



uOttawa

L'Université canadienne
Canada's university

FACULTÉ DES ÉTUDES SUPÉRIEURES
ET POSTDOCTORALES



FACULTY OF GRADUATE AND
POSTDOCTORAL STUDIES

Mo'Tassem Al-Arydah

AUTEUR DE LA THÈSE / AUTHOR OF THESIS

Ph.D (Mathematics)

GRADE / DEGREE

Department of Mathematics and Statistics

FACULTÉ, ÉCOLE, DÉPARTEMENT / FACULTY, SCHOOL, DEPARTMENT

Existence of a Positive Solution to a Nonlinear System of PDEs in a Domain with a Triple-phase
Boundary

TITRE DE LA THÈSE / TITLE OF THESIS

Arian Novruzi

DIRECTEUR (DIRECTRICE) DE LA THÈSE / THESIS SUPERVISOR

CO-DIRECTEUR (CO-DIRECTRICE) DE LA THÈSE / THESIS CO-SUPERVISOR

EXAMINATEURS (EXAMINATRICES) DE LA THÈSE / THESIS EXAMINERS

Nasir Uddin Ahmed

Lucy Campbell

Wlater Craig (McMaster University)

Paul-Eugène Parent

Gary W. Slater

Le Doyen de la Faculté des études supérieures et postdoctorales / Dean of the Faculty of Graduate and Postdoctoral Studies

EXISTENCE OF A POSITIVE SOLUTION TO A
NONLINEAR SYSTEM OF PDES IN A DOMAIN WITH A
TRIPLE-PHASE BOUNDARY

By
AL-ARYDAH MO'TASSEM, Candidate PhD
October 2009

A Thesis
submitted to the School of Graduate Studies and Research
in partial fulfillment of the requirements
for the degree of
PhD. of Science in Mathematics¹

© Copyright 2009
by AL-ARYDAH MO'TASSEM, Candidate PhD, Ottawa, Canada

¹The PhD. Program is a joint program with Carleton University, administered by the Ottawa-Carleton Institute of Mathematics and Statistics



Library and Archives
Canada

Published Heritage
Branch

395 Wellington Street
Ottawa ON K1A 0N4
Canada

Bibliothèque et
Archives Canada

Direction du
Patrimoine de l'édition

395, rue Wellington
Ottawa ON K1A 0N4
Canada

Your file *Votre référence*
ISBN: 978-0-494-61230-9
Our file *Notre référence*
ISBN: 978-0-494-61230-9

NOTICE:

The author has granted a non-exclusive license allowing Library and Archives Canada to reproduce, publish, archive, preserve, conserve, communicate to the public by telecommunication or on the Internet, loan, distribute and sell theses worldwide, for commercial or non-commercial purposes, in microform, paper, electronic and/or any other formats.

The author retains copyright ownership and moral rights in this thesis. Neither the thesis nor substantial extracts from it may be printed or otherwise reproduced without the author's permission.

In compliance with the Canadian Privacy Act some supporting forms may have been removed from this thesis.

While these forms may be included in the document page count, their removal does not represent any loss of content from the thesis.

AVIS:

L'auteur a accordé une licence non exclusive permettant à la Bibliothèque et Archives Canada de reproduire, publier, archiver, sauvegarder, conserver, transmettre au public par télécommunication ou par l'Internet, prêter, distribuer et vendre des thèses partout dans le monde, à des fins commerciales ou autres, sur support microforme, papier, électronique et/ou autres formats.

L'auteur conserve la propriété du droit d'auteur et des droits moraux qui protègent cette thèse. Ni la thèse ni des extraits substantiels de celle-ci ne doivent être imprimés ou autrement reproduits sans son autorisation.

Conformément à la loi canadienne sur la protection de la vie privée, quelques formulaires secondaires ont été enlevés de cette thèse.

Bien que ces formulaires aient inclus dans la pagination, il n'y aura aucun contenu manquant.


Canada

Abstract

We consider a system of nonlinear PDEs describing the reaction-diffusion dynamics near a triple-phase boundary in the catalyst layer of hydrogen fuel cells. The system involves bulk diffusion and surface reaction-diffusion processes and is an approximation of a model with a thin layer. The coupling of surface and bulk diffusion involves a nonlinear equation (adsorption-desorption process) and a singular boundary condition. Using certain a priori estimates, variational methods techniques and the fixed point theorem, we prove the existence of a positive bounded weak solution. Moreover, we prove the validity of the model by computing the solution numerically and comparing it with the solution obtained with a thin layer model.

Acknowledgements

I would like to express my deepest gratitude to my advisor, Dr. Arian Novruzi for his indispensable guidance and support. He has always been a source of inspiration during my studies. I would like to thank the other members of my committee, Dr. Nasir Uddin Ahmed, Dr. Lucy Campbell, and Dr. Walter Craig for their time and critical review of my work.

The financial support I received during my graduate study came from a teaching assistantship which I held in the Department of Mathematics at the University of Ottawa and from a scholarship granted by the University of Ottawa. I am most grateful to this department and university.

I also thank my parents and my wife for helping me all these years.

Dedication

I dedicate this work to my father, my mother and my wife.

Contents

Abstract	ii
Acknowledgements	iii
Dedication	iv
0 Introduction.	1
0.1 Mathematical model.	2
0.2 Purpose of the work. Organization of the thesis.	8
1 Water only model.	10
1.1 Introduction. Main result.	10
1.2 A fixed point approach.	14
1.2.1 A priori estimates.	15
1.3 Operator T is well-defined.	21
1.4 Existence of a fixed point.	27
1.4.1 Proof of the main result (Theorem 1.1.3).	31
1.5 Numerical results.	32
1.5.1 Numerical sensitivity analysis of the relative error.	37
2 Water-vapor system.	41
2.1 Introduction. Main result.	41
2.2 A fixed point approach.	45
2.2.1 A priori estimates.	47

2.3	Operator T is well-defined.	55
2.4	Existence of a fixed point.	59
2.4.1	Proof of the main result (Theorem 2.1.3).	62
2.5	Numerical results.	63
2.5.1	Numerical sensitivity analysis of the relative error.	65
3	Full model.	71
3.1	Introduction. Main result.	71
3.2	A fixed point approach.	76
3.2.1	A priori estimates	79
3.3	Operator T is well-defined.	83
3.4	Existence of a fixed point.	89
3.4.1	Proof of the main result (Theorem 3.1.3).	92
	Summary	93
	A Thin layer model.	95
	B Limit model (surface diffusion model).	98

Chapter 0

Introduction.

Proton exchange membrane (PEM) fuel cells, like other hydrogen fuel cells, are a candidate to be one of the future clean power sources, which can be used for transportation applications and some stationary applications. For this reason much research has been done aiming to increase the efficiency of fuel cells.

It is important to develop a model that can predict the performance of the fuel cell, as it helps to understand the internal phenomena that may not be observed experimentally. For this purpose, a microscopic mathematical model will be introduced to describe the reaction dynamics in the cathode catalyst layer, which is considered the heart of the fuel cell because it is where the main reactions take place. There are a number of studies describing the microscopic catalyst layer reaction, see [3], [14] and the references within, for example.

In this work we consider a physical model given by a nonlinear system of PDEs and we prove that this system has a positive bounded weak solution. By this we reach the validity of this model, at least from the mathematical point of view. The existence of a solution requires some conditions on some physical parameters, found experimentally. This emphasizes the importance of these parameters. Finally, we do some numerical computations to find solutions numerically and compare them with numerical results obtained from another model.

0.1 Mathematical model.

A PEM fuel cell consists of an anode, which is electrically negative, and a cathode, which is electrically positive, see Figure 1 which is taken from <http://www.hartisee.com/index.php?page=fuel-cell-vehicles>. The anode is composed of carbon particles whose surfaces are covered with platinum particles. The platinum acts as a catalyst, increasing the rate of the ionization process. The anode is porous so that hydrogen can pass through it. The cathode is also composed of carbon particles whose surfaces are covered with platinum particles. The platinum in the cathode acts as a catalyst, increasing the rate of the reaction. The cathode is porous so that oxygen can pass through it.

How do PEM fuel cells work? The hydrogen (H_2) is channeled to the anode where the platinum catalyst particles separate the hydrogen's electrons from the protons. The membrane allows the protons to pass through to the cathode, but not the electrons. At the cathode, oxygen (O_2) and electrons flowing in an external circuit combine with the hydrogen H^+ ions diffusing from the anode to form water (H_2O) and generate heat. The electrons flow in an external circuit. The flow of the electrons makes a current.

Since the main reaction takes place in the cathode catalyst layer it is worth examining its structure. A microscopic view inside the catalyst of the cathode side provides us with a picture like that in Figure 2, which shows a carbon particle (the ellipsoid) imbedded in an ionomer fiber (the rectangular plate). The remaining space is air. The reaction takes place where the ionomer, carbon and air meet (triple phase boundary), namely on the middle circle of the carbon particle. Due to symmetry of the reaction in the z direction, we consider a two dimensional domain at the point "O", in the plane π . The plane π is orthogonal to the triple phase boundary between carbon, ionomer and air. This domain is called a triple phase boundary domain because the point "O" contacts three different mediums: carbon, ionomer and air.

We will provide a mathematical model that describes the reaction kinetics near the triple phase point "O". The mathematical model represents diffusion and reaction phenomena of the chemical components in an under-saturated region. It is modeled

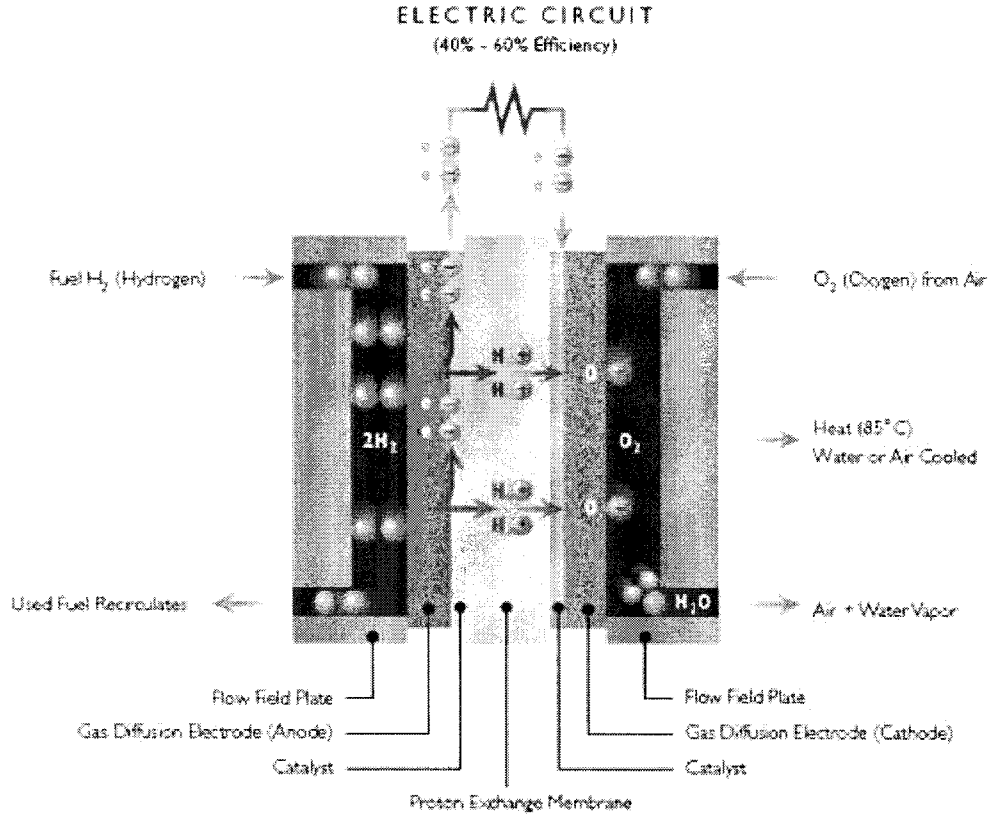


Figure 1: Hydrogen Fuel Cell. Picture is taken from <http://www.hartisee.com/index.php?page=fuel-cell-vehicles>.

under the assumption that the reduction reaction inside the catalyst layer can only take place near the contact point of the three regions that are air pores, carbon and ionomer portion, see [3]. The model we consider is introduced to approximate a model with a thin layer of water at the air pore/carbon particle interface, see [14]. The model of the reaction kinetics near the triple phase point “O” is given by a system of PDEs coupling six variables u , p , φ , v , w , and q as follows

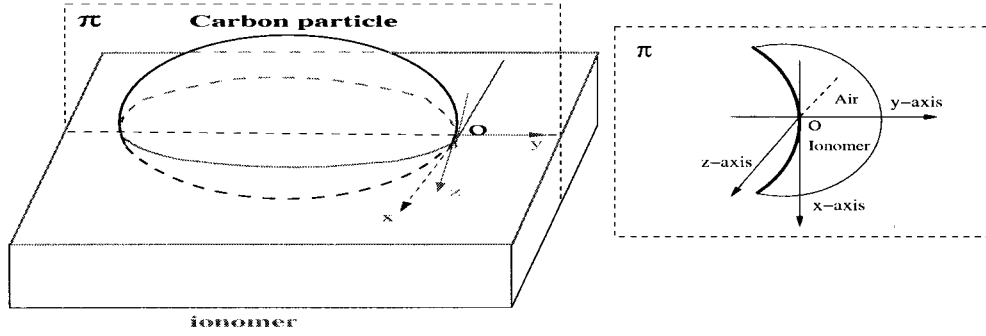


Figure 2: Triple phase boundary in catalyst layer.

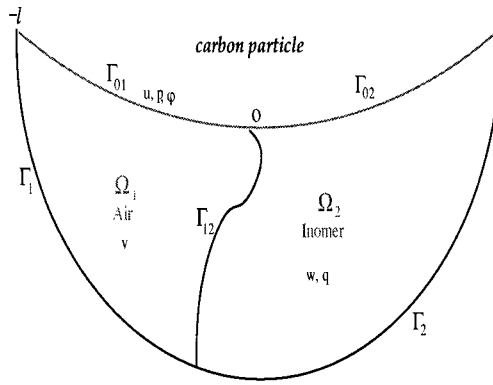


Figure 3: Triple phase boundary domain.

$$-(uu_s)_s + du - av(\bar{w} - u)^+ = f_u \quad \text{on } \Gamma_{01}, \quad (1)$$

$$-(ue^{p-\varphi}p_s)_s = -f_p \quad \text{on } \Gamma_{01}, \quad (2)$$

$$-\varphi_{ss} = f_\varphi \quad \text{on } \Gamma_{01}, \quad (3)$$

$$-\Delta v = 0 \quad \text{in } \Omega_1, \quad (4)$$

$$-\nabla \cdot ((w + |\nabla w|)\nabla w) = 0 \quad \text{in } \Omega_2, \quad (5)$$

$$-\nabla \cdot ((wc_+(w) + |\nabla q|)\nabla q) = 0 \quad \text{in } \Omega_2, \quad (6)$$

equipped with the following boundary conditions

$$(uu_s)(-l) = 0, \quad u(0) = w(0), \quad (7)$$

$$(ue^{p-\varphi}p_s)(-l) = 0, \quad p(0) = q(0), \quad (8)$$

$$\varphi_s(-l) = 0, \quad \varphi(0) = q(0) - \ln(c_+(w(0))), \quad (9)$$

$$v = g_v \quad \text{on } \Gamma_1, \quad (10)$$

$$\partial_\nu v = du - av(\bar{w} - u)^+ \quad \text{on } \Gamma_{01}, \quad (11)$$

$$w = \mathbf{i}(v) \quad \text{on } \Gamma_{12}, \quad (12)$$

$$(w + |\nabla w|)\partial_\nu w = -uu_s\delta_0 + h_w\chi_{\Gamma_{02}} + \partial_\nu v\chi_{\Gamma_{12}} \quad \text{on } \partial\Omega_2 \setminus \Gamma_2, \quad (13)$$

$$(wc_+(w) + |\nabla q|)\partial_\nu q = -ue^{p-\varphi}p_s\delta_0 - h_q\chi_{\Gamma_{02}} + 0\chi_{\Gamma_{12}} \quad \text{on } \partial\Omega_2 \setminus \Gamma_2, \quad (14)$$

$$w = g_w \quad \text{on } \Gamma_2, \quad (15)$$

$$q = g_q \quad \text{on } \Gamma_2. \quad (16)$$

Here, Ω_1, Ω_2 are open, simply connected and bounded sets with Lipschitz boundary, $\Gamma_{01}, \Gamma_{02}, \Gamma_1, \Gamma_2$ are connected parts of the boundaries as presented in Figure 3, $\Omega := \Omega_1 \cup \Gamma_{12} \cup \Omega_2$, $\Gamma := \Gamma_1 \cup \Gamma_2$ and ν the outward unit normal vector to $\partial\Omega$ or to $\partial\Omega_2$. Also

$$c_+ := c_+(w) = -\frac{1}{2}k_+w + \left(\frac{1}{4}(k_+w)^2 + k_+w\right)^{\frac{1}{2}} \quad \text{in } \Omega_2, \quad (17)$$

for some positive constant k_+ , and the reaction terms are given by

$$f_i = k_i(c_o - e^{-p})e^{p-5\varphi}, \quad i = u, p,$$

$$f_\varphi = k_\varphi e^{p-\varphi},$$

$$h_j = k_j(c_o - e^{-q})\sqrt{c_+(w)}e^{5q}, \quad j = w, q.$$

The physical parameters k_i for $i = u, p, \varphi$, and $j = w, q$ together with a, d, \bar{w}, k_+ and c_o are positive constants (positive quantities), here \bar{w} is the capacity of medium to contain water. Also the functions g_v, g_w and g_q (reference values for v, w and q) are nonnegative. These functions together with \mathbf{i} (represents equilibrium between vapor in Ω_1 and water in Ω_2) are assumed to be sufficiently smooth functions (to be

specified). δ_0 is the Dirac measure with support at the origin and $(\cdot)^+$ means the positive part of (\cdot) .

The variable u represents the water concentration attached to hydronium ions and defined on Γ_{01} , which approximates a thin layer. The variable v represents the concentration of vapor, defined in Ω_1 , which represents the air phase. The variable w is the concentration of water attached to hydronium ions and defined in Ω_2 , which is porous domain. The function p defined on Γ_{01} , and the function q defined in Ω_2 , are functions of φ . The function φ represents potential and c_+ represents the concentration of hydronium ions. In fact, we have

$$p = \varphi + \ln c_+ \text{ on } \Gamma_{01}, \quad q = \varphi + \ln(c_+(w)) \text{ in } \Omega_2,$$

so

$$c_+ := e^{p-\varphi} \chi_{\Gamma_{01}} + e^{q-\varphi} \chi_{\Omega_2}.$$

Here a change of variables is done on the original thin layer model (the thin layer model is given in φ and c_+ , see Appendix A). The following table helps in reading the boundary conditions (7)-(16).

Description of boundary conditions		
Variable(s)	Boundary condition	Description
u, v	$\partial_\nu v = du - av(\bar{w} - u)^+$	Adsorption-desorption phenomena on Γ_{01} .
u, w	$u(0) = w(0)$ $(w + \nabla w)\partial_\nu w(0) = -uu_s\delta_0$	Continuity at the origin, O . Continuity of fluxes at the origin. ¹
v, w	$w = i(v)$ $\partial_\nu v = (w + \nabla w)\partial_\nu w$	Jump discontinuity on Γ_{12} . Continuity of fluxes on Γ_{12} .
p, q	$p(0) = q(0)$ $(wc_+(w) + \nabla q)\partial_\nu q(0) = -ue^{p-\varphi}p_s\delta_0$	Continuity at the origin. Continuity of fluxes at the origin.
φ	$\varphi(0) = q(0) - \ln(c_+(w(0)))$	Continuity at the origin.

¹Note that there is no meaning to write the flux at the origin equal to a Dirac measure because the Dirac measure is defined on an interval containing the origin. However, we used this notation in the table to explain the coupling at the origin.

It is important to point out that the model (1)-(16) is an approximation of a model with a thin layer, see Figure 4, and a modified version of a thin layer model (see Appendix A and Appendix B). The thin layer model for water concentration w has only one equation in both thin layer and Ω_2 . The boundary Γ_{01} in our model approximates this thin layer and $u(s) := \frac{1}{\delta} \int_{-\delta}^0 w(s, \nu) d\nu$, where s varies on the curve Γ_{01} , and represents the arc length from the point “O” to the point (x, y) on Γ_{01} and ν is unit normal to Γ_{01} . δ is the thickness of the thin layer. In fact, this is not the only modification that we did on the thin layer model. The terms $|\nabla w|$ respectively $|\nabla q|$ in (5) and (6) are added to the diffusion coefficients of w and q equations to regularize the solution for the two equations near the origin. Note that without this modification of diffusion coefficients the solutions are singular. This is due to the existence of Dirac measures supported at the origin in (13) and (14), which leads to a logarithmic solution for w^2 and q^2 near “O”, see Page 148, [23]. The gradient terms $|\nabla w|$ and $|\nabla q|$ in (5) and (6) are added to regularize the solution around the origin to a continuous solution as we will see in the next chapters.

We will also provide some numerical computations to show that our model approximates well the thin layer model.

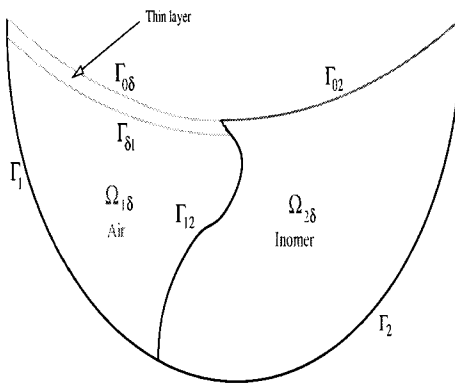


Figure 4: Domain with thin layer on air pore-carbon particle interface.

Remark 0.1.1. To make the work easier we assumed c_o , that is the concentration of oxygen, constant in Ω_1 and Ω_2 . This assumption agrees with the numerical results in Section 3, [3].

0.2 Purpose of the work. Organization of the thesis.

The purpose of this work is to prove the existence of a positive bounded weak solution for the system (1)-(16), which proves the validity of the mathematical model it represents. Next, to find the solution numerically and compare it with the numerical solution for the thin layer model. Note that having a numerical solution for the mathematical model helps in approximating the current in the fuel cell. In fact, the current (I) is defined in the following way

$$I := \int_{\Gamma_{01} \cup \Gamma_{02}} S_c ds$$

where

$$S_c = \begin{cases} f_i/k_i & \text{on } \Gamma_{01}. \\ h_j/k_j & \text{on } \Gamma_{02}. \end{cases}$$

for $i = u, p$ and for $j = w, q$, see [3].

The difficulties that we will deal with to reach our objective are multiple. Namely, the nonlinearity of the equations and boundary conditions, the boundary conditions' singularities of w and q at the origin due to the Dirac measure, the boundary conditions being of mixed type, that is Dirichlet, Neumann and Robin boundary conditions, and a large number of variables.

In order to concentrate on the structure of (1)-(16) and to better understand it, we decided to break the system into simplified, yet significative, problems beginning with a two variable system involving only u in Γ_{01} and w in Ω_2 , then a three variable system involving u in Γ_{01} , v in Ω_1 and w in Ω_2 , and finally the full model (1)-(16). During this work considering a smaller number of variables means the rest of variables are considered given. In conclusion, the existence of a positive bounded weak solution for the full model (1)-(16) is proved.

The work is divided into three chapters as follows:

In chapter one, a simplified version of the system (1)-(16) is considered, that is a system of PDEs in two variables u in Γ_{01} and w in Ω_2 . By fixing certain terms we transform the nonlinear PDEs system to a fixed point equation, the weak form of this new system (system with fixed terms) has a variational problem. We prove several L^∞ , H^1 and $W^{1,p}$ a priori estimates and using the Schauder fixed point theorem we prove the existence of a

positive bounded weak solution. Next we provide some numerical computations to validate our model and to compare our model with the thin layer model.

The importance of this model is that it explains the structure of the coupling around the triple phase point “O”, while it has a simple form (the equations (7) and (13)).

In chapter two, another simplified version with three variables u in Γ_{01} , v in Ω_1 and w in Ω_2 is considered (water-vapor system). Using the same approach as in chapter one, the existence of a positive bounded weak solution is proved, and numerical computations are provided.

The importance of this model is that, in addition to the coupling at “O”, it includes the coupling along air/ionomer interface and the air/carbon interface (the equations (12), (13) and (11)), which is given by mixed boundary conditions and describes the water/vapor adsorption-desorption equilibrium.

In chapter three, the same approach used for the simplified systems, that is the fixed point approach, is applied on the full model (1)-(16) and the existence of positive bounded solution is proved.

Note that in addition to the key difficulties in chapters one and two, the model of chapter three includes a nonlinear coupling at the origin (9) which increases the number of PDEs.

Remark 0.2.1. *Note that through out this thesis all the proofs introduced are done by us.*

Chapter 1

Water only model.

In this chapter a simplified version of (1)-(16) is considered, that is a model containing only two variables representing the concentration of water in two different mediums, namely u and w . The other variables in (1)-(16) are considered given. The fixed point approach is used to prove the existence of a positive bounded weak solution for this simple model, which proves mathematically the validity of the model. Next, some numerical computations are provided to approximate the solution (u, w) . Also, we will compare the numerical solution for this simplified system with the numerical solution for a corresponding simplified thin layer model.

1.1 Introduction. Main result.

In this section we will consider a system of PDEs coupling two variables u and w as follows.

$$\begin{cases} -(uu_s)_s + du - a(\bar{w} - u)^+ = f & \text{on } \Sigma, \\ -\nabla \cdot ((w + |\nabla w|)\nabla w) = 0 & \text{in } \Omega, \end{cases} \quad (18)$$

equipped with the following boundary conditions

$$\begin{cases} (uu_s)(-l) = 0, \quad u(0) = w(0), \\ (w + |\nabla w|)\partial_\nu w = -uu_s\delta_0 + h & \text{on } \partial\Omega \setminus \Gamma, \\ w = g & \text{on } \Gamma. \end{cases} \quad (19)$$

Here, Ω , Σ are open, connected, bounded sets and Ω with Lipschitz boundary. Γ is a connected part of the boundary, as presented in Figure 5, ν the outward unit normal vector to $\partial\Omega$.

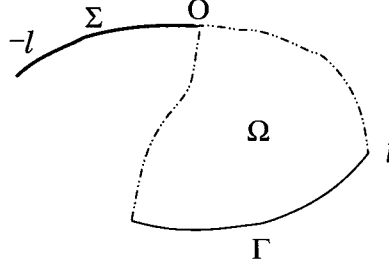


Figure 5: Triple phase boundary domain.

Moreover, the physical parameters a , d , \bar{w} are positive (positive quantities in physics). Also g and h are sufficiently smooth functions (to be determined later), δ_0 is the Dirac measure with support at the origin and $(\cdot)^+$ means positive part of (\cdot) .

The problem (18)-(19) represents a simplified model of (1)-(16). The variable u represents the water concentration and is defined on Σ , which approximates a thin layer. The variable w is the concentration of water and is defined in Ω , which is the porous domain. The other variables in (1)-(16) are considered given and absorbed by the functions f , h and g and by the physical parameters a and d .

The boundary conditions (19) are of the mixed type Dirichlet and Neumann with Dirac measure supported at the origin. Note also that u is coupled with w by continuity at the origin and w is coupled with u by the singular flux term $-uu_s\delta_0$ (note that nonlinear elliptic and parabolic equations with singular Dirichlet or Neumann boundary conditions have been considered in literature, see for example [4], [10], [21]).

Assuming that (18)-(19) has a smooth solution (u, w) , if we set

$$\begin{aligned} Y &= (H^1(\Sigma) \times W^{1,3}(\Omega)) \cap \{(u, w) : u(0) = w(0)\}, \\ Y_g &= Y \cap \{(u, w) : w = g \text{ on } \Gamma\}, \\ Y_0 &= Y \cap \{(\psi_u, \psi_w) : \psi_w = 0 \text{ on } \Gamma\}, \end{aligned} \tag{20}$$

and by multiplying the equations of (18) respectively by ψ_u and ψ_w , $(\psi_u, \psi_w) \in Y_0$, integrating by parts, and using (19), then adding the results we get

$$\int_{\Sigma} (uu'\psi'_u + (du - a(\bar{w} - u)^+ - f)\psi_u) - \int_{\partial\Omega \setminus \Gamma} h\psi_w + \int_{\Omega} (w + |\nabla w|)(\nabla w \cdot \nabla \psi_w) = 0, \quad \forall (\psi_u, \psi_w) \in Y_0. \quad (21)$$

Here $u' := \frac{du}{ds}$ and $\psi'_u := \frac{d\psi_u}{ds}$.

Definition 1.1.1. We say $(u, w) \in Y_g$ is a weak solution of (18)-(19) if (u, w) satisfies (21).

Remark 1.1.2. Note that if (u, w) is a weak solution of (18)-(19), then it is clear that by taking $(\psi_u, \psi_w) \in Y_0$ with compact support we find easily that (u, w) satisfies (18) in the sense of distributions.

If (u, w) is smooth enough, for example if u , resp. w , is a $H^2(\Sigma)$, resp. $W^{2,3}(\Omega)$, function, then u , resp. w satisfies (18) in $L^2(\Sigma)$, resp., $L^3(\Omega)$ sense. Furthermore, by taking $\psi_u(-l)$ and $\psi_w|_{\partial\Omega \setminus \Gamma}$ arbitrarily we find that (u, w) satisfies also (19).

Indeed, let $\psi_u \in H^1(\Sigma) \cap \{\psi_u(0) = 0\}$ and $u \in H^2(\Sigma)$. Next using $(\psi_u, 0) \in Y_0$ with (21) and integrating by parts imply

$$u(-l)u'(-l)\psi_u(-l) + \int_{\Sigma} (-(uu')' + (du - a(\bar{w} - u)^+ - f))\psi_u = 0.$$

Now, with ψ_u with compact support, we have

$$(uu')' = du - a(\bar{w} - u)^+ - f, \text{ in } L^2(\Sigma) \text{ sense,} \quad (22)$$

which implies

$$u(-l)u'(-l)\psi_u(-l) = 0,$$

then from arbitrariness of $\psi_u(-l)$ we have $uu'(-l) = 0$, see Lemma 3.9-3, [19].

In the same way, we have

$$\nabla \cdot ((w + |\nabla w|)\nabla w) = 0, \quad (23)$$

holds in $L^3(\Omega)$ sense.

Now let $(\psi_u, \psi_w) \in Y_0$. Note that with this choice of test function u is satisfying

$$-u(0)u'(0)\psi_u(0) + \int_{\Sigma} uu'\psi'_u + (N(u) - f)\psi_u = 0. \quad (24)$$

Now subtract this result from (21), we get

$$u(0)u'(0)\psi_u(0) - \int_{\partial\Omega \setminus \Gamma} h\psi_w + \int_{\Omega} (w + |\nabla w|)(\nabla w \cdot \nabla \psi_w) = 0, \quad \forall (\psi_u, \psi_w) \in Y_0. \quad (25)$$

However, $\psi_w(0) = \psi_u(0)$, which implies

$$u(0)u'(0)\psi_w(0) - \int_{\partial\Omega\setminus\Gamma} h\psi_w + \int_{\Omega} (w + |\nabla w|)(\nabla w \cdot \nabla \psi_w) = 0, \quad \forall (\psi_u, \psi_w) \in Y_0. \quad (26)$$

Also with $(\psi_u, \psi_w) \in Y_0$ we have the following weak form for (23)

$$- \int_{\partial\Omega\setminus\Gamma} (w + |\nabla w|)\partial_\nu w \psi_w + \int_{\Omega} (w + |\nabla w|)(\nabla w \cdot \nabla \psi_w) = 0, \quad \forall (\psi_u, \psi_w) \in Y_0. \quad (27)$$

Now subtracting (27) from (26) implies

$$u(0)u'(0)\psi_w(0) + \int_{\partial\Omega\setminus\Gamma} ((w + |\nabla w|)\partial_\nu w - h)\psi_w = 0.$$

which can be written as

$$\int_{\partial\Omega\setminus\Gamma} ((w + |\nabla w|)\partial_\nu w - h + u(0)u'(0)\delta_0)\psi_w = 0,$$

which implies

$$(w + |\nabla w|)\partial_\nu w = h - uu'\delta_0 \text{ on } \partial\Omega\setminus\Gamma,$$

because ψ_w is an arbitrarily function.

Note that the boundary condition $w = g$ on Γ is included in the space Y_g . \square

The main purpose of this chapter is to prove that (18)-(19) has positive, bounded weak solutions. Namely, we will prove the following theorem.

Theorem 1.1.3. *Assume that the constants and functions in (21) satisfy*

$$a, d, \bar{w} > 0, \quad (28)$$

$$0 \leq f \in L^\infty(\Sigma), \quad 0 \leq h \in L^{\frac{3}{2}}(\partial\Omega\setminus\Gamma), \quad (29)$$

$$0 < \inf_{\Gamma} g, \quad g = Tr(G)|_{\Gamma}, \quad G \in W_g^{1,3}(\Omega), \quad (30)$$

then

(i) (21) has at least one solution $(u, w) \in Y_g$. Moreover, (u, w) satisfies:

$$u_m \leq u \leq u^m, \quad (31)$$

$$u_m \leq w \leq w^m, \quad (32)$$

where u_m , u^m , and w^m are constants depending only on the given data and the domain.

(ii) Assume $\max\{\frac{1}{d}\|f\|_{L^\infty(\Sigma)}, \sup_{\Gamma} g\} < \bar{w}$. If a , d , $\|f\|_{L^1(\Sigma)}$ and $\|h\|_{L^{\frac{3}{2}}(\partial\Omega\setminus\Gamma)}$ are small enough, then u and w are “physically meaningful” solutions in the sense that

$$u \leq \bar{w} \text{ on } \Sigma, \quad w \leq \bar{w} \text{ in } \Omega. \quad (33)$$

Here $Tr(G)$ is the trace of the function G and $W_g^{1,3}(\Omega) := W^{1,3}(\Omega) \cap \{w|_{\Gamma} = g\}$.

Remark 1.1.4. The constant \bar{w} represents a “capacity” of the medium to absorb/contain water. A solution (u, w) is physically meaningful if u and w are smaller than \bar{w} . The conditions in (ii), physically, are translated as a “control” of water production (by reducing the reaction, the terms d , f and h , or by reducing adsorption, the term a). The conditions of Theorem 1.1.3 emphasize the importance of certain parameters/data of the system (18)-(19), in particular the parameters a and d , which are found experimentally. The theorem imposes conditions on these parameters in order to find physically meaningful solutions. \square

To prove Theorem 1.1.3 we use a fixed point approach. We transform the existence of a positive bounded weak solution problem for (21) to a fixed point problem for an operator S (to be defined). After fixing some variables in (21) it will be an Euler-Lagrange equation for a variational problem. An operator T that maps the fixed variables to the solution of this Euler-Lagrange equation is introduced. Next, we prove that T is well defined by proving that the variational problem has a unique solution. The operator S defined as a composition of T with imbedding map and projection map satisfies conditions of the well known Schauder fixed point theorem (to be stated), so it has a fixed point, which satisfies (21).

1.2 A fixed point approach.

Throughout this chapter we will assume that (28)-(30) hold. Let us start by introducing

$$X := C^0(\bar{\Sigma}) \times C^0(\bar{\Omega}). \quad (34)$$

For $(\hat{u}, \hat{w}) \in X$, consider $(u, w) \in Y_g$ solution (whenever it exists) of

$$\int_{\Sigma} \hat{u}(u' \psi'_u) + (N(u) - f)\psi_u - \int_{\partial\Omega \setminus \Gamma} h\psi_w + \int_{\Omega} (\hat{w} + |\nabla w|)(\nabla w \cdot \nabla \psi_w) = 0, \quad \forall (\psi_u, \psi_w) \in Y_0. \quad (35)$$

Here

$$N(u) := du - a(\bar{w} - u)^+. \quad (36)$$

For given (\hat{u}, \hat{w}) the solution (u, w) of (35) is different from the solution (u, w) of (21). However, if we consider the map:

$$\begin{aligned} T : B \subset X &\rightarrow C \subset Y \subset X \\ (\hat{u}, \hat{w}) &\rightarrow (u, w) = \text{solution of (35)} \end{aligned} \quad (37)$$

with B and C subsets to be defined in such way that $T(B) \subset C$. Next, any fixed point of T is a solution of (21). Note that $Y \subset X$ follows from the imbedding theorem [1].

Remark 1.2.1. Note that with this choice of fixing, the new weak form (35) is the Euler-Lagrange equation for a variational problem (to be shown later). \square

In the following remark we clarify what are the norms used with the spaces X and Y .

Remark 1.2.2.

(i) The spaces X and Y are equipped with the norms

$$\begin{aligned} \|(\hat{u}, \hat{w})\|_X &:= \|\hat{u}\|_{C^0(\bar{\Sigma})} + \|\hat{w}\|_{C^0(\bar{\Omega})}, \\ \|(u, w)\|_Y &:= \|u\|_{H^1(\Sigma)} + \|w\|_{W^{1,3}(\Omega)}. \end{aligned}$$

Note that equipped with these norms, both X and Y are Banach spaces and Y is reflexive (see Theorem III.16, [5]).

(ii) Note that the space X is chosen to have Y compactly imbedded in X (see [1]). So the choice of X is not unique. For example one can introduce $X := C^0(\bar{\Sigma}) \times H^{\frac{1}{2}+\epsilon}(\Omega)$ for $0 < \epsilon < \frac{1}{2}$, and still have Y compactly imbedded in X (see [1]), and with this new space X , the work can be repeated in a similar way. \square

1.2.1 A priori estimates.

In order to prove that T is well defined, a certain number of a priori estimates are needed. In particular we prove positivity and boundedness of (u, w) solution for (35). This will be done under some assumptions on (\hat{u}, \hat{w}) , which leads to definition of the sets B and C .

In the following lemma we will prove that the function $N(u)$ defined by (36) is a monotone increasing function. The monotonicity of N will help in proving that T is well defined as we will see.

Lemma 1.2.3. *The function $u \mapsto N(u)$ in (36) is increasing.*

Proof. The function $u \mapsto -(\bar{w} - u)^+$ is decreasing, therefore $u \mapsto -N(u)$ is increasing. \square

Let us begin by proving that solution of (35) is positive whenever the fixed variables are positive.

Proposition 1.2.4. *Assume (28)-(30) hold. Let $(u, w) \in Y_g$ be a solution of (35) corresponding to any $(\hat{u}, \hat{w}) \in X$. If $\hat{u} \geq 0$ and $\hat{w} \geq 0$ then:*

(i) $u \geq 0$ on Σ .

(ii) $w \geq 0$ in Ω .

Proof. Let u^- , respectively w^- , denote the negative part of u , respectively w . Note that $u^-(0) = w^-(0)$ and $w^- = 0$ on Γ because $\inf_{\Gamma} g > 0$. From Theorem 7.6, [15] and Proposition 3.5, [16], we have $(u^-, w^-) \in Y_0$ and

$$(u^-)' = \chi_{\{u \leq 0\}} u', \quad \nabla w^- = \chi_{\{w \leq 0\}} \nabla w.$$

Taking $(\psi_u, \psi_w) := (u^-, w^-)$ in (35) we get

$$\int_{\Sigma} \hat{u} ((u^-)')^2 + (N(u) - f) u^- - \int_{\partial\Omega \setminus \Gamma} h w^- + \int_{\Omega} (\hat{w} + |\nabla w|) |\nabla w^-|^2 = 0.$$

Note that from

$$\begin{aligned} N(u) u^- &= d u u^- + -a(\bar{w} - u)^+ u^- \geq d |u^-|^2 \geq 0, \\ f u^-, h u^- &\leq 0, \end{aligned}$$

it follows that

$$\int_{\Sigma} d |u^-|^2 + \int_{\Omega} |\nabla w^-|^3 = 0,$$

which implies $u^- = 0$ a.e. on Σ , so $u \geq 0$ a.e. on Σ . Also $\nabla w^- = 0$ a.e. in Ω , which implies that $w^- = K$ a.e. However, $w^-|_{\Gamma} = 0$ because $\inf_{\Gamma} g > 0$. Therefore, $w \geq 0$ a.e. in Ω . Since $Y \subset X$ (imbedding theorem), then the positivity holds everywhere. \square

The solution (u, w) depends strongly on the term $N(u)$ given by (36). It turns out that the solution of $N(z) = 0$ plays a key role to estimate u and w from below.

Note that

$$\begin{aligned} N : [0, \infty) &\rightarrow \mathbb{R} \\ z &\rightarrow N(z) := dz - a(\bar{w} - z)^+. \end{aligned} \tag{38}$$

In the following remark we prove the existence of positive root for N .

Remark 1.2.5. Set

$$m := \frac{a}{a+d} \bar{w} \in (0, \bar{w}) \tag{39}$$

Note that $N(m) = 0$. Moreover $N(z) < 0$ for $z \in [0, m)$ and $N(z) > 0$ for $z > m$. \square

Define

$$u_m := \min\{\inf_{\Gamma} g, m\}, \quad (40)$$

and note that as $u_m \leq m$, then Remark 1.2.5 implies

$$N(u_m) \leq 0. \quad (41)$$

In the following proposition we will show that if a solution (u, w) for (35) exists and the fixed variables are well chosen, then u and w are bounded below by u_m .

Proposition 1.2.6. *Assume (28)-(30) hold. Let $(\hat{u}, \hat{w}) \in X$ with $\hat{u} \geq u_m$, $\hat{w} \geq 0$ and assume $(u, w) \in Y_g$ solves (35), then $u \geq u_m$, $w \geq u_m$.*

Proof. Assume by absurd that the proposition does not hold, namely assume that $u < u_m$ or $w < u_m$ on a set of nonzero measure. Define $(\psi_u, \psi_w) := ((u - u_m)^-, (w - u_m)^-)$, so we have $(\psi_u, \psi_w) \neq (0, 0)$ on a set of nonzero measure. Since $(u, w) \in Y_g$ and $u_m \leq \inf_{\Gamma} g$ (equation (40)), then we have $(\psi_u, \psi_w) \in Y_0$ (see Theorem 7.6, [15] and Proposition 3.5, [16]). With this choice of (ψ_u, ψ_w) equation (35) gives

$$0 = \int_{\Sigma} \hat{u}(\psi'_u)^2 + (N(u) - f)\psi_u - \int_{\partial\Omega \setminus \Gamma} h\psi_w + \int_{\Omega} (\hat{w} + |\nabla w|)|\nabla \psi_w|^2.$$

Since $N(u)\psi_u \geq N(u_m)\psi_u = 0$, $f\psi_u \leq 0$ and $h\psi_w \leq 0$, then it follows that

$$\int_{\Sigma} \hat{u}|\psi'_u|^2 + \int_{\Omega} |\nabla \psi_w|^3 = 0,$$

which implies $\psi'_u = 0$ a.e. on Σ and $\nabla \psi_w = 0$ a.e. in Ω . However, $\psi_w|_{\Gamma} = 0$ and $\psi_u(0) = \psi_w(0)$. Therefore, Poincaré's inequality implies $\psi_u = 0$ a.e. on Σ and $\psi_w = 0$ a.e. Ω , which is impossible. By this contradiction, we reached the end of the proof. \square

Let $u^m > u_m$ be a constant (to be determined later), and set

$$\begin{aligned} B &= X \cap \{(\hat{u}, \hat{w}) : u_m \leq \hat{u} \leq u^m, u_m \leq \hat{w}\}, \\ C &= Y_g \cap \{(u, w) : u_m \leq u \leq u^m, u_m \leq w\}. \end{aligned} \quad (42)$$

Remark 1.2.7. Note that $C \subset B$ and the sets B and C are convex and closed. Indeed, the convexity of the sets is trivial. For the closedness, we consider a sequence (u_n, w_n) in B or C , converging to (u, w) in the respective norm. Up to a subsequence, the sequence (u_n, w_n) converges also a.e. in $\Sigma \times \Omega$, see Theorem IV.9, [5] and Page 88, [7], this proves the closedness of B and C . \square

In the following lemma, we introduce some estimations that we will use in the next work.

Lemma 1.2.8. *If $(\hat{u}, \hat{w}) \in X$, $(u, w) \in Y_g$ and $\psi_w := (w - \sup_{\Gamma} g)^+$, then Hölder, triangle, Poincaré and Young inequalities together with trace and imbedding theorems imply*

$$\|w\|_{W^{1,3}(\Omega)} \leq K_1(\|\nabla w\|_{L^3(\Omega)} + \|G\|_{W^{1,3}(\Omega)}), \quad (43)$$

$$\int_{\partial\Omega \setminus \Gamma} h|\psi_w| \leq K_2 \|h\|_{L^{\frac{3}{2}}(\partial\Omega \setminus \Gamma)}^{\frac{3}{2}} + \frac{1}{6} \|\nabla \psi_w\|_{L^3(\Omega)}^3, \quad (44)$$

$$\int_{\partial\Omega \setminus \Gamma} h|w| \leq K_3(\|h\|_{L^{\frac{3}{2}}(\partial\Omega \setminus \Gamma)}^{\frac{3}{2}} + \|h\|_{L^{\frac{3}{2}}(\partial\Omega \setminus \Gamma)} \|G\|_{W^{1,3}(\Omega)}) + \frac{1}{6} \|\nabla w\|_{L^3(\Omega)}^3, \quad (45)$$

$$\int_{\Sigma} (a\bar{w} + |f|)|u| \leq \frac{1}{d}(a\bar{w}|\Sigma| + \|f\|_{L^2(\Sigma)})^2 + \frac{d}{4} \|u\|_{L^2(\Sigma)}^2, \quad (46)$$

$$\int_{\Omega} \hat{w}|\nabla w|^2 \leq K_4 \|\hat{w}\|_{C^0(\bar{\Omega})} \|w\|_{W^{1,3}(\Omega)}^2, \quad (47)$$

$$\int_{\Omega} |\nabla w|^3 \leq \|w\|_{W^{1,3}(\Omega)}^3, \quad (48)$$

$$\int_{\Sigma} \hat{u}(u')^2 \leq \|\hat{u}\|_{C^0(\bar{\Sigma})} \|u\|_{H^1(\Sigma)}^2, \quad (49)$$

$$\left| \int_{\Sigma} \int_0^{u(s)} N(k) dk ds \right| \leq K_5 \|u\|_{H^1(\Sigma)}^2. \quad (50)$$

Here K_i , $i = 1, \dots, 5$ are constants.

Proof. For reader convenience, we will show (44). Indeed,

$$\begin{aligned} \int_{\partial\Omega \setminus \Gamma} h|\psi_w| &\leq \|h\|_{L^{\frac{3}{2}}(\partial\Omega \setminus \Gamma)} \|\psi_w\|_{L^3(\partial\Omega \setminus \Gamma)} \\ &\leq M_1 \|h\|_{L^{\frac{3}{2}}(\partial\Omega \setminus \Gamma)} \|\psi_w\|_{W^{1,3}(\Omega)} \\ &\leq M_2 \|h\|_{L^{\frac{3}{2}}(\partial\Omega \setminus \Gamma)} \|\nabla \psi_w\|_{L^3(\Omega)} \\ &\leq \frac{2\sqrt{2}}{3} (M_2 \|h\|_{L^{\frac{3}{2}}(\partial\Omega \setminus \Gamma)})^{\frac{3}{2}} + \frac{1}{6} \|\nabla \psi_w\|_{L^3(\Omega)}^3, \end{aligned}$$

for some constants M_1 and M_2 , where we have used Hölder's inequality, trace theorem, Poincaré and Young inequalities respectively. As a result (44) holds with $K_2 := \frac{2\sqrt{2}}{3} M_2^{\frac{3}{2}}$. \square

In the following proposition we will show that, if $(\hat{u}, \hat{w}) \in B$, then u is bounded above by u^m . With this result and the one in Proposition 1.2.6, we have a stable set B , that is, if $(\hat{u}, \hat{w}) \in B$, then (u, w) solution for (35) is also in B ($C \subset B$ (compact imbeddings)).

Proposition 1.2.9. *Assume (28)-(30) hold. There exists $u^m \geq \max\{\bar{w}, \sup_{\Gamma} g, \frac{\|f\|_{L^\infty(\Sigma)}}{d}\}$ such that if $(\hat{u}, \hat{w}) \in B$ and $(u, w) \in Y_g$ solves (35) then $(u, w) \in C$.*

Proof. Let u_m be given by (40) and $(\hat{u}, \hat{w}) \in B$. By Proposition 1.2.6 we have $u_m \leq u$, $u_m \leq w$.

Claim (i): There exists $\alpha_0 \in \mathbb{R}$ such that

$$\alpha_0 \psi_u(0) = \int_{\Sigma} \hat{u}(u' \psi'_u) + (du - a(\bar{w} - u)^+ - f) \psi_u =: L(\psi_u), \quad \forall \psi_u \in H^1(\Sigma). \quad (51)$$

Indeed, from (35) we have $L(\psi_u) = 0$ for all $\psi_u \in H^1(\Sigma) \cap \{\psi_u(0) = 0\}$ because $(\psi_u, 0) \in Y_0$. Next, for arbitrarily $\psi_u \in H^1(\Sigma)$, we have

$$\begin{aligned} L(\psi_u) &= L(\psi_u(0) + (\psi_u - \psi_u(0))) = L(1)\psi_u(0) + L(\psi_u - \psi_u(0)) \\ &= L(1)\psi_u(0), \end{aligned}$$

which proves the claim with $\alpha_0 = L(1)$, see Lemma 3.9-3, [19].

Note that by classical regularity results, see [5], if $\hat{u} \in H^1(\Sigma)$, then $u \in H^2(\Sigma) \subset C^1(\bar{\Sigma})$ and then $\alpha_0 = \hat{u}(0)u'(0)$. In general, we have $(\hat{u}u')' = -f - du + a(\bar{w} - u)^+$ in $\mathcal{D}'(\Sigma)$. Since $f, u \in L^2(\Sigma)$, then $\hat{u}u' \in H^1(\Sigma)$, which from the fact that $\hat{u} \in C^0(\bar{\Sigma})$ implies $u' \in C^0(\bar{\Sigma})$. Therefore, even with $\hat{u} \in C^0(\bar{\Sigma})$, we have $\alpha_0 = \hat{u}(0)u'(0)$, which can be obtained from (51) by taking $\psi_u(s) = \max\{0, 1 + \frac{s}{\epsilon}\}$ and letting $\epsilon \rightarrow 0$.

Claim (ii): If $\alpha_0 \leq 0$, then

$$u \leq \max\{\bar{w}, \frac{\|f\|_{L^\infty(\Sigma)}}{d}\}. \quad (52)$$

Indeed, taking $\psi_u := \left(u - \max\{\bar{w}, \frac{\|f\|_{L^\infty(\Sigma)}}{d}\}\right)^+ \in H^1(\Sigma)$ (see Theorem 7.6, [15] and Proposition 3.5, [16]) in (51), then we have

$$\begin{aligned} 0 \geq \hat{u}(0)u'(0)\psi_u(0) &= \int_{\Sigma} \hat{u}|\psi'_u|^2 + (du - a(\bar{w} - u)^+ - f)\psi_u \\ &= \int_{\Sigma} \hat{u}|\psi'_u|^2 + (du - f)\psi_u \\ &\geq \int_{\Sigma} \hat{u}|\psi'_u|^2 + (du - \|f\|_{L^\infty(\Sigma)})\psi_u \\ &\geq \int_{\Sigma} \hat{u}|\psi'_u|^2 + d\psi_u^2, \end{aligned} \quad (53)$$

which implies $\psi_u = 0$ and proves the claim.

Claim (iii): If $\alpha_0 > 0$, then $w \leq z^w$, for some constant z^w to be determined.

Indeed, set $(\psi_u, \psi_w) := ((u - \sup_{\Gamma} g)^+, (w - \sup_{\Gamma} g)^+) \in Y_0$. With this choice of test function in (35), then we obtain

$$\begin{aligned} \int_{\Omega} |\nabla \psi_w|^3 &\leq \int_{\Omega} (\hat{w} + |\nabla w|) |\nabla \psi_w|^2 \\ &\leq - \int_{\Sigma} (\hat{u}(u' \psi'_u) + (N(u) - f) \psi_u) + \int_{\partial\Omega \setminus \Gamma} h \psi_w \\ &= -\hat{u}(0)u'(0)\psi_u(0) + \int_{\partial\Omega \setminus \Gamma} h \psi_w \end{aligned} \quad (54)$$

$$\begin{aligned} &< \int_{\partial\Omega \setminus \Gamma} h \psi_w \\ &\leq K_2 \|h\|_{L^{\frac{3}{2}}(\partial\Omega \setminus \Gamma)}^{\frac{3}{2}} + \frac{1}{6} \|\nabla \psi_w\|_{L^3(\Omega)}^3, \end{aligned} \quad (55)$$

where we have used (44). From (55), we have

$$\frac{5}{6} \int_{\Omega} |\nabla \psi_w|^3 < K_2 \|h\|_{L^{\frac{3}{2}}(\partial\Omega \setminus \Gamma)}^{\frac{3}{2}}. \quad (56)$$

Again Poincaré's inequality and imbedding theorems, namely $W^{1,3}(\Omega) \subset C^0(\bar{\Omega})$ (see [1]), imply

$$K_3 \|(w - \sup_{\Gamma} g)^+\|_{C^0(\bar{\Omega})}^3 = K_3 \|\psi_w\|_{C^0(\bar{\Omega})}^3 \leq K_4 \|\psi_w\|_{W^{1,3}(\Omega)}^3 \leq \frac{5}{6} \int_{\Omega} |\nabla \psi_w|^3. \quad (57)$$

for some constants K_3 and K_4 . Now, from (56) and (57), we have

$$K_3 \|(w - \sup_{\Gamma} g)^+\|_{C^0(\bar{\Omega})}^3 < K_2 \|h\|_{L^{\frac{3}{2}}(\partial\Omega \setminus \Gamma)}^{\frac{3}{2}}. \quad (58)$$

This implies

$$\begin{aligned} w &\leq \sup_{\Gamma} g + K_5 \|h\|_{L^{\frac{3}{2}}(\partial\Omega \setminus \Gamma)}^{\frac{1}{2}} \\ &:= z^w. \end{aligned} \quad (59)$$

Here $K_5 = \frac{K_2}{K_3}$, a constant depending only on data and domain.

(iii) *Claim: Set*

$$u^m = \max\{z^w, \bar{w}, \frac{\|f\|_{L^\infty(\Sigma)}}{d}\}, \quad (60)$$

then the proposition holds.

Indeed, if $\alpha_0 \leq 0$ then the Claim (ii) holds and the proposition follows from (52). If $\alpha_0 > 0$ then as $u^m \geq z^w$ the proposition follows from Claim (iii). Indeed, first, note that $(\psi_u, \psi_w) := ((u - u_m)^+, 0) \in Y_0$, then with this test function in (35), then we have

$$\begin{aligned}
0 &= \int_{\Sigma} \hat{u} |\psi'_u|^2 + (du - a(\bar{w} - u)^+ - f)\psi_u \\
&\geq \int_{\Sigma} \hat{u} |\psi'_u|^2 + (du - f)\psi_u \\
&\geq \int_{\Sigma} \hat{u} |\psi'_u|^2 + (du - \|f\|_{L^\infty(\Sigma)})\psi_u \\
&\geq \int_{\Sigma} \hat{u} |\psi'_u|^2 + d\psi_u^2,
\end{aligned} \tag{61}$$

which implies that $\psi_u = 0$ and $u \leq u^m$. \square

From (59) it is clear that we have

Corollary 1.2.10. *Assume $\max\{\sup_{\Gamma} g, \frac{\|f\|_{L^\infty(\Sigma)}}{d}\} < \bar{w}$. If*

$$\|h\|_{L^{\frac{3}{2}}(\partial\Omega \setminus \Gamma)} \text{ is small enough,} \tag{62}$$

then $u^m \leq \bar{w}$.

Proof. From (59) and (60) we have the following. If $\sup_{\Gamma} g < \bar{w}$ and $\|h\|_{L^{\frac{3}{2}}(\partial\Omega \setminus \Gamma)} \ll 1$, then $z^w \leq \bar{w}$. If in addition, we have $\frac{\|f\|_{L^\infty(\Sigma)}}{d} < \bar{w}$, then the result follows from (60). \square

1.3 Operator T is well-defined.

To prove that T is well defined, it is enough to prove that (35) has a unique positive solution. In order to prove this we will look at (35) as the Euler-Lagrange equation of a functional I . Next we will show that the functional I has a unique minimizer, which will prove that T is well-defined.

Let $(\hat{u}, \hat{w}) \in B$ and consider

$$\begin{aligned}
I(u, w) &:= \int_{\Sigma} \frac{1}{2} \hat{u} (u')^2 ds + \int_{\Omega} \frac{1}{6} (3\hat{w} + 2|\nabla w|) |\nabla w|^2 dx dy \\
&+ \int_{\Sigma} \int_0^{u(s)} (N(k) - f(s)) dk ds - \int_{\partial\Omega \setminus \Gamma} h w ds,
\end{aligned} \tag{63}$$

then consider the problem

$$\text{Find } (u, w) \in Y_g : I(u, w) = \min\{I(\psi_u, \psi_w), (\psi_u, \psi_w) \in Y_g\}. \quad (64)$$

To emphasize the dependence of I on (\hat{u}, \hat{w}) , we will also use the notation $I(u, w; \hat{u}, \hat{w})$ instead of $I(u, w)$.

We will show that I is strictly convex, Gâteaux differentiable and that (35) is the Euler-Lagrange equation of I . Next, we will show that (64) has a unique minimizer and this will imply that the operator T is well-defined.

To proceed in our work we will prove the following proposition, which contains some important properties of the functional I .

Proposition 1.3.1. *The functional $I : (u, w) \in Y_g \rightarrow \mathbb{R}$ satisfies:*

- (i) $(u, w) \mapsto I(u, w)$ is strictly convex.
- (ii) $(u, w) \mapsto I(u, w)$ is lower semicontinuous with respect to weak topology in Y .
- (iii) For all $(\hat{u}_n, \hat{w}_n) \in X$ converging to (\hat{u}, \hat{w}) in X , for all $(u_n, w_n) \in Y_g$ bounded in Y and for any $(u, w) \in Y_g$ we have

$$\lim_{n \rightarrow \infty} (I(u_n, w_n; \hat{u}_n, \hat{w}_n) - I(u_n, w_n; \hat{u}, \hat{w})) = \lim_{n \rightarrow \infty} (I(u, w; \hat{u}_n, \hat{w}_n) - I(u, w; \hat{u}, \hat{w})) = 0. \quad (65)$$

Proof. (i) Note that the terms

$$\hat{u}(s)z_1^2, \quad (3\hat{w}(x) + 2|z_2|)z_2^2,$$

are strictly convex in z_1 and z_2 respectively, being of the form $|x|^p$, $p \geq 1$, see Exercise 4.24, [13]. Also the terms

$$f(s)u, \quad h(t)w$$

are convex (linear) in u and w . It remains only to deal with the term involving N . Since $N(k)$ is increasing in k , then Theorem 4, [2], implies that the term $\int_0^{u(s)} N(k)dk$ is convex. This shows that $I(u, w)$ is strictly convex being a sum of positive convex, strictly positive convex and linear functions.

(ii) The lower semicontinuity of I follows from Theorem 2.2.1, [24] (also one can apply Theorem 4.4 [16], to prove that each term of I is lower semicontinuous). Note that for $\lambda \in \mathbb{R}$

the set $\{(u, w), I(u, w) \leq \lambda\}$ is closed in Y with strong topology because $(u, w) \mapsto I(u, w)$ is continuous in Y with respect to strong topology.

(iii) We have

$$\begin{aligned} |(I(u, w; \hat{u}_n, \hat{w}_n) - I(u, w; \hat{u}, \hat{w}))| & \\ & \leq \int_{\Sigma} \frac{1}{2} |\hat{u}_n - \hat{u}| (u')^2 + \frac{1}{2} \int_{\Omega} |\hat{w}_n - \hat{w}| |\nabla w|^2 \\ & \leq \frac{1}{2} \|\hat{u}_n - \hat{u}\|_{C^0(\bar{\Sigma})} \|u'\|_{L^2(\Gamma_{01})}^2 + \frac{1}{2} \|\hat{w}_n - \hat{w}\|_{C^0(\bar{\Omega})} \|\nabla w\|_{L^2(\Omega)}^2. \end{aligned}$$

From convergence of (\hat{u}_n, \hat{w}_n) in X , we have that the right hand side of the last inequality tends to zero. The other limit is proven similarly. \square

Now we will introduce the relation between (35) and the minimization problem (64).

Proposition 1.3.2. *The functional $(u, w) \mapsto -I(u, w)$ is Gâteaux differentiable and for $(\psi_u, \psi_w) \in Y_0$ we have*

$$\begin{aligned} I'(u, w)(\psi_u, \psi_w) &= \int_{\Sigma} \hat{u} (u' \psi'_u) + (N(u) - f) \psi_u - \int_{\partial\Omega \setminus \Gamma} h \psi_w \\ &+ \int_{\Omega} (\hat{w} + |\nabla w|) (\nabla w \cdot \nabla \psi_w). \end{aligned} \quad (66)$$

Proof. For $\theta \in \mathbb{R}$, θ small. set $i(\theta) = I(u + \theta\psi_u, w + \theta\psi_w)$. Define κ as the right hand side in (66). We have

$$\left| \frac{i(\theta) - i(0)}{\theta} - \kappa \right| \leq \int_{\Sigma} |F_1(\theta, s)| ds + \int_{\partial\Omega \setminus \Gamma} |F_2(\theta, s)| ds + \int_{\Omega} |F_3(\theta, s)| dx, \quad (67)$$

where

$$\begin{aligned} F_1(\theta, s) &= \left(\frac{\hat{u}}{2\theta} ((u' + \theta\psi'_u)^2 - (u')^2) - \hat{u} u' \psi'_u \right) \\ &+ \left(\frac{d}{2\theta} ((u + \theta\psi_u)^2 - u^2) - u \psi_u \right) - \frac{1}{\theta} (f(u + \theta\psi_u) - f u) - f \psi_u \\ &- \frac{a}{\theta} \left(\int_{u(s)}^{u(s) + \theta\psi_u} ((\bar{w} - k)^+ - (\bar{w} - u)^+) dk \right), \\ F_2(\theta, x) &= \frac{h}{\theta} ((w + \theta\psi_w) - w) - h \psi_w, \\ F_3(\theta, x) &= \frac{1}{\theta} \left(\left(\frac{1}{2} \hat{w} + \frac{1}{3} |\nabla w + \theta \nabla \psi_w| \right) |\nabla w + \theta \nabla \psi_w|^2 \right. \\ &- \left. \left(\frac{1}{2} \hat{w} + \frac{1}{3} |\nabla w| |\nabla w|^2 \right) - (\hat{w} + |\nabla w|) (\nabla w \cdot \nabla \psi_w) \right). \end{aligned}$$

Note that $|F_i(\theta, z)| \rightarrow 0$ a.e. for $i = 1, 2, 3$ respectively, as $\theta \rightarrow 0$. Indeed, in F_1 the term

$$\frac{a}{\theta} \left(\int_{u(s)}^{u(s)+\theta\psi_u} ((\bar{w} - k)^+ - (\bar{w} - u)^+) dk \right) = a\psi_u((\bar{w} - u - \tilde{\theta}\psi_u)^+ - (\bar{w} - u)^+) \rightarrow 0,$$

as $\theta \rightarrow 0$, where we have used the Mean Value Theorem for integrals, with $\tilde{\theta}(s) \in (u(s), u(s) + \theta\psi_u(s))$. For other terms in F_1 and for F_2 and F_3 it is easy to prove that limits tend to zero directly using the definition of directional derivative for each of the integrals separately. As a result

$$\left| \frac{i(\theta) - i(0)}{\theta} - \kappa \right| \rightarrow 0, \quad (68)$$

which means (66) holds. \square

Therefore, it follows

Proposition 1.3.3. *The equation (35) is the Euler-Lagrange equation of $I(u, w)$.*

Now with all the previous preparation, we can prove that T is well defined.

Theorem 1.3.4. *Assume (28)-(30) hold. For any $(\hat{u}, \hat{w}) \in B \subset X$ the equation (35) has a unique solution $(u, w) \in C \subset Y$. Thus, the operator $T : B \subset X \rightarrow C \subset Y$, $T(\hat{u}, \hat{w}) = (u, w)$ is well-defined.*

Proof. The proof is in two parts.

Uniqueness. It follows from strict convexity of I . For the sake of completeness, let us show the proof in detail. Let (u_i, w_i) , $i = 1, 2$ be two solutions of (35). Recall Lemma 1.2.3 that $N(u)$ is increasing in u . Therefore,

$$(N(u_1) - N(u_2))(u_1 - u_2) \geq 0. \quad (69)$$

Consider (35) for each of the solutions (u_i, w_i) , $i = 1, 2$, with $(\psi_u, \psi_w) := (u_1 - u_2, w_1 - w_2) \in Y_0$, and subtract them. Next we obtain

$$\begin{aligned} 0 &= \int_{\Sigma} \hat{u}(\psi'_u)^2 + (N(u_1) - N(u_2))\psi_u + \int_{\Omega} \hat{w}|\nabla\psi_w|^2 \\ &+ \int_{\Omega} (|\nabla w_1|\nabla w_1 - |\nabla w_2|\nabla w_2) \cdot \nabla\psi_w. \end{aligned} \quad (70)$$

All the factors under this integral are nonnegative. Indeed, for the terms with N this follows from (69), while for the last term in (70), it follows from the fact that $z \in \mathbb{R}^n \rightarrow -H(z) = |z|^3$

is convex and then $(\nabla H(z) - \nabla H(y)) \cdot (z - y) \geq 0$, see Theorem 4.62, [13]. Therefore, $\psi'_u = 0$ on Σ , $\nabla \psi_w = 0$ in Ω and this implies $\psi_u = \psi_w = K$ a.e. constant ($\psi_u(0) = \psi_w(0)$). As $\psi_w|_\Gamma = 0$, it follows $\psi_u = \psi_w = 0$, which means $(u_1, w_1) = (u_2, w_2)$.

Existence. As (35) cannot have more than one solution, to prove the existence of a solution of (35) it is enough to show that the problem (64) has a solution. Next, from Proposition 1.3.3 the solution of (64) is the only solution of (35).

We can use Corollary III.20, [5] (note that we are using a result similar to that in Theorem 4.6, [16], but with more complicated domain). Indeed, we have $I : C \rightarrow -\mathbb{R}$ is a convex, lower semicontinuous function (Proposition 1.3.1), with C nonempty closed convex set. Note that $I(u, w) \neq \pm\infty$ because of the estimations (45)-(50).

It remains to show that: for $(u, w) \in C$, we have $I(u, w) \rightarrow \infty$ as $\|(u, w)\|_Y \rightarrow \infty$.

We have

$$\begin{aligned}
I(u, w) &= \int_\Sigma \frac{1}{2} \hat{u}(u')^2 + \frac{1}{6} \int_\Omega (3\hat{w} + 2|\nabla w|)|\nabla w|^2 \\
&+ \int_\Sigma \int_0^{u(s)} (N(k) - f(s)) dk ds - \int_{\partial\Omega \setminus \Gamma} h w ds \\
&\geq \int_\Sigma \frac{1}{2} (u_m(u')^2 + du^2) + \int_\Omega \frac{1}{3} |\nabla w|^3 \\
&- \int_\Sigma (a\bar{w} + |f|)|u| - \int_{\partial\Omega \setminus \Gamma} |h||w| \\
&\geq K_2 \left(\int_\Sigma (u')^2 + u^2 + \int_\Omega |\nabla w|^3 \right) + K_3,
\end{aligned} \tag{71}$$

where K_2, K_3 , are positive constants independent from (u, w) , and we have used (45) and (46). Now note that having $\|w\|_{W^{1,3}(\Omega)} \rightarrow \infty$ implies $\|\nabla w\|_{L^3(\Omega)} \rightarrow \infty$ because of (43). So, $I(u, w) \rightarrow \infty$ as $\|(u, w)\|_Y \rightarrow \infty$, $(u, w) \in C$. Next, Corollary III.20, [5] implies $I(u, w)$ has a minimizer in C , which end the proof. \square

In the following remark we will prove that $I(G(0), G; \hat{u}, \hat{w})$ is bounded whenever (\hat{u}, \hat{w}) is bounded in X , see section 1.2. This will help us in proving that the operator T given by (37) takes bounded sets to bounded sets, which will be used in proving the compactness of the operator S (to be defined).

Remark 1.3.5. For $(\hat{u}, \hat{w}) \in B$ satisfying $\|(\hat{u}, \hat{w})\|_X \leq M$, we have

$$\begin{aligned}
I(G(0), G; \hat{u}, \hat{w}) &\leq \frac{1}{2}|\Sigma|d(G(0))^2 + a\bar{w}|G(0)||\Sigma| + \int_{\partial\Omega \setminus \Gamma} |h||G| \\
&\quad + \int_{\Omega} \frac{1}{2}\hat{w}|\nabla G|^2 + \frac{1}{3}|\nabla G|^3 \\
&\leq K_1(G) + \frac{1}{2}\|\hat{w}\|_{C^0(\bar{\Omega})}\|\nabla G\|_{L^2(\Omega)}^2 \\
&\leq K_1(G) + \frac{1}{2}M\|\nabla G\|_{L^2(\Omega)}^2 \\
&= K(G, M),
\end{aligned} \tag{72}$$

for some constants $K_1(G)$ and $K(G, M)$ depending only on G and M . \square

Proposition 1.3.6. Assume (28)-(30) hold. For $M, N > 0$ set

$$B_M := B \cap \{\|(\hat{u}, \hat{w})\|_X \leq M\}, \quad C_N := C \cap \{\|(u, w)\|_Y \leq N\}. \tag{74}$$

There exists $N > 0$ such that for all $(\hat{u}, \hat{w}) \in B_M$, if $(u, w) \in C$ is the solution of (35), then $(u, w) \in C_N$.

Proof. Let $(\hat{u}, \hat{w}) \in B_M$, $(u, w) \in C$ be the solution of (35). As the solution $(u, w) \in C$ is unique, then it is the minimizer for $I(u, w)$. Now, as $(G(0), G) \in Y_g$, then we have

$$I(u, w; \hat{u}, \hat{w}) \leq I(G(0), G; \hat{u}, \hat{w}).$$

From this inequality, the definition of $I(u, w)$ in (63) and (73), we have

$$\begin{aligned}
\frac{1}{2} \int_{\Sigma} u_m(u')^2 + du^2 + \frac{1}{3} \int_{\Omega} |\nabla w|^3 \\
\leq I(G(0), G; \hat{u}, \hat{w}) - \int_{\Sigma} \int_0^{u(s)} (-a(\bar{w} - k)^+ - f(s)) dk ds + \int_{\partial\Omega \setminus \Gamma} hw, \\
\leq K(G, M) + \int_{\Sigma} (a\bar{w} + |f|)|u| + \int_{\partial\Omega \setminus \Gamma} h|w| \\
\leq K(G, M) + \frac{1}{d}(a\bar{w}|\Sigma| + \|f\|_{L^2(\Sigma)})^2 + \frac{d}{4}\|u\|_{L^2(\Sigma)}^2 \\
+ K_3(\|h\|_{L^{\frac{3}{2}}(\partial\Omega \setminus \Gamma)}^{\frac{3}{2}} + \|h\|_{L^{\frac{3}{2}}(\partial\Omega \setminus \Gamma)}\|G\|_{W^{1,3}(\Omega)} + \frac{1}{6}\|\nabla w\|_{L^3(\Omega)}^3),
\end{aligned} \tag{75}$$

where we have used (45) and (46). Rearranging (75) implies

$$\begin{aligned}
\int_{\Sigma} \frac{1}{2}u_m(u')^2 + \frac{1}{4}du^2 + \frac{1}{6} \int_{\Omega} |\nabla w|^3 &\leq K_2(G) + \frac{1}{d}(a\bar{w}|\Sigma| + \|f\|_{L^2(\Sigma)})^2 \\
&\quad + K_3(\|h\|_{L^{\frac{3}{2}}(\partial\Omega \setminus \Gamma)}^{\frac{3}{2}} + \|h\|_{L^{\frac{3}{2}}(\partial\Omega \setminus \Gamma)}\|G\|_{W^{1,3}(\Omega)}) \\
&=: L_1.
\end{aligned}$$

Now introducing $L_2 := L_1 / \min\{\frac{1}{2}u_m, \frac{d}{4}, \frac{1}{6}\}$, then we have

$$\|u\|_{H^1(\Sigma)}^2 + \|\nabla w\|_{L^3(\Omega)}^3 \leq L_2. \quad (76)$$

Using (43) and (76), we have that

$$\|w\|_{W^{1,3}(\Omega)} \leq K_1(L_2^{\frac{1}{3}} + \|G\|_{W^{1,3}(\Omega)}).$$

As a result

$$\begin{aligned} \|u\|_{H^1(\Sigma)} + \|w\|_{W^{1,3}(\Omega)} &\leq \sqrt{L_2} + K_1(L_2^{\frac{1}{3}} + \|G\|_{W^{1,3}(\Omega)}) \\ &=: N. \end{aligned}$$

By this we reach the end of the proof. \square

1.4 Existence of a fixed point.

We need the following well known *Schauder Fixed Point Theorem*, see Page 280, [15].

Theorem 1.4.1. *Let X be a Banach space and $S : X \rightarrow X$ compact, continuous and assume that the set*

$$\Lambda := \{x \in X : x = \lambda Sx, \lambda \in [0, 1]\} \quad (77)$$

is bounded in X , then S has a fixed point.

To apply this theorem, we consider $S = J \circ T \circ \Pi$

$$\begin{aligned} S : X &\xrightarrow{\Pi} B \xrightarrow{T} C \xrightarrow{J} X, \\ (\tilde{u}, \tilde{w}) &\rightarrow (\hat{u}, \hat{w}) \rightarrow (u, w) \rightarrow (u, w), \end{aligned} \quad (78)$$

where Π is a projection operator and J is the imbedding map. The operator Π is defined as follows. Consider

$$\begin{aligned} \Pi_u &: \mathbb{R}^+ \rightarrow [u_m, u^m] \\ s &\rightarrow \Pi_u(s) := \min\{u^m, \max\{u_m, s\}\}, \\ \Pi_w &: \mathbb{R}^+ \rightarrow [u_m, \infty) \\ s &\rightarrow \Pi_w(s) := \max\{s, u_m\}, \end{aligned}$$

and then set

$$\begin{aligned} \Pi : X &\longrightarrow B \\ (\tilde{u}, \tilde{w}) &\rightarrow (\hat{u}, \hat{w}) = \Pi(\tilde{u}, \tilde{w}) := (\Pi_u(\tilde{u}), \Pi_w(\tilde{w})). \end{aligned} \quad (79)$$

Remark 1.4.2. Note that $\Pi(X) \subset B = X \cap \{(\hat{u}, \hat{w}) : u_m \leq \hat{u} \leq u^m, u_m \leq \hat{w}\}$. We will show this in two steps. In the first step we will show that $\Pi(X) \subset X$ and in the second step we will prove that $u_m \leq \hat{u} \leq u^m$, $u_m \leq \hat{w}$.

Indeed, if $(\tilde{u}, \tilde{w}) \in X$, then $\Pi(\tilde{u}, \tilde{w}) = (\min\{u^m, \max\{u_m, \tilde{u}\}\}, \max\{\tilde{w}, u_m\}) \in C^0(\bar{\Sigma}) \times C^0(\bar{\Omega}) = X$ (maximum and minimum of two continuous functions are also continuous). Also $\Pi_u(\tilde{u}(s)) = \hat{u}(s) \in [u_m, u^m] \forall s \in \bar{\Sigma}$, and $\Pi_w(\tilde{w}(x)) = \hat{w}(x) \geq u_m \forall x \in \bar{\Omega}$, which by taking the maximum $\forall s \in \bar{\Sigma}$ and $\forall x \in \bar{\Omega}$ imply $u_m \leq \hat{u} \leq u^m$ and $u_m \leq \hat{w}$. \square

We have

Lemma 1.4.3. *The operator J is Lipschitz and compact, and the operator Π is Lipschitz. Both Π and J have Lipschitz constants equal to one.*

Proof. For properties of J , see [1]. The Lipschitz property of Π follows from the fact that $\Pi_u(s)$ and $\Pi_w(s)$ are continuous, piecewise linear functions with derivatives in $[-1, 1]$. For the sake of completeness, let us show the proof in detail.

Note that Π_u and Π_w can be written in the following way

$$\Pi_w(s) = \max(s, u_m) = u_m + (s - u_m)^+, \quad \Pi_u(s) = \min(\Pi_w(s), u^m) = u^m + (\Pi_w(s) - u^m)^-. \quad (80)$$

Now for $(\tilde{u}_1, \tilde{w}_1), (\tilde{u}_2, \tilde{w}_2) \in X$. We have

$$\begin{aligned} |\Pi_w(\tilde{w}_1(x)) - \Pi_w(\tilde{w}_2(x))| &\leq |(\tilde{w}_1(x) - u_m)^+ - (\tilde{w}_2(x) - u_m)^+| \\ &\leq |\tilde{w}_1(x) - \tilde{w}_2(x)|. \end{aligned}$$

Taking the maximum of both sides for all $x \in \bar{\Omega}$, we have

$$\|\Pi_w(\tilde{w}_1) - \Pi_w(\tilde{w}_2)\|_{C^0(\bar{\Omega})} \leq \|\tilde{w}_1 - \tilde{w}_2\|_{C^0(\bar{\Omega})},$$

which proves that Π_w is Lipschitz with Lipschitz constant equal to one.

Note that we can look at Π_u as a composition of two functions, that are the Lipschitz function Π_w and the function $u^m + (s - u^m)^-$, which is Lipschitz by a similar work to that done for Π_w . As a result the composition is Lipschitz and we have

$$\|\Pi_u(\tilde{u}_1) - \Pi_u(\tilde{u}_2)\|_{C^0(\bar{\Sigma})} \leq \|\tilde{u}_1 - \tilde{u}_2\|_{C^0(\bar{\Sigma})}.$$

Now $\Pi = (\Pi_u, \Pi_w)$ is Lipschitz because both Π_u and Π_w are Lipschitz (see the norm defined for the space X , Remark 1.2.2). \square

We need the following classical results. In the first result we have that for convex functions, lower semicontinuity with respect to strong topology implies lower semicontinuity with respect to weak topology. In the second result we have that a compact operator maps weakly convergent sequences to strongly convergent sequences. Let us state the two results.

Theorem 1.4.4. *Let E be a Banach space and $H : E \rightarrow \mathbb{R}$ be a convex, lower semicontinuous in E (for the strong topology in E), which implies that E is lower semicontinuous for the weak topology $\sigma(E, E')$, i.e. if $u_n \rightharpoonup u$ weakly in E , then*

$$H(u) \leq \liminf_{n \rightarrow \infty} H(u_n).$$

Proof. See Corollary III.8, [5]. □

Proposition 1.4.5. *Let E and F be two Banach spaces. Assume that $E \subset F$ with compact imbedding. If $E \ni u_n \rightharpoonup u$ weakly in E , then $u_n \rightarrow u$ strongly in F .*

Proof. See [8]. □

Let us prove that S satisfies the conditions of Theorem 1.4.1.

Proposition 1.4.6. *Assume (28)-(30) hold, then the operator S is compact and continuous.*

Proof. The proof is in two parts.

(i) *S is compact.* Let $(\tilde{u}_n, \tilde{w}_n) \in X$ be bounded, $\|(\tilde{u}_n, \tilde{w}_n)\| \leq K$, for some constant $K > 0$. As Π is Lipschitz continuous, Π takes bounded sets to bounded sets. So it is enough to prove that $\hat{T} = J \circ T : B \rightarrow \mathcal{X}$ is compact. From Proposition 1.3.6, the sequence $(u_n, w_n) = T(\hat{u}_n, \hat{w}_n)$ is bounded in Y . From the compactness of the map J it follows that (u_n, w_n) has a convergent subsequence in X , so \hat{T} is compact.

(ii) *S is continuous.* Again, as Π is continuous, it is enough to prove that \hat{T} is continuous. Furthermore, it is enough to prove that \hat{T} is sequentially continuous because the normed space X is a first countable space, see [22], then let $(\hat{u}_n, \hat{w}_n)_{n=1}^{\infty} \in B$, with $(\hat{u}_n, \hat{w}_n) \rightarrow (\hat{u}, \hat{w})$ in X as $n \rightarrow \infty$. We have to prove that $(u_n, w_n) := \hat{T}(\hat{u}_n, \hat{w}_n) \rightarrow (u, w) := \hat{T}(\hat{u}, \hat{w})$ in X as $n \rightarrow \infty$.

Let us note first that $(\hat{u}, \hat{w}) \in B$ because $B \subset X$ is closed (see Remark 1.2.7). It follows from Theorem 1.3.4 that $(u, w) := T(\hat{u}, \hat{w})$ is well defined. To prove that $\lim_{n \rightarrow \infty} (u_n, w_n) = (u, w)$ in X it is enough to prove that (see Proposition 1.4.5)

$$(u_n, w_n) \rightharpoonup (u, w) \quad \text{weakly in } Y. \tag{81}$$

The sequence $\{(u_n, w_n)\}_{n=1}^\infty$ is bounded in Y . As Y is a Banach reflexive space, the sequence (u_n, w_n) has a subsequence converging weakly in Y . Consider any converging subsequences and denote it with the same name (u_n, w_n) , and let $(u^*, w^*) \in Y$, $(u_n, w_n) \rightharpoonup (u^*, w^*)$ weakly in Y . In other words, $(u_n, w_n) \in Y$, $u_n \rightharpoonup u^*$ weakly in $H^1(\Sigma)$ and $w_n \rightharpoonup w^*$ weakly in $W^{1,3}(\Omega)$.

We will prove that $(u^*, w^*) = (u, w)$, where $(u, w) \in Y_g$ is the unique weak solution of (64). So, all the weak Y limit points of (u_n, w_n) are equal to (u, w) , which proves (81) (see Proposition 4.2, [17]). Indeed, from (iii) Proposition 1.3.1, we have

$$\begin{aligned} I(u^*, w^*; \hat{u}, \hat{w}) &\leq \liminf_{n \rightarrow \infty} I(u_n, w_n; \hat{u}, \hat{w}) \\ &= \liminf_{n \rightarrow \infty} (I(u_n, w_n; \hat{u}, \hat{w}) - I(u_n, w_n; \hat{u}_n, \hat{w}_n)) + \liminf_{n \rightarrow \infty} I(u_n, w_n; \hat{u}_n, \hat{w}_n) \\ &\leq \lim_{n \rightarrow \infty} I(u, w; \hat{u}_n, \hat{w}_n) \\ &= I(u, w; \hat{u}, \hat{w}), \end{aligned}$$

and then from the unique solvability of (64) it follows $(u^*, w^*) = (u, w)$, which completes the proof of the proposition. \square

In order to use Theorem 1.4.1, it remains to show first that S satisfies the property that the set (77) is bounded in X , which is proved in the following proposition.

Proposition 1.4.7. *The set $\Lambda := \{x \in X : x = \lambda Sx, \lambda \in [0, 1]\}$ is bounded in X , i.e there exists $C_\Lambda > 0$, such that $\|(u, w)\|_X \leq C_\Lambda$, for all $(u, w) \in \Lambda$.*

Proof. For $x = (\tilde{u}, \tilde{w}) \in \Lambda$ set $(\hat{u}, \hat{w}) = \Pi(\tilde{u}, \tilde{w}) \in B$ and $(u, w) = T(\hat{u}, \hat{w}) \in B$. So we have $(\tilde{u}, \tilde{w}) = \lambda(u, w)$. From the construction of operator Π , we have $\hat{w} = \max\{u_m, \tilde{w}\} = \max\{u_m, \lambda w\}$. Therefore, $|\nabla \hat{w}| \leq |\nabla w|$ a.e. in Ω . Now from (72) and (75) we have

$$\begin{aligned} \frac{1}{2} \int_\Sigma (u_m(u')^2 + du^2) &+ \frac{1}{6} \int_\Omega 2|\nabla w||\nabla w|^2 \\ &\leq \frac{1}{2} |\Sigma| d(G(0))^2 + \int_\Omega \frac{1}{2} \hat{w} |\nabla G|^2 + \frac{1}{3} |\nabla G|^3 \\ &- \int_\Sigma \int_0^{u(s)} (-a\bar{w} - f(s)) dk ds + \int_{\partial\Omega \setminus \Gamma} hw \\ &\leq \frac{1}{2} \|\hat{w}\|_{L^3(\Omega)} \|\nabla G\|_{L^3(\Omega)}^2 + K(G) + \frac{1}{d} (a\bar{w}|\Sigma| + \|f\|_{L^2(\Sigma)})^2 + \frac{d}{4} \|u\|_{L^2(\Sigma)}^2 \\ &+ K_3 (\|h\|_{L^{\frac{3}{2}}(\partial\Omega \setminus \Gamma)}^{\frac{3}{2}} + \|h\|_{L^{\frac{3}{2}}(\partial\Omega \setminus \Gamma)} \|G\|_{W^{1,3}(\Omega)}) + \frac{1}{6} \|\nabla w\|_{L^3(\Omega)}^3, \quad (82) \end{aligned}$$

for some constant $K(G)$, where we have used Hölder inequality, (45) and (46). Moreover, for $|\nabla \hat{w}| \leq |\nabla w|$ a.e. in Ω , we have

$$\begin{aligned}
\|\nabla G\|_{L^3(\Omega)}^2 \|\hat{w}\|_{L^3(\Omega)} &\leq \|\nabla G\|_{L^3(\Omega)}^2 (\|\hat{w} - \hat{G}\|_{L^3(\Omega)} + \|\hat{G}\|_{L^3(\Omega)}) \\
&\leq \|\nabla G\|_{L^3(\Omega)}^2 (M_1 \|\nabla \hat{w}\|_{L^3(\Omega)} + \|\hat{G}\|_{W^{1,3}(\Omega)}) \\
&\leq \|\nabla G\|_{L^3(\Omega)}^2 (M_1 \|\nabla w\|_{L^3(\Omega)} + \|\hat{G}\|_{W^{1,3}(\Omega)}) \\
&\leq \frac{1}{12} \|\nabla w\|_{L^3(\Omega)}^3 + M_2(G), \tag{83}
\end{aligned}$$

for some constants M_1 and $M_2(G)$, where we have used triangle, Poincaré and Young inequalities. Note that $\hat{G} := \max\{u_m, \lambda G\}$, bounded independently from λ . Now using (82) and (83) and rearranging, we have

$$\begin{aligned}
\int_{\Sigma} \left(\frac{1}{2} u_m (u')^2 + \frac{1}{4} du^2 \right) + \frac{1}{12} \int_{\Omega} |\nabla w|^3 \\
\leq M_2(G) + K(G) + \frac{1}{d} (a\bar{w}|\Sigma| + \|f\|_{L^2(\Sigma)})^2 \\
+ K_3 (\|h\|_{L^{\frac{3}{2}}(\partial\Omega \setminus \Gamma)}^{\frac{3}{2}} + \|h\|_{L^{\frac{3}{2}}(\partial\Omega \setminus \Gamma)} \|G\|_{W^{1,3}(\Omega)}), \tag{84}
\end{aligned}$$

which implies Λ is uniformly bounded in Y . As $Y \subset X$ (imbedding theorem [1]), this implies that (u, w) is uniformly bounded in X . \square

1.4.1 Proof of the main result (Theorem 1.1.3).

Proof. (i) The operator $S : X \rightarrow -X$ satisfies the conditions of Theorem 1.4.1 (Proposition 1.4.6 and 1.4.7). Therefore, Theorem 1.4.1 implies S has a fixed point denoted (u, w) . As $S = J \circ T \circ P$ and J is the imbedding map, it follows that $(u, w) \in C \subset Y_g$ and (u, w) satisfies (21).

The bound (31) is proven in Propositions 1.2.6 and 1.2.9. Also the lower bound of (32) is proven in Proposition 1.2.6. For the upper bound of (32) we proceed as follows. From (51), we have

$$|\hat{u}(0)u'(0)| = |L(1)| \leq (\|f\|_{L^1(\Sigma)} + a|\Sigma|\bar{w}) =: \phi_0, \tag{85}$$

then we can proceed exactly as in (iii) Proposition 1.2.9. We bound $\hat{u}(0)u'(0)$ by (85) and then we obtain

$$|\hat{u}(0)u'(0)\psi_u(0)| \leq (u^m - \sup_{\Gamma} g)\phi_0, \tag{86}$$

then, like in (iii) in the proof of Proposition 1.2.9, equations (54) and (86) can be used to show that

$$\begin{aligned} w &\leq \sup_{\Gamma} g + ((K_5 \|h\|_{L^{\frac{3}{2}}(\partial\Omega \setminus \Gamma)})^{\frac{3}{2}} + (u^m - \sup_{\Gamma} g)\phi_0)^{\frac{1}{3}} \\ &=: w^m. \end{aligned} \tag{87}$$

By this we proved (i).

(ii) If the data are “small” enough ((ii) Theorem 1.1.3 holds) then Corollary 1.2.10 implies that $u^m \leq \bar{w}$. Also it implies $\phi_0 \ll 1$, $\|h\|_{L^{\frac{3}{2}}(\partial\Omega \setminus \Gamma)} \ll 1$ and $\sup_{\Gamma} g < \bar{w}$, therefore $w^m \leq \bar{w}$. As a result, $\max\{u^m, w^m\} \leq \bar{w}$ and this ends the proof. \square

Remark 1.4.8. *Note that Theorem 1.1.3 implies the existence of a solution, but doesn't imply the uniqueness. In fact, we will not discuss the uniqueness of solution in the whole thesis and we will leave that as an open problem.*

1.5 Numerical results.

In this section we will present some numerical computations to find a solution for (18)-(19) and to verify that the system (18)-(19) approximates well the reaction-diffusion phenomena around the triple phase point by comparing our results with those obtained using the thin layer model (see Appendix A). We will give here the essentials of this thin layer model.

Another goal for this section is to show that the numerical computation agrees with the analysis result (ii), Theorem 1.1.3.

The numerical method used here is the finite element method with a finite space consisting of piecewise polynomials of degree two, see [20]. Since our model is nonlinear, Newton's method is applied too, see [9]. In fact, the COMSOL multiphysics software (matlab package) is used. The numerical issues are not the focus of our work, so we just provided a COMSOL script with the equations and the boundary conditions. The program transforms the system of PDEs into its weak form and then applies the finite element method and Newton's method to provide us with the numerical solution.

It is worth to point out that the system (18)-(19) involves a boundary condition that contains the Dirac measure supported at the origin, see the boundary condition on $\partial\Omega \setminus \Gamma$. So it is hard to deal with the strong form of this system to find the numerical solution. However, the weak form of this system, equation (21) does not contain the Dirac measure

boundary condition. Fortunately, the finite element method does not use the strong form to calculate the numerical solution, rather it uses the weak form of PDEs.

The same is true for all the systems in this thesis that is, by writing the weak form of any of these systems we get an equation that does not have most of the boundary coupling conditions, especially the boundary condition that contains the Dirac measure. So the only difficulty that remain is the nonlinearity of the equations in the system which implies that the system of algebraic equations that results from applying the finite element method is nonlinear too. This system of algebraic equations is solved using Newton's method.

Another important note to find a good numerical approximation for the solution of the system (18)-(19) and any other system in this thesis is to refine the mesh (increase the number of triangles used by the finite element method) on all edges where we have boundary coupling conditions, specially around the origin, in order to catch the behavior of the solution there.

Now let us go back to our goal. Consider the following thin layer model, in which the concentration of water w is the only variable. Namely

$$\begin{aligned}
 -\nabla \cdot (w_\delta \nabla w_\delta) &= 0 && \text{in } \Omega_\delta \cup \Omega_{2\delta} \cup \Gamma_\delta, \\
 w_\delta \partial_\nu w_\delta &= h_\delta && \text{on } \partial(\Omega_\delta \cup \Omega_{2\delta} \cup \Gamma_\delta) \setminus \Gamma, \\
 w_\delta &= g && \text{on } \Gamma.
 \end{aligned}
 \tag{88}$$

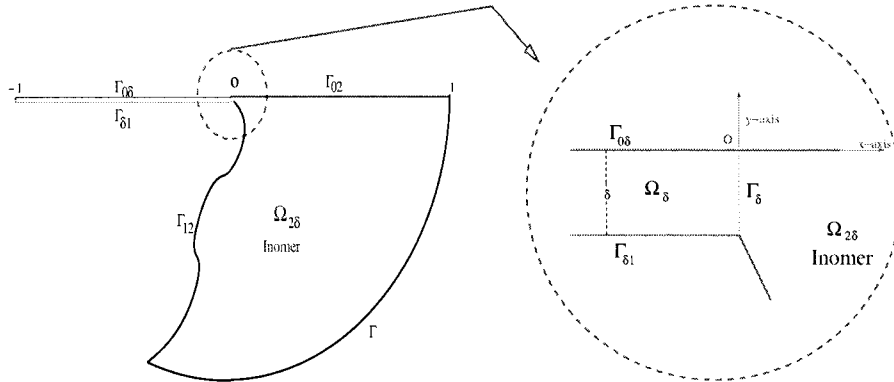


Figure 6: Domain with thin layer.

Here

$$h_\delta := \begin{cases} h_{0\delta}, & \text{on } \Gamma_{0\delta}, \\ dw_\delta - a(\bar{w} - w_\delta)^+, & \text{on } \Gamma_{\delta 1}, \\ h_{02}, & \text{on } \Gamma_{02}, \\ 0, & \text{on } \Gamma_{12} \cup \{(x, y) : x = -1\}. \end{cases} \quad (89)$$

Domains Ω_δ , $\Omega_{2\delta}$ and the boundary $\Gamma_{0\delta}$, $\Gamma_{\delta 1}$, Γ_δ , Γ_{12} , Γ_{02} , Γ are shown in Figure 6, in which picture on the right is a magnification of the picture on the left picture around the origin.

Our system of PDEs corresponding to this model can be found as follows. First, u equation can be found by taking the average integral for w_δ in equation (88) in Ω_δ (average integral with respect to y only, for more details see [3]), then define

$$u(x) := \frac{1}{\delta} \int_{-\delta}^0 w_\delta(x, y) dy. \quad (90)$$

Second step, in $\Omega_{2\delta}$ the equation of w is found by adding the gradient term $|\nabla w_\delta|$ to the diffusion coefficient of w_δ in equation (88), then calling the variable that satisfies this new equation w . As a result our system of PDEs and boundary conditions is the following

$$\begin{cases} -\delta(uu_x)_x + du - a(\bar{w} - u)^+ = h_{0\delta} & \text{on } \Gamma_{0\delta}, \\ -\nabla \cdot ((w + |\nabla w|)\nabla w) = 0 & \text{in } \Omega_{2\delta}, \end{cases} \quad (91)$$

equipped with the following boundary conditions

$$\begin{cases} (uu_x)(-1) = 0, \quad u(0) = w(0), \\ (w + |\nabla w|)\partial_\nu w = -\delta uu_x \delta_0 + h_{02}\chi_{\Gamma_{02}} + 0\chi_{\Gamma_{12} \cup \Gamma_\delta}, & \text{on } \Gamma_\delta \cup \Gamma_{02} \cup \Gamma_{12}, \\ w = g, & \text{on } \Gamma. \end{cases} \quad (92)$$

Note that Γ_{01} (represents Σ) is the line segment $[-1, 0]$ on the x -axis (this explains why we replace u_s , the derivative with respect to s , by u_x , the derivative with respect to x).

To compare the numerical solution for (91)-(92) with the numerical solution for (88), we define the extension function w_0 as follows

$$w_0(x, y) = \begin{cases} u(x, 0), & (x, y) \in \Omega_\delta, \\ w(x, y), & (x, y) \in \Omega_{2\delta}, \end{cases} \quad (93)$$

then we calculate the L^2 norm of the pointwise relative difference between w_δ and w_0 .

$$e := \left\| \frac{w_\delta - w_0}{w_\delta} \right\|_{L^2(\Omega_\delta \cup \Omega_{2\delta})}.$$

The following table of data will be used to find the numerical solutions for both the thin layer boundary value problem (BVP), the system (88), and the system (91)-(92).

Parameter	Value
g	5.607068
\bar{w}	21
$h_{0\delta}$	$0.82333 * (10^5/7)^x$
h_{02}	$0.074097 * (0.556)^x$

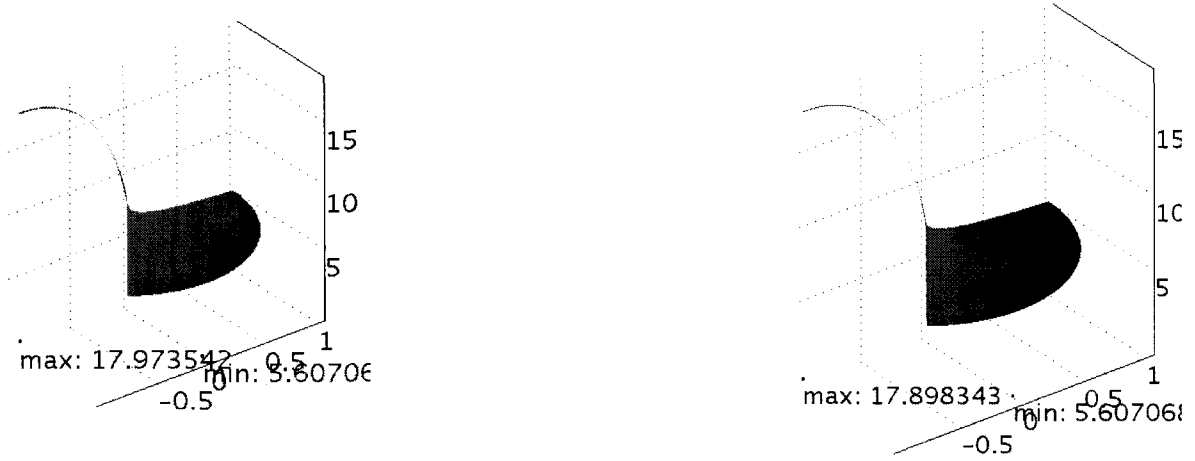


Figure 7: Solutions of the two systems for $a = 0.624981$ and $d = 9.982947 * 10^{-4}$. On the left we have the solution for the thin layer BVP (88). On the right we have the solution for (91)-(92) (surface diffusion model). The error $e = 0.024117$.

It is clear from the two figures, Figure 7 and Figure 8 that we have similar numerical solutions for both thin layer BVP (88) and surface diffusion model (91)-(92). In fact, the relative error between the solutions in Figure 7 is $e = 0.024117$ and in Figure 8 is $e = 0.018148$.

Also note that in the pictures on the right, Figures 7 and 8, we have $\max w \leq 17.89834 < \bar{w}$. A simple calculation implies $\|h_{0\delta}\|_{L^1(\Gamma_{01})} = 0.086052$, $\|h_{02}\|_{L^{\frac{3}{2}}(\Gamma_{02})} = 0.055681$. From the data used to find Figure 7, we have $\|h_{0\delta}\|_{\infty}/d = 824.736423 \gg \bar{w}$. However, the data used to find Figure 8 implies $\|h_{0\delta}\|_{\infty}/d = 0.824736 \ll \bar{w}$. Therefore, the picture on the right side of Figure 8, can only be presented as an agreement of the numerical results with what we have in (ii), Theorem 1.1.3.

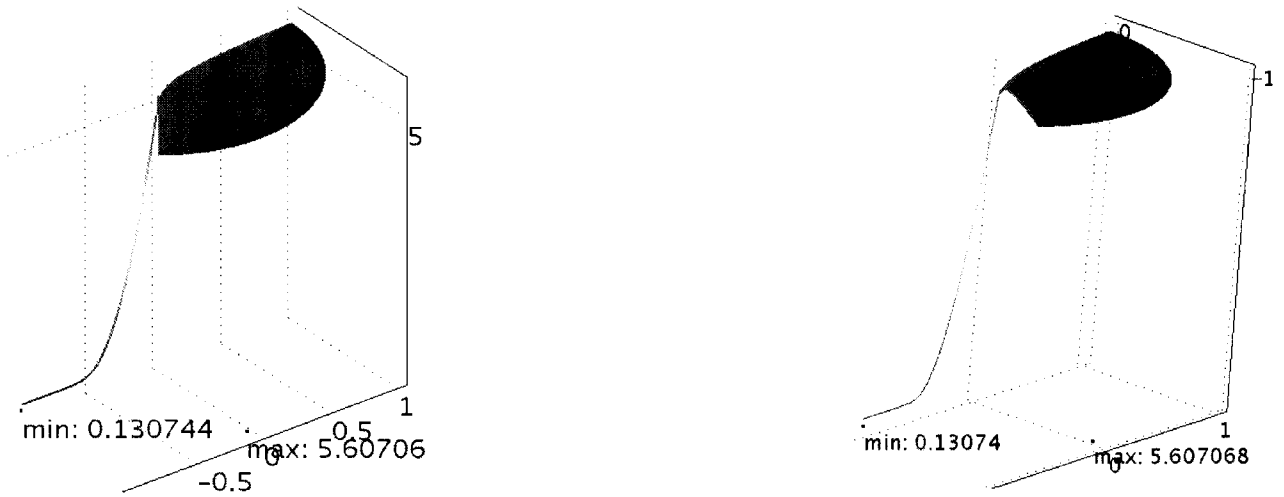


Figure 8: *Solutions of the two systems for $a = 0.00625$ and $d = 0.998295$. On the left we have solution for the thin layer BVP (88) (thin layer model). On the right we have solution for (91)-(92) (surface diffusion model). The error $e = 0.018148$.*

Remember that the gradient $|\nabla w|$ in the diffusion coefficient of w equation (91) is added to change the solution w from nonregular (w^2 is logarithmic around the origin) around the origin to regular solution (continuous), so we can make the coupling $u(0) = w(0)$. One can add $k|\nabla w|^{n-2}\chi_{\{(x,y):x^2+y^2\leq\epsilon\}}$ for some constants $k > 0$, $n > 2$ and $\epsilon > 0$, instead of adding $|\nabla w| = 1 \cdot |\nabla w|^{3-2}\chi_{\{(x,y):x^2+y^2\leq 1\}}$ and still get continuity around the origin. In other words, we will replace the water equation in (91)-(92) by

$$\nabla \cdot ((w + k|\nabla w|^{n-2}\chi_{\{(x,y):x^2+y^2\leq\epsilon\}})\nabla w) = 0 \text{ in } \Omega_{2\delta}. \quad (94)$$

for some constants $k > 0$, $n > 2$ and $\epsilon > 0$.

Can we minimize $e(k, n, \epsilon; a, d)$ with respect to the parameters k , n and ϵ , with $e(k, n, \epsilon; a, d)$ defined by

$$e(k, n, \epsilon; a, d) = \left\| \frac{w_\delta - w_0(k, n, \epsilon)}{w_\delta} \right\|_{L^2(\Omega_\delta \cup \Omega_{2\delta})}, \quad (95)$$

where $w_0(k, n, \epsilon)$ is the extension (see (93)) of the solution for the surface diffusion model (91)-(92) with (94) replacing w equation in this model.

1.5.1 Numerical sensitivity analysis of the relative error.

In this section we will find numerically an optimal (k, n, ϵ) at which the relative error $e(k, n, \epsilon; a, d)$ given by (95) has a relative minimum. We let the three parameters change in a small box, $(k, n, \epsilon) \in [10^{-5}, 22] \times [2.0001, 20] \times [10^{-3}, 1]$ and search for optimal (k, n, ϵ) in this box, at which e has a relative minimum.

Since we can not cover every point in this box, we will search for the minimum between finitely many $(k, n, \epsilon) \in [10^{-5}, 22] \times [2.0001, 20] \times [10^{-3}, 1]$, which are chosen to cover the whole box in the following way.

Suppose that $k_i \in [10^{-5}, 22]$, $i = 1, 2, \dots, m_k$, with equal distance, $n_j \in [2.0001, 20]$, $j = 1, 2, \dots, m_n$, with equal distance, $\epsilon_l \in [10^{-3}, 1]$, $l = 1, 2, \dots, m_\epsilon$, with equal distance, then the corresponding solution to (k_i, n_j, ϵ_l) is numbered using the following algorithm.

```

for l = 1, ..., mϵ,
  for i = 1, ..., mk,
    for j = 1, ..., mn,
      nijl = (l - 1) * mk * mn + (i - 1) * mn + j
    end
  end
end.

```

The following table helps in reading the figures below.

Relative error dependence on parameters.						
c(k,n,ϵ;a,d)	a	d	k	n	ϵ	Smallest relative error and optimal (k, n, ϵ)
Figures 9 and 10	0.624981	$9.982947 * 10^{-4}$	$k \in [10^{-5}, 22]$	$n \in [2.0001, 20]$	$\epsilon \in [10^{-3}, 1]$	$e = 0.010703$ at $(1.0009 * 10^{-2}, 4.73333, 0.0208)$
Figures 11 and 12	0.00625	0.998295	$k \in [10^{-5}, 22]$	$n \in [2.0001, 20]$	$\epsilon \in [10^{-3}, 1]$	$e = 0.005499$ at $(5.6733, 2.2508, 0.134)$

It is clear from the Figures 9-11 that the relative error e varies if one or more of the five parameters k , n , ϵ , a and d vary. Also the choice of the optimal (k, n, ϵ) depend on the physical parameters (a , d in particular). For example, the more the absorption parameter a the more the water on Γ_{01} .

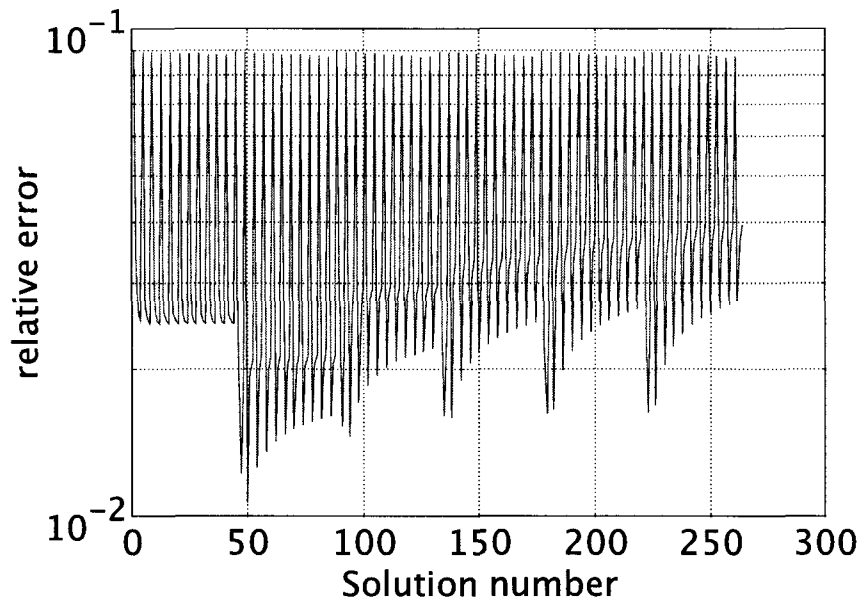


Figure 9: *Relative error function* $(k, n, \epsilon) \rightarrow \epsilon(k, n, \epsilon)$. *Smallest relative error* $e = 0.010703$ *at* $n_{ijl} = 50$, *where* $a = 0.624981$ *and* $d = 9.982947 * 10^{-4}$, *optimal* $(k, n, \epsilon) = (1.0009 * 10^{-2}, 4.73333, 0.0208)$.

Note that in the case $a = 0.624981$, $d = 9.982947 * 10^{-4}$, we can approximate an optimal $(k, n, \epsilon) = (1.0009 * 10^{-2}, 4.73333, 0.0208)$, where $e = 0.010703$. Also in the case $a = 0.00625$ and $d = 0.998295$, we can approximate an optimal $(k, n, \epsilon) = (5.6733, 2.2508, 0.134)$, where $e = 0.005499$.

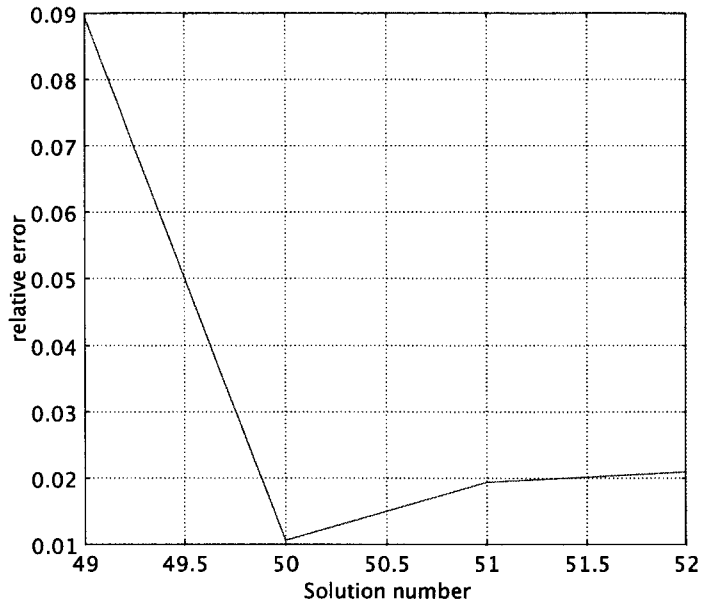


Figure 10: Relative error function $(k, n, \epsilon) \mapsto -e(k, n, \epsilon)$, near the optimal solution number $n_{ijl} = 50$.

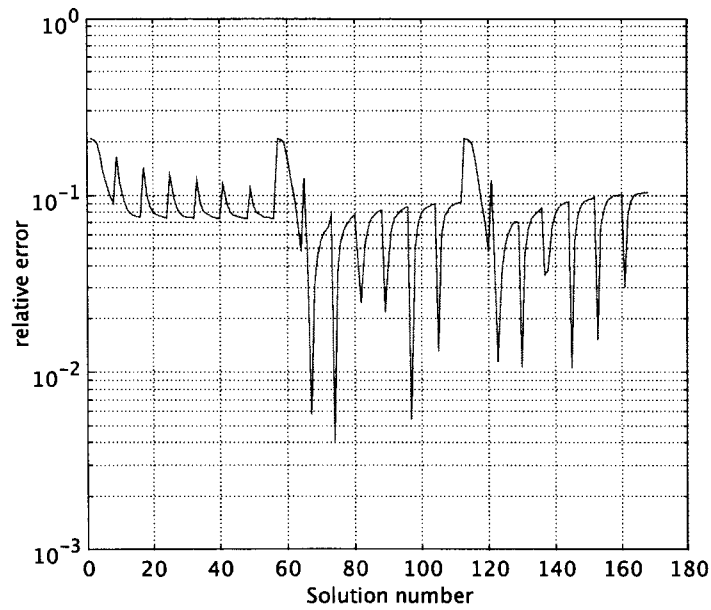


Figure 11: Relative error function $(k, n, \epsilon) \mapsto -e(k, n, \epsilon)$. Smallest relative error $e = 0.005499$ at $n_{ijl} = 74$, where $a = 0.00625$ and $d = 0.998295$, optimal $(k, n, \epsilon) = (5.6733, 2.2508, 0.134)$.

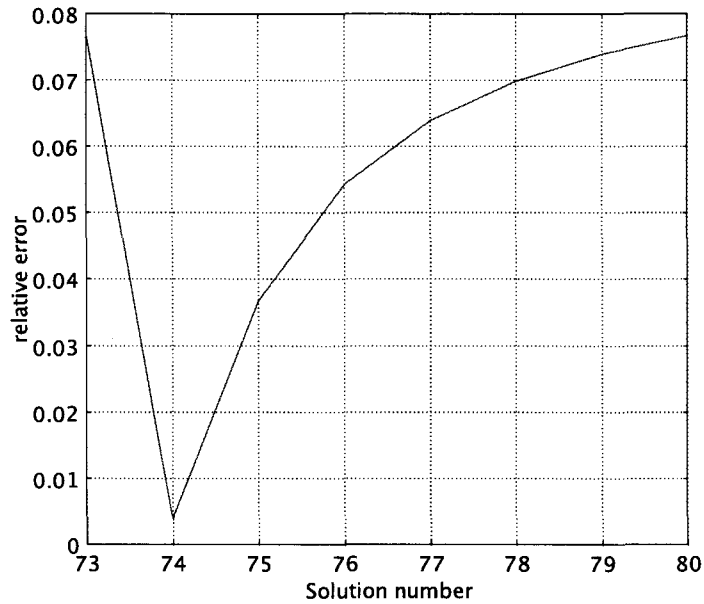


Figure 12: Relative error function $(k, n, \epsilon) \mapsto -e(k, n, \epsilon)$ near the optimal solution number $n_{ijl} = 74$.

Remark 1.5.1. As we see from these numerical computation, the best approximation of w is obtained for a particular choice of (k, n, ϵ) . The analysis of such (k, n, ϵ) is not the object of this thesis. \square

Chapter 2

Water-vapor system.

In this chapter we consider another simplified version of the system (1)-(16), but more complex than the one we considered in Chapter 1. This simplified system is related to the system (18)-(19), namely by considering water and vapor concentrations together, which are related with adsorption-desorption equilibrium on Γ_{01} , so in addition to the difficulties in the water only model, we have to deal with the adsorption-desorption (nonlinear) boundary condition on Γ_{01} , condition (11) and the jump continuity function between water and vapor on Γ_{12} condition (12). The same approach as in Chapter 1 is used here to prove the existence, that is the fixed point approach. Sections and theorems are presented in a similar way to that in Chapter 1, and the same way is used for presenting the numerical results.

2.1 Introduction. Main result.

In this chapter we will consider a system of PDEs coupling three variables u , v and w as follows

$$\begin{cases} -(uu_s)_s + du - av(\bar{w} - u)^+ = f & \text{on } \Gamma_{01}, \\ -\Delta v = 0 & \text{in } \Omega_1, \\ -\nabla \cdot ((w + |\nabla w|)\nabla w) = 0 & \text{in } \Omega_2, \end{cases} \quad (96)$$

equipped with the following boundary conditions

$$\left\{ \begin{array}{ll} (uu_s)(-l) = 0, \quad u(0) = w(0), & \\ v = g_v & \text{on } \Gamma_1, \\ \partial_\nu v = du - av(\bar{w} - u)^+ & \text{on } \Gamma_{01}, \\ w = i(v) & \text{on } \Gamma_{12}, \\ (w + |\nabla w|)\partial_\nu w = -uu_s\delta_0 + h\chi_{\Gamma_{02}} + \partial_\nu v\chi_{\Gamma_{12}} & \text{on } \partial\Omega_2 \setminus \Gamma_2, \\ w = g_w & \text{on } \Gamma_2. \end{array} \right. \quad (97)$$

Domains, boundaries, physical parameters and functions are as defined in Section 0.1, that is, Ω_1, Ω_2 are open, simply connected, bounded sets, with a Lipschitz boundary. $\Gamma_{01}, \Gamma_{02}, \Gamma_1, \Gamma_2$ are connected parts of the boundaries. $\Omega := \Omega_1 \cup \Gamma_{12} \cup \Omega_2$, $\Gamma := \Gamma_1 \cup \Gamma_2$ and ν the outward unit normal vector to $\partial\Omega$ or to $\partial\Omega_2$. Domains and boundaries are presented in Figure 13, which is more general than Figure 3.

Also we still have the same conditions on parameters and data, that is a, d and \bar{w} are positive. The functions g_v and g_w are nonnegative. These functions together with i are assumed to be sufficiently smooth functions (to be specified later). δ_0 is the Dirac measure with support at the origin, and $(\cdot)^+$ means positive part of (\cdot) .

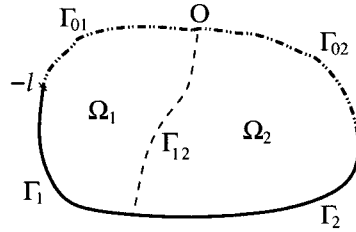


Figure 13: Domain with a triple phase boundary

The variables u, v and w are described in Section 0.1, that is u represents the water concentration and defined on Γ_{01} . The variable v represents the concentration of vapor, defined in Ω_1 , which represents the air phase. The variable w is the concentration of water and defined in Ω_2 , which is porous domain.

Assuming that (96)-(97) has a smooth solution (u, v, w) , if we set

$$\begin{aligned} Z &= H^1(\Gamma_{01}) \times H^1(\Omega_1) \times W^{1,3}(\Omega_2) \cap \{(u, v, w) : u(0) = w(0)\}, \\ Z_g &= Z \cap \{(u, v, w) : v = g_v \text{ on } \Gamma_1, w = g_w \text{ on } \Gamma_2, w = \dot{i}(v) \text{ on } \Gamma_{12}\}, \\ Z_0 &= Z \cap \{(\psi_u, \psi_v, \psi_w) : \psi_v = 0 \text{ on } \Gamma_1, \psi_w = 0 \text{ on } \Gamma_2, \psi_v = \psi_w \text{ on } \Gamma_{12}\}, \end{aligned} \quad (98)$$

then using the test functions ψ_u, ψ_v and ψ_w , $(\psi_u, \psi_v, \psi_w) \in Z_0$ with the equations of (96) respectively, adding the results, then using (97) we get

$$\begin{aligned} &\int_{\Gamma_{01}} (u(u' \psi'_u) + (-av(\bar{w} - u)^+ + du)(\psi_u - \psi_v) - f\psi_u) - \int_{\Gamma_{02}} (h\psi_w) \\ &+ \int_{\Omega_1} (\nabla v \cdot \nabla \psi_v) + \int_{\Omega_2} (w + |\nabla w|)(\nabla w \cdot \nabla \psi_w) = 0, \quad \forall (\psi_u, \psi_v, \psi_w) \in Z_0. \end{aligned} \quad (99)$$

Here $u' := \frac{du}{ds}$ and $\psi'_u := \frac{d\psi_u}{ds}$.

Definition 2.1.1. We say $(u, v, w) \in Z_g$ is a weak solution of (96)-(97) if (u, v, w) satisfies (99).

Remark 2.1.2. Note that if (u, v, w) is a weak solution of (96)-(97) then it is clear that by taking $(\psi_u, \psi_v, \psi_w) \in Z_0$ with compact support we find easily that (u, v, w) satisfies (96) in the sense of distributions.

If (u, v, w) is smooth enough, for example if u , resp. v, w , is a $H^2(\Gamma_{01})$, resp. $H^2(\Omega_1)$, $W^{2,3}(\Omega_2)$, functions, then u , resp. v, w satisfies (96) in $L^2(\Gamma_{01})$, resp. $L^2(\Omega_1)$, $L^3(\Omega_2)$ sense. Furthermore, by taking $\psi_u(-l), \psi_u(0) = \psi_w(0), \psi_v|_{\Gamma_{01}}, \psi_w|_{\Gamma_{02}}$ and $\psi_v|_{\Gamma_{12}} = \psi_w|_{\Gamma_{12}}$ arbitrarily we find that (u, v, w) satisfies also (97). \square

The main purpose of this chapter is to prove that (96)-(97) has a positive, bounded weak solution. Namely, we will prove the following theorem which is a generalized version of Theorem 1.1.3.

Theorem 2.1.3. *Assume that the constants and functions in (99) satisfy*

$$a, d, \bar{w} > 0, \quad (100)$$

$$i \in C^1(\mathbb{R}), \quad i(0) = 0 \text{ and } \exists 0 < \underline{\sigma} < \bar{\sigma} \text{ such that : } \underline{\sigma} \leq \sigma \leq \bar{\sigma}, \sigma := \frac{1}{i}, \quad (101)$$

$$0 \leq f \in L^\infty(\Gamma_{01}), \quad 0 \leq h \in L^{\frac{3}{2}}(\Gamma_{02}), \quad (102)$$

$$0 < \inf_{\Gamma} g, \quad g := \begin{cases} \mathbf{i}(g_v), & \Gamma_1, \\ g_w, & \Gamma_2, \end{cases} \quad g = \text{Tr}(G)|_{\Gamma}, \quad G \in H_g^1(\Omega) \cap W_g^{1,3}(\Omega_2). \quad (103)$$

(i) If the condition

$$\frac{d}{a} \frac{1}{\bar{w}} < \underline{\sigma} \leq \sigma \leq \bar{\sigma} < \frac{d}{a} \frac{1}{(\bar{w} - \inf_{\Gamma} g)^+}. \quad (104)$$

holds, then (99) has at least one solution $(u, v, w) \in Z_g$. Moreover, (u, v, w) satisfies

$$u_m \leq u \leq u^m, \quad (105)$$

$$\mathbf{i}^{-1}(u_m) \leq v, \quad \|v\|_{H^1(\Omega_1)} \leq v^m, \quad (106)$$

$$u_m \leq w \leq w^m, \quad (107)$$

where u_m, u^m, w^m and v^m are constants depending only on the given data and the domain.

(ii) Assume $\max\{\sup_{\Gamma} g, \frac{\|f\|_{L^\infty(\Gamma_{01})}}{d}\} < \bar{w}$. If $\frac{d^2}{\underline{\sigma}}, \frac{(a\bar{\sigma})^2}{\underline{\sigma}}, \|f\|_{L^1(\Gamma_{01})}, \|h\|_{L^{\frac{3}{2}}(\Gamma_{01})}$ are small enough, then u, v and w are “physically meaningful” solutions in the sense that

$$u \leq \bar{w} \text{ on } \Gamma_{01}, \quad w \leq \bar{w} \text{ in } \Omega_2. \quad (108)$$

Note that the results for u and w in this theorem are the same results that we already proved in Theorem 1.1.3, which make sense because the system (18)-(19) is a special case from the system (96)-(97), when v is considered constant.

Remark 2.1.4. The set of data satisfying (i) and (ii) of Theorem 2.1.3 is not empty. Indeed, let $0 < \underline{\sigma} < \bar{\sigma}$ be fixed and set $d = \alpha a$, for some constant α to be described, then

- 1) Choose $\alpha: \alpha < \bar{w}\underline{\sigma}$. Next consider $g: \sup_{\Gamma} g < \bar{w}$ and $\bar{w} - \inf_{\Gamma} g < \frac{\underline{\sigma}}{\alpha}$. So (i) is satisfied.
- 2) Take α and a small: $\frac{d^2}{\underline{\sigma}} = \frac{\alpha^2}{\underline{\sigma}^2} \frac{(a\bar{\sigma})^2}{\underline{\sigma}}$, and $\frac{(a\bar{\sigma})^2}{\underline{\sigma}}$ are small, and also $\|f\|_{L^\infty}$ and $\|h\|_{L^{3/2}}$ small, then (ii) is satisfied. \square

Remark 2.1.5. The constant \bar{w} still represents a “capacity” of the medium to contain water and we still need u and w smaller than \bar{w} to have a physically meaningful solution. Also the theorem imposes conditions on parameters, in particular the parameters a, d and the function \mathbf{i} , which are found experimentally, in order to find physically meaningful solutions. \square

To prove Theorem 2.1.3 we will use the same approach used to prove Theorem 1.1.3, that is the fixed point approach. In addition, a change of variables is needed before proceeding in this approach as we will see.

2.2 A fixed point approach.

Throughout this chapter we will assume that (100)-(103) hold. From (101), we have that i has an inverse, denoted by j and

$$j \in C^1(\mathbb{R}), \quad J(0) = 0, \quad \underline{\sigma} \leq j \leq \bar{\sigma}. \quad (109)$$

The weak form equation (99) can be written in a simpler form by introducing the function, still denoted by w , as follows

$$w(x, y) := \begin{cases} i \circ v(x, y), & (x, y) \in \Omega_1 \cup \Gamma_{12}, \\ w(x, y), & (x, y) \in \Omega_2. \end{cases} \quad (110)$$

Since

$$v = j(w), \quad \nabla v = \sigma(w)\nabla w, \quad \Delta v = \nabla \cdot (\sigma(w)\nabla w) \quad \text{in } \Omega_1, \quad (111)$$

then (99) is equivalent to find $(u, w) \in Y_g$ satisfying

$$\begin{aligned} \int_{\Gamma_{01}} (u(u'\psi'_u) + N(u, w)(\psi_u - \psi_w) - f\psi_u) - \int_{\Gamma_{02}} (h\psi_w) + \int_{\Omega} D(w, |\nabla w|)(\nabla w \cdot \nabla \psi_w) &= 0, \\ \forall (\psi_u, \psi_w) \in Y_0, \end{aligned} \quad (112)$$

where

$$N(u, w) = du - aj(w)(\bar{w} - u)^+, \quad (113)$$

$$D(v, w) = \sigma(v)\chi_{\Omega_1} + (v + w)\chi_{\Omega_2}, \quad (114)$$

$$Y = H^1(\Gamma_{01}) \times (H^1(\Omega) \cap W^{1,3}(\Omega_2)) \cap \{(u, w) : u(0) = w(0)\}, \quad (115)$$

$$Y_g = Y \cap \{(u, w), w|_{\Gamma} = g\}, \quad (116)$$

$$Y_0 = Y \cap \{(\psi_u, \psi_w) : \psi_w|_{\Gamma} = 0\}. \quad (117)$$

We will prove that (112) has a positive solution $(u, w) \in Y_g$ using a fixed point approach. Namely, set

$$X := C^0(\bar{\Gamma}_{01}) \times H^{\frac{1}{2}+\epsilon}(\Omega), \quad 0 < \epsilon < \frac{1}{2}, \quad (118)$$

then for $(\hat{u}, \hat{w}) \in X$, we consider $(u, w) \in Y_g$ solution of

$$\begin{aligned} \int_{\Gamma_{01}} (\hat{u}u'\psi'_u + (N(u, \hat{w}) - f)\psi_u - N(\hat{u}, w)\psi_w) - \int_{\Gamma_{02}} (h\psi_w) + \int_{\Omega} D(\hat{w}, |\nabla w|)(\nabla w \cdot \nabla \psi_w) &= 0, \\ \forall (\psi_u, \psi_w) \in Y_0. \end{aligned} \quad (119)$$

For given (\hat{u}, \hat{w}) the solution (u, w) of (119) and the solution (u, w) of (112) are different, but related. Indeed, if we consider the map:

$$\begin{aligned} T : B \subset X &\rightarrow C \subset Y \subset X \\ (\hat{u}, \hat{w}) &\rightarrow (u, w) = \text{solution of (119)} \end{aligned} \quad (120)$$

with B and C subsets to be defined in such way that $T(B) \subset C$. Next any fixed point of T is a solution of (112).

Remark 2.2.1. Note that with this choice of fixing, the new weak form (119) is the Euler-Lagrange equation for a variational problem (to be shown later). \square

Note that with $(\hat{u}, \hat{w}) \in X$, the integrals in (119) are finite. In the following remark we will show that $\mathbf{j}(\hat{w})$ is regular enough to have the integral on Γ_{01} , equation (119) well defined.

Remark 2.2.2. Note that if $(\hat{u}, \hat{w}) \in X$, then $\mathbf{j}(\hat{w}) \in L^2(\Gamma_{01})$. Fixed $s \in \Gamma_{01}$, then Taylor expansion theorem around 0 (see [6]) and (101) imply

$$\begin{aligned} |\mathbf{j}(\hat{w}(s))| &= |\mathbf{j}(0) + (\hat{w}(s) - 0)\sigma(c(s))| \\ &\leq \bar{\sigma}|\hat{w}(s)|, \end{aligned} \quad (121)$$

for some $c(s) \in (0, \hat{w}(s))$. Therefore, having $\hat{w} \in L^2(\Gamma_{01})$ (which is true because we have $\hat{w} \in H^\epsilon(\partial\Omega)$ from trace theorem, see Theorem 1.2.2., [18]) implies $\mathbf{j}(\hat{w}) \in L^2(\Gamma_{01})$. \square

In the following remark we will define the norms on the spaces X and Y .

Remark 2.2.3.

(i) The spaces X and Y are equipped with the norms

$$\begin{aligned} \|(\hat{u}, \hat{w})\|_X &:= \|\hat{u}\|_{C^0(\bar{\Gamma}_{01})} + \|\hat{w}\|_{H^{\frac{1}{2}+\epsilon}(\Omega)}, \\ \|(u, w)\|_Y &:= \|u\|_{H^1(\Gamma_{01})} + \|w\|_{H^1(\Omega_1)} + \|w\|_{W^{1,3}(\Omega_2)}. \end{aligned}$$

Note that equipped with these norms X and Y are Banach spaces and Y is reflexive.

(ii) The choice of X is not unique, we need Y compactly imbedded in X ,. One can look for any space X where Y is compactly imbedded in X and X satisfies all the convergent conditions needed in this chapter. \square

To prove that the operator T is well-defined and that it has a fixed point, again a certain number of a priori estimates are needed, in particular, the positiveness and boundedness of (u, w) .

2.2.1 A priori estimates.

Remember how important the one variable function N (38) was in estimating (u, w) solution for (35) from below, see Propositions 1.2.4 and 1.2.6. In the following work the same idea will be applied on the two variables function N (113).

In the following lemma we will prove monotonicity of N with respect to both u and w , which will help us in proving that solution of (119) is positive and bounded, and in proving that T is well defined.

Lemma 2.2.4. *The function $u \mapsto -N(u, w)$ is increasing and the function $w \mapsto -N(u, w)$ is decreasing.*

Proof. Since $u \mapsto -(\bar{w} - u)^+$ is decreasing, then $u \mapsto -N(u, w)$ is increasing. Also, as the function $w \mapsto -j(w)$ is increasing, then $w \mapsto -N(u, w)$ is decreasing. \square

A similar positive result to that in Proposition 1.2.4 still holds here for (u, w) satisfying (119). We have

Proposition 2.2.5. *Assume (100)-(103) hold. Let $(u, w) \in Y_g$ be a solution of (119) corresponding to any $(\hat{u}, \hat{w}) \in X$. If $\hat{u} \geq 0$ and $\hat{w} \geq 0$ then:*

(i) $u \geq 0$ on Γ_{01} .

(ii) $w \geq 0$ a.e. in Ω .

Proof. Let u^- , resp. w^- , denote the negative part of u , resp. w . Note that $u^-(0) = w^-(0)$ and $w^- = 0$ on Γ because $\inf_{\Gamma} g > 0$. From Theorem 7.6, [15] and Proposition 3.5, [16] we have $(u^-, w^-) \in Y_0$ and

$$(u^-)' = \chi_{\{u \leq 0\}} u', \quad \nabla w^- = \chi_{\{w \leq 0\}} \nabla w.$$

Taking $(\psi_u, \psi_w) := (u^-, w^-)$ in (119) we get

$$\int_{\Gamma_{01}} \hat{u}((u^-)')^2 + (N(u, \hat{w}) - f)u^- - N(\hat{u}, w)w^- - \int_{\Gamma_{02}} hw^- + \int_{\Omega} D(\hat{w}, |\nabla w|)|\nabla w^-|^2 = 0.$$

Note that from

$$\begin{aligned} N(u, \hat{w})u^- &= duu^- - aj(\hat{w})(\bar{w} - u)^+u^- \geq d|u^-|^2 \geq 0, \\ N(\hat{u}, w)w^- &= d\hat{u}w^- - aj(w)(\bar{w} - \hat{u})^+w^- \leq 0, \end{aligned}$$

it follows that

$$\int_{\Gamma_{01}} d|u^-|^2 + \int_{\Omega_1} \underline{\sigma} |\nabla w^-|^2 + \int_{\Omega_2} |\nabla w^-|^3 \leq 0,$$

which implies $u^- = 0$ a.e. on Γ_{01} , so $u \geq 0$ a.e. on Γ_{01} . Since $u \in H^1(\Gamma_{01}) \subset C^0(\bar{\Gamma}_{01})$, then $u \geq 0$ everywhere on Γ_{01} . Also $\nabla w^- = 0$ a.e. in Ω , which implies that $w^- = K$ a.e. in Ω , for some constant K . However, $w^-|_{\Gamma} = 0$ because $g > 0$. Therefore, $w \geq 0$ a.e. in Ω . \square

The root of the one variable function N (38) was very helpful in finding the lower bound u_m for the solution in Proposition 1.2.6. The same idea will be applied here on the following function.

$$\begin{aligned} \mathbf{n} : [0, \infty) &\rightarrow \mathbb{R} \\ z &\rightarrow \mathbf{n}(z) := dz - a\mathbf{j}(z)(\bar{w} - z)^+ (= N(z, z)). \end{aligned} \quad (122)$$

The following proposition contains a sufficient condition for the existence of roots for the function \mathbf{n} .

Proposition 2.2.6. *If $\underline{\sigma}$ satisfies (104), i.e. $\frac{d}{a} \frac{1}{\bar{w}} < \underline{\sigma}$, then there exists always a root u_m of \mathbf{n} such that*

$$\mathbf{n}(u_m) = 0, \quad \mathbf{n}(z) < 0 \text{ in } (0, u_m), \quad 0 < u_m < \bar{w}. \quad (123)$$

If moreover $\bar{\sigma}$ satisfies (104), i.e. $\bar{\sigma} < \frac{d}{a} \frac{1}{(\bar{w} - \inf_{\Gamma} g)^+}$, then

$$0 < u_m \leq \min\{\inf_{\Gamma} g, \bar{w}\}. \quad (124)$$

Proof. We have $\mathbf{n}(z) \in C^0([0, \infty))$, $\mathbf{n}(z) = dz - a\mathbf{j}(z)(\bar{w} - z)$ on $[0, \bar{w}]$. So $\mathbf{n} \in C^1([0, \bar{w}])$ and

$$\mathbf{n}'(0) = d - a\bar{w}\sigma(0) < 0, \quad \mathbf{n}(z) = dz > 0 \text{ on } [\bar{w}, +\infty).$$

Therefore, $\mathbf{n}(z) = 0$ has solutions in $(0, \bar{w})$. Let us denote

$$u_m = \inf\{z > 0, \mathbf{n}(z) = 0\}. \quad (125)$$

We have $0 < u_m < \bar{w}$ because $\mathbf{n}(0) < 0$ and $\mathbf{n}'(0) < 0$ and $\mathbf{n}(z) > 0$ on $[\bar{w}, \infty)$.

Let us prove that $u_m \leq \inf_{\Gamma} g$. We can assume that $0 < \inf_{\Gamma} g < \bar{w}$ as otherwise, so if $\inf_{\Gamma} g \geq \bar{w}$, the claim is proved. Assume, by absurd, that $\inf_{\Gamma} g < u_m$. From the choice of u_m we have $\mathbf{n}(\inf_{\Gamma} g) < 0$, which implies that

$$a\mathbf{j}(\inf_{\Gamma} g)(\bar{w} - \inf_{\Gamma} g) > d \inf_{\Gamma} g.$$

Hence applying the Taylor expansion theorem around 0 (see [6]) on the function \mathbf{j} , it follows

$$b\sigma(c) \inf_{\Gamma} g(\bar{w} - \inf_{\Gamma} g) > d \inf_{\Gamma} g, \quad (126)$$

for some constant $c \in (0, \inf_{\Gamma} g)$. Simplifying and rearranging implies

$$\bar{\sigma} \geq \sigma(c) > \frac{d}{a} \frac{1}{(\bar{w} - \inf_{\Gamma} g)},$$

which contradicts (104). □

A similar result to that in Proposition 1.2.6 can be proved for (u, w) satisfying (119), that is the boundedness of the solution (u, w) of (119) (if it exists) from below by the constant u_m .

Proposition 2.2.7. *Assume (100)-(104) hold. Let $(\hat{u}, \hat{w}) \in X$ with $\hat{u} \geq u_m$, $\hat{w} \geq u_m$ and assume $(u, w) \in Y_g$ solves (119), then $u \geq u_m$, $w \geq u_m$.*

Proof. Assume by absurd that the proposition does not hold, namely assume that $u < u_m$ or $w < u_m$ on a set of nonzero measure. Define $(\psi_u, \psi_w) := ((u - u_m)^-, (w - u_m)^-)$, so we have $(\psi_u, \psi_w) \neq (0, 0)$ on a set of nonzero measure. Since $(u, w) \in Y_g$ and $u_m \leq \inf_{\Gamma} g$ (124) we have $(\psi_u, \psi_w) \in Y_0$ (see Theorem 7.6 [15] and Proposition 3.5, [16]). With this choice of (ψ_u, ψ_w) the equation (119) gives

$$0 = \int_{\Gamma_{01}} \hat{u}(\psi'_u)^2 + ((N(u, \hat{w}) - f)\psi_u - N(\hat{u}, w)\psi_w) - \int_{\Gamma_{02}} h\psi_w + \int_{\Omega} D(\hat{w}, |\nabla w|)|\nabla \psi_w|^2.$$

Note that

$$\begin{aligned} N(u, \hat{w})\psi_u &\geq N(u_m, \hat{w})\psi_u \geq N(u_m, u_m)\psi_u = \mathbf{n}(u_m)\psi_u = 0, \\ N(\hat{u}, w)\psi_w &\leq N(u_m, w)\psi_w \leq N(u_m, u_m)\psi_w = \mathbf{n}(u_m)\psi_w = 0. \end{aligned}$$

As $f\psi_u \leq 0$, $h\psi_w \leq 0$ it follows that

$$\int_{\Gamma_{01}} \hat{u}|\psi'_u|^2 + \int_{\Omega_1} \underline{\sigma}|\nabla \psi_w|^2 + \int_{\Omega_2} |\nabla \psi_w|^3 \leq 0,$$

which implies $\psi'_u = 0$ a.e. on Γ_{01} and $\nabla \psi_w = 0$ a.e. in Ω . However, $\psi_w|_{\Gamma} = 0$ and $\psi_u(0) = \psi_w(0)$. Therefore, $\psi_u = 0$ a.e. on Γ_{01} and $\psi_w = 0$ a.e. Ω , which is impossible. By this contradiction we reached the end of the proof. □

Let $u^m > u_m$ be a constant that will be determined later, and set

$$\begin{aligned} B &= X \cap \{(\hat{u}, \hat{w}) : u_m \leq \hat{u} \leq u^m, u_m \leq \hat{w}\}, \\ C &= Y_g \cap \{(u, w) : u_m \leq u \leq u^m, u_m \leq w\}. \end{aligned} \quad (127)$$

Note that this is a similar setting to that done in (42). However, the spaces X and Y in this chapter are different from X and Y in Chapter 1.

Remark 2.2.8. The sets B, C are convex and closed for the same reasons given in Remark 1.2.7. Note that if $\hat{w}_n \rightarrow \hat{w}$ in $H^{\frac{1}{2}+\epsilon}(\Omega) \subset L^1(\Omega)$ (compact imbeddings, see [1]), then $\hat{w}_n \rightarrow \hat{w}$ a.e. in Ω (see Theorem IV.2, [5], convergence in $L^1(\Omega)$ implies convergence a.e. in Ω) (converging in $L^p(\Omega)$ implies converging a.e. in Ω), which helps in proving the closedness of B . \square

In the following lemma we will introduce some estimations that will be used in the next work.

Lemma 2.2.9. *If $(\hat{u}, \hat{w}) \in X$, $(u, w) \in Y_g$ and $\psi_w := (w - \sup_{\Gamma} g)^+$, then (101), Hölder, triangle, Poincaré and Young inequalities together with trace, imbedding and Taylor expansion theorems imply*

$$\int_{\Gamma_{01}} \hat{u}(u')^2 \leq \|\hat{u}\|_{C^0(\bar{\Gamma}_{01})} \|u\|_{H^1(\Gamma_{01})}^2, \quad (128)$$

$$\int_{\Omega_1} \sigma(\hat{w})|\nabla w|^2 \leq \bar{\sigma} \|w\|_{H^1(\Omega_1)}^2, \quad (129)$$

$$\int_{\Omega_2} \hat{w}|\nabla w|^2 \leq \|\hat{w}\|_{C^0(\bar{\Omega}_2)} \|w\|_{H^1(\Omega_2)}^2, \quad (130)$$

$$\int_{\Omega_2} |\nabla w|^3 = \|\nabla w\|_{L^3(\Omega_2)}^3 \leq \|w\|_{W^{1,3}(\Omega_2)}^3, \quad (131)$$

$$\left| \int_{\Gamma_{01}} \int_0^{u(x)} s ds dx \right| \leq \|u\|_{H^1(\Gamma_{01})}^2, \quad (132)$$

$$\left| \int_{\Gamma_{01}} \int_0^{w(s)} \mathbf{j}(k)(\bar{w} - \hat{u})^+ dk ds \right| \leq \bar{\sigma} \bar{w} \|w\|_{H^1(\Omega_1)}^2, \quad (133)$$

$$\left| \int_{\Gamma_{01}} \int_0^{u(t)} \mathbf{j}(\hat{w})(\bar{w} - k)^+ dk dt \right| \leq \bar{\sigma} \bar{w} \|\hat{w}\|_{H^{\frac{1}{2}+\epsilon}(\Omega)} \|u\|_{L^2(\Gamma_{01})}, \quad (134)$$

$$\int_{\Gamma_{01}} (a\bar{\sigma}\bar{w}|\hat{w}| + |f|)|u| \leq \frac{1}{d} \|a\bar{\sigma}\bar{w}|\hat{w}| + f\|_{L^2(\Gamma_{01})}^2 + \frac{d}{4} \|u\|_{L^2(\Gamma_{01})}^2, \quad (135)$$

$$\int_{\Gamma_{02}} h|w| \leq N_2 \|h\|_{L^{\frac{3}{2}}(\Gamma_{02})}^{\frac{3}{2}} + \frac{1}{6} \|\nabla w\|_{L^3(\Omega_2)}^3 + N_3 \|h\|_{L^{\frac{3}{2}}(\Gamma_{02})} \|G\|_{W^{1,3}(\Omega)}, \quad (136)$$

$$\int_{\Gamma_{02}} h|\psi_w| \leq N_1 \|h\|_{L^{\frac{3}{2}}(\Gamma_{02})}^{\frac{3}{2}} + \frac{1}{3} \|\nabla \psi_w\|_{L^3(\Omega)}^3, \quad (137)$$

$$du^m \|\psi_w\|_{L^1(\Gamma_{01})} \leq \frac{\sigma}{2} \|\nabla \psi_w\|_{L^2(\Omega_1)}^2 + \frac{L_1^2 d^2 u_m^2}{2\sigma}, \quad (138)$$

$$\|w\|_{L^1(\Gamma_{01})} \leq \frac{\sigma}{4} \|\nabla w\|_{L^2(\Omega_1)}^2 + L_2(G), \quad (139)$$

$$\|w\|_{H^1(\Omega_1)} \leq L_3(\|\nabla w\|_{L^2(\Omega_1)} + \|G\|_{H^1(\Omega_1)}), \quad (140)$$

$$\|w\|_{W^{1,3}(\Omega_2)} \leq L_4(\|\nabla w\|_{L^3(\Omega_2)} + \|G\|_{W^{1,3}(\Omega_2)}), \quad (141)$$

Here, L_* and N_* are constants independent from u and w .

A result like that in Proposition 1.2.9 still holds here, that is

Proposition 2.2.10. *Assume (100)-(104) hold. There exists*

$u^m \geq \max\{\bar{w}, \sup_{\Gamma} g, \frac{\|f\|_{L^\infty}}{d}\}$ such that if $(\hat{u}, \hat{w}) \in B$ and $(u, w) \in Y_g$ solves (119) then $(u, w) \in C$.

Proof. The proof is similar to that done for Proposition 1.2.9. However, details will be shown again here to be used in the future.

Let u_m be given by (124) and $(\hat{u}, \hat{w}) \in B$. By Proposition 2.2.7 we have $u_m \leq u$, $u_m \leq w$. Similar to the proof of Proposition 1.2.9 we have

$$\hat{u}(0)u'(0)\psi_u(0) = \int_{\Gamma_{01}} \hat{u}(u'\psi'_u) + (du - a\mathcal{J}(\hat{w})(\bar{w} - u)^+ - f)\psi_u, \quad \forall \psi_u \in H^1(\Gamma_{01}). \quad (142)$$

Claim (i): Assume $\hat{u}(0)u'(0) \leq 0$, then

$$u \leq \max\{\bar{w}, \frac{\|f\|_{L^\infty}}{d}\}. \quad (143)$$

Indeed, taking $\psi_u := \left(u - \max\{\bar{w}, \frac{\|f\|_{L^\infty}}{d}\}\right)^+$ in (142) gives

$$\begin{aligned} 0 \geq \hat{u}(0)u'(0)\psi_u(0) &\geq \int_{\Gamma_{01}} \hat{u}|\psi'_u|^2 + (-a\mathcal{J}(\hat{w})(\bar{w} - u)^+ + du - f)\psi_u \\ &= \int_{\Gamma_{01}} \hat{u}|\psi'_u|^2 + (du - f)\psi_u \\ &\geq \int_{\Gamma_{01}} \hat{u}|\psi'_u|^2 + (du - \|f\|_{L^\infty(\Gamma_{01})})\psi_u, \end{aligned}$$

which implies $\psi_u = 0$ a.e. on Γ_{01} , which proves the claim.

Claim (ii): Assume $\hat{u}(0)u'(0) > 0$. Set

$$\rho(z) = K_1^3 \|(z - \sup_{\Gamma} g)^+\|_{C^0(\bar{\Omega}_2)}^3 - \frac{L_1 d^2}{2\sigma} (z)^2 - \frac{5}{6} N_1 \|h\|_{L^{\frac{3}{2}}(\Gamma_{02})}^{\frac{3}{2}}, \quad (144)$$

with K_1, L_1 and N_1 , are constants to be determined later in this proof. The roots of this polynomial play a key role in proving boundedness above for u , as we will see.

Let z^u denote the largest positive root of ρ (to be proved as a part of this proof) i.e.

$$z^u = \max\{z : \rho(z) = 0\}, \quad (145)$$

then $z^u \geq \sup_{\Gamma} g$. If moreover $u^m \geq z^u$ then $w|_{\Omega_2} \leq u^m$.

Indeed, set $(\psi_u, \psi_w) := ((u - \sup_{\Gamma} g)^+, (w - \sup_{\Gamma} g)^+)$. Of course $(\psi_u, \psi_w) \in Y_0$ and then from (119) and (142) we obtain

$$\begin{aligned} & \underline{\sigma} \int_{\Omega_1} |\nabla \psi_w|^2 + \int_{\Omega_2} |\nabla \psi_w|^3 \\ & \leq \int_{\Omega_1} \sigma(\hat{w}) |\nabla \psi_w|^2 + \int_{\Omega_2} (\hat{w} + |\nabla w|) |\nabla \psi_w|^2 \\ & = - \int_{\Gamma_{01}} (\hat{u}(u' \psi'_u) + (N(u, \hat{w}) - f) \psi_u) + \int_{\Gamma_{01}} N(\hat{u}, w) \psi_w + \int_{\Gamma_{02}} (h \psi_w) \\ & = -\hat{u}(0)u'(0)\psi_u(0) + \int_{\Gamma_{01}} N(\hat{u}, w) \psi_w + \int_{\Gamma_{02}} h \psi_w \\ & < \int_{\Gamma_{01}} N(\hat{u}, w) \psi_w + \int_{\Gamma_{02}} h \psi_w \\ & \leq \int_{\Gamma_{01}} d\hat{u} \psi_w + \int_{\Gamma_{02}} h \psi_w \\ & \leq d u^m \|\psi_w\|_{L^1(\Gamma_{01})} + \int_{\Gamma_{02}} h \psi_w \\ & \leq \frac{\sigma}{4} \|\nabla \psi_w\|_{L^2(\Omega_1)}^2 + \frac{1}{6} \|\nabla \psi_w\|_{L^3(\Omega_2)}^3 + \frac{L_1 d^2}{2\sigma} (u^m)^2 + N_2 \|h\|_{L^{\frac{3}{2}}(\Gamma_{02})}^{\frac{3}{2}}. \end{aligned} \quad (146)$$

where we have used (137) and (138). From (146) we obtain

$$\frac{5}{6} \int_{\Omega_2} |\nabla \psi_w|^3 < \frac{L_1 d^2}{2\sigma} (u^m)^2 + \frac{5}{6} N_1 \|h\|_{L^{\frac{3}{2}}(\Gamma_{02})}^{\frac{3}{2}}.$$

As $W^{1,3}(\Omega_2) \subset C^0(\bar{\Omega}_2)$ (imbedding theorem, [1]), then we have

$$K_1^3 \|(w - \sup_{\Gamma} g)^+\|_{C^0(\bar{\Omega}_2)}^3 \leq \frac{5}{6} \int_{\Omega_2} |\nabla \psi_w|^3,$$

for some constant k_1 , which implies

$$K_1^3 \|(w - \sup_{\Gamma} g)^+\|_{C^0(\bar{\Omega}_2)}^3 < \frac{L_1 d^2}{2\sigma} (u^m)^2 + \frac{5}{6} N_1 \|h\|_{L^{\frac{3}{2}}(\Gamma_{02})}^{\frac{3}{2}}. \quad (147)$$

Now, we recall the function ρ given by (144). Note that $\rho(z)$ is obtained essentially from (147) by replacing w and u^m by z . We have

$$\rho \in C^0(\mathbb{R}), \quad \rho(\sup_{\Gamma} g) < 0, \quad \lim_{z \rightarrow +\infty} \rho(z) = +\infty.$$

Therefore, ρ has at least one real root greater than $\sup_{\Gamma} g$. So z^u given by (145) satisfies $z^u \geq \sup_{\Gamma} g$. Now, if by absurd $w|_{\Omega_2} > u^m$ for a point in Ω_2 , then from (147), which always holds because $(\hat{u}, \hat{w}) \in B$, we obtain

$$\begin{aligned} 0 &> K_1^3 \|(w - \sup_{\Gamma} g)^+\|_{C^0(\bar{\Omega}_2)}^3 - \left(\frac{K_1 d^2}{2\sigma} (u^m)^2 + \frac{2}{3} N_1 \|h\|_{L^{\frac{3}{2}}(\Gamma_{02})}^{\frac{3}{2}} \right) \\ &> \rho(u^m) \geq \rho(z^u) = 0, \end{aligned} \quad (148)$$

which is a contradiction and proves that $w|_{\Omega_2} \leq u^m$.

(iii) *Claim: Set*

$$u^m = \max\{z^u, \bar{w}, \sup_{\Gamma} g, \frac{\|f\|_{L^\infty}}{d}\}, \quad (149)$$

then the proposition holds.

Indeed, if $\hat{u}(0)u'(0) \leq 0$ then the Claim (i) holds and the proposition follows from (143). If $\hat{u}(0)u'(0) > 0$ then as $u^m \geq z^u$ the proposition follows from Claim (ii). \square

Note that (147), implies

Corollary 2.2.11. *Assume $\max\{\sup_{\Gamma} g, \frac{\|f\|_{L^\infty(\Gamma_{01})}}{d}\} < \bar{w}$. If*

$$\frac{d^2}{\sigma}, \quad \|h\|_{L^{\frac{3}{2}}(\Gamma_{02})} \quad \text{are small enough,} \quad (150)$$

then $u^m \leq \bar{w}$.

Proof. From (144), we have $z^u \rightarrow \sup_{\Gamma} g$, when $\frac{d^2}{\sigma}, \|h\|_{L^{\frac{3}{2}}(\Gamma_{02})} \ll 1$. If in addition $\max\{\sup_{\Gamma} g, \frac{\|f\|_{L^\infty(\Gamma_{01})}}{d}\} < \bar{w}$, then (149) implies $u^m \leq \bar{w}$ \square

Also we have the following.

Proposition 2.2.12. *Assume the conditions (100)-(104) hold. There exists $w^m > u_m$ such that if $(u, w) \in B$ is a fixed point of T then $\|w\|_{C^0(\bar{\Omega}_2)} \leq w^m$.*

If moreover $\max\{\sup_{\Gamma} g, \frac{\|f\|_{L^\infty(\Gamma_{01})}}{d}\} < \bar{w}$ and $\frac{d^2}{\underline{\sigma}}, \frac{(a\bar{\sigma})^2}{\underline{\sigma}}, \|f\|_{L^1(\Gamma_{01})}, \|h\|_{L^{\frac{3}{2}}(\Gamma_{02})}$ are small enough, then $w^m \leq \bar{w}$.

Proof. From (119) with $(\hat{u}, \hat{w}) = (u, w)$ and (ψ_u, ψ_w) as in Proposition 2.2.10, we obtain

$$\begin{aligned} u_m \int_{\Gamma_{01}} (\psi'_u)^2 &+ \underline{\sigma} \int_{\Omega_1} |\nabla \psi_w|^2 + \int_{\Omega_2} |\nabla \psi_w|^3 \\ &\leq \int_{\Gamma_{01}} (f - N(u, w))\psi_u + N(u, w)\psi_w + \int_{\Gamma_{02}} h\psi_w \\ &\leq \int_{\Gamma_{01}} u^m(f + a\bar{\sigma} \bar{w} + d\psi_w) + \int_{\Gamma_{02}} h\psi_w \end{aligned} \quad (151)$$

$$\begin{aligned} &\leq \int_{\Gamma_{01}} u^m(f + a\bar{\sigma} \bar{w} \sup_{\Gamma} g + (d + a\bar{\sigma} \bar{w})\psi_w) + \int_{\Gamma_{02}} h\psi_w \\ &\leq u^m \left(\|f + a\bar{\sigma} \bar{w} \sup_{\Gamma} g\|_{L^1(\Gamma_{01})} + (d + \bar{w}a\bar{\sigma})K_1 \|\nabla \psi_w\|_{L^2(\Omega_1)} \right) \\ &+ K_2 \|h\|_{L^{\frac{3}{2}}(\Gamma_{02})} \|\nabla \psi_w\|_{L^3(\Omega_2)} \\ &\leq u^m \|f + a\bar{\sigma} \bar{w} \sup_{\Gamma} g\|_{L^1(\Gamma_{01})} + \frac{(K_1 u^m)^2 (d + a\bar{\sigma} \bar{w})^2}{4\underline{\sigma}} + \frac{2}{3} K_2^{\frac{3}{2}} \|h\|_{L^{\frac{3}{2}}(\Gamma_{02})}^{\frac{3}{2}} \\ &+ \underline{\sigma} \int_{\Omega_1} |\nabla \psi_w|^2 + \frac{1}{3} \int_{\Omega_2} |\nabla \psi_w|^3, \end{aligned} \quad (152)$$

for some constant K_1 and K_2 , where we have used Hölder, Poincaré and Young inequalities together with trace theorem. Therefore, from (152) we get

$$\begin{aligned} \|w\|_{C^0(\bar{\Omega}_2)} &\leq \sup_{\Gamma} g + \|(w - \sup_{\Gamma} g)^+\|_{C^0(\bar{\Omega}_2)} \\ &\leq \sup_{\Gamma} g + K_3 \|\nabla \psi_w\|_{L^3(\Omega_2)} \\ &\leq \sup_{\Gamma} g \\ &+ K_3 \left[\frac{3}{2} \left(u^m \| \bar{w} a \bar{\sigma} \sup_{\Gamma} g + f \|_{L^1(\Gamma_{01})} + \frac{(K_1 u^m)^2 (d + \bar{w} a \bar{\sigma})^2}{4\underline{\sigma}} + \frac{2}{3} K_2^{\frac{3}{2}} \|h\|_{L^{\frac{3}{2}}(\Gamma_{02})}^{\frac{3}{2}} \right) \right]^{1/3} \\ &:= w^m, \end{aligned} \quad (153)$$

for some constant K_3 . Imbedding theorem and Poincaré's inequality are used here. Now from (153), it is clear that $w \leq \bar{w}$ when $\frac{d^2}{\underline{\sigma}}, \frac{(a\bar{\sigma})^2}{\underline{\sigma}}, \|f\|_{L^1(\Gamma_{01})}, \|h\|_{L^{\frac{3}{2}}(\Gamma_{02})}$ are small enough. \square

2.3 Operator T is well-defined.

To prove the existence of (u, w) that satisfies the weak form (119) we will look at this equation as the Euler-Lagrange equation of a functional I (the same technique used in Section 1.3). We will show that the functional I has a unique minimizer, which will prove that T is well-defined.

Let $(\hat{u}, \hat{w}) \in B$ and consider

$$\begin{aligned} I(u, w) &:= \int_{\Gamma_{01}} \frac{1}{2} \hat{u} (u')^2 + \frac{1}{6} \int_{\Omega} (3\sigma(\hat{w})\chi_{\Omega_1} + (3\hat{w} + 2|\nabla w|)\chi_{\Omega_2}) |\nabla w|^2 \\ &+ \int_{\Gamma_{01}} \int_0^{u(s)} (N(k, \hat{w}(s)) - f(s)) dk ds - \int_{\Gamma_{01}} \int_0^{w(s)} N(\hat{u}(s), k) dk ds - \int_{\Gamma_{02}} h w dt. \end{aligned} \quad (154)$$

$$\text{Find } (u, w) \in Y_g : \quad I(u, w) = \min\{I(\psi_u, \psi_w), (\psi_u, \psi_w) \in Y_g\}. \quad (155)$$

Note that $I(u, w; \hat{u}, \hat{w})$ will also be used instead of $I(u, w)$, to show the dependence of I on (\hat{u}, \hat{w}) .

In the following proposition we will prove some useful properties for the functional I .

Proposition 2.3.1. *The functional $I : (u, w) \in Y_g \rightarrow \mathcal{H}(u, w) \in \mathbb{R}$ satisfies:*

(i) $(u, w) \mapsto \mathcal{H}(u, w)$ is strictly convex.

(ii) $(u, w) \mapsto \mathcal{H}(u, w)$ is lower semicontinuous with respect to weak topology in Y .

(iii) For all $(\hat{u}_n, \hat{w}_n) \in X$ converging to (\hat{u}, \hat{w}) in X and $\hat{w}_n \rightarrow \hat{w}$ a.e., for all $(u_n, w_n) \in Y_g$ bounded in Y and for any $(u, w) \in Y_g$ we have

$$\lim_{n \rightarrow \infty} (I(u_n, w_n; \hat{u}_n, \hat{w}_n) - I(u_n, w_n; \hat{u}, \hat{w})) = \lim_{n \rightarrow \infty} (I(u, w; \hat{u}_n, \hat{w}_n) - I(u, w; \hat{u}, \hat{w})) = 0. \quad (156)$$

Proof. Note that (i) and (ii) have the same proof done for Proposition 1.3.1. For (iii) we have

$$\begin{aligned} |I(u, w; \hat{u}_n, \hat{w}_n) - I(u, w; \hat{u}, \hat{w})| &\leq \int_{\Gamma_{01}} \frac{1}{2} |\hat{u}_n - \hat{u}| (u')^2 + \int_{\Omega} \frac{1}{2} (|\sigma(\hat{w}_n) - \sigma(\hat{w})|\chi_{\Omega_1} + |\hat{w}_n - \hat{w}|\chi_{\Omega_2}) |\nabla w|^2 \\ &+ \int_{\Gamma_{01}} a |\mathcal{J}(\hat{w}_n) - \mathcal{J}(\hat{w})| \int_0^{|u(s)|} (\bar{w} - k)^+ dk ds \\ &+ \int_{\Gamma_{01}} \int_0^{|w(s)|} (d|\hat{u}_n - \hat{u}| + a|\mathcal{J}(k)| |(\bar{w} - \hat{u}_n)^+ - (\bar{w} - \hat{u})^+|) dk ds \end{aligned}$$

$$\begin{aligned} &\leq \int_{\Gamma_{01}} \frac{1}{2} |\hat{u}_n - \hat{u}| (u')^2 + \int_{\Omega} \frac{1}{2} (|\sigma(\hat{w}_n) - \sigma(\hat{w})| \chi_{\Omega_1} + |\hat{w}_n - \hat{w}| \chi_{\Omega_2}) |\nabla w|^2 \\ &+ \int_{\Gamma_{01}} a \bar{w} \bar{\sigma} |\hat{w}_n - \hat{w}| |u| ds + \int_{\Gamma_{01}} |\hat{u}_n - \hat{u}| \left(d|w| + \frac{1}{2} a \bar{\sigma} |w|^2 \right) ds. \end{aligned}$$

Note that from Sobolev imbedding we have $\hat{w}_n - \hat{w} \in L^q(\Omega)$, $1 \leq q \leq \frac{4}{1-2\epsilon}$, see [12] and [11]. From convergence of (\hat{u}_n, \hat{w}_n) in X and $\hat{w}_n \rightarrow \hat{w}$ a.e. in $\Gamma_{01} \times \Omega$, and from Lebesgue's dominated convergence theorem it follows that all the integrals on the right hand side of the last inequality tend to zero. The other limit is proven similarly. \square

Now let us prove the relation between (119) and the minimization problem (155).

Proposition 2.3.2. *The functional $(u, w) \mapsto \mathcal{I}(u, w)$ is Gâteaux differentiable and for $(\psi_u, \psi) \in Y_0$ we have*

$$\begin{aligned} I'(u, w)(\psi_u, \psi) &= \int_{\Gamma_{01}} \hat{u} (u_s \psi'_u) + \int_{\Omega} D(\hat{w}, w)(\nabla w \cdot \nabla \psi_w) \\ &+ \int_{\Gamma_{01}} ((N(u, \hat{w}) - f) \psi_u - N(\hat{u}, w) \psi_w) ds - \int_{\Gamma_{02}} h \psi_w, \end{aligned} \quad (157)$$

Proof. The proof is similar to that done for proposition 1.3.2. \square

As a result we have the following proposition.

Proposition 2.3.3. *The equation (119) is the Euler-Lagrange equation of $I(u, w)$.*

Now we can prove that T is well defined.

Theorem 2.3.4. *Assume (100)-(104) hold. For any $(\hat{u}, \hat{w}) \in B \subset X$ the equation (119) has a unique solution $(u, w) \in C \subset Y$. Thus, the operator $T : B \subset X \rightarrow C \subset Y$, $T(\hat{u}, \hat{w}) = (u, w)$ is well-defined.*

Proof. The idea of the proof is similar to that in the proof of Theorem 1.3.4. Again we have two parts.

Uniqueness. It follows from strict convexity of I . For the sake of completeness, we will show the proof in detail. Recall Lemma 2.2.4 that $N(u, w)$ is increasing in u and decreasing in w . Therefore

$$(N(u_1, w) - N(u_2, w))(u_1 - u_2) \geq 0, \quad (N(u, w_1) - N(u, w_2))(w_1 - w_2) \leq 0. \quad (158)$$

Let (u_i, w_i) , $i = 1, 2$ be two solutions of (119). Consider (119) for each solution (u_i, w_i) , with $(\psi_u, \psi_w) := (u_1 - u_2, w_1 - w_2) \in Y_0$, and subtract them, then we obtain

$$\begin{aligned} 0 &= \int_{\Gamma_{01}} \hat{u}(\psi'_u)^2 + (N(u_1, \hat{w}) - N(u_2, \hat{w}))\psi_u + (N(\hat{u}, w_2) - N(\hat{u}, w_1))\psi_w \\ &+ \int_{\Omega} (\sigma(\hat{w})\chi_{\Omega_1} + \chi_{\Omega_2}) |\nabla \psi_w|^2 + \int_{\Omega_2} (|\nabla w_1| |\nabla w_1| - |\nabla w_2| |\nabla w_2|) \cdot \nabla \psi_w. \end{aligned} \quad (159)$$

All the factors under this integral are nonnegative. Indeed, for the terms with N this follows from (158), for the nonnegativity of other terms see the proof of Theorem 1.3.4. Therefore, $\psi'_u = 0$ a.e. on Γ_{01} , $\nabla \psi_w = 0$ a.e. in Ω , which implies $\psi_u = \psi_w = K$ a.e. constant ($\psi_u(0) = \psi_w(0)$). Since $\psi_w|_{\Gamma} = 0$, then $\psi_u = \psi_w = 0$, which mean $(u_1, w_1) = (u_2, w_2)$.

Existence. As (119) cannot have more than one solution, to prove the existence of a solution of (119) it is enough to show that the problem (155) has a solution, then from Proposition 2.3.3 the solution of (155) is the only solution of (119).

As in the proof of existence in Theorem 1.3.4, Corollary III.35, [5] will be used here. Indeed, we have $I : C \rightarrow -\mathcal{R}$ is a convex function, with C nonempty closed convex set. Also $I(u, w) \neq \pm\infty$, which follows from (128)-(136).

$$\begin{aligned} I(u, w) &= \int_{\Gamma_{01}} \frac{1}{2} \hat{u}(u')^2 + \frac{1}{6} \int_{\Omega} (3\sigma(\hat{w})\chi_{\Omega_1} + (3\hat{w} + 2|\nabla w|)\chi_{\Omega_2}) |\nabla w|^2 \\ &+ \int_{\Gamma_{01}} \int_0^{u(s)} (N(k, \hat{w}(s)) - f(s)) dk ds - \int_{\Gamma_{01}} \int_0^{w(s)} N(\hat{u}(s), k) dk ds \\ &- \int_{\Gamma_{02}} h w dt \\ &\geq \int_{\Gamma_{01}} \frac{1}{2} u_m(u')^2 + \frac{1}{6} \int_{\Omega} (3\sigma\chi_{\Omega_1} + 2|\nabla w|\chi_{\Omega_2}) |\nabla w|^2 \\ &+ \int_{\Gamma_{01}} \int_0^{u(s)} (-a\hat{j}(\hat{w})(\bar{w} - k)^+ - |f(s)|) dk ds - \int_{\Gamma_{01}} \int_0^{w(s)} d\hat{u}(s) dk ds \\ &- \int_{\Gamma_{02}} h w dt \\ &\geq \int_{\Gamma_{01}} \frac{1}{2} (u_m(u')^2 + du^2) + \int_{\Omega_1} \frac{1}{2} \bar{\sigma} |\nabla w|^2 + \int_{\Omega_2} \frac{1}{3} |\nabla w|^3 \\ &- \int_{\Gamma_{01}} (a\bar{\sigma} \bar{w} |\hat{w}| + |f|) |u| - \int_{\Gamma_{01}} du^m |w| - \int_{\Gamma_{02}} |h| |w| \\ &\geq K_2 \left(\int_{\Gamma_{01}} (u')^2 + u^2 + \int_{\Omega_1} |\nabla w|^2 + \int_{\Omega_2} |\nabla w|^3 \right) + K_3, \end{aligned} \quad (160)$$

where K_2, K_3 are constants independent from (u, w) . Here we have used Taylor expansion theorem around 0 for \hat{j} . Also, we have used, (135), (136) and (139). Note that having

$\|w\|_{H^1(\Omega_1)} + \|w\|_{W^{1,3}(\Omega_2)} \rightarrow \infty$, implies $\|\nabla w\|_{L^2(\Omega_1)} + \|\nabla w\|_{L^3(\Omega_2)} \rightarrow \infty$ because of (140) and (141). Next, Corollary III.35, [5] implies $I(u, w)$ has a minimizer in C , which end the proof. \square

In the following remark we will prove that $I(|G(0)|, |G|; \hat{u}, \hat{w})$ is bounded, which will be used later to prove that T takes bounded sets to bounded sets. Note that $(|G(0)|, |G|) \in Y_g$ (see Theorem 7.6, [15] and Proposition 3.5, [16]).

Remark 2.3.5. For $(\hat{u}, \hat{w}) \in B$ satisfying $\|(\hat{u}, \hat{w})\|_X \leq M$, we have

$$\begin{aligned} I(|G(0)|, |G|; \hat{u}, \hat{w}) &\leq \int_{\Gamma_{01}} \frac{1}{2} (d(G(0))^2 + a\bar{\sigma} \bar{w}G^2) \\ &\quad + \int_{\Omega_1} \frac{1}{2} \bar{\sigma} |\nabla G|^2 + \int_{\Omega_2} \frac{1}{2} \hat{w} |\nabla G|^2 + \frac{1}{3} |\nabla G|^3 \\ &\leq K_1(G) + \frac{1}{2} \|\hat{w}\|_{L^3(\Omega_2)} \|\nabla G\|_{L^3(\Omega_2)}^2 \\ &\leq K_1(G) + K_2 \|\hat{w}\|_{H^{\frac{1}{2}+\epsilon}(\Omega)} \|\nabla G\|_{L^3(\Omega_2)}^2 \end{aligned} \quad (161)$$

$$\leq K_1(G) + \frac{1}{2} K_2 M \|\nabla G\|_{L^3(\Omega_2)}^2, \quad (162)$$

for some constants $K_1(G)$ and K_2 , with $K_1(G)$ depends on G ; where we have used Hölder's inequality, imbedding theorems $(H^{\frac{1}{2}+\epsilon}(\Omega) \subset L^3(\Omega_2))$, see [11] and (101). \square

In the following we will show that T takes bounded sets to bounded sets.

Proposition 2.3.6. *Assume (100)-(104) hold. Let $M > 0$ and consider $B_M := B \cap \{ \|(\hat{u}, \hat{w})\|_X \leq M \}$. There exists $N > 0$ such that $(u, w) = T(\hat{u}, \hat{w}) \in C_N := C \cap \{ \|(u, w)\|_Y \leq N \}$.*

Proof. As $T(\hat{u}, \hat{w}) = (u, w) \in Y_g$ is unique, then it is the minimizer for $I(u, w)$. Since $(G(0), G) \in Y_g$, then we have

$$I(u, w; \hat{u}, \hat{w}) \leq I(|G(0)|, |G|; \hat{u}, \hat{w}),$$

from this inequality and the definition of $I(u, w)$ (154), we have

$$\begin{aligned} \frac{1}{2} \int_{\Gamma_{01}} u_m(u')^2 + du^2 &+ \frac{1}{6} \int_{\Omega} (3\sigma\chi_{\Omega_1} + 2|\nabla w|\chi_{\Omega_2}) |\nabla w|^2 \\ &\leq \frac{1}{2} \int_{\Gamma_{01}} \hat{u}(u')^2 + du^2 + \frac{1}{6} \int_{\Omega} (3\sigma(\hat{w})\chi_{\Omega_1} + 2|\nabla w|\chi_{\Omega_2}) |\nabla w|^2 \end{aligned}$$

$$\begin{aligned}
&\leq I(|G(0)|, |G|; \hat{u}, \hat{w}) - \int_{\Gamma_{01}} \int_0^{u(s)} (-aj(\hat{w}(s))(\bar{w} - k)^+ - f(s)) dk ds \\
&+ \int_{\Gamma_{01}} \int_0^{w(s)} d\hat{u}(s) dk ds + \int_{\Gamma_{02}} h w, \\
&\leq I(|G(0)|, |G|; \hat{u}, \hat{w}) + \int_{\Gamma_{01}} (a\bar{\sigma}\bar{w}|\hat{w}| + |f|)|u| + d \int_{\Gamma_{01}} u^m |w| \\
&+ \int_{\Gamma_{02}} h |w| \\
&\leq I(|G(0)|, |G|; \hat{u}, \hat{w}) + (a\bar{\sigma}\bar{w} \|\hat{w}\|_{L^2(\Gamma_{01})} + \|f\|_{L^2(\Gamma_{01})}) \|u\|_{L^2(\Gamma_{01})} \\
&+ d \int_{\Gamma_{01}} u^m |w| + \int_{\Gamma_{02}} h |w| \tag{163} \\
&\leq K_1(G) + \frac{1}{2} K_2 M \|\nabla G\|_{L^3(\Omega_2)}^2 + \frac{1}{d} (a\bar{\sigma}\bar{w} M + \|f\|_{L^2(\Gamma_{01})})^2 + \frac{d}{4} \|u\|_{L^2(\Gamma_{01})}^2 \\
&+ \frac{d^2}{\sigma} (u^m)^2 + \frac{\sigma}{4} \|\nabla w\|_{L^2(\Omega_1)}^2 \\
&+ N_2 (\|h\|_{L^{\frac{3}{2}}(\Gamma_{02})}^{\frac{3}{2}} + \|h\|_{L^{\frac{3}{2}}(\Gamma_{02})} \|G\|_{W^{1,3}(\Omega)}) + \frac{1}{6} \|\nabla w\|_{L^3(\Omega_2)}^3, \tag{164}
\end{aligned}$$

where we have used Hölder and Young inequalities, together with trace theorem, (162) and inequalities similar to (135), (136) and (139). Rearranging implies

$$\|u\|_{H^1(\Gamma_{01})}, \quad \|\nabla w\|_{L^2(\Omega_1)}, \quad \|\nabla w\|_{L^3(\Omega_2)} \quad \text{are bounded uniformly,}$$

which proves the proposition (note that boundedness of gradients is enough because of (140) and (141)). \square

2.4 Existence of a fixed point.

To apply this theorem we consider the map $S = J \circ T \circ \Pi$ as follows

$$\begin{aligned}
S : \quad X &\xrightarrow{\Pi} B \xrightarrow{T} C \xrightarrow{J} X, \\
(\tilde{u}, \tilde{w}) &\rightarrow (\hat{u}, \hat{w}) \rightarrow (u, w) \rightarrow (u, w).
\end{aligned} \tag{165}$$

where Π is a projection operator and J is the imbedding map. The operator Π is defined as follows. Consider

$$\begin{aligned}
\Pi_u : \mathbb{R} \times \mathbb{R} &\rightarrow [u_m, u^m] & \Pi_w : \mathbb{R} \times \mathbb{R} &\rightarrow [u_m, \infty) \\
s &\rightarrow \Pi_u(s) := \min\{u^m, \max\{u_m, s\}\}, & s &\rightarrow \Pi_w(s) := \max\{s, u_m\},
\end{aligned}$$

and then set

$$\begin{aligned} \Pi : X &\longrightarrow B \\ (\tilde{u}, \tilde{w}) &\longrightarrow (\hat{u}, \hat{w}) = \Pi(\tilde{u}, \tilde{w}) := (\Pi_u(\tilde{u}), \Pi_w(\tilde{w})). \end{aligned} \quad (166)$$

Remark 2.4.1. Note that $\Pi(X) \subset B = X \cap \{(\hat{u}, \hat{w}) : u_m \leq \hat{u} \leq u^m, u_m \leq \hat{w}\}$. We will show this in two steps ($\Pi(X) \subset X$, then $u_m \leq \hat{u} \leq u^m, u_m \leq \hat{w}$).

Let $(\tilde{u}, \tilde{w}) \in X$, then $\Pi_u \tilde{u} = \min\{u^m, \max\{u_m, \tilde{u}\}\} \in C^0(\bar{\Gamma}_{01})$ (maximum and minimum of two continuous functions are also continuous). Also from (80), we have

$$\begin{aligned} \int_{\Omega} |\hat{w}(x)|^2 dx &= \int_{\Omega} |u_m + (\tilde{w}(x) - u_m)^+|^2 dx \\ &\leq \int_{\Omega} (|u_m|^2 + 2u_m|\tilde{w}(x) - u_m| + |\tilde{w}(x) - u_m|^2) dx < \infty, \end{aligned} \quad (167)$$

and

$$\begin{aligned} \int_{\Omega} \int_{\Omega} \frac{|\hat{w}(x) - \hat{w}(y)|^2}{|x - y|^{3+2\epsilon}} dx dy &= \int_{\Omega} \int_{\Omega} \frac{|(\tilde{w}(x) - u_m)^+ - (\tilde{w}(y) - u_m)^+|^2}{|x - y|^{3+2\epsilon}} dx dy \\ &\leq \int_{\Omega} \int_{\Omega} \frac{|\tilde{w}(x) - \tilde{w}(y)|^2}{|x - y|^{3+2\epsilon}} dx dy < \infty, \end{aligned} \quad (168)$$

which imply $\Pi(\tilde{w}) = \hat{w} \in H^{\frac{1}{2}+\epsilon}(\Omega)$. By this we proved $\Pi(\tilde{u}, \tilde{w}) \in C^0(\bar{\Gamma}_{01}) \times H^{\frac{1}{2}+\epsilon}(\Omega) = X$, which from the arbitrariness of (\tilde{u}, \tilde{w}) implies $P(X) \subset X$.

Now from the definition of Π we have $\Pi_u(\tilde{u}(s)) = \hat{u}(s) \in [u_m, u^m] \forall s \in \bar{\Gamma}_{01}$, and $\Pi_w(\tilde{w}(x)) = \hat{w}(x) \geq u_m$ a.e. $x \in \Omega$, which imply $u_m \leq \hat{u} \leq u^m$ and $u_m \leq \hat{w}$. \square

We have

Lemma 2.4.2. *The operator J is Lipschitz and compact, and the operator Π is Lipschitz. Both operators Π and J have Lipschitz constant equal to one.*

Proof. The properties of J are classical, see [1], [5], [15]. The Lipschitz property of Π follows from the fact that $\Pi_u(s)$ and $\Pi_w(s)$ are continuous, piecewise linear functions with derivatives in $[-1, 1]$. \square

Proposition 2.4.3. *Assume (100)-(104) hold, then the operator S is compact and continuous.*

Proof. The proof is similar to that done for Proposition 1.4.6, where we made use of the Lipschitz continuity of Π to prove the compactness and continuity for $\hat{T} := J \circ T$

instead of proving them for S . Note that T still takes bounded sets to bounded sets, see Proposition 2.3.6, so the proof of compactness is similar to that in Proposition 1.4.6. Also having $\hat{w}_n \rightarrow \hat{w}$ in $H^{\frac{1}{2}+\epsilon}(\Omega)$ implies $w_n \rightarrow w$ a.e., see Theorem IV.2, [5]. Next using (iii) in Proposition 2.3.1 we can prove the continuity of S exactly in the same way we did in Proposition 1.4.6. \square

In the following lemma we will prove that for $(\hat{u}, \hat{w}) \in B$ with $\hat{w} = \max\{u_m, \lambda w\}$, the solution of (155) is bounded. This result will be used later to prove the boundedness of the set Λ defined in (77).

Lemma 2.4.4. *For $(\hat{u}, \hat{w}) \in B$ let $(u, w) \in C$ be the solution of (155). Assume $\hat{w} = \max\{u_m, \tilde{w}\} = \max\{u_m, \lambda w\}$, for a certain $\lambda \in [0, 1]$, then there exists $C_\Lambda > 0$, independent of λ , such that*

$$\|(u, w)\|_Y \leq C_\Lambda. \quad (169)$$

Proof. From the assumptions we have $\hat{w} \in H^1(\Omega) \cap W^{1,3}(\Omega_2)$ and $|\nabla \hat{w}| \leq |\nabla w|$ a.e. in Ω . Next, using (161) and (163)

$$\begin{aligned} \frac{1}{2} \int_{\Gamma_{01}} u_m (u')^2 + du^2 &+ \frac{1}{6} \int_{\Omega} (3\sigma \chi_{\Omega_1} + 2|\nabla w| \chi_{\Omega_2}) |\nabla w|^2 \\ &\leq K_1(G) + \frac{1}{2} \|\hat{w}\|_{L^3(\Omega_2)} \|\nabla G\|_{L^3(\Omega_2)}^2 \\ &+ (a\bar{\sigma}\bar{w} \|\hat{w}\|_{L^1(\Gamma_{01})} + \|f\|_{L^2(\Gamma_{01})}) \|u\|_{L^2(\Gamma_{01})} \\ &+ d \int_{\Gamma_{01}} u^m |w| + \int_{\Gamma_{02}} h |w|. \end{aligned} \quad (170)$$

Also, we have

$$\begin{aligned} \|\hat{w}\|_{L^3(\Omega_2)} \|\nabla G\|_{L^3(\Omega_2)}^2 &\leq \|\hat{w} - \hat{G}\|_{L^3(\Omega_2)} \|\nabla G\|_{L^3(\Omega_2)}^2 + \|\hat{G}\|_{L^3(\Omega_2)} \|\nabla G\|_{L^3(\Omega_2)}^2 \\ &\leq M_1 (\|\nabla \hat{w}\|_{L^3(\Omega_2)} + \|\nabla \hat{G}\|_{L^3(\Omega_2)}) \|\nabla G\|_{L^3(\Omega_2)}^2 \\ &+ \|\hat{G}\|_{L^3(\Omega_2)} \|\nabla G\|_{L^3(\Omega_2)}^2 \\ &\leq \frac{1}{12} \|\nabla w\|_{L^3(\Omega_2)}^3 \left(\frac{4}{3} M_1^{\frac{3}{2}} + M_1 \right) \|\nabla G\|_{L^3(\Omega_2)}^3 + \|\hat{G}\|_{L^3(\Omega_2)} \|\nabla G\|_{L^3(\Omega_2)}^2 \\ &= \frac{1}{12} \|\nabla w\|_{L^3(\Omega_2)}^3 + K_3(G). \end{aligned} \quad (171)$$

where $\hat{G} := \max\{u_m, \lambda G\}$, which is bounded independently from λ . Also

$$\begin{aligned}
u^m a \bar{\sigma} \bar{w} \|\hat{w}\|_{L^1(\Gamma_{01})} &\leq u^m a \bar{\sigma} \bar{w} (M_2 \|\nabla(\hat{w} - G)\|_{L^2(\Omega_1)} + \|G\|_{L^1(\Gamma_{01})}) \\
&\leq u^m a \bar{\sigma} \bar{w} (M_2 (\|\nabla G\|_{L^2(\Omega_1)} + \|\nabla \hat{w}\|_{L^2(\Omega_1)}) + \|G\|_{L^1(\Gamma_{01})}) \\
&\leq u^m a \bar{\sigma} \bar{w} (M_2 \|\nabla G\|_{L^2(\Omega_1)} + \|G\|_{L^1(\Gamma_{01})}) + 2(\bar{w} M_2)^2 \frac{(u^m a \bar{\sigma})^2}{\sigma} \\
&\quad + \frac{\sigma}{8} \|\nabla w\|_{L^2(\Omega_1)}^2 \\
&=: \frac{\sigma}{8} \|\nabla w\|_{L^2(\Omega_1)}^2 + K_4(G).
\end{aligned} \tag{172}$$

for some constants M_1 , M_2 , $K_3(G)$ and $K_4(G)$.

Now the result follows from (136), (139), (170), (171) and (172). \square

In order to use Theorem 1.4.1, we need to show first that S satisfies the property that the set (77) is bounded. We have

Proposition 2.4.5. *The set $\Lambda := \{x \in X : x = \lambda Sx, \lambda \in [0, 1]\}$ is bounded in X , i.e there exists $C_\Lambda > 0$ constant, such that $\|(u, w)\|_X \leq C_\Lambda$, for all $(u, w) \in \Lambda$.*

Proof. For $x = (\tilde{u}, \tilde{w}) \in \Lambda$ set $(\hat{u}, \hat{w}) = \Pi(\tilde{u}, \tilde{w}) \in B$ and $(u, w) = T(\hat{u}, \hat{w}) \in C \subset B$. So we have $(\tilde{u}, \tilde{w}) = \lambda(u, w)$. From the construction of operator Π , we have $\hat{w} = \max\{u_m, \tilde{w}\} = \max\{u_m, \lambda w\}$. From Lemma 2.4.4 we have that Λ is bounded in Y . However, $Y \subset X$ (see [1]), so we have Λ bounded in X . \square

2.4.1 Proof of the main result (Theorem 2.1.3).

Proof. (i) The operator $S : X \rightarrow -X$ satisfies the conditions of Theorem 1.4.1 (Proposition 2.4.3, and 2.4.5). Therefore S has a fixed point denoted (u, w) . As $S = J \circ T \circ \Pi$ it follows that $(u, w) \in C \subset Y_g$ and (u, w) satisfies (112). By denoting $v = \mathbf{j}(w)|_{\Omega_1}$, then we have that $(u, v, w) \in Z_g$ and solves (99).

The bound (105) is proven in Propositions 2.2.7 and 2.2.10. The bound (107) is proven in Propositions 2.2.7 and 2.2.12.

The lower bound of (106) is proven in Proposition 2.2.7, with $v_m = J(u_m)$. For the upper bound for (106), we proceed as follows. From (152), we have

$$\int_{\Omega_1} |\nabla w|^2 \leq K^*, \tag{173}$$

for some constant K^* . As a result

$$\int_{\Omega_1} |\nabla v|^2 \leq \int_{\Omega_1} |\sigma(w)\nabla w|^2 \leq \bar{\sigma}^2 K^*. \quad (174)$$

From triangle and Poincaré inequalities we have

$$\|v\|_{H^1(\Omega_1)} \leq K_1(\|\nabla v\|_{L^2(\Omega_1)} + \|G_v\|_{H^1(\Omega_1)}),$$

for some constant K_1 . Here $G_v \in H^1(\Omega_1)$ with $G_v|_{\Gamma_1} = g_v$ (trace theorem). Therefore the bound (106) follows with $v^m := K_1(\bar{\sigma}^2 K^* + \|G_v\|_{H^1(\Omega_1)})$.

(ii) It follows from Corollary 2.2.11 and Proposition 2.2.12. \square

2.5 Numerical results.

As in Section 1.5, some numerical computations are presented here to find a numerical solution for (96)-(97) (the finite element method with a finite space consists of piecewise polynomials of degree two (see [20]) and the Newton's method are applied (see [9]), again by applying the COMSOL program) and to verify that this system approximates well the reaction-diffusion phenomena around the triple phase point. We will compare our results with those obtained using the thin layer model (see Appendix A) corresponding to our simplified model, that is

$$\begin{cases} -\Delta v = 0 & \text{in } \Omega_{\delta 1}, \\ -\nabla \cdot (w_\delta \nabla w_\delta) = 0 & \text{in } \Omega_\delta \cup \Omega_{2\delta} \cup \Gamma_\delta, \end{cases} \quad (175)$$

equipped with the following boundary conditions

$$\left\{ \begin{array}{ll} v = g_v & \text{on } \Gamma_1, \\ \partial_\nu v = dw_\delta - av(\bar{w} - w_\delta)^+ & \text{on } \Gamma_{\delta 1}, \\ w_\delta = i(v) & \text{on } \Gamma_{12}, \\ w_\delta \partial_\nu w_\delta = 0 & \text{on } \{(x, y) : x = -1\}, \\ w_\delta \partial_\nu w_\delta = h_{0\delta} \chi_{\Gamma_{0\delta}} + h_{02} \chi_{\Gamma_{02}} + \partial_\nu v \chi_{\Gamma_{12} \cup \Gamma_{\delta 1}} & \text{on } \Gamma_{0\delta} \cup \Gamma_{\delta 1} \cup \Gamma_{12} \cup \Gamma_{02}, \\ w_\delta = g_w & \text{on } \Gamma_2. \end{array} \right. \quad (176)$$

Domains Ω_δ , $\Omega_{1\delta}$, $\Omega_{2\delta}$ and the boundary $\Gamma_{0\delta}$, $\Gamma_{\delta 1}$, Γ_δ , Γ_{12} , Γ_{02} , Γ are shown in Figure 14, in which the picture on the right is a magnification picture of the picture on the right around the origin. Our system of PDEs corresponding to this model is

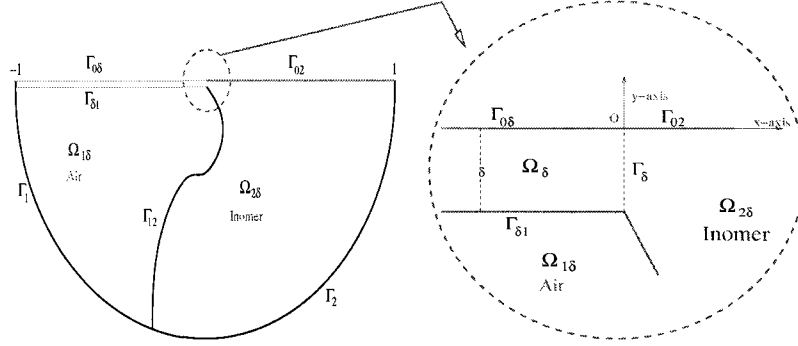


Figure 14: Thin layer.

$$\begin{cases} -\delta(uu_x)_x + du - av(\bar{w} - u)^+ = h_{0\delta} & \text{on } \Gamma_{0\delta}, \\ -\Delta v = 0 & \text{in } \Omega_{1\delta}, \\ -\nabla \cdot ((w + |\nabla w|)\nabla w) = 0 & \text{in } \Omega_{2\delta}, \end{cases} \quad (177)$$

equipped with the following boundary conditions

$$\begin{cases} (uu_x)(-1) = 0, & u(0) = w(0), \\ v = g_v & \text{on } \Gamma_1, \\ \partial_\nu v = du - av(\bar{w} - u)^+ & \text{on } \Gamma_{0\delta}, \\ w = \dot{i}(v) & \text{on } \Gamma_{12} \cup \Gamma_\delta, \\ (w + |\nabla w|)\partial_\nu w = -\delta uu_x \delta_0 + h_{02} \chi_{\Gamma_{02}} + \partial_\nu v \chi_{\Gamma_{12} \cup \Gamma_\delta}, & \text{on } \Gamma_\delta \cup \Gamma_{02} \cup \Gamma_{12}, \\ w = g_w & \text{on } \Gamma. \end{cases} \quad (178)$$

(note that the derivative with respect to x , u_x , is used instead of the derivative with respect to s , u_s , because Γ_{01} is a line segment parallel to the x -axis). To compare the numerical solution for (177)-(178) with the numerical solution for (175)-(176) we will do the following steps. We define the extension function w_0 , that is

$$w_0(x, y) = \begin{cases} u(x, 0), & (x, y) \in \Omega_\delta, \\ w(x, y), & (x, y) \in \Omega_{2\delta}. \end{cases} \quad (179)$$

Next we calculate the L^2 norm of the pointwise relative difference between w_δ and w_0 , that is

$$e = \left\| \frac{w_\delta - w_0}{w_\delta} \right\|_{L^2(\Omega_\delta \cup \Omega_{2\delta})}.$$

The following table of data is used here to find the numerical solutions for the two systems.

<i>Parameter</i>	<i>Value</i>
g_v	0.01
\bar{w}	21
$h_{0\delta}$	$0.82333 * (10^5/7)^x$
h_{02}	$0.074097 * (0.556)^x$
$i(v)$	$862.9v - 102146v^2 + 7192431v^3$
g_w	$i(g_v)$

Comparing Figure 15 and Figure 16, we conclude that numerical solutions for both thin layer system (175)-(176) and the system (177)-(178) are similar. The same is true if we compare Figure 17 and Figure 18. In fact, the relative error between the solutions in Figure 15 and Figure 16 is $e = 0.050133$ and the relative error between the solutions in Figure 17 and Figure 18 is $e = 0.08355$.

To reduce the relative error between the solutions of the two systems, we will use the same technique used in Section 1.5, that is, we will replace the water equation in (91)-(92) by

$$\nabla \cdot ((w + k|\nabla w|^{n-2}\chi_{\{(x,y):x^2+y^2 \leq \epsilon\}})\nabla w) = 0 \text{ in } \Omega_{2\delta}. \quad (180)$$

for some constants $k > 0$, $n > 2$ and $\epsilon > 0$. Next define

$$e(k, n, \epsilon; a, d) = \left\| \frac{w_\delta - w_0(k, n, \epsilon)}{w_\delta} \right\|_{L^2(\Omega_\delta \cup \Omega_{2\delta})}, \quad (181)$$

where $w_0(k, n, \epsilon)$ is an extension (see (179)) of solution for the surface diffusion model having (180) as an equation for w .

2.5.1 Numerical sensitivity analysis of the relative error.

In this section we will find an optimal (k, n, ϵ) at which the relative error $e(k, n, \epsilon; a, d)$ given by (181) has a relative minimum. The same technique as in the subsection 1.5.1 is used to number the solution corresponding to (k, n, ϵ) in (94).

The following table helps in reading the figures below.

Relative error dependence on parameters.						
$c(k,n,\epsilon;a,d)$	a	d	k	n	ϵ	Smallest relative error and optimal (k, n, ϵ)
Figures 19 and 20	49.998495	$9.982947 * 10^{-4}$	$k \in [10^{-5}, 22]$	$n \in [2.0001, 20]$	$\epsilon \in [10^{-3}, 1]$	$e = 0.003965$ at $(1.0009 * 10^{-2}, 4.73333, 0.0208)$
Figures 21 and 22	0.49998495	0.998295	$k \in [10^{-5}, 22]$	$n \in [2.0001, 20]$	$\epsilon \in [10^{-3}, 1]$	$e = 0.004209$ at $(4.4275, 2.4008, 0.3004)$

Again the Figures 19-22 imply that the relative error e depend on the five parameters k, n, ϵ, a and d as we expected and as we got Section 1.5.

Note that in the case $a = 49.998495, d = 9.982947 * 10^{-4}$, we can approximate an optimal $(k, n, \epsilon) = (1.0009 * 10^{-2}, 4.73333, 0.0208)$, where $e = 0.003965$. Also in the case 0.49998495 and $d = 0.998295$, we can approximate an optimal $(k, n, \epsilon) = (4.4275, 2.4008, 0.3004)$, where $e = 0.004209$.

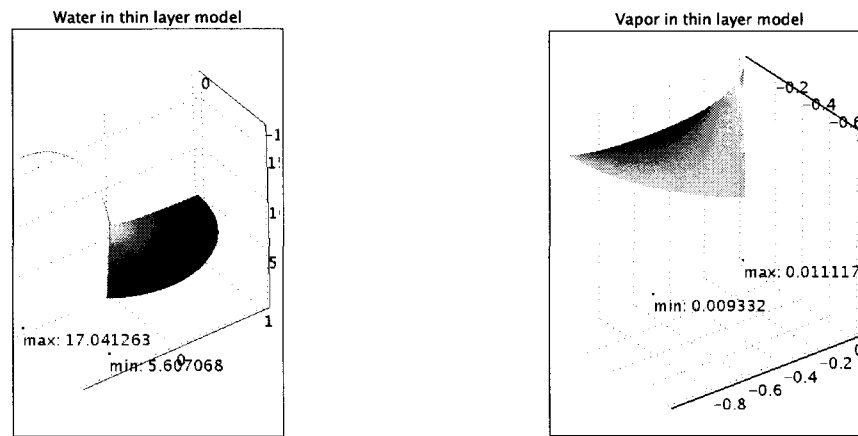


Figure 15: *Water and vapor solutions of the system (175)-(176) (thin layer model) for $a = 49.998495$ and $d = 9.982947 * 10^{-4}$.*

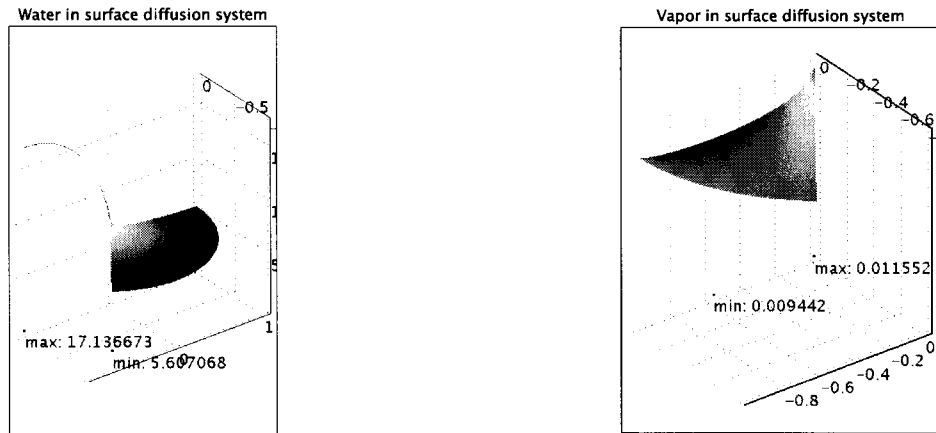


Figure 16: *Water and vapor solutions of the system (177)-(178) (surface diffusion model) for $a = 49.998495$ and $d = 9.982947 * 10^{-4}$.*

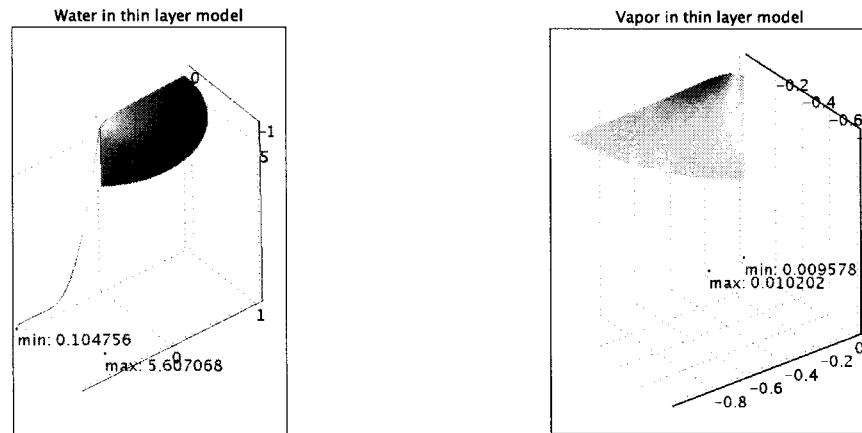


Figure 17: *Water and vapor solutions of the system (175)-(176) (thin layer model) for $a = 0.49998495$ and $d = 0.998295$.*

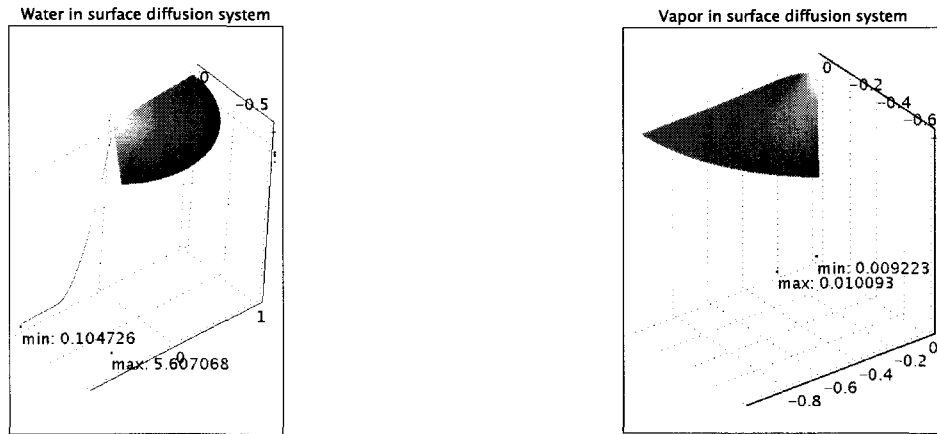


Figure 18: Water and vapor solutions of the system (177)-(178) (surface diffusion model) for $a = 0.49998495$ and $d = 0.998295$.

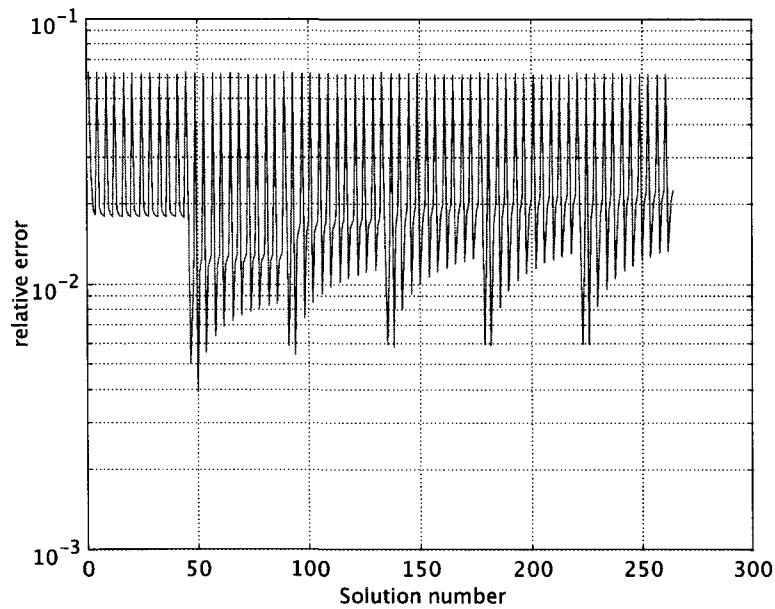


Figure 19: Relative error function $(k, n, \epsilon) \mapsto \epsilon(k, n, \epsilon)$. Smallest relative error $e = 0.003965$ at $n_{ijl} = 50$, where optimal $(k, n, \epsilon) = (1.0009 * 10^{-2}, 4.73333, 0.0208)$, $a = 49.998495$ and $d = 9.982947 * 10^{-4}$.

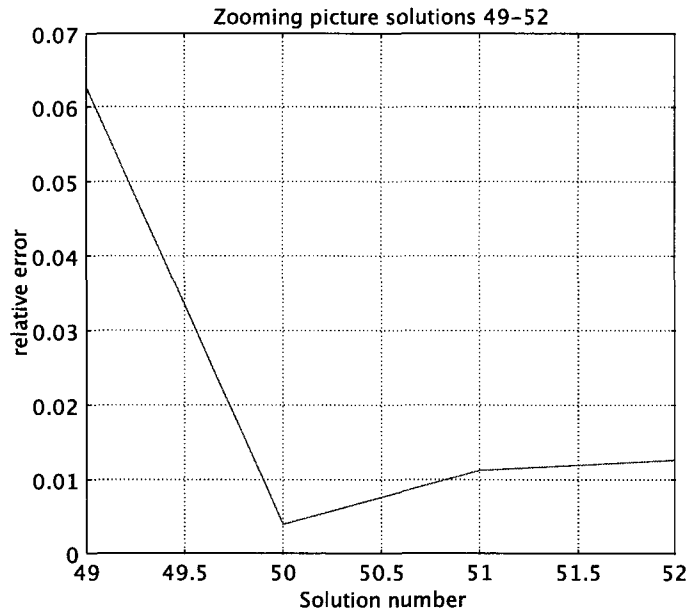


Figure 20: Relative error function $(k, n, \epsilon) \mapsto -e(k, n, \epsilon)$, near the optimal solution number $n_{ijl} = 50$.

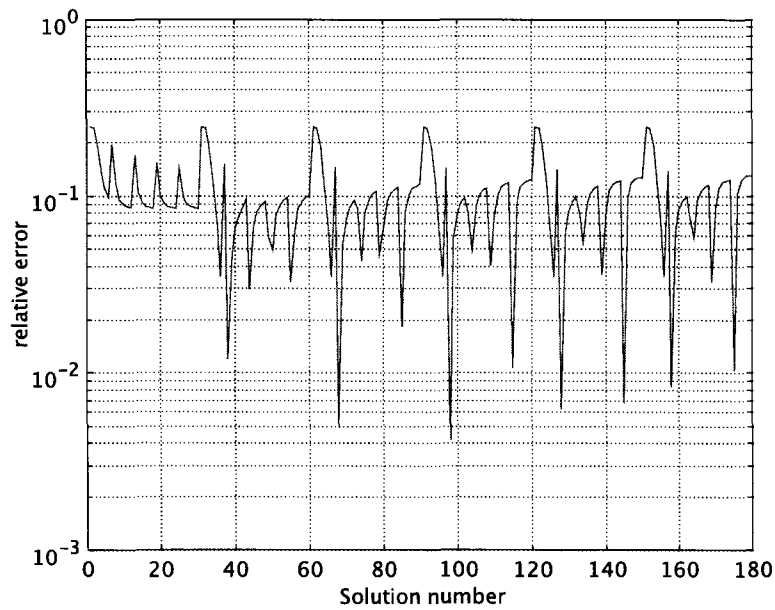


Figure 21: Relative error function $(k, n, \epsilon) \mapsto -e(k, n, \epsilon)$. Smallest relative error $e = 0.004209$ at $n_{ijl} = 98$, where optimal $(k, n, \epsilon) = (4.4275, 2.4008, 0.3004)$, $a = 0.49998495$ and $d = 0.998295$.

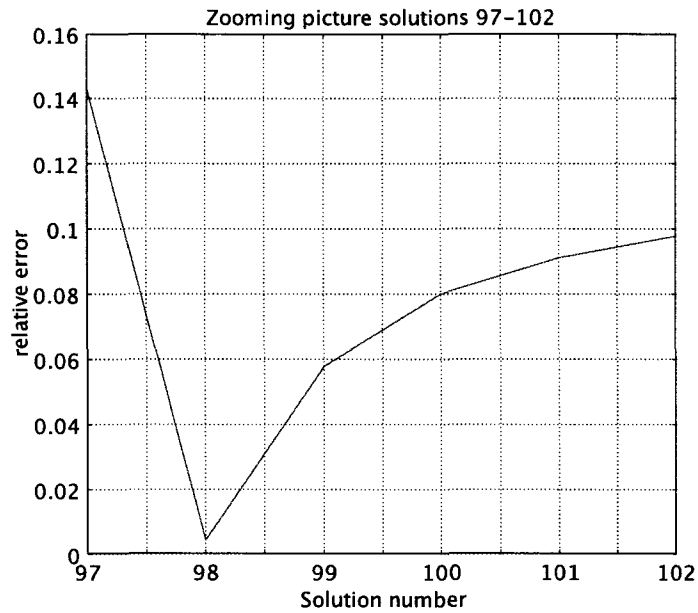


Figure 22: Relative error function $(k, n, \epsilon) \mapsto -e(k, n, \epsilon)$, near the optimal solution number $n_{ijl} = 98$.

Remark 2.5.1. As we see from these numerical computation, the best approximation of w is obtained for a particular choice of (k, n, ϵ) . The analysis of such (k, n, ϵ) is not the object of this thesis. \square

Chapter 3

Full model.

In this chapter the full model (1)-(16) is considered, which in addition to the difficulties in the systems in chapters one and two contains the nonlinear coupling at the origin, condition (9), and increases the number of PDEs. The same technique used in chapters one and two is used to prove the existence of a positive bounded weak solution for this system. In fact, the positivity will be proven only for u , v and w .

3.1 Introduction. Main result.

We will consider a system of PDEs coupling six variables u , p , ϕ , v , w , and q , which is obtained from system (1)-(16) with $\phi := \varphi - \varphi(0)$ and $\varphi(0) = q(0) - \ln c_+(w(0)) = p(0) - \ln c_+(u(0))$, see (9). The system (1)-(16) is equivalent to

$$-(uu_s)_s + du - av(\bar{w} - u)^+ = f_u \quad \text{on } \Gamma_{01}, \quad (182)$$

$$-(uc_+(u(0))e^{p-(\phi+p(0))}p_s)_s = -f_p \quad \text{on } \Gamma_{01}, \quad (183)$$

$$-\phi_{ss} = f_\phi \quad \text{on } \Gamma_{01}, \quad (184)$$

$$-\Delta v = 0 \quad \text{in } \Omega_1, \quad (185)$$

$$-\nabla \cdot ((w + |\nabla w|)\nabla w) = 0 \quad \text{in } \Omega_2, \quad (186)$$

$$-\nabla \cdot ((wc_+(w) + |\nabla q|)\nabla q) = 0 \quad \text{in } \Omega_2, \quad (187)$$

equipped with the following boundary conditions

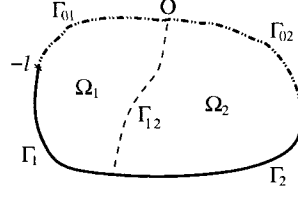


Figure 23: Triple phase boundary domain

$$(uu_s)(-l) = 0, \quad u(0) = w(0), \quad (188)$$

$$(uc_+(u(0))e^{p-(\phi+p(0))}p_s)(-l) = 0, \quad p(0) = q(0), \quad (189)$$

$$\phi_s(-l) = 0, \quad \phi(0) = 0, \quad (190)$$

$$v = g_v \quad \text{on } \Gamma_1, \quad (191)$$

$$\partial_\nu v = du - av(\bar{w} - u)^+ \quad \text{on } \Gamma_{01}, \quad (192)$$

$$w = i(v) \quad \text{on } \Gamma_{12}, \quad (193)$$

$$(w + |\nabla w|)\partial_\nu w = -uu_s\delta_0 + h_w\chi_{\Gamma_{02}} + \partial_\nu v\chi_{\Gamma_{12}} \quad \text{on } \partial\Omega_2 \setminus \Gamma_2, \quad (194)$$

$$(wc_+(w) + |\nabla q|)\partial_\nu q = -uc_+(u(0))e^{p-(\phi+p(0))}p_s\delta_0 - h_q\chi_{\Gamma_{02}} + 0\chi_{\Gamma_{12}} \quad \text{on } \partial\Omega_2 \setminus \Gamma_2, \quad (195)$$

$$w = g_w \quad \text{on } \Gamma_2, \quad (196)$$

$$q = g_q \quad \text{on } \Gamma_2. \quad (197)$$

Here domains, boundaries and data are as in Section 0.1, that is Ω_1, Ω_2 are open, simply connected and bounded sets with Lipschitz boundaries. $\Gamma_{01}, \Gamma_{02}, \Gamma_1, \Gamma_2$ are connected parts of the boundaries as presented in Figure 23, $\Omega := \Omega_1 \cup \Gamma_{12} \cup \Omega_2$, $\Gamma := \Gamma_1 \cup \Gamma_2$ and ν the outward unit normal vector to $\partial\Omega$ or to $\partial\Omega_2$. Also

$$c_+ := c_+(w) = -\frac{1}{2}k_+w + \left(\frac{1}{4}(k_+w)^2 + k_+w\right)^{\frac{1}{2}} \quad \text{in } \Omega_2, \quad (198)$$

for some positive constant k_+ ,

$$\begin{aligned} f_i &= k_i(c_o - e^{-p})\sqrt{c_+(u(0))}e^{p-0.5(\phi+p(0))}, \quad i = u, p, \quad f_\phi = k_\phi c_+(u(0))e^{p-(\phi+p(0))}, \\ h_j &= k_j(c_o - e^{-q})\sqrt{c_+(w)}e^{5q}, \quad j = w, q. \end{aligned}$$

The physical parameters k_i for $i = u, p, \phi$, and $j = w, q$ together with a, d, \bar{w}, k_+ and c_o are positive constants. Also the functions g_v, g_w and g_q are nonnegative. These functions

together with \mathbf{i} are assumed to be sufficiently smooth functions (to be defined later). δ_0 is the Dirac measure with support at the origin.

The system (182)-(197) represents the full model of reaction-diffusion processes near a triple phase boundary in fuel cell cathode catalyst layer. It is obtained from a thin layer model [3] by the change of variables

$$\phi = \varphi - \varphi(0) \text{ in } \Gamma_{01} \cup \Omega_2, \quad p = \phi + \varphi(0) + \ln c_+ \text{ on } \Gamma_{01}, \quad q = \phi + \varphi(0) + \ln c_+(w) \text{ in } \Omega_2.$$

Here Γ_{01} approximates a thin layer with thickness $\delta = 10^{-9}$ and $u(s) := \frac{1}{\delta} \int_{-\delta}^0 w(s, \nu) d\nu$, where s varies on the arc Γ_{01} and ν is perpendicular to Γ_{01} , for more details see [3].

Originally, the equations (186) and (187) did not have gradient terms in the diffusion coefficients, i.e. they are $\nabla \cdot (w \nabla w) = 0$ and $\nabla \cdot (w c_+(w) \nabla q) = 0$. Since (194) and (195) contain a Dirac singularity at the origin, we expect a non regular solution (logarithmic solution around the origin for w^2 and q^2). The terms $|\nabla w|$ and $|\nabla q|$ are added to the diffusion coefficients of w and q in (186) and (187) to have the system (182)-(197) with regular solution (the coupling $u(0) = w(0)$ and the coupling $p(0) = q(0)$ are true only after adding the gradient terms).

Assuming that (182)-(197) has a smooth solution (u, p, ϕ, v, w, q) , if we set

$$\begin{aligned} Z &= (H^1(\Gamma_{01}))^3 \times H^1(\Omega_1) \times (W^{1,3}(\Omega_2))^2 \\ &\cap \{(u, p, \phi, v, w, q) : u(0) = w(0), p(0) = q(0), \phi(0) = 0\}, \\ Z_g &= Z \cap \{(u, p, \phi, v, w, q) : v = g_v \text{ on } \Gamma_1, w = \mathbf{i}(v) \text{ on } \Gamma_{12}, w = g_w \text{ on } \Gamma_2, q = g_q \text{ on } \Gamma_2\}, \\ Z_0 &= Z \cap \{(\psi_u, \psi_p, \psi_\phi, \psi_v, \psi_w, \psi_q) : \psi_v = 0 \text{ on } \Gamma_1, \psi_v = \psi_w \text{ on } \Gamma_{12}, \psi_w = \psi_q = 0 \text{ on } \Gamma_2\}, \end{aligned} \tag{199}$$

then using the test function $(\psi_u, \psi_p, \psi_\phi, \psi_v, \psi_w, \psi_q) \in Z_0$, with the equations (182)-(187) respectively, integrating by parts, adding the results and using (188)-(197), we get

$$\begin{aligned} &\int_{\Gamma_{01}} (uu' \psi'_u + u c_+(u(0)) e^{p-(\phi+p(0))} p' \psi'_p + \phi' \psi'_\phi) \\ &+ \int_{\Gamma_{01}} (N(u, w)(\psi_u - \psi_v) - f_u \psi_u + f_p \psi_p - f_\phi \psi_\phi) + \int_{\Gamma_{02}} (-h_w \psi_w + h_q \psi_q) \\ &+ \int_{\Omega_1} (\nabla v \cdot \nabla \psi_v) + \int_{\Omega_2} (w + |\nabla w|)(\nabla w \cdot \nabla \psi_w) + (w c_+(w) + |\nabla q|)(\nabla q \cdot \nabla \psi_q) = 0, \\ &\quad \forall (\psi_u, \psi_p, \psi_\phi, \psi_v, \psi_w, \psi_q) \in Z_0. \end{aligned} \tag{200}$$

Here the function $N(u, w) = du - \mathbf{a}j(w)(\bar{w} - u)^+$ exactly as defined in (113). Note that $(\cdot)' := \frac{d(\cdot)}{ds}$.

Definition 3.1.1. We say $(u, p, \phi, v, w, q) \in Z_g$ is a weak solution of (182)-(197), if (u, p, ϕ, v, w, q) satisfies (200).

Remark 3.1.2. Note that if (u, p, ϕ, v, w, q) is a weak solution of (182)-(197), then it is clear that by taking $(\psi_u, \psi_p, \psi_\phi, \psi_v, \psi_w, \psi_q) \in Z_0$ with compact support we find easily that (u, p, ϕ, v, w, q) satisfies (182)-(187) in the sense of distributions.

If furthermore (u, p, ϕ, v, w, q) is smooth enough, for example if $(u, p, \phi) \in (H^2(\Gamma_{01}))^3$, $v \in H^2(\Omega_1)$, and $(w, q) \in (W^{2,3}(\Omega_2))^2$, then u , p and ϕ satisfy (182), (183) and (184) respectively in $L^2(\Gamma_{01})$ sense, v satisfies (185) in $L^2(\Omega_1)$ sense, w and q satisfy (186), (187) respectively in $L^3(\Omega_2)$ sense.

Furthermore, by taking $\psi_u(-l)$, $\psi_p(-l)$, $\psi_\phi(-l)$, $\psi_u(0) = \psi_w(0)$, $\psi_p(0) = \psi_q(0)$, $\psi_v|_{\Gamma_{01}}$, $\psi_w|_{\Gamma_{02}}$, $\psi_q|_{\Gamma_{02}}$ and $\psi_v|_{\Gamma_{12}} = \psi_w|_{\Gamma_{12}}$ arbitrarily we find that (u, p, ϕ, v, w, q) satisfies also (188)-(197). Indeed, using the test function $(\psi_u, \psi_p, \psi_\phi, \psi_v, \psi_w, \psi_q) := (0, 0, \psi_\phi, 0, 0, 0) \in Z_0$ in (200) and integrating by parts give

$$\phi'(-l)\psi_\phi(-l) - \int_{\Gamma_{01}} (\phi'' + f_\phi)\psi_\phi = 0, \quad (201)$$

If in addition $\psi_\phi \in H^1(\Gamma_{01})$ and with compact support and $\phi \in H^2(\Gamma_{01})$, then we have

$$-\phi'' = f_\phi, \quad \text{in } L^2(\Gamma_{01}) \text{ sense.} \quad (202)$$

It follows that $\phi'(-l)\psi_\phi(-l) = 0$, which by arbitrariness of ϕ implies $\phi'(-l) = 0$. The second boundary condition $\phi(0) = 0$ is included in the space Z_g .

Also using the test function $(\psi_u, \psi_p, \psi_\phi, \psi_v, \psi_w, \psi_q) := (\psi_u, 0, \psi_v, 0, \psi_w, 0) \in Z_0$, we have

$$\begin{aligned} \int_{\Gamma_{01}} (uu'\psi'_u) + N(u, w)(\psi_u - \psi_v) - f_u\psi_u - \int_{\Gamma_{02}} (h_w\psi_w) \\ + \int_{\Omega_1} (\nabla v \cdot \nabla \psi_v) + \int_{\Omega_2} (w + |\nabla w|)(\nabla w \cdot \nabla \psi_w) = 0, \end{aligned} \quad (203)$$

which is similar to (99), so by Remark 2.1.2, we have (182), (185), (186) and their boundary conditions (188), (191)-(194) and (196) hold.

Finally, using the test function $(\psi_u, \psi_p, \psi_\phi, \psi_v, \psi_w, \psi_q) := (0, \psi_p, 0, 0, 0, \psi_q) \in Z_0$, we have

$$\int_{\Gamma_{01}} (uc_+(u(0))e^{p-(\phi+p(0))}p'\psi'_p + f_p\psi_p) + \int_{\Gamma_{02}} (h_q\psi_q) + \int_{\Omega_2} (wc_+(w) + |\nabla q|)(\nabla q \cdot \nabla \psi_q) = 0, \quad (204)$$

then with a work similar to what we did in Remark 1.1.2, we can prove that (183), (187) and their boundary conditions (189), (195), (197) hold. \square

Define

$$H_w^1(\Omega) := H^1(\Omega) \cap \{w : w|_{\Gamma} = g_w\}, \quad W_j^{1,3}(\Omega_2) := W^{1,3}(\Omega_2) \cap \{w : w|_{\Gamma_2} = g_j\}, \quad \text{for } j = w, q. \quad (205)$$

The purpose of this work is to prove that (182)-(197) has a positive bounded weak solution (we will prove the positivity for u , v and w only). Namely, we will prove the following theorem.

Theorem 3.1.3. *Assume that the constants and functions in (200) satisfy*

$$a, d, \bar{w} > 0, \quad (206)$$

$$\mathbf{i} \in C^1(\mathbb{R}), \quad \mathbf{i}(0) = 0 \text{ and } \exists 0 < \underline{\sigma} < \bar{\sigma} < \infty \text{ such that : } \underline{\sigma} \leq \sigma \leq \bar{\sigma}, \sigma := \frac{1}{\mathbf{i}}, \quad (207)$$

$$0 < \inf g_w, \quad g_w := \begin{cases} \mathbf{i}(g_v), & \Gamma_1, \\ g_w, & \Gamma_2, \end{cases} \quad g_w = G_w|_{\Gamma}, \quad G_w \in H_w^1(\Omega) \cap W_w^{1,3}(\Omega_2). \quad (208)$$

$$-\ln c_o \leq \inf g_q, \quad g_q = G_q|_{\Gamma}, \quad G_q \in W_q^{1,3}(\Omega_2), \quad (209)$$

(i) *If the condition*

$$\frac{d}{a} \frac{1}{\bar{w}} < \underline{\sigma} \leq \sigma \leq \bar{\sigma} < \frac{d}{a} \frac{1}{(\bar{w} - \inf g_w)^+}. \quad (210)$$

holds, then (200) has at least one solution $(u, p, \phi, v, w, q) \in Z_g$. Moreover, (u, p, ϕ, v, w, q) satisfies:

$$u_m \leq u \leq u^m, \quad (211)$$

$$u_m \leq \mathbf{i}(v), \quad \|v\|_{H^1(\Omega_1)} \leq v^m, \quad (212)$$

$$u_m \leq w \leq w^m, \quad (213)$$

$$0 \leq \phi \leq \phi^m, \quad (214)$$

$$-\ln c_o \leq p, q \leq \sup_{\Gamma_2} g_q. \quad (215)$$

where u_m , u^m , w^m , v^m and ϕ^m are constants depending only on the given data and the domain.

(ii) *Assume $\max\{\sup_{\Gamma} g_w, \frac{\|f_u\|_{L^\infty(\Gamma_{01})}}{d}\} < \bar{w}$. If $\frac{d^2}{\underline{\sigma}}$, $\frac{(a\bar{\sigma})^2}{\underline{\sigma}}$, $\|f_u\|_{L^1(\Gamma_{01})}$, $\|h_w\|_{L^{\frac{3}{2}}(\Gamma_{02})}$ are small enough², then u , p , ϕ , v , w and q are “physically meaningful” solutions in the sense that*

$$u \leq \bar{w} \text{ on } \Gamma_{01}, \quad w \leq \bar{w} \text{ in } \Omega_2. \quad (216)$$

Remark 3.1.4. Note that

$$\begin{aligned} \|f_u\|_{L^\infty(\Gamma_{01})} &\leq k_u \sqrt{c_o} (c_o - e^{-\sup_{\Gamma_2} g_q}) e^{\sup_{\Gamma_2} g_q} = k_u \sqrt{c_o} (c_o e^{\sup_{\Gamma_2} g_q} - 1), \\ \|h_w\|_{L^\infty(\Gamma_{02})} &\leq k_w (c_o - e^{-\sup_{\Gamma_2} g_q}) e^{0.5 \sup_{\Gamma_2} g_q} = k_w (c_o e^{0.5 \sup_{\Gamma_2} g_q} - e^{-0.5 \sup_{\Gamma_2} g_q}). \end{aligned}$$

Both f_u and h_w are uniformly bounded (bounded by a constant independent from the variables u, p, ϕ, v, w and q) with bounds increasing functions in $\sup_{\Gamma_2} g_q$. In conclusion, one can replace the conditions on f_u and h_w in (ii), Theorem 3.1.3 by conditions on $\sup_{\Gamma_2} g_q$, that is, the conditions on f_u and h_w can be replaced by $\sup_{\Gamma_2} g_q$ small enough and satisfies $\frac{k_u |\Gamma_{01}| \sqrt{c_o} (c_o e^{\sup_{\Gamma_2} g_q} - 1)}{d} \leq \bar{w}$. \square

Note that the conditions on data and the results for u, w and v in this theorem are similar to those in Theorem 1.1.3 and Theorem 2.1.3 in Chapters one and two. This makes sense because the two systems (96)-(97) and (18)-(19) are special cases of the system (182)-(197), when p, ϕ, q (and v) are considered constants.

To prove Theorem 3.1.3 we will use the same approach used to prove Theorem 1.1.3 and Theorem 2.1.3, that is the fixed point approach. Again after changing of variables, fixing some variables in (200), the resulting weak equation will be the Euler-Lagrange equation for a variational problem. An operator T that maps fixed variables to the solution of this Euler-Lagrange equation is introduced. We prove that T is continuous and takes bounded sets to bounded sets. Next we introduce S as a composition of T with an imbedding map and a projection map and apply the Schauder fixed point theorem to S . The existence of fixed point for S implies the existence of solution for (200) as we will see.

3.2 A fixed point approach.

Throughout this chapter we will assume that (206)-(209) hold. Section 2.2 (equations (110) and (111)) eliminates the variable v and leads to the following problem, equivalent to (200),

²To better understanding the conditions in (ii), Theorem 3.1.3, see Remark 3.1.4.

that is to find $(u, p, \phi, w, q) \in Y_g$ satisfying

$$\begin{aligned} & \int_{\Gamma_{01}} (uu' \psi'_u + uc_+(u(0))e^{p-(\phi+p(0))} p' \psi'_p + \phi' \psi_\phi) \\ & + \int_{\Gamma_{01}} (N(u, w)(\psi_u - \psi_w) - f_u \psi_u + f_p \psi_p - f_\phi \psi_\phi) + \int_{\Gamma_{02}} (-h_w \psi_w + h_q \psi_q) \\ & + \int_{\Omega} D_w(w, |\nabla w|)(\nabla w \cdot \nabla \psi_w) + \int_{\Omega_2} D_q(w, |\nabla q|)(\nabla q \cdot \nabla \psi_q) = 0, \quad \forall (\psi_u, \psi_p, \psi_\phi, \psi_w, \psi_q) \in Y_0, \end{aligned} \quad (217)$$

where

$$D_w(v, w) = \sigma(v)\chi_{\Omega_1} + (v + w)\chi_{\Omega_2}, \quad (218)$$

$$D_q(w, q) = wc_+(w) + q, \quad (219)$$

$$\begin{aligned} Y &= (H^1(\Gamma_0))^3 \times (H^1(\Omega) \cap W^{1,3}(\Omega_2)) \times W^{1,3}(\Omega_2) \\ &\quad \cap \{(u, p, \phi, w, q) : u(0) = w(0), p(0) = q(0), \phi(0) = 0\}, \end{aligned} \quad (220)$$

$$Y_g = Y \cap \{(u, p, \phi, w, q) : w = g_w \text{ on } \Gamma, q = g_q \text{ on } \Gamma_2\}, \quad (221)$$

$$Y_0 = Y \cap \{(\psi_u, \psi_p, \psi_\phi, \psi_w, \psi_q) : \psi_w|_{\Gamma} = 0, \psi_q|_{\Gamma_2} = 0\}. \quad (222)$$

The fixed point approach will be used here in order to prove that (217) has a positive bounded solution $(u, p, \phi, w, q) \in Y_g$. We will begin by setting

$$X := (C^0(\bar{\Gamma}_{01}))^3 \times H^{\frac{1}{2}+\epsilon}(\Omega) \times C^0(\bar{\Omega}_2), \quad 0 < \epsilon < \frac{1}{2}, \quad (223)$$

then for $(\hat{u}, \hat{p}, \hat{\phi}, \hat{w}, \hat{q}) \in X$, we consider $(u, p, \phi, w, q) \in Y_g$ solution whenever it exists, of

$$\begin{aligned} & \int_{\Gamma_{01}} (\hat{u}u' \psi'_u + \hat{u}c_+(\hat{u}(0))e^{\hat{p}-\hat{\phi}-\hat{p}(0)} p' \psi'_p - \hat{f}_\phi \psi_\phi) \\ & + \int_{\Gamma_{01}} (N(u, \hat{w}) - f_u) \psi_u - N(\hat{u}, w) \psi_w + \hat{f}_p \psi_p - \hat{f}_\phi \psi_\phi) + \int_{\Gamma_{02}} (-\hat{h}_w \psi_w + \hat{h}_q \psi_q) \\ & + \int_{\Omega} D_w(\hat{w}, |\nabla w|)(\nabla w \cdot \nabla \psi_w) + \int_{\Omega_2} D_q(\hat{w}, |\nabla q|)(\nabla q \cdot \nabla \psi_q) = 0, \\ & \quad \forall (\psi_u, \psi_p, \psi_\phi, \psi_w, \psi_q) \in Y_0. \end{aligned} \quad (224)$$

where

$$\begin{cases} \hat{f}_u &= k_u(c_o - e^{-\hat{p}}) \sqrt{c_+(\hat{u}(0))} e^{\hat{p}-0.5(\hat{\phi}+\hat{p}(0))}, \\ \hat{f}_p &= k_p(c_o - e^{-\hat{p}}) \sqrt{c_+(\hat{u}(0))} e^{\hat{p}-0.5(\hat{\phi}+\hat{p}(0))}, \\ \hat{f}_\phi &= k_\phi c_+(\hat{u}(0)) e^{\hat{p}-\hat{\phi}-\hat{p}(0)}, \\ \hat{h}_w &= k_w(c_o - e^{-\hat{q}}) \sqrt{c_+(\hat{w})} e^{0.5\hat{q}}, \\ \hat{h}_q &= k_q(c_o - e^{-\hat{q}}) \sqrt{c_+(\hat{w})} e^{0.5\hat{q}}. \end{cases} \quad (225)$$

Remark 3.2.1.

- (i) The choice of the space X is not unique. We need Y compactly imbedded in X and we need sequences in the space X to satisfy uniform and a.e. convergent properties as we will see later. In fact, one can replace $H^{\frac{1}{2}+\epsilon}(\Omega)$ in the space X by $H^{\frac{1}{2}+\epsilon}(\Omega) \cap C^0(\overline{\Omega}_2)$ with $\|\hat{w}\| := \|\hat{w}\|_{H^{\frac{1}{2}+\epsilon}(\Omega)} + \|\hat{w}\|_{C^0(\overline{\Omega}_2)}$.
- (ii) Note that the fixing of variables (which mean replacing the original functions with hat functions) make (224) the Euler-Lagrange equation for a functional I (to be defined later). Also the fixing in the functions \hat{f}_i and \hat{h}_j , for $i = u, p, \phi$ and $j = w, q$ are not randomly taken. For $(\hat{u}, \hat{p}, \hat{\phi}, \hat{w}, \hat{q}) \in B$ (a set to be defined), we need $\hat{f}_u, \hat{f}_\phi, \hat{h}_w$ unconditionally positive and uniformly bounded with respect to $(\hat{u}, \hat{p}, \hat{\phi}, \hat{w}, \hat{q})$ and (u, p, ϕ, w, q) . Also, we need \hat{f}_p and \hat{h}_q positive and increasing in p, q satisfying the same bounds in the set B . For more details see next, Lemma 3.2.4. \square

Remark 3.2.2. The spaces X and Y are equipped with the norms

$$\begin{aligned} \|(\hat{u}, \hat{p}, \hat{\phi}, \hat{w}, \hat{q})\|_X &:= \|\hat{u}\|_{C^0(\overline{\Gamma}_{01})} + \|\hat{p}\|_{C^0(\overline{\Gamma}_{01})} + \|\hat{\phi}\|_{C^0(\overline{\Gamma}_{01})} + \|\hat{w}\|_{H^{\frac{1}{2}+\epsilon}(\Omega)} + \|\hat{q}\|_{C^0(\overline{\Omega}_2)}, \\ \|(u, p, \phi, w, q)\|_Y &:= \|u\|_{H^1(\Gamma_{01})} + \|p\|_{H^1(\Gamma_{01})} + \|\phi\|_{H^1(\Gamma_{01})} + \|w\|_{H^1(\Omega_1)} + \|w\|_{W^{1,3}(\Omega_2)} \\ &\quad + \|q\|_{W^{1,3}(\Omega_2)}. \end{aligned}$$

Note that equipped with these norms, X is a Banach space and Y is Banach reflexive space (see, Theorem III.16, [5]). \square

For given $(\hat{u}, \hat{p}, \hat{\phi}, \hat{w}, \hat{q})$, consider the map:

$$\begin{aligned} T : B \subset X &\quad \rightarrow C \subset Y \subset X \\ (\hat{u}, \hat{p}, \hat{\phi}, \hat{w}, \hat{q}) &\quad \rightarrow (u, p, \phi, w, q) = \text{solution of (224)} \end{aligned} \tag{226}$$

with B and C subsets to be defined in such way that $T(B) \subset C$. Next any fixed point of T is a solution of (217), which explains the relation between the solution of (224) and the solution of (217).

To prove that the operator T is well-defined and that it has a fixed point, a certain number of a priori estimates are needed.

3.2.1 A priori estimates

Let u_m, u^m, w^m and ϕ^m be positive constants (we will show how to choose these constants), such that $u^m > u_m$. Set

$$B = X \cap \{(\hat{u}, \hat{p}, \hat{\phi}, \hat{w}, \hat{q}) : u_m \leq \hat{u} \leq u^m, \quad u_m \leq \hat{w}, \quad -\ln c_o \leq \hat{p}, \hat{q} \leq \sup_{\Gamma_2} g_q, \quad 0 \leq \hat{\phi} \leq \phi^m\},$$

$$C = Y_g \cap \{(u, p, \phi, w, q) : u_m \leq u \leq u^m, \quad u_m \leq w, \quad -\ln c_o \leq p, q \leq \sup_{\Gamma_2} g_q, \quad 0 \leq \phi \leq \phi^m\}.$$

The main object of the section is to prove that $T(B) \subset C$.

Knowing the properties of the function $w \mapsto -e_+(w)$ defined in (198), will help us proving the existence of positive bounded weak solution for (182)-(197). In the following remark we will prove some useful properties for the function $w \mapsto -e_+(w)$.

Remark 3.2.3. The function $w \mapsto -e_+(w)$ given by (198) is an increasing function, $c_+([0, \infty)) = [0, 1)$. Indeed, $c'_+(w) = \frac{1}{2}k_+(-1 + \frac{\frac{1}{2}k_+w + 1}{\sqrt{\frac{1}{4}(k_+w)^2 + k_+w}}) \geq 0$ and $\lim_{w \rightarrow \infty} c_+(w) = 1$ (limit is evaluated by multiplying with conjugate).

As the domain of the function $w \mapsto -e_+(w)$ equals $[0, \infty)$, then we should have $w \geq 0$ in order to have a real solution for the system (182)-(197), which means not only we need $w \geq 0$ to have a physically meaningful solution, but also we need it to have a real mathematical solution. \square

The following remark shows some properties of the functions defined in (225).

Lemma 3.2.4. *Let $(\hat{u}, \hat{p}, \hat{\phi}, \hat{w}, \hat{q}) \in B$. We have*

(i) $\hat{f}_u, \hat{f}_\phi, \hat{h}_w$ are nonnegative uniformly bounded functions with respect to $(\hat{u}, \hat{p}, \hat{\phi}, \hat{w}, \hat{q})$.

(ii) $\hat{f}_p \cdot (p + \ln c_o) \geq 0$.

(iii) $\hat{h}_q \cdot (q + \ln c_o) \geq 0$. \square

Proof. Let $(\hat{u}, \hat{p}, \hat{\phi}, \hat{w}, \hat{q}) \in B$.

(i). Note that $0 \leq c_o - e^{-\hat{p}}, c_o - e^{-\hat{q}} \leq c_o - e^{-\sup_{\Gamma_2} g_p}$. Note also $0 \leq \sqrt{c_+(\hat{w})}e^{0.5\hat{q}} \leq e^{.5\sup_{\Gamma_2} g_q}$ and $0 \leq c_+(\hat{u}(0))e^{\hat{p}-\hat{\phi}-\hat{p}(0)}, \sqrt{c_+(\hat{u}(0))}e^{\hat{p}-0.5(\hat{\phi}+\hat{p}(0))} \leq e^{\sup_{\Gamma_2} g_p + |\ln c_o|}$. Therefore, \hat{f}_u, \hat{f}_ϕ and \hat{h}_w are nonnegative bounded functions being a product of the previous terms.

(ii) and (iii). Note that $(c_o - e^{-p}), (c_o - e^{-q})$ are increasing functions in p, q respectively, and they have the common root $-\ln c_o$, so we have $\text{sgn}((c_o - e^{-p})(p + \ln c_o)) = \text{sgn}(c_o - e^{-p})\text{sgn}(p + \ln c_o) = +1$, and $\text{sgn}((c_o - e^{-q})(q + \ln c_o)) = +1$ for the same reason (sgn is the

sign function). Moreover, $\sqrt{c_+(\hat{u}(0))}e^{\hat{p}-0.5(\hat{\phi}+\hat{p}(0))}$, $\sqrt{c_+(\hat{w})}e^{0.5\hat{q}} \geq 0$, which proves (ii) and (iii). \square

In the following lemma, we introduce some estimations that we will use in the next work.

Lemma 3.2.5. *If $(\hat{u}, \hat{p}, \hat{\phi}, \hat{w}, \hat{q}) \in B$ and $(u, p, \phi, w, q) \in C$, $\psi_w = (w - \sup_{\Gamma} g_w)^+$ and $\psi_q = (q - \sup_{\Gamma} g_q)^+$, then Hölder, triangle, Poincaré and Young inequalities together with the trace and imbedding theorems imply*

$$\int_{\Gamma_{01}} \hat{u}c_+(\hat{u}(0))e^{\hat{p}-\hat{\phi}-\hat{p}(0)}(p')^2 \leq u^m e^{\sup_{\Gamma_2} g_q + \ln c_o} \|p'\|_{L^2(\Gamma_{01})}^2, \quad (227)$$

$$\int_{\Gamma_{01}} (\phi')^2 \leq \|\phi\|_{H^1(\Gamma_{01})}^2, \quad (228)$$

$$\int_{\Gamma_{01}} |\hat{f}_{\phi}||\phi| \leq \frac{1}{2}\|\hat{f}_{\phi}\|_{L^2(\Gamma_{01})}^2 + \frac{1}{2}\|\phi'\|_{L^2(\Gamma_{01})}^2, \quad (229)$$

$$\int_{\Omega_2} (3\hat{w}c_+(\hat{w}) + 2|\nabla q|)|\nabla q|^2 \leq 3\|\hat{w}\|_{C^0(\bar{\Omega}_2)}\|\nabla q\|_{L^2(\Omega_2)}^2 + 2\|\nabla q\|_{L^3(\Omega_2)}^3, \quad (230)$$

$$\left| \int_{\Gamma_{01}} \int_0^{p(s)} \hat{u}\sqrt{c_+(\hat{u}(0))}e^{\hat{p}-0.5(\hat{\phi}+\hat{p}(0))} dk ds \right| \leq u^m e^{\sup_{\Gamma_2} g_q + 0.5 \ln c_o} \|p\|_{L^1(\Gamma_{01})}, \quad (231)$$

$$\left| \int_{\Gamma_{02}} \int_0^{q(t)} k_q(c_o - e^{-k})\sqrt{c_+(\hat{w}(t))}e^{.5\hat{q}(t)} dk dt \right| \leq k_q c_o e^{0.5 \sup_{\Gamma_2} g_q} \|q\|_{L^1(\Gamma_{02})}, \quad (232)$$

$$\int_{\Gamma_{01}} (a\bar{\sigma}w|\hat{w}| + |\hat{f}_u|)|u| \leq \frac{1}{d}\|a\bar{\sigma}w|\hat{w}| + \hat{f}_u\|_{L^2(\Gamma_{01})}^2 + \frac{d}{4}\|u\|_{L^2(\Gamma_{01})}^2, \quad (233)$$

$$\int_{\Gamma_{02}} |\hat{h}_w||w| \leq N_1\|\hat{h}_w\|_{L^{\frac{3}{2}}(\Gamma_{02})}^{\frac{3}{2}} + \frac{1}{6}\|\nabla w\|_{L^3(\Omega_2)}^3 + N_3\|\hat{h}_w\|_{L^3(\Gamma_{02})}\|G\|_{W^{1,3}(\Omega)}, \quad (234)$$

$$\int_{\Gamma_{02}} |h_w||\psi_w| \leq N_2\|h_w\|_{L^{\frac{3}{2}}(\Gamma_{02})}^{\frac{3}{2}} + \frac{1}{6}\|\nabla \psi_w\|_{L^3(\Omega)}^3, \quad (235)$$

$$\|w\|_{L^1(\Gamma_{01})} \leq \frac{\sigma}{4}\|\nabla w\|_{L^2(\Omega_1)}^2 + L_2(G_w), \quad (236)$$

$$\|p\|_{H^1(\Gamma_{01})} \leq L_3(\|p'\|_{L^2(\Gamma_{01})} + \|\nabla q\|_{L^3(\Omega_2)} + \|G_q\|_{W^{1,3}(\Omega_2)}), \quad (237)$$

$$\|\phi\|_{H^1(\Gamma_{01})} \leq L_4\|\phi'\|_{L^2(\Gamma_{01})}, \quad (238)$$

$$\|w\|_{H^1(\Omega_1)} \leq L_3(\|\nabla w\|_{L^2(\Omega_1)} + \|G_w\|_{H^1(\Omega_1)}), \quad (239)$$

$$\|w\|_{W^{1,3}(\Omega_2)} \leq L_4(\|\nabla w\|_{L^3(\Omega_2)} + \|G_w\|_{W^{1,3}(\Omega_2)}), \quad (240)$$

$$\|q\|_{W^{1,3}(\Omega_2)} \leq L_5(\|\nabla q\|_{L^3(\Omega_2)} + \|G_q\|_{W^{1,3}(\Omega_2)}). \quad (241)$$

Here, L_* and N_* are constants depending only on data and domain.

Proof. For reader convenience, we will show (237). Indeed,

$$\begin{aligned}
\|p\|_{H^1(\Gamma_{01})} &\leq \|p - q(0)\|_{H^1(\Gamma_{01})} + \|q(0)\|_{H^1(\Gamma_{01})} \\
&\leq M_1 \|p'\|_{L^2(\Gamma_{01})} + M_2 \|q\|_{C^0(\bar{\Omega}_2)} \\
&\leq M_1 \|p'\|_{L^2(\Gamma_{01})} + M_3 \|q\|_{W^{1,3}(\Omega_2)} \\
&\leq M_1 \|p'\|_{L^2(\Gamma_{01})} + M_3 (\|q - G_q\|_{W^{1,3}(\Omega_2)} + \|G_q\|_{W^{1,3}(\Omega_2)}) \\
&\leq M_1 \|p'\|_{L^2(\Gamma_{01})} + M_3 (M_4 \|\nabla(q - G_q)\|_{L^3(\Omega_2)} + \|G_q\|_{W^{1,3}(\Omega_2)}) \\
&\leq M_1 \|p'\|_{L^2(\Gamma_{01})} + M_3 (M_4 \|\nabla q\|_{L^3(\Omega_2)} \\
&\quad + (M_4 + 1) \|\nabla G_q\|_{L^3(\Omega_2)} + \|G_q\|_{L^3(\Omega_2)}), \\
&\leq L_3 (\|p'\|_{L^2(\Gamma_{01})} + \|\nabla q\|_{L^3(\Omega_2)} + \|G_q\|_{W^{1,3}(\Omega_2)})
\end{aligned} \tag{242}$$

for some constants M_i , for $i = 1, 2, 3, 4$ and $L_3 := \max(M_1, M_3 M_4, M_4 + 1)$, where we have used the triangle inequality, imbedding theorem and Poincaré's inequality. \square

In the following proposition we will prove that the set B is stable under the operator T , that is, if $(\hat{u}, \hat{w}, \hat{p}, \hat{q}, \hat{\phi}) \in B$, then $T(\hat{u}, \hat{w}, \hat{p}, \hat{q}, \hat{\phi}) \in C \subset B$.

Proposition 3.2.6. *Assume (206)-(210) hold. If $(\hat{u}, \hat{w}, \hat{p}, \hat{q}, \hat{\phi}) \in B$ and $(u, p, \phi, w, q) \in Y_g$ solves (224), then $(u, p, \phi, w, q) \in C$.*

Proof. The proof consists of three parts. In (i), we will prove that $u_m \leq u \leq u^m$ and $u_m \leq w$. In (ii), we will prove $-\ln c_o \leq p, q \leq \sup_{\Gamma_2} g_q$. Finally, in (iii) we will prove $0 \leq \phi \leq \phi^m$.

(i) *Claim:* $u_m \leq u \leq u^m$ and $u_m \leq w$.

Note that if $(\psi_u, \psi_p, \psi_\phi, \psi_w, \psi_q) \in Y_0$, then $(\psi_u, 0, 0, \psi_w, 0) \in Y_0$. Therefore, from (224) we have

$$\begin{aligned}
&\int_{\Gamma_{01}} (\hat{u} u' \psi'_u + (N(u, \hat{w}) - \hat{f}_u) \psi_u - N(\hat{u}, w) \psi_w) - \int_{\Gamma_{02}} (\hat{h}_w \psi_w) \\
&+ \int_{\Omega} D_w(\hat{w}, |\nabla w|) (\nabla w \cdot \nabla \psi_w) = 0,
\end{aligned} \tag{243}$$

which is similar to (119) ($\hat{f}_u, \hat{h}_w \in L^\infty(\Gamma_{01})$) (see (i), Lemma 3.2.4) in (243) instead of f and h in (119)). Next the result follows from Propositions 2.2.7 and 2.2.10.

(ii) *Claim:* $-\ln c_o \leq p, q \leq \sup_{\Gamma_2} g_q$.

This part consist of two parts. In (a) we will prove that $-\ln c_o \leq \hat{p}, \hat{q}$ and in (b) we will prove $\hat{p}, \hat{q} \leq \sup_{\Gamma_2} g_q$.

(a) Set $(\psi_u, \psi_p, \psi_\phi, \psi_w, \psi_q) := (0, (p + \ln c_o)^-, 0, 0, (q + \ln c_o)^-)$. Since $(u, w, p, q, \phi) \in Y_g$ and $\inf_{\Gamma_2} g_q \geq -\ln c_o$ (see (209)), then we have $(\psi_u, \psi_p, \psi_\phi, \psi_w, \psi_q) \in Y_0$ (see Theorem 7.6, [15] and Proposition 3.5, [16]). With this choice of $(\psi_u, \psi_p, \psi_\phi, \psi_w, \psi_q)$ the equation (224) gives

$$\int_{\Gamma_{01}} \hat{u}c_+(\hat{u}(0))e^{\hat{p}-\hat{\phi}-\hat{p}(0)}(\psi'_p)^2 + \hat{f}_p\psi_p + \int_{\Gamma_{02}} \hat{h}_q\psi_q + \int_{\Omega_2} D_q(\hat{w}, |\nabla q|)|\nabla\psi_q|^2 = 0. \quad (244)$$

Since $\hat{f}_p\psi_p \geq 0$, $\hat{h}_q\psi_q \geq 0$ ((ii) and (iii), Lemma 3.2.4), then it follows that all terms in (244) are nonnegative. Therefore, $\nabla\psi_q = 0$ a.e. in Ω_2 , and as $\psi_q|_{\Gamma_2} = 0$, it follows that $\psi_q = 0$ a.e. in Ω_2 . Also as $uc_+(\hat{u}(0)) \geq u_m c_+(u_m) > 0$, then $\psi'_p = 0$. However, $\psi_p(0) = \psi_q(0) = 0$, and so we have $\psi_p = 0$.

(b) Now, set $(\psi_u, \psi_p, \psi_\phi, \psi_w, \psi_q) := (0, (p - \sup_{\Gamma_2} g_q)^+, 0, 0, (q - \sup_{\Gamma_2} g_q)^+) \in Y_0$, then with this choice of test function in the equation (224), we have (244), with all terms nonnegative ((ii) and (iii), Lemma 3.2.4) and the result follows similarly to (a).

(iii) *Claim:* $0 \leq \phi \leq \phi^m$.

Also this part is divided into two parts. In (a) we will prove $0 \leq \phi$ and in (b) we will prove $\phi \leq \phi^m$ for some constant ϕ^m .

(a) Set $(\psi_u, \psi_p, \psi_\phi, \psi_w, \psi_q) := (0, 0, \phi^-, 0, 0) \in Y_0$ (Theorem 7.6, [15] and Proposition 3.5, [16]). With this choice of $(\psi_u, \psi_p, \psi_\phi, \psi_w, \psi_q)$ the equation (224) gives

$$\int_{\Gamma_{01}} (\psi'_\phi)^2 - \hat{f}_\phi\psi_\phi = 0, \quad (245)$$

Since $\hat{f}_\phi\psi_\phi \leq 0$ ((i), Lemma 3.2.4), then all terms are nonnegative. As a result $\psi'_\phi = 0$ a.e. on Γ_{01} , so $\psi_\phi = 0$ as $\psi_\phi(0) = 0$, which ends this part.

(b) Set $(\psi_u, \psi_p, \psi_\phi, \psi_w, \psi_q) := (0, 0, \phi, 0, 0) \in Y_0$, then with this test function, (224) implies

$$\int_{\Gamma_{01}} (\phi')^2 = \int_{\Gamma_{01}} \hat{f}_\phi\phi, \quad (246)$$

then using (229) and simplifying, we have

$$\|\phi'\|_{L^2(\Gamma_{01})} \leq \|\hat{f}_\phi\|_{L^2(\Gamma_{01})}. \quad (247)$$

Now, using the imbedding theorem, namely $H^1(\Gamma_{01}) \subset C^0(\bar{\Gamma}_{01})$, we have

$$\|\phi\|_{C^0(\bar{\Gamma}_{01})} \leq M\|\phi'\|_{L^2(\Gamma_{01})} \leq M\|\hat{f}_\phi\|_{L^2(\Gamma_{01})} =: \phi^m, \quad (248)$$

for some constants M . By this we reach the end of this proof. \square

3.3 Operator T is well-defined.

To prove the existence of (u, p, ϕ, w, q) that satisfies the weak form (224), it is easier to look at this equation as the Euler-Lagrange equation of a functional I , then prove that the functional I has a unique minimizer. This proves that T is well-defined.

Let $(\hat{u}, \hat{p}, \hat{\phi}, \hat{w}, \hat{q}) \in B$ and consider

$$\begin{aligned}
I(u, p, \phi, w, q) &:= \int_{\Gamma_{01}} \frac{1}{2} (\hat{u}(u')^2 + \hat{u}c_+(\hat{u}(0))e^{\hat{p}-\hat{\phi}-\hat{p}(0)}(p')^2 + (\phi')^2) \\
&+ \int_{\Gamma_{01}} \int_0^{u(s)} (N(k, \hat{w}(s)) - \hat{f}_u(s)) dk ds - \int_{\Gamma_{01}} \int_0^{w(s)} N(\hat{u}(s), k) dk ds \\
&+ \int_{\Gamma_{01}} \int_0^{p(s)} k_p(c_o - e^{-k}) \sqrt{c_+(\hat{u}(0))} e^{\hat{p}(s)-0.5(\hat{\phi}(s)+\hat{q}(s))} dk ds - \int_{\Gamma_{01}} \hat{f}_\phi \phi \\
&+ \int_{\Gamma_{02}} (-\hat{h}_w w + \int_0^{q(t)} k_q(c_o - e^{-k}) \sqrt{c_+(\hat{w}(t))} e^{5\hat{q}(t)} dk) dt \\
&+ \frac{1}{6} \int_{\Omega} (3\sigma(\hat{w})\chi_{\Omega_1} + (3\hat{w} + 2|\nabla w|)\chi_{\Omega_2}) |\nabla w|^2 \\
&+ \frac{1}{6} \int_{\Omega_2} (3\hat{w}c_+(\hat{w}) + 2|\nabla q|) |\nabla q|^2. \tag{249}
\end{aligned}$$

Find $(u, p, \phi, w, q) \in Y_g$:

$$I(u, p, \phi, w, q) = \min\{I(\psi_u, \psi_p, \psi_\phi, \psi_w, \psi_q), (\psi_u, \psi_p, \psi_\phi, \psi_w, \psi_q) \in Y_g\}. \tag{250}$$

Note that to show the dependence of I on $(\hat{u}, \hat{p}, \hat{\phi}, \hat{w}, \hat{q})$ we will also use the notation $I(u, p, \phi, w, q; \hat{u}, \hat{p}, \hat{\phi}, \hat{w}, \hat{q})$ instead of $I(u, p, \phi, w, q)$.

In the following proposition we will prove that the functional I is strictly convex, lower semicontinuous and satisfies a convergence property, which will help in proving the existence of a minimizer for I .

Proposition 3.3.1. *We have*

- (i) $(u, p, \phi, w, q) \mapsto \mathcal{H}(u, p, \phi, w, q)$ is strictly convex.
- (ii) $(u, p, \phi, w, q) \mapsto \mathcal{H}(u, p, \phi, w, q)$ is lower semicontinuous with respect to weak topology in Y .

(iii) For all sequences $(\hat{u}_n, \hat{p}_n, \hat{\phi}_n, \hat{w}_n, \hat{q}_n) \in X$ converging to $(\hat{u}, \hat{p}, \hat{\phi}, \hat{w}, \hat{q})$ in X and $\hat{w}_n \rightarrow \hat{w}$ a.e. $x \in \Omega$, for all $(u_n, p_n, \phi_n, w_n, q_n) \in Y_g$ bounded in Y and for any $(u, p, \phi, w, q) \in Y_g$ we have

$$\begin{aligned} & \lim_{n \rightarrow \infty} (I(u_n, p_n, \phi_n, w_n, q_n; \hat{u}_n, \hat{p}_n, \hat{\phi}_n, \hat{w}_n, \hat{q}_n) - I(u_n, p_n, \phi_n, w_n, q_n; \hat{u}, \hat{p}, \hat{\phi}, \hat{w}, \hat{q})) = \\ & \lim_{n \rightarrow \infty} (I(u, p, \phi, w, q; \hat{u}_n, \hat{p}_n, \hat{\phi}_n, \hat{w}_n, \hat{q}_n) - I(u, p, \phi, w, q; \hat{u}, \hat{p}, \hat{\phi}, \hat{w}, \hat{q})) = 0. \end{aligned} \quad (251)$$

Proof. (i) Note that all terms in $I(u, p, \phi, w, q)$ are convex for the same reasons mentioned in the proof of Proposition 1.3.1. The strict convexity of the terms corresponding to \hat{f}_p and \hat{h}_q , which are increasing in k follows from Theorem 4, [2].

(ii) Again the lower semicontinuity of I follows from Theorem 2.2.1, [24]. Note that for $\lambda \in \mathbb{R}$ the set $\{(u, p, \phi, w, q), I(u, p, \phi, w, q) \leq \lambda\}$ is closed in Y because $(u, p, \phi, w, q) \rightarrow I(u, p, \phi, w, q)$ is continuous in Y with respect to strong topology.

(iii) We have

$$\begin{aligned} & |I(u, p, \phi, w, q; \hat{u}_n, \hat{p}_n, \hat{\phi}_n, \hat{w}_n, \hat{q}_n) - I(u, p, \phi, w, q; \hat{u}, \hat{p}, \hat{\phi}, \hat{w}, \hat{q})| \leq \\ & \int_{\Gamma_{01}} \frac{1}{2} |\hat{u}_n - \hat{u}| (u')^2 + \frac{1}{2} \left| \hat{u}_n c_+(\hat{u}_n(0)) e^{\hat{p}_n - \hat{\phi}_n - \hat{p}_n(0)} - \hat{u} c_+(\hat{u}(0)) e^{\hat{p} - \hat{\phi} - \hat{p}(0)} \right| (p')^2 \\ & + \int_{\Gamma_{01}} a \bar{w} \bar{\sigma} |\hat{w}_n - \hat{w}| |u| ds \\ & + \int_{\Gamma_{01}} k_u |u| \left| (c_o - e^{-\hat{p}_n}) \sqrt{c_+(\hat{u}_n(0))} e^{\hat{p}_n - 0.5(\hat{\phi}_n + \hat{p}_n(0))} - (c_o - e^{-\hat{p}}) \sqrt{c_+(\hat{u}(0))} e^{\hat{p} - 0.5(\hat{\phi} + \hat{p}(0))} \right| \\ & + \int_{\Gamma_{01}} |\hat{u}_n - \hat{u}| (d|w| + \frac{1}{2} a \bar{\sigma} |w|^2) \\ & + \int_{\Gamma_{01}} \left| \sqrt{c_+(\hat{u}_n(0))} e^{\hat{p}_n - 0.5(\hat{\phi}_n + \hat{p}_n(0))} - \sqrt{c_+(\hat{u}(0))} e^{\hat{p} - 0.5(\hat{\phi} + \hat{p}(0))} \right| \left| \int_0^{p(s)} k_p (c_o - e^{-k}) dk \right| ds \\ & + \int_{\Gamma_{01}} k_\varphi |\phi| \left| \sqrt{c_+(\hat{u}_n(0))} e^{\hat{p}_n - \hat{\phi}_n - \hat{p}_n(0)} - \sqrt{c_+(\hat{u}(0))} e^{\hat{p} - \hat{\phi} - \hat{p}(0)} \right| \\ & + \int_{\Gamma_{02}} k_w \left| (c_o - e^{-\hat{q}_n(t)}) \sqrt{c_+(\hat{w}_n)} e^{0.5 \hat{q}_n(t)} - (c_o - e^{-\hat{q}(t)}) \sqrt{c_+(\hat{w})} e^{0.5 \hat{q}(t)} \right| dt \\ & + \int_{\Gamma_{02}} \left| \sqrt{c_+(\hat{w}_n)} e^{0.5 \hat{q}_n} - \sqrt{c_+(\hat{w})} e^{0.5 \hat{q}} \right| \left| \int_0^{q(t)} k_q (c_o - e^{-k}) dk \right| dt \\ & + \int_{\Omega_1} \frac{1}{2} |\sigma(\hat{w}_n) - \sigma(\hat{w})| |\nabla w|^2 + \int_{\Omega_2} \frac{1}{2} (|\hat{w}_n - \hat{w}| |\nabla w|^2 + |\hat{w}_n c_+(\hat{w}_n) - \hat{w} c_+(\hat{w})| |\nabla q|). \end{aligned}$$

From convergence of $(\hat{u}_n, \hat{p}_n, \hat{\phi}_n, \hat{w}_n, \hat{q}_n)$ in X and $\hat{w}_n \rightarrow \hat{w}$ a.e., we have that the right hand side of the last inequality tends to zero.

Indeed, using the fact that a composition of a continuous function with a uniform convergent sequence is also uniform convergent, then all the terms having one of the uniform convergent sequences \hat{u}_n , \hat{p}_n or $\hat{\phi}_n$ converge to zero.

Note that from compact imbeddings we have $\hat{w}_n \rightarrow \hat{w}$ in $L^3(\Omega)$ and a.e. (from the assumptions). For integral containing $|\hat{w}_n - \hat{w}|$ the result is cleared. Also for the integral having σ , which is bounded, the convergence follows from the Lebesgue dominated convergence theorem. Now for the integral

$$\begin{aligned} & \int_{\Gamma_{02}} \left| \sqrt{c_+(\hat{w}_n)} e^{5\hat{q}_n} - \sqrt{c_+(\hat{w})} e^{0.5\hat{q}} \right| \left| \int_0^{q(t)} k_q(c_o - e^{-k}) dk \right| dt \\ & \leq \int_{\Gamma_{02}} k_q c_o |q(t)| \left(\left| \sqrt{c_+(\hat{w}_n)} \right| e^{0.5\hat{q}_n(t)} - e^{0.5\hat{q}(t)} \right) \\ & \quad + \left| e^{0.5\hat{q}(t)} \right| \left| \sqrt{c_+(\hat{w}_n)} - \sqrt{c_+(\hat{w})} \right| dt, \end{aligned} \quad (252)$$

which converges to zero because we have $e^{0.5\hat{q}_n} - e^{0.5\hat{q}} \rightarrow 0$ uniformly being a composition of the continuous function $e^{-0.5x}$ and the uniform convergent sequence q_n , and we have also $\left| \sqrt{c_+(\hat{w}_n)} - \sqrt{c_+(\hat{w})} \right| \leq 2$, and converges to zero a.e., we can then use the Lebesgue's dominated convergence theorem on the last line of (252). Note that we have used the triangle inequality in (252).

For other integrals having \hat{q}_n , the same idea is applicable. Also the second limit in (iii) can be done in the same way. \square

Now let us prove the relation between (224) and the minimization problem (250).

Proposition 3.3.2. *For $(u, p, \phi, w, q) \in Y_g$, the functional $(u, p, \phi, w, q) \mapsto \mathcal{F}(u, p, \phi, w, q)$ is Gâteaux differentiable and for $(\psi_u, \psi_p, \psi_\phi, \psi_w, \psi_q) \in Y_0$ we have*

$$\begin{aligned} I'(u, p, \phi, w, q)(\psi_u, \psi_p, \psi_\phi, \psi_w, \psi_q) &= \int_{\Gamma_{01}} (\hat{u}u'\psi'_u + \hat{u}c_+(\hat{u}(0))e^{\hat{p}-\hat{\phi}-\hat{p}(0)}p'\psi'_p + \phi'\psi'_\phi) \\ &+ \int_{\Gamma_{01}} (N(u(s), \hat{w}(s)) - f(s))ds - \int_{\Gamma_{01}} N(\hat{u}(s), w(s))ds \\ &+ \int_{\Gamma_{01}} (\hat{f}_p\psi_p - \hat{f}_\phi\phi) + \int_{\Gamma_{02}} (-\hat{h}_w\psi_w + \hat{h}_q\psi_q) \\ &+ \int_{\Omega} (\sigma(\hat{w})\chi_{\Omega_1} + (\hat{w} + |\nabla w|)\chi_{\Omega_2}) \nabla w \cdot \nabla \psi_w \\ &+ \int_{\Omega_2} (\hat{w}c_+(\hat{w}) + |\nabla q|)\nabla q \cdot \nabla \psi_q. \end{aligned} \quad (253)$$

Proof. For $\theta \in \mathbb{R}$ (θ small) set

$i(\theta) = I(u + \theta\psi_u, p + \theta\psi_p, \phi + \theta\psi_\phi, w + \theta\psi_w, q + \theta\psi_q) - I(u, p, \phi, w, q)$ and set κ as the right hand side of (253). We have

$$\begin{aligned} \left| \frac{i(\theta) - i(0)}{\theta} - \kappa \right| &\leq \int_{\Gamma_{01}} |F_{01}(\theta, s)| ds + \int_{\Gamma_{02}} |F_{02}(\theta, s)| ds + \int_{\Omega_1} |F_1(\theta, x)| dx \\ &\quad + \int_{\Omega_2} |F_2(\theta, x)| dx, \end{aligned} \quad (254)$$

where

$$\begin{aligned} F_{01}(\theta, s) &= \hat{u} \left(\frac{1}{2\theta} ((u' + \theta\psi'_\theta)^2 - (u')^2) - u'\psi'_u \right) \\ &\quad + \hat{u}c_+(\hat{u}(0))e^{\hat{p}-\hat{\phi}-\hat{p}(0)} \left(\frac{1}{2\theta} ((p' + \theta\psi'_p)^2 - (p')^2) - p'\psi'_p \right) \\ &\quad + \frac{1}{\theta} ((\phi' + \theta\psi'_\phi)^2 - (\phi')^2) - \phi'\psi'_\phi \\ &\quad - \frac{a}{\theta} \left(\int_{u(s)}^{u(s)+\theta\psi_u} \mathbf{j}(\hat{w})((\bar{w} - k)^+ - (\bar{w} - u)^+) dk \right) \\ &\quad + d \left(\frac{1}{2\theta} ((u + \theta\psi_u)^2 - u^2) - u\psi_u \right) \\ &\quad - \frac{1}{\theta} (\hat{f}_u \cdot (u + \theta\psi_\theta) - \hat{f}_u \cdot u) - \hat{f}_u \cdot \psi_u \\ &\quad + \frac{a}{\theta} \left(\int_{w(s)}^{w(s)+\theta\psi_w} (\mathbf{j}(k) - \mathbf{j}(w))(\bar{w} - \hat{u})^+ dk \right) \\ &\quad + k_p \sqrt{c_+(\hat{u}(0))} e^{\hat{p}(s)-0.5(\hat{\phi}(s)+\hat{q}(0))} \left(\frac{1}{\theta} \int_{p(s)}^{p(s)+\theta\psi_p} (e^{-p(s)} - e^{-k}) dk \right) \\ &\quad - \frac{1}{\theta} (\hat{f}_\phi \cdot (u + \theta\psi_\theta) - \hat{f}_\phi \cdot u) - \hat{f}_\phi \cdot \psi_u, \\ F_{02}(\theta, s) &= -\hat{h}_w \left(\frac{1}{\theta} ((w + \theta\psi_w) - w) - \psi_w \right) \\ &\quad + k_q \sqrt{c_+(\hat{w}(t))} e^{.5\hat{q}(t)} \left(\frac{1}{\theta} \int_{q(t)}^{q(t)+\theta\psi_q} (e^{-q(t)} - e^{-k}) dk \right), \\ F_1(\theta, x) &= \frac{1}{2\theta} (\sigma(\hat{w})|\nabla w + \theta\nabla\psi_w|^2 - \sigma(\hat{w})|\nabla w|^2) - \sigma(\hat{w})(\nabla w \cdot \nabla\psi_w), \\ F_2(\theta, x) &= \frac{1}{\theta} \left(\left(\frac{1}{2}\hat{w} + \frac{1}{3}|\nabla w + \theta\nabla\psi_w| \right) |\nabla w + \theta\psi_w|^2 - \left(\frac{1}{2}\hat{w} + \frac{1}{3}|\nabla w| \right) |\nabla w|^2 \right) \end{aligned}$$

$$\begin{aligned}
& - (\hat{w} + |\nabla w|)(\nabla w \cdot \nabla \psi_w) + \frac{1}{\theta} \left(\frac{1}{2} \hat{w} c_+(\hat{w}) + \frac{1}{3} |\nabla q + \theta \psi_q| |\nabla q + \theta \psi_q|^2 \right. \\
& \left. - \left(\frac{1}{2} \hat{w} c_+(\hat{w}) + \frac{1}{3} |\nabla q| |\nabla q|^2 \right) - (\hat{w} c_+(\hat{w}) + |\nabla q|)(\nabla q \cdot \nabla \psi_q) \right).
\end{aligned}$$

Each of the functions F_* is bounded by an L^1 function (independently from θ) and $F_* \rightarrow 0$ a.e. as $\theta \rightarrow 0$. From Lebesgue's dominated convergence theorem and we have

$$\lim_{\theta \rightarrow 0} \left| \frac{i(\theta) - i(0)}{\theta} - \kappa \right| = 0,$$

which proves (253). □

As a result we have the following proposition.

Proposition 3.3.3. *The equation (224) is the Euler-Lagrange equation of $I(u, p, \phi, w, q)$.*

Now we will prove that T is well defined. In fact, we have the following theorem.

Theorem 3.3.4. *For any $(\hat{u}, \hat{p}, \hat{\phi}, \hat{w}, \hat{q}) \in B \subset X$ the equation (224) has a unique solution $(u, p, \phi, w, q) \in C \subset Y$. Thus, the operator $T : B \subset X \rightarrow C \subset Y$, $T(\hat{u}, \hat{p}, \hat{\phi}, \hat{w}, \hat{q}) = (u, p, \phi, w, q)$ is well-defined.*

Proof. As in the proofs of Theorem 1.3.4 and Theorem 2.3.4, the proof is two parts.

Uniqueness. Follows from the strict convexity of the functional I see (i), Proposition 3.3.1.

Existence. As in the proof of existence in the Theorems 1.3.4 and 2.3.4, Corollary III.20, [5] will be used here. Indeed, we have $I : C \rightarrow \mathbb{R}$ is a convex function, with C nonempty closed convex set. Also $I(u, p, \phi, w, q) \neq \pm\infty$, which follows from the estimations (227)-(234). Also for $(\hat{u}, \hat{p}, \hat{\phi}, \hat{w}, \hat{q}) \in B$, $(u, p, \phi, w, q) \in C$.

$$\begin{aligned}
I(u, p, \phi, w, q) & \geq \int_{\Gamma_{01}} \frac{1}{2} (u_m(u')^2 + du^2 + a\underline{\sigma} \bar{w} w^2 + \gamma(p')^2 + (\phi')^2) \\
& - \int_{\Gamma_{01}} ((a\underline{\sigma} \bar{w} |\hat{w}| + |\hat{f}_u|) |u| + d|\hat{u}| |w| + |\hat{f}_\phi| |\phi|) - \int_{\Gamma_{02}} |h_w| |w| \\
& + \int_{\Omega_1} \frac{1}{2} \underline{\sigma} |\nabla w|^2 + \int_{\Omega_2} \frac{1}{3} (|\nabla w|^3 + |\nabla q|^3) \\
& \geq M_1 \left(\int_{\Gamma_{01}} (u')^2 + u^2 + w^2 + (p')^2 + (\phi')^2 + \int_{\Omega_1} |\nabla w|^2 \right. \\
& \left. + \int_{\Omega_2} (|\nabla w|^3 + |\nabla q|^3) \right) + M_2, \tag{255}
\end{aligned}$$

with $\gamma := u_m c_+(u_m) e^{-\ln c_0 - \phi^m - \sup_{\Gamma_2} g_q}$ and M_1, M_2 constants independent of (u, p, ϕ, w, q) , and where we have used (229), (233), (234) and (236). Note that if $\|(u, p, \phi, w, q)\|_Y \rightarrow \infty$, then so is the right hand side of (255) because of the inequalities (237)-(241). Next, from Corollary III.20, [5], $I(u, p, \phi, w, q)$ has a minimizer in C which ends the proof. \square

In the following remark we prove an estimation for $I(u, p, \phi, w, q)$ for a specific $(u, p, \phi, w, q) \in Y_g$, which will help in proving that T is bounded.

Remark 3.3.5. For $(\hat{u}, \hat{p}, \hat{\phi}, \hat{w}, \hat{q}) \in B$, with $\|(\hat{u}, \hat{p}, \hat{\phi}, \hat{w}, \hat{q})\|_X \leq M$ and $G = (G_w(0), G_q(0), 0, G_w, G_q) \in Y_g$, we have

$$\begin{aligned}
I(|G|) &:= I(|G_w|(0), |G_q|(0), 0, |G_w|, |G_q|; \hat{u}, \hat{p}, \hat{\phi}, \hat{w}, \hat{q}) \\
&\leq \int_{\Gamma_{01}} \frac{1}{2} (d(G_w(0)))^2 + a\bar{\sigma} \bar{w} G_w^2 \\
&\quad + \int_{\Omega_1} \frac{1}{2} \bar{\sigma} |\nabla G_w|^2 + \int_{\Omega_2} \frac{1}{2} \hat{w} |\nabla G_w|^2 + \frac{1}{3} |\nabla G_w|^3 \\
&\quad + \int_{\Omega_2} \frac{1}{2} |\hat{w}| c_+(\hat{w}) |\nabla G_q|^2 + \frac{1}{3} |\nabla G_q|^3 \\
&\leq K_1(G_w, G_q) + \frac{1}{2} \|\hat{w}\|_{L^3(\Omega_2)} (\|\nabla G_w\|_{L^3(\Omega_2)}^2 + \|\nabla G_q\|_{L^3(\Omega_2)}^2) \\
&\leq K_1(G_w, G_q) + K_2 \|\hat{w}\|_{H^{\frac{1}{2}+\epsilon}(\Omega)} (\|\nabla G_w\|_{L^3(\Omega_2)}^2 + \|\nabla G_q\|_{L^3(\Omega_2)}^2) \\
&\leq K_1(G_w, G_q) + \frac{1}{2} K_2 M (\|\nabla G_w\|_{L^3(\Omega_2)}^2 + \|\nabla G_q\|_{L^3(\Omega_2)}^2) \\
&:= K(G_w, G_q, M),
\end{aligned} \tag{256}$$

for some constants K_1 and K_2 , where we have used Hölder's inequality and imbedding theorem, that is $H^{\frac{1}{2}+\epsilon}(\Omega) \subset L^3(\Omega)$ (see [1] and [12]), and $|c_+(\hat{w})| \leq 1$ (Remark 3.2.3). Note that if $G \in Y_g$, then $|G| := (|G_w|(0), |G_q|(0), 0, |G_w|, |G_q|) \in Y_g$ (see Theorem 7.6, [15] and Proposition 3.5, [16]). \square

In the following proposition we will prove that T takes bounded sets to bounded sets.

Proposition 3.3.6. *Assume (206)-(210) hold. Let $M > 0$ and consider $B_M = B \cap \{(\hat{u}, \hat{p}, \hat{\phi}, \hat{w}, \hat{q})\|_X \leq M\}$. There exists $N > 0$ such that $(u, p, \phi, w, q) = T(\hat{u}, \hat{p}, \hat{\phi}, \hat{w}, \hat{q}) \in C_N := C \cap \{(u, p, \phi, w, q)\|_Y \leq N\}$.*

Proof. As the solution $(u, p, \phi, w, q) \in Y_g$ of (224) is unique, then it is the minimizer for $I(u, p, \phi, w, q)$ in Y_g . Since $|G| \in Y_g$, then we have

$$I(u, p, \phi, w, q; \hat{u}, \hat{p}, \hat{\phi}, \hat{w}, \hat{q}) \leq I(|G|; \hat{u}, \hat{p}, \hat{\phi}, \hat{w}, \hat{q}).$$

From this inequality and (257), we have

$$\begin{aligned}
& \frac{1}{2} \int_{\Gamma_{01}} (u_m(u')^2 + du^2 + \gamma(p')^2 + (\phi')^2) + \frac{\sigma}{2} \int_{\Omega_1} |\nabla w|^2 + \frac{1}{3} \int_{\Omega_2} (|\nabla w|^3 + |\nabla q|^3) \\
& \leq I(|G|) + \int_{\Gamma_{01}} ((a\bar{\sigma}w|\hat{w}| + |\hat{f}_u|)|u| + d|\hat{u}||w| + \hat{f}_\phi\phi) + \int_{\Gamma_{02}} \hat{h}_w|w| \\
& \leq K(G_w, G_q, M) + \int_{\Gamma_{01}} \hat{f}_\phi\phi + \int_{\Gamma_{01}} (a\bar{\sigma}w|\hat{w}| + |\hat{f}_u|)|u| + d \int_{\Gamma_{01}} |\hat{u}||w| + \int_{\Gamma_{02}} \hat{h}_w|w|,
\end{aligned} \tag{258}$$

$$\tag{259}$$

where $\gamma := u_m c_+(u^m) e^{-\ln c_o - \phi^m - \sup_{\Gamma_2} g_q}$. Now using (229), (233), (234) and (236) and rearranging, then we have

$$\|u\|_{H^1(\Gamma_{01})} + \|p'\|_{L^2(\Gamma_{01})} + \|\phi'\|_{L^2(\Gamma_{01})} + \|\nabla w\|_{L^2(\Omega_1)} + \|\nabla w\|_{L^3(\Omega_2)} + \|\nabla q\|_{L^3(\Omega_2)} \leq N(G, M),$$

which is enough to prove the proposition because of the inequalities (237)-(241). \square

3.4 Existence of a fixed point.

To apply Theorem 1.4.1, we consider $S = J \circ T \circ \Pi$.

$$\begin{aligned}
S : X & \xrightarrow{\Pi} B \xrightarrow{T} C \xrightarrow{J} X, \\
(\tilde{u}, \tilde{p}, \tilde{\phi}, \tilde{w}, \tilde{q}) & \rightarrow (\hat{u}, \hat{p}, \hat{\phi}, \hat{w}, \hat{q}) \rightarrow (u, p, \phi, w, q) \rightarrow (u, p, \phi, w, q).
\end{aligned} \tag{260}$$

where $\Pi : X \rightarrow B$ is a projection operator and $J : Y \hookrightarrow X$ is the imbedding map.

The operator Π is defined as follows. Consider the operators

$$\begin{aligned}
\Pi_1 : \mathbb{R}^2 & \rightarrow [m, \infty) \\
(s; m) & \rightarrow \Pi_1(s; m) := \max\{s, m\}, \\
\Pi_2 : \mathbb{R}^3 & \rightarrow [m, n] \\
(s; m, n) & \rightarrow \Pi_2(s; m, n) := \min\{n, \max\{m, s\}\},
\end{aligned}$$

then define $\Pi_u(s) := \Pi_2(s; u_m, u^m)$, $\Pi_p(s) := \Pi_2(s; p_m, p^m)$, $\Pi_\phi(s) := \Pi_2(s; 0, \phi^m)$, $\Pi_w(s) := \Pi_1(s; u_m)$ and $\Pi_q(s) := \Pi_p(s)$. Here u_m, u^m are defined as in Propositions 3.2.6, and ϕ^m as in (248), $p_m := -\ln c_o$ and $p^m := \sup_{\Gamma_2} g_q$.

Now define

$$\begin{aligned}
\Pi : X & \longrightarrow B \\
(\tilde{u}, \tilde{p}, \tilde{\phi}, \tilde{w}, \tilde{q}) & \rightarrow (\hat{u}, \hat{p}, \hat{\phi}, \hat{w}, \hat{q}) := (\Pi_u(\tilde{u}), \Pi_p(\tilde{p}), \Pi_\phi(\tilde{\phi}), \Pi_w(\tilde{w}), \Pi_q(\tilde{q})).
\end{aligned} \tag{261}$$

Remark 3.4.1. Note that the projection maps Π_i for $i = u, p, \phi$ are similar to Π_u in Sections 1.4 and 2.4, Π_w is similar to that in Section 2.4 and Π_q is similar to Π_w in Section 1.4 (same domains, same ranges and same definitions). As a result we have $\Pi(X) \subset B$, which follows by a work similar to that done in Sections 1.4 and 2.4.

We have

Lemma 3.4.2. *The operator J is Lipschitz and compact, and the operator Π is Lipschitz, both Π and J have Lipschitz constant equal to one.*

Proof. The proof is similar to the proof of Lemma 1.4.3. The properties of J are classical, see [1], [5], [15]. The Lipschitz property of Π follows from the fact that $\Pi_i(s)$, for $i = u, p, \phi, w, q$ are continuous, piecewise linear functions with derivatives in $[-1, 1]$. \square

Proposition 3.4.3. *Assume (206)-(210) hold, then the operator S is compact and continuous.*

Proof. The proof is similar to that done for Proposition 1.4.6, in which we proved the compactness and the continuity for $\hat{T} := J \circ T$ because Π is Lipschitz continuous. Note that we still have T takes bounded sets to bounded sets, see Proposition 3.3.6, so the proof of compactness is similar to that in Proposition 1.4.6. Also having $\hat{w}_n \rightarrow w$ in $H^{\frac{1}{2}+\epsilon}(\Omega)$ implies $\hat{w}_n \rightarrow w$ a.e., see [5]. Next using (iii) in Proposition 3.3.1 we can prove the continuity of S exactly in the same way we did in Proposition 1.4.6. \square

In the following lemma we will prove that for any $(\hat{u}, \hat{p}, \hat{\phi}, \hat{w}, \hat{q}) \in B$, (u, p, ϕ, w, q) solution of (250) is bounded. This result will be used later to prove that the set Λ (see (77)) is bounded.

Lemma 3.4.4. *There exists $C_\Lambda > 0$ such that for $(\hat{u}, \hat{p}, \hat{\phi}, \hat{w}, \hat{q}) \in B$, if $(u, p, \phi, w, q) \in C$ is the solution of (250) and $\hat{w} = \max\{u_m, \lambda w\}$, for a certain $\lambda \in [0, 1]$, then*

$$\|(u, p, \phi, w, q)\|_Y \leq C_\Lambda. \quad (262)$$

Proof. From the assumptions we have $\hat{w} \in H^1(\Omega) \cap W^{1,3}(\Omega_2)$ and $|\nabla \hat{w}| \leq |\nabla w|$ a.e. in Ω . Also we have

$$I(u, p, \phi, w, q; \hat{u}, \hat{p}, \hat{\phi}, \hat{w}, \hat{q}) \leq I(|G|),$$

then, using (256) and (258)

$$\begin{aligned}
& \frac{1}{2} \int_{\Gamma_{01}} (u_m(u'))^2 + du^2 + \gamma(p')^2 + (\phi')^2 + \\
& + \frac{1}{6} \int_{\Omega} (3\underline{\sigma}\chi_{\Omega_1} + 2|\nabla w|\chi_{\Omega_2}) |\nabla w|^2 + \frac{1}{3} \int_{\Omega_2} |\nabla q|^3 \\
& \leq I(|G|) + \int_{\Gamma_{01}} ((a\overline{\sigma w}|\hat{w}| + |\hat{f}_u|)|u| + d|\hat{u}||w| + \hat{f}_\phi\phi) + \int_{\Gamma_{02}} \hat{h}_w|w| \\
& \leq K_1(G_w, G_q) + K_2\|\hat{w}\|_{L^3(\Omega)} (\|\nabla G_w\|_{L^3(\Omega_2)}^2 + \|\nabla G_q\|_{L^3(\Omega_2)}^2) \\
& + \int_{\Gamma_{01}} ((a\overline{\sigma w}|\hat{w}| + |\hat{f}_u|)|u| + d|\hat{u}||w| + \hat{f}_\phi\phi) + \int_{\Gamma_{02}} \hat{h}_w|w| \\
& \leq K_1(G_w, G_q) + K_2\|\hat{w}\|_{L^3(\Omega_2)} (\|\nabla G_w\|_{L^3(\Omega_2)}^2 + \|\nabla G_q\|_{L^3(\Omega_2)}^2) \\
& + \left(a\overline{\sigma w}\|\hat{w}\|_{L^1(\Gamma_{01})} + |\Gamma_{01}|^{\frac{1}{2}}k_u(c_o - e^{-\sup_{\Gamma_2} g_q})e^{\sup_{\Gamma_2} g_q + |\ln c_o|} \right) u^m \\
& + du^m|\Gamma_{01}|\|w\|_{L^1(\Gamma_{01})} + k_\varphi e^{\sup_{\Gamma_2} g_q + |\ln c_o|}\|\phi\|_{L^1(\Gamma_{01})} \\
& + |\Gamma_{02}|^{\frac{1}{2}}k_w(c_o - e^{-\sup_{\Gamma_2} g_q})e^{0.5\sup_{\Gamma_2} g_q}\|w\|_{L^1(\Gamma_{02})}. \tag{263}
\end{aligned}$$

A similar work to that in (171) implies

$$K_2\|\hat{w}\|_{L^3(\Omega_2)} (\|\nabla G_w\|_{L^3(\Omega_2)}^2 + \|\nabla G_q\|_{L^3(\Omega_2)}^2) \leq \frac{1}{12}\|\nabla w\|_{L^3(\Omega_2)}^3 + M_1(G_w), \tag{264}$$

for some constants $M_1(G_w)$. Also with a similar work we can prove

$$|\Gamma_{02}|^{\frac{1}{2}}k_w(c_o - e^{-\sup_{\Gamma_2} g_q})e^{0.5\sup_{\Gamma_2} g_q}\|w\|_{L^1(\Gamma_{02})} \leq \frac{1}{12}\|\nabla w\|_{L^3(\Omega_2)} + M_2(G_w), \tag{265}$$

$$k_\varphi e^{\sup_{\Gamma_2} g_q + |\ln c_o|}\|\phi\|_{L^1(\Gamma_{01})} \leq \frac{1}{2}\|\phi'\|_{L^2(\Gamma_{01})} + M_3, \tag{266}$$

for some constants $M_2(G_w)$ and M_3 .

Now using (172) (with G_w instead of G), (236), (264), (265) and (266) with (263) and rearranging the result, we have

$$\|u\|_{H^1(\Gamma_{01})}, \quad \|p'\|_{L^2(\Gamma_{01})}, \quad \|\phi'\|_{L^2(\Gamma_{01})}, \quad \|\nabla w\|_{L^2(\Omega_1)}, \quad \|\nabla w\|_{L^3(\Omega_2)}, \quad \|\nabla q\|_{L^3(\Omega_2)}$$

are uniformly bounded, which is enough to prove the proposition because of the inequalities (237)-(241). \square

In order to use Theorem 1.4.1, we need to show first that S satisfies the property that the set Λ given by (77) is bounded. We have

Proposition 3.4.5. *The set $\Lambda := \{x \in X : x = \lambda Sx, \lambda \in [0, 1]\}$ is bounded in X , i.e there exists $C_\Lambda > 0$ such that $\|(u, p, \phi, w, q)\|_X \leq C_\Lambda$, for all $(u, p, \phi, w, q) \in \Lambda$.*

Proof. For $x = (\tilde{u}, \tilde{p}, \tilde{\phi}, \tilde{w}, \tilde{q}) \in \Lambda$ set $(\hat{u}, \hat{p}, \hat{\phi}, \hat{w}, \hat{q}) = \Pi(\tilde{u}, \tilde{p}, \tilde{\phi}, \tilde{w}, \tilde{q}) \in B$ and $(u, p, \phi, w, q) = T(\hat{u}, \hat{p}, \hat{\phi}, \hat{w}, \hat{q})$. So if $x = \lambda Sx$, then $(\tilde{u}, \tilde{p}, \tilde{\phi}, \tilde{w}, \tilde{q}) = \lambda(u, p, \phi, w, q)$. From the construction of the operator Π , we have $\hat{w} = \Pi_w(\tilde{w}) = \max\{u_m, \tilde{w}\} = \max\{u_m, \lambda w\}$. From Lemma 3.4.4 we have that Λ is bounded in Y . However, $Y \subset X$ (compactly imbedded, see [11]), so we have (u, p, ϕ, w, q) is uniformly bounded in X . \square

3.4.1 Proof of the main result (Theorem 3.1.3).

Proof. (i) The operator $S : X \rightarrow -X$ satisfies the conditions of Theorem 1.4.1 (Proposition 3.4.3, and 3.4.5). Therefore S has a fixed point denoted (u, p, ϕ, w, q) . As $S = J \circ T \circ P$, it follows that $(u, p, \phi, w, q) \in C \subset Y_g$ and (u, p, ϕ, w, q) satisfies (217). If we set $v = \mathbf{j}(w)|_{\Omega_1}$, then we have that $(u, p, \phi, v, w, q) \in Z_g$ and solves (200).

All the bounds (211)-(215), except $w \leq w^m$, are proved in Proposition 3.2.6. Now for $w \leq w^m$, note that (u, w) satisfies the weak form (243), which is exactly the weak form (119) with $f = \hat{f}_u$, $h = \hat{h}_w$. When $(\hat{u}, \hat{p}, \hat{\phi}, \hat{w}, \hat{q}) = (u, p, \phi, w, q)$ we have $f_u = \hat{f}_u$, $h_w = \hat{h}_w$ and by Remark 3.1.4 we have f_u, h_w bounded in $L^\infty(\Gamma_{01}), L^\infty(\Gamma_{02})$ respectively (bounded uniformly in (u, p, ϕ, w, q)). Applying Proposition 2.2.12 on the weak form (243), we have $w \leq w^m$.

(ii) As we said in (i), (u, w) satisfies the weak form (243), with the uniformly bounded functions $\hat{f}_u = f_u$ and $\hat{h}_w = h_w$, when $(\hat{u}, \hat{p}, \hat{\phi}, \hat{w}, \hat{q}) = (u, p, \phi, w, q)$. From the uniform boundedness of f_u and h_w (see Remark 3.1.4), the result is proven similarly to one in Chapter 2, namely it follows by applying Corollary 2.2.11 and Proposition 2.2.12 on the weak form (243). \square

Summary

In this thesis we proved the existence of a positive bounded weak solution for a system of nonlinear PDEs (see (i), Theorem 3.1.3). This implies the validity of the mathematical model it represents.

The Schauder fixed point theorem is used in the proof as follows: we define the weak form for this system, then by fixing some variables, the weak form becomes the Euler-Lagrange equation for a variational problem, which has a unique solution. An operator that maps fixed variables (in a suitable space) into the solution of the Euler-Lagrange equation is defined. Next, the existence is a result of applying the Schauder fixed point theorem on this operator.

In this work we demonstrated the importance of some functions. For example, the isotherm function i (equation (12)), which describes the equilibrium between vapor in the air phase Ω_1 and water in the porous phase Ω_2 . The isotherm function i needs to be smooth and has a bounded derivative with positive bounds (see (104)). Also the term $N(u, w)$ (equation (113)), which represents the adsorption-desorption phenomena between vapor in Ω_1 and water on Γ_{01} . The adsorption-desorption term $N(u, w)$ plays a key role in estimating the solution from below, see section 2.2.1.

Another analysis result is proved, that is the water in the model can be controlled through some suitable conditions on data (see (ii), Theorem 3.1.3). This proves that the solution is physically meaningful (water can't exceed the capacity of the medium to contain water).

The essentials of the numerical method are given in sections 1.5 and 2.5.

The numerical computations show the importance of the parameters a and d in determining the behavior of the solution (by changing a and d , u changes from increasing Figure 7 to decreasing Figure 8)). In addition to this, the numerical work proves that in our surface diffusion model we need to add the gradient term $|\nabla w|$ only in a small neighborhood of

the origin O , to have a best approximation for the realistic thin layer model. In fact, we considered (94) as an equation for water in Ω_2 in the system (96)-(97) and we got that the best approximation for the simplified realistic thin layer model correspond to our model is obtained for a particular choice of (k, n, ϵ) with $\epsilon < 0.5$.

Similarly, we can repeat the numerical work after considering

$$\nabla \cdot ((wc_+(w) + \tau|\nabla q|^{m-2}\chi_{\{(x,y):x^2+y^2 \leq \rho\}})\nabla q) = 0 \text{ in } \Omega_2, \quad (267)$$

for some constants $\tau, \rho > 0$ and $m > 2$, as an equation for q in Ω_2 and (94) as an equation for water in Ω_2 . We expect that the surface diffusion model (1)-(16) has the best approximation of the thin layer model for a particular choice of $(k, n, \epsilon, \tau, m, \rho)$.

In conclusion, the surface diffusion model (1)-(16) (with the equations (94) and (267) resp. replacing equations (5) and (6) resp.) and with a particular choice of $(k, n, \epsilon, \tau, m, \rho)$ is a good approximation of the realistic thin layer model and can be used to describe the reaction kinetics near a triple phase boundary in the catalyst layer of a hydrogen fuel cell.

Appendix A

Thin layer model.

The thin layer model [3], which represents the reaction kinetics near the triple phase point “O” is given by a system of PDEs coupling five variables c_v (concentration of vapor), c_o (concentration of oxygen), c_w (concentration of water), c_+ (concentration of protons) and φ (potential) as follows

$$\left\{ \begin{array}{l} -\Delta c_v = 0 \quad \text{in } \Omega_{1\delta}, \\ -\Delta c_o = 0 \quad \text{in } \Omega_{1\delta} \cup \Omega_{2\delta}, \\ -\nabla \cdot ((D_w \chi_{\Omega_\delta} + \chi_{\Omega_{2\delta}}) c_w \nabla c_w) = 0 \quad \text{in } \Omega_\delta \cup \Omega_{2\delta} \cup \Gamma_\delta, \\ -\nabla \cdot c_w ((D_+ \chi_{\Omega_\delta} + \chi_{\Omega_{2\delta}}) (c_+ \nabla \varphi + c'_+ \nabla c_w)) = 0 \quad \text{in } \Omega_\delta \cup \Omega_{2\delta} \cup \Gamma_\delta, \\ -\Delta \varphi = 0 \quad \text{in } \Omega_\delta, \end{array} \right. \quad (268)$$

and

$$c_+ \equiv c_+(c_w) = -\frac{1}{2} k_H c_w + \left(\frac{1}{4} (k_H c_w)^2 + k_H c_w \right)^{\frac{1}{2}} \quad \text{in } \Omega_{2\delta}, \quad (269)$$

$c'_+ = \frac{dc_+(c_w)}{dc_w}$; D_w , D_+ and k_H are positive constant.

These PDEs are equipped with boundary conditions as follows

$$\left\{ \begin{array}{l} -c_w \partial_n c_w = 0, \\ -c_w (c_+ \partial_n \varphi + c'_+ \partial_n c_w) = 0, \\ -\partial_n \varphi = 0, \end{array} \right. \quad \text{on } \{(x, y) : x = -1\}, \quad (270)$$

$$c_o = g_o, \quad c_v = g_v, \quad \text{on } \Gamma_1, \quad (271)$$

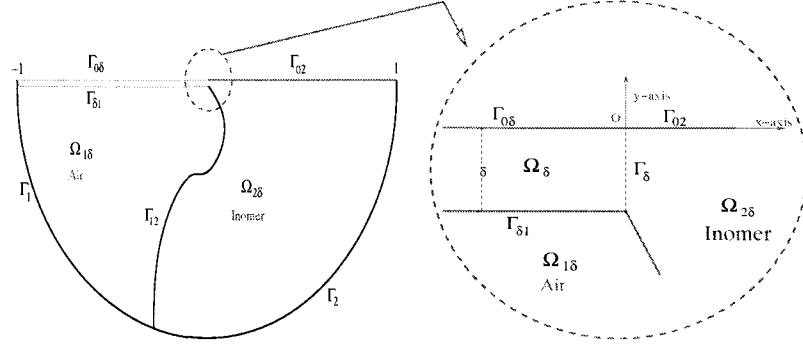


Figure 24: Thin layer.

$$\left\{ \begin{array}{l} -\partial_n c_o = k_o S_c, \\ -c_w \partial_n c_w = k_w S_c, \\ -c_w (c_+ \partial_n \varphi + c'_+ \partial_n c_w) = k_+ S_c, \\ -\partial_n \varphi = k_\varphi c_+, \end{array} \right. \quad \text{on } \Gamma_{0\delta}, \quad (272)$$

$$\left\{ \begin{array}{l} -\partial_n c_o = 0, \\ -\partial_n c_v = k_a c_v (c_w^{\max} - c_w) - k_d c_w, \\ -D_w c_w \partial_n c_w = -\partial_n c_v, \\ -c_w (c_+ \partial_n \varphi + c'_+ \partial_n c_w) = 0, \\ -\partial_n \varphi = 0, \end{array} \right. \quad \text{on } \Gamma_{\delta 1}, \quad (273)$$

where

$$S_c = c_o c_+ e^{0.5\varphi} - e^{-0.5\varphi}, \quad (274)$$

k_o , k_w , k_+ , k_φ , k_a , and k_d are positive constants, with $k_a \gg k_d$ and c_w^{\max} a constant representing the capacity of the medium to adsorb/contain water. On Γ_{12} we have:

$$\left\{ \begin{array}{l} c_w = c_w^*(r), \\ c_o |_{\Gamma_{12}^+} = c_o |_{\Gamma_{12}^-}, \\ \partial_n c_o |_{\Gamma_{12}^+} = D_o \partial_n c_o |_{\Gamma_{12}^-}, \end{array} \right. \quad \text{on } \Gamma_{12}, \quad (275)$$

where

$$\left\{ \begin{array}{l} c_w^*(r) = 0.3 + 10.8 r - 16 r^2 + 14.1 r^3, \\ r = 80 c_v, \end{array} \right. \quad (276)$$

and D_o is a positive constant, $c_o|_{\Gamma_{12}^+}$, respectively $c_o|_{\Gamma_{12}^-}$, means the value of c_o on Γ_{12} taken as a limit from Ω_2 , respectively Ω_1 . Finally, the remaining boundary conditions are

$$\begin{aligned}
\partial_n c_o &= 0, \quad \varphi = g, \quad c_w = g_w, & \text{on } \Gamma_2, \\
-\partial_n c_o &= k_o S_c & \text{on } \Gamma_{02}, \\
-c_w \partial_n c_w &= -k_w S_c \chi_{\Gamma_{02}} - D_v \partial_n c_v \chi_{\Gamma_{12}} & \text{on } \Gamma_{02} \cup \Gamma_{12}, \\
-c_w (c_+(c_w) \partial_n \varphi + c'_+(c_w) \partial_n c_w) &= k_+ S_c \chi_{\Gamma_{02}} + 0 \chi_{\Gamma_{12}} & \text{on } \Gamma_{02} \cup \Gamma_{12}, \\
[\varphi] &= 0, & \text{on } \Gamma_\delta,
\end{aligned}$$

for some positive constant D_v , $[\varphi]$ means the jump in φ .

Note that the domains Ω_δ , $\Omega_{1\delta}$, $\Omega_{2\delta}$ and the boundaries $\Gamma_{0\delta}$, $\Gamma_{\delta 1}$, Γ_δ , Γ_{12} , Γ_{02} , Γ are as presented in Figure 24.

Appendix B

Limit model (surface diffusion model).

The limit model corresponding to the thin layer model in Appendix A (see [3]), which represents the reaction kinetics near the triple phase point “O”, is given by a system of PDEs coupling five variables c_v (concentration of vapor), c_o (concentration of oxygen), c_w (concentration of water), c_+ (concentration of protons) and φ (potential) as follows

$$\left\{ \begin{array}{ll} -\delta\partial_x(c_w\partial_x c_w) = k_w^s S_c + k_a c_v (c_w^{\max} - c_w) - k_d c_w & \text{on } \Gamma_{01}, \\ -\delta\partial_x [c_w(\partial_x c_+ + c_+ \partial_x \varphi)] = -k_+^s S_c & \text{on } \Gamma_{01}, \\ -\partial_{xx}\varphi = k_\varphi c_+ & \text{on } \Gamma_{01}, \\ -\Delta c_v = 0 & \text{in } \Omega_1, \\ -\Delta c_o = 0 & \text{in } \Omega_1 \cup \Omega_2, \\ -\nabla \cdot (c_w \nabla c_w) = 0 & \text{in } \Omega_2, \\ -\nabla \cdot c_w (c_+ \nabla \varphi + c_+' \nabla c_w) = 0 & \text{in } \Omega_2, \end{array} \right. \quad (277)$$

and

$$c_+ \equiv c_+(c_w) = -\frac{1}{2}k_H c_w + \left(\frac{1}{4}(k_H c_w)^2 + k_H c_w \right)^{\frac{1}{2}} \quad \text{in } \Omega_2, \quad (278)$$

$$S_c = c_o c_+ e^{0.5\varphi} - e^{-0.5\varphi}, \quad (279)$$

with $k_a, k_d, k_\varphi, k_H, k_w^s$, and k_+^s , given constants with $k_a \gg k_d$. Also c_w^{\max} is a constant representing the capacity of the medium to adsorb/contain water.

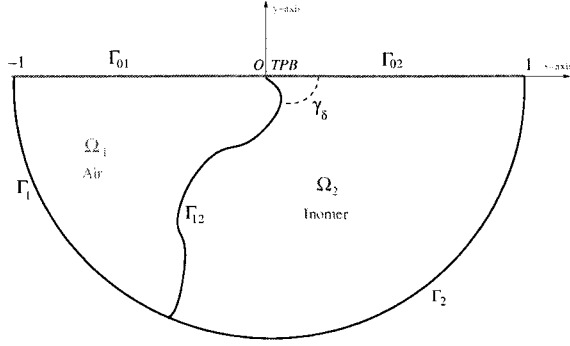


Figure 25: Triple phase boundary domain.

These PDEs in Ω_1 are equipped with boundary conditions as follows

$$c_o = g_o, \quad c_v = g_v, \quad \text{on } \Gamma_1, \quad (280)$$

$$\begin{cases} -\partial_n c_o = k_o S_c, \\ -\partial_n c_v = k_a c_v (c_w^{\max} - c_w) - k_d c_w, \end{cases} \quad \text{on } \Gamma_{01}, \quad (281)$$

$$\begin{cases} c_w = c_w^*(r), \\ c_o |_{\Gamma_{12}^+} = c_o |_{\Gamma_{12}^-}, \\ \partial_n c_o |_{\Gamma_{12}^+} = D_o \partial_n c_o |_{\Gamma_{12}^-} \end{cases} \quad \text{on } \Gamma_{12}, \quad (282)$$

where

$$\begin{cases} c_w^*(r) = 0.3 + 10.8r - 16r^2 + 14.1r^3, \\ r = 80c_v. \end{cases} \quad (283)$$

and D_o is a constant.

The PDEs in Ω_2 are equipped with

$$\partial_n c_o = 0, \quad \varphi = g, \quad c_w = g_w, \quad \text{on } \Gamma_2, \quad (284)$$

$$-\partial_n c_o = k_o S_c \quad \text{on } \Gamma_{02}, \quad (285)$$

$$\begin{aligned} -c_w \partial_n c_w &= -k_w S_c \chi_{\Gamma_{02}} - D_v \partial_n c_v \chi_{\Gamma_{12}} \\ &\quad + K_w c_w(0^-) \partial_x c_w(0^-) \delta_0 \end{aligned} \quad \text{on } \partial\Omega_2 \setminus \Gamma_2 \quad (286)$$

$$\begin{aligned} -c_w (c_+(c_w) \partial_n \varphi + c'_+(c_w) \partial_n c_w) &= k_+ S_c \chi_{\Gamma_{02}} + 0 \chi_{\Gamma_{12}} \\ &\quad + K_+ (c_w (c_+ \partial_x \varphi + c'_+ \partial_x c_w)(0^-) \delta_0 \end{aligned} \quad \text{on } \partial\Omega_2 \setminus \Gamma_2, \quad (287)$$

where D_v, K_w, K_+ are constants, δ_0 is the Dirac delta at the origin and $c'_+ = \frac{dc_+(c_w)}{dc_w}$.

For the differential equations in Γ_{01} we have the boundary conditions

$$c_w(0^-) = \langle c_w \rangle_{\gamma_\delta}, \quad c_+(0^-) = \langle c_+ \rangle_{\gamma_\delta}, \quad \varphi(0^-) = \langle \varphi \rangle_{\gamma_\delta}, \quad (288)$$

$$-(c_w \partial_x c_w)(-1) = 0, \quad -(c_w (\partial_x c_+ + c_+ \partial_x \varphi))(-1) = 0, \quad -\partial_x \varphi(-1) = 0, \quad (289)$$

where $\langle (\cdot) \rangle_{\gamma_\delta} := \frac{1}{|\gamma_\delta|} \int_{\gamma_\delta} (\cdot) d\sigma$.

Note that the domains Ω_1, Ω_2 and the boundaries $\Gamma_{01}, \Gamma_{02}, \Gamma_1, \Gamma_2$ are as presented in Figure 25.

Remark B.0.6. *Note that this limit model has a singular solution around the origin (w^2 and q^2 are logarithmic functions around the origin, due to the Dirac singularity at the origin in the boundary conditions (286) and (287). Moreover, a numerical computation in [3] shows that this limit model system has a solution only when $u_x(0) \leq 0$.*

Bibliography

- [1] **R. Adams**, Sobolev Spaces, Vol. 65, *Pure and Applied Mathematics, Academic Press, New York, 1973.*
- [2] **H. Alvarez**, On the Characterization of Convex Functions, Vol. 48, *Revista De La Union Mathematics, Argentina, 2007.*
- [3] **P. Berg, A. Novruzi and O. Volkov**, Reaction Kinetics at the Triple-phase Boundary in PEM Fuel Cell, *Journal of Fuel Cell Science and Technology, 5, No. 2, 021007.*
- [4] **L. Boccardo, T. Gallouët**, *Nonlinear Elliptic and Parabolic Equations Involving Measures Data*, Journal of Functional Analysis, vol. 87, no 1, pp. 149-169, 1989.
- [5] **H. Brézis**, Analyse Fonctionnelle. Théorie et Applications, *Paris, 1970.*
- [6] **R. Buck**, Advanced Calculus, *The Mcgraw-Hill book Company, Inc., USA, 1956.*
- [7] **D. Cohn**, *Measure Theorem*, Birkhäuser Bosten, 1980.
- [8] **L. Demkowics and J. Oden**, *Applied Functional Analysis, CRC Press, 1996.*
- [9] **P. Deuffhard**, *Newton Methods for Nonlinear Problems, Affine Invariance and Adaptive Algorithms*, Springer, 2004, ISBN 3540210997.
- [10] **J. Droniou**, *Solving Convection-Diffusion Equations with Mixed, Neumann and Fourier Boundary Conditions and Measures as Data, by a Duality Method*, Adv. Differential Equations 5 (2000), no. 10-12, 1341-1396.
- [11] **I. V. Egorov, J. V. Egorov, M. A. Shubin**, *Foundations of the Classical Theory of Partial Differential Equations*, Springer, 1998 ISBN 3540638253, 9783540638254.

- [12] **A. Ern and J. Guermond**, Theory and Practice of Finite Elements, *Springer Verlag, New York, 2004*.
- [13] **I. Fonseca and G. Leoni**, Modern Methods in the Calculus of Variations: Lp Spaces, *Springer, 2007*.
- [14] **S. Ge, X. Li, B. Yi, and Hsinga**, Absorption, Desorption, and Transport of Water in Polymer Electrolyte Membranes for Fuel Cells, *J. Electrochem. Soc.*, 152, pp. 11491157.
- [15] **D. Gilbarg, N. S. Trudinger**, Elliptic Partial Differential Equations of Second Order, *Springer Verlag, Berlin Heidelberg, New York, Tokyo, 1983*.
- [16] **E. Giusti**, *Direct Methods in the Calculus of Variations*, World Scientific Publishing, 2003.
- [17] **P. Hawkes**, Advances in Imaging and Electron Physics, *Published by Academic Press, 1996, ISBN 0120147378, 9780120147373*.
- [18] **B. Khoromskij**, Numerical Solution of Elliptic Differential Equations by Reduction to the Interface, *Springer, 2004 ISBN 3540204067, 9783540204060*.
- [19] **E. Kreyszig**, Introductory Functional Analysis with Application, *John Wiley and Sons, New York, Toronto, 1978*.
- [20] **S. Larsson and V. Thomée**, *Partial Differential Equations with Numerical Methods*, Springer, 2003, ISBN 9783540887058, 9783540887065.
- [21] **M. Marcus, L. Véron**, *Semilinear Parabolic Equations with Measure Boundary Data and Isolated Singularities*, *J. Anal. Math.* 85 (2001), 245–290.
- [22] **M. G. Murdeshwar**, General Topology, *New Age International, 1996*.
- [23] **M. Renardy and R. Rogers**, An Introduction to Partial Differential Equations, *Springer, VerLag, New York, Inc., 1993*.
- [24] **C. Zalinescu**, Convex Analysis in General Vector Spaces, *World Scientific, 2002 ISBN 9812380671, 9789812380678*.