.80063	-15.37970	6.68417
05	-6.84186	3.61967	0.68639	-10.86128	2.22686
06	-10.01866	-5.52664	-2.06763	-6.98254	-6.98320
07	11.84092	44.98795	-0.89605	-13.56726	3.96259
08	-1.38368	-4.52119	1.35758	7.52383	-2.32373
09	-1.32720	-5.64555	3.34971	0.22028	3.22187
10	4.00930	6.97537	-0.88763	-1.87394	-1.11151
11	1.87727	-4.01437	0.77757	3.12529	1.22471
12	0.66493	-0.07666	0.37848	3.12119	0.35903
13	0.55397	1.21004	3.90533	-0.17577	-0.50013
14	-2.78631	-13.32380	3.81951	-9.87365	2.15307
15	1.68048	-11.85732	-3.11734	23.65585	-10.50070
16	0.27217	1.50520	-0.23886	3.57256	-3.87425
17	-14.67356	-15.16207	4.21408	-5.87941	1.61149
18	9.42265	-13.35348	-9.62564	5.96297	-1.35368
19	-1.00207	16.75470	1.62470	-6.49090	0.92092
20	-10.40102	-1.01923	-8.08435	1.83541	-0.60854
C_k	-0.38603	-0.12697	0.42647	1.27889	-0.24928

TABLE 4.3 (continued)

No. (Source)	Y16	Y17	Y18	Y19	Y20
01	-0.10222	-0.15591	-0.06585	-0.06996	0.18493
02	-1.09061	0.59167	0.86490	2.46692	1.79687
03	-0.00371	-3.11292	-4.37128	-5.21531	-2.11334
04	2.45061	1.00522	2.47301	3.14464	0.61125
05	-0.18618	-2.20330	0.22922	1.21185	0.86517
06	-2.24326	2.47766	2.89069	-0.97649	7.59447
07	5.18982	-12.62910	-7.54632	0.57976	-11.85680
08	-1.36135	0.52783	-0.44158	-1.51050	-0.03961
09	1.02366	-0.71927	-1.77741	-4.98112	-5.17233
10	-1.61827	-0.06280	0.85769	3.82381	2.94339
11	0.21550	-0.07801	-0.62165	2.65722	-0.05965
12	-0.29501	0.59971	-0.57495	-0.98456	-0.72084
13	0.28700	-0.29490	-1.72619	-1.13799	-2.16357
14	1.25768	-0.10471	0.03102	4.50819	1.76421
15	-5.89683	7.93221	4.78475	-8.11335	2.89609
16	-1.85576	2.93684	0.56108	-4.06832	0.62221
17	0.91079	0.03131	0.94919	1.35909	-1.01346
18	-0.96158	1.54397	4.30418	-4.76372	3.79798
19	1.91940	-3.35813	-0.78566	-4.76372	-6.63768
20	-0.50978	0.34677	0.93439	-0.67593	4.89394
C_k	0.09493	0.11055	-0.09233	0.09591	-0.31170

TABLE 4.3 (continued)

No. (Source)	Y21	Y22	Y23	Y26	Y27
01	-0.03043	0.02554	0.03405	-0.09198	0.18691
02	0.41806	0.06657	0.01838	-0.48356	1.18202
03	-5.26491	-3.08316	0.82359	10.16859	-4.41708
04	4.01832	1.70280	-0.23009	-9.13160	7.25915
05	0.60023	1.51761	-0.63946	-2.17504	-3.92643
06	1.91637	1.66145	-0.86784	-5.97785	-1.13310
07	-4.37125	-3.99404	-0.77944	1.37181	-7.70965
08	0.15180	0.07383	0.09439	0.74591	-0.22180
09	-0.98746	-0.43122	0.24293	-0.72972	0.51858
10	0.00351	-0.19698	-0.06598	0.65099	0.06244
11	0.22313	-0.34263	0.23232	1.01155	2.86348
12	-0.27973	-0.39828	0.01254	1.50775	-0.25105
13	-0.58107	-0.40216	0.06275	-0.25919	-0.99410
14	1.17021	1.68128	0.05692	-3.50803	0.89579
15	2.60414	0.09753	-0.51339	4.11406	10.24922
16	0.00026	0.01951	-0.20313	-0.67089	-1.23476
17	0.41783	0.90701	-0.23039	-2.74897	-2.86271
18	3.34047	1.52215	0.17898	1.21731	14.06262
19	-2.29257	-0.18201	-0.10285	1.19519	-7.46334
20	0.86062	1.29123	0.03163	0.90614	-5.87256
C_k	-0.11590	-0.13236	-0.06555	0.16683	-0.62725

TABLE 4.3 (continued)

No. (Source)	Y28	Y29	Y30
01	-0.17633	-0.18835	-0.14559
02	0.02036	-1.12771	0.57832
03	2.69880	5.19812	2.48085
04	-2.23010	-6.05118	-0.30648
05	-3.32409	-2.24980	-1.45297
06	-6.62497	-4.18909	4.25830
07	4.67893	3.77055	-9.68320
08	0.59272	1.68584	2.19470
09	-0.54807	2.23644	-1.88467
10	0.53908	-1.79659	1.87103
11	1.94539	0.85882	-0.46297
12	1.30208	1.24266	1.27789
13	-0.10648	0.49729	0.65024
14	-3.85842	-3.13767	-0.62380
15	3.89896	3.27307	6.64568
16	-1.09263	-0.95102	3.22947
17	-1.36837	-2.00762	2.99500
18	-2.83422	-0.53362	-3.11700
19	-0.15331	0.81107	-3.87842
20	-1.29507	-0.55671	1.73648
C_k	0.36419	0.19146	-0.24686

TABLE 4.4 Truth table derived by invoking t-test on the coefficients of regression

X	01	02	03	04	05	06	07	08	09	10	11	12	13	14	15	16	17	18	19	20
Y01			T	T	F	F	F	T	F	F			F	F	T	F	T	T	F	F
Y02			T	F	F	F	F	F	F	F			F	F	F	F	F	F	F	F
Y03			F	F	F	F	F	T	F	F			T	F	F	F	F	F	F	F
Y04			T	T	F	F	F	T	F	F			F	T	F	F	F	F	F	F
Y05			F	F	F	F	F	F	F	F			T	F	F	F	F	T	F	F
Y06																				
Y07			F	F	F	F	T	F	F	F			F	F	T	F	F	F	F	F
Y08			F	F	F	T	T	T	F	F			F	F	T	T	F	F	T	F
Y09			T	F	F	F	F	T	T	F			T	F	F	F	F	T	F	F
Y10			T	T	F	F	F	T	F	F			F	T	F	T	F	T	F	F
Y11																				
Y12																				
Y13			F	F	F	F	F	F	F	F			T	F	F	F	F	T	F	T
Y14			F	F	F	F	F	T	F	F			F	T	F	F	F	F	F	F
Y15			F	F	F	F	F	T	F	F			F	F	F	F	F	F	F	F
Y16			F	F	F	F	T	T	F	T			F	F	T	F	F	F	F	F
Y17			T	F	F	F	F	F	F	F			F	F	T	T	F	F	T	F
Y18			T	F	F	T	T	F	T	F			T	F	T	F	F	T	F	F
Y19			F	F	F	F	F	F	T	T			F	T	F	F	F	F	F	F
Y20																				
Y21			F	T	F	F	T	F	F	F			F	F	F	F	F	T	T	F
Y22			T	F	T	T	T	F	F	F			F	T	F	F	F	F	F	T
Y23			T	F	T	T	F	F	F	F			F	F	F	F	F	F	F	F
Y26			T	F	F	F	F	F	F	F			F	F	F	F	F	F	F	F
Y27			F	F	F	F	F	F	F	F			F	F	F	F	F	T	T	T
Y28																				
Y29			F	F	F	F	F	T	F	F			F	F	F	F	F	F	F	F
Y30			F	F	F	F	T	T	F	F			F	F	F	T	F	F	T	F

T → The variable identified at the top of the column makes significant contribution in explaining the ECG identified at the left of the row

F → Not "T"

depolarization process and are independent of orientation or position of the heart inside the thorax as well as the body build.

By carrying out preceding analysis in a variety of normal cases, we can determine the equivalent sources that represent normal electrical activity of the heart.

4.2 Second Computation

In section 4.1, our analysis showed that variables X01, X02, X11, X12 should be excluded from the system. Thus we have re-computed all the results, now with 16 explanatory variables only. Table 4.5 (1) - (7) is drawn similar to table 4.1 (a) - (g). As should be expected, the order in which explanatory variable enter regression solution is the same as before until the excluded variable is encountered. The order changes thereafter. The coefficient of determination at the end of each regression solution has changed, but not greatly. However, the solution does not degenerate as much as compared to the first computation.

Table 4.6 shows the partial statistical evaluation of the regression solutions now obtained. This table should be compared with table 4.2.

TABLE 4.5 Sequence of inclusion of explanatory variables and the corresponding changes in the goodness of fit. (X01, X02, X11, X12 excluded compared to table 4.1)

(1)

Y01		Y02		Y03		Y04	
X	CD	X	CD	X	CD	X	CD
X07	.84026	X20	.79961	X20	.69354	X15	.78748
X08	.88439	X03	.92222	X08	.90507	X08	.87860
X18	.96927	X13	.96388	X10	.97299	X05	.97640
X03	.97908	X05	.97219	X14	.97874	X04	.98339
X04	.98540	X08	.97944	X13	.98415	X14	.99096
X10	.99150	X18	.98195	X18	.98606	X20	.99265
X06	.99343	X10	.98580	X03	.98885	X09	.99478
X17	.99408	X09	.98753	X04	.98863	X03	.99573
X19	.99489	X17	.98822	X05	.98845	X10	.99599
X13	.99509	X04	.98813	X16	.98822	X06	.99613
X15	.99660	X15	.98785	X09	.98801	X19	.99680
X09	.99687	X14	.98808	X07	.98765	X13	.99675
X20	.99690	X07	.98773	X15	.98706	X16	.99705
X16	.99692	X16	.98714	X06	.98633	X17	.99701
X05	.99676	X19	.98655	X17	.98551	X18	.99690
X14	.99654	X06	.98573	X19	.98456	X07	.99688

YK, X, CD same as defined under Table 4.1

(2)

Y05		Y06		Y07		Y08	
X	CD	X	CD	X	CD	X	CD
X20	.70736	X20	.88797	X18	.83411	X18	.85811
X19	.89375	X14	.91841	X14	.91059	X06	.90623
X08	.91723	X15	.94317	X07	.95465	X19	.95492
X15	.94514	X17	.94925	X08	.96871	X13	.96266
X13	.95675	X04	.97233	X17	.97836	X05	.96492
X18	.97321	X05	.97880	X06	.98474	X03	.96984
X06	.97681	X08	.98023	X09	.98649	X04	.97402
X16	.97890	X07	.98616	X19	.98813	X07	.97728
X09	.97855	X18	.98853	X04	.98886	X15	.98463
X14	.97831	X06	.98888	X03	.98920	X17	.98796
X17	.97921	X03	.98864	X15	.98975	X08	.98806
X05	.97901	X09	.98826	X13	.99188	X16	.98991
X10	.97837	X10	.98832	X05	.99193	X10	.99031
X03	.97719	X19	.98792	X16	.99207	X20	.99087
X07	.97580	X13	.98774	X10	.99183	X09	.99039
X04	.97419	X16	.98692	X20	.99131	X14	.98975

(3)

Y09		Y10		Y11		Y12	
X	CD	X	CD	X	CD	X	CD
X09	.84718	X20	.91084	X10	.73867	X10	.69887
X08	.87008	X07	.95053	X06	.79901	X14	.72540
X04	.93584	X14	.95625	X08	.87795	X07	.85469
X19	.96535	X18	.96204	X19	.90006	X15	.88995
X06	.97083	X19	.98427	X14	.91654	X05	.90994
X20	.97183	X08	.98776	X05	.93332	X17	.91936
X03	.97285	X06	.98922	X04	.94211	X04	.92294
X17	.97443	X13	.98974	X18	.94561	X13	.92645
X10	.97621	X05	.99092	X17	.95960	X06	.92555
X07	.97668	X09	.99115	X03	.95960	X19	.92433
X14	.97770	X04	.99338	X20	.96062	X09	.92466
X15	.97971	X03	.99561	X07	.96224	X03	.92381
X13	.98052	X15	.99602	X09	.96054	X08	.92185
X18	.98294	X16	.99652	X16	.95853	X20	.92554
X05	.98196	X17	.99661	X15	.95596	X18	.92456
X16	.98077	X10	.99640	X13	.95303	X16	.91953

(4)

Y13		Y14		Y15		Y16	
X	CD	X	CD	X	CD	X	CD
X08	.64235	X08	.74529	X14	.54871	X17	.84322
X18	.88401	X14	.80683	X05	.82816	X14	.87088
X13	.94401	X07	.88446	X06	.89068	X07	.89808
X09	.95280	X04	.94811	X18	.92591	X10	.96656
X10	.96921	X15	.94889	X10	.95093	X04	.96767
X20	.96988	X17	.95004	X08	.97504	X06	.96801
X14	.98521	X16	.94999	X04	.97774	X18	.97398
X07	.98866	X10	.94953	X15	.98333	X05	.98202
X16	.98919	X03	.94840	X09	.98597	X08	.98483
X17	.98886	X20	.94750	X20	.98646	X19	.98670
X03	.98991	X05	.95427	X16	.98615	X15	.98799
X19	.98975	X13	.95476	X07	.98600	X16	.98946
X05	.98968	X09	.95474	X19	.98637	X09	.98934
X15	.98915	X18	.95294	X13	.98618	X13	.98907
X04	.98853	X19	.95106	X17	.98535	X03	.98841
X06	.98782	X06	.94807	X03	.98437	X20	.98765

(5)

Y17		Y18		Y19		Y20	
X	CD	X	CD	X	CD	X	CD
X09	.69288	X09	.83846	X09	.84799	X09	.79337
X18	.81440	X19	.86510	X08	.86846	X19	.81028
X08	.87836	X15	.93970	X04	.92127	X06	.92751
X05	.92657	X07	.96747	X19	.94523	X16	.96172
X06	.95546	X18	.97272	X06	.96381	X05	.96504
X20	.96237	X13	.97926	X20	.96809	X04	.96516
X04	.96541	X06	.98762	X03	.96987	X03	.96628
X03	.97331	X14	.98919	X10	.97643	X10	.96698
X19	.97636	X10	.98999	X07	.97687	X07	.96654
X10	.97762	X03	.99126	X14	.97849	X14	.96964
X17	.97817	X17	.99298	X15	.98087	X20	.97516
X14	.97804	X08	.99317	X13	.98087	X13	.97597
X07	.98656	X04	.99368	X16	.98048	X08	.97496
X15	.98854	X20	.99339	X05	.98025	X17	.97364
X16	.99280	X16	.99302	X17	.97929	X18	.97214
X13	.99282	X05	.99258	X18	.97805	X15	.97071

(6)

Y21		Y22		Y23		Y26	
X	CD	X	CD	X	CD	X	CD
X14	.78432	X14	.78804	X14	.91712	X10	.54802
X08	.90712	X03	.90638	X15	.96076	X18	.60844
X15	.91322	X05	.94578	X06	.96126	X14	.78112
X05	.95219	X09	.96867	X09	.97202	X07	.93861
X09	.96483	X07	.98645	X08	.97809	X04	.94546
X04	.98103	X20	.98970	X03	.98218	X03	.94543
X13	.98336	X17	.99141	X19	.98686	X06	.95153
X03	.98455	X04	.99236	X05	.98915	X19	.95809
X06	.87665	X06	.99306	X17	.98898	X17	.96283
X19	.99043	X19	.99434	X20	.98877	X20	.96448
X20	.99165	X18	.99541	X07	.99128	X08	.96421
X07	.99292	X13	.99568	X18	.99109	X13	.96303
X18	.99502	X15	.99551	X10	.99066	X16	.96169
X17	.99495	X08	.99525	X04	.99020	X15	.95986
X10	.99472	X16	.99496	X13	.98960	X05	.95750
X16	.99443	X10	.99463	X16	.98891	X09	.95469

(7)

Y27		Y28		Y29		Y30	
X	CD	X	CD	X	CD	X	CD
X20	.67174	X09	.56083	X09	.68497	X17	.83287
X19	.70516	X18	.65739	X08	.72099	X18	.84631
X09	.76550	X14	.82767	X18	.78424	X09	.89041
X08	.85016	X07	.95258	X05	.86128	X06	.95781
X04	.89829	X06	.96038	X06	.92340	X05	.97726
X03	.93644	X05	.96673	X04	.93993	X08	.97949
X06	.94769	X15	.96728	X03	.95302	X19	.98269
X17	.95319	X19	.96886	X19	.95609	X04	.98518
X18	.95178	X13	.96885	X15	.95823	X13	.98601
X14	.96098	X20	.96840	X07	.97271	X20	.98681
X13	.96409	X03	.96761	X17	.97631	X03	.98638
X07	.96410	X17	.96846	X10	.97944	X07	.98588
X15	.96503	X08	.96763	X16	.97913	X15	.98767
X05	.96715	X16	.96592	X20	.97857	X16	.98794
X16	.96548	X04	.96407	X15	.97750	X10	.98799
X10	.96341	X10	.96169	X13	.97608	X14	.98721

A very noticeable change is seen in the von Neumann ratio of the least squares estimated disturbances. Moreover, now, the significance level Q(1%) = 2.706 only (section 3.2.2, Fig. 3.2). The solution now is fairly satisfactory except in a few cases.

One new column that has been added in this table is ABSE = Average absolute error

$$= \frac{1}{N} \sum_{i=1}^{N} \left|\frac{d_i}{Y_i}\right| = \frac{1}{N} \sum_{i=1}^{N} \left|\frac{Y_i - \hat{Y}_i}{Y_i}\right| \quad \text{(see section 3.2.2)} \qquad (4.1)$$

A look at the statistics in table 4.6 reveals that the regression solution of Y02, Y03, Y11, Y12, Y14, Y27, Y28 are still not very good.

Table 4.7 gives the new values of the regression coefficients and should be compared with table 4.3. We will again study the significance of explanatory variables on each regression solution by applying t-test on the coefficients shown in table 4.7. The outcome of this test is shown in table 4.8. We see a significant improvement in the behavior of explanatory variables as compared to table 4.4.

Best fit or regression solution has been obtained in the case of Y10, Y18, Y21, Y22, followed by Y04, Y29, Y01, Y17, Y08, Y09, Y16, etc. In Fig. 4.5, we have plotted the computed and given values of ECGs Y01, Y02, - --, Y30. The computer output for estimated values is given in appendix XIV.

TABLE 4.6 Partial statistical evaluation of regression solutions

	OVR	SDR	MSDR	CD	von Neumann Q	ABSE
Y01	-.12594	1.13857	1.29635	.99654	2.71262	.33632
Y02	-.27649	1.90777	3.63958	.98573	2.63291	.40895
Y03	-.75884	1.44653	2.09245	.98456	2.00747	.23935
Y04	-.23256	.61483	.37801	.99688	2.80902	.21796
Y05	-.78863	1.30125	1.69324	.97419	2.27697	.43667
Y06	-.16264	.64803	.41994	.98692	2.43059	.43891
Y07	-.11080	.73368	.53828	.99131	3.15474	.35755
Y08	-.12104	.92784	.86089	.98975	3.04099	.55455
Y09	-.16426	2.04354	4.17604	.98077	2.27589	.74307
Y10	-.08016	1.00972	1.01953	.99640	2.64548	.34421
Y11	.577797	3.07044	9.42760	.95303	2.25319	1.84857
Y12	.63481	4.99119	24.91197	.91953	2.27134	2.07064
Y13	.46565	1.55121	2.40625	.98782	2.63084	.33556
Y14	.82187	2.61456	6.83592	.94807	2.40331	.54042
Y15	-.48845	1.11581	1.24504	.98437	2.91632	.44506
Y16	-.11089	.52395	.27453	.98765	2.49044	.43458
Y17	-.09442	.43965	.19329	.99282	3.27101	.17397
Y18	-.08989	.47556	.22616	.99258	3.05304	.26520
Y19	-.17190	1.41710	2.00819	.97805	2.20959	.83893
Y20	-.23530	1.90335	3.62275	.97071	2.58723	1.64699
Y21	.411961	.37077	.13747	.99443	2.47505	.21518

TABLE 4.6 (continued)

	CVR	SDR	MSDR	CD	von Neumann Q	ABSE
Y22	.50350	.27063	.07324	.99463	2.83933	.29150
Y23	-.14670	.12974	.01683	.98891	3.05421	.11361
Y26	.52066	1.28783	1.65850	.95469	2.80032	.98585
Y27	.34000	1.31961	1.74137	.96341	2.26145	1.34212
Y28	.37827	1.19511	1.42829	.96169	2.61983	1.10525
Y29	.27969	1.04446	1.09090	.97608	2.37796	.83796
Y30	.15811	.79201	.62728	.98721	2.63887	1.14614

CVR, SDR, MSDR, CD, Q as defined under Table 4.2

ABSE = Av. absolute error

TABLE 4.7 Coefficients a_{ik} obtained from regression solution with 16 explanatory variables.

No. (source)	Y01	Y02	Y03	Y04	Y05	Y06	Y07
03	16.259460	13.718631	-3.977377	6.431388	-0.834413	-1.309666	-4.416696
04	-9.373561	5.401357	1.663485	-12.055565	-0.251498	-3.924326	1.093234
05	0.735418	-7.209711	-2.476460	-1.329238	-2.092554	-0.110438	-1.239059
06	6.434048	0.690439	-0.294067	-2.232809	-3.400624	1.332161	3.757969
07	5.473072	3.451221	-4.344385	2.512297	-0.896087	-3.348883	-18.228958
08	0.990227	-0.686324	3.380761	2.635116	0.479208	0.854375	0.491422
09	2.339661	-3.255872	0.638649	1.929381	1.397040	1.150069	-1.577189
10	0.102916	3.980207	-1.845399	-1.339429	-0.647751	-1.047070	0.684772
13	2.537192	2.076839	2.591044	1.067993	4.711365	0.437731	-0.979675
14	0.068830	2.115833	0.182568	-2.500145	1.916037	0.096537	1.126576
15	-12.033394	-9.021428	2.395637	0.790235	-3.165718	0.997255	10.547853
16	-2.110101	1.651035	-0.635876	1.276607	1.656150	0.061348	1.486438
17	-7.590570	-7.350227	-0.676090	-1.106965	2.885571	2.704203	2.695354
18	-20.314865	-8.882878	-5.526786	-1.903392	-9.291328	-2.781252	2.040933
19	-1.280163	-1.605373	-0.253966	0.422249	1.885232	-0.742239	-2.749220
20	1.584949	-3.070736	-0.942429	-1.858804	-4.068341	0.081232	-0.234109
C_k	0.355160	0.285230	0.050140	0.148480	-0.009050	-0.056570	0.016760

TABLE 4.7 (continued)

No. (Source)	Y08	Y09	Y10	Y11	Y12	Y13	Y14
03	-3.385444	-7.435447	8.067261	16.690414	13.683823	-5.885507	5.731200
04	-0.812040	10.057020	-13.691188	7.977569	10.453512	1.442839	-9.793079
05	-1.702873	-0.876348	-5.407178	-5.885508	1.582325	2.197230	-7.051311
06	7.462030	1.904889	1.819006	-6.525640	-11.133391	0.570851	-0.999186
07	-20.874207	-18.820023	-2.459458	8.747987	40.544830	-3.734816	-5.245607
08	1.443192	-1.889030	2.662272	-1.699635	-4.391827	1.144958	6.668009
09	0.428259	-10.074822	0.649975	-1.714082	-3.226580	3.137249	1.447226
10	-1.131981	5.212958	-0.335669	3.384260	6.066275	-0.787950	-3.632174
13	-0.364252	-3.180822	1.845421	0.114372	2.259348	3.580873	-1.212711
14	-0.116880	5.353054	-5.079570	-2.944308	-10.281041	4.724042	-12.813421
15	11.964867	0.480510	9.336521	0.793042	-11.649278	-2.780511	19.983459
16	3.211269	-0.368012	3.684730	1.848252	-0.305121	-0.596086	3.483159
17	0.839512	7.031960	-2.673420	-14.853641	-10.273379	6.184290	-9.542457
18	-0.998130	9.214229	-8.411729	8.769957	-11.181719	-11.075259	5.735633
19	-5.805529	-4.110693	-1.380697	0.918450	12.405289	2.510551	-1.802308
20	1.800283	4.028696	-2.498596	-7.252507	-6.999536	-7.218097	7.111877
C_k	-0.082030	0.932180	0.122250	0.829210	1.475900	0.302160	0.676230

TABLE 4.7 (continued)

No. (Source)	Y15	Y16	Y17	Y18	Y19	Y20	Y21
03	0.201271	0.285685	-3.448818	-4.418917	-3.440515	-1.848483	-5.022635
04	8.622565	1.686413	1.534677	1.607182	6.923099	1.224839	4.250925
05	2.667431	-0.677725	-1.406365	-0.198042	1.987188	0.647961	0.562204
06	-5.227158	-2.801842	3.086349	2.095900	3.439409	8.084367	2.263005
07	4.332172	7.473959	-12.173541	-9.930142	-5.495275	-17.545959	-5.502612
08	-2.466031	-1.241378	0.366205	-0.302243	-1.477134	0.053638	0.197514
09	2.904793	0.639842	-0.124907	-2.020845	-7.626001	-5.901307	-1.415545
10	-1.402280	-1.398444	-0.380102	1.284319	4.978767	3.457902	0.265231
13	-0.775654	0.292683	-0.371689	-1.561498	-1.743725	-2.120043	-0.622143
14	1.348386	0.295075	0.154662	0.965072	4.707165	3.284077	1.322925
15	-11.214617	-5.960109	7.562061	5.727839	-6.618234	4.175691	3.004097
16	-2.895923	-1.426591	2.216119	0.309681	-1.893729	0.988921	0.263942
17	0.259259	-0.924397	0.830286	2.392675	1.454947	1.362206	0.597859
18	-1.516261	-0.644712	1.423231	4.006810	1.244922	2.830426	2.964975
19	2.258501	2.108534	-3.157499	-1.703442	-2.686271	-7.200809	-2.213593
20	1.481491	-0.085405	0.173863	-0.237513	3.018403	4.522758	1.113473
C_k	0.100600	-0.233140	0.040000	0.083160	0.289240	0.223880	-0.113720

TABLE 4.7 (continued)

No. (Source)	Y22	Y23	Y26	Y27	Y28	Y29	Y30
03	-3.064137	0.931582	9.957313	-2.893688	3.054949	5.063386	1.617413
04	1.010289	0.130173	-6.993735	11.555909	1.058841	-4.843463	0.335616
05	1.044976	-0.564150	-0.554831	-2.950582	-1.385195	-1.077043	-0.164223
06	0.947954	-0.514599	-3.807599	3.420505	-3.028148	-2.813243	4.800179
07	-4.889464	-0.947208	5.191336	-11.255159	7.609695	8.954823	-8.222300
08	0.036474	-0.110767	0.350651	-0.343049	0.194367	1.399091	1.877407
09	-0.621513	0.121503	0.167589	-1.445377	-0.433733	2.979136	-0.393672
10	0.039732	-0.067059	-0.289779	0.513727	-0.085700	-2.554606	1.027587
13	-0.283294	0.010649	-0.635692	-1.652474	-0.703626	0.194449	0.597063
14	1.977920	-0.030661	-4.624347	0.276773	-5.228019	-4.691694	-0.190056
15	0.552862	-0.552573	2.351237	10.544175	2.417976	1.603393	5.625747
16	0.062426	-0.011282	-1.000631	1.036186	-0.758957	-1.203918	1.766586
17	1.242287	-0.385741	-4.022917	-4.015525	-3.109635	-4.166622	-1.604981
18	1.544073	0.078302	1.375385	12.218700	2.266044	-0.096484	-2.880431
19	-0.716605	0.123283	2.946497	-4.850730	2.433664	2.363859	-3.760198
20	0.707056	0.402744	2.702839	-1.533621	1.849686	1.033187	1.043497
C_k	-0.088570	-0.010040	0.022000	-0.189890	0.099130	-0.248620	-0.363220

TABLE 4.8 Truth table derived by invoking t-test on the coefficients of regression.

X	01	02	03	04	05	06	07	08	09	10	11	12	13	14	15	16	17	18	19	20
Y01			T	T	F	T	F	F	F	F			T	F	T	F	T	T	F	F
Y02			T	F	T	F	F	F	F	F			F	F	F	F	F	F	F	F
Y03			F	F	F	F	F	T	F	F			T	F	F	F	F	F	F	F
Y04			T	T	F	T	F	T	T	F			T	T	F	F	F	F	F	T
Y05			F	F	F	T	F	F	F	F			T	F	F	F	F	T	F	T
Y06			F	T	F	F	F	T	F	F			F	F	F	F	T	F	F	F
Y07			T	F	F	T	T	F	F	F			F	F	T	F	F	F	T	F
Y08			F	F	F	T	T	T	F	F			F	F	T	T	F	F	T	F
Y09			F	T	F	T	F	F	T	T			T	T	F	F	F	F	F	F
Y10			T	T	T	F	F	T	F	F			T	T	T	T	F	T	F	T
Y11			T	F	F	F	F	F	F	F			F	F	F	F	T	F	F	F
Y12			F	F	F	F	T	F	F	F			F	F	F	F	F	F	F	F
Y13			F	F	F	F	F	F	F	F			T	T	F	F	T	T	F	T
Y14			F	F	F	F	F	T	F	F			F	T	F	F	F	F	F	F
Y15			F	T	F	T	F	T	T	F			F	F	T	F	F	F	F	F
Y16			F	F	F	T	T	T	F	T			F	F	T	F	F	F	T	F
Y17			T	F	T	T	T	F	F	F			F	F	T	T	F	F	T	F
Y18			T	F	F	T	T	F	T	T			T	F	T	F	T	T	T	F
Y19			F	T	F	T	F	F	T	T			F	T	F	F	F	F	F	F
Y20			F	F	F	T	T	F	T	F			F	F	F	F	F	F	T	F
Y21			T	T	F	T	T	F	T	F			T	T	F	F	F	T	T	T
Y22			T	F	T	T	T	F	T	F			F	T	F	F	T	T	F	T
Y23			T	F	T	T	F	F	F	F			F	F	F	F	F	F	F	T
Y24																				
Y25																				
Y26			T	T	F	T	F	F	F	F			F	T	F	F	F	F	F	F
Y27			F	T	F	T	T	F	F	F			F	F	F	F	F	T	T	F
Y28			F	F	F	F	T	F	F	F			F	T	F	F	F	F	F	F
Y29			F	T	F	T	T	T	T	T			F	T	F	F	T	F	F	F
Y30			F	F	F	T	T	T	F	F			F	F	F	F	F	F	T	F

T, F as defined under Table 4.4

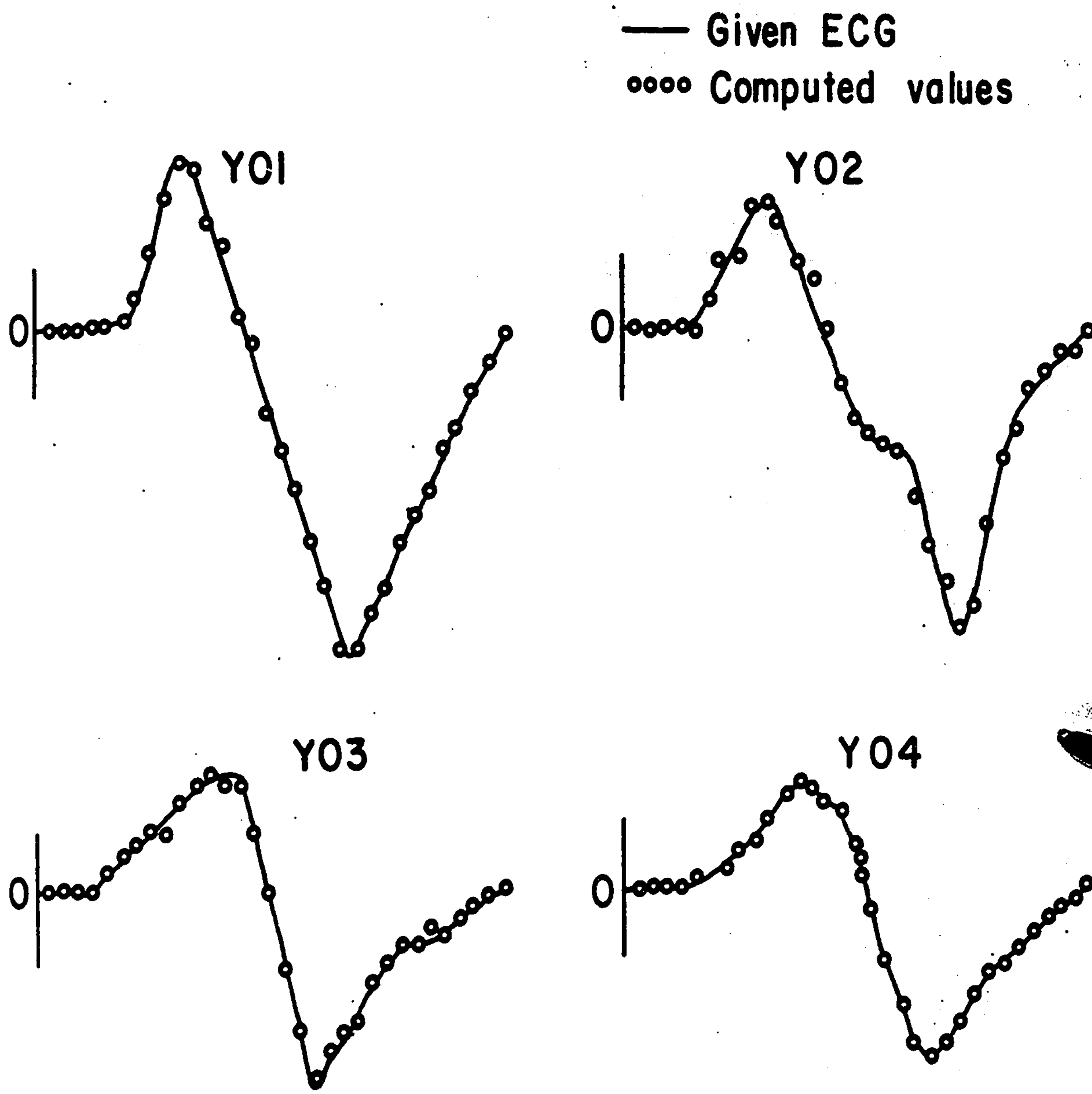


Fig. 4.5 Given electrocardiograms and their computed values. Y01, Y02, - - - - , Y30, refer to ECGs at positions 01, 02, - - - - , 30, respectively, as shown in Fig. 4.1. Computed values were obtained from the coefficients given in table 4.7 and the equivalent sources included in the solution (Appendix XIV).

Fig. 4.5 (continued)

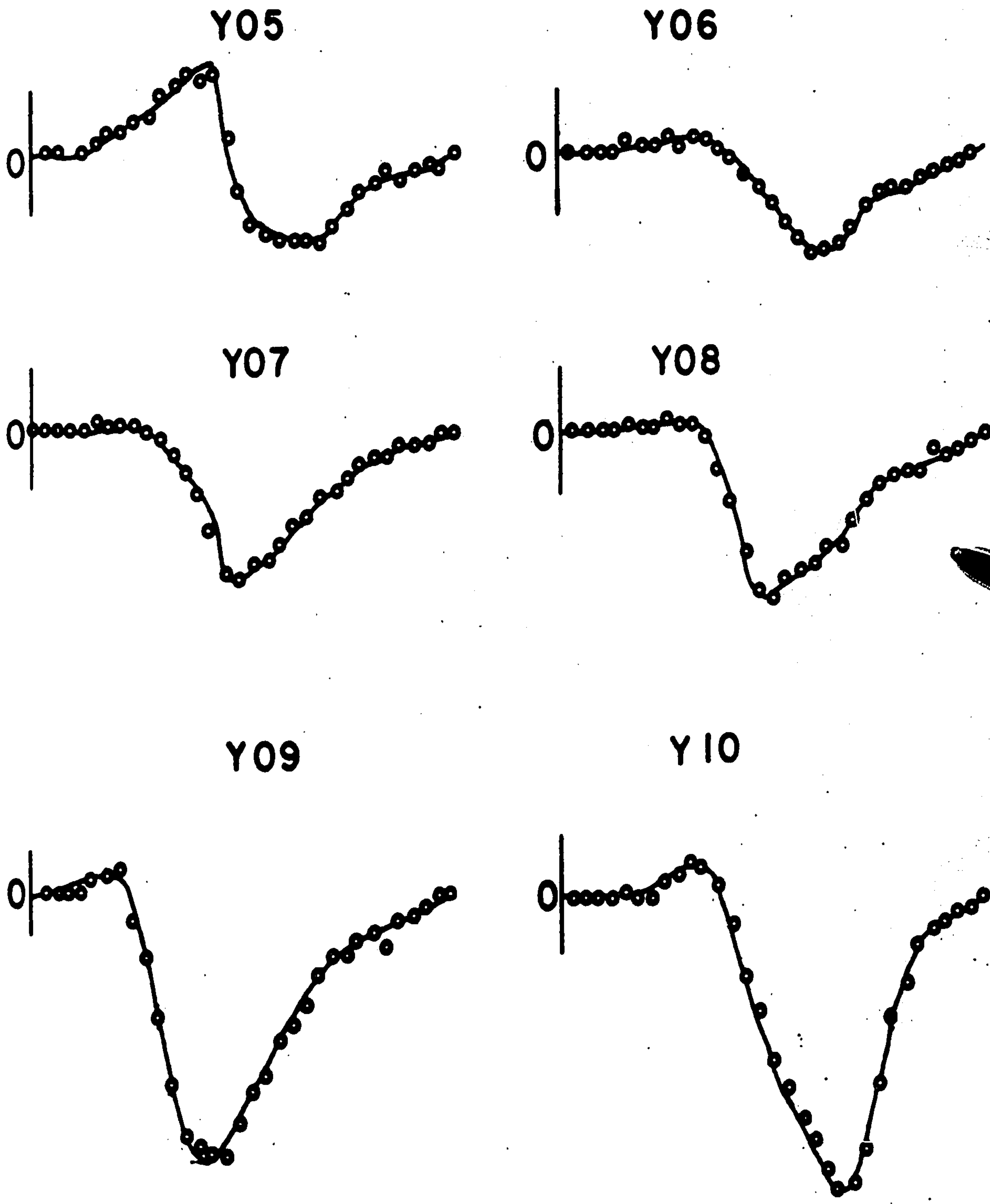


Fig. 4.5 (continued)

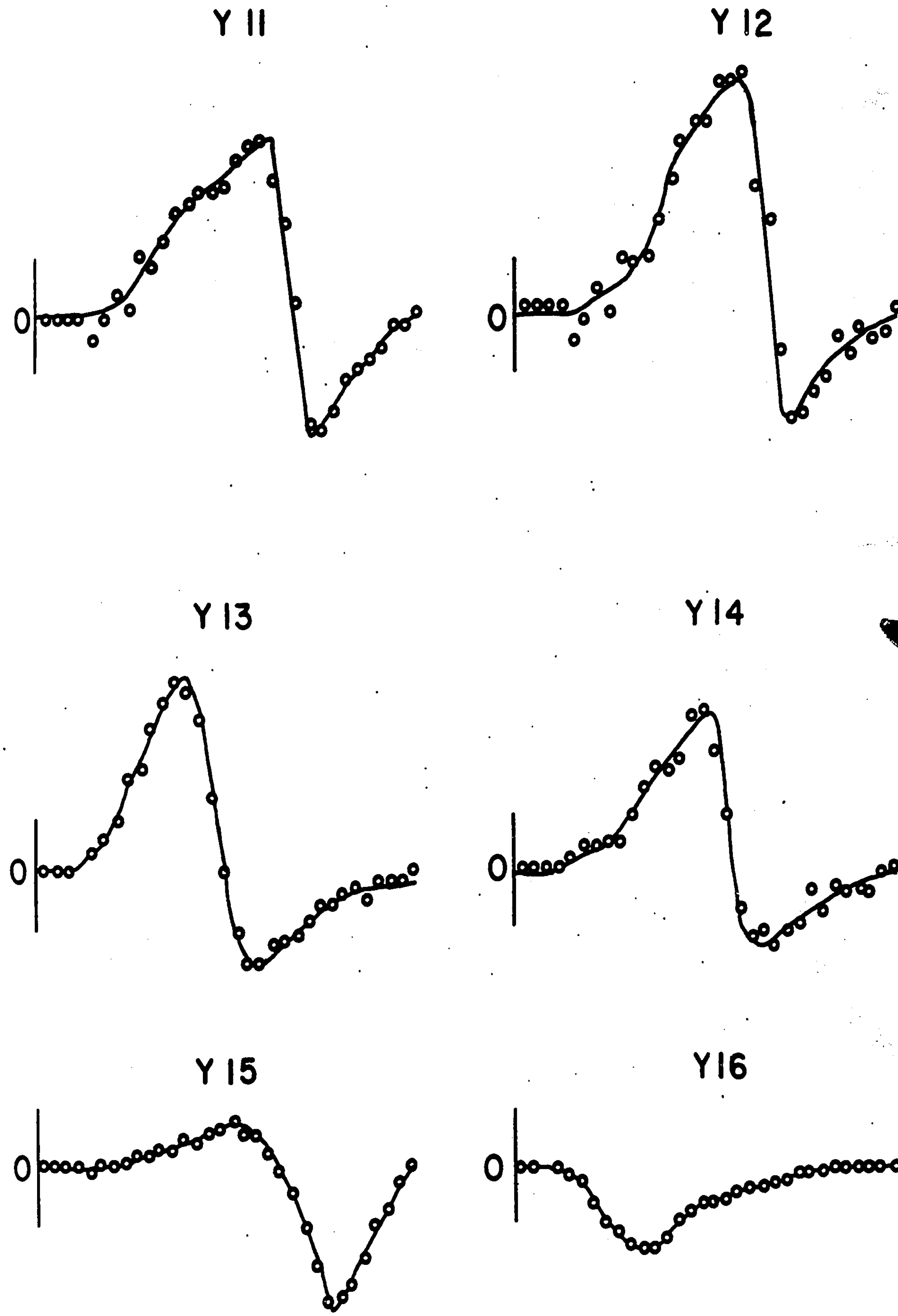


Fig. 4.5 (continued)

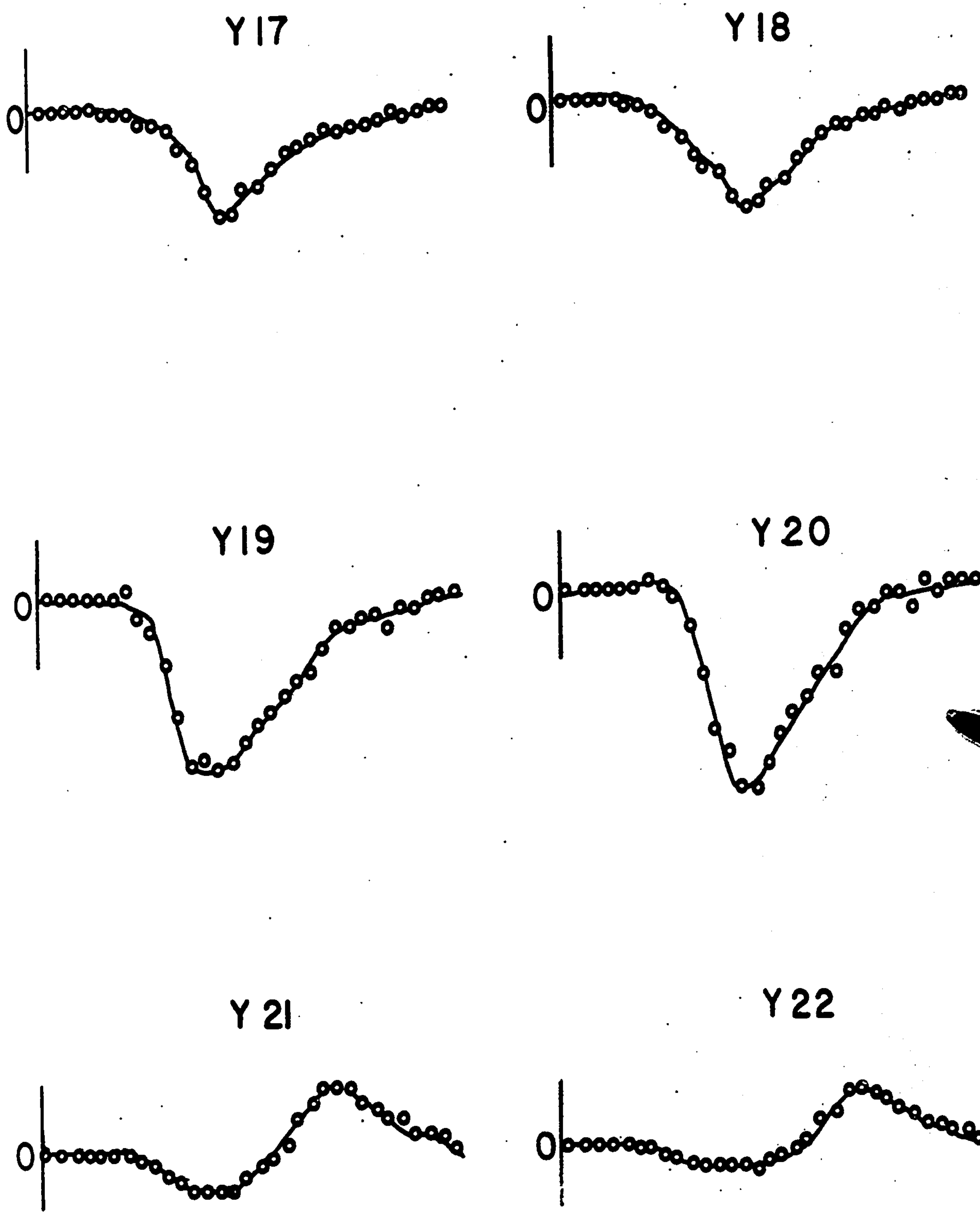
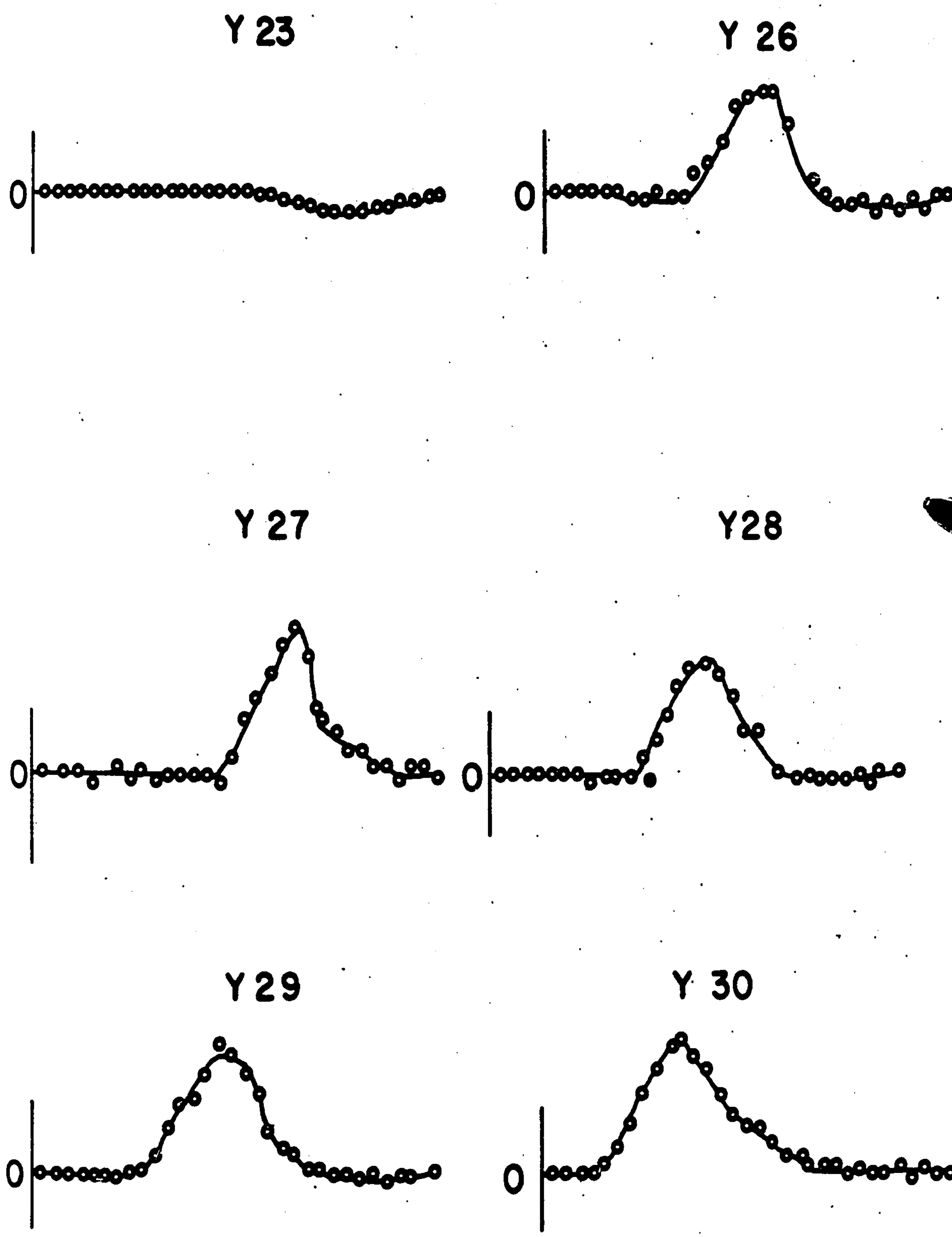


Fig. 4.5 (continued)



4.4 Inverse problem

The inverse problem can be stated as follows: Given surface electrocardiograms $\{Y_k(t)\}$ and system parameters a_{ik}, determine equivalent sources $\{X_i(t)\}$ inside the heart.

We have already shown that a given electrocardiogram $Y_k(t)$ can be explained in terms of equivalent sources $\{X_i(t)\}$ by the relationship

$$\hat{Y}_k(t) = C_k + \sum_{i=1}^{M} a_{ik} X_i(t) \simeq Y_k(t) \tag{4.2}$$

By solving the model we obtained coefficients a_{ik} these coefficients are functions of the medium between the heart and the k-th point on the body and also the position of the point K. The coefficients a_{ik} are thus invariant if the position of the points on the body is kept fixed. But a change in the normal electrical activity of the heart would change $\{X_i(t)\}$ and $\{Y_k(t)\}$.

At an instant of time $t = t_j$, we can write for point K

$$\hat{Y}_k(t_j) = C_k + \sum_{i=1}^{M} a_{ik} X_i(t_j) \tag{4.3}$$

If the potentials are recorded at total number L of points on the body, we obtain a set of L simultaneous equations like (4.3) with $K = 1, 2, ----, L$. This in effect is the same problem as in equation (3.1) with the exception that

(i) the sources $\{X_i(t)\}$ are now unknown and sum squares deviation $\sum_{i=1}^{L} (Y_i - \hat{Y}_i)^2$ is to be minimized with respect to $\{X_i(t_j)\}$ instead of a_{ik}.

(ii) there are a total of L simultaneous equations for $K = 1, 2, ----, L$ for all the positions of points on the body and fixed instant of time $t = t_j$, whereas in the previous case there were N simultaneous equations for t_j, $j = 1, 2, ----, N$ and fixed position on the body.

In order to solve equation (4.3) for M-unknowns $\{X_i(t), i = 1, 2, ----, M\}$, we require that ECGs should be recorded on the body at number "L" of points, where $L > M$.

Each regression solution will now give the values of sources $\{X_i(t)\}$ for instant $t = t_j$. If $j = 1, 2, ----, N$, the regression solution will be carried out N times to find the complete history of the sources. In the previous case we had carried out regression "L" times for $K = 1, 2, ----, L$, to determine the set of coefficients for all the points on the body.

For the present analysis, we will use the same ECGs $\{Y_k(t)\}$ that were used for direct solution in section 4.1 and 4.2. The set of coefficients has been chosen from section 4.2 for 21 positions out of 28 discussed there. The 21 positions chosen are as follows:

01, 05 - 10, 13, 15 - 23, 26 - 27, 29 - 30

If we were able to obtain a perfect fit for electrocardiograms at these positions in the direct solution, then the inverse solution should produce equivalent sources identical to the original ones. A perfect fit is never a case in experimental analysis. However, the quality of the equivalent sources that the inverse solution will produce, will, by implication, indicate the quality of the direct solution in this case.

Fig. 4.5, shows the outline of the computational procedure for obtaining inverse solution. In this figure M1 is the reduced number of equivalent sources and L1 is the number of points on the body selected for obtaining inverse solution.

Table 4.9 shows the statistical evaluation of solutions obtained for 29 out of 32 sample points taken over 80 msec duration. The sample points 01, 02, have not been shown because at these instants the electrocardiograms chosen show zero potential except for a very slight deflection in Y09,

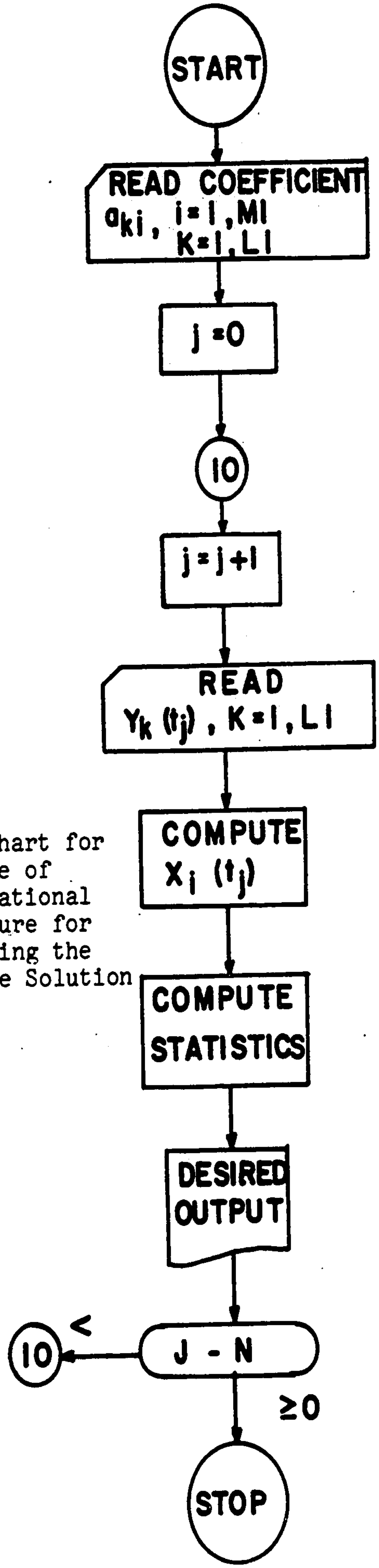


Fig. 4.5 Flow Chart for Outline of Computational Procedure for obtaining the Inverse Solution

TABLE 4.9 Partial statistical evaluation of regression solution (inverse problem)

	CVR	SDF	MSDR	CD	von Neumann Q	ABSE
YT03	1.56512	.15279	.02334	.91500	2.24275	.08821
YT04	.94318	.26499	.07022	.90133	2.59036	.19278
YT05	.75193	.36522	.13339	.91912	2.41902	.24664
YT06	.22869	.17642	.03112	.99133	2.18211	.10502
YT07	.15497	.19113	.03653	.99687	2.97170	.09211
YT08	.30449	.48864	.23877	.99136	2.06796	.14669
YT09	.21387	.38496	.14819	.99771	2.68319	.11860
YT10	.13530	.15269	.02332	.99981	2.78772	.03295
YT11	2.58767	.28341	.08032	.99954	2.17107	.03387
YT12	-.11544	.19516	.03809	.99984	2.31750	.02958
YT13	-.08547	.28125	.07910	.99972	2.58784	.03689
YT14	-.04849	.25769	.06641	.99976	2.98242	.01857
YT15	-.06298	.49213	.24219	.99900	2.16917	.04144
YT16	-.01280	.10825	.01172	.99995	2.41141	.00472
YT17	-.01573	.13577	.01843	.99991	3.32583	.01311
YT18	-.02206	.18896	.03571	.99983	2.96960	.02735
YT19	-.01036	.0944	.00891	.99995	3.37879	.00914
YT20	-.05840	.50389	.25391	.99879	2.00061	.03451
YT21	-.02888	.24206	.05859	.99976	2.38479	.02835
YT22	-.02778	.21195	.04492	.99980	2.73086	.02862

TABLE 4.9 (continued)

	CVR	SDR	MSDR	CD	von Neumann Q	ABSE
YT23	-.00427	.02923	.00085	.99999	6.70785	.00534
YT24	-.02302	.13844	.01917	.99983	2.58364	.04653
YT25	-.04370	.23346	.05450	.99937	2.12050	.04412
YT26	-.02834	.13143	.01727	.99972	2.35186	.03396
YT27	-.04698	.17539	.03076	.99921	2.76482	.05082
YT28	-.07491	.22616	.05115	.99788	2.40310	.06880
YT29	-.06643	.15468	.02393	.99820	2.51226	.06494
YT30	-.06659	.09957	.00991	.99829	2.36105	.07188
YT31	-.09184	.06560	.00430	.99639	2.75562	.05404

YTJ indicates potentials Y01, Y02, - - - -, Y30 at instant $t = t_J$ where J = 1, 2, - - - -, 32, the sample points over 80 msec duration.

CVR, SDR, MSDR, CD, Q and ABSE are defined under table 4.6.

Y20, Y26 and Y30. At sample point 32, all the ECGs have zero potential.

A look at table 4.9, will show that the coefficient of determination in all the cases is high, except for YT03, YT04 and YT05. This indicates in general, a good fit. von Neumann ratio of the least squares estimated disturbances, Q is not high enough and this would, to some extent, affect our estimates. However, at YT23, Q is as high as 6.7.

In table 4.10, we have shown the outcome of t-test on the values estimated by regression solutions. As would be seen, a large number of estimated values is significant. We will use this table to decide whether an estimate is acceptable or not. If we consider the sample points, where a large number of "F" appears in the table and compare them with the values of original sources at these points, we will notice that the original sources are zero or near zero.

Table 4.11 shows the time-history of equivalent sources as obtained from regression analysis. These values should be compared with the original values given in appendix XIII. It will be noticed that the estimates compare favorably, with few exceptions. However, the time-history magnitude is not identical to the original sources. This should be

TABLE 4.10 Truth table derived by invoking t-test on the estimated values of sources.

X	01	02	03	04	05	06	07	08	09	10	11	12	13	14	15	16	17	18	19	20
Y03			F	F	F	F	T	T	T	T			F	F	T	F	F	F	F	T
Y04			T	T	F	F	T	T	T	T			F	F	F	F	F	F	F	T
Y05			T	F	F	F	F	T	F	F			F	F	F	F	F	F	F	T
Y06			T	T	F	F	F	T	F	F			F	F	F	F	T	F	F	F
Y07			T	T	F	F	F	T	T	T			T	F	F	T	T	F	F	F
Y08			T	T	F	F	F	T	T	T			F	F	F	F	T	F	F	F
Y09			T	T	T	F	F	T	T	T			F	F	F	F	T	F	F	T
Y10			T	T	T	T	F	T	T	T			T	F	F	F	T	T	F	F
Y11			T	T	T	T	F	T	T	T			F	F	F	T	T	T	T	F
Y12			T	T	T	F	F	T	T	T			T	F	F	F	T	T	T	F
Y13			T	T	T	T	F	T	T	T			T	F	F	F	T	T	T	F
Y14			T	T	T	T	F	T	T	T			T	F	F	F	T	T	T	T
Y15			T	T	T	T	T	T	T	T			T	F	F	F	T	T	T	T
Y16			T	T	T	T	T	T	T	T			T	F	F	T	T	T	T	T
Y17			T	T	T	T	T	T	T	T			T	F	T	T	T	T	T	T
Y18			T	T	T	T	T	T	T	T			T	T	T	T	T	T	T	T
Y19			T	T	T	T	T	F	T	T			T	T	T	T	T	T	T	T
Y20			T	T	T	T	T	F	T	T			T	T	T	T	F	T	T	T
Y21			T	T	T	T	T	F	T	T			T	T	T	T	F	T	T	T
Y22			T	T	T	T	T	F	T	T			T	T	T	T	F	T	T	T
Y23			F	T	T	T	T	T	T	T			T	T	T	T	T	T	T	T
Y24			T	T	T	T	T	F	F	T			T	T	T	T	F	T	T	T
Y25			F	F	T	T	T	F	F	F			T	T	T	T	F	T	T	T
Y26			F	F	T	T	T	F	F	F			T	T	T	T	F	T	T	T
Y27			F	F	T	T	T	T	T	T			T	T	T	T	F	T	T	T
Y28			F	F	F	T	T	F	T	T			T	T	T	T	F	T	T	T
Y29			F	F	F	T	T	F	F	F			T	T	T	T	F	T	T	T
Y30			F	F	T	F	T	F	T	T			F	T	T	T	F	F	T	T
Y31			F	F	F	T	T	F	F	F			T	T	T	F	F	T	F	T

T → The value of source (identified at the top of the column, for sample point indicated at the left of row is significant.

F → Not "T"

TABLE 4.11 Time history of equivalent dipole sources as obtained from the inverse solution. Values are given at 2.5 msec interval i.e. 01 corresponds to 2.5 msec, 02 to 5.0 msec etc.

NO	X03	X04	X05	X06	X07	X08	X09	X10
01	-	-	-	-	-	-	-	-
02	-	-	-	-	-	-	-	-
03	-	-	-	-	-00.18*	00.49	-00.51*	-00.57*
04	00.32	00.15	-	-	-00.28*	1.13	-00.90*	- 1.06*
05	00.46	00.27*	-	-	-	1.66	-	-
06	00.32	00.65	-	-	-	2.44	-	-
07	1.27	1.16	-	-	-	4.54	00.77	1.17
08	1.55	1.34	-	-	-	6.38	1.35	1.99
09	1.41	1.24	1.24	-	-	5.78	2.67	3.90
10	1.93	1.73	0.92	0.59	-	7.73	4.38	4.91
11	2.32	2.09	0.80	0.57	-	9.86	5.34	5.22
12	2.06	1.88	1.12	0.25*	-	9.89	6.75	6.64
13	1.97	2.25	1.11	1.77	-	9.85	6.29	6.22
14	1.88	2.06	1.97	0.94	-	9.63	5.81	6.19
15	2.41	2.10	1.78	1.17	0.70	11.07	5.89	5.69
16	1.39	2.00	1.50	1.68	0.83	8.32	6.63	6.35
17	1.35	1.95	2.33	1.63	0.90	5.17	5.91	6.22
18	1.93	1.[illegible]	2.50	1.64	1.02	3.11	5.60	[illegible]
19	1.67	1.69	1.78	2.00	1.78	-	6.73	6.77
20	1.34	1.70	2.00	1.93	1.75	-	5.71	5.37
21	0.77	1.43	2.41	1.91	1.81	-	4.51	3.94
22	0.45	0.95	2.62	2.38	1.72	-	3.23	2.50
23	-	0.47	2.31	2.14	1.93	0.16*	2.27	1.50
24	0.13*	0.29	2.27	2.19	1.86	-	- -	0.52
25	-	-	0.64	3.19	1.99	-	-	-
26	-	-	0.45	2.39	1.29	-	-	-
27	-	-	0.38	1.28	1.69	0.62*	-0.73*	-0.77*
28	-	-	-	0.78	1.46	-	-0.67*	-0.80*
29	-	-	-	0.54	1.21	-	-	-
30	-	-	0.22*	-	0.88	-	-0.34*	-0.38*
31	-	-	-	- 0.27*	0.54	-	-	-
32	-	-	-	-	-	-	-	-

Values marked * are incorrect values as given by the inverse solution, when these values are compared with the time history of assumed equivalent sources in the case of normal cardiac activity (Appendix XIII).

TABLE 4.11 (continued)

NO	X13	X14	X15	X16	X17	X18	X19	X20
01	-	-	-	-	-	-	-	-
02	-	-	-	-	-	-	-	-
03	-	-	-0.32*	-	-	-	-	-0.34*
04	-	-	-	-	-	-	-	-0.70*
05	-	-	-	-	-	-	-	-0.77*
06	-	-	-	-	0.98	-	-	-
07	-0.34*	-	-	0.54*	1.56	-	-	-
08	-	-	-	-	1.64	-	-	-
09	-	-	-	-	1.25	-	-	-0.69*
10	0.63*	-	-	-	1.71	0.18	-	-
11	-	-	-	0.75*	2.06	0.33	0.19	-
12	1.32	-	-	-	1.79	0.78	0.73	-
13	3.98	-	-	-	1.90	1.60	1.00	-
14	6.70	-	-	-	1.55	2.30	3.75	0.82
15	4.07	-	-	-	1.88	2.78	2.89	1.36
16	5.30	-	-	0.74	1.56	2.85	3.45	2.81
17	6.47	-	0.46	1.21	1.24	3.10	3.22	3.34
18	7.10	0.85	1.03	1.47	0.96	3.22	3.36	3.71
19	6.17	3.07	1.89	2.89	0.93	3.37	3.26	4.35
20	5.50	4.41	2.18	3.18	-	3.17	3.94	4.31
21	6.41	4.30	2.17	2.46	-	3.01	3.12	3.83
22	7.54	4.37	2.06	2.11	-	3.26	3.37	3.84
23	6.48	5.07	2.37	2.50	0.08*	3.44	3.44	4.59
24	5.60	4.74	2.22	2.62	-	2.59	3.21	3.41
25	5.54	5.18	2.27	2.74	-	2.20	4.39	3.45
26	3.68	4.96	2.22	2.63	-	2.15	3.44	2.73
27	1.53	4.47	1.94	2.55	-	1.47	2.21	1.84
28	0.64	3.98	1.71	2.12	-	0.77	1.57	1.49
29	0.40	3.20	1.51	1.56	-	0.41	1.18	1.39
30	-	2.05	1.09	0.84	-	-	0.27	0.77
31	-0.30*	1.56	0.82	-	-	-0.02*	-	0.53
32	-	-	-	-	-	-	-	-

acceptable, especially in view of the fact that the solutions obtained now have used estimates a_{ik} from the direct solution.

In this analysis of the inverse problem, we took ECGs for the case of normal cardiac activity. This was done to make a comparison of estimates obtained with the original values. For abnormal activity of the heart, the new ECGs at the same reference positions 01, 02, - - - -, 30 could be fed as input data. The inverse problem would, then, give the time history of new equivalent sources inside the heart at the locations as before. Further work would indicate the type of abnormalities that can be determined through the solution of the inverse problem.

CONCLUSION

We have carried out an analysis of the unipolar electrocardiograms, recorded on the body surface encompassing the entire cardiac region, in terms of a number of proposed equivalent dipole sources representing ventricular depolarization of the heart. The equivalent sources which were found to be making no significant contribution, were deleted. A transfer matrix, representing relationship between the significant equivalent sources and body surface electrocardiograms, was then determined. This transfer matrix was later used to determine the inverse solution.

We have thus shown that the multivariate regression analysis can be applied to electrocardiography with promising results. We have discussed justification of the method and provided necessary material for evaluation and analysis of results. What remains to be done is to apply the proposed method to a large number of normal cases. This would determine the equivalent sources that can be taken, in general, to represent normal electrical activity of the heart. Large scale studies would also reveal the variability of coefficients in the transfer matrix from one subject to another.

The material presented in connection with regression analysis is quite general. It can be usefully applied to statistical analysis of data e.g., in economic studies, medicine and biology.

APPENDICES

APPENDIX I

Potential due to a Single Dipole

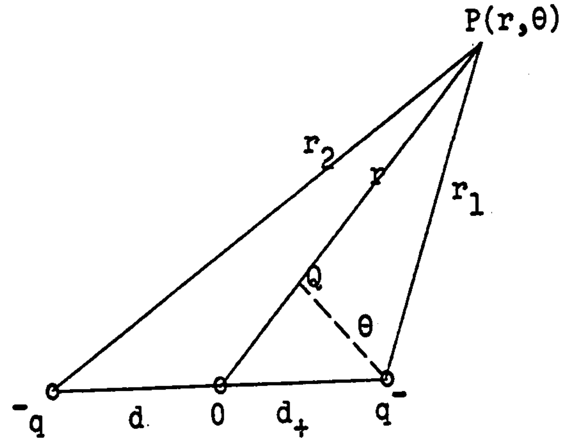


Fig. AI.1

As shown in the Fig. AI.1, the potential at point P due to the charges +q and -q at radial distances r_1 and r_2 is

$$V_p = \frac{1}{4\pi\varepsilon}\left\{\frac{q}{r_1} - {}^{q}/_{r_2}\right\}$$

Now

$$r_1 = OP - OQ \quad \text{(approximately)}$$

$$= r - d\cos\theta$$

Similarly

$$r_2 = r + d\cos\theta$$

$$\therefore \quad V_p = \frac{q}{4\pi\varepsilon}\left[\frac{1}{r - d\cos\theta} - \frac{1}{r + d\cos\theta}\right]$$

$$= \frac{q}{4\pi\varepsilon}\left[\frac{2d\cos}{r^2 - d^2\cos^2\theta}\right] \simeq \frac{q2d}{4\pi\varepsilon r^2}$$

where 2dq = moment of dipole

$$\therefore \quad V_p = \frac{M}{4\pi\varepsilon}\cdot\frac{\cos\theta}{r^2}$$

APPENDIX II

Potential due to Multiple Dipoles

Let us consider a probe at point P (Fig. AII.1). The dipole is located on the ring of radius "R", with the dipole moment directed radially outward.

Let

M = dipole moment.

ϕ = angle that the probe makes with reference to the horizontal axis.

θ = angle that the position of dipole makes with reference to horizontal axis.

Then by definition (Appendix I)

V_p = potential at P due to dipole

$$= \left(\frac{1}{4\pi\varepsilon}\right)\frac{M\cos\lambda}{AP^2} \qquad \text{(AII.1)}$$

and V_Q = potential at Q due to dipole

$$= \left(\frac{1}{4\pi\varepsilon}\right)\frac{M\cos\lambda}{AQ^2} \qquad \text{(AII.2)}$$

where ε = electric permittivity of the medium.

Now $AP^2 = AB^2 + BP^2$

$$= (OP\cos\phi - R\cos\theta)^2 + (OP\sin\phi - R\sin\theta)^2$$

$$= OP^2 + R^2 - 2(OP)(R)(\cos\theta\cos\phi + \sin\theta\sin\phi)$$

$$= OP^2 + R^2 - 2(OP)(R)\cos(\theta - \phi) \qquad \text{(AII.3)}$$

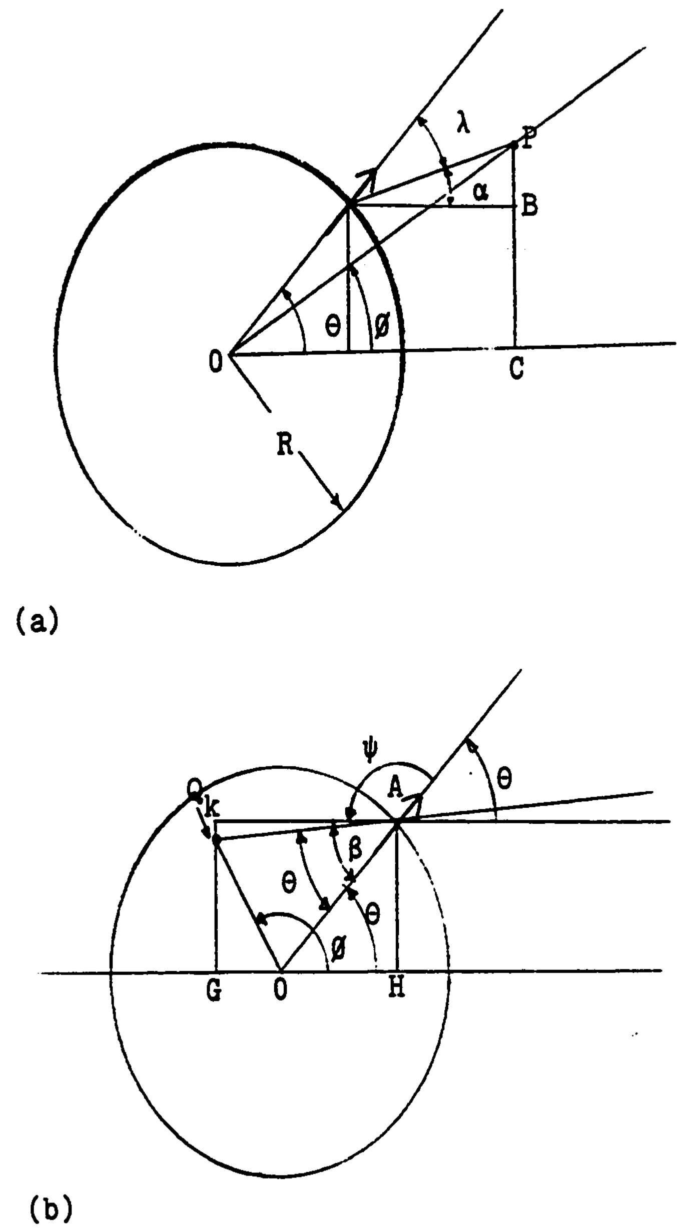


Fig. Al.1

Let

$$\frac{OP}{R} = F$$

$$\therefore \quad AP^2 = R^2[F^2 + 1 - 2F \text{ Cos } (\theta - \emptyset)] \qquad \text{(AII.4)}$$

$$\text{Cos}\lambda = \text{Cos}(\theta - \alpha) = \text{Cos}\theta\text{Cos}\alpha + \text{Sin}\theta\text{Sin}\alpha \qquad \text{(AI.5)}$$

But

$$\text{Cos}\lambda = \frac{AB}{AP} = \frac{OP\text{Cos}\emptyset - R\text{Cos}\theta}{AP} = \frac{R(F\text{Cos}\emptyset - \text{Cos}\theta)}{AP}$$

and $$\text{Sin}\alpha = \frac{BP}{AP} = \frac{OP\text{Sin}\emptyset - R\text{Sin}\theta}{AP} = \frac{R(F\text{Sin}\emptyset - \text{Sin}\theta)}{AP}$$

$$\therefore \quad \text{Cos}\lambda = \frac{R[\text{Cos}\theta(F\text{Cos}\emptyset - \text{Cos}\theta) + \text{Sin}\theta(F\text{Sin}\emptyset - \text{Sin}\theta)]}{AP}$$

$$= \frac{R[F\text{Cos}(\theta - \emptyset) - 1]}{AP} \qquad \text{(AI.6)}$$

$$V_p = \left(\frac{1}{4\pi\varepsilon}\right) \frac{MR[F\text{Cos}(\theta - \emptyset) - 1]}{AP^3} \qquad \text{(AII.7)}$$

Also

$$AQ^2 = (QK)^2 + (AK)^2$$

$$= (AH - GQ) + (OH + GH)^2 \qquad \text{(AII.8)}$$

$$= [R\text{Sin}\theta - PQ\text{Sin } (\pi - \emptyset)]^2 + [R\text{Cos}\theta + OQ\text{Cos}(\pi - \emptyset)]^2$$

Let

$$\frac{OQ}{R} = E$$

$\therefore$ QQ = RE substituting in AII.8

$$AQ^2 = R^2(\text{Sin}\theta - E\text{Sin}\emptyset)^2 + R^2(\text{Cos}\theta - E\text{Cos}\emptyset)^2$$

$$= R^2[1 + E^2 - 2E\text{Cos}(\theta - \emptyset)] \qquad \text{(AII.9)}$$

$$\psi = \pi - \beta$$

$$\text{Cos}\psi = - \text{Cos}\beta \qquad \text{(AII.10)}$$

But

$$\cos\beta = \frac{AQ^2 + R^2 - OQ^2}{2(AQ)(R)}$$

$$= \frac{R^2[1 + E^2 - 2E\cos(\theta - \phi)] + R^2 - R^2E^2}{2(AQ)R}$$

$$= \frac{R}{AQ}[1 - E\cos(\theta - \phi)]$$

$$\therefore \quad \cos\psi = \frac{R}{AQ}[E\cos(\theta - \phi) - 1]$$

$$\therefore \quad V_Q = \left(\frac{1}{4\pi\varepsilon}\right)\frac{MR[E\cos(\theta - \phi) - 1]}{AQ^3} \qquad \text{(AII.11)}$$

Fortran II
Dec Computer PDP-8

```
C; PROGRAM FOR CALCULATING POTENTIALS
C; SPECIFY THE INPUTS AS FOLLOWS
C; 1. POSITION OF PROBE IN TERMS OF ANGLE IN DEGREES
C; 2. ANGLE SUBTENDED AT THE ORIGIN BY THE ARC CONTAINING DIPOLES
C; 3. INITIAL, FINAL, AND INCREMENTAL VALUES OF RADIUS TIMES 100
C; 4. INITIAL, FINAL, AND INCREMENTAL VALUES OF FACTOR F TIMES 100
C; 5. INITIAL, FINAL, AND INCREMENTAL VALUES OF THE NUMBER OF DIPOLES
2;FORMAT(E)
3;FORMAT(/)
4;FORMAT(I,I,I)
5;FORMAT(/,"R=",I,"F=",I,"N=",I,"POTENTIAL=",E,/)
1;ACCEPT 2,PHI
TYPE 3
ACCEPT 2,ARC
TYPE 3
ACCEPT 4,IR,JR,KR
TYPE 3
ACCEPT 4,IF,JF,KF
TYPE 3
ACCEPT 4,IN,JN,KN
TYPE 3
CRAD=22./(7.*180.)
DO 50 I=IR,JR,KR
R=I
R=R/100.
DO 50 J=IF,JF,KF
F=J
F=F/100.
DO 50 K=IN,JN,KN
POT=0.
SK=K
SM=1./SK
DELT=ARC/SK
DELTI=DELT/2.
THETA=DELTI
DO 40 L=1,K
GO TO(10),L
THETA=THETA+DELT
10;CONTINUE
ANG=THETA-PHI
ANG=ANG*CRAD
COANG=COSF(ANG)
AP=R*SQTF(F*F+1.-2.*F*COANG)
POT=POT+SM*R*(F*COANG-1.)/(AP**3)
40;CONTINUE
TYPE 5,I,J,K,POT
50;CONTINUE
GO TO 1
END
```

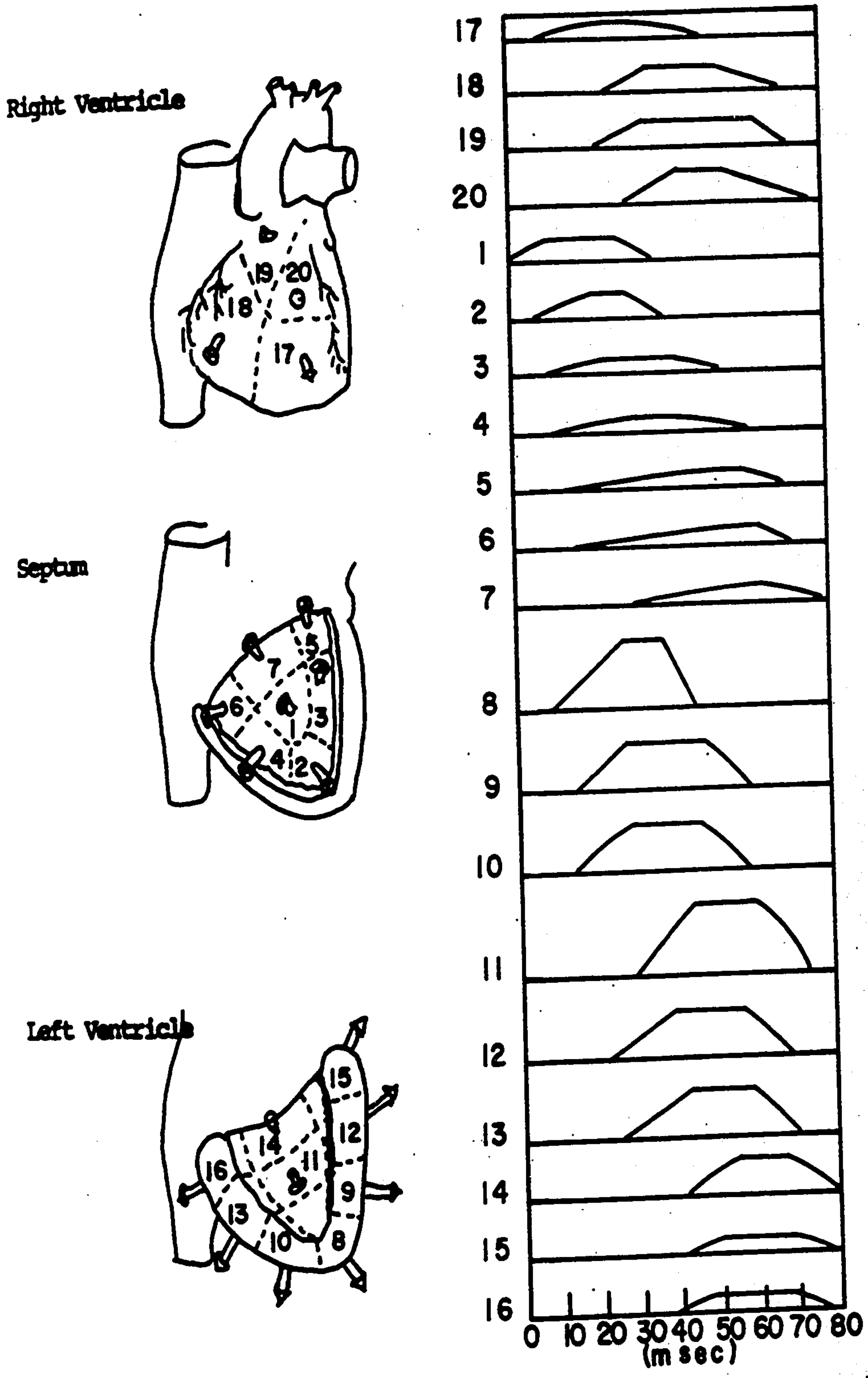


Fig. IV.1 On the left the heart is represented by the twenty myocardial segments. These segments are in turn replaced by equivalent dipoles placed at the centre of each segment. The time history of dipole source currents is shown on the right. [47]

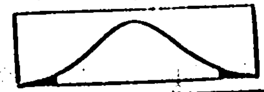
Correlation Coefficients [52]

The table gives values of the correlation coefficient r which have certain probabilities of being *exceeded* for observations of variables whose parent distributions are independent. The number of pairs of observations is N. To illustrate: for a sample of 10 pairs of observations on unrelated variables, the probability is 0.10 that it will have $r \geq 0.549$, and the probability is 0.001 that $r \geq 0.875$.

N	Probability				
	0.10	0.05	0.02	0.01	0.001
3	0.988	0.997	0.999	1.000	1.000
4	0.900	0.950	0.980	0.990	0.999
5	0.805	0.878	0.934	0.959	0.992
6	0.729	0.811	0.882	0.917	0.974
7	0.669	0.754	0.833	0.874	0.951
8	0.621	0.707	0.789	0.834	0.925
10	0.549	0.632	0.716	0.765	0.872
12	0.497	0.576	0.658	0.708	0.823
15	0.441	0.514	0.592	0.641	0.760
20	0.378	0.444	0.516	0.561	0.679
30	0.307	0.362	0.423	0.464	0.572
40	0.264	0.312	0.367	0.403	0.502
60	0.219	0.259	0.306	0.337	0.422
80	0.188	0.223	0.263	0.291	0.366
100	0.168	0.199	0.235	0.259	0.327

Values of t [53]

For Given Degrees of Freedom (n) and at Specified Levels of Significance (P)

This table shows the black areas: 

n	.90	.80	.70	.60	.50	.40	.30	.25	.20	.10	.05	.025	.02	.01	.005	.001	n
	Level of significance (P)								Level of significance (P)								
1	.158	.325	.510	.727	1.000	1.376	1.963	2.414	3.078	6.314	12.706	25.452	31.821	63.657	127.32	636.619	1
2	.142	.289	.445	.617	.816	1.061	1.386	1.604	1.886	2.920	4.303	6.205	6.965	9.925	14.089	31.598	2
3	.137	.277	.424	.584	.765	.978	1.250	1.423	1.638	2.353	3.182	4.176	4.541	5.841	7.453	12.941	3
4	.134	.271	.414	.569	.741	.941	1.190	1.344	1.533	2.132	2.776	3.495	3.747	4.604	5.598	8.610	4
5	.132	.267	.408	.559	.727	.920	1.156	1.301	1.476	2.015	2.571	3.163	3.365	4.032	4.773	6.859	5
6	.131	.265	.404	.553	.718	.906	1.134	1.273	1.440	1.943	2.447	2.969	3.143	3.707	4.317	5.959	6
7	.130	.263	.402	.549	.711	.896	1.119	1.254	1.415	1.895	2.365	2.841	2.998	3.499	4.029	5.405	7
8	.130	.262	.399	.546	.706	.889	1.108	1.240	1.397	1.860	2.306	2.752	2.896	3.355	3.832	5.041	8
9	.129	.261	.398	.543	.703	.883	1.100	1.230	1.383	1.833	2.262	2.685	2.821	3.250	3.690	4.781	9
10	.129	.260	.397	.542	.700	.879	1.093	1.221	1.372	1.812	2.228	2.634	2.764	3.169	3.581	4.587	10
11	.129	.260	.396	.540	.697	.876	1.088	1.214	1.363	1.796	2.201	2.593	2.718	3.106	3.497	4.437	11
12	.128	.259	.395	.539	.695	.873	1.083	1.209	1.356	1.782	2.179	2.560	2.681	3.055	3.428	4.318	12
13	.128	.259	.394	.538	.694	.870	1.079	1.204	1.350	1.771	2.160	2.533	2.650	3.012	3.372	4.221	13
14	.128	.258	.393	.537	.692	.868	1.076	1.200	1.345	1.761	2.145	2.510	2.624	2.977	3.326	4.140	14
15	.128	.258	.393	.536	.691	.866	1.074	1.197	1.341	1.753	2.131	2.490	2.602	2.947	3.286	4.073	15
16	.128	.258	.392	.535	.690	.865	1.071	1.194	1.337	1.746	2.120	2.473	2.583	2.921	3.252	4.015	16
17	.128	.257	.392	.534	.689	.863	1.069	1.191	1.333	1.740	2.110	2.458	2.567	2.898	3.222	3.965	17
18	.127	.257	.392	.534	.688	.862	1.067	1.189	1.330	1.734	2.101	2.445	2.552	2.878	3.197	3.922	18
19	.127	.257	.391	.533	.688	.861	1.066	1.187	1.328	1.729	2.093	2.433	2.539	2.861	3.174	3.883	19
20	.127	.257	.391	.533	.687	.860	1.064	1.185	1.325	1.725	2.086	2.423	2.528	2.845	3.153	3.850	20
21	.127	.257	.391	.532	.686	.859	1.063	1.183	1.323	1.721	2.080	2.414	2.518	2.831	3.135	3.819	21
22	.127	.256	.390	.532	.686	.858	1.061	1.182	1.321	1.717	2.074	2.406	2.508	2.819	3.119	3.792	22
23	.127	.256	.390	.532	.685	.858	1.060	1.180	1.319	1.714	2.069	2.398	2.500	2.807	3.104	3.767	23
24	.127	.256	.390	.531	.685	.857	1.059	1.179	1.318	1.711	2.064	2.391	2.492	2.797	3.090	3.745	24
25	.127	.256	.390	.531	.684	.856	1.058	1.178	1.316	1.708	2.060	2.385	2.485	2.787	3.078	3.725	25
26	.127	.256	.390	.531	.684	.856	1.058	1.177	1.315	1.706	2.056	2.379	2.479	2.779	3.067	3.707	26
27	.127	.256	.389	.531	.684	.855	1.057	1.176	1.314	1.703	2.052	2.373	2.473	2.771	3.056	3.690	27
28	.127	.256	.389	.530	.683	.855	1.056	1.175	1.313	1.701	2.048	2.368	2.467	2.763	3.047	3.674	28
29	.127	.256	.389	.530	.683	.854	1.055	1.174	1.311	1.699	2.045	2.364	2.462	2.756	3.038	3.659	29
30	.127	.256	.389	.530	.683	.854	1.055	1.173	1.310	1.697	2.042	2.360	2.457	2.750	3.030	3.646	30
40	.126	.255	.388	.529	.681	.851	1.050	1.167	1.303	1.684	2.021	2.329	2.423	2.704	2.971	3.551	40
60	.126	.254	.387	.527	.679	.848	1.046	1.162	1.296	1.671	2.000	2.299	2.390	2.660	2.915	3.460	60
120	.126	.254	.386	.526	.677	.845	1.041	1.156	1.289	1.658	1.980	2.270	2.358	2.617	2.860	3.373	120
∞	.126	.253	.385	.524	.674	.842	1.036	1.150	1.282	1.645	1.960	2.241	2.326	2.576	2.807	3.291	∞

APPENDIX VII

Determination of Q(s) for testing independence of regression disturbances [section 3.3.2]

The program has been written in fortran II and computations were carried out on the Digital Equipment Corporation Computer PDP-8.

Computation procedure

1. Determine the range of beta variate at the desired significance level.
2. Determine the value of beta variate for desired accuracy.
3. Using program B, evaluate Q(s).

Symbols:

N = Number of observations

J = Number of parameters to be adjusted (No. of explanatory variables + 1)

Examples:

(i)	(ii)	(iii)	(iv)
N = 32	N = 32	N = 69	N = 17
J = 17	J = 21	J = 3	J = 13

Computed values:

Example No.	Significance level			
	1%		5%	
	x	Q	x	Q
(i)	. 55	2.706	. 61	2.878
(ii)	. 62	3.132	. 68	3.269
(iii)	.375	1.500	.415	1.664
(iv)	.295	1.262	.365	1.533

PROGRAM A

```
C; PROGRAM FOR INCOMPLETE BETA FUNCTION RATIO
C; COMPLETE BETA FUNCTION APPEARING AS FIRST OUUTPUT
C; INTEGRATION IS CARRIED OUT USING SIMPSONS RULE
21;FORMAT(I)
22;FORMAT(E)
23;FORMAT(/)
24;FORMAT(/,/,/," PARAMETER-P    PARAMETER-Q  ",
" BETA VARIATE   FUNCTION RATIO ",/,/)
25;FORMAT(/,E,E,E,E,/)
C; ICAT IS AN OPTION FOR INDIRECT OR DIRECT FORM OF P AND Q
ACCEPT 21,ICAT
TYPE 23
GO TO(1,2),ICAT
1;ACCEPT 21,N,J
PN=N
PJ=J
P=(PN+PJ)/2.
Q=(PN-PJ+2.)/2.
TYPE 23
GO TO 3
2;ACCEPT 22,P,Q
TYPE 23
3;CONTINUE
C; NP IS THE NUMBER OF POINTS OVER UNIT INTERVAL
ACCEPT 21,NP
FACTOR=NP
C; Y IS THE INTEGRATION INTERVAL
ACCEPT 22,FIY,FJY,FKY
C; CALCULATION OF COMPLETE BETA FUNCTION
Y=1.
N=FACTOR*Y
LIM=N/2
N=2*LIM
C; INITIAL VALUE OF FUNCTION IS F0
F0=0.
XN=N
H=Y/XN
X=H
T=0.
F=0.
C; FN IS THE GIVEN FUNCTION OF X
```

```
DO 10 I=1,LIM,1
FN=(X**(P-1.))*((1.-X)**(Q-1.))
F=F+FN
X=X+H
FN=(X**(P-1.))*((1.-X)**(Q-1.))
T=T+FN
X=X+H
10;CONTINUE
R=(F0+4.*F+2.*T-FN)*H/3.
RC=R
TYPE 24
TYPE 25,P,Q,Y,RC
C; CALCULATION OF INCOMPLETE BETA FUNCTION RATIO
IY=1000.*FIY
JY=1000.*FJY
KY=1000.*FKY
DO 15 J=IY,JY,KY
Y=J
Y=Y/1000.
N=FACTOR*Y
LIM=N/2
N=2*LIM
C; INITIAL VALUE OF FUNCTION IS F0
F0=0.
XN=N
H=Y/XN
X=H
T=0.
F=0.
C; FN IS THE GIVEN FUNCTION OF X
DO 35 I=1,LIM,1
FN=(X**(P-1.))*((1.-X)**(Q-1.))
F=F+FN
X=X+H
FN=(X**(P-1.))*((1.-X)**(Q-1.))
T=T+FN
X=X+H
35;CONTINUE
R=(F0+4.*F+2.*T-FN)*H/3.
R=R/RC
TYPE 25,P,Q,Y,R
15;CONTINUE
GO TO(1,2),ICAT
END
```

1
32 17
1ØØ .2 1. .1

PARAMETER-P	PARAMETER-Q	BETA VARIATE	FUNCTION RATIO
+Ø.245ØØØE+2	+Ø.85ØØØØE+1	+Ø.1ØØØØØE+1	+Ø.67199ØE-8
+Ø.245ØØØE+2	+Ø.85ØØØØE+1	+Ø.2ØØØØØE+Ø	+Ø.93Ø261E-11
+Ø.245ØØØE+2	+Ø.85ØØØØE+1	+Ø.3ØØØØØE+Ø	+Ø.739277E-7
+Ø.245ØØØE+2	+Ø.85ØØØØE+1	+Ø.4ØØØØØE+Ø	+Ø.28914ØE-4
+Ø.245ØØØE+2	+Ø.85ØØØØE+1	+Ø.5ØØØØØE+Ø	+Ø.195628E-2
+Ø.245ØØØE+2	+Ø.85ØØØØE+1	+Ø.6ØØØØØE+Ø	+Ø.384847E-1
+Ø.245ØØØE+2	+Ø.85Ø2ØØE+1	+Ø.7ØØØØØE+Ø	+Ø.274349E+Ø

1
32 17
1ØØ .5 .6 .Ø1

PARAMETER-P	PARAMETER-Q	BETA VARIATE	FUNCTION RATIO
+Ø.245ØØØE+2	+Ø.85ØØØØE+1	+Ø.1ØØØØØE+1	+Ø.67199ØE-8
+Ø.245ØØØE+2	+Ø.85ØØØØE+1	+Ø.5ØØØØØE+Ø	+Ø.195628E-2
+Ø.245ØØØE+2	+Ø.85ØØØØE+1	+Ø.51ØØØØE+Ø	+Ø.277Ø94E-2
+Ø.245ØØØE+2	+Ø.85ØØØØE+1	+Ø.52ØØØØE+Ø	+Ø.38792ØE-2
+Ø.245ØØØE+2	+Ø.85ØØØØE+1	+Ø.53ØØØØE+Ø	+Ø.536815E-2
+Ø.245ØØØE+2	+Ø.85ØØØØE+1	+Ø.54ØØØØE+Ø	+Ø.7345Ø8E-2
+Ø.245ØØØE+2	+Ø.85ØØØØE+1	+Ø.55ØØØØE+Ø	+Ø.99399ØE-2
+Ø.245ØØØE+2	+Ø.85ØØØØE+1	+Ø.56ØØØØE+Ø	+Ø.133055E-1

32 17
1ØØ .55 .56 .ØØ1

PARAMETER-P	PARAMETER-Q	BETA VARIATE	FUNCTION RATIO
+Ø.245ØØØE+2	+Ø.85ØØØØE+1	+Ø.1ØØØØØE+1	+Ø.67199ØE-8
+Ø.245ØØØE+2	+Ø.85ØØØØE+1	+Ø.549ØØØE+Ø	+Ø.964369E-2
+Ø.245ØØØE+2	+Ø.85ØØØØE+1	+Ø.55ØØØØE+Ø	+Ø.99399ØE-2
+Ø.245ØØØE+2	+Ø.85ØØØØE+1	+Ø.551ØØØE+Ø	+Ø.102386E-1

1
32 17
100 .6 .65 .01

PARAMETER-P	PARAMETER-Q.	BETA VARIATE	FUNCTION RATIO
+0.245000E+2	+0.850000E+1	+0.100000E+1	+0.671990E-8
+0.245000E+2	+0.850000E+1	+0.600000E+0	+0.384847E-1
+0.245000E+2	+0.850000E+1	+0.610000E+0	+0.489306E-1
+0.245000E+2	+0.850000E+1	+0.620000E+0	+0.615943E-1

1
32 17
100 .61 .62 .001

PARAMETER-P	PARAMETER-Q	BETA VARIATE	FUNCTION RATIO
+0.245000E+2	+0.850000E+1	+0.100000E+1	+0.671990E-8
+0.245000E+2	+0.850000E+1	+0.609000E+0	+0.477898E-1
+0.245000E+2	+0.850000E+1	+0.610000E+0	+0.489306E-1
+0.245000E+2	+0.850000E+1	+0.611000E+0	+0.500871E-1

1
32 21
100 .2 1. .1

PARAMETER-P	PARAMETER-Q	BETA VARIATE	FUNCTION RATIO
+0.264999E+2	+0.650000E+1	+0.100000E+1	+0.861181E-7
+0.264999E+2	+0.650000E+1	+0.200000E+0	+0.411190E-13
+0.264999E+2	+0.650000E+1	+0.300000E+0	+0.938605E-9
+0.264999E+2	+0.650000E+1	+0.400000E+0	+0.863856E-6
+0.264999E+2	+0.650000E+1	+0.500000E+0	+0.126078E-3
+0.264999E+2	+0.650000E+1	+0.600000E+0	+0.519442E-2
+0.264999E+2	+0.650000E+1	+0.700000E+0	+0.776975E-1

1
32 21
100 .6 .7 .01

PARAMETER-P	PARAMETER-Q	BETA VARIATE	FUNCTION RATIO
+0.264999E+2	+0.650000E+1	+0.100000E+1	+0.861181E-7
+0.264999E+2	+0.650000E+1	+0.600000E+0	+0.519442E-2

+0.264999E+2	+0.650000E+1	+0.610000E+0	+0.710988E-2
+0.264999E+2	+0.650000E+1	+0.620000E+0	+0.963551E-2
+0.264999E+2	+0.650000E+1	+0.630000E+0	+0.129320E-1

1
32 21
100 .62 .63 .001

PARAMETER-P	PARAMETER-Q	BETA VARIATE	FUNCTION RATIO
+0.264999E+2	+0.650000E+1	+0.100000E+1	+0.861181E-7
+0.264999E+2	+0.650000E+1	+0.620000E+0	+0.963551E-2
+0.264999E+2	+0.650000E+1	+0.621000E+0	+0.992704E-2
+0.264999E+2	+0.650000E+1	+0.622000E+0	+0.102269E-1

1
32 21
100 .6 .7 .01

PARAMETER-P	PARAMETER-Q	BETA VARIATE	FUNCTION RATIO
+0.264999E+2	+0.650000E+1	+0.100000E+1	+0.861181E-7
+0.264999E+2	+0.650000E+1	+0.600000E+0	+0.519442E-2
+0.264999E+2	+0.650000E+1	+0.610000E+0	+0.710988E-2
+0.264999E+2	+0.650000E+1	+0.620000E+0	+0.963551E-2
+0.264999E+2	+0.650000E+1	+0.630000E+0	+0.129320E-1
+0.264999E+2	+0.650000E+1	+0.640000E+0	+0.171903E-1
+0.264999E+2	+0.650000E+1	+0.650000E+0	+0.226305E-1
+0.264999E+2	+0.650000E+1	+0.660000E+0	+0.295138E-1
+0.264999E+2	+0.650000E+1	+0.670000E+0	+0.381303E-1
+0.264999E+2	+0.650000E+1	+0.680000E+0	+0.488001E-1
+0.264999E+2	+0.650000E+1	+0.690000E+0	+0.618616E-1

1
32 21
100 .68 .69 .001

PARAMETER-P	PARAMETER-Q	BETA VARIATE	FUNCTION RATIO
+0.264999E+2	+0.650000E+1	+0.100000E+1	+0.861181E-7
+0.264999E+2	+0.650000E+1	+0.679000E+0	+0.476258E-1
+0.264999E+2	+0.650000E+1	+0.680000E+0	+0.488001E-1
+0.264999E+2	+0.650000E+1	+0.681000E+0	+0.499883E-1

1
69 3 .2 1. .1
100

PARAMETER-P	PARAMETER-Q	BETA VARIATE	FUNCTION RATIO
+0.360000E+2	+0.340000E+2	+0.100000E+1	+0.524354E-21
+0.360000E+2	+0.340000E+2	+0.200000E+0	+0.301700E-8
+0.360000E+2	+0.340000E+2	+0.300000E+0	+0.977778E-4
+0.360000E+2	+0.340000E+2	+0.400000E+0	+0.270762E-1
+0.360000E+2	+0.340000E+2	+0.500000E+0	+0.404971E+0

1
69 3 .4 .45 .01
100

PARAMETER-P	PARAMETER-Q	BETA VARIATE	FUNCTION RATIO
+0.360000E+2	+0.340000E+2	+0.100000E+1	+0.524354E-21
+0.360000E+2	+0.340000E+2	+0.400000E+0	+0.270762E-1
+0.360000E+2	+0.340000E+2	+0.410000E+0	+0.397093E-1
+0.360000E+2	+0.340000E+2	+0.420000E+0	+0.566540E-1

1
69 3 .41 .42 .001
100

PARAMETER-P	PARAMETER-Q	BETA VARIATE	FUNCTION RATIO
+0.360000E+2	+0.340000E+2	+0.100000E+1	+0.524354E-21
+0.360000E+2	+0.340000E+2	+0.409000E+0	+0.382657E-1
+0.360000E+2	+0.340000E+2	+0.410000E+0	+0.397093E-1
+0.360000E+2	+0.340000E+2	+0.411000E+0	+0.411946E-1
+0.360000E+2	+0.340000E+2	+0.412000E+0	+0.427263E-1
+0.360000E+2	+0.340000E+2	+0.413000E+0	+0.443029E-1
+0.360000E+2	+0.340000E+2	+0.414000E+0	+0.459254E-1
+0.360000E+2	+0.340000E+2	+0.415000E+0	+0.475908E-1
+0.360000E+2	+0.340000E+2	+0.416000E+0	+0.493071E-1
+0.360000E+2	+0.340000E+2	+0.417000E+0	+0.510694E-1

1
69 3 .35 .4 .01
100

PARAMETER-P	PARAMETER-Q	BETA VARIATE	FUNCTION RATIO
+0.360000E+2	+0.340000E+2	+0.100000E+1	+0.524354E-21
+0.360000E+2	+0.340000E+2	+0.350000E+0	+0.253590E-2
+0.360000E+2	+0.340000E+2	+0.360000E+0	+0.434436E-2
+0.360000E+2	+0.340000E+2	+0.370000E+0	+0.719524E-2
+0.360000E+2	+0.340000E+2	+0.380000E+0	+0.115389E-1

1
69 3
100 .37 .38 .001

PARAMETER-P	PARAMETER-Q	BETA VARIATE	FUNCTION RATIO
+0.360000E+2	+0.340000E+2	+0.100000E+1	+0.524354E-21
+0.360000E+2	+0.340000E+2	+0.370000E+0	+0.719524E-2
+0.360000E+2	+0.340000E+2	+0.371000E+0	+0.755419E-2
+0.360000E+2	+0.340000E+2	+0.372000E+0	+0.792824E-2
+0.360000E+2	+0.340000E+2	+0.373000E+0	+0.831835E-2
+0.360000E+2	+0.340000E+2	+0.374000E+0	+0.872487E-2
+0.360000E+2	+0.340000E+2	+0.375000E+0	+0.914796E-2
+0.360000E+2	+0.340000E+2	+0.376000E+0	+0.958898E-2
+0.360000E+2	+0.340000E+2	+0.377000E+0	+0.100479E-1

1
17 3
100 .2 1. .1

PARAMETER-P	PARAMETER-Q	BETA VARIATE	FUNCTION RATIO
+0.100000E+2	+0.800000E+1	+0.100000E+1	+0.514191E-5
+0.100000E+2	+0.800000E+1	+0.200000E+0	+0.493292E-3
+0.100000E+2	+0.800000E+1	+0.300000E+0	+0.126925E-1
+0.100000E+2	+0.800000E+1	+0.400000E+0	+0.918987E-1

1
17 3
100 .3 .4 .01

PARAMETER-P	PARAMETER-Q	BETA VARIATE	FUNCTION RATIO
+0.100000E+2	+0.800000E+1	+0.100000E+1	+0.514191E-5
+0.100000E+2	+0.800000E+1	+0.300000E+0	+0.126925E-1

+0.100000E+2	+0.800000E+1	+0.310000E+0	+0.161755E-1
+0.100000E+2	+0.800000E+1	+0.320000E+0	+0.203818E-1
+0.100000E+2	+0.800000E+1	+0.330000E+0	+0.254092E-1
+0.100000E+2	+0.800000E+1	+0.340000E+0	+0.313561E-1
+0.100000E+2	+0.800000E+1	+0.350000E+0	+0.383258E-1
+0.100000E+2	+0.800000E+1	+0.360000E+0	+0.464183E-1
+0.100000E+2	+0.800000E+1	+0.370000E+0	+0.557312E-1

1
17 3
100

PARAMETER-P	PARAMETER-Q	BETA VARIATE	FUNCTION RATIO
	.36	.37	.001
+0.100000E+2	+0.800000E+1	+0.100000E+1	+0.514191E-5
+0.100000E+2	+0.800000E+1	+0.360000E+0	+0.464183E-1
+0.100000E+2	+0.800000E+1	+0.361000E+0	+0.472931E-1
+0.100000E+2	+0.800000E+1	+0.362000E+0	+0.481804E-1
+0.100000E+2	+0.800000E+1	+0.363000E+0	+0.490796E-1
+0.100000E+2	+0.800000E+1	+0.364000E+0	+0.499916E-1
+0.100000E+2	+0.800000E+1	+0.365000E+0	+0.509161E-1

1
17 3
100

PARAMETER-P	PARAMETER-Q	BETA VARIATE	FUNCTION RATIO
	.25	.3	.01
+0.100000E+2	+0.800000E+1	+0.250000E+0	+0.312080E-2
+0.100000E+2	+0.800000E+1	+0.260000E+0	+0.423263E-2
+0.100000E+2	+0.800000E+1	+0.270000E+0	+0.568775E-2
+0.100000E+2	+0.800000E+1	+0.280000E+0	+0.753259E-2
+0.100000E+2	+0.800000E+1	+0.290000E+0	+0.984085E-2
+0.100000E+2	+0.800000E+1	+0.300000E+0	+0.126925E-1

17 3
100

PARAMETER-P	PARAMETER-Q	BETA VARIATE	FUNCTION RATIO
	.29	.295	.001
+0.100000E+2	+0.800000E+1	+0.100000E+1	+0.514191E-5
+0.100000E+2	+0.800000E+1	+0.290000E+0	+0.984085E-2
+0.100000E+2	+0.800000E+1	+0.291000E+0	+0.101000E-1
+0.100000E+2	+0.800000E+1	+0.292000E+0	+0.103648E-1
+0.100000E+2	+0.800000E+1	+0.293000E+0	+0.106353E-1

PROGRAM B

```
C;PROGRAM FOR CALCULATING QU(S) GIVEN BETA VARIATE X
ACCEPT 3,N,INTV,NTV,KNTV,IX,JX,KX
3;FORMAT(I)
DO 1 I=INTV,NTV,KNTV
AN=N
AI=I
ALPHA=(4.*AI**2-1.)/(AN**2)
BETA=4.-(4.*AI**2+2.)/(AN**2)
DO 1 J=IX,JX,KX
AJ=J
AJ=AJ/100.
QU=ALPHA+BETA*AJ
TYPE 100,N,I,ALPHA,BETA,AJ,QU
100;FORMAT(/,I,I,E,E,E,E,/)
1;CONTINUE
END
```

```
32     17     17      1                     55      55      1
+32    +17    +0.112792E+1    +0.286914E+1    +0.550000E+0    +0.270595E+1
!
32     17     17      1                     61      61      1
+32    +17    +0.112792E+1    +0.286914E+1    +0.610000E+0    +0.287310E+1
!

32     21     21      1                     62      62      1
+32    +21    +0.172168E+1    +0.227538E+1    +0.620000E+0    +0.313242E+1
!
32     21     21      1                     68      68      1
+32    +21    +0.172168E+1    +0.227538E+1    +0.680000E+0    +0.326894E+1
!

69     3      3       1                     41      41      1
+69    +3     +0.735142E-2    +0.399201E+1    +0.410000E+0    +0.164407E+1
!
69     3      3       1                     42      42      1
+69    +3     +0.735142E-2    +0.399201E+1    +0.420000E+0    +0.168399E+1
!
69     3      3       1                     37      37      1
+69    +3     +0.735142E-2    +0.399201E+1    +0.370000E+0    +0.148439E+1
!
69     3      3       1                     38      38      1
+69    +3     +0.735142E-2    +0.399201E+1    +0.380000E+0    +0.152431E+1
!

17     3      3       1                     36      36      1
+17    +3     +0.121107E+0    +0.386851E+1    +0.360000E+0    +0.151377E+1
!
17     3      3       1                     37      37      1
+17    +3     +0.121107E+0    +0.386851E+1    +0.370000E+0    +0.155245E+1
!
17     3      3       1                     29      29      1
+17    +3     +0.121107E+0    +0.386851E+1    +0.290000E+0    +0.124297E+1
!
17     3      3       1                     30      30      1
+17    +3     +0.121107E+0    +0.386851E+1    +0.300000E+0    +0.128166E+1
!
```

APPENDIX VIII

Program for regression of Y on X

The program has been written in accordance with the discussion of the regression model given in sections 3.2 and 3.3.

The program is in FORTRAN II and computations were carried out on the Digital Equipment Corporation computer PDP-8 with typewriter input. Dimension statements have been ovoided so that each step could be compared with details given in section 3.3.

Symbols:

N = Number of observations

SX = ΣX where summation is carried out over N

SY = ΣY

SSX = ΣX^2

SXY = ΣXY

SSY = ΣY^2

SSDX = Sum squares deviation of X

SSDY = Sum squares deviation of Y

SDX = standard deviation of X

SDY = standard deviation of Y

R = correlation coefficient

SSE = sum squares explained from regression

CD = coefficient of determination

B = regression coefficient

SEB = standard error of B

T = t-value for the coefficient of regression (TEE)

CVR = coefficient of variation

Q = von Neumann ratio of the least squares estimated disturbances

```
C; PROGRAM FOR REGRESSION OF Y ON X
C; GIVEN N SETS OF OBSERVATIONS
11;FORMAT(I)
12;FORMAT(E)
13;FORMAT(/)
14;FORMAT(/,"END OF INPUT DATA",/)
15;FORMAT(/,/,/,"OUTPUT BEGINS",/,/,/)
16;FORMAT(/,"STANDARD DEVIATION OF X=",E,/,/,"STANDARD DEVIATION'
OF Y=",E,/,/,"CORRELATION COEFFICIENT=",E,/,/,/)
17;FORMAT(/,"REGRESSION COEFFICIENT=",E,/,/)
37;FORMAT(/,"CONSTANT=",E,/,"STANDARD ERROR OF COEFFICIENT=",E,/)
47;FORMAT(/,"COEFFICIENT OF VARIATION=",E,/)
18;FORMAT(/,"SUM SQUARES DEVIATIONS EXPLAINED=",E,/,/,"SUM SQUARES'
DEVIATIONS OF Y=",E,/)
33;FORMAT(/,"TEE VALUE=",E,/,"COEFFICIENT OF DETERMINATION=",E,/)
SX=0.
SY=0.
SSX=0.
SXY=0.
SSY=0.
ACCEPT 11,N
TYPE 13
DO 10 I=1,N
ACCEPT 12,X,Y
SX=SX+X
SY=SY+Y
SSX=SSX+X**2
SXY=SXY+X*Y
SSY=SSY+Y**2
10;CONTINUE
TYPE 14
XN=N
SSDX=SSX-SX**2/XN
SSDY=SSY-SY**2/XN
SDX=SQTF(SSDX/(XN-1.))
SDY=SQTF(SSDY/(XN-1.))
R=(SXY-SX*SY/XN)/(SDX*SDY*(XN-1.))
SSE=R**2*SSDY
ASDR=(SSDY-SSE)/(XN-2.)
```

```
CD=1.-ASDR*(XN-1.)/SSDY
B=R*SDY/SDX
A=(SY-B*SX)/XN
SEB=SQTF(ASDR/SSDX)
T=B/SEB
CVR=SQTF(ASDR)*XN/SY
TYPE 15
TYPE 16,SDX,SDY,R
TYPE 17,B
TYPE 37,A,SEB
TYPE 47,CVR
TYPE 18,SSE,SSDY
TYPE 38,T,CD
TYPE 30
30;FORMAT(/,"READ INPUT DATA AGAIN",/)
V2=0.
V4=0.
ACCEPT 12,X,Y
K=1
YC1=B*X+A
V1=Y-YC1
TYPE 32,Y,YC1,V1
20;V2=V2+V1*V1
IF(N-K)22,22,21
21;ACCEPT 12,X,Y
K=K+1
YC2=B*X+A
V3=Y-YC2
V4=V4+(V3-V1)*(V3-V1)
TYPE 32,Y,YC2,V3
V1=V3
GO TO 20
22;CONTINUE
G=V4/V2
TYPE 33,G
32;FORMAT(/,"Y-GIVEN=",E,"Y-CALC=",E,"ERROR=",E,/,/)
33;FORMAT(/,"VON NEUMANNS RATIO OF THE LEAST SQUARES",/,"ESTIMATED'
DISTURBANCES=",E,/,/)
END
```

APPENDIX IX

MULTIPLE REGRESSION ANALYSIS : DETAILS OF COMPUTATION PROCEDURE

(see chapter 3 for discussion)

Example

Given:

Observation No.	X_1	X_2	X_3	Y
1	32.	48.	54.	15.
2	36.	33.	19.	16.
3	3.	28.	30.	14.
4	12.	33.	64.	22.
5	36.	34.	60.	24.
6	24.	36.	53.	19.
7	19.	42.	29.	13.
8	20.	33.	55.	15.
9	27.	36.	62.	23.
10	15.	22.	33.	12.
11	45.	46.	68.	25.
12	9.	28.	42.	17.
13	11.	32.	45.	18.
14	33.	34.	39.	19.
15	21.	45.	39.	18.

n = No. of observations

= 15

m = No. of independent variables

= 3

Preliminary calculations and development of (normal equations) basic matrix:

(i) Sum squares and products (summation over the total observations)

$\Sigma X_1X_1 = 9817.000000$	$\Sigma X_1X_2 = 12793.000000$	$\Sigma X_1X_3 = 16561.000000$	$\Sigma X_1Y = 6511.000000$
$\Sigma X_2X_1 = 12793.000000$	$\Sigma X_2X_2 = 19456.000000$	$\Sigma X_2X_3 = 24935.000000$	$\Sigma X_2Y = 9657.000000$
$\Sigma X_3X_1 = 16561.000000$	$\Sigma X_3X_2 = 24935.000000$	$\Sigma X_3X_3 = 34936.000000$	$\Sigma X_3Y = 13080.000000$
$\Sigma YX_1 = 6511.000000$	$\Sigma YX_2 = 9657.000000$	$\Sigma YX_3 = 13080.000000$	$\Sigma Y^2 = 5088.000000$

(ii) Standard deviations

$\sigma_1 = 11.873540$ $\quad\sigma_2 = 7.217702$ $\quad\sigma_3 = 14.667099$ $\quad\sigma_y = 4.035555$

(iii) Average values

$\overline{X}_1 = 22.86667$ $\quad\overline{X}_2 = 35.333333$ $\quad\overline{X}_3 = 46.133333$ $\quad\overline{Y} = 18.000000$

(iv) Residual sum of squares

$S_1^2 = \Sigma(X_1 - \overline{X}_1)^2 = 1973.733333$ $\quad S_2^2 = 729.333333$ $\quad S_3^2 = 3011.733333$ $\quad S_y^2 = 228.000000$

(v) Correlation coefficients

r_{11} = 1.000000	r_{12} = .561485	r_{13} = .302393	r_{1Y} = .502364
	r_{22} = 1.000000	r_{23} = .326793	r_{2Y} = .286917
		r_{33} = 1.000000	r_{3Y} = .753024

(vi) Normalized form of basic matrix

1.000000	.561485	.302393	.502364
.561485	1.000000	.326793	.286917
.302393	.326793	1.000000	.753024

STEPWISE REGRESSION

STEP - I

(i) Calculate the ratio

$R = \frac{r_{iy}}{r_{ii}}$ for i = 1, 2, 3 to decide which of the variables is to be included first

$R_m = \frac{r^2_{3y}}{r_{33}} = .567046 > \frac{r^2_{2y}}{r_{22}} > \frac{r^2_{1y}}{r_{11}}$

∴ Variable X_3 to be included first, put m = 3 and identify it to be the first in the matrix. (i.e. k = 1) Interchange row 3 vs row 1 and column 3 vs column 1

New matrix is

1.000000	.326793	.302393	.753024
.326793	1.000000	.561485	.286917
.302393	.561485	1.000000	.502364

STEP - II

$R_m = \frac{r_{1y}^2}{r_{kk}} = .083027 > \frac{r_{2y}^2}{r_{kk}}$

∴ Variable to be included at this stage is X_1

m = 1

k = 2

Interchange row 2 vs row 3 and column 2 vs column 3

New matrix is

1.000000	.302393	.326793	.753024
-.302393	.908558	.462665	.274654
-.326793	.462665	.893206	.040833

STEP - III

$R_m = .014913$

m = 2

k = 3 = total number of parameters

New matrix is same as the recalculated matrix in the last stage.

(iv) Recalculate matrix according to algorithm

$$r_{ij}\ (\text{New}) = r_{ij} - \frac{r_{ik} r_{kj}}{r_{kk}} \quad \text{for } \begin{matrix} i \neq k \\ j \neq k \end{matrix}$$

$$r_{kj}\ (\text{New}) = \frac{r_{kj}}{r_{kk}} \quad \text{for } \begin{matrix} i = k \\ j \neq k \end{matrix}$$

$$r_{ik}\ (\text{New}) = - {r_{ik}}/{r_{kk}} \quad \begin{matrix} i \neq k \\ j = k \end{matrix}$$

$$r_{kk}\ (\text{New}) = \frac{1}{r_{kk}} \quad i = k = j$$

Recalculated matrix

1.000000	.326793	.302393	<u>.753024</u>
-.326793	.893206	.462665	.040833
-.302393	.462665	.908558	.274654

Recalculated matrix - (STEP II)

1.100645	-.332828	.172806	<u>.661612</u>
-.332828	1.100645	.509229	<u>.302297</u>
-.172806	-.509229	.657604	-.099029

Recalculated matrix - (STEP III)

1.146055	-.199012	-.262781	<u>.687635</u>
-.199012	1.494977	-.774371	<u>.378982</u>
-.262781	-.774371	1.520672	-<u>.150590</u>

(v) $S^2_{y/x3}$ = Sum squares deviation explained by regression of y on x3 = $R_m S^2_y$

= 129.286438

Mean squares deviation from regression

$\sigma^2_{y/x3} = (S^2_y - S^2_{y/x3}) / (n - 2)$

= 7.59335

Coefficient of determination

$CD = 1. - \sigma^2_{y/x3} / \sigma^2_y = .533742$

value of coefficient

$b_3 = r_{ky} \frac{\sigma_y}{\sigma_m} = r_{1y} \frac{\sigma_y}{\sigma_3}$

= .207189

Standard error of the coefficient

$S_{b3} \sqrt{\frac{\sigma^2_{y/x}}{S^2_3}} \cdot r_{11} = .050212$

$t_{b3} = b_3 / S_{b3} = 4.126$

Constant

$b_o = \overline{Y} - b_3 \overline{X}_3 = 8.441649$

Coefficient of variation

$= \frac{\sigma_{y/x}}{\overline{Y}} = .15309$

$S^2_{y/x1} = 18.930176$

$S^2_{y/x3,x1} = 148.216614$

$\sigma^2_{y/x3,x1} = 6.64861$

$CD = 1. - \frac{\sigma^2_{y/x3,x2}}{\sigma^2_y} = .591752$

$b_3 = .182038$

$b_4 = .102744$

$S_{b3} = .049293$

$S_{b1} = .060890$

$t_{b3} = 3.693$

$t_{b1} = 1.687$

$b_o = \overline{Y} - (b_3 \overline{X}_3 + b_2 \overline{X}_2)$

= 7.252559

Coefficient of variation

= .14325

$S^2_{y/x2} = 3.400115$

$S^2_{y/x3,x1,x2} = 151.616714$

$\sigma^2_{y/x3,x2,x1} = 6.94393$

$CD = 1. - \frac{\sigma^2_{y/x3,x2,x1}}{\sigma^2_y} = .57362$

$b_3 = .189198$

$b_1 = .128807$

$b_2 = -.084197$

$S_{b3} = .051404$

$S_{b1} = .072523$

$S_{b2} = .120325$

$t_{b3} = 3.681$

$t_{b1} = 1.776$

$t_{b2} = -.700$

$b_o = 9.301251$

Coefficient of variations

= .14640

TERMINAL CALCULATIONS

von Neumann Ratio of the Least Square Estimated

Disturbances

$$Q = \frac{\sum_{j=2}^{n} [(\hat{Y}_j - Y_j) - (\hat{Y}_{j-1} - Y_{j-1})]^2}{\sum_{j=2}^{N} (\hat{Y}_j - Y_j)^2}$$

$$= 1.82588$$

Average Absolute error

$$\sum_{i=1}^{N} \left| \frac{\hat{Y}_i - Y_i}{Y_i} \right| = 0.56757 \quad \text{where } \hat{Y}_i = \text{estimated value of } Y_i$$

Sum squares deviation explained from regression

$$S^2_{y/x3,x1,x2} = 151.61671$$

Sum squares deviation of dependent viariable

$$S^2_y = 228.00000$$

APPENDIX X

Digital Computer Program for Multiple Regression Analysis

This program was written in Fortran II for use on IBM System 1620 digital computer. The program was translated into Fortran IV when the Computing Centre (Univ. of Ottawa) acquired IBM System 360/Model 40. All single precision statements were changed to double precision for better accuracy. Some new statements e.g. S.0001, S.0003, S.0026 and S.0028 were added and corresponding changes were made to facilitate handling of multiple sets of input data.

Comments are included in the program to make it self-explanatory. For discussion and algorithm see section 3.6.3 and Appendix IX.

```
LEVEL0 1JUL66                              IBM OS/360 BASIC FORTRAN IV (E) COMPILATION
                C
                C       MULTIVARIATE REGRESSION USING STEPWISE PROCEDURE
                C       STATISTICAL EVALUATION OF REGRESSION SOLUTION
                C
S.0001                  DIMENSION XXNA(22)
S.0002                  DIMENSION NREG(22),IREG(22),XNAME(22)
S.0003                  DOUBLE PRECISION XXX(22,33)
S.0004                  DOUBLE PRECISION SSDIV(22),SSDE(22),X(22,33),S(22,22),SUM(22),
                       1SD(22),B(22),SAVE,P,Y,ABSE
S.0005            100   FORMAT(A4,6X,7F10.6/(10X,7F10.6))
S.0006            200   FORMAT('0',A4,5X,7F12.6,/,(10X,7F12.6))
S.0007            500   FORMAT(2I5,3F5.2)
S.0008            600   FORMAT(7I3,16F7.3,(/3X,16F7.3))
S.0009            601   FORMAT(1H0,24HCORRELATION COEFFICIENTS,/)
S.0010            700   FORMAT(1H ,4X,I2,F15.5,E20.8,F12.3,F17.3,4X,A8)
S.0011            701   FORMAT(1H1,2X,20HDEPENDENT VARIABLE  ,A8//9H  PAR NO.,4X,9HPAR VAL
                       1UE,9X,9HPAR VALUE,7X,9HTEE VALUE,5X,12HCONTRIBUTION,4X,23HVARIABLE
                       2 IDENTIFICATION//)
S.0012            702   FORMAT(1H ,4X,I2,F15.5,E20.8,F12.3///,1H ,8X,5HCVR =,F10.5,5X,5HSD
                       1R =,F10.5,5X,6HMSDR =,F10.5,5X,4HCD =,F8.5)
S.0013            703   FORMAT(48H OOPS REACH HAS EXCEEDED GRASP   CONGRATULATIONS//5HMSDR
                       1=,F15.5,5X,2HI=,I2,2X,7HS I I =,F15.5,5X,6HSSDER=,F15.5,5X,6HSSDDV
                       2=,F20.5/9HSSDIV K =,F20.5,10X,8HOFFENDER,2X,A8)
S.0014            800   FORMAT('1',9X,'VON NEUMANN RATIO OF THE LEAST SQUARES',/,
                       110X,'ESTIMATED DISTURBANCES =',F10.5,//)
S.0015            801   FORMAT(10X,'AVERAGE ABSOLUTE ERROR=',F10.5,
                       1/,/,10X,'SUM SQUARES DEVIATION EXPLAINED FROM REGRESSION=',F20.5,/
                       2,/,10X,'SUM SQUARES DEVIATION OF DEPENDENT VARIABLE     =',F20.5)
S.0016            900   FORMAT(1H1,4X,15HOBSERVATION NO.,16X,7HY-GIVEN,13X,6HY-CALC,14X,5H
                       1ERROR,10X, 13HERROR/Y-GIVEN//(9X,I3,12X,3F20.6,F17.6))
S.0017            967   FORMAT(1H1)
S.0018            968   FORMAT(10X,F10.6)
                C
                C       N **** NUMBER OF OBSERVATIONS
                C       NIV **** NUMBER OF INDEPENDENT VARIABLES
                C       REJECT **** F TEST LEVEL BELOW WHICH THE VARIABLE IS REJECTED
                C       ADMIT **** F TEST LEVEL ABOVE WHICH THE VARIABLE IS ADMITTED
                C       TOL **** TOLERANCE VALUE FOR SIGNIFICANCE
                C
S.0019                  READ(1,500   )N,NIV,REJECT,ADMIT,TOL
                C
                C       NTV **** NUMBER OF TOTAL VARIABLES
                C       XXNA(K) **** TEMPORARY STORAGE OF VARIABLE NAMES
                C       XXX(I,J) **** TEMPORARY STORAGE OF VARIABLE VALUES
                C
S.0020                  NTV=NIV&1
S.0021                  DO 971 K=1,NIV
S.0022            971   READ(1,100)XXNA(K),(XXX(K,JI),JI=1,N)
S.0023            969   WRITE(3,967)
S.0024                  READ(1,100)XXNA(NTV),(XXX(NTV,JI),JI=1,N)
S.0025                  DO 69 K=1,NTV
S.0026                  XNAME(K)=XXNA(K)
S.0027                  DO 772 JI=1,N
```

```
S.0028        772 X(K,JI)=XXX(K,JI)
            C
            C      X(I,J) **** J TH OBSERVED VALUE OF I TH VARIABLE
            C
S.0029         69 WRITE(3,200) XNAME(K),(X(K,JI),JI=1,N)
S.0030            DO 1 I=1,NTV
S.0031            SUM(I)=0.0
S.0032            DO 1 J=1,NTV
S.C033          1 S(I,J)=0.0
            C
            C      SUM(I) **** SUM OF OBSERVED VALUES OF I TH VARIABLE
            C      S(I,J) **** SUM OF CROSS PRODUCTS OF I TH AND J TH VARIABLE
            C
S.0034            DO 5 I=1,NTV
S.0035            DO 5 K=1,N
S.0036            SUM(I)=SUM(I)&X(I,K)
S.0037            DO 5 J=1,NTV
S.0038          5 S(I,J)=S(I,J)&X(I,K)*X(J,K)
S.0039            FN=N
            C
            C      SSDDV **** SUM SQUARES DEVIATION OF DEPENDENT VARIABLE
            C
S.0040            SSDDV=S(NTV,NTV)-SUM(NTV)**2/FN
            C
            C      NDFT **** NUMBER OF DEGREES OF FREEDOM TOTAL
            C      ADFT **** AVAILABLE DEGREES OF FREEDOM TOTAL = NDFT
            C      SD(I) **** SAMPLE STANDARD DEVIATION OF I TH VARIABLE
            C
S.0041            NDFT=N-1
S.C042            ADFT=NDFT
S.0043            DO 7 I=1,NTV
S.0044          7 SD(I)=SQRT((S(I,I)-(SUM(I)**2)/FN)/(ADFT))
            C
            C      NREG(I) **** IDENTIFICATION FOR THE I TH VARIABLE IN REGRESSION
            C      SUM(I) **** AVERAGE VALUE OF I TH VARIABLE
            C      SSDIV(I) **** SUM SQUARES DEVIATION OF I TH VARIABLE
            C      S(I,J) **** CORRELATION COEFFICIENT BETWEEN I TH AND J TH VARIABLE
            C
S.CC45            DO 8 I=1,NIV
S.0046            NREG(I)=I
S.0047            SUM(I)=SUM(I)/FN
S.C048            SSDIV(I)=S(I,I)-(SUM(I)**2)*FN
S.0049            S(I,I)=1.0
S.0050            K=I&1
S.0051            DO 8 J=K,NTV
S.0052          8 S(I,J)=(S(I,J)-SUM(I)*SUM(J))/((ADFT)*SD(I)*SD(J))
S.0053            NIVM=NIV-1
S.0054            DO 10 I=1,NIVM
S.0055            K=I&1
S.0056            DO 10 J=K,NIV
S.0057         10 S(J,I)=S(I,J)
S.0058            WRITE(3,601   )
S.0059            DO 27 I=1,NIV
S.0060         27 WRITE(3,600   )I,(S(I,J),J=1,I)
S.0061            I=NTV
```

```
S.0062          WRITE(3,600   )I,(S(J,I),J=1,NIV)
S.0063          TOT=0.0
        C
        C       SSDER **** SUM SQUARES DEVIATION EXPLAINED FROM REGRESSION
        C
S.0064          SSDER=0.0
        C
        C       NIVINC **** NUMBER OF INDEPENDENT VARIABLES INCLUDED IN REGRESSION
        C       SOLUTION
        C
        C       NTVINC **** NUMBER OF TOTAL VARIABLES IN REGRESSION SOLUTION
        C
S.CC65          NIVINC=0
S.C066       11 NTVINC=NIVINC&1
S.C067          RMAX=1.0E30
        C
        C       ADF **** AVAILABLE DEGREES OF FREEDOM
        C       R **** SUM SQUARES DEVIATION EXPLAINED BY THE NEW VARIABLE
        C       INCLUDED DIVIDED BY TOTAL SUM SQUARES DEVIATION OF DEPENDENT
        C       VARIABLE
        C
S.CC68          ADF=NDFT-NIVINC
S.0069          DO 43 I=1,NIVINC
S.0070          R=S(I,NTV)**2/S(I,I)
S.0071          IF(R-RMAX)44,43,43
S.0072       44 RMAX=R
S.0073          NEXT=I
S.C074       43 CONTINUE
        C
        C       F TEST FOR REJECTING AVARIABLE
        C
S.0075          IF(RMAX*ADF/(1.-TOT)-REJECT)45,45,41
        C
        C       IREJ **** CONTROL FOR REJECTION OF THE VARIABLE
        C
S.0076       45 IREJ=1
S.C077          SSDER=SSDER-RMAX*SSDDV
S.0078          TOT=TOT-RMAX
        C
        C       SSDUER **** SUM SQUARES DEVIATION UNEXPLAINED FROM REGRESSION
        C
S.0079          SSDUER=SSDDV-SSDER
S.CC8C          K=NREG(NEXT)
S.0081          NREG(NEXT)=NREG(NIVINC)
S.0082          NREG(NIVINC)=K
S.C083          IREG(NIVINC)=K
S.0084          GO TO 83
S.0085       41 RMAX=0.0
S.0086          DO 46 I=NTVINC,NIV
S.0087          IF(S(I,I)-TOL)46,30,30
S.0088       30 R=S(I,NTV)**2/S(I,I)
S.0089          IF(R-RMAX)46,46,47
S.0090       47 RMAX=R
S.C091          NEXT=I
S.0092       46 CONTINUE
```

```
C
C     IREG(I) **** IDENTIFICATION FOR THE I TH STEP IN REGRESSION
C
      K=NREG(NEXT)
      NREG(NEXT)=NREG(NTVINC)
      NREG(NTVINC)=K
      IREG(NTVINC)=K
C
C     F TEST FOR ADMITTING A VARIABLE
C
      IF(RMAX*(ACF-1.)/(1.-TOT-RMAX)-ADMIT)73,73,56
   56 NIVINC=NIVINC&1
      IREJ=0
C
C     SSDE(I) **** SUM SQUARES DEVIATION EXPLAINED BY INCLUSION OF I TH
C     VARIABLE
C
      SSDE(NIVINC)=RMAX*SSDDV
      TOT=TOT&RMAX
      SSDER=SSDER&SSDE(NIVINC)
      SSDUER=SSDDV-SSDER
   83 CONTINUE
C
C     INTERCHANGING OF ROWS
C
      DO 14 J=1,NTV
      SAVE=S(NEXT,J)
      S(NEXT,J)=S(NIVINC,J)
   14 S(NIVINC,J)=SAVE
C
C     INTERCHANGING OF COLUMNS
C
      DO 15 J=1,NIV
      SAVE=S(J,NEXT)
      S(J,NEXT)=S(J,NIVINC)
   15 S(J,NIVINC)=SAVE
C
C     P **** PIVOTAL TERM CORRESPONDING TO THE VARIABLE ADMITTED
C
      P=S(NIVINC,NIVINC)
      S(NIVINC,NIVINC)=1.0
      DO 16 J=1,NTV
C
C     DIVISION BY PIVOTAL TERM
C
   16 S(NIVINC,J)=S(NIVINC,J)/P
      DO 17 J=1,NIV
      IF(NIVINC-J)59,17,59
   59 P=S(J,NIVINC)
      S(J,NIVINC)=0.0
      DO 18 K=1,NTV
   18 S(J,K)=S(J,K)-P*S(NIVINC,K)
   17 CONTINUE
      NIVINC=NIVINC-IREJ
C
```

```
                C
                C       BO **** AVERAGE VALUE OF Y
                C
S.0125                  BO=SUM(NTV)/FN
S.0126                  ADF=NDFT-NIVINC
                C
                C       ASDR **** AVERAGE SQUARES DEVIATION FROM REGRESSION
                C
S.0127                  ASDR=SSDUER/ADF
                C
                C       CD **** COEFFICIENT OF DETERMINATION
                C
S.0128                  CD=1.-ASDR/SSDDV*ADFT
S.0129                  WRITE(3,701   )XNAME(NTV)
S.0130                  Y=SD(NTV)
S.0131                  DO 21 I=1,NIVINC
S.0132                  K=IREG(I)
S.0133                  P=Y*S(I,NTV)/SD(K)
                C
                C       B(I) **** REGRESSION COEFFICIENT FOR THE I TH VARIABLE IN
                C       REGRESSION
                C
S.0134                  B(I)=P
S.0135                  SEB=ASDR*S(I,I)/SSDIV(K)
S.0136                  IF(SEB)37,37,36
S.0137               37 WRITE(3,703   )ASDR,I,S(I,I),SSDIV(K),SSDER,SSDDV,XNAME(K)
S.0138                  GO TO 969
                C
                C       SEB **** STANDARD ERROR OF COEFFICIENT
                C
S.0139               36 SEB=SQRT(SEB)
                C
                C       T **** TEE VALUE
                C
S.0140                  T=P/SEB
S.0141                  WRITE(3,700   )I,P,P,T,SSDE(I),XNAME(K)
                C
                C       BO **** CONSTANT TERM
                C
S.0142               21 BO=BO-P*SUM(K)
S.0143                  SEB=0.0
S.0144                  DO 25 I=1,NIVINC
S.0145                  P=0.0
S.0146                  I1=IREG(I)
S.0147                  DO 26 J=1,NIVINC
S.0148                  J1=IREG(J)
S.0149               26 P=P&S(I,J)*SUM(J1)
S.0150               25 SEB=SEB&P*SUM(I1)
S.0151                  SEB=SEB&SSDUER/(FN**2)
S.0152                  IF(SEB)37,37,38
S.0153               38 I=0
S.0154                  T=BO/SQRT(ASDR*SEB)
S.0155                  SQ=SQRT(ASDR)
                C
                C       CV **** COEFFICIENT OF VARIATION
                C
```

```
      CV=SQ/SUM(NTV)*FN
      WRITE(3,702   )I,BO,BO,T,CV,SQ,ASDR,CD
      GO TO 11
   73 Q=0.0
      ABSE=0.0
      DO 22 J=1,N
C
C     YC **** CALCULATED VALUE OF Y
C
      YC=BO
      DO 23 I=1,NIVINC
      K=IREG(I)
      NTV=NIV&1
   23 YC=YC&B(I)*X(K,J)
C
C     X(1,J) **** ERROR BETWEEN CALCULATED AND GIVEN VALUE OF Y
C     X(3,J) **** ERROR DIVIDED BY GIVEN VALUE OF Y
C     ABSE **** AVERAGE ABSOLUTE ERROR
C
      X(1,J)=X(NTV,J)-YC
      X(3,J)=X(1,J)/X(NTV,J)
      X(2,J)=YC
   22 ABSE=ABSE&DABS(X(3,J))
C
C     Q **** VON NEUMANN RATIO OF THE LEAST SQUARES ESTIMATED
C     DISTURBANCES
C
      DO 24 I=2,N
   24 Q=Q&(X(1,I)-X(1,I-1))**2
      Q=Q/SSDUER
      FN=NIVINC
      ABSE=ABSE/FN
      WRITE(3,800)Q
      WRITE(3,801)ABSE,SSDER,SSDDV
      WRITE(3,900   )(J,X(NTV,J),X(2,J),X(1,J),X(3,J),J=1,N)
      WRITE(2,968)(B(I),I=1,NIV)
      GO TO 969
      END

         SIZE OF COMMON  000000   PROGRAM  023296
                         END OF COMPILATION  MAIN
```

X1	32.000000	36.000000	3.000000	12.000000	36.000000	24.000000	19.000000
	20.000000	27.000000	15.000000	45.000000	9.000000	11.000000	33.000000
	21.000000						
X2	48.000000	33.000000	28.000000	33.000000	34.000000	36.000000	42.000000
	33.000000	36.000000	22.000000	46.000000	28.000000	32.000000	34.000000
	45.000000						
X3	54.000000	19.000000	30.000000	64.000000	60.000000	53.000000	29.000000
	55.000000	62.000000	33.000000	68.000000	42.000000	45.000000	39.000000
	39.000000						
Y	15.000000	16.000000	14.000000	22.000000	24.000000	19.000000	13.000000
	15.000000	23.000000	12.000000	25.000000	17.000000	18.000000	19.000000
	18.000000						

CORRELATION COEFFICIENTS

1	1.000			
2	0.561	1.000		
3	0.302	0.327	1.000	
4	0.502	0.287	0.753	

DEPENDENT VARIABLE Y

PAR NO.	PAR VALUE	PAR VALUE	TEE VALUE	CONTRIBUTION	VARIABLE IDENTIFICATION
1	0.20719	0.20718967D 00	4.126	129.286	X3
0	8.44165	0.84416494E 01	0.066		

CVR = 0.15309 SDR = 2.75560 MSDR = 7.59335 CD = 0.53374

DEPENDENT VARIABLE Y

PAR NO.	PAR VALUE	PAR VALUE	TEE VALUE	CONTRIBUTION	VARIABLE IDENTIFICATION
1	0.18204	0.182038120 00	3.693	129.286	X3
2	0.10274	0.102744050 00	1.687	18.930	X1
0	7.25256	0.72525597E 01	0.060		

CVR = 0.14325 SDR = 2.57849 MSDR = 6.64861 CD = 0.59175

DEPENDENT VARIABLE Y

PAR NO.	PAR VALUE	PAR VALUE	TEE VALUE	CONTRIBUTION	VARIABLE IDENTIFICATION
1	0.18920	0.189198150 00	3.681	129.286	X3
2	0.12881	0.128807610 00	1.776	18.930	X1
3	-0.08420	-0.84197969D-01	-0.700	3.400	X2
0	9.30125	0.93012514E 01	0.069		

CVR = 0.14640 SDR = 2.63513 MSDR = 6.94393 CD = 0.57362

VON NEUMANN RATIO OF THE LEAST SQUARES
ESTIMATED DISTURBANCES = 1.82588

AVERAGE ABSOLUTE ERROR= 0.56757

SUM SQUARES DEVIATION EXPLAINED FROM REGRESSION= 151.61671

SUM SQUARES DEVIATION OF DEPENDENT VARIABLE = 228.00000

OBSERVATION NO.	Y-GIVEN	Y-CALC	ERROR	ERROR/Y-GIV
1	15.000000	19.598282	-4.598282	-0.306552
2	16.000000	14.754547	1.245453	0.077841
3	14.000000	13.006074	0.993926	0.070995
4	22.000000	20.177078	1.822922	0.082860
5	24.000000	22.427475	1.572525	0.065522
6	19.000000	19.388992	-0.388992	-0.020473
7	13.000000	13.699021	-0.699021	-0.053771
8	15.000000	19.504745	-4.504745	-0.300316
9	23.000000	21.478195	1.521805	0.066165
10	12.000000	15.624542	-3.624542	-0.302045
11	25.000000	24.089951	0.910049	0.036402
12	17.000000	16.049271	0.950729	0.055925
13	18.000000	16.537689	1.462311	0.081239
14	19.000000	18.067886	0.932114	0.049059
15	18.000000	15.596025	2.403975	0.133554

APPENDIX XI

Digital Computer Program using Subroutines for Correlation Coefficients and Solution by Gauss-Jordan Method

Subroutine Corre transforms a given set of observations (on a number of explanatory variables and the dependent variable) into normalized augmented matrix. The element r_{ij} in this matrix represents the correlation coefficient between the i-th and j-th variables. Subroutine Simult solves the system, represented by the augmented matrix, by Gauss-Jordan elimination method (described in section 3.6.1). The elements of the resulting solution vector are multiplied by the ratio of standard deviations of the dependent variable and the corresponding explanatory variable to determine the solution vector for the original system.

For simplicity standard deviation of the i-th variable has been calculated as

$$\sigma_i = \sqrt{(\Sigma X_i^2) - N(\bar{X})^2}$$

The example taken here for computing results is the same as discussed in Appendices IX and X. Only part of the computer printout is shown.

```
LEVELO 1JUL66                          IBM OS/360 BASIC FORTRAN IV (E) COMPILATION
                  C         MAIN PROGRAM USING SUBROUTINES CORRE AND SIMULT
S.0001                      DIMENSION X( 21,50),XNAME(21)
S.0002                      DIMENSICN CC (20,21),S(21),SD(21),SSD(21)
S.0003                      DIMENSION A(20,21)
S.0004                      COMMON N,X,N1,A
S.0005                      COMMON CC,S,SD,SSD,NTO,FNTO
S.0006                      NRED=1
S.0007                      NPUD=2
S.0008                      NPRD=3
S.0009                   50 FORMAT(5X,I5,5X,I5)
S.0010                   51 FORMAT('1')
S.0011                   52 FORMAT(A5,5X,7F10.5/(10X,7F10.5))
S.0012                   53 FORMAT('0',2X,A5,2X,7F10.5/(10X,7F10.5))
S.0013                   54 FORMAT('1','VARIABLE NUMBER',5X,'STANDARD DEVIATION',4X,
                           1'SUM SQUARES DEVIATION',//)
S.0014                   55 FORMAT('0',4X,I5,2(15X,F10.5))
S.0015                   56 FORMAT('1','CORRELATION COEFFICIENTS-NORMALIZED AUGMENTED MATRIX',
                           1//)
S.0016                   57 FORMAT('0',8F10.5/(10X,8F10.5))
S.0017                   58 FORMAT('1',' NUMBER',7X,'COEFFICIENT',//)
S.0018                   59 FORMAT('0',2X,I5,7X,F10.5)
S.0019                   60 FORMAT('1',' NUMBER',7X,'Y-GIVEN',11X,'Y-CALCULATED',12X,'ERROR',
                           1//)
S.0020                   61 FORMAT('0',2X,I5,3(10X,F10.5))
S.0021                   62 FORMAT(///,' CONSTANT=',F10.6)
S.0022                      READ(NRED,50)N,NTO
S.0023                      N1=N+1
S.0024                      WRITE(NPRD,51)
S.0025                      DO 1 I=1,N1
S.0026                      READ(NRED,52)XNAME(I),(X(I,J),J=1,NTO)
S.0027                      WRITE(NPRD,53)XNAME(I),(X(I,J),J=1,NTO)
S.0028                    1 CONTINUE
S.0029                      FNTO=NTO
S.0030                      CALL CORRE
S.0031                      WRITE(NPRD,54)
S.0032                      DO 2 I=1,N1
S.0033                      WRITE(NPRD,55)I,SD(I),SSD(I)
S.0034                    2 CONTINUE
S.0035                      WRITE(NPRD,56)
S.0036                      DO 3 I=1,N
S.0037                      DO 13  J=1,N1
S.0038                      A(I,J)=CC(I,J)
S.0039                   13 CONTINUE
S.0040                      WRITE(NPRD,57)(A(I,J),J=1,N1)
S.0041                    3 CONTINUE
S.0042                      CALL SIMULT
S.0043                      WRITE(NPRD,58)
S.0044                      DO 4 I=1,N
S.0045                      A(I,N1)=A(I,N1)*SD(N1)/SD(I)
S.0046                      WRITE(NPRD,59)I,A(I,N1)
S.0047                    4 CONTINUE
S.0048                      WRITE(NPRD,60)
S.0049                      SUM=0.
```

```
      DO 5 I=1,N
      SUM=SUM+A(I,N1)*S(I)
    5 CONTINUE
      S(N1)=S(N1)/FNTO
      CONST=S(N1)-SUM
      DO 6 J=1,NTO
      YC1=0.
      DO 16 I=1,N
      YC1=YC1+A(I,N1)*X(I,J)
   16 CONTINUE
      YC2=YC1+CONST
      ERROR=X(N1,J)-YC2
      WRITE(NPRD,61)J,X(N1,J),YC2,ERROR
    6 CONTINUE
      WRITE(NPRD,62)CONST
      STOP
      END
         SIZE OF COMMON   007828    PROGRAM   002186
                            END OF COMPILATION  MAIN
```

LEVELO 1JUL66 IBM OS/360 BASIC FORTRAN IV (E) COMPILATION

```
C           SOLUTION OF SIMULTANEOUS EQUATIONS
C           GAUSS-JORDAN ELIMINATION METHOD
C
            SUBROUTINE SIMULT
            DIMENSION X(21,50),XNAME(21)
            DIMENSION CC(20,21),S(21),SD(21),SSD(21)
            DIMENSION A(20,21)
            COMMON N,X,N1,A
            COMMON CC,S,SD,SSD,NTO,FNTO
      107   FORMAT('1','DIAGONAL ELEMENT IN EQUATION',I4,' IS ZERO',/
           1' TRANSPOSE EQUATION WITH A PREVIOUS ONE',/)
            NPRD=3
            DO 105 I=1,N
            IF(A(I,I))100,106,100
      100   P=1./A(I,I)
            DO 101 J=1,N1
            A(I,J)=P*A(I,J)
      101   CONTINUE
            DO 102 K=1,N
            IF(I-K)103,102,103
      103   Q=A(K,I)
            DO 104 J=1,N1
            A(K,J)=A(K,J)-Q*A(I,J)
      104   CONTINUE
      102   CONTINUE
      105   CONTINUE
            INDIC=0
            RETURN
      106   IF(N -I)200,204,200
      200   I1=I+1
            IF(A(L,I))201,203,201
      201   DO 202 J=I,N1
            HOLD=A(I,J)
            A(I,J)=A(L,J)
            A(L,J)=HOLD
      202   CONTINUE
            GO TO 100
      203   CONTINUE
      204   WRITE(NPRD,107)I
            INDIC=1
            RETURN
            END
```

SIZE OF COMMON 007828 PROGRAM 001182

END OF COMPILATION SIMULT

LEVELO 1JUL66 IBM OS/360 BASIC FORTRAN IV (E) COMPILATION

```
C     SUBROUTINE FOR CORRELATION COEFFICIENTS
      SUBROUTINE CORRE
      DIMENSION X(21,50),XNAME(21)
      DIMENSION CC(20,21),S(21),SD(21),SSD(21)
      DIMENSION A(20,21)
      COMMON N,X,N1,A
      COMMON CC,S,SD,SSD,NTO,FNTO
      DO 1 I=1,N1
      S(I)=0.
      DO 1 J=1,N1
      CC(I,J)=0.
    1 CONTINUE
      DO 2 I=1,N1
      DO 2 K=1,NTO
      S(I)=S(I)+X(I,K)
      DO 2 J=1,N1
      CC(I,J)=CC(I,J)+X(I,K)*X(J,K)
    2 CONTINUE
      DO 3 I=1,N1
      SSD(I)=CC(I,I)-S(I)*S(I)/FNTO
      SD(I)=SQRT(SSD(I))
    3 CONTINUE
      DO 4 I=1,N
      CC(I,I)=1.
      S(I)=S(I)/FNTO
      K=I+1
      DO 4 J=K,N1
      CC(I,J)=(CC(I,J)-S(I)*S(J))/(SD(I)*SD(J))
    4 CONTINUE
      NM1=N-1
      DO 5 I=1,NM1
      K=I+1
      DO 5 J=K,N
      CC(J,I)=CC(I,J)
    5 CONTINUE
      RETURN
      END
```

SIZE OF COMMON 007828 PROGRAM 001292

END OF COMPILATION CORRE

VARIABLE NUMBER	STANDARD DEVIATION	SUM SQUARES DEVIATION
1	44.42673	1973.73437
2	27.00621	729.33594
3	54.87926	3011.73437
4	15.09967	228.00000

CORRELATION COEFFICIENTS-NORMALIZED AUGMENTED MATRIX

1.00000	0.56149	0.30240	0.50237
0.56149	1.00000	0.32680	0.28693
0.30240	0.32680	1.00000	0.75303

NUMBER	COEFFICIENT
1	0.12881
2	-0.08420
3	0.18920

CONSTANT= 9.301162

APPENDIX XII

Digital Computer Program using Subroutines for Correlation Coefficients and Solution by Gauss-Seidel Method

For Subroutine Corre and other details see Appendix XI.

Subroutine solvgs solves the system represented by augmented matrix by Gauss-Seidel iterative method (described in section 3.6.2).

Only part of the computer printout is shown. The results given here should be compared with those in Appendix XI.

NTIT = Total number of iterations if the solution does not converge to preset tolerance.

INDIC = 0, when solution converges to within preset tolerance.

ICT = number of iteration needed for convergence to desired level.

LEVELC 1JUL66 IBM OS/360 BASIC FORTRAN IV (E) COMPILATION

```
C       MAIN PROGRAM USING CORRE AND SOLVGS
        DIMENSION X(21,50),XNAME(21)
        DIMENSION CC(20,21),S(21),SD(21),SSD(21)
        DIMENSION A(20,21)
        DIMENSION XX(20)
        COMMON N,X,N1,A
        COMMON CC,S,SD,SSD,NTO,FNTO
        COMMON XX,NTIT,TOL,INDIC,ICT
        NRED=1
        NPUD=2
        NPRD=3
   50   FORMAT(5X,I5,5X,I5)
   51   FORMAT('1')
   52   FORMAT(A5,5X,7F10.5/(10X,7F10.5))
   53   FORMAT('0',2X,A5,2X,7F10.5/(10X,7F10.5))
   54   FORMAT('1','VARIABLE NUMBER',5X,'STANDARD DEVIATION',4X,
       1'SUM SQUARES DEVIATION',//)
   55   FORMAT('0',4X,I5,2(15X,F10.5))
   56   FORMAT('1','CORRELATION COEFFICIENTS-NORMALIZED AUGMENTED MATRIX',
       1//)
   57   FORMAT('0',8F10.5/(10X,8F10.5))
   58   FORMAT('1',' NUMBER',7X,'COEFFICIENT',//)
   59   FORMAT('0',2X,I5,7X,F10.5)
   60   FORMAT('1',' NUMBER',7X,'Y-GIVEN',11X,'Y-CALCULATED',12X,'ERROR',
       1//)
   61   FORMAT('0',2X,I5,3(10X,F10.5))
   62   FORMAT(///,' CONSTANT=',F10.6)
   65   FORMAT(///)
   70   FORMAT(15X,I5,E20.8)
   75   FORMAT('0',' INDIC=',I5,5X,'ICT=',I5)
        READ(NRED,50)N,NTO
        N1=N+1
        WRITE(NPRD,51)
        DO 1 I=1,N1
        READ(NRED,52)XNAME(I),(X(I,J),J=1,NTO)
        WRITE(NPRD,53)XNAME(I),(X(I,J),J=1,NTO)
    1   CONTINUE
        FNTO=NTO
        CALL CORRE
        WRITE(NPRD,54)
        DO 2 I=1,N1
        WRITE(NPRD,55)I,SD(I),SSD(I)
    2   CONTINUE
        WRITE(NPRD,56)
        DO 3 I=1,N
        DO 13 J=1,N1
        A(I,J)=CC(I,J)
   13   CONTINUE
        WRITE(NPRD,57)(A(I,J),J=1,N1)
    3   CONTINUE
        READ(NRED,70)NTIT,TOL
        CALL SOLVGS
        DO 10 I=1,N
```

```
      A(I,N1)=XX(I)
   10 CONTINUE
      WRITE(NPRD,69)
      WRITE(NPRD,75)INDIC,ICT
      WRITE(NPRC,58)
      DO 4 I=1,N
      A(I,N1)=A(I,N1)*SD(N1)/SD(I)
      WRITE(NPRD,59)I,A(I,N1)
    4 CONTINUE
      WRITE(NPRD,60)
      SUM=0.
      DO 5 I=1,N
      SUM=SUM+A(I,N1)*S(I)
    5 CONTINUE
      S(N1)=S(N1)/FNTO
      CONST=S(N1)-SUM
      DO 6 J=1,NTO
      YC1=0.
      DO 16 I=1,N
      YC1=YC1+A(I,N1)*X(I,J)
   16 CONTINUE
      YC2=YC1+CONST
      ERROR=X(N1,J)-YC2
      WRITE(NPRD,61)J,X(N1,J),YC2,ERROR
    6 CONTINUE
      WRITE(NPRC,62)CONST
      STOP
      END
```

SIZE OF COMMON 007924 PROGRAM 002394

END OF COMPILATION MAIN

LEVELO 1JUL66 IBM OS/360 BASIC FORTRAN IV (E) COMPILATION

```
C     SUBROUTINE FOR CORRELATION COEFFICIENTS
      SUBROUTINE CORRE
      DIMENSION X(21,50),XNAME(21)
      DIMENSION CC(20,21),S(21),SD(21),SSD(21)
      DIMENSION A(20,21)
      DIMENSION XX(20)
      COMMON N,X,N1,A
      COMMON CC,S,SD,SSD,NTO,FNTO
      COMMON XX,NIT,TOL,INDIC,ICT
      DO 1 I=1,N1
      S(I)=0.
      DO 1 J=1,N1
      CC(I,J)=0.
    1 CONTINUE
      DO 2 I=1,N1
      DO 2 K=1,NTO
      S(I)=S(I)+X(I,K)
      DO 2 J=1,N1
      CC(I,J)=CC(I,J)+X(I,K)*X(J,K)
    2 CONTINUE
      DO 3 I=1,N1
      SSD(I)=CC(I,I)-S(I)*S(I)/FNTO
      SD(I)=SQRT(SSD(I))
    3 CONTINUE
      DO 4 I=1,N
      CC(I,I)=1.
      S(I)=S(I)/FNTO
      K=I+1
      DO 4 J=K,N1
      CC(I,J)=(CC(I,J)-S(I)*S(J))/(SD(I)*SD(J))
    4 CONTINUE
      NM1=N-1
      DO 5 I=1,NM1
      K=I+1
      DO 5 J=K,N
      CC(J,I)=CC(I,J)
    5 CONTINUE
      RETURN
      END
```

SIZE OF COMMON 007924 PROGRAM 001292

END OF COMPILATION CORRE

```
LEVELO 1JUL66                           IBM OS/360 BASIC FORTRAN IV (E) COMPILATION
C                  SUBROUTINE FOR SOLVING SIMULTANEOUS EQUATIONS
C                  GAUSS-SEIDEL ITERATIVE METHOD
C
C                  SOLUTION WILL CONVERGE      WHEN       DIAGONAL TERMS ARE LARGER
C                  IN MAGNITUDE THAN OTHER TERMS IN THEIR ROW
C
C                  A SUFFICIENT CONDITION FOR CONVERGENCE IS THAT ABSOLUTE VALUE OF
C                  DIAGONAL TERM IN A ROW IS GREATER THAN THE SUM OF ABSOLUTE
C                  VALUES OF OTHER TERMS IN THAT ROW
C
                   SUBROUTINE SOLVGS
                   DIMENSION X(21,50),XNAME(21)
                   DIMENSION CC(20,21),S(21),SD(21),SSD(21)
                   DIMENSION A(20,21)
                   DIMENSION XX(20)
                   COMMON N,X,N1,A
                   COMMON CC,S,SD,SSD,NTO,FNTO
                   COMMON XX,NTIT,TOL,INDIC,ICT
                   DO 1 I=1,N
                   XX(I)=A(I,N1)/A(I,I)
                1  CONTINUE
                   ICT=0
                5  INDIC=0
                   DO 2 I=1,N
                   SS=A(I,N1)
                   DO 3 J=1,N
                   SS=SS-A(I,J)*XX(J)
                3  CONTINUE
                   SS=(SS+A(I,I)*XX(I))/A(I,I)
                   IF(ABS(SS-XX(I))-TOL)6,7,7
                7  INDIC=1
                6  CONTINUE
                   XX(I)=SS
                2  CONTINUE
                   ICT=ICT+1
                   IF(ICT-NTIT)9,9,99
                9  CONTINUE
                   IF(INDIC-1)4,5,4
                4  CONTINUE
                   RETURN
               99  IF(INDIC)10,4,10
               10  INDIC=-INDIC
                   RETURN
                   END
                      SIZE OF COMMON  007924   PROGRAM  000918
                                END OF COMPILATION  SOLVGS
```

VARIABLE NUMBER	STANDARD DEVIATION	SUM SQUARES DEVIATION
1	44.42673	1973.73437
2	27.00621	729.33594
3	54.87926	3011.73437
4	15.09967	228.00000

CORRELATION COEFFICIENTS-NORMALIZED AUGMENTED MATRIX

1.00000	0.56149	0.30240	0.50237
0.56149	1.00000	0.32680	0.28693
0.30240	0.32680	1.00000	0.75303

NUMBER	COEFFICIENT
1	0.12881
2	-0.08420
3	0.18920

INCIC= 0 ICT= 12

CONSTANT= 9.301163

gitized form of equivalent dipole sources representing electrical activity of the heart (Appendix IV)

lagnitude is given in arbitrary units)

X01	X02	X03	X04	X05	X06	X07	X08	X09	X10
C.8C	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
1.80	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
2.3C	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
3.00	0.70	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
3.50	1.20	0.25	0.20	0.0	0.0	0.0	1.20	0.0	0.0
3.50	1.80	0.80	0.70	0.18	0.0	0.0	2.50	0.0	0.0
3.5C	2.00	1.00	1.0C	0.25	0.0	0.0	3.80	0.80	1.60
3.50	3.00	1.30	1.20	0.50	0.25	0.0	5.30	2.30	3.00
3.50	3.50	1.70	1.50	0.60	0.30	0.0	6.10	3.40	4.30
3.5C	3.50	1.85	1.75	C.70	0.50	0.0	7.80	4.50	5.20
3.50	3.50	2.00	1.90	0.90	0.60	0.0	8.80	5.70	5.80
3.00	3.40	2.00	2.00	1.00	0.80	0.0	9.80	6.70	6.70
2.50	3.C0	2.00	2.15	1.10	1.00	0.15	9.80	6.70	6.70
1.60	2.00	2.00	2.15	1.50	1.25	0.35	9.70	6.50	6.70
0.70	1.20	2.00	2.00	1.60	1.30	0.70	9.70	6.50	6.70
C.0	0.20	1.90	2.00	1.70	1.60	0.90	8.50	6.50	6.70
0.0	0.0	1.90	2.00	1.90	1.70	1.00	5.20	6.50	6.70
C.0	0.0	1.90	1.80	2.00	1.80	1.50	2.50	6.50	6.70
C.0	0.0	1.60	1.70	2.00	1.90	1.60	0.0	6.50	6.70
C.0	0.0	1.00	1.50	2.15	2.00	1.80	0.0	6.00	5.80
0.0	0.0	0.70	1.50	2.15	2.00	2.00	0.0	5.00	4.50
0.0	0.0	0.50	1.00	2.15	2.30	2.00	0.0	4.00	3.30
C.0	0.0	0.0	0.50	2.15	2.30	2.00	0.0	2.50	1.80
0.0	0.0	0.0	0.25	2.00	2.50	2.00	0.0	0.70	0.0
0.0	0.0	0.0	0.0	1.00	2.50	2.00	0.0	0.0	0.0
C.0	0.0	C.0	0.0	0.20	2.50	2.00	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	1.80	1.90	0.0	0.0	0.0
C.0	0.0	0.0	0.0	0.0	1.00	1.70	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.20	1.30	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	1.00	0.0	0.0	0.0
C.C	0.0	0.0	0.0	0.0	0.0	0.50	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0

X11	X12	X13	X14	X15	X16	X17	X18	X19	X20
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
C.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.60	0.0	0.0	0.0
C.0	0.0	0.0	0.0	0.0	0.0	1.00	0.0	0.0	0.0
C.0	0.0	0.0	0.0	0.0	0.0	1.20	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	1.60	0.0	0.0	0.0
0.0	0.20	0.0	0.0	0.0	0.0	1.60	0.0	0.0	0.0
0.0	1.50	0.30	0.0	0.0	0.0	1.80	0.0	0.30	0.0
0.0	2.00	1.20	0.0	0.0	0.0	1.80	0.40	0.70	0.0
C.0	3.10	2.20	0.0	0.0	0.0	1.80	1.00	1.80	0.0
1.50	4.00	2.80	0.0	0.0	0.0	1.80	1.30	2.30	0.30
3.40	5.00	6.00	0.0	0.0	0.0	1.80	2.30	3.10	1.20
4.5C	5.50	5.50	0.0	0.0	0.0	1.70	2.70	3.20	2.00
6.70	6.70	5.50	0.0	0.0	0.70	1.60	3.20	3.30	2.80
8.50	6.70	6.00	0.30	0.60	1.30	1.30	3.20	3.30	3.50
9.50	6.70	6.70	1.90	1.50	1.90	1.00	3.25	3.30	4.30
9.5C	6.70	6.70	2.50	1.80	2.50	0.80	3.20	3.30	4.30
9.60	6.70	6.70	4.20	2.30	2.50	0.0	3.15	3.50	4.40
9.80	6.70	6.70	4.50	2.50	2.50	0.0	3.15	3.50	4.40
9.8C	6.50	6.70	5.20	2.50	2.50	0.0	3.15	3.50	4.40
9.80	6.00	6.70	5.20	2.40	2.50	0.0	2.70	3.50	4.70
9.80	5.25	6.00	5.00	2.40	2.50	0.0	2.30	3.50	3.70
9.40	3.10	5.50	5.00	2.40	2.50	0.0	1.90	3.50	3.50
8.50	2.80	3.70	5.20	2.30	2.50	0.0	1.50	3.50	2.80
7.50	1.30	2.50	4.80	2.30	2.50	0.0	0.90	3.00	2.50
5.0C	0.0	0.50	4.30	2.00	2.30	0.0	0.30	1.50	2.00
2.7C	0.0	0.0	3.50	1.70	1.50	0.0	0.0	1.00	1.50
0.0	0.0	0.0	2.50	1.20	0.80	0.0	0.0	0.0	0.80
C.0	0.0	0.0	1.30	C.70	0.0	0.0	0.0	0.0	0.50
C.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0

Digitized form of given unipolar electrocardiograms (Fig. 4.2) and their computed values

(See Fig. 4.5)

Y01		Y02		Y03		Y04	
GIVEN	CALCULATED	GIVEN	CALCULATED	GIVEN	CALCULATED	GIVEN	CALCULATED
0.0	0.355163	0.0	0.285227	0.0	0.050140	0.0	0.148484
0.0	0.355163	0.0	0.285227	0.0	0.050140	0.0	0.148484
0.0	0.355163	0.0	0.285227	0.0	0.050140	0.0	0.148484
0.40	0.355163	0.0	0.285227	0.80	0.050140	0.50	0.148484
1.00	-0.820755	1.00	-0.438569	3.00	3.039752	1.20	1.843176
1.80	1.818608	5.00	4.677294	4.50	5.362728	2.00	2.096257
3.00	4.115473	9.00	9.938116	6.00	6.711000	3.20	2.277494
10.00	11.013652	12.00	10.223056	8.30	8.332966	4.50	5.434045
20.00	18.600174	16.00	17.688171	10.00	7.986813	7.00	6.634520
24.00	23.264679	19.00	18.018814	12.00	12.854466	9.00	9.628024
22.00	22.593109	16.00	14.982519	14.00	15.042197	13.00	12.810734
16.00	15.378730	10.00	9.560989	15.00	16.256210	15.00	14.775105
10.00	11.417269	4.00	7.163289	17.00	15.034533	13.00	12.487411
3.00	2.054109	0.0	-0.422867	16.00	14.348953	11.50	11.429647
-3.00	-1.724566	-7.00	-7.185253	6.00	8.098592	10.50	11.243252
-10.00	-11.175931	-11.00	-12.724925	-1.00	-0.685551	6.00	5.745488
-17.00	-16.916031	-15.50	-14.181307	-10.00	-10.851908	-3.00	-3.133277
-22.00	-21.942017	-16.00	-15.320014	-18.00	-19.708923	-10.00	-9.962581
-30.00	-29.468262	-16.80	-16.377884	-28.00	-26.739487	-17.00	-17.426712
-35.00	-35.672379	-20.00	-23.183350	-24.00	-22.433640	-22.00	-21.529663
-44.00	-44.315094	-30.00	-29.696609	-20.00	-19.815048	-24.00	-23.736755
-44.00	-43.364960	-36.00	-34.973190	-17.00	-18.235931	-22.00	-21.737274
-40.00	-39.651276	-43.00	-41.641495	-13.00	-13.304138	-19.00	-19.590256
-35.00	-35.776840	-37.00	-38.329163	-10.00	-9.932316	-15.50	-15.512867
-30.00	-29.266296	-28.00	-27.061813	-8.00	-7.215103	-12.00	-11.921207
-26.00	-26.187988	-19.00	-18.004364	-7.00	-7.230453	-11.00	-11.296741
-21.50	-22.104874	-11.00	-13.675786	-6.00	-5.551373	-9.00	-8.512160
-17.00	-16.106812	-9.50	-8.480604	-5.30	-6.143896	-7.00	-7.168633
-13.00	-13.526788	-7.00	-6.756025	-3.50	-3.566223	-5.00	-4.890245
-9.00	-8.859881	-4.00	-2.935438	-2.00	-2.225703	-3.00	-3.107059
-4.00	-4.449722	-2.00	-3.088944	-0.80	-0.678982	-1.50	-2.221793
0.0	0.355163	0.0	0.285227	0.0	0.050140	0.0	0.148484

Y05		Y06		Y07		Y08	
GIVEN	CALCULATED	GIVEN	CALCULATED	GIVEN	CALCULATED	GIVEN	CALCULATED
C.C	-0.C09C48	0.0	-0.C56566	0.0	C.C16756	C.C	-0.C82C27
C.C	-C.C09048	0.C	-0.C56566	0.0	0.C16756	0.C	-0.C82C27
0.C	-0.C09C48	0.C	-0.C56566	0.0	0.C16756	0.0	-0.C82027
0.0	-0.C09C48	0.30	-0.C56566	0.5C	0.016756	0.10	-0.082027
1.CC	2.C3844C	0.5C	1.478522	0.80	1.338146	0.30	1.144742
2.5C	2.8543CC	0.80	0.968934	1.CO	0.949541	0.70	0.782167
4.CO	3.7468C6	1.CO	0.418243	1.2C	1.319241	1.CO	0.317753
5.CO	5.134639	1.50	2.168360	1.1C	1.250776	1.20	2.137709
6.CC	5.424172	1.80	1.110147	0.5C	-0.5755C6	1.40	0.896796
8.CC	8.671272	2.00	2.412644	-2.CO	-1.199806	1.60	1.730785
10.CC	1C.7946C0	2.CO	2.329452	-4.CC	-3.725706	0.0	-0.264583
11.5C	12.383664	1.50	1.207009	-6.00	-6.237425	-4.50	-5.520203
13.2C	11.C83673	0.C	-0.547068	-9.CC	-9.6C0465	-11.CC	-10.331678
14.CC	12.521658	-3.CO	-3.144622	-16.CO	-16.C40176	-19.CO	-18.740005
0.0	2.439568	-6.CO	-5.283545	-23.CO	-21.945816	-26.CO	-25.162735
-6.CC	-6.301767	-8.CO	-8.C86340	-22.50	-23.406937	-25.CO	-26.138016
-10.CC	-10.420C94	-11.CO	-11.C00946	-21.5C	-20.627289	-23.50	-22.685486
-11.5C	-12.737739	-13.00	-13.565277	-19.50	-20.4275C5	-22.50	-22.889267
-13.50	-13.C08905	-15.CC	-15.125408	-18.CC	-17.796448	-21.CC	-21.413528
-14.5C	-13.605928	-16.50	-15.242443	-15.CO	-14.291128	-19.CO	-17.918335
-15.CC	-14.148125	-15.CC	-15.C79784	-13.CO	-13.477382	-17.CC	-17.676315
-13.50	-14.154195	-12.00	-12.282068	-10.80	-10.469226	-14.00	-13.506340
-11.00	-11.458016	-8.CC	-8.643367	-9.CO	-9.5123C0	-10.CC	-10.559252
-9.CO	-9.C06418	-7.OC	-6.859137	-7.50	-7.363623	-8.30	-8.4644C9
-6.00	-5.654406	-6.CO	-5.695278	-5.50	-5.573575	-7.CO	-6.636989
-5.00	-5.196654	-5.30	-5.419624	-4.5C	-4.3C0870	-6.30	-6.699845
-4.CC	-2.875481	-4.5C	-4.543551	-3.5C	-3.915462	-5.5C	-6.C49852
-3.5C	-4.556904	-3.5C	-3.433188	-2.80	-2.325313	-4.5C	-2.882379
-2.5C	-2.262723	-2.80	-2.638837	-2.00	-1.925642	-3.50	-4.C83104
-2.CC	-1.843658	-1.50	-1.853336	-1.50	-1.736484	-2.50	-2.881352
-1.CO	-2.216415	-0.80	-0.866815	-0.40	-0.366736	-1.50	-1.395530
0.C	-0.CC9C48	0.C	-0.C56566	0.C	0.C16756	0.C	-0.C82C27

Y09		Y10		Y11		Y12	
GIVEN	CALCULATED	GIVEN	CALCULATED	GIVEN	CALCULATED	GIVEN	CALCULATED
0.50	0.932176	0.0	0.122252	0.0	0.829212	0.0	1.475899
1.10	0.932176	0.0	0.122252	0.0	0.829212	0.0	1.475899
2.00	0.932176	0.0	0.122252	0.0	0.829212	0.0	1.475899
2.80	0.932176	0.0	0.122252	0.0	0.829212	0.0	1.475899
3.30	3.037060	0.0	0.991504	0.30	-4.354415	0.30	-4.446662
3.80	4.175376	0.0	0.001200	1.00	-0.396274	1.50	-1.227714
3.50	4.875580	1.00	0.037974	1.50	3.786390	3.00	4.116584
1.00	-2.921335	2.00	2.251939	3.50	0.962184	4.50	0.880467
-9.00	-8.687476	3.50	3.330128	7.00	10.271120	5.30	9.915054
-19.00	-17.378220	4.30	5.483958	10.00	8.485478	8.00	9.301564
-27.00	-28.642120	5.50	4.725573	13.50	12.608312	11.00	10.024177
-37.00	-36.789307	2.50	1.469315	16.50	17.531128	16.00	16.038437
-40.00	-37.801361	-5.00	-3.986370	18.20	19.129349	25.00	23.723618
-40.50	-39.862946	-12.00	-12.442253	20.00	20.749557	29.00	30.439514
-38.00	-40.566101	-19.00	-17.856171	22.00	20.925903	33.00	33.729599
-35.00	-34.377686	-25.00	-25.840637	23.50	21.953720	36.00	33.422363
-30.50	-29.217163	-29.50	-29.506989	25.50	26.745636	38.50	40.228027
-26.50	-27.311279	-33.50	-33.384674	27.50	29.156464	40.00	40.700348
-23.00	-21.787262	-38.00	-38.236145	29.00	29.935410	40.20	42.237869
-19.00	-19.861725	-43.00	-43.415878	30.00	22.645126	35.00	22.579987
-15.00	-16.395096	-46.50	-46.198135	15.00	15.977490	14.00	16.510086
-12.00	-11.798657	-45.50	-44.223160	-1.00	2.284905	-15.00	-6.043139
-9.50	-8.802513	-39.00	-39.780640	-21.00	-18.755844	-18.50	-18.274429
-8.50	-8.611421	-29.50	-30.161301	-19.00	-19.925507	-15.00	-16.516403
-7.00	-6.097980	-20.00	-18.844620	-15.00	-16.949219	-11.50	-13.710592
-6.00	-5.154646	-14.00	-14.676495	-12.00	-11.546090	-9.00	-10.562182
-4.50	-7.436836	-8.00	-8.318157	-10.00	-9.180827	-7.00	-3.132318
-3.70	-2.959020	-6.00	-5.603066	-8.00	-8.087456	-5.20	-6.556055
-2.80	-2.219975	-4.50	-4.219117	-5.00	-5.248367	-4.00	-2.381599
-1.60	-1.000051	-2.80	-2.883392	-2.00	-1.155324	-2.50	-3.504727
-1.00	0.831840	-1.10	-2.424649	-1.00	-1.695518	-1.00	-3.271299
0.0	0.932176	0.0	0.122252	0.0	0.829212	0.0	1.475899

Y13		Y14		Y15		Y16	
GIVEN	CALCULATED	GIVEN	CALCULATED	GIVEN	CALCULATED	GIVEN	CALCULATED
0.0	0.302162	0.0	0.676232	0.0	0.100600	0.0	-0.233140
0.0	0.302162	0.30	0.676232	0.0	0.100600	0.0	-0.233140
0.0	0.302162	1.00	0.676232	0.0	0.100600	-0.30	-0.233140
2.00	0.302162	1.50	0.676232	0.0	0.100600	-1.00	-0.233140
4.00	4.203877	2.00	2.426551	0.0	-0.928252	-2.00	-1.868726
5.50	6.045932	2.80	4.264363	0.30	0.871728	-2.50	-2.973931
10.00	9.429872	4.00	4.085298	1.00	0.611670	-6.00	-5.982618
15.00	16.438675	5.00	5.104241	1.30	0.555291	-9.50	-9.659405
20.00	18.108185	7.00	5.908237	2.00	2.627434	-11.70	-11.354326
24.50	25.672470	10.00	10.260270	2.50	2.272243	-13.50	-13.647621
29.00	29.736389	14.00	14.848069	3.00	3.383758	-14.50	-14.231227
32.50	33.484955	16.00	18.056137	3.70	3.442973	-15.00	-14.325618
34.40	31.390839	19.00	17.120850	4.50	5.260935	-11.50	-12.568863
27.00	26.819244	22.00	18.858963	5.30	4.694988	-10.00	-10.147611
12.00	13.181543	24.00	27.100998	6.00	6.089678	-8.50	-8.161825
0.0	-0.056566	26.00	28.241791	7.00	7.193661	-7.30	-7.200273
-10.00	-9.438102	27.00	21.673279	8.00	8.285196	-6.50	-6.751686
-14.00	-15.387202	10.00	9.764188	7.30	5.944433	-5.70	-5.714522
-16.00	-15.985101	-8.00	-6.251692	5.50	6.828431	-4.70	-4.648041
-14.00	-11.934924	-12.00	-11.486371	4.00	2.689842	-4.20	-4.398616
-12.00	-12.168014	-13.50	-10.827558	0.0	0.575663	-3.50	-2.914907
-9.50	-10.425987	-11.50	-13.435053	-6.00	5.622202	-3.00	-3.410553
-8.00	-8.632096	-10.00	-10.572530	-10.00	-10.039639	-2.50	-2.398589
-6.00	-5.240191	-8.50	-9.328061	-16.00	-16.947067	-2.00	-1.834029
-5.00	-5.910926	-7.00	-3.951824	-25.00	-23.105438	-1.50	-1.897169
-4.00	-3.408652	-5.80	-7.961493	-23.00	-22.882599	-1.20	-0.909121
-3.50	-2.505570	-4.50	-3.420403	-19.00	-20.462814	-1.00	-0.670746
-2.80	-4.297317	-3.50	-3.725345	-15.00	-15.545589	-0.80	-1.316885
-2.30	-1.842372	-2.60	-3.127715	-11.00	-9.521645	-0.50	-0.336244
-1.50	-1.210507	-1.90	-4.146722	-6.50	-6.785345	-0.30	-0.383221
-0.70	-0.979397	-1.00	-1.059649	-3.00	-3.089897	-0.10	-0.327342
0.0	0.302162	0.0	0.676232	0.0	0.100600	0.0	-0.233140

Y17		Y18		Y19		Y20	
GIVEN	CALCULATED	GIVEN	CALCULATED	GIVEN	CALCULATED	GIVEN	CALCULATED
0.0	-0.039998	0.0	-0.083158	0.0	0.289241	0.15	0.223878
0.0	-0.039998	0.0	-0.083158	0.40	0.289241	0.50	0.223878
0.0	-0.039998	0.0	-0.083158	0.80	0.289241	0.70	0.223878
0.0	-0.039998	0.0	-0.083158	1.00	0.289241	0.90	0.223878
0.0	0.342350	0.0	0.206463	0.90	-0.085859	1.00	0.888416
-0.20	-0.232127	-0.70	-0.891843	0.30	0.502808	1.20	1.215415
-0.60	-0.625900	-1.00	-0.783481	0.0	2.266682	1.50	2.412297
-1.00	-0.771696	-1.90	-2.042771	-1.00	-2.126614	1.00	0.900244
-1.70	-2.015706	-3.50	-4.038303	-2.00	-4.153113	0.40	-0.955916
-2.50	-2.422366	-6.50	-5.981771	-10.00	-9.508655	-5.00	-5.057243
-3.60	-3.722095	-8.50	-8.674215	-18.00	-18.031815	-12.00	-12.822120
-5.80	-6.183789	-10.00	-10.312512	-25.00	-25.026810	-20.00	-22.096207
-9.00	-8.625621	-12.00	-11.819391	-27.00	-25.036407	-30.00	-25.529129
-13.00	-12.998622	-16.00	-15.492756	-26.00	-26.575287	-31.60	-31.227036
-17.50	-16.980423	-17.00	-17.340530	-25.00	-25.795959	-30.30	-32.128738
-16.00	-16.721207	-16.00	-16.292435	-22.50	-22.250397	-28.00	-28.157990
-14.00	-13.519611	-14.50	-13.870868	-20.00	-20.074509	-24.00	-23.370285
-12.50	-12.650095	-13.00	-13.067080	-18.50	-18.417160	-21.00	-20.186829
-10.00	-10.139284	-10.00	-10.125610	-16.00	-15.240751	-17.00	-17.432495
-8.00	-7.614952	-8.00	-7.721707	-13.00	-12.454533	-15.00	-13.881796
-6.50	-6.837173	-6.00	-6.595739	-10.00	-11.279335	-11.50	-13.610059
-5.50	-5.299545	-5.00	-4.731565	-7.30	-8.074486	-7.50	-7.376798
-4.80	-4.529470	-4.50	-4.668004	-5.50	-4.837816	-4.00	-3.734305
-4.00	-4.089805	-4.00	-3.760361	-4.60	-4.650988	-2.60	-2.950460
-3.50	-3.397887	-3.00	-3.323994	-4.00	-3.260530	-2.00	-0.750403
-3.00	-3.020027	-2.50	-2.171102	-3.50	-2.719183	-1.60	-1.511593
-2.20	-2.625026	-2.00	-2.598953	-2.80	-5.074403	-1.10	-3.769869
-1.70	-0.909822	-1.70	-1.159638	-2.30	-1.455545	-0.90	1.260951
-1.10	-1.424029	-1.10	-1.053273	-1.70	-1.941915	-0.70	-1.309350
-0.80	-0.840426	-0.60	-0.669480	-1.00	-0.480265	-0.30	0.308284
-0.50	-0.545333	-0.30	0.097094	-0.50	0.537355	-0.10	0.904558
0.0	-0.039998	0.0	-0.083158	0.0	0.289241	0.0	0.223878

Y21		Y22		Y23		Y26	
GIVEN	CALCULATED	GIVEN	CALCULATED	GIVEN	CALCULATED	GIVEN	CALCULATED
0.0	-0.113724	0.0	-0.088571	0.0	-0.010045	-0.10	-0.022004
0.0	-0.113724	0.0	-0.088571	0.0	-0.010045	-0.20	-0.022004
0.0	-0.113724	0.0	-0.088571	0.0	-0.010045	-0.25	-0.022004
0.0	-0.113724	0.0	-0.088571	0.0	-0.010045	-0.30	-0.022004
0.0	0.076535	-0.10	-0.136594	0.0	-0.115479	-0.50	-0.924392
0.0	0.036656	-0.30	-0.311108	0.0	0.062138	-0.60	-0.197928
-0.30	0.015035	-0.60	-0.685464	0.0	0.016779	-0.80	-1.021734
-1.00	-1.151846	-1.30	-1.229433	0.0	-0.179473	-0.90	-1.761395
-2.50	-2.770540	-2.50	-2.602939	0.0	0.107929	-0.90	-0.034300
-3.60	-3.666150	-3.20	-3.153309	0.0	-0.031097	-0.70	-0.697827
-5.00	-5.044541	-3.80	-3.767075	0.0	0.048896	0.0	0.789502
-6.30	-6.368058	-4.00	-4.066353	0.0	0.046203	2.00	2.961247
-6.70	-6.303537	-4.10	-4.207080	0.0	-0.023343	6.00	4.189184
-6.30	-6.144697	-4.00	-3.708737	0.0	-0.006842	9.00	8.115956
-5.90	-6.432175	-3.90	-4.290669	0.0	-0.040943	13.00	14.463333
-4.50	-4.255453	-3.60	-3.431540	0.0	0.002934	15.30	15.733760
-2.50	-2.472076	-3.00	-2.793905	0.0	0.164909	17.70	16.808090
-1.00	-0.921436	-1.50	-1.568324	0.0	-0.247388	18.00	16.529236
1.00	0.928046	0.0	-0.171047	-0.70	-0.526849	10.00	11.569380
5.00	5.563724	3.00	3.509267	-1.30	-1.396552	6.50	5.068417
8.50	8.038433	5.00	4.724424	-2.00	-2.019507	1.00	2.411575
10.00	9.619716	7.30	7.074869	-2.50	-2.487782	-0.70	-0.282196
10.30	10.430433	8.50	8.436461	-3.00	-2.959481	-1.30	-1.623912
9.50	9.677979	7.50	7.741976	-3.50	-3.543709	-1.70	-2.216962
8.00	7.946311	6.50	6.262092	-3.30	-3.214349	-1.80	-0.803894
6.50	6.615167	5.00	5.163770	-3.10	-3.046314	-2.00	-2.817914
5.00	4.680058	3.60	3.548682	-2.60	-2.708507	-1.90	-1.057518
4.30	4.583651	3.00	2.962430	-2.30	-2.378601	-1.80	-2.567803
3.00	2.775207	2.00	2.044918	-1.80	-1.680543	-1.40	-0.723083
2.30	2.397820	1.20	1.245783	-1.40	-1.383821	-1.00	-2.208282
1.00	1.514374	0.50	0.778524	-0.80	-0.708937	-0.50	-0.440699
0.0	-0.113724	0.0	-0.088571	0.0	-0.010045	0.0	-0.022004

Y27 GIVEN	Y27 CALCULATED	Y28 GIVEN	Y28 CALCULATED	Y29 GIVEN	Y29 CALCULATED	Y30 GIVEN	Y30 CALCULATED
0.0	-0.189891	0.0	0.099133	0.0	-0.248621	-0.50	-0.363221
0.0	-0.189891	0.0	0.099133	0.0	-0.248621	-0.90	-0.363221
0.0	-0.189891	0.0	0.099133	-0.50	-0.248621	-0.40	-0.363221
0.0	-0.189891	0.0	0.099133	-0.80	-0.248621	0.0	-0.363221
0.0	-1.423105	0.0	-0.557902	-1.00	-0.772533	1.00	1.398215
0.0	0.180042	0.0	0.411227	-1.00	-0.451101	4.00	4.224717
0.0	1.278129	-0.10	0.389549	-1.00	-1.685426	9.00	8.086244
0.0	-0.730947	-0.40	-1.308415	-0.80	-0.783432	12.00	12.819852
0.0	0.257807	-0.70	-0.491732	0.0	0.615821	16.00	16.195831
-0.20	-1.363036	-0.90	-0.839658	3.00	2.784617	20.00	19.879425
-0.30	-0.621076	-0.70	-0.066646	6.20	6.844110	21.30	20.522217
-0.60	-0.059551	1.00	2.444915	9.50	10.505297	19.50	18.640533
-0.90	0.163276	7.00	5.030419	14.00	12.031257	15.00	16.328217
-1.00	-0.419534	10.00	8.934642	16.00	15.299074	12.50	12.670266
0.0	-1.812970	13.00	14.441586	19.00	20.255142	9.00	9.134269
3.00	2.892408	16.50	17.013641	19.00	19.497803	7.50	6.914474
7.50	8.986986	19.00	17.856171	17.00	16.182220	6.00	6.232801
12.00	12.007735	17.00	15.872540	14.00	12.883563	4.50	4.858345
16.00	15.937600	11.00	12.464265	6.00	6.747104	3.00	2.402486
20.00	20.491943	8.00	7.092340	3.50	3.575502	2.30	2.717847
23.50	22.078384	6.00	7.158126	2.00	3.102459	1.00	0.714109
20.00	18.927994	2.20	1.986242	1.00	0.469787	0.30	0.690404
7.50	8.981483	0.0	0.001881	0.20	-0.083921	0.0	-0.190314
6.30	6.643673	-0.70	-0.943650	-0.10	-0.230676	-0.30	-0.701850
5.00	3.962523	-1.00	-0.444104	-0.80	-0.293433	-0.30	0.298973
3.40	4.484438	-1.00	-1.558015	-1.00	-1.565112	-0.30	-0.823218
2.00	1.232209	-1.00	-0.118059	-1.10	-0.066575	-0.30	-0.673419
1.00	1.254587	-1.00	-1.704776	-1.10	-1.894728	-0.30	0.837556
0.0	-0.840566	-0.90	-0.731632	-1.00	-0.757401	-0.20	-0.738670
0.0	1.501945	-0.70	-1.587063	-0.50	-1.235546	-0.10	-0.061696
0.0	1.156446	-0.50	-0.275017	-0.20	-0.231442	0.0	-0.261671
0.0	-0.189891	0.0	0.099133	0.0	-0.248621	0.0	-0.363221

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