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by

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ABSTRACT

In this study we offer proofs for two basic results in the theory of almost periodic functions; the propositions in question are Bohr's Approximation Theorem and the Uniqueness Theorem for almost periodic functions.

By using the analytic representation theory of positive definite sequences, we carry out a special type of Diophantine analysis and derive from a part of this a well-known theorem concerning almost periods. We then deduce Bohr's theorem directly from the theorem concerning almost periods. The noteworthy feature in this step is that we avoid considering the so-called limit periodic functions.

In the presentation of the uniqueness theorem, we have followed H. Weyl and based our proof on the theory of compact normal operators in a pre-Hilbert space. The eigenvalue problem which arises in this way, is resolved in terms of elementary linear algebra.

At the end of our discussion an interpretation of these two theorems in terms of the notions pertaining to the theory of Banach algebras is briefly indicated.

In the quest for a self-contained presentation we were obliged to include in our discussions several results belonging to the firmly established portions of the theory in question.

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I

PRELIMINARIES

We commence with a brief outline of notions and results used in our study.

POSITIVE DEFINITE SEQUENCES

A sequence (finite or infinite) of complex numbers c_n $(-\infty \leq n \leq \infty)$ is called positive definite if for any finite set of complex numbers $\lambda_1, \ldots, \lambda_N$, the inequality

$$\sum_{k,j=1}^{N} c_{k-j} \lambda_k \overline{\lambda}_j \geq 0$$

holds; $\overline{\lambda}_j$ denotes the complex conjugate of λ_j.

ANALYTIC REPRESENTATION THEOREM (G. Herglotz): A necessary and sufficient condition for a sequence to be positive definite is that its terms $(c_n)_{n=1}^{\infty}$ have the representation

$$c_n = \int_{-\pi}^{\pi} e^{i\pi x} \, dg(x) \qquad (-\infty \leq n \leq \infty)$$

where g is a non-decreasing function of bounded variation.

SELECTION PRINCIPLE OF D. HILBERT

THEOREM: Let $(g_n)_{n=1}^{\infty}$ be a sequence of real valued functions defined on some set E so that $\sup_{x \in E} |g_n(x)| \leq K$ for all n. Then, given any countable subset D of E, there exists a subsequence $(g_{n_i})_{n=1}^{\infty}$ which converges at every point of D.

Proof: Let $D = \{x_1, x_2, \ldots\}$. Then $(g_n(x_1))_{n=1}^{\infty}$ is a bounded sequence of real numbers; hence by Bolzano-Weierstrauss' theorem we may choose a subsequence $(g_n^{(1)})_{n=1}^{\infty}$ from $(g_n)_{n=1}^{\infty}$ such that the sequence $(g_n^{(1)}(x_1))_{n=1}^{\infty}$ converges to some A_1. Next, we consider the sequence $(g_n^{(1)}(x_2))_{n=1}^{\infty}$. Again by Bolzano-Weierstrauss' theorem we can select a subsequence $(g_n^{(2)})_{n=1}^{\infty}$ from $(g_n^{(1)})_{n=1}^{\infty}$ such that it converges to A_2. Continuing this process indefinitely, we obtain a sequence of subsequences $(g_n^{(m)})_{n=1}^{\infty}$ $(m = 1, 2, \ldots)$ of $(g_n)_{n=1}^{\infty}$ such that

$$\lim_{n\to\infty} g_n^{(m)}(x_m) = A_m$$

exists. We then consider the diagonal sequence $(g_n^{(n)})_{n=1}^{\infty}$. For a fixed k, $(g_n^{(n)}(x_k))_{n\geq k}$ is a subsequence of $(g_n^{(k)}(x_k))_{n\geq k}$, and therefore converges to A_k. Thus $(g_n^{(n)})_{n=1}^{\infty}$ converges at every point of D and the proof is complete.

SPECTRAL THEORY OF COMPACT NORMAL OPERATORS

Let H be a pre-Hilbert space and T a non-zero compact normal operator in H. Then T has at least one non-zero eigenvalue. The eigenspace corresponding to any non-zero eigenvalue of a compact (not necessarily normal) operator in H is finite dimensional and the set of its eigenvalues is at most countable with zero as the only possible accumulation point. If $\lambda \neq 0$ is not an eigenvalue of a compact operator in H, then λ is in the resolvent set of the operator. Finally, a non-zero number is an eigenvalue of a compact operator if and only if it is the eigenvalue of the adjoint operator.

Remark: For an exposition of positive definite sequences and compact normal operators see [1] pages 144 and 123 respectively.

Chapter I

BASIC DEFINITIONS AND PROPERTIES OF UNIFORMLY ALMOST PERIODIC FUNCTIONS

A continuous complex-valued function f defined on the entire real line R is called uniformly almost periodic (UAP for short), if for every $\varepsilon > 0$, there is a number $L = L(\varepsilon) > 0$ such that in each interval of length L there is an ε-almost period of f. That is, there is a number $\tau = \tau(\varepsilon)$ so that for all $x \in R$ we have

$$|f(x + \tau) - f(x)| < \varepsilon$$

The symbol $E\{\varepsilon; f\}$ will be employed to denote the set of ε-almost periods of the function f and $\overline{E}\{\varepsilon; f\}$ is used to denote the set of integers of $E\{\varepsilon; f\}$.

It is customary to call a set E of real numbers relatively dense, if there exists a number $T > 0$ such that any interval of length T contains at least one element of E.

In terms of these conventions the definition of a UAP function reads as follows: A continuous complex-valued function defined on R is a UAP function if for any $\varepsilon > 0$, the set $E\{\varepsilon; f\}$ is relatively dense.

The concept of almost periodicity is a generalization of pure periodicity. It was originally introduced and heavily investigated by Harold Bohr. In this section we shall look at some basic propositions in the theory of UAP functions.

Proposition 1 - A UAP function is bounded on the entire real line R

Proof. Let f be a UAP function and choose $\varepsilon = 1$. Since f is continuous, the function $|f|$ has a maximum M on the closed interval

$[0, L(1)]$, where $L(1)$ is obtained through the above definition. Suppose x_o is an arbitrary real number. We select an almost period $\tau = \tau(1)$ in the interval $[-x_o, -x_o + L(1)]$. Thus $0 \leq x_o + \tau \leq L(1)$ and we have

$$|f(x_o)| \leq |f(x_o) - f(x_o + \tau)| + |f(x_o + \tau)| \leq 1 + M$$

which finishes the proof.

<u>Proposition 2</u> - <u>A UAP function is uniformly continuous on the entire real line R</u>

<u>Proof</u>: Denote by ε an arbitrary positive number and choose $L = L(\varepsilon/3)$. The UAP function f is uniformly continuous on the closed interval $[-1, 1 + L]$. Hence we can find a positive number $\delta < 1$ such that for any y_1, y_2 in $[-1, 1 + L]$ we have

$$|f(y_2) - f(y_1)| < \varepsilon/3 .$$

whenever $|y_2 - y_1| < \delta$. Suppose x_1 and x_2 is an arbitrary pair of real numbers for which $|x_2 - x_1| < \delta$. Let τ be an $\varepsilon/3$-almost period of f included in the interval $[-x_1, -x_1 + L]$. Since $|x_2 - x_1| < \delta$ and $0 \leq x_1 + \tau \leq L$, we see that $-1 < x_2 + \tau < 1 + L$. Therefore

$$|f(x_2) - f(x_1)| \leq |f(x_2) - f(x_2 + \tau)| + |f(x_2 + \tau) - f(x_1 + \tau)|$$

$$+ |f(x_1 + \tau) - f(x_1)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

and since the choice of ε was arbitrary, the proof is finished.

<u>Proposition 3</u> - <u>Let f be a UAP function. For any positive number ε it is possible to select numbers $L = L(\varepsilon)$ and $\delta = \delta(\varepsilon)$ such that in each interval of length L in R we can find a subinterval of</u>

length δ all points of which are ε-almost periods of f.

Proof: Suppose $T = T(\varepsilon/2)$ is a positive number with the property that in each interval of length T in R there is at least one $\varepsilon/2$-almost period of f. Furthermore denote by $\tilde{\delta} = \tilde{\delta}(\varepsilon/2)$ a positive number such that

$$\sup_{x \in R} |f(x + h) - f(x)| < \frac{\varepsilon}{2}$$

whenever $|h| < \tilde{\delta}$. ($\tilde{\delta}$ exists by Proposition 2). Let τ be an $\varepsilon/2$-almost period of f contained in the interval $(\alpha + \tilde{\delta}, \alpha + T + \tilde{\delta})$. If $|h| < \tilde{\delta}$, then $\tau + h$ is in the interval $(\alpha, \alpha + T + 2\tilde{\delta})$ and is an ε-almost period of f; the latter follows from the relation

$$|f(x + \tau + h) - f(x)| \leq |f(x + h + \tau) - f(x + h)| + |f(x + h) - f(x)| < \varepsilon.$$

Hence the numbers $L = T + 2\tilde{\delta}$ and $\delta = 2\tilde{\delta}$ satisfy the claim made above.

Proposition 4 - Let f and g be two UAP functions. Then for arbitrary $\varepsilon > 0$ there is a number $M(\varepsilon)$ such that every interval of length $M(\varepsilon)$ on R contains at least one number $\gamma(\varepsilon)$ which is a common ε-almost period of both functions f and g.

Proof: For $\varepsilon > 0$ let δ be as in Proposition 3. Let M be a positive integer so that every interval of length $L = M(\delta)$ contains an $\varepsilon/2$-almost period of f as well as an $\varepsilon/2$-almost period of g. Divide the real line into intervals $I_n = [(n - 1)L, nL]$. Then for each integer n, I_n contains a $\sigma_n \in E\{\varepsilon/2; f\}$ and a $\tau_n \in E\{\varepsilon/2; g\}$. Now divide the interval $[-L, L]$ into $2M$ intervals J_k, each of length δ. Since $\sigma_n, \tau_n \in I_n$ we have $|\sigma_n - \tau_n| < L$. Hence $\sigma_n - \tau_n \in J_k$ for some k. This k may be called the index of n.

As only finitely many indices are available, there exists a positive integer N such that every index is attained as n runs from $-N$ to N. Therefore, there exists an m_n, $-N < m_n < N$, such that n and m_n have the same index k, that is $\sigma_n - \tau_n \in J_k$ and $\sigma_{m_n} - \tau_{m_n} \in J_k$. Let $\bar{\sigma}_n = \sigma_n - \sigma_{m_n}$, $\bar{\tau}_n = \tau_n - \tau_{m_n}$. Clearly $\bar{\sigma}_n \in E\{\varepsilon;\ f\}$, $\bar{\tau}_n \in E\{\varepsilon;\ g\}$. Also, $|\bar{\sigma}_n - \bar{\tau}_n| = |(\sigma_n - \tau_n) - (\sigma_{m_n} - \tau_{m_n})| < \delta$. Since $\bar{\sigma}_n$ $E\{\varepsilon;\ f\}$, it follows from Proposition 3 that $\bar{\tau}_n \in E\{\varepsilon;\ f\}$, that is, $\bar{\tau}_n$ is a common ε-almost period of f and g. It is easy to see that the set $\{\bar{\sigma}_n : n = 0, \pm 1, \pm 2, \ldots\}$ is relatively dense: for any n

$$|\bar{\sigma}_{n+1} - \bar{\sigma}_n| = |(\sigma_{n+1} - \sigma_{m_{n+1}}) - (\sigma_n - \sigma_{m_n})| \leq$$

$$\leq |\sigma_{n+1} - \sigma_n| + |\sigma_{m_{n+1}} - \sigma_{m_n}| < 2L + (L + 2NL).$$

As each $\tilde{\tau}_n$ differs from the corresponding $\bar{\sigma}_n$ by less than δ, the set $\{\bar{\tau}_n : n = 0, \pm 1, \pm 2, \ldots\}$ is likewise relatively dense.

<u>Remark:</u> Proposition 4 evidently holds for any finite system of UAP functions.

<u>Proposition 5</u> - <u>The set of integral ε-almost periods of a UAP function f is relatively dense.</u>

<u>Proof:</u> Let $g(x) = \sin 2\pi x$. Taking $\varepsilon_1 < \varepsilon$, there is by Proposition 3 a $\delta > 0$ such that all numbers having distances less than δ from $E\{\varepsilon_1;\ f\}$ belong to $E\{\varepsilon;\ f\}$. Choose $\varepsilon_2 > 0$ such that the elements of the set $E\{\varepsilon_2;\ g\}$ differ from integers by less than δ. Pick

$\varepsilon_3 = \min(\varepsilon_1, \varepsilon_2)$. Then the set ε_3-almost periods common to f and g, namely $E\{\varepsilon_3; f, g\}$, is relatively dense by Proposition 4 and its elements differ from integers by less than δ, because g is periodic with period 1. Let J denote the set of integers nearest to each number of $E\{\varepsilon_3; f, g\}$. Then J is relatively dense. But $E\{\varepsilon_3; f, g\} \subset E\{\varepsilon_1; f\}$. Therefore the distance of each number in J from the set $E\{\varepsilon_1; f\}$ is less than δ. Consequently each number of J belongs to $E\{\varepsilon; f\}$, and clearly to $\overline{E}\{\varepsilon; f\}$. Thus J being relatively dense, $\overline{E}\{\varepsilon; f\}$ is also relatively dense and the proof is complete.

Consider the complete metric space M_∞ of bounded complex-valued functions on $(-\infty, \infty)$, where distance between two elements f and g is defined by

$$\rho(f, g) = \sup_{x \in R} |f(x) - g(x)|.$$

By Proposition 1 each UAP function belongs to M_∞.

We shall call a set Q in a metric space X <u>sequentially compact</u> if every infinite subset of Q contains a convergent sequence with limit in X (not necessarily belonging to the set Q).

<u>Proposition 6</u> - <u>A set S of UAP functions is sequentially compact in M_∞ if and only if:</u>

(i) <u>the functions in the set S are uniformly bounded and equi-continuous and</u>

(ii) <u>the functions in the set S are equi-almost periodic</u>. That is, for every $\eta > 0$, <u>there exists an $L = L(\eta)$ such that each interval</u>

of length L contains a number p which is an η-almost period for all functions of the set S.

Proof: Necessity. Consider condition (i) first. Suppose S is sequentially compact. The uniform boundedness of the functions in S follows from the fact that every sequentially compact subset S of a metric space is totally bounded. We now show that S is an equicontinuous set of functions. For a given $\varepsilon > 0$ we construct a finite $\varepsilon/3$-net $u_1, u_2, \ldots, u_n$ for the set S. Since the u_k's of $\varepsilon/3$-net can be so chosen as belonging to S, then from proposition 2 it follows that the u_k's are uniformly continuous on $(-\infty, \infty)$. For each u_k we select a δ_k such that $|u_k(x_1) - u_k(x_2)| < \varepsilon/3$ whenever $|x_1 - x_2| < \delta_k$. Let $\delta = \min_{1 \leq k \leq n} \delta_k$. If $|x_1 - x_2| < \delta$, then for each function $u \in S$ we have

$$|u(x_1) - u(x_2)| \leq \sup_{x_1 \in R} |u(x_1) - u_k(x_1)| + |u_k(x_1) - u_k(x_2)| +$$

$$+ \sup_{x_2 \in R} |u_k(x_2) - u(x_2)| < 2\rho(u, u_k) + \varepsilon/3.$$

If we choose from the net, those functions u_k for which $\rho(u_k, u) < \varepsilon/3$, then

$$|u(x_1) - u(x_2)| < \varepsilon.$$

Now, since $\varepsilon > 0$ was chosen arbitrarily and the above estimate does not depend on the position of points x_1 and x_2 nor the choice of the function u in S, we conclude that the set S of functions is equicontinuous.

Next we consider condition (ii). Since S is sequentially compact, for each $\eta > 0$ there exists a finite $\eta/3$-net for the set S consisting of elements $v_1, \ldots, v_n$; all these functions we can consider as belonging to the set S. By the remark following Proposition 4 a number $L > 0$ with the property that each interval $(\alpha, \alpha + L)$ contains a number t which is a common $\eta/3$-almost period for all $v_i \in S$ $(1 \leq i \leq n)$. Then $|v_i(x + t) - v_i(x)| < \eta/3$, $(1 \leq i \leq n)$; $x \in R$. On the other hand $(v_i)_{i=1}^n$ constitutes an $\eta/3$-net. Thus, for each function $v \in S$ there exists some v_i for which

$$|v(x) - v_i(x)| < \eta/3; \quad x \in R.$$

From the last two inequalities it follows that

$$\begin{aligned} |v(x + t) - v(x)| &\leq |v(x + t) - v_i(x + t)| + \\ &+ |v_i(x + t) - v_i(x)| + |v_i(x) - v(x)| < \\ &< 3\eta/3 = \eta \quad \text{for } x \in R. \end{aligned}$$

Hence t is an η-almost period for all $v \in S$ and the necessity of condition (ii) is verified.

Sufficiency. We assume that a set S of UAP functions satisfies conditions (i) and (ii). Next select an $\eta > 0$ and let $L = L(\eta)$ be so determined that each interval of length L has an η-almost period for all $w \in S$. For each function $w \in S$ we define a function $\overline{w}$ by

$$\overline{w}(x) = \begin{cases} w(x) & \text{for } x \in [-L, L] \\ w(x - r_n) & \text{for } \begin{cases} x \in (nL, (n + 1)L] & n = 1, 2, \ldots \\ x \in [nL, (n + 1)L) & n = -2, -3, \ldots \end{cases} \end{cases}$$

Here r_n is an η-almost period for all $w \in S$ which is in the interval $(nL, (n + 1)L)$. We denote the set of all functions $\overline{w}$ by S_η.

The functions $\overline{w}$ in S_η satisfy the conditions of Ascoli's theorem on the interval $[-L, L]$. Namely, if Q is compact, then a set in $C(Q)$ is sequentially compact if and only if it is bounded and equicontinuous. Thus S_η is sequentially compact in the sense of uniform convergence on this interval. Since $x - r_n \in [-L, L]$, by the definition of $\overline{w}$, a sequence of these functions which converges uniformly on the interval $[-L, L]$, also converges uniformly on the entire real line. Hence S_η is sequentially compact in the sense of uniform convergence on the entire real line. That is, in the sense of the metric of M_∞. For arbitrary $w \in S$ and the corresponding $\overline{w} \in S$ we have $w(x) - \overline{w}(x) = 0$ for $x \in [-L, L]$. And $w(x) - \overline{w}(x) = w(x) - w(x - r_n)$ for

$$\begin{cases} x \in (nL, (n+1)L] & n = 1, 2, \ldots \\ x \in [nL, (n+1)L) & n = -2, -3, \ldots \end{cases}$$

Since r_n is an η-almost period of w, we have for arbitrary x:

$$|w(x) - \overline{w}(x)| < \eta$$

Thus the sequentially compact set S_η forms an η-net for S in the space M_∞. Consequently, S is sequentially compact and conditions (i) and (ii) are evidently sufficient.

Note: For $\lambda \in R$, the <u>translate</u> f_λ of the function f is defined by

$$f_\lambda(x) = f(x + \lambda).$$

<u>Proposition 7</u> - <u>A continuous function is UAP if and only if the set of its translates is sequentially compact in</u> M_∞.

<u>Proof</u>: Let f be UAP. Evidently the set of its translate satisfies both conditions of Proposition 6.

Conversely, suppose the set $\{f_\lambda : \lambda \in R\}$ is sequentially compact. Then it contains a finite ε-net $f_{\lambda_1}, \ldots, f_{\lambda_n}$. We order the f_{λ_i} according to the rising index $\lambda_1 < \lambda_2 < \ldots < \lambda_n$. For each f_λ there exists an f_{λ_i} such that

$$\rho(f_\lambda, f_{\lambda_i}) = \sup_{x \in R} |f_\lambda(x) - f_{\lambda_i}(x)| < \varepsilon$$

or

$$|f(x + \lambda) - f(x + \lambda_i)| < \varepsilon \quad \text{for all } x.$$

If we put $x + \lambda_i = x'$, we get

$$|f(x' + \lambda - \lambda_i) - f(x')| < \varepsilon, \quad x' \in R$$

Thus, for an arbitrary real λ one of the numbers $\lambda - \lambda_i$, $(1 \leq i \leq n)$, is an ε-almost period. It follows that each interval $(a, a + \lambda_n - \lambda_1)$ contains an ε-almost period. For, if we let $a + \lambda_n = \lambda$, we obtain as almost period one of the numbers $a + \lambda_n - \lambda_i$, $(1 \leq i \leq n)$. Since $\lambda_1 \leq \lambda_i \leq \lambda_n$, we have $a + \lambda_n - \lambda_i \in [a, a + \lambda_n - \lambda_1]$ and the proof is complete.

Remark: Proposition 7 can serve as an alternate definition for UAP functions; this was done by S. Bochner. Using this definition of UAP functions one can easily verify that the sum of two UAP functions is again a UAP function. Indeed, if f and g are two UAP functions, then by Proposition 7, any sequence $(f + g)_{\lambda_i} = f_{\lambda_i} + g_{\lambda_i}$ has $\mu_n = \lambda_{i_n}$ such that (f_{μ_n}) and (g_{μ_n}) are both uniformly convergent.

$((f+g)_{\mu_n}) = (f_{\mu_n} + g_{\mu_n})$ is also uniformly convergent by Proposition 7. Therefore $f + g$ is a UAP function as well. It is now trivial that if f and g are UAP, then the functions af, f_c, (with a real and c complex), $|f|$ and $f.g$ are also UAP. Indeed,

$$|f^2(x+t) - f^2(x)| = |f(x+t) + f(x)||f(x+t) - f(x)|$$

and

$$f(x)g(x) = \frac{1}{4}[(f(x)+g(x))^2 - (f(x)-g(x))^2].$$

<u>Proposition 8</u> - <u>The uniform limit f of a sequence $(f_n)_{n=1}^{\infty}$ of UAP functions is again a UAP function.</u>

<u>Proof</u>: Given any $\varepsilon > 0$, choose an $N = N(\varepsilon)$ such that

$$\sup_{x \in R} |f(x) - f_N(x)| < \varepsilon/3.$$

Let τ be an $\varepsilon/3$-almost period of f_N. We then have

$$|f(x+\tau) - f(x)| \leq |f(x+\tau) - f_N(x+\tau)| + |f_N(x+\tau) - f_N(x)| + + |f_N(x) - f(x)| < 3\varepsilon/2$$

which proves the proposition. For, $E\{\varepsilon/3; f_N\}$ is relatively dense, and because of the above inequality each $\varepsilon/3$-almost period of f_N is an ε-almost period of f, $E\{\varepsilon; f\}$ is relatively dense.

<u>Remark</u>: We may summarize some of the above results in the following statement: <u>The set of UAP functions forms a complex Banach space under the norm</u>

$$||f|| = \sup_{x \in R} |f(x)|.$$

An important consequence of Proposition 8 is the following. Consider the set of all exponential polynomials

$$s_n(x) = \sum_{k=1}^{n} a_k e^{i\lambda_k x},$$

where λ_k's are real numbers and a_k's are complex numbers. Each summand in the above expression is a periodic function with period $2\pi/|\lambda_k|$ if $\lambda_k \neq 0$, or constant, and therefore a UAP function. Thus the sum s_n is a UAP function as well. From Proposition 8 it follows that the class of all uniform limits of exponential polynomials are also UAP functions.

In Chapter IV we shall occupy ourselves with establishing the converse result, namely that each UAP function is the uniform limit of exponential polynomials. This result gives a deep characterization of the space of UAP functions and is the content of H. Bohr's

APPROXIMATION THEOREM:

Every UAP function f can be approximated uniformly for $-\infty < x < \infty$ by finite sums of the form

$$s(x) = \sum_{k=1}^{n} a_k e^{i\lambda_k x},$$

that is, for each $\varepsilon > 0$ there exists a sum s such that

$$|f(x) - s(x)| \leq \varepsilon \quad \text{for all } x.$$

In conclusion of this chapter we mention the following:

If $\sum_{k=1}^{\infty} |a_k| < \infty$ (where a_k's are complex numbers) and $\{\lambda_k\}_{k=1}^{\infty}$ is a set of real numbers, then

$$\sum_{k=1}^{\infty} a_k e^{i\lambda_k x}$$

is a UAP function. The foregoing statement is an immediate consequence of Proposition 8 and will be used in the next chapter.

Chapter II

A THEOREM IN ADDITIVE NUMBER THEORY

In this chapter we employ the representation theory of positive definite sequences to derive some propositions of number theoretic character. We shall call a set E of positive integers r e l a t i v e l y d e n s e, if there exists a number $T > 0$ such that any interval of length T on the positive part of the real line contains at least one element of the set E. We prove first the following proposition:

Proposition 1 - For any relatively dense set of positive integers E it is possible to exhibit real numbers $\lambda_1, \ldots, \lambda_m$ such that all integers n, for which the numbers $\frac{\lambda_k n}{2\pi}$ $(k = 1, 2, \ldots, m)$ differ from integers by not more than 1/4, are representable in the form

$$n = n_p + n_q - n_r - n_s,$$

where n_p, n_q, n_r, and n_s belong to the set E.

The foregoing proposition will be used in the next chapter to show that the almost periods of a UAP function coincide with the solutions of a system of inequalities of the form

$$|\lambda_k t| < \delta \pmod{2\pi}; \; k = 1, \ldots, m.$$

These inequalities signify that there exist integers n_k for which the ordinary inequalities $|\lambda_k t - 2\pi n_k| < \delta$, $k = 1, \ldots, m$, are satisfied.

We now turn to the proof of Proposition 1, stated on the preceding page. Suppose that E is a relatively dense set of positive integers and let χ_E signify the characteristic function of the set E. Denote by A_N the following

$$A_N = \sum_{(0<n<N)} \chi_E(n)$$

and for all integers n let

$$\phi_N(n) = \frac{1}{A_N} \sum_{(0<n_1<N)} \chi_E(n + n_1)\, \chi_E(n_1).$$

In the sequel we consider only those natural numbers N' for which the foregoing function ϕ_N is defined, that is for which $A_N \neq 0$. Note that

$$0 \leq \phi_{N'}(n) \leq 1.$$

By the selection principle of Hilbert we see that from the sequence $(\phi_{N'})$ we can select a subsequence $(\phi_{N''})$ which converges pointwise, that is for each integer n, to some limit function ϕ_ω. Moreover we have:

$$0 \leq \phi_\omega(n) \leq 1.$$

We observe: If $\phi_\omega(n) > 0$ for some n, then this n can be represented as difference of two elements of the set E. Indeed, if $\phi_\omega(n) > 0$ for some n, then there exists an N for which $\phi_N(n) > 0$. Thus for one of the numbers $n_1 = 1, \ldots, N - 1$ we have $\chi_E(n + n_1)\, \chi_E(n_1) > 0$ and consequently $n + n_1 = n_2 \in E$ and $n_1 \in E$;

but $n = n_2 - n_1$.

We now verify that $\phi_\omega(n)$ is a positive definite sequence for $n = 0, \pm 1, \pm 2, \ldots$; we show that for any complex numbers $\rho_0, \rho_1, \ldots, \rho_m$ $(m < \infty)$ we have

$$H = \sum_{\substack{0 \le n_1 \le m \\ 0 \le n_2 \le m}} \phi_\omega(n_1 - n_2)\, \rho_{n_1} \overline{\rho_{n_2}} \ge 0.$$

Consider an approximating sequence $(\phi_{N'})$ for ϕ_ω. Then

$$H = \lim_{N' \to \infty} \sum_{\substack{0 \le n_1 \le m \\ 0 \le n_2 \le m}} \phi_{N'}(n_1 - n_2)\, \rho_{n_1} \overline{\rho_{n_2}}.$$

But

$$\sum_{\substack{0 \le n_1 \le m \\ 0 \le n_2 \le m}} \phi_{N'}(n_1 - n_2)\, \rho_{n_1} \overline{\rho_{n_2}} =$$

$$= \frac{1}{A_{N'}} \sum_{\substack{-n_2 < n_3 < N'-n_2 \\ 0 \le n_1 \le m \\ 0 \le n_2 \le m}} \chi_E(n_1 + n_2)\, \chi_E(n_2 + n_3)\, \rho_{n_1} \overline{\rho_{n_2}}.$$

Hence

$$\sum_{\substack{0 \le n_1 \le m \\ 0 \le n_2 \le m}} \phi_{N'}(n_1 - n_2)\, \rho_{n_1} \overline{\rho_{n_2}} =$$

$$= \frac{1}{A_{N'}} \sum_{(0 < n_3 < N')} \Big| \sum_{(0 \le n \le m)} \chi_E(n + n_3)\, \rho_n \Big|^2 + \mathcal{R}_{N'}$$

where

$$\mathcal{R}_{N'} = \frac{1}{A_{N'}} \sum_{\substack{-n_2 < n_3 \leq 0 \\ 0 \leq n_1 \leq m \\ 0 \leq n_2 \leq m}} \chi_E(n_1 + n_2)\, \chi_E(n_2 + n_3)\, \rho_{n_1} \overline{\rho_{n_2}}$$

$$- \frac{1}{A_{N'}} \sum_{\substack{N'-n_2 \leq n_3 < N' \\ 0 \leq n_1 \leq m \\ 0 \leq n_2 \leq m}} \chi_E(n_1 + n_2)\, \chi_E(n_2 + n_3)\, \rho_{n_1} \overline{\rho_{n_2}}.$$

Since $A_{N'} \to \infty$ as $N' \to \infty$ we have that $\mathcal{R}_{N'} \to 0$ as $N' \to \infty$.

Therefore: $H \geq 0$.

In view of the analytic representation theorem for positive definite sequences we can write

$$\phi_\omega(n) = \int_{-\pi}^{\pi} e^{int}\, dg(t), \quad (n = 0, \pm 1, \pm 2, \ldots)$$

where g is a non-decreasing function with total variation

$$\int_{-\pi}^{\pi} dg(t) = \phi_\omega(0) \leq 1.$$

Decomposing the function g into its monotone increasing jump-function g_d and its monotone increasing continuous part g_c, we define

$$\Psi(n) = \int_{-\pi}^{\pi} e^{int}\, dg_d(t)$$

and

$$\Phi(n) = \int_{-\pi}^{\pi} e^{int}\, dg_c(t).$$

We note that $\Psi(n)$ represents an absolutely convergent series with non-negative coefficients:

$$\Psi(n) = \sum C_k e^{i\lambda_k n}.$$

We also see that $\phi(n)$ possesses the following asymptotic behaviour:

$$\phi(n) \to 0 \quad \text{as} \quad n \to \infty.$$

The latter assertion is a consequence of the Riemann-Lebesgue theorem; $\phi(n)$ represents Fourier coefficients of a summable function.

We now show that $\Psi(n)$ is non-negative.

The proof conveniently splits into two steps; first we show that the assumption $\Psi(n_0) < 0$ for some n_0 leads to a contradiction and then we show that the case $\operatorname{Im}(\Psi(n_0)) \neq 0$ for some n_0 is impossible as well.

Assume that for a certain n_0 we have $\Psi(n_0) < 0$. Then we can find a real number $\delta > 0$ such that the set

$$\mathcal{E}_\delta = \{n: \Psi(n) < -\delta\}$$

is not empty. Since $\Sigma|C_k| < \infty$, the function Ψ is UAP by what was said at the end of the last section. Take $\varepsilon = -(\Psi(n_0) + \delta)$. The set $\mathcal{E}$ of integral ε-almost periods of Ψ is relatively dense by Prop. 4, pages 3-4. Take any $m \in \mathcal{E}$. Then $|\Psi(n_0 + m) - \Psi(n_0)| < \varepsilon$ so that $\Psi(n_0 + m) < \Psi(n_0) + \varepsilon = -\delta$. Hence $n_0 + m \in \mathcal{E}_\delta$, and since $\mathcal{E}$ is relatively dense, so is $\mathcal{E}_\delta$. We can find a number $\tilde{n} \in \mathcal{E}_\delta$ such that $|\phi(\tilde{n})| < \delta/2$ on account of the asymptotic behaviour of $\phi(n)$ as $n \to \infty$. Therefore we have $\phi_\omega(\tilde{n}) \leq \Psi(\tilde{n}) + |\phi(\tilde{n})| < -\delta/2 < 0$. But this contradicts the fact that $0 \leq \phi_\omega(n) \leq 1$.

In a similar manner we convince ourselves that $\Psi(n)$ is not complex. For, if there was a number n_0 such that $\Psi(n_0)$ was a proper complex number, then we could find a disk in the complex plane with center at the point $\Psi(n_0)$ such that all points of this disk would be at a distance, say $\delta > 0$ or more, from the real axis. The set of numbers n for which $\Psi(n)$ are located inside the mentioned disk would constitute a non-empty set which would in fact be relatively dense. In this relatively dense set we could pick an element $\tilde{n}$ such that $|\Phi(\tilde{n})| < \delta/2$ holds. Therefore the sum $\Psi(\tilde{n}) + \Phi(\tilde{n}) = \phi_\omega(\tilde{n})$ would have to be a proper complex number and we once again have reached a contradiction.

Next we want to show that the average

$$C_0 = \lim_{K\to\infty} \frac{1}{K} \sum_{(0<n<K)} \Psi(n)$$

is strictly positive. From the asymptotic behaviour of $\Phi(n)$ we have

$$\frac{1}{N} \sum_{(0<n<N)} \Phi(n) \to 0 \quad \text{as} \quad N \to \infty.$$

Thus

$$C_0 = \lim_{K\to\infty} \frac{1}{K} \sum_{(0<n<K)} \phi_\omega(n).$$

On the other hand

$$\frac{1}{K} \sum_{(0<n<K)} \phi_N(n) = \frac{1}{K\, A_N} \sum_{\substack{0 < n_1 < N \\ 0 < n < K}} \chi_E(n + n_1)\, \chi_E(n_1) =$$

$$= \frac{1}{\Lambda_N} \sum_{(0 < n_1 < N)} \chi_E(n_1)\{ \frac{1}{K} \sum_{(n_1 < n_2 < n_1 + K)} \chi_E(n_2)\}.$$

Since E is a relatively dense set of positive integers, one can find numbers $a > 0$ and $K_0 > 0$ such that the number of elements of the set E situated in any interval of length K, where $K \geq K_0$, will be larger than aK.

Consequently,

$$\frac{1}{K} \sum_{(n_1 < n_2 < n_1 + K)} \chi_E(n_2) \geq a$$

and therefore

$$\frac{1}{K} \sum_{(0<n<K)} \phi_N(n) \geq a.$$

Passage to the limit as $N \to \infty$ gives

$$\frac{1}{K} \sum_{(0<n<K)} \phi_\omega(n) \geq a$$

so that we get the desired result, namely: $C_0 \geq a > 0$.

We consider the convolution

$$\Lambda_N(n) = \frac{1}{N} \sum_{(0 < n_1 < N)} \phi_\omega(n + n_1)\, \phi_\omega(n_1).$$

Since $\phi_\omega(n) = \Psi(n) + \phi(n)$ and making use of the asymptotic behaviour of $\phi(n)$ as $n \to \infty$, it is seen that the following limit exists:

$$\Lambda(n) = \lim_{N\to\infty} \Lambda_N(n) = \lim_{N\to\infty} \frac{1}{N} \sum_{(0 < n_1 < N)} \Psi(n + n_1)\, \Psi(n_1);$$

we obtain

$$\Lambda(n) = C_0^{\,2} + \sum_{\lambda_k \neq 0} C_k^{\,2} e^{i\lambda_k n}.$$

If for some n_0 we have the inequality $\Lambda(n_0) > 0$, then we can find a number n_1 such that

$$\phi_\omega(n_0 + n_1) > 0 \text{ and } \phi_\omega(n_1) > 0$$

holds. In this case however the numbers $n_0 + n_1$ and n_1 can be represented as difference of two elements of the set E. Thus every n, for which $\Lambda(n) > 0$ is satisfied, has a representation of the type

$$n = n_p + n_q - n_r - n_s,$$

where n_p, n_q, n_r, and n_s belong to the set E.

The series

$$\Lambda(n) = C_0^{\,2} + \sum_{\lambda_k \neq 0} C_k^{\,2} e^{i\lambda_k n}$$

converges absolutely; we also know that $C_0 > 0$, $C_k \geq 0$ and $\Lambda(n) \geq 0$ because $\Psi(n) \geq 0$. We write

$$\Lambda(n) = C_0^{\,2} + \sum_{k=1}^{m} C_k^{\,2} \cos \lambda_k n + \sum_{k=m+1}^{\infty} C_k^{\,2} \cos \lambda_k n,$$

where it can be assumed that

$$\left| \sum_{k=m+1}^{\infty} C_k^{\,2} \cos \lambda_k n \right| < C_0^{\,2}/2$$

upon suitable choice of the number m. This then means that

$$\Lambda(n) > C_0^{\,2}/2 + \sum_{k=1}^{m} C_k^{\,2} \cos \lambda_k n.$$

The quantity on the right side of the last inequality is larger than zero provided

$$\sum_{k=1}^{m} C_k^2 \cos \lambda_k n \geq 0.$$

This will be the case when $\cos \lambda_k n \geq 0$; the latter condition amounts to the requirement that

$$|\lambda_k n| \leq \pi/2 \pmod{2\pi}.$$

The foregoing therefore answers the question, when is $\Lambda(n) > 0$, considering the series expansion for $\Lambda(n)$.
Evidently we are now done with the proof of the proposition stated at the beginning of this chapter.

In the foregoing proof we established that $\phi_\omega(n) > 0$ implies that n permits a representation as the difference of two elements of the relatively dense set E of positive integers. We wish to find out next what analogous claim can be made regarding the expression $\Psi(n)$. We commence with a definition.

If S is any set of positive integers, let $\Pi_n(S)$ denote the number of elements of S less than the natural number n. We say that a relatively dense set E of positive integers satisfies a certain property P for <u>n e a r l y a l l</u> elements of E, if the subset $E_1 \subset E$ of elements not satisfying property P is negligible in the sense that the ratio $\Pi_n(E_1) / \Pi_n(E)$ tends to zero as $n \to \infty$.

We show that if $\delta > 0$ is a sufficiently small fixed number, then nearly all n for which $\Psi(n) > \delta$ can be represented as difference of two elements of the relatively dense set E of positive integers. Consider the set of positive integers

$$\mathfrak{A}_\delta = \{n : \Psi(n) > \delta\}.$$

That the set $\mathfrak{A}_\delta$ is infinite for some $\delta > 0$ follows from the fact that

$$\lim_{K\to\infty} \frac{1}{K} \sum_{0<n<K} \Psi(n) = C_0 > 0.$$

Let $\mathfrak{A}'_\delta$ denote the subset of $\mathfrak{A}_\delta$ whose elements are not representable as difference of numbers in E, then by what we know about ϕ_ω,

$$\phi_\omega(n) = 0 \text{ if and only if } n \in \mathfrak{A}'_\delta.$$

Whence

$$\sum_{\substack{0 < n < N \\ n \in \mathfrak{A}'_\delta}} \phi(n) = - \sum_{\substack{0 < n < N \\ n \in \mathfrak{A}'_\delta}} \Psi(n) < - \delta\Pi_N(\mathfrak{A}'_\delta)$$

and therefore

$$\sum_{\substack{0 < n < N \\ n \in \mathfrak{A}_\delta}} |\phi(n)| \geq \sum_{\substack{0 < n < N \\ n \in \mathfrak{A}'_\delta}} |\phi(n)| > \delta\Pi_N(\mathfrak{A}'_\delta).$$

Thus we get

$$\delta \frac{\Pi_N(\mathfrak{A}'_\delta)}{\Pi_N(\mathfrak{A}_\delta)} \leq \frac{1}{\Pi_N(\mathfrak{A}_\delta)} \sum_{\substack{0 < n < N \\ n \in \mathfrak{A}_\delta}} |\phi(n)|.$$

But for any sequence of numbers converging to zero, the sequence of consecutive arithmetic means converges to zero. Since $\mathfrak{A}_\delta$ is infinite, the asymptotic behaviour of $\phi(n)$ as $n \to \infty$ implies

$$\frac{1}{\Pi_N(\mathfrak{A}_\delta)} \sum_{\substack{0 < n < N \\ n \in \mathfrak{A}_\delta}} |\phi(n)| \to 0 \quad \text{as} \quad N \to \infty$$

and therefore

$$\frac{\Pi_N(\mathfrak{A}_\delta')}{\Pi_N(\mathfrak{A}_\delta)} \to 0 \quad \text{as} \quad N \to \infty.$$

But this is what we set out to do.

By an argument completely analogous to the one given further above, we observe that the consideration of the series

$$\Psi(n) = C_0 + \sum_{\lambda_k \neq 0} C_k e^{i\lambda_k n}$$

leads to the following statement: It is possible to exhibit real numbers $\lambda_1, \ldots, \lambda_m$, $\eta > 0$, $\delta > 0$ such that $\Psi(n) > \delta$ for any positive integer n, for which all numbers

$$\frac{\lambda_k n}{2\pi} \qquad (k = 1, 2, \ldots, m)$$

differ from integers by not more than η.

It is now easy to see the validity of the following proposition:

Proposition 2 - For any relatively dense set E of positive integers we can find real numbers $\lambda_1, \ldots, \lambda_m$ and $\eta > 0$ such that nearly all integers n, for which the numbers

$$\frac{\lambda_k n}{2\pi} \qquad (k = 1, \ldots, m)$$

differ from integers by not more than η, are representable as the difference of two elements of the given set E.

Chapter III

A THEOREM CONCERNING ALMOST PERIODS

Using the results of the last chapter we now prove a theorem concerning the almost periods of a UAP function. In the next chapter we shall derive from this result Bohr's approximation theorem without the use of limit periodic functions.

THEOREM CONCERNING ALMOST PERIODS: *If f is a UAP function, then for any $\varepsilon > 0$ we can find a $\delta > 0$, and real numbers* $\lambda_0, \lambda_1, \ldots, \lambda_m$ *such that all solutions t of the system of inequalities*

$$|\lambda_k t| < \delta \pmod{2\pi}; \quad k = 0, 1, \ldots, m$$

are ε-almost periods of the function f.

The proof of the foregoing theorem conveniently decomposes into two lemmas.

Lemma 1 - *Let* $\{t_1, t_2, \ldots\}$ *be a relatively dense set of positive reals. Moreover, suppose that there is a positive real α such that for any distinct indices* n_1 *and* n_2

$$|t_{n_1} - t_{n_2}| > \alpha > 0; \quad n_1, n_2 = 1, 2, \ldots$$

holds. Then for any given $\beta > 0$ we can find a $\delta > 0$, and reals $\lambda_0, \lambda_1, \ldots, \lambda_m$ *so that all solutions t of the system of inequalities*

$$|\lambda_k t| < \delta \pmod{2\pi}; \quad k = 1, 2, \ldots, m$$

satisfy as well an inequality of the form

$$|t - (t_p + t_q - t_r - t_s)| < \beta$$

with suitable elements t_p, t_q, t_r, and t_s, dependent on t and belonging to the relatively dense set $\{t_1, t_2, \ldots\}$.

Proof of the Lemma: We first determine a natural number M so large that

$$(1/M) < \alpha \quad \text{and} \quad (1/M) < (\beta/5)$$

holds. Next we select natural numbers n_i by the following rule:

$$n_i = [M\, t_i],\ i = 1, 2, \ldots,$$

where $[\quad]$ signifies the integral part of the number so enclosed. We now observe that for $i \neq k$ and $t_i > t_k$ we have

$$n_i - n_k = [M\, t_i] - [M\, t_k] \geq [M(t_i - t_k)] \geq [\tfrac{1}{\alpha}\, \alpha] = 1.$$

Thus the numbers n_i form a relatively dense set of distinct positive integers and we can apply Proposition 1, Chapter II. That is, there are reals $\lambda_1, \ldots, \lambda_m$ so that all numbers n which solve

$$|\lambda_k n| < \pi/2 \pmod{2\pi};\ k = 1, 2, \ldots, m$$

are of the form $n = n_p + n_q - n_r - n_s$. Since we operate with residue classes mod 2π, we may assume that the reals $\lambda_1, \ldots, \lambda_m$ lie in the interval $[0, 2\pi)$. We let

$$\lambda_0 = 2\pi M \quad \text{and} \quad \delta = \pi/(4M)$$

and show that the numbers $\lambda_0, \lambda_1, \ldots, \lambda_m$ and δ satisfy the assertions of Lemma 1. Let t be a solution of

$$|\lambda_k t| < \delta \pmod{2\pi};\quad k = 0, 1, \ldots, m.$$

Then there exists a number n such that

$$|2\pi Mt - 2\pi n| < \pi/(4M)$$

or

$$|t - \frac{n}{M}| < 1/(8M^2) < \beta/5.$$

In addition, there exist integers $n_1, \ldots, n_m$ such that

$$|\lambda_k \frac{n}{m} + 2\pi n_k| \leq |\lambda_k t + 2\pi n_k| + \lambda_k |t - \frac{n}{m}| < \pi/(4M) + 2\pi/(8M^2) \leq \pi/(2M)$$

holds for $k = 1, 2, \ldots, m$. Therefore the number n satisfies the inequalities

$$|\lambda_k n| < \pi/2 \pmod{2\pi}$$

for $k = 1, 2, \ldots, m$.

But $n = n_p + n_q - n_r - n_s$, hence

$$|t - (t_p + t_q - t_r - t_s)| \leq |t - \frac{n}{M}| + |\frac{n}{M} - (\frac{n_p}{M} + \frac{n_q}{M} - \frac{n_r}{M} - \frac{n_s}{M})| +$$

$$+ |\frac{n_p}{M} - t_p| + |\frac{n_q}{M} - t_q| + |\frac{n_r}{M} - t_r| + |\frac{n_s}{M} - t_s| <$$

$$< \beta/5 + 0 + \beta/5 + \beta/5 + \beta/5 + \beta/5$$

and the proof is complete.

Remark: In Lemma 1 we can delete the condition: "suppose that there is a positive real α such that for any distinct indices n_1 and n_2

$$|t_{n_1} - t_{n_2}| > \alpha > 0; \quad n_1, n_2 = 1, 2, \ldots \text{ holds}".$$

This is a bonus due to the fact that the set $\{t_1, t_2, \ldots\}$ is relatively dense. Indeed, if $\{t_1, t_2, \ldots\}$ is relatively dense, then we can select a $T > 0$ such that every interval of length $T/2$ contains an element of the set $\{t_1, t_2, \ldots\}$. If we choose from each interval $((i - 1/2)T, iT)$, $i = 1, 2, \ldots$, an element t_{n_i}, then

$$|t_{n_i} - t_{n_k}| > T/2 > 0 \qquad i \neq k.$$

In view of the preceding remark we have:

Lemma 2 - Let $E = \{t_1, t_2, \ldots\}$ be a relatively dense set of positive reals. Then for any given $\beta > 0$ we can find a $\delta > 0$ and reals $\lambda_0, \ldots, \lambda_m$ so that all solutions t of

$$|\lambda_k t| < \delta \pmod{2\pi}; \quad k = 0, 1, \ldots, m$$

satisfy an inequality of the form

$$|t - (t_p + t_q - t_r - t_s)| < \beta$$

with suitable elements t_p, t_q, t_r, and t_s depending on t and belonging to the set E.

Before we start with the verification of the theorem concerning almost periods stated earlier in Chapter III, we consider the following two observations. First, it is not necessary to get into a separate discussion of positive and negative almost periods because if t is an ε-almost period of a UAP function f, then so is $-t$. Next, if t_1 and t_2 are ε_1- and ε_2-almost periods of a UAP function f respectively, then $t_1 \pm t_2$ are $(\varepsilon_1 + \varepsilon_2)$-almost periods of f.

We select some relatively dense set $E = \{t_1, t_2, \ldots\}$ of $\varepsilon/8$-almost periods of the UAP function f and apply to it Lemma 2. To begin with, choose an arbitrary β. We get that the solutions t of the system of inequalities

$$|\lambda_k t| < \delta \pmod{2\pi}; \quad k = 0, 1, \ldots, m$$

with the numbers $\lambda_0, \ldots, \lambda_m$ and δ depending on β also satisfy an inequality of the form

$$|t - (t_p + t_q - t_r - t_s)| < \beta.$$

Since the elements of the set E are $\varepsilon/8$-almost periods of f, we see that

$$t_p + t_q - t_r - t_s = t(\tfrac{\varepsilon}{2})$$

is an $\varepsilon/2$-almost period of f. The number $t(\frac{\varepsilon}{2})$ differs from t by at most β. The function f is uniformly continuous on the entire real line by Proposition 2, Chapter I. Thus we can take β so small that every number t which differs from an almost period $t(\frac{\varepsilon}{2})$ of f by less than β,

$$|t - t(\tfrac{\varepsilon}{2})| < \beta$$

is an ε-almost period of the function f. Thus, if $\varepsilon > 0$ is given, we select $\beta > 0$ as was just explained. Then by Lemma 2 we choose the numbers $\delta, \lambda_0, \lambda_1, \ldots, \lambda_m$ and note that these numbers are precisely those numbers the existence of which was claimed by the theorem concerning almost periods. This completes the proof of the theorem.

Chapter IV

PROOF OF BOHR'S APPROXIMATION THEOREM

Keeping in mind that our problem is essentially the construction of Fourier series for UAP functions, we shall begin with a theorem which is the key to all these considerations.

THEOREM CONCERNING MEAN VALUE: For each UAP function f there exists the m e a n v a l u e defined by

$$M\{f(x)\} = \lim_{T\to\infty} \frac{1}{T} \int_0^T f(x)\, dx.$$

Moreover, the limit

$$\lim_{T\to\infty} \frac{1}{T} \int_a^{a+T} f(x)\, dx = M\{f(x+a)\}$$

exists uniformly for a.

Proof: Let $\varepsilon > 0$ be given. Let

$$L = L(\varepsilon/2) \quad \text{and} \quad A = \sup_{-\infty<x<\infty} |f(x)|.$$

Denote by α an arbitrary real number and by t an $\varepsilon/2$-almost period of f situated in the interval $(\alpha, \alpha + L)$. Then

$$\left|\frac{1}{T}\int_0^T f(x)dx - \frac{1}{T}\int_\alpha^{\alpha+T} f(x)dx\right| \le \left|\frac{1}{T}\int_0^T f(x)dx - \frac{1}{T}\int_t^{t+T} f(x)dx\right| +$$

$$+ \left|\frac{1}{T}\int_\alpha^t f(x)dx\right| + \left|\frac{1}{T}\int_{t+T}^{\alpha+T} f(x)dx\right| \le$$

$$\le \frac{1}{T}\int_0^T |f(x) - f(x+t)|\, dx + \frac{1}{T}\int_\alpha^t |f(x)|\, dx + \frac{1}{T}\int_{t+T}^{\alpha+T} |f(x)|dx$$

$$< \varepsilon/2 + (2AL)/T \tag{1}$$

because

$$\int_{\alpha}^{\alpha+T} = \int_{\alpha}^{t} + \int_{t}^{t+T} + \int_{t+T}^{\alpha+T}$$

and

$$\int_{t}^{t+T} f(y)dy = \int_{0}^{T} f(x+t)dx$$

when we substitute $y = x + t$.

Considering the arithmetic average of the n differences

$$\frac{1}{T}\int_{0}^{T} f(x)dx - \frac{1}{T}\int_{(m-1)T}^{mT} f(x)dx; \qquad m = 1, 2, \ldots, n$$

we get from (1):

$$\left|\frac{1}{T}\int_{0}^{T} f(x)dx - \frac{1}{nT}\int_{0}^{nT} f(x)dx\right| < \varepsilon/2 + (2AL)/T. \tag{2}$$

Let T_1 and T_2 be positive numbers such that $m_1T_1 = m_2T_2$, where m_1 and m_2 are integers. From (2) it follows that

$$\left|\frac{1}{T_1}\int_{0}^{T_1} f(x)dx - \frac{1}{T_2}\int_{0}^{T_2} f(x)dx\right| < \varepsilon + 2AL\left(\frac{1}{T_1} + \frac{1}{T_2}\right). \tag{3}$$

The last inequality carries over to arbitrary positive numbers T_1 and T_2 by continuity consideration. If T_1 and T_2 are strictly greater than $(4AL)/\varepsilon$, then we see from (3) that

$$\left|\frac{1}{T_1}\int_{0}^{T_1} f(x)dx - \frac{1}{T_2}\int_{0}^{T_2} f(x)dx\right| < 2\varepsilon$$

which proves the existence of the limit

$$\lim_{T\to\infty} \frac{1}{T}\int_{0}^{T} f(x)dx = M\{f(x)\}.$$

Taking $n \to \infty$ in inequality (2) we get:

$$\left|\frac{1}{T}\int_0^T f(x)dx - M\{f(x)\}\right| \leq \varepsilon/2 + (2AL)/T. \qquad (4)$$

To get the second assertion of the theorem, we note first that for a constant a:

$$M\{f(x+a)\} = M\{f(x)\}.$$

This follows from

$$\frac{1}{T}\int_0^T f(x+a)dx = \frac{1}{T}\int_a^{a+T} f(x)dx =$$

$$= \frac{1}{T}\int_a^0 f(x)dx + \frac{1}{T}\int_0^T f(x)dx + \frac{1}{T}\int_T^{a+T} f(x)dx.$$

Now as $T \to \infty$ the second term on the right hand side of the last equation tends to $M\{f(x)\}$, whereas the other two terms tend to zero because the absolute value of each does not exceed $|a|\ A/T$.

It remains to show that for each $\varepsilon > 0$ there exists a number $T_0 = T_0(\varepsilon)$, independent of the number a and such that for $T > T_0$ the inequality

$$\left|\frac{1}{T}\int_0^T f(x+a)dx - M\{f(x+a)\}\right| < \varepsilon$$

holds. But this follows directly from inequality (4) because the numbers A and L are independent of the number a. In particular, letting $a = -T$, we get

$$M\{f(x)\} = \lim_{T\to\infty} \frac{1}{T}\int_{-T}^0 f(x)dx = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^T f(x)dx,$$

which completes the proof of the theorem.

In the interval $-\infty < x < \infty$ the system $\{e^{i\lambda x}:\ \lambda \text{ real}\}$ is an orthonormal system in the sense that

$$M\{e^{i\lambda_1 x}\, e^{-i\lambda_2 x}\} = \begin{cases} 0 & \text{if } \lambda_1 \neq \lambda_2 \\ 1 & \text{if } \lambda_1 = \lambda_2. \end{cases}$$

We let

$$a(\lambda) = M\{f(x)\, e^{-i\lambda x}\}.$$

The non-zero $a(\lambda)$'s are at most countable and are called the Fourier coefficients of the function f. Indeed, let $\lambda_1, \ldots, \lambda_N$ be distinct real numbers and $c_1, \ldots, c_N$ arbitrary complex numbers. Then

$$M\{|f(x) - \sum_{n=1}^{N} c_n e^{i\lambda_n x}|^2\} =$$

$$= M\{|f(x)|^2 - \sum_{n=1}^{N} |a(\lambda_n)|^2 + \sum_{n=1}^{N} |c_n - a(\lambda_n)|^2\}.$$

First of all it is clear that the mean values appearing in the foregoing equality exist because the functions involved are UAP. The equality is verified as follows:

$$M\{|f(x) - \sum_{n=1}^{N} c_n e^{i\lambda_n x}|^2\} =$$

$$= M\{(f(x) - \sum_{n=1}^{N} c_n e^{i\lambda_n x})(\overline{f(x)} - \sum_{n=1}^{N} \overline{c_n} e^{-i\lambda_n x})\} =$$

$$= M\{f(x)\overline{f(x)}\} - \sum_{n=1}^{N} \overline{c_n} M\{f(x) e^{-i\lambda_n x}\} - \sum_{n=1}^{N} c_n M\{\overline{f(x)} e^{i\lambda_n x}\} +$$

$$+ \sum_{n_1=1}^{N} \sum_{n_2=1}^{N} c_{n_1} \overline{c_{n_2}} M\{e^{i\lambda_{n_1} x} e^{-i\lambda_{n_2} x}\} =$$

$$= M\{|f(x)|^2\} - \sum_{n=1}^{N} c_n\, a(\lambda_n) - \sum_{n=1}^{N} c_n\, \overline{a(\lambda_n)} + \sum_{n=1}^{N} c_n\, \bar{c}_n =$$

$$= M\{|f(x)|^2\} - \sum_{n=1}^{N} (c_n - a(\lambda_n))(\overline{c_n - a(\lambda_n)}) - \sum_{n=1}^{N} a(\lambda_n)\, \overline{a(\lambda_n)} =$$

$$= M\{|f(x)|^2\} - \sum_{n=1}^{N} |a(\lambda_n)|^2 + \sum_{n=1}^{N} |c_n - a(\lambda_n)|^2.$$

Taking in particular $c_n = a(\lambda_n)$, $n = 1, \ldots, N$, and since

$M\{|f(x) - \sum_{n=1}^{N} c_n e^{i\lambda_j x}|^2\} \geq 0$, we get Bessel's inequality

$$\sum_{n=1}^{N} |a(\lambda_n)|^2 \leq M\{|f(x)|^2\}.$$

From this it follows that the number of λ's for which $|a(\lambda)| > d$, is less than $M\{|f(x)|^2\}/d^2$. Taking $d_n = 1/n$ $(n = 1, 2, \ldots)$ and considering the sets $B_1 = \{\lambda: |a(\lambda)| > 1\}$ and $B_n = \{\lambda: d_n \geq a(\lambda) > d_{n+1}\}$, $n = 2, 3, \ldots$, we see that the set of λ for which $a(\lambda) \neq 0$ is at most countable.

Proof of Bohr's Approximation Theorem: Let f be a UAP function. Denote by M_f the vector space over the rationals generated by $\lambda_1, \lambda_2, \ldots$, the Fourier exponents of f. M_f has a basis $\beta_1, \beta_2, \ldots$, and each β_k may be chosen from the set $\lambda_1, \lambda_2, \ldots$ Since the β_k's form a basis there exist positive integers $m = m(\lambda_1, \ldots, \lambda_n)$ and $q = q(\lambda_1, \ldots, \lambda_n)$ such that $\lambda_j = (s_{j1}\beta_1 + \ldots + s_{jm}\beta_m)/q$, $j = 1, \ldots, n$, where s_{jk} is an integer, j = 1, . . . , n; k = 1, . . . , m. Let

$$\delta = 1/(4m \max_{\substack{1 \leq j \leq n \\ 1 \leq k \leq m}} |s_{jk}|).$$

It follows that every t which satisfies the inequalities

$$|\beta_k t/q| < \delta \pmod{2\pi}; \quad k = 1, \ldots, m \tag{5}$$

also satisfies the inequalities

$$|\lambda_j t| < \pi/2 \pmod{2\pi}; \quad j = 1, \ldots, n. \tag{6}$$

Let N be an arbitrary natural number, and consider the Bochner-Fejér kernel defined by

$$K_N(\alpha_k t) = \frac{1}{N} [(\sin (N\alpha_k t)/2)/(\sin (\alpha_k t)/2)]^2 =$$

$$= \sum_{|\nu|<N} (1 - \frac{|\nu|}{N}) e^{-i\nu\alpha_k t},$$

where $\alpha_k = \beta_k/q$, $k = 1, \ldots, m$. If $|\alpha_k t| \geq \delta \pmod{2\pi}$, $(\delta < \pi)$, then $|\sin (\alpha_k t/2)| \geq |\sin (\delta/2)|$. Thus for such a t,

$$K_N(\alpha_k t) < (N \sin^2(\delta/2))^{-1} \tag{7}$$

We note two properties of the Bochner-Fejér kernel: It is never negative and its mean value equals 1 because it is equal to the constant term of K_N.

We consider the composite Bochner-Fejér kernel:

$$K^N(t) = K_N(\alpha_1 t) K_N(\alpha_2 t) \ldots K_N(\alpha_m t).$$

It is again seen that it is never negative and its mean value equals 1. Let

$$E = \{t: |\alpha_k t| < \delta \pmod{2\pi}, k = 1, \ldots, m\}$$

$$E_k = \{t: |\alpha_k t| < \delta \pmod{2\pi)}\}; \; k = 1, \ldots, m,$$

where $\alpha_k = \beta_k/q$, $k = 1, \ldots, m$. Then $E = E_1 \cap \ldots \cap E_m$.

For $T > 0$, put $E^{(T)} = E \cap (-T, T)$, $E_k^{(T)} = E_k \cap (-T, T)$, $k = 1, \ldots, m$.

Let g be any non-negative UAP function and $A = \sup_t g(t)$.

Then by (7):

$$\frac{1}{2T} \int_{-T}^{T} g(t)\, K^N(t)dt =$$

$$= \frac{1}{2T} \int_{E_1^{(T)}} g(t)\, K^N(t)dt +$$

$$+ \frac{1}{2T} \int_{(-T,T)-E_1^{(T)}} g(t)K_N(\alpha_1 t)K_N(\alpha_2 t) \ldots K_N(\alpha_m t)dt \leq$$

$$\leq \frac{1}{2T} \int_{E_1^{(T)}} g(t)\, K^N(t)dt +$$

$$+ \frac{1}{N \sin^2(\delta/2)} \int_{(-T,T)-E_1^{(T)}} g(t)K_N(\alpha_2 t) \ldots K_N(\alpha_m t)dt \leq$$

$$\leq \frac{1}{2T} \int_{E_1^{(T)}} g(t)\, K^N(t)dt + \frac{1}{N \sin^2(\delta/2)} \int_{-T}^{T} g(t)\, \frac{K^N(t)}{K_N(\alpha_1 t)}\, dt.$$

Separating the points of the set $E_2^{(T)}$ from the set $E_1^{(T)}$ we get

$$\frac{1}{2T} \int_{-T}^{T} g(t)\, K^N(t)\, dt \leq \frac{1}{2T} \int_{E_1^{(T)} \cap E_2^{(T)}} g(t)\, K^N(t)\, dt +$$

$$+ \frac{A}{N \sin^2(\delta/2)}\, \frac{1}{2T} \int_{-T}^{T} \left(\frac{K^N(t)}{K(\alpha_1 t)} + \frac{K^N(t)}{K_N(\alpha_2 t)}\right)dt.$$

Continuing this process and passing to the limit as $T \to \infty$, we obtain, using the fact that $E = E_1 \cap \ldots \cap E_m$,

$$M\{g(t)K^N(t)\} \le \lim_{T\to\infty} \frac{1}{2T} \int_{E(T)} g(t)\, K^N(t)\, dt + \frac{mA}{N \sin^2(\delta/2)} \tag{8}$$

Now we invoke the theorem concerning almost periods: Every $t \in E$ is an $\varepsilon/2$-almost period of f;

$$|f(x+t) - f(x)| < \varepsilon/2. \tag{9}$$

We consider the Bochner-Fejér polynomial

$$P_N(x) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(x+t)\, K^N(t)\, dt.$$

Applying (8) to the UAP function $g(t) = |f(x+t) - f(x)|$ we get

$$|P_N(x) - f(x)| \le \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |f(x+t) - f(x)|\, K^N(t)\, dt$$

$$\le \lim_{T\to\infty} \frac{1}{2T} \int_{E(T)} |f(x+t) - f(x)|\, K^N(t) dt + \frac{2Cm}{N \sin^2(\delta/2)}$$

where $C = \sup_t |f(t)|$. Hence by (9)

$$|P_N(x) - f(x)| \le \frac{\varepsilon}{2} \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} K^N(t) dt + \frac{2Cm}{N \sin^2(\delta/2)}$$

$$= (\varepsilon/2) + (2Cm)/(N \sin^2(\delta/2)). \tag{10}$$

For fixed m and ε, N can be taken so large that the inequality

$$\frac{2Cm}{N \sin^2(\delta/2)} < \frac{\varepsilon}{2}$$

is satisfied. We see therefore that the estimate (10) holds uniformly in x and Bohr's approximation theorem is proved. The Bochner-Fejér polynomial P_N is of the form

$$\sum_{|\nu_k|<N, 1\le k\le m} B_{1,\ldots,m}\ a(\nu_1\ \alpha_1 + \ldots + \nu_m\ \alpha_m)\ e^{i(\nu_1\alpha_1 + \ldots + \nu_m\alpha_m)x}$$

where

$$B_{1,\ldots,m} = (1 - \frac{|\nu_1|}{N})\ .\ .\ .\ (1 - \frac{|\nu_m|}{N})$$

and $$a(\alpha) = M\{f(t)\ e^{-i\alpha t}\}.$$

Chapter V

THE UNIQUENESS THEOREM FOR UNIFORMLY ALMOST PERIODIC FUNCTIONS

In this section, we are concerned with the proof of the uniqueness theorem for UAP functions. It asserts:

If all the Fourier coefficients of a UAP function are zero, then the function must vanish everywhere.

Following H. Weyl [9], we base the proof of the foregoing theorem on the theory of compact normal operators in a pre-Hilbert space. The first step in our proof will therefore consist in the construction of a suitable pre-Hilbert space from the available data.

Consider the set of all UAP functions; it forms a vector space over the complex field under the usual algebraic operations. Using the concept of the mean value of a UAP function introduced earlier, we define an inner product on the vector space of UAP functions as follows

$$\langle f, g \rangle = M\{f(x)\,\overline{g(x)}\}$$

To see that this is indeed an inner product, it suffices to verify that

$$M\{|f(x)|^2\} = 0 \quad \text{implies} \quad f(x) \equiv 0.$$

To this end, we prove the following

Proposition 1 - If g is a non-negative UAP function and $g(x_o) > 0$ for some $x_o = x$, then

$$M\{g(x)\} > 0$$

Proof: Let $g(x_o) = a > 0$. Since g is continuous, we can choose a $\delta = \delta(\frac{a}{3})$ such that for $|x - x_o| < \delta$ we have

$$|g(x) - g(x_o)| < \frac{a}{3}.$$

We denote, as usual, by $L = L(\frac{a}{3})$ the greatest distance between two $\frac{a}{3}$ - almost periods of g. We show that in each interval of length $K = L + 2\delta$, namely $(\alpha, \alpha + L + 2\delta)$ with α arbitrary, there is a subinterval of length 2δ, such that for all points x in this subinterval, we have

$$g(x) > \frac{a}{3} .$$

Let τ be an $\frac{a}{3}$ - almost period of g contained in the interval $(\alpha + \delta - x_o, \alpha + L + \delta - x_o)$. Then the number $x_o + \tau$ is contained in the interval $(\alpha + \delta, \alpha + \delta + L)$, and if $|x - x_o| \leq \delta$, then $x + \tau$ runs through an interval of length 2δ, where

$$g(x + \tau) = g(x_o) + [g(x) - g(x_o)] + [g(x + \tau) - g(x)]$$

$$> a - \frac{a}{3} - \frac{a}{3} = \frac{a}{3} .$$

Thus

$$\frac{1}{2nK} \int_{-nK}^{nK} g(x)dx = \frac{1}{2nK} \sum_{j=-(n-1)}^{n} \int_{(j-1)K}^{jK} g(x)dx$$

$$> \frac{1}{2nK} 2n\ 2\delta\ \frac{a}{3} = \frac{2a\delta}{3K} .$$

Letting $n \to \infty$, we get

$$M\{g(x)\} = \frac{2a\delta}{3K} > 0,$$

which is what we set out to prove.

In the sequel below, we denote by U the non-separable pre-Hilbert space which we just constructed. For a fixed $f \in U$, the equations

$$v(s) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(s - t)\, u(t)dt = \underset{t}{M}\{f(s - t)\, u(t)\} \qquad (1)$$

and

$$w(s) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \overline{f(t - s)}\, u(t)dt = \underset{t}{M}\{\overline{f(t - s)}\, u(t)\} \qquad u \in U$$

define linear operators A and T from U into itself, respectively, because the convolution of two UAP functions is again a UAP function. We now show that A is a normal operator. That is if $u_1, u_2 \in U$, then,

(i) $\langle Au_1, u_2\rangle = \langle u_1, Tu_2\rangle$

and

(ii) $AT = TA$

To verify (i) consider the following relation:

$$\langle Au_1, u_2\rangle = \underset{s}{M}\{\underset{t}{M}\{f(s - t)\, u_1(t)\}\, \overline{u_2(s)}\}$$

Since the mean value is a uniform limit, we have

$$\langle Au_1, u_2\rangle = \underset{t}{M}\{u_1(t)\, \underset{s}{M}\{f(s - t)\, \overline{u_2(s)}\}\}$$

$$= \underset{t}{M}\{u_1(t)\, \overline{\underset{s}{M}\{\overline{f(s - t)}\, u_2(s)\}}\} = \langle u_1, Tu_2\rangle$$

To show (ii) is also true, we begin with

$$ATu = \underset{t}{M}\{f(s - t)\, \underset{\tau}{M}\{\overline{f(\tau - t)}\, u(\tau)\}\}$$

$$= \underset{\tau}{M}\{u(\tau)\, \underset{t}{M}\{f(s - t)\, \overline{f(\tau - t)}\}\}$$

After changing the variable from t to s by $\tau - t = \sigma - s$, we get

$$\underset{t}{M}\{f(s - t)\, \overline{f(\tau - t)} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T+\tau+s}^{T+\tau+s} f(\sigma - \tau)\, \overline{f(\sigma - s)}d\sigma =$$

$$= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(\sigma - \tau)\, \overline{f(\sigma - s)}\, d\sigma$$

$$= \underset{\sigma}{M}\{f(\sigma - \tau)\, \overline{f(\sigma - s)}\}$$

Hence

$$ATu = \underset{\tau}{M}\{u(\tau)\, \underset{\sigma}{M}\{f(\sigma - \tau)\, \overline{f(\sigma - s)}\}\}$$

$$= \underset{\sigma}{M}\{\overline{f(\sigma - s)}\, \underset{\tau}{M}\{f(\sigma - \tau)\, u(\tau)\}\} = TAu$$

Next we show that A is compact. That is, the image under A of a bounded set is relatively compact. Let $Q = \{u \in U : ||u||^2 = M\{|u(t)|^2\} \leq 1\}$. We must now prove that if $(u_n(t))_{n=1}^{\infty}$ is an infinite sequence in Q, then its image under A contains a subsequence $(v_n(t))_{n=1}^{\infty}$ which converges strongly in U to some $v_o \in U$. The function $u \in Q$ is transformed by equation (1) page 40 into the function

$$v(s) = \underset{t}{M}\{f(s - t)\, u(t)\}.$$

But

$$|v(s)|^2 \leq \underset{t}{M}\{|f(s - t)|^2\}\, \underset{t}{M}\{|u(t)|^2\}$$

$$\leq \underset{t}{M}\{|f(t)|^2\} = K < \infty \qquad (2)$$

Hence the functions v are bounded in norm. Now, from equations (1) of this chapter, we condlude that for $s_1, s_2 \in R$ and $u \in Q$,

$$v(s_1) - v(s_2) = \underset{t}{M}\{[f(s_1 - t) - f(s_2 - t)]\, u(r)\}.$$

or

$$|v(s_1) - v(s_2)|^2 \leq \underset{t}{M}\{|f(s_1 - t) - f(s_2 - t)|^2\} < \infty;$$

which shows that the functions $v(\cdot)$ are equi-continuous. Indeed, if f is UAP, then f is uniformly continuous, thus, for each $\varepsilon > 0$, there is a $\delta > 0$ such that

$$|f(s_1 - t) - f(s_2 - t)| \leq \varepsilon \qquad t \in R$$

whenever $|s_1 - s_2| \leq \delta$. Consequently, for $|s_1 - s_2| \leq \delta$, we get $|v(s_1) - v(s_2)|^2 \leq M\{\varepsilon^2\} = \varepsilon^2$. Hence $|v(s_1) - v(s_2)| \leq \varepsilon$. The last inequality on page 41 also shows that the set of functions v are equi-almost periodic (every ε-almost period of f is an ε-almost period for all v simultaneously) and we can use Proposition 6, page 5. Indeed, for all functions v,

$$|v(s_1) - v(s_2)|^2 \leq M_t\{|f(s_1 - t) - f(s_2 - t)|^2\}.$$

So, if τ is a translation number of f belonging to ε, and if $s_1 - s_2 = \tau$, then $|v(s_1) - v(s_2)| \leq \varepsilon$. Hence we can pick from the set of functions v a sequence v_n which converges uniformly on the real line to a UAP function v_o by Proposition 8, page 10. Since we also have that

$$M_t\{|w_1(t) - w_2(t)|^2\}^{1/2} \leq \sup_{x \quad R} |w_1(t) - w_2(t)|$$

the sequence v_n converges not only uniformly on R, but in the norm of U as well. Whence A is compact. (Incidentally, recalling the definitions of operators A and T, we can easily see that for every $u \in U$, if τ is an ε-almost period of f, then τ is an $\varepsilon[M_t\{|u(t)|^2\}]^{1/2}$-almost period of both v and w).

Since A is a non-zero compact normal operator on a pre-Hilbert space, it has a non-zero eigenvalue μ and the eigenspace corresponding to μ is of finite dimension. If we can show that an exponential function of the form $e^{-i\lambda x}$ is in the eigenspace of μ, then the relation

$$\mu e^{-i\lambda t} = \underset{x}{M}\{f(t - x)\, e^{-i\lambda x}\} = e^{-i\lambda t}\, \underset{x}{M}\{f(t - x)\, e^{-i\lambda(t-x)}\} \qquad (t,\ x \in R)$$

gives the desired contradiction, and the uniqueness theorem is proved.

Now we show that, if $a_1, \ldots, a_n$ is a base of the eigenspace corresponding to μ, then there is an exponential of the above type which can be expressed linearly in terms of the eigenvectors $a_1, \ldots, a_n$. To do this, consider the following equation

$$\mu\, a_r(t) = \underset{x}{M}\{f(t - x)\, a_r(x)\} \qquad (1 \leq r \leq n)$$

Furthermore, if $\sigma \in R$, then

$$A a_r(x + \sigma) = \mu\, a_r(x + \sigma)$$

Hence, $a_r(x + \sigma)$ belongs to the space of eigenvectors corresponding to μ. Therefore,

$$a_r(x + \sigma) = \sum_{k=1}^{n} c_{rk}(\sigma)\, a_k(x) \tag{3}$$

It remains to show that for $\sigma = 0$, equation (3) reduces to

$$a_r(x) = \sum_{k=1}^{n} b_{rk}\, e^{i\lambda_k x} \tag{4}$$

where b_{rk}'s are complex numbers. We shall do this after discussing the following two lemmas.

<u>Lemma _ - An arbitrary pairwise commutative set of unitary matrices can be reduced to diagonal form by a unique unitary matrix.</u>

Proof: The proof conveniently falls into three parts.

Part (i): Suppose V and W are two unitary matrices and E_λ the subspace of eigenvectors of the matrix V corresponding to the eigenvalue λ. It follows that E_λ is invariant with respect to the matrix W. Indeed, if $x \in E_\lambda$ (i.e. $Vx = \lambda x$), then $VWx = WVx = \lambda Wx$. That is $Wx \in E_\lambda$.

Part (ii): Next, we show that an arbitrary set of pairwise commutative matrices acting on a vector space E have a common eigenvector. The proof is by induction on the dimension of E. If E is of dimension one, the assertion is trivially true. We assume that it is also true for dimension $k - 1$, and prove the assertion for dimension k. We observe that if every vector in E is an eigenvector of the set of matrices under consideration, the lemma is verified. For, it then follows that these matrices are multiples of the identity matrix. We shall assume to the contrary that there exists in E a vector which is not eigenvector to some matrix A in our set. Let E_1 be the set of all eigenvectors of A corresponding to an eigenvalue λ. By part (i) of this lemma, E_1 is invariant with respect to the remaining matrices and A. Since E_1 is different from the null space and the whole space, it is of dimension less than k, say $k - 1$. But by the inductive assumption, the lemma is true for dimension $k - 1$. Thus E_1 must contain a vector which is a common eigenvector of the matrices in question.

Part (iii): According to part (ii) of this proof there exists a vector e_1 in E which is a common eigenvector of all these matrices.

We consider the $k - 1$ dimensional space E_1 orthogonal to e_1. The space E_1 is invariant under all these matrices. Indeed, if V is any of these matrices, λ an eigenvalue of V and $x \in E_1$, then the inner product $\langle Vx, e_1 \rangle$ is:

$$\langle Vx, e_1 \rangle = \langle x, V^{-1} e_1 \rangle = \langle x, \frac{1}{\lambda} e_1 \rangle = \langle \frac{1}{\lambda} x, e_1 \rangle = 0.$$

Now consider only those matrices acting on E_1. Again by part (ii) E_1 contains a vector e_2 which is a common eigenvector to all these matrices. The set of all vectors in E_1 which are orthogonal to e_2 form a $k - 2$ dimensional subspace invariant under all these matrices. Continuing this process we obtain a set of k mutually orthogonal vectors $e_1, e_2, \ldots, e_k$ where each e_i $(1 \leq i \leq k)$ is an eigenvector of the pairwise commutative matrices under consideration. Furthermore, $e_1, e_2, \ldots, e_n$ form a basis in E and relative to $e_1, e_2, \ldots, e_n$, the commutative matrices in question are diagonal. This completes the proof.

Lemma 2 - Let f be a continuous, complex valued function of the real variable x $(-\infty < x < \infty)$. If

(i) $f(x + y) = f(x) f(y)$ for all $x, y \in R$ and

(ii) $|f(x)| = 1$,

then $$f(x) = e^{i\lambda x} \qquad \lambda \in R.$$

This Lemma is a direct consequence of Cauchy's functional equation and will not be proved.

Now we are in a position to deduce equation (4) from equation (3). Let $E(\sigma) = \{c_{rk}(\sigma)\}_{r,k=1}^{n}$ and $E(\sigma)^*$ denote the transposed conjugate of $E(\sigma)$. Recall that the $a_r(x)$ were orthonormal. Consequently, so are the $a_r(x + \sigma)$, $(\sigma \in R)$. Then

$$E(\sigma)\, E(\sigma)^* = [c_{rk}(\sigma)\, a_k(x)][\overline{c_{kr}(\sigma)}\, \overline{a_k(x)}] = 1.$$

Thus the matrices $E(\sigma)$ are unitary. To show they are also commutative, substitute $\sigma + \eta$ instead of σ in (3). We then get

$$\begin{aligned} a_r(x + (\sigma + \eta)) &= \sum_{k=1}^{n} c_{rk}(\eta + \sigma)\, a_k(x) \\ &= a_r((x + \eta) + \sigma) \\ &= \sum_{k=1}^{n} c_{rk}(\sigma)\, a_k(x + \eta) \\ &= \sum_{k=1}^{n} c_{rk}(\sigma) \sum_{j=1}^{n} c_{rj}(\eta)\, a_j(x) \\ &= \sum_{k=1}^{n} a_k(x) \sum_{j=1}^{n} c_{rj}(\sigma)\, c_{jr}(\eta) \end{aligned}$$

Since the $a_r(x)$ are linearly independent, we get

$$\sum_{k=1}^{n} c_{rk}(\sigma + \eta) = \sum_{j=1}^{n} c_{rj}(\sigma)\, c_{jr}(\eta)$$

That is,

$$E(\sigma + \eta) = E(\sigma)\, E(\eta) = E(\eta)\, E(\sigma).$$

By Lemma 1 there exists a constant unitary matrix V, such that for $\sigma \in R$,

$$V\, E(\sigma)\, V^{-1} = \begin{bmatrix} \delta_1(\sigma) & \cdots & & 0 \\ \vdots & \delta_2(\sigma) & & \vdots \\ \vdots & & & \vdots \\ 0 & \cdots & & \delta_n(\sigma) \end{bmatrix} \qquad (5)$$

We shall now show that the functions $\delta_r(\sigma)$ $(1 \leq r \leq n)$ satisfy the conditions of Lemma 2. Indeed, the continuity of the $\delta_r(\sigma)$ follows from the continuity of the $c_{rk}(\sigma)$. Moreover, $E(\sigma)$ is unitary implies $|\delta_r(\sigma)| = 1$. Finally, we have

$$V\,E(\sigma + \eta)\,V^{-1} = \begin{bmatrix} \delta_1(\sigma + \eta) & \cdots & & 0 \\ \vdots & \delta_2(\sigma + \eta) & & \vdots \\ 0 & \cdots & & \delta_n(\sigma + \eta) \end{bmatrix}$$

$$= V\,E(\sigma)V^{-1}[V\,E(\eta)\,V^{-1}]$$

$$= \begin{bmatrix} \delta_1(\sigma)\delta_1(\eta) & \cdots & & 0 \\ \vdots & \delta_2(\sigma)\delta_2(\eta) & & \vdots \\ 0 & \cdots & & \delta_n(\sigma)\delta_n(\eta) \end{bmatrix}$$

Thus, the functional relation $\delta_r(\sigma + \eta) = \delta_r(\sigma)\,\delta_r(\eta)$ is satisfied. Hence $\delta_r(\sigma) = e^{i\lambda_r\sigma}$. From the matrix equation (5) it follows that the functions $c_{rk}(\sigma)$ are also linear combinations of the functions $e^{i\lambda_r\sigma}$. Substituting $\sigma = 0$ in

$$a_r(x + \sigma) = \sum_{k=1}^{n} c_{rk}(\sigma)\,a_k(x)$$

we express the functions $a_r(x)$ linearly by the functions $e^{i\lambda_r x}$. Indeed, let

$$a_r(x) = \sum_{k=1}^{n} b_{rk}\,e^{i\lambda_k x} \;;$$

it suffices to show that the $\det[b_{rk}] \neq 0$. If $\det[b_{rk}] = 0$, then there exists numbers $u_1, u_2, \ldots, u_n$ not all equal to zero and such that for $r = 1, 2, \ldots, n$ we have

$$\sum_{k=1}^{n} u_k b_{kr} = 0$$

This implies that

$$\sum_{k=1}^{n} u_k a_k(x) = 0$$

which contradicts the linear independence of the functions $a_r(x)$.

SOME FINAL REMARKS

Recall that each UAP function was found to be uniformly continuous and bounded on R. Thus, for each UAP function f, we can define

$$||f|| = \sup_{x \in R} |f(x)|.$$

Under this norm, the set of all UAP functions forms a Banach space B. If we introduce convolution multiplication defined by

$$(f * g)(x) = \underset{t}{M}\{f(x - t)\ g(t)\} \qquad f, g \in B$$

We will get a commutative Banach algebra B . Using the notions and results established in the theory of Banach algebras, our two basic theorems have the following interpretations.

Bohr's Approximation Theorem - Every closed ideal in B is the intersection of the regular maximal ideals containing it.

The Uniqueness Theorem for UAP Functions - The Banach algebra B is semi-simple.

In addition to the two foregoing theorems, the following two theorems also have a basic role in the theory of UAP functions.

Parseval's Equality - If a UAP function f has the Fourier series

$$f(x) \sim \sum_{\lambda} a(\lambda)\ e^{i\lambda x}$$

then the following Parseval's equality, or completeness relation holds:

$$M\{|f(x)|^2\} = \sum_{\lambda} |a(\lambda)|^2 .$$

Multiplication Theorem - If f and g are two UAP functions such that

$$f(x) \sim \sum_{\lambda} a(\lambda)\, e^{i\lambda x} \quad \text{and} \quad g(x) \sim \sum_{\lambda} b(\lambda)\, e^{i\lambda x}$$

then fg has the Fourier expansion

$$f(x)\, g(x) \sim \sum_{M} \Big(\sum_{\lambda+\lambda'=M} a(\lambda)\, b(\lambda') \Big) e^{iMx}$$

A prominent fact in the theory of UAP functions is the equivalence of these four theorems in the sense that one can be proved directly from the other.

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