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BENACERRAF'S DILEMMA AND NATURAL REALISM FOR ARITHMETIC

By

Anoop K. Gupta B.A., M.A.

A Thesis

Submitted to the School of Graduate Studies  
in Partial Fulfilment of the Requirements

for the Degree

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AUTHOR: Anoop Gupta B.A., M.A., (McMaster  
University)

SUPERVISOR: Professor Mathieu Marion

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Nature never did betray the heart that loved her.

- William Wordsworth, "Tintern Abbey"

(Wordsworth 1983, 276)

### **Abstract**

A natural realist approach to the philosophy of arithmetic is defended by way of considering and arguing against contemporary attempts to solve Paul Benacerraf's dilemma (1973). The first horn of the dilemma concerns the existence of abstract mathematical objects, which seems necessitated by a desire for a unified semantics. Benacerraf adopts an extensional semantics whereby the reference of terms for natural numbers must be abstract objects. The second horn concerns a desirable causal constraint on knowledge, according to which "for X to know that S is true requires some causal relation to obtain between X and the referents of the names, predicates, and quantifiers of S". Within the philosophical tradition, Benacerraf's dilemma crystallizes the tension between realists (roughly the first horn) and empiricists (roughly the second horn). It is shown that natural realism and naturalism meet.

Both horns of the dilemma are amended. Abstract objects are conceived along conceptualist lines such that their existence is not mind-independent. The causal constraint is refined by drawing upon the work of Mark Steiner (1973). Natural realism is the notion that the truth-values of arithmetical statements are recognition-transcendent. Natural realism explains why one and only one arithmetic is applicable. It is in some sense discovered, which is captured by the idea that arithmetical statements have truth-values

even if one is not able to adduce what they are.

It is argued that with proper emendation, like that of Philip Kitcher (1984), empiricism as envisioned by J.S. Mill is defensible. Furthermore, the epistemology of arithmetical knowledge is divided into two tiers. The first principles of arithmetic are acquired by causal interaction with physical objects. As Kitcher has shown, empiricism functions at the first-tier. The *a priori* is revised such that arithmetical knowledge generated from the first-tier is considered non-empirical.

It is argued that pragmatists' indispensability argument, once qualified, allows that arithmetic's applicability is one reason to consider natural realism for that domain (Putnam 1979a). Furthermore, this thesis provides a case where aspects of Hilary Putnam's early and later views are rendered consistent. By way of utilizing both Putnam's writing in favour of realism (1979a), and those that break with them (e.g., 1981), it is suggested that he should not have abandoned his earlier view. In a nutshell, the realization of knowledge can depend upon agents' values, methods, and so on, without forcing the abandonment of realism (provided one has the correct values and methods). Putnam's early view sets the criteria for when one can be a realist about a given domain.

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## Preface

Bertrand Russell explained that mathematical investigations can move in two opposite directions. Philosophy of mathematics moves backwards, to ever greater abstraction, while mathematics forwards, generating ever increasing complexity (Russell 1919, 1-2). That is, the philosophy of mathematics does not promise to offer advances in the domain in which it is about. It has a philosophical value. For example, getting clear about the realist debate demands testing one's ideas in a variety of domains, like that of arithmetic. Historically, in fact, many philosophical issues, like realism, have been shaped by investigations into mathematics.

Benacerraf writes:

The philosophy of mathematics is philosophy in a pure state, stripped of all worldly appendages; austere; philosophy without the sugar coating of a pretense to Relevance to Life; but also heady, perhaps best taken in short sips. (Benacerraf 1996, 10)

In its pure state, one can see the limits of philosophical prose to give expression to practices (e.g., mathematical practices).<sup>1</sup> Philosophers may propose metaphysical views that

are epistemologically implausible. For example, one may claim that the number "3" has a mind-independent existence. Since one cannot obviously interact with mind-independent abstract objects, one is forced to a dubious epistemology. One's metaphysics outstrips one's epistemology; realism is maintained at the cost of rationality. Going to the other extreme, one may recoil to idealism. For instance, one may claim that arithmetic is an arbitrary construction, which, though plausible, deflates what one means when one says that arithmetical statements are true. One's epistemology determines one's metaphysics; rationality restrains realism. Both extremes are unacceptable: they are both counter-intuitive.

More generally, an important philosophical lesson in the twentieth century (admittedly an old lesson) is the idea that one should not easily accept philosophical theories that are at odds with things one needs to believe in order to live. Relevance to life can not be escaped. One needs both realism and rationality. The aim of this thesis is to demonstrate that truth (i.e., a realist metaphysics) can be reconciled with a desire for plausibility (i.e., a rational epistemology). The reconciliation of realism with rationality harkens back to the idea of a pre-established harmony. As Bob Dylan says, "Like every sparrow falling, like every grain of sand."<sup>2</sup>

## TABLE OF CONTENTS

|                                                    |               |
|----------------------------------------------------|---------------|
| <b>Abstract.....</b>                               | <b>iv</b>     |
| <b>Acknowledgments.....</b>                        | <b>vi</b>     |
| <b>Preface.....</b>                                | <b>vii</b>    |
| <br><b>(0) Introduction.....</b>                   | <br><b>1</b>  |
| (0.1) Plan Of The Argument.....                    | 5             |
| (0.2) Natural Realism.....                         | 7             |
| <br><b>PART I: ANALYSIS OF THE DILEMMA</b>         |               |
| <br><b>(1) Abstract Objects.....</b>               | <br><b>11</b> |
| (1.1) David Lewis' Analysis.....                   | 13            |
| (1.2) Problems With Nominalism And Platonism.....  | 17            |
| (1.3) Conceptualism: The Best Of Both Worlds.....  | 23            |
| <br><b>(2) The Causal Constraint.....</b>          | <br><b>29</b> |
| (2.1) Mark Steiner's Analysis.....                 | 30            |
| (2.2) Against The Causal Constraint.....           | 36            |
| (2.3) In Defense Of The Causal Constraint.....     | 41            |
| <br><b>(3) The Indispensability Argument.....</b>  | <br><b>49</b> |
| (3.1) Quine's Pragmatism.....                      | 50            |
| (3.2) Putnam's Pragmatism.....                     | 59            |
| (3.3) Defending The Indispensability Argument..... | 64            |

|                                                   |    |
|---------------------------------------------------|----|
| (3.4) Revising The Indispensability Argument..... | 69 |
|---------------------------------------------------|----|

## **PART II: PLATONISM**

|                                                   |               |
|---------------------------------------------------|---------------|
| <b>(4) Neo-Fregean Realism.....</b>               | <b>74</b>     |
| (4.1) Frege's Gatekeeper.....                     | 76            |
| (4.2) Trouble On The Farm.....                    | 80            |
| (4.3) The Linguistic Wrong Turn.....              | 84            |
| <br><b>(5) Maddy's Set Theoretic Realism.....</b> | <br><b>88</b> |
| (5.1) Maddy's Naturalized Realism.....            | 89            |
| (5.2) Ontological Double-Vision.....              | 100           |
| (5.3) The Empiricist Legacy.....                  | 105           |

## **PART III: NOMINALISM**

|                                        |                |
|----------------------------------------|----------------|
| <b>(6) Kitcher's Empiricism.....</b>   | <b>108</b>     |
| (6.1) Playing With Pebbles.....        | 110            |
| (6.2) The Origin Of Number.....        | 119            |
| (6.3) Kitcher's Naturalism.....        | 125            |
| (6.4) Naturalism At Work.....          | 135            |
| <br><b>(7) Field's Nominalism.....</b> | <br><b>140</b> |
| (7.1) What Is Good For The Goose.....  | 141            |
| (7.2) Spinning This Web.....           | 153            |
| (7.3) Hubris: Back To Field.....       | 158            |

**PART IV: NATURAL REALISM**

|                                                     |                |
|-----------------------------------------------------|----------------|
| <b>(8) Natural Realism.....</b>                     | <b>162</b>     |
| (8.1) Naturalism.....                               | 162            |
| (8.2) Natural Realism.....                          | 167            |
| (8.3) The Stronger Program.....                     | 183            |
| <br><b>(9) Concluding Remarks.....</b>              | <br><b>198</b> |
| (9.2) Retrospective.....                            | 196            |
| (9.2) Benacerraf's Dilemma and Natural Realism..... | 200            |
| <br>-----                                           |                |
| Endnotes.....                                       | 206            |
| Bibliography.....                                   | 252            |

## 0. Introduction

Paul Benacerraf (1973) presented a dilemma in his paper "Mathematical Truth". He writes:

It is my contention that two quite distinct kinds of concerns have separately motivated accounts of the nature of mathematical truth: (1) the concern for having a homogeneous semantical theory in which semantics for the propositions of mathematics parallel the semantics for the rest of language, and (2) the concern that the account of mathematical truth mesh with a reasonable epistemology. It will be my general thesis that almost all accounts of the concept of mathematical truth can be identified with serving one or another of these masters *at the expense of the other*. (Benacerraf 1973, 661)

The first horn deals with the existence of abstract mathematical objects, which, as Benacerraf explained, seems necessitated by a desire for a unified semantics. He adopts an extensional semantics whereby the reference of terms for natural numbers must be abstract objects. The second horn concerns a desirable causal constraint on knowledge, according to which "for X to know that S is true requires some causal relation to obtain between X and the referents of the names, predicates, and quantifiers of S" (Benacerraf 1973, 669).

The first horn of the dilemma conflicts with the second one

that requires truth to be knowable. According to Benacerraf, "...[A]n account of mathematical truth, to be acceptable, must be consistent with the possibility of having mathematical knowledge" (Benacerraf 1973, 667). On the one hand, positing numbers as abstract objects - to satisfy the first horn - raises the epistemological problem of cognitive access. Assuming that numbers exist, does not explain, for instance, how one gets to know "[t]here are at least three perfect numbers greater than 17" (Benacerraf 1973, 663), and to affirm that knowledge must fulfil the causal constraint cuts off the possibility of the knowledge of abstract mathematical objects. One does not causally interact with such abstract objects.

Epistemologies that include the causal constraint - e.g., scientific realism propounded along the lines of Devitt (Devitt 1991) - must confront Benacerraf's dilemma for several reasons. First, commitment to an empiricist epistemology is threatened. Second, confronting the dilemma clarifies the epistemological status of a tool, - mathematics - which scientists often use in forming descriptions of the world. Third, a verdict on the status of mathematical abstract objects clarifies the scientific realists' over-all epistemological picture. As Benacerraf puts it, "We should always bear in mind that what is really at issue is our over-all philosophical view" (Benacerraf 1973, 662). The letter of Benacerraf's dilemma shall be abandoned but not the tension between a realist metaphysics (the first horn) and a

reasonable epistemology (the second horn).

One can view the positions in the philosophy of mathematics since (1973) as a response to Benacerraf's dilemma. In this thesis, the most important of these views shall be discussed, with a view to developing an alternative one that draws upon Putnam's (1979a) realism and Kitcher's (1984) empiricism .

For the purpose of this thesis, limitations are placed upon the scope of the discussion of Benacerraf's dilemma. Though it is assumed that the view to be developed applies generally to mathematical truth, an argument for that shall not be presented. Arithmetical truth, as defined for Peano arithmetic, i.e., the theory which is constituted by first-order logic, with identity, and the following five axioms, shall be the focus (Halmos 1974, 46; Engeler 1983, 38):

(i)  $N(0)$

("0" is a number.)

(ii)  $N(x) \rightarrow N(x+1)$

(every number has a successor.)

(iii)  $N(x) \ \& \ N(y) \ \& \ (x+1 = y+1) \rightarrow x=y$

(no two numbers have the same successor.)

(iv)  $N(x) \rightarrow 0 \neq x+1$

("0" is not the successor of any number.)

(v)  $\forall x((A(x) \rightarrow N(x)) \ \& \ (A(0) \ \& \ \forall x(A(x) \rightarrow A(x+1))) \rightarrow \forall y(N(y) \rightarrow A(y)))$

(a property of "0" and of every successor of "0" belongs to every number.)

Peano arithmetic codifies our knowledge of numbers in an axiomatic system. In broadest terms, the axiomatic (or deductive) method consists in accepting without proof certain postulates, then deriving from them by one or many rules of inference (e.g., modus ponens) all the other true statements of the theory.<sup>1</sup>

One may wish to keep in mind that foundations can themselves be deemed axiomatic or reductive. Stewart Shapiro remarks:

Let  $P$  be a body of knowledge or a field of study...A *foundation* for  $P$  is a reconstruction of its principles, either its truths or its knowable propositions. There are at least two forms that a foundation can take. One of them, the *axiomatic method*, consists of collecting of propositions, the *axioms*, together with a demonstration that all, or many, of the accepted truths of  $P$  can be derived from them...The other sort of foundation may be called *reductive*. The basic subject matter of  $P$  is cast, or recast, in terms of another mathematical theory. At most foundational activity is contemporaneous with the development of  $P$ . To take just one example, the axiomatization of arithmetic by, say, Peano, Dedekind or even Euclid came after the substantial knowledge of the natural numbers. (Shapiro 1991, 26)

For example, one may reconstruct arithmetic along the lines of the Peano axioms or reduce it to another theory, whose terms are considered to be more primitive, as was attempted by logicism or Hilbert's program of the 1930's.

Both the axiomatic and reductive method could function in strong or moderate foundational programs, which can be defined following Shapiro:

Define *strong foundationalism* for  $P$  to be the view that there is a *single* foundation for  $P$ , one that is absolutely secure, or, failing that, as secure as is humanely possible...Define *moderate foundationalism* for

*P* to be the view that it is possible to provide at least one foundation for *P*, again either absolutely secure or as good as possible...A moderate foundationalist program is in effect, a reconstruction of *P*...Since, on such views, there may be more than one foundation, alternatives are not necessarily competitors. For all we know, the field can be rebuilt in more than one way. (Shapiro 1991, 27)

Basically, the difference between a strong and moderate foundationalism hinges on whether one accepts only one foundation or many, respectively. There shall be no defense of the Peano axioms as the sole foundation for arithmetic since a moderate foundationalism is endorsed (though it is expected that all foundations of arithmetic will be isomorphic (in second-order logic)). Peano arithmetic is used here for the purpose of illustration, though examples will be sometimes drawn, more generally, from mathematics.

#### **(0.1) Plan Of The Argument.**

Benacerraf's dilemma is only viable as a problem if its two horns obtain. In Chapter 1, a conceptualist account of abstract objects will be advocated. In Chapter 2, it shall be argued that one can speak of proximities to experiential evidence, where the first principles of arithmetic are empirical generalizations; what one derives from them will be argued to count as *a priori* knowledge. (Also, epistemic realism can range over both empirical and *a priori* knowledge claims.)

In Chapter 3, the first type of answer to the dilemma is considered. As Hart (1998) has shown, the indispensability

argument tries to bypass Benacerraf's dilemma. There is no need to worry about access to arithmetical objects since one can indirectly infer their existence by their applicability in scientific practice. In outline, the goal will be to defend the pragmatists.

In addition to the pragmatist's solution, however, since each of the horns can be dealt with in at least two ways - by assenting or dissenting to each one - there are four possible strategies for dealing with the dilemma: affirming the existence of abstract objects but denying the causal constraint; affirming both the existence of arithmetical objects and the causal constraint; denying the existence of abstract objects but affirming the causal constraint; or denying both the existence of arithmetical objects and the causal constraint.

In Chapter 4, the second strategy is considered. The context principle is supposed to explain how one is able successfully to refer to numbers as abstract objects without the need to postulate a special faculty of perception (Wright 1983; Hale, 1987; Demopoulos 1995; Boolos 1997).<sup>2</sup>

In Chapter 5, the third strategy will be discussed. Following Quine's (1969) proposal for a naturalized epistemology, Penelope Maddy (1990), (1997) has proposed a naturalist's defense of set-theoretic platonism. By way of perception of sets, she hopes to succeed in justifying belief in the axioms for set theory.

In Chapter 6, the fourth strategy is considered. It allows for arithmetical knowledge to be based on empirical generalizations (Kitcher, 1984). As far apart as Maddy's platonism and Kitcher's empiricism may be, both accept the idea at the core of structuralist accounts, whereby arithmetic is about a set of equivalent (isomorphic) structures. For arithmetic, the axioms which encode the rules of construction (what Kitcher calls first principles) require explanation.

In Chapter 7, the fifth option is considered. Taking their lead from an idea of Dedekind (1963), Benacerraf (1965), Putnam (1979a), are the many structuralist views, e.g., Hellman (1989), Chihara (1990), Shapiro (1991), (1997), and Resnik (1997). Harty Field (1980), (1989), is the focus. In a nutshell, he argues, in Science Without Numbers for instance, that the reference to abstract objects in (Newtonian) physics can be eliminated.

Structuralists, who usually adopt the language of set theory, contend that one does not need to refer to abstract objects to account for arithmetical truth. Statements of a theory such as Peano arithmetic are not about numbers as objects but a set of structures that serves as models for the theory.

### **(0.2) Natural Realism.**

Realism captures one's naive intuitions that  $P$  being "true" means that  $P$  is so even if one does not go and look. Traditional realists hold that statements of a given class

have a truth-value even if one is not able to adduce what the truth-values are. One problem with realism is that one may be forced to the actual recognition-transcendence of the truth-values of statements, i.e., skepticism. Anti-realism offers a safeguard from skepticism: it rejects the recognition-transcendence of the truth-values of statements (Griffin 1995, 1). According to the radical anti-realist, for instance, the truth of " $124 \div 4 = 31$ ", depends upon the procedure for doing long division, which per se, is agent-dependent.

In Chapter 8, a new, alternative strategy is propounded that aims to avoid the traditional problems of metaphysical realism and anti-realism. It will be argued, one does not have to choose between skeptical and anti-realist doctrines, that is, between assuming knowledge is either actually recognition-transcendent or agent-dependent. The view to be propounded shall henceforth be called "natural realism".<sup>3</sup> Natural realism is the view that the truth-values of arithmetical statements are recognition-transcendent and recognizable.

Putnam's (1979a) justification for realism will be relied upon. As he points out, (1) mathematics is unique - there is only one mathematics and (2), it is largely applicable to the world. Both points - that concerning its uniqueness and applicability - underscore the necessity of mathematics (Putnam 1979a, 70).<sup>4</sup>

If one is so constrained by necessity in mathematics - as discussed - natural realism, it shall be argued, is a

plausible hypothesis because according to it, arithmetical statements have truth-values regardless whether one can adduce what they are or not - the truth-values are recognition-transcendent.

As Frege has pointed out what is important is the justification for the criteria by which one adjudicates the truth-values of statements (Frege 1953, sec. 3). According to the natural realist, statements' truth-values are in principle independent of either the procedures being enacted (at any one time) or of their existence. Finally, in Chapter 9, natural realism is shown to be the optimum solution to Benacerraf's dilemma.

Following the lead of Putnam's (1979a) account, the natural realist advocates that recognition-transcendence applies in two senses. That is to say, arithmetical truths are independent of procedures being enacted or of their existence. Natural realism aims to account for the necessity of arithmetical truth. It aims to account for the fact that arithmetic has to be constructed, but one cannot make it up any way one wants.

This thesis is novel in two ways. One the first score, naturalism's greatest contemporary exponent, Willard van Orman Quine, has not sufficiently extended naturalism to arithmetic (making arithmetical truth contingent upon its utilization in scientific practice). As Putnam comments, "The philosophy of logic and mathematics is the area in which the notion of

'naturalizing epistemology' seems most obscure" (Putnam 1994a, 260). Furthermore, in this thesis, naturalism, like that elaborated by Kitcher (1984), is shown to be consistent natural realism. As Putnam writes, "Platonism is itself a research program" (Putnam 1979a, xii).

On the second score, by utilizing both Putnam's writing in favor of realism (1979a), and those that are supposed to break with them (e.g., 1981), it shall be suggested that he should not have abandoned his earlier view. In a nutshell, the realization of knowledge can depend upon agents' values, methods, and so on, but that does not force the abandonment of realism (provided one has the correct values and methods). Putnam's early view sets the criteria for when one can be a realist about a given domain (Section 8.2). This thesis provides a case where his early and later views are rendered consistent.

# PART I

## ANALYSIS OF THE DILEMMA

## 1. Abstract Objects

Though one is trained since at least Kant to separate questions of existence from conceptual issues (e.g., to separate existence from the concept of God), the connection can yield clarity.<sup>1</sup> For instance, discerning the ontological status of unicorns helps one define unicorn as mythical animals.

There are, nowadays, three generic ways to deal with the ontological status of abstract objects - nominalism, platonism, and conceptualism.<sup>2</sup> Nominalism is the doctrine that there are no abstract objects or universals; only individuals exist. According to Quine, nominalism is "the philosophy according to which there are really no universals at all" (Quine 1953, 118). For example, " $\epsilon$ " - the notation for membership - does not represent the abstract idea "membership" that has some mind-independent existence, but is a mark or inscription (Goodman 1972, 183).<sup>3</sup>

Platonism in arithmetic can be understood as assent to the existence of abstract objects; numbers, for a platonist, are outside of space, time, and are mind-independent (Quine 1953, 14). Brown writes of platonism, "*Mathematical objects are perfectly real and exist independently of us*. Mathematical objects are no different than everyday objects (pine trees) or the exotic entities of science (positrons). We don't in any way create them; we discover them" (Brown 1999, 11).

Finally, conceptualism is the doctrine that perception provides only particulars (tokens) and thought universals (types); abstract objects are outside of space but not time, and are mind-dependent (Quine 1953, 14). Quine writes, for example, that "[conceptualism] treats classes as constructions rather than discoveries" (Quine 1953, 125).<sup>4</sup>

In what follows, Lewis's (1986) attempt to define abstract objects is considered first. Second, nominalist and platonist solutions to the ontological problem of universals are considered. Finally, a conceptualist definition of abstract objects is advocated, which captures a feature of nominalism (abstract objects do not exist mind-independent) and realism (the truth-values of statements about abstract objects are recognition-transcendent).

### **(1.1) David Lewis' Analysis.**

Benacerraf assumes that reference is required for truth.<sup>5</sup> Having knowledge about a variety of tree, a planet, or nebula, usually means that such things exist. Simply transferring such

reasoning to the language of arithmetic takes the least work. One does not have to invent an ad hoc epistemology for arithmetic. There are things out there in the world; one discovers them and describes their different properties.

Maintaining a unified semantics means that the language of mathematics - which contains predicates, singular terms, quantifiers etc. - would require objects that are the reference of these singular terms. "Caesar" refers to Caesar (who did exist), and the number "13" denotes that arithmetical object (which for a platonist always exists) (Benacerraf 1973, 668). As Benacerraf puts it, abstract objects are required because of a "hermeneutics" of the language of mathematics (Benacerraf 1999, 36).

As Hart explains, in commenting upon Benacerraf's dilemma, "Just as truths of history require the existence of historical figures like Caesar, so truths of mathematics require the existence of prime numbers like 13" (Hart 1998, 2). Yet as Hart also observes that the requirement for a platonist ontology conflicts with the requirement for an empiricist epistemology (Hart 1998, 5). One becomes stuck with the peculiar idea that abstract objects exist mind-independently.

As Hart recalls, an idea which goes back to at least Frege, namely, that everything is either physical, mental or abstract. He contends that mathematical entities are abstract in order to maintain necessity, since he believes that

physical and mental entities are contingent (Hart 1998, 60). According to him, only abstract objects have necessity; so, for instance, Boyle's law<sup>6</sup> is not necessary if it is either physical (e.g., refers to the physical properties of things), or mental (e.g., is only an idea). It must be an abstract object to be necessary. As Hart remarks, however, "What is it to be abstract anyway?" (Hart 1998, 3)

In The Plurality of Worlds, David Lewis shows that one can draw a line between abstract and concrete objects in practice, though there is perhaps no hard and fast line between them (there will remain difficult cases). It should be kept in mind, however, that just because one cannot offer a definition as exacting as one may want, does not mean the term in question should be considered meaningless.<sup>7</sup>

Lewis cites five ways to define abstract objects and considers the first four: exemplification, conflation, negation, and abstraction.<sup>8</sup> Exemplification is the idea that one can define abstract objects by providing examples: "Concrete entities are things like donkeys and puddles and protons and stars, whereas abstract entities are like numbers" (Lewis 1986, 81). But, as he points out, "...[T]here are too many ways that numbers differ from donkeys *et al.* and we still are none the wiser about where to put a border between donkey-like and number-like" (Lewis 1986, 82). Citing examples alone will not provide the definition of "abstract object".

Roughly put, conflation is the idea that the distinction

between abstract and non-abstract entities can be drawn along the lines of the ontological debate between nominalists and platonists.<sup>9</sup> Lewis writes, "[T]he distinction between concrete and abstract entities is just the distinction between individuals and sets, or between particulars and universals, or perhaps particular individuals and everything else" (Lewis 1986, 83). Universals (e.g. "white") have problems that resemble those of abstract objects.<sup>10</sup> What is the ontological status of a universal? Where are they? How does one get to know them? How could anything (e.g., a particular) relate to a universal? This particular definition by conflation postpones the issue because it requires a definition of a "universal".

The negative definition is the idea that abstract objects can be defined by what they are not, for example: "[A]bstract entities have no spatio-temporal location" (Lewis 1986, 83). A negative definition may not decide hard cases: Wavicles and minds might be considered abstract objects and objective features of the world (they are non-local but in time).<sup>11</sup> In short, the negative definition is vague.

Finally, the definition by abstraction is the idea that abstract objects can be defined by how they are generated: "[A]bstract entities are abstractions from concrete entities" (Lewis 1986, 83). Abstract objects are abstracted in thought - like colours and shapes, so that they exist independently of other features. Yet an abstract object does not have a

separate existence. The definition by abstraction blurs the distinction between mental entities and abstract objects.

Lewis concludes his discussion with this negative statement:

I don't really know what is meant by someone who says that mathematical objects are abstract while donkeys, even other-worldly ones, are concrete...I listed four ways, by no means equivalent, in which such a statement might be meant... Whichever [way] is meant, I can more or less agree... (Lewis 1986, 111)

The definition of abstract objects can, however, be given gleanings aspects of the four strategies Lewis outlines: The number "3" is an abstract object (an example); abstract objects are like universals (conflation); they are not in space (though it could be in time when thought of) (the negative way); and finally, their distinguishing mark, they are abstracted from concrete things (abstraction) (Kitcher 1984, 70-1). In what follows, the definition by conflation (abstract objects are like universals) and abstraction (abstract objects are derived from concrete things) shall be utilized.

### **(1.2) Problems With Nominalism and Platonism.**

In accordance with the idea of conflation, the terms "abstract object" and "universal" shall be used interchangeably. In the following discussion, one shall assume that the world can be metaphysically divided up in different ways, where whatever is ontologically reified could be treated as a particular or universal (for example, a colour, idea, or

set could be treated as particulars or universals). The point of the discussion shall be to rehearse the problems with these doctrines writ large, as it were, as to bring into stark relief the conceptualist alternative. Moreover, the goal is to show that the conceptualist's position allows a better fit with one's natural ways of speaking and acting. Detailed refutations of nominalism and platonism occur in the chapters dedicated to those views (Chapters 6-7 & 4-5).

The nominalist faces problems of accounting for cognitive access if there are only individuals. One is able to pick out a chair only because one has familiarized oneself with the type "chair". Chair-type things have certain qualities that qualifies them as belonging to that designation. One could not pick out anything if there were no common properties (because all properties would be unique, too). Nominalism raises epistemological problems accounting for the similarity of things. A nominalized world would be unintelligible (more shall be said about the problems with nominalism in chapter 7). As Quine notes, however, types (kinds) are necessary to learning in the widest sense (Quine 1969, 129).

Nominalism is motivated by a perceived ontological worry - about positing too many entities - and epistemological problems - about access to abstract objects. Even if one did not adopt the extreme form discussed, at the very least, a nominalized world requires thinking of universals as illusions of sorts, i.e., they are eliminated, which is counter-

intuitive. As George Boolos contends, it is nominalism that flies in the face of common sense. Abstract objects are not as esoteric as nominalists may think. Abstract objects do not "twinkle" - they do not go out of existence when one erases the blackboard (Boolos 1998, 128). He writes, "But we twentieth-century city dwellers deal with abstract objects all of the time. We note with horror our 'bank balances'. We listen to 'radio programs': 'All Things Considered' is an abstract object [and so on]" (Boolos 1998, 128-9). He goes on:

It is thus no surprise that we should be able to reason mathematically about many of the things we experience, for they are already 'abstract'. It is very much a philosopher's view that the only objects there are are physical or material objects, or regions of space-time, or whatever it is that philosophers tell us the latest version of physical theory proclaims to be the ultimate constituents of matter. (Boolos 1998, 129)

The existence of infinitely many natural numbers seems to me not more troubling than that of infinitely many computer programs or the existence of infinitely many sentences of English...Irrealism about numbers seems no more tenable than irrealism about programs or sentences. It is an odd view, to say the least, that there are infinitely many programs but not, or only finitely many, natural numbers. (Boolos 1998, 129)

In a nutshell, Boolos rejects the nominalist strategy because: if common sense is one's guide, according to him, abstract objects neither pose the problems of an inflated ontology nor the problem of cognitive access. The underlying motivation for nominalism is thus questionable. Boolos concedes, however, his intuition is not conclusive.<sup>12</sup>

Going to the other end of the spectrum, platonism leads to an ontological infinite regress which entails a problem of

cognitive access.<sup>13</sup> Epistemologically, one could never have access to universals (in a world where there were only universals), because there would be no basic properties. In order to comprehend a universal one must be able to compare common properties (e.g. chairs share a certain set of properties). Yet one could never make the comparison because the properties - also universals - would be ontologically, hence semantically, types, *ad infinitum*. One could never learn the basic bits which make up things (because ontologically there may not be any semantic building blocks). There would be an infinite cognitive regress. Even if one does not adopt the extreme form of platonism about universals discussed, one is still left with reifying, for example, the existence of numbers as mind-independent abstract objects and perhaps, thinking of individuals as illusions, all of which is counter-intuitive.

So far in the discussion, it has been assumed there is a correlation between one's ontological commitments and what it is possible to experience. Quine and Armstrong, however, advise one to separate ontology from semantics.<sup>14</sup> Also, and more generally, John Leslie, a contemporary neo-platonist, does not think semantical analysis can resolve ontological disputes. He remarks, "You can set up more or less what linguistic techniques you please for dealing with this messy area [of ontology]. Don't imagine you are thereby settling whether Neo-platonism is right" (Leslie 1989, 173).

Yet there is a moral to the discussion since both positions, it has been suggested, are counter-intuitive; they do not seem to capture how we in fact think of abstract objects. When one gets lost in nominalist or platonist ideologies, one should turn to practice. As Peter Strawson, who wanted to give common sense a philosophical priority, remarked:

The rejectionist's [e.g., conceptualist's] task, as he sees it, must be to replace such pictures [e.g., of the nominalist and platonist] with an account of the natural realities which underlie and legitimize, or give sense to, those perfectly acceptable ways of talking about meanings, concepts, necessities, etc. which I have mentioned. (Strawson 1983, 75)

One's ontology should not *a priori* exclude what is required to get cognition off the ground. Both extreme nominalism and platonism fail because they posit a world where cognition is impossible (according to what is empirically known about cognition, Section 2.3).

As Armstrong notes, one cannot reduce the one (nominalism or platonism) to the other (Armstrong 1978a, 80; 1978b, 2-3). One needs both universals and particulars. On the one hand, to avoid the epistemological problem of the nominalist would require the ability to grasp the identity of different instances of particulars (i.e., universals). On the other hand, to avoid the problem of cognitive access faced by the platonist about universals requires nominalism for a privileged class of properties (i.e., particulars with the cognitive ability to access them). In sum, one has relearned

Kant's dictum.<sup>15</sup> Paraphrasing (and reversing the order): Cognition with particulars but no universals is blind. Cognition with universals but no particulars is empty. Nominalism and platonism entail cognitive blindness and emptiness, respectively.

Robert M. Francescotti, for example, attempts to explain cognition in terms of the relationship between higher- and lower-order thoughts. Saying, "there is a book on the table" is empirical (a lower-order thought). Yet remembering seeing the book on the table is not as immediate (Francescotti 1995, 239). Like memories, thoughts of greater abstraction are further removed from experience (they are higher-order thoughts). The relationship between thoughts of different orders, the lower and higher ones, frames the model of how he explains cognition.

According to Francescotti, cognitive development consists of a negotiated, organic process between nominalistic-type sensory inputs, which are close to experience (bottom-up processing) and universal-type abstract concepts which are not (top-down processing). He writes:

...[T]he common consensus today is that perceptual activity is more accurately viewed as the result of concurrent and interactive bottom-up and top-down processing. Moreover, since the background information that influences perception has propositional content, it approaches the status of genuine beliefs and expectations. (Francescotti 1995, 241)

Conceptual schemes (top-down processing) are wrung from the tides of experience (bottom-up processing), as it were. Both

concepts and data are required in order to be able to think in the first place.<sup>16</sup> Francescotti's account is consistent with the conceptualist who claims that concepts are thought-dependent.

### **(1.3) Conceptualism: The Best of Both Worlds.**

Quine writes:

Tactically, conceptualism is no doubt the strongest position of the three [nominalism, conceptualism, platonism]; for the tired nominalist can lapse into conceptualism and still allay his puritanical conscience with the reflection that he has not quite taken to eating lotus with the platonists. (Quine 1953, 129)

Conceptualism tallies with common sense. Abstract objects do not exist mind-independently. They are a type of mental object. The degree of generality of a mental object can be gauged, as it were, from its distance from the initial perception of the concrete object (the greater the distance from perception the more abstract it is). For instance, one can think of a particular chair (that would be more abstract than an actual chair, which is in space), but not as abstract as the concept "chair" (which has a greater generality ranging over all tokens of the chair-type).

There was a reason, however, that had driven Hart, for instance, away from views of the conceptualist's ilk. Abandoning platonism, according to Hart, is tantamount to the loss of arithmetical necessity.<sup>17</sup> One may recall that in his elucidation of Benacerraf's dilemma, the necessity of a statement required that its singular terms refer to mind-

independent objects. If one can maintain realism, without positing mind-independent abstract objects, however, the conceptualist can have the best of both worlds. Is Hart right to think that the loss of a platonist conception of abstract objects takes realism with it?

Following G. Kreisel, Michael Dummett and Putnam, for instance, agree that what is at issue is objectivity, not in terms of the essence of existence (Kreisel 1959, 138; Dummett 1978, 228; Putnam 1979a, 70). Dummett writes, "'The point is not the existence of mathematical objects, but of mathematical truth'" (Dummett 1978, 228).

Dummett conceives of realism in terms of the recognition-transcendence of the truth or falsity of a statement, not entities (because he thinks that objectivity is what is at issue in considering mathematical truth) (Dummett 1978, 146).<sup>18</sup> Yet he notes that it is in virtue of an independent reality that a statement would be rendered true or false for a realist. Conversely, according to Dummett, anti-realism is the idea that bivalence is only to be maintained for the class of disputed statements when there is a way, in principle, of deciding them.

Putnam concurs with Dummett's definition. In "What is Mathematical Truth", he contends that traditional realism consists of two claims: (1) statements of a given theory are true or false because of (2) something external to the mind (Putnam 1979a, 70). The change from entity-realism to that of

epistemic-realism, does not of itself dissolve the ontological problems that surrounds abstract objects' mind-independent existence.

Putnam demonstrates, however, that one can be a realist without assenting to the mind-independent existence of abstract entities. His strategy goes further than Dummett in emphasizing objectivity over existence. Putnam writes:

Not only are 'objects' conditional upon material objects; they are, in a sense, mere abstract possibilities. Studying how mathematical objects behave might better be described as studying what structures are abstractly possible and what structures are not abstractly possible. (Putnam 1979a, 60)<sup>19</sup>

Putnam does not think that platonism is the only way to account for arithmetic's necessity (Putnam 1979a, 72).<sup>20</sup> For Putnam, arithmetic's objectivity is not existential but modal; it is about what sort of structures are (and are not) possible. Putnam's idea that objectivity does not require mind-independent objects is developed in Chapter 8.

Neil Tennant agrees with Dummett (and Putnam) that entity-realism must be separated from other varieties (Tennant 1987, 3; 1997, 19-20). He demonstrates that ontological and semantic realism can be related to each other in four ways. His analysis can be extended in order to distinguish between ontological, epistemic and semantic realism. Different questions are posed, respectively: What is there (ontology)? Is the statement *X* true or false (epistemic)? What does the term *Y* mean (semantics)? The three types of realism can be related to each other in six ways that are coherent.<sup>21</sup> "+"

shall indicate realism and "-" anti-realism. Consider the first possibility.

|    | Ontology | Epistemics | Semantics |
|----|----------|------------|-----------|
| 1. | +        | +          | +         |

One can be a realist in all domains. One could be an ontological realist about the moon, statements pertaining to it, and the meaning of the term "moon". Putnam and Devitt hold some such view for natural kind terms (Putnam 1979b, 30, 267; Devitt 1981, 200).<sup>22</sup>

Also one could be a realist in one domain and not the other.

|    | Ontology | Epistemics | Semantics |
|----|----------|------------|-----------|
| 2. | +        | +          | -         |
| 3. | +        | -          | -         |

The second and third cases are forms of traditional metaphysical realism with respect to the recognition-transcendence of either objects or truth-values. For instance, the second case is the view that one maintains ontological and epistemic realism, and semantic anti-realism.<sup>23</sup> The third case is the view that one is a realist about entities, but not statements about them or their meanings - another variety of metaphysical realism (Section 8.2).

|    | Ontology | Epistemics | Semantics |
|----|----------|------------|-----------|
| 4. | -        | -          | -         |

The fourth case is the idea that one can be an ontological, epistemic and semantic anti-realist. For example, one may be an anti-realist about the existence of chess pieces (an

ontological question), the rules of chess (an epistemic question), and the definition of chess pieces (a semantic question).<sup>24</sup>

|    | Ontology | Epistemics | Semantics |
|----|----------|------------|-----------|
| 5. | -        | +          | -         |

The fifth case is the idea that one can be a ontological and semantic anti-realist while an epistemic-realist (e.g., the correct definition of "3" is undecidable).

|    | Ontology | Epistemics | Semantics |
|----|----------|------------|-----------|
| 6. | -        | +          | +         |

Finally, the ideal position (which shall be defended in Chapter 8) is the sixth. The sixth case is the notion that one adopt ontological anti-realism (e.g., "3" does not have a mind-independent existence) but an epistemic and semantic realism (about say, the truths and definitions of the number "3").

Maddy, for instance, regards the debate concerning the ontological status of abstract objects as: an "intramural squabble between metaphysicians, and a squabble in which it isn't clear, what, if anything, is really at stake" (Maddy 1991, 158). She writes:

The new consensus, as I see it, is this: some form of ontological tinkering can defuse Benacerraf's dilemma without sacrificing standard mathematics...the most basic form of epistemic access to these regions is perceptual. (Maddy 1991, 156-7)

The consensus Maddy refers to masks a great deal of disagreements (Maddy 1991, 156). Dummett concurs, however,

"The existence of abstract objects was never more than a pseudo-problem, and, when we have recognized it as such, the real problems remain" (Dummett 1991, 240). As Benacerraf had forecasted, "Given such an account [that denies the first horn of the dilemma], the task of accounting for mathematical truth would remain" (Benacerraf 1973, 669).

The moral: the conceptualist can make good, at least in principle, on Quine's challenge (i.e., one can avoid both nominalism and platonism and account for necessity). Platonists do not have the monopoly on realism, and therefore, those who deny the mind-independent existence of abstract arithmetical entities are not forced to adopt epistemic or semantic anti-realism. To solve the issue one must get a sense of conceptualism countenancing realism (see the discussion in chapter 8).

One should notice, however, that Benacerraf's dilemma takes on a new form: can one, at the same time, be an epistemic-realist about arithmetical truth and maintain a commitment to the causal constraint requirement?<sup>25</sup> In the next chapter, the second horn of the dilemma is considered.

## 2. The Causal Constraint

As Crispin Wright, a critic of the causal constraint requirement for arithmetic remarks, "The most simple causal theory of knowledge would try to see the knowledge that *P* as an informational state induced in the mind of a sufficiently intelligent and perceptive, appropriately placed subject by the very state of affairs that *P*" (Wright 1983, 85). For example, one can judge the statement "There is a book on the table" simply by looking at the table and seeing that there is a book on it or not. One causally interacts with the book and table by seeing them.

In this chapter, Mark Steiner's attempt to precisely define the causal constraint requirement is considered; second, accounts of knowledge that reject that requirement are considered and eventually rejected. Finally, accounts of knowledge that accept the causal constraint are discussed. Quinean naturalism is defended by showing that one can accept the causal constraint requirement for the principles of arithmetic, the *a prioriticity* of what one derives from them,

and the necessity of the entire theory.

### **(2.1) Mark Steiner's Analysis.**

Benacerraf looked to the causal constraint requirement as a way of making the link between belief and truth because of a view popularized at the time, in particular by that of Alvin Goldman (Burgess and Rosen 1997, 36). Goldman linked the belief that  $P$  to the fact that  $P$  by a causal chain, terminating in the state of affairs that induces the mental state  $P$  (which could include perception, memory, reconstruction by inference, or a combination of these ways) (Goldman 1967, 304, 358, 370). Benacerraf writes, "[I]n mathematics it must be possible to link up what it is for  $p$  to be true with my belief in  $p$ " (Benacerraf 1973, 667). He evokes the causal constraint to link belief with truth, which, he claims, is required for both reference and knowledge (Benacerraf 1973, 671).

The causal constraint requirement shall be discussed by drawing upon examples from the domains of both semantics and epistemics. The reason is as follows.<sup>1</sup> The causal constraint requirement is supposed to link a name with its bearer so that one can be sure one has the correct definition of a term. Similarly, the causal constraint requirement is supposed to link an appropriately constructed statement with a state of affairs so that one can know if the claim is true.<sup>2</sup> For example, if the meaning of the term "gold" requires causal interaction with gold, so do some knowledge claims about it.<sup>3</sup>

But how should the causal constraint be defined?

In a study of the causal constraint, Mark Steiner considers five formulations beginning with the simplest one:

- (0) One cannot know that  $p$  unless  $p$  causes this knowledge (or belief) that  $p$ . (Steiner 1973, 59)

The sentence "snow is white", however, is not the cause of snow being white (or "snow being white"). Using a word is different from mentioning it. For example, a dog may be friendly and fluffy (using the word "dog"), but "dog" is neither friendly nor fluffy (mentioning the word). Steiner notes that substituting an English sentence for the second occurrence of " $p$ " is a violation of the use-mention distinction.

Steiner then considers a reformulation of the causal constraint requirement that does not violate the use-mention distinction:

- (1) One cannot know that  $p$  unless the fact that  $p$  causes one's knowledge (or belief) that  $p$ . (Steiner 1973, 59)

Reference to facts, however, raises the issue of their existence. If facts exist, they do not have any more causal efficacy than abstract objects (one cannot, for instance, weigh facts). Boiling water can burn one's hand, but the fact "that water is boiling" cannot. Facts are causally inert.

Steiner's third formulation of the causal constraint requirement thus avoids mentioning facts:

- (2) One cannot know that a sentence  $S$  is true, unless  $S$  must be used in a causal explanation of one's knowing (or believing) that  $S$  is true. (Steiner 1973,

60)

As Steiner explains, "Now it is generally accepted that, in order to provide an explanation, a sentence must at least be true. And the Tarski-platonist interpretation of mathematical propositions is the only such interpretation known to guarantee the truth of the axioms and theorems" (Steiner 1973, 61). What Steiner has in mind is the idea that a statement about "p" is true because of p, which is consistent with arithmetical platonism.

The causal theorist can restate her claim so that the issue becomes explicitly one about the existence of objects.

- (3) One cannot know anything about  $F$ 's, unless this knowledge (or belief) is caused by the  $F$ 's, or some of the  $F$ 's, etc. (Steiner 1973, 61)

Steiner claims that this formulation raises numerous problems: first, is the sentence, "There are no  $F$ 's" about  $F$ 's? The question leads to paradox. One cannot know that there are no  $F$ 's unless there are some  $F$ 's.<sup>4</sup> Second, causal efficacy is imputed to  $F$ 's, that is, to all sorts of abstract objects. Finally,  $F$ 's are treated as agents (as if they intended to cause certain effects).

Steiner considers a final formulation of the causal constraint.

- (4) One cannot know anything about  $F$ 's unless this knowledge (belief) is caused by at least one event in which at least one  $F$  [or one correlate of  $F$ ] participates. (Steiner 1973, 62)

According to him, number and functions are excluded since they cannot participate in events. Abstract objects, he contends,

are excluded.

Also, Steiner thinks that (4) cannot be used to account for say, a fossilized footprint since an animal did not participate in the event which led to its creation. Steiner writes, "What happened is that an event led, not to another event, but to a condition; and the condition was partly 'responsible' for another event, an event which in turn caused certain beliefs to form in the mind of the zoologist" (Steiner 1973, 62).

Steiner contends that it is arbitrary to limit the causal constraint to a certain type of knowledge, that is, to exclude its application to abstract objects (Steiner 1973, 63). His conclusion is thus that one should accept (2), and reject the other accounts (e.g., because they exclude causal knowledge of abstract objects). According to him, the most plausible version of the causal constraint requirement, i.e., (2), is consistent with mathematical platonism, and the one most antagonistic to it, i.e., (4), is implausible (Steiner 1973, 63).

Yet the issue of cognitive access to abstract objects will arise if, as platonists contend, the truth of a statement requires it. To ascertain the truth-values of arithmetical statements requires causal interaction with numbers. The problem of cognitive access to arithmetical objects is thus a problem for Steiner's preferred definition as much as the subsequent versions he considers.

Yet one can embrace a causal constraint requirement that applies to abstract objects without embracing platonism if they have physical instantiations. Mathematical platonism is not the only way to guarantee the truth of statements that contain claims, for example, about numbers.

According to the empiricist, numerical knowledge is acquired from events in which the properties of physical objects participate. One's knowledge of numbers is, in principle, causally constrained. As Brown comments upon the empiricist's strategy for arithmetic: "And John Stuart Mill held that numbers are a kind of very general property that objects possess. A four-legged, blue, wooden chair has the property *four* just as it has the properties *blue* and *wooden*" (Brown 1999, 55). The idea is to relate a number (a type) to a property of a physical object (a token) mediated, causally, by an act of collecting (an event).

Steiner's claim against (4), therefore, evaporates: knowledge of abstract objects does not require causal interaction with them, but with their instantiations. Also, (4) can be stretched to account for fossils (by blurring the distinction between conditions and events). That is, fossils provide knowledge about prehistoric life because they are caused by events in which the life forms in question participate. Finally, the idea that the instantiations of numbers are being treated as agents is not a fatal criticism. The language is figurative: numerical instantiations do not

have intentions anymore than the fire that burns one's hand. That is, being causally efficacious does not imply agency. (4) is superior to Steiner's choice because it makes the role of objects explicit. Henceforth, in discussing the causal constraint requirement, (4) shall serve as the definition.

According to the empiricists, for example, the fact that there are no *F*'s is caused by an event, e.g., seeing no *F*'s (in which no *F*'s participate, as it were), which is consistent with Kitcher's account of how one acquires the concept "0" (Section 6.3). According to him, numerical knowledge is derived from events, like collecting physical objects (Section 6.3).

Kitcher, one may want to notice, contends that numbers are predicates of collecting operations (Section 6.3). His view can be explained by looking to what Aristotle says about sound. As Aristotle explained, "[B]oth the sound and the hearing so far as it is actual must be found in that which has the faculty of hearing...Now the actuality of that which can sound is just sound or sounding..." (426a5-10). In broad outline, sound is produced by the activity of an agent (i.e., hearing) interacting with some object in the world (a correlate of sound). Sound is best understood as a verb (sounding), similarly numbers should be understood as being formed by an activity (i.e., numbering). Numbers are produced by collecting operations (in the same way sound is a matter of hearing). Both sound and numbers are the result of an agent

engaging in some sort of activity (hearing or counting, respectively) which requires interacting with their correlates.

For both Mill and Kitcher, numerical knowledge is causally constrained insofar as it is an activity that is acquired from practices that require interaction with physical objects (Section 6.2). Attempting to decide if numbers are properties (Mill) or predicates (Kitcher) is moot for the present purposes because both accounts fulfil the causal constraint requirement (though the reader should keep in mind that it is Kitcher's view that shall be subsequently defended). Kitcher's account fits the fourth formulation. For example:

One cannot know the second axiom of Peano number theory [every number has a successor] unless this knowledge (belief) is caused by at least one event (e.g., an act of collecting) in which some physical objects (number-correlates) participate.

The axiom can be formulated as a statement which makes reference to abstract objects, namely, numbers (Section 6.3).

### **(2.2) Against The Causal Constraint.**

Yet are the Peano axioms causally constrained as so defined? In epistemological discussions, defeasibility is essentially the idea that knowledge could be otherwise (and indefeasibility is the idea that knowledge could not be otherwise). The causal theory of knowledge has been part of a debate that includes epistemic defeasibility. The causal theory of knowledge shall, therefore, be discussed in

conjunction with the issue of epistemic defeasibility. There are four options when considering appropriate arithmetical statements (one that could have a truth-value). The Peano axioms can be: (1) non-empirical and indefeasible; (2) non-empirical and defeasible; and (3) empirical and defeasible. The goal will be to reject the first three options and defend the final option: (4), the Peano axioms are empirical and indefeasible.

The view that arithmetical knowledge is non-empirical and indefeasible is the first option and defended by Wright and Lewis. According to Wright, the causal constraint requirement does not always allow one to distinguish between: (1) the belief in *P* and (2), the fact of *P* (Wright 1983, 84-5, 92, 103).

He provides two counter-examples to the causal constraint requirement of which only the first shall be considered here. He asks one to suppose that one was watching a television broadcast of a sports event where the commentator announces that Borg has broken a string on his tennis racket. The match, however, had been delayed; one was watching a game from a previous year.

One's knowledge - "Borg has broken a string on his tennis racket" - is causally disconnected from the event (one did not see Borg practicing indoors). The truth of *P* is not conferred by an actual state of affairs *P*. Wright's counter-example to the causal constraint requirement, he contends, also plagues

claims about the future and the laws of nature (because one does not causally interact with future events in the present).<sup>5</sup>

According to Wright, in the case of arithmetical knowledge the causal constraint requirement cannot bridge the gap between belief and knowledge. He remarks:

The truth-conferring 'circumstances' and 'states of affairs' talked of have evidently to be interpreted as complex physical, spatio-temporally located entities, if they are to play the relevant sort of role in causal explanation of the genesis is of a knowing subject's beliefs. (Wright 1983, 85)

As he points out, it has not been explained how causal interaction with abstract objects would lead to knowledge (Wright 1983, 86).<sup>6</sup> He surveys several variations on the formulation of the causal constraint by Steiner and suggests the following as the most plausible version (the one endorsed in Section 2.1):

None of one's beliefs constitute knowledge about  $F$ 's unless any complete causal explanation of these beliefs must advert to at least one event in which at least one  $F$  [or one correlate of  $F$ ] participates. (Wright 1983, 92)

According to Wright, knowledge about abstract objects are excluded from the causal constraint requirement. Since abstract objects are obviously not located in space-time, the problem arises of how one can causally interact with them.<sup>7</sup> A dilemma parallel to Benacerraf's arises: either abstract objects do not exist (in which case there is no mathematical knowledge) or they are unknown. Both options, says Wright, are unacceptable.

Furthermore, empirical knowledge, according to Wright, is defeasible (a further fact could change one's mind), yet arithmetical knowledge seems absolute (no further fact could change one's mind). Wright remarks:

An immediate and simple countervailing thought is: can room be made by such a picture for any form of knowledge *a priori*? More specifically, what room can be made for knowledge *a priori* of necessary truth?...[W]hat is distinctive about the acquisition of any piece of knowledge *a priori* is precisely that it has no essential causal antecedents save a training in certain relevant concepts. (Wright 1983, 95)

He goes on:

Even assuming the global correctness of some form of the causal connection, however, the real problem is to cope with the apparently *a priori* character of pure mathematics, and this problem would remain even after a thorough going nominalist reconstruction. (Wright 1983, 96)

According to him, Frege has an alternative to an epistemology of arithmetic that relies upon the causal constraint requirement. Frege asks about the truth or falsity of statements dealing with abstract objects whose existence is settled by a syntactic criteria. Wright contends that the onus lies with the nominalist to explain why a solution does not rely on the syntactic criteria (Wright 1983, 87, 96). In Chapter 4, the appeal to a syntactic criteria as espoused by Wright will be criticized.<sup>8</sup>

According to Wright, a "related" challenge arises in justifying the truth of a statement or the meaning of a term (Wright 1983, 97). As he says, "[I]n short, that a causal conception of reference is called for only where it is

appropriate to seek a causal theory of knowledge" (Wright 1983, 102). According to him, however, the causal constraint applies to some of, but not all, object-directed thought. He claims, more generally, that it does not apply to arithmetical knowledge. As he remarks, "Thus the platonist must, sooner or later, fight a potentially critical battle in the theatre of the philosophy of mind" (Wright 1983, 103).

In outline, David Lewis concurs with Wright. Lewis claims that though the causal constraint applies to some of our knowledge (e.g., some of our scientific knowledge), it is a "hasty generalization" to think it applies to all of it (Lewis 1986, 110, 112). According to him, for instance, knowledge could fall into one of three categories: knowledge that obtains in a possible world (e.g., life on earth is carbon based); knowledge which is not necessary in every possible world (e.g., a world where life is silicon based); and finally, knowledge which is necessary in every possible world (e.g., arithmetical knowledge) (Lewis 1986, 112-3).<sup>9</sup> According to Wright and Lewis, to repeat, arithmetical knowledge is non-empirical and indefeasible.

The view that arithmetical knowledge is non-empirical and defeasible is a second option and is defended by Hartry Field.<sup>10</sup> A body of knowledge that is not causally constrained may be constituted by stipulations (as with, say, statements about the rules of chess) (Kripke 1972, 39). Field claims that the mathematical belief in  $P$  has to do with being derived from

axioms:

For as mathematics has become reduced to the truth of a smaller and smaller set of basic axioms; so we could explain the fact that they have been logically deduced from axioms, if we could just explain the fact that what mathematicians take as axioms tend to be true. (Field 1989, 231)

He writes, however:

[T]here is a big gap between the consistency of an axiomatic theory [like arithmetic] and its truth. In the case of physics we can presumably fill this gap at least in sketch: we can sketch the route whereby the assumed properties of, say, the electromagnetic field lead to various observable physical phenomenon, and thereby affect our perceptual beliefs, and thereby indirectly affect our beliefs about the electromagnetic field. (Field 1989, 232)

According to Field, the causal constraint does not apply to arithmetic which is defeasible (Chapter 7).<sup>11</sup> At any rate, the most important challenge to causal constraint proponents is the possibility of *a priori* knowledge, which is often claimed characterizes arithmetical statements (regardless of whether they are considered indefeasible or defeasible). So, in the next section, the idea that all of arithmetical knowledge is non-empirical shall be criticized.

### **(2.3) In Defense of The Causal Constraint.**

The third option is the idea that arithmetical knowledge is empirical and defeasible which is a consequence of Quine's dissolution of the analytical-synthetic distinction in "Two Dogmas of Empiricism" (1953). He denies the possibility of *a priori* knowledge (as it has been traditionally understood).

As Quine pointed out, the distinction between analytical and synthetic statements mirrors Hume's separation of the

relation of ideas, on the one hand and, on the other hand, matters of fact (Quine 1953, 20). On his analysis analytical knowledge requires understanding the truth of a statement in virtue of the meaning of its parts.<sup>12</sup> Yet on his account, all statements are empirical because one cannot separate their linguistic and factual components (Quine 1953, 42).

Quine claims that the loss of distinction between non-empirical and empirical knowledge has the following consequence. According to him, all statements are synthetic and open to revision (Quine 1953, 20, 42, 43).<sup>13</sup> A revision is not a change in the use of language (where two words exchange referents). A revision implies a substantial change: e.g., water is H<sub>2</sub>O. If all statements are open to revision, the possibility of analytic statements is even more implausible. One cannot maintain the synonymy of terms if their meanings can be revised. Moreover, according to him, all of knowledge would be empirical to different degrees by tracking down the justificatory origin of causal networks. He writes:

One can ask in the same [empirical] spirit, how we developed our religious talk, and our talk of witchcraft, and our talk of analyticity, and logical modalities. If we managed to reconstruct these causal chains of language and learning, we would find that here and there the learner had made a little leap on the strength of analogy or conjecture or confusion; but then the same seemed to be true of our learning to talk of bodies. (Quine 1973, 136)

Quine's point is consistent with his epistemological holism (see Section 3.1). Just as all concepts are acquired empirically so too are all experiences conceptually

contaminated. Therefore, all of knowledge is empirical in the sense that it is so at its base. All knowledge, however, is also non-empirical in the sense that there are no pure experiences that are not shaped in one way or another (see the paraphrase of Kant in Section 2.2). In slightly radical terms, all of theory and data are part of a form of life.

The two most important objections to Quine's picture can be gleaned from Wright and Lewis. The problem simply amounts to this: contrary to what Quine seems forced to claim in the wake of his "Two Dogmas of Empiricism" (1953), arithmetical knowledge does seem *a priori* and indefeasible. How can the statement " $5+7=12$ " be empirical or revisable? In defending Quinean naturalism, an attempt shall be made to explain why extending his views to arithmetic are not counter-intuitive. There are two reasons: he revises the notion of *a priori* knowledge (he does not abandon it) and, second, indefeasibility need not be given up by the naturalist.

On the issue of *a priori* knowledge, one may recall that Kant (1781) defined the *a priori* against experience:

We shall understand by *a priori* knowledge, not knowledge which is independent of this or that experience, but knowledge absolutely independent of all experience.<sup>14</sup>

Devitt notes, for instance, that any naturalist view, like his own or Quine's, must, therefore, reject the *a priori* (Quine 1953, 20). Devitt says:

[T]hat there is only one way of knowing, the empirical way and that it is the basis of science...So I reject '*a priori* knowledge'. (Devitt 1996, 2)

If there is to be a priori knowledge, then, at least, there must be some way of justification other than the empirical way. (Devitt 1996, 34)

Quine and Putnam, however, have suggested that the notion of a *a priori* knowledge should be reformulated (Quine 1960, 55, 65-9; Putnam 1979b, 40): as knowledge which is far removed from experience but not completely independent of it. Quine calls the naturalized notion of a *a priori* knowledge "stimulus-analytic" (Quine 1960, 55, 65-9). He writes, "[A] sentence is analytic if everybody learns that it is true by learning its words. Analyticity, like observability, hinges on social uniformity" (Quine 1973, 79). That is, he suggests that a priori knowledge includes those statements that have been well established, solidified. He, therefore, suggests drawing a "rough line" between the empirical and non-empirical (Quine 1973, 80). A *a priori* statements are more orthodox than empirical ones.

Similarly, Putnam has suggested that one needs to view the distinction between analytic and synthetic statements as a "continuum" (Putnam 1979b, 40). According to him, for example, assenting to "there is a book on this table" is empirical, i.e., the causal constraint requirement holds (Putnam 1979b, xiv, 41).<sup>15</sup>

In the wake of Quine, therefore, one can still draw a distinction between analytical and synthetic statements though the distinction is one of degree. For example, assenting to "the sun will rise tomorrow" is further removed from immediate

experience, but still empirical since it requires an inductive inference (e.g., assuming that the sun's participation in past events, all things being equal, shall reoccur in the future).<sup>16</sup>

Not all scientific knowledge, however, allows for causal verification. For instance, Michael Resnik remarks:

Dirac posited anti-matter in attempting to make physical sense of the negative square roots in the equations of special relativity. Group-theoretic reasoning led contemporary particle physicists to posit certain bosons. Yet it is hard to think of any causal process involving anti-matter or bosons that would 'appropriately' connect them with the physicists who posited them.<sup>17</sup> (Resnik 1997, 191)

Yet these are perhaps cases of abduction where the objects are inferred because they serve an explanatory function in a theory, which is, per se, causally constrained (Chapter 3).

Some recent work in cognition supports this idea of an *a priori* knowledge that designates statements removed from experience by degree. As Sam Inglis says, "[E]xperiential dependence [is] a matter of degree" (Inglis 1996/7, 215). He explains:

Some perceptual element features in the causal history of every instance of knowledge; apriority, I suggest, is a matter of filtering out particular experiential causes which are pertinent to whichever epistemological question we are concerned with in a given case. (Inglis 1996/7, 222)

Some beliefs (e.g., in induction) are required for intelligible perceptual experience in general. According to Inglis, the propositional content of conceptual schemes, for instance, are *a priori*. For example, the statement "objects

are extended" counts as a *a priori* knowledge. A *a priori* truths which are necessary for possible experience may be acquired at a young age, but once one has them they are indispensable for higher-order thought (i.e., thoughts of greater abstraction).

As George Rey notes, the naturalist's *a priori* differs from traditional formulation. Yet if one can find an example of a *a priori* knowledge - that is, a statement whose justification does not require consulting the tribunal of experience - it would have been empirically established. As he explains, "...[W]hether or not there is a *a priori* knowledge is an empirical issue..." (Rey 1998, 25). He provides a lengthy example, but a shorter one will suffice.<sup>18</sup>

If (1), John is to the right of Tom, and (2), Tom to the right of Betty, one can deduce, (3), that John is to the right of Betty.

The third statement is a *a priori* because its truth depends on the relations in the first two propositions, not on experience. Of course, to understand what it means to be "to the right of" does require experience (similarly, to understand a logical law one has to learn it). The conclusion of the example, though not independent of experience, is well established; it is far removed from experience. As Rey remarks, "We thus arrive, perhaps with some irony, at a defense of the very *a priori* knowledge Quine and Putnam abjure, but by using the very empirical means that they favour" (Rey 1998, 40 and 39).<sup>19</sup> Quine and Putnam, however, would agree.

The application of the causal constraint can be circumscribed as follows:

One cannot know anything about *F*'s unless this knowledge (belief) is caused by at least one event in which at least one *F* [or one correlate of *F*] participates. The proximity between *F*'s and a *F* must be close, as judged by practitioners in the relevant field.

The causal constraint requirement holds for knowledge about a fossil because zoologists would assent to it. Claiming that the Peano axioms are empirical generalizations is consistent with practice. Kitcher's account, for example, is supposed to be an accurate description of how the foundations of arithmetic are acquired (Chapter 6). Mathematicians, presumably, would not assent to the causal constraint requirement applying to most arithmetical statements because it conflicts with practice: that is not, for example, the way they do long division (Section 3.3). The causal constraint requirement does not hold for the revised version of *a priori* knowledge.

On the issue of necessity, logico-mathematical knowledge has been thought to be - in addition to being *a priori* - eternal (and empirical ones to be tentative) (Tennant 1997, 406, 434). Both Field and Devitt, for instance, claim that the naturalist must reject *a priori* knowledge because all of knowledge is deemed defeasible.<sup>20</sup>

Yet even after one has been convinced by Quine that all of knowledge must be empirical, one need not accept that it is defeasible. The view that arithmetical knowledge is empirical

and indefeasible is defended, for example, by Maddy. Her argument is considered in Chapter 5, so it shall not be rehearsed here. Suffice it to note that Maddy's view is in the tradition of the scientific realist, which has been defended by Putnam and Kripke for natural kind terms.<sup>21</sup> An example from science can illustrate the fourth option. According to the scientific realist, "electricity":

is connected by a certain causal chain to a situation in which a *description* of electricity is given, and generally a *causal* description - that is, one which singles out electricity as the physical magnitude responsible for certain effects in a certain way. (Putnam 1979b, 200)<sup>22</sup>

Empirical knowledge can, in principle, be necessary (Kripke 1972, 35, 38).

It has been argued that Quinean naturalism provides the warrant for extending naturalism to arithmetic. Quine's views have been defended by showing, first, that he revises the notion of *a priori* knowledge and, second, does not have to abandon arithmetical necessity. A defense that one can embrace both epistemic-realism for arithmetic and maintain the causal constraint for its axioms is provided in Chapter 8. The authors of recent attempts to account for arithmetical truth, however, often assume the validity of either one, or both, of the horns of Benacerraf's dilemma. Recent attempts to account for arithmetical truth, therefore, shall be criticized in relation to Benacerraf's dilemma. The indispensability argument is considered in the next chapter.

### 3. The Indispensability Argument

The indispensability argument, roughly put, is the idea that mathematics is true because it is indispensable to scientific descriptions which are already taken to be so.<sup>1</sup> The idea is captured by C. S. Peirce's notion of "abduction", which could be formulated thus: if our best scientific theories of  $q$ , presupposes the existence of  $p$ , then observations of  $q$  gives us good reason to believe  $p$ .<sup>2</sup> Truth, from the pragmatist point of view, is what works within the totality of the collective enterprise of science. Since mathematics is part of the scientific enterprise, it inherits truth, as it were. It should be pointed out that since the views of Quine and Putnam are to be developed as a solution to Benacerraf's dilemma, the discussion will be lengthy.

In the first two sections, Quine's and Putnam's pragmatism is considered. In the third section, reasons to reject the indispensability argument are outlined and criticized. In the final section, a revised version of the indispensability argument is defended.

### **(3.1) Quine's Pragmatism.**

According to Hans Reichenbach, epistemology has two features. One aspect requires describing how knowledge is acquired. The other aspect requires criticizing practice, i.e., laying out how it ought to be justified (Reichenbach 1938, 3). Reichenbach's distinction parallels a separation between the context of discovery from justification that goes back at least to Frege. According to Frege, one must separate how one discovers X (the descriptive task) from how one justifies X (the critical task) (Frege 1953, sec.3).

One must also distinguish between the first-level of justification which concerns the content of a discipline.<sup>3</sup> At the first-level of justification an epistemological account must be faithful to practice otherwise it could affect the content of a discipline (which of itself, is usually considered a warrant for emending one's epistemology).<sup>4</sup>

The second-level of justification concerns the foundations of the first principles or methods required for practice. At the second-level, different types of justification for arithmetic, both platonist and formalist, make "no difference to their [mathematician's] practice" (Maddy 1990, 3). That is, one can offer an epistemological account at the second-level of justification that is separated from how the first principles of knowledge are acquired. Whether one accepts the Peano axioms because they describe the nature of mind-independent numbers or as conventions makes no

difference. What matters is that one accept them.

Yet traditional foundationalist epistemologies claim that describing how the Peano axioms are acquired is not to offer a justification for them. Separating the context of discovery from that of the context of justification entails the following: Empiricists who, for example, rest knowledge upon sense-data, cannot know that what titillates them refers to an external world (Sellars 1997, 13). That is, they cannot utilize method *P* to justify *P*. According to the foundationalist, the methodology that one takes to be the ground of knowledge cannot be justified by its employment. First philosophy, thus, has often been described as leaving one with the problem of trying to pull oneself up by one's bootstraps.<sup>5</sup>

Generally, pragmatists reject the distinction between the context of discovery from justification. They hold that at the second-level justification there can be nothing deeper than describing how the methodologies one employs are acquired. They are motivated by a desire to escape foundational epistemology by emphasizing practice. Instead of trying to pull themselves up by their bootstraps, pragmatists explain how one bought the boots, as it were.

Quine's view shall now be considered in detail. His naturalized epistemology grew out of the ruins of its foundationalist predecessor. Foundational epistemology, according to Quine, attempts to justify knowledge on a model

akin to an axiomatic system like that of Euclid. Knowledge rests upon a finite number of first principles, which are often taken to be self-evident, intuitive truths. In the foundations of mathematics, for example, he distinguishes the conceptual from the doctrinal; the former concerns meaning (clarifying and defining concepts) and the latter concerns truth (establishing laws by proving them) (Quine 1969, 69).<sup>6</sup>

Quine says that the two tenets of empiricism are unassailable. First, whatever evidence there is for science is empirical (the doctrinal), and second, the inculcation of the meanings of words (the conceptual) must ultimately rest on sensory evidence (Quine 1969, 75). He contends that science - i.e., empirical psychology - explains how one acquires basic concepts which serve as the foundation of knowledge.<sup>7</sup> According to him, however, the structure of foundational epistemology is not completely lost in the naturalist's alternative. There is still what is foundational (i.e., acquired, basic concepts) and what rests upon that (i.e., the doctrinal). Quine's naturalist program, therefore, depends on two arguments, one negative (i.e., a rejection of first philosophy) and one positive (i.e., an attempt to utilize scientific methodology *P* in justifying *P*).

An empiricist epistemology seems prey, however, to the charge of circularity. Utilizing science to explain the connection between evidence and knowledge seems to beg the question about the reliability of empiricism per se.

Naturalized epistemology, however, is the idea that one does not view science from an extra-scientific perspective. Quine says the worry of circularity is annulled "once we have stopped dreaming of deducing science from observations" (Quine 1969, 76). As he puts it, "Better to discover how science is in fact developed and learned than to fabricate a fictitious structure to a similar effect" (Quine 1969, 78). He goes on, "Epistemology, or something like it, simply falls into place as a chapter of psychology and hence natural science" (Quine 1969, 82).<sup>8</sup> His naturalized epistemology starts with science because it the best (i.e., most successful) theory available (Quine 1969, 69-90; 1992, 19). According to the pragmatist, a reason that the scientific methodology produces successful - i.e., true - knowledge is that it is self-correcting. As Sellars says:

For empirical knowledge, like its sophisticated extension, science, is rational, not because it has a *foundation* but because it is a self-correcting enterprise which can put any claim in jeopardy, though not all at once. (Sellars 1997, 79)

For the naturalist, scepticism about the entire scientific enterprise is banned. Rather, scepticism functions as part of the scientific methodology.

Quine's argument for the indispensability of abstract objects requires a standard of ontological commitment. He advocates the following standard:

[A] theory is committed to those and only those entities to which the bound variables of the theory must be capable of referring in order that that the affirmations made in the theory or true. (Quine 1953, 13)

Quine, however, is not slipping into some sort of linguistic or methodological idealism.<sup>9</sup> He writes:

It is no wonder, then, that ontological controversy should end in controversy over language. But we must not jump to the conclusion that what there is depends on words. Translatability of a question into semantical terms is no indication that the question is linguistic. To see Naples is to bear a name which, when prefixed to the words 'see Naples', yields a true sentence; still there is nothing linguistic about seeing Naples. (Quine 1953, 16)

At any rate, Quine finds his standard of ontological commitment in the notion of quantification. Quite generally, when the terms of one's theory quantify over some objects, they must exist. He says that one must distinguish between explicitly presupposing  $X$  and not explicitly presupposing  $X$  (Quine 1953, 102). His argument draws upon an analysis of language, which is often used to explicitly quantify over numbers. As he writes:

When we say, for example,  $(\exists x)(x \text{ is a prime} \ \& \ x > 1,000,000)$ , we are saying that *there is* something which is prime and exceeds a million; and any such entity is a number, hence a universal. In general, *entities of a given sort are assumed by a theory if and only if some of them must be counted among the values of the variables in order that the statements affirmed in the theory are true.* (Quine 1953, 103)

Quine is not, for example, advocating that when one tells the story of Cinderella, she must exist. Quine writes, "What there is does not depend on one's use of language, but what one says there is does" (Quine 1953, 103). He reifies numbers because they, for example, are required by an arithmetical discourse necessary for a theory that one already considers true, namely, science. As he writes, "For I deplore that facile

line of thought according to which we may freely use abstract terms, in all the ways terms are used, without thereby acknowledging the existence of abstract objects" (Quine 1960, 119). For him, the existence of arithmetical objects is justified because they are quantified over in a theory which is indispensable to scientific practice. His reference to the existence of objects casts the issue in terms of entity-realism.<sup>10</sup>

Quine's argument for the indispensability of abstract entities is modeled upon the process by which one assents to the existence of objects in everyday life. According to him, humans assent to the existence of physical objects because they are basic to our language; the focus of successful communication and they allow for fairly direct conditioning (Quine 1960, 234, 238).

Physical and abstract objects seem to be on the same footing insofar as both are common to linguistic practices. But the suspicion, as he points out, is that physical objects are "better attested to" than abstract ones (Quine 1960, 234). One can imagine having experiences whereby one can pick out a cookie, apple or chair, but the same is not the case for unobservables or arithmetical objects.

Quine, however, notes that in order to assent to the existence of an object, one needs, first, comparative directness with sensory stimulation and second, utility for theory. For example, he says that when one points to a rabbit

and announces "rabbit", a non-English speaker cannot know if one is referring to the rabbit or rabbit parts (Quine 1969, 46). This example of radical translation was supposed to show that sensory stimulation alone is not enough to know something. To pick out a rabbit requires a shared meaning embedded in language that supervenes upon sensory stimulations. One way to put it is that connotation is required for denotation. As Quine writes, "Talk of external things, our very notion of things, is just the conceptual apparatus that helps us to foresee and control the triggering of out sensory receptors" (Quine 1981, 1).

His argument for the indispensability of abstract objects depends upon his epistemological holism. Epistemological holism can be understood as the idea that knowledge is verified as much by theory as by data. Theory helps in decisions about the acceptance and interpretation of data as much as data helps in choosing a theory (Quine 1981, 1). As he says, "Physical objects are postulated entities which round out and simplify the flux of experience just as the introduction of irrational numbers simplifies the laws of arithmetic" (Quine 1953, 19).

On Quine's analysis, the (epistemological) difference between physical and abstract objects is "illusory" (Quine 1981, 16). Again, empirical science is supposed to provide the explanatory bridge between how one gets from sensory stimulations to the recognition of objects (Quine 1981, 2, 22-

3). According to Quine, the process by which one comes to know tables and chairs applies, more generally, to abstract arithmetical objects. With all objects - physical or abstract - one has to operate within a conceptual scheme that begins with experience. If one pictures knowledge as contained in a circle, one can say that the core contains solidified parts of the theory which is indirectly assumed (stimulus-analytic statements), while the circumference is in contact with experience.

The first theory (or scheme), however, is our everyday world upon which science builds. The origin of everyday concepts is the basis for more abstract ones. As he puts it, ontology is an "outgrowth" of lay culture (Quine 1981, 9). He portrays our epistemic situation thus:

The naturalistic philosopher begins his reasoning within the inherited world of theory as a going concern. He tentatively believes all of it, but believes also some unidentified portions are wrong. He tries to improve, clarify, and understand the system from within. He is the busy sailor adrift on Neurath's boat. (Quine 1981, 72)

All knowledge becomes part of one holistic, and many faceted epistemic enterprise (Quine 1992, 16). Moreover, the scientific enterprise is continuous with the inductive reasoning used to establish everyday knowledge, where one learns, for instance, that fire burns.

Knowledge of abstract objects is merely a further extension of science. As Quine writes:

At any rate the ontology of abstract objects is part of the ship which, in Neurath's figure, we are rebuilding at sea. (Quine 1953, 16)

The ontology of abstract objects is part of the ship too.  
(Quine 1960, 124).

More specifically:

[S]ince mathematics is an integral part of this higher myth, the utility of this myth for the physical sciences is evident enough. (Quine 1953, 18)<sup>11</sup>

Epistemologically these [mathematical objects] are myths on the same footing with physical objects and gods, neither better or worse except for the difference in degree to which they expedite our dealings with sensory experiences. (Quine 1953, 45)

Arithmetical objects, according to Quine, fair no worse than everyday objects; they are all relative to our epistemological point of view, our "interests and purposes" (Quine 1953, 18-19).

Quine, though putting all knowledge on the same footing, does concede that there are degrees of closeness to experience. Most scientific and mathematical knowledge is not confirmed by experience (at least not directly). According to him, abstract objects exist because the language of arithmetic, for example, commits one to the numbers which it quantifies over (Quine 1992, 30-1).<sup>12</sup> Mathematics, on his account of it, is empirical by its application in science (Quine 1992, 55). Decisions about what counts as real are made from within a theory, and this is supposed to be as true of physical objects as it is for abstract ones (Quine 1953, 102). Quine leaves decisions on what abstract objects to reify up to mathematicians and scientists (Quine 1960, 275).<sup>13</sup> Since Quine justifies mathematics by its application in scientific practice, one may wonder what happens to the

unapplied parts? (see Section 3.3) Stretched to its limits, it is Quine's holism that saves the unapplied parts of mathematics. (Also, one could easily extend it to large cardinals which Quine does not do.) The unapplied parts of mathematics are true because they are "couched in the same grammar and vocabulary that generate the applied parts of mathematics" (Quine 1992, 94).

At any rate, the talk of epistemological holism, however, indicates anti-realism because proponents of that doctrine emphasize the idea that truth is linked to certain methods. Yet Quine avoids extreme conceptual relativity (any view of the world is as good as any other) and the idea that as one's methods change so does truth. As he writes, "'Truth' is one thing, warranted belief another. We can gain clarity and enjoy the sweet simplicity of two valued logic by heeding the distinction" (Quine 1992, 94). Suffice it to say that whether Quine is, generally, a realist or anti-realist is open to interpretation. It has been suggested, however, that he is committed to the mind-independent existence of abstract objects like numbers.

### **(3.2) Putnam's Pragmatism.**

In the wake of Quine, it is useful to consider Putnam's views in relation to the four points just discussed: foundationalist epistemology, ontological commitment, epistemological holism and realism. First, Putnam, like Quine, is an anti-foundationalist. In "Mathematics Without

Foundations" (1979a) he expresses scepticism about foundationalist epistemology (Section 7.1). For him, philosophy, generally speaking, should describe scientific methodology and not criticize it. Putnam's view is epitomized in his claim that when science and philosophy meet the latter changes, not the former (Putnam 1979a, 44). Science is more secure than epistemology. Thus, accepted knowledge, like mathematics, should not be abandoned for philosophical reasons. According to Putnam, one must begin with the truth of scientific and mathematical knowledge. At both the first- and second-level of justification, how knowledge is acquired (in domain S) should be a guide to constructing the epistemology of S (Putnam 1979a, 11).

Second, describing practice is the basis for Putnam's realism about mathematics. To understand his argument, one must first briefly look at how he justifies the empirical sciences. Putnam argued, in his early writings, that realism does not make the success of science a miracle (Putnam 1979a, 73). The reason science is successful is because it gets at how things are. Crudely put, if the collective enterprise of science works, that is evidence for its truth. Mathematical knowledge is a further extension of science. Not all knowledge, however, is directly verified. He notes that many theoretical claims can be accepted because they are useful, before any substantive proof is forthcoming. As he writes:

We will be justified in accepting classical propositional calculus or Peano number theory not because the relevant

statements are 'unrevisable in principle', but because a great deal of science presupposes these statements, and because no real alternative is in the field. (Putnam 1998, 175)

According to Putnam, one has a right to take mathematics to be true because it is required for scientific practice.<sup>14</sup>

It should be emphasized, however, that Putnam's and Quine's employment of the indispensability argument differs in terms of their respective ontological commitments. Quine reifies mind-independent abstract objects and Putnam does not. In Section 1.3, it was shown that for Putnam, arithmetical knowledge is justified in terms of what it is possible (and not possible) to construct, and not the existence of mind-independent abstract objects. He accounts for mathematical necessity without platonism. According to him, the focus is upon "the truth of  $p$ " (not the mind-independent existence of numbers). Putnam gave up an earlier view where he held that mathematical entities are mind-independent (Putnam 1971).

Third, like Quine's efforts in the "Two Dogmas of Empiricism" (1953), Putnam's work criticizes the distinction between analytic-synthetic statements. He takes his departure from attacking the positivists' verificationist doctrine. Positivists, such as Carnap, had distinguished mathematical assertions from empirical ones, which are supposed to be verifiable by experience (Carnap 1935, 36).<sup>15</sup> Putnam rejects the idea that there is, on the one hand, empirical knowledge and, on the other, the formal sciences (which are *a priori*) (Putnam 1979a, 1). As discussed in relation to Quine in

Section 2.3, all knowledge has both empirical and non-empirical aspects.

Putnam's attempt to follow practice results in an epistemologically egalitarian outlook. The process by which one forms beliefs and tests them (which is essentially empirical) is refined in scientific methodology. His epistemology is continuous with basic human functioning (seeing tables and chairs), theoretical developments, unto the formal sciences like mathematics. Ultimately all knowledge must be justified in the same way, that is, by describing how it is acquired. Moreover, the acquisition of knowledge is dependent on an entire world-view, which is just another way of saying that data is theory-laden (Putnam 1981, 215).

His epistemology (internal realism) can be sketched in slightly more detail as follows. According to Putnam, theory contains facts (particular assertions that are true or false). Facts (and theory) are determined according to a criterion of rationally acceptable (under idealized epistemic conditions (Putnam 1981, 49, 55)). As he writes:

The view which I shall defend holds, to put it very roughly, that there is 'rationality'; thus, to put it even more crudely, the only criterion for what is a fact is what it is rational to accept. (Putnam 1981, x)

A criterion of rational acceptability, in turn, rests upon shared values. Values are present in the questions one asks and what one finds interesting or relevant.<sup>16</sup> In scientific practice, there is a reciprocal relationship between the accepted theory and the data which illuminates how values

function. Putnam writes:

Even our description of our own sensations, so dear as a starting point for knowledge to generations of epistemologists, is heavily affected (as are the sensations themselves for that matter) by a host of conceptual choices. The very inputs upon which our knowledge is based are conceptually contaminated; but contaminated inputs are better than none. (Putnam 1981, 54)

The concepts and shared values that one comes to adopt render the world intelligible.<sup>17</sup> Just as one learns that walking on thin ice is unsafe, much of knowledge, somewhere down the line, rests upon experience, e.g., induction. Furthermore, according to Putnam, values such as believing in induction are, ultimately, predispositions grounded in human nature. He writes:

There is only one possible explanation [why there is a great deal of epistemic consensus]: human interests, human salience, human cognitive processes, must have a structure which is heavily determined by innate or constitutional factors. Human nature isn't all that plastic. (Putnam 1978, 56)

To summarize, facts sit within a theory, which rests upon values grounded, in turn, in human nature.<sup>18</sup>

Finally, like Quine, Putnam's epistemological holism - i.e., his internal realism - seems anti-realist. According to Putnam, knowledge cannot be justified from outside of a perspective per se. One has no access to the world as it is in-itself. Truth and falsity function within different theories that provide various descriptions of the world. There are, for example, the representations of physics, biology, and so on. According to Putnam, there are different ways to

conceive the world. As he writes, however, "But the question 'which kind of 'true' is really Truth' is one that internal realism rejects" (Putnam 1990, 96).

Putnam's internal realism (i.e., his emphasis upon the agent-dependence of knowledge) and his attack upon metaphysical realism (Section 8.2) are often seen as a radical break with his earlier, realist views (1979a). Therefore, one may think that the justification of arithmetic that Putnam provides cannot be given a realist interpretation without sticking exclusively to his early views. In Section 8.2, however, it shall be explained how his early views are consistent with aspects of internal realism.<sup>19</sup>

### **(3.3) Defending The Indispensability Argument.**

It is worthwhile considering some classic problems with the indispensability argument, which has been raised by Penelope Maddy, a proponent of set-theoretic realism. Her criticisms must be refuted if the indispensability argument is to be a viable option. Maddy asks, first, if there is such a thing as an accepted theory - science - which arithmetic could be indispensable to? (Maddy 1992, 280)

Yet Maddy assumes certain scientific standards of evidence - e.g., the causal constraint requirement - which she seeks to extend to sets. She cannot call into question the idea that there is such a thing as scientific standards of justification without undermining her set-theoretic realism. Her criticism is self-defeating.<sup>20</sup>

Second, according to Maddy, being useful is not equivalent to being true (Maddy 1992, 281). Mathematics' successful application in scientific inquiry may not warrant an epistemological pay-off. More generally, one cannot identify *P*'s success - e.g., *P*'s predictive success or application - with *P*'s truth. Maddy writes: "In short, legitimate choice of method in the foundations of set-theory does not seem to depend on physical facts in the way indispensability theory requires" (Maddy 1992, 289).<sup>21</sup>

Maddy contends, however, that mathematics' applicability is an argument against considering it a formal game (Maddy 1992, 275). According to her, mathematics' applicability is a reason to believe that one cannot make up mathematics anyway one wants. Mathematics' applicability is a reason to be a realist. She, therefore, must agree that being useful is a quality that is at least indicative of truth. Her criticism is at odds with what is required to motivate her set-theoretic realism.

Third, both Quine and Putnam must confront the issue of the unapplied parts of mathematics. As Maddy points out, justification in terms of abduction does not extend to the unapplied parts of mathematics (Maddy 1992, 278). She writes:

The trouble is that this [the indispensability argument] does not square with the actual mathematical attitude towards unapplied mathematics...Here simple indispensability theory rejects accepted mathematical practices on non-mathematical grounds, thus ruling itself out as the desired philosophical account of mathematics as practised. (Maddy 1992, 278-9)

In addition, Anthony Peressini contends that one cannot know if the indispensability argument works even for the applied parts of mathematics until it is made clear what "units (theory, object, theorem, axiom)...what parts of pure mathematical theory are confirmed" (Peressini 1997, 216-7). He writes:

It is not clear, however, that unitary operators have the same 'indispensable' status as do Hermitian operators, since unitary operators do not correspond to an aspect of physical reality in the way Hermitian operators do. (Peressini 1997, 221)

Quine can respond to these concerns. He clarifies an earlier view that would render the unapplied parts of mathematics unjustified. In recent writings, Quine casts the net wide, including the unapplied parts within the already utilized sections. According to Quine, non-applied parts of mathematics are true - i.e., the objects those parts of mathematical theory quantify over can be inferred. That is, if one part of mathematics is true (the terms it quantifies over exist), it is safe to say the rest is (the rest of the terms germane to the unapplied parts refer). As Peressini, notes, for Quine, mathematics get to be part of the boat, as it were (Peressini 1997, 226).

Putnam's talk of possible application would, presumably, follow Quine. For Putnam, the non-applied parts of mathematics are true because they could be possibly applied (they could be applied for the same reason Quine cites: they are couched in the same grammar and vocabulary as the applied parts) (Putnam

1979a, 60). Putnam says, for example, that he regards sets of very high cardinality as "speculative and daring extensions of the basic mathematical apparatus of science" (Putnam 1979a, 56).

In addition to Maddy's criticism, a number of more recent concerns have been raised with the indispensability argument and they shall be considered next. Fourth, as Peressini explains, mathematics is supposed to be indispensable to a more secure body of knowledge. For a pragmatist, scientific methodologies must be more secure than mathematical ones because mathematics' principles can be justified by the former (but not vice versa). As Peressini remarks, however, when scientific theories are discarded, the mathematics thus employed remains. The notion that scientific knowledge is more secure than mathematics is counter-intuitive and is therefore a reason to reject the indispensability argument (Peressini 1999, 258).<sup>22</sup>

Yet in the pragmatist's defense it can be argued that one must distinguish between epistemological and metaphysical security. Even though mathematics is justified by way of its application in science, that does not make its truths less secure. Domains about which one is a realist have an equal claim to necessity. What differs between the two is priority of justification - how one justifies one by way of the other. It can be argued that science, ultimately (at the second-level of justification), is epistemologically more secure than

mathematics, but their truths are metaphysically on an equal footing.

Fifth, Sober notes that mathematics has been used for a variety of theories, ones that are true and ones that are false (Sober 1993, 43, 55; Maddy 1997, 143). Abduction applied in science differs from the case of mathematics. He explains, "It is less often noticed that mathematics allows us to construct theories that make *false* predictions and that we could not construct such predictively *unsuccessful* theories without mathematics" (Sober 1993, 53). Sober claims that mathematics is the background and cannot inherit support from the theories it participates in (Sober 1993, 53).

Mark Colyvan has responded to Sober by pointing out that since mathematics is not responsible for giving rise to false predictions or hypotheses, it can share in the credit for correct ones (Colyvan 1999, 330). Colyvan illustrates his point with an analogy:

Blaming the mathematics is like a programmer blaming the programming language. And similarly, claiming that mathematics cannot share in the credit is like claiming that the programming language cannot share the credit for the successful program. (Colyvan 1999, 329)

One must, for example, distinguish between the fault of the scientist (e.g., in coming up with a false theory), and the credit to be given to mathematics (in the case of a successful scientific theory that utilizes mathematics).

Finally, the merit of the indispensability argument has been undermined because of nominalist accounts that also take

into account mathematics' applicability (e.g., Field 1989; Feferman 1998, 207). Accounting for mathematics' applicability without needing to infer the existence of abstract objects undermines the idea that justification in terms of abduction is warranted for arithmetic. Yet, Field's program, for instance, is unacceptable until he can refute Shapiro (1998), and show how his account can eliminate reference to abstract objects (see Chapter 7). Since the language of arithmetic in the twentieth century has been in terms of set-theory, Feferman's alternative, for instance, is not constraining.<sup>23</sup> There is no compelling reason to accept a nominalization.

#### **(3.4) Revising The Indispensability Argument.**

Thus far, the indispensability argument has been defended. Several further criticisms of the indispensability argument, however, require revising it, and they shall be considered next. First, if abduction is to apply generally it may allow for the countenancing of improper mathematical or scientific theories.<sup>24</sup> One may want to note that sometimes improper mathematical theories, for example, are also found useful in science:

Newton introduced fluxions to perform the seemingly impossible role of being both zero and not zero: he had to assume that a variable was not zero in order to avoid dividing by zero, but then assumed that it was zero to get his results for instantaneous velocity...[Also, Dirac's delta function] was nonsense, and yet in spite of this it worked brilliantly in a successful physical theory [quantum mechanics]...These techniques actually 'fly in the face' of pure mathematics in the same way that dividing by zero 'flies in the face' of ordinary arithmetic. (Peressini 1997, 224)

Successful application of Newton's calculus would lead one to believe in the mind-independent existence of infinitesimals. Abduction can be revised to meet these challenges. It should be reformulated thus:

If our best scientific practice  $q$  presupposes  $p$ , then  $q$  gives one good reason to believe  $p$ , only if  $p$  is countenanced by the practitioners in the relevant field.<sup>25</sup>

Undesirable objects can be excluded from reification in terms of abduction by heeding the proviso implicit in the revised formulation: one should not countenance the existence of objects or infer truths unless they are sanctioned by practitioners in the relevant field. Scientists are likely to have a great deal of consensus over unobservables (eventually). Conversely, arithmetical objects would not fare well because consensus upon, for example, countenancing the mind-independent existence of numbers would likely not obtain (consensus may never obtain). Inferring the mind-independent existence of arithmetical objects would be excluded but not its truth.

The revised formulation is faithful to Quine's intentions. Quine had claimed that decisions on what abstract objects to reify would be left up to practitioners (Quine 1960, 275). According to him, for example, practice is the final court of appeal. Quine, however, had also decided that numbers should be reified (Colyvan 1998, 50). Quine was not as faithful to practice as he could have been. If one was to radically follow practice, only mathematicians would decide if

numbers exist. Upon the revised formulation of abduction, arithmetical knowledge, for example, does receive an internal justification since one follows the practice of mathematicians (not scientists or philosophers).

Finally, Peressini points out that the use of abduction in science differs from that in mathematics because "the physical application requires empirical bridge principles to underwrite physical interpretation" (Peressini 1999, 214).<sup>26</sup> When deciding to assent to unobservables, for instance, rules of inference relating to some observations are necessary. For instance, if doing  $X$  results in observations  $Y$ , under conditions  $C$ , physical objects (unlike arithmetical ones) form part of the causal story about phenomenon  $Y$ .

Mathematical objects reified by abduction are further removed from empirical confirmation than even unobservables in the following sense. Since physical objects causally function in explanations, a description of those objects are provided in scientific theories, but the same is not true, say, for the number three. Mathematics does not yield predictions (e.g., " $124 \div 4 = 31$ " is not a prediction) nor (for Quine) do their entities stimulate our sensory receptors. As Peressini points out, indispensability proponents who faithfully follow the analogy between science and mathematics, must justify mathematics internally (as was the case in science) (Peressini 1999, 267). Numbers, for example, are often derived from premises not new observations (Field 1980, 10-11, 40). As

Sober remarks, therefore, "Indispensability is not a synonym for empirical confirmation, but its very antithesis" (Sober 1993, 44).<sup>27</sup>

Justifying the existence of arithmetical objects in terms of abduction gives rise to what shall henceforth be dubbed the "so-what" problem, i.e., the idea that the existence of mathematical objects are epistemologically irrelevant.<sup>28</sup> The indispensability argument would not lead one to know what mathematical objects are only that they exist (Feferman 1998, 297).

The "so-what" problem is a challenge posed to Quine because he attempts to infer the existence of abstract objects. The fatal problems with the indispensability argument are related to Quine's attempt to infer the existence of abstract objects. The revised version of the indispensability argument evades the "so-what" problem and is faithful to the pragmatist's maxim. Putnam's tack escapes the "so-what" problem because he does not posit the existence of abstract objects. (He focuses upon the objectivity of arithmetical knowledge.) Putnam's views will be subsequently defended in Chapter 8.

Being faithful to the pragmatist's maxim - follow practice - supports an epistemological division of labour. First-level justification would require describing how mathematics is justified according to the first principles. Second-level justification would require describing how the

first principles of mathematics are acquired - i.e., naturalism. Such an epistemological division of labour is advocated by Kitcher's naturalism, who is defended in Chapter 6. Furthermore, consistent with Putnam's aim to account for objectivity (and not existence), the indispensability argument has been revised to exclude inferring the mind-independent existence of arithmetical objects. It shall be subsequently argued that upon the revised formulation of the indispensability argument, arithmetic's applicability is one reason to be a realist about the truth-values of appropriately formed statements in that domain. Realism ranges over both levels of justification (discussion in Chapter 8). In the next chapter, neo-Fregean attempts to justify arithmetical knowledge shall be considered.

## PART II

### PLATONISM

## 4. Neo-Fregean Realism

Neo-Fregean strategies, like those of Bostock (1974), Wright (1983), and Hale (1994b) are based upon the "linguistic turn", according to which an analysis of language is to be a guide to one's ontology. The linguistic turn, which can be traced back to the writings of Frege, was an attempt to avoid empiricist theories of meaning (whereby, for example, meaning is explained in terms of mental images) and Kantian representations.

Neo-Fregeans rely upon the context principle, which is in its broadest formulation the idea that words have meaning only in the context of a proposition (Frege 1953, sec. 46, 55, 60). It is supposed to set the criteria for when singular terms refer. Hume's principle is to tell one what numbers are:

the number of Fs is equal to the number of Gs if and only if the Fs are in an one to one correlation with Gs. (Frege 1953, x, sec. 63)<sup>1</sup>

In this chapter, problems with the context principle are considered, though all the debates surrounding it are not entered into. It is argued that, for the Fregean, once the

context principle is rejected (as it must be) the threat of skepticism looms large, i.e., the notion that real knowledge or truth is impossible.

#### (4.1) Frege's Gatekeeper.

One may recall that Benacerraf's dilemma pits a realist ontology against an empiricist epistemology, which is the division Frege insists upon. Benacerraf's dilemma, therefore, seems custom made with Frege in mind. In the preface to The Foundations of Arithmetic, Frege notes that he has adhered to three fundamental principles in his book:

Always to separate sharply the psychological from the logical, the subjective from the objective; never to ask for the meaning of a word in isolation, but only in the context of a proposition; never to lose sight of the distinction between concept and object. (Frege 1953, x)

Following the empiricist tradition, Frege uses "idea" always in a psychological sense, distinguishing ideas from "thoughts" and objects. If something depends upon the human mind, or mental processes, it is subjective. Frege remarks, "If there is no owner of ideas then there are also no ideas, for ideas need an owner and without one they cannot exist. If there is no ruler, there are also no subjects" (Frege 1977, 21).<sup>2</sup>

But arithmetic cannot be grounded in psychology because it is objective.<sup>3</sup> He writes, "If number were an idea, then arithmetic would be psychology" (Frege 1953, sec. 26), and, "Mathematics is not concerned with the nature of the mind, and the answer to any question whatsoever in psychology must be for mathematics a matter of complete indifference" (Frege

1953, sec. 93). As he says:

No, sensations are absolutely no concern of arithmetic. No more are mental pictures formed from the amalgamated traces of earlier sense-impressions. All these phases of consciousness are characteristically fluctuating and indefinite, in strong contrast to the definiteness and fixing of the concepts and objects of mathematics. (Frege 1953, v)

For him, something is either subjective or objective.<sup>4</sup>

Frege writes:

'We' is not the object of mathematics at all, just as little as our ideas are. Truths in mathematics are eternal and not dependent on whether we are alive or dead or become aware of them. (Frege 1979, 78)

Empiricist claims, cast in terms of ideas, are subjective and lack a truth-value.<sup>5</sup> According to Frege, for example, beauty and jurisprudence, are subjective - truth and falsity do not apply to them (the objects in their respective domains are not independent of subjects).<sup>6</sup>

What is objective is not determined by human cognitive processes, but recognized by it. Frege writes, "Yet there is something objective in it [one's intuitions] all the same; everyone recognizes the same geometrical axioms, even if only by his behaviour, and must do so if he is to find his way about in the world" (Frege 1953, sec. 26). According to him, for example, the concept "point" has an "objective meaning" (Frege 1953, sec. 26).<sup>7</sup>

Frege begins with a linguistic analysis that assumes that reference is the essence of language.<sup>8</sup> The context principle is among other things an attempt to avoid the empiricist theory of meaning. According to the empiricist, the

meanings of words are understood in terms of "mental pictures or acts of the individual mind" (Frege 1953, x). For an empiricist, according to Frege, numbers are treated as predicates. For empiricists like Locke, for example, predicates were often subjective qualities attributed to an object. Frege writes:

For this reason I have avoided calling a number such as 0 or 1 or 2 a *property* of a concept. Precisely because it forms only an element in what is asserted, the individual number shows itself for what it is, a self-subsistent object. (Frege 1953, sec. 55)

He also writes:

Only in a proposition have the words really a meaning. It may be that mental pictures float before us all the while, but these need not correspond to the logical elements in a judgement. It is enough if the proposition taken as a whole has a sense; it is this that confers on its parts also their content. (Frege 1953, sec. 60)

The context principle means that one must understand the role a term plays in the context of sentences in which it is used in order to set its meaning.<sup>9</sup> According to Frege, an analysis of number terms shows that they do not function as predicates (standing for properties) but as proper names (standing for "self-subsisting objects"), asserting something objective of a concept (Frege 1980, 73).<sup>10</sup> If number terms are to be understood as standing for self-subsisting objects, the problem, says Frege is one of specifying the identity conditions: "And that is enough to give us a class of propositions which must have a sense, namely those which express our recognition of a number as the same again" (Frege 1953, sec. 62). Frege presents Hume's principle as providing

a criterion of identity for number terms.

For Frege the requirement for self-subsisting objects is epistemological as well as metaphysical because there is an assumed link between language and the world.<sup>11</sup> Mathematical objects are metaphysically prior (they exist whether one inquires about them or not). Syntactic categories are epistemologically prior (one does not know if mathematical objects exist without examining language) (Wright 1983, 13; Wetzel 1990, 239).<sup>12</sup>

Neo-Fregeans generally follow Frege's strategy. Arithmetical truth requires mind-independent objects, whose existence can be ascertained by an analysis of language. Neo-Fregeans, however, sanction mind-independent numbers but not thoughts.<sup>13</sup> The context principle remains their way to justify the existence of mind-independent numbers.

According to Wright, for instance, the context principle presupposes a correspondence, under certain specified conditions, between singular terms and reality.

An expression's candidacy to refer to an object is a matter of its syntax: that once it has been settled that a class of expressions function as singular terms by syntactic criteria, there can be no further question about whether they succeed in objectual reference which can be raised by someone who is prepared to allow that appropriate contexts in which they do so feature true. (Wright 1992, 28)

Hale writes, "The effect of viewing the matter in this [Frege's] way is to reduce the question of whether numbers are among the objects that there are to questions about logical form of certain statements and their truth values" (Hale

1994b, 17). He says that the context principle is "an advantage" - one does not have to worry about the usual metaphors of platonism, like grasping thoughts.<sup>14</sup> Hale writes of the context principle:

Provision of such a criteria [like the context principle] is an essential prerequisite, not only to any full defense of Frege's own view, but to progress on a wide range of issues in the philosophies of language and mathematics and in general metaphysics.<sup>15</sup>

However, the more one relies on the context principle, the more one is liable to fall if it does.

#### **(4.2) Trouble on the Farm.**

Though debates surrounding the context principle shall not be entered into, an argument against it shall be rehearsed.<sup>16</sup> Neo-Fregeans are faced with the so-called Caesar problem: how does one know that Caesar is not a number? Wright thinks that Hume's principle offered a response to the Caesar problem because it lays down identity conditions. Introducing numbers following Hume's principle sets identity conditions so that different occurrences of the term "2" refer to the same content.

Numbers introduced by Hume's principle, however, are indeterminate (one is only told that the content referred to must be the same, not what it is). The problem with Hume's principle, according to Sullivan and Potter, concerns circularity. One cannot be confident that Hume's principle is fully adequate to its objects unless one is already so assured. Hume's principle does not tell one that Caesar is

not a number unless one already assumes that. In Frege's terms, Hume's principle only fixed the sense of numerical singular terms, leaving undermined their reference (not even showing they are terms in fact) (Sullivan and Potter 1997, 138).

As Sullivan and Potter explain (making reference to Locke), "gold" can designate gold, but it does not tell one about it. They write, "Our conceptions fix what (if anything) we are pointing at, but cannot settle its nature: that is a matter of what is out there" (Sullivan and Potter 1997, 151-2). Perhaps in the case of natural kinds like gold one identifies some class of properties that allow it to be picked out (as it actually is). One cannot, however, go and look at abstract objects.

One should note that Hale and Wright invoke a further restraint - the sortal inclusion principle (SIP) - so that numbers introduced by Hume's principle are not people. Hale states the sortal inclusion principle thus: .

A sort of objects F is included within a sort G only if the content of a suitable range of identity statements about G's - those linking terms denoting G's that are candidates to be F's - is the same as that asserting satisfaction of the criterion of identity for the corresponding F's. (Hale 1994, 131)

According to Sullivan and Potter, the combination of Hume's principle with SIP is an attempt to combine realism (the self-subsistence of numbers) with epistemological transparency (Sullivan and Potter 1997, 142, 149, 152).

The introduction of SIP may solve the Caesar problem by

requiring that a suitable range of identity statements that link a term with its reference apply to both Fs and Gs. One may think SIP works as follows. The italicized part of the statement "water is *H<sub>2</sub>O*" would have to apply to *pani* (Hindi for "water") to know that *pani* = water. Yet water and *pani* may be defined differently in practice (perhaps *pani* means a thirst quencher). Thus the two terms - F and G - need not be intentionally isomorphic because meaning is to be given in terms of reference. So, the identity must have to do with the their instances (the tokens). Each instance of *pani* must yield recognition that it is water. Ostensive definition, however, is not helpful when dealing with abstract objects (one cannot point to them). Sullivan and Potter note that SIP implies the following:

*f* = *g* only if, for some *f*<sub>1</sub> and *g*<sub>1</sub>, *f*<sub>1</sub> has the same content as *g*<sub>1</sub>. (Sullivan and Potter 1997, 150)

In a nutshell, the sortal inclusion principle does not allow one to assume that two terms have the same content "unless one's understanding of this way of referring to instances of these concepts is sufficient to assure one that they are co-instantiated" (Sullivan and Potter 1997, 150-1). "*Pani*" always refers to water. Similarly, number terms always refer to the same concept. When one refers to the number "3", for example, in different contexts, it always denotes the same content; it never denotes "Caesar". No instantiation of Caesar can be linked with appropriate identity statements to the content of "3". For example, Caesar's mother was so-and-so, was born

about 100 B.C., was a Roman general, and so on; yet none of these statements applies to the content of "3".

Adopting SIP, however, has unpleasant consequences when applied to all terms. Consider "apple"; if the identity conditions of terms are to be understood by co-instantiation, one can prove that no apple was ever a meal because some instances of meals will not be apples (Sullivan and Potter 1997, 151). Sullivan and Potter remark, "Perhaps Wright and Hale would be content to accept that consequence; but should it really be necessary to cut things quite so finely to be confident that Caesar is no number?" (Sullivan and Potter 1997, 151) They conclude:

Numbers surely are *just* what arithmetic requires them to be. In criticizing Wright and Hale's solution to the Caesar problem we do not suggest that it is, after all, intelligible that 'Caesar might really be...behind our backs, as it were' (Wright 1983, 113). What we have been concerned to show is that Wright and Hale do not win their way to the right conclusion; further, that they fail to do so because they cannot regard Caesar problem as 'daft' (ibid.) - as, in the end, it *must* be seen to be. Their advocacy of SIP [the sortal inclusion principle] shows that their instincts are in the right direction...But...SIP is *just* too blunt a tool to do the job. (Sullivan and Potter 1997, 152)

According to Sullivan and Potter, as well as Dummett and Wetzel, the amendment (the sortal inclusion principle) is an incongruous addition to the picture.<sup>17</sup>

The worry about the complexity of SIP is not without reason since, as Sullivan and Potter also observe, for Frege, numbers were transparent (Sullivan and Potter 1997, 149).<sup>18</sup> Frege had expected that the context principle's application

would be straightforward. Dummett concludes that the context principle does not justify the existence of mathematical objects - despite Wright's defense (Dummett 1991, 292, 307).<sup>19</sup>

#### **(4.3) The Linguistic Wrong Turn.**

According to neo-Fregeans, there is an affinity between language and reality. Number terms refer to mind-independent abstract objects, which can be inferred from an analysis of language (Frege, however, does not explain the affinity between language and the world, i.e., why the context principle works) (Wetzel 1990, 239). Whether the analysis of language is a guide to what there is (or vice versa), is not addressed because, at certain junctures, an isomorphism is assumed, for example, in the case of singular terms.

The context principle, however, as a guide to what there is, underdetermines objects.<sup>20</sup> What Wright's defense of the context principle shows is that as a realist must find a different avenue than attempting to uncover certain semantical rules that are supposed - as Putnam says in a different context - an ad hoc "magical theory of reference" plausible for singular terms (Putnam 1990, 51). The assumption that syntax can be a substantive guide to what there is, one's ontology, - of the original, linguistic turn - is dubious.<sup>21</sup> With the loss of a link between linguistic terms and mind-independent objects, the specter of scepticism looms. (knowledge is language-transcendent.)

The empiricist take a different tack. In broad outline,

according to a Quinian naturalist, concepts are acquired - in terms of an entire social-biological context - and in combination with other concepts generate linguistic possibilities, various meanings (e.g., see Chapter 1). The mind develops along with its interactions with its environment. Meaning is an empirical matter: discovering the reference of terms or how they are used. A naturalist's account of meanings violates Frege's first principle, which requires separating what is objective from what is psychological.<sup>22</sup> How arithmetical knowledge is acquired (which is social and psychological) can have no part in its justification (because arithmetical knowledge is objective). For Frege there are only two ways to go.

As Frege saw, the danger of anti-platonism (like empiricism) is a lapse into subjectivism. According to a subjectivist (or inter-subjective) view of language, meaning is grounded in social practices (e.g., an interpretation of Wittgenstein) or human physiology (Dummett 1993, 448). Card-carrying idealists go a step further, contending that conceptual schemes embedded in a language create reality (Goodman 1972, 9). According to an idealist, the semantics and epistemics of arithmetic are a matter of conventions, which are either socially or biologically grounded (e.g., one could claim that the Peano axioms are conventions).

A plausible epistemology of arithmetical knowledge can be gleaned from Wright's remark about language acquisition. He

says that no philosophy of language should make acquisition a mystery (Wright 1983, 43). According to the context principle, one has to look to the linguistic environment in which the term in question is seated to understand its meaning; but the context principle raises the issue of how one learns the other terms (an infinite regress analogous to that which is discussed in Chapter 1 could ensue).

Wright concedes that his neo-Fregean strategy offers no "effective response" to the acquisition challenge (Wright 1983, 49). Wright claims that avoiding a semantic, infinite regress requires a developmental model, which is implicit in the context principle for singular terms:

It is not ostensive definition which really breaks the circle, or short-circuits the regress, in the case of singular terms standing for concrete objects. It is background training in the use of whole sentences involving singular terms of the relevant sort. At any rate, this is the model of the situation which the empiricist is forced to adopt. (Wright 1983, 46)

An account of language acquisition (even though Wright restricted his comments to concrete objects) propounded in terms of development resonates with Wetzel (Wetzel 1990, 294).<sup>23</sup>

Wetzel rejects both the idea that syntax can be the guide to what there is (e.g., Wright's strategy), and its inverse, the idea that ontology can tell one what a singular term means (e.g., Putnam's account of natural kinds):

Rather, it is to suggest that there is more give and take between syntax and ontology than either approach allows. A plausible theory could be constructed, I believe, which explains the relationship between the two in terms

of stagewise development, such that sometimes our ontological intuitions inform our view of syntax, and sometimes syntactic principles determine our view of ontology. Eventually we achieve 'reflective equilibrium' between the two. Needless to say, this must remain the subject for another paper. (Wetzel 1990, 254)

One is stuck with conceptual schemes, that many, like Wetzel, have tried to conceive of as entailing neither realism or idealism (arithmetical knowledge is neither discovered nor created). Yet claiming, like some (e.g., those in the German idealists tradition, or some Wittgensteinians), that thought and being are the same thing is a mystery. As Wetzel points out, rejecting the context principle does not determine the realist debate. One still wants to know what there is? What is true? And what does "true" mean?

It has been argued that the context principle must be rejected. The way to make headway must begin - taking one's leave from Frege - by clarifying realism for arithmetic - and departing from him - reconsidering empiricism (i.e., an alternative to Fregean epistemology). The next strategy to be considered is Maddy's attempts to naturalize her realism.

## 5. Maddy's Set-Theoretic Realism

In Section 3.1, it was pointed out that the indispensability argument, according to Maddy, is counter-intuitive because the type of justification is external, i.e., mathematics is justified by appeal to scientific practice).<sup>1</sup> Her naturalism is supposed to provide an internal justification for mathematics. According to her, mathematical truth relies for its justification upon the perception of its own objects. Her naturalism is an attempt to offer an alternative to the indispensability argument (Maddy 1992, 278-9). She contends that her account provides an explanation of mathematical truth that has a priority, certainty and necessity which the Quine and Putnam strategy does not have (Maddy 1990, 177). Her account has a priority to that of the pragmatists because it does not require an appeal to mathematics' applicability and is more certain and necessary because one directly verifies the truths of set theory.

Since it was also pointed out in Section 3.1, however, that Maddy relies on the fact of mathematics' applicability to

argue against, for example, formalists, it seems that she is both against and for the indispensability argument.<sup>2</sup> Yet it can be argued in Maddy's defense that she relies on the fact of mathematics' applicability to motivate her realism and not to offer the justification of mathematical knowledge: her project takes care of that.

In this chapter, first, criticisms of Maddy's set-theoretic strategy, its relationship to the work of Kurt Gödel and her account of perception are considered. Second, the problem of the mind-independence of sets will be scrutinized. Third, it shall be argued that mathematical realism can survive without mind-independent objects (like sets). (The naturalist can evade the threat of skepticism that the platonist faces.) The naturalist can also avoid subjectivism (and inter-subjectivism), i.e., the idea that truth is dependent upon, and relative to, one's personal (or group) point of view.

#### **(5.1) Maddy's Naturalized Realism.**

Maddy looks to set theory as the basis of mathematics for pragmatic reasons. According to her, in the foundation of mathematics, set theory plays a unifying role (Maddy 1997, 34).<sup>3</sup> Sets are "simple and manageable entities that form the basis for a surprisingly effective and efficient mathematical theory" (Maddy 1990, 62). According to her, however, a set-theoretic foundation does not require that all legitimate techniques are included in the usual methods of set theory

(other methods may be set theory surrogates). It should be pointed out that her realism encompasses parts of set theory that go beyond the means required to account for Peano arithmetic, and therefore falls outside the scope of this thesis; her view, however, must be considered because it is an important response to Benacerraf's dilemma.

According to Maddy, Benacerraf brought into focus what is required for a successful account in the philosophy of mathematics by showing how two competing goals - a unified semantics and an empiricist epistemology - were resulting in a dilemma (Maddy 1991, 158). She attempts to solve the dilemma by marrying realism about sets to a naturalistic epistemology, thus attempting to satisfy both horns of the dilemma (Maddy 1991, 158).

Yet, before examining Maddy's naturalized realism, caution requires that one confront the tenability of using set theory as the foundation of mathematics. First, Michael Hallett, a scholar of Cantorian set theory, claims that the realist needs a better definition of a set than a collection, gathering, and so on (which is all Cantor provided) (Hallett 1984, xii).<sup>4</sup> He points out that set theory is not more obvious than what it is supposed to support (e.g., an account of natural numbers) because the definition of "set" is unclear.<sup>5</sup> Also, he points out that for Cantor, sets were formed by putting things together; yet since God did that for him, one could be a realist about sets (they existed in the eyes of

God, as it were) (Hallett 1984, 301). Dispensing with theology, however, Hallett asks: "When does it make sense to say that a collection forms 'one thing' [a set] as opposed to a mere aggregate?" (Hallett 1984, 299). He goes on to say: "[B]ut if set theory is to be the ultimate framework one is still left with the conceptual mystery of why all mathematical objects should be sets (unities out of manifolds)" (Hallett 1984, 305). Also, using set theory as the basis of mathematics conflicts with the historical fact that mathematics pre-dates set theory. If mathematics were founded on set theory, what was it before that? (Balaguer 1991, 104).

In Maddy's defense, however, it could be claimed that the definition of "set" is as clear as many other concepts in science, like "force".<sup>6</sup> That is to say, it is as clear as a concept utilized in a scientific practice needs to be. In the twentieth century, the foundations of arithmetic have been given in term of set theory. So, in choosing set theory she can claim to be following the practice of mathematicians. Also, in providing the foundations of arithmetic, one need not require that one has to know the Peano axioms to do arithmetic. The Peano axioms are implicit in practice. The Peano axioms captures the meaning of numbers and Maddy attempts to show that rudimentary arithmetical knowledge is acquired from perceiving sets. The idea of providing foundations for a theory like arithmetic in naturalistic terms is to describe how the encoded truths are acquired.

Second, however, as Adrian Riskin claims, there is no one set theory; various ones give rise to different theorems (thus they are not isomorphic) (Riskin 1994, 113, 117-9).<sup>7</sup> Benacerraf (1965), in "What Numbers Can't Be", for example, showed that if numbers are sets, one does not know what kind of sets they are. Consider two types of sets.

Zermelo sets

von Neumann sets

$\{0\}, \{\{0\}\}, \{\{\{0\}\}\}, \dots$        $\{0\}, \{0, \{0\}\}, \{0, \{0\}, \{0, \{0\}\}\}, \dots$

The statement "a pair is included in a set with three members" is true for both Zermelo and von Neumann sets. Yet, the statement, "a singleton is included in a three member set" is false for Zermelo sets and true for von Neumann sets. The statement can be generalized as a theorem and formalized as follows:

$$(\forall x)(\forall y)((x > y) \leftrightarrow (y \in x))$$

That is, all x's are greater than all y's if and only if, y's are members of x's. The theorem applies to von Neumann sets but not Zermelo sets.

Aldo Antonelli sums up the worry of Riskin and Benacerraf. He contends that there are:

...[S]everal, mutually incompatible conceptions of the set-theoretic universe...Different conceptions of sets are good for different purposes, allow for the modeling of different parts of mathematics, classical or otherwise, and respond to different motivations. (Antonelli 1999, 138-139)

He concludes:

But now the coexistence of several set-theoretic universes renders the many possible reductions of mathematics to set theory essentially incommensurable, a

realization that turns the set theorist's monotheism into a chaotic pantheon. (Antonelli 1999, 161)

Maddy, however, may be able to offer the same response she did to Benacerraf's challenge. She had argued that though Zermelo and von Neumann sets are not the same they are co-extensive: they can be mapped on to each other (Maddy 1990, 93). She has admitted that there could be other possible foundations but required that they be set-theoretic "surrogates"; they must be equivalent descriptions, equal, not the same. As Frege had pointed out, there is a difference between two things being equal and the same (Frege 1953, sec. 65). Things which are equal can be put in a one to one correspondence, while being the same requires that they are so in every respect. A six member Zermelo set is equal to a six member von Neumann set (they are of the cardinality) but they are not the same (the theorem applies to one not the other).

There shall be no decision whether there is one set theory to which mathematics can be successfully reduced. There is always the possibility with the advancement of knowledge that a better foundation or that a multitude of them will be found suitable for various parts of mathematics (e.g. analysis, topology, etc.).<sup>8</sup> Yet this discussion goes beyond the case of arithmetical knowledge and thus falls outside the limits of this thesis.

Maddy's naturalist strategy must now be considered in more detail. It is useful to consider Gödel's work because Maddy follows his lead in three respects: a commitment to

entity-realism, an attempt to account for cognitive access and the idea of a two-tier epistemology. According to Gödel, abstract objects exist mind-independently. He appeals to acts of intuition to gain cognitive access to them.

But, despite their remoteness from sense experience, we do have something like a perception of the objects of set-theory, as is seen from the fact that the axioms force themselves upon us as being true. (Gödel 1990b, 268)

Of course, he does not require that one use one's faculty of vision. For him, the idea of perceiving the objects of set theory is metaphorical and is intended to suggest some super-sensory faculty. Mathematical knowledge, at any rate, is justified internally since one has direct access to objects (they need not be inferred as indispensability theorists suggest). Yet intuition cannot account for all of the principles of set theory. Gödel, therefore, embraces a two-tier epistemology. He writes:

[E]ven in case it [an axiom] had no intrinsic necessity at all, a decision about its truth is possible also in another way, namely, inductively by studying its 'success', that is, its fruitfulness in consequences and in particular in 'verifiable' consequences, i.e., consequences demonstrable without the new axiom, whose proofs by means of the new axiom, however, are considerably simpler and easier to discover, and make it possible to condense into one proof many different proofs. (Gödel 1990b, 182)

A two-tier epistemology engenders a justificatory division of labour. The first-tier concerns what is foundational (e.g., the Peano axioms), which, according to Gödel, are acquired by intuitions of the objects of set-theory. The second-tier concerns what is not obvious like, for example, the axiom of

choice.<sup>9</sup>

Broadly, Maddy's realism, first, is consistent with Gödel, whereby both the domain of mathematics and science, one is constrained by a reality not of one's own making (Maddy 1990, 33). She writes:

Realism, then (at first approximation) is the view that mathematics is the science of numbers, sets, functions, etc., just as physical science is the study of ordinary physical objects, astronomical bodies, subatomic particles, and so on. (Maddy 1990, 2)<sup>10</sup>

To account for objectivity, according to her, it is not good enough that mathematical objects exist since even an idealist can assent to that. Sets must exist mind-independently (Maddy 1990, 6). Metaphysically, entity realism holds for both science and mathematics, and epistemologically, the causal constraint applies more or less across the board (it is supposed to apply to the axioms of set theory) (Maddy 1990, 15). As she explains:

The Platonist's hope is that an account of what makes one pattern of sensory stimulation into a perception of a physical object might also provide an account of what makes another pattern of stimulation into a perception of a set of physical objects. (Maddy 1990, 49)

Just as one can know about apples from looking at them, eating them, and so on, according to Maddy, some casual interactions with sets are to provide knowledge of them. Furthermore, she claims that there is no ontological priority of everyday objects to sets because the world is inherently physical and mathematical (Maddy 1990a, 273). She clarifies her view vis-a-vis structuralism:

For the set-theoretic realist, this 'structure' consists of real objects, the sets; these are the bed rock, the things that instantiate various mathematical universals. For the structuralist, it is just one more structure, made up of featureless points in certain relations. (Maddy 1990, 173)

Just as a type, "apple" is instantiated by particular apples, so too, according to her, is "set" learned from its mind-independent tokens.

Second, however, one must confront the problem of cognitive access to sets. As Maddy says, "...[W]e need to explain what intuition is and why it works; we need to catalogue extrinsic methods and explain why they are rational methods in the pursuit of truth" (Maddy 1990, 35). She contends that the idea of intuition of which Gödel spoke must be made concrete (Maddy 1990, 41).<sup>11</sup> Unlike Gödel's, Maddy's sets are located in space-time (Maddy 1990, 78, 178). On her account, therefore, one can see sets as one sees an apple. Talk of perception must be taken literally.

Finally, Maddy also follows Gödel in embracing a two-tier epistemology for mathematics to account for where perception falls off (Maddy 1990, 173).<sup>12</sup> She writes, "[T]he most primitive truths are intuitively given, obvious; the more theoretical hypotheses are justified extrinsically by their consequences, by their ability to systematize and explain lower-level theory, and so on" (Maddy 1990, 106). Elementary set theory (the first-tier) will be explained naturalistically. What cannot be generated from the first-tier can be justified by its usefulness to the theory itself.<sup>13</sup>

Maddy (like Kitcher 1984) concentrates on the first-tier, working out her naturalism.

Maddy's naturalized epistemology is motivated by the assumption that concepts simplify the world (Maddy 1990, 7). Concepts allow one to see things (like sets) that one may not otherwise notice. She says, "What actually happens is developing neurological mediation between purely sensory inputs and our own primitive beliefs about physical objects" (Maddy 1990, 7). According to Maddy, just as one can see gold, one can perceive a set (e.g., a table, chair, and inkwell). She contends that by isolating samples of the "kind 'set'...the word then refers to the kind of which these samples are members" (Maddy 1990, 48). By interaction with particular sets one acquires the concept "set", (which connotes that kind), and helps one, in turn, see their instances.

Maddy claims that when one sees three things, one acquires perceptual beliefs about them, in which the concept of set participates. Learning to perceive a set, for example, entails the knowledge that things can be grouped together (Maddy 1990, 58). She concedes, however:

...the amount we know about things by perception is very limited...the same goes for sets...[For example] nailing down this number-bearer's more esoteric properties is a theoretical matter. (Maddy 1990, 61)

One cannot, for example, perceive a set being a member of itself. One cannot learn everything about set theory from perception, yet Maddy takes the case to be analogous to seeing

other medium-sized objects (where one may have to infer, for instance, that they have "back sides").

According to Maddy, perceiving objects results in changes in one's brain (neural pathways are developed). Seeing a triangle as one (not just, say, sense data) requires focusing the eyes in a certain way and habituation in general, which leaves its mark upon the brain, as it were. The brain is physically changed as one learns, from the most basic activities (like seeing an object) to complex ones (playing drums, doing gymnastics, mathematics, and so on). Aristotle was right in pointing out the importance of habit in human development.<sup>14</sup> Maddy brings him up to date:

In other words, cell-assembly is what permits the subject to see a triangle with identity, to acquire perceptual beliefs about it...Crudely put, human beings develop neural object detectors which allow them to perceive independent, physical objects. (Maddy 1990, 58-9)<sup>15</sup>

According to her, one also develops neural object detectors for sets, which she leaves to cognitive scientists to explain (Maddy 1990, 181):

...[I]nteractions with sets of physical objects bring about structural changes in the brain...and that the resulting 'set detector' is what enables adults to acquire perceptual beliefs about sets. (Maddy 1990, 65)

She contends, for instance, that by seeing an apple one "can hardly help seeing the apple as a unit" (Maddy 1988, 281). Maddy's program can be looked upon as a Post-Quinian project which attempts to extend naturalism to mathematics.

Maddy claims that it is an additional cognitive ability to be able to see a pair of apples, that is, things can be

grouped together in sets, as opposed to just seeing two separate things. A set, unlike an aggregate, can uniquely divide into numbers.<sup>16</sup> For instance, an apple could be one thing, or many (a stem, body, and so on). Conversely, a set which is a pair can only contain two members.

According to Maddy, when one sees an apple, one can also see a set (a singleton): there are two objects, the apple and the set. A deck of cards can be one thing or fifty-two cards; similarly, according to her, a set can be in the same place as other objects and be a separate entity (Maddy 1990, 59). She claims that seeing sets is like the picture that is often shown in psychology courses of the old and young women (Maddy 1990, 64-5). If one looks at the picture one way, one sees the young women, and in another way, the old one. Similarly, one may see, for example, an apple or singleton (a set with one member).

The ability to see sets also allows one to group together different things (a table, chair and inkwell). Maddy claims that the concept "set" does not create sets, but allows for their perception. She claims that a set and an object are separate entities though there is no perceptible difference between them (Maddy 1988, 281). As she says, "The existence of mathematical entities is just the existence of physical stuff with certain structural properties; the existence of numerable physical objects is the existence of mathematical ones" (Maddy 1988, 283).

Maddy contends, however, that the naturalist can "ignore" realist talk of mind-independent sets and can focus on the:

advantages and disadvantages of these statements as a means towards particular mathematical goals. If and when the naturalistic philosopher does turn to metaphysical questions, she is constrained by the conclusions of her naturalistic methodological inquiries. (Maddy 1997, 233)

But alas, like Gödel, Maddy's early account did seek an account of mathematical truth that will depend upon set theory (where sets exist mind-independently) (Maddy 1990, 75).<sup>17</sup>

### (5.2) Ontological Double-Vision.

Maddy's program falters on a philosophically far more fundamental issue than debates over the foundations of arithmetic. The fatal problems with Maddy's program which all surround her attempt to establish the mind-independence of sets.

First, Maddy's card example - a pack of cards could be one thing or fifty-two cards - does not provide evidence for several objects, but rather two descriptions of one thing. Chihara sums up the problem:

Evidently, it [the set] looks exactly like the apple itself. After all, I don't see anything at that exact region in space that looks different from the apple, since it has exactly the same shape and colour. Perhaps it feels different. Let's touch it. But I can't feel anything there other than the apple. Evidently, this strange entity feels no different from the apple. How about its smell or taste? Again, it would seem that the set must be identical in smell and taste to the apple. So it looks, feels, smells, and tastes exactly like the apple and is located in exactly the same spot and at exactly the same time - yet it is a distinct entity! (Chihara 1982, 223-4).<sup>18</sup>

Set-theoretic realism is peculiar, which is not an argument

itself, but provides the motivation to scrutinize it. One must heed Goodman's generating principle. The principle is the idea that one should never generate two things from the very same material (Goodman 1964, 200). One should notice that not heeding Goodman's generating principles leads to absurd consequences, i.e., the very peculiarity already alluded to. One could generate an infinite ontology from one apple. Following Maddy's line of reasoning, one could claim, for instance, that a set is co-extensive with a set, gremlins, Santa Claus, and so on. Each entity may have its own unperceptible qualities (e.g., Santa Claus is concerned for children).<sup>19</sup> The problem of an overblown ontology is raised by claiming that sets exist mind-independently because one could generate an infinite amount of them.

Second, according to Hale, sets do not admit of disconfirmation.<sup>20</sup> Disconfirmation presupposes some standard by which a group comes to the same conclusion and it is evidenced by a resulting consensus. With a medium-sized object, the standard of disconfirmation (let us suppose) says that when one sees properties X, Y and Z, one should assert "apple". There is widespread agreement in everyday life when attesting to an apple's existence.

Similarly, one could say that when one is confronted with properties A, B and C, one should assert "set". Yet one could not prove to someone that seeing a singleton was wrong. In the case of a disagreement about the mind-independent existence of

different sets, there will be no consensus because there is no perceptual difference among the sets and the objects they are dependent upon. Assenting to all the sets that a group of people see does not provide consensus among the group either (unless they all agree with the totality of perceptions).

With say, an apple, there will be widespread agreement when it is present, save the case where one argues that there are only time-slices of elementary particles.<sup>21</sup> The case of atomistic reductionism is a debate about eliminating different levels of description in favour of one. The debate about the mind-independent existence of sets, however, can be fought at one level of description - that of everyday objects - and does not require an ontological elimination. Thus there is disagreement over the existence for sets that has no parallel for other objects. Hale concludes, "[T]aking the analogy seriously involves...holding lower level set theoretic propositions to be vulnerable to empirical disconfirmation in ways that they are not" (Hale 1993, 59).<sup>22</sup>

Finally, Emily Carson has meticulously challenged two of the key aspects of Maddy's set-theoretic realism, which concern the first-tier and its relationship to the second one.<sup>23</sup> Carson points out that for Maddy perception of sets does not provide knowledge that they can be members of themselves. One may see two apples as a set. Yet one cannot see a set of the unit set. Carson writes:

But the property of being able to be a member of another set is surely not an esoteric feature of sets; it is an

*essential* feature, basic to the very theory of sets...Until one has shown that we perceive something as an object with *this* property, one has not shown that we perceive sets...Because we do not perceive these things as objects capable of being members of other sets, they simply are not sets, but at best mere aggregates" (Carson 1996, 7).

According to Carson, because Maddy's perceived sets lack the quality of being able to be members of themselves, they are not sets. Carson contends that what one can perceive is aggregates not sets.<sup>24</sup> The first-tier of Maddy's epistemology fails.

Carson adds, however, that even if we grant that one does perceive sets, it leaves one stuck with sets of a low rank. She writes, "There remains an epistemological gap between knowledge of the things we might be said to perceive and general knowledge about sets, including those higher up in the hierarchy" (Carson 1996, 10). She contends that one has no way of perceiving higher order sets.

At the second-tier, inferring the existence of higher order sets - because doing so has useful consequences - gives rise to Carson's next argument. She writes, "What is lacking thus far is an explanation of the relation of these perceivable sets to the sets postulated at the theoretical level" (Carson 1996, 13). Carson says that a two-tier epistemology depends on a natural kind claim (sets all share the same essential properties), but, "[T]he 'natural kind' claim fails, and there is no intuitive conception of set that might link the finite sets to the infinite ones" (Carson 1996,

16).<sup>25</sup> According to her, sets inferred (by projection occurring in the second-tier) may not be of the same natural kind as the first ones (which were perceived).

Carson concludes that the first-tier without the second is not an epistemology of set theory (one is stuck with sets of a low rank). The second-tier without the first is not a realist epistemology (one no longer perceives mind-independent sets). Moreover, Maddy must show that the sets perceived at the first-tier are of the same natural kind as the ones endorsed for extrinsic reasons at the second-tier (Carson 1996, 16).

The notion of sets being members of themselves is in analogy with physical objects. The analogy breaks down when one tries to imagine seeing a set of a unit set. The reason one cannot see stockpiles of sets is because they are abstract objects.<sup>26</sup> Carson's attack on the first-tier is unassailable. As Boolos says, "It is haywire to think that when you have some Cheerios, you are eating a set - what you're doing is: eating THE CHEERIOS" (Boolos 1998, 72).

One should keep in mind, however, that Carson's criticism of Maddy does not throw into disrepute mathematical realism or the idea of a two-tier epistemology itself. Putnam has shown that mathematical realism can be maintained by looking to the consequences of a body of knowledge. Also, the problem of the relationship between what is justified by two-tiers does not, for example, threaten Kitcher's account. According to him,

since sets are constructed by a rule, one can be assured that the product would be of the same kind.<sup>27</sup> By embracing a two-tier epistemology, therefore, an empiricist can, in principle, account for both the acquisition of basic mathematical concepts and what is thereafter generated (Chapter 7 & Section 8.1).

### **(5.3) The Empiricist Legacy.**

In this section, it shall be shown that Maddy does not need the ontological thesis to be a realist. Naturalism could be consistent with three possibilities: (1) naturalism and idealism (anti-realism); (2), naturalism plus the existence of mind-independent sets (set-theoretic realism); and (3), naturalism and epistemic-realism (natural realism).

Idealism, for present purposes, is the view that truth is mind-dependent. (The recognition-transcendence of the truth-values of statements is denied.) One of the species of idealism is subjectivism. A conventional epistemology is a form of subjectivism; it is the idea that knowledge is relative to a group consensus (see discussion of conventionalism in Section 8.3).

Thomas Norton-Smith, in "An Arithmetic of Action Kinds: Kitcher Gone Mad(dy)", utilizes the naturalism of Kitcher and Maddy, while exorcizing the claim that sets are mind-independent. He replaces Maddy's sets with Kitcher's act of collecting. He provides an analogy for sets, which captures the idealist position:

In viewing a rock as a tool, and using it as such, I have not discovered or recognized a property of the rock; in addition to having a certain density and composition, properties which make the rock suitable for use as a tool, the rock does not also have the property of *being* a tool...there is only the activity - viewing the rock as a tool. (Norton-Smith 1991, 219)<sup>28</sup>

The world (the rock) is amenable to being viewed as a tool but it is not so in-itself. He is a realist about the rock and a idealist about artifacts. On his account, the epistemological difference between objects and sets is that the former are mind-independent and the later are not. Similarly, one could see a picture of the old and young woman (referred to earlier), for example, as that of lines on the page (Balaguer 1994, 105). According to the idealist being considered, there is only one object there (the lines). The picture is constructed.

Though the act of collecting presupposes the world is amenable to that activity, the agent-dependency of the product indicates idealism. (e.g., according to Kitcher, numbers are predicates of collective operations (not properties of mind-independent sets) (Kitcher 1980, 225; Norton-Smith 1991, 218)). A idealist would claim that sets are produced by cognitive process (such as the ones Maddy alludes to) and the truths about them are relative to the creator.

Maddy's naturalism may collapse into idealism of some form, which is the reason she had earlier posited mind-independent objects. Traditionally, a realist wants one's statements to be true or false because of some independently

existing reality (say, the surface of Mars). According to Maddy, one sees a set, as one perceives gold, and articulates its properties. Yet if one cannot verify the existence of mind-independent abstract objects, one is forced to an extreme form of metaphysical realism which is consistent with a skeptical epistemologist, i.e., the idea that the truth-values of arithmetical statements are actually recognition-transcendent (Section 8.2).

Maddy was prompted to assent to the mind-independence of sets because of a false dichotomy: either idealismivism or entity-realism. There is a third option which she must consider: an optimistic epistemology, i.e., the idea that even if knowledge (about subject S) is a subjective creation, one could still be an epistemic-realist about S (the statements of S have truth-values if one can adduce them or not).<sup>29</sup> As the pragmatists argued, one can rely upon the practical consequences of arithmetic to justify that body of knowledge (Chapters 3 & 8). If Maddy followed Putnam and advocated an epistemic-realism, her problems would evaporate. In the next chapter, Kitcher's empiricist strategy shall be considered.

## PART III

### NOMINALISM

## 6. Kitcher's Empiricism

In The Nature of Mathematical Knowledge, Kitcher defends a naturalist approach to foundations of mathematics that builds upon the ideas of J.S. Mill. According to Kitcher, Mill's empiricism for mathematics attempted to attack transcendentalism in favour of a naturalism for all of knowledge (Mill 1846, Bk. II, chap. 5, sec. 4.)<sup>1</sup> and argued against the necessity of mathematics ((Mill 1846, Bk. II, chap. 5, sec. 1; Kitcher 1998, 100)). The two points are related. Mill and Kitcher assume that denying the mind-independence of abstract objects entails the rejection of mathematics' necessity.

First, Mill's empiricism shall be considered and defended against Frege. Second, historical and anthropological evidence is marshaled in the defense of empiricism. Third, Kitcher's attempt to build on Mill is considered. Fourth, contemporary naturalistic programs are considered and the rejection of arithmetical necessary is criticized. Furthermore, it is pointed out that naturalism need not slide into subjectivism.

### (6.1) Playing With Pebbles.

Kitcher points out that Pre-Fregean philosophy (like that of Mill) is peculiar not only in its epistemological preoccupations but the "willingness to draw on the ideas of the emerging sciences, to cull concepts from ventures in psychology and physics, later still to find inspiration in Darwin" (Kitcher 1992, 54). He goes on:

Twentieth-century historians of philosophy would ultimately reclaim the great early moderns by sanitizing their psychological and other scientific references...In recent years, confidence in conceptual analysis and in 'first philosophy' has begun to waver...Some of their endeavours mark the return of epistemological naturalism, scorned by Frege and labelled as illicit by Wittgenstein. (Kitcher 1992, 55)

Kitcher traces empiricism to Aristotle, and more recently, to the British empiricists (Kitcher 1998, 57).<sup>2</sup> Naturalism has returned with Quine (Kitcher 1992, 114).

Yet of the British empiricists (e.g. Hume and Locke) confronted with the case of mathematics, it was only Mill that did not succumb to transcendentalism. Mill's empiricism won him the notoriety for having attracted more "ridicule and disdain than the positions of any other thinker in history" (Kitcher 1998, 57).

According to Mill, abstract mathematical objects, often appealed to by platonists to account for necessity, do not exist - he did not have to worry about access to mind-independent abstract objects (Mill 1846, Bk. II, chap. 5, sec. 2; Bk. II, chap. 6, sec. 2). Mill's philosophy illustrates the type of epistemology which Benacerraf thought was desirable:

that is, one where the causal constraint requirement applied to knowledge.

Mill had to choose between regarding mathematical propositions as being necessary (and *a priori*) or empirically well-founded (and not necessary). He chose empiricism. He writes:

A principle ascertained by experience is more than a summing up of what we have specifically observed in the individual cases that we have examined; it is a generalization grounded on those cases, and expressive of our belief, that what we there found to be true is true of an indefinite number of cases which we have not examined, and are never likely to examine (Mill 1846, Bk. II, chap. 1, sec. 3).

He goes on to note, that such inference "is not susceptible of question" (Mill 1846, Bk. II, chap. 1, sec. 3).

Mill is aware that points without magnitude, straight lines, lines without breadth, circles with all radii equal, squares with perfect angles, and so on, do not exist (Mill 1846, Bk. II, chap. 5, sec. 1). Yet since they cannot be nothing, he says, "The definitions, as they are called, must be regarded as some of our first and most obvious generalizations concerning natural objects" (Mill 1846, Bk. II, chap. 5., sec. 1). For example, one can take two pieces of wood and notice one is straighter than another - thus achieving the idea of "straightness" by abstraction.<sup>3</sup>

Consider the idea that two straight lines cannot enclose a (Euclidian) space. Mill writes, "experimental proof crowds in upon us in such endless profusion...that we should soon have a stronger ground for believing the axioms [that anything

else derived from the senses]" (Mill 1846, Bk. II, chap. 5, sec. 4). He calls mathematical axioms experimental truths - they are verified (Mill 1846, Bk. II, chap. 5, sec. 4). The axioms (or definitions) of mathematics have no certainty beyond the evidence of observation and experience (Mill 1846, Bk. II, chap. 6, sec. 1).<sup>4</sup> The rules that apply to numbers are the same for magnitudes that one can encounter in the world (Mill 1846, Bk. II, chap. 6, sec. 2).<sup>5</sup> He writes, "The fact asserted in the definition of a number is a physical fact" (Mill 1846, Bk. III, chap. 24, sec. 5).

According to Mill, induction is the foundation of the science of numbers (Mill 1846, Bk. II, chap. 6, sec. 2).<sup>6</sup> Ascertaining a principle from experience means making a generalization. According to him, induction moves from particular cases towards a general truth (Mill 1846, Bk. III, chap. 1, sec. 1). As he writes, "All inference is from particulars to particulars: General propositions are merely registers of such inferences already made, and a sort of formula for making more..." (Mill 1846, Bk. II, chap. 3, sec. 4). For example, once one has a definition for a triangle, one can construct more of them (say, by checking that all the angles add up to a hundred and eighty degrees) (Mill 1846, Bk. III, chap. 2, sec. 2). The general truth is in turn verified by its application to specific cases (Mill 1846, Bk. III, chap. 3, sec. 1).

Empirical truths, however, have often been taken to be

tentative (not necessary). An empirical truth (so the claim goes) always allows the possibility of refutation, revision, or amendment. Mill claims that the particular certainty attributed to mathematics "is an illusion" (Mill 1846, Bk. II, chap. 5, sec. 1). Mill says that if one defines "necessity" as what cannot be conceived otherwise, one will notice (let us suppose) that this has to do with habits of the mind not how things are in themselves (Mill 1846, Bk. II, chap. 5, sec. 6).

Frege claims that Mill's naturalism has several problems. Frege's criticism have been influential, for instance, in the Vienna circle. The logical positivists considered an empirical justification of arithmetical knowledge untenable because of criticisms like that of Frege's. In order to defend Mill, one must, therefore, tackle Frege's criticisms. A first criticism is based on the fact that all the properties of magnitudes do not apply to arithmetical concepts. For example, Frege claims that addition is not equal to "heaping up" (Frege 1953, sec. 9).<sup>7</sup> For example, numbers are not extended. Also, the spatial relation of an apple, orange and pear can change, but this does not effect the number "three". Conversely, numbers possess properties - like being odd or even - that magnitudes do not have (Frege 1953, sec. 10). Also, numbers conceived of as collections - in the physical world - do not tell one how to divide them up. A tree with one thousand leaves or a deck of cards could count as one thing or many (Frege 1953, sec. 22, 25).<sup>8</sup>

Kitcher, like Mill, however, does not require that arithmetical concepts, such as addition, be the same as the physical objects from which they are derived. Rather, Kitcher seeks to explain how, on the basis of interactions with physical objects, one acquires arithmetical concepts such as addition (which is formalized in the eleventh and twelfth axioms of *Mill Arithmetic* (see Section 6.3)). Speaking in terms of the act of collecting, he puts emphasis on how one acquires arithmetical concepts (Kitcher 1980, 224; 1984, 110). Empiricists need not claim, for instance, that all properties of magnitudes apply to numbers or vice versa.

Second, Frege claims that the definitions of numbers that are very large cannot state a matter of fact (Frege 1953, sec. 7).<sup>9</sup> Even if one were able to observe very large finite numbers, the series of whole numbers is infinite (Frege 1979, 277), and one cannot see an infinite group of objects. He claims, therefore, that mathematics cannot depend on experience because there are not enough things. Indeed, in a similar vein, one may wonder how one acquires the concept "0".

Yet both Putnam (1979a) and Kitcher (1984) advocate the idea of possible constructions. Very large numbers need not be observed, since they can be constructed along the lines of the Peano axioms. Transfinite numbers can be possible constructions of an ideal agent (justified by their usefulness within mathematical theory). Suffice it to say that the contentious issue of accounting for transfinite numbers will

be taken up in considering Kitcher in Section 6.3. Also considered in Section 6.3 is Kitcher's account of "0" which can be hinted at here by considering the conjecture of Daniel Isaacson:

We mark the beginning of that process [of forming the number sequence] by counting 1, and the further natural numbers arise by iteration of the process of counting the next one (by a slight increase in sophistication, we may think of 0 as counting the empty collection, and in this way as constituting the smallest natural number). (Isaacson 1998, 204)

Today's empiricists accept that much of mathematics is not empirical. As Kitcher points out, Mill (or any empiricist) can take a holistic line such that not each proposition must be verified empirically. They acknowledge that part of mathematics is derived from principles that are acquired in the manner that Mill allowed for with, for example, the concept of "straightness"; it was said by him to be acquired by acts of abstraction from observation.

By developing a two-tier epistemology like Maddy, Kitcher limits his naturalism to principles (Kitcher 1980, 102, 215, 220).<sup>10</sup> Carrying out a mathematical proof (say, a case of long division) does not require going and checking with pebbles (Kitcher 1980, 218). The fact that division is defined for all of the natural numbers is explained by the fact that the rules are the same for all of them. As Dummett writes, "If Mill had allowed that such equations [ $7+5=12$ ] did follow from the definitions, but had claimed for them an empirical status on the ground that definitions themselves rested on empirical

facts, his position would have been stronger" (Dummett 1991, 58).

Third, Frege says that Mill confuses the laws of psychology with those of mathematics. According to Frege, running together how one acquires and justifies knowledge leads to the threat of psychologism, which has been discussed in Section 4.1. According to Frege, the origin and application of arithmetic must be separated from its justification (Frege 1953, v, ix, sec.9, 17, 26, 93).<sup>11</sup> Frege writes:

It not uncommonly happens that we first discover the content of a proposition, and only later give the rigorous proof of it, on other and more difficult lines; and often this same proof also reveals more precisely the conditions restricting the validity of the original proposition. In general, therefore, the question of how we arrive at the content of a judgement should be kept distinct from the other question, Whence do we derive the justification for its [a judgment's] assertion? (Frege 1953, sec. 3)

Justification entails a judgement that, in turn, must be defended. Frege writes:

Now these distinctions between a *priori* and a *posteriori*, synthetic and analytic, concern, as I see it, not the content of the judgment but the justification of making the judgment. Where there is no justification, the possibility of drawing the distinction vanishes...When a proposition is called a *posteriori* or analytic in my sense, this is not a judgment about the conditions, psychological, physiological and physical, which have made it possible to form the content of the proposition in our consciousness; nor is it a judgement about the way in which some other man has come, perhaps erroneously, to believe it true; rather, it is a judgement about the ultimate ground upon which rests the justification for holding it to be true. (Frege 1953, sec. 3)

According to Frege, justification for *P* must be separated from the conditions that gave rise to *P*.<sup>12</sup>

As pointed out in Section 3.1, however, the Quinean naturalist uses the description of psychological process as an account of the foundations of knowledge (there is nothing epistemologically deeper than acquisition). Similarly, Mill shows how the principles of arithmetic are acquired (and can explain arithmetic's utility since physical reality is mathematically amenable). A naturalist rejects the distinction between justification and discovery (Section 2.3). Galloway comments on traditional epistemology (like Frege's):

...[N]o purely descriptive account of the process by which a particular belief was acquired can by itself provide a justification for that belief. Justification always requires, over and above the descriptive account, some further, non-empirical account of the circumstances under which true beliefs acquired in that kind of way count as knowledge...[Naturalized epistemology] rejects as utterly misconceived [traditional philosophical epistemology's] aspiration to provide human knowledge with a justificatory grounding that is independent of our actual cognitive practices. (Galloway 1992, 334-6)<sup>13</sup>

The naturalist's rejection of Frege's alternative rests on the implausibility of providing a non-empirical account of the principles of arithmetic. One may wish to recall that it was argued in Section 4.2, the Fregean platonist comes to grief on the second horn of Benacerraf's dilemma.

Fourth, Frege says that arithmetic cannot rest upon induction. According to him, induction can only be justified by a theory of probability or habit (Frege 1953, sec. 10). On the one hand, if it rests upon probability, that already presupposes arithmetic because one has to talk of numbers. On the other hand, if arithmetic rests upon habituation it is

subjective and therefore cannot be a foundation for arithmetic.

Both of Frege's assumptions can be challenged. The justification of induction does not depend on arithmetic in any epistemologically significant way. Arithmetic is required to talk about probabilities not to justify them. For example, one could be a realist about the truth-values of probability statements without maintaining that numbers exist mind-independently. The statements in question would have a truth-value independent of arithmetic because they are about states of affairs.

Also, habit does not necessarily entail subjectivism. For the naturalist, induction is as much a part of everyday life as arithmetical truth. Inductive reasoning may be a biologically ingrained disposition, as it were, due to habituation to probability events. As Quine remarks, "[E]volution helps to clarify...induction, now that we are allowing epistemology the resources of natural science" (Quine 1969, 90).

Finally, according to Frege's principles (see Section 4.1), realism demands the exclusion of psychological accounts because they are bound to collapse into subjectivism and thus cannot serve as the foundation of arithmetic.<sup>14</sup> Basically, if one is describing certain mental processes or states, the consequence is that arithmetical knowledge may not be about mind-independent truths but the nature of the subject (Frege

1979, 144).<sup>15</sup> It suffices to say, however, that naturalism does not logically entail subjectivism (Sections 2.3, 5.4, 8.2). Moreover, subjectivism is not a threat unless one assumes that all habits (i.e., beliefs) are as good as any other (which is denied in Sections 5.4). Habituation need not entail subjectivism if one has the correct habits.

### **(6.2) The Origin of Number.**

Historical and psychological evidence suggests that various aspects of arithmetical knowledge actually originated in ways that Mill proposed. In a nutshell, numerical knowledge came about from manipulating physical objects.<sup>16</sup> Karl Menninger contends, in a study of the cultural history of mathematics, that "Early man's environment determined his thinking and actions, and also his counting" (Menninger 1969, 9). According to him, both numbers and their notation arose by a series of abstractions from a lived-world (Menninger 1969, 86). Counting practices freed themselves from this origin. Primitive counting was sometimes carried out by making notches on tally sticks (Menninger 1969, 240). The Fiji Islanders, for example, placed notches on their clubs to record how many they had killed (Menninger 1969, 39).<sup>17</sup> There was also body-counting:

The term 'body-counting' designates the number sequences arrived at by some...peoples in which the parts of the human body - the head, the eyes, the arms, and so forth - were arranged in a certain order. (Menninger 1969, 34)

For example, a tribe on the island of Papua counts the right eye as ten and the left as eleven. As with finger-counting,

numbers are related to parts of the body (as opposed to being completely abstract) (Crossley 1987, 12-17).

Yet counting systems that relied upon physical objects were linked to specific applications (like counting one's tomatoes). In carving notches or associating numbers with physical objects, uses were limited to numbers of a small cardinality. One did not harvest one million tomatoes. Development of the number sequence requires a separation from physical objects. Here, Menninger provides a linguistic study:

...Lithuanian, for instance, has different words for the 'grey' of geese and of horses, or wool, of human hair, and so forth, it has not a separate word for the generic concept 'grey' which is abstract or 'empty', and must be embodied or 'filled' by actual concrete objects. (Menninger 1969, 10)

Menninger remarks, "Some...peoples have completely fused the number and the object in a single entity. The Fiji Islanders, for example, call 10 boats *bola*, 10 coconuts *koro* and 1000 coconuts *saloro*" (Menninger 1969, 11). The link between number words and objects is also expressed in English:

...[A] 'last' in English (last, load) may have greater or lesser value according to the thing specified (one last of meal or herring to 12, of salt to 18 tons, of gunpowder to 24 kegs, of bricks to 500, of tiles to 1000, of hides to 144 pieces, to wool to 12 sacks). (Menninger 1969, 30)<sup>18</sup>

Here, numbers are abstract in that they can apply to any object, though they are inculcated in practices (like business) that deal with the trade (in physical objects).

Menninger observes:

At first the supplementary quality insinuated itself between the amount of objects to which numbers were

applied and the quantity of number words; once these have been formed, the supplementary quality has done its job and disappears. The number sequence is born! (Menninger 1969, 37)

Being completely abstract, the concept of numbers (like "grey") applies to any object (or none); the only practice they need participate in is that of arithmetic.

According to Menninger, the next step - and of equal importance for arithmetic - was the abstraction in notation. Notation can be formed by idiographic representations (as in China) where the concept forms the word (e.g., the word "dog" may resemble a dog). Notation can also be formed by phonetic representations where a symbol, with no pictorial significance, represents the sound of words (e.g., the number "one hundred and twelve" is not represented by strokes) (Menninger 1969, 466).

Numeral systems can be further subdivided according to use of place value notation (e.g., Chinese) or abstract gradational numerals as is the case with, e.g., the Indian notation.<sup>19</sup> Menninger comments:

[A]ll written numerals based on the law of ordering, and also the 'named' gradational numerals, require an abacus or counting board for computations: they are merely 'number representing' and not 'computational numerals'. The only numeral with which it is possible to calculate freely - that is, without a counting board or numeral tables - are the 'unnamed' gradational numerals, the abstract place-value notation with 9 digits and a zero sign like those that arose in India. (Menninger 1969, 466)

Indian notation, however, also bears idiographic traces like those found on ancient tally sticks. For example, the notation

for "one" looks like one thing (a stroke), and so on, for "two" (two strokes). The strokes, "-", "=", etc., of the Brahmi script, as well as the intermediate Gvalior, Sanskrit, and the East and West Arabic one, bear a corresponding resemblance to "1", "2", and "3" (Menninger 1969, 418). According to Menninger, the number notation (like the number sequence), was acquired by abstraction from dealing with the observation and manipulation of physical objects.

As Crossley, more generally, comments on arithmetical knowledge: "Linguistic evidence suggests that the idea of number was only gradually abstracted from the particular objects being counted" (Crossley 1987, 156). He writes:

Certainly the view that [numerical knowledge is] innate has been held by distinguished anthropologists...On reflection they appear [like their notation] in a very different light as a phenomenon which has slowly grown and developed as the need has arisen...[T]hey have developed from quite concrete beginnings into the sophisticated abstract concepts we have today...In our present culture they are highly abstract objects described by axioms but the move from concreteness to abstraction seems to have been a sudden one which took place due to Dedekind in the later half of the nineteenth century. (Crossley 1987, 1)

For example:

Euler's theorem says (roughly) that the number of vertices plus the number of faces minus the number of edges of a convex polyhedron is 2. The modern version of the theorem is completely abstract while the earliest version seems to deal with objects in the real world. I believe that similar changes have taken place in our conception of the natural numbers and irrationals. In our study of the real numbers we have seen that the irrational numbers entered Greek mathematics not as numbers but as lengths and other magnitudes. The Greeks manipulated these with great skill. (Crossley 1987, 154)

Once one had acquired the concept of number per se, one may

think that it is a self-subsistent object. Numbers are no longer about physical things, though they can be applied to them, but mind-independent abstract objects.

Newton's conception of numbers represents a transitional stage between numbers as linked to physical objects and numbers as abstract objects. Newton, for example, defines number in his Cambridge lectures of arithmetic: "...By *Number* we understand not so much the Multitude of unities, as the abstracted Ratio of any Quantity to another Quantity of the same kind, which we take for unity" (Mayberry 1988, 321). Newton's definition abstracts ratio from any quantity of the same kind as a thing. As John Mayberry observes:

...[I]n Euclidian mathematics ratios are not mathematical objects, but *relationships* between mathematical objects. Newton's numbers have a decidedly insubstantial air about them for anyone steeped in the classical, Euclidian tradition. Not only are they abstractions, but they are abstractions from abstractions, from 'things' which are themselves already unsubstantial, indeed quite literally so, since ratios belong to the category of *relation* rather than to the category of *substance*. (Mayberry 1988, 323)<sup>20</sup>

Mayberry adds that, "In fact, it was the 'quantities' to which Newton referred in his definition which were the objects that formed the subject matter of Euclidian mathematics: lines, surfaces, solids, weights, times, and finite multitudes..." (Mayberry 1988, 323). According to him, the nineteenth century took abstraction further into formalism, leaving behind the reification of abstract objects, and the concrete beginnings from which they (and numerical notation) arose (Mayberry 1988, 352).

Historical evidence (e.g., Menninger 1967; Crossley 1987; Mayberry 1988) tallies with the analysis of J.J. Gibson, the theorist of perception who offers an account of how the acquisition of abstract concepts takes place. That is to say, not only do historical records suggest that arithmetical knowledge was acquired by acts of collecting physical objects together, but that is also how individuals acquire basic arithmetical knowledge (like the concept "one").

Gibson's view is often called "ecological realism" because he emphasizes direct perception of objects in an "environment" (Gibson 1979, 7). The use of the term "realism" is suggestive of a naive realism, that is, the idea that seeing an object does not require inferring its existence. Gibson's ecological approach can be extended to origin of the formal sciences. He writes of faces, edges and vertices:

These components have meaning for environmental objects because, for example, the edge is characteristic of a cutting tool and the vertex is characteristic of a piercing tool. (Gibson 1979, 29)

According to Gibson, an environment is ecological because one encounters a meaningful world which one has a relationship to (not sense data) (Gibson 1979, 53-4). In an environment, according to him, one does not encounter sensory stimulations, space<sup>21</sup> or time.<sup>22</sup> One has, for example, good, bad or wasted time. As Gibson writes:

*Direct perception* is the activity of getting information from the ambient array of light. I call this a process of *information pick-up* that involves the explanatory activity of looking around, getting around, and looking at things. This is quite different from the supposed

activity of gathering information from the inputs of the optic nerves, whatever they may prove to be. (Gibson 1979, 147)

Gibson rejected what was the received view in the theory of perception that made reference to sensory stimulation (Gibson 1979, 304). He claims that a species perceives a thing in terms of how it is useful to that organism.<sup>23</sup> Gibson's views are, of course, controversial, but a discussion of these issues is beyond the scope of this thesis.<sup>24</sup>

One need merely emphasize here that Gibson's theory is consistent with Mill's idea of collecting, which is used to account for small numerosities, because the activity has utility (e.g., counting one's food supply or one's enemies). The ability to acquire numerical knowledge is useful for human beings in order to get around (e.g., knowing one has two tomatoes) and in their everyday dealings (e.g., when one wants to trade the tomatoes). Counting practices explain how and why one develops numerical knowledge.

### **(6.3) Kitcher's Naturalism.**

Kitcher's naturalistic story - which he refers to as "an evolutionary theory of mathematical knowledge" (Kitcher 1984, 92) - draws on Gibson's work. As Kitcher writes:

Some of the central ideas of [Gibson's] ecological realism can be used to add further detail to my account of mathematical knowledge. From a different perspective, my account may be seen as resolving a problem for ecological realism, the problem of how to fit mathematical knowledge into the ecological approach. (Kitcher 1984, 11)

One should recall that an empiricist, like Kitcher, attempts

to explain the acquisition of numerical knowledge based on the manipulation of physical objects. Kitcher, however, makes amendments to Mill's project in order to strengthen it against Frege's criticisms.<sup>25</sup>

Mathematical knowledge, according to Kitcher, is usually passed on from others (e.g., past generations). He writes:

More exactly, I shall suppose that the knowledge of an individual is grounded in the knowledge of community authorities. The knowledge of the authorities of later communities is grounded in the authorities of earlier communities. Putting these two points together, we can envisage the mathematical knowledge of someone at the present day to be explained by reference to a chain of prior knowers. (Kitcher 1984, 5)

To this, he adds that, "Mathematical knowledge arises from rudimentary knowledge acquired in perception" (Kitcher 1984, 5).<sup>26</sup> His view is supposed to account for mathematical objectivity, utility, and at the same time not to curtail classical mathematics (Kitcher 1984, 6, 115).<sup>27</sup>

One should note, however, that Kitcher is not preoccupied with the foundations of mathematics, deciding for instance if logic, definitions, axioms of geometry, arithmetic, or set theory, are the basis on which mathematics is founded (Kitcher 1984, 47). He focuses on naturalism. According to him, all knowledge rests on "simple observations" (Kitcher 1984, 91), for example, "Sets can be recast as statements asserting the existence of operations performed by the ideal subject" (Kitcher 1984, 132).

Kitcher's slogan is constituted by the empirical element (psychological acquisition), inherited mathematics (historical

transmission) and its future development (extension) (Kitcher 1984, 95). One could see the process (as described by Mill and Kitcher) unfold in young children which, by the way, resembles the historical accounts considered in Section 6.2 (Kitcher 1984, 96).<sup>28</sup> His account, for example, is consistent with historical evidence that suggests that ancient counting systems were ways of forming collections of physical objects. He writes, therefore, "[Platonism] errs by adopting a picture of mathematical reality without recognizing the route through which that picture emerged" (Kitcher 1984, 142).

At the very least, Kitcher's account is to be consistent with how arithmetical knowledge is acquired. He writes:

[W]e recognize, for example, that if one performs the collective operation called 'making two', then performs on different objects the collective operation called 'making three', then performs the collective operation of combining, the total operation is an operation of 'making five'. Knowledge of such properties of such operations is relevant to arithmetic because arithmetic is concerned with collective operations. (Kitcher 1984, 108)<sup>29</sup>

Kitcher contends that mathematics is not about an independently existing reality, but what the world will let us do (Kitcher 1984, 108). Mathematics, according to him, is the result of idealizations from observations, provided the activity is carried out by the ideal agent. He writes:

Arithmetic owes its truth not to the actual operations of human agents but to the ideal operations performed by ideal agents. In other words, I construe arithmetic as an *idealizing theory*: the relationship between arithmetic and the actual operations of human agents parallels that between the laws of ideal gases and the actual gases which exist in our world. (Kitcher 1984, 109)

He claims that the ideal subject should be tested by

"manipulations of reality" (Kitcher 1984, 110). That is, one could find out if one was doing arithmetic the way everyone else does.

The principles of mathematics, which Kitcher encodes in fifteen axioms, are idealizations from experience. Kitcher rewrites the first-order statements of additive arithmetic in a first order language called *Mill Arithmetic*, using non-logical first-order logical primitives and predicates and identity (Kitcher 1984, 112-114).<sup>30</sup> The primitive notions are those of *one-operation*; of a one-operation being the *successor* of another; of an operation being an *addition* to another one; and of the *matchability* of operations (he ignores multiplication for the sake of simplicity, but assumes that his account can be easily extended to it). Therefore, he uses the following primitive predicates: "Ux", "Sxy", "Axyz", "Mxy" ("x is a one-operation"; "x is a successor operation of y"; "x is an addition on y and z"; "x and y are matchable"). The first seven axioms define matchability. The first three encode reflexivity, symmetry, and transitivity.

$$(1) (\forall x) Mxx$$

For all x, x's are matchable unto themselves.

$$(2) (\forall x)(\forall y) (Mxy - Myx)$$

For all x's that are matchable to all y's, y's are matchable with x's.

$$(3) (\forall x)(\forall y)(\forall z) (Mxy - (Myz - Mxz))$$

For all x's that are matchable to y's, then all y's are

matchable to z's, then, x's are matchable with z's.

The fourth and fifth axioms defines the variables in terms of operations.

$$(4) (\forall x)(\forall y) ((Ux \& Mxy) - Uy)$$

For all x's that are one-operations and are matchable with all y's, y's are one-operations.

$$(5) (\forall x)(\forall y) ((Ux \& Uy) - Mxy)$$

For all x's that are one-operations and all y's that are one-operations, x's are matchable with y's.

$$(6) (\forall x)(\forall y)(\forall z)(\forall w) ((Sxy \& Szw \& Myw) - Mxz)$$

That is, if two operations are successors of matchable operations then the successors are matchable.

$$(7) (\forall x)(\forall y)(\forall z) ((Sxy \& Mxz) - (\exists w)(Myw \& Szw))$$

As Kitcher writes, "If an operation *a* is matchable with a successor of some operation *b* then there is an operation matchable with *b* of which *a* is a successor" (Kitcher 1984, 113).

In axioms eight to fifteen, Kitcher embodies the five Peano axioms within the language of *Mill Arithmetic*. The eight axiom works in conjunction with the sixth one to capture the third postulate of Peano number theory, which states that no two numbers have the same successor.

$$(8) (\forall x)(\forall y)(\forall z)(\forall w) (Sxy \& Szw \& Mxz) - Myw$$

As Kitcher writes, "...[I]f two operations are successor operations and are matchable then the operations of which they are successors are matchable" (Kitcher 1984, 113).

It is not clear how one could perceive "0" objects. In the defense of empiricists, however, it can be argued that one can acquire the concept "0" in terms of the abstract notion of "negation" (of having nothing to collect).<sup>31</sup> In defining "0" in *Mill Arithmetic*, Kitcher relies upon negation. For example, the ninth axiom captures the fourth postulate of Peano arithmetic, which states that "0" is not the successor of any number.

$$(9) (\forall x)(\forall y) \neg(Ux \ \& \ Sxy)$$

For all x's and all y's, it is not the case that x's are one-operations and the successor of y operations.

The tenth axiom embodies the fifth postulate of Peano arithmetic, which states that any property of "0" and every successor belongs to every number.

$$(10) ((\exists x) (Ux \rightarrow \Psi x) \ \& \ (\forall x)(\forall y)((\Psi y \ \& \ Sxy) \rightarrow \Psi x)) \rightarrow (\forall x)\Psi x$$

(for all open sentences " $\Psi x$ " of the language).

As Kitcher remarks, "Finally, the induction principle is glossed as the assertion that whatever property is shared by all one-operations and which is such that if an operation has the property then all successor operations of that operation have the property is a property which holds universally" (Kitcher 1984, 113).

The recursive definition of addition is added in axioms eleven and twelve.

$$(11) (\forall x)(\forall y)(\forall z)(\forall w)(Axyz \ \& \ Uz \ \& \ Swy) \rightarrow Mxw$$

$$(12) (\forall x)(\forall y)(\forall z)(\forall u)(\forall v)(\forall w)((Axyz \ \& \ Szu \ \& \ Svw \ \& \ Awyu) \rightarrow Mxv)$$

As Kitcher writes, these axioms state that, "[T]he result of adding one to the number  $n$  is the successor of  $n$ ; and the result of adding the successor of  $m$  to  $n$  is the successor of the result of adding  $m$  to  $n$ " (Kitcher 1984, 114).

The first two of the Peano axioms are captured by the last three axioms of *Mill Arithmetic*.

$$(13) (\exists x) Ux$$

That is, there exists one-operation (e.g., "0").

The fourteenth axiom captures the second postulate of Peano number theory, which states that every number has a successor.

$$(14) (\forall x)(\exists y) Syx$$

That is, all operations have a successor.

$$(15) (\forall x)(\forall y)(\exists z) Azxy$$

For all successor operations,  $x$ 's and  $y$ 's, are an addition of a operation.

Following Mill, Kitcher inherits a division of labour in the epistemology of mathematics. On Kitcher's account, there are first principles, which are encoded in the language of *Mill Arithmetic*. The observations from which the principles of mathematics are formed, depends on the activity of collecting, carried out by the ideal agent and they are public (i.e., forming collections is an activity that can be witnessed by others) (Kitcher 1984, 65, 109, 110, 139, 161; 1980, 224). Empiricism functions at the first level.

Most of mathematics, however, is obtained by deductions

from first principles (Kitcher 1980, 215). Consider the second axiom of Peano number theory, which states that every number has a successor (and is encoded in axiom fourteen of *Mill Arithmetic*). One has learned, for example, adding an apple to a set of them increases the membership by one. One makes an inductive inference, "membership can go on increasing by one (because it has so in the past), and in this way one forms a rule encoded in an axiom.<sup>32</sup> One may wonder how one can carry out the rule to construct the natural number series which is infinite. According to Kitcher, however, an ideal agent could carry out operations indefinitely. As he writes:

When we envisage the collective activity of a human agent who is bent on iterated performance of successor operations, we recognize that the iteration must eventually stop - at the agent's death, if not before. Despite this, we conceive of the performance as one that could easily have been continued. (Kitcher 1984, 119)

Kitcher can explain the infinity of the natural numbers in term of what operations could be carried out. One may wonder if the talk of an ideal agent is not just as obscure as the platonist alternative. Yet the of notion of 'idealization is commonplace in the pragmatist tradition (e.g., Peirce's ideal end of inquiry and Putnam's idealized epistemic conditions). As Kitcher points out, idealization is common place is science, and thus gains pragmatic assent for being a notion part of a successful practice. Also, the Platonist idea that one can perceive (grasp or intuit) mind-independent abstract objects is speculation drawn upon the analogy based upon how one interacts with physical objects. For obvious reason, there

has never been any scientific consensus that one can interact with mind-independent abstract objects. Yet Kitcher's ideal agent can be claimed to be less obscure than platonism because it is an extension of what it is already agreed upon that real agents can do.

Yet what does objectivity of the axioms of *Mill Arithmetic*, and hence arithmetic, amount to for Kitcher? According to Kitcher, arithmetic's utility and objectivity can be explained by reference to a structure that is common to all physical objects (Kitcher 1984, 107, 109). He writes, "I suggest that Mill's insight is that what we call the mathematical structure of reality consists in the way that nature permits human rearrangement..." (Kitcher 1998, 69).

The world circumscribes how one can arrange things but that only tells one that the physical world is extended and can be measured. The permanent possibility of rearrangement, for Kitcher, is defined by the ideal collector. Arithmetic, according to him, has objectivity in that it is true for all human beings. It is not only a habit; it is biologically ingrained in our species (and perhaps others, too). As Kitcher writes:

I propose that the view that mathematics describes the structure of reality should be articulated as the claim that mathematics describes the operational activity of an ideal subject...[T]o say that mathematics is true in virtue of ideal operations is to explicate the thesis that mathematics describes the structure of the world. (Kitcher 1984, 111)

Kitcher accepts that mathematics is normative because it is

applicable and he sets about to explain that.<sup>33</sup> His account looks to the nature of the (ideal) subject. But Kitcher is not claiming the ideal agent is privy to a mind-independent reality (Kitcher 1978, 134). The agent he discusses is the ideal human agent. Nonetheless, if human cognitive processes had been different, an alternative mathematics would be possible (something one cannot even conceive because that perspective is unavailable). On his account, therefore, mathematics is not necessary.

According to Kitcher either the truth-value of mathematical statements is ultimately derived from experience (in which case one could be wrong) or they are *a priori* depending upon intuition (Kitcher 1984, 64). The largest part of mathematical practice, however, is deductive and hence bound to appear to be certain. According to him, the assurance that comes with following formal rules is deflated because the principles that provide the foundations are empirically based. By rejecting the mind-independence of the axioms and adopting an empiricist epistemology, Kitcher loses the necessity of arithmetic.

Mill, and empiricists after him, like, for example, Kitcher, Lakatos, and Quine, all (at least on occasion) deny that any type of knowledge is necessary.<sup>34</sup> The empiricist can explain the history of mathematics and science - so the claim goes - because they accept in principle that all of knowledge is a tentative construction, some of them are more

established, others less so. If one had *a priori* knowledge, why have humans got it wrong so many times?<sup>35</sup> Why has it taken so long to acquire the knowledge of "0"? The *a priorist* makes mathematical (and scientific progress) a miracle. Empiricists, therefore, look upon revisability as an asset because they think it explains epistemic progress (Kitcher 1980, 219; 1984, 8).

#### (6.4) Naturalism At Work.

The challenge of naturalized epistemology - using science to justify itself - has been (perhaps inadvertently) taken up by scientists. In a recent symposium, whose proceedings are published in Mind and Language, Karen Wynn, a psychologist, proposes that some mathematical ability is innate and thus attacks mathematical naturalism like that of Kitcher. First, Wynn points out that if empiricists were to claim, for example, that seeing is required for numerical knowledge and those blind from birth possess such knowledge, their thesis is refuted. She claims that empiricists need "some detailed specification of exactly what kinds of actions, combined with what forms of observation are sufficient for acquiring numerical truths" (Wynn 1992, 316).

Contra Wynn, however, one does not just have to see blocks (a blind person could feel them, for example). The empiricist can allow that numerical knowledge is gained in more than one way. Moreover, as Galloway, a defender of naturalism, points out, Kitcher allows mathematical knowledge

to be passed on from previous generations (Galloway 1992, 343, 344). Moreover, if seeing three dots is the basis for the number "three", it does not matter if those dots are real or imagined (of course one would to have acquired the concept of one dot).

Second, however, Wynn claims that an innate mental mechanism - the accumulator mechanism - is what allows the possibility of numerical knowledge (Wynn 1992a, 370). Wynn and Paul Bloom remark: "Axioms of arithmetic are posited to involve relations inherent in the output of the accumulator mechanism" (Wynn and Bloom 1992, 414). Wynn presents evidence that "suggests" sensitivity to "numerosity" in infants (and other species) is innate and not gained by induction over experience.<sup>36</sup> She writes, "Our intuition of the inevitability of one plus one equalling two goes far beyond our experience, even if our experience has been quite consistent in yielding the correct results" (Wynn 1992, 317).

According to Galloway, however, being able to see objects allows numerical knowledge of the kind Wynn discusses:

For if we attribute to the animal or child an ability to think of dolls, or beach balls, or whatever it is that is used in the experimental set-up, we are *already* attributing an ability to tell one doll from two, two from three, and so on for at least small-sized aggregates of dolls. (Galloway 1992, 350)

According to Galloway, acquiring the concept of an object entails rudimentary numerical knowledge (of the type Wynn cites); it is thus not innate but acquired.

Moreover, as Kris Kirby points out, empirical data is not

"decisive" (Kirby 1992, 362-3). Evidence for innate knowledge can often also support the ability-hypothesis. The ability hypothesis is the idea that knowledge is acquired due to innate faculties. For example, one cannot distinguish if learning, say, the number "two", was innate (it could be learned, for instance, during the course of the experiments she conducts).

There are others from various disciplines that advocate the idea that brain architecture, which is the result of at least 3,500,000 years of evolution, explains arithmetic's uniqueness. Stanislas Dehaene is a psychologist who claims that, "Numbers, like other mathematical objects, are mental constructions whose roots are to found in the adaptation of the human brain to the regularities of nature" (Dehaene 1997, 252). He goes on, "Some mathematical objects now seem very intuitive only because their structure is well adapted to our brain architecture" (Dehaene 1997, 7).<sup>37</sup> Brian Butterworth, who studies cognitive neurophysiology, writes: "It may turn out that there are basic capabilities that are indeed innate and universal, and that the differences in the level of adult performance will depend on experience and education" (Butterworth 1999, xiii). He claims that a number module for small numerosities is innate (which he locates in the left parietal lobe) (Butterworth 1999, xii, 19, 190). Keith Devlin, a mathematician, also speaks of "an innate facility for mathematical thought" (Devlin 2000, vii and 270), which he

claims is results from the ability to engage in abstract thinking that arises with (or is amplified by) the use of language (Devlin 2000, 285). He claims that, "The mathematical world is a product of the way the human mind encounters the physical world. Thus, mathematics is determined by both the world around us and the structure of our brains" (Devlin 2000, 142).<sup>38</sup>

On the accounts of Wynn (1992a), Dehaene (1997), Butterworth (1999), and Devlin (2000), however, it is not clear if what is established is the innateness or ability hypothesis (i.e., numerical knowledge is innate, or the ability to acquire it is innate). The innateness hypothesis is harder to defend.<sup>39</sup> The following picture is at least more plausible. As Galloway points out, if Wynn were to claim that humans have an innate disposition or ability to gain mathematical knowledge, her view comes close to that of Mill and Kitcher (Galloway 1992, 333, 358). The development of the brain functions at two stages: first, what is innately given (e.g., the ability to acquire small numerosities), and second, what thereafter becomes a possibility with learning (e.g., long division). As the field of neurological psychology, and the cognitive sciences, generally, expand, a clearer picture should emerge (Dehaene 1997, x). Philosophically, however, there is nothing to be gained from adjudicating the dispute between the innateness of ability hypotheses because both positions are naturalist variants. Arithmetic's principles

are explained naturalistically in both cases.

It should also be pointed out that, ultimately, neither the innateness nor ability hypothesis fares better in accounting for arithmetical necessity (Wynn 1992a, 370, 373). Naturalism, in the tradition of Mill's empiricist legacy, may explain arithmetical necessity say, if there is a uniformity in human nature (e.g., Putnam 1981; Wynn 1992; Dehaene 1997; Butterworth 1999; Devlin 2000). Yet changes in human nature (or that of the universe) could, in principle, result in a new set of arithmetical truths (today's arithmetical truths would be false) or conventionalism collapsing into scepticism.<sup>40</sup> That is to say, arithmetical knowledge rests upon standards that could vary among agents depending, for example, upon their biology or cultural differences. Realism would be lost and whether innate or learned, the threat of subjectivism looms large.<sup>41</sup> As Galloway and Wynn point out, the loss of necessity is a high price to pay for naturalism: the idea that mathematics could be otherwise (e.g.,  $1+1=3$ ), is not conceivable (Galloway 1992, 348; Wynn 1992, 317).<sup>42</sup>

It was pointed out in Section 5.3, however, that there are, generally, three possible views that are consistent with naturalism (roughly, idealism, metaphysical realism and natural realism). It was also suggested that natural realism must be considered because it can avoid the deficiencies of the other two accounts (also see Section 8.2).<sup>43</sup> In the next chapter, Field's nominalist program shall be considered.

## 7. Field's Nominalism

Quine writes, "[T]he nominalist of old, objects to admitting abstract entities at all, even in the restrained sense of mind-made ones" (Quine 1953, 15). Similarly, as was discussed in Chapter 1, nominalism in the twentieth century is the doctrine that there are no abstract objects or universals (Goodman 1972, 183). As Goodman writes, "Nominalism...consists of the refusal to countenance any entities other than individuals. Its opposite, platonism, recognizes at least some non-individuals" (Goodman 1977, 26). Field concurs, "Nominalism is the doctrine that there are no abstract entities...In defending nominalism therefore I am denying that numbers, functions, sets or any similar entities exist" (Field 1980, 1 and 227). Conversely, realism, "is the Platonic doctrine that universals or abstract entities have being independently of the mind; the mind may discover them but we cannot create them" (Quine 1953, 14). A mathematical realist, according to Field, is committed to mind-independent entities. He writes, "[T]he sentence 'there are prime numbers greater

than seventeen' is true only if there is at least one entity with the properties of being a prime number, being prime, and being greater than seventeen" (Field 1989, 53).

Field does not consider the third option of conceptualism or epistemic realism (Chapter 1), which he calls mathematical idealism:

[I]t is the view that mathematical entities exist but are somehow mind-dependent or language-dependent. But I find the idea that mathematical entities are mind-dependent or language-dependent obscure, and am not tempted to take this way out in opposing mathematical realism. In what follows I will simply ignore mathematical idealism as an option. (Field 1989, 228)<sup>1</sup>

It shall be argued that this third option must be considered.

First, the motivation and elaboration of Field's nominalism are sketched; second, variants of nominalism are considered. Finally, it shall be argued that Field avoids Benacerraf's dilemma at a high cost - he lapses into subjectivism - because his notion of realism is antiquated.

#### **(7.1) What Is Good For The Goose.**

Field's nominalism shares with formalists an attempt to justify mathematics without appeal to mind-independent abstract objects. As Brown writes of early formalism:

Chess and other board games consist of symbols or tokens and of rules for moving them around. No one would take the chess pieces as denoting anything. Formalists love the analogy: mathematics is just a game; mathematical objects are like chess pieces and mathematical rules are like the arbitrary rules of a game. (Brown 1999, 63)

Shapiro distinguish's between term and game formalism, which are both implicit in Brown's characterization. Shapiro writes, "Term *formalism* is the view that mathematics is about

characters or symbols - the systems of numerals and other linguistic forms" (Shapiro 2000, 142). Game formalism is the idea that "[t]he 'content' of mathematics is exhausted by the rules for operating with its language" (Shapiro 2000, 144). Game formalism, like Hilbert's program, is an attempt to account for mathematics in terms of rules alone. Shapiro comments on projects like Hilbert: "What is mathematics about? Nothing, or it can be regarded as about nothing. What is mathematical knowledge? It is knowledge of what follows from what" (Shapiro 2000, 150). As Quine writes:

[T]he formalist keeps classical mathematics as a play of insignificant notations. This play of notations can still have utility - whatever utility it has already shown itself to have as a crutch for physicists and technologists. But utility need not imply significance in any literal linguistic sense. (Quine 1953, 15)

To understand Field's motivation, one must look to the historical context. His pyrrhic victory (eliminating abstract objects and truth) is motivated by the desire to avoid foundationalist epistemology, which often entails platonism (and shall be explained in turn). One may want to recall from Section 3.1, that foundationalist epistemology is the attempt to represent the ground of knowledge in terms of the axiomatic method. One has some set of axioms or first principles, and knowledge is derived from them. According to the foundationalist epistemologist, what one takes to be the methods that serve as the foundation of knowledge cannot be justified by their employment.

Foundationalist epistemology is a problem because it has

not been explained how one can have commerce with the required objects (the axioms) one must be a platonist about and upon which the rest of knowledge is supposed to rest. As Benacerraf explained, for example, formalists postpone his dilemma. He remarked:

One obvious answer - that some of these [arithmetical] propositions are true if and only if they are derived from certain axioms via certain rules of inference - will not help here...But the regress that this invites is transparent, for the same questions must then be asked about the set theory in terms of which the answers are couched. (Benacerraf 1973, 673)

One has not explained the link between truth and proof. He writes:

...[W]hat grip we have on the theory cannot *simply* be through them [the axioms]...That still leaves it mysterious just what that grip [of the axioms] is, but that is nothing new. (Benacerraf 1999, 45)

Are the axioms of set theory abstract objects (the first horn) and, if so, how does one have access to them (the second horn)? If the rules of inference are embedded in the axioms of a system, Brown's remark, in a different context, is apt: "So it would seem that conventionalism has to make the kind of Platonistic assumptions about consequences that it so desperately wants to avoid - the consequences are 'already there' just waiting to be discovered" (Brown 1999, 140).

For a foundationalist epistemologist, however, one may never know if cognitive access obtains. The problem of justifying knowledge can be considered by way of an analogy. In the domain of mathematics, "[Gödel demonstrated]...that the axiomatic method has certain inherent limitations, which rule

out the possibility that even ordinary arithmetic of the integers can ever be fully axiomatized" (Nagel and Newman 1958, 6). The idea of incompleteness - that from any system strong enough to represent Peano arithmetic one can derive a proposition that is undecidable - has its parallel in traditional problems of foundational epistemology: the infinite regress of justification (where one can keep digging deeper, as it were) which was discussed in Section 3.1.

Anyhow, at some point, justifications run out and one is faced with the prospect of utilizing method *P* to justify *P* (i.e., of falling into circularity). Foundational epistemology, thus, has often been characterized as one trying to pull oneself up by one's bootstraps. The enlightened desire to put knowledge on a firm footing - foundationalist epistemology - gave way by its failure, in the later half of the twentieth century, to scepticism.

Goodman remarks: "Any effort in philosophy to make the obscure obvious [e.g., to do foundational epistemology] is likely to be unappealing, for the penalty of failure is confusion while the reward of success is banality" (Goodman 1977, L). On the one hand, if foundationalism fails, one may lapse into scepticism (i.e., confusion may set in: how does one know the axioms are true?). On the other hand, success may be banal (i.e., it may result in trivial truths).

Putnam, in "Mathematics Without Foundations", echoes the idea that accepted knowledge, like mathematics, should not be

abandoned for epistemological reasons. Epistemology should not force epistemic confusion because the specter of certitude cannot be achieved. He says, "I don't think mathematics is unclear; I don't think mathematics has a crisis in its foundations; indeed, I don't believe it has or needs foundations" (Putnam 1979a, 43).<sup>2</sup> According to Putnam, when philosophy meets science, it usually changes (not science) (Putnam 1979a, 44).<sup>3</sup> He thinks, however, that there is a modest gain in clarity to be achieved by extricating oneself from the foundationalist's query (Putnam 1979a, 45). He suggests an alternative:

In short, if one fastens on the first picture (the 'object' picture), then mathematics is wholly extensional, but presupposes a vast totality of external objects; while if one fastens on to the second picture (the 'modalist' picture), then mathematics has no special objects of its own, but simply tell us what follows from what. (Putnam 1979a, 48)

He claims that modality can be clarified by the set-theoretic notion of a model. Necessary means true in all models, possible, true in some. According to him, modality is as clear as notions utilized in scientific practice (Putnam 1979a, 49, 51).<sup>4</sup> He remarks, "The heart of pragmatism is the idea that notions that are inescapable to our best practice, are justified by that very fact; and in this respect, I am a pragmatist" (Putnam 1994b, 260). He writes:

'Numbers exist'; but all this comes to, for mathematics anyways is that (1)  $\omega$ -sequences are possible mathematically speaking; and (2) there are necessary truths of the form 'if  $\alpha$  is a  $\omega$ -sequence...' (whether any concrete example of an  $\omega$ -sequence exists or not). (Putnam

1979a, 49)<sup>5</sup>

On his account, mathematics is about what is and is not possible (not mind-independent objects).

According to him, platonism and modalism are equivalent descriptions. As was pointed out in Section 3.1, second-level justification does not effect practice (nor is the mass of the universe changed by either account). Nothing is lost or gained in scientific or mathematical practice from either account.

Putnam's work influenced those who picked up on the idea that one needs to reject platonism, while taking into account mathematics' applicability.<sup>6</sup> Modalism, for instance, has been elaborated, in different ways, by Hellman and Field (Hellman 1989, 61; Field 1989, 227).<sup>7</sup>

In seeking to avoid foundational epistemology and platonism, Field's account aims at the virtues Putnam hinted at in "Mathematics Without Foundations": first, Field wields Ockham's razor; mathematics has an internal justification that does not depend on positing mind-independent objects (Field 1980, 45). Field remarks:

In short, what raises the really serious epistemological problems is not merely the postulation of causally inaccessible entities; rather, it is the postulation of entities that are causally inaccessible and can't fall within the field of vision and do not bear any physical relation to us that could possibly explain how we have reliable information about them. (Field 1989, 69)

That is, platonism must be avoided at all costs. Second, Field does not view competing theories as right or wrong (they are equivalent descriptions) (Field 1989, 242). For example, as

shown in Section 3.1, various philosophies of mathematics - both platonist and nominalist - need not effect the practice of mathematics (so nothing important is lost for practice in abandoning platonism). Finally, Field accounts for mathematics' applicability (Field 1989, 1).<sup>8</sup> Though Field's account of mathematics is fictionalist, unlike fictions (say, claims about *Oliver Twist*), one can say that mathematics is valuable because it is applicable (Field 1989, 3-4).

According to Field, the indispensability of theoretical entities rests on two claims: they play a powerful role in scientific practice from which we deduce a wide range of phenomenon and no other theories with these entities can explain these phenomenon (Field 1980, 7). Field contends that mathematical objects play a powerful role in scientific practice but not that platonism is the only way to account for arithmetic's applicability. His program is an attempt to demonstrate that an elimination of mathematical entities allows for applicability (Field 1980, vii; 1989, 17, 65, 230).<sup>9</sup>

Field's strategy is: (i) to provide a nominalized formulation of science (for which he offers Newtonian gravitational theory as a paradigm to be extended); and (ii), to take mathematics as an addition to scientific theory that need only be conservative (Shapiro 1998, 233).

Mathematics is included in the body of nominalistic assertions. Field explains that nominalistic formulations of

physics and platonistic formulations "have precisely the same nominalistically-stable consequences; and so mathematical entities are theoretically dispensable in the theory of gravitation" (Field 1980, 90).<sup>10</sup> What is good for the goose (the nominalization of science) is so for the gander (mathematics).

Mathematical entities do not provide new knowledge of Newtonian gravitational theory. One can eliminate appeal to mind-independent objects (like numbers) since, according to Field, all mathematical theories are conservative. He concedes, however, that mathematical objects have a limited role insofar as they are required in order to deduce nominalistically stated conclusions from nominalistically stated premises (Field 1980, 14). Field writes:

I do not of course claim that the nominalistic concepts are any where near as convenient to work with in solving problems or performing computations: for these purposes, the usual numerical apparatus is a practical necessity. But it is a necessity that the nominalist need not forgo: he can treat the apparatus in the way suggested earlier in the book, i.e. as a useful instrument for making deductions from the nominalistic system that is ultimately of interest; an instrument which yields no conclusions not obtainable without it, but which yields them more easily. (Field 1980, 91)

Field denies that mathematics is true: "And here 'true' is being used in a purely disquotational sense [If mathematicians accept 'p' then p (Field 1989, 230)]" (Field 1989, 240). That is, he accepts the consequences of his rejection platonism. He writes:

Now, it seems to me that the issue of what is and what not 'nominalist' is totally without interest. I have no

attachment to the label. The real issue is: is there good reason to prefer a metaphysics that does without mathematical entities to one that includes them.... (Field 1989, 45)

He clarifies the issue: either you are on the platonist's side or the nominalist's side.

Field only requires that mathematics be conservative.<sup>11</sup>

The conservativeness principle is the idea that:

A mathematical theory  $S$  is conservative if, for any nominalistic assertion  $A$  and any body of assertions  $N$ ,  $A$  is not a consequence of  $N+S$  unless  $A$  is a consequence of  $N$  alone. (Field 1989, 125)

Less formally:

To say that a theory is conservative is to say that it is consistent with every internally consistent theory that is 'purely about the physical world' in that it invokes no reference to mathematical objects. (Field 1989, 240)

Conservativeness requires that a theory is truth preserving (he describes conservativeness as "necessary truth without truth") (Field 1989, 241). According to him, mathematics' applicability depends on conservativeness and not truth (truth does not entail conservativeness) (Field 1989, 243).

According to Field, there is a difference between mathematics and science. Physical theories are not conservative (Field 1980, 12). Field writes, "Physical theories about unobservables are certainly not conservative, they give rise to genuinely new conclusions about observables" (Field 1980, 14). Physical theory about unobservables provides new knowledge of observables.

At first glance, Field's division between mathematical and scientific theories tallies with the traditional idea that

empirical postulates are revisable in a way that mathematics is not. One may discover, so the reasoning goes, a mind-independent physical object that revises one's beliefs. According to him, however, since there are no mind-independent mathematical objects, there are no facts about them. He takes the final step to rid himself of platonism. For him, mathematics is, in principle, revisable (Field 1980, 34; 1989, 241 fn.13). Instead of truth, he advocates a reliability requirement:

...[W]e should view with suspicion any claim to know facts about a certain domain if we believe it impossible in principle to explain the reliability of our beliefs about that domain. (Field 1989, 233)

One may recall that Lewis draws a distinction between empirical knowledge (which obtains in a possible world), logical possibility (true in a possible world), and necessary knowledge (which is true in every possible world) (Section 2.2). Field points out that mathematics' reliability is often considered to follow from its necessity. As he writes, "[S]ince mathematics consists entirely of necessary truths, there can be no sensible problem of explaining why it is that our mathematical beliefs are a reliable indicator of the mathematical facts" (Field 1989, 233). Since, however, Field has denied mathematical necessity, the onus falls upon him to refute views such as Lewis'. His argument against mathematical necessity is composed of four points. First, Field points out that there are statements about the applicability of arithmetic, which may not be purely physical or mathematical

(he calls them "mixed statement") (Field 1989, 233-4, 253).<sup>12</sup>

For example:

[F]or some natural number  $n$  there is a function that maps the natural numbers less than  $n$  onto the set of all particles of matter. (Field 1989, 233)

It is not clear if these statements should be considered necessary or not because they are mixed statements. Field notes that one may be tempted to claim that the purely mathematical aspects could be separated from the empirical elements (hence drawing Lewis' distinction between what is necessary and defeasible). Field writes, however:

*[T]he task of splitting up all such assertions into two components is precisely the same as the task of showing that mathematics is dispensable in the empirical sciences; i.e., the task of showing that in any application of a mixed assertion...a purely non-mathematical statement could take its place. (Field 1989, 235)*

That is, separating the mathematics from the science requires demonstrating that the former is dispensable. If Field's nominalization is successful, all knowledge is defeasible according to Lewis' standard, which, one should recall, required that necessary statements be independent of all experience.

Field can account for mixed statements since their reliability is on par with empirical statements (they obtain in a possible world). According to him, mathematics is reliable because it is about what follows from what. Mathematics cannot be revisable because of empirical evidence. He writes, "They [mathematical statements] are of course

mathematically necessary in the sense that they follow from basic laws of mathematics" (Field 1989, 235).

Second, accounting for mathematical necessity in terms of rules of inference encoded in axioms postpones the issue: "[I]n particular, we need an argument that the epistemological problem does not arise for the mathematical axioms themselves" (Field 1989, 235). Field writes: "So the objectivity of mathematics (if interpreted as requiring at least the disquotational truth of acceptable mathematical axioms) requires the existence of mathematical objects" (Field 1989, 272). The spectre of platonism looms large if one wants to maintain as Lewis does that mathematics is necessary.

Third, Field criticizes Lewis' assumption that there is a link between counterfactuality and necessity. One may advocate the necessity of mathematics and claim that counterfactuals are intelligible. Field remarks, "There is a strong case to be made for saying that if the axioms were false, mathematicians' opinions would be just as they are now..." (Field 1989, 237). One must recall that for Field, truth and falsity are related to existing and non-existing mind-independent objects. According to Field, however, it would make no difference if the Peano axioms, for instance, were false (all that matters is that they are consistent). Rejecting platonism requires, for him, abandoning talk of truth.

Finally, according to Field, there is no reason to

explain mathematical reliability in counterfactual terms in the first place: "[T]here is still the problem of explaining the actual correlation between our believing 'p' and its being the case p" (Field 1989, 238). Since the link between mind-independent abstract objects and truth has not been explained satisfactorily by platonists, it would make no obvious difference if mathematical objects existed or not to those who think mathematics is absolutely necessary. One cannot explain the reliability of mathematics by appeal to mind-independent abstract objects because there is no commerce with them. Platonism may be epistemologically irrelevant:

[I]f one postulates mathematical entities one is going to have a serious problem explaining the correlation between our mathematical beliefs and the facts about those entities; and I have tried to argue that *to the extent that providing such an explanation appears impossible*, believers in a mathematical realm face a genuine epistemological problem. (Field 1989, 238)

In order to avoid the platonist's plight, Field abandons mathematical necessity.

## **(7.2) Spinning This Web..**

Field's nominalization of mathematics is structuralist. For a structuralist, a mathematical object is a featureless point formed by its relations - a node in a web of relations - and does not have an internal nature.<sup>13</sup> Richard Dedekind can be said to be structuralism's first proponent, when he defined natural numbers, in terms of the theory of structural properties (Dedekind 1963, 95). The relations in a formal system are defined in terms of logical consequence, in

mathematics by deriving theorems from axioms (or in science, by deducing testable consequences from hypotheses) (Resnik 1997, 139, 142). Burgess and Rosen remark:

[S]tructuralism is, at first approximation, the view that theorems of mathematics like [for the natural order of the natural numbers it is the case that...] should be constructed not as being specifically about some one structure of a given type... but rather as being generalizations about all structures of the given type, like [for any progressive order (and there are some), it is the case that...]. (Burgess and Rosen 1997, 146)

Structuralists, who adopt the language of set-theory, are usually opposed to nominalists, since they are committed to abstract objects like structures (Burgess and Rosen 1997, 147). A structuralist may also be a nominalist, however, if mixed with modalism about possible structures (Putnam 1979a), or if structures are conceived of as concrete models (Maddy 1997; Resnik 1997).<sup>14</sup>

Also, nominalism and structuralism can, in turn, be linked to constructivism, if, for instance, numbers are understood to be embedded in structures constructed according to effective procedures. Burgess and Rosen write that the constructivist:

[r]equires, before the existence of a real number will be admitted, that a purely mathematical specification of the number, such as will in principle permit the effective generation of the successive digits of its decimal expansion, must be given. (Burgess and Rosen 1997, 173)<sup>15</sup>

Weyl writes, for example:

In any of the numerous existential theorems in mathematics, what is valuable in each case is not the theorem as such but the construction carried out in its proof; without it the theorem is an empty shadow. (Weyl 1949, 51)<sup>16</sup>

According to the constructivist, without the required procedure that provides the proof, arithmetical truth stands to disappear into the shadowy world of undecidability, which for Brouwer and Dummett, means the rejection of LEM and bivalence (Dummett 1978, xix).<sup>17</sup>

Of course, Field's nominalism differs from constructivism because he abandons platonism and hence truth (whereas constructivists try and explain how truth is possible without platonism). Yet they are similar in a number of ways. First, constructivists emphasize the importance of procedures and proofs. According to Field, mathematics is reliable because it is about following rules. Second, constructivists hold that mathematics is agent dependent. Similarly, Field holds that mathematical objects are fictions, i.e., they are constructed. Finally, they both abandon mathematical necessity insofar as it implies mind-independence (thus, on their own terms, both must deny mathematical necessity).

Nominalist, structuralist and constructivist views have the problem accounting for questions that concern the infinity of the natural numbers: the world, one's representations of it, and one's abilities to carry out procedures are finite.<sup>18</sup> The infinity of natural numbers, however, need not be a problem for a structuralist program according to Chihara, Hellman and Resnik (Chihara 1973, 173; Hellman 1989, 140; Resnik 1997, 232).<sup>19</sup> The structuralist is capable of drawing upon realism,<sup>20</sup> empiricism,<sup>21</sup> a two-tier epistemology and

modalism,<sup>22</sup> (Sections 5.1, 6.3).<sup>23</sup> Hybrid strategies, like Resnik's, embrace realism, empiricism, and a two-tier epistemology (Resnik 1997, vii). He accepts bivalence but not recognition-transcendence, making truth, as he puts it, immanent. Resnik denies that mathematics is true independently of our beliefs and practices, yet, by making truth immanent, maintains bivalence (Resnik 1997, 171). He, therefore, is what Tennant calls a Gödelian optimist.

As Tennant has pointed out, realism has often been defined in terms of recognition-transcendence and bivalence (Tennant 1997, 179). Conversely, anti-realism has been conceived of as the doctrine that rejects the recognition-transcendence requirement and bivalence (Section 8.2). Yet Tennant shows that there is another possibility which he calls Gödelian optimism (Tennant 1997, 179). A Gödelian optimist separates the issues of recognition-transcendence and bivalence. He claims that in the domain of arithmetic, for example, bivalence holds but not the recognition-transcendence requirement. It will be explained in Section 8.3, that Gödelian optimism, however, differs from natural realism which allows for recognition-transcendence in principle and the rejection of bivalence (at least for domains other than arithmetic).

Resnik claims that concepts are first acquired from a *that*. The concept "pebble", for instance, is derived from the resemblance of pebbles so that it is individuated. Particular

things can, then, be grouped together. (He uses the word "structure" interchangeably with "pattern" (Resnik 1997, 202)). A template facilitates the perception of patterns among objects (Resnik 1997, 202, 232).<sup>24</sup> A template is a more sophisticated conceptual scheme. One can not only pick out pebbles but patterns which they may form (e.g., two sets of pebbles may form patterns that are roughly isomorphic to each other).<sup>25</sup> So, from interaction with physical objects one can obtain knowledge of abstract structures. He writes:

[I]f the point of positing mathematical objects is to describe certain patterns, then it is plausible to allow that systems of physical objects instantiating these patterns can inform us of properties of mathematical objects. (Resnik 1997, 224)

He points out, however, "But I am not claiming that we can mirror the entire mathematical universe with diagrammatic templates. After all, the latter are always finite while the former is vastly infinite" (Resnik 1997, 232).

Resnik advocates a two-tier epistemology. According to him, the axioms of a formal system are ultimately abstractions from the structure of concrete patterns instantiated by physical objects. Abstracting from the structure of patterns gives rise to formal systems, according to which the axioms, for instance, can be used to construct numbers of high cardinalities. He can account for the infinity of natural numbers. The higher reaches of mathematics are the places one has yet to discover (the relation of Resnik's project to natural realism is discussed in Chapter 9).<sup>26</sup> As with Maddy's

two-tier epistemology previously discussed in Chapter 5, Resnik claims that one can generate theorems from the first principles (Resnik 1997, 232). So, roughly, the axioms of a formal system like mathematics are empirically justified and what one derives from them is justified in terms of them. Thus, there is also an epistemological division of labour for Resnick.

### **(7.3) Hubris: Back To Field.**

Several of the problems with Field's program shall be rehearsed, though the focus shall be on criticizing the motivation for his project.<sup>27</sup> First, Field's nominalization is incomplete. Shapiro points out that Field's physical theory ranges over both points and regions (the latter is like a class). Shapiro writes, "The nominalistic physics is formulated in a second-order language, whose first-order variables range over space-time points and whose second-order variables range over regions" (Shapiro 1998, 226). Both first and second-order variables make reference to abstract objects. Since points are not extended, do not move or change, they qualify as abstract objects (they are objects of thought not perception, Chapter 1) (Shapiro 2000, 233). Moreover, according to Shapiro, for second-order languages, deductive and semantic conservativeness are not co-extensive (Shapiro 1998, 231). Deductive conservativeness does not hold for second-order languages.<sup>28</sup> Semantic conservativeness alone is relevant for second-order languages. When one's theory

mentions "regions", it denotes.<sup>29</sup> Field, therefore, is committed to platonism about points and regions.

Second, according to Shieber, presupposing a notion of consistency puts Field's notion of reliability (e.g., " $2+3=5$ ") at odds with his non-factualism (the *a priori* status of mathematics) (Shieber 1999, 389). The *a priori* is usually understood in one of two ways: either strongly, when no empirical evidence can count for or against  $X$  or weakly, when no empirical evidence can count for  $X$  (Shieber 1999, 379). Field construes the *a priori* in a manner even weaker than the weak version. According to Field's non-factualism,  $X$  is a *priori* if it is not revisable within belief system  $P$  (Shieber 1999, 382).

For Field, when a belief system changes, so does what is necessary (Shieber 1999, 383). Shieber writes:

[E]ither he [Field] gives reliability and power [e.g., arithmetic's uniqueness] a special status and thereby rejects non-factualism, or he clings to non-factualism and admits that reliability and power [e.g., arithmetic's uniqueness and applicability] just happen to be the criteria for epistemic goodness that he himself *prefers* - a preference based on little more than a matter of taste. (Shieber 1999, 389)<sup>30</sup>

On the one hand, if arithmetic's reliability and power are a measure of its epistemic merit, independent of any belief system, the non-factualist thesis fails, which is untenable because it forces Field to contradict himself. He must contradict his claim to being a non-factualist.

On the other hand, if the measure of arithmetic's epistemic merit is indexed to a belief system, the non-

factualist thesis holds. If Field maintains non-factualism for the epistemic criteria, he does not explain why there is only one arithmetic which is consistently applicable. He does not explain why only one arithmetic is conservative.

Finally, according to Field, the truth of nominalized sciences cannot pass on to mathematics because there are no mind-independent objects that belong to mathematics. Field's argument requires that one assumes (as Benacerraf did) that platonism is the only way to truth. Yet mind-independent objects are not (always) required for truth. Ontological- and epistemic-realism must be separated (Chapter 1). Field's project is ill-motivated because he ignores what he calls mathematical idealism (the third option between platonism and nominalism). The third way - realism without mind-independent entities - must be considered.

While abandoning truth, Field accounts for arithmetic's applicability. Yet he does not explain why there is only one and only arithmetic that is consistently applicable. It has been argued that Field pursues a project that has high costs with little returns. Arithmetic's uniqueness and applicability should breed humility. Natural realism shall be considered in the next chapter.

## PART IV

### NATURAL REALISM

## 8. Natural Realism

Why should one be a naturalist or realist? Definitions and defense will first be offered for naturalism, and second, natural realism.<sup>1</sup> Furthermore, it will be argued that naturalism and natural realism are consistent doctrines.

### **(8.1) Naturalism.**

In Chapter 1, it was argued that a conceptualist's denial of abstract objects' mind-dependence does not effect epistemic realism for arithmetic. The natural realism to be developed for arithmetic is a form of epistemic realism. In Chapters 2, the causal constraint was defined drawing upon Steiner's analysis. Steiner's definition (4) was shown to be consistent with Mill's empiricism: numbers (the types) are encoded in the principles of arithmetic, which are derived from events, like collecting, that the properties of physical objects (the tokens) participate in. Also in Chapter 2, it was pointed out that whether one adopts Mill's or Kitcher's empiricism misses the mark: numbers are abstractions from counting practices. Numerical knowledge, therefore, is causally constrained

insofar as it is an activity that is acquired from interaction with physical objects. It was further argued that the principles of arithmetic can be justified empirically while what one derives from them need not be. *A priori* knowledge can be discussed in terms of proximity to experience.

In Chapter 2, it was argued that Quine's "Two Dogmas of Empiricism" (1953) provides the basis for the extension of naturalism to arithmetic. Furthermore, in Chapter 6, naturalism has been discussed in relation to Mill and Kitcher. One may recall that both Maddy and Kitcher advocate a two-tier epistemology and focus on the explanation of the acquisition of the principles of arithmetic. Yet it should be pointed out that the natural realist's two-tier epistemology differs from both Maddy and Kitcher. Like them, the first-tier is to be explained naturalistically. Like Kitcher, the second-tier involves what one can derive from the principles. The natural realist follows Kitcher in explaining the principles of arithmetic as empirical generalizations, and in recognizing that what one derives from them is not empirical. Yet unlike Maddy and Kitcher, both tiers are justified, ultimately, by their consequences, i.e., by mathematics' application in scientific practice (as discussed in Chapter 3).

At any rate, all empiricist epistemologists found knowledge on sense experience (Brook and Stainton 2000, 3). The empiricist justifies knowledge from the ground up. Yet they separate knowledge's acquisition from its justification:

thus bowing to face with the problems that surround any foundationalist epistemology. For example, empiricist programs risk collapsing into idealism if they cannot guarantee a link to the external world.

Naturalism is a specific variety of empiricism, but as pointed out in Section 3.1, it avoids the traditional problems of empiricism. Naturalists blur the distinction between justification and discovery (Section 6.3). They begin with an epistemic content and utilize science to explain how it is acquired, thus avoiding the problems that surround traditional foundationalist epistemology. It was shown, again in Chapter 3, that Quine's "Naturalized Epistemology" (1969) explains that embracing naturalism avoids traditional worries about circularity (i.e., using science to justify itself). The focus shall be on defining the type of naturalism that has been advocated.

Andrew Brook and Robert Stainton distinguish four types of naturalism. Of the first two, they write (Brook and Stainton 2000, 192-3):

*Weak naturalism* is the idea that philosophical theories should be *consistent* with what science tells us about the world.

*Moderate naturalism* is the idea that the philosophy of knowledge and language should be *informed about* what empirical scientists have discovered about knowledge and mind.

Weak and moderate naturalism, are inconsistent with the naturalism of Quine. They do not avoid traditional problem with foundationalism (because they allow an epistemological

discourse that escapes the purview of possible scientific explanation).<sup>2</sup> They do not exclude foundational epistemological programs.

Skipping to the last option, "strongest naturalism", Brook and Stainton write:

*Strongest naturalism* is the idea that philosophical problems about knowledge and the mind (and almost everything else) are really scientific ones and can be adequately answered by using only the methods of science, natural science in particular.

The strongest naturalism has the problem of reductionism. It requires that epistemological accounts be given in neurophysiological terms (or ones reducible to neurophysiology). It excludes descriptions of human behaviour (e.g., ones that include free will) that are not given in physicalistic terms, thus being ruled out as a desirable option.

The third option, yet to be considered, is stronger naturalism:

*Stronger naturalism* is the idea that one accepts stronger naturalism [see below] but goes one step further. It holds that neuroscience is the only justifiable approach to cognition.

Stronger naturalism and functionalism are consistent with the naturalism propounded in Chapters 2,6 (as envisioned by Quine 1969).<sup>3</sup> It is the most viable option.

Stronger naturalism, however, has two problems. These debates it should be pointed out, go beyond the scope of this thesis but deserve to be mentioned. First, stronger naturalism

suffers under the weight of the *is-ought* distinction. Stronger naturalism does not tell one what one ought to do, what practices to adopt, but only perhaps how one acquires them. One can inquire into how one acquired, for example, the principles of mathematics, but that does not speak to their justification. Yet suffice it to say, that in this thesis justification was defended, ultimately, in terms of consequences, i.e., the applicability of mathematics. Furthermore, it may be argued in defense of naturalism, that empirical inquiry may aid in a justification of practices or values, by describing the process - the *is* - by which the *ought* comes about.<sup>4</sup> According to Quine, the practice and values which give rise to a body of knowledge may be submitted to empirical investigation.

Second, stronger naturalism is in conflict with functionalism, which provides non-physical types of descriptions (Brook and Stainton 2000, 206). Brook and Stainton write of functionalism:

The mind is certain functions of a complex system, for example, the brain; each and every particular mental state or event is some physical state or event of that system; but states described as mental cannot be reduced to states described any other way, so the mental requires its own propriety language. (Brook and Stainton 2000, 232)

Functionalism is desirable because it avoids the counter-intuitive conclusions that require ontological reductionism, which was considered in relation to nominalism and platonism in Chapter 1. For example, if the mind was considered an

abstract object, contemporary nominalists like Field would eliminate it. As it was also pointed out in Chapter 1, such types of reductionism are undesirable. Yet if the concept of "science" is broadened to include functionalist psychologies, stronger naturalism can avoid, for example, the type of physicalistic reductionism that was advocated by Field.<sup>5</sup> Yet, as shown in Chapter 6, embracing naturalism does not determine the realist debate.

### **(8.2) Natural Realism.**

The issue to be settled is thus: can one be an epistemic-realist about arithmetic once one has embraced an empiricist epistemology? Resnik, who was discussed in Section 7.2, aims in outline at the natural realism to be developed in this chapter. First, he conceives of mathematics as the study of structures about which he is a Gödelian optimist. (One should recall that a Gödelian optimist accepts bivalence but rejects recognition-transcendence.) Second, Resnik naturalizes the principles of arithmetic in terms of the perception of patterns that instantiate the structure and, finally, he advocates a two-tier epistemology.

Yet what sort of realism, if any, should one embrace anyways? Realism has often been considered untenable because of the critiques of metaphysical realism such as Putnam's (Putnam 1981, 49-50, 54). Here, the purpose shall be to explain why metaphysical realism is untenable and suggest that a revised version of realism may be tenable. Putnam offers

this definition of metaphysical realism:

On this perspective, the world consists of some fixed totality of mind-independent objects. There is exactly one true and complete description of 'the way the world is'. Truth involves some sort of correspondence relation between words or thought-signs and external set of things. I shall call this perspective the externalist perspective, because its favourite point of view is the God's Eye point of view. (Putnam 1981, 49)

There are three characteristics of realism to be dealt with here: (1) mind-independent objects; (2) one correct description of the world; and (3), a correspondence theory of truth. It has already been pointed out in Section 1.3, that the realist debate has shifted from a controversy about the mind-independence of objects to one about the recognition-transcendence of the truth-values of statements. The claim that the truth-values of statements are recognition-transcendent is the following: a proposition may be true even though no one is able (even in principle) to recognize that it is true. It is important to emphasize that realism merely asserts that these truth-values may transcend one's ability to recognize them.

There are two reasons why the term "recognition-transcendent" is preferable to "mind-independent" when discussing realism. Claiming that truth is mind-independent may mistakenly suggest that some entity is mind-independent. Also, the term "recognition-transcendence" allows one to avoid the appearance of paradox when discussing the truth-values of statements that are about the mind. That is, the truth-value of a statement which is about the mind need not be mind-

dependent (i.e., there could be recognition-transcendent truths about the mind). The term "recognition-transcendent", thus, is more precise than "mind-independent". The term "recognition-transcendence", therefore, shall be used henceforth. Furthermore, realism has been defined in terms of recognition-transcendence by Dummett, Wright, and Tennant (Dummett 1977, 7; Wright 1993, 296; Tennant 1997, 167).

What is of essence in the realist debate, however, has not changed: can one know the way the world is?

At any rate, and in a nutshell, two of the tenants of metaphysical realism were rejected by Putnam for the following reasons: first, the one correct description of the world requirement was rejected by Putnam because it requires an elimination of an everyday world for one taken to be more fundamental (Putnam 1981, 73; 1994, 492) (also see the discussion of reductive-eliminativism in Section 8.1). Second, the correspondence theory of truth was rejected by Putnam because it requires an external point of view to confirm the correspondence between the mind and world which he claims is not achievable (a God's eye point of view) (Putnam 1981, 49, 73). The reasons for rejecting the correspondence theory of truth, in essence, parallels the problem of foundationalist epistemology (Section 3.1). That is, if method *P* is to guarantee the correspondence between a proposition and the world, one has to justify method *P* and thus the problem of an infinite regress of justification becomes apparent. One never

hits rock bottom unless one assumes some sort of correspondence that justifies a certain method (see Section 7.1).

In lieu of a known correspondence, the truth-values of statements about the world are rendered actually recognition-transcendent. The requirement of recognition-transcendence is consistent with scepticism:

- A. One's best theory of the world could be wrong (metaphysical realism).
- B. Judgements are suspended, since the world may be other than it is conceived to be (scepticism).

For a Quinean naturalist, however, the idea of massive error - global scepticism - is ruled out (scepticism only functions within the scientific epistemic enterprise, not about it per se). At best, the sort of realism the sceptic could be left with is left with is called "fig-leaf realism" by Devitt. He writes:

The very weakest form of realism is completely unspecific about what exists; it requires only that *something* does. When the independence dimension is added, this 'weak realism' amounts simply to the claim that something objectively exists independently of the mental. This commits realism only to an undifferentiated, uncategorized, external world, a Kantian 'thing-in-itself'...It is a world we cannot know about or talk about. It cannot play a role in explaining any phenomenon. It is an ideal addition to idealism: anti-realism with a fig-leaf. (Devitt 1991, 17)

Put crudely, the idea is that one is a realist about some stuff regardless of whether realism holds for specific objects.<sup>6</sup>

Yet the idea of recognition-transcendence is absurd if it throws truth about specific things into the nether world of

unknowability (i.e., if it results in scepticism) (Tennant 1997, 50). Moreover, there is reason to think that metaphysical realism will lead to scepticism.<sup>7</sup> That is, following Putnam, once one rejects the correspondence theory of truth, one is forced to the actual recognition-transcend of the truth-values of statements, unless one adopts anti-realism. The anti-realist restricts truth to what can be known. The possibility that truth could be other than it is determined to be is ruled out.<sup>8</sup> As Griffin says, "The great advantage of anti-realism, especially for a empiricist, is that it offers a secure defense against scepticism" (Griffin 1995, 14).

Brouwer, who was a mathematician, by talking about the limits of the applicability of the law of the excluded middle (LEM), provided the outline of contemporary ontological and epistemic anti-realism.<sup>9</sup> He characterizes realism in terms of recognition-transcendence and discusses it in terms of three tenets (Brouwer 1975, 551):<sup>10</sup> (1) truth is independent of human thought (like an undiscovered continent); (2) knowledge can be extended by the powers of thought; and (3), "false" is the converse of true.

For an intuitionist, however, according to Brouwer, an assertion in any domain is true if it is realized, i.e., "if these truths have been experienced" (Brouwer 1975, 490).<sup>11</sup> Brouwer writes:

Only after intuitionism had recognized mathematics as a autonomic, interior constructional mental activity which

although it has found extremely useful linguistic expression and can be applied to an exterior world, nevertheless neither in its origin nor in the essence of its method has anything to do with language or an exterior world, on the one hand, axioms become illusory, on the other hand the criterion of truth and falsehood of a mathematical assertion was confined to mathematical activity itself, without appeal to logic or to a hypothetical omniscient being. (Brouwer 1975, 551)

According to him, bivalence only holds for realizable statements (Brouwer 1975, 524).<sup>12</sup>

According to Brouwer, there are four possibilities for an assertion (Brouwer 1975, 552): (i) it is proved true; (ii) it is proved false, i.e., absurd; (iii) it is neither true or false, but an algorithm is not known to lead to a decision, and (iv), it is neither true nor false, "*nor do we know an algorithm leading to the statement either that A is true or that A is absurd.*" According to Brouwer, (i)-(iii) fall within the domain of the judgeable - they are recognizable. On his account, (iv) is the class of assertions that are undecidable and for which LEM does not hold - they are not recognizable (Brouwer 1975, 490, 492, 551, 552).<sup>13</sup>

Anti-realism, which had been proposed as an alternative to metaphysical realism, has also been defined in terms of the recognizability of the truth and falsity of propositions. A statement whose truth is not in principle recognizable is undecidable. The anti-realist rejects that the law of the excluded middle (LEM) applies to domains that are undecidable.<sup>14</sup> Dummett writes:

From an intuitionistic standpoint, therefore, an understanding of a mathematical statement consists in the

capacity to recognize a proof of it when presented with one; and the truth of such a statement can consist only in the existence of such a proof. (Dummett 1977, 6)

He goes on to say that mathematical theory and truth "exist only in virtue of our mathematical activity, which consists in mental operations, and have only those properties which they can be recognized as having (Dummett 1977, 7). A statement that is not decidable in virtue of such activities will be neither true nor false.

Dummett requires that different classes of statements - the "disputed class" - be dealt with on a case by case basis (Dummett 1978, 146; 1991, 16; 1993, 230, 378, 380, 409).<sup>15</sup> As he writes, "Of course, there is no question of anyone's being a realist *tout court*...one may adopt a realist interpretation of a certain class of statements..." (Dummett 1993, 378). And again: "There is little likelihood of a uniform solution to all of them" (Dummett 1991, 15). It suffices to say that it may not be easy to decide when LEM applies in one domain and not another because what is recognizable is contentious.

Embracing anti-realism raises the spectre of strict actualism. Strict actualism is the idea that a statement is true or false only at the time it is evidenced (an absurd view that could not even account for linguistic competence since our use of many terms requires assuming some statements about them are invariant over time) (Wright 1993, 108; Dummett 1993, 447).<sup>16</sup> In order to avoid strict actualism, for an anti-realist, a statement must be decidable in principle. Dummett

writes:

[One is] likely to agree that there are true statements whose truth we do not at present recognize and *shall not in fact ever recognize*; to deny this would appear to be to espouse a constructivism altogether too extreme. One surely cannot crudely equate truth with being recognized, or with being treated, as true. (Dummett 1993, 447)

He only requires that a statement be in principle recognizable to be deemed determinate.<sup>17</sup> According to Dummett, recognizability is defined in terms of what it is possible for human beings to do. He writes:

The fundamental difference between the anti-realist and the realist lies in this: that...the anti-realist interprets 'capable of being known' to mean 'capable of being known by us [human beings (1978, 6)]', whereas the realist interprets it to mean 'capable of being known by some hypothetical being whose intellectual capacities and powers of observation may exceed our own.' (Dummett 1978, 24)

What human beings cannot do delimits the bounds of knowledge. A statement that cannot be decided by human beings is not one that can ever count as part of the body of knowledge.<sup>18</sup>

The idea of being "in principle recognizable", however, is obscure for the same reason that talk of what "human beings are capable of" was (Dummett 1978, 24). That is, one cannot provide any uncontroversial content to these phrases. The anti-realist is in danger of sliding into strict actualism since that may, ironically, be less controversial if there is agreement upon what constitutes deciding a statement at a particular time. As Wright remarks, the anti-realist shows a failure of nerve by separating "in principle" from "actual" (Wright 1993, 32). Yet even if one grants that the anti-

realist can avoid strict actualism, he may not avoid realism. As Dummett comments in his postscript to "Truth": "[I said] in effect, that a realist interpretation is possible only for those statements which are in principle effectively decidable (i.e., those for which there is no serious issue between the realist and anti-realist)" (Dummett 1978, 24). For statements that are accepted as decidable, there is an agreement between the realist and anti-realist on the application of LEM. As Dummett remarks, "For decidable statements the assumption of the principle of bivalence does little or no harm, since, by hypothesis, we can at will determine the truth-value of those statements" (Dummett 1993, 62). A natural realist, anti-realist, and the Gödelian optimist - that is, everyone except those committed to actual recognition-transcendence of the truth-values of statements - can admit the statements in question have a decidable truth-value (Tennant 1997, 7, 166).

Anti-realist and realist views sit on a continuum; their extreme forms occupying opposite ends of the spectrum.

| (i)              | (ii)                | (iii)                |
|------------------|---------------------|----------------------|
| Strict Actualism | Anti-Realism        | Metaphysical Realism |
| //////////       | / / / / / / / / / / | / / / / /            |

One must distinguish between at least three cases. First, for the metaphysical realist a statement's truth or falsity is recognition-transcendent. Second, according to the strict actualist, a statement is true or false at the moment it is recognized as such. From the extreme of strict actualism one

can locate, in the following order, radical anti-realism and Dummett's anti-realism.<sup>19</sup> Finally, according to a natural realist, a statement is true or false just in case it is in principle decidable.

The natural realist is distinguished from the anti-realist by embracing a more robust notion of decidability in principle. For example, according to a natural realist decidability in principle only requires an abidance of the law of non-contradiction.<sup>20</sup> That is to say, "in principle" is thus given a radical interpretation that any anti-realist would reject (the difference between natural realism and anti-realism is discussed in Section 8.3).

Dummett writes:

But it seems that we ought to interpose between the platonist and constructivist picture, say of objects springing into being in response to our probing. We do not *make* the objects but must accept them as we find them (this corresponds to the proof imposing itself on us); but they were not already there for our statements to be true or false before we carried out the investigations that brought them into being. (This is of course intended only as a picture; but its point is to break what seems to me the false dichotomy between the platonism and the constructivist pictures which surreptitiously dominates our thinking about the philosophy of mathematics. (Dummett 1978, 185)

Dummett aims at a middle road between the metaphor of discovery (platonism) and invention (constructivism). Though Dummett talks of the being of abstract objects, what is important is the idea of a middle way between the platonist and constructivist. Waismann was aiming at the idea of a middle road when he said, "We make, and we do not make

mathematics. We cannot control mathematics. The creation is stronger than the creator" (Waismann 1982, 33). As Kitcher remarks, in a different context, "My interpretation so far might suggest that Mill is a peculiar type of constructivist...However, it would be more accurate to regard him as concerned less with what we do to the world than with what the world will let us do to it" (Kitcher 1980, 224).

Natural realism stands in between metaphysical realism and anti-realism. The natural realist does not accept the dichotomy between omniscience and human finitude. The natural realist links truth to evidence, but requires it be decidable in principle.<sup>21</sup> Natural realism allows that a proposition can be decidable, even if (by our present lights) one does not know what its truth-value is (in both Brouwer's sense (iii) and (iv)). Not deciding the truth-value of a proposition does not bear on its status nor does determining it. Truth is separated from what is decidable now. For example, asserting "the moon is made of cheese", according to natural realism, grants the statement is determinate as if it were recognition-transcendent (whether or not one has the means to render a decision upon the assertion). Bivalence is maintained one way or the other (just in case they are decided at this time, and if they are not). As Putnam writes:

Why does the fact that the truth value of a proposition may be undiscoverable by us suggest to some philosophers - indeed, why does it count as *proof* for some philosophers - that the proposition in question doesn't have a truth value? Surely, some kind of idealistic metaphysics must be lurking in the underbrush!...Surely,

the mere fact that we may never know whether the continuum hypothesis is true or false is by itself just no reason to think that it doesn't have a truth value! (Putnam 1998, 178)

Putnam considers the idea of rejecting bivalence because the truth-value of a statement in question is recognition-transcendent at present implausible. Natural realism captures the idea that a statement will have a truth-value if even one were not to render a judgment upon it. It salvages the realistic spirit.

Natural realism is the idea expressed by three principles, which can be drawn from Putnam's writings. First, Putnam linked truth to a criteria of rational acceptability (Putnam 1981, 49, 51). A proposition is decidable when a judgement is rendered according to a standard.<sup>22</sup> The idea can be expressed as follows.

Principle 1: A statement has a truth-value just in case it is a member of rationally acceptable statements.

According to the natural realist, however, a statement need only be in principle recognizable. That is, an arithmetical statement has a determinate truth-value before one renders a judgment.

Second, one needs the correct criterion by which to render a verdict on a truth-candidate.

Principle 2: Rational acceptability must be correct.

Like Frege, Putnam acknowledges that what is epistemologically significant is not the judgement, but the justification of the criterion (or procedure) on which it is based (Frege 1953,

sec.3). Finally, a criterion of rational acceptability, in turn, is justified by yielding knowledge that is unique and successful.

Principle 3: Rational acceptability is correct just in case it produces knowledge that is unique and successful.

Scientific and mathematical knowledge, for example, are unique because they beat competitors (on evidential grounds). Uniqueness for a body of knowledge entails that most statements that are included in that set are determinately true or false and in principle decidable. There is only one arithmetic (and arithmetical statements are determinately true and false and possibly recognition-transcendent). For example, according to a natural realist, Goldbach's conjecture - i.e., every even number greater than 2 is the sum of two primes - is either true or false even though human beings may never find out.

As Nagel and Newman point out, Goldbach's conjecture is an example of Gödel's incompleteness theorem.<sup>23</sup> That is, Goldbach's conjecture "is an example of an arithmetical statement that may be true, but may be non-derivable from the axioms of arithmetic" (Nagel and Newman 1958, 59). Even if one augmented the axioms of arithmetic to decide Goldbach's conjecture, "there will always be further arithmetical truths that are not formally derivable from the augmented set" (Nagel and Newman 1958, 59). The incompleteness results can be construed as a challenge to the anti-realist, i.e., they

provide the possibility of a true but undecidable statement (Gardiner 2000, 97). Dummett's solution is to reject that there are true but undecidable statements. Gödel's solution is to accept that there are some undecidable statements (those which are undecidable now) have truth-values and are meaningful (Gödel 1990b, 268). Gödel's incompleteness theorem does not offer a knock-down argument against anti-realism (or realism).<sup>24</sup>

So far it has been suggested that Putnam's early and later views are consistent (Section 3.2). That is, even once one embraces all of Putnam's internalism - e.g., the realizability of truth depends on methods, values, and so on - his early view provide the criteria for when has got the right values and methods. Mathematics (hence arithmetic) is successful (Putnam 1979a, 73-4). The qualities of uniqueness and success tell the naturalized epistemologist where to begin - why to be optimistic, e.g., about arithmetical truth. However, till knowledge is produced - till the standard is enacted, as it were - then its reliability is unknown. Once arithmetic knowledge is accepted to be true in terms of the natural realist criteria propounded, the only justificatory work to be done is to offer a naturalistic account. As with Quine, one begins with knowledge and works backwards to explain how one acquires it.

One may wish to compare natural realism in relation to the distinctions that Tennant has drawn (Tennant 1997, 159-

173). He separates four cases based on the acceptance or rejection of bivalence and recognition-transcendence. The first two positions are traditional views which he calls the "Yes, Yes" and "No, No" positions (i.e., they either accept both bivalence and recognition-transcendence or they do not) (Tennant 1997, 159). There are two other options which have not been traditionally considered. As Tennant remarks, "Dummett appears to be blind to the prima-facie distinct possibilities of M-realism and of Gödelian optimism (the 'Yes, No' and 'No, Yes' options respectively)" (Tennant 1997, 160).

The first case is orthodox realism which accepts both bivalence and recognition-transcendence. Orthodox realism is just metaphysical realism. It is undesirable if metaphysical realism collapses into scepticism. Yet if one follows the optimistic view propounded truth can be secured by looking to the consequences of a body of knowledge. Even the optimistic-orthodox realist, however, believes that truth is recognition-transcendent. The fact that a statement's truth-values are decidable is not what makes propositions true; that it is possible that one did not know the truth-values and this would not affect what the values were - one is just incredibly fortunate in knowing many of them.

According to Tennant, the second option is moderate anti-realism which rejects bivalence and abandons recognition-transcendence, e.g., Dummett's view. As it has already been

pointed out, all views sit on a continuum, i.e., they differ by degree. If anti-realism is liberalized to require that statements be decidable in principle it comes, as Dummett points out, quite close to realism. There is, however, a difference. On the one hand, anti-realism is idealist: it is unattractive to the natural realist because the anti-realist attempts to impose methodological limits on truth, which is captured by Mark Gardiner. Gardiner, in an detailed study of anti-realist arguments, summarizes his findings thus:

[My] main argument advanced was to the effect that the co-extensiveness of truth and verification is no argument for the claim that truth is reduced to (or is dependent upon) verification...But at a deeper level, it [the anti-realist's strategy] is fallacious because it assumes what it purports to establish - that the limits of what we can establish is the case are the limits of what *is* the case. (Gardiner 2000, 221)

Anti-realism is an unsatisfactory account of arithmetical necessity.

For the realist, on the other hand, just because one can know the truth-value of a statement is not what makes it so. Truth is discovered in the sense that the truth-values of statements are so regardless whether one can adduce what they are (a view which the anti-realist will likely reject, Sections 8.3 & 9.1).

The third option is M-realism which is the idea that one rejects bivalence and accepts recognition-transcendence. Expanding the realist's portfolio, the M-realist maintains that statements have a truth-value even if one cannot adduce

them but rejects bivalence. M-realism could apply to "non-sense" (see the fifth response in section 8.3).

The final option is Gödelian optimism which countenances bivalence and not recognition-transcendence. It is the view that truth:

would always be in principle detectable, even though our best systems of proof at any one time may be incomplete, and even though we have theorems for certain (incomplete!) theories and their consistent extensions. (Tennant 1997, 179)

Heeding Dummett's advice means that one must deal with the applicability of any one view based upon the disputed class of statements in question. For the domain of arithmetic, there is perhaps an overlap between the Gödelian optimist and the orthodox realist, as bivalence holds for both views.

### **(8.3) The Stronger Program.**

Several threats to natural realism shall be considered in this section. The first problem concerns access to a "given" (like sense data), such that there is an unmediated correspondence between the mind and world (required by a foundationalist, empiricist epistemology) which may plague realism (Sellars 1997, 13).<sup>25</sup> According to Donald Davidson, for instance, empiricism has been unpalatable because, among other reasons, it requires an untenable distinction between form and content (Davidson 1986, 309). The form is regarded as a conceptual scheme. Some content, for example, sense data, is postulated to serve as the foundations of knowledge.

Yet, according to the naturalist, what counts as given at

any stage of cognitive development is determined by empirical investigation. Science is employed to account for how knowledge is attained. There is no privileged class of data - like patches of colours - and especially not a class that is ontologically reified which is required for cognition to get off the ground. A naturalist account of how knowledge is acquired - e.g., does sense data play a role in one's epistemology? - must be established by scientific inquiry. Pragmatists avoid problems with traditional empiricism (and foundationalist epistemology). The natural realist account proposed follows the lead of the naturalized epistemologist. The problem of a given is avoided.

Second, one may complain that key terms which the natural realist relies upon are vague. For example, what constitutes rational acceptability and applicability are not well-defined. Though a standard must be used to judge a truth-candidate, what that is has not been spelled out. Also mathematics' applicability can be understood in various ways. Brown, for example, distinguishes between mathematics representing a non-mathematical realm or describing a world that is already mathematical (Brown 1999, 55). As Brown writes, "Looking back on the debate, Field versus Quine and Putnam, we can see it as an implicit debate about whether mathematics *represents* (Field) or *describes* (Putnam and Quine)" (Brown 1999, 56).

One may recall that in Chapter 3, it was argued that Quine's indispensability argument is unpalatable because in

Brown's terms, it leads Quine to conclude that mathematics is descriptive of mind-independent abstract objects. The descriptions in turn are used to represent the physical world. One may also recall that according to Putnam, mathematics is about describing the structures one creates which can be used to represent the physical world. In this thesis it has been suggested that mathematics is both representative and descriptive. Mathematics is representative in the sense that its principles are acquired from interaction with physical objects as Kitcher had explained (see Section 2.1). What one generates from the principles constitutes the basis upon which one can describe the created structures as Putnam had explained (Sections 1.3 & 7.1). It suffices to say, roughly, that the first-tier is representative and the second-tier descriptive. For a more detailed discussion of what it means for mathematics to hook onto the world one is referred to Brown (1999, 46-61).<sup>26</sup>

Defining rational acceptability and applicability in more detail is a task for naturalized epistemology and as such lies beyond the scope of this thesis.<sup>27</sup> The reason is as follows: the naturalist emphasizes the descriptive element of epistemology because criticism external to scientific practice is suspect (asking if the criteria is correct from outside of scientific practice is banned). Getting clear on the definitions of rational acceptability and applicability is a descriptive task, working out, for example, what standards

scientists use in accepting a theory.

One should note that, as is discussed in chapter 3, Quine allows for the justification of unapplied parts of mathematics because they are couched in the same grammar as the affected part. Similarly, once a criteria of rational acceptability is justified for a certain class of statements (principle 3), determinately true statements lose nothing by not being applied; they are part of a class of statements that are applicable.

Third, false theories have allowed for successful application. How does one know that our current standards of success - i.e., uniqueness and applicability - are correct? Mathematical necessity may be an illusion which results from long held habits (perhaps biologically ingrained ones), and hence, not a reason to consider natural realism.

The problem is captured in Benacerraf's recent reflections upon the dilemma, which appear in a paper (1999) that comments upon the writings of Boolos. Benacerraf points out that Boolos had assented to the existence of abstract objects that mathematical theories quantify over. Yet as Benacerraf also points out, Boolos had appealed to common sense to reject the existence of sets of large cardinalities (Benacerraf 1999, 37).

According to Benacerraf, Boolos' position falters, in broad outline, upon his dilemma. On the one hand, there is the issue of truth (for Boolos, abstract objects exist). On the

other hand, there is the issue of plausibility (for Boolos, the number  $\kappa$  and whatever comes above it does not exist). As Benacerraf explains, "The claim of George's that I found so startling was that some cardinals whose existence were implied by this theory were so *huge* that George [Boolos], for one, simply couldn't believe they existed" (Benacerraf 1999, 33).

Yet why should one think that truth has anything to do with common sense? As Benacerraf writes, "And what does 'plausibility-to-us' have to do with truth?" (Benacerraf 1999, 38) Benacerraf identifies the problem:

[Y]et without some link between plausibility and truth this position [Boolos' rejection of  $\kappa$ ] is very hard to occupy for a 'hard-core' realist about the meaning of set-theoretic language. This is, and continues to be, the agonizing pull of 'Mathematical Truth', at least for me. (Benacerraf 1999, 38)

Benacerraf goes on:

The reason why listening to George [Boolos] made the author of 'Mathematical Truth' squirm is that in that article, I brand holistic (and conventionalist, and related accounts) of truth in mathematics as defective - precisely because they don't help us understand why the features we identify with truth [like applicability] is correctly so identified. (Benacerraf 1999, 44)

What has remained at issue in the generically realist and empiricist views considered is the link between truth and plausibility.

Yet according to a natural realist knowledge rests upon a criteria of rational acceptability. Collectively - uniqueness and applicability - provide the case for natural realism and they are not without reason: for example, Maddy asks, following Frege, if mathematics is just a formal game,

why is it so useful? (Maddy 1989, 1123) As Wright points out, constructivists can explain why  $2+3=5$  but not why it is always so (Wright 1993, 180).

According to the natural realist, arithmetic is discovered in the sense that one is forced to recognize its truths when one wants to acquire that body of knowledge. The natural realist explains arithmetic's uniqueness in term of recognizability in principle. The reason there is consensus about arithmetical knowledge is because it is true in the natural realist sense of the term. Arithmetic realism explains why one and only one arithmetic is applicable.<sup>28</sup> True accounts stand to provide more accurate and reliable epistemic applications. Burgess and Rosen remark:

[T]he answers science gives to the questions that it considers are only constrained by a very slight degree by political and economic power, but are constrained to a very high degree by regularities in the world no political or economic power can change; and it is for this reason that the answers science delivers are applicable to the world to such a high degree...<sup>29</sup> (Burgess and Rosen 1997, 224)

Furthermore, the qualities of uniqueness and applicability are co-dependent. On the one hand, arithmetic is applicable to the world. On the other hand, since one and only one arithmetic is applicable, that is where a naturalist's investigation must begin. Truth and plausibility-to-us meet. Adopting the phraseology of Putnam, and Wittgenstein before him, practice is where the spade turns (Putnam 1987, 85). One may want to recall that Putnam emphasizes the importance of following practice in constructing an epistemology:

[One should not] give up on that [certain views], whatever our philosophical convictions, we employ and must employ when we live our lives. Until now, I have not mentioned the word 'pragmatism' in these Dewey lectures. But if there was one great insight in pragmatism, it was the insistence that what has weight in our lives should also have weight in philosophy. (Putnam 1994a, 517)

All arguments rest upon premises, so Putnam makes plain where the pragmatist begins. One may recall that the pragmatist attempts to follow practice at both the first and second-level of justification (Section 3.1). Putnam follows in the footsteps of his teacher, Reichenbach: "A philosopher who is to put aside his principles any time he steers a motor car is a bad philosopher" (Reichenbach 1938, 347).

Fourth, uniqueness - so the charge goes - is an illusion. It has been demonstrated that, historically, mathematical (Kitcher 1984) and scientific knowledge (Kuhn 1962) have changed (Kitcher 1984, 229-272).<sup>30</sup> Cheryl Misak, a contemporary scholar of Peirce, says, "The following pessimistic induction seems to be warranted: every theory in the past has eventually shown to be false, so our current theories may well be false" (Misak 1991, 120).

Yet epistemic change, presumably, moves from worse to better accounts. "Better" can perhaps be explicated by a realist in terms of approximation to the truth.<sup>31</sup> Putnam's strategy is in the pragmatist tradition of Peirce. Misak writes of the Peircian strategy, "Truth is the property of hypotheses that would be believed if inquiry were pursued as far as it could fruitfully go. To suggest that there is more

to truth than that is to abandon the pragmatic methodology in favour of transcendental metaphysics" (Misak 1991, 44).<sup>32</sup> Misak goes on:

True beliefs are those which would, in the end get along with experience and one explanation of our beliefs achieving better and better fit with experience [and application to the world] is that a good number of them are true. A good number of them would be permanently doubt resistant. (Misak 1991, 124)

One's epistemology may require being a realist about some scientific facts (e.g., arithmetical knowledge), while allowing that other ones are being refined. One should note, in passing, a blanket approach - being a realist about a domain based on its success - may have the problem (in domains other than arithmetic) of ascribing truth to a specific proposition that may be over turned (i.e., the uniqueness of individual statements is called into question).

Fifth, the natural realist view proposed, may entail a counter-intuitive conclusion. There may be domains where principle one obtains but not two, that is to say, where a standard of rational acceptability exists, but does not yield a body of knowledge that is either unique or successful.<sup>33</sup>

In order to avoid eliminating conventional truths, however, distinctions must be made between: (1) non-sense, (2) conventionalism (c-truths) and (3), natural realism (r-truths). Non-sense ranges over cases where all principles discussed in Section 8.2 fail.<sup>34</sup> It ranges over, say, the domain of humour where bivalence does not hold. One has to be a realist about the fact that there is no standard yet to be

discovered in that domain. In outline, non-sense is consistent with what Tennant called M-realism. Any non-sense claim may be transformed into a c-truth with the introduction of a standard.

C-truths are conventional; they meet principles one but not two or three (again from Section 8.2). In the case of conventional truths, there is a standard but it does not yield knowledge that is unique or successful. C-truths, therefore, are contingent upon a standard that is not transcendent of what is decidable now and thus suggests a radically anti-realist position. Knowledge like say, the rules that would determine when one is in checkmate in the game of chess, would count as a c-truth. Also, what base system one uses for counting would generate c-truths. For instance, Butterworth observes:

English contains residues of other bases that have been used in parallel with base 10, and which historically may have preceded it in Europe. Base 12 was used for money (12 pennies to the shilling), weight of precious metals (12 Troy ounces to the pound), length (12 inches to the foot), and quantity (dozen, gross). The word *twelve* reflects its origins much less transparently than 'teen' words, which are still clearly in the form 'unit+ten'. The division of the day and night, and our clock face, into 12 hours has an ancient Egyptian origin, and is possibly the source of twelve-counting. (Butterworth 1999, 63-4)

Base systems, like the game of soccer or chess, can be revised. Grouping of a whale as a mammal is a c-truth (e.g., the platypus challenged necessity of the distinction between mammals and non-mammals because it lays eggs, has a beak, but is not a bird).<sup>35</sup> Some of the claims made by the social

sciences may be c-truths if the domain fails to maintain uniqueness. If enough statements turn out to be unstable the domain itself slides into c-realism.

R-truths are defined in terms of all principles, i.e., there is a standard that yields knowledge that is unique and successful (bivalence holds). Appropriate arithmetical statements are r-truths, which it has already been pointed out is consistent with Gödelian optimism. The claim that blue whales give live births, have hair, nurse their young, and so on, could also count as a r-truth. One may hope that statements that are scientific, ethical, aesthetic, and so on, fall within the natural realist's scope. Natural realism can apply, for instance, to universal claims that make reference to human behaviour.

Sixth, it has been pointed out that natural realism can be distinguished from anti-realism and metaphysical realism. Yet it is not clear exactly how natural realism is different. So, perhaps natural realism adds nothing to any of the views already proposed for statements deemed to fall within the domain of r-realism.<sup>36</sup>

Yet according to the natural realist, in principle decidability ranges over the standard used to make determinations. Consider an analogy. Suppose that the standard used to determine the statement "the cat is on the mat" is the flipping of a coin: tails renders the statement true and heads determines it as false. Also one could suppose that the

knowledge produced is unique and successful. The nature of the universe is such that "tails" will only obtain when the cat is on the mat (there is some causal relation between flipping the coin and the cat being on the mat). The knowledge is applicable: it allows one to increase one's chances of catching the cat.

There are four possibilities for the statement in question when the standard is known: (i) true (tails obtain); (ii) false (heads obtain); (iii) true (and undetermined); and (iv), false (and undetermined). In the cases of the first two, the statements have been determined (tails and heads, respectively).

The anti-realist can accept that the third and fourth cases have truth-values if one is yet to carry out the procedure. If one is yet to discover the flipping-the-coin-method, for the anti-realist, LEM is rejected (because the anti-realist has linked truth to methods). For the realist the truth and falsity of the statement in question remains determinately true or false before the flipping-the-coin-method is enacted or discovered (e.g., if there are no rational agents and only a cat on a mat). In all cases, the status of the statement is unaffected by either carrying out the procedure or having yet discovered it.<sup>37</sup> Thus, the natural realist distinguishes himself from the anti-realist.

When the standard is unknown, however, it may be because there is not one. According to a natural realist, LEM does not

apply to statements where there is no standard in principle: they are (v), statements for which bivalence does not hold. For the domain of arithmetic, however, Gödelian optimism is indistinguishable from natural realism because the truth-values of statements are decidable (the Gödelian position is that optimistic).<sup>38</sup>

The naturalized epistemologist, for instance, is highly optimistic about human cognitive abilities for evolutionary reasons. Similarly, for domains where r-realism applies, the natural realist is highly optimistic that agents can come to know the methods that lead to uncovering the truth-values of statements. Perhaps the natural realist must have eschatological views about the unfolding of the universe, such that evolution will produce organisms with faculties and behaviours that allow knowledge.<sup>39</sup>

Finally, Dummett proposed anti-realism because he thought that realism attained victory too easily (Dummett 1993, 410). How can one adjudicate between the finite (i.e., constructivist) and eternal (i.e., realist) accounts of arithmetical truth (since constructivists can also account for the applicability of arithmetic, as seen in Chapters 3&7)?

The natural realist's tack, could be negative, trotting out the usual complaints about constructivism (e.g., it curtails mathematics and conflicts with what mathematicians think they are doing, though Dummett points out that the latter is not an argument) (Dummett 1978, 207; 1993, 442). Yet

what practitioners think they are doing may be important to pragmatists who try to adhere their epistemology to practice.<sup>40</sup>

As with Quine, and the pragmatist tradition at large, doubts make sense within an enterprise. Deciding between natural realism and anti-realism from outside of science is banned; yet that is where the debate must be fought (Misak 1991, 50). Hacking remarks:

It [scientific knowledge] is empirically adequate - wonderfully so. The realist asks why is it empirically adequate - is that not because there just are molecules? The anti-realist retorts that explanation is no hall-mark of truth, and all our evidence points only to empirical adequacy. In short the argument goes around in circles (as, I contend, do all arguments conducted at this level of discussion). (Hacking 1983, 55)

Hacking is right in the sense that one cannot, of course, empirically adjudicate the realist debate. That is why it is philosophy. Perhaps one has to choose between natural realism and anti-realism - which is not to say that one cannot amend one's view based on cogent argumentation - but it is to concede that reason has limits and epistemology is a vague science. Anti-realism looks wrong (i.e., it does not account for arithmetic's uniqueness). Yet extending natural realism to domains other than arithmetic will be problematic since it is harder to satisfy that they contain knowledge which is unique and successful (though it is not impossible to do).<sup>41</sup>

The natural realist for arithmetic begins with the working hypothesis that arithmetical knowledge is true and makes plain what that consists in, as well as working backwards to explain why it is so. One ought to take seriously



the necessity of arithmetic which natural realism accounts for. It has already been argued that natural realism is the optimal characterization of arithmetical knowledge. In the next chapter, natural realism shall be shown to solve Benacerraf's dilemma.

## 9. Concluding Remarks

In this chapter, first, a retrospective is provided whereby Benacerraf's dilemma is placed in a historical context and the steps leading to natural realism are reviewed; second, the results of the thesis are made plain.

### (9.1) Retrospective.

Though not a historian, Gödel provides a sketch which shows how Benacerraf's dilemma can be seen as a result of a tension within the history of philosophy. Gödel divides up different views within the philosophy of mathematics by their proximity or distance from metaphysics (or religion) (Gödel 1990c, 375). On the one hand, (the left one), there is scepticism, materialism, and positivism, e.g., empiricism, which is pessimistic and relativistic. On the other hand, (the right one), there is spiritualism, idealism, and theology, e.g., *a priorism*, which is optimistic and realist.

Gödel acknowledges that there are positions that have features of both (e.g., empirical theology and Schopenhauer's pessimistic idealism) (Gödel 1990c, 375). Nonetheless,

generally, he claims that the distinction holds.

The left hand view is relativistic and empirical, which he takes to be the spirit of the times: "[T]hese nihilistic consequences are very well in accord with the spirit of the time..." (Gödel 1990c, 379). Under it the antinomies of set-theory have been "exaggerated" by empiricists; but he laments that arguments are "of no use against the spirit of the time" (Gödel 1990c, 377).

Gödel sits on the right hand.<sup>1</sup> He thinks that there is a fact of the matter independently of whether one recognizes it or not. Gödel claims that one should "believe that a question not decidable now has meaning and may be decidable in the future" (Gödel 1990b, 268).<sup>2</sup>

There is a middle way. Gödel writes, "[T]he correct attitude appears to me to be that the truth lies in the middle or consists of a combination of the two conceptions...In any case there is no reason to trust blindly in the spirit of the time..." (Gödel 1990c, 381). Gödel has an account of necessity that attempts to integrate empiricism (he appeals to intuition to account for access to mathematical truths). Gödel conjectures, for instance, "Now one may view the whole development of empirical science as a systematic and conscious extension of what the child does when it develops in the first direction" (Gödel 1990c, 385). He attempted to find a way between realism and empiricism, roughly, the first and second horn of Benacerraf's dilemma.

As Kreisel had remarked, in another context, "I regard the 'rival' philosophies of mathematics in this light: not as contradictory in substance, but as emphasizing different aspects of mathematics" (Kreisel 1959, 145).<sup>3</sup> Natural realism is consistent with the middle road that Gödel outlines and draws upon rival philosophies as Kreisel had suggested.

In the Introduction, moderate foundationalism was advocated, which allows different (yet equivalent) representations of the same content. In Chapter 1, conceptualism was defended: abstract objects exist but are not mind-independent. It was argued that ontological and epistemic-realism must be separated (one can be an epistemic-realist without positing mind-independent abstract objects). Natural realism for arithmetic is framed in terms of the recognition-transcendence of the truth-values of statements of that class, not the existence of mind-independent abstract objects.

In Chapter 2, it was argued that arithmetical knowledge is not categorically different from the empirical sciences (as the logical positivists held).<sup>4</sup> The causal constraint requirement was defined drawing upon Steiner's analysis. As Kitcher had shown, the definitions that form the principles (e.g., the Peano axioms) are empirical; they are derived from acts of collecting physical objects (Section 6.3). Furthermore, a revised version of *a priori* knowledge was advocated. What one derives from the principles is *a priori*.

The natural realist also adopts a two-tier epistemology for arithmetic such that both empirical and non-empirical aspects are included. Tennant's slogan is apt: "Logical positivism was almost right!" (Tennant 1997, 10)

In Chapter 3, it was argued that the indispensability argument must be revised. Pragmatists need not infer the existence of arithmetical objects but only its truth. It was argued that inferring the truth of arithmetical claims requires, in addition to being applicable, that they must be accepted by practitioners in the relevant field (which it was suggested will prohibit the ontological reification of mind-independent arithmetical objects). Pragmatists are right to think that mathematics' applicability is an important argument for arithmetical realism.

In Chapter 4, it was argued that in virtue of positing the mind-independent existence of abstract objects, neo-Fregeans cannot explain what they are. Their attempt to avoid the Caesar problem is considered unacceptable and thus poses the specter of skepticism. According to the natural realist, however, those influenced by the linguistic turn, like Wright (1983) were correct to require that one should begin by defining one's terms when solving philosophical-ontological problems.

In Chapter 5, it was argued that Maddy (1990; 1997), does not establish the mind-independence of sets. Furthermore, it was argued that she is stuck with a false dichotomy: idealism

or mind-independent sets. The natural realist does not require assenting to the mind-independence of objects like sets, thus avoids the bulk of Maddy's problems.

In Chapter 6, Kitcher's naturalism was defended from criticisms that Frege had launched against Mill. For example, it was explained why justification of arithmetical knowledge can be provided in terms of its acquisition. Kitcher demonstrates how the principles of arithmetic can be acquired from acts of collecting physical objects. It was argued, however, that Kitcher (1984) cannot account for arithmetic's uniqueness. The abandonment of realism was criticized. Furthermore, Kitcher's naturalism was suggested to be, in principle, consistent with natural realism for arithmetic.

In Chapter 7, it was argued that Field does not explain why there is one and only one arithmetic that is applicable. Also, Field's strategy relies on the possibility of an elimination of abstract objects which it has already been claimed is not realizable (Shapiro 1998).

Structuralism has been utilized in forming hybrids with some of the other strategies discussed, which has also been considered, for example, in the case of Resnik's work (1997). One may recall that he advocates Gödelian optimism, where empiricism is used to explain the justification of the principles of mathematics, and a two-tier epistemology. As explained in section 8.2, for the domain of arithmetic, Gödelian optimism is indistinguishable from natural realism:

bivalence holds for that class of statements. Resnik, furthermore, marries realism with naturalism. His view, therefore, points the way towards natural realism.

**(9.2) Benacerraf's Dilemma and Natural Realism.**

A satisfactory solution to Benacerraf's dilemma should have the following virtues. First, it ought to maintain a commitment to the existence of arithmetical abstract objects. Second, it should maintain the causal constraint for empirically based knowledge. Third, it should not unnecessarily expand one's ontology. Fourth, it should resist positing special faculties of apprehension. Finally, it will explain arithmetic's necessity, taking into account its uniqueness and applicability.<sup>5</sup>

Natural realism fulfils these criteria. First, abstract arithmetical objects are posited as interpreted by conceptualists, without either assuming weighty metaphysical commitments or positing special faculties of apprehension. Second, natural realism does not challenge the epistemology of empirical knowledge; the causal constraint requirement applies to tokens of physical types and the acquisition of basic arithmetical concepts (and legitimate cases of abduction). Finally, natural realism explains arithmetic's uniqueness. Natural realism is a plausible hypothesis because it emphasizes that truth is in principle decidable (arithmetic is discovered, which is explained in terms of the recognition-transcendence requirement).

The thesis has two virtues. On the first score, Quine, naturalism's greatest contemporary exponent, has not sufficiently extended naturalism to arithmetic (making its epistemology wholly contingent upon arithmetic's utilization in scientific practice). Kitcher's extension of naturalism to arithmetic was defended.

On the second score, in Chapter 8, natural realism was defended. It avoids the traditional problems of metaphysical realism and anti-realism. Natural realism accounts for the necessity of arithmetical truth while avoiding skepticism. Arithmetical statements are discovered in the sense that they have truth-values whether one can adduce what they are or not. For the domain of arithmetic, however, the natural realist has no quarrel with the Gödelian optimist because arithmetical statements are decidable in principle.

Following Putnam's (1979a) lead, it was argued that arithmetic's uniqueness and applicability are reasons to embrace natural realism. An attempt was made to codify the principles for determining what one can be a natural realist about (which is shown to allow natural realism to range over the domain of arithmetical truths).

Furthermore, this thesis provides a case where Putnam's early and later views are rendered consistent. By utilizing both Putnam's writing in favour of realism (1979a), and those that break with them (e.g., 1981), suggests that he should not have abandoned his earlier view. In a nutshell, the

realization of knowledge can depend upon agents' values, methods and so on, without forcing the abandonment of realism (provided one has the correct values and methods) (Section 3.1). It has been argued that Putnam's early view provides the criteria for when one can be a realist about a given domain (Section 8.2). Traditionally, however, empiricism and realism have been pitted against each other. A virtue of this thesis is that it provides a case study where naturalism and natural realism meet. It is essentially mystical.

### **Notes to the Preface**

1. Ian Hacking points out the tension between realism and rationality (Hacking 1983, 2).

2. The song "Every Grain of Sand" appears on the album, Shot of Love, 1973. Its transcription can be found in (? 1990, 462).

### Notes to the Introduction

1. Nagel and Newnan write, "[an axiomatic system] consists in accepting *without proof* certain propositions as axioms or postulates (e.g., the axiom that through two points just one straight line can be drawn), and then deriving from the axioms all other propositions of the system as theorems. The axioms constitute 'foundations' of the system; the theorems are the 'superstructure', and are obtained from the axioms with the exclusive help of principles of logic" (Nagel and Newnan 1958, 4-5).
2. The discovery in the 1980's of Frege's theorem, i.e., arithmetic can be derived in second order logic with the operator "the number *F*" and the so-called "Hume's principle" (i.e., the number of *F*s is equal to the number of *G*s if and only if the *F*s are in an one to one correlation with the *G*s) has sparked renewed interest in Frege's foundational system and its philosophical underpinnings.
3. The term "natural realism" is used by Putnam in his Dewey Lectures (Putnam 1999, xii), but the usage here is completely independent of that and must not be confused with what he intends.
4. Hardy writes, "It seems to me that no philosophy can possibly be sympathetic to a mathematician which does not admit, in one manner or another, the immutable and unconditional validity of mathematical truth. Mathematical theorems are true or false, their truth or falsity is absolute and independent of our knowledge of them. In some sense, mathematical truth is part of objective reality" (Hardy 1925, 4).

### Notes to Abstract Objects

1. Immanuel Kant, Critique of Pure Reason: B 622-3.

2. The terms "nominalism" and "platonism" are not an exact fit with how they are used in contemporary debates in the philosophy of mathematics. As Hacking points out, idealism is about existence and nominalism is a classification (Hacking 1983, 108). Therefore, one could be a nominalist and a realist (about individuals) or embrace Berkeley's objective idealism. Yet in the philosophy of mathematics, nominalists are idealists about universals, and platonists are realists about them: so, the use of the terms "nominalism" and "platonism" has a justification.

3. Also see: (Field 1980, 1). Nelson Goodman, at one point, denies this, only requiring that everything is individual (Goodman 1972, 157). First, the revised view does not seem to give one any more than an identity criteria (the definition is too broad). Second, this was not always his view. As Goodman and Quine said, "We do not believe in abstract entities" (Goodman 1972, 173).

4. Quine goes on, "The conceptualist theory of classes requires no classes to exist beyond those corresponding to [in principle] expressible conditions of membership" (Quine 1953, 126). "In principle" has been added to avoid rejecting higher infinities or parts of the higher theory of numbers, that is, to allow for progressive creation (Quine 1953, 126). Kitcher writes, "Conceptualists claim that we have basic *a priori* knowledge of mathematical axioms in virtue of our possession of mathematical concepts" (Kitcher 1984, 65). Conceptualism is also consistent with Strawson's suggestion that abstract objects are a matter of analytic definition. Thought provides abstract objects (e.g., *a priori* truths), while perception only particulars (particular truths) (Strawson 1983, 72). Conversely, according to a platonist, an independent reality of abstract objects like "prime numbers" renders propositions about them true or false (Putnam 1979a, 7, 74).

5. Benacerraf writes, "Reference is what is presumably most closely connected with truth, and it is for *this* reason that I will limit my attention to reference" (Benacerraf 1973, 662). He also remarks, "I believe in a causal theory of reference..." (Benacerraf 1973, 671).

6. Boyle's law (advanced by Robert Boyle in 1662) states that the pressure and volume of a gas are inversely proportional to one another, or  $PV=k$ , where  $P$  is pressure,  $V$  is volume, and  $k$  is a constant of proportionality. One could also take the example of Archimedes' principle that states that a body immersed in a fluid is buoyed up by a force equal in weight of

the displaced fluid.

7. Hale notes that there is an indeterminacy in discussing abstract objects (Hale 1987, 194, 199). As Quine notes, one need only offer definition to the degree of precision as required by practice. He notes, chemistry does not go into neutrons or electrons, and in the case of the taxonomy used in zoology, similarity is based on what is suitable (practical) for that science (Quine 1963, 137).

8. The Frege-Dummett way (which is a type of abstraction) is the idea that X is derived from an abstraction of concepts (according to Fregean abstraction principles). Suffice it to say that Lewis and Lowe find the fifth strategy insufficient for defining abstract objects (Lowe 1982, 514). Lewis writes, "I shall pass over a fifth way, offered by Dummett in chapter 14 his (1981), in which the distinction between abstract and concrete entities is drawn in terms of how we could understand their names. Even if this fifth way succeeds in drawing a border, as for all I know it may, it tells us nothing directly about how the entities on opposite sides of that border differ in their nature. It is like saying that snakes are the animals we instinctively fear - maybe so, but it tells us nothing about the nature of snakes" (Lewis 1986, 82 fn.56). And, Lowe dismisses it as "altogether too problematic" (Lowe 1982, 520).

9. Universal types are not convertible with classes. The type "tiger" is unaffected by the tiger population (though a class would be) (Armstrong 1989, 27). Universals are non-spacial but need not be atemporal. Though not admitting of ostensive definition, universals are objects of thought and reference (Hale 1987, 189, 191).

10. Conflation may be useful for pragmatic reasons: more has been written about universals. See: (Quine 1953, 14; Armstrong 1989, 54).

11. See: (Burgess and Rosen 1997, 23).

12. Boolos writes, "What the most effective rebuttal to the view [that abstract objects are not a problem] might be could well depend on how the position was articulated" (Boolos 1998, 129). Also, one may wish to note that David Armstrong raises a similar question about the nominalist's motivation (Armstrong 1978a, 12).

13. See: (Russell 1919, 142; Goodman 1977, 106; Armstrong 1983, 100). Plato had already considered the problem by asking if one needed a form for the relationship between a form and a particular (which would reify relations and result in an

infinite regress). The problem of the ontological status of relations is raised in Plato's Parmenides, 133b.

14. Also see: (Armstrong 1978b, 12).

15. Kant writes, "Thoughts without content are empty, intuitions without concepts are blind" (Kant 1965, B 75).

16. One may worry about the chicken-egg problem: which comes first concepts of data. This is responded to in Chapter 5, fn.23.

17. Reference does not have to be in terms of existing objects alone. Steven J. Wagner notes that following Heyting, one can refer to mental constructions or proofs. Both platonism (limited by paradox) and conceptualism (limited by progressive creation) assume universals and classes (Quine 1953, 127).

18. Dummett, however, had first conceived of realism in terms of entities (Dummett 1977, ix, 7). He writes, "The dispute between intuitionists and platonists relates to the acceptability of a realistic interpretation of mathematical statements as referring to an independently existing objective reality" (Dummett 1977, ix). Again: "Thinking of a statement as true or false independently of our knowledge involves a supposition of some external mathematical reality, whereas thinking of it as being rendered true, if at all, only by a mathematical construction does not (Dummett 1977, 12). Dummett's view, however changed the issue to one of epistemic realism (Dummett 1978, 358-61; 1991). L.E.J. Brouwer had introduced the rejection of the law of the excluded middle prior to Dummett (Brouwer 1975, 109-10, 443, 488, 490, 492, 511, 524-6, 551-2). Though Russell - as early of 1903 - had said so (though he did not link it to the realism debate that was still in terms of types of entities) (Russell 1912, 69-70). For Russell's discussion of truth see: (Russell 1940, 233), and his view that the law of the excluded middle must be maintained for realism (Russell 1940, 258-171) (personal communication from Nicholas Griffin).

19. Putnam goes on, "The important thing is that the mathematician is studying something objective, even if he is not studying and unconditional 'reality' of non-material things, and the physicist who states a law of nature with the aid of mathematical formula is abstracting a real feature of a real material world, even if he has to speak of numbers, vectors, tensors, state-functions, or whatever to make the abstraction" (Putnam 1979a, 60).

20. Putnam's talk of possibility is not perhaps helpful because one still has to worry about the impending existence of mathematical objects (Burgess and Rosen 1997, 140).

However, Crispin Wright points out that both Benacerraf and Dedekind would concur with Putnam; there are other ways to be a realist than reify entities (Wright 1983, 117). Putnam's suggestion may be an advance because, first, numbers are not mind-independent and second, they are generated by rules so that one can know about them (See the response provided to Carson in Section 6.3).

21. Here are the incoherent cases:

|          |            |           |
|----------|------------|-----------|
| Ontology | Epistemics | Semantics |
| 7. +     | -          | +         |

One should notice that the seventh case is incoherent since if one is a semantic realist about the number "3" there is at least one statement that one was true independently if one could recognize it or not (i.e., the statement about the definition of "3").

|          |            |           |
|----------|------------|-----------|
| Ontology | Epistemics | Semantics |
| 8. -     | -          | +         |

The eighth case is incoherent for the same reasons as the seventh case.

22. Putnam writes of the distinctions, "'Electrical charge' refers to the same magnitude even if our theory of that magnitude has changed drastically. And we can identify that magnitude in a way that is independent of the most violent theory change by, for example, singling it out as the magnitude which is causally responsible for certain effects" (Putnam 1979b, ix). Devitt says, "If we do not view reference as an objective relationship between words and the world, we cannot view truth as a property a sentence has in virtue of an objective correspondence with the world" (Devitt 1981, xii).

23. For example, "X is gold" (Putnam 1979b, 269) or "water is H<sub>2</sub>O" (Putnam 1979b, 237), could entail a denial of semantic realism.

24. Putnam cites Wittgenstein as an example, and the logical positivists that adopted a version of it. The logical positivists made human knowledge dependent on observables and combined it with a theory of meaning (verificationism), which, he remarks, seems close to idealism (Putnam 1979b, vii). Putnam writes, "In my opinion, verificationism and behaviourism are fundamentally misguided doctrines" (Putnam 1979b, viii). Both verificationism and behaviourism limit what one can legitimately talk about (the unverifiable and mental states, respectively) in order to impose an methodological regiment.

25. Lewis (1986), for example, captures a widely held assumption when he maintains that necessary statements must be

*a priori* and empirical ones defeasible. Kitcher (1984), operating with the same assumption, embraces empiricism and abandons arithmetical necessity.

### Notes to The Causal Constraint

1. Semantic theorists can distinguish between, on the one hand, an object, its internal constituents (roughly, the reference), and on the other hand, the properties it gives rise to (roughly, the object's representation, sense, and name) (Stanford and Kitcher 2000, 105). The problem of linking a name with its bearer parallels the issue of linking a belief with a state of affairs. For example, Saul Kripke, Putnam and Michael Devitt have defended the causal constraint for a certain class of names (e.g., natural kind terms) (Kripke 1972, 137; Putnam 1979b, xiii; Devitt 1981, ix, 126). According to Kripke and Devitt, most of the time questions of meaning and reference require empirical investigation (Kripke 1972, 137; Devitt 1981, 20). One's stand on the causal constraint can apply to semantics and epistemics in two ways that are viable, that is, when it applies or does not.

2 Kripke proposes the rigid designator. A rigid designator is a name or description that designates the same thing in every possible world (Kripke 1972, 3, 24, 48). Rigid designation is where 'a' and 'b' means if 'a=b' then 'a=b' is necessary (Kripke 1972, 3). Kripke distinguishes (1) all identical objects are necessarily so; (2) identity statements between two rigid designators are necessarily so; and (3), that identity statements between what we call 'names' in actual languages are necessary (Kripke 1972, 4). He notes that the truth conditions for (1) and (2) are obvious. Yet thinks (3), rigid designators, allow counter-factuals to be necessary (true in every possible world, even in ones where descriptions differ) (Kripke 1972, 6-7). The name "Hitler", in some possible world, may not have been that person that killed more Jews than any other person in history. Yet "Hitler" would still designate the same person (Kripke 1972, 75). Kripke separates the issue of reference from meaning (or description) (Kripke 1972, 5). According to Kripke, reference can be fixed but that does not entail the same for meaning. For instance, the description of Columbus as the one to first discovered the Americas may be wrong, yet the speaker's intention may designate the right person by "Columbus" (Kripke 1972, 84). The name is linked to the reference correctly, but the description is wrong.

3 Kripke and Putnam link the meaning of terms to an initial baptism (Kripke 1972, 96; Putnam 1979b, 200). Stanford and Kitcher remark "...[A] principal motivation for causal theories lies in the possibility of discovering that some members of a natural kind lack properties originally used in picking out the kind, and just this happened in the history of chemistry with respect to acids" (Stanford and Kitcher 2000, 117). Once one has causally interacted with an object, one's "stereotype" of it forms the content (or description) to which

the name is attached (Putnam 1979b, 256, 271). According to Kripke and Devitt, the causal theory of reference, does not work for all terms but is the best over-all semantics (Kripke 1972, 21, 94; Devitt 1981, 203).

4 A causal theory of reference has the challenge of accounting for empty terms, ones that do not refer (Devitt 1981, 167). Yet this challenge can be met (the issue of "0", See Section 6.3).

5 See: (Resnik 1997, 106). Some wrong beliefs (e.g., "the earth is flat") stand to fulfil the causal constraint requirement and are not knowledge. One, however, can maintain that many initial posits by science are latter detected (Resnik 1997, 195) and, that the casual constraint is necessary for the attainment of that knowledge (see sec. 8.3).

6 Wright points out there are two opinions: One can claim, as the platonist's have, that abstract objects are visible or scepticism. Again, both options, according to Wright, are unacceptable.

7 Wright remarks, "There is then no way of explaining our capacity to know about the natural numbers, e.g., conceived as objects, save be recourse to Gödelian platonism, that is, in effect, by denying their causal insularity" (Wright 1983, 87).

8 Wright puts two requirements on the proponent of the causal constraint: (1) to define it and (2) define "abstract" and "concrete".

9 Of the second choice, Lewis writes, "[N]othing can depend counterfactually on non-contingent matters. For instance nothing can depend counterfactually on what objects there are...Nothing sensible can be said about how our opinions would be different if there were no number seventeen" (Lewis 1986, 111). Conversely, for example, "electron beliefs counterfactually depend on the existence and nature of electrons" (Field 1989, 233). The concept of necessity as that which cannot be otherwise is not clear (Kripke 1972, 36, 41). Common sense tells one that claims like "9 is greater than 7" are necessary whereas ones like "the number of planets are greater than seven" are not so (Kripke 1972, 48).

10 For instance, the causal connection of a particular belief to an agent, according to Hartry Field, is not at issue but the more general question of reliability (that may or may not include the causal constraint) (Field 1989, 233).

11 According to Field, the *a priori* is not interesting without the claim of indefeasibility (Field 1998, 2); but he only claims that *a priori* statements be indefeasible on empirical

grounds. So, for example, one could make changes to one's logic, but the reasons for doing so, according to Field, would not be because of new observations (thought that may be a motivation). And against Field, Devitt defers to Quine's philosophy (Devitt 1998, 60). Field criticizes Lewis as follows: (1) the premise that all facts of the mathematical realm are necessary is questionable; (2) Lewis' claim is too easy; (3) he assumes a controversial link between necessity and counterfactuality; and (4), there is no obvious reason to explain mathematical reliability in counterfactual or modal terms (Field 1989, 233-239). Ascension to Lewis' possible worlds is barred by a desire for an economical ontology

12 According to Quine, for example, "bachelor is an unmarried man" can only be true as an empirical fact (Quine 1953, 31). There is no intrinsic relationship between "bachelor" and "unmarried" that allows one to claim that the statement under consideration is true in virtue of the meaning of its components.

13 Also see: (Kripke 1972, 77).

14 I. Kant, Critique of Pure Reason, B2.

15 Also, for example, Kripke claims that, in some cases, a term - the statement defining it - is a *a priori* in virtue of a baptism that is a stipulation (Kripke 1972, 135). Also, Devitt concedes, for instance, that the notion of proximity to experience, is consistent with Quine, Putnam and his own naturalism (Devitt 1998, 62).

16 Putnam writes, "[T]he essence of idealism is to view scientific [and mathematical] theories and concepts as instruments for predicting sensations [or nothing more than one's constructions] and not as representative of real things and magnitudes" (Putnam 1979b, 198). For example, Bohr's use of "electron" in 1900 and 1934, differed in both theory and description; according to idealism, all one has is different descriptions, theories, and structures. Yet Putnam remarks, "Someone who identifies conceptualization with linguistic activity and who identifies linguistic activity with response to observable situations in accordance with rules of language which are themselves no more than implicit conventions or implicit stipulations (in the ordinary unphilosophical sense of 'stipulation' and 'convention') will, it seems to me, have a deeply distorted conception of human knowledge and, indirectly, of some of the objects of human knowledge" (Putnam 1979b, 41).

17 For Quine and Putnam in virtue of being part of an empirical enterprise, abduction can apply to claims like the one Resnik cites.

18 Rey writes, "[S]uppose that as a matter of contingent fact our brains are so organized that we have a little sub-system that is capable of grinding out the theorems of first-order logic. For simplicity, suppose it is a non-axiomatic system of natural deduction, relying entirely on the operation of standard rules like *modus ponens*, universal generalization, conditionalization, etc., so that theorems may be identified as lines of a proof with null premise numbers. And suppose that someone, Ellen, was caused to believe, say: '(R) Nothing bites all and only those things that don't bite themselves' purely as a result of the operation of this system. Now, why shouldn't this belief of Ellen's count as a *a priori* knowledge?" (Rey 1998, 33).

19 Devitt claims that Rey's example is not a way of knowing at all, and if it was, one could make anything true (Devitt 1998, 51, 53).

20 According to traditional empiricists, like J.S. Mill, for example, arithmetical statements are empirical and open to revision (Kitcher 1980; 1984, 70-1). Devitt rejects the *a priori* because all knowledge is empirical (and since he finds the *a priori* to be obscure) (Devitt 1998, 45, 46). Devitt seems to think it obscure because there is debate about what constitutes it. He also rejects *a priori* knowledge because he does not believe in innate knowledge (which would not, of course, requiring looking to experience) (Devitt 1998, 46). One may wish to note, however, assent or dissent to the causal constraint for domain *P* does not determine whether realism holds for *P*.

21 For example, according to Kripke, Putnam and Devitt, individuals using terms do not have to know the technical laws that define natural kind terms or be personally acquainted with the references of terms; one can gain their knowledge of words through a causal chain, such as books written by people who were causally connected to the objects in question (Kripke 1972, 94; Putnam 1979b], 200, 281). Each network terminates in causal relations (even if that spans many centuries). Kripke adds that the present intention overrides the historical one (Kripke 1972, 163). So, for instance, what one means by "Columbus" today (the wrong description, say), does designate Columbus. For Devitt, however, even a fictitious character, is causally interacted with in the work in which it appears. Fictional characters are first baptized as stipulations, and subsequently causal interactions with them are possible. Also see: (Devitt 1981; Inglis 1996/7, 216; Rey 1998, 27).

22 Koppelberg writes "...[R]eliabilists hold that it is the causal origin or the causal sustenance of a belief which is responsible for its epistemic justification" (Koppelberg 1999, 447).

### **Notes to The Indispensability Argument**

1. There is a historical precedent for the pragmatists. G. Frege writes, "It is applicability alone which elevates arithmetic from a game to the rank of science" (Frege 1970, 187). Gödel held that view (Gödel 1990b, 269), and more recently, Maddy (Maddy 1992, 275). Also see: (Kitchen 1980, 219). Finally, P. Garden remarks, "[Jean Baptist Joseph Fourier (1768-1830)] was first and foremost a physicist, and he expressed very definitely his view that mathematics only justifies itself by the help it gives towards the solution of physical problems..." - from the introduction (Cantor 1918, 1). Brown defines applicability: "Mathematics hooks onto the world by providing representations in the form of structurally similar models" (Brown 1999, 49; also see 46-9).

2. See: (Putnam 1971, 73-4).

3. For example, Popper's epistemology was emended idea because the idea of the falsifiability of individual claims stood to throw into disrepute large parts of science. Also, one reason that abduction was proposed was that it was to account for what scientists actually do, i.e., the acceptance of knowledge claims that lack direct observation. See Brown's discussion (Brown 1989, 133-151); he claims that history can be a guide to a normative methodology (Brown 1989, 151).

4. The descriptive aspect deals with the first-level of justification which concerns mathematicians' object of study (e.g., theorems) and the content of a discipline (e.g., proving theorems). For astronomy, for example, one may think of planetary movements (the objects of study) and describing those movements (the content of astronomy). For example, if a philosopher of mathematics came up with say, a consistent account whereby truth-candidates were determined by the reading tea leaves (in accord with a book of rules), it would be odd because mathematicians do not make a habit of checking their views in this way. In the case of mathematics, the first-level of justification "asks about the mathematics required to do science, the other asks about the foundational underpinnings of said mathematics" (Peressini 1997, 217).

5 If one asks "how do you know there is a book on the table?", one may retort, "I see it". Why should one believe what one sees? Perhaps because of one's past experience - perception has an inductive justification. Yet why should one believe in induction? At some point justifications will run out, and one will be left with, in this case, the attempt to justify induction inductively thus, the problem referred to.

6 Logicism attempted to reduce mathematical concepts to logical ones which was supposed to have a doctrinal pay-off.

Similarly, for the logical positivists, natural knowledge was to be based on sense experience (Quine 1969, 71). Quine writes, "Just as mathematics is to be reduced to logic, or logic to set theory, so natural knowledge is to be based somehow on sense experience. This means explaining the notion of body in sensory terms; here is the conceptual side. And it means justifying our knowledge of truths of nature in sensory terms; here is the doctrinal side of the bifurcation" (Quine 1969, 71). Quine writes, "To endow the truths of nature with the full authority of immediate experience was as forlorn a hope as hoping to endow the truths of mathematics with the potential obviousness of elementary logic" (Quine 1969, 74). It was not that experimental implications were too complicated to trace. The problem was that large blocks of a theory may match sensory statements, but individual statements in the block may not (Quine 1969, 79).

7 It can be argued on behalf of the naturalized epistemologist, in the case of basic arithmetic for instance, the logic of discovery (e.g., as explained by Philip Kitchen 1984) does count as a justification (See Section 3.1).

8 Just as one's eyes are irradiated in two dimensions and we see in three, similarly, concepts are used in constructing the world (Quine 1969, 84). Quine even suggests that some structural traits of colour perception - and induction itself - may have an evolutionary explanation (Quine 1969, 90). Also see: (Maddy 1990a, 620).

9 This is what I call a c-truth in Section 8.3. For instance, as Quine notes, for Dewey, knowledge, mind and meaning are part of the same world (Quine 1969, 26). Yet talk of a system, holism, and so on, can give the impression that Quine is lapsing into idealism. See, however, Quine's remarks: (Quine 1953, 16).

10 Hacking has employed the phrase but my usage is completely independent of his.

11 Quine writes: "The reason for admitting numbers as objects is precisely their efficacy in organizing and expediting the sciences. The reason for admitting classes is much the same" (Quine 1960, 237). In fact, he notes that mathematics did develop along side science (Quine 1981, 154).

12 Quine writes: "But to view classes, numbers, and the rest in this instrumental way is not to deny having reified them; it is to explain why" (Quine 1981, 15).

13 Quine writes: "Each reform is an adjustment of the scientific scheme, comparable to the introduction or repudiation of some category of elementary physical particles"

(Quine 1960, 123). The same values that care for ontological economy in science apply to mathematics (Quine 1960, 269).

14. Following practice is reflected in his epistemology (regardless of the domain of inquiry). He remarks, the axiom of choice was defended because it was widely used (Putnam 1979a, 66-7, 76). Also see (Putnam 1978, 76; 1981, 73). It may seem that Putnam's holism is more splintered than Quine's, because he often speaks of different inquiries, levels, and descriptions, which are epistemologically tightly bound together. Rather than viewing science as one block, Quine says, "more modest chunks suffice" (Quine 1981, 71). So, one can say some parts of knowledge are more closely tied together (say, within one domain), than the entire picture which may be not as closely conjoined.

15 Mathematical statements, Carnap says, do not "possess any factual content" (Carnap 1939, 2); for example, they do not yield any predictions as to be testable. Yet positivism suffers from several generic problems (arguably, from taking their thesis too far): (1) Claiming their division - the meaningless, analytic, and empirical - is semantic; rooting verification in a theory of meaning (statements about the past, future, present, and even concerning the external world once one dies) may be deemed vacuous (Reichenbach 1938, 73, 135). (2) They may have a simplified notion of science that excludes speculation as meaningless. And, (3), they abandon the realist hypothesis because it is empirically unverifiable. Finally, there own thesis seems empirically unverifiable, i.e., meaningless.

16 For example, Putnam notes, inductive logics all depend on some *a priori* ordering of hypothesis (e.g. by simplicity). In mathematics the analogy would be, say, the axioms of set theory. As he notes, in Newton's Principia, "rule 4" tells one that when there are two hypotheses one should choose the one that is accepted and *a priori* more plausible (Putnam 1979a, 66-7, 75).

17 Putnam writes, "And I argued that being rational invokes having criteria of relevance as well as criteria of rational acceptability, and that all of our values are involved in our criteria of relevance. The decision that a picture of the world is true (or true by our present lights, or 'as true as anything is') and answers the relevant questions (as well as we are able to answer them) rests on and reveals our total system of value commitments. A being with no values would have no facts either" (Putnam 1981, 201).

18 His justification of one account of human nature over another rests in the fact that he starts from one. One must distinguish between science and how it is used.

19 The possibility arises that Putnam may disagree with the interpretation to be provided, but that need not be constraining because the logical possibility arises that he could be wrong.

20 See: (Quine 1969, 87; Putnam 1981, 55; 1990, 178).

21 Quine writes, "The fundamental use of natural numbers is in measuring classes; in saying that a class has 'n' members. Other serious uses prove to be reducible to this use" (Quine 1981, 15).

22 One may wish to notice that Field would agree that science may be more secure than mathematics (see Chapter 7). Mathematics, according to him, is defeasible but scientific knowledge need not be because the causal constraint requirement holds for that domain. Roughly put, according to him, scientific beliefs can be verified, in principle, by investigating mind-independent objects.

23 In this thesis I have focused upon Field, so regrettably, for reasons of scope, have less to say about Feferman who is a constructivist.

24 The problem of abduction justifying false hypotheses also occurs in science. For example, Maddy notes: (i) in computing trajectories we treat sections of the earth as flat; (ii) we assume the ocean infinitely deep when we analyze waves on the surface; (iii) we use continuous functions to represent quantities like energy, charge, and angular momentum when we know they are quantized; (iv) we take liquids to be fluid substances in fluid dynamics, despite atomic theory. Maddy notes that her examples are, of course, cases of idealizations. One could stipulate that abduction does not cover idealizations. But it is not always clear - especially when considering mathematics' application - when one is dealing with an idealization and when not (Maddy 1997, 152).

25 Jody Azzouni's distinction between thin (e.g., mathematical) posits and thick (e.g., scientific) posits is another way of drawing the same line between when abduction should be utilized and when not (Azzouni 1994, 65).

26 Also see: (Parson 1983, 195).

27 Abduction applied within science will be representative of cases where the objects in question function causally in the theory: that is why there is often a consensus about inferring

their existence. Arithmetical objects do not causally functioning in scientific theory: there is no scientific reason to infer their existence.

28 For Quine, since the indispensability argument involves inferring the existence of abstract objects without any related observations, Russell's words, written in a different context, are apt, "[It] has many advantages; there are the same as the advantages of theft over honest toil" (Russell 1919, 71). Things are even worse than Russell suggests. Benacerraf remarks, "For with theft at least you come away with the loot, whereas implicit definition, conventional postulation, and their cousins [like the indispensability argument] are incapable of bringing truth" (Benacerraf 1973, 679).

### Notes to Neo-Fregean Realism

1. Hume's principle showed how to form numbers without sets, and was not supposed to be based on experience (Demopoulos 1995, 5).
2. Yet, he notes that, "no two men have the same idea" (Frege 1977, 15). Also see: (Frege 1953, sec. 24).
3. Also, according to Frege, truth cannot be a matter of correspondence (of ideas with reality) because, in a nutshell, there is no way to compare the ideas with an external object (Frege 1977, 3-4).
4. See: (Dummett 1973, 495; 1991, 77).
5. Frege writes, "Without offering this as a definition, I mean by 'a thought' something for which the question of truth can arise at all. So I count what is false among thoughts no less than what is true" (Frege 1977, 4).
6. Frege says, "A proposition no more ceases to be true when I cease to think of it than the sun ceases to exist when I shut my eyes (Frege 1953, vi). Beauty (Frege 1979, 3) and jurisprudence (Frege 1979, 203) are subjective matters, but truth is not (Frege 1979, 132). Thus he writes, "Psychology should not imagine that it can contribute anything whatever to the foundation of arithmetic" (Frege 1953, vi). He also thinks that saying X is neither true nor false is idealism (Frege 1964, 20). So, Frege's positivism - to speak anachronistically - accepts bivalence (e.g., he defends LEM by saying one cannot serve two masters (Frege 1979, 169)) for mathematics, and multi-valence for what he considers subjective domains of inquiry.
7. Frege writes, "No description of mental processes which precede the forming of a judgement of number...can ever take the place of the definition of the concept" (Frege 1953, sec. 26). Also see: (Frege 1964, 13). Frege wrote, "One cannot define everything, any more than one can decompose every chemical substance. To do either presupposes that we are dealing with something composite. In many cases one has to be satisfied with leading the reader, by means of hints, to understand the word as it is intended" (Frege 1979, 89). He writes, "Definitions show their worth by proving useful" (Frege 1953, sec. 70).
8. Frege writes, "How, then, are numbers to be given to us, if we cannot have any ideas or intuitions of them? Since it is only in the context of a proposition that words have meaning,

our problem becomes this: To define the sense of a proposition in which number words occur" (Frege 1953, sec. 62). The foundations of arithmetic had shifted from epistemology to semantics. Dummett remarks that with Descartes, epistemology was made the basis of inquiry (Dummett 1973, 667). Descartes was not interested, first, in explaining what there was, or how things worked, but how one could know anything at all. For Frege, one has to take one step back, beginning with meanings. An analysis of the meaning of sentences was required to achieve Frege's goal of making proof rigorous (Dummett 1973, 2). For example, though Euclid was a paragon in showing how to derive an infinite system from finite axioms, even he was not rigorous by Frege's standards because he did not define the rules of inference. Though, Frege notes, one cannot prove everything, at least, like Euclid, one can lay bare one's assumptions (Frege 1964, 3).

9. Frege says, "it is a mere illusion to suppose that a concept can be made an object without altering it" (Frege 1953, x). He disagrees with nominalists who claim, for instance, that numbers are nothing more than the rules of formation stipulate because they convert concepts into objects (e.g., marks upon the page).

10. Also see: (Frege 1964, 31; 1980, 115).

11. Thus Frege tell one not to confuse a sign with the thing signified (two signs can refer to the same thing) (Frege 1970, 22). In addition, he tells one that numbers are not ideas any more than astronomy is not about planets (Frege 1953, 37). Properties belong to the object not the concept (Frege 1964, 11). For Frege, abstract objects are given in thought and not created by them (Dummett 1991, 240). There are such things as numbers (Wright 1983, 129). He separates objects from concepts (Wright 1983, 7) (a number-object is an extension of a concept) (Frege 1964, 6). Number theory is the science where one discovers properties of, say, zero (Wright 1983, xiii).

12. Frege says that the nature and laws of arithmetic rest on settling if numbers are definable (Frege 1953, sec. 3). Definition serves both in the logicist project (as to make the reduction rigorous) and realism (to know what there is). Yet, since Frege distinguishes the logical from the psychological, meaning cannot be in terms of mental images. Frege tells us that a great part of the work of a philosopher is struggle against language (Frege 1979, 270). Wright says, "For Frege, syntactic categories are prior to ontological ones, and it is by reference to the syntactic structure of true statements that ontological questions are to be understood and settled" (Wright 1983, 25).

13. David Bostock says, numbers refer as logical objects under certain conditions, when theory "contains non-eliminable subject expressions purporting to mention the thing, or it employs subject-variables which are intended to include that thing within their range" (Bostock 1974, 55).
14. The context principle is viable if a "satisfactory criteria of the kind Dummett envisages can be provided" (Hale 1994b, 18). Hale's formulation of an amended context principle - the sortal inclusion principle - is to be defined later.
15. Bob Hale, "Singular Terms" in (McGuinness and Oliveri 1994, 17).
16. As Wetzel points out, the context principle "gives equivocal results when applied to terms in sentences containing more than two candidates for proper namehood" (Wetzel 1990, 249). In some cases X will denote, and other times not.
17. Wetzel writes, "...Dummett's criteria for delimiting English singular terms, even as amended by Crispin Wright and Bob Hale, fail to do so" (Wetzel 1990, 254). Also see: (Sullivan and Potter 1997, 146).
18. Frege writes, "In arithmetic we are not concerned with objects which we come to know as something alien from without through the medium of the senses, but with objects given directly to our reason and, as its nearest kin, utterly transparent to it" (Frege 1953, sec. 105).
19. Dummett writes, "I do not propose to contest the view, which seems to be agreed on all sides, that the criteria I proposed were not adequate as they stood. What perturbs me is how complicated it appears that the emendations must be if the purpose is to be achieved" (McGuinness and Oliveri, 1994, 269).
20. The context principle would exclude qualification over "nothing" and "something" - as in "Nothing is wise" (Wetzel 1990, 240). At any rate, one way out of the overblown ontology problem is to adhere to the "Frege-Dummett thesis that singular terms can only be introduced with criteria of identity for the references...Then it might be urged that 'grins' and the like can be provided with no such criterion, thus deflating what would otherwise be an overblown ontology" (Lowe 1995, 511). Lowe writes, "[T]his move already appears to be a concession in the direction of admitting that metaphysical considerations independent of the theory of meaning are relevant to the question of what an object is and what objects there are - for identity criteria are precisely metaphysical principles, telling us (as Locke would put it)

what identity consists in for objects of given kinds" (Lowe 1995, 511). Also, one cannot just defer to intuition as to know when the context principle demands denotation, because, as Wetzel points out, people will have different intuitions (Wetzel 1990, 252).

21. Hahn observes, "...[I]t is a big mistake to infer the structure of the world from the structure of language" (Hahn 1980, 8).

22. According to Frege, the laws of thought are the laws of logic (Putnam 1979b, xiii).

23. Wright says, "For to give an account of how a concept can be acquired, that is, a model of explanation of it, is a method par excellence both of distinctively displaying one's understanding of the concept and of bringing its content into sharp relief" (Wright 1983, 86).

### Notes to Maddy's Set-Theoretic Realism

1. See: (Maddy 1990, 3).

2 For a discussion concerning Maddy's problem of being both for and against the indispensability argument see (Dieterle 1999, 131). According to J.M. Dieterle, Maddy had used the indispensability argument to limit extending naturalism to other domains (like astrology) (Dieterle 1999, 131). Yet Dieterle thinks she is inconsistent to argue against indispensability and then go on to use it to exclude any unpalatable naturalism (Dieterle 1999, 131). Maddy, however, could always just stipulate that mathematics' indispensability to science is not where its justification lies (her project takes care of that); rather indispensability is a limiting condition on any naturalism (as to exclude unpalatable ones - e.g., astrology is not indispensable to science).

3 The naturalism to be advocated, of course, is not Quine's view (he used the indispensability argument). But what I am employing is the idea in his naturalized epistemology: One begins with science (in this case arithmetic), and uses scientific resources to work out one's epistemology.

4 G.A. Antonelli concurs: "Thus, the iterative conception of set, although perfectly consistent and adequate for classical mathematics, is seen on closer scrutiny to depend on mathematical intuitions that are far from basic and unrefined" (Antonelli 1999, 150).

5 Hallett writes, "But understanding the set concept is anything but simple: there is a mystery surrounding it which easy familiarity with the axioms conceal" (Hallett 1984, ix). Cantor, the founder of the mathematical theory of the infinite (Hallett 1984, 1), was never clear what a set was (Hallett 1984, xi). Hallett writes, "It is not good philosophically speaking reducing to set theory something so basic to human thought as the elementary theory of natural numbers if you cannot also explain why numbers are sets, and why the set concept is even more fundamental. But the set concept is far too unclear for any such explanation to be given...Is this not really reduction without explanation, or at least does it not seem in the end that we are explaining the relatively well-understood (natural numbers) with the less well-understood (set)?" (Hallett 1984, 300-1)

6 For example, Putnam says a notion is clear when it is already accepted or adequately explained in terms of antecedent ones (Putnam 1979a, 18; Frege 1953, sec. 70; 1964, 13; 1979, 89). So, if a concept - e.g. "force" - has sufficient practical value either because it already functions in a theory (like Newtonian mechanics), or is explained in

terms of ones that do, any further questions are idle. Just as science extends itself by hypotheses, so too, with mathematics generated from primitive axioms.

7 Riskin writes, "When mathematicians prove that different theorems hold in different set theories, they consider both of those theorems to be real, and they will never give up either" (Riskin 1994, 117). Riskin's own view is that a subjective element functions in choosing between which set theory one adopts (Riskin 1994, 120).

8 Maddy may be open to the idea that not all of mathematics must embrace set theory (though she would probably want to make the judgement based on the merits of the specific case).

9 **Definition** (Every set has a choice function). Let  $S$  be a set and  $F$  be a local function. Then by definition  $f$  is a choice function for  $S$  : iff:  $[f : P(S) - \{0\} \rightarrow S$  and  $\forall x \in P(S) - \{0\} [f(x) \in x]]$ . Thus a choice function  $f$  for a non-empty set  $S$  "chooses" an element  $f(T)$  from every non-empty subset  $T \subseteq S$ . Of course every such  $T$  has such an element. The problem is that, in the general case, we have no way of uniformly describing a particular element in each such set. This is why we must posit a choice function which makes the selection for us, so to speak. Of course the empty set is a choice function for the empty set (Mayberry 2000, 163).

10 Maddy says, "[M]ost garden-variety statements of set theory do have unambiguous and objective truth-values" (Maddy 1988, 275).

11 See: (Gödel 1990b, 268). According to Maddy, both a set-theoretic realist and a structuralist can agree that "the most basic level of knowledge must be accounted for in perceptual terms" (Maddy 1988, 280).

12 The second-tier explains those aspects of mathematics that are accepted for extrinsic ones (e.g., simplifies theory; yields a new proof; coherent with past conjectures; new insight into old theorems) (Maddy 1990, 144).

13 Maddy writes, "Notice first that every structure studied by the local structuralist has an instantiation within the set-theoretic hierarchy. Where the set-theoretic realist studies a certain set, the local structuralist studies the structure instantiated by that set, but methodologically there is no difference...Thus, the global structuralist is faced with a choice between postulating the necessary uninstantiated structures and finding something to instantiate those structures" (Maddy 1988, 278).

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15 Basically, the picture Maddy paints goes thus: (1) an object stimulates phase sequences of certain assemblies; (2) those help see the object; till (3) it is fixed (it hard not to see the object) (Maddy 1990, 57). Maddy claims that object detectors are the result of evolutionary pressures and early childhood experiences (Maddy 1990, 58-9).

16 See: (Kremer 1991, 258).

17 As Maddy says, "[I]f you want to know if there is a mathematical object of a certain sort you ask (ultimately) if there is a set theoretic surrogate of that sort; if you want to know if a given statement is provable or disprovable, you mean (ultimately), from the axioms of set theory" (Maddy 1997, 26).

18 Lewis says that we need to separate Mr. Hyde from Dr. Jekyll (Lewis 1991, 61). Also see: (Hallett 1984, 3).

19 According to Michael Kremer, Maddy's idea that sets have an "unpreceptible difference" (Maddy 1990, 152) from objects entails "that one could perceive distinct objects without being able to perceive any difference between them" (Kremer 1991, 257). Balaguer remarks, "Since the set and the aggregate are made of the same matter, both lead to the same retinal stimulation...But if we receive only one retinal stimulation, then the perceptual data about the set is identical to the perceptual data about the aggregate. Thus, we cannot perceive the difference between the aggregate and the set" (Balaguer 1996, 104). Balaguer also says that Maddy's thesis that sets are perceptible is at odds with the fact they are abstract (Balaguer 1994, 98, 104). He writes, "[W]hile the set of eggs and the aggregate of egg-stuff are made of the same matter and share the same location, they are not identical, they are structured differently" (Balaguer 1994, 100). He rejects Maddy's claim that sets are not abstract. An aggregate, for instance, could form a multitude of different sets (while a set is unique).

20 According to Hale, science uses experimental confirmation and disconfirmation for low level generalizations and that does not hold for sets (Hale 1993, 53). He writes, "All that is required for my argument is that the axioms adopted for set theory will have such elementary set theoretic generalizations as consequences - which should, according to the species of empiricism under discussion be empirically testable via their instantiations with respect to small finite sets of physical objects" (Hale 1993, 58).

21 Dieterle says, Maddy requires cell-assembly for perception of sets, and cell-assembly is the result of perception (Balaguer 1994, 105). But, the circularity is not vicious if conceived of empirically in developmental stages. In addition, when he requires that platonism explain how we get to mathematical truth (Balaguer 1991, 107), he does not seem to take the naturalist notion that we start with it and work backwards seriously.

22 Also, the thesis that sets can be perceived leaves one to wonder how one interacts with an empty set, which is presumably not in space-time (Kremer 1991, 259). Kremer writes, "...[O]ne might suggest something like the following: the empty set detector is a sort of higher order cell assembly. It is activated when an attempt to activate some other detector (e.g. an apple detector) fails (after some fixed time, say)...All this is just speculation, of course. All it is meant to show is that the idea that one can perceive the empty set is not just crazy" (Kremer 1991, 266).

23 It has already been pointed out that a set includes the one it is an immediate successor of. See: (Maddy 1990, 14).

24 Carson raises the issue that one that there is not principled way of drawing a distinction between sets that one can perceive and one's that cannot be so verified (Carson 1996, 8). Frege also said that drawing a distinction between big and small numbers is arbitrary, therefore, foreshadowing the problem of a two-tier epistemology: where does one tier stop and another begins (Frege 1953, sec. 5). Kremer writes, however, "Consequently, we ought to admit that there is a perceivable difference between an object and its singleton, and that this difference consists in the possession by the latter of a number property that the former lacks" (Kremer 1991, 269).

25 Balaguer echoes the worry noting that it is not clear what the relationship between impure sets and pure ones (Balaguer 1994, 101).

26 It has been remarked by another of Maddy's critics, Balaguer, for instance, "Indeed, it seems to me that in claiming that sets are structurally different from aggregates, Maddy commits herself to the claim that sets are abstract, in some relevant sense" (Balaguer 1994, 100). He adds, "And it is worth emphasizing here that the invisibility of sets is a direct result of their abstractness. The reason we can't see them is that there is something abstract about them, over and above the aggregates they correspond to" (Balaguer 1994, 106).

27 For example, a pattern, understood broadly as a regularity (which could be generated by an algorithm (Devlin 2000, 90)), could be produced by the activity of collecting. The point is that once a rule is obtained, it will not allow equivocal results, thus avoiding Carson's worry.

28 Maddy writes, "My central point here has been that various naturalistic versions of mathematical realism are more similar than they are different...we would do better on...more important issues that we face together" (Maddy 1988, 283). As Norton-Smith retorts, "While Maddy holds that sets form a natural kind, I interpret arithmetic to be about human action kinds" (Norton-Smith 1991, 218). He goes on, "There is no independently existing collection before me to perceive" (Norton-Smith 1991, 226).

29 Optimistic epistemology run the gauntlet from scientific and mathematic realism (like Putnam's scientific realism or Russell's logicism) to mysticism.

### Notes to Kitcher's Naturalism

1. In the most general terms empiricism claims: "[T]hat experience is the only source capable of furnishing us with knowledge of the world...[A]ll knowledge originates in what is immediately experienced" (Hahn 1980, 39). Second, transcendental philosophy attempts to identify the conditions of human thought (Kitcher 1998, 97). Those conditions are then deemed *a priori*. Generally, Kitcher uses transcendentalism to deem non-empirical - i.e., *a priori* - types of approaches to mathematics. Third, the naturalist uses the description of psychological process as an account of the foundations of knowledge - there is nothing epistemologically deeper than acquisition (Kitcher 1980, 220).

2. Aristotle writes, "Of things defined, i.e. 'whats', some are like 'snub', and some like 'concave'. And these differ because 'snub' is bound up with matter (for what is snub is a concave nose), while concavity is independent of perceptible matter" (1025b30). Aristotle thinks physical objects are separable (divisible) but moveable; mathematical objects not separable but immovable; and metaphysics deals with what is both separable and immovable. So, for example, the number one is perceptible (1054a10), yet "mathematics is theoretical and is a science that deals with things that are at rest, but its subject matter cannot exist apart" (1064a30). Aristotle's main point hinges on what can be conceived separable from matter and movement (1026a5) - say, in thought (293b55) - and what can not be. It seems as though Aristotle wants to be empiricist - founding mathematics in the world - but want to maintain (like Plato) their immovability (necessity).

3. See: (Kitcher 1998, 86-7).

4. Mill gives examples of where we have been wrong (Mill 1846, Bk. II, chap. 3, sec. 2). He reinforces his empiricist demand: "In distant parts of the stellar regions where the phenomenon may be entirely unlike those which we are acquainted, it would be folly to affirm confidently that this general law [causality] prevails, any more than those special ones which we have found to hold universally for our planet" (Mill 1846, Bk. III, chap. 21, sec. 5). It seems Mill's view could easily collapse into some form of actual realism (X is T or F only if verified at this moment), since there is not much more reason to assume phenomenon is different in stellar regions, to other parts of the world, or at different times (Mill 1846, Bk. III, chap. 22, sec. 1).

5. Mill writes, "The existence, therefore, of a phenomenon is but another word for its being perceived, or for the inferred possibility of perceiving it" (Mill 1846, Bk. III, chap. 24, sec. 1). Large numbers can be differentiated because they

allow the possibility of a perceptible difference (Mill 1846, Bk. III, chap. 24, sec. 5).

6. Number only differ for Mill in that they do not rest on hypothesis (except for  $1=1$ , the same equal units) (Mill 1846, Bk. III, chap. 24, sec. 6). One needs the identity assumption so that "equals added to equals make equal sums" (Mill 1846, Bk. III, chap. 24, sec. 4).

7. Frege points out that objects possess properties that concepts like numbers do not possess (Frege 1979, 76). He also claims that if one was to embrace empiricism then even fictional characters would be considered empirical (which he presumably assumes is absurd) (Frege 1953, sec. 8). Yet, one should recall that in the wake Quine's "Two Dogmas", many like Devitt espouse the empirical bases of fictional characters (see Section 2.2).

8. First, Frege's claim, on occasion, makes reference to the nature of mathematics (which I emphasize in italics). For example, Frege writes, "But it is in the nature of mathematics always to prefer proof, where proof is possible, to any confirmation by induction" (Frege 1953, sec. 2). Similarly: "It does not make sense that what is by nature sensible should occur in what is non-sensible" (Frege 1953, sec. 24). He goes on, "How can a science which bases its claim to fame on being as definite and accurate as possible repose on a concept as hazy as this?" (Frege 1953, sec. 30). Some of Frege's claims about the nature of mathematics, however, do not constitute fatal blows against Mill, because, in a nutshell, not all qualities of physical objects need apply to numbers. Second, Frege's idea of the nature of mathematics is shown in how he distinguishes brutes from human beings (which he does by higher intellectual abilities). He does not want to consider an ability of say, dogs (who can presumably distinguish the difference between the size of aggregates), as equal to men: "Consequently, such properties of things as being undivided or being isolated which animals can perceive quite as well as we do, cannot be essential to our concept" (Frege 1953, sec. 31). These are weak arguments because they beg the question (e.g., the justification of mathematics must be mathematical, i.e., non-empirical, which is precisely what is at issue); or they have weak premises (e.g., if animals can do X, X cannot be important to the foundations of arithmetic). Third, in fact, some of what Frege says is a polemic. For example, Frege speaks of the empiricist as engaging in a "psychological peep-show that reveal process of ideation (how they come about)" (Frege 1964, 24). He writes: "All this [empiricism] is pure childishness, of course, and that we have to bother with such things is a sufficient indictment of the times we live it" (Frege 1979, 73).

9. Dummett also notes that one cannot verify (or falsify) the continuum hypothesis by perception (Dummett 1991, 310). This objection can be dealt with by employing a two-tier epistemology as has been described.

10. Frege may be sceptical of a two-tier epistemology as is indicated in his comment that one cannot distinguish between small and large numbers (Frege 1953, sec. 5). Yet as shall be pointed out in Sections 6.3 & 8.1, this does not effect natural realism. Analogously, one may not know where the first tier stops and the second begins.

11. Frege writes that nothing can be learned about natural numbers from natural processes like counting peas, "from the way they originate psychologically" (Frege 1979, 276). He says, "number is not abstracted from things in the way that colour, weight and hardness are" (Frege 1953, sec. 45). Yet Frege's claim that numbers are not derived by abstraction is beside the point, since he separates acquisition from justification.

12. Frege says that though mathematics can be applied to objects, they "do not have any role in constructing mathematics as a system" (Frege 1979, 238). As Frege claims, "Mill always confuses the applications that can be made of arithmetical propositions, which often are physical and do presuppose observed facts, with the pure mathematical proposition" (Frege 1953, sec. 13). Though he notes that arithmetic is applicable, he claims that has nothing to do with its justification (Frege 1953, sec. 19). That is to say, mathematics cannot be justified by its use (as pragmatists would subsequently do) because arithmetical objectivity requires mind-independent abstract objects (see Section 4.1).

13. Also see: (Kitcher 1980, 220, 223).

14. As Frege notes, evolutionary explanations can lead to subjectivism (Frege 1953, vii, sec. 10). Also, speaking of collecting makes reference to time (one operation after another), which also, according to Frege, leads to subjectivism: "Time is only a psychological necessity for numbering...it has nothing to do with the concept of number" (Frege 1953, sec. 40). Yet it shall be argued that naturalism does not entail subjectivism (Sections 5.3 & 8.2).

15. Mill writes, "All attributes [quality, relation, quantity], therefore, are to us nothing but our sensations, and other states of feeling, or something inextricably involved therein...(Mill 1846, Bk. II, chap. 3, sec. 15).

16. P. Jourdain remarks, for instance: "[T]he light that was thrown on the general conception of a function and its

'continuity', of the 'convergence' of infinite series, and of the integral, first began to shine as a result of Fourier's original and bold treatment of the problems of the conduction of heat" (Cantor 1919, 1). Empiricism, however, is not always involved in advanced mathematical discoveries. Bundle theory (mathematics) was discovered independently of science (though Yang subsequently discovered its connection to gauge field theory (science)) (Steiner 1988, 34).

17 The Russian ruble (*rublis*) is derived from *rubitj*, which means to cut a notch (Menninger 1969, 225).

18 For example, a 'ton', for instance, was a measure of grain (Menninger 1969, 119). During the seventeenth century the English developed a taste for a drink from the West Indies with five ingredients (arrak, sugar, lemon, seasoning and water); the so-called "punch" bears a resemblance to the Sanskrit *panca*, meaning five (Menninger 1969, 180). Aristotle spoke of the fifth element (ether). "Quint" was an old word for a violin (because they used to have five strings). In French, "quinte" is a jurisdictional boundary of a city with a periphery of five miles (Menninger 1969, 180).

19 Encipherment can take two forms: (1), as ordered numerals grouped in units or ranks, like the Roman ones (e.g., "V" = 5; "X" = 10; "L" = 50; "C" = 100; "(I)" = 1000; "((I))" = 10,000) (Menninger 1969, 466); or (2), as gradational numerals. For example, in the Indian system, the first nine digits are enciphered, numbered and ranked (Menninger 1969, 396). Graduation is achieved by giving names to elements of the sequence (Menninger 1969, 45).

20 As Newton says in Principia, "Those defile the purity of mathematical and philosophical truth who confound real qualities with their relations and sensible measures" (Putnam 1979a, 80).

21 Gibson writes, "The [Kantian] doctrine that we could not perceive the world around us unless we already had the concept of space is non-sense. It is quite the other way around: We could not conceive of empty space unless we could see the ground under our feet and the sky above. Space is a myth, a ghost, a fiction of geometers" (Gibson 1979, 3). According to Gibson, we live in a medium - like air - not space (Gibson 1979, 32). Gibson also gives examples such as persistence and permanence which he uses to replace invariance in mathematics (Gibson 1979, 13). For instance, when ice begins to melt, one says it 'ceases to exist' - the idea of variance and invariance is abstracted from such concrete encounters (Gibson 1979, 13). According to Gibson, one encounters surfaces in a medium, not a plane in space (as discussed in geometry) (Gibson 1979, 33).

22 Time is always first measured in relation to our practices (i.e., an animal's span of life), in years and seconds (not millions of years) (Gibson 1979, 11). Use, Gibson conjectures, is linked to survival. Time is used to measure change (Gibson 1979, 12). The look, for instance, of some objects suggests harm, and others, a use (Gibson 1979, 20). Positive and negative utilization of things is the guiding force of locomotion, and hence perception (Gibson 1979, 232).

23 See: (Kitcher 1984, 12). "Calculus", after all, does come from the Latin, meaning a pebble used in counting.

24 Though Gibson's attack on stimulus-response behaviourism is praiseworthy, he is still, a behaviourist, i.e., his view is at odds with the naturalist account of cognition proposed because he denies that anything goes on in the head. Gibson writes, "This is not a process of sensory inputs, however, but the extracting of invariants from perceptions" (Gibson 1979, 2). He goes on, "[A]ll we ever see is the environment or facts about the environment, never photons or waves or radiant energy" (Gibson 1979, 55). Gibson credits Gestalt psychologists for reacting against the sensory stimulation picture (Gibson 1979, 138, 140). On Gibson's account, "It [perception] begins with the flowing array of the observer who walks from one vista to another, moves around an object of interest, and can approach for scrutinizing, thus exacting the invariants that underlie the changing perspective structure and seeing the connections between hidden and unhidden surfaces" (Gibson 1979, 303). Suffice his work can be attached to a functionalist account of cognition. Also note, Gibson suggests physics is about a real, meaningless world (e.g., photons), and perception about a meaningful environment (Gibson 1979, 33). Physical theory, however, has its own environment, to use Gibson's terminology.

25 Kitcher writes: "My quarrel with earlier empiricists is, for the most part, that their accounts have been incomplete rather than mistaken" (Kitcher 1984, 4). Kitcher, for example, claims that Frege's point that Mill misses the associativity of addition requires an amendment to Mill's work (Kitcher 1998, 77).

26 Kitcher notes the a priorist can either take the analytic route (statement are true in virtue of their content), or posit a faculty that allows access to a mathematical reality (Kitcher 1984, 46).

27 Kitcher does point out, however, that knowledge can be necessary and not a *priori* (also, contingent truths can be a *priori*) (Kitcher 1984, 16-7).

28 For example, see: (Crossley 1987, 1-58).

29 Also see: (Kitcher 1984, 110-1, 145-6).

30 My discussion is taken directly from these pages.

31 Also see: (Kremer 1991, 269).

32 The second axiom:  $N(x) \rightarrow N(x+1)$ . For the naturalist, induction can rest upon itself. Note, intuition need not always have a proximate empirical analogy as has been chosen here. One can have intuitions when playing chess, for example; one makes discoveries. Notice, the induction is also at work in the tenth axiom of *Mill Arithmetic* which captures the fifth postulate of Peano (any property of "0" and every successor belongs to every number). For example, "all numbers are odd or even" is an inductive inference from "some numbers are odd and even."

33 In fact, against the constructivist who presuppose *a priori* one can know what can and cannot be constructed, Kitcher writes, "But there is no reason to bind this epistemological claim [*a priori* limits] to the basic ontological thesis [mathematical results of ideal subject] of constructivism" (Kitcher 1984, 110).

34

Examiner's report: "Lakatos is an 'empiricist' in the sense of being a fallibilist, but not in the sense (of Quine and Kitcher) that sensory experience is the source of mathematical knowledge" (Personal communication from James Brown, edited by A. Gupta).

35 Kitcher points out that some "self-evident" principles have turned out false (e.g., Frege, Cantor and Dedekind accepted comprehensiveness: Any property can determine a set) (Kitcher 1984, 63). Lakatos writes, "Definitions are frequently proposed and argued about when counterexamples emerge" (Kitcher 1984, 16). The *a priori*ist, Kitcher claims, cannot explain mathematical progress (Kitcher 1978, 90; 1984, 92, 100, 228; 1992, 113; Lakatos 1979, 2).

36 Wynn also claims that children know numbers before counting, thus suggesting they are not generated by counting in that case (Wynn 1992, 329). As Marcus Giaguinto points out, the strength of Wynn's argument, "[T]he visual faculty of infants tested had not been developed long enough for the expectation revealed to have been inferred from past observations" (Giaguinto 1992, 364-5). For reasons to be pointed out, I shall not adjudicate this debate.

37 Also see (Dehaene 1997, 252; Wynn and Bloom 1992, 409).

38 Devlin's remark echoes Aristotle's account of human perception in general: "...[B]oth the sound [the world] and the hearing [perception] so far as it is actual must be found in that which has the faculty of hearing [the brain]...Now the actuality of that which can sound is just sound or sounding [i.e., the activity of perceiving]..." (426a5-10).

39 See: (Gibson 1979, 158).

40 For the difference between scepticism and conventionalism see the discussion in Section 8.3.

41 Mayberry writes, "But it can no longer impose itself upon us as *Idea*, as something given and objective which we must simply accept and adapt to" (Mayberry 1988, 321).

42 Putnam provides the slogan: "Reality is not a part of the human mind; rather the human mind is part - and a small part at that - of reality" (Putnam 1979a, vii).

43 One should note that except in the first case, strong and weak foundationalism are compatible with any of the types of epistemological programs to be considered (Putnam's 1998) (see Introduction and Chapter 7).

### Notes to Field's Nominalism

1. He does not explain where the obscurity lies.
2. Putnam counsils that not all problems are so, and agreement that a term is unclear does not mean it is (Putnam 1979a, 44).
3. As Hardy also notes, "Mathematicians have always resented attempts by philosophers or logicians to lay down dogmas imposing limitations on mathematical truth or thought...That 'the finite cannot understand the infinite' should surely be a theological and not a mathematical war-cry" (Hardy 1925, 5). Lewis concurs, "Our knowledge of mathematics is ever so much more secure than our knowledge of the epistemology that seeks to cast doubt on mathematics" (Lewis 1986, 109). One has to pay attention to one's intuitions. As Lewis - of all people - puts it, "For it is pointless to build a theory, however nicely systematized it might be, that it would be unreasonable to believe. And a theory cannot earn credence just by its unity and economy" (Lewis 1986, 134).
4. Quine and Chihara would disagree, for example, that one can avoid existence by appeal to possibilities (Quine 1953, 17; Chihara 1973, 77). Shapiro explains, "To say that a sentence is logically possible is to say that there is a certain set that satisfies it...It does no good to render mathematical 'existence' in terms of logical possibility if the latter is to be rendered in terms of existence in the set-theoretic hierarchy. Putting the views together, the statement that a sentence is logically possible is really a statement about all set-theoretic models of set theory. Who says there are such models?" (Shapiro 2000, 275) Shapiro comments, however, "Hellman...demurs from the standard, model-theoretic accounts of the logical modalities. Instead, he takes the logical notions as primitive, not to be reduced to set theory" (Shapiro 2000, 275). As Hellman points out, possibility itself is not an actual object (Hellman 1989, 59).
- 5  
Griffin writes, "In fact, ' $\omega$ ' appeared for the first time in the French translations of his [Cantor's] writings. Previously he has used ' $\aleph$ '...Cantor has changed the notation in order to distinguish the transfinite ordinals represented by  $\omega$  as actual infinities from the potential or false infinite traditionally represented by  $\aleph$ " (Griffin 1991, 241 fn.19).
- 6 Curry remarks, "Mathematics is the science of formal systems...Moreover the arbitrary nature of definitions which can constitute the frame of a formal system show that, in principle, at least all formal systems stand on par" (Curry 1951, 56). He goes on, "The formalist conception of mathematics is thus free from any metaphysical bias, and is

therefore compatible with practically any sort of philosophy" (Curry 1951, 58).

7 Field claims that if it was an "equivalent description" of mathematical objects the indispensability argument would support modalized mathematics (Field 1989, 269). But, Field says, the indispensability argument supports mathematical objects not modalized mathematics. Next, Field does not think modalism can explain applicability (Field 1989, 281). Modalism utilized in a realist program could.

8 Frege also has a polemic against the formalist. Frege writes, "These numerical signs do not really designate anything: they are themselves the things that we are inquiring about: Quite a dodge, a degree of cunning amounting, one might say, to genius" (Frege 1979, 275). Frege argues that nominalists cannot explain application, confuse formal theory with meta-theory, and have no coherent explanation of an infinite sequence" (Dummett 1991, 252).

9 Field cites four issues that have occupied philosophers of mathematics: (a) How much mathematics is true? (b) What entities do we have to postulate for mathematical truth? (c) What account can we give of mathematical truth? And (d) What account can we give of mathematics applicability? (Field 1980, vii) He says (d) is the important one: According to Field, if applicability is not a reason for truth "then there is no reason to regard any part of mathematics as true" (Field 1980, viii).

10 For a platonist conservativeness follows from necessary truth; yet, Field says, one needs a reason for necessary truths (or one could deduce anything by picking and choosing one's necessary truths) (Field 1989, 44).

11 Field claims that mathematical statements are contingently false because there are no mathematical objects. See: (Colyan 2000, 87).

12 Incidentally, his argument against modalism rests on the idea that applying possibility for mathematical statements would also have to apply to scientific ones (Field 1989, 268).

13 For example, Weyl remarks, "For a mathematician it is irrelevant what circles are" (Weyl 1949, 8). More generally, Weyl remarks, "Perhaps the philosophically most relevant feature of modern science is the emergence of abstract symbolic structures as the core of behind - as Eddington put it - the colourful tale of the subjective storyteller mind" (Weyl 1949, 237). Curry adds, "[A]lthough a formal system may be represented in various ways, yet the theorems derived according to the specifications of the primitive frame remain

true without regard to changes in representation. There is, therefore, a sense in which the primitive frame defines a formal system as unique object of thought" (Curry 1951, 30). Resnik also points out that doing a proof is not empirical or everything would be so (Resnik 1997, 154). Resnik says, "However, for some time the practice of pure mathematics has reflected the idea that mathematics is concerned with structures involving mathematical objects and not with the 'internal' nature of the objects themselves" (Resnik 1997, 201). He writes, "Mathematical objects are featureless, abstract positions in structures (or more suggestively patterns)" (Resnik 1997, 4.). For instance, David Lewis points out that a set theory structuralist quantifies over relations before set theory (Lewis 1991, 53).

14 See: (Burgess and Rosen 1997, 146-7; Hellman 1989, 61; Field 1989, 227; Putnam 1979a).

15 Constructivism "rejects existence proofs of a certain kind, so-called 'non-constructive' existence proofs, which purport to establish that there exists a mathematical entity with some mathematical property, but do not even implicitly identify any specific instance of such an entity" (Burgess and Rosen 1997, p.173). Dummett echoes the idea: "A proof of a closed statement of the form  $\exists x A(x)$ , say one in which the variable ranges over the natural numbers, is constructive just in case it either itself provides a specific instance of  $[A(n+1)]$  or yields a effective means, at least in principle, for finding a proof of such an  $[A(n+1)]$ " (Dummett 1977, 9 and 1978, 207). More generally, Parsons remarks, "The philosophical interest of constructivism is that it is the expression of a thoroughgoing idealist or anti-realist conception of mathematical existence and truth" (Parsons 1983, 30).

16 Weyl writes, "Without claiming to give a mechanically applicable criterion, our description bears out the essential fact that objectivity is an issue decidable on the ground of experience only" (Weyl 1949, 72). In fact, Weyl thinks that it is the exact natural science that is distinctive of our culture, i.e., the procedure employed for attaining truth (Weyl 1949, 216).

17 Kreisel writes, "...[T]he constructive aspect of mathematics which Bernays calls, epigrammatically, a mathematics of doing in contrast to a mathematics of being, or, more formally, an idealization of process an idealization of what is the case" (Kreisel 1959, 146).

18 Parsons 1983, 21. As Parsons puts it, the price of consistency is incompleteness (Parsons 1983, 267). Weyl writes, more generally, "Mathematics is the science of the infinite" (Weyl 1949, 66).

19 Lewis says it would be absurd to reject mathematics for philosophical reasons (Lewis 1991, 58). Yet constructivism does not logically entail anti-realism; one could link it with modalism or a realism by which entities come into existence as one probes (Dummett 1978, 18, 185).

20 See: (Resnik 1997, 272). Though he does point out that a realist can deny the causal constraint (Resnik 1997, 175).

21 See: (Field 1989, 68).

22 Hellman writes, "Thus, actual platonist mathematical objects are dispensable after all" (Hellman 1989, 143). According to Hellman mathematical statements are hypothetical "as to what would hold in any structure of the appropriate type" (Hellman 1989, 53). For example, sets are the result of a process and a possible one (Hellman 1989, 61).

23 See: (Resnik 1997, 232). For example, since Field's nominalistic physical theory does range over space-time points and regions (Field 1989, 68) - which are infinite - the problem of a higher mathematics need not arise (Hellman 1989, 140).

24 A "template" can be understood by turning to Goodman's, more general, epistemology. According to Goodman, categories are required for the construction of a description (whether one sees something as a criminal, man, or moving object) (Goodman 1972, 9, 358, 398, 401).

25 Isomorphic domains possess the same structure if nothing could be asserted in P that could not be asserted in L (Weyl 1949, 25).

26 See: (Curry 1951, 30). Shapiro writes, "[A] formal theory is a rather crude model of various mathematical fields..." (Shapiro 1991, 4).

27 Chihara identifies the points at which Field can be attacked: (1) doubt that there is no scientific ground for preferring a platonist particle physics over a nominalist first-order version (Chihara asks why a scientist should adopt a nominalized science (Chihara 1990, 182); (2) doubt present day science can be put in nominalistic terms; (3) doubt Field's nominalistic ontology of space-time; (4) doubt his account of cardinality; (5) doubt model-theoretic terms; (6) doubt if conservativeness explains usefulness; and (7), doubt if reasonable theorems of mathematics are false (Chihara 1990, 173).

28 Shapiro explains, "The theorems for the application of set theory - the existence of representing homomorphism - are not possible unless all models of the original theory are 'standards'. This, in turn, requires a second-order axiomatization, and, in this case, deductive conservativeness does not hold" (Shapiro 1996, 234).

29 Completeness requires that every pertinent general proposition " $(\forall x) (Px \vee \neg Px)$ ", be decidable by logical inference based on the axioms (Weyl 1949, 24). Also, Field's program depends upon assumptions that are not themselves nominalizable. Shapiro remarks, "In short, the foundational 'goals' for which higher-order terminology is invoked are regarded as unattainable in semantics, either because they cannot be attained at all - there is nothing unequivocal to describe - or because communication is attained only informally" (Shapiro 1991, 197). Shapiro contends that Field must choose between deductive conservativeness or applicability. On the one hand, if Field abandons deductive conservativeness, he could end up a platonist (which has already been denied, Section 3.4). On the other hand, not being able to explain applicability is (because platonism is abandoned), by his own yard stick, unsatisfactory (Shapiro 1996, 226, 234).

30 Shieber makes the often made assumption that factualism requires mind-independence (i.e., independence from any belief system). The issue, however, is about if one's belief system is correct.

### Notes to Natural Realism

1. Putnam writes, "Science at best is a way of coming to know, and hopefully a way of acquiring some reverence for, the wonders of nature" (Putnam 1979a, xiv).

2 Moderate naturalism cannot tell one what values and practices one ought to adopt. Philosophical reflection that meditates upon what ought to be the case escapes empirical investigation. Moderate naturalism - although it requires that one's epistemology should be informed by science - may not be prescriptive. That is to say, a moderate naturalist can utilize science to help describe why one has the values one does (upon which one's science is founded), but it cannot prescribe what one ought to subscribe to (Brook and Stainton 2000, 194, 197).

3 Brook and Stainton remark, "It is really too early to tell, but two recent developments mildly favour merely moderate or stronger naturalism and functionalism" (Brook and Stainton 2000, 205).

4 Anoop Gupta, "Wilson-Kitcher Ecological Behaviourism, The *Is-Ought* Problem, and Naturalized Epistemology" (forthcoming).

5 As Putnam writes, "I cannot follow 'physicalists (for example, Hartry Field) who would agree that 'intentional' or semantical properties (for example, reference) can be reduced to physical ones. A fortiori, I cannot agree that all properties are physical. If scientific realism is scientific imperialism - physicalism and materialism - I am not a scientific realist" (Putnam 1994a, 492). On a functionalist view, one may want to note in passing, biology may determine what abilities are humanely possible (yet the development of abilities are contingent upon, for example, psychological, social, and economic factors).

6. Fig-leaf realism is consistent with scepticism about any body of knowledge. Analogical epistemology is an example of fig-leaf realism. It draws upon the writings of Aquinas, whereby knowledge of the empirical world (God's creation) are a symbol of God (Gupta 1997, 295-6). So, if one was an anti-realist or realist about a body of knowledge (e.g., today's arithmetic), it would, insofar as it is part of knowledge of the creation, be a symbol of God. Yet a realism that is weak enough to be consistent with subjectivism for any particular body of knowledge, is, as Devitt said, "anti-realism with a fig-leaf" (Devitt 1991, 17).

7 Of course, this puts things in terms of entity-realism, yet that has been presupposed for objectivity (Chapter 1).

Personal Communication from Nicholas Griffin, 18 September 1999: "Realism is consistent with scepticism, but does not entail it. Scepticism (about subject S) asserts that we know nothing about S, i.e., that the truth-values of all propositions about S *actually* transcend our ability to recognize them. Realism merely asserts that these truth-values *may* transcend our ability to recognize them. (You can't derive a claim that something *is* the case from a claim that it *may be* the case.) Most realists would claim that we do know the truth-values of some propositions about S, but not the truth-values of others. But two limiting cases are possible: (i) the sceptical realist who says that in fact we don't know the truth-values of any of them; (ii) the 'optimistic' optimistic realist, however believes that truth is recognition-transcendent; that the fact that we do know the truth-values is not what makes the propositions true; that it is possible that we didn't know them and this would not affect what the values were - we're just incredibly fortunate in knowing all of them. Neither of these extreme positions is very plausible. Anti-realism, by contrast, is incompatible with scepticism. This is its one big attraction." First, I am following Putnam, such that by rejecting the correspondence theory of truth (which Putnam takes to be an aspect of the metaphysical realist doctrine) forces scepticism. With no possibility of knowing if correspondence fulfils, the truth-values of propositions are rendered *actually* recognition-transcendence (unless one embraces anti-realism or natural realism). Of course, the issue of correspondence phrases the issue in terms of entity-realism; in the case of mathematics, however, one can replace "correspondence theory of truth" with talk of having (or not having) a canonical method for attaining the truth of a proposition. Second, Griffin is correct to say that both extremes (the complete transcendence or immanence of truth) is implausible. The advantage of natural realism is that it allows that for domain S some truths are known and others not so (but bivalence holds in both cases). Finally, the anti-realist can avoid scepticism but not subjectivism (which seems like a high price to pay to avoid subjectivism). The natural realist can avoid (for domain S) both scepticism and subjectivism (Section 8.2).

8 Griffin writes, "The correspondence theory of truth states that a proposition is true if and only if it 'corresponds' to the way the world is. A proposition whose truth **actually** transcends our ability to recognize it, may still correspond to the world, but, of course, we will not, in this case, be able to recognize how the world is in this respect" (Private Communication: 18 September 1999).

9 Dummett, for example, says there are two issues for intuitionism (the second one being infinite sequences), and the first: "...[I]s the general theory of meaning for a mathematical language, according to which the understanding of a mathematical statement is to be thought of given in terms of the mental constructions which may serve to prove the statement, rather than in terms of the state of affairs within objective mathematical reality which render it true or false: it is this which causes the principles of intuitionistic reasoning to deviate from those of classical logic" (Dummett 1977, v and ix). Some propositions, like universal ones - of the form "all  $x$  is  $A$ " - may be neither true nor false because there is no way to decide them (Dummett 1977, 4-5). Constructivism, however, need not be linked to a theory of meaning at all as Dummett suggests. A constructivist could assert the epistemic claim concerning the rejection of the law of the excluded middle for a certain class of statement without rejecting bivalence for them (i.e., without rejecting that the meaning of "truth" must be given in terms of use). Brown points out, however, "Brouwer's intuitionism is based on a Kantian view of our intuition of time while Dummett's is based on a Wittgensteinian account of meaning which leads him to a kind of verificationism. The sources of their respective constructive mathematics is quite different, but the result is the same" (Brown 1999, 120). The motivation for Brouwer's and Dummett's intuitionism was to avoid platonism, and thus, the anti-realist result is the same (Brouwer 1975, 551; Dummett 1977, v, ix). Yet the different origins shows itself in the result: Dummett's (Wittgensteinian) view is not psychologistic as is Brouwer's (Kantian view) (Sundholm 1984-5, 271-6).

10 Realism was a received doctrine for those developing the classical algebra of logic. Brouwer writes, "Classical algebra of logic, founded by Boole, developed by De Morgan, Jevons, Peirce, and perfected by Schröder, furnishes a formal image of the laws of common-sensical thought. This common-sensical thought is based on the following, conscious or subconscious, threefold belief: First, in a truth existing independently of human thought and expressible by means of sentences called 'true assertions', mainly assigning certain properties to certain objects or stating that objects possessing certain properties exist or that certain phenomena behave according to certain laws. Furthermore in the possibility of extending one's knowledge of truth by the mental process of thinking, in particular thinking accompanied by linguistic operations independent of experience called 'logical reasoning', which to a limited stock of 'evidently' true assertions mainly founded on experience and sometimes called *axioms*, contrives to add an abundance of further truths. Finally, using the term 'false' for the converse of true, in the so-called 'principle of the excluded third' saying that each assertion, in particular each existence assertion and each assignment of a property to an

object or of a behaviour to phenomenon, is either true or false, independently of human beings knowing about this falsehood or truth..." (Brouwer 1975, 551).

11 Brouwer writes, "Obviously the field of validity of the principle of the excluded third [the law of the excluded middle] is identical with the intersection of the fields of validity of the principle of testability and the principle of reciprocity of complementarity [i.e., if a statement is proved to be non-contradictory, it can be proved true (Brouwer 1975, 524)]" (Brouwer 1975, 492). Brouwer goes on, "In intuitionism, of course, all these principles [judgeability, testability, and reciprocity of complementarity], being, assertions about assertions are only 'realized', i.e., only then convey truths, when these truths have been experienced. On this basis it can be proved that the extended principles are not only not true, but even contradictory. On the other hand, in their simple form, all three principles are, although not true, at least non-contradictory" (Brouwer 1975, 524). Dummett notes that intuitionists, however, are wary of extending ideas about meaning beyond mathematics (Dummett 1977, 138).

12 See: (Wright 1993, 85).

13 See: (Dummett 1977, 17; Wright 1993, 190).

14 According to Dummett - for intuitionists - one need not hold the radical doctrine that LEM only holds for statements that have been decided, but rather, that one can "prove A" (Dummett 1977, 18-9). Nonetheless, Dummett writes, "'We can prove A' must be understood as being rendered true only by our actually proving A, but as being rendered false only by our finding a purely mathematical obstacle to proving it. From any standpoint therefore, there can, again, be no guarantee that every mathematical statement is either true or false" (Dummett 1977, 19). However, who are "we"? Wright remarks that anti-realism is a new name for idealism and the opposite of realism (Wright 1992, 3; 1993, 2). Wright notes, something of logical positivism lives on in anti-realism, since they constrain truth and label some statements, if not senseless, neither true or false (Wright 1993, 279).

15 Wright says that "various regions of discourse" may be more successful for realism than others (Wright 1993, 2).

16 Strict actualism has the same problem as nominalism; universals are banned and the world becomes unintelligible (Chapter 1).

17 Brown writes, "For example, there is a short (hence surveyable) rule for calculating the value of  $\pi$  to any decimal place. Thus the infinite expansion is perfectly real and

already exists independently of our actually carrying out the calculation. By contrast, there is no rule for locating the twin primes - we simply have to work out the instances. Thus by this [constructivist] criterion, there is no fact of the matter about twin primes, though there is about the decimal expansion of  $\pi$ " (Brown 1999, 151).

18 "Human beings", however, is still vague, since what is recognizable to alchemists in the sixteenth century is going to differ from chemists in the nineteenth. That is to say, as knowledge, technology, and in the long run, evolution, change human beings, what is recognizable is going to change. Claiming that LEM does not apply to a class of statements today could require revision of that verdict if what it is possible for human beings to do changes. Conflicting verdicts on the application of LEM is absurd and highlights the problem of attempting to lay down *a priori* what is and is not in possible for human beings to do.

19. The difference between can be explicated in modal terms for the purpose of illustration. For a radical anti-realist, X must be necessarily decidable (e.g., strict actualist) (Also see Marion and Dubucs 1999). For a natural realist, X must be possibly decidable (e.g., natural realism). For a metaphysical realist, X is necessarily not decidable but bivalence is maintained (e.g., the sceptic).

20 The law of non-contradiction:  $\neg(\exists x)(Px \ \& \ \neg Px)$ ; and LEM:  $(\forall x)(Px \vee \neg Px)$ , with an exclusive disjunction. Natural realism is thus more liberal (more realist) than Dummett's notion of "decidable in principle" (Dummett 1977, v, 19, 24; 1978, xxvii, 6, 248).

21 See: (Wright 1993, 240).

22 See: (Wright 1992, 109, 148).

23 Tennant provides a usefully summary of the results: "[N]o acceptable formal system can capture all the truths of classical arithmetic" (Tennant 1997, 189):  $\neg(\exists S)(\forall \phi) (\phi \text{ is true} \rightarrow S \rightarrow \phi)$ .

24 As Gardiner points out, Dummett realized that Gödel's challenge can be understood as a conflict of three principles (Gardiner 2000, 98): (i) truth is co-extensive with provability; (ii) the identification of meaning with use; (iii) the existence of true but undecidable sentences. Generally, (i) and (ii) is at odds with (iii). On the one hand, if one accepts that there are true statements that are undecidable, truth transcends provability and meaning (if one

accepts the manifestation argument, e.g., to manifest knowledge of "square" means one can pick out a square (Gardiner 2000, 17)). The incompleteness theorem entails that there will be a mathematical statement which is not provable nor which one could manifest knowledge of. Thus, the debate must be fought according to the following criterion: one can accept only (i) and (ii) or (iii) (but not both). Concerning (i), Gardiner rejects it. Concerning (ii), Gardiner offers detailed refutations of what Dummett thought was entailed by his semantic strategy. (e.g., Gardiner accepts that a theory of meaning must be harmonized with a theory of understanding but rejects that the realist cannot do this.) He opens the way for accepting (iii).

25 Also see: (Hellman 1977, xx).

26 Brown writes, "More generally, a mathematical representation of a non-mathematical realm occurs when there is a homomorphism between a relational system  $P$  and a mathematical system  $M$ .  $P$  will consist of a domain  $D$  and the relations  $R_1, R_2, \dots$  defined in that domain;  $M$  similarly consists of a domain  $D^*$  and relations  $R^*_1, R^*_2, \dots$  on its domain. A homomorphism is a mapping from  $D$  to  $D^*$  that preserves the structure in the appropriate way (Brown 1999, 47).

27 Rational acceptability, for instance, may include explanatory value, coherence, simplicity, and so on.

28 Resnik writes, "It would not be surprising if the solutions to these two problems [of reference and epistemology] come together, for if we can determine how we know mathematical objects and what we can know of them, then we should have some idea of how we relate to them. That, in turn, could provide a clue to how we refer to them" (Resnik 1980, 234).

29 Arthur Fine asks, first, why such a small set of truths have been successful (Fine 1986, 119). Second, he claims realism adds nothing to a theory (except perhaps providing the psychological motivation for one, thus speaking of "motivational realism") (Fine 1986, 111, 129). First, however, it seems he may mean "small" in relation to all theories ever advocated: science's success is noteworthy. Second, any epistemological theory - like realism - need not add something to science (the issue is what is an accurate description of a body of knowledge). Note, Fine also engages in the polemic, for instance, claiming anti-realism is consistent with what most scientists think (Fine 1986, 11, 111, 123); a claim that is either wrong or not a strong reason (it is an improper appeal to authority: scientists are not epistemologists) (see Chapter 3). What has been said here is programmatic and

demands further thought (to be taken up in a future work).

30 If science is a self-correcting enterprise, that also entails endorsing the idea of epistemic change (Reichenbach 1938, 364; Putnam 1979b, ix, xv; Misak 1991, 54; Sellars 1997, 79).

31 Fine does not think that approximation to the truth can be justified in terms of success (Fine 1986, 121). He is wrong, in principle: A theory may consist of a number of claims. As they become refined (say, rejecting some), more detailed (expanded), or added to (say, linked to other theories), there is reason to believe that technical success will increase. For instance, though the steam engine was known before the second law of thermodynamics, its efficiency was subsequently improved with as understanding of how gases, like oxygen, function at different temperatures.

32 Also see: (Misak 1991, 118). Misak remarks, "[A] person does not have a complete grasp of a predicate *F* if she is unable to say what would be the consequences of hypotheses of the sort '*a* is *F*'" (Misak 1991, 4).

33. Also see: (Wright 1992, 7; 1993, 86, 425). Note, as an aside, however, for the claims to which natural realism applies, one must distinguish between historical ones and laws (which may not be easy, for instance, in the social sciences). Consider a scientific law, *X* (*P* causes *Q* under conditions *C*). What is *X*'s scope? (a) *X* is true on 23 June 1942; (b) *X* is true in 1942; (c) *X* has been true in the past; (d) *X* is true in our universe; or (e) *X* is true. The first three cases are historical claims and not scientific laws. The fourth and fifth are, for example, scientific laws and have a counterfactual truth (they are to be true in the future) (Cartwright 1983, 46). Cartwright remarks, for example, "Probabilities are not data but ideal theoretic representations" (Cartwright 1989, 35). Being theory-laden, however, does not effect natural realism. Natural realism can apply to both historical facts, scientific laws and arithmetical truths. For the natural realist, at the very least, scientific and arithmetical truths are true in our universe. (Lewis's scheme is used but not the way he divides up different types of knowledge (Lewis 1986, 112-3)). The natural realist tentatively circumscribes possibility to what is conceivable. Yet suffice it to say, since the notion of possibility is obscure (it requires a metaphysical debate), the natural realist interprets the possible recognition-transcendence of arithmetical facts as entailing truth in our universe, and not in every possible one.

34 "Non-sense" should be understood epistemically not semantically. That is to say, it is not necessary that a claim

which qualifies as epistemic non-sense (e.g., does God wear a green hat?) is semantic non-sense.

35 Consider a taxonomy: "The animals themselves can be divided in different ways. Zoology classifies them by heredity and autonomy, by phylum, class, order, genus, and species, but psychology can classify them by their way of life, as predatory or preyed upon, terrestrial or aquatic, crawling or walking, flying or non-flying, and arboreal or ground living" (Gibson 1979, 7).

36 For example, the scientific realist may contend that the truth-value of a statement about the position of a quantum particle is either true or false; an anti-realism may deny the application of LEM to high level physics. The cracks in the alliance, thus, come over when the application of LEM must be excluded.

37. It is not required that there is only one standard, but they must all yield the same verdicts upon truth-candidates to be included in the domain of r-realism.

38 The Gödelian is perhaps too optimistic in not allowing that bivalence may have to be rejected for some classes of statements. According to the natural realist, the truth-values of statements are, therefore, recognition-transcendent, which allows the possibility that bivalence does not hold for some class of statements. Furthermore, even though recognition-transcendence was taken to be essential to metaphysical realism, it is worth pointing out that natural realism does not require that there be one correct description of the world. That is, one can be a natural realist about different descriptions of the world (Gupta 2000, 76).

39 See (Dennett 1998, 66).

40 As Misak remarks, "The pragmatic account of truth pays special attention to the fact that, pre-theoretically, we think that inquiry aims at truth" (Misak 1991, 162). Dummett also says that, "For the realist, actual practice has to be explained not corrected, and his explanation is the only one which will fit the facts of practice" (Dummett 1993, 318).

41 See: (Cartwright 1983, 19). The instrumentalist, for example, attempts to render theory consistent with phenomenon (observations) and not a real world (Cartwright 1983, 56-7). Anti-realism, so the claim goes, is a better description of the relationship between theory and data (e.g. it account for under determination of theory by data) as oppose to the superfluous contention that realism explains why theory and data match-up (which is not always the case).

### Notes to Concluding Remarks

1. Gödel chastises, for example, Bertrand Russell for backing away from realism (a fault he attributes to the influence of Wittgenstein). He writes, "When he [Russell] started on a concrete problem, the objects to be analyzed (e.g., the classes or propositions) soon for the most part turned into 'logical fictions'" (Gödel 1990b, 121).
2. Gödel says that just because abstract objects cannot be associated with any "actions of certain things in our sense organs, [they] are not purely subjective, as Kant asserted" (Gödel 1990b, 268). He goes on, "Rather, they, too, may represent an aspect of objective reality, but, as opposed to the sensations, their presence is in us may be due to another kind of relationship between ourselves and reality" (Gödel 1990b, 268). Gödel writes to Tarski, "One of my reasons [for belief in CH] is that I don't believe in any kind of irrationality such as, e.g. random sequences in any absolute sense" (Gödel 1990b, 175). There is a reality - that is a certain way, orderly - and, we have a faculty to perceive them. He also writes, "[Intuitions] are necessary not only for obtaining unambiguous answers to the question of transfinite set theory, but also for the solution of the problems of finitary number theory (of the type of Goldback's conjecture)" (Gödel 1990b, 269).
3. Benacerraf seems to agree: Ultimately I will argue that each kind of account has its merits and defects: each address itself to an important component of a coherent over-all philosophical account of truth and knowledge" (Benacerraf 1973, 666). Yet he does not think the standard (platonist) and combinatorial views are coherent, hence the dilemma. Furthermore, he does not take seriously the idea that one can reject the standard view - though he notes that he thinks Putnam maintains its rejection (Benacerraf 1973, 669 and fn. 6).
4. Putnam articulates one of his aims: "...[M]athematics is not an *a priori* science, and an attempt to spell out what its empirical and quasi-empirical aspects really are, historically and methodologically" (Putnam 1979a, vii).
5. Hardy writes, "It seems to me that no philosophy can possibly be sympathetic to a mathematician which does not admit, in one manner or another, the immutable and unconditional validity of mathematical truth. Mathematical theorems are true or false, their truth or falsity is absolute and independent of our knowledge of them. In some sense, mathematical truth is part of objective reality" (Hardy 1925, 4).

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