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Polynomial Identities for Skew-Symmetric Matrices

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Faculty of Graduate and Postdoctoral Studies
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For the M.Sc. degree in Mathematics

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Table of Contents

Introduction

This thesis is concerned with polynomial identities (PI's) for K_n, the subspace of skew-symmetric matrices of M_n over an algebraically closed field F of characteristic 0. These PI's are polynomials in a finite number of non-commuting variables (i.e., elements in $F\langle x_1, x_2, ..., x_k\rangle$, the free associative algebra) that vanish for all specializations of the variables to elements of K_n.

A major starting point in the study of PI's is the work of Amitsur and Levitski. These authors have shown (in [AL]) that in the case of our field F (characteristic 0, algebraically closed) the *standard polynomial*

$$s_{2n} = \sum_{\sigma \in S_{2n}} (-1)^{\sigma} x_{\sigma(1)} x_{\sigma(2)} ... x_{\sigma(2n)},$$

where S_{2n} is the symmetric group on $2n$ elements, is a polynomial identity for M_n, that M_n has no polynomial identity of degree less than $2n$, and that any identity of M_n of degree $2n$ is a scalar multiple of s_{2n}. These standard identities appear to play a dominant role in the theory of PI's, but it has been found that there are limits to this role; namely, Amitsur has shown (in [A]) that s_{2n} does not generate $I(M_n)$ (the ideal of identities for M_n) as a T-ideal:

Definition. In an algebra R, a *T-ideal* I is an ideal of R such that $\phi(I) \subset I$ for all algebra endomorphisms ϕ of R.

In fact, only for M_2 do we have a complete characterization of the T-ideal of PI's, $I(M_2)$ (see [DG],[L1],[L2]).

Given an algebra over F with involution $(A, *)$, we may consider $*$-*polynomials*: elements of $F\langle x_1, x_2, ..., x_k, x_1^*, x_2^*, ..., x_k^*\rangle$. Of course, we call $*$-*polynomial identities* those polynomials which vanish for all specializations of the variables $x_1, ..., x_k$ to elements $a_1, ..., a_k$ of A, where if $x_i = a_i \in A$ then $x_i^* = a_i^*$. Because of the equations

$$x = \frac{1}{2}(x + x^*) + \frac{1}{2}(x - x^*)$$

and

$$x^* = \frac{1}{2}(x + x^*) - \frac{1}{2}(x - x^*)$$

we may think of an arbitrary $f \in F\langle x_1, x_2, ..., x_k, x_1^*, x_2^*, ..., x_k^*\rangle$ as a polynomial in skew and symmetric variables and write it as $f(x_1 + x_1^*, ... x_k + x_k^*, x_1 - x_1^*, ... x_k - x_k^*)$. We may therefore consider polynomials $f(x_1, ..., x_r, y_{r+1}, ..., y_s)$ with x_i a "skew-symmetric" variable for $1 \le i \le r$, and y_j a "symmetric" variable for $r \le j \le s$; i.e., when we substitute we must have

$$x_i = a_i \in K_n := \{a \in A | a^* = -a\}$$

and

$$y_j = b_j \in H_n := \{b \in A | b^* = b\}.$$

Over our algebraically closed field F, $M_n(F)$ has at most two involutions up to isomophism: the transpose involution t and the symplectic involution s (see [R1,Thm 3.1.62]). In the latter case n must be even. In both cases the minimal degree of a $*$-polynomial identity for M_n is unknown, but it has been shown by Giambruno in [G, Thm. 1] that for $n > 2$ this minimal degree must be greater than n.

Some results have been obtained for unmixed identities with $* = t$: identities in which the variables are all symmetric ($\in H_n$) or all skew ($\in K_n$) with respect to the transpose involution. It has been found that when only symmetric variables are considered, the minimal degree is $2n$ (see [SL]), and a full treatment of these identities of minimal degree has been given by Ma and Racine in [MR]. Analogous results for the strictly skew case have not yet been found, but significant progress has been made by Kostant in a theorem that bears his name. Following in the same vein as Amitsur and Levitski, he has shown (see [Kos1]) that s_{2n-2} is a PI for K_n, n even. Next, Rowen [R1] strengthened this result to include all n, and also showed that s_{2n-3} is not a PI for K_n. Finally, and of most relevance to this particular thesis, Racine and D'Amour (in [DR]) have described all PI's of minimal degree for $K_n, n < 5$. Our research begins where these last authors left off, at $n = 5$.

Though Racine and D'Amour found PI's of degree $2n - 3$, it was expected that this only occurs for low n, and that the minimal degree, in general, will be $2n - 2$. In this thesis we'll describe the progress we have made towards an understanding of $I(K_n)$, the space of PI's of K_n, $n \geq 5$. This research has a large computational aspect, so a full outline both of our programs and the results of those programs will be given. Perhaps most interesting among these results is the sheer volume of identities that we have found ($I_8(K_5)$, the space of multilinear identities of K_5 of degree 8, has dimension 1756). This explains why progress on the skew case has not kept up with the symmetric case. We will then introduce a family of PI's for K_n, n even, of degree $2n - 3$ that we have found and we will outline a method which we hope to apply towards proving that these are indeed PI's. In a final chapter we will give a partial treatment of PI's for K_5 involving only 2 variables.

1. Preliminaries

1.1. Multilinear Polynomials

Consider an element in $F\langle x_1, x_2, ..., x_k\rangle$. Such a polynomial is said to be *multilinear* if it is linear in each variable, and a simple argument shows that any multilinear polynomial is also *homogeneous*: all of its monomials are of the same overall degree and are of the same degree in each variable. Every such polynomial can be written as a linear combination of length $k!$:

$$\sum_{\sigma \in S_k} a_\sigma x_{\sigma(1)} x_{\sigma(2)} ... x_{\sigma(k)}, \quad a_\sigma \in F,$$

where S_k is the symmetric group on k elements.

We may completely linearize any non-multilinear polynomial. Namely, given a polynomial $g(y_1, ..., y_l) \in F\langle y_1, ..., y_l\rangle$, and assuming further that g is of degree $r > 1$ in y_1, we may look instead at the polynomial

$$g(x+y, y_2, ..., y_l) - g(x, y_2, ..., y_l) - g(y, y_2, ..., y_l).$$

This is a polynomial in $l+1$ variables with degree in x and y lower than the degree of g in y_1. Continuing in this way we obtain a multilinear polynomial.

Example. Let $g(x, y) = x^2 y \in F\langle x, y\rangle$, and by using the above method we get

$$g_1(x, y, z) := g(x+z, y) - g(x, y) - g(z, y) = (x+z)^2 y - x^2 y - z^2 y = xzy + zxy.$$

Moreover,

$$g_1(x, y, x) = 2x^2 y = 2g(x, y).$$

$\square$

A multilinear polynomial $p(x_1, x_2, ..., x_k)$ is said to be *alternating* in two variables, say x_i and x_j, if

$$p(x_1, x_2, ..., x_i, ..., x_j, ..., x_k) = -p(x_1, x_2, ..., x_j, ..., x_i, ..., x_k)$$

and it is said to be *symmetric* in those two variables if

$$p(x_1, x_2, ..., x_i, ..., x_j, ..., x_k) = p(x_1, x_2, ..., x_j, ..., x_i, ..., x_k).$$

It is possible for a polynomial to be alternating in more than two variables, say $x_{i_1}, x_{i_2}, ..., x_{i_r}$, and this is the case if for any $\sigma \in S_r$

$$p(x_1, x_2, ..., x_{\sigma(i_1)}, ..., x_{\sigma(i_2)}, ..., x_{\sigma(i_r)}, ..., x_k) = (-1)^\sigma p(x_1, x_2, ..., x_{i_1}, ..., x_{i_2}, ..., x_{i_r}, ..., x_k), \quad (1)$$

where $(-1)^\sigma$ denotes the sign of the permutation σ. Similarly, a polynomial $p(x_1, ..., x_k)$ is symmetric in $x_{i_1}, x_{i_2}, ..., x_{i_r}$ if for any $\sigma \in S_r$

$$p(x_1, x_2, ..., x_{\sigma(i_1)}, ..., x_{\sigma(i_2)}, ..., x_{\sigma(i_r)}, ..., x_k) = p(x_1, x_2, ..., x_{i_1}, ..., x_{i_2}, ..., x_{i_r}, ..., x_k).$$

Example. Let $p(x_1, x_2, x_3, x_4, x_5, x_6; y_1, y_2)$ be a multilinear polynomial that is alternating in the x_i's and symmetric in the y_j's, and assume further that the monomial $x_1x_2x_3x_4x_5x_6y_1y_2$ appears in p with coefficient 1. From (1) we see that the term

$$-x_1x_2x_3x_4x_6x_5y_1y_2$$

appears in p, and that, in general, the term

$$(-1)^\sigma x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)}x_{\sigma(4)}x_{\sigma(5)}x_{\sigma(6)}y_{\tau(1)}y_{\tau(2)}$$

appears in p for any $\sigma \in S_6$, $\tau \in S_2$. This implies that the alternating sum

$$\sum_{\substack{\sigma\in S_6\\ \tau\in S_2}} (-1)^\sigma x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)}x_{\sigma(4)}x_{\sigma(5)}x_{\sigma(6)}y_{\tau(1)}y_{\tau(2)}$$

appears in p, and that when this alternating sum, which is a polynomial alternating in 6 variables and symmetric in 2, is subtracted from p we are left with a polynomial that is still alternating in 6 variables and symmetric in 2. Continuing in this way we see that p is a linear combination of such alternating sums, with the y_i's appearing at various positions in the monomial.

A piece of notation useful for expressing these "sums of alternating sums" is that of a *pattern* (introduced in [V], though we use a slightly modified version). A *pattern* is a finite sequence of the letters A and B (e.g. ABBA), and we use these patterns, as functions, in the following manner:

$$ABBA(x_1, x_2; y_1, y_2) := x_1y_1y_2x_2$$

Let

$$P = \{BBAAAAAA, BABAAAAA, BAABAAAA, ..., AAAAAABB\}$$

(i.e., the set of all possible sequences for this case). We may now write p as a linear combination of alternating sums:

$$p(x_1, x_2, x_3, x_4, x_5, x_6; y_1, y_2) = \sum_{\pi\in P} a_\pi\Big(\sum_{\substack{\sigma\in S_6\\ \tau\in S_2}} (-1)^\sigma \pi(x_{\sigma(1)}, ..., x_{\sigma(6)}; y_{\tau(1)}, y_{\tau(2)})\Big),\ a_\pi \in F,$$

which, when the context is clear, we will shorten to

$$p(x_1, x_2, x_3, x_4, x_5, x_6; y_1, y_2) = \sum_{\pi\in P} a_\pi\Big(\sum_{\substack{\sigma\in S_6\\ \tau\in S_2}} (-1)^\sigma \pi\Big)$$

or, as is often convenient,

$$p(x_1, x_2, x_3, x_4, x_5, x_6; y_1, y_2) = \sum_{\substack{\sigma \in S_6 \\ \tau \in S_2}} (-1)^\sigma \sum_{\pi \in P} a_\pi \pi.$$

Now, it frequently happens that we would like to equate all of the y_j's. When we do so (i.e., when we let $y_1 = y_2 = y$ in the above), we will write the result as

$$p(x_1, x_2, x_3, x_4, x_5, x_6; y, y) = 2 \sum_{\sigma \in S_6} (-1)^\sigma \sum_{\pi \in P} a_\pi \pi.$$

□

Example. Let $p(x_1, x_2, x_3, x_4, x_5, x_6; y_1, y_2)$ be a multilinear polynomial that is symmetric in the x_i's and symmetric in the y_j's, and assume further that the monomial $x_1x_2x_3x_4x_5x_6y_1y_2$ appears in p with coefficient 1. Using (2) and proceeding as above we see that the sum

$$\sum_{\substack{\sigma \in S_6 \\ \tau \in S_2}} x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)}x_{\sigma(4)}x_{\sigma(5)}x_{\sigma(6)}y_{\tau(1)}y_{\tau(2)}$$

appears in p, and that p may be expressed as

$$p(x_1, x_2, x_3, x_4, x_5, x_6; y_1, y_2) = \sum_{\pi \in P} a_\pi \Big(\sum_{\substack{\sigma \in S_6 \\ \tau \in S_2}} \pi(x_{\sigma(1)}, ..., x_{\sigma(6)}; y_{\tau(1)}, y_{\tau(2)}) \Big), \quad a_\pi \in F.$$

If we now let all of the x_i's equal x, and let all of the y_j's be y, we get the polynomial

$$6!2! \sum_{\pi \in P} a_\pi \pi(x, x, x, x, x, x; y, y).$$

Since we are in characteristic 0 we may divide and obtain a 2-variable identity of the form

$$\sum_{\pi \in P} a_\pi \pi(x, x, x, x, x, x; y, y).$$

□

An identity need not be multilinear, but it is enough to consider polynomials of this type. To see this, consider first an arbitrary polynomial $p(x_1, ..., x_k)$. If p is a PI, then all of its homogeneous components are PI's as well (See [Kap, Lemma 4]), so we may assume that p is homogeneous. Next, p, since it may be multilinearized, gives rise to a multilinear identity, say $q(y_1, ..., y_l)$. We may equate appropriate y_j's in q to obtain a multiple of p, and then since we are in characteristic 0 we may divide to obtain p itself. Thus, p is a PI if and only if its q is a PI, and so, at least in characteristic

0, the polynomial identities of for some fixed subspace $V \subset M_n(F)$ are determined by the multlinear identities of V.

PI Spaces as S_k-Modules

Fix a subspace $H \subset M_n$, and let $I_k(H)$ be the space of all multilinear PI's for H of degree k (it is clear that this collection is indeed a vector space). Let $p \in I_k(H)$, say

$$p(x_1, x_2, ..., x_k) = \sum_{\sigma \in S_k} a_\sigma x_{\sigma(1)} x_{\sigma(2)} ... x_{\sigma(k)}, \quad a_\sigma \in F.$$

Now, for $\tau \in S_k$, we define

$$\begin{aligned} \tau(p(x_1, x_2, ..., x_k)) &:= p(x_{\tau(1)}, x_{\tau(2)}, ..., x_{\tau(k)}) \\ &= \sum_{\sigma \in S_k} a_\sigma x_{\sigma(\tau(1))} x_{\sigma(\tau(2))} ... x_{\sigma(\tau(k))} \\ &= \sum_{\sigma \in S_k} a_{\sigma\tau^{-1}} x_{\sigma(1)} x_{\sigma(2)} ... x_{\sigma(k)} \end{aligned} \tag{2}$$

where we are multiplying permutations right-to-left. Since $p(x_{\sigma(1)}, ..., x_{\sigma(k)}) \in I_k(H)$, this gives $I_k(H)$ an S_k-module structure.

1.2. 2-variable identities

When trying to prove that a given 2-variable polynomial is a PI, an obvious method is to substitute generic matrices for the variables and compute. Of course, this is often unmanageably complex, but in some cases we may make the method workable by a simplification of one of our variables. Traditionally, when working with the full matrix algebra (or symmetric matrices), one considers matrices with n distinct eigenvalues. In our particular case of PI's for K_n, we may assume that one of our variables is a *triagonal* element, and in this section we define triagonal elements and explain the validity of this assumption.

The Zariski Topology and Polynomial Maps

Let F be an algebraically closed field and consider F^n and $F[x_1, ..., x_n]$ (the polynomial algebra in commuting variables). For a subset $S \subset F[x_1, ..., x_n]$, define $V(S)$ to be the set of all n-tuples $(a_1, ..., a_n) \in F^n$ which satisfy $f(a_1, ..., a_n) = 0$ for all $f \in S$. We call $V(S)$ the *variety* defined by S. Now, a simple argument shows that these varieties satisfy the axioms for closed subsets of a topological space, and we have therefore obtained a topology on F^n with our open sets taking the form $F^n - V(S)$, for some $S \subset F[x_1, ..., x_n]$. This is called the *Zariski* topology.

We now define *polynomial maps* on F^n to be maps from F^n to F^m of the form

$$(a_1, ..., a_n) \mapsto (f_1(a_1, ..., a_n), ..., f_m(a_1, ..., a_n))$$

where $f_i \in F[x_1, ..., x_n]$, $1 \leq i \leq m$. These maps are continuous when both F^n and F^m are endowed with the Zariski topology (see [Jac2, p.429]), and hence to show that such a map is the zero map we need only show that it is zero on a Zariski-dense subset of F^n.

We may now reinterpret our PI's in terms of polynomial maps. Let $g(x_1, ..., x_k) \in F\langle x_1, ..., x_k\rangle$ (the free associative algebra) be a multilinear polynomial, and fix $k-1$ matrices in $K_n(F)$: $A_2, A_3, ..., A_k$. Now consider the map

$$\begin{aligned} P_{A_2,A_3,\ldots,A_k} : K_n &\to K_n \\ A &\mapsto g(A, A_2, A_3, ..., A_k). \end{aligned}$$

When we identify K_n with a subspace of F^{n^2} it is clear that a map of this form is a polynomial map from F^{n^2} to F^{n^2}, and thus to show that g is a PI it is enough to show that all such polynomial maps (for different choices for the A_j's) are zero on a Zariski-dense subset of K_n. This shows that when trying to prove by direct calculation that g is a PI for K_n we may replace one of the variables with a generic element from a Zariski-dense subset of K_n (and the rest by generic skews). In our case, the Zariski-dense subset will be the set of *triagonals.*

Definition. A *Jordan triple system* is a vector space V over a field F endowed with a quadratic map $P : V \to End_F(V)$ which satisfies

$$\begin{aligned} &(1) \quad L(x,y)P(x) = L(y,x)P(x) \\ &(2) \quad L(P(x)y, y) = L(x, P(y)x) \\ &(3) \quad P(P(x)y) = P(x)P(y)P(x) \end{aligned}$$

where $L : V \times V \to End_F(V)$ is defined by

$$L(x,y)z := P(x+z)y - P(x)y - P(z)y.$$

For a Jordan triple system V we define an ideal to be a subspace $W \subset V$ such that

$$P(V)W + P(W)V + L(W,V)V \subset W.$$

V is *simple* if it has no proper ideals and $L(V,V)V \neq 0$.

Example. With

$$P(x)y := -xyx,$$

K_n is a Jordan triple system. □

Definition. An element $e \in V$ satisfying $P(e)e = e$ is said to be a *tripotent* element. Two nonzero tripotents, say e, f, are said to be *orthogonal* if they satisfy

$$\begin{aligned} L(e,e)f &= 0 \\ L(f,f)e &= 0. \end{aligned}$$

For later use we note that in K_n the left hand sides of these equations simplify to

$$\begin{aligned} &-fe^2 - e^2 f \\ &-ef^2 - f^2 e. \end{aligned} \tag{1}$$

A collection of pairwise orthogonal tripotents, say $(e_1, ..., e_r)$ is said to be an *orthogonal system.*

From orthogonal systems we obtain the notion of a *frame.* We start with an orthogonal system and then refine it by either splitting elements into more "primitive" orthogonal tripotents (for example, f and g are taken instead of e, where $e = f + g$), or by adding tripotents to the system which obey the orthogonality condition. A frame is then defined to be an orthogonal system that cannot be further refined in this way. Since we are working over finite-dimensional vector spaces, any orthogonal system may be extended to a frame. The following proposition gives us a criteria under which an orthogonal system becomes a frame.

Proposition 1 ([N3, remarks before Frame Thm. p.2628-2629]. Let $(e_1, ..., e_r)$ be an orthogonal system in a Jordan triple system V and define $c = e_1 + ... + e_r$. $(e_1, ..., e_r)$ is a frame if and only if given a tripotent $f \in V$,

$$L(e_j, e_j)f = 2f \Rightarrow f = 0, \pm e_j, \quad 1 \leq j \leq r,$$

and

$$L(c, c)f = 0 \Rightarrow f = 0.$$

□

Example. For the Jordan triple system K_5 one may check that the following are tripotent elements:

$$e_1 := \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad e_2 := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

□

Proposition 2. (e_1, e_2) is a frame for K_5.

PROOF. We wish to apply Proposition 1. For $i = 1, 2$ and

$$f = \begin{pmatrix} 0 & f_{12} & f_{13} & f_{14} & f_{15} \\ -f_{12} & 0 & f_{23} & f_{24} & f_{25} \\ -f_{13} & -f_{23} & 0 & f_{34} & f_{35} \\ -f_{14} & -f_{24} & -f_{34} & 0 & f_{45} \\ -f_{15} & -f_{25} & -f_{35} & -f_{45} & 0 \end{pmatrix} \in K_5$$

an arbitrary tripotent,

$$L(e_i,e_i)f = 2f \Rightarrow -fe_i^2 - e_i^2 f = 2f \quad (from\ (1)).$$

But, $e_1^2 = diag\{-1,-1,0,0,0\}$ and $e_2^2 = diag\{0,0,-1,-1,0\}$, so these two equations become

$$\begin{pmatrix} 0 & 2f_{12} & f_{13} & f_{14} & f_{15} \\ -2f_{12} & 0 & f_{23} & f_{24} & f_{25} \\ -f_{13} & -f_{23} & 0 & 0 & 0 \\ -f_{14} & -f_{24} & 0 & 0 & 0 \\ -f_{15} & -f_{25} & 0 & 0 & 0 \end{pmatrix} = 2f, \quad \begin{pmatrix} 0 & 0 & f_{13} & f_{14} & 0 \\ 0 & 0 & f_{23} & f_{24} & 0 \\ -f_{13} & -f_{23} & 0 & 2f_{34} & f_{35} \\ -f_{14} & -f_{24} & -2f_{34} & 0 & f_{45} \\ 0 & 0 & -f_{35} & -f_{45} & 0 \end{pmatrix} = 2f.$$

Thus we see that in both cases f must be zero except possibly in the 2×2 block along the diagonal where e_i is non-zero. This implies that in the first case $f = f_{12}e_1$ and that in the second case that $f = f_{34}e_2$. But f is a tripotent so $P(f)f = f \Rightarrow -f^3 = f$, and this last condition requires that, respectively, $f_{12}^3 = f_{12}$, $f_{34}^3 = f_{34}$. Thus we have shown that in the first case $f_{12} = 0, \pm 1$ and in the second case $f_{34} = 0, \pm 1$, so the first condition of Proposition 1 is satisfied.

Next, consider $c = e_1 + e_2$ and again f, an arbitrary tripotent in K_5.

$$L(c,c)f = 0 \Rightarrow -c^2 f - fc^2 = 0 \ (from\ (1)),$$

but we have $c^2 = diag\{-1,-1,-1,-1,0\}$ which yields

$$\begin{pmatrix} 0 & 2f_{12} & 2f_{13} & 2f_{14} & f_{15} \\ -2f_{12} & 0 & 2f_{23} & 2f_{24} & f_{25} \\ -2f_{13} & -2f_{23} & 0 & 2f_{34} & f_{35} \\ -2f_{14} & -2f_{24} & -2f_{34} & 0 & f_{45} \\ -f_{15} & -f_{25} & -f_{35} & -f_{45} & 0 \end{pmatrix} = 0 \Rightarrow f = 0.$$

Thus the second condition of Proposition 1 is satisfied. □

Our next step will be to show that we can use this frame to prove identities, but first we will need some definitions.

Definition. A group G is said to be a *topological group* if it is a topological space and if both the group multiplication $G \times G \to G$ and the inverse operation $G \to G$ are continuous. We denote by G^0 the connected component of G that includes the identity, and we call it the *identity component.*

Proposition 3 (Private communication, E. Neher). Let H be the group of automorphisms of K_n as a Jordan triple system. Then H is a topological group and H^0 is induced by conjugation with special orthogonal matrices, i.e., conjugation by matrices A ($x \mapsto AxA^{-1}$) of determinant 1 such that $AA^t = A^tA = I_n$. □

Theorem 1 ([N4, Thm. 2.15]). Let V be a simple Jordan triple system over an algebraically closed field F, and let H be the group of automorphisms of V. Any two frames, say $(e_1, ..., e_r)$ and $(f_1, ..., f_s)$, are conjugate: there exists $t \in H^0$ with

$$t(Fe_1 + .. + Fe_r) = Ff_1 + ... + Ff_s$$

and in particular $r = s$, and

$$t((e_1, ..., e_r)) = (\pm f_1, ..., \pm f_s).$$

□

By ([N2, lemma 1.3, chapter 4]) we have that the only ideals for $K_5(F)$ are subspaces of the form $K_5(E)$ for E an ideal of F, so K_5 does not have any nontrivial ideals. A simple calculation shows that $L(K_5, K_5)K_5 \neq 0$, and thus $K_5(F)$ is simple, and we may make use of the theorem.

To do so, we first define a *triagonal* element to be a F-linear combination of orthogonal tripotents. Now, given an arbitrary triagonal element in K_5, we take the set of tripotents involved in its sum (an orthogonal system) and extend it to a frame. By the theorem, this frame is conjugate to (e_1, e_2).

Neher has proven the following theorem:

Theorem 2 ([N1, Thm. 1.4]). In a simple, finite-dimensional Jordan triple system over an algebraically closed field F of characteristic not 2, the set of triagonal elements is Zariski-dense. □

Applying the above arguments we may, when attempting to prove directly that a given polynomial is a PI for K_n, replace one of our variables with a generic triagonal matrix and the remaining variables with generic skew matrices. We then apply to this polynomial our automorphism $t \in H^0$, and, since t is merely conjugation by a matrix (and thus preserves the associative product on K_n), we obtain for our generic triagonal matrix a computationally optimal form:

$$\begin{pmatrix} 0 & d_1 & 0 & 0 & 0 \\ -d_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & d_2 & 0 \\ 0 & 0 & -d_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \end{pmatrix}, \quad d_1, d_2 \in F.$$

Lie Polynomials

That we are able to make progress by interpreting K_n as a Jordan triple system leads us to ask whether similar progress can be made by considering the Lie properties of K_n (it is of classical type B or D, depending on the parity of n). We thus consider *Lie polynomials*:

Definition. An element $f(x_1, ..., x_k) \in F\langle x_1, ..., x_k\rangle$ is said to be a Lie polynomial if it may be written

$$f(x_1, ..., x_k) = \sum_{\sigma \in S_k} a_\sigma[[...[x_{\sigma(1)}, x_{\sigma(2)}], x_{\sigma(3)}], ...], x_{\sigma(k)}], \quad a_\sigma \in F,$$

where $[x, y] := xy - yx$.

Now, not much is known about Lie polynomials (though some notable results have been obtained by Bahturin (see [B])), and in our experience the Jordan approach has born more fruit; however, one Lie polynomial did arise and we will make note of it below.

1.3. The Pfaffian

The Cayley-Hamilton theorem tells us that an $n \times n$ matrix satisfies its degree n characteristic polynomial. This polynomial may be used to derive PI's, but in the skew case we can actually do better. What's been found is that for $2m \times 2m$ skew-symmetric matrices, $m \in \mathbf{N}$, the determinant turns out to be a perfect square, and we may hence consider its square root. We call one of these square roots the *Pfaffian*, and we work with this to derive the *generic minimal polynomial*: a polynomial of degree m analagous to the characteristic polynomial.

The Generic Minimal Polynomial

Proposition([R2,Prop 2.5.8]). For $r \in K_{2m}(F[\lambda])$, $\det(r)$ is a perfect square in $F[\lambda]$.

Definition. For $r \in K_{2m}(F[\lambda])$, define Pf(r) to be the square root of det(r) satisfying Pf(a)=1 with

$$a = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}.$$

From this we may derive a polynomial analogous to the characteristic polynomial, but first we need one final definition.

Definition. The symplectic involution on $2m \times 2m$ matrices is defined by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^s = \begin{pmatrix} D^t & -B^t \\ -C^t & A^t \end{pmatrix},$$

where the matrix has been divided into $m \times m$ blocks.

Theorem 1 ([R2, Thm 2.5.10]). For $x \in M_n(F)$ satisfying $x^s = x$ and $b := \sum_{i=1}^{m}(e_{i,i+m} - e_{i+m,i})$, $(\lambda - x)b$ is skew-symmetric and the polynomial $p(\lambda) := Pf((\lambda - x)b) \in F[\lambda]$ has x as a root. □

We call $p(\lambda)$ the *generic minimal polynomial for x*. Now, let A be an invertible element in $K_n(F)$, and notice that the map $* : x \mapsto Ax^tA^{-1}$ is an involution of $M_n(F)$. It is well-known that involutions (of the first kind) of $M_n(F)$ are either of symplectic (s) or orthogonal (t) type, and it turns out that this particular involution is symplectic. This we may see by calculating the dimension of the space of symmetric elements and noticing that it matches that of the symplectic case not the orthogonal case. What this means for us is summed up in the following theorem.

Theorem 2 ([R2, Thm 3.1.61]). Over an algebraically closed field F of characteristic 0, $(M_n(F), *) \simeq (M_n(F), s)$ (as algebras with involution). □

This is actually a special case of the theorem proved in [R2]. Putting this all together, we see that an element x with $x^* = x$ will satisfy, from Theorem 1, $p(x) = 0$.

1.4. Computations

We are looking for multilinear PI's for K_5, and we did this in what we call here the "Main Computation". Afterwards, we performed computations that were motivated by the results of the "Main", such as computing characters or investigating specific identity types. This section outlines these computations, and then concludes with a quick note about an application of probability to "substitution" computations.

Main Computation

As mentioned above, a multilinear identity of degree k for a subspace $H \subset M_n$ may be written

$$\sum_{\sigma \in S_k} a_\sigma x_{\sigma(1)} x_{\sigma(2)} ... x_{\sigma(k)}, \quad a_\sigma \in F,$$

so we assume this form and perform matrix substitutions to obtain conditions on the a_σ. A substitution here is a choice of k matrices, $A_1, ..., A_k \in H$, that are substituted in for the x_i's, and the result of such a substitution is a matrix condition involving $n \times n$ matrices. From this matrix condition we can extract n^2 scalar conditions. We illustrate all of this with the following example.

Example. If we were to perform the substitution in which all of the x_i are I_5, the 5×5 identity matrix, we would obtain the matrix condition

$$(\sum_{\sigma \in S_k} a_\sigma) I_5 = 0$$

which yields 20 null conditions $(0 = 0)$ and 5 copies of the condition

$$\sum_{\sigma \in S_k} a_\sigma = 0.$$

□

In our computation we were looking for PI's for K_n, so the substitutions were taken from the canonical basis elements for K_n,

$$e_{ij} - e_{ji}, \quad 1 \leq i < j \leq n, \quad i \neq j.$$

These were performed in either a lexicographic or random order depending on whether completeness, or respectively, speed was our main concern. In other words, we would either count our way through all possible choices for the x_i, or take random choices. Once a condition was obtained it was converted into a matrix row of length k! and appended to a master matrix. When a suitable number of rows had been added, the master was row-reduced, and substitutions resumed. This process looped until the master's rank stabilized (at either k! rows, indicating that the identity space was zero, or at some number of rows less than k!) or we ran out of substitutions to perform. Finally, the master's null space was computed, yielding a basis for the space of multilinear identities of K_n of degree k. This computation was programmed in C++ and ran on HPCVL (High Perfomance Computing Virtual Laboratory).

For polynomial degree less than 8, we used infinite precision integer arithmetic. It was not always possible to divide our rows by GCDs, so our entries grew rather quickly into integers with hundreds and even thousands of digits. By the end of the process the numbers came back down, but this intermediate explosion still required, at higher degrees, far too much memory, and for this reason degree 8 was performed over prime fields, $p = 199$ and $p = 211$.

The next task was to convert the prime-field PI's to PI's over F, and we accomplished this with the following algorithm. Let C_{199} be the matrix (the null space) generated by the $p = 199$ computation, and let C_{211} be the matrix generated by the $p = 211$ computation. By inspection, we noticed that the rows corresponded directly (i.e., the first row of C_{199} had non-zero entries in the same positions as those in the first row of C_{211}), and to illustrate we'll take the first row of C_{199} and the first row of C_{211} and compare them. Now assume that in one of these non-zero entries the former row has an 87 and latter row has a 93. We search for a fraction $i/j \in \mathbf{Q}$ such that

$$ij^{-1} \equiv 87 \quad (mod\ 199)$$

and

$$ij^{-1} \equiv 93 \quad (mod\ 211).$$

This search proceeds in a direct fashion: we search through fractions with numerators between $-M$ and M, and denominators between 1 and N, and, in our case, it was necessary to choose $M > 500$ and $N > 20$.

In this case the fraction is found fairly quickly, and it is $-25/2$:

$$(-25)(2)^{-1} \equiv (-25)(100) \equiv 87 \pmod{199}$$

$$(-25)(2)^{-1} \equiv (-25)(106) \equiv 93 \pmod{211}$$

Continuing in this way, we convert all of the other coefficients, and then test, via a substitution, whether our new polynomial is a PI or not(see below for more on PI testing by computation).

A final step in this computation was to determine whether our polynomials were alternating or symmetric in any variables. This was achieved via a direct application of equation 1.1.1, and the equation following it, and we simply looked to see which polynomials had variables that satisfied these equations. It was found that many of the PI's found here do exhibit symmetric/alternating behaviour, and our next computation took a closer look at some of these.

The 2-Symmetric, (2n-4)-Alternating Computation

The main computation yields PI's for K_5 that are symmetric in 2 variables and alternating in the remaining 6 variables, and the following computation was pursued in order to determine whether there are any more polynomials of this type.

Recall the definition of a pattern (e.g. ABBA). To illustrate our method we'll look at the case of 6 A's and 2 B's. Let

$$P = \{BBAAAAAA, BABAAAAA, BAABAAAA, ..., AAAAAABB\}$$

(i.e., all possible sequences for this case). We assume a polynomial identity of the form

$$\sum_{\pi \in P} a_\pi \Big(\sum_{\substack{\sigma \in S_6 \\ \tau \in S_2}} (-1)^\sigma \pi(x_{\sigma(1)}, ..., x_{\sigma(6)}; y_{\tau(1)}, y_{\tau(2)}) \Big)$$

and as mentioned in the introduction, a multilinear identity that is symmetric in two variables and alternating in the remaining 6 variables must be expressible in this fashion. Now, as in the main computation, we perform substitutions to get conditions on the a_π. There is, however, an important difference: in the main computation the coefficients were coefficients for monomials, and here they are coefficients for polynomials.

For spaces of polynomials of lower degrees (< 9) like this one, our computation was performed more or less directly. We first generated the $\binom{8}{2} = 28$ polynomials (one for each $\pi \in P$) of the form

$$\sum_{\substack{\sigma \in S_6 \\ \tau \in S_2}} (-1)^\sigma \pi(x_{\sigma(1)}, ..., x_{\sigma(6)}; y_{\tau(1)}, y_{\tau(2)})$$

and then made a substitution, $A_1, ..., A_8 \in K_5$, into each polynomial. The result was 28 matrices, call them the A_π, and hence a matrix condition:

$$\sum_{\pi \in P} a_\pi A_\pi$$

which yielded 25 scalar conditions on the a_π.

For polynomials of higher degrees $(9, 10)$, memory constraints came into play, and it was found that we could only fit one polynomial into memory at a time. Therefore, a given substitution was performed by computing each of the A_π one at a time (and thus requiring only one polynomial in memory at a time), and as result, spending more time generating, discarding, and regenerating the polynomials.

This computation was programmed in Mathematica (see Appendix A for a sample program) and ran on a desktop computer.

The 2-Variable Computation

The main computation yields polynomials that are symmetric in 2 variables and also symmetric in the remaining variables, and, as noted in the introduction, we can use these k-variable identities to obtain 2-variable identities that are of degree 2 in one variable and degree $k - 2$ in the other. This led us to look at 2-variable identities in general: identies that are of degree l in one variable and of degree $k - l$ in the other variable.

To illustrate the method, we will use patterns and take the case of 6 A's and 2 B's. Let P be as above, and assume a polynomial of the form

$$\sum_{\pi \in P} a_\pi \pi(x, x, x, x, x, x; y, y).$$

Just as in the previous computations, we now perform substitutions, obtain conditions on the a_π, and row-reduce. These PI's form a vector space over F, and this computation gives us a basis.

This computation was programmed in Mathematica (see Appendix B for a sample program) and ran on a desktop computer.

The Character Computation

We recall the definition of a character:

Definition. Given a representation ρ of a group G, i.e. a group homomorphism $G \to GL(V)$ where V is a finite dimensional vector space over a field F, we define the *character* of this representation to be the map

$$\chi : G \to F$$
$$g \mapsto Tr(\rho(g))$$

where Tr is the usual matrix trace. □

$I_8(K_5)$ is a S_8-module, so we may compute its character. It is, in our case, a F-valued trace function on S_8 that is constant on the conjugacy classes. S_8 has 22 conjugacy classes, so we take a permutation from each: $\tau_1, \tau_2, ..., \tau_{22}$. We then take the matrix found in the main computation, say C, and interpret its rows ($c_1, c_2, ..., c_d$, where $d = 1756$ is the number of rows in C) as basis elements for $I_8(K_5)$, our space of identities.

We can now get the required traces,

$$Tr(\tau_i|_{I_8(K_5)}), 1 \leq i \leq 22,$$

as follows. We take the polynomial corresponding to each c_i and act on it with τ_1 as in (2) to get $\tau_1(c_i)$, a row vector. It is important to note that we are just rearranging the coefficients of these c_i, and that for all of the PI's that we are dealing with, an order of the monomials is fixed. Now we form the matrix

$$[\, c_1{}^t \quad c_2{}^t \quad ... \quad c_d{}^t \quad \tau_1(c_1)^t \quad \tau_1(c_2)^t \quad ... \quad \tau_1(c_d)^t \,]$$

which we row-reduce. What we are in effect doing here is expressing each $\tau_1(c_i)$ as a linear combination of the c_i's. The last step is to take the trace of the upper right $d \times d$ sub-matrix (this sub-matrix is $\tau_1|_{I_8(K_5)}$) , and this trace, call it t_1, is the value of the character on τ_1's conjugacy class. We proceed in this fashion until we have all of the t_i's, and these we put into a vector $\mathbf{t} = (t_1, t_2, ..., t_{22})$ which represents our character.

Now we would like to express this character as a linear combination of S_8's irreducible characters $\chi_1, \chi_2, ..., \chi_{22}$ (see [JK, p.351]). This we do by forming the matrix

$$[\, \chi_1^t \quad \chi_2^t \quad ... \quad \chi_{22}^t \quad \mathbf{t}^t \,]$$

and row-reducing.

A Note on Probability and PI's

When trying to decide whether a given polynomial is a PI, an obvious first step is to perform random substitutions and see whether the resulting expression evaluates to 0. Of course, if we instead find that the result is nonzero, then what we have is not a PI; but if all of our substitutions yield 0, and we perform enough of them, then it is "likely" that the polynomial is indeed a PI. We may make this more precise, and we start by quoting a result from [S] on polynomials in commuting variables.

Theorem([S, Cor.1]). *Let $Q(x_1, ..., x_m) \in F[x_1, ..., x_m]$ be a nonzero polynomial of total degree d over a field F and let S be a finite subset of F. Then the proportion of m-tuples $(a_1, ..., a_m) \in S^m$ such that $Q(a_1, ..., a_m) = 0$ is at most $d/|S|$.*

This theorem does not apply directly to our non-commutative case, but we may make use of it as follows. Consider a multilinear polynomial

$p(x_1, ..., x_k) \in F\langle x_1, x_2, ..., x_k\rangle$, and assume that this is not a PI for K_n. What is the probability that this polynomial will vanish for a given random substitution from K_n? To answer this question we replace the x_i's with generic matrices (i.e., matrices whose entries are $n(n-1)/2$ independent indeterminates). We may now evaluate $p(x_1, ..., x_k)$, and the result is a matrix whose n^2 entries are all polynomials in the $kn(n-1)/2$ commuting indeterminates. If p is not a PI then at least one of these polynomial entries, say h, is nonzero and the above theorem applies.

If we now pick our substitution to come from matrices with integer entries in the interval $[-50k, 50k]$, and hence our $kn(n-1)/2$ indeterminates from the interval $[-50k, 50k]$, we see that the probability that a given random substitution evaluates to 0 (i.e., that h evaluates to 0) is at most $d/(100k+1) = k/(100k+1) < 1/100$. Thus, if just one such substitution evaluates to 0, it is very likely that our original assumption (that p is not a PI) was wrong, and that in fact p is a PI. For future reference, we summarize this in the following lemma.

Lemma 1. Let $f \in F\langle x_1, ..., x_k\rangle$ and let $X_1, ..., X_k \in K_n(S)$, where S is the set of integers in the interval $[-50k, 50k]$. Assuming that f is not an identity for K_n, the probability that $f(X_1, ..., X_k) = 0$ is less than $\frac{1}{100}$. □

Thus, if t random substitutions are performed and all yield 0, the probability that what we have is not a PI is strictly less than $(\frac{1}{100})^t$. In other words: extremely unlikely. This argument was suggested to us by John Dixon.

2. Identities

Our main computation, in which we were looking for multilinear PI's for K_5, yielded nothing for degrees less than 8. Therefore the minimal degree of a PI for K_5 is at least 8 and, since $s_8 \in I(K_5)$ by the result of Kostant and Rowen, the minimal degree is 8. For degree 8 our main computation yielded a space of multilinear identities with dimension 1756 and character

$$\begin{aligned}\chi = &\chi_3 + \chi_5 + \chi_6 + \chi_8 + 2\chi_9 + 4\chi_{10} + 4\chi_{11} + 3\chi_{12} + \chi_{13} + \chi_{14} + 3\chi_{15} \\ &+ 4\chi_{16} + 4\chi_{17} + \chi_{18} + 3\chi_{19} + 2\chi_{20} + 2\chi_{21} + \chi_{22},\end{aligned}$$

where the χ_i's are the irreducible characters of S_8 (χ_1 the trivial character, χ_{22} the alternating character) as found in [JK, p351]. The computation of this character, like the main computation, was performed over two prime fields ($p = 199, 211$); and as expected, both computations gave the same result. Also, it should be noted that, again as expected, the spaces found in the two main computations ($p = 199, 211$) both contained s_8.

It was soon found that several of the 1756 polynomials are symmetric in 2 variables and alternating in the remaining 6, and it is these that we will discuss first.

2.1. 2-Symmetric, (2n-5)-Alternating Identities

From the main computation we obtained a PI for K_5 alternating in 6 variables and symmetric in 2. This led us to look at PI's for $K_n, n = 4, 5, 6$, alternating in $2n-4$ variables and symmetric in 2. We performed the 2-Symmetric, $(2n-4)$-Alternating Computation and found spaces of dimension 6 in all cases. This computation also yielded bases for all three spaces, and the bases for K_4 and K_6 were noticeably similar. In particular, they both contain a polynomial in which only the coefficients 0,1 and -1 appear, and it was found that the former, say $\kappa_4(x_1, x_2, x_3, x_4; y_1, y_1)$, can be written

$$\begin{aligned}\sum_{\substack{\sigma\in S_4\\ \tau\in S_2}} (-1)^\sigma(- ABAABA + ABBAAA + AABABA \\ - AABBAA + BAABAA - BABAAA)\end{aligned}$$

and the latter, say $\kappa_6(x_1, ..., x_8; y_1, y_1)$, can be written

$$\begin{aligned}\sum_{\substack{\sigma\in S_8\\ \tau\in S_2}} (-1)^\sigma(&- AAAAAABBAA - AAAAABAABA - AAAABABAAA \\ &+ AAAAAABABA + AAAAABBAAA + AAAABAABAA \\ &- AABAAAABAA - ABAAAAAABA - BAAAAABAAA \\ &+ AABAAAAABA + ABAAAABAAA + BAAAAAABAA \\ &- AABBAAAAAA - ABAABAAAAA - BABAAAAAAA \\ &+ AABABAAAAA + ABBAAAAAAA + BAABAAAAAA).\end{aligned}$$

In [DR] the polynomial

$$\kappa(x_1,x_2,x_3;y)=\sum_{\sigma\in S_3}(-1)^\sigma(\{\{y,y,x_{\sigma(1)}\},x_{\sigma(2)},x_{\sigma(3)}\}-\{\{y,x_{\sigma(1)},x_{\sigma(2)}\}x_{\sigma(3)},y\}$$
$$-\{x_{\sigma(1)}\{y,x_{\sigma(2)},x_{\sigma(3)}\},y\})$$

where the brackets denote the triple product $\{a,b,c\}=abc+cba$, is shown to be a PI for K_4. It turns out that κ_4 is a consequence of κ, and this we may see by first expanding κ's triple products to get

$$\kappa=\sum_{\sigma\in S_3}(-1)^\sigma(-x_{\sigma(1)}x_{\sigma(3)}x_{\sigma(2)}yy-x_{\sigma(1)}yx_{\sigma(2)}x_{\sigma(3)}y+x_{\sigma(1)}yyx_{\sigma(2)}x_{\sigma(3)}$$
$$-x_{\sigma(2)}x_{\sigma(1)}yx_{\sigma(3)}y+x_{\sigma(3)}x_{\sigma(2)}x_{\sigma(1)}yy+x_{\sigma(3)}x_{\sigma(2)}yyx_{\sigma(1)}$$
$$-yx_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)}y-yx_{\sigma(3)}x_{\sigma(2)}x_{\sigma(1)}y-yx_{\sigma(3)}x_{\sigma(2)}yx_{\sigma(1)}$$
$$-yx_{\sigma(3)}yx_{\sigma(1)}x_{\sigma(2)}+yyx_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)}-yyx_{\sigma(2)}x_{\sigma(3)}x_{\sigma(1)})$$

and now, since this sum is alternating in x_1, x_2, and x_3, we may rearrange the $x_{\sigma(*)}$'s (while keeping track of signs of permutations) to obtain

$$\kappa=\sum_{\sigma\in S_3}(-1)^\sigma(+x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)}yy-x_{\sigma(1)}yx_{\sigma(2)}x_{\sigma(3)}y+x_{\sigma(1)}yyx_{\sigma(2)}x_{\sigma(3)}$$
$$+x_{\sigma(1)}x_{\sigma(2)}yx_{\sigma(3)}y-x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)}yy-x_{\sigma(1)}x_{\sigma(2)}yyx_{\sigma(3)}$$
$$-yx_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)}y+yx_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)}y+yx_{\sigma(1)}x_{\sigma(2)}yx_{\sigma(3)}$$
$$-yx_{\sigma(1)}yx_{\sigma(2)}x_{\sigma(3)}+yyx_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)}-yyx_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)}).$$

In this expression we may cancel terms of the form $x_*x_*x_*yy, yyx_*x_*x_*$, and $yx_*x_*x_*y$ to get

$$\kappa=\sum_{\sigma\in S_3}(-1)^\sigma(-x_{\sigma(1)}yx_{\sigma(2)}x_{\sigma(3)}y+x_{\sigma(1)}yyx_{\sigma(2)}x_{\sigma(3)}+x_{\sigma(1)}x_{\sigma(2)}yx_{\sigma(3)}y$$
$$-x_{\sigma(1)}x_{\sigma(2)}yyx_{\sigma(3)}+yx_{\sigma(1)}x_{\sigma(2)}yx_{\sigma(3)}-yx_{\sigma(1)}yx_{\sigma(2)}x_{\sigma(3)}).$$

Linearizing in y yields

$$\kappa(x_1,x_2,x_3;y_1,y_2)=$$
$$\sum_{\substack{\sigma\in S_3\\ \tau\in S_2}}(-1)^\sigma(-x_{\sigma(1)}y_{\tau(1)}x_{\sigma(2)}x_{\sigma(3)}y_{\tau(2)}+x_{\sigma(1)}y_{\tau(1)}y_{\tau(2)}x_{\sigma(2)}x_{\sigma(3)}$$
$$+x_{\sigma(1)}x_{\sigma(2)}y_{\tau(1)}x_{\sigma(3)}y_{\tau(2)}-x_{\sigma(1)}x_{\sigma(2)}y_{\tau(1)}y_{\tau(2)}x_{\sigma(3)}$$
$$+y_{\tau(1)}x_{\sigma(1)}x_{\sigma(2)}y_{\tau(2)}x_{\sigma(3)}-y_{\tau(1)}x_{\sigma(1)}y_{\tau(2)}x_{\sigma(2)}x_{\sigma(3)})$$

which, using our pattern notation, is

$$\sum_{\substack{\sigma\in S_3\\ \tau\in S_2}}(-1)^\sigma(-ABAAB+ABBAA+AABAB-AABBA+BAABA-BABAA).$$

This is κ_4 with "an A pulled off of the right," and we may write κ_4 as

$$\kappa(x_1,x_2,x_3;y_1,y_2)x_4-\kappa(x_1,x_2,x_4;y_1,y_2)x_3 \\ +\kappa(x_1,x_3,x_4;y_1,y_2)x_2-\kappa(x_2,x_3,x_4;y_1,y_2)x_1,$$

thus showing that $\kappa_4\in I(K_4)$ is a consequence of $\kappa\in I(K_4)$.

This correspondence between κ_4 and κ led us to ask if we could "take an A off of the right" of κ_6 to obtain an PI for K_6 of degree $2n-3=2(6)-3=9$. This we did, and it turns out that the polynomial we obtain,

$$\begin{aligned}\sum_{\substack{\sigma\in S_8\\ \tau\in S_2}}(-1)^\sigma(&-AAAAAABBA-AAAAABAAB-AAAABABAA\\ &+AAAAAABAB+AAAAABBAA+AAAABAABA\\ &-AABAAAABA-ABAAAAAAB-BAAAAABAA\\ &+AABAAAAAB+ABAAAABAA+BAAAAAABA\\ &-AABBAAAAA-ABAABAAAA-BABAAAAAA\\ &+AABABAAAA+ABBAAAAAA+BAABAAAAA),\end{aligned}$$

is, with a high degree of probability (see Lemma 1.4.1), a PI of degree $2n-3$ for K_6. This indicates that the minimal degree for a PI of K_6 is at most 9.

The similarity between κ and this last polynomial led us to wonder whether these two were part of some larger family of PI's. A family was found but it requires that we let the "last x_j", y_1, and y_2 in each of these polynomials be y. For example, instead of $\kappa(x_1,x_2,x_3;y_1,y_2)$ we work with $\kappa(x_1,x_2,y;y,y)$. When we do this we obtain the following:

$$s_3(y^2,y,x_1)\in I(K_3)$$

$$s_4(y^2,y,x_1,x_2)\in I(K_4)$$

$$s_7(y^2,y,x_1,\ldots,x_5)+\frac{1}{2}\sum_{\sigma\in S_6}s_3(x_{\sigma(1)},x_{\sigma(2)},yx_{\sigma(3)}x_{\sigma(4)}x_{\sigma(5)}x_{\sigma(6)}y)|_{x_6=y}\in I(K_5)$$

$$s_8(y^2,y,x_1,\ldots,x_5,x_6) \\ +\frac{1}{6}\sum_{\sigma\in S_7}s_4(x_{\sigma(1)},x_{\sigma(2)},x_{\sigma(3)},yx_{\sigma(4)}x_{\sigma(5)}x_{\sigma(6)}x_{\sigma(7)}y)|_{x_7=y}\in I(K_6),$$

where the second polynomial is in fact $\frac{1}{2}\kappa(x_1,x_2,y;y,y)$ (see below). Also, this second polynomial is Lie with

$$s_4(y^2,y,x_1,x_2)=\frac{1}{2}\sum_{\tau\in S_2}([[[[x_{\tau(1)},y],y],y],x_{\tau(2)}]-[[[[x_{\tau(1)},y],y],x_{\tau(2)}],y]).$$

Now, all of the polynomials in the above family have been verified (by computation and with a high degree of probability, see lemma 1.4.1) to be PI's, and proofs for the first two are given in ([DR]). In what follows we will give an alternate proof for the second polynomial, and in doing so we will outline a method of proof that we hope will extend to the general case for n even. We believe that the general case is, for n even,

$$\begin{aligned} &s_{2n-3}(y^2, y, x_1, ..., x_{2n-5}) \\ &+ \tfrac{1}{(2n-8)!} \sum_{\sigma \in S_{2n-4}} s_{2n-7}(x_{\sigma(1)}, ..., x_{\sigma(2n-8)}, y x_{\sigma(2n-7)} x_{\sigma(2n-6)} ... x_{\sigma(2n-4)} y)|_{x_{2n-4}=y} \\ &\in I(K_n) \end{aligned}$$

and for n odd

$$\begin{aligned} &s_{2n-4}(y^2, y, x_1, ..., x_{2n-6}) \\ &+ \tfrac{1}{(2n-9)!} \sum_{\sigma \in S_{2n-4}} s_{2n-8}(x_{\sigma(1)}, ..., x_{\sigma(2n-9)}, y x_{\sigma(2n-8)} x_{\sigma(2n-7)} ... x_{\sigma(2n-5)} y)|_{x_{2n-5}=y} \\ &\in I(K_n). \end{aligned}$$

Therefore, what we have conjectured here is a family of PI's for K_n with degree $2n-2$ for odd n and degree $2n-3$ for even n, and as mentioned, we hope to extend the following proof to all even n.

ALTERNATE PROOF FOR $s_4(y^2, y, x_1, x_2) \in I(K_4)$:

Let $A, B \in K_4$ with A invertible. As noted above (1.3), we have that every element of K_4 that is symmetric with respect to the involution $* : x \mapsto Ax^tA^{-1}$ satisfies the polynomial $p(\lambda) = Pf((\lambda - x)b)$. Thus, by noting that the element AB is symmetric with respect to * we have that $p(AB) = 0$ and hence, by explicit computation,

$$(AB)^2 - \frac{1}{2}tr(AB)AB + \frac{1}{8}(tr(AB))^2 - \frac{1}{4}tr((AB)^2) = 0, \tag{1}$$

where if $C = (c_{ij}) \in M_n$ then $tr(C) = c_{11} + ... + c_{nn}$, the usual matrix trace. That the generic minimal polynomial for x takes this form is not obvious, but the coefficients can be found in a manner analogous to the way in which the coefficients for the characteristic polynomial are found using Newton's identities (see [R3]).

We would like to remove the condition "A is invertible". We do this by first noting, in a way analogous to the triagonal discussion, that the invertible elements are Zariski-dense in $K_n(F)$ (see [Jac2], exercise 7.11.1). It then follows that since the left hand side of the above equation (a polynomial map) is zero for A invertible in K_n, it is also zero for A arbitrary in K_n.

Take a multilinearized version of the left hand side of (1) and call it $GM(A, B, C, D)$. Now let

$$P(A, B, C, D) := GM(A, B, C, D) - GM(A, B, C, D)^t = 0,$$

and notice that those terms which are of overall degree 0 have cancelled, and we are left with a degree 4 part and a degree 2 part. Since $A, B, C, D \in K_4$, the degree 4 part of $P(A, B, C, D)$ is

$$\begin{aligned}
&\textit{degree 4 part of } GM(A,B,C,D) - \textit{degree 4 part of } GM(A,B,C,D)^t \\
&= ABCD + CBAD + ADCB + CDAB - (ABCD + CBAD + ADCB + CDAB)^t \\
&= ABCD + CBAD + ADCB + CDAB - DCBA - DABC - BCDA - BADC
\end{aligned}$$

and the degree 2 part of $P(A, B, C, D)$ is:

$$\begin{aligned}
&-\frac{1}{2}tr(CD)AB - \frac{1}{2}tr(AD)CB - \frac{1}{2}tr(CB)AD - \frac{1}{2}tr(AB)CD \\
&\quad - (-\frac{1}{2}tr(CD)BA - \frac{1}{2}tr(AD)BC - \frac{1}{2}tr(CB)DA - \frac{1}{2}tr(AB)DC).
\end{aligned}$$

Thus,

$$P(x,y,y,y) = 2(xyyy + yyxy - yyyx - yxyy) - [x,y]tr(y^2)$$

Now we'll see how this relates to $s_4(y^2, y, x_1, x_2)$. It has been shown above that κ can be written:

$$\begin{aligned}
&\kappa(x_1,x_2,x_3;y_1,y_2) = \\
&\quad \sum_{\substack{\sigma\in S_3 \\ \tau\in S_2}} (-1)^\sigma(-ABAAB + ABBAA + AABAB - AABBA + BAABA - BABAA)\cdot
\end{aligned}$$

If we now let $y_1 = y, y_2 = y$, and $x_3 = y$, we get (after some calculation):

$$\begin{aligned}
&\kappa(x_1,x_2,y;y,y) = \\
&2\sum_{\sigma\in S_2} (-1)^\sigma(-BBAAB + BAABB + ABBAB - ABABB + BBABA - BABBA),
\end{aligned}$$

and this is actually $2s_4(y^2, y, x_1, x_2)$ since the terms of $s_4(y^2, y, x_1, x_2)$ where y^2 and y are adjacent vanish.

We now make use of P to get $\kappa(x_1, x_2, y; y, y)$ $(= 2s_4(y^2, y, x_1, x_2))$:

$$\begin{aligned}
&P([x_1,x_2],y,y,y) \\
&\quad = [x_1,x_2]yyy + yy[x_1,x_2]y + [x_1,x_2]yyy + yy[x_1,x_2]y \\
&\qquad - yyy[x_1,x_2] - y[x_1,x_2]yy - yyy[x_1,x_2] - y[x_1,x_2]yy - [[x_1,x_2],y]tr(y^2) \\
&\quad = 2[x_1,x_2]yyy - 2yyy[x_1,x_2] + 2yy[x_1,x_2]y - 2y[x_1,x_2]yy - [[x_1,x_2],y]tr(y^2)
\end{aligned}$$

Which in our notation is:

$$2\sum_{\sigma\in S_2}(-1)^\sigma(AABBB - BBBAA + BBAAB - BAABB) - [[x_1,x_2],y]tr(y^2). \quad (2)$$

Also:

$$
\begin{aligned}
[x_2, P(x_1,y,y,y)] &= [x_2, x_1yyy + yyx_1y + x_1yyy + yyx_1y - yyyx_1 \\
&\quad - yx_1yy - yyyx_1 - yx_1yy - [x_1,y]tr(y^2)] \\
&= x_2x_1yyy + x_2yyx_1y + x_2x_1yyy + x_2yyx_1y \\
&\quad - x_2yyyx_1 - x_2yx_1yy - x_2yyyx_1 - x_2yx_1yy - x_2[x_1,y]tr(y^2) \\
&\quad - x_1yyyx_2 - yyx_1yx_2 - x_1yyyx_2 - yyx_1yx_2 \\
&\quad + yyyx_1x_2 + yx_1yyx_2 + yyyx_1x_2 + yx_1yyx_2 + [x_1,y]x_2tr(y^2)
\end{aligned}
$$

Subtracting $[x_1, P(x_2,y,y,y)]$ from this, we get

$$
\begin{aligned}
2\sum_{\sigma\in S_2}(-1)^\sigma(-AABBB - ABBAB + ABABB - BBABA + BBBAA + BABBA) \\
+ ([[x_1,y],x_2] - [[x_2,y],x_1])tr(y^2)
\end{aligned}
\tag{3}
$$

Adding (2) and (3) (and using the Jacobi identity to clear the degree 2 parts) yields:

$$
2\sum_{\sigma\in S_2}(BBAAB - BAABB - ABBAB + ABABB - BBABA + BABBA)
$$

Which is $-\kappa(x_1,x_2,y;y,y)$. □

2.2. 2-Variable Identities

We performed the 2-variable computation for $k = 8$ and $n = 5$, and it was found that there is a one-dimensional space for the 2-variable PI's of the form (6,2) (i.e., symmetric and of degree 6 in one variable and symmetric and of degree 2 in the other). Dimensions for all (i,j) combinations were found as well:

Degree	**Dimension**
$(8,0)$	0
$(7,1)$	0
$(6,2)$	1
$(5,3)$	2
$(4,4)$	3

For $k = 10$ and $n = 6$, we found that there were no 2-variable PI's, and we therefore concluded that there is also nothing for $k = 9$, $n = 6$. We will now attempt to characterize all 2-variable identities for K_5.

Theorem 1. *The polynomials*

$$B_1(x,y) = x[[xyx,x],[x,y]]x$$

and

$$B_2(x,y) = -[x,y][xyx,x][x,y] + [xyx,x][x,y]xy - yx[x,y][xyx,x].$$

are PI's for K_5.

PROOF. As outlined in 1.2, we may work directly and take

$$X = \begin{pmatrix} 0 & d_1 & 0 & 0 & 0 \\ -d_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & d_2 & 0 \\ 0 & 0 & -d_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad d_1, d_2 \in F,$$

a triagonal element, and

$$Y = \begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} & a_{15} \\ -a_{12} & 0 & a_{23} & a_{24} & a_{25} \\ -a_{13} & -a_{23} & 0 & a_{34} & a_{35} \\ -a_{14} & -a_{24} & -a_{34} & 0 & a_{45} \\ -a_{15} & -a_{25} & -a_{35} & -a_{45} & 0 \end{pmatrix}, \quad a_{ij} \in F,$$

an arbitrary skew element. For the first polynomial we compute:

$$XYX = \begin{pmatrix} 0 & -a_{12}d_1^2 & -a_{24}d_1d_2 & a_{23}d_1d_2 & 0 \\ a_{12}d_1^2 & 0 & a_{14}d_1d_2 & -a_{13}d_1d_2 & 0 \\ a_{24}d_1d_2 & -a_{14}d_1d_2 & 0 & -a_{34}d_2^2 & 0 \\ -a_{23}d_1d_2 & a_{13}d_1d_2 & a_{34}d_2^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad (1)$$

$[XYX, X] =$

$$\begin{pmatrix} 0 & 0 & -a_{14}d_1^2d_2 - a_{23}d_1d_2^2 & a_{13}d_1^2d_2 - a_{24}d_1d_2^2 & 0 \\ 0 & 0 & -a_{24}d_1^2d_2 + a_{13}d_1d_2^2 & a_{23}d_1^2d_2 + a_{14}d_1d_2^2 & 0 \\ a_{14}d_1^2d_2 + a_{23}d_1d_2^2 & a_{24}d_1^2d_2 - a_{13}d_1d_2^2 & 0 & 0 & 0 \\ -a_{13}d_1^2d_2 + a_{24}d_1d_2^2 & -a_{23}d_1^2d_2 - a_{14}d_1d_2^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$[X, Y] =$

$$\begin{pmatrix} 0 & 0 & a_{23}d_1 + a_{14}d_2 & a_{24}d_1 - a_{13}d_2 & a_{25}d_1 \\ 0 & 0 & -a_{13}d_1 + a_{24}d_2 & -a_{14}d_1 - a_{23}d_2 & -a_{15}d_1 \\ -a_{23}d_1 - a_{14}d_2 & a_{13}d_1 - a_{24}d_2 & 0 & 0 & a_{45}d_2 \\ -a_{24}d_1 + a_{13}d_2 & a_{14}d_1 + a_{23}d_2 & 0 & 0 & -a_{35}d_2 \\ -a_{25}d_1 & a_{15}d_1 & -a_{45}d_2 & a_{35}d_2 & 0 \end{pmatrix},$$

$$[[XYX,X],[X,Y]] = \begin{pmatrix} 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & * \\ * & * & * & * & 0 \end{pmatrix},$$

and finally (by (1)),

$$X\,[[XYX,X],[X,Y]]\,X = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and $B_1(x,y) \in I(K_5)$.

Now, for the second polynomial, we in addition let $a = [XYX, X]$, and thus consider

$$-[X,Y]a[X,Y] + a[X,Y]XY - YX[X,Y]a.$$

One can check that

$$[X,Y]a = \lambda I_4 + Q, \tag{2}$$

and by transposing we get

$$a[X,Y] = \lambda I_4 + Q^t, \tag{3}$$

where

$$I_4 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, Q = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ q_{51} & q_{52} & q_{53} & q_{54} & 0 \end{pmatrix}, \lambda, q_{5j} \in F, 1 \le j \le 4.$$

Therefore, our polynomial becomes

$$\begin{aligned} &-(\lambda I_4 + Q)[X,Y] + (\lambda I_4 + Q^t)XY - YX(\lambda I_4 + Q) \\ &= (\lambda I_4)YX - Q[X,Y] + Q^tXY - YX(\lambda I_4) - YXQ. \end{aligned}$$

We now note that XY and YX have (respectively) the forms

$$\begin{pmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} * & * & * & * & 0 \\ * & * & * & * & 0 \\ * & * & * & * & 0 \\ * & * & * & * & 0 \\ * & * & * & * & 0 \end{pmatrix},$$

and also that $I_4 = I - e_{55}$ where I is the 5×5 identity matrix and e_{55} is a canonical basis element. Thus, we may derive the equations

$$YXQ = 0, \quad Q^tXY = 0, \quad e_{55}XY = 0, \quad YXe_{55} = 0$$

and combining these with our new form for I_4 brings our polynomial to

$$\lambda(I - e_{55})YX - Q[X,Y] - YX\lambda(I - e_{55}) = -Q[X,Y] - \lambda e_{55}YX.$$

We can show that this is zero by working with equations (2) and (3). Taking (2)$\times[X,Y]$ and subtracting $[X,Y] \times$ (3) gives us

$$\begin{aligned} 0 &= (\lambda I_4 + Q)[X,Y] - [X,Y](\lambda I_4 + Q^t) \\ &= \lambda[X,Y] - \lambda e_{55}[X,Y] + Q[X,Y] - \lambda[X,Y] + \lambda[X,Y]e_{55} - [X,Y]Q^t, \\ &= (Q[X,Y] + \lambda e_{55}YX) + (\lambda XYe_{55} - [X,Y]Q^t) \end{aligned}$$

but these two bracketed terms have the forms

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * \end{pmatrix},$$

and

$$\begin{pmatrix} 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & * \end{pmatrix},$$

respectively; therefore, to conclude that they are both zero it is enough to show that the $(5,5)$ entry of the first is zero. We have,

$$\begin{aligned} 0 &= ((Q[X,Y] + \lambda e_{55}YX) + (\lambda XYe_{55} - [X,Y]Q^t))_{55} \\ &= (Q[X,Y] - [X,Y]Q^t)_{55} + (\lambda e_{55}YX)_{55} + (\lambda XYe_{55})_{55} \\ &= (Q[X,Y] - [X,Y]Q^t)_{55}, \end{aligned}$$

since $(\lambda e_{55}YX)_{55} = (\lambda XYe_{55})_{55} = 0$. But $(Q[X,Y])^t = -[X,Y]Q^t$ so

$$0 = (Q[X,Y])_{55} = (Q[X,Y] + \lambda e_{55}YX)_{55}.$$

Thus,

$$-Q[X,Y] - \lambda e_{55}YX = 0,$$

proving that $B_2(x,y)$ is a PI for K_5. □

Now, when we consider along with these two PI's (the first is a (6,2), the second is a (5,3)) the polynomial

$$\begin{aligned} B_3(x,y) = 2(&y[[xyx,x],[x,y]]y + x[[yxy,y],[y,x]]x) \\ &- y[[y,x^3],[x,y]]y - x[[x,y^3],[y,x]]x - \{x,[[xyx,y],[x,y]],y\} \end{aligned}$$

(a (4,4), it has been computationally verified but not yet proven) we get a full description of the 2-variable identities for K_5 (i.e., all 2-variable identities for K_5 are consequences of these three). To see this, first define $B(x, y) := B_1(x + y, y) \in I(K_5)$ and notice that this polynomial has an (i,j) component (a homogeneous component symmetric and of degree i in x and symmetric and of degree j in y) for each $0 \leq i, j \leq 8$, $i + j = 8$. By looking at $B(2x, y) - 2^3 B(x, y) \in I(K_5)$, we see that we have gotten rid of the (3,5) component of $B(x, y)$, and we may in fact use this method to isolate any (i,j) component of $B(x, y)$ that we please. Thus, all of the (i,j) components of $B(x, y)$ are identities, and we have obtained them from our original $B_1(x, y)$.

By working in this way it was found that B_1 generates a (5,3) identity that is independent from B_2, and a (4,4) identity independent from B_3. And it was also found that B_2 generates a (4,4) identity independent from B_3 and the (4,4) identity generated by B_1. Thus we have a obtained bases for the all three spaces.

Summary

As mentioned in the introduction, Racine and D'Amour have described all identities for K_n, $n < 5$. In our research we began at $n = 5$, and immediately found that 8 is the minimal degree, and that there is a large space of identities for degree 8. More precisely, a space of degree 8 multilinear identities for K_5 was computed and it has dimension 1756. A character for this space was also computed and it involves, in its decomposition into irreducible characters, all but 4 of S_8's irreducible characters.

Our next step was to look into this massive space for "simple" identities: identities that may be written in a closed form. We knew by [R1] that s_8 should be present (and we verified that it is), so we next went looking (1.4) for PI's alternating not in all 8 variables (like s_8) but alternating in 6 variables and symmetric in 2. We discovered that there is a space of dimension 6 for this type, and we also found a space of dimension 6 for the K_4 (PI's alternating in 4 variables, symmetric in 2) and K_6 (PI's alternating in 8 variables, symmetric in 2) cases. We found similarities between these last two spaces, and from this we were able to conjecture the family of polynomials for K_n described in 2.1. This conjecture has been verified (via Lemma 1.4.1) for $n < 7$, and we have proofs (see above, and also [DR]) for $n < 5$.

PI's involving only 2 variables are also "simple", so we looked at those as well. We found that K_5 does indeed have PI's of this type, and we were able to first find them (1.4) and then obtain a characterization (2.2). We then repeated these computations for K_6 and found that K_6 does not satisfy any 2-variable identities.

Appendix A

```
(* This sample program investigates PI's for K4 that are alternating in
4 variables (xi's) and symmetric in two (yj's).  We start by generating the
15 polynomials corresponding to the (6 2) = 15 possible placements of the yj's among
the xi's (see 1.4).  *)
pr1 = Permutations[x1,x2,x3, x4];
m[x_] := Dot @@ x;
sigs = Map[Signature, pr1]; pols = Table[0, 15];
a = 0;
Do[
  Do[
    a = a + 1;
    perms = pr1;
    Do[
      perms[[k]] = Insert[perms[[k]], y1, i];
      perms[[k]] = Insert[perms[[k]], y2, j];
    , { k, 24} ];
    perms = Map[m, perms];
    pols[[a]] = pols[[a]] + perms.sigs;
    perms = pr1;
    Do[
      perms[[k]] = Insert[perms[[k]], y2, i];
      perms[[k]] = Insert[perms[[k]], y1, j];
    , { k, 24 } ];
    perms = Map[m, perms];
    pols[[a]] = pols[[a]] + perms.sigs;
  , { j, i + 1, 6 } ];
, { i, 5 } ];

(*Here we give the 15 polynomials names so that we can substitute into them
later.*)
p1[x1_, x2_, x3_, x4_, y1_, y2_] = pols[[1]];
p2[x1_, x2_, x3_, x4_, y1_, y2_] = pols[[2]];
p3[x1_, x2_, x3_, x4_, y1_, y2_] = pols[[3]];
p4[x1_, x2_, x3_, x4_, y1_, y2_] = pols[[4]];
p5[x1_, x2_, x3_, x4_, y1_, y2_] = pols[[5]];
p6[x1_, x2_, x3_, x4_, y1_, y2_] = pols[[6]];
p7[x1_, x2_, x3_, x4_, y1_, y2_] = pols[[7]];
p8[x1_, x2_, x3_, x4_, y1_, y2_] = pols[[8]];
p9[x1_, x2_, x3_, x4_, y1_, y2_] = pols[[9]];
p10[x1_, x2_, x3_, x4_, y1_, y2_] = pols[[10]];
p11[x1_, x2_, x3_, x4_, y1_, y2_] = pols[[11]];
p12[x1_, x2_, x3_, x4_, y1_, y2_] = pols[[12]];
p13[x1_, x2_, x3_, x4_, y1_, y2_] = pols[[13]];
```

```
p14[x1_, x2_, x3_, x4_, y1_, y2_] = pols[[14]];
p15[x1_, x2_, x3_, x4_, y1_, y2_] = pols[[15]];

(* Here we start generating rows for my matrix (eq)*)
d = 0;
eq = Table[0, 15];
Do[
  (*Setting up the substitution A1, ..., A6*)
  A1 = Table[Random[], 4, 4];
  A1 = Floor[2*A1];
  A1 = A1 - Transpose[A1];
  A2 = Table[Random[], 4, 4];
  A2 = Floor[2*A2];
  A2 = A2 - Transpose[A2];
  A3 = Table[Random[], 4, 4];
  A3 = Floor[2*A3];
  A3 = A3 - Transpose[A3];
  A4 = Table[Random[], 4, 4];
  A4 = Floor[2*A4];
  A4 = A4 - Transpose[A4];
  A5 = Table[Random[], 4, 4];
  A5 = Floor[2*A5];
  A5 = A5 - Transpose[A5];
  A6 = Table[Random[], 4, 4];
  A6 = Floor[2*A6];
  A6 = A6 - Transpose[A6];

  (*Here we plug in the sub, and extract from it the rows for our matrix (eq).
  Then we append these rows to eq and row reduce.*)
  f610 = RowReduce[ Transpose[ Partition[ Flatten[p1[A1, A2, A3,A4, A5, A6],
  p2[A1, A2, A3, A4, A5, A6], p3[A1, A2, A3, A4, A5, A6],
  p4[A1, A2, A3, A4, A5, A6], p5[A1, A2,A3, A4, A5, A6],
  p6[A1, A2, A3, A4, A5, A6], p7[A1, A2, A3, A4, A5, A6],
  p8[A1, A2, A3, A4, A5, A6], p9[A1, A2, A3, A4, A5, A6],
  p10[A1, A2, A3, A4, A5, A6], p11[A1, A2, A3, A4, A5, A6],
  p12[A1, A2, A3, A4, A5, A6], p13[A1, A2, A3, A4, A5, A6],
  p14[A1, A2, A3, A4, A5, A6], p15[A1, A2, A3, A4, A5, A6]], 16] ] ];
  eq = Join[eq, f610];
  eq = RowReduce[eq];
  eq = Complement[eq, Table[0, 15]];
  If[Dimensions[eq][[1]] > d, d = Dimensions[eq][[1]]];
  Print[d, Max[eq]];
, { 10 } ]

(*Print the resulting nullspace.*)
MatrixForm[NullSpace[eq]]
```

Appendix B

```
(* This sample program investigates two-variable identities for K6 that involve
7 x's and 3 y's.  First we set up p5(x,y).  This is a vector of length 120, each
entry is one of the monomials xxxxxxxxyy, xxxxxxxyxy, xxxxxxxyyx,...*)

m[{x1_, x2_, x3_, x4_, x5_, x6_, x7_, x8_, x9_, x10_}]
    := x1.x2.x3.x4.x5.x6.x7.x8.x9.x10;
per5[x_, y_] := Permutations[x, x, x, x, x, x, x, y, y, y];
p5[x_, y_] = Map[m, per5[x, y]];

(*Now we perform substitutions into p5(x,y) to obtain conditions on the a_σ
(see 1.4), put the conditions into a matrix, and row-reduce*)
d = 0;
eq = Table[0, 120];
Do[
  A1 = Table[Random[], 6, 6];
  A1 = Floor[2*A1];
  A1 = A1 - Transpose[A1];
  A2 = Table[Random[], 6, 6];
  A2 = Floor[2*A2];
  A2 = A2 - Transpose[A2];
  f610 = RowReduce[ Transpose[ Partition[ Flatten[p5[A1, A2]], 36] ] ];
  eq = Join[eq, f610];
  eq = RowReduce[eq];
  eq = Complement[eq, Table[0, 120]];
  If[Dimensions[eq][[1]] > d, d = Dimensions[eq][[1]]];
  Print[d, Length[eq[[1]]], Max[eq[[1]]]];
, 40]

(*Print the resulting null space.*)
MatrixForm[NullSpace[eq]]
```

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