Schreier graphs and ergodic properties of boundary actions

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Abstract

This thesis is broadly concerned with two problems: investigating the ergodic properties of boundary actions, and investigating various properties of Schreier graphs. Our main result concerning the former problem is that, in a variety of situations, the action of an invariant random subgroup of a group G on a boundary of G (e.g. the hyperbolic boundary, or the Poisson boundary) is conservative (there are no wandering sets). This addresses a question asked by Grigorchuk, Kaimanovich, and Nagnibeda and establishes a connection between invariant random subgroups and normal subgroups. We approach the latter problem from a number of directions (in particular, both in the presence and the absence of a probability measure), with an emphasis on what we term Schreier structures (edge-labelings of a given graph Γ which turn Γ into a Schreier coset graph). One of our main results is that, under mild assumptions, there exists a rich space of invariant Schreier structures over a given unimodular graph structure, in that this space contains uncountably many ergodic measures, many of which we are able to describe explicitly.

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Finally, I would like to thank the many people who have helped and befriended me along the way. The list of names is so long that, were I to attempt to write it, I would inevitably leave someone out and suffer a pang of regret upon realizing my mistake. I hope, therefore, that those whom I would like to thank know who they are and know that they have my gratitude.

Dedication

To my mother, Marie-Louise,

who would have been proud

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Chapter 1

Preliminaries

1.1 Summary

The results of this thesis can be grouped into two basic categories: results concerning ergodic properties of boundary actions, and results concerning what we call *Schreier structures*. The former are contained in Chapters 3 and 4 and the latter in Chapters 5 and 6.

To be more precise, this thesis is organized as follows. In Chapter 2, we provide background on the fundamental objects and tools that we will employ, namely Schreier graphs, Lebesgue spaces, ergodic theory, and the notion of soficity. Chapter 3 is concerned with the boundary action of an *invariant random subgroup* of the free group and leads up to a proof that the boundary action of a *sofic* random subgroup of the free group is conservative. Using results from [31], this allows us to characterize the possible measures of various *limit sets* associated to an invariant random subgroup. Chapter 4 (the results of which are joint with Kaimanovich) contains a proof that the boundary action of an invariant random subgroup of a group G on the *Poisson boundary* of G is conservative; using results in [40], this has immediate applications to hyperbolic groups. Chapter 5 moves on to the study of *invariant Schreier structures*, where we prove, among other things, that under mild assumptions there exist uncountably many ergodic Schreier structures over a given unimodular graph structure. Finally, in Chapter 6, we consider Schreier structures in the absence of a probability measure, proving some basic observations along the way. Chapters 3, 4, and 5 have been adapted from the papers [15], [17], and [16], respectively.

In the course of reading a mathematical text, it is not uncommon to find oneself annoyed at how the author will at times go to great lengths to describe basic notions, yet at others throw about seemingly difficult concepts without explanation. We are afraid that the reader of this thesis may fall victim to this frustration as well, but hope to mitigate it slightly with a short word on some of the background knowledge that we will assume, as well as on notation.

We assume basic knowledge of graph theory, group theory, and measure theory. In particular, we will not spend time defining what a graph is, although it should be noted that the definition of Serre [61] serves our purposes well. We also find it convenient to use the language of category theory and will thus frequently speak of categories, morphisms, functors, and the like. We have aimed to keep our notation simple and clean, and also, since we are constantly switching between graphs and groups, to adhere to the principle that objects in the graph-theoretic world be denoted by Greek letters and objects in the group-theoretic world be denoted by Latin letters. Thus, Γ will always denote a graph, and G will always denote a group, $\Lambda(G)$ will denote the space of Schreier graphs of a group G, and $L(G)$ will denote its lattice of subgroups, etc.

1.2 Introduction

As is typical of much of contemporary mathematics, this thesis touches upon a number of areas, in particular graph theory, group theory, and ergodic theory. The fundamental object of interest that ties our work together is arguably the *Schreier graph*. Discovered by Schreier in his landmark 1921 paper [60], Schreier graphs are (as Schreier himself noted) natural generalizations of Cayley graphs, discovered by Cayley in 1878 [18] and used with great success by Dehn [21], who was the first to consider infinite Cayley graphs and termed them *Gruppenbilder*, or "group pictures." Indeed, a Cayley graph is precisely a picture of a group, and thus allows one to give geometric meaning to what is by nature an algebraic object. This point of view—the basis of geometric group theory—has proved to be extremely fruitful. Following Dehn, Schreier referred to his graphs as *Nebengruppenbilder* [60], roughly translatable as "coset pictures." Just as Cayley graphs are pictures of how the elements of a given group G stand in geometric relation to one another, Schreier graphs are pictures of how the cosets of a subgroup $H \leqslant G$ stand in geometric relation to one another within the ambient group G (in particular, if $N \trianglelefteq G$ is normal, then the Schreier graph of N will correspond to the Cayley graph of the quotient group G/N).

Consider now the following very general problem, doubtless of fundamental interest in algebra and many other branches of mathematics: *Given a group G, describe its subgroups and their properties.* Taking the geometric point of view described above, this problem can be rephrased as: *Given a group G, describe its Schreier graphs and their properties.* Among the things that makes this problem difficult in general is the fact that the lattice of subgrups of a given group may be intractably large (e.g. if G is a free group of rank $n > 1$), so that one might not know where to begin or, say, how to find interesting examples. A general technique that can be useful in overcoming such obstacles is the introduction of a notion of *randomness*.

Generally speaking, randomness can enter the picture in a number of ways, for instance via a model which generates random structures of the sort one might be interested in. Famous examples include the *Erdős-Rényi random graph*, introduced in [27] and responsible for an enormous amount of research, or, more recently, *Gromov's random groups* (see [36]), responsible for producing new examples of groups with interesting properties. But another approach—and it is the one which we will take—is to begin with a large space of objects and equip it with an *invariant measure*. The idea here is that, a priori, objects sampled with respect to an arbitrary measure will be too disorderly to be interesting. Invariance, however, imposes *stochastic homogeneity* (see [41] for the origin of the term) on the random object, permitting one to strike a balance between the highly symmetric and the hopelessly wild and apply the powerful tools of ergodic theory.

Vershik recently called for a description of the invariant measures on the lattice of subgroups, denoted $L(G)$, of a given group G (see [66]), going on to provide such a description in the case when G is the infinite symmetric group [67]. Invariant measures on $L(G)$ (to be more precise, probability measures on $L(G)$ invariant under the action of G by conjugation) were later termed *invariant random subgroups* in [5], and their study has been the focus of a large amount of research (see, for instance, [5], [6], [10], [11], [12], [13], [26], and [68]). Via the map that assigns to a given subgroup its Schreier graph, one may also speak of invariant measures on the *space of Schreier graphs* of G, denoted $\Lambda(G)$. This point of view connects the theory of random subgroups directly with the theory of *random rooted graphs*.

Research in the field has thus far proceeded along two general lines: elucidating, on the one hand, the connection between invariant random subgroups and normal subgroups (or, what is the same, the connection between invariant random Schreier graphs and Cayley graphs), and investigating spaces of invariant measures themselves. The motivation for the former stems from the fact that invariant random subgroups are a probabilistic generalization of normal subgroups, whereby spatial homogeneity is replaced by stochastic homogeneity. Highlights here include the results of Stuck and Zimmer [63], who, working in the context of Lie groups, generalized Margulis's normal subgroup theorem to invariant random subgroups, and the recent work of Abért, Glasner, and Virág [5], who, developing the spectral theory of random graphs, generalized Kesten's theorem to invariant random subgroups. On the other hand, Bowen, Grigorchuk, and Kravchenko have recently investigated the spaces of invariant measures on the lattices of subgroups of free groups and lamplighter groups (see [12] and [13]), showing in each case that the space of such measures has a rich structure, in that the invariant measures supported on infinite-index subgroups comprise a *Poulsen simplex* (this implies that the ergodic measures are dense).

In their groundbreaking 1977 paper, Feldman and Moore [28] showed that the notion of invariance need not be defined with respect to a group action but may be intrinsic to the *discrete measured equivalence relation* underlying a given space. This insight was later given a geometric flavor by Adams [3], Gaboriau [29], and Kaimanovich, who realized that the space of *rooted graphs* (that is, graphs endowed with a distinguished base point, called the *root*) possesses a natural equivalence relation (the relation of being isomorphic as unrooted graphs, sometimes called the *root-moving equivalence relation*), and in turn considered random rooted graphs in the presence of an invariant measure and the associated problem of stochastic homogenization [41]. The theory of random rooted graphs became, and continues to be, a very active area of research (see, for instance, [4], [8], [9], [20], [25] [45], [46], and [51]).

Indeed, the theory of random rooted graphs predates and subsumes the theory of random Schreier graphs (and itself descends from the theory of holonomy-invariant measures on foliated spaces—see, for instance, [55]). Yet there is a significant difference in viewpoint: whereas the theory of random rooted graphs (or, more generally, leafwise graph structures on a discrete measured equivalence relation) has, a priori, little to do with group actions, the theory of random Schreier graphs has an explicit algebraic slant. A further difference to take note of is the existence of two notions of invariance for random rooted graphs, namely invariance in the classical sense of Feldman and Moore and *unimodularity*, e.g. in the sense of Aldous and Lyons [4]. The relationship between these notions was recently clarified by Kaimanovich [44], but exploring the extent to which random rooted graphs and random Schreier graphs can be reconciled remains an interesting problem.

One of the main results of this thesis links the theory of invariant random subgroups with hyperbolic geometry. To be more precise, any Gromov hyperbolic space X possesses a *hyperbolic boundary* ∂X upon which the isometry group $\mathrm{Iso}(X)$ acts by homeomorphisms. The study of such boundary actions has a long history. In the classical setting when $X = \mathbb{H}^2$ is the hyperbolic plane, there is a large literature on *Fuchsian groups*—discrete groups of isometries of $\mathbb{H}^2$—and their actions on the boundary $\partial\mathbb{H}^2 \cong \mathbb{S}^1$, both in the topological and measure-theoretic setting, where it is natural to equip $\partial\mathbb{H}^2$ with the Lebesgue (or *visibility*) measure.

The most natural discrete analogue of the hyperbolic plane is a finitely generated free group $\mathbb{F}_n$ of rank $n > 1$, which becomes a hyperbolic space upon identifying it with its Cayley graph, the regular tree. Moreover, the hyperbolic boundary $\partial\mathbb{F}_n$ comes equipped with a natural *uniform measure* $\mathfrak{m}$, and subgroups $H \leqslant \mathbb{F}_n$ play the role of discrete groups of isometries of $\mathbb{F}_n$. Grigorchuk, Kaimanovich, and Nagnibeda [31] recently studied the ergodic properties of boundary actions $H \circlearrowright (\partial\mathbb{F}_n, \mathfrak{m})$, exploiting the combinatorial structure of the free group and, in particular, the Schreier graph of the subgroup H to draw many parallels with the classical theory. They also raised the question of what happens when H is an invariant random subgroup, rather than an individual subgroup. Our first result is the following.

Theorem 1.2.1. *Let μ be a sofic measure on $L(\mathbb{F}_n)$, the lattice of subgroups of the finitely generated free group of rank n. Then the boundary action $H \circlearrowright (\partial\mathbb{F}_n, \mathfrak{m})$ of a μ-random subgroup $H \leqslant \mathbb{F}_n$ is almost surely conservative.*

Recall that an action $G \circlearrowright (X, \mu)$ is conservative if every subset $A \subseteq X$ is *recurrent* (cf. the classical Poincaré recurrence theorem). This generalizes a theorem of Kaimanovich [39], who showed (albeit in a considerably broader context) that the boundary action of a normal subgroup is necessarily conservative. Theorem 1.2.1 thus provides a further connection between invariant random subgroups and normal subgroups (let us also note that, as shown in [31], the boundary action of an individual subgroup of the free group need not be conservative). We go on to prove the following theorem, which is joint with Kaimanovich.

Theorem 1.2.2. *Let G be a countable group, μ_0 a nondegenerate probability measure on G, and μ an invariant measure on the lattice of subgroups of G. Then the action $H \circlearrowright (\partial G, \nu)$ of a μ-random subgroup $H \leqslant G$ on the Poisson boundary of G determined by μ_0 is almost surely conservative.*

This generalizes Theorem 1.2.1, as the boundary $(\partial\mathbb{F}_n, \mathfrak{m})$ is naturally isomorphic to the Poisson boundary of the simple random walk on $\mathbb{F}_n$. Nevertheless, we include discussions of both theorems, as the approaches taken to solve them differ considerably. In proving Theorem 1.2.1, we exploit the structure of a random Schreier graph which is *sofic*, and also consider the existence of a certain *ergodic theorem* for random Schreier graphs (of which there are not many—see [14] for an overview of ergodic theorems in other contexts). In proving Theorem 1.2.2, we draw upon the theory of random walks and employ the notion of a *Poisson bundle*, which is, roughly speaking, the object attained upon attaching to a random Schreier graph its Poisson boundary (and which is analogous to the unit tangent bundle of a negatively curved manifold). Such bundles were introduced by Kaimanovich in [42] and also recently considered by Bowen [11].

Our next results concern the relationship between random rooted graphs and random Schreier graphs. Denote by Ω the space of $2n$-regular graphs and by Λ the space of Schreier graphs of the free group $\mathbb{F}_n$. There is a natural forgetful map $f : \Lambda \to \Omega$ that sends a Schreier graph to its underlying unlabeled graph, and the basic problem we consider is that of describing the *Schreier structures*—that is, elements of the fibers $f^{-1}(\Gamma)$—with which unlabeled graphs may be endowed. More generally, there is an induced map $f : \mathcal{P}(\Lambda) \to \mathcal{P}(\Omega)$ between the spaces of probability measures on Ω and Λ, and it is natural to ask about the behavior of invariant measures under the map f, as well as to describe the fiber of invariant measures over a given invariant measure on Ω.

We prove a number of folklore results concerning these questions, such that every regular graph of even degree admits a Schreier structure, and that the image of an invariant measure on Λ under f is a unimodular measure on Ω (this latter observation allows us to exhibit closed invariant subspaces of Λ which do not support an invariant measure). Our main result here is the following.

Theorem 1.2.3. *Under mild assumptions, the fiber $f^{-1}(\mu)$ of invariant measures* *over a given unimodular measure μ on Ω contains uncountably many ergodic measures.*

This shows that Schreier structures are far from trivial decorations but themselves possess a rich structure. Here many questions present themselves, such as whether it is feasible to obtain a complete description of the space of Schreier structures over a given unlabeled graph in certain nontrivial cases (even the question of determining the number of Cayley structures over a given graph is a nontrivial problem—see Chapter IV of [38] for more on this point). We also know of no complete description of the space of invariant Schreier structures over a given unimodular random graph except in trivial cases.

The final chapter of this thesis consists of results and observations which are, admittedly, not fully formed but which we hope are nonetheless of interest. Borrowing from category theory, we show that it is possible to speak (in a rigorous way) of a *sheaf* of Schreier structures, and that the arising topological considerations lead naturally to a notion of asymptotic dimension for arbitrary metric spaces. We also ruminate on the connection between the Schreier structures over a given graph Γ and covering maps going into and out of Γ, proving some basic observations along the way.

Chapter 2

Background

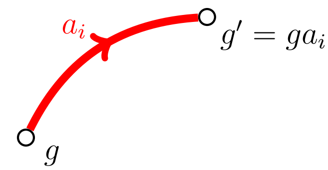

Figure 2.1: Constructing a Cayley graph.

2.1 Cayley graphs and Schreier graphs

A fundamental—indeed, perhaps *the* fundamental—object of interest for us is the *Schreier graph*. Introduced by Schreier in his landmark paper [60] (under the German name *Nebengruppenbild*), it is, as Schreier himself noted, a natural generalization of the *Cayley graph* introduced by Cayley in [18] and used with great success by Dehn [21] (who termed it the *Gruppenbild*). The idea behind the Cayley graph is both simple and profound: given a group G—a priori an algebraic object—one would like, without imposing any assumptions on G, to associate to it a geometric object. Moreover, this object should be a *G-torsor*, thus serving as a true geometric realization of our group (recall that a G-torsor is a space upon which G acts freely and transitively). It turns out that this is always possible: the sought-after G-torsor is precisely the Cayley graph, and it is defined as follows.

Definition 2.1.1. (Cayley graph) Let G be a group and $\mathcal{A} = \{a_i\}_{i \in I}$ a generating set for G. The *Cayley graph* of G constructed with respect to $\mathcal{A}$ is the graph Γ whose vertex set is identified with G and such that two elements g and g' are connected with an edge directed from g to g' and labeled with the generator $a_i \in \mathcal{A}$ if and only if $ga_i = g'$.

Let us immediately make a few observations about the Cayley graph of a given group G. Note, for instance, that it is indeed a G-torsor: Let $x \in \Gamma$ be any vertex and $g \in G$ any group element. Then we may write g as a (possibly empty) word in the symmetrized generating set $\mathcal{A} \cup \mathcal{A}^{-1}$ (i.e. $\mathcal{A}$ together with the inverses of each of its

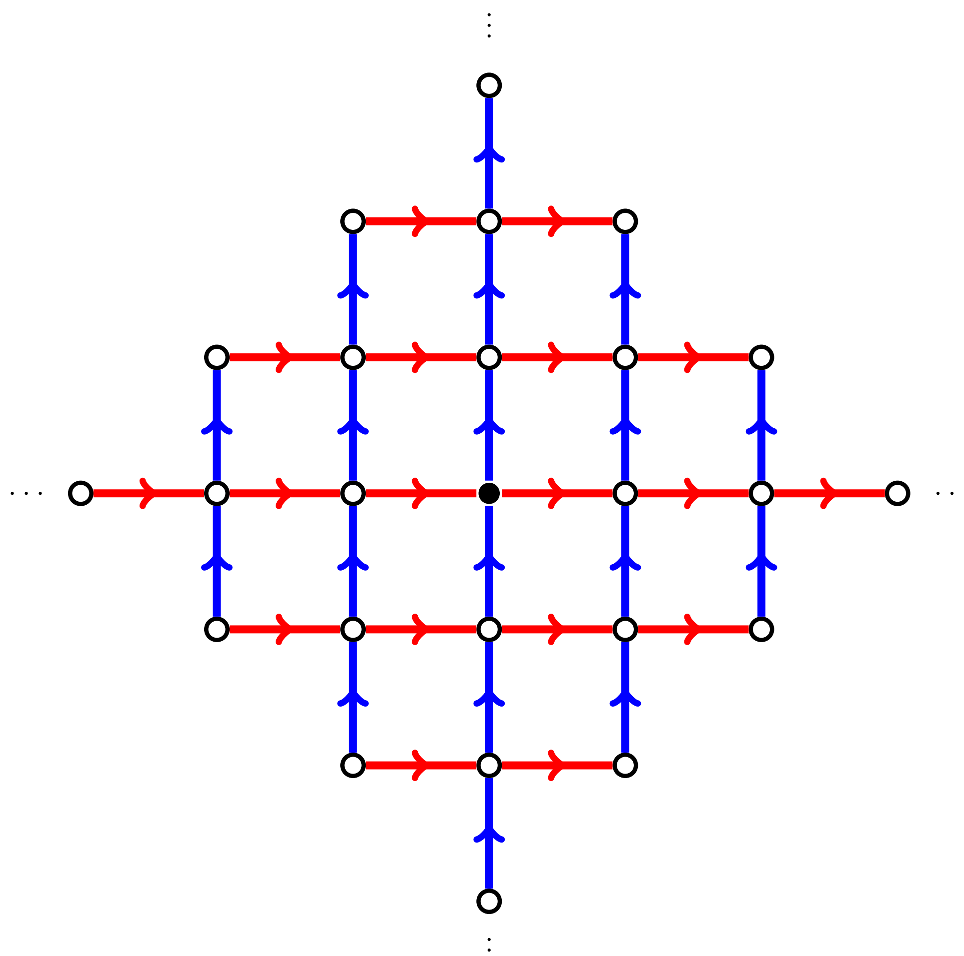

Figure 2.2: The Cayley graph of the group $\mathbb{Z}^2$ constructed with respect to the standard generators $(1, 0)$ and $(0, 1)$.

elements), and this word determines a unique path γ in Γ starting from x. To be more precise, γ is the path obtained by starting at x and following, in the obvious order, the sequence of edges corresponding to the sequence of generators in our presentation of g (note that following a generator a_i^{-1} is tantamount to traversing a directed edge labeled with a_i in the direction opposite to which the edge is pointing). We then declare the endpoint of the path γ to be gx. It is easy to see that this determines a group action which, via the identification between the vertices of Γ and G, corresponds precisely to the action of G on itself; hence, Γ is a G-torsor. (Going the other way around, a graph Γ is a Cayley graph of G—and here we do not assume that the edges of Γ are labeled—if and only if it admits a free and transitive action of G by graph automorphisms. This is the content of Sabidussi's theorem [59].)

Notice also that the Cayley graph is certainly not unique: indeed, its definition depends on a choice of generating set of the corresponding group G, of which there are in general many. As will become evident later, however, two Cayley graphs are Cayley graphs of the same group G if and only if their automorphism groups (in the category of edge-labeled graphs) coincide with G. Moreover, if G is finitely generated, then any two Cayley graphs of G constructed with respect to finite generating sets will exhibit the same coarse geometry—to be more precise, they will be *quasi-isometric* (quasi-isometry being the fundamental notion of equivalence in geometric group theory).

Let us now introduce the Schreier graph. The starting point is again a group G together with a generating set. What is added to the picture is a subgroup $H \leqslant G$. One may then consider the coset space

$$G/H = \{Hg \mid g \in G\}$$

and endow it with a graph structure just as we endowed G with a graph structure to produce a Cayley graph.

Definition 2.1.2. (Schreier graph) Let G be a group and $\mathcal{A} = \{a_i\}_{i \in I}$ a generating set for G, and let $H \leqslant G$ be a subgroup. The *Schreier graph* of H constructed with respect to $\mathcal{A}$ is the graph Γ whose vertex set is identified with G/H and such that two cosets Hg and Hg' are connected with an edge directed from Hg to Hg' and labeled with the generator $a_i \in \mathcal{A}$ if and only if $Hga_i = Hg'$.

Schreier graphs may be thought of as pictures of how the cosets of a given subgroup tile the ambient group. Moreover, they are indeed immediate and natural generalizations of Cayley graphs. To see this one need only notice that, if $N \trianglelefteqslant G$ is a normal subgroup, then the Schreier graph of N is precisely the Cayley graph of the quotient group G/N. The geometry of a Schreier graph, however, may be far wilder than that of a Cayley graph, i.e. it may be very far from being vertex-transitive.

There are a number of facts about Schreier graphs which we would like to mention, and which may give the reader a better idea of how to think about these objects. Let us list these facts now. In what follows, Γ will denote the Schreier graph of a subgroup $H \leqslant G$, and $\mathcal{A}$ the choice of generating set of the group G.

i. Schreier graphs are naturally *rooted graphs*, namely graphs endowed with a distinguished base point, called the *root*. The natural choice of root is the trivial coset H (or, if our graph is being thought of as a Cayley graph, the group identity). We often denote the root by a lowercase Latin letter as well, e.g. by x.

ii. Schreier graphs are *regular*, meaning that each of their vertices has the same degree. This follows from the fact that, for each vertex $x \in \Gamma$ and each generator $a \in \mathcal{A}$, there is precisely one *incoming* and one *outgoing* edge labeled with a attached to x. Note that a *loop*—a cycle of length one—counts as two edges, corresponding to its two orientations. It follows that the degree of vertices in a Schreier graph is always an even number $2n$ (provided $\mathcal{A}$ is finite). Accordingly,

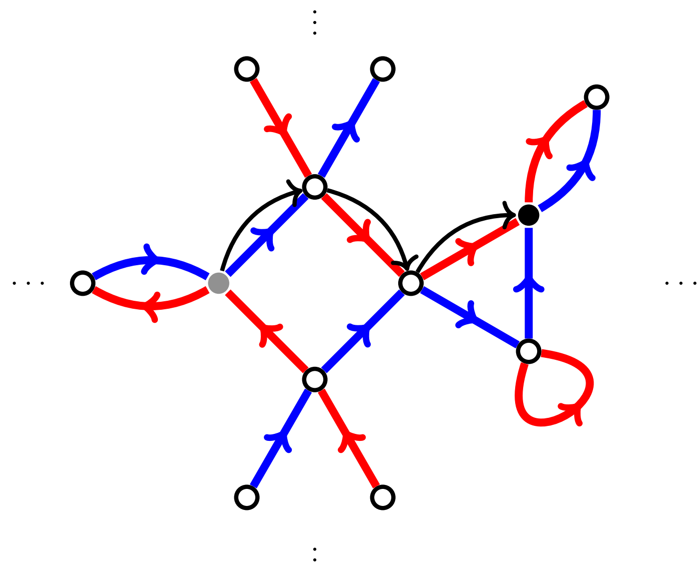

Figure 2.3: A Schreier graph of the free group $\mathbb{F}_2 = \langle a, b \rangle$, with red edges representing the generator a and blue edges the generator b. Shown here is conjugation by the element $ba^2 \in \mathbb{F}_2$, which entails starting at the root (the gray vertex), then following the edges corresponding to the generators b, a, and a again (in that order), and declaring their endpoint to be the new root (the black vertex).

we will often speak of $2n$-*regular* graphs.

iii. Schreier graphs are connected (as the action of G on G/H is transitive). As already mentioned, Schreier graphs may have loops (cycles of length one). They may also have *multi-edges* (multiple edges that join the same pair of vertices).

iv. Let $G \circlearrowright X$ denote an action of G on a space X, and denote by

$$\mathcal{O}_x = \{gx \mid g \in G\}$$

the orbit of a point $x \in X$. Then $\mathcal{O}_x$ is naturally given the structure of a Schreier graph by joining two points $y, z \in \mathcal{O}_x$ with an edge directed from y to z and labeled with a generator $a \in \mathcal{A}$ if and only if $z = ay$. The root of this Schreier graph is the point x. In light of this observation, the Schreier graph of Definition 2.1.2 is precisely the Schreier graph of the action of G on the coset space G/H.

v. The edge-labeling of a Schreier graph Γ by the generators of a group, though often not emphasized in the literature, will for us be a feature of fundamental importance (we will later refer to it as a *Schreier structure*). The synesthesiate will notice that Schreier graphs are *colorful*. We invite the reader to assign a color to each of the generators $a \in \mathcal{A}$ and paint the edges of Γ accordingly. It is not difficult to see that if one begins at an arbitrary vertex $x \in \Gamma$ and follows the edges corresponding to a particular color "in both directions as far as one can go," then one will necessarily trace out a (possibly infinite) cycle. By induction, cycles of this sort completely decompose Γ. A Schreier graph is thus, to wax poetic, a collection of colorful, carefully interlocking cycles.

vi. The edge-labeling of Γ just described serves as a sort of road map, in that it tells us how the group G acts on Γ (recall that this is precisely how we defined on action of G on its Cayley graph). Another way to think of it is this: if $g \in G$ is a group element (which, as before, we present as a word in the alphabet

$\mathcal{A} \cup \mathcal{A}^{-1}$), then it naturally transforms the Schreier graph Γ by *rerooting* it, i.e. by shifting the location of the root to the endpoint of the path which begins at the old root and corresponds to the element g (see Figure 2.3). The resulting rerooted Schreier graph is nothing else than the Schreier graph of the *conjugated subgroup* gHg^{-1}. In particular, if $G \circlearrowright X$ is a group action and $x, y \in X$ belong to the same orbit, then the Schreier graphs on the orbits $\mathcal{O}_x$ and $\mathcal{O}_y$ will be identical save for the positions of their roots (cf. the orbit-stabilizer theorem).

vii. Recall that a *covering map* between graphs Γ and Δ is a graph homomorphism $p : \Gamma \to \Delta$ which is locally an isomorphism, in the sense that, for any vertex $x \in \Gamma$, its *star*, namely the set of edges incident with x, is mapped bijectively onto the star of $p(x) \in \Delta$. The theory of covering graphs is analogous to the classical theory of covering spaces (see, for instance, the classic paper of Stallings [62] for more on covering graphs). It is not difficult to see that the quotient map $p : G \to G/H$ that collapses the cosets of H induces a covering of Schreier graphs $p : G \to \Gamma$ (here we also denote by G the Cayley graph of the group G constructed with respect to $\mathcal{A}$).

viii. Denote by $L(G)$ the lattice of subgroups of G, and by $\Lambda(G)$ the corresponding *space of Schreier graphs* of G. The space $L(G)$ is a lattice ordered by inclusion, and we may regard $\Lambda(G)$ as a lattice ordered by coverings, i.e. by saying that $\Gamma \leqslant \Delta$ if there exists a covering map $p : \Gamma \to \Delta$ that sends the root of Γ to the root of Δ. The map $f : L(G) \to \Lambda(G)$ that sends a subgroup $H \leqslant G$ to its Schreier graph is then a lattice isomorphism that sends an ordered pair $H \leqslant H'$ to the covering map $p : \Gamma \to \Gamma'$, where Γ and Γ' are the Schreier graphs of H and H', respectively.

ix. The inverse map $f^{-1} : \Lambda(G) \to L(G)$ can be thought of as follows. If Γ is a Schreier graph of G, then the subgroup H to which it corresponds is the stabilizer of Γ under the action of G, i.e. the set of elements of G which fix the root of Γ. Equivalently, H may be recovered from Γ by passing to the *fundamental group* $\pi_1(\Gamma, x)$, namely the set of homotopy classes of closed paths that begin and end at the root $x \in \Gamma$. Each homotopy class of paths in $\pi_1(\Gamma, x)$ corresponds to a unique reduced word in the alphabet $\mathcal{A} \cup \mathcal{A}^{-1}$, and the collection of all such words together with the operation of concatenation (followed by free reduction) comprises a subgroup $\widetilde{H}$ of $\mathbb{F}_\mathcal{A}$, the free group generated by $\mathcal{A}$. The image of $\widetilde{H}$ under the canonical epimorphism $\phi : \mathbb{F}_\mathcal{A} \to G$ is precisely H.

x. Building off of the previous two points, the *universal cover* of any Schreier graph is the free group $\mathbb{F}_\mathcal{A}$ (or, rather, its Cayley graph, the regular tree). Indeed, in light of the previous discussion, any Schreier graph of any group with generating set $\mathcal{A}$ is at once a Schreier graph of $\mathbb{F}_\mathcal{A}$. This simple but important insight allows one to speak of Schreier graphs *abstractly*, i.e. without appealing to the coset structure determined by a subgroup. That is, we could define a Schreier graph to be a (connected and rooted) $2n$-regular graph Γ whose edges come in n different colors and is such that, for each vertex of Γ, there is exactly one incoming and one outgoing edge of a given color attached to it (this graph is then automatically a Schreier graph of $\mathbb{F}_\mathcal{A}$, namely the one corresponding to the fundamental group of Γ).

2.2 Lebesgue spaces

The probability spaces with which we will work are, as a rule, *Lebesgue spaces*, also referred to in the literature as *standard probability spaces*. In this section, we define what these spaces are and discuss their basic properties. Lebesgue spaces were axiomatized and classified in the fundamental work of Rokhlin [58], and we refer the reader to [58] for a full treatment of the results discussed here. Note also that, as is customary when working with measure spaces, we will take all equalities, inclusions, etc. between measure spaces to hold *modulo zero*, that is, up to the inclusion or exclusion of null sets (sets of measure zero). Accordingly, we may at times refrain from using qualifying expressions such as "almost everywhere."

Recall that any measure space (X, μ) naturally splits into a *nonatomic part* and an *atomic part*:

$$X = X_0 \sqcup X_1.$$

The nonatomic part X_0 is the collection of all points $x \in X$ such that $\mu(x) = 0$, and the atomic part X_1 is the collection of all *atoms*, i.e. points $x \in X$ such that $\mu(x) > 0$. A Lebesgue space may now be defined as a measure space whose nonatomic part is isomorphic to an interval of the real line. More precisely, we have the following definition.

Definition 2.2.1. (Lebesgue probability space) Let (X, μ) be a probability space whose nonatomic part has measure $\mu(X_0) =: t$, and denote by μ_0 the restriction of μ to X_0. Then (X, μ) is said to be a *Lebesgue space* if (X_0, μ_0) is isomorphic to the interval $[0, t] \subset \mathbb{R}$ endowed with the usual Lebesgue measure.

The isomorphism referred to in the above definition is an isomorphism in the category of measure spaces. That is, two measure spaces (X, μ) and (Y, ν) are isomorphic if there exist null sets $X' \subset X$ and $Y' \subset Y$ such that there is a measurable isomorphism (that is, a bimeasurable bijection) $f : X \backslash X' \to Y \backslash Y'$ with the property that $f_* \mu = \nu$, i.e. the image of μ under f is precisely ν (see also [48]). Focusing our attention on Lebesgue spaces is hardly restrictive. Remarkably, it turns out that every reasonable measure space is a Lebesgue space. In particular, every *Polish space* (one which is separable, metrizable, and complete) equipped with a Borel probability measure is a Lebesgue space.

It follows that, unlike most categories (e.g. the category of topological spaces, or the category of groups), the category of Lebesgue spaces—which, once again, contains nearly every probability space which one is likely to encounter—is rather manageable. Indeed, if one throws out measure spaces with a nontrivial atomic part (the atomic part being an at most countable set), then there is only *one* Lebesgue space up to isomorphism. (The real line $\mathbb{R}$ and the Euclidean plane $\mathbb{R}^2$, though certainly not isomorphic as, say, topological spaces, or as real vector spaces, are isomorphic as measure spaces when equipped with nonatomic probability measures.)

By a *quotient map*, or *projection*, between Lebesgue spaces, we will mean simply a morphism in the category of Lebesgue spaces—in other words, a measurable map $\pi : (X, \mu) \to (Y, \nu)$ such that $\pi_* \mu = \nu$. Notice that, modulo zero, any such map is surjective, which justifies our terminology. A fundamental and important feature of Lebesgue spaces is contained in the following theorem.

Theorem 2.2.2. (Rokhlin) *Let (X, μ) be a Lebesgue space. There are natural one-to-one correspondences between the following objects:*

i. Quotient maps $\pi : (X, \mu) \to (Y, \nu)$.

ii. Measurable partitions of (X, μ).

iii. Complete sub-σ-algebras of the σ-algebra on X.

It is not difficult to see that items i. and ii. are in one-to-one correspondence: Given a quotient map $\pi : (X, \mu) \to (Y, \nu)$, there is an associated measurable partition, namely the preimage partition, whose elements are the fibers $\pi^{-1}(y)$, where $y \in Y$. Conversely, given a measurable partition ξ of X, there is a corresponding quotient map, namely the one that collapses the elements of ξ. In similar fashion, any quotient map π determines a sub-σ-algebra of the σ-algebra on X, namely the *preimage sub-σ-algebra*, whose elements are the preimages $\pi^{-1}(B)$ of measurable sets $B \subseteq Y$. The main difficulty in proving Theorem 2.2.2 is showing that, given any sub-σ-algebra, it is possible to build a quotient map whose preimage sub-σ-algebra coincides with it.

Another important feature of Lebesgue spaces—and one which we will make use of repeatedly—is the existence of *conditional measures* defined on the fibers of any quotient map. The standard picture to have in mind here is that of the usual projection $\pi : [0,1]^2 \to [0,1]$ of the unit square (equipped with two-dimensional Lebesgue measure) onto the unit interval (equipped with one-dimensional Lebesgue measure), i.e. the projection given by $\pi(x,y) = x$. See also Figure 2.4.

Theorem 2.2.3. (Rokhlin) *Let $\pi : (X,\mu) \to (Y,\nu)$ be a quotient map of Lebesgue spaces. Then for almost every $y \in Y$, there exists a conditional measure μ_y defined on the fiber $\pi^{-1}(y)$, and the measure μ decomposes as an integral of the system of measures $\{\mu_y\}_{y\in Y}$ with respect to the measure ν. In the language of differentials, this means that*

$$d\mu(x,y) = d\mu_y(x)\, d\nu(y).$$

More concretely, we have

$$\mu(A) = \int \mu_y \left(A \cap \pi^{-1}(y)\right) d\nu(y)$$

for any measurable set $A \subseteq X$.

Notice that (as is the case with our projection $\pi : [0,1]^2 \to [0,1]$), the fibers of a quotient map may well be null sets. The conditional measures whose existence is guaranteed by Theorem 2.2.3 thus considerably generalize the conditional measures which one learns about in an introductory course on probability theory. Note also that Theorem 2.2.3 subsumes the classical Fubini theorem. In fact, any two quotient maps each of whose conditional measures is nonatomic are isomorphic as morphisms in the category of Lebesgue spaces.

2.3 The space of rooted graphs

We will often be concerned with *random rooted graphs*, be they Schreier graphs or unlabeled graphs, and will therefore require a measurable structure on the *space of*

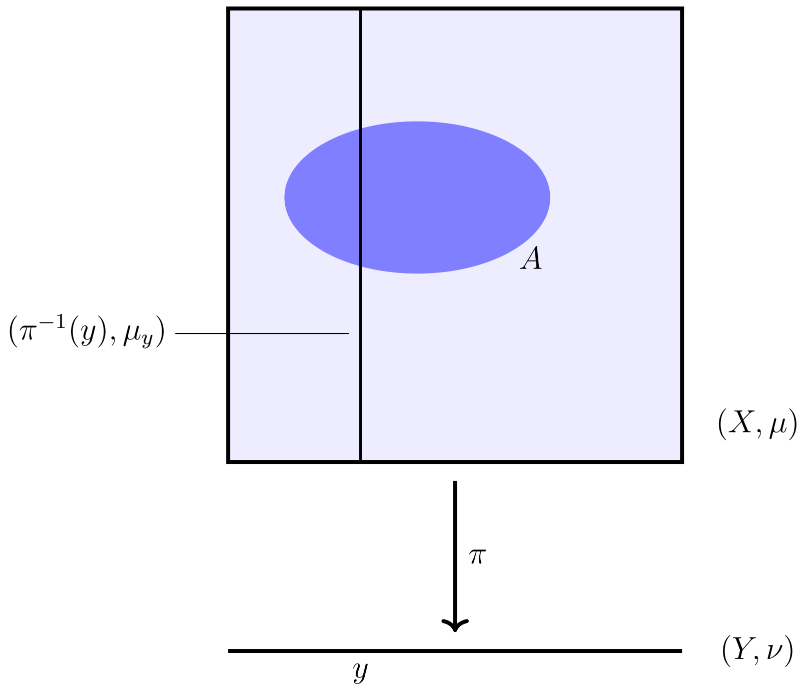

Figure 2.4: Conditional measures are defined on the fibers of any quotient map between Lebesgue spaces.

rooted graphs. To this end, define Ω to be the space of connected rooted graphs of uniformly bounded geometry, i.e. the space of connected graphs $\Gamma = (\Gamma, x)$ each of which is equipped with a distinguished vertex x, called its root, and for which there exists a number d (whose precise value will not presently concern us) such that

$$\max_{y \in \Gamma} \deg(y) \leqslant d$$

for all $\Gamma \in \Omega$. The space Ω may naturally be realized as the projective limit

$$\Omega = \varprojlim \Omega_r, \tag{2.3.1}$$

where Ω_r is the set of (isomorphism classes of) r-neighborhoods centered at the roots of elements of Ω and the connecting morphisms $\pi_r : \Omega_{r+1} \to \Omega_r$ are restriction maps that send an $(r+1)$-neighborhood V to the r-neighborhood U of its root. (Looking at things the other way around, $\pi_r(V) = U$ only if there exists an embedding $U \hookrightarrow V$

that sends the root of U to the root of V.) Endowing each of the sets Ω_r with the discrete topology turns Ω into a compact Polish space. Note that compactness comes from the fact that our graphs are of uniformly bounded geometry, which implies that each of the sets Ω_r is finite.

There is no canonical metric on the space Ω, but the idea here is that two graphs $\Gamma, \Delta \in \Omega$ are close together if they agree on large neighborhoods of their roots. Thus, the metric

$$\rho(\Gamma, \Delta) := 2^{-r},$$

where r is the largest radius such that the r-neighborhoods of the roots of Γ and Δ are isomorphic (and where we set $\rho(\Gamma, \Delta) = 0$ if $r = \infty$), is an example of a metric that generates the topology on Ω. Note that our projective limit topology, or variants of it, arise in many other situations as well, e.g. when one topologizes sequence spaces, profinite groups, spaces of tilings, etc. In fact there is a natural correspondence between projective systems of finite discrete sets and locally finite trees, and the projective limit of such a system may be identified with the *boundary* of the corresponding tree. This set is often homeomorphic to a Cantor set, although there may exist isolated points as well.

Throughout this thesis, we may regard an r-neighborhood $U \in \Omega_r$ both as a rooted graph and as the *cylinder set*

$$U = \{(\Gamma, x) \in \Omega \mid U_r(x) \cong U\},$$

where $U_r(x)$ denotes the r-neighborhood of the point $x \in \Gamma$. A cylinder set U is the "shadow" of the r-neighborhood U in the projective system (2.3.1). Note that a finite Borel measure μ on Ω is the same thing as a family of measures $\mu_r : \Omega_r \to \mathbb{R}$ that satisfies

$$\mu_r(U) = \sum_{V \in \pi_r^{-1}(U)} \mu_{r+1}(V)$$

for all $U \in \Omega_r$ and for all r. We will be interested primarily in the space of $2n$*-regular* rooted graphs, namely rooted graphs each of whose vertices has degree $2n$, and for the sake of notational simplicity we will also denote this space by Ω. As an aside, note that imposing regularity on our graphs still leaves us with an enormous space: every graph of bounded geometry d, for instance, can be embedded into a regular graph (e.g. by attaching branches of the d-regular tree to vertices whose degrees are less than d).

We may topologize the *space of Schreier graphs* $\Lambda(G)$ of a finitely generated group G along precisely the same lines as indicated above, namely by defining $\Lambda_r(G)$ to be the set of (isomorphism classes of) r-neighborhoods centered at the roots of Schreier graphs in $\Lambda(G)$ and putting

$$\Lambda(G) = \varprojlim \Lambda_r(G).$$

Something to take note of here is that our isomorphism classes of r-neighborhoods are defined with respect to the category of edge-labeled graphs, where morphisms between graphs must respect not only the graph structure but edge-labelings as well. Via the identification between the space of Schreier graphs $\Lambda(G)$ and the lattice of subgroups $L(G)$ of G, we are able to topologize $L(G)$. The resulting topology on $L(G)$ is an instance of the *Chabauty topology*, introduced in [19].

2.4 Ergodic theory

Denote by $G \circlearrowright (X, \mu)$ the action of a group G on a measure space. The measure μ is said to be *invariant* with respect to the action of G if for any measurable set $A \subseteq X$ and any $g \in G$, one has

$$\mu(g^{-1}A) = \mu(A).$$

That is to say, the action of G preserves the measure μ, in that $g_*\mu = \mu$ for any $g \in G$. The measure μ is said to be *quasi-invariant* if for any $g \in G$, the measure $g_*\mu$ is equivalent to μ, i.e. the action of G preserves the measure class of μ. The measure μ is said to be *ergodic* if whenever $gA = A$ for all $g \in G$, then $\mu(A) = 0$ or $\mu(A) = 1$, i.e. there are no nontrivial invariant sets.

We cannot hope to give a satisfactory overview of ergodic theory or even the basics of ergodic theory here. Instead, we will make mention only of some of the key aspects of the theory that we will draw upon throughout this thesis. The first is *ergodic decomposition*. To be more precise, ergodic measures are, as it were, the basic building blocks of quasi-invariant measures, in that any quasi-invariant measure on a Lebesgue space can be decomposed into an integral of ergodic measures in an essentially unique way. For a very simple example of this, suppose that μ is an ergodic measure on a Lebesgue space X (with respect to the action of a group G), and consider the space $X' := X \sqcup X$ obtained by lumping two copies of X together. The space X' comes with an obvious G-action and an obvious probability measure μ' (obtained by equipping each copy of X with the scaled measure $\mu/2$). Now, the measure μ' is certainly not ergodic—each copy of X is a nontrivial invariant set—but it can be decomposed into *ergodic components*, which are precisely the copies of X equipped with their associated conditional measures. The quotient of (X', μ') obtained by collapsing these components to points is the *space of ergodic components*—in our case, a two-point set, each of whose elements has measure $1/2$.

It turns out that any quasi-invariant measure can be decomposed into ergodic measures in this way. In general, however, the ergodic decomposition of a space (X, μ) may consist of sets which are null sets with respect to μ. Let us return, for instance, to our picture of the projection of $[0,1]^2$ onto $[0,1]$, and consider the action of $\mathbb{S}^1$ on the torus $\mathbb{S}^1 \times \mathbb{S}^1$ given by $t(x,y) \mapsto (x, y+t)$, where the torus is equipped with two-dimensional Lebesgue measure $\lambda \otimes \lambda$. The ergodic components of this action are sets of the form $\{x\} \times \mathbb{S}^1$, which are clearly null sets with respect to $\lambda \otimes \lambda$. In order to describe ergodic components in full generality, we therefore make use of Theorem 2.2.2 by noting that the invariant subsets of a Lebesgue space comprise a σ-algebra; consequently, there is a unique quotient map corresponding to this σ-algebra.

Theorem 2.4.1. (Rokhlin) *Let $G \circlearrowright (X, \mu)$ denote a quasi-invariant action of a group on a Lebesgue space, and let $\pi : (X, \mu) \to (Y, \nu)$ denote the quotient map associated to the sub-σ-algebra of invariant sets on X. Then for almost every $y \in Y$, the conditional measure μ_y on the fiber $\pi^{-1}(y)$ is ergodic with respect to the action of G. The decomposition of (X, μ) into the system of fibers $\{\pi^{-1}(y)\}_{y \in Y}$ is its ergodic decomposition, and the quotient space (Y, ν) is the space of ergodic components.*

One take-away of Theorem 2.4.1 is that "one can always pass to an ergodic measure." When studying a dynamical system with a quasi-invariant measure, one can hope to understand it by passing to its ergodic components and understanding them.

Consider next the action $G \circlearrowright X$ of a countable group on a standard Borel space X by measurable transformations. There is a natural equivalence relation associated to this action, namely the *orbit equivalence relation* $\mathcal{O} \subseteq X \times X$ whereby $(x, y) \in \mathcal{O}$, i.e. two points $x, y \in X$ are equivalent, if and only if they belong to the same orbit of G. The equivalence relation $\mathcal{O}$ has two basic features:

i. It is *discrete*, in the sense that each equivalence class, i.e. orbit, $\mathcal{O}_x$ is an at most countable set.

ii. $\mathcal{O} \subseteq X \times X$ is a Borel subset of the product of X with itself.

It is natural to abstract away our group action and take the above two properties as a definition.

Definition 2.4.2. We define a *discrete measured equivalence relation* to be a Borel equivalence relation $\mathcal{E} \subseteq X \times X$ on a standard Borel space X such that for each $x \in X$, its equivalence class, denoted $\mathcal{E}_x$, is at most countable.

In their groundbreaking 1977 paper, Feldman and Moore [28] realized that invariance, quasi-invariance, and ergodicity of a measure μ on X could be interpreted solely in terms of a discrete measured equivalence relation. What's more, any such equivalence relation can be realized as the orbit equivalence relation of the action of a countable group, whence the usual notions of quasi-invariance, invariance, and ergodicity are recovered.

To be more precise, let $\mathcal{E} \subseteq X \times X$ be a discrete measured equivalence relation and μ a probability measure X, and consider the *left projection* $\pi_\ell : \mathcal{E} \to X$ that sends a pair $(x, y) \in \mathcal{E}$ to its first coordinate. The projection π_ℓ naturally determines the *left counting measure* $\widetilde{\mu}_\ell$ on $\mathcal{E}$ with differential $d\widetilde{\mu}_\ell = d\nu_x \, d\mu$, where ν_x is the counting measure on the equivalence class of x. In other words, $\widetilde{\mu}_\ell$ is defined on measurable sets $A \subseteq \mathcal{E}$ as

$$\widetilde{\mu}_\ell(A) = \int \nu_x(A \cap \pi_\ell^{-1}(x)) \, d\mu = \int |A \cap \pi_\ell^{-1}(x)| \, d\mu.$$

In analogous fashion, the *right projection* $\pi_r : \mathcal{E} \to X$ that sends an element of $\mathcal{E}$ to its second coordinate determines the *right counting measure* $\widetilde{\mu}_r$ on $\mathcal{E}$. We now say that the measure μ is *invariant (with respect to $\mathcal{E}$)* if the lift $\widetilde{\mu}_\ell$ (or $\widetilde{\mu}_r$) is invariant under the involution ι given by $(x, y) \mapsto (y, x)$; see the following diagram.

$$\begin{tikzcd}
(\mathcal{E}, \widetilde{\mu}_\ell) \arrow[rr, leftrightarrow, "\iota"] \arrow[dr, "\pi_\ell"'] & & (\mathcal{E}, \widetilde{\mu}_r) \arrow[dl, "\pi_r"] \\
& (X, \mu) &
\end{tikzcd}$$

Similarly, we say that μ is *quasi-invariant (with respect to $\mathcal{E}$)* if the left and right counting measures $\widetilde{\mu}_\ell$ and $\widetilde{\mu}_r$ are equivalent. The measure μ is *ergodic (with respect to $\mathcal{E}$)* if, given any measurable subset $A \subseteq X$, its *saturation*

$$[A] := \bigcup_{x \in A} \mathcal{E}_x$$

has either full or zero measure.

Theorem 2.4.3. (Feldman and Moore) *Any discrete measured equivalence relation $\mathcal{E} \subseteq X \times X$ can be realized as the orbit equivalence relation of an action of a countable group G on X (one may always assume that $G = \mathbb{F}_\infty$, the free group of rank $\aleph_0$). Moreover, a measure μ on X is quasi-invariant (or invariant, or ergodic) with respect to $\mathcal{E}$ if and only if it is quasi-invariant (or invariant, or ergodic, respectively) with respect to the action of any countable group on X whose induced orbit equivalence relation coincides with $\mathcal{E}$.*

Theorem 2.4.3 allows one to speak of invariant measures in situations when there is no group action in sight. As noticed by Kaimanovich [41], for example, the space of rooted graphs possesses a natural discrete measured equivalence relation, namely the one whereby two graphs are deemed equivalent if they are isomorphic as unrooted graphs (this is sometimes called the *root-moving equivalence relation*). Accordingly, one may speak of rooted graphs in the presence of an invariant measure, or, if one likes, *invariant random graphs*. Here Theorem 2.4.3 guarantees the existence of a group action whose orbit equivalence relation coincides with the root-moving equivalence relation, but the point is that, a priori, the group action and graph structure have nothing to do with one another. (Chapter 5, however, can be viewed as an attempt to reconcile these two notions.)

Finally, let us recall the notions of *conservativity* and *dissipativity*, two of the most fundamental properties in ergodic theory. The definition goes as follows.

Definition 2.4.4. (Conservative and dissipative actions) An action $G \circlearrowright (X, \mu)$ of a group on a measure space by measurable transformations is said to be *conservative* if every subset $A \subseteq X$ of positive measure is *recurrent*, meaning that for almost every $x \in A$, there exists a nontrivial group element $g \in G$ such that $gx \in A$. The action is said to be *dissipative* if X is the union of the translates of a *wandering set*, namely a measurable subset $A \subseteq X$ whose G-translates are pairwise disjoint.

For examples, consider the classical Poincaré recurrence theorem, which asserts that

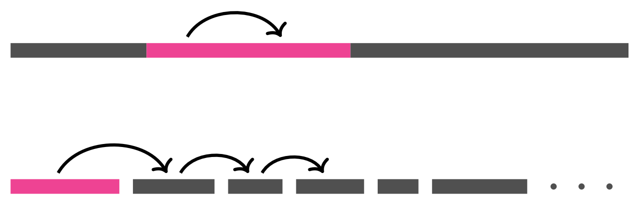

Figure 2.5: An action is conservative (above) if every set is recurrent. An action is dissipative (below) if there exists a *wandering set*, whose translates are pairwise disjoint.

any $\mathbb{Z}$-action on a finite measure space by measure preserving transformations is conservative. A simple example of a dissipative action is the usual action by translations of $\mathbb{Z}$ on the real line $\mathbb{R}$ equipped with Lebesgue measure, where a (maximal) wandering set is given by the unit interval. In fact there always exists a maximal wandering set (see [1]), which one may think of as a measurable fundamental domain of the action space. See Figure 2.5 for an illustration of the notions of conservativity and dissipativity.

Conservativity and dissipativity are, in a sense, opposite. In fact it is a classical result that any quasi-invariant action $G \circlearrowright (X, \mu)$ of a countable group on a Lebesgue space admits a decomposition

$$X = \mathcal{C} \sqcup \mathcal{D} \tag{2.4.1}$$

into conservative and dissipative parts $\mathcal{C}$ and $\mathcal{D}$, respectively, so that the action of G restricted to $\mathcal{C}$ is conservative and the action of G restricted to $\mathcal{D}$ is dissipative. The decomposition (2.4.1) is called the *Hopf decomposition* of the action $G \circlearrowright (X, \mu)$, and it is unique (modulo zero). We refer the reader to [1] and the references therein for more background.

A dissipative action is by nature essentially free. We will later make use of the following fact (see, for instance, [43]).

Proposition 2.4.5. *A (quasi-invariant) action $G \circlearrowright (X, \mu)$ of a countable group on a Lebesgue space is dissipative if and only if its ergodic components consist of single*

G-orbits, in which case the associated conditional measures are atomic.

It follows that in seeking to describe the Hopf decomposition of an action $G \circlearrowright (X, \mu)$, it is useful to look at its ergodic components. In particular, Proposition 2.4.5 gives us the following criterion for conservativity.

Corollary 2.4.6. *Let $G \circlearrowright (X, \mu)$ denote the action of an infinite countable group on a Lebesgue space. If the conditional measures on the ergodic components of this action are nonatomic, then the action is conservative.*

2.5 Soficity

A recurring theme in this thesis will be the notion of *soficity*. The story begins with Gromov [35], who defined the notion of a *sofic group* in 1999, going on to show that sofic groups satisfy Gottschalk's surjunctivity conjecture in topological dynamics. Sofic groups were given their name by Weiss [69] (the word "sofic" is derived from the Hebrew word for "finite"). Later, Bowen defined a notion of *sofic entropy* for actions of sofic groups [10], vastly extending the classical theory of entropy developed for $\mathbb{Z}$-actions by Kolmogorov and Sinai.

Roughly speaking, a finitely generated group is sofic if it is possible to approximate its Cayley graph to arbitrary precision by finite graphs. Amenable groups, for which *Følner sets* serve as approximations, and residually finite groups, for which finite quotients serve as approximations, are immediate examples of sofic groups. In sharpening one's understanding of a mathematical property, it is typical to give examples both of objects which satisfy the property and objects which do not. Remarkably, however, the following basic question is open.

Question 2.5.1. *Is every group sofic?*

For much more on sofic groups, we refer the reader to the survey of Pestov [53].

It was soon realized that the notion of soficity can be extended to objects other than groups. Employing the notion of so-called *Benjamini-Schramm convergence* developed in [9], Aldous and Lyons were able to define sofic random rooted graphs [4], the definition of which naturally applies to random Schreier graphs as well. Working in greater generality, Elek and Lippner went on to define sofic discrete measured equivalence relations [26], and working in greater generality still, Dykema, Kerr, and Pichot have recently defined the notion of a sofic groupoid [23]. "The sofic question" remains open here as well: as of this writing, there is no example of a nonsofic object in any of the aforementioned contexts.

Let us give the definition of sofic random Schreier graphs, which we will later use. In order to make sense of the definition, note that a finite Schreier graph Γ determines an invariant measure μ on the space of Schreier graphs $\Lambda(\mathbb{F}_n)$ in a natural way, namely the finitely supported probability measure equidistributed on the set of *rerootings* of Γ. In other words, μ is the probability measure one obtains by choosing a vertex of Γ uniformly at random and declaring it the new root.

Definition 2.5.2. (Sofic random Schreier graph) An invariant measure μ on the space of Schreier graphs $\Lambda(\mathbb{F}_n)$ is *sofic* if there exists a sequence of finite Schreier graphs $\{\Gamma_i\}_{i\in\mathbb{N}}$ such that the sequence $\{\mu_i\}_{i\in\mathbb{N}}$ of finitely supported invariant measures determined by the graphs Γ_i converges to μ in the weak-$*$ topology.

Let us mention that the above definition differs slightly from the usual definitions of soficity, according to which elements of the approximating sequence $\{\Gamma_i\}_{i\in\mathbb{N}}$ need not consist of bona fide Schreier graphs. We will show that our definition is in fact equivalent to the usual one in Chapter 3.

The weak-$*$ convergence in Definition 2.5.2 is precisely Benjamini-Schramm convergence. It can be defined with respect to unlabeled rooted graphs as well, but in this context invariance must be replaced with the related notion of *unimodularity* (we discuss the relationship between invariance and unimodularity in greater detail in

Chapter 5). In any case, soficity of a (random) rooted graph entails being able to realize its *neighborhood statistics* via finite graphs, so that if, say, an invariant measure μ on $\Lambda(\mathbb{F}_n)$ assigns mass $t \in [0,1]$ to a given r-neighborhood $U \in \Lambda_r(\mathbb{F}_n)$, then soficity of the measure μ means that it must be possible to construct a finite graph such that the proportion of its r-neighborhoods which are isomorphic to U is $t \pm \varepsilon$, with $\varepsilon > 0$ arbitrarily small (and likewise for all other r-neighborhoods charged by μ).

Let us also point out that the weak-$*$ limit of measures which come from finite graphs is necessarily an invariant measure (or, in the case of unlabeled graphs, a unimodular measure, which follows from the fact that the space of unimodular measures is closed). Accordingly, it only makes sense to speak of soficity in the context of invariant (or unimodular) measures.

We cannot resist posing a question to conclude this section: Although simple measure-theoretic arguments show that a nonunimodular measure cannot admit approximations by unimodular measures supported on finite graphs, it would be interesting to have a better geometric understanding of this phenomenon. If, for instance, Γ is a vertex-transitive nonunimodular graph (so that the Dirac measure δ_Γ is nonunimodular), why, from a geometric point of view, should it be impossible to construct a finite graph that looks like Γ at almost all of its points?

Chapter 3

The boundary action of a sofic random subgroup of the free group

Consider the finitely generated free group of rank n, namely

$$\mathbb{F}_n = \langle a_1, \ldots, a_n \rangle.$$

The free group has a natural *boundary*, denoted $\partial\mathbb{F}_n$, and it admits a number of interpretations. Viewing elements of $\mathbb{F}_n$ as finite reduced words in the alphabet

$$\mathcal{A} \cup \mathcal{A}^{-1} = \{a_1^{\pm 1}, \ldots, a_n^{\pm 1}\},$$

the boundary $\partial\mathbb{F}_n$ is the space of infinite reduced words in the alphabet $\mathcal{A} \cup \mathcal{A}^{-1}$ endowed with the topology of pointwise convergence. Equivalently, $\partial\mathbb{F}_n$ is the projective limit of the spheres $\partial U_r(\mathbb{F}_n, e)$, i.e. the sets of words in $\mathbb{F}_n$ of length r, where each such set is given the discrete topology and the connecting maps serve to delete the last symbol of a given word (the space $\partial\mathbb{F}_n$ is thus a Cantor set provided $n > 1$). Taking a more geometric view, $\partial\mathbb{F}_n$ is naturally homeomorphic to the *space of ends* of the Cayley graph of $\mathbb{F}_n$, the $2n$-regular tree. The latter object being a Gromov hyperbolic space, $\partial\mathbb{F}_n$ may be viewed as the *hyperbolic boundary* of $\mathbb{F}_n$ (so that $\mathbb{F}_n \cup \partial\mathbb{F}_n$ is its *hyperbolic compactification*). And when equipped with the uniform measure $\mathfrak{m}$ (which we will define in a moment), $(\partial\mathbb{F}_n, \mathfrak{m})$ is naturally isomorphic to the *Poisson boundary* of the simple random walk on $\mathbb{F}_n$, a fact first established by Dynkin and Malyutov [24].

Grigorchuk, Kaimanovich, and Nagnibeda [31] recently studied the ergodic properties of the action of a subgroup $H \leqslant \mathbb{F}_n$ on the boundary of $\mathbb{F}_n$ equipped with the uniform measure $\mathfrak{m}$. To be explicit, $\mathfrak{m}$ is the probability measure given by

$$\mathfrak{m}(g) = \frac{1}{2n(2n-1)^{|g|-1}}, \tag{3.0.1}$$

where we allow g to represent both an element of $\mathbb{F}_n$ (here $|g|$ is the word length of g) and the cylinder set consisting of those infinite words whose truncations to their first $|g|$ symbols are equal to g. Of course, the denominator of (3.0.1) is just the cardinality of the sphere $\partial U_{|g|}(\mathbb{F}_n, e)$.

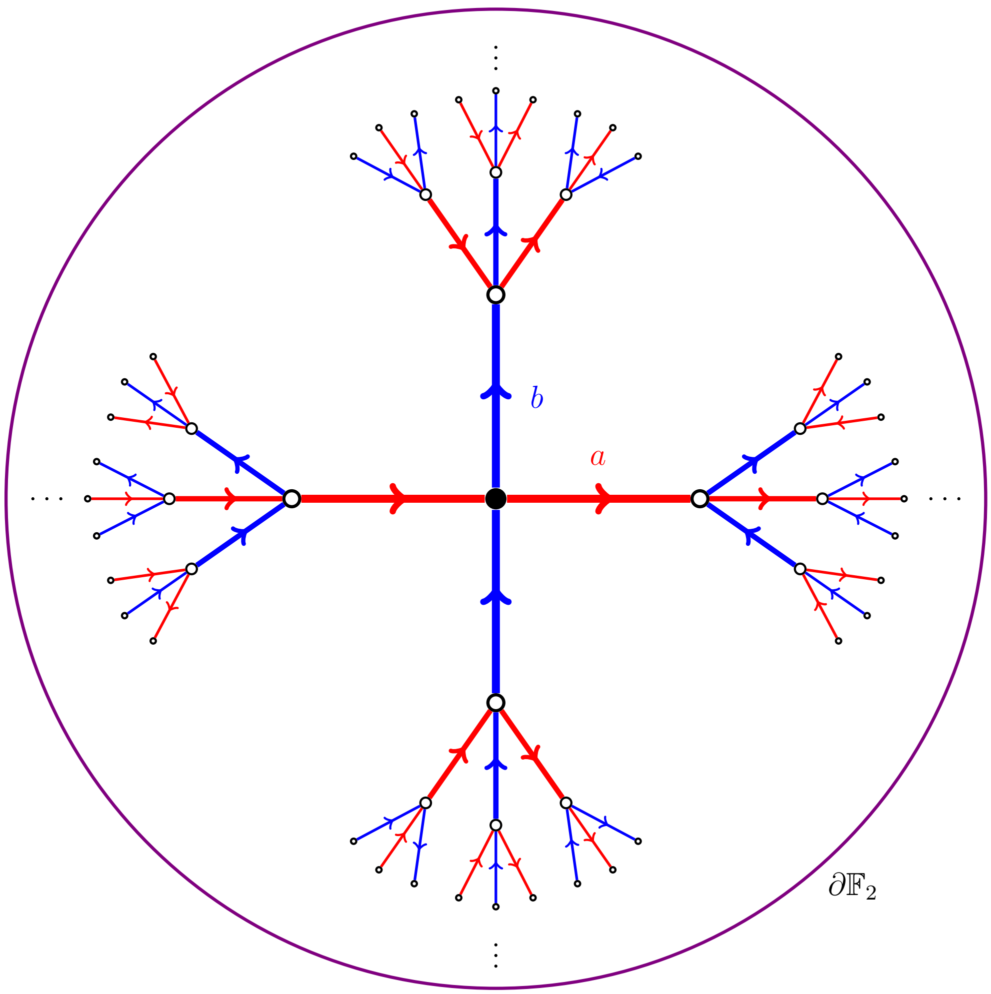

Figure 3.1: The free group $\mathbb{F}_2$ (presented as a Cayley graph), together with its boundary $\partial\mathbb{F}_2$.

The aforementioned boundary action, which we denote by $H \circlearrowright (\partial\mathbb{F}_n, \mathfrak{m})$, is analogous to the action of a Fuchsian group on the boundary of the hyperbolic plane $\partial\mathbb{H}^2 \cong \mathbb{S}^1$ equipped with Lebesgue measure: both actions, the latter being a classical object of study, are boundary actions of discrete groups of isometries of a Gromov hyperbolic space. In [31], the combinatorial structure of the space $\mathbb{F}_n$, and especially the Schreier graphs corresponding to its subgroups, are exploited in order to investigate the action $H \circlearrowright (\partial\mathbb{F}_n, \mathfrak{m})$. In particular, Theorem 2.12 of [31] gives a combinatorial characterization of the Hopf decomposition of this action. Let us review this result.

Recall that a quasi-invariant action $G \circlearrowright (X, \mu)$ of a countable group on a Lebesgue space admits a unique decomposition

$$X = \mathcal{C} \sqcup \mathcal{D},$$

called the Hopf decomposition, into conservative and dissipative parts, so that the action of G restricted to $\mathcal{C}$ is conservative and the action of G restriced to $\mathcal{D}$ is dissipative. Turning our attention to the action $H \circlearrowright (\partial\mathbb{F}_n, \mathfrak{m})$, consider the Schreier graph (Γ, H) of H, and let $T \subseteq \Gamma$ be a *geodesic spanning tree*, i.e. a spanning tree such that $d_T(H, Hg) = d_\Gamma(H, Hg)$ for all vertices (cosets) Hg (see Figure 3.2). Such a spanning tree always exists. Let $\Omega_H \subseteq \partial\mathbb{F}_n$ denote the *Schreier limit set*. It is the set of infinite words (which of course correspond to infinite paths in Γ) that pass through edges not in T infinitely often. Let $\Delta_H \subseteq \mathbb{F}_n$ denote the *Schreier fundamental domain*. It is the set of infinite words that remain in T. We then have the following boundary decomposition:

$$\partial\mathbb{F}_n = \Omega_H \sqcup \bigsqcup_{h \in H} h\Delta_H. \tag{3.0.2}$$

That is, $\partial\mathbb{F}_n$ is the disjoint union of the Schreier limit set and the H-translates of the Schreier fundamental domain. It is shown in [31] (see Theorem 2.12) that the decomposition (3.0.2) is in fact the Hopf decomposition of the action $H \circlearrowright (\partial\mathbb{F}_n, \mathfrak{m})$.

Theorem 3.0.3. (Grigorchuk, Kaimanovich, and Nagnibeda) *Let $H \leqslant \mathbb{F}_n$ be a non-*

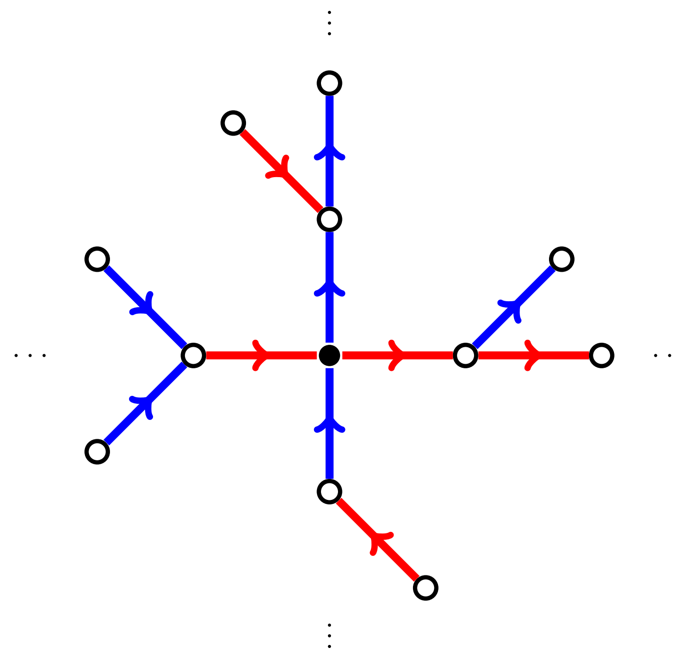

Figure 3.2: A geodesic spanning tree inside a Schreier graph.

trivial subgroup. The conservative part of the boundary action $H \circlearrowright (\partial\mathbb{F}_n, \mathfrak{m})$ coincides with the Schreier limit set Ω_H. The dissipative part coincides with the H-translates of the Schreier fundamental domain Δ_H.

Moreover, Theorem 4.10 of [31] shows that the measure of the Schreier fundamental domain is related to the growth of the Schreier graph (Γ, H) of H.

Theorem 3.0.4. (Grigorchuk, Kaimanovich, and Nagnibeda) *The measure of the Schreier fundamental domain Δ_H determined by a nontrivial subgroup $H \in L(\mathbb{F}_n)$ is equal to*

$$\mathfrak{m}(\Delta_H) = \lim_{r\to\infty} \frac{|\partial U_r(\Gamma, H)|}{|\partial U_r(\mathbb{F}_n, e)|},$$

and the above sequence of ratios is nonincreasing.

Remark 3.0.5. *Note that Theorem 3.0.4 remains valid if one replaces the spheres ∂U_r with neighborhoods U_r.*

To get a feel for why this is so, note that $|\partial U_r(\Gamma, H)|$ coincides with the size of the r-

neighborhood of the root of the geodesic spanning tree T (see, once again, Figure 3.2). Since the Schreier fundamental domain Δ_H corresponds to the set of infinite words which remain in T, it will only have positive measure if T is sufficiently large, in the sense of Theorem 3.0.4. Incidentally, an analogous result was proved by Sullivan [64] for Fuchsian groups, where $|U_r(\mathbb{F}_n, e)|$ is replaced with the volume of the ball of radius r in the hyperbolic plane and $|\partial U_r(\Gamma, H)|$ with the volume of the ball of radius r in the quotient surface.

As shown in [31], the action $H \circlearrowleft (\partial\mathbb{F}_n, \mathfrak{m})$ can in general exhibit any type of behavior. It will be conservative, for example, whenever H is of finite index, or when H is a normal subgroup of $\mathbb{F}_n$. The action may also be dissipative, which is the case, for instance, whenever H is finitely generated and of infinite index. It may also be the case that both the conservative and dissipative parts of the action have positive measure: see, for instance, Example 4.27 of [31].

3.1 In search of an ergodic theorem

Our main result is that the boundary action of a sofic random subgroup is conservative. By Theorem 3.0.4, this assertion is equivalent to the assertion that

$$\lim_{r\to\infty} \frac{|U_r(\Gamma, H)|}{|U_r(\mathbb{F}_n, e)|} = 0, \tag{3.1.1}$$

where the numerator of the above fraction is the size of the r-neighborhood of the root of our random Schreier graph and the denominator is the size of the r-neighborhood of the identity of the Cayley graph of $\mathbb{F}_n$. In proving this result, our focus will first be on a considerably more general question regarding the asymptotic density of a given set inside of neighborhoods centered at the root of a random graph. This latter question can be formulated as follows. Given a Borel subset $A \subseteq \Lambda(\mathbb{F}_n)$, consider the densities $\rho_{A,r} : \Lambda(\mathbb{F}_n) \to \mathbb{Q}$ given by

$$\rho_{A,r}(\Gamma, x) := \frac{|A \cap U_r(x)|}{|U_r(x)|},$$

i.e. $\rho_{A,r}$ describes how dense the set A is inside of the r-neighborhood of the root of a Schreier graph. To be a bit more precise, for any $A \subseteq \Lambda(\mathbb{F}_n)$ there is an induced Borel embedding

$$\Theta_A : \Lambda(\mathbb{F}_n) \to \bigcup_{\Gamma \in \Lambda(\mathbb{F}_n)} \{0,1\}^{\Gamma} \tag{3.1.2}$$

which sends a Schreier graph Γ to the *binary field* $\mathcal{F} : \Gamma \to \{0,1\}$ given by

$$\mathcal{F}(x) = \begin{cases} 1, & (\Gamma, x) \in A \\ 0, & (\Gamma, x) \notin A \end{cases}, \tag{3.1.3}$$

where (Γ, x) is the Schreier graph obtained from Γ by *rerooting* Γ at the vertex x. The resulting space of binary configurations over elements of $\Lambda(\mathbb{F}_n)$ (namely, the image of Θ_A) serves to highlight the set A, and the corresponding functions $\rho_{A,r}$ may now be written as

$$\rho_{A,r}(\mathcal{F}) = \frac{1}{|U_r(x)|} \sum_{y \in U_r(x)} \mathcal{F}(y). \tag{3.1.4}$$

Note that if μ is an invariant measure on $\Lambda(\mathbb{F}_n)$, then $(\Theta_A)_*\mu$ is an invariant measure on $\bigcup_{\Gamma \in \Lambda(\mathbb{F}_n)} \{0,1\}^{\Gamma}$. From now on, when talking about the density of a given set A inside of r-neighborhoods, we will refer to the functions $\rho_{A,r}$ defined over the binary field constructed as per (3.1.3), without necessarily making mention of the map Θ_A. We are now ready to formulate our question:

Question 3.1.1. *Let μ be an invariant random Schreier graph and $A \subseteq \Lambda(\mathbb{F}_n)$ a Borel set, and consider the average densities*

$$\mathbb{E}(\rho_{A,r}) = \int \rho_{A,r}\, d\mu.$$

Then supposing $\mathbb{E}(\rho_{A,0}) > 0$, are the averages $\mathbb{E}(\rho_{A,r})$ bounded away from zero?

A related question which is interesting but which we will not address is the following.

Question 3.1.2. *Let μ be an invariant random Schreier graph and $A \subseteq \Lambda(\mathbb{F}_n)$ a Borel set. Do the averages $\mathbb{E}(\rho_{A,r})$ converge as $r \to \infty$?*

More generally, consider an $\mathbb{F}_n$-invariant measure μ on $\{0,1\}^{\Lambda(\mathbb{F}_n)}$ (which needn't necessarily come from a Borel set $A \subseteq \Lambda(\mathbb{F}_n)$ as above). The following example shows that if such an invariant random binary field has a fixed geometry, meaning that it is supported on a common underlying graph, then it must answer Question 3.1.1 in the positive.

Example 3.1.3. Let $\Gamma \in \Lambda(\mathbb{F}_n)$ be a Cayley graph, i.e. the Schreier graph of a normal subgroup of $\mathbb{F}_n$, and μ an invariant measure on $\{0,1\}^{\Gamma}$. Then one readily verifies that the average densities $\mathbb{E}(\rho_r)$ are all the same. Indeed, we have

$$\begin{aligned}\int \rho_r \, d\mu &= \int \left(\frac{1}{|U_r(x)|} \sum_{y \in U_r(x)} \mathcal{F}(y) \right) d\mu \\ &= \frac{1}{|U_r(x)|} \sum_{y \in U_r(x)} \left(\int \mathcal{F}(y) \, d\mu \right) \\ &= \frac{1}{|U_r(x)|} \sum_{y \in U_r(x)} \mathbb{E}(\rho_0) = \mathbb{E}(\rho_0).\end{aligned}$$

If, however, our random invariant Schreier graph ceases to be so nice (e.g. if it ceases to be vertex-transitive), then the averages $\mathbb{E}(\rho_r)$ can be expected to vary considerably from $\mathbb{E}(\rho_0)$. Our question is: How much? Can they be arbitrarily close to zero if $\mathbb{E}(\rho_0)$ is not zero?

Question 3.1.1 asks whether invariance implies that, given a subset of the space of Schreier of positive measure (in other words, a nontrivial property of the root of our random graph), it will be asymptotically dense inside of large r-neighborhoods. For the sake of brevity, let us give this property a name.

Definition 3.1.4. (Property D) We say that an invariant random Schreier graph μ has *property D* if it answers Question 3.1.1 in the positive, in the sense that, if $A \subseteq \Lambda(\mathbb{F}_n)$ is any subset with $\mu(A) > 0$, then the average densities $\mathbb{E}(\rho_{A,r})$ of A

inside of r-neighborhoods are bounded away from zero.

We will show that, upon placing a mild condition on our random invariant Schreier graph Γ—namely *soficity*—the averages $\mathbb{E}(\rho_{A,r})$ can get arbitrarily small only if Γ exhibits a rather wild geometry. To be a little more precise, we will introduce a notion which we call *relative thinness* and show that the average densities $\mathbb{E}(\rho_{A,r})$ can get arbitrarily small only if Γ is arbitrarily relatively thin at different scales. We are then able to deduce the conservativity of the boundary action of a sofic random subgroup via the following argument:

i. If Γ satisfies property D (and is not the Dirac measure concentrated on the Cayley graph of $\mathbb{F}_n$, a case which is easily dealt with), then there exists a number $k \in \mathbb{N}$ such that the set of Schreier graphs whose roots belong to a cycle of length k has positive measure, and whose density inside of r-neighborhoods is therefore bounded away from zero. The fact that cycles of bounded length are sufficiently dense inside of Γ is in turn enough for us to show that Γ must satisfy (3.1.1).

ii. If Γ does not satisfy property D, then its geometry is such that it cannot grow too quickly; in particular, we are again able to show that Γ must satisfy the condition (3.1.1).

3.2 Sofic invariant subgroups

In order to apply our ideas, we will require that our invariant random subgroup μ be *sofic*. Note that this is a bit of a misnomer, as we are not requiring that a μ-random subgroup of $\mathbb{F}_n$ be sofic (which, given that subgroups of free groups are themselves free, is automatic), but rather that the corresponding μ-random Schreier graph be sofic. Let us recall the definition. Recall also that the uniform probability measure on a finite Schreier graph Γ determines an invariant measure on $\Lambda(\mathbb{F}_n)$, namely the uniform measure supported on the set of rerootings of Γ.

Definition 3.2.1. (Sofic random Schreier graph) An invariant random Schreier graph μ is *sofic* if there exists a sequence of finite Schreier graphs $\{\Gamma_i\}_{i\in\mathbb{N}}$ such that $\mu_i \to \mu$ weakly, where μ_i is the invariant measure on $\Lambda(\mathbb{F}_n)$ determined by Γ_i.

Let us note (as is also done in [9]) that the weak convergence of measures in Definition 3.2.1 can be thought of in more geometric terms as follows: Suppose first that Γ is a Cayley graph (which becomes an invariant random Schreier graph when identified with the Dirac measure concentrated on itself). We say that a finite graph (Γ', μ) equipped with the uniform probability measure is an (r, ε)-*approximation* to Γ if there exists a set $A \subseteq \Gamma'$ of measure $\mu(A) > 1 - \varepsilon$ such that for all $x \in A$, the r-neighborhood of x in Γ' is isomorphic (in the category of edge-labeled graphs) to the r-neighborhood of the identity (or, indeed, of any other vertex) in Γ. The graph Γ is sofic precisely if it admits an (r, ε)-approximation for any pair (r, ε), where $r \in \mathbb{N}$ and $\varepsilon > 0$. A group G is thus sofic if, given any Cayley graph Γ of G, it is possible to construct finite graphs which locally look like Γ at almost all of their points. More generally, suppose that Γ is a random invariant Schreier graph. The distribution of Γ naturally determines a probability measure μ_r on $\Lambda_r(\mathbb{F}_n)$, the set of r-neighborhoods of Schreier graphs of $\mathbb{F}_n$, and we again say that a finite graph (Γ, μ) equipped with the uniform probability measure is an (r, ε)-approximation to Γ if for all $U \in \Lambda_r(\mathbb{F}_n)$ we have $|\mu(U) - \mu_r(U)| < \varepsilon$. Then, as before, a random invariant Schreier graph is sofic precisely if it admits finite (r, ε)-approximations for any pair (r, ε).

As mentioned in the previous chapter, our definition does not take exactly the same form as the ones given, for instance, in [26] or [35]. The main difference is that we require our approximating sequence to consist of bona fide Schreier graphs, and not, as is usually the case, of graphs which need not have the structure of a Schreier graph at all of their points. Let us therefore quickly show that our definition—which we feel is a bit cleaner—is in fact equivalent to the usual one.

Theorem 3.2.2. *If there exist finite graphs (Γ_i, μ_i) which are a sofic approximation to μ, then they may be modified to create finite Schreier graphs (Γ_i', μ_i') which are a sofic approximation to μ.*

Remark 3.2.3. *Here the graphs Γ_i need not have the structure of a Schreier graph at each of their points, i.e. there may exist points whose degree is not $2n$ or are such that the edges attached to them do not have a Schreier labeling. Another caveat that should be pointed out is that a Schreier graph is by definition connected and rooted, although we do not actually impose these conditions in Definition 3.2.1 or the above proposition: there is no sense in assigning a root to the graphs of a sofic approximation (as every vertex is effectively treated as a root), and it is often natural for such graphs to have several connected components (e.g. if the measure they approximate is supported on a set of several distinct Cayley graphs).*

Proof: Let Γ_i be an (r, ε)-approximation to μ and $A \subseteq \Gamma_i$ the set of points at which Γ_i does not have the structure of a Schreier graph. Let Γ_i' be the subgraph of Γ_i induced by the set $\Gamma_i \backslash A$ and $A' \subseteq \Gamma_i'$ the set of points at which Γ_i' does not have the structure of a Schreier graph. Note that A' is a subset of the set of neighbors of the removed set A, and that therefore $\mu_i(A \cup A') < \varepsilon$ (since the r-neighborhood $U_r(x) \subseteq \Gamma_i$ of any point $x \in A$ does not approximate μ, neither does the r-neighborhood of any neighbor of x, provided $r > 1$).

Now, the edges attached to points $x \in A'$ are properly labeled with the generators $a_1, \ldots, a_n$ of $\mathbb{F}_n$—the only problem is that some generators may be missing, i.e. it may be that $\deg(x) < 2n$. We thus "stitch up" the graph Γ_i' as follows: for every generator a_i which does not label any of the edges (neither incoming nor outgoing) attached to a given point $x \in A'$, add a loop to x and label it with a_i. If, on the other hand, there exists precisely one edge (assume without loss of generality that it is outgoing) attached to x and labeled with a generator a_i, then consider the longest path γ whose edges are labeled only with a_i and which is attached to x. The endpoint of γ will be a vertex $y \in A'$ distinct from x; to "complete the cycle," we thus need only join x and y with an edge and label this edge with a_i in the obvious way. By repeating this procedure for every vertex in A', we ensure that Γ_i' has the structure of a Schreier graph at every point while modifying it only on a set of very small measure. It follows that the sequence of Schreier graphs (Γ_i', μ_i') is a sofic approximation to μ. ∎

Note that Definition 3.2.1 readily generalizes to invariant random fields, i.e. invariant measures on $\bigcup_{\Gamma \in \Lambda(\mathbb{F}_n)} \{0,1\}^{\Gamma}$: one must simply define convergence with respect to finite $\{0,1\}$-labeled Schreier graphs. We will make use of the following lemma later.

Lemma 3.2.4. *Let μ be a sofic random Schreier graph and $A \subseteq \Lambda(\mathbb{F}_n)$ a Borel set. Then the invariant random field $(\Theta_A)_*\mu$, where Θ_A is the embedding (3.1.2), is also sofic.*

Proof: Denote by A_r the collection of cylinder sets $U \in \Lambda_r(\mathbb{F}_n)$ such that $\mu(A \cap U) > 0$. Clearly, $A \subseteq A_r$, and moreover $\mu(A_r \backslash A) =: \varepsilon_r \to 0$, i.e. the sets A_r approximate A. Let Γ be a finite (r, ε)-approximation to μ, and construct a binary field $\mathcal{F} : \Gamma \to \{0,1\}$ by assigning to a given vertex $x \in \Gamma$ the value 1 if the cylinder set corresponding to its r-neighborhood $U_r(x)$ belongs to A_r and the value 0 otherwise. Then $\mathcal{F}$ is an (r, ε)-approximation to μ_r and hence an $(r, \varepsilon + \varepsilon_r)$-approximation to μ. By constructing fields $\mathcal{F}_i$ in this way for a sequence of finite graphs Γ_i which are (r_i, ε_i)-approximations to μ, with $r_i \to \infty$ and $\varepsilon_i \to 0$, we obtain a sofic approximation to $(\Theta_A)_*\mu$. ∎

Morally speaking, Lemma 3.2.4 allows us to phrase Question 3.1.1 in terms of finite graphs, namely those which come from a sofic approximation. Working with finite graphs in turn has several advantages, as we show in the next section.

3.3 Relative thinness

In order to investigate Question 3.1.1, we would like to introduce a notion which we call *relative thinness*. To be more precise, let Γ be a Schreier graph, and consider the functions $\tau_r : \Gamma \to \mathbb{Q}$ defined by

$$\tau_r(x) := \sum_{y \in U_r(x)} \frac{1}{|U_r(y)|}.$$

Note that if, say, all of the r-neighborhoods of Γ have the same size (as is the case, for instance, when Γ is a Cayley graph), then $\tau_r \equiv 1$. If, on the other hand, the r-neighborhood of a point $x \in \Gamma$ is small compared to the r-neighborhoods near it, then one will have $\tau_r(x) < 1$ (and if it is large compared to the r-neighborhoods near it, then one will have $\tau_r(x) > 1$). We thus say that a Schreier graph Γ is *relatively thin at scale* r at a point $x \in \Gamma$ if $\tau_r(x) < 1$ (if a piece of cloth is worn down at a particular spot, then the regions surrounding that spot will have more mass than is to be found at the spot itself).

One feature of relative thinness is that it is "tempered," meaning that if Γ is very thin at x and y is a neighbor of x, then Γ will be thin at y as well. To be more precise, let us say that a function $f : \Gamma \to \mathbb{R}$ is C*-Lipschitz* if whenever $x, y \in \Gamma$ are neighbors,

$$f(x) \leqslant Cf(y)$$

for some constant $C \geqslant 1$. Likewise, we say that a family of functions $\{f_i : \Gamma_i \to \mathbb{R}\}_{i \in I}$ is *uniformly* C*-Lipschitz* over the family of graphs $\{\Gamma_i\}_{i \in \mathbb{N}}$ if each f_i is C-Lipschitz for some constant $C \geqslant 1$ that does not depend on i. We now have the following lemma.

Lemma 3.3.1. *Let $\Gamma \in \Lambda$ be a Schreier graph of $\mathbb{F}_n$. Then there exists a constant $C \geqslant 1$ such that the family of functions $\{\tau_r\}_{r \in \mathbb{N}}$ is uniformly C-Lipschitz over Γ.*

Proof: Note first that if x and y are neighbors in Γ, then we have the bound

$$|U_r(x)| \geqslant \frac{1}{2n-1}|U_r(y)|. \tag{3.3.1}$$

Put $S := U_r(y)\backslash U_r(x)$, and let S' denote a choice, for each vertex $z \in S$, of a neighbor z' which belongs to $U_r(x)$. Then

$$\begin{aligned}
\tau_r(y) - \tau_r(x) &\leqslant \sum_{z \in S} \frac{1}{|U_r(z)|} \\
&\leqslant (2n-1)^2 \sum_{z \in S'} \frac{1}{|U_r(z)|} \\
&\leqslant (2n-1)^2 \tau_r(x).
\end{aligned}$$

Here the second line is obtained by applying the inequality (3.3.1) and using the fact that points in S' may have at most $2n-1$ neighbors in S. It follows that each τ_r is C-Lipschitz with $C = (2n-1)^2 + 1$. ∎

Moreover, it turns out that, at least in the model case of a finite Schreier graph (which carries a unique invariant probability measure), thinness and the densities $\rho_{A,r}$ given by (3.1.4) are directly related to one another.

Proposition 3.3.2. *Let (Γ, A, μ) be a finite Schreier graph Γ equipped with the uniform probability measure, together with a subset $A \subseteq \Gamma$. Then*

$$\int_\Gamma \rho_{A,r}\, d\mu = \int_A \tau_r\, d\mu,$$

where $\rho_{A,r}$ is the r-neighborhood density of the set A.

Proof: One must simply observe that, whether summing $\rho_{A,r}$ over Γ or τ_r over A, for a given point $x \in \Gamma$ the quantity $1/|U_r(x)|$ is summed exactly once for every point $y \in A$ such that $x \in U_r(y)$. ∎

As a corollary, we obtain:

Corollary 3.3.3. *Given a finite Schreier graph (Γ, μ) equipped with the uniform probability measure, τ_r integrates to 1 over Γ.*

Proof: Simply choose $A = \Gamma$ in the hypotheses of Proposition 3.3.2. Then

$\rho_{A,r} \equiv 1$, so that we have

$$\int_\Gamma \tau_r \, d\mu = \int_\Gamma \rho_{A,r} \, d\mu = \int_\Gamma 1 \, d\mu = 1.$$

∎

We thus find that the *average thinness* of a finite Schreier graph is always one. Proposition 3.3.2 can therefore be interpreted as saying that, if the average of $\rho_{A,r}$ over a finite Schreier graph Γ is small relative to $\mathbb{E}(\rho_{A,0}) = \mu(A)$, then the set A must be concentrated at points where Γ is relatively thin (at scale r).

Corollary 3.3.3 tells us that, if Γ is a finite Schreier graph, then by integrating the functions τ_r against the uniform probability measure on Γ, we obtain a new probability measure ν_r. Suppose now that μ is a sofic random Schreier graph, and let $\{\Gamma_i\}_{i\in\mathbb{N}}$ be a sofic approximation to μ. Then one readily verifies that the sequence of probability measures $\nu_{r,i}$—those obtained by integrating τ_r against the uniform measures μ_i—converges weakly to a probability measure ν_r on $\Lambda(\mathbb{F}_n)$. That is, soficity implies that τ_r is a density with respect to μ.

Proposition 3.3.4. *Let μ be a sofic random Schreier graph which is ergodic and which does not satisfy property D. Then there exist finite Schreier graphs (Γ_i, A_i, μ_i) together with subsets $A_i \subseteq \Gamma_i$ such that the Γ_i are a sofic approximation to μ, $\mu_i(A_i) \to 1$, and $\mathbb{E}(\tau_i \mid A_i) \to 0$.*

Proof: If μ does not satisfy property D, then there exists a set $A \subseteq \Lambda(\mathbb{F}_n)$ with $\mu(A) > 0$ such that $\mathbb{E}(\rho_{A,r}) \to 0$ along some subsequence of radii $r \in \mathbb{N}$, and hence such that $\mathbb{E}(\tau_r \mid A) \to 0$. Let $\{g_i\}_{i\in\mathbb{N}}$ be an enumeration of $\mathbb{F}_n$ (e.g. the lexicographic order), and put

$$A_k := A \cup g_1 A \cup \ldots \cup g_k A.$$

It follows from the fact that the τ_r are uniformly C-Lipschitz (Proposition 3.3.1) that

$\mathbb{E}(\tau_r \mid A_k) \to 0$ for any k. Indeed, putting

$$m := \max_{1 \leqslant i \leqslant k} |g_i|,$$

we have $\mathbb{E}(\tau_r \mid A_k) \leqslant C^m \mathbb{E}(\tau_r \mid A) \to 0$. Moreover, by ergodicity, $\mu(A_k) \to 1$. By Lemma 3.2.4, there exists a sofic approximation $\{\mathcal{F}_{i,k}\}_{i \in \mathbb{N}}$ for each invariant random field $(\Theta_{A_k})_* \mu$, which is the same thing as a sequence of finite Schreier graphs $(\Gamma_{i,k}, A_{i,k}, \mu_{i,k})$ such that the $A_{i,k}$ approximate A_k (just take $A_{i,k} = \{x \in \Gamma_{i,k} \mid \mathcal{F}_{i,k}(x) = 1\}$). By choosing an appropriate diagonal sequence, we prove our claim. ∎

Suppose again that μ is a sofic random Schreier graph which is ergodic and does not satisfy property D. Our next goal is to show that the geometry of μ must be quite peculiar. To do so, we will look at the sofic approximation to μ guaranteed by Proposition 3.3.4, i.e. the sequence of finite Schreier graphs (Γ_i, A_i, μ_i), with $\mu_i(A_i) \to 1$ and $\mathbb{E}(\tau_i \mid A_i) \to 0$. A trick we will emply is the following: instead of working with the functions τ_r and letting r vary, we may instead modify the structure of our Schreier graphs and work only with the function τ_1. Thus if Γ_i is one of our Schreier graphs (constructed, by default, with respect to the standard generating set $\mathcal{A} = \{a_1, \ldots, a_n\}$), denote by $\Gamma_i^{(r)}$ what we call the *r-contraction* of Γ_i obtained by regarding it as a Schreier graph of $\mathbb{F}_n$ constructed with respect to the generating set consisting of all group elements of length less than or equal to r. One readily verifies that τ_r over Γ agrees with τ_1 over $\Gamma^{(r)}$, in the sense that the diagram

$$\begin{tikzcd} \Gamma_i \arrow[r, hook] \arrow[dr, "\tau_r"'] & \Gamma_i^{(r)} \arrow[d, "\tau_1"] \\ & \mathbb{Q} \end{tikzcd}$$

commutes (here the upper arrow is the obvious identification between the vertices of Γ_i and the vertices of $\Gamma_i^{(r)}$). By modifying the structure of our graphs in this way (for ever larger values of r) and choosing an appropriate diagonal sequence, our sofic approximation now takes the form of a sequence of finite Schreier graphs (Γ_i, A_i, μ_i) such that $\mu_i(A_i) \to 1$ and $\mathbb{E}(\tau_1 \mid A_i) \to 0$.

We do not know of any invariant random Schreier graph which fails to have property D. In order to get a sense of what a sequence of graphs satisfying the aforementioned conditions might look like, however, consider the following example.

Example 3.3.5. Let X_N be a set of 2^N points and Y_N a set of N points, and let Γ_N denote the complete bipartite graph between X_N and Y_N, i.e. the graph obtained by adding to the set $X_N \sqcup Y_N$ all possible edges (x, y) such that $x \in X_N$ and $y \in Y_N$. Then the sequence of graphs (Γ_N, X_N, μ_N) has the property that $\mathbb{E}(\tau_1 \mid X_N) \to 0$. Indeed, it is easy to see that for fixed N, τ_1 is constant over each of X_N and Y_N, and that $\tau_1|_{X_N} \to 0$ whereas $\tau_1|_{Y_N} \to \infty$. At the same time, we have $\mu_N(X_N) \to 1$.

Note, however, that the graphs constructed in Example 3.3.5 cannot be realized as a sequence of contracted Schreier graphs. Indeed, suppose that, possibly upon adding loops to the vertices of the bipartite graphs of Example 3.3.5 and turning some of their edges into multi-edges, we were able to label their edges with generators of $\mathbb{F}_n$. Then for each vertex $x \in X_N$, it must be the case that one of its external edges, meaning an edge (x, y) with $y \in Y_N$, is labeled with one of the standard generators $a_1, \ldots, a_n$ (or one of their inverses)—were this not the case, x would be fixed by every a_i and hence by $\mathbb{F}_n$ itself, a contradiction, since x has $\mathbb{F}_n$-labeled external edges attached to it. By the pigeonhole principle, there must thus exist a generator $a_i^{\pm 1}$ and a subset $X'_N \subseteq X_N$ of measure $\mu_N(X'_N) \geqslant \mu_N(X_N)/2n$ such that $a_i^{\pm 1} X'_N \subseteq Y_N$. But this is again a contradiction, since $\mu_N(Y_N) \to 0$ and μ_N is an invariant measure. Alternatively, note that there is an ever widening gap between the values of τ_1 over X_N and Y_N, which violates the fact that τ_1 is C-Lipschitz (Proposition 3.3.1).

The family of graphs constructed in Example 3.3.5 has what one might call a lopsided structure. That is to say, graphs in the family split into a set of large measure and a set of small measure in such a way that all of the neighbors of a given vertex in the large set belong to the small set. The next proposition shows that, despite the fact that the bipartite graphs considered above cannot be realized as Schreier graphs, a

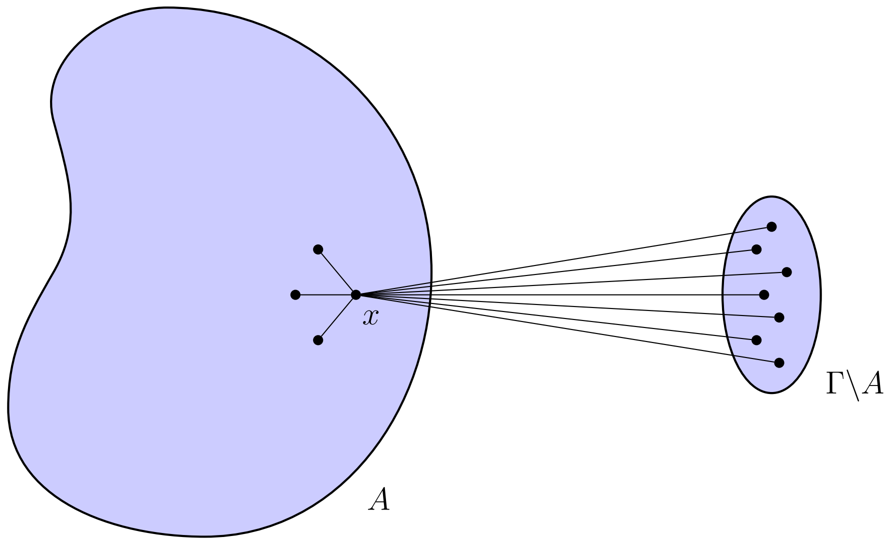

Figure 3.3: Finite (contracted) Schreier graphs Γ that approximate invariant random subgroups which do not satisfy property D have a subset $A \subseteq \Gamma$ of large measure such that, for a random point $x \in A$, the large majority of its neighbors belong to the complement $\Gamma\backslash A$.

version of this phenomenon must occur whenever μ is a sofic random Schreier graph which is ergodic and does not satisfy property D (see also Figure 2).

Proposition 3.3.6. *Let μ be a sofic random Schreier graph which is ergodic and does not satisfy property D. Then there exists a sequence of finite (contracted) Schreier graphs (Γ_i, A_i, μ_i) such that the Γ_i are a sofic approximation to μ, $\mu_i(A_i) \to 1$, and*

$$\lim_{i\to\infty} \mathbb{E}\left(\frac{\deg_{A_i}(x)}{\deg_{\Gamma_i\backslash A_i}(x)} \,\middle|\, A_i \right) = 0,$$

where $\deg_A(x)$ denotes the number of neighbors of x in the set A.

Proof: Let (Γ_i, A_i, μ_i) be finite contracted Schreier graphs that are a sofic approximation to μ and such that $\mu_i(A_i) \to 1$ and $\mathbb{E}(\tau_1 \mid A_i) \to 0$. We have

$$\begin{aligned} \mathbb{E}(\tau_1 \mid A_i) &= \frac{1}{|A_i|} \sum_{x \in A_i} \frac{1 + \deg_{A_i}(x)}{1 + \deg(x)} \\ &= \frac{1}{|A_i|} \sum_{x \in A_i} \frac{1 + \deg_{A_i}(x)}{1 + \deg_{A_i}(x) + \deg_{\Gamma_i\backslash A_i}(x)} < \varepsilon_i, \end{aligned}$$

with $\varepsilon_i \to 0$. It follows that, for any $K > 0$, the subsets $A_{i,K} \subset A_i$ over which $\deg_{\Gamma_i \setminus A_i}(x) \leqslant K \deg_{A_i}(x)$ satisfy $\mu_i(A_{i,K}) \to 0$. Indeed, were this not the case, we would have

$$\begin{aligned}\mathbb{E}(\tau_1 \mid A_i) &\geqslant \mathbb{E}(\tau_1 \mid A_{i,K})\mu_i(A_{i,K}) \\ &\geqslant \frac{1}{|A_{i,K}|} \sum_{x \in A_{i,K}} \frac{1 + \deg_{A_i}(x)}{1 + (K+1)\deg_{A_i}(x)} \mu_i(A_{i,K}) \\ &\geqslant \frac{\delta}{K+1}\end{aligned}$$

for all $i \in \mathbb{N}$, where $\delta > 0$ is a fixed lower bound of the values $\mu_i(A_{i,K})$. We therefore find that the ratio of the expected number of internal neighbors to external neighbors of points in A_i tends to zero, as desired. ∎

3.4 Conservativity of the boundary action

We are now in a position to prove our main result. To this end, let us understand a *k-cycle* to be a closed path which is isomorphic to a k-sided polygon. Our main idea is that an invariant random Schreier graph which satisfies property D must have a certain density of k-cycles, i.e. that there exists a k such that a given vertex of an invariant random Schreier graph belongs to a k-cycle with positive probability, and that this in turn restricts the growth of our random graph enough to render Δ_H a null set.

Theorem 3.4.1. *The boundary action $H \circlearrowright (\partial\mathbb{F}_n, \mathfrak{m})$ of a sofic random subgroup of the free group is conservative.*

Proof: Suppose first that μ is an invariant random Schreier graph that satisfies property D. It is not difficult to see that, with the exception of one trivial case, there must always exist a number k such that the Borel set A of Schreier graphs whose roots belong to a k-cycle has positive measure. Indeed, if this were not the case, then μ would be the Dirac measure concentrated on the Cayley graph of $\mathbb{F}_n$ (whose boundary action is of course conservative). By assumption, there thus exists an $\varepsilon > 0$ such that $\mathbb{E}(\rho_{A,r}) \geqslant \varepsilon$ for all r. Put $f(r) := 2n(2n-1)^{r-1}$, let X_r denote the size of the radius-r sphere centered at the root of a μ-random Schreier graph, let $\ell = \lfloor k/2 \rfloor$, and let $r \geqslant 1$ be an initial radius. Trivially, $\mathbb{E}(X_r) \leqslant f(r)$. We are then able to bound $\mathbb{E}(X_{r+\ell})$ as

$$\mathbb{E}(X_{r+\ell}) \leqslant f(r+\ell) - \varepsilon f(r)$$

and, continuing inductively, to obtain the general bound

$$\begin{aligned}\mathbb{E}(X_{r+(m+1)\ell}) &\leqslant (2n-1)^{\ell}\mathbb{E}(X_{r+m\ell}) - \varepsilon\left(\mathbb{E}(X_{r+m\ell}) - \varepsilon\mathbb{E}(X_{r+(m-1)\ell})\right) \\ &= \left((2n-1)^{\ell} - \varepsilon\right)\mathbb{E}(X_{r+m\ell}) + \varepsilon^2\mathbb{E}(X_{r+(m-1)\ell}),\end{aligned} \tag{3.4.1}$$

since each k-cycle that passes through the boundary of an $(r+m\ell)$-neighborhood allows us to decrease the trivial bound on the size of the boundary of an $(r+(m+1)\ell)$-neighborhood by one. Note that (3.4.1) is a linear homogenous recurrence relation with characteristic polynomial

$$\chi(t) = t^2 - \left((2n-1)^{\ell} - \varepsilon\right)t - \varepsilon^2.$$

It is easy to see that χ has distinct real roots. The general solution of the recurrence relation (3.4.1) thus yields the bound

$$\begin{aligned}\mathbb{E}(X_{r+m\ell}) \leqslant C_0&\left((2n-1)^{\ell} - \varepsilon + \sqrt{\left((2n-1)^{\ell}-\varepsilon\right)^2 + 4\varepsilon^2}\right)^m \\ &+ C_1\left((2n-1)^{\ell} - \varepsilon - \sqrt{\left((2n-1)^{\ell}-\varepsilon\right)^2 + 4\varepsilon^2}\right)^m,\end{aligned} \tag{3.4.2}$$

whereupon applying initial conditions readily gives $C_0 = C_1 = f(r)/2$ (in order to simplify notation, we have doubled the roots of χ). By Theorem 3.0.4, we have

$$\mathbb{E}(\mathfrak{m}(\Delta_H)) = \int \mathfrak{m}(\Delta_H)\, d\mu$$

$$
\begin{aligned}
&= \int \lim_{r\to\infty} \frac{|\partial U_r(\Gamma,H)|}{|\partial U_r(\mathbb{F}_n,e)|}\,d\mu \\
&= \lim_{r\to\infty} \frac{1}{f(r)} \int |\partial U_r(\Gamma,H)|\,d\mu \\
&= \lim_{r\to\infty} \frac{1}{f(r)} \mathbb{E}(X_r).
\end{aligned}
$$

Passing to the subsequence $\{r+m\ell\}_{m\in\mathbb{N}}$ and replacing the second (and clearly smaller) term of (3.4.2) with the first, we see that

$$
\begin{aligned}
\lim_{r\to\infty} \frac{\mathbb{E}(X_r)}{f(r)} &\leqslant \lim_{m\to\infty} \frac{f(r)}{f(r+m\ell)} \left((2n-1)^\ell - \varepsilon + \sqrt{\left((2n-1)^\ell-\varepsilon\right)^2 + 4\varepsilon^2}\right)^m \\
&= \lim_{m\to\infty} \left(1 - \frac{\varepsilon}{(2n-1)^\ell} + \sqrt{1 - \frac{2\varepsilon}{(2n-1)^\ell} + \frac{5\varepsilon^2}{(2n-1)^{2\ell}}}\right)^m.
\end{aligned}
$$

But a simple calculation shows that what is inside the parentheses is less than one, so that the above limit is zero. It follows that $\mathbb{E}(\Delta_H) = 0$ and therefore that the boundary action of our invariant random subgroup is conservative.

Suppose next that μ is a sofic random subgroup which is ergodic and does not satisfy property D. Then by Proposition 3.3.6, it has a lopsided sofic approximation, i.e. a sofic approximation consisting of contracted Schreier graphs (Γ_i, A_i, μ_i) such that $\mu_i(A_i) \to 1$ and the average external degree of vertices in A_i is much smaller than their average external degree, in the sense that their ratio tends to zero. Since $\mu_i(\Gamma_i \backslash A_i) \to 0$, this implies that

$$
\mathbb{E}(\deg(x) \mid A_i) \ll \mathbb{E}(\deg(x) \mid \Gamma_i \backslash A_i),
$$

again in the sense that the ratio of these two quantities tends to zero. But the vertex degree of a point in a contracted Schreier graph $\Gamma^{(r)}$ is precisely one less than the size of the r-neighborhood of the corresponding uncontracted graph Γ. We thus find that, over a set of arbitrarily large measure, the ratio of the average size of (arbitrarily large) r-neighborhoods in our Schreier graphs to $|U_r(\mathbb{F}_n, e)|$ is arbitraily small, which proves our claim. ∎

To conclude this section, let us remark that, although Theorem 4.3.3 says, in effect, that sofic random subgroups cannot grow as quickly as the free group, it is reasonable to expect that they can still grow very quickly: it is proved in [5] (see Theorem 40) that there exists a (nonatomic) regular unimodular random graph whose exponential growth rate is maximal.

3.5 Cogrowth and limit sets

It is interesting to examine other questions considered in [31] for sofic random subgroups. Note, for example, that Theorem 4.3.3 immediately implies that a sofic random Schreier graph $\Gamma \in \Lambda(\mathbb{F}_n)$ cannot contain a *branch of* $\mathbb{F}_n$, i.e. a subgraph isomorphic to the unique tree one of whose vertices has degree one and all of whose other vertices have degree $2n$, since the presence of a branch implies the existence of a nontrivial wandering set (another way to say this is that every edge of an invariant random Schreier graph must belong to a cycle). Recall, moreover, that the *cogrowth* of a subgroup $H \leqslant \mathbb{F}_n$ (i.e. the "growth of H inside of $\mathbb{F}_n$") is defined to be

$$v_H := \limsup_{r \to \infty} \sqrt[r]{|H \cap U_r(\mathbb{F}_n, e)|} \leqslant 2n - 1.$$

By Theorem 4.2 of [31], if $v_H < \sqrt{2n-1}$, then the action $H \circlearrowleft (\partial\mathbb{F}_n, \mathfrak{m})$ is dissipative. We therefore have the following corollary of Theorem 4.3.3.

Corollary 3.5.1. *The cogrowth of a sofic random subgroup $H \in L(\mathbb{F}_n)$ must satisfy $v_H \geqslant \sqrt{2n-1}$.*

Alternatively, a Schreier graph is *Ramanujan* if and only if its cogrowth does not exceed $\sqrt{2n-1}$, and it is proved in [5] (see Theorem 5) that random unimodular d-regular graphs are Ramanujan if and only if they are trees, which shows that an invariant random subgroup H satisfies $v_H > \sqrt{2n-1}$.

There are various limit sets associated to a subgroup $H \leqslant \mathbb{F}_n$ (most of which descend from the general theory of discrete groups of isometries of Gromov hyperbolic spaces).

The *radial limit set*, denoted Λ_H^{rad}, is the set of limit points (in $\partial\mathbb{F}_n$) of sequences of elements of H which are contained within a tubular neighborhood of a certain geodesic ray in $\mathbb{F}_n$. There are the *small horospheric limit set*, denoted $\Lambda_H^{\mathrm{hor},s}$, which is the set of boundary points $\omega \in \partial\mathbb{F}_n$ such that any *horosphere* centered at ω contains infinitely many elements of H, the Schreier limit set Ω_H, and the *big horospheric limit set*, denoted $\Lambda_H^{\mathrm{hor},b}$, which is the set of boundary points $\omega \in \partial\mathbb{F}_n$ such that a certain horosphere centered at ω contains infinitely many elements of H. There are also the divergence set of the *Poincaré series* of H, denoted Σ_H, and the *full limit set*, denoted Λ_H, which is the set of all limit points (in $\partial\mathbb{F}_n$) of elements of H. We refer the reader to [31] for the precise definitions of these sets.

As is shown in [31], there is a certain amount of flexibility in the $\mathfrak{m}$-measures of the aforementioned limit sets for arbitrary subgroups $H \leqslant \mathbb{F}_n$: although several of these sets necessarily have the same measure, the measure of the full limit set Λ_H may take on a range of values (and may well be a null set). Once again, however, the situation for sofic random subgroups is more rigid, as the following theorem shows.

Theorem 3.5.2. *Let H be a sofic random subgroup. Then the limit sets $\Lambda_H^{\mathrm{hor},s}$, Ω_H, $\Lambda_H^{\mathrm{hor},b}$, Σ_H, and Λ_H all have full $\mathfrak{m}$-measure.*

Proof: By Theorems 3.20 and 3.21 of [31], the aforementioned limit sets are contained in one another in the order in which we have listed them, i.e.

$$\Lambda_H^{\mathrm{rad}} \subseteq \Lambda_H^{\mathrm{hor},s} \subseteq \Omega_H \subseteq \Lambda_H^{\mathrm{hor},b} \subseteq \Sigma_H \subseteq \Lambda_H,$$

and the middle four of these have the same $\mathfrak{m}$-measure. By Theorem 4.3.3, $\mathfrak{m}(\Omega_H) = 1$. These facts taken together imply the claim. ∎

By Theorem 3.35 of [31] (which is an analogue of the *Hopf-Tsuji-Sullivan theorem*, valid for discrete groups of isometries of n-dimensional hyperbolic space), either $\mathfrak{m}(\Lambda_H^{\mathrm{rad}}) = 1$ or $\mathfrak{m}(\Lambda_H^{\mathrm{rad}}) = 0$, the former occurring when the simple random walk on (Γ, H) is recurrent and the latter when the simple random walk on (Γ, H) is transient. The following examples show that the $\mathfrak{m}$-measure of the radial limit set of a nonatomic invariant random subgroup may be either zero or one.

Example 3.5.3. (An invariant random subgroup with the property that $\mathfrak{m}(\Lambda_H^{\mathrm{rad}}) = 1$) Consider the Cayley graph Γ of the group $\mathbb{Z}^2$ constructed with respect to the standard generators $a = (1, 0)$ and $b = (0, 1)$. It is a classical result that the simple random walk on $\mathbb{Z}^2$ is recurrent [57], so Theorem 3.35 of [31] implies that $\mathfrak{m}(\Lambda_H^{\mathrm{rad}}) = 1$, where H is the fundamental group of Γ. The graph Γ contains infinitely many "a-chains," i.e. bi-infinite geodesics labeled with the generator a, and by independently reversing the orientations of these a-chains or leaving their orientations fixed, we generate a large space of Schreier graphs each of whose underlying unlabeled graphs is isomorphic to the two-dimensional integer lattice (in particular, the simple random walk on these graphs remains recurrent). There is a natural uniform measure on this space (the uniform measure on its projective structure), and it is not difficut to see that this measure is invariant.

Example 3.5.4. (An invariant random subgroup with the property that $\mathfrak{m}(\Lambda_H^{\mathrm{rad}}) = 0$) Consider the Cayley graph Γ of the group $\mathbb{Z}^3$ constructed with respect to the standard generators $a = (1, 0, 0)$, $b = (0, 1, 0)$, and $c = (0, 0, 1)$. It is again a classical result that the simple random walk on $\mathbb{Z}^3$ (or, indeed, on $\mathbb{Z}^n$ for $n \geqslant 3$) is transient, so that $\mathfrak{m}(\Lambda_H^{\mathrm{rad}}) = 0$, where H is the fundamental group of Γ. By employing the same trick as in the previous example, we again generate a large space of Schreier graphs for which the uniform measure is a nonatomic invariant probability measure.

Chapter 4

Random walks and Poisson bundles

The main result of this chapter is a generalization of the main result of the previous chapter. Our approach, however, is very different. Rather than exploiting soficity or the combinatorial structure of a random Schreier graph, we will exploit the theory of random walks and, in particular, properties of the *Poisson boundary*—a natural "space at infinity" associated to any Markov chain. To be more precise, if G is a group and $H \leqslant G$ a subgroup, then H acts naturally on the Poisson boundary $(\partial G, \nu)$ of any random walk on G. We show that if μ is an invariant measure on $L(G)$, then the action $H \circlearrowleft (\partial G, \nu)$ is almost surely conservative. As the boundary of the free group $(\partial \mathbb{F}_n, \mathfrak{m})$ may be identified with the Poisson boundary of the simple random walk on $\mathbb{F}_n$, this is indeed a generalization of our previous result. In fact, as shown in [40], the hyperbolic boundary of a hyperbolic group can, under natural assumptions, be identified with the Poisson boundary, which allows us to establish conservativity of the boundary action of an invariant random subgroup in a wide variety of contexts. In fact, although the results of this section are phrased in terms of invariant random subgroups, they remain valid for *stationary random subgroups*, i.e. when our invariant measure on the space of subgroups is replaced with a stationary measure of the random walk on leafwise Schreier graphs (invariant measures are automatically stationary, but unlike invariant measures, stationary measures always exist).

Our main idea is to work with the *Poisson bundle* associated to an invariant random subgroup μ. Intuitively speaking, this is the object obtained by attaching to a μ-random subgroup H the Poisson boundary of the Schreier graph of H (this is analogous to considering the unit tangent bundle of a negatively curved manifold). Such bundles were introduced by Kaimanovich [42] and also recently considered by Bowen [11], who used them to solve an instance of the Furstenberg entropy realization problem. In our case, the existence of nonatomic conditional measures associated to a certain quotient of Poisson bundles allows us to deduce conservativity of the boundary action. The results of this section are joint with Kaimanovich and can also be found in [17].

4.1 The Poisson boundary of a Schreier graph

Let G be a countable group and μ_0 a *nondegenerate* probability measure on G, meaning that $\mathrm{supp}(\mu_0)$, the support of μ_0, generates G as a semigroup. Then given an initial distribution on G, which we will take to be the Dirac measure δ_e concentrated at the identity, we may speak of the *random walk* on G determined by μ_0 and δ_e, namely the Markov chain whose state space is G, whose state at time zero is $e \in G$, and whose transition probabilities are given by

$$\widetilde{p}(g, gh) = \mu_0(h).$$

This can be thought of in geometric terms by considering the Cayley graph of G constructed with respect to the generating set $\mathrm{supp}(\mu_0)$ (for the sake of notational simplicity, we will also denote this Cayley graph by G). Our random walk then describes the trajectory of a Markov particle which begins at $e \in G$, then visits a neighbor x of e with the probability assigned by μ_0 to the group element which labels the edge joining e to x, then visits a neighbor y of x with the probability assigned by μ_0 to the group element which labels the edge joining x to y, and so on. Note that, a priori, $\mathrm{supp}(\mu_0)$ may well be infinite, in which case our Cayley graph will not be locally finite.

Now let H be a subgroup of G and $\Gamma = (\Gamma, x)$ the corresponding Schreier graph, which we also construct with respect to the generating set $\mathrm{supp}(\mu_0)$. There is a unique *covering map* $f : G \to \Gamma$ such that $f(e) = x$, which is induced by the quotient map obtained by collapsing the cosets of H. Recall that a covering of graphs is a graph homomorphism $f : \Gamma \to \Gamma'$ such that for each vertex $x \in \Gamma$, its *star*, i.e. the set of edges incident with x, is mapped bijectively onto the star of $f(x) \in \Gamma'$.

(In the category of Schreier graphs, homomorphisms must also have the property that Schreier structures, namely edge-labelings by the generators of G, are respected, which implies that our covering map f is indeed unique. In fact, it is an immediate but important consequence of this property that, in the category of Schreier graphs, every morphism is a covering map.)

The random walk on G determined by μ_0 and δ_e naturally descends to Γ via the covering map $f : G \to \Gamma$. To be more precise, it determines the random walk on Γ with initial distribution $f_*\delta_e = \delta_x$ and whose transition probabilities $p(y, \cdot)$ are distributed among the neighbors of y in accordance with the edges joining them to y. That is, if $\{e_i\}_{i \in I}$ is the set of (oriented) edges that join a vertex y to a vertex z, then we have

$$p(y, z) = \sum_{i \in I} \mu_0(g_i),$$

where $g_i \in \mathrm{supp}(\mu_0)$ is the generator that labels the edge e_i. Note that a loop, i.e. a cycle of length one, attached to a vertex y counts as two edges (which correspond to its two orientations).

The *Poisson boundary* of the Schreier graph Γ, which we denote by $\partial\Gamma$, is a measure space which describes the stochastically significant *behavior at infinity* of our random walk. It is defined as follows. Let X^∞ denote the space of infinite sample paths of our random walk, which may be thought of as the projective limit of the spaces X^t of length t sample paths. As these spaces come equipped with natural probability measures ν^t (the *t-dimensional distributions* of the random walk), X^∞ comes equipped with a natural probability measure as well, which we denote by ν^∞. The space X^∞ is too large to meaningfully describe the behavior at infinity of our random walk. In order to define an appropriate boundary at infinity, one would like to quotient X^∞ by the equivalence relation $\mathcal{E}$ defined as follows: say that two infinite sample paths $(x_t)_{t \geqslant 0}$ and $(y_t)_{t \geqslant 0}$ are equivalent (i.e. approach the same point at infinity) if $x_t = y_{t+k}$ for all sufficiently large t and some integer $k \in \mathbb{Z}$. Thus, sample paths are equivalent if they eventually coincide, up to a shift in time. A priori, the equivalence relation $\mathcal{E}$ will not be measurable; to overcome this difficulty, we therefore pass to its *measurable envelope* $M(\mathcal{E})$, namely the finest measurable equivalence relation on X^∞ whose equivalence classes are unions of equivalence classes of $\mathcal{E}$.

Definition 4.1.1. (Poisson boundary) The *Poisson boundary* of the random walk on a Schreier graph $\Gamma \in \Lambda(G)$ determined by a nondegenerate measure μ_0 on G (and initial distribution δ_e) is the quotient

$$\partial\Gamma := X^\infty / M(\mathcal{E}),$$

where X^∞ is the space of infinite sample paths of the random walk and $M(\mathcal{E})$ denotes the measurable envelope of the equivalence relation defined in the previous paragraph. The Poisson boundary $\partial\Gamma$ comes equipped with the *harmonic measure* ν, which is the image of ν^∞ under the quotient map $\pi : X^\infty \to \partial\Gamma$.

Equivalently (and more succinctly), the Poisson boundary may be defined as the space of ergodic components of the *time shift* $T : X^\infty \to X^\infty$ given by $T(x_t)_{t\geqslant 0} = (x_{t+1})_{t\geqslant 0}$. Note that, although we sometimes suppress mention of measures in our notation, we are working in the category of Lebesgue spaces. It is in this context that our claims are validated (cf. Chapter 2).

Our group G acts on its Poisson boundary $(\partial G, \nu)$ in a natural way. Indeed, this action corresponds exactly to the time shift in the space of sample paths. To be more precise, let $g \in G$ be a group element, and let $(x_t)_{t\geqslant 0}$ be an infinite sample path of our random walk. Then we may obtain a new sample path by first traversing a path γ which begins at e and ends at g (so that the group element read upon traversing γ is precisely g), and thereafter traversing the path $(x_t)_{t\geqslant 0}$ (which must be done in a unique way, as $(x_t)_{t\geqslant 0}$ is but a sequence of elements in our generating set $\mathrm{supp}(\mu_0)$, which unambiguously labels our Cayley graph). As this operation clearly respects the equivalence relation $\mathcal{E}$ and, moreover, does not depend on the choice of γ, we obtain an action $G \circlearrowright (\partial G, \nu)$.

Recall that if $G \circlearrowright X$ is an action of a locally compact group on a space X and μ is a Borel probability measure on G, then there exists a *convolution operator* $P_\mu : \mathcal{P}(X) \to \mathcal{P}(X)$, where $\mathcal{P}(X)$ denotes the space of Borel probability measures on X, given by

$$P_\mu \nu := \mu * \nu = \int_G g_* \nu \, d\mu(g).$$

A measure $\nu \in \mathcal{P}(X)$ is called *μ-stationary* if it satisfies $\mu * \nu = \nu$. Note that in our situation (of a countable group G equipped with a nondegenerate measure μ_0), convolution takes the form

$$\mu_0 * \nu = \sum_{g \in G} \mu_0(g) \, g_* \nu.$$

It is an essential feature of the Poisson boundary that the harmonic measure ν is indeed μ_0-stationary. Moreover, the space $L^\infty(\partial\Gamma, \nu)$ is naturally isometric to the space of *bounded harmonic functions* on Γ, namely bounded functions $f : \Gamma \to \mathbb{R}$ whose values at a given point x may be obtained by averaging its values over the neighbors of x, so that one has

$$f(x) = \sum_{y \in \Gamma} p(x, y) f(y)$$

for all $x \in \Gamma$. In particular, triviality of the Poisson boundary is therefore equivalent to the absence of nonconstant bounded harmonic functions (the *Liouville property*).

If $H \leqslant G$ is a subgroup, then we may speak of the action $H \circlearrowright (\partial\Gamma, \nu)$. As one might expect, the Poisson boundary of the Schreier graph of H is the quotient of the Poisson boundary of G by this action, as the following proposition shows.

Proposition 4.1.2. *Let $H \leqslant G$ be a subgroup and Γ the corresponding Schreier graph of H. Then $\partial\Gamma = \partial G / H$.*

Proof: The covering map $f : G \to \Gamma$ induces an isomorphism $f : X^\infty(G) \to X^\infty(\Gamma)$ between the spaces of infinite sample paths in G and in Γ, and it is easy to see that $\mathcal{E}(G)$ is finer than $\mathcal{E}(\Gamma)$, i.e. if two sample paths are equivalent in $X^\infty(G)$, then their images under f are equivalent in $X^\infty(\Gamma)$. It follows that there exists a quotient map $f' : \partial G \to \partial\Gamma$ between Poisson boundaries:

$$\begin{array}{ccc} X^\infty(G) & \xrightarrow{f} & X^\infty(\Gamma) \\ \downarrow & & \downarrow \\ \partial G & \xrightarrow{f'} & \partial\Gamma \end{array}$$

One readily verifies that the natural actions of H on the spaces in our diagram are equivariant with respect to all of the maps involved. Now, it is easy to see that the action $H \circlearrowright X^\infty(\Gamma)$ preserves $\mathcal{E}(\Gamma)$: composing a sample path $(x_t)_{t\geqslant 0} \in X^\infty(\Gamma)$ with an element $h \in H$ entails following a closed path (one that corresponds to h) that begins and ends at the root $x \in \Gamma$, and then following the path $(x_t)_{t\geqslant 0}$, which obviously implies that $(x_t)_{t\geqslant 0}$ and $h(x_t)_{t\geqslant 0}$ are $\mathcal{E}(\Gamma)$-equivalent. Consequently, the action of H on $\partial\Gamma$ is trivial.

Conversely, it is clear that if $(\widetilde{x}_t)_{t\geqslant 0}, (\widetilde{x}'_t)_{t\geqslant 0} \in X^\infty(G)$ belong to separate H-orbits, then their images under f must belong to separate H-orbits as well (as otherwise, the inverse map f^{-1} would establish their H-equivalence in $X^\infty(G)$). It follows that f' is indeed the quotient map $f' : \partial G \to \partial G/H$. ∎

4.2 Poisson bundles

If $N \trianglelefteqslant G$ is normal, then the quotient map from the Poisson boundary of G to the Poisson boundary of the Schreier graph Γ of N (see Proposition 4.1.2) is G-equivariant, so that the G-action on ∂G descends to a natural action on $\partial\Gamma$. This is no longer the case, however, for a subgroup which is not normal. Nevertheless, it is possible to obtain a G-equivariant quotient map by passing from an individual subgroup to an invariant random subgroup.

Thus, let μ be an invariant measure on the space of Schreier graphs $\Lambda(G) = \Lambda$ of our countable group G. Central to our approach is the notion of a *Poisson bundle*. Intuitively speaking, it is the object obtained by attaching to each Schreier graph $\Gamma \in \Lambda$ its Poisson boundary $\partial\Gamma$. To be more precise, denote by $X^\infty(G)$ the space of sample paths in the Cayley graph G issued from the identity, and consider the bundle

$$\widetilde{\mathcal{Y}} := \bigcup_{\Gamma\in\Lambda} (\{\Gamma\} \times X^\infty(G)) \cong \Lambda \times X^\infty(G).$$

We may equip the fibers $\{\Gamma\} \times X^\infty(G)$ with a system of measures $\{\widetilde{\nu}_\Gamma^\infty\}_{\Gamma\in\Lambda}$ defined as follows. If $H \leqslant G$ is the subgroup whose Schreier graph is Γ, then $\widetilde{\nu}_\Gamma^\infty$ is the measure whose image under the quotient of $X^\infty(G)$ by H is ν_Γ^∞, the natural measure on the space of infinite sample paths of the random walk on Γ. We are then able to equip $\widetilde{\mathcal{Y}}$ with a measure $\widetilde{\theta}$ by integrating the base measure μ on Λ against the system of fiberwise measures $\{\widetilde{\nu}_\Gamma^\infty\}_{\Gamma\in\Lambda}$.

The space of ergodic components of the action $G \circlearrowright (\widetilde{\mathcal{Y}}, \widetilde{\theta})$ is the *covering Poisson bundle* $(\widetilde{\mathcal{X}}, \widetilde{\eta})$, whose fibers are Λ-indexed copies of the Poisson boundary ∂G. In analogous fashion, we denote by $X^\infty(\Gamma)$ the space of infinite sample paths in the Schreier graph Γ which are issued from the root and consider the bundle

$$\mathcal{Y} := \bigcup_{\Gamma\in\Lambda} (\{\Gamma\} \times X^\infty(\Gamma)),$$

whose Borel structure comes from realizing $\mathcal{Y}$ as the projective limit

$$\mathcal{Y} = \varprojlim \mathcal{Y}_r,$$

where $\mathcal{Y}_r$ is the set of r-neighborhoods which are centered at the roots of Schreier graphs $\Gamma \in \Lambda$ and endowed with a distinguished sample path of length r. We then equip the bundle $\mathcal{Y}$ with the measure θ obtained by integrating the base measure μ against the system of fiberwise measures $\{\nu_\Gamma^\infty\}_{\Gamma\in\Lambda}$. The space of ergodic components of the action $G \circlearrowleft (\mathcal{Y}, \theta)$ is the Poisson bundle $(\mathcal{X}, \eta)$, whose fibers are the Poisson boundaries of μ-random Schreier graphs.

Moreover, there exist natural quotient (or covering) maps $\Sigma : (\widetilde{\mathcal{Y}}, \widetilde{\theta}) \to (\mathcal{Y}, \theta)$ and $\Pi : (\widetilde{\mathcal{X}}, \widetilde{\eta}) \to (\mathcal{X}, \eta)$, which can be interpreted as the systems of fiberwise quotient maps $\Sigma_\Gamma : \{\Gamma\} \times X^\infty(G) \to \{\Gamma\} \times X^\infty(\Gamma)$ and $\Pi_\Gamma : \{\Gamma\} \times \partial G \to \{\Gamma\} \times \partial\Gamma$, respectively, which quotient by the action of the subgroup $H_\Gamma = \pi_1(\Gamma, x)$. (Recall that $\partial\Gamma \cong \partial G / H_\Gamma$ by Proposition 4.1.2.) Putting everything together, we thus have the following diagram.

$$\begin{tikzcd} (\widetilde{\mathcal{Y}}, \widetilde{\theta}) \arrow[r] \arrow[d, "\Sigma"'] & (\widetilde{\mathcal{X}}, \widetilde{\eta}) \arrow[d, "\Pi"] \\ (\mathcal{Y}, \theta) \arrow[r] & (\mathcal{X}, \eta) \end{tikzcd} \tag{4.2.1}$$

In terms of the theory of discrete measured equivalence relations, we are dealing with a *random walk along equivalence classes* of the discrete measured equivalence relation on (Λ, μ) whose classes (or *leaves*) consist of the Schreier graphs corresponding to the conjugates of a subgroup $H \leqslant G$ (this equivalence relation coincides with the orbit equivalence relation determined by the action of G). The random walk is determined by a measurable system of leafwise transition probabilities, and $(\mathcal{X}, \eta)$ is the bundle of Poisson boundaries of leafwise random walks.

Note that if Γ and $\Gamma' := g\Gamma g^{-1}$ are conjugate (equivalent) Schreier graphs, then there exists a canonical isomorphism $\phi_{\Gamma,g} : \partial\Gamma \to \partial\Gamma'$ between the Poisson boundaries of Γ and Γ'. Indeed, let γ be a path in Γ joining y to x. Then given a sample path $(x_t)_{\geqslant 0}$ beginning at x, composing it with γ yields a sample path beginning at y. It is clear that this association descends to a map between Poisson boundaries and is independent of the choice of γ. In light of this observation, it is not difficult to see that the quotient maps in diagram (4.2.1) are G-equivariant, with a given element $g \in G$ serving to send almost every leafwise boundary $\{\Gamma\} \times \partial\Gamma$ to the leafwise boundary $\{\Gamma'\} \times \partial\Gamma'$ via the map $\phi_{\Gamma,g}$. Our setup is illustrated in Figure 4.1.

We will later make use of the following lemma, which also appears in [11].

Lemma 4.2.1. *If the measure μ on Λ is ergodic, then the measures $\widetilde{\eta}$ and η on the bundles $\widetilde{\mathcal{X}}$ and $\mathcal{X}$ are ergodic as well.*

Proof: Since the action $G \circlearrowleft (\partial G, \widetilde{\nu})$ is weakly mixing [2] and μ is ergodic, the measure $\widetilde{\eta} = \mu \otimes \widetilde{\nu}$ is ergodic as well. Since the system $G \circlearrowleft (\mathcal{X}, \eta)$ is the quotient of an ergodic system, the measure η is ergodic. ∎

4.3 Conservativity

Let G be a group and X a set on which G acts. Recall that a positive function $\alpha : G \times X \to \mathbb{R}$ is called a (multiplicative) *cocycle* (see, for instance, [71]) if it satisfies the cocycle identity

$$\alpha(gh, x) = \alpha(g, hx)\alpha(h, x)$$

for all $x \in X$ and $g, h \in G$. Two cocycles α and β are said to be *cohomologous* if there exists a positive function $\varphi : X \to \mathbb{R}$ such that

$$\beta(g, x) = \frac{\varphi(gx)}{\varphi(x)}\alpha(g, x).$$

Given a quasi-invariant action of a group G on a Lebesgue space (X, μ), one may naturally associate to it the *Radon-Nikodym cocycle*

$$\Delta(g, x) := \frac{d(g\mu)}{d\mu}(x).$$

Finally, given a G-invariant measurable partition ξ of the space (X, μ), i.e. a measurable partition with the property that whenever $x \sim_\xi y$, we have $gx \sim_\xi gy$ as well (for any $g \in G$), there exists a natural G-equivariant quotient map $\pi_\xi : (X, \mu) \to (\overline{X}, \overline{\mu})$

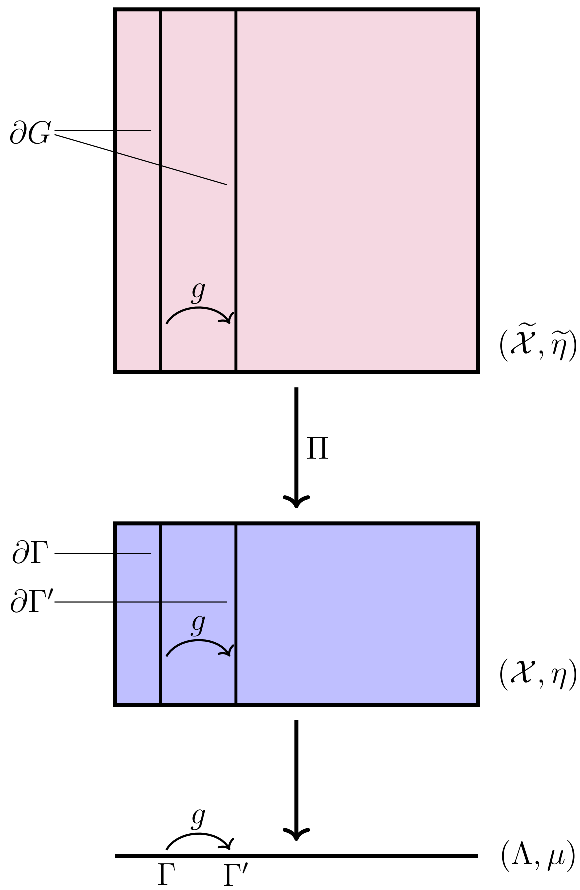

Figure 4.1: Two Poisson bundles sit atop the space (Λ, μ): the bundle $(\widetilde{\mathcal{X}}, \widetilde{\eta})$, whose fibers are copies of ∂G, and the bundle $(\mathcal{X}, \eta)$, whose fibers are the Poisson boundaries of Schreier graphs $\Gamma \in (\Lambda, \mu)$. If the conditional measures of the bundle $(\widetilde{\mathcal{X}}, \widetilde{\eta})$ with respect to Π are nonatomic, then the boundary action of a μ-random subgroup is conservative.

which assigns to each point its equivalence class. One may then speak of the associated Radon-Nikodym cocycle

$$\Delta_\xi(g,x) := \frac{d(g\overline{\mu})}{d\overline{\mu}}(\overline{x}),$$

where $x \in \pi_\xi^{-1}(\overline{x})$. Note that we define Δ_ξ as a function on $G \times X$ (and not $G \times \overline{X}$).

We will need the following lemma, which is adapted from [39].

Lemma 4.3.1. *Let ξ be a G-invariant measurable partition of the bundle $\widetilde{\mathcal{X}}$, and let $\varphi : \mathcal{X} \to \mathbb{R}$ be a nonnegative measurable function on the quotient bundle $\mathcal{X}$ such that the measure $\varphi\eta$ is μ_0-stationary. Then φ is constant with respect to η.*

Proof: Since the measure $\varphi\eta$ is μ_0-stationary, we have that for η-almost every $x \in \mathcal{X}$,

$$\begin{aligned}\varphi(x) = \frac{d(\varphi\eta)}{d\eta}(x) &= \frac{d(\mu_0 * \varphi\eta)}{d\eta}(x)\\ &= \sum_{g\in G} \frac{d(g\varphi\eta)}{d\eta}(x)\mu_0(g)\\ &= \sum_{g\in G} \varphi(g^{-1}x)\frac{d(g\eta)}{d\eta}(x)\mu_0(g).\end{aligned}$$

This observation can be interpreted as saying that φ is a harmonic function of the Markov chain on $\mathcal{X}$ whose transition probabilities are given by

$$p(x, g^{-1}x) = \mu_0(g)\frac{d(g\eta)}{d\eta}(x). \tag{4.3.1}$$

Notice that this chain preserves the measure η. For a fixed $s > 0$, consider now the set

$$A_s := \{x \in \mathcal{X} \mid \varphi(x) \geqslant s\},$$

and suppose that $\eta(A_s) > 0$. Denote by X^∞ the space of infinite sample paths of the Markov chain determined by (4.3.1). Then by the Poincaré recurrence theorem applied to the time shift $T : X^\infty \to X^\infty$, η-almost every sample path $(x_t)_{t\geqslant 0}$ eventually lands in A_s. Let $\tau : X^\infty \to \mathbb{Z}_{\geqslant 0}$ denote the associated stopping time, namely

$$\tau(x_t)_{t\geqslant 0} := \min\{t \geqslant 0 \mid x_t \in A_s\},$$

and put $\tau \wedge t := \min\{\tau, t\}$. The fact that φ is harmonic implies that for any $t \geqslant 0$, we have

$$\varphi(x) = \mathbb{E}_x(\varphi(x_{\tau\wedge t})),$$

which implies that

$$\varphi(x) \geqslant \mathbb{E}_x(\varphi(x_\tau)) \geqslant s.$$

Consequently, φ is almost surely constant. ∎

Theorem 4.3.2. *Let ξ be a measurable G-invariant partition of the boundary bundle $\widetilde{\mathcal{X}}$. Then the Radon-Nikodym cocycles $\Delta, \Delta_\xi : G \times \widetilde{\mathcal{X}} \to \mathbb{R}$ are cohomologous if and only if ξ is the point partition.*

Proof: Suppose that Δ and Δ_ξ are cohomologous. Then there exists a positive measurable function $\varphi : \widetilde{X} \to \mathbb{R}$ such that

$$\frac{d(g\widetilde{\eta})}{d\widetilde{\eta}}(\widetilde{x}) = \frac{d(g\varphi\eta)}{d(\varphi\eta)}(x)$$

almost everywhere and for every $g \in G$. But the measure η is μ_0-stationary, so that, by Lemma 4.3.1, $\varphi \equiv 1$. It follows that the conditional random walk determined by the quotient ξ coincides with the original random walk, so that ξ is indeed the point partition. ∎

We are now ready to prove our main result.

Theorem 4.3.3. *Let μ be an invariant measure on $L(G) = L$. Then the boundary action $H \circlearrowright (\partial G, \nu)$ of a μ-random subgroup $H \leqslant G$ is almost surely conservative.*

Proof: Assume without loss of generality that μ is ergodic. Then the measure $\widetilde{\eta}$ on the bundle $\widetilde{\mathcal{X}}$ is ergodic as well by Lemma 4.2.1. Denote by $\mathcal{C}_H$ and $\mathcal{D}_H$ the conservative and dissipative parts, respectively, of the boundary action $H \circlearrowleft (\partial G, \nu)$ of a μ-random subgroup H. It is not difficult to see that the subset

$$E := \bigcup_{H \in L} \mathcal{D}_H \subseteq \widetilde{\mathcal{X}}$$

is invariant. Indeed, the canonical isomorphism between the Poisson boundaries of Schreier graphs corresponding to conjugate subgroups H and H' is an orbit equivalence and hence maps $\mathcal{C}_H$ onto $\mathcal{C}_{H'}$ and $\mathcal{D}_H$ onto $\mathcal{D}_{H'}$. By ergodicity, either $\widetilde{\eta}(E) = 0$ or $\widetilde{\eta}(E) = 1$. If the former is the case, then the boundary action of a μ-random subgroup H is almost surely conservative, proving our claim. Suppose, therefore, that $\widetilde{\eta}(E) = 1$. It then follows from Corollary 2.4.6 that the conditional measures on the ergodic components of the quotient map $\Pi : (\widetilde{\mathcal{X}}, \widetilde{\eta}) \to (\mathcal{X}, \eta)$ are atomic.

Consider now the positive function $\varphi : \widetilde{\mathcal{X}} \to \mathbb{R}$ defined as

$$\varphi(\widetilde{x}) := \eta_x(\widetilde{x}).$$

That is, φ assigns to a point $\widetilde{x} \in \widetilde{\mathcal{X}}$ its measure as an atom of the corresponding conditional measure η_x. But φ also shows the Radon-Nikodym cocycles Δ and Δ_Π to be cohomologous, as

$$\begin{aligned} \frac{d(g\widetilde{\eta})}{d\widetilde{\eta}}(\widetilde{x}) &= \frac{d(g\eta_{g^{-1}x})}{d\eta_x}(\widetilde{x}) \frac{d(g\eta)}{d\eta}(x) \\ &= \frac{\varphi(gx)}{\varphi(x)} \frac{d(g\eta)}{d\eta}(x). \end{aligned}$$

By Theorem 4.3.2, this is possible only if Π is trivial, i.e. if the preimage partition of Π is the point partition. It follows that, if ∂G is nontrivial, then the conditional measures $\{\eta_x\}_{x \in \mathcal{X}}$ are nonatomic, and hence that the boundary action of a μ-random subgroup of G is almost surely conservative. ∎

4.4 Applications to hyperbolic groups

Let G be a finitely generated group. Recall that G is said to be *hyperbolic* if its Cayley graph (constructed with respect to a finite generating set and regarded as a simplicial 1-complex) is a Gromov hyperbolic space. We refer the reader to [33] and [30] for the relevant background and definitions. In the case of geodesic metric spaces X (which include Cayley graphs), hyperbolicity can be characterized by a "skinny triangles" condition. That is, let $x, y, z \in X$ be three arbitrary points joined with pairwise geodesics γ_{xy}, γ_{xz}, and γ_{yz}. Then X is Gromov hyperbolic if there exists a global *hyperbolicity constant* $\delta \geqslant 0$ such that

$$\gamma_{yz} \subseteq U_\delta(\gamma_{xy} \cup \gamma_{xz}),$$

i.e. the geodesic segment γ_{yz} is contained in the δ-neighborhood of the union of γ_{xy} and γ_{xz}. Hyperbolicity is a quasi-isometric invariant and hence does not depend on the choice of finite generating set with which we construct our Cayley graph.

Any Gromov hyperbolic space X possesses a *hyperbolic boundary*, which we denote by $\partial_h X$ and upon which the isometry group $\mathrm{Iso}(X)$ acts by homeomorphisms, and the space $X \cup \partial_h X$, suitably topologized, becomes a compactification (the *hyperbolic compactification*) of X. Moreover, in many situations the hyperbolic boundary may be equipped with a natural boundary measure ν_h, and it is of fundamental interest to study the ergodic properties of the boundary action $H \circlearrowright (\partial_h X, \nu_h)$, where $H \leqslant \mathrm{Iso}(X)$ is a (discrete) group of isometries of X. (Consider, for instance, the classical situation when $X = \mathbb{H}^2$ is the hyperbolic plane and H is a Fuchsian group acting on the boundary sphere $\partial\mathbb{H}^2 \cong \mathbb{S}^1$ equipped with Lebesgue measure.)

In [40], Kaimanovich proved that the hyperbolic boundary $(\partial_h G, \nu_h)$ of a hyperbolic group G, where the measure ν_h is determined by a distribution on G which has a finite first moment and whose support generates a nonelementary subgroup, coincides with its Poisson boundary $(\partial G, \nu)$. Moreover, any subgroup $H \leqslant G$ acts on the Cayley graph of G by isomotries and thus extends to a boundary action. Accordingly, actions of subgroups $H \leqslant G$ on the Poisson boundary translate to actions on the hyperbolic boundary, so that Theorem 4.3.3 gives us the following result.

Theorem 4.4.1. *Let G be a hyperbolic group and μ an invariant measure on its lattice of subgroups. Then the action $H \circlearrowright (\partial_h G, \nu_h)$ of a μ-random subgroup $H \leqslant G$ on the hyperbolic boundary of G is almost surely conservative.*

An important special case of Theorems 4.3.3 and 4.4.1 is the situation when G is the finitely generated free group of rank $n > 1$, namely $\mathbb{F}_n = \langle a_1, \ldots, a_n \rangle$, and μ_0 is the uniform probability measure supported on the symmetric generating set $\mathcal{A} \cup \mathcal{A}^{-1} = \{a_1^{\pm 1}, \ldots, a_n^{\pm 1}\}$. The random walk determined by μ_0 and δ_e is then the *simple random walk* on $\mathbb{F}_n$. As was first established by Dynkin and Malyutov [24], its Poisson boundary $(\partial \mathbb{F}_n, \mathfrak{m})$ is naturally isomorphic to the space of infinite reduced words in the alphabet $\mathcal{A} \cup \mathcal{A}^{-1}$ endowed with the topology of pointwise convergence and equipped with the uniform measure given by

$$\mathfrak{m}(g) = \frac{1}{2n(2n-1)^{|g|-1}}, \tag{4.4.1}$$

where we again use g to denote both a nontrivial element of $\mathbb{F}_n$ and the cylinder set comprised of all infinite reduced words which begin with the word g. As $\mathbb{F}_n$ is a hyperbolic group (indeed, its Cayley graph constructed with respect to the generating set $\mathcal{A} \cup \mathcal{A}^{-1}$ is the $2n$-regular tree), it possesses a hyperbolic boundary which, when equipped with its corresponding boundary measure, is isomorphic to $(\partial \mathbb{F}_n, \mathfrak{m})$.

It was shown in the previous chapter that, assuming μ is *sofic*, the boundary action of a μ-random subgroup of $\mathbb{F}_n$ is almost surely conservative. Theorem 4.3.3 allows us to prove this result in full generality (removing the assumption of soficity), thus giving us the following.

Theorem 4.4.2. *The boundary action $H \circlearrowright (\partial \mathbb{F}_n, \mathfrak{m})$ of an invariant random sub-*

group of the free group is almost surely conservative.

Recall the *limit sets* associated to discrete groups of isometries $H \leqslant \mathrm{Iso}(X)$ of a hyperbolic space X: The *radial limit set*, denoted Λ_H^{rad}, is the set of limit points in $\partial_h X$ of sequences of elements of H which are contained within a tubular neighborhood of a certain geodesic ray in X. The *small horospheric limit set*, denoted $\Lambda_H^{\mathrm{hor},s}$, is the set of boundary points $\omega \in \partial_h X$ such that any *horosphere* centered at ω contains infinitely many elements of H, and the *big horospheric limit set*, denoted $\Lambda_H^{\mathrm{hor},b}$, is the set of boundary points $\omega \in \partial_h X$ such that a certain horosphere centered at ω contains infinitely many elements of H. There are also the divergence set of the *Poincaré series* of H, denoted Σ_H, and the *full limit set*, denoted Λ_H, which is the set of all limit points in $\partial_h X$ of elements of H.

In the case when $X = \mathbb{F}_n$ is the free group, we are now able to extend our result on limit sets established in the previous chapter to invariant random subgroups in full generality.

Theorem 4.4.3. *The limit sets $\Lambda_H^{\mathrm{hor},s}$, $\Lambda_H^{\mathrm{hor},b}$, Σ_H, and Λ_H of an invariant random subgroup all have full $\mathfrak{m}$-measure.*

Moreover, as was shown in [31], the radial limit set $\Lambda_H^{\mathrm{rad}} \subseteq \partial\mathbb{F}_n$ has either full or zero measure, depending on whether the simple random walk on the Schreier graph of H is recurrent or transient, respectively. Either of these situations is possible for an invariant random subgroup μ (see the previous chapter for explicit examples). This fact combined with Theorem 4.4.3 therefore completely characterizes the possible measures of the limit sets of an invariant random subgroup. (Note, however, that once again this characterization ceases to hold for individual subgroups—see [31].)

We now also obtain the following result on the growth of r-neighborhoods of the root of the Schreier graph of an invariant random subgroup.

Theorem 4.4.4. *The growth of an invariant random Schreier graph of the free group*

$\mathbb{F}_n$ is dominated by the growth of $\mathbb{F}_n$, in the sense that

$$\lim_{r\to\infty} \frac{|U_r(\Gamma, x)|}{|U_r(\mathbb{F}_n, e)|} = 0$$

almost surely.

Chapter 5

Invariant Schreier structures

A Schreier graph Γ possesses two kinds of structures, which we will for the moment refer to as a *geometric structure* and an *algebraic structure*. The former is the underlying graph structure, which determines the geometry of Γ, in particular allowing one to equip Γ with a metric. The latter is the labeling of edges of Γ with the generators of a group G, which one may always assume to be the free group $\mathbb{F}_n := \langle a_1, \ldots, a_n \rangle$. The algebraic structure is not an arbitrary labeling: each vertex $x \in \Gamma$ must be attached to precisely one incoming and one outgoing edge labeled with a given generator a_i. Each such labeling, together with a choice of root, identifies Γ as a particular subgroup of $\mathbb{F}_n$, and in general a given unlabeled graph may possess many—indeed, even uncountably many—distinct algebraic structures.

This chapter is, broadly speaking, an investigation of the algebraic structures—which we will henceforth call *Schreier structures*—with which $2n$-regular graphs may be endowed (recall that a graph is $2n$-regular if each of its vertices has degree $2n$). We are especially interested in random Schreier structures which are *invariant* in some sense. To be more precise, denote by Λ the space of Schreier graphs of $\mathbb{F}_n$ and by Ω the space of $2n$-regular rooted graphs, and consider the forgetful map $f : \Lambda \to \Omega$ that sends a Schreier graph to its underlying unlabeled graph. There is an induced map $f : \mathcal{P}(\Lambda) \to \mathcal{P}(\Omega)$ from the space of probability measures on Λ to the space of probability measures on Ω, and moreover the space $\mathcal{P}(X)$, where $X = \Lambda$ or Ω, contains several subspaces of "nice" measures, namely: $\mathcal{C}(\Lambda)$, the space of probability measures on Λ invariant under the action of $\mathbb{F}_n$ by conjugation; $\mathcal{I}(X)$, the space of measures invariant with respect to the discrete measured equivalence relation underlying X; $\mathcal{U}(X)$, the space of *unimodular* measures; and $\mathcal{S}(X) \subseteq \mathcal{U}(X)$, the space of *sofic* measures.

Our results may be summarized as follows:

i. The map $f : \Lambda \to \Omega$ is surjective, i.e. every $2n$-regular graph admits a Schreier structure (Theorem 5.2.4).

ii. $\mathcal{C}(\Lambda) = \mathcal{I}(\Lambda) = \mathcal{U}(\Lambda)$, i.e. the spaces of conjugation-invariant, invariant, and unimodular measures on Λ coincide (Theorem 5.3.2).

iii. $f_*\mathcal{U}(\Lambda) \subseteq \mathcal{U}(\Omega)$, i.e. the image of a unimodular (equivalently, invariant) measure on Λ is a unimodular measure on Ω (Proposition 5.3.3).

iv. The induced map $f : \mathcal{S}(\Lambda) \to \mathcal{S}(\Omega)$ is surjective, i.e. any sofic measure on Ω can be lifted to a sofic measure on Λ (Proposition 5.4.1).

v. Assuming it is nonempty, the fiber $f^{-1}(\mu)$ of invariant measures over a unimodular measure $\mu \in \mathcal{U}(\Omega)$ supported on *rigid graphs* is very large, in that it contains an uncountable family of ergodic measures, many of which we are able to describe explicitly (Theorem 5.4.3).

vi. For a large class of groups G, the Dirac measure δ_G concentrated on an unlabeled Cayley graph of G can be lifted to a nonatomic invariant measure on Λ (Theorem 5.4.4).

The first three of these statements are certainly known to experts, yet they might best be described as folklore—though they are often used, we know of no sources that give explicit and general proofs. Moreover, we are able to use statement iii. to exhibit closed invariant subspaces of Λ which do not support an invariant measure (see Corollary 5.3.4 and Example 5.3.6). The latter three statements comprise the main results of the chapter. Morally speaking, they show that there exists a wealth of invariant algebraic structures sitting atop a given invariant geometric structure. This is line with and expands upon recent work by Bowen [12], who showed that the subspace of $\mathcal{I}(\Lambda)$ consisting of measures supported on infinite Schreier graphs is a *Poulsen simplex* (which implies that the set of extremal points, i.e. ergodic measures, is dense). Indeed, some of our work is inspired by his. Note also that, via the correspondence between the Schreier graphs of a given group G and the lattice of subgroups $L(G)$ of that group, an invariant Schreier structure determines an invariant random subgroup.

5.1 Invariant and unimodular measures

As is detailed in [44], there are two notions of invariance for measures μ on Ω. There is invariance in the classical sense of Feldman and Moore [28], according to which invariance is defined with respect to the underlying discrete measured equivalence relation of Ω, and there is *unimodularity* in the sense of Benjamini and Schramm [9] (see also [2]). Let us go over these notions in turn.

Consider first the equivalence relation $\mathcal{E} \subset \Omega \times \Omega$ whereby $(\Gamma, x) \sim (\Delta, y)$ if and only if there exists an isomorphism $\phi : \Gamma \to \Delta$ of unrooted graphs. The *left projection* $\pi_\ell : \mathcal{E} \to \Omega$ that sends an element of $\mathcal{E}$ to its first coordinate determines a *left counting measure* $\widetilde{\mu}_\ell$ on $\mathcal{E}$ with differential $d\widetilde{\mu}_\ell = d\mu\, d\nu_\Gamma$, where ν_Γ is the counting measure on the equivalence class of Γ. In other words, $\widetilde{\mu}_\ell$ is defined on Borel sets $A \subseteq \mathcal{E}$ as

$$\widetilde{\mu}_\ell(A) = \int \nu_\Gamma(A \cap \pi_\ell^{-1}(\Gamma))\, d\mu = \int |A \cap \pi_\ell^{-1}(\Gamma)|\, d\mu.$$

In analogous fashion, the *right projection* $\pi_r : \mathcal{E} \to \Omega$ that sends an element of $\mathcal{E}$ to its second coordinate determines a *right counting measure* $\widetilde{\mu}_r$ on $\mathcal{E}$. We now say that the measure μ is invariant if the lift $\widetilde{\mu}_\ell$ (or $\widetilde{\mu}_r$) is invariant under the involution ι given by $(\Gamma, \Delta) \mapsto (\Delta, \Gamma)$; see the following diagram.

$$\begin{tikzcd}
(\mathcal{E}, \widetilde{\mu}_\ell) \arrow[rr, leftrightarrow, "\iota"] \arrow[dr, "\pi_\ell"'] & & (\mathcal{E}, \widetilde{\mu}_r) \arrow[dl, "\pi_r"] \\
& (\Lambda, \mu) &
\end{tikzcd}$$

Definition 5.1.1. (Invariance) A measure μ on Ω is *invariant* if $\widetilde{\mu}_\ell = \widetilde{\mu}_r$, i.e. if the left and right counting measures on the equivalence relation $\mathcal{E}$ coincide. We denote the space of invariant measures on Ω by $\mathcal{I}(\Omega)$.

Consider next the space $\widetilde{\Omega}$ of *doubly rooted graphs*, whose elements are graphs (Γ, x, y) with a distinguished *principal root* x and *secondary root* y. The left projection $\pi_x : \widetilde{\Omega} \to \Omega$ given by $(\Gamma, x, y) \mapsto (\Gamma, x)$ determines a measure $\widetilde{\mu}_x$ on $\widetilde{\Omega}$ with differential $d\widetilde{\mu}_x = d\mu\, dw_\Gamma$, where w_Γ is the *weighted counting measure* on Γ given by

$$w_\Gamma(y) = |\mathcal{O}_y(\mathrm{Aut}_x(\Gamma))|,$$

i.e. the mass assigned to a vertex $y \in \Gamma$ is the cardinality of its orbit under the action of the stabilizer $\mathrm{Aut}_x(\Gamma) \leqslant \mathrm{Aut}(\Gamma)$. Thus, $\widetilde{\mu}_x$ is defined on Borel sets $A \subseteq \widetilde{\Omega}$ as

$$\widetilde{\mu}_x(A) = \int w_\Gamma(A \cap \pi_x^{-1}(\Gamma))\, d\mu.$$

Here as before there is a second projection, namely the right projection $\pi_y : \widetilde{\Omega} \to \Omega$ given by $(\Gamma, x, y) \mapsto (\Gamma, y)$, which, again in analogous fashion, determines a measure $\widetilde{\mu}_y$ on $\widetilde{\Omega}$. We say that the measure μ is unimodular if the lift $\widetilde{\mu}_x$ (or $\widetilde{\mu}_y$) is invariant under the natural involution given by $(\Gamma, x, y) \mapsto (\Gamma, y, x)$.

Definition 5.1.2. (Unimodularity) A measure μ on Ω is *unimodular* if $\widetilde{\mu}_x = \widetilde{\mu}_y$, i.e. if the left and right weighted counting measures on the space of doubly rooted graphs coincide. We denote the space of unimodular measures on Ω by $\mathcal{U}(\Omega)$.

Unimodularity can also be described as follows. Let $\widetilde{\Omega}^1 \subset \widetilde{\Omega}$ denote the space of doubly rooted graphs (Γ, x, y) whose principal and secondary roots are at unit distance from one another. We present $\widetilde{\Omega}^1$ as the projective limit

$$\widetilde{\Omega}^1 = \varprojlim \widetilde{\Omega}^1_r, \tag{5.1.1}$$

where $\widetilde{\Omega}^1_r$ is the set of (isomorphism classes of) r-neighborhoods of edges that connect the principal and secondary roots of graphs $(\Gamma, x, y) \in \widetilde{\Omega}^1$. A measure μ on the projective system (2.3.1) may be lifted to a measure $\widetilde{\mu}$ on (5.1.1) by putting

$$\widetilde{\mu}(U, x, y) = w_U(y)\mu(U, x),$$

where, as above, $w_U(y) = |\mathcal{O}_y(\mathrm{Aut}_x(U))|$, and the measure μ is unimodular precisely if $\widetilde{\mu}(U, x, y) = \widetilde{\mu}(U, y, x)$ for all $(U, x, y) \in \widetilde{\Omega}^1_r$ and for all r.

Finally, there is the space of *sofic measures*, denoted $\mathcal{S}(\Omega)$. We refer the reader to the previous sections for the definition of soficity. Let us remark that the uniform measure supported on the set of rerootings of a finite graph is always unimodular (and need not be invariant). Since the space of unimodular measures is closed in the weak-$*$ topology (see [44] for a proof), the space of sofic measures is by definition contained in the space of unimodular measures, i.e. $\mathcal{S}(\Omega) \subseteq \mathcal{U}(\Omega)$ (the question of whether these spaces coincide is of course an open problem).

5.2 Existence of Schreier structures

Let us be clear about what we mean by a Schreier structure. The definition goes as follows.

Definition 5.2.1. (Schreier structure) Let $\Gamma \in \Omega$ be a $2n$-regular rooted graph. A *Schreier structure* Σ on Γ is a labeling of its edges by the generators of the free group $\mathbb{F}_n = \langle a_1, \ldots, a_n \rangle$ that turns Γ into a Schreier graph, i.e. a map $\Sigma : E_0(\Gamma) \to \mathcal{A}$, where $E_0(\Gamma)$ denotes a choice of orientation for each edge $(x, y) \in \Gamma$, such that for each $x \in \Gamma$ and each $1 \leqslant i \leqslant n$, there is precisely one incoming edge labeled with a_i and one outgoing edge labeled with a_i attached to x.

It is natural to ask whether any (connected and rooted) $2n$-regular graph admits a Schreier structure, i.e. whether the forgetful map $f : \Lambda \to \Omega$ that sends a Schreier graph to its underlying unlabeled graph is surjective. It is well-known that this question has a positive answer, but the literature on Schreier graphs is a bit fuzzy on this point. A statement of the result (in various forms) is to be found, for example, in [37], [50], [38], [31], and [32], the latter four of which cite one another on this question, but the only proof of the claim in these sources is the one due to Gross [37], who showed in 1977 that every finite $2n$-regular graph can be realized as a Schreier graph of the symmetric group (this proof is reproduced in [50]). In fact, seeing that every $2n$-regular graph can be realized as a Schreier graph of $\mathbb{F}_n$ requires nothing but classical results from graph theory that go back much further than the aforementioned sources. Let us go over the argument here.

A graph possesses a Schreier structure if and only if it is 2-*factorable* (see [56] for a recent survey on graph factorization). Recall that a 2-*factor* of a graph Γ is a 2-regular subgraph of Γ whose vertex set coincides with that of Γ. Note that a 2-factor needn't be connected (if it were, it would be a Hamiltonian cycle). A graph is 2-factorable if it can be decomposed into 2-factors whose edge sets are mutually disjoint, whence the connection with Schreier structures becomes plain: if Γ has a Schreier structure, then the subgraph Γ_i of Γ consisting of those edges labeled with the generator a_i is a 2-factor, and $\Gamma = \Gamma_1 \cup \ldots \cup \Gamma_n$ is a 2-factorization of Γ. Conversely, if $\Gamma = \Gamma_1 \cup \ldots \cup \Gamma_n$ is a 2-factorization of Γ, one need only give an orientation to the components of each Γ_i and label their edges with the generator a_i to obtain a labeling of Γ. The following result was proved by Petersen [54] in 1891.

Theorem 5.2.2. (Petersen) *Every finite $2n$-regular graph is 2-factorable.*

Theorem 5.2.2 can be proved by using the fact that a finite graph has an *Euler tour*, i.e. a closed path that visits every edge exactly once, if and only if each of its vertices is of even degree. One can then split any finite $2n$-regular graph into a certain bipartite graph and apply Hall's theorem (also known as the marriage lemma) to extract a 2-factor; by induction, one obtains a 2-factorization (see Chapter 2.1 of [22] for the full argument). By the above discussion, we have the following corollary.

Corollary 5.2.3. *Every finite $2n$-regular graph admits a Schreier structure.*

Passing to the infinite case is made possible via an application of the *infinity lemma*, which asserts that every infinite locally finite tree contains a geodesic ray; it appears in König's classical text on graph theory [49], first published in 1936 (see Chapter 6.2), or in Chapter 8.1 of [22].

Theorem 5.2.4. *Every $2n$-regular graph admits a Schreier structure.*

Proof: Let Γ be an infinite $2n$-regular graph (the finite case has already been taken care of by Theorem 5.2.2). Assume that Γ is connected, and let $x_0 \in \Gamma$ be an arbitrarily chosen root. Consider U_r, the r-neighborhood centered at x_0, and note that the cardinality of its *cut set* $\mathcal{C}$, i.e. the set of edges that connect vertices in U_r to vertices not in U_r, is even. This follows from the equation

$$\sum_{x \in U_r} \deg(x) = 2|E(U_r)| + |\mathcal{C}|,$$

given that the left hand side and the first term in the right hand side are even numbers. Consider now the graph $U_r \cup \mathcal{C}$. By grouping the edges in $\mathcal{C}$ into pairs, removing each pair from $U_r \cup \mathcal{C}$, and connecting the vertices in U_r to which the elements of each pair were attached by a new edge, we "close up" the neighborhood U_r and turn it into a $2n$-regular graph. By Corollary 5.2.3, this graph admits a Schreier structure, which in turn determines a labeling of U_r.

We now employ the infinity lemma. Let Σ_r denote the set of Schreier structures of U_r (we have just shown that Σ_r is nonempty), and construct a tree by regarding the elements of each Σ_r as vertices and connecting every vertex in Σ_{r+1} by an edge to the vertex in Σ_r that represents the Schreier structure obtained by restricting the structure on U_{r+1} to U_r. It follows that there exists a geodesic ray in our tree, i.e. an infinite sequence of Schreier structures on the neighborhoods $\{U_r\}_{r \in \mathbb{N}}$ each of which is an extension of the last and which exhaust Γ. This implies the claim. ∎

Remark 5.2.5. *The infinity lemma (equivalently, the use of a projective system of Schreier structures) is necessary in the proof of Theorem 5.2.4, since it is in general not the case that a Schreier structure defined on a given neighborhood will extend to a Schreier structure on a bigger neighborhood. Alternatively, one can prove Theorem 5.2.4 by assuming that every $2n$-regular graph covers the bouquet of n loops (which is the case—in fact we have essentially just proved this) and noting that Schreier struc-*

tures can be pulled back along covering maps. See the next chapter for more on this point.

5.3 Schreier graphs versus unlabeled graphs

In this section, we compare Schreier graphs and unlabeled graphs, focusing on the spaces of invariant and unimodular measures on these two classes of graphs and how such measures behave under the forgetful map that sends a Schreier graph to its underlying unlabeled graph. Note that a homomorphism of Schreier graphs is a homomorphism of graphs that respects the additional structure carried by a Schreier graph, i.e. that maps one edge to another only if both edges have the same label and orientation. An important feature of Schreier graphs is that this additional structure lends them a certain rigidity which is not generally enjoyed by unlabeled graphs.

Proposition 5.3.1. *The vertex stabilizer* $\mathrm{Aut}_x(\Gamma) \leqslant \mathrm{Aut}(\Gamma)$ *of a Schreier graph* (Γ, x) *is always trivial.*

Proof: Let $(\Gamma, x) \in \Lambda$ be an arbitrary Schreier graph, and suppose that $\phi \in \mathrm{Aut}_x(\Gamma)$ is a nontrivial automorphism, so that there exist distinct points $y, z \in \Gamma$ (which are necessarily equidistanced from x) such that $\phi(y) = z$. If y and z are at unit distance from x, then ϕ obviously fixes each of them, since, by definition, the edges (x, y) and (x, z) have different labels. If y and z are at distance $r \geqslant 1$ from x, then consider a geodesic $\gamma : [0, r] \to \Gamma$ that joins x to y. Since ϕ is an isometry, the image $\phi_*\gamma$ is a geodesic that joins x to z. Now let $0 \leqslant t < r$ be a value such that $\gamma(t) = \phi_*\gamma(t)$ but $\gamma(t+1) \neq \phi_*\gamma(t+1)$ (since $y \neq z$, such a value must exist). Then ϕ must send $\gamma(t+1)$ to $\phi_*\gamma(t+1)$, but this is impossible, since the edges $(\gamma(t), \gamma(t+1))$ and $(\gamma(t), \phi_*\gamma(t+1))$ again have different labels. ∎

The spaces $\mathcal{I}(\Omega)$ and $\mathcal{U}(\Omega)$ are not the same. The Dirac measure concentrated on an infinite vertex-transitive nonunimodular graph (such as the *grandfather graph*, first constructed by Trofimov [65]) is an example of a measure that is invariant but not unimodular. Conversely, taking an invariant measure supported on rigid graphs, i.e. graphs whose automorphism groups are trivial, and multiplying each of these graphs by a finite nonunimodular graph (such as the segment of length two) yields a measure which is unimodular but not invariant (see [44]). As we will soon show, however, the notions of invariance and unimodularity coincide for Schreier graphs, and both can be viewed in terms of conjugation-invariance.

The action of $\mathbb{F}_n$ on $L(\mathbb{F}_n)$ by conjugation, i.e. the action given by $(g, H) \mapsto gHg^{-1}$ is easily seen to be continuous when thought of as an action on $\Lambda(\mathbb{F}_n)$ and, as we have already seen, has the effect of shifting the root of a Schreier graph $\Gamma \in \Lambda(\mathbb{F}_n)$ in the manner prescribed by the group element $g \in \mathbb{F}_n$. Denote the space of *conjugation-invariant* measures on $\Lambda(\mathbb{F}_n)$ by $\mathcal{C}(\Lambda)$.

Theorem 5.3.2. *The spaces of invariant, unimodular, and conjugation-invariant measures on the space of Schreier graphs coincide.*

Proof: Proposition 5.3.1 implies that $\mathcal{I}(\Lambda) = \mathcal{U}(\Lambda)$. Indeed, since the vertex stabilizer of a Schreier graph Γ is always trivial, the spaces $\mathcal{E}$ and $\widetilde{\Lambda}$ may be identified, and the weighted counting measure w_Γ is precisely the counting measure ν_Γ. To see that $\mathcal{C}(\Lambda) = \mathcal{I}(\Lambda)$, it is enough to know that, by the classical theory (see Corollary 1 of [28] or Proposition 2.1 of [48]), a measure is invariant in the sense of Definition 5.3.2 if and only if it is invariant with respect to the action of a countable group whose induced orbit equivalence relation coincides with the equivalence relation $\mathcal{E} \subset \Omega \times \Omega$. Since $\mathbb{F}_n$ is clearly such a group, it follows that $\mathcal{C}(\Lambda) = \mathcal{I}(\Lambda) = \mathcal{U}(\Lambda)$. ∎

Let $f : \Lambda \to \Omega$ be the forgetful map that sends a Schreier graph to its underlying unlabeled graph. The following proposition shows that the image of a unimodular (equivalently, invariant) measure on Λ is necessarily a unimodular measure on Ω. Since the notions of invariance and unimodularity are not the same downstairs, i.e. on the space of unlabeled graphs Ω, it can therefore be said that the forgetful map f respects unimodularity.

Proposition 5.3.3. *The image of a unimodular measure under f is unimodular, i.e. $f_*\mathcal{U}(\Lambda) \subseteq \mathcal{U}(\Omega)$.*

Proof: Lift μ to $\widetilde{\Lambda}^1$, and consider the map $\widetilde{f} : \widetilde{\Lambda}^1_r \to \widetilde{\Omega}^1_r$ that sends a neighborhood $(U, x, y) \in \widetilde{\Lambda}^1_r$ to its underlying unlabeled neighborhood. It is easy to see that both f and $\widetilde{f}$ extend to homomorphisms of projective systems and therefore that $\nu := f_*\mu$ and $\widetilde{\nu} := \widetilde{f}_*\widetilde{\mu}$ are measures. We thus have a diagram

$$\begin{array}{ccc} (\widetilde{\Lambda}^1_r, \widetilde{\mu}) & \xrightarrow{\widetilde{f}} & (\widetilde{\Omega}^1_r, \widetilde{\nu}) \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\pi} \\ (\Lambda_r, \mu) & \xrightarrow{f} & (\Omega_r, \nu) \end{array} \tag{5.3.1}$$

for each r, where Λ_r and Ω_r are the images of $\widetilde{\Lambda}^1_r$ and $\widetilde{\Omega}^1_r$, respectively, under the natural projection $(U, x, y) \mapsto (U, x)$. To see that the measure $\widetilde{\nu}$ satisfies the unimodularity condition, note that for any $(U, x, y) \in \widetilde{\Omega}^1_r$, there is a one-to-one correspondence between the preimages $\widetilde{f}^{-1}(U, x, y)$ and $\widetilde{f}^{-1}(U, y, x)$, which is given simply by exchanging the principal and secondary roots of the distinguished edges of neighborhoods in $\widetilde{\Lambda}^1_r$. (This correspondence is one-to-one by Proposition 5.3.1.) It is now straightforward that, since the measure $\widetilde{\mu}$ is unimodular, the aforementioned preimages have the same mass and therefore that $\widetilde{\nu}(U, x, y) = \widetilde{\nu}(U, y, x)$.

It remains to check that $\widetilde{\nu}$ is in fact the lift of ν. To see this, note that, again by Proposition 5.3.1,

$$|\widetilde{f}^{-1}(U, x, y)| = w_U(y)|f^{-1}(U, x)|.$$

Moreover, we have

$$\pi_*\widetilde{f}^{-1}(U, x, y) = f^{-1}(U, x).$$

A bit of diagram chasing now yields the result. Starting from the upper right hand corner of our diagram, we have

$$\begin{aligned}\widetilde{\nu}(U,x,y) &= \widetilde{\mu}(\widetilde{f}^{-1}(U,x,y)) \\ &= \frac{1}{w_U(y)}\mu(\pi_*\widetilde{f}^{-1}(U,x,y)) \\ &= \frac{1}{w_U(y)}\mu(f^{-1}(U,x)) \\ &= \frac{1}{w_U(y)}\nu(U,x),\end{aligned}$$

so that $\nu(U,x) = w_U(y)\widetilde{\nu}(U,x,y)$, as desired. ∎

An interesting consequence of Proposition 5.3.3 is that it allows one to exhibit closed invariant subspaces of Λ which do not support an invariant measure.

Corollary 5.3.4. *Let $\Gamma \in \Omega$ be an infinite vertex-transitive nonunimodular graph. Then $f^{-1}(\Gamma)$, the space of Schreier structures over Γ, is a closed invariant subspace of Λ which does not support an invariant measure.*

Proof: Let $X := f^{-1}(\Gamma)$. It is easy to see that X is closed and invariant (as the equivalence class of Γ consists of a single point). Suppose that μ is an invariant measure supported on X. Then its image $f_*\mu$ is the Dirac measure on Γ. But this is a nonunimodular measure, contradicting Proposition 5.3.3. ∎

Remark 5.3.5. *More generally, Corollary 5.3.4 can be applied to nonunimodular graphs whose equivalence classes are finite.*

Example 5.3.6. Given a rooted d-regular tree T, where $d \geqslant 3$, together with a *boundary point* $\omega \in \partial T$, one constructs the grandfather graph Γ of Trofimov [65] as follows: note first that the boundary point ω allows one to assign an orientation to each edge of T, namely the orientation that points to ω, i.e. given an edge (x,y), there is a unique geodesic ray $\gamma : \mathbb{Z}_{\geqslant 0} \to T$ beginning either at x or at y and such that $\lim_{t\to\infty}\gamma(t) = \omega$, and it is the orientation of this ray that determines the orientation of (x, y). Next, connect each vertex $x \in T$ to its grandfather, namely the vertex one arrives at by moving two steps towards ω with respect to the orientation just defined. The result is a $(d^2 - d + 2)$-regular vertex-transitive nonunimodular graph, and moreover it is not difficult to see that $X := f^{-1}(\Gamma)$ is a large (uncountable) space (e.g. see Theorem 5.4.4 below). By Corollary 5.3.4, X does not support an invariant measure.

5.4 Invariant Schreier structures over unlabeled graphs

It would be interesting to fully understand the relationship between unimodular measures on Λ and unimodular measures on Ω. We do not know, for instance, whether the induced map $f : \mathcal{U}(\Lambda) \to \mathcal{U}(\Omega)$ is surjective, i.e. whether, given a unimodular measure ν on the space of rooted graphs, there always exists a unimodular measure μ on the space of Schreier graphs such that $f_*\mu = \nu$. Something quite close to this statement, however, is indeed true; namely, the induced map between the spaces of sofic measures on Λ and Ω is surjective (note that this map is well-defined, as applying the forgetful map f to a sofic approximation of a measure $\mu \in \mathcal{S}(\Lambda)$ yields a sofic approximation of the measure $f_*\mu$).

Proposition 5.4.1. *The induced map $f : \mathcal{S}(\Lambda) \to \mathcal{S}(\Omega)$ is surjective.*

Proof: Let $\mu \in \mathcal{S}(\Omega)$ be a sofic measure and $\{\Gamma_i\}_{i \in \mathbb{N}}$ a sofic approximation of μ consisting of $2n$-regular graphs. By Theorem 5.2.4, each Γ_i may be endowed with a Schreier structure Σ_i. We thus obtain a sequence of measures $\nu_i \in \mathcal{S}(\Lambda)$, namely those arising from the graphs (Γ_i, Σ_i). By compactness, this sequence has a convergent subsequence whose limit measure ν is obviously sofic and, moreover, must map to μ under f. ∎

A further natural question of interest is to describe the fiber of invariant measures $f^{-1}(\mu)$ over a given unimodular measure $\mu \in \mathcal{U}(\Omega)$. Although we are unable to answer this question in full generality, we are able to show that, under mild assumptions, this fiber is very large, in that it contains uncountably many ergodic measures. Invariant Schreier structures, in other words, are not trivial decorations but themselves possess a rich structure. The aforementioned mild assumption is rigidity. To be more precise, a graph is said to be *rigid* if its automorphism group is trivial, and we require that our unimodular measure μ be supported on rigid graphs. Such an assumption is not very restrictive and, indeed, even natural, as essentially all known examples of invariant measures on the space of rooted graphs (such as random *Galton-Watson trees*—see [51]—or their horospheric products [45]) are supported on rigid graphs.

In proving the following results, we will understand an *a_i-cycle* to be any graph obtained by choosing a vertex x in a Schreier graph and, with x as our starting point, following the generator a_i in both directions as far as one can go. An a_i-cycle is thus always isomorphic to the Cayley graph of a cyclic group with generating set $\mathcal{A} = \{a_i\}$. A fundamental operation on a_i-cycles for us will be *reversal*; that is, given an a_i-cycle, one may always reverse its orientation by applying the formal inversion $a_i \mapsto a_i^{-1}$ to its labels. Note that this operation does not destroy the Schreier structure of a graph (although it may well yield a new Schreier structure). We first establish a lemma.

Lemma 5.4.2. *Let Γ be a Schreier graph whose underlying unlabeled graph is rigid, and let a_i be a fixed generator of $\mathbb{F}_n$. Then the space X of Schreier graphs obtained by independently reversing the orientations of a_i-cycles in Γ or keeping their orientations fixed is either finite or uncountable.*

Proof: Let $\{C_j\}_{j\in J}$, where $J \subseteq \mathbb{N}$, be an enumeration of the a_i-cycles in Γ, and consider the space $\{0,1\}^J$. For each $\omega = (\omega_j)_{j\in J} \in \{0,1\}^J$, denote by Γ_ω the Schreier graph obtained from Γ by fixing the orientation of the a_i-cycle C_j if $\omega_j = 0$ and reversing it if $\omega_j = 1$. The space X is in one-to-one correspondence with $\{0,1\}^J$:

on the one hand, each $\Gamma \in X$ can be realized as some Γ_ω (by recording the orientation of each of its a_i-cycles), and if Γ and Δ are distinct elements of X, then clearly $\Gamma_\omega \neq \Delta_{\omega'}$. Conversely, if $\omega \neq \omega'$, then $\Gamma_\omega \neq \Gamma_{\omega'}$. Indeed, let $j \in J$ be an index for which $\omega_j \neq \omega'_j$. Then if Γ_ω and $\Gamma_{\omega'}$ have isomorphic Schreier structures, there must exist a nontrivial automorphism $\phi : \Gamma \to \Gamma$ of the underlying unlabeled graph (as the identity map preserves the orientation of C_j), which contradicts the fact that our Schreier graph is rigid. We thus find that X is finite if and only if J is finite and uncountable otherwise. ∎

Theorem 5.4.3. *Let $\mu \in \mathcal{U}(\Omega)$ be a nonatomic ergodic measure supported on rigid graphs. Then provided it is nonempty, the fiber $f^{-1}(\mu)$ of invariant measures over μ contains uncountably many ergodic measures.*

Proof: Let $\nu \in f^{-1}(\mu)$ be a lift of μ to a (necessarily nonatomic) invariant measure on Λ; assume, moreover, that ν is also ergodic. Put $X := \operatorname{supp}(\nu)$, and let $p \in (0,1)$ be a fixed probability. By the pigeonhole principle, there must exist a generator a_i of $\mathbb{F}_n$ such that a ν-random Schreier graph Γ contains infinitely many a_i-cycles with positive probability, since otherwise Γ would be finite almost surely and ν would be an atomic measure. By ergodicity, it must in fact be the case that almost every $\Gamma \in X$ contains infinitely many a_i-cycles. For each Schreier graph $\Gamma \in X$, denote by $\nu_{\Gamma,p}$ the Bernoulli measure over $f(\Gamma)$—the underlying unlabeled graph—obtained by independently reversing the orientation of each a_i-cycle (for our chosen index i) of Γ with probability p. By Lemma 5.4.2, these measures are nonatomic. Denote by ν_p the measure obtained by integrating the measures $\nu_{\Gamma,p}$ against the base measure ν.

The measure ν_p can be described explicitly as follows. Let $U \in \Lambda_r$ be a cylinder set for which $\nu(U) > 0$. The graph U has an obvious "cycle decomposition," namely the 2-factorization that comes from its Schreier structure; independently reversing (with probability p) the orientations of the a_i-cycles in this factorization yields a (conditional) Bernoulli measure on the set of neighborhoods $U_1, \ldots, U_k$ with the same cycle decomposition as U. Since reversing the orientation of a cycle in U may yield a neighborhood isomorphic to U, we must quotient isomorphic neighborhoods $U_i \cong U_j$. Doing this for all $U \in \Lambda_r$ determines the measures that ν_p assigns to cylinder sets and also makes plain that, if $p \neq q$, then $\nu_p \neq \nu_q$.

It is not difficult to see that ν_p is invariant; indeed, passing to the space $\widetilde{\Lambda}^1$ of doubly rooted graphs, it is obvious that, for a given doubly rooted neighborhood $(U, x, y) \in \widetilde{\Lambda}^1_r$, we have $\nu_p(U, x, y) = \nu_p(U, y, x)$, since the cycle decomposition of a neighborhood is independent of a choice of basepoint(s). Moreover, the measure ν_p is ergodic: Put $\widetilde{X} := \mathrm{supp}(\nu_p)$ and denote by $\pi : \widetilde{X} \to X$ the obvious projection of $\widetilde{X}$ onto X, and suppose that $A \subset \widetilde{X}$ is a nontrivial invariant set. Assume for the moment that A is a union of cylinder sets. Then there exists a cylinder set $U \subset \widetilde{X}$ such that $A \cap U = \emptyset$, and by ergodicity of the measure ν, for every $\Gamma = (\Gamma, H) \in A$ there exist infinitely many $g \in \mathbb{F}_n$ (corresponding to infinitely many distinct positions of the root of Γ) such that $(\Gamma, gHg^{-1}) \in \pi(U)$. On the other hand, the set of Schreier graphs (Γ, H) such that $(\Gamma, gHg^{-1}) \notin U$ for all $g \in \mathbb{F}_n$ is a null set with respect to any conditional measure $\nu_{\Gamma,p}$ and hence a null set with respect to ν_p. It follows that $\nu_p(A) = 0$, a contradiction. Since A can be approximated to arbitrary accuracy by unions of cylinder sets (i.e. for any $\varepsilon > 0$, there exists a union of cylinder sets A_ε with $\nu_p(A \triangle A_\varepsilon) < \varepsilon$), we find that A must be trivial. ∎

To conclude, let us consider highly nonrigid graphs as well. Any unlabeled Cayley graph obviously supports an invariant Schreier structure, namely the one which comes from the associated group, but it is interesting to ask about the existence of other invariant measures. The following theorem shows that in the case when ν is the Dirac measure concentrated on an unlabeled Cayley graph, it can very often be lifted to a nonatomic measure in $\mathcal{I}(\Lambda) = \mathcal{U}(\Lambda)$.

Theorem 5.4.4. *Let G be an infinite noncyclic group, and suppose $\mathcal{A} = \{a_1, \ldots, a_n\}$ is a generating set for G such that none of the elements $a_i a_j \in G$, for distinct indices $1 \leqslant i, j \leqslant n$, is of order two. Then there exists a nonatomic measure $\mu \in \mathcal{I}(\Lambda)$ such that $f_* \mu = \delta_G$, where δ_G is the Dirac measure concentrated on an unlabeled Cayley graph of G.*

Proof: Assume, without loss of generality, that $\mathcal{A}$ does not contain the identity, and let $a_i \in \mathcal{A}$ be a generator such that G (which we think of as the Cayley graph determined by $\mathcal{A}$) contains infinitely many a_i-cycles. Since G is infinite and noncyclic, such an a_i must exist. Now let $a_j \in \mathcal{A}$ be a generator distinct from a_i (such a generator must again exist, since otherwise $G \cong \mathbb{Z}$), and put $a_{n+1} := a_i a_j$. Let $\mathcal{A}_0 = \mathcal{A} \cup \{a_{n+1}\}$, and let G_0 be the Cayley graph of G determined by our new generating set.

Consider now the space $X \subset \Lambda(\mathbb{F}_{n+1})$ obtained from G_0 by independently reversing the orientation of each a_{n+1}-cycle contained in G_0 or leaving it the same. We claim that the space X is uncountable: Let $\Gamma, \Gamma' \in X$ be two relabelings of G_0 such that Γ keeps the orientation of a particular a_{n+1}-cycle C the same whereas Γ' reverses it.

Next, choose a vertex x in $C \subset \Gamma$, and let y denote the vertex reached upon traversing the outgoing edge labeled with a_{n+1} attached to x. Let $\gamma, \gamma' : [0, r] \to G$ be geodesics in G (and not in G_0) that connect the origin to x and to y, respectively (note that γ and γ' may be empty), and denote by H and H' the subgroups corresponding to the graphs Γ and Γ', respectively. Then

$$w(\gamma) a_{n+1} w(\gamma')^{-1} =: h \in H,$$

where $w(\gamma)$ and $w(\gamma')$ are the words read upon traversing γ and γ'. But it is not difficult to see that $h \in H'$ if and only if $a_{n+1} = a_{n+1}^{-1}$, i.e. if and only if a_{n+1} has order two, a contradiction (see Figure 5.1). It follows that if $\Gamma, \Gamma' \in X$ assign different orientations to a particular a_{n+1}-cycle, then they are not equal. On the other hand, the number of ways to assign orientations to the a_{n+1}-cycles in G_0 is clearly uncountable.

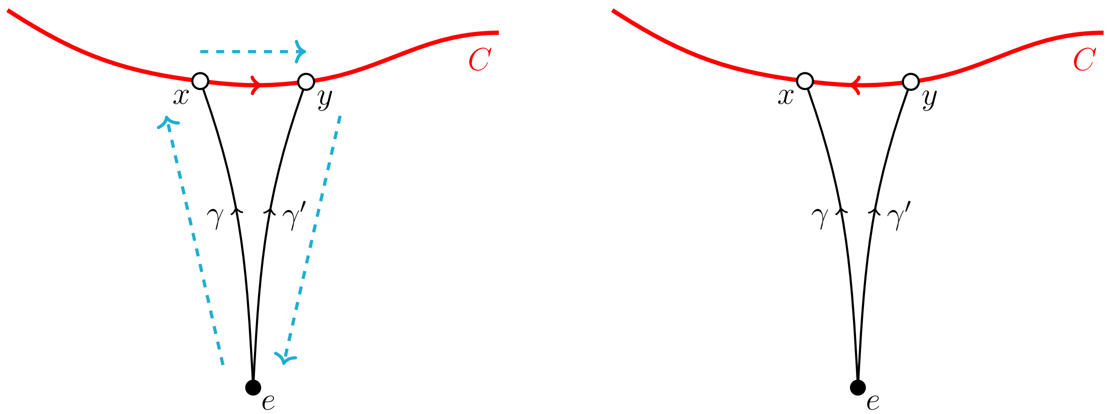

Figure 5.1: If two elements of X assign different orientations to a particular a_{n+1}-cycle C in G_0, then they must represent distinct subgroups of $\mathbb{F}_{n+1}$, as the word read upon traversing the path γ, then following the outgoing edge labeled with a_{n+1}, and then traversing the inverse of γ' (left) cannot belong to both subgroups unless a_{n+1} has order two.

Therefore, X is uncountable.

By choosing to reverse the orientations of a_{n+1}-cycles independently of one another with a fixed probability $p \in (0, 1)$, we obtain a measure μ whose support is X and which, in light of the fact that X is uncountable, is nonatomic. The measure μ is ergodic by the same argument given in Theorem 5.4.3. ∎

Remark 5.4.5. *Theorem 5.4.4 certainly applies to a large class of groups. Even so, the conditions of the theorem can be weakened. Indeed, the theorem holds whenever G has a Cayley graph that contains infinitely many a_i-cycles (for some i) such that its fundamental group changes upon reversing the orientation of one (and hence any) such cycle. On the other hand, note that one cannot in general insist on a minimal generating set. This is impossible, for example, when G is a free product of cyclic groups.*

We conclude this chapter with a few open problems. One is the following.

Question 5.4.6. *Is the map $f : \mathcal{U}(\Lambda) \to \mathcal{U}(\Omega)$ surjective, i.e. is it always possible to lift a unimodular measure on Ω to a unimodular measure on Λ?*

Here we suspect that the answer is positive and, in particular, that the assumption of soficity, though useful (recall that the map $f : \mathcal{S}(\Lambda) \to \mathcal{S}(\Omega)$ is indeed surjective), is not necessary. One way to try to attack this question is via a recent result of Csóka and Lippner [20], who, under mild assumptions, construct invariant random perfect matchings in Cayley graphs. Since "splicing together" two edge-disjoint perfect matchings yields a 2-factor, there is hope that an inductive procedure consisting of repeatedly extracting perfect matchings may yield an invariant 2-factorization, whence passing to an invariant Schreier structure may be quite easy. We thank Lewis Bowen for bringing this point to our attention.

Another natural problem is to explicitly describe the fiber of invariant Schreier structures over a given unimodular random graph. This problem would seem to be very difficult in general, but it would be nice to have answers for certain concrete and nontrivial cases. Consider, for example, the following problem.

Question 5.4.7. *Describe the invariant Schreier structures on $\mathbb{Z}^2$, the standard two-dimensional lattice.*

It is not difficult to see that there exists a large number of invariant Schreier structures on $\mathbb{Z}^2$. Consider, for instance, the random Schreier structures one obtains by taking the standard Cayley structure on $\mathbb{Z}^2$ and randomly reversing the orientations of a_i-chains (that is, horizontal or vertical copies of $\mathbb{Z}$). Yet there are doubtless many more invariant Schreier structures, e.g. ones where a_i-cycles consist of "infinite staircases," or finite-length cycles. It would be nice to have a full description of the geometric possibilities. Here is an even simpler question to which we do not know the answer:

Question 5.4.8. *Describe the periodic Schreier structures on $\mathbb{Z}^2$.*

By a periodic Schreier structure, we mean one whose orbit under the action of the free group is finite. It would of course be interesting to consider other Cayley graphs as well.

Chapter 6

Schreier structures in the absence of a probability measure

In this final chapter, we move away from randomness and attempt to consider Schreier structures on their own, in the absence of a probability measure. We must confess that the considerations of this chapter do not lead to particularly clear-cut results or, indeed, to anything that might be regarded as a considerable advance. Nevertheless, we hope that some of ideas sketched here may offer new insights into how one might proceed in better understanding Schreier structures and their behavior.

In the first section, we toy with the idea of a *sheaf of Schreier structures*. Although the assignment of Schreier structures to the subgraphs of a given graph is not a sheaf in the classical (topological) sense, we show that it can nevertheless be regarded as a sheaf by passing to the notion of a *Grothendieck topology*. This in turn leads us to consider a notion of dimension for arbitrary metric spaces which is a quasi-isometric invariant and which may be a rediscovery of Gromov's *asymptotic dimension*, introduced in [34]. Finally, we consider the relationship between the space of Schreier structures over a given graph and the covering maps which go into and out of that graph, proving some basic observations.

6.1 The sheaf of Schreier structures

We hold the following (admittedly somewhat vague) question to be of fundamental interest:

Question 6.1.1. *Let Γ be a $2n$-regular graph. Is it possible to attain a description of the space of Schreier structures $\mathrm{Sch}(\Gamma)$ over Γ?*

One way to try to attack this question is to take a cue from topology and ask whether it is possible to talk of the *sheaf of Schreier structures* on a graph Γ. The idea would thus be the following: regard Γ as a topological space (e.g. a metric space) and consider the functor that assigns to any open set (subgraph) $U \subseteq \Gamma$ the set of Schreier structures over U. Recall that a *presheaf* of sets on a topological space X is a contravariant functor $\mathcal{F} : \mathcal{O}(X) \to \mathbf{S}$ from the category of open subsets of X—which we denote by $\mathcal{O}(X)$, hopefully without creating any confusion with orbits of a group action, and where the morphisms are inclusions—to the category of sets, denoted $\mathbf{S}$. The image of an inclusion $U \hookrightarrow V$ under $\mathcal{F}$ is a *restriction map* $\mathrm{res}_{V,U} : \mathcal{F}(V) \to \mathcal{F}(U)$, which serves to restrict a given object in $\mathcal{F}(V)$ (e.g. a continuous function) to the subset U. A presheaf is a *sheaf* if it satisfies the following two conditions:

i. If $\{U_i\}_{i\in I}$ is an open cover of an open set $U \subseteq X$, and if $f, g \in \mathcal{F}(U)$ have the property that their restrictions to U_i agree for all $i \in I$, then $f = g$.

ii. If $\{U_i\}_{i\in I}$ is an open cover of an open set $U \subseteq X$, and if for each $i \in I$ there exists an object $f_i \in \mathcal{F}(U_i)$ such that for all pairs $(i, j) \in I \times I$, the restrictions of f_i and f_j agree on the intersection $U_i \cap U_j$, then there exists an object $f \in \mathcal{F}(U)$ such that its restriction to U_i equals f_i for all $i \in I$.

Property ii. is the *gluing axiom*, which asserts that as long as they agree on the overlaps, objects defined over the elements of an open cover can be glued together into a global object. By Property i., this global object is unique.

But a problem quickly becomes apparent when one tries to make the assignment of Schreier structures to subgraphs a sheaf. It manifests itself, for instance, when one places the natural metric topology on a graph Γ. This is of course the discrete topology, so that every subset is open. To see what goes wrong, consider the segment graph of length two consisting of three vertices x, y, and z and two edges (x, y) and (y, z). The edges (x, y) and (y, z) comprise an open cover of the whole graph, and their intersection is the edgeless graph consisting just of the vertex y. Now assign a Schreier structure to (x, y) by directing it from x to y and labeling it with a generator a, and assign a Schreier structure to (y, z) by directing it from z to y and also labeling it with a. These structures vacuously agree on the overlap $y = (x, y) \cap (y, z)$, but they certainly don't determine a global Schreier structure, as both of our edges are directed towards y and bear the same label. In other words, the gluing axiom is violated: the assignment of Schreier structures to subgraphs, though a presheaf, is not a sheaf. Things are no better if one instead endows a graph with the topology of a simplicial 1-complex.

The difficulty, in a nutshell, is that open sets need not overlap in a sufficiently nice way. To remedy this situation, we therefore redefine what it means for a family of subsets of a graph to serve as an open cover. Our starting point is to consider only sets which are *r-neighborhoods*, i.e. sets $U \subseteq \Gamma$ of the form

$$U = \{x \in \Gamma \mid \rho(K, x) \leqslant r\},$$

where K is an arbitrary subset of Γ and ρ is the standard graph metric. Let us say that a subset $K \subseteq U$ of an r-neighborhood $U \subseteq \Gamma$ in an ambient graph Γ is an *r-basis* of U if the r-neighborhood of K (taken inside of Γ) is equal to U. Our definition of covers now goes as follows.

Definition 6.1.2. (Cover) Let Γ be a graph equipped with the usual graph metric. We say that a system of r-neighborhoods $\{U_i\}_{i \in I}$ is a *cover* of Γ if there exist bases $K_i \subseteq U_i$ such that

$$\bigcup_{i \in I} K_i = \Gamma,$$

i.e. that cover Γ in the usual sense.

Definition 6.1.2, as we shall soon see, does what we need—the open sets of covers as we have defined them must overlap in a nice way, so that the assignment of Schreier structures to sets really is a sheaf. There is, however, a price to pay: the collection of r-neighborhoods of a graph is by no means a topology (it is not closed under intersections). This issue can be dealt with by passing to the more general notion of a *Grothendieck topology*, which we will discuss now. Grothendieck topologies originated from algebraic geometry, via the consideration of certain maps which resembled fiber bundles, as well as in making precise the duality between the Galois group of a field extension and the fundamental group of a topological space (see [52], to which we also refer the reader for a full treatment). Broadly speaking, the idea behind Grothendieck topologies is to generalize the notion of a topology by abstracting the notion of an open cover. One thus begins with a category $\mathbf{C}$ (in the classical situation, this is the category of open subsets of a given topological space) and, for each object U of $\mathbf{C}$, specifies what it means for a collection of morphisms with codomain U to serve as a cover of U (in the classicial situation, a collection of morphisms $\{U_i \hookrightarrow U\}_{i\in I}$ is a cover precisely when $\bigcup_{i\in I} U_i = U$). Such a specification is a Grothendieck topology on $\mathbf{C}$ and in turn allows one to define a general notion of a sheaf.

Given an object U in a category $\mathbf{C}$, recall that $\mathrm{Hom}(\cdot, U)$ is the contravariant functor that maps each object V of $\mathbf{C}$ to the set of morphisms $f : V \to U$ and that maps each morphism $f : V \to W$ to the function given by $g \mapsto g \circ f$. A *sieve* $\mathcal{S}$ on an object U is a subfunctor of $\mathrm{Hom}(\cdot, U)$, i.e. a functor from $\mathbf{C}$ to the category of sets such that $\mathcal{S}(V) \subseteq \mathrm{Hom}(V, U)$ for all objects V and such that $\mathcal{S}(f : V \to W)$ is the restriction of the function $g \mapsto g \circ f$ to $\mathcal{S}(V)$. The intuition here is that, under certain conditions, a sieve will specify a *cover* of a given object. A basic operation on sieves is the pullback. Given a morphism $f : V \to W$ and a sieve $\mathcal{S}$ on W, the pullback of $\mathcal{S}$ under f, denoted $f^*\mathcal{S}$, is the sieve on V with the following property: for any object U, we have $f^*\mathcal{S}(U) = \{g : U \to V \mid f \circ g \in \mathcal{S}(U)\}$.

Definition 6.1.3. (Grothendieck topology) A *Grothendieck topology* J on a category $\mathbf{C}$ is a collection $J(U)$ of sieves, called *covering sieves*, for each object U of $\mathbf{C}$ such that the following conditions are met.

i. If $\mathcal{S}$ is a covering sieve of U and $f : V \to U$ is a morphism, then $f^*\mathcal{S}$ is a covering sieve of V.

ii. If $\mathcal{S}$ is a covering sieve of U and $\mathcal{T}$ is any sieve on U, and for every object V of $\mathbf{C}$ and every morphism $f : V \to U$ in $\mathcal{S}(V)$ the pullback $f^*\mathcal{T}$ is a covering sieve of V, then $\mathcal{T}$ is a covering sieve of U.

iii. For any object U of $\mathbf{C}$, $\mathrm{Hom}(\cdot, U)$ is a covering sieve of U.

A pairing $(\mathbf{C}, J)$ of a small category and a Grothendieck topology is called a *site*.

In the general context of Grothendieck topologies, it also becomes possible to define a general notion of a sheaf. The definition goes as follows (see also [52]).

Definition 6.1.4. (Sheaf on a Grothendieck topology) Let $(\mathbf{C}, J)$ be a site, and suppose that the category $\mathbf{C}$ has fibered products. A *presheaf* on $\mathbf{C}$ is a contravariant functor $\mathcal{F} : \mathbf{C} \to \mathbf{S}$ from $\mathbf{C}$ to the category of sets. A presheaf $\mathcal{F}$ is a *sheaf* if for any object U in $\mathbf{C}$ and any covering sieve $\mathcal{S} \in J(U)$ given by $\mathcal{S}(U) = \{f_i : U_i \to U\}_{i \in I}$, the diagram

$$\mathcal{F}(U) \longrightarrow \prod_{i \in I} \mathcal{F}(U_i) \rightrightarrows \prod_{i,j \in I} \mathcal{F}(U_i \times_U U_j) \tag{6.1.1}$$

is an equalizer. Here the morphism on the left sends objects over U to the Cartesian product of their restrictions to the U_i. The morphisms on the right map to the Cartesian product of restrictions to all pairwise fibered products, where one map works by assigning to $U_i \times_U U_j$ the objects obtained by restricting U_i and the other by restricting U_j.

Remark 6.1.5. *Recall that in a diagram of the form*

$$E \xrightarrow{\mathrm{Eq}} X \underset{g}{\overset{f}{\rightrightarrows}} Y,$$

the morphism Eq *is the equalizer of f and g if it satisfies $f \circ \mathrm{Eq} = g \circ \mathrm{Eq}$ and is the universal morphism with this property. The example to have in mind is the case when f and g are functions; then E can be taken to be the subset of X over which f and g agree and* Eq *the inclusion map.*

Now let Γ be a graph. We define a category $\mathbf{C}(\Gamma)$ associated to Γ whose objects are the 1-neighborhoods in Γ and whose morphisms are inclusions. We define a Grothendieck topology J_1 on $\mathbf{C}$ as follows: if U is an object of $\mathbf{C}(\Gamma)$, then $J_1(U)$, the collection of covering sieves of U, is the collection of all sets of inclusions $\{U_i \hookrightarrow U\}_{i\in I}$ with the property that each U_i is a 2-neighborhood and the system $\{U_i\}_{i\in I}$ covers U in the sense of Definition 6.1.2 (that is, with $r = 2$). Before proceeding, let us pause to describe what fibered products look like in the category $\mathbf{C}(\Gamma)$.

Lemma 6.1.6. (Fibered products in $\mathbf{C}(\Gamma)$) *Let $U \hookrightarrow W$ and $V \hookrightarrow W$ be two morphisms in $\mathbf{C}(\Gamma)$ with a common codomain, and let $X_{\max}$ and $Y_{\max}$ be maximal bases of U and V, respectively. Then*

$$U \times_W V = U_1(X_{\max} \cap Y_{\max}).$$

That is, the fibered product of the inclusions is the 1-neighborhood of the intersection of the maximal bases of U and V.

Proof: The 1-neighborhood $U(X_{\max} \cap Y_{\max})$ is obviously a sub-neighborhood of both U and V. It therefore follows that the compositions of its inclusions into U and V with the inclusions of U and V into W are equal. To see that $U_1(X_{\max} \cap Y_{\max})$ is universal, let U' be another 1-neighborhood that maps into both U and V. By virtue of mapping into U, the maximal basis of U' is a subset of $X_{\max}$, and by virtue of mapping into V, it is a subset of $Y_{\max}$. Its intersection with $X_{\max} \cap Y_{\max}$ therefore determines an inclusion $U' \hookrightarrow U_1(X_{\max} \cap Y_{\max})$, which is obviously unique. ∎

We are now ready to show that, armed with our Grothendieck topology, the functor that assigns to a given 1-neighborhood U the set of Schreier structures over U is indeed a sheaf.

Theorem 6.1.7. *Let Γ be a graph and $(\mathbf{C}(\Gamma), J)$ its associated site. Then the contravariant functor* $\mathrm{Sch} : \mathbf{C}(\Gamma) \to \mathbf{S}$ *that assigns to a 1-neighborhood U in $\mathbf{C}(\Gamma)$ the set of Schreier structures* $\mathrm{Sch}(U)$ *over U is a sheaf.*

Proof: It is obvious that the functor Sch is a presheaf. Suppose now that U is an object (that is, a 1-neighborhood) in $\mathbf{C}(\Gamma)$ and $\mathcal{S} \in J(U)$ a covering sieve consisting of inclusions $\{U_i \hookrightarrow U\}_{i\in I}$. To see that Sch is a sheaf, it is actually enough to verify the classical sheaf axioms. Thus, suppose $\Sigma, \Sigma' \in \mathrm{Sch}(U)$ are two Schreier structures on U whose restrictions to U_i agree for all $i \in I$, and suppose that $\Sigma \neq \Sigma'$. Then there exists an edge $(x, y) \subseteq U$ which is given different labels by Σ and Σ'. But by the definition of our covering sieves, the point x belongs to the basis of some U_i; since U_i is a neighborhood of this basis, it must also contain the point y and hence the entire edge (x, y), a contradiction.

To verify the second sheaf axiom, suppose that $\{\Sigma_i \in \mathrm{Sch}(U_i)\}_{i\in I}$ is a system of Schreier structures with the property that, for all $i, j \in I$, the restrictions of Σ_i and Σ_j to $U_i \times_U U_j$ agree. Suppose next that $x \in U$ is an arbitrary point. If $\mathrm{star}(x)$, the set of edges incident with x, is contained in the 1-basis of some U_i, then the Schreier structure of $\mathrm{star}(x)$ inherited from Σ_i passes unproblematically to the full neighborhood U. The other possibility is that x lies on the boundary of the 1-basis K_i of some U_i, so that $\mathrm{star}(x)$ contains a point y which is not in K_i. But by our construction, the point y then is contained in the 1-basis K_j of some U_j. Accordingly, the edge (x, y) belongs to the fibered product $U_i \times_U U_j$ (recall that, by Lemma 6.1.6, this is a 1-neighborhood) and therefore inherits a well-defined Schreier structure which again passes unproblematically to U. ∎

Theorem 6.1.7 is a bit of abstract nonsense that tells us that the functor that assigns Schreier structures to subgraphs of a given graph can indeed be regarded as a sheaf, provided the subgraph in question is sufficiently nice (that is, provided it is an r-neighborhood), and provided one defines an appropriate Grothendieck topology associated to Γ. Note also that we could have consolodated the space Ω of all graphs (or any subspace of interest thereof) into a single category consisting of all possible 1-neighborhoods, and thus spoken of Schreier structures over a large class of graphs, rather than a single graph. Furthermore, note that there is nothing intrinsically special about 1-neighborhoods, other than the fact that they are the smallest neighborhoods that do the job. That is to say, we could have defined everything in terms of larger neighborhoods and achieved the same result. We will return to this observation in a moment.

In sheaf-theoretic language, Question 6.1.1 may be reformulated as follows: *Can one describe the global sections of the sheaf* Sch*?* We must confess that we have not used the point of view developed in this section to make significant headway in answering this question. Our goal was rather to make precise a certain observation, and also to make a connection between Schreier structures and sheaves on a site. To our knowledge, this connection has not been considered before. There are a doubtless a number of questions one may ask here, e.g. how our particular sheaf behaves inside the *topos* of all sheaves on the site $(\mathbf{C}(\Gamma), J_1)$. We hope that some of them may prove interesting.

There is another silver lining. It is natural to ask how the covers which we have considered in this section behave with respect to different underlying graphs, and moreover to ask what happens if general r-neighborhoods, rather than 1-neighborhoods, are used to construct covers. As we show in the next section, it makes sense to ask these questions in the context of arbitrary metric spaces, whence it is possible to define a notion of dimension for metric spaces which is a quasi-isometric invariant.

6.2 The upper covering dimension

Let (X, ρ), or just X for short, be a metric space. The *topological* or *covering dimension* (calculated with respect to the topology induced by ρ) is among the most basic dimensions that one might assign to X. It was discovered by Lebesgue [47] in 1911 and serves as an important topological invariant. The covering dimension does not, however, say much about the metric properties of a space: consider the fact that the unit interval and the real line have the same covering dimension, despite the fact that the diameter of the former space is finite and the diameter of the latter space is infinite. And although the covering dimension is obviously an isometric invariant, it fails, for instance, to be a quasi-isometric invariant: the spaces $\mathbb{R}$ and $\mathbb{Z}$, to take just one example, are quasi-isometric but have different covering dimensions. Our goal in this section is to develop a notion of dimension for metric spaces which is inspired by the covering dimension and based on the ideas of the previous section. The idea is to cover a metric space X in a certain way with increasingly fatter sets, all the while—just as with the covering dimension—minimizing the largest number of sets to which points in X may belong. A key feature of this dimension is that it is indeed a quasi-isometric invariant, a fact which has immediate and interesting consequences. In particular, it becomes possible to talk about the dimension of a finitely generated group G, given that any two Cayley graphs of G constructed with respect to finite generating sets are quasi-isometric.

It must be pointed out that our notion of dimension, which we call the *upper covering dimension*, may coincide with Gromov's *asymptotic dimension* introduced in [34]. It too is a quasi-isometric invariant and has generated a significant amount of research (see, for instance, the survey of Bell and Dranishnikov [7]), in particular serving as a vital tool in the work of Yu on the Novikov conjecture [70]. This arguably takes some of the air out of our results (we were unaware of Gromov's asymptotic dimension when the work in this section was first completed), but we record them here nonetheless.

Recall that a *hypergraph* $\mathcal{H}$ on a set X is a collection of subsets of X. Given a set with a hypergraph structure $\mathcal{H}$ and a point $x \in X$, the *degree* of x, denoted $\deg_{\mathcal{H}}(x)$, is the number of hyperedges (i.e. sets) in $\mathcal{H}$ to which x belongs. The classical *covering dimension* of a topological space X is the smallest number d such that, given any open cover $\mathcal{U}$ of X, there exists a refinement $\mathcal{U}'$ of $\mathcal{U}$ with

$$\max_{x \in X} \deg_{\mathcal{U}'}(x) \leqslant d + 1.$$

(A refinement of an open cover $\mathcal{U}$ is an open cover each of whose elements is contained in an element of $\mathcal{U}$.) The topological dimension of $\mathbb{R}$ is thus one, as it is always possible to refine an open cover until no point $x \in \mathbb{R}$ belongs to more than two open sets (and although there exists an open cover such that every point belongs to only one open set—just take the real line itself—one obviously cannot do any better in general). Likewise, the topological dimension of $\mathbb{R}^d$ is d, the topological dimension of a d-dimensional manifold is d, and any discrete space is zero-dimensional.

Our idea is to define a notion of dimension for a metric space (X, ρ) by mimicking the idea behind the covering dimension. To do so, we will use the metric ρ to strengthen what it means for a system of subsets to cover X, just as we did in the previous section. The fundamental subsets of X that interest us will be its *r-neighborhoods*. That is, if $r \geqslant 0$ is a real number and K is any subset of X, then the associated r-neighborhood is the set

$$U_r(K) = \{x \in X \mid \rho(x, K) \leqslant r\}.$$

As before, we call the set K a *basis* of the r-neighborhood $U_r(K)$. Note that every r-neighborhood has a unique maximal basis: the r-neighborhood of a given point in an r-neighborhood U is either an r-neighborhood in X or it is not; the union of those vertices whose r-neighborhoods in U are r-neighborhoods in X is a maximal basis. We now define covers of X as follows.

Definition 6.2.1. *(Covers) By a* cover *of a metric space X, we mean a collection of subsets $\mathcal{U} = \{U_i\}_{i \in I}$ of X each of which is an r-neighborhood and which have bases that cover X in the usual sense, i.e.*

$$X = \bigcup_{i \in I} K_i,$$

where K_i is a basis of the r-neighborhood U_i. We denote the collection of covers of X by r-neighborhoods (for a fixed $r \geqslant 0$) by $J_r(X)$.

A crucial feature of covers $\mathcal{U} \in J_r(X)$ is that they often force sets to overlap when they might otherwise be disjoint. It is impossible, for example, to cover $\mathbb{Z}$ with finite r-neighborhoods, where $r > 1$, in such a way that every point belongs to only one r-neighborhood. Note also that a system of r-neighborhoods $\mathcal{U}$ of X is a cover if and only if the union of the maximal bases of the neighborhoods $U \in \mathcal{U}$ is equal to X.

We now wish to mimic the covering dimension by choosing covers $\mathcal{U} \in J_r(X)$ that minimize the largest number of r-neighborhoods to which points in X may belong. It would be cheating, however, to allow the neighborhoods in our covers to grow too large (the space X itself, for example, is always an r-neighborhood and hence a cover on its own). Given a cover $\mathcal{U} \in J_r(X)$, define its *norm* to be

$$\|\mathcal{U}\| := \sup_{U \in \mathcal{U}} \operatorname{diam}(U).$$

The key will be to work with covers of finite norm. Thus, we put

$$\dim_r(X) := \min_{\|\mathcal{U}\| < \infty} \left(\max_{x \in X} \deg_{\mathcal{U}}(x) - 1 \right), \tag{6.2.1}$$

where $\mathcal{U}$ ranges over all finite-norm covers in $J_r(X)$. And since it is always possible, for instance, to rescale a metric, it makes sense to pass to the extremal case rather than focus on a particular r.

Definition 6.2.2. (Upper covering dimension) Given a metric space X, we define its *upper covering dimension* to be

$$\dim(X) = \lim_{r \to \infty} \dim_r(X), \tag{6.2.2}$$

where $\dim_r(X)$ is given by (6.2.1).

Remark 6.2.3. *Note that the limit (6.2.2) exists, since* $\dim_r(X)$ *is nondecreasing.*

From now on, unless stated otherwise we will always refer to the upper covering dimension of a space when we talk about its dimension.

Recall that a map $f : (X, \rho_X) \to (Y, \rho_Y)$ between metric spaces is a *quasi-isometry* if there exist constants $C_1 \geqslant 1$ and $C_2 \geqslant 0$ such that for all $x, y \in X$,

$$\frac{1}{C_1}\rho_X(x,y) - C_2 \leqslant \rho_Y(f(x), f(y)) \leqslant C_1\rho_X(x,y) + C_2$$

and there exists a constant $C_3 \geqslant 0$ such that the image of f is C_3-dense in Y, in the sense that any point in Y is no farther than a distance C_3 from the image of f. A quasi-isometry is thus an isometry up to uniform perturbations or, to put it more intuitively still, two spaces are quasi-isometric if they look the same from a distance. Moreover, being quasi-isometric is an equivalence relation: every metric space is (trivially) quasi-isometric to itself, if $f : X \to Y$ is a quasi-isometry, then there exists a quasi-isometry $g : Y \to X$, and the composition of quasi-isometries is again a quasi-isometry. Since the upper covering dimension describes the dimension of a space attained upon covering it with increasingly fatter sets and thus, as it were, upon zooming out as far as one can, one might expect it to be a quasi-isometric invariant. As we will presently show, this is indeed the case.

Theorem 6.2.4. *If two metric spaces (X, ρ_X) and (Y, ρ_Y) are quasi-isometric, then $\dim(X) = \dim(Y)$.*

Proof: Let $f : X \to Y$ be a quasi-isometry, and let $\{\mathcal{V}_r\}_{r\geqslant 0}$ be a system of finite-norm covers $\mathcal{V}_r \in J_r(Y)$ such that $\mathcal{V}_r$ realizes $\dim_r(Y)$. Define $q(r)$ to be the supremum of the distances r' such that the preimages of any two points at distance greater than r from one another in Y are at distance greater than r' from one another in X. Note that, since f is a quasi-isometry, $q(r)$ is nondecreasing and

$$\lim_{r\to\infty} q(r) = \infty.$$

Now construct a system of covers $\{\mathcal{U}_{q(r)}\}_{r\geqslant 0}$ on X by pulling back the covers $\mathcal{V}_r$ as follows. For each $V \in \mathcal{V}_r$ whose basis L contains an element from the image of f, let $U \in \mathcal{U}_{q(r)}$ be the $q(r)$-neighborhood of $f^{-1}(L)$ in X, and let $\mathcal{U}_{q(r)}$ be the collection of all such neighborhoods. It is easy to see that each $\mathcal{U}_{q(r)}$ is indeed a cover in $J_{q(r)}(X)$:

every $U \in \mathcal{U}_{q(r)}$ is a $q(r)$-neighborhood, and every point $x \in X$ belongs to the basis of at least one such neighborhood, since the bases of the neighborhoods $V \in \mathcal{V}_r$ cover Y. Moreover, the fact that f is a quasi-isometry implies $\|\mathcal{U}_{q(r)}\| < \infty$.

Let $y \in Y$ be a point in the image of f with $\deg_{\mathcal{V}_r}(y) =: d \in \mathbb{Z}_{\geqslant 0} \cup \{\infty\}$, and let $\{V_i\}_{i \in I}$ be the neighborhoods in $\mathcal{V}_r$ to which y belongs. Given a point $x \in f^{-1}(y)$, the pullbacks of each of the neighborhoods V_i may contribute to $\deg_{\mathcal{U}_{q(r)}}(x)$. Suppose, however, that $\deg_{\mathcal{U}_{q(r)}}(x) > d$. Then there exists a neighborhood $V \in \mathcal{V}_r$ which does not contain y, whose basis L contains an element from the image of f, and whose pullback U contains x. But $\rho_Y(y, L) > r$, which implies that $\rho_X(x, f^{-1}(L)) > q(r)$, a contradiction. It follows that $\deg_{\mathcal{U}_{q(r)}}(x) \leqslant d$ and hence, since y was arbitrary, that for every point $x \in X$, we have

$$\deg_{\mathcal{U}_{q(r)}}(x) \leqslant \deg_{\mathcal{V}_r}(f(x)).$$

We thus find that $\dim(X) \leqslant \dim(Y)$. By applying our argument with respect to a quasi-isometry $g : Y \to X$, we attain the opposite inequality and may conclude that $\dim(X) = \dim(Y)$. ∎

Another feature of the upper covering dimension is that it behaves as one would expect with respect to subspaces.

Proposition 6.2.5. *The upper covering dimension is monotonic, i.e. if $X' \subseteq X$, then* $\dim(X') \leqslant \dim(X)$.

Proof: Any cover $\mathcal{U} = \{U_i\}_{i \in I} \in J_r(X)$ of X naturally restricts to a cover of X' by letting U_i' be the r-neighborhood (in X') of $K_i \cap X'$, where K_i is a basis of U_i. The collection $\mathcal{U}' = \{U_i'\}_{i \in I}$ belongs to $J_r(X')$, and we clearly have $\deg_{\mathcal{U}'}(x) \leqslant \deg_{\mathcal{U}}(x)$ for all $x \in X'$. This implies the claim. ∎

We now pause to give a few examples.

Example 6.2.6. Any metric space of finite diameter is obviously zero-dimensional:

for any $r \geqslant 0$, one may cover the space with a single bounded r-neighborhood by choosing the space itself as a basis. The converse, however, is not true: consider the space $X = \{2^n\}_{n\in\mathbb{N}}$, whose metric is inherited from $\mathbb{R}$. It is not difficult to see that, no matter the size of r, it is always possible to cover X with uniformly bounded r-neighborhoods which do not overlap.

Example 6.2.7. As the previous example already shows, the upper covering dimension is not a complete quasi-isometric invariant: to take another simple example, the spaces $\mathbb{R}_{\geqslant 0}$ and $\mathbb{R}$ are both one-dimensional, but they are not quasi-isometric.

We also show that, as one would expect, the Euclidean spaces satisfy $\dim(\mathbb{R}^d) = d$. Here the argument is somewhat more involved, so that we will present the claim in the form of a proposition.

Proposition 6.2.8. *The upper covering dimension of $\mathbb{R}^d$ is d.*

Proof: It is not difficult to construct an open cover (in the topological sense) $\mathcal{U}$ of $\mathbb{R}^d$ such that it has a positive Lebesgue number, the diameters of its elements are uniformly bounded from above and below, and no point in $\mathbb{R}^d$ belongs to more than $d+1$ sets in $\mathcal{U}$. One can then find an $\varepsilon > 0$ such that each set $U \in \mathcal{U}$ is an ε-neighborhood and $\mathcal{U} \in J_\varepsilon(\mathbb{R}^d)$. By scaling the cover $\mathcal{U}$, one shows that $\dim(\mathbb{R}^d) \leqslant d$.

Conversely, let $\mathcal{V}$ be an open cover (again in the topological sense) of the d-dimensional sphere $\mathbb{S}^d$ (which we equip with its usual metric). By compactness, $\mathcal{V}$ has a positive Lebesgue number $\delta > 0$. Suppose now that $\mathcal{U}$ is an open cover of $\mathbb{R}^d$ which satisfies the following conditions:

i. The diameters of elements of $\mathcal{U}$ are uniformly bounded from above by some constant $C > 0$.

ii. No point in $\mathbb{R}^d$ belongs to more than d' elements of $\mathcal{U}$, where $d' < d+1$.

Denote by $\hat{\mathbb{R}}^d = \mathbb{R}^d \cup \{\infty\}$ the one-point compactification of $\mathbb{R}^d$ and by $\theta : \hat{\mathbb{R}}^d \to \mathbb{S}^d$ the homeomorphism whose restriction to $\mathbb{R}^d$ is (inverse) stereographic projection.

Let $R > 0$ be large enough that the complement of $B_R(0)$, the ball of radius R centered at the origin, is mapped onto a set of diameter less than δ via θ. Next, scale down the cover $\mathcal{U}$ until every element of the scaled cover contained in $B_{R+C}(0)$ maps onto a set of diameter less than δ via θ.

We now define a new open cover $\mathcal{U}'$ of $\mathbb{R}^d$ as follows. Let $\{U_i\}_{i\in I}$ be the collection of elements of $\mathcal{U}$ which nontrivially intersect the complement of $B_{R+C}(0)$, let U_R be the open set which is the union of the U_i, and put

$$\mathcal{U}' := \mathcal{U} \backslash \{U_i\}_{i\in I} \cup \{U_R\}.$$

Note that $\mathcal{U}'$, which we extend to a cover of $\hat{\mathbb{R}}^d$ in the obvious way, still satisfies condition ii. above. The image $\mathcal{V}' := \theta_* \mathcal{U}'$ of $\mathcal{U}'$ under θ is an open cover of $\mathbb{S}^d$ each of whose elements has diameter less than δ and which therefore refines $\mathcal{V}$. But since θ is a homeomorphism, no point of $\mathbb{S}^d$ belongs to more than $d' < d+1$ elements of $\mathcal{V}'$, contradicting the fact that the topological dimension of $\mathbb{S}^d$ is d. This shows that $\dim(\mathbb{R}^d) \geqslant d$, completing the proof. ∎

It is also interesting to apply the ideas considered here to graphs—metric spaces which are ubiquitous in mathematics yet often regarded as zero-dimensional or perhaps one-dimensional objects. In fact, let us immediately consider finitely generated groups and their Cayley graphs. Let G be a finitely generated group. As is well known, the identity map $\mathrm{Id} : G \to G$ induces a quasi-isometry between any two Cayley graphs of G constructed with respect to finite generating sets. Theorem 6.2.4 thus permits us to give the following definition.

Definition 6.2.9. (Dimension of a group) We define the *dimension* of a finitely generated group G, denoted $\dim(G)$, to be the upper covering dimension of any Cayley graph of G constructed with respect to a finite generating set.

Let us again give some basic examples.

Example 6.2.10. Unlike the case for arbitrary metric spaces (see Example 6.2.6), a group is zero-dimensional if and only if it is finite. One direction is obvious. Conversely, suppose G is an infinite zero-dimensional group. Then there exists an isometric embedding $\mathbb{Z}_{\geqslant 0} \hookrightarrow \Gamma$ for any Cayley graph Γ of G (every infinite, connected, and locally finite graph contains a geodesic ray). But this contradicts the monotonicity of the upper covering dimension (Proposition 6.2.5), given that $\mathbb{Z}_{\geqslant 0}$ is one-dimensional.

Example 6.2.11. The free Abelian groups satisfy $\dim(\mathbb{Z}^d) = d$. This follows immediately from Proposition 6.2.8 and the fact that $\mathbb{R}^d$ and $\mathbb{Z}^d$ are quasi-isometric.

Example 6.2.12. Infinite, locally finite trees—in particular the free group $\mathbb{F}_n$—are one-dimensional. This can be seen by splitting a given tree T into annuli of width $r \geqslant 1$ with respect to a fixed root, then dividing the annuli into connected components. It is not difficult to see that these connected components have uniformly bounded diameter and that, upon choosing as a cover the collection of, say, $r/4$-neighborhoods of the connected components, no point in T will belong to more than two sets as $r \to \infty$.

Note that the upper covering dimension is in general not well-behaved under quotient maps. That is, if $\xi = \{\xi_i\}_{i \in I}$ is a partition of a metric space (X, ρ) such that ρ descends to a metric on the quotient space $\overline{X} := X/\xi$ (which implies in particular that $\rho(\xi_i, \xi_j) > 0$ for all $i \neq j$), then it is in general not the case that $\dim(X) \geqslant \dim(\overline{X})$. To see this, consider simply the canonical epimorphism $\phi : \mathbb{F}_n \to \mathbb{Z}^d$, where $d > 1$. The map ϕ determines a quotient map of Cayley graphs, yet the dimension goes up, since $\dim(\mathbb{F}_n) = 1$. It would be interesting to understand the circumstances under which quotient maps do behave as one would expect, i.e. cause a drop in dimension. Let us formulate a question regarding this.

Question 6.2.13. *Let Γ be a graph of bounded geometry and $\xi = \{\xi_i\}_{i \in I}$ a partition of Γ such that each ξ_i is connected. Is it then true that $\dim(\Gamma) \geqslant \dim(\Gamma/\xi)$? Suppose that $\dim(\Gamma) = n$ and $\dim(\xi_i) = k$ for all $i \in I$, and moreover that the ξ_i are mutually quasi-isometric with uniformly bounded constants. Do we then have $\dim(\Gamma/\xi) = n - k$?*

These questions are related to a question recently asked by Benjamini (personal communication) concerning invariant partitions of Cayley graphs.

6.3 Schreier structures and covering maps

This section is a first attempt to explore the connection between Schreier structures on a given $2n$-regular rooted graph Γ and covering maps going into and out of Γ. In what follows, we will slightly alter our definition of a Schreier graph from the one given previously. That is, we will work with *half-loops* rather than loops (half-loops are also considered in [5]). By a half-loop, we mean a cycle of length one which possesses a single orientation, rather than two orientations. A half-loop thus looks just like a loop but only contributes one to the degree of the vertex to which it is attached, as it may only be traversed in one direction. Accordingly, the bouquet of n loops—the Schreier graph of the free group $\mathbb{F}_n$—becomes the bouquet of $2n$ half-loops (to take just one example). Note also that when considering a graph Γ, we will not work with its isomorphism class but with a fixed representative of this class and may therefore regard two Schreier structures on Γ as distinct even if they correspond to the same subgroup of $\mathbb{F}_n$. The $2n$-regular tree thus possesses uncountably many Schreier structures, even though they all correspond to the same subgroup (namely the trivial subgroup). Let us begin with the following simple observation.

Proposition 6.3.1. *Schreier structures are contravariant. That is, if* $p : \widetilde{\Gamma} \to \Gamma$ *is a covering map between* $2n$*-regular graphs and* $\Sigma \in \mathrm{Sch}(\Gamma)$ *is a Schreier structure on* Γ*, then* Σ *naturally lifts to a Schreier structure* $p^*\Sigma \in \mathrm{Sch}(\widetilde{\Gamma})$.

Proof: Let $x \in \Gamma$ be an arbitrary vertex and $\mathrm{star}(x)$ the star of x (that is, the set of edges incident with x). Let $p^{-1}(x) = \{x_i\}_{i\in I}$ be the preimage of x and

$$U_x := \bigsqcup_{i\in I} \mathrm{star}(x_i)$$

the corresponding local trivialization. The restriction of Σ to star(x) obviously pulls back to a Schreier structure on each star(x_i) and therefore to a Schreier structure on U_x. The fact that p is a graph homomorphism guarantees that the Schreier structures on each U_x (as x varies over Γ) paste together to yield a global Schreier structure. ∎

Given $2n$-regular rooted graphs Γ and Δ, denote by $\mathrm{Cov}(\Gamma, \Delta)$ the (possibly empty) set of covering maps $p : \Gamma \to \Delta$ which send the root of Γ to the root of Δ. We have the following corollary.

Corollary 6.3.2. *Let Γ be a $2n$-regular graph and B the bouquet of $2n$ half-loops. Then the space of Schreier structures on Γ may be identified with the set of covering maps $p : \Gamma \to B$, i.e.* $\mathrm{Sch}(\Gamma) \cong \mathrm{Cov}(\Gamma, B)$*.*

Proof: Fix an initial Schreier structure $\Sigma_0 \in \mathrm{Sch}(B)$, i.e. label the $2n$ edges of B with the generators $a_1^{\pm 1}, \dots, a_n^{\pm 1}$ in some way. If $p : \Gamma \to B$ is a covering map, then by Proposition 6.3.1 it lifts to a Schreier structure $p^*\Sigma_0 \in \mathrm{Sch}(\Gamma)$. Conversely, if $\Sigma \in \mathrm{Sch}(\Gamma)$, then one may read off a covering map $p \in \mathrm{Cov}(\Gamma, B)$ by comparing the labelings of Γ and B. That is, the edges incident with the root of Γ must map to the edges of B in a unique way in order to preserve the labeling. The edges incident with the neighbors of the root of Γ must then also map to the edges of B in a unique way, and so on. It is clear that these associations are inverses of one another, which establishes the claim. ∎

There is also a connection from above, namely a connection between $\mathrm{Sch}(\Gamma)$ and covering maps from the $2n$-regular rooted tree T to Γ (recall that T is the universal cover of any $2n$-regular graph). In order to describe this connection, consider the group $\mathrm{Aut}_0(T)$ of all automorphisms of T which fix the root. The group $\mathrm{Aut}_0(T)$ acts on $\mathrm{Cov}(T, \Gamma)$ in a natural way: given an automorphism ϕ and a covering map p, composition with ϕ yields a new covering map $p \circ \phi$. Moreover, this action is free and transitive, as the following proposition shows.

Proposition 6.3.3. *The space* $\mathrm{Cov}(T,\Gamma)$ *is an* $\mathrm{Aut}_0(T)$*-torsor.*

Proof: It is easy to see that the action $\mathrm{Aut}_0(T) \circlearrowright \mathrm{Cov}(T,\Gamma)$ is free. Indeed, given a covering map p, composition with any nontrivial automorphism $\phi \in \mathrm{Aut}_0(T)$ obviously determines a covering map distinct from p. Suppose next that p and q are two covering maps. There is clearly a unique automorphism ϕ_1 of the 1-neighborhood U_1 of the root of T such that the restrictions of $p \circ \phi_1$ and q to U_1 agree. One then continues inductively: there is a unique extension of ϕ_1 to an automorphism of U_2, the 2-neighborhood of the root of T, such that the restrictions of $p \circ \phi_2$ and q to U_2 agree, and so on. In this way, one constructs an automorphism ϕ of the full $2n$-regular tree with the property that $p \circ \phi = q$, which proves transitivity. ∎

Consider next the group $\mathrm{Aut}_1(T) \cong \mathrm{Aut}_0(T)/\mathfrak{S}_{2n}$ of all automorphisms of T which fix its first level (equivalently, the star of the root). Denote the root of T by x_0. The group $\mathrm{Aut}_1(T)$ is naturally isomorphic to the infinite iterated *wreath product* of the symmetric group $\mathfrak{S}_{2n-1}$ with itself. Seeing what this means is tantamount to seeing that any automorphism $\phi \in \mathrm{Aut}_1(T)$ can be realized—in a unique way—as a collection of $T\backslash\{x_0\}$-indexed permutations in $\mathfrak{S}_{2n-1}$. To make matters simpler, order the elements of $T\backslash\{x_0\}$ "level by level," i.e. by labeling the first level of T with the numbers $1,\dots,2n$, the second level of T with the numbers $2n+1,\dots,4n^2$, and so on. Then we may put $\phi = (\sigma_i)_{i\geqslant 1}$, whence applying ϕ to T amounts to first permuting the $2n-1$ branches of T attached to the vertex x_1 as per the permutation σ_1, then permuting the $2n-1$ attached to the vertex x_2 as per the permutation σ_2, and so on. Conversely, any system of permutations $(\sigma_i)_{i\geqslant 1}$ clearly determines an automorphism of T.

We now order the group $\mathrm{Aut}_1(T)$ by fixing an ordering of the symmetric group $\mathfrak{S}_{2n-1}$, i.e. by fixing a bijection $f : \mathfrak{S}_{2n-1} \to \{1,\dots,(2n-1)!\}$, and noting that, in terms of sets, we have the identifications

$$\begin{aligned}\mathrm{Aut}_1(T) &\cong \prod_{i\in\mathbb{N}} \mathfrak{S}_{2n-1} \\ &\cong \prod_{i\in\mathbb{N}} \{1,\ldots,(2n-1)!\}.\end{aligned}$$

We endow $\mathrm{Aut}_1(T)$ with the lexicographic ordering inherited from the above product space. For simplicity, let us assume that $f(\mathrm{Id}) = 1$. Let us also say that a covering map $p : T \to \Gamma$ is *compatible* with a Schreier structure $\widetilde{\Sigma} \in \mathrm{Sch}(T)$ if $\widetilde{\Sigma}$ descends to a structure $p_*\widetilde{\Sigma} \in \mathrm{Sch}(\Gamma)$. This means that, given any edge in Γ, each of its preimages under p is labeled with the same generator of $\mathbb{F}_n$ in the same way (i.e. with the same orientation). The sought-after connection between $\mathrm{Sch}(\Gamma)$ and $\mathrm{Cov}(T,\Gamma)$ goes as follows.

Proposition 6.3.4. *Let Γ be a $2n$-regular graph. Then there exists a natural identification* $\mathrm{Cov}(T,\Gamma) \cong \mathrm{Sch}(\Gamma) \times \mathrm{Aut}_1(T)$.

Proof: Fix an initial Schreier structure $\widetilde{\Sigma}_0 \in \mathrm{Sch}(T)$. Given a covering map $p \in \mathrm{Cov}(T,\Gamma)$, there exists an automorphism $\phi \in \mathrm{Aut}_1(T)$ such that the covering map $q := p \circ \phi$ is compatible with the structure $\widetilde{\Sigma}_0$. To see that such an automorphism exists, consider our enumeration $\{x_i\}_{i\geqslant 0}$ of the vertices of T, and put

$$\Gamma_k := \bigcup_{i=0}^{k} \mathrm{star}(x_i).$$

Then define $\mathcal{S}_k$ to be the set of automorphisms $\phi \in \mathrm{Aut}_1(\Gamma_k)$ such that the restriction of p to Γ_k composed with ϕ is compatible with $\widetilde{\Sigma}_0$ (or, rather, the restriction of $\widetilde{\Sigma}_0$ to Γ_k). The sets $\mathcal{S}_k$ are finite and nonempty and determine a projective system, i.e. there exist natural maps $f_k : \mathcal{S}_{k+1} \to \mathcal{S}_k$ obtained by restricting automorphisms of Γ_{k+1} to Γ_k. Elements of the corresponding projective limit $\mathcal{S}$ are readily seen to be automorphisms of T whose composition with p is compatible with $\widetilde{\Sigma}_0$. In order to obtain a canonical automorphism in $\mathcal{S}$, we choose the automorphism $\phi_{\min} \in \mathcal{S}$ which is minimal with respect to our ordering on $\mathrm{Aut}_1(T)$. This determines a map

$$\eta : \mathrm{Cov}(T, \Gamma) \to \mathrm{Sch}(\Gamma) \times \mathrm{Aut}_1(T)$$

given by $\eta(p) = ((\phi_{\min})_* \widetilde{\Sigma}_0, \phi_{\min})$. That is, we associate to a covering map p the minimal automorphism $\phi \in \mathrm{Aut}_1(T)$ such that $p \circ \phi$ is compatible with the structure $\widetilde{\Sigma}_0$, together with the image of $\widetilde{\Sigma}_0$ under $p \circ \phi$.

Conversely, let $(\Sigma, \phi) \in \mathrm{Sch}(\Gamma) \times \mathrm{Aut}_1(T)$ be an arbitrary pairing of a Schreier structure with an automorphism, and consider the structure $\phi_* \widetilde{\Sigma}_0 \in \mathrm{Sch}(T)$. By starting from the roots of Γ and T, the structures Σ and $\phi_* \widetilde{\Sigma}_0$ again allow us to read off a canonical covering map $p : T \to \Gamma$. As before, we define p to be the covering map that maps the star of $x_0 \in T$ onto the star of the root x of Γ in a way that respects our Schreier structures, then pass to the stars adjacent to x_0 and x and do the same for them, and so on. The image of $\phi_* \widetilde{\Sigma}_0$ under p is thus precisely Σ, and we put $\eta^{-1}(\Sigma, \phi) = p \circ \phi^{-1}$. One readily verifies that η^{-1} is indeed the inverse of η, which establishes our claim. ∎

Remark 6.3.5. *Note that we have implicitly used the fact that we are working with half-loops rather than loops in the proof of Proposition 6.3.4, as a half-loop can be mapped onto with complete freedom (unlike a regular loop, which is such that if an edge labeled with a generator a_i is mapped onto it via a graph homomorphism, then the corresponding edge labeled with a_i and possessing the opposite orientation must be mapped onto it as well).*

Propositions 6.3.3 and 6.3.4 immediately lead to the following question: How may the action of $\mathrm{Aut}_0(T)$ on $\mathrm{Cov}(T, \Gamma)$ be interpreted in terms of the identification given to us by Proposition 6.3.4? In particular, for a given $\Sigma \in \mathrm{Sch}(\Gamma)$, let us put

$$A_\Sigma := \{(\Sigma, \phi) \mid \phi \in \mathrm{Aut}_1(T)\},$$

i.e. A_Σ is the fiber over Σ of the projection of $\mathrm{Cov}(\mathbb{F}_n, \Gamma)$ onto $\mathrm{Sch}(\Gamma)$. Then what can be said of the set $\mathrm{stab}(A_\Sigma) \subseteq \mathrm{Aut}_0(\mathbb{F}_n)$ of automorphisms that fix A_Σ, and how does this set depend on Σ?

Note also that, if $p : \Gamma \to \Delta$ is a covering map between $2n$-regular rooted graphs, then the following picture emerges.

$$\begin{tikzcd}
\Gamma \arrow[r, dash] \arrow[d, "p"] & \mathrm{Cov}(T,\Gamma) \arrow[r, dash] \arrow[d, "\chi_p"] & \mathrm{Sch}(\Gamma) \\
\Delta \arrow[r, dash] & \mathrm{Cov}(T,\Delta) \arrow[r, dash] & \mathrm{Sch}(\Delta) \arrow[u, "\psi_p"']
\end{tikzcd}$$

That is to say, the space of covering maps $\mathrm{Cov}(T, \Gamma)$ embeds into $\mathrm{Cov}(T, \Delta)$ via the map χ_p that composes a given map in $\mathrm{Cov}(T, \Gamma)$ with p, whereas the space of Schreier structures $\mathrm{Sch}(\Delta)$ embeds (in the other direction) into $\mathrm{Sch}(\Gamma)$ via the map ψ_p that lifts structures from Δ to Γ along p. As stated at the beginning of this chapter, we are unable to offer much in the way of clear results here but hope that our observations may light a way towards them.

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