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ORIENTATION OF ORDERED SETS:

THEIR STRUCTURE AND ENUMERATION

BY

WEI PING LIU

**A Thesis submitted to the
School of Graduate Studies and Research
in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy
in
Electrical Engineering**

**Ottawa-Carleton Institute for Electrical Engineering
Faculty of Engineering
University of Ottawa
Ottawa, Ontario**



Wei Ping Liu, Ottawa, Canada, 1991



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ISBN 0-315-75638-7

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ACKNOWLEDGMENTS

I would like to thank my supervisor, Professor I. Rival, for his guidance and great help during the development of this thesis.

This thesis is dedicated to all those who, in one way or another, helped and encouraged me during my studies and research, to my friends, to my professors, and also to my family and relatives.

Orientation of ordered sets: their structure and enumeration

W. P. Liu

Abstract

Order diagrams, as fundamental representations of ordered sets, provide important computational models for the investigation of sorting and searching problems. Diagrams are needed for decision making in operations research, where hierarchical data structures are used. In practice, diagrams are used to display hierarchic structures. Examples include PERT networks, ISA hierarchies in knowledge representation diagrams and subroutine call graphs. Also, diagrams and their theory may tell us in what conditions we can draw a "good" diagram and how to draw it. For example, our characterization of bipartite ordered sets (Theorem 3.2.9) provides an efficient algorithm (with complexity $O(|S|)$) to test whether a bipartite ordered set is planar. Our new results about the enumeration of blocking relations may have application to visibility and separability.

The aim of this thesis is to enumerate and classify order diagrams. Some new characterizations will be proved. We also obtain new upper bounds and lower bounds, asymptotic estimation and even precise formulas. The proofs use deep ideas from combinatorial optimization, graph theory and the theory of ordered sets.

After a brief account of some basic definitions, notation and terminology that will be used in this thesis, we investigate, in Chapter 2, an important operation - inversion. A new and efficient characterization for inversions will be given, which, in turn, implies the first polynomial algorithm to test if an ordered set is an inversion of another. In this chapter, we also consider the complexity of inversions.

In Chapter 3, we study two important properties of ordered sets - planarity and s-genus. A necessary and sufficient condition for planar bipartite ordered sets is given, with its proof implying a method to construct a planar embedding of a planar ordered set. For outerplanar graphs, we will prove not only that an outerplanar graph has many planar orientations but also any orientation is planar provided it is "short" enough. As for s-genus of ordered sets, although we have not obtained any bound of it, our examples do tell us that some approaches to the s-genus problem do not work.

In Chapter 4, we enumerate the number of reorientations of an ordered set. Some interesting counter examples are provided.

In the last chapter, we will consider some problems in computational geometry, such as problems related to blocking relations, and to guarding problems.

To summarize, here are the main new results that will be proved in this thesis.

Theorem 2.2.1. *An ordered set Q is an inversion of an ordered set P if and only if the reversed edges can be partitioned into cuts.*

Theorem 2.2.2. *There is an algorithm with complexity $O(|S|^5)$ to test whether an ordered set is an inversion of another.*

Theorem 3.2.9. *A bipartite ordered set is planar if and only if its covering graph is planar.*

Theorem 3.3.1. *Any n -element outerplanar covering graph has at least $2^{n/2}$ planar orientations.*

Theorem 4.2.1. *There is a covering graph G some of whose independent sets cannot be the set of maximal elements of some orientation of G .*

Theorem 4.2.4. *There is an ordered set such that for some matching M and a subset M' of M , there is no reorientation which reverses the edges of M' and none of $M-M'$.*

Theorem 4.4.1. *Almost any n -element covering graph has at least $2^{n/3}$ orientations.*

Theorem 4.4.3. *Any n -element planar covering graph has at least $2^{n/3}$ orientations.*

Theorem 4.4.7. *Any n -element lattice with girth at least seven has at least $2^{n/3}$ orientations.*

Theorem 5.1.1. *Any n -element planar truncated lattice has at most $n(n-1)+1$ reorientations. Furthermore, the bound is tight.*

Theorem 5.3.1. *$n-2$ points ($n > 3$) on a (fixed) convex polygon containing n line segments are occasionally necessary and always sufficient to see the n segments.*

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CHAPTER 1 INTRODUCTION

Ordered sets occur widely in science, technology and even in our daily lives. In particular, orders are vitally important in almost all areas of discrete mathematics, such as optimization, operations research, scheduling, computation and geometry.

Consider the following problem. Suppose there are n "jobs" J_1, J_2, \dots, J_n and m identical "machines"; each job J_i has to be processed by one of the m machines in a unit of time. Furthermore, there are some technical restrictions on the jobs, that is, one job has to be processed before some others. For example, the job of washing dishes must precede the job of drying the dishes. The goal is to find a "schedule" of the n jobs, minimizing the completion time of the last job. The scheduling problem, that is, to find an optimal schedule, is easy if $m = 1$ since "any" schedule is optimal; the two machine problem is also solvable. Indeed, there is a polynomial algorithm which always yields an optimal schedule for $m = 2$ (cf. E. G. Coffman and R. L. Graham [1972]). If $m > 4$, the problem is "NP-complete", and for $m = 3$ the complexity status is unknown. On the other hand, if there is no restriction at all among jobs, then the scheduling problem turns out to be very easy, since any reasonable schedule is optimal. So, it seems that, at least in this case, the difficulty is caused by the restrictions on the jobs.

Let S be the set of jobs and two jobs J_i and J_k two jobs. We say that job J_i is "less than" job J_k if job J_i has to be processed before J_k . Then, the set of jobs with the restrictions on them is an ordered set. How hard a scheduling problem for S is, mainly depends on how complicated the ordered set representing the restrictions among the jobs is. If the ordered set is simple, the scheduling problem may have a solution, that is, an optimal scheduling can be easily found. A good example of this is that if the ordered set is a "down tree", then there is an efficient algorithm which always generates an optimal schedule (cf. T. C. Hu [1961]). If the ordered set is complicated, then it may be difficult to find an optimal scheduling.

In the past two decades, ordered set theory has seen great development; many beautiful results have been obtained and many new branches have been created. On the other hand, although great effort has been made, there are still many open problems in ordered sets, such as the problem of whether there is any efficient characterization for "covering graphs", the problem of whether the "fixed-point property" is preserved under directed product and the problem of whether any ordered set (with at least three elements) contains a pair of elements whose removal decreases the "order-dimension" by at most one.

In order to discuss ordered sets further, let us give some basic definitions, notations and terminology of ordered sets.

An *ordered set* $P = (S, \geq)$ consists of a set S (sometimes, $S(P)$) and a binary relation \geq on S satisfying the following conditions; for a, b and c in S :

- 1) $a \geq a$;
- 2) if $a \geq b$ and $b \geq c$, then $a \geq c$;
- 3) if $a \geq b$ and $b \geq a$, then $a = b$.

For convenience, we also use \leq ; $a \leq b$ if and only if $b \geq a$. The relation $a > b$ means $a \geq b$ and $a \neq b$ and $a \geq b$ (P) means $a \geq b$ in P . We say an ordered set Q is a *subset* of an ordered set P if $S(P) \supseteq S(Q)$ and $a \geq b$ in Q if and only if $a \geq b$ in P . For two elements a and b , we say a *covers* b (in P), or b is *covered* by a , denoted by $a >- b$ (P) or, simply, $a >- b$, if $a > x \geq b$ implies $x = b$. If $a >- b$, then the element a is an *upper cover* of the element b and b is a *lower cover* of a . The set of upper (lower) covers of a is denoted by $U(a)$ ($L(a)$). An element a is *maximal* (*minimal*) if $U(a) = \emptyset$ ($L(a) = \emptyset$). If a is the only maximal (minimal) element of an ordered set, we call the element a the *top* (*bottom*) of the ordered set. By $\max P$ ($\min P$), we denote the set of maximal (minimal) elements of P .

Consider an ordered set $P = \{a > b > c > d, a > e > f > d, b > f, e > c\}$. In this ordered set P , $a >- b$ and $b >- c$; the element a is maximal, and indeed, the top, and the

element d is minimal, and indeed, the bottom, elements b and e are the lower covers of the element a and elements c and f are the upper covers of the element d .

Associated with P , there is an ordered set with the same element set as P , the *dual* of P , denoted by P^d , in which for two elements a and b , $a > b$ (P^d) if and only if $b > a$ (P). We will use an important ordered set, the *n-dimensional (hyper)cube*, which is denoted by 2^n , consisting of all subsets of an n -element set and for two elements A and B , $A \leq B$ (2^n) if and only if A is a subset of B .

All ordered sets discussed here are finite, that is, we only discuss those ordered sets whose underlying sets are finite.

We will make use of many definitions, notation and terminology in graph theory. Here, we summarize some of them and those which are not mentioned can be found in *Graphs and Hypergraphs* (C. Berge [1973]).

Let G be a graph. The vertex set of the graph G is denoted by $V(G)$, or simply, V and the edge set of G by $\varepsilon(G)$. Let U be a subset of $V(G)$. $G - U$ stands for the subgraph of G induced by $V(G) - U$, in particular, for a vertex v , $G - v$ stands for the subgraph of G induced by vertex set $V(G) - \{v\}$. A vertex v is a *cut vertex* if the removal of v from a connected graph G yields a non-connected graph, that is, $G - v$ is not connected. The *girth* of G , denoted by $\text{girth}(G)$, is the length of a smallest cycle of G ; G is *triangle-free* if $\text{girth}(G) > 3$. (All graphs discussed here are triangle-free.)

A graph G is *planar* if there is a representation of G on the plane in which no pairs of edges intersect except possibly at an end of the edges, and such a representation is called a *planar embedding* of the graph; G is *outerplanar* if it is planar and the *exterior face* (cf. F. Harary [1972]) of some planar embedding is a Hamiltonian cycle (cf. Figure 1.1).

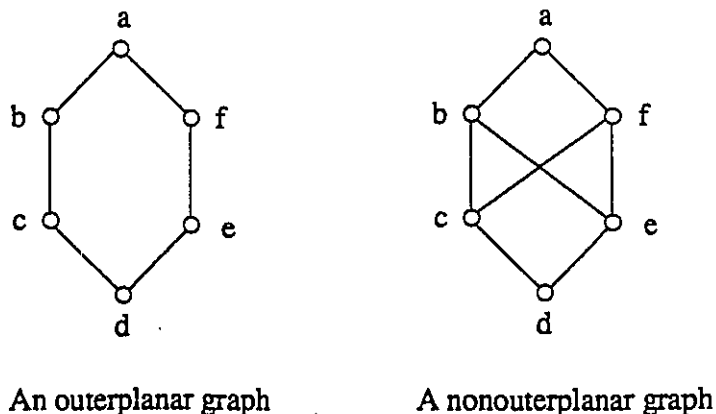


Figure 1.1

As we have seen from the example given on page 2, the relations of a simple ordered set are difficult to handle and read. To make use of ordered sets efficiently, we need some good methods to represent ordered sets, among which graphical ones play decisive roles.

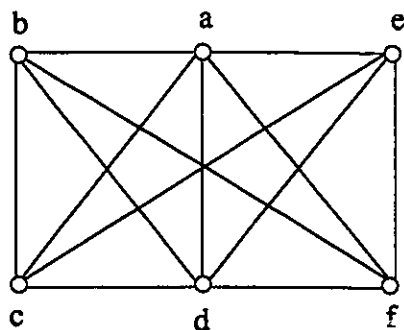
There are three basic but important graphical representations for ordered sets: compatibility graph, covering graph and diagram. Each graphical scheme uses vertices (little circles in the plane) for elements of the ordered sets. The *comparability graph* of an ordered set P is an (undirected) graph in which an edge joins two vertices a and b if and only if, either $a < b (P)$ or $b < a (P)$ (Figure 1.2). (Two assumptions we would like to make here: one is that we ignore reflexive edges, that is, loops at the vertices; another is that, for simplicity, we use letters "a" and "b" to denote both vertices of the comparability graph (and later, of "covering graphs" and "diagrams") and the corresponding elements, and we will do so as long as no confusion is created.) Although comparability graphs are only incidental in this thesis, we make two observations. One is that, generally speaking, a comparability graph has many edges which not only make the graph difficult to handle but also useless in practice for reading; another is that because of the abundance of edges, they are "easy" to investigate. In fact, there are characterizations of comparability graphs: *a graph G is a comparability graph if and only if every odd cycle of length at least five*

has a chord (A. Ghouila-Houri [1962], P. C. Gilmore and A. G. Hoffman [1964]). Furthermore, T. Gallai [1967] gave a complete list of forbidden induced subgraphs of comparability graphs. In addition, there are many papers dealing with comparability graphs, and even surveys (cf. D. Kelly [1985] and R. H. Mohring [1985]).

What is an efficient graphical representation of an ordered set? The *covering graph* of P is a graph whose vertices are the elements of P and in which an edge joins two vertices a and b precisely if a covers b or b covers a . The covering graph (each ordered set has only one covering graph) of an ordered set gives almost all information about the ordered set except that we do not know which element is bigger. A *diagram* of P is a pictorial representation of P on the plane in which small circles, corresponding to the elements of P are arranged in such a way that for elements a and b in P , the circle corresponding to the element a is higher than the circle corresponding to the element b whenever $a > b$ and a straight line segment is drawn between the two circles if a covers b (Figure 1.2).

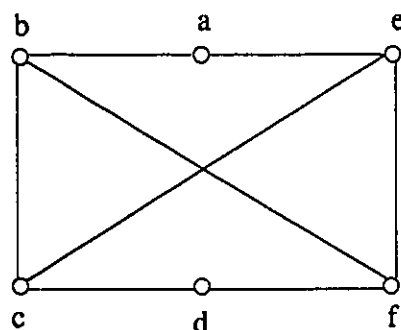
An ordered set is (k -)connected if its covering graph is (k -)connected. All ordered sets discussed here, unless otherwise stated, are connected.

Example. Let P be the ordered set: $\{a > b > c > d, a > e > f > d, b > f, e > c\}$, then its comparability graph, covering graph and diagram are given as follows (Figure 1.2).



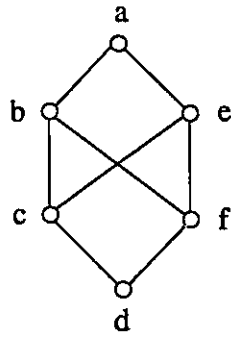
The comparability graph of P

(1)



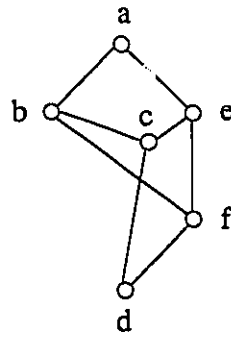
The covering graph of P

(2)



A diagram of P

(3)

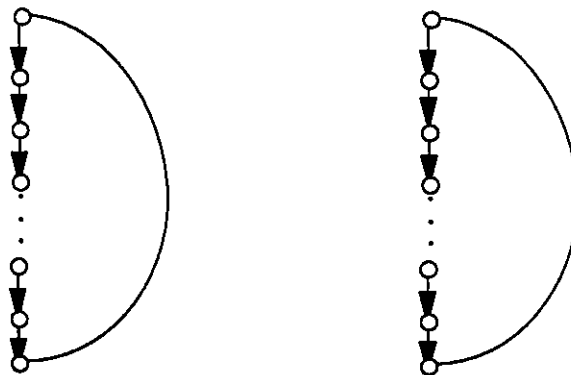


Another diagram of P

(4)

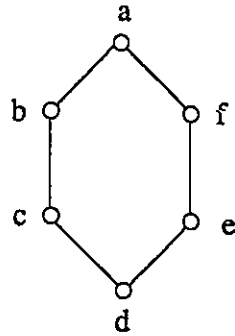
Figure 1.2

One observation about diagrams is that there is a one-to-one correspondence between ordered sets and their diagrams, though, just like graphs, an ordered set may have many different "isomorphic" diagrams (cf. Figure 1.2). Therefore, diagrams and ordered sets may be used interchangeably. Another observation about diagrams is that a diagram of an ordered set may be viewed as a directed graph, in which there are no subgraphs "isomorphic" to one of the directed graph illustrated in Figure 1.3, whose underlying graph is the covering graph of the ordered set, if we assign each straight line of the diagram an arrow directing from the higher endpoint of the line to the lower endpoint of the line (Figure 1.4). So, we can regard a diagram of an ordered set as an orientation of the covering graph.

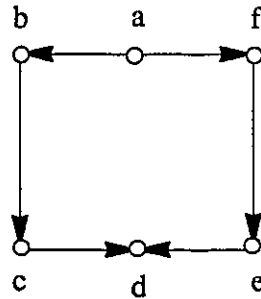


Two types of forbidden subdigraphs for a diagram

Figure 1.3



A diagram P

The directed graph corresponding to P
Figure 1.4

We have mentioned that if two ordered sets P and Q have the same diagram (that is, their diagrams are isomorphic) then the two ordered sets are identical, that is, there is a one-to-one mapping f from P to Q such that for any two elements a and b of P , $a > b$ (P) if and only if $f(a) > f(b)$ (Q). But two different ordered sets (not identical) may have the same covering graph. For example, the two ordered sets illustrated in Figure 1.5 (1) and (2) have the same covering graph which is illustrated in Figure 1.5 (3). Call an ordered set (a diagram) P an *orientation* of a covering graph G if G is the covering graph of P . An ordered set (a diagram) Q is a *reorientation* of an ordered set P if they have the same covering graph.

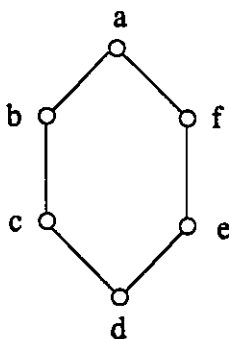
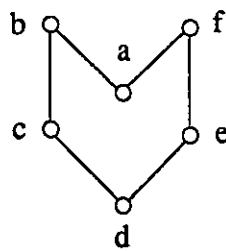
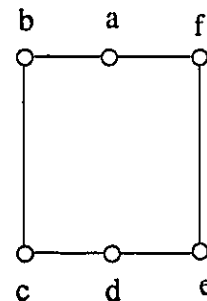
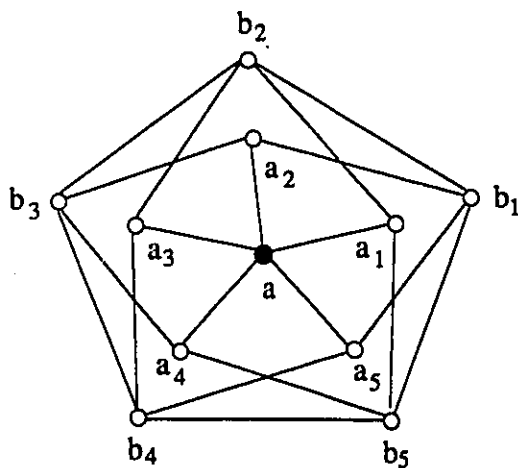
An ordered set P
(1)Another ordered set Q
(2)The common covering graph
(3)

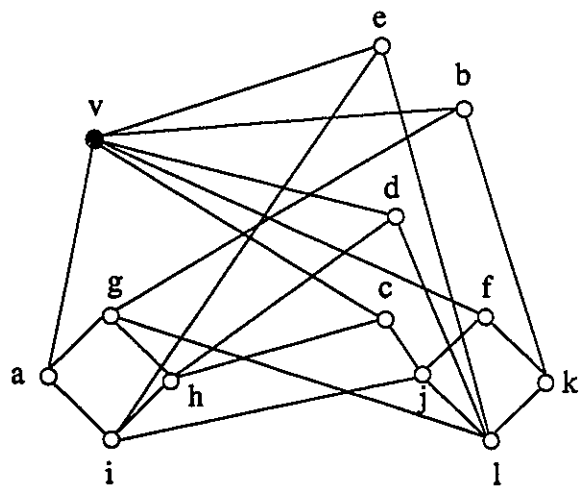
Figure 1.5

One of the fundamental open problems in ordered set theory is to find necessary and sufficient conditions for covering graphs, in other words, to characterize covering graphs of ordered sets. Until recently, little was known (cf. L. Alvarez [1965] and D. Duffus and I. Rival [1983]). In fact, J. Nešetřil and V. Rödl [1987] have shown that *the problem to decide whether a triangle-free graph is a covering graph, or equivalently, to decide whether a triangle-free graph can be oriented as a diagram, is NP-complete.* Therefore, it seems unlikely that we can find an efficient characterization for covering graphs.

Quite some time ago, perhaps, it was thought that every triangle-free graph was a covering graph. J. Mycielski [1955] proved that this is not true by providing a triangle-free *noncovering* graph (which is not a covering graph) (Figure 1.6 and cf. O. Pretzel [1985]). Moreover, we cannot hope that every graph with large girth is a covering graph. In fact, by probability methods, J. Nešetřil and V. Rödl [1978] proved that *for any positive integer k , there is a noncovering graph with girth at least k* , which answered



M_5 , the smallest noncovering graph



Another noncovering graph

Figure 1.6

P.Erdős' tantalizing problem negatively: *is it true that any graph with large girth is a covering graph* [1971]. J. Nešetřil and V. Rödl's result is important theoretically, but the problem of how to explicitly give a noncovering graph with girth at least k (> 4) remains a challenge. Moreover, it is difficult just to give a triangle-free noncovering graph.

Let us consider another difficult problem, that is, the problem of whether there is a nontrivial "invariant" of all diagrams with the same covering graph, in other words, a nontrivial property that all orientations of a covering graph have, a property that all reorientations of the ordered set have. M. Pouzet and I. Rival [1988] conjectured that there is no nontrivial diagram invariant at all (cf. K. Ewacha, W. Li and I. Rival [1990]). One of the reasons for this conjecture may be that a covering graph has many orientations, making it unlikely that all these orientations have a common property. (Recalling that a comparability graph only has few "reorientations", that is, not many different ordered sets have the same comparability graph, it is perhaps not surprising that there are comparability invariants.)

How many reorientations does an ordered set have or, equivalently, how many orientations does a covering graph have? This is a difficult question. An *antichain* (an ordered set whose covering graph has no edge) has only one reorientation. O. Pretzel [1986] shown that *any n -element ordered set has at least $n^2/2 + n$ reorientations*. It seems that an ordered set has many more reorientations. For example, *an n -element ordered set whose covering graph is a tree has 2^{n-1} reorientations; an n -element bipartite ordered set has at least $2^{n/2}$ reorientations and an n -element ordered set whose covering graph is planar has at least $2^{n/3}$ reorientations*. Indeed, we conjecture that *any n -element covering graph has exponentially many, say $2^{\sqrt{n}}$, orientations*. We have strong evidence to support our conjecture (cf. G. Brightwell and J. Nešetřil [1991]).

THEOREM (W. P. Liu and I. Rival [1991]). *Almost all n -element ordered sets have at least 2^{n^3} reorientations.*

We will also discuss two problems related to the number of reorientations. Is it true that for any independent set I of a covering graph G , there is an orientation of G in which I is the set of maximal elements? Is it true that for any orientation Q of a matching M of a covering graph G , there is an orientation of G which is an "extension" of Q ?

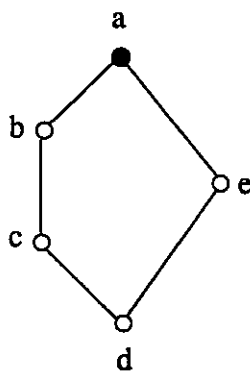
Here is the motivation. While counting the reorientations of an ordered set, the set of maximal elements plays an important role. First, *any antichain of an ordered set can be the set of maximal elements of some of its reorientations* (cf. Chapter 2); second, *an antichain is an independent set of the covering graph of an ordered set*; third, if we could prove that any independent set of a covering graph could be an antichain of some of its orientations, we could estimate the number of orientations of a covering graph since *any n -element covering graph has independent set of size at least $\sqrt{n/2}$* (for details, see Chapter 4).

Similarly, we can consider the problem about matchings.

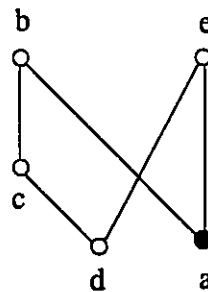
Almost all problems concerning diagrams are very hard. Perhaps, as we have said, one reason is that a diagram has many reorientations; another is that so far we do not have many efficient methods to orient a covering graph, or, in other words, to obtain a new diagram from the given one by reversing the directions of some edges (remember a diagram is considered to be a directed graph). We do not even know if we can reverse an edge of a diagram, because the reversal of an edge may cause the reversals of some others.

K. M. Mosesjan [1972] introduced an operation called *pushdown* to yield a new diagram from the original. Pushdown is so far the only general and nontrivial method to generate a new diagram.

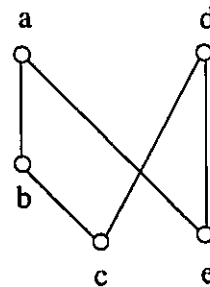
For any maximal element a of an ordered set P , say an ordered set Q is obtained from P by *pushing down* the element a , or, Q is a *pushdown* of P , if all (order) relations of Q are the same as that of P except that the element a is minimal in Q and the upper covers of the element a in Q are exactly the lower covers of a in P (Figure 1.7). Obviously, we can do pushdown operation again on the new ordered set. An ordered set Q is an *inversion* of an ordered set P if Q can be obtained from P by a sequence of pushdowns (Figure 1.8).



a is a maximal element of P



Q is obtained from P by pushing down a



W is an inversion of P , by pushing down elements a , b , c and e successively

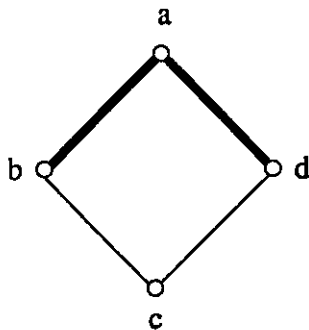
Figure 1.7

Figure 1.8

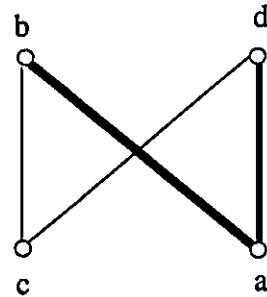
Applying pushdown, K. M. Mosesjan proved that *any element of an ordered set can be the top of some inversion of the ordered set*. By applying Mosesjan's result, we can easily prove the graph M_5 is not a covering graph by showing that the element a of the graph cannot be the top of some orientation of the graph (Figure 1.6).

O. Pretzel [1986] first gave an interesting characterization of inversions (for detail see Chapter 2). Following is another characterization of inversions .

THEOREM (W. P. Liu and I. Rival [1991]). *An ordered set Q is an inversion of an ordered set P if and only if the reversed edges can be partitioned into cuts (cf. Figure 1.9 and Figure 1.10).*

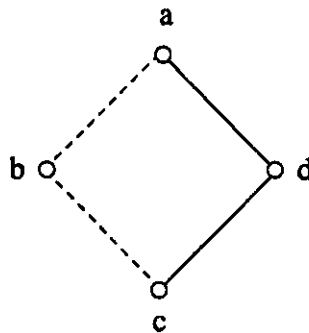


$\{a >- b, a >- d\}$ is a cut of P



An inversion of P which reverses these edges

Figure 1.9



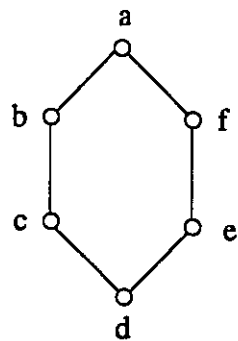
Although $\{a >- b, b >- c\}$ is an edge cut of the covering graph of P , it is not a cut of P . Thus, there is no inversion of P reversing the edges.

Figure 1.10

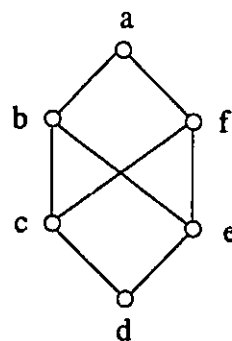
One of the important differences between inversions and other reorientations is that an ordered set may have many reorientations, (although we do not know how many) while it may have as few as $O(n^2)$ inversions. (Of course, it could be true that any reorientation of some ordered set is an inversion of it. For example, for any ordered set whose covering graph is a tree, any reorientation of it is an inversion of the ordered set. Another

such an ordered set is an ordered set whose covering graph is planar and each face is of length four (cf. Lemma 3.3.4). In fact, W. P. Liu and I. Rival [1991] have proved that *any n -element ordered set has at least $(n^2 + 2n)/2 - n \log_2 n$ inversions*, and an ordered set which has exactly $(n^2 - n)/2$ inversions is also given. It will be interesting to know if there is a polynomial algorithm to check if an ordered set is an inversion of another, which is why the theorem above is significant; second, the theorem above also tells us the complexity to obtain the inversion from the given ordered set; finally, from the theorem above, we know, at least, one family of edges - the disjoint union of *cuts* - are "reversible".

Planarity of graphs is an important property of graphs. There are many papers studying them. Both elegant combinatorial characterizations (e.g. K. Kuratowski [1930]) of planar graphs and efficient algorithms to check whether a graph is planar have been found (e.g. J. Hopcroft and R. E. Tarjan [1974]). Similarly, we can define planarity of ordered sets. We say an ordered set P is *planar* if it has a diagram in which no two of the lines corresponding to the covering pairs intersect, except possibly at an endpoint, where they meet in a small circle corresponding to an element of P (Figure 1.11), such rendering is called a *planar embedding* of the ordered set.



A planar ordered set

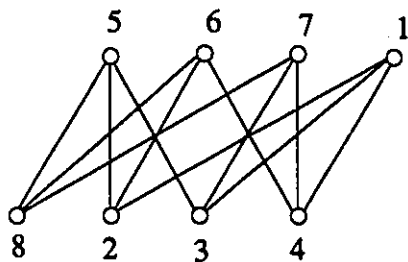


A nonplanar ordered set

Figure 1.11

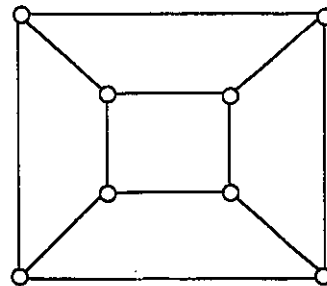
Although the planarity of an ordered set plays an important role in the study of ordered sets, little is known, except some special cases. An ordered set P is a *lattice* if for any two elements x and y , there is a *greatest lower bound* b and *least upper bound* a , i.e. $a \geq x, y \geq b$ and $z \geq x, y$ implies that $z \geq a$, and $z \leq x, y$ implies that $z \leq b$. There are some characterizations of planar lattices (cf. D. Kelly and I. Rival [1975]) and efficient algorithms for testing whether a lattice is planar (cf. C. R. Platt [1976]).

Which ordered sets are planar, generally? A necessary condition for a planar ordered set is that its covering graph is planar. But the planarity of a covering graph of an ordered set does not always imply that the ordered set is planar. For example, the ordered set 2^3 (see Figure 1.12 (4)) is not planar, although its covering graph is planar (Figure 1.12 (2)). What interests us is that the ordered set illustrated in Figure 1.12 (1) "looks" even more nonplanar than the ordered set 2^3 (in other words, the diagram has more edges crossing) is planar. Figure 1.12 (3) gives one of the planar embeddings of the ordered set illustrated in Figure 1.12 (1). Since we cannot characterize planar ordered sets efficiently, we may try another approach - classify those planar covering graphs which have planar orientations. Indeed, we have the following conjectures.



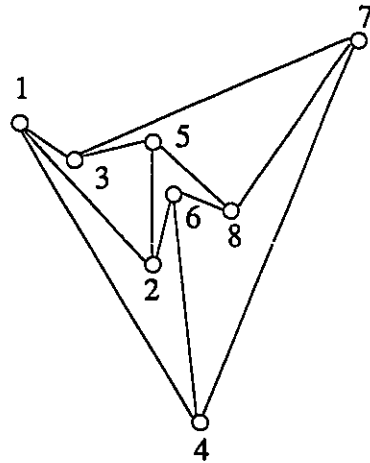
A bipartite ordered set P

(1)



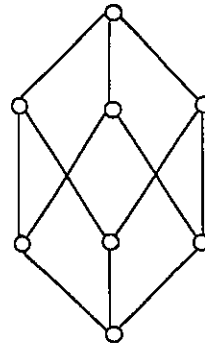
The covering graph of P

(2)



A planar embedding of P

(3)

The ordered set 2^3

(4)

Figure 1.12

CONJECTURE 1 (G. di Battista, W. P. Liu and I. Rival [1990]). *Every planar covering graph has a planar orientation*

We even hope that any planar covering graph has a planar orientation which is the "shortest" among all orientations. A *chain* (with the top t and the bottom b) of *length* m of an ordered set is a sequence of elements $t = a_1, a_2, a_3, \dots, a_m = b$ such that $a_1 > a_2, a_2 > a_3, \dots, a_{m-1} > a_m$. A chain C of an ordered set P is *maximum* if no other chain of P is longer than C .

CONJECTURE 2 (W. P. Liu). *Every planar covering graph has a planar orientation whose maximum chain is of length at most three.*

Conjecture 2 is true if the graph is bipartite. An ordered set is *bipartite* if it can be partitioned into two "levels", that is, any element of the ordered set is either maximal or minimal. For example, the ordered set illustrated in Figure 1.12 (1) is bipartite but 2^3 is not (Figure 1.12 (4)).

THEOREM (G. di Battista, W. P. Liu and I. Rival [1990]). *Every planar bipartite covering graph has a planar bipartite orientation.*

Recall that *any n -element planar covering graph has at least $2^{n/3}$ orientations.* Furthermore, some of its orientations may be planar and some of them may not. Therefore, to prove Conjecture 1 is true, we have to discover a smart criterion to select our favourite one among at least $2^{n/3}$ orientations. So far, we have no idea how to establish such a criterion.

Diagrams are important representations of ordered sets, but they are not the only ones. Another quite different way to represent an ordered set is to use disjoint convex figures on the plane and directions of motion associated with them. Such a collection of convex figures on the plane may model the problem of separating clusters of figures on a computer screen, the problem of guarding collections of "objects d'art", or even the problem of navigating an iceberg field in our far north.

Suppose each figure in a collection of disjoint, convex figures is assigned a single direction of motion, not necessarily identical. For figures A and B we say that B *obstructs* A and write $A \rightarrow B$ if there is a line joining a point of A to a point of B along the direction assigned to A . We write $A < B$ if there is a sequence $A = A_1 \rightarrow A_2 \rightarrow A_3 \rightarrow \dots \rightarrow A_k = B$. This transitive relation $<$ is antisymmetric, provided it has no "directed cycle", in which case it becomes an order on the set of these figures. Call this ordered set a *blocking relation*. If each figure is assigned one of the m directions, then the "representation" (that is, the convex figures together with the directions of motion) is called a *m -directional representation* of the corresponding order (Figure 1.13).

An ordered set is a *planar truncated lattice* if it is obtained from a planar lattice by deleting the top, or the bottom, or both or neither of these. Here, we are interested in only one-directional representations, because *there is a one-to-one correspondence between*

one-directional representations and planar truncated lattices (I. Rival and J. Urrutia [1987]).

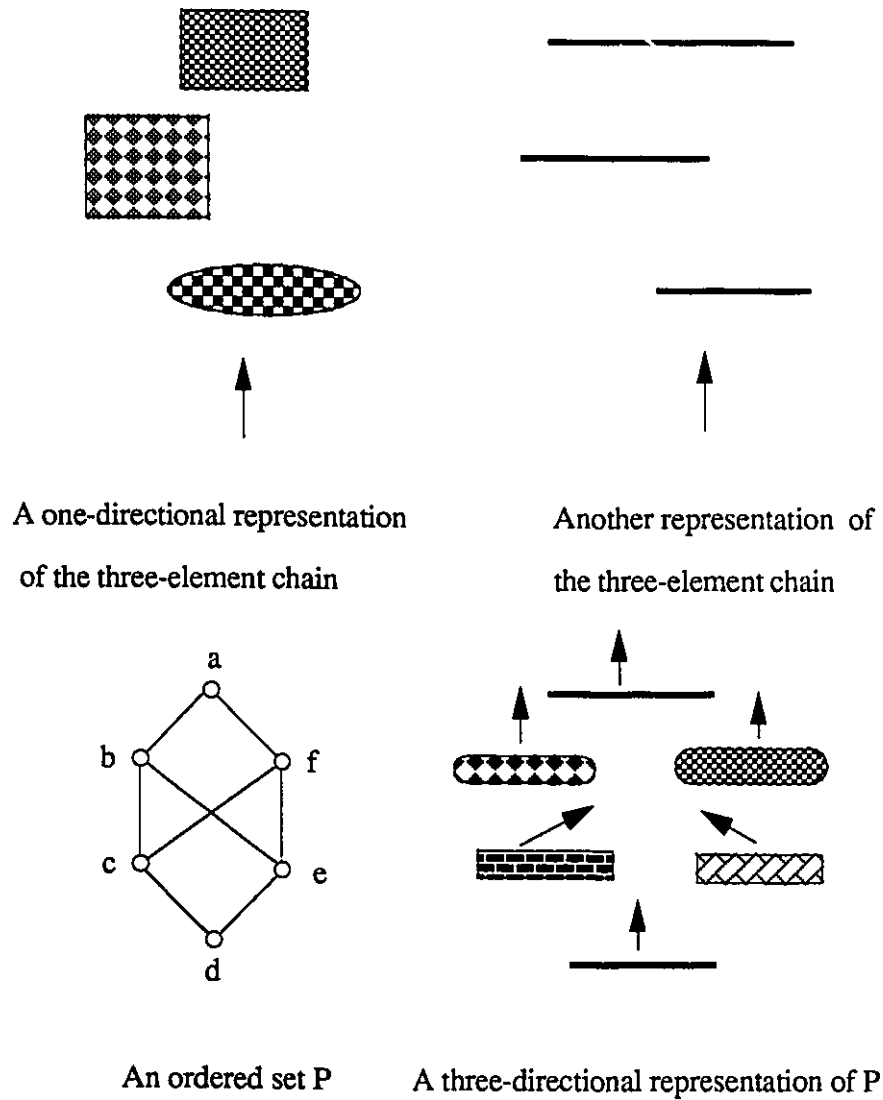


Figure 1.13

From the definition, the representation of a blocking relation depends on not only the convex figures but also the direction of motion. By changing the direction of motion but fixing the convex figures, we may obtain a representation of a new blocking relation (Figure 1.14); the new blocking relation is called a *reorientation* of the original with respect

to the representation. How many reorientations does a planar truncated lattice have with respect to a representation of it? Here is our answer.

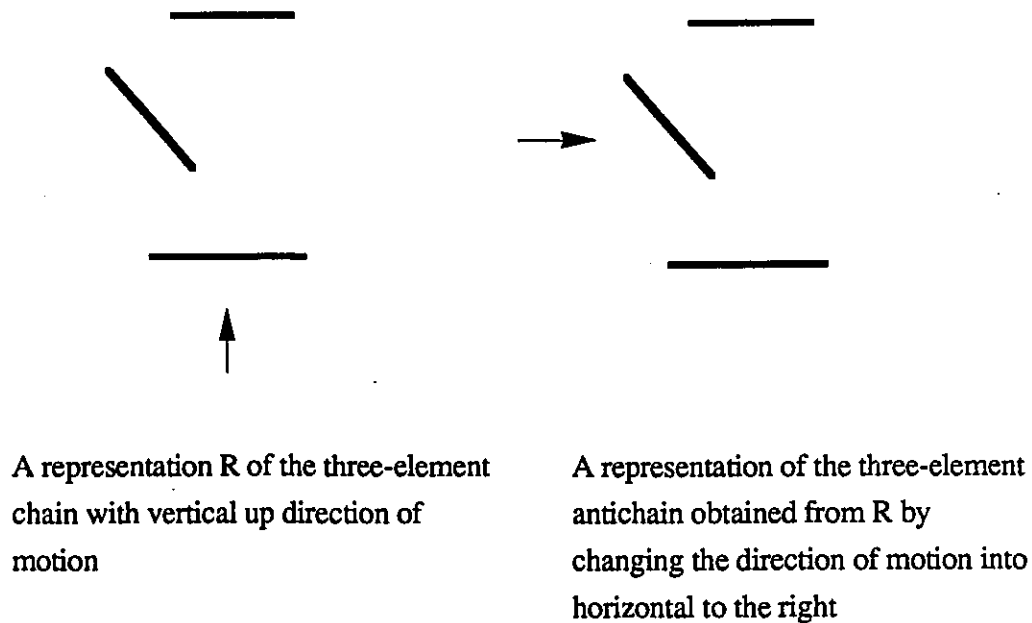


Figure 1.14

THEOREM (W. P. Liu and I. Rival [1990]). *Any n -element truncated planar lattice has at most $n(n - 1) + 1$ reorientations with respect to any of its representations; moreover, the bound is best possible.*

We are interested in special representations of planar truncated lattices in which all figures are horizontal line segments.

There are papers dealing with how to represent a planar graph in the smallest area using only vertical line segments or horizontal line segments with integer coordinates to represent edges and points to represent vertices (cf. W. T. Tutte [1960] and [1963] and N. Chiba et al [1984] and [1985]). Also, there are some results about representing a planar graph in the smallest area using horizontal line segments with integer length to represent

vertices of the graph and vertical line segments with integer length its edges (G. di Battista and R. Tamassia [1986]). We discuss the same problem.

THEOREM (W. P. Liu and I. Rival [1990]). *For any n -element planar truncated lattice, there is a one-directional representation of it in which all objects are horizontal segments with integer lengths such that the upright rectangle enclosing the segments has area at most $n(n - 1)/2$; moreover, the bound is best possible.*

How many guards are necessary, and how many are sufficient, to patrol the n paintings of an art gallery? This interesting problem was posted by Victor Klee in 1973 (cf. Honsberger, [1976]). Since V. Chvatal [1975] published what has become known as the "Chvatal's Art Gallery Theorem": $\lfloor n/3 \rfloor$ guards are occasionally necessary and always sufficient to cover a polygon of n vertices, there was a rush of papers, surveys and even a book (J. O'Rourke [1987]).

There are numerous other interesting versions of the "Art Gallery Problem" (cf. A. Aggarwal [1984]): traditional orthogonal art galleries all of whose walls are parallel to axes, in which case $\lfloor n/4 \rfloor$ guards are necessary and sufficient (J. Kahn, M. Klawe, and D. Kleitman [1983]); mobile guards, each of whom patrols along a line segment with an n -vertex polygon, of whom $\lfloor n/4 \rfloor$ are necessary and sufficient (O'Rourke [1983]); no wall at all, that is, guards can be put at any position except on a segment, in which case $\lceil 3n/4 \rceil$ guards are occasionally necessary and always sufficient (J. Czyzowicz, I. Rival and J. Urrutia [1989]). We consider another version in which guards are put at some positions on a convex polygon containing n line segments (paintings). This model has its usefulness. In this case guards cannot go inside the polygon because, perhaps, it is impossible or dangerous to go inside.

THEOREM (W. P. Liu and I. Rival). $n - 2$ ($n > 3$) guards are sometimes necessary and always sufficient to see n segments on a convex polygon containing the segments.

We can consider the same problem in space, that is, how many points are necessary and how many are sufficient to see n line segments, no two of which intersect each other, in space? If the points are restricted to a ball containing the segment, how many points are necessary and how many are sufficient to see the n line segments? These two problems are so hard that we have not even solved nontrivial cases. However, we give two surprising examples showing that $\lfloor 2n/3 \rfloor$ *points are sometimes necessary to see n line segments in space generally*, and n *points on a ball containing n line segments are sometimes necessary to see them*.

The remaining chapters are organized as follows. In Chapter 2, we will fully investigate inversion; in Chapter 3, we will study two important properties of ordered sets - planarity and s -genus; in Chapter 4, we will estimate the number of reorientations of ordered sets; and in Chapter 5, some problems in computational geometry will be considered.

CHAPTER 2 INVERSION

2.1. INTRODUCTION. Diagrams and covering graphs are important graphical representations of ordered sets. Moreover, since a diagram is an orientation of a covering graph, the study of covering graphs will highlight diagrams.

What graphs are covering graphs? or equivalently, what graphs can be oriented as diagrams? On the one hand, not every triangle-free graph is a covering graph; even *a graph with large girth may not be a covering graph* (J. Nešetřil and V. Rödl [1978]); on the other hand, not only cannot we explicitly give a noncovering graph with girth at least five, but also it is difficult to exhibit a noncovering graph. Therefore, it is no wonder that, in spite of great effort, the following problem still remains one of the hardest problems in ordered set theory.

PROBLEM 1. *Find necessary and sufficient conditions for a graph to be a covering graph.*

So far, only for some special ordered sets – lattices – are necessary and sufficient conditions for covering graphs known (cf. H. J. Bandelt [1989], H. J. Bandelt and I. Rival [1989], D. Duffus and I. Rival [1983], J. Nešetřil and V. Rödl [1987], O. Pretzel [1985] and [1986], and I. Rival [1985] and [1987]). One of the earliest results was due to L. Alvarez [1965] who has proved the following theorem.

THEOREM 2.1.1. *A finite graph G is the covering graph of a finite "distributive" lattice if and only if G satisfies the following conditions:*

- 1) G is connected and contains no odd cycles;
- 2) *there are a_1, a_2 in $V(G)$ such that $\text{diam}(G) = d(a_1, a_2)$ and if $\{c, e\}$ and $\{c, d\}$ belong to $\varepsilon(G)$ with $d(a_i, b) = d(a_i, e) = d(a_i, c) + 1$, then there is an unique f_i in $V(G)$ such that $d(a_i, f_i) = d(a_i, c) = 2$ and $\{f_i, e\}, \{f_i, d\}$ are in $\varepsilon(G)$, where*

$i = 1$ or 2 ;

- 3) if G contains a subgraph S isomorphic to F (Figure 2.1.1), then S is contained in a subgraph T isomorphic to the covering graph of 2^3 ;
- 4) G contains no subgraph isomorphic to $K_{2,3}$ (Figure 2.1.2).

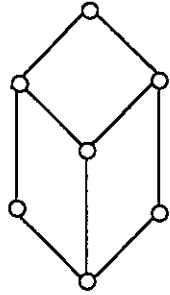


Figure 2.1.1

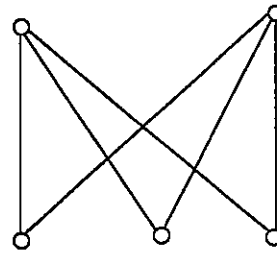


Figure 2.1.2

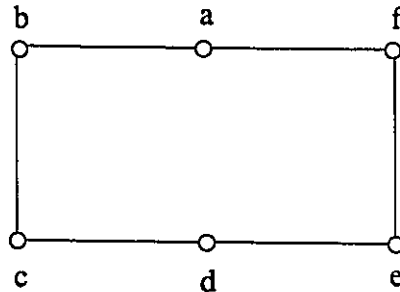
Here, $d(u,v)$ denotes the distance of two vertices of u and v , and $\text{diam}(G) = \max\{d(u,v) : \text{all } u \text{ and } v \text{ in } G\}$.

J. Jakubik [1975] proved that if two lattices L and L' have the same covering graph and L is "modular" (distributive) then L' is also modular (distributive).

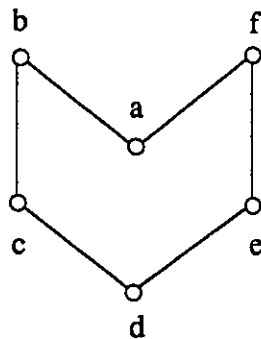
A map f from a graph G to a graph G' is *edge-preserving* if for any edge uv of G , $f(u)f(v)$ is an edge of G' . A graph H is a *retract* of G if there are edge-preserving maps f of $V(H)$ to $V(G)$ and g of $V(G)$ to $V(H)$ such that $g \circ f(a) = a$ for each vertex a in $V(H)$. D. Duffus and I. Rival [1983] characterized those covering graphs with distributive lattice orientations. They proved *a finite graph G is the covering graph of a distributive lattice of length n if and only if G is a retract of the n -dimensional hypercube 2^n and $\text{diam}(G) = n$* . H. J. Bandelt [1984] generalized this result: *a graph is the covering graph of a "median graph" if and only if it is a retract of a hypercube*.

Generally, to check if a graph is covering graph, the only method so far is to see if the graph can be oriented as a diagram. However, it is not easy to orient a graph as a

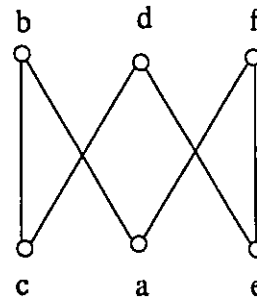
diagram even if we do know the graph is a covering graph, since a covering graph may have many orientations (Figure 2.1.3).



A covering graph G



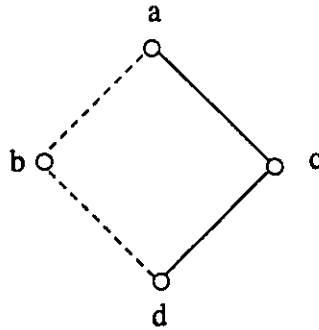
An orientation of G



Another orientation of G

Figure 2.1.3

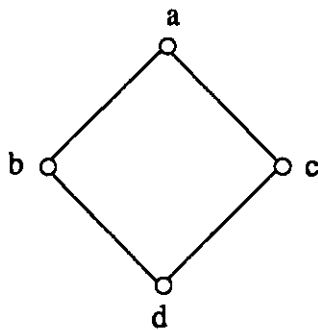
What are the possible orientations of the covering graph of an ordered set, then? or equivalently, what are the possible reorientations of an ordered set? From the definition, an orientation of a covering graph can differ from another orientation of the covering graph only in the directions of some of the edges, in other words, a reorientation of an ordered set P can differ from the original only in the reversed edges - if a covers b in P then a reversal makes b an upper cover of a . Therefore, the possible reorientations of an ordered set correspond exactly to those subsets of edges of P which can be reversed (are *reversible*). It is easy to see that some of the subsets of the edges of an ordered set P may be not reversed (Figure 2.1.4).



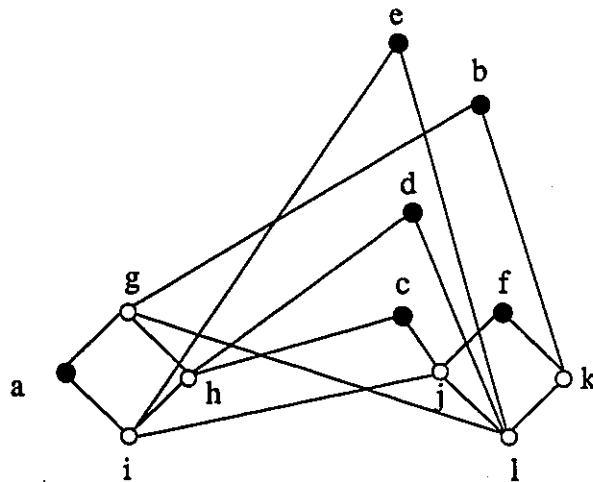
The edge set $\{a > b, b > d\}$ of an ordered set is not reversible

Figure 2.1.4

Which edges of an ordered set can be reversed? For example, for the simple ordered set illustrated in Figure 2.1.5 (1), the reversal of the edge $a > b$ forces us to reverse either the edge $a > c$ or the edge $c > d$; and if the edges $a > b$ and $b > d$ are reversed, then we have to reverse the edges $a > c$ and $c > d$. Consider a more complicated example illustrated in Figure 2.1.5 (2). If, for some purpose, we want to



An ordered set
(1)



Another ordered set
(2)

Figure 2.1.5

keep the elements b, c, d, e, f as maximal and reverse the edge $g > a$, then the edge $a > i$ has to be reversed (for details see Chapter 4). Therefore, the reversal of one edge could

lead to the reversals of some other edges. (But we can prove that for any edge of an ordered set, there is a reorientation of it which reverses the edge and fixes at least one edge, provided the ordered set has at least two edges.)

As we have seen from the above, the problem of which edges of an ordered set can be reversed is closely related to the problem of characterizing covering graphs. Until now little was known (e.g. J. Jakubik [1954] and [1975], and I. Rival [1985]). All we have known is that a certain type of set of edges of an ordered set, a "cut", is reversible, in other words, the reversal of a cut gives a diagram, and, indeed, the reversal of a cut is the result of a sequence of "pushdowns".

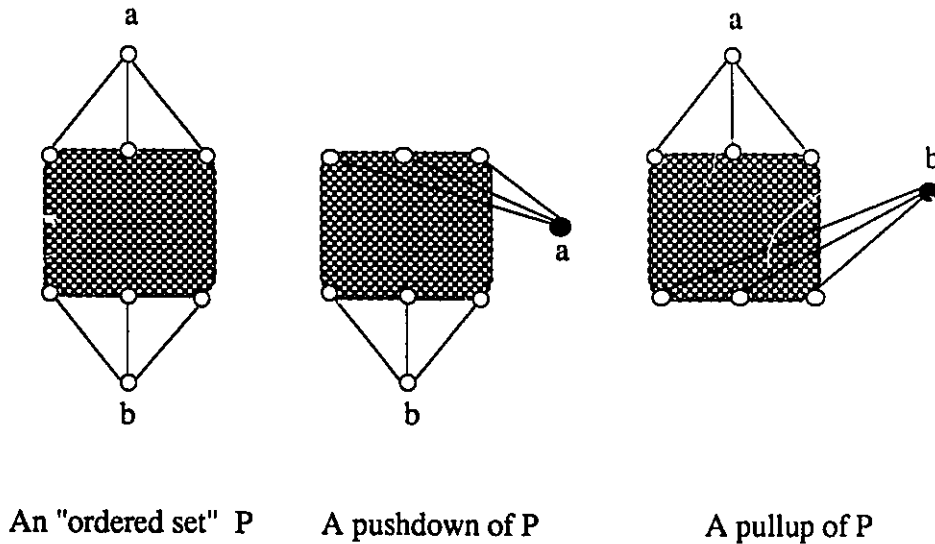
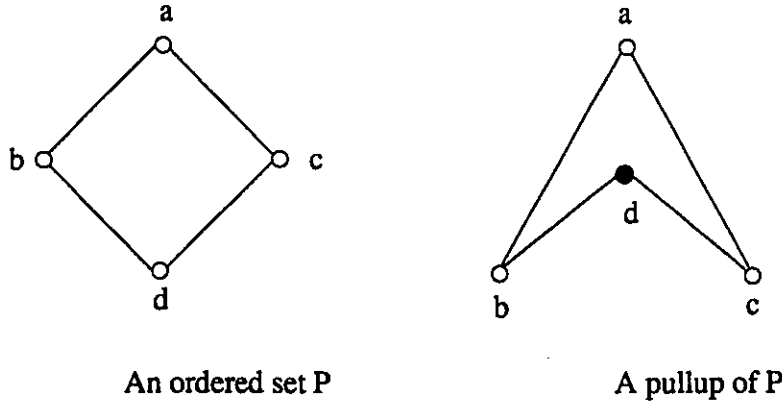


Figure 2.1.6

For a maximal element a of P , construct an ordered set Q such that $Q - a = P - a$, the element a is minimal in Q and its upper covers are the elements of $L(a)$ in P , where $P - a$ denotes the subset of P with $S(P - a) = S(P) - \{a\}$ for a maximal (minimal) element a . We say that Q is obtained from P by *pushing down* a , or Q is a *pushdown* of P (Figure 2.1.6). The dual construction (*pullup*) is also used. For any minimal element b of P , construct an ordered set Q such that $P - b = Q - b$, the element b is maximal in Q and its lower covers

are the elements of $U(b)$ in P (Figure 2.1.6). A *pullup* can be replaced by a sequence of *pushdowns* (cf. O. Pretzel [1986]) (Figure 2.1.7). An ordered set Q is an *inversion* of an ordered set P if Q can be obtained from P by a sequence of pushdowns or pullups (Figure 2.1.8).



A pullup Q of P which can be obtained by a sequence of pushdowns of a , b and c

Figure 2.1.7

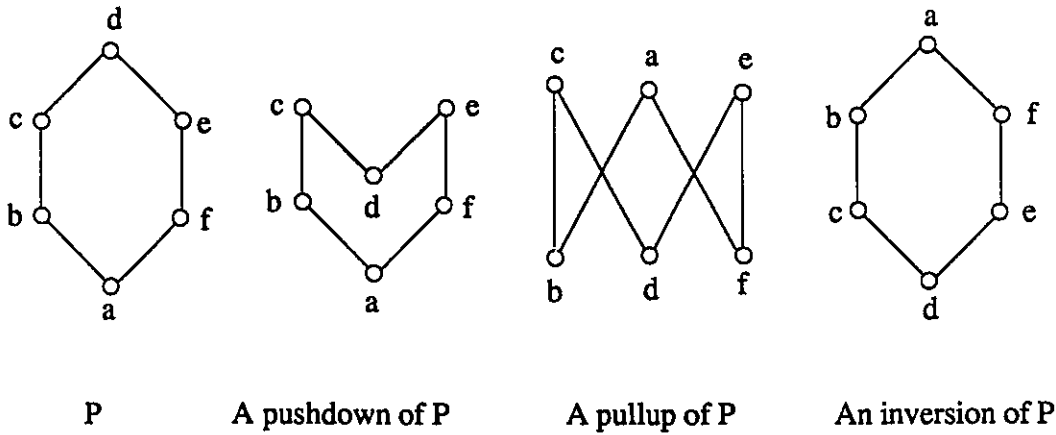
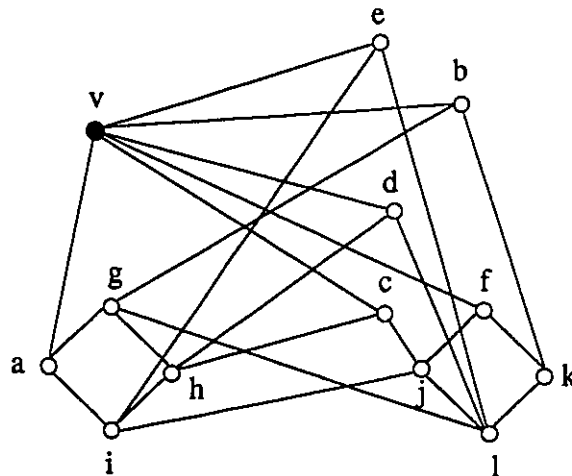


Figure 2.1.8

For an element a of an ordered set P , if a is not its top, then we can successively push down every maximal element which is not the element a until the element a is the top. We have proved an important result: *for an ordered set P and any element a in P there is an inversion Q of P such that a is the top of Q* (K. M. Mosesjan [1972]).

By applying Mosesjan's result, we can easily show that the graph M_5 illustrated in Figure 1.6 is not a covering graph. Suppose M_5 is a covering graph. According to Mosesjan's result, the element a can be the top of an orientation P of it, which implies that the elements a_1, a_2, a_3, a_4, a_5 are the lower covers of a in P . Since b_1, b_2, b_3, b_4 and b_5 constitute a pentagon, at least three consecutive elements, say b_1, b_2 and b_3 , of them constitute a chain. Without loss of generality, we can assume $b_1 > b_2 > b_3$ in P . Hence, $a_2 > b_1$ and $a_2 > b_3$, which is impossible in a diagram. So M_5 is not a covering graph. Similarly, by showing that the element v cannot be the top of some orientation, we can prove that the graph illustrated in Figure 2.1.9 is not a covering graph, either.



Another noncovering graph

Figure 2.1.9

O. Pretzel also investigated the pushdown operation and obtained some interesting results (cf. [1985] and [1986]).

Let P be the diagram of an ordered set. Associated with every cycle C of the covering graph G of the ordered set is a direction. Let $a > b (P)$ belong to the cycle C . We say the edge $a > b (P)$ is *forward* if the direction of C is from a to b , otherwise, that is,

if the direction of C is from b to a , then we say the edge $a \rightarrow b$ is *backward*. For example, let P be the ordered set illustrated in Figure 2.1.10 and let the direction of the cycle be clockwise (that is, the direction sequence is (a, f, e, d, c, b)). Then the edge $a \rightarrow f$ is forward and the edge $c \rightarrow d$ is backward. Define the *flow-difference* $f_P(C)$ of a cycle C for an orientation P of a covering graph G by

$f_P(C) = \text{the number of forward edges in } C - \text{the number of backward edges in } C.$

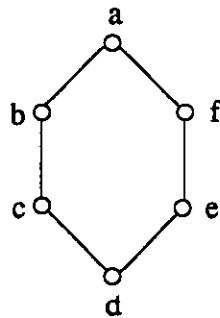


Figure 2.1.10

Let P and Q be two orientations of a covering graph. If for any cycle C of the covering graph, $f_P(C) = f_Q(C)$, then we say P and Q have the *same flow-difference*. For example, the ordered set P_1 and the ordered set P_2 illustrated in Figure 2.1.11 have the same flow-difference but P_1 and P_3 do not have the same flow-difference. Flow-difference is an inversion invariant, that is, *a diagram Q is an inversion of a diagram P if and only if P and Q have the same flow-difference* (O. Pretzel [1986]). So the ordered set P_3 is not an inversion of the ordered set P_1 , while P_2 is an inversion of P_1 (Figure 2.1.11).

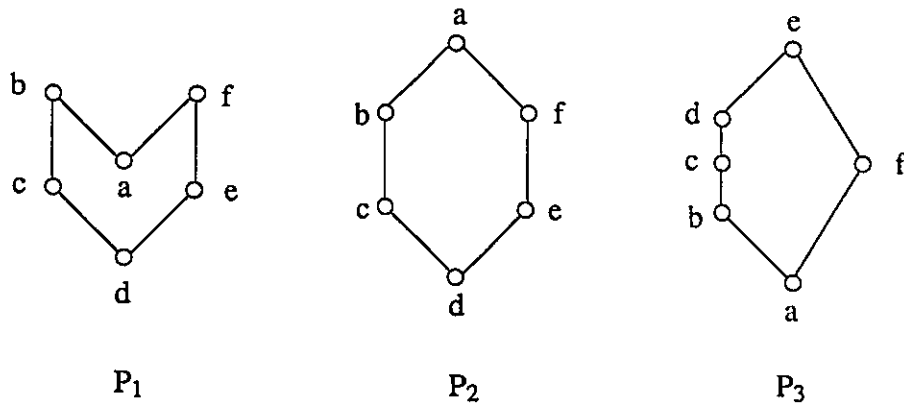


Figure 2.1.11

Pretzel's flow-difference theorem about inversion is the first interesting characterization of inversion. But it does not give a polynomial algorithm to test whether an ordered set is an inversion of another, since, according to Pretzel's theorem, to know if an ordered set is an inversion of another, we have to check each cycle of their common covering graph.

Pushdown is simple and important, for, if we consider any problem about orientations, it seems that so far pushdown is the only efficient tool to make use of. For example, in their paper which gives an upper bound of the number of orientations of a covering graph, G. Brightwell and J. Nešetřil [1991] used pushdown. O. Pretzel also used pushdown to investigate noncovering graphs, the relation between chromatic number of a graph and the "k-good" orientations of the graph (cf. O. Pretzel [1985] and [1986]). In Chapter 3 and Chapter 4, we will make repeated use of pushdown.

The following proposition will be used later.

PROPOSITION 2.1.3 (O. Pretzel [1986]). *Let P be an orientation of a covering graph G and let Q be an inversion of P by pushing down the vertices $v_1, v_2, v_3, \dots, v_n$ in that order (this includes the assumption that after v_1, \dots, v_{i-1} have been pushed down v_i is maximal, but a vertex can occur several times in the sequence).*

- 1) *If x and y are adjacent in G and $x = v_i = v_j$ for $i \neq j$, then $y = v_k$ for some $i < k < j$. So the numbers of occurrences of x and y differ by at most one and the edge xy has the same direction in P and in Q if and only if x and y occur equally often.*
- 2) *$P = Q$ if and only if all vertices of G occur equally often in the sequence v_1, \dots, v_n . The sequence in which they all occur once are precisely that obtained by taking the vertices of G in descending order according to some linear extension of P .*
- 3) *If x and z are arbitrary vertices of G the numbers of occurrences of x and z differ by at most $d(x, z)$.*

2.2. A NECESSARY AND SUFFICIENT CONDITION FOR INVERSIONS

In this section, we will give a necessary and sufficient condition for an ordered set to be an inversion of another, which, in turn, gives the first polynomial algorithm to test if an ordered set is an inversion of another.

A *zigzag* connecting two elements a and b of a diagram P is a sequence $a = x_0, x_1, x_2, \dots, x_m = b$ such that x_k covers x_{k+1} or x_{k+1} covers x_k in P . Let E consist of the edges $a_1 >- b_1, a_2 >- b_2, \dots$ and let $P - E$ denote the diagram whose edge set consists of the edges belonging to P but not to E (Figure 2.2.1). Let U_E denote the subset of P with $S(U_E)$ consisting of those elements of P connected to some a_i in $P - E$ (that is, the elements a in P such that there is a zigzag connecting a and a_i in $P - E$) (cf. Figure 2.2.1). Then U_E is an *upper set* of P , that is, if x belongs to U_E and $y > x$, then y belongs to U_E , too. Let D_E denote the subset of P with $S(D_E)$ consisting of those elements of P connected to some b_i in $P - E$ (Figure 2.2.1). Similarly, D_E is a *down set*, that is, if x belongs to D_E and $y < x$, then y belongs to D_E . We call E a *cut* of P if $S(D_E) \cap S(U_E) = \emptyset$ (Figure 2.2.2). It should be noticed that the definition of cut here is different from the *edge cut* used in graph theory, although, according to the definition, a cut of a diagram is an edge cut of its covering graph. An edge cut of the covering graph of a diagram may not necessarily be a cut (of the diagram) (cf. Figure 2.2.2).

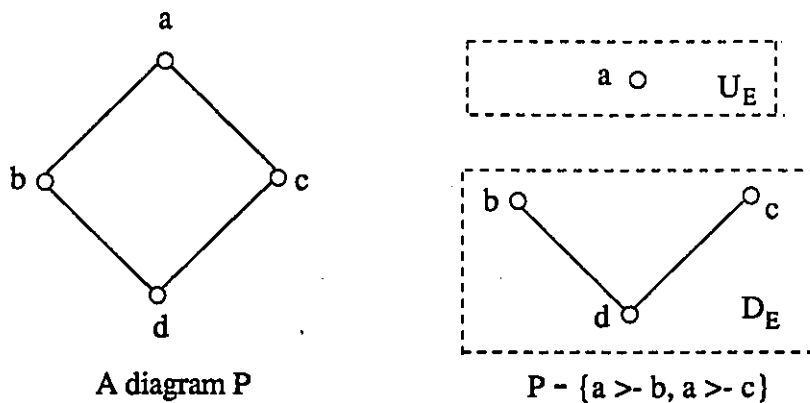


Figure 2.2.1

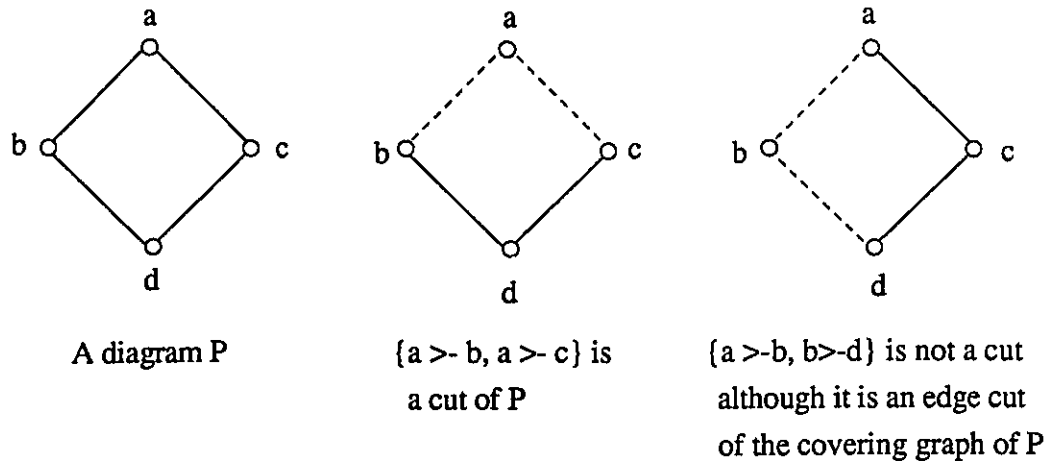


Figure 2.2.2

The following theorem establishes the relation between inversion and cuts.

THEOREM 2.2.1 (W. F. Liu and I. Rival [1991]). *An ordered set Q is an inversion of an ordered set P if and only if the reversed edges can be partitioned into cuts.*

Before proving the theorem, we emphasize that Theorem 2.2.1 implies a polynomial algorithm to check whether an ordered set Q is an inversion of an ordered set P .

THEOREM 2.2.2. *There is an algorithm with complexity $O(|S(P)|^5)$ to test if an ordered set Q is an inversion of an ordered set P .*

Proof. Consider the following algorithm.

An algorithm:

- 1) If the covering graph of the ordered Q is different from that of the ordered set P , then Q is not an inversion of P .
- 2) Let E be the set of the reversed edges by Q , that is,

$$E = \{a >- b (P): b >- a (Q)\}.$$
- 3) If E is empty, then Q is an inversion of P .

- 4) If, for any edge $a >- b$ of E , there is a zigzag in $P - E$ connecting a and d for some edge $c >- d$ belonging to E (it is possible that the edge $a >- b$ is identical to the edge $c >- d$), then Q is not an inversion of P . Otherwise, let

$$E_0 = \{x >- y (P) \in E : \text{there is no zigzag in } P - E \text{ connecting } x \text{ and } v \\ \text{for some edge } u >- v (P) \text{ belonging to } E\}.$$

- 5) Replace E and P by $E - E_0$ and the ordered set P_0 consisting of two subsets U_{E_0} and D_{E_0} of P plus the edge set $E_0^r = \{y >- x (Q) : x >- y (P) \in E_0\}$, that is, for two elements u and v in $U_{E_0} (D_{E_0})$, $u > v$ in P_0 if and only if $u > v (U_{E_0})$ ($u > v (D_{E_0})$); for two elements u in U_{E_0} and v in D_{E_0} , $v > u$ in P_0 if and only if for some edge $x >- y (P)$ belonging to E_0 , $v \geq y (D_{E_0})$ and $x \geq u (U_{E_0})$.

Go to Step 3.

First, we consider the complexity of the algorithm.

Let $n = |S(P)|$. Step 1 takes at most n^2 units of time; Step 2 takes at most n^2 units of time; the first part of Step 4 takes at most n^3 units of time, since it takes at most n units of time to check, for an edge $a >- b$ of E , if there is a zigzag in $P - E$ connecting the elements a and d for some edge $c >- d$ belonging to E (for example, by applying depth first search strategy), and there are at most n^2 edges and n elements in $S(P)$; each of Step 3, Step 4 and Step 5 iterates at most n^2 times since $|E| < n^2$.

Now, we prove the correctness of the algorithm.

Obviously, Step 3 is true. Since E_0 is a cut, P_0 is an inversion P which reverses exactly the edges of E_0 . Therefore, all we have to prove is that the first half of Step 4 of the algorithm is correct. To this end, suppose, for any edge $a >- b (P)$ of E , there is a zigzag in $P - E$ connecting a and d for some edge $c >- d (P)$ belonging to E . Then, there is a sequence of edges $a_1 >- b_1, a_2 >- b_2, \dots, a_k >- b_k$ belonging to E such that there is a zigzag in $P - E$ connecting a_{i-1} and b_i , $2 \leq i \leq k+1$, where, $b_{k+1} = b_1$. Therefore, it is

neither the case that any two edges of $a_i > - b_j$ are in the same cut, nor that each of them belongs to a different cut, which means that E cannot be partitioned into cuts. We have a contradiction to Theorem 2.2.1.

Q. E. D.

Following are two examples to illustrate how the above algorithm works.

Example 2.2.1. Let P and Q be the ordered sets given in Figure 2.2.3.

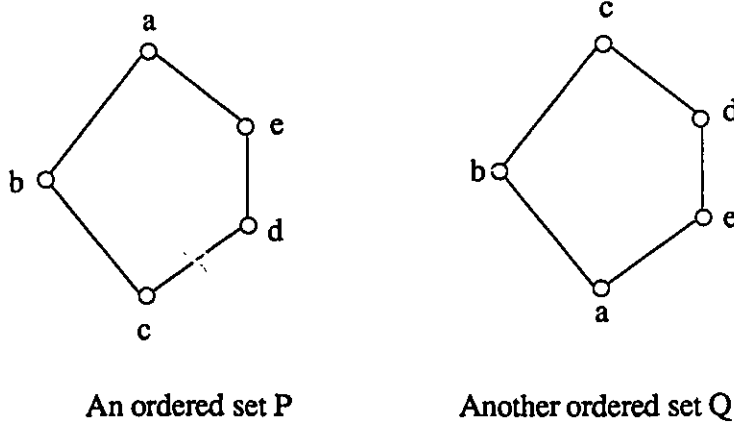


Figure 2.2.3

First iteration:

$$E = \{a > - b, b > - c, d > - c, e > - d, a > - e\}; E_0 = \{a > - b, a > - d\};$$

P_0 is the ordered set illustrated in Figure 2.2.4 (1).

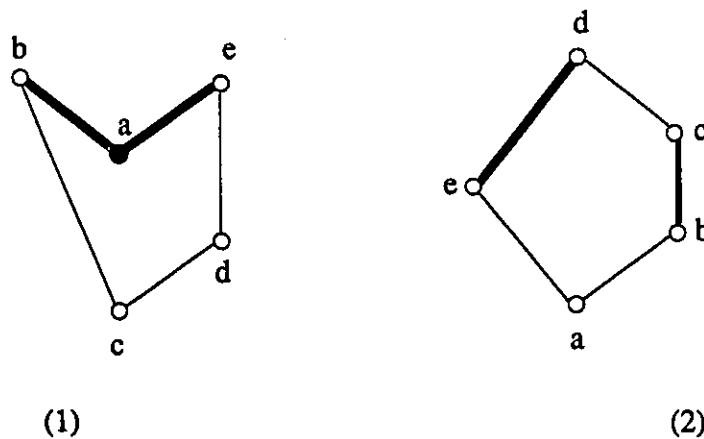


Figure 2.2.4

Second iteration:

$$E = \{b >- c, d >- c, e >- d\}; E_0 = \{b >- c, e >- d\};$$

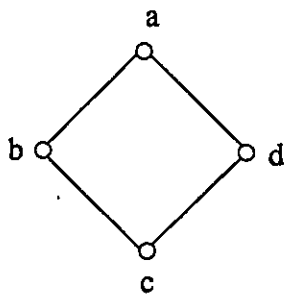
P_0 is the ordered set illustrated in Figure 2.2.4 (2).

Third iteration:

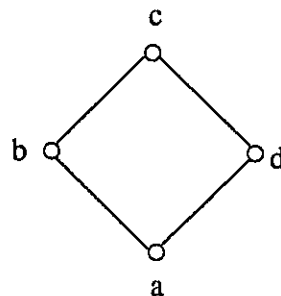
$$E = \{d >- c\};$$

since there is a zigzag in $P - E$ connecting d and c , so, Q is not an inversion of P .

Example 2.2.2. Let P and Q be the ordered sets given in Figure 2.2.5.



An ordered set P



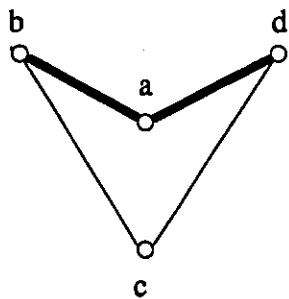
Another ordered set Q

Figure 2.2.5

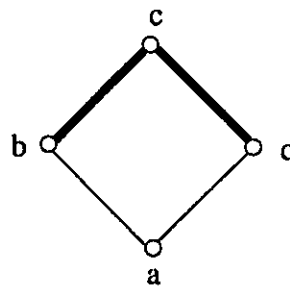
First iteration:

$$E = \{a >- b, b >- c, d >- c, a >- d\}; E_0 = \{a >- b, a >- d\};$$

P_0 is the ordered set illustrated in Figure 2.2.6 (1).



(1)



(2)

Figure 2.2.6

Second iteration:

$$E = \{b > c, d > c\};$$

$$E_0 = E;$$

P_0 is the ordered set illustrated in Figure 2.2.6 (2).

Third iteration:

E is empty;

so, the ordered set Q is an inversion of the ordered set P .

Associated with any inversion Q of an ordered set P is a (mixed) sequence of pushdowns and pullups. Since a pullup can be replaced by a sequence of pushdowns, we may associate with any inversion Q a sequence (a_1, a_2, \dots) of pushdowns (here we assume that after pushing down a_i , a_{i+1} is maximal, $1 \leq i$). The sequence of pushdowns need not be unique. For example, if P and Q are the ordered sets illustrated in Figure 2.2.7, then both (b, c) and (a, b, c, d, c, b) are sequences of pushdowns producing the same inversion Q of P . Some elements of the sequence may occur several times. Let $r_Q(x)$ stand for the number of occurrences of the element x in the inversion sequence for Q (cf. the remark at the end of this section). For example, see the ordered set Q illustrated in Figure 2.2.7, which is obtained from P by the pushdown sequence (a, b, c, d, c, b) , $r_Q(a) = r_Q(d) = 1$ and $r_Q(b) = r_Q(c) = 2$.

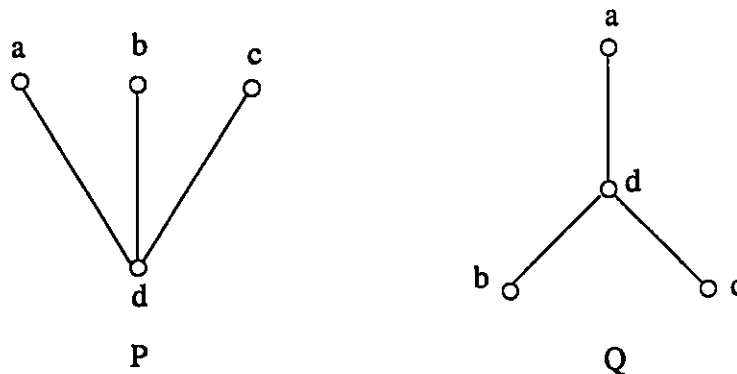


Figure 2.2.7

Now, we can begin to prove Theorem 2.1.1. To this end, we first establish a sequence of lemmas, which are of independent importance.

LEMMA 2.2.3 (cf. O. Pretzel [1986]). *Let P be an ordered set and let Q be an inversion of P . Let $a >- b$ in P .*

- 1) $a >- b$ in Q if and only if $r_Q(a) = r_Q(b)$.
- 2) $b >- a$ in Q if and only if $r_Q(a) = r_Q(b) + 1$.

LEMMA 2.2.4 (cf. O. Pretzel [1986]). *Let P be an ordered set and let Q be an inversion of P . Let $a > b$ in P . Then $r_Q(a) \geq r_Q(b)$ and, if, moreover, $r_Q(a) = r_Q(b)$ then $a > b$ in Q .*

LEMMA 2.2.5. *Suppose Q is the inversion of an ordered set P in whose pushdown sequence each element a occurs $r_Q(a)$ times. Then the inversion of Q in whose pushdown sequence each element a occurs $(\max\{r_Q(x): x \text{ in } Q\} - r_Q(a))$ times is exactly the ordered set P .*

Proof. Let R be the inversion of Q . Suppose $a >- b$ (P). If $r_Q(a) = r_Q(b)$, then

$$\begin{aligned} r_R(a) - r_R(b) &= [\max\{r_Q(x): x \text{ in } Q\} - r_Q(a)] - [\max\{r_Q(x): x \text{ in } Q\} - r_Q(b)] \\ &= 0. \end{aligned}$$

According to Lemma 2.2.3, $a >- b$ (Q) and therefore $a >- b$ (R). If $r_Q(a) = r_Q(b) + 1$, then $b >- a$ (Q), by Lemma 2.2.3. From the following equality

$$\begin{aligned} r_R(a) - r_R(b) &= [\max\{r_Q(x): x \text{ in } Q\} - r_Q(a)] - [\max\{r_Q(x): x \text{ in } Q\} - r_Q(b)] \\ &= r_Q(b) - r_Q(a) = -1, \end{aligned}$$

and Lemma 2.2.3, we know $a >- b$ (R). In any event, $a >- b$ (P) implies $a >- b$ (R). So $P = R$ since R and P have the same covering graph. **Q. E. D.**

LEMMA 2.2.6. *Let P be an ordered set, Q an inversion of P . Let S be a subset of P such that for each a, b in S , $r_Q(a) = r_Q(b)$. Then the subset S in Q has the same order as it has in P .*

Proof. Suppose Q is the inversion of the ordered set P in whose pushdown sequence each element a occurs $r_Q(a)$ times. Suppose $a > b (P)$. By Lemma 2.2.4, we have $a > b (Q)$. Suppose $a > b (Q)$. By Lemma 2.2.5, the inversion of Q in whose pushdown sequence each element a occurs $r(a) = \max\{r_Q(x) : x \text{ in } Q\} - r_Q(a)$ times is the ordered set P . Since for any two elements a and b in Q , $r(a) = r(b)$, so, by Lemma 2.2.4, $a > b$ in P . That is, $a > b (P)$ if and only if $a > b (Q)$.

Q.E.D.

From the construction of an inversion as a sequence of pushdowns we have the following fact.

LEMMA 2.2.7. *Let U be an upper set of an ordered set P . Then there is an inversion Q of P such that*

$$r_Q(a) = \begin{cases} 1 & \text{if } a \text{ belongs to } U \\ 0 & \text{otherwise} \end{cases}$$

LEMMA 2.2.8. *Let P be an ordered set. For any cut E of P there is an inversion which reverses precisely that edges of E .*

Proof. Let E consist of the edges $a_1 > b_1, a_2 > b_2, \dots$ in P . Then U_E is an upper set, so, according to Lemma 2.2.7, there is an inversion Q in which each element a of U_E satisfies $r_Q(a) = 1$. From Lemma 2.2.4, only the edges of the cut E are reversed.

Q.E.D.

From the definition of cut and Lemma 2.2.8, we have the following lemma .

LEMMA 2.2.9. *Let P be an ordered set and let Q be an inversion of P . Then any cut of P either all of whose edges are reversed, or none of whose edges is reversed, in Q , is also a cut of Q .*

Now, we are ready to prove Theorem 2.2.1.

Proof of Theorem 2.2.1. First, we establish the sufficiency. To this end let

$$E = E_1 \cup E_2 \cup \dots \cup E_k$$

be a disjoint union of cuts of P . If $k = 1$, then the conclusion follows from Lemma 2.2.8. Let $k > 1$. In view of Lemma 2.2.8 there is an inversion Q_1 of P which reverses precisely the edges of E_1 . According to Lemma 2.2.9, each of E_2, E_3, \dots, E_k is a cut of Q_1 . Then, by induction on the number of cuts, there is an inversion Q of Q_1 which reverses precisely the edges of $E_2 \cup E_3 \cup \dots \cup E_k$. So, Q is the inversion of P which reverses exactly the edges of E .

We turn to the necessity. To this end it is convenient to formulate and prove a "cut partition" as follows.

Every inversion Q of P induces a partition of P into subsets A_0, A_1, \dots defined by

$$S(A_i) = \{x : r_Q(x) = r_Q(a_0) + i\},$$

where, $r_Q(a_0) = \min\{r_Q(x) : x \in P\}$, and the reversed edges are precisely those between successive pairs of A_i 's (this is illustrated schematically in Figure 2.2.8).

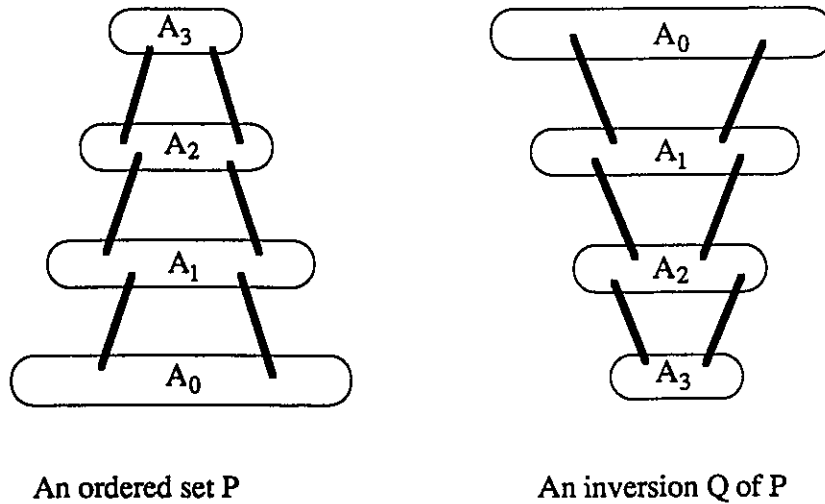


Figure 2.2.8

Choose a_0 in P such that $r_Q(a_0) = \min\{r_Q(x) : x \text{ in } P\}$ ($r_Q(a_0)$ may be 0). Then let

$$S(A_0) = \{x : r_Q(x) = r_Q(a_0)\}.$$

The following properties of A_0 are straightforward to verify.

For each x not in A_0 and y in A_0 , $x \not\leq y$ in Q [Lemma 2.2.4].

A_0 in Q is identically ordered to A_0 in P [Lemma 2.2.6].

There are elements a, a_1 with a in A_0 and a_1 not, such that $a > a_1$ in Q and $r_Q(a_1) = r_Q(a) + 1$ [P is connected].

Now, let

$$S(A_1) = \{x : r_Q(x) = r_Q(a_1) = r_Q(a_0) + 1\}.$$

Suppose that the subsets A_j , $j = 0, 1, \dots, m$, have been defined satisfying the following properties.

$$S(A_j) = \{x : r_Q(x) = r_Q(a_j) = r_Q(a_0) + j\};$$

A_j in Q is identically ordered to A_j in P ;

if x belongs to A_j and y to A_{j-1} then $x \not\leq y$ in Q ;

if y belongs to A_{j-1} and x to $P - (A_0 \cup A_1 \cup \dots \cup A_{j-1})$ with $y > x$ in Q then x belongs to A_j and $x > y$ in P .

In the light of Lemma 2.2.4 again, for each x not in $A_0 \cup A_1 \cup \dots \cup A_m$, and an element y in A_m , $x \not\geq y$ in Q . As P is connected, there are elements a and a_{m+1} with a in $A_0 \cup A_1 \cup \dots \cup A_m$ and a_{m+1} not, such that $a > a_{m+1}$ in Q . Since a_{m+1} is not in $A_0 \cup A_1 \cup \dots \cup A_m$ and a is in A_m , so, $r_Q(a_{m+1}) = r_Q(a) + 1 = r_Q(a_0) + m + 1$. Let

$$S(A_{m+1}) = \{ x : r_Q(x) = r_Q(a_0) + m + 1 \}.$$

Then, A_{m+1} satisfies these properties, too:

A_{m+1} in Q is identically ordered to A_{m+1} in P ;

if x belongs to A_m and y to A_{m+1} then $y \not\geq x$ in Q ;

if y belongs to A_{m+1} and x to $P - (A_0 \cup A_1 \cup \dots \cup A_m)$ with $y > x$ in Q then x belongs to A_m and $x > y$ in P .

In this way we produce a decomposition, that is, a "cut partition", of P into blocks A_0, A_1, \dots . Let E_i stand for the set of all the edges $x > y$ in P such that x belongs to A_i and y to A_{i-1} (Figure 2.2.8). It is now evident that each E_i is a cut of P , and Q just reverses the edges of $E_1 \cup E_2 \cup \dots$. This completes the proof of Theorem 2.2.1.

Q.E.D.

Here is an application of Theorem 2.2.1.

COROLLARY 2.2.10. *For any edge $a > b$ of an ordered set P with at least two edges, there is an inversion Q of P which reverses the edge and $Q \neq P^d$.*

Proof. In fact, if the covering graph of P is a tree, then the edge $a > b$ is a cut; otherwise, the edges, each of which joins an element of the down set $\{v : v \leq b(P)\}$ of P and an element of the upper set W with $S(W) = S(P) - \{v : v \leq b(P)\}$ of P , constitute a cut which does not contain all edges of P . In any case, the cut obtained above does not contain all the edges of P . According to Theorem 2.2.1, the reversal of the cut leads an inversion Q of P with $Q \neq P^d$.

Q. E. D.

Let Q be an inversion of an ordered set P . It is well known that even if the maximal elements of P are identical to the maximal elements of Q , the two ordered sets P and Q need not be identical. For example, both the ordered sets P and Q illustrated in Figure 2.2.9 are inversions of P with elements a and b as their maximal elements, although P is not equal to Q . If, however, P and Q have just one maximal element, the top, the same for both, then $P = Q$ (O. Pretzel [1986]). We have a general result. Before giving our result, we make the following observation. Let Q be an inversion of an ordered set P and let $\min \{r_Q(x) : x \text{ in } P\} = m$, then, from the cut partition in the proof of Theorem 2.2.1, the ordered set obtained from P by pushing down each element x of P $(r_Q(x) - m)$ times is exactly Q .

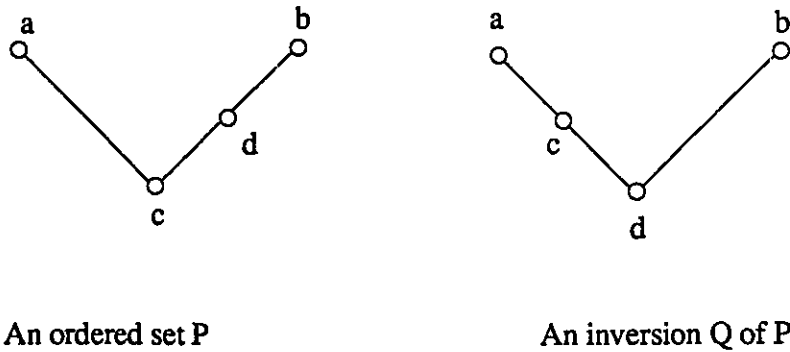


Figure 2.2.9

THEOREM 2.2.11 (W. P. Liu and I. Rival [1991]). *Let P and Q be inversions of an ordered set. Let $\max P = \max Q$ and let $r_P(a) = r_P(b)$ for any two elements a and b in $\max P$. Then $P = Q$ if and only if $r_Q(x) = r_Q(y)$ for any two elements x and y in $\max Q$.*

Proof. According to the above observation, we can assume that $r_P(a) = 0$ for any element a in $\max P = \max Q$. Let $P_1 = P$ and $Q = P_2$ be inversions of R . In terms of the cut partition as in the proof of Theorem 2.2.1, each inversion P_i induces a partition $A_0^i, A_1^i, \dots, i = 1, 2$. Suppose there are maximal elements a, b of P_2 such that $r_{P_2}(a) < r_{P_2}(b)$. Then a and b lie in different A_k^2 sets, although both a and b lie in A_0^1 . As R is

connected there is a zigzag $Z = \{a = x_1, x_2, \dots, x_m = b\}$ such that, for each k , $x_k > x_{k+1}$ or $x_{k+1} > x_k$. We can count the number of edges reversed, in P_1 and in P_2 , along Z , which, in view of the hypothesis that $P_1 = P_2$, must be identical, too. Thus, let

$$q_i = |\{x_{k+1} > x_k \text{ in } Z : x_k > x_{k+1} \text{ in } P_i\}| - |\{x_k > x_{k+1} \text{ in } Z : x_{k+1} > x_k \text{ in } P_i\}|.$$

As a, b both belong to A_0 it follows from the cut partition that $q_1 = 0$. On the other hand, $q_2 \neq 0$. This is a contradiction.

To prove the converse, let us suppose that $r_{P_2}(a) = r_{P_2}(b)$ for any two maximal elements a, b of P_2 . Let x belong to A_0^2 , say $x \leq a$ in P_2 for some maximal element a of P_2 . By the cut partition, $x \leq a$ in R . By hypothesis, $r_{P_1}(a) = 0$, so, $r_{P_1}(x) = 0$, too. Thus, x belongs to A_0^1 . Now let x belong to A_0^1 . Then for some maximal element a of P_1 , $x \leq a$ in A_0^1 , so $x \leq a$ in R . Again, $r_{P_2}(x) = r_{P_2}(a)$, by Lemma 2.2.4 and the assumption that $r_{P_2}(a) = r_{P_2}(b)$ for elements a and b in $\max Q$, whence x belongs to A_0^2 . Therefore, $A_0^1 = A_0^2$. We may then apply the same argument to conclude that $A_1^1 = A_1^2, A_2^1 = A_2^2$, etc. In particular, $P_1 = P_2$.

Q.E.D.

Remark. We know that the sequence of pushdowns associated with an inversion of an ordered set need not be unique. Suppose the sequences S_1 and S_2 produce the same inversion Q of an ordered set P . Let $r_{S_i}(x)$ stand for the number of occurrences of x in the sequence S_i and

$$A_k^i = \{x \text{ in } P : r_{S_i}(x) = \min\{r_{S_i}(y) : y \text{ in } P\} + k\},$$

$i = 1, 2$. According to the argument of the first part of the proof of Theorem 2.2.11, (with S_1 corresponding to P_1 , P corresponding to R and S_2 to P_2) $A_k^1 = A_k^2$ for all k , although $r_{S_1}(x)$ need not be equal to $r_{S_2}(x)$. That is the reason that we use $r_Q(x)$ instead of $r_S(x)$ for an inversion Q of an ordered set P .

2.3. THE COMPLEXITY OF INVERSIONS AND THE NUMBER OF INVERSIONS. In this section, we use Theorem 2.2.1 to investigate the complexity and the number of inversions.

Let P be an n -element ordered set and let Q be an inversion of P . Let $p(n)$ be the smallest number such that any inversion of P can be produced with at most $p(n)$ pushdowns. Let $e(P, Q)$ be the smallest number of reversals of edges such that Q can be produced by pushdowns (each edge may be reversed many times). Let

$$e(n) = \max\{e(P, Q) : P \text{ is an } n\text{-element ordered set and } Q \text{ an inversion of } P\}.$$

Here is the result which is an improvement of that of W. P. Liu and I. Rival [1991].

THEOREM 2.3.1 (W. P. Liu). $p(n) \leq n(n-1)/2$ and $e(n) \leq n^3/12 + 5n^2/8 + 17n/15$.

Proof. First, we prove the first inequality. Let $\deg(a)$ stand for the *degree* of an element a in P , that is, the total number of upper covers and lower covers of a in P . Let Q be an inversion of P and let

$$r_Q(a_0) = \min\{r_Q(a) : a \text{ in } P\}.$$

Then, according to the cut partition, proved in Section 2.2,

$$p(n) = \sum (r_Q(a_0) + i) |A_i|.$$

where $|A|$ denotes the cardinality of the set A . As any inversion amounts, according to Theorem 2.2.1, to reversing the edges of a disjoint union of cuts, we may suppose, by the argument given before Theorem 2.2.11, that $r_Q(a_0) = 0$. Therefore,

$$p(n) = |A_1| + 2|A_2| + \dots + k|A_k|.$$

To obtain an upper bound for $p(n)$ observe that $1 \leq |A_i| \leq n - i$, for each $i = 1, 2, \dots, k$, so

$$p(n) \leq 1 + 2 + 3 + \dots + (k - 1) + k(n - k)$$

$$\begin{aligned} &= nk - (k^2 + k)/2 \\ &\leq (n^2 - n)/2. \end{aligned}$$

In fact, to see that this upper bound can be attained it is enough to take an n -element chain $a_0 < a_1 < a_2 < \dots < a_{n-1}$. It is easy to verify that the inversion in which a_0 is the top requires $(n^2 - n)/2$ individual pushdowns. (Each element a_i occurs at least i times in any inversion sequence.)

Now, we establish the other inequality, $e(n)$, by induction on the number of elements. If $n = 2$, the inequality is obviously true. Let $e(n) = e(P, Q)$. We may assume that any minimal element a of the diagram $(A_i)_P$ is an upper cover of an element b of the diagram $(A_{i-1})_P$, where,

$$A_i = \{x \text{ in } P : r_Q(x) = r_Q(a_0) + i, \text{ where, } r_Q(a_0) = \min\{r_Q(v) : v \text{ in } P\}\}$$

$0 \leq i \leq k$, and $(A_i)_P$ denotes the subset of P with $S((A_i)_P) = A_i$. (Notice that the edges of the diagram of $(A_i)_P$ are those of the diagram of P whose end elements belong to $S(A_i)$. For convenience, say such a subset $(A_i)_P$ of P a *subdiagram* of P induced by A_i .) For otherwise we can obtain an inversion P' from P by pulling up the element a . Obviously, Q is an inversion of P' and

$$A_j' = A_j, j \neq i, i+1,$$

$$A_i' = A_i - \{a\}$$

and

$$A_{i+1}' = A_{i+1} \cup \{a\}.$$

Furthermore, $e(P', Q) = e(P, Q) + \deg(a) > e(P, Q)$, which is a contradiction.

Take a maximal element u in $(A_k)_P$.

Case 1. $\deg(u) \leq (n - k + 1)/2$.

Let $P_1 = P - u$ and let $Q_1 = Q - u$. Then

$$e(P, Q) \leq e(P_1, Q_1) + 2k(n - k + 1)/2.$$

By induction hypothesis, $e(P_1, Q_1) \leq (n - 1)^3/12 + 5(n - 1)^2/8 + 17n/15$. So

$$\begin{aligned} e(P, Q) &\leq (n - 1)^3/12 + 5(n - 1)^2/8 + 17(n - 1)/15 + 2k(n - k + 1)/2 \\ &\leq n^3/12 + 5n^2/8 + 17n/15. \end{aligned}$$

Case 2 $\deg(u) \geq (n - k)/2 + 1$

Choose a lower cover v of u . Let $P_1 (Q_1)$ be a diagram whose edge set consists of those edges of $P (Q)$ which are not "adjacent" to the element v . If v belongs to A_k , then any element in $A_0 \cup A_1 \cup \dots \cup A_{k-2}$ is not a lower cover of v . So,

$$\deg(v) \leq n - (n - k)/2 - 1 - (k - 1) = (n - k)/2.$$

Therefore

$$\begin{aligned} e(P, Q) &\leq e(P_1, Q_1) + 2k[(n - k)/2] \\ &\leq (n - 1)^3/12 + 5(n - 1)^2/8 + 17(n - 1)/15 + 2k[(n - k)/2] \\ &\leq n^3/12 + 5n^2/8 + 17n/15. \end{aligned}$$

If v belongs to A_{k-1} , then any element in $A_0 \cup A_1 \cup \dots \cup A_{k-3}$ is not a lower cover of v .

So,

$$\deg(v) \leq n - (n - k)/2 - 1 - (k - 2) = (n - k)/2 + 1.$$

Therefore

$$\begin{aligned} e(P, Q) &\leq e(P_1, Q_1) + (2k - 1)[(n - k)/2 + 1] \\ &\leq (n - 1)^3/12 + 5(n - 1)^2/8 + 17(n - 1)/15 + (2k - 1)[(n - k)/2 + 1] \\ &\leq n^3/12 + 5n^2/8 + 17n/15. \end{aligned}$$

Q.E.D.

The bound for $e(n)$ given in Theorem 2.3.1 is not sharp. We have an example (Figure 2.3.1) showing that

$$e(n) \geq (2n^3 + n^2 + 31n + 50)/27.$$

In fact, we tentatively conjecture that $(2n^3 + n^2 + 31n + 50)/27$ is close to the sharp bound of $e(n)$.

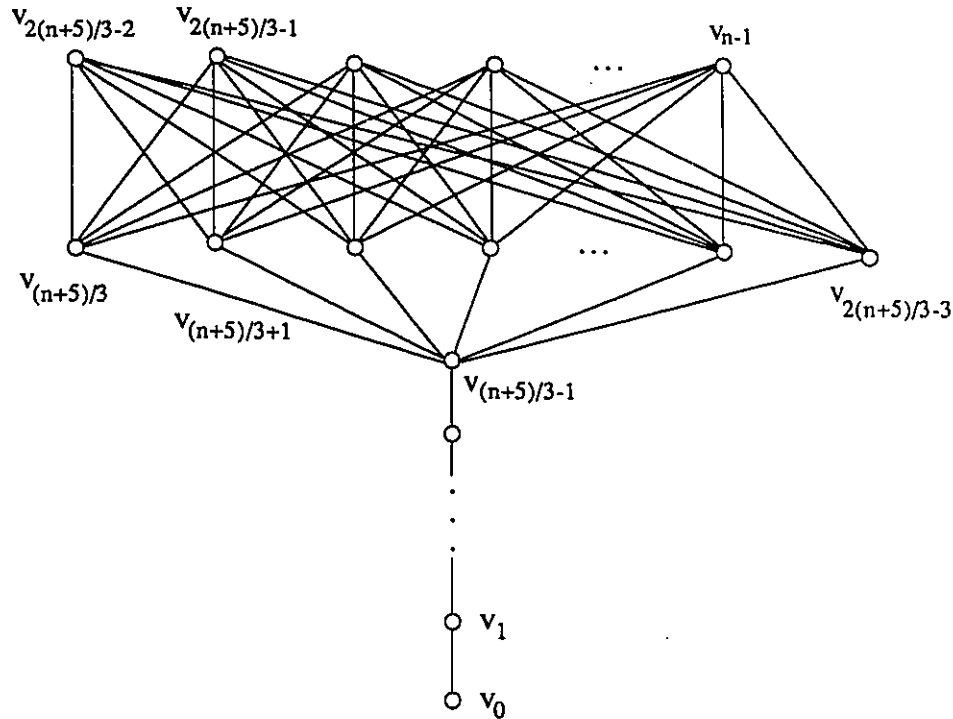


Figure 2.3.1

Now, we turn to consider the number of inversions of an ordered set. How many distinct orientations are there for a covering graph? or equivalently, how many reorientations does an ordered set have? We will consider this problem in general in Chapter 4. However, we may estimate a lower bound by counting the minimum number of distinct inversions.

THEOREM 2.3.2 (O. Pretzel [1986]). *Any n -element ordered set has at least*

$$n^2/4 + n/2$$

inversions.

Here is an improvement of Theorem 2.3.2.

THEOREM 2.3.3 (W. P. Liu and I. Rival [1991]). *Any n -element ordered set has at least $(n^2 + 2n)/2 - n \log_2 n$ inversions.*

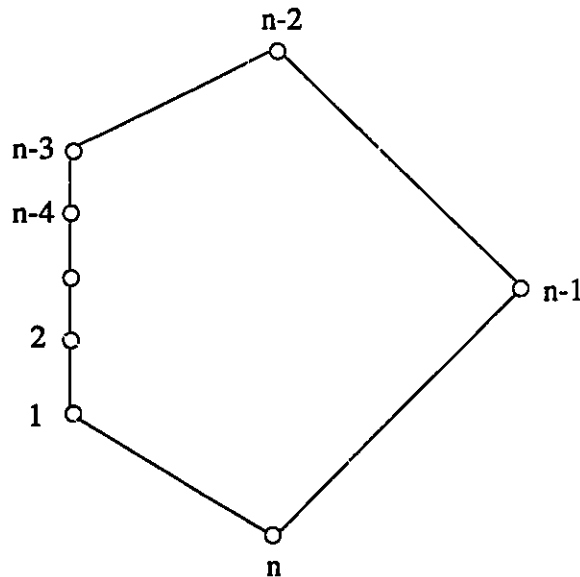


Figure 2.3.2

Before proving Theorem 2.3.3, we give an example. Let P be the ordered set illustrated in Figure 2.3.2. Using Theorem 2.2.1, it is easy to verify that the ordered set has precisely

$$(n - 2)(n - 3)/2 + 2(n - 2) + 1 = (n^2 - n)/2$$

distinct inversions.

LEMMA 2.3.4 (O. Pretzel [1986]). *For any two elements in an ordered set, neither of which covers the other, there is an inversion with just these two as the maximal elements.*

Proof. Let a and b be distinct elements of an ordered set P such that neither a covers b nor b covers a . (It may be, however, that $a > b$ or $b > a$.) Let $U = \{x : x \leq a \text{ in } P\}$. By Mosesjan's theorem, there is an inversion of P_1 in which $\max P_1 = \{a\}$. Now consider the pushdown P_2 of P_1 obtained by pushing down a . In P_2 , the elements a and b are *noncomparable*, that is, neither $a \geq b$ nor $b \geq a$. Let U_1 be the upper set of P_2 consisting of all elements x such that $x \leq a$ or $x \leq b$. Applying Lemma 2.2.7 will produce an inversion P_3 of P_2 (of course, of P) in which $\max P_3 = \{a, b\}$.

Q.E.D.

Now, we are ready to prove Theorem 2.3.3.

Proof of Theorem 2.3.3. Suppose P contains an element a satisfying

$$\deg(a) \geq \log_2(n^2/2).$$

Consider the inversion P_1 of P in which a is the top element. (There is such an inversion, according to Mosesjan's theorem.) Then, in P_1 , a has at least $\log_2(n^2/2)$ lower covers. Evidently, for every distinct subset S of the $\log_2(n^2/2)$ lower covers of a there is an inversion in which the set of maximal elements is precisely S . Adding the inversion with a as the top, this gives, in all, at least

$$2^{\log_2(n^2/2)} - 1 + 1 = n^2/2 \geq (n^2 + 2n)/2 - n \log_2 n$$

inversions.

Let us suppose that, for each a in P , $\deg(a) \leq \log_2(n^2/2)$. If P_1 is an inversion of P in which a is the top, then there are at least $n - 1 - \deg(a)$ elements not covered by a . According to Lemma 2.3.4, the element a together with any one of these $n - 1 - \deg(a)$ elements is a subset for which there is an inversion with just these two as the maximal

elements. This holds for every element a . There are also n inversions with a top. Therefore, in all, there are at least

$$\sum_{a \text{ in } P} [n - 1 - \text{deg}(a)]/2 + n$$

inversions of P ,

$$\sum_{a \text{ in } P} [n - 1 - \text{deg}(a)]/2 + n \geq (n^2 + n)/2 - n \times \max\{\text{deg}(x): x \in P\}/2$$

$$\geq (n^2 + n)/2 - [n \log_2 (n^2/2)]/2$$

$$=(n^2 + 2n)/2 - n \log_2 n.$$

Q.E.D.

CHAPTER 3. PLANAR COVERING GRAPHS

3.1. INTRODUCTION. What is a "good" diagram? By far the best known criterion is planarity. An ordered set P is *planar* if it has a plane representation of its diagram in which no pair of lines corresponding to two edges of P intersect, except possibly at an endpoint. For example, the ordered set 2^3 is not planar and the ordered set illustrated in Figure 3.1.1 (1) is planar (Figure 3.1.1 (2)).

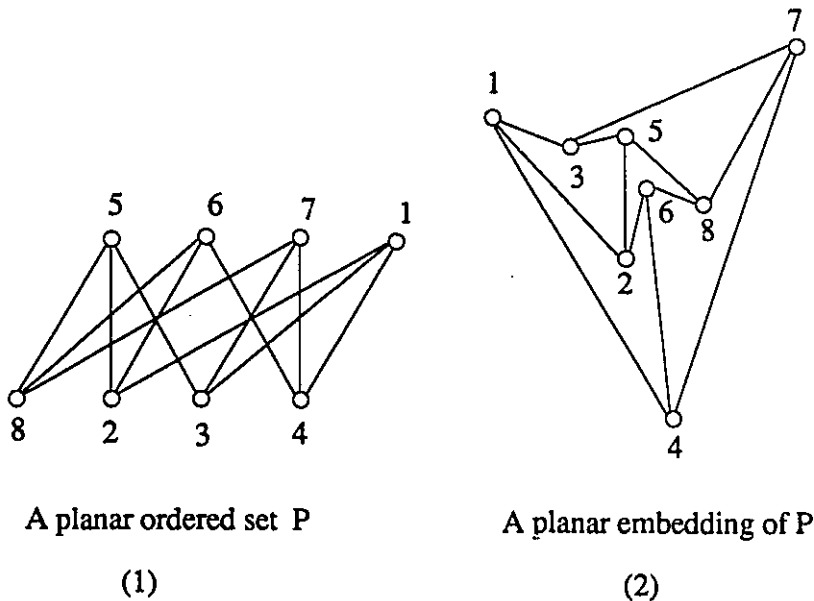


Figure 3.1.1

We know that in graph theory there are well known results, showing that both elegant combinatorial characterizations of planar graphs (e.g. K. Kuratowski [1930]) and efficient algorithms (e.g. J. Hopcroft and R. E. Tarjan [1974]) for testing if a graph is planar. If we consider special kinds of ordered sets—lattices—then, similarly, there are beautiful combinatorial characterizations for them and efficient algorithms to detect if a lattice is planar. The planarity of an ordered set is closely related to its "order dimension": *a lattice is planar if and only if it has dimension two* (cf. G. Birkhoff [1967]);

D. Kelly and I. Rival [1975] gave a list of forbidden diagrams of a planar lattice: *a lattice is planar if and only if it contains no subset illustrated in Figure 3.1.2.* An efficient

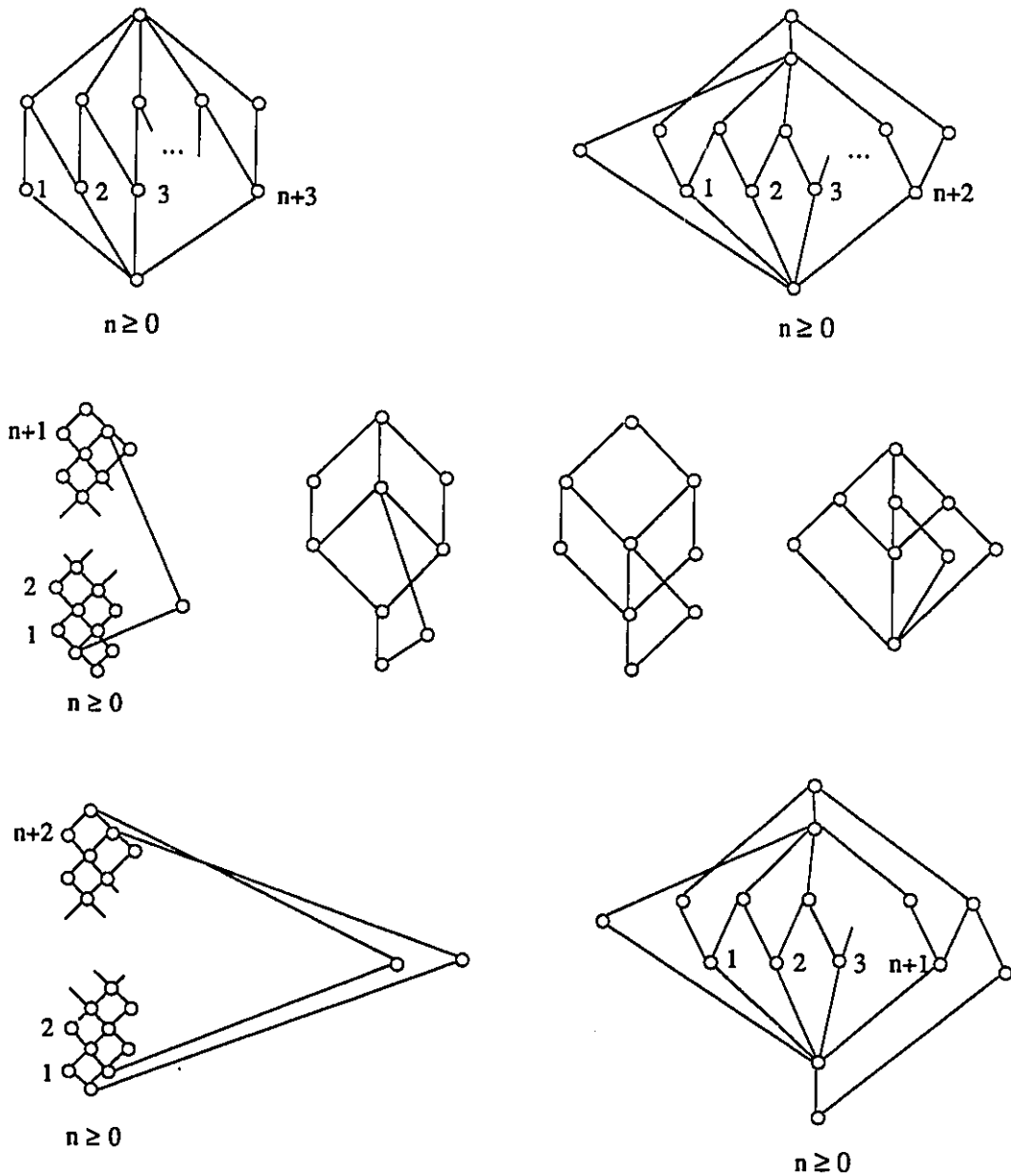


Figure 3.1.2

algorithm (with complexity $O(n)$) of testing whether an n -element lattice is planar is due to C. R. Platt [1976]: *a lattice is planar if and only if the graph obtained from its covering*

graph by adding the edge joining the top and the bottom of the lattice is planar, since, according to J. Hopcroft and R. E. Tarjan [1974], the complexity to test if a graph is planar is $O(n)$.

An ordered set with top and bottom is not necessarily a lattice (Figure 3.1.3). But, *if an ordered set with top and bottom is planar, then the ordered set must be a lattice* (D. Kelly and I. Rival [1975]).

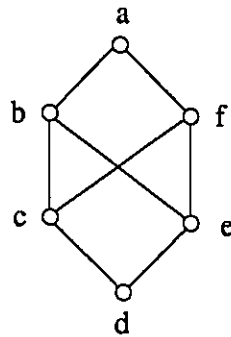


Figure 3.1.3

Generally, in spite of all results about planar lattices, the problem of characterizing planar ordered sets remains open. We know few general classes of planar ordered sets. Since it seems unlikely that we can characterize planar ordered sets efficiently, we want to know what covering graphs have planar orientations. We may ask whether any planar covering graph has a planar orientation. Indeed, G. di Battista, W. P. Liu and I. Rival [1990] have the following conjecture.

CONJECTURE 1. *Any planar covering graph has a planar orientation.*

Furthermore, we hope that some "shortest" orientation of a planar covering graph is planar. The length of a maximum chain of an ordered set P is called the *length* of P .

CONJECTURE 2 (W. P. Liu). *Any planar covering graph has a planar orientation with length at most three .*

In Chapter 4, we will prove that *any n -element planar (covering) graph has at least $2^{n/3}$ orientations*. It seems that not every orientation of a planar covering graph is planar. (Maybe, it is also interesting to classify those planar covering graphs which have no nonplanar orientation. Obviously, any orientation of a cycle is planar and any orientation of a tree is planar.) In order to prove Conjecture 1 is true, we have to decide, among at least $2^{n/3}$ orientations of an n -element planar covering graph, which one is planar, that is, we have to invent an algorithm to select our favourite orientations.

Let us consider another property of ordered sets - s -genus.

The "genus" of a graph is closely related to the planarity of the graph. In fact, *a graph is planar if and only if its genus is zero* (cf. F. Harary [1972]). The sphere genus of an ordered set can be defined similarly, that is, the *sphere genus*, or simply, *s-genus* of an ordered set P is the minimum number of handles needed to draw the diagram of P on the surface of a sphere with handles, without any crossing edges, in such a way that, whenever $a > b$ in P , the z -coordinate of a is larger than the z -coordinate of b , and all edges of P are monotonic with respect to the z -coordinate (cf. K. Ewacha, W. Li and I. Rival [1990]). An element of an ordered set is *irreducible* if the element has only one upper cover or one lower cover. Presently, K. Reuter and I. Rival [1990] proved a theorem which establishes a relation between the degree of an irreducible element and the s -genus of an ordered set.

THEOREM 3.1.1 (K. Reuter and I. Rival [1990]). *Every lattice contains an irreducible element of degree at most*

$$4 \text{ s-genus} + 3.$$

And also in their paper, K. Reuter and I. Rival obtained a bound of the number of the edges of a lattice using s-genus (cf. B. Bollobas and I. Rival [1979]).

THEOREM 3.1.2 (K. Reuter and I. Rival [1990]). *For any lattice L ledge set of $L \leq (4 \times \text{s-genus}(L) + 3) \times |L|$.*

The following theorems establish a relation between the s-genus of an ordered set and the planarity of its covering graph.

THEOREM 3.1.3 (S. Foldes, I. Rival and J. Urrutia [1988]). *A lattice has s-genus zero if and only its covering graph is planar.*

THEOREM 3.1.4 (S. Foldes, I. Rival and J. Urrutia [1988]). *Any ordered set with top and bottom whose covering graph is planar has s-genus zero.*

According to the definition of s-genus, if an ordered set is planar, then the ordered set has s-genus zero. But the converse is not true, an ordered set with s-genus zero is not necessarily planar (recall that *a graph has genus zero if and only if the graph is planar*). Here is another conjecture .

CONJECTURE 3 (W. P. Liu and I. Rival [1990]). *Any planar graph has an orientation with s-genus zero.*

All in all, in this chapter, we will study two properties of ordered sets - planarity and s-genus. In sections 3.2 and 3.3, we consider the planarity and in section 3.4, we study the s-genus.

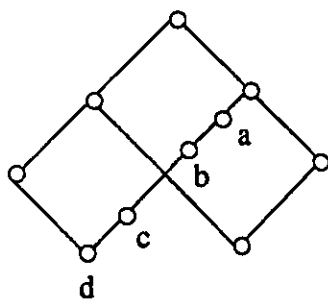
3.2 PLANAR BIPARTITE GRAPHS. Recall, using pushdown, *an n -element bipartite graph has at least $2^{n/2}$ orientations.* If the graph is planar, then among those orientations, there is at least one planar bipartite orientation. Namely, we have the following theorem, which claims that Conjecture 2 is true if the graph is bipartite.

THEOREM 3.2.1 (G. di Battista, W. P. Liu and I. Rival [1990]). *Any planar bipartite graph has a planar bipartite orientation.*

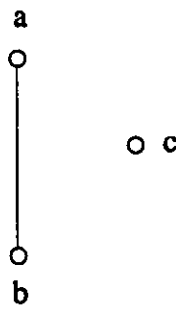
(In fact, *any planar bipartite graph has at least two planar bipartite orientations* since if an ordered set P is planar and connected, then its dual P^d is also planar and $P \neq P^d$.)

To prove Theorem 3.2.1, we introduce some definitions and notations which will be used later.

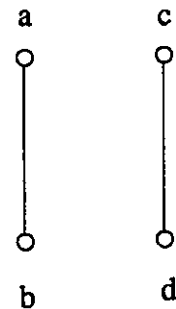
A path C of a graph G is called a *2-path* if C consists of degree two vertices, except possibly the two end vertices. For a 2-path C , let $G - C$ stands for the subgraph of G induced by those vertices which are not *internal vertices* of the path C (a vertex of a path is a *internal vertex* if it is not an end vertex of the path (cf. W. T. Tutte [1984]).



A diagram P
Figure 3.2.1



$((a, b, c))_P$
Figure 3.2.2



$((a >- b, c >- d))_P$
Figure 3.2.3

Let P be a diagram. By $\epsilon(P)$, we denote the edge set of P (recall that a diagram can be regarded as a special digraph); for a subset $S_1 (\epsilon_1)$ of $S(P) (\epsilon(P))$, let $(S_1)_P ((\epsilon_1)_P)$

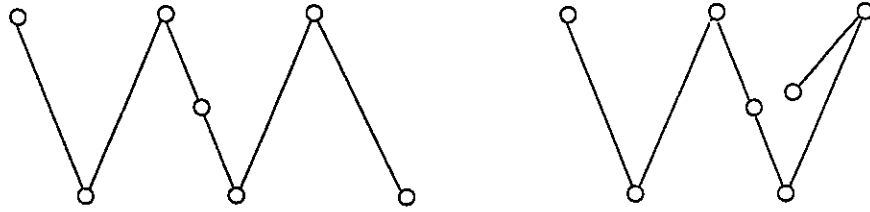
denote a diagram whose edges are those both of whose elements belong to S_1 (those of ϵ_1). For example, if P is the diagram illustrated in Figure 3.2.1, then $((a, b, c))_P$ is the diagram illustrated in Figure 3.2.2 and $((a \succ b, c \succ d))_P$ is the diagram illustrated in Figure 3.2.3.

For convenience, the term "a face of a planar graph" means the subgraph of the graph induced by the edges constituting the face in a fixed underlying planar embedding. The *boundary* of a planar graph is the exterior face of the planar embedding of the graph.

Let $F(Z)$ be a face (path) of a planar covering graph G and let P be a planar orientation of G . When we say the face (zigzag) $F(Z)$ of P , we mean the diagram $(\epsilon(F))_P$ ($(\epsilon(Z))_P$) in an underlying planar embedding of P (here, we assume that F is a face of an underlying planar embedding of P). So we use the edges or the vertices of $F(Z)$ to denote the face (zigzag). We also say "the boundary of P " which is the "exterior face" (that is, the unbounded face) of the planar embedding of P .

A planar embedding of zigzag $Z = \{v_1, v_2, v_3, \dots, v_k\}$ is *simple* if the line segment representing $v_{i-1} \succ v_i$ or $v_i \succ v_{i-1}$ is on the left (or on the extension) of the line segment representing $v_i \succ v_{i+1}$ or $v_{i+1} \succ v_i$, $2 \leq i \leq k-1$ (Figure 3.2.4); a zigzag is *maximal* if the two end elements are maximal (Figure 3.2.5); a planar embedding of a face is *simple* if it consists of two simple embeddings of two maximal zigzags, one of which is called the *lower part* of the face and the other one the *upper part* (Figure 3.2.6).

For simplicity, we directly call a zigzag (face) simple instead of calling some planar embedding of the zigzag (face) simple.



A simple zigzag

A nonsimple zigzag

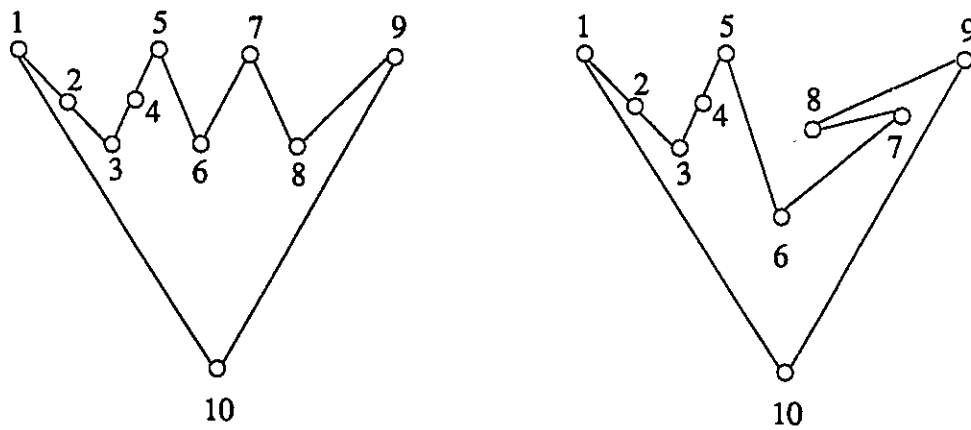
Figure 3.2.4



A maximal zigzag

A nonmaximal zigzag

Figure 3.2.5

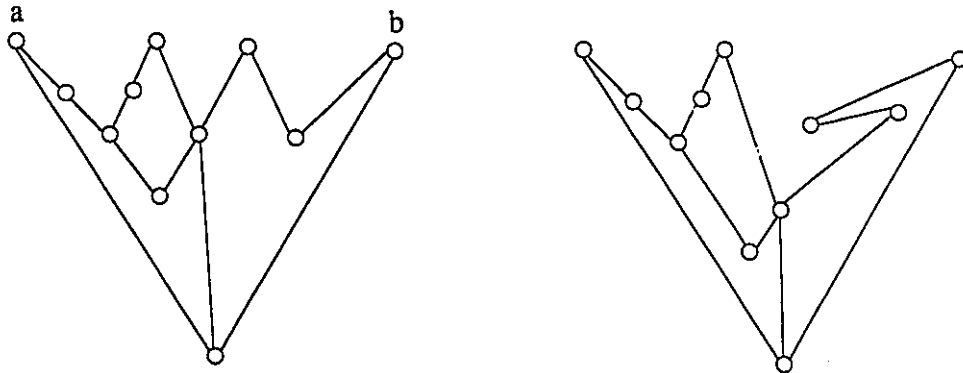


A simple face consisting of the lower part {1, 10, 9} and the upper part {1, 2, 3, 4, 5, 6, 7, 8, 9}

A nonsimple face

Figure 3.2.6

Call a planar embedding of a planar orientation *simple* if all the faces of the embedding are simple (Figure 3.2.7). Let F be the boundary of a simple planar embedding of an ordered set P . The two common maximal elements of the two maximal zigzags constituting F are called *turning vertices* of the planar embedding (Figure 3.2.7).



A simple planar embedding of an ordered set P with a and b as the turning vertices

A nonsimple planar embedding of the ordered set P

Figure 3.2.7

Now, we establish some lemmas.

LEMMA 3.2.2 (cf. W. T. Tutte [1984]). *Let G be a 2-connected planar graph whose maximum degree is at least three. Then there is a 2-path Z such that $G - Z$ is 2-connected.*

Proof. By induction on the number of edges of G . Let uv be an edge of G . If the graph $G - \{uv\} = G_1$ is 2-connected, then we are done. Suppose that G_1 is not 2-connected. Let H_1, H_2, \dots, H_k be 2-connected components of G_1 and let v_1, v_2, \dots, v_m be the cut vertices of G_1 . Then $m = k - 1$ since G is 2-connected. Furthermore, we can assume that G_1 looks like the graph illustrated in Figure 3.2.1, since G is 2-connected. Since the maximum degree of G is at least three, there is a 2-connected component, say

H_i , that contains at least two vertices. If the maximum degree of H_i is at least three, by induction, there is a 2-path Z_1 in H_i such that $H_i - Z_1$ is 2-connected. Let Z be the subpath of Z_1 with v_{i-1} or v_i , $1 \leq i \leq k$ ($v_0 = v$ and $v_k = u$) or both as the end vertices of Z if Z_1 contains v_{i-1} or v_i or both, otherwise let $Z = Z_1$. Then, obviously $G - Z$ is 2-connected and Z is a 2-path. If all vertices of H_i have degree two, then let Z be a path connecting v_{i-1} and v_i . Then Z is a 2-path and $G - Z$ is 2-connected.

Q.E.D.

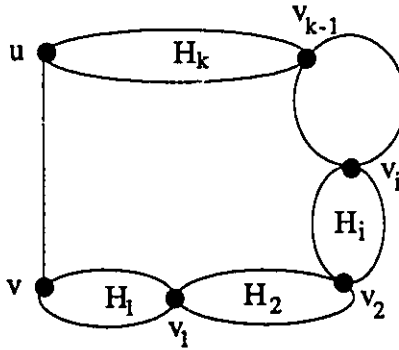


Figure 3.2.8

COROLLARY 3.2.3. *Let Z be the path described in Lemma 3.2.2 and let F be a face of G . Then $\epsilon(Z) \cap \epsilon(F) \neq \emptyset$ implies $\epsilon(F) \supset \epsilon(Z)$.*

Proof. Let v_1, v_2, \dots, v_k be the path Z . Suppose $\epsilon(Z) - \epsilon(F) \neq \emptyset$. Then, for some i , $1 \leq i \leq k$, either the edge $v_{i-1}v_i$ or v_iv_{i+1} does not belong to F . Without loss of generality, we can assume that $v_{i-1}v_i$ is not an edge of F and $v_{i-2}v_{i-1}$ is an edge of F . Notice that F is a face, v_{i-1} is of degree at least three and not an end vertex of Z , which is a contradiction since Z is a 2-path.

Q.E.D.

LEMMA 3.2.4 (cf. W. T. Tutte [1984]). *Let H be a 2-connected subgraph of G and let L be a 2-path of G . If $|V(L) \cap V(H)| = 2$, then $H \cup L$ is also 2-connected.*

LEMMA 3.2.5. *Let $F = \{a_1, a_2, \dots, a_m\}$ be a face (clockwise order) of a 2-connected planar graph whose maximum degree is at least three. Then there is a 2-path Z such that either $\varepsilon(F) \cap \varepsilon(Z) = \emptyset$ or $Z = \{a_i, a_{i+1}, \dots, a_j\}$, where, $1 \leq i < j \leq m$, and $G - Z$ is 2-connected.*

Proof. Let $W = \{a_p, a_{p+1}, \dots, a_q\}$ be a 2-path of G , $p \leq m < q$, where $a_i = a_{i-m}$ if $i > m$ and let $[\varepsilon(F) - \varepsilon(W)] \cup \varepsilon(U)$ be a face of $G_1 = G - \varepsilon(W)$, where $U = \{u_1 = a_p, u_2, \dots, u_r = a_q\}$. Let $a_t = a_i$, $t < p$. If G_1 has maximum degree two, then $\{a_t, a_{t+1}, \dots, a_p\}$ is a 2-path with $t < p \leq m$. Suppose G_1 has maximum degree at least three. Let T be a 2-path of G_1 such that $G_1 - T$ is 2-connected.

Case 1. $[[\varepsilon(F) - \varepsilon(W)] \cup \varepsilon(U)] \cap \varepsilon(T) = \emptyset$

Then T is a 2-path of G , $\varepsilon(F) \cap \varepsilon(T) = \emptyset$, and $G - T$ is 2-connected.

Case 2. $\varepsilon(U) \supseteq \varepsilon(T)$

Then T is a 2-path of G , $\varepsilon(F) \cap \varepsilon(T) = \emptyset$, and $G - T$ is 2-connected.

Case 3. $\varepsilon(F) - \varepsilon(W) \supseteq \varepsilon(T)$

Then, T is a 2-path of G , $T = \{a_i, a_{i+1}, \dots, a_j\}$, where $t \leq i < j \leq p$, and $G - T$ is 2-connected.

Case 4. $\varepsilon(T) \cap \varepsilon(U) \neq \emptyset$ and $[\varepsilon(F) - \varepsilon(W)] \cap \varepsilon(T) \neq \emptyset$

If $u_1 u_2$ and $u_r u_{r-1}$ belong to $\varepsilon(T)$, then, according to Lemma 3.2.4, either the path $\varepsilon(F) - \varepsilon(W) = \{a_t, a_{t+1}, \dots, a_p\}$ is a 2-path and the graph $G - [\varepsilon(F) - \varepsilon(W)]$ is 2-connected if $\varepsilon(T) \supseteq \varepsilon(F) - \varepsilon(W)$, or the path U is a 2-path of G , and the graph $G - U$ is 2-connected if

$\varepsilon(T) \supseteq \varepsilon(U)$. Otherwise, without loss of generality, we can assume u_1u_2 belongs to $\varepsilon(T)$ but $u_r u_{r-1}$ does not. Then, according to Lemma 3.2.4, the edges of $\varepsilon(T) - \varepsilon(U)$ constitute a 2-path of the graph G ; furthermore, $\varepsilon(F) - \varepsilon(W) \supseteq \varepsilon(T) - \varepsilon(U)$ and the graph $G - [\varepsilon(T) - \varepsilon(U)]$ is 2-connected.

Q.E.D.

LEMMA 3.2.6. *Let G be a bipartite graph consisting of a cycle C and a path Z with $|V(C) \cap V(Z)| = 2$. Then for any simple planar embedding of a bipartite orientation Q of C , there is a simple planar embedding of a bipartite orientation of G such that the zigzag consisting of the edges of Z divides the interior face of Q into two faces.*

Proof. This is evident.

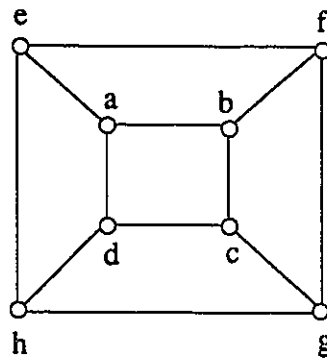
Q.E.D.

Let $F = \{a_1, a_2, \dots, a_m\}$ be a face of a bipartite planar graph (clockwise order). Then m is even. Furthermore, $\{a_1, a_3, \dots, a_{m-1}\}$ is an independent set. For otherwise, if a_{2i+1} is adjacent to a_{2k+1} ($k > i$), then the edges $a_{2i+1}a_{2i+2}, \dots, a_{2k}a_{2k+1}, a_{2i+1}a_{2k+1}$ constitute an odd cycle. Similarly, $\{a_2, a_4, \dots, a_m\}$ is an independent set.

THEOREM 3.2.7. *Let $F = \{a_1, a_2, \dots, a_{2k}\}$ be the boundary (clockwise order) of a 2-connected planar bipartite graph G . Then there is a simple planar embedding E of a bipartite orientation P of G satisfying the following conditions:*

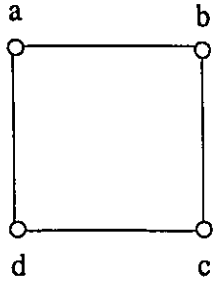
- 1) $\max P \supseteq \{a_1, a_3, \dots, a_{2k-1}\}$;
- 2) F is the boundary of E ;
- 3) $a_1, a_3, \dots, \text{ and } a_{2k-1}$ belong to the upper part of the boundary of E ;
- 4) a_1 and a_{2k-1} are the turning vertices;
- 5) f is a face of G if and only if f is a face of E .

Theorem 3.2.7 tells us that there is a one-to-one mapping between the faces of an planar embedding of planar bipartite graph G and the faces of the corresponding planar embedding a bipartite orientation of G . Before giving the proof of Theorem 3.2.7, we use an example to describe how the proof of Theorem 3.2.7 will work. Let G be our favourite planar graph illustrated in Figure 3.2.9. We want to obtain a planar representation of a bipartite orientation P of G such that the face set of P is the same as that of G . To this end, first find a 2-path $C_1 = \{ef\}$ such that $G_1 = G - C_1$ is 2-connected; second, find a 2-path $C_2 = \{ae, eh\}$ of G_1 such that $G_2 = G_1 - C_2$ is 2-connected; third, find a 2-path $C_3 = \{dh, hg\}$ of G_2 such that $G_3 = G_2 - C_3$ is 2-connected; finally, find a 2-path $C_4 = \{cg, gf, fb\}$ of G_3 such that $G_4 = G_3 - C_4$ is 2-connected. We can produce the planar bipartite orientations P_4, P_3, P_2, P_1 and P of G_4, G_3, G_2, G_1 and G as those illustrated in Figure 3.2.10.

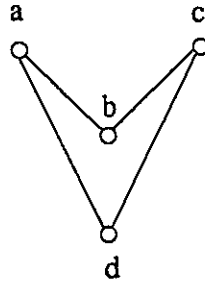


A planar bipartite graph G

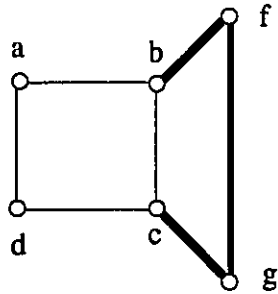
Figure 3.2.9



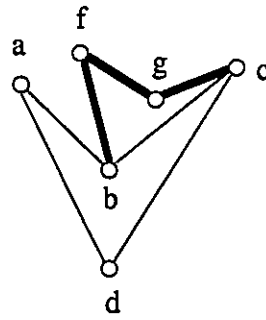
A planar embedding of G_4



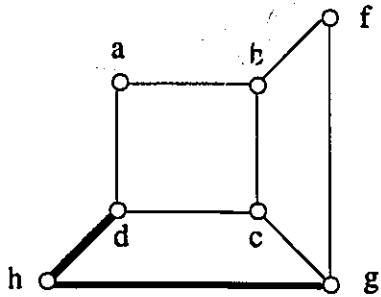
A planar embedding of P_4



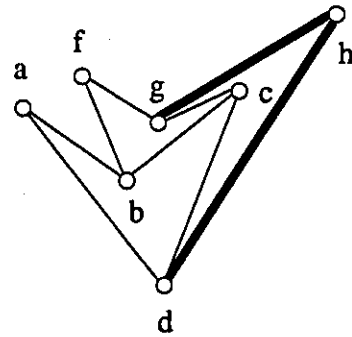
A planar embedding of G_3



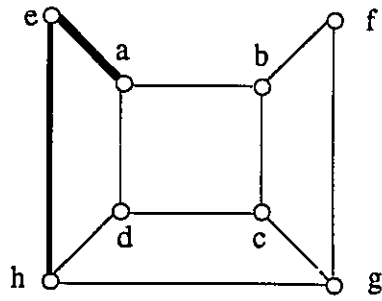
A planar embedding of P_3



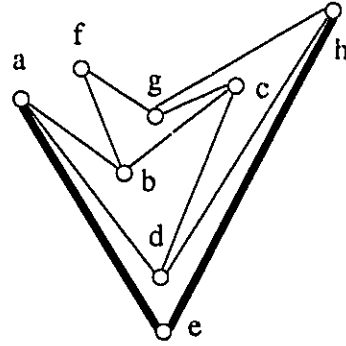
A planar embedding of G_2



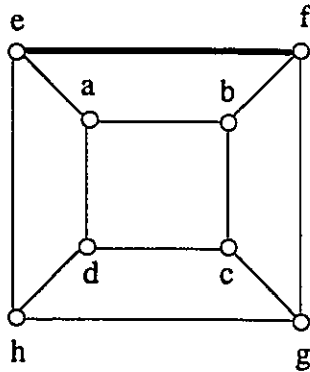
A planar embedding of P_2



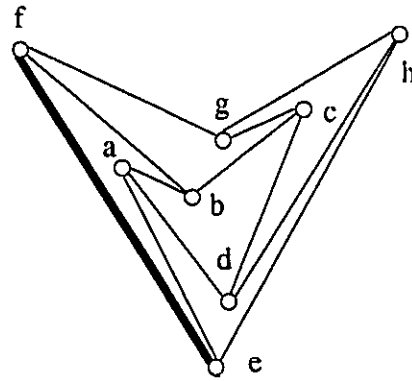
A planar embedding of G_1



A planar embedding of P_1



A planar embedding of G



A planar embedding of P

Figure 3.2.10

Proof of Theorem 3.2.7. By induction on the number of edges. If G is a cycle, then, obviously, there is a simple embedding of a bipartite orientation of G satisfying the conditions. Suppose that G has maximum degree at least three. According to Lemma 3.2.3, there is a 2-path Z such that $G_1 = G - Z$ is 2-connected.

First, assume $\epsilon(Z) \cap \epsilon(F) = \emptyset$. Let F_1 and F_2 be the two faces of G such that $\epsilon(F_1) \cap \epsilon(Z) \neq \emptyset$ and $\epsilon(F_2) \cap \epsilon(Z) \neq \emptyset$. Then, according to Corollary 3.2.3, the graph

$F' = [\varepsilon(F_1) \cup \varepsilon(F_2)] - \varepsilon(Z)$ is a face of G_1 . By induction hypothesis, there is a simple planar embedding E_1 of a bipartite orientation Q of G_1 satisfying the following conditions:

- 1) $\max Q \supseteq \{a_1, a_3, \dots, a_{2k-1}\}$;
- 2) F is the boundary of E_1 ;
- 3) $a_1, a_3, \dots, a_{2k-1}$ belong to the upper part of the boundary of E_1 ;
- 4) a_1 and a_{2k-1} are the turning vertices;
- 5) f is a face of G_1 if and only if f is a face of E_1 .

Notice that F' is a simple interior face of Q . By Lemma 3.2.6, we can add the zigzag Z to Q to obtain a simple planar embedding of a bipartite orientation P of G which satisfies all the five conditions, too.

Now, according to Lemma 3.2.5, we can assume that $Z = \{a_p, a_{p+1}, \dots, a_q\}$ is a 2-path of G , where, $1 \leq p < q \leq 2k$, and $G - Z$ is 2-connected. Let $\{a_p, a_{p+1}, \dots, a_q, w_{r-1}, \dots, w_2\} = F_3$ be the other face of G containing Z . Then $F'' = \varepsilon(F) \cup \varepsilon(F_3) - \varepsilon(Z)$ is a face of $G_1 = G - Z$. Write $a_p = w_1$, $a_q = w_r$ and $m = q - p + 1$.

Case 1. $p = 2i - 1$ and $q = 2j - 1$ (Figure 3.2.11)

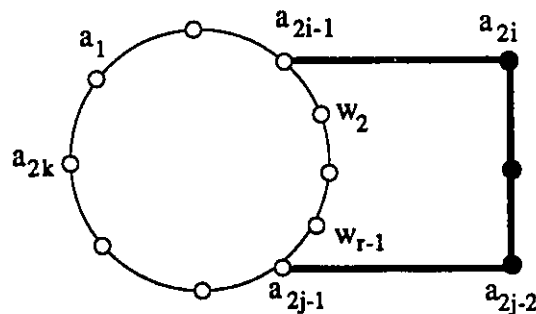


Figure 3.2.11

Then m and r are odd since G is bipartite. By induction hypothesis, there is a simple planar embedding E_1 of a bipartite orientation Q of G_1 satisfying the following conditions:

- 1) $\max Q \supseteq \{a_1, \dots, a_{2i-1}, w_3, \dots, w_{r-2}, a_{2j-1}, \dots, a_{2k-1}\} = I$;
- 2) F'' is the boundary of E_1 ;
- 3) I belongs to the upper part of the boundary of E_1 ;
- 4) a_1 and a_{2k-1} are the turning vertices;
- 5) f is a face of G_1 if and only if f is a face of E_1 (Figure 3.2.12).

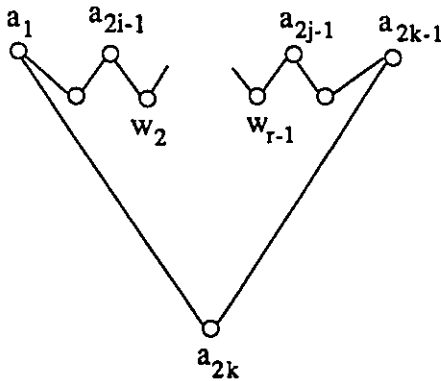


Figure 3.2.12

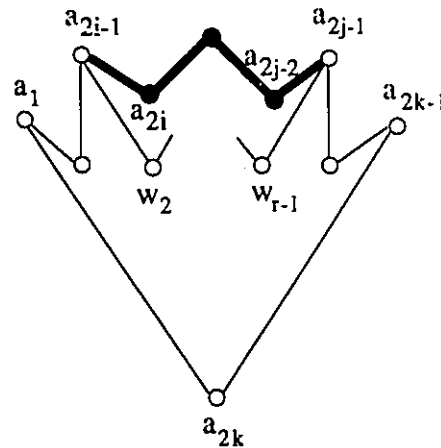


Figure 3.2.13

We can obtain a simple planar embedding E of a bipartite orientation P of G satisfying the following conditions by adding the zigzag Z to E_1 :

- 1) $\max P \supseteq \{a_1, a_3, \dots, a_{2k-1}\}$;
- 2) F is the boundary of E ;
- 3) $a_1, a_3, \dots, a_{2k-1}$ belong to the upper part of the boundary of E ;
- 4) a_1 and a_{2k-1} are the turning vertices;
- 5) f is a face of G if and only if f is a face of E (Figure 3.2.13).

Case 2. $p = 2i - 1$ and $q = 2j$, $j < k$ (Figure 3.2.14)

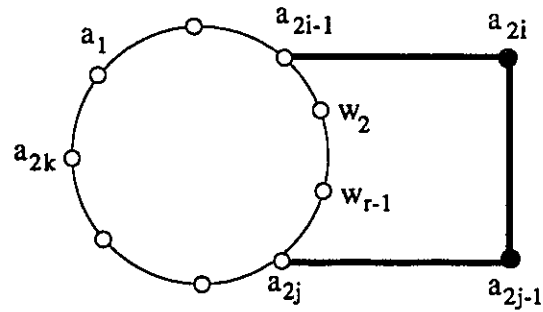


Figure 3.2.14

Then m and r are even since G is bipartite. By induction hypothesis, there is a simple planar embedding E_1 of a bipartite orientation Q of G_1 satisfying the following conditions:

- 1) $\max Q \supseteq \{a_1, a_3, \dots, a_{2i-1}, w_3, \dots, w_{r-1}, a_{2j+1}, \dots, a_{2k-1}\} = I$;
- 2) F'' is the boundary of E_1 ;
- 3) I belongs to the upper part of the boundary of E_1 ;
- 4) a_1 and a_{2k-1} are the turning vertices;
- 5) f is a face of G_1 if and only if f is a face of E_1 (Figure 3.2.15).

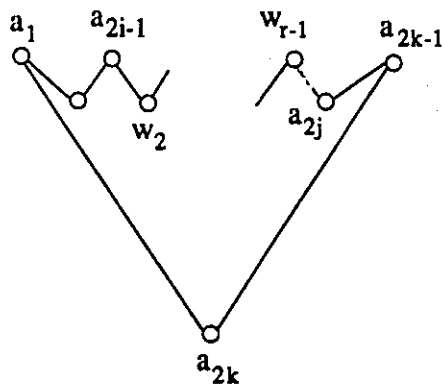


Figure 3.2.15

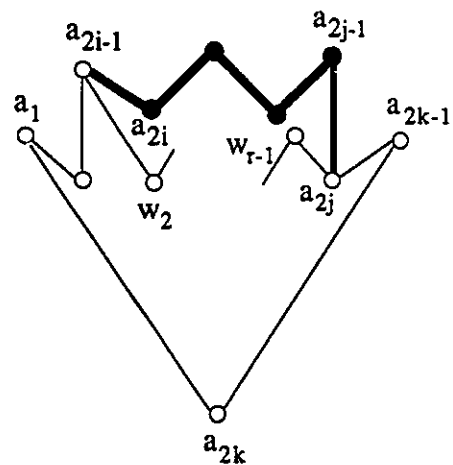


Figure 3.2.16

We can obtain a simple planar embedding E of a bipartite orientation P of G satisfying the following conditions by adding the zigzag Z to E_1 :

- 1) $\max P \supseteq \{a_1, a_3, \dots, a_{2k-1}\}$;
- 2) F is the boundary of E ;
- 3) $a_1, a_3, \dots, a_{2k-1}$ belong to the upper part of the boundary of E ;
- 4) a_1 and a_{2k-1} are the turning vertices;
- 5) f is a face of G if and only if f is a face of E (Figure 3.2.16).

Case 3. $p = 2i - 1$ and $q = 2k$ (Figure 3.2.17)

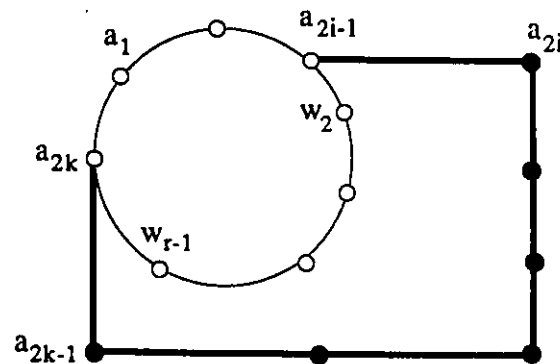


Figure 3.2.17

Then m and r are even since G is bipartite. By induction hypothesis, there is a simple planar embedding E_1 of a bipartite orientation Q of G_1 satisfying the following conditions:

- 1) $\max Q \supseteq \{a_1, a_3, \dots, a_{2i-1}, w_3, \dots, w_{r-1}\} = I$;
- 2) F'' is the boundary of E_1 ;
- 3) I belongs to the upper part of the boundary of E_1 ;
- 4) a_1 and w_{r-1} are the turning vertices;
- 5) f is a face of G_1 if and only if f is a face of E_1 (Figure 3.2.18).

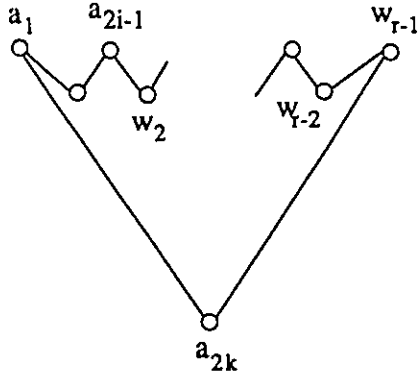


Figure 3.2.18

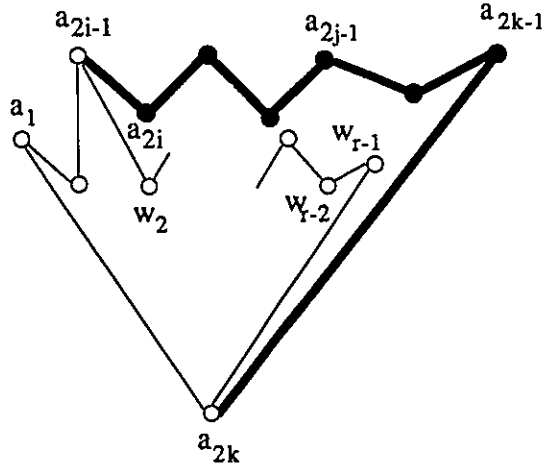


Figure 3.2.19

We can obtain a simple planar embedding E of a bipartite orientation P of G satisfying the following conditions by adding the zigzag Z to E_1 :

- 1) $\max P \supseteq \{a_1, a_3, \dots, a_{2k-1}\}$;
- 2) F is the boundary of E ;
- 3) $a_1, a_3, \dots, a_{2k-1}$ belong to the upper part of the boundary of E ;
- 4) a_1 and a_{2k-1} are the turning vertices;
- 5) f is a face of G if and only if f is a face of E (Figure 3.2.19).

Case 4. $p = 2i$ and $q = 2j - 1 < 2k$ (Figure 3.2.20)

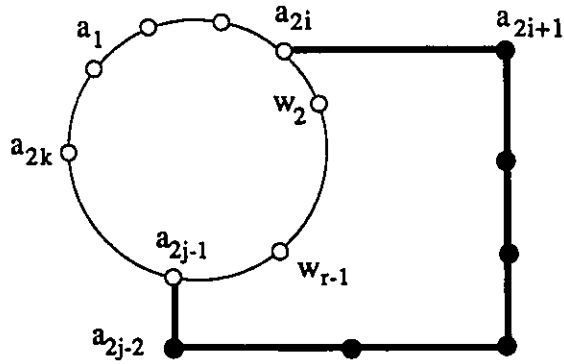


Figure 3.2.20

By the same argument as that in Case 2.

Case 5. $p = 2i$ and $q = 2j$ (Figure 3.2.21)

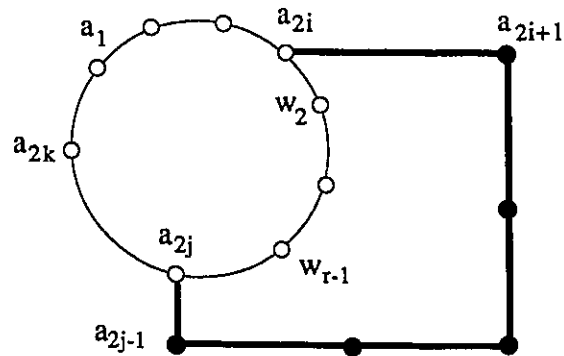


Figure 3.2.21

Then r and m are odd. By induction hypothesis, there is a simple planar embedding E_1 of a bipartite orientation Q of G_1 satisfying the following conditions:

- 1) $\max Q \supseteq I = \{a_1, a_3, \dots, a_{2i-1}, w_2, \dots, w_{r-1}, a_{2j+1}, \dots, a_{2k-1}\}$ (if $q < 2k$)

or

$$\max Q \supseteq I = \{a_1, \dots, a_{2i-1}, w_2, \dots, w_{r-1}\} \text{ (if } q = 2k\text{);}$$

- 2) F'' is the boundary of E_1 ;
- 3) I belongs to the upper part of the boundary of E_1 ;
- 4) a_1 and a_{2k-1} or a_1 and w_{r-1} ;
- 5) f is a face of G_1 if and only if f is a face of E_1 (Figure 3.2.22).

We can obtain a simple planar embedding E of a bipartite orientation of G satisfying the following conditions by adding the zigzag Z to E_1 :

- 1) $\max P \supseteq \{a_1, a_3, \dots, a_{2k-1}\}$;
- 2) F is the boundary of E ;
- 3) $a_1, a_3, \dots, a_{2k-1}$ belong to the upper part of the boundary of E ;
- 4) a_1 and a_{2k-1} are the turning vertices;

5) f is a face of G if and only if f is a face of E (Figure 3.2.23).

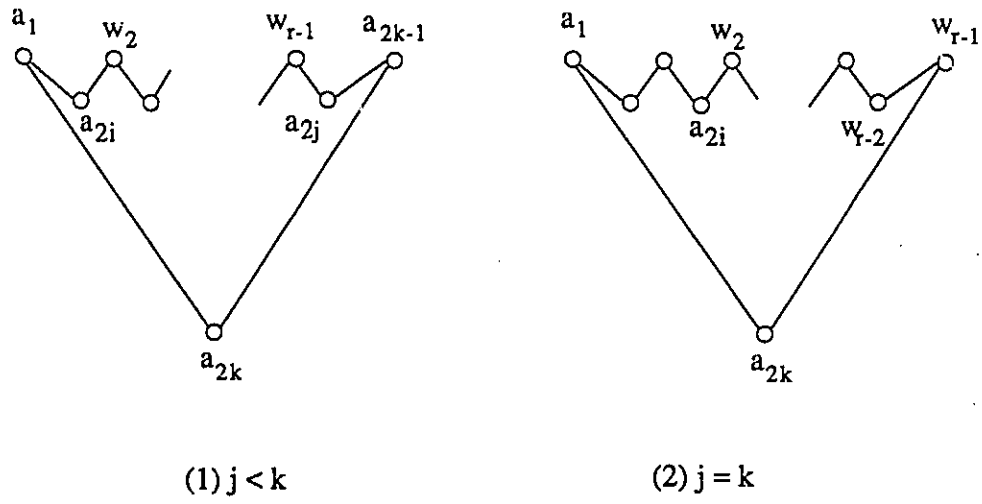


Figure 3.2.22

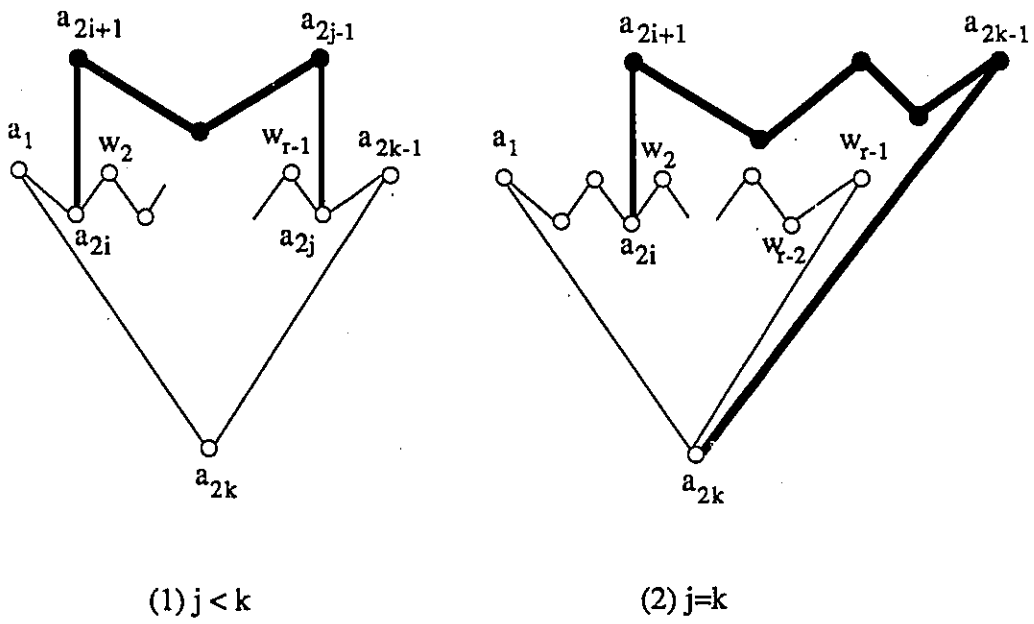


Figure 3.2.23

This completes proof of Theorem 3.2.7.

Q.E.D.

By induction on the number of cut vertices, we can easily prove the following lemma.

LEMMA 3.2.8. *Let G be a planar bipartite graph. Then there is a 2-connected planar bipartite graph containing G as a subgraph.*

Now we are ready to prove our main theorem in this section.

Proof of Theorem 3.2.1. Let G be a planar bipartite graph. If G is 2-connected, then the theorem follows from Theorem 3.2.7; if G is not 2-connected, then, according to Lemma 3.2.8, there is a planar bipartite extension H of G . By Theorem 3.2.7, there is a planar bipartite orientation Q of H . The theorem follows from the simple fact that any diagram obtained from a planar bipartite diagram by deleting some edges is planar and bipartite.

Q.E.D.

Remark. The proof of Theorem 3.2.1 is "constructive", that is, the proof tells us how to obtain a planar embedding of a bipartite orientation from a planar embedding of a bipartite planar covering graph, although it is complicated. The construction of planar embedding of a planar orientation of a planar covering graph may be useful. There is a simpler proof which is not constructive (G. di Battista, W. P. Liu and I. Rival [1990]).

THEOREM 3.2.9. *A bipartite ordered set is planar if and only its covering graph is planar.*

Proof. It is enough to prove the sufficiency. Let P be a bipartite orientation of a connected planar bipartite graph G . Consider a planar bipartite orientation Q of G . Without loss of generality, we can assume that

$$\max P \cap \max Q \neq \emptyset.$$

For otherwise, we can consider Q^d , the dual of Q , which is planar. Let v be an element of $\max P \cap \max Q$ and let u be a maximal element of Q . Then, any zigzag connecting u and v contains odd number of elements. Notice that P and Q have the same covering graph G which is bipartite. So u is maximal in P . Similarly, if u is maximal in P , then u is maximal in Q . That is, $\max P = \max Q$. So $P = Q$ since both P and Q are bipartite.

Q.E.D.

If an orientation P of a planar bipartite graph is not the "shortest", then P may not be planar.

THEOREM 3.2.10 (G. di Battista, W. P. Liu and I. Rival [1990]). *There is a planar bipartite covering graph having an orientation of length three which is not planar.*

Proof. Let G be the graph illustrated in Figure 3.2.24 and let P be the ordered set illustrated in Figure 3.2.25. Then G is bipartite and planar and P is an orientation of G with length three. It is easy to verify that P is not planar.

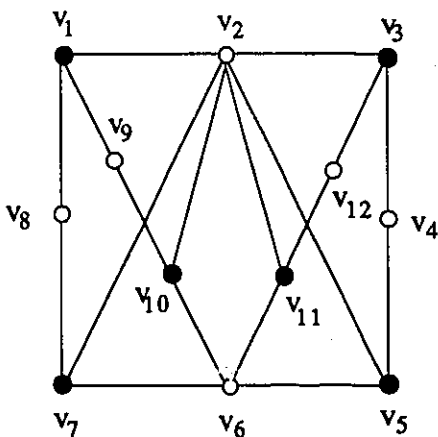


Figure 3.2.24

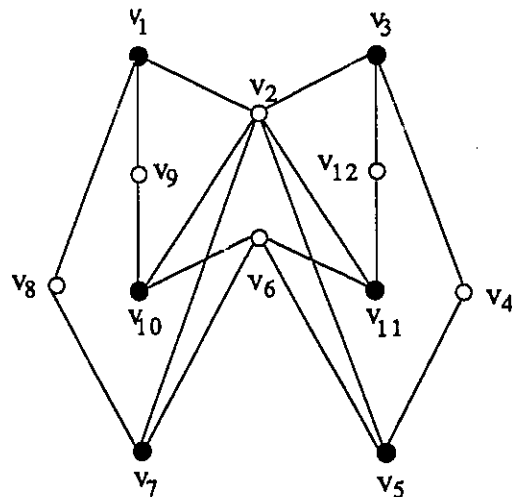


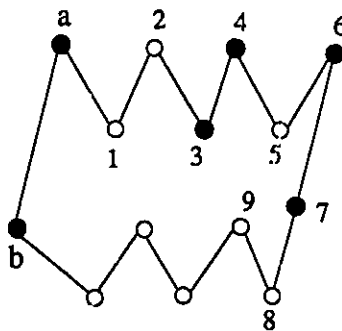
Figure 3.2.25

3.3 OUTERPLANAR COVERING GRAPHS. A graph is *outerplanar* if it has a planar embedding in which the external face is a Hamiltonian cycle. With this special condition, it is no wonder that an outerplanar graph has almost all the properties that we expect. Indeed, in this section we shall see that *any outerplanar covering graph has many planar orientations, any orientation with minimum length of an outerplanar covering graph is planar, and any independent set of an outerplanar graph can be an antichain of a planar orientation.* (In Chapter 4, we will give an example to show that some independent set of a covering graph cannot be an antichain of an orientation of the covering graph).

The following theorem is our first result in this section.

THEOREM 3.3.1 (W. P. Liu). *Any n -element outerplanar covering graph has at least $2^{n/2}$ planar orientations.*

Before proving the theorem, we give a definition. Let R be a planar embedding of an ordered set P and let $a > b$ (P). We say that in R the edge $a > b$ is *on the left (right)* if all of the edges of P , except $a > b$, and elements, except the elements a and b , are on the right (left) of the extension of the line segment presenting the edge $a > b$ (Figure 3.3.1).



The edge $a > b$ is on the left and the edge $6 > 7$ is on the right, but the edge $4 > 3$ is neither on the left nor the right

Figure 3.3.1

Let

$L_G(a, ab) = \{P: P \text{ is an orientation of } G \text{ with the element } a \text{ as maximal such that in some planar embedding of } P \text{ the edge } a \rightarrow b \text{ is on the left } \};$

$l_G(a, ab) = \{P: P \text{ is an orientation of } G \text{ with the element } a \text{ as minimal such that in some planar embedding of } P \text{ the edge } b \rightarrow a \text{ is on the left } \};$

$R_G(a, ab) = \{P: P \text{ is an orientation of } G \text{ with the element } a \text{ as maximal such that in some planar embedding of } P \text{ the edge } a \rightarrow b \text{ is on the right } \};$

$r_G(a, ab) = \{P: P \text{ is an orientation of } G \text{ with the element } a \text{ as minimal such that in some planar embedding of } P \text{ the edge } b \rightarrow a \text{ is on the right } \}.$

Obviously, we the following relation is true

$$|L_G(a, ab)| = |R_G(a, ab)| = |r_G(a, ab)| = |l_G(a, ab)|.$$

Proof of Theorem 3.3.1. Actually, we prove the following stronger result.

THEOREM 3.3.1'. *Let G be an outerplanar covering graph with C as the Hamiltonian cycle. Then for any edge ab belonging to C ,*

$$|r_G(a, ab)| + |R_G(a, ab)| \geq 2^{|G|/2},$$

where, $|G|$ is the number of vertices of the graph G .

Proof By induction on the number of the vertices of G to show that

$$|R_G(a, ab)| \geq 2^{|G|/2-1}.$$

If G is a cycle, then Theorem 3.3.1' is obviously true, since at least $(|G| - 2)/2$ edges of G can be independently oriented, each of which induces a member of $R_G(a, ab)$. Assume that the face $F \neq C$ containing the edge ab contains two edges cd and ef , where cd is a chord and ef is also a chord if F contains two chords. Let Z_1 be the path consisting of the edges of F between c and e and let Z_2 be the path consisting of the edges of F between

d and f (Z_2 may consist of only one vertex d if $d = f$). Let G_1 and G_2 be the two components of $G - (\varepsilon(Z_1) \cup \varepsilon(Z_2))$ (Figure 3.3.2) (G_2 may consist of only the edge ef).

First, suppose $|Z_1| + |Z_2| = 4$.

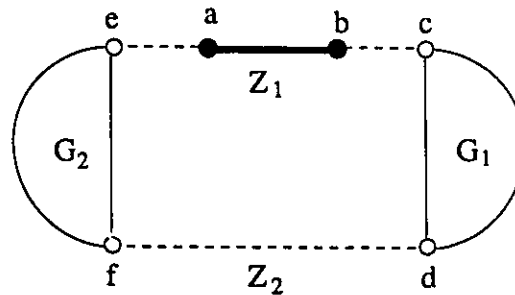


Figure 3.3.2

Case 1. $a = e$ and $b = c$ (Figure 3.3.3)

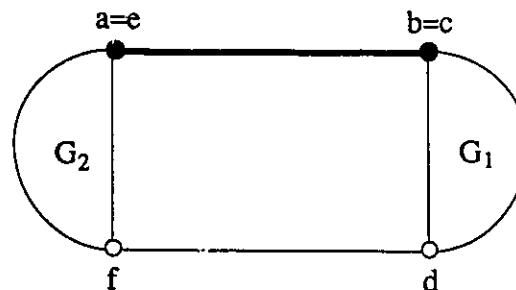


Figure 3.3.3

Then, associated with an orientation in $L_{G_1}(c, cd)$ and an orientation in $R_{G_2}(e, ef)$, there is an orientation in $R_G(a, ab)$ (Figure 3.3.4 (1)); associated with an orientation in $L_{G_1}(c, cd)$ and an orientation in $R_{G_2}(e, ef)$, there is an orientation in $R_G(a, ab)$ (Figure 3.3.4 (2)). So

$$|R_G(a, ab)| \geq |R_{G_2}(e, ef)| \times (|L_{G_1}(c, cd)| + |L_{G_1}(c, cd)|)$$

$$\geq 2^{|G_2|/2-1} 2^{|G_1|/2} = 2^{|G|/2-1},$$

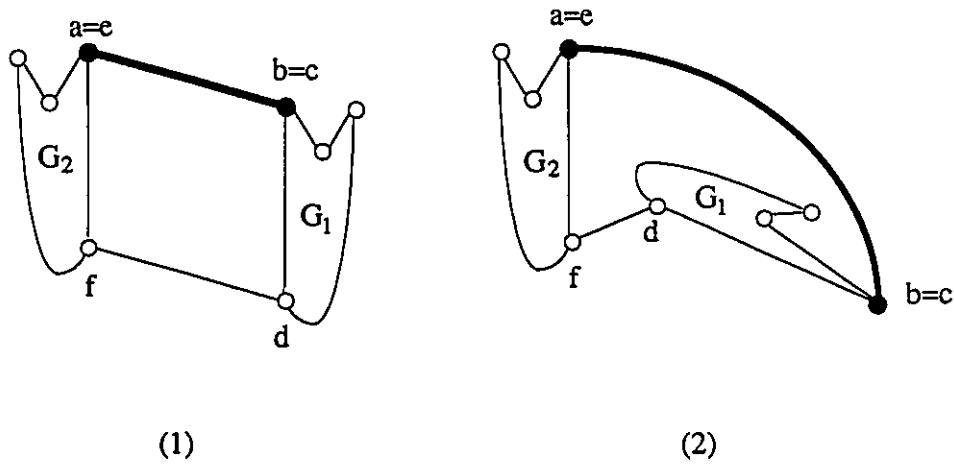


Figure 3.3.4

Case 2. $c = b$ and $f = d$ (Figure 3.3.5)

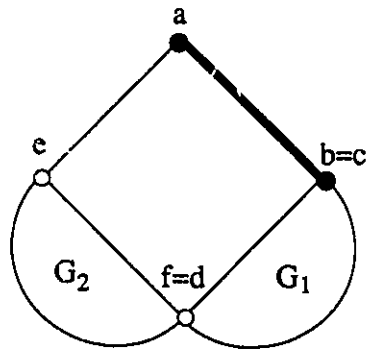


Figure 3.3.5

Then, associated with an orientation in $L_{G_1}(c, cd)$ and an orientation in $R_{G_2}(e, ef)$, there is an orientation in $R_G(a, ab)$ (Figure 3.3.6 (1)); associated with an orientation in $l_{G_1}(c, cd)$ and an orientation in $r_{G_2}(e, ef)$, there is an orientation in $R_G(a, ab)$ (Figure 3.3.6 (2)). So

$$\begin{aligned} |R_G(a, ab)| &\geq |L_{G_1}(c, cd)| \times |R_{G_2}(e, ef)| + |l_{G_1}(c, cd)| \times |r_{G_2}(e, ef)| \\ &= |R_{G_2}(e, ef)| \times [|L_{G_1}(c, cd)| + |l_{G_1}(c, cd)|] \end{aligned}$$

$$= |R_{G_2}(e, ef)| \times [|L_{G_1}(c, cd)| + |l_{G_1}(c, cd)|]$$

$$\geq 2^{|G_2|/2-1} 2^{|G_1|/2} = 2^{|G|/2-1}.$$

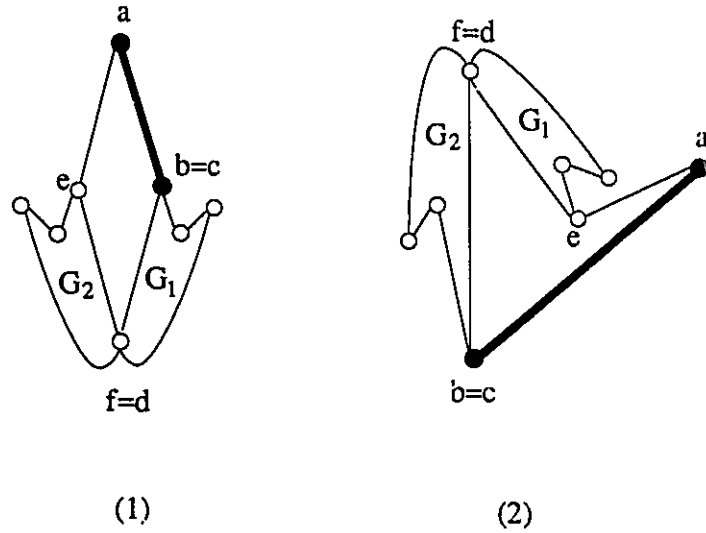


Figure 3.3.6

Case 3. $a = e$ and $f = d$ (Figure 3.3.7)

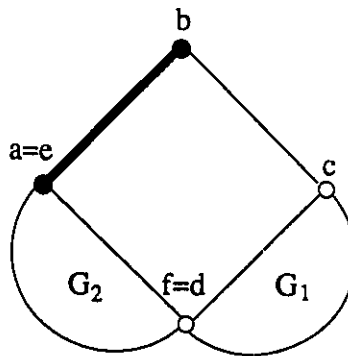


Figure 3.3.7

Then, associated with an orientation in $L_{G_1}(c, cd)$ and an orientation in $R_{G_2}(e, ef)$, there is an orientation in $R_G(a, ab)$ (Figure 3.3.8 (1)); associated with an orientation in $l_{G_1}(c, cd)$ and an orientation in $R_{G_2}(e, ef)$, there is an orientation in $R_G(a, ab)$ (Figure 3.3.8 (2)). So

$$\begin{aligned}
 |R_G(a, ab)| &\geq |L_{G_1}(c, cd)| \times |R_{G_2}(e, ef)| + |L_{G_1}(c, cd)| \times |R_{G_2}(e, ef)| \\
 &= |R_{G_2}(e, ef)| \times [|L_{G_1}(c, cd)| + |L_{G_1}(c, cd)|] \\
 &\geq 2^{|G_2|/2-1} 2^{|G_1|/2} = 2^{|G|/2-1}.
 \end{aligned}$$

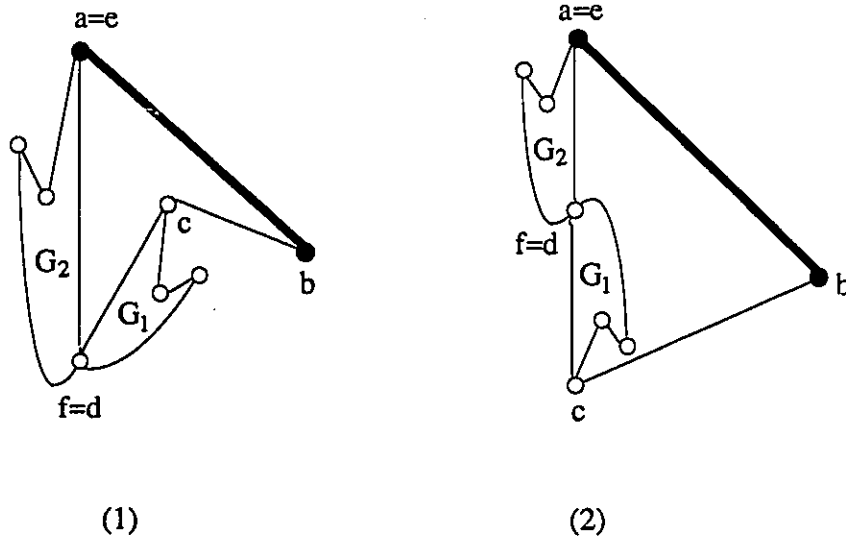


Figure 3.3.8

Now, assume $|Z_1| + |Z_2| \geq 5$. Let H be the outerplanar covering graph obtained from G by deleting an edge, say xy , of Z_2 (or an edge of Z_1 which is not adjacent to a , if $d = f$) and identifying the vertices x and y . For each ordered set in $R_H(a, ab)$, obviously, we can obtain two ordered sets in $R_G(a, ab)$, in one of which $x > y$ and $y > x$ in the other. So

$$|R_G(a, ab)| \geq 2 \times |R_H(a, ab)| \geq 2 \times 2^{|H|/2-1} \geq 2^{|G|/2-1}.$$

Q.E.D.

Given an orientation P of a covering graph G , if G is bipartite, then P may be bipartite, too. If G is not bipartite, then any orientation P of G has length at least three. An orientation P of a covering graph is said to have *minimum length* if for any orientation Q of the graph, the length of Q is not less than the length of P . According to Theorem 3.2.9, *any orientation with minimum length of a planar bipartite graph is planar.*

Naturally, we may ask whether it is true that any orientation with length three of a planar covering graph is planar, if the graph is not bipartite. (In Chapter 4, we will prove that any planar covering graph has an orientation with length three.)

THEOREM 3.3.2 (cf. G. di Battista, W. P. Liu and I. Rival [1990]). *There is a planar covering graph such that not every of its orientations with minimum length is planar.*

Proof. Let P be the ordered set illustrated in Figure 3.3.9 whose covering graph is planar (Figure 3.3.10). Then, P is of minimum length (three) since G is not bipartite. It is routine to check that P is not planar.

Q.E.D

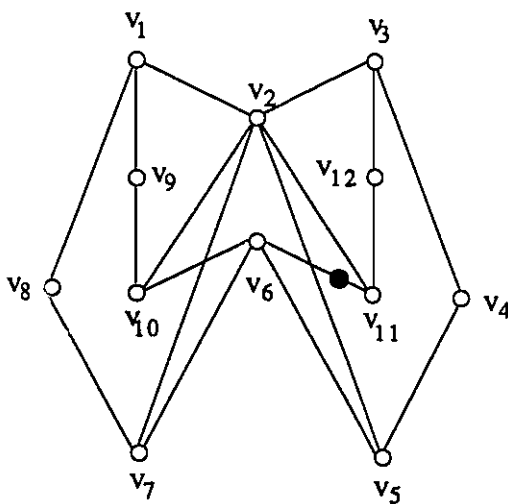


Figure 3.3.9

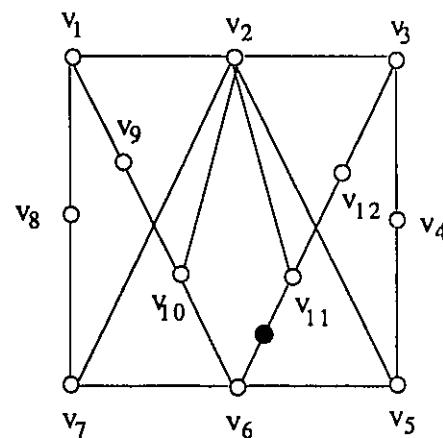


Figure 3.3.10

Despite Theorem 3.3.2, we have following the positive theorem.

THEOREM 3.3.3 (W. P. Liu). *Any orientation with minimum length of an outerplanar covering graph is planar.*

Proof. We will prove the theorem by showing the following technical result.

THEOREM 3.3.3'. *Let P be an orientation with minimum length of an outerplanar graph G . Then, for any edge $a \succ b(P)$ which is not a chord, there is a planar embedding of P such that all of the vertices and the edges of P are inside a convex polygon with the edge $a \succ b$ on the boundary of the polygon and the elements a and b are two vertices of the polygon.*

Proof. First, for convenience, we introduce a notation. Let P be an ordered set and let Z be a zigzag consisting of elements with degree two, except possibly the end elements. Let P_1 denote the ordered set whose diagram consists of the edges of $\epsilon(P) - \epsilon(Z)$. P^*Z stands for the directed graph (P^*Z is not necessarily a diagram) obtained from P_1 by identifying u and v where v and u are the end vertices of Z . It is easy to verify the following lemma.

LEMMA 3.3.4. *If P^*Z is an ordered set and has a planar embedding satisfying the conditions given in Theorem 3.3.3', then P has a planar embedding satisfying all of the conditions, too.*

Now, we are ready to prove Theorem 3.3.3'. If G is bipartite, then it follows from Theorem 3.2.7. Suppose that G is not bipartite. Let F be an odd face of G , that is, the length of F is odd. If $a \succ b(P)$ does not belong to F , then we can choose an edge $x \succ y(P)$ belonging to F which is not a chord such that $Q = P^*(x \succ y)$ is an ordered set. By induction, there is a planar embedding of the ordered set Q satisfying all the conditions of Theorem 3.3.3'. Applying Lemma 3.3.4, we know that P has a planar embedding satisfying all of the conditions mentioned in Theorem 3.3.3'. Now, assume that $a \succ b$ belongs to F . Choose a chord c_1d_1 and an edge c_2d_2 belonging to F (if F contains two chords, then choose another chord as c_2d_2). Without loss of generality, we can assume P looks like the "ordered set" illustrated in Figure 3.3.11.

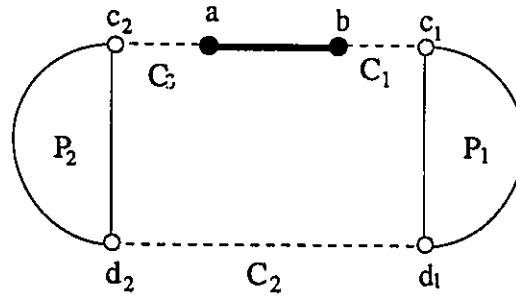


Figure 3.3.11

Let

$$\varepsilon(F) = \{a > b\} \cup C_1 \cup \{c_1 d_1\} \cup C_2 \cup \{c_2 d_2\} \cup C_3$$

(cf. Figure 3.3.11). If we can find zigzags Z_1, Z_2, \dots, Z_k such that $((P * Z_1) * Z_2) * \dots * Z_k$ is an ordered set whose covering graph is outerplanar, then by induction and Lemma 3.3.4, the ordered set P has a planar embedding satisfying all of the conditions described in Theorem 3.3.3'. To this end, let P_1 and P_2 be the two connected subsets of $P - \{\varepsilon(C_1) \cup \varepsilon(C_2) \cup \varepsilon(C_3) \cup \{a > b\}\}$ containing the edges $c_1 > d_1$ and $c_2 > d_2$, respectively (cf. Figure 3.3.11). By induction, there are a planar embedding of P_1 and a planar embedding of P_2 which satisfy all of the conditions given in Theorem 3.3.3'.

Case 1. $d_1 \neq d_2$.

Subcase 1.1. $c_1 > d_1 (P)$ and $c_2 > d_2 (P)$

If $d_1 > d_2$, then, since P is of length three, $c_1 > d_3 (P)$ belonging to C_1 . So $((P * C_1') * C_2) * C_3$ is an ordered set whose covering graph is outerplanar, where $C_1' = C_1 - \{c_1 > d_3\}$ (Figure 3.3.12).

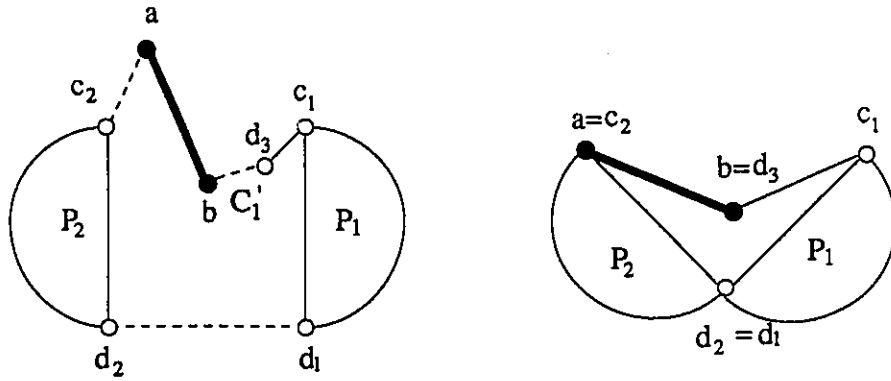


Figure 3.3.12

If $d_1 \not> d_2 (P)$, there is an edge $c_3 > d_3$ belonging to C_2 . So

$$(((P * C_1) * C_2') * C_2'') * C_3$$

is an ordered set whose covering graph is outerplanar, where $C_2' \cup C_2'' = C_2 - \{c_3 > d_3\}$

(Figure 3.3.13).

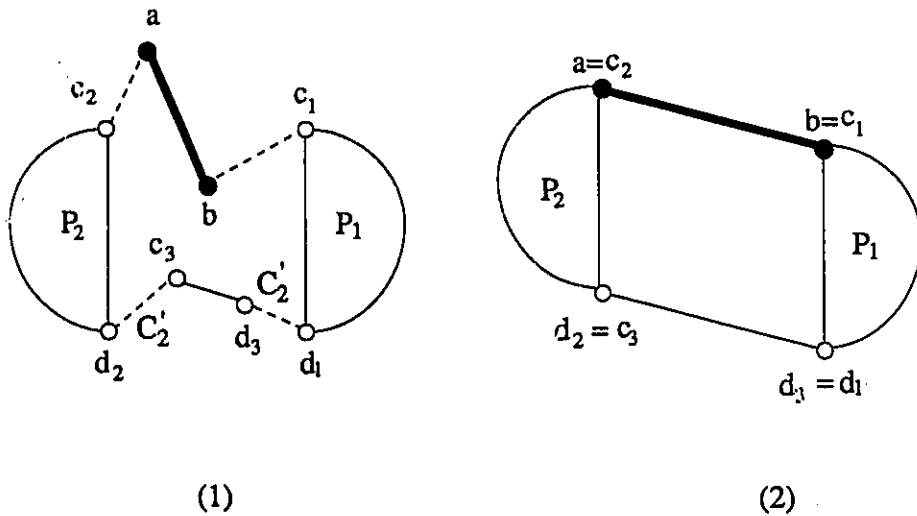


Figure 3.3.13

Subcase 1.2. $d_2 > c_2 (P)$ and $c_1 > d_1 (P)$

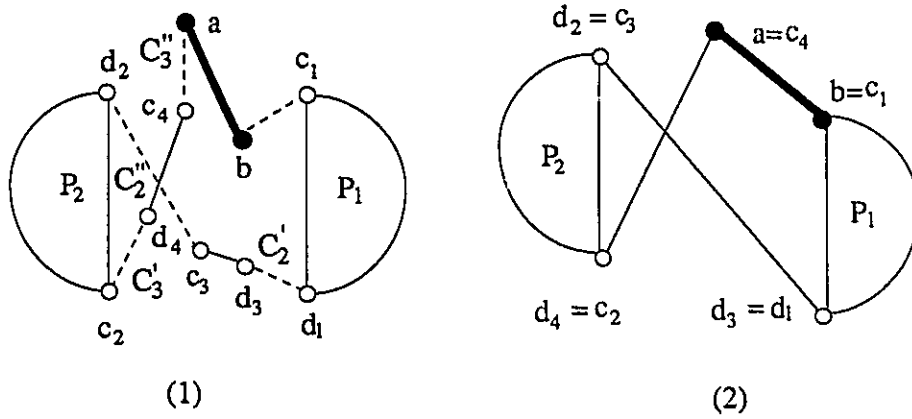


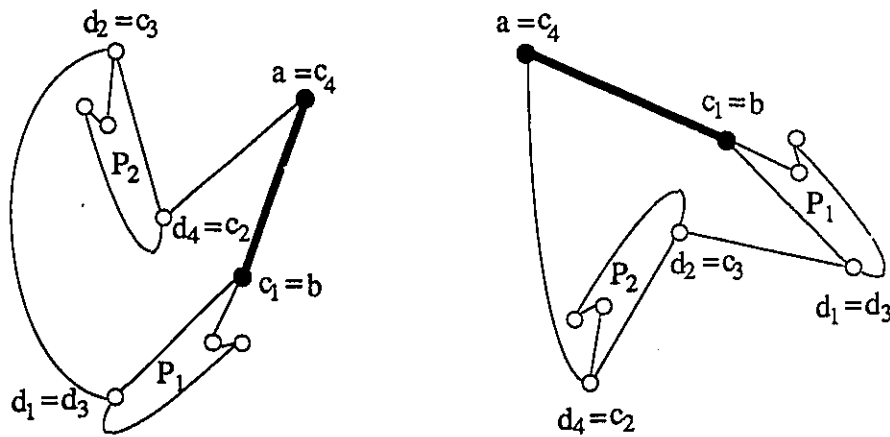
Figure 3.3.14

There is an edge $c_3 >- d_3 (P)$ belonging to C_2 since P has length three and $d_1 \neq d_2$. If $a \neq c_2$, then there is an edge $c_4 >- d_4 (P)$ belonging to C_2 since P is of length three (Figure 3.3.14). We can now construct a planar embedding of the ordered set

$$(((P * C_1) * C_2') * C_2'') * C_3') * C_3'',$$

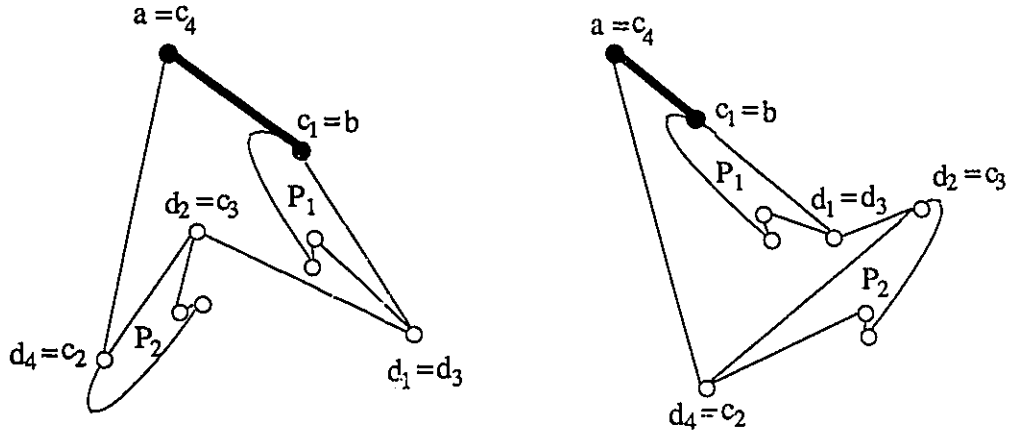
where $C_2 = C_2' \cup \{c_3 >- d_3\} \cup C_2''$ and $C_3 = C_3' \cup \{c_4 >- d_4\} \cup C_3''$ (Figure 3.3.15).

(Notice that $\{d_2, c_2\} \cap [\max P_2 \cup \min P_2] \neq \emptyset$.) If $a = c_2$, then there are an edge $c_4 >- b$ belonging to C_1 and an edge $d_2 >- d_3$ belonging to C_3 since P is of length three



$d_2 \in \max P_2$ and $c_1 \in \max P_1$

$c_2 \in \min P_2$ and $c_1 \in \max P_1$



$d_2 \in \max P_2$ and $d_1 \in \min P_1$

$c_2 \in \min P_2$ and $d_1 \in \min P_1$

Figure 3.3.15

(Figure 3.3.16 (1)). We can now construct a planar embedding of the ordered set $(P * C_1') * C_2'$ (Figure 3.3.16 (2)), where, $C_2 = C_2' \cup \{d_2 > d_3\}$ and $C_1 = C_1' \cup \{c_4 > b\}$. (Notice that $d_2 \in \max P_2$) (Figure 3.3.17)

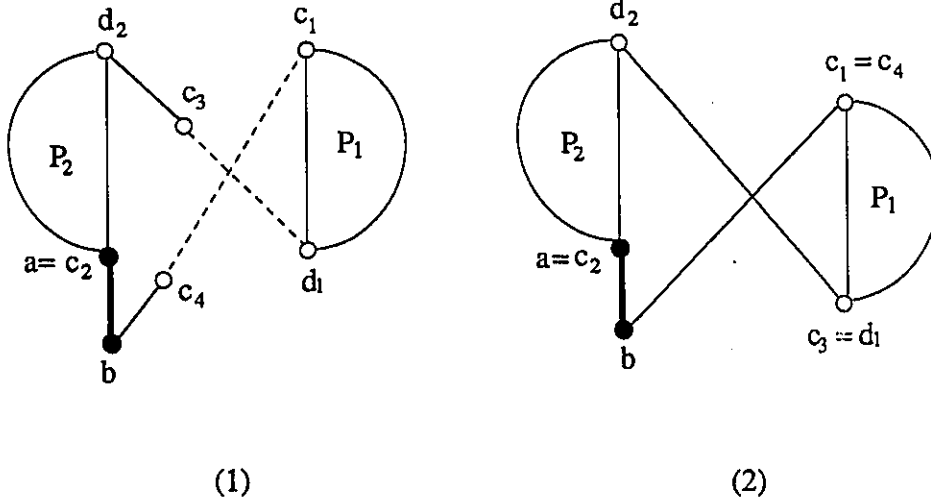


Figure 3.3.16

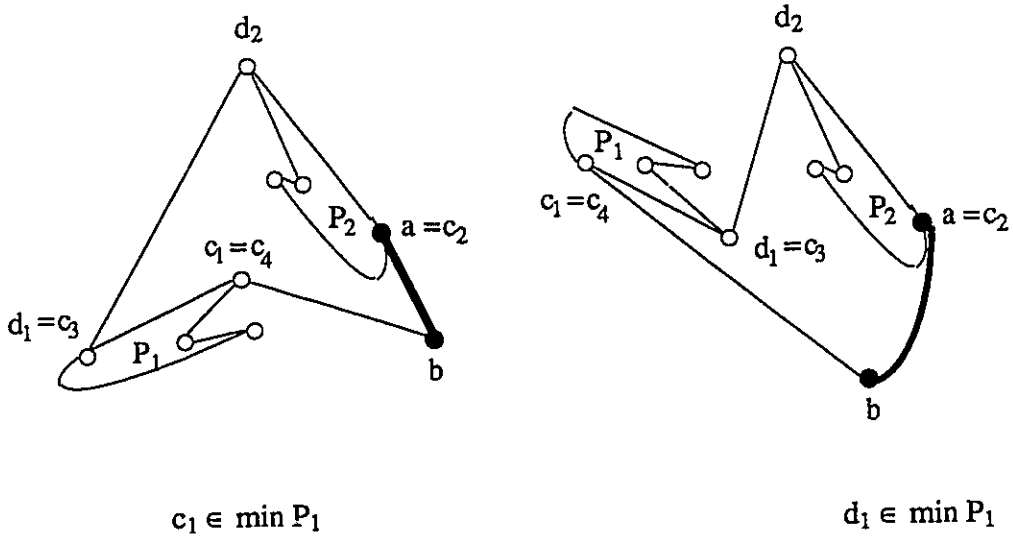


Figure 3.3.17

Subcase 1.3. $d_1 > c_1$ and $c_2 > d_2$

Then there is an edge $c_3 > d_3$ belonging to C_2 since the ordered set P is of length three and $d_1 \neq d_2$. $P' = (((P * C_1) * C_2') * C_2'') * C_3$ is an ordered set whose covering graph is outerplanar and P' has fewer elements than P , where $C_2 = C_2' \cup C_2'' \cup \{c_3 > d_3\}$ (Figure 3.3.18).

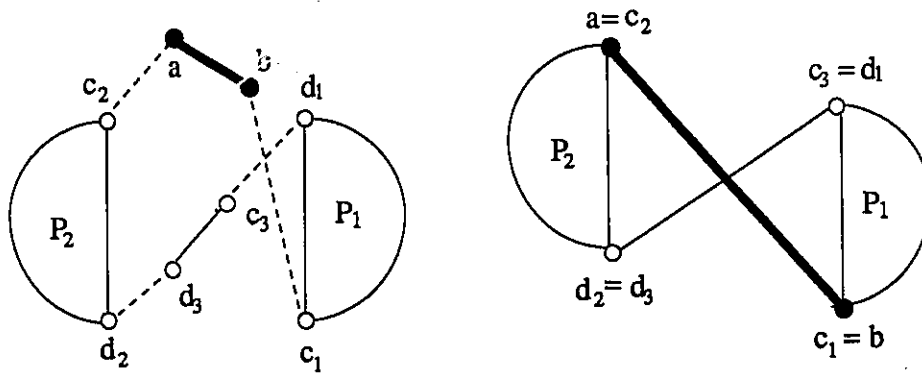


Figure 3.3.18

Subcase 1.4. $d_1 > c_1(P)$ and $d_2 > c_2(P)$

If $d_1 \not\geq d_2$, let $c_3 \succ d_3(P)$ belong to C_2 . Then $P' = (((P * C_1) * C_2') * C_2'') * C_3$ is an ordered set with outerplanar covering graph, where, $C_2' \cup C_2'' = C_2 - \{c_3 \succ d_3\}$ (Figure 3.3.19).

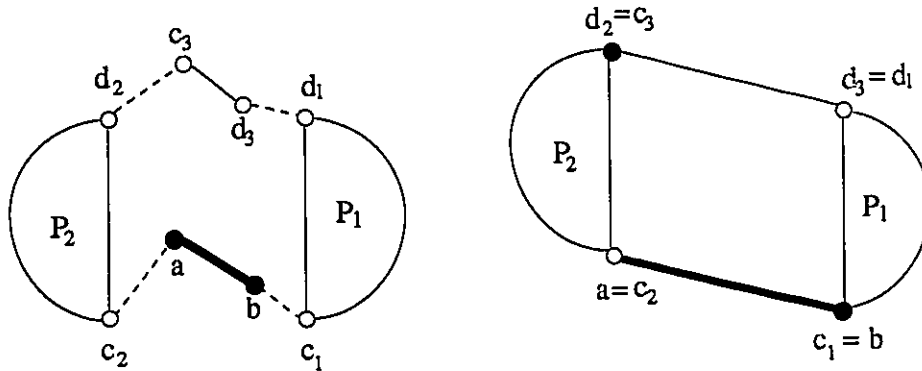


Figure 3.3.19

Otherwise, i.e. $d_1 > d_2$, so, there is an edge $d_3 \succ c_2$ belonging to C_3 . Then $P' = ((P * C_1) * C_2) * C_3'$ is an ordered set with outerplanar covering graph, where, $C_3' = C_3 - \{d_3 \succ c_2\}$ (Figure 3.3.20). In any event, P' has fewer elements than P .

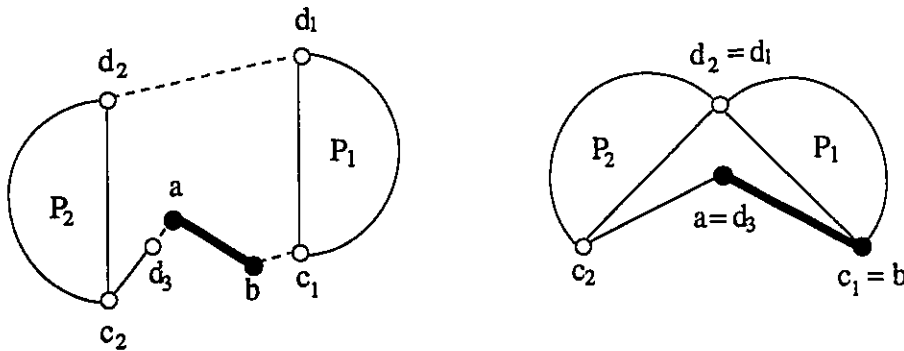


Figure 3.3.20

Case 2. $d_1 = d_2$

Subcase 2.1. $c_1 \succ d_1(P)$ and $c_2 \succ d_2(P)$

If $c_2 \not\geq a$, then there is an edge $c_3 \succ d_3(P)$ in C_3 . So $P' = ((P * C_1) * C_3') * C_3''$ is an ordered set with an outerplanar covering graph and P' has fewer elements than P , where, $C_3 = C_3' \cup \{c_3 \succ d_3\} \cup C_3''$ (Figure 3.3.21). If $c_2 \geq a$, then $b \not\geq c_1$. So there

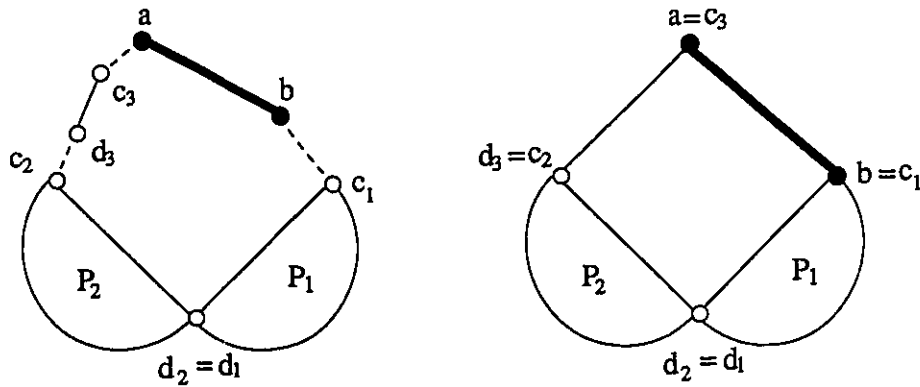


Figure 3.3.21

is an edge $c_3 \succ d_3(P)$ in C_1 (Figure 3.3.22). $P' = ((P * C_1') * C_1'') * C_3$ is an ordered set with an outerplanar covering graph and P' has fewer elements than P , where $C_1 = C_1' \cup \{c_3 \succ d_3\} \cup C_1''$ (Figure 3.3.23).

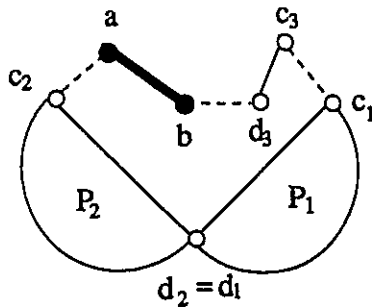


Figure 3.3.22

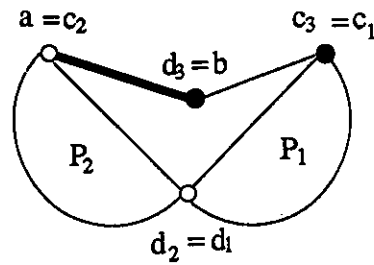


Figure 3.3.23

Subcase 2.2. $c_1 \succ d_1(P)$ and $d_2 \succ c_2(P)$

Then $c_4 \succ c_2$ (P) and $c_1 \succ d_3$ (P), since P is of length three. $(P * C_1) * C_3'$ is an ordered set with an outerplanar covering graph, where, $C_3 = C_3' \cup \{c_4 \succ c_2\}$ and $C_1 = C_1' \cup \{c_1 \succ d_3\}$ (Figure 3.3.24). By induction, P_1 and P_2 have planar embeddings

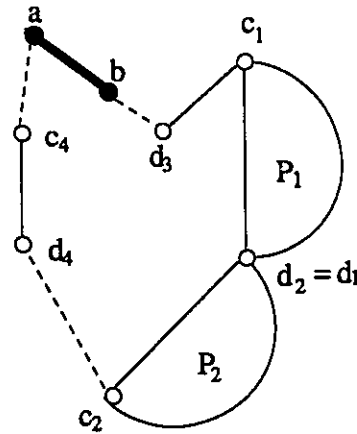


Figure 3.3.24

satisfying all the conditions given in Theorem 3.3.3'. We can construct a planar embedding of P (Figure 3.3.25). (Notice that c_1 is in $\max P_1$ and c_2 is in $\min P_2$ since P has length at most three.)

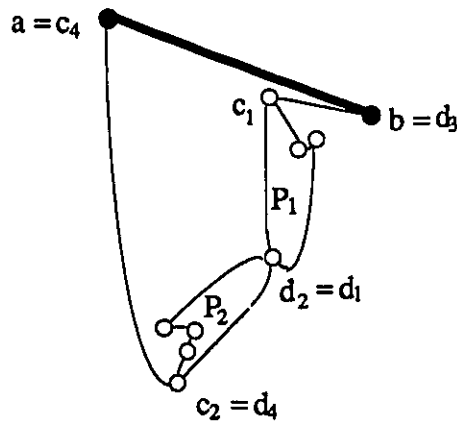


Figure 3.3.25

Subcase 2.3. $d_1 \succ c_1$ (P) and $c_2 \succ d_2$ (P)

If $a = c_2$, then $d_3 \succ c_1 (P)$ since P is of length at most three (Figure 3.3.26),

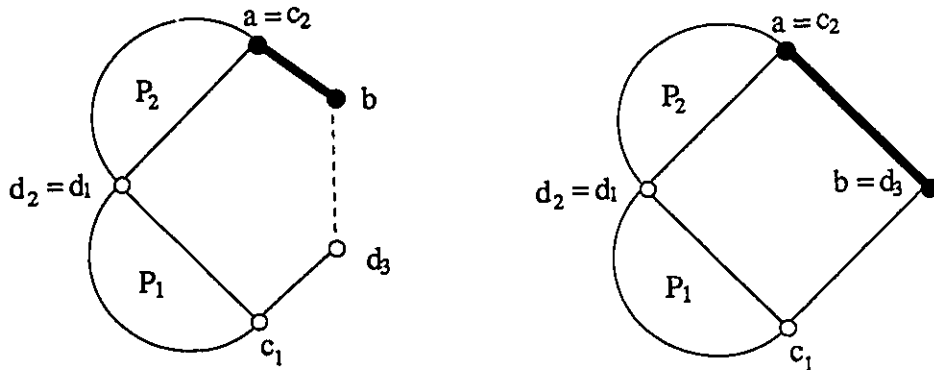


Figure 3.3.26

otherwise $c_2 \succ d_3$ (Figure 3.3.27). So, either $P * C_1'$ or $(P * C_1) * C_3'$ is an ordered set with an outerplanar covering graph and has fewer elements than P , where $C_1' = C_1 - \{d_3 \succ c_1\}$ and $C_3' = C_3 - \{c_2 \succ d_3\}$

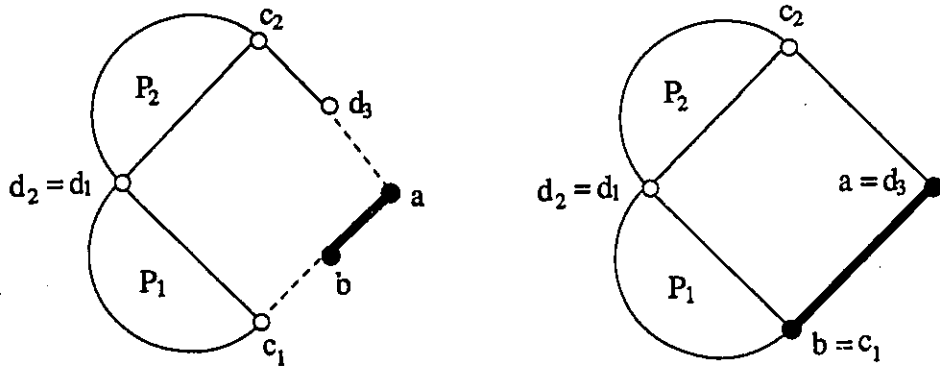


Figure 3.3.27

Subcase 2.4. $d_1 \succ c_1 (P)$ and $d_2 \succ c_2 (P)$. By the same argument as that in Subcase 2.1.

This completes the proof of Theorem 3.3.3'.

Q.E.D.

THEOREM 3.3.5 (W. P. Liu). *There is a nonplanar ordered set with length five whose covering graph is outerplanar.*

Proof. The ordered set illustrated in Figure 3.3.28 (1) is nonplanar and has length five, although its covering graph is outerplanar (Figure 3.3.28 (2)).

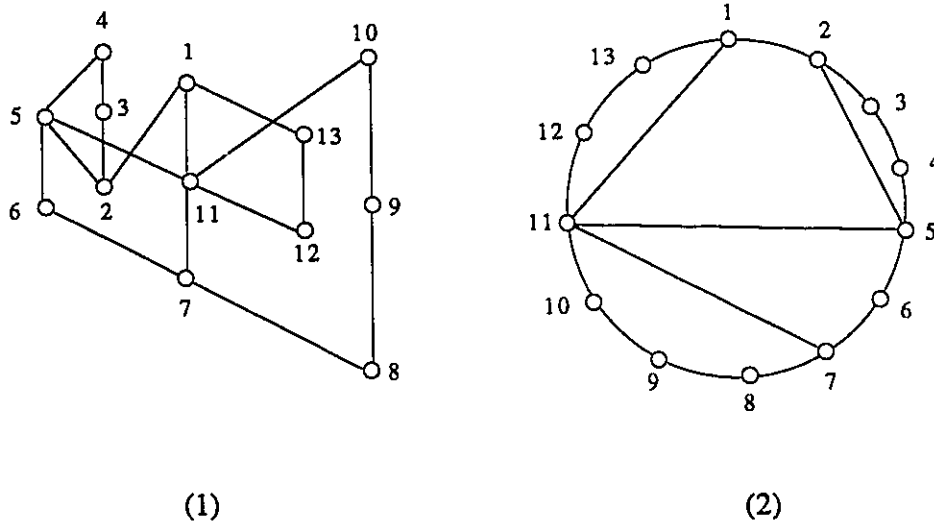


Figure 3.3.28

THEOREM 3.3.6 (W. P. Liu [1990]). *Suppose G is an outerplanar covering graph. Then any independent set of G can be an antichain of a planar orientation of G .*

Proof. It is enough to show the following technical theorem.

THEOREM 3.3.6'. *For any independent set $I = \{a_1, a_2, \dots, a_k\}$ (clockwise order along the Hamiltonian cycle C) of an outerplanar covering graph G , there is a planar embedding of an orientation P of G such that*

- 1) $\max P \supseteq I$;
- 2) the boundary $B(P)$ of P consists of the edges of C ;
- 3) $B(P)$ is simple;
- 4) I belongs to the upper part of $B(P)$;

5) a_1 and a_k are the turning vertices.

Proof. Without loss of generality, we can assume that for any xy not in $\varepsilon(G)$, either $G \cup \{xy\}$ is not outerplanar or not triangle-free or I is not independent in $G \cup \{xy\}$. Then it is easy to verify that any interior face of G contains at most five vertices. For convenience, let C_1 denote the half of C from a_1 to a_k clockwise and C_2 the other half of C (Figure 3.3.29). $B(Q)$ ($B(G)$) denotes the boundary of a planar embedding of an orientation Q (the Hamiltonian cycle of the outerplanar graph G).

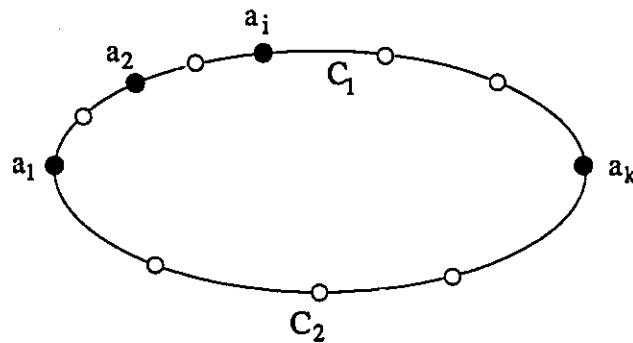


Figure 3.3.29

Case 1. $a_i v \in \varepsilon(G)$ is a chord for some a_i in $V(C_1)$ and some v in $V(C_2) - \{a_1, a_k\}$

(Figure 3.3.30)

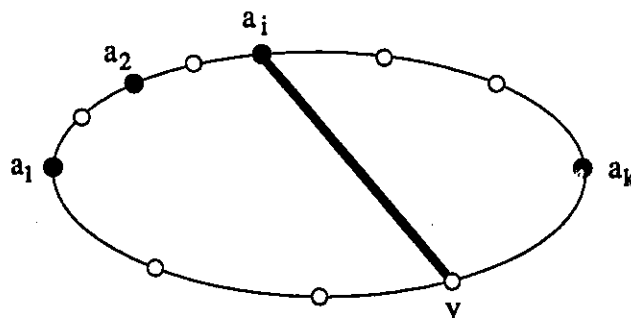


Figure 3.3.30

Let G_1' (G_2') be the component of $G - \{a_i, v\}$ containing a_1 (a_k) and let

$$G_1 = G_1' \cup \{a_i, v\} \text{ and } G_2 = G_2' \cup \{a_i, v\}.$$

By induction hypothesis, there are planar embeddings of orientations P_1 and P_2 of G_1 and G_2 such that

- 1) $\max P_1 \supseteq I \cap V(G_1)$ and $\max P_2 \supseteq I \cap V(G_2)$;
- 2) $B(P_1) = B(G_1)$ and $B(P_2) = B(G_2)$;
- 3) $B(P_1)$ and $B(P_2)$ are simple;
- 4) $I \cap V(G_1)$ ($I \cap V(G_2)$) belongs to the upper part of $B(P_1)$ ($B(P_2)$);
- 5) a_1 and a_i (a_i and a_k) are the turning vertices of P_1 (P_2) (Figure 3.3.31).

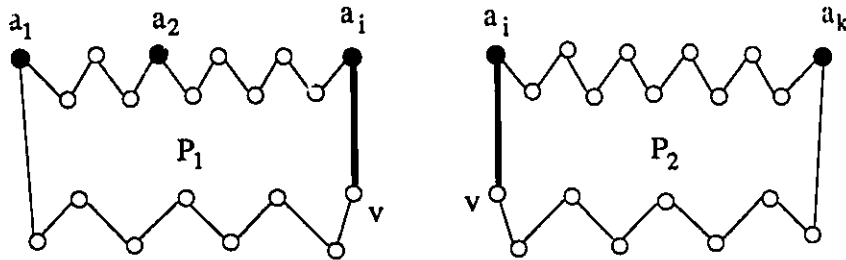


Figure 3.3.31

Then the planar embedding of P obtained from P_1 and P_2 by identifying the edge $a_i \rightarrow v$ satisfies the five conditions (Figure 3.3.32).

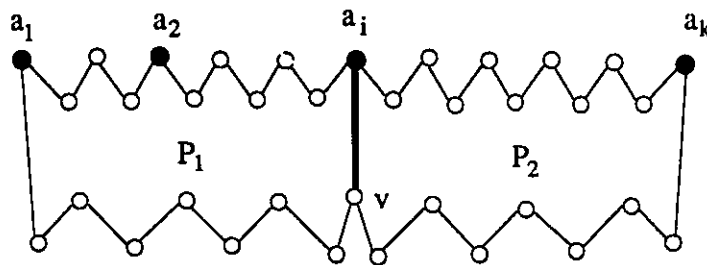


Figure 3.3.32

Case 2. $a_i v \in \epsilon(G)$ is a chord for some a_i in $V(C_1)$ and some v in $V(C_1)$ (Figure 3.3.33)

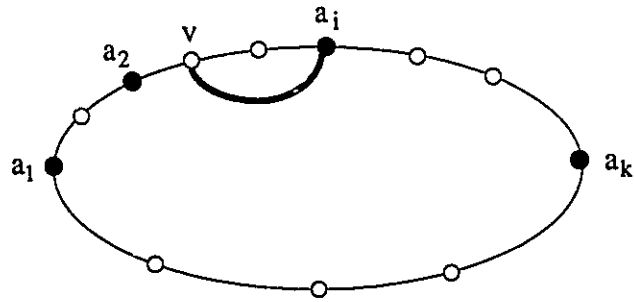


Figure 3.3.33

Without loss of generality, we can assume $\{a_i v\} \cup \varepsilon(C')$ is a face, where, C' is a subpath of C_1 between v and a_i , and v is between a_i and a_1 . By induction, there is a planar orientation P_1 of $G - C'$ which satisfies the five conditions (Figure 3.3.34). Then one of the orientations of G illustrated in Figure 3.3.35 satisfies the five conditions, too.

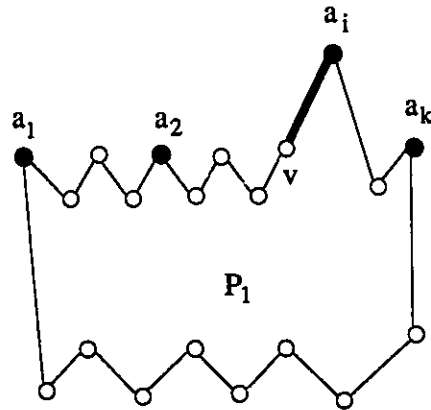
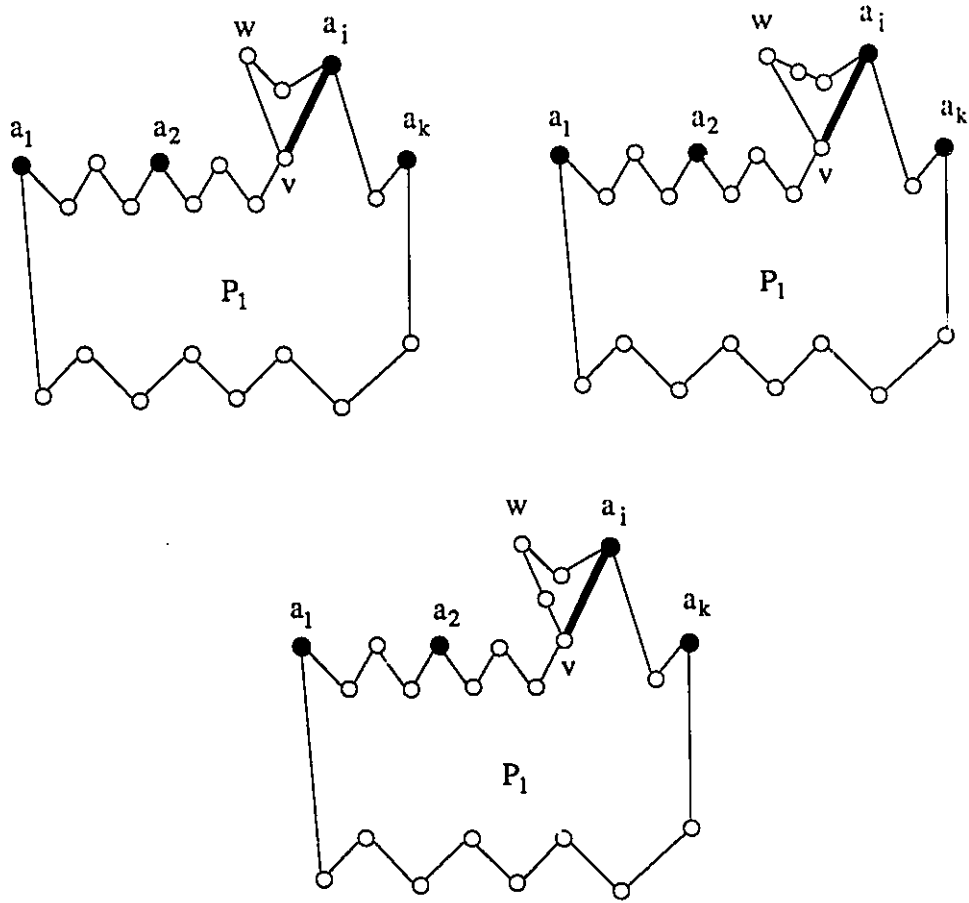


Figure 3.3.34



w may be in I

Figure 3.3.35

Case 3. $uv \in \epsilon(G)$ is a chord for some u and v in $V(C_1)$ (Figure 3.3.36)

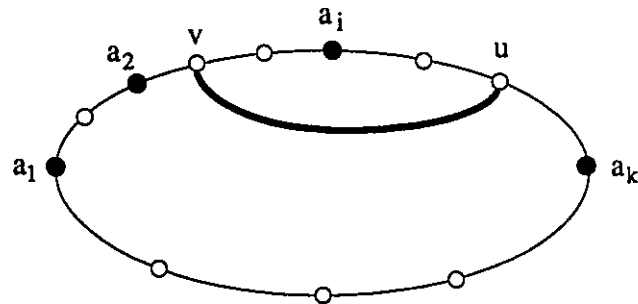


Figure 3.3.36

We can assume that $\{uv\} \cup \varepsilon(C')$ is a face of G , where C' is a subpath of C_1 between u and v . Furthermore, by symmetry and induction, we can assume that there is a planar embedding P_1 of an orientation of $G - C'$ satisfying the five conditions and looks like the one illustrated in Figure 3.3.37. Then one of the embeddings illustrated in Figure 3.3.38 is what we need.

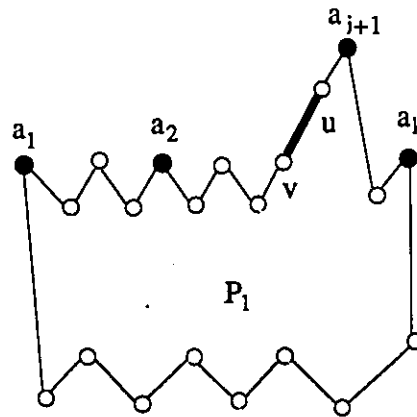
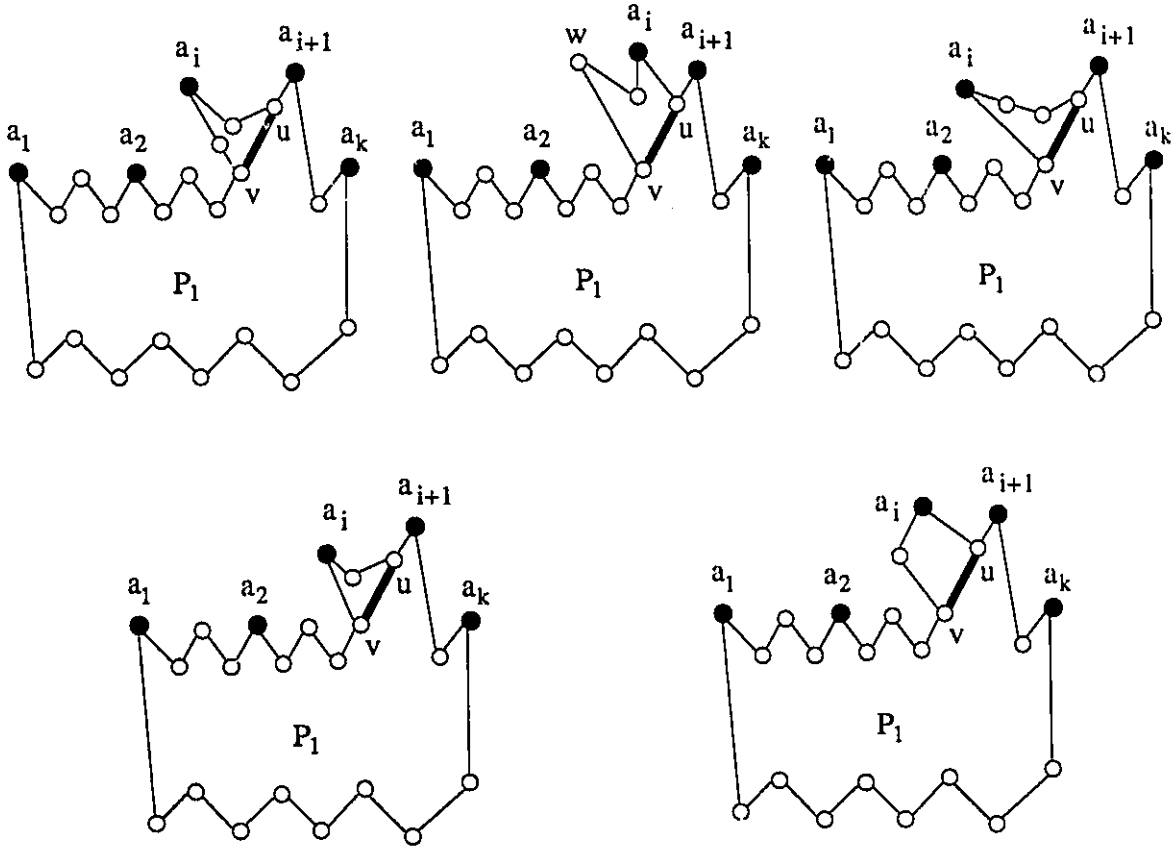


Figure 3.3.37



w may be in I
Figure 3.3.38

Case 4. $uv \in \varepsilon(G)$ is a chord for some u and v in $V(C_2) - \{a_1, a_k\}$
(Figure 3.3.39)

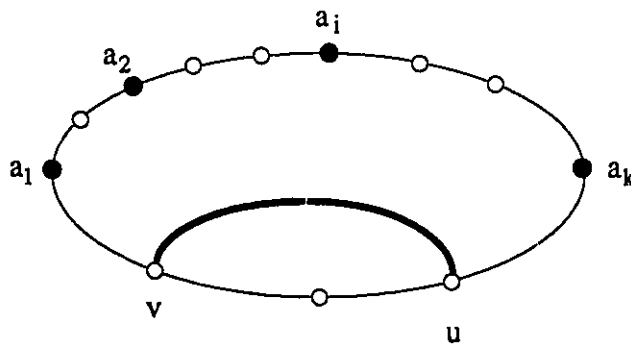


Figure 3.3.39

We can assume that $\{uv\} \cup \epsilon(C'')$ is a face of G , where C'' is a subpath of C_2 between u and v . Furthermore, by symmetry and induction, we can assume that there is a planar embedding P_1 of an orientation of $G - C''$ which satisfies the five conditions and looks like the one illustrated in Figure 3.3.40. Then one of the embeddings illustrated in Figure 3.3.41 satisfies the five conditions.

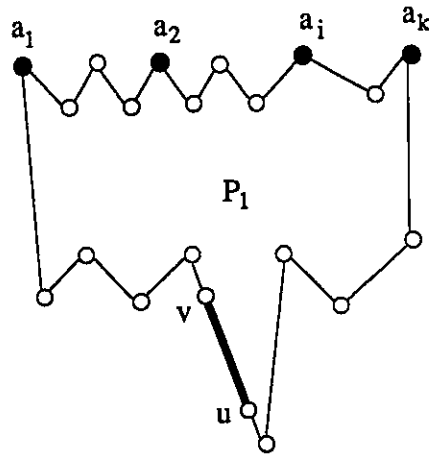


Figure 3.3.40

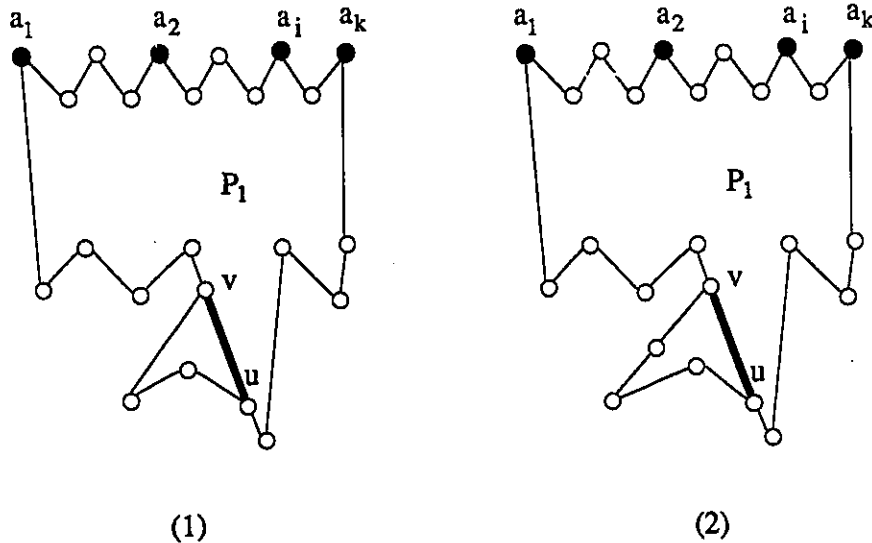


Figure 3.3.41

Case 5. no vertex in I is adjacent to a chord and no two vertices in $V(C_1)$ ($V(C_2)$)

are joined by a chord

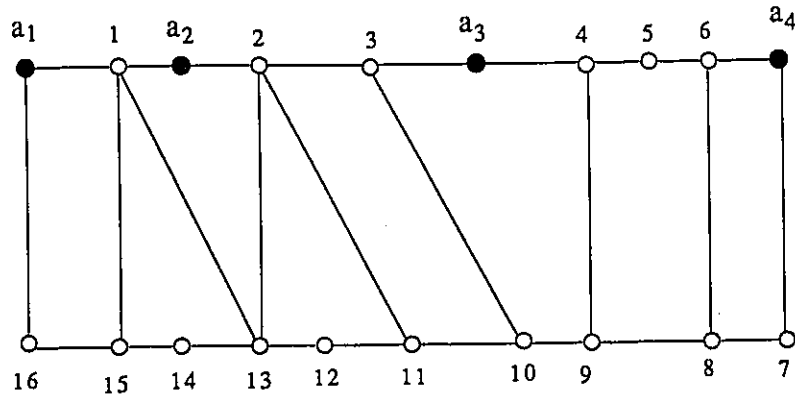
Construct a planar embedding of an orientation P of G as follows. First, draw the path C_1 as a zigzag with length (of the ordered set) at most three such that all the vertices of I are maximal; second, for any vertex u in $V(C_1) - I$, draw the subpath of C_2 between the most left vertex and most right vertex of $N(u) \cap (V(C_2) - \{a_1, a_k\})$ (recall that G is outerplanar) as a zigzag; third, for a 2-path (with u and v as its endpoints) in C_1 with $v > u$ in P and u being between v and a_k (a_1), let u_1 be the most left (right) vertex in

$$N(u) \cap (V(C_2) - \{a_1, a_k\})$$

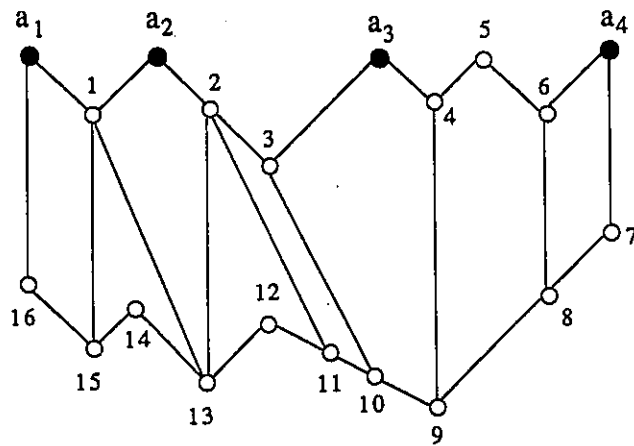
and v_1 the most right (left) vertex in

$$N(v) \cap (V(C_2) - \{a_1, a_k\}),$$

if $u_1 v_1 \notin \epsilon(C_2)$, then draw the subpath of C_2 between v_1 and u_1 as a zigzag, if $u_1 v_1$ is an edge of C_2 , then $v_1 > u_1$ (P); finally, draw each of the rest of the edges of C_2 arbitrarily (Figure 3.3.42).



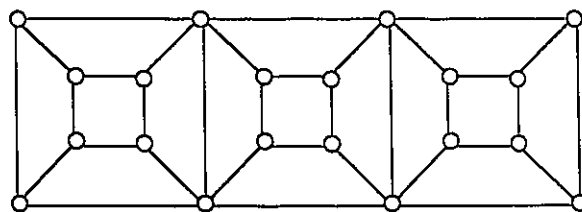
An outerplanar graph G



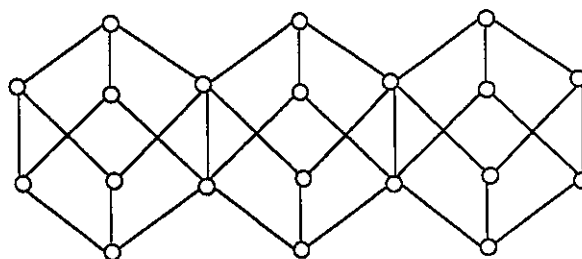
A planar embedding of an orientation of G

Figure 3.3.42

3.4. THE S-GENUS OF AN ORDERED SET WITH PLANAR COVERING GRAPH. The theorems in section 3.3 only partly answered Conjecture 1. What about Conjecture 3? We may even ask a weaker one; *is there a constant k such that any planar covering graph has a planar orientation whose s -genus is less than k ?* Just as in the bipartite case, a natural approach is to prove every orientation of a planar covering graph has s -genus less than k . Unfortunately, the problem is not so simple. Reuter and Rival [1990] showed that *for any k , there is an ordered set with planar covering graph whose s -genus is at least k .* Figure 3.4.1 gives an ordered set with s -genus two whose covering graph is planar. It seems that Conjecture 3 is as hard as Conjecture 1.



A planar graph G



An orientation of G with s -genus two

Figure 3.4.1

S. Foldes, I. Rival and J. Urrutia [1988] proved the following theorem .

THEOREM 3.4.1. *Any ordered set with top and bottom whose covering graph is planar has s -genus zero.*

Theorem 3.4.1 reminds us of an approach to Conjecture 3 .

CONJECTURE 4 (W. P. Liu and I. Rival [1990]). *Any 2-connected planar covering graph has an orientation with top and bottom.*

The following conjecture is more interesting .

CONJECTURE 5. *Any planar covering graph has a planar extension with an orientation having top and bottom.*

Recall that a graph H is an *extension* of a graph G if the graph G is a subgraph of H . If Conjecture 5 were true, then Conjecture 4 would be true, too, since, according to Theorem 3.4.1, the graph H has an orientation with genus zero, and so does G . But Conjecture 5 is **not** true. Before describing the main theorem of this section, we list two of Pretzel's theorems which will be used in this section.

THEOREM 3.4.2 (O. Pretzel [1986]). *Let Q be a reorientation of an ordered set P . Then Q is an inversion of Q if and only if P and Q have the same flow-difference.*

THEOREM 3.4.3 (O. Pretzel [1986]). *Let P and Q be inversions of an ordered sets. If P and Q have the same top, then $P = Q$.*

LEMMA 3.4.4. *Consider a planar graph whose faces, except possibly the external face, are four cycles and two orientations P and Q of it. Then, P is an inversion of Q .*

Proof. By Theorem 3.4.2, it is enough to show that for any orientation P of G , any cycle C of P has flow-difference zero (here, we assume the direction of each cycle is clockwise). Since G is planar, C is the union of four cycles C_1, C_2, \dots, C_m . So

$$f(C) = f(C_1) + f(C_2) + \dots + f(C_m),$$

where $f(C)$ and $f(C_i)$ are the flow differences of the cycles C and C_i , respectively. Notice that $f(C_i) = 0$, $1 \leq i \leq m$, since C_i is a four cycle. Thus, $f(C) = 0$.

Q.E.D.

COROLLARY 3.4.5. *Suppose G is a planar graph whose faces, except possibly the external face, are four cycles. Then for any vertex v of G there is only one orientation of G with v as the top.*

Proof. Let P_1 and P_2 be two orientations of the covering graph G with the element v as the top. By Lemma 3.4.4, P_1 is an inversion of P_2 . Of course, P_2 is an inversion of itself. According to Theorem 3.4.3, we have $P_1 = P_2$ since they have the same top v .

Q.E.D.

LEMMA 3.4.6. *Suppose H is a planar extension of a planar covering graph G whose faces are four cycles. Let P_1 and P_2 be orientations of H with the vertex v in $V(G)$ as the top. Then*

$$(V(G))_{P_1} = (V(G))_{P_2}.$$

Proof. If $(V(G))_{P_1} \neq (V(G))_{P_2}$, then, according to Corollary 3.4.5, one of them, say, $(V(G))_{P_1}$, has no top. Assume $v_1 \in \max(V(G))_{P_1} - \{v\}$. Since P_1 has top v , there is a chain L in P_1 joining v_1 and v_2 with $V(L) \cap V(G) = \{v_1, v_2\}$, where, $v_1 < v_2 \leq v$ (P_1). The path consisting of the edges of L is inside the face of G containing v_1 and v_2 , since H is a planar extension of G . Since v_1 is noncomparable to v_2 in $(V(G))_{P_1}$ and the face containing v_1 and v_2 is four cycle $\{v_1, v_2, v_3, v_4\}$, either $v_1 > v_3$ in P_1 and $v_2 > v_3$ in P_1 or $v_3 > v_1$ in P_1 and $v_3 > v_2$ in P_1 . In the first case, we have a chain consisting of the edges of L and an edge $v_1 > v_3$ (P_1). At the same time, there is a covering relation $v_2 > v_3$ (P_1) in P_1 , which violates that P_1 is a diagram (Figure 3.4.2). In the second case, that is, $v_3 > v_1$ (P_1) and $v_3 > v_2$ (P_1), we have a chain consisting of the edges of

L and the edge $v_3 > v_2$, and also, there is a covering relation $v_3 > v_1$ (P_1), which contradicts that P_1 is a diagram (Figure 3.4.3).

Q.E.D.

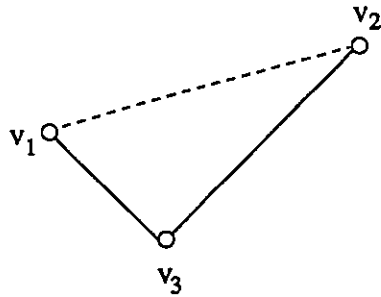


Figure 3.4.2

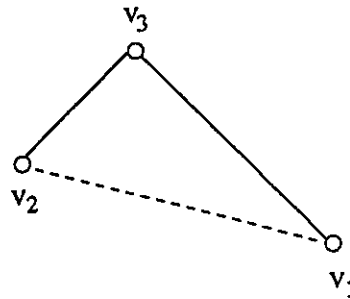


Figure 3.4.3

The following theorem tells us that Conjecture 4 is false .

THEOREM 3.4.7 (W. P. Liu and I. Rival). *There is a 2-connected planar covering graph any of whose orientations with top has no bottom.*

Proof. Let G be the planar covering graph illustrated in Figure 3.4.4 which is 2-connected. We show that any orientation of G with top has no bottom. By symmetry, it is enough to show that any orientation with v_1 or v_{12} as the top has no bottom. If v_1 (v_{12}) is the top, then, by Corollary 3.4.5, the orientation P of G must be the one illustrated in Figure 3.4.5 (Figure 3.4.6). Obviously, P has no bottom.

Q.E.D.

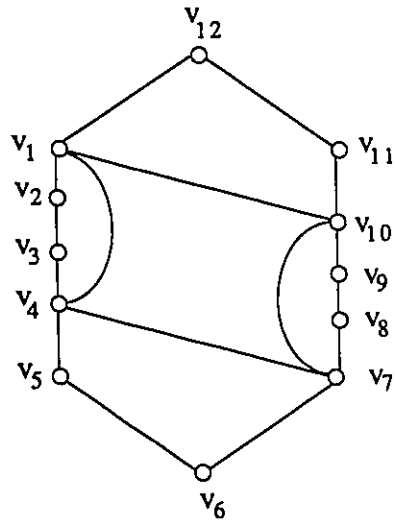


Figure 3.4.4

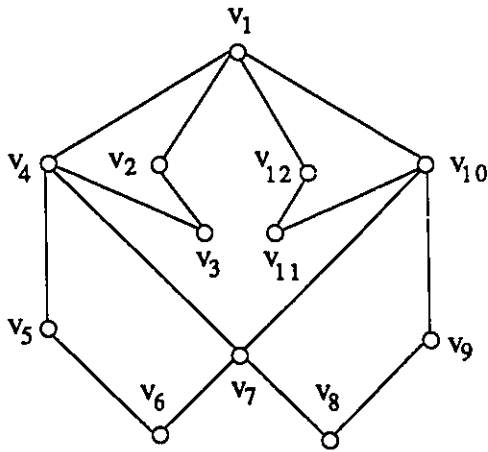


Figure 3.4.5

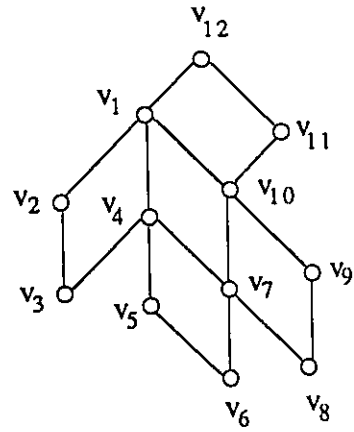


Figure 3.4.6

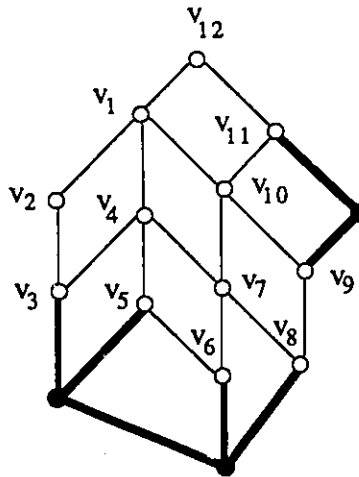


Figure 3.4.7

The planar covering graph G illustrated in Figure 3.4.4 is interesting. However we can construct a planar extension H which has an orientation with top and bottom (Figure 3.4.7). The following theorem is much stronger than Theorem 3.4.7 .

THEOREM 3.4.8 (W. P. Liu and I. Rival). *There is a planar covering graph G such that for any planar extension H of G , any orientation of H with top has no bottom.*

In order to prove Theorem 3.4.8, we need another lemma.

LEMMA 3.4.9. *Let the elements t and b be the top and bottom of an orientation of a minimum planar extension H of a planar covering graph G . If $t(b) \notin V(G)$, then any lower (upper) cover of $t(b)$ belongs to $V(G)$.*

Proof. It is sufficient to consider the case that the element t is the top. To this end, let P be an orientation with t as the top of H and let

$$A = \{a_1, a_2, \dots, a_k\} = \max_P \{v \in V(G) : v < t(P)\}.$$

Then, a_i is noncomparable to a_j if $a_i \neq a_j$. We claim that the graph H' obtained from the graph $H - \{u \in V(H) - V(G) : u \neq t, u > a_j(P) \text{ for some } a_j \in A\}$ by adding edges $ta_1, ta_2,$

..., ta_k is a planar covering graph. We only need to prove that H' is planar, since, obviously, it is triangle-free (as $\{a_1, a_2, \dots, a_k\}$ is an antichain of P).

Consider a planar embedding of H . For vertices a_i and a_j in A , there is a path C whose interior vertices belong to $V(H) - V(G)$ connecting a_i and a_j with $V(C) \cap V(G) = \{a_i, a_j\}$, since $t > a_i$, $t > a_j$ and any vertex bigger than a_i or a_j in P belongs to $V(H) - V(G)$. So, a_i and a_j belong to a same face of a planar embedding of G , which implies that all the vertices of A belong to the same face of G . Therefore, H' is planar. This leads to a contradiction to the assumption that H is minimum.

Q.E.D.

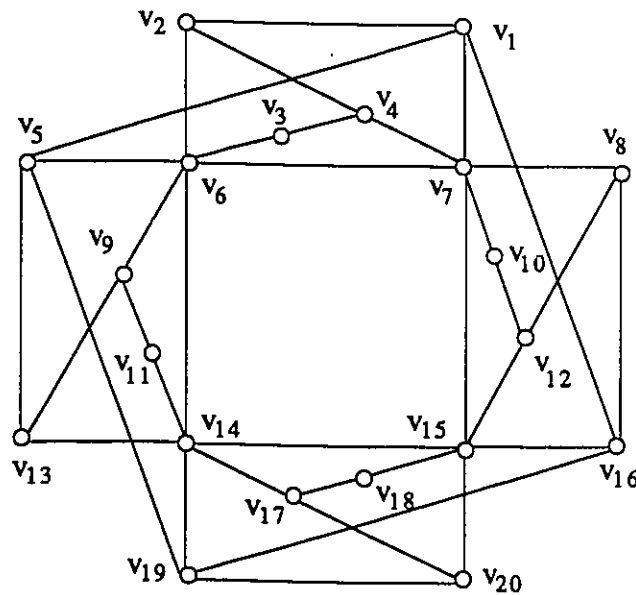


Figure 3.4.8

LEMMA 3.4.10. *Let G be the graph illustrated in Figure 3.4.8 and let C be a path connecting v_i and v_j with $V(C) \cap V(G) = \{v_i, v_j\}$. Then the graph $G \cup C$ is planar if and only if v_i and v_j belong to the same face of the planar embedding of G illustrated in Figure 3.4.8.*

Proof. By symmetry, it is enough to consider those cases that $i = 1, 2, 3, 4, 6, 7$. We will prove that if v_i and v_j do not belong to the same face of G , then a subdivision of

$$K_{3,3} = (\{v_{i_1}, v_{i_2}, v_{i_3}\}, \{v_{k_1}, v_{k_2}, v_{k_3}\})$$

is a subgraph of $G \cup C$, which is denoted by

$$(\{v_{i_1}, v_{i_2}, v_{i_3}\}, \{v_{k_1}, v_{k_2}, v_{k_3}\}) \leq G \cup C.$$

For convenience, by $(v_i, v_j) = (v_p, v_q)$, we mean that the graph $G \cup C_1$ is planar if and only if the graph $G \cup C_2$ is planar, where, C_1 (C_2) is a path connecting v_i (v_p) and v_j (v_q) with $V(C_1) \cap V(G) = \{v_i, v_j\}$ ($V(C_2) \cap V(G) = \{v_p, v_q\}$).

Case 1. $i = 1$

- If $j = 3$, then $(\{v_2, v_3, v_7\}, \{v_1, v_4, v_6\}) \leq G \cup C$;
 if $j = 9$, then $(\{v_5, v_7, v_9\}, \{v_1, v_6, v_{13}\}) \leq G \cup C$;
 if $j = 10$, then $(\{v_8, v_{10}, v_{15}\}, \{v_1, v_7, v_{12}\}) \leq G \cup C$;
 if $j = 11$, then $(\{v_6, v_{11}, v_{13}\}, \{v_1, v_9, v_{14}\}) \leq G \cup C$;
 if $j = 12$, then $(\{v_7, v_{12}, v_{16}\}, \{v_1, v_8, v_{15}\}) \leq G \cup C$;
 if $j = 13$, then $(\{v_6, v_{13}, v_{19}\}, \{v_1, v_5, v_{14}\}) \leq G \cup C$;
 if $j = 14$, then $(\{v_7, v_{14}, v_{19}\}, \{v_1, v_6, v_{15}\}) \leq G \cup C$;
 if $j = 15$, then $(\{v_6, v_{15}, v_{16}\}, \{v_1, v_7, v_{14}\}) \leq G \cup C$;
 if $j = 17$ or 18 , then $(\{v_{15}, v_{17}, v_{19}\}, \{v_1, v_{20}, v_{14}\}) \leq G \cup C$;
 if $j = 20$, then $(\{v_{14}, v_{16}, v_{20}\}, \{v_1, v_{15}, v_{19}\}) \leq G \cup C$.

Case 2. $i = 2$

- If $j = 9$ or 11 or 13 or 14 , then $(\{v_1, v_4, v_{14}\}, \{v_2, v_6, v_7\}) \leq G \cup C$;
 if $j = 8$ or 10 or 12 or 15 or 16 or 17 or 18 or 19 or 20 , then

$$(\{v_1, v_4, v_{15}\}, \{v_2, v_6, v_7\}) \leq G \cup C.$$

Case 3. $i = 3$

Then $(\{v_2, v_3, v_7\}, \{v_1, v_4, v_6\}) \leq G \cup C$.

Case 4. $i = 4$

Then $(\{v_2, v_7, v_{14}\}, \{v_1, v_4, v_6\}) \leq G \cup C$.

Case 5. $i = 6$

Then since $(v_6, v_8) = (v_2, v_4)$, $(v_6, v_{10}) = (v_3, v_{14})$, $(v_6, v_{12}) = (v_4, v_{14})$, $(v_6, v_{16}) = (v_1, v_{14})$, $(v_6, v_{17}) = (v_4, v_{15})$, $(v_6, v_{18}) = (v_3, v_{15})$, $(v_6, v_{19}) = (v_1, v_{15})$, $(v_6, v_{20}) = (v_2, v_{15})$, from case 1, case 2, case 3 and case 4, we know that for any path connecting v_6 and v_j , $G \cup C$ is planar if and only if v_6 and v_j belong to a same face of G .

Case 6. $i = 7$

Then since $(v_7, v_5) = (v_1, v_{15})$, $(v_7, v_9) = (v_4, v_{15})$, $(v_7, v_{11}) = (v_3, v_{15})$, $(v_7, v_{13}) = (v_2, v_{15})$, $(v_7, v_{17}) = (v_4, v_{14})$, $(v_7, v_{18}) = (v_3, v_{14})$, $(v_7, v_{19}) = (v_1, v_{14})$, $(v_7, v_{20}) = (v_2, v_{14})$, from cases 1, 2, 3 and 4, we know that for any path C joining v_7 and v_j , $G \cup C$ is planar if and only if v_7 and v_j belong to a same face of G .

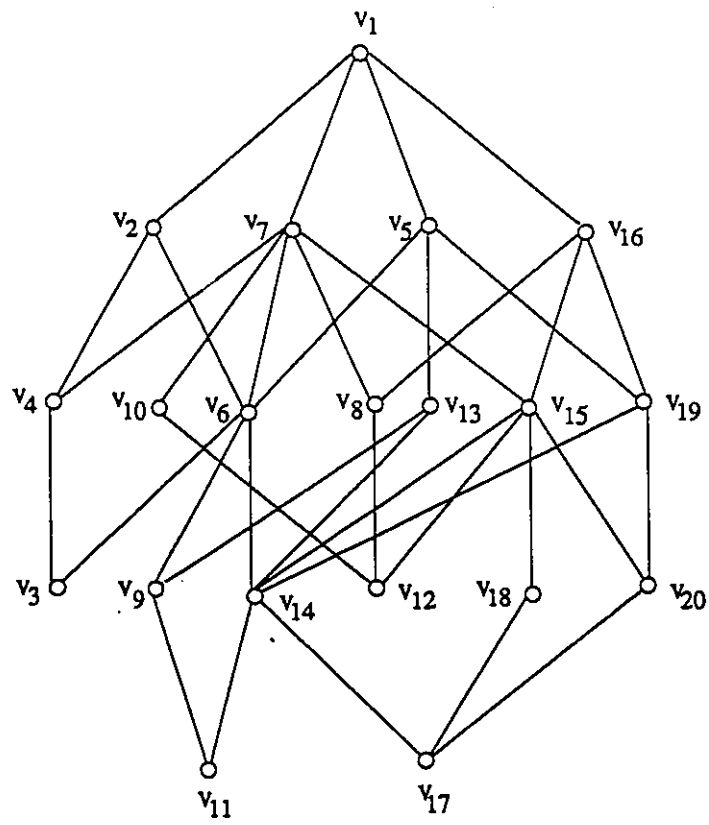
This completes the proof of Lemma 3.4.10.

Q.E.D.

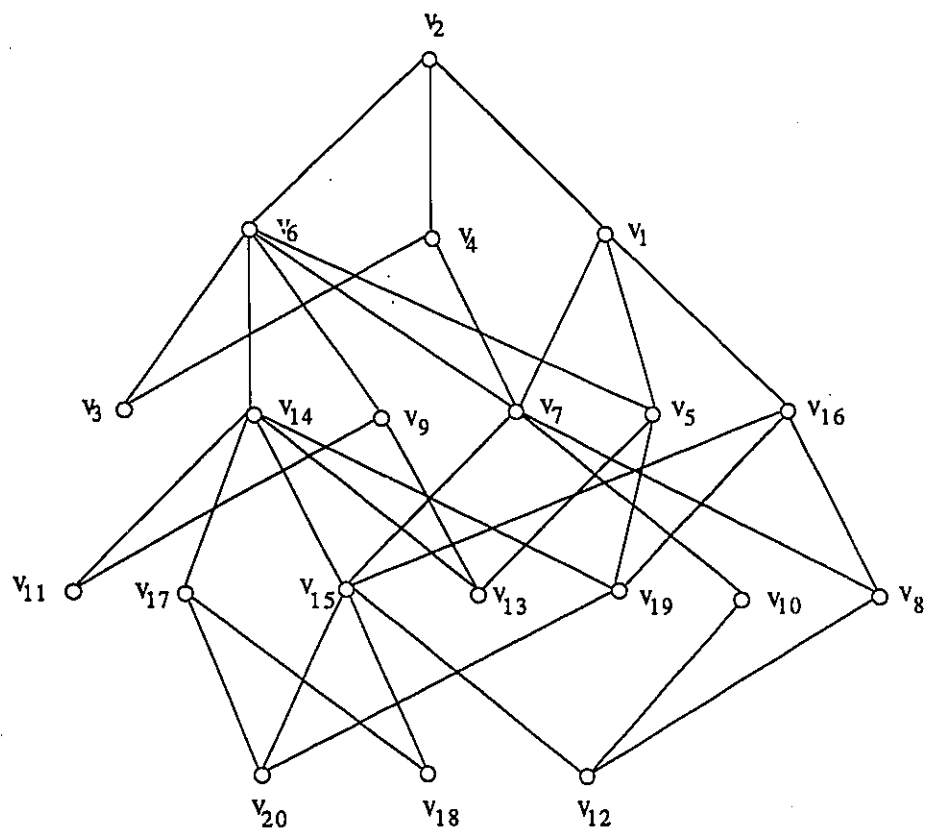
Proof of Theorem 3.4.5. Suppose the theorem is not true. Let H be a minimum triangle-free planar extension of G such that some orientation, P , of H has top u and bottom, where G is the graph given in Figure 3.4.8.

For simplicity, we use the following notation.

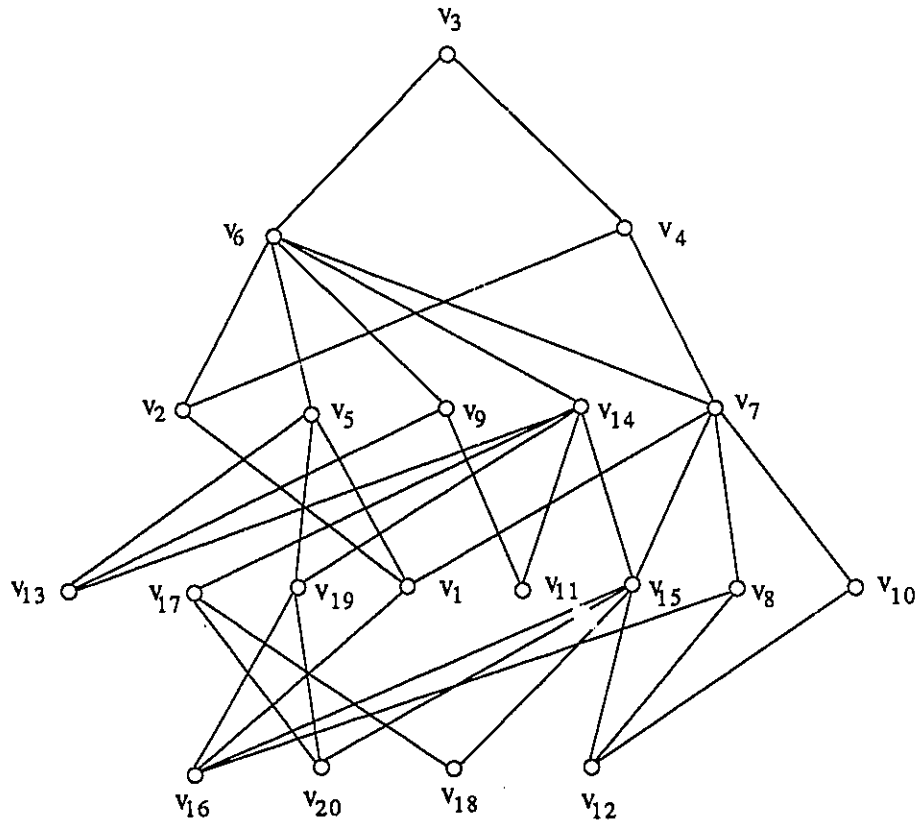
$v_1, v_2, \dots, v_k \succ (>, \neq) u_1, u_2, \dots, u_m$ means that $v_i \succ (>, \neq) u_j$, $1 \leq i \leq k$, $1 \leq j \leq m$.



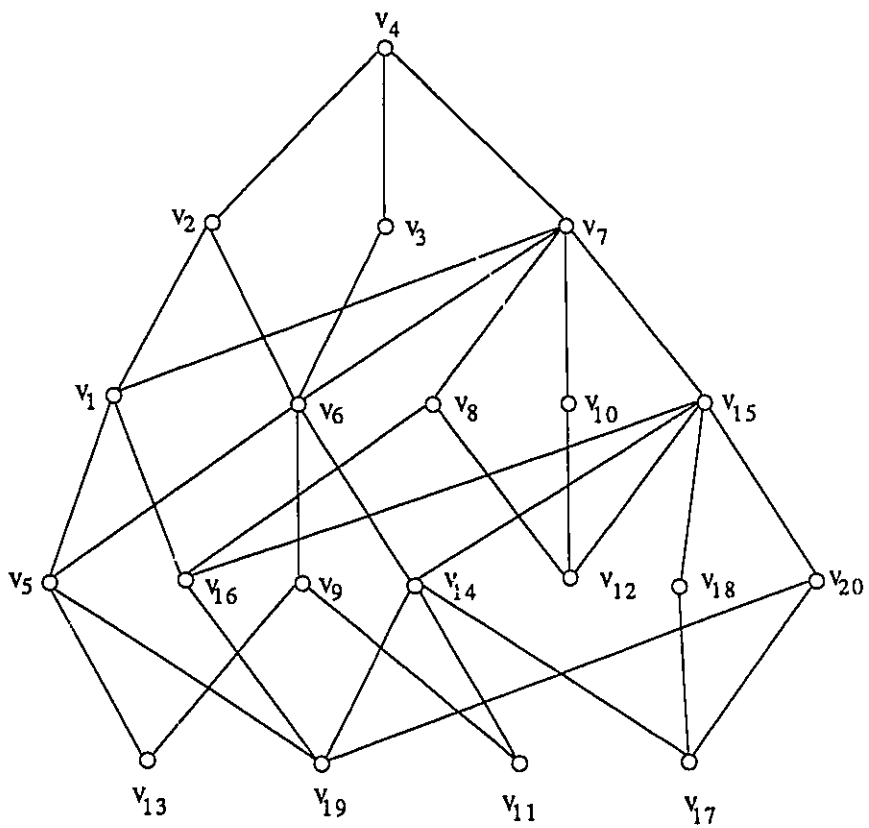
T_1



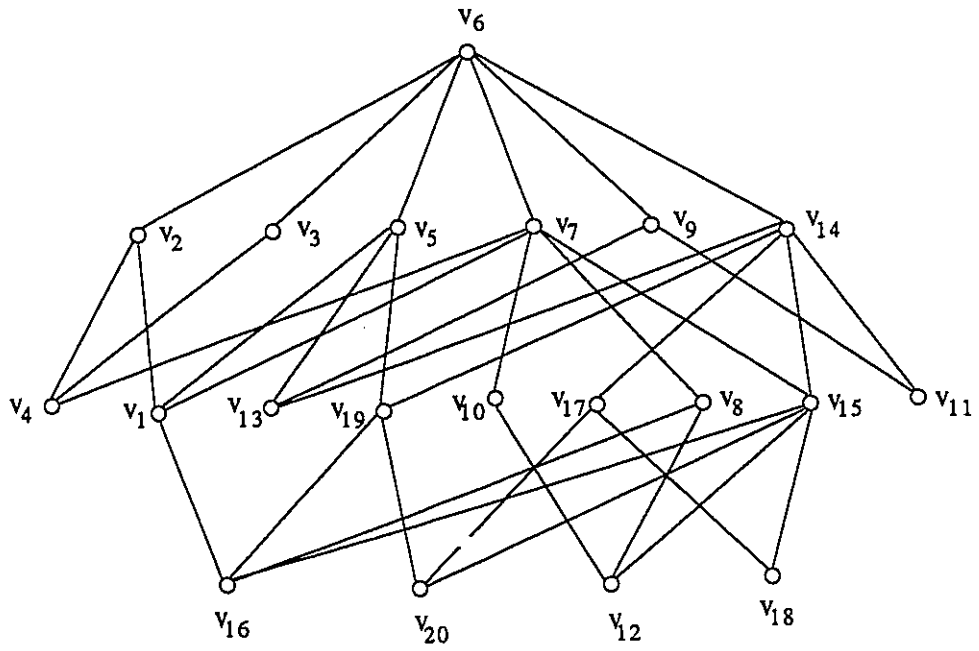
T_2



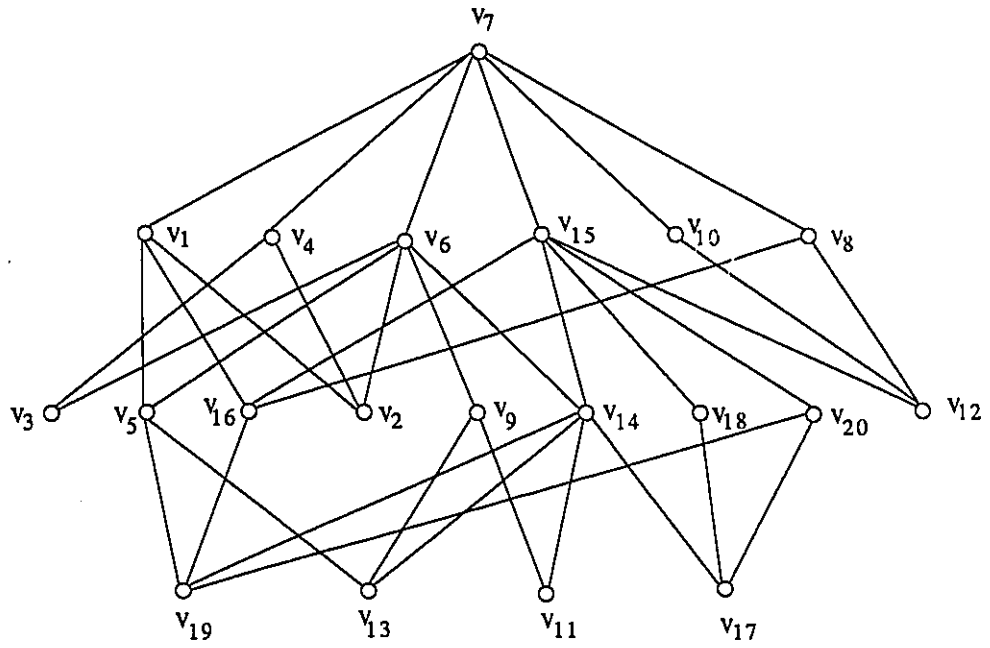
T_3



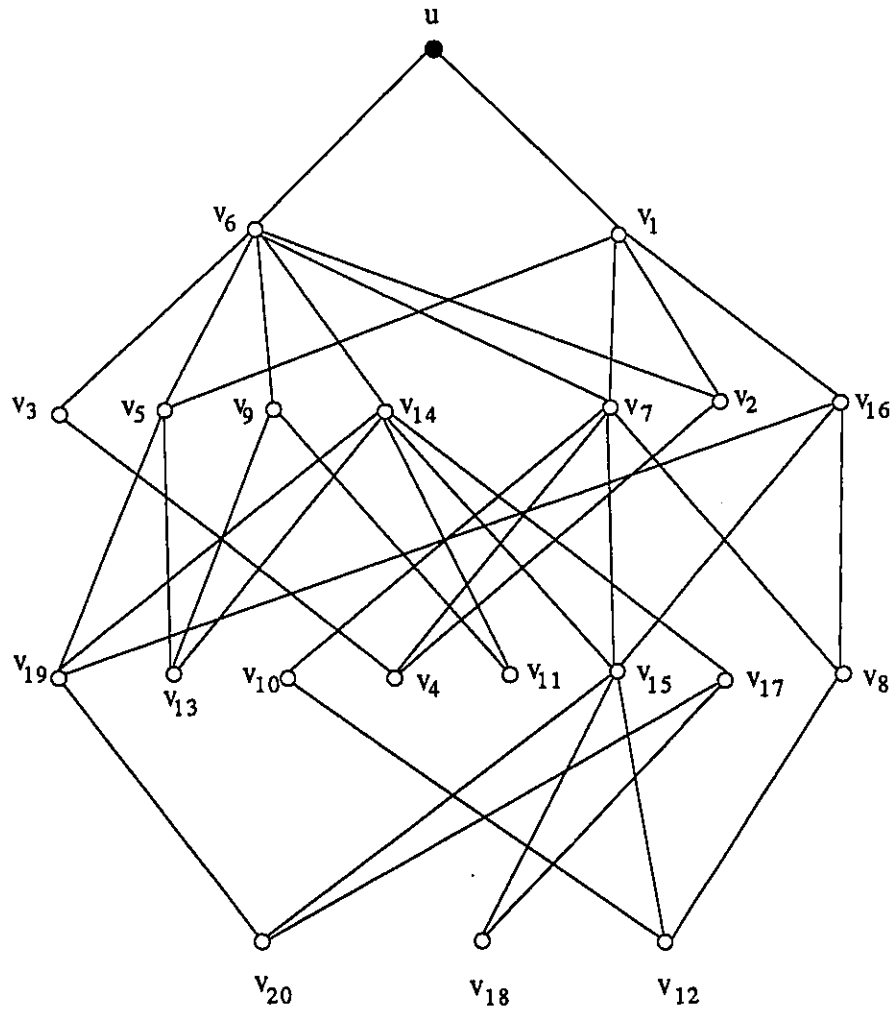
T_4



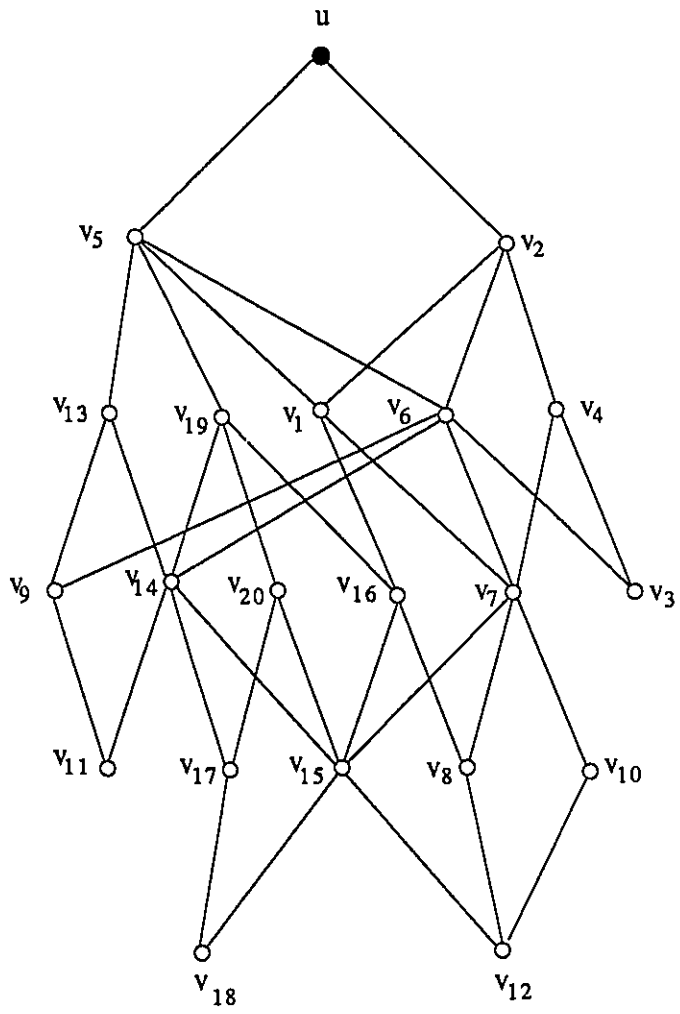
T_5



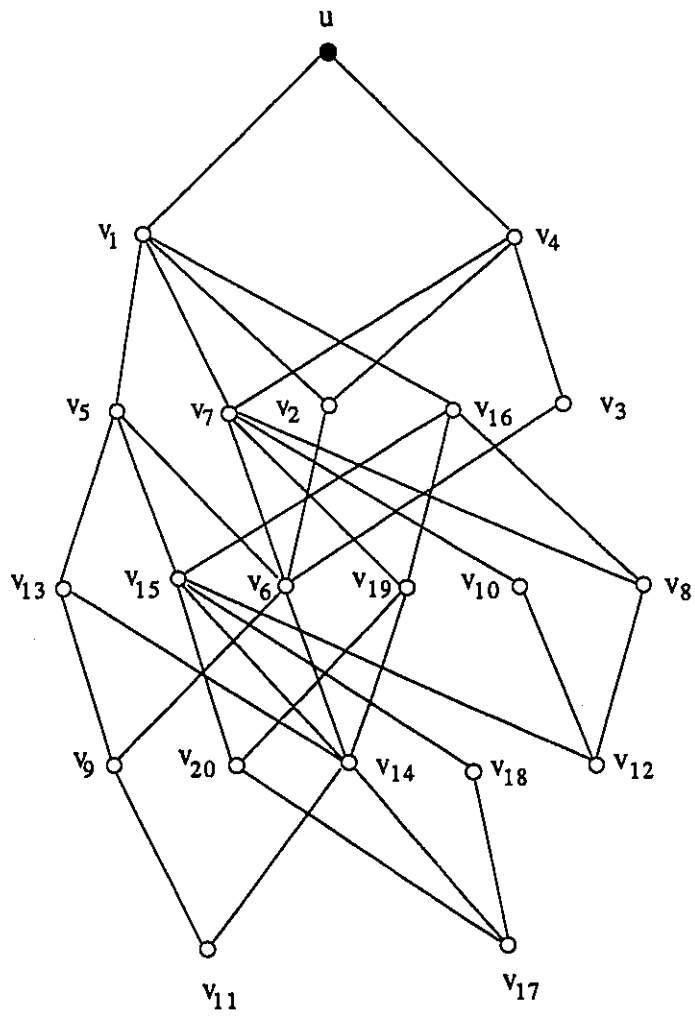
T_6



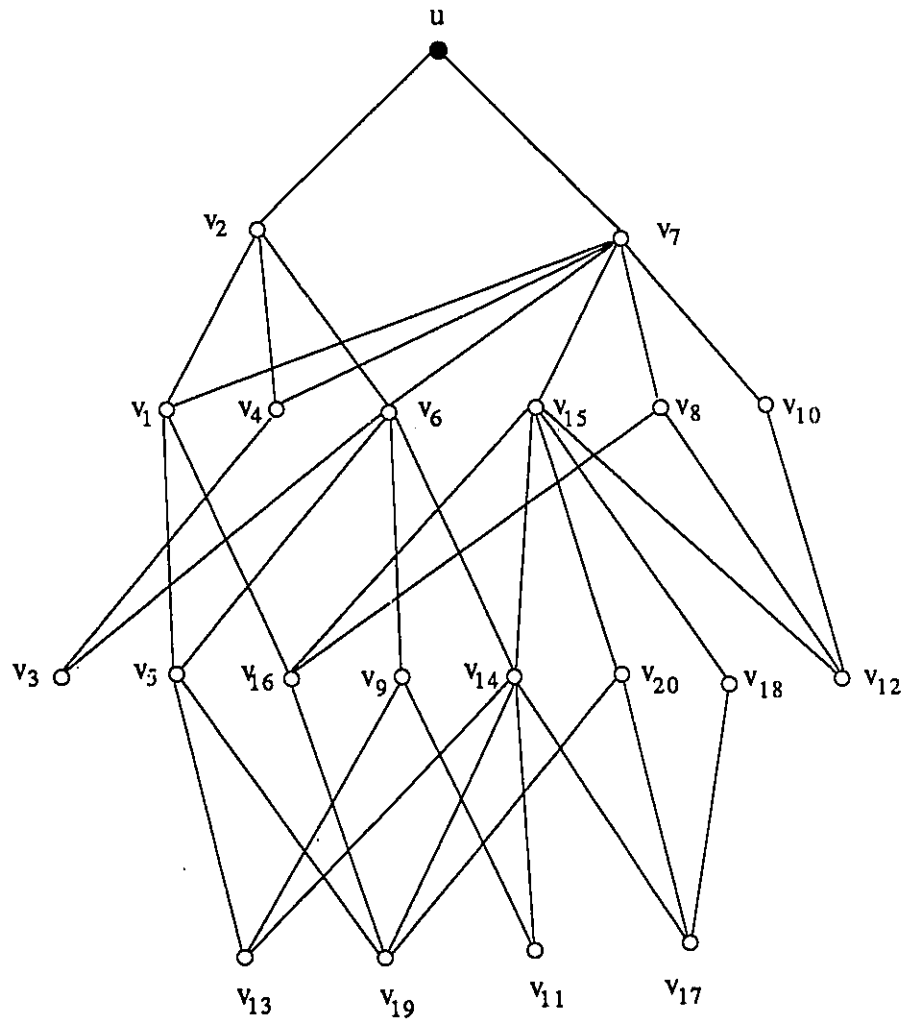
T_7



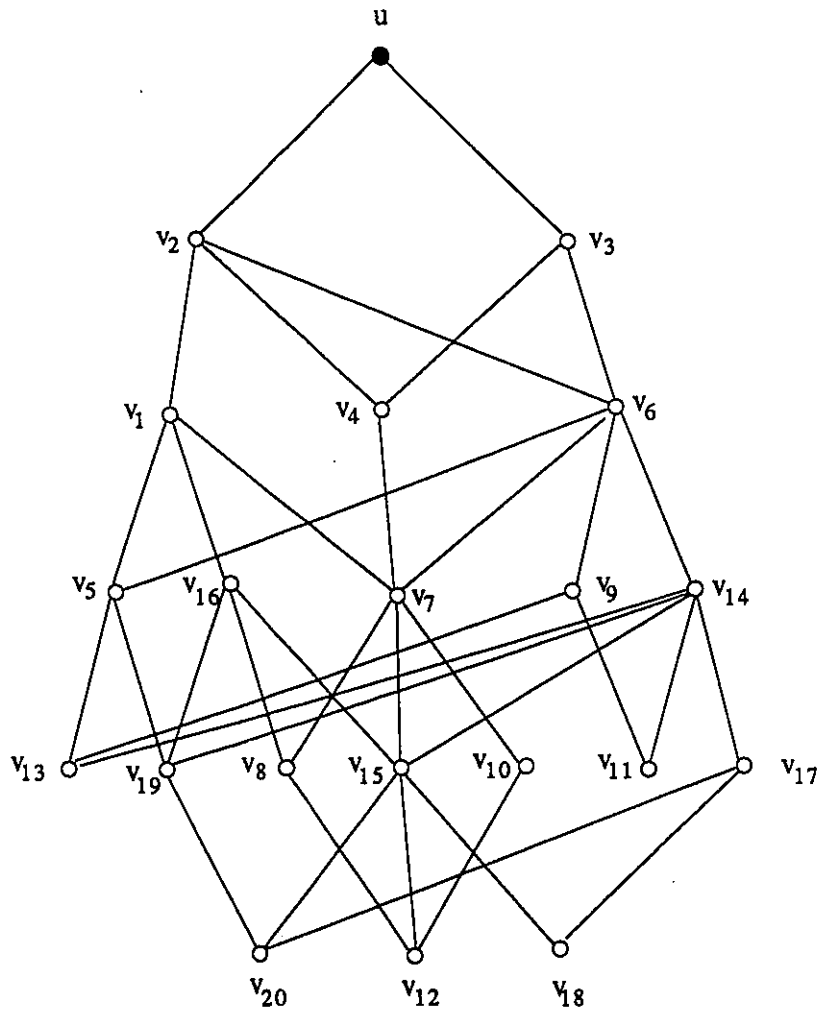
T_8



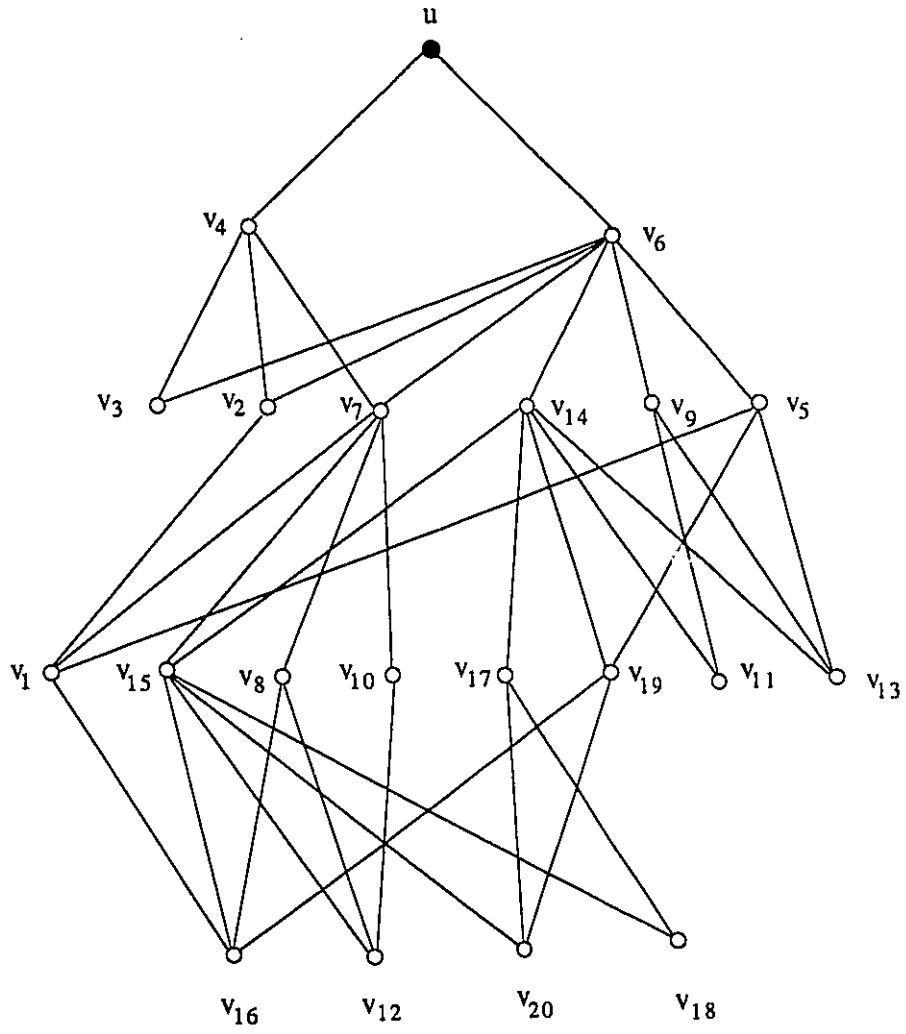
T_9



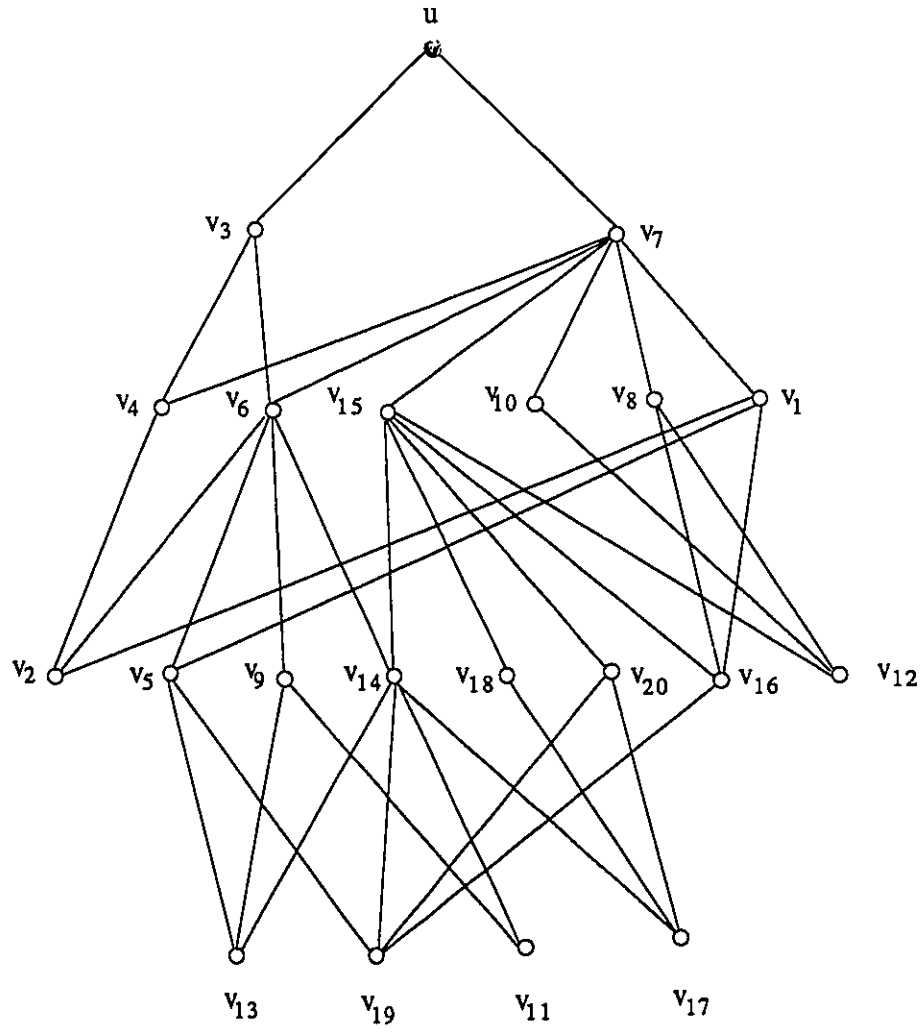
T_{10}



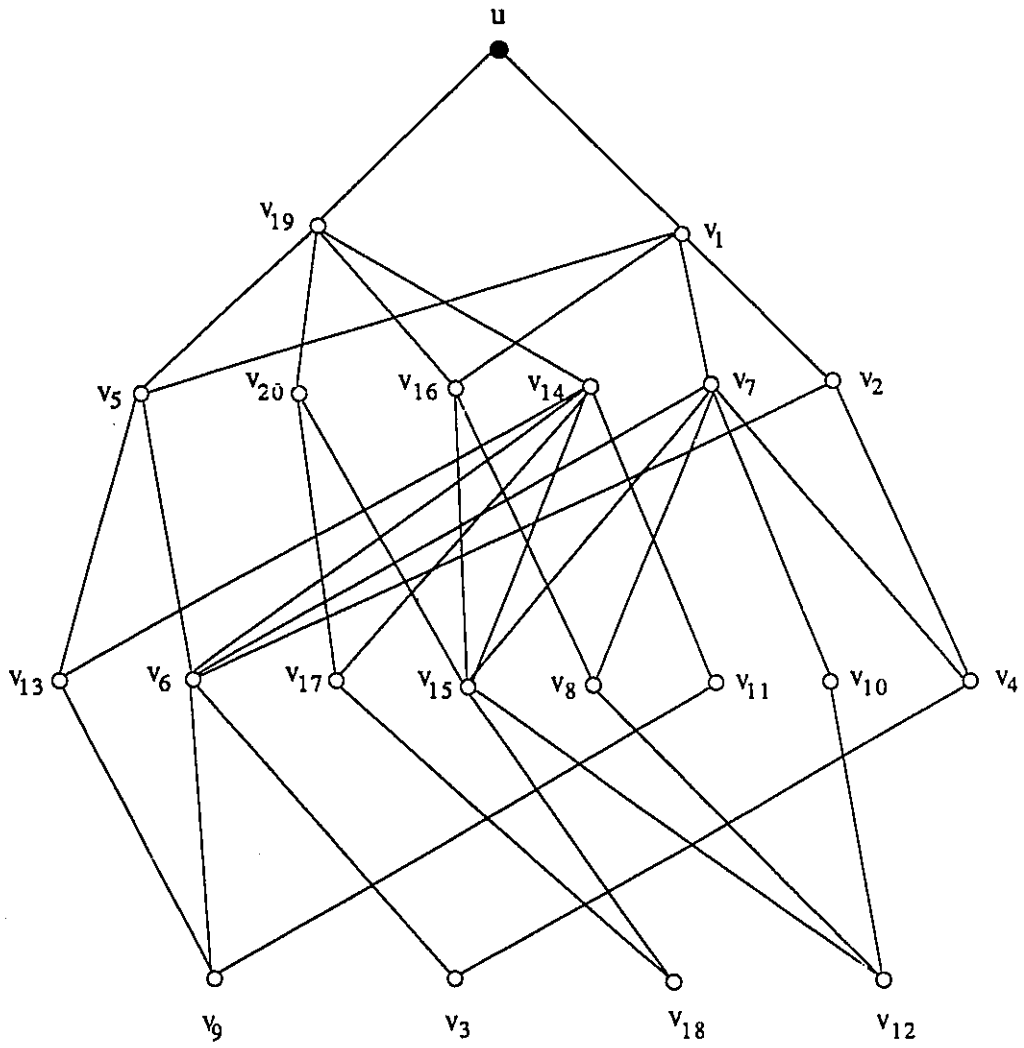
T_{11}



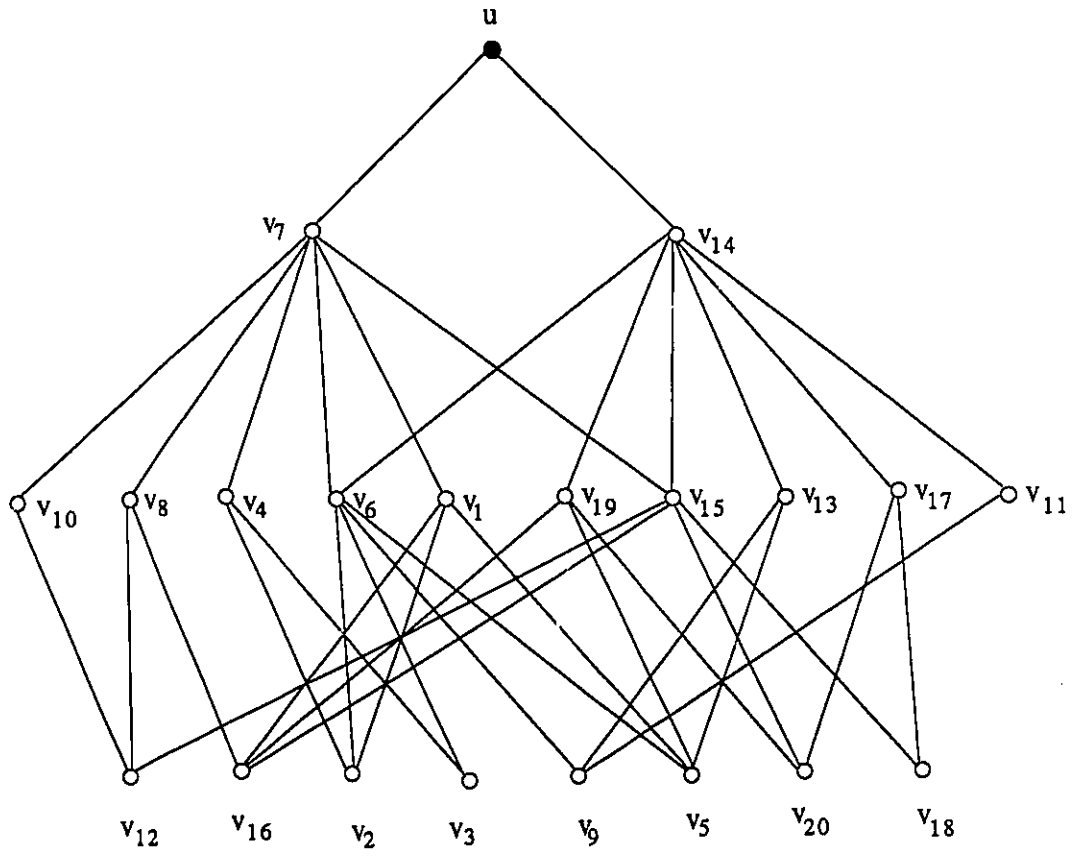
T_{12}



T_{13}



T_{14}



T_{15}

First, assume that $u = v_i$ for some i , $1 \leq i \leq 20$. By symmetry, we need only consider the cases $i = 1, 2, 3, 4, 6$ and 7 . According to Corollary 3.4.8, $(V(G))_P$ must be one of the diagrams T_1, T_2, T_3, T_4, T_5 , and T_6 . Without loss of generality, suppose $(V(G))_{P_i} = T_i$, $1 \leq i \leq 6$. We have the following observations:

- B1 $v_{11}, v_{17} \not\geq v_3, v_{12} (P_1)$;
- B2 $v_{12}, v_{20} \not\geq v_3, v_{11}, v_{13}, v_{18} (P_2)$;
- B3 $v_{12}, v_{18} \not\geq v_{11}, v_{13}, v_{16}, v_{20} (P_3)$;
- B4 $v_{11}, v_{19} \not\geq v_{12}, v_{13}, v_{17} (P_4)$;
- B5 $v_{12}, v_{18} \not\geq v_4, v_{11}, v_{13}, v_{16}, v_{20} (P_5)$;
- B6 $v_{11}, v_{19} \not\geq v_2, v_3, v_{12}, v_{13}, v_{17} (P_6)$.

B1 is true since $v_6 > v_3 (P_1)$, $v_{15} > v_{12} (P_1)$ and $v_6, v_{15} > v_{11}, v_{17} (P_1)$; B2 is true since $v_{14} > v_{11}, v_{13} (P_2)$, $v_6 > v_3 (P_2)$, $v_{15} > v_{12}, v_{18}, v_{20} (P_2)$ and $v_4, v_6, v_{14} > v_{12}, v_{20} (P_2)$; B3 is true since $v_{14} > v_{11}, v_{13} (P_3)$, $v_{15} > v_{12}, v_{16}, v_{18}, v_{20} (P_3)$ and $v_{14} > v_{12}, v_{18} (P_3)$; B4 is true since $v_{15} > v_{12} (P_4)$, $v_{14} > v_{11}, v_{13}, v_{17}, v_{19} (P_4)$ and $v_{15} > v_{11}, v_{19} (P_4)$; B5 is true since $v_7 > v_4 (P_5)$, $v_{14} > v_{13}, v_{11} (P_5)$, $v_{15} > v_{12}, v_{16}, v_{18}, v_{20} (P_5)$, $v_7 > v_{12}, v_{18} (P_5)$ and $v_{14} > v_{12}, v_{18} (P_5)$; B6 is true since $v_6 > v_2, v_3 (P_6)$, $v_{15} > v_{12} (P_6)$, $v_{14} > v_{11}, v_{13}, v_{17}, v_{19} (P_6)$, $v_6 > v_{11}, v_{19} (P_6)$, $v_{15} > v_{11}, v_{19} (P_6)$.

If $P = P_1$ has bottom, then, by B1, there must be a path C_1 in H connecting v_{11} and v_{17} with $V(C_1) \cap V(G) = \{v_{11}, v_{17}\}$. According to Lemma 3.4.10, H is not planar, which is a contradiction. Similarly, P_2, P_3, P_4, P_5 and P_6 have no bottom.

Now, assume u belongs to $V(H) - V(G)$. By Lemma 3.4.9 and Lemma 3.4.15, the neighbours of u in H must belong to the same face of G . Notice that each face of G is of length four, and u has two neighbors which belong to $V(G)$ (if u has only one neighbour v_i , then v_i is the top of $P - \{u\}$ which has bottom, contradicting the assumption that H is minimum). Hence $G \cup \{u\}$ is planar and each of its faces consists of

four vertices. By symmetry, we only need to consider those cases that the neighbour set of u is either $\{v_1, v_6\}$ or $\{v_2, v_5\}$ or $\{v_1, v_4\}$ or $\{v_2, v_7\}$ or $\{v_2, v_3\}$ or $\{v_4, v_6\}$ or $\{v_3, v_7\}$ or $\{v_1, v_{19}\}$ or $\{v_7, v_{14}\}$. By Lemma 3.4.6, $(V(G) \cup \{u\})_P$ must be one of the ordered sets $T_7, T_8, T_9, T_{10}, T_{11}, T_{12}, T_{13}, T_{14}$ and T_{15} .

Let $(V(G) \cup \{u\})_{P_j} = T_j$, $6 < j < 16$. As before, we have the following observations:

$$B7 \quad v_{12}, v_{18} \not\leq v_4, v_{11}, v_{13}, v_{20} (P_7);$$

$$B8 \quad v_{12}, v_{18} \not\leq v_3, v_{11} (P_8);$$

$$B9 \quad v_{11}, v_{17} \not\leq v_{12} (P_9);$$

$$B10 \quad v_{11}, v_{17} \not\leq v_3, v_{12}, v_{13}, v_{19} (P_{10});$$

$$B11 \quad v_{12}, v_{18} \not\leq v_{11}, v_{13}, v_{20} (P_{11});$$

$$B12 \quad v_{12}, v_{18} \not\leq v_3, v_{11}, v_{13}, v_{16}, v_{20} (P_{12});$$

$$B13 \quad v_1, v_{17} \not\leq v_2, v_{13}, v_{12}, v_{19} (P_{13});$$

$$B14 \quad v_3 \text{ is noncomparable to } v_9 \text{ in } P_{14} \text{ and } v_{18} \text{ is not comparable with } v_{12} \text{ in } P_{14};$$

$$B15 \quad \{v_{12}, v_{16}, v_{18}, v_{20}\} \text{ and } \{v_2, v_3, v_5, v_9\} \text{ are antichains in } P_{15}.$$

By the same argument as before, it is easy to check that if $(V(G) \cup \{u\})_P = T_j$, $6 < j < 14$, then P has no bottom. What we have to prove is that if

$$(V(G) \cup \{u\})_P = T_{14} \text{ or } T_{15},$$

then P has no bottom.

By the argument given in the first part of this proof, we know that if P has bottom b , then b is not in $V(G)$. According to Lemma 3.4.9, all the upper covers of b (actually, b has only two upper covers, since any face of G is of length four) must be in $V(G)$, which implies that if $(V(G) \cup \{u\})_P = T_{14}$, then

$$|\{v_3, v_9, v_{12}, v_{18}\} \cap \{v \in V(G) : v > b(P)\}| = 2 \quad (1)$$

and if $(V(G) \cup \{u\})_P = T_{15}$, then

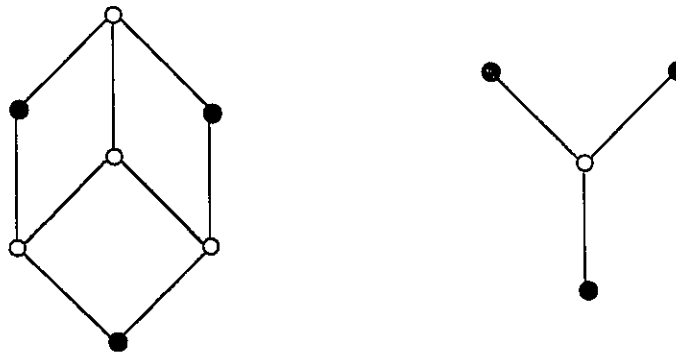
$$|\{v_2, v_3, v_5, v_9, v_{12}, v_{16}, v_{18}, v_{20}\} \cap \{v \in V(G) : v \succ b(P)\}| = 2. \quad (2)$$

In the first case, by B14, the upper covers of b must be either $\{v_3, v_9\}$ or $\{v_{12}, v_{18}\}$, which means that there is a path C in H connecting v_3 and v_9 or v_{12} and v_{18} with $|V(C) \cap V(G)| = 2$. By Lemma 3.4.10, H is not planar. By the equality (2), the second case violates the observation B15.

This completes the proof of Theorem 3.4.5.

Q.E.D.

We end this section by introducing a new definition - "thickness" - of ordered sets. Define the *thickness* of an ordered set P , denoted by $\theta(P)$, to be the minimum number of planar diagrams whose union is the diagram of P .



Two diagrams whose union is 2^3

Figure 3.4.9

Example 3.4.1. 1) If P is planar, then $\theta(P)=1$. 2) $\theta(2^3) = 2$, since the ordered set 2^3 is not planar and is the union of two diagrams (Figure 3.4.9).

The thickness of a diagram could be arbitrarily large. For example, let $K_{n,n}$ be the complete bipartite ordered set, then (G. Ringel [1965])

$$\theta(K_{n,n}) = \lceil n^2/4(n-1) \rceil.$$

We are interested in those diagrams whose covering graphs are planar. From the fact that $\theta(2^3)=2$, we know that for a planar covering graph G , the following inequality can be true:

$$\theta^*(G) = \max\{\theta(P) : P \text{ is an orientation of } G\} \geq 2.$$

An ordered set P whose covering graph is planar may have arbitrary large s -genus, while its thickness cannot be too big.

LEMMA 3.4.11 (Nash-Williams [1961]). *Let G be a nontrivial graph with p vertices and q edges and let q_m be the maximum number of edges in a subgraph with m vertices of G . Then G is the disjoint union of $\max\{q_m/(m-1) : 0 < m < p+1\}$ spanning forests.*

LEMMA 3.4.12. *Any n -element triangle-free planar graph has at most $2n-4$ edges.*

COROLLARY 3.4.13. *Any triangle-free planar graph is the disjoint union of two spanning forests.*

Proof. Let G be a triangle-free planar graph with n vertices. By Lemma 3.4.12, any subgraph with m vertices of G has at most $2m - 4$ edges. According to Lemma 3.4.11, G is the disjoint union of $\max\{q_m/(m-1) : 0 < m < n+1\} \leq \max\{(2m-4)/(m-1) : 0 < m < n+1\} \leq 2$ spanning forests.

Q. E. D.

THEOREM 3.4.14. *Any ordered set with planar covering graph has thickness at most two.*

Proof. Let P be an ordered set with the planar covering graph G . Then, according to Corollary 3.4.13,

$$G = F_1 \cup F_2,$$

where F_1 and F_2 are spanning forests of G . So,

$$P = P_1 \cup P_2,$$

where, P_i is the diagram with the covering graph F_i , $i = 1, 2$. Obviously, P_1 and P_2 are planar. Thus, $\theta^*(G) = \max\{\theta(P) : P \text{ is an orientation of } G\} \leq 2$.

Q. E. D.

3.5. MORE ABOUT PLANAR ORIENTATIONS. In this section, we explore some properties of planar orientations. From sections 3.2 and 3.3, we know that any independent set belonging to a face of a planar bipartite graph or an outerplanar graph can be an antichain of a planar orientation of the graph. Indeed, the assumption that *any independent set belonging to a face of planar covering graph can be an antichain of a planar orientation* is reasonable if we assume that any planar covering graph has planar orientation.

THEOREM 3.5.1 (W. P. Liu). *Let I be an independent set belonging to a face of planar covering graph G . Then there is a planar embedding of an orientation P of G such that I is an antichain of P and belongs to the boundary of the embedding.*

Before proving Theorem 3.5.1, we give one more definition. Let H be a planar extension of a planar covering graph G and let E be a planar embedding of an orientation Q of H . Then the (planar) embedding consisting of the line segments representing the edges of the orientation $P = (V(G))_Q$ of G is called a *subembedding* of E induced by P .

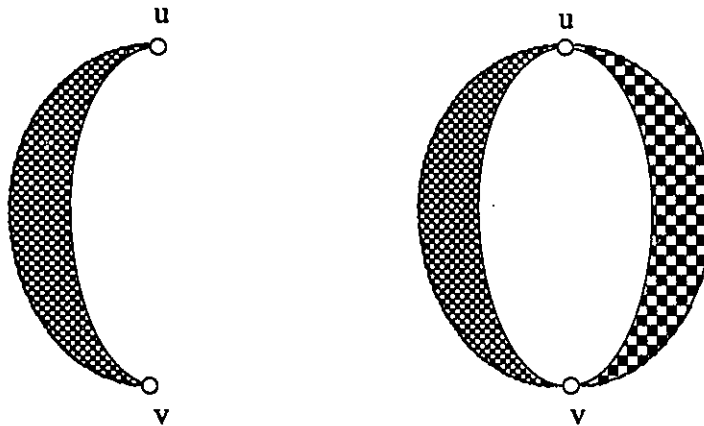
In this section, we assume that any planar covering graph has a planar orientation.

LEMMA 3.5.2. *Let v be a vertex of a planar covering graph G . Then there is a planar embedding E of an orientation of G in which v belongs to the boundary of E .*

Proof. Choose a vertex u not adjacent to v which belongs to the same face as v . Let H be the graph obtained from G and G' by identifying v with v' , and u with u' , where G' is a copy of G and u' (v') the copy of u (v) (Figure 3.5.1). Consider a planar embedding E' of an orientation R of H . If v belongs to the boundary of E' , then v belongs to the boundary of the subembedding E_1 of E' induced by $(V(G))_R$. Otherwise, the subembedding E_2 of E' induced by $(V(G'))_R$ must be contained in an interior face (that is, not in the external face) of E_1 . So v' must belong to the boundary of E_2 . Since G' is a

copy of G , v belongs to the boundary of a planar embedding of an orientation. In any case, there is a planar embedding E of an orientation of G such that v belongs to the boundary of E .

Q.E.D.



A planar "graph" G

The "graph" H obtained from G and its copy by identifying two vertices u and v

Figure 3.5.1

LEMMA 3.5.3. *For any vertex v of a planar covering graph G , there is a planar embedding R of an orientation of G such that v is maximal and belongs to the boundary of R .*

Proof. Let v and u be two independent vertices belonging to the same face of G . Let H' be the graph obtained from the graphs $G_0 = G$, G_1 , G_2 and G_3 by identifying the vertex $v_0 = v$ with the vertices v_1 , v_2 and v_3 , where G_i is a copy of the graph G , the vertex v_i a copy of the vertex v , $0 \leq i \leq 3$. Let $H = H' \cup \{u_0u_1, u_1u_2, u_2u_3, u_3u_0\}$, u_1 , u_2 and u_3 are copies of $u = u_0$. Then H is a planar covering graph. By Lemma 3.5.2, there is a planar embedding E of an orientation H such that v belongs to the boundary of E .

Notice that the four edges u_0u_1 , u_1u_2 , u_2u_3 and u_3u_0 constitute a cycle C which divides the plane in two parts - the interior part and the external part. Since v is a cut vertex of H' , the subembedding of E' induced by $(V(H'))_R$ must be either inside the external face of C or the interior face of C , and furthermore, the subembedding of E' induced by $(V(G_i))_R$ must be inside the external face of the subembedding of E' induced by $(V(G_j))_R$ for $i \neq j$. Therefore, v must be maximal in of the ordered sets $(V(G_i))_R$, $i = 0, 1, 2, 3$, which implies that v is maximal in a planar orientation of G since G_i is a copy of G .

Q.E.D.

Now, we are ready to prove Theorem 3.5.1. To this end, let $I = \{a_1, \dots, a_k\}$ be an independent set belonging to a face G . Let $H = G \cup \{va_1, \dots, va_k\}$, where v does not belong to $V(G)$. Then H is a planar covering graph. By Lemma 3.5.3, there is a planar embedding E' of an orientation Q of H such that v belongs to the boundary of E' and v is maximal in Q . So I is an antichain of $P = (V(G))_Q$ and belongs to the boundary of a planar embedding of P which is a subembedding of E' induced by P .

CHAPTER 4. ENUMERATION

4.1. INTRODUCTION. For an ordered set P , the reversals of some of the edges of P may produce a new ordered set. How many reorientations does an ordered set have? We do not know. O. Pretzel [1986] proved that *any n -element ordered set has at least $n^2/2 + n$ reorientations*. It seems that an ordered set has many more orientations.

The problem of enumerating the reorientations of an ordered set is of fundamental importance.

As we know, one of the most difficult problems in ordered set theory is to find a nontrivial order theoretical property of an ordered set such that any reorientation of it satisfies the property – diagram invariants. There are few results in this area. An element is *doubly irreducible* if the element has one upper cover and one lower cover. *If P and P' are finite lattices with the same covering graph and P is planar, then P' contains doubly irreducible elements* (R. Jégou, R. Nowakowski and I. Rival [1985]). *If P and P' are finite lattices with the same covering graph and P has a planar embedding $e(P)$ in which all its doubly irreducible elements lie on the boundary, then for any planar embedding $e(P')$ of P' , the set of faces of $e(P')$ is the same as the set of faces of $e(P)$* (R. Jégou, R. Nowakowski and I. Rival [1985]). So far the only known nontrivial diagram invariant – genus – is due to K. Ewacha, W. Li and I. Rival [1990].

Every surface is topologically equivalent to a sphere with handles; the *genus* of the surface is the number of handles that must be added to obtain its homeomorphism type. The *genus* of an ordered set is the smallest integer g such that its diagram can be drawn, without edge crossings, on a surface with genus g , in such a way that, whenever $a > b$ in P , the z -coordinate of a is larger than the z -coordinate of b , and all edges of P are monotonic with respect to the z -coordinate. (We should notice the difference between the

genus here and the **sphere** genus defined in Chapter 3.) K. Ewacha, W. Li and I. Rival [1990] proved that *the (graph) genus of a covering graph is always equal to the genus of any its orientations*. (Recall that although the genus of a covering graph is zero, some of its orientations may have arbitrarily large **sphere** genus.) It is reasonable that only rarely is a property a diagram invariant for, as we shall show, *almost every ordered set has many reorientations*.

Another interesting problem is which elements of an ordered set P can be the set of maximal elements of a reorientation of P . O. Pretzel found such a set and proved the following theorem.

THEOREM (O. Pretzel [1986]). *Let I be an independent set of the covering graph G of an ordered set P such that for any cycle C of G , $|I \cap C|$ is at most the number of forward edges of C . Then I can be the set of maximal elements of a reorientation of P .*

Can we generalize Pretzel's result? Is it true that any independent set of the covering graph of an ordered set P can be an antichain of some of the reorientations of P ? This question is closely related to the problem of enumerating the reorientations of an ordered set for the following reasons. First, it is quite easy to show that any *antichain of an ordered set P is an independent set of its covering graph*. Second, *any n -element covering graph has an independent set of size at least $\sqrt{n/2}$* . Here is the argument. If the ordered set P has an antichain of size at least $\sqrt{n/2}$, then, obviously, the covering graph has an independent set of size at least $\sqrt{n/2}$; otherwise, the ordered set P has a chain $a_1 > a_2 > \dots > a_k$, where $k \geq \sqrt{2n}$; so, $\{a_1, a_3, a_5 \dots\}$ is an independent set of size at least $\sqrt{n/2}$. Third, *if $A(P)$ is a subset of an ordered set P which can be the set of the maximal elements of a reorientation of P (for example, an antichain of P), then, P has at least $2^{|A(P)|}$ reorientations*, since any subset of $A(P)$ can be the set of maximal elements of a reorientation of P by pushdown.

In the remaining sections, we first study the possibility that an independent set of a covering graph G can be the set of maximal elements of some orientation of G , then enumerate the number of orientations of a covering graph.

4.2 TWO EXAMPLES. An antichain of an ordered set P is an independent set of the covering graph of P . Theorem 3.2.9 and Theorem 3.3.6 tell us that any independent set of a covering graph can be the set of maximal elements of an orientation of the covering graph if the graph is outerplanar or planar and bipartite. We have the following two questions.

QUESTION 1 (cf. W. P. Liu and I. Rival [1991]). *Is it true that any independent set of a covering graph G can be the set of maximal elements of an orientation of G ?*

QUESTION 2 (cf. W. P. Liu and I. Rival [1991]). *Is it true that, for any "matching" M of an ordered set P and a subset M' of M , there is an orientation of P which reverses the edges in M' and none in $M - M'$?*

If both of the two problems had positive answers, then we could easily prove that any n -element ordered set has at least $2^{n/3}$ reorientations. The reason is as follows. First, it is quite easy to verify the following inequality:

the size of a maximum independent set + $2 \times$ the size of a maximum matching $\geq n$.

In fact, let M be a maximum matching of P . Then $V(G) - V(M)$ is an independent set, where, G is the covering graph of P . The inequality follows from the fact that the size of a maximum independent set is not less than $|V(G) - V(M)|$. Second, from the inequality above, we know that either

the size of a maximum independent set $\geq n/3$

or

the size of a maximum matching $\geq n/3$.

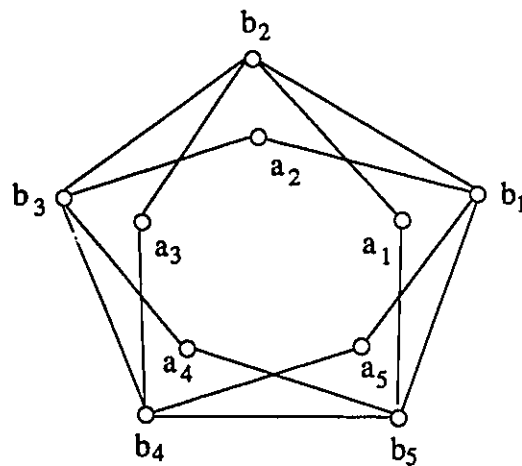
Finally, if P contains an independent set I with $|I| \geq n/3$, then, according to the assumption, there is a reorientation Q of P such that I is an antichain. Possibly by pushdown, we can obtain a reorientation P' of Q (of course, of P) with I as the set of maximal elements. Since any subset of I can be the set of maximal elements of a

reorientation of P' , the ordered set P has at least $2^{|M|} \geq 2^{n/3}$ reorientations. Otherwise, the ordered set P contains a matching M with $|M| \geq n/3$. By the assumption again, for each subset M' of M , there is a reorientation P' of P which reverses the edges in M' none in $M - M'$. Again, P has at least $2^{|M|} (\geq 2^{n/3})$ reorientations.

Unfortunately, the answers to both questions are negative.

THEOREM 4.2.1 (W. P. Liu and I. Rival [1991]). *There is a covering graph G with an independent set which is not the set of maximal elements of an orientation of G .*

Proof. Let G be the covering graph illustrated in Figure 4.2.1 (since M_5 is the smallest noncovering graph). We prove the theorem by showing that the independent set $\{a_1, a_2, a_3, a_4, a_5\}$ cannot be an antichain of an orientation of G .



A covering graph G

Figure 4.2.1

Suppose there does exist an orientation P of G with the independent set as an antichain. Then, from P , we can obtain an ordered set with the independent set as the set of its maximal elements, from which we can get an ordered set Q with the element a as the top by adding the covering relation $a >- a_1, a >- a_2, a >- a_3, a >- a_4$ and $a >- a_5$. The

ordered set Q is an orientation of M_5 (see Figure 4.2.2). We obtain a contradiction since M_5 is not a covering graph.

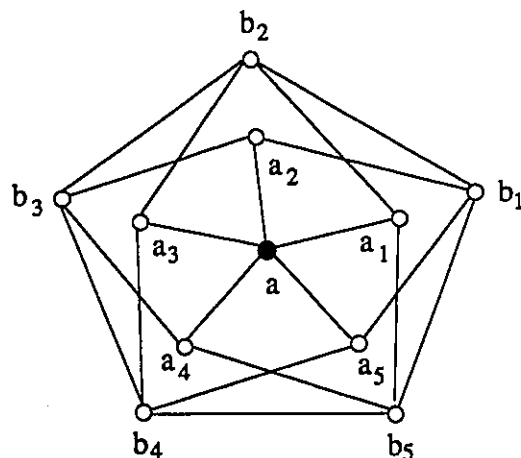


Figure 4.2.2

Similarly, the independent set $\{a, b, c, d, e, f\}$ of the covering graph G illustrated in Figure 4.2.3 cannot be an antichain of an orientation of it. Notice that the graph G is the covering graph of the ordered set P illustrated in Figure 2.1.7 (2), so this proves that the reversal of the edge $g > a$ forces us to reverse the edge $a > i$, if we want to keep the elements $b, c, d, e,$ and f as maximal elements.

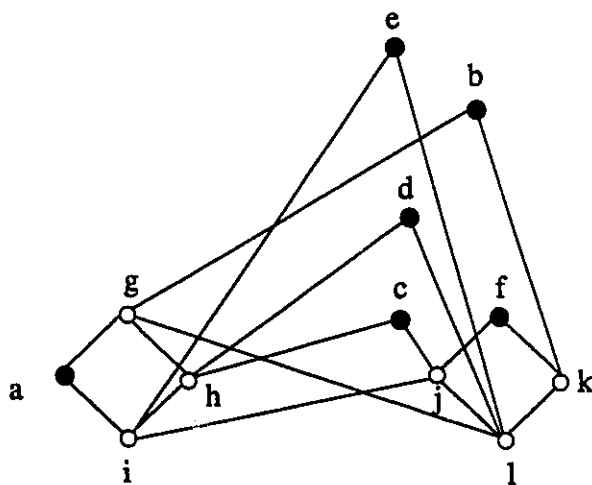


Figure 4.2.3

We can prove a result stronger than Theorem 4.2.1, that is, we cannot expect that any independent set of a covering graph with large girth is the set of maximal elements of an orientations of the graph.

THEOREM 4.2.2 (J. Nešetřil and V. Rödl [1978]). *For any integer k , there is a non-covering graph with girth at least k .*

THEOREM 4.2.3 (W. P. Liu). *For any integer k , there is a covering graph G with girth at least k having an independent set which cannot be the set of maximal elements of any orientation of G .*

Proof. Let H be a minimum (with respect to the number of vertices) noncovering graph with girth at least k (according to Theorem 4.2.2, there is a such graph). Choose any vertex v in H . Let $G = H - v$. Then, by the assumption that H is minimum, the graph G is a covering graph with girth at least k . The neighbour set $I = \{v_i : 1 \leq i \leq m\}$ of v in H is an independent set which cannot be the set of maximal elements of an orientation of G . For otherwise, let Q be the ordered set obtained from the ordered set P by adding covering relations $v \succ v_i$, $1 \leq i \leq m$. Then Q is an orientation of H , which contradicts the assumption that H is not a covering graph.

Q.E.D.

We turn to consider the second question.

THEOREM 4.2.4 (W. P. Liu and I. Rival [1991]). *There is an ordered set P such that for some matching M and a subset M' of M , there is no reorientation which reverses the edges of M' and none of $M - M'$.*

Proof. Let P be the ordered set illustrated in Figure 4.2.4 and let M be the matching in bold. Let M' consist of the edge $b \succ a$ (P). We show that if $b \succ a$ is reversed then some other edge of M must also be reversed. To see this, let $b \succ a$ be reversed. By not

reversing any edges of $M - M'$, we have the following sequence of reversals. Either the edge $a_1 \succ a$ or the edge $b \succ b_1$ must be reversed. If $b \succ b_1$ is reversed, then also $b_2 \succ b_3, b_{10} \succ b_3$, then $b_4 \succ b_5$ and $b_6 \succ b_7$, then $b_8 \succ b_9$ from which $b_8 \succ b_5$ too must be reversed, although it belongs to $M - M'$. Similarly, if the edge $a_1 \succ a$ (P) is reversed, eventually, the edge $a_4 \succ a_{10}$ (P) which belong to $M - M'$, must be reversed.

Q.E.D.

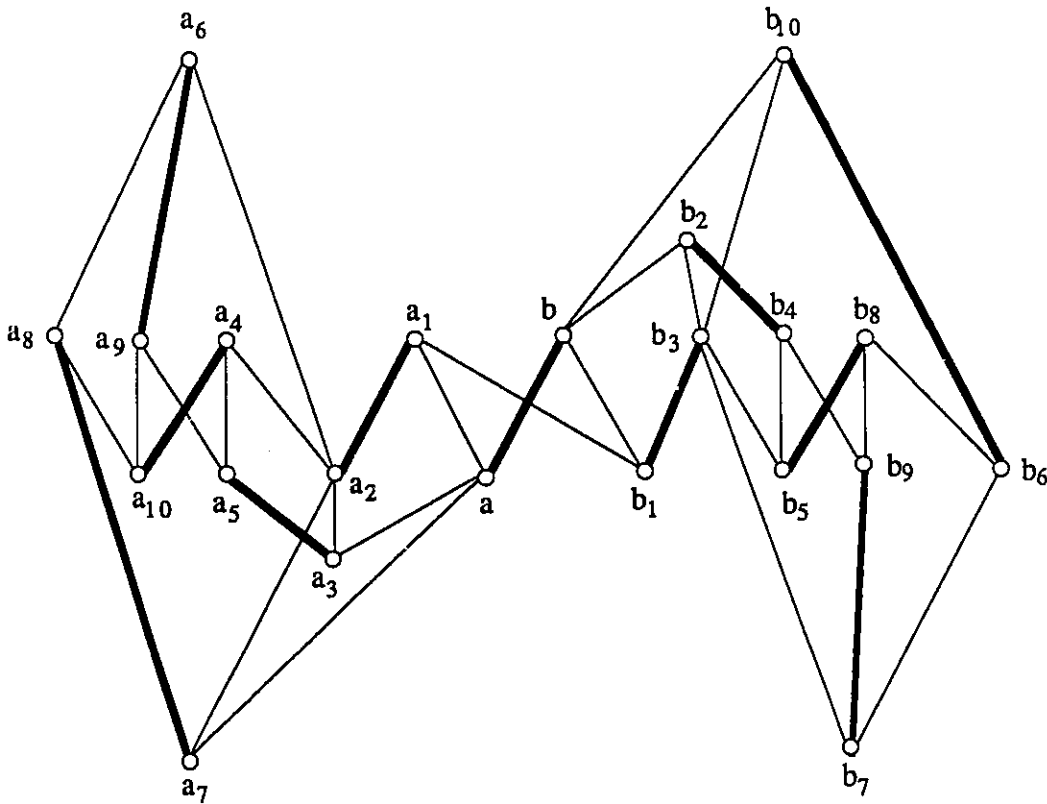


Figure 4.2.4

Remark: In spite of Theorem 4.2.3, the problem of how to explicitly give a covering graph G and an independent set which cannot be an antichain of any orientation of G remains open.

Here is a positive result.

THEOREM 4.2.5 (W. P. Liu). *Any independent set of a planar lattice can be the set of maximal elements of a reorientation of the lattice.*

Proof. We need the following well known result.

THEOREM 4.2.6 (R. Jégou, R. Nowakowski and I. Rival [1985]). *Any planar lattice has a doubly irreducible element.*

Let P be a planar lattice. Without loss of generality, we may assume that the set I is a maximal independent set and P is 2-connected. By induction on the number of elements, we prove that *for any independent set I there is a reorientation Q of P such that I is the set of maximal elements of Q which has length at most three.* To this end, let $C: a = a_1 > a_2 > \dots > a_k = b$ be a maximal chain of P whose internal elements have degree two (according to Theorem 4.2.6). Let P_1 be a subset of P with $S(P_1) = S(P) - \{ \text{the internal elements of } C \}$. Then P_1 is a planar lattice. Let D be the other chain in P with the element a as the top and the element b as the bottom such that $D \cup C$ is a face of P . By induction hypothesis, there is a reorientation Q_1 of P_1 in which any maximal independent set containing $I - S(C)$ is the set of maximal elements of Q_1 which has length at most three. It is easy to add a zigzag consisting of the edges of C to Q_1 to obtain a reorientation Q of P such that $I = \max Q$ and Q has length at most three.

Q.E.D.

4.3. A SUFFICIENT CONDITION. In this section, we give a necessary and sufficient condition for a directed graph obtained from a diagram to produce a diagram by reversing some of its edges. The theorem is important theoretically; it gives a method to prove if a directed graph is a diagram.

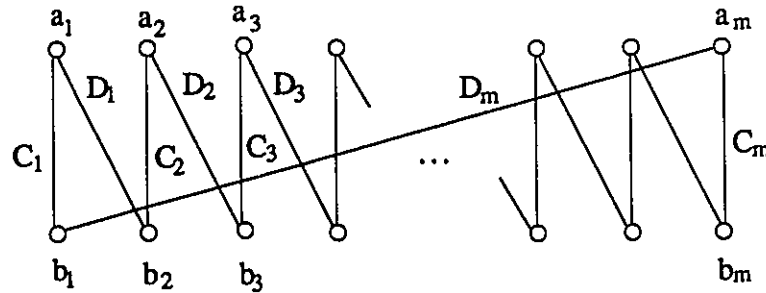


Figure 4.3.1

In this section, a *cycle* of a diagram P is a subset $\{a_1, a_2, \dots, a_m, b_1, b_2, \dots, b_m\}$ of P such that there are chains $C_1, C_2, \dots, C_m, D_1, D_2, \dots, D_m$ satisfying, for each $i = 1, 2, \dots, m$,

- 1) $a_i = \text{top } C_i = \text{top } D_i$;
- 2) $b_i = \text{bottom } C_i = \text{bottom } D_{i-1}$;
- 3) $C_i \cap D_i = \{a_i\}$;
- 4) $C_{i+1} \cap D_i = \{b_{i+1}\}$;
- 5) $\{a_1, a_2, \dots, a_m, b_1, b_2, \dots, b_m\} \cap (C_i \cup D_i) = \{a_i, b_i, b_{i+1}\}$;
- 6) $\{a_1, a_2, \dots, a_m, b_1, b_2, \dots, b_m\} \cap (C_{i+1} \cup D_i) = \{a_i, a_{i+1}, b_{i+1}\}$,

with $b_{m+1} = b_1$, where top C (bottom C) denotes the top (bottom) of the chain C (Figure 4.3.1). A cycle of P in which $u > v (P)$ belongs to the chain C_1 is *irreversible* with respect to Q , a directed graph obtained from P by reversing some of the edges of P , if the reversed edges of the cycle are precisely those of

- 1) $(C_1 \cup C_2 \cup \dots \cup C_m) - \{u > v\}$

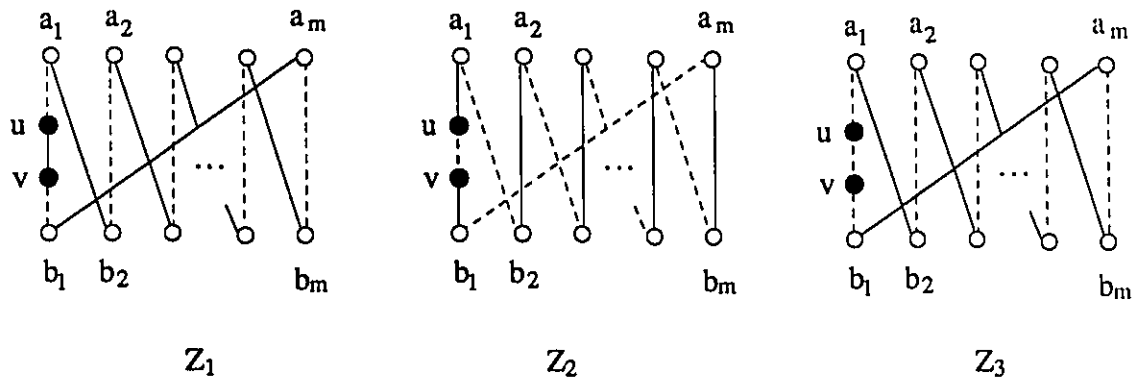
or those of

$$2) (D_1 \cup D_2 \cup \dots \cup D_m) \cup \{u > v\}$$

or those of

$$3) (C_1 \cup C_2 \cup \dots \cup C_m),$$

where a_1 may be u and b_1 may be v (Figure 4.3.2).



The three types of irreversible cycles (the dashed edges are the reversed ones)

Figure 4.3.2

THEOREM 4.3.1 (W. P. Liu and I. Rival [1991]). *A directed graph Q constructed from the diagram of an ordered set P is itself a reorientation of P if and only if P has no irreversible cycle with respect to Q .*

Proof. Let Q be a reorientation of an ordered set P and let P contain an irreversible cycle with respect to Q , either Z_1 , Z_2 or Z_3 . If it is Z_1 , then $u > v$ (Q) and there is another chain in Q

$$u > \dots > a_1 D_1 C_2^d D_2 C_3^d \dots D_{m-1} C_m^d D_m b_1 > \dots > v,$$

where C_i^d is the dual of C_i . If the irreversible cycle is Z_2 then $v > u$ (P) and, at the same time, there is another chain in Q

$$v > \dots > b_1 D_m^d C_m D_{m-1}^d C_{m-1} \dots C_2 D_1^d a_1 > \dots > u,$$

where D_1^d is the dual of D_1 . If the irreversible cycle is Z_3 then $v > u (P)$ and, at the same time, there another chain in Q

$$u > \dots > a_1 D_1 C_2^d D_2 C_3^d \dots D_{m-1} C_m^d D_m b_1 > \dots > v.$$

In any case, the existence of a nontrivial chain between a covering pair is impossible as Q is the diagram of an ordered set.

Conversely, suppose P has no irreversible cycle with respect to Q . For contradictions, suppose that Q is not a diagram. In this case, there is a directed edge in Q joining u and v , and a directed path C not containing this edge with one end v and other end u . We may assume that $u > v (P)$. We shall construct an irreversible cycle in P following the directed path joining v to u in Q .

Suppose first that the directed edge points from v to u in Q and the directed path C is from v to u in Q . Let C_1 consist of the consecutive strings of vertices x of C such that either $x \geq u (P)$ or $x \leq v (P)$. Put

$$a_1 = \max_P C_1 \text{ and } b_1 = \min_P C_1.$$

Next, let D_1 consist of the next consecutive string of vertices satisfying $x < a_1 (P)$ and put

$$b_2 = \min_P D_1.$$

Then C_2 consists of the next part satisfying $x > b_2 (P)$ and

$$a_2 = \max_P C_2.$$

Eventually, there is a integer m , and a string D_m of vertices x satisfying $x < a_m (P)$ and

$$b_1 = \min_P D_m.$$

Then $\{a_1, a_2, \dots, a_m, b_1, b_2, \dots, b_m\}$ is an irreversible cycle of type Z_2 .

Suppose now that the directed edge points from u to v in Q while the directed path is again from v to u in Q . Let C_1 consist of the consecutive strings of vertices x of C such that $x \geq u(P)$ or $x \leq v(P)$. Set

$$a_1 = \max_P C_1 \text{ and } b_1 = \min_P C_1.$$

As before, define the consecutive chains D_i, C_{i+1} with endpoint a_i, b_{i+1} to produce an irreversible cycle in P ; this time of type Z_3^d , which is equivalent to the type Z_3 .

Now, suppose the directed path C is from u to v in Q . Then construct C_1 , the consecutive strings consisting of the vertices x of C , such that $x \geq u(P)$ or $x \leq v(P)$. Put

$$a_1 = \max_P C_1 \text{ and } b_1 = \min_P C_1.$$

As before, we may again construct chains D_i, C_{i+1} with endpoint a_i, b_{i+1} to produce an irreversible cycle in P which, if the directed edge is from v to u in Q , is of the type Z_3 and, if the directed edge is from u to v in Q , is of the type Z_1 .

Q.E.D.

Remark. Z_2 is the "complement" of Z_1 in the sense that interchanging the reversed edges and non-reversed edges of Z_1 produces Z_2 .

4.4. ENUMERATING REORIENTATIONS OF ORDERED SETS. An ordered set has many reorientations, although we do not yet know how many. In fact, although G. Brightwell and J. Nešetřil [1991] have proved that *there is an n -element covering graph which has at most $2^{O(n \log_2 \log_2 \log_2 n / \log_2 \log_2 n)}$ orientations*, the following conjecture remains open.

CONJECTURE 7 (cf. W. P. Liu and I. Rival [1991]). *Any covering graph has exponentially many orientations.*

We have the following theorem to support our conjecture.

THEOREM 4.4.1 (W. P. Liu and I. Rival [1991]). *Almost every n -element covering graph has $2^{n/3}$ orientations.*

Proof. To prove our theorem, we make use of the following well known result.

THEOREM 4.4.2 (D. J. Kleitman and B. L. Rothschild [1975]). *Almost every covering graph has chromatic number at most three.*

Let us first be sure that an ordered set with an m -element antichain A has at least 2^m reorientations. To see this we apply the pushdown operation in two different ways. First, P has a reorientation in which the set of maximal elements is precisely A . Indeed, as long as some element a in A is not maximal there is a maximal element a' not in A such that $a' > a$, to which we may apply the pushdown operation. For essentially the same reason, for every subset B of A , there is an inversion whose maximal elements are just the elements of B (apply pushdown operation repeatedly to maximal elements not belonging to B). Thus there are 2^m different labeled reorientations.

According to Theorem 4.4.2, for any n -element ordered set P whose covering graph has chromatic number three, there is a reorientation Q of P with length at most three

(say, we can reorient P as Q such that $u > v$ in Q if and only if the chromatic number of u is greater than that of v). Thus, of three "levels" of Q , one, at least, is an antichain with at least $n/3$ elements. In view of the remark above, the ordered set will have at least $2^{n/3}$ distinct reorientations.

Q.E.D.

COROLLARY 4.4.3 (W. P. Liu and I. Rival [1991]). *Every n -element planar covering graph has at least $2^{n/3}$ orientations.*

Proof It follows from Theorem 4.4.1 and a well known result that *every triangle-free planar graph has chromatic number at most three* (cf. H. Grotzsch [1958]).

Q.E.D.

Let $\text{girth}(P)$ and $\text{color}(P)$ stand for the girth and the chromatic number of the covering graph of an ordered set P . In spite of Theorem 4.2.1 and Theorem 4.2.4, we have following results.

THEOREM 4.4.4 (W. P. Liu and I. Rival [1991]). *Let I be an independent set of the covering graph of an ordered set P . If $\text{girth}(P) > \text{color}(P) + 1$, then there is a reorientation Q of P such that $\max Q = I$.*

Proof (O. Pretzel). Let $P' = (S(P) - I)_P$. Color P' with $\text{color}(P)$ colors. Extend this coloring to P by giving each element of the set I a new color $\text{color}(P)+1$. Orient the covering graph of P in the way by directing each edge to the element with high color. As $\text{girth}(P) > \text{color}(P) + 1$, this directed graph Q is a reorientation of P . Now $\max Q \supseteq I$. As in the proof of Theorem 4.4.1, we can push down the undesirable maximal elements until $\max Q = I$.

Q. E. D.

THEOREM 4.4.5 (W. P. Liu and I. Rival [1991]). *Let M be a matching of the covering graph of an ordered set P . If $\text{girth}(P) > 2 \times \text{color}(P) - 2$, then for any subset S of M , there is an orientation Q of P which reverses the edges of S and none of $M-S$.*

Proof. Let C be a $\text{color}(P)$ -coloring of the covering graph G of P and let Q be the orientation of G induced by C , that is, $a > b$ (Q) if and only if $C(a) > C(b)$ (Q is an orientation since $\text{girth}(P) > 2 \times \text{color}(P) - 2$). Let

$$S_1 = S - \{a > b (Q) : b > a (P) \text{ and } b > a (P) \in S\} \\ \cup \{a > b (Q) : b > a (P) \text{ and } b > a (P) \in M - S\}.$$

Partition the set S_1 into two parts by letting E_1 consist of those edges of S_1 not adjacent to a maximal element of Q and $E_2 = S_1 - E_1$. For every edge $a > b$ (Q) in E_1 , reverse it by recoloring the element a by a new color $1/C(a)$ (suppose color 1 is not used here). This gives at most $\text{color}(P) - 2$ new colors. The directed graph R given by the resulting coloring reverses all the edges of E_1 and none of $M - S_1$ and is an orientation, because $\text{girth}(P) > 2 \times \text{color}(P) - 2$. Now, consider the edges of E_2 . An edge e in E_2 has its head v colored $\text{color}(P)$. Thus the element v is maximal in R . As the edges of M are independent, pushing down a maximal element reverses at most one edge of E_2 . So push down the heads of the edges of E_2 . That gives the desired reorientation which reverses all the edges of S and none of $M - S$.

Q.E.D.

COROLLARY 4.4.6 (W. P. Liu and I. Rival [1991]) *Let P be an n -element ordered set with $\text{girth}(P) > 2 \times \text{color}(P) - 2$. Then P has at least $2^{n/3}$ reorientations.*

Proof. The corollary follows from the argument given in the beginning of Section 4.2, Theorems 4.4.4 and 4.4.5.

Q.E.D.

THEOREM 4.4.7 (W. P. Liu). *Any n -element lattice with girth at least seven has at least $2^{\lfloor n/2 \rfloor}$ reorientations.*

Proof. Suppose P is a lattice with girth at least seven. If there is an antichain consisting of at least $\lfloor n/2 \rfloor$ elements, then, according to the argument that we often used before (pushdown), P has at least $2^{\lfloor n/2 \rfloor}$ reorientations. Now, assume that P has no $\lfloor n/2 \rfloor$ element antichain. Then the maximum chain $v_1 > v_2 > v_3 > \dots > v_k$ is of length at least $\lfloor \sqrt{2n} \rfloor$.

Let $E = \{v_1 > v_2, v_3 > v_4, v_5 > v_6, \dots\}$. If we can prove that for any subset S of E , there is a reorientation Q of P which reverses all the edges of S and none of $E - S$, then P has at least $2^{\lfloor \sqrt{2n} \rfloor / 2} = 2^{\lfloor n/2 \rfloor}$ reorientations.

Inductively define a directed graph Q_i as follows. Let Q_1 be the directed graph obtained from the diagram of P by reversing all the edges of S . Suppose that Q_{i-1} has been defined which reverses all the edges of S and none of $E - S$. If Q_{i-1} is a diagram then we are done. Suppose that Q_{i-1} is not a diagram. Then, according to Theorem 4.3.1, P contains an irreversible cycle with respect to Q_{i-1} .

Case 1. the cycle is of the type Z_1 (Figure 4.4.1)

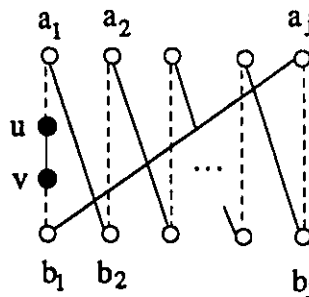


Figure 4.4.1

Subcase 1.1. $j > 1$

It is easy to verify that $\{u > v\} \cup \varepsilon(D_1) \cup \varepsilon(D_2) \cup \dots \cup \varepsilon(D_m) - E \neq \emptyset$. (Here, we use the notation in section 4.3.) For otherwise, we will obtain a contradiction that P is not a diagram. So we can reverse an edge of $\{u > v\} \cup \varepsilon(D_1) \cup \varepsilon(D_2) \cup \dots \cup \varepsilon(D_m) - E$ and obtain a directed graph Q_i in which none of edges of $E - S$ is reversed.

Subcase 1.2. $j = 1$ (Figure 4.4.2)

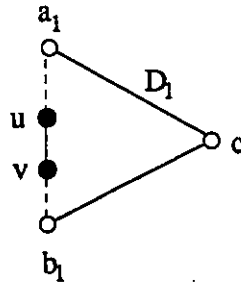


Figure 4.4.2

Let $a_1 > c (P)$ and $c > d (P)$ belong to D_1 , where d may be b_1 . Then either the edge $a_1 > c (P)$ or the edge $c > d (P)$ belongs to $E - S$. By reversing the edge belonging to $E - S$, we obtain a directed graph Q_i in which none of edges of $E - S$ is reversed.

Case 2. the cycle is of the type Z_3 (Figure 4.4.3)

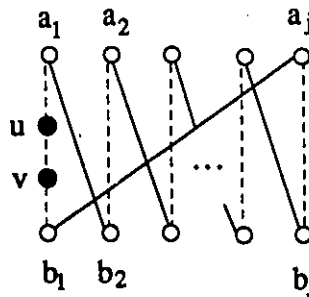


Figure 4.4.3

Subcase 2.1 $j > 1$

As in Subcase 1.1, we have $\epsilon(D_1) \cup \epsilon(D_2) \cup \dots \cup \epsilon(D_m) - E \neq \emptyset$. By reversing one of the edges of $\epsilon(D_1) \cup \epsilon(D_2) \cup \dots \cup \epsilon(D_m) - E$, we can obtain a directed graph Q_i in which none of edges of $E - S$ is reversed.

Subcase 2.2. $j = 1$ (Figure 4.4.4)

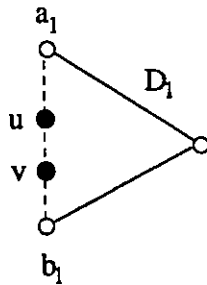


Figure 4.4.4

By the same argument as that in subcase 1.2.

Case 3. the cycle is of the type Z_2 (Figure 4.4.5)

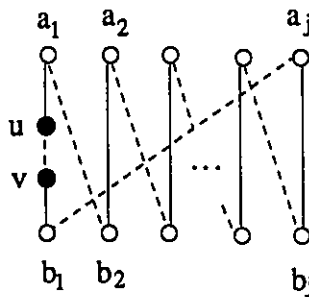


Figure 4.4.5

Subcase 3.1. $j > 2$

By the same argument as that in Subcase 1.1.

Subcase 3.2. $j = 1$ (Figure 4.4.6)

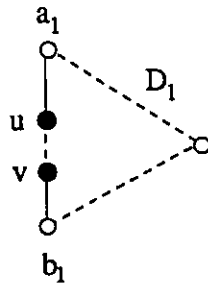


Figure 4.4.6

From the assumption that P has girth at least 7, we know that either $\epsilon(C_1') - E \neq \emptyset$ or $\epsilon(C_1'') - E \neq \emptyset$, where $\epsilon(C_1') \cup \epsilon(C_1'') = C_1 - \{u \rightarrow v\}$ since either $|\epsilon(D_1)| \leq |\epsilon(C_1)|$. By reversing an edge in $\epsilon(C_1') \cup \epsilon(C_1'') - E$, we obtain a directed graph Q_i in which none of the edges of $E - S$ is reversed.

Subcase 3.3. $j = 2$

Since P is a lattice, we can obtain an irreversible cycle which is same as that in Subcase 3.2.

Thus, we get a sequence of directed graphs Q_1, \dots, Q_m which satisfy the following conditions:

- 1) Q_i does not reverse any edge of $E - S$;
- 2) Q_{i+1} reverses at least one more edge than Q_i does.

Since that P is finite, for some Q_j , P has no irreversible cycle respect to Q_j , that is, Q_j is a diagram which reverses all the edges of S and none of $E - S$.

Q.E.D.

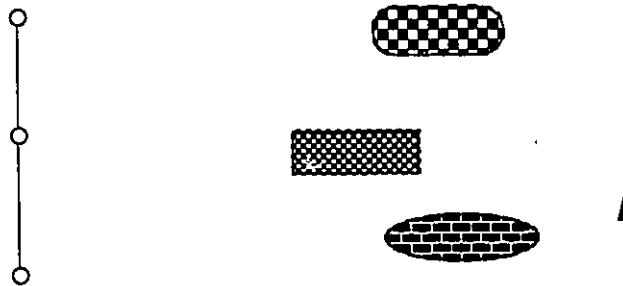
We end this chapter with two more conjectures:

CONJECTURE 8 (W. P. Liu). *Any independent set of a planar covering graph can be the set of maximal elements of an orientation of the graph.*

CONJECTURE 9 (W. P. Liu). *Let $\{a_1b_1, a_2b_2, \dots, a_kb_k\}$ be a matching of a planar covering graph G . Then there is an orientation of G in which $a_i > b_i, 1 \leq i \leq k$.*

CHAPTER 5. ENUMERATION AND STRUCTURE OF BLOCKING RELATIONS

5.1. ENUMERATING ONE-DIRECTIONAL BLOCKING RELATIONS. In the last three chapters (Chapters 2, 3, 4), we concentrated on the study of diagrams and covering graphs. Although diagrams and covering graphs are important representations of ordered sets, there are other ways to represent an ordered set. For example, we can use convex figures with directions of motion. More specifically, consider a set of convex figures on the plane, and associate with each of them a direction of motion, not necessarily identical. For two figures A and B , we say A *obstructs* B , denoted by $B \rightarrow A$ if there is a line joining a point of A to a point of B following the direction of motion of B . We write $B < A$ if there is a sequence of figures $B = A_1 \rightarrow A_2 \rightarrow A_3 \dots \rightarrow A_k = A$. The transitive relation $<$ on the convex figures is antisymmetric as long as there is no "cycle" $A = A_1 \rightarrow A_2 \rightarrow A_3 \dots \rightarrow A_k = A$. In this case, the relation $<$ is an order, which is called a blocking relation, in other words, an ordered set is a *blocking relation* if it can be represented by convex figures with directions of motion on the plane (Figure 5.1.1), and the set of convex figures with the directions of motion is called a *representation* of the ordered set (blocking relation).

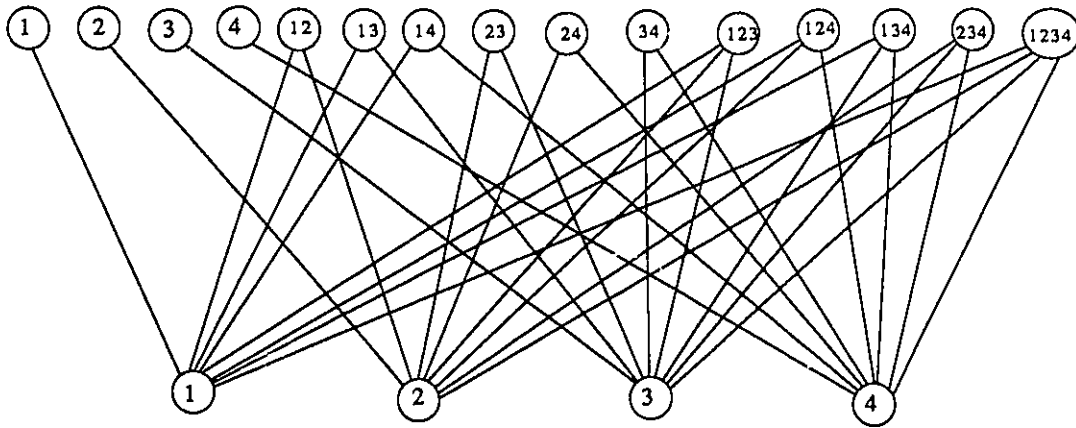


An ordered set P

A representation of P with one direction of motion

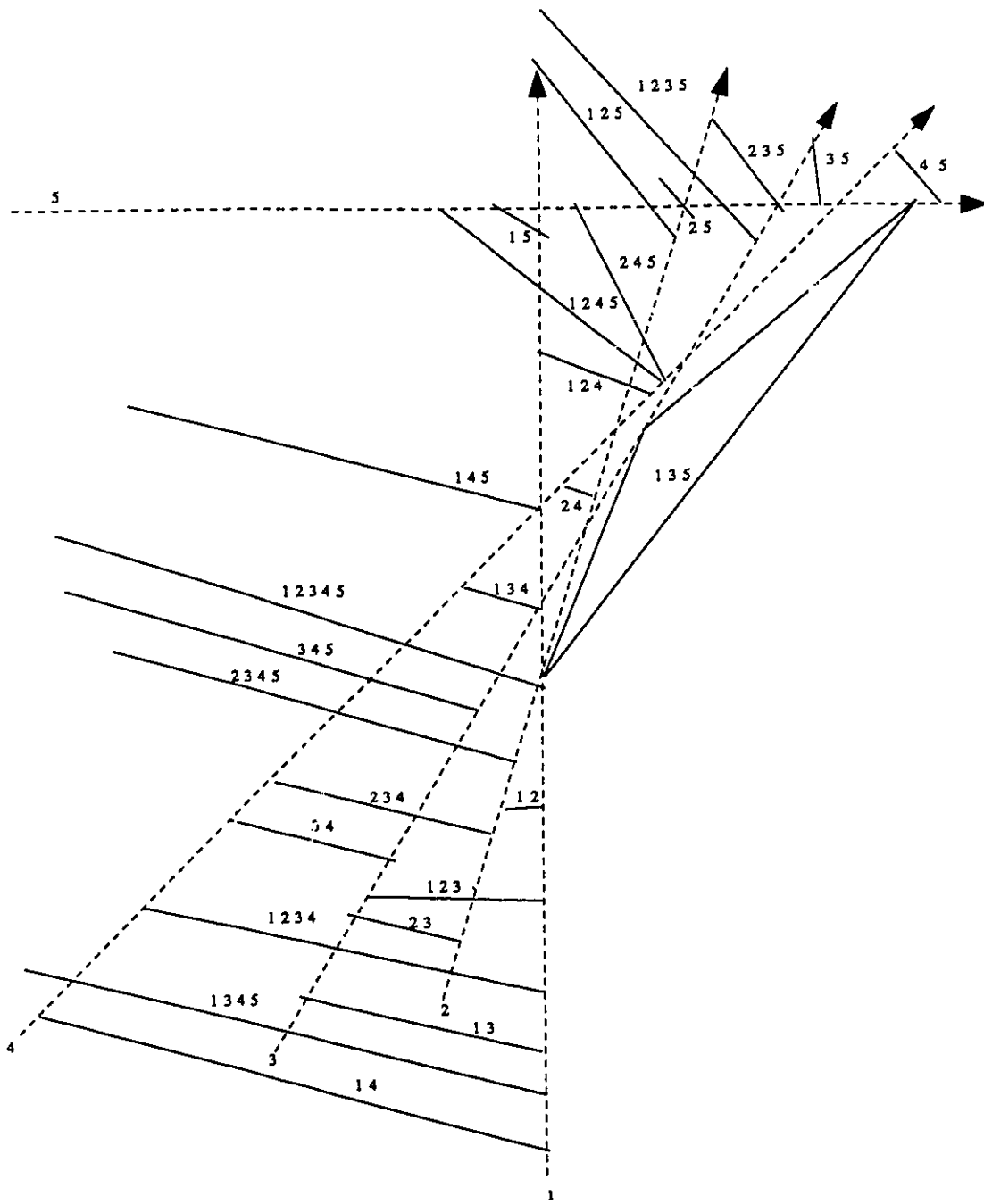
Figure 5.1.1

Is it true that any ordered set is a blocking relation? I. Rival and J. Urrutia [1988] have answered this question negatively. Let H_m be a bipartite ordered set which has m minimal elements $1, 2, \dots, m$ and $2^m - 1$ maximal elements A_{i_1, i_2, \dots, i_k} , where, $1 \leq i_1 < i_2 < \dots < i_k \leq m$, with $j < A_{i_1, i_2, \dots, i_k}$ if and only if $j \in \{i_1, i_2, \dots, i_k\}$ (Figure 5.1.2 (1)). They have proved that *the ordered set H_m is not a blocking relation, as long as m is big enough*. Also in their paper, Rival and Urrutia proved that for $m \leq 4$, H_m is a blocking relation and left the problem open of whether H_5 is a blocking relation. We can prove that H_5 is indeed a blocking relation (Figure 5.1.2 (2)). However, we do not know what is the smallest number m such that H_m is not blocking relation (what about H_6 ?).



H_4

(1)



A representation of H_5
 Figure 5.1.2 (2)

I. Rival and J. Urrutia [1988] also characterized those ordered sets which are (one-directional) blocking relations, that is, all the figures have the same direction of motion: *an ordered set is a (one-directional) blocking relation if and only if the ordered set is a planar truncated lattice.* There is no known characterization for m -directional blocking relations ($m \geq 2$), although *any ordered set has a subdivision which has a 2-directional representation* (I. Rival and J. Urrutia [1988]).

For a given representation of a one-directional blocking relation (from now on, we only consider one-directional blocking relations), by changing the direction of motion but fixing the convex figures, we may obtain a representation of a new blocking relation. Call the new blocking relation a *reorientation* of the original with respect to the representation.

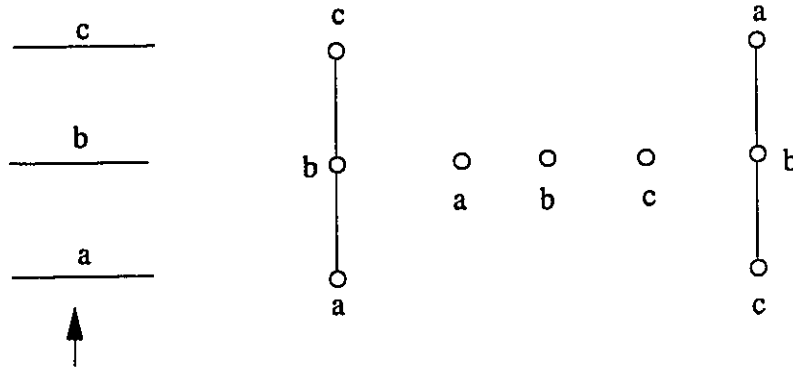
For convenience, we always assume that the original representation of a blocking relation has the vertical up direction of motion.

For a representation R of a planar truncated lattice and a number α , $0 \leq \alpha \leq \pi$, let R_α stand for the representation obtained from R such that the angle between the directions of R and R_α is α . Obviously, if $\pi < \beta \leq 2\pi$, then, $R_\beta = (R_{\beta-\pi})^d$, that is, the blocking relation representing R_β is the dual of that represented by $R_{\beta-\pi}$. So it is enough to consider those reorientations R_α with $0 \leq \alpha \leq \pi$.

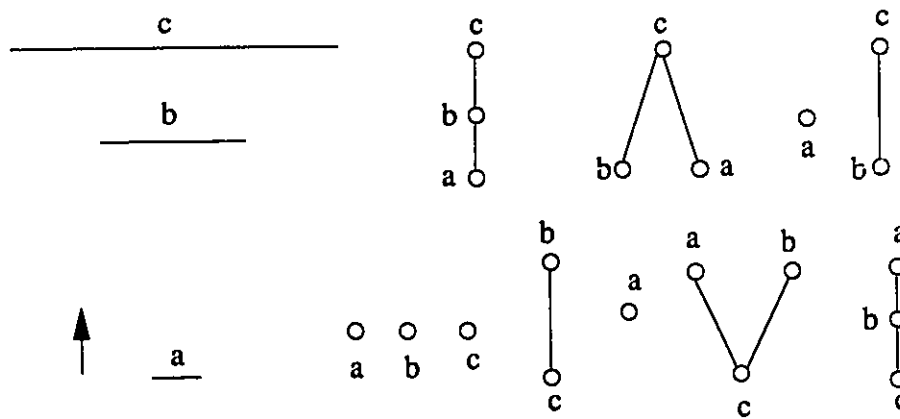
We are interested in the number of reorientations of planar truncated lattices. Since different representations of the same planar truncated lattice may have different numbers of reorientations (Figure 5.1.3), we adopt the following notations. For a planar truncated lattice P , let

$$\text{Reor}(R,P) = |\{Q: Q \text{ is a reorientation of } P \text{ with respect to the representation } R \text{ of } P\}|$$

$$\text{Reor}(P) = \max\{\text{Reor}(R,P): R \text{ is a representation of } P\}.$$



A representation of a 3-element chain and the 3 reorientations with respect to it



Another representation of a 3-element chain and the 7 reorientations with respect to it
Figure 5.1.3

(Here, we have no restriction on the shape of each figure of representations as long as each figure is convex.) Here is our main result in this section.

THEOREM 5.1.1 (W. P. Liu and I. Rival [1990]). *For any n -element planar truncated lattice P , $\text{Reor}(P) \leq n(n-1) + 1$.*

Let R be a representation of a planar truncated lattice P . For two figures A and B , we write $A \succ B(R)$ ($A > B(R)$) if the element represented by A is an upper cover of

(bigger than) the element represented by B. $A \parallel B (R)$ means that the elements represented by figures A and B are noncomparable.

For any figure A and an angle α , define

$$S_{\alpha}'(A) = \begin{cases} \{(x,y): \text{for some point } (x_A, y_A) \text{ in } A, x = x_A \text{ and } y \geq y_A\} & \text{if } \alpha = 0 \\ \{(x, y): \text{for some point } (x_A, y_A) \text{ in } A, x \geq x_A \\ \text{and } y - y_A = (x - x_A)\tan(\pi/2 - \alpha)\} & \text{if } 0 < \alpha \leq \pi/2 \\ \{(x, y): \text{for some point } (x_A, y_A) \text{ in } A, x \geq x_A \\ \text{and } y - y_A = -(x - x_A)\tan(\pi - \alpha)\} & \text{if } \pi/2 < \alpha < \pi \\ \{(x,y): \text{for some point } (x_A, y_A) \text{ in } A, x = x_A \text{ and } y \leq y_A\} & \text{if } \alpha = \pi, \end{cases}$$

and $S_{\alpha}(A) = \cup \{S_{\alpha}'(B): B \geq A (R_{\alpha})\}$.

LEMMA 5.1.2. $B > A (R_{\alpha})$ if and only if $S_{\alpha}(A) \supseteq B$.

LEMMA 5.1.3. Suppose $A \not\geq B (R_{\alpha})$. Then there is a positive number ϵ_0 such that if $0 \leq \epsilon < \epsilon_0$, then, $A \not\geq B (R_{\alpha+\epsilon})$ and $A \not\geq B (R_{\alpha-\epsilon})$.

Proof. Since R has only finitely many figures, the set (as a subset of the plane)

$$\zeta = \cup \{X: X \text{ is a figure of } R \text{ and } X \cap S_{\alpha}(B) = \emptyset\}$$

is a bounded and closed set which is a subset of the set ζ . Suppose there is no such number ϵ_0 . Then, for any $\epsilon_m > 0$ (ϵ_m converges to 0), there are two sequences of points (x_m, y_m) in $\zeta \cap S_{\alpha+\epsilon_m}(B)$ and (x_m', y_m') in $\zeta \cap S_{\alpha-\epsilon_m}(B)$, $1 \leq m$, respectively. On the other hand, since ζ is bounded and closed set (as a subset of the whole plane), the sequences $\{(x_m, y_m): 1 \leq m\}$ and $\{(x_m', y_m'): 1 \leq m\}$ converge to points (x_0, y_0) in X_1

which is a subset of ζ and (x_0', y_0') in X_2 which is a subset of ζ , respectively. Since the sequence $\{\epsilon_m\}$ converges to 0, for any large enough integer m , we have

$$X_1 \cap S_{\alpha+\epsilon_m}(B) = \emptyset$$

and

$$X_2 \cap S_{\alpha-\epsilon_m}(B) = \emptyset.$$

Since the sets $S_{\alpha+\epsilon_m}(B)$, $S_{\alpha-\epsilon_m}(B)$, X_1 and X_2 are closed, the point (x_m, y_m) does not belong to $S_{\alpha+\epsilon_m}(B)$, and neither does the point (x_m', y_m') to $S_{\alpha-\epsilon_m}(B)$. This leads to a contradiction.

Q.E.D.

LEMMA 5.1.4. *Suppose $A > B (R_\alpha)$ and $B > A (R_\beta)$, $\pi \geq \beta > \alpha \geq 0$. Then for some γ , $\beta > \gamma > \alpha$, $A \parallel B (R_\gamma)$.*

Proof. Let

$$\alpha_0 = \sup\{\psi: A > B (R_\tau) \text{ for all } \tau \leq \psi\}$$

and

$$\alpha_1 = \inf\{\psi: B > A (R_\tau) \text{ for all } \tau \geq \psi\}.$$

Suppose the lemma is not true, that is, for any angle φ , $\alpha \leq \varphi \leq \beta$, either $A > B (R_\varphi)$ or $B > A (R_\varphi)$. Then $\alpha_0 = \alpha_1$.

Case 1. $A > B (R_{\alpha_1})$

Then $B \not> A (R_{\alpha_1})$. According to Lemma 5.1.3, there exists a positive ϵ_0 such that $B \geq A (R_{\alpha_1+\epsilon})$ for all ϵ , $0 \leq \epsilon \leq \epsilon_0$. Since either $A > B (R_\varphi)$ or $B > A (R_\varphi)$, $A > B (R_{\alpha_1+\epsilon})$ for all ϵ , $0 \leq \epsilon \leq \epsilon_0$. Therefore,

$$\sup\{\psi: A > B (R_\tau) \text{ for all } \tau \leq \psi\} \geq \alpha_1 + \varepsilon_0 = \alpha_0 + \varepsilon_0,$$

which contradicts the definition of α_0 .

Case 2. $B > A (R_{\alpha_1})$

Then, $A \not\geq B (R_{\alpha_1})$. According to Lemma 5.1.3 again, there exists a positive ε_0 such that $A \not\geq B (R_{\alpha_1 - \varepsilon})$ for all ε , $0 \leq \varepsilon \leq \varepsilon_0$. So, $B > A (R_{\alpha_1 - \varepsilon})$ for all ε , $0 \leq \varepsilon \leq \varepsilon_0$, which means that

$$\inf\{\psi: B > A (R_\tau) \text{ for all } \tau \geq \psi\} \leq \alpha_1 - \varepsilon_0,$$

which contradicts the definition of α_1 .

Q.E.D.

LEMMA 5.1.5. *Suppose $A > B (R)$. If, for some α , $0 \leq \alpha \leq \pi$, $A \parallel B (R_\alpha)$, then $A \not\geq B (R_\beta)$ for any β , $\alpha \leq \beta \leq \pi$.*

Proof. Let $y = f_1(S_\alpha(B), x)$ and $y = f_2(S_\alpha(B), x)$ be the two boundary curves of $S_\alpha(B)$ with

$$f_1(S_\alpha(B), u) \leq v \leq f_2(S_\alpha(B), u)$$

for all points (u, v) in $S_\alpha(B)$. Since $A > B (R)$ and $A \parallel B (R_\alpha)$, $v > f_2(S_\alpha(B), u)$ for any point (u, v) in figure A. Suppose for some $\beta > \alpha$, $A > B (R_\beta)$. Then there is a chain $A = A_0 > A_1 > A_2 > \dots > A_{m+1} = B (R_\beta)$. Notice that B (as a subset of the plane) is a subset of $S_\alpha(B)$ and $A \cap S_\alpha(B) = \emptyset$; there are figures A_i and A_{i+1} such that

$$A_i \cap S_\alpha(B) = \emptyset \text{ and } S_\alpha(B) \supset A_{i+1}.$$

Make a "highest" tangent line L of A_{i+1} which is parallel to the direction of motion of R_β (here, assume L is tangent to A_{i+1} at the point (x_1, y_1)) (Figure 5.1.4). The line L

intersects the figure A_i since $A_i > A_{i+1} (R_\beta)$. Obviously, the line L_1 which passes through the point (x_1, y_1) and is parallel to the direction of motion of R_α either intersects the figure A_i or A_{i+1} is below L_1 . Notice that A_i is "above" the curve $y = f_1(S_\alpha(B), x)$ (Figure 5.1.4). So $A_i \cap S_\alpha(B) \neq \emptyset$, which is a contradiction (Figure 5.1.4).

Q.E.D.

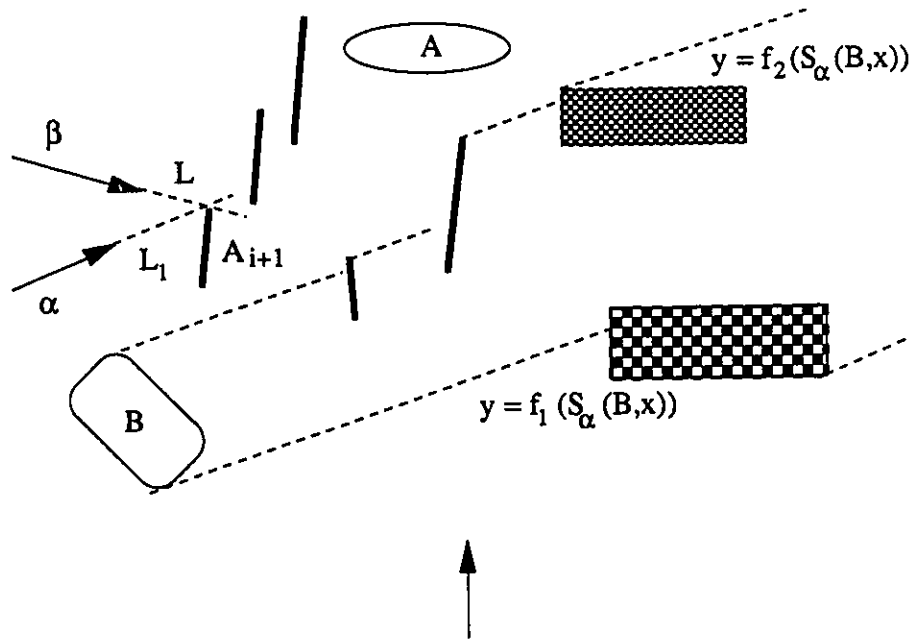


Figure 5.1.4

LEMMA 5.1.6. *Suppose $A \parallel B (R)$ for two figures A and B of a representation R . If for some $\alpha, 0 < \alpha < \pi$, $A > B (R_\alpha)$, then $B \geq A (R_\beta)$ for all $\beta, \alpha \leq \beta \leq \pi$.*

Proof. If $B > A (R_\beta)$ for some $\beta, \alpha \leq \beta \leq \pi$, then $A > B (R_{\beta+\pi})$. Let $Q = R_{\beta+\pi}$, then $R_\alpha = Q_{\pi-\beta+\alpha}$. Furthermore, $A \parallel B (Q_{\pi-\beta})$, $A > B (Q)$ and $A > B (Q_{\pi-\beta+\alpha})$, which contradicts Lemma 5.1.5 since $\pi - \beta + \alpha < \pi$ (Figure 5.1.5).

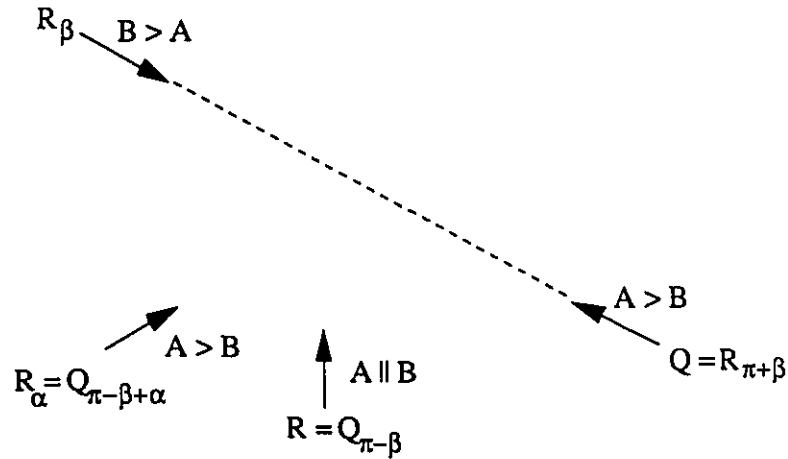


Figure 5.1.5

LEMMA 5.1.7. *The relation between any two figures A and B of a representation changes at most twice as the direction of motion changes.*

Proof. First, assume that $A \parallel B (R)$, where R is a representation of a planar truncated lattice. Suppose $A > B (R_\alpha)$ for some α , $0 < \alpha < \pi$. According to Lemma 5.1.6,

$$B \succcurlyeq A (R_\beta)$$

for all β , $\alpha \leq \beta \leq \pi$. According to Lemma 5.1.5, if $A \parallel B (R_{\beta_0})$ for some $\beta_0 > \beta$, then

$$A \parallel B (R_\tau)$$

for any $\tau \geq \beta_0$ (Figure 5.1.6). That is, the relation between A and B changes at most twice: from $A \parallel B$ into $A > B$ or $B > A$ and from $A > B$ or $B > A$ into $A \parallel B$.

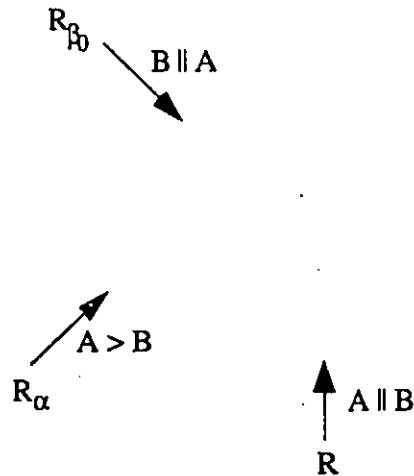


Figure 5.1.6

Similarly, if $A > B$ (R), the relation between the figures A and B changes at most twice: from $A > B$ into $B \parallel A$, and then from $B \parallel A$ into $B > A$.

Q. E. D.

Proof of Theorem 5.1.1. It follows from Lemma 5.1.7.

Q. E. D.

The following theorem shows that the right bound given in Theorem 5.1.1 is best possible.

THEOREM 5.1.8 (W. P. Liu and I. Rival [1990])

- 1) Let C be an n -element chain, then $\text{Reor}(C) = n(n-1) + 1$;
- 2) Let A be an n -element antichain, then $\text{Reor}(A) \geq n(n-1)$.

Proof. Here, we only prove Part (1), and Part (2) can be proved similarly. To this end, let s_i be horizontal segments with endpoints

$$\left(-\sum_{1 \leq m \leq i} 1/2^m, y_i\right) \text{ and } \left(\sum_{1 \leq m \leq i} 1/2^m, y_i\right),$$

$1 \leq i \leq n$, where $y_n = 0$, $y_{n-1} = -1$, and y_j satisfies the following condition: for $1 \leq j \leq r \leq k \leq n$,

$$\left(\sum_{1 \leq m \leq n} 1/2^m + \sum_{1 \leq m \leq j} 1/2^m \right) / (-y_j) < \left(\sum_{1 \leq m \leq k} 1/2^m + \sum_{1 \leq m \leq r} 1/2^m \right) / (y_k - y_r).$$

Then, the segments with the vertical up direction of motion constitute a representation of the chain C . It is easy to check (but some calculation is needed) that

$$s_r > s_t (R_\alpha)$$

with $t < r - 1$ and $0 \leq \alpha \leq \pi/2$ implies that

$$s_n > s_{n-1} > \dots > s_{t+1}$$

and

$$s_i \parallel s_j (R_\alpha)$$

$1 \leq i, j < t$. Similarly,

$$s_r > s_t (R_\beta)$$

with $\pi/2 \leq \beta \leq \pi$ and $r > t + 1$ implies that

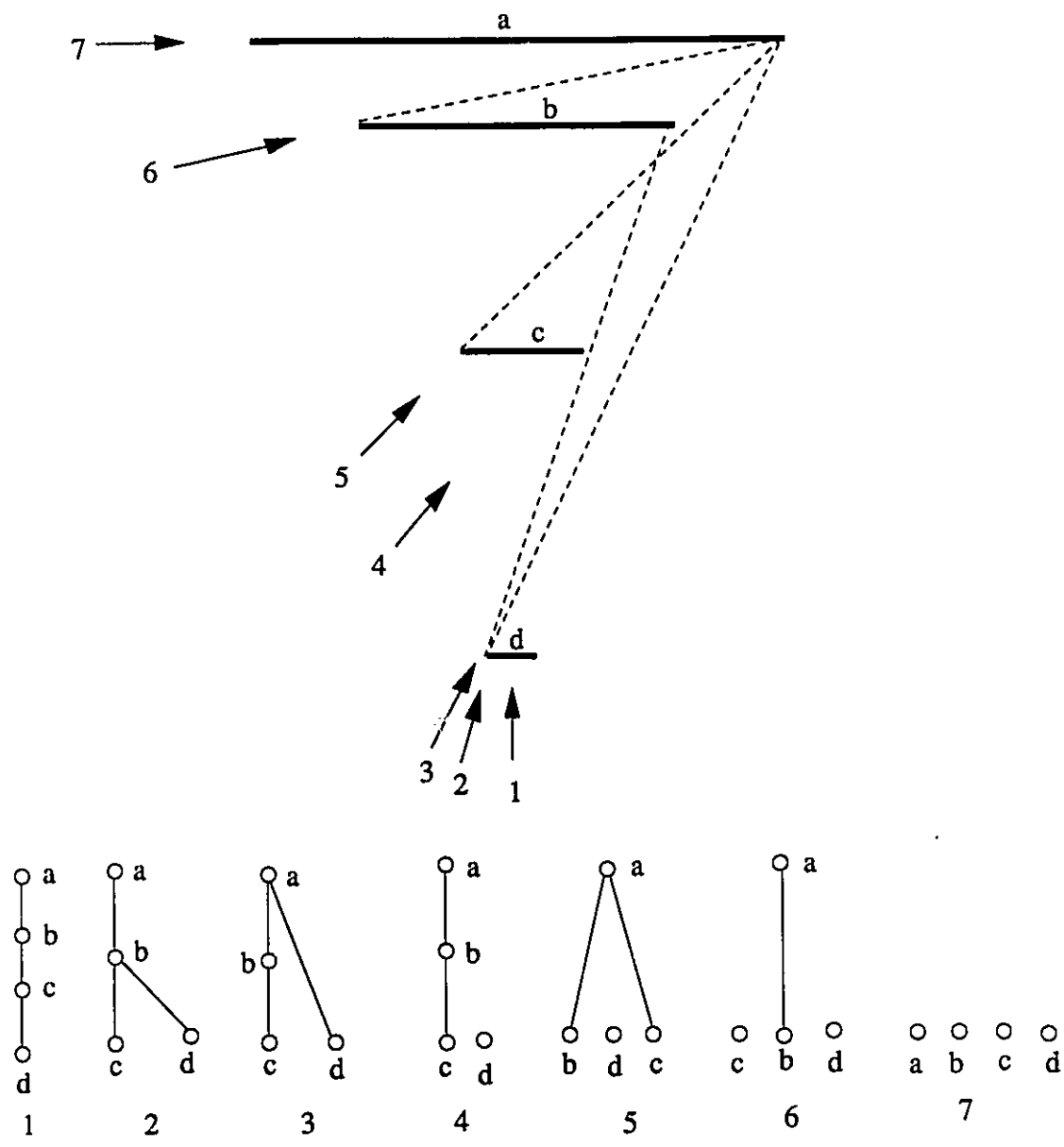
$$s_{t+1} > s_{t+2} > \dots > s_n (R_\beta)$$

and

$$s_i \parallel s_j (R_\beta)$$

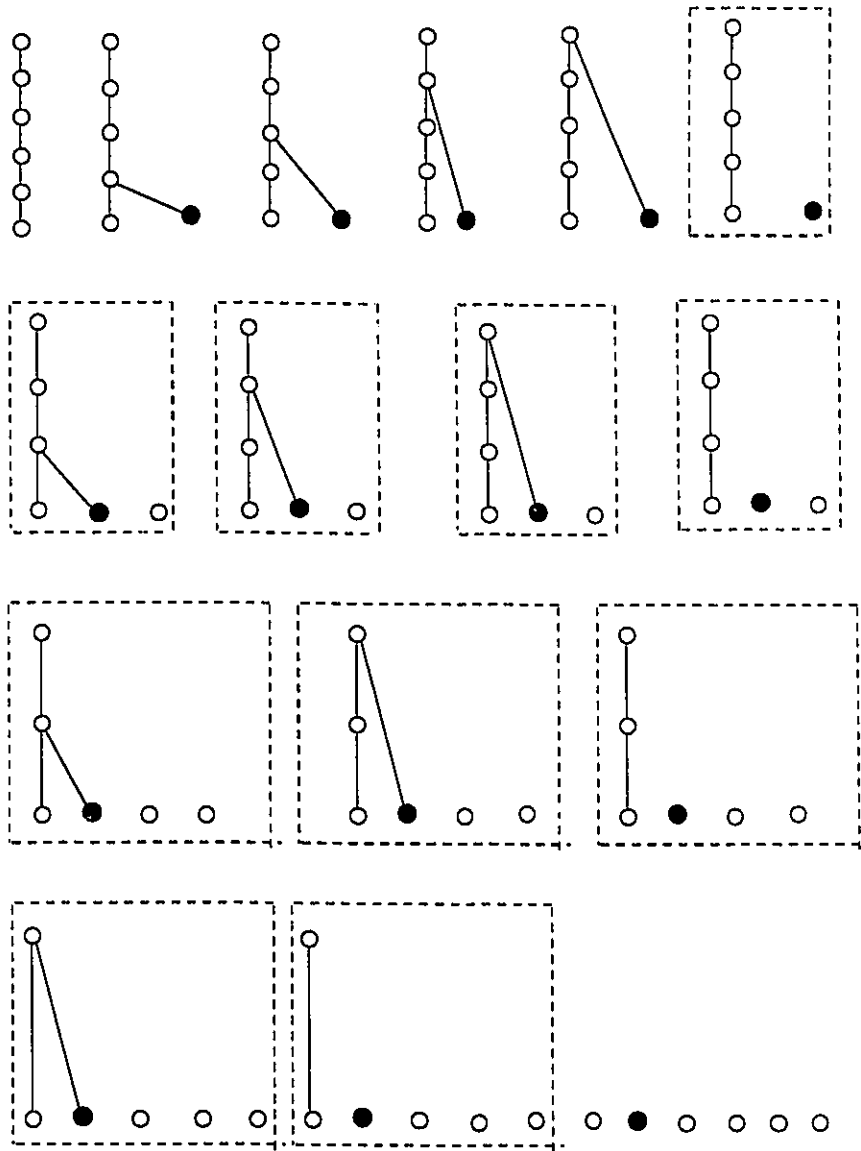
for all i, j with $1 \leq i, j < t$. That is, $|\text{Reor}(R, C)| \geq n(n - 1) + 1$. (Figure 5.1.7 (1) illustrates a representation of the 4-element chain and 7 reorientations with respect to it (the other 6 reorientations are their duals except the antichain); Figure 5.1.7 (2) gives the 16 reorientations of the 6-element chain with respect to the representation described in the proof of the theorem.) So, $\text{Reor}(C) = n(n - 1) + 1$.

Q. E. D.



The seven orientations of a 4-element chain and the corresponding directions of motion, the other six reorientations are their duals except the antichain

(1)



The $n(n-1)/2 + 1$ reorientations of an n -element chain (the other $n(n-1)/2$ reorientations are their duals except the antichain), where $n = 6$

(2)

Figure 5.1.7

5.2 EMBEDDING BLOCKING RELATIONS IN SMALL AREAS. Here, we study a more common representation, *segment representation*, in which all figures are horizontal segments of integer length, each with integer coordinate endpoints.

One of the reasons for the popularity of segment representations is that *any planar truncated lattice P has a segment representation*. The argument is as follows. Let R be any representation of P. First, move each figure vertically up such that if we change the direction of motion into horizontal to the right, then the figures with the new direction of motion form an antichain (Figure 5.2.1 (1), (2)); second, replace each figure by a horizontal segment whose left (right) endpoint has the same horizontal coordinate as

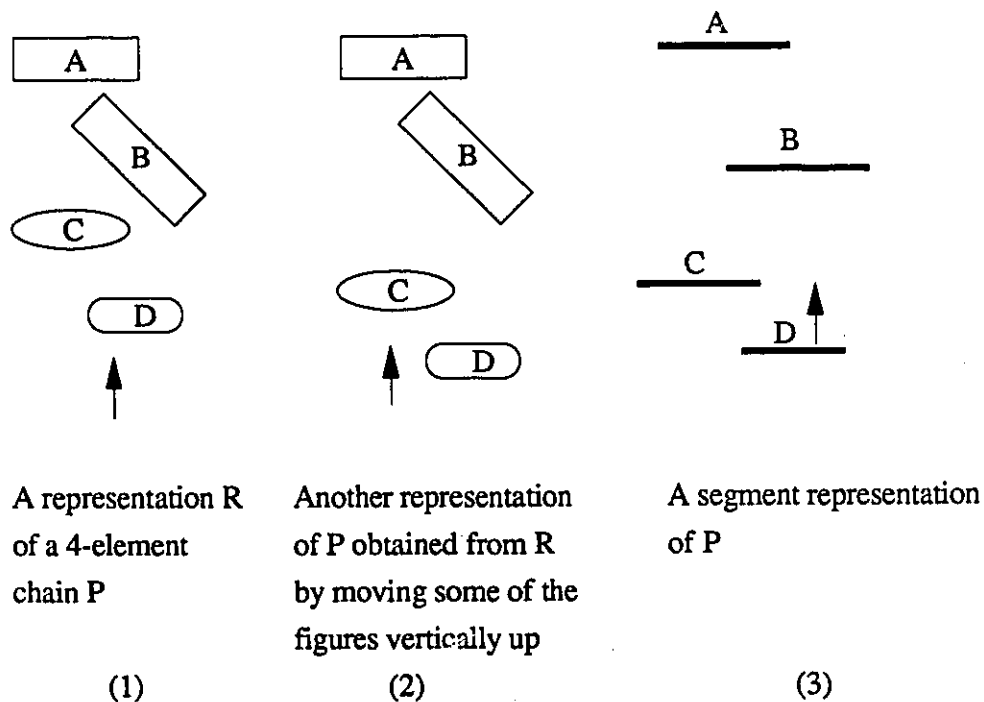


Figure 5.2.1

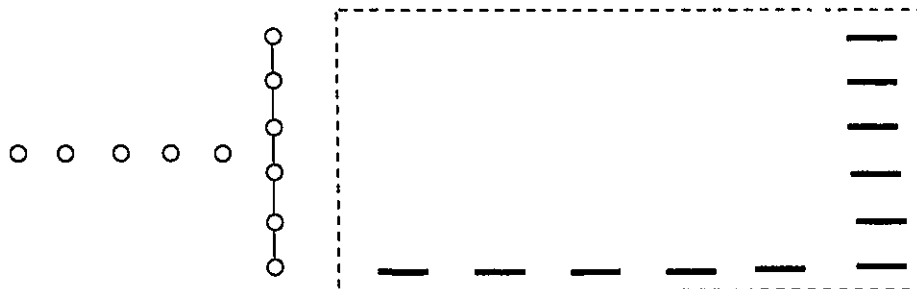
the most left (right) point of the figure (Figure 5.2.1 (3)); third, if necessary, extend each segment such that the length of each new one is rational; and finally, change the scale of coordinates so that each segment has integer length and its endpoints have integer

coordinates. Obviously, the segments with the vertical direction of motion give a segment representation of P .

How to obtain a segment representation with small area? For a segment representation R of a planar truncated lattice P , consider the area of the smallest upright rectangle enclosing all of its segments and denote it by $\text{Area}(R, P)$, and write

$$\text{Area}(P) = \min\{\text{Area}(R, P) : R \text{ is a segment representation } R \text{ of } P\}.$$

For example, for the representation R illustrated in Figure 5.2.2 (2) of the planar truncated lattice P illustrated in Figure 5.2.2 (1), $\text{Area}(R, P) = 55$, and indeed, $\text{Area}(P) = \text{Area}(R, P)$.



A planar truncated lattice P

(1)

A representation R of P

(2)

Figure 5.2.2

There are similar ideas for graphs. For example, a *grid drawing* of a planar graph G is a drawing of G such that the vertices are placed at grid points on the plane (a point is *grid point* if its two coordinates are integers), and the edges are drawn as polygonal lines that bend only at grid points. D. Woods [1982] proved that *every n -vertex planar graph has a grid drawing with $O(n^2)$ area and $O(n^2)$ bends*. More recently, R. Tamassia, I. G. Tollis and J. S. Vitter [1991] proved that *there is an $O(n)$ time algorithm that constructs a planar orthogonal grid drawing of n -vertex, 2-connected (multi)graphs with $O(n)$ maximum edges length, $O(n^2)$ area and at most $2n + 4$ bends*.

Another representation of a planar graph G is its *visibility representation* which consists of horizontal segments representing the vertices of G and vertical segments representing edges of G such that the edge-segment associated with an edge uv has its endpoints on the vertex-segments associated with u and v , and does not intersect any other vertex-segments (cf. B. Grunbaum and G. Shephard [1981]). G. di Battista and R. Tamassia [1986] proved *for any n -vertex planar graph G , there is an algorithm that constructs, in $O(n)$ time, a visibility representation for G .*

As for segment representations, we have the following theorem.

THEOREM 5.2.1 (W. P. Liu and I. Rival [1990]). *Let P be an n -element planar truncated lattice. Then*

$$\text{Area}(P) \leq n(n-1)/2;$$

moreover, the bound is best possible.

Let $c(P)$ denote the number of elements of a maximum chain of a planar truncated lattice P . For a segment representation R of P , let $h(R)$ ($w(R)$) be the height (width) of the smallest upright rectangle enclosing R . It is easy to show that there is a *perfect* segment representation R such that $h(R) = c(P) - 1$. In fact, any segment representation R with the smallest $h(R)$ is perfect.

To prove Theorem 5.2.1, we make use of the following lemma.

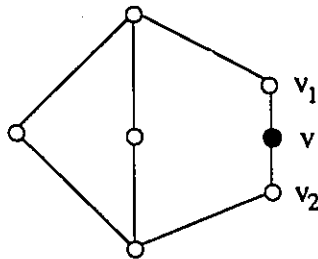
LEMMA 5.2.2. *For any planar truncated lattice P , there is a perfect segment representation R of P such that*

$$w(R) \leq 2[|P| - h(R)] - 1.$$

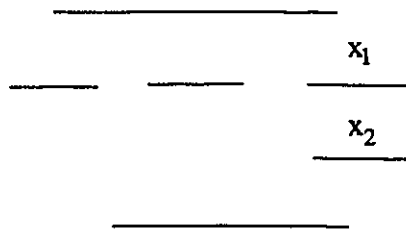
Proof. We will prove our lemma by induction on $|P|$. If $|P| = 1$, obviously, $h(R) = 0$ and $w(R) = 1$. So, $w(R) \leq 2[|P| - h(R)] - 1$. Now, assume $|P| > 1$.

Suppose v is an element on the boundary of a planar embedding of P corresponding to R such that v has at most one upper cover v_1 and at most one lower cover v_2 (cf. Jégou, Nowakowski and Rival [1985]). Consider a perfect segment representation R_1 of the subset Q of P with $S(Q) = S(P) - \{v\}$. Let the segments x_1 and x_2 represent the elements v_1 and v_2 , respectively.

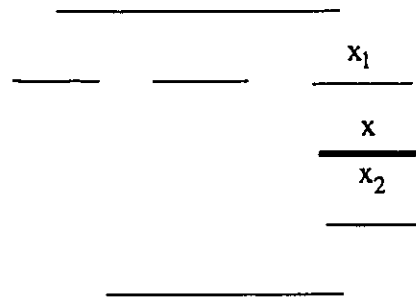
Case 1. $v_1 > v_2$ (Q) or the degree of v is one



A truncated planar lattice P



A perfect representation R_1 of Q



A perfect representation R of P

Figure 5.2.3

First, assume $v_1 > v_2$ (Q). Let R_1 be a perfect segment representation which satisfies the conditions of the lemma. Construct R as follows. First, move vertically down by one all the segments which represent elements of $\{u : u < v_1(P)\}$, then put a unit segment x representing the element v with vertical coordinate same as that of x_1 in R_1 minus one and $\text{right}_R(x) = \max\{\text{right}_{R_1}(x_1), \text{right}_{R_1}(x_2)\}$, where $\text{right}_R(x)$ denotes the

horizontal coordinate of the right endpoint of the segment x in R (Figure 5.2.3). Since $v_1 > v_2(Q)$ or equivalently, x_1 obstructs x_2 , therefore, since

$$h(R) = \begin{cases} h(R_1)+1 & \text{if any maximum chain contains } v \\ h(R_1) & \text{otherwise} \end{cases}$$

R is a perfect segment representation of P and

$$\begin{aligned} w(R) &= w(R_1) \leq 2[|P| - 1] - h(R_1) - 1 \\ &\leq 2[|P| - h(R) + 1] - 3 = 2[|P| - h(R)] - 1. \end{aligned}$$

Now, suppose that v has degree one. Without loss of generality, we may assume that v has one lower cover and no upper cover. By the same argument as above, we may obtain a perfect segment representation satisfying the condition of the lemma.

Case 2. v_1 is not an upper cover of v_2

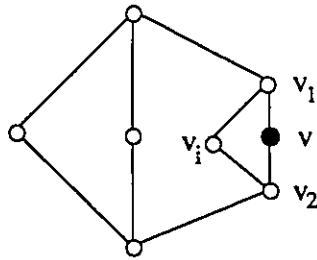
Since $v_1 > v_2(Q)$, there are segments x_i, \dots, x_j in the perfect parallel segment representation R_1 such that the segment x_1 obstructs the segment x_j , the segment x_i obstructs the segment x_{i+1}, \dots , and the segment x_j obstructs the segment x_2 . Without loss of generality, we may assume that

$$\min\{\text{right}_{R_1}(x_1), \text{right}_{R_1}(x_2)\} \geq \max\{\text{right}_{R_1}(x_k) : i \leq k \leq j\},$$

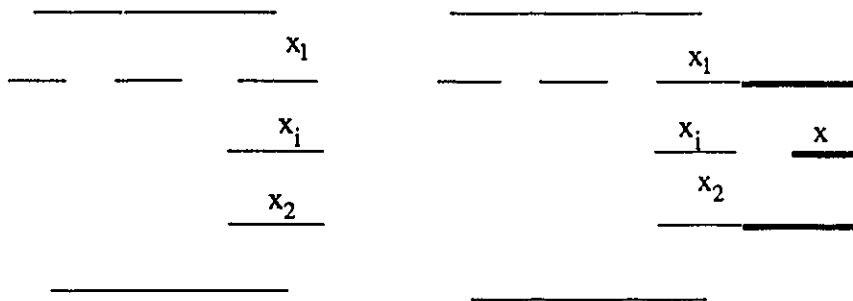
for otherwise, we can always extend the segments x_1 and x_2 . Construct R as follows. First, extend segments x_1 and x_2 towards the right by one, then, put a unit segment x with vertical coordinate as that of x_i and the horizontal coordinate of the left endpoint of the segment x is $\min\{\text{right}_{R_1}(x_1) + 1, \text{right}_{R_1}(x_2) + 1\}$ (Figure 5.2.4). Then R is a perfect segment representation of the ordered set P with $h(R_1) = h(R)$. Furthermore,

$$\begin{aligned} w(R) &= w(R_1) + 2 \leq 2[|P| - 1] - h(R_1) - 1 + 2 \\ &= 2[|P| - h(R)] - 1 + 2 - 2 = 2[|P| - h(R)] - 1. \end{aligned}$$

Q.E.D



A planar truncated lattice P



A perfect representation R_1 of Q

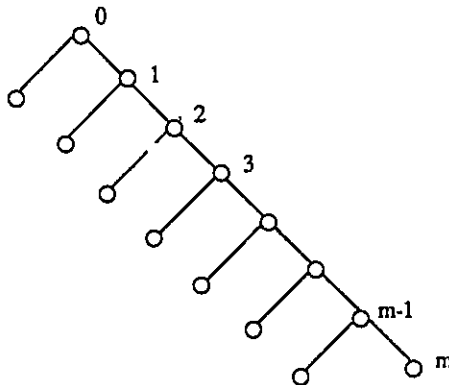
A perfect representation R of P

Figure 5.2.4

Now, we can prove Theorem 5.2.1 .

Proof of Theorem 5.2.1. Let R be a perfect segment representation of P described in Lemma 5.2.2. By Lemma 5.2.2, we have

$$\begin{aligned} \text{Area}(P) &\leq \text{Area}(R, P) \leq w(R)h(R) \leq \{2[n-c(P)+1]-1\}[c(P)-1] \\ &\leq n(n-1)/2. \end{aligned}$$



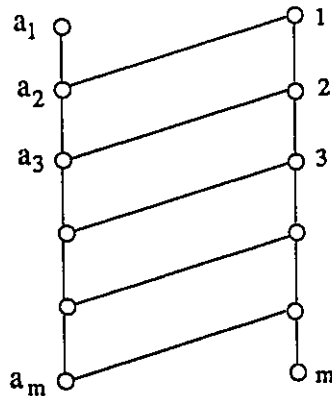
To see that the bound in Theorem 5.2.1 is best possible, consider a simple ordered set P illustrated in Figure 5.2.5. We show that $\text{Area}(P) \geq n(n-1)/2$, where, $n = 2m+1$. In fact, for any segment representation R of P , the smallest rectangle enclosing R has height at least m and width at least $2(m+1) - 1$, since P has an $(m+1)$ -element chain and an $(m+1)$ -element antichain. Therefore,

$$\text{Area}(P) \geq m(2m+2-1) = [2 \times (n-1)/2 + 1](n-1)/2 = n(n-1)/2.$$

Q.E.D.

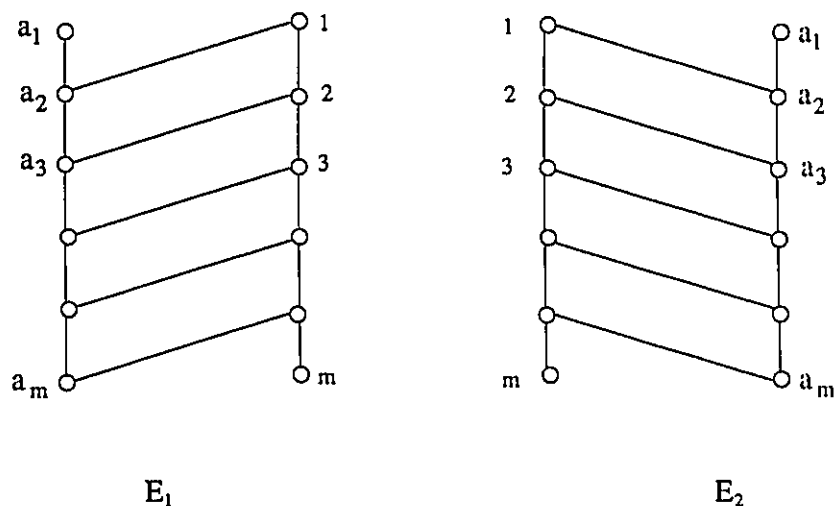
Remark. It can be that for any representation R of a planar truncated lattice P , $w(R)$ is not equal to the width of P . For example, the ordered set P illustrated in Figure 5.2.6 has width 2, although for any segment representation R , the upright rectangle enclosing R has width at least $n/2 + 2 = m + 2$. The argument is as follows. Observe that P has only two planar embeddings E_1 and E_2 (Figure 5.2.7). By induction on $n/2$, it is easy to show that for any segment representation R_i of E_i ,

$$|\text{left}_{R_i}(s_k) - \text{left}_{R_i}(t_k)| \geq n/2 + 1,$$



A planar truncated lattice P

Figure 5.2.6



The only two planar embeddings E_1 and E_2 of P

Figure 5.2.7

$i=1$ or 2 , where, $\text{left}_{R_i}(s)$ is the horizontal coordinate of the left endpoint of the segment s in the representation R_i , and s_k is the segment representing the element a_k and t_k is the segment representing the element k , $1 \leq k \leq n/2$. So, $w(R) \geq n/2 + 2$.

Now, we consider the relation between $\text{Reor}(P)$ and $\text{Area}(P)$ for a planar truncated lattice P . Recall that in Section 5.1, the segment representation of a chain is of very large area. The question of whether there is a segment representation whose segments are in the smallest area having "many" reorientations sounds interesting. Our second theorem in this section answers this question positively.

Theorem 5.2.3 (W. P. Liu and I. Rival [1990]). *For any odd integer n there is an n -element truncated lattice P such that a segment representation R of P satisfies the following conditions:*

- 1) $\text{Area}(R, P) = \text{Area}(P)$;
- 2) $\text{Reor}(R, P) \geq$

$$(n-1) \left\{ \frac{19(n-1) - 20 \ln^2[(n-1)/2] - 20 \ln[(n-1)/2]}{80 \ln[(n-1)/2]} \right\}.$$

$$(n-1)\{[19(n-1) - 20\ln^2[(n-1)/2] - 20\ln[(n-1)/2]]/[80\ln[(n-1)/2]]\}.$$

Proof. Let P be the ordered set illustrated in Figure 5.2.8 ($m = (n-1)/2$). Let R be a representation of P consisting of segments $s_1, s_2, \dots, s_m, q_0, q_1, \dots, q_m$, where, s_i is the segment with end points $(-2m - 3 + 2i, 0)$ and $(-2m - 3 + 2i + 1, 0)$, $1 \leq i \leq m$, and q_j is the segment with end points $(0, j)$ and $(-1, j)$, $1 \leq j \leq m$ (Figure 5.2.8). Then

$$\text{Area}(R, P) = m(2m + 1) = n(n-1)/2 = \text{Area}(P)$$

since P has an m -element chain and an $(m+1)$ -element antichain.

To prove the second part of theorem, we need the following well known result in number theory.

LEMMA 5.2.4 (cf. L. E. Dickson [1952]). *The number of prime numbers which are not bigger than an integer m , $\phi(m)$, is at least $0.95m/\ln m$.*

Let

$$\tan \alpha_{i,j} = (2m + 3 - 2i)/(m - j),$$

for all i and j , $1 \leq i \leq m$, $0 \leq j \leq m - 1$, and let $R_{i,j}$ denote the representation obtained from R such that the direction of motion of $R_{i,j}$ and the vertical axis constitute the angle $\alpha_{i,j}$. Then, $q_i \succ s_j$ ($R_{i,j}$). Furthermore, if $\alpha > \alpha_{i,j}$, then s_i is noncomparable to q_j in the representation whose direction of motion and vertical axis constitute the angle α . From the definition of $\alpha_{i,j}$, we know that the number of different angles among the angles $\alpha_{1,1}, \alpha_{1,2}, \dots, \alpha_{1,m}, \alpha_{2,1}, \alpha_{2,2}, \dots, \alpha_{2,m}, \dots, \alpha_{m,1}, \alpha_{m,2}, \dots, \alpha_{m,m}$ is at least

$$(m - m/2) + (m - m/3) + (m - m/5) + \dots + (m - m/p(m)),$$

where, $p(m)$ is the maximum prime not bigger than m . Therefore,

$$\begin{aligned} \text{Reor}(R, P) &\geq m \sum_{q \leq m, \text{ is prime}} (1 - 1/q) \geq m\phi(m) - m \sum_{q \leq m, \text{ is prime}} 1/q \\ &\geq m\phi(m) - (m/2) \sum_{1 \leq q \leq p(m)} 1/i. \end{aligned}$$

$$\text{Reor}(R,P) \geq 0.95m^2/\ln m - (\ln m + 1)m/2$$

$$= (n - 1) \{ 19(n - 1) - 20\ln^2[(n-1)/2] - 20\ln[(n-1)/2] \} / \{ 80\ln[(n-1)/2] \}.$$

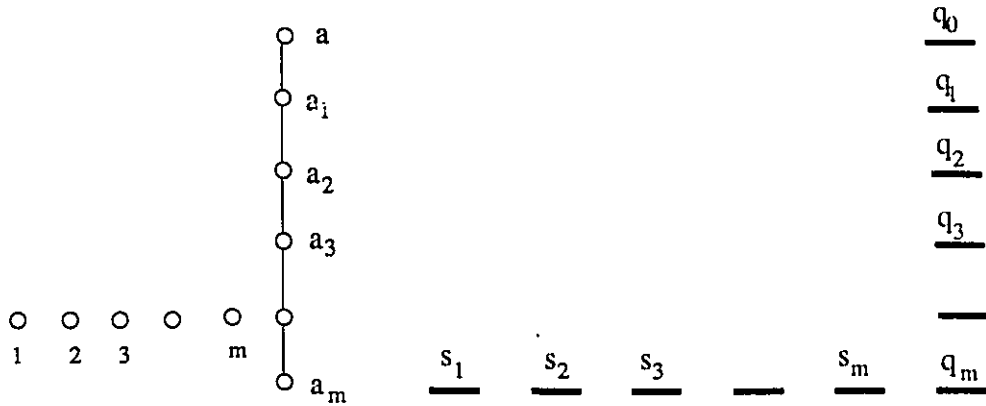
A truncated planar lattice P A segment representation of P

Figure 5.2.8

5.3 SEEING SEGMENTS FROM THE OUTSIDE. In 1973, V. Klee posed the following problem. How many guards are necessary, and how many are sufficient, to patrol the n paintings of an art gallery? Say k points inside a polygon H *guard* the polygon, if, for any point v inside the polygon, there is a point u of the k points such that the line segment joining u and v does not intersect any line segment constituting H . V. Chvatal [1975] proved that $\lfloor n/3 \rfloor$ *points are occasionally necessary and always sufficient to guard a polygon of n vertices* (the polygon consists of n line segments). There are many interesting variations of the Chvatal's Art Gallery Theorem, such as traditional orthogonal gallery (a gallery is *orthogonal* if any segment of its boundary is parallel to the coordinate axes) in which case $\lfloor n/4 \rfloor$ *are necessary and always sufficient* (J. Kahn, M. Klawe and D. Kleitman [1983]); mobile guards, each of whom patrols along a line segment with an n -point polygon, in which case, $\lfloor n/4 \rfloor$ *are necessary and always sufficient* (J. O'Rourke [1983]).

Consider a slightly different version. Suppose there are n line segments on the plane, no two of which intersect. We say a point p not on any segment *sees* a segment s , or s is *visible* from the point p , if the triangle, denoted by $T(s, p)$, decided by the segment s and the point p does not intersect any other segment. The question is: how many points are necessary and how many are sufficient to see n segments on the plane? J. Czyzowicz, I. Rival and J. Urrutia [1989] have the following conjecture.

CONJECTURE 10 (J. Czyzowicz, I. Rival and J. Urrutia [1989]). $\lceil n/2 \rceil$ *points are enough to see n segments, where the points can be anywhere provided they are not on any segment.*

Furthermore, they provided an example to show that $\lceil n/2 \rceil$ *points are sometimes necessary to see n segments.*

If, for some reason, the guards cannot go inside a polygon containing n segments (practically, maybe, it is dangerous or impossible to go inside the polygon), how many points are necessary and how many are sufficient to see the segments? For our purpose, we must suppose each segment is visible from a point on the boundary D of the convex polygon (here, we assume the set of segments and the convex polygon are given). Here is our answer.

THEOREM 5.3.1 (W. P. Liu and I. Rival [1990]). *$n-2$ points on the convex polygon are always sufficient and sometimes necessary to see n (> 3) segments.*

To prove the theorem, we need some definitions, notations and a series of lemmas.

First, observe a well known fact: *any point v on D induces an order on the segments*, that is, for two segments a and b , $b > a(v)$ or $a < b(v)$, if there is a sequence of segments $a = b_1, b_2, \dots, b_k = b$ such that $T(b_i, v) \cap b_{i-1} \neq \emptyset, 2 \leq i \leq k$.

In this section, two orders will be used: an order on points on D with respect to a selected point v_0 on D - for two points u and v on D , $u < v(v_0)$ ($u \leq v(v_0)$) or $v > u(v_0)$ ($v \geq u(v_0)$) exactly means that the length of the curve from v_0 to v along D clockwise is longer (longer or equal) than that of the curve from v_0 to u along D clockwise; and another order on segments with respect to a point v on D , that is, the order induced by the point v .

For any segment a and a point v on D , we distinguish the two endpoints of the segment a , the left endpoint, denoted by λ_a , which is the left side endpoint of the segment (or the closer endpoint to v if $T(a, v)$ is a line) if we stand at the point v , and the right endpoint, denoted by ρ_a , which is the right side endpoint of the segment (or the farther endpoint to v if $T(a, v)$ is a line) if we stand at the point v . Of course, λ_a depends on a point v . But, since we use λ_a or ρ_a only when talking about a line passing through two

points $\lambda_a(\rho_a)$ and v on D , denoted by (λ_a, v) and (ρ_a, v) , in which case the point that we stand at is v .

For a segment a , the extension of the segment a is denoted by $E(a)$.

LEMMA 5.3.2. *Any point on the convex polygon D sees at least one segment.*

Proof. It follows from the observation that any point on D induces an order and the fact that any ordered set has a minimal element.

LEMMA 5.3.3. *For any segment a , there is a point on D seeing at least two segments, including the segment a .*

Proof. Suppose v_0 sees the segment a . Let

$$v_1 = \sup\{u \in D: \text{if } v_0 \leq t < u \text{ (} v_0), \text{ then, } t \text{ sees } a\}.$$

We prove that the point v_1 does not see the segment a . Suppose v_1 sees a . Then

$$T(a, v_1) \cap c = \emptyset,$$

for any segment c other than the segment a . Let v_2 be a point on D which is very close to v_1 with $v_1 < v_2$ (v_0). Then for any u , $v_1 \leq u < v_2$ (v_0) and any segment c other than the segment a , $T(a, u) \cap c = \emptyset$ since $T(a, v_1)$ and c are closed sets (as subsets of the plane) and there are only finitely many segments. So,

$$\sup\{u \in D: \text{if } v_0 \leq t < u \text{ (} v_0), \text{ then } t \text{ sees } a\} \geq v_2 > v_1,$$

which contradicts the choice of the point v_1 . Therefore, v_1 sees a segment b other than a . Then, for any point v on D very close to v_1 with $v < v_1$ (v_0), v sees b . By the definition of v_1 , v sees the segment a , too. That is, v sees both segments a and b .

Q.E.D.

For simplicity of the proofs in the later lemmas and Theorem 5.3.1, we make the following observation. Suppose the sequence of segments (b_1, b_2, \dots) is a chain with respect to a point v on D , i.e.

$$b_1 > b_2 > b_3 > \dots (v).$$

Then, we can obtain a subsequence of it

$$b_{i_1} > b_{i_2} > b_{i_3} > \dots (v)$$

such that

$$T(b_{i_{t-1}}, v) \cap b_{i_t} \neq \emptyset$$

for all $t > 1$, and

$$T(b_{i_{t-1}}, v) \cap b_{i_k} = \emptyset$$

for all $k > t$, by letting

$$b_{i_1} = b_1$$

and

$$b_{i_t} = \max\{b_k: k > i_{t-1} \text{ and } T(b_{i_{t-1}}, v) \cap b_k \neq \emptyset\}.$$

From now on, if a chain $b_1 > b_2 > b_3 > \dots (v)$ is considered, we always assume

$$T(b_{i-1}, v) \cap b_i \neq \emptyset$$

for all $i > 1$, and

$$T(b_{i-1}, v) \cap b_k = \emptyset$$

for all $k > i$.

LEMMA 5.3.4. *Suppose $b_1 > b_2 > b_3 > \dots > b_k (v)$. Then*

$$(1) (\rho_{b_2}, v) \cap b_3 \neq \emptyset \text{ implies } (\rho_{b_{i-1}}, v) \cap b_i \neq \emptyset \text{ and } (\lambda_{b_{i-1}}, v) \cap b_i = \emptyset;$$

$$(2) (\lambda_{b_2}, v) \cap b_3 \neq \emptyset \text{ implies } (\lambda_{b_{i-1}}, v) \cap b_i \neq \emptyset \text{ and } (\rho_{b_{i-1}}, v) \cap b_i = \emptyset,$$

$$3 \leq i \leq k - 1.$$

Proof. Since the proofs are identical, we only prove (1). To this end, it is enough to show that for $i < k - 2$

$$(\rho_{b_{i-1}}, v) \cap b_i \neq \emptyset \text{ implies } (\rho_{b_i}, v) \cap b_{i+1} \neq \emptyset.$$

If $(\rho_{b_i}, v) \cap b_{i+1} = \emptyset$, then b_{i+1} is contained in $T(b_i, v) - T(b_{i-1}, v)$, which, in turn, is contained in $T(b_i, v)$ (Figure 5.3.1). So

$$T(b_i, v) \cap b_{i+2} \neq \emptyset,$$

which contradicts the assumption.

Q.E.D.

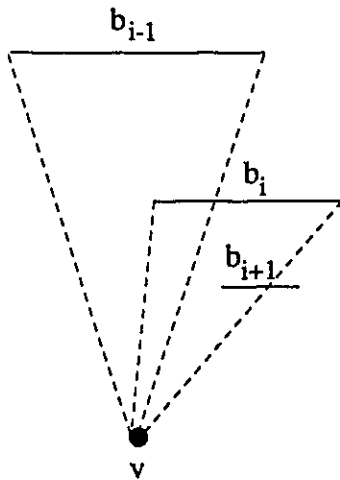


Figure 5.3.1

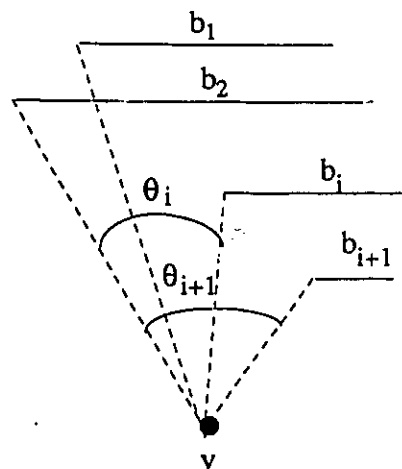


Figure 5.3.2

The following corollary can be induced from Lemma 5.3.4 directly.

COROLLARY 5.3.5 *If $b_1 > b_2 > b_3 > \dots > b_k(v)$ and $(\rho_{b_2}, v) \cap b_3 \neq \emptyset$, then*

$$\theta_k > \theta_{k-1} > \theta_{k-2} > \dots > \theta_4 > \theta_3,$$

where θ_i is the clockwise angle from the line (λ_{b_2}, v) to the line (λ_{b_i}, v) (Figure 5.3.2).

COROLLARY 5.3.6. *Suppose the point v sees a segment a and for a segment b , $b = b_1 > b_2 > b_3 > \dots > b_k = a(v)$. If the triangle $T(b, v)$ is on the "left" ("right") of the triangle $T(a, v)$, then*

$$(\rho_{b_i}, v) \cap b_{i+1} \neq \emptyset \quad ((\lambda_{b_i}, v) \cap b_{i+1} \neq \emptyset).$$

Proof. If $k = 2$, then, the corollary is obviously true. Otherwise, it follows from Corollary 5.3.5.

Q.E.D.

LEMMA 5.3.7. *Assume a point u sees a segment a and a point v sees a segment b . If $T(a, u) \cap T(b, v) \neq \emptyset$ (Figure 5.3.3), then, $a \not\leq b(u)$ and $b \not\leq a(v)$.*

Proof. It follows from Corollary 5.3.6.

Q.E.D.

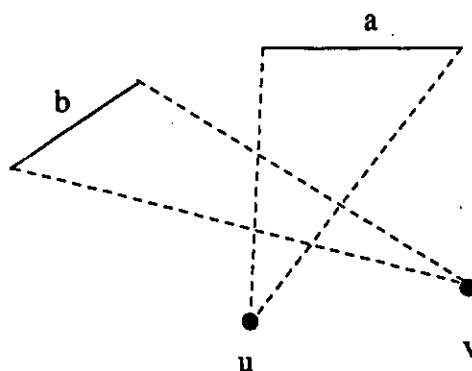


Figure 5.3.3

LEMMA 5.3.8. *Suppose any point sees at most two segments and v sees segments a and b . If u sees a segment c , then $T(c, u) \cap T(a, v) \neq \emptyset$ implies $T(c, u) \cap T(b, v) = \emptyset$ and vice versa.*

Proof. Suppose

$$T(c, u) \cap T(a, v) \neq \emptyset \text{ and } T(c, u) \cap T(b, v) \neq \emptyset.$$

By symmetry, we may assume $T(c, u)$, $T(a, v)$ and $T(b, v)$ intersect as shown in Figure 5.3.4. Then, by Lemma 5.3.7, $c \not\leq a(v)$ and $c \not\leq b(v)$. So there is a segment d other than the segments a and b which is visible from v , since the ordered set induced by the segment set $\{s: s \text{ is segment and } s \leq c(v)\}$ and the point v does not contain the segments a

and b. Namely, the point v sees at least three segments a , b and d . This leads to a contradiction.

Q.E.D.

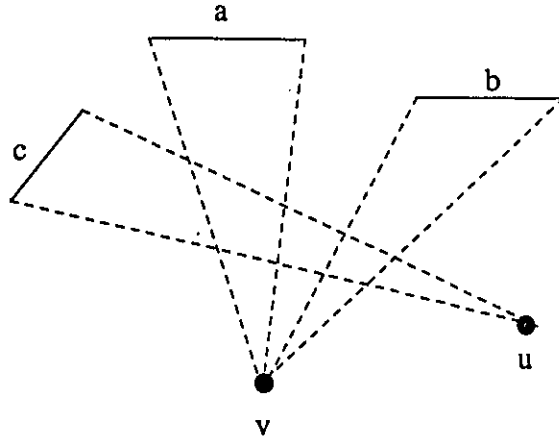


Figure 5.3.4

Proof of Theorem 5.3.1. First, we prove the necessity.

Consider segments a_1, a_2, \dots, a_{n-3} on the plane whose positions are illustrated in Figure 5.3.5. To see any segment a_i inside the "triangle" consisting of the segments a_{n-2}, a_{n-1} and a_n , a point must belong to the curve segments $I_i, 1 \leq i \leq n-3$. Since $I_i \cap I_j = \emptyset$ if $i \neq j$, at least $n - 3$ points are needed to see segments a_1, a_2, \dots, a_{n-3} . Notice that any point of $I_1 \cup I_2 \cup \dots \cup I_{n-3}$ does not see the segment a_n . So, one more point is needed. Therefore, at least $n - 2$ points are needed to see the n segments.

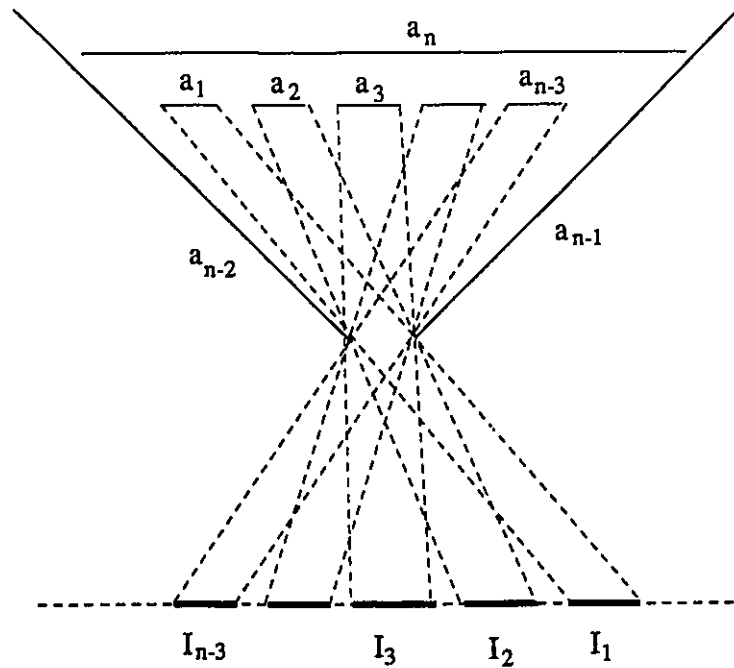


Figure 5.3.5

Now, we consider the sufficiency.

If there is a point which sees at least three segments, then, obviously, $n - 2$ points are enough to see all the segments, since each other segment is visible from a point on D. Now, assume each point sees at most two segments. We will find two points which see four segments a, b, c and d.

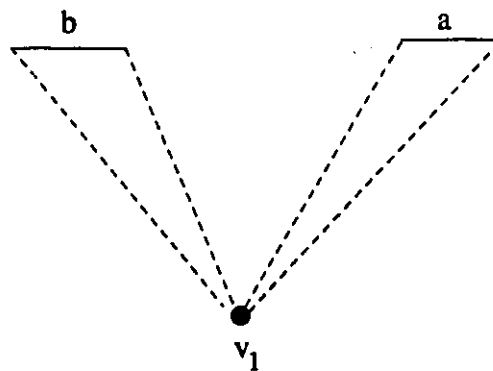


Figure 5.3.6

Suppose the point v_1 sees segments a and b (according to Lemma 5.3.3, there is such a point) with $T(b, v_1)$ on the left of $T(a, v_1)$ (Figure 5.3.6). We claim that if v_2 sees a segment c , $c \neq a, b$, then

$$T(c, v_2) \cap [T(b, v_1) \cup T(a, v_1)] = \emptyset. \quad (1)$$

Suppose (1) is not true. By symmetry, we may assume

$$T(c, v_2) \cap T(a, v_1) = \emptyset \text{ and } T(c, v_2) \cap T(b, v_1) \neq \emptyset,$$

since, according to Lemma 5.3.8, it is impossible that

$$T(c, v_2) \cap T(a, v_1) \neq \emptyset \text{ and } T(c, v_2) \cap T(b, v_1) \neq \emptyset.$$

(Figure 5.3.7) Without loss of generality, we may assume that if $v_1 < w < v_2$ (v_1), and w sees a segment $e \neq a, b$, then

$$T(e, w) \cap T(b, v_1) = \emptyset.$$

By the assumption that no point sees three segments and Lemma 5.3.8, v_1 does not see the segment c and v_2 does not see the segment a . Furthermore, according to Lemma 5.3.7, $c \not> b(v_1)$. So $c > a(v_1)$, which will be used later.

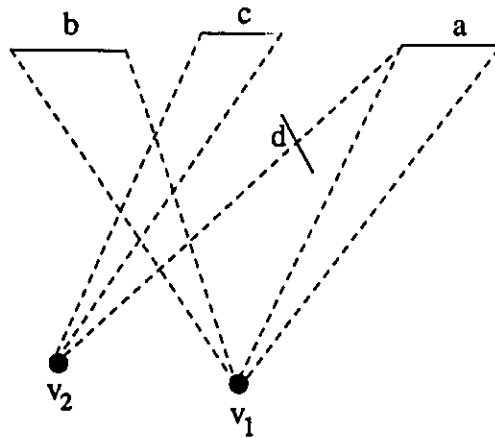


Figure 5.3.7

Let $a >_d(v_2)$ (since v_2 does not see a). We prove that for one of the two points of $E(d) \cap D$, say v_3 ,

$$v_1 < v_3 < v_2(v_1). \quad (2)$$

Suppose (2) is not true. First, the point ρ_d (with respect to v_2) must be inside the area surrounded by the two lines (λ_a, v_2) , (λ_a, v_1) and the curve segment $\{x \in D: v_1 \leq x \leq v_2(v_1)\}$ (Figure 5.3.7). Notice that the point v_1 does not see the segment d , which implies that $d = d_1 > d_2 > \dots > d_m = a(v_1)$. (If $d > b(v_1)$, then $d > c(v_1)$ since v_2 sees c . So, by the result that $c > a(v_1)$ proved before, $d > c > a(v_1)$.) By Corollary 5.3.6,

$$(\rho_{d_i}, v_2) \cap d_{i+1} \neq \emptyset,$$

for all i , $1 \leq i < m$. So, $a > d_i > d(v_2)$ for some i , which contradicts that $a > d(v_2)$. Hence, (2) is true.

According to the choice of the point v_2 , the point v_3 does not see d . Let $d > d_1(v_3)$ (Figure 5.3.8). By the same argument as above, we can prove that for one of the two points of $E(d_1) \cap D$, say, v_4 , we have $v_1 < v_4 < v_3(v_1)$. For otherwise, we will obtain the contradiction that either $a > d(v_2)$ or $d > d_1(v_3)$. In this way, we obtain a sequence of segments $d = d_0, d_1, d_2, \dots$ and a sequence of points v_3, v_4, \dots such that

$$d_i > d_{i+1}(v_{i+3})$$

and

$$v_1 < v_{i+3} < v_{i+2}(v_1)$$

for all $i \geq 0$. Moreover, if $i \neq j$, then $d_i \neq d_j$. But, this is impossible, since there are only finitely many segments. Thus, (1) must be true.

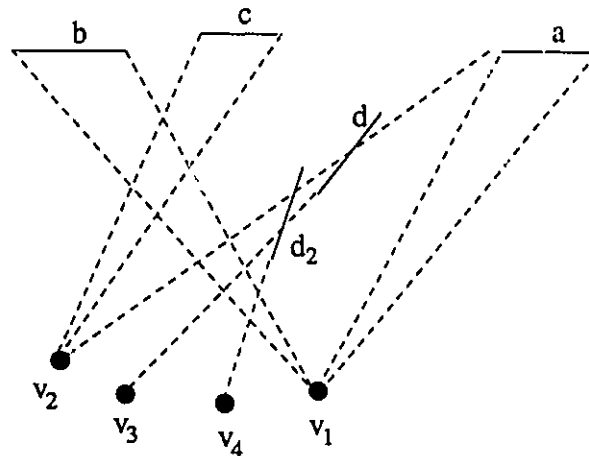


Figure 5.3.8

Now, we are ready to complete the proof of the theorem. Choose a point v_3 which sees a segment $c \neq a, b$ such that if $v_3 < v(v_1)$, then v does not see any other segment than the segments a or b or c . Let

$$w = \inf\{x: \text{if } x < y \leq v_3(v_1), \text{ then } y \text{ sees } c\},$$

$$w' = \sup\{x: \text{if } v_3 \leq y < x(v_1), \text{ then } y \text{ sees } c\}.$$

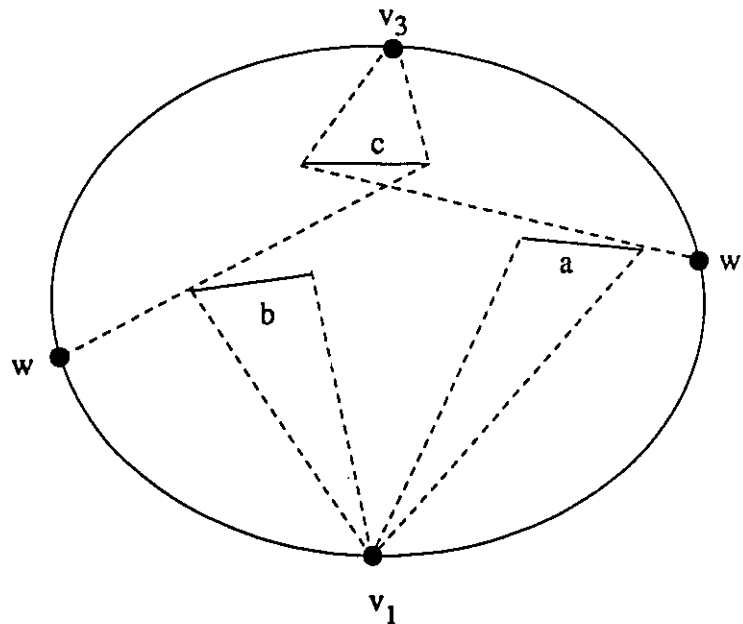
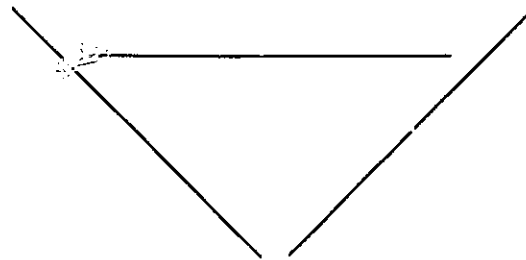


Figure 5.3.9

(Figure 5.3.9) Then neither the point w nor the point w' sees the segment c . If w or w' sees a segment s other than the segments a , b and c , then a point x , $w < x < w'$ (v_1), very close to either w or w' sees the segments s and c , which implies that we find two points v_1 and x which sees four segments a , b , c and s . Suppose the points w and w' see only the segment a or b . Furthermore, we can assume that any point x , $w < x < w'$, cannot see any segment other than segments a , b and c , which, according to (1), implies that any segment (at least one more segment since there are at least four segments) other than the segments a , b and c must be inside either the area A surrounded by the lines (ρ_c, w) , (λ_b, v_1) and the curve segment $\{x: w \leq x \leq v_1 (v_1)\}$ or the area B surrounded by the lines (λ_c, w') , (ρ_a, v_1) and the curve segment $\{x: w' \leq x (v_1)\}$. Therefore, the point v_1 sees at least three segments, which is a contradiction, since the point v_1 sees at least one segment in the area $A \cup B$. This completes the proof of the theorem.

Remark: It is possible that two points are necessary to see three segments. To see the three segments on the polygon containing them in Figure 5.3.10, two points are needed.



Two points are necessary to see three segments

Figure 5.3.10

5.4. SEGMENTS IN SPACE. In this section, we consider segments in space. Unfortunately, we have not succeeded to get any nontrivial number of points which are sufficient to see n segments in space. (Obviously, n points are always sufficient to see n segments.) We even do not know whether $n - 1$ points are always sufficient to see n segments. But, our interesting examples tell us that the problem of how many points are needed to see segments in the space is not only quite different from but also much harder than the corresponding problem on the plane.

In this section, all points and segments, unless otherwise stated, are in space. Recall that $\lceil n/2 \rceil$ points may be sufficient to see n segments on the plane (cf. J. Czyzowicz, I. Rival and J. Urrutia [1989]) and notice the fact that we have more freedom in space than that on the plane to choose our favourite points to see the given segments. The following theorem is striking.

THEOREM 5.4.1 (W. P. Liu and I. Rival [1990]). *There are $3n + 6$ line segments in space such that at least $2n$ points are needed to see the segments.*

Proof. Before giving the example, we need some notation. Let ab stand for a segment with a and b as its endpoints. For a point a , (x_a, y_a, z_a) denotes its three coordinates; and sometimes, we also use $v = (x, y, z)$ to denote a point v with coordinates x , y and z .

Consider $3n$ segments $a_k b_k$, $c_k d_k$ and $e_k f_k$ with the following coordinates:

$$(x_{a_k}, y_{a_k}, z_{a_k}) = (m, 0, (k - 1)m),$$

$$(x_{b_k}, y_{b_k}, z_{b_k}) = (-m, 0, (k - 1)m - 2),$$

$$(x_{c_k}, y_{c_k}, z_{c_k}) = (t_k/2 - s_k - m/2, \sqrt{3}(m - t_k)/2, (k - 1)m),$$

$$(x_{d_k}, y_{d_k}, z_{d_k}) = ((t_k + m)/2 - s_k, -\sqrt{3}(m + t_k)/2, (k - 1)m - 2),$$

$$(x_{c_k}, y_{c_k}, z_{c_k}) = (t_k - (m + s_k)/2, -\sqrt{3}(m + s_k)/2, (k - 1)m),$$

$$(x_{f_k}, y_{f_k}, z_{f_k}) = (t_k + (m - s_k)/2, \sqrt{3}(m - s_k)/2, (k - 1)m - 2),$$

where, $1 > s_k > t_k > 0$, $1 \leq k \leq n$.

First, we prove that if a point $v = (\alpha, \beta, \gamma)$ with $(\alpha, \beta) \in \Gamma$ sees a segment $a_k b_k$ or $c_k d_k$ or $e_k f_k$, then,

$$(k - 1)m - 11 \leq \gamma \leq (k - 1)m + 11 \quad (1)$$

where,

$$\Gamma = \{(x, y): -m - 1 \leq x, y \leq m + 1\}.$$

For convenience, we give the functions of following lines:

$$E(a_k b_k): \begin{cases} (X + m)/(2m) = [Z - (k - 1)m + 2]/2 \\ Y = 0 \end{cases}$$

$$E(c_k d_k): [X - t_k/2 + s_k + m/2]/m = [Y - \sqrt{3}(m - t_k)/2]/(-\sqrt{3}m) = [Z - (k - 1)m]/(-2);$$

$$E(e_k f_k): [X - t_k - (-s_k + m)/2]/(-m) = [Y - \sqrt{3}(m - s_k)/2]/(-\sqrt{3}m) \\ = [Z - (k - 1)m + 2]/2;$$

$$\text{the XY-projection of } E(c_k d_k): Y = -\sqrt{3}(X + s_k);$$

$$\text{the XY-projection of } E(e_k f_k): Y = \sqrt{3}(X - t_k);$$

$$\text{the XY-projection of } E(b_k c_k): Y = \sqrt{3}(m - t_k)(X + m)/(t_k - 2s_k + m);$$

$$\text{the XY-projection of } E(a_k f_k): Y = \sqrt{3}(m - s_k)(X - m)/(2t_k - s_k - m);$$

the XY-projection of $E(b_k f_k)$: $Y = \sqrt{3}(m - s_k)(X + m)/(2t_k - s_k + 3m)$;

the XY-projection of $E(a_k c_k)$: $Y = \sqrt{3}(m - t_k)(X - m)/(t_k - 2s_k - 3m)$;

the XY-projection of $E(b_k d_k)$: $Y = -\sqrt{3}(m + t_k)(X + m)/(t_k - 2s_k + 3m)$;

the XY-projection of $E(a_k e_k)$: $Y = \sqrt{3}(m + s_k)(X - m)/(-2t_k + s_k + 3m)$;

the XY-projection of $E(a_k d_k)$: $Y = \sqrt{3}(m + t_k)(X - m)/(-t_k + 2s_k + m)$;

the XY-projection of $E(b_k e_k)$: $Y = -\sqrt{3}(m + s_k)(X + m)/(2t_k - s_k + m)$,

where, $E(ab)$ denotes the line which is the extension of the segment ab . Partition Γ into A, B, D and E or A', B', D' , and E' as follows:

$$A = \{(x, y) : \max\{-m - 1, -\sqrt{3}(m + t_k)(x + m)/(t_k - 2s_k + 3m), -\sqrt{3}(x + s_k)\} \leq y \leq \min\{m + 1, \sqrt{3}(m - t_k)(x + m)/(t_k - 2s_k + m)\}, x \leq m + 1\},$$

$$B = \{(x, y) : \max\{\sqrt{3}(m - t_k)(x + m)/(t_k - 2s_k + m), \sqrt{3}(m - t_k)(x - m)/(t_k - 2s_k - 3m)\} \leq y \leq m + 1\},$$

$$D = \{(x, y) : \max\{-m - 1, \sqrt{3}(m + t_k)(x - m)/(-t_k + 2s_k + m)\} \leq y \leq \min\{m + 1, \sqrt{3}(m - t_k)(x - m)/(t_k - 2s_k - 3m), -\sqrt{3}(x + s_k)\}, x \geq -m - 1\},$$

$$E = \{(x, y) : -m - 1 \leq y \leq \min\{-\sqrt{3}(m + t_k)(x + m)/(t_k - 2s_k + 3m), \sqrt{3}(m + t_k)(x - m)/(-t_k + 2s_k + m)\},$$

$$A' = \{(x, y) : \max\{-m - 1, \sqrt{3}(m + t_k)(x - m)/(-2t_k + s_k + 3m), \sqrt{3}(x - t_k)\} \leq y \leq \min\{m + 1, \sqrt{3}(m - s_k)(x - m)/(2t_k - s_k - m)\}\},$$

$$B' = \{(x, y) : m + 1 \geq y \geq$$

$$\max\{\sqrt{3}(m - t_k)(x - m)/(2t_k - s_k - m), \sqrt{3}(m - s_k)(x + m)/(2t_k - s_k + 3m)\}$$

$$D' = \{(x, y) : -m - 1 \leq y \leq$$

$$\max\{-\sqrt{3}(m + s_k)(x + m)/(t_k - 2s_k + 3m), \sqrt{3}(m + t_k)(x - m)/(-t_k + 2s_k + m)\},$$

$$E' = \{(x, y) : \max\{-m - 1, -\sqrt{3}(m + t_k)(x + m)/(2t_k - s_k + m)\} \leq y \leq$$

$$\min\{m + 1, \sqrt{3}(m - s_k)(x + m)/(2t_k - s_k + 3m), \sqrt{3}(x - t_k)\}, \quad x \leq m + 1\}.$$

We begin with the left half of the inequality (1).

Case 1. $w = (\alpha, \beta) \in A$

The XY-projection of $E(vb_k)$ intersects the XY-projection of $E(c_k d_k)$ at the point (x_0, y_0) , where

$$x_0 = -[\beta m + \sqrt{3}s_k(\alpha + \beta)]/[\beta + \sqrt{3}s_k(\alpha + m)] \quad (2)$$

and

$$y_0 = \sqrt{3}\beta(m - s_k)/[\beta + \sqrt{3}(\alpha + m)]. \quad (3)$$

The point M_0 on $E(c_k d_k)$ with X-coordinate x_0 and Y-coordinate y_0 has the Z-coordinate

$$z_0 = (k - 1)m + [2y_0 - \sqrt{3}(m - t_k)]/(\sqrt{3}m).$$

To find the upper bound of γ , it is enough in this case to consider those points such that u , M_0 and b_k are collinear. So, from the following equation

$$[Z - (k - 1)m + 2]/[z_0 - (k - 1)m + 2] = Y/y_0,$$

we can get

$$\begin{aligned}
\gamma &= \beta[z_0 - (k-1)m + 2]/y_0 + (k-1)m - 2 \\
&= \beta\{[2y_0 - \sqrt{3}(m - t_k)]/(\sqrt{3}m) + 2\}/y_0 + (k-1)m - 2 \\
&= 2\beta/(\sqrt{3}m) - (m - t_k)\beta/(my_0) + (k-1)m - 2 + 2\beta/y_0 \\
&\quad \text{(By } \beta \leq m + 1, \text{ formulas (2) and (3))} \\
&\leq 2(m+1)/(\sqrt{3}m) - [(m - t_k)/m]\{[\beta + \sqrt{3}(\alpha + m)]/[\sqrt{3}(m - s_k)]\} + \\
&\quad (k-1)m - 2 + 2[\beta + \sqrt{3}(\alpha + m)]/[\sqrt{3}(m - s_k)] \\
&\quad \text{(By } -m - 1 \leq \beta \leq m + 1) \\
&\leq 2(m+1)/(\sqrt{3}m) + [(m - t_k)/m]\{(\sqrt{3} + m + 1)/[\sqrt{3}(m - s_k)]\} + \\
&\quad (k-1)m - 2 + 2[m + 1 + \sqrt{3}(2m + 1)]/[\sqrt{3}(m - s_k)] \\
&\leq 2 - 2 + (m - t_k)/(m - s_k) + (k-1)m + 10m/[\sqrt{3}(m - s_k)] \\
&\leq (k-1)m + 11.
\end{aligned}$$

Case 2. $(\alpha, \beta) \in B$

To find the upper bound of γ , it is enough in this case to consider those points such that v, c_k, a_k and b_k are coplanar; the plane is

$$X + pY - mZ + (k-1)m^2 - m = 0 \quad (4)$$

where $p = (3m + 2s_k - t_k)/[\sqrt{3}(m - t_k)]$. Since the point v is on the plane (4), so

$$\gamma = \alpha/m + p\beta/m + (k-1)m - 1.$$

Hence,

$$\gamma = \alpha/m + p\beta/m + (k-1)m - 1 \quad (\text{By } \alpha, \beta \leq m+1)$$

$$\leq (m+1)/m + 2p + (k-1)m - 1 \quad (\text{By } 0 \leq t_k < s_k \leq 1)$$

$$\leq (k-1)m + 1 + 2(3m+4)/[\sqrt{3}(m-1)] \leq (k-1)m + 7.$$

Case 3. $u = (\alpha, \beta) \in D$

The XY-projection of $E(va_k)$ intersects the XY-projection of $E(c_k d_k)$ at the point (x_1, y_1) , where

$$x_1 = [\beta m - \sqrt{3}s_k(\alpha - m)]/[\beta + \sqrt{3}(\alpha - m)] \quad (5)$$

and

$$y_1 = -\sqrt{3}\beta(m + s_k)/[\beta + \sqrt{3}(\alpha - m)]. \quad (6)$$

The point M_1 on $E(c_k d_k)$ with X-coordinate x_1 and Y-coordinate y_1 has the Z-coordinate

$$z_1 = (k-1)m + [2y_1 - \sqrt{3}(m - t_k)]/(\sqrt{3}m). \quad (7)$$

In order to find the upper bound of γ , it is enough to consider those points v such that v , M_1 and a_k are collinear. From the following equation

$$[Z - (k-1)m]/[z_0 - (k-1)m] = Y/y_1,$$

we can get

$$\gamma = \beta[z_1 - (k-1)m]/y_0 + (k-1)m \quad (\text{By (7)})$$

$$= \beta\{[2y_1 - \sqrt{3}(m - t_k)]/(\sqrt{3}m)\}/y_1 + (k-1)m$$

$$= 2\beta/(\sqrt{3}m) - \beta(m - t_k)/(my_1) + (k-1)m$$

(By $\beta \leq m+1$ and formula (6))

$$\leq 2(m+1)/(\sqrt{3}m) + (m-t_k)\{[\beta + \sqrt{3}(\alpha-m)]/[\sqrt{3}m(m+s_k)]\} + (k-1)m$$

(By $\alpha, \beta \leq m+1$)

$$\leq 2 + (k-1)m + (m-t_k)(1+m+\sqrt{3})/[\sqrt{3}m(m+s_k)]$$

(By $0 \leq t_k < s_k \leq 1$)

$$\leq (k-1)m + 3.$$

Case 4. $(\alpha, \beta) \in E$

By the exactly same argument as that in Case 1, we can prove that $\gamma \leq (k-1)m + 7$.

Now, we can turn to the lower bound of γ .

Case 1. $(\alpha, \beta) \in A'$

The XY-projection of $E(va_k)$ intersects the XY-projection of $E(e_k f_k)$ at the point (x_0', y_0') , where

$$x_0' = [\beta m - \sqrt{3}t_k(\alpha - m)]/[\beta - \sqrt{3}(\alpha - m)] \quad (8)$$

and

$$y_0' = \sqrt{3}\beta(m - t_k)/[\beta - \sqrt{3}(\alpha - m)]. \quad (9)$$

The point M_0' on $E(e_k f_k)$ with X-coordinate x_0' and Y-coordinate y_0' has the Z-coordinate

$$z_0' = (k-1)m - 2 + [2y_0' - \sqrt{3}(m - s_k)]/(-\sqrt{3}m). \quad (10)$$

In this case, we only need consider those points v such that v, M_0' and a_k are collinear.

From the equation

$$[Z - (k-1)m]/[z_0' - (k-1)m] = Y/y_0',$$

we can get

$$\gamma \geq \beta[z_0' - (k-1)m]/y_0' + (k-1)m \quad (\text{By (10)})$$

$$= \beta\{[2y_0' - \sqrt{3}(m - s_k)]/(-\sqrt{3}m) - 2\}/y_0' + (k-1)m$$

$$= 2\beta/(-\sqrt{3}m) + \beta(m - t_k)/my_0' + (k-1)m - 2\beta/y_0'$$

(By $\beta \geq -m - 1$, formula (9))

$$\geq -2(m+1)/(\sqrt{3}m) + (m - s_k)\{[\beta - \sqrt{3}(\alpha - m)]/[\sqrt{3}m(m - t_k)]\} + (k-1)$$

(By $\beta \geq -m - 1$ and $0 \leq t_k < s_k \leq 1$)

$$\geq -2 + (k-1)m - (1 + m + \sqrt{3})/(\sqrt{3}m) - 2[m + 1 + \sqrt{3}(1 + 2m)]/[\sqrt{3}(m - t_k)]$$

$$\geq (k-1)m - 11.$$

Case 2. $(\alpha, \beta) \in B'$

To find the lower bound of γ , it is enough in this case to consider those points such that v , f_k , a_k and b_k are coplanar, the plane is

$$X + qY - mZ + (k-1)m^2 - m = 0 \quad (11)$$

where $p = (3m + 2t_k - s_k)/[\sqrt{3}(m - s_k)]$. Since v is on the plane (11), so

$$\gamma = \alpha/m + q\beta/m + (k-1)m - 1.$$

Hence,

$$\gamma = \alpha/m + p\beta/m + (k-1)m - 1 \quad (\text{By } \alpha, \beta \geq -m - 1)$$

$$\geq -(m+1)/m - (m+1)q/m + (k-1)m - 1 \quad (\text{By } 0 \leq t_k < s_k \leq 1)$$

$$\geq (k-1)m - 1 - 4(m+1)/m \geq (k-1)m - 6.$$

Case 3. $(\alpha, \beta) \in D'$

By the same argument as that in Case 1, we can prove that $\gamma \geq (k-1)m - 3$.

Case 4. $(\alpha, \beta) \in E'$

By the same argument as that in Case 2, we can prove that $\gamma \geq (k-1)m - 7$.

Therefore, we have proved that (1) is true. By symmetry, we know that if

$(\alpha, \beta) \in \Gamma$ and the point v sees the segment $c_k d_k$ or the segment $e_k f_k$, then

$$(k-1)m - 11 \leq \gamma \leq (k-1)m + 11$$

Finally, we can prove our theorem. To this end, let $a_k b_k$, $c_k d_k$ and $e_k f_k$ be the segments obtained from the segments $a_k' b_k'$, $c_k' d_k'$ and $e_k' f_k'$ by turning around the axes

$$\begin{cases} X=0 \\ Y=0 \\ Z=t \end{cases} \quad \begin{cases} X=-1/4 \\ Y=-\sqrt{3}/4 \\ Z=t \end{cases} \quad \begin{cases} X=1/4 \\ Y=-\sqrt{3}/4 \\ Z=t \end{cases} \quad -\infty < t < \infty$$

respectively, the same angle θ such that the XY-projection of $a_k b_k$ intersects the line

$$\begin{cases} X = m+1 \\ Y = t \\ Z = 0 \end{cases} \quad -\infty < t < \infty$$

at the point $(m+1, (k-1)(m+1)/(2n))$, where

$$(x_{a_k'}, y_{a_k'}, z_{a_k'}) = (m, 0, (k-1)m),$$

$$(x_{b_k'}, y_{b_k'}, z_{b_k'}) = (-m, 0, (k-1)m - 2),$$

$$(x_{c_k}, y_{c_k}, z_{c_k}) = (-m/2 - 1/4, \sqrt{3}(2m - 1)/4, (k - 1)m),$$

$$(x_{d_k}, y_{d_k}, z_{d_k}) = (m/2 - 1/4, -\sqrt{3}(2m + 1)/4, (k - 1)m - 2),$$

$$(x_{e_k}, y_{e_k}, z_{e_k}) = (-m/2 + 1/4, -\sqrt{3}(2m + 1)/4, (k - 1)m),$$

$$(x_{f_k}, y_{f_k}, z_{f_k}) = (m/2 + 1/4, \sqrt{3}(2m - 1)/4, (k - 1)m - 2),$$

$1 \leq k \leq n$. Assume that the XY-projection of $a_k b_k$ intersects the XY-projection of $c_k d_k$ at the point u_1 , assume that the XY-projection of $c_k d_k$ intersects the XY-projection of $e_k f_k$ at the point u_2 and that the XY-projection of $a_k b_k$ intersects the XY-projection of $e_k f_k$ at the point u_3 . Then, it is easy to show that the triangle decided by the three points u_1 , u_2 and u_3 is equiangular. Furthermore,

$$t_k = \{[\sqrt{3} + j_2(k)]\sqrt{[1 + j_1^2(k)]}\}/\{4[j_2(k) - j_1(k)]\},$$

$$s_k = \{[\sqrt{3} - j_3(k)]\sqrt{[1 + j_1^2(k)]}\}/\{4[j_2(k) - j_3(k)]\},$$

where $j_1(k) = (k - 1)/(2n)$,

$$j_2(k) = \tan\{\pi/3 + \arctan[(k - 1)/(2n)]\}$$

and

$$j_3(k) = \tan\{2\pi/3 + \arctan[(k - 1)/(2n)]\}.$$

Let $u_1 w_1$, $u_2 w_2$, $u_3 w_3$, $u_4 w_4$, $u_5 w_5$ and $u_6 w_6$ be the segments the coordinates of whose endpoints are

$$(X_{u_1}, Y_{u_1}, Z_{u_1}) = (0, \tau, nm),$$

$$(X_{w_1}, Y_{w_1}, Z_{w_1}) = (0, \tau, -m),$$

$$(X_{u_2}, Y_{u_2}, Z_{u_2}) = (0, -\tau, nm),$$

$$(X_{w_2}, Y_{w_2}, Z_{w_2}) = (0, -\tau, -m),$$

$$(X_{u_3}, Y_{u_3}, Z_{u_3}) = (-(1 + 2\tau\sqrt{3})/4, -(\sqrt{3} + 2\tau)/14, nm),$$

$$(X_{w_3}, Y_{w_3}, Z_{w_3}) = (-(3 + 6\tau\sqrt{3})/12, -(3\sqrt{3} + 6\tau)/12, -m),$$

$$(X_{u_4}, Y_{u_4}, Z_{u_4}) = (-(3 - 6\tau\sqrt{3})/12, -(3\sqrt{3} - 6\tau)/12, nm),$$

$$(X_{w_4}, Y_{w_4}, Z_{w_4}) = (-(3 - 6\tau\sqrt{3})/12, -(3\sqrt{3} - 6\tau)/12, -m),$$

$$(X_{u_5}, Y_{u_5}, Z_{u_5}) = ((3 + 6\tau\sqrt{3})/12, (3\sqrt{3} + 6\tau)/12, nm),$$

$$(X_{w_5}, Y_{w_5}, Z_{w_5}) = ((3 + 6\tau\sqrt{3})/12, (3\sqrt{3} + 6\tau)/12, -m),$$

$$(X_{u_6}, Y_{u_6}, Z_{u_6}) = ((3 - 6\tau\sqrt{3})/12, -(3\sqrt{3} - 6\tau)/12, nm),$$

$$(X_{w_6}, Y_{w_6}, Z_{w_6}) = ((3 - 6\tau\sqrt{3})/12, -(3\sqrt{3} - 6\tau)/12, -m),$$

respectively, where τ is a very small positive number.

Consider a point $v = (\alpha, \beta, \gamma)$. By what we proved before, if $(\alpha, \beta) \in \Gamma$ and the point v sees at least two segments, then the segments must be some of the segments $a_k b_k$, $c_k d_k$ and $e_k f_k$ for some k . Furthermore, by the construction, the point v can not see $a_k b_k$, $c_k d_k$ and $e_k f_k$ simultaneously. If $(\alpha, \beta) \notin \Gamma$ and the point v sees, say, $a_k b_k$, then, by the construction, if we choose the positive τ very small, either the Y -coordinate of the intersection point of the XY -projection of the segment vb_k and the line

$$\begin{cases} X = m+1 \\ Y = t & -\infty < t < \infty \\ Z = 0 \end{cases}$$

is between $(k-1)(m+1)/(2n) - (m+1)/(8n)$ and $(k-1)(m+1)/(2n) + (m+1)/(8n)$, or the Y -coordinate of the intersection point of the XY -projection of the segment va_k and the line

$$\begin{cases} X = -m - 1 \\ Y = t & -\infty < t < \infty \\ Z = 0 \end{cases}$$

is between $-(k - 1)(m + 1)/(2n) - (m + 1)/(8n)$ and $-(k - 1)(m + 1)/(2n) + (m + 1)/(8n)$. Thus, the point v sees at most one segment. That is, we have proved that any point v sees at most two segments and if it does see two segments, then the two segments must be two of the segments $a_k b_k$, $c_k d_k$ and $e_k f_k$ for some k . Hence, at least $2n$ points are needed to see all the $2n+6$ segments.

Q.E.D.

THEOREM 5.4.2 (W. P. Liu and I. Rival [1990]). *There is a "box" containing twelve segments, each of which is visible from a point of the box, such that twelve points on the box are necessary to see the segments.*

Proof. Let $a_1 b_1$, $a_2 b_2$, $a_3 b_3$, $a_9 b_9$, $a_{10} b_{10}$, $a_{11} b_{11}$ and $a_{12} b_{12}$ be the segments the coordinates of whose endpoints are

$$(x_{a_1}, y_{a_1}, z_{a_1}) = (m, 0, m+1),$$

$$(x_{b_1}, y_{b_1}, z_{b_1}) = (-m, 0, m-1),$$

$$(x_{a_2}, y_{a_2}, z_{a_2}) = (-(1+m)/2, \sqrt{3}(m-1)/2, m+1),$$

$$(x_{b_2}, y_{b_2}, z_{b_2}) = ((-1+m)/2, -\sqrt{3}(m+1)/2, m-1),$$

$$(x_{a_3}, y_{a_3}, z_{a_3}) = ((1-m)/2, -\sqrt{3}(m+1)/2, m+1),$$

$$(x_{b_3}, y_{b_3}, z_{b_3}) = ((1+m)/2, \sqrt{3}(m-1)/2, m-1),$$

$$(x_{a_{10}}, y_{a_{10}}, z_{a_{10}}) = (m, 0, 1),$$

$$(x_{b_{10}}, y_{b_{10}}, z_{b_{10}}) = (-m, 0, -1),$$

$$(x_{a_{11}}, y_{a_{11}}, z_{a_{11}}) = (-(1+m)/2, \sqrt{3}(m-1)/2, 1),$$

$$(x_{b_{11}}, y_{b_{11}}, z_{b_{11}}) = ((-1+m)/2, -\sqrt{3}(m+1)/2, -1),$$

$$(x_{a_{12}}, y_{a_{12}}, z_{a_{12}}) = ((1-m)/2, -\sqrt{3}(m+1)/2, 1),$$

$$(x_{b_{12}}, y_{b_{12}}, z_{b_{12}}) = ((1+m)/2, \sqrt{3}(m-1)/2, -1).$$

respectively. Let $a_k b_k$ be the segments passing through the points u_k and v_k with $z_{u_k} = 2m$ and $z_{v_k} = -m$, $4 \leq k \leq 9$, where

$$(x_{u_4}, y_{u_4}, z_{u_4}) = (1 + \tau/2, \sqrt{3}\tau/2, m - (1 + \tau)/m - \tau),$$

$$(x_{v_4}, y_{v_4}, z_{v_4}) = (1 - \tau, 0, (1 - \tau)/m + \tau),$$

$$(x_{u_5}, y_{u_5}, z_{u_5}) = (1 - \tau/2, -\sqrt{3}\tau/2, m - (1 - \tau)/m - \tau),$$

$$(x_{v_5}, y_{v_5}, z_{v_5}) = (1 + \tau, 0, (1 + \tau)/m + \tau),$$

$$(x_{u_6}, y_{u_6}, z_{u_6}) = (1 + \tau, 0, m - (1 + \tau)/m - \tau),$$

$$(x_{v_6}, y_{v_6}, z_{v_6}) = ((1 - \tau)/2, -\sqrt{3}\tau/2, (1 - \tau)/m + \tau),$$

$$(x_{u_7}, y_{u_7}, z_{u_7}) = (1 - \tau, 0, m - (1 - \tau)/m - \tau),$$

$$(x_{v_7}, y_{v_7}, z_{v_7}) = ((1 + \tau)/2, \sqrt{3}\tau/2, (1 + \tau)/m + \tau),$$

$$(x_{u_8}, y_{u_8}, z_{u_8}) = (\tau/2, -\sqrt{3}(2 + \tau)/2, m - (1 + \tau)/m - \tau),$$

$$(x_{v_8}, y_{v_8}, z_{v_8}) = (\tau/2, \sqrt{3}(-2 + \tau)/2, (1 - \tau)/m + \tau),$$

$$(x_{u_9}, y_{u_9}, z_{u_9}) = (-\tau/2, \sqrt{3}(-2 + \tau)/2, m - (1 - \tau)/m - \tau),$$

$$(x_{v_9}, y_{v_9}, z_{v_9}) = (-\tau/2, -\sqrt{3}(2 + \tau)/2, (1 + \tau)/m + \tau).$$

Consider the box whose boundary B consists of the following planes:

$$\pi_1: Y = m + 1/2,$$

$$\pi_2: Y = -m - 1/2,$$

$$\pi_3: X = m + 1/2,$$

$$\pi_4: X = -m - 1/2,$$

$$\pi_5: Z = 2m + 1/2,$$

$$\pi_6: Z = -m - 1/2.$$

We will show that any point $v = (\alpha, \beta, \gamma)$ on B sees at most one segment. By symmetry, there are only two cases to deal with: (1) v sees a_1b_1 and (2) v sees a_4b_4 .

Case 1. v sees a_1b_1

According to the fact that we proved in Theorem 1, v does not see $a_{10}b_{10}$, $a_{11}b_{11}$ and $a_{12}b_{12}$. Furthermore, by the construction of a_4b_4 and a_5b_5 (i.e. u_4 and u_5 are close to the point $(1, 0, -1/m + m)$, v_4 and v_5 to the point $(1, 0, 1/m)$, if $\tau > 0$ is close to 0), we know that if $\tau > 0$ is close to 0, then,

$$(\alpha, \beta, \gamma) \in \{(x, y, z): x = -m - 1/2, -1 \leq y \leq 1 \text{ and } m - 11 \leq z \leq m + 11\} \cup$$

$$\{(x, y, z): x = m + 1/2, -1 \leq y \leq 1 \text{ and } m - 11 \leq z \leq m + 11\}.$$

Without loss of generality, we may assume

$$(\alpha, \beta, \gamma) \in \{(x, y, z): x = m + 1/2, -1 \leq y \leq 1 \text{ and } m - 11 \leq z \leq m + 11\}.$$

So

$$T(a_5b_5, v) \cap a_3b_3 \neq \emptyset, \quad T(a_6b_6, v) \cap a_3b_3 \neq \emptyset, \quad T(a_7b_7, v) \cap a_3b_3 \neq \emptyset,$$

$$T(a_8b_8, v) \cap a_3b_3 \neq \emptyset, \quad T(a_9b_9, v) \cap a_3b_3 \neq \emptyset;$$

$$T(a_4b_4, v) \cap a_{12}b_{12} \neq \emptyset;$$

$$T(a_3b_3, v) \cap a_4b_4 \neq \emptyset;$$

$$T(a_2b_2, v) \cap a_9b_9 \neq \emptyset.$$

Hence, the point v sees only the segment a_1b_1 .

Case 2. the point v sees the segment a_4b_4

If the positive number τ is sufficiently small, then, either the point v is very close to the point $(1, 0, 2m + 1/2)$ with $0 < \beta < \sqrt{3}(\alpha - 1)$ or the point v is very close to the point $(1, 0, -m - 1/2)$ with $\sqrt{3}(\alpha - 1) < \beta < 0$. By the proof of Theorem 5.5.1, the point v does not see the segments $a_1b_1, a_2b_2, a_3b_3, a_{10}b_{10}, a_{11}b_{11}$ and a_4b_4 . Furthermore, the point v does not see the segment a_5b_5 . Similarly, if the point v sees the segment a_6b_6 or a_7b_7 , then v must be close to the point $(-1, 0, 2m + 1/2)$ or the point $(-1, 0, -m - 1/2)$; if v sees the segment a_8b_8 or a_9b_9 , then v is close to either the point $(0, -\sqrt{3}, 2m + 1/2)$ or the point $(0, -\sqrt{3}, -m - 1/2)$. So the point v does not see segments a_6b_6, a_7b_7, a_8b_8 and a_9b_9 . Namely, the point v sees only the segment a_4b_4 .

Q.E.D.

Remark. We can change the number 12 into $3n + 6$ in Theorem 5.4.2. The proof idea is the same as that in Theorem 5.4.2 but some more complicated calculation is needed.

CONCLUSION

In this thesis, we have considered four types of problems: problems related to inversions; problems related to planarity; problems related to enumerations and problems related to blocking relations in computational geometry.

In Chapter 2, we gave a new characterization of inversion, which, in turn, implies the first efficient algorithm to test if an ordered set is an inversion of another; we have also used this characterization to estimate the number of inversions. In Chapter 3, we investigated two important properties of ordered sets – planarity and s-genus. A characterization for planar bipartite ordered sets has been given, with its proof describing a way to draw a planar bipartite ordered set. An interesting counter example about s-genus was also given. In Chapter 4, we enumerated the orientations of a covering graph, which highlights why some problems in ordered set theory are so hard. Some related problems - independent set problem and matching problem have been discussed, too. In Chapter 5, we considered the so-called blocking relations and guarding problem. Although the proofs in this chapter are very long, they are very interesting. We also have listed some open problems, some of which may be easy or, at least, not too hard, others may be very hard.

In summary, here are the main results we have contributed.

Theorem 2.2.1. *An ordered set Q is an inversion of an ordered set P if and only if the reversed edges can be partitioned into cuts.*

Theorem 2.2.2. *There is an algorithm with complexity $O(|S|^5)$ to test whether an ordered set is an inversion of another.*

Theorem 3.2.9. *A bipartite ordered set is planar if and only if its covering graph is planar.*

Theorem 3.3.1. *Any n -element outerplanar covering graph has at least $2^{n/2}$ planar orientations.*

Theorem 3.3.3. *Any orientation of an outerplanar covering graph is planar provided the orientation is the shortest.*

Theorem 3.3.5. *There is a nonplanar ordered set with length five whose covering graph is outerplanar.*

Theorem 3.3.6. *Suppose G is an outerplanar covering graph. Then any independent set of G can be an antichain of a planar orientation of G .*

Theorem 3.4.8. *There is a planar covering graph G such that for any planar extension H of G , any orientation of H with top has no bottom.*

Theorem 3.4.14. *Any ordered set with planar covering graph has thickness at most two.*

Theorem 4.2.1. *There is a covering graph G some of whose independent sets cannot be the set of maximal elements of some orientation of G .*

Theorem 4.2.4. *There is an ordered set such that for some matching M and a subset M' of M , there is no reorientation which reverses the edges of M' and none of $M - M'$.*

Theorem 4.2.5. *Any independent set of a planar lattice can be the set of the maximal elements of a reorientation of the lattice.*

Theorem 4.3.1. *A directed graph Q constructed from the diagram of an ordered set P is itself a reorientation of P if and only if P has no irreversible cycle with respect to Q .*

Theorem 4.4.1. *Almost any n -element covering graph has at least $2^{n/3}$ orientations.*

Theorem 4.4.3. *Any n -element planar covering graph has at least $2^{n/3}$ orientations.*

Theorem 4.4.7. *Any n -element lattice with girth at least seven has at least $2^{n/3}$ orientations.*

Theorem 5.1.1. *Any n -element planar truncated lattice has at most $n(n-1)+1$ reorientations. Furthermore, the bound is tight.*

Theorem 5.2.1. *Let P be an n -element planar truncated lattice. Then*

$$\text{Area}(P) \leq n(n-1)/2;$$

moreover, the bound is best possible.

Theorem 5.2.3. *For any odd integer n there is an n -element truncated lattice P such that a segment representation R of P satisfies the following conditions:*

- 1) $\text{Area}(R, P) = \text{Area}(P)$;
- 2) $\text{Reor}(R, P) = O(n^2/\ln(n))$.

Theorem 5.3.1. *$n-2$ points ($n > 3$) on a (fixed) convex polygon containing n line segments are occasionally necessary and always sufficient to see the n segments.*

Theorem 5.4.1. *There are $3n + 6$ line segments in space such that at least $2n$ points are necessary to see the segments.*

Theorem 5.4.2. *There is a box containing twelve segments, each of which is visible from a point on the box, such that twelve points on the box are necessary to see the segments.*

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