A study of Parabolic and Hyperbolic Anderson models driven by fractional Brownian sheet with spatial Hurst index in $(0, 1)$

Yiping Ma

Thesis submitted to the Faculty of Science in partial fulfillment of the requirements
for the degree of
Master of Science Mathematics and Statistics[1]

Department of Mathematics and Statistics
Faculty of Science
University of Ottawa

---

[1] The M.Sc. program is a joint program with Carleton University, administered by the Ottawa-Carleton Institute of Mathematics and Statistics

Abstract

The goal of this thesis is to present a comprehensive study of the parabolic and hyperbolic Anderson models with constant initial condition, driven by a Gaussian noise which is fractional in space with index $H > \frac{1}{2}$ or $H < \frac{1}{2}$, and is either white in time, or fractional in time with index $H_0 > \frac{1}{2}$. As a preliminary step, we study the linear stochastic heat and wave equations with the same type of noise. In the case $H_0 > \frac{1}{2}$ and $H < \frac{1}{2}$, we present a new result, regarding the solution of the parabolic Anderson model with general initial condition given by a measure.

Acknowledgement

Firstly, I would like to express my special gratitude to my supervisor Professor Raluca Balan from the Department Mathematics and Statistics at University of Ottawa. Professor Balan offered me continuous support and professional knowledge to help me with my research, as well as guidance for writing this thesis. Despite her busy schedule, she constantly steered me in the right direction.

Besides, I am also extremely grateful to the experts who have done extensive research in this area. Without their efforts and accomplishments, this thesis could not have been written.

Finally, I would like to thank my parents for providing me with unfailing support and great encouragement throughout my years of study and through the process of doing research and writing this thesis. This accomplishment would not have been possible without them. Thank you.

Contents

Chapter 1

Introduction

Stochastic Partial Differential Equations (SPDEs) have become increasingly popular in the literature in the last few decades, since they can be used as models in a wide variety of applications. In the original lecture notes of Walsh [29], the noise perturbing such an equation was given by a Gaussian space-time white noise, which behaves like Brownian motion in space and time. In the recent years, many articles have been dedicated to the study of SPDEs driven by a more sophisticated Gaussian noise, and in particular a noise which behaves like fractional Brownian motion (fBm) in space and time. Recall that the fBm is a zero-mean Gaussian process $\{B_t^{(H)}\}_{t\in\mathbb{R}}$ with covariance

$$E[B_t^{(H)}B_s^{(H)}] = R_H(t,s) = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}).$$

The parameter $H \in (0,1)$ is called the Hurst index. If $H > \frac{1}{2}$, the covariance R_H admits the following representation: for any $t, s > 0$,

$$R_H(t,s) = \alpha_H \int_0^t \int_0^s |u-v|^{2H-2} du dv, \tag{1.0.1}$$

where $\alpha_H = H(2H-1)$. For any $H \in (0,1)$, this covariance has the spectral representation:

$$R_H(t,s) = C_H \int_{\mathbb{R}} \mathcal{F}1_{[0,t]}(\xi)\overline{\mathcal{F}1_{[0,s]}}(\xi)|\xi|^{1-2H} d\xi,$$

where $C_H = \Gamma(2H-1)\sin(\pi H)/(2\pi)$ and $\mathcal{F}$ is the Fourier transform. More details about stochastic analysis with respect to fBm can be found in [21, 23, 24]. In the case $H = \frac{1}{2}$, the fBm becomes the classical Brownian motion. As fBm of index H has sample paths which are $(H-\varepsilon)$-Hölder continuous, the fBm with index $H > \frac{1}{2}$ has **smoother** paths than Brownian motion, whereas the fBm with index $H < \frac{1}{2}$ has **rougher** paths than Brownian motion. The multi-parameter generalization of fBm is called a fractional Brownian sheet. This is a zero-mean Gaussian process $\{W(t,x); t \geq 0, x \in \mathbb{R}\}$ with covariance

$$E[W(t,x)W(s,y)] = R_{H_0}(t,s)R_H(x,y)$$

for some $H_0, H \in (0,1)$.

Important examples of SPDEs are the *Parabolic Anderson Model* (PAM):

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) &= \dfrac{1}{2}\dfrac{\partial^2 u}{\partial x^2}(t,x) + u(t,x)\dot{W}(t,x), \quad t>0, x\in\mathbb{R} \\ u(0,x) &= 1, \quad x\in\mathbb{R} \end{cases} \tag{1.0.2}$$

and the *Hyperbolic Anderson Model* (HAM):

$$\begin{cases} \dfrac{\partial^2 u}{\partial^2 t}(t,x) &= \dfrac{\partial^2 u}{\partial x^2}(t,x) + u(t,x)\dot{W}(t,x), \quad t>0, x\in\mathbb{R} \\ u(0,x) &= 1, \quad x\in\mathbb{R} \\ \dfrac{\partial u}{\partial t}(0,x) &= 0, \quad x\in\mathbb{R} \end{cases} \tag{1.0.3}$$

In this thesis, we study the existence, uniqueness and the exponential growth of the second moment for the solutions of equations (1.0.2) and (1.0.3) driven by a fractional Brownian sheet with temporal index $H_0 = \frac{1}{2}$ or $H_0 > \frac{1}{2}$ and spatial index $H > \frac{1}{2}$ or $H < \frac{1}{2}$.

By definition, the **solution** to equation (1.0.2), respectively equation (1.0.3), is a collection $u = \{u(t,x); t \geq 0, x \in \mathbb{R}\}$ of random variables which satisfies the following integral equation: for any $t \geq 0$ and $x \in \mathbb{R}$, with probability 1,

$$u(t,x) = 1 + \int_0^t \int_{\mathbb{R}} G(t-s, x-y) u(s,y) W(ds,dy),$$

where

$$G(t,x) = G^h(t,x) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{x^2}{2t}\right) \tag{1.0.4}$$

is the fundamental solution of the heat equation on $\mathbb{R}_+ \times \mathbb{R}$ (for equation (1.0.2)), respectively

$$G(t,x) = G^w(t,x) = \frac{1}{2} 1_{\{|x|<t\}} \tag{1.0.5}$$

is the fundamental solution of the wave equation on $\mathbb{R}_+ \times \mathbb{R}$ (for equation (1.0.3)).

There are many references which studied the solutions to equations (1.0.2) and (1.0.3) with Gaussian noise which is white in time (i.e. behaves in time like Brownian motion) and is colored in space, for instance behaves in space like fractional Brownian motion with index $H > \frac{1}{2}$. One of the first articles in this direction is Dalang's seminal article [13]. Other important references dedicated to SPDEs with white noise in time are: [14, 20, 22, 25, 27].

A first step in this analysis of equations (1.0.2) and (1.0.3) is to study the linear stochastic heat equation:

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = \dfrac{1}{2}\dfrac{\partial^2 u}{\partial x^2}(t,x) + \dot{W}(t,x), \quad t>0, x\in\mathbb{R} \\ u(0,x) = 0, \quad x\in\mathbb{R} \end{cases} \tag{1.0.6}$$

respectively the linear stochastic wave equation:

$$\begin{cases} \dfrac{\partial^2 u}{\partial^2 t}(t,x) = \dfrac{\partial^2 u}{\partial x^2}(t,x) + \dot{W}(t,x), \quad t>0, x\in\mathbb{R} \\ u(0,x) = 0, \quad x\in\mathbb{R} \\ \dfrac{\partial u}{\partial t}(0,x) = 0, \quad x\in\mathbb{R} \end{cases} \tag{1.0.7}$$

In this thesis, we will prove the well-known fact that the solutions of equations (1.0.6) and (1.0.7) with noise which is white in time and fractional in space exist for any $H \in (0,1)$.

When it comes to SPDEs with a Gaussian noise which is white in time and fractional in space with $H < \frac{1}{2}$, some important contributions are contained in articles [5, 6, 16] which specify the restriction $H > \frac{1}{4}$. Paper [16] deals with equation (1.0.2) and proves that the solution exists and its p-th moments are bounded. In [6], it is shown that the restriction $H > \frac{1}{4}$ ensures the existence and uniqueness of the solution to equation (1.0.3).

In recent years, there has been a rising number of articles dedicated to SPDEs with a Gaussian noise, which is colored in both space and time. In the case of linear equations, the problem of existence of solutions to equations (1.0.6) and (1.0.7) with this noise were solved in [10] where it was demonstrated that the necessary and sufficient conditions for the existence of solutions were quite different for the two equations. We will review these proofs in Sections 4.2 and 4.3.

Article [2] studied equations (1.0.2) and (1.0.3) with a Gaussian noise which is fractional in time with index $H_0 > \frac{1}{2}$ and colored in space. This article gives the necessary condition for the existence of solution and exponential bounds for the moments of the solution. There investigation were continued in article [4]. Article [19] proved that the existence of solution is equivalent to showing that the Wiener Chaos expansion $\sum_{n\geq 0} I_n(f_n(\cdot,t,x))$ converges in $L^2(\Omega)$, where

$$f_n(t_1,x_1,\cdot,t_n,x_nt,x) = G(t-t_n,x-x_n)\cdots G(t_2-t_1,x_2-x_1)1_{\{0<t_1<\cdots<t_n<t\}},$$

and $I_n(f_n(\cdot,t,x))$ is the multiple Wiener integral of order n with respect to W. We will review this proof in Section 4.4. The authors of [17] established the Feynman-Kac formula for the moments of the solution of equation (1.0.2), proved the Hölder continuity and gave exponential bounds for the moments of this solution.

Article [7] considered the case of equation (1.0.2) driven by the same noise as in [2], but with a general initial condition given by a measure.

However, only few articles studied models driven by a Gaussian noise which is fractional in time with index $H_0 > \frac{1}{2}$ and fractional in space with index $H < \frac{1}{2}$. This is the case of the recent papers [12, 18] related to the parabolic case. More precisely, [12] is devoted to the existence of solution and the exponential growth of its moments, and [18] focuses on the topic of joint Hölder continuity of the solution. The hyperbolic Anderson model with this type of noise has been studied in the recent preprint [28].

In this thesis, we will discuss at length the proofs contained in these articles, by providing all the details which are missing from the papers. The thesis is organized as follows. Chapters 2 and 3 introduce the Gaussian noise which is white in time and fractional in space with index $H > \frac{1}{2}$, respectively $H < \frac{1}{2}$. First, we prove the existence of solutions for the linear stochastic heat and wave equations with this type of noise. In the final sections of these two chapters, we show that the parabolic and hyperbolic Anderson models with this type of noise have unique solutions and obtain upper and lower bounds for the second moments of the solutions, which have the form of an exponential function of t. In the case $H < \frac{1}{2}$, the existence of solution holds only for $H > \frac{1}{4}$.

In Chapter 4, we discuss the case of a Gaussian noise which is fractional in time with index $H_0 > \frac{1}{2}$ and fractional in space with index $H > \frac{1}{2}$. Following the same procedure as that in Chapter 2 and 3, we examine first the linear equations with this type of noise. In the case of equations (1.0.2) and (1.0.3), we show that the second moment of the solution satisfies

$$E|u(t,x)|^2 \leq C_1 \exp(C_2 t^{\rho}) \quad \text{for all } t > 0, x \in \mathbb{R},$$

where C_1, C_2 are constants depending on H_0 and H, and

$$\begin{aligned} \rho = \rho^h &= \frac{2H_0 + H - 1}{H} \quad \text{for equation (1.0.2)} , \\ \rho = \rho^w &= \frac{2H_0 + 2H}{2H + 1} \quad \text{for equation (1.0.3)} . \end{aligned}$$

Next we focus on the parabolic Anderson model with a general initial condition given by a measure μ_0, which satisfies the condition

$$\int_{\mathbb{R}} e^{-ax^2} \mu_0(dx) < \infty \quad \text{for all } a > 0. \tag{1.0.8}$$

In this case, following the approach of [7], we show that the second moment of the solution satisfies:

$$E|u(t,x)|^2 \leq C_1 J_0^2(t,x) \exp(C_2 t^{\rho^h}),$$

where $J_0(t,x) = \int_{\mathbb{R}} G^h(t, x-y) \mu_0(dy)$ is the solution of the deterministic heat equation.

Finally in Chapter 5, we consider a Gaussian noise which is fractional in time with index $H_0 > \frac{1}{2}$ and fractional in space with index $H < \frac{1}{2}$. As in the previous chapters, we first study the existence of solution for the linear stochastic heat and wave equations. Then, we show that the parabolic Anderson model with this type of noise has a unique solution, if the parameters of the noise satisfy

$$H_0 + H > \frac{3}{4},$$

and we obtain the following upper bound for the second moment of the solution:

$$E|u(t,x)|^2 \leq C_1 \exp(C_2 t^{\rho^h}).$$

In the case of hyperbolic Anderson model, we obtain the existence of the solution under the condition $H > \frac{1}{4}$, and we prove that the solution satisfies:

$$E|u(t,x)|^2 \leq C_1 \exp(C_2 t^{\rho^w}).$$

There results are consistent with those obtained in references [18] (for the parabolic Anderson model), respectively [28] (for the hyperbolic Anderson model), but are derived in this thesis using different methods.

Finally, in Section 5.3, we present a result which is new in the literature, and gives the existence of the solution of the parabolic Anderson model driven by a Gaussian noise which is fractional in time with index $H_0 > \frac{1}{2}$ and fractional in space with index $H < \frac{1}{2}$, and initial condition given by a measure μ_0 which satisfies condition (1.0.8). Unfortunately, for this result, we had to assume that $H > \frac{1}{3}$. Moreover, we show that the solution satisfies

$$E|u(t,x)|^2 \leq C_1 J_0^2(t,x) \exp(C_2 t^{\frac{2H_0+2H}{3H-1}}).$$

Condition (1.0.8) is very general and is satisfied by a wide range of measures μ_0 (for instance, the Lebesgue measure or the Dirac delta measure). The existence of solution of Hyperbolic Anderson Model with a general initial condition satisfying condition (1.0.8) is an open problem in the literature. The difficulty is due to the fact that the fundamental solution of the wave equation does not satisfy a key semigroup-type property as the heat kernel (see Lemma 4.5.2 below).

The results presented in Chapter 2-5 of the thesis hold also for $H = 1/2$ (i.e in the case when the noise is white in space). In this case, the proofs are similar to (but simpler than) those presented in the thesis. To avoid unnecessary repetitions, we decided to focus in this thesis only on the cases $H > 1/2$ and $H < 1/2$.

The study of equation (1.0.2) and (1.0.3) with noise W which is fractional in time with $H_0 < 1/2$ remains an open problem in the literature.

In Chapter 6, we included a table summarizing the results of the thesis. The appendices contains some auxiliary results which were used in the thesis.

Chapter 2

The case $H_0 = \frac{1}{2}$ and $H > \frac{1}{2}$

In this chapter, we introduce a Gaussian noise which is white in time (i.e. behaves in time like Brownian motion) and fractional in space with index $H > \frac{1}{2}$. This noise is associated with a fractional Brownian sheet on $\mathbb{R}_+ \times \mathbb{R}$, with index $H_0 = \frac{1}{2}$ in time and index $H > \frac{1}{2}$ in space. Then, we study the linear stochastic heat and wave equations, as well as equation (1.0.2) and (1.0.3) with this type of noise. These results are particular cases of the results contained in Dalang's seminal article [13], which are obtained here using a different method.

2.1 The noise

In this section, we introduce the noise and we construct the Wiener integral with respect to the noise. Let $W = \{W([0,t] \times A); t \geq 0, A \in \mathcal{B}_b(\mathbb{R})\}$ be a zero-mean Gaussian process defined on a probability space $(\Omega, \mathcal{F}, P)$, with covariance given by:

$$E[W([0,t] \times A) \cdot W([0,s] \times B)] = (t \wedge s)\alpha_H \int_A \int_B |x-y|^{2H-2} dxdy, \tag{2.1.1}$$

where $\alpha_H = H(2H-1)$ and $H \in (1/2, 1)$. Note that the process $W(t,x) = W([0,t] \times [0,x])$ is a fBs (see Remark 2.1.5 below). Here $\mathcal{B}_b(\mathbb{R})$ denotes the class of bounded Borel sets of $\mathbb{R}$. Note that this noise has the covariance structure of Brownian motion in time, and that of fractional Brownian motion in space. We have

$$\begin{aligned}
&E[W([0,t] \times A) \cdot W([0,s] \times B)] \\
&\qquad = \left(\int_0^\infty 1_{[0,t]}(u) 1_{[0,s]}(u) du\right) \left(\alpha_H \int_{\mathbb{R}} \int_{\mathbb{R}} 1_A(x) 1_B(y) |x-y|^{2H-2} dxdy\right) \\
&\qquad = \int_0^\infty \int_{\mathbb{R}} \int_{\mathbb{R}} 1_{[0,t]}(u) 1_A(x) 1_{[0,s]}(u) 1_B(y) \alpha_H |x-y|^{2H-2} dxdydu \\
&\qquad = \int_0^\infty \int_{\mathbb{R}} \int_{\mathbb{R}} 1_{[0,t]\times A}(u,x) 1_{[0,s]\times B}(u,y) \alpha_H |x-y|^{2H-2} dxdydu
\end{aligned}$$

$$:= \langle 1_{[0,t]\times A}, 1_{[0,s]\times B}\rangle_{\mathcal{H}}.$$

Set $W(1_{[0,t]\times A}) = W([0,t]\times A)$. Let $\mathcal{E}$ be the set of finite linear combinations of indicator functions of the form $1_{[0,t]\times A}$ with $t>0$ and $A \in \mathcal{B}_b(\mathbb{R})$. By linearity, we extend W to the set $\mathcal{E}$.

Lemma 2.1.1. *W is an isometry from $\mathcal{E}$ to $L^2(\Omega)$, i.e. any $\varphi, \psi \in \mathcal{E}$,*

$$E[W(\varphi)W(\psi)] = \langle \varphi, \psi\rangle_{\mathcal{H}} = \alpha_H \int_0^\infty \int_{\mathbb{R}} \int_{\mathbb{R}} \varphi(u,x)\psi(u,y)|x-y|^{2H-2} dxdydu. \quad (2.1.2)$$

Proof: Let $\varphi = \sum_{i=1}^n a_i 1_{[0,t_i]\times A_i}$ and $\psi = \sum_{j=1}^m b_j 1_{[0,s_j]\times B_j}$. Then

$$\begin{aligned}
E[W(\varphi)W(\psi)] &= E\Big[W\Big(\sum_{i=1}^n a_i 1_{[0,t_i]\times A_i}\Big) W\Big(\sum_{j=1}^m b_j 1_{[0,s_j]\times B_j}\Big)\Big] \\
&= E\Big[\Big(\sum_{i=1}^n a_i W(1_{[0,t_i]\times A_i})\Big)\Big(\sum_{j=1}^m b_j W(1_{[0,s_j]\times B_j})\Big)\Big] \\
&= \sum_{i=1}^n \sum_{j=1}^m a_i b_j E[W(1_{[0,t_i]\times A_i}) W(1_{[0,s_j]\times B_j})] \\
&= \sum_{i=1}^n \sum_{j=1}^m a_i b_j \int_0^\infty \int_{\mathbb{R}} \int_{\mathbb{R}} 1_{[0,t_i]\times A_i}(u,x) 1_{[0,s_j]\times B_j}(u,y) \alpha_H |x-y|^{2H-2} dxdydu \\
&= \int_0^\infty \int_{\mathbb{R}} \int_{\mathbb{R}} \left(\sum_{i=1}^n a_i 1_{[0,t_i]\times A_i}\right) \left(\sum_{j=1}^m b_j 1_{[0,s_j]\times B_j}\right) \alpha_H |x-y|^{2H-2} dxdydu \\
&= \int_0^\infty \int_{\mathbb{R}} \int_{\mathbb{R}} \varphi(u,x)\psi(u,y)\alpha_H |x-y|^{2H-2} dxdydu \\
&= \langle \varphi, \psi\rangle_{\mathcal{H}}.
\end{aligned}$$

∎

We denote by $\mathcal{S}(\mathbb{R})$ the set of rapidly decreasing functions on $\mathbb{R}$, i.e. infinitely differentiable functions on $\mathbb{R}$ such that $\sup_{x\in\mathbb{R}} |x|^n \varphi^{(k)}(x) < \infty$ for any $n, k \in \mathbb{N}$. We know that for any $\varphi \in \mathcal{S}(\mathbb{R})$,

$$\alpha_H \int_{\mathbb{R}} \varphi(x)|x|^{2H-2} dx = C_H \int_{\mathbb{R}} \mathcal{F}\varphi(\xi)|\xi|^{1-2H} d\xi, \quad (2.1.3)$$

where $\mathcal{F}\varphi$ is the Fourier transform of φ, $\alpha_H = H(2H-1)$ and

$$C_H = \frac{\Gamma(2H-1)\sin(\pi H)}{2\pi}. \quad (2.1.4)$$

Relation (2.1.3) is the particular case of the following identity: for any $\varphi \in \mathcal{S}(\mathbb{R}^d), \alpha \in (0, d)$, we have

$$\int_{\mathbb{R}^d} \varphi(x)|x|^{-\alpha} dx = C_{d,\alpha} \int_{\mathbb{R}^d} \mathcal{F}\varphi(\xi)|\xi|^{-(d-\alpha)} d\xi \tag{2.1.5}$$

where

$$C_{d,\alpha} = \pi^{-\frac{d}{2}} 2^{-\alpha} \frac{\Gamma(\frac{d-\alpha}{2})}{\Gamma(\frac{\alpha}{2})}.$$

To obtain relation (2.1.3), we take $d = 1$ and $\alpha = 2 - 2H$.

Using relation (2.1.3), we obtain the following result:

Theorem 2.1.2. *For any $H \in (\frac{1}{2}, 1)$ and for any $\varphi, \psi \in \mathcal{S}(\mathbb{R})$,*

$$\alpha_H \int_{\mathbb{R}} \int_{\mathbb{R}} \varphi(x)\psi(y)|x - y|^{2H-2} dx dy = C_H \int_{\mathbb{R}} \mathcal{F}\varphi(\xi)\overline{\mathcal{F}\psi(\xi)}|\xi|^{1-2H} d\xi. \tag{2.1.6}$$

Proof: We let $z = x - y$. Hence using Fubini's theorem and letting $\widetilde{\psi}(y) = \psi(-y)$,

$$\begin{aligned} &\alpha_H \int_{\mathbb{R}} \varphi(x) \left(\int_{\mathbb{R}} \psi(y)|x - y|^{2H-2} dy \right) dx = \alpha_H \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \varphi(x)\psi(x - z)|z|^{2H-2} dz \right) dx \\ &= \alpha_H \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \varphi(x)\psi(x - z) dx \right) |z|^{2H-2} dz = \alpha_H \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \varphi(x)\widetilde{\psi}(z - x) dx \right) |z|^{2H-2} dz \\ &= \alpha_H \int_{\mathbb{R}} (\varphi * \widetilde{\psi})(z)|z|^{2H-2} dz. \end{aligned}$$

Since $\varphi * \widetilde{\psi} \in \mathcal{S}(\mathbb{R})$,we can apply (2.1.3) to the function $\varphi * \widetilde{\psi}$, and we obtain:

$$\alpha_H \int_{\mathbb{R}} (\varphi * \widetilde{\psi})(z)|z|^{2H-2} dz := C_H \int_{\mathbb{R}} |\mathcal{F}(\varphi * \widetilde{\psi})(\xi)|^2 |\xi|^{1-2H} d\xi.$$

The conclusion follows using Lemma C.1. ∎

Remark 2.1.3. *Relation (2.1.6) also holds for any functions $\varphi, \psi \in L^1(\mathbb{R})$ such that $\mathcal{E}_H(|\varphi|) < \infty$ and $\mathcal{E}_H(|\psi|) < \infty$, where*

$$\mathcal{E}_H(|\varphi|) = \int_{\mathbb{R}} \int_{\mathbb{R}} |\varphi(x)||\varphi(y)||x - y|^{2H-2} dx dy$$

(see Lemma A.1 of [8]).

Let $\mathcal{H}$ be the completion of the $\mathcal{E}$ with respect to the inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$. By Theorem 2.1.2, the map $\varphi \mapsto W(\varphi)$ is an isometry from $\mathcal{E}$ to $L^2(\Omega)$ which we extend to $\mathcal{H}$. For any $\varphi \in \mathcal{H}$, we say that $W(\varphi)$ is the *Wiener integral* of φ with respect to W and we denote

$$W(\varphi) = \int_0^\infty \int_{\mathbb{R}} \varphi(t,x) W(dt,dx).$$

Note that $W(\varphi)$ is well-defined if and only if $\varphi \in \mathcal{H}$.

Theorem 2.1.4. *a) If $\varphi : [0,T] \times \mathbb{R} \mapsto \mathbb{R}$ is so that*

$$\|\varphi\|^2_{|\mathcal{H}|} \overset{def}{:=} \alpha_H \int_0^T \int_{\mathbb{R}} \int_{\mathbb{R}} |\varphi(t,x)||\varphi(t,y)||x-y|^{2H-2} dx dy dt < \infty,$$

then $\varphi \in \mathcal{H}$.
b) Let $\varphi : [0,T] \times \mathbb{R} \mapsto \mathbb{R}$ be so that $\varphi(t,\cdot) \in L^1(\mathbb{R})$ for all $t \in [0,T]$, and $\mathcal{F}\varphi(t,\cdot)(\xi) = \int_{\mathbb{R}} e^{-i\xi \cdot x} \varphi(t,x) dx$, $\xi \in \mathbb{R}$ be the Fourier transform of $\varphi(t,\cdot)$. If

$$\|\varphi\|^2_o := C_H \int_0^T \int_{\mathbb{R}} |\mathcal{F}\varphi(t,\cdot)(\xi)|^2 |\xi|^{1-2H} d\xi dt < \infty,$$

then $\varphi \in \mathcal{H}$, and $\|\varphi\|^2_{\mathcal{H}} = \|\varphi\|^2_o$.

Remark 2.1.5. Clearly, the indicator function $1_{[0,t]\times[0,x]}$ belongs to the space $\mathcal{E}$, (since $[0,x]$ is a bounded set in $\mathbb{R}$), hence it belongs to the space $\mathcal{H}$, for any $t > 0$ and $x \in \mathbb{R}$. Letting $W(t,x) := W(1_{[0,t]\times[0,x]})$, it can be proved that the process $\{W(t,x); t \geq 0, x \in \mathbb{R}\}$ is a fractional Brownian sheet with index $H_0 = \frac{1}{2}$ in time and index H in space.

2.2 Linear equations

In this section, we study the stochastic linear heat equation (1.0.6) and stochastic linear wave equation (1.0.7). We begin with the linear stochastic heat equation (1.0.6) with noise $\dot{W}$ as in Section 2.1.

Definition 2.2.1. *We say that a process $\{u(t,x), t \geq 0, x \in \mathbb{R}\}$ is a* **solution** *to* (1.0.6) *if*

$$u(t,x) = \int_0^t \int_{\mathbb{R}} G^h(t-s, x-y) W(ds,dy)$$

where G^h is the fundamental solution of the heat equation on $\mathbb{R}_+ \times \mathbb{R}$ given by (1.0.4).

Note that the solution exists if and only if $g_{tx} \in \mathcal{H}$ for any $t > 0, x \in \mathbb{R}$ where $g_{tx}(s, y) = G^h(t-s, x-y)1_{[0,t]}(s)$. Also note that

$$\mathcal{F}g_{tx}(s, \cdot)(\xi) = 1_{[0,t]}(s)\mathcal{F}G^h(t-s, x-\cdot)(\xi) = 1_{[0,t]}(s)e^{-i\xi\cdot x}\overline{\mathcal{F}G^h(t-s, \cdot)(\xi)} \quad (2.2.1)$$

using Lemma C.2, since $G^h(t, \cdot) \in L^1(\mathbb{R})$.

Since $G^h(t, x) = \frac{1}{\sqrt{2\pi t}}e^{-\frac{|x|^2}{2t}}$ is the density of a random variable X with a normal distribution $N(0, t)$ and $G^h(t, x) = G^h(t, -x)$, we get:

$$\begin{aligned}\mathcal{F}G^h(t, x)(\xi) &= \int_{\mathbb{R}} e^{-i\xi\cdot x}G^h(t, x)dx = \int_{\mathbb{R}} e^{i\xi\cdot(-x)}G^h(t, -x)dx = \int_{\mathbb{R}} e^{i\xi\cdot x}G^h(t, x)dx \\ &= E(e^{i\xi\cdot X}) = \exp\left\{-\frac{1}{2}|\xi|^2 t\right\}.\end{aligned}$$

The content of the following result is given by Lemma 6.1.(a) of [26].

Theorem 2.2.2. *For any $H \in (\frac{1}{2}, 1)$, equation* (1.0.6) *with noise $\dot{W}$ as in Section 2.1 has a unique solution.*

Proof: **Step 1** We first show that the solution exists if and only if

$$\int_{\mathbb{R}} \frac{1}{1+|\xi|^2}\mu(d\xi) < \infty \quad (2.2.2)$$

where $\mu(d\xi) = C_H|\xi|^{1-2H}d\xi$.

By Theorem 2.1.4 *b*), it is enough to check that

$$I_t := \int_0^t \int_{\mathbb{R}} |\mathcal{F}g_{tx}(s, \cdot)(\xi)|^2\mu(d\xi)ds < \infty.$$

By Fubini's theorem,

$$I_t = \int_{\mathbb{R}} \left(\int_0^t |\mathcal{F}g_{tx}(s, \cdot)(\xi)|^2 ds\right)\mu(d\xi).$$

Note that, by (2.2.1) and the change of variable $s' = t - s$,

$$\begin{aligned}\int_0^t |\mathcal{F}g_{tx}(s, \cdot)(\xi)|^2 ds &= \int_0^t |\mathcal{F}G^h(t-s, \cdot)(\xi)|^2 ds = \int_0^t |\mathcal{F}G^h(s', \cdot)(\xi)|^2 ds' \\ &= \int_0^t \exp(-|\xi|^2 s)ds = \frac{1-\exp(-|\xi|^2 t)}{|\xi|^2}.\end{aligned}$$

We will prove that

$$\frac{1\wedge t}{2(1+|\xi|^2)} \le \int_0^t |\mathcal{F}g_{tx}(s, \cdot)(\xi)|^2 ds \le \frac{2}{1+|\xi|^2}(t\vee 1). \quad (2.2.3)$$

For the upper bound, we consider two cases.
If $|\xi|^2 \geq 1$,

$$\frac{1-\exp\{-|\xi|^2 t\}}{|\xi|^2} \leq \frac{1}{|\xi|^2} \leq \frac{2}{1+|\xi|^2}.$$

If $|\xi|^2 < 1$, since for any $x \geq 0, 1 - e^{-x} \leq x$, then

$$\frac{1-\exp\{-|\xi|^2 t\}}{|\xi|^2} \leq t \leq \frac{2}{1+|\xi|^2} t.$$

We get

$$\int_0^t |\mathcal{F} g_{tx}(s,\cdot)(\xi)|^2 ds \leq \frac{2}{1+|\xi|^2}(t \vee 1).$$

For the lower bound, we use the fact that for any $x \geq 0, 1 - e^{-x} \geq \frac{x}{1+x}$. Hence

$$\int_0^t |\mathcal{F} g_{tx}(s,\cdot)(\xi)|^2 ds \geq \frac{t}{1+|\xi|^2 t}.$$

If $|\xi|^2 t \geq 1$,

$$\frac{t}{1+|\xi|^2 t} \geq \frac{t}{2|\xi|^2 t} \geq \frac{1}{2(1+|\xi|^2)}.$$

If $|\xi|^2 t < 1$,

$$\frac{t}{1+|\xi|^2 t} \geq \frac{t}{2} \geq \frac{t}{2(1+|\xi|^2)}.$$

We get

$$\frac{t}{1+|\xi|^2 t} \geq \frac{1 \wedge t}{2(1+|\xi|^2)}.$$

This proves (2.2.3). Then

$$\frac{1 \wedge t}{2} \int_{\mathbb{R}} \frac{1}{1+|\xi|^2} \mu(d\xi) \leq I_t \leq 2(t \vee 1) \int_{\mathbb{R}} \frac{1}{1+|\xi|^2} \mu(d\xi).$$

Hence $I_t < \infty$ if and only if (2.2.2) holds.
Step 2 In this step, we show that condition (2.2.2) holds for any $H \in (\frac{1}{2}, 1)$. Note that $I = I_1 + I_2$, where $I_1 = \int_{|\xi|<1} \frac{1}{1+|\xi|^2} \mu(d\xi), I_2 = \int_{|\xi|\geq 1} \frac{1}{1+|\xi|^2} \mu(d\xi)$. If $|\xi| < 1$, then

$$\frac{1}{2} \leq \frac{1}{1+|\xi|^2} \leq 1,$$

so

$$I_1 \leq \int_{|\xi|<1} \mu(d\xi) = C_H \int_{|\xi|<1} |\xi|^{1-2H} (d\xi) < \infty$$

since $1 - 2H + 1 > 0$. On the other hand, if $|\xi| \geq 1$, then

$$\frac{1}{2|\xi|^2} \leq \frac{1}{1+|\xi|^2} \leq \frac{1}{|\xi|^2},$$

so

$$I_2 \leq \int_{|\xi|\geq 1} \frac{1}{|\xi|^2}\mu(d\xi) = C_H \int_{|\xi|\geq 1} \frac{1}{|\xi|^2}|\xi|^{1-2H}(d\xi) = C_H \int_{|\xi|\geq 1} |\xi|^{-2H-1}(d\xi) < \infty$$

since $-2H - 1 + 1 < 0$. ∎

Next we consider the linear stochastic wave equation (1.0.7) with the same noise $\dot{W}$ as in Section 2.1.

Definition 2.2.3. *We say that a process* $\{u(t,x), t \geq 0, x \in \mathbb{R}\}$ *is a* **solution** *to* (1.0.7) *if*

$$u(t,x) = \int_0^t \int_{\mathbb{R}} G^w(t-s, x-y) W(ds, dy)$$

where G^w *is the fundamental solution of the wave equation on* $\mathbb{R}_+ \times \mathbb{R}$ *given by* (1.0.5).

Note that

$$\begin{aligned}\mathcal{F}G^w(t,\cdot)(\xi) &= \int_{\mathbb{R}} e^{-i\xi\cdot x}\frac{1}{2}1_{\{|x|<t|\}}dx = \int_{-t}^t \frac{1}{2}e^{-i\xi\cdot x}dx = \frac{e^{-i|\xi|t} - e^{i|\xi|t}}{-2i|\xi|} \\ &= \frac{\cos(t|\xi|) - i\sin(t|\xi|) - \cos(t|\xi|) - i\sin(t|\xi|)}{-2i|\xi|} \\ &= \frac{\sin(t|\xi|)}{|\xi|}.\end{aligned}$$

The following result corresponds to Lemma 6.1 (b) of [26].

Theorem 2.2.4. *For any* $H \in (\frac{1}{2}, 1)$*, equation* (1.0.7) *with noise* $\dot{W}$ *as in Section 2.1 has a unique solution.*

Proof: As in the proof of Theorem 2.2.2, we will prove that

$$\frac{\cos^2 1}{3}(t^3 \wedge t)\frac{1}{1+|\xi|^2} \leq \int_0^t |\mathcal{F}g_{tx}(s,\cdot)(\xi)|^2 ds \leq 2(t \vee t^3)\frac{1}{1+|\xi|^2}. \tag{2.2.4}$$

where $g_{tx}(s,y) = G^w(t-s, x-y)1_{[0,t]}(s)$. For the upper bound, we consider two cases. If $|\xi| \geq 1$, then

$$\int_0^t |\mathcal{F}g_{tx}(s,\cdot)(\xi)|^2 ds = \int_0^t \frac{\sin^2(s|\xi|)}{|\xi|^2}ds \leq \int_0^t \frac{1}{|\xi|^2}ds = \frac{t}{|\xi|^2} \leq \frac{2}{1+|\xi|^2}t.$$

If $|\xi| < 1$, since $\sin^2 x \leq x^2$ for any $x \geq 0$ and $1 \leq \frac{2}{1+|\xi|^2}$, we have:

$$\int_0^t |\mathcal{F}g_{tx}(s,\cdot)(\xi)|^2 ds = \int_0^t \frac{\sin^2(s|\xi|)}{|\xi|^2} ds \leq \int_0^t \frac{s^2|\xi|^2}{|\xi|^2} ds \leq \int_0^t \frac{2s^2}{1+|\xi|^2} ds = \frac{2}{1+|\xi|^2}\frac{t^3}{3}.$$

For the lower bound, we consider two cases. If $t|\xi| < 1$, since for $x \in [0,1], \sin x \geq x \cos 1$,then

$$\int_0^t |\mathcal{F}g_{tx}(s,\cdot)(\xi)|^2 ds = \int_0^t \frac{\sin^2(s|\xi|)}{|\xi|^2} ds \geq \frac{\cos^2 1}{|\xi|^2}\int_0^t s^2|\xi|^2 ds = \frac{t^3 \cos^2 1}{3} \geq \frac{t^3\cos^2 1}{3(1+|\xi|^2)}.$$

If $t|\xi| \geq 1$, since for $x \geq 1, \sin(2x) < x$, we have:

$$\begin{aligned}\int_0^t |\mathcal{F}g_{tx}(s,\cdot)(\xi)|^2 ds &= \frac{1}{|\xi|^2}\int_0^t \sin^2(s|\xi|) ds = \frac{1}{2|\xi|^2}\int_0^t (1-\cos(2s|\xi|)) ds \\ &= \frac{1}{2|\xi|^2}\left(t - \frac{\sin(2t|\xi|)}{2|\xi|}\right) \geq \frac{1}{2|\xi|^2}\left(t - \frac{t|\xi|}{2|\xi|}\right) = \frac{t}{4|\xi|^2} \geq \frac{t}{4(1+|\xi|^2)}.\end{aligned}$$

This proves (2.2.4). Then

$$\frac{\cos^2 1}{3}(t^3 \wedge t)\int_{\mathbb{R}} \frac{1}{1+|\xi|^2}\mu(d\xi) \leq I_t \leq 2(t \vee t^3)\int_{\mathbb{R}} \frac{1}{1+|\xi|^2}\mu(d\xi),$$

where I_t is the same as in the proof of Theorem 2.2.2. The conclusion follows. ∎

2.3 The Parabolic and Hyperbolic Anderson models

In this section, we consider equations (1.0.2) and (1.0.3) with noise $\dot{W}$ as in Section 2.1.

The goal of this section is to prove that the solutions to equations (1.0.2) and (1.0.3) exist and to show that the second moments of these solutions can be bounded by an exponential function of t. We will also discuss a Feynman-Kac formula for the second moment of the solution to equation (1.0.2).

Definition 2.3.1. *We say that a process* $\{u(t,x); t \geq 0, x \in \mathbb{R}\}$ *is a* **solution** *to equation* (1.0.2)*, respectively equation* (1.0.3)*, if for any* $t > 0, x \in \mathbb{R}$*, with probability 1,*

$$u(t,x) = 1 + \int_0^t \int_{\mathbb{R}} G(t-s, x-y)u(s,y)W(ds,dy), \tag{2.3.1}$$

where $G = G^h$ *for equation* (1.0.2)*, respectively* $G = G^w$ *for equation* (1.0.3)*.*

According to an argument developed in [19], it is known that the solution to (1.0.2) (or (1.0.3)) exists if and only if the series $\sum_{n\geq 1} I_n(f_n(\cdot,t,x))$ converges in $L^2(\Omega)$, and in this case the solution is given by

$$u(t,x) = 1 + \sum_{n\geq 1} I_n(f_n(\cdot,t,x))$$

where I_n is the multiple integral of order n with respect to $\dot{W}$, and

$$f_n(t_1,x_1,\cdots,t_n,x_n,t,x) = G(t-t_n,x-x_n)\cdots G(t_2-t_1,x_2-x_1)1_{\{0<t_1<\cdots<t_n<t\}}.$$

Note that the terms of the series $\sum_{n\geq 1} I_n(f_n(\cdot,t,x))$ are orthogonal in $L^2(\Omega)$. Observe also that the first term of this series is

$$I_1(f_1(\cdot,t,x)) = \int_0^t \int_{\mathbb{R}} G(t-s,x-y)W(ds,dy)$$

which is exactly the solution of the linear equation studied in Section 2.2.

We begin to introduce some basic elements of Malliavin calculus. Let $\widetilde{f_n}(\cdot,t,x)$ be the symmetrization of $f_n(\cdot,t,x)$, i.e.

$$\widetilde{f_n}(t_1,x_1,\cdots,t_n,x_n,t,x) = \frac{1}{n!}\sum_{\rho\in\mathcal{S}_n} f_n(t_{\rho(1)},x_{\rho(1)},\cdots,t_{\rho(n)},x_{\rho(n)},t,x),$$

where $\mathcal{S}_n$ is the set of all permutation of $1,2,\cdots,n$. For instance, for $n=2$,

$$\begin{aligned}\widetilde{f_2}(t_1,x_1,t_2,x_2,t,x) &= \frac{1}{2}[f_2(t_1,x_1,t_2,x_2,t,x) + f_2(t_2,x_2,t_1,x_1,t,x)]\\ &= \frac{1}{2}[G(t-t_2,x-x_2)G(t_2-t_1,x_2-x_1)1_{\{0<t_1<t_2<t\}}\\ &+ G(t-t_1,x-x_1)G(t_1-t_2,x_1-x_2)1_{\{0<t_2<t_1<t\}}].\end{aligned}$$

Moreover, the solution is unique and given by:

$$\begin{aligned}u(t,x) &= 1 + \sum_{n\geq 1} I_n(f_n(\cdot,t,x))\\ &= 1 + \sum_{n\geq 1}\int_0^t\int_{\mathbb{R}}\int_0^{t_n}\int_{\mathbb{R}}\cdots\int_0^{t_2}\int_{\mathbb{R}} G(t-t_n,x-x_n)\cdots G(t_2-t_1,x_2-x_1)\\ &\quad W(dt_1,dx_1)\cdots W(dt_n,dx_n).\end{aligned}$$

We first need to estimate separately $E|I_n(f_n(\cdot,t,x))|^2$.

One first result gives the form of the Fourier transform of the kernel $f_n(\cdot,t,x)$.

Lemma 2.3.2. *For both heat and wave equations, for any* $0 < t_1 < \cdots < t_n < t$, *the function* $(x_1, \cdots, x_n) \mapsto f_n(t_1, x_1, \cdots, t_n, x_n, t, x)$ *is in* $L^1(\mathbb{R}^n)$ *and its Fourier transform is :*

$$\begin{aligned}
&\mathcal{F}f_n(t_1, \cdot, \cdots, t_n, \cdot, t, x)(\xi_1, \cdots, \xi_n) \\
&\quad = \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} e^{-i(\xi_1 x_1 + \cdots + \xi_n x_n)} f_n(t_1, x_1, \cdots, t_n, x_n, t, x) dx_1 \cdots dx_n \\
&\quad = e^{-i(\xi_1 + \cdots + \xi_n) \cdot x} \overline{\mathcal{F}G(t_2 - t_1, \cdot)(\xi_1)} \cdots \overline{\mathcal{F}G(t - t_n, \cdot)(\xi_1 + \cdots + \xi_n)}. \qquad (2.3.2)
\end{aligned}$$

Proof: We prove this lemma by induction on n. We fix $0 < t_1 < \cdots < t_n < t$. First, we take $n = 2$ and let $x_2 - x_1 = y_1$, $x - x_2 = y_2$. We obtain:

$$\begin{aligned}
&\int_{\mathbb{R}} \int_{\mathbb{R}} e^{-i(\xi_1 x_1 + \xi_2 x_2)} G(t - t_2, x - x_2) G(t_2 - t_1, x_2 - x_1) dx_1 dx_2 \\
&= \int_{\mathbb{R}} e^{-i\xi_2 x_2} G(t - t_2, x - x_2) \left(\int_{\mathbb{R}} e^{-i\xi_1 x_1} G(t_2 - t_1, x_2 - x_1) dx_1 \right) dx_2 \\
&= \int_{\mathbb{R}} e^{-i(\xi_1 + \xi_2) x_2} G(t - t_2, x - x_2) \left(\int_{\mathbb{R}} e^{i\xi_1 (x_2 - x_1)} G(t_2 - t_1, x_2 - x_1) dx_1 \right) dx_2 \\
&= \int_{\mathbb{R}} e^{-i(\xi_1 + \xi_2) x_2} G(t - t_2, x - x_2) \left(\int_{\mathbb{R}} e^{i\xi_1 y_1} G(t_2 - t_1, y_1) dy_1 \right) dx_2 \\
&= \overline{\mathcal{F}G(t_2 - t_1, \cdot)(\xi_1)} \int_{\mathbb{R}} e^{-i(\xi_1 + \xi_2) x_2} G(t - t_2, x - x_2) dx_2 \\
&= e^{-i(\xi_1 + \xi_2) x} \overline{\mathcal{F}G(t_2 - t_1, \cdot)(\xi_1)} \int_{\mathbb{R}} e^{i(\xi_1 + \xi_2)(x - x_2)} G(t - t_2, x - x_2) dx_2 \\
&= e^{-i(\xi_1 + \xi_2) x} \overline{\mathcal{F}G(t_2 - t_1, \cdot)(\xi_1)} \int_{\mathbb{R}} e^{i(\xi_1 + \xi_2) y_2} G(t - t_2, y_2) dy_2 \\
&= e^{-i(\xi_1 + \xi_2) x} \overline{\mathcal{F}G(t_2 - t_1, \cdot)(\xi_1)} \cdot \overline{\mathcal{F}G(t - t_2, \cdot)(\xi_1 + \xi_2)}.
\end{aligned}$$

Second, we assume that the equation (2.3.2) holds for $n - 1$ and we prove it for n. Note that

$$f_n(t_1, x_1, \cdots, t_n, x_n, t, x) = G(t - t_n, x - x_n) f_{n-1}(t_1, x_1, \cdots, t_n, x_n) 1_{[0,t]}(t_n).$$

We obtain:

$$\begin{aligned}
&\mathcal{F}f_n(t_1, \cdot, \cdots, t_n, \cdot, t, x)(\xi_1, \cdots, \xi_n) \\
&= \int_{\mathbb{R}^n} e^{-i(\xi_1 x_1 + \cdots + \xi_n x_n)} G(t - t_n, x - x_n) f_{n-1}(t_1, x_1, t_2, x_2, \cdots, t_n, x_n) dx_1 \cdots dx_n \\
&= \int_{\mathbb{R}} e^{-i\xi_n x_n} G(t - t_n, x - x_n) \mathcal{F}f_{n-1}(t_1, \cdot, \cdots, t_{n-1}, \cdot, t_n, x_n)(\xi_1, \cdots, \xi_{n-1}) dx_n \\
&= \overline{\mathcal{F}G(t_2 - t_1, \cdot)(\xi_1)} \cdots \overline{\mathcal{F}G(t_n - t_{n-1}, \cdot)(\xi_1 + \cdots + \xi_{n-1})}
\end{aligned}$$

$$\begin{aligned}
&\int_{\mathbb{R}} e^{-i(\xi_1+\cdots+\xi_n)x_n} G(t-t_n, x-x_n)dx_n \\
&= e^{-i(\xi_1+\cdots+\xi_n)x}\overline{\mathcal{F}G(t_2-t_1,\cdot)(\xi_1)}\cdots\overline{\mathcal{F}G(t_n-t_{n-1},\cdot)(\xi_1+\cdots+\xi_{n-1})} \\
&\int_{\mathbb{R}} e^{i(\xi_1+\cdots+\xi_n)(x-x_n)} G(t-t_n, x-x_n)dx_n \\
&= e^{-i(\xi_1+\cdots+\xi_n)x}\overline{\mathcal{F}G(t_2-t_1,\cdot)(\xi_1)}\cdots\overline{\mathcal{F}G(t_n-t_{n-1},\cdot)(\xi_1+\cdots+\xi_{n-1})} \\
&\int_{\mathbb{R}} e^{i(\xi_1+\cdots+\xi_n)y_n} G(t-t_n, y_n)dy_n \\
&= e^{-i(\xi_1+\cdots+\xi_n)x}\overline{\mathcal{F}G(t_2-t_1,\cdot)(\xi_1)}\cdots\overline{\mathcal{F}G(t-t_n,\cdot)(\xi_1+\cdots+\xi_n)}.
\end{aligned}$$

∎

The following result is an extension of Theorem 2.1.2 to the n-dimensional case. We will use this result for computing $\mathcal{H}^{\bigotimes n}$-norm of the kernel $f_n(\cdot, t, x)$, where $\mathcal{H}^{\bigotimes n}$ is the n-th tensor product of the space $\mathcal{H}$.

Lemma 2.3.3. *For any $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$,*

$$\begin{aligned}
\alpha_H^n &\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \varphi(x_1,\cdots,x_n)\psi(y_1,\cdots,y_n)\prod_{i=1}^n |x_i-y_i|^{2H-2}dx_1\cdots dx_n dy_1\cdots dy_n \\
&= C_H^n \int_{\mathbb{R}^n} \mathcal{F}\varphi(\xi_1,\cdots,\xi_n)\overline{\mathcal{F}\psi(\xi_1,\cdots,\xi_n)}\prod_{i=1}^n |\xi_i|^{1-2H}d\xi_1\cdots d\xi_n.
\end{aligned}$$

Proof: Suppose first that $\varphi(x_1,\cdots,x_n) = \varphi_1(x_1)\cdots\varphi_n(x_n)$ for some $\varphi_1,\cdots,\varphi_n \in \mathcal{S}(\mathbb{R})$, and $\psi(y_1,\cdots,y_n) = \psi_1(y_1)\cdots\psi_n(y_n)$ for some $\psi_1,\cdots,\psi_n \in \mathcal{S}(\mathbb{R})$.

Using Fubini's theorem and Theorem 2.1.2, we get:

$$\begin{aligned}
\alpha_H^n &\int_{\mathbb{R}}\int_{\mathbb{R}} \varphi(x_1,\cdots,x_n)\psi(y_1,\cdots,y_n)\prod_{i=1}^n |x_i-y_i|^{2H-2}dx_1\cdots dx_n dy_1\cdots dy_n \\
&= \prod_{j=1}^n \left(\alpha_H \int_{\mathbb{R}}\int_{\mathbb{R}} \varphi_j(x_j)\psi_j(y_j)|x_j-y_j|^{2H-2}dx_j dy_j\right) \\
&= \prod_{j=1}^n \left(C_H \int_{\mathbb{R}} \mathcal{F}\varphi_j(\xi_j)\overline{\mathcal{F}\psi_j(\xi_j)}|\xi_j|^{1-2H}d\xi_j\right) \\
&= C_H^n \int_{\mathbb{R}} \mathcal{F}\varphi(\xi_1,\cdots,\xi_n)\overline{\mathcal{F}\psi(\xi_1,\cdots,\xi_n)}\prod_{i=1}^n |\xi_i|^{1-2H}d\xi_1\cdots d\xi_n
\end{aligned}$$

since

$$\mathcal{F}\varphi(\xi_1, \cdots, \xi_n) = \mathcal{F}\varphi(\xi_1)\cdots\mathcal{F}\varphi(\xi_n), \quad \overline{\mathcal{F}\psi(\xi_1, \cdots, \xi_n)} = \overline{\mathcal{F}\psi(\xi_1)}\cdots\overline{\mathcal{F}\psi(\xi_n)}.$$

In the general case, for any $\varphi \in \mathcal{S}(\mathbb{R})$, there exists a sequence $(\varphi_n)_n$ of finite linear combinations of functions of the product form mentioned above such that $\|\varphi_n - \varphi\| \to 0$. The general case is proved by approximation. We omit the details. ∎

Remark 2.3.4. *Lemma 2.3.3 can be extended to more general functions φ and ψ on $\mathbb{R}^n$ (similarly to Remark 2.1.3).*

The following lemma gives the exact formula for the second moment of $I_n(f_n(\cdot, t, x))$. We denote

$$T_n(t) = \{0 < t_1 < \cdots < t_n < t\}.$$

Lemma 2.3.5. *For both heat and wave equation,*

$$\begin{aligned} E|I_n(f_n(\cdot, t, x))|^2 &= \alpha_H^n \int_{T_n(t)} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \prod_{i=1}^n |x_i - y_i|^{2H-2} \\ &\qquad f_n(t_1, x_1, \cdots, t_n, x_n, t, x) f_n(t_1, y_1, \cdots, t_n, y_n, t, x) d\mathbf{x} d\mathbf{y} d\mathbf{t} \quad (2.3.3) \\ &= C_H^n \int_{T_n(t)} \int_{\mathbb{R}^n} |\mathcal{F}G(t_2 - t_1, \cdot)(\xi_1)|^2 \cdots |\mathcal{F}G(t - t_n, \cdot)(\xi_1 + \cdots + \xi_n)|^2 \\ &\qquad \times |\xi_i|^{1-2H} \cdots |\xi_n|^{1-2H} d\xi_1 \cdots d\xi_n dt_1 \cdots dt_n. \quad (2.3.4) \end{aligned}$$

In particular,

$$E|I_n(f_n(\cdot, t, x))|^2 \leq C_H^n \int_{T_n(t)} \prod_{j=1}^n \left(\sup_{\eta \in \mathbb{R}} \int_{\mathbb{R}} |\mathcal{F}G(t_{j+1} - t_j, \cdot)(\xi_j + \eta)|^2 |\xi_j|^{1-2H} d\xi_j \right) dt_j, \tag{2.3.5}$$

where by convention we let $t_{n+1} = t$.

Proof: It is known that the multiple integral is not an isometry. More precisely, $E|I_n(f)|^2 = n!|\widetilde{f}|^2_{\mathcal{H}^{\otimes n}}$ for any $f \in \mathcal{H}^{\otimes n}$. Denoting $\mathbf{x} = (x_1, \cdots, x_n), \mathbf{y} = (y_1, \cdots, y_n), \mathbf{t} = (t_1, \cdots, t_n)$, we have:

$$\begin{aligned} E|I_n(f_n(\cdot, t, x))|^2 &= n! \|\widetilde{f}_n(\cdot, t, x)\|^2_{\mathcal{H}^{\otimes n}} \\ &= n! \int_{[0,t]^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \prod_{i=1}^n (\alpha_H |x_i - y_i|^{2H-2}) \\ &\qquad \widetilde{f}_n(t_1, x_1, \cdots, t_n, x_n, t, x) \widetilde{f}_n(t_1, y_1, \cdots, t_n, y_n, t, x) d\mathbf{x} d\mathbf{y} d\mathbf{t} \end{aligned}$$

$$= n!\alpha_H^n \int_{[0,t]^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\prod_{i=1}^n |x_i - y_i|^{2H-2}$$
$$\left(\frac{1}{n!}\sum_{\rho\in\mathcal{S}_n} f_n(t_{\rho(1)}, x_{\rho(1)}, \cdots, t_{\rho(n)}, x_{\rho(n)}, t, x)\right)$$
$$\left(\frac{1}{n!}\sum_{\sigma\in\mathcal{S}_n} f_n(t_{\sigma(1)}, y_{\sigma(1)}, \cdots, t_{\sigma(n)}, y_{\sigma(n)}, t, x)\right) d\mathbf{x}d\mathbf{y}d\mathbf{t}. \tag{2.3.6}$$

Note that

$$\left(\sum_{\rho\in\mathcal{S}_n} f_n(t_{\rho(1)}, x_{\rho(1)}, \cdots, t_{\rho(n)}, x_{\rho(n)}, t, x)\right)\left(\sum_{\sigma\in\mathcal{S}_n} f_n(t_{\sigma(1)}, y_{\sigma(1)}, \cdots, t_{\sigma(n)}, y_{\sigma(n)}, t, x)\right)$$
$$= \sum_{\rho,\sigma\in\mathcal{S}_n} G(t - t_{\rho(n)}, x - x_{\rho(n)})\cdots G(t_{\rho(2)} - t_{\rho(1)}, x_{\rho(2)} - x_{\rho(1)})1_{\{0<t_{\rho(1)}<\cdots<t_{\rho(n)}<t\}}$$
$$G(t - t_{\sigma(n)}, x - y_{\sigma(n)})\cdots G(t_{\sigma(2)} - t_{\sigma(1)}, x_{\sigma(2)} - y_{\sigma(1)})1_{\{0<t_{\sigma(1)}<\cdots<t_{\sigma(n)}<t\}}.$$

If $\rho \neq \sigma$, then

$$1_{\{0<t_{\rho(1)}<\cdots<t_{\rho(n)}<t\}}1_{\{0<t_{\sigma(1)}<\cdots<t_{\sigma(n)}<t\}} = 0.$$

Hence

$$E|I_n(f_n(\cdot,t,x))|^2 = n!\alpha_H^n \sum_{\rho\in\mathcal{S}_n}\int_{\{0<t_{\rho(1)}<\cdots<t_{\rho(n)}<t\}}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\prod_{i=1}^n |x_i - y_i|^{2H-2}$$
$$\frac{1}{(n!)^2} f_n(t_{\rho(1)}, x_{\rho(1)}, \cdots, t_{\rho(n)}, x_{\rho(n)}, t, x) f_n(t_{\rho(1)}, y_{\rho(1)}, \cdots, t_{\rho(n)}, y_{\rho(n)}, t, x) d\mathbf{x}d\mathbf{y}d\mathbf{t}$$
$$= \frac{1}{n!}\alpha_H^n \sum_{\rho\in\mathcal{S}_n}\int_{\{0<t_{\rho(1)}<\cdots<t_{\rho(n)}<t\}}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\prod_{i=1}^n |x_i - y_i|^{2H-2}$$
$$f_n(t_{\rho(1)}, x_{\rho(1)}, \cdots, t_{\rho(n)}, x_{\rho(n)}, t, x) f_n(t_{\rho(1)}, y_{\rho(1)}, \cdots, t_{\rho(n)}, y_{\rho(n)}, t, x) d\mathbf{x}d\mathbf{y}d\mathbf{t}.$$

Using the change of variables: $t_{\rho(i)} = t_i', x_{\rho(i)} = x_i', y_{\rho(i)} = y_i'$, it follows that

$$E|I_n(f_n(\cdot,t,x))|^2 = \frac{1}{n!}\alpha_H^n \sum_{\rho\in\mathcal{S}_n}\int_{\{0<t_1'<\cdots<t_n'<t\}}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\prod_{i=1}^n |x_i' - y_i'|^{2H-2}$$
$$f_n(t_1', x_1', \cdots, t_n', x_n', t, x) f_n(t_1', y_1', \cdots, t_n', y_n', t, x) d\mathbf{x}'d\mathbf{y}'d\mathbf{t}'.$$

Note that all the integrals appearing above are the same. Therefore, relation (2.3.3) follows. By Remark 2.3.4 and relation (2.3.3),

$$E|I_n(f_n(\cdot,t,x))|^2 = C_H^n \int_{T_n(t)}\int_{\mathbb{R}^n} |\mathcal{F}f_n(t_1, \cdot, \cdots, t_n, \cdot, t, x)(\xi_1, \cdots, \xi_n)|^2$$

$$\prod_{i=1}^{n} |\xi_i|^{1-2H} d\xi_1 \cdots d\xi_n dt_1 \cdots dt_n.$$

Using Lemma 2.3.2 and noting that $|e^{-i(\xi_1+\cdots+\xi_n)x}| = 1$, we obtain relation (2.3.4). Then we obtain:

$$E|I_n(f_n(\cdot,t,x))|^2 \leq C_H^n \int_{T_n(t)} \int_{\mathbb{R}^{n-1}} \prod_{i=1}^{n-1} \mathcal{F}G(t_{i+1}-t_i,\cdot)(\sum_{k=1}^{i} \xi_k)$$
$$\left(\sup_{\eta\in\mathbb{R}} \int_{\mathbb{R}} \mathcal{F}G(t-t_n,\cdot)(\xi_n+\eta)|\xi_n|^{1-2H} d\xi_n\right) \prod_{i=1}^{n-1} |\xi_i|^{1-2H} d\xi_1 \cdots d\xi_{n-1} dt_1 \cdots dt_{n-1}$$
$$\leq C_H^n \int_{T_n(t)} \prod_{i=1}^{n} \left(\sup_{\eta\in\mathbb{R}} \int_{\mathbb{R}} |\mathcal{F}G(t_{i+1}-t_i,\cdot)(\xi_i+\eta)|^2 |\xi_i|^{1-2H} d\xi_i\right) dt_1 \cdots dt_n.$$

∎

To evaluate the supremum appearing in relation (2.3.5), we use the following result. Part *b*) of this result is taken from Lemma 3.1 of [2]. Part *a*) is new, but was used implicitly in [15].

Lemma 2.3.6. *a) For any* $\alpha \in (0,d)$,

$$\sup_{\eta\in\mathbb{R}^d} \int_{\mathbb{R}^d} |\mathcal{F}G^h(t,\cdot)(\xi+\eta)|^2 |\xi|^{-\alpha} d\xi \leq C_{\alpha,d} t^{-(d-\alpha)/2}, \tag{2.3.7}$$

where $C_{\alpha,d}$ *is a constant depending on* α *and* d. *In particular for* $d=1$ *and* $\alpha = 2H-1$ *with* $H \in (\frac{1}{2},1)$,

$$\sup_{\eta\in\mathbb{R}} \int_{\mathbb{R}} |\mathcal{F}G^h(t,\cdot)(\xi+\eta)|^2 |\xi|^{1-2H} d\xi \leq C_H t^{H-1},$$

where C_H *is a constant depending on* H.
b) For any $\alpha \in (0,d)$,

$$\sup_{\eta\in\mathbb{R}^d} \int_{\mathbb{R}^d} |\mathcal{F}G^w(t,\cdot)(\xi+\eta)|^2 |\xi|^{-\alpha} d\xi \leq C'_{\alpha,d} t^{\alpha-d+2}, \tag{2.3.8}$$

where $C'_{\alpha,d}$ *is a constant depending on* α *and* d. *In particular for* $d=1$ *and* $\alpha = 2H-1$ *with* $H \in (\frac{1}{2},1)$,

$$\sup_{\eta\in\mathbb{R}} \int_{\mathbb{R}} |\mathcal{F}G^w(t,\cdot)(\xi+\eta)|^2 |\xi|^{1-2H} d\xi \leq C'_H t^{2H},$$

where C'_H *is a constant depending on* H.

Proof: a) Recall that $\mathcal{F}G^h(t,\cdot)(\xi) = \exp(-\frac{1}{2}|\xi|^2 t)$. We use the change of variables $\xi' = \sqrt{t}(\xi + \eta)$. Then $\xi = \xi'/\sqrt{t} - \eta$ and $d\xi = t^{-d/2} d\xi'$. We obtain:

$$\begin{aligned}
\int_{\mathbb{R}^d} |\mathcal{F}G^h(t,\cdot)(\xi+\eta)|^2 |\xi|^{-\alpha} d\xi &= \int_{\mathbb{R}^d} e^{-|\xi+\eta|^2 t} |\xi|^{-\alpha} d\xi \\
&= t^{-d/2} \int_{\mathbb{R}^d} e^{-|\xi'|^2} \left| \frac{\xi'}{\sqrt{t}} - \eta \right|^{-\alpha} d\xi' = \left(\frac{1}{\sqrt{t}} \right)^{-\alpha} t^{-d/2} \int_{\mathbb{R}^d} e^{-|\xi'|^2} |\xi' - \sqrt{t}\eta|^{-\alpha} d\xi' \\
&= t^{-(d-\alpha)/2} \int_{\mathbb{R}^d} e^{-|\xi'|^2} |\xi' - \sqrt{t}\eta|^{-\alpha} d\xi' = t^{-(d-\alpha)/2} I(\sqrt{t}\eta),
\end{aligned}$$

where for any $a \in \mathbb{R}^d$, we define

$$I(a) = \int_{\mathbb{R}^d} e^{-|\xi'|^2} |\xi' - a|^{-\alpha} d\xi' = \int_{\mathbb{R}^d} e^{-|\xi+a|^2} |\xi|^{-\alpha} d\xi.$$

We use the inequality $e^{-x} \leq \frac{1}{1+x}$ for all $x \geq 0$. Hence

$$I(a) \leq \int_{\mathbb{R}^d} \frac{1}{1+|\xi+a|^2} |\xi|^{-\alpha} d\xi \leq \sup_{a \in \mathbb{R}^d} \int_{\mathbb{R}^d} \frac{1}{1+|\xi+a|^2} |\xi|^{-\alpha} d\xi = \int_{\mathbb{R}^d} \frac{1}{1+|\xi|^2} |\xi|^{-\alpha} d\xi$$

where the last equation follows by Lemma 4.1 of [9]. Note that since $d-2 < \alpha < d$,

$$\int_{\mathbb{R}^d} \frac{1}{1+|\xi|^2} |\xi|^{-\alpha} d\xi < \infty.$$

To see this, we consider separately integrals on $|\xi| \leq 1$ and $\{|\xi| > 1\}$. When $|\xi| \leq 1$, we use the inequality $\frac{1}{1+|\xi|^2} \leq 1$, and hence

$$\int_{|\xi| \leq 1} \frac{1}{1+|\xi|^2} |\xi|^{-\alpha} d\xi \leq \int_{|\xi| \leq 1} |\xi|^{-\alpha} d\xi = C_d \int_0^1 r^{-\alpha} r^{d-1} dr < \frac{1}{d-\alpha}$$

since $\alpha < d$. Here C_d is the area of the unit sphere $S_1(0) = \{z \in \mathbb{R}^d; |z| = 1\}$ in $\mathbb{R}^d$. When $|\xi| > 1$, we use the inequality $\frac{1}{1+|\xi|^2} \leq \frac{1}{|\xi|^2}$, and hence

$$\int_{|\xi| > 1} \frac{1}{1+|\xi|^2} |\xi|^{-\alpha} d\xi \leq \int_{|\xi| > 1} |\xi|^{-\alpha-2} d\xi = C_d \int_1^\infty r^{-\alpha-2} r^{d-1} dr < \frac{1}{\alpha - d + 2}$$

since $\alpha > d-2$.

b) Recall that $\mathcal{F}G^w(t,x) = \frac{\sin(t|\xi|)}{|\xi|}$. We use the change of variable $\xi' = t|\xi + \eta|$. Then $\xi = \frac{\xi'}{t} - \eta$ and $d\xi = t^{-d} d\xi'$. Then we obtain:

$$\int_{\mathbb{R}^d} |\mathcal{F}G^w(t,\cdot)(\xi+\eta)|^2 |\xi|^{-\alpha} d\xi = \int_{\mathbb{R}^d} \frac{\sin^2(t|\xi+\eta|)}{|\xi+\eta|^2} |\xi|^{-\alpha} d\xi$$

$$\begin{aligned}
&= t^{-d}\int_{\mathbb{R}^d} \frac{\sin^2(|\xi'|)}{|\xi'/t|^2}\left|\frac{\xi'-t\eta}{t}\right|^{-\alpha} d\xi' = \left(\frac{1}{t}\right)^{-\alpha} t^2 t^{-d}\int_{\mathbb{R}^d}\frac{\sin^2(|\xi'|)}{|\xi'|^2}|\xi'-t\eta|^{-\alpha}d\xi' \\
&= t^{\alpha-d+2}\int_{\mathbb{R}^d}\frac{\sin^2(|\xi'|)}{|\xi'|^2}|\xi'-t\eta|^{-\alpha}d\xi' = t^{\alpha-d+2}J(t\eta),
\end{aligned}$$

where for any $a \in \mathbb{R}^d$, we define

$$J(a) = \int_{\mathbb{R}^d}\frac{\sin^2(|\xi'|)}{|\xi'|^2}|\xi'-a|^{-\alpha}d\xi' = \int_{\mathbb{R}^d}\frac{\sin^2(|\xi+a|)}{|\xi+a|^2}|\xi|^{-\alpha}d\xi.$$

We use the inequality $\frac{\sin^2 x}{x^2} \leq \frac{2}{1+x^2}$ for all $x > 0$. Hence,

$$\begin{aligned}
J(a) &\leq 2\int_{\mathbb{R}^d}\frac{1}{1+|\xi+a|^2}|\xi|^{-\alpha}d\xi \leq 2\sup_{a\in\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{1}{1+|\xi+a|^2}|\xi|^{-\alpha}d\xi \\
&= 2\int_{\mathbb{R}^d}\frac{1}{1+|\xi|^2}|\xi|^{-\alpha}d\xi < \infty.
\end{aligned}$$

For the last equality, we used Lemma 4.1 of [9]. ∎

We are now ready to state main result about the existence of solution.

Theorem 2.3.7. *For any $H \in (\frac{1}{2}, 1)$, equations* (1.0.2) *and* (1.0.3) *with noise $\dot{W}$ as in Section 2.1 have unique solutions. Moreover,*

$$E|u(t,x)|^2 \leq C_1 \exp(C_2 t)$$

where $C_1 > 0$ and $C_2 > 0$ are constants depending on H.

Proof: We need to prove that the series $\sum_{n\geq 1} I_n(f_n(\cdot,t,x))$ converges in $L^2(\Omega)$ (see page 14). Since $E[I_n(f_n(\cdot,t,x))I_m(f_m(\cdot,t,x))] = 0$ for any $n \neq m$, it is enough to prove that:

$$\sum_{n\geq 1} E|I_n(f_n(\cdot,t,x))|^2 < \infty.$$

By Lemmas 2.3.5 and 2.3.6, we have:

$$E|I_n(f_n(\cdot,t,x))|^2 \leq C^n \int_{\{0<t_1<\cdots<t_n<t\}}\prod_{j=1}^n (t_{j+1}-t_j)^a dt_1\cdots dt_n,$$

where C is a constant depending on H and

$$a = \begin{cases} H-1 & \text{for heat equation} \\ 2H & \text{for wave equation.}\end{cases}$$

Therefore, using Lemma A.3 with $\beta_j = a$ for all $j = 1, \cdots, n$, it follows that

$$E|I_n(f_n(\cdot, t, x))|^2 \leq C^n \frac{\Gamma(1+a)^n}{\Gamma(n(1+a)+1)} t^{n(1+a)}.$$

Using inequality (A.9),

$$\begin{aligned} E|u(t,x)|^2 &= \sum_{n\geq 0} E|I_n(f_n(\cdot,t,x))|^2 \leq \frac{1}{C'} \sum_{n\geq 0} \frac{C^n \Gamma(1+a)^n}{\Gamma(n(1+a)+1)} t^{n(1+a)} \\ &\leq C_1 \exp\left\{(C\Gamma(1+a)t^{1+a})^{1/(1+a)}\right\} = C_1 \exp(C_2 t), \end{aligned}$$

where C_1 and C_2 are constants depending on H. Therefore, the series $I_n(f_n(\cdot, t, x))$ converges in $L^2(\Omega)$, and equations (1.0.2) and (1.0.3) have unique solutions. ▮

In the remaining part of this section, we discuss a Feynman-Kac representation for the second moment of the solution of equation (1.0.2). This result is in fact valid for a spatially homogeneous Gaussian noise $\dot{W}$ with a general covariance function f.

Theorem 2.3.8. *(Feynman-Kac Representation for Second Moment) If u^h is the solution of equation* (1.0.2)*, then*

$$E|u^h(t,x)|^2 = E[\exp(L(t))]$$

where $L(t) = \int_0^t f(B_s^1 - B_s^2)ds$, $(B_s^1)_{s\geq 0}$ and $(B_s^2)_{s\geq 0}$ are independent Brownian motions and $f(x) = \alpha_H |x|^{2H-2}$.

Proof: By equation (2.3.3), we have

$$\begin{aligned} E|u(t,x)|^2 = \sum_{n\geq 0} E|I_n(f_n(\cdot,t,x))|^2 = \sum_{n\geq 0} \int_{T_n(t)} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \prod_{i=1}^n f(x_i - y_i) \\ f_n(t_1, x_1, \cdots, t_n, x_n, t, x) f_n(t_1, y_1, \cdots, t_n, y_n, t, x) d\mathbf{x} d\mathbf{y} d\mathbf{t}. \end{aligned}$$

For any $0 < t_1 < \cdots < t_n < t$ fixed,

$$f_n(t_1, x_1, \cdots, t_n, x_n, t, x) = G^h(t - t_n, x - x_n) \cdots G^h(t_2 - t_1, x_2 - x_1)$$

and

$$f_n(t_1, y_1, \cdots, t_n, y_n, t, x) = G^h(t - t_n, x - y_n) \cdots G^h(t_2 - t_1, y_2 - y_1).$$

Thus,

$$E|u(t,x)|^2 = \sum_{n\geq 0} \int_{T_n(t)} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \prod_{i=1}^n f(x_i - y_i) G^h(t - t_n, x - x_n) \cdots G^h(t_2 - t_1, x_2 - x_1)$$

$$G^h(t-t_n, x-y_n)\cdots G^h(t_2-t_1, y_2-y_1)d\mathbf{x}d\mathbf{y}d\mathbf{t}.$$

We use the change of variables $t - t_{n+1-i} = t'_i$, $x - x_{n+1-i} = x'_i$, $y - y_{n+1-i} = y'_i$ for $i = 1, \cdots, n$. Note that for the new variables, $0 < t'_1 < \cdots < t'_n < t$. We obtain:

$$\begin{aligned} E|u(t,x)|^2 = \sum_{n\geq 0} \int_{T_n(t)} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \prod_{k=1}^n f(x'_k - y'_k) \\ G^h(t'_1, x'_1)G^h(t'_2 - t'_1, x'_2 - x'_1)\cdots G^h(t'_n - t'_{n-1}, x'_n - x'_{n-1}) \\ G^h(t'_1, y'_1)G^h(t'_2 - t'_1, y'_2 - y'_1)\cdots G^h(t'_n - t'_{n-1}, y'_n - y'_{n-1})d\mathbf{x}'d\mathbf{y}'d\mathbf{t}'. \end{aligned}$$

Then we denote $t'_i = t_i$, $x'_i = x_i$, $y'_i = y_i$. According to Lemma B.1,

$$G^h(t_1, x_1)G^h(t_2 - t_1, x_2 - x_1)\cdots G^h(t_n - t_{n-1}, x_n - x_{n-1}) = f_{\mathbf{X}}(x_1, \cdots, x_n),$$
$$G^h(t_1, y_1)G^h(t_2 - t_1, y_2 - y_1)\cdots G^h(t_n - t_{n-1}, y_n - y_{n-1}) = f_{\mathbf{Y}}(y_1, \cdots, y_n),$$

are the density functions of vectors $\mathbf{X} = (B^1_{t_1}, \cdots, B^1_{t_n})$ and $\mathbf{Y} = (B^2_{t_1}, \cdots, B^2_{t_n})$, respectively. Then we have

$$\begin{aligned} E|u(t,x)|^2 = \sum_{n\geq 0} \int_{T_n(t)} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \prod_{k=1}^n f(x_i - y_i) \\ f_{B^1_{t_1},\cdots,B^1_{t_n}}(x_1, \cdots, x_n) f_{B^2_{t_1},\cdots,B^2_{t_n}}(y_1, \cdots, y_n)d\mathbf{x}d\mathbf{y}d\mathbf{t}. \end{aligned}$$

We use the fact that for any independent vectors $\mathbf{X}$, $\mathbf{Y}$ of dimension n, and for any function $h : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$,

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} h(\mathbf{x}, \mathbf{y}) f_{\mathbf{X}}(\mathbf{x}) f_{\mathbf{Y}}(\mathbf{y})d\mathbf{x}d\mathbf{y} = E[h(\mathbf{X}, \mathbf{Y})].$$

Applying this to the function $h(\mathbf{x}, \mathbf{y}) = \prod_{i=1}^n f(x_i - y_i)$, we obtain

$$E|u(t,x)|^2 = \sum_{n\geq 0} \int_{T_n(t)} E\left[\prod_{k=1}^n f(B^1_{t_k} - B^2_{t_k})\right] d\mathbf{t} = E\left[\sum_{n\geq 0} \int_{T_n(t)} \prod_{k=1}^n f(B^1_{t_k} - B^2_{t_k})d\mathbf{t}\right].$$

Let $F(t_1, \cdots, t_n) = \prod_{k=1}^n f(B^1_{t_k} - B^2_{t_k})$. Note that F is symmetric. By Lemma A.8, we obtain

$$\begin{aligned} E|u(t,x)|^2 &= E\left[\sum_{n\geq 0} \frac{1}{n!} \int_{[0,t]^n} \prod_{k=1}^n f(B^1_{t_k} - B^2_{t_k})d\mathbf{t}\right] = E\left[\sum_{n\geq 0} \frac{1}{n!} \left(\int_0^t f(B^1_s - B^2_s)ds\right)^n\right] \\ &= E\left[\exp\left\{\int_0^t f(B^1_s - B^2_s)ds\right\}\right] = E[\exp(L(t))]. \end{aligned}$$

∎

Chapter 3

The case $H_0 = \frac{1}{2}$ and $H < \frac{1}{2}$

In this chapter, we consider a Gaussian noise which is white in time and fractional in space with index $H < \frac{1}{2}$. This noise corresponds to a fractional Brownian sheet on $\mathbb{R}_+ \times \mathbb{R}$, with the index $H_0 = \frac{1}{2}$ in time and index $H < \frac{1}{2}$ in space. First, we examine the question of existence of solution for the linear stochastic heat and wave equations (1.0.6) and (1.0.7) with this noise. Next, we consider equations (1.0.2) and (1.0.3) with this noise and we show that the second moments of these solutions admit upper and lower bounds which are exponential function of t.

The results presented in this chapter are taken from references [6] and [16]. The case $H < 1/2$ is more difficult than the case $H > 1/2$ because in this case, we cannot use the representation (1.0.1) of the covariance of the fBm.

3.1 The noise and the linear equations

In this section, we introduce the noise perturbing the equations, and we discuss the linear equations. This section is similar to Section 2.1.

Let $W = \{W([0,t] \times A); t \geq 0, A \in \mathcal{B}_b(\mathbb{R})\}$ be a zero-mean Gaussian process with covariance

$$E[W([0,t] \times A) \cdot W([0,s] \times B)] = (t \wedge s) C_H \int_{\mathbb{R}} \mathcal{F}1_A(\xi)\overline{\mathcal{F}1_B(\xi)}|\xi|^{1-2H} d\xi \qquad (3.1.1)$$
$$:= \langle 1_{[0,t]\times A}, 1_{[0,s]\times B}\rangle_{\mathcal{H}}.$$

where $H \in (0, \frac{1}{2})$ and C_H is given by (2.1.4).

Set $W(1_{[0,t]\times A}) = W([0,t] \times A)$. Let $\mathcal{E}$ be the set of linear combinations of indicator functions of the form $1_{[0,t]\times A}$ with $t > 0$. By linearity, we extend W to $\mathcal{E}$.

Lemma 3.1.1. *W is an isometry from $\mathcal{E}$ to $L^2(\Omega)$, i.e. for any $\varphi, \psi \in \mathcal{E}$,*

$$E[W(\varphi)W(\psi)] = \langle \varphi, \psi \rangle_{\mathcal{H}} = C_H \int_0^\infty \int_{\mathbb{R}} \mathcal{F}\varphi(t,\cdot)(\xi)\overline{\mathcal{F}\psi(t,\cdot)(\xi)}|\xi|^{1-2H} dt d\xi.$$

Proof: Let

$$\varphi(t,x) = \sum_{i=1}^{n} a_i 1_{[0,t_i]}(t) 1_{A_i}(x), \quad \psi(t,x) = \sum_{j=1}^{m} b_j 1_{[0,s_j]}(t) 1_{B_j}(x).$$

Then

$$W(\varphi) = \sum_{i=1}^{n} a_i W(1_{[0,t_i]\times A_i}), \quad W(\psi) = \sum_{j=1}^{m} b_j W(1_{[0,s_j]\times B_j}),$$

and

$$\mathcal{F}\varphi(t,\cdot)(\xi) = \sum_{i=1}^{n} a_i 1_{[0,t_i]}(t) \mathcal{F}1_{A_i}(\xi), \quad \mathcal{F}\psi(t,\cdot)(\xi) = \sum_{j=1}^{m} b_j 1_{[0,s_j]}(t) \mathcal{F}1_{B_j}(\xi).$$

Therefore, we obtain:

$$\begin{aligned}
E[W(\varphi)W(\psi)] &= E\left[\left(\sum_{i=1}^{n} a_i W(1_{[0,t_i]\times A_i})\right)\left(\sum_{j=1}^{m} b_j W(1_{[0,s_j]\times B_j})\right)\right] \\
&= \sum_{i=1}^{n}\sum_{j=1}^{m} a_i b_j E[W(1_{[0,t_i]\times A_i}) \cdot W(1_{[0,s_j]\times B_j})] \\
&= \sum_{i=1}^{n}\sum_{j=1}^{m} a_i b_j \left(\int_0^\infty 1_{[0,t_i]}(t) 1_{[0,s_j]}(t) dt\right)\left(C_H \int_{\mathbb{R}} \mathcal{F}1_{A_i}(\xi)\overline{\mathcal{F}1_{B_j}(\xi)}|\xi|^{1-2H} d\xi\right) \\
&= C_H \int_0^\infty \int_{\mathbb{R}} \left(\sum_{i=1}^{n} a_i 1_{[0,t_i]}(t)\mathcal{F}1_{A_i}(\xi)\right)\left(\sum_{j=1}^{m} b_j 1_{[0,s_j]}(t)\overline{\mathcal{F}1_{B_j}(\xi)}\right)|\xi|^{1-2H} dt d\xi \\
&= C_H \int_0^\infty \int_{\mathbb{R}} \mathcal{F}\varphi(t,\cdot)(\xi)\overline{\mathcal{F}\psi(t,\cdot)(\xi)}|\xi|^{1-2H} d\xi.
\end{aligned}$$

∎

Let $\mathcal{H}$ be the completion of the $\mathcal{E}$ with respect to the inner product $\langle\cdot,\cdot\rangle_{\mathcal{H}}$. By Lemma 3.1.1, the map $\varphi \mapsto W(\varphi)$ is an isometry from $\mathcal{E}$ to $L^2(\Omega)$ which we extend to $\mathcal{H}$. For any $\varphi \in \mathcal{H}$, we say that $W(\varphi)$ is the *Wiener integral* of φ with respect to W and we denote

$$W(\varphi) = \int_0^\infty \int_{\mathbb{R}} \varphi(t,x) W(dt,dx).$$

The following result is the analogue of Theorem 2.1.4 for the case $H < \frac{1}{2}$.

Theorem 3.1.2. *Let* $\varphi : [0,T] \times \mathbb{R} \mapsto \mathbb{R}$ *be such that* $\varphi(t,\cdot) \in L^1(\mathbb{R})$ *for all* $t \in [0,T]$ *and* $\mathcal{F}\varphi(t,\cdot)$ *be the Fourier transform of* $\varphi(t,\cdot)$*. If*

$$\|\varphi\|_o^2 := C_H \int_0^T \int_{\mathbb{R}} |\mathcal{F}\varphi(t,\cdot)(\xi)|^2 |\xi|^{1-2H} d\xi dt < \infty,$$

then $\varphi \in \mathcal{H}$*, and* $\|\varphi\|_{\mathcal{H}}^2 = \|\varphi\|_o^2$*.*

Next we consider the linear heat equation (1.0.6) and the linear wave equation (1.0.7) with noise $\dot{W}$ above. The solution to these equations are defined as in Definition 2.2.1, respectively Definition 2.2.3.

The following theorem is proved similarly to Theorem 2.2.2 and Theorem 2.2.4; we omit the details.

Theorem 3.1.3. *For any* $H \in (0, \frac{1}{2})$*, equations* (1.0.6) *and* (1.0.7) *with noise* $\dot{W}$ *with covariance given by* (3.1.1) *have unique solutions.*

Remark 3.1.4. Similarly to Remark 2.1.5, the indicator function $1_{[0,t]\times[0,x]}$ belongs to the space $\mathcal{E}$, (since $[0,x]$ is a bounded set in $\mathbb{R}$), hence it belongs to the space $\mathcal{H}$ for any $t > 0$ and $x \in \mathbb{R}$. Letting $W(t,x) := W(1_{[0,t]\times[0,x]})$, it can be proved that the process $\{W(t,x); t \geq 0, x \in \mathbb{R}\}$ is a fractional Brownian sheet with index $H_0 = \frac{1}{2}$ in time and index H in space.

3.2 The Parabolic and Hyperbolic Anderson models

In this section, we consider equations (1.0.2) and (1.0.3) with noise $\dot{W}$ as in Section 3.1. The solutions to these equations are defined as in Definition 2.3.1. We follow the discussion in [16] for (1.0.2) and [6] for (1.0.3). (Note that reference [16] considers a more general initial condition u_0.)

We begin by recalling an elementary result taken from [5].

Lemma 3.2.1. *(Lemma 3.1 of [5]) a) The integral* $\int_{\mathbb{R}} |\mathcal{F}G^h(t,\cdot)(\xi)|^2 |\xi|^{\alpha} d\xi$ *is finite if and only if* $\alpha > -1$*, and in this case,*

$$\int_{\mathbb{R}} |\mathcal{F}G^h(t,\cdot)(\xi)|^2 |\xi|^{\alpha} d\xi = \Gamma\left(\frac{1+\alpha}{2}\right) t^{-(1+\alpha)/2}. \tag{3.2.1}$$

b) The integral $\int_{\mathbb{R}} |\mathcal{F}G^w(t,\cdot)(\xi)|^2 |\xi|^{\alpha} d\xi$ *is finite if and only if* $\alpha \in (-1,1)$*, and in this case*

$$\int_{\mathbb{R}} |\mathcal{F}G^w(t,\cdot)(\xi)|^2 |\xi|^{\alpha} d\xi = 2^{1-\alpha} C_\alpha t^{1-\alpha}, \tag{3.2.2}$$

where

$$C_\alpha = \begin{cases} (1-\alpha)^{-1}\Gamma(\alpha)\sin(\pi\alpha/2) & if \alpha \in (0,1) \\ \alpha^{-1}(1-\alpha)^{-1}\Gamma(1+\alpha)\sin(\pi\alpha/2) & if \alpha \in (-1,0) \\ \pi/2 & if \alpha = 0 \end{cases}$$

Proof: a) We change the variable of $\xi' = t\xi$ and $x^2 = y$. We get:

$$\begin{aligned}\int_{\mathbb{R}} |\mathcal{F}G^h(t,\cdot)(\xi)|^2|\xi|^\alpha d\xi &= \int_{\mathbb{R}} e^{-t|\xi|^2}|\xi|^\alpha d\xi = \frac{1}{\sqrt{t}}\int_{\mathbb{R}} e^{-|\xi'|^2}\Big|\frac{\xi'}{\sqrt{t}}\Big|^\alpha d\xi' \\ &= t^{-(1+\alpha)/2}\int_{\mathbb{R}} e^{-|\xi'|^2}|\xi'|^\alpha d\xi' = 2t^{-(1+\alpha)/2}\int_0^\infty e^{-x^2}x^\alpha dx \\ &= 2t^{-(1+\alpha)/2}\int_0^\infty e^{-y}y^{\alpha/2}\frac{1}{2\sqrt{y}}dy = t^{-(1+\alpha)/2}\int_0^\infty e^{-y}y^{(\alpha-1)/2}dy \\ &= t^{-(1+\alpha)/2}\Gamma\left(\frac{1+\alpha}{2}\right) = C_\alpha t^{-(1+\alpha)/2}.\end{aligned}$$

b) Using the change of variable $\xi' = t\xi$, we have:

$$\begin{aligned}\int_{\mathbb{R}} |\mathcal{F}G^w(t,\cdot)(\xi)|^2|\xi|^\alpha d\xi &= \int_{\mathbb{R}} \frac{\sin^2(|t\xi|)}{|\xi|^2}|\xi|^\alpha d\xi = \int_{\mathbb{R}} \frac{\sin^2(|t\xi|)}{|\xi|^{2-\alpha}}d\xi = t^{-1}\int_{\mathbb{R}} \frac{\sin^2(|\xi'|)}{|\xi'/t|^{2-\alpha}}d\xi' \\ = t^{1-\alpha}\int_{\mathbb{R}} \frac{\sin^2(|\xi'|)}{|\xi'|^{2-\alpha}}d\xi' &= 2t^{1-\alpha}\int_0^\infty \frac{\sin^2 x}{x^{2-\alpha}}dx = t^{1-\alpha}\int_0^\infty \frac{1-\cos(2x)}{x^{2-\alpha}}dx = C'_\alpha t^{1-\alpha},\end{aligned}$$

where C'_α is a constant depending on α. Note that by Lemma D.1 of [5]

$$\int_0^\infty \frac{1-\cos(x)}{x^a}dx < \infty$$

if and only if $a \in (1,3)$. In our case, $2-\alpha \in (1,3)$ since $\alpha \in (-1,1)$. ▮

We are now ready to give the main results of this section, regarding the existence of the solution. For the proof of this result, we use the argument contained in [6] and [16].

Theorem 3.2.2. *If $H \in (\frac{1}{4}, \frac{1}{2})$, then equations* (1.0.2) *and* (1.0.3) *have unique solutions. Moreover, if $u(t,x)$ is the solution to either one of these equations, then*

$$C'_1 \exp\{C'_2 t\} \le E|u(t,x)|^2 \le C_1 \exp\{C_2 t\}$$

for some positive constants C_1, C_2, C'_1, C'_2 depending on H.

Proof: **Step 1** (Upper bound): We use equation (2.3.4) which also holds in the case of the noise which is white in time and fractional in space with index $H < \frac{1}{2}$. We let $T_n(t) = \{0 < t_1 < \cdots < t_n < t\}$ and $t_{n+1} = t$. Then

$$\begin{aligned} E|I_n(f_n(\cdot,t,x))|^2 &= C_H^n \int_{T_n(t)} \int_{\mathbb{R}^n} |\mathcal{F}G(t_2-t_1,\cdot)(\xi_1)|^2 \cdots \\ &\qquad \times |\mathcal{F}G(t-t_n,\cdot)(\xi_1+\cdots+\xi_n)|^2 \prod_{i=1}^n |\xi_i|^{1-2H} d\xi_1 \cdots d\xi_n dt_1 \cdots dt_n \\ &= C_H^n \int_{T_n(t)} \int_{\mathbb{R}^n} \prod_{i=1}^n |\mathcal{F}G(t_{i+1}-t_i,\cdot)(\eta_i)|^2 \\ &\qquad \times |\eta_1|^{1-2H} |\eta_2-\eta_1|^{1-2H} \cdots |\eta_n-\eta_{n-1}|^{1-2H} d\eta_1 \cdots d\eta_n dt_1 \cdots dt_n, \end{aligned} \tag{3.2.3}$$

where we use the change of variables $\eta_i = \xi_1 + \cdots + \xi_i$ for $i = 1, 2, \cdots, n$. Using the fact that $(a+b)^p \le a^p + b^p$ for any $p \in (0,1)$ and $a, b > 0$, we obtain

$$|\eta_j - \eta_{j-1}|^{1-2H} \le (|\eta_j| + |\eta_{j-1}|)^{1-2H} \le |\eta_j|^{1-2H} + |\eta_{j-1}|^{1-2H} \tag{3.2.4}$$

since $H < 1/2$. We use the following fact: for any finite set $\mathcal{S}$ and positive numbers $(a_i)_{i\in\mathcal{S}}$ and $(b_j)_{j\in\mathcal{S}}$,

$$\prod_{j\in\mathcal{S}} (a_j + b_j) = \sum_{I\subset\mathcal{S}} \Big(\prod_{j\in I} a_j\Big)\Big(\prod_{j\in S\setminus I} b_j\Big).$$

Hence

$$\begin{aligned} \prod_{j=2}^n |\eta_j - \eta_{j-1}|^{1-2H} &\le \prod_{j=2}^n (|\eta_j|^{1-2H} + |\eta_{j-1}|^{1-2H}) \\ &= \sum_{I\subset\{2,\cdots,n\}} \left(\prod_{j\in I} |\eta_{j-1}|^{1-2H}\right)\left(\prod_{j\in\{2,\cdots,n\}\setminus I} |\eta_j|^{1-2H}\right) \\ &= \sum_{I\subset\{2,\cdots,n\}} \left(\prod_{j\in I-1} |\eta_j|^{1-2H}\right)\left(\prod_{j\in\{2,\cdots,n\}\setminus I} |\eta_j|^{1-2H}\right) \end{aligned}$$

where $I-1 = \{j-1; j \in I\}$. Note that the last sum can be written as $\sum_{\alpha\in D_n} \prod_{j=1}^n |\eta_j|^{\alpha_j}$ where D_n is a set of cardinality 2^{n-1} of multi-indices $\alpha = (\alpha_1, \cdots, \alpha_n)$ with the following properties:

$$\alpha_1, \alpha_n \in \{0, 1-2H\}, \quad \alpha_j \in \{0, 1-2H, 2(1-2H)\}, j = 2, \cdots, n-1,$$

and

$$|\alpha| = \sum_{j=1}^n \alpha_j = (n-1)(1-2H).$$

The exact description of the set D_n is not necessary. So

$$\prod_{j=2}^{n} |\eta_j - \eta_{j-1}|^{1-2H} \leq \sum_{\alpha \in D_n} \prod_{j=1}^{n} |\eta_j|^{\alpha_j}. \tag{3.2.5}$$

We obtain

$$\begin{aligned} E|I_n(f_n(\cdot,t,x))|^2 &\leq C_H^n \int_{T_n(t)} \int_{\mathbb{R}^n} \prod_{i=1}^{n} |\mathcal{F}G(t_{i+1}-t_i,\cdot)(\eta_i)|^2 |\eta_1|^{1-2H} \\ &\qquad \sum_{\alpha \in D_n} \prod_{i=1}^{n} |\eta_i|^{\alpha_i} d\eta_1 \cdots d\eta_n dt_1 \cdots dt_n \\ &= C_H^n \sum_{\alpha \in D_n} \int_{T_n(t)} \left(\int_{\mathbb{R}} |\mathcal{F}G(t_2-t_1,\cdot)(\eta_1)|^2 |\eta_1|^{1-2H+\alpha_1} d\eta_1 \right) \\ &\qquad \left(\prod_{i=2}^{n} \int_{\mathbb{R}} |\mathcal{F}G(t_{i+1}-t_i,\cdot)(\eta_i)|^2 |\eta_i|^{\alpha_i} d\eta_i \right) dt_1 \cdots dt_n. \end{aligned}$$

We consider separately the *Parabolic Anderson Model* and *Hyperbolic Anderson Model*.

a) For the *Parabolic Anderson Model*, using Lemma 3.2.1 a), we get

$$E|I_n(f_n(\cdot,t,x))|^2 \leq C_{H,1}^n \sum_{\alpha \in D_n} \int_{T_n(t)} (t_2-t_1)^{-\frac{2-2H+\alpha_1}{2}} \prod_{i=2}^{n} (t_{i+1}-t_i)^{-\frac{1+\alpha_i}{2}} dt_1 \cdots dt_n,$$

where $C_{H,1} > 0$ is a constant depending on H. To apply this lemma, we need $1-2H+\alpha_1 > -1$ and $\alpha_i > -1$ for all $i = 2, \cdots, n$. Since $\alpha_i > 0$ for all $i = 1, \cdots, n$, this imposes no restrictions on H. For every $\alpha \in D_n$ fixed, we now use Lemma A.3 with

$$\beta_1 = -\frac{2-2H+\alpha_1}{2}, \quad \beta_j = -\frac{1+\alpha_j}{2}, j = 2, \cdots, n.$$

Hence

$$|\beta| = \sum_{j=1}^{n} \beta_j = -\frac{2-2H+(n-1)+|\alpha|}{2} = Hn - n.$$

To apply Lemma A.3, we need $\beta_j > -1$ for all $j = 1, \cdots, n$. When $\alpha_j = 2(1-2H)$, this imposes the restriction $H > \frac{1}{4}$. We obtain:

$$\begin{aligned} &\int_{T_n(t)} (t_2-t_1)^{-\frac{2-2H+\alpha_1}{2}} \prod_{j=2}^{n} (t_{j+1}-t_j)^{-\frac{1+\alpha_j}{2}} d\mathbf{t} \\ &\quad = t^{Hn} \frac{\Gamma(-\frac{2-2H+\alpha_1}{2}+1) \prod_{j=2}^{n} \Gamma(-\frac{1+\alpha_j}{2}+1)}{\Gamma(Hn+1)} \leq C_{H,2}^n \frac{t^{Hn}}{\Gamma(Hn+1)}, \end{aligned}$$

where

$$C_{H,2} = \max\left\{\Gamma(H), \Gamma\left(\frac{4H-1}{2}\right), \sqrt{\pi}\right\}.$$

Since the previous estimate does not depend on α and $\text{card}(D_n) = 2^{n-1}$,

$$E|I_n(f_n(\cdot,t,x))|^2 \le 2^{n-1} C_{H,1}^n C_{H,2}^n \frac{t^{nH}}{\Gamma(Hn+1)} \le \frac{C^n t^{nH}}{\Gamma(Hn+1)},$$

where $C = 2C_{H,1}C_{H,2}$ is a constant depending on H. Finally using inequality (A.9), we obtain:

$$\begin{aligned} E|u(t,x)|^2 &= \sum_{n\ge 0} E|I_n(f_n(\cdot,t,x))|^2 \le \sum_{n\ge 0} \frac{(Ct^H)^n}{\Gamma(Hn+1)} \\ &\le C_1^h \exp\{C^{1/H} t\} = C_1^h \exp\{C_2^h t\}, \end{aligned}$$

where C_1^h and C_2^h are constants depending on H.

b) We consider next the *Hyperbolic Anderson Model*. By Lemma 3.2.1 b),

$$E|I_n(f_n(\cdot,t,x))|^2 \le C_{H,3}^n \sum_{\alpha\in D_n} \int_{T_n(t)} (t_2-t_1)^{2H-\alpha_1} \prod_{i=2}^n (t_{i+1}-t_i)^{1-\alpha_i} dt_1 \cdots dt_n,$$

where $C_{H,3} > 0$ is a constant depending on H. To apply this lemma, we need $1 - 2H + \alpha_1 \in (-1,1)$ and $\alpha_j \in (-1,1)$ for $j = 2, \cdots, n$. When $\alpha_j = 2(1-2H)$, the condition $\alpha_j < 1$ imposes the restriction $H > \frac{1}{4}$. For every $\alpha \in D_n$ fixed, we use Lemma A.3 with

$$\beta_1 = 2H - \alpha_1, \quad \beta_j = 1 - \alpha_j, j = 2, \cdots, n.$$

Hence

$$|\beta| = \sum_{j=1}^n \beta_j = 2H + (n-1) - |\alpha| = 2Hn.$$

To apply Lemma A.3, we need $\beta_j > -1$ for all $j = 1, \cdots, m$, which imposes no restriction on H. We obtain

$$\begin{aligned} &\int_{T_n(t)} (t_2-t_1)^{2H-\alpha_1} \prod_{j=2}^n (t_{j+1}-t_j)^{1-\alpha_j} d\mathbf{t} \\ &\quad = t^{2Hn+n} \frac{\Gamma(2H-\alpha_1+1)\prod_{j=2}^n \Gamma(2-\alpha_j)}{\Gamma(2Hn+n+1)} \le C_{H,4}^n \frac{t^{n(2H+1)}}{\Gamma((2H+1)n+1)}, \end{aligned}$$

where

$$C_{H,4} = \max\left\{\Gamma(2H+1), \Gamma(4H), 1\right\}.$$

Since the previous estimate does not depend on α and $\text{card}(D_n) = 2^{n-1}$,

$$E|I_n(f_n(\cdot,t,x))|^2 \leq 2^{n-1} C_{H,3}^n C_{H,4}^n \frac{t^{n(2H+1)}}{\Gamma((2H+1)n+1)} \leq \frac{C^n t^{n(2H+1)}}{\Gamma((2H+1)n+1)},$$

where $C = 2C_{H,3}C_{H,4}$ is a constant depending on H, which is different than in part a). Finally using inequality (A.9), we obtain

$$\begin{aligned} E|u(t,x)|^2 = \sum_{n\geq 0} E|I_n(f_n(\cdot,t,x))|^2 &\leq \sum_{n\geq 0} \frac{(Ct^{2H+1})^n}{\Gamma((2H+1)n+1)} \\ &\leq C_1^w \exp\{C^{1/(2H+1)}t\} = C_1^w \exp\{C_2^w t\}, \end{aligned}$$

where C_1^w and C_2^w are constants depending on H.

Step 2 (lower bound): We consider the set $A = \{(\eta_1, \cdots, \eta_n) \in \mathbb{R}^n; \eta_1 \in \mathbb{R}_+, \eta_2 \in \mathbb{R}_-, \cdots, \eta_n \in \mathbb{R}_-\}$ if n is even and $A = \{(\eta_1, \cdots, \eta_n) \in \mathbb{R}^n; \eta_1 \in \mathbb{R}_+, \eta_2 \in \mathbb{R}_-, \cdots, \eta_n \in \mathbb{R}_+\}$ if n is odd. For any $(\eta_1, \cdots, \eta_n) \in A$,

$$|\eta_j - \eta_{i-1}| = |\eta_i| + |\eta_{i-1}| \geq |\eta_i|$$

for all $j = 2, \cdots, n$. Since the function $\xi \to |\xi|^{1-2H}$ is increasing on $\mathbb{R}_+$ for all $(\eta_1, \cdots, \eta_n) \in A$.

$$|\eta_j - \eta_{i-1}|^{1-2H} \geq |\eta_i|^{1-2H} \tag{3.2.6}$$

According to (3.2.3) and (3.2.6),

$$\begin{aligned} E|I_n(f_n(\cdot,t,x))|^2 &\geq C_H^n \int_{T_n(t)} \int_A \prod_{i=1}^n |\mathcal{F}G(t_{i+1}-t_i,\cdot)(\eta_i)|^2 |\eta_j|^{1-2H} d\eta_1 \cdots d\eta_n dt_1 \cdots dt_n \\ &= \Big(\frac{C_H}{2}\Big)^n \int_{T_n(t)} \prod_{i=1}^n \Big(\int_{\mathbb{R}} |\mathcal{F}G(t_{i+1}-t_i,\cdot)(\eta_i)|^2 |\eta_j|^{1-2H} d\eta_j \Big) dt_1 \cdots dt_n, \end{aligned}$$

where for the last line we use the fact that:

$$\int_{\mathbb{R}_+} |\mathcal{F}G(t,\cdot)(\eta)|^2 |\eta|^{1-2H} d\eta = \int_{\mathbb{R}_-} |\mathcal{F}G(t,\cdot)(\eta)|^2 |\eta|^{1-2H} d\eta = \frac{1}{2} \int_{\mathbb{R}} |\mathcal{F}G(t,\cdot)(\eta)|^2 |\eta|^{1-2H} d\eta.$$

Applying Lemma 3.2.1 with $\alpha = 1 - 2H$, we get

$$\int_{\mathbb{R}} |\mathcal{F}G(t,\cdot)(\xi)|^2 |\xi|^{1-2H} d\xi = C_H^{(1)} t^a,$$

where $C_H^{(1)}$ is a constant depending on H and

$$a = \begin{cases} H-1 & \text{for heat equation} \\ 2H & \text{for wave equation} \end{cases}.$$

Therefore, by Lemma A.3 with $\beta_j = a$ for all $j = 1, \cdots, n$,

$$E|I_n(f_n(\cdot, t, x))|^2 \geq \Big(\frac{C_H C_H^{(1)}}{2}\Big)^n \int_{T_n(t)} \prod_{i=1}^n (t_{j+1} - t_j)^a dt_1 \cdots dt_n$$
$$= \Big(\frac{C_H C_H^{(1)}}{2}\Big)^n \frac{\Gamma(1+a)^n}{\Gamma(n(1+a)+1)} t^{n(1+a)} = \frac{(C_H^{(2)})^n t^{n(1+a)}}{\Gamma(n(1+a)+1)},$$

where $C_H^{(2)}$ is a constant depending on H. Using inequality (A.10), we infer that

$$E|u(t,x)|^2 = \sum_{n\geq 0} E|I_n(f_n(\cdot, t, x))|^2 \geq \sum_{n \geq 0} \frac{(C_H^{(2)} t^{1+a})^n}{\Gamma(n(1+a)+1)}$$
$$\geq C_1' \exp\Big((C_H^{(3)})^{1/1+a} t\Big) = C_1' \exp\{C_2' t\}.$$

where C_1' and C_2' are constants depending on H. ∎

Chapter 4

The case $H_0 > \frac{1}{2}$ and $H > \frac{1}{2}$

In this chapter, we study the case when the noise behaves in time like fractional Brownian motion with index $H_0 > \frac{1}{2}$ and in space like fractional Brownian motion with index $H > \frac{1}{2}$. In other words, the noise is associated with a fractional Brownian sheet with index H_0 in time and index H in space.

The results presented in this chapter parallel those of Chapter 2, but the fact that the noise is colored in time introduces significant new difficulties, compared to the case of the white noise in time. As in the previous two chapters, we first introduce the noise, then we discuss the linear equations, followed by the study of equations (1.0.2) and (1.0.3). Finally, at the end of this chapter, we include a section dedicated to equation (1.0.2) with a very general initial equation condition (given by a measure), as presented in reference [7].

The results presented in this chapter are particular cases of some results from references [2, 3, 7, 10].

4.1 The noise

In this section, we introduce the Gaussian noise $\dot{W}$ which is fractional in time with index $H_0 > \frac{1}{2}$ and fractional in space with index $H > \frac{1}{2}$.

Let $W = \{W([0,t] \times A); t \geq 0, A \in \mathcal{B}_b(\mathbb{R})\}$ be a zero-mean Gaussian process with covariance:

$$E[W([0,t] \times A) \cdot W([0,s] \times B)] = R_{H_0}(t,s)\alpha_H \int_A \int_B |x-y|^{2H-2} dxdy, \qquad (4.1.1)$$

where $\alpha_H = H(2H-1)$ and

$$R_{H_0}(t,s) = \frac{1}{2}(t^{2H_0} + s^{2H_0} - |t-s|^{2H_0}) \qquad (4.1.2)$$

is the covariance of the fractional Brownian motion with index H_0. We assume that $H_0 > \frac{1}{2}$ and $H > \frac{1}{2}$. Note that

$$R_{H_0}(t,s) = \alpha_{H_0} \int_0^t \int_0^s |u-v|^{2H_0-2} du dv. \tag{4.1.3}$$

Hence

$$\begin{aligned} &E[W([0,t] \times A) \cdot W([0,s] \times B)] \\ &\qquad = \alpha_{H_0}\alpha_H \int_{[0,t]\times A} \int_{[0,s]\times B} |u-v|^{2H_0-2}|x-y|^{2H-2} dxdydudv \\ &\qquad := \langle 1_{[0,t]\times A}, 1_{[0,s]\times B} \rangle_{\mathcal{H}}. \end{aligned}$$

Set $W(1_{[0,t]\times A}) = W([0,t] \times A)$. Let $\mathcal{E}$ be the set of linear combinations of indicator functions of the form $1_{[0,t]\times A}$, with $t > 0$ and $A \in \mathcal{B}_b(\mathbb{R})$. We extend W by linearity to $\mathcal{E}$.

Lemma 4.1.1. *W is an isometry from $\mathcal{E}$ to $L^2(\Omega)$, i.e. for any $\varphi, \psi \in \mathcal{E}$,*

$$\begin{aligned} E[W(\varphi)W(\psi)] &= \langle \varphi, \psi \rangle_{\mathcal{H}} \\ &= \alpha_{H_0}\alpha_H \int_0^\infty \int_0^\infty \int_{\mathbb{R}} \int_{\mathbb{R}} \varphi(t,x)\psi(s,y)|t-s|^{2H_0-2}|x-y|^{2H-2} dxdydtds. \end{aligned}$$

Proof: Let $\varphi = \sum_{i=1}^n a_i 1_{[0,t_i]\times A_i}$ and $\psi = \sum_{j=1}^m b_j 1_{[0,s_j]\times B_j}$. Then

$$\begin{aligned} E[W(\varphi)W(\psi)] &= E\Big[W\Big(\sum_{i=1}^n a_i 1_{[0,t_i]\times A_i}\Big) W\Big(\sum_{j=1}^m b_j 1_{[0,s_j]\times B_j}\Big)\Big] \\ &= \sum_{i=1}^n \sum_{j=1}^m a_i b_j E[W(1_{[0,t_i]\times A_i}) W(1_{[0,s_j]\times B_j})] \\ &= \sum_{i=1}^n \sum_{j=1}^m a_i b_j \alpha_{H_0}\alpha_H \int_{[0,t_i]\times A_i} \int_{[0,s_j]\times B_j} |t-s|^{2H_0-2}|x-y|^{2H-2} dsdydtdx \\ &= \alpha_{H_0}\alpha_H \sum_{i=1}^n \sum_{j=1}^m a_i b_j \int_0^\infty \int_0^\infty \int_{\mathbb{R}} \int_{\mathbb{R}} 1_{[0,t_i]\times A_i}(t,x) 1_{[0,s_j]\times B_j}(s,y) \\ &\qquad |t-s|^{2H_0-2}|x-y|^{2H-2} dxdydtds \\ &= \alpha_{H_0}\alpha_H \int_0^\infty \int_0^\infty \int_{\mathbb{R}} \int_{\mathbb{R}} \left(\sum_{i=1}^n a_i 1_{[0,t_i]\times A_i}\right) \left(\sum_{j=1}^m b_j 1_{[0,s_j]\times B_j}\right) \\ &\qquad |t-s|^{2H_0-2}|x-y|^{2H-2} dxdydtds \\ &= \alpha_{H_0}\alpha_H \int_0^\infty \int_0^\infty \int_{\mathbb{R}} \int_{\mathbb{R}} \varphi(t,x)\psi(s,y)|t-s|^{2H_0-2}|x-y|^{2H-2} dxdydtds \end{aligned}$$

$$= \langle \varphi, \psi \rangle_{\mathcal{H}}.$$

∎

Let $\mathcal{H}$ be the completion of $\mathcal{E}$ with respect to $\langle \cdot, \cdot \rangle_{\mathcal{H}}$. We extend the isometry map from Lemma 4.1.1 to $\mathcal{H}$. We denote this map by $W(\varphi), \varphi \in \mathcal{H}$ and we say that $W(\varphi)$ is the *Wiener integral* of φ with respect to W. We use the notation

$$W(\varphi) = \int_0^\infty \int_{\mathbb{R}} \varphi(t,x) W(dt,dx).$$

Remark 4.1.2. *By Remark 2.1.3, for any $\varphi, \psi \in \mathcal{E}$,*

$$\begin{aligned}\langle \varphi, \psi \rangle_{\mathcal{H}} &= \alpha_{H_0} \int_0^\infty \int_0^\infty |t-s|^{2H_0-2} \left(\alpha_H \int_{\mathbb{R}} \int_{\mathbb{R}} \varphi(t,x)\psi(s,y)|x-y|^{2H-2} dx dy \right) dt ds \\ &= \alpha_{H_0} \int_0^\infty \int_0^\infty |t-s|^{2H_0-2} \left(C_H \int_{\mathbb{R}} \mathcal{F}\varphi(t,\cdot)(\xi) \overline{\mathcal{F}\psi(s,\cdot)(\xi)} |\xi|^{1-2H} d\xi \right) dt ds.\end{aligned}$$

We recall the following result which is a particular case of Theorem 2.6 (c) of [9].

Theorem 4.1.3. *Let $\varphi : [0,T] \times \mathbb{R} \to \mathbb{R}$ be such that $\varphi(t,\cdot) \in L^1(\mathbb{R})$ for all $t \in [0,T]$ and let*

$$\mathcal{F}\varphi(t,\cdot)(\xi) = \int_{\mathbb{R}} e^{-i\xi \cdot x} \varphi(t,x) dx, \quad \xi \in \mathbb{R}$$

be the Fourier transform of $\varphi(t,\cdot)$. Suppose that $\int_0^T |\mathcal{F}\varphi(t,\cdot)(\xi)| dt < \infty$. If

$$\|\varphi\|_o^2 := \alpha_{H_0} C_H \int_0^T \int_0^T \int_{\mathbb{R}} |t-s|^{2H_0-2} \mathcal{F}\varphi(t,\cdot)(\xi) \overline{\mathcal{F}\varphi(s,\cdot)(\xi)} |\xi|^{1-2H} d\xi dt ds < \infty,$$

then $\varphi \in \mathcal{H}$ and $\|\varphi\|_{\mathcal{H}}^2 = \|\varphi\|_o^2$.

Remark 4.1.4. Note that $1_{[0,t]\times[0,x]} \in \mathcal{H}$ and the process $\{W(t,x); t \geq 0, x \in \mathbb{R}\}$ given by $W(t,x) = W(1_{[0,t]\times[0,x]})$ is a fractional Brownian sheet of index H_0 in time and index H in space.

4.2 Linear heat equation

In this section, we consider the linear heat equation (1.0.6) with noise $\dot{W}$ as in Section 4.1. The solution to this equation is defined as in Definition 2.2.1.

The results that we present below are particular cases of the results given in Section 4 of [10], obtained for $d = 1$ and $\mu(d\xi) = C_H |\xi|^{1-2H} d\xi$. The **solution** of (1.0.6) is the process $\{u(t,x); t \geq 0, x \in \mathbb{R}\}$ defined by:

$$u(t,x) = \int_0^t \int_{\mathbb{R}} G^h(t-s, x-y) W(ds,dy),$$

where G^h is given by (1.0.4). This solution exists if and only if the stochastic integral is well-defined, i.e $g_{tx} \in \mathcal{H}$ where $g_{tx}(s,y) = 1_{[0,t]}(s)G^h(t-s, x-y)$.

Theorem 4.2.1. *For all* $H_0 \in (\frac{1}{2}, 1)$ *and* $H \in (\frac{1}{2}, 1)$*, equation* (1.0.6) *with noise* $\dot{W}$ *as in Section 4.1 has a unique solution.*

Proof: **Step 1:** In this step, we show that $g_{tx} \in \mathcal{H}$ for any $t > 0, x \in \mathbb{R}$ if and only if

$$I = \int_{\mathbb{R}} \left(\frac{1}{1+|\xi|^2}\right)^{2H_0} \mu(d\xi) < \infty,$$

where $\mu(d\xi) = C_H|\xi|^{1-2H}d\xi$. By Theorem 4.1.3, to show that $g_{tx} \in \mathcal{H}$, it is enough to show that $I_t < \infty$, where

$$I_t = \alpha_{H_0} \int_0^t \int_0^t \int_{\mathbb{R}} \mathcal{F}g_{tx}(r,\cdot)(\xi)\overline{\mathcal{F}g_{tx}(s,\cdot)(\xi)}|r-s|^{2H_0-2}\mu(d\xi)drds.$$

Note that by relation (2.2.1) and the change of variables $r' = t-r$ and $s' = t-s$,

$$\begin{aligned} I_t &= \alpha_{H_0} \int_{\mathbb{R}} \int_0^t \int_0^t \mathcal{F}G^h(t-r,\cdot)(\xi)\overline{\mathcal{F}G^h(t-s,\cdot)(\xi)}|r-s|^{2H_0-2}drds\mu(d\xi) \\ &= \int_{\mathbb{R}} \left(\alpha_{H_0} \int_0^t \int_0^t \mathcal{F}G^h(r,\cdot)(\xi)\overline{\mathcal{F}G^h(s,\cdot)(\xi)}|r-s|^{2H_0-2}drds\right)\mu(d\xi). \end{aligned}$$

Let

$$\begin{aligned} N_t(\xi) &= \alpha_{H_0} \int_0^t \int_0^t \mathcal{F}G^h(r,\cdot)(\xi)\overline{\mathcal{F}G^h(s,\cdot)(\xi)}|r-s|^{2H_0-2}drds \\ &= \alpha_{H_0} \int_0^t \int_0^t \exp\left(-\frac{1}{2}|\xi|^2 r\right)\exp\left(-\frac{1}{2}|\xi|^2 s\right)|r-s|^{2H_0-2}drds \end{aligned}$$

We will prove that

$$\frac{1}{4}(t^{2H_0} \wedge 1)\left(\frac{1}{1+|\xi|^2}\right)^{2H_0} \leq N_t(\xi) \leq b_{H_0}(4H_0)^{2H_0}(t^{2H_0} \vee 1)\left(\frac{1}{1+|\xi|^2}\right)^{2H_0}. \quad (4.2.1)$$

For the upper bound, we consider two cases. In the case $|\xi| \leq 1$, using Corollary D.3 and the fact $e^{-x} \leq 1$ for any $x > 0$, followed by the inequality $1 \leq \frac{2}{1+|\xi|^2}$, we obtain:

$$\begin{aligned} N_t(\xi) &\leq b_{H_0}t^{2H_0-1}\int_0^t \exp(-|\xi|^2 s)ds \leq b_{H_0}t^{2H_0-1}\int_0^t 1ds = b_{H_0}t^{2H_0} \\ &\leq b_{H_0}t^{2H_0}\left(\frac{2}{1+|\xi|^2}\right)^{2H_0} = b_{H_0}2^{2H_0}t^{2H_0}\left(\frac{1}{1+|\xi|^2}\right)^{2H_0} \end{aligned}$$

$$\leq b_{H_0}(4H_0)^{2H_0}t^{2H_0}\left(\frac{1}{1+|\xi|^2}\right)^{2H_0}.$$

In the case $|\xi| \geq 1$, we use Lemma D.1 and the fact that $1-e^{-x} \leq 1$ and $\frac{1}{|\xi|^2} \leq \frac{2}{1+|\xi|^2}$ for any $x > 0$, we get:

$$\begin{aligned} N_t(\xi) &\leq b_{H_0}\left(\int_0^t \exp\Big\{-\frac{|\xi|^2}{2H_0}s\Big\}ds\right)^{2H_0} = b_{H_0}\left(\frac{2H_0}{|\xi|^2}\right)^{2H_0}\left(1-\exp\Big\{-\frac{|\xi|^2}{2H_0}t\Big\}\right)^{2H_0} \\ &\leq b_{H_0}\left(\frac{2H_0}{|\xi|^2}\right)^{2H_0} = b_{H_0}(2H_0)^{2H_0}\left(\frac{1}{|\xi|^2}\right)^{2H_0} \leq b_{H_0}(4H_0)^{2H_0}\left(\frac{1}{1+|\xi|^2}\right)^{2H_0}. \end{aligned}$$

Therefore we get

$$N_t(\xi) \leq b_{H_0}(4H_0)^{2H_0}(t^{2H_0} \vee 1)\left(\frac{1}{1+|\xi|^2}\right)^{2H_0}.$$

As for the lower bound, we also consider two cases. The first case is when $t|\xi|^2 \leq 1$. Using the fact that $e^{-x} \geq 1-x$ for all $x > 0$, then we obtain:

$$\exp\Big(-\frac{t|\xi|^2}{2}\Big) \geq 1-\frac{t|\xi|^2}{2} \geq \frac{1}{2}.$$

So using Lemma E.1, we have:

$$N_t(\xi) \geq \frac{1}{4}\alpha_{H_0}\int_0^t\int_0^t |r-s|^{2H_0-2}drds = \frac{1}{4}t^{2H_0} \geq \frac{1}{4}t^{2H_0}\left(\frac{1}{1+|\xi|^2}\right)^{2H_0}.$$

The second case is when $t|\xi|^2 \geq 1$. Using the change of variable $u = \frac{t|\xi|^2}{2}$ and $v = \frac{s|\xi|^2}{2}$, we have:

$$\begin{aligned} N_t(\xi) &= \alpha_{H_0}\Big(\frac{2}{|\xi|^2}\Big)^2\int_0^{\frac{t|\xi|^2}{2}}\int_0^{\frac{t|\xi|^2}{2}} e^{-u}e^{-v}\left|\frac{2u}{|\xi|^2}-\frac{2v}{|\xi|^2}\right|^{2H_0-2}dudv \\ &= \alpha_{H_0}\Big(\frac{2}{|\xi|^2}\Big)^{2H_0}\int_0^{\frac{t|\xi|^2}{2}}\int_0^{\frac{t|\xi|^2}{2}} e^{-u}e^{-v}|u-v|^{2H_0-2}dudv. \end{aligned}$$

Since the integral is nonnegative, using the fact that $\frac{t|\xi|^2}{2} \geq \frac{1}{2}$ and $e^{-u} > 1-u \geq \frac{1}{2}$ for $u \in [0,\frac{1}{2}]$, we obtain:

$$\begin{aligned} N_t(\xi) &\geq \alpha_{H_0}\Big(\frac{2}{|\xi|^2}\Big)^{2H_0}\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}} e^{-u}e^{-v}|u-v|^{2H_0-2}dudv \\ &\geq \frac{1}{4}\Big(\frac{2}{|\xi|^2}\Big)^{2H_0}\alpha_{H_0}\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}} |u-v|^{2H_0-2}dudv = \frac{1}{4}\Big(\frac{2}{|\xi|^2}\Big)^{2H_0}\Big(\frac{1}{2}\Big)^{2H_0} \end{aligned}$$

$$= \frac{1}{4}\Big(\frac{1}{|\xi|^2}\Big)^{2H_0} \geq \frac{1}{4}\Big(\frac{1}{1+|\xi|^2}\Big)^{2H_0}.$$

Note that for the equality above, we used Lemma E.1 again. So we get:

$$N_t(\xi) \geq \frac{1}{4}(t^{2H_0} \wedge 1)\left(\frac{1}{1+|\xi|^2}\right)^{2H_0}.$$

Hence we proved (4.2.1). Therefore, we can see

$$\frac{t^{2H_0} \wedge 1}{4}\int_{\mathbb{R}}\left(\frac{1}{1+|\xi|^2}\right)^{2H_0}\mu(d\xi) \leq I_t \leq b_{H_0}(4H_0)^{2H_0}(t^{2H_0} \vee 1)\int_{\mathbb{R}}\left(\frac{1}{1+|\xi|^2}\right)^{2H_0}\mu(d\xi).$$

Step 2: In this step, we prove that $I < \infty$ if and only if $2H_0 + H > 1$, where

$$I = \int_{|\xi|\leq 1}\left(\frac{1}{1+|\xi|^2}\right)^{2H_0}\mu(d\xi) + \int_{|\xi|\geq 1}\left(\frac{1}{1+|\xi|^2}\right)^{2H_0}\mu(d\xi) := I_1 + I_2.$$

We note that the condition $2H_0 + H > 1$ is automatically satisfied since $H_0 > \frac{1}{2}$. Note that $I_1 < \infty$ for any $H_0, H \in (\frac{1}{2}, 1)$ since $\frac{1}{1+|\xi|^2} \leq 1$ and hence

$$I_1 \leq C_H \int_{|\xi|\leq 1}|\xi|^{1-2H}d\xi < \infty.$$

To treat I_2, we use the fact that $\frac{1}{2|\xi|^2} \leq \frac{1}{1+|\xi|^2} \leq \frac{1}{|\xi|^2}$ if $|\xi| \geq 1$. We have:

$$C_H\frac{1}{2^{2H_0}}\int_{|\xi|\geq 1}|\xi|^{1-2H-4H_0}d\xi \leq I_2 \leq C_H\int_{|\xi|\geq 1}|\xi|^{1-2H-4H_0}d\xi.$$

Hence $I_2 < \infty$ if and only if $\int_{|\xi|\geq 1}|\xi|^{1-2H-4H_0}d\xi < \infty$ which is equivalent to saying that $2H_0 + H > 1$. ∎

4.3 Linear wave equation

In this section, we consider the linear wave equation (1.0.7) with noise $\dot{W}$ as in Section 4.1. The solution to this equation is defined as in Definition 2.2.3.

This solution exists if and only if the stochastic integral is well-defined, i.e $g_{tx} \in \mathcal{H}$ where $g_{tx}(s,y) = 1_{[0,t]}(s)G^w(t-s, x-y)$, and G^w is given by (1.0.5).

The proof of the following result is based on a combination of techniques borrowed from the proofs of Proposition 3.7 of [10] (for the upper bound) and Theorem 4.3 of [3] (for the lower bound).

Theorem 4.3.1. *For all* $H_0 \in (\frac{1}{2}, 1)$ *and* $H \in (\frac{1}{2}, 1)$*, equation* (1.0.7) *with noise* $\dot{W}$ *as in the Section 4.1 has a unique solution.*

Proof: **Step 1** We show that $g_{tx} \in \mathcal{H}$ for any $t > 0, x \in \mathbb{R}$ if and only if

$$I = \int_{\mathbb{R}} \left(\frac{1}{1 + |\xi|^2} \right)^{H_0 + \frac{1}{2}} \mu(d\xi) < \infty,$$

where $\mu(d\xi) = C_H |\xi|^{1-2H} d\xi$. By Theorem 4.1.3, to show that $g_{tx} \in \mathcal{H}$, it is enough to show that $I_t < \infty$, where

$$I_t = \alpha_{H_0} \int_0^t \int_0^t \int_{\mathbb{R}} \mathcal{F} g_{tx}(r, \cdot)(\xi) \overline{\mathcal{F} g_{tx}(s, \cdot)(\xi)} |r - s|^{2H_0 - 2} \mu(d\xi) dr ds.$$

We use a similar argument as in the proof of Theorem 4.2.1. Note that by relation (2.2.1) and the change of variables $r' = t - r$ and $s' = t - s$,

$$\begin{aligned} I_t &= \alpha_{H_0} \int_{\mathbb{R}} \int_0^t \int_0^t \mathcal{F} G^w(t - r, \cdot)(\xi) \overline{\mathcal{F} G^w(t - s, \cdot)(\xi)} |r - s|^{2H_0 - 2} dr ds \mu(d\xi) \\ &= \int_{\mathbb{R}} \left(\alpha_{H_0} \int_0^t \int_0^t \mathcal{F} G^w(r, \cdot)(\xi) \overline{\mathcal{F} G^w(s, \cdot)(\xi)} |r - s|^{2H_0 - 2} dr ds \right) \mu(d\xi). \end{aligned}$$

Let

$$\begin{aligned} N_t(\xi) &= \alpha_{H_0} \int_0^t \int_0^t \mathcal{F} G^w(r, \cdot)(\xi) \overline{\mathcal{F} G^w(s, \cdot)(\xi)} |r - s|^{2H_0 - 2} dr ds \\ &= \frac{\alpha_{H_0}}{|\xi|^2} \int_0^t \int_0^t \sin(r|\xi|) \sin(s|\xi|) |r - s|^{2H_0 - 2} dr ds, \end{aligned}$$

we will prove that

$$D_{H_0,H} (t^{2H_0+2} \wedge t) \left(\frac{1}{1 + |\xi|^2} \right)^{H_0 + \frac{1}{2}} \leq N_t(\xi) \leq B_{H_0,H} (t^{2H_0+2} \vee 1) \left(\frac{1}{1 + |\xi|^2} \right)^{H_0 + \frac{1}{2}}, \tag{4.3.1}$$

where

$$D_{H_0,H} = \min \left\{ \alpha_{H_0} \cos^2 1 \frac{\beta(2, 2H_0 - 1)}{H_0 + 1}, C_{H_0} 4^{1 - 2H_0} \left(\frac{\pi}{2} - \frac{4}{3} \right) \right\},$$

and

$$B_{H_0,H} = \max \left\{ \frac{2^{H_0 + \frac{1}{2}}}{3} b_{H_0}, 2^{3H_0 - \frac{3}{2}} C_{H_0} \left(\frac{100}{9} \frac{1}{1 - H_0} + 8\pi \right) \right\}.$$

Here $\beta(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ is the beta function and C_{H_0} is given by (2.1.4).

For the upper bound, we consider two cases. In the case $|\xi| \leq 1$, using Corollary D.3 and the fact $|\sin(x)| \leq x$ for any $x > 0$, followed by the inequality $1 \leq \frac{2}{1+|\xi|^2}$, we obtain:

$$\begin{aligned} N_t(\xi) &\leq b_{H_0} t^{2H_0-1} \frac{1}{|\xi|^2} \int_0^t \sin^2(s|\xi|) ds \leq b_{H_0} t^{2H_0-1} \int_0^t s^2 ds = \frac{1}{3} b_{H_0} t^{2H_0+2} \\ &\leq \frac{1}{3} b_{H_0} t^{2H_0+2} \left(\frac{2}{1+|\xi|^2} \right)^{H_0+\frac{1}{2}} = \frac{2^{H_0+\frac{1}{2}}}{3} b_{H_0} t^{2H_0+2} \left(\frac{1}{1+|\xi|^2} \right)^{H_0+\frac{1}{2}}. \end{aligned} \tag{4.3.2}$$

In the case $|\xi| \geq 1$, we use the change of variable $r' = r|\xi|$ and $s' = s|\xi|$ and then we get:

$$\begin{aligned} N_t(\xi) &= \frac{\alpha_{H_0}}{|\xi|^2} \int_0^{t|\xi|} \int_0^{t|\xi|} \sin r \sin s \left| \frac{r}{|\xi|} - \frac{s}{|\xi|} \right|^{2H_0-2} |\xi|^{-2} dr ds \\ &= \frac{\alpha_{H_0}}{|\xi|^{2H_0+2}} \int_0^{t|\xi|} \int_0^{t|\xi|} \sin r \sin s |r-s|^{2H_0-2} dr ds = \frac{1}{|\xi|^{2H_0+2}} \|\sin(\cdot)\|^2_{\mathcal{H}(0,t|\xi|)}, \end{aligned}$$

where

$$\begin{aligned} \|\sin(\cdot)\|^2_{\mathcal{H}(0,t|\xi|)} &= \alpha_{H_0} \int_0^{t|\xi|} \int_0^{t|\xi|} \sin r \sin s |r-s|^{2H_0-2} dr ds \\ &= C_{H_0} \int_{\mathbb{R}} \frac{(\sin(\tau t|\xi|) - \tau \sin(t|\xi|))^2 + (\cos(\tau t|\xi|) - \cos(t|\xi|))^2}{|\tau^2 - 1|^2} |\tau|^{-(2H_0-1)} d\tau, \end{aligned}$$

which we obtain by Lemma C.5. Therefore,

$$N_t(\xi) = \frac{C_{H_0}}{|\xi|^{2H_0+2}} \int_{\mathbb{R}} \frac{|\tau|^{-(2H_0-1)}}{|\tau^2-1|^2} [f_t^2(|\xi|,\tau) + g_t^2(|\xi|,\tau)] d\tau, \tag{4.3.3}$$

where

$$f_t(|\xi|,\tau) = \sin(\tau t|\xi|) - \tau \sin(t|\xi|), \quad g_t(|\xi|,\tau) = \cos(\tau t|\xi|) - \cos(t|\xi|).$$

We split the integral into two regions $|\tau| \leq \frac{1}{2}$ and $|\tau| > \frac{1}{2}$, and we denote the corresponding integrals by $N_t^{(1)}(\xi)$ and $N_t^{(2)}(\xi)$. Since $|f_t(|\xi|,\tau)| \leq 1+\tau$ and$|g_t(|\xi|,\tau)| \leq 2$ for any $|\xi| > 1, \tau > 0$, we obtain:

$$\begin{aligned} N_t^{(1)}(\xi) &\leq \frac{C_{H_0}}{|\xi|^{2H_0+2}} \int_{|\tau|\leq\frac{1}{2}} \frac{|\tau|^{-(2H_0-1)}}{(\tau^2-1)^2} [(1+\tau)^2 + 4] d\tau \\ &\leq \frac{C_{H_0}}{|\xi|^{2H_0+2}} \frac{(1+\frac{1}{2})^2 + 4}{(1-(\frac{1}{2})^2)^2} \int_{|\tau|\leq\frac{1}{2}} |\tau|^{-(2H_0-1)} d\tau \end{aligned}$$

$$\begin{aligned}
&\leq \frac{100}{9}\frac{C_{H_0}}{|\xi|^{2H_0+2}}2\int_0^{\frac{1}{2}}|\tau|^{-(2H_0-1)}d\tau = \frac{100}{9}\frac{C_{H_0}}{|\xi|^{2H_0+2}}2\frac{(1/2)^{2-2H_0}}{2-2H_0}\\
&= \frac{100}{9}\frac{C_{H_0}2^{2H_0-2}}{1-H_0}\frac{1}{|\xi|^{2H_0+2}} \leq \frac{100}{9}\frac{C_{H_0}2^{2H_0-2}}{1-H_0}\frac{1}{|\xi|^{2H_0+1}}.
\end{aligned}$$

Noting that $|\tau|^{-(2H_0-1)} \leq (\frac{1}{2})^{-(2H_0-1)}$ if $|\tau| > \frac{1}{2}$, we get:

$$\begin{aligned}
N_t^{(2)}(\xi) &\leq \frac{C_{H_0}}{|\xi|^{2H_0+2}}\Big(\frac{1}{2}\Big)^{-(2H_0-1)}\int_{|\tau|>\frac{1}{2}}\frac{1}{(\tau^2-1)^2}[f_t^2(|\xi|,\tau)+g_t^2(|\xi|,\tau)]d\tau\\
&\leq \frac{C_{H_0}2^{2H_0-1}}{|\xi|^{2H_0+2}}\int_{\mathbb{R}}\frac{1}{(\tau^2-1)^2}[f_t^2(|\xi|,\tau)+g_t^2(|\xi|,\tau)]d\tau\\
&\leq C_{H_0}2^{2H_0-1}\frac{4\pi(t\vee t^3)}{|\xi|^{2H_0+2}}\frac{|\xi|^3}{1+|\xi|^2} = C_{H_0}2^{2H_0-1}\frac{4\pi(t\vee t^3)}{|\xi|^{2H_0+1}}\frac{|\xi|^2}{1+|\xi|^2}\\
&\leq C_{H_0}2^{2H_0-1}4\pi(t\vee t^3)\frac{1}{|\xi|^{2H_0+1}},
\end{aligned}$$

where the second last inequality is obtained by using Lemma C.6. Using the fact that $\frac{1}{|\xi|^2} \leq \frac{2}{1+|\xi|^2}$ for $|\xi| \geq 1$, we obtain:

$$\begin{aligned}
N_t(\xi) &= N_t^{(1)}(\xi) + N_t^{(2)}(\xi)\\
&\leq \frac{100}{9}\frac{C_{H_0}2^{2H_0-2}}{1-H_0}\frac{1}{|\xi|^{2H_0+1}} + C_{H_0}2^{2H_0-1}4\pi(t\vee t^3)\frac{1}{|\xi|^{2H_0+1}}\\
&= 2^{2H_0-2}C_{H_0}\left(\frac{100}{9}\frac{1}{1-H_0}+8\pi(t\vee t^3)\right)\frac{1}{|\xi|^{2H_0+1}}\\
&\leq 2^{2H_0-2}C_{H_0}\left(\frac{100}{9}\frac{1}{1-H_0}+8\pi(t\vee t^3)\right)\left(\frac{2}{1+|\xi|^2}\right)^{H_0+\frac{1}{2}}\\
&= 2^{3H_0-\frac{3}{2}}\left(\frac{100}{9}\frac{1}{1-H_0}+8\pi\right)(1\vee t\vee t^3)\left(\frac{2}{1+|\xi|^2}\right)^{H_0+\frac{1}{2}}. \qquad (4.3.4)
\end{aligned}$$

Relations (4.3.2) and (4.3.4) give the upper bound in (4.3.1) using the fact that, if $t \geq 1$, then $t^{2H_0+2} \geq t^3 \geq t \geq 1$, and if $t < 1$, then $t^{2H_0+2} < t^3 < t < 1$.

As for the lower bound, we also consider two cases. The first case is when $t|\xi| \leq 1$. Using Lemma E.4 and the fact that $\sin x \geq x\cos 1$ for all $x \in (0,1)$ and $1 \geq \left(\frac{1}{1+|\xi|^2}\right)^{2H_0+1}$, we obtain:

$$\begin{aligned}
N_t(\xi) &\geq \alpha_{H_0}\cos^2 1\int_0^t\int_0^t rs|r-s|^{2H_0-2}drds\\
&= \alpha_{H_0}\cos^2 1\frac{\beta(2,2H_0-1)}{H_0+1}t^{2H_0+2}
\end{aligned}$$

$$\geq \alpha_{H_0} \cos^2 1 \frac{\beta(2, 2H_0 - 1)}{H_0 + 1} t^{2H_0+2} \left(\frac{1}{1 + |\xi|^2} \right)^{H_0 + \frac{1}{2}}. \tag{4.3.5}$$

The second case is when $t|\xi| \geq 1$. Using Lemma C.4 and Plancherel theorem (Theorem C.3), we get

$$\begin{aligned} J(t|\xi|) &:= \int_{\mathbb{R}} \frac{1}{(\tau^2 - 1)^2} [f_t^2(|\xi|, \tau) + g_t^2(|\xi|, \tau)] d\tau = \int_{\mathbb{R}} |(\mathcal{F}_{0, t|\xi|} \sin)(\tau)|^2 d\tau \\ &= 2\pi \int_0^{t|\xi|} \sin^2 x dx = \pi \int_0^{t|\xi|} (1 - \cos 2x) dx = t|\xi|\pi \left(1 - \frac{\sin(2t|\xi|)}{2t|\xi|} \right) \geq \frac{t|\xi|\pi}{2}, \end{aligned}$$

where for the last inequality, we used the fact that $\sin 2x < x$ for $x \geq 1$. Let $\rho > 1$ be a constant, which will be specified below. Then we get $|\tau|^{-(2H_0-1)} \geq |\rho|^{-(2H_0-1)}$ in the region $|\tau| \leq \rho$. Note that the integrand of $N_t(\xi)$ is nonnegative. Using relation (4.3.3), it follows that:

$$\begin{aligned} N_t(\xi) &= \frac{C_{H_0}}{|\xi|^{2H_0+2}} \int_{\mathbb{R}} \frac{|\tau|^{-(2H_0-1)}}{(\tau^2 - 1)^2} [f_t^2(|\xi|, \tau) + g_t^2(|\xi|, \tau)] d\tau \\ &\geq \frac{C_{H_0} \rho^{-(2H_0-1)}}{|\xi|^{2H_0+2}} \int_{|\tau| \leq \rho} \frac{f_t^2(|\xi|, \tau) + g_t^2(|\xi|, \tau)}{(\tau^2 - 1)^2} d\tau \\ &= \frac{C_{H_0} \rho^{-(2H_0-1)}}{|\xi|^{2H_0+2}} \left(J(t|\xi|) - \int_{|\tau| > \rho} \frac{f_t^2(|\xi|, \tau) + g_t^2(|\xi|, \tau)}{(\tau^2 - 1)^2} d\tau \right) \\ &\geq \frac{C_{H_0} \rho^{-(2H_0-1)}}{|\xi|^{2H_0+2}} \left(\frac{t|\xi|\pi}{2} - \int_{|\tau| > \rho} \frac{f_t^2(|\xi|, \tau) + g_t^2(|\xi|, \tau)}{(\tau^2 - 1)^2} d\tau \right). \end{aligned}$$

Since

$$f_t^2(|\xi|, \tau) + g_t^2(|\xi|, \tau) \leq (1 + |\tau|)^2 + 2^2 \leq 2(1 + |\tau|)^2 \leq 2t|\xi|(1 + |\tau|)^2,$$

letting $A_\rho = 2 \int_{|\tau| > \rho} \frac{(1+|\tau|)^2}{(\tau^2-1)^2} d\tau$, we obtain:

$$\begin{aligned} N_t(\xi) &\geq \frac{C_{H_0} \rho^{-(2H_0-1)}}{|\xi|^{2H_0+2}} \left(\frac{t|\xi|\pi}{2} - \int_{|\tau| > \rho} \frac{2t|\xi|(1 + |\tau|)^2}{(\tau^2 - 1)^2} d\tau \right) \\ &\geq \frac{C_{H_0} \rho^{-(2H_0-1)}}{|\xi|^{2H_0+2}} \left(\frac{t|\xi|\pi}{2} - t|\xi| A_\rho \right) \geq \frac{C_{H_0} \rho^{-(2H_0-1)}}{|\xi|^{2H_0+2}} \left(\frac{\pi}{2} - A_\rho \right) t|\xi| \\ &\geq C_{H_0} \rho^{1-2H_0} \left(\frac{\pi}{2} - A_\rho \right) \frac{t}{|\xi|^{2H_0+1}} \geq C_{H_0} \rho^{1-2H_0} \left(\frac{\pi}{2} - A_\rho \right) t \left(\frac{1}{1 + |\xi|^2} \right)^{H_0 + \frac{1}{2}}. \end{aligned}$$

Now we choose ρ such that $\frac{\pi}{2} - A_\rho > 0$. For instance, we take $\rho = 4$. Then

$$A_\rho = 4 \int_4^\infty \frac{(1 + \tau)^2}{(\tau^2 - 1)^2} d\tau = 4 \int_4^\infty \frac{(1 + \tau)^2}{(\tau + 1)^2 (\tau - 1)^2} d\tau = 4 \int_4^\infty \frac{1}{(\tau - 1)^2} d\tau = \frac{4}{3}.$$

Hence,

$$N_t(\xi) \geq C_{H_0} 4^{1-2H_0} \left(\frac{\pi}{2} - \frac{4}{3}\right) t \left(\frac{1}{1+|\xi|^2}\right)^{H_0+\frac{1}{2}}. \tag{4.3.6}$$

Relation (4.3.5) and (4.3.6) give the lower bound in (4.3.1).

Step 2: In this step, we prove that $I < \infty$ if and only if $2H_0 + 2H > 1$, where

$$I = \int_{|\xi|\leq 1} \left(\frac{1}{1+|\xi|^2}\right)^{H_0+\frac{1}{2}} \mu(d\xi) + \int_{|\xi|\geq 1} \left(\frac{1}{1+|\xi|^2}\right)^{H_0+\frac{1}{2}} \mu(d\xi) := I_1 + I_2.$$

We note that the condition $2H_0 + 2H > 1$ is automatically satisfied since $H_0 > \frac{1}{2}$. Note that $I_1 < \infty$ since $\frac{1}{1+|\xi|^2} \leq 1$ and hence

$$I_1 \leq C_H \int_{|\xi|\leq 1} |\xi|^{1-2H} d\xi < \infty.$$

To treat I_2, we use the fact that $\frac{1}{2|\xi|^2} \leq \frac{1}{1+|\xi|^2} \leq \frac{1}{|\xi|^2}$ if $|\xi| \geq 1$. We have:

$$C_H \frac{1}{2^{H_0+\frac{1}{2}}} \int_{|\xi|\geq 1} |\xi|^{-2H-2H_0} d\xi \leq I_2 \leq C_H \int_{|\xi|\geq 1} |\xi|^{-2H-2H_0} d\xi.$$

Hence $I_2 < \infty$ if and only if $\int_{|\xi|\geq 1} |\xi|^{-2H-2H_0} d\xi < \infty$ which is equivalent to saying that $2H_0 + 2H > 1$. ∎

4.4 The Parabolic and Hyperbolic Anderson models

In this section, we consider equations (1.0.2) and (1.0.3) with noise $\dot{W}$ introduced in Section 4.1. The solution is defined as in Definition 2.3.1.

Theorem 4.4.1. *For any $H_0 \in (\frac{1}{2}, 1)$ and $H \in (\frac{1}{2}, 1)$, equations* (1.0.2) *and* (1.0.3) *with noise $\dot{W}$ as in Section 4.1 have unique solutions. Moreover,*

$$E|u(t,x)|^2 \leq C_1 \exp(C_2 t^{\rho}), \tag{4.4.1}$$

where $C_1 > 0$ and $C_2 > 0$ are constants depending on H_0 and H, and

$$\rho = \rho^h = \frac{2H_0 + H - 1}{H} \quad \textit{for equation } (1.0.2)\ ,$$
$$\rho = \rho^w = \frac{2H_0 + 2H}{2H+1} \quad \textit{for equation } (1.0.3)\ .$$

Proof: Similarly to (2.3.6), we have

$$\begin{aligned}
E|I_n(f_n(\cdot,t,x))|^2 &= n!\|\tilde{f}_n(\cdot,t,x)\|^2_{\mathcal{H}^{\otimes n}}\\
&= \frac{1}{n!}\alpha^n_{H_0}\int_{[0,t]^{2n}}\prod_{i=1}^n |t_i-s_i|^{2H_0-2}\Bigg(\alpha^n_H\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\prod_{i=1}^n|x_i-y_i|^{2H-2}\\
&\qquad n!\tilde{f}_n(t_1,x_1,\cdots,t_n,x_n,t,x)n!\tilde{f}_n(s_1,y_1,\cdots,s_n,y_n,t,x)d\mathbf{x}d\mathbf{y}\Bigg)d\mathbf{t}d\mathbf{s}\\
&= \frac{1}{n!}\alpha^n_{H_0}\int_{[0,t]^{2n}}\prod_{i=1}^n|t_i-s_i|^{2H_0-2}\psi_n(\mathbf{t},\mathbf{s})d\mathbf{t}d\mathbf{s} \qquad (4.4.2)
\end{aligned}$$

where

$$\begin{aligned}
\psi_n(\mathbf{t},\mathbf{s}) =& C^n_H(n!)^2\int_{\mathbb{R}^n}\mathcal{F}\tilde{f}_n(t_1,\cdot,\cdots,t_n,\cdot,t,x)(\xi_1,\cdots,\xi_n)\\
&\overline{\mathcal{F}\tilde{f}_n(s_1,\cdot,\cdots,s_n,\cdot,t,x)(\xi_1,\cdots,\xi_n)}\prod_{i=1}^n|\xi_i|^{1-2H}d\xi_1\cdots d\xi_n.
\end{aligned}$$

By the Cauchy-Schwartz inequality,

$$\psi_n(\mathbf{t},\mathbf{s}) \le \psi_n(\mathbf{t},\mathbf{t})^{1/2}\psi_n(\mathbf{s},\mathbf{s})^{1/2}.$$

Hence

$$E|I_n(f_n(\cdot,t,x))|^2 \le \frac{1}{n!}\alpha^n_{H_0}\int_{[0,t]^{2n}}\prod_{i=1}^n|t_i-s_i|^{2H_0-2}\psi_n(\mathbf{t},\mathbf{t})^{1/2}\psi_n(\mathbf{s},\mathbf{s})^{1/2}d\mathbf{t}d\mathbf{s}.$$

Using the Littlewood-Hardy inequality given by Lemma D.4,

$$\begin{aligned}
E|I_n(f_n(\cdot,t,x))|^2 &\le \frac{1}{n!}b^n_{H_0}\left(\int_{[0,t]^n}\psi_n(\mathbf{t},\mathbf{t})^{\frac{1}{2H_0}}d\mathbf{t}\right)^{2H_0}\\
&= \frac{1}{n!}b^n_{H_0}\left(\sum_{\rho\in S_n}\int_{0<t_{\rho(1)}<\cdots<t_{\rho(n)}<t}\psi_n(\mathbf{t},\mathbf{t})^{\frac{1}{2H_0}}d\mathbf{t}\right)^{2H_0}. \qquad (4.4.3)
\end{aligned}$$

Let $\rho \in S_n$ and $(t_1,\cdots,t_n)\in[0,t]^n$ such that $0<t_{\rho(1)}<\cdots<t_{\rho(n)}<t$. Using the change of variable $y_i = x_{\rho(i)}$, we note that

$$\begin{aligned}
&\mathcal{F}\tilde{f}_n(t_1,\cdot,\cdots,t_n,\cdot,t,x)(\xi_1,\cdots,\xi_n)\\
&=\int_{\mathbb{R}^n}e^{-i(\xi_1x_1+\cdots+\xi_nx_n)}\tilde{f}_n(t_1,x_1,\cdots,t_n,x_n,t,x)dx_1\cdots dx_n
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{n!}\int_{\mathbb{R}^n} e^{-i(\xi_{\rho(1)}x_{\rho(1)}+\cdots+\xi_{\rho(n)}x_{\rho(n)})} f_n(t_{\rho(1)},x_{\rho(1)},\cdots,t_{\rho(n)},x_{\rho(n)},t,x)dx_{\rho(1)}\cdots dx_{\rho(n)}\\
&= \frac{1}{n!}\int_{\mathbb{R}^n} e^{-i(\xi_{\rho(1)}y_1+\cdots+\xi_{\rho(n)}y_n)} f_n(t_{\rho(1)},y_1,\cdots,t_{\rho(n)},y_n,t,x)dy_1\cdots dy_n\\
&= \frac{1}{n!}\mathcal{F}f_n(t_{\rho(1)},\cdot,\cdots,t_{\rho(n)},\cdot,t,x)(\xi_{\rho(1)},\cdots,\xi_{\rho(n)}). \qquad (4.4.4)
\end{aligned}$$

By Lemma 2.3.2,

$$\begin{aligned}
&\mathcal{F}f_n(t_{\rho(1)},\cdot,\cdots,t_{\rho(n)},\cdot,t,x)(\xi_{\rho(1)},\cdots,\xi_{\rho(n)})\\
&\quad = e^{-i(\xi_{\rho(1)}+\cdots+\xi_{\rho(n)})\cdot x}\overline{\mathcal{F}G(t_{\rho(2)}-t_{\rho(1)},\cdot)(\xi_{\rho(1)})}\cdots\overline{\mathcal{F}G(t-t_{\rho(n)},\cdot)(\xi_{\rho(1)}+\cdots+\xi_{\rho(n)})}.
\end{aligned}$$

Letting $u_i = t_{\rho(i+1)} - t_{\rho(i)}$ and using the change of variable $\xi_i' = \xi_{\rho(i)}$, we have

$$\begin{aligned}
\psi_n(\mathbf{t},\mathbf{t}) &= C_H^n (n!)^2 \int_{\mathbb{R}^n} |\mathcal{F}\tilde{f}_n(t_1,\cdot,\cdots,t_n,\cdot,t,x)|^2 \prod_{i=1}^n |\xi_i|^{1-2H} d\xi_i\\
&= C_H^n \int_{\mathbb{R}^n} |\mathcal{F}G(u_1,\cdot)(\xi_1)|^2 \cdots |\mathcal{F}G(u_n,\cdot)(\xi_1+\cdots+\xi_n)|^2 \prod_{i=1}^n |\xi_i|^{1-2H} d\xi_i \qquad (4.4.5)\\
&\le C_H^n \int_{\mathbb{R}^{n-1}} \prod_{i=1}^{n-1} |\mathcal{F}G(u_i,\cdot)(\sum_{k=1}^{i}\xi_k)|^2\\
&\qquad \left(\sup_{\eta\in\mathbb{R}} \int_{\mathbb{R}} |\mathcal{F}G(u_n,\cdot)(\xi_n+\eta)|^2 |\xi_n|^{1-2H} d\xi_n\right) \prod_{i=1}^{n-1} |\xi_i|^{1-2H} d\xi_1 \cdots d\xi_{n-1}\\
&\le C_H^n \prod_{i=1}^{n} \left(\sup_{\eta\in\mathbb{R}} \int_{\mathbb{R}} |\mathcal{F}G(u_i,\cdot)(\xi_i+\eta)|^2 |\xi_i|^{1-2H} d\xi_i\right). \qquad (4.4.6)
\end{aligned}$$

According to Lemma 2.3.6, we have

$$\sup_{\eta\in\mathbb{R}} \int_{\mathbb{R}} |\mathcal{F}G(t,\cdot)(\xi+\eta)|^2 |\xi|^{1-2H} d\xi \le C_H' t^a$$

where C_H' is a constant depending on H and

$$a = \begin{cases} H-1 & \text{for heat equation,}\\ 2H & \text{for wave equation.} \end{cases}$$

Therefore, we obtain that if $0 < t_{\rho(1)} < \cdots < t_{\rho(n)} < t$, then

$$\psi_n(\mathbf{t},\mathbf{t}) \le (C_H C_H')^n \prod_{i=1}^n u_i^a = C_{H,1}^n \prod_{i=1}^n (t_{\rho(i+1)} - t_{\rho(i)})^a,$$

where $C_{H,1} = C_H C'_H$. Coming back to (4.4.3), we let $T_{n,\rho}(t) = \{0 < t_{\rho(1)} < \cdots < t_{\rho(n)} < t\}$ and $T_n(t) = \{0 < t_1 < \cdots < t_n < t\}$. Then we obtain:

$$\begin{aligned} E|I_n(f_n(\cdot,t,x))|^2 &\leq \frac{1}{n!} b_{H_0}^n \left(\sum_{\rho \in S_n} \int_{T_{n,\rho}(t)} C_{H,1}^{\frac{n}{2H_0}} \prod_{i=1}^n |t_{\rho(i+1)} - t_{\rho(i)}|^{\frac{a}{2H_0}} dt_{\rho(1)} \cdots dt_{\rho(n)} \right)^{2H_0} \\ &= (n!)^{2H_0-1} C_{H,H_0,1}^n \left(\int_{T_n(t)} \prod_{i=1}^n |t_{(i+1)} - t_i|^{\frac{a}{2H_0}} dt_1 \cdots dt_n \right)^{2H_0}, \end{aligned}$$

where we used the change of variable $t'_i = t_{\rho(i)}$ and we let $C_{H,H_0,1} = b_{H_0} C_{H,1}$. By Lemma A.3, we obtain:

$$\begin{aligned} E|I_n(f_n(\cdot,t,x))|^2 &\leq (n!)^{2H_0-1} C_{H,H_0,2}^n \left(\frac{t^{n(\frac{a}{2H_0}+1)}}{\Gamma\left(n\left(\frac{a}{2H_0}+1\right)+1\right)} \right)^{2H_0} \\ &= (n!)^{2H_0-1} C_{H,H_0,2}^n t^{n(a+2H_0)} \left(\frac{1}{\Gamma(n(\frac{a}{2H_0}+1)+1)} \right)^{2H_0}. \end{aligned}$$

According to Corollary A.5, for any $a > 0$, there exist constants $C_1, C_2, C_{a,1}, C_{a,2} > 0$ such that

$$C_1 C_{a,1}^n (n!)^a \leq \Gamma(an+1) \leq C_2 C_{a,2}^n (n!)^a.$$

Then we get

$$E|I_n(f_n(\cdot,t,x))|^2 \leq (n!)^{2H_0-1} C_{H,H_0}^n t^{n(a+2H_0)} \frac{1}{(n!)^{(\frac{a}{2H_0}+1)2H_0}} = C_{H,H_0}^n t^{n(a+2H_0)} \frac{1}{(n!)^{a+1}},$$

where C_{H,H_0} is a constant depending on H and H_0. Note that for equation (1.0.2), we have $a = H - 1$ and

$$E|I_n(f_n(\cdot,t,x))|^2 \leq C_{H,H_0}^n t^{n(H+2H_0-1)} \frac{1}{(n!)^H}, \tag{4.4.7}$$

while for equation (1.0.3), we have $a = 2H$ and

$$E|I_n(f_n(\cdot,t,x))|^2 \leq C_{H,H_0}^n t^{n(2H+2H_0)} \frac{1}{(n!)^{2H+1}}. \tag{4.4.8}$$

Therefore,

$$E|u(t,x)|^2 = \sum_{n\geq 0} E|I_n(f_n(\cdot,t,x))|^2 = C_{H,H_0}^n \sum_{n \geq 0} \frac{(t^{a+2H_0})^n}{(n!)^{a+1}} \leq C_3 \exp\left(C_4 t^{\frac{a+2H_0}{a+1}}\right).$$

where for the last inequality we used Lemma A.7. Relation (4.4.1) follows, since for equation (1.0.2), $a = H - 1$ and $\frac{a+2H_0}{a+1} = \frac{2H_0+H+1}{H}$, whereas for equation (1.0.3), $a = 2H$ and $\frac{a+2H_0}{a+1} = \frac{2H_0+2H}{2H+1}$. ∎

Remark 4.4.2. Note that if $H_0 = \frac{1}{2}$, the exponents ρ^h and ρ^w given by Theorem 4.4.1 are equal to 1, which is consistent with the statement of Theorem 2.3.7.

Remark 4.4.3. Equation (1.0.2) with Gaussian noise with covariance function

$$f(x) = C_{\alpha,d}|x|^{-(d-\alpha)} \qquad \text{for } 0 < \alpha < d, \tag{4.4.9}$$

was considered in [2]. In this chapter, $d = 1$ and $f(x) = \alpha_H |x|^{2H-2}$ for $H \in (\frac{1}{2}, 1)$, which corresponds to the function f given by (4.4.9) with $\alpha = 2H - 1$. Proposition 3.3 of [2] gives for equation (1.0.2):

$$E|I_n(f_n(\cdot, t, x))|^2 = \frac{1}{n!}\alpha_n(t) \leq C^n t^{(2H_0-(d-\alpha)/2)n} \frac{1}{(n!)^{1-(d-\alpha)/2}}$$

which coincides with (4.4.7) in the case $d = 1$ and $\alpha = 2H - 1$. On the other hand, for equation (1.0.3), Remark 3.5 of [2] gives

$$E|I_n(f_n(\cdot, t, x))|^2 \leq C^n t^{(2H_0+\alpha-d+2)n} \frac{1}{(n!)^{\alpha-d+3}}$$

which coincides with (4.4.8) in the case $d = 1$ and $\alpha = 2H - 1$.

Remark 4.4.4. Note that the value of the exponent ρ of t obtained in relation (4.4.1) for the solution to equation (1.0.2) is strictly larger than the value of this exponent for the solution to equation (1.0.3), since

$$\rho^w := \frac{2H_0 + 2H}{2H + 1} = 1 + \frac{2H_0 - 1}{2H + 1} < 1 + \frac{2H_0 - 1}{H} = \frac{2H_0 + H - 1}{H} =: \rho^h.$$

Remark 4.4.5. Theorem 6.4 of [17] shows that if u is the solution of equation (1.0.2) with noise $\dot{W}$ as in Section 4.1, then we also have the lower bound:

$$E|u(t, x)|^2 \geq C_1' \exp(C_2' t^{\rho^h}).$$

A similar result exists for equation (1.0.3); see Theorem 2.1.(c) of [4].

Remark 4.4.6. Similarly to Theorem 2.3.8, it can be proved that the second moment of the solution to equation (1.0.2) with noise $\dot{W}$ as in Section 4.1 admits the Feynmac-Kac representation $E|u(t, x)|^2 = E[\exp(L(t))]$, where

$$L(t) = \alpha_{H_0}\alpha_H \int_0^t \int_0^t |r - s|^{2H_0-2} |B_r^1 - B_s^2|^{2H-2} dr ds,$$

and $(B_r^1)_{r\geq 0}$ and $(B_s^2)_{s\geq 0}$ are independent Brownian motions.

4.5 The Parabolic Anderson model with general initial condition

In this section, we consider the *Parabolic Anderson Model* with general initial condition given by a measure:

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) &= \dfrac{1}{2}\dfrac{\partial^2 u}{\partial x^2}(t,x) + u(t,x)\dot{W}(t,x), \quad t>0, x\in\mathbb{R} \\ u(0,\cdot) &= \mu_0(\cdot), \end{cases} \tag{4.5.1}$$

with the same noise $\dot{W}$ as in Section 4.1.

We assume that μ_0 is a measure on $\mathbb{R}$ which satisfies the following condition:

$$\int_{\mathbb{R}} e^{-ax^2}\mu_0(dx) < \infty, \quad \text{for all } a>0. \tag{4.5.2}$$

Note that condition (4.5.2) is equivalent to

$$\int_{\mathbb{R}} G^h(t,x-y)\mu_0(dy) < \infty, \quad \text{for all } t>0 \text{ and } x\in\mathbb{R},$$

where G^h is given by (1.0.4).

Let $J_0(t,x)$ be the solution of the deterministic heat equation:

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) &= \dfrac{1}{2}\dfrac{\partial^2 u}{\partial x^2}(t,x), \quad t>0, x\in\mathbb{R} \\ u(0,\cdot) &= \mu_0(\cdot). \end{cases} \tag{4.5.3}$$

The initial condition in (4.5.3) is a formal writing, since in fact the function $u(0,x)$ may be not well defined. By convention, we say that the following function is the solution to (4.5.3):

$$J_0(t,x) = \int_{\mathbb{R}} G^h(t,x-y)\mu_0(dy).$$

An important particular case is when $\mu_0(dx) = u_0(x)dx$ for a non-negative function u_0. In this case, the initial condition in (4.5.3) is defined rigourously as $u(0,x) = u_0(x)$ for all $x\in\mathbb{R}$. The case $u_0(x)=1$ is considered in equation (1.0.2). In this case, $J_0(t,x)=1$ for all $t>0$ and $x\in\mathbb{R}$.

Similarly to Definition 2.3.1, we have the following definition.

Definition 4.5.1. *We say that a process $\{u(t,x); t>0, x\in\mathbb{R}\}$ is a* **solution** *to equation* (4.5.1) *if for any $t>0$ and $x\in\mathbb{R}$, with probability 1,*

$$u(t,x) = J_0(t,x) + \int_0^t \int_{\mathbb{R}} G^h(t-s,x-y)u(s,y)W(ds,dy). \tag{4.5.4}$$

Note that the solution u is not defined at time $t = 0$, unless μ_0 is absolutely continuous with respect to the Lebesgue measure.

If it exists, the solution of (4.5.4) has the series representation:

$$u(t,x) = J_0(t,x) + \sum_{n\geq 1} I_n(f_n(\cdot,t,x)),$$

where the kernel function $f_n(\cdot,t,x)$ is given by:

$$\begin{aligned} f_n(t_1,x_1,\cdots,t_n,x_n,t,x) = G^h(t-t_n,x-x_n)\cdots G^h(t_2-t_1,x_2-x_1) \\ J_0(t_1,x_1)1_{\{0<t_1<\cdots<t_n<t\}}. \end{aligned}$$

The solution exists if and only if $\sum_{n\geq 1} I_n(f_n(\cdot,t,x))$ converges in $L^2(\Omega)$.

Note that

$$\begin{aligned} &f_n(t_1,x_1,\cdots,t_n,x_n,t,x) \\ &\quad = \int_{\mathbb{R}} G^h(t-t_n,x-x_n)\cdots G^h(t_2-t_1,x_2-x_1)G^h(t_1,x_1-x_0)\mu_0(dx_0). \end{aligned}$$

To obtain an alternative formula for $f_n(\cdot,t,x)$, we use the following lemma:

Lemma 4.5.2. *(Lemma A.4 of [11]) For any $t,s>0$ and $x,y\in\mathbb{R}$,*

$$G^h(t,x)G^h(s,y) = G^h(t+s,x-y)G^h\left(\frac{ts}{t+s},\frac{sx+ty}{t+s}\right).$$

Using Lemma 4.5.2, we have:

$$\begin{aligned} G^h(t_2-t_1,x_2-x_1)G^h(t_1,x_1-x_0) &= G^h(t_2-t_1,x_2-x_1)G^h(t_1,x_0-x_1) \\ &= G^h(t_2,x_2-x_0)G^h\left(\Big(1-\frac{t_1}{t_2}\Big)t_1,\Big(1-\frac{t_1}{t_2}\Big)x_0+\frac{t_1}{t_2}x_2-x_1\right). \end{aligned}$$

Then,

$$\begin{aligned} G^h(t_3-t_2,x_3-x_2)G^h(t_2,x_2-x_0) &= G^h(t_3-t_2,x_3-x_2)G^h(t_2,x_0-x_2) \\ &= G^h(t_3,x_3-x_0)G^h\left(\Big(1-\frac{t_2}{t_3}\Big)t_2,\Big(1-\frac{t_2}{t_3}\Big)x_0+\frac{t_2}{t_3}x_3-x_2\right). \end{aligned}$$

By induction, letting $t_{n+1}=t$ and obtain the following result:

Lemma 4.5.3. *For any $0<t_1<\cdots<t_n<t$ and $x_1,\cdots,x_n,x\in\mathbb{R}$,*

$$\begin{aligned} &f_n(t_1,x_1,\cdots,t_n,x_n,t,x) \\ &\quad = \int_{\mathbb{R}} G^h(t,x-x_0)\prod_{j=1}^n G^h\left(\Big(1-\frac{t_j}{t_{j+1}}\Big)t_j,\Big(1-\frac{t_j}{t_{j+1}}\Big)x_0+\frac{t_j}{t_{j+1}}x_{j+1}-x_j\right)\mu_0(dx_0). \end{aligned}$$

In particular, the previous lemma shows that $f_n(t_1, \cdot, \cdots, t_n, \cdot, t, x)$ is integrable on $\mathbb{R}^n$. The next result gives the Fourier transform of this function. We omit its proof.

Lemma 4.5.4. *(Lemma 2.5 of [7]) For any* $0 < t_1 < \cdots < t_n < t = t_{n+1}$ *and for any* $\xi_1, \cdots, \xi_n \in \mathbb{R}$, *we have*

$$\mathcal{F}f_n(t_1, \cdot, \cdots, t_n, \cdot, t, x)(\xi_1, \cdots, \xi_n) = \prod_{k=1}^{n} \exp\left\{ -\frac{1}{2} \cdot \frac{t_{k+1} - t_k}{t_k t_{k+1}} \Big| \sum_{j=1}^{k} t_j \xi_j \Big|^2 \right\}$$
$$\times \exp\left\{ -i \frac{\sum_{j=1}^{n} t_j \xi_j}{t} \cdot x \right\} \int_{\mathbb{R}} \exp\left\{ -i \Big[\sum_{j=1}^{n} \big(1 - \frac{t_j}{t}\big) \xi_j \Big] \cdot x_0 \right\} G(t, x - x_0) \mu_0(dx_0)$$

For any $\mathbf{t} = (t_1, \cdots, t_n) \in [0, t]^n, \mathbf{s} = (s_1, \cdots, s_n) \in [0, t]^n, n \geq 1, t > 0$ and $x \in \mathbb{R}$, we define

$$\begin{aligned} \psi_{t,x}^{(n)}(\mathbf{t}, \mathbf{s}) =& (n!)^2 \int_{\mathbb{R}^n} \mathcal{F}\tilde{f}_n(t_1, \cdot, \cdots, t_n, \cdot, t, x)(\xi_1, \cdots, \xi_n) \\ & \overline{\mathcal{F}\tilde{f}_n(s_1, \cdot, \cdots, s_n, \cdot, t, x)(\xi_1, \cdots, \xi_n)} \mu(d\xi_1) \cdots \mu(d\xi_n), \end{aligned} \tag{4.5.5}$$

where $\mu(d\xi) = C_H |\xi|^{1-2H} d\xi$.

The following result gives an upper bound for $\psi_{t,x}^{(n)}(\mathbf{t}, \mathbf{s})$, which will be used in Theorem 4.5.6.

Lemma 4.5.5. *(Lemma 3.2 of [7]) If* $0 < t_{\rho(1)} < \cdots < t_{\rho(n)} < t := t_{\rho(n+1)}$, *then*

$$\psi_{t,x}^{(n)}(\mathbf{t}, \mathbf{t}) \leq J_0^2(t, x) \int_{\mathbb{R}^n} \prod_{k=1}^{n} \exp\left\{ -\frac{t_{\rho(k+1)} - t_{\rho(k)}}{t_{\rho(k)} t_{\rho(k+1)}} \Big| \sum_{j=1}^{k} t_{\rho(j)} \xi_j \Big|^2 \right\} \mu(d\xi_1) \cdots \mu(d\xi_n).$$

Proof: By relation (4.4.4), we have:

$$\mathcal{F}\tilde{f}_n(t_1, \cdots, t_n, t, x)(\xi_1, \cdots, \xi_n) = \frac{1}{n!} \mathcal{F}f_n(t_{\rho(1)}, \cdots, t_{\rho(n)}, t, x)(\xi_1, \cdots, \xi_n).$$

Applying Lemma 4.5.4, using the fact $| \int_{\mathbb{R}} \cdots \mu_0(d\xi)| \leq \int_{\mathbb{R}} | \cdots | \mu_0(d\xi)$ and $|e^{-i\xi \cdot x}| = 1$, we get

$$\begin{aligned} \psi_{t,x}^{(n)}(\mathbf{t}, \mathbf{t}) &= \int_{\mathbb{R}^n} (n!)^2 |\mathcal{F}\tilde{f}_n(t_1, \cdots, t_n, t, x)(\xi_1, \cdots, \xi_n)|^2 \mu(d\xi_1) \cdots \mu(d\xi_n) \\ &= \int_{\mathbb{R}^n} |\mathcal{F}f_n(t_{\rho(1)}, \cdots, t_{\rho(n)}, t, x)(\xi_1, \cdots, \xi_n) \mu(d\xi_1) \cdots \mu(d\xi_n) \\ &= \int_{\mathbb{R}^n} \prod_{k=1}^{n} \exp\left\{ -\frac{t_{\rho(k+1)} - t_{\rho(k)}}{t_{\rho(k)} t_{\rho(k+1)}} \Big| \sum_{j=1}^{k} t_{\rho(j)} \xi_j \Big|^2 \right\} \left| \exp\left\{ -i \frac{\sum_{j=1}^{n} t_{\rho(j)} \xi_j}{t} \cdot x \right\} \right|^2 \end{aligned}$$

$$\times \left| \int_{\mathbb{R}} \exp\left\{ -i\Big[\sum_{j=1}^n \big(1-\frac{t_j}{t}\big)\xi_j\Big]\cdot x_0 \right\} G^h(t,x-x_0)\mu_0(dx_0)\right|^2 \mu(d\xi_1)\cdots\mu(d\xi_n)$$

$$\leq J_0^2(t,x)\int_{\mathbb{R}^n}\prod_{k=1}^n \exp\left\{ -\frac{t_{\rho(k+1)}-t_{\rho(k)}}{t_{\rho(k)}t_{\rho(k+1)}}\Big|\sum_{j=1}^k t_{\rho(j)}\xi_j\Big|^2\right\}\mu(d\xi_1)\cdots\mu(d\xi_n),$$

where we recall that $J_0(t,x) = \int_{\mathbb{R}} G^h(t,x-x_0)\mu_0(dx_0)$. ∎

The following result is the main result of this section. This result is a particular case of Theorem 1.2 of [7].

Theorem 4.5.6. *For $H_0 \in (\frac{1}{2},1)$ and $H \in (\frac{1}{2},1)$, equation (4.5.1) has a unique solution. Moreover,*

$$E|u(t,x)|^2 \leq C_1 J_0^2(t,x)\exp\left\{C_2 t^{\rho^h}\right\},$$

where $\rho^h = \frac{2H_0+H-1}{H}$ and C_1 and C_2 are constants depending on H_0 and H.

Proof: We denote $\mu(d\xi) = C_H|\xi|^{1-2H}d\xi$. By (4.4.2), we have

$$\begin{aligned} E|I_n(f_n(\cdot,t,x))|^2 &= n!\|\tilde{f}_n(\cdot,t,x)\|^2_{\mathcal{H}^{\otimes n}} \\ &= \frac{1}{n!}\alpha_{H_0}^n \int_{[0,t]^{2n}}\prod_{i=1}^n |t_i-s_i|^{2H_0-2}\psi_{t,x}^{(n)}(\mathbf{t},\mathbf{s})d\mathbf{t}d\mathbf{s}, \end{aligned} \tag{4.5.6}$$

where $\psi_{t,x}^{(n)}(\mathbf{t},\mathbf{s})$ is given by (4.5.5).

By the Cauchy-Schwartz inequality,

$$\psi_{t,x}^{(n)}(\mathbf{t},\mathbf{s}) \leq \psi_{t,x}^{(n)}(\mathbf{t},\mathbf{t})^{1/2}\psi_{t,x}^{(n)}(\mathbf{s},\mathbf{s})^{1/2}.$$

Using this inequality and applying Lemma D.4, we obtain:

$$\begin{aligned} E|I_n(f_n(\cdot,t,x))|^2 &\leq \frac{1}{n!}b_{H_0}^n\left(\int_{[0,t]^{2n}}\psi_{t,x}^{(n)}(\mathbf{t},\mathbf{t})^{\frac{1}{2H_0}}d\mathbf{t}\right)^{2H_0} \\ &= \frac{1}{n!}b_{H_0}^n\left(\sum_{\rho\in\mathcal{S}_n}\int_{\{0<t_{\rho(1)}<\cdots<t_{\rho(n)}<t\}}\psi_{t,x}^{(n)}(\mathbf{t},\mathbf{t})^{\frac{1}{2H_0}}d\mathbf{t}\right)^{2H_0}, \end{aligned} \tag{4.5.7}$$

By Lemma 4.5.5, for any fixed $\rho \in \mathcal{S}_n$,

$$\int_{\{0<t_{\rho(1)}<\cdots<t_{\rho(n)}<t\}}\psi_{t,x}^{(n)}(\mathbf{t},\mathbf{t})^{\frac{1}{2H_0}}d\mathbf{t} \leq J_0^{\frac{1}{H_0}}(t,x)\int_{\{0<t_{\rho(1)}<\cdots<t_{\rho(n)}<t\}}$$

$$\left(\int_{\mathbb{R}^n}\prod_{k=1}^{n}\exp\left\{-\frac{t_{\rho(k+1)}-t_{\rho(k)}}{t_{\rho(k)}t_{\rho(k+1)}}\Big|\sum_{j=1}^{k}t_{\rho(j)}\xi_j\Big|^2\right\}\mu(d\xi_1)\cdots\mu(d\xi_n)\right)^{\frac{1}{2H_0}}d\mathbf{t}.$$

Using the change of variables $t'_i = t_{\rho(i)}$ for $i = 1, \cdots, n$ in the dt integral above, we see that this integral has the same value for all $\rho \in \mathcal{S}_n$. We obtain:

$$\int_{\{0<t_{\rho(1)}<\cdots<t_{\rho(n)}<t\}}\psi_{t,x}^{(n)}(\mathbf{t},\mathbf{t})^{\frac{1}{2H_0}}d\mathbf{t} \le J_0^{\frac{1}{H_0}}(t,x)\int_{\{0<t_1<\cdots<t_n<t\}}I_t^{(n)}(t_1,\cdots,t_n)^{\frac{1}{2H_0}}d\mathbf{t},$$

where

$$I_t^{(n)}(t_1,\cdots,t_n)=\int_{\mathbb{R}^n}\exp\left\{-\sum_{k=1}^{n}\frac{t_{k+1}-t_k}{t_kt_{k+1}}\Big|\sum_{i=1}^{k}t_i\xi_i\Big|^2\right\}\mu(d\xi_1)\cdots\mu(d\xi_n). \quad (4.5.8)$$

Taking the sum over all $\rho \in \mathcal{S}_n$ and coming back to (4.5.7), we obtain:

$$E|I_n(f_n(\cdot,t,x))|^2 \le \frac{1}{n!}b_{H_0}^n J_0^2(t,x)(n!)^{2H_0}\left(\int_{\{0<t_1<\cdots<t_n<t\}}I_t^{(n)}(t_1,\cdots,t_n)^{\frac{1}{2H_0}}d\mathbf{t}\right)^{2H_0}, \quad (4.5.9)$$

By Lemma 3.4 of [7], we know that for any function $\psi \in \mathcal{S}(\mathbb{R})$ such that $\psi * \widetilde{\psi} \ge 0$ (where $\widetilde{\psi}(x) = \psi(-x)$ for all $x \in \mathbb{R}$) and for any tempered measure μ on $\mathbb{R}$ whose Fourier transform (in the sense of distributions) is a locally integrable function, we have:

$$\sup_{\eta\in\mathbb{R}}\int_{\mathbb{R}}|\mathcal{F}\psi(\xi+\eta)|^2\mu(d\xi)=\int_{\mathbb{R}}|\mathcal{F}\psi(\xi)|^2\mu(d\xi).$$

We can use this lemma since in the case

$$H > \frac{1}{2},$$

the Fourier transform of $\mu(d\xi) = C_H|\xi|^{1-2H}$ in the space $\mathcal{S}'(\mathbb{R})$ of tempered distributions on $\mathbb{R}$ is the locally integrable function $f(x) = H(2H-1)|x|^{2H-2}$. Then we obtain:

$$\begin{aligned}I_t^{(n)}(t_1,\cdots,t_n) &\le \left(\int_{\mathbb{R}}\exp\left(-\frac{t_2-t_1}{t_1t_2}|t_1\xi_1|^2\right)\mu(d\xi_1)\right)\\ &\qquad\times\left(\prod_{i=2}^{n}\sup_{\eta\in\mathbb{R}}\int_{\mathbb{R}}\exp\left(-\frac{t_{i+1}-t_i}{t_it_{i+1}}\Big|t_i\xi_i+\eta\Big|^2\right)\mu(d\xi_i)\right)\\ &=\prod_{i=1}^{n}\int_{\mathbb{R}}\exp\left(-\frac{t_{i+1}-t_i}{t_it_{i+1}}|t_i\xi_i|^2\right)\mu(d\xi_i). \quad (4.5.10)\end{aligned}$$

Note that using the change of variable $\xi' = t^{1/2}\xi$,

$$\int_{\mathbb{R}} e^{-t|\xi|^2}\mu(d\xi) = C_{H,1}t^{H-1} \quad \text{for all } t > 0,$$

where $C_{H,1} = \int_{\mathbb{R}} e^{-|\xi|^2}\mu(d\xi)$. Using inequality (4.5.10), it follows that

$$\begin{aligned} I_t^{(n)}(t_1, \cdots, t_n) &\le C_{H,1}^n \left(\frac{t_2 - t_1}{t_1 t_2} \frac{t_3 - t_2}{t_2 t_3} \cdots \frac{t - t_n}{t_n t} t_1^2 t_2^2 \cdots t_n^2 \right)^{H-1} \\ &= C_{H,1}^n t^{1-H} [t_1(t_2 - t_1) \cdots (t - t_n)]^{H-1}. \end{aligned} \tag{4.5.11}$$

Using relations (4.5.9) and (4.5.11), and applying Lemma A.3, we see that:

$$\begin{aligned} E|I_n(f_n(\cdot, t, x))|^2 &\le b_{H_0}^n J_0^2(t,x)(n!)^{2H_0-1} C_{H,1}^n t^{1-H} \\ &\quad \left(\int_{\{0<t_1<\cdots<t_n<t\}} [t_1(t_2 - t_1) \cdots (t - t_n)]^{\frac{H-1}{2H_0}} dt_1 \cdots dt_n \right)^{2H_0} \\ &= b_{H_0}^n J_0^2(t,x)(n!)^{2H_0-1} C_{H,1}^n t^{1-H} \left(\frac{\Gamma\left(\frac{H-1}{2H_0} + 1\right)^{n+1}}{\Gamma\left(\frac{H-1}{2H_0}(n+1) + n + 1\right)} t^{\frac{H-1}{2H_0}(n+1)+n} \right)^{2H_0}. \end{aligned}$$

We use the fact that for any $a > 0$ and $b \in \mathbb{R}$, there exists $N_{a,b} \in \mathbb{N}$ and $C_{a,b} > 0$ such that

$$\Gamma(an + 1 + b) \ge C_{a,b}^n (n!)^a \quad \text{for all } n \ge N_{a,b}$$

(see Lemma A.6.)

In our case, there exists $N_1 \in \mathbb{N}$ and $C_2 > 0$ depending on H_0, H such that

$$\Gamma\left(\frac{H-1}{2H_0}(n+1) + n + 1\right) = \Gamma\left(n\frac{2H_0 + H - 1}{2H_0} + \frac{H-1}{2H_0} + 1\right) \ge C_2^n (n!)^{\frac{2H_0+H-1}{2H_0}}$$

for all $n \ge N_1$. Let $C_3 = b_{H_0} C_{H,1} \left[\frac{1}{C_2}\Gamma\left(\frac{H-1}{2H_0} + 1\right)\right]^{2H_0}$ and $C = \Gamma\left(\frac{H-1}{2H_0} + 1\right)^{2H_0}$. Then

$$\begin{aligned} E|I_n(f_n(\cdot, t, x))|^2 &\le C J_0^2(t,x)(n!)^{2H_0-1} t^{1-H} C_3^n t^{(H-1)(n+1)+2H_0 n} \frac{1}{(n!)^{2H_0+H-1}} \\ &= C J_0^2(t,x) t^{n(2H_0+H-1)} C_3^n \frac{1}{(n!)^H}, \end{aligned}$$

for all $n \ge N_1$. Therefore, using Lemma A.7, we obtain:

$$E|u(t,x)|^2 = \sum_{n\ge 0} E|I_n(f_n(\cdot, t, x))|^2 \le \sum_{n\ge 0} C J_0^2(t,x) t^{n(2H_0+H-1)} C_3^n \frac{1}{(n!)^H}$$

$$\leq C_4 J_0^2(t,x) \exp\left\{ C_5 t^{\frac{2H_0+H-1}{H}} \right\}.$$

where C_4 and C_5 are constants depending on H_0 and H. ∎

Chapter 5

The case $H_0 > \frac{1}{2}$ and $H < \frac{1}{2}$

In this chapter, we consider the more difficult case of a Gaussian noise which is fractional in time with index $H_0 > \frac{1}{2}$ and fractional in space with index $H < \frac{1}{2}$, a case which has been studied in the recent references [12, 18, 28]. As in the previous chapters, we begin by investigating the linear stochastic heat and wave equations, before studying equations (1.0.2) and (1.0.3) with this noise. The existence of solution is obtained under the condition $H_0 + H > \frac{3}{4}$ for equation (1.0.2), respectively $H > \frac{1}{4}$ for equation (1.0.3). We present arguments that are slightly different than those of [12, 18, 28].

In the final section of this chapter, we present a result which is new in the literature regarding the existence of the solution of the parabolic Anderson model with this noise and general initial condition given by a measure μ_0 (as in Section 4.5). Unfortunately, for this result we had to impose the additional condition $H > \frac{1}{3}$. Moreover, we obtained a power of the exponent of t (in the exponential function which gives the upper bound for the second moment of the solution) which is different than the exponent ρ^h given by Theorem 4.5.6. We believe that this result is not optimal, but we could not find an alternative proof yielding the exponent ρ^h, and under the optimal condition $H_0 + H > \frac{3}{4}$.

5.1 The noise and the linear equations

In this section, we introduce the noise and we study the linear equations (1.0.6) and (1.0.7) with this noise. Note that the covariance of the noise is a combination of the spatial covariance of the noise given in Chapter 3 (since $H < \frac{1}{2}$) with the temporal covariance structure of the noise given in Chapter 4 (since $H_0 > \frac{1}{2}$).

Let $W = \{W([0,t] \times A); t \geq 0, A \in \mathcal{B}_b(\mathbb{R})\}$ be a zero-mean Gaussian noise with covariance:

$$E[W([0,t] \times A) \cdot W([0,s] \times B)] = R_{H_0}(t,s) C_H \int_{\mathbb{R}} \mathcal{F}1_A(\xi)\overline{\mathcal{F}1_B(\xi)}|\xi|^{1-2H} d\xi \quad (5.1.1)$$

$$:= \langle 1_{[0,t]\times A}, 1_{[0,s]\times B}\rangle_{\mathcal{H}}.$$

where $H_0 \in (\frac{1}{2}, 1), H \in (0, \frac{1}{2})$, C_H is given by (2.1.4) and $R_{H_0}(t,s)$ is given by (4.1.2).

Set $W(1_{[0,t]\times A}) = W([0,t]\times A)$. By linearity, we extend W to the set $\mathcal{E}$ of linear combinations of indicator functions $1_{[0,t]\times A}$ with $t \geq 0$ and $A \in \mathcal{B}_b(R)$.

Lemma 5.1.1. *W is an isometry from $\mathcal{E}$ to $L^2(\Omega)$, i.e. for any $\varphi, \psi \in \mathcal{E}$,*

$$\begin{aligned} E[W(\varphi)W(\psi)] &:= \langle \varphi, \psi\rangle_{\mathcal{H}} \\ &= \alpha_{H_0} C_H \int_0^\infty \int_0^\infty \int_{\mathbb{R}} \mathcal{F}\varphi(t,\cdot)(\xi)\overline{\mathcal{F}\psi(s,\cdot)(\xi)}|t-s|^{2H_0-2}|\xi|^{1-2H} d\xi dt ds. \end{aligned}$$

Proof: Let $\varphi = \sum_{i=1}^n a_i 1_{[0,t_i]\times A_i}$ and $\psi = \sum_{j=1}^m b_j 1_{[0,s_j]\times B_j}$. As in the Lemma 3.1.1, we have:

$$\begin{aligned} E[W(\varphi)W(\psi)] &= E\left[\left(\sum_{i=1}^n a_i W(1_{[0,t_i]\times A_i})\right)\left(\sum_{j=1}^m b_j W(1_{[0,s_j]\times B_j})\right)\right] \\ &= \sum_{i=1}^n \sum_{j=1}^m a_i b_j E[W(1_{[0,t_i]\times A_i}) \cdot W(1_{[0,s_j]\times B_j})] \\ &= \sum_{i=1}^n \sum_{j=1}^m a_i b_j \left(R_{H_0}(t_i, s_j) C_H \int_{\mathbb{R}} \mathcal{F}1_{A_i}(\xi)\overline{\mathcal{F}1_{B_j}(\xi)}|\xi|^{1-2H} d\xi \right) \\ &= \sum_{i=1}^n \sum_{j=1}^m a_i b_j \left(\alpha_{H_0} \int_0^{t_i}\int_0^{s_j} |t-s|^{2H_0-2} ds dt \right)\left(\int_{\mathbb{R}} C_H \mathcal{F}1_{A_i}(\xi)\overline{\mathcal{F}1_{B_j}(\xi)}|\xi|^{1-2H} d\xi \right) \\ &= \sum_{i=1}^n \sum_{j=1}^m a_i b_j \left(\alpha_{H_0} C_H \int_0^{t_i}\int_0^{s_j}\int_{\mathbb{R}} \mathcal{F}1_{A_i}(\xi)\overline{\mathcal{F}1_{B_j}(\xi)}|t-s|^{2H_0-2}|\xi|^{1-2H} d\xi ds dt \right) \\ &= \alpha_{H_0} C_H \int_0^\infty \int_0^\infty \int_{\mathbb{R}} \sum_{i=1}^n \sum_{j=1}^m a_i b_j 1_{[0,t_i]}(t) 1_{[0,s_j]}(s) \\ &\qquad \mathcal{F}1_{A_i}(\xi)\overline{\mathcal{F}1_{B_j}(\xi)}|t-s|^{2H_0-2}|\xi|^{1-2H} d\xi dt ds \\ &= \alpha_{H_0} C_H \int_0^\infty \int_0^\infty \int_{\mathbb{R}} \left(\sum_{i=1}^n a_i 1_{[0,t_i]}(t)\mathcal{F}1_{A_i}(\xi)\right)\left(\sum_{j=1}^m b_j 1_{[0,s_j]}(s)\overline{\mathcal{F}1_{B_j}(\xi)}\right) \\ &\qquad |t-s|^{2H_0-2}|\xi|^{1-2H} dt ds d\xi \\ &= \alpha_{H_0} C_H \int_0^\infty \int_0^\infty \int_{\mathbb{R}} \mathcal{F}\varphi(t,\cdot)(\xi)\mathcal{F}\psi(s,\cdot)(\xi)|t-s|^{2H_0-2}|\xi|^{1-2H} d\xi dt ds. \end{aligned}$$

where for the fourth equality, we used relation (4.1.3) for expressing $R_{H_0}(t_i, s_j)$. ■

Let $\mathcal{H}$ be the completion of $\mathcal{E}$ with respect to $\langle \cdot, \cdot \rangle_{\mathcal{H}}$. We extend the isometry map from Lemma 5.1.1 to $\mathcal{H}$. We denote this map by $W(\varphi), \varphi \in \mathcal{H}$ and we say that $W(\varphi)$ is the *Wiener integral* of φ with respect to W. We use the notation

$$W(\varphi) = \int_0^\infty \int_{\mathbb{R}} \varphi(t,x) W(dt,dx).$$

The following result is a particular case of Theorem 2.6 c) of [9].

Theorem 5.1.2. *Let* $\varphi : [0,T] \times \mathbb{R} \to \mathbb{R}$ *be such that* $\varphi(t,\cdot) \in L^1(\mathbb{R})$ *for all* $t \in [0,T]$ *and let*

$$\mathcal{F}\varphi(t,\cdot)(\xi) = \int_{\mathbb{R}} e^{-i\xi \cdot x} \varphi(t,x) dx, \quad \xi \in \mathbb{R}$$

be the Fourier transform of $\varphi(t,\cdot)$. *Suppose that* $\int_0^T |\mathcal{F}\varphi(t,\cdot)(\xi)| dt < \infty$. *If*

$$\|\varphi\|_o^2 := \alpha_{H_0} C_H \int_0^T \int_0^T \int_{\mathbb{R}} |t-s|^{2H_0-2} \mathcal{F}\varphi(t,\cdot)(\xi) \overline{\mathcal{F}\varphi(s,\cdot)(\xi)} |\xi|^{1-2H} d\xi dt ds < \infty,$$

then $\varphi \in \mathcal{H}$ *and* $\|\varphi\|_{\mathcal{H}}^2 = \|\varphi\|_o^2$.

Remark 5.1.3. The process $\{W(t,x); t \geq 0, x \in \mathbb{R}\}$ given by $W(t,x) = W(1_{[0,t]\times[0,x]})$ is a fractional Brownian sheet of index H_0 in time and index H in space.

Next we consider equations (1.0.6) and (1.0.7) with noise $\dot{W}$ as above. The solutions to these equations are defined as in Definition 2.2.1, respectively Definition 2.2.3.

The following result has the same proof as Theorem 4.2.1 (for heat equation) and Theorem 4.3.1 (for wave equation).

Theorem 5.1.4. *For any* $H_0 \in (\frac{1}{2}, 1)$ *and* $H \in (0, \frac{1}{2})$, *equations* (1.0.6) *and* (1.0.7) *with a Gaussian noise with covariance given by* (5.1.1) *have unique solutions.*

5.2 The Parabolic and Hyperbolic Anderson models

In this section, we consider equations (1.0.2) and (1.0.3) with noise $\dot{W}$ introduced in Section 5.1. The solution is defined as in Definition 2.3.1.

Theorem 5.2.1. *a) For any* $H_0 \in (\frac{1}{2}, 1)$ *and* $H \in (0, \frac{1}{2})$, *such that*

$$H_0 + H > \frac{3}{4},$$

equation (1.0.2) *with noise* $\dot{W}$ *as in Section 5.1 has a unique solution. Moreover,*

$$E|u(t,x)|^2 \leq C_1^h \exp(C_2^h t^{\rho^h}),$$

where

$$\rho^h = \frac{2H_0 + H - 1}{H},$$

and C_1^h, C_2^h *are positive constants depending on* H_0 *and* H.

b) For any $H_0 \in (\frac{1}{2}, 1)$ *and* $H \in (\frac{1}{4}, \frac{1}{2})$, *equation* (1.0.3) *with noise* $\dot{W}$ *as in Section 5.1 has a unique solution. Moreover,*

$$E|u(t,x)|^2 \leq C_1^w \exp(C_2^w t^{\rho^w}),$$

where

$$\rho^w = \frac{2H_0 + 2H}{2H + 1},$$

and C_1^w, C_2^w *are positive constants depending on* H_0 *and* H.

Proof: We proceed similarly to the proof of the Theorem 4.4.1. In this case,

$$E|I_n(f_n(\cdot,t,x))|^2 = n!\|\tilde{f}_n(\cdot,t,x)\|^2 = \frac{1}{n!}\alpha_{H_0}^n \int_{[0,t]^{2n}} \prod_{i=1}^n |t_i - s_i|^{2H_0-2} \psi_n(\mathbf{t},\mathbf{s}) d\mathbf{t} d\mathbf{s}$$

where

$$\begin{aligned}\psi_n(\mathbf{t},\mathbf{s}) =& C_H^n (n!)^2 \int_{\mathbb{R}^n} \mathcal{F}\tilde{f}_n(t_1,\cdot,\cdots,t_n,\cdot,t,x)(\xi_1,\cdots,\xi_n) \\ & \overline{\mathcal{F}\tilde{f}_n(s_1,\cdot,\cdots,s_n,\cdot,t,x)(\xi_1,\cdots,\xi_n)} \prod_{i=1}^n |\xi_i|^{1-2H} d\xi_1 \cdots d\xi_n.\end{aligned}$$

By relation (4.4.3) (which is a consequence of Littlewood-Hardy inequality given by Lemma D.4),

$$E|I_n(f_n(\cdot,t,x))|^2 \leq \frac{1}{n!} b_{H_0}^n \left(\sum_{\rho \in S_n} \int_{0 < t_{\rho(1)} < \cdots < t_{\rho(n)} < t} \psi_n(\mathbf{t},\mathbf{t})^{\frac{1}{2H_0}} d\mathbf{t} \right)^{2H_0}. \qquad (5.2.1)$$

Let $\rho \in S_n$ and $(t_1,\cdots,t_n) \in [0,t]^n$ such that $0 < t_{\rho(1)} < \cdots < t_{\rho(n)} < t$. Denote $u_i = t_{\rho(i+1)} - t_{\rho(i)}$ for all $i = 1,\cdots,n$ where $t_{\rho(n+1)} = t$. We will use relation (4.4.5) for calculating $\psi_n(\mathbf{t},\mathbf{t})$. However, we will not be able to use estimate (4.4.6) since Lemma 2.3.6 cannot be applied for the case $H < \frac{1}{2}$. Using the change of variable $\eta_i = \xi_1 + \cdots + \xi_i$ for all $i = 1,\cdots,n$, we obtain:

$$\psi_n(\mathbf{t},\mathbf{t}) = C_H^n \int_{\mathbb{R}^n} |\mathcal{F}G(u_1,\cdot)(\xi_1)|^2 \cdots |\mathcal{F}G(u_n,\cdot)(\xi_1 + \cdots + \xi_n)|^2 \prod_{i=1}^n |\xi_i|^{1-2H} d\xi_1 \cdots d\xi_n$$

$$= C_H^n \int_{\mathbb{R}^n} |\mathcal{F}G(u_1,\cdot)(\eta_1)|^2 \cdots |\mathcal{F}G(u_n,\cdot)(\eta_n)|^2 |\eta_1|^{1-2H} \prod_{j=2}^n |\eta_j - \eta_{j-1}|^{1-2H} d\eta_1 \cdots d\eta_n.$$

For the estimate of the last integral, we proceed as in the proof of Theorem 3.2.2. More precisely, by relation (3.2.5), we have:

$$\psi_n(\mathbf{t},\mathbf{t}) \leq C_H^n \sum_{\alpha \in D_n} \int_{\mathbb{R}^n} |\mathcal{F}G(u_1,\cdot)(\eta_1)|^2 \cdots |\mathcal{F}G(u_n,\cdot)(\eta_n)|^2 |\eta_1|^{1-2H} \prod_{i=1}^n |\eta_i|^{\alpha_i} d\eta_1 \cdots d\eta_n$$

$$= C_H^n \sum_{\alpha \in D_n} \left(\int_{\mathbb{R}} |\mathcal{F}G(u_1,\cdot)(\eta_1)|^2 |\eta_1|^{1-2H+\alpha_1} d\eta_1 \right) \prod_{i=2}^n \left(\int_{\mathbb{R}} |\mathcal{F}G(u_i,\cdot)(\eta_i)|^2 |\eta_i|^{\alpha_i} d\eta_i \right). \tag{5.2.2}$$

At this point, we need to consider separately the parabolic case and the hyperbolic case.

a) In the case of equation (1.0.2), using (5.2.2) and Lemma 3.2.1 a), we get:

$$\psi_n(\mathbf{t},\mathbf{t}) \leq C_H^n \sum_{\alpha \in D_n} \Gamma\left(\frac{2-2H+\alpha_1}{2}\right) u_1^{-\frac{2-2H+\alpha_1}{2}} \prod_{i=2}^n \left(\Gamma\left(\frac{1+\alpha_i}{2}\right) u_i^{-\frac{1+\alpha_i}{2}} \right)$$

$$\leq C_{H,1}^n \sum_{\alpha \in D_n} u_1^{-\frac{2-2H+\alpha_1}{2}} \prod_{i=2}^n u_i^{-\frac{1+\alpha_i}{2}},$$

where $C_{H,1} > 0$ is a constant depending on H. To apply Lemma 3.2.1 a), we need $1 - 2H + \alpha_1 > -1$ and $\alpha_i > -1$ for $i = 2, \ldots, n$, which is clearly true since $\alpha_i \geq 0$ for all $i = 1, \cdots, n$. Coming back to (5.2.1), using the change of variable $t_{\rho(i)} = t_i'$ and the inequality $(\sum_{i=1}^N a_i)^{\frac{1}{2H_0}} \leq \sum_{i=1}^N a_i^{\frac{1}{2H_0}}$, we obtain:

$$E|I_n(f_n(\cdot,t,x))|^2 \leq \frac{1}{n!} b_{H_0}^n \left[C_{H,1}^n n! \int_{T_n(t)} \left(\sum_{\alpha \in D_n} (t_2 - t_1)^{-\frac{2-2H+\alpha_1}{2}} \prod_{j=2}^n (t_{j+1} - t_j)^{-\frac{1+\alpha_j}{2}} \right)^{\frac{1}{2H_0}} d\mathbf{t} \right]^{2H_0}$$

$$= (n!)^{2H_0 - 1} C_{H,1}^n b_{H_0}^n \left[\int_{T_n(t)} \left(\sum_{\alpha \in D_n} (t_2 - t_1)^{-\frac{2-2H+\alpha_1}{2}} \prod_{j=2}^n (t_{j+1} - t_j)^{-\frac{1+\alpha_j}{2}} \right)^{\frac{1}{2H_0}} d\mathbf{t} \right]^{2H_0}$$

$$\leq (n!)^{2H_0 - 1} C_{H,1}^n b_{H_0}^n \left(\sum_{\alpha \in D_n} \int_{T_n(t)} (t_2 - t_1)^{-\frac{2-2H+\alpha_1}{4H_0}} \prod_{j=2}^n (t_{j+1} - t_j)^{-\frac{1+\alpha_j}{4H_0}} d\mathbf{t} \right)^{2H_0}.$$

For each fixed $\alpha \in D_n$, we apply Lemma A.3 with

$$\beta_1 = -\frac{2-2H+\alpha_1}{4H_0}, \quad \beta_j = -\frac{1+\alpha_j}{4H_0}, j = 2, \cdots, n.$$

Hence

$$|\beta| = \sum_{i=1}^{n} \beta_i = -\frac{2-2H+(n-1)+|\alpha|}{4H_0} = \frac{Hn-n}{2H_0},$$

since $|\alpha| = (n-1)(1-2H)$. (To apply Lemma A.3, we need $\beta_j > -1$ for all $j = 1, \cdots, n$. When $\alpha_j = 2(1-2H)$, this imposes the restriction $H_0 + H > \frac{3}{4}$. We encounter this condition also for $\beta_1 > -1$ when $\alpha_1 = 1-2H$.) We obtain:

$$\int_{T_n(t)} (t_2-t_1)^{-\frac{2-2H_0+\alpha_1}{4H_0}} \prod_{j=2}^{n} (t_{j+1}-t_j)^{-\frac{1+\alpha_j}{4H_0}} d\mathbf{t}$$

$$= t^{\frac{Hn-n}{2H_0}+n} \frac{\Gamma(-\frac{2-2H_0+\alpha_1}{4H_0}+1)\prod_{j=2}^{n}\Gamma(-\frac{1+\alpha_j}{4H_0}+1)}{\Gamma(\frac{Hn-n}{2H_0}+n+1)} \leq C_{H,2}^{n} \frac{t^{n\frac{2H_0+H-1}{2H_0}}}{\Gamma(n\frac{2H_0+H-1}{2H_0}+1)},$$

where

$$C_{H,2} = \max\left\{\Gamma\Big(-\frac{2-2H}{4H_0}+1\Big), \Gamma\Big(-\frac{3-4H}{4H_0}+1\Big), \Gamma\Big(-\frac{1}{4H_0}+1\Big)\right\}.$$

Since the previous estimate does not depend on α and $\mathrm{card}(D_n) = 2^{n-1}$,

$$E|I_n(f_n(\cdot,t,x))|^2 \leq (n!)^{2H_0-1} C_{H,1}^{n} b_{H_0}^{n} \left\{2^{n-1} C_{H,2}^{n} \frac{t^{n\frac{2H_0+H-1}{2H_0}}}{\Gamma(n\frac{2H_0+H-1}{2H_0}+1)}\right\}^{2H_0}$$

$$\leq C_{H,H_0,1}^{n} (n!)^{2H_0-1} \frac{t^{n(2H_0+H-1)}}{(\Gamma(n\frac{2H_0+H-1}{2H_0}+1))^{2H_0}},$$

where $C_{H,H_0,1} = C_{H,1} b_{H_0} (2C_{H,2})^{2H_0}$. Finally by Corollary A.5, we obtain:

$$E|I_n(f_n(\cdot,t,x))|^2 \leq C_{H,H_0,2}^{n} (n!)^{2H_0-1} \frac{t^{n(2H_0+H-1)}}{(n!)^{\frac{2H_0+H-1}{2H_0}\cdot 2H_0}} = \frac{(C_{H,H_0,2} t^{2H_0+H-1})^n}{(n!)^H},$$

where $C_{H,H_0,2}$ is a constant depending on H and H_0. By Lemma A.7,

$$E|u(t,x)|^2 = \sum_{n\geq 0} E|I_n(f_n(\cdot,t,x))|^2 \leq \sum_{n\geq 0} \frac{(C_{H,H_0,2} t^{2H_0+H-1})^n}{(n!)^H}$$

$$\leq C_1^h \exp\left\{C_2^h t^{\frac{2H_0+H-1}{H}}\right\},$$

where $C_1^h > 0$ and $C_2^h > 0$ are constants depending on H and H_0.

b) We consider next equation (1.0.3). By relation (5.2.2) and Lemma 3.2.1 b),

$$\psi_n(\mathbf{t},\mathbf{t}) \leq C_{H,3}^{n} \sum_{\alpha\in D_n} u_1^{2H-\alpha_1} \prod_{i=2}^{n} u_i^{1-\alpha_i},$$

where $C_{H,3} > 0$ is a constant depending on H. To apply this lemma, we need $1 - 2H + \alpha_1 \in (-1, 1)$ and $\alpha_j \in (-1, 1)$ for $j = 2, \cdots, n$. When $\alpha_j = 2(1 - 2H)$, the condition $\alpha_j < 1$ imposes the restriction $H > \frac{1}{4}$. Coming back to (5.2.1), using the same change of variable as in the proof of a) and the inequality $(\sum_{i=1}^N a_i)^{\frac{1}{2H_0}} \leq \sum_{i=1}^N a_i^{\frac{1}{2H_0}}$, we obtain:

$$
\begin{aligned}
E|I_n(f_n(\cdot, t, x))|^2 &\leq \frac{1}{n!} b_{H_0}^n \Bigg[C_{H,3}^n n! \\
&\qquad \int_{T_n(t)} \left(\sum_{\alpha \in D_n} (t_2 - t_1)^{2H - \alpha_1} \prod_{j=2}^n (t_{j+1} - t_j)^{1 - \alpha_j} \right)^{\frac{1}{2H_0}} d\mathbf{t} \Bigg]^{2H_0} \\
&= (n!)^{2H_0 - 1} C_{H,3}^n b_{H_0}^n \left[\int_{T_n(t)} \left(\sum_{\alpha \in D_n} (t_2 - t_1)^{2H - \alpha_1} \prod_{j=2}^n (t_{j+1} - t_j)^{1 - \alpha_j} \right)^{\frac{1}{2H_0}} d\mathbf{t} \right]^{2H_0} \\
&\leq (n!)^{2H_0 - 1} C_{H,3}^n b_{H_0}^n \left(\sum_{\alpha \in D_n} \int_{T_n(t)} (t_2 - t_1)^{\frac{2H - \alpha_1}{2H_0}} \prod_{j=2}^n (t_{j+1} - t_j)^{\frac{1 - \alpha_j}{2H_0}} d\mathbf{t} \right)^{2H_0}.
\end{aligned}
$$

For any $\alpha \in D_n$ fixed, we apply Lemma A.3 with

$$
\beta_1 = \frac{2H - \alpha_1}{2H_0}, \quad \beta_j = \frac{1 - \alpha_j}{2H_0}, j = 2, \cdots, n.
$$

Hence

$$
|\beta| = \sum_{j=1}^n \beta_j = \frac{2H + (n-1) - |\alpha|}{2H_0} = \frac{Hn}{H_0}.
$$

To apply Lemma A.3, we need $\beta_j > -1$ for all $j = 1, \cdots, m$, which imposes no restriction on H. We obtain:

$$
\begin{aligned}
&\int_{T_n(t)} (t_2 - t_1)^{\frac{2H - \alpha_1}{2H_0}} \prod_{j=2}^n (t_{j+1} - t_j)^{\frac{1 - \alpha_j}{2H_0}} d\mathbf{t} \\
&\qquad = t^{\frac{Hn}{H_0} + n} \frac{\Gamma(\frac{2H - \alpha_1}{2H_0} + 1) \prod_{j=2}^n \Gamma(\frac{1 - \alpha_j}{2H_0} + 1)}{\Gamma(\frac{Hn}{H_0} + n + 1)} \leq C_{H,4}^n \frac{t^{n \frac{H_0 + H}{H_0}}}{\Gamma(n \frac{H_0 + H}{H_0} + 1)},
\end{aligned}
$$

where

$$
C_{H,4} = \max \left\{ \Gamma\left(\frac{H}{H_0} + 1 \right), \Gamma\left(\frac{4H - 1}{2H_0} + 1 \right), \Gamma\left(\frac{1}{2H_0} + 1 \right) \right\}.
$$

Since the previous estimate does not depend on α and $\text{card}(D_n) = 2^{n-1}$,

$$
E|I_n(f_n(\cdot, t, x))|^2 \leq (n!)^{2H_0 - 1} C_{H,3}^n b_{H_0}^n \left\{ 2^{n-1} C_{H,4}^n \frac{t^{n \frac{H_0 + H}{H_0}}}{\Gamma(n \frac{H_0 + H}{H_0} + 1)} \right\}^{2H_0}
$$

$$\leq C^n_{H,H_0,3}(n!)^{2H_0-1}\frac{t^{n(2H_0+2H)}}{(\Gamma(n\frac{H_0+H}{H_0}+1))^{2H_0}},$$

where $C_{H,H_0,3} = C_{H,3}b_{H_0}(2C_{H,4})^{2H_0}$. By Corollary A.5, we obtain:

$$E|I_n(f_n(\cdot,t,x))|^2 \leq C^n_{H,H_0,4}(n!)^{2H_0-1}\frac{t^{n(2H_0+2H)}}{(n!)^{\frac{H_0+H}{H_0}\cdot 2H_0}} = \frac{(C_{H,H_0,4}t^{2H_0+2H})^n}{(n!)^{2H+1}},$$

where $C_{H,H_0,4}$ is a constant depending on H and H_0. Finally by Lemma A.7,

$$\begin{aligned} E|u(t,x)|^2 &= \sum_{n\geq 0} E|I_n(f_n(\cdot,t,x))|^2 \leq \sum_{n\geq 0}\frac{(C_{H,H_0,4}t^{2H_0+2H})^n}{(n!)^{2H+1}} \\ &\leq C^w_1 \exp\left\{C^w_2 t^{\frac{2H_0+2H}{2H+1}}\right\}, \end{aligned}$$

where $C^w_1 > 0$ and $C^w_2 > 0$ are constants depending on H and H_0. ∎

Remark 5.2.2. Note that if $H_0 = \frac{1}{2}$ then the exponents ρ^h and ρ^w given by Theorem 5.2.1 are equal to 1, which is consistent with the statement of Theorem 3.2.2.

Remark 5.2.3. Condition $H_0 + H > \frac{3}{4}$ that is encountered for equation (1.0.2) is natural since it coincides with condition $H > \frac{1}{4}$ when $H_0 = \frac{1}{2}$. On the other hand, for equation (1.0.3), we believe that the condition $H > \frac{1}{4}$ given by Theorem 5.2.1 *b*) is not optimal since it does not involve the Hurst index H_0. This is an open problem in the literature, (see also the recent preprint [28]). The problem is due to the use of the Littlewood-Hardy inequality given by Lemma D.4 leading to estimate (5.2.1). As we have seen in the proof of Theorem 4.3.1, the existence of the solution for the linear wave equation is obtained using different methods than for the linear heat equation, in particular methods which *do not rely* on the Littlewood-Hardy inequality.

5.3 The Parabolic Anderson model with general initial condition

In this section, we consider equation (4.5.1) with noise introduced in Section 5.1. We recall this equation below for the sake of completeness:

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) &= \dfrac{1}{2}\dfrac{\partial^2 u}{\partial x^2}(t,x) + u(t,x)\dot{W}(t,x), \quad t>0, x\in\mathbb{R} \\ u(0,\cdot) &= \mu_0(\cdot). \end{cases}$$

The solution is defined as in Definition 4.5.1. The initial condition is given by a measure μ_0 which satisfies the condition (4.5.2).

In the recent article [18], the authors proved that that solution u of equation (4.5.1) satisfies:

$$E|u(t,x)|^2 \leq C_1 t^{-2\beta} \exp\left\{C_2 t^{\frac{2H_0+H-1}{H}}\right\}$$

under the condition $H_0 + H > \frac{3}{4}$, assuming that the initial condition μ_0 satisfies the condition:

$$\int_{\mathbb{R}} \left(1 + |\xi|^{\frac{1}{2}-H}\right) e^{-t|\xi|^2} |\mathcal{F}\mu_0(\xi)| d\xi \leq C t^{-\beta} \tag{5.3.1}$$

for some $\beta < H_0$. In this section, we prove a slightly different result, under a more general condition on μ_0 but a more restrictive condition on H, namely $H > 1/3$. An example when condition (4.5.2) is satisfied but condition (5.3.1) dees not hold is when μ_0 is the Lebesgue measure.

The following result is new in the literature and is the main contribution of the thesis.

Theorem 5.3.1. *If $H_0 \in (\frac{1}{2}, 1)$, $H \in (\frac{1}{3}, \frac{1}{2})$, and μ_0 satisfies* (4.5.2)*, then equation* (4.5.1) *with noise $\dot{W}$ given in Section 5.1 has a unique solution. Moreover,*

$$E|u(t,x)|^2 \leq C_1 J_0^2(t,x) \exp\left\{C_2 t^{\frac{2H_0+H-1}{3H-1}}\right\},$$

where C_1 and C_2 are constants depending on H_0 and H.

Proof: We begin by arguing as in the first part of the proof of Theorem 4.5.6. Note that relation (4.5.9), is still valid in the case $H < \frac{1}{2}$, i.e,

$$E|I_n(f_n(\cdot,t,x))|^2 \leq b_{H_0}^n (n!)^{2H_0-1} J_0^2(t,x) \left(\int_{\{0<t_1<\cdots<t_n<t\}} I_t^{(n)}(t_1,\cdots,t_n)^{\frac{1}{2H_0}} d\mathbf{t}\right)^{2H_0}. \tag{5.3.2}$$

where $I_t^{(n)}(t_1,\cdots,t_n)$ is given by (4.5.8) and $\mu(d\xi) = C_H|\xi|^{1-2H}d\xi$.

In the case $H < 1/2$, we estimate the integral $I_t^{(n)}(t_1,\cdots,t_n)$ differently than in the case $H > 1/2$ since relation (4.5.10) may not hold in the case $H < 1/2$. First, using the change of variables $t_j\xi_j = z_j$ for $j = 1,\cdots,n$, we obtain:

$$I_t^{(n)}(t_1,\cdots,t_n) = \int_{\mathbb{R}^n} \prod_{k=1}^n \exp\left\{-\frac{t_{k+1}-t_k}{t_k t_{k+1}}\left|\sum_{j=1}^k z_j\right|^2\right\}\left|\frac{z_k}{t_k}\right|^{1-2H} \frac{1}{t_1}dz_1\cdots\frac{1}{t_n}dz_n$$

$$= (t_1 \cdots t_n)^{2H-2} \int_{\mathbb{R}^n} \prod_{k=1}^{n} \exp\left\{ - \frac{t_{k+1} - t_k}{t_k t_{k+1}} \left| \sum_{j=1}^{k} z_j \right|^2 \right\} |z_k|^{1-2H} dz_1 \cdots dz_n,$$

where $t_{n+1} = t$. Then as in the proof of the Theorem 3.2.2, we use the change of variables $\eta_i = z_1 + \cdots + z_i$ for $i = 1, \cdots, n$, followed by inequality (3.2.5). We obtain:

$$I_t^{(n)}(t_1, \cdots, t_n) \leq (t_1 \cdots t_n)^{2H-2} \int_{\mathbb{R}^n} \exp\left(-\frac{t_2 - t_1}{t_1 t_2} |\eta_1|^2 \right) \cdots \exp\left(-\frac{t - t_n}{t_n t} |\eta_n|^2 \right)$$
$$|\eta_1|^{1-2H} \prod_{i=2}^{n} |\eta_i - \eta_{i-1}|^{1-2H} d\eta_1 \cdots d\eta_n$$
$$\leq (t_1 \cdots t_n)^{2H-2} \sum_{\alpha \in D_n} \int_{\mathbb{R}^n} \exp\left(-\frac{t_2 - t_1}{t_1 t_2} |\eta_1|^2 \right) \cdots \exp\left(-\frac{t - t_n}{t_n t} |\eta_n|^2 \right)$$
$$|\eta_1|^{1-2H} \prod_{i=1}^{n} |\eta_i|^{\alpha_i} d\eta_1 \cdots d\eta_n$$
$$= (t_1 \cdots t_n)^{2H-2} \sum_{\alpha \in D_n} \left(\int_{\mathbb{R}} \exp\left(-\frac{t_2 - t_1}{t_1 t_2} |\eta_1|^2 \right) |\eta_1|^{1-2H+\alpha_1} d\eta_1 \right)$$
$$\prod_{i=2}^{n} \left(\int_{\mathbb{R}} \exp\left(-\frac{t_i - t_{i-1}}{t_{i-1} t_i} |\eta_i|^2 \right) |\eta_i|^{\alpha_i} d\eta_i \right),$$

where D_n is a set of multi-indices $\alpha = (\alpha_1, \cdots, \alpha_n)$ with the following properties:

$$\alpha_1, \alpha_n \in \{0, 1-2H\}, \quad \alpha_j \in \{0, 1-2H, 2(1-2H)\} \text{ for } j = 2, \cdots, n,$$

$$|\alpha| = \sum_{j=1}^{n} \alpha_j = (n-1)(1-2H).$$

The exact description of the set D_n is not needed.

Each of the integrals above can be computed explicitly using Lemma 3.2.1 *a*). To apply this lemma, we need $1 - 2H + \alpha_1 > -1$ and $\alpha_i > -1$ for $i = 2, \ldots, n$, which is clearly true since $\alpha_i \geq 0$ for all $i = 1, \ldots, n$. We obtain:

$$I_t^{(n)}(t_1, \cdots, t_n) \leq (t_1 \cdots t_n)^{2H-2} \sum_{\alpha \in D_n} \Gamma\left(\frac{2 - 2H + \alpha_1}{2} \right) \left(\frac{t_2 - t_1}{t_1 t_2} \right)^{-\frac{2-2H+\alpha_1}{2}}$$
$$\prod_{i=2}^{n} \left\{ \Gamma\left(\frac{1 + \alpha_i}{2} \right) \left(\frac{t_{i+1} - t_i}{t_i t_{i+1}} \right)^{-\frac{1+\alpha_i}{2}} \right\}$$
$$\leq C_{H,1}^{n} (t_1 \cdots t_n)^{2H-2} \sum_{\alpha \in D_n} \left(\frac{t_2 - t_1}{t_1 t_2} \right)^{-\frac{2-2H+\alpha_1}{2}} \prod_{i=2}^{n} \left(\frac{t_{i+1} - t_i}{t_i t_{i+1}} \right)^{-\frac{1+\alpha_i}{2}},$$

where

$$C_{H,1} = \max\left\{\Gamma\left(1-H\right), \Gamma\left(\frac{3-4H}{2}\right), \Gamma\left(\frac{1}{2}\right)\right\},$$

Then we can see that:

$$\begin{aligned}
I_t^{(n)}(t_1,\cdots,t_n) \le{}& C_{H,1}^n (t_1\cdots t_n)^{2H-2} \sum_{\alpha\in D_n} (t_1t_2)^{\frac{2-2H+\alpha_1}{2}} (t_2t_3)^{\frac{1+\alpha_2}{2}} \cdots (t_nt)^{\frac{1+\alpha_n}{2}} \\
&(t_2-t_1)^{-\frac{2-2H+\alpha_1}{2}} \prod_{i=2}^{n} (t_{i+1}-t_i)^{-\frac{1+\alpha_i}{2}} \\
={}& C_{H,1}^n (t_1\cdots t_n)^{2H-2} \sum_{\alpha\in D_n} t_1^{\frac{2-2H+\alpha_1}{2}} t_2^{\frac{2-2H+\alpha_1+1+\alpha_2}{2}} t_3^{\frac{1+\alpha_2+1+\alpha_3}{2}} \cdots t_n^{\frac{1+\alpha_{n-1}+1+\alpha_n}{2}} t^{\frac{1+\alpha_n}{2}} \\
&(t_2-t_1)^{-\frac{2-2H+\alpha_1}{2}} \prod_{i=2}^{n} (t_{i+1}-t_i)^{-\frac{1+\alpha_i}{2}} \\
={}& C_{H,1}^n \sum_{\alpha\in D_n} t_1^{\frac{2H-2+\alpha_1}{2}} t_2^{\frac{2H-1+\alpha_1+\alpha_2}{2}} t_3^{\frac{4H-2+\alpha_2+\alpha_3}{2}} \cdots t_n^{\frac{4H-2+\alpha_{n-1}+\alpha_n}{2}} t^{\frac{1+\alpha_n}{2}} \\
&(t_2-t_1)^{-\frac{2-2H+\alpha_1}{2}} \prod_{i=2}^{n} (t_{i+1}-t_i)^{-\frac{1+\alpha_i}{2}}.
\end{aligned}$$

We take power $\frac{1}{2H_0}$. We use inequality $(\sum_{i=1}^N a_i)^{\frac{1}{2H_0}} \le \sum_{i=1}^N a_i^{\frac{1}{2H_0}}$ for $H_0 \in (\frac{1}{2}, 1)$. Then we obtain:

$$\begin{aligned}
I_t^{(n)}(t_1,\cdots,t_n)^{\frac{1}{2H_0}} \le{}& C_{H,1}^{\frac{n}{2H_0}} \sum_{\alpha\in D_n} t^{\frac{1+\alpha_n}{4H_0}} t_1^{\frac{2H-2+\alpha_1}{4H_0}} t_2^{\frac{2H-1+\alpha_1+\alpha_2}{4H_0}} \prod_{j=3}^{n} t_j^{\frac{4H-2+\alpha_{j-1}+\alpha_j}{4H_0}} \\
&(t_2-t_1)^{-\frac{2-2H+\alpha_1}{4H_0}} \prod_{j=2}^{n} (t_{j+1}-t_j)^{-\frac{1+\alpha_j}{4H_0}},
\end{aligned}$$

Coming back to (5.3.2), it follows that

$$\begin{aligned}
E|I_n(f_n(\cdot,t,x))|^2 \le{}& J_0^2(t,x) b_{H_0}^n (n!)^{2H_0-1} C_{H,1}^n \Bigg(\sum_{\alpha\in D_n} t^{\frac{1+\alpha_n}{4H_0}} \int_{\{0<t_1<\cdots<t_n<t\}} \\
& t_1^{\frac{2H-2+\alpha_1}{4H_0}} t_2^{\frac{2H-1+\alpha_1+\alpha_2}{4H_0}} \prod_{j=3}^{n} t_j^{\frac{4H-2+\alpha_{j-1}+\alpha_j}{4H_0}} (t_2-t_1)^{-\frac{2-2H+\alpha_1}{4H_0}} \prod_{j=2}^{n} (t_{j+1}-t_j)^{-\frac{1+\alpha_j}{4H_0}} \, d\mathbf{t} \Bigg)^{2H_0}.
\end{aligned} \tag{5.3.3}$$

To compute the integrals above, we use Lemma A.1. We fix $\alpha \in D_n$ and denote

$$\widetilde{\alpha}_1 = \frac{2H-2+\alpha_1}{4H_0}, \widetilde{\alpha}_2 = \frac{2H-1+\alpha_1+\alpha_2}{4H_0}, \widetilde{\alpha}_j = \frac{4H-2+\alpha_{j-1}+\alpha_j}{4H_0}, j = 3,\ldots,n.$$

and

$$\widetilde{\beta}_1 = -\frac{2-2H+\alpha_1}{4H_0}, \quad \widetilde{\beta}_j = -\frac{1+\alpha_j}{4H_0}, j = 2, \dots, n.$$

We verify that the parameters $\widetilde{\alpha}_j, \widetilde{\beta}_j, j = 1, \cdots, n$ satisfy the conditions of Lemma A.1. The condition $\widetilde{\alpha}_1 > -1$ is equivalent to $\alpha_1 > 2 - 4H_0 - 2H$ which is clearly satisfied since $\alpha_1 > 0$. The condition $\widetilde{\beta}_1 > -1$ is equivalent to $\alpha_1 < 4H_0 + 2H - 2$, and when $\alpha_1 = 1 - 2H$, this becomes

$$H_0 + H > \frac{3}{4} \tag{5.3.4}$$

which holds since $H_0 > \frac{1}{2}$ and $H > \frac{1}{3}$. The condition $\widetilde{\beta}_j > -1$ for $j = 2, \cdots, n$ is equivalent to $\alpha_j < 4H_0 - 1$, and when $\alpha_j = 2(1-2H)$, this becomes (5.3.4).

Next, we verify the condition

$$\sum_{i=1}^{k} (\widetilde{\alpha}_i + \widetilde{\beta}_i) + k + 1 + \widetilde{\alpha}_{k+1} > 0 \quad \text{for all } k = 1, \dots, n-1 \tag{5.3.5}$$

by induction on k. Note that

$$\widetilde{\alpha}_1 + \widetilde{\beta}_1 = \frac{(2H-2+\alpha_1) - (2-2H+\alpha_1)}{4H_0} = \frac{H-1}{H_0},$$

$$\widetilde{\alpha}_2 + \widetilde{\beta}_2 = \frac{2H-1+\alpha_1+\alpha_2-(\alpha_2+1)}{4H_0} = \frac{2H-2+\alpha_1}{4H_0},$$

$$\widetilde{\alpha}_k + \widetilde{\beta}_k = \frac{4H-2+\alpha_{k-1}+\alpha_k-(\alpha_k+1)}{4H_0} = \frac{4H-3+\alpha_{k-1}}{4H_0}, \text{ for } k = 3, \dots, n.$$

Hence

$$\widetilde{\alpha}_1 + \widetilde{\beta}_1 + \widetilde{\alpha}_2 + \widetilde{\beta}_2 = \frac{H-1}{H_0} + \frac{2H-2+\alpha_1}{4H_0} = \frac{6H-6+\alpha_1}{4H_0},$$

and for $k = 3, \cdots, n$,

$$\begin{aligned} \sum_{i=1}^{k} (\widetilde{\alpha}_i + \widetilde{\beta}_i) &= \frac{6H-6+\alpha_1}{4H_0} + \sum_{j=3}^{k} \frac{4H-3+\alpha_{j-1}}{4H_0} \\ &= \frac{6H-6+(k-2)(4H-3)+\sum_{i=1}^{k-1}\alpha_i}{4H_0} \\ &= k\frac{4H-3}{4H_0} - \frac{H}{2H_0} + \frac{1}{4H_0}\sum_{i=1}^{k-1}\alpha_i, \end{aligned} \tag{5.3.6}$$

In particular, for $k = n$,

$$|\widetilde{\alpha}| + |\widetilde{\beta}| = \sum_{k=1}^{n} \widetilde{\alpha}_k + \sum_{k=1}^{n} \widetilde{\beta}_k = n\frac{4H-3}{4H_0} - \frac{H}{2H_0} + \frac{1}{4H_0}\sum_{i=1}^{n-1} \alpha_i$$

$$= n\frac{4H-3}{4H_0} - \frac{H}{2H_0} + \frac{(n-1)(1-2H)-\alpha_n}{4H_0} = n\frac{H-1}{2H_0} - \frac{1+\alpha_n}{4H_0} \quad (5.3.7)$$

since $|\alpha| = (n-1)(1-2H)$.

We verify condition (5.3.5) by induction. To see that condition (5.3.5) holds for $k = 1$, note that

$$\widetilde{\alpha}_1 + \widetilde{\beta}_1 + \widetilde{\alpha}_2 + 2 = \frac{H-1}{H_0} + \frac{2H-1+\alpha_1+\alpha_2}{4H_0} + 2 = \frac{6H-5+8H_0+\alpha_1+\alpha_2}{4H_0} > 0,$$

since $6H-5+8H_0 > 6H-5+6-8H = 1-2H > 0$ as long as (5.3.4) holds. Now assuming the condition (5.3.5) holds for $k-1$, we prove it for k:

$$\sum_{i=1}^{k}(\widetilde{\alpha}_i + \widetilde{\beta}_i) + \widetilde{\alpha}_{k+1} + k + 1 = \left[\sum_{i=1}^{k-1}(\widetilde{\alpha}_i + \widetilde{\beta}_i) + \widetilde{\alpha}_k + k\right] + \left(\widetilde{\alpha}_{k+1} + \widetilde{\beta}_k + 1\right) > 0,$$

since the first term is positive by the induction hypothesis, and

$$\widetilde{\alpha}_{k+1} + \widetilde{\beta}_k + 1 = \frac{4H-2+\alpha_k+\alpha_{k+1}}{4H_0} - \frac{\alpha_k+1}{4H_0} + 1 = \frac{4H-3+\alpha_{k+1}+4H_0}{4H_0} > 0$$

using the fact that $4H_0 + 4H - 3 > 0$ and $\alpha_{k+1} \geq 0$. This proves (5.3.5).

We compute the integral on the right-hand side of (5.3.3). By Lemma A.1,

$$\int_{0<t_1<\cdots<t_n<t} \prod_{i=1}^{n} t_i^{\widetilde{\alpha}_i} \prod_{i=1}^{n} (t_{i+1}-t_i)^{\widetilde{\beta}_i}\, d\mathbf{t}$$

$$= \frac{\Gamma(\widetilde{\alpha}_1+1)\prod_{i=1}^{n}\Gamma(\widetilde{\beta}_i+1)}{\Gamma(|\widetilde{\alpha}|+|\widetilde{\beta}|+n+1)} \prod_{k=1}^{n-1} \frac{\Gamma(\sum_{i=1}^{k}(\widetilde{\alpha}_i+\widetilde{\beta}_i)+k+1+\widetilde{\alpha}_{k+1})}{\Gamma(\sum_{i=1}^{k}(\widetilde{\alpha}_i+\widetilde{\beta}_i)+k+1)} t^{|\widetilde{\alpha}|+|\widetilde{\beta}|+n}. \quad (5.3.8)$$

Our first task is to give an upper bound for the constant

$$\gamma_n := \prod_{k=1}^{n-1} \frac{\Gamma(\sum_{i=1}^{k}(\widetilde{\alpha}_i+\widetilde{\beta}_i)+k+1+\widetilde{\alpha}_{k+1})}{\Gamma(\sum_{i=1}^{k}(\widetilde{\alpha}_i+\widetilde{\beta}_i)+k+1)}.$$

Note that by (5.3.6), we have:

$$\sum_{i=1}^{k}(\widetilde{\alpha}_i + \widetilde{\beta}_i) + k + 1 = k\frac{4H_0+4H-3}{4H_0} + 1 - \frac{H}{2H_0} + \frac{1}{4H_0}\sum_{i=1}^{k-1}\alpha_i,$$

for $k = 3, \cdots, n$. We use the following asymptotic property of the Gamma function: for any $a > 0$,

$$\lim_{n\to\infty} \frac{\Gamma(n+a)}{\Gamma(n)n^a} = 1.$$

It follows that for any $a > 0$, there exists $N_a^* \in \mathbb{N}$ such that

$$\Gamma(n+a) \le 2\Gamma(n)n^a \quad \text{for all } n \ge N_a^*. \tag{5.3.9}$$

In our case, using the fact that Gamma function is increasing (for k large) and employing inequality (5.3.9) with $a = \frac{1-2H}{2H_0}$ and $n = k\frac{4H_0+4H-3}{4H_0} + 1 - \frac{H}{2H_0} + \frac{\sum_{i=1}^{k-1}\alpha_i}{4H0}$, we have

$$\begin{aligned}
&\Gamma(\sum_{i=1}^{k}(\widetilde{\alpha}_i + \widetilde{\beta}_i) + k + 1 + \widetilde{\alpha}_{k+1}) \\
&= \Gamma\left(k\frac{4H_0+4H-3}{4H_0} + 1 - \frac{H}{2H_0} + \frac{\sum_{i=1}^{k-1}\alpha_i}{4H_0} + \frac{4H-2+\alpha_k+\alpha_{k+1}}{4H_0}\right) \\
&\le \Gamma\left(k\frac{4H_0+4H-3}{4H_0} + 1 - \frac{H}{2H_0} + \frac{\sum_{i=1}^{k-1}\alpha_i}{4H_0} + \frac{4H-2+4(1-2H)}{4H_0}\right) \\
&= \Gamma\left(k\frac{4H_0+4H-3}{4H_0} + 1 - \frac{H}{2H_0} + \frac{\sum_{i=1}^{k-1}\alpha_i}{4H_0} + \frac{1-2H}{2H_0}\right) \\
&\le 2\Gamma\left(\sum_{i=1}^{k}(\widetilde{\alpha}_i + \widetilde{\beta}_i) + k + 1\right)\left(k\frac{4H_0+4H-3}{4H_0} + 1 - \frac{H}{2H_0} + \frac{\sum_{i=1}^{k-1}\alpha_i}{4H_0}\right)^{\frac{1-2H}{2H_0}},
\end{aligned}$$

if $k\frac{4H_0+4H-3}{4H_0} \ge N^*_{\frac{1-2H}{2H_0}}$, i.e. $k \ge N_1$, for some N_1 depending on H_0 and H. It follows that for any $k \ge N_1$

$$\begin{aligned}
&\frac{\Gamma(\sum_{i=1}^{k}(\widetilde{\alpha}_i + \widetilde{\beta}_i) + k + 1 + \widetilde{\alpha}_{k+1})}{\Gamma(\sum_{i=1}^{k}(\widetilde{\alpha}_i + \widetilde{\beta}_i) + k + 1)} \le 2\left(k\frac{4H_0+4H-3}{4H_0} + 1 - \frac{H}{2H_0} + \frac{\sum_{i=1}^{k-1}\alpha_i}{4H_0}\right)^{\frac{1-2H}{2H_0}} \\
&\le 2\left\{k\frac{4H_0+4H-3}{4H_0} + 1 - \frac{H}{2H_0} + \frac{1-2H+2(k-2)(1-2H)}{4H_0}\right\}^{\frac{1-2H}{2H_0}} \\
&= 2\left\{k\frac{4H_0-1}{4H_0} + \frac{4H_0+4H-3}{4H_0}\right\}^{\frac{1-2H}{2H_0}} \\
&= 2\left\{\frac{1}{4H_0}[4H_0+4H-3+(4H_0-1)k]\right\}^{\frac{1-2H}{2H_0}} \\
&\le 2\left\{\frac{1}{4H_0}2(4H_0-1)k\right\}^{\frac{1-2H}{2H_0}} =: C_0 k^{\frac{1-2H}{2H_0}}.
\end{aligned}$$

Hence, for all $n \geq N_1$, we obtain the following upper bound for γ_n:

$$\gamma_n = \prod_{k=1}^{n-1} \frac{\Gamma(\sum_{i=1}^{k}(\widetilde{\alpha}_i + \widetilde{\beta}_i) + k + 1 + \widetilde{\alpha}_{k+1})}{\Gamma(\sum_{i=1}^{k}(\widetilde{\alpha}_i + \widetilde{\beta}_i) + k + 1)} \leq C_1 \prod_{k=N_1}^{n-1} \left(C_0 k^{\frac{1-2H}{2H_0}} \right)$$

$$= C_1 C_0^n \left(\prod_{k=N_1}^{n-1} k \right)^{\frac{1-2H}{2H_0}} \leq C_1 C_0^n (n!)^{\frac{1-2H}{2H_0}}, \tag{5.3.10}$$

where $C_1 = \prod_{k=1}^{N_1-1} \frac{\Gamma(\sum_{i=1}^{k}(\widetilde{\alpha}_i+\widetilde{\beta}_i)+k+1+\widetilde{\alpha}_{k+1})}{\Gamma(\sum_{i=1}^{k}(\widetilde{\alpha}_i+\widetilde{\beta}_i)+k+1)}$ is a constant depending on H_0 and H.

We now give a lower bound for $\Gamma(|\widetilde{\alpha}| + |\widetilde{\beta}| + n + 1)$. Recall that by (5.3.7),

$$|\widetilde{\alpha}| + |\widetilde{\beta}| + n + 1 = n\frac{H-1}{2H_0} - \frac{1+\alpha_n}{4H_0} + n + 1 = n\frac{2H_0+H-1}{2H_0} - \frac{1+\alpha_n}{4H_0} + 1. \tag{5.3.11}$$

Since $\alpha_n \leq 1 - 2H$,

$$|\widetilde{\alpha}| + |\widetilde{\beta}| + n + 1 \geq n\frac{2H_0+H-1}{2H_0} + \frac{H-1}{2H_0} + 1.$$

We now use the fact from Lemma A.6 that for any $a > 0$ and $b \in \mathbb{R}$, there exists $N_{a,b} \in \mathbb{N}$ and $C_{a,b} > 0$ such that

$$\Gamma(an + 1 + b) \geq C_{a,b}^n (n!)^a \quad \text{for all } n \geq N_{a,b}. \tag{5.3.12}$$

Since $\Gamma(x)$ is increasing for $x > x_0$ (for some $x_0 \in (1,2)$), using inequality (5.3.12) with $a = \frac{2H_0+H-1}{2H_0}$ and $b = \frac{H-1}{2H_0}$, we obtain:

$$\Gamma(|\widetilde{\alpha}| + |\widetilde{\beta}| + n + 1) \geq \Gamma\left(n\frac{2H_0+H-1}{2H_0} + \frac{H-1}{2H_0} + 1 \right) \geq C_2^n (n!)^{\frac{2H_0+H-1}{2H_0}}, \tag{5.3.13}$$

for $n \geq N_2$, where C_2 and N_2 are constants depending on H_0 and H.

Coming back to (5.3.8) and using relations (5.3.10), (5.3.11) and (5.3.13), we obtain:

$$\int_{0<t_1<\cdots<t_n<t} \prod_{i=1}^{n} t_i^{\widetilde{\alpha}_i} \prod_{i=1}^{n} (t_{i+1} - t_i)^{\widetilde{\beta}_i} \, dt_1 \cdots dt_n$$

$$\leq C_3^n \frac{(n!)^{\frac{1-2H}{2H_0}}}{(n!)^{\frac{2H_0+H-1}{2H_0}}} t^{n\frac{2H_0+H-1}{2H_0} - \frac{1+\alpha_n}{4H_0}} = C_3^n \frac{t^{n\frac{2H_0+H-1}{2H_0}}}{(n!)^{\frac{3H+2H_0-2}{2H_0}}} t^{-\frac{1+\alpha_n}{4H_0}},$$

for any $n > N_3$, where C_3 and N_3 are constants depending on H_0 and H.

We now return to (5.3.3). Since the previous estimate does not depend on $\alpha = (\alpha_1, \ldots, \alpha_n) \in D_n$ and $\text{card}(D_n) = 2^{n-1}$,

$$\begin{aligned} E|I_n(f_n(\cdot,t,x))|^2 &\leq J_0^2(t,x) b_{H_0}^n (n!)^{2H_0-1} C_{H,1}^n \left(C_3^n t^{\frac{1+\alpha_n}{4H_0}} \sum_{\alpha \in D_n} \frac{t^{n\frac{2H_0+H-1}{2H_0}}}{(n!)^{\frac{3H+2H_0-2}{2H_0}}} t^{-\frac{1+\alpha_n}{4H_0}} \right)^{2H_0} \\ &\leq J_0^2(t,x) C_4^n \frac{t^{n(2H_0+H-1)}}{(n!)^{3H-1}} = J_0^2(t,x) \frac{(C_4 t^{2H_0+H-1})^n}{(n!)^{3H-1}}, \end{aligned}$$

where $C_4 = (2C_3)^{2H_0} b_{H_0} C_{H,1}$. By Lemma A.7,

$$\begin{aligned} E|u(t,x)|^2 = \sum_{n \geq 0} E|I_n(f_n(\cdot,t,x))|^2 &\leq J_0^2(t,x) \sum_{n \geq 0} \frac{(C_4 t^{2H_0+H-1})^n}{(n!)^{3H-1}} \\ &\leq C_5 J_0^2(t,x) \exp\left\{ C_6 t^{\frac{2H_0+H-1}{3H-1}} \right\}, \end{aligned}$$

where $C_5 > 0$ and $C_6 > 0$ are constants depending on H and H_0. ∎

Remark 5.3.2. If instead of (5.3.10), one could prove that $\gamma_n \leq C^n$ for some constant $C > 0$, then this would yield the optimal exponent ρ^h in Theorem 5.3.1 and would eliminate the restriction $H > \frac{1}{3}$.

Chapter 6

Summary of results

We give below a summary of the results presented in the thesis.

	$H > \frac{1}{2}$		$H < \frac{1}{2}$	
	PAM	HAM	PAM	HAM
$H_0 = \frac{1}{2}$	solution exists for any $H \in (\frac{1}{2}, 1)$ (Theorem 2.3.7)	solution exists for any $H \in (\frac{1}{2}, 1)$ (Theorem 2.3.7)	solution exists for any $H \in (\frac{1}{4}, \frac{1}{2})$ (Theorem 3.2.2)	solution exists for any $H \in (\frac{1}{4}, \frac{1}{2})$ (Theorem 3.2.2)
$H_0 > \frac{1}{2}$	solution exists for any $H_0 \in (\frac{1}{2}, 1)$ and $H \in (\frac{1}{2}, 1)$ (Theorem 4.4.1)	solution exists for any $H_0 \in (\frac{1}{2}, 1)$ and $H \in (\frac{1}{2}, 1)$ (Theorem 4.4.1)	solution exists for any $H_0 \in (\frac{1}{2}, 1)$ and $H \in (0, \frac{1}{2})$ with $H_0 + H > \frac{3}{4}$ (Theorem 5.2.1)	solution exists for any $H_0 \in (\frac{1}{2}, 1)$ and $H \in (\frac{1}{4}, \frac{1}{2})$ (Theorem 5.2.1)

Table 6.1: Summary of the results about existence of solutions to equations (1.0.2) and (1.0.3)

Appendix A

Some auxiliary results

In this appendix section, we include some auxiliary results which are used in the thesis.

The following result is new in the literature.

Lemma A.1. *Let* $\alpha_1 > -1, \beta_j > -1$ *for any* $j = 1, \cdots, n$*. Suppose that for any* $k = 1, \ldots, n-1$,

$$\sum_{j=1}^{k} (\alpha_i + \beta_i) + \alpha_{k+1} + k + 1 > 0. \tag{A.1}$$

Then

$$\int_{\{0<t_1<\cdots<t_n<t\}} \prod_{j=1}^{n} t_j^{\alpha_j} \prod_{j=1}^{n} (t_{j+1} - t_j)^{\beta_j} dt_1 \ldots dt_n$$
$$= \frac{\Gamma(\alpha_1 + 1) \prod_{j=1}^{n} \Gamma(\beta_j + 1)}{\Gamma(|\alpha| + |\beta| + n + 1)} \prod_{k=1}^{n-1} \frac{\Gamma(\sum_{i=1}^{k} (\alpha_i + \beta_i) + \alpha_{k+1} + k + 1)}{\Gamma(\sum_{i=1}^{k} (\alpha_i + \beta_i) + k + 1)} t^{|\alpha| + |\beta| + n}. \tag{A.2}$$

where $t_{n+1} = t$, $|\alpha| = \sum_{j=1}^{n} \alpha_j$ *and* $|\beta| = \sum_{j=1}^{n} \beta_j$.

Proof: We prove (A.2) by induction. We denote by $I_n(t, \beta_0, \cdots, \beta_n)$ the integral on the left-hand side of (A.2). For $n = 1$, using the change of variable $z = t_1/t$,

$$\begin{aligned} I_1(t, \alpha_1, \beta_1) &= \int_0^t t_1^{\alpha_1} (t - t_1)^{\beta_1} dt_1 = t^{\alpha_1 + \beta_1} \int_0^t \left(\frac{t_1}{t}\right)^{\alpha_1} \left(1 - \frac{t_1}{t}\right)^{\beta_1} dt_1 \\ &= t^{\alpha_1 + \beta_1 + 1} \int_0^1 z^{\alpha_1} (1 - z)^{\beta_1} dz. \end{aligned}$$

Using the fact that for any $\alpha > 0, \beta > 0$,

$$\int_0^1 (1 - z)^{\alpha - 1} z^{\beta - 1} dz = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)},$$

we obtain:

$$I_1(t,\alpha_1,\beta_1)=\frac{\Gamma(\alpha_1+1)\Gamma(\beta_1+1)}{\Gamma(\alpha_1+\beta_1+2)}t^{\alpha_1+\beta_1+1},$$

since $\alpha_1>-1,\beta_1>-1$, and consequently, $\alpha_1+\beta_1+2>0$.

For $n=2$, using the change of variable $z=t_2/t$, and the calculation for $n=1$, we obtain:

$$\begin{aligned}
&I_2(t,\alpha_1,\alpha_2,\beta_1,\beta_2)=\int_0^t t_2^{\alpha_2}(t-t_2)^{\beta_2}\left(\int_0^{t_2} t_1^{\alpha_1}(t_2-t_1)^{\beta_1}dt_1\right)dt_2\\
&=\int_0^t t_2^{\beta_2}(t-t_2)^{\beta_2}I_1(t_2,\alpha_1,\beta_1)dt_2=\frac{\Gamma(\alpha_1+1)\Gamma(\beta_1+1)}{\Gamma(\alpha_1+\beta_1+2)}\int_0^t t_2^{\alpha_1+\alpha_2+\beta_1+1}(t-t_2)^{\beta_2}dt_2\\
&=\frac{\Gamma(\alpha_1+1)\Gamma(\beta_1+1)}{\Gamma(\alpha_1+\beta_1+2)}\cdot\frac{\Gamma(\beta_2+1)\Gamma(\alpha_1+\alpha_2+\beta_1+2)}{\Gamma(\alpha_1+\alpha_2+\beta_1+\beta_2+3)}t^{\alpha_1+\alpha_2+\beta_1+\beta_2+2}.
\end{aligned}$$

since $\alpha_1>-1,\beta_1>-1,\beta_2>-1,\alpha_1+\alpha_2+\beta_1+2>0$, and consequently $\alpha_1+\alpha_2+\beta_1+\beta_2+3>0$.

Next we assume that (A.3) holds for $n-1$ and we prove it for n. By the induction hypothesis,

$$\begin{aligned}
&I_{n-1}(t_n,\alpha_1,\cdots,\alpha_{n-1},\beta_1,\cdots,\beta_{n-1})=\\
&\qquad\frac{\Gamma(\alpha_1+1)\prod_{j=1}^{n-1}\Gamma(\beta_j+1)}{\Gamma(\sum_{j=1}^{n-1}(\alpha_j+\beta_j)+n)}\prod_{k=1}^{n-2}\frac{\Gamma(\sum_{i=1}^k(\alpha_i+\beta_i)+\alpha_{k+1}+k+1)}{\Gamma(\sum_{i=1}^k(\alpha_i+\beta_i)+k+1)}t^{\sum_{j=1}^{n-1}(\alpha_j+\beta_j)+n-1}.
\end{aligned}$$

Hence, using the change of variable $z=t_n/t$,

$$\begin{aligned}
&I_n(t,\alpha_1,\cdots,\alpha_n,\beta_1,\cdots,\beta_n)=\int_0^t t_n^{\alpha_n}(t-t_n)^{\beta_n}I_{n-1}(t_n,\alpha_1,\cdots,\alpha_{n-1},\beta_1,\cdots,\beta_{n-1})dt_n\\
&\quad=\frac{\Gamma(\alpha_1+1)\prod_{j=1}^{n-1}\Gamma(\beta_j+1)}{\Gamma(\sum_{j=1}^{n-1}(\alpha_j+\beta_j)+n)}\prod_{k=1}^{n-2}\frac{\Gamma(\sum_{i=1}^k(\alpha_i+\beta_i)+\alpha_{k+1}+k+1)}{\Gamma(\sum_{i=1}^k(\alpha_i+\beta_i)+k+1)}\\
&\qquad\qquad\int_0^t t_n^{\sum_{j=1}^{n-1}(\alpha_j+\beta_j)+\alpha_n+n-1}(t-t_n)^{\beta_n}dt_n\\
&\quad=\frac{\Gamma(\alpha_1+1)\prod_{j=1}^{n}\Gamma(\beta_j+1)}{\Gamma(|\alpha|+|\beta|+n+1)}\prod_{k=1}^{n-1}\frac{\Gamma(\sum_{i=1}^k(\alpha_i+\beta_i)+\alpha_{k+1}+k+1)}{\Gamma(\sum_{i=1}^k(\alpha_i+\beta_i)+k+1)}t^{|\alpha|+|\beta|+n},
\end{aligned}$$

since $\alpha_1>-1,\beta_j>-1$ for any $j=1,\cdots,n$ and $\sum_{j=1}^k(\alpha_i+\beta_i)+\alpha_{k+1}+k+1>0$ for any $k=1,\ldots,n-1$. Therefore (A.2) is proved. ∎

Remark A.2. *Note that condition* (A.1) *holds if* $\alpha_i \geq 0$ *for all* $i = 1, \ldots, n$.

The following results is a particular case of Lemma A.1 if $\alpha_i = 0$ for all $i = 2, \ldots, n$ and α_1 is denoted β_0.

Lemma A.3. *Let* $\beta_j > -1$, *for any* $j = 1, \cdots, n$. *Then*

$$\int_{\{0<t_1<\cdots<t_n<t\}} \prod_{j=0}^{n} (t_{j+1} - t_j)^{\beta_j} dt_1 \cdots dt_n \quad = t^{|\beta|+n} \frac{\prod_{j=0}^{n} \Gamma(\beta_j + 1)}{\Gamma(|\beta| + n + 1)}, \tag{A.3}$$

where $t_0 = 0, t_{n+1} = t$ *and* $|\beta| = \sum_{j=0}^{n} \beta_j$. *Consequently, if there exist* $M > \varepsilon > 0$, *so that* $\varepsilon \leq \beta_j + 1 \leq M$, *for any* $j = 0, \cdots, n$, *then*

$$\int_{\{0<t_1<\cdots<t_n<t\}} \prod_{j=0}^{n} (t_{j+1} - t_j)^{\beta_j} dt_1 \cdots dt_n \leq \frac{C_0^{n+1}}{\Gamma(|\beta| + n + 1)} t^{|\beta|+n}, \tag{A.4}$$

where $C_0 = \sup_{x \in [\varepsilon, M]} \Gamma(x)$.

We use the notation $a_n \sim b_n$ if $\lim_{n\to\infty} a_n/b_n \to 1$.

Lemma A.4. *For any* $a > 0$, $\Gamma(an+1) \sim (n!)^a a^{an+\frac{1}{2}} (2\pi n)^{\frac{1-a}{2}}$, *that is,*

$$\lim_{n\to\infty} \frac{\Gamma(an+1)}{(n!)^a a^{an+\frac{1}{2}} (2\pi n)^{(1-a)/2}} = 1. \tag{A.5}$$

Proof: By Stirling Formula, we know that $\Gamma(n+1) = n! \sim n^n e^{-n} (2\pi n)^{1/2}$. So, we get $(n^n e^{-n})^a \sim (n!)^a (2\pi n)^{-a/2}$. Moreover, $\Gamma(x+1) \sim x^x e^{-x} (2\pi x)^{1/2}$ as $x \to \infty$. Therefore,

$$\Gamma(an+1) \sim (an)^{an} e^{-an} (2\pi an)^{1/2} \sim (n^n e^{-n})^a (2\pi an)^{1/2} a^{an} \sim (n!)^a a^{an+\frac{1}{2}} (2\pi n)^{(1-a)/2}.$$

Therefore equation (A.5) is proved. ∎

Corollary A.5. *For any* $a > 0$, *there exist some constants* $C_1 > 0$ *and* $C_2 > 0$ *depending on* a *so that*

$$C_1 C_{a,1}^n (n!)^a \leq \Gamma(an+1) \leq C_2 C_{a,2}^n (n!)^a.$$

where $C_{a,1} = \min\{a^a, a^a 2^{(1-a)/2}\}$ *and* $C_{a,2} = \max\{a^a, a^a 2^{(1-a)/2}\}$.

Proof: Form the equation (A.5), we can infer that for any $a > 0$, there exists some constants $C_1 > 0$ and $C_2 > 0$ depending on a so that

$$C_1 K_n (n!)^a \leq \Gamma(an+1) \leq C_2 K_n (n!)^a$$

where $K_n = a^{an} n^{(1-a)/2}$. We consider separately the cases $a < 1$ and $a > 1$. If $a \in (0,1)$, then $1-a > 0$. For the upper bound, noting that $n \leq 2^n$ for any $n \geq 1$, we have

$$n^{(1-a)/2} \leq (2^n)^{(1-a)/2} = (2^{(1-a)/2})^n.$$

Therefore we get

$$K_n \leq a^{an} (2^{(1-a)/2})^n = (a^a 2^{(1-a)/2})^n.$$

For the lower bound, we have $n^{(1-a)/2} \geq 1$ since $1-a > 0$. Hence

$$K_n \geq a^{an} = (a^a)^n.$$

Next we consider the case $a > 1$. For the upper bound, we have $n^{(1-a)/2} \leq 1$ since $1-a < 0$. Hence

$$K_n \leq a^{an} = (a^a)^n.$$

For the lower bound, since $1-a < 0$, we have $n^{\frac{1-a}{2}} \geq (2^n)^{\frac{1-a}{2}} = (2^{\frac{1-a}{2}})^n$. Therefore we get

$$K_n \geq a^{an} (2^{(1-a)/2})^n = (a^a 2^{(1-a)/2})^n.$$

∎

Lemma A.6. *For any $a > 0$ and $b \in \mathbb{R}$, there exists $N_{a,b} \in \mathbb{N}$ and $C_{a,b} > 0$ such that*

$$\Gamma(an+1+b) \geq C_{a,b}^n (n!)^a \quad \textit{for all } n \geq N_{a,b} \tag{A.6}$$

Proof: Since

$$\lim_{n\to\infty} \frac{\Gamma(an+1+b)}{\Gamma(an+1) n^b} = 1,$$

there exists $N_1 \in N$ depending on a, b such that

$$\Gamma(an+1+b) \geq \frac{1}{2} \Gamma(an+1) n^b \quad \text{for all } n \geq N_1.$$

By Corollary A.5, $\Gamma(an+1) \geq C_1 C_2^n (n!)^a$ where C_1 and C_2 are constants depending on a. Hence

$$\Gamma(an+1+b) \geq \frac{1}{2} C_1 C_2 (n!)^a n^b \quad \text{for all } n \geq N_1.$$

The conclusion follows since $n^b \geq 1$ if $b \geq 0$ and $n^b \geq 2^{nb}$ if $b < 0$. ∎

We consider that Mittag-Leffer function:

$$E_{a,b}(x) = \sum_{n>0} \frac{x^n}{\Gamma(an+b)}, \quad x > 0, a > 0, b > 0.$$

We let $E_a(x) = E_{a,1}(x)$. It is known that

$$\lim_{x\to\infty} \frac{E_{a,b}(x)}{x^{\frac{1-b}{a}} \exp(x^{1/a})} = \frac{1}{a}. \tag{A.7}$$

In particular, for $b = 1$,

$$\lim_{x\to\infty} \frac{E_a(x)}{\exp(x^{1/a})} = \frac{1}{a}. \tag{A.8}$$

From here, we deduce that there exists a constant $C_a > 0$ depending on a such that

$$E_a(x) \leq C_a \exp(x^{1/a}) \quad \text{for all } x > 0 \tag{A.9}$$

(see relation (70) of [4]). Similarly, there exists a constant $C'_a > 0$ such that

$$E_a(x) \geq C'_a \exp(x^{1/a}) \quad \text{for all } x > 0. \tag{A.10}$$

Lemma A.7. *Let $x > 0$ be arbitrary. For any $a > 0$, there exist some positive constants C_1, C_2, C'_1, C'_2 depending on a such that*

$$C'_1 \exp(C'_2 x^{1/a}) \leq \sum_{n\geq 0} \frac{x^n}{(n!)^a} \leq C_1 \exp(C_2 x^{1/a})$$

.

Proof: By Corollary A.5, there exist some positive constants $C_1^*, C_2^*, C_{a,1}, C_{a,2}$ depending on a, so that for all $n \geq 0$,

$$C_1^* C_{a,1}^n \leq \frac{\Gamma(an+1)}{C_n (n!)^a} \leq C_2^* C_{a,2}^n.$$

So we get

$$C_1^* \frac{C_{a,1}^n}{\Gamma(an+1)} \leq \frac{1}{(n!)^a} \leq C_2^* \frac{C_{a,2}^n}{\Gamma(an+1)},$$

and hence taking the sum over $n \geq 0$, we obtain:

$$C_1^* E_a(C_{a,1}x) \leq \sum_{n\geq 0} \frac{x^n}{(n!)^a} \leq C_2^* E_a(C_{a,2}x). \tag{A.11}$$

By relation (A.9), there exists a constant $C_2 > 0$ depending on a such that

$$E_a(C_{a,2}x) \leq C_2 \exp\{(C_{a,2}x)^{1/a}\}.$$

Coming back to (A.11), we obtain:

$$\sum_{n\geq 0} \frac{x^n}{(n!)^a} \leq C_2^* C_2 \exp(C_{a,2}^{1/a} x^{1/a}).$$

The lower bound is obtained similarly. By relation (A.10), there exists a constant C_1 depending on a such that

$$E_a(C_{a,1}x) \geq C_1 \exp\{(C_{a,1}x)^{1/a}\}.$$

Using (A.11), we obtain:

$$\sum_{n\geq 0} \frac{x^n}{(n!)^a} \geq C_1^* C_1 \exp(C_{a,1}^{1/a} x^{1/a}).$$

∎

We say that a function $f : \mathbb{R}^n \to \mathbb{R}$ is *symmetric* if

$$f(t_{\rho(1)}, \cdots, t_{\rho(n)}) = f(t_1, \cdots, t_n) \quad \text{for any } \rho \in S_n,$$

where S_n is the set of permutations of $\{1, \cdots, n\}$.

Lemma A.8. *If $f : [0,t]^n \to \mathbb{R}$ is a symmetric function, then*

$$\int_{[0,t]^n} f(t_1, \cdots, t_n) dt_1 \cdots dt_n = n! \int_{\{0<t_1<\cdots<t_n<t\}} f(t_1, \cdots, t_n) dt_1 \cdots dt_n.$$

Proof: Suppose first that $n = 2$. Since $[0,t]^2 = \{0 \leq t_1 < t_2 \leq t\} \cup \{0 \leq t_2 < t_1 \leq t\} \cup \{0 \leq t_1 = t_2 \leq t\}$,

$$\int_{[0,t]^2} f(t_1, t_2) dt_1 dt_2 = \int_{\{0<t_1<t_2<t\}} f(t_1, t_2) dt_1 dt_2 + \int_{\{0<t_2<t_1<t\}} f(t_1, t_2) dt_1 dt_2.$$

For the second integral, we use the change of variables $s_1 = t_2$, $s_2 = t_1$. Using the fact that $f(s_2, s_1) = f(s_1, s_2)$ and Fubini's theorem,

$$\int_{\{0<t_2<t_1<t\}} f(t_1, t_2)dt_1dt_2 = \int_{\{0<s_1<s_2<t\}} f(s_2, s_1)ds_2ds_1 = \int_{\{0<s_1<s_2<t\}} f(s_1, s_2)ds_1ds_2.$$

Hence

$$\int_{[0,t]^2} f(t_1, t_2)dt_1dt_2 = 2! \int_{\{0<t_1<t_2<t\}} f(t_1, t_2)dt_1dt_2.$$

Next we consider the case of general n. In this case,

$$\int_{[0,t]^n} f(t_1, \cdots, t_n)d\mathbf{t} = \sum_{\rho\in\mathcal{S}_n} \int_{\{0<t_{\rho(1)}<\cdots<t_{\rho(n)}<t\}} f(t_1, \cdots, t_n)dt_1 \cdots dt_n.$$

Note that all the integrals appearing in the sum above are the same. To see this, we use the change of variables $s_i = t_{\rho(i)}$ for $i = 1, \cdots, n$. Then $t_i = s_{\sigma(i)}$ for $i = 1, \cdots, n$ where $\sigma = \rho^{-1}$. By the symmetry of f and Fubini's theorem,

$$\begin{aligned}\int_{\{t_{\rho(1)}<\cdots<t_{\rho(n)}\}} f(t_1, \cdots, t_n)dt_1 \cdots dt_n &= \int_{\{s_1<\cdots<s_n\}} f(s_{\sigma(1)}, \cdots, s_{\sigma(n)})ds_{\sigma(1)} \cdots ds_{\sigma(n)} \\ &= \int_{\{s_1<\cdots<s_n\}} f(s_1, \cdots, s_n)ds_1 \cdots ds_n.\end{aligned}$$

∎

Appendix B

A result about Brownian motion

In this Appendix, we prove an elementary result about the marginal density of Brownian motion, which is used in the proof of Theorem 2.3.8, for the Feynman-Kac formula for the second moment of the solution of the parabolic Anderson model with white noise in time.

Lemma B.1. *If* $(B_t)_{t\geq 0}$ *is a Brownian motion and* $0 < t_1 < \cdots < t_n < t$*, then the density of* $(B_{t_1}, \cdots, B_{t_n})$ *is*

$$f_{B_{t_1},\cdots,B_{t_n}}(x_1, \cdots, x_n) = G^h(t_1, x_1)G^h(t_2 - t_1, x_2 - x_1) \cdots G^h(t_n - t_{n-1}, x_n - x_{n-1}),$$

where G^h *is the fundamental solution of the heat equation on* $\mathbb{R}_+ \times \mathbb{R}$*, given by* (1.0.4).

Proof: Let $\mathbf{X} = (B_{t_1}, B_{t_2} - B_{t_1}, \cdots, B_{t_n} - B_{t_{n-1}})$. Since the increments of Brownian motion are independent and $B_t - B_s$ has density $G^h(t-s, \cdot)$, $\mathbf{X}$ has the density function:

$$f_{\mathbf{X}}(x_1, \cdots, x_n) = G^h(t_1, x_1)G^h(t_2 - t_1, x_2 - x_1) \cdots G^h(t_n - t_{n-1}, x_n - x_{n-1}). \quad \text{(B.1)}$$

Let $\mathbf{Y} = (B_{t_1}, B_{t_2}, \cdots, B_{t_n})$. Then $\mathbf{Y} = g(\mathbf{X})$ where $g = (g_1, \cdots, g_n)$ and

$$g_j(\mathbf{x}) = \sum_{i=1}^{j} x_i, \quad \text{for all } j = 1, \cdots, n.$$

Hence $\mathbf{X} = h(\mathbf{Y})$ where $h = (h_1, \cdots, h_n)$ is the inverse transform of g:

$$h_1(\mathbf{y}) = y_1,$$

$$h_i(\mathbf{y}) = y_i - y_{i-1}, \quad \text{for } i = 2, \cdots, n.$$

It follows that the density of $\mathbf{Y}$ is given by

$$f_{\mathbf{Y}}(\mathbf{y}) = f_{\mathbf{X}}(h(\mathbf{y}))|\mathbf{J}(\mathbf{y})|$$

where $\mathbf{J}(\mathbf{y})$ is the Jacobian:

$$\mathbf{J}(\mathbf{y}) = \begin{vmatrix} \frac{\partial h_1}{\partial y_1} & \frac{\partial h_1}{\partial y_2} & \cdots & \frac{\partial h_1}{\partial y_n} \\ \frac{\partial h_2}{\partial y_1} & \frac{\partial h_2}{\partial y_2} & \cdots & \frac{\partial h_2}{\partial y_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial h_n}{\partial y_1} & \frac{\partial h_n}{\partial y_2} & \cdots & \frac{\partial h_n}{\partial y_n} \end{vmatrix} = \begin{vmatrix} 1 & 0 & \dots & 0 & 0 \\ -1 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & -1 & 1 \end{vmatrix} = 1$$

Thus,

$$f_{\mathbf{Y}}(\mathbf{y}) = f_{\mathbf{X}}(h(\mathbf{y})) = f_{\mathbf{X}}(y_1, y_2 - y_1, \cdots, y_n - y_{n-1}). \tag{B.2}$$

Using (B.1) and (B.2), it follows that

$$f_{\mathbf{Y}}(y_1, \cdots, y_n) = G^h(t_1, y_1) G^h(t_2 - t_1, y_2 - y_1) \cdots G^h(t_n - t_{n-1}, y_n - y_{n-1}).$$

∎

Appendix C

Fourier transform

In this section, we include some auxiliary results about the Fourier transform which were used in the thesis. Recall that the Fourier transform of a function $\varphi \in L^1(\mathbb{R})$ is defined by:

$$\mathcal{F}\varphi(\xi) = \int_{\mathbb{R}} e^{-i\xi\cdot x}\varphi(x)dx, \quad \text{for all } \xi \in \mathbb{R},$$

where $e^{ix} = \cos x + i\sin x$. We denote by $\varphi * \psi$ the convolution of functions of φ and ψ:

$$(\varphi * \psi)(x) = \int_{\mathbb{R}} \varphi(x-y)\psi(y)dy, \quad \text{for all } x \in \mathbb{R}$$

Recall that $\mathcal{S}(\mathbb{R})$ is the space of rapidly decreasing functions on $\mathbb{R}$, i.e. infinitely differentiable functions on $\mathbb{R}$ whose derivatives decay faster than any polynomial.

Lemma C.1. *For any functions $\varphi, \psi \in \mathcal{S}(\mathbb{R})$,*

$$\mathcal{F}(\varphi * \widetilde{\psi})(\xi) = \mathcal{F}\varphi(\xi)\overline{\mathcal{F}\psi(\xi)},$$

where $\widetilde{\psi}(x) = \psi(-x)$ for all $x \in \mathbb{R}$.

Proof: Using Fubini's theorem and the change of variable $y = x - z$, we have:

$$\begin{aligned}
\mathcal{F}(\varphi * \widetilde{\psi})(\xi) &= \int_{\mathbb{R}} e^{-i\xi\cdot z}(\varphi * \widetilde{\psi})(z)dz = \int_{\mathbb{R}} e^{-i\xi z}\left(\int_{\mathbb{R}} \varphi(x)\widetilde{\psi}(z-x)dx\right)dz \\
&= \int_{\mathbb{R}} e^{-i\xi\cdot z}\left(\int_{\mathbb{R}} \varphi(x)\psi(x-z)dx\right)dz = \int_{\mathbb{R}} e^{-i\xi x}\varphi(x)\left(\int_{\mathbb{R}} e^{-i\xi\cdot(z-x)}\psi(x-z)dz\right)dx \\
&= \left(\int_{\mathbb{R}} e^{-i\xi\cdot x}\varphi(x)dx\right)\left(\int_{\mathbb{R}} e^{i\xi\cdot y}\varphi(y)dy\right) = \mathcal{F}\varphi(\xi)\overline{\mathcal{F}\psi(\xi)}.
\end{aligned}$$

∎

Lemma C.2. *For any function $\varphi \in L^1(\mathbb{R})$, and for any $x \in \mathbb{R}$,*

$$\mathcal{F}\varphi_x(\xi) = e^{-i\xi\cdot x}\overline{\mathcal{F}\varphi(\xi)} \quad \text{for all } \xi \in \mathbb{R}$$

where $\varphi_x(y) = \varphi(x-y)$ for all $y \in \mathbb{R}$. (Sometimes we denote φ_x by $\varphi(x-\cdot)$ and φ_x is called the shift *of φ by x.)*

Proof: Letting $z = x - y$, we get:

$$\begin{aligned}\mathcal{F}\varphi_x(\xi) &= \int_{\mathbb{R}} e^{-i\xi\cdot y}\varphi_x(y)dy = \int_{\mathbb{R}} e^{-i\xi\cdot x}e^{i\xi\cdot(x-y)}\varphi(x-y)dy \\ &= e^{-i\xi\cdot x}\int_{\mathbb{R}} e^{i\xi\cdot z}\varphi(z)dz = e^{-i\xi\cdot x}\overline{\mathcal{F}\varphi(\xi)}.\end{aligned}$$

∎

Theorem C.3. *(Plancherel Theorem) For any functions $\varphi, \psi \in L^2(\mathbb{R})$,*

$$\int_{\mathbb{R}} \varphi(x)\psi(x)dx = \frac{1}{2\pi}\int_{\mathbb{R}} \mathcal{F}\varphi(\xi)\overline{\mathcal{F}\psi(\xi)}d\xi.$$

In particular, for any $\varphi \in L^2(\mathbb{R})$,

$$\int_{\mathbb{R}} |\varphi(x)|^2dx = \frac{1}{2\pi}\int_{\mathbb{R}} |\mathcal{F}\varphi(\xi)|^2d\xi.$$

The remaining results of this section are taken from reference [10] and were used in Section 4.3 for the study of the linear stochastic wave equation with Gaussian noise with index $H_0 > \frac{1}{2}$ in time and index $H > \frac{1}{2}$ in space.

Lemma C.4. *For any $\tau > 0$, we let $(\mathcal{F}_{0,T}\sin)(\tau) = \int_0^T e^{-i\tau T}\sin t dt$. Then*

$$|(\mathcal{F}_{0,T}\sin)(\tau)|^2 = \frac{1}{(\tau^2-1)^2}[(\sin\tau T - \tau\sin T)^2 + (\cos\tau T - \cos T)^2].$$

Proof: Using the fact that $\sin t = (e^{it} - e^{-it})/(2i)$ and $\cos t = (e^{it} + e^{-it})/2$, we obtain:

$$\begin{aligned}(\mathcal{F}_{0,T}\sin)(\tau) &= \int_0^T e^{-i\tau t}\sin t dt = \int_0^T e^{-i\tau t}\frac{e^{it}-e^{-it}}{2i}dt \\ &= \frac{1}{2i}\left(\int_0^T e^{it(-\tau+1)} - e^{-it(\tau+1)}dt\right) = \frac{1}{2i}\left(\frac{e^{iT(-\tau+1)}-1}{i(-\tau+1)} - \frac{e^{-iT(\tau+1)}-1}{-i(\tau+1)}\right)\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2i}\left(\frac{e^{-i\tau T}e^{iT}-1}{i(-\tau+1)} + \frac{e^{-i\tau T}e^{-iT}-1}{i(\tau+1)}\right) \\
&= \frac{(e^{-i\tau T}e^{iT}-1)(1+\tau)+(e^{-i\tau T}e^{-iT}-1)(1-\tau)}{2i^2(1-\tau)(1-\tau)} \\
&= \frac{e^{-i\tau T}(e^{iT}+e^{-iT})+\tau e^{-i\tau T}(e^{iT}-e^{-iT})-2}{2(\tau^2-1)} \\
&= \frac{\cos Te^{-i\tau T}+i\tau\sin Te^{-i\tau T}-1}{\tau^2-1} = \frac{(\cos T+i\tau\sin T)e^{-i\tau T}-1}{\tau^2-1} \\
&= \frac{(\cos T+i\tau\sin T)(\cos\tau T-i\sin\tau T)-1}{\tau^2-1} \\
&= \frac{\cos\tau T\cos T+\tau\sin\tau T\sin T-1+i(\tau\cos\tau T\sin T-\sin\tau T\cos T)}{\tau^2-1}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
|(\mathcal{F}_{0,T}\sin)(\tau)|^2 &= \mathrm{Re}^2[(\mathcal{F}_{0,T}\sin)(\tau)] + \mathrm{Im}^2[(\mathcal{F}_{0,T}\sin)(\tau)] \\
&= \frac{(\cos\tau T\cos T+\tau\sin\tau T\sin T-1)^2}{(\tau^2-1)^2} + \frac{(\tau\cos\tau T\sin T-\sin\tau T\cos T)^2}{(\tau^2-1)^2} \\
&= \frac{1}{(\tau^2-1)^2}\Big\{\cos^2\tau T\cos^2 T+\tau^2\sin^2\tau T\sin^2 T+1 \\
&\qquad +2\tau\sin\tau T\cos\tau T\sin T\cos T-2\cos\tau T\cos T-2\tau\sin\tau T\sin T+ \\
&\qquad \tau^2\cos^2\tau T\sin^2 T+\sin^2\tau T\cos^2 T-2\tau\sin\tau T\cos\tau T\sin T\cos T\Big\} \\
&= \frac{1}{(\tau^2-1)^2}\big(1+\tau^2\sin^2 T-2\tau\sin\tau T\sin T+\cos^2 T-2\cos\tau T\cos T\big) \\
&= \frac{1}{(\tau^2-1)^2}\Big\{(\sin\tau T-\tau\sin T)^2+(\cos\tau T-\cos T)^2\Big\}.
\end{aligned}
$$

■

Lemma C.5. *for any $t, s \in [0,T]$ and $\tau > 0$, we have*

$$
\begin{aligned}
\|\sin(\cdot)\|^2_{\mathcal{H}(0,T)} &:= \alpha_H \int_0^T \int_0^T \sin t\sin s|t-s|^{2H-2}dtds \\
&= C_H \int_{\mathbb{R}} \frac{|\tau|^{-(2H-1)}}{(\tau^2-1)^2}[(\sin\tau T-\tau\sin T)^2+(\cos\tau T-\cos T)^2]d\tau,
\end{aligned}
$$

where $\alpha_H = H(2H-1)$ and C_H is given by (2.1.4).

Proof: By Theorem 2.1.2, we get

$$\alpha_H \int_0^T \int_0^T \sin t \sin s |t-s|^{2H-2} dt ds = C_H \int_{\mathbb{R}} |(\mathcal{F}_{0,T} \sin)(\tau)|^2 |\tau|^{1-2H} d\tau,$$

where $(\mathcal{F}_{0,T} \sin)(\tau) = \mathcal{F}(1_{[0,T]} \sin)(\tau) = \int_0^T e^{-i\tau t} \sin t dt$. This conclusion follows by Lemma C.4. ∎

We introduce the following functions: for any $\tau \in \mathbb{R}, \lambda > 0$ and $t > 0$, we let

$$f_t(\lambda,\tau) = \sin(\tau\lambda t) - \tau \sin(\lambda t), \qquad g_t(\lambda,\tau) = \cos(\tau\lambda t) - \cos(\lambda t).$$

Lemma C.6. *For any $\tau \in \mathbb{R}, \lambda > 0$ and $t > 0$,*

$$C_1(t \wedge t^3)\frac{\lambda^3}{1+\lambda^2} \le \int_{\mathbb{R}} \frac{1}{(\tau^2-1)^2}[f_t^2(\lambda,\tau) + g_t^2(\lambda,\tau)]d\tau \le C_2(t \vee t^3)\frac{\lambda^3}{1+\lambda^2},$$

where $C_1 = (2\pi \sin^2 1)/3$ and $C_2 = 4\pi$.

Proof: Using Lemma C.5 with $T = \lambda t$, Plancherel theorem and the change of variable $x = \lambda s$, we obtain:

$$\begin{aligned}\int_{\mathbb{R}} \frac{1}{(\tau^2-1)^2}[f_t^2(\lambda,\tau) + g_t^2(\lambda,\tau)]d\tau &= \int_{\mathbb{R}} |(\mathcal{F}_{0,\lambda t} \sin)(\tau)|^2 d\tau = 2\pi \int_0^{\lambda t} |\sin x|^2 dx \\ &= 2\pi\lambda \int_0^t |\sin \lambda s|^2 ds = 2\pi\lambda^3 \int_0^t \frac{|\sin \lambda s|^2}{\lambda^2} ds.\end{aligned}$$

Then the result follows using relation (2.2.4). ∎

Appendix D

Some important inequalities

In this appendix section, we discuss some important inequalities used in the thesis.

Lemma D.1. *For any* $H \in (\frac{1}{2}, 1)$ *and for any function* $\varphi \in L^{1/H}[0,T]$,

$$\alpha_H \int_0^T \int_0^T |\varphi(t)||\varphi(s)||t-s|^{2H-2} dt ds \leq b_H \left(\int_0^T |\varphi(t)|^{1/H} dt \right)^{2H}$$

for some constant $b_H > 0$.

Proof: For any $\varphi \in L^{1/H}[0,T]$, using the Hölder's inequality and letting $q = \frac{1}{1-H}$ and $p = \frac{1}{H}$, we obtain:

$$\begin{aligned}
\|\varphi\|^2_{|\mathcal{H}[0,T]|} &:= \alpha_H \int_0^T \int_0^T |\varphi(t)||\varphi(s)||t-s|^{2H-2} dt ds \\
&= \alpha_H \int_0^T |\varphi(t)| \left(\int_0^T |\varphi(s)||t-s|^{2H-2} ds \right) dt \\
&\leq \alpha_H \left(\int_0^T |\varphi(t)|^{1/H} dt \right)^H \left(\int_0^T \left(\int_0^T |\varphi(s)||t-s|^{2H-2} ds \right)^{1/(1-H)} dt \right)^{1-H} .
\end{aligned}$$

Next we applied the Hardy-Littlewood-Sobolev inequality (see relation (12) of [1]):

$$\left\| \int_0^T f(y)|x-y|^{\alpha-1} dy \right\|_{L^q[0,T]} \leq A_{p,q} \|f\|_{L^p[0,T]},$$

where $A_{p,q}$ is a positive constant depending on p and q, and $0 < \alpha < 1$, $1 < p < q < \infty$ satisfy $\frac{1}{q} = \frac{1}{p} - \alpha$. We pick the particular values $\alpha = 2H - 1$, $p = \frac{1}{H}$ and $q = \frac{1}{1-H}$. In our case, we get:

$$\left(\int_0^T \left(\int_0^T |\varphi(s)||t-s|^{2H-2} ds \right)^{1/(1-H)} dt \right)^{1-H} \leq A_H \left(\int_0^T |\varphi(t)|^{1/H} dt \right)^H ,$$

where A_H is a constant depending on H. Therefore,

$$\|\varphi\|^2_{|\mathcal{H}[0,T]|} \leq b_H \left(\int_0^T |\varphi(t)|^{1/H} dt \right)^{2H},$$

where $b_H = \alpha_H A_H$. ∎

Lemma D.2. *For any $H \in (\frac{1}{2}, 1)$ and for any function $\varphi \in L^2[0,T]$,*

$$\left(\int_0^T |\varphi(t)|^{1/H} dt \right)^{2H} \leq T^{2H-1} \int_0^T |\varphi(t)|^2 dt.$$

Proof: For any $\varphi \in L^2[0,T]$, using the Hölder's inequality with $p = 2H$ and $q = \frac{2H}{2H-1}$, we obtain:

$$\int_0^T |\varphi(t)|^{1/H} dt \leq \left(\int_0^T |\varphi(t)|^{\frac{1}{H} \cdot 2H} dt \right)^{\frac{1}{2H}} \left(\int_0^T 1 dt \right)^{\frac{2H-1}{2H}} = \left(\int_0^T |\varphi(t)|^2 dt \right)^{\frac{1}{2H}} T^{\frac{2H-1}{2H}}.$$

Taking power $2H$, we obtain:

$$\left(\int_0^T |\varphi(t)|^{1/H} dt \right)^{2H} \leq T^{2H-1} \int_0^T |\varphi(t)|^2 dt.$$

∎

Corollary D.3. *For any $H \in (\frac{1}{2}, 1)$ and for any function $\varphi \in L^2[0,T]$,*

$$\alpha_H \int_0^T \int_0^T |\varphi(t)||\varphi(s)||t-s|^{2H-2} dt ds \leq b_H T^{2H-1} \int_0^T |\varphi(t)|^2 dt,$$

where $b_H > 0$ is the constant from Lemma D.1.

Proof: This follows immediately from Lemmas D.1 and D.2. ∎

The next result is an extension of Lemma D.1 to higher dimensions.

Lemma D.4. *For any $H \in (\frac{1}{2}, 1)$ and for any function $\varphi \in L^{1/H}([0,T]^n)$,*

$$\alpha_H^n \int_{[0,T]^{2n}} |\varphi(t_1, \cdots, t_n)||\varphi(s_1, \cdots, s_n)| \prod_{i=1}^n |t_i - s_i|^{2H-2} d\mathbf{t} d\mathbf{s}$$
$$\leq b_H^n \left(\int_{[0,T]^n} |\varphi(t_1, \cdots, t_n)|^{1/H} d\mathbf{t} \right)^{2H}.$$

Proof: We use the same argument as for Lemma D.1, based on the Hardy-Littlewood-Sobolev inequality in higher dimensions. ∎

Appendix E

Some useful identities

Lemma E.1. *For any* $H \in (\frac{1}{2}, 1)$*,*

$$H(2H-1)\int_0^T \int_0^T |t-s|^{2H-2} dt ds = T^{2H}.$$

Proof: Using the change of variable $s' = t - s$, we have:

$$\begin{aligned}
&H(2H-1)\int_0^T \int_0^T |t-s|^{2H-2} dt ds \\
&= H(2H-1)\left(\int_0^T \int_0^T (t-s)^{2H-2} 1_{\{t>s\}} dt ds + \int_0^T \int_0^T (s-t)^{2H-2} 1_{\{s>t\}} dt ds\right) \\
&= 2H(2H-1)\int_0^T \left(\int_0^t (t-s)^{2H-2} ds\right) dt = 2H(2H-1)\int_0^T \left(\int_0^t s^{2H-2} ds\right) dt \\
&= 2H(2H-1)\int_0^T \frac{1}{2H-1} t^{2H-1} dt = 2H \int_0^T t^{2H-1} dt = T^{2H}.
\end{aligned}$$

∎

Lemma E.2. *If* $H \in (\frac{1}{2}, 1)$*, then for any* $t > 0$ *and for any* $s \in (0, t)$*,*

$$\int_0^t |r-s|^{2H-2} dr = \frac{2}{2H-1}(t-s)^{2H-1} \leq \frac{2}{2H-1} t^{2H-1}.$$

Proof:

$$\int_0^t |r-s|^{2H-2} dr = \int_0^t (r-s)^{2H-2} 1_{\{r>s\}} dr + \int_0^t (s-r)^{2H-2} 1_{\{s>r\}} dr$$

$$
\begin{aligned}
&= 2\int_s^t (r-s)^{2H-2}dr = 2\int_0^{t-s} r^{2H-2}dr \\
&= \frac{2}{2H-1}(t-s)^{2H-1} \le \frac{2}{2H-1}t^{2H-1}.
\end{aligned}
$$

∎

Lemma E.3. *If $H \in (\frac{1}{2}, 1)$, then for any $t > 0$,*

$$\int_{[0,t]^{2n}} \prod_{i=1}^n |t_i - s_i|^{2H-2}\psi(\mathbf{t})d\mathbf{s}d\mathbf{t} \le \Big(\frac{2}{2H-1}\Big)^n t^{n(2H-1)} \int_{[0,t]^n} \psi(\mathbf{t})d\mathbf{t}.$$

Proof: Using Lemma E.2,

$$
\begin{aligned}
\int_{[0,t]^{2n}} \prod_{i=1}^n |t_i - s_i|^{2H-2}\psi(\mathbf{t})d\mathbf{s}d\mathbf{t} &= \int_{[0,t]^n} \psi(\mathbf{t})\Bigg(\int_{[0,t]^n} \prod_{i=1}^n |t_i - s_i|^{2H-2}d\mathbf{s}\Bigg)d\mathbf{t} \\
&= \int_{[0,t]^n} \psi(\mathbf{t})\prod_{i=1}^n \Bigg(\int_0^t |t_i - s_i|^{2H-2}ds_i\Bigg)d\mathbf{t} \\
&\le \int_{[0,t]^n} \psi(\mathbf{t})\Big(\frac{2}{2H-1}t^{2H-1}\Big)^n d\mathbf{t} \\
&= \Big(\frac{2}{2H-1}\Big)^n t^{n(2H-1)} \int_{[0,t]^n} \psi(\mathbf{t})d\mathbf{t}.
\end{aligned}
$$

∎

Lemma E.4. *For any $H \in (\frac{1}{2}, 1)$,*

$$\int_0^T \int_0^T ts|t-s|^{2H-2}dtds = \frac{\beta(2, 2H-1)}{H+1}T^{2H+2},$$

where β denotes the beta function: $\beta(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$.

Proof: Using the change of variable $s' = \frac{s}{t}$, we have:

$$
\begin{aligned}
&\int_0^T \int_0^T ts|t-s|^{2H-2}dtds \\
&= \Bigg(\int_0^T \int_0^T ts(t-s)^{2H-2}1_{\{t>s\}}dtds + \int_0^T \int_0^T ts(s-t)^{2H-2}1_{\{s>t\}}dtds\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&= 2\int_0^T t\left(\int_0^t s(t-s)^{2H-2}ds\right)dt = 2\int_0^T t\left(\int_0^1 t^2 s'(t-ts')^{2H-2}ds'\right)dt\\
&= 2\int_0^T t\left(\int_0^1 t^{2H}s(1-s)^{2H-2}ds\right)dt = 2\int_0^T t^{2H+1}\left(\int_0^1 s(1-s)^{2H-2}ds\right)dt\\
&= 2\beta(2,2H-1)\int_0^T t^{2H+1}dt = \frac{\beta(2,2H-1)}{H+1}T^{2H+2}.
\end{aligned}
$$

∎

Bibliography

[1] Alòs, E. and Nualart, D. (2003). Stochastic integration with respect to the fractional Brownian motion. *Stochastics and Stochastic Reports* **75**, 129-152.

[2] Balan, R. M. (2012). The stochastic wave equation with multiplicative fractional noise: a Malliavin calculus approach. *Potential Analysis* **36**, 1-34.

[3] Balan, R. M. (2012). Some linear SPDEs driven by a fractional noise with Hurst index greater than 1/2. *Infinite Dimensional Analysis, Quantum Probability and Related Topics* **15**, no.4, 1250023, 27.

[4] Balan, R. M. and Conus, D. (2016). Intermittency for the wave and heat equations with fractional noise in time. *Annals of Probability* **44**, 1488-1534.

[5] Balan, R. M., Jolis, M. and Quer-Sardanyons, L. (2015). SPDEs with affine multiplicative fractional noise in space with index H in (1/4, 1/2). *Electronic Journal of Probability* **20**, no. 54, 1-36.

[6] Balan, R. M., Jolis, M. and Quer-Sardanyons, L. (2017). Intermittency for the Hyperbolic Anderson Model with rough noise in space. *Stochastic Processes and Their Applications* **127**, 2316-2338.

[7] Balan, R. M. and Chen, L. (2018). Parabolic Anderson Model with space-time homogeneous Gaussian noise and rough initial condition. *Journal of Theoretical Probability* **31**, 2216-2265.

[8] Balan, R. M., Quer-Sardanyons, L. and Song, J. (2019). Existence of density for the stochastic wave equation with space-time homogeneous Gaussian noise. *Electronic Journal of Probability* **24**, no. 106, 1-43.

[9] Balan, R. M. and Song, J. (2017). Hyperbolic Anderson Model with space-time homogeneous Gaussian noise. *ALEA, Latin American Journal of Probability and Mathematical Statistics* **14**, 799-849.

[10] Balan, R. M. and Tudor, C. (2010). The stochastic wave equation with fractional noise: a random field approach. *Stochastic Processes and their Applications* **120**, 2468-2494.

[11] Chen, L. and Dalang, R. C. (2015). Moments and growth indices for the nonlinear stochastic heat equation with rough initial conditions. *Annals of Probabability* **6**, 3006–3051

[12] Chen, X. (2019). Parabolic Anderson Model with rough or crititial Gaussian noise. *Annales de l'Institut Henri Poincaré (B) Probability and Statistics* **55**, 941-976.

[13] Dalang, R. C. (1999). Extending the martigale measure stochastic integral with application to spatially homogeneous S.P.D.E.'s. *Electronic Journal of Probability* **4**, no. 6, 1-29.

[14] Dalang, R. C. and Frangos, N. (1998). The stochastic wave equation in two spatial dimensions. *Annals of Probability* **26**, 187-212.

[15] Hu, Y. (2001). Heat equations with frational noise potentials. *Applied Mathematics and Optimization* **43**, 221-243.

[16] Hu, Y., Huang, J., Lê, K., Nualart, D. and Tindel S. (2016). Parabolic Anderson Model with rough dependence in space. In :"Computation and Combinations in Dynamics Stochastics and Control ". *Abel Symposia* **vol 13**, Springer, Cham.

[17] Hu, Y., Huang, J., Nualart, D. and Tindel S. (2015). Stochastic heat equations with general multiplicative Gaussian noises: Hölder continuity and intermittency. *Electronic Journal of Probability* **20**, no. 55, 1-50.

[18] Hu, Y. and Lê, K. (2019). Joint Hölder continuity of parabolic Anderson model. *Acta Mathematica Scientia* **39**, 764-780.

[19] Hu, Y. and Nualart, D. (2009). Stochastic heat equation driven by fractional noise and local time. *Probability Theory and Related Fields* **143**, 285-328.

[20] Millet, A. and Sanz-Solé, M (1999). A stochastic wave equation in two space dimension: smoothness of the law. *Annals of Probability* **27**, 803-844.

[21] Nualart, D. (2003). Stochastic integration with respect to fractional Brownian motion and applications. *Contemporary Mathematics* **336**, 3-39.

[22] Nualart, D. and Quer-Sardanyons, L. (2007). Existence and smoothness of the density for spatially homogeneous SPDEs. *Potential Analysis* **27**, 281-299.

[23] Pipiras, V. and Taqqu, M. (2000). Integrations questions related to fractional Brownian motion. *Probability Theory and Related Fields***118**, 251–291.

[24] Pipiras, V. and Taqqu, M. (2001). Are classes of deterimistic integrands for the fractional Brownian motion on a finite interval complete? *Bernoulli* **7**, 873–897.

[25] Quer-Sardanyons, L. and Sanz-Solé, M. (2004). Absolute continuity of the law of the solution of the 3-dimensional stochastic wave equation. *Jounal of Functional Analysis* **206**, 1-32.

[26] Sanz-Solé, M. (2005). *Malliavin Calculus with Applications to Stochastic Partial Differential Equations.* EPFL Press, Boca Ratou.

[27] Sanz-Solé, M. and Sarra, M. (2002). Hölder continuity for the stochastic heat equation with spatially correlated noise. In:" Progess in Probabability." **vol 52**, 259-268, Birkhäuser.

[28] Song, J., Song, X. and Xu, F. (2019). Fractional stochastic wave equation driven by a Gaussian noise rough in space. Preprint aviable on arXiv.1904.09905.

[29] Walsh. J. B. (1986). An introduction to stochastic particial differencial equations. Ecole d'Eté de Probabilités de Sanit-Flour XIV. *Lecture Notes in Mathematics* **1180**, 265-439, Springer-Verlag.