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Common stationary point of multivalued asymptotically regular mappings

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Abstract We establish a relationship between asymptotic regularity and common stationary points of multivalued mappings on a metric space. As a consequence of our results, we obtain a new common fixed point result for two asymptotically regular single-valued mappings. Our work significantly improves and complements comparable results in the literature.

Mathematical Subject Classification Primary 47H04 · 47H09 · 47H10

1 Introduction

In 1966, Browder and Petryshyn [7] introduced the concept of asymptotic regularity for single-valued mappings. This notion is significant since several contractive mappings are asymptotically regular (see [8,11]). Abbas et al. [2] studied asymptotically regular mappings in complex-valued metric spaces.

Recently, Górnicki [12] introduced Boyd-Wong-type self-mappings [8] and obtained the following generalization of a result of Reich [24]:

Theorem 1.1 *Let* (X, d) *be a complete metric space and* $G : X \to X$ *be an asymptotically regular mapping. Suppose there exists* $\varphi \in \mathcal{J}$ *(see Definition 2.5) and* $K \in [0, \infty)$ *such that for each* $m, w \in X$,

$$d(Gm, Gw) \leq \varphi(d(m, w)) + K[d(m, Gm) + d(w, Gw)].$$

If G *is orbitally continuous or* k*-continuous, then* G *has a unique fixed point* $z \in X$*. Moreover, for each* $w \in X$, $G^n w \to z$ *as* $n \to \infty$.

Markin [18] initiated the metric fixed point theory of multivalued mappings. For some applications of multivalued mappings, we refer to Khan et al. [14], Balaj and Khamsi [5] and Corley [9]. Petrusel and Rus [21], Nguyen [17] and Singh and Mishra [27] have studied multivalued asymptotically regular mappings. Various stationary point results and their applications in optimization problems and control theory appeared in [1,3,4,9].

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Aubin and Siegel [4] established the existence of stationary points for a dissipative lower semicontinuous multifunction. They used their results to prove some variants of the Caristi theorem and Ekeland variational principle. Corley [9] showed that maximization in relation to a cone is comparable to the case of finding stationary points of certain multivalued mappings. In 2020, Panyanak [19] studied the existence of stationary points for semicompact lower semicontinuous multivalued mappings in a metric space.

We establish a connection between asymptotic regularity and approximate stationary point property for certain multivalued mappings. We extend the recent work of Górnicki [12] for multivalued asymptotically regular mappings. Then, we establish some unique common stationary point results for a pair of multivalued asymptotically regular mappings. We point out that our Corollary 3.9 is new even for single-valued mappings.

2 Preliminaries

Let (X, d) be a metric space. We denote the family of nonempty subsets of X and the family of nonempty, closed and bounded subsets of X by $P(X)$ and $CB(X)$, respectively. For $\emptyset \neq \mathcal{U} \subset X$, we adopt the following notations and definitions:

$$\textit{The distance from } m \in X \textit{ to } \mathcal{U};\ d(m, \mathcal{U}) := \inf\{d(m, w) : w \in \mathcal{U}\}.$$

$$\textit{The diameter of } \mathcal{U};\ D(\mathcal{U}) := \sup\{d(m, w) : m, w \in \mathcal{U}\}.$$

$$\textit{The radius of } \mathcal{U} \textit{ relative to } m \in X;\ \delta(m, \mathcal{U}) := \sup\{d(m, w) : w \in \mathcal{U}\}.$$

$$\textit{The Hausdorff metric on } CB(X);\ H(\mathcal{U}, \mathcal{V}) := \max\left\{\sup_{m\in\mathcal{U}} d(m, \mathcal{V}), \sup_{q\in\mathcal{V}} d(q, \mathcal{U})\right\},$$

for all $\mathcal{U}, \mathcal{V} \in CB(X)$. Clearly, $\delta(m, \mathcal{U}) = H(\{m\}, \mathcal{U})$.

Let $G : X \to 2^X$ be a multivalued mapping. We say $m \in X$ is: (i) a fixed point of G if $m \in Gm$. (ii) a stationary point of G if $Gm = \{m\}$. Stationary point is also referred to as endpoint [3] or strict fixed point [21]. By $S(G)$ and $F(G)$, we mean the set of stationary points and fixed points of G, respectively. Note that $S(G) \subseteq F(G)$. Moreover, the two notions coincide for a single-valued mapping. Furthermore, we note from [20] that $m \in F(G)$ if and only if $d(m, Gm) = 0$, and $m \in S(G)$ if and only if $\delta(m, Gm) = 0$.

Definition 2.1 [19] A multivalued mapping $G : X \to CB(X)$ has approximate stationary point property if $\inf_{w\in X} \delta(w, Gw) = 0$. Equivalently, G has approximate stationary point property if it has an approximate stationary sequence $\{w_n\}$ (i.e. $\lim_{n\to\infty} \delta(w_n, Gw_n) = 0$).

Definition 2.2 [4] A sequence $\{w_n\}$ in X is called an orbital sequence (or trajectory) of a multivalued mapping $G : X \to 2^X$ starting at w_0 if $w_{n+1} \in Gw_n$ for all $n \geq 0$.

Definition 2.3 [27] A multivalued mapping $G : X \to CB(X)$ is said to be asymptotically regular at w_0 if for each orbital sequence $\{w_n\}$, we have

$$\lim_{n\to\infty} d(w_n, w_{n+1}) = 0.$$

G is called asymptotically regular mapping if it is asymptotically regular at each element of X.

We now present some examples of multivalued asymptotically regular mappings:

(1) Let $X = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$ be endowed with the usual metric. Define $G : X \to CB(X)$ by

$$Gm = \begin{cases} \{0\}, & m = 0 \\ \{0, \frac{1}{n+2}\}, & m = \frac{1}{n}. \end{cases}$$

Then, G is asymptotically regular.

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(2) Let $X = \{1, 2, 3, 8\}$ be endowed with the usual metric. Define $G : X \to CB(X)$ by

$$G1 = \{1\}, \quad G2 = \{8\}, \quad G3 = \{1\} \text{ and } G8 = \{3\}.$$

Then, G is asymptotically regular (see [23, Example 4.5]).
We have the following forms of continuity for a multivalued mapping:

Definition 2.4 [19] Let $G : X \to CB(X)$ be a multivalued mapping and $z \in X$. We say:

1. G is upper semi-continuous if for any sequences $\{m_n\}$ and $\{w_n\}$ with $m_n \in X$ and $w_n \in G(m_n)$, the conditions $\lim_{n\to\infty} m_n = m$ and $\lim_{n\to\infty} w_n = w$ imply $w \in Gm$.
2. G is lower semi-continuous if for any sequence $m_n \in X$ with $\lim_{n\to\infty} m_n = m$ and $w \in Gm$, there exists a sequence $\{w_n\}$ such that $w_n \in Gm_n$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} w_n = w$.
3. G is continuous if it is both upper and lower semi-continuous.
4. G is Hausdorff-continuous(or simply H-continuous) if $\lim_{n\to\infty} H(Gw_n, Gz) = 0$ whenever a sequence $\{w_n\}$ in X converges to z.
5. G is orbital H-continuous if $\lim_{n\to\infty} H(Gw_n, Gz) = 0$ whenever an orbital sequence $\{w_n\}$ converges to z.

From [10], we note the following:

(i) If G is H-continuous, then it is continuous and orbital H-continuous.
(ii) If G is continuous with compact values, then it is H-continuous, and hence orbital H-continuous.

Definition 2.5 1. Let $\mathcal{S}$ be the family of functions $\alpha : [0, \infty) \to [0, 1)$ satisfying the condition $\alpha(t_n) \to 1$ implies $t_n \to 0$.
2. Let $\varphi : [0, \infty) \to [0, \infty)$ be a mapping satisfying the condition $\varphi(t) < t$ for all $t > 0$.
   (i) Let $\mathcal{J}$ be the family of functions φ which are upper semi-continuous (equivalently, the family of functions φ satisfying $t_n \to t \geq 0$ implies $\limsup_{n\to\infty} \varphi(t_n) \leq \varphi(t)$).
   (ii) Let $\mathcal{N}$ be the family of functions φ which are increasing, right continuous with $\limsup_{t\to\infty} \frac{\varphi(t)}{t} < 1$.

We state some needed technical results.

Lemma 2.6 [25] *Let (X, d) be a metric space.*

1. *Let $\mathcal{U}, \mathcal{V} \in CB(X)$. Then for every $s > 1$ and $u \in \mathcal{U}$, there exists $q \in \mathcal{V}$ such that $d(u, q) \leq sd(u, \mathcal{V}) \leq sH(\mathcal{U}, \mathcal{V})$.*
2. *Let $\mathcal{U}$ be a nonempty bounded subset of X and $0 < p < 1$ be given. Then for every $x \in X$, there exists $u \in \mathcal{U}$ such that $d(x, u) \geq p\delta(x, \mathcal{U})$.*

Lemma 2.7 [26] *Let $G : X \to CB(X)$ be a multivalued mapping and $m, w \in X$. If $w' \in Gw$, then we have*

$$d(m, w') \leq \delta(m, Gm) + H(Gm, Gw).$$

Lemma 2.8 [22] *Let $G : X \to CB(X)$ be a multivalued mapping. Then for $m, w \in X$, we have*

$$d(m, w) \leq d(m, Gm) + H(Gm, Gw) + d(w, Gw) + D(Gw).$$

Lemma 2.9 [13] *Let $\{a_n\}$ and $\{b_n\}$ be sequences of non-negative real numbers satisfying $a_n \leq \varphi(a_{n-1}) + b_n$, where $\varphi \in \mathcal{N}$. If $b_n \to 0$, then $a_n \to 0$.*

Lemma 2.10 [19] *Let $G : X \to CB(X)$ be a multivalued mapping and $\{w_n\}$ be a sequence in X. Then $\delta(w_n, Gw_n) \to 0$ if and only if $d(w_n, Gw_n) \to 0$ and $D(Gw_n) \to 0$.*

3 Main results

We assume X is a complete metric space unless stated otherwise.

Lemma 3.1 *Let (X, d) be a metric space. A multivalued asymptotically regular mapping $G : X \to CB(X)$ has approximate stationary point property.*

Proof Let w_0 be an arbitrary element of X and $s > 1$. By Lemma 2.6 (2), there exists $w_1 \in Gw_0$ such that $\delta(w_0, Gw_0) \leq sd(w_0, w_1)$. If $w_0 = w_1$, then w_0 is a stationary point of G and we are done. Suppose otherwise; we can find $w_2 \in Gw_1$ such that $\delta(w_1, Gw_1) \leq sd(w_1, w_2)$. With the assumption that $w_n \neq w_{n+1}$ for all $n \geq 0$, we can construct an orbital sequence $\{w_n\}$ such that

$$\delta(w_n, Gw_n) \leq sd(w_n, w_{n+1}).$$

Taking limit as $n \to \infty$, asymptotic regularity of G implies $\lim_{n\to\infty} \delta(w_n, Gw_n) = 0$. Hence $\{w_n\}$ is an approximate stationary point sequence as required. □

The following example shows that the converse of Lemma 3.1 is not true.

Example 3.2 Let $X = [-3, 2]$ be endowed with the usual metric. Define

$$Gm = \begin{cases} [-2, -1], & m \in [-3, 0) \\ [\frac{m}{2}, m], & m \in [0, 2] \end{cases} \quad \text{and} \quad w_n = \begin{cases} -2, & n \text{ is an even integer} \\ -1, & n \text{ is an odd integer} \end{cases}.$$

Observe that G has a stationary point 0 and thus has approximate stationary point property. However, $\{w_n\}$ is an orbital sequence of G with $d(w_n, w_{n+1}) = 1$. Hence, G is not asymptotically regular.

We follow Geraghty [11] to give conditions on G such that asymptotic regularity coincides with approximate stationary point property.

Theorem 3.3 *Let $G : X \to CB(X)$ be a multivalued mapping satisfying*

$$H(Gm, Gw) \leq \alpha(d(m, w))d(m, w) \text{ for all } m, w \in X,$$

where $\alpha \in \mathcal{S}$. If G is H-continuous, then the following are equivalent:

(i) G is asymptotically regular;
(ii) G has an approximate stationary point property.

Proof (i) ⇒ (ii) follows by Lemma 3.1.
(ii) ⇒ (i): Let $\{w_n\}$ be an approximate stationary point sequence of G. Then $\delta(w_n, Gw_n) \to 0$. By Lemma 2.10, we get

$$d(w_n, Gw_n) \to 0 \text{ and } D(Gw_n) \to 0. \tag{3.1}$$

Suppose $\{w_n\}$ is not a Cauchy sequence. Then, $\limsup_{n,m\to\infty} d(w_n, w_m) > 0$.

Now, Lemma 2.8 implies

$$\begin{aligned} d(w_n, w_m) &\leq d(w_n, Gw_n) + H(Gw_n, Gw_m) + d(w_m, Gw_m) + D(Gw_m) \\ &\leq d(w_n, Gw_n) + \alpha(d(w_n, w_m))d(w_n, w_m) + d(w_m, Gw_m) + D(Gw_m). \end{aligned}$$

Then,

$$1 - \alpha(d(w_n, w_m)) \leq \frac{d(w_n, Gw_n) + d(w_m, Gw_m) + D(Gw_m)}{d(w_n, w_m)}. \tag{3.2}$$

Using the assumption that $\limsup_{n,m\to\infty} d(w_n, w_m) > 0$, (3.1) and (3.2), we have

$$\limsup_{n,m\to\infty} (1 - \alpha(d(w_n, w_m))) = 0.$$

This implies $\limsup_{n,m\to\infty} \alpha(d(w_n, w_m)) = 1$. Consequently, $\limsup\limits_{n,m\to\infty} d(w_n, w_m) = 0$ since $\alpha \in \mathcal{S}$. This is a contradiction. Hence, $\{w_n\}$ is a Cauchy sequence. Completeness of X implies $\{w_n\}$ converges to $z \in X$.

Using Lemma 2.7, we have

$$\delta(z, Gz) \le d(z, w_n) + \delta(w_n, Gw_n) + H(Gw_n, Gz). \tag{3.3}$$

Taking limit as $n \to \infty$, H-continuity of G and (3.3) give $\delta(z, Gz) = 0$. Thus, z is a stationary point of G. If v is a stationary point of G different from z, then $d(v, z) = H(Gv, Gz) \le \alpha(d(v, z))d(v, z) < d(v, z)$. This is a contradiction. Hence G has a unique stationary point.
For $y_0 \in X$, let $\{y_n\}$ be an orbital sequence. The inequality

$$d(y_{n+1}, z) \le H(Gy_n, Gz) \le \alpha(d(y_n, z))d(y_n, z) < d(y_n, z)$$

implies that $d(y_n, z) \to \lambda$. Suppose $\lambda > 0$. Then, $d(y_{n+1}, z) \le \alpha(d(y_n, z))d(y_n, z)$ and $\alpha(d(y_n, z)) \to 1$. Consequently, $d(y_n, z) \to 0$ since $\alpha \in \mathcal{S}$. This is a contraction. Hence, any orbital sequence of G starting from y_0 converges to z. It follows that $d(y_{n+1}, y_n) \to 0$ and G is asymptotically regular. □

We are now in a position to extend the work of Górnicki [12] to multivalued mappings.

Theorem 3.4 *Let $G : X \to CB(X)$ be a multivalued asymptotically regular mapping. Suppose there exists $\alpha \in \mathcal{S}$ and $K \in [0, \infty)$ such that for each $m, w \in X$,*

$$H(Gm, Gw) \le \alpha(d(m, w))d(m, w) + K[d(m, Gm) + d(w, Gw)]. \tag{3.4}$$

If G is orbitally H-continuous, then G has a unique stationary point.

Proof Let $s > 1$. Then by Lemma 2.6 and Lemma 3.1, there exists an orbital sequence $\{w_n\}$ such that $\delta(w_n, Gw_n) \le sd(w_n, w_{n+1})$ and $\lim_{n\to\infty} \delta(w_n, Gw_n) = 0$. Using arguments similar to those in the proof of the second part of Theorem 3.4, $d(w_n, Gw_n)$ replaced by $(1 + K)d(w_n, Gw_n)$, $d(w_m, Gw_m)$ replaced by $(1 + K)d(w_m, Gw_m)$ and H-continuity replaced by orbitally H-continuity, we can easily show that G has a unique stationary point z. □

Here is a generalization of Theorem 1.1.

Theorem 3.5 *Let $G : X \to CB(X)$ be a multivalued asymptotically regular mapping. Suppose there exists $\varphi \in \mathcal{J}$ and $K \in [0, \infty)$ such that for each $m, w \in X$,*

$$H(Gm, Gw) \le \varphi(d(m, w)) + K[d(m, Gm) + d(w, Gw)]. \tag{3.5}$$

If G is orbitally H-continuous, then G has a unique stationary point.

Proof Let $s > 1$. Then by Lemma 3.1, there exists an orbital sequence $\{w_n\}$ such that $\delta(w_n, Gw_n) \le sd(w_n, w_{n+1})$ and $\lim_{n\to\infty} \delta(w_n, Gw_n) = 0$.

We show that $\{w_n\}$ is a Cauchy sequence. Suppose not. Then there exists an $\epsilon > 0$ and sequences of integers $\{m(k)\}, \{n(k)\}$ with $m(k) > n(k) \ge k$ such that for $k = 1, 2, \cdots$, we have

$$d(w_{m(k)}, w_{n(k)}) \ge \epsilon. \tag{3.6}$$

By choosing $m(k)$ to be the smallest number exceeding $n(k)$ for which (3.6) holds, we may assume that $d(w_{m(k)-1}, w_{n(k)}) < \epsilon$. Now,

$$\begin{aligned}\epsilon \le d(w_{m(k)}, w_{n(k)}) &\le d(w_{m(k)}, w_{m(k)-1}) + d(w_{m(k)-1}, w_{n(k)})\\ &< d(w_{m(k)}, w_{m(k)-1}) + \epsilon.\end{aligned}$$

Letting $k \to \infty$, it follows by asymptotic regularity of G that

$$\lim_{k\to\infty} d(w_{m(k)}, w_{n(k)}) = \epsilon. \tag{3.7}$$

Using Lemma 2.8, we have

$$\begin{aligned}d(w_{n(k)}, w_{m(k)}) &\le d(w_{n(k)}, Gw_{n(k)}) + H(Gw_{n(k)}, Gw_{m(k)})\\ &\quad + d(w_{m(k)}, Gw_{m(k)}) + \delta(Gw_{m(k)})\\ &\le (K + 1)[d(w_{n(k)}, Gw_{n(k)}) + d(w_{m(k)}, Gw_{m(k)})]\\ &\quad + \varphi(d(w_{n(k)}, w_{m(k)})) + \delta(Gw_{m(k)}).\end{aligned}$$

Taking limit as $k \to \infty$, it follows from upper semi-continuity of φ, (3.7) and asymptotic regularity of G that

$$\epsilon = \lim_{k\to\infty} d(w_{n(k)}, w_{m(k)}) \leq \limsup_{k\to\infty} \varphi(d(w_{n(k)}, w_{m(k)})) \leq \varphi(\epsilon) < \epsilon.$$

This is a contradiction. Hence $\{w_n\}$ is a Cauchy sequence and thus converges to $z \in X$.

By Lemma 2.7, we have

$$\delta(z, Gz) \leq d(z, w_n) + \delta(w_n, Gw_n) + H(Gw_n, Gz). \tag{3.8}$$

Taking limit as $n \to \infty$, orbital H-continuity of G and (3.8) give $\delta(z, Gz) = 0$. Thus, z is a stationary point of G. If v is a stationary point of G different from z, we have

$$d(v, z) = H(Gv, Gz) \leq \varphi(d(v, z)) + K[d(v, Gv) + d(z, Gz)] < d(v, z).$$

This is a contradiction. Hence G has a unique stationary point z. □

As a consequence of Theorems 3.4 and 3.5, we get the following extension of ([6], Theorem 2.1) for multivalued mappings.

Corollary 3.6 *Let $G : X \to CB(X)$ be a multivalued asymptotically regular mapping satisfying the inequality*

$$H(Gm, Gw) \leq Md(m, w) + K[d(m, Gm) + d(w, Gw)] \text{ for all } m, w \in X, \tag{3.9}$$

where $0 \leq M < 1$ and $0 \leq K < \infty$. If G is orbitally H-continuous, then G has a unique stationary point.

Next, we establish unique common stationary point results for a pair of multivalued asymptotically regular mappings to obtain two variants of Theorem 3.3 of Khan and Oyetunbi [15].

Theorem 3.7 *Let $G : X \to CB(X)$ and $R : X \to CB(X)$ be multivalued asymptotically regular mappings satisfying the inequality*

$$H(Gm, Rw) \leq Md(m, w) + K[d(m, Gm) + d(w, Rw)] \text{ for all } m, w \in X, \tag{3.10}$$

where $0 \leq M < 1$ and $0 \leq K < \infty$. If G and R are orbitally H-continuous, then G and R have a unique common stationary point.

Proof Let $s > 1$ be such that $M_1 = sM < 1$. Then by Lemma 3.1, there exists orbital sequence of G starting at $w_0 \in X$ such that $\delta(w_n, Gw_n) \leq sd(w_n, w_{n+1})$ and $\lim_{n\to\infty} \delta(w_n, Gw_n) = 0$. Let $q_0 = w_0$. Then, for each $w_{n+1} \in Gw_n$, Lemma 2.6 guarantees the existence of $q_{n+1} \in Rq_n$ such that

$$d(w_{n+1}, q_{n+1}) \leq sH(Gw_n, Rq_n).$$

Step 1: $\lim_{n\to\infty} d(w_n, q_n) = 0$.

If $G = R$, we are done. Suppose otherwise. Assume $M = 0$. Then for all $m, w \in X$, we have

$$\begin{aligned} d(w_{n+1}, q_{n+1}) &\leq sH(Gw_n, Rq_n) \\ &\leq sK[d(w_n, Gw_n) + d(q_n, Rq_n)] \\ &\leq sK[d(w_n, w_{n+1}) + d(q_n, q_{n+1})]. \end{aligned}$$

It follows from asymptotic regularity of G and R that $\lim_{n\to\infty} d(w_{n+1}, q_{n+1}) = 0$.

Suppose $G \neq R$ and $M \neq 0$. Then,

$$\begin{aligned} d(w_{n+1}, q_{n+1}) &\leq sH(Gw_n, Rq_n) \\ &\leq sMd(w_n, q_n) + sK[d(w_n, Gw_n) + d(q_n, Rq_n)] \\ &\leq M_1 d(w_n, q_n) + sK[d(w_n, w_{n+1}) + d(q_n, q_{n+1})]. \end{aligned}$$

Let $a_n = d(w_{n+1}, q_{n+1})$, $\varphi = M_1 t$ and $b_n = sK[d(w_n, w_{n+1}) + d(q_n, q_{n+1})]$. Then, asymptotic regularity of G and R and Lemma 2.9 imply $\lim_{n\to\infty} d(w_{n+1}, q_{n+1}) = 0$.

In general, asymptotic regularity of G and R, $\lim_{n\to\infty} d(w_{n+1}, q_{n+1}) = 0$ and the inequality

$$d(w_n, q_n) \leq d(w_n, w_{n+1}) + d(w_{n+1}, q_{n+1}) + d(q_{n+1}, q_n)$$

imply

$$\lim_{n\to\infty} d(w_n, q_n) = 0. \tag{3.11}$$

Step 2: $\{w_n\}$ is a Cauchy sequence and thus converges to $z \in X$. Moreover, z is a unique common stationary point of G and R.

By Lemma 2.8, we have

$$\begin{aligned} d(w_n, w_m) &\le d(w_n, Gw_n) + H(Gw_n, Gw_m) + d(w_m, Gw_m) + \delta(Gw_m) \\ &\le d(w_n, w_{n+1}) + d(w_m, w_{m+1}) + \delta(Gw_m) \\ &\quad + H(Gw_n, Rq_n) + H(Rq_n, Gw_m) \\ &\le (K+1)[d(w_n, w_{n+1}) + d(w_m, w_{m+1})] + \delta(Gw_m) \\ &\quad + 2Kd(q_n, q_{n+1}) + M[d(w_n, q_n) + d(q_n, w_m)]. \end{aligned}$$

Thus,

$$\begin{aligned} (1-M)d(w_n, w_m) \le{}& (K+1)[d(w_n, w_{n+1}) + d(w_m, w_{m+1})] + \delta(Gw_m) \\ &+ 2Kd(q_n, q_{n+1}) + 2Md(w_n, q_n). \end{aligned}$$

Taking limit as $n, m \to \infty$, it follows from asymptotic regularity of G and R and (3.11) that $\lim_{n\to\infty} d(w_n, w_m) = 0$. Hence $\{w_n\}$ is a Cauchy sequence. The completeness of X implies $\{w_n\}$ converges to $z \in X$. Moreover,

$$d(q_n, z) \le d(q_n, w_n) + d(w_n, z).$$

Then, we get from Step 1 and convergence of $\{w_n\}$ that $\lim_{n\to\infty} q_n = z$.
Using similar argument as in the proof of Theorem 3.5 orbital continuity of G and R, it is easy to show that z is a common stationary point of G and R. Suppose there is $p \neq z$ in $S(G) \cap S(R)$. Let $m = p$ and $w = z$. Then (3.10) becomes $d(p, z) \le Md(p, z)$. Hence a contradiction. In conclusion, we have that z is a unique common stationary point of G and R. □

The following result is a significant generalization of Theorem 2.2 of Khan and Oyetunbi [15].

Theorem 3.8 *Let $G : X \to CB(X)$ and $R : X \to CB(X)$ be multivalued asymptotically regular mappings satisfying the inequality*

$$\delta(Gm, Rw) \le \varphi(d(m, w)) + K[\delta(m, Gm) + \delta(w, Rw)] \text{ for all } m, w \in X, \tag{3.12}$$

where $\varphi \in \mathcal{N}$, $\varphi(0) = 0$ and $0 \le K < \infty$. If G and R are orbitally H-continuous, then G and R have a unique common stationary point.

Proof Since G and R are asymptotically regular, it follows by Lemma 3.1 that for $s > 1$, there exist orbital sequences $\{w_n\}$ and $\{q_n\}$ of G and R, respectively starting at $w_0 = q_0$ such that

$$\lim_{n\to\infty} \delta(w_n, Gw_n) = \lim_{n\to\infty} \delta(q_n, Rq_n) = 0. \tag{3.13}$$

Now,

$$\begin{aligned} d(w_{n+1}, q_{n+1}) &\le \delta(Gw_n, Rq_n) \\ &\le \varphi(d(w_n, q_n)) + K[\delta(w_n, Gw_n) + \delta(q_n, Rq_n)]. \end{aligned}$$

Using argument similar to Step 1 in the proof of Theorem 3.7, we get

$$\lim_{n\to\infty} d(w_n, q_n) = 0. \tag{3.14}$$

Next, we show that $\{w_n\}$ is Cauchy. Suppose $\{w_n\}$ is not a Cauchy sequence. As in the proof of Theorem 3.5 there exists $\epsilon > 0$ and sequences of integers $\{m(k)\}$, $\{n(k)\}$ such that $m(k) > n(k) \ge k$ and

$$\lim_{k\to\infty} d(w_{m(k)}, w_{n(k)}) = \epsilon. \tag{3.15}$$

Using (3.14), (3.15), asymptotic regularity of G and R, we get

$$\lim_{k\to\infty} d(w_{m(k)-1}, q_{n(k)-1}) = \epsilon. \tag{3.16}$$

Now, in view of

$$\begin{aligned} d(w_{n(k)}, w_{m(k)}) &\le \delta(Gw_{n(k)-1}, Gw_{m(k)-1}) \\ &\le \delta(Gw_{n(k)-1}, Rq_{n(k)-1}) + \delta(Rq_{n(k)-1}, Gw_{m(k)-1}) \\ &\le K[\delta(w_{n(k)-1}, Gw_{n(k)-1}) + \delta(q_{n(k)-1}, Rq_{n(k)-1})] \\ &\quad + K[\delta(w_{m(k)-1}, Gw_{m(k)-1}) + \delta(q_{n(k)-1}, Rq_{n(k)-1})] \\ &\quad + \varphi(d(w_{n(k)-1}, q_{n(k)-1})) + \varphi(d(w_{m(k)-1}, q_{n(k)-1})), \end{aligned}$$

it follows from (3.13), (3.15), (3.16) and the right continuity of φ that $\epsilon \le \varphi(\epsilon) < \epsilon$. This is a contradiction. Hence $\{w_n\}$ is a Cauchy sequence. Since X is a complete metric space, $\{w_n\}$ converges to $z \in X$. Also, we have that $\{q_n\}$ converges to $z \in X$. Using similar argument as in the proof of Theorem 3.5 and orbital continuity of G and R, it is easy to show that z is a unique common stationary point of G and R. □

The following common fixed point result is new for single-valued mappings:

Corollary 3.9 *Suppose that $G : X \to X$ and $R : X \to X$ are asymptotically regular mappings satisfying*

$$d(Gm, Rw) \le \varphi(d(m, w)) + K[d(m, Gm) + d(w, Rw)] \text{ for all } m, w \in X, \tag{3.17}$$

where $\varphi \in \mathcal{N}$, $\varphi(0) = 0$ and $0 \le K < \infty$. Suppose further that G and R are orbitally continuous. Then G and R have a unique common fixed point z. Additionally, $\lim_{n\to\infty} G^n w = z = \lim_{n\to\infty} R^n w$ for any $w \in X$.

The following example illustrates that Corollary 3.9 is a genuine improvement of Theorem 2.2 of Khan and Oyetunbi [15].

Example 3.10 Let $X = [0, \infty)$ be endowed with the usual metric. Suppose that $G : X \to X$ and $R : X \to X$ are mappings defined by:

$$Gm = \frac{m}{m+1} \quad \text{and} \quad Rm = \begin{cases} 0, & m < \frac{3\sqrt{73}-9}{32} \\ \frac{1}{2}, & m \ge \frac{3\sqrt{73}-9}{32}. \end{cases}$$

We note that G and R are asymptotically regular and orbitally continuous. For sufficiently small $m = \epsilon > 0$ and $w = 0$, there is no $K \ge 0$ such that $|Gm - Rw| \le K[|m - Gm| + |w - Rw|]$. Hence, Theorem 3.3 in [15] is not applicable. On the other hand, it is easy to show that

$$|Gm - Rw| \le \varphi(|m - w|) + 8[|m - Gm| + |w - Rw|],$$

where $\varphi = G$. All the conditions of Corollary 3.10 hold, so G and R have 0 as their unique common fixed point. Moreover, $\lim_{n\to\infty} G^n w = 0 = \lim_{n\to\infty} R^n w$ for any $w \in X$.

Definition 3.11 Two multivalued mappings $G : X \to CB(X)$ and $R : X \to CB(X)$ are said to have approximate common stationary point property if there exists a sequence $\{w_n\}$ in X such that $\lim_{n\to\infty} \delta(w_n, Gw_n) = \lim_{n\to\infty} \delta(w_n, Rw_n) = 0$.

The following common stationary point result can be easily established on the lines of proof of Theorem 3.4 and the inequality

$$\delta(w_n, w_m) \le \delta(w_n, Gw_n) + \delta(Gw_n, Rw_m) + \delta(w_m, Rw_m).$$

Theorem 3.12 *Let $G : X \to CB(X)$ and $R : X \to CB(X)$ be two H-continuous multivalued mappings satisfying one of the following conditions for any $m, w \in X$:*

1. $H(Gm, Rw) \le \alpha(d(m, w))d(m, w) + K[d(m, Gm) + d(w, Rw)]$,
2. $\delta(Gm, Rw) \le \alpha(d(m, w))d(m, w) + K[\delta(m, Gm) + \delta(w, Rw)]$,

where $0 \leq K < \infty$ *and* $\alpha \in \mathcal{S}$. *Then* G *and* R *have a unique common stationary point if and only if* G *and* R *have approximate common stationary point property.*

Remark 3.13 Theorems 3.7 and 3.8 are independent of the condition $M + 2K < 1$ and thus improve and complement the work of Khan [16].

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