Frobenius Brauer Categories

Saima Samchuck-Schnarch

Thesis submitted in partial fulfillment of the requirements for the degree of
Master of Science Mathematics and Statistics[1]

Department of Mathematics and Statistics
Faculty of Science
University of Ottawa

[1] The M.Sc. program is a joint program with Carleton University, administered by the Ottawa-Carleton Institute of Mathematics and Statistics

Abstract

Given a symmetric Frobenius superalgebra A equipped with a compatible involution, we define the associated *Frobenius Brauer category* $\mathcal{B}(A)$ and *affine Frobenius Brauer category* $\mathcal{AB}(A)$, generalizing the plain Brauer category $\mathcal{B}$ and affine Brauer category $\mathcal{AB}$. We define the *orthosymplectic Lie superalgebra* $\mathfrak{osp}_{m|2n}(A)$ and a functor from $\mathcal{B}(A)$ to $\mathfrak{osp}_{m|2n}(A)$-mod, the category of supermodules over $\mathfrak{osp}_{m|2n}(A)$. We also define a functor from $\mathcal{AB}(A)$ to the endofunctor supercategory of $\mathfrak{osp}_{m|2n}(A)$-mod. We prove that these two functors are well-defined and use the former functor to prove a basis result for $\mathcal{B}(A, \delta)$, a specialized version of $\mathcal{B}(A)$. Prior to defining these categories and functors, we provide the background information on super-mathematics and Frobenius superalgebras needed to understand the new results.

Dedications

For Brian, Kai-Lei, Zephyr, and Ari.

Acknowledgements

Thanks to my advisor, Alistair Savage, for his support, guidance, and mentorship throughout my graduate and undergraduate studies. I'm not sure whether I would have pursued graduate studies at all if not for your encouragement – thank you!

Thanks to my family for always supporting and believing in me.

Thanks to Alex, Anhelina, Carly, Devlin, Maribeth, Noah, and Raj for all of the fun times and good memories, both over the past couple of socially-distant years and beforehand.

Contents

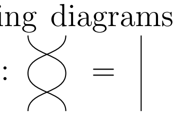
Chapter 1

Introduction

1.1 Intended Audience

This thesis is designed to be accessible to any reader familiar with basic abstract algebra (specifically: vector spaces, algebras, modules, and tensor products) and symmetric monoidal categories. Familiarity with super-mathematics (e.g. superalgebras, supermodules, and supercategories) could be helpful, but we provide fairly detailed introductions to these topics in Chapters 2, 3, and 5.

1.2 Overview and History

Richard Brauer first defined *Brauer algebras* in his 1937 paper [Bra37]. These algebras have been studied extensively since then, and are of particular interest to representation theorists due to their connections to the representations of orthogonal and symplectic groups and Lie algebras. In their 2012 paper [LZ15], Lehrer and Zhang defined the *Brauer category* $\mathcal{B}$, whose endomorphism algebras are Brauer algebras. Brauer categories consist of string diagrams subject to natural geometric relations like the *double crossing-relation*: . More abstractly, $\mathcal{B}$ is the free linear symmetric monoidal category generated by a single symmetrically self-dual object $\Big|$. Many results regarding Brauer algebras can be recast in terms of $\mathcal{B}$; for instance, Schur-Weyl duality for the orthogonal and symplectic groups corresponds to constructing full functors from $\mathcal{B}$ to the categories of finite-dimensional representations of those groups.

Many variations on $\mathcal{B}$ have been studied, including the oriented Brauer category $\mathcal{OB}$, which corresponds to the general linear group/Lie algebra. In recent years, a new type of Brauer category has appeared in the literature: *oriented Frobenius Brauer (super)categories*, denoted $\mathcal{OB}(A)$. These categories (studied, for instance, in

[McS21]) generalize the ordinary oriented Brauer category by introducing *Frobenius tokens*, a new type of generating morphism drawn as $\uparrow\!\bullet\, a$, where a is an element of some fixed *Frobenius superalgebra* A. Frobenius superalgebras are $\mathbb{Z}_2$-graded algebras equipped with a special kind of bilinear form called a *Frobenius form*; notable examples of Frobenius (super)algebras include matrix superalgebras, finite group algebras, quaternion algebras, and zigzag superalgebras. Many constructions involving $\mathcal{OB}$ can be naturally extended to $\mathcal{OB}(A)$, yielding connections to the representation theory of Frobenius superalgebras.

In this thesis, we define and study the *(non-oriented) Frobenius Brauer categories* $\mathcal{B}(A)$. In Chapter 2, we cover preliminary information regarding super-mathematics, including super vector spaces, super category theory, and superalgebras. In Chapter 3, we introduce string diagrams in the super-setting and prove some fundamental results regarding categorical duality. In Chapter 4, we define Frobenius superalgebras and related notions, establish some basic results regarding Frobenius superalgebras, and give many examples. In Chapter 5, we provide a brief overview of the theory of sesquilinear and Hermitian forms, and then define and study $\mathfrak{osp}_{m|2n}(A)$, the *orthosymplectic Lie algebra* over a Frobenius superalgebra A. Finally, in Chapter 6, we define $\mathcal{B}(A)$ and the closely-related *affine Frobenius Brauer category* $\mathcal{AB}(A)$, and prove that both categories are rigid and pivotal. We then define functors $F\colon \mathcal{B}(A) \to \mathfrak{osp}_{m|2n}(A)\text{- mod}$ and $\mathcal{F}\colon \mathcal{AB}(A) \to \mathrm{End}(\mathfrak{osp}_{m|2n}(A)\text{- mod})$ (where $\mathfrak{osp}_{m|2n}(A)\text{- mod}$ denotes the category of supermodules over $\mathfrak{osp}_{m|2n}(A)$) and prove that these functors are well-defined. We also prove a basis result for $\mathcal{B}(A,\delta)$, which is a specialized version of $\mathcal{B}(A)$.

1.3 Originality of Results

The content of Chapters 2 and 3 would be familiar to those well-versed in super category theory; in these chapters and throughout the rest of the thesis, we have included many details in the definitions and proofs, with the aim of making the content more accessible to those not already familiar with the topics. The definitions and results in Chapter 4 are standard. Most of the examples in Section 4.3 are well-known, though the zigzag superalgebra trace map and associated involution defined in Example 4.3.12 do not seem to have appeared in the literature to date. The general definitions and results regarding supermodules and Lie superalgebras in Chapter 5 are standard, but the content regarding $\mathfrak{osp}_{m|2n}(A)$ in Section 5.5 is novel; the author was unable to find any explicit mentions of orthosymplectic Lie superalgebras over Frobenius superalgebras in the literature. The generalized orthosymplectic involution discussed in Sections 5.2, 5.3, and 5.4 has been studied before, but the author is not aware of any pre-existing detailed derivations of its properties starting from the abstract definition in 5.2.7. Non-oriented Frobenius Brauer categories have not appeared in the literature to date, and as such, most of Chapter 6 is novel. In particular, the definitions of $\mathcal{B}(A)$, $\mathcal{AB}(A)$, their specialized counterparts, the proofs that these categories are rigid and pivotal, the definitions of the functors F and $\mathcal{F}$, and the proofs that these functors are well-defined are all original.

1.4 Further Directions

- Throughout this thesis, we assume that our Frobenius superalgebras are symmetric. It is known that if A is any Frobenius superalgebra, there is an automorphism $\psi\colon A \to A$, known as the *Nakayama automorphism*, such that $\mathrm{tr}(ab) = (-1)^{\bar{a}\bar{b}}\,\mathrm{tr}(b\psi(a))$ for all $a, b \in A$; if A is symmetric, one can take $\psi = \mathrm{id}_A$. It seems plausible that many of the results in this thesis could be generalized to non-symmetric Frobenius superalgebras by using the Nakayama automorphism whenever symmetry is needed.

- Similarly, we consider only even trace maps in this thesis. One particularly notable Frobenius superalgebra, the two-dimensional Clifford superalgebra Cl, has no even symmetric trace map, but it does have a natural odd symmetric trace map. It seems plausible that many of the results in this thesis could be generalized to allow non-even trace maps by introducing appropriate sign terms.

- It seems like the marked Brauer category studied in [KT17] should correspond to a Frobenius Brauer category over Cl, and that the periplectic Lie superalgebra $\mathfrak{p}_n$ should correspond to an orthosymplectic Lie superalgebra over Cl. This would be a natural analogue of the fact that $\mathfrak{gl}_{m|n}(\mathrm{Cl})$ is isomorphic to the queer Lie superalgebra $\mathfrak{q}_{m+n}$ – see [MS21, Ex. 3.5] for details. However, the definitions of $\mathcal{B}(A)$ and $\mathfrak{osp}_{m|2n}(A)$ given in this thesis make essential use of an involution $-^*$ on A that is compatible with the trace map for A. It is straightforward to show that no involution can be compatible with the odd trace map for Cl, and thus it is not possible to define $\mathcal{B}(\mathrm{Cl})$ or $\mathfrak{osp}_{m|2n}(\mathrm{Cl})$ using the methods in this thesis. It would be interesting to investigate possible modifications to the definitions of $\mathcal{B}(A)$ and $\mathfrak{osp}_{m|2n}(A)$ that might allow the Frobenius Brauer category to also cover the periplectic case.

- Frobenius Heisenberg categories (see, for example, [BSW21]) are a generalization of the oriented Frobenius Brauer category $\mathcal{OB}(A)$; one recovers $\mathcal{OB}(A)$ by setting the central charge to be $k = 0$. The quantum analogue of these categories, introduced in [BSW22], thus yield a quantum oriented Frobenius Brauer category when one sets $k = 0$. Quantum analogues of the non-oriented Frobenius Brauer category and $\mathfrak{osp}_{m|2n}(A)$ have not yet been defined. The author plans to define and study these structures in his future PhD research.

- The arguments used to show that the functor $\mathcal{F}$ is well-defined in Section 6.5 involve a significant amount of computation. A more conceptual approach might yield more elegant proofs.

Chapter 2

Introduction to Super-mathematics

Throughout this thesis, we fix a ground field $\Bbbk$ of characteristic not equal to 2. Unless stated otherwise, all tensor products are taken over $\Bbbk$.

2.1 Super Vector Spaces

Definition 2.1.1 (Super vector space)**.** A *super vector space* (also known as a *vector superspace*) is a $\mathbb{Z}_2$-graded vector space. Explicitly, a super vector space is a vector space V together with a direct sum decomposition $V = V_0 \oplus V_1$. In the context of super vector spaces, we use the term *parity* instead of grade. That is, for $i \in \mathbb{Z}_2$, we call the vectors in V_i *homogeneous elements of parity* i. For all $v \in V_i$, we write $\overline{v} = i$ to indicate its parity. Elements of parity 0 are called *even*, and those of parity 1 are called *odd*. Note that the zero vector belongs to both V_0 and V_1, so it does not have a well-defined parity. We adopt the convention that the zero vector is both even and odd; this is often convenient when stating definitions and results.

Remark 2.1.2. When working with individual vector spaces and the linear maps between them, there is no difference between the terms "super vector space" and "$\mathbb{Z}_2$-graded vector space". In other words, the category of super vector spaces is identical to the category of $\mathbb{Z}_2$-graded vector spaces. However, as we will discuss later in this chapter, these two categories have different symmetric monoidal structures; see Remark 2.1.16. (For an introduction to symmetric monoidal categories, see e.g. [ML98, Ch. XI].)

Remark 2.1.3. In general, not all nonzero elements of a super vector space are even or odd. For example, one can turn $V = \mathbb{R}^2$ into a super vector space over $\mathbb{R}$ by setting $V_0 = \operatorname{Span}\left\{\begin{bmatrix}1\\0\end{bmatrix}\right\}$ and $V_1 = \operatorname{Span}\left\{\begin{bmatrix}0\\1\end{bmatrix}\right\}$. Then $\begin{bmatrix}1\\1\end{bmatrix}$ is inhomogeneous, i.e. neither even nor odd.

Definition 2.1.4. Let $V = V_0 \oplus V_1$ be a finite-dimensional super vector space and let $m = \dim(V_0), n = \dim(V_1)$. We say that the *(super)dimension* of V is $m \mid n$.

Definition 2.1.5. A linear map $T\colon V \to W$ between super vector spaces is called *parity-preserving* or *even* if $\overline{T(v)} = \overline{v}$ for all homogeneous $v \in V$. The map T is called *parity-reversing* or *odd* if we instead have $\overline{T(v)} = \overline{v} + 1$. Writing $\overline{T}$ for the parity of a homogeneous linear map, we thus have $\overline{T(v)} = \overline{T} + \overline{v}$ in general.

Proposition 2.1.6. *Let V and W be super vector spaces. The space of linear maps* $\mathrm{Hom}(V, W)$ *is a super vector space with respect to the decomposition into even and odd maps discussed in Definition 2.1.5.*

Proof: It is immediate that a linear combination of even maps is even, and the same for odd maps. It only remains to show that $\mathrm{Hom}(V, W)$ is the direct sum of the space of even maps and the space of odd ones. First, let $f\colon V \to W$ be an arbitrary linear map. Let $f_0^0\colon V_0 \to W_0$ be the restriction of f to V_0 followed by the projection $W \to W_0$. Define $f_0^1\colon V_0 \to W_1, f_1^0\colon V_1 \to W_0$, and $f_1^1\colon V_1 \to W_1$ analogously. We can see that both f_0^0 and f_1^1 preserve parity based on their domains and codomains. Thus $f_0^0 \oplus f_1^1$ is even. Similarly, f_0^1 and f_1^0 reverse parity based on their domains and codomains, so $f_0^1 \oplus f_1^0$ is odd. A straightforward computation shows that $(f_0^0 \oplus f_1^1) + (f_0^1 \oplus f_1^0) = f$, and thus every linear map decomposes as a sum of an even map and an odd map. To show that the sum is direct, suppose that $z\colon V \to W$ is both even and odd. Let $v \in V$ be homogeneous. Then $\overline{z(v)}$ must be both even and odd. The only such element of W is the zero vector, so z is the zero map, as desired. ∎

Remark 2.1.7. It is easy to see that the composition of two even linear maps yields another even linear map. The composition of two odd linear maps also yields an even linear map. The composition of an odd and an even linear map (in either order) yields an odd linear map. In other words, composition is a parity-preserving operation, i.e. $\overline{f \circ g} = \bar{f} + \bar{g}$. This is analogous to the way that integer parity behaves under addition, which explains the terminology.

Lemma 2.1.8. *Let $f\colon V \to W$ be an invertible homogeneous linear map. Then* $\bar{f} = \overline{f^{-1}}$.

Proof: Using Proposition 2.1.6, we may decompose the inverse as $f^{-1} = g_0 + g_1$, where g_i has parity $\bar{f} + i$. We have:

$$\begin{aligned} \mathrm{id}_V &= f^{-1} \circ f \\ &= (g_0 + g_1) \circ f \\ &= (g_0 \circ f) + (g_1 \circ f). \end{aligned}$$

Since composition is parity-preserving, we have $\overline{g_i \circ f} = \bar{f} + i + \bar{f} = i$. We find that $g_1 \circ f$ must be zero, as otherwise id_V would be either inhomogeneous or parity-reversing, and id_V is obviously parity-preserving. Thus $\mathrm{id}_V = g_0 \circ f$. A totally analogous argument shows that $\mathrm{id}_W = f \circ g_0$. Thus g_0 is the inverse of f, and so $\overline{f^{-1}} = \overline{g_0} = \bar{f}$. ∎

Definition 2.1.9. We write $\mathcal{SVec}$ for the category whose objects are super vector spaces and whose morphisms are even linear maps.

Remark 2.1.10. Unless stated otherwise, isomorphisms of super vector spaces take place in $\mathcal{SVec}$. That is, all super vector space isomorphisms are assumed to be even. The same is true for isomorphisms of superalgebras and supermodules, which will be defined in Section 2.3 and Chapter 5, respectively.

Example 2.1.11. Let V be an ordinary vector space. Then V can also be viewed as a super vector space by setting $V_0 = V$ and $V_1 = \{0\}$, i.e. declaring that all vectors are even. If we turn two vector spaces V and W into super vector spaces in this way, then even linear maps $V \to W$ are just ordinary linear maps $V \to W$. Thus the category of ordinary vector spaces is a full subcategory of $\mathcal{SVec}$.

Definition 2.1.12. Let V and W be super vector spaces. The tensor product $V \otimes W$ is naturally a super vector space, with parities given by:

$$(V \otimes W)_0 = (V_0 \otimes W_0) \oplus (V_1 \otimes W_1), \qquad (V \otimes W)_1 = (V_0 \otimes W_1) \oplus (V_1 \otimes W_0).$$

More compactly, this says that $\overline{v \otimes w} = \bar{v} + \bar{w}$ for homogeneous $v \in V, w \in W$.

Definition 2.1.13. Let V, W, X, and Y be super vector spaces, and let $f \colon V \to X$ and $g \colon W \to Y$ be linear maps. We define the linear map $f \otimes g \colon V \otimes W \to X \otimes Y$ by setting $(f \otimes g)(u \otimes v) = (-1)^{\bar{g}\bar{u}} f(u) \otimes g(v)$. See Remark 2.1.16 for a brief discussion of why the sign term $(-1)^{\bar{g}\bar{u}}$ appears in this definition.

Remark 2.1.14. When working with super vector spaces, it is often convenient to give definitions and state results in terms of just homogeneous elements, as was done in the previous definition. For the sake of brevity, we adopt the convention that whenever we write $\bar{v}$, we are implicitly assuming v is homogeneous. Whenever we give a definition in terms of homogeneous elements, the full definition is obtained by linear extension to the full super vector space.

Definition 2.1.15. There is a natural symmetric monoidal structure on $\mathcal{SVec}$:

- The tensor product for objects is the one given in Definition 2.1.12;

- The tensor product for morphisms is the one given in Definition 2.1.13;
- The unit object is the ground field $\Bbbk$ as a purely even one-dimensional space;
- The unitors are the same as those for the canonical monoidal structure on the category of vector spaces;
- For all objects X, Y, the symmetric braiding $\sigma_{X,Y} \colon X \otimes Y \to Y \otimes X$ is given by $\sigma_{X,Y}(x \otimes y) = (-1)^{\bar{x}\bar{y}} y \otimes x$.

Remark 2.1.16. One can define an alternate symmetric monoidal structure on $\mathcal{SVec}$ by setting $\sigma_{X,Y}(x \otimes y) = y \otimes x$, i.e. using the same braiding as in the category of vector spaces. By convention, the term "super vector space" implies that one is using the braiding that involves the sign term $(-1)^{\bar{x}\bar{y}}$. The term "$\mathbb{Z}_2$-graded vector space" is more appropriate when using the signless braiding.

The sign term in the braiding encodes a rule that pervades super-mathematics in general: whenever two odd elements exchange places, a factor of -1 should be applied. This mathematical framework was originally developed to reflect properties of supersymmetry in theoretical physics.

2.2 Monoidal Supercategories

Definition 2.2.1 (Supercategory)**.** A *supercategory* is a category enriched in $\mathcal{SVec}$. More concretely, a category $\mathcal{C}$ is a supercategory if:

- For all objects X, Y in $\mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is a super vector space;
- Composition of morphisms is a parity-preserving bilinear operation. Explicitly, this means that:
  - $(\alpha f + \beta g) \circ h = \alpha(f \circ h) + \beta(g \circ h)$;
  - $f \circ (\alpha g + \beta h) = \alpha(f \circ g) + \beta(f \circ h)$;
  - $\overline{f \circ g} = \bar{f} + \bar{g}$,

  for all scalars $\alpha, \beta \in \Bbbk$ and morphisms f, g, h such that the indicated compositions and additions are defined.

Example 2.2.2. Every category enriched in the category of vector spaces can be viewed as a supercategory by declaring all morphisms to be even.

Example 2.2.3. The category $\mathcal{SVec}$ is a supercategory, but only in the sense of the previous example since it contains only even morphisms by definition. For a more interesting example, one can also consider the supercategory $\mathcal{SVec}_{\mathrm{sup}}$, which is defined in the same way as $\mathcal{SVec}$, but taking *all* linear maps between super vector spaces for the morphisms, not just even maps.

Definition 2.2.4 (Tensor product supercategory)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be supercategories. The *tensor product supercategory* $\mathcal{C} \otimes \mathcal{D}$ is defined as follows:

- The objects are pairs (A, X), with A being an object in $\mathcal{C}$ and X being an object in $\mathcal{D}$;
- We set $\mathrm{Hom}_{\mathcal{C}\otimes\mathcal{D}}((A, X), (B, Y)) = \mathrm{Hom}_{\mathcal{C}}(A, B) \otimes \mathrm{Hom}_{\mathcal{D}}(X, Y)$ for all objects (A, X) and (B, Y) in $\mathcal{C} \otimes \mathcal{D}$, using the tensor product discussed in Definition 2.1.12;
- Let $f\colon A \to B$ and $g\colon B \to C$ be morphisms in $\mathcal{C}$, and $\alpha\colon X \to Y$ and $\beta\colon Y \to Z$ morphisms in $\mathcal{D}$. Then we define $(g \otimes \beta) \circ (f \otimes \alpha) = (-1)^{\bar{\beta}\bar{f}}(g \circ f) \otimes (\beta \circ \alpha)$.

Intuitively, the sign term in the definition for composition in $\mathcal{C} \otimes \mathcal{D}$ arises because we need to use the $\mathcal{SVec}$ braiding to swap the positions of β and f in the expression before composing coordinate-wise. This idea is made formal in the setting of general enriched category theory in [Kel05, § 1.4].

Definition 2.2.5 (Superfunctor, superbifunctor)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be supercategories. A *superfunctor* from $\mathcal{C}$ to $\mathcal{D}$ is a $\Bbbk$-linear functor $F\colon \mathcal{C} \to \mathcal{D}$ that preserves the parity of morphisms. In other words, a superfunctor is a functor between supercategories that is an even linear map when restricted to any given hom-set in $\mathcal{C}$. A *superbifunctor* is a superfunctor whose domain is a tensor product supercategory.

Definition 2.2.6 (Supernatural transformation)**.** Let $F, G\colon \mathcal{C} \to \mathcal{D}$ be superfunctors and $i \in \mathbb{Z}_2$. A *supernatural transformation of parity* i from F to G, denoted $\alpha\colon F \to G$, is a collection of $\mathcal{D}$-morphisms $\alpha_X \in \mathrm{Hom}_{\mathcal{D}}(F(X), G(X))$, where X ranges over $\mathrm{ob}(\mathcal{C})$, such that $\overline{\alpha_X} = i$ for all $X \in \mathrm{ob}(\mathcal{C})$ and the following diagram commutes for all homogeneous $f\colon X \to Y$ in $\mathcal{C}$:

$$\begin{tikzcd} F(X) \arrow[r, "(-1)^{i\bar{f}}F(f)"] \arrow[d, "\alpha_X"'] & F(Y) \arrow[d, "\alpha_Y"] \\ G(X) \arrow[r, "G(f)"'] & G(Y) \end{tikzcd} . \tag{2.2.1}$$

A *supernatural transformation* $\alpha\colon F \to G$ is the sum of an even and an odd supernatural transformation. A *supernatural isomorphism* is a supernatural transformation for which each of the components is an isomorphism in $\mathcal{D}$.

Definition 2.2.7 (Monoidal supercategory)**.** A *monoidal supercategory* is a supercategory $\mathcal{C}$ equipped with the following structure:

- A superbifunctor $\otimes\colon \mathcal{C} \otimes \mathcal{C} \to \mathcal{C}$, called the *tensor product* for $\mathcal{C}$;

- A unit object $\mathbb{1}$;
- A supernatural isomorphism α called the associator, with components of the form $\alpha_{X,Y,Z}\colon (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$;
- A supernatural isomorphism λ called the left unitor, with components of the form $\lambda_X\colon \mathbb{1} \otimes X \to X$;
- A supernatural isomorphism ρ called the right unitor, with components of the form $\rho_X\colon X \otimes \mathbb{1} \to X$,

such that the following two diagrams, respectively called the triangle and pentagon diagrams, commute for any objects W, X, Y, Z in $\mathcal{C}$:

$$\begin{tikzcd}
(X \otimes \mathbb{1}) \otimes Y \arrow[r, "\rho_X \otimes \mathrm{id}_Y"] \arrow[d, "\alpha_{X,\mathbb{1},Y}"'] & X \otimes Y \\
X \otimes (\mathbb{1} \otimes Y) \arrow[ur, "\mathrm{id}_X \otimes \lambda_Y"'] &
\end{tikzcd} , \tag{2.2.2}$$

$$\begin{tikzcd}
& (W \otimes X) \otimes (Y \otimes Z) \arrow[dr, "\alpha_{W,X,Y \otimes Z}"] & \\
((W \otimes X) \otimes Y) \otimes Z \arrow[ur, "\alpha_{W \otimes X,Y,Z}"] \arrow[d, "\alpha_{W,X,Y} \otimes \mathrm{id}_Z"'] & & (W \otimes (X \otimes (Y \otimes Z))) \\
(W \otimes (X \otimes Y)) \otimes Z \arrow[rr, "\alpha_{W,X \otimes Y,Z}"'] & & W \otimes ((X \otimes Y) \otimes Z) \arrow[u, "\mathrm{id}_W \otimes \alpha_{X,Y,Z}"']
\end{tikzcd} . \tag{2.2.3}$$

Definition 2.2.8 (Symmetric monoidal supercategory)**.** A *symmetric monoidal supercategory* is a monoidal supercategory $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$ that is additionally equipped with a supernatural isomorphism σ, called the symmetric braiding, with components of the form $\sigma_{X,Y}\colon X \otimes Y \to Y \otimes X$, such that the following diagrams commute for all objects X, Y, Z in $\mathcal{C}$:

$$\begin{tikzcd}
& X \otimes (Y \otimes Z) \arrow[r, "\sigma_{X,Y \otimes Z}"] & (Y \otimes Z) \otimes X \arrow[dr, "\alpha_{Y,Z,X}"] & \\
(X \otimes Y) \otimes Z \arrow[ur, "\alpha_{X,Y,Z}"] \arrow[dr, "\sigma_{X,Y} \otimes \mathrm{id}_Z"'] & & & Y \otimes (Z \otimes X) \\
& (Y \otimes X) \otimes Z \arrow[r, "\alpha_{Y,X,Z}"] & Y \otimes (X \otimes Z) \arrow[ur, "\mathrm{id}_Y \otimes \sigma_{X,Z}"'] &
\end{tikzcd} , \tag{2.2.4}$$

$$\begin{tikzcd} X \otimes Y \arrow[rr, "\mathrm{id}_{X\otimes Y}"] \arrow[dr, "\sigma_{X,Y}"'] & & X \otimes Y \\ & Y \otimes X \arrow[ur, "\sigma_{Y,X}"'] & \end{tikzcd} \quad . \tag{2.2.5}$$

Remark 2.2.9. The definition for a (symmetric) monoidal supercategory is almost identical to that of a (symmetric) monoidal category; the only difference is that the tensor product here is a *super*bifunctor and the associator, unitors, and braiding are *super*natural isomorphisms.

Example 2.2.10. The supercategory $\mathcal{SVec}_{\text{sup}}$ has a natural symmetric monoidal structure defined identically to the symmetric monoidal structure on $\mathcal{SVec}$ discussed in Definition 2.1.15.

Definition 2.2.11 (Strict monoidal supercategory)**.** A (symmetric) monoidal supercategory is called *strict* if the associator and both unitors are the identity supernatural transformation. Note that this implies we have:

$$(X \otimes Y) \otimes Z = X \otimes (Y \otimes Z), \qquad \mathbb{1} \otimes X = X = X \otimes \mathbb{1}, \qquad f \otimes \mathrm{id}_{\mathbb{1}} = f = \mathrm{id}_{\mathbb{1}} \otimes f$$

for all objects X, Y, Z and morphisms f in $\mathcal{C}$.

Definition 2.2.12 (Monoidal superfunctor)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be monoidal supercategories with tensor products and units $\otimes_{\mathcal{C}}, \mathbb{1}_{\mathcal{C}}$ and $\otimes_{\mathcal{D}}, \mathbb{1}_{\mathcal{D}}$, respectively. Let $\alpha^{\mathcal{C}}, \rho^{\mathcal{C}}$, and $\lambda^{\mathcal{C}}$ denote the associator and unitors for $\mathcal{C}$, and $\alpha^{\mathcal{D}}, \rho^{\mathcal{D}}$, and $\lambda^{\mathcal{D}}$ the associator and unitors for $\mathcal{D}$. A *monoidal superfunctor from $\mathcal{C}$ to $\mathcal{D}$*, sometimes called a *lax monoidal superfunctor*, is a superfunctor $F\colon \mathcal{C} \to \mathcal{D}$ together with an even supernatural isomorphism with components $\phi_{X,Y}\colon F(X) \otimes_{\mathcal{D}} F(Y) \to F(X \otimes_{\mathcal{C}} Y)$ (with X, Y ranging over the objects of $\mathcal{C}$) and an even isomorphism $\iota\colon \mathbb{1}_{\mathcal{D}} \to F(\mathbb{1}_{\mathcal{C}})$ such that the following diagrams commute for all objects X, Y, Z in $\mathcal{C}$:

$$\begin{tikzcd} (F(X) \otimes_{\mathcal{D}} F(Y)) \otimes_{\mathcal{D}} F(Z) \arrow[r, "\alpha^{\mathcal{D}}_{F(X),F(Y),F(Z)}"] \arrow[d, "\phi_{X,Y} \otimes_{\mathcal{D}} \mathrm{id}_{F(Z)}"'] & F(X) \otimes_{\mathcal{D}} (F(Y) \otimes_{\mathcal{D}} F(Z)) \arrow[d, "\mathrm{id}_{F(X)} \otimes_{\mathcal{D}} \phi_{Y,Z}"] \\ F(X \otimes_{\mathcal{C}} Y) \otimes_{\mathcal{D}} F(Z) \arrow[d, "\phi_{X \otimes_{\mathcal{C}} Y, Z}"'] & F(X) \otimes_{\mathcal{D}} F(Y \otimes_{\mathcal{C}} Z) \arrow[d, "\phi_{X, Y \otimes_{\mathcal{C}} Z}"] \\ F((X \otimes_{\mathcal{C}} Y) \otimes_{\mathcal{C}} Z) \arrow[r, "F(\alpha^{\mathcal{C}}_{X,Y,Z})"'] & F(X \otimes_{\mathcal{C}} (Y \otimes_{\mathcal{C}} Z)) \end{tikzcd} \quad , \tag{2.2.6}$$

$$\begin{tikzcd} \mathbb{1}_{\mathcal{D}} \otimes_{\mathcal{D}} F(X) \arrow[r, "\iota \otimes_{\mathcal{D}} \mathrm{id}_{F(X)}"] \arrow[d, "\lambda^{\mathcal{D}}_{F(X)}"'] & F(\mathbb{1}_{\mathcal{C}}) \otimes_{\mathcal{D}} F(X) \arrow[d, "\phi_{\mathbb{1}_{\mathcal{C}}, X}"] \\ F(X) & F(\mathbb{1}_{\mathcal{C}} \otimes_{\mathcal{C}} X) \arrow[l, "F(\lambda^{\mathcal{C}}_X)"] \end{tikzcd} \quad , \tag{2.2.7}$$

$$\begin{tikzcd}
F(X)\otimes_{\mathcal{D}}\mathbb{1}_{\mathcal{D}} \arrow[r, "\mathrm{id}_{F(X)}\otimes_{\mathcal{D}}\iota"] \arrow[d, "\rho^{\mathcal{D}}_{F(X)}"'] & F(X)\otimes_{\mathcal{D}}F(\mathbb{1}_{\mathcal{C}}) \arrow[d, "\phi_{X,\mathbb{1}_{\mathcal{C}}}"] \\
F(X) & F(X\otimes_{\mathcal{C}}\mathbb{1}_{\mathcal{C}}) \arrow[l, "F(\rho^{\mathcal{C}}_X)"]
\end{tikzcd} . \tag{2.2.8}$$

A monoidal superfunctor is called *strict* if $\phi_{X,Y}$ and ι are identity morphisms (for all objects X, Y in $\mathcal{C}$).

Definition 2.2.13 (Endofunctor supercategory)**.** Let $\mathcal{C}$ be a supercategory. The *endofunctor supercategory of* $\mathcal{C}$ is denoted $\mathrm{End}(\mathcal{C})$ and defined as follows:

- The objects of $\mathrm{End}(\mathcal{C})$ are superfunctors $F\colon \mathcal{C}\to\mathcal{C}$, also called *superendofunctors on* $\mathcal{C}$;
- The morphisms of $\mathrm{End}(\mathcal{C})$ are supernatural transformations;
- If $\alpha\colon F\to G$ and $\beta\colon G\to H$ are supernatural transformations, their composition is given by $(\beta\circ\alpha)_X=\beta_X\circ\alpha_X$ for all objects X in $\mathcal{C}$ (this is known as the *vertical composition* of β and α);
- The identity morphisms in $\mathrm{End}(\mathcal{C})$ are the identity supernatural transformations.

This endofunctor supercategory has the following natural strict monoidal structure:

- The tensor product on objects is given by composition. That is, if F and G are superendofunctors on $\mathcal{C}$, we set $F\otimes G=F\circ G$, where this denotes the ordinary composition of (super)functors;
- The tensor product on morphisms is given by the *horizontal composition* of supernatural transformations, defined as follows. If $\alpha\colon F\to F'$ and $\beta\colon G\to G'$ are supernatural transformations, then $(\alpha\otimes\beta)_X=\alpha_{G'(X)}\circ F(\beta_X)$ for all objects X in $\mathcal{C}$. Note that, as per the previous bullet point, $\alpha\otimes\beta$ is a supernatural transformation from $F\circ G$ to $F'\circ G'$, and accordingly $(\alpha\otimes\beta)_X$ is a morphism from $F(G(X))$ to $F'(G'(X))$;
- The unit object is the identity superfunctor $\mathrm{id}_{\mathcal{C}}\colon\mathcal{C}\to\mathcal{C}$;
- The associator is the identity;
- The unitors are both the identity;

Lemma 2.2.14. *As defined above,* $\mathrm{End}(\mathcal{C})$ *is indeed a monoidal supercategory.*

Proof: It is straightforward to verify that $\mathrm{End}(\mathcal{C})$ is a category. Towards showing that composition yields a superbifunctor on $\mathrm{End}(\mathcal{C})$, let F and G be superendofunctors on $\mathcal{C}$ and let X be an object of $\mathcal{C}$. We have:

$$\begin{aligned}
(\mathrm{id}_F \otimes \mathrm{id}_G)_X &= (\mathrm{id}_F)_{G(X)} \circ F((\mathrm{id}_G)_X) && \text{By the definition of } \otimes \text{ on morphisms} \\
&= \mathrm{id}_{F(G(X))} \circ F(\mathrm{id}_{G(X)}) \\
&= \mathrm{id}_{F(G(X))} \circ \mathrm{id}_{F(G(X))} && \text{Since } F \text{ is a functor} \\
&= \mathrm{id}_{(F\otimes G)(X)} \circ \mathrm{id}_{(F\otimes G)(X)} && \text{By the definition of } \otimes \text{ on objects} \\
&= \mathrm{id}_{(F\otimes G)(X)} && \text{By the definition of identity morphisms (in } \mathcal{C}) \\
&= (\mathrm{id}_{F\otimes G})_X,
\end{aligned}$$

which shows that $\otimes$ respects identity morphisms. Next, let

$$\begin{gathered}
F \xrightarrow{\alpha} G \xrightarrow{\beta} H \\
F' \xrightarrow[\alpha']{} G' \xrightarrow[\beta']{} H'
\end{gathered}$$

be a collection of composable supernatural transformations. By the definition of a monoidal supercategory, we need to show that:

$$(\beta \otimes \beta') \circ (\alpha \otimes \alpha') = (-1)^{\overline{\beta'}\bar{\alpha}}(\beta \circ \alpha) \otimes (\beta' \circ \alpha').$$

Let X be an object in $\mathcal{C}$. We then compute:

$$\begin{aligned}
&((\beta \otimes \beta') \circ (\alpha \circ \alpha'))_X \\
&= (\beta \otimes \beta')_X \circ (\alpha \circ \alpha')_X \\
&= (\beta_{H'(X)} \circ G(\beta'_X)) \circ (\alpha_{G'(X)} \circ F(\alpha'_X)) && \text{By the definition of } \otimes \text{ on morphisms} \\
&= \beta_{H'(X)} \circ (G(\beta'_X)) \circ \alpha_{G'(X)}) \circ F(\alpha'_X) \\
&= \beta_{H'(X)} \circ ((-1)^{\overline{\beta'_X}\bar{\alpha}} \alpha_{H'(X)} \circ F(\beta'_X) \circ F(\alpha'_X) && \text{Since } \alpha \text{ is a supernatural transformation} \\
&= (-1)^{\overline{\beta'}\bar{\alpha}}(\beta_{H'(X)} \circ \alpha_{H'(X)}) \circ (F(\beta'_X) \circ F(\alpha'_X)) && \text{Since } \overline{\beta'} = \overline{\beta'_X} \text{ by definition} \\
&= (-1)^{\overline{\beta'}\bar{\alpha}}(\beta \circ \alpha)_{H'(X)} \circ F((\beta' \circ \alpha')_X) && \text{Since } F \text{ is a functor} \\
&= (-1)^{\overline{\beta'}\bar{\alpha}}((\beta \circ \alpha) \otimes (\beta' \circ \alpha'))_X, && \text{By the definition of } \otimes \text{ on morphisms}
\end{aligned}$$

as desired. The triangle and pentagon diagrams commute trivially since the associator and both unitors are identities. ∎

2.3 Superalgebras

Definition 2.3.1 (Superalgebra)**.** A *superalgebra* is a super vector space A together with a parity-preserving bilinear multiplication map $A \times A \to A$, typically written as juxtaposition. (The product being parity-preserving means that $\overline{ab} = \bar{a} + \bar{b}$ for all $a, b \in A$.) Unless stated otherwise, all superalgebras in this document are unital and associative.

Remark 2.3.2. Similar to what was discussed in Remark 2.1.2, the distinction between superalgebras and ordinary $\mathbb{Z}_2$-graded algebras only arises when one considers the symmetric monoidal structures for these two kinds of algebras. This difference will play an important role in, for example, Definitions 2.3.3 and 4.2.7.

Definition 2.3.3 (Commutative superalgebra)**.** A superalgebra A is called *commutative* if $ab = (-1)^{\bar{a}\bar{b}} ba$ for all $a, b \in A$. Some authors call such superalgebras *supercommutative* to emphasize the inclusion of the sign term $(-1)^{\bar{a}\bar{b}}$. Note that a commutative superalgebra is precisely a commutative monoid object in $\mathcal{SVec}$.

Example 2.3.4. Every ordinary algebra A can be viewed as a superalgebra by declaring A to be purely even. The algebra multiplication is automatically parity-preserving in this case. One particularly important case is given by taking A to be the one-dimensional algebra $\Bbbk$. Indeed, Proposition 2.3.17 tells us that the *only* possible $\Bbbk$-superalgebra structure on $\Bbbk$ is given by declaring $\Bbbk$ to be purely even.

Definition 2.3.5 (Opposite superalgebra)**.** Let A be a superalgebra. The *opposite superalgebra* A^{op} is defined as follows: as a set, $A^{\mathrm{op}} = \{a^{\mathrm{op}} \mid a \in A\}$, where $-^{\mathrm{op}}$ is a formal symbol used to indicate that the element lies in A^{op} rather than A itself. The $\mathbb{Z}_2$-grading is given by $\overline{a^{\mathrm{op}}} = \bar{a}$. The addition in A^{op} is the same as that in A, i.e. $a^{\mathrm{op}} + b^{\mathrm{op}} = (a + b)^{\mathrm{op}}$. The multiplication is given by $a^{\mathrm{op}} b^{\mathrm{op}} = (-1)^{\bar{a}\bar{b}} (ba)^{\mathrm{op}}$, where ba denotes multiplication in A. It is straightforward to verify that A^{op} is a superalgebra.

Definition 2.3.6. Let A and B be superalgebras. Then $A \otimes B$ is a superalgebra, with the multiplication given by the linear extension of $(a \otimes b)(\alpha \otimes \beta) = (-1)^{\bar{b}\bar{\alpha}} (a\alpha) \otimes (b\beta)$. The identity in $A \otimes B$ is $1_A \otimes 1_B$, and the $\mathbb{Z}_2$-grading is given by $\overline{a \otimes b} = \bar{a} + \bar{b}$.

Example 2.3.7 (Path algebra)**.** Let Q be a quiver, i.e. a directed graph where parallel edges and loops are permitted. The *path algebra of* Q, denoted $\Bbbk Q$, is the vector space with a basis given by the set of paths in Q, with multiplication defined as follows: let $P_1 = e_1 e_2 \cdots e_n$ and $P_2 = f_1 f_2 \cdots f_m$ be paths in Q, with the e_i and f_j being edges in Q. Let v be the terminal vertex of P_1 and w the initial vertex of P_2. If $v = w$, we define $P_1 P_2$ to be the concatenation of the paths, i.e. $P_1 P_2 = e_1 \cdots e_n f_1 \cdots f_m$. If $v \neq w$, we set $P_1 P_2 = 0$. We include one empty path i_v for each vertex $v \in V(Q)$. The initial and terminal vertex of i_v is v. If Q has finitely many vertices, $\Bbbk Q$ is a unital algebra, with the identity element being $\sum_{v \in V(Q)} i_v$. If Q has infinitely many vertices, $\Bbbk Q$ does not contain an identity element. As such, we will only consider path algebras on finitely many vertices.

Path algebras are naturally $\mathbb{N}$-graded. The grade of a path P is simply its length, i.e. the number of edges in the path. It is easy to see that the multiplication in $\Bbbk Q$ preserves the $\mathbb{N}$-grading. Reducing the $\mathbb{N}$-grades mod 2 induces a $\mathbb{Z}_2$-grading on $\Bbbk Q$, making $\Bbbk Q$ into a superalgebra. This $\mathbb{Z}_2$-grading on $\Bbbk Q$ says that a path P has the same parity as its length. For instance, paths of the form $e_1 e_2$ are even, and those of the form $e_1 e_2 e_3$ are odd.

Definition 2.3.8 (Double quiver)**.** Let Γ be a graph. (Unless specified otherwise, we use the term "graph" to mean a simple undirected graph throughout this thesis.) We write $D\Gamma$ for the *double quiver of* Γ. The vertices of $D\Gamma$ are the same as the vertices of Γ. For each edge e in Γ joining vertices u and v, there are two directed edges in $D\Gamma$: one edge from u to v, denoted $u \to v$, and one edge from v to u, denoted $v \to u$. Because Γ is simple, the edges of $D\Gamma$ can be uniquely specified by listing their source and target vertices in this way. We denote paths similarly; for instance, $u \to v \to w$ is the path of length 2 going from u to v to w.

Definition 2.3.9 (Oriented graph)**.** Let Γ be a graph. An *orientation of* Γ is a subquiver O of $D\Gamma$ such that for each pair of vertices u, v that are adjacent in Γ, either the edge $u \to v$ is in O or $v \to u$ is in O, but not both. The subquiver O amounts to choosing an orientation for each edge in Γ. Such a subquiver O is also called an *oriented graph*. We write $\overline{O}$ for the underlying undirected graph of O. That is, $\overline{O} = \Gamma$. Given an oriented graph O, we define the *orientation function* $\epsilon \colon V(O) \times V(O) \to \{1, -1, 0\}$ by

$$\epsilon_{u,v} = \begin{cases} 1 & \text{if } (u \to v) \text{ is in } O, \\ -1 & \text{if } (v \to u) \text{ is in } O, \\ 0 & \text{if } u \text{ and } v \text{ are not connected in } O. \end{cases}$$

Example 2.3.10 (Zigzag superalgebra)**.** Let O be a connected oriented graph with at least 3 vertices. (Unless specified otherwise, we use the term "connected oriented graph" to refer to an oriented graph whose underlying undirected graph is connected throughout this thesis.) The *zigzag superalgebra* $A(O)$ is the quotient of $\Bbbk D\overline{O}$ by the ideal generated by the following elements:

- All paths of the form $u \to v \to w$ with $u \neq w$;
- All elements of the form $\epsilon_{u,v}(u \to v \to u) - \epsilon_{u,w}(u \to w \to u)$.

Put another way, $A(O)$ is obtained from $\Bbbk D\overline{O}$ by declaring that all paths of length 2 going between three distinct vertices are zero, and all 2-cycles with the same initial/terminal vertex are equal or differ by a sign, depending on the orientations of the constituent edges in O. Note that both of these identifications respect path length parity (where 0 is considered to be a path with arbitrary length), and thus $A(O)$ inherits the $\mathbb{Z}_2$-grading from $\Bbbk D\overline{O}$. We will discuss bases for zigzag superalgebras in Lemma 4.3.11.

It is also possible to define zigzag superalgebras over graphs with fewer than 3 vertices; see, for instance, [Cou16, Def. 3.2].

Example 2.3.11 (Endomorphism superalgebra)**.** Let $\mathcal{C}$ be a supercategory and X an object in $\mathcal{C}$. Then $\mathrm{Hom}_{\mathcal{C}}(X, X)$ is a superalgebra called the *endomorphism superalgebra over* X, with the operations being the addition and composition of linear maps.

Definition 2.3.12 (Supermatrices)**.** Let $k, l, m, n \in \mathbb{N}$ and let A be a superalgebra. We define the super vector space of *supermatrices* $\mathrm{Mat}_{k|l,m|n}(A)$ as follows:

- The elements are $(k + l) \times (m + n)$ matrices with entries in A, which we view as block matrices of the form $M = \begin{bmatrix} M_{00} & M_{01} \\ M_{10} & M_{11} \end{bmatrix}$, where M_{00} is $k \times m$, M_{01} is $k \times n$, M_{10} is $l \times m$, and M_{11} is $l \times n$.
- Addition and scaling by elements of $\Bbbk$ are the ordinary operations on matrices.
- The $\mathbb{Z}_2$-grading on $\mathrm{Mat}_{k|l,m|n}(A)$ is as follows: for $a \in A$, $i \in \{1, \ldots, k + l\}$, and $j \in \{1, \ldots, m + n\}$, let $a_{(i,j)}$ denote the $(k + l) \times (m + n)$ matrix with a in position (i, j) and all other entries equal to 0. We set $\overline{a_{(i,j)}} = \bar{a} + \epsilon$, where $\epsilon = \begin{cases} 0 & \text{if } (i, j) \text{ lies in one of the main diagonal blocks,} \\ 1 & \text{if } (i, j) \text{ lies in one of the off-diagonal blocks.} \end{cases}$

It is clear that $\{a_{(i,j)} \mid a \in A \text{ is homogeneous}, i \in \{1, \ldots, k + l\}, j \in \{1, \ldots, m + n\}\}$ spans $\mathrm{Mat}_{k|l,m|n}(A)$, so this definition totally specifies the $\mathbb{Z}_2$-grading. For an alternate perspective on this grading, define a function $p_{d|e} \colon \{1, \ldots, d + e\} \to \mathbb{Z}_2$ by:

$$p_{d|e}(i) = \begin{cases} 0 & \text{if } i \leq d, \\ 1 & \text{if } i \geq d + 1. \end{cases}$$

Then $\overline{a_{(i,j)}} = \bar{a} + p_{k|l}(i) + p_{m|n}(j)$. When the dimensions are clear from context, we will suppress the subscripts and write just $p(i)$ and $p(j)$ instead of $p_{k|l}(i)$ and $p_{m|n}(j)$.

Definition 2.3.13. Let $m, n \in \mathbb{N}$ and let A be a superalgebra. We define $A^{m|n}$ to either be $\mathrm{Mat}_{1|0,m|n}(A)$ or $\mathrm{Mat}_{m|n,1|0}(A)$, depending on what is convenient in a given context. This is analogous to how the vector space $\Bbbk^n$ can be realized as either $1 \times n$ matrices, i.e. row vectors, or as $n \times 1$ matrices, i.e. column vectors.

Remark 2.3.14. Just as every n-dimensional vector space over $\mathbb{k}$ is isomorphic to $\mathbb{k}^n$, every $(m \mid n)$-dimensional super vector space over $\mathbb{k}$ is isomorphic to $\mathbb{k}^{m|n}$. Explicitly, if we write e_i for the vector with a 1 in position i and zeroes elsewhere, and if $\{v_1, v_2, \ldots, v_{m+n}\}$ is a basis of an $(m \mid n)$-dimensional super vector space such that $v_1, \ldots, v_m$ are even and $v_{m+1}, \ldots, v_{m+n}$ are odd, the map $e_i \mapsto v_i$ is an isomorphism.

Similarly, just as ordinary $m \times n$ matrices over A correspond to linear maps from A^n to A^m, $(k \mid l) \times (m \mid n)$ supermatrices over A correspond to linear maps from $A^{m|n}$ to $A^{k|l}$. Given a matrix $M \in \mathrm{Mat}_{k|l,m|n}(A)$, the associated linear map $M\colon A^{m|n} \to A^{k|l}$ is simply given by left multiplication, i.e. $x \mapsto Mx$. It is easy to confirm that every linear map $A^{m|n} \to A^{k|l}$ is of this form. Here, we have realized $A^{m|n}$ and $A^{k|l}$ as column vectors. If we instead use row vectors, the matrix M corresponds to a linear map $A^{k|l} \to A^{m|n}$ given by right multiplication, i.e. $x \mapsto xM$. This perspective can serve to motivate the definition of the $\mathbb{Z}_2$-grading on $\mathrm{Mat}_{k|l,m|n}(A)$; one can confirm that the supermatrix parities specified in Definition 2.3.12 agree with the parities of the associated linear maps.

Lemma 2.3.15. *Fix a superalgebra A and $k, l, m, n, r, s \in \mathbb{N}$, and let $S \in \mathrm{Mat}_{k|l,m|n}(A)$ and $T \in \mathrm{Mat}_{m|n,r|s}(A)$. Then $\overline{ST} = \bar{S} + \bar{T}$. In other words, supermatrix multiplication is parity-preserving.*

Proof: Using the correspondence between supermatrices and linear maps, this follows immediately from the fact that the composition of linear maps is a parity-preserving operation. It is also straightforward to verify the identity directly in the case $S = a_{(i,j)}, T = b_{(k,l)}$; the full result follows from linearity. ∎

Example 2.3.16 (Matrix superalgebra)**.** Let $m, n \in \mathbb{N}$, and let A be a superalgebra. We define the *matrix superalgebra* $\mathrm{Mat}_{m|n}(A)$ to be $\mathrm{Mat}_{m|n,m|n}(A)$, with the product being the ordinary matrix product. Lemma 2.3.15 tells us that this product is parity-preserving, and thus $\mathrm{Mat}_{m|n}(A)$ is a superalgebra.

Proposition 2.3.17. *Let A be a superalgebra with unit 1_A. Then $\overline{1_A} = 0$, i.e. superalgebra units are always even.*

Proof: Since A is a super vector space, 1_A decomposes uniquely into a sum of homogeneous elements, say $1_A = i_0 + i_1$, with $\overline{i_j} = j$. We then compute:

$$a = 1_A a = (i_0 + i_1)a = i_0 a + i_1 a.$$

Since multiplication in A is parity-preserving, $i_0 a$ has the same parity as a and $i_1 a$ has the opposite parity. This implies that $i_1 a = 0$, since otherwise a would be equal to an element whose parity is opposite to that of a (if $i_0 a = 0$) or is inhomogeneous (if $i_0 a \neq 0$). Thus $a = i_0 a$, and this holds for all homogeneous $a \neq 0$. Since all elements in A are linear combinations of homogeneous elements, this implies that $a = i_0 a$ for all $a \in A$. A totally analogous argument shows that i_0 is also a right identity for A. Units in algebras are unique, so $i_0 = 1_A$, which says that 1_A is even. ∎

Corollary 2.3.18. *Let $\mathcal{C}$ be a supercategory and X an object in $\mathcal{C}$. Then the identity morphism id_X is even.*

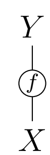
Chapter 3

String Diagrams and Categorical Duality

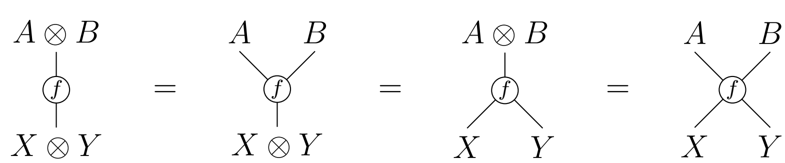
Throughout this chapter, let $\mathcal{C}$ be a strict monoidal supercategory with tensor product $\otimes$ and unit object $\mathbb{1}$.

3.1 Introduction to String Diagrams

Morphisms in $\mathcal{C}$ can be drawn as *string diagrams*, which allow one to use geometric intuition when proving results and investigating new structures. The string diagram associated to a morphism $f\colon X \to Y$ is drawn as follows:

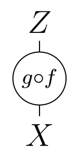


Note that the domain of the morphism appears at the bottom of the diagram, and the codomain at the top. Put another way, string diagrams should be read bottom-to-top. The circle labelled f is called a *coupon*. The line going from X to Y is called a *string* or *strand*. String diagrams for morphisms whose domain and/or codomain are tensor products of objects are often drawn with multiple strands going into or out of the coupon. For instance, a morphism $f\colon X \otimes Y \to A \otimes B$ could be drawn in any of the following ways, depending on the context:

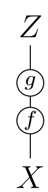


It is typically most useful to draw each tensor factor as its own strand, as was done in the fourth diagram, and this is the convention we use throughout most of the thesis. That being said, it is sometimes more convenient to consider products like $A \otimes B$ as a single object, ignoring the fact that it happens to be a product. In these cases one would represent $A \otimes B$ with a single strand, as was done in the first and third diagrams.

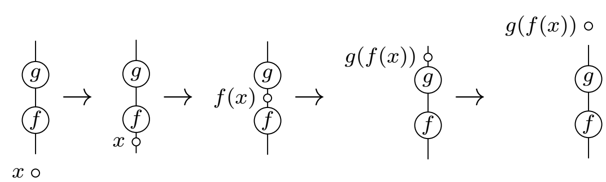
Let $f\colon X \to Y$ and $g\colon Y \to Z$ be a pair of composable morphisms. Using the convention we just defined, $g \circ f$ could be drawn as the following string diagram:



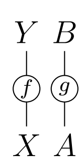
That being said, composite string diagrams can also be drawn in a much more visually meaningful way. We use vertical juxtaposition of string diagrams to represent the composition of morphisms. For f and g as before, the composite $g \circ f$ is drawn as the following string diagram:



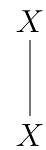
If $\mathcal{C}$ is a concrete category (one where the objects are sets and the morphisms are functions between those sets, e.g. the category of super vector spaces), one can imagine string diagrams as a path that elements travel along, with functions being applied as the element passes through each coupon. To illustrate, let $x \in X$. We can picture the action of the morphism $g \circ f$ as follows (omitting object labels to reduce clutter):



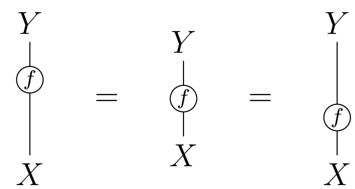
Horizontal juxtaposition of string diagrams represents the tensor product of morphisms. For instance, if $f\colon X \to Y$ and $g\colon A \to B$ are morphisms, then the tensor product $f \otimes g\colon X \otimes A \to Y \otimes B$ is drawn as:



Identity morphisms are drawn as strands without coupons. For instance, the identity on X is drawn as follows:

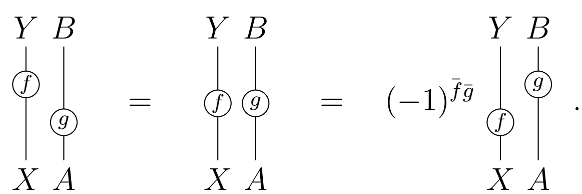


The identity axioms $f \circ \mathrm{id}_X = f = \mathrm{id}_Y \circ f$ correspond to these equalities of string diagrams:



This tells us that we can lengthen or shorten strands without changing the represented morphism. The following result gives another fundamental coherence property for string diagrams, allowing us to draw diagrams in different ways without changing their meaning.

Proposition 3.1.1. *The* superinterchange law *holds in $\mathcal{C}$: for arbitrary morphisms $f\colon X \to Y$ and $g\colon A \to B$, we have:*

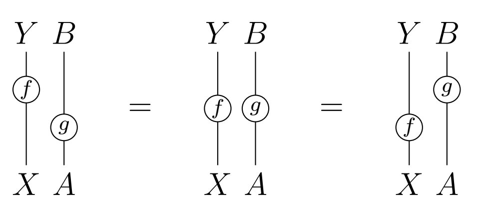


(SINTER)

Thus we may slide coupons past each other, picking up a sign term if both coupons represent odd morphisms. In algebraic notation, this says:

$$(f \otimes \mathrm{id}_B) \circ (\mathrm{id}_X \otimes g) = f \otimes g = (-1)^{\bar{f}\bar{g}}(\mathrm{id}_Y \otimes g) \circ (f \otimes \mathrm{id}_A).$$

If at least one of f and g are even, the sign term vanishes and we obtain the interchange law*:*



(INTER)

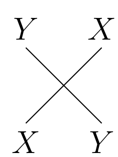
Proof: We compute:

$(f \otimes \mathrm{id}_B) \circ (\mathrm{id}_X \otimes g) = (-1)^{\overline{\mathrm{id}_B}\ \overline{\mathrm{id}_X}}(f \circ \mathrm{id}_X) \otimes (\mathrm{id}_B \circ g)$ Since $\otimes$ is a superbifunctor

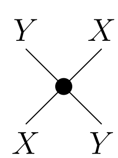
$$
\begin{aligned}
&= (f \circ \mathrm{id}_X) \otimes (\mathrm{id}_B \circ g) && \text{Since identity morphisms are even} \\
&= f \otimes g \\
&= (\mathrm{id}_Y \circ f) \otimes (g \circ \mathrm{id}_A) \\
&= (-1)^{\bar{f}\bar{g}} (\mathrm{id}_Y \otimes g) \circ (f \otimes \mathrm{id}_A). && \text{Since } \otimes \text{ is a superbifunctor}
\end{aligned}
$$

∎

Some categories – such as the Brauer categories introduced in Section 6.1 – are defined directly in terms of string diagrams. In this setting, generating morphisms are given by string diagrams with particular shapes. Such morphisms are usually referred to by simply drawing the corresponding diagram. The domain and codomain of such a morphism are specified the labels on the strands. For instance, this diagram represents a morphism from $X \otimes Y$ to $Y \otimes X$:

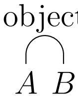


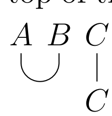
If there are multiple generating morphisms with the same domain and codomain, one can attach dots or other decorations to the strands to differentiate the diagrams. For instance, another morphism from $X \otimes Y$ to $Y \otimes X$ could be drawn as follows:



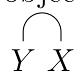
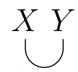
.

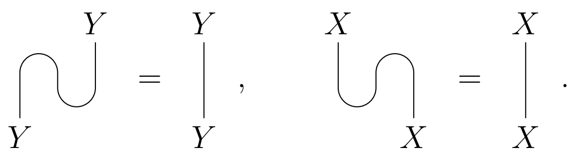
3.2 Categorical Duality

The concept of *duality* appears in many branches of mathematics. The precise definition of duality depends on the context, but there is a categorical definition which covers many cases of interest. In the following (and throughout the rest of the thesis), we adopt the common convention of drawing the unit object $\mathbb{1}$ in string diagrams as empty space, i.e. the lack of a strand. For instance, 

represents a morphism from $A \otimes B$ to $\mathbb{1}$ since the top of the diagram has no open strands (and thus no object labels), and the diagram 

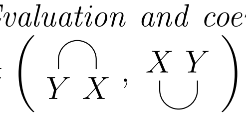
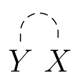
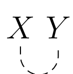
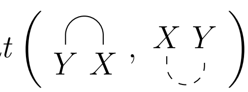
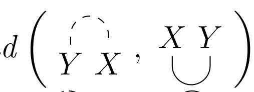
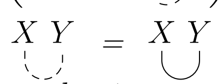
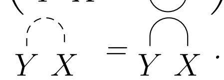
represents a morphism from $\mathbb{1} \otimes C$ to $A \otimes B \otimes C$. Since $\mathcal{C}$ is strict, we have $\mathbb{1} \otimes C = C$, so the latter diagram represents a morphism from C to $A \otimes B \otimes C$. Using this convention, the domain and codomain of a string diagram can be determined just by looking at the labels on the strands; the portions of the diagram corresponding to the unit object may be disregarded.

Definition 3.2.1 (Categorical dual)**.** Let X and Y be objects in $\mathcal{C}$. We say that Y is *the left dual of* X if there exist two even morphisms,  and  , called the *evaluation* and *coevaluation*, respectively, such that the *zigzag equations* hold:

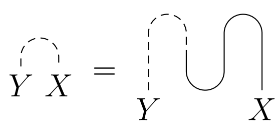


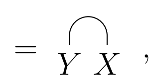
(ZIGZAG)

If this is the case, we also say that X is *the right dual of* Y. Due to their shapes, evaluation morphisms are also called *caps* and coevaluation morphisms are also called *cups*.

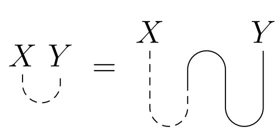
Proposition 3.2.2. *Evaluation and coevaluation morphisms form unique pairs, in the following sense: let*  *be a evaluation-coevaluation pair. Suppose*  *and*  *are two morphisms such that*  *and*  *are also evaluation-coevaluation pairs. Then*  *and*  *. As such, we may unambiguously speak of* the *coevaluation morphism associated to an evaluation morphism, and vice versa.*

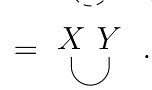
Proof: We have:

 Using (ZIGZAG) on the right

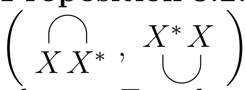
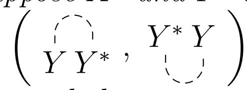
 Using (ZIGZAG) on the left

and similarly,

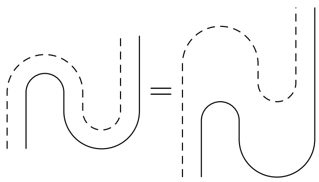
 Using (ZIGZAG) on the right

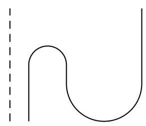
 Using (ZIGZAG) on the left

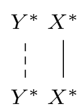
∎

Proposition 3.2.3. *Suppose X^* and Y^* are right duals of X and Y, respectively. Let*  *and*  *be evaluation-coevaluation pairs for these objects. To reduce notational clutter, we omit the object labels for these morphisms in the following. The morphisms*  *form an evaluation-coevaluation pair, showing that $Y^* \otimes X^*$ is right dual to $X \otimes Y$.*

Proof: We need to show that the zigzag equations hold for the nested cup and cap morphisms. Indeed, we have:

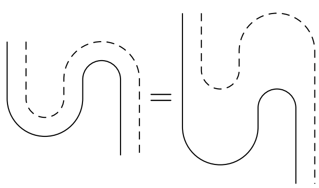
 Using (INTER)

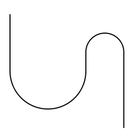
$=$  Using (ZIGZAG) on the dashed strand

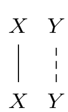
$=$  Using (ZIGZAG) on the solid strand

$= \mathrm{id}_{Y^* \otimes X^*}$.

The other equation is similar:

 Using (INTER)

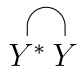
$=$  Using (ZIGZAG) on the dashed strand

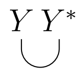
$=$  Using (ZIGZAG) on the solid strand

$= \mathrm{id}_{X \otimes Y}$.

∎

Theorem 3.2.4. *Let X, Y, and Z be objects in $\mathcal{C}$. Suppose Y^* is left dual to Y. Then $\mathrm{Hom}_{\mathcal{C}}(X \otimes Y, Z)$ is isomorphic to $\mathrm{Hom}_{\mathcal{C}}(X, Z \otimes Y^*)$, and this isomorphism is linear and parity-preserving.*

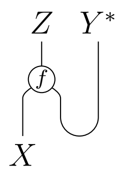
Proof: Let (

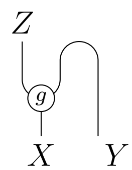
, 

) be an evaluation-coevaluation pair for Y. We define maps

$$\textsf{Cup}_{X,Y,Z} \colon \mathrm{Hom}_{\mathcal{C}}(X \otimes Y, Z) \to \mathrm{Hom}_{\mathcal{C}}(X, Z \otimes Y^*),$$
$$\textsf{Cap}_{X,Y,Z} \colon \mathrm{Hom}_{\mathcal{C}}(X, Z \otimes Y^*) \to \mathrm{Hom}_{\mathcal{C}}(X \otimes Y, Z)$$

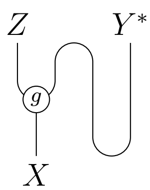
by the following, for all $f \in \mathrm{Hom}_{\mathcal{C}}(X \otimes Y, Z)$ and $g \in \mathrm{Hom}_{\mathcal{C}}(X, Z \otimes Y^*)$:

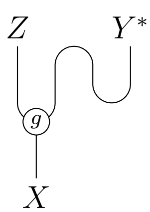
$\textsf{Cup}_{X,Y,Z}(f) =$ 

, $\textsf{Cap}_{X,Y,Z}(g) =$ 

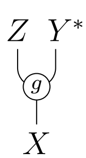
.

We claim that $\textsf{Cup}_{X,Y,Z}$ and $\textsf{Cap}_{X,Y,Z}$ are inverse maps. For one direction of composition, we have:

$\textsf{Cup}_{X,Y,Z}(\textsf{Cap}_{X,Y,Z}(g)) =$ 

$=$ 

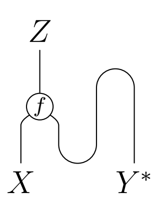
Using (INTER)

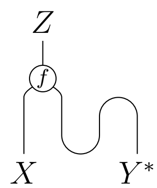
$=$ 

Using (ZIGZAG)

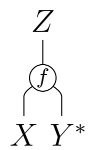
$= g,$

which shows that $\textsf{Cup}_{X,Y,Z} \circ \textsf{Cap}_{X,Y,Z}$ is the identity. The other direction of composition is similar:

$\textsf{Cap}_{X,Y,Z}(\textsf{Cup}_{X,Y,Z}(f)) =$ 

$=$ 

Using (INTER)

$=$ 

Using (ZIGZAG)

$= f,$

and so $\mathbf{Cup}_{X,Y,Z}$ and $\mathbf{Cap}_{X,Y,Z}$ are indeed mutual inverses. This establishes the desired isomorphism of hom-sets. Next, note that both $\mathbf{Cap}_{X,Y,Z}$ and $\mathbf{Cap}_{X,Y,Z}$ are given by taking the tensor product of their arguments with a fixed even morphism and then composing that product with another fixed even morphism. Tensor products and composition are both parity-preserving bilinear operations by the definition of a monoidal supercategory, so $\mathbf{Cup}_{X,Y,Z}$ and $\mathbf{Cap}_{X,Y,Z}$ are linear and parity-preserving. ∎

Chapter 4

Frobenius Superalgebras

From this point on, we adopt the convention that $\Bbbk$ is always taken to be a purely even super vector space unless otherwise indicated. If V is a (super) vector space, we write V' for the dual space of V, i.e. the space of linear maps from V to $\Bbbk$.

4.1 Nondegenerate Bilinear Forms

Definition 4.1.1 (Nondegenerate bilinear form)**.** Let V be a super vector space. A *nondegenerate bilinear form* on V is a bilinear map $\sigma\colon V \times V \to \Bbbk$ satisfying the following equivalent conditions:

(1) $x \mapsto \sigma(-, x)$ is an isomorphism $V \cong V'$.

(2) $x \mapsto \sigma(x, -)$ is an isomorphism $V \cong V'$.

(3) The matrix associated to σ (relative to any basis of V) is invertible.

(4) If $\sigma(y, x) = 0$ for all $y \in V$, then $x = 0$.

(5) If $\sigma(x, y) = 0$ for all $y \in V$, then $x = 0$.

Unless stated otherwise, all nondegenerate bilinear forms in this document are required to be even, meaning that $\overline{\sigma(x, y)} = \bar{x} + \bar{y}$ for all $x, y \in V$.

Lemma 4.1.2. *Let $\sigma\colon V \times V \to \Bbbk$ be a nondegenerate bilinear form. The isomorphisms $V \cong V'$ given by $x \mapsto \sigma(x, -)$ and $x \mapsto \sigma(-, x)$ are both parity-preserving.*

Proof: Let $x, y \in V$. Since σ is even, we have $\overline{\sigma(x, y)} = \bar{x} + \bar{y}$. This holds for all y, so $\sigma(x, -)$ has parity $\bar{x}$. Thus $x \mapsto \sigma(x, -)$ is parity-preserving. The proof for the other isomorphism is totally analogous. ∎

Lemma 4.1.3. *Let $\sigma\colon V \times V \to \Bbbk$ be a nondegenerate bilinear form. Let $v, w \in V$ be homogeneous elements with opposite parities. Then $\sigma(v, w) = 0$.*

Proof: Since σ is even, $\overline{\sigma(v,w)} = \bar{v} + \bar{w} = 1$. The only odd element in $\Bbbk$ is 0, so we must have $\sigma(v, w) = 0$. ∎

For the rest of this section, fix a finite-dimensional super vector space V, a nondegenerate bilinear form $\sigma\colon V \times V \to \Bbbk$, and a homogeneous basis B for V.

Proposition 4.1.4. *There exists a basis for V, called the* left dual basis of B *and denoted $B^\vee := \{b^\vee \mid b \in B\}$, that satisfies the following duality equation (for all $b, c \in B$):*

$$\sigma(b^\vee, c) = \delta_{bc}.$$

Moreover, we have $\overline{b^\vee} = \bar{b}$ for all $b \in B$.

Proof: Since σ is nondegenerate, $x \mapsto \sigma(x, -)$ is an isomorphism $\phi\colon V \to V'$. Let $B' = \{b' \mid b \in B\}$ denote the basis (of V') that is dual to B. That is, we have $b'(c) = \delta_{bc}$ for all $b, c \in B$. For each $b \in B$, define $b^\vee = \phi^{-1}(b')$. For all $b, c \in B$, we then have:

$$\begin{aligned}
\sigma(b^\vee, c) &= (\phi(b^\vee))(c) \\
&= (\phi(\phi^{-1}(b')))(c) \\
&= b'(c) \\
&= \delta_{bc},
\end{aligned}$$

as desired. Recalling that σ is even, $\sigma(b^\vee, b) = 1$ implies $\overline{b^\vee} + \bar{b} = \bar{1} = 0$, so $\overline{b^\vee} = \bar{b}$. ∎

Lemma 4.1.5. *Let $x \in V$. Then $x = \sum_{b \in B} \sigma(b^\vee, x) b = \sum_{b \in B} \sigma(x, b) b^\vee$.*

Proof: Since B is a basis for A, we have $x = \sum_{c \in B} x_c c$ for some scalars $x_c \in \Bbbk$. We then have:

$$\begin{aligned}
\sum_{b \in B} \sigma(b^\vee, x) b &= \sum_{b \in B} \sum_{c \in B} x_c \sigma(b^\vee, c) b && \text{Since } \sigma \text{ is bilinear} \\
&= \sum_{b \in B} \sum_{c \in B} x_c \delta_{bc} b && \text{Since } B \text{ and } B^\vee \text{ are dual} \\
&= \sum_{b \in B} x_b b && \text{Since all other terms vanish} \\
&= x.
\end{aligned}$$

Similarly, since $B^\vee$ is a basis for A, we have $x = \sum_{c \in B} x_c^\vee c^\vee$ for some scalars $x_c^\vee \in \Bbbk$. We then have:

$$\begin{aligned}
\sum_{b\in B} \sigma(x,b)b^\vee &= \sum_{b\in B}\sum_{c\in B} x_c^\vee \sigma(c^\vee, b) b^\vee && \text{Since } \sigma \text{ is bilinear}\\
&= \sum_{b\in B}\sum_{c\in B} x_c^\vee \delta_{bc} b^\vee && \text{Since } B \text{ and } B^\vee \text{ are dual}\\
&= \sum_{b\in B} x_b^\vee b^\vee && \text{Since all other terms vanish}\\
&= x.
\end{aligned}$$

∎

Lemma 4.1.6. *Let $S \subseteq V$ be a set in bijection with B, and write s_b for the element of S corresponding to $b \in B$. The identity $\sigma(s_c, b) = \delta_{bc}$ holds for all $b, c \in B$ if and only if $S = B^\vee$ and $s_b = b^\vee$ for all $b \in B$. In other words, when checking if S is the left dual basis of B, one does not have to explicitly prove that S is a basis.*

Proof: If $S = B^\vee$ and $s_b = b^\vee$ for all $b \in B$, the desired identity holds by definition. So suppose $\sigma(s_c, b) = \delta_{bc}$ holds for all $b, c \in B$. Let $f\colon V \to V$ denote the linear map given by the extension of $f(b) = s_b$ for all $b \in B$. We claim that f is an isomorphism. Using Lemma 4.1.3, the identity $\sigma(s_c, b) = \delta_{bc}$ tells us that $\overline{s_b} = \bar{b}$ for all $b \in B$, so f is parity-preserving. To see that f is injective, let $x \in V$ and suppose $f(x) = 0$. Express x in terms of the basis B as $x = \sum_{b\in B} \alpha_b b$. Let $c \in B$. We then compute:

$$\begin{aligned}
0 &= \sigma\left(f(x), c\right)\\
&= \sum_{b\in B} \alpha_b \sigma\left(f(b), c\right) && \text{Since } f \text{ is linear and } \sigma \text{ is bilinear}\\
&= \sum_{b\in B} \alpha_b \sigma(s_b, c)\\
&= \sum_{b\in B} \alpha_b \delta_{bc}\\
&= \alpha_c,
\end{aligned}$$

which shows that $\alpha_b = 0$ for all $b \in B$, and so $x = 0$. Thus f is injective. Since f is a map from V to itself, this implies that f is an isomorphism. Isomorphisms of vector spaces send bases to bases, and thus $f(B) = S$ is a basis, as desired. ∎

Proposition 4.1.7. *A bilinear form $\sigma\colon V \times V \to \Bbbk$ is nondegenerate if and only if a left dual basis (in the sense defined in Proposition 4.1.4) exists.*

Proof: One direction was proved in Proposition 4.1.4. For the other direction, suppose B is a basis and $B^\vee$ is its left dual. Let $y \in V$ and suppose that $\sigma(x, y) = 0$ for all $x \in V$. Express y in terms of the basis B as $y = \sum_{b \in B} \alpha_b b$. Then, for all $c \in B$, we have:

$$\begin{aligned} 0 &= \sigma(c^\vee, y) \\ &= \sum_{b \in B} \alpha_b \sigma(c^\vee, b) && \text{Since } \sigma \text{ is bilinear} \\ &= \sum_{b \in B} \alpha_b \delta_{bc} && \text{Since } B^\vee \text{ is dual to } B \\ &= \alpha_c. \end{aligned}$$

Thus $y = 0$, and so σ is nondegenerate. ∎

The following result will be used many times in Chapter 6.

Proposition 4.1.8. *The sum $\sum_{b \in B} b \otimes b^\vee$ is independent of the choice of the basis B. That is, if C is an arbitrary second basis for V, then we have $\sum_{b \in B} b \otimes b^\vee = \sum_{c \in C} c \otimes c^\vee$.*

Proof: We compute:

$$\begin{aligned} \sum_{b \in B} b \otimes b^\vee &= \sum_{b \in B} \left(\sum_{c \in C} \sigma(c^\vee, b) c \right) \otimes b^\vee && \text{Using Lemma 4.1.5} \\ &= \sum_{b \in B} \sum_{c \in C} c \otimes \sigma(c^\vee, b) b^\vee && \text{Since } \otimes \text{ is bilinear} \\ &= \sum_{c \in C} c \otimes \sum_{b \in B} \sigma(c^\vee, b) b^\vee && \text{Since } \otimes \text{ is bilinear} \\ &= \sum_{c \in C} c \otimes c^\vee, && \text{Using Lemma 4.1.5} \end{aligned}$$

as desired.

For an alternate, more conceptual proof, recall that $V' \otimes V$ is isomorphic to $\mathrm{Hom}(V, V)$ via the map $f \otimes v \mapsto f(-)v$. Since σ is nondegenerate, we also have

$V \otimes V \cong V' \otimes V$ via $v \otimes w \mapsto \sigma(-, v) \otimes w$. Composing these isomorphisms, we find that $V \otimes V \cong \operatorname{Hom}(V, V)$ via the isomorphism $\alpha\colon v \otimes w \mapsto \sigma(-, v)w$. We then have:

$$\begin{aligned}\alpha\left(\sum_{b\in B} b \otimes b^\vee\right)(x) &= \sum_{b\in B} \sigma(x, b)b^\vee \\ &= x, && \text{Using Lemma 4.1.5}\end{aligned}$$

which shows that $\sum_{b\in B} b \otimes b^\vee = \alpha^{-1}(\mathrm{id}_V)$. Since id_V is obviously independent of the basis B, so is its image in α^{-1}. ∎

4.2 Definitions and Basic Properties

In this section, we present several key definitions and results relating to Frobenius superalgebras. Many examples will be provided in the following section.

Definition 4.2.1 (Frobenius superalgebra)**.** A *Frobenius superalgebra* is a superalgebra A equipped with a nondegenerate bilinear form $\sigma\colon A \times A \to \Bbbk$ satisfying $\sigma(xy, z) = \sigma(x, yz)$ for all $x, y, z \in A$. The map σ is called the *Frobenius form* of A.

Equivalently (as we will prove in Proposition 4.2.4), one may require that A is equipped with a linear functional $\mathrm{tr}\colon A \to \Bbbk$ such that $\ker(\mathrm{tr})$ contains no nonzero left ideals of A. The map tr is called the *trace map* for A. As with nondegenerate bilinear forms, we require trace maps to be even unless stated otherwise.

Definition 4.2.2 (Frobenius algebra)**.** A *Frobenius algebra* is a Frobenius superalgebra whose underlying vector space is purely even. The super-structure plays no role in this case.

Remark 4.2.3. All Frobenius superalgebras must be finite-dimensional. This follows from the fact that the Frobenius form yields an isomorphism between the vector spaces A and A' based on the definition of nondegeneracy. It is well-known that such an isomorphism exists if and only if A is finite-dimensional.

Proposition 4.2.4. *The two definitions of a Frobenius superalgebra in Definition 4.2.1 are indeed equivalent.*

Proof: First, suppose that A is equipped with a Frobenius form σ. Define the associated trace map by $\mathrm{tr}_\sigma(x) = \sigma(x, 1_A)$. This is clearly a linear map since σ is bilinear, and it is even since 1_A and σ are even. Let I be an arbitrary left ideal contained in $\ker(\mathrm{tr}_\sigma)$, and let $a \in I$. Then $ba \in I$ for all $b \in A$. Since $I \subseteq \ker(\mathrm{tr}_\sigma)$, this means that $0 = \mathrm{tr}_\sigma(ba) = \sigma(ba, 1_A)$. Using the defining property of a Frobenius form, we find:

$$\begin{aligned} 0 &= \sigma(ba, 1_A) \\ &= \sigma(b, a1_A) \\ &= \sigma(b, a), \end{aligned}$$

and this holds for all b. Since σ is nondegenerate, this implies $a = 0$. But a was arbitrary, so we conclude that I is the zero ideal, as desired.

In the other direction, suppose A is equipped with a trace map tr. Define the associated Frobenius form by $\sigma_{\mathrm{tr}}(x, y) = \mathrm{tr}(xy)$. This is clearly a bilinear map since tr is linear, and it is even since tr and the superalgebra multiplication are both parity-preserving. Towards showing that σ_{tr} is nondegenerate, suppose that $\sigma_{\mathrm{tr}}(-, y)\colon A \to \Bbbk$ is the zero map. By definition, this means that $\mathrm{tr}(xy) = 0$ for all $x \in A$. Thus $xy \in \ker(\mathrm{tr})$ for all x. Thus $\langle y \rangle$ is a left ideal of A contained in $\ker(\mathrm{tr})$. Since tr is a trace map, we must have $\langle y \rangle = \{0\}$, and thus $y = 0$. This shows that σ_{tr} is nondegenerate, as desired. Moreover, it is easy to see that the maps $\mathrm{tr} \mapsto \sigma_{\mathrm{tr}}$ and $\sigma \mapsto \mathrm{tr}_\sigma$ are inverses, establishing the desired equivalence of definitions. ∎

Remark 4.2.5. Throughout the rest of this document, we primarily use the definition of Frobenius superalgebras in terms of trace maps. Since $(x, y) \mapsto \mathrm{tr}(xy)$ is nondegenerate, all of the results from Section 4.1 can be interpreted in terms of trace maps. For instance, Lemma 4.1.5 tells us that $x = \sum\limits_{b \in B} \mathrm{tr}(b^\vee x)b = \sum\limits_{b \in B} \mathrm{tr}(xb)b^\vee$.

Proposition 4.2.6. *Let A be a superalgebra. Suppose that tr_1 and tr_2 are trace maps for A. Then there exists some invertible $u \in A$ such that $\mathrm{tr}_2(a) = \mathrm{tr}_1(ua)$ for all $a \in A$. Moreover, $a \mapsto \mathrm{tr}_1(ua)$ is always a valid trace map for any choice of invertible u. The parity of the trace map $a \mapsto \mathrm{tr}_1(ua)$ is $\overline{\mathrm{tr}_1} + \bar{u}$.*

Proof: (This argument is adapted from the proof of Proposition 2.1.6 in [Abr97].)

We will prove the claims in reverse order. First, let $u \in A$ be invertible and $\mathrm{tr}_1\colon A \to \Bbbk$ a trace map. Since multiplication in A is bilinear, the map $\mathrm{tr}_2\colon A \to \Bbbk$ given by $\mathrm{tr}_2(x) = \mathrm{tr}_1(ux)$ is linear. Since the multiplication in A is parity-preserving, we have $\overline{\mathrm{tr}_2} = \overline{\mathrm{tr}_1} + \bar{u}$. To see that tr_2 is a valid trace map, let I be an arbitrary left ideal contained in $\ker(\mathrm{tr}_2)$. Let $a \in I$. Then, for all $b \in A$, we have $(u^{-1}b)a \in I$. This then implies that:

$$\begin{aligned} \mathrm{tr}_1(ba) &= \mathrm{tr}_1(uu^{-1}ba) \\ &= \mathrm{tr}_2((u^{-1}b)a) && \text{By the definition of } \mathrm{tr}_2 \\ &= 0. && \text{Since } (u^{-1}b)a \in I \text{ and } I \subseteq \ker(\mathrm{tr}_2) \end{aligned}$$

Thus the left ideal $\langle a\rangle$ is contained in $\ker(\mathrm{tr}_1)$. Since tr_1 is a trace map, we must have $a = 0$. We conclude that $\ker(\mathrm{tr}_2)$ contains no nonzero left ideals of A, as desired.

For the other direction, suppose that $\mathrm{tr}_1, \mathrm{tr}_2 \colon A \to \Bbbk$ are arbitrary trace maps. By the definition of a Frobenius superalgebra and the relationship between trace maps and Frobenius forms, $a \mapsto (b \mapsto \mathrm{tr}_1(ab))$ is an isomorphism $\phi_1 \colon A \to A'$. The map tr_2 is an element of A', so it has a preimage $u \in A$ satisfying $\phi_1(u) = \mathrm{tr}_2$. By the definition of ϕ_1, we thus have $\mathrm{tr}_2(a) = \mathrm{tr}_1(ua)$ for all $a \in A$. It just remains to prove that u is invertible.

Analogously to the previous construction, $a \mapsto (b \mapsto \mathrm{tr}_2(ab))$ is an isomorphism $\phi_2 \colon A \to A'$, so there exists some $v \in A$ satisfying $\phi_2(v) = \mathrm{tr}_1$. This tells us that $\mathrm{tr}_1(a) = \mathrm{tr}_2(va)$ for all $a \in A$. We thus have:

$$\begin{aligned}\mathrm{tr}_1(a) &= \mathrm{tr}_2(va)\\ &= \mathrm{tr}_1(uva),\\ \mathrm{tr}_2(a) &= \mathrm{tr}_1(ua)\\ &= \mathrm{tr}_2(vua),\end{aligned}$$

and so $\phi_1(uv) = \phi_1(1_A)$ and $\phi_2(vu) = \phi_2(1_A)$. Since ϕ_1 and ϕ_2 are isomorphisms, we conclude that $uv = 1_A = vu$, and thus that u is invertible. ∎

Definition 4.2.7 (Symmetric Frobenius superalgebra)**.** We call a bilinear form σ *symmetric* if it satisfies $\sigma(x, y) = (-1)^{\bar{x}\bar{y}}\sigma(y, x)$ for all $x, y \in A$. Similarly, a trace map tr is called symmetric if it satisfies $\mathrm{tr}(xy) = (-1)^{\bar{x}\bar{y}}\,\mathrm{tr}(yx)$ for all $x, y \in A$. A Frobenius superalgebra equipped with a symmetric Frobenius form or trace map is called a *symmetric Frobenius superalgebra.*

If A is an ordinary Frobenius algebra (i.e. a purely even Frobenius superalgebra), the symmetry condition becomes $\sigma(x, y) = \sigma(y, x)$ for all $x, y \in A$, or equivalently $\mathrm{tr}(xy) = \mathrm{tr}(yx)$ for all $x, y \in A$. We call such algebras *symmetric Frobenius algebras.*

Lemma 4.2.8. *Let A be a symmetric Frobenius superalgebra with trace map* tr*, and let B be a homogeneous basis for A. For all $b \in B$, we have $(b^\vee)^\vee = (-1)^{\bar{b}}b$. More concretely, this asserts that* $\mathrm{tr}((-1)^{\bar{b}}bc^\vee) = \delta_{bc}$ *for all $b, c \in B$.*

Proof: We have:

$$\begin{aligned}\mathrm{tr}((-1)^{\bar{b}}bc^\vee) &= (-1)^{\bar{b}}\,\mathrm{tr}(bc^\vee)\\ &= (-1)^{\bar{b}+\overline{bc^\vee}}\,\mathrm{tr}(c^\vee b) && \text{Since tr is symmetric}\\ &= (-1)^{\bar{b}+\overline{bc^\vee}}\delta_{bc}\\ &= (-1)^{\bar{b}+\overline{bb^\vee}}\delta_{bc}\\ &= \delta_{bc}, && \text{Since } \overline{b^\vee} = \bar{b} \text{ and } \bar{b}\bar{b} = \bar{b}\end{aligned}$$

as desired. ∎

Proposition 4.2.9. *Let A be a symmetric Frobenius superalgebra with trace map* tr. *Then the following identities hold for all $x, y \in A$:*

$$\operatorname{tr}(xy) = (-1)^{\bar{x}\bar{y}} \operatorname{tr}(yx) = (-1)^{\bar{x}} \operatorname{tr}(yx) = (-1)^{\bar{y}} \operatorname{tr}(yx).$$

Proof: The first equality restates the definition of a symmetric trace map. If $\bar{x} = \bar{y}$, then $\bar{x} = \bar{y} = \bar{x}\bar{y}$, so the remaining equalities hold. If $\bar{x} \neq \bar{y}$, then $\operatorname{tr}(xy) = \operatorname{tr}(yx) = 0$ as proved in Lemma 4.1.3, and so the identities hold in this case as well. ∎

Definition 4.2.10 (Involution)**.** Let A be a superalgebra. An *involution on A* is an even $\Bbbk$-linear map $-^*\colon A \to A$ that is its own inverse and satisfies $(xy)^* = (-1)^{\bar{x}\bar{y}} y^* x^*$ for all $x, y \in A$. This identity is called the anti-homomorphism property. Note that some authors use the term "involution" to refer to maps satisfying $(xy)^* = x^* y^*$, i.e. ordinary homomorphisms rather than anti-homomorphisms, and would instead call the involutions in this thesis *anti-involutions* or *super-involutions.*

Lemma 4.2.11. *Let A be a superalgebra and $-^*$ an involution on A. Then $1_A^* = 1_A$. Thus involutions on A are precisely self-inverse superalgebra homomorphisms from A to A^{op}.*

Proof: We compute:

$$\begin{aligned}
1_A^* &= 1_A^* 1_A \\
&= ((1_A^* 1_A)^*)^* && \text{Since } -^* \text{ is self-inverse} \\
&= (1_A^* (1_A^*)^*)^* && \text{Using the anti-homomorphism property and the fact that } 1_A \text{ is even} \\
&= (1_A^* 1_A)^* && \text{Since } -^* \text{ is self-inverse} \\
&= (1_A^*)^* \\
&= 1_A. && \text{Since } -^* \text{ is self-inverse}
\end{aligned}$$

∎

Lemma 4.2.12. *Involutions are unique up to isomorphism, in the following sense: let A be a superalgebra and let $-^*$ and $-^\star$ be involutions on A. Then there exists a (parity-preserving) superalgebra automorphism $f\colon A \to A$ such that $f(a^*) = a^\star$ for all $a \in A$.*

Proof: Set $f = -^\star \circ -^*$. Then f is a composition of parity-preserving superalgebra isomorphisms $A \to A^{\mathrm{op}} \to (A^{\mathrm{op}})^{\mathrm{op}} \cong A$, so it's a parity-preserving superalgebra automorphism. Using the fact that $-^*$ is self-inverse, we have $f(a^*) = ((a^*)^*)^\star = a^\star$ for all $a \in A$, as desired. ∎

Definition 4.2.13. Let A be a Frobenius superalgebra with trace map tr. An involution $-^*$ on A is called *compatible with* tr or *trace-compatible* if $\operatorname{tr}(a^*) = \operatorname{tr}(a)$ for all $a \in A$.

Lemma 4.2.14. *Let A be a symmetric Frobenius superalgebra with trace map* tr *and a trace-compatible involution $-^*$. For all $x, y \in A$, we have $\operatorname{tr}(xy) = \operatorname{tr}(x^*y^*)$ and $\operatorname{tr}(xy^*) = \operatorname{tr}(x^*y)$.*

Proof: We compute:

$$\begin{aligned} \operatorname{tr}(xy) &= \operatorname{tr}((xy)^*) && \text{Since tr is compatible with } -^* \\ &= \operatorname{tr}((-1)^{\bar{x}\bar{y}} y^* x^*) && \text{Since } -^* \text{ is an involution} \\ &= \operatorname{tr}(x^* y^*). && \text{Since } -^* \text{ is even and tr is symmetric} \end{aligned}$$

The other identity follows immediately from the first since $-^*$ is self-inverse. ∎

Lemma 4.2.15. *Let A be a Frobenius superalgebra with trace map* tr*, and fix a basis B of A. For all $b \in B$, we have $(b^*)^\vee = (b^\vee)^*$. Here, $(b^*)^\vee$ denotes the dual of b^* relative to the involuted basis $B^* = \{b^* \mid b \in B\}$.*

Proof: Let $b, c \in B$. We compute:

$$\begin{aligned} \operatorname{tr}((c^\vee)^* b^*) &= \operatorname{tr}(c^\vee b) && \text{Using Lemma 4.2.14} \\ &= \delta_{bc}, \end{aligned}$$

which yields the desired result in light of Lemma 4.1.6. ∎

4.3 Examples

Example 4.3.1. The one-dimensional algebra $\Bbbk$ becomes a Frobenius algebra when equipped with the trace map $\operatorname{tr} = \operatorname{id}$. Since $\Bbbk$ is a field, it has no proper ideals. Thus id is a valid trace map. Moreover, id is a symmetric trace map since $\Bbbk$ is commutative. For the same reason, id is an involution on $\Bbbk$, and this involution is obviously trace-compatible.

Example 4.3.2. Consider $\mathbb{C}$ as a 2-dimensional algebra over $\mathbb{R}$. When equipped with the trace map that extracts the real part of a complex number, i.e. $\operatorname{tr}(a+bi) = a$, $\mathbb{C}$ becomes a Frobenius algebra. This map is clearly $\mathbb{R}$-linear, and since $\mathbb{C}$ is a field, tr is a symmetric trace map, as in the previous example. Complex conjugation, given by $(a+bi)^* = a - bi$, is a well-known involution on $\mathbb{C}$ that is compatible with tr.

Example 4.3.3 (Quaternion algebra)**.** Let $a, b \in \Bbbk$ be nonzero. The *quaternion algebra* $(a,b)_{\Bbbk}$ is the 4-dimensional associative unital algebra over $\Bbbk$ generated by the elements $\{i,j,k\}$ subject to the relations $i^2 = a, j^2 = b, ij = k$, and $ji = -k$. The two most widely-studied quaternion algebras are the *(Hamiltonian) quaternions* $\mathbb{H} = (-1,-1)_{\mathbb{R}}$ and the *split quaternions* $(-1,1)_{\mathbb{R}} \cong \operatorname{Mat}_{2,2}(\mathbb{R})$. Using Frobenius's theorem on division algebras and the Wedderburn-Artin theorem, one can prove that all quaternion algebras over $\mathbb{R}$ are isomorphic to either $\mathbb{H}$ or the split quaternions (see e.g. [ZY08, §2.3] for more details).

When equipped with the trace map given by $\operatorname{tr}(c + di + ej + fk) = c$, the quaternion algebra $(a,b)_{\Bbbk}$ becomes a Frobenius algebra. This map is clearly $\Bbbk$-linear, and it is straightforward to show that its kernel contains no nontrivial left ideals. This trace map is symmetric. To prove this, it suffices to show that $\operatorname{tr}(xy) = \operatorname{tr}(yx)$ for all $x, y \in \{1, i, j, k\}$. This is trivial if $x = y$ or if either of x or y is equal to 1. Due to the defining relations for $(a,b)_{\Bbbk}$, all other products of basis elements have zero $\Bbbk$ component, and thus if $x, y \in \{i,j,k\}$ are distinct, they satisfy $\operatorname{tr}(xy) = 0 = \operatorname{tr}(yx)$.

Quaternionic conjugation, given by $(c + di + ej + fk)^* = c - di - ej - fk$, is an involution on $(a,b)_{\Bbbk}$ that is compatible with tr. One can easily verify that this is an involution by direct computation, i.e. by computing $(xy)^*$ and y^*x^* for all $x, y \in \{1, i, j, k\}$ and showing that they are equal.

Definition 4.3.4 (Supertrace)**.** Let A be a superalgebra and $m, n \in \mathbb{N}$. The *supertrace* of a matrix $M = \begin{bmatrix} M_{00} & M_{01} \\ M_{10} & M_{11} \end{bmatrix} \in \operatorname{Mat}_{m|n}(A)$, denoted $\operatorname{str}(M)$, is defined to be:

$$\operatorname{str}(M) = \operatorname{Tr}(M_{00}) - (-1)^{\overline{M_{11}}} \operatorname{Tr}(M_{11}), \tag{4.3.1}$$

where Tr is the ordinary matrix trace. This definition can be equivalently stated as:

$$\operatorname{str}(M) = \sum_{i=1}^{m+n} (-1)^{p(i)(\overline{m_{ii}}+1)} m_{ii},$$

where m_{ij} denotes the (i,j)-entry of M. In particular, this tells us that for all $a \in A$ and $i, j \in \{1, \dots, m+n\}$ we have $\operatorname{str}(a_{(i,j)}) = (-1)^{p(i)(\bar{a}+1)} \delta_{ij} a$.

Example 4.3.5. Let A be a Frobenius superalgebra with trace map tr_A. For any $m, n \in \mathbb{N}$, $\operatorname{Mat}_{m|n}(A)$ becomes a Frobenius superalgebra when equipped with the trace map given by $\operatorname{tr}(S) = \operatorname{tr}_A(\operatorname{str}(S))$. This trace map is symmetric if and only if tr_A is. In the special case $A = \Bbbk$ (with $\mathrm{tr}_A = \mathrm{id}_\Bbbk$), we find that $\mathrm{Mat}_{m|n}(\Bbbk)$ is a symmetric Frobenius superalgebra, with its trace map being the supertrace.

Definition 4.3.6. Let A be a superalgebra. We define an even linear map $\rho\colon A \to A$ by $\rho(a) = (-1)^{\bar{a}}a$. If M is a (super)matrix with entries from A, we write $\rho(M)$ for the matrix obtained from M by applying ρ entrywise.

Example 4.3.7 (Generalized orthosymplectic involution)**.** Let A be a Frobenius superalgebra with trace map tr_A and a compatible involution $-^*$. There is an involution on $\mathrm{Mat}_{m|2n}(A)$ that is compatible with the trace map given by $\mathrm{tr} = \mathrm{tr}_A \circ \mathrm{str}$ called the *generalized orthosymplectic involution*. It is denoted $-^\dagger$ and has the following action:

$$\begin{bmatrix} X_{00} & X_{01} & X_{02} \\ X_{10} & X_{11} & X_{12} \\ X_{20} & X_{21} & X_{22} \end{bmatrix}^\dagger = \begin{bmatrix} X_{00}^{\top *} & -\rho(X_{20}^{\top *}) & \rho(X_{10}^{\top *}) \\ \rho(X_{02}^{\top *}) & X_{22}^{\top *} & -X_{12}^{\top *} \\ -\rho(X_{01}^{\top *}) & -X_{21}^{\top *} & X_{11}^{\top *} \end{bmatrix},$$

where $-^{\top *} = -^* \circ -^\top$. We will define $-^\dagger$ in a more abstract way in Section 5.2, prove that it is an involution in Section 5.3, and show that $-^\dagger$ can be computed as indicated above in Section 5.5.

Example 4.3.8 (Transpose involution)**.** Let A be a Frobenius superalgebra with trace map tr_A and an involution $-^*$. There is an involution on $\mathrm{Mat}_{n|n}(A)$ called the *transpose involution*, denoted $-^{\mathrm{trp}}$, and defined by:

$$X^{\mathrm{trp}} = \begin{bmatrix} X_{00} & X_{01} \\ X_{10} & X_{11} \end{bmatrix}^{\mathrm{trp}} := \begin{bmatrix} X_{11}^{\top *} & -\rho(X_{01}^{\top *}) \\ \rho(X_{10}^{\top *}) & X_{00}^{\top *} \end{bmatrix}.$$

This involution is generally not compatible with the trace map given by $\mathrm{tr} = \mathrm{tr}_A \circ \mathrm{str}$, since we have:

$$\begin{aligned} \mathrm{tr}(X^{\mathrm{trp}}) &= \mathrm{tr}_A(\mathrm{Tr}(X_{11}^{\top *}) - (-1)^{\overline{X_{00}}}\,\mathrm{Tr}(X_{00}^{\top *})), \\ \mathrm{tr}(X) &= \mathrm{tr}_A(\mathrm{Tr}(X_{00}^{\top *}) - (-1)^{\overline{X_{11}}}\,\mathrm{Tr}(X_{11}^{\top *})), \end{aligned}$$

and these two expressions are usually not equal, even if $-^*$ is compatible with tr_A.

Remark 4.3.9. As proved in [Rac98, Prop. 13, Prop. 14], the generalized orthosymplectic involution and transpose involution are the only involutions on $\mathrm{Mat}_{m|2n}(D)$ (up to isomorphism), where D is a central division superalgebra with involution over $\Bbbk$ and $m, n \neq 0$. In particular, this implies that there are no involutions on $\mathrm{Mat}_{m|2n}(\Bbbk)$ in the case where $m, n \neq 0$, $m \neq n$, and n is odd. In the case where either m or n is zero, $-^{\mathrm{st}\,*}$ is an involution on $\mathrm{Mat}_{m|2n}(D)$. Note that $\mathrm{Mat}_{n|0}(D)$ and $\mathrm{Mat}_{0|n}(D)$ are equal as superalgebras, but their natural supertrace maps (as specified in Definition 4.3.4) differ by a sign term.

Example 4.3.10 (Finite group algebra)**.** Let G be a finite group with identity 1_G, and let $\Bbbk[G]$ denote the group algebra of G. This algebra becomes a symmetric Frobenius algebra when equipped with the trace map $\mathrm{tr}\colon \Bbbk[G] \to \Bbbk$ given by $\mathrm{tr}\left(\sum_{g\in G} \alpha_g g\right) = \alpha_{1_G}$, where $\alpha_g \in \Bbbk$ are scalars. Note that the finiteness of G is required for $\Bbbk[G]$ to be a Frobenius algebra in light of Remark 4.2.3. There is a natural involution on $\Bbbk[G]$ given by the $\Bbbk$-linear extension of $g \mapsto g^{-1}$. This involution is compatible with tr since for all $g \in G$, we have $g = 1_G$ if and only if $g^{-1} = 1_G$.

Lemma 4.3.11. *Recall the zigzag superalgebra $A(O)$ defined in Example 2.3.10. Let O be a connected oriented graph with 3 or more vertices. Any subset of the double quiver $D\overline{O}$ consisting of all empty paths, paths of length 1, and an arbitrary choice of 2-cycle $u \to v \to u$ for each vertex $u \in V(O)$ is a basis of $A(O)$.*

Proof: See the proof of Proposition 3.7 in [Cou16]; that paper deals with non-super skew zigzag algebras, but the same arguments work for the zigzag superalgebras studied in this document. ∎

Example 4.3.12. Let O be a connected oriented graph with 3 or more vertices. Define a linear map $\mathrm{tr}\colon A(O) \to \Bbbk$ by the linear extension of

$$\mathrm{tr}(P) = \begin{cases} \epsilon_{u,v} & \text{if } P = (u \to v \to u) \text{ for some } u, v \in V(O), \\ 0 & \text{otherwise} \end{cases}$$

for all paths $P \in A(O)$, where ϵ is the orientation function from Definition 2.3.9. Note: tr is well-defined because if $(u \to v \to u) = (a \to b \to a)$ in $A(O)$, we must have $\epsilon_{u,v} = \epsilon_{a,b}$ based on the definition of $A(O)$. It is straightforward to verify that tr is a valid trace map. Moreover, one can check that tr is symmetric by computing $\mathrm{tr}(PQ)$ and $\mathrm{tr}(QP)$ in the cases where P and Q are empty paths, paths of length 1, and/or 2-cycles.

Definition 4.3.13. Let O be an oriented graph. An *involution on O* is a self-inverse function $-^*\colon V(O) \to V(O)$ such that $\epsilon_{u^*,v^*} = -\epsilon_{u,v}$ for all $u, v \in V(O)$. That is, $u \to v$ is an edge in O if and only if $v^* \to u^*$ is an edge in O. Note that this involution condition can be equivalently expressed as: $\epsilon_{u^*,v^*} = \epsilon_{v,u}$ for all $u, v \in V(O)$.

Example 4.3.14. For all $n \in \mathbb{N}$, the directed path graph P_n and directed cycle graph C_n both have a natural involution. Write $\{1, 2, \dots, n\}$ for the vertices of P_n and C_n, with the edges of P_n being $E := \{i \to (i+1) \mid i \in \{1, \dots, n-1\}\}$ and the edges of C_n

being $E \cup \{n \to 1\}$. The involution $-^*$ on both P_n and C_n is given by $i^* = n+1-i$. Written another way, $-^*$ maps vertices as follows:

$$\begin{cases} 1 \mapsto n, \\ 2 \mapsto n-1, \\ 3 \mapsto n-2, \\ \vdots \\ n \mapsto 1. \end{cases}$$

Let O be any oriented graph. The *opposite graph* O^{op} is obtained from O by reversing the direction of all of the edges in O. For each vertex $v \in V(O)$, we write v^{op} for the associated vertex in O^{op}. The disjoint union $O + O^{\mathrm{op}}$ has a natural involution given by $v^* = v^{\mathrm{op}}$ and $(v^{\mathrm{op}})^* = v$.

Example 4.3.15. Let O be an oriented graph with 3 or more vertices. Let $-^*$ be an involution on O. This involution naturally induces an involution on the zigzag superalgebra $A(O)$ given by the linear extension of:

$$\begin{aligned} i_v^* &= i_{v^*}, \\ (u \to v)^* &= v^* \to u^*, \\ (u \to v \to u)^* &= -(u^* \to v^* \to u^*), \end{aligned}$$

recalling that i_v denotes the empty path at vertex v. It is easy to see that this map is self-inverse since $-^*$ is self-inverse on vertices. By Lemma 4.3.11, $A(O)$ is spanned by the empty paths, paths of length 1, and 2-cycles in $D\overline{O}$, so this specifies $-^*$ on the entire zigzag superalgebra. To see that $-^*$ is well-defined, note that we have:

$$\begin{aligned} &(\epsilon_{u,v}(u \to v \to u) - \epsilon_{u,w}(u \to w \to u))^* \\ &= -\epsilon_{u,v}(u^* \to v^* \to u^*) + \epsilon_{u,w}(u^* \to w^* \to u^*) \\ &= \epsilon_{u^*,v^*}(u^* \to v^* \to u^*) - \epsilon_{u^*,w^*}(u^* \to w^* \to u^*) && \text{Since } -^* \text{ is an involution on } O \\ &= 0, && \text{By the definition of } A(O) \end{aligned}$$

and so $-^*$ respects the relations that define $A(O)$. To prove that $-^*$ is an involution, one needs to show that $(PQ)^* = (-1)^{\bar{P}\bar{Q}}QP$, where P and Q range over all empty paths, paths of length 1, and 2-cycles in $D\overline{O}$. This is a straightforward but tedious verification.

The involution $-^*$ on $A(O)$ is compatible with the trace map defined in Example 4.3.12. To see why this is the case, note that we have:

$$\begin{aligned} \operatorname{tr}((u \to v \to u)^*) &= \operatorname{tr}(-(u^* \to v^* \to u^*)) \\ &= -\epsilon_{u^*,v^*} && \text{By the definition of tr} \end{aligned}$$

$$\begin{aligned} &= \epsilon_{u,v} && \text{Since } -^* \text{ is an involution on } O \\ &= \operatorname{tr}(u \to v \to u). \end{aligned}$$

Since tr is 0 on paths that are not 2-cycles, this shows that $-^*$ is compatible with tr.

Example 4.3.16 (Zigzag algebra)**.** Given a graph Γ, one can define the (non-super) *zigzag algebra* $A(\Gamma)$ by replacing the relation $\epsilon_{u,v}(u \to v \to u) = \epsilon_{u,w}(u \to w \to u)$ in Example 2.3.10 with the unsigned equivalent, i.e. $(u \to v \to u) = (u \to w \to u)$. Note that zigzag algebras are not simply purely even zigzag superalgebras, as is the case for matrix algebras and superalgebras; all zigzag superalgebras have odd elements. When equipped with the trace map given by the linear extension of

$$\operatorname{tr}(P) = \begin{cases} 1 & \text{if } P \text{ is a 2-cycle in } \Gamma, \\ 0 & \text{otherwise,} \end{cases}$$

$A(\Gamma)$ becomes a symmetric Frobenius algebra. There is a natural involution on $A(\Gamma)$ given by reversing the direction of the paths. That is, the involution $-^*$ is the identity on paths of length 0 or 2 (since the only nonzero paths of length 2 in $A(\Gamma)$ are cycles), and $(u \to v)^* = v \to u$. It is easy to see that this action yields a trace-compatible involution.

Example 4.3.17 (Truncated polynomial algebra)**.** The truncated polynomial algebra $P = \Bbbk[x]/(x^n)$ becomes a symmetric Frobenius algebra when equipped with the trace map given by the linear extension of $\operatorname{tr}(x^i) = \delta_{i,n-1}$. That is, tr extracts the coefficient on the x^{n-1} term of its argument. This trace map is symmetric and compatible with the involution $-^* = \mathrm{id}$ since P is commutative.

Chapter 5

Supermodules and Lie Superalgebras

Throughout this chapter, fix a symmetric Frobenius superalgebra A with trace map tr and an involution $-^*\colon A \to A$ that is compatible with tr.

5.1 Supermodules

Definition 5.1.1 (Supermodule)**.** A *left supermodule over* A (resp. *right supermodule over* A) is a $\mathbb{Z}_2$-graded left (resp. right) module over A whose A-action is parity-preserving. That is, if V is a left supermodule over A, we must have $\overline{av} = \bar{a} + \bar{v}$ for all $a \in A$ and $v \in V$, and similarly for right supermodules.

Definition 5.1.2 (Supermodule homomorphism)**.** Let V and W be left supermodules over A. An *A-supermodule homomorphism* from V to W is a function $f\colon V \to W$ that satisfies $f(av_1 + v_2) = (-1)^{\bar{a}\bar{f}} af(v_1) + f(v_2)$ for all $a \in A$ and $v_1, v_2 \in V$. If V and W are instead right supermodules over A, an A-supermodule homomorphism from V to W is a function $f\colon V \to W$ that satisfies $f(v_1 a + v_2) = f(v_1)a + f(v_2)$ for all $a \in A$ and $v_1, v_2 \in V$.

Example 5.1.3. The superalgebra A is naturally both a left and right supermodule over itself, with the module action coinciding with the algebra multiplication. More generally, for any $m, n \in \mathbb{N}$, $A^{m|n}$ is both a left and right supermodule over A, with the actions given by $a \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{m+n} \end{bmatrix} = \begin{bmatrix} ab_1 \\ ab_2 \\ \vdots \\ ab_{m+n} \end{bmatrix}$ and $\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{m+n} \end{bmatrix} a = \begin{bmatrix} b_1 a \\ b_2 a \\ \vdots \\ b_{m+n} a \end{bmatrix}$, respectively.

Definition 5.1.4. Let V be a right A-supermodule. We define $\mathrm{Hom}_A(V, A)$ to be the set of all A-supermodule homomorphisms from V to A (considered as a right supermodule over itself). This set has a natural *left* A-supermodule structure, with the A-action given by $(af)(v) = af(v)$ for all $f \in \mathrm{Hom}_A(V, A), v \in V, a \in A$. From this point forward, we will write $\mathrm{Hom}^L_A(V, A)$ for this supermodule to emphasize the fact that it is a left supermodule and to disambiguate it from the supermodule specified in the following definition.

Definition 5.1.5. Let W be a left A-supermodule. There is a natural right A-supermodule structure on W given by $wa := (-1)^{\bar{a}\bar{w}}a^*w$ for all $a \in A, w \in V$. For all $a, b \in A$ and $w \in W$ we have:

$$\begin{aligned}
w(ab) &= (-1)^{\overline{ab}\bar{w}}(ab)^*w \\
&= (-1)^{\bar{a}\bar{w}+\bar{b}\bar{w}+\bar{a}\bar{b}}(b^*a^*)w \\
&= (-1)^{\bar{a}\bar{w}+\bar{b}\bar{w}+\bar{a}\bar{b}}b^*(a^*w) && \text{Since } W \text{ is a left } A\text{-supermodule} \\
&= (-1)^{\bar{b}\bar{w}+\bar{a}\bar{b}}b^*(wa) && \text{By the definition of the right action on } W \\
&= (wa)b. && \text{By the definition of the right action on } W
\end{aligned}$$

The fact that this right action is distributive follows from the fact that $-^*$ is linear and the left action on W is distributive. Thus $(w, a) \mapsto (-1)^{\bar{a}\bar{w}}a^*w$ is a valid right A-action.

The case $W = \mathrm{Hom}^L_A(V, A)$ for some right A-supermodule V is of particular interest. We write $\mathrm{Hom}^R_A(V, A)$ for the right supermodule obtained from $\mathrm{Hom}^L_A(V, A)$ via the above definition.

Definition 5.1.6 (Lie superalgebra)**.** A *Lie superalgebra* is a super vector space $\mathfrak{g}$ together with a parity-preserving bilinear operation $[-, -]\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ called the *Lie superbracket*. This superbracket must be super skew-symmetric, meaning that $[X, Y] = -(-1)^{\bar{X}\bar{Y}}[Y, X]$ for all $X, Y \in \mathfrak{g}$, and it must satisfy the super Jacobi identity:

$$[X, [Y, Z]] = [[X, Y], Z] + (-1)^{\bar{X}\bar{Y}}[Y, [X, Z]]$$

for all $X, Y, Z \in \mathfrak{g}$.

If A is an associative superalgebra, one can define a Lie superbracket on A called the *supercommutator* by setting $[X, Y] = XY - (-1)^{\bar{X}\bar{Y}}YX$. It is straightforward to verify that this definition yields a valid superbracket.

Definition 5.1.7. Let $\mathfrak{g}$ be a Lie superalgebra. A *left supermodule over* $\mathfrak{g}$ is a super vector space V together with a bilinear map $\cdot\colon \mathfrak{g} \times V \to V$ that satisfies

$$[X, Y] \cdot v = X \cdot (Y \cdot v) - (-1)^{\bar{X}\bar{Y}}Y \cdot (X \cdot v) \tag{5.1.1}$$

for all $X, Y, \in \mathfrak{g}$ and $v \in V$. When the context is clear, we sometimes omit the $\cdot$ and write the $\mathfrak{g}$-action as juxtaposition.

If A is an associative superalgebra and $\mathfrak{g}$ is the Lie superalgebra obtained by equipping A with the supercommutator, it is straightforward to show that any left supermodule over A is also a left supermodule over $\mathfrak{g}$.

Example 5.1.8. Let $\mathfrak{g}$ be a Lie superalgebra. Any super vector space V becomes a left $\mathfrak{g}$-supermodule when equipped with the trivial action, namely $X \cdot v = 0$ for all $X \in \mathfrak{g}, v \in V$.

Definition 5.1.9. Let $\mathfrak{g}$ be a Lie superalgebra, and let V and W be left $\mathfrak{g}$-supermodules. Then $V \otimes W$ becomes a left $\mathfrak{g}$-supermodule when equipped with the following action (for all $X \in \mathfrak{g}$ and $v \otimes w \in V \otimes W$):

$$X \cdot (v \otimes w) = (X \cdot v) \otimes w + (-1)^{\bar{X}\bar{v}} v \otimes (X \cdot w).$$

It is straightforward to verify that this definition yields a valid supermodule action.

Definition 5.1.10. Let $\mathfrak{g}$ be a Lie superalgebra. We define $\mathfrak{g}$-mod to be the monoidal supercategory whose objects are left $\mathfrak{g}$-supermodules and whose morphisms are $\mathfrak{g}$-equivariant $\Bbbk$-linear maps. That is, if V and W are $\mathfrak{g}$-supermodules, a morphism $f \colon V \to W$ is a linear map such that for all $X \in \mathfrak{g}$ and $v \in V$ we have:

$$f(X \cdot v) = (-1)^{\bar{X}\bar{f}} X \cdot f(v). \tag{5.1.2}$$

The tensor product for $\mathfrak{g}$-mod is given by Definition 5.1.9. The unit object is $\Bbbk$, equipped with trivial action specified in Example 5.1.8. The associator and unitors are the same as those for the category of super vector spaces.

5.2 Sesquilinear and Hermitian Forms

Throughout the rest of the chapter, fix a right A-supermodule V.

Definition 5.2.1 (Sesquilinear form, Hermitian form)**.** A parity-preserving $\Bbbk$-bilinear map $\varphi \colon V \times V \to A$ is said to be a *sesquilinear form* if it satisfies the following identity for all $a, b \in A$ and $v, w \in V$:

$$\varphi(va, wb) = (-1)^{\bar{a}\bar{v}} a^* \varphi(v, w) b.$$

A *Hermitian form* on V is a sesquilinear form φ that additionally satisfies the following identity for all $v, w \in V$:

$$\varphi(v, w) = (-1)^{\bar{v}\bar{w}} \varphi(w, v)^*.$$

Definition 5.2.2 (Nondegenerate form, unimodular form)**.** Let $\varphi \colon V \times V \to A$ be a sesquilinear form. For each $v \in V$, define a function $\varphi_v \colon V \to A$ by $\varphi_v(w) = \varphi(v, w)$. Note that each φ_v has parity $\bar{v}$ since φ is parity-preserving. Moreover, each φ_v is a homomorphism of right A-supermodules since the map is clearly additive and for any $a \in A, w \in V$ we have:

$$\varphi_v(wa) = \varphi(v, wa)$$

$$
\begin{aligned}
&= \varphi(v,w)a && \text{Since } \varphi \text{ is sesquilinear}\\
&= \varphi_v(w)a.
\end{aligned}
$$

Thus $v \mapsto \varphi_v$ is a map from V to $\mathrm{Hom}^R_A(V,A)$. In fact, this map is a morphism of right A-supermodules since:

$$
\begin{aligned}
\varphi_{va+w}(u) &= \varphi(va+w,u)\\
&= (-1)^{\bar{v}\bar{a}}a^*\varphi(v,u)+\varphi(w,u) && \text{Since } \varphi \text{ is sesquilinear}\\
&= (-1)^{\bar{v}\bar{a}}a^*\varphi_v(u)+\varphi_w(u)\\
&= (\varphi_v a)(u)+\varphi_w(u).
\end{aligned}
$$

The form φ is said to be *nondegenerate* if the map $v \mapsto \varphi_v$ is injective, and *unimodular* if the map is bijective.

Lemma 5.2.3. *Let $\varphi\colon V\times V\to A$ be a nondegenerate $\Bbbk$-bilinear map. An element $w\in V$ satisfies $\varphi(v,w)=0$ for all $v\in V$ if and only if $v=0$.*

Proof: If $v=0$, the result follows immediately from the bilinearity of φ. In the other direction, suppose that $\varphi(v,w)=0$ for all $v\in V$. In other words, φ_v is the zero map from V to A. Since φ is nondegenerate, the map $v\mapsto\varphi_v$ is injective, and thus we conclude that $v=0$. ∎

Lemma 5.2.4. *Let $\varphi\colon V\times V\to A$ be a nondegenerate $\Bbbk$-bilinear map. Let $v,w\in V$. If $\varphi(v,u)=\varphi(w,u)$ for all $u\in V$, then $v=w$.*

Proof: In such a case, we have:

$$
\begin{aligned}
0 &= \varphi(v,u)-\varphi(w,u)\\
&= \varphi(v-w,u). && \text{Since } \varphi \text{ is sesquilinear}
\end{aligned}
$$

Lemma 5.2.3 then implies that $v-w=0$, which yields $v=w$. ∎

Lemma 5.2.5. *If $\varphi\colon V\times V\to A$ is unimodular, then the composite map*

$$\mathrm{tr}\circ\varphi\colon V\times V\to\Bbbk$$

is a nondegenerate $\Bbbk$-bilinear form on V. If φ is Hermitian, then $\mathrm{tr}\circ\varphi$ is symmetric.

Proof: Bilinearity follows from the bilinearity of φ and the linearity of tr. Suppose first that φ is unimodular. Towards showing that $\operatorname{tr} \circ \varphi$ is nondegenerate, let $v \in V$ be nonzero. By Lemma 5.2.3, there exists some $w \in V$ such that $\varphi(v, w) \in A$ is nonzero. In turn, since tr is a trace map, there exists some $a \in A$ such that $\operatorname{tr}(\varphi(v, w)a) \neq 0$. Using such an a, we have:

$$\begin{aligned} \operatorname{tr}(\varphi(v, wa)) &= \operatorname{tr}(\varphi(v, w)a) && \text{Since } \varphi \text{ is sesquilinear} \\ &\neq 0, \end{aligned}$$

and thus $v \mapsto \operatorname{tr}(\varphi(v, -))$ is an injective map from V to V'. Since V and V' have the same dimension as $\Bbbk$-vector spaces, we find that this map must also be surjective, and thus $\operatorname{tr} \circ \varphi$ is nondegenerate. Suppose now that φ is Hermitian. To see that the composite form is symmetric, let $v, w \in V$ be homogeneous. Then we have:

$$\begin{aligned} \operatorname{tr}(\varphi(v, w)) &= \operatorname{tr}((-1)^{\bar{v}\bar{w}}(\varphi(w, v))^*) && \text{Since } \varphi \text{ is Hermitian} \\ &= (-1)^{\bar{v}\bar{w}} \operatorname{tr}(\varphi(w, v)), && \text{Since } -^* \text{ is compatible with tr} \end{aligned}$$

as desired. ∎

From this point forward, fix a unimodular Hermitian form $\varphi \colon V \times V \to A$.

Definition 5.2.6. Let $\operatorname{End}_A(V)$ denote the associative superalgebra consisting of all A-module endomorphisms of V. Note that V is naturally a left $\operatorname{End}_A(V)$-supermodule, with the action being function application.

Lemma 5.2.7. *For all $X \in \operatorname{End}_A(V)$, there exists a unique $X^\dagger \in \operatorname{End}_A(V)$ such that the following identity holds for all $v, w \in V$:*

$$\varphi(v, Xw) = (-1)^{\bar{X}\bar{v}}\varphi(X^\dagger v, w). \tag{DAGGER}$$

Proof: Fix $X \in \operatorname{End}_A(V)$. For each $v \in V$, define a map $X_v \colon V \to A$ by

$$X_v(w) = (-1)^{\bar{X}\bar{v}}\varphi(v, Xw).$$

It is easy to see that each X_v is a homomorphism of right A-supermodules. Since φ is unimodular, $v \mapsto \varphi_v$ is an isomorphism. Label the inverse map (i.e. $\varphi_v \to v$) $\alpha \colon \operatorname{Hom}_A^R(V, A) \to V$. For use in the following computations, note that for any $f \in \operatorname{Hom}_A^R(V, A)$, $\alpha(f)$ is the unique element of V such that $\varphi(\alpha(f), w) = f(w)$ holds for all $w \in V$. Define $X^\dagger \in \operatorname{End}_A(V)$ by setting $X^\dagger v = \alpha(X_v)$ for each $v \in V$. Towards showing that this indeed defines an element of $\operatorname{End}_A(V)$, let $u, v, w \in V$ and $a \in A$. We have:

$$\varphi(\alpha(X_{va}), w) = X_{va}(w)$$

$$
\begin{aligned}
&= (-1)^{\bar{X}\bar{v}+\bar{X}\bar{a}}\varphi(va, Xw) && \\
&= (-1)^{\bar{X}\bar{v}+\bar{X}\bar{a}+\bar{a}\bar{v}}a^*\varphi(v, Xw) && \text{Since } \varphi \text{ is sesquilinear} \\
&= (-1)^{\bar{a}\overline{X_v}}a^*X_v(w) && \\
&= (X_va)(w) && \text{By the definition of the right action on } \mathrm{Hom}_A^R(V, A) \\
&= \varphi(\alpha(X_va), w). &&
\end{aligned}
$$

Lemma 5.2.4 then allows us to conclude that $\alpha(X_{va}) = \alpha(X_va)$. Next, we have:

$$
\begin{aligned}
X_{v+w}(u) &= \varphi(v + w, Xu) \\
&= \varphi(v, Xu) + \varphi(w, Xu) \\
&= X_v(u) + X_w(u),
\end{aligned}
$$

and

$$
\begin{aligned}
X^\dagger(va + w) &= \alpha(X_{va+w}) && \\
&= \alpha(X_{va} + X_w) && \\
&= \alpha(X_{va}) + \alpha(X_w) && \text{Since } \alpha \text{ is } \Bbbk\text{-linear} \\
&= \alpha(X_va) + \alpha(X_w) && \\
&= \alpha(X_v)a + \alpha(X_w) && \text{Since } \alpha \text{ is a homomorphism of right } A\text{-supermodules} \\
&= (X^\dagger v)a + X^\dagger w. &&
\end{aligned}
$$

This shows that $X^\dagger$ is indeed an element of $\mathrm{End}_A(V)$. Finally, we have:

$$
\begin{aligned}
\varphi(X^\dagger v, w) &= \varphi(\alpha(X_v), w) \\
&= X_v(w) \\
&= (-1)^{\bar{X}\bar{v}}\varphi(v, Xw),
\end{aligned}
$$

which shows that (DAGGER) holds. To see that $X^\dagger$ is uniquely determined by this identity, suppose $Y \in \mathrm{End}_A(V)$ satisfies $\varphi(v, Xw) = (-1)^{\bar{X}\bar{v}}\varphi(Yv, w)$ for all $v, w \in V$. Then $\varphi(X^\dagger v, w) = \varphi(Yv, w)$ for all $v, w \in V$. Lemma 5.2.4 allows us to conclude that $X^\dagger v = Yv$ for all $v \in V$, i.e. $X^\dagger = Y$. ∎

Definition 5.2.8 (Orthosymplectic Lie superalgebra)**.** Define

$$\mathfrak{osp}(V) = \{X \in \mathrm{End}_A(V) \mid \varphi(Xv, w) = -(-1)^{\bar{X}\bar{v}}\varphi(v, Xw) \text{ for all } v, w \in V\}.$$

We call $\mathfrak{osp}(V)$ the *orthosymplectic Lie superalgebra over* V. The Lie superbracket in $\mathfrak{osp}(V)$ is given by the supercommutator $[X, Y] = XY - (-1)^{\bar{X}\bar{Y}}YX$. Note that

V is naturally a left supermodule over $\mathfrak{osp}(V)$ since V is a left supermodule over the superalgebra $\mathrm{End}_A(V)$, with the action being function application. We will verify that $\mathfrak{osp}(V)$ is indeed a Lie superalgebra in Corollary 5.3.5.

Lemma 5.2.9. *We have*

$$\mathfrak{osp}(V) = \{X \in \mathrm{End}_A(V) \mid X^\dagger = -X\}.$$

Proof: Suppose that $X \in \mathrm{End}_A(V)$ satisfies $X^\dagger = -X$. Then, for all $v, w \in V$ we have:

$$\begin{aligned} \varphi(Xv, w) &= (-1)^{\bar{X}\bar{v}}\varphi(v, X^\dagger w) && \text{Using (DAGGER)} \\ &= -(-1)^{\bar{X}\bar{v}}\varphi(v, Xw), \end{aligned}$$

and so $X \in \mathfrak{osp}(V)$. For the other direction, let $X \in \mathfrak{osp}(V)$. For all $v, w \in V$ we then have:

$$\begin{aligned} \varphi(Xv, w) &= -(-1)^{\bar{X}\bar{v}}\varphi(v, Xw) && \text{Using the defining property of } \mathfrak{osp}(V) \\ &= (-1)^{\bar{X}\bar{w}}\varphi(v, (-X)w). \end{aligned}$$

Since $X^\dagger$ is the unique element that makes (DAGGER) hold for all $v, w \in V$, we conclude that $X^\dagger = -X$, as desired. ∎

Lemma 5.2.10. *Let $C, D, E, F \in \mathrm{End}_A(V)$. The identity $\varphi(Cv, Dw) = \varphi(Ev, Fw)$ holds for all $v, w \in V$ if and only if $\mathrm{tr}(\varphi(Cv, Dw)) = \mathrm{tr}(\varphi(Ev, Fw))$ for all $v, w \in V$.*

Proof: The first condition trivially implies the second. For the other direction, suppose that $\mathrm{tr}(\varphi(Cv, Dw)) = \mathrm{tr}(\varphi(Ev, Fw))$ for all $v, w \in V$. Let B be a basis of A. We then have:

$$\begin{aligned} & \text{For all } b \in B, \mathrm{tr}(\varphi(Cv, Dwb)) = \mathrm{tr}(\varphi(Ev, Fwb)) \\ \implies & \text{For all } b \in B, \mathrm{tr}(\varphi(Cv, Dw)b) = \mathrm{tr}(\varphi(Ev, Fw)b) && \text{Since } \varphi \text{ is sesquilinear} \\ \implies & \sum_{b \in B} \mathrm{tr}(\varphi(Cv, Dw)b)b^\vee = \sum_{b \in B} \mathrm{tr}(\varphi(Ev, Fw)b)b^\vee \\ \implies & \varphi(Cv, Dw) = \varphi(Ev, Fw), && \text{Using Lemma 4.1.5} \end{aligned}$$

as desired. ∎

Corollary 5.2.11. *We have*

$$\mathfrak{osp}(V) = \{X \in \mathrm{End}_A(V) \mid \mathrm{tr}(\varphi(Xv, w)) = -(-1)^{\bar{X}\bar{v}}\, \mathrm{tr}(\varphi(v, Xw)) \text{ for all } v, w \in V\}.$$

Note that this is Lemma 5.2.9 with $\mathrm{tr} \circ \varphi$ *being used in the place of* φ*.*

Proof: For any given $X \in \mathrm{End}_A(V)$, we define $S_X \in \mathrm{End}_A(V)$ by the linear extension of $S_X v = (-1)^{\bar{X}\bar{v}} v$. Using the notation of Lemma 5.2.10, we take $C = X, D = \mathrm{id}_V, E = -S_X$, and $F = \mathrm{id}_V$. That lemma allows us to conclude that $\mathrm{tr}(\varphi(Xv, w)) = -(-1)^{\bar{X}\bar{v}}\, \mathrm{tr}(\varphi(v, Xw))$ holds for all $v, w \in V$ if and only if $\varphi(Xv, w) = -(-1)^{\bar{X}\bar{v}} \varphi(v, Xw)$ holds for all $v, w \in V$, as desired. ∎

5.3 Properties of $-^\dagger$

In this section, we will study the properties of $-^\dagger \colon \mathrm{End}_A(V) \to \mathrm{End}_A(V)$.

Lemma 5.3.1. *For all* $X, Y \in \mathrm{End}_A(V)$ *and* $\lambda \in \Bbbk$*, we have* $(\lambda X + Y)^\dagger = \lambda X^\dagger + Y^\dagger$*. That is,* $-^\dagger$ *is a linear operator on* $\mathrm{End}_A(V)$*.*

Proof: We assume that $\bar{X} = \bar{Y}$. The full result follows directly from this case. Let $v, w \in V$. We compute:

$$\begin{aligned}
\varphi(v, (\lambda X + Y)w) &= \varphi(v, \lambda Xw + Yw) \\
&= \lambda\varphi(v, Xw) + \varphi(v, Yw) && \text{Since } \varphi \text{ is } \Bbbk\text{-bilinear} \\
&= (-1)^{\bar{X}\bar{v}} \lambda\varphi(X^\dagger v, w) + (-1)^{\bar{Y}\bar{v}} \varphi(Y^\dagger v, w) && \text{Using (DAGGER)} \\
&= (-1)^{\overline{\lambda X + Y}\bar{v}} \varphi((\lambda X^\dagger + Y^\dagger)v, w). && \text{Since } \varphi \text{ is } \Bbbk\text{-bilinear}
\end{aligned}$$

Thus $\lambda X^\dagger + Y^\dagger$ satisfies the instance of (DAGGER) that defines $(\lambda X + Y)^\dagger$. We conclude the two endomorphisms are equal. ∎

Lemma 5.3.2. *The operator* $-^\dagger$ *is parity-preserving.*

Proof: Let $X \in \mathrm{End}_A(V)$. The case $X^\dagger = 0$ is trivial. If $X^\dagger \neq 0$, there exists some $v \in V$ such that $X^\dagger v \neq 0$. Fix such a v. Since φ is unimodular, $\varphi_{X^\dagger v}$ is not the zero map. Thus there exists some $w \in V$ such that $\varphi(X^\dagger v, w) \neq 0$. Next, (DAGGER) implies $\varphi(v, Xw) = (-1)^{\bar{X}\bar{v}} \varphi(X^\dagger v, w)$, and both sides of this equality have the same well-defined parity since they are nonzero. Since φ is parity-preserving, the expression on the left has parity $\bar{v} + \bar{X} + \bar{w}$ and the expression on the right has parity $\bar{v} + \bar{w} + \overline{X^\dagger}$. We conclude that $\bar{X} = \overline{X^\dagger}$. ∎

Lemma 5.3.3. *The operator $-^\dagger$ is self-inverse.*

Proof: Let $X \in \mathrm{End}_A(V)$. By the definition of $(X^\dagger)^\dagger$ and the fact that $-^\dagger$ is parity-preserving, it suffices to show that $\varphi(v, X^\dagger w) = (-1)^{\bar{X}\bar{v}}\varphi(Xv, w)$ for all $v, w \in V$. Indeed, we have:

$$\begin{aligned}
\varphi(Xv, w) &= (-1)^{\bar{X}\bar{w}+\bar{v}\bar{w}}(\varphi(w, Xv))^* && \text{Since } \varphi \text{ is Hermitian}\\
&= (-1)^{\bar{v}\bar{w}}(\varphi(X^\dagger w, v))^* && \text{Using (DAGGER)}\\
&= (-1)^{\bar{X}\bar{v}}((\varphi(v, X^\dagger w))^*)^* && \text{Since } \varphi \text{ is Hermitian}\\
&= (-1)^{\bar{X}\bar{v}}\varphi(v, X^\dagger w), && \text{Since } -^* \text{ is self-inverse}
\end{aligned}$$

as desired. ∎

Lemma 5.3.4. *The operator $-^\dagger$ is an involution on* $\mathrm{End}_A(V)$*.*

Proof: By the definition of $-^\dagger$, it suffices to show that

$$\varphi(v, XYw) = (-1)^{\overline{XY}\bar{v}}\varphi((-1)^{\bar{X}\bar{Y}}Y^\dagger X^\dagger v, w)$$

holds for all $v, w \in V$. Indeed, we have:

$$\begin{aligned}
\varphi(v, XYw) &= (-1)^{\bar{X}\bar{v}}\varphi(X^\dagger v, Yw) && \text{Using (DAGGER)}\\
&= (-1)^{\bar{X}\bar{v}+\bar{Y}\bar{X}+\bar{Y}\bar{v}}\varphi(Y^\dagger X^\dagger v, w) && \text{Using (DAGGER) and the fact that } -^\dagger \text{ is parity-preserving}\\
&= (-1)^{\overline{XY}\bar{v}}\varphi((-1)^{\bar{X}\bar{Y}}Y^\dagger X^\dagger v, w),
\end{aligned}$$

as desired. ∎

Corollary 5.3.5. *The orthosymplectic Lie superalgebra* $\mathfrak{osp}(V)$ *is indeed a Lie superalgebra.*

Proof: Since the Lie superbracket in $\mathfrak{osp}(V)$ is given by the supercommutator, all that remains to be shown is that $\mathfrak{osp}(V)$ is closed under this superbracket. Let $X, Y \in \mathfrak{osp}(V)$. We then have:

$$\begin{aligned}
[X, Y]^\dagger &= (XY - (-1)^{\bar{X}\bar{Y}}YX)^\dagger\\
&= (-1)^{\bar{X}\bar{Y}}Y^\dagger X^\dagger - X^\dagger Y^\dagger && \text{Since } -^\dagger \text{ is an involution}\\
&= (-1)^{\bar{X}\bar{Y}}YX - XY && \text{Using Lemma 5.2.9}\\
&= -[X, Y],
\end{aligned}$$

which, in light of Lemma 5.2.9, shows that $[X, Y] \in \mathfrak{osp}(V)$. ∎

5.4 The Generalized Orthosymplectic Involution

Fix $m, n \in \mathbb{N}$. In this section, we will define a unimodular Hermitian form on $A^{m|2n}$ (realized as column vectors). As shown previously, such a form induces an involution on $\mathrm{End}_A(A^{m|2n}) \cong \mathrm{Mat}_{m|2n}(A)$. Our goal is to find an explicit formula for this supermatrix involution.

Definition 5.4.1. For all $a \in A$ and $i \in \{1, \ldots, m+2n\}$, let a_i denote the element of $A^{m|2n}$ with an a in position i and zeroes elsewhere. For any basis B of A, define $B_{m|2n} = \{b_i \mid b \in B, i \in \{1, \ldots, m+2n\}\}$. It is easy to see that $B_{m|2n}$ is a basis of $A^{m|2n}$ (as a vector space over $\Bbbk$).

Definition 5.4.2. Define a function $\iota\colon \{1, \ldots, m+2n\} \to \mathbb{Z}_2$ by

$$\iota(i) = \begin{cases} 1 & \text{if } i \in \{m+1, \ldots, m+n\}, \\ 0 & \text{otherwise,} \end{cases}$$

and a permutation $-^{\pi}$ on $\{1, \ldots, m+2n\}$ by

$$i^{\pi} = \begin{cases} i & \text{if } i \in \{1, \ldots, m\}, \\ i+n & \text{if } i \in \{m+1, \ldots, m+n\}, \\ i-n & \text{if } i \in \{m+n+1, \ldots, m+2n\}. \end{cases}$$

These definitions immediately imply that $-^{\pi}$ is self-inverse, that $p(i^{\pi}) = p(i)$, and that $\iota(i) + \iota(i^{\pi}) = p(i)$.

Definition 5.4.3. Define a $(m \mid 2n) \times (m \mid 2n)$ supermatrix by the following block form:

$$P = \begin{bmatrix} I_m & 0 & 0 \\ 0 & 0 & I_n \\ 0 & -I_n & 0 \end{bmatrix}.$$

For any $k, l \in \mathbb{N}$, define a $(k \mid l) \times (k \mid l)$ supermatrix by the following block form:

$$S_{k|l} = \begin{bmatrix} I_k & 0 \\ 0 & -I_l \end{bmatrix}.$$

Here, I_j denotes the $j \times j$ identity matrix. It is easy to verify that $Pa_i = (-1)^{\iota(i)} a_{i^{\pi}}$ for any $a \in A$ and $i \in \{1, \ldots, m+2n\}$.

Definition 5.4.4 (Supertranspose)**.** For any $m, n, k, l \in \mathbb{N}$, we define the *supertranspose* operation

$$-^{\mathrm{st}}\colon \mathrm{Mat}_{k|l,m|n}(A) \to \mathrm{Mat}_{m|n,k|l}(A)$$

by:

$$M^{\mathrm{st}} = \begin{bmatrix} M_{00} & M_{01} \\ M_{10} & M_{11} \end{bmatrix}^{\mathrm{st}} = \begin{bmatrix} M_{00}^{\top} & -\rho(M_{10}^{\top}) \\ \rho(M_{01}^{\top}) & M_{11}^{\top} \end{bmatrix},$$

where ρ is the map from Definition 4.3.6. The supertranspose is clearly $\Bbbk$-linear and parity-preserving. Note, however, that unlike the transpose $-^{\top}$ for ordinary matrices, $-^{\mathrm{st}}$ is *not* self-inverse. Instead, we have:

$$\left(\begin{bmatrix} M_{00} & M_{01} \\ M_{10} & M_{11} \end{bmatrix}^{\mathrm{st}}\right)^{\mathrm{st}} = \begin{bmatrix} M_{00} & -M_{01} \\ -M_{10} & M_{11} \end{bmatrix} = S_{k|l} \begin{bmatrix} M_{00} & M_{01} \\ M_{10} & M_{11} \end{bmatrix} S_{m|n}. \tag{2ST}$$

To reduce notational clutter, we write $-^{\mathrm{st}\,*} = -^{*} \circ -^{\mathrm{st}}$ and $-^{\top *} = -^{*} \circ -^{\top}$. It is easy to see that we also have $-^{\mathrm{st}\,*} = -^{\mathrm{st}} \circ -^{*}$ and $-^{\top *} = -^{\top} \circ -^{*}$, i.e. it doesn't matter whether one applies the (super)transpose first and then the involution $-^{*}$ or vice versa.

Definition 5.4.5. For all $a \in A, i, j \in \{1, \dots, m+2n\}$, set $\sigma^{a}_{i,j} = (-1)^{(p(i)+p(j))(\bar{a}+p(i))}$. It is straightforward to verify that $a^{\mathrm{st}\,*}_{(i,j)} = \sigma^{a}_{i,j} a^{*}_{(j,i)}$ and $a^{\mathrm{st}\,*}_{i} = (-1)^{p(i)(\bar{a}+1)}(a^{*})^{\top}_{i}$.

Lemma 5.4.6. *The supermatrices P and $S_{k|l}$ are invertible, with*

$$\begin{aligned} P^{-1} &= PS_{m|2n} = S_{m|2n}P = P^{\mathrm{st}\,*}, \\ P &= S_{m|2n}P^{-1} = P^{-1}S_{m|2n} = (P^{-1})^{\mathrm{st}\,*}, \\ S_{k|l}^{-1} &= S_{k|l}. \end{aligned}$$

Proof: All of these identities follow immediately from direct computation. For instance,

$$\begin{aligned} S_{k|l}S_{k|l} &= \begin{bmatrix} I_k & 0 \\ 0 & -I_l \end{bmatrix} \begin{bmatrix} I_k & 0 \\ 0 & -I_l \end{bmatrix} \\ &= \begin{bmatrix} I_k & 0 \\ 0 & I_l \end{bmatrix} \\ &= I_{k|l}, \end{aligned}$$

which verifies that $S_{k|l}^{-1} = S_{k|l}$. Similarly,

$$\begin{aligned} (PS_{m|2n})P &= \begin{bmatrix} I_m & 0 & 0 \\ 0 & 0 & I_n \\ 0 & -I_n & 0 \end{bmatrix} \begin{bmatrix} I_m & 0 \\ 0 & -I_{2n} \end{bmatrix} \begin{bmatrix} I_m & 0 & 0 \\ 0 & 0 & I_n \\ 0 & -I_n & 0 \end{bmatrix} \\ &= \begin{bmatrix} I_m & 0 & 0 \\ 0 & 0 & I_n \\ 0 & -I_n & 0 \end{bmatrix} \begin{bmatrix} I_m & 0 & 0 \\ 0 & -I_n & 0 \\ 0 & 0 & -I_n \end{bmatrix} \begin{bmatrix} I_m & 0 & 0 \\ 0 & 0 & I_n \\ 0 & -I_n & 0 \end{bmatrix} \end{aligned}$$

$$
\begin{aligned}
&= \begin{bmatrix} I_m & 0 & 0 \\ 0 & 0 & -I_n \\ 0 & I_n & 0 \end{bmatrix} \begin{bmatrix} I_m & 0 & 0 \\ 0 & 0 & I_n \\ 0 & -I_n & 0 \end{bmatrix} \\
&= \begin{bmatrix} I_m & 0 & 0 \\ 0 & I_n & 0 \\ 0 & 0 & I_n \end{bmatrix} \\
&= I_{m|2n},
\end{aligned}
$$

which shows that $P^{-1} = PS_{m|2n}$. The rest of the computations are analogous. ∎

Lemma 5.4.7. *Let $o, p, q, r \in \mathbb{N}, X \in \mathrm{Mat}_{o,p}(A)$, and $Y \in \mathrm{Mat}_{q,r}(A)$. (Note that these are non-super matrices over A.) Then $(XY)^{\top *} = (-1)^{\bar{X}\bar{Y}} Y^{\top *} X^{\top *}$.*

Proof: By linearity, it suffices to consider the case where $X = a_{(i,j)}$ and $Y = b_{(k,l)}$ for some $a, b \in A$, $i \in \{1, \ldots, o\}$, $j \in \{1, \ldots, p\}$, $k \in \{1, \ldots, q\}$, and $l \in \{1, \ldots, r\}$. In this case, we have:

$$
\begin{aligned}
(XY)^{\mathrm{st}\,*} &= (a_{(i,j)} b_{(k,l)})^{\top *} \\
&= \delta_{jk} (ab)^{\top *}_{(i,l)} \\
&= \delta_{jk} (ab)^{*}_{(l,i)} \\
&= (-1)^{\bar{a}\bar{b}} \delta_{jk} (b^* a^*)_{(l,i)} \\
&= (-1)^{\bar{a}\bar{b}} (b^*)_{(l,k)} (a^*)_{(j,i)} \\
&= (-1)^{\bar{a}\bar{b}} b^{\top *}_{(k,l)} a^{\top *}_{(i,j)} \\
&= (-1)^{\bar{X}\bar{Y}} Y^{\top *} X^{\top *}.
\end{aligned}
$$

∎

Lemma 5.4.8. *Let $o, p, q, r \in \mathbb{N}$, $X \in \mathrm{Mat}_{o|p,m|n}(A)$, and $Y \in \mathrm{Mat}_{m|n,q|r}(A)$. Then $(XY)^{\mathrm{st}\,*} = (-1)^{\bar{X}\bar{Y}} Y^{\mathrm{st}\,*} X^{\mathrm{st}\,*}$.*

Proof: Write X and Y in block form as $X = \begin{bmatrix} X_{00} & X_{01} \\ X_{10} & X_{11} \end{bmatrix}, Y = \begin{bmatrix} Y_{00} & Y_{01} \\ Y_{10} & Y_{11} \end{bmatrix}$. Recall that, by the definition of the $\mathbb{Z}_2$-grading on supermatrices, we have $\overline{X_{00}} = \overline{X_{11}} = \bar{X}$, $\overline{Y_{00}} = \overline{Y_{11}} = \bar{Y}$, $\overline{X_{01}} = \overline{X_{10}} = \bar{X} + 1$ and $\overline{Y_{01}} = \overline{Y_{10}} = \bar{Y} + 1$. With this in mind, we compute:

$$
\begin{aligned}
&(XY)^{\mathrm{st}\,*} \\
&= \begin{bmatrix} X_{00}Y_{00} + X_{01}Y_{10} & X_{00}Y_{01} + X_{01}Y_{11} \\ X_{10}Y_{00} + X_{11}Y_{10} & X_{10}Y_{01} + X_{11}Y_{11} \end{bmatrix}^{\mathrm{st}\,*} \\
&= \begin{bmatrix} (X_{00}Y_{00} + X_{01}Y_{10})^{\top *} & -\rho(X_{10}Y_{00} + X_{11}Y_{10})^{\top *} \\ \rho(X_{00}Y_{01} + X_{01}Y_{11})^{\top *} & (X_{10}Y_{01} + X_{11}Y_{11})^{\top *} \end{bmatrix} \\
&= (-1)^{\bar{X}\bar{Y}} \begin{bmatrix} Y_{00}^{\top *}X_{00}^{\top *} + (-1)^{\bar{X}+\bar{Y}+1}Y_{10}^{\top *}X_{01}^{\top *} & (-1)^{\bar{X}}Y_{00}^{\top *}X_{10}^{\top *} + (-1)^{\bar{Y}}Y_{10}^{\top *}X_{11}^{\top *} \\ (-1)^{\bar{Y}+1}Y_{01}^{\top *}X_{00}^{\top *} + (-1)^{\bar{X}+1}Y_{11}^{\top *}X_{01}^{\top *} & (-1)^{\bar{X}+\bar{Y}+1}Y_{01}^{\top *}X_{10}^{\top *} + Y_{11}^{\top *}X_{11}^{\top *} \end{bmatrix} \\
&= (-1)^{\bar{X}\bar{Y}} \begin{bmatrix} Y_{00}^{\top *} & (-1)^{\bar{Y}}Y_{10}^{\top *} \\ (-1)^{\bar{Y}+1}Y_{01}^{\top *} & Y_{11}^{\top *} \end{bmatrix} \begin{bmatrix} X_{00}^{\top *} & (-1)^{\bar{X}}X_{10}^{\top *} \\ (-1)^{\bar{X}+1}X_{01}^{\top *} & X_{11}^{\top *} \end{bmatrix} \\
&= (-1)^{\bar{X}\bar{Y}} \begin{bmatrix} Y_{00}^{\top *} & -\rho(Y_{10}^{\top *}) \\ \rho(Y_{01}^{\top *}) & Y_{11}^{\top *} \end{bmatrix} \begin{bmatrix} X_{00}^{\top *} & -\rho(X_{10}^{\top *}) \\ \rho(X_{01}^{\top *}) & X_{11}^{\top *} \end{bmatrix} \\
&= (-1)^{\bar{X}\bar{Y}} Y^{\mathrm{st}\,*} X^{\mathrm{st}\,*},
\end{aligned}
$$

using Lemma 5.4.7 in the third equality. ∎

Definition 5.4.9. Define a map $\varphi\colon A^{m|2n} \times A^{m|2n} \to A$ by $\varphi(x, y) = (Px)^{\mathrm{st}\,*}y$. Using Lemma 5.4.8, we have $\varphi(x, y) = x^{\mathrm{st}\,*}P^{-1}y$.

Proposition 5.4.10. *The map φ is a unimodular Hermitian form.*

Proof: It is clear that φ is parity-preserving since P^{-1} is even and both $-^{\mathrm{st}\,*}$ and supermatrix multiplication are parity-preserving. Likewise, φ is clearly bilinear. Let $a, b \in A$, $v, w \in A^{m|2n}$. We compute:

$$
\begin{aligned}
\varphi(va, wb) &= (va)^{\mathrm{st}\,*}P^{-1}(wb) \\
&= (v(aI_{1|0}))^{\mathrm{st}\,*}P^{-1}wb \\
&= (-1)^{\bar{a}\bar{v}}(aI_{1|0})^{\mathrm{st}\,*}v^{\mathrm{st}\,*}P^{-1}wb \qquad \text{Using Lemma 5.4.8} \\
&= (-1)^{\bar{a}\bar{v}}a^{*}I_{1|0}v^{\mathrm{st}\,*}P^{-1}wb \\
&= (-1)^{\bar{a}\bar{v}}a^{*}(v^{\mathrm{st}\,*}P^{-1}w)b \\
&= (-1)^{\bar{a}\bar{v}}a^{*}\varphi(v, w)b,
\end{aligned}
$$

which shows that φ is sesquilinear. We also have:

$$
\begin{aligned}
(-1)^{\bar{v}\bar{w}}(\varphi(w, v))^{*} &= (-1)^{\bar{v}\bar{w}}(w^{\mathrm{st}\,*}P^{-1}v)^{*} \\
&= (-1)^{\bar{v}\bar{w}}(w^{\mathrm{st}\,*}P^{-1}v)^{\mathrm{st}\,*} \qquad \text{Since } -^{\mathrm{st}} \text{ is the identity on } (1 \mid 0) \times (1 \mid 0) \text{ supermatrices}
\end{aligned}
$$

$$
\begin{aligned}
&= v^{\mathrm{st}\,*}(P^{-1})^{\mathrm{st}\,*}(w^{\mathrm{st}\,*})^{\mathrm{st}\,*} && \text{Using Lemma 5.4.8} \\
&= v^{\mathrm{st}\,*}PS_{m|2n}wS_{1|0} && \text{Using Lemma 5.4.6 and (2ST)} \\
&= v^{\mathrm{st}\,*}P^{-1}wS_{1|0} && \text{Using Lemma 5.4.6} \\
&= v^{\mathrm{st}\,*}P^{-1}w && \text{Since } S_{1|0} \text{ is the } (1 \mid 0) \times (1 \mid 0) \text{ identity supermatrix} \\
&= \varphi(v, w),
\end{aligned}
$$

which shows that φ is Hermitian. Finally, recall that every map in $\mathrm{Hom}_A(A^{m|2n}, A)$ is given by $w \mapsto vw$ for some fixed (unique) row vector $v \in A^{m|2n}$. With this in mind, let $f \in \mathrm{Hom}_A(A^{m|2n}, A)$, and let v_f denote the associated column vector. For any $v \in A^{m|2n}$, we then have:

$$
\begin{aligned}
\varphi((v_f P^{-1})^{\mathrm{st}\,*}, w) &= ((v_f P^{-1})^{\mathrm{st}\,*})^{\mathrm{st}\,*}P^{-1}w \\
&= S_{1|0}v_f P^{-1}S_{m|2n}P^{-1}w && \text{Using (2ST)} \\
&= v_f P^{-1}S_{m|2n}P^{-1}w && \text{Since } S_{1|0} \text{ is the } (1 \mid 0) \times (1 \mid 0) \text{ identity supermatrix} \\
&= v_f PP^{-1}w && \text{Using (2ST) and Lemma 5.4.6} \\
&= v_f w,
\end{aligned}
$$

and thus $(v_f P^{-1})^{\mathrm{st}\,*}$ is the unique column vector such that $f(w) = \varphi((v_f P^{-1})^{\mathrm{st}\,*}, w)$ for all $w \in A^{m|2n}$. We conclude that φ is unimodular. ∎

Proposition 5.4.11. *For all $v, w \in A^{m|2n}$ and $X \in \mathrm{Mat}_{m|2n}(A)$, we have*

$$\varphi(v, Xw) = (-1)^{\bar{X}\bar{v}}\varphi(PX^{\mathrm{st}\,*}P^{-1}v, w).$$

Proof: We compute:

$$
\begin{aligned}
(-1)^{\bar{X}\bar{v}}\varphi(PX^{\mathrm{st}\,*}P^{-1}v, w) &= (-1)^{\bar{X}\bar{v}}(PX^{\mathrm{st}\,*}P^{-1}v)^{\mathrm{st}\,*}P^{-1}w \\
&= v^{\mathrm{st}\,*}(P^{-1})^{\mathrm{st}\,*}(X^{\mathrm{st}\,*})^{\mathrm{st}\,*}P^{\mathrm{st}\,*}P^{-1}w && \text{Using Lemma 5.4.8} \\
&= v^{\mathrm{st}\,*}PS_{m|2n}XS_{m|2n}P^{-1}P^{-1}w && \text{Using Lemma 5.4.6 and (2ST)} \\
&= v^{\mathrm{st}\,*}P^{-1}XPP^{-1}w && \text{Using Lemma 5.4.6} \\
&= v^{\mathrm{st}\,*}P^{-1}Xw \\
&= \varphi(v, Xw),
\end{aligned}
$$

as desired. ∎

Corollary 5.4.12. *The involution* $-^\dagger$ *on* $\mathrm{Mat}_{m|2n}(A)$ *is given by* $X^\dagger = PX^{\mathrm{st}\,*}P^{-1}$*. This involution can be computed in block form as:*

$$\begin{bmatrix} X_{00} & X_{01} & X_{02} \\ X_{10} & X_{11} & X_{12} \\ X_{20} & X_{21} & X_{22} \end{bmatrix}^\dagger = \begin{bmatrix} X_{00}^{\top *} & -\rho(X_{20}^{\top *}) & \rho(X_{10}^{\top *}) \\ \rho(X_{02}^{\top *}) & X_{22}^{\top *} & -X_{12}^{\top *} \\ -\rho(X_{01}^{\top *}) & -X_{21}^{\top *} & X_{11}^{\top *} \end{bmatrix}.$$

Proof: Since $X^\dagger$ is the *unique* matrix that makes $\varphi(v, Xw) = (-1)^{\bar{X}\bar{v}}\varphi(X^\dagger v, w)$ hold, we must have $X^\dagger = PX^{\mathrm{st}\,*}P^{-1}$. ∎

Lemma 5.4.13. *Recalling that we have assumed* $-^*$ *is compatible with* tr_A*,* $-^\dagger$ *is compatible with the trace map on* $\mathrm{Mat}_{m|2n}(A)$ *given by* $\mathrm{tr}(X) = \mathrm{tr}_A(\mathrm{str}(X))$*, as defined in Example 4.3.5.*

Proof: Write $X \in \mathrm{Mat}_{m|2n}(A)$ in block form as $X = \begin{bmatrix} X_{00} & X_{01} & X_{02} \\ X_{10} & X_{11} & X_{12} \\ X_{20} & X_{21} & X_{22} \end{bmatrix}$. Recall that we write Tr for the ordinary matrix trace. By the definition of str, we have:

$$\begin{aligned}
\mathrm{tr}(X^\dagger) &= \mathrm{tr}_A\left(\mathrm{str}\left(\begin{bmatrix} X_{00}^{\top *} & -\rho(X_{20}^{\top *}) & \rho(X_{10}^{\top *}) \\ \rho(X_{02}^{\top *}) & X_{22}^{\top *} & -X_{12}^{\top *} \\ -\rho(X_{01}^{\top *}) & -X_{21}^{\top *} & X_{11}^{\top *} \end{bmatrix}\right)\right) \\
&= \mathrm{tr}_A(\mathrm{Tr}(X_{00}^{\top *} - (-1)^{\bar{X}}X_{22}^{\top *} - (-1)^{\bar{X}}X_{11}^{\top *})) \\
&= \mathrm{tr}_A(\mathrm{Tr}(X_{00} - (-1)^{\bar{X}}X_{22} - (-1)^{\bar{X}}X_{11})^*) && \text{Since } \mathrm{Tr}\circ -^\top = \mathrm{Tr} \text{ and } -^* \text{ is linear} \\
&= \mathrm{tr}_A(\mathrm{Tr}(X_{00} - (-1)^{\bar{X}}X_{11} - (-1)^{\bar{X}}X_{22})) && \text{Since } -^* \text{ is compatible with } \mathrm{tr}_A \\
&= \mathrm{tr}_A\left(\mathrm{str}\left(\begin{bmatrix} X_{00} & X_{01} & X_{02} \\ X_{10} & X_{11} & X_{12} \\ X_{20} & X_{21} & X_{22} \end{bmatrix}\right)\right) \\
&= \mathrm{tr}(X),
\end{aligned}$$

as desired. ∎

5.5 Orthosymplectic Lie Superalgebras

Definition 5.5.1 (Orthosymplectic Lie superalgebra)**.** We write $\mathfrak{osp}_{m|2n}(A) = \mathfrak{osp}(A^{m|2n})$. We call $\mathfrak{osp}_{m|2n}(A)$ the *orthosymplectic Lie superalgebra over* A *of superdimension* $(m \mid 2n)$.

Remark 5.5.2. As the name suggests, the orthosymplectic Lie superalgebra generalizes the well-known orthogonal and symplectic Lie algebras. Consider first the case $A = \Bbbk$ (with $-^* = \mathrm{id}$) and $n = 0$. Since $n = 0$, $\mathfrak{osp}_{m|0}(\Bbbk)$ has the trivial $\mathbb{Z}_2$-grading; that is, it is a (non-super) Lie algebra. Also note that in this case, the unimodular form φ from Definition 5.4.9 is given by $\varphi(v, w) = v^\top w$, i.e. the canonical nondegenerate symmetric bilinear form on $\Bbbk^m$. With this in mind, $\mathfrak{osp}_{m|0}(\Bbbk)$ consists of the $m \times m$ matrices X that satisfy the following identity for all $v, w \in \Bbbk^m$:

$$v^\top X^\top w = -v^\top X w,$$

which is equivalent to saying that $X^\top = -X$ (which one could also obtain from Lemma 5.2.9, noting that $-^\dagger = -^\top$ in this case). In other words, $\mathfrak{osp}_{m|0}(\Bbbk)$ consists of the skew symmetric matrices of dimension m. We conclude that $\mathfrak{osp}_{m|0}(\Bbbk) = \mathfrak{o}_m$, where $\mathfrak{o}_m$ denotes the orthogonal Lie algebra of rank m over $\Bbbk$.

In the case $A = \Bbbk$ and $m = 0$, the unimodular form φ is the nondegenerate skew-symmetric form on $\Bbbk^{2n}$ given by $\varphi(v, w) = v^\top \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix} w$, and $\mathfrak{osp}_{0|2n}(\Bbbk)$ is the Lie algebra consisting of symplectic $2n \times 2n$ matrices. Thus $\mathfrak{osp}_{0|2n}(\Bbbk) = \mathfrak{sp}_{2n}$, where $\mathfrak{sp}_{2n}$ denotes the symplectic Lie algebra of rank n over $\Bbbk$.

These connections between the orthosymplectic, orthogonal, and symplectic Lie (super)algebras are already well-known over $\Bbbk$. Working by analogy, one can define $\mathfrak{o}_m(A) = \mathfrak{osp}_{m|0}(A)$ and $\mathfrak{sp}_{2n}(A) = \mathfrak{osp}_{0|2n}(A)$ for any Frobenius superalgebra A equipped with a compatible involution.

Lemma 5.5.3. *Let $M \in \mathrm{Mat}_{m|2n}(A)$. Then $M \in \mathfrak{osp}_{m|2n}(A)$ if and only if $M = N - N^\dagger$ for some matrix $N \in \mathrm{Mat}_{m|2n}(A)$.*

Proof: By Lemma 5.2.9, M lies in $\mathfrak{osp}_{m|2n}(A)$ precisely when $M^\dagger = -M$. We have:

$$\begin{aligned}(N - N^\dagger)^\dagger &= N^\dagger - (N^\dagger)^\dagger \\ &= N^\dagger - N \\ &= -(N - N^\dagger),\end{aligned}$$

and so $N - N^\dagger \in \mathfrak{osp}_{m|2n}(A)$. Conversely, if $M \in \mathfrak{osp}_{m|2n}(A)$, then

$$M = \frac{1}{2}M + \frac{1}{2}M = \frac{1}{2}M - \left(\frac{1}{2}M\right)^\dagger,$$

as desired. ∎

Lemma 5.5.4. *For all $X \in \mathrm{Mat}_{m|2n}(A)$, we have $(X^\dagger)^{\mathrm{st}\,*} = (X^{\mathrm{st}\,*})^\dagger$.*

Proof: We compute:

$$\begin{aligned}(X^\dagger)^{\mathrm{st}\,*} &= (PX^{\mathrm{st}\,*}P^{-1})^{\mathrm{st}\,*} \\ &= (P^{-1})^{\mathrm{st}\,*}(X^{\mathrm{st}\,*})^{\mathrm{st}\,*}P^{\mathrm{st}\,*} \\ &= P(X^{\mathrm{st}\,*})^{\mathrm{st}\,*}P^{-1} && \text{Using Lemma 5.4.6} \\ &= (X^{\mathrm{st}\,*})^\dagger,\end{aligned}$$

as desired. ∎

Corollary 5.5.5. *The Lie superalgebra $\mathfrak{osp}_{m|2n}(A)$ is closed under $-^{\mathrm{st}\,*}$.*

Proof: Using Lemma 5.5.3, it suffices to consider elements of the form $N - N^\dagger$. We compute:

$$\begin{aligned}(N - N^\dagger)^{\mathrm{st}\,*} &= N^{\mathrm{st}\,*} - (N^\dagger)^{\mathrm{st}\,*} \\ &= N^{\mathrm{st}\,*} - (N^{\mathrm{st}\,*})^\dagger. && \text{Using Lemma 5.5.4}\end{aligned}$$

Again using Lemma 5.5.3, we find that this matrix is in $\mathfrak{osp}_{m|2n}(A)$. ∎

Proposition 5.5.6. *Write $\sigma(X, Y) = \mathrm{tr}(XY)$ for the nondegenerate bilinear form on $\mathrm{Mat}_{m|2n}(A)$ inherited from the trace map given by $\mathrm{tr}(X) = \mathrm{tr}_A(\mathrm{str}(X))$. The form σ is also nondegenerate when restricted to $\mathfrak{osp}_{m|2n}(A)$.*

Proof: Let $X \in \mathfrak{osp}_{m|2n}(A)$ be some nonzero element. We need to show that there exists some $Y \in \mathfrak{osp}_{m|2n}(A)$ such that $\sigma(X, Y) \neq 0$. Since $X \in \mathfrak{osp}_{m|2n}(A)$, we have $X - X^\dagger = 2X$, and thus $X - X^\dagger$ is a nonzero supermatrix. Since σ is nondegenerate on $\mathrm{Mat}_{m|2n}(A)$, there exists some $y \in \mathrm{Mat}_{m|2n}(A)$ such that $\sigma(X - X^\dagger, y) \neq 0$. Set $Y = y - y^\dagger$. By Lemma 5.5.3, $Y \in \mathfrak{osp}_{m|2n}(A)$. We then have:

$$\begin{aligned}\sigma(X, Y) &= \mathrm{tr}(XY) \\ &= \mathrm{tr}(X(y - y^\dagger)) \\ &= \mathrm{tr}(Xy) - \mathrm{tr}(Xy^\dagger) \\ &= \mathrm{tr}(Xy) - \mathrm{tr}(X^\dagger y) && \text{Using Lemma 4.2.14} \\ &= \mathrm{tr}((X - X^\dagger)y) \\ &= \sigma(X - X^\dagger, y) \\ &\neq 0,\end{aligned}$$

as desired. ∎

Lemma 5.5.7. *There exists a basis B of A such that $b^* = \pm b$ for all $b \in B$.*

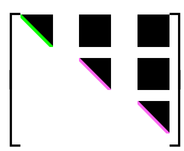
Proof: Recall that $-^*$ is a self-inverse linear map from A to itself. Thus its minimal polynomial divides $x^2 - 1 = (x-1)(x+1)$. As such, A's minimal polynomial is a product of distinct linear factors, and so $-^*$ is diagonalizable. Thus there exists some basis B of A such that $b^* = \alpha_b b$ for all $b \in B$, for some scalars $\alpha_b \in \Bbbk$. Applying $-^*$ to both sides, we have $b = \alpha_b b^* = \alpha_b^2 b$, and thus $\alpha_b^2 = 1$. We conclude that $\alpha_b = \pm 1$, as desired. ∎

Throughout the rest of the chapter, fix a basis B of A such that $b^* = \pm b$ for all $b \in B$.

Definition 5.5.8. Set $C = \{1, \dots, m+2n\}$ for brevity. Define sets $I_1, I_2, I_3, I \subseteq C^2$ by:

$$\begin{aligned} I_1 &= \{(i,j) \in C^2 \mid i < j\}, \\ I_2 &= \{(i,j) \in C^2 \mid i = j, p(i) = 0\}, \\ I_3 &= \{(i,j) \in C^2 \mid i = j, p(i) = 1\} \\ I &= I_1 \cup I_2 \cup I_3. \end{aligned}$$

These indices correspond to the positions marked in the following block matrix, with I_1 drawn in black, I_2 in green, and I_3 in pink:



. (5.5.1)

Definition 5.5.9. For all $i, j \in \{1, \dots, m+2n\}$ and $a \in A$, define $G^a_{i,j} = a_{(i,j)}P - (a_{(i,j)}P)^\dagger$. Lemma 5.5.3 tells us that $G^a_{i,j} \in \mathfrak{osp}_{m|2n}(A)$.

Lemma 5.5.10. *For all $a \in A$, we have*

$$G^a_{i,i} = a_{(i,i)}P - (-1)^{p(i)} a^*_{(i,i)}P = (a - (-1)^{p(i)}a^*)_{(i,i)}P.$$

Proof: Based on the definition of $G^a_{i,i}$, it suffices to show $(a_{(i,i)}P)^\dagger = (-1)^{p(i)} a^*_{(i,i)}P$. We compute:

$$\begin{aligned} (a_{(i,i)}P)^\dagger &= P(a_{(i,i)}P)^{\mathrm{st}\,*}P^{-1} \\ &= PP^{\mathrm{st}\,*}(a_{(i,i)})^{\mathrm{st}\,*}P^{-1} \\ &= PP^{-1}a^*_{(i,i)}P^{-1} && \text{Using Lemma 5.4.6} \\ &= a^*_{(i,i)}SP && \text{Using Lemma 5.4.6} \\ &= (-1)^{p(i)}a^*_{(i,i)}P, && \text{By the definition of } S \end{aligned}$$

as desired. ∎

Lemma 5.5.11. *Define*

$$\begin{aligned}
\mathcal{B}_1 &= \{G^b_{i,j} \mid b \in B, (i,j) \in I_1\},\\
\mathcal{B}_2 &= \{G^b_{i,i} \mid b \in B, (i,i) \in I_2, b - b^* \neq 0\},\\
\mathcal{B}_3 &= \{G^b_{i,i} \mid b \in B, (i,i) \in I_3, b + b^* \neq 0\},\\
B^{\mathfrak{osp}}_{m|2n} &= \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3.
\end{aligned}$$

Then $B^{\mathfrak{osp}}_{m|2n}$ is a basis of $\mathfrak{osp}_{m|2n}(A)$.

Note that $\mathcal{B}_2$ is empty if and only if $-^* = \mathrm{id}$ (or $m = 0$). To contrast, $\mathcal{B}_3$ is never empty (unless $n = 0$), since this would imply that $-^*$ acts on all of A by negation, and involutions preserve 1_A.

Proof: Recall that $X \in \mathfrak{osp}_{m|2n}(A)$ precisely when $X^\dagger = -X$. Written in block form, this says:

$$\begin{bmatrix} X_{00}^{\top *} & -\rho(X_{20}^{\top *}) & \rho(X_{10}^{\top *}) \\ \rho(X_{02}^{\top *}) & X_{22}^{\top *} & -X_{12}^{\top *} \\ -\rho(X_{01}^{\top *}) & -X_{21}^{\top *} & X_{11}^{\top *} \end{bmatrix} = \begin{bmatrix} -X_{00} & -X_{01} & -X_{02} \\ -X_{10} & -X_{11} & -X_{12} \\ -X_{20} & -X_{21} & -X_{22} \end{bmatrix},$$

which yields the following equivalent conditions on the individual blocks (after removing redundant equations):

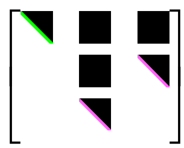
$$\begin{cases} X_{00}^{\top *} &= -X_{00}, \\ \rho(X_{20}^{\top *}) &= X_{01}, \\ \rho(X_{10}^{\top *}) &= -X_{02}, \\ X_{22}^{\top *} &= -X_{11}, \\ X_{12}^{\top *} &= X_{12}, \\ X_{21}^{\top *} &= X_{21}. \end{cases}$$

The equation $X_{00}^{\top *} = -X_{00}$ tells us that X_{00} is determined by its upper triangular entries. The entries strictly above the main diagonal may be chosen arbitrarily, but the entries a on the main diagonal must satisfy $a^* = -a$.

Similarly, the identities $X_{12}^{\top *} = X_{12}$ and $X_{21}^{\top *} = X_{21}$ tell us that X_{12} and X_{21} are each determined by their upper triangular entries. The entries above the main diagonal may be chosen arbitrarily, but the entries a on the main diagonal must satisfy $a^* = a$.

The identities $\rho(X_{20}^{\top *}) = X_{01}, \rho(X_{10}^{\top *}) = -X_{02}$, and $X_{22}^{\top *} = -X_{11}$ tell us that X_{20}, X_{10}, and X_{22} are determined by X_{01}, X_{02}, and X_{11}, respectively. Since there are no other conditions that involve those matrices, X_{01}, X_{02}, and X_{11} may be chosen arbitrarily.

Putting these conditions together graphically, a matrix $X \in \mathfrak{osp}_{m|2n}(A)$ is completely determined by its entries in the following positions:


,

where the entries in black positions may be chosen arbitrarily, those in green positions must satisfy $a^* = -a$, and those in pink positions must satisfy $a^* = a$. Note that this diagram can be obtained from (5.5.1) by swapping the last two columns. Ignoring signs, this is precisely the action of multiplying a matrix by P on the right. Given that B is a basis of A, we conclude that $\mathcal{B}_1$ spans the subspace of $\mathfrak{osp}_{m|2n}(A)$ whose only nonzero entries are in the black positions. Lemma 5.5.10 tells us that the matrices in $\mathcal{B}_2$ all have the form $(b - b^*)_{(i,i)}P$, and those in $\mathcal{B}_3$ have the form $(b + b^*)_{(i,i)}P$. It is easy to see that any $a \in A$ satisfying $a^* = -a$ can be expressed in a unique way as a linear combination of the $b - b^*$. Similarly, those satisfying $a^* = a$ can be expressed in a unique way as a linear combination of the $b + b^*$. Thus $\mathcal{B}_2$ and $\mathcal{B}_3$ span the subspaces of $\mathfrak{osp}_{m|2n}(A)$ whose only nonzero entries are in the green and pink positions, respectively. We conclude that $B^{\mathfrak{osp}}_{m|2n}$ spans $\mathfrak{osp}_{m|2n}(A)$. Linear independence follows from the fact that B is a basis of A and that $G^b_{i,j}$ and $G^c_{k,l}$ are disjoint matrices whenever $(i,j),(k,l) \in I$ and $(i,j) \neq (k,l)$. ∎

Lemma 5.5.12. *Let $(i,j),(k,l) \in I$. Then* $\delta_{il}\delta_{jk} = \begin{cases} 1 & \text{if } i = j = k = l, \\ 0 & \text{otherwise.} \end{cases}$

Proof: If $i = j = k = l$, then clearly $\delta_{il}\delta_{jk} = 1$. In the other direction, suppose that $\delta_{il}\delta_{jk} = 1$. Suppose by way of contradiction that $i \neq j$. Since $(i,j) \in I$, we must have $i < j$, which implies $l < k$. This is impossible since $(k,l) \in I$. Thus we must have $i = j$. Given that $\delta_{il}\delta_{jk} = 1$, we have $i = j = k = l$, as desired. ∎

Proposition 5.5.13. *For all $b \in B$ and $(i,j) \in I$, we have* $(G^b_{i,j})^\vee = \frac{\sigma^b_{i,j}(-1)^{p(j)}}{2+2\delta_{ij}}(G^{b^{\vee *}}_{i,j})^{\mathrm{st}\,*}$. *(This dual is taken with respect to the nondegenerate bilinear form $\sigma(X,Y) = \mathrm{tr}(XY)$.)*

Proof: First, note that Corollary 5.5.5 tells us $(G^{b^{\vee *}}_{i,j})^{\mathrm{st}\,*} \in \mathfrak{osp}_{m|2n}(A)$. Let $(i,j),(k,l) \in I$ and $b,c \in B$, and suppose that $G^b_{i,j} \in B^{\mathfrak{osp}}_{m|2n}$. Based on the definition of $B^{\mathfrak{osp}}_{m|2n}$, this amounts to assuming that either $i \neq j$ or $b - (-1)^{p(i)}b^* \neq 0$. Recalling that $b^* = \pm b$ for all $b \in B$, we either have $i \neq j$ or $b^* = -(-1)^{p(i)}b$. With this in mind, we compute:

$$
\begin{aligned}
G^c_{k,l} &= c_{(k,l)}P - (c_{(k,l)}P)^\dagger \\
&= c_{(k,l)}P - P(c_{(k,l)}P)^{\mathrm{st}\,*}P^{-1} \\
&= c_{(k,l)}P - PP^{\mathrm{st}\,*}c^{\mathrm{st}\,*}_{(k,l)}P^{-1} \\
&= c_{(k,l)}P - \sigma^c_{k,l}PP^{-1}c^*_{(l,k)}P^{-1} && \text{Using Lemma 5.5.4} \\
&= c_{(k,l)}P - \sigma^c_{k,l}c^*_{(l,k)}P^{-1}, \\
(G^{b^\vee *}_{i,j})^{\mathrm{st}\,*} &= (b^{\vee *}_{(i,j)}P - (b^{\vee *}_{(i,j)}P)^\dagger)^{\mathrm{st}\,*} \\
&= (b^{\vee *}_{(i,j)}P)^{\mathrm{st}\,*} - ((b^{\vee *}_{(i,j)}P)^{\mathrm{st}\,*})^\dagger && \text{Using Lemma 5.5.4} \\
&= P^{\mathrm{st}\,*}(b^{\vee *})^{\mathrm{st}\,*}_{(i,j)} - (P^{\mathrm{st}\,*}(b^{\vee *})^{\mathrm{st}\,*}_{(i,j)})^\dagger \\
&= \sigma^b_{i,j}P^{-1}(b^\vee)_{(j,i)} - P(P^{-1}(b^{\vee *})^{\mathrm{st}\,*}_{(i,j)})^{\mathrm{st}\,*}P^{-1} \\
&= \sigma^b_{i,j}P^{-1}(b^\vee)_{(j,i)} - P((b^{\vee *})^{\mathrm{st}\,*}_{(i,j)})^{\mathrm{st}\,*}(P^{-1})^{\mathrm{st}\,*}P^{-1} \\
&= \sigma^b_{i,j}P^{-1}(b^\vee)_{(j,i)} - PSb^{\vee *}_{(i,j)}SPP^{-1} \\
&= \sigma^b_{i,j}P^{-1}(b^\vee)_{(j,i)} - P^{-1}b^{\vee *}_{(i,j)}S.
\end{aligned}
$$

We thus have:

$$
\begin{aligned}
(G^{b^\vee *}_{i,j})^{\mathrm{st}\,*}G^c_{k,l} &= \left(\sigma^b_{i,j}P^{-1}(b^\vee)_{(j,i)} - P^{-1}b^{\vee *}_{(i,j)}S\right)\left(c_{(k,l)}P - \sigma^c_{k,l}c^*_{(l,k)}P^{-1}\right) \\
&= \sigma^b_{i,j}P^{-1}(b^\vee)_{(j,i)}c_{(k,l)}P - \sigma^b_{i,j}\sigma^c_{k,l}P^{-1}(b^\vee)_{(j,i)}c^*_{(l,k)}P^{-1} \\
&\quad - P^{-1}b^{\vee *}_{(i,j)}Sc_{(k,l)}P + \sigma^c_{k,l}P^{-1}b^{\vee *}_{(i,j)}Sc^*_{(l,k)}P^{-1}.
\end{aligned}
$$

We compute the trace of this expression in parts. Recalling that $\mathrm{tr} = \mathrm{tr}_A \circ \mathrm{str}$, we have:

$$
\begin{aligned}
\mathrm{tr}(\sigma^b_{i,j}P^{-1}(b^\vee)_{(j,i)}c_{(k,l)}P) &= \sigma^b_{i,j}\,\mathrm{tr}((b^\vee)_{(j,i)}c_{(k,l)}PP^{-1}) && \text{Using the symmetry of tr} \\
&= \sigma^b_{i,j}\delta_{ik}\,\mathrm{tr}_A(\mathrm{str}((b^\vee c)_{(j,l)})) \\
&= \sigma^b_{i,j}\delta_{ik}\delta_{jl}(-1)^{p(j)(\bar{b}+\bar{c}+1)}\,\mathrm{tr}_A(b^\vee c) \\
&= \sigma^b_{i,j}\delta_{ik}\delta_{jl}(-1)^{p(j)}\delta_{bc}.
\end{aligned}
$$

Next,

$$
\begin{aligned}
&\mathrm{tr}(\sigma^b_{i,j}\sigma^c_{k,l}P^{-1}(b^\vee)_{(j,i)}c^*_{(l,k)}P^{-1}) \\
&= \sigma^b_{i,j}\sigma^c_{k,l}\,\mathrm{tr}(SPP^{-1}(b^\vee)_{(j,i)}c^*_{(l,k)}) && \text{Using Lemma 5.4.6 and the symmetry of tr} \\
&= \sigma^b_{i,j}\sigma^c_{k,l}\delta_{il}\,\mathrm{tr}(S(b^\vee c^*)_{(j,k)}) \\
&= \sigma^b_{i,j}\sigma^c_{k,l}\delta_{il}(-1)^{p(j)}\,\mathrm{tr}_A(\mathrm{str}((b^\vee c^*)_{(j,k)})) && \text{By the definition of } S
\end{aligned}
$$

$$
\begin{aligned}
&= \sigma^b_{i,j}\sigma^c_{k,l}\delta_{il}\delta_{jk}(-1)^{p(j)+p(j)(\bar{b}+\bar{c}+1)}\operatorname{tr}_A(b^\vee c^*) \\
&= \sigma^b_{i,i}\sigma^c_{i,i}\delta_{ij}\delta_{jk}\delta_{kl}(-1)^{p(j)(\bar{b}+\bar{c})}\operatorname{tr}_A(b^\vee c^*) && \text{Using Lemma 5.5.12} \\
&= -\sigma^b_{i,i}\sigma^c_{i,i}\delta_{ij}\delta_{jk}\delta_{kl}(-1)^{p(j)(\bar{b}+\bar{c})+p(k)}\operatorname{tr}_A(b^\vee c) && \text{Since } k=l \text{ implies } c^* = -(-1)^{p(k)}c \\
&= -\sigma^b_{i,i}\sigma^b_{i,i}(-1)^{p(j)}\delta_{ij}\delta_{jk}\delta_{kl}\delta_{bc} \\
&= -\sigma^b_{i,j}(-1)^{p(j)}\delta_{ik}\delta_{jl}\delta_{ij}\delta_{bc}. && \text{Since } \sigma^b_{i,i}=1
\end{aligned}
$$

Next,

$$
\begin{aligned}
\operatorname{tr}(P^{-1}b^{\vee *}_{(i,j)}Sc_{(k,l)}P) &= (-1)^{p(k)}\operatorname{tr}(PP^{-1}b^{\vee *}_{(i,j)}c_{(k,l)}) && \text{Using the definition of } S \text{ and the symmetry of tr} \\
&= (-1)^{p(k)}\delta_{jk}\operatorname{tr}((b^{\vee *}c)_{(i,l)}) \\
&= (-1)^{p(k)+p(i)(\bar{b}+\bar{c}+1)}\delta_{jk}\delta_{il}\operatorname{tr}_A(b^{\vee *}c) \\
&= (-1)^{p(k)+p(i)(\bar{b}+\bar{c}+1)}\delta_{jk}\delta_{il}\operatorname{tr}_A(b^{*\vee}c) && \text{Using Lemma 4.2.15} \\
&= \delta_{ij}\delta_{jk}\delta_{kl}(-1)^{p(i)(\bar{b}+\bar{c})}\operatorname{tr}_A(b^{*\vee}c) && \text{Using Lemma 5.5.12} \\
&= -\delta_{ij}\delta_{jk}\delta_{kl}(-1)^{p(i)(\bar{b}+\bar{c}+1)}\operatorname{tr}_A(b^{\vee}c) && \text{Since } i=j \text{ implies } b^* = -(-1)^{p(i)}b \\
&= -\delta_{ij}\delta_{jk}\delta_{kl}(-1)^{p(j)}\delta_{bc} \\
&= -\sigma^b_{i,j}(-1)^{p(j)}\delta_{ik}\delta_{jl}\delta_{ij}\delta_{bc}. && \text{Since } \sigma^b_{i,i}=1
\end{aligned}
$$

Finally,

$$
\begin{aligned}
&\operatorname{tr}(\sigma^c_{k,l}P^{-1}b^{\vee *}_{(i,j)}Sc^*_{(l,k)}P^{-1}) \\
&= \sigma^c_{k,l}\operatorname{tr}(SPP^{-1}b^{\vee *}_{(i,j)}Sc^*_{(l,k)}) && \text{Using Lemma 5.4.6 and the symmetry of tr} \\
&= \sigma^c_{k,l}\operatorname{tr}(Sb^{\vee *}_{(i,j)}Sc^*_{(l,k)}) \\
&= \sigma^c_{k,l}(-1)^{p(i)+p(l)}\operatorname{tr}(b^{\vee *}_{(i,j)}c^*_{(l,k)}) && \text{Using the definition of } S \\
&= \sigma^c_{k,l}(-1)^{p(i)+p(l)+p(k)(\bar{b}+\bar{c}+1)}\delta_{jl}\delta_{ik}\operatorname{tr}_A(b^{\vee *}c^*) \\
&= \sigma^c_{i,j}(-1)^{p(j)+p(k)(\bar{b}+\bar{c})}\delta_{ik}\delta_{jl}\operatorname{tr}_A(b^\vee c) && \text{Using Lemma 4.2.14} \\
&= \sigma^b_{i,j}(-1)^{p(j)}\delta_{ik}\delta_{jl}\delta_{bc}.
\end{aligned}
$$

Putting these results together, we have:

$$
\begin{aligned}
\operatorname{tr}((G^{b^{\vee *}}_{i,j})^{\mathrm{st}\,*}G^c_{k,l}) &= \sigma^b_{i,j}\delta_{ik}\delta_{jl}\delta_{bc}(-1)^{p(j)}\,(1+\delta_{ij}+\delta_{ij}+1) \\
&= \frac{2+2\delta_{ij}}{\sigma^b_{i,j}(-1)^{p(j)}}\delta_{ik}\delta_{jl}\delta_{bc},
\end{aligned}
$$

which establishes the claimed dual formula. ∎

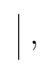
Chapter 6

Frobenius Brauer Categories

Throughout this chapter, fix $m, n \in \mathbb{N}$, a symmetric Frobenius superalgebra A with trace map tr, an involution $-^*$ that is compatible with tr, and a choice of homogeneous basis B for A such that $b^* = \pm b$ for all $b \in B$ (see Lemma 5.5.7).

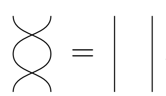
6.1 Non-Affine Frobenius Brauer Categories

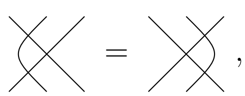
Definition 6.1.1 (Brauer category)**.** The *Brauer category*, denoted $\mathcal{B}$, is the strict $\Bbbk$-linear monoidal category with one generating object, written as an un-oriented string:

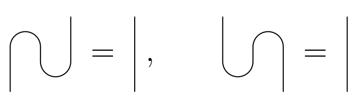


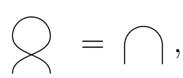
the following generating morphisms:

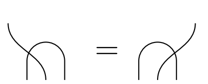
respectively called the *crossing*, *cup*, and *cap*, subject to the following relations:

 (DCROSS)

 (BRAID)

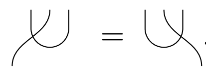
 (ZIGZAG)

 (∩DELOOP)

 (∩SLIDE)

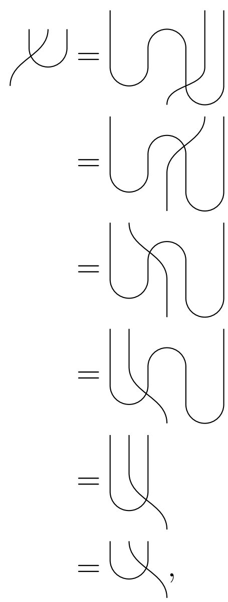
Note that this presentation of $\mathcal{B}$ is similar to the one given in [LZ15, Thm. 2.6] and [RS18, Def. 1.1], but has fewer defining relations. The aforementioned presentations include reflected versions of (∩SLIDE) and (∩DELOOP); in the following two lemmas, we will show that those reflected relations follow from the relations listed above.

Lemma 6.1.2. *The cup analogue of* (∩SLIDE) *also holds in* $\mathcal{B}$*. Namely, we have:*



(∪SLIDE)

Proof:



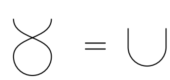
Using (ZIGZAG)

Using (∩SLIDE)

Using (ZIGZAG)

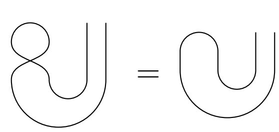
as desired. ∎

Lemma 6.1.3. *The cup analogue of* (∩DELOOP) *also holds in* $\mathcal{B}$*. Namely, we have:*



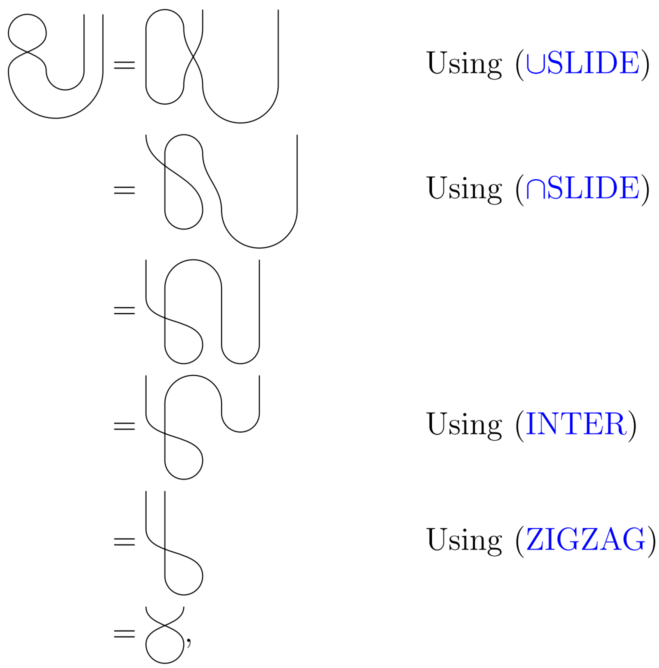
(∪DELOOP)

Proof: Starting from the equality given by (∩DELOOP), we adjoin two nested cups to the bottom of each diagram, yielding:



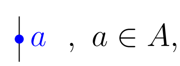
(6.1.1)

Applying (ZIGZAG) to the top of the diagram on the right yields. Thus it suffices to show that the diagram on the left of (6.1.1) is equal to. Indeed, we have:

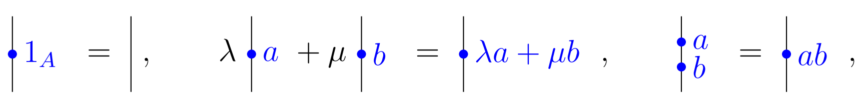


as desired. ∎

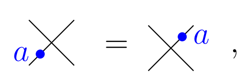
Definition 6.1.4 (Frobenius Brauer category)**.** The *Frobenius Brauer category associated to the Frobenius superalgebra A*, denoted $\mathcal{B}(A)$, is the supercategory defined using the same generators and relations as $\mathcal{B}$ (see Definition 6.1.1), plus a new family of generating morphisms:



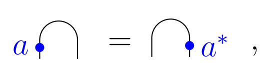
together with the following new relations, for all $a, b \in A$ and $\lambda, \mu \in \Bbbk$:



(TOKEN)



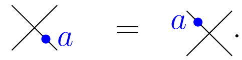
(A×SLIDE1)



(A∩SLIDE)

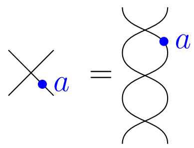
The decoration on a is called a *Frobenius token*. For each $a \in A$, the generating morphism a has parity $\bar{a}$. All other generators are even.

Lemma 6.1.5. *The mirrored analogue of* (A×SLIDE1) *also holds in* $\mathcal{B}(A)$*. Namely, we have:*

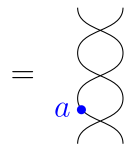


(A×SLIDE2)

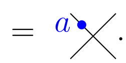
Proof:



Using (DCROSS)



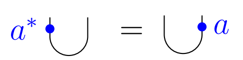
Using (A×SLIDE1)



Using (DCROSS)

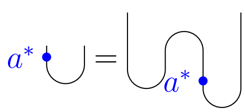
∎

Lemma 6.1.6. *The cup analogue of* (A∩SLIDE) *also holds in* $\mathcal{B}(A)$*. Namely, we have:*

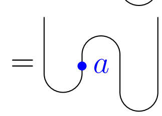


(A∪SLIDE)

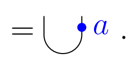
Proof:



Using (ZIGZAG) on the left

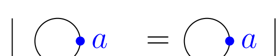


Using (A∩SLIDE)

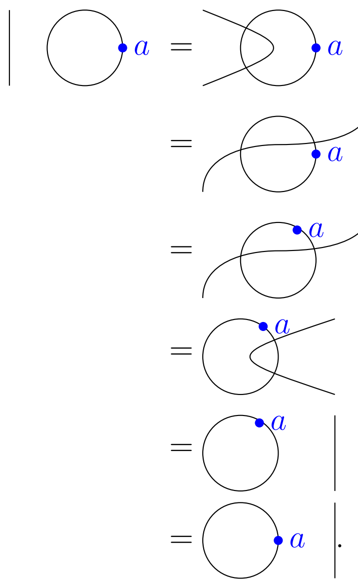


Using (ZIGZAG) on the right

∎

Lemma 6.1.7. *For all* $a \in A$*, we have*  *In other words, bubbles with tokens on them are central with respect to* $\otimes$*. Taking* $a = 1_A$ *and using* (TOKEN)*, this also shows that tokenless bubbles are central.*

Proof: We have:



Using (DCROSS)

Using (∩SLIDE)

Using (A×SLIDE2)

Using (∪SLIDE)

Using (DCROSS)

∎

Definition 6.1.8 (Center, cocenter)**.** The *center of* A, denoted $Z(A)$, is the subalgebra of A generated by all homogeneous elements $a \in A$ such that $ab = (-1)^{\bar{a}\bar{b}}ba$ for all $b \in A$.

The *cocenter of* A, denoted $C(A)$, is the quotient of A by the subspace generated by $\{ab - (-1)^{\bar{a}\bar{b}}ba \mid a, b \in A,\ \text{both } a \text{ and } b \text{ are homogeneous}\}$. Put another way, $C(A)$ is the super vector space obtained from A by identifying ab and $(-1)^{\bar{a}\bar{b}}ba$ for all homogeneous $a, b \in A$. Note that $C(A)$ is in general only a super vector space; the product from A may not induce a superalgebra structure on $C(A)$.

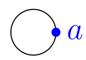
Lemma 6.1.9. *The super vector spaces $Z(A)$ and $C(A)'$ are isomorphic.*

Proof: The trace map for A yields such an isomorphism. Recall that, by the definition of a Frobenius superalgebra, $x \mapsto (y \mapsto \mathrm{tr}(xy))$ is an isomorphism between A and A'. Thus it suffices to show that $\tau_x := y \mapsto \mathrm{tr}(xy)$ is an element of $C(A)'$ whenever $x \in Z(A)$. Let $x \in Z(A)$ and $y, z \in A$. Then we have:

$$\begin{aligned}
\tau_x(yz) &= \mathrm{tr}(xyz) && \\
&= (-1)^{\overline{xy}\cdot\bar{z}}\,\mathrm{tr}(zxy) && \text{Since tr is symmetric} \\
&= (-1)^{\bar{x}\bar{z}+\bar{y}\bar{z}}\,\mathrm{tr}(zxy) && \\
&= (-1)^{\bar{y}\bar{z}}\,\mathrm{tr}(xzy) && \text{Since } x \in Z(A) \\
&= (-1)^{\bar{y}\bar{z}}\tau_x(zy), &&
\end{aligned}$$

which shows that we indeed have $\tau_x \in C(A)'$. ∎

Definition 6.1.10. Let $\delta\colon A \to \Bbbk$ be a linear map that satisfies $\delta(ab) = (-1)^{\bar{a}\bar{b}}\delta(ba)$ for all $a, b \in A$. Note that this is the same as saying that δ is well-defined as a linear map from $C(A)$ to $\Bbbk$. The *specialized Frobenius Brauer category associated to A with parameter δ*, denoted $\mathcal{B}(A,\delta)$, is $\mathcal{B}(A)$ with one additional relation imposed:

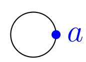


$$= \delta(a), \qquad \text{(BUBBLE)}$$

Where we write just $\delta(a)$ for the automorphism of the unit object obtained by scaling the identity map by $\delta(a)$.

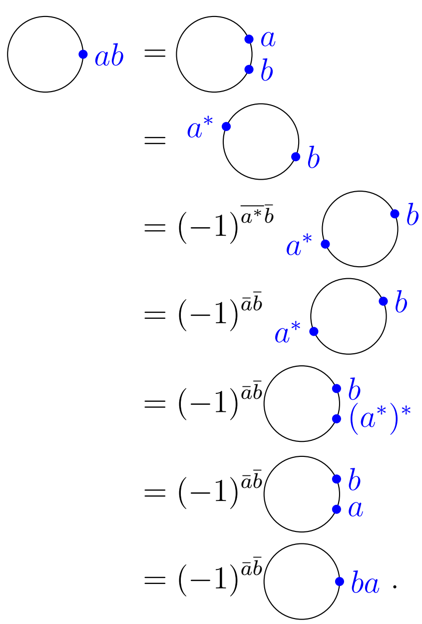
Note that all of the additional relations that we have shown to hold in $\mathcal{B}(A)$ also hold in $\mathcal{B}(A,\delta)$ since all of the defining relations for $\mathcal{B}(A)$ are also part of the definition of $\mathcal{B}(A,\delta)$.

Remark 6.1.11. Lemma 6.1.9 tells us that picking a linear map $\delta\colon C(A) \to \Bbbk$ in the definition of $\mathcal{B}(A,\delta)$ is equivalent to choosing an element of $Z(A)$. With this in mind, we could equivalently define $\mathcal{B}(A,d)$, for some $d \in Z(A)$, by imposing the following additional relation on $\mathcal{B}(A)$:



$$= \operatorname{tr}(da). \qquad \text{(BUBBLE}'\text{)}$$

Remark 6.1.12. The condition $\delta(ab) = (-1)^{\bar{a}\bar{b}}\delta(ba)$ in the definition of $\mathcal{B}(A,\delta)$ is required to prevent the category from becoming degenerate. For $a, b \in A$, we have the following chain of equalities:



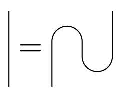
Using (TOKEN)
Using (A∩SLIDE)
Using SINTER
Since $-^*$ is even
Using (A∪SLIDE)
Since $-^*$ is self-inverse
Using (TOKEN)

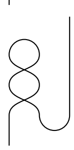
Applying (BUBBLE) to the first and last diagrams in this calculation, we obtain the identity $\delta(ab) \cdot \mathrm{id} = (-1)^{\bar{a}\bar{b}}\delta(ba) \cdot \mathrm{id}$ in $\mathcal{B}(A, \delta)$, where id denotes the identity map on the unit object. This implies that:

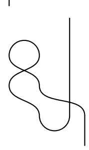
$$(\delta(ab) - (-1)^{\bar{a}\bar{b}}\delta(ba)) \cdot \mathrm{id} = 0, \tag{6.1.2}$$

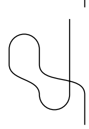
where 0 is the zero map from the unit object to itself. If $\delta(ab) \neq (-1)^{\bar{a}\bar{b}}\delta(ba)$ as elements of $\Bbbk$, then $\delta(ab) - (-1)^{\bar{a}\bar{b}}\delta(ba)$ is invertible. Scaling each side of (6.1.2) by this inverse yields $\mathrm{id} = 0$. Since $\mathcal{B}(A, \delta)$ is a strict monoidal supercategory, we have $f = f \otimes \mathrm{id}$ for all morphisms f, as noted in Definition 2.2.11. By the elementary properties of a $\Bbbk$-linear category, this forces all morphism spaces in $\mathcal{B}(A, \delta)$ to be degenerate, in the sense that they consist of only the zero map.

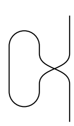
Remark 6.1.13. There are several possible alternate presentations of $\mathcal{B}$, $\mathcal{B}(A)$ and $\mathcal{B}(A, \delta)$. For instance, one could assume the cup version of (∩SLIDE) rather than the cap version. Looking at all of the diagrams in the proof of Lemma 6.1.2 upside-down yields a proof that the cup relation implies the cap relation. Similarly, one could assume the cup versions of (∩DELOOP) and (A∩SLIDE) rather than the cap versions. Another option is to assume only one of the self-dual equations from (ZIGZAG), but both of the sliding relations, i.e. (∩SLIDE) and (∪SLIDE). The other self-dual equation is implied by these assumptions in the following way:

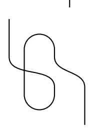
 Using the assumed half of (ZIGZAG)

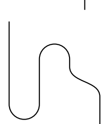
$=$  Using (DCROSS)

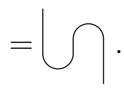
$=$  Using (∪SLIDE)

$=$  Using (∩DELOOP)

$=$ 

$=$  Using (∩SLIDE)

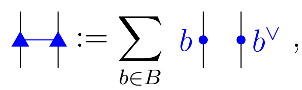
$=$  Using (∪DELOOP)



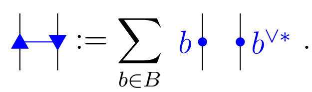
One could also assume the other half of (ZIGZAG) together with both sliding relations and the cup version of (∩DELOOP). The above proof, viewed upside-down, shows that the other self-dual equation follows from these assumptions.

6.2 Affine Frobenius Brauer Categories

Definition 6.2.1 (Teleporters)**.** We define the *ordinary teleporter* to be following morphism:

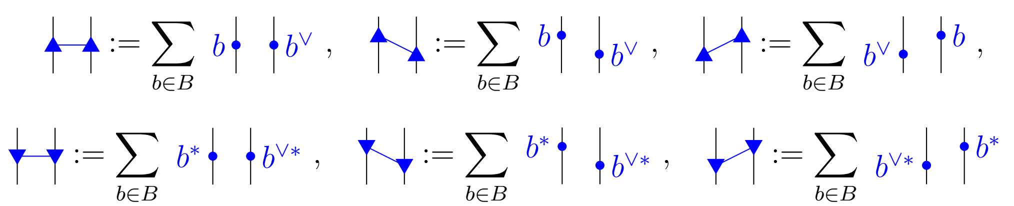


and the *reflecting teleporter* to be the following morphism:

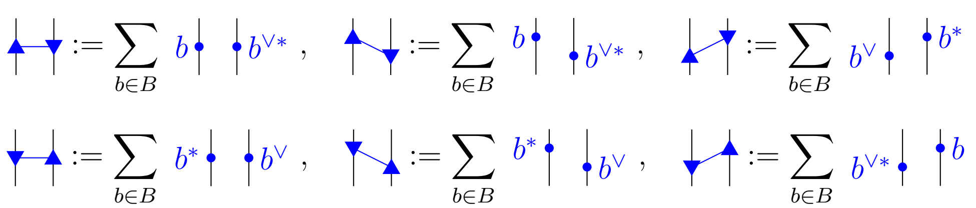


The reason for these names will be explained in Lemma 6.2.6. The triangles are called *teleporter endpoints*.

Definition 6.2.2. There are many alternate ways to draw the two teleporters defined above; using the following alternate forms, results like Corollaries 6.2.7 and 6.2.8 can be stated in a much more visually intuitive way. In the following several lemmas, we will show that all of the morphisms on the following two lines are equal (note that the first morphism is the ordinary teleporter):

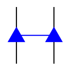
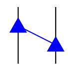
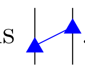


We will also show that all of the following morphisms are equal (note that the first morphism is the reflecting teleporter):



Note that the dual tokens always appear on the strand whose teleporter endpoint is vertically lower (or on the right strand, if the two endpoints are on the same horizontal line). Also note that upwards-facing endpoints correspond to plain tokens, i.e. those labelled with b or $b^\vee$, whereas downwards-facing endpoints correspond to involuted tokens, i.e. those labelled with b^* or $b^{\vee *}$.

Lemma 6.2.3. *Teleporters are basis-independent. Moreover, each of the teleporters on the first line of diagrams in Definition 6.2.2 are equal to those directly below them on the second line, and similarly for the third and fourth lines.*

Proof: Recall that Proposition 4.1.8 tells us that $\sum_{b\in B} b \otimes b^\vee$ is basis-independent. Using (TOKEN), all of the steps in that proof can be repeated in terms of string diagrams to show that  and  are basis-independent. An almost-identical proof shows that $\sum_{b\in B} b^\vee \otimes b$ is basis-independent, and thus so is . Using Lemma 4.2.15, the sums defining the teleporters on the second line can be obtained from those on the first line by summing over the involuted basis $B^* = \{b^* \mid b \in B\}$. Due to the aforementioned basis-independence, this shows that the teleporters on second line are equal to those immediately above them.

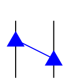
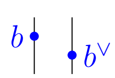
The proof for the third and fourth line of teleporters is analogous, using the fact that $\sum_{b\in B} b \otimes b^{\vee *}$ is basis-independent in place of Proposition 4.1.8. To see why this result holds, let B and C be arbitrary bases of A. Then:

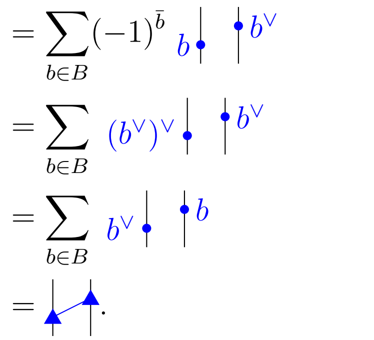
$$\begin{aligned}
\sum_{b\in B} b \otimes b^{\vee *} &= \sum_{b\in B}\sum_{c\in C} \mathrm{tr}(c^\vee b) c \otimes b^{\vee *} && \text{Using Lemma 4.1.5}\\
&= \sum_{c\in C} c \otimes \sum_{b\in B} \mathrm{tr}(c^{\vee *} b^*) b^{\vee *} && \text{Using Lemma 4.2.14}\\
&= \sum_{c\in C} c \otimes \sum_{b\in B} \mathrm{tr}(c^{\vee *} b^*) b^{* \vee} && \text{Using Lemma 4.2.15}\\
&= \sum_{c\in C} c \otimes c^{\vee *}, && \text{Using Lemma 4.1.5}
\end{aligned}$$

as desired. ∎

Lemma 6.2.4. *The three teleporters on each line of diagrams in Definition 6.2.2 are all equal.*

Proof: In light of the previous lemma, it suffices to show that the teleporters on line 1 are all equal, and so are those on line 3. The first two teleporters on each line are equal due to the first equality in (SINTER). For the remaining equality on line 1, we have:

 $= \sum_{b\in B}$ 

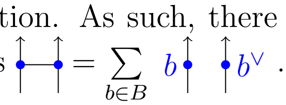


Using (SINTER)

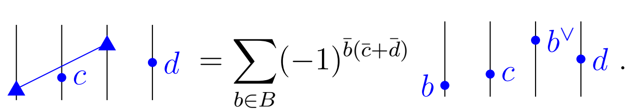
Using Lemma 4.2.8

By switching the sum to be over $B^\vee$

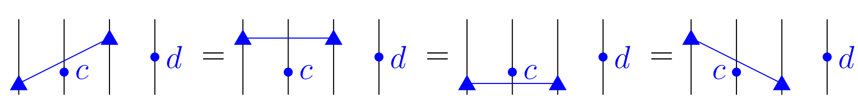
The proof for the teleporters on line 3 is totally analogous. ∎

The previous two lemmas tell us that we may slide the endpoints of teleporters past each other and/or simultaneously flip the direction of both endpoints without changing the represented morphism. Thus the only feature of a teleporter that ultimately matters is whether its endpoints point in the same direction or not; this is what determines whether a teleporter is ordinary or reflecting. We will see in Lemma 6.2.6 that ordinary and reflecting teleporters have different properties when $-^* \neq \mathrm{id}$, and so they are generally not equal. Of course, when $-^* = \mathrm{id}$, these two kinds of teleporters *are* equal since the applications of $-^*$ in Definition 6.2.1 vanish. In oriented Frobenius Brauer categories (see [MS21]), the Frobenius algebra A does not come equipped with an involution. As such, there is only one type of teleporter in the oriented setting, defined as  .

Remark 6.2.5. While we will not need this level of generality, one can also use teleporters within larger diagrams. When doing so, one should add the sign term $(-1)^{\bar{b}x}$ to the sum defining the teleporter, where x is the sum of the parities of all tokens in the diagram vertically between the two teleporter endpoints. For example,

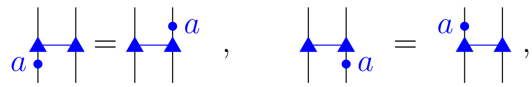


This convention ensures that one can freely slide the endpoints of teleporters along strands; the sign terms arising from (SINTER) do not need to be actively tracked since they are incorporated into the definition of the teleporters. For instance, we have:

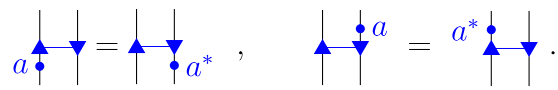


Lemma 6.2.6. *Tokens teleport across teleporters, moving from the base of one triangle to the tip of the other (or vice versa). If the teleporter endpoints point in the same direction, the label on the token is unchanged. If the endpoints point in opposite*

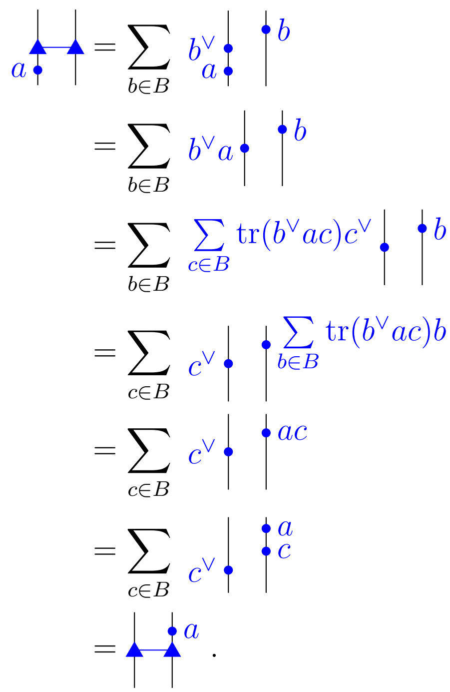
directions, $-^*$ must be applied to the label after teleportation. That is, for any $a \in A$, we have:



and



Proof: For the first identity, we have:



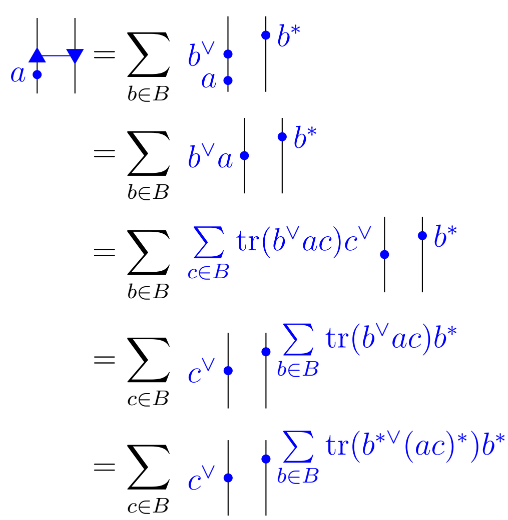
Using (TOKEN)

Using Lemma 4.1.5

Using Lemma 4.1.5

Using (TOKEN)

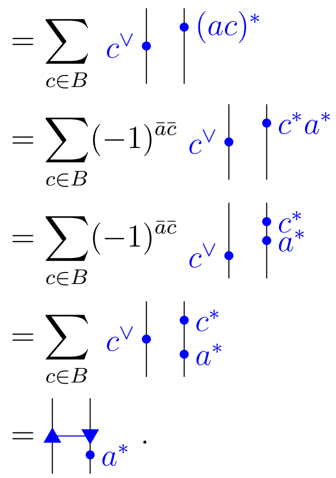
The proof for the second identity is similar. For the third identity, we have:



Using (TOKEN)

Using Lemma 4.1.5

Using Lemmas 4.2.14 and 4.2.15



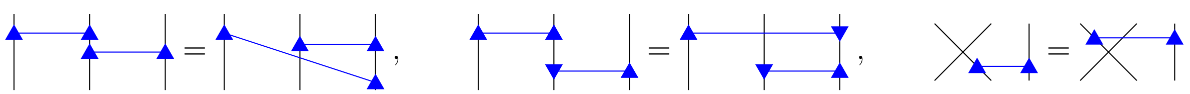
Using Lemma 4.1.5

Using (TOKEN)

Using (SINTER)

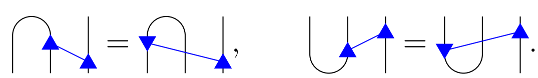
The proof for the fourth identity is similar to that for the third one. ∎

Corollary 6.2.7. *The endpoints of teleporters can teleport across other teleporters and slide through crossings. When an endpoint teleports across an ordinary teleporter or slides through a crossing, its orientation is maintained. When teleporting across a reflecting teleporter, the orientation of the endpoint must be flipped. For instance, we have:*



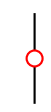
Proof: Recall that teleporters are defined via a sum of tokens. Using Lemma 6.2.6 and (A×SLIDE1), one can slide these tokens through crossings and teleport them across a teleporter, picking up an application of $-^*$ on each token if it is a reflecting teleporter. The result then follows from Definition 6.2.1. ∎

Corollary 6.2.8. *Teleporter endpoints slide through cups and caps, flipping orientation in the process. For instance,*

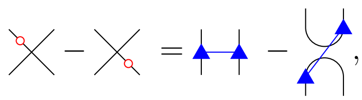


Proof: The proof is analogous to that for the previous corollary, using (A∩SLIDE) and (A∪SLIDE) rather than Lemma 6.2.6 and (A×SLIDE1). ∎

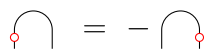
Definition 6.2.9 (Affine Frobenius Brauer category)**.** The *affine Frobenius Brauer category associated to* A, denoted $\mathcal{AB}(A)$, is obtained from $\mathcal{B}(A)$ by adjoining a new even generating morphism, written as



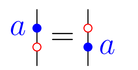
and called an *(affine) dot*, together with the following relations, for all $a \in A$:



(DOT×SLIDE1)



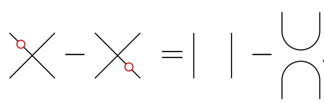
(DOT∩SLIDE)



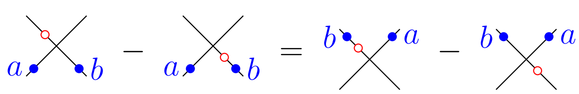
(DOTSWAP)

For any linear map $\delta\colon A \to \Bbbk$ satisfying $\delta(ab) = (-1)^{\bar{a}\bar{b}}\delta(ba)$ for all $a, b \in A$, we also define the *specialized affine Frobenius Brauer category associated to* A *with parameter* δ, denoted $\mathcal{AB}(A, \delta)$, by imposing (BUBBLE) on $\mathcal{AB}(A)$.

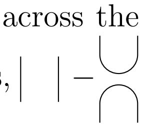
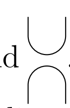
Remark 6.2.10. In the ordinary (i.e. non-Frobenius) affine Brauer category $\mathcal{AB}$, which was studied, for instance, in [RS18], the dot satisfies the following sliding relation:



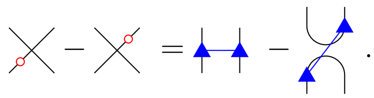
(Note: due to differing conventions, the relation in [RS18, Def 1.2] has the right hand side negated.) The Frobenius generalization of this relation is (DOT×SLIDE1). To see why the introduction of teleporters is natural, consider sliding two tokens from the bottom to the top of the left hand side of the relation:



Using (A×SLIDE1) and (DOTSWAP)

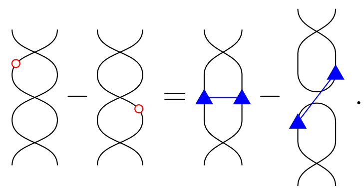
Note that the tokens have swapped positions horizontally. In order for the Frobenius generalization of the dot sliding relation to be coherent, tokens must slide in the same way across the left and right hand sides of the relation. Without introducing teleporters,  clearly demonstrates a different token sliding behaviour; indeed, tokens do not even slide in the same way across  and . In order to get the correct sliding behaviour, tokens on the bottom left of these diagrams need to teleport to the top right, and tokens on the bottom right need to teleport to the top left. The teleporters featured in (DOT×SLIDE1) allow precisely this kind of teleportation.

Lemma 6.2.11. *A mirrored analogue of* (DOT×SLIDE1) *also holds in* $\mathcal{AB}(A)$. *Namely, we have:*



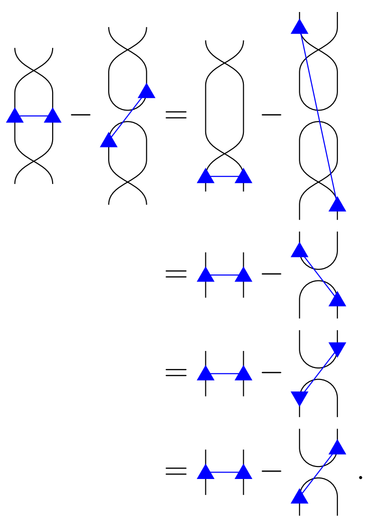
(DOT×SLIDE2)

Proof: Adjoining a crossing to the top and bottom of (DOT×SLIDE1) yields:



(6.2.1)

Working with the diagram on the right, we have:



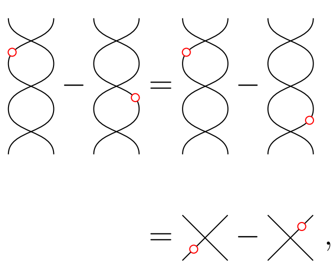
Using Corollary 6.2.7

Using (DCROSS) on the left and (∩DELOOP) and (∪DELOOP) on the right

Using Corollary 6.2.8

Using Lemma 6.2.3

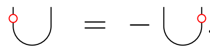
Working with the left hand side of (6.2.1), we have:



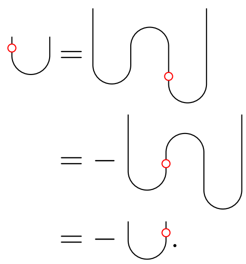
Using (DCROSS) twice

which proves that (DOT×SLIDE2) holds. ∎

Lemma 6.2.12. *The cup analogue of* (DOT∩SLIDE) *also holds in* $\mathcal{AB}(A)$*. Namely, we have:*


(DOT∪SLIDE)

Proof:



Using (ZIGZAG) on the left

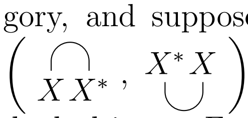
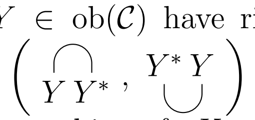
Using (DOT∩SLIDE)

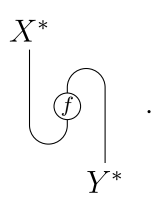
Using (ZIGZAG) on the right

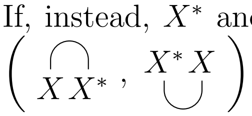
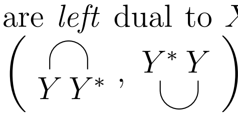
∎

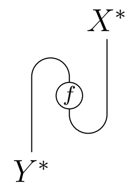
6.3 Rigidity and Pivotality of $\mathcal{AB}(A)$

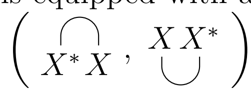
Definition 6.3.1 (Rigid supercategory)**.** A *rigid* supercategory is a monoidal supercategory in which every object has both a right and left dual. Note that being rigid is a property that a supercategory may have, rather than a structure it may be equipped with. The definition does not involve choosing particular dual objects or (co)evaluation morphisms – it just asserts that every object has at least one right dual, and at least one left dual. One can also define *left rigid* and *right rigid* categories, wherein only left (resp. right) duals are assumed to exist for all objects.

Definition 6.3.2 (Left mate, right mate)**.** Let $\mathcal{C}$ be a strict monoidal supercategory, and suppose $X, Y \in \mathrm{ob}(\mathcal{C})$ have right duals X^* and Y^*, respectively. Let  and  be evaluation-coevalution pairs for these dual objects. For any morphism $f\colon X \to Y$, the *right mate* of f is the following morphism:



If, instead, X^* and Y^* are *left* dual to X and Y, with evaluation-coevalution pairs  and , then the *left mate* of f is the following morphism:



Definition 6.3.3 (Strict pivotal supercategory)**.** A *strict pivotal* supercategory is a strict monoidal supercategory wherein every object X is equipped with a right dual X^* together with a fixed evaluation-coevaluation pair  , with this data satisfying the following additional conditions, for all objects X and Y (with $\mathbb{1}$ denoting the monoidal unit in the category):

(a) $\mathbb{1}^* = \mathbb{1}$;

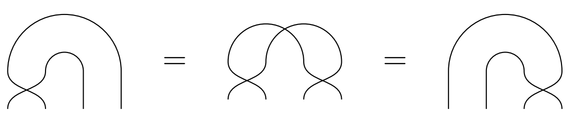
(b) $(X^*)^* = X$;

(c) The evaluation-coevaluation pair for $X \otimes Y$ is the nested one constructed in the proof of Proposition 3.2.3 (which necessitates that $(X \otimes Y)^* = Y^* \otimes X^*$);

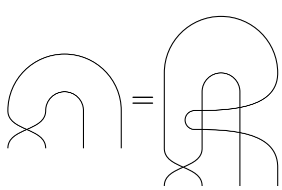
(d) For every morphism $f\colon X \to Y$, the right and left mates of f are equal.

Note that the condition $(X^*)^* = X$ says that X^* – the chosen right dual for X – is also left dual to X. Thus left and right duals coincide in a strict pivotal supercategory. Also note that every strict pivotal supercategory is rigid.

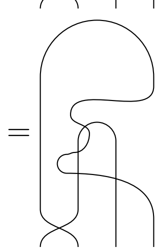
Lemma 6.3.4. *In $\mathcal{B}$, we have*

 (6.3.1)

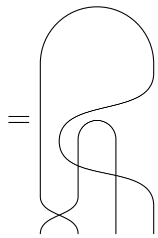
Proof:



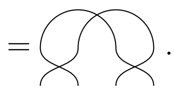
Using (DCROSS)



Using ($\cap$SLIDE)

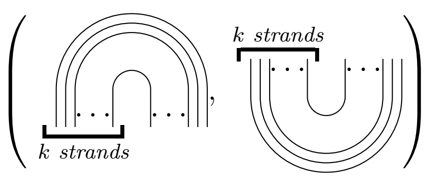


Using (DCROSS)



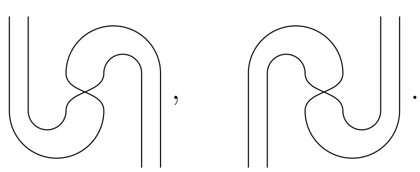
The other equality is proved in the same way; simply mirror all of the manipulations horizontally. ∎

Proposition 6.3.5. *The category $\mathcal{AB}(A)$ is strict pivotal, as are $\mathcal{AB}(A,\delta), \mathcal{B}(A)$, $\mathcal{B}(A,\delta)$, and $\mathcal{B}$. All of the objects in these categories are self-dual. The evaluation-coevaluation pair for the k'th tensor power of the generating object is:*

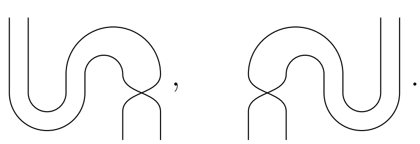


In the special case $k = 0$, this pair should be interpreted as $(\mathrm{id}_{\mathbb{1}}, \mathrm{id}_{\mathbb{1}})$.

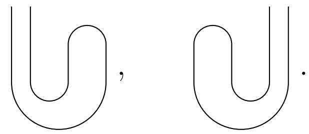
Proof: Relation (ZIGZAG) precisely says that the generating object is self-dual. Repeated applications of Proposition 3.2.3 show that all nontrivial tensor powers of the generating object are also self-dual via the nested cups and caps defined above. The fact that $\mathbb{1}$ is self-dual follows from the fact that $\mathrm{id}_{\mathbb{1}} = \mathrm{id}_{\mathbb{1}} \otimes \mathrm{id}_{\mathbb{1}}$ in any strict monoidal category. All that remains is to verify the condition on mates. Based on the definition of $\mathcal{AB}(A)$, it suffices to check that the left and right mate of each generating morphism coincide. For all generators except the affine dot, this follows immediately from Proposition 6.6.7. In the following, we give a more elementary and direct proof. The right and left mates of the crossing are:



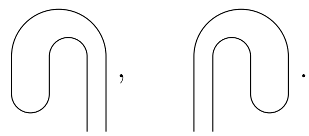
Using Lemma 6.3.4, these two diagrams are equal to, respectively, the following diagrams:



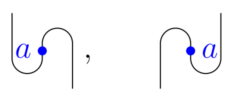
Using Proposition 3.2.3 (i.e. the comment above regarding the self-duality of powers of the generating object), we may replace the nested cups and caps in each diagram with the identity on two strands. Thus both diagrams are equal to , and so the two mates coincide. Next, the right and left mates of the cap morphism are:



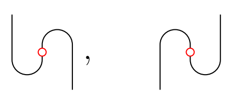
Using (ZIGZAG) on each diagram, we see that they are both equal to , and so they coincide. The right and left mates of the cup morphism are:



Using (ZIGZAG) on each diagram, we see that they are both equal to , and so they coincide. The right and left mates of a Frobenius token morphism are:



Using (A∩SLIDE) and then (ZIGZAG) on each diagram, we see that they are both equal to a^* , and thus they coincide. Finally, the right and left mates of the affine dot morphism are:



Using (DOT∩SLIDE) and then (ZIGZAG) on each diagram, we see that they are both equal to −, and thus they coincide. ∎

6.4 Functor

Throughout the rest of the chapter, we write $V = A^{m|2n}$ for the sake of brevity.

Definition 6.4.1. Define $\phi\colon V \times V \to \Bbbk$ by $\phi = \mathrm{tr} \circ \varphi$, where φ is the unimodular Hermitian form specified in Definition 5.4.9 and tr is the trace map on A. Lemma 5.2.5 tells us that ϕ is a symmetric nondegenerate bilinear form on V.

Proposition 6.4.2. *With respect to ϕ, the dual basis of $B_{m|2n}$ (see Definition 5.4.1) is given by*

$$(b_i)^\vee = (-1)^{p(i)(\bar{b}+p(i))+\iota(i^\pi)}(b^{\vee *})_{i^\pi}.$$

Proof: Let $b, c \in B$ and $i, j \in \{1, \dots, m+2n\}$. We compute:

$$\begin{aligned}
\phi((-1)^{p(j)(\bar{c}+p(j))+\iota(j^\pi)}(c^{\vee *})_{j^\pi}, b_i) &= (-1)^{p(j)(\bar{c}+p(j))+\iota(j^\pi)} \operatorname{tr}((P(c^{\vee *})_{j^\pi})^{\mathrm{st}\,*} b_i) \\
&= (-1)^{p(j)(\bar{c}+p(j))} \operatorname{tr}((c^{\vee *})_j^{\mathrm{st}\,*} b_i) \\
&= \operatorname{tr}((c^\vee)_j^\top b_i) \\
&= \delta_{ij} \operatorname{tr}(c^\vee b) \\
&= \delta_{ij}\delta_{bc},
\end{aligned}$$

as desired. ∎

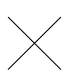
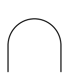
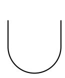
Lemma 6.4.3. *For all $v, w \in V$ and $a \in A$, we have $\phi(va, w) = (-1)^{\bar{a}\bar{w}}\phi(v, wa^*)$.*

Proof: We compute:

$$\begin{aligned}
\phi(va, w) &= \operatorname{tr}(\varphi(va, w)) \\
&= (-1)^{\bar{a}\bar{v}} \operatorname{tr}(a^*\varphi(v, w)) && \text{Since } \varphi \text{ is sesquilinear} \\
&= (-1)^{\bar{a}\bar{w}} \operatorname{tr}(\varphi(v, w)a^*) && \text{Since tr is symmetric} \\
&= (-1)^{\bar{a}\bar{w}} \operatorname{tr}(\varphi(v, wa^*)) && \text{Since } \varphi \text{ is sesqilinear} \\
&= (-1)^{\bar{a}\bar{w}}\phi(v, wa^*).
\end{aligned}$$

∎

Definition 6.4.4. Define a $\Bbbk$-linear monoidal superfunctor $F\colon \mathcal{B}(A) \to \mathfrak{osp}_{m|2n}(A)$- mod (see Definition 5.1.10) by mapping the generating object of $\mathcal{B}(A)$ to V and its generating morphisms as follows:

f	$F(f)$
	$X\colon v \otimes w \mapsto (-1)^{\bar{v}\bar{w}} w \otimes v$
	$\cap\colon v \otimes w \mapsto \phi(v, w)$
	$\cup\colon \alpha \mapsto \alpha \sum_{b \in B_{m\|2n}} b \otimes b^\vee$
	$\tilde{a}\colon x \mapsto (-1)^{\bar{a}\bar{x}} xa^*$

In the definition of $\cup$, $b^\vee$ refers to the dual of b with respect to ϕ. We check that all of the specified images are valid $\mathfrak{osp}_{m|2n}(A)$-mod morphisms in Proposition 6.4.6. In Proposition 6.4.7, we show that F is well-defined. We show that this superfunctor is also well-defined as a superfunctor from $\mathcal{B}(A,\delta)$ to $\mathfrak{osp}_{m|2n}(A)$-mod (for a particular choice of δ) in Corollary 6.4.9.

Lemma 6.4.5. *Let $T \in \mathfrak{osp}_{m|2n}$. Then* $T \cdot \cup(1) = T \cdot \left(\sum_{b\in B_{m|2n}} b \otimes b^\vee\right) = 0.$

Proof: We first compute:

$$\begin{aligned}
\sum_{b\in B_{m|2n}} Tb\otimes b^\vee &= \sum_{b\in B_{m|2n}}\sum_{c\in B_{m|2n}} \phi(c^\vee, Tb)c\otimes b^\vee && \text{Using Lemma 4.1.5}\\
&= \sum_{b\in B_{m|2n}} c\otimes \sum_{c\in B_{m|2n}} -(-1)^{\bar{T}\bar{c}}\phi(Tc^\vee, b)b^\vee && \text{Using the defining property of } \mathfrak{osp}_{m|2n}(A)\\
&= -\sum_{c\in B_{m|2n}} (-1)^{\bar{T}\bar{c}} c\otimes \left(\sum_{b\in B_{m|2n}} \phi(Tc^\vee, b)b^\vee\right)\\
&= -\sum_{c\in B_{m|2n}} (-1)^{\bar{T}\bar{c}} c\otimes Tc^\vee. && \text{Using Lemma 4.1.5}
\end{aligned}$$

Using this identity, we obtain:

$$\begin{aligned}
T\cdot\left(\sum_{b\in B_{m|2n}} b\otimes b^\vee\right) &= \sum_{b\in B_{m|2n}}\left(Tb\otimes b^\vee + (-1)^{\bar{T}\bar{b}} b\otimes Tb^\vee\right)\\
&= \left(\sum_{b\in B_{m|2n}} Tb\otimes b^\vee\right) + \sum_{b\in B_{m|2n}} (-1)^{\bar{T}\bar{b}} b\otimes Tb^\vee\\
&= 0.
\end{aligned}$$

∎

Proposition 6.4.6. *All of the image morphisms specified in Definition 6.4.4 are indeed morphisms in $\mathfrak{osp}_{m|2n}(A)$-mod. That is, they are all $\mathfrak{osp}_{m|2n}$-equivariant linear maps.*

Proof: Let $T \in \mathfrak{osp}_{m|2n}$, $a \in A$, $\alpha \in \Bbbk$, and $x, y \in V$. Then we have:

$$X(T\cdot x\otimes y) = X(Tx\otimes y + (-1)^{\bar{T}\bar{x}} x\otimes Ty)$$

$$
\begin{aligned}
&= (-1)^{\bar{T}\bar{y}+\bar{x}\bar{y}} y \otimes Tx + (-1)^{\bar{x}\bar{y}} Ty \otimes x && \text{By the definition of } X \\
&= (-1)^{\bar{x}\bar{y}} \left(Ty \otimes x + (-1)^{\bar{T}\bar{y}} y \otimes Tx \right) \\
&= (-1)^{\bar{x}\bar{y}} (T \cdot y \otimes x) \\
&= T \cdot X(x \otimes y), && \text{By the definition of } X
\end{aligned}
$$

so X is $\mathfrak{osp}_{m|2n}$-equivariant. Next:

$$
\begin{aligned}
\cap(T \cdot x \otimes y) &= \cap(Tx \otimes y + (-1)^{\bar{T}\bar{x}} x \otimes Ty) \\
&= \phi(Tx, y) + (-1)^{\bar{T}\bar{x}} \phi(x, Ty) && \text{By the definition of } \cap \\
&= \phi(Tx, y) - \phi(Tx, y) && \text{Using the defining property of } \mathfrak{osp}_{m|2n}(A) \\
&= 0 \\
&= T \cdot \phi(x, y) && \text{By the definition of } \cdot \text{ on } \Bbbk \\
&= T \cdot \cap(x \otimes y),
\end{aligned}
$$

so $\cap$ is $\mathfrak{osp}_{m|2n}$-equivariant. Next:

$$
\begin{aligned}
\cup(T \cdot \alpha) &= \cup(0) && \text{By the definition of } \cdot \text{ on } \Bbbk \\
&= 0 \cdot \sum_{b \in B_{m|2n}} b^\vee \otimes b && \text{By the definition of } \cup \\
&= 0 \\
&= 0 \cdot \alpha \\
&= T \cdot \cup(1)\alpha && \text{By Lemma 6.4.5} \\
&= T \cdot \cup(\alpha), && \text{Since } \cup \text{ is linear}
\end{aligned}
$$

so $\cup$ is $\mathfrak{osp}_{m|2n}$-equivariant. Finally:

$$
\begin{aligned}
\tilde{a}(T \cdot x) &= \tilde{a}(Tx) \\
&= (-1)^{\bar{a}\bar{T}+\bar{a}\bar{x}} Txa^* && \text{By the definition of } \tilde{a} \\
&= (-1)^{\bar{a}\bar{T}} T((-1)^{\bar{a}\bar{x}} xa^*) \\
&= (-1)^{\bar{a}\bar{T}} T\tilde{a}(x) \\
&= (-1)^{\bar{a}\bar{T}} T \cdot \tilde{a}(x).
\end{aligned}
$$

Keeping in mind that $\tilde{a}$ has parity $\bar{a}$, this shows that $\tilde{a}$ is $\mathfrak{osp}_{m|2n}$-equivariant. ∎

Proposition 6.4.7. *All of the relations that define $\mathcal{B}(A)$ are satisfied by the appropriate F-images in $\mathfrak{osp}_{m|2n}(A)$-mod. Thus F is well-defined. For brevity, we will refer to the F-images of these relations as if they were the relations themselves, e.g. (DCROSS) asserts that $X \circ X = \mathrm{id}_{V \otimes V}$.*

Proof: Relations (DCROSS) and (BRAID) are straightforward to verify by direct computation. The first part of (TOKEN) says that $\tilde{1_A} = \mathrm{id}_V$, which is true since $1_A^* = 1_A$. The second part of the relation says that $\widetilde{(\lambda a + \mu b)} = \lambda\tilde{a} + \mu\tilde{b}$. This property follows immediately from the fact that $-^*$ is linear. Finally, the last part of the relation says that $\tilde{a} \circ \tilde{b} = \widetilde{ab}$, which follows from the calculation in Definition 5.1.5.

Relation ZIGZAG follows from the fact that ϕ is a nondegenerate bilinear form; the desired identities are the ones established in Lemma 4.1.5.

Relation (∩DELOOP) says that ϕ is symmetric, which was noted to be the case in Definition 6.4.1.

Relation (∩SLIDE) asserts that $(-1)^{\bar{x}\bar{y}}\phi(x,z)y = (-1)^{\bar{y}\bar{z}}\phi(x,z)y$. If $\bar{x} \neq \bar{z}$, both sides of the identity are 0 since ϕ is even. If $\bar{x} = \bar{z}$, then the identity also clearly holds.

Relation (A×SLIDE1) holds since we have:

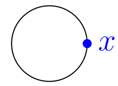
$$\begin{aligned} X((\tilde{a} \otimes \mathrm{id})(x \otimes y)) &= (-1)^{\bar{a}\bar{x}+\bar{x}\bar{y}+\bar{a}\bar{y}} y \otimes xa^* \\ &= (-1)^{\bar{x}\bar{y}+\bar{a}\bar{y}} y \otimes \tilde{a}(x) \\ &= (-1)^{\bar{x}\bar{y}}(\mathrm{id} \otimes \tilde{a})(y \otimes x) \quad \text{Using the sign term from Definition 2.1.13} \\ &= (\mathrm{id} \otimes \tilde{a})(X(x \otimes y)). \end{aligned}$$

Finally, relation (A∩SLIDE) holds since we have:

$$\begin{aligned} \cap((\tilde{a} \otimes \mathrm{id})(x \otimes y)) &= (-1)^{\bar{a}\bar{x}}\phi(xa^*, y) \\ &= (-1)^{\bar{a}\bar{x}+\bar{a}\bar{y}}\phi(x, y(a^*)^*) \quad \text{Using Lemma 6.4.3} \\ &= \cap((\mathrm{id} \otimes \widetilde{a^*})(x \otimes y)). \end{aligned}$$

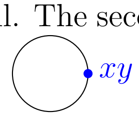
∎

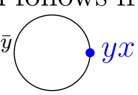
Lemma 6.4.8. *Define a map $t\colon A \to \Bbbk$ by $t(x) = \sum_{b \in B_{m|2n}} (-1)^{\bar{b}}\phi(b^\vee, bx)$. Then t is independent of the choice of basis $B_{m|2n}$ and satisfies $t(xy) = (-1)^{\bar{x}\bar{y}}t(yx)$ for all $x, y \in A$.*

Proof: Note that the F-image of the bubble 

is the linear map $\Bbbk \to \Bbbk$ given by $(\cap \circ (\mathrm{id} \otimes \tilde{x}) \circ \cup)$. We can evaluate this map as follows:

$$\begin{aligned}
(\cap \circ (\mathrm{id} \otimes \tilde{x}) \circ \cup)(1) &= \sum_{b \in B_{m|2n}} \phi(b, b^\vee x^*) \\
&= \sum_{b \in B_{m|2n}} (-1)^{\bar{b}\bar{x}} \phi(bx, b^\vee) \quad \text{Using Lemma 6.4.3} \\
&= \sum_{b \in B_{m|2n}} (-1)^{\bar{b}} \phi(b^\vee, bx) \quad \text{Since } \phi \text{ is symmetric} \\
&= t(x).
\end{aligned}$$

In other words, $t(x)$ is just the F-image of the bubble drawn above evaluated at 1. We've already shown that all of the generating morphisms in $\mathcal{B}(A)$ are basis-independent, so t is as well. The second claim follows from the calculation in Remark 6.1.12, which tells us that 

$= (-1)^{\bar{x}\bar{y}}$ 

in $\mathcal{B}(A)$. ∎

Corollary 6.4.9. *The action of the functor F given in Definition 6.4.4 also defines a functor from $\mathcal{B}(A, t)$ to $\mathfrak{osp}_{m|2n}(A)$-*mod*.*

Proof: Lemma 6.4.8 shows that t satisfies the property required by the definition of $\mathcal{B}(A, t)$ and that the F-image of (BUBBLE) is satisfied in $\mathfrak{osp}_{m|2n}(A)$- mod. All of the other required properties and relations were checked above when we showed that $F \colon \mathcal{B}(A) \to \mathfrak{osp}_{m|2n}(A)$- mod is well-defined. ∎

6.5 Affine Functor

Definition 6.5.1 (Universal enveloping superalgebra)**.** Let $\mathfrak{g}$ be a Lie superalgebra with basis $B_{\mathfrak{g}}$. The *universal enveloping superalgebra of* $\mathfrak{g}$, denoted $U(\mathfrak{g})$, is the free (unital, associative) superalgebra on $B_{\mathfrak{g}}$ subject to the following relation for all $x, y \in \mathfrak{g}$:

$$xy - (-1)^{\bar{x}\bar{y}} yx = [x, y],$$

where juxtaposition denotes multiplication in $U(\mathfrak{g})$ and $[x, y]$ is the Lie superbracket in $\mathfrak{g}$. There is a natural comultiplication map $\Delta \colon U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$ given by $\Delta(x) = x \otimes 1 + 1 \otimes x$ for all $x \in \mathfrak{g}$, where 1 denotes the formal identity in $U(\mathfrak{g})$. The action of Δ on the rest of $U(\mathfrak{g})$ is determined by the fact that it is a superalgebra homomorphism, i.e. $\Delta(uv) = \Delta(u)\Delta(v)$ for all $u, v \in U(\mathfrak{g})$.

If W is a left $\mathfrak{g}$-supermodule, then W is also a left $U(\mathfrak{g})$-supermodule with the following action: let $w \in W$ and $u \in U(\mathfrak{g})$. By the definition of $U(\mathfrak{g})$, we have $u = x_1 x_2 \cdots x_n$ for some $x_i \in \mathfrak{g}$. We then define:

$$uw = (x_1 x_2 \cdots x_n)w = x_1 \cdot (x_2 \cdot (\cdots (x_{n-1} \cdot (x_n \cdot w)))),$$

where $\cdot$ denotes the action of $\mathfrak{g}$ on W. That is, the formal multiplication in $U(\mathfrak{g})$ corresponds to the composition of $\mathfrak{g}$-actions. The relation $xy - (-1)^{\bar{x}\bar{y}} yx = [x, y]$ ensures that this $U(\mathfrak{g})$-action yields a well-defined supermodule structure. For more details on universal enveloping superalgebras, see e.g. [CW12, Subsection 1.4].

Definition 6.5.2. Define $\Omega = \sum_{M \in B^{\mathfrak{osp}}_{m|2n}} M \otimes M^\vee \in \mathfrak{osp}_{m|2n}(A) \otimes \mathfrak{osp}_{m|2n}(A)$ and $C = \sum_{M \in B^{\mathfrak{osp}}_{m|2n}} MM^\vee \in U(\mathfrak{osp}_{m|2n}(A))$. The proof of Proposition 4.1.8 shows that both Ω and C are independent of the basis $B^{\mathfrak{osp}}_{m|2n}$. We call C the *quadratic Casimir element* for $\mathfrak{osp}_{m|2n}(A)$. Note that both Ω and C are even.

Lemma 6.5.3. *For all $T \in \mathfrak{osp}_{m|2n}(A)$, we have $CT = TC$ as elements of $U(\mathfrak{osp}_{m|2n}(A))$ acting on $A^{m|2n}$.*

Proof: In the following, we write tr for the Frobenius trace map on $\mathrm{Mat}_{m|2n}(A)$ specified in Example 4.3.5. Recall that the bilinear form $(S, T) \mapsto \mathrm{tr}(ST)$ is nondegenerate on $\mathfrak{osp}_{m|2n}(A)$. Note that the formal multiplication in $U(\mathfrak{osp}_{m|2n}(A))$ coincides with matrix multiplication when acting on $A^{m|2n}$, as so we may calculate as follows:

$$\begin{aligned}
CT &= \sum_{M \in B^{\mathfrak{osp}}_{m|2n}} MM^\vee T \\
&= \sum_{M \in B^{\mathfrak{osp}}_{m|2n}} M \sum_{N \in B^{\mathfrak{osp}}_{m|2n}} \mathrm{tr}(M^\vee TN)N^\vee && \text{Using Lemma 4.1.5} \\
&= \sum_{N \in B^{\mathfrak{osp}}_{m|2n}} \sum_{M \in B^{\mathfrak{osp}}_{m|2n}} \mathrm{tr}(M^\vee TN)MN^\vee \\
&= \sum_{N \in B^{\mathfrak{osp}}_{m|2n}} TNN^\vee && \text{Using Lemma 4.1.5} \\
&= TC,
\end{aligned}$$

as desired. ∎

Corollary 6.5.4. *For all $T \in \mathfrak{osp}_{m|2n}(A)$, we have $\Omega\Delta(T) = \Delta(T)\Omega$ as elements of $U(\mathfrak{osp}_{m|2n}(A))$ acting on $A^{m|2n} \otimes A^{m|2n}$.*

Proof: A straightforward computation shows $\Omega = \frac{\Delta(C)-C\otimes 1-1\otimes C}{2}$. We then have:

$$\begin{aligned}
&\Omega\Delta(T)\\
&= \frac{\Delta(CT) - CT\otimes 1 - C\otimes T - T\otimes C - 1\otimes CT}{2}\\
&= \frac{\Delta(TC) - TC\otimes 1 - C\otimes T - T\otimes C - 1\otimes TC}{2} && \text{Using Lemma 6.5.3}\\
&= \frac{\Delta(T)\Delta(C) - (T\otimes 1 + 1\otimes T)(C\otimes 1) - (T\otimes 1 + 1\otimes T)(1\otimes C)}{2}\\
&= \Delta(T)\Omega.
\end{aligned}$$

∎

Lemma 6.5.5. *Let $c, d \in B$ and $k, l \in \{1, \ldots, m+2n\}$. Then*

$$\sum_{b\in B}(-1)^{\bar{b}(1+\overline{d_l})}(cb^\vee)_k \otimes (db)_l = (-1)^{\overline{c_k}\,\overline{d_l}}\sum_{b\in B}(-1)^{(\bar{b}+p(k)+p(l))\overline{d_l}+p(l)}(bd)_k \otimes (b^\vee c)_l.$$

Proof: We compute:

$$\begin{aligned}
&\sum_{b\in B}(-1)^{\bar{b}(1+\overline{d_l})}(cb^\vee)_k \otimes (db)_l\\
&= \sum_{b\in B}(-1)^{\bar{b}(1+\overline{d_l})}\sum_{e\in B}(cb^\vee)_k \otimes \operatorname{tr}(e^\vee db)(e)_l && \text{Using Lemma 4.1.5}\\
&= \sum_{e\in B}(-1)^{(\bar{e}+\bar{d})(1+\overline{d_l})}\sum_{b\in B}(c\operatorname{tr}(e^\vee db)b^\vee)_k \otimes (e)_l && \text{Since the summand is 0 if } \bar{b}+\bar{d}+\bar{e}\neq 0\\
&= \sum_{e\in B}(-1)^{(\bar{e}+\bar{d})(1+\overline{d_l})}(ce^\vee d)_k \otimes (e)_l && \text{Using Lemma 4.1.5}\\
&= \sum_{e\in B}(-1)^{(\bar{e}+\bar{d})(1+\overline{d_l})}\sum_{b\in B}\operatorname{tr}(b^\vee ce^\vee)(bd)_k \otimes (e)_l && \text{Using Lemma 4.1.5}\\
&= \sum_{b,e\in B}(-1)^{(\bar{e}+\bar{d})(1+\overline{d_l})+\bar{e}(\bar{b}+\bar{c})}\operatorname{tr}(e^\vee b^\vee c)(bd)_k \otimes (e)_l && \text{Using the symmetry of tr}\\
&= \sum_{b\in B}(-1)^{(\bar{b}+\bar{c}+\bar{d})(1+\overline{d_l})+(\bar{b}+\bar{c})^2}(bd)_k \otimes \left(\sum_{e\in B}\operatorname{tr}(e^\vee b^\vee c)e\right)_l && \text{Since the summand is 0 if } \bar{b}+\bar{c}+\bar{e}\neq 0\\
&= \sum_{b\in B}(-1)^{(\bar{b}+\bar{c}+\bar{d})(1+\overline{d_l})+\bar{b}+\bar{c}}(bd)_k \otimes (b^\vee c)_l && \text{Using Lemma 4.1.5}\\
&= (-1)^{\overline{c_k}\,\overline{d_l}}\sum_{b\in B}(-1)^{(\bar{b}+p(k)+p(l))\overline{d_l}+p(l)}(bd)_k \otimes (b^\vee c)_l.
\end{aligned}$$

∎

The following result will be used to prove Lemma 6.5.12.

Lemma 6.5.6. *Let $u, v \in V$. Then*

$$2(-1)^{\bar{u}\bar{v}}\Omega(v \otimes u) = \sum_{b \in B}(-1)^{\bar{b}+\bar{v}\bar{b}} ub^{\vee} \otimes vb - (-1)^{\bar{u}\bar{b}} \sum_{x \in B_{m|2n}} \phi(ub^{\vee}, v)x \otimes x^{\vee}b.$$

Proof: By linearity, it suffices to consider the case $u = c_k$ and $v = d_l$, where we have $c, d \in B$ and $k, l \in \{1, \dots, m+2n\}$. By Proposition 5.5.13, we have $\Omega = \sum_{b \in B} \sum_{j=1}^{m+2n} \sum_{i=1}^{j} \frac{\sigma^b_{i,j}(-1)^{p(j)}}{2+2\delta_{ij}} G^b_{i,j} \otimes (G^{b^{\vee *}}_{i,j})^{\mathrm{st}\,*}$. Note that this expression includes some summands that do not appear in the definition $\Omega = \sum_{M \in B^{\mathfrak{osp}}_{m|2n}} M \otimes M^{\vee}$, as $G^b_{i,j}$ may not be an element of $B^{\mathfrak{osp}}_{m|2n}$ when $i = j$; see Lemma 5.5.11. However, all of these extra summands are 0, so the value of Ω remains unchanged. Using the definition of $G^b_{i,j}$, one can directly compute:

$$\begin{aligned}(G^b_{i,j} \otimes (G^{b^{\vee *}}_{i,j})^{\mathrm{st}\,*})(d_l \otimes c_k) = {} & \frac{\delta_{jl^\pi}\delta_{ik}}{2+\delta_{kl^\pi}}(-1)^{(\overline{b_k}+p(l))\overline{d_l}+p(l)}(bd)_k \otimes (b^{\vee}c)_l \\ & - \frac{\delta_{jl^\pi}\delta_{jk}\sigma^b_{i,k}}{2+2\delta_{ik}}(-1)^{(\overline{b_k}+p(i))\overline{d_k}+\iota(k)+\iota(i^\pi)+p(k)}(bd)_i \otimes (b^{\vee *}c)_{i^\pi} \\ & - \frac{\delta_{il^\pi}\delta_{ik}\sigma^b_{k,j}}{2+\delta_{jk}}(-1)^{(\overline{b_k}+p(j))\overline{d_k}+\iota(k)+\iota(j^\pi)+p(j)}(b^* d)_j \otimes (b^{\vee}c)_{j^\pi} \\ & + \frac{\delta_{il^\pi}\delta_{jk}}{2+\delta_{kl^\pi}}(-1)^{(\overline{b_k}+p(l))\overline{d_l}+p(l)}(b^* d)_k \otimes (b^{\vee *}c)_l.\end{aligned}$$

Call the four terms on the right hand side t_1, t_2, t_3, and t_4, respectively (that is, the right hand side is $t_1 - t_2 - t_3 + t_4$). Recall that $b^* = \pm b$ for all $b \in B$ and $b^{\vee *} = b^{*\vee}$. Thus $(b^* d)_j \otimes (b^{\vee}c)_{j^\pi} = (bd)_j \otimes (b^{\vee *}c)_{j^\pi}$ and $(b^* d)_j \otimes (b^{\vee *}c)_{i^\pi} = (bd)_j \otimes (b^{\vee}c)_{i^\pi}$. Using the latter equality, we find that

$$t_1 + t_4 = \left(\frac{\delta_{jl^\pi}\delta_{ik} + \delta_{il^\pi}\delta_{jk}}{2+2\delta_{kl^\pi}}\right)(-1)^{(\overline{b_k}+p(l))\overline{d_l}+p(l)}(bd)_k \otimes (b^{\vee}c)_l.$$

Since we always have $i \le j$ in the sum defining Ω, $\delta_{jl^\pi}\delta_{ik} = 1$ is only possible when $k \le l^\pi$, and $\delta_{il^\pi}\delta_{jk}$ is only possible when $k \ge l^\pi$. Thus $\sum_{j=1}^{m+2n}\sum_{i=1}^{j} \delta_{jl^\pi}\delta_{ik} + \delta_{il^\pi}\delta_{jk} = 1 + \delta_{kl^\pi}$, and so

$$\sum_{b \in B}\sum_{j=1}^{m+2n}\sum_{i=1}^{j} t_1 + t_4 = \frac{1}{2}\sum_{b \in B}(-1)^{(\overline{b_k}+p(l))\overline{d_l}+p(l)}(bd)_k \otimes (b^{\vee}c)_l,$$

regardless of the value of δ_{kl^π}. Next, note that t_2 and t_3 are zero unless $\delta_{kl^\pi} = 1$. When we do have $k = l^\pi$, t_2 contributes a nonzero term whenever $j = k$. Since we must have $i \leq j$, we get nonzero terms corresponding to $i = 1, 2, \ldots, k$. Similarly, t_3 contributes nonzero terms corresponding to $i = k$ and $j = k, k+1, \ldots, m+2n$. Thus:

$$\begin{aligned}
\sum_{b\in B}\sum_{j=1}^{m+2n}\sum_{i=1}^{j} t_2 + t_3 &= \delta_{kl^\pi}\sum_{b\in B}\sum_{i=1}^{k}\frac{\sigma^b_{i,k}}{2+2\delta_{ik}}(-1)^{(\overline{b_k}+p(i))\overline{d_k}+\iota(k)+\iota(i^\pi)+p(k)}(bd)_i\otimes(b^{\vee *}c)_{i^\pi} \\
&\quad + \sum_{j=k}^{m+2n}\frac{\sigma^b_{k,j}}{2+2\delta_{jk}}(-1)^{(\overline{b_k}+p(j))\overline{d_k}+\iota(k)+\iota(j^\pi)+p(j)}(bd)_j\otimes(b^{\vee *}c)_{j^\pi} \\
&= \delta_{kl^\pi}\sum_{b\in B}\sum_{i=1}^{k-1}\frac{\sigma^b_{i,k}}{2}(-1)^{(\overline{b_k}+p(i))\overline{d_k}+\iota(k)+\iota(i^\pi)+p(k)}(bd)_i\otimes(b^{\vee *}c)_{i^\pi} \\
&\quad + 2\frac{\sigma^b_{k,k}}{2+2}(-1)^{(\overline{b_k}+p(k))\overline{d_k}+\iota(k)+\iota(k^\pi)+p(k)}(bd)_k\otimes(b^{\vee *}c)_{k^\pi} \\
&\quad + \sum_{j=k+1}^{m+2n}\frac{\sigma^b_{k,j}}{2}(-1)^{(\overline{b_k}+p(j))\overline{d_k}+\iota(k)+\iota(j^\pi)+p(j)}(bd)_j\otimes(b^{\vee *}c)_{j^\pi} \\
&= \delta_{kl^\pi}\sum_{b\in B}\sum_{i=1}^{m+2n}\frac{\sigma^b_{i,k}}{2}(-1)^{(\overline{b_k}+p(i))\overline{d_k}+\iota(k)+\iota(i^\pi)+p(k)}(bd)_i\otimes(b^{\vee *}c)_{i^\pi},
\end{aligned}$$

where in the last equality we combine the sums by re-indexing $j \to i$ and using the fact that $\sigma^b_{i,k}(-1)^{p(k)} = \sigma^b_{k,i}(-1)^{p(i)}$. In total, we find

$$\begin{aligned}
2(-1)^{\bar u\bar v}\Omega(v\otimes u) &= (-1)^{\bar u\bar v}\sum_{b\in B}\sum_{j=1}^{m+2n}\sum_{i=1}^{j}\frac{\sigma^b_{i,j}(-1)^{p(j)}}{2+2\delta_{ij}}(G^b_{i,j}\otimes(G^{b^{\vee *}}_{i,j})^{\mathrm{st}\,*})(d_l\otimes c_k) \\
&= (-1)^{\bar u\bar v}\Bigg((-1)^{(\overline{b_k}+p(l))\overline{d_l}+p(l)}(bd)_k\otimes(b^\vee c)_l \\
&\quad - \delta_{kl^\pi}\sum_{b\in B}\sum_{i=1}^{m+2n}\sigma^b_{i,k}(-1)^{(\overline{b_k}+p(k))\overline{d_k}+\iota(k)+\iota(i^\pi)+p(k)}(bd)_i\otimes(b^{\vee *}c)_{i^\pi}\Bigg).
\end{aligned}$$

We now work to show that the right hand side of the desired identity is equal to the above expression. Recalling that $u = c_k$ and $v = d_l$ and using Lemma 6.5.5, we have:

$$\begin{aligned}
\sum_{b\in B}(-1)^{\bar b+\bar v\bar b}ub^\vee\otimes vb &= \sum_{b\in B}(-1)^{\bar b(1+\overline{d_l})}(cb^\vee)_k\otimes(d)_l \\
&= (-1)^{\bar u\bar v}\sum_{b\in B}(-1)^{(\overline{b_k}+p(l))\overline{d_l}+p(l)}(bd)_k\otimes(b^\vee c)_l,
\end{aligned}$$

which is equal to the first summand in the expression for $2(-1)^{\bar{u}\bar{v}}\Omega(v \otimes u)$ computed above. All that remains is to show that $(-1)^{\bar{u}\bar{b}} \sum_{x \in B_{m|2n}} \phi(ub^\vee, v)x \otimes x^\vee b$ is equal to the other term. First, we compute

$$\begin{aligned}
\phi(ub^\vee, v) &= \phi((cb^\vee)_k, d_l) \\
&= \operatorname{tr}((P(cb^\vee)_k)^{\mathrm{st}\,*} d_l) && \text{By the definition of } \phi \\
&= (-1)^{\iota(k)} \operatorname{tr}((cb^\vee)^{\mathrm{st}\,*}_{k^\pi} d_l) \\
&= (-1)^{\iota(k)+p(k)(\bar{b}+\bar{c}+1)} \operatorname{tr}(((cb^\vee)^*_{k^\pi})^\top d_l) \\
&= (-1)^{\iota(k)+p(k)(\bar{b}+\bar{c}+1)} \delta_{lk^\pi} \operatorname{tr}((cb^\vee)^* d) \\
&= (-1)^{\iota(k)+p(k)(\bar{b}+\bar{c}+1)} \delta_{kl^\pi} \operatorname{tr}(cb^\vee d^*) && \text{Using Lemma 4.2.14 and the fact that } -^\pi \text{ is self-inverse} \\
&= \delta_{kl^\pi} (-1)^{\iota(k)+p(k)(\bar{b}+\bar{c}+1)+\bar{c}(\bar{b}+\bar{d})} \operatorname{tr}(b^\vee d^* c)
\end{aligned}$$

Using Lemmas 4.1.5, 4.2.14, 6.4.2, and the fact that $\operatorname{tr}(xy) = 0$ unless $\bar{x} + \bar{y} = 0$, we then have:

$$\begin{aligned}
&\sum_{b \in B} (-1)^{\bar{u}\bar{b}} \sum_{x \in B_{m|2n}} \phi(ub^\vee, v)x \otimes x^\vee b \\
&= \delta_{kl^\pi} \sum_{b \in B} (-1)^{\overline{c_k}\bar{b}+\iota(k)+p(k)(\bar{b}+\bar{c}+1)+\bar{c}(\bar{b}+\bar{d})} \sum_{a \in B} \sum_{i=1}^{m+2n} \operatorname{tr}(b^\vee d^* c) a_i \otimes (-1)^{p(i)(\bar{a}+p(i))+\iota(i^\pi)} (a^{\vee *} b)_{i^\pi} \\
&= \delta_{kl^\pi} \sum_{a \in B} \sum_{i=1}^{m+2n} (-1)^{\overline{c_k}(\bar{c}+\bar{d})+\iota(k)+p(k)(\bar{d}+1)+\bar{c}+p(i)\overline{a_i}+\iota(i^\pi)} a_i \otimes (a^{\vee *} \sum_{b \in B} \operatorname{tr}(b^\vee d^* c) b)_{i^\pi} \\
&= \delta_{kl^\pi} \sum_{a \in B} \sum_{i=1}^{m+2n} (-1)^{\overline{c_k}(\bar{c}+\bar{d})+\iota(k)+p(k)(\bar{d}+1)+\bar{c}+p(i)\overline{a_i}+\iota(i^\pi)} a_i \otimes (a^{\vee *} d^* c)_{i^\pi} \\
&= \delta_{kl^\pi} \sum_{a \in B} \sum_{i=1}^{m+2n} (-1)^{\overline{c_k}(\bar{c}+\bar{d})+\iota(k)+p(k)(\bar{d}+1)+\bar{c}+p(i)\overline{a_i}+\iota(i^\pi)} a_i \otimes \sum_{b \in B} \operatorname{tr}(a^{\vee *} d^* b^*)(b^{\vee *} c)_{i^\pi} \\
&= \delta_{kl^\pi} \sum_{b \in B} \sum_{i=1}^{m+2n} \sum_{a \in B} (-1)^{\overline{c_k}(\bar{c}+\bar{d})+\iota(k)+p(k)(\bar{d}+1)+\bar{c}+p(i)(\bar{b}+\overline{d_i})+\iota(i^\pi)+\bar{b}\bar{d}} \operatorname{tr}(a^{\vee *}(bd)^*) a_i \otimes (b^{\vee *} c)_{i^\pi} \\
&= \delta_{kl^\pi} \sum_{b \in B} \sum_{i=1}^{m+2n} \sum_{a \in B} (-1)^{\overline{c_k}(\bar{c}+\bar{d})+\iota(k)+p(k)(\bar{d}+1)+\bar{c}+p(i)(\bar{b}+\overline{d_i})+\iota(i^\pi)+\bar{b}\bar{d}} \operatorname{tr}(a^\vee bd) a_i \otimes (b^{\vee *} c)_{i^\pi} \\
&= \delta_{kl^\pi} \sum_{b \in B} \sum_{i=1}^{m+2n} (-1)^{\overline{c_k}(\bar{c}+\bar{d})+\iota(k)+p(k)(\bar{d}+1)+\bar{c}+p(i)(\bar{b}+\overline{d_i})+\iota(i^\pi)+\bar{b}\bar{d}} (bd)_i \otimes (b^{\vee *} c)_{i^\pi} \\
&= \delta_{kl^\pi} \sum_{b \in B} \sum_{i=1}^{m+2n} \sigma^b_{i,k} (-1)^{(\bar{b}+p(i)+p(k))\overline{d_k}+\iota(k)+\iota(i^\pi)+p(k)} (bd)_i \otimes (b^{\vee *} c)_{i^\pi},
\end{aligned}$$

where in the last equality we use the fact that the term is 0 unless $k = l^\pi$ to simplify the exponent on (-1). This is precisely the second term in the expression for $2(-1)^{\bar{u}\bar{v}}\Omega(v \otimes u)$ that was computed above, as desired. ∎

Definition 6.5.7. Define a monoidal superfunctor $\mathcal{F}\colon \mathcal{AB}(A) \to \mathrm{End}(\mathfrak{osp}_{m|2n}(A)\text{- mod})$ as follows: $\mathcal{F}$ sends the generating object of $\mathcal{AB}(A)$ to the tensor endomorphism $V \otimes -\colon \mathfrak{osp}_{m|2n}(A)\text{- mod} \to \mathfrak{osp}_{m|2n}(A)\text{- mod}$. For each generator $f \in \{\times, \cap, \cup, \bullet a\}$, let $\mathcal{F}(f)$ be the supernatural transformation with components $\mathcal{F}(f)_W = F(f) \otimes \mathrm{id}_W$, where $W \in \mathfrak{osp}_{m|2n}(A)\text{- mod}$ and $F\colon \mathcal{B}(A) \to \mathfrak{osp}_{m|2n}(A)\text{- mod}$ is the functor specified in Definition 6.4.4. Finally, we define $\mathcal{F}(\circ) : V \otimes - \to V \otimes -$ to be the even supernatural transformation with the following components, for $W \in \mathfrak{osp}_{m|2n}(A)\text{- mod}$:

$$\mathcal{F}(\circ)_W : V \otimes W \to V \otimes W, \quad v \otimes w \mapsto (C \otimes 1 + 2\Omega)(v \otimes w).$$

Ignoring the dot, we have a map $\mathcal{F}_{\mathcal{B}}\colon \mathcal{B}(A) \to \mathrm{End}(\mathfrak{osp}_{m|2n}(A)\text{- mod})$. A standard result implies that $\mathcal{F}_{\mathcal{B}}$ is a monoidal superfunctor – see, for instance, [McS21, Thm 5.3.9] for a proof. Thus all that remains is for us to show that $\mathcal{F}(\circ)$ is an even supernatural transformation and that $\mathcal{F}$ respects the defining relations for $\mathcal{AB}(A)$ that involve the dot. We will do so in the following several lemmas.

Lemma 6.5.8. *As defined above, $\mathcal{F}(\circ)$ is indeed an even supernatural transformation from $V \otimes -$ to $V \otimes -$.*

Proof: First, we need to show that each $\mathcal{F}(\circ)_W$ is a morphism in $\mathfrak{osp}_{m|2n}(A)\text{- mod}$, i.e. an $\mathfrak{osp}_{m|2n}(A)$-equivariant linear map. It is straightforward to see that $\mathcal{F}(\circ)_W$ is an even linear map since C and Ω are both even. Let $v \in V, w \in W$, and $T \in \mathfrak{osp}_{m|2n}(A)$. We then have:

$$\begin{aligned}
\mathcal{F}(\circ)_W(T \cdot (v \otimes w)) &= \mathcal{F}(\circ)_W(\Delta(T)(v \otimes w)) \\
&= (C \otimes 1 + 2\Omega)\Delta(T)(v \otimes w) \\
&= \Delta(T)(C \otimes 1 + 2\Omega)(v \otimes w) && \text{Using Lemma 6.5.3 and Corollary 6.5.4} \\
&= \Delta(T)\mathcal{F}(\circ)_W(v \otimes w) \\
&= T \cdot \mathcal{F}(\circ)_W(v \otimes w),
\end{aligned}$$

showing that the map is $\mathfrak{osp}_{m|2n}(A)$-equivariant. Next, let $f\colon W \to W'$ be a morphism in $\mathfrak{osp}_{m|2n}(A)\text{- mod}$. We need to show that the following diagram commutes:

$$\begin{tikzcd}
V \otimes W \arrow[r, "\mathrm{id}_V \otimes f"] \arrow[d, "\mathcal{F}(\circ)_W"'] & V \otimes W' \arrow[d, "\mathcal{F}(\circ)_{W'}"] \\
V \otimes W \arrow[r, "\mathrm{id}_V \otimes f"'] & V \otimes W'
\end{tikzcd} . \tag{6.5.1}$$

Let $v \otimes w \in V \otimes W$. Taking the top path yields:

$$\begin{aligned}\mathcal{F}(\;)_{W'}((\mathrm{id}_V \otimes f)(v \otimes w)) &= (-1)^{\bar{f}\bar{v}}\mathcal{F}(\;)_{W'}(v \otimes f(w)) \\ &= (-1)^{\bar{f}\bar{v}}(C \otimes 1 + 2\Omega)(v \otimes f(w)) \\ &= (-1)^{\bar{f}\bar{v}}Cv \otimes f(w) + 2(-1)^{\bar{f}\bar{v}}\Omega(v \otimes f(w)),\end{aligned}$$

whereas the bottom path yields:

$$\begin{aligned}&(\mathrm{id}_V \otimes f)(\mathcal{F}(\;)_{W}(v \otimes w)) \\ &= (\mathrm{id}_V \otimes f)((C \otimes 1 + 2\Omega)(v \otimes w)) \\ &= (\mathrm{id}_V \otimes f)(Cv \otimes w + 2\sum_{M \in B^{\mathfrak{osp}}_{m|2n}} (-1)^{\bar{v}\bar{M}} Mv \otimes M^\vee w) \\ &= (-1)^{\bar{f}\bar{v}}Cv \otimes f(w) + 2\sum_{M \in B^{\mathfrak{osp}}_{m|2n}} (-1)^{\bar{v}\bar{M}+\bar{f}\bar{M}+\bar{f}\bar{v}} Mv \otimes f(M^\vee w) \\ &= (-1)^{\bar{f}\bar{v}}Cv \otimes f(w) + 2\sum_{M \in B^{\mathfrak{osp}}_{m|2n}} (-1)^{\bar{v}\bar{M}+\bar{f}\bar{v}} Mv \otimes M^\vee f(w) \qquad \text{Since } f \text{ is } \mathfrak{osp}_{m|2n}(A)\text{-equivariant} \\ &= (-1)^{\bar{f}\bar{v}}Cv \otimes f(w) + 2(-1)^{\bar{f}\bar{v}}\Omega(v \otimes f(w)),\end{aligned}$$

as desired. ∎

Lemma 6.5.9. *For all $v, w \in V$, we have $\phi(Cv, w) = \phi(v, Cw)$.*

Proof: We compute:

$$\begin{aligned}\phi(v, Cw) &= \sum_{M \in B^{\mathfrak{osp}}_{m|2n}} \phi(v, MM^\vee w) \\ &= \sum_{M \in B^{\mathfrak{osp}}_{m|2n}} -(-1)^{\bar{u}\bar{M}}\phi(Mv, M^\vee w) \qquad \text{Using the defining property of } \mathfrak{osp}_{m|2n}(A) \\ &= \sum_{M \in B^{\mathfrak{osp}}_{m|2n}} (-1)^{\bar{M}}\phi(M^\vee Mv, w) \qquad \text{Using the defining property of } \mathfrak{osp}_{m|2n}(A) \\ &= \sum_{M \in B^{\mathfrak{osp}}_{m|2n}} \phi(M^\vee((-1)^{\bar{M}}M)v, w) \\ &= \sum_{M \in B^{\mathfrak{osp}}_{m|2n}} \phi(M^\vee(M^\vee)^\vee v, w)\end{aligned}$$

$$= \sum_{M \in B^{\mathfrak{osp}}_{m|2n}} \phi(MM^\vee v, w)$$ By switching the sum to be over the dual basis

$$= \phi(Cv, w).$$

∎

Lemma 6.5.10. *The $\mathcal{F}$-image of relation* (DOT∩SLIDE) *holds in* $\mathfrak{osp}_{m|2n}(A)$- mod. *That is,*

$$\mathcal{F}\left(\cap\right) = -\mathcal{F}\left(\cap\right).$$

Proof: By definition, $\mathcal{F}\left(\cap\right)$ is the supernatural transformation from $V \otimes V \otimes -$ to the identity functor on $\mathfrak{osp}_{m|2n}(A)$- mod (that is, the monoidal identity object in $\mathrm{End}(\mathfrak{osp}_{m|2n}(A)\text{- mod}))$ with its component at $W \in \mathfrak{osp}_{m|2n}(A)$- mod given by the following, for $u, v \in V$ and $w \in W$:

$$\begin{aligned}
&u \otimes v \otimes w \\
\mapsto\ & \left(\mathcal{F}\left(\cap\right) \circ \left(\mathcal{F}\left(|\right) \otimes \mathrm{id}_{V\otimes -}\right)\right)_W (u \otimes v \otimes w) \\
=\ & \mathcal{F}\left(\cap\right)_W \left((\mathcal{F}\left(|\right) \otimes \mathrm{id}_{V\otimes -})_W (u \otimes v \otimes w)\right) \\
=\ & \mathcal{F}\left(\cap\right)_W \left((\mathcal{F}(|)_{V\otimes W} \circ (V \otimes -)(\mathrm{id}_{V\otimes W}))(u \otimes v \otimes w)\right) && \text{By the definition of } \otimes \text{ on morphisms} \\
=\ & \mathcal{F}(\cap)_W \left(\mathcal{F}(|)_{V\otimes W} \circ \mathrm{id}_{V\otimes(V\otimes W)}\right)(u \otimes v \otimes w) && \text{By the definition of } V \otimes - \text{ on morphisms} \\
=\ & \mathcal{F}(\cap)_W (\mathcal{F}(|)_{V\otimes W}(u \otimes (v \otimes w))) \\
=\ & \mathcal{F}(\cap)_W ((C \otimes 1 + 2\Omega)(u \otimes (v \otimes w))) \\
=\ & \mathcal{F}(\cap)_W \Bigg(Cu \otimes v \otimes w + 2 \sum_{M \in B^{\mathfrak{osp}}_{m|2n}} (-1)^{\bar{M}\bar{u}} Mu \otimes M^\vee v \otimes w \\
&\quad + (-1)^{\bar{M}\bar{u} + \bar{M}\bar{v}} Mu \otimes v \otimes M^\vee w \Bigg) \\
=\ & \phi(Cu, v)w \\
&\quad + 2(-1)^{\bar{M}\bar{u}} \sum_{M \in B^{\mathfrak{osp}}_{m|2n}} \phi(Mu, M^\vee v)w + (-1)^{\bar{M}\bar{v}} \phi(Mu, v)M^\vee w.
\end{aligned}$$

On the other hand, $\mathcal{F}\left(\cap\right)$ is the supernatural transformation with component at W given by the following:

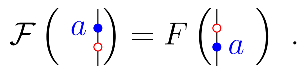
$$
\begin{aligned}
&u \otimes v \otimes w \\
&\mapsto \left(\mathcal{F}\left(\bigcap\right) \circ \left(\mathrm{id}_{V\otimes -} \otimes \mathcal{F}\left(\textcolor{red}{\phi}\right)\right)\right)_W (u \otimes v \otimes w) \\
&= \mathcal{F}(\bigcap)_W((\mathrm{id}_{V\otimes -} \otimes \mathcal{F}(\textcolor{red}{\phi}))_W(u \otimes v \otimes w)) \\
&= \mathcal{F}(\bigcap)_W((\mathrm{id}_{V\otimes(V\otimes W)} \circ (V \otimes -)(\mathcal{F}(\textcolor{red}{\phi})_W))(u \otimes v \otimes w)) && \text{By the definition of } \otimes \text{ on morphisms} \\
&= \mathcal{F}(\bigcap)_W((\mathrm{id}_V \otimes \mathcal{F}(\textcolor{red}{\phi})_W)(u \otimes (v \otimes w))) && \text{By the definition of } V \otimes - \text{ on morphisms} \\
&= \mathcal{F}(\bigcap)_W(u \otimes \mathcal{F}(\textcolor{red}{\phi})_W(v \otimes w)) \\
&= \mathcal{F}(\bigcap)_W(u \otimes (C \otimes 1 + 2\Omega)(v \otimes w)) \\
&= \mathcal{F}(\bigcap)_W \left(u \otimes Cv \otimes w + 2 \sum_{M \in B^{\mathfrak{osp}}_{m|2n}} (-1)^{\bar{M}\bar{v}} u \otimes Mv \otimes M^\vee w \right) \\
&= \phi(u, Cv)w + 2 \sum_{M \in B^{\mathfrak{osp}}_{m|2n}} (-1)^{\bar{M}\bar{v}} \phi(u, Mv) M^\vee w \\
&= \phi(u, Cv)w - 2 \sum_{M \in B^{\mathfrak{osp}}_{m|2n}} (-1)^{\bar{M}\bar{v} + \bar{M}\bar{u}} \phi(Mu, v) M^\vee w. && \text{Using the defining property of } \mathfrak{osp}_{m|2n}(A)
\end{aligned}
$$

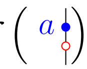
Comparing this result to the previous computation, we find that the desired identity reduces to showing that $\phi(Cu, v) + 2 \sum_{M \in B^{\mathfrak{osp}}_{m|2n}} (-1)^{\bar{M}\bar{u}} \phi(Mu, M^\vee v) = -\phi(u, Cv)$ for all $u, v \in V$. Indeed, we have:

$$
\begin{aligned}
&\phi(Cu, v) + 2 \sum_{M \in B^{\mathfrak{osp}}_{m|2n}} (-1)^{\bar{M}\bar{u}} \phi(Mu, M^\vee v) \\
&= \phi(Cu, v) - 2 \sum_{M \in B^{\mathfrak{osp}}_{m|2n}} \phi(u, MM^\vee v) && \text{Using the defining property of } \mathfrak{osp}_{m|2n}(A) \\
&= \phi(Cu, v) - 2\phi(u, Cv) \\
&= \phi(u, Cv) - 2\phi(u, Cv) && \text{Using Lemma 6.5.9} \\
&= -\phi(u, Cv),
\end{aligned}
$$

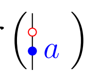
as desired. ∎

Lemma 6.5.11. *The $\mathcal{F}$-image of relation* (DOTSWAP) *holds in* $\mathfrak{osp}_{m|2n}(A)$-mod. *That is,*



Proof: Let W be an $\mathfrak{osp}_{m|2n}(A)$-supermodule. Then $\mathcal{F}$  is the supernatural transformation whose W component is given by:

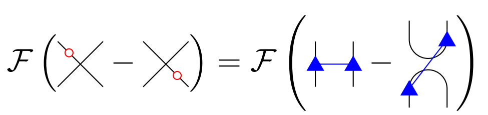
$$\begin{aligned} v \otimes w &\mapsto (C \otimes 1 + 2\Omega)(v \otimes w) \\ &= Cv \otimes w + 2 \sum_{M \in B^{\mathfrak{osp}}_{m|2n}} (-1)^{\bar{M}\bar{v}} Mv \otimes M^{\vee} w \\ &\mapsto (-1)^{\bar{a}\bar{v}} Cva^* \otimes w + 2 \sum_{M \in B^{\mathfrak{osp}}_{m|2n}} (-1)^{\bar{M}\bar{v}+\bar{a}\bar{M}+\bar{a}\bar{v}} Mva^* \otimes M^{\vee} w, \end{aligned}$$

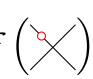
where $v \in V$ and $w \in W$. On the other hand, $\mathcal{F}$  is the supernatural transformation whose W component is given by:

$$\begin{aligned} v \otimes w &\mapsto (-1)^{\bar{a}\bar{v}} va^* \otimes w \\ &\mapsto (-1)^{\bar{a}\bar{v}} (C \otimes 1 + 2\Omega)(va^* \otimes w)) \\ &= (-1)^{\bar{a}\bar{v}} Cva^* \otimes w + 2 \sum_{M \in B^{\mathfrak{osp}}_{m|2n}} (-1)^{\bar{a}\bar{v}+\bar{M}\bar{a}+\bar{M}\bar{v}} Mva^* \otimes M^{\vee} w, \end{aligned}$$

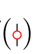
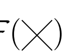
confirming the desired identity. ∎

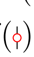
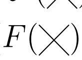
Lemma 6.5.12. *The $\mathcal{F}$-image of relation* (DOT×SLIDE1) *holds in* $\mathfrak{osp}_{m|2n}(A)$-mod. *That is,*

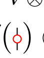


Proof: Let W be an $\mathfrak{osp}_{m|2n}(A)$-supermodule. Then $\mathcal{F}$  is the supernatural transformation from $V \otimes V \otimes -$ to $V \otimes V \otimes -$ with its component at W given by the following, for $u, v \in V$ and $w \in W$:

$u \otimes v \otimes w$

$\mapsto ((\mathcal{F}($$) \otimes \mathrm{id}_{V\otimes -}) \circ \mathcal{F}($$))_W (u \otimes v \otimes w)$

$= (\mathcal{F}($$) \otimes \mathrm{id}_{V\otimes -})_W \, (F($$)_W (u \otimes v \otimes w))$

$= (-1)^{\bar{u}\bar{v}} (\mathcal{F}($$) \otimes \mathrm{id}_{V\otimes -})_W \, (v \otimes u \otimes w)$

$$= (-1)^{\bar{u}\bar{v}}(\mathcal{F}(\mid)_{V\otimes W} \circ (V \otimes -)(\mathrm{id}_{V\otimes W}))(v \otimes u \otimes w)$$

By the definition of $\otimes$ on morphisms in $\mathrm{End}(\mathfrak{osp}_{m|2n}(A)\text{- mod})$

$$= (-1)^{\bar{u}\bar{v}}(\mathcal{F}(\mid)_{V\otimes W} \circ \mathrm{id}_{V\otimes V\otimes W})(v \otimes u \otimes w)$$

By the definition of $V \otimes -$ on morphisms

$$\begin{aligned}
&= (-1)^{\bar{u}\bar{v}}\mathcal{F}(\mid)_{V\otimes W}(v \otimes u \otimes w) \\
&= (-1)^{\bar{u}\bar{v}}(C \otimes 1 + 2\Omega)(v \otimes (u \otimes w)) \\
&= (-1)^{\bar{u}\bar{v}}Cv \otimes u \otimes w + 2 \sum_{M \in B^{\mathfrak{osp}}_{m|2n}} (-1)^{\bar{u}\bar{v}+\bar{M}\bar{v}} Mv \otimes M^\vee u \otimes w \\
&\quad + (-1)^{\bar{u}\bar{v}+\bar{M}\bar{v}+\bar{M}\bar{u}} Mv \otimes u \otimes M^\vee w.
\end{aligned}$$

On the other hand, $\mathcal{F}\left(\times\right)$ is the supernatural transformation with component at W given by:

$$\begin{aligned}
&u \otimes v \otimes w \\
&\mapsto (\mathcal{F}(\times) \circ (\mathrm{id}_{V\otimes -} \otimes \mathcal{F}(\mid)))_W(u \otimes v \otimes w) \\
&= \mathcal{F}(\times)_W\left((\mathrm{id}_{V\otimes V\otimes W} \circ (V \otimes -)(\mathcal{F}(\mid)_W))(u \otimes v \otimes w)\right)
\end{aligned}$$

By the definition of $\otimes$ on morphisms

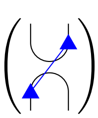
$$= \mathcal{F}(\times)_W\left((\mathrm{id}_V \otimes \mathcal{F}(\mid)_W)(u \otimes v \otimes w)\right)$$

By the definition of $V \otimes -$ on morphisms

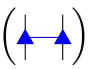
$$\begin{aligned}
&= \mathcal{F}(\times)_W\left(u \otimes (C \otimes 1 + 2\Omega)(v \otimes w)\right) \\
&= \mathcal{F}(\times)_W\left(u \otimes Cv \otimes w + 2 \sum_{M \in B^{\mathfrak{osp}}_{m|2n}} (-1)^{\bar{M}\bar{v}} u \otimes Mv \otimes M^\vee w\right) \\
&= (-1)^{\bar{u}\bar{v}}Cv \otimes u \otimes w + 2 \sum_{M \in B^{\mathfrak{osp}}_{m|2n}} (-1)^{\bar{M}\bar{v}+\bar{u}\bar{M}+\bar{u}\bar{v}} Mv \otimes u \otimes M^\vee w.
\end{aligned}$$

Thus $\mathcal{F}\left(\times - \times\right)$ is the supernatural transformation with component at W given by:

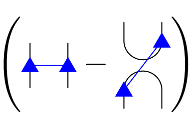
$$u \otimes v \otimes w \mapsto 2 \sum_{M \in B^{\mathfrak{osp}}_{m|2n}} (-1)^{\bar{u}\bar{v}+\bar{M}\bar{v}} Mv \otimes M^\vee u \otimes w = (-1)^{\bar{u}\bar{v}}(2\Omega(v \otimes u)) \otimes w.$$

Next, $\mathcal{F}$  is the supernatural transformation with component at W given by:

$$
\begin{aligned}
&u \otimes v \otimes w \\
&\mapsto \sum_{b \in B} \left(\left(\mathrm{id}_{V \otimes -} \otimes \mathcal{F}\left(\big|\!\!\bullet\, b \right) \right) \circ \mathcal{F}(\cup) \circ \mathcal{F}(\cap) \circ \left(\mathcal{F}\left(\big|\!\!\bullet\, b^{\vee} \right) \circ \mathrm{id}_{V \otimes -} \right) \right)_W (u \otimes v \otimes w) \\
&= \sum_{b \in B} \left(\mathrm{id}_{V \otimes V \otimes W} \circ (V \otimes -) \left(\mathcal{F}\left(\big|\!\!\bullet\, b \right)_W \right) \right) \circ \mathcal{F}(\cup)_W \circ \mathcal{F}(\cap)_W \\
&\quad \circ \left(\mathcal{F}(\big|\!\!\bullet\, b^{\vee})_{V \otimes W} \circ (V \otimes -)(\mathrm{id}_{V \otimes W}) \right) (u \otimes v \otimes w) \\
&= \sum_{b \in B} \left((\mathrm{id}_V \otimes \mathcal{F}\left(\big|\!\!\bullet\, b \right)_W) \right) \circ \mathcal{F}(\cup)_W \circ \mathcal{F}(\cap)_W \circ (\mathcal{F}(\big|\!\!\bullet\, b^{\vee})_{V \otimes W})(u \otimes v \otimes w) \\
&= \sum_{b \in B} (-1)^{\bar{u}\bar{b}} \left((\mathrm{id}_V \otimes \mathcal{F}\left(\big|\!\!\bullet\, b \right)_W) \right) \circ \mathcal{F}(\cup)_W \circ \mathcal{F}(\cap)_W (u(b^{\vee})^* \otimes v \otimes w) \\
&= \sum_{b \in B} (-1)^{\bar{u}\bar{b}} \left((\mathrm{id}_V \otimes \mathcal{F}\left(\big|\!\!\bullet\, b \right)_W) \right) \circ \mathcal{F}(\cup)_W \, (\phi(u(b^{\vee})^*, v) w) \\
&= \sum_{b \in B} (-1)^{\bar{u}\bar{b}} \left((\mathrm{id}_V \otimes \mathcal{F}\left(\big|\!\!\bullet\, b \right)_W) \right) \left(\sum_{x \in B_{m|2n}} \phi(u(b^{\vee})^*, v) x \otimes x^{\vee} \otimes w \right) \\
&= \sum_{b \in B} (-1)^{\bar{u}\bar{b}} \sum_{x \in B_{m|2n}} \phi(u(b^{\vee})^*, v) x \otimes x^{\vee} b^* \otimes w.
\end{aligned}
$$

Finally, $\mathcal{F}$ () is the supernatural transformation with component at W given by:

$$
\begin{aligned}
&u \otimes v \otimes w \\
&\mapsto \sum_{b \in B} \left(\left(\mathrm{id}_{V \otimes -} \otimes \mathcal{F}\left(\big|\!\!\bullet\, b \right) \right) \circ \left(\mathcal{F}\left(\big|\!\!\bullet\, b^{\vee} \right) \otimes \mathrm{id}_{V \otimes -} \right) \right)_W (u \otimes v \otimes w) \\
&= \sum_{b \in B} \left(\left(\mathrm{id}_V \otimes \mathcal{F}(\big|\!\!\bullet\, b)_W \right) \circ \left(\mathcal{F}(\big|\!\!\bullet\, b^{\vee})_{V \otimes W} \right) \right) (u \otimes v \otimes w) \\
&= \sum_{b \in B} (-1)^{\bar{u}\bar{b}} \left(\mathrm{id}_V \otimes \mathcal{F}(\big|\!\!\bullet\, b)_W \right) (u(b^{\vee})^* \otimes v \otimes w) \\
&= \sum_{b \in B} (-1)^{\bar{b} + \bar{v}\bar{b}} u(b^{\vee})^* \otimes v b^* \otimes w,
\end{aligned}
$$

and so $\mathcal{F}$ () is the supernatural transformation with component at W

given by:

$$u \otimes v \otimes w \mapsto \sum_{b \in B} (-1)^{\bar{b}+\bar{v}\bar{b}} u(b^\vee)^* \otimes vb^* \otimes w - (-1)^{\bar{u}\bar{b}} \sum_{x \in B_{m|2n}} \phi(u(b^\vee)^*, v)x \otimes x^\vee b^* \otimes w.$$

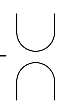
The desired equality thus reduces to showing that

$$(-1)^{\bar{u}\bar{v}} 2\Omega(v \otimes u) = \sum_{b \in B} (-1)^{\bar{b}+\bar{v}\bar{b}} ub^\vee \otimes vb - (-1)^{\bar{u}\bar{b}} \sum_{x \in B_{m|2n}} \phi(ub^\vee, v)x \otimes x^\vee b$$

for all $u, v \in V$. (Note that we have removed the applications of $-^*$ by summing over the involuted basis $B^* = \{b^* \mid b \in B\}$.) This is precisely what we proved in Lemma 6.5.6, so we conclude that the image of (DOT×SLIDE1) holds.

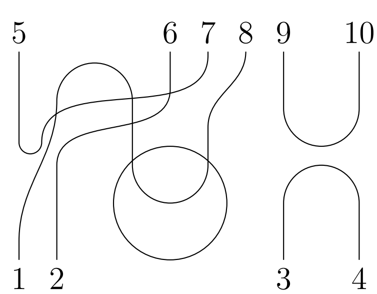
∎

6.6 Bases of Morphism Spaces

Definition 6.6.1 (String diagram)**.** Let $\mathcal{C}$ be one of the Brauer categories discussed in this thesis, i.e. $\mathcal{B}, \mathcal{B}(A), \mathcal{B}(A,\delta), \mathcal{AB}, \mathcal{AB}(A)$, or $\mathcal{AB}(A,\delta)$. In the context of this section, a *string diagram* in $\mathcal{C}$ is a morphism obtained by composing and/or taking tensor products of the identity and/or the generating morphisms for $\mathcal{C}$, but without taking any nontrivial linear combinations. For instance, and  are both string diagrams in $\mathcal{B}$, but the morphisms 3 and  $-$  are not string diagrams.

Definition 6.6.2 (Brauer diagram)**.** Let $s, r \in \mathbb{N}$. An (s,r)-*Brauer diagram* is a string diagram in $\mathrm{Hom}_{\mathcal{B}}(|^{\otimes s}, |^{\otimes r})$. Let $\mathbb{B}_{s,r}$ denote the set of (s,r)-Brauer diagrams, and $\overline{\mathbb{B}_{s,r}}$ the set of (s,r)-Brauer diagrams with no closed loops.

We sometimes label the ends of the strands in a Brauer diagram with natural numbers, increasing from left-to-right and bottom-to-top. For example, here is a labelled $(4,6)$-Brauer diagram:



Each (s,r)-Brauer diagram induces a perfect matching of $\{1,\dots,s+r\}$ by pairing the endpoints of each strand. For instance, the matching given by the above diagram is $\{\{1,8\},\{2,6\},\{3,4\},\{5,7\},\{9,10\}\}$. We define an equivalence relation $\sim$ on $\mathbb{B}_{s,r}$ (and, by extension, $\overline{\mathbb{B}_{s,r}}$) by setting $d \sim d'$ (for two diagrams $d, d' \in \mathbb{B}_{s,r}$) precisely when they induce the same matching of $\{1,\dots,s+r\}$ and they contain the same number of closed loops.

Proposition 6.6.3 ([RS18, Prop. 2.5])**.** *If $d \sim d'$ in $\mathbb{B}_{s,r}$, then the morphisms represented by d and d' are equal in $\mathcal{B}$. Thus equivalence classes of $\sim$ may be interpreted as morphisms in $\mathcal{B}$.*

Corollary 6.6.4. *If $d \sim d'$ in $\mathbb{B}_{s,r}$, then the morphisms represented by d and d' are equal in $\mathcal{B}(A)$ and $\mathcal{B}(A,\delta)$. Thus equivalence classes of $\sim$ may be interpreted as morphisms in $\mathcal{B}(A)$ and $\mathcal{B}(A,\delta)$.*

Proof: The result follows from Proposition 6.6.3 by using the evident inclusions $\mathcal{B} \to \mathcal{B}(A)$ and $\mathcal{B} \to \mathcal{B}(A,\delta)$. ∎

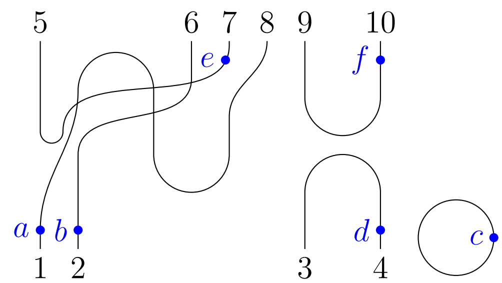
Theorem 6.6.5 ([RS18, Thm. A])**.** *For all $s,r \in \mathbb{N}$, the equivalence classes $\mathbb{B}_{s,r}/\sim$ form a basis of $\mathrm{Hom}_{\mathcal{B}}(|^{\otimes s}, |^{\otimes r})$.*

Definition 6.6.6 (Brauer diagram with tokens)**.** An *(s,r)-Brauer diagram with tokens* is a string diagram in the morphism space $\mathrm{Hom}_{\mathcal{C}}(|^{\otimes s}, |^{\otimes r})$, where $\mathcal{C}$ is $\mathcal{B}(A)$, $\mathcal{B}(A,\delta)$, $\mathcal{AB}(A)$, or $\mathcal{AB}(A,\delta)$. We say such a diagram is in *normal form* if:

- Each strand has exactly one token on it;
- All tokens on non-closed strands appear just before one of the ends of the string (i.e. there are no crossings appearing vertically between the token and the relevant end of the strand);
- All tokens at the top of the diagram lie on the same horizontal line, as do all tokens at the bottom of the diagram;
- If a strand has ends at both the top and bottom of the diagram, its token appears at the bottom of the strand;
- If a strand has both of its ends on the same side of the diagram, its token appears on the right of the strand;
- All closed loops are completely separated from other strands (i.e. not intersecting with any other strands or nested within other closed loops) and are in the shape of a bubble;

- All bubbles appear strictly to the right of all non-closed strands;
- All tokens on bubbles are on the right side of the bubble, and are vertically aligned with the bottommost line of tokens on non-closed strands.

For example, here is a labelled $(4, 6)$-Brauer diagram with tokens in normal form:



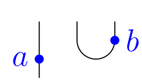
.

Let $\mathbb{B}_{s,r}^{A,B}$ denote the set of (s, r)-Brauer diagrams with tokens in normal form whose tokens are all labelled with one of the elements from the basis B of A, and $\overline{\mathbb{B}_{s,r}^{A,B}}$ the set of such diagrams with no closed loops. We define an equivalence relation $\doteqdot$ on $\mathbb{B}_{s,r}^{A,B}$ (and, by extension, $\overline{\mathbb{B}_{s,r}^{A,B}}$) by setting $d \doteqdot d'$ (for two diagrams $d, d' \in \mathbb{B}_{s,r}^{A,B}$) precisely when:

- After removing all tokens, d and d' are $\sim$-equivalent as elements of $\mathbb{B}_{s,r}$;
- For each non-closed string in d, the token on that string has the same label as the token on the associated string in d' (i.e. the string whose endpoints have the same labels as the string in d);
- The multiset of labels of tokens on closed loops is the same for both d and d'.

Proposition 6.6.7. *If $d \doteqdot d'$ in $\mathbb{B}_{s,r}^{A,B}$, then the morphisms represented by d and d' are equal in $\mathcal{B}(A)$ (and thus also in $\mathcal{B}(A, \delta)$, $\mathcal{AB}(A)$, and $\mathcal{AB}(A, \delta)$). Thus equivalence classes of $\doteqdot$ may be interpreted as morphisms in (affine) Frobenius Brauer categories.*

Proof: We will first suppose that d and d' have no bubbles. Consider the diagrams e and e' obtained from d and d' by removing the tokens from each strand. By the definition of $\doteqdot$, we have $e \sim e'$ as elements of $\mathbb{B}_{s,r}$. Corollary 6.6.4 tells us that e and e' are equal as morphisms in $\mathcal{B}(A)$. Note that placing the previously-removed tokens back onto e and e' amounts to pre- and post-composing these morphisms with a tensor product of identity morphisms and/or token morphisms. For instance, if our diagrams d and e were as follows,

$d =$ 

, $e =$  , placing the tokens back on e is equivalent to precomposing with $\big|\!\bullet a$ and postcomposing with $\big| \;\; \big|\!\bullet b \; \big|$, yielding d.

By the definition of $\doteq$, the original diagrams d and d' have exactly the same tokens in the same positions. Thus d and d' are obtained from the equal morphisms e and e' by pre- and post-composing with the same morphisms. Since composition is well-defined, this implies that $d = d'$ in $\mathcal{B}(A)$.

Now we consider the general case. If d and d' have any bubbles, the definition of $\doteq$ guarantees that they have the same number of them, say n, and the multiset M of the token labels on those bubbles is the same for both diagrams. Consider the diagrams f and f' obtained from d and d' by removing all of their bubbles. Clearly $f \doteq f'$ since $d \doteq d'$. By the special case of the present proposition proved above, $f = f'$ as morphisms in $\mathcal{B}(A)$. Let g denote an automorphism of the unit object in $\mathcal{B}(A)$ given by a diagram composed of n horizontally-juxtaposed bubbles with the token labels from M distributed arbitrarily among them (placed on the right sides of the bubbles). We then have $f \otimes g = f' \otimes g$. Using Lemma 6.1.7, we may freely reorder the bubbles in each diagram until they reside in the same places that they did in d and d'. Thanks to Definition 6.6.6, all of the tokens on bubbles in both d and d' are horizontally aligned with the bottommost line of tokens on non-closed strands, so reordering the bubbles will not cause any sign terms to arise due to (SINTER). We thus find that $d = f \otimes g = f' \otimes g = d'$, as desired. ∎

Theorem 6.6.8. *For all $s, r \in \mathbb{N}$, the morphism space $\mathrm{Hom}_{\mathcal{B}(A)}(|^{\otimes s}, |^{\otimes r})$ is spanned by the set of equivalence classes $\mathbb{B}^{A,B}_{s,r}/\doteq$, and $\mathrm{Hom}_{\mathcal{B}(A,\delta)}(|^{\otimes s}, |^{\otimes r})$ is spanned by $\overline{\mathbb{B}^{A,B}_{s,r}}/\doteq$.*

Proof: First, let f be an arbitrary morphism in $\mathrm{Hom}_{\mathcal{B}(A)}(|^{\otimes s}, |^{\otimes r})$. By the definition of $\mathcal{B}(A)$, f can be written as a linear combination of string diagrams. Let f' denote the linear combination of string diagrams obtained by removing all tokens from those diagrams. We can naturally view f' as a morphism in $\mathrm{Hom}_{\mathcal{B}}(|^{\otimes s}, |^{\otimes r})$. By Theorem 6.6.5, f' is equal to a linear combination of representatives from $\mathbb{B}_{s,r}/\sim$ that we will denote g'. This means that f' can be transformed into g' by applying a sequence of the relations that define $\mathcal{B}$. All of those relations also hold in $\mathcal{B}(A)$, so we can apply the same sequence of relations to f and obtain a linear combination of diagrams, call it g, that has the same shape as g'. By construction, f and g are equal as morphisms in $\mathcal{B}(A)$. Since morphisms corresponding to equivalent elements of $\mathbb{B}_{s,r}/\sim$ are equal, we may stipulate that all of the representatives chosen for the diagrams in f' have all closed loops completely separated from the other strands, in the sense of Definition 6.6.6; it's clear that every equivalence class of $\sim$ must contain such a representative, so we can always do this.

A priori, the tokens on diagrams in f could get in the way of applying our sequence of relations, but we proved in Section 6.1 that tokens slide over all caps, cups, and crossings in $\mathcal{B}(A)$ (potentially picking up applications of $-^*$ and/or a sign term along the way), so we may slide any tokens out of the way whenever necessary.

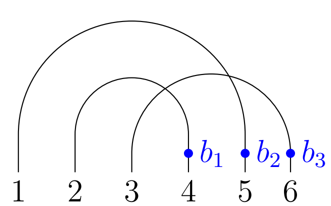
Using the aforementioned token-sliding properties, we may slide all of the tokens on the constituent diagrams of g and combine all of the tokens on each string into one token using (TOKEN), absorbing all sign terms into the token label in the process. If there are any strings without a token, we may use the same relation to add a token with label 1_A to those strings. Next, we write the elements of A labelling each of the tokens as a linear combination of the basis vectors from B, and use (TOKEN) once more to replace each diagram in g with a linear combination of diagrams with one token on each string, each labelled with an element of B. We may then slide the tokens on each string into the position dictated by the definition of normal forms for Brauer diagrams. We thus conclude that f is equal to a linear combination of diagrams from $\mathbb{B}_{s,r}^{A,B}$ (i.e. representatives of equivalence classes), as desired.

The proof for $\mathcal{B}(A,\delta)$ starts with an arbitrary $f \in \mathrm{Hom}_{\mathcal{B}(A,\delta)}(|^{\otimes s}, |^{\otimes r})$ and repeats the above argument to show that f is equal to a linear combination of diagrams from $\mathbb{B}_{s,r}^{A,B}$. At that point, we simply apply relation (BUBBLE) to eliminate each bubble and replace it with a scalar. This shows that f is equal to a linear combination of diagrams in normal form with no closed loops, i.e. representatives from $\overline{\mathbb{B}_{s,r}^{A,B}}/\simeq$. ∎

From this point forward, let $\mathcal{C}$ denote either $\mathcal{B}(A)$ or $\mathcal{B}(A,\delta)$. We adopt the following convention: whenever working with a perfect matching M, we write the pairs in M with the smaller element on the left. That is, whenever we write $\{i,j\} \in M$, we always have $i < j$.

Definition 6.6.9. Let $k \in \mathbb{N}$, M be a perfect matching of $\{1,\dots,2k\}$, and $T\colon M \to B$ a function. We write $C_M(T)$ for the $(2k,0)$-Brauer diagram with tokens in normal form which induces the matching M, has no closed loops, and, for each pair $p = \{i,j\} \in M$, has the label $T(p)$ on the strand joining positions i and j. By the definition of $\simeq$, this specifies $C_M(T)$ up to equivalence. Proposition 6.6.7 then allows us to unambiguously interpret $C_M(T)$ as a morphism in $\mathcal{C}$ independent of the way we choose to draw the diagram.

Example 6.6.10. Take $k = 3$, and let M be the matching $\{\{1,5\},\{2,4\},\{3,6\}\}$. Define a function $T\colon M \to B$ by $T(\{2,4\}) = b_1, T(\{1,5\}) = b_2$, and $T(\{3,6\}) = b_3$, where b_1, b_2, b_3 are some fixed elements of B. Then $C_M(T)$ is the following diagram:



.

Remark 6.6.11. Let $k \in \mathbb{N}$, M be a perfect matching of $\{1, \ldots, 2k\}$, and $T \colon M \to B$ a function. By the definition of F, we can see that $F(C_M(T))$ is the linear map from $V^{\otimes 2k}$ to $\Bbbk$ given by the linear extension of:

$$x_1 \otimes x_2 \otimes x_3 \otimes x_4 \otimes \cdots \otimes x_{2k-1} \otimes x_{2k} \mapsto \epsilon_{M,T,x_1,\ldots,x_{2k}} \prod_{p=\{i,j\} \in M} \phi(x_i, x_j(T(p))^*),$$

where $\epsilon_{M,T,x_1,\ldots,x_{2k}} = \pm 1$ denotes a sign term depending on the number and positions of string crossings in $C_M(T)$ and the parities of the x_i and the $T(p)$. For instance, using the diagram from Example 6.6.10, $F(C_M(T))$ is, up to a sign term, the linear map from $V^{\otimes 6}$ to $\Bbbk$ given by the linear extension of:

$$x_1 \otimes x_2 \otimes x_3 \otimes x_4 \otimes x_5 \otimes x_6 \mapsto \phi(x_1, x_5 b_2^*)\phi(x_2, x_4 b_1^*)\phi(x_3, x_6 b_3^*).$$

Definition 6.6.12. Let $k \in \mathbb{N}$, M be a perfect matching of $\{1, \ldots, 2k\}$, and $T \colon M \to B$ be a function. Let E denote the standard basis of $A^k = A^{k|0}$ as an A-module. Let $e \colon M \to E$ be a bijection – note that both M and E have cardinality k, so such a bijection always exists. For use in the following lemma, we define $x_T^e = x_1 \otimes x_2 \otimes \cdots \otimes x_{2k}$ to be the simple tensor in $(A^k)^{\otimes 2k}$ specified by setting $x_i = T(p)^\vee e(p)$ and $x_j = 1_A e(p)$ for all pairs $p = \{i, j\} \in M$. Since M is a perfect matching, this uniquely defines x_T^e.

Lemma 6.6.13. *Let $k \in \mathbb{N}, M$ and M' be perfect matchings of $\{1, \ldots, 2k\}$, and $T, T' \colon M \to B$ be arbitrary functions. Let x_T^e be the vector specified in Definition 6.6.12 (with an arbitrary choice for the bijection e). We then have the following:*

$$F(C_{M'}(T'))(x_T^e) = \epsilon \delta_{MM'} \delta_{TT'},$$

where $\epsilon = \pm 1$.

Proof: First, suppose $M \neq M'$. Then there exists some pair $p = \{i, j\}$ in M' that is not a pair in M. As per the action mentioned in Remark 6.6.11, $\phi(x_i, x_j(T'(p))^*)$ is a factor in $F(C_{M'}(T'))(x_T^e)$. Since p is not a pair in M, i and j belong to different pairs in M, say p_i and p_j. By the definition of x_T^e, we have $x_i = \alpha e(p_i)$ and $x_j = \beta e(p_j)$ for some $\alpha, \beta \in A$. Since e is a bijection, $e(p_i) \neq e(p_j)$. Thus:

$$\begin{aligned}
\phi(x_i, x_j(T'(p))^*) &= \operatorname{tr}((\alpha e(p_i))^{\top *}(\beta e(p_j)(T'(p))^*)) && \text{By the definition of } \phi \text{, recalling that } V = A^k \text{ and so } P = I, -^{\mathrm{st}} = -^{\top} \\
&= \operatorname{tr}(\alpha^* e(p_i)^\top \beta (T'(p))^* e(p_j)) \\
&= \delta_{e(p_i)e(p_j)} \operatorname{tr}(\alpha^* \beta (T'(p))^*) \\
&= 0. && \text{Since } e(p_i) \neq e(p_j)
\end{aligned}$$

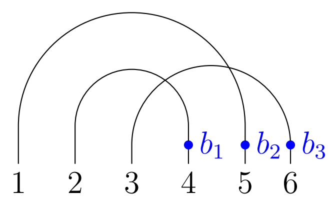
Thus $F(C_{M'}(T'))(x_T^e) = 0$ in this case. Suppose now that $M = M'$. Working up to a sign in the following, we have:

$$F(C_{M'}(T'))(x_T^e) = \prod_{p=\{i,j\} \in M} \phi(x_i, x_j(T'(p))^*)$$

$$
\begin{aligned}
&= \prod_{p=\{i,j\}\in M} \phi((T(p)^\vee)^* e(p), (T'(p))^* e(p)) && \text{By the definition of } x_T^e \\
&= \prod_{p=\{i,j\}\in M} \operatorname{tr}((T(p)^\vee)^* (T'(p))^*) \\
&= \prod_{p=\{i,j\}\in M} \operatorname{tr}(T(p)^\vee T'(p)) && \text{Using Lemma 4.2.14} \\
&= \prod_{p=\{i,j\}\in M} \delta_{T(p)T'(p)} && \text{Since } T(p), T'(p) \in B \\
&= \delta_{TT'},
\end{aligned}
$$

which establishes the desired result. ∎

Example 6.6.14. Consider the following diagram once more:



,

with $k = 3$ and both M and T as specified in Example 6.6.10. Define $e\colon M \to E$ by $e(\{1,5\}) = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $e(\{2,4\}) = \begin{bmatrix}0\\1\\0\end{bmatrix}$, and $e(\{3,6\}) = \begin{bmatrix}0\\0\\1\end{bmatrix}$. Using the notation from the previous lemma, we have:

$$
x_T^e = x_1 \otimes x_2 \otimes x_3 \otimes x_4 \otimes x_5 \otimes x_6 = \begin{bmatrix}b_2^\vee\\0\\0\end{bmatrix} \otimes \begin{bmatrix}0\\b_1^\vee\\0\end{bmatrix} \otimes \begin{bmatrix}0\\0\\b_3^\vee\end{bmatrix} \otimes \begin{bmatrix}0\\1\\0\end{bmatrix} \otimes \begin{bmatrix}1\\0\\0\end{bmatrix} \otimes \begin{bmatrix}0\\0\\1\end{bmatrix},
$$

and, up to a sign,

$$
\begin{aligned}
F(C_M(T))(x_T^e) &= \phi(x_1, x_5 b_2^*)\phi(x_2, x_4 b_1^*)\phi(x_3, x_6 b_3^*) \\
&= \phi\left(\begin{bmatrix}b_2^\vee\\0\\0\end{bmatrix}, \begin{bmatrix}b_2^*\\0\\0\end{bmatrix}\right) \phi\left(\begin{bmatrix}0\\b_1^\vee\\0\end{bmatrix}, \begin{bmatrix}0\\b_1^*\\0\end{bmatrix}\right) \phi\left(\begin{bmatrix}0\\0\\b_3^\vee\end{bmatrix}, \begin{bmatrix}0\\0\\b_3^*\end{bmatrix}\right) \\
&= \operatorname{tr}\left(\begin{bmatrix}b_2^{\vee *} & 0 & 0\end{bmatrix} \begin{bmatrix}b_2^*\\0\\0\end{bmatrix}\right) \operatorname{tr}\left(\begin{bmatrix}0 & b_1^{\vee *} & 0\end{bmatrix} \begin{bmatrix}0\\b_1^*\\0\end{bmatrix}\right) \operatorname{tr}\left(\begin{bmatrix}0 & 0 & b_3^{\vee *}\end{bmatrix} \begin{bmatrix}0\\0\\b_3^*\end{bmatrix}\right) \\
&= \operatorname{tr}(b_2^{\vee *} b_2^*) \operatorname{tr}(b_1^{\vee *} b_1^*) \operatorname{tr}(b_3^{\vee *} b_3^*)
\end{aligned}
$$

$$= 1 \cdot 1 \cdot 1$$
$$= 1.$$

Lemma 6.6.15. *Let $k \in \mathbb{N}$. The $\doteqdot$-equivalence classes of $\overline{\mathbb{B}_{2k,0}^{A,B}}$ are linearly independent in $\mathcal{C}$.*

Proof: For each perfect matching M of $\{1, \ldots, 2k\}$, let $\mathcal{T}_M$ denote the set of functions from M to B. We can see that the following is a complete set of representatives of $\doteqdot$-equivalence classes in $\overline{\mathbb{B}_{2k,0}^{A,B}}$:

$$\mathcal{D} = \{C_M(T) \mid M \text{ is a perfect matching of } \{1, \ldots, 2k\}, T \in \mathcal{T}_M\}.$$

No two of these diagrams can be $\doteqdot$-equivalent because they either have different token labels on at least one strand or induce a different matching of $\{1, \ldots, 2k\}$. Conversely, by the definition of $\doteqdot$, any diagram d in $\overline{\mathbb{B}_{2k,0}^{A,B}}$ is $\doteqdot$-equivalent to $C_M(T)$, where M is the matching induced by d and T is the function which sends a pair $\{i, j\}$ to the token label on the strand joining positions i and j in d. It thus suffices to show that $\mathcal{D}$ is linearly independent in $\mathcal{C}$.

Suppose that we have $\sum_{d \in \mathcal{D}} \alpha_d d = 0$ for some scalars $\alpha_d \in \Bbbk$. Write f for the F-image of this linear combination of diagrams, i.e. $f = \sum_{d \in \mathcal{D}} \alpha_d F(d)$. Then $f = 0$. Let M be a perfect matching of $\{1, \ldots, 2k\}$ and $T \in \mathcal{T}_M$. Let $x_T^e \in (A^k)^{\otimes 2k}$ be the element specified in Definition 6.6.12 (again with the bijection e being arbitrary). As computed in Lemma 6.6.13, $F(d)(x_T^e) = \pm 1$ exactly when d induces the matching M and has token labels given by T, and $F(x_T^e)(d) = 0$ otherwise. Thus $f(x_T^e) = \pm \alpha_{C_M(T)}$. Since f is the zero map, this shows that $\alpha_{C_M(T)} = 0$. This works for all choices of M and T, so we conclude that $\alpha_d = 0$ for all $d \in \mathcal{D}$. Thus $\mathcal{D}$ is indeed linearly independent. ∎

Corollary 6.6.16. *The $\doteqdot$-equivalence classes of $\overline{\mathbb{B}_{2k,0}^{A,B}}$ form a basis of* $\mathrm{Hom}_{\mathcal{B}(A,\delta)}(|^{\otimes 2k}, |^{\otimes 0})$.

Proof: The result follows immediately from Theorem 6.6.8 and Lemma 6.6.15. ∎

Lemma 6.6.17. *Let $s, r \in \mathbb{N}$ with $s + r = 2k$. Then there exists a linear isomorphism* $\mathrm{Hom}_{\mathcal{C}}(|^{\otimes s}, |^{\otimes r}) \cong \mathrm{Hom}_{\mathcal{C}}(|^{\otimes 2k}, |^{\otimes 0})$.

Proof: Noting that the generating object $|$ is self-dual, the result follows from applying Theorem 3.2.4 r times. ∎

Theorem 6.6.18. *Let $s, r \in \mathbb{N}$ with $s + r = 2k$. The equivalence classes $\overline{\mathbb{B}_{s,r}^{A,B}} / \doteqdot$ form a basis of* $\mathrm{Hom}_{\mathcal{B}(A,\delta)}(|^{\otimes s}, |^{\otimes r})$.

Proof: Write $\mathcal{D} = \overline{\mathbb{B}^{A,B}_{s,r}} / \doteqdot$ for brevity. Theorem 6.6.8 tells us that $\mathcal{D}$ is spanning. By the definition of $\doteqdot$, two diagrams in $\overline{\mathbb{B}^{A,B}_{s,r}}$ are equivalent precisely when they induce the same matching of $\{1, \dots, 2k\}$ and have all of the same token labels. It is well-known that the number of perfect matchings of $\{1, \dots, 2k\}$ is given by the double factorial $(2k-1)!!$. As for token labels, there are k strands and $|B| = m+2n$ choices for labels on each strand. We conclude that $\mathcal{D}$ contains $(m+2n)^k(2k-1)!!$ elements. Using Corollary 6.6.16, we find that $\mathrm{Hom}_{\mathcal{B}(A,\delta)}(|^{\otimes 2k}, |^{\otimes 0})$ has dimension $(m+2n)^k(2k-1)!!$. By Lemma 6.6.17, we have $\mathrm{Hom}_{\mathcal{B}(A,\delta)}(|^{\otimes 2k}, |^{\otimes 0}) \cong \mathrm{Hom}_{\mathcal{B}(A,\delta)}(|^{\otimes s}, |^{\otimes r})$, so the latter also has dimension $(m+2n)^k(2k-1)!!$. This implies that $\mathcal{D}$ must be a basis, as desired. ∎

Remark 6.6.19. Recall that Theorem 6.6.5 gives bases for the morphism spaces in the plain Brauer category $\mathcal{B}$, which immediately yields a similar result for the specialized category $\mathcal{B}(\delta)$ as well. To contrast, Theorem 6.6.18 only gives a result for $\mathcal{B}(A,\delta)$, not $\mathcal{B}(A)$. Based on the basis result for $\mathcal{B}$, we conjecture that $\mathbb{B}^{A,B}_{s,r} / \doteqdot$ is a basis of $\mathrm{Hom}_{\mathcal{B}(A)}(|^{\otimes s}, |^{\otimes r})$. That being said, it seems unlikely that the techniques employed in this section could be used to prove such a result. Our proof of Theorem 6.6.18 makes essential use of the functor $F \colon \mathcal{B}(A) \to \mathfrak{osp}_{m|2n}(A)\text{-}\mathrm{mod}$. As outlined in Lemma 6.4.8, the F-image of a bubble in $\mathcal{B}(A)$ is a scalar in $\mathfrak{osp}_{m|2n}(A)\text{-}\mathrm{mod}$. Thus diagrams that differ only in their number of bubbles or bubble token labels will have linearly dependent F-images, even though we conjecture such diagrams to be linearly independent in $\mathcal{B}(A)$. In [BSW21], a basis result was proved for Frobenius Heisenberg categories, which are a generalization of the oriented Frobenius Brauer categories $\mathcal{OB}(A)$. It may be possible to adapt the arguments from that paper to work with the non-oriented Frobenius Brauer categories studied in this thesis.

Bibliography

[Abr97] Lowell Abrams. *Frobenius algebra structures in topological quantum field theory and quantum cohomology*. PhD thesis, 1997. URL: https://home.gwu.edu/~labrams/docs/thesis.pdf.

[Bra37] Richard Brauer. On algebras which are connected with the semisimple continuous groups. *Ann. of Math. (2)*, 38(4):857–872, 1937. doi:10.2307/1968843.

[BSW21] Jonathan Brundan, Alistair Savage, and Ben Webster. Foundations of Frobenius Heisenberg categories. *Journal of Algebra*, 578:115–185, jul 2021. URL: https://doi.org/10.1016%2Fj.jalgebra.2021.02.025, doi:10.1016/j.jalgebra.2021.02.025.

[BSW22] Jonathan Brundan, Alistair Savage, and Ben Webster. Quantum Frobenius Heisenberg categorification. *Journal of Pure and Applied Algebra*, 226(1):106792, jan 2022. URL: https://doi.org/10.1016%2Fj.jpaa.2021.106792, doi:10.1016/j.jpaa.2021.106792.

[Cou16] Chad Couture. Skew-zigzag algebras. *Symmetry, Integrability and Geometry: Methods and Applications*, jun 2016. URL: https://doi.org/10.3842%2Fsigma.2016.062, doi:10.3842/sigma.2016.062.

[CW12] Shun-Jen Cheng and Weiqiang Wang. *Dualities and representations of Lie superalgebras*, volume 144 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2012. URL: https://doi-org.proxy.bib.uottawa.ca/10.1090/gsm/144, doi:10.1090/gsm/144.

[Kel05] Gregory M. Kelly. Basic concepts of enriched category theory. *Repr. Theory Appl. Categ.*, (10), 2005. Reprint of the 1982 original [Cambridge Univ. Press, Cambridge; MR0651714]. URL: http://www.tac.mta.ca/tac/reprints/articles/10/tr10.pdf.

[KT17] Jonathan R. Kujawa and Benjiman C. Tharp. The marked Brauer category. *Journal of the London Mathematical Society*, 95(2):393–413, jan 2017. URL: https://doi.org/10.1112%2Fjlms.12015, doi:10.1112/jlms.12015.

[LZ15] Gus I. Lehrer and Ruibin B. Zhang. The Brauer category and invariant theory. *J. Eur. Math. Soc. (JEMS)*, 17(9):2311–2351, 2015. doi:10.4171/JEMS/558.

[McS21] Alexandra McSween. Affine oriented Frobenius Brauer categories and general linear Lie superalgebras. Master's thesis, 2021. URL: http://hdl.handle.net/10393/42342, doi:http://dx.doi.org/10.20381/ruor-26564.

[ML98] Saunders Mac Lane. *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1998.

[MS21] Alexandra McSween and Alistair Savage. Affine oriented Frobenius Brauer categories. 2021. arXiv:2101.04582.

[Rac98] Michel L. Racine. Primitive superalgebras with superinvolution. *Journal of Algebra*, 206(2):588–614, 1998. URL: https://www.sciencedirect.com/science/article/pii/S0021869397974123, doi:https://doi.org/10.1006/jabr.1997.7412.

[RS18] Hebing Rui and Linliang Song. Affine Brauer category and parabolic category O in types B, C, D. *Mathematische Zeitschrift*, 293, 12 2018. doi:10.1007/s00209-018-2207-x.

[ZY08] Sham. Zi Yang. Quaternion algebras and quadratic forms, 2008. URL: http://hdl.handle.net/10012/3656.