OPTIMAL CONTROL OF SYSTEMS GOVERNED BY ITO STOCHASTIC DIFFERENTIAL EQUATIONS

by

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TO

MY GRANDMOTHER

MY PARENTS

AND

MY WIFE

ABSTRACT

In this thesis, we consider the optimization problems of systems governed by stochastic Ito differential equation in which both controls and parameters are to be chosen optimally with respect to certain performance criterion. In these optimization problems, the class of admissible controls to be considered consists of bounded measurable functions which depend only on partially observed current state.

In Chapter 1, a short survey of existing results on problems of optimal control of systems governed by stochastic Ito differential equations is presented.

In Chapter 2, the optimization problem of stochastic systems with Markov terminal time is considered. It is shown [section 2.3] that this stochastic optimization problem can be converted into an optimization problem of an equivalent class of parabolic partial differential systems with first boundary condition. For this reduced problem,we note that both the controls and parameters appear in the coefficients of its differential operator. In section 2.4, some preparatory material, necessary for the subsequent section, is presented. Based on these results, a necessary condition for both optimal controls and optimal parameters is derived in Theorem 2.5.1. Further, results on a necessary condition for optimal controls and a necessary condition for optimal parameters are given in Theorem 2.5.2 and Corollary 2.5.3 respectively. For linear time optimal control problems, it is shown in section 2.6 that the bang bang principle holds true. Under certain additional

assumptions, the existence of an optimal control is proved in section 2.7. At the end of the chapter, conclusions and suggestions for further studies are presented.

In Chapter 3, the optimization problem of Ito stochastic differential systems with fixed terminal time is considered. In section 3.3, the above stochastic optimization problem is converted into an equivalent optimization problem of a class of distributed parameter systems with Cauchy condition in which both controls and parameters appear in its differential operator. Based on the preparatory results given in section 3.5, a necessary condition for optimality for both controls and parameters considered together is developed [Theorem 3.6.3]. This result is then used to obtain individual necessary conditions, one for controls [Corollary 3.6.4] and one for the parameters [Corollary 3.6.7]. Further, the bang bang principle is shown to hold true for a class of linear Ito stochastic differential systems in section 3.7. Finally, conclusions and suggestions for further studies for the optimization problems considered in this chapter is given in the last section.

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LIST OF NOTATIONS FOR CHAPTER 2 and CHAPTER 3

$I \,\underline{\Delta}\, [0, T]$, (p. 17).

ξ - dynamic state vector, (p. 16).

$\hat{\xi}$ - the first n_1-components of the vector ξ that the controller can observe , (p. 17).

$\hat{\hat{\xi}}$ - the rest $n-n_1$ components of ξ, (p. 17).

Σ - parameter space, (p. 16).

π_o - initial probability measure, (p. 16).

Ω - supporting set for π_o .

$Q \,\underline{\Delta}\, (0, T) \times \Omega$, (p. 17).

$\hat{Q}$ - The projection onto $(t, \hat{x})$-space of Q , (p. 18).

D - the class of admissible controls, (Chapter 2, p. 18 and Chapter 3, p. 38).

U - control restraint set , (Chapter 2, p. 18 and Chapter 3, p. 38).

$C^{\ell}(K)$ - as defined in Section 2. 2 , (p. 18).

$C_o^{\ell}(K)$ - as defined in Section 3. 2, (p. 40).

$W^1(K)$ - as defined in Section 3. 2 , (p. 40).

$a_{ij} \,\underline{\Delta}\, \frac{1}{2} (c \cdot c^*)_{ij}$, (i, j=1, ..., n), (p. 20).

$a_i \,\underline{\Delta}\, \sum_{j=1}^{n} \frac{\partial a_{ij}}{\partial x_j} - b_i$, (i=1, ..., n), (p. 21) .

$\mathcal{F}$ - as defined in Section 2. 4 , (p. 21).

$\hat{\hat{\Omega}}$ - the projection onto $\hat{\hat{x}}$-spce of the set Ω, (p. 31).

$\Theta \triangleq [0, T) \times R^n$, (p. 42).

$O_k \triangleq \{x : |x| < k\}$, (p. 45).

$\Theta_k \triangleq O_k \times [0, T)$, (p. 45).

$H^{\lambda, \lambda/2}(\Theta_k)$ - as defined in Section 3.2. , (pp. 40-41).

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CHAPTER I

A SHORT REVIEW OF SOME EXISTING RESULTS ON OPTIMAL CONTROL OF SYSTEMS GOVERNED BY ITO STOCHASTIC DIFFERENTIAL EQUATIONS

1.1. INTRODUCTION

There is ample evidence in History of early man's attempt to find the best way to accomplish a given task. Trial-and-error methods were the basic tools for most of their decisions. Much later, mathematicians such as Euler, Lagrange, and others systematized this procedure by developing appropriate mathematical tools for solving such problems. It was the development of the calculus of variations which first placed these problems on a rigorous mathematical basis.

In the last few decades, these methods have been extensively generalized and applied by McShane [37], Bellman [9], Pontryagin, Boltyanski, Gamkrelidze and Mishchenko [39], Neustadt [38] Cesari [15], [16], Warga [50], Oğuztöreli [41] and many others. It appears to the author that the work of Pontryagin and his coworkers [39] is the one which aroused the interest of many investigators from different disciplines of natural and social sciences. However, most of this work centered around ordinary dynamical systems. The effects of stochastic characteristics exhibited in practice were not taken into account. It is quite well known that the behavior of natural phenomena is governed by chances and does not follow strict deterministic laws. For example, electric current exhibits visible chance fluctuations; the trajectory of a projectile is influenced by the uncertainty of the initial speed; and the demand for goods and services varies in some random manner. Thus, the general control problem is to find the control of a noisy nonlinear dynamical system in some optimal fashion, given only accessible noisy observations of the system state.

In this thesis, we will consider a class of stochastic control systems whose dynamics can be described by the well-known Ito differential equation. For this class of control systems, a number of developments have been reported by Kushner [31], [32], [33], [34], Fleming [21], [22], [23], [24], Fleming and Nisio [25], Sworder [45], [46], [47], Wonham [51], [52], [53], Ahmed [1], [2], Davis [17], Davis and Varaiya [18], Duncan and Varaiya [19], Beneš [10], [11], Ahmed and Teo [4], and others. An excellent bibliography can be found in [20].

However, it may be noted that the existence and uniqueness of solutions of systems governed by stochastic integral equations have been extensively studied by Bharucha-Reid [12], [13], [14]. Problems of optimal control of these systems were reported in [3] and others.

Since numerous results on optimal control problems for stochastic systems are now available in the literature, it is not possible to give an exhaustive survey of this subject. However, a brief review of results on stochastic optimal control problems directly related to the subject matter of this thesis will be presented.

1.2. STOCHASTIC APPROACH

In reference [31], Kushner has shown that for almost all sample paths the necessary condition for optimality of Pontryagin type is valid for the class of stochastic systems described by the following Ito differential equation

$$(1.2.1)\quad d\xi(t) = b(\xi(t), t, u(t))dt + dw(t), \qquad 0 \le t \le T,$$

with the cost functional $\varepsilon\{h^* \cdot \xi(T)\}$, where w is the Brownian motion;

$\mathcal{E}$ denotes the mathematical expectation; and * denotes vector transpose. Further, he has demonstrated that the differential equation for the adjoint variables are identical to those of Pontryagin. However, it may be noted that these results hold only for open loop control systems with additive white noise.

Recently, Kushner [34] applied the abstract variational theory of Neustadt [38] to obtain a stochastic maximum principle for the problem in which the system is governed by Ito differential equation subject to a general state space constraint. However, he considered open loop controls which appear only in the drift coefficient of the system and the cost integrands. In the last section of the paper, Kushner has indicated that similar results remain valid for stochastic closed loop control systems if the control variables are assumed to satisfy Lipschitz conditions.

Note that problems of optimal control for systems governed by sufficiently general stochastic nonlinear Ito differential equation without state space constraint can be reduced to equivalent optimal control problems of distributed parameter systems. The development of necessary conditions for optimality for these reduced problems is the main subject matter of this thesis. It appears to the author that the advantage with this approach in contrast to Kushner's lies in the fact that the optimal feedback controllers can be computed using this method. Thus, it would be interesting to obtain an equivalent optimal control problem of a distributed parameter system, corresponding to the stochastic problem considered by Kushner [34] and derive a necessary condition for optimality. Such a result would be of greater usefulness because of its inherent computational possibility. This is an open problem.

In reference [2], Ahmed considered an optimal control problem similar to that of Fleming and Nisio [25] for a class of stochastic control problems with finite memory. He used certain results of Prohorov [40] and Skorokhod [42] to prove [2, Lemma 1, p. 16] that the class of admissible random controls D is compact and closed in a Prohorov metric. For each admissible random control, the existence and uniqueness of solution of the stochastic system was proven [2, Prop. 3, p. 16] using the well known successive approximation method [25, p.783 and 29, p. 41]. Further, it was shown [2, Lemma 2, p. 23] that the solution x of the system is continuously dependent on the control u and thus the stochastic trajectory set X is also compact and closed in the Prohorov topology [2, Lemma 3, p. 25]. Based on these results, he has shown [2, Prop 4, p. 26] that if the cost functional J is taken as a real valued non-negative lower semicontinuous function on the product space $X \times D$, then there exists a control $u^o \in D$ and a corresponding trajectory $x^o \in X$ such that the expected value $\mathcal{E}\{J(x^o, u^o)\}$ is the minimum. At the end of his paper, the problem on synthesis of optimal feedback controller based on complete or partial observation of the past states was also briefly discussed. This is an interesting open problem.

In reference [11], Beneš considered the problem of optimal control of systems governed by the following stochastic Ito functional-differential equations

$$(1.2.2) \qquad d\xi(t) = b(t, \xi, u(t,\xi))\,dt + dw(t), \qquad 0 \le t \le 1,$$

with cost functional

(1.2.3) $$J(u) = \mathcal{E}\{\int_0^1 f(t,\xi,u(t,\xi))\,dt\},$$

where w is the Brownian motion, b and f are nonanticipative functionals describing system dynamics and cost integrand respectively, and $u(t,\cdot)$ is a control with values in a set U, and depending at time t only on given information about the past $\{\xi(s), s \leq t\}$.

Let $C \triangleq C[0,1]$ denote the space of continuous functions $y : [0,1] \longrightarrow R^n$ and S_t the σ-algebra of events $\{y \in C : y(s) \in A, 0 \leq s \leq t \leq 1\}$ with A Borel subset of R^n. For each t, let G_t be a sub σ-algebra of the σ-algebra S_t. The set D of admissible controls consists of functions $u : [0,1] \times C \rightarrow U$ which are Lebesgue measurable in t and G_t - measurable in y for each t. There is a σ-algebra G over $[0,1] \times C$ [11, p. 450] such that admissibility is equivalent to G-measurability.

Girsanov proved [26, Thm. 1, p. 287] that if φ is a nonanticipative Brownian functional with $\int_0^1 |\varphi|^2\, dt < \infty$ almost surely and if $\mathcal{E}\{\exp. \mathcal{L}(\varphi)\} = 1$, then the process W defined by

(1.2.4) $$W(t) = w(t) - \int_0^t \varphi\, d\tau$$

is a Wiener process under the measure $\tilde{P} \triangleq \{\exp. \mathcal{L}(\varphi)\} \cdot dP$, where $\mathcal{L}(\varphi) \triangleq \int_0^1 \varphi\, dw - \frac{1}{2}\int_0^1 |\varphi|^2\, dt$.

With measure $\tilde{P}$ this asserts that w is the solution of the stochastic equation

(1.2.5) $$dw = \varphi \cdot dt + dW$$

Following this approach with $\varphi \triangleq b(t, w, u(t,w))$, Beneš obtained the following expression

$$\mathcal{E}\{e^{\mathcal{L}(b)} \cdot \int_0^1 f(t,w,u(t,w))\,dt\} \tag{1.2.6}$$

for the cost functional corresponding to the control u. Then Beneš showed [11, pp. 451-453] that the control problem can be reformulated as a search for admissible u that achieves

$$\inf_{u \in D} \mathcal{E}\{e^{\mathcal{L}(b)} \cdot \int_0^1 f(t,w,u(t,w))\,dt\} .$$

For this reduced problem, Beneš proved [11, Thm. 8, pp. 471] that if U is compact, $b(t, y, v)$ grows at most linearly with y, and $G_t = S_t$, then the set of densities $\{\exp. \mathcal{L}(b) : u \in D\}$ attainable by the asmissible controls is convex, and there exists an optimal control $u^o \in D$ achieving $\inf_{u \in D} \mathcal{E}\{e^{\mathcal{L}(b)} \int_0^1 f \; dt\}$.

1.3 ANALYTICAL APPROACH

In this section, we will discuss results on problems of optimal control of stochastic systems in which necessary conditions for optimality are given in deterministic forms using the theory of partial differential equations. This approach reduces the stochastic optimal control problem into an optimal control problem of distributed parameter systems (partial differential equations).

When the states of the Markov process to be controlled are completely observable by the controller, the relevant partial differential equation and boundary data can be formally deduced by well-known Bellman dynamic programming technique.

By the application of this technique, Fleming [22] has developed some interesting results for the stochastic optimal control problems in which the plant is described by the following Ito differential equation

$$(1.3.1)\quad \begin{cases} d\,\xi(t) = b(t, \xi(t), u(t, \xi(t)))\,dt + c(t, \xi(t))\,dw(t), \\ \xi(s) = x, \quad 0 \leq s \leq t \leq T, \end{cases}$$

and the cost functional is given by

$$(1.3.2)\quad J(u) \triangleq \mathcal{E}\left\{ \int_0^{\tau} f(t, \xi(t), u(t, \xi(t)))\,dt \right\},$$

where $\xi \triangleq (\xi_1, \ldots, \xi_n)$; $w \triangleq (w_1, \ldots, w_m)$; b and c are assumed to have appropriate dimensions ; and τ is the Markov time as defined in section 2.1 [p. 17].

This result is briefly discussed below.

Let U be a compact and convex subset of a r-dimensional Euclidean space R^r. Let $T > 0$ be fixed, and let $Q_T \triangleq [0, T] \times R^n$. By admissible control we mean a function u on the strip Q_T with values in U such that for each $T' < T$, u satisfies a uniform Hölder condition in t and uniform Lipschitz conditions in x on $Q_{T'}$. Let D denote the set of all admissible controls. Let $b \triangleq (b_1, \ldots, b_n)$ be a bounded continuous function from $Q_T \times U$ into R^n and $c \triangleq (c_{ij})$, $i = 1, \ldots, n$, $j = 1, \ldots, m$ a bounded matrix-valued function on Q_T such that b and c satisfy uniform Hölder conditions in t and uniform Lipschitz conditions in x on $Q_T \times U$.

Let $a(t,x) \triangleq \frac{1}{2}(c \cdot c^*)(t,x)$ satisfy uniform parabolicity condition [condition (iii) of the assumption A_1 in section 2.2, p. 19], where $*$ denotes matrix transpose. Let f be real-valued, bounded, uniformly Hölder continuous in t and uniformly Lipschitz continuous in x on $Q_T \times U$.

Let $\mathcal{F}_o$ denote the set of all real-valued functions ψ which satisfy all the properties given in the definition of $\mathcal{F}_1$ [section 2.4, p. 21] and in addition : the partial derivatives ψ_t, $\psi_{x_i x_j}$, $i, j = 1, \ldots, n$ are continuous on $Q \setminus \{T\} \times \partial\Omega$, where Q and $\partial\Omega$, the boundary of Ω, are as defined in section 2.1.

Define

$$\phi^o(t,x) \triangleq \operatorname*{Inf}_{u \in D} \psi_u(t,x) , \tag{1.3.3}$$

where $\psi_u(t,x) \triangleq \mathcal{E}\{ \int_t^T f(s, \xi(s), u(s, \xi(s)))ds \mid \xi(t) = x \}$.

Then, it is shown [22, Thm. 2.1, p. 260] that $\phi^o \in \mathcal{F}_o$ and satisfies the following parabolic differential equation

$$\phi_t + \sum_{i,j=1}^{n} a_{ij}(t,x) \cdot \phi_{x_i x_j} + \min_{v \in U} [f(t,x,v) + \sum_{i=1}^{n} b_i(t,x,v) \cdot \phi_{x_i}] = 0 \tag{1.3.4}$$

on Q with the boundary data

$$\phi(t,x) = 0 \qquad \text{for} \quad (t,x) \in [0,T] \times \partial\Omega \cup \{T\} \times \Omega . \tag{1.3.5}$$

It is clear from the expression (1.3.4) that there may be no optimal control in the class D. For instance, when $f \equiv 1$ and b linear in v, then the extremal control u^o may have discontinuities. However, suppose b and f are of class $C^{(2)}(Q_T \times U)$; b is linear in v ; and

$$\sum_{i,j=1}^{r} \frac{\partial^2 f}{\partial v_i \partial v_j} \cdot z_i \cdot z_j \geq \beta_\ell \cdot |z|^2$$

for every $z \in R^r$, where $\beta_\ell > 0$, then, as shown in [22, Thm. 2.2, p. 261], there exists a $u^o \in D$ such that, for all $(t,x) \in Q$, $\psi_{u^o}(t,x) = \phi^o(t,x)$ and $f(t,x,v) + b(t,x,v) \cdot \phi^o_x(t,x)$ is minimum when $v = u^o(t,x)$.

In many problems of practical interest the states of the process being controlled are only partially observable. This leads to both mathematical and computational difficulties.

In reference [53], Wonham considered the dynamical equations of the form

$$(1.3.6) \quad \begin{cases} d\,\xi(t) = A(t) \cdot \xi(t)dt + b(t,u(t))\,dt + c(t)\,dw^1(t), & 0 \le t \le T, \\ \xi(o) = \xi_o ; & \end{cases}$$

with the observation equations

$$(1.3.7) \quad \begin{cases} d\,\eta(t) = F(t)\,\xi(t)\,dt + G(t) \cdot dw^2(t), & 0 \le t \le T, \\ \eta(o) = 0 . & \end{cases}$$

where $\xi \in R^n$ is dynamic state ; u is the control vector taking values in a convex compact subset $U \subset R^r$; $\eta \in R^n$ is channel output; and w^1, w^2 are independent, standard Wiener processes in R^{m_1} and R^{m_2}, respectively.

An underlying probability triple (Ω, B, P) which carries ξ_o and the Wiener processes $w^1(t)$ and $w^2(t)$ is assumed to be given. If R is a family of random variables, $\sigma(R) \subset B$ denotes the smallest σ-algebra relative to which R is measurable.

Let C denote the space of continuous functions from $[0,T]$ into R^n, with the usual sup norm. An admissible control is considered as a measurable function V from $[0,T] \times C$ into U satisfying uniform Lipschitz conditions in $y \in C$ on $[0,T] \times C$. For $y \in C$ let

$$(1.3.8) \qquad \pi_t\, y(s) = \begin{cases} y(s) \quad , & 0 \le s \le t \quad , \\ y(t) \quad , & t \le s \le T \quad . \end{cases}$$

The control applied at time t has the form

$$(1.3.9) \qquad u(t) = V(t, \pi_t \eta) \ .$$

The problem is to find a control u of this kind that will minimize the cost functional

$$(1.3.10) \qquad J(u) = \varepsilon \left\{ \int_0^T f(t, \xi(t), u(t))\, dt \right\} .$$

Let $S_t \triangleq \sigma\{\eta(s), 0 \le s \le t\}$ and let $\tilde{\xi}(t) \triangleq \varepsilon\{\xi(t) \mid S_t\}$.

Then, Wonham showed [53, pp. 316-319] that $\tilde{\xi}$ is the solution of the following stochastic differential equation

$$(1.3.11) \quad \begin{cases} d\tilde{\xi} = A \cdot \tilde{\xi}\, dt + b(t, \tilde{u}(t, \tilde{\xi}))\, dt + Q \cdot F^* \cdot (G \cdot G^*)^{-\frac{1}{2}} \cdot d\tilde{w} \\ \tilde{\xi}(0) = \varepsilon\{\xi_0\} \ , \\ \tilde{u} \in \tilde{D} \quad , \end{cases}$$

where $\tilde{D}$ is the class of functions

$$\tilde{u} : [0,T] \times R^n \longrightarrow U$$

satisfying Hölder conditions in $t \in [0, T]$ and Lipschitz conditions in $x \in R^n$; and $\tilde{w}$ is the Wiener process induced by $d\eta - F \cdot \tilde{\xi}\, dt = [G(t) \cdot G(t)^*]^{1/2} \cdot d\tilde{w}$. Further, the conditional covariance matrix Q can be obtained by solving the following matrix differential equation

$$(1.3.12) \quad \begin{cases} \dfrac{dQ}{dt} = A \cdot Q + Q \cdot A^* + c \cdot c^* - Q \cdot F^* \cdot (G \cdot G^*)^{-1} \cdot F \cdot Q \\ Q(0) \triangleq Q_0 \ , \ (Q_0 \triangleq \mathcal{E}\{[\xi_0 - \mathcal{E}(\xi_0)] \cdot [\xi_0 - \mathcal{E}(\xi_0)]^*\}), \end{cases}$$

where the superscripts '$*$' and '-1' denote transpose and matrix inversion respectively. Based on these results, the problem of optimal control of partially observable states was reduced [53, p. 321] to an equivalent optimal control problem which consists of the dynamical system (1.3.11) and the cost functional

$$(1.3.13) \qquad J(\tilde{u}) \triangleq \mathcal{E}\left\{ \int_0^T \tilde{f}(s, \tilde{\xi}(s), \tilde{u}(s, \tilde{\xi}(s)))\, ds \;\middle|\; \tilde{\xi}(0) \right\} ,$$

where

$$\tilde{f}(t, x, u) \triangleq \mathcal{E}\{ f(t, \tilde{\xi}(t), u) \mid \tilde{\xi}(t) = x \} .$$

For this reduced problem, Wonham showed [53, Lemma 6.1, p. 323; Lemma 5.1, p. 321] that there exists a function $\mu(t, x, P)$ satisfying uniform Hölder conditions in t and uniform Lipschitz conditions in (x, P) on $[0, T] \times R^n \times \{P \in R^n : |P| \leq \rho\}$ with values in U such that

$$(1.3.14) \quad \begin{aligned} & b(t, \mu(t, x, P))^* \cdot P + \tilde{f}(t, x, \mu(t, x, P)) \triangleq \lambda(t, x, P, \mu(t, x, P)) \\ & \leq b(t, v)^* \cdot P + \tilde{f}(t, x, v) \end{aligned}$$

for all $(t, x, P, v) \in [0,T] \times R^n \times \{P \in R^n : |P| \leq \rho\} \times U$,

where ρ is a constant and $|\cdot|$ denotes the usual Euclidean norm; and that

$$\text{(1.3.15)} \quad \begin{cases} \phi_t(t,x) + \frac{1}{2}\, \mathrm{tr} \cdot \{\tilde{c}(t)^* \cdot \phi_{xx}(t,x) \cdot \tilde{c}(t)\} + x^* \cdot A(t)^* \cdot \phi_x(t,x) \\ \qquad + \lambda(t, x, \phi_x(t,x), \mu(t, x, \phi_x(t,x))) = 0 \qquad (t,x) \in [0,T] \times R^n \\ \qquad \phi(T,x) = 0 \qquad\qquad x \in R^n, \end{cases}$$

where $\tilde{c} \; \underline{\Delta} \; Q \cdot F^* \cdot (G \cdot G^*)^{-\frac{1}{2}}$.

Further, $\mu(t,x, \phi_x(t,x)) \; \underline{\Delta} \; \tilde{u}^o$ is an optimal control in $\tilde{D}$ and is related to the optimal control of the original partially observable problem by

$$\text{(1.3.16)} \quad \tilde{u}^o(t, \tilde{\xi}(t)) = u^o(t), \; (u^o(t) \; \underline{\Delta} \; V^o(t, \pi_t \eta)) .$$

In reference [24, section 6, pp. 226-227], Fleming has extended this result to cover the case in which the control restraint set U can be taken unbounded and the conditions imposed on the cost integrand f can be weakened.

In the rest of this section, we will discuss some recent results of Fleming [23], and Ahmed and the author [4] on the subject of stochastic optimal control problems with controls based only on partially observed current states.

Consider the stochastic control system described by the following general nonlinear stochastic Ito differential equation

$$(1.3.17)\quad \begin{cases} d\xi(t) = b(t, \xi(t), u(t, \hat{\xi}(t)))dt + c(t, \xi(t), u(t, \hat{\xi}(t)))\, dw(t); \quad 0 \le t \le T \\ \xi(o) = \xi_o, \quad (\pi_o - \text{initial probability measure}) , \end{cases}$$

with the cost functional

$$(1.3.18) \qquad J(u) = \mathcal{E}\left\{ \int_0^{\tau} f(t, \xi(t), u(t, \hat{\xi}(t)))\, dt \right\} ,$$

where for each $t \in [0,T]$, $\xi(t) \in R^n$ is the dynamic state of the system ; $\hat{\xi}(t) \in R^{n_1}$, $0 \le n_1 \le n$, is the first n_1-components of this vector that the controller can observe ; $\{ w(t),\ t \in [0,T] \}$ is the m-dimensional Wiener process independent of ξ_o ; and τ is the Markov time as defined in section 2.1.

It is clear from the system equation (1.3.17) that the admissible controls are based only on the partially observed current states. In this case, Fleming has noted [23] that this class of stochastic optimal control problems can be transformed into an equivalent class of optimal control problems of linear second order first boundary value problems of parabolic type with controls appearing in the coefficients of the differential operator. This result is reported formally by the author and Ahmed in Theorem 3 [48, p. 363]. For this reduced problem, Fleming has derived [23, Thm. 1, p. 201] a necessary condition for optimality under the assumptions that the control variables satisfy Lipschitz conditions and the coefficients of the parabolic partial differential operator are of class $C^{(2)}([0,T] x R^n x U)$. and $a(t,x,u) \triangleq \frac{1}{2}(c \cdot c^*)(t,x,u)$ satisfies a uniform parabolicity condition [condition (iii) of the assumption (A_1), p. 19]. Under the assumption that the diffusion coefficient is independent of the control variables, Fleming has derived [23, Thm. 2, p. 202] a necessary condition for optimality for the corresponding class of stochastic control problems in which admissible controls are assumed only measurable with values in a compact convex subset U of R^r. Further, the drift coefficient and cost integrand are only measurable in the time variable and continuous in both the space and control variables. Under the additional linearity assumption of the drift coefficient in the control variables, Fleming has also given [23, Thm. 3, p. 205] an existence theorem of optimal controls for this problem. However, as far as the existence theorem is concenred, Ahmed and the author have shown [4, Thm. 1, pp.] that the conditions imposed on the drift coefficient b and the control restraint set U can be substantially relaxed at the expense of others. Their result is based on Fillipov technique [28, Thm. 3', p. 281] rather than the lower semicontinuity arguments as used by Fleming [23, Appendix 3, p. 213]. Further, it may be noted that if the cost integrand is independent of the control variables then this result contains Fleming's as special case.

Consider the stochastic control system (1.3.17) and the cost functional (1.3.18) with τ taken as a positive fixed quantity. In this situation, it is shown [48, Thm. 1, pp. 359-361] that this class of stochastic optimal control problems reduces to the optimal control of an equivalent class of Cauchy problems with control variables appearing in the coefficients of the differential operator. For this reduced problem, necessary conditions for determination of the optimal control will be derived in Chapter 3.

Unfortunately, existence theorems of optimal controls similar to [23; Thm. 3, p. 205] and [4, Thm. 1, p.] for the above problem are not available. This is an open problem.

1.4. CONCLUSION

In general, results on the existence of optimal controls can not be used directly to compute the optimal controller. In fact, these results provide only an important information that the system under consideration can be controlled optimally with respect to the given performance index and the given class of controls. Thus, as far as the existence of optimal controls is concerned, it appears to the author that neither the stochastic approach nor the analytical approach is superior over the other. However, we note that the analytical approach gives rise to a deterministic necessary condition for optimality for stochastic control problems in contrast to a random necessary condition for optimality obtained by using stochastic approach. Thus, it is clear that the analytical approach dominates the stochastic approach in the sense that the former provides the possibility for the computation of the optimal controller whereas the latter does not.

CHAPTER 2

OPTIMAL CONTROL OF ITO STOCHASTIC DIFFERENTIAL SYSTEMS WITH MARKOV TERMINAL TIME

2. 1. INTRODUCTION

In this chapter, we consider a class of stochastic systems described by Ito differential equation for which both the parameters and controls are to be chosen optimally with respect to certain performance criterion. The controls to be optimized are taken as unknown functions which depend only on partially observed current states. It is shown [Theorem 2. 3. 2.] that this stochastic optimization problem can be reduced to an equivalent optimization problem of distributed parameter systems. To this reduced problem, a necessary condition for optimality for both controls and parameters combined together is given in Theorem 2. 5. 1. Results on individual necessary conditions for optimal controls and optimal parameters are presented in Theorem 2. 5. 2 and Corollary 2. 5. 3 respectively. The bang bang principle is shown to hold true for linear time optimal control problems in Theorem 2. 6. 1. Results on the existence of optimal controls are given in Theorem 2. 7. 1 and Theorem 2. 7. 2. The other theorems and lemmas presented in this Chapter are only the preparatory materials for the development of the above mentioned results.

Consider the system described by the following Ito differential equation

$$S_1 : \begin{cases} d\xi(t) = b(t, \xi(t), \sigma, u(t, \hat{\xi}(t)))dt + c(t, \xi(t), \sigma)dw(t), t \in [0, T], \\ \qquad\qquad 0 < T < \infty , \\ \xi(o) = \xi_o \quad , (\pi_o \text{ - initial probability measure}), \\ \sigma \in \Sigma \quad , (\Sigma\text{- parameters space}), \\ u \in D \quad , (D \text{ - class of admissible controls}), \end{cases}$$

where for each $t \in [0,T] \triangleq I$

$$\xi(t) = (\xi_1(t), \ldots, \xi_n(t)) \in R^n$$

is the dynamic state of the system,

$$\hat{\xi}(t) = (\xi_1(t), \ldots, \xi_{n_1}(t)) \in R^{n_1}, \quad 0 \le n_1 \le n,$$

is (without loss of generality) the first n_1 components of this vector that the controller can observe, and $\hat{\xi}$ (t) is the rest $n-n_1$ components;

$$b : I \times R^n \times R^{r_1} \times R^{r_2} \longrightarrow R^n ;$$

$$u : I \times R^{n_1} \longrightarrow R^{r_2} \text{ is the control law ;}$$

$$c : I \times R^n \times R^{r_1} \longrightarrow n \times m \text{ matrices ;}$$

$\{w(t), t \in I\}$ is the m-dimensional Wiener process independent of ξ_0; Σ is a convex subset of R^{r_1}; and D, to be defined later, is the set of admissible controls.

Let Ω be an open set in R^n (with compact closure) supporting the initial probability measure π_0 and let the boundary $\partial\Omega$ of Ω satisfy the following property: each point of $\partial\Omega$ is locally representable by functions with Hölder continuous second order partial derivatives [22, p. 257].

Given the data $\xi(s) = x$, $s \in (0,T)$, let us stop the process ξ at the first time $\tau \in [s,T)$ when $\xi(\tau) \in \partial\Omega$; if $\xi(t) \in \Omega$ for all $t \in [s,T)$, then we set $\tau = T$. The random variable τ is the first exit time from the cylinder $Q \triangleq (0,T) \times \Omega$ and is known as Markov time.

Let us define the set of admissible controls on $\hat{Q}$ by

$D \triangleq \{u : u \text{ measurable on } \hat{Q},\ u(t,\hat{x}) \in U,\ (t,\hat{x}) \in \hat{Q}\}$,

where $\hat{Q}$ is the projection onto $(t,\hat{x})$ - space of the cylinder Q and U is a compact convex subset of R^{r_2}.

For convenience, the elements of the set $\Sigma \times D$ will be called 'POLICIES'.

With the above preparations, we may state the problem P_1 as: subject to the dynamic constraints S_1, find a policy $(\sigma, u) \in \Sigma \times D$ that minimizes the cost functional

$$J(\sigma, u) = \mathcal{E}\left\{\int_0^{\tau} f(t, \xi(t), \sigma, \underline{u}(t, \hat{\xi}(t)))\,dt\right\},$$

where f is a real-valued Carathéodory function defined on $I \times (\bar{\Omega} \times \Sigma \times U)$, (i.e. measurable in t for each $(x, \sigma, v) \in \bar{\Omega} \times \Sigma \times U$ and continuous in (x, σ, v) for almost all $t \in I$, $\bar{\Omega}$ the closure of the set Ω and τ the Markov time.

2.2 NOTATIONS AND BASIC ASSUMPTIONS

Before describing the basic assumptions to be imposed for the coefficients of the system S_1, we introduce some useful notations.

$|B|$ denotes the Lebesgue measure of the measurable set B of any finite dimensional Euclidean space. ∂B denotes the boundary of the set B and $\bar{B}$ its closure. Let K be any connected subset of a n-dimensional Euclidean space and denote by $C^{\ell}(K)$ $1 \leq \ell \leq \infty$, the class of all ℓ times differentiable real valued functions on K.

Throughout this chapter, the following assumptions will be referred to collectively as (A_1).

(i) b and f are bounded measurable in I for each $(x, \sigma, v) \in \overline{\Omega} \times \Sigma \times U$ and bounded continuous in $\overline{\Omega} \times U$ for each $\sigma \in \Sigma$ and for almost all t [illegible] I ; and $b(t,x,\cdot,u)$, $f(t,x,\cdot,u) \in C^1(\Sigma)$ almost everywhere on $\overline{Q}$ for every $u \in D$;

(ii) $a(t,x,\sigma) = \frac{1}{2}(c \cdot c^*)(t,x,\sigma)$ is continuous and bounded on $\overline{Q} \times \Sigma$, where * denotes matrix transpose ;

(iii) there exists a number $\alpha_\ell > 0$ such that

$$a_{ij}(t,x,\sigma) \cdot z_i \, z_j \geq \alpha_\ell \, |z|^2 \quad \text{for all } z \in R^n$$

uniformly on $\overline{Q} \times \Sigma$ (uniformly parabolic) ;

(iv) $$\frac{|a(t,x,\sigma) - a(t_1,x_1,\sigma)|}{|t - t_1| + |x - x_1|} \leq M ,$$

where $t, t_1 \in I$; $x, x_1 \in \overline{\Omega}$; $\sigma \in \Sigma$ and M is a constant.

Note that unless otherwise stated the convention adopted above and throughout the thesis is to take summation up to n over repeated indices.

2.3. TRANSFORMATION OF STOCHASTIC OPTIMIZATION PROBLEMS TO EQUIVALENT DISTRIBUTED PARAMETER OPTIMIZATION PROBLEMS.

Throughout this chapter, we denote by $L(\sigma,u)$ the following differential operator :

$$(2.3.1) \quad L(\sigma,u) \cdot \psi \triangleq \psi_t + [a_{ij}(t,x,\sigma) \cdot \psi_{x_i x_j} + b_i(t,x,\sigma,u) \cdot \psi_{x_i}] ,$$

where $u \in D$, $\psi_{x_i} \triangleq \frac{\partial \psi}{\partial x_i}$, $\psi_{x_i x_j} \triangleq \frac{\partial^2 \psi}{\partial x_i \partial x_j}$ and

$a_{ij}(t,x,\sigma) = \frac{1}{2} (c \cdot c^*)_{ij} (t,x,\sigma)$.

In the sequel, we need the following theorem [4 , Thm. 1, p.].

Theorem 2.3.1. Consider the stochastic system S_1 [section 2.1, p. 16]. Suppose that the coefficients b and c satisfy the conditions given in (A_1). Then, for each $(\sigma, u) \in \Sigma \times D$ the system under consideration has a unique solution $\xi(\sigma,u)$ which is a strong Markov process [44, p. 388].

The proof follows from Corollary 3.2 and Theorem 6.2 of Stroock and Varadhan [44, pp. 366-367 and p. 392].

Using Ito's Lemma [44, Thm. 2.5, p. 352], it is shown [48, Thm. 3, p. 363] that the problem P_1 can be reduced to the optimization problem of an equivalent distributed parameter system.

Theorem 2.3.2 : The problem P_1 is equivalent to problem P'_1 which consists of the first boundary value problem S'_1 and the cost functional (2.3.2) :

$$S'_1 : \begin{cases} L(\sigma,u) \cdot \phi(\sigma,u)(t,x) + f(t,x,\sigma,u(t,x)) = 0 , & (t,x) \in [0,T) \times \Omega \\ \phi(\sigma,u)(t,x) = 0 & (t,x) \in (0,T] \times \partial\Omega \\ \phi(\sigma,u)(T,x) = 0 & x \in \Omega \\ (\sigma,u) \in \Sigma \times D & \end{cases}$$

$$\min_{(\sigma,u) \in \Sigma \times D} J(\sigma,u) = \min_{(\sigma,u) \in \Sigma \times D} \int_{\Omega} \phi(\sigma,u)(0,x) \, d\pi_0(x) , \tag{2.3.2}$$

where π_o is the initial probability measure defined on Ω .

The proof is given in [48, Thm. 3, p. 363] .

2.4. EXISTENCE AND UNIQUENESS OF SOLUTIONS OF RELATED SYSTEMS

For the proof of necessary conditions for optimality for the problem P'_1, it would be required to consider the existence and uniqueness of solutions of the first boundary value problem S'_1 and its adjoint system.

For convenience of further references, let $\mathcal{F}_1$ denote the class of all real-valued functions ϕ on the cylinder $\overline{Q}$ having the following properties [23, p. 202] :

(i) ϕ and ϕ_{x_i} , $i = 1, 2, \ldots, n$, are Hölder continuous on $\overline{Q}$;

(ii) the partial derivatives ϕ_t, $\phi_{x_i x_j}$, $i, j = 1, \ldots, n$, are square integrable on Q ;

(iii) $\phi(t,x) = 0$ for all $(t,x) \in I \times \partial\Omega \cup \{T\} \times \Omega$.

Definition 2.4.1 : For each $(\sigma,u) \in \Sigma \times D$, the function $\phi(\sigma,u) \in \mathcal{F}_1$ is said to be a solution of the first boundary value problem S'_1 if it satisfies the corresponding equations of S'_1 on $I \times \overline{\Omega} \underset{a}{=} \overline{Q}$.

The existence and uniqueness of solution in $\mathcal{F}_1$ of the system S'_1 are known [23, Appendix 1, pp. 209-210] .

Using the property (iv) of the assumption (A_1) and denoting $\sum_{j=1}^{n} \frac{\partial a_{ij}(t,x,\sigma)}{\partial x_j} - b_i(t,x,\sigma,u)$ by $a_i(t,x,\sigma,u)$, it follows easily that the system adjoint to S'_1, corresponding to each $(\sigma,u) \in \Sigma \times D$, may be written in the form

$$AS'_1 : \begin{cases} L^*(\sigma, u) \cdot q(\sigma, u) = 0 & (t,x) \in Q \\ q(t,x) = 0 & (t,x) \in I \times \partial\Omega \\ q(0,x) = q_o(x) & x \in \Omega \end{cases}$$

where, for each $(\sigma, u) \in \Sigma \times D$,

$$L^*(\sigma,u) \cdot \psi \triangleq -\frac{\partial \psi}{\partial t} + (a_{ij}(t,x,\sigma) \cdot \psi_{x_j} + a_i(t,x,\sigma,u) \cdot \psi)_{x_i} ;$$

and for any measurable subset of Ω, $\pi_o(E) = \int_E q_o(x) \; dx$.

For the weak solution of the system AS'_1, we use the definition of Fleming [23, p. 199].

Definition 2.4.3 : For each $(\sigma, u) \in \Sigma \times D$, an integrable function $q(\sigma, u)$ on Q is said to be a weak solution of the problem AS'_1, if, for every $\psi \in \mathcal{F}_1$ for which $L(\sigma, u)\,\psi$ is bounded,

$$(2.4.1) \quad \int_Q [\, L(\sigma,u)\, \psi(t,x) \,] \cdot q(\sigma,u)(t,x) \, dt \; dx = - \int_\Omega \psi(o,x) \; \pi_o(dx) .$$

The existence and uniqueness of the weak solution are also reported by Fleming [23, Appendix 2, pp. 211-213].

2.5. NECESSARY CONDITIONS FOR OPTIMALITY

For the determination of the optimal policy (control and parameter combined), we have the following result.

Theorem 2.5.1 : Consider the problem P'_1. Suppose that the assumption (A_1) is satisfied and that $a_{ij}(t,x,\cdot)$, $b_i(t,x,\cdot,\cdot)$, $(i,\ j=1,\ldots,n)$, $f(t,x,\cdot,\cdot)$ belong to $C^1(\Sigma \times U)$ almost everywhere in Q with gradient bounded in $R^{r_1+r_2}$ for almost all $(t,x) \in Q$ and for every $(\sigma, v) \in \Sigma \times U$. Then, if $(\sigma^o, u^o) \in \Sigma \times D$ is an optimal policy (whose existence is assumed), it is necessary that there exists a weak solution $q(\sigma^o, u^o)$ of the adjoint system AS'_1 corresponding to $(\sigma^o, u^o) \in \Sigma \times D$ such that for every $(\sigma, u) \in \Sigma \times D$

.5.1)

$$\begin{aligned}
&\sum_{k=1}^{r_1} \Big\{ \int_Q \big[a_{ij;\sigma_k}(t,x,\sigma^o) \cdot \phi_{x_i x_j}(\sigma^o,u^o)(t,x) + b_{i,\sigma_k}(t,x,\sigma^o,u^o) \cdot \phi_{x_i}(\sigma^o,u^o)(t,x) \\
&\qquad + f_{\sigma_k}(t,x,\sigma^o,u^o)\big] \cdot q(\sigma^o,u^o)(t,x) \cdot (\sigma_k - \sigma_k^o)\, dt\, dx \Big\} \\
&\qquad\qquad + \sum_{\ell=1}^{r_2} \Big\{ \int_Q \big[b_{i,u_\ell}(t,x,\sigma^o,u^o) \cdot \phi_{x_i}(\sigma^o,u^o)(t,x) \\
&\qquad\qquad + f_{u_\ell}(t,x,\sigma^o,u^o)\big] \cdot q(\sigma^o,u^o)(t,x) \cdot (u_\ell - u_\ell^o)\, dt\, dx \Big\} \\
&\geq 0 ,
\end{aligned}$$

where $\alpha'(y)$ denotes the Gateaux differential of α at $y \in \Sigma \times D$ defined as follows.:

$$\lim_{\varepsilon \downarrow 0} \frac{\alpha(y+\varepsilon h) - \alpha(y)}{\varepsilon} = (\alpha'(y), h) \text{ with } (\alpha'(y), h) \triangleq \sum_{k=1}^{r_1+r_2} \alpha'_{y_k}(y) \cdot h_k$$

<u>Proof :</u> For each $(\sigma, u) \in \Sigma \times D$, let $\phi(\sigma, u)$ be the corresponding solution of the system S_1' and let $(\sigma^o, u^o) \in \Sigma \times D$ be the optimal policy.

Clearly,

2.5.2)

$$\begin{aligned}
&\int_Q L(\sigma^o,u^o)\,[\phi(\sigma^o,u^o)(t,x) - \phi(\sigma,u)(t,x)] \cdot q(\sigma^o,u^o)(t,x)\,dt\,dx \\
&= \int_Q [(L(\sigma,u) - L(\sigma^o,u^o)) \cdot \phi(\sigma,u)(t,x) + f(t,x,\sigma,u) - f(t,x,\sigma^o,u^o)]\, q(\sigma^o,u^o)(t,x)\,dt\,dx .
\end{aligned}$$

Thus, by using the relation (2.4.1) to its left hand side with ψ replaced by $\phi(\sigma^o, u^o) - \phi(\sigma, u)$, we obtain

$$(2.5.3) \qquad -\int_\Omega \phi(\sigma^o, u^o)(o,x) q_o(x)dx + \int_\Omega \phi(\sigma, u)(o,x)\, q_o(x)\, dx$$

$$= \int_Q [(L(\sigma,u) - L(\sigma^o,u^o))\phi(\sigma,u)(t,x) + f(t,x,\sigma,u) - f(t,x,\sigma^o,u^o)]\, q(\sigma^o,u^o)(t,x) dt\, dx.$$

Since

$$(2.5.4) \qquad 0 \leq J(\sigma,u) - J(\sigma^o,u^o) = \int_\Omega \phi(\sigma,u)(o,x) q_o(x)dx - \int_\Omega \phi(\sigma^o,u^o)(o,x) q_o(x) dx,$$

it is clear that the expression (2.5.3) reduces to

$$(2.5.5) \qquad \int_Q \{(a_{ij}(t,x,\sigma) - a_{ij}(t,x,\sigma^o))\phi_{x_i x_j}(\sigma,u) + (b_i(t,x,\sigma,u) - b_i(t,x,\sigma^o,u^o))\cdot \phi_{x_i}(\sigma,u)$$

$$+ f(t,x,\sigma,u) - f(t,x,\sigma^o,u^o)\}\, q(\sigma^o,u^o)(t,x) dt\, dx \geq 0,$$

for all $(\sigma, u) \in \Sigma \times D$.

For any $(\sigma,u) \in \Sigma \times D$ let $\varepsilon \in [0,1]$ and let $(\sigma,u) - (\sigma^o,u^o) \triangleq (\sigma', u')$. Since $\Sigma \times D$ is convex, $(\sigma^o,u^o) + \varepsilon(\sigma',u') \in \Sigma \times D$. Thus, dividing the inequality (2.5.5) by ε and replacing (σ, u) by $(\sigma^o,u^o) + \varepsilon(\sigma', u')$, we obtain

$$(2.5.6) \qquad \frac{1}{\varepsilon}\Big[\int_Q \{(a_{ij}(t,x,\sigma^o + \varepsilon\sigma') - a_{ij}(t,x,\sigma^o))\cdot \phi_{x_i x_j}(\sigma^o + \varepsilon\sigma', u^o + \varepsilon u')$$

$$+ (b_i(t,x,\sigma^o + \varepsilon\sigma', u^o + \varepsilon u') - b_i(t,x,\sigma^o,u^o))\cdot \phi_{x_i}(\sigma^o + \varepsilon\sigma', u^o + \varepsilon u')$$

$$+ f(t,x,\sigma^o + \varepsilon\sigma', u^o + \varepsilon u') - f(t,x,\sigma^o,u^o)\}\, q(\sigma^o,u^o)(t,x)\, dt \cdot dx\Big] \geq 0.$$

Since, by hypothesis, the coefficients of the system S'_1 belong to $C^1(\Sigma \times U)$ almost everywhere in Q with bounded gradient for almost every $(t,x) \in Q$ and every $(\sigma, u) \in \Sigma \times U$, it is clear that their incremental ratios converge to the corresponding Gateaux differentials almost everywhere in Q as $\varepsilon \downarrow 0$. Denoting by a any of the coefficients a_{ij}, b_i, f and noting that $a(t,x,\sigma^o + \varepsilon\sigma', u^o + \varepsilon u') \longrightarrow a(t,x,\sigma^o,u^o)$ a.e. in Q as $\varepsilon \downarrow 0$, it follows from [23, paragraph 2 of Appendix 1, p. 210] that $\phi(\sigma^o + \varepsilon\sigma', u^o + \varepsilon u')$ and $\phi_x(\sigma^o + \varepsilon\sigma', u^o + \varepsilon u')$ converge to $\phi(\sigma^o, u^o)$ and $\phi_x(\sigma^o, u^o)$ uniformly on $\bar{Q}$ while $\phi_t(\sigma^o + \varepsilon\sigma', u^o + \varepsilon u')$ and $\phi_{xx}(\sigma^o + \varepsilon\sigma', u^o + \varepsilon u')$ converge, respectively, to $\phi_t(\sigma^o, u^o)$ and $\phi_{xx}(\sigma^o, u^o)$ weakly in $L^2(Q)$. Thus by letting $\varepsilon \downarrow 0$ in the inequality (2.5.6), we obtain the condition (2.5.1). This completes the proof.

In the rest of this section, we will discuss individual results on pointwise necessary conditions for optimal controls [Theorem 2.5.2] and necessary conditions for optimal parameters [Corollary 2.5.3].

Consider the problem P'_1 with all the coefficients independent of the parameter vector $\sigma \in \Sigma$. Clearly, its equivalent optimal control problem of distributed parameter system, which is denoted by P'_2, is the problem P'_1 with σ deleted. Call the corresponding first boundary value problem S'_2. Further, it is understood that the system AS'_2, which is adjoint to the system S'_2, is the system AS'_1 with σ deleted.

In the absence of the parameter vector σ, necessary conditions for optimality for the problem P'_1 can be proven under weaker conditions. This result was given by Fleming [23, Thm. 2, p. 202] and is presented below with a different proof.

<u>Theorem 2.5.2</u>: Consider the problem $P^{'}_{2}$. Suppose that $u^o \in D$ is an optimal control (whose existence is assumed) and that the assumption (A_1) is satisfied. Then there exists a weak solution $q(u^o)$ of the adjoint system $AS^{'}_{2}$ corresponding to the optimal control $u^o \in D$ so that for almost every $(t, \hat{x}) \in \hat{Q}$ and every $u \in D$

$$\begin{aligned} &\int_{\hat{\hat{\Omega}}} [b_i(t,\hat{x},\hat{\hat{x}},u^o(t,\hat{x}))\cdot \phi_{x_i}(u^o)+f(t,\hat{x},\hat{\hat{x}},u^o(t,\hat{x}))]\cdot q(u^o)(\hat{\hat{x}} \mid t,\hat{x})\cdot d\hat{\hat{x}} \\ (2.5.7)\quad &\le \int_{\hat{\hat{\Omega}}} [b_i(t,\hat{x},\hat{\hat{x}},u(t,\hat{x}))\cdot \phi_{x_i}(u^o)+f(t,\hat{x},\hat{\hat{x}},u(t,\hat{x}))]\cdot q(u^o)(\hat{\hat{x}} \mid t,\hat{x})\cdot d\hat{\hat{x}} \end{aligned}$$

where $\hat{\hat{\Omega}}$ is the projection onto $\hat{\hat{x}}$-space of the set $\ddot{\Omega}$, $\phi(u^o)$ the solution of the system $S^{'}_{2}$ corresponding to $u^o \in D$ and $q(u^o)(\hat{\hat{x}} \mid t, \hat{x}) \triangleq$

$$\frac{q(u^o)(t,\hat{x},\hat{\hat{x}})}{\int_{\hat{\hat{\Omega}}} q(u^o)(t,\hat{x},\hat{\hat{x}})\cdot d\hat{\hat{x}}}$$ is the conditional density.

<u>Outline of the Proof :</u> Using the inequality (2. 5. 5) instead of (16.11) [36, p. 249], we would obtain the proof of the condition (2. 5. 7) similar to that given in the subsections 16. 2 and 16. 3 [36, pp. 250-251] .

Consider the problem P_1 with its coefficients dependent only of the parameter vector σ. Clearly, the equivalent optimal parameter selection problem of the distributed parameter system, which is denoted by P_3', is the problem P_1' with u deleted. Further, the corresponding first boundary problem and its adjoint system are to be denoted by S_3' and AS_3' respectively.

By the application of Theorem 2.5.1, we have the following result for the optimal parameter (if one exists).

Corollary 2.5.3 : Consider the problem P_3'. Suppose that the assumption (A_1) is satisfied. Then, for $\sigma^o \in \Sigma$ to be an optimal parameter with $\phi(\sigma^o) \in \mathcal{F}_1$ the corresponding solution of the system S_3', it is necessary that

$$(2.5.8) \quad \int_Q \left\{ \frac{\partial a_{ij}(t,x,\sigma^o)}{\partial \nu} \cdot \phi_{x_i x_j}(\sigma^o) + \frac{\partial b_i(t,x,\sigma^o)}{\partial \nu} \cdot \phi_{x_i}(\sigma^o) + \frac{\partial f(t,x,\sigma^o)}{\partial \nu} \right\} q(\sigma^o)(t,x)\, dt \cdot dx \geq 0$$

(for Σ closed) ;

$$(2.5.9) \quad \int_Q \left\{ \frac{\partial a_{ij}(t,x,\sigma^o)}{\partial \sigma_k} \cdot \phi_{x_i x_j}(\sigma^o) + \frac{\partial b_i(t,x,\sigma^o)}{\partial \sigma_k} \cdot \phi_{x_i}(\sigma^o) + \frac{\partial f(t,x,\sigma^o)}{\partial \sigma_k} \right\} q(\sigma^o)(t,x)\, dt\, dx = 0 ,$$

$k = 1, \ldots, r_1$, (for Σ open) ;

where $\frac{\partial a(t,x,\sigma^o)}{\partial \nu} \triangleq \lim_{\varepsilon \downarrow 0} \frac{a(t,x,\sigma^o+\varepsilon \nu) - a(t,x,\sigma^o)}{\varepsilon}$, $\sigma^o + \varepsilon \nu \in \Sigma$

$0 \leq \varepsilon \leq 1$; and ν any vector directed inward into Σ emanating from σ^o; and $q(\sigma^o)$ is the weak solution of the adjoint system AS'_3 corresponding to the optimal parameter vector $\sigma^o \in \Sigma$.

Remark 2.5.4: Assume that the coefficients a_{ij}, b_i, $(i,j = 1, \ldots, n)$ and f as functions of the parameter σ are defined on, and continuously differentiable in, an open set containing Σ^o. Then, it is easily verified that the necessary condition (2.5.8) is equivalent to the condition

$$\sum_{k=1}^{r_1} \Big[\int_Q \Big\{ \frac{\partial a_{ij}(t,x,\sigma^o)}{\partial \sigma_k} \cdot \phi_{x_i x_j}(\sigma^o) + \frac{\partial b_i(t,x,\sigma^o)}{\partial \sigma_k} \cdot \phi_{x_i}(\sigma^o) + \frac{\partial f(t,x,\sigma^o)}{\partial \sigma_k} \Big\} \cdot q(\sigma^o) \cdot dt \cdot dx \Big] \cdot [\sigma_k - \sigma_k^o] \geq 0 \tag{2.5.10}$$

for all $\sigma \in \Sigma$.

As a consequence of Corollary 2.5.3, we have

Corollary 2.5.5. Consider the problem P'_3.
Suppose that all the hypotheses given in Corollary 2.5.3 are satisfied and that the coefficients a_{ij}, b_i; $(i, j = 1, \ldots, n)$, and f are linear in the parameter σ. Then, if the parameter restraint set Σ

is a compact and convex polyhedron in R^{r_1}, the optimal parameter vector σ^o takes its value at one of the vertices of Σ.

The results given for the problem P_3' are also reported in [6] and illustrated by a numerical example given in [49, p.287].

2.6 BANG BANG PRINCIPLE.

In ordinary differential systems, LaSalle introduced the concept of Bang Bang Principle [27, p. 46] for the study of linear time optimal control problems. Recently, many investigators have also attempted to prove Bang Bang Principle for stochastic systems.

Using the well known Dynamic Programming Principle, Katayama proved [30] that for linear time invariant stochastic systems the control, that is assumed based on completely observed current states and that maximizes the expectation of the first exit time, is of "bang bang" type. In fact, it has been shown [5] recently that by the application of Corollary 2.5.2, this result can be greatly generalized to cover a class of nonlinear stochastic systems with controls using only partially information about the current states. This class of system is described by

$$S_4 : \begin{cases} d\,\xi(t) = b(t, \xi(t))dt + g(t, \xi(t)) \cdot u(t, \hat{\xi}(t))dt + c(t, \xi(t))\, dw(t) , \\ t \in I' , \\ \xi(o) = \xi_o \sim (\pi_o \text{ - initial probability measure }) \\ u \in D \end{cases}$$

where $\xi(t)$, $\hat{\xi}(t)$, $\{ w(t),\ t \in I \}$ and D are as defined in section 2.1;

$b \ : \ I \times R^n \longrightarrow R^n$;

$g \ : \ I \times R^n \longrightarrow n \times r_2$ matrices;

and $c \ : \ I \times R^n \longrightarrow n \times m$ matrices.

The problem to be called P_4 is stated as : Given the system S_4, find a control $u \in D$ that maximizes the expectation of the first exit time from the set Ω as defined in section 2.1 (i.e. $\max \ \mathcal{E}\{ \tau : u \in D \}$).

The optimal control problem of the equivalent distributed parameter system is the problem P_1' with the coefficients redefined as $a(t,x,\sigma) \equiv a(t,x)$, $b(t,x,\sigma,u) \equiv b(t,x) + g(t,x) \cdot u$ and $f \equiv 1$. Call this problem P_4'. The corresponding first boundary value problem and its adjoint system will be denoted by S_4' and AS_4' respectively.

With these preparations, we have the following result.

Theorem 2.6.1 : Consider the problem P_4'. Suppose that the assumption (A_1) holds with $a(t,x,\sigma) = a(t,x)$, $b(t,x,\sigma,u) = b(t,x) + g(t,x) \cdot u$ and $f = 1$. Then there exists a control $\hat{u} \in D$ that maximizes the expectation of the first passage time and is necessarily of 'bang bang' type.

The proof follows from the direct application of Theorem 2.7.1 (Theorem 2.7.2) and Corollary 2.5.2 with 'min' replaced by 'max'.

Before we present a special case of Theorem 2.6.1, we need the following definition.

Definition 2.6.2 : The system S_4' is said to be normal if to every admissible control u the solution $\phi(u)$ and any weak solution $q(u)$ of the system AS_4' have the property that for no $j = 1, 2, \ldots, r_2$

$$z_j(t, \hat{x}) \triangleq \{ \int_{\hat{\hat{\Omega}}} (g_{ij}(t, x, \hat{\hat{x}}) \cdot \phi_{x_i}(u) \cdot q(u)(\hat{\hat{x}} \mid t, \hat{x}) \, . \, d\hat{\hat{x}} \}$$

is identically zero over a set of positive measure,

where $\hat{\hat{\Omega}}$ is the projection onto $\hat{\hat{x}}$ - space of the set Ω and

$$q(u)(\hat{\hat{x}} \mid t, \hat{x}) \triangleq \frac{q(u)(t, x, x)}{\int_{\Omega} q(u)(t, \hat{x}, \hat{\hat{x}}) \, d\hat{\hat{x}}}$$

As a consequence of Theorem 2.6.1, we have

Corollary 2.6.3. If the control restraint set U is a unit hypercube in R^{r_2}, D is the corresponding class of admissible controls and the system S_4' is normal, then the optimal control is of the form

$$u_j^o(t, \hat{x}) = \text{sign.} \left[\int_{\hat{\hat{\Omega}}} g_{ij}(t, \hat{x}, \hat{\hat{x}}) . \phi_{x_u}(u^o) . q(u^o)(\hat{\hat{x}} \mid t, \hat{x}) \, d\hat{\hat{x}} \right] \tag{2.6.1}$$

$j = 1, \ldots, r_2$;

where $q(u^o)$ satisfies the adjoint equations AS_4' in the weak sense. Further, $\phi(u^o)$ is the solution of the following nonlinear integro-partial differential equation with the given boundary conditions

$$(2.6.2)\quad \begin{cases} -\dfrac{\partial\phi(u^o)}{\partial t} = (b_i(t,x)\cdot\phi_{x_i}(u^o) + \sum_{j=1}^{r_2}(g_{ij}(t,x)\cdot\phi_{x_i}(u^o))\,\mathrm{sign.}\left(\int_{\hat\Omega} g_{ij}(t,x)\cdot\phi_{x_i}(u^o)\cdot q(u^o)(\hat{\hat{x}}|t,\hat{x})d\hat{\hat{x}}\right) \\ \qquad + a_{ij}(t,x)\cdot\phi_{x_ix_j}(u^o) + 1, & (t,x)\in[o,T)\times\Omega \\ \phi(u^o)(t,x) = 0 & (t,x)\in(o,T]\times\partial\Omega \\ \phi(u^o)(T,x) = 0 & x\in\Omega \end{cases}$$

Remark 2.6.4 : In the case of partial information available to the controller, the optimal control law is given by (2.6.1). It is interesting to mention the following two extreme cases :

(i) $n_1 = n$ (complete information). In this case, the controller uses complete information. Setting $n_1 = n$ in (2.6.1) and interpreting

$$(2.6.3)\quad \lim_{n_1\to n}\int_{\Omega\setminus\hat\Omega} h(\hat{x},\hat{\hat{x}})\cdot d\hat{\hat{x}} = h(\hat{x},\hat{\hat{x}}),$$

we obtain for $(t,x)\in Q$

$$(2.6.4)\quad u_j^o(t,x) = \mathrm{sign.}\,[g_{ij}(t,x)\cdot\phi_{x_i}(u^o)],\quad j=1,\ldots,r_2;$$

and the differential equation in (2.6.2) takes the form

$$(2.6.5)\quad \begin{aligned} -\frac{\partial\phi(u^o)}{\partial t} &= b_i(t,x)\cdot\phi_{x_i}(u^o) + \sum_{j=1}^{r_2}|g_{ij}(t,x)\cdot\phi_{x_i}(u^o)(t,x)| \\ &\quad + a_{ij}(t,x)\cdot\phi_{x_ix_j}(u^o) + 1 \end{aligned}$$

This result indicates that finding the optimal controller does not involve the adjoint system (AS'_4) in contrast to the other cases $(0 \leq n_1 < n)$ as mentioned in [5].

(ii). $n_1 = 0$ (no information.) The controller uses no information. Setting $n_1 = 0$ in (2.6.1) we obtain

$$u_j^o(t) = \text{sign} \cdot \left[\int_\Omega g_{ij}(t,x) \cdot \phi_{x_i}(u^o) \cdot q(u^o)(t,x)\, dx \right], \tag{2.6.6}$$

$$j = 1, \ldots, r_2 .$$

This modifies the integral - partial differential equations (2.6.2) and (AS'_4) accordingly.

The preceding results have important implications. In order to solve for the optimal controller u^o, in general, with the exception of the case $n_1 = n$, it is necessary to solve the two point boundary value problem (TPBVP) consisting of the system (2.6.2) and the system (AS'_4).

2.7 EXISTENCE OF OPTIMAL CONTROLS

In section 2.5, we have derived necessary conditions for optimality for the problems $P'_1 - P'_3$. Then a question naturally arises as to whether or not the above mentioned problems have their corresponding extremals This question is to be discussed in this section.

Without loss of generality, we will only investigate the existence of controls that solve the problem P'_2(involving only controls).

However, it may be mentioned that it does not present any additional difficulty toward solving the general problem P'_1 involving both controls and parameters.

As stated in chapter 1, we know that a numerous developments concerning the existence of optimal controls for open loop systems have been reported in Fleming and Nisio [25], Ahmed [2] and others . However, our attention is on the problem of existence of optimal feedback controllers unlike open loop controls. Thus, the above mentioned results can not be used directly to answer our question. However, under certain additional properties of the coefficients of the operator L(u) and the cost integrand f, Fleming has reported [23, Thm. 3, p. 205] an existence theorem on optimal control for the problem P'_2. This result is presented in the following theorem.

Theorem 2.7.1 : Consider the problem P'_2. Suppose that the assumption (A_1) is satisfied and that b is linear in u. Further, f is assumed to be convex in u. Then, there exists a control $u^o \in D$ that minimizes the cost functional J(u) .

Proof : The proof is given in [23, Thm. 3, p. 205] .

Based on Fillippov technique [28, Thm 3' , p. 281] , Ahmed and Teo [4, Thm. 1, p.] have recently presented an existence theorem of optimal controls for the problem P'_2 under the following assumptions :

(i) the hypotheses (ii) —— (iv) in (A_1) are satisfied ;

(ii) the control restraint set is taken as a measurable set valued function $U(t,\hat{x})$ defined in $\hat{Q}$ with values nonempty compact convex subset of a fixed compact set $\tilde{U} \subset R^{r}$, where $\hat{Q}$ is the projection onto $(t, \hat{x})$ - space of the cylinder Q ; and

(iii) b and f are bounded measurable on $\bar{Q}$ for each $u \in \tilde{U}$ and continuous on $\tilde{U}$ for all $(t, x) \in \bar{Q}$. Further the set valued function $\Gamma(t,x)$, $(t, x) \in Q$, defined by $\Gamma(t,x) \triangleq \{ \bar{b}(t,x,u) : u \in U(t, \hat{x}) \}$, is convex for each $(t,x) \in Q$ where $\bar{b} \triangleq \begin{pmatrix} f \\ b \end{pmatrix}$ is the $(n+1)$ - vector constructed by adjointing f to the n-vector b, and $\hat{x}$ is the projection of x. The class of admissible controls considered is defined by the set $D \triangleq \{ u : u \text{ measurable on } \hat{Q},\ u(t, \hat{x}) \in U(t, \hat{x}),\ (t,\hat{x}) \in \hat{Q} \}$. These hypotheses will be referred to collectively as (H) .

Note that in the hypothesis (H) , b is not necessary linear in u and that the admissible controls are more general than those in D.

Based on the assumptions mentioned above, the following result was presented in [4, Thm. 1, p.   ] which is quoted here without proof.

Theorem 2.7.2 : Consider the problem P_2' and suppose that the hypothesis (H) is satisfied. Then there exists a control $u^o \in D$ that minimizes the cost functional $J(u)$.

It should be mentioned that neither Theorem 2.7.1 nor Theorem 2.72 admits controls in the diffusion coefficient a_{ij}. This is an open problem.

2.8 CONCLUSIONS AND SUGGESTIONS FOR FURTHER STUDY.

In this chapter, the stochastic optimization problem was reduced to the optimization problem of an equivalent class of first boundary value problems in which the coefficients of the differential operator involve both control and parameter variables. To this reduced problem, necessary conditions for determination of the optimal control as well as the optimal parameter have been developed. This result was also used to obtain necessary conditions for optimality for some special but important cases.

In view of the assumption (A_1, sec. 2.2), we note that the conditions imposed on the diffusion coefficient are rather strong and further it is also assumed to be independent of the control variables. Thus, it would be interesting to obtain necessary conditions for optimality and existence of optimal controls under weaker conditions. Therefore, we will list some open problems. This list is, of course, not intended to be complete.

(i) If we consider the dynamical system

$$d\xi(t) = b(t, \xi(t), u(t, \eta(t)))\, dt + c(t, \xi(t))\, dw_1(t)$$

with controls based on the observation $\eta(t)$ which is governed by the following stochastic Ito differential equation

$$d\eta(t) = b'(t, \xi(t), \eta(t))\, dt + c'(t, \xi(t), \eta(t))\, dw_2(t) ,$$

then this problem can be transformed into the problem of the type considered in this chapter by regarding $(\eta(t), \xi(t))$ as the state of the system at time t. However, if the observation channel is not noisy, then the diffusion coefficient of the resultant stochastic control system will fail to satisfy the condition of ' Uniform Parabolicity '. Thus, the results obtained in this chapter do not apply to this situation. Therefore the solution to this problem will be of great interest.

(ii) To derive necessary conditions for optimality and show existence of optimal controls for the control problem P_2' [section 2.5, p. 25] in which the diffusion coefficient depends also on the control variables u .

(iii). In general, for the determination of optimal policy, optimal control or optimal parameter, it is required to solve a Two-Point-Boundary-Value-Problem (TPBVP) arising from the corresponding necessary conditions for optimality. This problem, as it arises in this work, involves two nonlinear parabolic partial differential equations. For instance, in Corollary 2.6.3, the TPBVP, corresponding to the determination of optimal feedback controller of the time optimal control problem P_4, has been explicitly stated in terms of the systems (2.6.2) and $(A\text{-}S_4')$. Thus, it is an interesting open problem to investigate numerical aspects of their solutions.

CHAPTER 3

OPTIMAL CONTROL OF ITO STOCHASTIC DIFFERENTIAL SYSTEMS WITH FIXED TERMINAL TIME

3.1 INTRODUCTION

In this chapter, we consider a class of stochastic systems described by Ito differential equation for which both parameters and controls are to be chosen optimally with respect to certain performance index over a fixed time interval. It is shown [Theorem 3.3.1] that this stochastic optimization problem can be reduced to an equivalent optimization problem of distributed parameter system with Cauchy conditions. To this reduced problem, a necessary condition for optimality for both controls and parameters combined together is presented in Theorem 3.6.3. This result is then used to derive necessary conditions for optimal control [Corollaries 3.6.4 and 3.6.3] and a necessary condition for optimal parameter [Corollary 3.6.7]. Further, the bang bang principle is shown to hold true for linear optimal control problems in Theorem 3.7.2. The other results presented in this chapter are only the preparatory materials for deriving the necessary conditions for optimality.

Consider the system S_1 [section 2.1, p. 16] and let Σ be a bounded convex subset of R^{r_1}, T a fixed time and D the set of admissible controls on $[0,T] \times R^{n_1}$, $[0,T] \triangleq I$, given by

$$D \triangleq \{u : u \text{ measurable on } I \times R^{n_1}, u(t,\hat{x}) \in U, (t,\hat{x}) \in I \times R^{n_1}\},$$

where U is a compact and convex subset of R^{r_2}.

For brevity, the elements of the set $\Sigma \times D$ will be called 'POLICIES' as in chapter 2.

With the above preparations, let us state the problem P_5 as.:

Subject to the dynamical constraint S_1, find a policy $(\sigma, u) \in \Sigma \times D$ that minimizes the cost functional J given by

$$J(\sigma, u) = \mathcal{E}\left\{\int_0^T f(t, \xi(t), \sigma, u(t, \hat{\xi}(t)))dt + \eta(\xi(T))\right\},$$

where f and η are real-valued functions defined on $I \times R^n \times R^{r_1} \times R^{r_2}$ and R^n respectively.

Throughout this chapter, the following assumptions will be referred to collectively as (A_2) :

(i). $a_{ij}(t, x, \sigma) = \frac{1}{2}(c \cdot c^*)_{ij}(t, x, \sigma)$, $(i, j = 1, \ldots, n)$, are measurable on I for each $(x, \sigma) \in R^n \times R^{r_1}$ and continuous on $R^n \times R^{r_1}$ for almost all $t \in I$, where $*$ denotes matrix transpose. Further, there exist constants a_ℓ, $a_u > 0$ so that $a_\ell |z|^2 \leq a_{ij}(t, x, \sigma) \cdot z_i \cdot z_j \leq a_u |z|^2$ almost everywhere on $I \times R^n$ for every $\sigma \in \Sigma$ and $z \in R^n$;

(ii). $\frac{\partial a_{ij}}{\partial x_j}$, $(i, j = 1, \ldots, n)$, are bounded measurable on I for each $(x, \sigma) \in R^n \times R^{r_1}$ and continuous bounded on $R^n \times R^{r_1}$ for almost all $t \in I$; and b_i, $(i = 1, \ldots, n)$, are bounded measurable on I for each $(x, \sigma, v) \in R^n \times R^{r_1} \times R^{r_2}$ and continuous bounded on $R^n \times R^{r_1} \times R^{r_2}$ for almost all $t \in I$;

(iii). f is measurable on I for each $(x, \sigma, v) \in R^n \times R^{r_1} \times R^{r_2}$ and continuous on $R^n \times R^{r_1} \times R^{r_2}$ for almost all $t \in I$ and $f(\cdot, \cdot, \sigma, u) \in L^q(I, L^2(R^n))$ for every $(\sigma, u) \in \Sigma \times D$, where $q \in (1, 2]$.

(iv) $\eta : R^n \longrightarrow R^1$ (real line) is continuous so that $\eta(\cdot) \in L^2(R^n)$.

As it is stated in the previous chapter, the convention adopted above and throughout the rest of the thesis is to take summation up to n over repeated indices

3.2 NOTATIONS

For brevity, notations given in section 2.2 will be used in this chapter without further mention. Besides, we will introduce some additional ones as defined below.

Let $C_o^{\ell}(K)$, $0 \leq \ell \leq \infty$, denote the class of all functions in $C^{\ell}(K)$ with compact support on K, where $C^{\ell}(K)$ is as defined in section 2.2. $W^1(K)$ is the completion of $C_o^{\infty}(K)$ in the norm

$$\| z \|_2 + \| z_x \|_2$$

where $\| z \|_2 \triangleq \left(\int_K | z |^2 dx \right)^{1/2}$ and $\| z_x \|_2 \triangleq \left(\int_K \sum_{i=1}^{n} | z_{x_i} |^2 dx \right)^{1/2}$

$Y(I, X)$ denotes the class of functions defined on the interval I with values in a normed space X, for example, $L^p(I, L^q(R^n))$, $p, q \geq 1$.

For any nonintegral positive number λ and for any measurable subset $\Theta_k \subset [0,T) \times R^n \triangleq \Theta$, $H^{\lambda, \lambda/2}(\Theta_k)$ denotes the Banach space of functions z that are continuous on Θ_k and have derivatives of the form

$$D_t^{a} \cdot D_x^{\beta} \, z \;\Delta\; \frac{\partial^{a}}{\partial t^{a}} \cdot \left(\frac{\partial^{\beta_1 + \cdots + \beta_n}}{\partial x_1^{\beta_1} \cdots \partial x_n^{\beta_n}} \cdot z \right), \qquad \sum_{i=1}^{n} \beta_i = \beta ,$$

β_i nonnegative integer, $2a + \beta \leq \lambda$, and have a finite norm

$$| z |^{(\lambda)}_{\Theta_k} \;\Delta\; \| z \|^{(\lambda)}_{\Theta_k} + \sum_{j=o}^{[\lambda]} \| z \|^{(j)}_{\Theta_k} .$$

Note that $[\lambda]$ denotes the largest integer of λ and

$$\| z \|^{(o)}_{\Theta_k} \;\Delta\; | z |^{(o)}_{\Theta_k} = \max_{\Theta_k} | z |$$

$$\| z \|^{(j)}_{\Theta_k} \;\Delta\; \sum_{(2a+\beta=j)} | D_t^{a} \cdot D_x^{\beta} \cdot z |^{(o)}_{\Theta_k}$$

$$\| z \|^{(\lambda)}_{\Theta_k} \;\Delta\; \| z \|^{(\lambda)}_{x,\Theta_k} + \| z \|^{(\lambda/2)}_{t,\Theta_k} ,$$

$$\| z \|^{(\lambda)}_{x,\Theta_k} \;\Delta\; \sum_{(2a+\beta=[\lambda])} \| D_t^{a} \cdot D_x^{\beta} \cdot z \|^{(\lambda-[\lambda])}_{x,\Theta_k} ,$$

$$\| z \|^{(\lambda/2)}_{t,\Theta_k} \;\Delta\; \sum_{o<\lambda-2a-\beta<2} | D_t^{a} \cdot D_x^{\beta} \cdot z \|^{\left(\frac{\lambda-2a-\beta}{2}\right)}_{t,\Theta_k} ,$$

$$\| z \|^{(\gamma)}_{x,\Theta_k} \;\Delta\; \sup_{(x,t),(x',t')\in\bar{\Theta}_k} \frac{| z(x,t) - z(x',t') |}{| x - x' |^{\gamma}} , \qquad o < \gamma < 1 ,$$

$$\| z \|^{(\gamma)}_{t,\Theta_k} \;\Delta\; \sup_{(x,t),(x',t')\in\bar{\Theta}_k} \frac{| z(x,t) - z(x',t') |}{| t - t' |^{\gamma}} , \qquad o < \gamma < 1 .$$

3.3 TRANSFORMATION OF STOCHASTIC OPTIMIZATION PROBLEMS TO EQUIVALENT DISTRIBUTED PARAMETER OPTIMIZATION PROBLEMS.

In this section, it is shown that the problem P_5 can be reduced to an optimization problem combining both controls and parameters of an equivalent class of Cauchy problems. Throughout this chapter, we denote by $L(\sigma, u)$ the following differential operator

$$L(\sigma,u)\cdot\psi \;\Delta\; \psi_t + (a_{ij}(t,x,\sigma)\cdot\psi_{x_i x_j} - a_i(t,x,\sigma,u(t,\hat{x}))\cdot\psi_{x_i}.$$

where $(\sigma,u)\in\Sigma \times D$, $a_{ij}(t,x,\sigma) = \frac{1}{2}(c\cdot c^{*})_{ij}(t,x,\sigma)$, $(i,j=1,\ldots n)$,

$$a_i(t,x,\sigma,u(t,\hat{x})) = \sum_{j=1}^{n}\frac{\partial a_{ij}(t,x,\sigma)}{\partial x_j} - b_i(t,x,\sigma,u(t,\hat{x}))$$ and

$*$ denotes matrix transpose.

Using the well known Backward Kolmogorov equation, it is shown (48, Thm. 1, p.359) that the stochastic optimization problem reduces to the optimization problem of an equivalent distributed parameter system as stated in

Theorem 3.3.1. The problem P_5 is equivalent to problem P'_5 :

$$P'_5\begin{cases} S'_5\begin{cases} L(\sigma,u)\cdot\phi(\sigma,u) + f(t,x,\sigma,u(t,\hat{x})) = 0, & (t,x)\in[0,T)\times R^n \;\Delta\; \Theta \\ \phi(\sigma,u)(T,x) = \eta(x) & x\in R^n \\ (\sigma,u)\in\Sigma\times D \end{cases} \\ (3.3.1)\quad \min\limits_{(\sigma,u)\in\Sigma\times D} J(\sigma,u) = \min\limits_{(\sigma,u)\in\Sigma\times D}\int_{R^n}\phi(\sigma,u)(o,x)\,\pi_o(dx), \end{cases}$$

where π_o is the distribution of the initial state ξ_o.

For convenience, whenever there is no confusion, the variable (t, x) will be suppressed and $a(\sigma, u)$ will be used to denote the function $a(t, x, \sigma, u)$, where $a(\sigma, u)$ stands for any of the coefficients or solution of the system S'_5.

Note that results of the following sections 3.4 and 3.5 are minor modification of known results. These are used for the proof of necessary conditions for optimality in section 3.6 and thus may be skipped in the first reading.

3.4. EXISTENCE AND UNIQUENESS OF SOLUTIONS OF RELATED SYSTEMS.

In order to find a $(\sigma, u) \in \Sigma \times D$ that solves the problem P'_5, it is necessary to show the existence and uniqueness of solutions of the Cauchy problem S'_5 and it's adjoint system.

For brevity, the statement: " C depends on the structure of the differential equation of the system S'_5 " will be used to mean that C is determined by the quantities a_ℓ, a_u, q and the bounds of the functions b_i and $\frac{\partial a_{ij}}{\partial x_j}$, $(i, j = 1, \ldots, n)$, where a_ℓ, a_u and q are as defined in (A_2).

In the sequel, we need

<u>Definition 3.4.1.</u> For each $(\sigma, u) \in \Sigma \times D$, a function $\phi(\sigma, u)$ is said to be a weak solution of the Cauchy problem S'_5 if

$\phi(\sigma, u) \in L^\infty(I, L^2(R^n)) \cap L^2(I, W^1(R^n))$.

and

$$(3.4.2) \quad \int_{\Theta} [-\phi(\sigma,u) \cdot \varphi_t - a_{ij}(t,x,\sigma) \cdot \phi_{x_i}(\sigma,u) \cdot \varphi_{x_j} - a_i(t,x,\sigma,u) \cdot \phi_{x_i}(\sigma,u) \cdot \varphi + f(t,x,\sigma,u)\varphi]\,dt\,dx = 0$$

for every $\varphi \in C_o^1(\Theta^o)$, Θ^o interior of the set $\Theta \,\underline{\Delta}\, [0,T) \times R^n$, and if

$$(3.4.3) \quad \lim_{t \to T} \int_{R^n} \phi(\sigma,u)(t,x) \cdot z(x)\,dx = \int_{R^n} \eta(x) \cdot z(x)\,dx$$

for all $z \in C_o^1(R^n)$.

In reference [7], it is shown that every weak solution of the Cauchy problem S'_5 has a representation which is continuous on Θ . Thus, ϕ will always be assumed to denote the continuous representative of the weak solution. Hence there is no difficulty in talking about the value of ϕ at any point of its domain of definition.

For the proof of the existence and uniqueness of solutions of the Cauchy problem S'_5, we need to construct a sequence of first boundary value problems from the Cauchy problem S'_5. Further, it should be mentioned that for any first boundary value problem or Cauchy problem considered in this chapter, we will adopt the convention that $L(\sigma,u) \cdot \psi \,\underline{\Delta}\, \psi_t + \Delta\psi$, $L^*(\sigma,u) \cdot \psi \,\underline{\Delta}\, -\psi_t + \Delta\psi$ and $f(t,x,\sigma,u) \equiv 0$ for all (t,x) outside their corresponding domain of definition, where Δ is the Laplacian operator.

With these preparations, we have

Theorem 3.4.2. Consider the Cauchy problem S'_5 [p. 42] . Suppose that the assumption (A_2) [pp. 39-40] is satisfied. Then for any $(\sigma,u) \in \Sigma \times D$, there exists a unique weak solution $\phi(\sigma,u)$

of this problem so that

$$(3.4.4)\quad \|\phi(\sigma,u)\|^2_{2,\infty} + \|\phi_x(\sigma,u)\|^2_{2,2} \le C\{\|\eta\|^2_2 + \|f(\sigma,u)\|^2_{2,q}\},$$

where $\|\cdot\|_{2,q} \triangleq \{\int_I (\int_{R^n} |\cdot|^2 dx)^{q/2} dt\}^{1/q}$; $\|\cdot\|_{2,\infty} \triangleq \sup_{t\in I} \{\int_{R^n} |\cdot|^2 dx)^{1/2}\}$;

$\|\cdot\|_2 \Delta \{\int_{R^n} |\cdot|^2 dx\}^{1/2}$; $q \in (1, 2]$; and C is a positive constant which depends only on T and the structure of differential equation of the system S'_5 (p. 43).

<u>Proof:</u> Let $\Theta_k \triangleq O_k \times [0,T)$, where $O_k = \{x : |x| < k\}$ for integer $k \ge 1$, and for any $(\sigma,u) \in \Sigma \times D$ let us consider the first boundary value problems

$$(3.4.5)\quad \begin{cases} L(\sigma,u)\cdot\phi = f(t,x,\sigma,u) & (t,x) \in \Theta_k \\ \phi(T,x) = \eta(x) & x \in O_k \\ \phi(t,x) = 0 & (t,x) \in I \times \partial O_k . \end{cases}$$

By Theorem 1 [8, p. 634], we note that for each value of k there exists a unique weak solution $\phi_k(\sigma,u)$ of problem (3.4.5). Thus, it follows from letting $\zeta = 1$, $s = 0$ and $\mu = \infty$ in [8, Lemma 1, p. 623] that

$$(3.4.6)\quad \|\phi_k(\sigma,u)\|^2_{2,\infty,\Theta_k} + \|(\phi_k(\sigma,u))_x\|^2_{2,2,\Theta_k} \le e^{\beta T} C_1 \{\|\eta\|^2_{2,O_k} + \|f(\sigma,u)\|^2_{2,q,\Theta_k}\},$$

where $\|\cdot\|_{2,q,\Theta_k} \underline{\Delta} \{\int_I (\int_{O_k} |\cdot|^2\, dx)^{q/2} dt\}^{1/q}$; $\|\cdot\|_{2,O_k} \Delta (\int_{O_k} |\cdot|^2 dx)^{1/2}$;

$\|\cdot\|_{2,\infty,\Theta_k} \underline{\Delta} \sup_{t \in I} \{(\int_{O_k} |\cdot|^2 dx)^{1/2}\}$; $q \in (1, 2]$; and C_1 is independent of k.

Clearly,

$$\|\eta\|^2_{2,O_k} \leq \|\eta\|^2_2 \quad \Delta \quad \int_{R^n} |\eta(x)|^2\, dx \quad ,$$

and $\|f(\sigma,u)\|_{2,q,\Theta_k} \leq \|f(\sigma,u)\|_{2,q} \Delta \{\int_I (\int_{R^n} |f(t,x,\sigma,u(t,\hat{x}))|^2 dx)^{q/2} dt\}^{1/q}$.

Thus, the estimate (3.4.6) can be reduced to

$$(3.3.7) \quad \|\phi_k(\sigma,u)\|^2_{2,\infty,\Theta_k} + \|(\phi_k(\sigma,u))_x\|^2_{2,2,\Theta_k} \leq C_2 ,$$

where C_2 depends only on T and the structure of the differential equation of the system S'_5. If we extend the domain of definition of $\phi_k(\sigma,u)$ by setting $\phi_k(\sigma,u) = 0$, for $|x| > k$ and $0 \leq t \leq T$, then

$$(3.4.8) \qquad \|\phi_k(\sigma,u)\|^2_{2,\infty} + \|(\phi_k(\sigma,u))_x\|^2_{2,2} \leq C_2 .$$

Furthermore,

$$(3.4.9) \qquad \|\phi_k(\sigma,u)\|_{2,2} \leq (T^{1/2} \cdot C_2) .$$

In view of (3.4.8) and (3.4.9) there exists a subsequence of the vectors $(\phi_k(\sigma,u), (\phi_k(\sigma,u))_{x_1}, \ldots, (\phi_k(\sigma,u))_{x_n})$, which we again index by k, and a vector $(\phi(\sigma,u), \phi^1(\sigma,u), \ldots, \phi^n(\sigma,u))$, such that $\phi_k(\sigma, u) \longrightarrow \phi(\sigma, u)$ and $(\phi_k(\sigma, u))_{x_i} \longrightarrow \phi^i(\sigma, u)$, $i = 1, \ldots, n$, weakly in $L^2(I, L^2(R^n))$. In particular, for all $\varphi \in C^1(\overline{\Theta})$ with compact support in R^n, we have

$$\iint_{\Theta} \varphi \cdot \phi_k(\sigma, u)\, dt\, dx \longrightarrow \iint_{\Theta} \varphi \cdot \phi(\sigma, u)\, dt\, dx$$

and $\displaystyle\iint_{\Theta} \varphi \cdot (\phi_k(\sigma, u))_{x_i}\, dt\, dx \longrightarrow \iint_{\Theta} \varphi \cdot \phi^i(\sigma, u)\, dt\, dx$

for all $i = 1, \ldots, n$.

Since

$$\iint_{\Theta} \varphi \cdot (\phi_k(\sigma, u))_{x_i}\, dt\, dx = -\iint_{\Theta} \varphi_{x_i} \cdot \phi_k(\sigma, u)\, dt\, dx$$

and

$$\iint_{\Theta} \varphi_{x_i} \cdot \phi_k(\sigma, u)\, dt\, dx \longrightarrow \iint_{\Theta} \varphi_{x_i} \cdot \phi(\sigma, u)\, dt\, dx,$$

it follows that

$$\iint_{\Theta} \varphi \cdot \phi^i(\sigma, u)\, dt\, dx = -\iint_{\Theta} \varphi_{x_i} \cdot \phi(\sigma, u)\, dt\, dx.$$

for all $\varphi \in C^1(\overline{\Theta})$ with compact support in R^n.

This implies that $\phi^i(\sigma, u)$ is the distributional derivative of $\phi(\sigma, u)$ with respect to x_i and we will write $\phi_{x_i}(\sigma, u)$ instead of $\phi^i(\sigma, u)$. Clearly, for the limiting function we have

$$(3.4.10) \qquad \|\phi(\sigma, u)\|_{2,2} \leq T^{1/2} \cdot C_2$$

$$(3.4.11) \qquad \|\phi_x(\sigma, u)\|_{2,2} \leq C_2.$$

This implies that $\phi(\sigma, u) \in L^2(I, W^1(R^n))$. Moreover, it follows from (3.4.8) and lemma 3 [8, p. 633] that

$$\| \phi(\sigma,u) \|_{2,\infty} \leq C_2 .$$

For any positive integer k and for any $(\sigma,u) \in \Sigma \times D$, let $\phi_k(\sigma,u)$ be the weak solution of the system (3.4.5). Then, if $\varphi = \varphi(t,x)$ is an arbitrary $C^1(\bar{\Theta})$ function with compact support in R^n which vanishes near $t = o$, it follows from the substitution of $T-t$ for t in the relation (2.2) [8, p. 622] that

$$\begin{aligned} &\int_{R^n} \eta(x)\cdot\varphi(T,x)\,dx + \\ &+ \iint_{\Theta} [-\phi_k(\sigma,u)\cdot\varphi_t - a_{ij}(t,x,\sigma)\cdot(\phi_k(\sigma,u))_{x_i}\cdot\varphi_{x_j} - a_i(t,x,\sigma,u)\cdot(\phi_k(\sigma,u))_{x_i}\cdot\varphi + \\ &+ f(t,x,\sigma,u)\cdot\varphi]\,dt\,dx = 0, \end{aligned} \tag{3.4.12}$$

for all k sufficiently large so that $(\text{sup.}\ \varphi) \cap \Theta \subset \Theta_k$.

Thus, letting $k \to \infty$ through the appropriate subsequence, it follows from the weak convergence in $L^2(I, W^2(R^n))$ of the sequence $\{\phi_k(\sigma,u)\}_{k=1}^{\infty}$ that

$$\begin{aligned} &\int_{R^n} \eta(x)\cdot\varphi(T,x)\,dx + \\ &+ \iint_{\Theta} [-\phi(\sigma,u)\cdot\varphi_t - a_{ij}(t,x,\sigma)\cdot\phi_{x_i}(\sigma,u)\cdot\varphi_{x_j} - a_i(t,x,\sigma,u)\cdot\phi_{x_i}(\sigma,u)\cdot\varphi + \\ &+ f(t,x,\sigma,u)\cdot\varphi]\,dt\,dx = 0 . \end{aligned} \tag{3.4.13}$$

Since φ is an arbitrary $C^1(\bar{\Theta})$ function with compact support in R^n which vanishes near $t = 0$, it is clear that the relation (3.4.13) holds true for all $\varphi \in C^1(\overline{\Theta})$ with compact support in R^n vanishing near $t = 0$. In particular, if $\varphi \in C_o^1(\Theta)$ then the first term on the right-hand side of the expression (3.4.13) equals zero. Thus, in view of Definition 3.4.1, it remains to show that $\phi(\sigma, u)$ satisfies the condition (3.4.3). For this, we first note that by substituting $T-$ for t in the relation (1.3) [8, p. 619] and using its following statement, we have

$$\begin{aligned} &\int_{R^n} \phi(\sigma, u)(t,x) \cdot \varphi(t,x)\, dx \Big|_{t=\tau} + \\ (3.4.14)\quad &\iint_{(o,\tau)xR^n} [-\phi(\sigma,u)\cdot\varphi_t - a_{ij}(t,x,\sigma)\cdot\phi_{x_j}(\sigma,u)\cdot\varphi_{x_i} - a_i(t,x,\sigma,u)\cdot\phi_{x_i}(\sigma,u)\cdot\varphi + \\ &\quad + f(t,x,\sigma,u)\cdot\varphi]\, dt\, dx = \int_{R^n} \phi(\sigma,u)(o,x)\cdot\varphi(o,x)\, dx \end{aligned}$$

for all $\varphi \in C^1(\overline{\Theta})$ with compact support in R^n, where $\tau \in [o, T)$. Let $T > \delta > o$ and let ζ be a C^1-function defined on I so that $\zeta(t) = 1$ for all $t \in [\delta, T]$ and vanishes near $t=0$. Then it is clear that for any $z \in C_o^1(R^n)$, $\zeta(t)\cdot z(x) \in C^1(\overline{\Theta})$ with compact support in R^n which vanishes near $t = 0$. Thus, replacing $\varphi(\cdot,\cdot)$ in the expressions (3.4.13) and (3.4.14) by $\zeta(\cdot)\cdot z(\cdot)$ and comparing their results, we obtain

$$(3.4.15)\quad \lim_{t \to T} \int_{R^n} \phi(\sigma,u)(t,x) \cdot z(x)\, dx = \int_{R^n} \eta(x) \cdot z(x)\, dx$$

for all $z \in C_o^1(R^n)$.

Therefore, $\phi(\sigma, u)$ is a weak solution of the system S_5' corresponding to $(\sigma, u) \in \Sigma \times D$.

Uniqueness follows from Theorem 2 [8, p. 639]. Since $\phi(\sigma, u)$ is unique, it follows that the sequence $\{\phi_k(\sigma,u)\}_{k=1}^{\infty}$ converges to $\phi(\sigma, u)$. This completes the proof.

Let $(\sigma^o, u^o) \in \Sigma \times D$ be the policy solving the problem $P_5^{\prime}$ (called the optimal policy) and considered $AS_5^{\prime}$ to be the differential system adjoint to the system $S_5^{\prime}$ corresponding to this optimal policy.

$$AS_5^{\prime} \begin{cases} L^*(\sigma^o, u^o)\cdot q = 0 & (t,x) \in (o,T] \times R^n, \\ q(o,x) = q_o(x) & x \in R^n, \end{cases}$$

where

$$L^*(\sigma^o, u^o)\cdot \varphi \triangleq -\psi_t + \{a_{ij}(t,x,\sigma^o)\cdot \psi_{x_j} + a_i(t,x,\sigma^o,u^o(t,\hat{x}))\cdot\psi\}_{x_i}$$; and

for any measurable subset E of R^n, $\pi_o(E) = \int_E q_o(x)\, dx$;

Remark 3.4.3: Theorem 3.4.2 remains valid for the adjoint system $AS_5^{\prime}$.

3[illegible] PREPARATORY RESULTS FOR THE DEVELOPMENT OF NECESSARY CONDITIONS FOR OPTIMALITY.

For the proof of necessary conditions for optimality for the problem $P_5^{\prime}$, we need to consider this problem with its coefficients and data replaced by its corresponding integral averages. For this, let K[illegible]) be a sufficiently smooth non-negative function defined on R^m for each positive integer s so that $K(y; s) = 0$ for $|y| \geq \frac{1}{s}$ and $\int_{|y| < \frac{1}{s}} K(y\,; s)dy = 1$ for all positive integers s.

For any real valued measurable function ζ on R^p and for any positive integer s, let us define on R^p the function ζ^s, called the integral average of ζ, by

$$\zeta^s(y) \triangleq \int_{R^p} K(y-y'\,;\; s)\cdot \zeta(y')\; dy' .$$

For each $(\sigma, u) \in \Sigma \times D$ and for every positive integer s, let $a^s_{ij}(\sigma)$, $a^s_i(\sigma, u)$, $(i, j = 1, \ldots, n)$, and $f^s(\sigma, u)$ denote, respectively, the integral averages of the functions $a_{ij}(\sigma)$, $a_i(\sigma, u)$, $(i, j = 1, \ldots, n)$, and $f(\sigma, u)$.

Similarly, let η^s be the integral average of η.

With these preparations, we consider the sequence of Cauchy problems

$$(3.5.1) \quad \begin{cases} L^s(\sigma, u)\,\phi + f^s(t, x, \sigma, u) = 0 & (t, x) \in \Theta \\ \phi(T, x) = \eta^s(x) & x \in R^n \\ (\sigma, u) \in \Sigma \times D, & \end{cases}$$

where, for each $(\sigma, u) \in \Sigma \times D$ and for each positive integer s, the operator $L^s(\sigma, u)$ may be written as

$$L^s(\sigma, u) \cdot \psi \;\Delta\; \psi_t + (a^s_{ij}(t, x, \sigma) \cdot \psi_{x_i})_{x_j} - a^s_i(t, x, \sigma, u) \cdot \psi_{x_i} .$$

Corresponding to each $(\sigma, u) \in \Sigma \times D$ and each positive integer s, the weak solution of the Cauchy problem (3.5.1) will be denoted by $\phi^s(\sigma, u)$.

Suppose that $(\sigma^o, u^o) \in \Sigma \times D$ solves the problem P_5' (called the optimal policy). Corresponding to this optimal policy and for each positive integer s, let q^s be the weak solution of the following system which is adjoint to the system (3.5.1).

$$(3.5.2) \quad \begin{cases} L^{*s}(\sigma^o, u^o) \cdot q = 0 & (t, x) \in (o, T] \times R^n, \\ q(o, x) = q^s_o(x) & x \in R^n . \end{cases}$$

Since the coefficients and data of the integral averaged problems obviously satisfy the assumptions of Theorem 3.4.2, the existence and uniqueness of the weak solution of these problems follow from that theorem.

In the sequel, we need the following lemma.

Lemma 3.5.1 : Consider the system S_5'. Suppose that the assumption (A_2) is satisfied. Then for each $(\sigma, u) \in \Sigma \times D$ the weak solution $\phi(\sigma, u)$ of the Cauchy problem S_5' is the weak limit in $L^2(I, W^1(R^n))$ of the sequence $\{\phi^s(\sigma, u)\}_{s=1}^{\infty}$ of the weak solutions of the Cauchy problem (3.5.1). Moreover, $\{\phi^s(\sigma, u)\}_{s=1}^{\infty}$ converges uniformly to $\phi(\sigma, u)$ in any compact subset of $\Theta \triangleq [o, T) \times R^n$.

Proof : For each integer $s \geq 1$ and for any $(\sigma, u) \in \Sigma \times D$ the problem (3.5.1) has a unique weak solution $\phi^s(\sigma, u)$ [Theorem 3.4.2]. Thus, by the application of the estimate (3.4.4) and the fact that integral averaging on Θ does not increase norm, we have

$$(3.5.3) \qquad \|\phi^s(\sigma, u)\|^2_{2,\infty} + \|\phi^s_x(\sigma, u)\|^2_{2,2} \leq C\{\|\eta\|^2_2 + \|f(\sigma, u)\|^2_{2,q}\},$$

where C and q are as defined in Theorem 3.4.2.

Using the above estimate instead of the estimate (3.4.6), we obtain the proof of convergence of $\{\phi^s(\sigma, u)\}_{s=1}^{\infty}$ in the weak sense to the weak solution $\phi(\sigma, u)$ similar to that given in the proof of Theorem 3.4.2.

To prove the second part of the lemma, let K be any compact subset of Θ. By the estimate (3.5.3) and Theorem B [8, p. 616], the sequence $\{\phi^s(\sigma,u)\}_{s=1}^{\infty}$ is uniformly bounded in K for any given $(\sigma,u) \in \Sigma \times D$. Obviously, for each positive integer s and for any $(\sigma,u) \in \Sigma \times D$, $\phi^s(\sigma,u)$ satisfies the condition (3.4.2) in K. Thus, it follows from Theorem C [8, p. 616] and the uniform boundedness of the family $\{\phi^s(\sigma,u)\}_{s=1}^{\infty}$ that this family is also equicontinuous in K for any $(\sigma,u) \in \Sigma \times D$. Therefore, it follows from Arzela-Ascoli's Theorem that there is a subsequence which converges uniformly in K. However, $\phi(\sigma,u)$ is unique. Thus, the sequence $\{\phi^s(\sigma,u)\}_{s=1}^{\infty}$ converges to $\phi(\sigma,u)$ uniformly in K. This completes the proof.

<u>Remark 3.5.2</u> : Lemma 3.5.1 remains valid for the adjoint systems AS_5' and (3.5.2).

For the proof of the necessary condition for optimality for the problem P'_5, we need Lemma 3.6.1 [p. 58] and Lemma 3.6.2 [p. 62]. However, in the proof of these two lemmas, the relation 1.4 [8, p. 620] will be used several times. In some places, this relation is to be used with φ replaced by the solutions of the first boundary value problems constructed from the Cauchy problems 3.5.1 ; whereas in the other places, φ is replaced by the solutions of the first boundary value problems corresponding to the adjoint system 3.5.2. For this, we will construct first boundary value problems from the Cauchy problems 3.5.1 and 3.5.2 as follows.

Let $\Theta_k = [0,T) \times O_k$, where $O_k = \{x : |x| < k\}$ for integers $k \geq 1$ and let us consider for any $(\sigma,u) \in \Sigma \times D$ the following first boundary value problems

$$(3.5.4)\quad \begin{cases} L^{s}(\sigma,u)\ \phi(t,x) + f^{s}(t,x,\sigma,u(t,x)) \cdot g_k(t,x) = 0 & (t,x) \in \Theta_k \\ \phi(T,x) = \eta^{s}(x)\cdot h_k(x) & x \in O_k \\ \phi(t,x) = 0 & (t,x) \in I \times \partial O_k , \end{cases}$$

where $L^{s}(\sigma,u)$, $f^{s}(t,x,\sigma,u)$ and η^{s} are as defined for the system (3.5.1) ; and for each integer $k \geq 1$ the function g_k belongs to $C_o^{\infty}(\Theta_k)$ so that $g_k = 1$ on Θ_{k-1} and $0 \leq g_k(t,x) \leq 1$ elsewhere. Similarly, let $h_k \in C_o^{\infty}(O_k)$ so that $h_k = 1$ on $\overline{O}_{k-1}$ and $0 \leq h_k(x) \leq 1$ elsewhere.

The definition of the weak solution for the system 3.5.4 in the sense of [8, p. 633] is quoted as follows :

<u>Definition 3.5.3</u> :- For each pair of positive integers s, k and for any $(\sigma, u) \in \Sigma \times D$, a function ϕ_k^{s} is said to be a weak solution of the first boundary value problem (3.5.4) if $\phi_k^{s} \in L^{\infty}(I, L^{2}(O_k)) \cap L^{2}(I, W^{1}(O_k))$ and

$$\int_{\Theta_k} [-\phi_k^{s}(t,x)\cdot \varphi_t(t,x) - a_{ij}^{s}(t,x,\sigma)\cdot(\phi_k^{s}(t,x))_{x_i}\cdot \varphi_{x_j}(t,x) - a_i^{s}(t,x,\sigma,u)\cdot(\phi_k^{s}(t,x))_{x_i} \cdot \varphi(t,x) + f(t,x,\sigma,u)\cdot \varphi(t,x)]\ dt\ dx = 0 .$$

for every $\varphi \in C_o^{1}(\Theta_k^{o})$, Θ_k^{o} interior of the set $\Theta_k \triangleq [0,T) \times O_k$, and if

$$\lim_{t \to T^{0}} \int_{O_k} \phi_k^{s}(t,x) \cdot z(x)\ dx = \int_{O_k} \eta^{s}(x)\cdot h_k(x) \cdot z(x)\ dx$$

for all $z \in C_o^{1}(O_k)$.

Similarly, corresponding to the adjoint system (3.5.2), let us consider the sequence of the following first boundary value problems

$$(3.5.5)\quad \begin{cases} L^{*s}(\sigma^{o},u^{o})\cdot q(t,x) = 0 & (t,x) \in (0,T] \times O_k \\ q(o,x) = q_o^{s}(x)\cdot h_k(x) & x \in O_k \\ q(t,x) = 0 & (t,x) \in I \times \partial O_k \end{cases}$$

where $L^{*s}(\sigma^{o},u^{o})$ and q_o^{s} are as defined for the system (3.5.2) and h_k is as defined for the system (3.5.4).

Remark 3.5.4: Since the integral averages are known to have derivatives of arbitrary order [43, p. 14], they obviously satisfy the assumptions of Theorem 5.2 [35, p. 320]. Thus, it follows from that theorem that for each pair of ~~positive~~ integers s and k the system (3.5.4) ((3.5.5)) has a unique classical solution $\phi_k^{s}(\sigma,u)$ (q_k^{s}) belonging to $H^{\lambda,\lambda/2}(\bar{\Theta}_k)$ [section 3.2] with λ any non integral positive number. In particular, it is also a weak solution. Thus, it follows from [8, Thm. 1, p. 634] that $\phi_k^{s}(\sigma,u)$ (q_k^{s}) is the only weak solution of the system (3.5.4) ((3.5.5)).

For the Cauchy problem (3.5.1) and the first boundary value problems (3.5.4), we have the following result.

Lemma 3.5.5: Consider the system (3.5.1). Suppose that for each integer ~~$s \geq$~~ 1 the assumption (A_2) is satisfied. Then, for each integer $s \geq 1$ and for any $(\sigma,u) \in \Sigma \times D$ the weak solution $\phi^{s}(\sigma,u)$ of the Cauchy problem (3.5.1) is the weak limit in $L^2(I, W^1(R^n))$ of the sequence $\{\phi_k^{s}(\sigma,u)\}_{k=1}^{\infty}$ of the solutions of the first boundary value problems (3.5.4). Moreover, $\{\phi_k^{s}(\sigma,u)\}_{k=1}^{\infty}$ converges uniformly to $\phi^{s}(\sigma,u)$ in any compact subset of Θ.

Proof : Since for each positive integer s the coefficients and data of the system 3.5.1 obviously satisfy the assumptions given for those of the original Cauchy problem S_5^1, the proof of convergence of $\{\phi_k^s(\sigma, u)\}_{k=1}^{\infty}$ in the weak sense to the weak solution $\phi^s(\sigma, u)$ is a direct consequence of that given in the proof of Theorem 3.4.2. The proof of the second part of the lemma follows from similar argument as given for Lemma 3.5.1. This completes the proof of the lemma.

Remark 3.5.6 : Lemma 3.5.5 remains valid for the systems (3.5.2) and (3.5.5) .

Remark 3.5.7 : In view of the proof of Theorem 3.4.2, we note that the sequence $\{\phi_k(\sigma, u)\}_{k=1}^{\infty}$ of the weak solutions of the first boundary value problems (3.4.5) converges in the weak sense to the weak solution $\phi(\sigma, u)$ of the original Cauchy problem S_5^1. Further, it follows from similar argument as given for the proof of the second part of Lemma 3.5.1 that $\{\phi_k(\sigma, u)\}_{k=1}^{\infty}$ converges uniformly to $\phi(\sigma, u)$ on any compact subset of Θ. Clearly, the above mentioned results remain valid for the weak solutions $\{q_k\}_{k=1}^{\infty}$ of the first boundary value problems corresponding to the original adjoint system AS_5^1.

In the proof of the desired necessary conditions for optimality for the problem P_5^1, we need the property of the weak convergence in $L^2(I, W^1(R^n))$ of the sequence $\{\phi^k\}_{k=1}^{\infty}$ of the weak solutions of the system S_5^1 whenever the sequence of the corresponding coefficients of the system converges almost everywhere in Θ as $k \to \infty$. For this, we have the following lemma.

Lemma 3.5.8 : Consider the Cauchy problems

$$(3.5.6) \qquad \begin{cases} -\dfrac{\partial \phi}{\partial t} = (a^k_{ij}(t,x) \cdot \phi_{x_i})_{x_j} - a^k_i(t,x) \cdot \phi_{x_i} + f^k(t,x) & (t,x) \in \Theta \\ \phi(T,x) = \eta(x) & x \in R^n \end{cases}$$

Suppose that a^k_{ij}, $(i,j=1,\ldots,n)$, satisfy the inequality in (A_2) independent of k and that a^k_i, $(i=1,\ldots,n)$, are bounded on Θ uniformly with respect to k and that $\| f^k \|_{2,q}$ is bounded independent of k, where $q \in (1,\infty]$. Further it is assumed that $\eta \in L^2(R^n)$ and that a^k_{ij}, a^k_i, $(i, j=1,\ldots,n)$, and f^k converge, respectively, to a_{ij}, a_i, $(i,j=1,\ldots,n)$, and f almost everywhere in Θ. Then the sequence of the weak solutions $\{\phi^k\}$ of the Cauchy problems (3.5.6) converges to ϕ weakly in $L^2(I, W^1(R^n))$, where ϕ is the weak solution of the system (3.5.6) with its coefficients replaced by their corresponding limits .

Proof: From the estimate (3.4.4) and the hypothese given for the coefficients and the data of the Cauchy problems (3.5.6), we have

$$(3.5.7) \qquad \begin{aligned} \| \phi^k \|^2_{2,\infty} + \| \phi^k_x \|^2_{2,2} &\leq C \{ \| \eta \|^2_2 + \| f^k \|^2_{2,q} \} \\ &\leq C_1 , \end{aligned}$$

where C is as defined in Theorem 3.4.2 ; $q \in (1,\infty]$; and $C_1 \triangleq C \cdot \{ \| \eta \|^2_2 + \sup_k \| f^k \|^2_{2,q} \}$.

Thus, using the above estimate instead of the estimate (3.4.6), we obtain the proof of convergence of $\{ \phi^k \}^\infty_{k=1}$ in the weak sense to the weak solution $\phi \in L^\infty(I, L^2(R^n)) \cap L^2(I, W^1(R^n))$ similar to that as given in the proof of Theorem 3.4.2. This completes the proof.

3.6 NECESSARY CONDITIONS FOR OPTIMALITY

In order to prove the necessary condition for optimal policy, we need the following two additional lemmas.

For the sake of brevity, let $\int_{\Theta} f(t,x)\cdot g(t,x)dt\,dx$ be denoted by $< f, g >$. This abbreviation will be used throughout the rest of this chapter.

Lemma 3.6.1: Consider the problem P_5'. Suppose that $(\sigma^o, u^o) \in \Sigma \times D$ is an optimal policy (whose existence is assumed) and that the assumption (A_2) holds. Then there exists a sequence of solutions q_k^s of the system (3.5.5) such that

$$\lim_{s\to\infty} \cdot \lim_{k\to\infty} < L(\sigma^o, u^o) \cdot (\phi(\sigma^o, u^o) - \phi(\sigma, u)), q_k^s > = 0 \tag{3.6.1}$$

independently of the order of taking the limit.

Proof: In the proof of this lemma, ϕ_k^s (q_k^s) is to denote the solution of the first boundary value problem defined on the set Θ_k with its coefficients and data replaced by their integral averages [p. 50], corresponding to the positive integer s.

For any pair of integers $s, k \geq 1$ and for any $(\sigma, u) \in \Sigma \times D$, we have

$$\begin{aligned} &< L(\sigma^o, u^o)(\phi(\sigma^o, u^o) - \phi(\sigma, u)), q_k^s > \\ &= < L(\sigma^o, u^o)(\phi(\sigma^o, u^o) - \phi(\sigma, u) - \phi^r(\sigma^o, u^o) + \phi^r(\sigma, u)), q_k^s > \\ &+ < L(\sigma^o, u^o)(\phi^r(\sigma^o, u^o) - \phi^r(\sigma, u) - \phi_\ell^r(\sigma^o, u^o) + \phi_\ell^r(\sigma, u)), q_k^s > \\ &+ < L(\sigma^o, u^o)(\phi_\ell^r(\sigma^o, u^o) - \phi_\ell^r(\sigma, u)), q_k^s > , \end{aligned} \tag{3.6.2}$$

where,

r and ℓ are any two positive integers; $\phi^r(\sigma,u)$ and $\phi^r_\ell(\sigma,u)$ are, respectively, the weak solution and the solution of the systems (3.5.1) and (3.5.4) both corresponding to the policy $(\sigma,u) \in \Sigma \times D$; and q^s_k is the solution of the system (3.5.5).

By defining $L(\sigma,u)\cdot\psi = \psi_t - \Delta\psi$ and $f(t,x,\sigma,u) = 0$ for all $x \notin O_k$ the first boundary value problem (3.5.4) with $s = r$ and $k = \ell$ is converted into an equivalent Cauchy problem on Θ. If, for the first boundary value problem, we set $\phi^r_\ell(\sigma,u) = 0$ for $(t,x) \notin \Theta_k$ then it is clear that the solution $\phi^r_\ell(\sigma,u)$ of this problem is also the weak solution of the (extended) Cauchy problem constructed above. Similarly, corresponding to the adjoint system (3.5.5) if we set its solution $q^s_k(t,x) = 0$ for all $(t,x) \notin \Theta_k$, then this solution is also necessarily the weak solution of the (extended) Cauchy problem constructed by adjoining $L^*(\sigma^o,u^o)\cdot\psi = -\psi_t - \Delta\psi$ for $t \in (0,T]$ and for all $x \notin O_k$ to the adjoint system (3.5.5). Thus, integrating by parts the last term in the right hand side of (3.6.2) and noting that $\phi^r_\ell(\sigma^o,u^o)(T,\cdot) = \phi^r_\ell(\sigma,u)(T,\cdot) = \eta(\cdot)\cdot h_k(\cdot)$, it is reduced to

$$\begin{aligned} &< L(\sigma^o,u^o)(\phi^r_\ell(\sigma^o,u^o) - \phi^r_\ell(\sigma,u)),\ q^s_k > \\ (3.6.3)\qquad &= < \phi^r_\ell(\sigma^o,u^o) - \phi^r_\ell(\sigma,u),\ L^*(\sigma^o,u^o)\cdot q^s_k > \\ &\quad - \int_{R^n}[\phi^r_\ell(\sigma^o,u^o)(o,x) - \phi^r_\ell(\sigma,u)(o,x)]\cdot q^s_o(x)\cdot h_k(x)\cdot dx \ . \end{aligned}$$

Writing the first term appearing in the right hand side of the above expression as

$$< \phi_\ell^r(\sigma^o,u^o) - \phi_\ell^r(\sigma,u),\ L^*(\sigma^o,u^o) q_k^s >$$

$$(3.6.4) \qquad = < \phi_\ell^r(\sigma^o,u^o) - \phi_\ell^r(\sigma,u),\ (L^*(\sigma^o,u^o) - L^{*s}(\sigma^o,u^o)) q_k^s >$$

$$+ < \phi_\ell^r(\sigma^o,u^o) - \phi_\ell^r(\sigma,u),\ L^{*s}(\sigma^o,u^o) \cdot q_k^s > ,$$

it follows from the relation (1.4) [8, p. 620] that the last term of the above expression reduces to

$$< \phi_\ell^r(\sigma^o,u^o) - \phi_\ell^r(\sigma,u),\ L^{*s}(\sigma^o,u^o) \cdot q_k^s >$$

$$(3.6.5) \qquad = \int_{R^n} [\phi_\ell^r(\sigma^o,u^o)(o,x) - \phi_\ell^r(\sigma,u)(o,x)] \cdot q_o^s(x) \cdot h_k(x) \cdot dx .$$

Combining (3.6.3), (3.6.4) and (3.6.5), and substituting their final relation for the last term in the right hand side of the expression (3.6.2), we have

$$< L(\sigma^o,u^o)(\phi(\sigma^o,u^o) - \phi(\sigma,u)),\ q_k^s >$$

$$(3.6.6) \qquad = < L(\sigma^o,u^o)(\phi(\sigma^o,u^o) - \phi(\sigma,u) - \phi^r(\sigma^o,u^o) + \phi^r(\sigma,u)), q_k^s >$$

$$+ < L(\sigma^o,u^o)(\phi^r(\sigma^o,u^o) - \phi^r(\sigma,u) - \phi_\ell^r(\sigma^o,u^o) + \phi_\ell^r(\sigma,u)),\ q_k^s >$$

$$+ < \phi_\ell^r(\sigma^o,u^o) - \phi_\ell^r(\sigma,u), (L^*(\sigma^o,u^o) - L^{*s}(\sigma^o,u^o)) \cdot q_k^s >$$

Note that $q_k^s \in H^{\lambda,\lambda/2}(\overline{\Theta}_k)$ [Remark 3.5.4] and $q_k^s(t,x) \equiv 0$ for $(t,x) \notin \Theta_k$. Thus by integrating by parts the t-differentials and those terms with coefficients appearing under the x-differentials of the expression in the right hand side of (3.6.6) and taking limit with respect to ℓ and then r, we deduce from Lemmas 3.5.5 and

3.5.1 that

$$< L(\sigma^o, u^o)(\phi(\sigma^o, u^o) - \phi(\sigma, u)), q_k^s >$$

$$(3.6.7) \quad = - < (a_{ij}(\sigma^o) - a_{ij}^s(\sigma^o))(q_k^s)_{x_j}, \phi_{x_i}(\sigma^o, u^o) - \phi_{x_i}(\sigma, u) >$$

$$- < (a_i(\sigma^o, u^o) - a_i^s(\sigma^o, u^o)) \cdot q_k^s, \phi_{x_i}(\sigma^o, u^o) - \phi_{x_i}(\sigma, u) > .$$

(Note that the order of taking limit with respect to r or ℓ is immaterial due to Lemma 3.5.1, Theorem 1 [8, p. 634] and Remark 3.5.7).

Letting $k \to \infty$ in (3.6.7), we have from Remark 3.5.6

$$\lim_{k \to \infty} < L(\sigma^o, u^o) \cdot (\phi(\sigma^o, u^o) - \phi(\sigma, u)), q_k^s >$$

$$(3.6.8) \quad = - < (a_{ij}(\sigma^o) - a_{ij}^s(\sigma^o)) \cdot q_{x_j}^s, \phi_{x_i}(\sigma^o, u^o) - \phi_{x_i}(\sigma, u) >$$

$$- < (a_i(\sigma^o, u^o) - a_i^s(\sigma^o, u^o)) q^s, \phi_{x_i}(\sigma^o, u^o) - \phi_{x_i}(\sigma, u) > .$$

Since the integral averages converge almost everywhere on Θ and the coefficients $a_{ij}(\sigma^o)$ and $a_i(\sigma^o, u^o)$, $(i, j = 1 \ldots n)$, are bounded on Θ, it follows from Hölder's Inequality and Lebesgue Dominated Convergence Theorem that (3.6.8) in the limit with respect to s converges to zero. On the other hand, if the limit is taken first with respect to s instead of k, then it follows from similar argument that the equation (3.6.7) reduces to

$$(3.6.9) \quad \lim_{s \to \infty} < L(\sigma^o, u^o)(\phi(\sigma^o, u^o) - \phi(\sigma, u)), q_k^s >$$

$$= 0 .$$

Therefore, we obtain the expression (3.6.1) independently of the order of taking the limit. This completes the proof.

Lemma 3.6.2 : Consider the problem P'_5. Suppose that $(\sigma^o, u^o) \in \Sigma \times D$ is the optimal policy (whose existence is assumed) and that the assumption (A_2) holds. Then there exists a weak solution q of the adjoint system AS'_5 such that

$$\begin{aligned}(3.6.10)\quad &\int_\Theta [(a_{ij}(t,x,\sigma) - a_{ij}(t,x,\sigma^o)) \cdot \phi_{x_i}(\sigma,u)(t,x) \cdot q_{x_j}(t,x) + (a_i(t,x,\sigma,u) - \\ &- a_i(t,x,\sigma^o,u^o)) \cdot \phi_{x_i}(\sigma,u)(t,x) \cdot q(t,x)]\,dt\,dx \\ &\leq \int_\Theta (f(t,x,\sigma,u) - f(t,x,\sigma^o,u^o)) \cdot q(t,x)\,dt\,dx\end{aligned}$$

for all $(\sigma, u) \in \Sigma \times D$.

Proof: Clearly for any pair of integers $s, k \geq 1$ and for any $(\sigma, u) \in \Sigma \times D$,

$$\begin{aligned}&< L(\sigma^o,u^o)\,\phi(\sigma^o,u^o),\, q_k^s > - < L(\sigma,u)\,\phi(\sigma,u),\, q_k^s > \\ (3.6.11)\quad &= < (L(\sigma^o,u^o) - L(\sigma,u))\,\phi(\sigma,u),\, q_k^s > + < L(\sigma^o,u^o)(\phi(\sigma^o,u^o) \\ &\qquad - \phi(\sigma,u)),\, q_k^s > ,\end{aligned}$$

where $\phi(\sigma, u)$ is the solution of the system S'_5 corresponding to $(\sigma, u) \in \Sigma \times D$ and q_k^s is the solution of the system (3.5.5). Since $q_k^s(t,x) \equiv 0$ for all $(t,x) \notin \Theta_k$, it follows from the relation (1.4) [8, p. 620] with the substitution of T-t for t that the left hand side of the expression (3.6.11) reduces to

$$(3.6.12)\quad \begin{aligned} &< L(\sigma^{o},u^{o})\cdot\phi(\sigma^{o},u^{o}), q_k^{s} > - < L(\sigma,u)\cdot\phi(\sigma,u), q_k^{s} > \\ &= < f(\sigma,u)-f(\sigma^{o},u^{o}), q_k^{s} > + \int_{R^n}[\phi(\sigma^{o},u^{o})(o,x)-\phi(\sigma,u)(o,x)]\cdot q_o^{s}(x)\cdot h_k(x)\,dx .\end{aligned}$$

Therefore,

$$(3.6.13)\quad \begin{aligned} &<(L(\sigma^{o},u^{o})-L(\sigma,u))\phi(\sigma,u), q_k^{s}> + <L(\sigma^{o},u^{o})(\phi(\sigma^{o},u^{o}) - \phi(\sigma,u)), q_k^{s} > \\ &= <f(t,x,\sigma,u)-f(t,x,\sigma^{o},u^{o}), q_k^{s}> + \int_{R^n}[\phi(\sigma^{o},u^{o})(o,x)-\phi(\sigma,u)(o,x)]\, q_o^{s}(x)\, h_k(x)\,dx .\end{aligned}$$

By the construction of the function h_k, it is clear that $\lim_{k\to\infty} h_k(x) = 1$ for every $x \in R^n$ and $|h_k(x)| \leq 1$ on R^n for all integers $k \geq 1$.

Further, since the weak solution is continuous in Θ, it follows that for every $(\sigma,u) \in \Sigma \times D$ $\phi(\sigma,u)(o,\cdot) \in L^2(R^n)$. Thus, it is easily verified with the help of Hölder's Inequality, Lebesgue Dominated Convergence Theorem and the fact that $q_o^{s} \to q_o$ almost everywhere in R^n that

$$(3.6.14)\quad \begin{aligned} &\lim_{s\to\infty}\,\lim_{k\to\infty} \int_{R^n}[\phi(\sigma^{o},u^{o})(o,x)-\phi(\sigma,u)(o,x)]\cdot q_o^{s}(x)\cdot h_k(x)\,dx \\ &= \int_{R^n}[\phi(\sigma^{o},u^{o})(o,x)-\phi(\sigma,u)(o,x)]\cdot q_o(x)\,dx \end{aligned}$$

independently of the order of taking limit.

Thus, by taking the limit with respect to s and k, we obtain from Lemma 3.6.1 that the expression (3.6.13) reduces to

$$(3.6.15)\quad \begin{aligned} &-\int_{\Theta}[(a_{ij}(t,x,\sigma^{o})-a_{ij}(t,x,\sigma))\cdot \phi_{x_i}(\sigma^{o},u^{o})(t,x)\cdot q_{x_j}(t,x)+(a_i(t,x,\sigma^{o},u^{o}) - \\ &\quad - a_i(t,x,\sigma,u))\cdot \phi_{x_i}(\sigma,u)\,(t,x)\cdot q(t,x)\,]\;dt\cdot dx \\ &= \int_{\Theta}[(f(t,x,\sigma,u)-f(t,x,\sigma^{o},u^{o}))\cdot q(t,x)]\,dt\cdot dx + \int_{R^n}[\phi(\sigma^{o},u^{o})(o,x) - \\ &\quad - \phi(\sigma,u)(o,x)]\cdot q_o(x)\,dx .\end{aligned}$$

Since $(\sigma^o, u^o) \in \Sigma \times D$ is the optimal policy by hypothesis, we obtain the condition (3.6.10) from the condition (3.6.15). This completes the proof.

Based on the above results, necessary condition for optimal policy (control and parameter combined) is presented in Theorem 3.6.3. The individual necessary conditions for optimal control and for optimal parameter are reported in Corollaries 3.6.4 and 3.6.7 respectively.

Note that all the necessary conditions for optimality mentioned above are given in integral form. However, the pointwise necessary condition for optimal control is also reported in Corollary 3.6.6.

For determination of optimal policies (controls and parameters combined), we have the following result.

Theorem 3.6.3: Consider the problem P_5. Suppose that the assumption (A_2) holds and that $a_{ij}(t,x,\cdot)$, $a_i(t,x,\cdot,\cdot)$, $(i,j=1,\ldots,n)$, belong to $C^1(\Sigma \times U)$, $\Sigma \times U \subset R^{r_1} \times R^{r_2} \triangleq R^{r_1+r_2}$, almost everywhere in Θ with the gradients bounded in $R^{r_1+r_2}$ for almost every $(t,x) \in \Theta$ and every $(\sigma, v) \in \Sigma \times U$. Further, it is assumed that the coefficient f is weakly Gateaux differentiable in the sense of $L^2(\Theta)$ at each point of $\Sigma \times D$. Then, if $(\sigma^o, u^o) \in \Sigma \times D$ is an optimal policy (whose existence is assumed), it is necessary that there exists a weak solution q of the adjoint system AS_5' so that for all $(\sigma, u) \in \Sigma \times D$

$$\begin{aligned}&\sum_{k=1}^{r_1} < a_{ij,\sigma_k}(\sigma^o)\cdot\phi_{x_i}(\sigma^o,u^o)\cdot q_{x_j}\; a_{i,\sigma_k}(\sigma^o,u^o)\cdot\phi_{x_i}(\sigma^o,u^o)\cdot q\\ &\qquad - f_{\sigma_k}(\sigma^o,u^o)\cdot q,(\sigma_k-\sigma_k^o) > \\ &- \sum_{\ell=1}^{r_2} < b_{i,u_\ell}(\sigma^o,u^o)\cdot\phi_{x_i}(\sigma^o,u^o)\cdot q+f_{u_\ell}(\sigma^o,u^o)\cdot q,\ (u_\ell-u_\ell^o) > \;\leq\; 0\,,\end{aligned}\tag{3.6.16}$$

where $\alpha'(y)$ denotes the Gateaux differential of α at $y \in \Sigma \times D$ defined as follows :

$$\lim_{\varepsilon \downarrow 0} \frac{\alpha(y+\varepsilon h)-\alpha(y)}{\varepsilon} = (\alpha'(y),\ h)$$

with $(\alpha'(y),\ h) \;\underline{\Delta}\; \sum_{i=1}^{r_1+r_2} \alpha_{y_i}(y)\cdot h_i$.

Proof : For any $(\sigma,u) \in \Sigma \times D$; let $\varepsilon \in [0,1]$ and let $(\sigma,u) - (\sigma^o,u^o) \;\underline{\Delta}\; (\sigma',u')$. Since $\Sigma \times D$ is convex $(\sigma^o,u^o) + \varepsilon\cdot(\sigma',u') \in \Sigma \times D$. Thus, dividing the inequality (3.6.10) [Lemma 3.6.2] by ε and replacing (σ,u) by $(\sigma^o,u^o)+\varepsilon\cdot(\sigma',u')$, we obtain

$$\begin{aligned}&\frac{1}{\varepsilon}\cdot\Big\{\int_\Theta\big[(a_{ij}(t,x,\sigma^o+\varepsilon\cdot\sigma')-a_{ij}(t,x,\sigma^o))\cdot\phi_{x_i}(\sigma+\varepsilon\sigma',u^o+\varepsilon u')(t,x)\cdot q_{x_j}(t,x)\ +\\ &+(a_i(t,x,\sigma+\varepsilon\sigma',u^o+\varepsilon\cdot u')-a_i(t,x,\sigma^o,u^o))\cdot\phi_{x_i}(\sigma^o+\varepsilon\cdot\sigma',u^o+\varepsilon u')(t,x)\,q(t,x)\big]\cdot\\ &\qquad\cdot dt\cdot dx\Big\}\\ &\leq\frac{1}{\varepsilon}\cdot\Big\{\int_\Theta(f(t,x,\sigma^o+\varepsilon\sigma',u^o+\varepsilon\cdot u')-f(t,x,\sigma^o,u^o))\cdot q(t,x)\cdot dt\cdot dx\Big\}\ .\end{aligned}\tag{3.6.17}$$

Note that, by hypotheses, the incremental ratio of the coefficient f conveges to its Gateaux differential weakly in $L^2(\Theta)$. Since the other coefficients of the Cauchy problem S_5' belong to $C^1(\Sigma \times U)$ almost everywhere in Θ with the gradients bounded in $R^{r_1+r_2}$ for almost every $(t,x) \in \Theta$ and every $(\sigma,v) \in \Sigma \times U$, it is clear that their incremental ratios converge to the corresponding Gateaux differentials almost everywhere in Θ as $\varepsilon \downarrow 0$. Denoting by α any of the coefficients a_{ij}, a_i, f and noting that $\alpha(\sigma^o+\varepsilon\sigma', u^o+\varepsilon\sigma')(t,x) \longrightarrow \alpha(\sigma^o, u^o)(t,x)$ almost everywhere in Θ as $\varepsilon \downarrow 0$, it follows from Lemma 3.5.8 that $\phi(\sigma^o+\varepsilon\sigma', u^o+\varepsilon\sigma')$ converges to $\phi(\sigma^o, u^o)$ weakly in $L^2(I, W^1(R^n))$ as $\varepsilon \downarrow 0$. Thus, by taking the limit with respect to ε in the inequality (3.6.17) and using the definition of a_i (Section 3.3) and the facts just mentioned, we obtain the condition (3.6.16). This completes the proof.

Consider the problem P_5 with all the coefficients independent of the parameter vector $\sigma \in \Sigma$. Clearly, its equivalent optimal control problem of distributed parameter system, which is denoted by P_6', is the problem P_5' with σ deleted. Call the corresponding Cauchy problem S_6'. Further, it is understood that the system AS_6', which is adjoint to the system S_6', is the system AS_5' with σ deleted.

With these preparations, we have the following result for the determination of optimal controllers.

Corollary 3.6.4 : Consider the problem P_6'. Suppose that $u^o \in D$ is an optimal control (if one exists) and that the corresponding assumptions of Theorem 3.6.3 are satisfied. Then there exists a weak solution q of the adjoint system AS_6' (corresponding to the optimal control $u^o \in D$) so that

$$(3.6.18) \quad \sum_{\ell=1}^{r_2} < b_{i,u_\ell}(u^o) \cdot \phi_{x_i}(u^o) \cdot q + f_{u_\ell}(u^o) \cdot q \, , \, (u_\ell - u_\ell^o) > \geq 0$$

for all $u \in D$,

where $\alpha_{u_\ell}(u^o)$, $(\ell = 1, \ldots, r_2)$, are as defined in Theorem 3.6.3 ; and

$\phi(u^o)$ is the weak solution of the system S_6' corresponding to $u^o \in D$.

The proof is a direct consequence of Theorem 3. 6. 3.

In order to obtain the pointwise necessary conditions for optimality for the problem P_6', we need the following well known result which is presented in the form of a lemma.

Lemma 3. 6. 5 : Let γ be a Lebesgue integrable function defined on $\hat{\Theta} \triangleq [o, T) \times R^{n_1}$, y a regular point in $\hat{\Theta}$ and let $E \subset \hat{\Theta}$ be any measurable set containing y and contracting to the one point set $\{y\}$. Then,

$$\lim_{|E| \to o} \left\{ \frac{1}{|E|} \int_E \gamma(\theta) \cdot d\theta \right\} = \gamma(y) .$$

Corollary 3. 6. 6 : Consider the problem P_6'. Suppose that all the hypotheses of Corollary 3. 6. 4 are satisfied. Then

$$\sum_{\ell=1}^{r_2} \left\{ \int_{R^{n-n_1}} [b_{i,u_\ell}(t, \hat{x}, \hat{\hat{x}}, u^o(t, \hat{x}))\, \phi_{x_i}(u^o)(t, \hat{x}, \hat{\hat{x}}) + f_{u_\ell}(t, \hat{x}, \hat{\hat{x}}, u^o(t, \hat{x}))] \cdot q(\hat{\hat{x}} | t, \hat{x})\, d\hat{\hat{x}} \right\} \cdot \{ v_\ell - u_\ell^o(t, \hat{x}) \} \geq 0 \tag{3.6.19}$$

for almost all $(t, \hat{x}) \in [o, T) \times R^{n_1}$ and every $v \in U$, where $\phi(u^o)$ and q are as defined in Corollary 3. 6. 4 ; $q(\hat{\hat{x}} | t, \hat{x}) \triangleq \dfrac{q(t, \hat{x}, \hat{\hat{x}})}{\int_{R^{n-n_1}} q(t, \hat{x}, \hat{\hat{x}})\, d\hat{\hat{x}}}$; $a_{u_\ell}(u^o)$, $(\ell = 1, \ldots, r_2)$, are as defined in Theorem 3. 6. 3; and $a_{u_\ell} \triangleq b_{i,u_\ell}$ $(i = 1, \ldots, n)$ or f_{u_ℓ} .

Proof : Let $(t^o, \hat{x}^o)$ be a regular point contained in the interior of $\hat{\Theta} \triangleq [o, T) \times R^{n_1}$ corresponding to the optimal control u^o and suppose that E is a measurable subset of $\hat{\Theta}$ containing $\{(t^o, \hat{x}^o)\}$ and contracting to the point $\{(t^o, \hat{x}^o)\}$ as $|E| \to 0$. Then, dividing the expression (3.6.18) by $|E|$ and replacing the control u by the one defined below

$$u(t,\hat{x}) = \begin{cases} v & \text{for } (t,\hat{x}) \in E \\ u^o(t,\hat{x}) & \text{elsewhere} \end{cases},$$

we have

$$\sum_{\ell=1}^{r_2} \Big[\frac{1}{|E|} \int_E \Big\{ \int_{R^{n-n_1}} \big(b_{i,u_\ell}(t,\hat{x},\hat{\hat{x}},u^o(t,\hat{x})) \cdot \phi_{x_i}(u^o)(t,\hat{x},\hat{\hat{x}}) + f_{u_\ell}(t,\hat{x},\hat{\hat{x}},u(t,\hat{x})) \big) \cdot q(t,\hat{x},\hat{\hat{x}})\, d\hat{\hat{x}} \Big\} \cdot \{ (v_\ell - u^o_\ell(t,\hat{x})) \}\, dt\, d\hat{x} \Big] \geq 0 \tag{3.6.20}$$

Dividing the above expression by $\int_{R^{n-n_1}} q(t,\hat{x},\hat{\hat{x}})\, d\hat{\hat{x}}$ and letting $|E| \to 0$, we obtain the condition (3.6.19) from Lemma 3.6.5 and the fact that $\int_{R^{n-n_1}} q(t,\hat{x},\hat{\hat{x}})\, d\hat{\hat{x}}$ is positive a.e. . This completes the proof.

Note that $q(\hat{\hat{x}} \mid t, \hat{x})$ ~~is to~~ denote the conditional probability density.

In some cases, the structure of a stochastic system is given and we will design the system by choosing values of its parameters so that the performance of the system is optimum with respect to a given performance criterion. Clearly, this kind of problem is the optimal parameter selection problem.

Let the above mentioned problem be described by the problem P_5 given in section 3.1 [pp. 38-39] with the coefficients of the corresponding system and the cost integrand independent of ~~the~~ control variables u. Call this problem P_7. As a direct consequence of Theorem 3.3.1, the problem P_7 reduces to an equivalent problem P'_7 which consists of the Cauchy problem S_5' and the cost functional (3.3.1) with u deleted. For convenience, this Cauchy problem and its adjoint system will be denoted by S_7' and AS_7' respectively.

With these preparations, we present below a necessary condition for the optimal parameter vector (if one exists).

Corollary 3.6.7: Consider the problem P_7'. Suppose that the corresponding assumptions in (A_2) are satisfied and that the coefficient f is Gateaux differentiable in the weak sense of $L^2(\Theta)$ at each point of Σ. Then, for $\sigma^o \in \Sigma$ to be an optimal parameter (if one exists) with $\phi(\sigma^o)$ the corresponding solution of the system S_7', it is necessary that

$$\int_\Theta \left\{ \frac{\partial a_{ij}(t,x,\sigma^o)}{\partial \nu} \cdot \phi_{x_i}(\sigma^o) \cdot q_{x_j} + \frac{\partial a_i(t,x,\sigma^o)}{\partial \nu} \cdot \phi_{x_i}(\sigma^o) \cdot q - \frac{\partial f(t,x,\sigma^o)}{\partial \nu} \cdot q \right\} dt\, dx \leq 0 \tag{3.6.21}$$

(for Σ closed);

$$\int_\Theta \left\{ \frac{\partial a_{ij}(t,x,\sigma^o)}{\partial \sigma_k} \cdot \phi_{x_i}(\sigma^o) \cdot q_{x_j} + \frac{\partial a_i(t,x,\sigma^o)}{\partial \sigma_k} \cdot \phi_{x_i}(\sigma^o) \cdot q - \frac{\partial f(t,x,\sigma^o)}{\partial \sigma_k} \cdot q \right\} dt\, dx = 0, \tag{3.6.22}$$

$k = 1, \ldots, r_1$, (for Σ open);

where $\dfrac{\partial \alpha(t,x,\sigma^o)}{\partial \nu} \triangleq \lim\limits_{\varepsilon \to 0} \dfrac{\alpha(t,x,\sigma^o + \varepsilon\nu) - \alpha(t,x,\sigma^o)}{\varepsilon}$,

$\sigma^o + \varepsilon\nu \in \Sigma$; $0 \leq \varepsilon \leq 1$; and ν any vector directed inward into Σ emanating from σ^o; and q is the weak solution of the adjoint system AS_7' corresponding to the parameter vector σ^o.

<u>Proof :</u> (i) <u>Σ closed</u> : Let ν be a directional vector from σ^o toward the interior of Σ so that $\sigma^o + \varepsilon\nu \in \Sigma$, $\varepsilon \in [0, 1]$. Then, it follows from condition 3.6.10 with σ replaced by $\sigma^o + \varepsilon\nu$ and u deleted that

$$\int_\Theta \{(a_{ij}(t,x,\sigma^o+\varepsilon\nu)-a_{ij}(t,x,\sigma^o))\cdot\phi_{x_i}(\sigma^o+\varepsilon\nu)\cdot q_{x_j}+(a_i(t,x,\sigma^o+\varepsilon\nu)-a_i(t,x,\sigma^o))\cdot\phi_{x_j}(\sigma^o+\varepsilon\nu)\cdot q-(f(t,x,\sigma^o+\varepsilon\nu)-f(t,x,\sigma^o))q\}\,dt\,dx \le 0. \tag{3.6.23}$$

Since the coefficients $a_{ij}(\sigma^o+\varepsilon\nu)$, $a_i(\sigma^o+\varepsilon\nu)$, $(i,j=1,\ldots,n)$, and $f(\sigma^o+\varepsilon\nu)$ converges, respectively, to $a_{ij}(\sigma^o)$, $a_i(\sigma^o)$, $(i,j=1,\ldots,n)$, and $f(\sigma^o)$ almost everywhere in Θ as $\varepsilon \downarrow 0$; and since these coefficients obviously satisfy the assumptions of Lemma 3.5.8, it follows from that lemma that $\phi(\sigma^o+\varepsilon\nu) \longrightarrow \phi(\sigma^o)$ weakly in $L^2(I, W^1(R^n))$ as $\varepsilon \downarrow 0$. Further, by the properties (assumed) of these coefficients considered as functions of σ in Σ, we note that the incremental ratios of the coefficients a_{ij}, a_i, $(i,j=1,\ldots,n)$, appearing in the expression (3.6.23) converge to their corresponding Gateaux differentials almost everywhere in Θ as $\varepsilon \downarrow 0$, while the incremental ratio of the coefficient f converges to its Gateaux differentials weakly in $L^2(\Theta)$. From these we obtain the condition (3.6.21) by dividing (3.6.23) by $\varepsilon > 0$ and letting $\varepsilon \downarrow 0$.

(ii) <u>Σ open :</u> Since σ^o is an interior point of Σ, the condition (3.6.22) follows from simple manipulation of the condition (3.6.21).

<u>Remark 3.6.8:</u> Assume that the coefficients a_{ij}, a_i, $(i,j=1,\ldots,n)$ and f as functions of the parameter σ are defined on and continuously differentiable in, an open set containing Σ. Then, it is easily verified

that the necessary condition (3.6.21) is equivalent to the condition

$$\sum_{k=1}^{r_1} \Big[\int_{\Theta} \Big\{ \frac{\partial a_{ij}(t,x,\sigma^o)}{\partial \sigma_k} \cdot \phi_{x_i}(\sigma^o) \cdot q_{x_j} + \frac{\partial a_i(t,x,\sigma^o)}{\partial \sigma_k} \cdot \phi_{x_i}(\sigma^o) \cdot q - \frac{\partial f(t,x,\sigma^o)}{\partial \sigma_k} \cdot q \Big\} dt \, dx \Big] \cdot [\sigma_k - \sigma_k^o] \leq 0 \qquad (3.6.24)$$

for all $\sigma \in \Sigma$.

As a consequence of Corollary 3.6.7 we have the following result.

Corollary 3.6.9 : Consider the problem P_7. Suppose that all the hypotheses given in Corollary 3.6.7 are satisfied and that the coefficients a_{ij}, a_i $(i,j=1,\ldots,n)$, and f are linear in the parameter σ. Then, if the parameter restraint set Σ is a compact and convex polyhedron in R^{r_1}, the optimal parameter vector σ^o, (if one exists), takes its value at one of the vertices of Σ.

3.7. BANG BANG PRINCIPLE

In certain problems of economics and pollution control it is required that the system trajectory spends maximum time in a desirable region of the state space. For example, in the problems of regulation of national economic growth it is required to adopt economic policies that ensure the desired economic growth without violating certain prespecified acceptable limits of inflation and unemployment in the country.

This class of problems can be formulated as follows : Consider the system S_1 with σ deleted and consider the set of admissible controls D. Let B be a closed bounded region in the state space R^n. It is required that trajectory ξ spends maximum time in the given region B. Define $\ell(\xi(u)) = \{ t \in I_V : \xi(u)(t) \in B \}$ and let ν be the Lebesgue measure on the real line. The problem is : find a control $u^o \in D$ that maximizes $\mathcal{E}\{\nu(\ell(\xi(u)))\}$. Call this problem P_8.

Consider the system S_1 [p.16] with σ deleted and let B be a compact subset of R^n. Let $\chi_B(x)$ be the characteristic function of the set B. Then $\nu(\ell(\xi(u))) = \int_o^T \chi_B(\xi(u)(t))\,dt$, and consequently it follows from Theorem 3.3.1 that the problem P_8 is equivalent to the problem

$$P_8' \left\{ \begin{array}{lll} S_8' \left\{ \begin{array}{l} L(u)\phi + \chi_B(x) = 0 \\ \phi(T,x) = 0 \\ u \in D \end{array} \right. & & \begin{array}{l} (t,x) \in [o,T) \times R^n \\ x \in R^n. \\ \\ \end{array} \\ & & \\ (3.7.1) \quad \max\limits_{u\in D} J(u) = \max\limits_{u \in D} \int_{R^n} \phi(u)(o,x)\,\pi_o(dx) & & \end{array} \right.$$

Clearly, the system adjoint to the system S_8' is AS_6' with $f = \chi_B$. Call this adjoint system AS_8'.

Remark 3.7.1 : If the system S_1 with σ deleted is required to spend maximum length of time inside the allowed region B during any interval of time of finite or infinite length then one may choose a suitable bounded measurable and Lebesgue integrable positive weighting function ς and define

$$\nu(\ell(\xi(u))) = \int_o^\infty \varsigma_o(t)\,\chi_B(\xi(u)(t))\,dt.$$

In this case the problem P_8' remains valid with $\chi_B(x)$ replaced by $\varsigma(t)\,\chi_B(x)$ and $T = \infty$.

The results mentioned above are also reported in [48, pp. 361-362].

Clearly the problem P_8' is only a special case of the problem P_6'. However, under the linearity property of the function b in the control variables u, we can show that the 'bang bang principle' holds true. For convenience, the problem P_8', with the coefficient b linearly dependent of the control variables u, will be called P_9'. Further, the Cauchy problem and its adjoint system corresponding to the problem P_9' will be denoted by S_9' and AS_9' respectively.

With these preparations, we have

<u>Theorem 3.7.2.</u> Consider the problem P_9'. Suppose that the corresponding assumptions in (A_2) is satisfied, where $b(t,x,u) \triangleq b(t,x) + g(t,x)\,u$. Then, if there exists a control maximizing the cost functional (3.7.1), it is necessarily of bang bang type.

The proof follows from the application of Corollary (3.6.4) with 'min' replaced by 'max'.

Before we present a special case of Theorem 3.7.2, we need

<u>Definition .7.3.</u> The system S_9' is said to be normal if to every admissible control u its corresponding weak solution $\phi(u)$ and the weak solution q of the system AS_9' have the property that for no $j = 1, \cdots, r_2$

$$(3.7.2)\qquad z_j(t,\hat{x}) \triangleq \left\{ \int_{R^{n-n_1}} \left(g_{ij}(t,\hat{x},\hat{\hat{x}})\,\phi_{x_i}(u)\right) q(\hat{\hat{x}}\,|\,t,\hat{x})\,d\hat{\hat{x}} \right\}$$

is identically zero over a set of positive measure, contained in $\hat{Q}$

where $$q(\hat{\hat{x}}\,|\,t,\hat{x}) \triangleq \frac{q(t,\hat{x},\hat{\hat{x}})}{\int_{R^{n-n_1}} q(t,\hat{x},\hat{\hat{x}})\,d\hat{\hat{x}}} \Big\}.$$

As a consequence of Theorem 3.7.2, we have the following result.

Corollary 3.7.4 : If the control restraint set U is a unit hypercube in R^{r_2}, D is the corresponding class of admissible controls and the system S_9' is normal, then the optimal control is of the form

$$(3.7.3)\quad u_j^o(t,\hat{x}) = \operatorname{sign}\left\{\int_{R^{n-n_1}} g_{ij}(t,\hat{x},\hat{\hat{x}})\cdot \phi_{x_i}(u^o)\cdot q(\hat{\hat{x}} \mid t,\hat{x})\, d\hat{\hat{x}}\right\},$$

$j = 1, \ldots, r_2$;

where q is the weak solution of the adjoint system AS_9'. ~~Further~~,
$\phi(u^o)$ is the weak solution of the following nonlinear integro-partial differential equation :

$$(3.7.4)\quad \begin{cases} -\dfrac{\partial \phi(u^o)}{\partial t} = (b_i(t,x)\cdot \phi_{x_i}(u^o) + \sum\limits_{j=1}^{r_2} (g_{ij}(t,x)\cdot \phi_{x_i}(u^o))\operatorname{sign}\left(\int_{R^{n-n_2}} g_{ij}(t,x)\cdot \phi_{x_i}(u^o)\cdot\right. \\ \left.\cdot\, q(\hat{x} \mid t,\hat{x})\cdot d\hat{x}\right) + a_{ij}(t,x)\cdot \phi_{x_i x_j}(u^o) + {}_B(x) \qquad (t,x)\in\Theta \\ \phi(u^o)(T,x) = 0 \qquad\qquad x \in R^n \end{cases}$$

3.8. CONCLUSIONS AND SUGGESTIONS FOR FURTHER STUDY

In this chapter, the class of stochastic optimization problems with fixed terminal time was reduced to the optimization problem of an equivalent class of Cauchy problems with both control and parameter variables appearing in the coefficients of the differential operator. To this reduced problem, necessary conditions for determination of both the optimal control and the optimal parameter have been developed. Further, this result was also applied to derive necessary conditions for optimality for some special problems.

To the knowledge of the author, it appears that the following related interesting problems still remain unsolved.

(i) Necessary conditions for optimality for the problem P'_5 in which the diffusion coefficient depends also on the control variables.

(ii) Necessary conditions for optimality for the problem P'_5 in which the diffusion coefficient is not necessarily uniformly parabolic.

(iii) Necessary conditions for optimality for the problem P'_6 in which the coefficients are only continuous in the control variables.

(iv) Existence Theorem on optimal controls for the problem P'_6 similar to [23, Thm. 3, p. 205] and [4, Thm. 1, p.].

APPENDIX A[**]

CERTAIN COMMENTS ON COMPUTATIONAL ASPECTS

General Case :

Using the necessary conditions (2. 5.1) (Chapter 2, p. 23), the form of the extremal policy is obtained. The resulting expression , as a functional of the state ϕ and the adjoint state q , is then substituted in the system equation S_1' (p. 20) and the adjoint AS_1' (p. 22). This gives rise to a two-point-boundary value problem (TPBVP) involving a coupled systems of nonlinear parabolic partial differential equations associated with the boundary conditions as given in the system equations S_1 (p. 20) and AS_1' (p. 22).

(a) At this stage , by discretization of the spatial domain Ω , the above infinite dimensional TPBVP can be converted into a finite dimensional TPBVP similar to that arising in the case of ordinary dynamical systems. The corresponding two-point -boundary -value -problem may then be solved using Davidon-Fletcher-Powell method. [2] along with Fibonacci Search technique [4]. A closer approximation to the actual solution of the problem may be achieved by refinement of the discretization. However, this procedure will require large computer memory and computational time.

(b) A direct approach would be to choose the missing initial condition $\varphi(0, \cdot) \underset{=}{\Delta} \psi(\cdot)$ in the system equation S_1 (p. 20) and solve the two coupled equations (S_1' and AS_1') forward in time to minimize an error function of the form

$$\mathcal{E}(\psi) \underset{=}{\Delta} \int_{\Omega} |\varphi(T,x)|^2 dx + \int_I \int_{\partial\Omega} |\varphi(t,x)|^2 dt\, dS_x + \int_I \int_{\partial\Omega} |q(t,x)|^2 dt\, dS_x .$$

[**] All the references used in this Appendix are to be given in page 78.

An algorithm like the Steepest Descent approach [4] may be used to compute successively the minimizing sequence of initial conditions $\varphi^{(k)}(0,x) \triangleq \psi^{(k)}(x)$.

Special Cases:

(a) In the case of stochastic systems with controls appearing linearly, it has been shown in Theorem 2.6.1' (Chapter 2, p. 30) that the bang bang principle holds. Using the form of the extremal control given in the expression 2.6.1 (p31), one obtains a nonlinear parabolic partial differential equation 2.6.5 (p. 32). This equation is derived on the assumption that complete information about the current state is available to the controller. In this case it is only necessary to solve this particular equation 2.6.5 for the state $\varphi(u^{0})$. This is then substituted into the equation 2.6.1. to obtain the optimal control. The equation 2.6.5 can be solved using Gauss Seidel method [1] or Crank-Nicolson method [3].

(b) In the problems of selection of parameters, the necessary condition given in Corollary 2.5.5 (Chapter 2, p. 28) states that the optimal parameter takes its values at one of the vertices of the given polyhedra. Therefore it is only essential to solve the original system equation $S_1{}'$ (p. 20) (with u deleted and σ appearing linearly) for each of the vertices of the polyhedra. This result has been used to find the optimal parameter by solving the parabolic partial differential equation using Crank - Nicolson method [3]. Numerical results are given in reference [5].

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APPENDIX B

CERTAIN COMMENTS ON THE COEFFICIENTS OF THE SYSTEM EQUATION

It may be noted that the diffusion coefficient c appearing in the system equation S_1 (p. 16) does not contain the control variable in its arguments. Since the admissible controls are assumed to be bounded and measurable, inclusion of such controls in the coefficient c would result in a measurable function. From the theory of parabolic partial differential equations as well as stochastic differential equations, existence and uniqueness of solution are known only for smooth diffusion coefficient.

In the presentation of necessary condition for optimality, it is essential that the system has a unique solution. In the presence of measurable diffusion coefficients, equivalent results do not exist. However, in the case of parabolic systems described in the divergence form, measurable coefficients are allowed. But the stochastic systems give rise to parabolic problems which are not in the divergence form. This is a fundamental mathematical problem and is the reason why control was not included in the diffusion coefficient.

However, it should be mentioned that no such restriction is placed on the drift coefficient b.

The author regrets any omission of references to significant contributions in the broad area of stochastic control theory. Such omissions are either inadvertent or because the subject matter of this thesis is not directly related to them.

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