Existence of Optimal Controls for Second-Order Nonlinear Evolution Equations

by

Sebti Kerbal

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Abstract

In this thesis we study the question of existence of optimal controls for systems governed by second order nonlinear evolution equations.
Let $I = [0,T]$, (X, H, X^*) be an evolution triple, with compact embedding $X \hookrightarrow H \hookrightarrow X^*$ and Y a separable, reflexive Banach space, modeling the control space. Here X^* denote the dual of the Banach space X .
Let $t \longrightarrow U(t)$ be a measurable set-valued map with values $U(t) \in 2^Y$. For admissible controls , we introduce the class $\mathcal{U}_{ad}$ given by
$\mathcal{U}_{ad} \equiv \{u : I \mapsto Y,\ \textit{strongly measurable},\ \textit{and}\ u(t) \in U(t)\ a.e.\}$.
We consider the following Lagrange type optimal control problem:

$$\left.\begin{cases} J(x,u) = \int_0^T L(t,x(t),\dot{x}(t),u(t))dt \rightarrow \inf \\ \textit{subject to the following state and control constraints :} \\ \ddot{x}(t) + A(t,\dot{x}(t)) + Bx(t) = f(t,x(t))u(t), \\ x(0) = x_0 \in X,\ \dot{x}(0) = x_1 \in H,\ u(t) \in U(t)\ a.e. \end{cases}\right\} (P)$$

To establish the existence of an optimal pair $\{x, u\}$ for the problem (P), an appropriate hypotheses on the data have been introduced and some apriori bounds for the admissible trajectories of (P) have been derived.

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Contents

Chapter 1

Introduction

1.1 A Brief Review of Literature

For any problem posed , one has to ask the following questions:

(i) existence of solutions,

(ii) uniqueness of solutions,

(iii) continuous dependence of solutions on the initial data.

Control problems are also concerned with the above questions.

Many physical systems are in general governed by first or second order linear or nonlinear evolution equations of the form:

$$\begin{cases} \frac{d\phi}{dt} = A(t,\phi) + f(t,\phi,u) \\ \phi(0) = \phi_0 \\ u \in \mathcal{U}_{ad} \end{cases} \tag{1.1}$$

or

$$\begin{cases} \frac{d^2\phi}{dt^2} = A(t,\phi,\dot{\phi}) + f(t,\phi,u) \\ \phi(0) = \phi_0 \\ \dot{\phi}(0) = \phi_1 \\ u \in \mathcal{U}_{ad} \end{cases} \tag{1.2}$$

where u is the control from a given admissible class $\mathcal{U}_{ad}$.

Systems (1.1) and (1.2) are defined in suitable Banach spaces .

One of the fundamental concerns of control theory is the question of existence of optimal controls. For systems governed by differential equations in finite or infinite dimensional Banach spaces , covering ordinary or delay differential equations. or

partial differential equations, this question has been the subject of great interest over the last decades [6,8,10,3,11,2, 21,22,26].
For finite dimensional problems much progress has been made. For a comprehensive treatment of the existing theories and techniques we refer to the books by Cesari [19],Clark [23] and Ioffe-Tichimirov [24].
The theory of distributed parameter systems (i.e. infinite dimensional systems),presents more difficulties and is lagging behind. An outline of the results existing in that direction can be found in books by Ahmed-Teo [2] and Lions [26].
We mention here, that the system **(1.1)** covers a wide variety of nonlinear [6,7] and linear [26]parabolic systems. On the other hand, **(1.2)** covers hyperbolic and other systems.
In many diffusion and wave propagation problems, the system is essentially governed by a nonlinear partial differential equation. Another interesting example, the Navier-Stokes equation arising in hydrodynamical problems, is a system of nonlinear diffusion equations.
Similarly, in other problems, one has to deal with a system of nonlinear diffusion equations, such as the **KdV** equation describing the dynamics of water waves.
The subject of control theory for nonlinear problems is very sketchy [20,21,16,26,6,4] and is wide open. Recently, interesting work has been done by Ahmed [12] and Papageorgiou [30], related to the question of existence of optimal controls for infinite dimensional systems governed by integro-differential equations, and for systems governed by nonlinear second order evolution equations .
In this thesis we present some recent results in this direction , extending the work of Papageorgiou [30].

1.2 Formulation of the Control Problem

Let $I = [0,T]$, (X, H, X^*) be an evolution triple, with compact embedding $X \hookrightarrow H \hookrightarrow X^*$ and Y a separable, reflexive Banach space, modeling the control space, where X^* denotes the dual of the Banach space X .
Let $t \mapsto U(t)$ be a measurable multifunctions with values $U(t) \in P_{wkc}(Y)$. For

admissible controls , we introduce the class $\mathcal{U}_{ad}$ given by

$$\mathcal{U}_{ad} \equiv \{u : I \to Y,\ strongly\ \ measurable,\ \ and\ \ u(t) \in U(t)\ a.e.\} \tag{1.3}$$

and

$$P_{wkc}(Y) = \{A \subseteq Y : nonempty,\ weakly\ compact\ and\ convex\} \tag{1.4}$$

We wish to consider the following Lagrange type optimal control problem:

$$\left\{ \begin{array}{l} J(x,u) = \int_0^T L(t,x(t),\dot{x}(t),u(t))dt \to\ \inf = m \\ subject\ to\ the\ following\ state\ and\ control\ constraints: \\ \ddot{x}(t) + A(t,\dot{x}(t)) + Bx(t) = f(t,x(t))u(t), \\ x(0) = x_0 \in X, \dot{x}(0) = x_1 \in H, u(t) \in U(t)\ a.e. \end{array} \right\} (P)$$

By an **admissible "state-control"** pair $\{x,u\}$ for (P), we understand a pair of state trajectory $x(\cdot) \in C(I,X)$ and control function $u(\cdot)\in L^{\infty}(I,Y)$ such that $\dot{x}(\cdot)$ $\in W_{p,q}(I)$ and both functions $x(\cdot), u(\cdot)$ satisfy the constraints of problem (P), where $W_{p,q}(I)$ is the Sobolev space defined by

$$W_{p,q}(I) = \{x : x \in L^p(X)\ and\ \dot{x} \in L^q(X^*)\}. \tag{1.5}$$

In (P) the infimum is takes as follows :

$$\begin{aligned} m \equiv \inf\{J(x,u), for\ admissible\ state - control\ pair\{x,u\}\} \\ = lim_{n\to\infty} J(x_n,u_n) \end{aligned} \tag{1.6}$$

Clearly, problem (P) cannot be solved without further assumptions on the data. These assumptions will be introduced in Chapter 3.

The fundamental questions of control problems are various and they sometimes present serious difficulties.

In fact, the difficult question of the existence of an optimal state-control pair $\{x,u\}$ arises in nonlinear systems because of the interaction between some topological properties and the nonlinearities.

Also the optimal state- control pair $\{x,u\}$ may be very difficult, if not impossible, to determine numerically. There are no easy computational methods for this purpose. However, for The existence of optimal control one can use two techniques:

- Classical method (the most popular technique used).

-Selections theorems (or Cesari's approch) wich is very difficult , specially to prove that the optimal control has a measurable selections (see the book of Ahmed-Teo [2, Th.5.4.9, pp.391.]. However some results in this direction have been obtained recently in [8,9].

1.3 Outline of the Thesis and Contributions

The thesis contains four chapters under the following headings:

Chapter 1. Introduction

Chapter 2. Nonlinear First-Order Evolution Equations.

Chapter 3. Optimal Control for Nonlinear Second- Order Evolution Equations.

Chapter 4. An Example.

In Chapter 2, we present the necessary background materials from mathematics to be used in the next chapters. The existence of solutions and their properties are given for first-order nonlinear evolution equations and parabolic problems.

Chapter 3 constitutes the most important part of the thesis. It is devoted to the study of the question of existence of optimal controls for second-order nonlinear evolution equations.

In the first part of this chapter, we discuss Papageorgiou's results and, in the second part , we present the main results of this thesis which extends Papageorgiou's results.

Finally in the last chapter we give an example , wich illustrates our main results (Theorem 3.3).

Chapter 2

Nonlinear First-Order Evolution Equations

2.1 Introduction

Nonlinear systems are more frequently encontered in practical problems than linear ones. Also the question of existence of optimal controls for such systems (first or second order) is of fundamental importance. In fact it is required to ensure that the evolution equation describing the control system has a unique solution for each admissible control. For the existence of a solution one can use several methods : semigroup theory, time discretization or the Galerkin's approach. Here we will use the last method which converts infinite- dimensional system into its finite-dimensional approximation and then by using a priori estimates and compactness arguments, one shows that the approximating sequence has a subsequence that converges to the solution of the original problem.
In view of this objective, in this chapter we shall present a basic review of the tools one can use to prove the existence of solutions.
The next chapter will be concerned with the question of existence of optimal control of second-order nonlinear evolution equations.

2.2 Monotonocity, Hemicontinuity and Demicontinuity

Definition 2.1

(i) *An operator* $A : V \longrightarrow V^*$ *is monotone on the real Banach space* V *with dual* V^* *if and only if,*

$< Au - Av, u - v >_{V^*,V} \geq 0$ *for all* $u, v \in V$.

where $< .,. >_{V,V^*}$ *denote the duality for the pair* (V, V^*).

(ii) A *is hemicontinuous if it is continuous from every closed line segment in* V *to the weak topology in* V^*. *That is if and only if the map*

$$t \longrightarrow < A(u + tv), w >$$

is continuous for all $u, v, w \in V, t \in R$.

(iii) A *is strictly monotone if:*

$$< Au - Av, u - v >_{V^*,V} > 0$$

for all $u, v \in V$, *with* $u \neq v$.

(iv) A *is said to be demicontinuous if and only if* A *transforms a strongly convergent sequence into a weakly convergent one . That is:* $u_n \xrightarrow{s} u$ *as* $n \to \infty$ *implies* $Au_n \xrightarrow{w} Au$ *as* $n \to \infty$.

Remark 2.1

(a) In(i) if A is linear then A is called positive definite.

i.e. $< Au, u > \geq 0$ for all $u \in V$.

(b) If A is multivalued, the Definition1(i) is replaced by $< y_1 - y_2, u - v >_{V,V^*} \geq 0$ for all $y_1 \in Au$, $y_2 \in Av$.

(c) Demicontinuity $\Longrightarrow$ Hemicontinuity (see Zeidler [33]).

2.3 Weak Convergence and Weak Compactness

Here we shall assume thay V is a reflexive Banach space and $1 \leq p < \infty$, $\frac{1}{p}+\frac{1}{q}=1$.

Definition 2.2 *A sequence* $\{f_k\}_{k=1}^{\infty} \subset L^p(I,V)$ *converges weakly to* $f \in L^p(I,V)$, *written*
$f_k \xrightarrow{w} f$ *in* $L^p(I,V)$,
provided

$$\int_I < f_k(t), g(t) >_{V,V^*} dt \longrightarrow \int_I < f(t), g(t) >_{V,V^*} dt$$

for each $g \in L^q(I,V^*)$.

Theorem 2.1 (Boundedness of weakly convergent sequences). *Assume that* $f_k \xrightarrow{w} f \in L^p(I,V)$, *then*

(i) $\{f_k\}_{k=1}^{\infty}$ *is bounded in* $L^p(I,V)$. *and*

(ii) $\| f \|_{L^p(I,V)} \leq \liminf \| f_k \|_{L^p(I,V)}$.

Theorem 2.2 (Weak compactness). *Assume* $1 < p < \infty$ *and the sequence* $\{f_k\}_{k=1}^{\infty}$ *is bounded in* $L^p(I,V)$. *Then there exists a subsequence* $\{f_{k_j}\}_{j=1}^{\infty} \subset \{f_k\}_{k=1}^{\infty}$ *and a function* $f \in L^p(I,V)$ *with* $f_{k_j} \xrightarrow{w} f \in L^p(I,V)$.

Remark 2.2

In the case $p = 1$ the terminology is slightly different. We say that $\{f_k\}_{k=1}^{\infty} \subset L^{\infty}(I,V)$converges to $f \in L^{\infty}(I,V)$ in the weak-star topology, written

$$f_k \xrightarrow{w^*} f \in L^{\infty}(I,V)$$

provided

$$\int_I < f_k(t), g(t) >_{V,V^*} dt \longrightarrow \int_I < f(t), g(t) >_{V,V^*} dt$$

as $k \to \infty$ for each $g \in L^1(I, V^*)$.

2.4 Monotone Operator and the Galerkin Method

We consider the operator equation

$$Ax = f \text{ for } x \in V \tag{2.1}$$

with the corresponding Galerkin equations

$$a(x_n, w_k) = < f, w_k > \text{ for } k = 1, 2, ..n \tag{2.2}$$

where

$a(x, w) =< Ax, w >$, $x_n = span\{w_1, ..., w_n\}$.

We seek $x_n \in V$ such that

$V_n = \sum_{k=1}^{n} c_{kn} w_k$,

where the coefficients c_{kn} are unknown real numbers .

Theorem 2.3 (Browder,Minty (1963)). *Let $A : V \longrightarrow V^*$ be a monotone , coercive and hemicontinuous operator on a real , separable reflexive Banach space X . Assume $\{w_1, w_2, ...w_n\}$ is a basis in V .Then the following assertion holds:*

(a) **Galerkin method.** *If dim $V=\infty$,then for each $n \in N$, the Galerkin equation (2.2) has a solution $x_n \in V_n$ and the sequence (x_n) has a weakly convergent subsequence*

$x_{n_k} \xrightarrow{w} x \in V$ as $n \to \infty$,

where x is a solution of the original equation (2.1).

(b) **Uniqueness.** *If A is strictly monotone , then equation (2.1) (resp.eq.(2.2))is uniquely solvable in V(resp.V_n).*

(c) **Strong convergence of the Galerkin method** *.Let dim $V=\infty$. If the operator A is strictly monotone , then the sequence of Galerkin solutions (x_n) converges weakly in V to the unique solution x of equation (2.1).*

If A is uniformly monotone, then (x_n) converges strongly in V to the unique solution of (2.1).

Proof: see Zeidler [33]

Remark2.4

Let $A : V \longrightarrow V^*$ on real reflexive Banach space V : If the operator A is monotone and hemicontinuous , then A satisfies the following condition

$x_n \xrightarrow{w} x$, $Ax_n \xrightarrow{w} y$, $\limsup < Ax_n, x_n > \leq \; < y, x >$

implies $Ax = y$.

2.5 First-Order Evolution Equations

2.5.1 Existence and uniquenes of solutions

A large class of physical systems are governed by monotone operators, they arise in applications to the nonlinear partial differential equations. In nonlinear problems an essential tool we use is the Galerkin method and so monotonicity is required. The technique used is to write the initial value problem into a functional equation. Let $V \hookrightarrow H \hookrightarrow V^*$ be an evolution triple and let $\{w_1, w_2, ...\}$ be a basis in V . We set $H_n = \text{span}\{w_1, w_2, \ldots, w_n\}$ and introduce on the n-dimentionnal space H_n the scalar product of the Hilbert space H. Note that $H_n \hookrightarrow V \hookrightarrow H$.

Let $M(V, V^*)$ denotes the space of nonlinear monotone hemicontinuous operators from V to V^* .

Suppose $A : I \longrightarrow M(V, V^*)$ and satisfies the following propperties :

(A1)**Coerciveness.** There exists a constant $c > 0$, $h(\cdot) \in L^1(I)$ and $1 < p < \infty$ such that

$< A(t)v, v >_{(V^*,V)} \geq h(t) + c \parallel v \parallel^p ,t \in I$,

for all $v \in V$.

$(A2)$ **Monotonocity**

$< A(t)u - A(t)v, u - v >_{V^*,V} \geq 0,\ t \in I,$

for all $u, v \in V$.

$(A3)$ **Growth condition.**

$\| A(t)v \|_{V^*} \leq c(1 + \| v \|^{p-1}_{L^p(I,V)})\ , t \in I$

for all $v \in V$. Let p, q satisfy the condition $\frac{1}{p} + \frac{1}{q} = 1$, that is , q is the conjugate to p

Under the above assumptions, one can prove the following result.

Theorem 2.4 [5]. *Consider the nonlinear evolution equation*

$$\begin{cases} \dot{\varphi} + A(t)\phi = f \\ \phi(0) = \phi_0 \end{cases} \tag{2.3}$$

and suppose the operator $A : I \longrightarrow M(V, V^*)$ *satisfies the assymptions* $(A1) - (A3)$. *Then for each* $\phi_0 \in H$ *and* $f \in L^q(I, V^*)$, *the equation (2.3) has a weak solution* $\phi \in L^p(I, V) \bigcap C(I, H)$.

Further if A is strictly monotone , then the solution is unique.

Proof: The proof will be based on Galerkin's approximations , for wich we need an apriori bound.If ϕ is any solution of (2.3) , then for each $t \in I$,

$$\int_0^t < \dot{\phi}, \phi >_{V^*,V} d\theta + \int_0^t < A(\theta)\phi, \phi >_{V^*,V} d\theta = \int_0^t < f, \phi >_{V^*,V} d\theta,$$

giving

$$| \phi(t) |_H^2 + 2\int_0^t < A(\theta)\phi(\theta), \phi(\theta) >_{V^*,V} d\theta = | \phi_0 |_H^2 + 2\int_0^t < f(\theta), \phi(\theta) >_{V^*,V} d\theta.$$

Using assumption $(A1)$ we obtain

$$| \phi(t) |_H^2 + 2c\int_0^t \| \phi(\theta) \|_V^p d\theta \leq | \phi_0 |_H^2 + 2\int_0^t | h(\theta) | d\theta + 2\int_0^T |< f(\theta), \phi(\theta) >_{(V^*,V)}| d\theta$$

$\leq | \phi_0 |_H^2 + 2\int_0^t | h(\theta) | d\theta + 2(\int_0^t \| f(\theta) \|_{V^*}^q d\theta)^{\frac{1}{q}} (\int_0^t \| \phi(\theta) \|_V^p)^{\frac{1}{p}}$ for all $t \in I$.

Hence using the Cauchy inequality ,

$$a.b \leq \frac{\epsilon^p}{p} \mid a \mid^p + \frac{\epsilon^{-q}}{q} \mid b \mid^q, \epsilon > 0, a, b \in R,$$

we have,

$$\mid \phi(t) \mid_H^2 + 2(c - \frac{\epsilon^p}{p}) \int_0^t \parallel \phi(\theta) \parallel_V^p d\theta \leq \mid \phi_0 \mid_H^2 + 2 \int_0^t \mid h(\theta) \mid d\theta + 2\frac{\epsilon^{-q}}{q} \int_0^t \parallel f(\theta) \parallel_{V^*}^q d\theta$$

and , for sufficiently small $\epsilon > 0$ so that $c > \epsilon^p \frac{\epsilon^p}{p}$, we obtain

$$\mid \phi(t) \mid_H^2 + c_1 \int_0^t \parallel \phi(\theta) \parallel_V^p d\theta \leq \mid \phi_0 \mid_H^2 + c_2 \int_0^t \mid h(\theta) \mid d\theta + c_3 \int_0^t \parallel f \parallel_{V^*}^q d\theta \quad (2.4)$$

for all $t \in I$, where c_1, c_2, c_3 are suitable positive constants . Hence , it follows that $\phi \in L^\infty(I, H) \cap L_p(I, V)$ and using $(A3)$ there exist constants c_5 and c_6 such that

$$\parallel \dot{\phi} \parallel_{L^q(I,V^*)} \leq c_5 + c_6 \parallel \phi \parallel_{L^p(I,V)}^{p-1} + \parallel f \parallel_{L^q(I,V^*)} . \quad (2.5)$$

Then from (2.4) and (2.5) it is not difficult to verify that there exist a constants c_7 and c_8 such that

$$\parallel \dot{\phi} \parallel_{L^q(I,V^*)} \leq c_7 + c_8(\mid \phi_0 \mid_H^2 + \parallel h \parallel_{L^1(I)} + \parallel f \parallel_{L^q(I,V^*)}) . \quad (2.6)$$

Then using inequalities (2.4) and (2.6), it follows that there exists a finite positive number M such that

$$\begin{aligned} &\parallel \phi \parallel_{L^p(I,V)} \leq M \\ &\parallel \dot{\phi} \parallel_{L^q(I,V^*)} \leq M \end{aligned} \quad (2.7)$$

where ϕ is any solution of (2.3) .

Let $W_{p,q} \equiv \{\phi : \phi \in L^p(I, V) \; and \, \dot{\phi} \in L^q(I, V^*)\}$

where the derivative is understood in the sense of distribution .

Endowed with the norm

$$\parallel \phi \parallel_{W_{p,q}} \equiv (\parallel \phi \parallel_{L^p(I,V)}^2 + \parallel \dot{\phi} \parallel_{L^q(I,V^*)}^2)^{\frac{1}{2}}, \quad (2.8)$$

$W_{p,q}$ is a Banach space , and by virtue of Th1.2.15 (1) one can conclude that $W_{p,q}$ can be embedded in the Banach space $C(\overline{I}, H)$. Hence it follows from (2.7) that

$\phi \in C(\overline{I}, H)$. With the a priori bounds we just established and Galerkin's approach we can now prove the existence of a solution of the Cauchy problem (2.3).
Since V is dense in the Hilbert space H , there exists a sequence $\{e_i\} \in V$ such that $(e_i, e_j)_H = \delta_{ij}$ and for an $x \in H$, x is the strong limit over n of the finite sum $\sum_{i=1}^n (x, e_i)e_i$ denoted by
$x = s.lim_n \sum_{i=1}^n (x, e_i)e_i$.
Consider the sequence

$$\phi^n(t) \equiv \sum_{i=1}^n \alpha_i^n(t)e_i \tag{2.9}$$

for suitable $\{\alpha_i^n$, i=1,...,n $\}$ so that $\alpha_i^n(0) = \beta_i$ where $\{\beta_i\}$ are the Fourier coefficients of ϕ_0 with respect to the fundamental sequence $\{v_i\}$ and

$$\phi_0 = s.lim \sum_{i=1}^n \beta_i e_i \equiv s.lim \phi_0^n. \tag{2.10}$$

For notational simplicity , we write $A(t)v \equiv A(t, v)$, and let $\{\phi^n\}$ be chosen such that the following system of equations holds

$$< \dot{\phi}^n, e_i >_{(V^*,V)} = - < A(t, \phi^n), e_i >_{(V^*,V)} + < f, e_i >_{(V^*,V)}, i = 1, \ldots, n. \tag{2.11}$$

The system (2.11) can be writen in the following equivalent system of ordinary differential equations in R^n

$$\begin{cases} \dot{\alpha}_i = F^n(t, \alpha^n) + g^n \\ \alpha^n(0) = \beta^n \end{cases} \tag{2.12}$$

where $F^n \equiv \{F_i^n, i = 1, 2, ...n\}$
with
$F_i^n = F_i^n(t, \alpha_1^n, \alpha_2^n, .., \alpha_n^n) = - < A(t, \sum_{j=1}^n \alpha_j^n e_j, e_i >, i = 1, 2, .., n.$
and
$g^n \equiv \{f_i, i = 1, 2, ...n\}$
with $f_i =< f, e_i >$,$i = 1, 2, ..n$
and
$\beta^n = \{\beta_i, i = 1, 2, ...n\}$.

Since the mapping $v \longrightarrow A(t)v$ is monotone and hemicontinuous from V to V^*, $\alpha \longrightarrow F^n(t,\alpha)$ is continuous from R^n to R^n and $-F^n$ is monotone. Hence it follows from the theory of ordinary differential equations that for each positive integer n, the system (2.12) has a unique absolutely continuous solution α^n on I with $\alpha^n(0) = \beta^n$. Thus ϕ^n , in (2.11) is well defined and $\{\phi^n\}$ is contained in a bounded subset of $W^{p,q}$.

In fact it follows from the equality

$$< \dot{\phi}^n, \phi^n > + < A(t,\phi^n), \phi^n > = < f, \phi^n > \tag{2.13}$$

and the apriori bounds established in (2.7) that

$$\{\| \phi^n \|_{L^p(I,V)}, \| \dot{\phi}^n \|_{L^q(I,V^*)}\} \leq M \tag{2.14}$$

for all positive integers n.

Since $L^p(I,V)$,$1 < p < \infty$, is a reflexive Banach space so is $L^q(I,V^*)$, then there exists a subsequence $\{\phi^{n_k}\} \in \{\phi^n\}$ and $\phi^0 \in L^p(I,V)$, with its distributiona l derivative $\dot{\phi}^0 \in L^q(I,V^*)$, such that

$$\phi^{n_k} \xrightarrow{w} \phi^0 \in L^p(I,V) \tag{2.15}$$

$$\dot{\phi}^{n_k} \xrightarrow{w} \dot{\phi}^0 \in L^q(I,V*) \tag{2.16}$$

Taking $n = n_k$ in (2.11) , multiplying by $\psi \in L^p(I)$, and integrating over [0,T], we obtain

$$\int_0^T < \dot{\phi}^{n_k}, \psi e_i > dt + \int_0^T < A(t,\phi^{n_k}), \psi e_i > dt = \int_0^T < f, \psi e_i > dt \tag{2.17}$$

for $1 < i < n_k$. By virtue of the monotonicity and hemicontinuity of A it follows from lemma 3.3 [6] that

whenever $v_n \xrightarrow{w} v_0 \in L^p(I,V)$, then $A(\cdot,v_n) \xrightarrow{w^*} A(\cdot,v_0)$ in $L^p(I,V^*)$ through subsequences if many.

Hence using (2.15), it follows from (2.16), and letting $k \to \infty$ that

$$\int_0^T < \dot{\phi}^0, \psi e_i > dt + \int_0^T < A(t,\phi^0), \psi e_i > dt = \int_0^T < f, \psi e_i > dt \tag{2.18}$$

for all integers $i \geq 1$ and $\psi \in L^p(I)$.

Therefore, ϕ^0 satisfies

$\dot{\phi}^0 + A(t, \phi^0) = f$

in the sense of distribution as elements of $L^q(I, V^*)$.

From (2.15), it follows that $\phi^0 \in W^{p,q}$ and hence $\phi^0 \in C(\overline{I}, H)$. Thus $\phi^0(0)$ is a well defined element of H. We show that

$\phi^0(0) = \phi_0$ as given.

For any $\psi \in C^1(\overline{I})$ with $\psi(T) = 0$, scalar multiply

$\dot{\phi}^0 + A(t, \phi^0) = f$ by $\psi(t)e_i$ to obtain

$$-(\phi^0(0), e_i)\psi(0) - \int_0^T < \phi^0, \dot{\psi}^0 e_0 > dt + \int_0^T < A(t, \phi^0), \psi(t)e_i > dt = \int_0^T < f(t), \psi(t)e_i > dt \qquad (2.19)$$

Integrating by parts the first term in (2.17) and comparing with (2.18), one can easily conclude that

$\lim_{k \to \infty}(\phi^0(0) - \phi_0^{n_k}, e_i)_H = 0$.

Since $\{e_i\}$ is a basis of V and V is dense in H, we conclude that $\phi_0^{n_k} \xrightarrow{w} \phi^0(0)$ in H. Furthermore, we recall that $\phi_0^{n_k} \longrightarrow \phi_0$ strongly in H. Hence $\phi^0(0) = \phi_0$, and ϕ^0 is a weak solution of (2.3).

For the uniqueness, if $\phi^* \in W^{p,q}$ is another solution of (2.3), then in $L^q(I, V^*)$ we have

$$\begin{cases} \frac{d}{dt}(\phi^0 - \phi^*) + A(t, \phi^0) - A(t, \phi^*) = 0, \\ \phi^0(0) - \phi^*(0) = 0 \end{cases} \qquad (2.3.0)$$

Since ϕ^0,$\phi^* \in C(\overline{I}, H)$, scalar multiplying (2.3.0) by $(\phi^0 - \phi^*)$ and integrating over the interval [0,t], $t \in I$, we have

$| \phi^0(t) - \phi^*(t) |_H^2 + 2\int_0^t < A(\theta, \phi^0(\theta)) - A(\theta, \phi^*(\theta)), \phi^0(\theta) - \phi^*(\theta) > d\theta = 0.$

Hence one can conclude from assumption (A2) that

$\phi^0(t) = \phi^*(t)$ for all $t \in I$, thus proving uniqueness.

2.6 Control Problem

The subject of control theory for nonlinear problems is vast and wide open . The nonlinearities usually appear in lower -order terms as in systems of the form:

$$\begin{cases} \frac{d\phi}{dt} + A(t,\phi) + B(t)\phi = f(t,\phi,u) \\ \phi(0) = \phi_0 \\ u \in \mathcal{U}_{ad} \end{cases} \tag{2.20}$$

for the first order and

$$\begin{cases} \frac{d^2\phi}{dt^2} + A(t,\phi) + B(t)\phi = f(t,\phi,u) \\ \phi(0) = \phi_0 \\ \dot{\phi}(0) = \phi_1 \\ u \in \mathcal{U}_{ad} \end{cases} \tag{2.21}$$

for the second order.

Systems of the form (2.21) will be considered in the third chapter , however, in the present chapter, we discuss control systems governed by the nonlinear evolution equation:

$$\begin{cases} \frac{d\phi}{dt} = A(t)\phi + f(t,\phi,u(t)), t \in I \\ \phi(0) = \phi_0 \\ u \in \mathcal{U}_{ad} \end{cases} \tag{2.22}$$

where $\{A(t), t \in I\}$is a family of densely defined linear operators, not necessarily bounded, with domain $D(A(t)) \subset V$ and range $R(A(t)) \subset V^*$ and $f : I \times V \times U \longrightarrow V^*$.

This system covers a wide variety of nonlinear systems.

We consider here an example of control of nonlinear parabolic system in the weak form.

We start with

2.6.1 Notation and formulation of the problem

Let Ω be an open bounded connected subset of the Euclidean space R^n.

The points of Ω are denoted by $x \equiv (x_1, x_2, ..., x_n)$ and the derivatives with respect to x_j-variables by $D_j \equiv \frac{\partial}{\partial x_j}$, $j = 1, 2, ..., n$; $D_k^j \equiv \frac{\partial^k}{\partial x_j^k}$, k any positive integer; and $D^\alpha \equiv D_1^{\alpha_1} \cdots D_n^{\alpha_n}$ for an n-vector $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n$ with α_i positive integers and $| \alpha |=$

$\sum_{i=1}^{n} \alpha_i$. We write integrals in Lebesgue n-measure over Ω as $\int_\Omega f(x)dx$. By functions we mean r-vector functions $\phi \equiv (\phi_1, \phi_2, ..., \phi_r)$ for some fixed $r \geq 1$, where each ϕ_i is a real-valued function on Ω or on $I \times \Omega$. Thus $D^\alpha \phi = (D^\alpha \phi_1, D^\alpha \phi_2, ..., D^\alpha \phi_r)$ and $\frac{\partial \phi}{\partial t} = \left(\frac{\partial \phi_1}{\partial t}, \frac{\partial \phi_2}{\partial t}, ... \frac{\partial \phi_r}{\partial t}\right)$
Let $1 < p < \infty$, where m be any fixed positive integer, and let $W^{m,p}$ be the Sobolev space defined by

$$W^{m,p}(\Omega) \equiv \{\phi \in L^p(\Omega) : D^\alpha \phi \in L^p(\Omega) for \mid \alpha \mid \leq m\}$$

with the norm

$$\mid \phi \mid_{m,p} \equiv \left(\sum_{|\alpha| \leq m} \| D_\alpha \phi \|_{L^p(\Omega)}^p \right)^{\frac{1}{p}}$$

$W^{m,p}(\Omega)$ is a reflexive Banach space. Let V be a closed subspace of $W^{m,p}(\Omega)$ with $C_0^\infty(\Omega) \subset V$ where $C_0^\infty(\Omega)$ is a family of infinitely differentiable functions with compact support in Ω . Since V is a closed subspace of a reflexive Banach space $W^{m,p}(\Omega)$, it is a reflexive Banach space with the norm $\mid \phi \mid_V \equiv \mid \phi \mid_{m,p}$.
We consider the Banach space $L^p(I, V)$ of equivalence classes of functions ϕ from I to V with

$$\int_I \mid \phi \mid_V^p dt < \infty$$

and the norm defined by $\| \phi \|_{L^p(I,V)} \equiv (\int_I \mid \phi \mid_V^p dt)^{\frac{1}{p}}$. $L^p(I, V)$ is itself a Banach space and further, its (topological) dual is $L^q(I, V^*)$; that is , $(L^p(I, V))^* = L^q(I, V^*)$, where $\frac{1}{p} + \frac{1}{q} = 1$ and V^* is the dual of V. Let H denote the Hilbert space $L^2(\Omega)$ and $A(t)$, $t \in I$, be a system of partial differential operators on Ω, with coefficients depending on $t \in I$, of the form

$$A(t)\phi \equiv \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha A_\alpha(t, x, \phi, D_1\phi, D_2\phi, ..., D_m\phi)$$

where each A_α is an r-vector function of $(t, x) \in I \times \Omega$, the r-vector function ϕ, and the values of all the derivatives $D^\beta \phi$ for $\mid \beta \mid \leq m$.

We use (f, g) for the duality product between $g \in L^p(I, V)$ and $f \in L^q(I, V^*)$ and $< f, g >$ for the duality product between $g \in V$ and $f \in V^*$. We shall consider the control problem for the nonlinear parabolic system of the form

$$\begin{cases} \frac{\partial\phi}{\partial t} + A(t)\phi = g, t \in (t_0, T], \\ \phi(t_0) = \phi_0 \in H, \end{cases} \tag{2.23}$$

where g is any measurable r-vector-valued function on $I \times \Omega$. This is an initial value problem .

For $\phi, \psi \in L^p(I, V)$ define

$$h(\phi, \psi) \equiv \sum_{|\alpha|\leq m} \int_{t_0}^{t_1} < A_\alpha(t, \phi, D_1\phi, ...D_m\phi, D^\alpha\psi > dt, \tag{2.24}$$

and let $g \in L^q(I, V^*)$ and $\phi_0 \in H$.

<u>Definition 2.3</u> *A function $\phi \in L^p(I, V)$ is said to be a weak solution of system* (2.23) *if it satisfies*

$(L\phi, \psi) + h(\phi, \psi) = (g, \psi)$

where L denotes the closure of the operator $L_0 = \frac{\partial}{\partial t}$ from the space $L^p(I, V)$ to the space $L^q(I, V^*)$.

or

$$-(\phi, L\psi) + h(\phi, \psi) =< \phi_0, \psi(0) > +(g, \psi)$$
$$for\ all\ \psi\ in\ L^p(I, V) \cap C^1(I, H). \tag{2.25}$$

with $\psi(t_1) = 0$.

Let E be a separable reflexive Banach space of r-vector-valued functions on Ω and Γ a closed subset of E and $\mathcal{U}$ the family of all strongly measurable functions on I with values in Γ. Let $f : I \times \Gamma \to V^*$ so that:

1. for each $t \in I$,$f(t, \cdot)$ is weakly continuous on Γ
2. for each $v \in \Gamma, f(\cdot, v)$ is measurable on I
3. for each $u \in \mathcal{U}_{ad}, \hat{f}(u) \in L^q(I, V^*)$ where $\hat{f}(u)(t) = f(t, u(t))$.

<u>Definition 2.4</u> *A function $f(t, u)$ of the two variables $u \in R, t \in G$ is said to satisfy Caratheodory's conditions, if it is continuous with respect to u for almost all $t \in G$*

and measurable with respect to t for all values of u.

Here G denotes a set of finite or infinite measure in n-dimensional space.

Definition 2.5 *An linear operator from a normed linear space X into a normed linear space Y is bounded if it transforms any bounded set into another bounded set.*

We shall denote by $\hat{f}$ the Nemyckii operator defined on the set of real functions on G ,and where $f(t,u)$ satisfies the Caratheodory conditions.
The nonlinear operator $\hat{f}$ plays a fundamental role in questions related to the existence of solutions of a vast class of nonlinear systems.
It turns out that the continuity and boundedness of the operator $\hat{f}$ follow from the fact that $\hat{f}$ acts from $L^p(G,R)$ into $L^q(G,R)$. In fact, one has the following properties:

Theorem 2.5 ([25],Th. 2.1-2.2 ,pp.22-26) *Suppose that the operator $\hat{f}$ transforms every function in $L^p(G,R)$ into a function in $L^q(G,R)$ $(1 \leq p,q < \infty, \frac{1}{p}+\frac{1}{q}=1)$. Then the operator $\hat{f}$ is countinuous and bounded.*

Remark 2.4 The boundedness of a nonlinear operator does not , in general, follow from its continuity.
For an example, see ([25] , p.13).

Theorem 2.6 ([25], Th.2.3 , p.27) *If the operator $\hat{f}$ acts from $L^p(G,R)$ into $L^q(G,R)$ $(1 \leq p,q < \infty, \frac{1}{p}+\frac{1}{q}=1)$ then*

$$| f(t,u) |_R \leq a(t) + b \mid u \mid^{\frac{p}{q}},$$

where b is a positive constant, $a(\cdot) \in L^q(G,R^+)$ and $| \cdot |_R$ denots the usual norm in R.

For a proof of Theorems 2.5-2.6 , we refer the reader to [25] .

For a general case (i.e. $\hat{f}$ acting from $L^p(G,V)$ into $L^q(G,V^*)$ $(1 \leq p,q < \infty, \frac{1}{p}+\frac{1}{q} = 1)$, where V is a reflexive Banach space with reflexive dual V^* .), see [13] (Lemma 1-2, pp.4–6).

Remark 2.5. Note that for functions with values in a separable Banach space , weak and strong measurability are the same. We now consider the following control problem of the nonlinear parabolic system

$$(S)\begin{cases} \frac{\partial\phi}{\partial t} + A(t)\phi = f(t,u(t)), t \in (t_0,t_1], \\ \phi(t_0) = \phi_0 \in H, \\ u \in \mathcal{U}_{ad}, \end{cases}$$

in the weak form

$$(S_w)\begin{cases} (L\phi,\psi) + h(\phi,\psi) = f(t,u(t)) \text{ for all } \psi \in L^p(I,V)\cap C(I,H), \\ \phi(t_0) = \phi_0 \in H, \\ u \in \mathcal{U}_{ad}, \end{cases}$$

2.6.2 Existence of solutions of the parabolic system S_w

For the existence of solutions for the problem S_w one needs the following fundamental assumption on the operator A.

(A_1) Each A_α is an r-vector-valued measurable function in all variables and continuous in ξ where

$\xi(\phi) = \{\xi^\alpha(\phi) \mid \alpha \mid\leq m\}$ and $\xi^\alpha(\phi) = D^\alpha\phi$.

For a fixed p, $1 < p < \infty$, there exist a constant $c^0 > 0$ and a fuction $g \in L^q(I\times\Omega, R)$ where $\frac{1}{p}+\frac{1}{q} = 1$, such that ,for all $(t,x) \in I\times\Omega$ and each vector $\xi = \{\xi^\alpha \mid \alpha \mid\leq m\}$

,

$$\mid A_\alpha(t,x,\xi) \mid\leq\mid g(t,x) \mid + c^0 \sum_{|\beta|\leq m} \mid \xi^\alpha \mid^{p-1}, \tag{2.26}$$

for all $\mid \alpha \mid\leq m$.

Note that $A\phi \in L^q(I,V^*)$ for each $\phi \in L^p(I,V)$. The following lemma, essentialy due to Browder, gives us the existence of solutions to the parabolic problem S_w.

Lemma 2.1

Suppose $A(t)$, $t \in I$, satisfies the basic assumption $(A1)$ and the following conditions are satisfied :

$(A2)$ for all $\phi, \psi \in L^p(I,V)$, $h(\phi, \phi - \psi) - h(\psi, \phi - \psi) = (A\phi - A\psi, \phi - \psi) \geq 0$, i.e., the operator A is monotone.

$(A3)$ There exists a nonnegative function $c : R^+ \longrightarrow R^+$ with $c(x) \to +\infty$ as $x \to +\infty$ such that
$h(\phi, \psi) = (A\phi, \phi) \geq c(\| \phi \|) \| \phi \|$ *for all* $\phi \in L^p(I,V)$.

$(A4)$ For each control $u \in \mathcal{U}$, $f(u) \in L^q(I, V^*)$.

Then for each control $u \in \mathcal{U}$ there exists one and only one solution $\phi(u) \in L^p(I,V)$ of the weak boundary value problem (S_w) corresponding to the parabolic system $\frac{\partial\phi}{\partial t} + A(t)\phi = f(u)$ with $\phi(t_0) = \phi_0 \in H$.
Furthermore , this solution $\phi = \phi(u) \in C(I,H)$.

__Proof__:the proof follows from [17],(Theorem1, p.343), (see also [18],Theorem 3.5, pp.35–38.) Now one can introduce the concept of admissible trajectory for the system (S)
Let $\mathcal{F} \equiv \{\phi : \phi \in L^p(I,V) \cap C^1(I,H)$ *such that* $\phi(t_1) = 0\}$.

__Definition 2.6__. *A function $\phi \in L^p(I,V)$ is said to be an admissible trajectory of system (S) if and only if*
(i) $\phi(t_0) = \phi_0$ *and there exists an admissible control $u \in U$ so that*
(ii) $(L\phi, \psi) + h(\phi, \psi) = (f(u), \psi)$ *for all* $\psi \in \mathcal{F}$.

Let $\mathcal{H} \subset L^p(I,V)$ denote the set of all functions $\{\phi\}$ that satisfy the above definition and corespond to some control $u \in H$.
For the next lemmas , we assume that assumption $(A3)$ has been replaced by $(A3)'$. There exists a finite number k_0 and a pair of finite positive numbers k_1 and k_2 such that the function c appearing in condition $(A3)$ satisfies
$c(x) \geq k_0 + k_1(x)^{p-1}$ for all $x \geq k_2$.

Lemma 2.2. *Suppose $A(t), t \in I$, satisfies Assumptions $(A1), (A2), (A3)'$ and suppose for each $u \in \mathcal{U}, f(u) \in L^q(I, V^*)$ with $\sup_{u \in U}(\int_I \mid f(u) \mid_{V^*}^q dt) \leq \beta$ for some $\beta \in (0, \infty)$ and $\phi_0 \in H$. Then the set of admissible trajectories $\mathcal{H}$ of system (S) is a bounded subset of $L^p(I, V)$ and is also conditionally weakly sequentially compact.*

Proof. Here we note that if we assume $\mathcal{H}$ is a bounded subset of $L^p(I, V)$, and since $L^p(I, V)$, $1 < p < \infty$, is a reflexive Banach space , then $\mathcal{H}$ is conditionally weakly sequentially compact.
For more details see [6],(Lemma 3.1,p.193).

Lemma 2.3. *Under the hypotheses of the previous lemma the set $Y = \{y : y = L\phi = \dot{\phi}, \phi \in \mathcal{H}\}$ is a bounded and conditionally weakly sequentially compact subset of $L^q(I, V^*)$.*

Proof: For the complete proof see [6],(Lemma 3.2 , p.194.)
Denote by A the operator determined by

$$(A\phi, \psi) = h(\phi, \psi) \;\; for \;\; \phi, \psi \in L^p(I, V).$$

From the property $(A1)$ one can easily see that $A : L^p(I, V) \longrightarrow L^q(I, V^*)$.
Under appropriate assumptions on the function f appearing in the system of equations (S) or (S_w), one can prove the weak closur of the set $\mathcal{H}$ by using of the following lemma.

Lemma 2.4. *Suppose A satisfies properties (A1), and (A2) and let*
(i) *A be hemicontinuous from the dense linear subset $D(A)$ of $L^p(I, V)$ to $L^q(I, V^*)$,*
(ii) $(A\phi_n - A\phi, \phi_n - \phi) = h(\phi_n, \phi_n - \phi) - h(\phi, \phi_n - \phi) \to 0$ *as* $n \to \infty$
whenever $\phi_n \xrightarrow{w} \phi$ in $L^q(I, V)$ and $A\phi_n \xrightarrow{w} A\phi$ in $L^q(I, V^*)$
Then $\eta = A\phi$, whenever $A\phi_n \longrightarrow \eta$ weakly in $L^q(I, V^*)$.

Proof. Here we shall follow the same proof of [6],(Lemma 3.3 , p.195).
Suppose $\phi_n \xrightarrow{w} \phi$ in $L^p(I, V)$ and $A\phi_n \xrightarrow{w} \eta$ in $L^q(I, V^*)$. For any $\psi \in D(A)$ we

can write, by virtue of property $(A1)$,

$(A\psi - \eta, \psi - \phi) = (A\psi - A\phi_n, \psi - \phi) + (A\phi_n - \eta, \psi - \phi)$

$(A\psi - A\phi_n, \psi - \phi_n) + (A\psi - A\phi_n, \phi_n - \phi) + (A\phi_n - \eta, \psi - \phi)$.

By hypothesis $(A2)$, $(A\psi - A\phi_n, \psi - \phi_n) \geq 0$ and consequently it follows from the above equality that

$(A\psi - \eta, \psi - \phi) \geq (A\psi - A\phi_n, \phi_n - \phi) + (A\phi_n - \eta, \psi - \phi)=$

$(A\psi - A\phi, \phi_n - \phi) - (A\phi_n - A\phi, \phi_n - \phi) + (A\phi_n - \eta, \psi - \phi)$.

The first term on the right hand side of the above inequality converges to zero as $\phi_n \xrightarrow{w} \phi$, the second term converges to zero by hypothesis (ii) of the lemma, and the third term converges to zero since $A\phi_n \xrightarrow{w} \eta$ in $L^q(I, V^*)$.

Thus

$$(A\psi - \eta, \psi - \phi) \geq 0 \; for \; all \; \psi \in D(A). \tag{2.27}$$

Suppose $\eta - A\phi \neq 0$; then , since $D(A)$ is dense in $L^p(I, V)$, there exists a $\psi^* \in D(A)$ such that

$$(\eta - A\phi, \psi^*) > 0.$$

Define $\psi_\theta = \phi + \theta\psi^*$ for $\theta > 0$. Then $\psi_\theta \in D(A)$ for all θ , and replacing ψ by ψ_θ in inequality (2.27) , we have

$$\theta(A\psi_\theta - \eta, \psi^*) \geq 0.$$

Canceling θ , we have

$$(\eta - A\psi_\theta, \psi^*) \leq 0,$$

which can be written as

$$(\eta - A\phi, \psi^*) \leq (A\psi_\theta - A\phi, \psi_*).$$

Letting $\theta \to 0$, the right-hand side goes to zero by the hemicontinuity of A.

Hence

$$0 < (\eta - A\phi, \psi^*) \leq 0,$$

which is a contradiction . Thus $\eta = A\phi$ or $A\phi_n \xrightarrow{w} A\phi$ in $L^q(I, V^*)$.

We conclude this section by mentioning that in the last lemmas it is shown that the family of admissible trajectories is a weakly closed and weakly sequentially compact subset of a reflexive Banach space and that the set of attainable states at any given time is a weakly compact subset of a Hilbert space.
Using theses basic results one can prove the existence of optimal controls for first-order nonlinear evolution equations.
The next chapter will be devoted with the existence of optimal controls for second-order nonlinear evolution equations and it will conclude with the main results of this thesis.

Chapter 3

Optimal Control of Nonlinear Second-Order Evolution Equations

3.1 Introduction

In this chapter we shall consider, the control problem of second-order nonlinear evolution equations .

First , we present a recent result of Papageorgiou [30], which establishes the existence of optimal controls for a class of systems governed by second-order nonlinear evolution equations. His results extend the works of Lions [26](Chapter IV) who limited his investigations to linear systems. An example of nonlinear hyperbolic distributed parameter optimal control problem is also presented in detail in the end of his work .

Second , the main results of this thesis will be presented.It is related to the problem of existence of optimal control of Lagrange type.

Our result extends Papageorgiou's earlier result mentioned above. We introduce more general conditions admitting stronger nonlinearities. In fact, Papageorgiou's results follows from our general results.

We mention here that our result will appear in [1].

3.2 Basic assumptions

Let Y be a separable reflexive Banach space.

Let H be a separable Hilbert space and X a dense subspace of H, carrying the structure of a separable reflexive Banach space, such that X embeds in H continuously. Identifying H with its dual (pivot space), we have $X \hookrightarrow H \hookrightarrow X^*$, with all embeddings being continuous and dense. We shall also assume that all the embeddings are compact. By $\|\cdot\|_X$ (*resp.* $|\cdot|_H, \|\cdot\|_{X^*}$). we will denote the norm of X (resp. of H,X^*). Also by $<\cdot,\cdot>$, we shall denote the duality brackets for the pair (X, X^*) and by $(\cdot,\cdot)$, the inner product of H. The two are compatible in the sense that $<\cdot,\cdot>_{X\times H}=(\cdot,\cdot)$. Let $W_{p,q}(I)=\{x \in L^p(X) : \dot{x} \in L^q(X^*)\}$. The derivative in this definition is understood in the sense of vector-valued distributions. This is a separable, reflexive Banach space with the norm $\| x \|_{W_{p,q}(I)}=(\| x \|^2_{L^p(X)} + \| \dot{x} \|^2_{L^q(X^*)})^{1/2}$. Recall that $W_{p,q}(I)$ embeds into $C(I,H)$ continuously (see Ahmed and Teo [2]). Therfore, every equivalence class in $W_{p,q}(I)$ has a unique representative in $C(I,H)$. Furthermore, since we have assumed that X embeds into H compactly, we have that $W_{p,q}(I)$ embeds into $L^p(H)$, compactly too. Finally, Nagy [28] proved that if X is a Hilbert space , then the injection $W_{p,q}(I) \hookrightarrow C(I,H)$ is compact. For further detail on evolution triples and the Banach space $W_{p,q}(I)$, we refer to Zeidler [33], chapter 23.

3.3 Existence of Optimal Controls

Let (X,H,X^*) be an evolution triple, with compact embeding $X \hookrightarrow H \hookrightarrow X^*$, and Y a separable reflexive Banach space, modeling the control space.

We consider the following Lagrange type optimal control problem:

$$(P)\left\{\begin{array}{l} J(x,u) = \int_0^T L(t,x(t),\dot{x}(t),u(t))dt \rightarrow \ inf = m \\ subject\ to\ the\ following\ state\ and\ control\ constraints: \\ \ddot{x}(t) + A(t,\dot{x}(t)) + Bx(t) = f(t,x(t))u(t), \\ x(0) = x_0 \in X, \dot{x}(0) = x_1 \in H, u(t) \in U(t)\ a.e. \end{array}\right\}$$

Definition 3.1 *By an admissible "state-control" pair $\{x,u\}$ for (P), we understand a pair of a state trajectory $x(\cdot) \in C(I,X)$ and a control function $u(\cdot) \in L^\infty(Y)$ such that $\dot{x}(\cdot) \in W_{p,q}(I)$ and both functions $x(\cdot), u(\cdot)$ satisfy the constraints of problem*

(P).

Definition 3.2 *An admissible state-control pair* $\{x, u\}$ *is said to be optimal if* $J(x,u) = m$.

Remark 3.1 Recall that $W_{p,q}(I)$ embeds into $C(I,H)$ continuously, and so the initial condition $\dot{x}(0) = x_1 \in H$ makes sense.
We are now ready to discus the following result:

3.3.1 Papageorgiou's result

To establish the existence of an optimal control pair for problem (P) Papageorgiou ([30]) considers the following hypotheses on the data:
H(A): $A : I \times X \to X^*$ is a map such that

1. $t \mapsto A(t,v)$ is measurable,
2. $v \mapsto A(t,v)$ is monotone and hemicontinuous
3. $< A(t,v), v > \geq c \| v \|_X^2$ *a.e. with* $c > 0$,
4. $\| A(t,v) \|_{X^*} \leq a(t) + b \| v \|_X$ a.e. with $a(\cdot) \in L^2(I), b > 0$

H(B): $B \in \mathcal{L}(X, X^*)$ (i.e. B is continuous, linear) is symmetric (i.e. $< Bx, z > = < x, Bz >$ for all $x, z, \in X$) and $< Bx, x > \geq c' \| x \|_X^2, c' > 0$ (i.e. $B(\cdot)$ is coercive)
H(f): $f : I \times H \to \mathcal{L}\ (Y, H)$ is a map such that

1. $t \mapsto f(t,x)u$ is measurable for every $(x,u) \in H \times Y$,
2. $x \mapsto f(t,x)^* h$ is continuous for every $(t,h) \in I \times H$,
3. $\| f(t,x) \|_{\mathcal{L}(Y,H)} \leq a_1(t) + b_1 \mid x \mid_H$ a.e. with $a_1(\cdot) \in L_+^2$, $b_1 > 0$.

H(U): $U : I \to P_{wkc}(Y)$ is a measurable multifunction so that $t \mapsto \mid U(t) \mid = \sup\{\| u \|_Y :\ u \in U(t)\} \equiv g(t)$, $g \in L_+^\infty$.
H(L): $L : I \times H \times H \times Y \to \bar{R} = R \cup \{+\infty\}$ is an integrand such that

1. $(t,x,y,u) \mapsto L(t,x,y,u)$ is Borel measurable,

2. $(x,y,u) \mapsto L(t,x,y,u)$ is lower semicontinuous ,

3. $u \mapsto L(t,x,y,u)$ is convex ,

4. $\varphi(t) - \hat{M}(|\, x\, |_H + |\, y\, |_H + \|\, u\, \|_Y)) \leq L(t,x,y,u)$ a.e. with $\varphi(\cdot) \in L^1, \hat{M} > 0$.

<u>$\mathrm{H_0}$:</u> There exists an admissible "state-control" pair (x,u) such that $J(x,u) < \infty$.

Under the above assumptions Papageourgiou establishes the following result:

<u>Theorem 3.1:</u> *If hypothesis $H(A)$, $H(B)$, $H(U)$, $H(L)$, H_0 hold and $x_0 \in X$, $x_1 \in H$, then problem (P) admits an optimal pair.*

<u>Proof</u>: For the complete proof, see Papageorgiou [30]

Now the following example of a nonlinear , hyperbolic optimal control problem will illustrate Papageorgiou's result (Theorem 3.1).

An example

Let Ω be a bounded domain in R^N, with smooth boundary $\Gamma = \partial\Omega$. Now one considers the following Lagrange control problem:

$$\left.\begin{cases} J(x,u) = \int_o^T \int_\Omega L(t,z,x(t,z),x_t(t,z),u(t,z))dzdt \to \inf = m' \\ \text{subject to } \{x,u\} \text{ satisfying the following constraints :} \\ \frac{\partial^2 x}{\partial t^2} - \Delta x - \sum_{i=1}^N D_i(k(t, |\, Dx_t\,|^2))D_i x_t = f(t,z,x(t,z))u(t,z) \text{ a.e.on } I \times \Omega \\ x\,|_{I\times\Gamma} = 0, x(0,z) = x_0(z), x_t(0,z) = x_1(z) \text{ and } \|\, U(t,.)\, \|_{L^2(\Omega)} \leq \eta(t) \text{ a.e.} \end{cases}\right\} (P')$$

Here $D_i = \frac{\partial}{\partial z_i}$, $\quad i = 1,2,\ldots,N$, $Dx = (D_1 x, \ldots, D_N x) = \textit{gradient of } x$, $DxDy = \sum_{i=1}^N D_i x D_i y$ and $|\, Dx\,|^2 = \sum_{i=1}^N |\, D_i x\,|^2$. We shall need the following hypotheses on the data of (P')

<u>H(k):</u> k is a function from $I \times R_+ \to R_+$ such that:

1. $t \mapsto k(t,\mu)$ is measurable,

2. $\mu \mapsto k(t,\mu)$ is continuous,

3. $0 \leq k(t,\lambda^2) \leq L$ for all $(t,\lambda) \in I \times R_+$,with $L > 0$ and $k(t,0) = 0$.

4. $< k(t,\lambda^2)\lambda - k(t,\mu^2)\mu \geq d(\lambda - \mu)$ for all $\lambda, \mu \in R_+$, $\lambda \geq \mu$ and for some $d > 0$

H(f)$_1$: $f : I \times \Omega \times R \to R$ is a function satisfying

1. $(t,z) \mapsto f(t,z,x)$ is measurable,

2. $x \mapsto f(t,z,x)$ is continuous,

3. $| f(t,z,x) | \leq a_1(t,z) + b_1(z) | x |$ a.e. with $a_1(\cdot,\cdot) \in L^2(I \times \Omega), b_1(\cdot) \in L^\infty(\Omega)$.

H(η): $\eta(\cdot) \in L^2_+$.
H$_0$: $x_0 \in H^1_0(\Omega), x_1 \in L^2(\Omega)$ and $m' < \infty$.
H(L): $L : I \times \Omega \times R \times R \times R \to \bar{R} = R \cup \{+\infty\}$ is an integrand such that

1. $(t,z,x,y,u) \mapsto L(t,z,x,y,u)$ is measurable,

2. $(x,y,u) \mapsto L(t,z,x,y,u)$ is lower semicontinuous,

3. $u \mapsto L(t,z,x,y,u)$ is convex,

4. $\varphi(t,z) - \hat{M}(x)(| x |_R + | y |_R + | u |_R) \leq L(t,z,x,y,u)$ a.e. with $\varphi(\cdot,\cdot) \in L^1(I \times \Omega)$, and $\hat{M}(.) \in L^\infty_+(\Omega)$.

In this case , $X = H^1_0(\Omega)$, $H = L^2(\Omega)$ and $X^* = H^{-1}(\Omega)$. It is well known that the triple (X, H, X^*) has all embedings compact (Sobolev embedding theorem).
We shall Consider the following Dirichlet forms:

$$a_1(t,x,y) = \int_\Omega \sum_{i=1}^N k(t, | Dx |^2) D_i D_i y dz = \int_\Omega k(t, | Dx |^2) Dx Dy dz$$

and

$$a_2(t,x,y) = \int_\Omega \sum_{i=1}^N D_i D_i y dz = \int_\Omega Dx Dy dz$$

for all $x, y \in H^1_0(\Omega)$.
One can find a nonlinear operator $A : I \times X \to X^*$ such that

$< A(t,x), y >= a_1(t,x,y)$,

$B \in \mathcal{L}(X, X^*)$ such that

$a_2(t,x,y) =< Bx, y >$, for all $x, y \in H_0^1$

and

$\hat{f} : I \times L^2(\Omega) \to L^2(\Omega)$ defined by

$$\hat{f}(t,x)(z) = f(t,z,x(z))$$

.

satisfying the hypotheses $H(A)$, $H(B)$ and $H(f)_1$.

Finaly let $\hat{f} : I \times L^2(\Omega) \to \mathrm{L}^2(\Omega)$ be defined by

$$\hat{f}(t,x)(z) = f(t,z,x(z))$$

and $\hat{L} : I \times L^2(\Omega) \times L^2(\Omega) \times L^2(\Omega) \to \bar{R}$ be defined by

$$\hat{L}(t,x,y,z,u) = \int_\Omega L(t,z,x(z),y(z),u(z))dz.$$

satisfying the hypotheses $H(f)$ and $H(L)$.

For the control space we put $Y = L^2(\Omega)$ and $U(t) = \{u \in L^2(\Omega) : \| u \|_{L^2(\Omega)} \leq \eta(t)\}$.

$U(\cdot)$ is mesurable and satisfies hypotheses $H(U)$.

Also let $\hat{x}_0 = x_0(\cdot) \in H_0^1$ and $\hat{x}_1 = x_1(\cdot) \in L^2(\Omega)$.

For more detail we refer the reader to [30].

Now problem (P') can be rewritten in the following equivalent abstract form :

$$(P') \left\{ \begin{array}{l} \hat{J}(x,u) = \int_0^T \hat{L}(t,x(t),\dot{x}(t),u(t))dt \to \inf = m' \\ \textit{subject to the following state and control constraints :} \\ \ddot{x}(t) + A(t,\dot{x}(t)) + Bx(t) = \hat{f}(t,x(t))u(t) \ a.e. \\ x(0) = \hat{x}_0 \in X, \dot{x}(0) = \hat{x}_1 \in H, u(t) \in U(t) \ a.e \end{array} \right\}$$

This has the form of problem (P) and one can check (see [30]) that it satisfies all the hypotheses of Theorem 3.1 thus we have the following Theorem:

<u>Theorem 3.2</u> *If hypotheses $H(k), H(f)_1, H(\eta), H_0$ and $H(L)$ hold, then (P') admits an optimal pair $\{x,u\} \in C(I, H_0^1(\Omega)) \times L^2(\Omega)$ such that*

$$\frac{\partial x}{\partial t} \in L^2(I, H_0^1(\Omega)) \cap C(I, L^2(\Omega)) \text{ and } \frac{\partial^2 x}{\partial t^2} \in L^2(I, H^{-1}(\Omega)).$$

Hence Theorem 3.2 concludes Papageorgiou's result and at the same time the first part of this chapter.

Now we go to the most important part of this chapter and present the main result of this thesis.

3.3.2 The Main Results of the Thesis

To establish the existence of an optimal pair $\{x, u\}$ for problem (P), we shall need the following hypotheses on the data:

H(A): $A : I \times X \to X^*$ is a map such that

1. $t \mapsto A(t, v)$ is measurable,
2. $v \mapsto A(t, v)$ is monotone and hemicontinuous,
3. $< A(t,v), v > \geq c \| v \|_X^p - d \mid v \mid_H^2$ a.e. with $c > 0, d \geq 0$.
4. $\| A(t,v) \|_{X^*} \leq a(t) + b \| v \|_X^{p-1}$ a.e. with $a(\cdot) \in L^q(I), b > 0$ or b $\in L_+^\infty(I)$.

H(B): $B \in \mathcal{L}(X, X^*)$ (i.e. B is continuous and linear) is symmetric (i.e. $< Bx, z > = < x, Bz >$ for all $x, z \in X$) and $< Bx, x > \geq c' \| x \|_X^2$, $c' > 0$, (i.e. $B(\cdot)$ is coercive)

H(f): $f : I \times H \to \mathcal{L}\ (Y, H)$ is a map such that

1. $t \mapsto f(t, x)u$ is measurable for every $(x, u) \in H \times Y$,
2. $x \mapsto f(t, x)^* h$ is continuous for every $(t, h) \in I \times H$,
3. $\| f(t, x) \|_{\mathcal{L}(Y,H)}^q \leq a_1(t) + b_1 \mid x \mid_H^2$ for $2 \leq p < \infty$ and $1 < q \leq 2$.

H(U): $U : I \to P_{wkc}(Y)$ is a measurable multifunction such that $t \mapsto \mid U(t) \mid = \sup\{\| u \|_Y : u \in U(t)\} \equiv g(t)$, $g \in L_+^\infty$,

H(L): $L : I \times H \times H \times Y \to \bar{R} = R \cup \{+\infty\}$ is an integrand such that

1. $(t, x, y, u) \mapsto L(t, x, y, u)$ is Borel measurable,

2. $(x, y, u) \mapsto L(t, x, y, u)$ is lower semicontinuous,

3. $u \mapsto L(t, x, y, u)$ is convex,

4. $\varphi(t) - \hat{M}(|x|_H + |y|_H + \|u\|_Y)) \leq L(t, x, y, u)$ a.e. with $\varphi(\cdot) \in L^1, \hat{M} > 0$. Finally since our cost-functional is $\bar{R}$-valued, $\inf\{J(x, u), u \in \mathcal{U}_{ad}\} = m$ may be ∞, and in this case there is nothing to prove. So we shall need the following feasibility hypothesis.

$\underline{H_0:}$ there exists an admissible "state-control" pair (x, u) such that $J(x, u) < \infty$.

<u>Lemma 3.1</u>: *Under assumptions $H(A)$, $H(B)$,$H(f)$ and $H(U)$, for each $x_0 \in X$, $x_1 \in H$ and $u \in \mathcal{U}_{ad}$ the evolution equation of problem (P) has a unique solution x satisfiying*
(a) $x \in L^\infty(X)$
(b) $\dot{x} \in L^\infty(H) \cap L^p(X)$
(c) $\ddot{x} \in L^q(X^*)$
(d) (b) and (c) $\Rightarrow \dot{x} \in W_{p,q}$
(e) $A(\cdot, \dot{x}(\cdot)) \in L^q(X^*)$.

The proof follows from standard application of Galarkin's technique and the apriori estimates given in Lemma 3.2 ; see also the proof of Theorem 2.4 .
Before studying the problem of existence of optimal controls, we shall start by deriving some apriori bounds for the admissible trajectories of (P).
Denote by S the set of solution trajectories of the evolution equation of problem (P) corresponding to the admissible set of controls as defined above.

<u>lemma 3.2</u> (A priori estimates): *Under assumptions $H(A3)$, $H(f)$, $H(B)$ and $H(U)$ the set $Z \equiv \{\dot{x}, x \in S\}$ is a bounded subset of $W_{p,q}(I)$.*

<u>Proof:</u> Let x be any solution trajectory of the evolution equation in problem (P), corresponding to an admissible control $u(\cdot) \in L^\infty(Y)$. By Lemma 2.1 , the following

scalar multiplication is well defined,

$$<\ddot{x}(t),\dot{x}(t)>+<A(t,\dot{x}(t)),\dot{x}(t)>+<Bx(t),\dot{x}(t)>=(f(t,x(t))u(t),\dot{x}(t)) \ a.e.$$

Since $\dot{x}\in W_{p,q}(I)$ it follows from Proposition 23.23 (iv), p.422 of Zeidler [33], that

$$<\ddot{x}(t),\dot{x}(t)>=\frac{1}{2}\frac{d}{dt}\mid \dot{x}(t)\mid_H^2 \ a.e.$$

Furthermore, because of hypotheses H(A3), we have

$$c\parallel \dot{x}(t)\parallel_X^p - d\mid \dot{x}(t)\mid_H^2 \leq <A(t,\dot{x}(t)),\dot{x}(t)> \ a.e.$$

Also using the product rule and exploiting the symmetry of the operator $B\in \mathcal{L}(X,X^*)$ (see hypothesis H(B)), we obtain

$$\begin{aligned}\frac{d}{dt}<Bx(t),x(t)> &=<B\dot{x}(t),x(t)>+<Bx(t),\dot{x}(t)>\\ &=2<B\dot{x}(t),x(t)> \ a.e.\end{aligned}$$

So finally we can write that

$$\frac{1}{2}\frac{d}{dt}\mid \dot{x}(t)\mid_H^2 + c\parallel \dot{x}(t)\parallel_X^p + \frac{1}{2}\frac{d}{dt}<Bx(t),x(t)> \leq d\mid \dot{x}(t)\mid_H^2 + (f(t,x(t))u(t),\dot{x}(t)) \ a.e.$$

Integrating the above inequality, we have

$$\begin{aligned}&\frac{1}{2}\mid \dot{x}(t)\mid_H^2 - \frac{1}{2}\mid x_1\mid_H^2 + c\int_0^t \parallel \dot{x}(s)\parallel_X^p ds + \frac{1}{2}<Bx(t),x(t)> - \frac{1}{2}<Bx_0,x_0>\\ &\leq d\int_0^t \mid \dot{x}(s)\mid_H^2 ds + \int_0^t (f(s,x(s))u(s),\dot{x}(s))ds,\\ \Rightarrow &\mid \dot{x}(t)\mid_H^2 + 2c\int_0^t \parallel \dot{x}(s)\parallel_X^p ds + c'\parallel x(t)\parallel_X^2\\ &\leq \mid x_1\mid_H^2 + \parallel B\parallel_{\mathcal{L}(X,X^*)}\parallel x_0\parallel_X^2\\ &+2d\int_0^t \mid \dot{x}(s)\mid_H^2 ds + 2\int_0^t (f(s,x(s))u(s),\dot{x}(s))ds \qquad (3.1)\end{aligned}$$

Note that by applying Cauchy's inequality,

$$a\cdot b \leq \frac{\epsilon^p}{p}\mid a\mid^p + \frac{\epsilon^{-q}}{q}\mid b\mid^q, \ \epsilon>0, \ a,b\in R,$$

to the last integral on the right-hand side and using $H(f)$,$H(U)$ we obtain

$$\int_0^t (f(s,x(s))u(s),\dot{x}(s))ds \le \int_0^t \mid f(s,x(s))u(s) \mid_H . \mid \dot{x}(s) \mid_H ds$$
$$\le (\int_0^t \mid \dot{x}(s) \mid_H^p ds)^{\frac{1}{p}} (\int_0^t \mid f(s,x(s))u(s) \mid_H^q ds)^{\frac{1}{q}}$$
$$\le \frac{\epsilon^p}{p}\int_0^t \mid \dot{x}(s) \mid_H^p ds + \frac{\epsilon^{-q}}{q}\int_0^t \mid f(s,x(s))u(s) \mid_H^q ds$$
$$\le \beta\frac{\epsilon^p}{p}\int_0^t \parallel \dot{x}(s) \parallel_X^p ds + \frac{\epsilon^{-q}}{q} \parallel g \parallel_\infty^q \parallel a_1 \parallel_{L^1} + b_1\frac{\epsilon^{-q}}{q} \parallel g \parallel_\infty^q \int_0^t \mid x(s) \mid_H^2 ds$$

where $\beta > 0$ is the embedding constant $X \hookrightarrow H$.
Hence

$$\mid \dot{x}(t) \mid_H^2 + 2(c - \beta\frac{\epsilon^p}{p})\int_0^t \parallel \dot{x}(s) \parallel_X^p ds + c' \parallel x(t) \parallel_X^2$$
$$\le M + 2d\int_0^t \mid \dot{x}(s) \mid_H^2 ds + 2\frac{\epsilon^{-q}}{q} \parallel g \parallel_\infty^q \parallel a_1 \parallel_{L^1} + 2b_1\frac{\epsilon^{-q}}{q} \parallel g \parallel_\infty^q \int_0^t \mid x(s) \mid_H^2 ds$$

with $M = \mid x_1 \mid_H^2 + \parallel B \parallel_{\mathcal{L}(X,X^*)} \parallel x_0 \parallel_X^2$,
and consequently, for a sufficiently small $\epsilon > 0$, such that $c > \beta\frac{\epsilon^p}{p}$, we obtain

$$\mid \dot{x}(t) \mid_H^2 + c_1\int_0^t \parallel \dot{x}(s) \parallel_X^p ds + c' \parallel x(t) \parallel_X^2 \le$$
$$c_2 + 2d\int_0^t \mid \dot{x}(s) \mid_H^2 ds + c_2 + c_3\int_0^t \mid x(s) \mid_H^2 ds \ a.e, \tag{3.2}$$

where c_1, c_2, c_3 are suitable positive constants.
Observe that since $\dot{x} \in W_{p,q}(I)$, from theorem 22 p.19 of Barbu [18], we have $x(s) = x_0 + \int_0^s \dot{x}(\tau)d\tau \ in \ X$ (hence in H too),

$$\Rightarrow \mid x(s) \mid_H^2 \le 2 \mid x_0 \mid_H^2 + 2(\int_0^s \mid \dot{x}(\tau) \mid_H d\tau)^2 \le 2 \mid x_0 \mid_H^2 + 2T\int_0^s \mid \dot{x}(\tau) \mid_H^2 d\tau.$$

Substituting this estimate in the inequality (3.2), we obtain

$$\mid \dot{x}(t) \mid_H^2 + c_1\int_0^t \parallel \dot{x}(s) \parallel_X^p ds + c' \parallel x(t) \parallel_X^2 \le c_4 + c_5\int_0^t \mid \dot{x}(\tau) \mid_H^2 d\tau,$$

where c_4 and c_5 are positive constants depending on c_2, c_3 ,d and $\mid x_0 \mid_H$. Hence by Gronwall's inequality, there exists a constant $M_2 > 0$ so that for every admissible trajectory $x(\cdot) \in C(I,X)$ and all $t \in I$, we have

$$\mid \dot{x}(t) \mid_H \le M_2. \tag{3.3}$$

But recall that $x(t)=x_0+\int_0^t \dot{x}(s)ds$ in H, for all $t\in I$. So for every trajectory $x(\cdot)$ of (P) and every $t\in I$, we have

$$|x(t)|_H \le |x_0|_H + \int_0^t |\dot{x}(s)|_H\, ds \le |x_0|_H + M_2T = M_3. \tag{3.4}$$

Using estimates (3.3) and (3.4) in inequality (3.2), we obtain:

$$|\dot{x}(t)|_H^2 + c_1\int_0^t \|\dot{x}(s)\|_X^p\, ds + c'\|x(t)\|_X^2 \le M_4,$$

where M_4 is a postive constant depending on c_5, M_1 and M_2. Then from the last inequality, it follows that

$$\dot{x}\in L^\infty(H)\cap L^p(X), x\in L^\infty(X). \tag{3.5}$$

Finally let $z\in L^p(X)$, and by $((\cdot,\cdot))_0$ denote the duality brackets for the pair $(L^p(X), L^q(X^*))$ (i.e., if $v\in L^q(X^*), z\in L^P(X)$, then $((v,z))_0=\int_0^T < v(t), z(t) > dt$). Also, let $\hat{A}: L^p(X)\to L^q(X^*)$ be the Nemyckii operator corresponding to $A(t,x)$; i.e. $\hat{A}(y)(t)=A(t,y(t))$ a.e. and similarly for evry $u\in S_U^\infty$, $(\hat{f}(x)u)(t)=f(t,x(t))u(t)$. Clearly , by assumption H(f3), $\hat{f}(x)u(\cdot)\in L^q(H)$.
With this notation we can rewrite the evolution equation of problem (P) as an abstract equation in $L^q(X^*)$:

$$\ddot{x}+\hat{A}(\dot{x})+Bx=\hat{f}(x)u.$$

Scalar multiplying this by $z\in L^p(X)$, we have

$$\begin{aligned}((\ddot{x},z))_0 &\le |((\hat{A}(\dot{x}),z))_0| + |((Bx,z))_0| + ((\hat{f}(x)u,z))_0\\ &\le \left[\|\hat{A}(\dot{x})\|_{L^q(X^*)} + \|Bx\|_{L^q(X^*)} + \|\hat{f}(x)u\|_{L^q(X^*)}\right]\|z\|_{L^p(X)}\\ &\le \left[\|a\|_{L^q} + bM_5 + \|B\|_{\mathcal{L}(X,X^*)}M_6 + \beta'\|g\|_\infty\|\tilde{a}_1\|_{L^q} + \tilde{b}_1M_2)\right]\|z\|_{L^p(X)}\end{aligned}$$

where β' is the embedding constant $H\hookrightarrow X^*$, and the existence of M_5, M_6 follows from (3.5) and (3.4). Since $z(\cdot)\in L^p(X)$ was arbitrary, we deduce that there exists $M_7>0$ such that, for all arbitrary trajectories $x(\cdot)$ of (P), we have

$$\|\ddot{x}\|_{L^q(X^*)} \le M_7. \tag{3.6}$$

Thus, the assertion of Lemma 3.1 follows from (3.5) and (3.6).

Theorem 3.3: *If the hypotheses $H(A)$, $H(B)$, $H(U)$, $H(L)$, H_0 hold and $x_0 \in X$, $x_1 \in H$, then problem (P) admits an optimal pair.*

Proof: From lemma 3.2 it follows that Z is a bounded subset of the reflexive Banach space $W_{p,q}(I)$. Therfore, Z is a relatively weakly compact subset of $W_{p,q}(I)$. Now let $\{(x_n, u_n)\}_{n\geq 1}$ be a minimizing sequence of admissible "state-control" pairs for problem (P); i.e.
$\lim_{n\to\infty} J(x_n, u_n) = \inf\{J(x,u), for\ admissible"state - control"\ pair(x,u)\} \equiv m.$
Since $\{x_n\}_{n\geq} \subseteq S$, by passing to a subsequence if necessary, we may assume that $\dot{x}_n \xrightarrow{w} y$ in $W_{p,q}(I)$. Hence one can easily see that $x \in C(I, X)$ and that $y = \dot{x}$ in the distribution sense. But recall that $W_{p,q}(I)$ embeds compactly into $L^p(H)$. Thus $\dot{x}_n \xrightarrow{s} y$ in $L^p(H)$ and clearly $x_n(t) \xrightarrow{s} x(t)$ in H uniformly on I. Furthermore, from hypotheses $H(U)$ and proposition 3.1 of [29] we have that $S_U^{\infty} \equiv \{u \in L^\infty(Y) : u(t) \in U(t) a.e\}$ is w_*-compact in $L^\infty(Y)$. Hence we may assume that $u_n \xrightarrow{w_*} u$ in $L^\infty(Y)$. Then invoking theorem 2.1 of Blader [14], we conclude that $J(x,u)$ is strong-w_* l.s.c. i.e. $J(x,u) \leq \liminf J(x_n, u_n) = m$, whenever $x_n \xrightarrow{s} x$ in $L^1(H)$ and $u_n \xrightarrow{w_*} u\ in\ L^\infty(Y)$. It suffices to show that (x,u) is an admissible "state-control" pair for (P). To this end, we have

$$((\ddot{x}_n, \dot{x}_n - \dot{x}))_0 + ((\hat{A}(\dot{x}_n), \dot{x}_n - \dot{x})) + ((Bx_n, \dot{x}_n - \dot{x}))_0 = ((\hat{f}(x_n)u_n, \dot{x}_n - \dot{x}))_0 \quad (3.7)$$

by integrating by parts formula (3.7) for functions in $W_{p,q}(I)$ (see Zeidler [33],Proposition 23.23(iv), pp. 422-423), we have:

$$((\ddot{x}_n, \dot{x}_n - \dot{x}))_0 = \frac{1}{2} \mid \dot{x}_n(T) - \dot{x}(T) \mid_H^2 - \frac{1}{2} \mid \dot{x}_n(0) - x_1 \mid_H^2 + ((\ddot{x}, \dot{x}_n - \dot{x}))_0. \quad (3.8)$$

Since $\dot{x}_n(0) = x_1$, the second term vanishes and $\mid \dot{x}_n(T) - \dot{x}(T) \mid_H \to 0$ as $n \to \infty$, ($\dot{x}_n \in C(I, H)$) and also since $\dot{x}_n \xrightarrow{w} y = \dot{x}$ in $L^p(X)$ we have $((\ddot{x}, \dot{x}_n - \dot{x}))_0 \to 0$ as $n \to \infty$. Then by passing to the limit as $n \to \infty$ in (3.8) we have

$$((\ddot{x}_n, \dot{x}_n - \dot{x}))_0 \to 0\ as\ n \to \infty. \quad (3.9)$$

Note that for every $h \in L^p(H)$, we have

$$\int_0^T (f(t, x_n(t))u_n(t), h(t))dt = \int_0^T (u_n(t), (f(t, x_n(t))^* h(t))dt.$$

But since $x_n(t) \xrightarrow{s} x(t)$ *in* H, then $(f(t, x_n(t)))^* h(t) \xrightarrow{s} (f(t, x(t)))^* h(t)$ in Y^* for almost all $t \in I$ (see hypotheses $H(f)2$). Also by $H(f)3$, we have

$$\| (f(t, x_n(t)))^* h(t) \|_{Y^*} \leq \| (f(t, x_n))^* \|_{\mathcal{L}(H, Y^*)} | h(t) |_H$$
$$\leq \tilde{a}(t) + (\tilde{b} M_3^{2/q}) \mid h(t) \mid_H \tag{3.10}$$

So there exists $\eta(\cdot) \in L^1(Y^*)$ so that $\| (f(t, x_n(t)))^* h(t) \|_{Y^*} \leq \| \eta(t) \|_{Y^*}$ a.e. and therefore it follows from dominated convergence theorem that $(\hat{f}(x_n))^* h \xrightarrow{s} (\hat{f}(x))^* h$ in $L^1(Y^*)$.

Hence

$$\int_0^T (u_n(t), (f(t, x_n(t))^* h(t))dt \to \int_0^T (u(t), (f(t, x(t))^* h(t))dt$$
$$\Rightarrow \int_0^T (f(t, x_n(t))u_n(t), h(t))dt \to \int_0^T (f(t, x(t))u(t), h(t))dt$$
$$\Rightarrow \hat{f}(x_n)u_n \xrightarrow{w} \hat{f}(x)u \text{ in } L^q(H).$$

On the other hand, since $\dot{x}_n \xrightarrow{w} \dot{x}$ in $W_{p,q}(I)$ and since the embedding $W_{p,q}(I) \hookrightarrow L^p(H)$ is compact, we have that $\| \dot{x}_n - \dot{x} \|_{L^p(H)} \to 0$ and hence

$$((\hat{f}(x_n)u_n, \dot{x}_n - \dot{x}))_{L^q(H), L^p(H)} \to 0. \tag{3.11}$$

Exploiting the symmetry of the operator B, we have

$$\frac{d}{dt} < B(x_n(t) - x(t)), x_n(t) - x(t) >= 2 < B(x_n(t) - x(t)), \dot{x}_n(t) - \dot{x}(t) > a.e.$$

Integrating the above equality, we get

$$< B(x_n(T) - x(T)), x_n(T) - x(T) >= 2((B(x_n - x), \dot{x}_n - \dot{x}))_0$$
$$\Rightarrow c' \| x_n(T) - x(T) \|_X^2 + 2((Bx, \dot{x}_n - \dot{x}))_0 \leq 2((Bx_n, \dot{x}_n - \dot{x}))_0$$

Note that since $\dot{x}_n \xrightarrow{w} \dot{x}$ in $L^p(X)$, we have $x_n(T) \xrightarrow{w} x(T)$ in X. Obviously

$$0 \leq \liminf \| x_n(T) - x(T) \|_X^2,$$

and clearly $((Bx, \dot{x}_n - \dot{x}))_0 \to 0$. Thus we have

$$c' \liminf \| x_n(T) - x(T) \|_X^2 + 2lim((Bx, \dot{x}_n - \dot{x}))_0 \leq 2 \liminf((Bx_n, \dot{x}_n - \dot{x}))_0$$

$$\Rightarrow 0 \leq \liminf((Bx_n, \dot{x}_n - \dot{x}))_0. \tag{3.12}$$

Now passing to the limit as $n \to \infty$ in (3.7) and using (3.9), (3.10) and (3.11) above we get that

$$\limsup((\hat{A}(\dot{x}_n), \dot{x}_n - \dot{x}))_0 \leq 0.$$

Also note that because of hypothesis H(A)4, $\{\hat{A}(\dot{x_n})\}_{n\geq 1} \subseteq L^q(X^*)$ is bounded and so, by passing to a subsequence, we may assume that $\hat{A}(\dot{x}_n) \xrightarrow{w} v \ inL^q(X^*)$. But $\hat{A}(\cdot)$ is hemicontinuous and monotone (since $A(t, \cdot)$ is), hence $\hat{A}(\dot{x}) = v$; i.e. $\hat{A}(\dot{x_n}) \xrightarrow{w} \hat{A}(\dot{x})$ in $L^q(X^*)$. Then for any $z \in L^p(X)$, we have :

$((\ddot{x}_n, z))_0 + ((\hat{A}(\dot{x}_n), z))_0 + ((Bx_n, z))_0 = ((\hat{f}(x_n)u_n, z))_0$

$\to ((\ddot{x}, z))_0 + ((\hat{A}(\dot{x}), z))_0 + ((Bx, z))_0 = ((\hat{f}(x)u, z))_0 \ as \ n \ \to \infty$

$\Rightarrow \ddot{x}(t) + A(t, \dot{x})(t) + Bx(t) = f(t, x(t))u(t) \ a.e.$

$x(0) = x_0 \in X, \dot{x}(0) = x_1 \in H, \ u \in S_U^\infty$

$\Rightarrow (x, u) \ is \ an \ admissible \ "state - control" \ pair \ \ for(P). \ So$

$J(x, u) = m$

$\Rightarrow (x, u)$ is the desired optimal pair. **Q.E.D**

3.4 Concluding remarks

One can see that if we put $p = 2$ and the constant $d = 0$ in our hypothesi $H(A)3$ (3.3.2) then hypotheses $H(A)3$ (3.3.1) of Papageorgiou follows.

Moreover the other assumptions on the data are a particular case of ours In fact , it suffices to choose $p = q = 2$.

Hence the more general conditions we have introduced arise in the strong nonlinearities $H(A)$. However hypotheses $H(L)$ on the integrand are the same.

We mention that Papageorgiou in [30] assumes the existence of the solution of problem (P), but here we proves its existence.

Chapter 4

An Example

The following example illustrates our results (Theorem 3.3) given in chapter 3.

4.1 An example.

In this section we work out in detail an example of a nonlinear hyperbolic optimal control problem.

Let Ω be a bounded domain in R^n, with smooth boundary $\Gamma = \partial\Omega$. We consider the following Lagrange control problem:

$$\left.\begin{cases} J(\phi,u) = \int_0^T \int_\Omega L(t,z,\phi(t,z),u(t,z))dzdt \to \inf = m' \\ \text{subject to } \{\phi,u\} \text{ satisfying the folloing constraints} \\ \frac{\partial^2\phi}{\partial t^2} - \Delta\phi - \sum_{i,j=1}^N D_i(k_{i,j}(t, \mid D\phi_t \mid^{p-2}))D_j\phi_t = f(t,z,\phi(t,z))u(t,z) a.e. on\ I\times\Omega \\ \phi \mid_{I\times\Gamma} = 0, \phi(0,z) = \phi_0(z), \phi_t(0,z) = \phi_1(z) and \parallel U(t,.) \parallel_{L^\infty(\Omega)} \leq \eta(t)\ a.e. \end{cases}\right\} (P')$$

Here $D_i = \frac{\partial}{\partial z_i}$, $i = 1,2,\ldots,N$, $D\phi(D_1\phi,\ldots,D_N\phi) = gradient\ of\ \phi, D\phi D\psi = \sum_{i,j=1}^N D_i\phi D_j\psi$ and $\mid D\psi \mid^2 = \sum_{i=1}^N \mid D_i\psi \mid^2$. We will need the following hypotheses on the data of (P')

<u>H(k):</u> k is a matrix from $I \times R_+ \to \mathcal{L}^+(R^n)$ such that:

1. $t \mapsto k(t,\mu)$ is measurable ,
2. $\mu \mapsto k(t,\mu)$ is continuous ,
3. $\mid k(t,\xi) \mid_{\mathcal{L}(R^n)} < \alpha + \beta \mid \xi \mid$ for all $(t,\xi) \in T \times R^n$ with $\beta > 0$ and $\alpha \geq 0$,

4. $< k(t, | \xi |^{p-2})\xi - k(t, | \eta |^{p-2})\eta, \xi - \eta >_{R^n} \geq 0$ for all $(t, \xi, \eta) \in I \times R^n \times R^n$,

5. $< k(t, | \xi |^{p-2})\xi, \xi >_{R^n} \geq \beta | \xi |^p_{R^n}$ for all $(t, \xi) \in I \times R^n$ and $\beta > 0$.

H(f)$_1$: $f : I \times \Omega \times R \to R$ is a function satisfying

1. $(t, z) \mapsto f(t, z, x)$ is measurable ,

2. $x \mapsto f(t, z, x)$ is continuous ,

3. $| f(t, z, x) | \leq a_1(t, z) + b_1(z) | x |$ a.e. with $a_1(\cdot, \cdot) \in L^2(I \times \Omega), b_1(\cdot) \in L^\infty(\Omega)$.

H(η): $\eta(\cdot) \in L^1_+$.
H$_0$: $\phi_0 \in W_0^{1,p}(\Omega), \phi_1 \in L^2(\Omega)$ and $m' < \infty$.
H($\hat{L}$): $\hat{L} : I \times \Omega \times R \times R \times R \to \bar{R} = R \cup \{+\infty\}$ is an itegrand such that

1. $(t, z, x, y, u) \mapsto \hat{L}(t, z, x, y, u)$ is measurable ,

2. $(x, y, u) \mapsto \hat{L}(t, z, x, y, u)$ is lower semicontinuous ,

3. $u \mapsto \hat{L}(t, z, x, y, u)$ is convex ,

4. $\varphi(t, z) - \hat{M}(x)(| x |_R + | y |_R + | u |_R) \leq L(t, z, x, y, u)$a.e. with $\varphi(.,.) \in L^1(I \times \Omega)$, and $\hat{M}(.) \in L^\infty_+(\Omega)$.

We denote by $W_0^{1,p}(\Omega)$ and $W^{1,p}(\Omega)$ the usual Sobolev spaces defined by

$$W_0^{1,p}(\Omega) = \{\phi \in W^{1,p}(\Omega) : \phi |_{\partial\Omega} \equiv 0\}$$

and

$$W^{1,p}(\Omega) = \{\phi \in L^p(\Omega) : D^\beta \phi \in L^p(\Omega), | \beta | \leq 1\}$$

respectively. $W^{1,p}(\Omega)$ is a Banach space with respect to the norm
$\| \phi \|^{(1)}_{p,\Omega} \equiv \sum_{|\beta| \leq 1} \| D^\beta \phi \|_{L^p(\Omega)}$.
It is separable for $1 \leq p < \infty$ and reflexive for $1 < p < \infty$.

Consider the following Dirichlet forms:

$$a_1(t, \phi, \psi) = \int_\Omega \sum_{i,j=1}^N k_{i,j}(t, | D\phi |^{p-2}) D_i\phi D_j\psi dz = \int_\Omega < k(t, | D\phi |^{p-2}) D\phi, D\psi >_{R^n} dz$$

and

$$a_2(\phi,\psi)=\int_\Omega \sum_{i,j=1}^{N} D_i\phi D_j\psi dz=\int_\Omega D\phi D\psi dz$$

for all $\phi,\psi\in W_0^{1,p}(\Omega)$. Using hypotheses $H(k)3$, we get

$$|\ a_1(t,\phi,\psi)\ |\le \left(\zeta \parallel \phi \parallel_{W_0^{1,p}(\Omega)} +\beta \parallel \phi \parallel^{p-1}_{W_0^{1,p}(\Omega)}\right) \parallel \psi \parallel_{W_0^{1,p}(Z)}$$

where ζ is a positive constant depending on the embedding constant $W_0^{1,p}(\Omega)\hookrightarrow W_0^{1,2}(\Omega)$ and α is as defined in $H(k)3$.
Therefore, there exists an operator $A: I\times X\to X^*$ such that

$$< A(t,\phi),\psi >= a_1(t,\phi,\psi).$$

Note that by Fubini's theorem ,
$t\mapsto a_1(t,\phi,\psi)$ is measurable for all $\phi,\psi\in W_0^{1,p}(\Omega)$. Hence, $t\mapsto A(t,\phi)$ is weakly measurable from I into $W^{-1,q}(\Omega)$.
But recall that $W^{-1,q}(\Omega)$ is a separble Hilbert space. Thus Pettis' measurability theorem tells us that
$t\mapsto A(t,\phi)$ is measurable.
Also let $\phi_n \xrightarrow{s} \phi$ in $W_0^{1,p}(\Omega)$. Then $D\phi_n \xrightarrow{s} D\phi$ in $L^p(\Omega,R^N)$ and since by hypotheses $H(k)2$, $k(t,\cdot)$ is continuous , we have

$$k(t,|\ D\phi_n(z)\ |^{p-2})\to k(t,|\ D\phi(z)\ |^{p-2})\ \ a.e.$$

$$\Rightarrow \int_\Omega < k(t,|\ D\phi_n\ |^{p-2})D\phi_n, D\psi >_{R^n} dz \to \int_\Omega < k(t,|\ D\phi\ |^{p-2})D\phi, D\psi >_{R^n} dz$$

$\Rightarrow A(t,\phi_n) \xrightarrow{w} A(t,\phi)\ in\ W^{-1,q}(\Omega) \Rightarrow A(t,\cdot)$ is demicontinuous, hence hemicontinuous (by Remark 2.1(c)). Also for every $\phi,\psi\in W_0^{1,p}(\Omega)$, we have

$$< A(t,\phi)-A(t,\psi),\phi-\psi >= \int_\Omega < k(t,|\ D\phi\ |^{p-2})D\phi - k(t,|\ D\psi\ |^{p-2})D\psi, (D\phi-D\psi) >_{R^n} dz.$$

Therefore, the monotonocity of $A(t,\cdot)$ follows from hypothesis H(k)4 . Furthermore, from hupothesis H(k)5 we obtain

$$< A(t,\phi),\phi >\ge \beta \parallel \phi \parallel^p_{W_0^{1,p}(\Omega)},\ with\ \beta>0.$$

Thus we have satisfied hypotheses H(A).

Next note that by the Cauchy-Schwartz inequality, we get

$$| a_2(\phi,\psi) | \leq \mu(\Omega)^{\frac{p-2}{p}} \| \phi \|_{W_0^{1,p}(\Omega)} \| \psi \|_{W_0^{1,p}(\Omega)} .$$

Thus there exists $B \in \mathcal{L}(X, X^*)$ such that

$$a_2(\phi,\psi) = < B\phi, \psi >$$

for all $\phi, \psi \in W_0^{1,p}(\Omega)$. Clearly B is symmetric and using Poincare's inequality, we obtain

$$< B\phi, \phi > \geq c' \| \phi \|^2_{W_0^{1,p}(\Omega)}, \; c' > 0.$$

Thus we have satisfied hypotheses H(B).

Let $\hat{f} : I \times L^2(\Omega) \to L^2(\Omega)$ be defined by

$$\hat{f}(t,\phi)(z) = f(t,z,\phi(z)).$$

In this case, $H = L^2(\Omega)$. Thus $\hat{f}(t,\phi)$ is the Nemyckii operator corresponding to f, acting from $L^2(\Omega)$ into $L^2(\Omega)$ and so by Theorem 2.5 and Theorem 2.6 it satisfies hypotheses $H(f)$.

For the control space we put $Y = L^\infty(\Omega)$ and $U(t) = \{u \in L^\infty(\Omega) : \| u \|_\infty \leq \eta(t)\}$. Note that GrU (graph of U) $= \{(t,u) \in I \times L^\infty(\Omega) : u(t) \in U(t) \; a.e.\}$. Observe that the function $(t,u) \mapsto (\eta(t) - \| u \|_\infty)$ is measurable in t and continuous in u , thus jointly measurable. Hence $GrU \in B(I) \times B(L^\infty(\Omega))$ with $B(I)(resp. B(L^\infty(\Omega))))$, being the Borel $\sigma-$ field of I (resp. of $L^\infty(\Omega)$).Then by theorem 4.2 of Wagner [32] $U(\cdot)$ is measurable, while from hypotheses $H(U)$, we deduce that $t \to | U(t) | \in L^\infty_+$. So we have satisfied hypothesis $H(U)$.

Also let $\hat{\phi}_0 = \phi_0(\cdot) \in W_0^{1,p}(\Omega)$ and $\hat{\phi}_1 = \phi_1(.) \in L^2(\Omega)$ (see hypotheses H_0). Finally let $\hat{L} : I \times L^2(\Omega) \times L^2(\Omega) \times L^\infty(\Omega) \to \bar{R}$ be defined by

$$L(t,\phi,\psi,u) = \int_\Omega \hat{L}(t,z,\phi(z),\psi(z),u(z))dz, \;\; \phi,\psi \in L^2(\Omega), u \in L^\infty(\Omega).$$

Invoking theorem 1 of Pappas [31], we can find Caratheodory's integrands $\hat{L}_k : I \times \Omega \times \mathbf{R} \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}, k \geq 1$ (*i.e.* $(t,z) \mapsto \hat{L}_k(t,z,\phi,\psi,u)$ is measurable and $(\phi, \psi, u) \mapsto \hat{L}_k(t, \phi, z, \psi, u)$ is continuous), so that $\hat{L}_k \uparrow \hat{L}$ and $\varphi(t,z) - M(z)(\mid \phi \mid_R + \mid \psi \mid_R + \mid u \mid_R) \leq \hat{L}_k(t, z, \phi, \psi, u,) \leq k$ a.e. where $k \geq 1$.

Set $L_k(t, \phi, \psi, u) = \int_\Omega \hat{L}_k(t, z, \phi(z), \psi(z), u(z))dz$. It is easy to check that $t \mapsto L_k(t, \phi, \psi, u)$ is mesurable, while $(\phi, \psi, u) \mapsto L_k(t, \phi, \psi, u)$ is continuous, thus $L_k(\cdot, \cdot, \cdot, \cdot)$ is jointly measurable.

Furthermore , from the monotone convergence theorem, we get $L_k \uparrow L$, hence $L(\cdot, \cdot, \cdot, \cdot)$ is measurable.

Also from Blader [14], we know that

$(\phi, \psi, z) \mapsto L(t, \phi, \psi, z)$ is lower semicontinuous while $L(t, \phi, \psi, \cdot)$ is clearly convex and $\hat{\varphi}(t) - \hat{M}(\| \phi \|_{L^2(\Omega)} + \| \psi \|_{L^2(\Omega)} + \| u \|_\infty) \leq L(t, \phi, \psi, u)$, with $\hat{\varphi}(t) = \| (\varphi(t, \cdot) \|_{L^2(\Omega)}$ and $\hat{M} = \| M(\cdot) \|_\infty$.

So we have satisfyied hypotheses $H(L)$.

In this case, $X = W_0^{1,p}(\Omega)$, $H = L^2(\Omega)$ and $X^* = W^{-1,q}(\Omega)$. We know that (X, H, X^*) is an evolution triple, whith all embeddings being compact (Sobolev embedding theorem).

Defining $x(t) = \phi(t, \cdot)$, it easy to check that problem (P') of this example is a special case of the abstract problem (P).

__Theorem 4.1:__

__If__ *hypotheses* $H(k), H(f)_1, H(\eta), H_0, H(L)$ *hold,* __then__ *(P') admits an optimal pair* $[x, u] \in C(I, W_0^{1,p}(\Omega)) \times L^\infty(I \times \Omega)$ *such that*

$$\frac{\partial x}{\partial t} \in L^\infty(I, W_0^{1,p}(\Omega)) \cap C(I, L^2(\Omega)) \text{ and } \frac{\partial^2 x}{\partial t^2} \in L^q(I, W^{-1,q}(\Omega)).$$

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