Common Cause Failure Analysis
of
Redundant Systems

THESIS SUBMITTED
TO THE SCHOOL OF GRADUATE STUDIES AND RESEARCH
AS PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE
MASTER OF APPLIED SCIENCES

By
Viswanath C. Hassan

DEPARTMENT OF MECHANICAL ENGINEERING
UNIVERSITY OF OTTAWA

ISBN 0-315-68113-6

To My Parents

Acknowledgements

I would like to express my sincere gratitude to my supervisor Prof. B.S. Dhillon for his continuous encouragement, advice and supervision throughout the course of this work.

I wish to thank the Natural Sciences and Engineering Research Council of Canada for the financial support provided. Sincere thanks to the Department of Mechanical Engineering for the various teaching assistantships provided.

Finally, I wish to thank my family, back home, for those letters of moral support and encouragement, without which this work would not have been possible.

Abstract

This study is concerned with the reliability analysis of commonly used repairable and non-repairable redundant systems (with common-cause failures) such as parallel, k-out-of-n and standby with identical and non-identical units. Formulas for steady state system availability, system reliability and system mean time to failure are developed. The variation of steady state system availability, system reliability and system mean time to failure with common-cause failures is shown by means of plots for the above mentioned configurations. This study clearly shows that the occurrence of common-cause failures has a negative effect on system reliability parameters.

Contents

List of Figures

Chapter 1

Introduction

Modern technology has generated a tendency to design and manufacture equipment and systems of greater sophistication, complexity and capacity. Serious implications of unreliable behaviour of equipment and systems have led to the desire for higher reliability. Thus, reliability has become one of the vital ingredients in system planning, design, development and operation. Redundancy plays a dominant role in increasing a system's reliability, especially where the system reliability must be greater than that of the components used.

In classical reliability modelling, it is assumed that the redundant units fail independently. However, in a complex redundant system, the occurrence of common-cause failures easily violates the assumption of independent failure of redundant units. A common-cause failure is defined as any instance where multiple units fail due to a single underlying effect. When this type of failure occurs, all other failures are triggered to constitute a complete system failure. Some of the reasons for the occurrence of common-cause failures are design deficiencies, operator and maintenance errors, operating environment, and external catastrophe. A reliability analysis of redundant systems which does not take into consideration the occurrence of common-cause failures may lead to optimistic prediction in system reliability.

1.1 General Overview of Common-cause Failures

Generally, in the reliability analysis of any complex engineering system an often significant and sometimes dominant aspect is a phenomenon that is commonly known as 'Common-cause Failure'. The term Common-cause Failure (CCF) is widely employed in the technical community to describe events involving multiple failures that are the result of the same (single) cause. Equipment redundancy, which has been a long-standing practice to avoid system and function level failures, is a potential victim of this kind of event. This is because in a redundant system it cannot be assumed that the alternate channels of the system, or of its sub-systems, are completely independent. If such an assumption is made, it leads to unrealistic and optimistic values of predicted failure probability.

An important feature that makes common-cause failures hard to guard against is that the original connection between the particular units that failed almost in unison may not be obvious at first sight. The fundamental defect that brought about the common-cause failure of two or more units may have arisen in the design phase of the equipment or the system. It may be due to unsuspected interdependence between various subsystems or components in a system where safety was sought by building in redundancy. Due to this uncertainty there is a definite requirement for further study of this reliability aspect, which has become more onerous as the need for systems having high reliability has increased.

We have seen an explosive trend in technological evolution and complexity, (e.g. space crafts, computerization, power production and distribution, automization etc.) where the consequences of failure can become catastrophic and create international concern. One facet of this problem has received considerable attention recently. This is the possibility that product or mission failure may occur due to some form of 'common-cause' - that is, the simultaneous loss of multiple redundant paths due to an underlying common mechanism, fault, or phenomena. In such instances the avoidance measures may be the very victims of such an occurrence.

Systems using redundancy techniques can tolerate a certain number and/or types of failure while continuing to maintain the required relationship between input and output conditions. This is so when the failures of individual components are independent. However, since the assumption of statistically-independent failure of redundant units is easily violated in practice, it is necessary to include the occurrence of common-cause failures in the reliability analyses of redundant systems

1.1.1 Definition of a Common-cause Failure and Possible Reasons

A common-cause failure is defined as any instance where multiple units or components fail due to a single underlying effect. Some of the reasons for the occurrence of common-cause failures are

1. Equipment Design Deficiency : The failures included under this category are those which may have been overlooked during the design phase of the system and may be due to interdependence between the electrical and mechanical subsystems of a redundant system.

2. Operations and Maintenance Errors : Improper maintenance, improper adjustment or calibration, and carelessness are some of the reasons for this type of failure.

3. Functional Deficiency : Inadequacy of designed protective action and inappropriate instrumentation are the prime reasons for functional deficiency.

4. Common Manufacturer : The same design or fabrication errors may be present in all the units of the redundant system if they are procured from the same manufacturer, resulting in a potential common-cause failure.

5. Common External Power Source : The failure of all the units in the redundant system may be due to a common external power source.

6. External Normal Environment : Examples of typical extremes of environmental conditions which could cause common-cause failures are temperature, pressure, humidity, moisture, vibration, acceleration, stress, corrosion, and contamination.

7. External Catastrophe : In addition to the man-made sources, common-cause failures can be caused from natural phenomena to which the general environment is subjected. Examples of events that can cause common-cause failures, are fire, flood, earthquake, tornado, explosion, missiles, and chemical sources.

In some cases, instead of a total redundant system failure, which is the extreme case, the common cause may produce a less severe but common degradation of the redundant units. This might result in an increase of the joint probability of failure of the units. In this degradation state, the redundant unit may fail at a time later than the first unit failure. However, the second unit failure is considered to be dependent and coupled to the first unit failure. Comprehensive lists of literature on common-cause failures are presented in Refs. [53, 85].

1.2 Literature Review

General system reliability analysis assumes "independence" of the system units and purely random failure processes. These assumptions are very helpful because of the easy probabilistic calculations they make possible. However, these assumptions are very far from reality.

The unavoidable existence of more or less hidden dependency structures is the limiting factor which impedes a system to achieve unlimited reliability. The awareness of this problem and the need to implement adequate defenses in a system against the occurrence of common cause failures have encouraged several proposals for analysis procedures and mathematical models. A discussion of the models presented by various researchers is given below.

Epler, E.P. [100], in one of the earliest publications (1969), emphasized that if both the control system and the instrumentation fail in a nuclear power plant, a potentially hazardous situation exists. Similar, and particularly identical instruments, are susceptible to such common failure modes. Examination of simultaneous failure incidents in the reactor protection systems indicated that the use of identical channels of instruments was only marginally acceptable for protection alone. Identical channels in both the protection and control systems are unacceptable if protection against failures of the control channels is provided solely by the identical channels in the system.

Jacobs, I.M. [154], in 1970, emphasized the importance of taking into account common-cause failures for the analysis of system reliability. He described how a study was organized and ground rules were formulated in the common-mode failure study of a reactor protection system at General Electric. He classified the reasons for common-mode failures as

- functional deficiency
- maintenance error
- design deficiency
- external event.

The 1970s saw a steady increase in the number of publications addressing common-cause failures, indicating a growing concern in the increase in probability of failure due to CCF events.

Gangloff, W.C. [117], in 1972, explained the causative factors for common-cause failures and the possible preventive measures in order to emphasize that common-cause failure analysis is 'in'. Fleming, K.N. [103] presented a method for computing the reliability of redundant safety systems, considering both the independent and common mode types of failure, in 1975. The probability of failure of a typical diesel-generator emergency power system was computed and compared with the reliability predictions based on the assumption that all failures were independent, and the comparison showed a remarkable increase in the probability of system failure when common failure modes were considered. A study of the analyses on the causes and consequences of failure mode on abnormal occurrences submitted to United States Atomic Energy Council (USAEC) was presented by Taylor, J.R. [211] in 1975. This study indicated that a large portion of failures involved design, installation and operation errors, and an unexpectedly large portion of incidents involved multiple failures.

Gangloff, W.C. [118], in 1975, summarized the reliability techniques to include multiple failures from a common-cause using qualitative extensions of both the fault tree and reliability block diagrams. He observed that existing methods were qualitative in nature in addition to being preventive rather than predictive in nature. He extended the categories of common-cause failures and possible preventive measures from his earlier paper [117]. Taylor, J.R. [212], in his report, in 1975, gave the classification of common mode or coupled failures based on the data from abnormal reports on nuclear reactors. Apostolakis, G.E. [6], in 1976, proposed a theoretical investigation of the importance of common mode failures on the reliability of redundant systems.

The failures were assumed to be due to fatal shocks like earthquake, explosion etc. He also emphasized that since most reliability analyses of redundant systems do not include potential common-cause failures in the probability calculations, more sophisticated methods of analyses are required, because common-mode failures cannot be ignored.

Hayden, K.C. [133], in 1976, discussed a broad category of failure mechanisms that can cause common-mode failures in nuclear plant redundant systems and other high-reliability systems. Several categories of multiple failure mechanisms were proposed for reducing the probability of common-mode failures. He separated common-cause failures into component and systematic failures, and categorized them in terms of generic failure mechanisms. Worrell, R.B. and Burdick, G.R. [235], in the same year, discussed the qualitative evaluation of the system logic models and the advantages inherent in qualitative analysis. The common-cause analysis for studying the effect of common-causes failure on system behaviour was explained through the use of a computer program called COMCAN.

Dhillon, B.S., Proctor, C.L. [73], in 1977, discussed the reliability hazard rate and mean time to failure for parallel, k-out-of-n, parallel-series and bridge networks of two state identical components subjected to common-mode failures. It was shown from the plots that there would be a significant increase in the probability of redundant system failure when common-mode failures were considered. It also compared the reliability of independent and common-mode failure networks to the reliability obtained from conventional independent failure mode network.

In 1977, Epler, E.P. [99] discussed diversity and periodic testing in protecting against common-cause failures. He cited ten examples where diversity and frequent periodic testing would not assure success. It was also noted that out of ten failures only one was a common-cause failure discoverable by periodic testing. Wagner, D.P., Cate, C.L., Fussell, J.B. [224] proposed an extension of the existing methods of computer aided common-cause failure analysis by allowing analysis of the complex systems often encountered in practice.

Jolly, M.E., Wreathall, J. [159], in 1977, described the principles adopted by the

United Kingdom Electricity Board in tackling the hazards of common-mode failures in reactor safety equipment. It was pointed out on similar lines to [99] that diversity is not an absolute measure, but one which has varying degree of depth. It was concluded that there is no objective way of establishing the probability of common-mode failure for high integrity equipment and there is no substitute for the use of engineering experience in depth.

In 1978, Dhillon, B.S. [61] presented an optimal preventive maintenance policy for a system with common-cause failures using Markov technique for both parallel and k-out-of-n configurations. In the same year Johnson, J.W., Vesely, W.E. [158] used Marshall-Olkin based approach to estimate common-mode probabilities using the recorded failure data, thereby showing the importance of the usage of current data sources in identifying and quantifying common mode failure probabilities. Easterling, R.G. [94] discussed three types of statistical dependence involved: dependence among failure events themselves, dependence of failure events on the conditions under which an item is asked to perform, and dependence among these conditions. He also pointed out that many statements about common-mode failures have either been omitted or blurred with respect to these distinctions. Fleming, K.N., Raabe, P.H. [109] compared three methods to predict the reliability characteristics of redundant systems subject to independent and common-cause failures using Markov models, and showed that the beta factor method is theoretically consistent with alternative approaches. Dhillon, B.S [56] in the same year presented a mathematical model to find the availability of a two non-identical units system with common-cause failures and also [55] presented formulas for system mean life, variance of time to failure, hazard rate and reliability of parallel, k-out-of-n, parallel-series and bridge networks with common-cause failures. These formulas were for identical components characterized by Weibull time to failure density function and was an extension of [73]. Another article by Dhillon, B.S. [54] presented a repairable k-out-of-n three state device network Markov model with common-cause failures, in which a device was said to be a three state device if it operated normally in its operational mode, but failed in either opened or short (closed) modes.

Watson, I.A. in 1979 [228] introduced the limitations of reliability assessment with particular reference to common-mode failures by taking the major findings of NCSR's

(National Council for Safety Research) as a means of bringing focus and perspective to bear on the subject. Heikkila, M. [135] reviewed some of the requirements that a common-mode failure should fulfill and presented a mathematical model for common-mode failures. In the same year, Smith, A.M. [205] critically reviewed the common-cause failure definitions and the inclusion of some types of secondary and/or cascade failures as common-cause failures. Chung, W.K. [36] presented a model representing a N-unit redundant system of three state devices with common-cause failures. Dhillon, B.S., et. al. [84] derived formulas for system reliability, MTTF and system hazard rate for a redundant system with 4-units in parallel. Watson, I.A., Edwards, G.T. [230] tried to rectify the lack of a uniform and comprehensive definition for common-cause failure by proposing a definition to explicitly summarize significant characteristics of common-cause failures. They also proposed a classification system for common-cause failures. In the same year, Chung, W.K. [42] extended his earlier work [36] to include k-out-of-n redundant system with replacement and common-cause failures. He also presented equations for steady-state probability, system availability, and frequency of encountering a particular state. Dhillon, B.S. [60] presented two detailed models, one with two identical three state redundant system and the other with two non-identical two state active unit parallel system with common-cause failures. The same researcher presented [53] an extensive bibliography on common-cause failures from journals and conference proceedings. The period between 1965-1978 was covered in this publication.

During 1980, after much debate about a universally accepted definition for common cause failures, leading to a divergence of interpretation and confusion that obscured meaningful progression in common-cause failure analysis, Smith, A.M. and Watson, I.A. [206] identified a broad spectrum of definitions used by various researchers on common-cause failures and proposed a new definition. On the same lines Watson, I.A. [226] described the historical concepts of common-cause/mode failures and their recognition, leading to the discussion of present day high reliability system management, design, maintenance and operation. Vaurio, J.K. [215] developed analytical equations for the unavailability of m-out-of-n systems, after taking into consideration common-cause failures and failures that escape detection by regular periodic tests. An overview of a unique approach developed to address susceptibility of electrical control systems to combined multiple failures of components and operators was given by

Rankin, J.P. [195].

Hudson, J.M. and Jean. G. [149] in 1981 proposed a methodology for the prediction of accident sequence probabilities involving hazardous operations resulting from the events that can cause simultaneous failures of several systems whose random failures would otherwise be independent. Rankin, J.P. [194] extended the principle [195] to identify the susceptibility of redundant systems to specific common-cause failures. Singh, C. [200] analyzed a TMR (Tripler Modular Redundancy) computer configuration and showed that the effect of repair and common-mode failures on the reliability and mean up time of the system are significant. A mathematical model for predicting the reliability of a non-identical three state active units redundant system with common-cause failures and one standby unit was developed by Chung, W.K. [37]. Dhillon, B.S. [58] presented an availability analysis of a two identical unit redundant system where each unit could fail in n-mutually exclusive failure modes. Govil, A.K. [123] investigated the reliability of a stand-by-system which experiences dual failure modes: first, failure resulting from the change in operating characteristics, and second, failure resulting due to common-cause. Chung, W.K. extended his model [37] to accommodate K-out-of-N:G redundant system with common-cause failures in [38]. Billington, R., et. al., [25] analyzed and compared several possible common-cause outage models and discussed the effects of common-cause failure rate and repair rate on system reliability. The same researchers [22] combined the effects of common-mode failures and adverse weather conditions on system reliability.

In 1982, Singh, C., Reza Ebrahmian, M. [201] extended their earlier model [202] and discussed modeling of common-mode failures in the reliability evaluation of transmission lines. Specifically, the effect of probability distribution of state residence times on reliability indices were investigated. Govil, A.K. [122] considered a complex system comprising of two types of systems. Type I consisted of N components in series and type II consisted of standby redundancy. It was also taken into consideration that a standby redundant unit may fail due to completed shelf-life. Two types of failure modes were considered one due to performance characteristics and the other due to common-cause. Under these conditions the availability of the system was investigated. In the same year Dhillon, B.S., Natesan, J. [72] presented a stochastic model which represented 'n' number of active redundant pulverizer system with common-cause failures.

Hirschmann, H., Nicolescu, T., Weber, R. [147], in 1983, proposed a model for the steady state unavailability of a k-out-of-n system with constant transition rates, with total repair from every failed state and a common-cause transition into a totally failed state due to common-mode failures. Matsuoka, T. [171] constructed a model in order to treat the effects of extreme environmental conditions and the common-mode failure on system reliability. In this model, he considered the failure rate as a function of cause, of severity of cause, of cause acting duration, as well as the failure mode. Virolainen, R. [220] discussed a connection between common-cause failures, statistical independence of components and the calculation of uncertainty. He also compared this idea with the statistical correlation model of Hartung, J. [132]. Platz, O. [187] presented a Markov model for common-cause failures and showed that his model covers several of the models that have been used to describe dependencies between the failure of components in redundant systems. Some results of the study, sponsored by the Electric Power Research Institute to develop an improved data classification scheme were summarized by Fleming, K.N. et. al. [108]. In addition, this study provided insight to the importance of interface between data analysis and common-cause modeling. Evans, M.G.K., Parry, G.W., Wreathall, J. [101] discussed the problem of quantitative treatment of common-cause failures. In relation to the data available, two models of common-cause failure were discussed with respect to estimating model parameters.

In 1984, Goel, L.R., Gupta, R. Singh, S.K. [120] computed several reliability characteristics from their model, which was composed of two non-identical active parallel units with one cold standby. In addition, each unit was assumed to have N components with each having a constant failure and repair rates. This model was a generalization of the model presented by Dhillon, B.S. [59]. Chung, W.K. [35] presented the reliability analysis of a K-out-of-N:G redundant system with M units acting as cold standby with 'r' repair facilities and common-cause failures. Natesan, J., Jardine, A.K.S. [181] studied a single-server n-unit active redundant pulverizer system with common-cause failures on similar grounds as that of [72] but under the assumption that the repair times distribution was general, while the failure time distribution was exponential. In another interesting paper by Heising, C.D., Guey, C.N. [137] the difference in results that can arise from applying the beta factor method and multiple dependent fraction method for calculating the system unavailability due to common-cause failures was demonstrated.

During 1985, Games, A.M., Breewood, M., Amendola, A., Martin, P., Keller, A.Z. [115] investigated the data from the European Reliability Data System (ERDS) for common-cause failures in nuclear power plants and demonstrated that different types of common-cause failures necessitated different search algorithms. Garg, R., Goel, L.R. [119], in one of the rare cost analysis papers, presented a mathematical model to predict the cost involved to run a n-component single unit system which could fail in n-mutually exclusive ways or due to a common-cause. Bickel, J.H. and Caivano, J.D. [21] provided a model for common-cause failure point estimates for usage in fault tree quantification. The European Reliability Data System's component event data bank was further investigated by Games, A.M., Amendola, A., Martin, P. [114] for common-cause failure analysis. Based on this experience a classification system was also proposed, defining multiple failure event space, wherein each category defined causal, modal, temporal and structural links between failures, thereby showing that the classification scheme and the search algorithm are useful organizational tools in common-cause failure studies. Page, R.J., et. al. [182] discussed the steps followed to identify the effect of common-cause initiators which could affect the functioning of a triply redundant reactor scram system (R.S.S) and to subsequently modify the design of the R.S.S. so that the effect is minimized. In the same year, Virolainen, R. [221] discussed the comparison of Fleming, K.N.'s β-factor method [103] and Hirschberg, S. [143] on similar lines compared the methods for quantitative analysis of common-cause failures. Guey, C.N., Heising, C.D. [126] focussed on the application of Inverse Stress-Strength Interference (ISST) technique for estimating common-cause failure probability and this method was compared with other commonly used models.

Pulkkinen, U., Hirschberg, S. [145], in 1985, compared the studies concerned with the estimation of common-cause failure contributions for systems of large diesel generators used in nuclear power plants in the United States, Finland and Sweden. The article described the discrepancies and similarities among the results and also provided possible reasons for the discrepancies. Wright, R.I. [239] examined the examples of common-cause failures in practice and suggested how they can be modelled for reliability studies. The results of a survey of common-cause failures in redundant industrial computer systems showed that common-cause failures rates are much higher than is generally expected in hardwired systems. Sharma, G.C., et. al. [199] studied a two unit (identical) parallel system which facilitated preventive maintenance and two types of repair, those being, regular and occasional, with common-cause failures. In the same year Dhillon, B.S., Rayapati, S.N. [74] presented two Markov models for evaluating the reliability and availability of two unit (non-identical) parallel system with human errors and common-cause failures.

In 1986, Fleming, K.N., Mosleh, A., Deremer, R.K. [106] discussed the advances made in the development of a systematic approach for incorporating common-cause events in reliability evaluations. They also described a general frame work for system level common-cause failure analysis. In the same year Atwood, C.L. [11] presented the Binomial Failure Rate common-cause model for estimating the rate of simultaneous failure of more than one component of a system and showed how the important parameters are estimated. The statistical and computational ideas were emphasized in this article. Yun, W.Y., Bai, D.S. [243] considered the problem of finding out the optimum number of units (redundant) needed in a parallel system with common-cause failures. Using cost models considering both common-cause as well as random failures, it was shown that the number of redundant units for minimizing the expected cost rates were finite and unique. It was also shown that the presence of common-cause failures tend to reduce the optimum number of units using the expected cost rate as the criterion for optimization. Dhillon, B.S., Rayapati, S.N. [77] proposed four different Markov models for performing reliability and availability analysis of three unit redundant system with common-cause failures and human error. Harris, B. [130] described various stochastic models for common-cause failures modelled by researchers and proposed a model for common-cause failure along the lines of stress-strength models in reliability. This model was an extension of common-load model of Mankamo, T. [166] and [167], in addition to being related to binomial failure rate model.

Dhillon, B.S., Rayapati, S.N. [76] in 1987 presented two models representing redundant systems with common-cause failures, human errors and partially energized standby systems. The effect of varying human errors and common-cause failures on system reliability, mean time to failure and steady state probabilities were also demonstrated by means of plots.

Chung, W.K. [40] proposed a mathematical model for repairable parallel system with standby units involving human errors and common-cause failures. Because of the need for a standard method for treating common-cause failures, so that a consistent approach could be followed, Martin, B.R., Wright, R.I. [170] proposed a model by combining the β-factor model with limiting values so that the model fulfills the requirement of acceptability, applicability, flexibility and consistency. The MOCUS-Bacfire Beta Factor (MOBB) code developed for fault tree analysis was illustrated by Heising, C.D., Luciani, M. [138] to be superior to other common-cause procedures. In the same year Humphreys, R.A. [152] allowed for the estimation of a Beta factor through his model which was intended to aid both design and assessment. The mathematical model of Mahmoud, M., Mokhles, M.A., Saleh, E.H. [165] applied the supplementary variable technique to investigate a mathematical model for a system with non-identical parallel redundant active units with a cold standby and with common-cause failures. This model was an extension of a similar model developed by Dhillon, B.S. [60] but with a difference in the repair policy. It was assumed that whenever a unit fails it would be repaired unlike in [60], where repair was carried out only when all the units including the standby unit failed.

In the subsequent year, 1988, Hokstad, P. [148] dealt with modeling of common-cause failures in redundant systems. He proposed a new parametric model for common-cause failures, denoted as the Random Probability Shock (RPS) model. He also discussed two of the well known models for common-cause failures, the β-factor and Binomial Failure Rate model, and evaluated their applicability and shortcomings. It was also shown that the β-factor and Binomial Failure Rate models were special cases of the RPS model. In the same year Dhillon, B.S., Rayapati, S.N. [75] presented three stochastic models representing standby redundant systems with common-cause failures.

In 1989, Singh, J. [203] formulated a problem using supplementary variable technique for a system with N operating units and M warm standby units having 'r' repair facilities. The system under consideration was assumed to fail either when there was a common-cause failure or when there were only K-1 good units. A trinomial failure rate model that allowed careful and elaborate treatment of common-cause failures was developed by Han, S.G., et. al. [129]. The trinomial failure rate model was developed to explicitly analyze ambiguous events such as partial failures, incipient failures, and potential failures.

Chung, W.K. [41] presented a reliability analysis of repairable and non-repairable systems with N redundant units, 'r' repair facilities and common-cause failures. In the same year Dhillon, B.S. [70] presented a stochastic analysis of a parallel system with common-cause failures and human error which are critical, and Yuan, J., et. al. [242] evaluated an efficient method to calculate system reliability with common-cause failures when the system and its associated class of common-cause failures were both arbitrary.

Dhillon, B.S., Viswanath, H.C. [85], in 1989, presented an extensive list of selective publications on common-cause failures covering the period between 1965-1989. This publication included those references in [53]. The same authors presented [86] the common-cause failure analysis of a two identical unit redundant system using Markov state space technique. The system reliability, steady-state availability, mean time to failure and variance of time to failure formulas were presented. In addition, system reliability, availability, and mean time to failure plots were shown. Furthermore, a common-cause failure analysis of a two non-identical unit parallel system was presented [87] by Dhillon, B.S. and Viswanath, H.C. The expressions for system reliability, time dependent system availability, and system mean time to failure were developed and it was shown that common-cause failures have a negative impact on system reliability by means of plots.

The same authors, in a recent publication, [88] presented three newly developed Markov models for identical unit parallel, k-out-of-n and standby arrangements. Generalized expressions for system reliability and mean time to failure were developed. In addition, the variation in system reliability and mean time to failure with common-cause failures was shown. Dhillon, B.S., Viswanath, H.C. in their publications [86], [87] and [88] assumed, unlike the earlier researchers, that a common-cause failure can occur when the system is in any one of its good states.

1.3 Objectives

The main objective of this study is to analyze the effects of common-cause failures on system availability, system reliability, and system mean time to failure. To meet this objective three of the most commonly used repairable and non-repairable systems, such as, Parallel, k-out-of-n and Standby are considered. Detailed reliability analyses using Markov state space technique has been carried out both when the system is comprised of identical units as well as when they are non-identical.

The following is the main difference between the models considered in this study and those proposed by previous researchers :

- The models proposed by previous researchers consider the occurrence of common-cause failures only when at least two units in the system are good. The models which are analyzed in this study consider the common-cause failure to occur from any good system state, that is, a common-cause failure can occur irrespective of the number of good units in the system.

Under the above mentioned condition, generalized expressions for system reliability and system mean time to failure have been developed for identical unit parallel, k-out-of-n and standby systems in addition to the non-identical unit standby system. Since the number of terms become too large for non-identical units parallel and k-out-of-n systems, generalized expressions have not been developed for these cases. However, specific cases have been considered under these configurations. Furthermore, the variation of steady state system availability, system reliability and mean time to failure with common-cause failures is shown by means of plots for all the models.

1.4 Organization of Study

This study is presented in 6 chapters.

Chapter 1 gives a brief introduction to common-cause failures and presents the literature review.

Chapter 2 is concerned with the reliability analysis of repairable and non-repairable parallel system with identical as well as non-identical units. In addition to the generalized expressions for system reliability and mean time to failure, specific case models are also presented in this chapter.

In Chapter 3 a repairable and non-repairable k-out-of-n system is considered and the effects of common-cause failures on such systems are studied. Generalized expressions for system reliability and mean time to failure are developed and specific case models such as (n-1)- out-of-n and (n-2)-out-of-n are analyzed.

Chapter 4 deals with both repairable and non-repairable standby systems. Expressions for system reliability and mean time to failure are presented for both identical unit as well as non-identical unit standby systems.

Chapter 5 discusses the results of Chapters 2, 3, and 4. The constants associated with the various models are defined in Appendices A, B and C.

Chapter 6 gives a list of most of the references on common-cause failures published between 1965 and 1989.

Chapter 2

Parallel System

This system represents a redundant unit network whose active units are connected in parallel. This means that if any one of the components fails, the system would still operate successfully. In other words, at least one component is required for the system success. This type of redundancy is widely used in industries. A typical example of such a system is an aircraft whose two independent active engines, enable it to fly normally when at least one engine is functioning successfully. The reliability analysis of such a system with identical as well as non-identical units with common-cause failures is presented in the proceeding sections.

2.1 Identical Unit Parallel System

This section presents the reliability analysis of a repairable as well as non-repairable identical unit parallel system using Markov state space technique. At time $t = 0$ all the units start working simultaneously. At least one of the units must function normally to ensure the overall system success. The failure rate of each unit is independent of time. Furthermore, the occurrence of a common-cause failure can cause the total system to fail. A common-cause failure can occur when the system is in any one of its operating states (eg. $0, 1, 2, \ldots, n-1$).

The following repair policies are considered in the analysis of identical unit parallel system :

1. Type I Repair : In this type of repair policy, in addition to the partially failed (1

unit failed, n-1 units operating; 2 units failed, n-2 units operating etc.) system being repaired to its previous state and to state 0, the completely failed system is also repaired back to state 0. Furthermore, the completely failed system due to the occurrence of a common-cause failure at state n is also restored back to state n ($n = 1, 2, 3, \ldots, n-1$).

2. Type II Repair : Under this repair policy the partially failed system (1 unit failed, n-1 units operating; 2 units failed, n-2 units operating etc) is repaired to its previous state and to state 0, but the completely failed system is never repaired.

Assumptions

The following assumptions are associated with all the models under this configuration :

1. The system has n identical and active units.
2. A system can fail either due to a common-cause failure or other failures.
3. Common-cause and other failures are statistically independent.
4. A common-cause failure can occur irrespective of the number of units operating in the system.
5. The system is said to be down when all the units in the system are non-operative.
6. Common-cause and other failure rates are constant.
7. Repair rates are constant.
8. A repaired unit is as good as new.

Notation

The following symbols are associated with all the models under this section :

n	Number of units in the system.
t	time.
λ_i	Constant failure rate from the $(i-1)^{th}$ system state; for $i = 1, 2, 3, \ldots, n$.

μ_i	Constant repair rate from the i^{th} system state; for $i = 1, 2, 3, \ldots, n$.
μ_{io}	Constant repair rate from the i^{th} system state to system state 0; for $i = 2, 3, 4, \ldots, n$.
$\mu_{c_{i+1}}$	Constant repair rate of the failed system from state $n+1$ to state i; for $i = 0, 1, 2, \ldots, n-1$.
$\lambda_{c_{i+1}}$	Constant common-cause failure rate of the system from state i; for $i = 0, 1, 2, 3, \ldots, n-1$.
j	j^{th} state of the system for $j = 0, 1, 2, 3, \ldots, n+1$.
$p_j(t)$	Probability that the system is in state j at time t; for $j = 0, 1, 2, 3, \ldots, n+1$.
$R(t)$	Reliability of the system in $[0, t]$.
s	Laplace transform variable.
CCF	Common-cause failure.
$MTTF$	System mean time to failure.
AV_{ss}	Steady state system availability.
UV_{ss}	Steady state system unavailability.
$\dot{p}_j(t)$	Derivative of $p_j(t)$ with respect to time t; for $j = 0, 1, 2, \ldots, n+1$.
p_i	Steady state probability, that the system is in state i; for $i = 0, 1, 2, \ldots, n+1$.

2.1.1 Parallel System With Type I Repair

The state space diagram for an Identical unit parallel system is shown in Figure 2.1. The numerals plus a letter in the boxes of the figure denote the system state numbers.

The corresponding differential equations for the model described in Figure 2.1 are

$$\dot{p}_0(t) = -(\lambda_1 + \lambda_{c_1})p_0(t) + \mu_1 p_1(t) + \sum_{i=2}^{n} \mu_{io} p_i(t) + \mu_{c_1} p_{n+1}(t) \tag{2.1}$$

$$\dot{p}_1(t) = \lambda_1 p_0(t) - (\mu_1 + \lambda_2 + \lambda_{c_2})p_1(t) + \mu_2 p_2(t) + \mu_{c_2} p_{n+1}(t) \tag{2.2}$$

$$\begin{aligned} \dot{p}_r(t) &= \lambda_r p_{r-1}(t) - \{\mu_{ro} + \mu_r + \lambda_{r+1} + \lambda_{c_{r+1}}\} p_r(t) + \\ &\quad (\mu_{r+1}) p_{r+1}(t) + (\mu_{c_{r+1}}) p_{n+1}(t) \end{aligned} \tag{2.3}$$

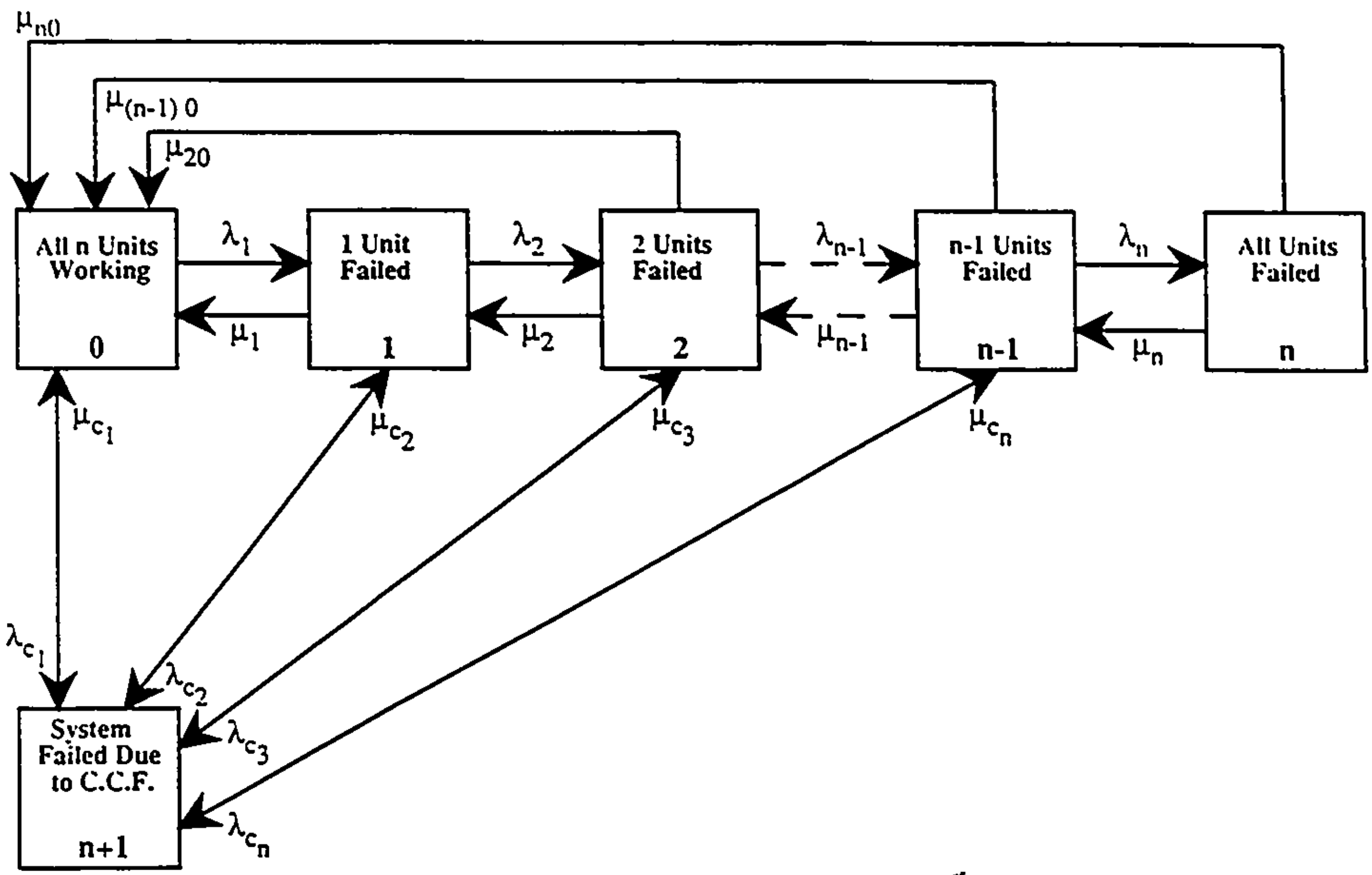

Figure 2.1: State Space Diagram for a n Identical Unit Parallel System

for $r = 2, 3, 4, \ldots, n-1$

$$\dot{p}_n(t) = \lambda_n p_{n-1}(t) - (\mu_n + \mu_{n_c}) p_n(t) \tag{2.4}$$

$$\dot{p}_{n+1}(t) = \sum_{i=0}^{n-1} \lambda_{c_{i+1}} p_i(t) - \{\sum_{i=1}^{n} \mu_{c_i}\} p_{n+1}(t) \tag{2.5}$$

At time $t = 0$, $p_0(0) = 1$ and all other initial condition probabilities are equal to zero.

By setting the derivatives of Equations (2.1) – (2.5) equal to zero and utilizing the relationship $\sum_{i=0}^{n+1} p_i = 1$ lead to the steady state probabilities.

The steady state system availability and unavailabilities can be determined by using the following expressions :

$$AV_{ss} = \sum_{i=0}^{n-1} p_i$$
$$UV_{ss} = \sum_{i=n}^{n+1} p_i$$

The steady state system availability plots for specified values of model parameters (from special case models) are shown in Figure 2.2. The plots show that the steady state system availability increases with increase in number of units in the parallel system. In addition, it may also be noted that an increase in common-cause failures results in a decrease in steady state system availability. The plots of steady state system unavailability are shown in Figure 2.3.

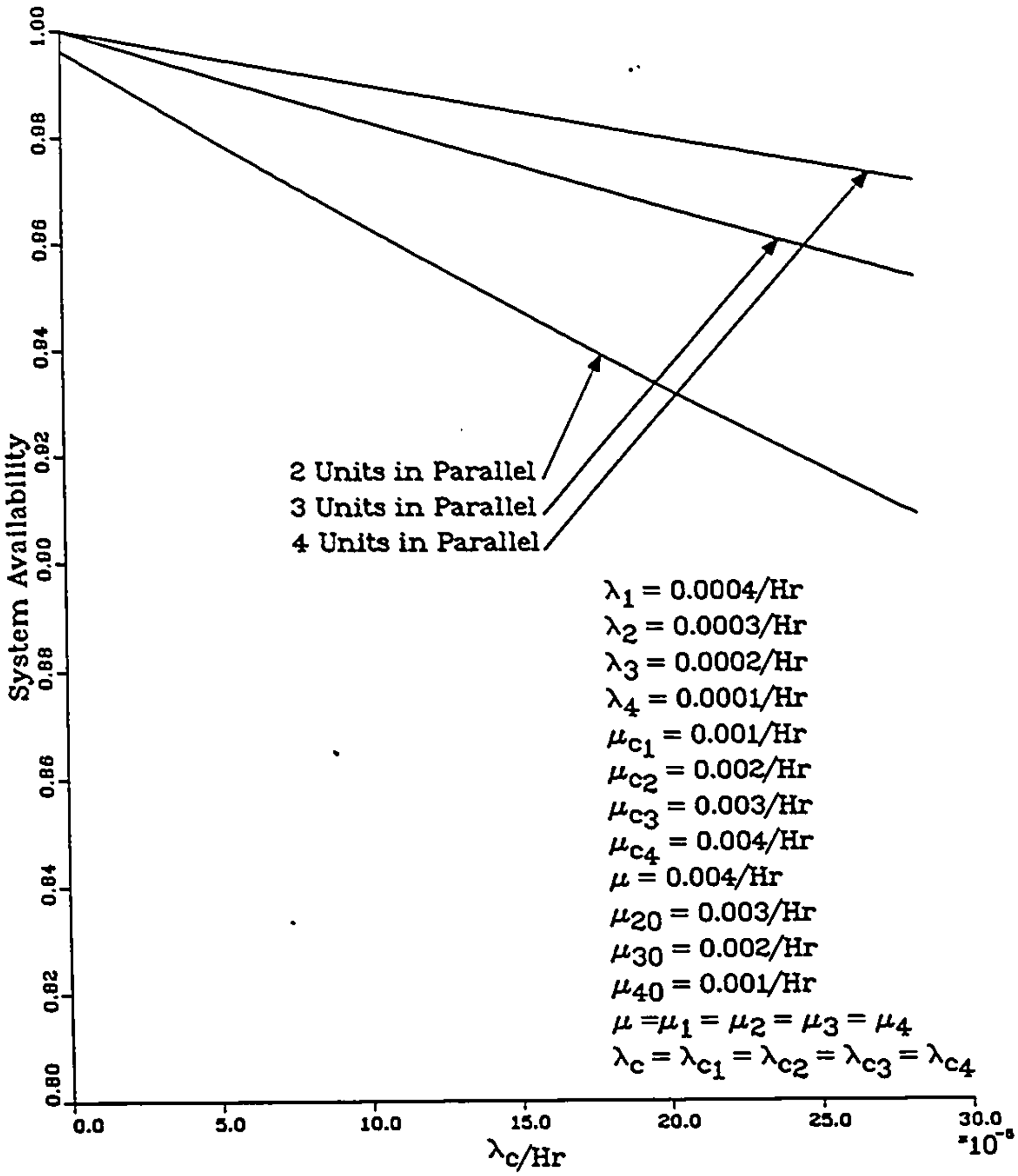

Figure 2.2: Typical Steady State System Availability Plots for an Identical Unit Parallel System with Type I Repair

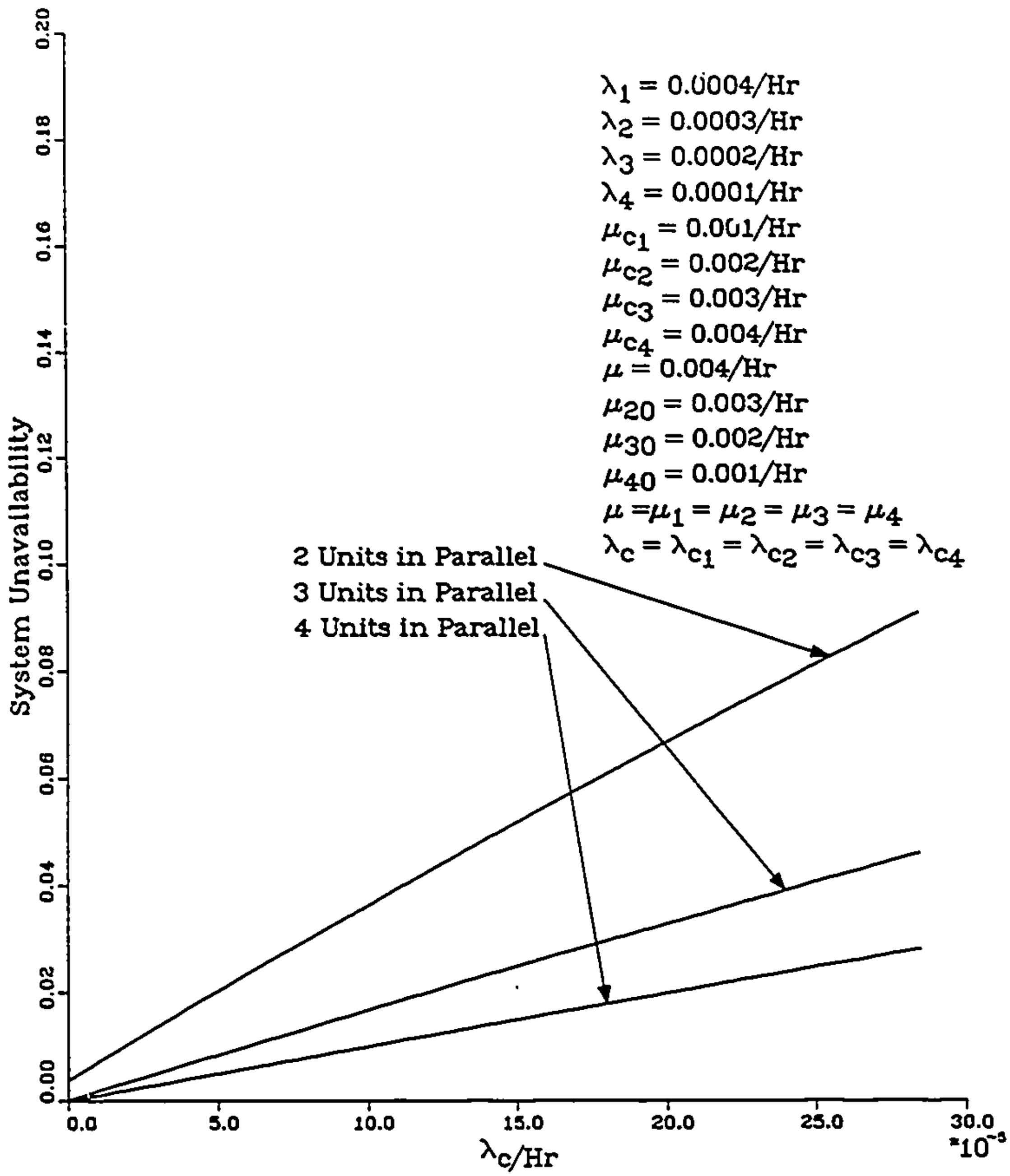

Figure 2.3: Typical Steady State System Unavailability Plots for an Identical Unit Parallel System with Type I Repair

Special Case Model I

Setting $n = 2$ in Figure 2.1 results in the following system of differential equations for a 2 identical unit parallel system :

$$\dot{p}_0(t) = -(\lambda_1 + \lambda_{c_1})p_0(t) + \mu_1 p_1(t) + \mu_{20}p_2(t) + \mu_{c_1}p_3(t) \quad (2.6)$$

$$\dot{p}_1(t) = \lambda_1 p_0(t) - (\mu_1 + \lambda_2 + \lambda_{c_2})p_1(t) + \mu_2 p_2(t) + \mu_{c_2}p_3(t) \quad (2.7)$$

$$\dot{p}_2(t) = \lambda_2 p_1(t) - (\mu_2 + \mu_{20})p_2(t) \quad (2.8)$$

$$\dot{p}_3(t) = \lambda_{c_1}p_0(t) + \lambda_{c_2}p_1(t) - (\mu_{c_1} + \mu_{c_2})p_3(t) \quad (2.9)$$

At time $t = 0$, $p_0(0) = 1$ and all other initial condition probabilities are equal to zero.

By setting the derivatives of the above equations equal to zero and by using the relationship $\sum_{i=0}^{3} p_i = 1$, we get the following steady state probabilities :

$$p_0 = \frac{\mu_{c_1}\lambda_2\mu_{20} + \mu_{c_2}\mu_1\mu_2 + \mu_{c_1}\mu_1\mu_2 + \mu_{c_2}\mu_1\mu_{20} + \mu_{c_1}\mu_1\mu_{20}}{D_1} + \frac{\mu_{c_1}\lambda_{c_2}\mu_2 + \mu_{c_1}\lambda_{c_2}\mu_{20} + \mu_{c_2}\lambda_2\mu_{20}}{D_1} \quad (2.10)$$

where the constant D_1 is defined in Appendix A.

$$p_1 = \frac{(\mu_2 + \mu_{20})(\lambda_1\mu_{c_2} + \lambda_{c_1}\mu_{c_2} + \lambda_1\mu_{c_1})}{D_1}$$

$$p_2 = \frac{\lambda_2(\lambda_1\mu_{c_2} + \lambda_{c_1}\mu_{c_2} + \lambda_1\mu_{c_1})}{D_1}$$

$$p_3 = \frac{\lambda_{c_1}\mu_1\mu_2 + \lambda_{c_1}\mu_1\mu_{20} + \lambda_{c_1}\lambda_2\mu_{20} + \lambda_{c_1}\lambda_{c_2}\mu_2 + \lambda_{c_1}\lambda_{c_2}\mu_{20} + \lambda_1\lambda_{c_2}\mu_2 + \lambda_1\lambda_{c_2}\mu_{20}}{D_1}$$

The system steady state availability and unavailability are given by

$$AV_{ss} = p_0 + p_1 \quad (2.11)$$

$$UV_{ss} = p_2 + p_3 \quad (2.12)$$

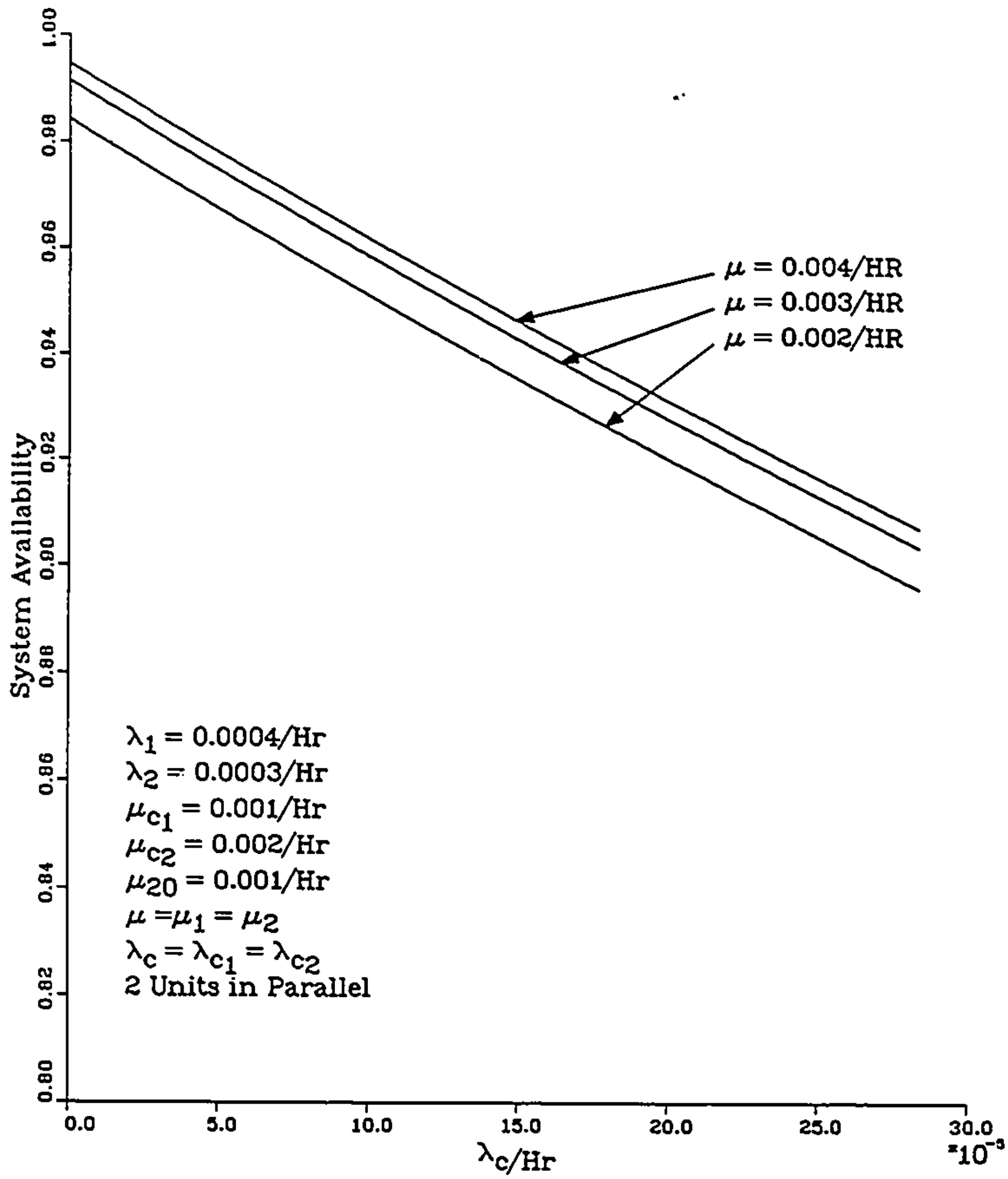

Figure 2.4: Steady State System Availability Plots for Special Case Model I ($n = 2$) with Type I Repair

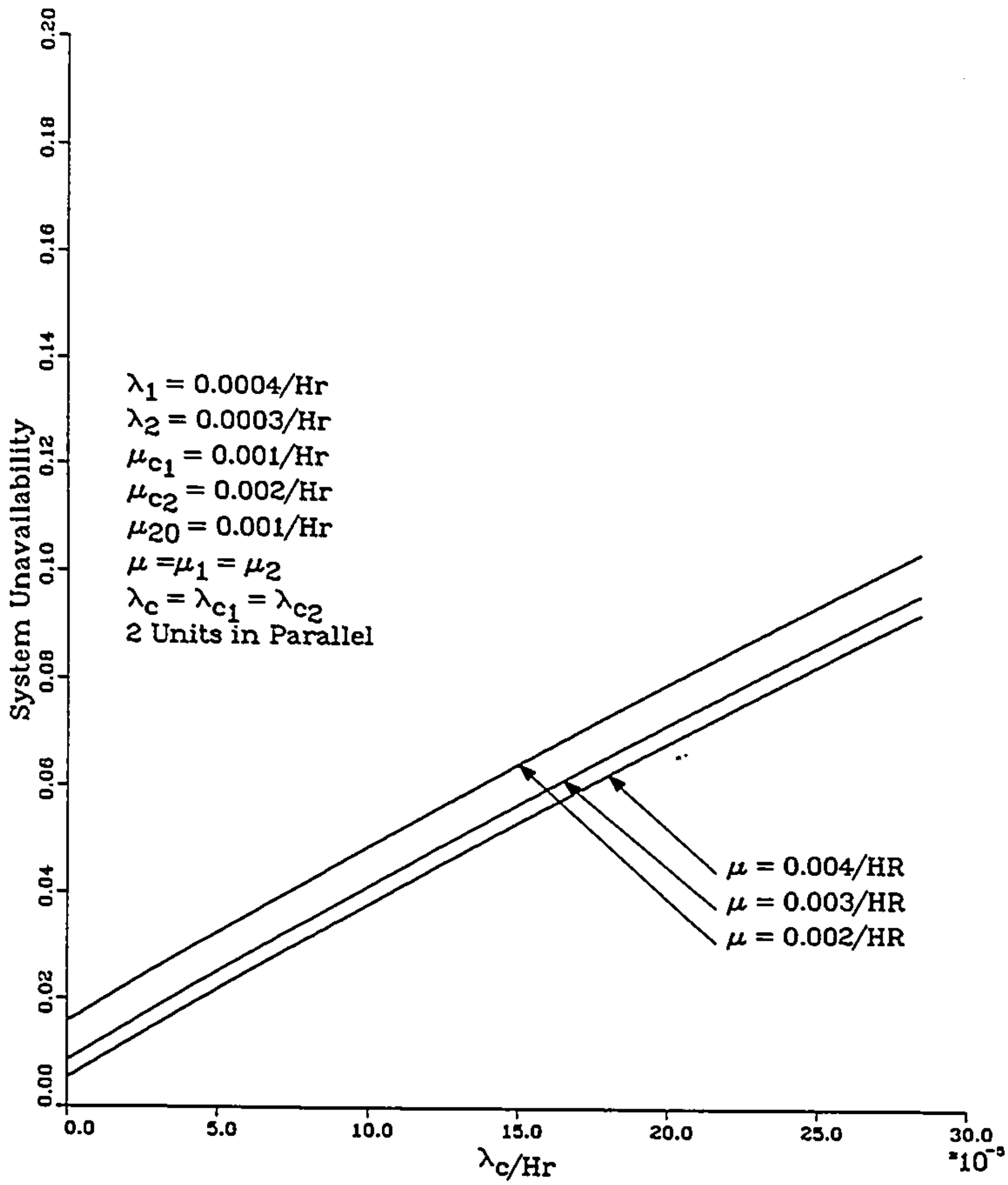

Figure 2.5: Steady State System Unavailability Plots for Special Case Model I ($n = 2$) with Type I Repair

The plots for the Equations (2.11) and (2.12) for specified values of model parameters are given in Figures 2.4 and 2.5 respectively. It can be clearly seen from the plots that the steady state system availability decreases with increasing values of common-cause failures. It can also be noted that the steady state system availability increases with an increase in the repair rate. The increase in steady state system unavailability with increase in the number of common-cause failures can be noted from Figure 2.5.

Special Case Model II

The following system of differential equations is obtained by setting $n = 3$ in Figure 2.1 :

$$\dot{p}_0(t) = -(\lambda_1 + \lambda_{c_1})p_0(t) + \mu_1 p_1(t) + \mu_{20}p_2(t) + \mu_{30}p_3(t) + \mu_{c_1}p_4(t) \quad (2.13)$$

$$\dot{p}_1(t) = \lambda_1 p_0(t) - (\mu_1 + \lambda_2 + \lambda_{c_2})p_1(t) + \mu_2 p_2(t) + \mu_{c_2}p_4(t) \quad (2.14)$$

$$\dot{p}_2(t) = \lambda_2 p_1(t) - (\mu_2 + \mu_{20} + \lambda_3 + \lambda_{c_3})p_2(t) + \mu_3 p_3(t) + \mu_{c_3}p_4(t) \quad (2.15)$$

$$\dot{p}_3(t) = \lambda_3 p_2(t) - (\mu_3 + \mu_{30})p_3(t) \quad (2.16)$$

$$\dot{p}_4(t) = \lambda_{c_1}p_0(t) + \lambda_{c_2}p_1(t) + \lambda_{c_3}p_2(t) - (\mu_{c_1} + \mu_{c_2} + \mu_{c_3})p_4(t) \quad (2.17)$$

At time $t = 0$, $p_0(0) = 1$, $p_1(0) = 0$, $p_2(0) = 0$, $p_3(0) = 0$ and $p_4(0) = 0$.

Setting the derivatives of Equations (2.13) – (2.17) equal to zero and using the relationship $\sum_{i=0}^{4} p_i = 1$, results in the following steady state probabilities :

$$p_0 = \frac{D_2}{D_7} \quad (2.18)$$

$$p_1 = \frac{D_3}{D_7} \quad (2.19)$$

$$p_2 = \frac{D_4}{D_7} \quad (2.20)$$

$$p_3 = \frac{D_5}{D_7} \quad (2.21)$$

$$p_4 = \frac{D_6}{D_7} \quad (2.22)$$

The constants D_2, D_3, D_4, D_5, D_6 *and* D_7 are defined in Appendix A.

The steady state system availability and steady state system unavailability expressions are given by

$$AV_{ss} = p_0 + p_1 + p_2 \quad (2.23)$$

$$UV_{ss} = p_3 + p_4 \quad (2.24)$$

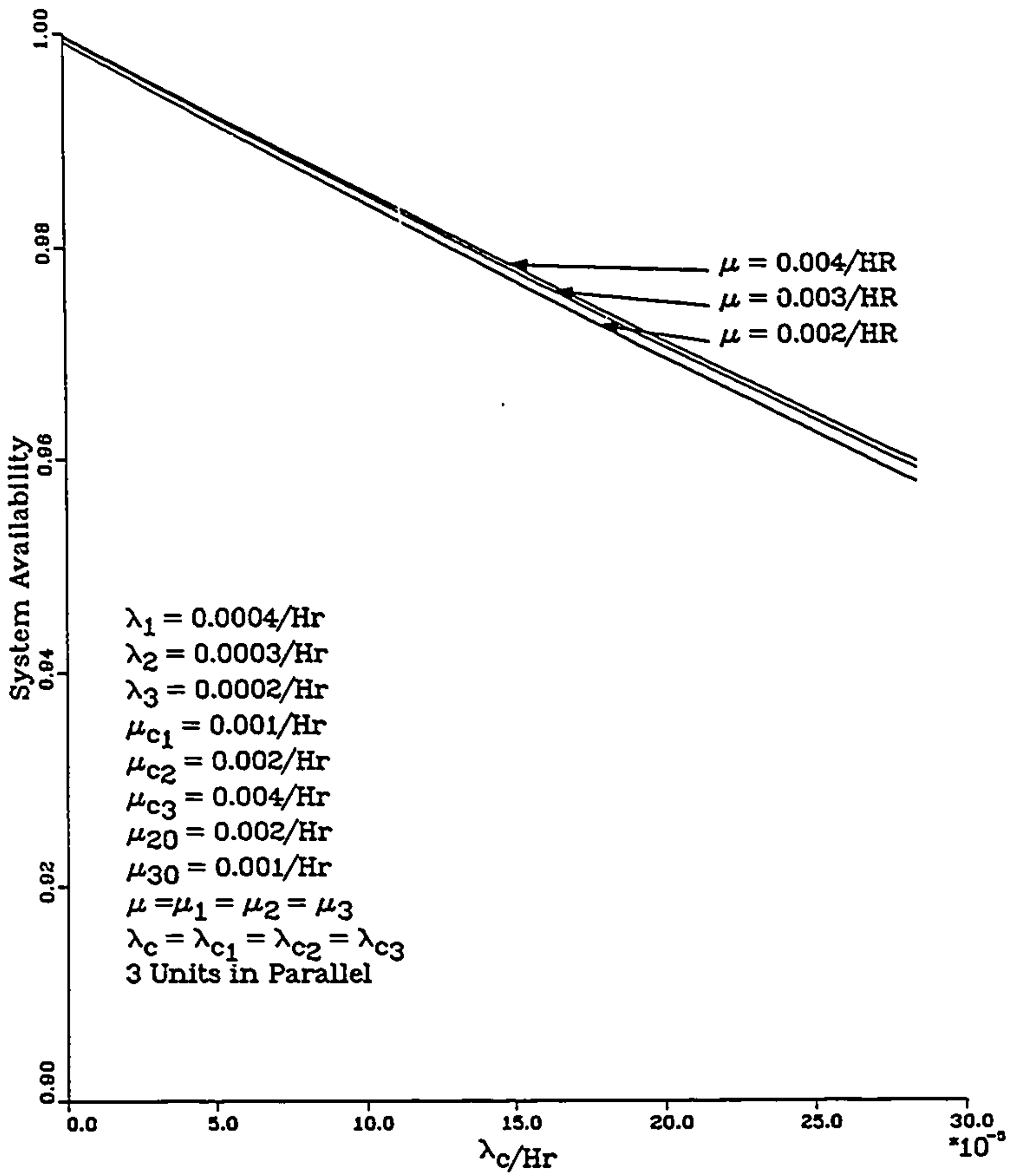

Figure 2.6: Steady State System Availability Plots for Special Case Model II ($n = 3$) with Type I Repair

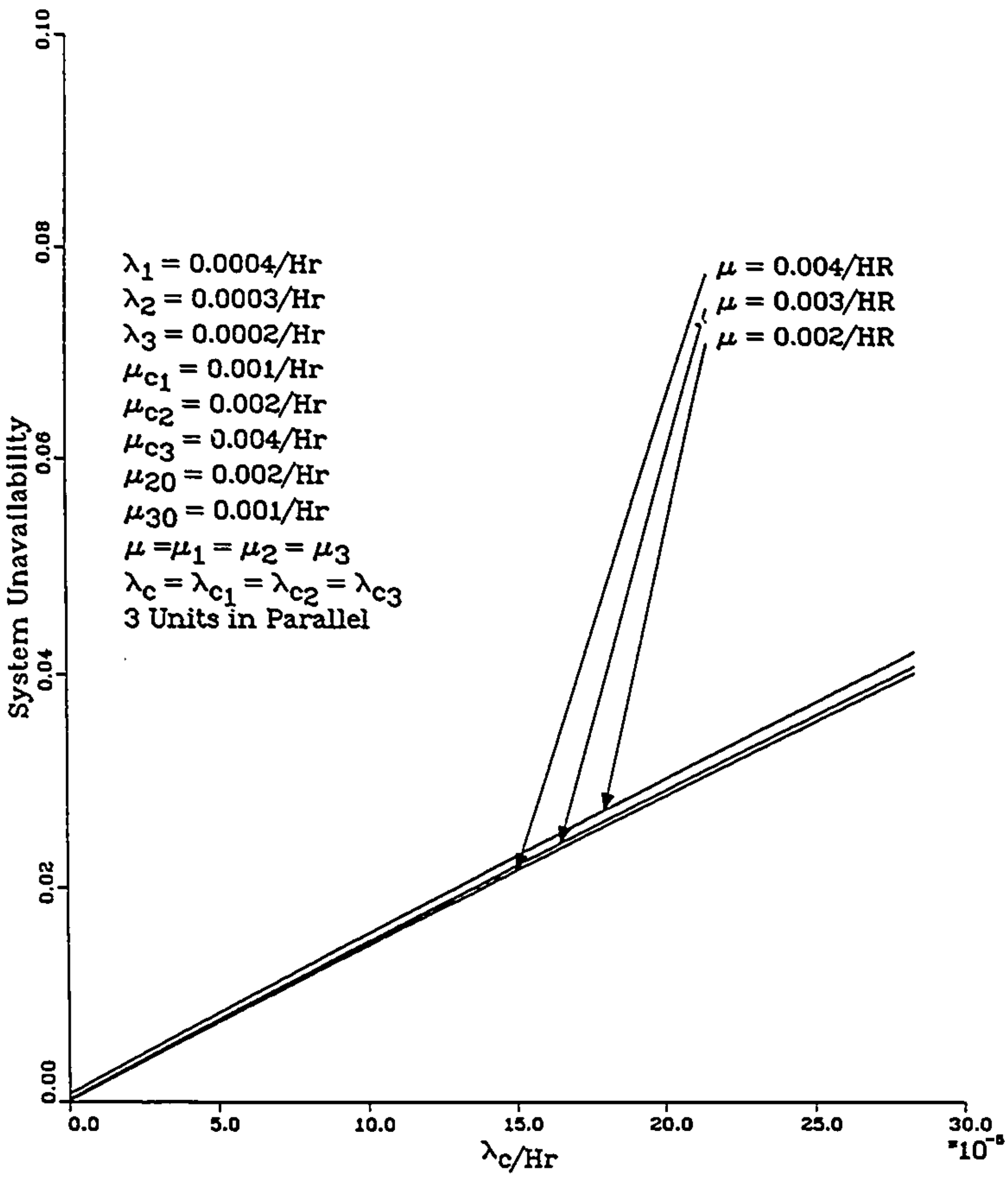

Figure 2.7: Steady State System Unavailability Plots for Special Case Model II ($n = 3$) with Type I Repair

The steady state system availability and steady state system unavailability are plotted in Figures 2.6 and 2.7, respectively, for specified values of model parameters. The plots indicate an increase in steady state system availability with an increase in repair rate. Furthermore, the plots also show that the steady state system availability decreases with an increase in common-cause failures.

2.1.2 Parallel System With Type II Repair

By setting the repair rates $\mu_{c_{i+1}}$; for $i = 0, 1, 2, \ldots, n-1$, μ_{n_o} and μ_n equal to zero, in Figure 2.1, results in a n identical unit parallel system with Type II repair whose system of differential equations can be expressed as follows :

$$\dot{p}_0(t) = -(\lambda_1 + \lambda_{c_1})p_0(t) + \mu_1 p_1(t) + \sum_{i=2}^{n-1} \mu_{io} p_i(t) \tag{2.25}$$

$$\dot{p}_1(t) = \lambda_1 p_0(t) - (\mu_1 + \lambda_2 + \lambda_{c_2})p_1(t) + \mu_2 p_2(t) \tag{2.26}$$

$$\dot{p}_{k-1}(t) = (\lambda_{k-1})p_{k-2}(t) - \{\mu_{(k-1)o} + \mu_{k-1} + \lambda_k + \lambda_{c_k}\}p_{k-1}(t) + \mu_k p_k(t) \tag{2.27}$$

for $k = 3, 4, 5 \ldots, (n-1)$

$$\dot{p}_j(t) = \lambda_j p_{j-1}(t) - \{\mu_{jo} + \mu_j + \lambda_{j+1} + \lambda_{c_{j+1}}\}p_j(t) \tag{2.28}$$

for $j = n-1$

$$\dot{p}_n(t) = \lambda_n p_{n-1}(t) \tag{2.29}$$

$$\dot{p}_{n+1}(t) = \sum_{i=0}^{n-1} \lambda_{c_{i+1}} p_i(t) \tag{2.30}$$

At time $t = 0$, $p_0(0) = 1$ and all other initial condition probabilities are equal to zero.

Solving Equations (2.25) – (2.30) with the aid of Laplace transforms, we get the following Laplace transforms of the state probabilities :

$$p_0(s) = \frac{1 + \mu_1 p_1(s) + \sum_{i=2}^{n-1} \mu_{io} p_i(s)}{s + \lambda_1 + \lambda_{c_1}} \tag{2.31}$$

$$p_1(s) = \frac{\lambda_1 p_0(s) + \mu_2 p_2(s)}{s + \mu_1 + \lambda_2 + \lambda_{c_2}} \tag{2.32}$$

$$p_{k-1}(s) = \frac{\lambda_{k-1} p_{k-2}(s) + \mu_k p_k(s)}{s + \mu_{(k-1)o} + \mu_{k-1} + \lambda_k + \lambda_{c_k}} \tag{2.33}$$

for $k = 3, 4, 5, \ldots, (n-1)$

$$p_j(s) = \frac{\lambda_j p_{j-1}(s)}{s + \mu_{jo} + \mu_j + \lambda_{j+1} + \lambda_{c_{j+1}}} \tag{2.34}$$

for $j = n - 1$

$$p_n(s) = \frac{\lambda_n p_{n-1}(s)}{s} \tag{2.35}$$

$$p_{n+1}(s) = \frac{\sum_{i=0}^{n-1} \lambda_{c_{i+1}} p_i(s)}{s} \tag{2.36}$$

The system mean time to failure for a n unit parallel system is given by

$$\begin{aligned} MTTF_n &= \lim_{s \to 0} R(s) \\ &= \lim_{s \to 0} [p_0(s) + p_1(s) + p_2(s) + \ldots + p_{n-2}(s) + p_{n-1}(s)] \end{aligned} \tag{2.37}$$

where $R(s)$ is the Laplace transform of the system reliability.

Thus, for a single unit system and two unit parallel system, respectively, we have

$$MTTF_1 = \frac{1}{\lambda_1 + \lambda_{c_1}}$$

$$MTTF_2 = \frac{\mu_1 + \lambda_2 + \lambda_{c_2} + \lambda_1}{(\lambda_2 \lambda_{c_1} + \mu_1 \lambda_{c1} + \lambda_{c_2} \lambda_{c_1} + \lambda_2 \lambda_1 + \lambda_{c_2} \lambda_1)}$$

Similarly, a generalized formula for the n unit parallel system mean time to failure ($MTTF_n$) is

$$MTTF_n = \frac{A_{n-1}\{D_8\} - \mu_{n-1}\lambda_{n-1}\{D_9\} + \prod_{i=1}^{n-1} \lambda_i}{A_{n-1}\{D_{10}\} - \mu_{n-1}\lambda_{n-1}\{D_{11}\} - \mu_{(n-1)o} \prod_{i=1}^{n-1} \lambda_i} \tag{2.38}$$

where

$$\begin{aligned} A_i &= \{\mu_{i+1} + \mu_{(i+1)o} + \lambda_{i+2} + \lambda_{c_{i+2}}\} \\ D_8 &= Numerator\ of\ MTTF_{n-1} \\ D_9 &= Numerator\ of\ MTTF_{n-2} \\ D_{10} &= Denominator\ of\ MTTF_{n-1} \\ D_{11} &= Denominator\ of\ MTTF_{n-2} \end{aligned}$$

Figure 2.8 shows the plots of system mean time to failure given by Equation (2.38) for specified values of model parameters. It can be noted from these plots that the system mean time to failure is directly proportional to the number of units in the parallel system and is inversely proportional to the number of common-cause failures.

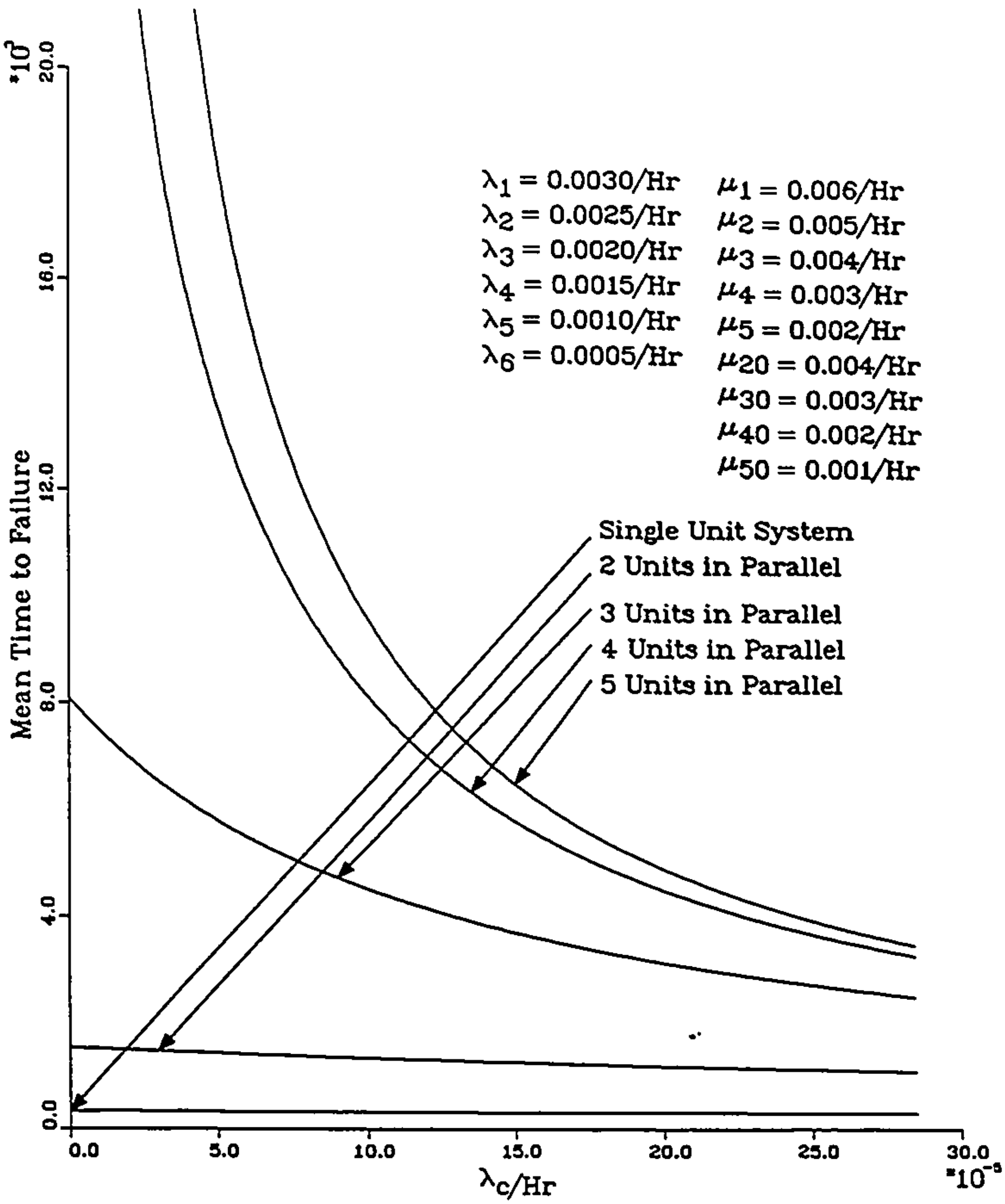

Figure 2.8: Typical System Mean Time to Failure Plots for an Identical Unit Parallel System with Type II Repair

Special Case Model I

The system of differential equations for a 2 unit parallel system can be obtained by setting $n = 2$ in Equations (2.25) – (2.30).

$$\dot{p}_0(t) = -(\lambda_1 + \lambda_{c_1})p_0(t) + \mu_1 p_1(t) \tag{2.39}$$

$$\dot{p}_1(t) = \lambda_1 p_0(t) - (\mu_1 + \lambda_2 + \lambda_{c_2})p_1(t) \tag{2.40}$$

$$\dot{p}_2(t) = \lambda_2 p_1(t) \tag{2.41}$$

$$\dot{p}_3(t) = \lambda_{c_1} p_0(t) + \lambda_{c_2} p_1(t) \tag{2.42}$$

At time $t = 0$, $p_0(0) = 1$, $p_1(0) = 0$, $p_2(0) = 0$, *and* $p_3(0) = 0$.

Solving Equations (2.39) – (2.42) with the aid of Laplace transforms the following time dependent state probabilities are obtained :

$$p_0(t) = \frac{\frac{1}{2}(\mu_1 + \lambda_2 - \lambda_1 - \lambda_{c_1} + \lambda_{c_2} + D_{12})e^{\{-\frac{1}{2}[D_{12}+\lambda_2+\lambda_{c_2}+\mu_1+\lambda_1+\lambda_{c_1}]+D_{12}\}t}}{D_{12}} - \frac{\frac{1}{2}(\mu_1 + \lambda_2 - \lambda_1 - \lambda_{c_1} + \lambda_{c_2} - D_{12})e^{-\frac{1}{2}[D_{12}+\lambda_2+\lambda_{c_2}+\mu_1+\lambda_1+\lambda_{c_1}]t}}{D_{12}} \tag{2.43}$$

$$p_1(t) = \frac{\lambda_1[e^{D_{12}t} - 1]e^{\{-\frac{1}{2}[D_{12}+\lambda_2+\lambda_{c_2}+\mu_1+\lambda_1+\lambda_{c_1}]t\}}}{D_{12}} \tag{2.44}$$

where the constant D_{12} is defined in Appendix A.

The system reliability is given by

$$R(t) = p_0(t) + p_1(t) \tag{2.45}$$

The plots of the Equation (2.45) are shown in Figure 2.9. From the plots it is evident that the system reliability decreases with increase in the number of common-cause failures.

The system mean time to failure of a two unit parallel system is given by

$$MTTF_2 = \frac{\mu_1 + \lambda_2 + \lambda_{c_2} + \lambda_1}{(\lambda_2\lambda_{c_1} + \mu_1\lambda_{c1} + \lambda_{c_2}\lambda_{c_1} + \lambda_2\lambda_1 + \lambda_{c_2}\lambda_1)} \tag{2.46}$$

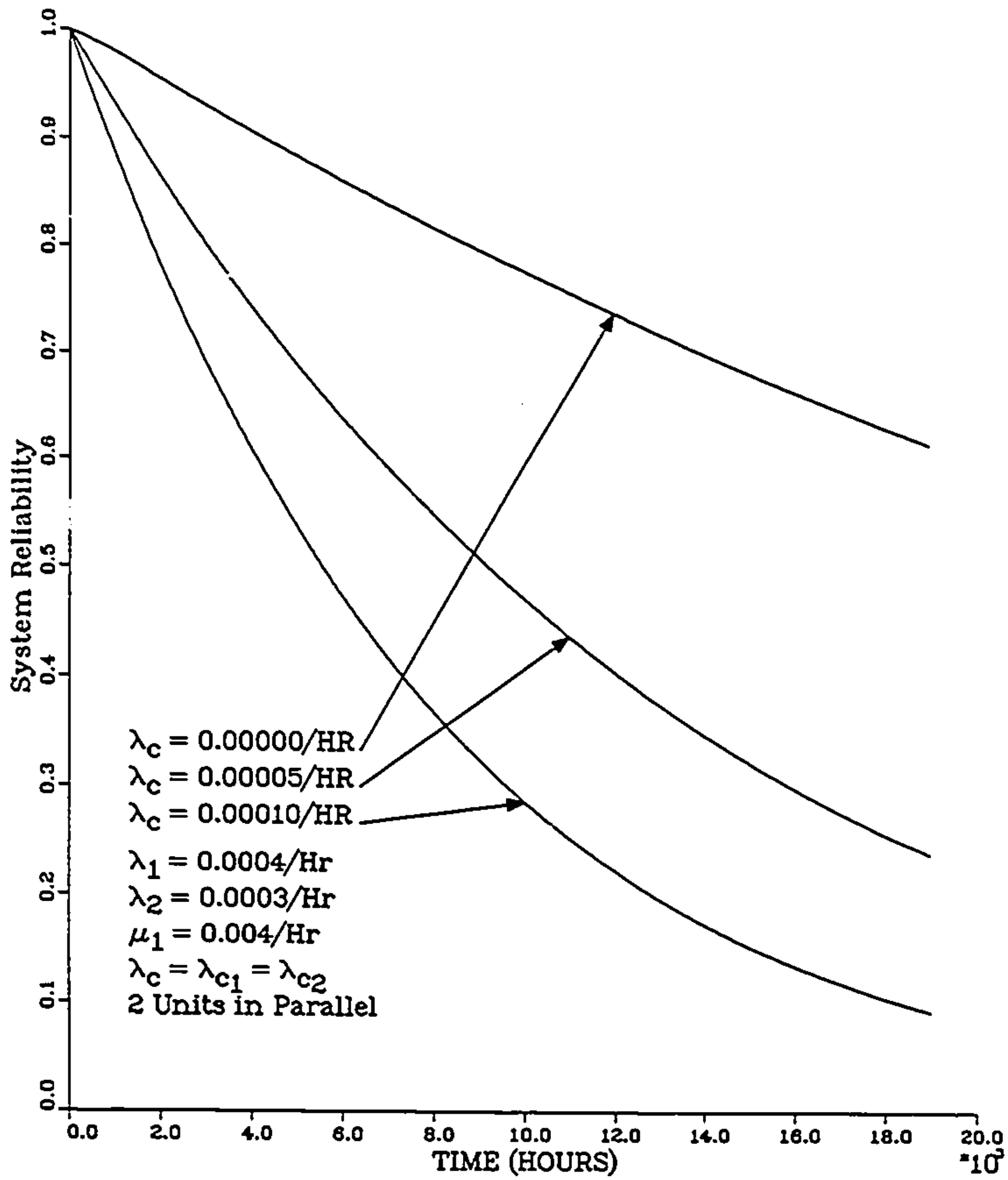

Figure 2.9: System Reliability Plots for Special Case Model I ($n = 2$) with Type II Repair

Special Case Model II

By setting $n = 3$ in Equations $(2.25) - (2.30)$ results in a 3 identical unit parallel system which is governed by the following differential equations :

$$\dot{p}_0(t) = -(\lambda_1 + \lambda_{c_1})p_0(t) + \mu_1 p_1(t) + \mu_{20}p_2(t) \tag{2.47}$$

$$\dot{p}_1(t) = \lambda_1 p_0(t) - (\mu_1 + \lambda_2 + \lambda_{c_2})p_1(t) + \mu_2 p_2(t) \tag{2.48}$$

$$\dot{p}_2(t) = \lambda_2 p_1(t) - (\mu_2 + \mu_{20} + \lambda_3 + \lambda_{c_3})p_2(t) \tag{2.49}$$

$$\dot{p}_3(t) = \lambda_3 p_2(t) \tag{2.50}$$

$$\dot{p}_4(t) = \lambda_{c_1}p_0(t) + \lambda_{c_2}p_1(t) + \lambda_{c_3}p_2(t) \tag{2.51}$$

At time $t = 0$, $p_0(0) = 1$, $p_1(0) = 0$, $p_2(0) = 0$, $p_3(0) = 0$ *and* $p_4(0) = 0$.

Solving Equations $(2.47) - (2.51)$ using Laplace transforms yields the following state probability expressions :

$$\begin{aligned} p_0(t) &= \frac{(s_1^2 + D_{13}s_1 + D_{14})e^{s_1 t}}{(s_1 - s_2)(s_1 - s_3)} + \frac{(s_2^2 + D_{13}s_2 + D_{14})e^{s_2 t}}{(s_3 - s_2)(s_1 - s_2)} + \\ &\quad \frac{(s_3^2 + D_{13}s_3 + D_{14})e^{s_3 t}}{(s_3 - s_2)(s_3 - s_1)} \end{aligned} \tag{2.52}$$

$$\begin{aligned} p_1(t) &= \frac{\lambda_1(s_1 + \mu_2 + \lambda_{c_3} + \lambda_3 + \mu_{20})e^{s_1 t}}{(s_3 - s_1)(s_2 - s_1)} + \\ &\quad \frac{\lambda_1(s_2 + \mu_2 + \lambda_{c_3} + \lambda_3 + \mu_{20})e^{s_2 t}}{(s_2 - s_1)(s_2 - s_3)} + \\ &\quad \frac{\lambda_1(s_3 + \mu_2 + \lambda_{c_3} + \lambda_3 + \mu_{20})e^{s_3 t}}{(s_3 - s_1)(s_3 - s_2)} \end{aligned} \tag{2.53}$$

$$\begin{aligned} p_2(t) &= \frac{\lambda_1\lambda_2 e^{(s_1 t)}}{(s_2 - s_1)(s_3 - s_1)} + \frac{\lambda_1\lambda_2 e^{(s_2 t)}}{(s_1 - s_2)(s_3 - s_2)} + \\ &\quad \frac{\lambda_1\lambda_2 e^{(s_3 t)}}{(s_3 - s_2)(s_3 - s_1)} + \end{aligned} \tag{2.54}$$

where s_1, s_2, *and* s_3 are the roots of the cubic equation and are determined as follows:

$$s^3 + D_{15}s^2 + D_{16}s + D_{17} = 0$$

$$\begin{aligned}
Let\ \alpha &= \frac{3D_{16} - (D_{15})^2}{9} \\
\beta &= \frac{9D_{15}D_{16} - 27D_{17} - 2(D_{15})^3}{54} \\
\Phi &= \sqrt[3]{\beta + \sqrt{\alpha^3 + \beta^2}} \\
\Omega &= \sqrt[3]{\beta - \sqrt{\alpha^3 + \beta^2}} \\
Thus,\ s_1 &= \Phi + \Omega - \frac{1}{3}D_{15} \\
s_2 &= -\frac{1}{2}(\Phi + \Omega) - \frac{1}{3}D_{15} + \frac{1}{2}i\sqrt{3}(\Phi - \Omega) \\
s_3 &= -\frac{1}{2}(\Phi + \Omega) - \frac{1}{3}D_{15} - \frac{1}{2}i\sqrt{3}(\Phi - \Omega)
\end{aligned}$$

The above i is associated with complex numbers.

$$\begin{aligned}
-D_{15} &= s_1 + s_2 + s_3 \\
D_{16} &= s_1s_2 + s_2s_3 + s_3s_1 \\
-D_{17} &= s_1s_2s_3
\end{aligned}$$

where the constants D_{13}, D_{14}, ..., D_{17} are defined in Appendix A.

The system reliability is given by

$$R(t) = p_0(t) + p_1(t) + p_2(t) \tag{2.55}$$

The plots of Equation (2.55) for specified values of model parameters is shown in Figure 2.10. It can be clearly seen from the plots that the system reliability decreases with an increase in the number of common-cause failures.

The system mean time to failure for a 3 unit parallel system is given below.

$$MTTF_3 = \frac{A_2\{\mu_1 + \lambda_2 + \lambda_{c_2} + \lambda_1\} - \mu_2\lambda_2\{1\} + \lambda_1\lambda_2}{[A_2\{\lambda_2\lambda_{c_1} + \mu_1\lambda_{c_1} + \lambda_{c_2}\lambda_{c_1} + \lambda_2\lambda_1 + \lambda_{c_2}\lambda_1\} - \mu_2\lambda_2\{\lambda_1 + \lambda_{c_1}\} - \mu_{20}\lambda_1\lambda_2]} \tag{2.56}$$

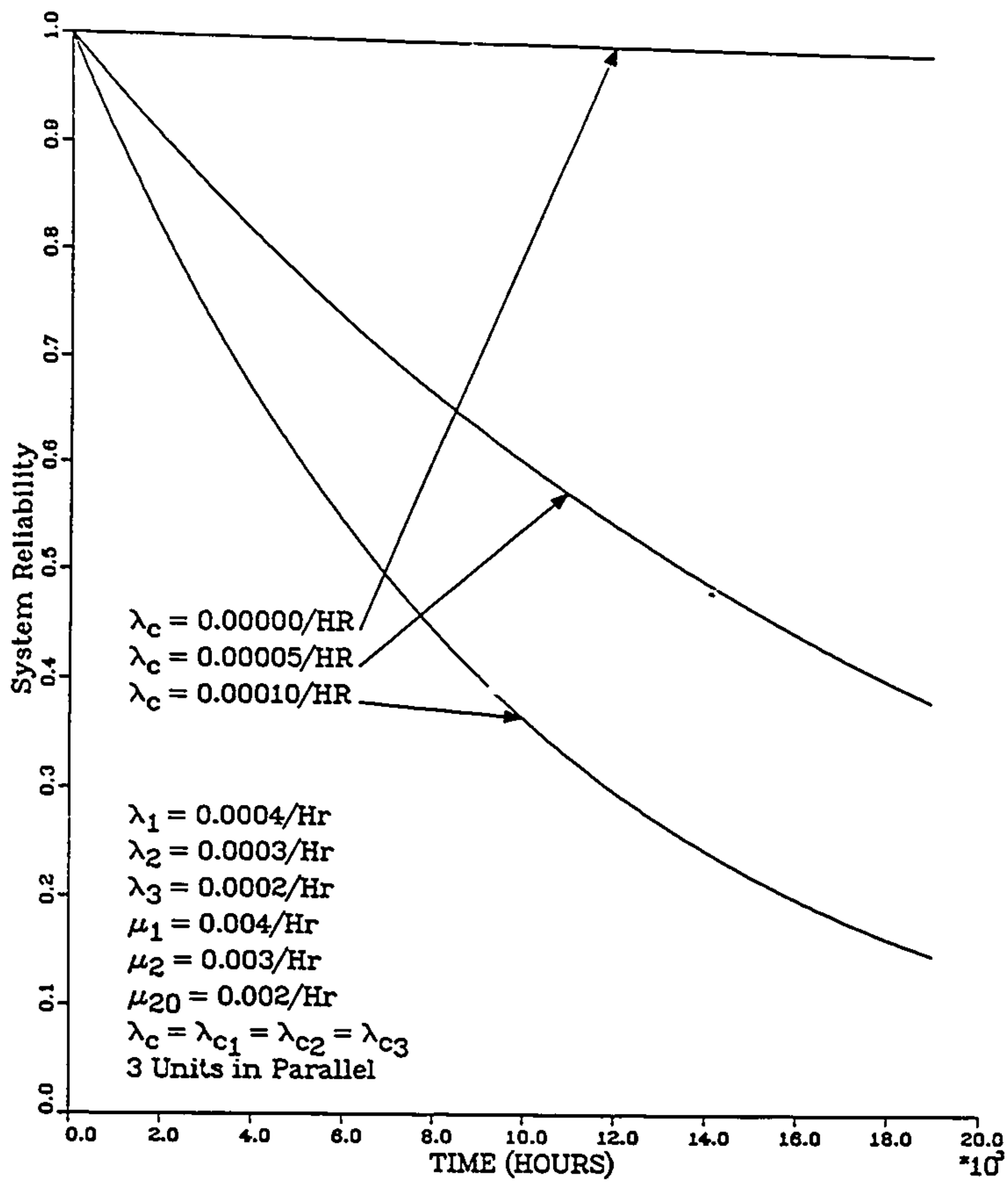

Figure 2.10: System Reliability Plots for Special Case Model II ($n = 3$) with Type II Repair

Special Case Model III

If we set $n = 4$ in Equations (2.25) – (2.30), the following system of differential equations is obtained :

$$\dot{p}_0(t) = -(\lambda_1 + \lambda_{c_1})p_0(t) + \mu_1 p_1(t) + \mu_{20}p_2(t) + \mu_{30}p_3(t) \tag{2.57}$$

$$\dot{p}_1(t) = \lambda_1 p_0(t) - (\mu_1 + \lambda_2 + \lambda_{c_2})p_1(t) + \mu_2 p_2(t) \tag{2.58}$$

$$\dot{p}_2(t) = \lambda_2 p_1(t) - (\mu_2 + \mu_{20} + \lambda_3 + \lambda_{c_3})p_2(t) + \mu_3 p_3(t) \tag{2.59}$$

$$\dot{p}_3(t) = \lambda_3 p_2(t) - (\mu_3 + \mu_{30} + \lambda_4 + \lambda_{c_4})p_3(t) \tag{2.60}$$

$$\dot{p}_4(t) = \lambda_4 p_3(t) \tag{2.61}$$

$$\dot{p}_5(t) = \lambda_{c_1}p_0(t) + \lambda_{c_2}p_1(t) + \lambda_{c_3}p_2(t) + \lambda_{c_4}p_3(t) \tag{2.62}$$

At time $t = 0$, $p_0(0) = 1$, $p_1(0) = 0$, $p_2(0) = 0$, $p_3(0) = 0$, $p_4(0)$ *and* $p_5(0) = 0$.

Solving Equations (2.57) – (2.62) using Laplace transforms, yields the following state probability expressions :

$$\begin{aligned} p_0(t) \;=\; & \frac{[s_1^3 + D_{18}s_1^2 + D_{19}s_1 + D_{20}]e^{s_1 t}}{(s_1 - s_2)(s_1 - s_3)(s_1 - s_4)} + \frac{[s_2^3 + D_{18}s_2^2 + D_{19}s_2 + D_{20}]e^{s_2 t}}{(s_2 - s_3)(s_4 - s_2)(s_1 - s_2)} + \\ & \frac{[s_3^3 + D_{18}s_3^2 + D_{19}s_3 + D_{20}]e^{s_3 t}}{(s_3 - s_2)(s_4 - s_3)(s_1 - s_3)} + \\ & \frac{[s_4^3 + D_{18}s_4^2 + D_{19}s_4 + D_{20}]e^{s_4 t}}{(s_2 - s_4)(s_4 - s_3)(s_1 - s_4)} \end{aligned} \tag{2.63}$$

$$\begin{aligned} p_1(t) \;=\; & \frac{[\lambda_1 s_1^2 + D_{25}s_1 + D_{26}]e^{s_1 t}}{(s_1 - s_4)(s_3 - s_1)(s_2 - s_1)} + \frac{[\lambda_1 s_2^2 + D_{25}s_2 + D_{26}]e^{s_2 t}}{(s_2 - s_1)(s_2 - s_4)(s_2 - s_3)} + \\ & \frac{[\lambda_1 s_3^2 + D_{25}s_3 + D_{26}]e^{s_3 t}}{(s_1 - s_3)(s_3 - s_4)(s_2 - s_3)} + \frac{[\lambda_1 s_4^2 + D_{25}s_4 + D_{26}]e^{s_4 t}}{(s_4 - s_1)(s_3 - s_4)(s_2 - s_4)} \end{aligned} \tag{2.64}$$

$$\begin{aligned} p_2(t) \;=\; & \frac{\lambda_1\lambda_2[s_1 + \mu_3 + \mu_{30} + \lambda_4 + \lambda_{c_4}]e^{s_1 t}}{(s_3 - s_1)(s_1 - s_4)(s_2 - s_1)} + \frac{\lambda_1\lambda_2[s_2 + \mu_3 + \mu_{30} + \lambda_4 + \lambda_{c_4}]e^{s_2 t}}{(s_2 - s_3)(s_2 - s_4)(s_2 - s_1)} + \\ & \frac{\lambda_1\lambda_2[s_3 + \mu_3 + \mu_{30} + \lambda_4 + \lambda_{c_4}]e^{s_3 t}}{(s_3 - s_4)(s_1 - s_3)(s_2 - s_3)} + \\ & \frac{\lambda_1\lambda_2[s_4 + \mu_3 + \mu_{30} + \lambda_4 + \lambda_{c_4}]e^{s_4 t}}{(s_4 - s_3)(s_1 - s_4)(s_2 - s_4)} \end{aligned} \tag{2.65}$$

$$
\begin{aligned}
p_3(t) \;=\; & \frac{\lambda_3\lambda_2\lambda_1 e^{s_1 t}}{(s_1 - s_4)(s_3 - s_1)(s_2 - s_1)} + \frac{\lambda_3\lambda_2\lambda_1 e^{s_2 t}}{(s_2 - s_1)(s_2 - s_4)(s_2 - s_3)} + \\
& \frac{\lambda_3\lambda_2\lambda_1 e^{s_3 t}}{(s_1 - s_3)(s_3 - s_4)(s_2 - s_3)} + \frac{\lambda_3\lambda_2\lambda_1 e^{s_4 t}}{(s_4 - s_1)(s_3 - s_4)(s_2 - s_4)} \qquad (2.66)
\end{aligned}
$$

where s_1, s_2, s_3 *and* s_4 are the roots of the quartic equation and can be determined as explained below.

$$s^4 + D_{21}s^3 + D_{22}s^2 + D_{23}s + D_{24} = 0$$

Let y_1 be a real root of the cubic equation

$$y^3 - D_{22}y^2 + (D_{21}D_{23} - 4D_{24})y + (4D_{22}D_{24} - D_{23}^2 - D_{21}^2 D_{24}) = 0$$

s_1, s_2, s_3 *and* s_4 are the roots of

$$z^2 + \frac{1}{2}\{D_{21} \pm \sqrt{D_{21}^2 - 4D_{22} + 4y_1}\}z + \frac{1}{2}\{y_1 \pm \sqrt{y_1^2 - 4D_{24}}\} = 0$$

$$
\begin{aligned}
-D_{21} &= s_1 + s_2 + s_3 + s_4 \\
D_{22} &= s_1s_2 + s_2s_3 + s_3s_4 + s_4s_1 + s_1s_3 + s_2s_4 \\
-D_{23} &= s_1s_2s_3 + s_2s_3s_4 + s_1s_2s_4 + s_1s_3s_4 \\
D_{24} &= s_1s_2s_3s_4
\end{aligned}
$$

and the constants $D_{18}, D_{19}, \ldots, D_{26}$ are defined in Appendix A.

The system reliability is given by

$$R(t) = p_0(t) + p_1(t) + p_2(t) + p_3(t) \qquad (2.67)$$

The plots of the Equation (2.67) are shown in Figure 2.11. From the plots the decreasing effect of common-cause failures on system reliability can be clearly seen.

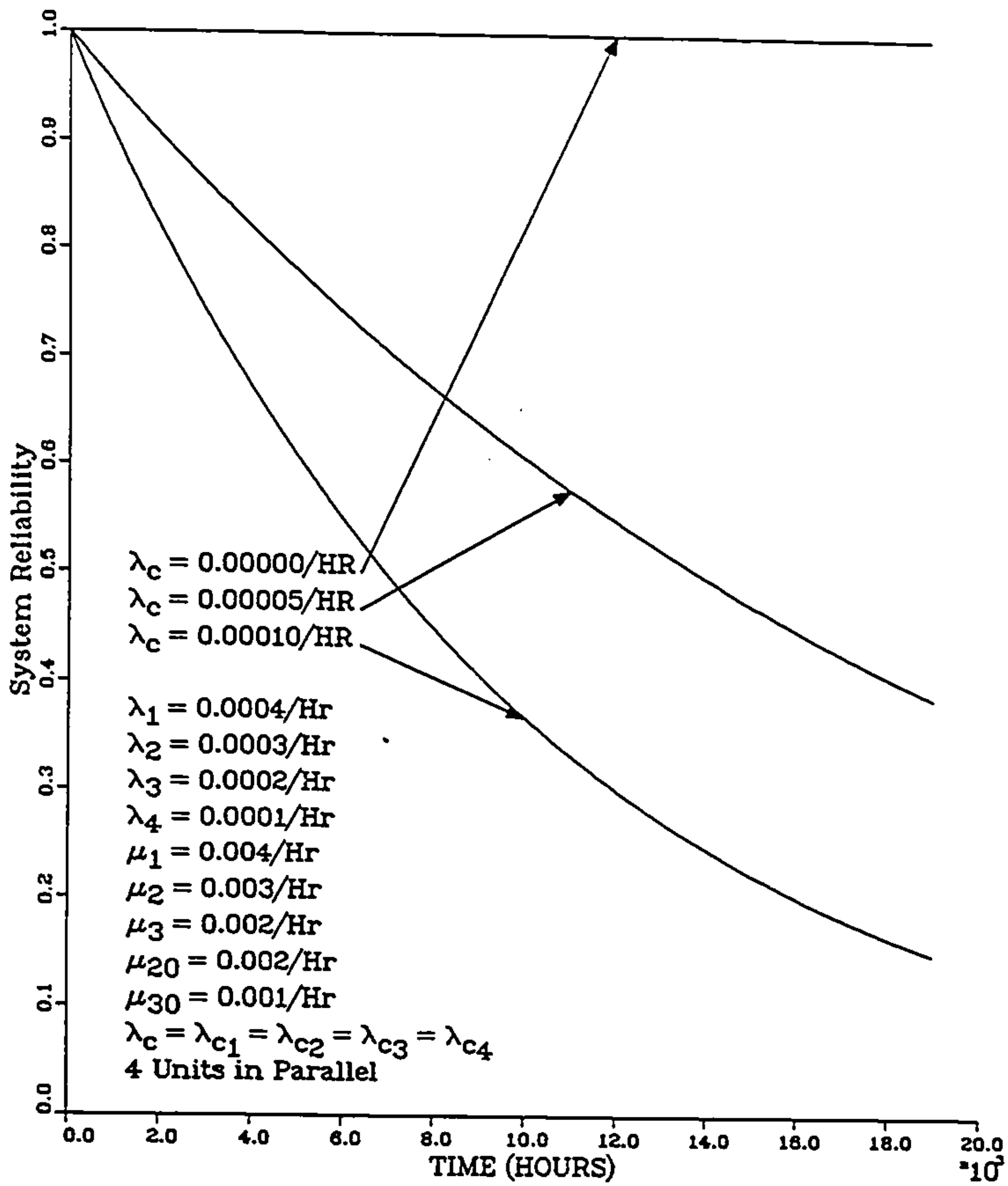

Figure 2.11: System Reliability Plots for Special Case Model III ($n = 4$) with Type II Repair

2.1.3 Parallel System Without Repair

Setting the repair rates μ_i; for $i = 1, 2, 3, \ldots, n$, μ_{i_b}; for $i = 2, 3, 4, \ldots, n$ and $\mu_{c_{i+1}}$; for $i = 0, 1, 2, \ldots, n-1$ equal to zero in Figure 2.1, yields a n identical unit parallel system without repair. The system of differential equations for such a system is as follows :

$$\dot{p}_0(t) = -(\lambda_1 + \lambda_{c_1})p_0(t) \tag{2.68}$$

$$\dot{p}_i(t) = \lambda_i p_{i-1}(t) - (\lambda_{i+1} + \lambda_{c_{i+1}})p_i(t) \tag{2.69}$$

for $i = 1, 2, 3, \ldots n-1$

$$\dot{p}_j(t) = \lambda_j p_{j-1}(t) \tag{2.70}$$

for $j = n$

$$\dot{p}_{n+1}(t) = \sum_{i=0}^{n-1} \lambda_{c_{i+1}} p_i(t) \tag{2.71}$$

At time $t = 0$, $p_0(0) = 1$ and all other initial condition probabilities are equal to zero.

The following Laplace transforms of the state probability expressions can be obtained by solving Equations (2.68) – (2.71) :

$$p_0(s) = \frac{1}{s + \lambda_1 + \lambda_{c_1}} \tag{2.72}$$

$$p_i(s) = \frac{\prod_{j=1}^{i} \lambda_j}{\prod_{j=1}^{i+1}(s + \lambda_j + \lambda_{c_j})} \tag{2.73}$$

for $i = 1, 2, 3, \ldots, n-1$

Taking inverse Laplace transforms of the above expressions, we get the following time dependent state probabilities :

$$p_0(t) = e^{-(\lambda_1 + \lambda_{c_1})t} \tag{2.74}$$

$$p_1(t) = \lambda_1 e^{-\frac{1}{2}(\lambda_2+\lambda_{c_2}+\lambda_{c_1}+\lambda_1)t} \times \frac{\{e^{\frac{1}{2}(\lambda_2+\lambda_{c_2}-\lambda_{c_1}-\lambda_1)t} - e^{-\frac{1}{2}(\lambda_2+\lambda_{c_2}-\lambda_{c_1}-\lambda_1)t}\}}{(\lambda_2+\lambda_{c_2}-\lambda_{c_1}-\lambda_1)} \tag{2.75}$$

$$p_r(t) = \sum_{k=1}^{r+1} \frac{\prod_{j=1}^{r} \lambda_j e^{-(\lambda_k+\lambda_{c_k})t}}{\prod_{i=1, i\neq k}^{r+1}(\lambda_i+\lambda_{c_i}-\lambda_k-\lambda_{c_k})} \tag{2.76}$$

for $r = 2, 3, 4, \ldots, n-1$

The system reliability is given by

$$R(t) = p_0(t) + p_1(t) + \sum_{i=2}^{n-1} p_i(t) \tag{2.77}$$

The plots of Equation (2.77) are given in Figure 2.12 for specified values of the model parameters and from the plots it is apparent that an increase in number of common-cause failures results in a decrease in the system reliability.

The n unit parallel system mean time to failure ($MTTF_n$) is given by

$$\begin{aligned} MTTF_n &= \lim_{s\to 0} R(s) \\ &= \lim_{s\to 0}\{p_0(s) + \sum_{i=1}^{n-1} p_i(s)\} \\ &= \left\{\frac{1}{(\lambda_1+\lambda_{c_1})} + \sum_{i=1}^{n-1} \frac{\prod_{j=1}^{i}\lambda_j}{\prod_{j=1}^{i+1}(\lambda_j+\lambda_{c_j})}\right\} \end{aligned} \tag{2.78}$$

The plots of Equation (2.78) are shown in Figure 2.13 for specified values of model parameters. The plots show a decreasing trend in the system mean time to failure with an increasing number of common-cause failures, as well as with a decrease in the number of units in the parallel system.

The variance of time to failure of the redundant system is given by

$$\begin{aligned} \sigma^2 &= -2\lim_{s\to 0} R'(s) - (MTTF)^2 \\ &= 2\left\{\frac{1}{(\lambda_1+\lambda_{c_1})^2} + \sum_{j=1}^{n-1}\sum_{k=1}^{j+1} \frac{\prod_{i=1}^{j}\lambda_i}{\left\{\prod_{i=1}^{j+1}(\lambda_i+\lambda_{c_i})\right\}(\lambda_k+\lambda_{c_k})}\right\} - \end{aligned}$$

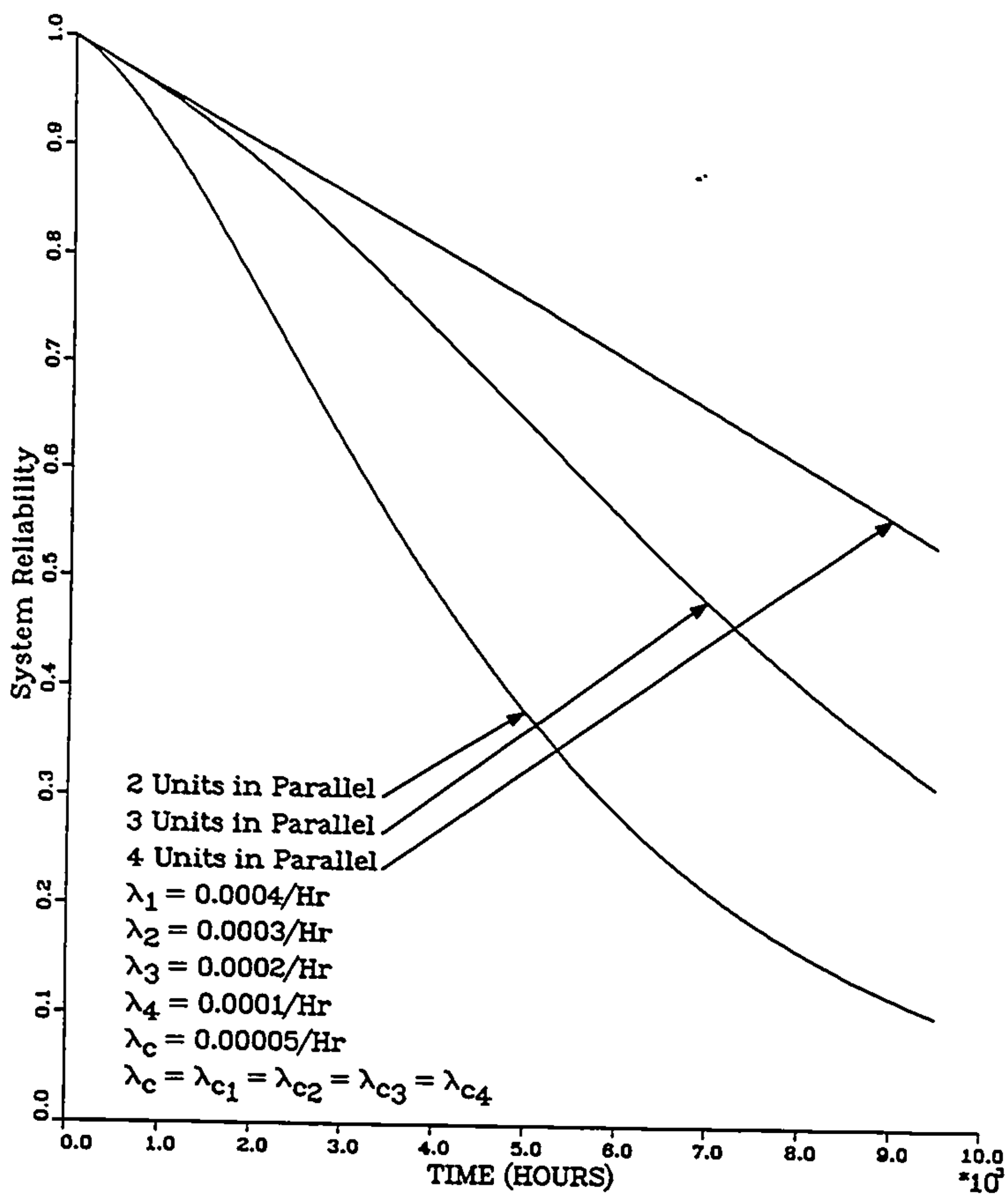

Figure 2.12: Typical System Reliability Plots for an Identical Unit Parallel System without Repair

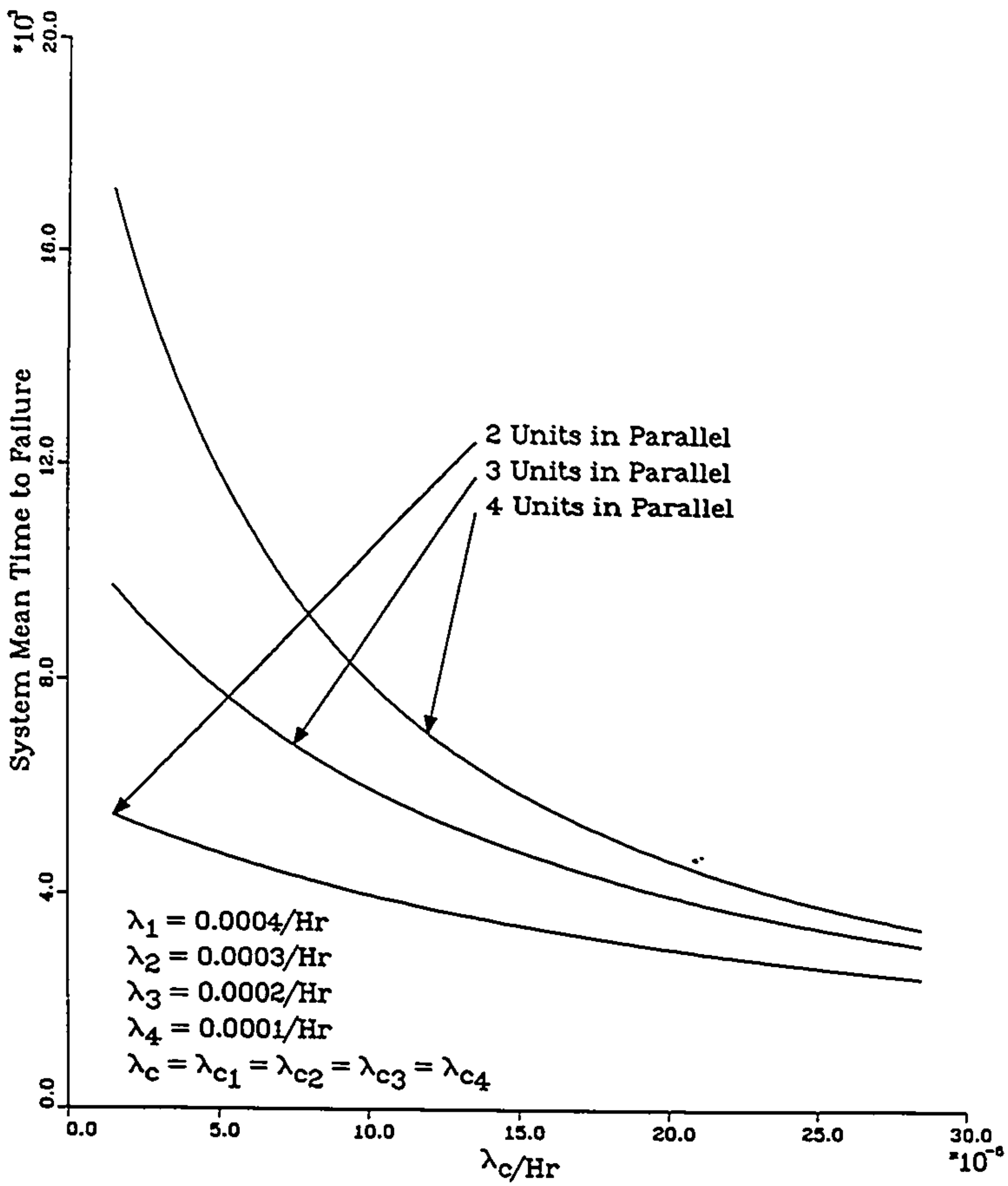

Figure 2.13: Typical System Mean Time to Failure Plots for an Identical Unit Parallel System without Repair

$$\left\{\frac{1}{\lambda_1 + \lambda_{c_1}} + \sum_{i=1}^{n-1} \frac{\prod_{j=1}^{i} \lambda_j}{\prod_{j=1}^{i+1} \lambda_j + \lambda_{c_j}}\right\}^2 \tag{2.79}$$

where $R'(s)$, is the derivative of Laplace transform of system reliability, $R(s)$ with respect to s.

Special Case Model I

The following system of differential equations is obtained by setting $n = 2$ in Equations (2.68) – (2.71) :

$$\dot{p}_0(t) = -(\lambda_1 + \lambda_{c_1})p_0(t) \tag{2.80}$$

$$\dot{p}_1(t) = \lambda_1 p_0(t) - (\lambda_2 + \lambda_{c_2})p_1(t) \tag{2.81}$$

$$\dot{p}_2(t) = \lambda_2 p_1(t) \tag{2.82}$$

$$\dot{p}_3(t) = \lambda_{c_1} p_0(t) + \lambda_{c_2} p_1(t) \tag{2.83}$$

At time $t = 0$, $p_0(0) = 1$, $p_1(0) = 0$, $p_2(0) = 0$ *and* $p_3(0) = 0$.

Solving the above Equations with the aid of Laplace transforms, we get the following time dependent state probability expressions :

$$p_0(t) = e^{-(\lambda_1+\lambda_{c_1})t} \tag{2.84}$$

$$p_1(t) = \lambda_1 e^{-\frac{1}{2}(\lambda_2+\lambda_{c_2}+\lambda_{c_1}+\lambda_1)t} \times \frac{\{e^{\frac{1}{2}(\lambda_2+\lambda_{c_2}-\lambda_{c_1}-\lambda_1)t} - e^{-\frac{1}{2}(\lambda_2+\lambda_{c_2}-\lambda_{c_1}-\lambda_1)t}\}}{(\lambda_2+\lambda_{c_2}-\lambda_{c_1}-\lambda_1)} \tag{2.85}$$

The system reliability given by

$$R(t) = p_0(t) + p_1(t) \tag{2.86}$$

The plots of the above Equations are shown in Figure 2.14. The decreasing effect of common-cause failures on system reliability can easily be seen from these plots.

The mean time to failure of a 2 unit parallel system can be expressed as

$$MTTF_2 = \frac{1}{\lambda_1 + \lambda_{c_1}} + \frac{\lambda_1}{(\lambda_1 + \lambda_{c_1})(\lambda_2 + \lambda_{c_2})} \tag{2.87}$$

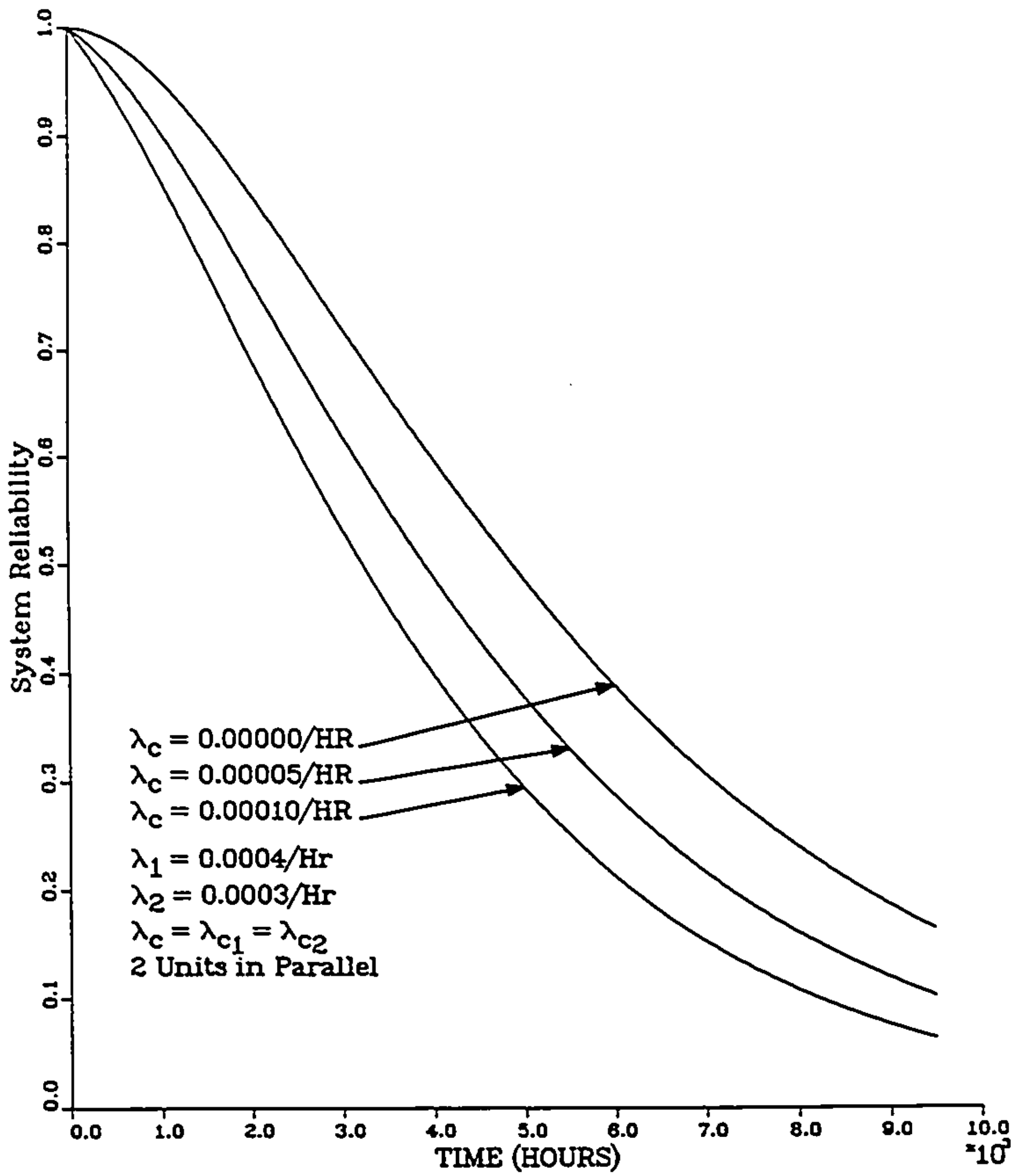

Figure 2.14: System Reliability Plots for Special Case Model I (n = 2) without Repair

Special Case Model II

Setting $n = 3$ in Equations (2.68) – (2.71), results in the following system of differential equations for a 3 identical unit parallel system :

$$\dot{p}_0(t) = -(\lambda_1 + \lambda_{c_1})p_0(t) \tag{2.88}$$

$$\dot{p}_1(t) = \lambda_1 p_0(t) - (\lambda_2 + \lambda_{c_2})p_1(t) \tag{2.89}$$

$$\dot{p}_2(t) = \lambda_2 p_1(t) - (\lambda_3 + \lambda_{c_3})p_2(t) \tag{2.90}$$

$$\dot{p}_3(t) = \lambda_3 p_2(t) \tag{2.91}$$

$$\dot{p}_4(t) = \lambda_{c_1} p_0(t) + \lambda_{c_2} p_1(t) + \lambda_{c_3} p_2(t) \tag{2.92}$$

At time $t = 0$, $p_0(0) = 1$, $p_1(0) = 0$, $p_2(0) = 0$, $p_3(0) = 0$ *and* $p_4(0) = 0$.

Solving the above Equations with the aid of Laplace transforms, we get the following state probability expressions :

$$p_0(t) = e^{-(\lambda_1+\lambda_{c_1})t} \tag{2.93}$$

$$p_1(t) = \lambda_1 e^{-\frac{1}{2}(\lambda_2+\lambda_{c_2}+\lambda_{c_1}+\lambda_1)t} \times \frac{\{e^{\frac{1}{2}(\lambda_2+\lambda_{c_2}-\lambda_{c_1}-\lambda_1)t} - e^{-\frac{1}{2}(\lambda_2+\lambda_{c_2}-\lambda_{c_1}-\lambda_1)t}\}}{(\lambda_2 + \lambda_{c_2} - \lambda_{c_1} - \lambda_1)} \tag{2.94}$$

$$p_2(t) = \frac{\lambda_1\lambda_2 e^{-(\lambda_1+\lambda_{c_1})t}}{(\lambda_2 + \lambda_{c_2} - \lambda_1 - \lambda_{c_1})(\lambda_3 + \lambda_{c_3} - \lambda_1 - \lambda_{c_1})} + \frac{\lambda_1\lambda_2 e^{-(\lambda_2+\lambda_{c_2})t}}{(\lambda_1 + \lambda_{c_1} - \lambda_2 - \lambda_{c_2})(\lambda_3 + \lambda_{c_3} - \lambda_2 - \lambda_{c_2})} + \frac{\lambda_1\lambda_2 e^{-(\lambda_3+\lambda_{c_3})t}}{(\lambda_1 + \lambda_{c_1} - \lambda_3 - \lambda_{c_3})(\lambda_2 + \lambda_{c_2} - \lambda_3 - \lambda_{c_3})} \tag{2.95}$$

The system reliability given by $p_0(t) + p_1(t) + p_2(t)$, is plotted for specified values of model parameters in Figure 2.15, which shows a decreasing effect of common-cause failures on system reliability.

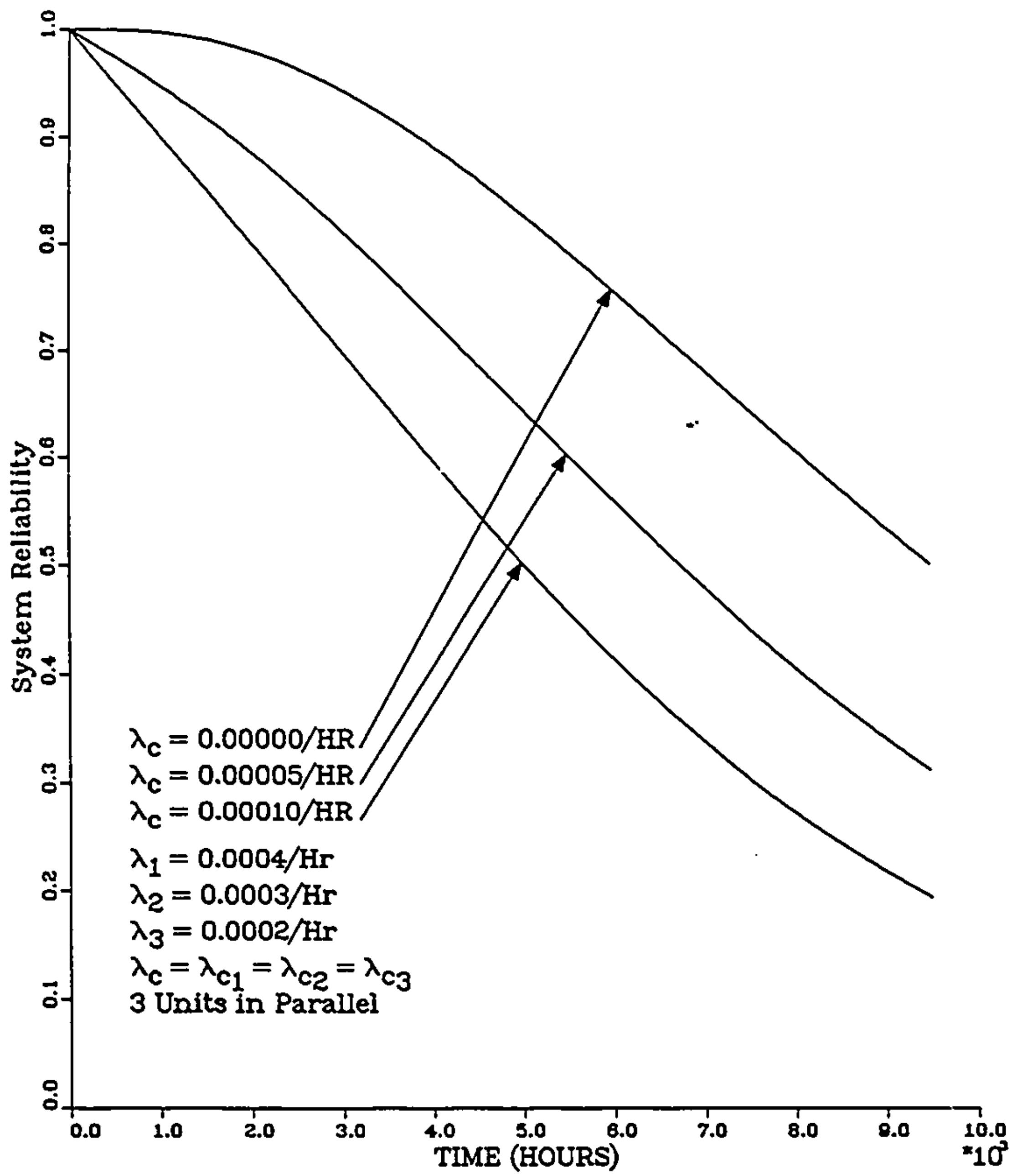

Figure 2.15: System Reliability Plots for Special Case Model II ($n = 3$) without Repair

The system mean time to failure of a 3 unit parallel system can be obtained by substituting $n = 3$ in Equation (2.78).

$$MTTF_3 = \frac{1}{\lambda_1 + \lambda_{c_1}} + \frac{\lambda_1}{(\lambda_1 + \lambda_{c_1})(\lambda_2 + \lambda_{c_2})} + \frac{\lambda_1 \lambda_2}{(\lambda_1 + \lambda_{c_1})(\lambda_2 + \lambda_{c_2})(\lambda_3 + \lambda_{c_3})} \qquad (2.96)$$

Special Case Model III

The following differential equations are obtained by setting $n = 4$ in Equations (2.68) – (2.71) :

$$\dot{p}_0(t) = -(\lambda_1 + \lambda_{c_1})p_0(t) \tag{2.97}$$

$$\dot{p}_1(t) = \lambda_1 p_0(t) - (\lambda_2 + \lambda_{c_2})p_1(t) \tag{2.98}$$

$$\dot{p}_2(t) = \lambda_2 p_1(t) - (\lambda_3 + \lambda_{c_3})p_2(t) \tag{2.99}$$

$$\dot{p}_3(t) = \lambda_3 p_2(t) - (\lambda_4 + \lambda_{c_4})p_3(t) \tag{2.100}$$

$$\dot{p}_4(t) = \lambda_4 p_3(t) \tag{2.101}$$

$$\dot{p}_5(t) = \lambda_{c_1}p_0(t) + \lambda_{c_2}p_1(t) + \lambda_{c_3}p_2(t) + \lambda_{c_4}p_3(t) \tag{2.102}$$

At time $t = 0$, $p_0(0) = 1$, $p_1(0) = 0$, $p_2(0) = 0$, $p_3(0) = 0$, $p_4(0) = 0$ *and* $p_5(0) = 0$.

The following state probabilities are obtained by solving the above differential equations with the aid of Laplace transforms :

$$p_0(t) = e^{-(\lambda_1+\lambda_{c_1})t} \tag{2.103}$$

$$p_1(t) = \lambda_1 e^{-\frac{1}{2}(\lambda_2+\lambda_{c_2}+\lambda_{c_1}+\lambda_1)t} \times \frac{\{e^{\frac{1}{2}(\lambda_2+\lambda_{c_2}-\lambda_{c_1}-\lambda_1)t} - e^{-\frac{1}{2}(\lambda_2+\lambda_{c_2}-\lambda_{c_1}-\lambda_1)t}\}}{(\lambda_2+\lambda_{c_2}-\lambda_{c_1}-\lambda_1)} \tag{2.104}$$

$$p_2(t) = \frac{\lambda_1\lambda_2 e^{-(\lambda_1+\lambda_{c_1})t}}{(\lambda_2+\lambda_{c_2}-\lambda_1-\lambda_{c_1})(\lambda_3+\lambda_{c_3}-\lambda_1-\lambda_{c_1})} + \frac{\lambda_1\lambda_2 e^{-(\lambda_2+\lambda_{c_2})t}}{(\lambda_1+\lambda_{c_1}-\lambda_2-\lambda_{c_2})(\lambda_3+\lambda_{c_3}-\lambda_2-\lambda_{c_2})} + \frac{\lambda_1\lambda_2 e^{-(\lambda_3+\lambda_{c_3})t}}{(\lambda_1+\lambda_{c_1}-\lambda_3-\lambda_{c_3})(\lambda_2+\lambda_{c_2}-\lambda_3-\lambda_{c_3})} \tag{2.105}$$

$$p_3(t) = \frac{\lambda_3\lambda_2\lambda_1 e^{-(\lambda_1+\lambda_{c_1})t}}{(\lambda_2+\lambda_{c_2}-\lambda_1-\lambda_{c_1})(\lambda_4+\lambda_{c_4}-\lambda_1-\lambda_{c_1})(\lambda_3+\lambda_{c_3}-\lambda_1-\lambda_{c_1})} + \frac{\lambda_3\lambda_2\lambda_1 e^{-(\lambda_2+\lambda_{c_2})t}}{(\lambda_2+\lambda_{c_2}-\lambda_1-\lambda_{c_1})(\lambda_4+\lambda_{c_4}-\lambda_2-\lambda_{c_2})(\lambda_2+\lambda_{c_2}-\lambda_3-\lambda_{c_3})} +$$

$$
\frac{\lambda_1\lambda_2\lambda_3 e^{-(\lambda_3+\lambda_{c_3})t}}{(\lambda_1+\lambda_{c_1}-\lambda_3-\lambda_{c_3})(\lambda_2+\lambda_{c_2}-\lambda_3-\lambda_{c_3})(\lambda_4+\lambda_{c_4}-\lambda_3-\lambda_{c_3})}+
\frac{\lambda_1\lambda_2\lambda_3 e^{-(\lambda_4+\lambda_{c_4})t}}{(\lambda_1+\lambda_{c_1}-\lambda_4-\lambda_{c_4})(\lambda_2+\lambda_{c_2}-\lambda_4-\lambda_{c_4})(\lambda_3+\lambda_{c_3}-\lambda_4-\lambda_{c_4})}
\tag{2.106}
$$

The system reliability is given by

$$
R(t) = p_0(t) + p_1(t) + p_2(t) + p_3(t) \tag{2.107}
$$

The system reliability given by Equation (2.107) is plotted in Figure 2.16 and the plots clearly show the inverse relationship between system reliability and common-cause failures.

The mean time to failure for a 4 unit parallel system is

$$
\begin{aligned}
MTTF_4 = {} & \frac{1}{\lambda_1+\lambda_{c_1}} + \frac{\lambda_1}{(\lambda_1+\lambda_{c_1})(\lambda_2+\lambda_{c_2})} + \\
& \frac{\lambda_1\lambda_2}{(\lambda_1+\lambda_{c_1})(\lambda_2+\lambda_{c_2})(\lambda_3+\lambda_{c_3})} + \\
& \frac{\lambda_1\lambda_2\lambda_3}{(\lambda_1+\lambda_{c_1})(\lambda_2+\lambda_{c_2})(\lambda_3+\lambda_{c_3})(\lambda_4+\lambda_{c_4})}
\end{aligned}
\tag{2.108}
$$

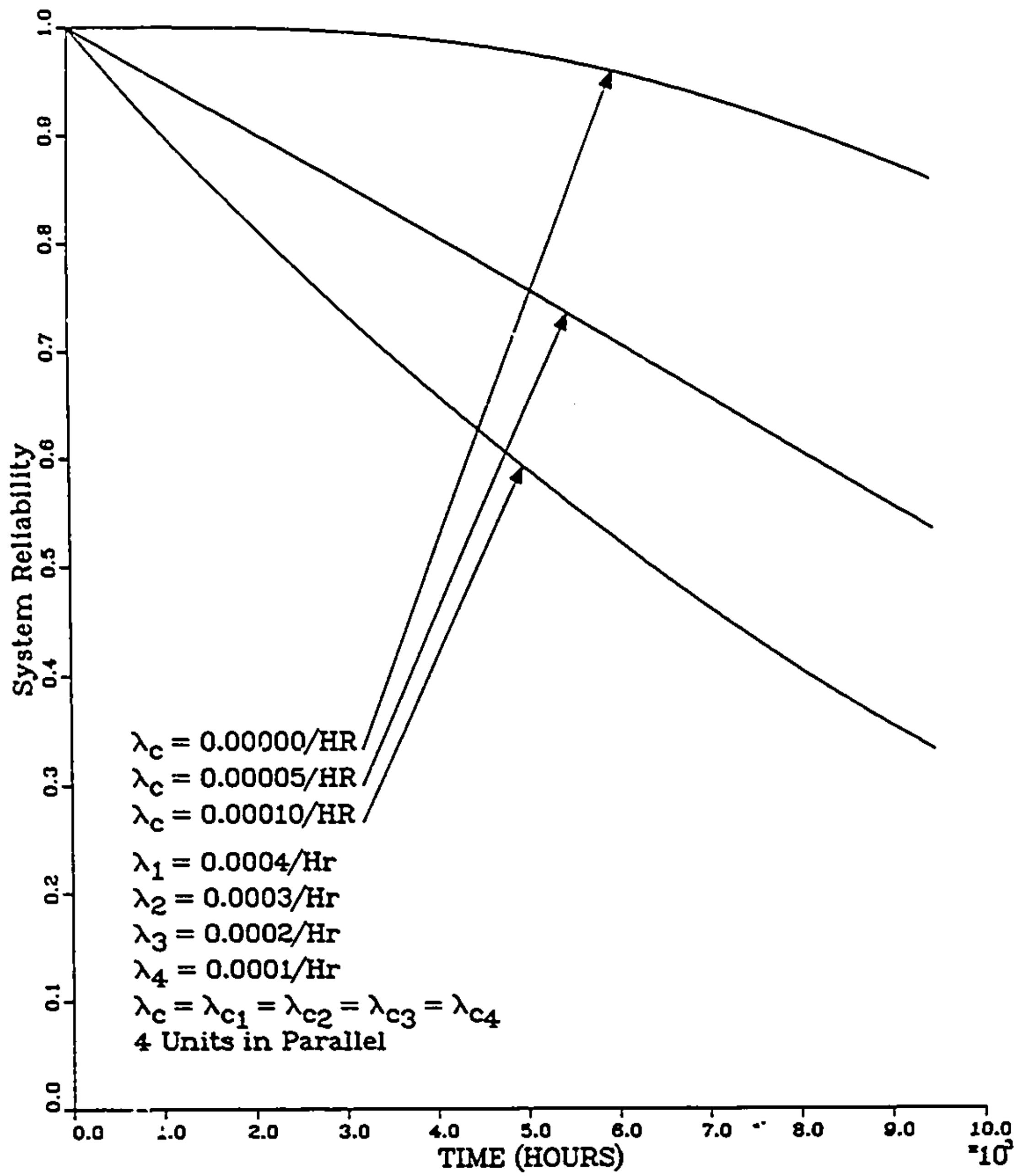

Figure 2.16: System Reliability Plots for Special Case Model III ($n = 4$) without Repair

2.1.4 Comparison of Markov Method Results with that of Block Diagram Method

To compare the results obtained through the Markov method we consider a modified identical unit parallel system shown in Figure 2.17 (block diagram method). It is simply a parallel network with a hypothetical unit in series [73]. The parallel stage, labelled 1 of Figure 2.17, represents all the independent failures for any n unit system. The series unit stage, labelled 2 in Figure 2.17, represents all the common-cause failures of the system.

Let λ be the constant failure rate of each unit and λ_c be the constant common-cause failure rate. The common-cause failure probability hypothetical unit is connected in series with the independent failure mode units. A failure of the hypothetical series unit (i.e., the occurrence of a common-cause failure) will cause the system failure. It is assumed that all common-cause failures are completely coupled.

The system reliability can be written as

$$\begin{aligned} R_p(t) &= R_1(t)R_2(t) \\ &= \{1-\prod_{i=1}^{n}(1-R_i)\}R_2(t) \end{aligned}$$

For a 3 identical unit parallel system

$$R_p(t) = \{1-(1-e^{-\lambda t})^3\}e^{-\lambda_c t} \qquad (2.109)$$

$$= \{3e^{(\lambda+\lambda_c)t} - 3e^{(2\lambda+\lambda_c)t} + e^{(3\lambda+\lambda_c)t}\} \qquad (2.110)$$

$$\begin{aligned} MTTF &= \int_0^\infty R(t) \\ &= \frac{11\lambda^2+6\lambda\lambda_c+\lambda_c^2}{(3\lambda+\lambda_c)(2\lambda+\lambda_c)(\lambda+\lambda_c)} \end{aligned} \qquad (2.111)$$

By substituting λ_1 with 3λ, λ_2 with 2λ, λ_3 with λ and letting $\lambda_{c_1} = \lambda_{c_2} = \lambda_{c_3} = \lambda_c$ in Equations (2.74) – (2.78), it can be seen that the resulting system reliability and system mean time to failure expressions are the same as given in Equations (2.110) and (2.111) respectively.

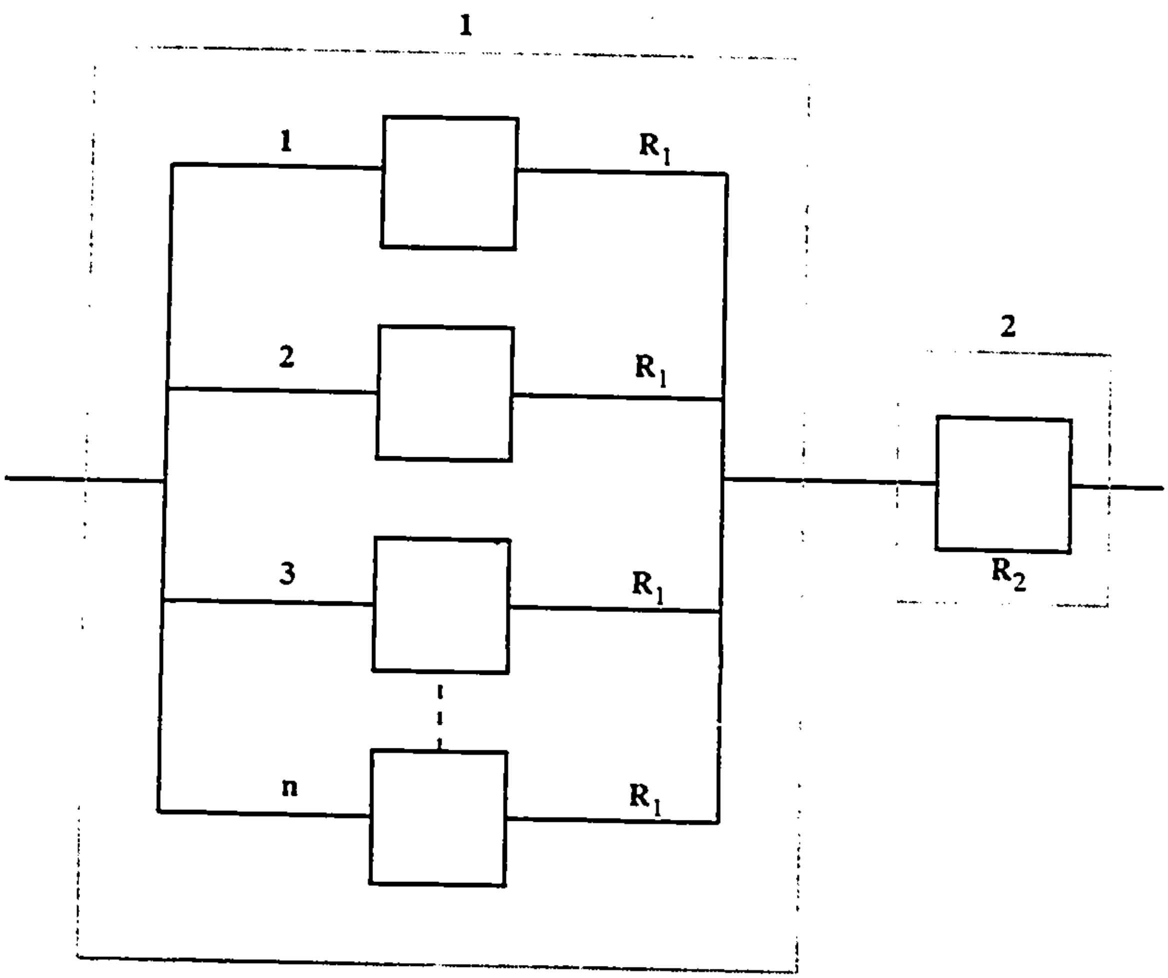

Figure 2.17: Modified Identical Units Parallel Network

2.2 Non-identical Unit Parallel System

The reliability analysis of a repairable and non-repairable, 2 and 3 non-identical unit parallel system with common-cause failures is presented in this section. At time $t = 0$ all the units start working simultaneously. At least one of the units must function properly to ensure that the system operates normally. The occurrence of a common-cause failure can cause the total system failure. In addition, a common-cause failure can occur irrespective of the number of units operating in the system.

Two types of repair policies are considered for the analyses of non-identical unit parallel system.

1. Type I Repair : Under the Type I Repair policy, the partially failed system, such as 1 unit failed and the other two operating, is repaired back to its normal operating state. In addition the system from its failed state (both due to common-cause failures as well as other types of failures) is restored back to its normal operating state 0 (where all the units in the system are operating).
2. Type II Repair : In this case the completely failed system is not repaired. However, the partially failed system (one unit failed and other operating) is repaired back to its normal operating state.

The assumptions associated with the identical unit parallel system hold good for this category as well, except that the system in this category has a specified number of non-identical units.

2.2.1 A Two Non-identical Unit Parallel System

<u>Notation</u>

The following symbols are associated with a two non-identical unit parallel system with Type I Repair, Type II Repair and without Repair :

t	time.
λ_a	Constant failure rate of unit A.
λ_b	Constant failure rate of unit B.

μ_a	Constant repair rate of the unit A.
μ_b	Constant repair rate of the unit B.
μ_3	Constant repair rate of the failed system from state 3 to state 0.
μ_c	Constant repair rate of the failed system from state 4 to state 0.
$\lambda_{c_{i+1}}$	Constant common-cause failure rate from state i; for $i = 0, 1, 2$.
j	j^{th} state of the system; for $j = 0, 1, 2, 3, 4$.
$p_j(t)$	Probability that the system is in state j at time t; for $j = 0, 1, 2, 3, 4$.
$R(t)$	Reliability of the system in $[0, t]$.
s	Laplace transform variable.
CCF	Common-cause failure.
$MTTF$	System mean time to failure.
AV_{ss}	Steady state system availability.
UV_{ss}	Steady state system unavailability.
$\dot{p}_j(t)$	Derivative of $p_j(t)$ with respect to time t; for $j = 0, 1, 2, 3, 4$.
p_i	Steady state probability, that the system is in state i; for $i = 0, 1, 2, 3, 4$.

Parallel System With Type I Repair

The state space diagram for a two non-identical unit parallel system is shown in Figure 2.18. The numerals in the boxes of the figure denote the system state numbers.

The system of differential equations associated with Figure 2.18 is

$$\dot{p}_0(t) = -(\lambda_a + \lambda_b + \lambda_{c_1})p_0(t) + \mu_a p_1(t) + \mu_b p_2(t) + \mu_3 p_3(t) + \mu_c p_4(t) \quad (2.112)$$

$$\dot{p}_1(t) = \lambda_a p_0(t) - (\mu_a + \lambda_b + \lambda_{c_3})p_1(t) \quad (2.113)$$

$$\dot{p}_2(t) = \lambda_b p_0(t) - (\mu_b + \lambda_a + \lambda_{c_2})p_2(t) \quad (2.114)$$

$$\dot{p}_3(t) = \lambda_b p_1(t) + \lambda_a p_2(t) - \mu_3 p_3(t) \quad (2.115)$$

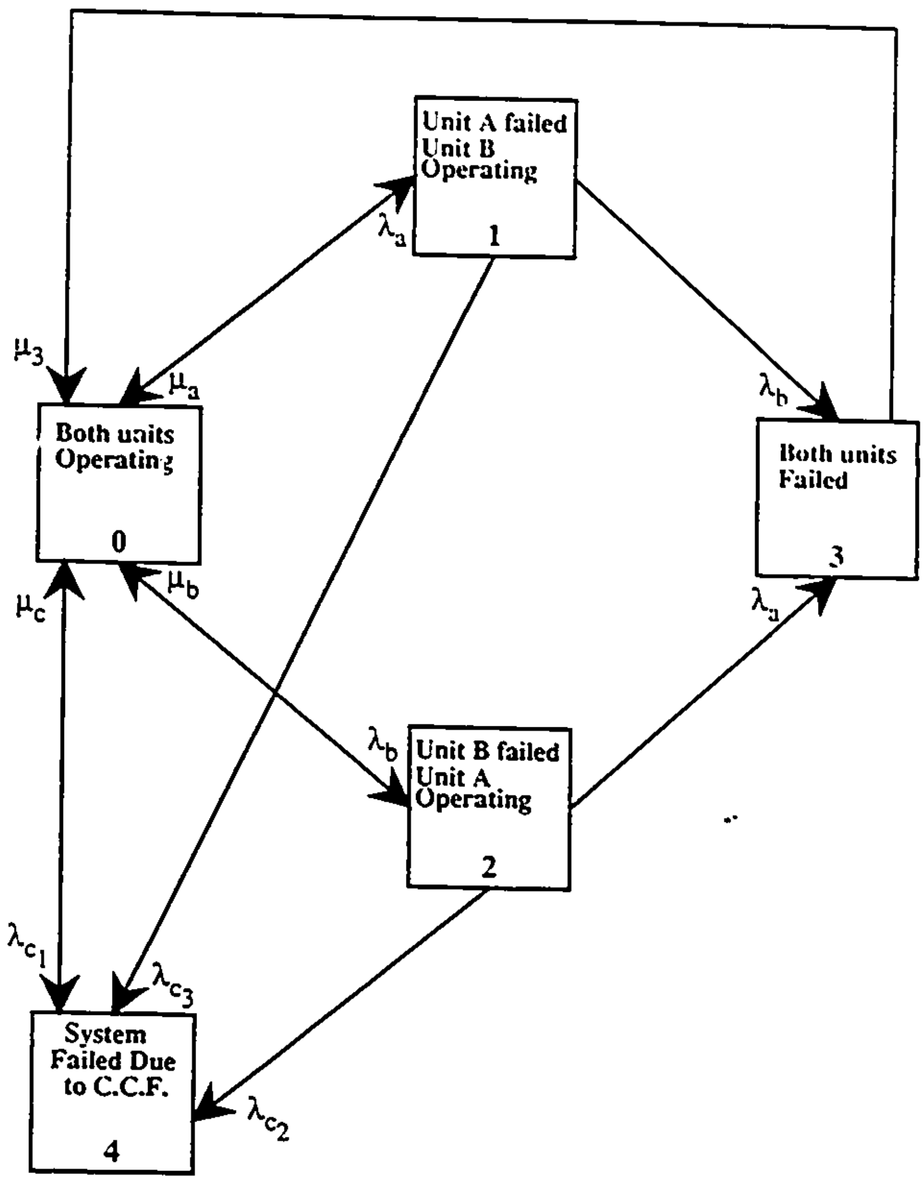

Figure 2.18: State Space Diagram for a two Non-identical Unit Parallel System

$$\dot{p}_4(t) = \lambda_{c_1} p_0(t) + \lambda_{c_2} p_1(t) + \lambda_{c_2} p_2(t) - \mu_c p_4(t) \tag{2.116}$$

At time $t = 0$, $P_0(0) = 1$, $p_1(0) = 0$, $p_2(0) = 0$, $p_3(0) = 0$ *and* $p_4(0) = 0$.

Solving Equations (2.112) – (2.116) using Laplace transforms results in the following state probability expressions :

$$\begin{aligned}
p_0(t) \;=\; & \frac{-D_{27}e^{s_4 t}}{s_4(s_3 - s_4)(s_2 - s_4)(s_1 - s_4)} + \frac{D_{28}e^{s_3 t}}{s_3(s_3 - s_4)(s_2 - s_3)(s_1 - s_3)} - \\
& \frac{D_{29}e^{s_2 t}}{s_2(s_2 - s_4)(s_2 - s_3)(s_1 - s_2)} + \frac{D_{30}e^{s_1 t}}{s_1(s_1 - s_4)(s_1 - s_3)(s_1 - s_2)} + \\
& \frac{D_{31}D_{34}}{s_1 s_2 s_3 s_4}
\end{aligned} \tag{2.117}$$

where s_1, s_2, s_3 *and* s_4 are roots of the quartic equation and are determined as follows:

$$s^4 + D_{35}s^3 + D_{36}s^2 + D_{37}s + D_{38} = 0$$

Let y_1 be a real root of the cubic equation

$$y^3 - D_{36}y^2 + (D_{35}D_{37} - 4D_{38})y + (4D_{36}D_{38} - D_{37}^2 - D_{35}^2 D_{38}) = 0$$

s_1, s_2, s_3 *and* s_4 are the roots of

$$\begin{aligned}
& z^2 + \frac{1}{2}\{D_{35} \pm \sqrt{D_{35}^2 - 4D_{36} + 4y_1}\}z + \frac{1}{2}\{y_1 \pm \sqrt{y_1^2 - 4D_{38}}\} = 0 \\
-D_{35} \;&=\; s_1 + s_2 + s_3 + s_4 \\
D_{36} \;&=\; s_1 s_2 + s_2 s_3 + s_3 s_4 + s_4 s_1 + s_1 s_3 + s_2 s_4 \\
-D_{37} \;&=\; s_1 s_2 s_3 + s_2 s_3 s_4 + s_1 s_2 s_4 + s_1 s_3 s_4 \\
D_{38} \;&=\; s_1 s_2 s_3 s_4
\end{aligned}$$

where the constants D_{27}, D_{28}, D_{29}, ..., D_{48} are defined in Appendix A.

$$\begin{aligned}
p_1(t) \;=\; & \frac{-[s_4^3 + D_{39}s_4^2 + D_{40}s_4 + D_{41}]\lambda_a e^{s_4 t}}{s_4(s_3 - s_4)(s_2 - s_4)(s_1 - s_4)} + \frac{[s_3^3 + D_{39}s_3^2 + D_{40}s_3 + D_{41}]\lambda_a e^{s_3 t}}{s_3(s_3 - s_4)(s_2 - s_3)(s_1 - s_3)} - \\
& \frac{[s_2^3 + D_{39}s_2^2 + D_{40}s_2 + D_{41}]\lambda_a e^{s_2 t}}{s_2(s_2 - s_4)(s_2 - s_3)(s_1 - s_2)} + \frac{[s_1^3 + D_{39}s_1^2 + D_{40}s_1 + D_{41}]\lambda_a e^{s_1 t}}{s_1(s_1 - s_4)(s_1 - s_3)(s_1 - s_2)} + \\
& \frac{D_{41}\lambda_a}{s_1 s_2 s_3 s_4}
\end{aligned} \tag{2.118}$$

$$p_2(t) = \frac{-[s_4^3 + D_{42}s_4^2 + D_{43}s_4 + D_{44}]\lambda_b e^{s_4 t}}{s_4(s_3 - s_4)(s_2 - s_4)(s_1 - s_4)} + \frac{[s_3^3 + D_{42}s_3^2 + D_{43}s_3 + D_{44}]\lambda_b e^{s_3 t}}{s_3(s_3 - s_4)(s_2 - s_3)(s_1 - s_3)} - \frac{[s_2^3 + D_{42}s_2^2 + D_{43}s_2 + D_{44}]\lambda_b e^{s_2 t}}{s_2(s_2 - s_4)(s_2 - s_3)(s_1 - s_2)} + \frac{[s_1^3 + D_{42}s_1^2 + D_{43}s_1 + D_{44}]\lambda_b e^{s_1 t}}{s_1(s_1 - s_4)(s_1 - s_3)(s_1 - s_2)} + \frac{D_{44}\lambda_b}{s_1 s_2 s_3 s_4} \quad (2.119)$$

$$p_3(t) = \frac{-[2s_4^2 + D_{45}s_4 + D_{46}]\lambda_a\lambda_b e^{s_4 t}}{s_4(s_3 - s_4)(s_2 - s_4)(s_1 - s_4)} + \frac{[2s_3^2 + D_{45}s_3 + D_{46}]\lambda_a\lambda_b e^{s_3 t}}{s_3(s_3 - s_4)(s_2 - s_3)(s_1 - s_3)} - \frac{[2s_2^2 + D_{45}s_2 + D_{46}]\lambda_a\lambda_b e^{s_2 t}}{s_2(s_2 - s_4)(s_2 - s_3)(s_1 - s_2)} + \frac{[2s_1^2 + D_{45}s_1 + D_{46}]\lambda_a\lambda_b e^{s_1 t}}{s_1(s_1 - s_4)(s_1 - s_3)(s_1 - s_2)} + \frac{\lambda_a\lambda_b D_{46}}{s_1 s_2 s_3 s_4} \quad (2.120)$$

$$p_4(t) = \frac{-[\lambda_{c_1}s_4^3 + (\lambda_{c_1}\mu_3 + D_{47})s_4^2 + (D_{47}\mu_3 + D_{48})s_4 + \mu_3 D_{48}]e^{s_4 t}}{s_4(s_3 - s_4)(s_2 - s_4)(s_1 - s_4)} + \frac{[\lambda_{c_1}s_3^3 + (\lambda_{c_1}\mu_3 + D_{47})s_3^2 + (D_{47}\mu_3 + D_{48})s_3 + \mu_3 D_{48}]e^{s_3 t}}{s_3(s_3 - s_4)(s_2 - s_3)(s_1 - s_3)} - \frac{[\lambda_{c_1}s_2^3 + (\lambda_{c_1}\mu_3 + D_{47})s_2^2 + (D_{47}\mu_3 + D_{48})s_2 + \mu_3 D_{48}]e^{s_2 t}}{s_2(s_2 - s_4)(s_2 - s_3)(s_1 - s_2)} + \frac{[\lambda_{c_1}s_1^3 + (\lambda_{c_1}\mu_3 + D_{47})s_1^2 + (D_{47}\mu_3 + D_{48})s_1 + \mu_3 D_{48}]e^{s_1 t}}{s_1(s_1 - s_4)(s_1 - s_3)(s_1 - s_2)} + \frac{D_{48}\mu_3}{s_1 s_2 s_3 s_4} \quad (2.121)$$

The availability of the redundant system is given by

$$AV_s(t) = p_0(t) + p_1(t) + p_2(t) \quad (2.122)$$

The plots of Equation (2.122) are shown in Figure 2.19. It is evident from the plots that the system availability decreases with an increase in the values of the common-cause failure rate.

By setting the derivatives in Equations (2.112) – (2.116) equal to zero and by using the relationship $\sum_{i=0}^{4} p_i = 1$, we get the following expressions for steady state probabilities :

$$p_0 = \frac{(\mu_a + \lambda_b + \lambda_{c_3})(\mu_b + \lambda_a + \lambda_{c_2})\mu_3\mu_c}{D_{49}} \quad (2.123)$$

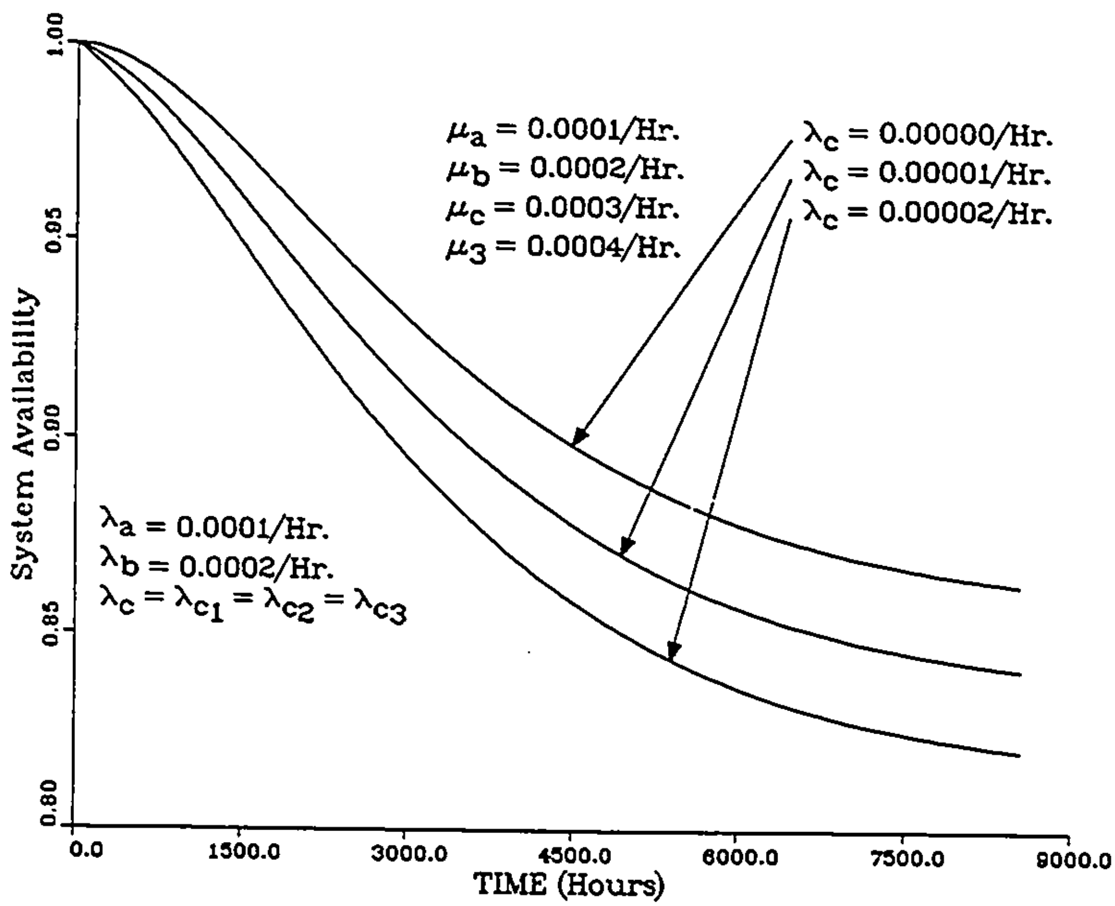

Figure 2.19: Time Dependent System Availability Plots for a two Non-identical Unit Parallel System with Type I Repair

where the constant D_{49} is defined in Appendix A.

$$p_1 = \frac{\lambda_a \mu_3 \mu_c(\mu_b + \lambda_a + \lambda_{c_2})}{D_{49}} \tag{2.124}$$

$$p_2 = \frac{\mu_c \mu_3 \lambda_b(\mu_a + \lambda_b + \lambda_{c_1})}{D_{49}} \tag{2.125}$$

$$p_3 = \frac{\lambda_a \mu_c \lambda_b(\mu_b + \lambda_a + \lambda_{c_2} + \mu_a + \lambda_b + \lambda_{c_1})}{D_{49}} \tag{2.126}$$

$$\begin{aligned} p_4 \ = \ & \frac{\mu_3(\lambda_{c_2}\lambda_{c_1}\mu_a + \mu_b\lambda_{c_1}\lambda_{c_3} + \mu_b\lambda_{c_1}\mu_a + \mu_b\lambda_a\lambda_{c_3} + \mu_b\lambda_{c_1}\lambda_b + \lambda_{c_2}\lambda_{c_1}\lambda_{c_3})}{D_{49}} + \\ & \frac{\mu_3(\lambda_{c_2}\lambda_b\mu_a + \lambda_a\lambda_{c_1}\lambda_{c_3} + \lambda_a\lambda_{c_1}\lambda_b + \lambda_a\lambda_{c_1}\mu_a + \lambda_{c_2}\lambda_b\lambda_{c_3} + \lambda_{c_2}\lambda_a\lambda_{c_3})}{D_{49}} + \\ & \frac{\mu_3(\lambda_{c_2}\lambda_{c_1}\lambda_b + \lambda_{c_2}\lambda_b^2 + \lambda_a^2\lambda_{c_3})}{D_{49}} \end{aligned} \tag{2.127}$$

The steady state system availability is given by

$$AV_{ss} = p_0 + p_1 + p_2 \tag{2.128}$$

The plots of the steady state system availability are given in Figure 2.20 and show that the increase in the values of common-cause failure rates decreases the system availability.

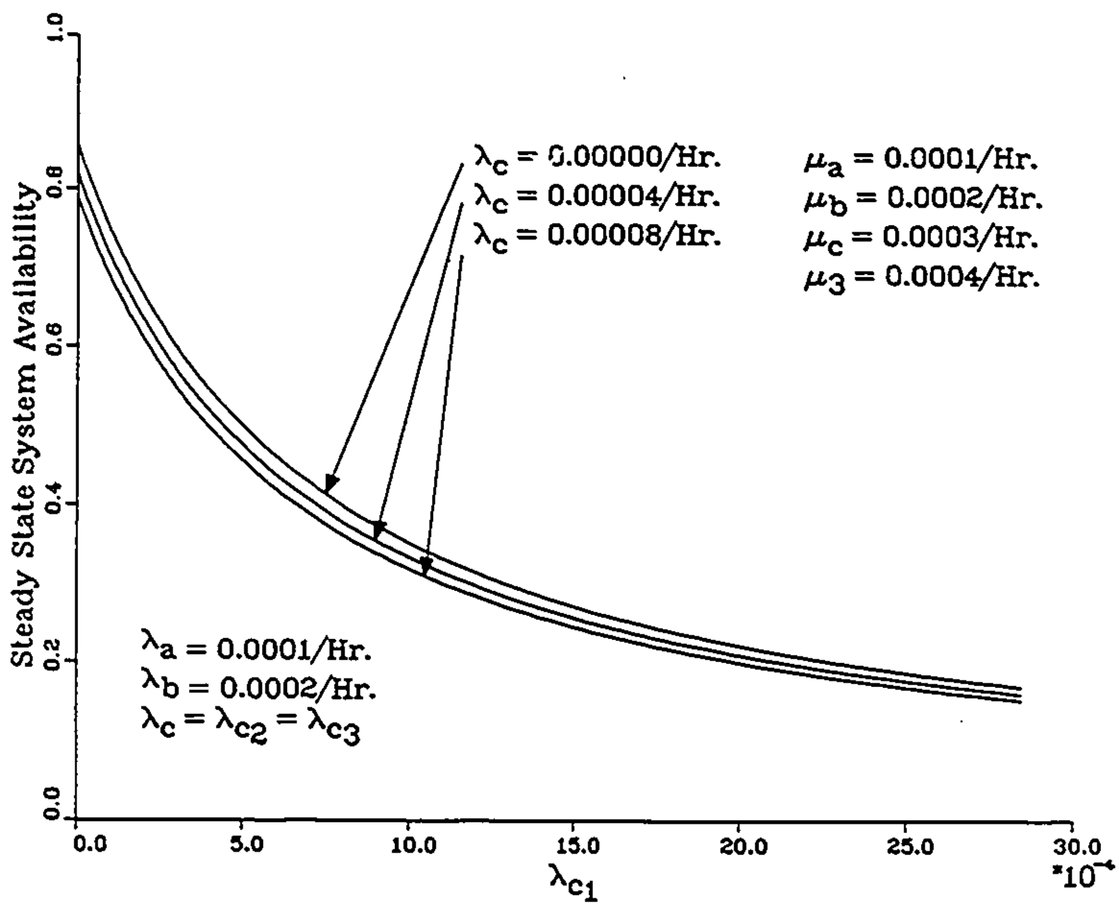

Figure 2.20: Steady State System Availability for a two Non-identical Unit Parallel System with Type I Repair

Parallel System With Type II Repair

Setting the repair rates μ_3 *and* μ_c equal to zero in Figure 2.18, results in the following system of differential equations :

$$\dot{p}_0(t) = -(\lambda_a + \lambda_b + \lambda_{c_1})p_0(t) + \mu_a p_1(t) + \mu_b p_2(t) \tag{2.129}$$

$$\dot{p}_1(t) = \lambda_a p_0(t) - (\mu_a + \lambda_b + \lambda_{c_3})p_1(t) \tag{2.130}$$

$$\dot{p}_2(t) = \lambda_b p_0(t) - (\mu_b + \lambda_a + \lambda_{c_2})p_2(t) \tag{2.131}$$

$$\dot{p}_3(t) = \lambda_b p_1(t) + \lambda_a p_2(t) \tag{2.132}$$

$$\dot{p}_4(t) = \lambda_{c_1} p_0(t) + \lambda_{c_3} p_1(t) + \lambda_{c_2} p_2(t) \tag{2.133}$$

The initial conditions of the system are, at time $t = 0$, $p_0(0) = 1$, $p_1(0) = 0$, $p_2(0) = 0$, $p_3(0) = 0$, *and* $p_4(0) = 0$.

The following time dependent state probabilities can be obtained by solving Equations (2.129) – (2.133) with Laplace transforms :

$$p_0(t) = \frac{[s_3^2 + D_{50}s_3 + D_{51}]e^{s_3 t}}{(s_3 - s_2)(s_3 - s_1)} - \frac{[s_2^2 + D_{50}s_2 + D_{51}]e^{s_2 t}}{(s_3 - s_2)(s_2 - s_1)} + \frac{[s_1^2 + D_{50}s_1 + D_{51}]e^{s_1 t}}{(s_2 - s_1)(s_3 - s_1)} \tag{2.134}$$

where s_1, s_2, s_3 are the roots of the cubic equation and are determined as follows :

$$s^3 + D_{52}s^2 + D_{53}s + D_{54} = 0$$

$$Let\ \alpha = \frac{3D_{53} - (D_{52})^2}{9}$$

$$\beta = \frac{9D_{52}D_{53} - 27D_{54} - 2(D_{52})^3}{54}$$

$$\Phi = \sqrt[3]{\beta + \sqrt{\alpha^3 + \beta^2}}$$

$$\Omega = \sqrt[3]{\beta - \sqrt{\alpha^3 + \beta^2}}$$

$$Thus,\ s_1 = \Phi + \Omega - \frac{1}{3}D_{52}$$

$$s_2 = -\frac{1}{2}(\Phi + \Omega) - \frac{1}{3}D_{52} + \frac{1}{2}i\sqrt{3}(\Phi - \Omega)$$

$$s_3 = -\frac{1}{2}(\Phi + \Omega) - \frac{1}{3}D_{52} - \frac{1}{2}i\sqrt{3}(\Phi - \Omega)$$

The above i is associated with complex numbers.

$$-D_{52} = s_1 + s_2 + s_3$$
$$D_{53} = s_1 s_2 + s_2 s_3 + s_3 s_1$$
$$-D_{54} = s_1 s_2 s_3$$

where D_{50}, D_{51}, D_{52}, ..., D_{59} are defined in Appendix A.

$$p_1(t) = \frac{(s_3\lambda_a + D_{55})e^{s_3 t}}{(s_3 - s_2)(s_3 - s_1)} - \frac{(s_2\lambda_a + D_{55})e^{s_2 t}}{(s_3 - s_2)(s_2 - s_1)} + \frac{(s_1\lambda_a + D_{55})e^{s_1 t}}{(s_3 - s_1)(s_2 - s_1)} \tag{2.135}$$

$$p_2(t) = \frac{(s_3\lambda_b + D_{56})e^{s_3 t}}{(s_3 - s_2)(s_3 - s_1)} - \frac{(s_2\lambda_b + D_{56})e^{s_2 t}}{(s_3 - s_2)(s_2 - s_1)} + \frac{(s_1\lambda_b + D_{56})e^{s_1 t}}{(s_3 - s_1)(s_2 - s_1)} \tag{2.136}$$

$$p_3(t) = \frac{(2\lambda_a\lambda_b s_3 + D_{57})e^{s_3 t}}{s_3(s_3 - s_1)(s_3 - s_2)} - \frac{(2\lambda_a\lambda_b s_2 + D_{57})e^{s_2 t}}{s_2(s_3 - s_2)(s_2 - s_1)} + \frac{(2\lambda_a\lambda_b s_1 + D_{57})e^{s_1 t}}{s_1(s_3 - s_1)(s_2 - s_1)} - \frac{D_{57}}{s_1 s_2 s_3} \tag{2.137}$$

$$p_4(t) = \frac{(\lambda_{c_1} s_3^2 + D_{58} s_3 + D_{59})e^{s_3 t}}{s_3(s_3 - s_1)(s_3 - s_2)} - \frac{(\lambda_{c_1} s_2^2 + D_{58} s_2 + D_{59})e^{s_2 t}}{s_2(s_3 - s_2)(s_2 - s_1)} + \frac{(\lambda_{c_1} s_1^2 + D_{58} s_1 + D_{59})e^{s_1 t}}{s_1(s_3 - s_1)(s_2 - s_1)} - \frac{D_{59}}{s_1 s_2 s_3} \tag{2.138}$$

The reliability of the redundant system is given by

$$\begin{aligned} R(t) &= p_0(t) + p_1(t) + p_2(t) \\ &= \frac{[s_3^2 + (\lambda_a + \lambda_b + D_{50})s_3 + (D_{55} + D_{51} + D_{56})]e^{s_3 t}}{(s_3 - s_2)(s_3 - s_1)} - \\ &\quad \frac{[s_2^2 + (\lambda_a + \lambda_b + D_{50})s_2 + (D_{55} + D_{51} + D_{56})]e^{s_2 t}}{(s_3 - s_2)(s_2 - s_1)} + \\ &\quad \frac{[s_1^2 + (\lambda_a + \lambda_b + D_{50})s_1 + (D_{55} + D_{51} + D_{56})]e^{s_1 t}}{(s_2 - s_1)(s_3 - s_1)} \end{aligned} \tag{2.139}$$

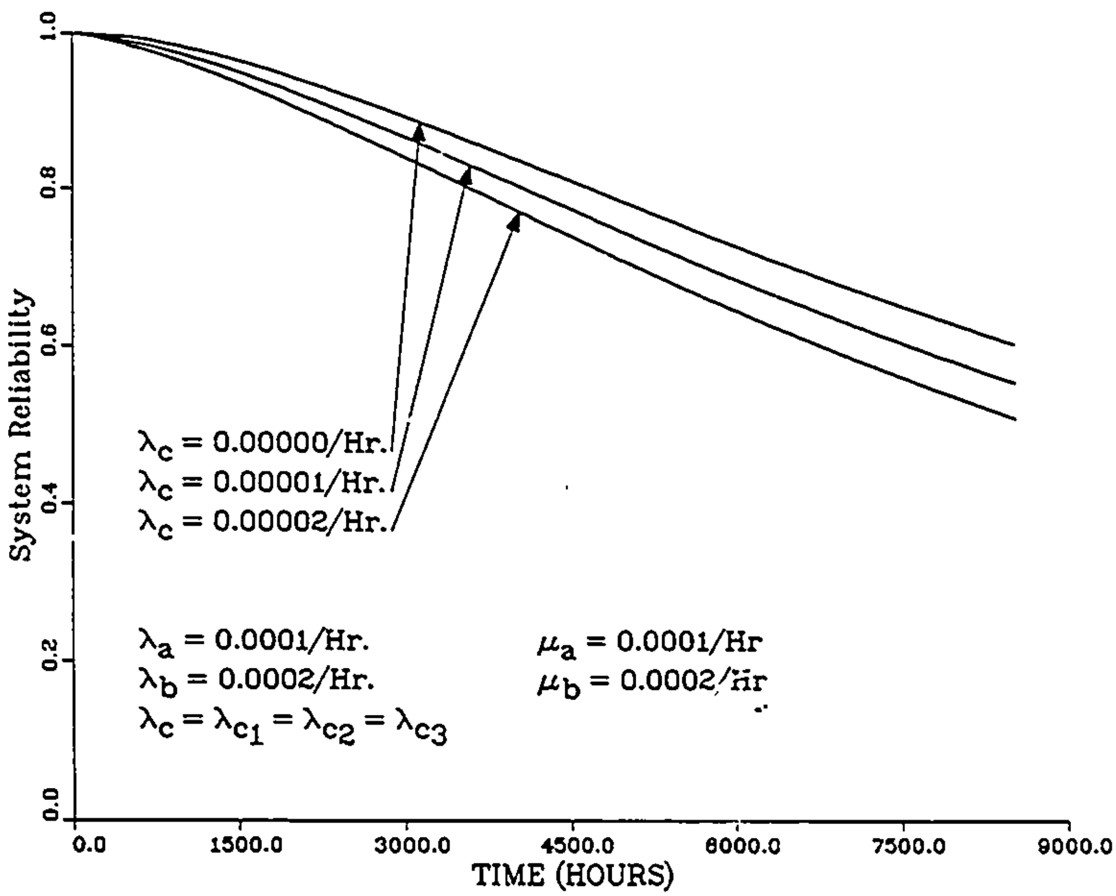

Figure 2.21: System Reliability Plots for a two Non-identical Unit Parallel System with Type II Repair

The plots of the system reliability given by Equation (2.139) are shown in Figure 2.21. The plots show a decreasing trend in system reliability with an increase in common-cause failures.

The system mean time to failure is given by

$$
\begin{aligned}
MTTF &= \lim_{s \to 0} R(s) = [p_0(s) + p_1(s) + p_2(s)] \\
&= \frac{-[D_{55} + D_{56} + D_{51}]}{s_1 s_2 s_3} \qquad (2.140)
\end{aligned}
$$

The plots of Equation (2.140) are given in Figure 2.22. The plots clearly show the decrease in $MTTF$ with an increase in common-cause failure rates.

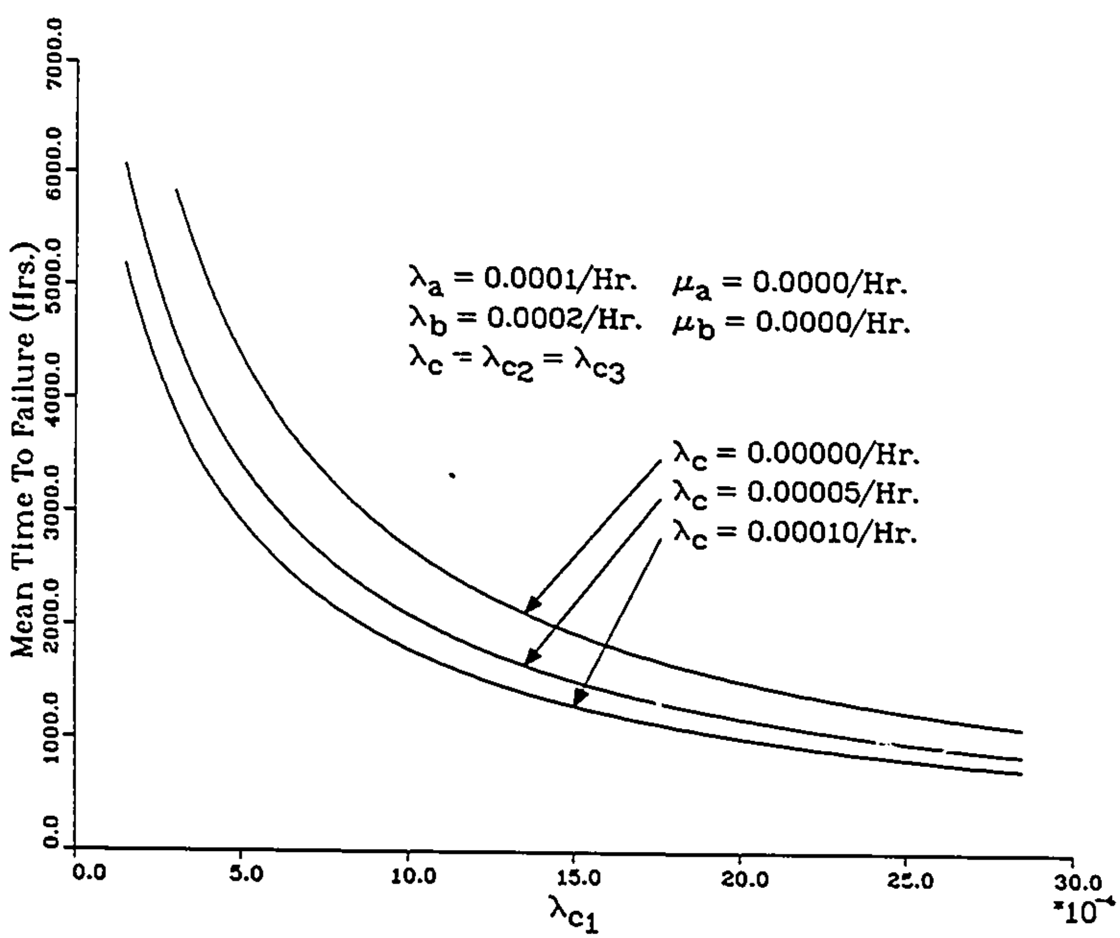

Figure 2.22: System Mean Time to Failure Plots for a two Non-identical Unit Parallel System with Type II Repair

Parallel System Without Repair

Setting the repair rates μ_a, μ_b, μ_c *and* μ_3 equal to zero in Figure 2.18 results in the following system of differential equations :

$$\dot{p}_0(t) = -(\lambda_a + \lambda_b + \lambda_{c_1})p_0(t) \quad (2.141)$$

$$\dot{p}_1(t) = \lambda_a p_0(t) - (\lambda_b + \lambda_{c_3})p_1(t) \quad (2.142)$$

$$\dot{p}_2(t) = \lambda_b p_0(t) - (\lambda_a + \lambda_{c_2})p_2(t) \quad (2.143)$$

$$\dot{p}_3(t) = \lambda_b p_1(t) + \lambda_a p_2(t) \quad (2.144)$$

$$\dot{p}_4(t) = \lambda_{c_1} p_0(t) + \lambda_{c_3} p_1(t) + \lambda_{c_2} p_2(t) \quad (2.145)$$

At time $t = 0$, $p_0(0) = 1, p_1(0) = 0, p_2(0) = 0, p_3(0) = 0,$ *and* $p_4(0) = 0$.

Solving Equations (2.141) – (2.145) we get

$$p_0(t) = e^{[-(\lambda_{c_1}+\lambda_a+\lambda_b)t]} \quad (2.146)$$

$$p_1(t) = \frac{-\lambda_a\{e^{[-(\lambda_{c_1}+\lambda_a+\lambda_b)t]} - e^{[-(\lambda_b+\lambda_{c_3})]t}\}}{\lambda_a - \lambda_{c_3} + \lambda_{c_1}} \quad (2.147)$$

$$p_2(t) = \frac{\lambda_b\{e^{[-(\lambda_{c_1}+\lambda_a+\lambda_b)t]} - e^{[-(\lambda_a+\lambda_{c_2}]t}\}}{\lambda_{c_2} - \lambda_{c_1} - \lambda_b} \quad (2.148)$$

$$\begin{aligned} p_3(t) = {} & [\lambda_{c_3}^2(\lambda_a + \lambda_{c_2}) - \lambda_b^2(\lambda_a + \lambda_{c_2}) + \lambda_{c_2}^2(\lambda_b + \lambda_{c_3}) - \lambda_a^2(\lambda_b + \lambda_{c_3})] \times \\ & \frac{e^{[-(\lambda_{c_1}+\lambda_a+\lambda_b)t]}\lambda_b\lambda_a}{[(\lambda_a + \lambda_{c_2})(\lambda_{c_1} + \lambda_a + \lambda_b)(\lambda_b + \lambda_{c_3})(\lambda_a - \lambda_{c_3} + \lambda_{c_1})(\lambda_{c_2} - \lambda_{c_1} - \lambda_b)]} - \\ & \frac{2\lambda_{c_1}(\lambda_{c_2}\lambda_b + \lambda_{c_3}\lambda_a + \lambda_{c_2}\lambda_{c_3} + \lambda_a\lambda_b)e^{[-(\lambda_{c_1}+\lambda_a+\lambda_b)t]}\lambda_b\lambda_a}{[(\lambda_a + \lambda_{c_2})(\lambda_{c_1} + \lambda_a + \lambda_b)(\lambda_b + \lambda_{c_3})(\lambda_a - \lambda_{c_3} + \lambda_{c_1})(\lambda_{c_2} - \lambda_{c_1} - \lambda_b)]} - \\ & [\lambda_{c_3}^2(\lambda_a + \lambda_b + \lambda_{c_1}) + \lambda_b^2(\lambda_{c_3} - \lambda_{c_1} - \lambda_a) - (\lambda_b + \lambda_{c_3})(\lambda_a^2 + \lambda_{c_1}^2)] \times \\ & \frac{e^{[-(\lambda_a+\lambda_{c_2})t]}\lambda_b\lambda_a}{[(\lambda_a + \lambda_{c_2})(\lambda_{c_1} + \lambda_a + \lambda_b)(\lambda_b + \lambda_{c_3})(\lambda_a - \lambda_{c_3} + \lambda_{c_1})(\lambda_{c_2} - \lambda_{c_1} - \lambda_b)]} - \\ & \frac{2\lambda_{c_1}\lambda_a(\lambda_b + \lambda_{c_3})]e^{[-(\lambda_a+\lambda_{c_2})t]}\lambda_b\lambda_a}{[(\lambda_a + \lambda_{c_2})(\lambda_{c_1} + \lambda_a + \lambda_b)(\lambda_b + \lambda_{c_3})(\lambda_a - \lambda_{c_3} + \lambda_{c_1})(\lambda_{c_2} - \lambda_{c_1} - \lambda_b)]} + \end{aligned}$$

$$
\frac{[(\lambda_{c_2}+\lambda_a)(\lambda_a^2+\lambda_{c_1}^2+2\lambda_{c_1}\lambda_b)+\lambda_a^2(\lambda_b+\lambda_{c_1}-\lambda_{c_2})]e^{-(\lambda_b+\lambda_{c_3})t}\lambda_b\lambda_a}{[(\lambda_a+\lambda_{c_2})(\lambda_{c_1}+\lambda_a+\lambda_b)(\lambda_b+\lambda_{c_3})(\lambda_a-\lambda_{c_3}+\lambda_{c_1})(\lambda_{c_2}-\lambda_{c_1}-\lambda_b)]}
$$
$$
\frac{\lambda_{c_1}^2(\lambda_b+\lambda_a+\lambda_{c_1})e^{-(\lambda_a+\lambda_{c_2})t}\lambda_b\lambda_a}{[(\lambda_a+\lambda_{c_2})(\lambda_{c_1}+\lambda_a+\lambda_b)(\lambda_b+\lambda_{c_3})(\lambda_a-\lambda_{c_3}+\lambda_{c_1})(\lambda_{c_2}-\lambda_{c_1}-\lambda_b)]}+
$$
$$
\frac{\lambda_a\lambda_b(\lambda_b+\lambda_{c_2}+\lambda_a+\lambda_{c_1})}{[(\lambda_b+\lambda_{c_3})(\lambda_{c_1}+\lambda_a+\lambda_b)(\lambda_a+\lambda_{c_2})]} \quad (2.149)
$$

$$
\begin{aligned}
p_4(t) &= \frac{D_{60}e^{-(\lambda_{c_1}+\lambda_a+\lambda_b)t}}{(\lambda_a+\lambda_{c_2})(\lambda_{c_1}+\lambda_a+\lambda_b)(\lambda_b+\lambda_{c_3})(\lambda_a-\lambda_{c_3}+\lambda_{c_1})(\lambda_{c_2}-\lambda_{c_1}-\lambda_b)}+ \\
&\frac{D_{61}e^{-(\lambda_a+\lambda_{c_2})t}}{(\lambda_a+\lambda_{c_2})(\lambda_{c_1}+\lambda_a+\lambda_b)(\lambda_b+\lambda_{c_3})(\lambda_a-\lambda_{c_3}+\lambda_{c_1})(\lambda_{c_2}-\lambda_{c_1}-\lambda_b)}+ \\
&\frac{D_{62}e^{-(\lambda_b+\lambda_{c_3})t}}{(\lambda_a+\lambda_{c_2})(\lambda_{c_1}+\lambda_a+\lambda_b)(\lambda_b+\lambda_{c_3})(\lambda_a-\lambda_{c_3}+\lambda_{c_1})(\lambda_{c_2}-\lambda_{c_1}-\lambda_b)}+ \\
&\frac{D_{63}}{(\lambda_a+\lambda_{c_2})(\lambda_{c_1}+\lambda_a+\lambda_b)(\lambda_b+\lambda_{c_3})} \quad (2.150)
\end{aligned}
$$

where D_{60}, D_{61}, D_{62} *and* D_{63} are defined in Appendix A.

The system reliability is given by

$$
\begin{aligned}
R(t) &= p_0(t)+p_1(t)+p_2(t) \\
&= \frac{[\lambda_b\lambda_a+\lambda_{c_1}\lambda_{c_3}+\lambda_{c_2}\lambda_{c_1}-\lambda_{c_1}^2-\lambda_{c_3}\lambda_{c_2}]e^{-(\lambda_{c_1}+\lambda_a+\lambda_b)t}}{(\lambda_{c_2}-\lambda_{c_1}-\lambda_b)(\lambda_a-\lambda_{c_3}+\lambda_{c_1})}+ \\
&\frac{[\lambda_b\lambda_{c_2}-\lambda_{c_1}\lambda_b-\lambda_b\lambda_a]e^{-(\lambda_a+\lambda_{c_2})t}}{(\lambda_{c_2}-\lambda_{c_1}-\lambda_b)(\lambda_a-\lambda_{c_3}+\lambda_{c_1})}+ \\
&\frac{[\lambda_a\lambda_{c_2}-\lambda_a\lambda_{c_1}-\lambda_b\lambda_a]e^{-(\lambda_b+\lambda_{c_3})t}}{(\lambda_{c_2}-\lambda_{c_1}-\lambda_b)(\lambda_a-\lambda_{c_3}+\lambda_{c_1})} \quad (2.151)
\end{aligned}
$$

The plots of Equation (2.151) are shown in Figure 2.23. These plots clearly show the reduction in system reliability with increasing values of common-cause failure rate λ_c and time t.

The system mean time to failure is given by

$$
\begin{aligned}
MTTF &= \lim_{s\to 0} R(s) = [p_0(s)+p_1(s)+p_2(s)] \\
&= \frac{(\lambda_a\lambda_{c_2}+\lambda_a^2+\lambda_{c_3}\lambda_a+\lambda_b\lambda_a+\lambda_{c_3}\lambda_{c_2}+\lambda_b^2+\lambda_b\lambda_{c_3}+\lambda_b\lambda_{c_2})}{(\lambda_b+\lambda_{c_3})(\lambda_{c_1}+\lambda_a+\lambda_b)(\lambda_a+\lambda_{c_2})} \quad (2.152)
\end{aligned}
$$

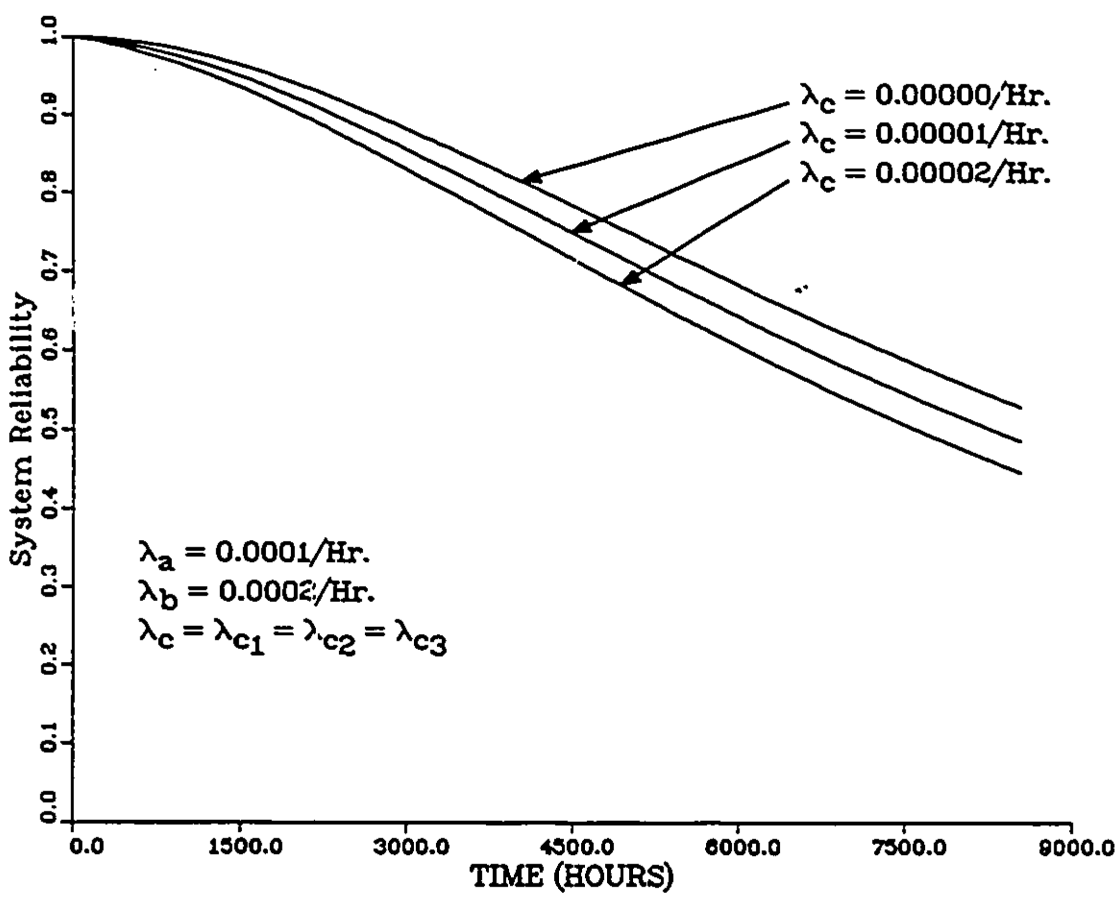

Figure 2.23: System Reliability Plots for a two Non-identical Unit Parallel System without Repair

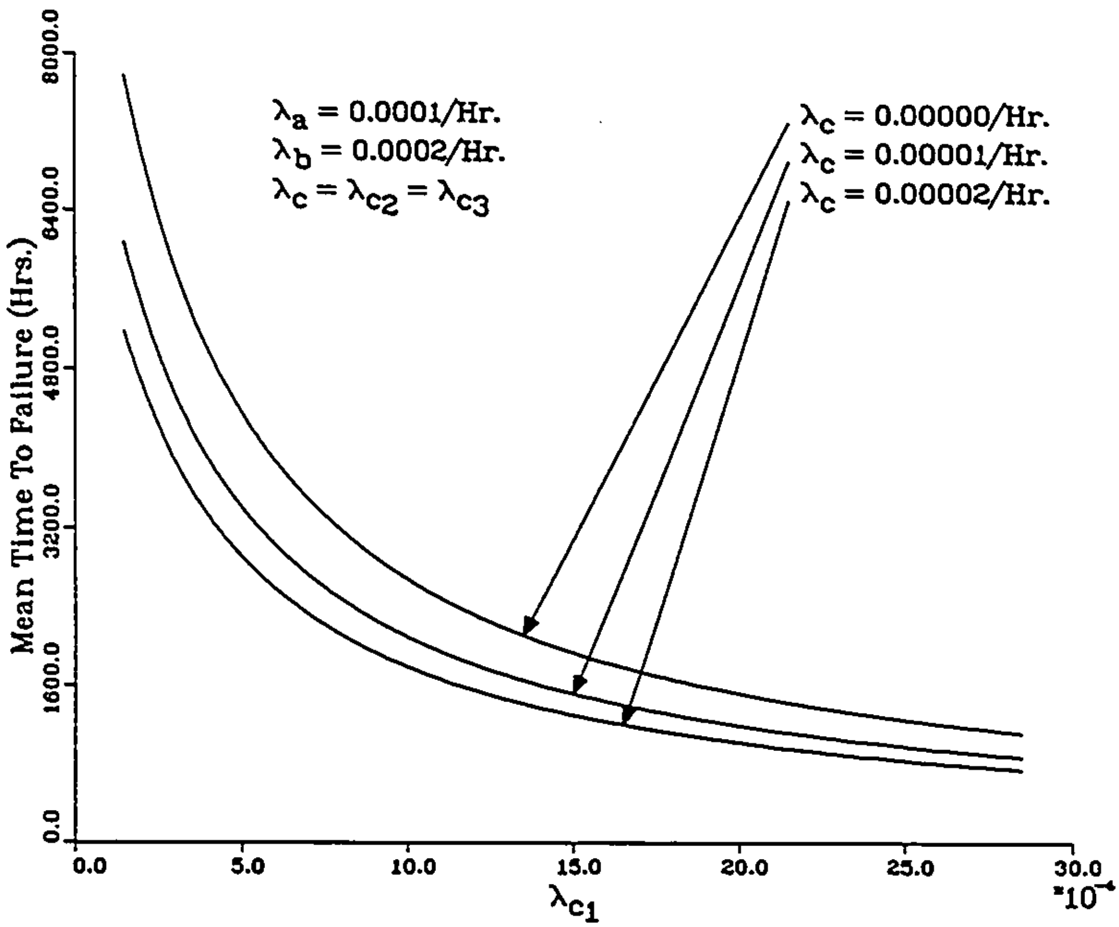

Figure 2.24: System Mean Time to Failure Plots for a two Non-identical Unit Parallel System without Repair

The decreasing effect of common-cause failures on system mean time to failure can be clearly noted from Figure 2.24.

2.2.2 A Three Non-identical Unit Parallel System

Notation

The following symbols are associated with this model :

t	time.
λ_a	Constant failure rate of unit A.
λ_b	Constant failure rate of unit B.
λ_c	Constant failure rate of unit C.
μ_a	Constant repair rate of the unit A.
μ_b	Constant repair rate of the unit B.
μ_c	Constant repair rate of the unit C.
μ_2	Constant repair rate of the failed system from state 7 to state 0.
μ_1	Constant repair rate of the failed system from state 8 to state 0.
$\lambda_{c_{i+1}}$	Constant common-cause failure rate from state i; for $i = 0, 1, 2, \ldots, 6$.
j	j^{th} state of the system; for $j = 0, 1, 2, \ldots, 8$.
$p_j(t)$	Probability that the system is in state j at time t; for $j = 0, 1, 2, \ldots, 8$.
$R(t)$	Reliability of the system in $[0, t]$.
s	Laplace transform variable.
CCF	Common-cause failure.
$MTTF$	System mean time to failure.
AV_{ss}	Steady state system availability.
UV_{ss}	Steady state system unavailability.
$\dot{p}_j(t)$	Derivative of $p_j(t)$ with respect to time t; for $j = 0, 1, 2, \ldots, 8$.
p_i	Steady state probability, that the system is in state i; for $i = 0, 1, 2, \ldots, 8$.

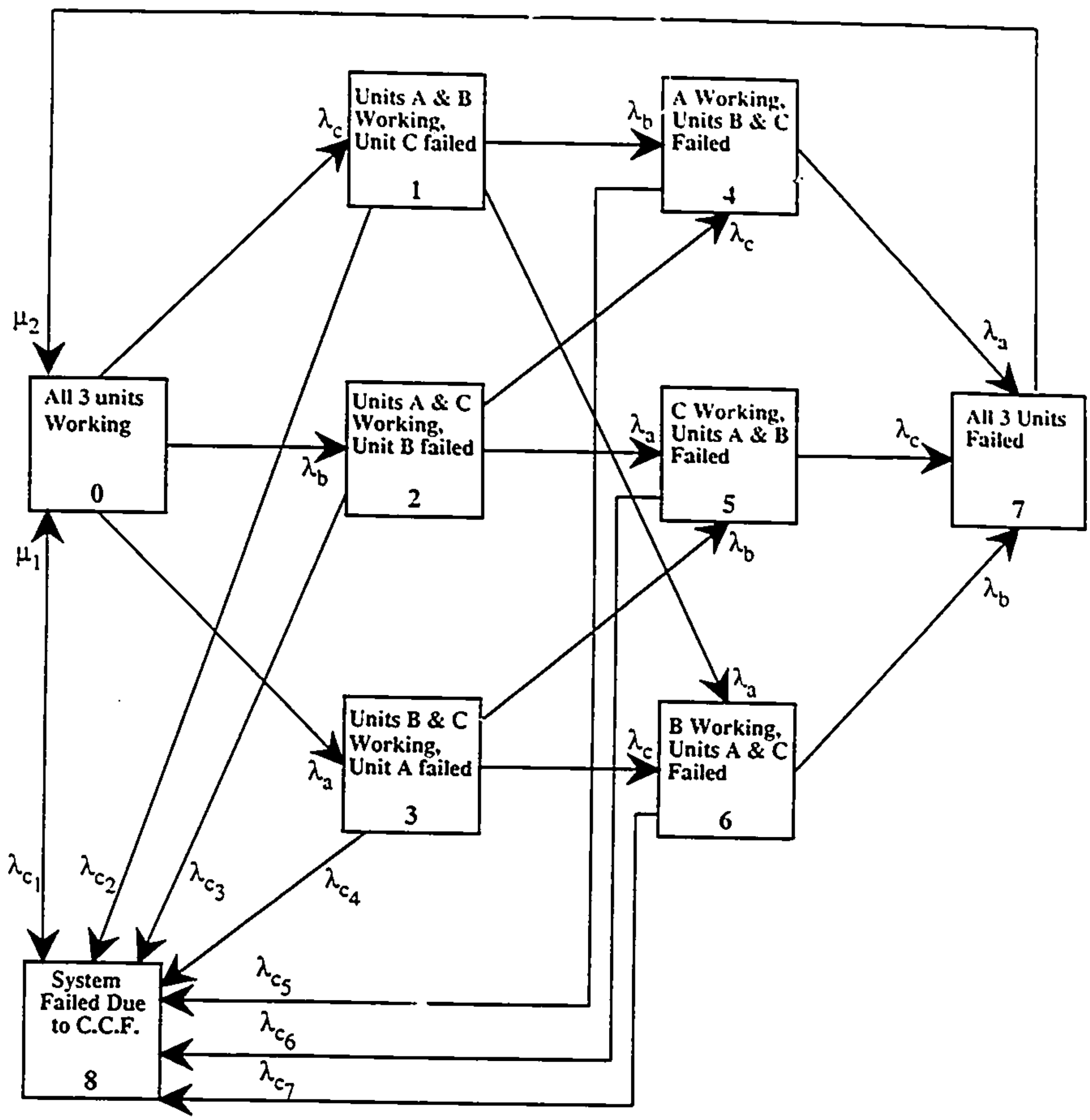

Figure 2.25: State Space Diagram for a Three Non-identical Unit Parallel System

Parallel System With Type I Repair

The state space diagram for this model is shown in Figure 2.25. The system state numbers are denoted by the numerals within each box in the figure.

The system of differential equations associated with this model is

$$\dot{p}_0(t) = -(\lambda_a + \lambda_b + \lambda_c + \lambda_{c_1})p_0(t) + \mu_2 p_7(t) + \mu_1 p_8(t) \tag{2.153}$$

$$\dot{p}_1(t) = \lambda_c p_0(t) - (\lambda_a + \lambda_b + \lambda_{c_2})p_1(t) \tag{2.154}$$

$$\dot{p}_2(t) = \lambda_b p_0(t) - (\lambda_a + \lambda_c + \lambda_{c_3})p_2(t) \tag{2.155}$$

$$\dot{p}_3(t) = \lambda_a p_0(t) - (\lambda_b + \lambda_c + \lambda_{c_4})p_3(t) \tag{2.156}$$

$$\dot{p}_4(t) = \lambda_b p_1(t) + \lambda_c p_2(t) - (\lambda_a + \lambda_{c_5})p_4(t) \tag{2.157}$$

$$\dot{p}_5(t) = \lambda_a p_2(t) + \lambda_b p_3(t) - (\lambda_c + \lambda_{c_6})p_5(t) \tag{2.158}$$

$$\dot{p}_6(t) = \lambda_a p_1(t) + \lambda_c p_3(t) - (\lambda_b + \lambda_{c_7})p_6(t) \tag{2.159}$$

$$\dot{p}_7(t) = \lambda_a p_4(t) + \lambda_c p_5(t) + \lambda_b p_6(t) - \mu_2 p_7(t) \tag{2.160}$$

$$\begin{aligned} \dot{p}_8(t) \quad = \quad & \lambda_{c_1} p_0(t) + \lambda_{c_2} p_1(t) + \lambda_{c_3} p_2(t) + \lambda_{c_4} p_3(t) + \lambda_{c_5} p_4(t) + \\ & \lambda_{c_6} p_5(t) + \lambda_{c_7} p_6(t) - \mu_1 p_8(t) \end{aligned} \tag{2.161}$$

At time $t = 0$, $p_0(0) = 1$ and all other initial condition probabilities are equal to zero.

Solving the Equations (2.153) – (2.161) by setting their derivatives equal to zero and by making use of the relationship $\sum_{i=0}^{8} p_i = 1$, results in the following steady state probabilities :

$$p_0 = \frac{D_{64}}{D_{73}} \tag{2.162}$$

$$p_1 = \frac{D_{65}}{D_{73}} \tag{2.163}$$

$$p_2 = \frac{D_{66}}{D_{73}} \tag{2.164}$$

$$p_3 = \frac{D_{67}}{D_{73}} \tag{2.165}$$

$$p_4 = \frac{D_{68}}{D_{73}} \tag{2.166}$$

$$p_5 = \frac{D_{69}}{D_{73}} \tag{2.167}$$

$$p_6 = \frac{D_{70}}{D_{73}} \tag{2.168}$$

$$p_7 = \frac{D_{71}}{D_{73}} \tag{2.169}$$

$$p_8 = \frac{D_{72}}{D_{73}} \tag{2.170}$$

where the constants D_{64}, D_{65}, ..., D_{73} are defined in Appendix A. The steady state system availability and steady state system unavailability can be expressed as

$$AV_{ss} = p_0 + p_1 + p_2 + p_3 + p_4 + p_5 + p_6 \tag{2.171}$$

$$UV_{ss} = p_7 + p_8 \tag{2.172}$$

The steady state system availability and steady state system unavailability are plotted in Figures 2.26 and 2.27 respectively. From these plots it can be seen that the increase in common-cause failures decreases the steady state system availability.

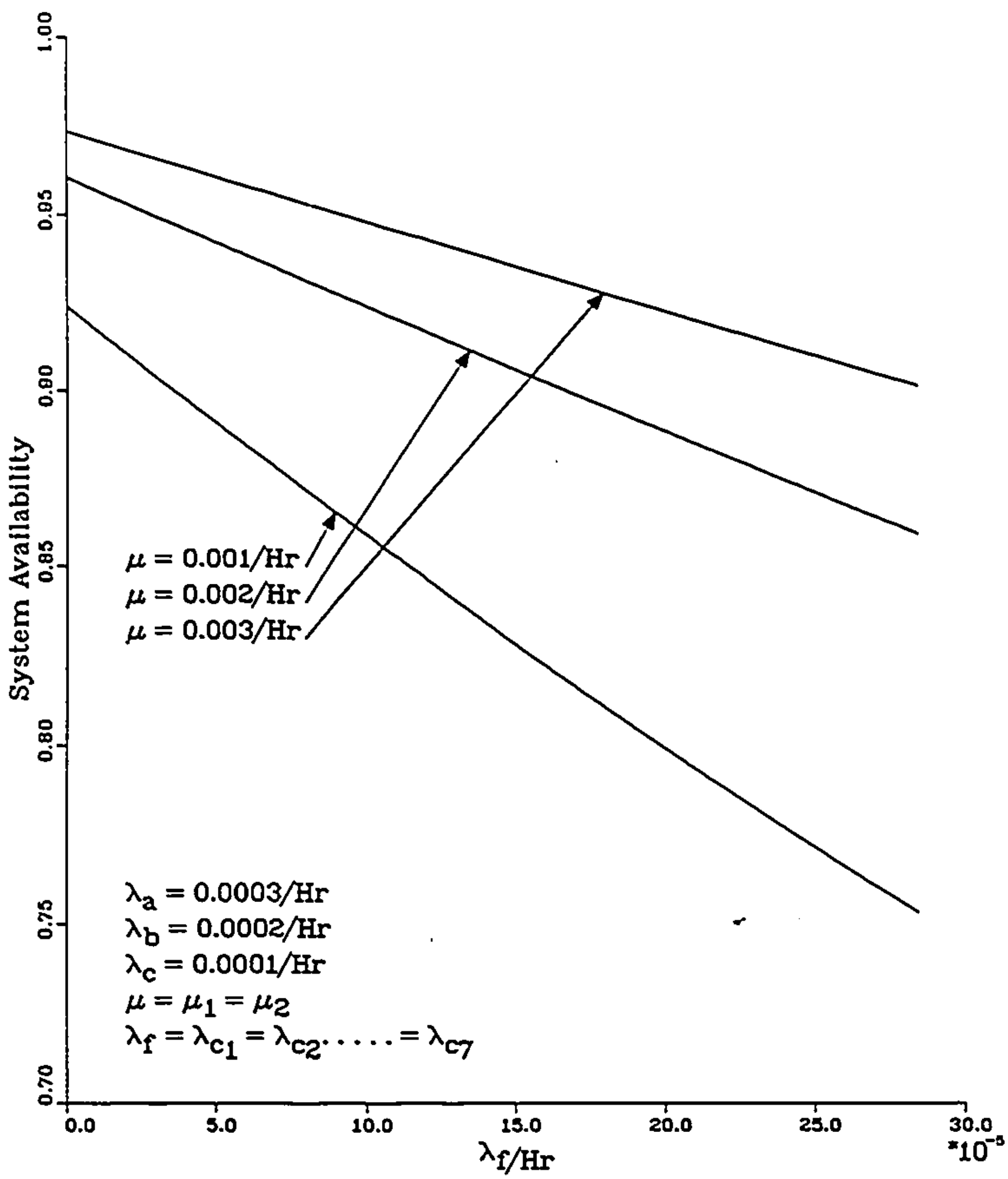

Figure 2.26: Steady State System Availability Plots for a Three Non-identical Unit Parallel System with Type I Repair

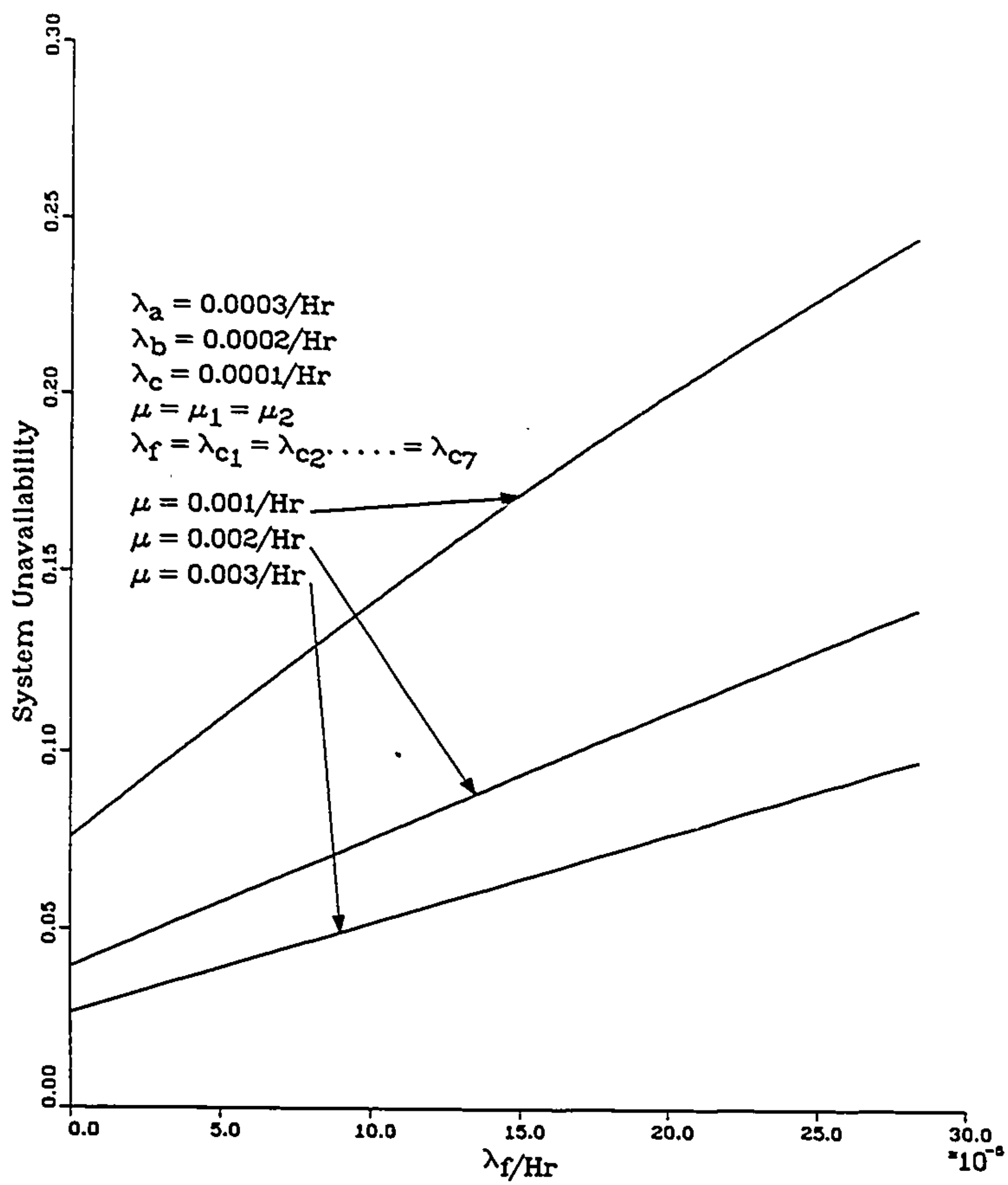

Figure 2.27: Steady State System Unavailability Plots for a Three Non-identical Unit Parallel System with Type I Repair

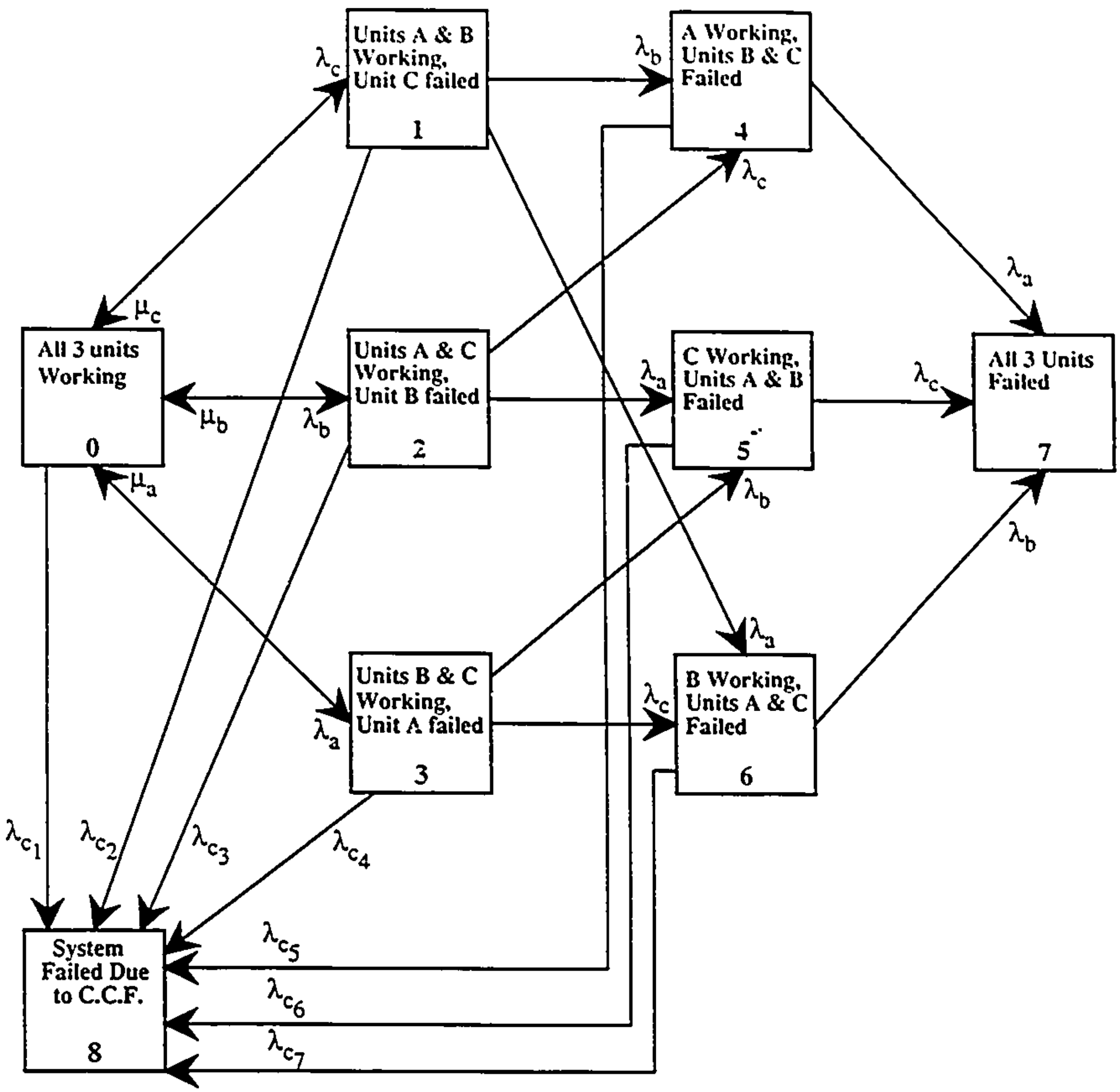

Figure 2.28: State Space Diagram for a Three Non-identical Unit Parallel System with Type II Repair

where the constants $D_{88}, D_{89}, \ldots, D_{103}$ are defined in Appendix A and s_1, s_2, s_3 *and* s_4 are the roots of the quartic equation $s^4 + D_{90}s^3 + D_{91}s^2 + D_{92}s + D_{94} = 0$, and can be determined as explained in Section 2.2.1.

$$
\begin{aligned}
p_1(t) \;=\; & \frac{[s_1^2 + s_1 D_{95} + D_{84}D_{83}]\lambda_c e^{s_1 t}}{(s_1 - s_2)(s_3 - s_1)(s_4 - s_1)} + \frac{[s_2^2 + s_2 D_{95} + D_{84}D_{83}]\lambda_c e^{s_2 t}}{(s_2 - s_1)(s_2 - s_3)(s_2 - s_4)} + \\
& \frac{[s_3^2 + s_3 D_{95} + D_{84}D_{83}]\lambda_c e^{s_3 t}}{(s_2 - s_3)(s_3 - s_1)(s_4 - s_3)} + \frac{[s_4^2 + s_4 D_{95} + D_{84}D_{83}]\lambda_c e^{s_4 t}}{(s_4 - s_3)(s_2 - s_4)(s_4 - s_1)}
\end{aligned}
\qquad (2.183)
$$

$$
\begin{aligned}
p_2(t) \;=\; & \frac{[s_1^2 + D_{96}s_1 + D_{82}D_{84}]\lambda_b e^{s_1 t}}{(s_1 - s_2)(s_3 - s_1)(s_4 - s_1)} + \frac{[s_2^2 + D_{96}s_2 + D_{82}D_{84}]\lambda_b e^{s_2 t}}{(s_2 - s_1)(s_2 - s_3)(s_2 - s_4)} + \\
& \frac{[s_3^2 + D_{96}s_3 + D_{82}D_{84}]\lambda_b e^{s_3 t}}{(s_3 - s_2)(s_3 - s_1)(s_3 - s_4)} + \frac{[s_4^2 + D_{96}s_4 + D_{82}D_{84}]\lambda_b e^{s_4 t}}{(s_2 - s_4)(s_3 - s_4)(s_4 - s_1)}
\end{aligned}
\qquad (2.184)
$$

$$
\begin{aligned}
p_3(t) \;=\; & \frac{[s_1^2 + s_1 D_{97} + D_{83}D_{82}]\lambda_a e^{s_1 t}}{(s_1 - s_2)(s_3 - s_1)(s_4 - s_1)} + \frac{[s_2^2 + s_2 D_{97} + D_{83}D_{82}]\lambda_a e^{s_2 t}}{(s_1 - s_2)(s_3 - s_2)(s_2 - s_4)} + \\
& \frac{[s_3^2 + s_3 D_{97} + D_{83}D_{82}]\lambda_a e^{s_3 t}}{(s_3 - s_2)(s_3 - s_1)(s_3 - s_4)} + \frac{[s_4^2 + s_4 D_{97} + D_{83}D_{82}]\lambda_a e^{s_4 t}}{(s_3 - s_4)(s_2 - s_4)(s_4 - s_1)}
\end{aligned}
\qquad (2.185)
$$

$$
\begin{aligned}
p_4(t) \;=\; & \frac{\lambda_b\lambda_c[2s_1^2 + D_{98}s_1 + D_{99}]e^{s_1 t}}{(s_1 + D_{85})(s_1 - s_4)(s_1 - s_3)(s_1 - s_2)} + \\
& \frac{\lambda_b\lambda_c[2s_2^2 + D_{98}s_2 + D_{99}]e^{s_2 t}}{(s_2 + D_{85})(s_4 - s_2)(s_2 - s_3)(s_1 - s_2)} + \\
& \frac{\lambda_b\lambda_c[2s_3^2 + D_{98}s_3 + D_{99}]e^{s_3 t}}{(s_3 + D_{85})(s_3 - s_4)(s_2 - s_3)(s_1 - s_3)} + \\
& \frac{\lambda_b\lambda_c[2s_4^2 + D_{98}s_4 + D_{99}]e^{s_4 t}}{(s_4 + D_{85})(s_4 - s_3)(s_2 - s_4)(s_1 - s_4)} + \\
& \frac{\lambda_b\lambda_c[2D_{85}^2 - D_{98}D_{85} + D_{99}]e^{-D_{85}t}}{(s_4 + D_{85})(s_3 + D_{85})(s_2 + D_{85})(s_1 + D_{85})}
\end{aligned}
\qquad (2.186)
$$

$$
\begin{aligned}
p_5(t) \;=\; & \frac{\lambda_b\lambda_a[D_{101} - D_{86}D_{100} + 2D_{86}^2]e^{-D_{86}t}}{(s_1 + D_{86})(s_3 + D_{86})(s_4 + D_{86})(s_2 + D_{86})} + \\
& \frac{\lambda_b\lambda_a[2s_1^2 + D_{100}s_1 + D_{101}]e^{s_1 t}}{(s_1 + D_{86})(s_1 - s_3)(s_4 - s_1)(s_2 - s_1)} + \\
& \frac{\lambda_b\lambda_a[2s_2^2 + D_{100}s_2 + D_{101}]e^{s_2 t}}{(s_2 + D_{86})(s_2 - s_1)(s_2 - s_3)(s_2 - s_4)} +
\end{aligned}
$$

$$
\frac{\lambda_b\lambda_a[2s_3^2 + D_{100}s_3 + D_{101}]e^{s_3 t}}{(s_3 + D_{86})(s_3 - s_1)(s_4 - s_3)(s_2 - s_3)} +
$$

$$
\frac{\lambda_b\lambda_a[2s_4^2 + D_{100}s_4 + D_{101}]e^{s_4 t}}{(s_4 + D_{86})(s_1 - s_4)(s_4 - s_3)(s_2 - s_4)} \tag{2.187}
$$

$$
\begin{aligned}
p_6(t) \;=\; & \frac{\lambda_a\lambda_c[D_{103} - D_{87}D_{102} + 2D_{87}^2]e^{-D_{87}t}}{(D_{87} + s_3)(D_{87} + s_1)(D_{87} + s_4)(s_2 + D_{87})} + \\
& \frac{\lambda_a\lambda_c[2s_1^2 + s_1 D_{102} + D_{103}]e^{s_1 t}}{(s_1 - s_3)(s_4 - s_1)(s_2 - s_1)(s_1 + D_{87})} + \\
& \frac{\lambda_a\lambda_c[2s_2^2 + s_2 D_{102} + D_{103}]e^{s_2 t}}{(s_2 - s_3)(s_2 - s_1)(s_2 - s_4)(s_2 + D_{87})} + \\
& \frac{\lambda_a\lambda_c[2s_3^2 + s_3 D_{102} + D_{103}]e^{s_3 t}}{(s_3 - s_1)(s_4 - s_3)(s_2 - s_3)(s_3 + D_{87})} + \\
& \frac{\lambda_a\lambda_c[2s_4^2 + s_4 D_{102} + D_{103}]e^{s_4 t}}{(s_3 - s_4)(s_4 - s_1)(s_2 - s_4)(s_4 + D_{87})}
\end{aligned} \tag{2.188}
$$

The system reliability is given by

$$
R(t) = p_0(t) + p_1(t) + p_2(t) + p_2(t) + p_3(t)p_4(t) + p_5(t) + p_6(t) \tag{2.189}
$$

The system reliability shown in Figure 2.29 indicates that the system reliability decreases with an increase in common-cause failures.

Parallel System Without Repair

Setting the repair rates μ_a, μ_b, *and* μ_c equal to zero in Figure 2.28 yields the following system of differential equations :

$$
\dot{p}_0(t) = -(\lambda_a + \lambda_b + \lambda_c + \lambda_{c_1})p_0(t) \tag{2.190}
$$

$$
\dot{p}_1(t) = \lambda_c p_0(t) - (\lambda_a + \lambda_b + \lambda_{c_2})p_1(t) \tag{2.191}
$$

$$
\dot{p}_2(t) = \lambda_b p_0(t) - (\lambda_a + \lambda_c + \lambda_{c_3})p_2(t) \tag{2.192}
$$

$$
\dot{p}_3(t) = \lambda_a p_0(t) - (\lambda_b + \lambda_c + \lambda_{c_4})p_3(t) \tag{2.193}
$$

$$
\dot{p}_4(t) = \lambda_b p_1(t) + \lambda_c p_2(t) - (\lambda_a + \lambda_{c_5})p_4(t) \tag{2.194}
$$

$$
\dot{p}_5(t) = \lambda_a p_2(t) + \lambda_b p_3(t) - (\lambda_c + \lambda_{c_6})p_5(t) \tag{2.195}
$$

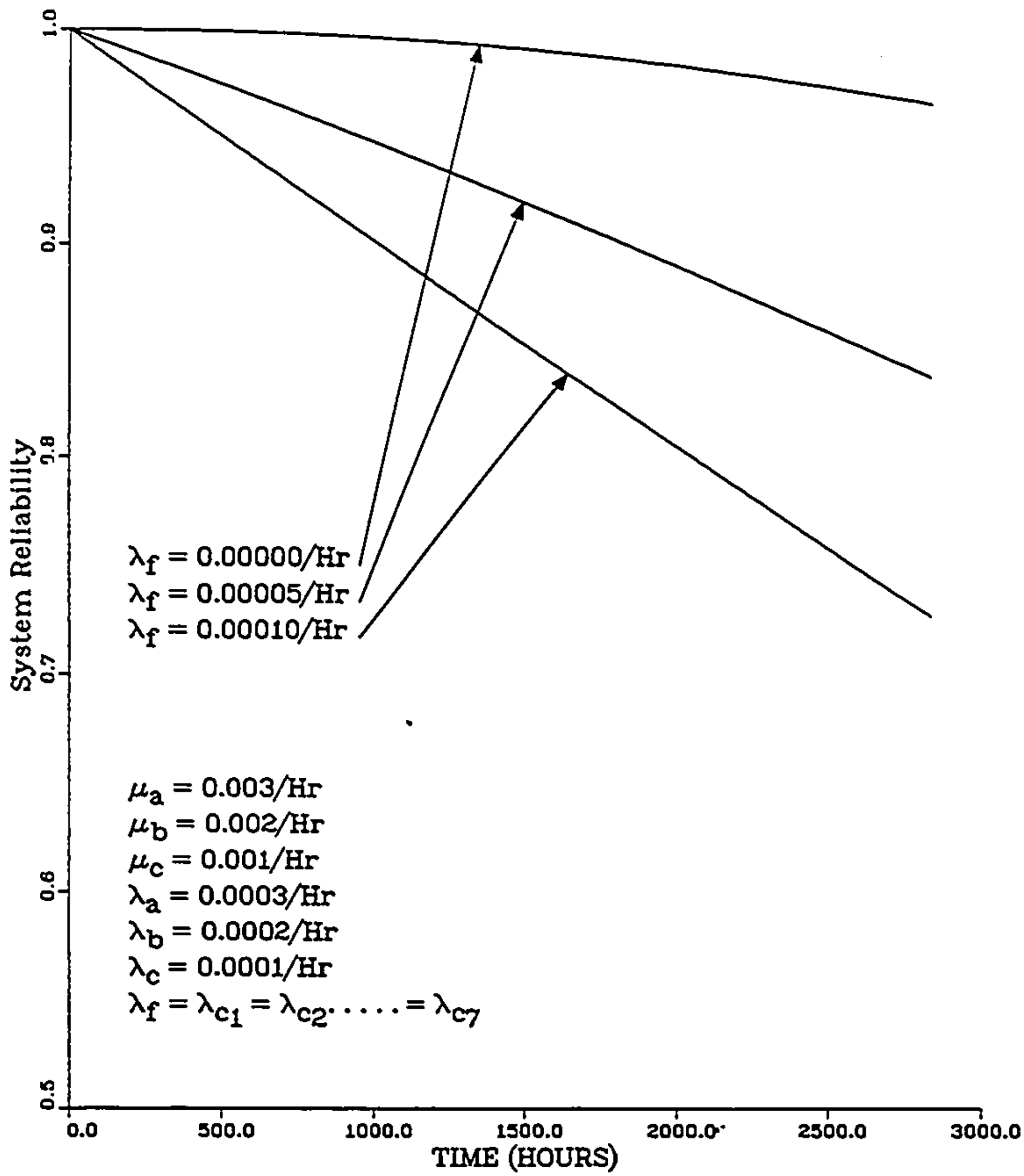

Figure 2.29: System Reliability Plots for a Three Non-identical Unit Parallel System with Type II Repair

$$\dot{p}_6(t) = \lambda_a p_1(t) + \lambda_c p_3(t) - (\lambda_b + \lambda_{e_7}) p_6(t) \tag{2.196}$$

$$\dot{p}_7(t) = \lambda_a p_4(t) + \lambda_c p_5(t) + \lambda_b p_6(t) \tag{2.197}$$

$$\begin{aligned} \dot{p}_8(t) &= \lambda_{e_1} p_0(t) + \lambda_{e_2} p_1(t) + \lambda_{e_3} p_2(t) + \lambda_{e_4} p_3(t) + \lambda_{e_5} p_4(t) + \\ &\quad \lambda_{e_6} p_5(t) + \lambda_{e_7} p_6(t) \end{aligned} \tag{2.198}$$

At time $t = 0$, $p_0(0) = 1$ and all other initial condition probabilities are equal to zero. The following state probabilities are obtained by solving Equations (2.190) – (2.198) with the aid of Laplace transforms :

$$p_0(t) = \epsilon^{-D_{104}t} \tag{2.199}$$

$$p_1(t) = \frac{\lambda_c(-\epsilon^{-D_{104}t} + \epsilon^{-D_{105}t})}{D_{104} - D_{105}} \tag{2.200}$$

$$p_2(t) = \frac{\lambda_b(-\epsilon^{-D_{104}t} + \epsilon^{-D_{106}t})}{D_{104} - D_{106}} \tag{2.201}$$

$$p_3(t) = \frac{\lambda_a(-\epsilon^{-D_{104}t} + \epsilon^{-D_{107}t})}{D_{104} - D_{107}} \tag{2.202}$$

$$\begin{aligned} p_4(t) &= \frac{(-2D_{108} + D_{105} + D_{106})\lambda_c\lambda_b e^{-D_{108}t}}{(D_{104} - D_{108})(D_{106} - D_{108})(D_{105} - D_{108})} + \\ &\quad \frac{\lambda_c\lambda_b e^{-D_{106}t}}{(D_{106} - D_{104})(D_{106} - D_{108})} + \\ &\quad \frac{\lambda_c\lambda_b e^{-D_{105}t}}{(D_{105} - D_{104})(D_{105} - D_{108})} + \\ &\quad \frac{(-D_{105} + 2D_{104} - D_{106})\lambda_c\lambda_b e^{-D_{104}t}}{(D_{104} - D_{106})(D_{104} - D_{105})(D_{104} - D_{108})} \end{aligned} \tag{2.203}$$

$$\begin{aligned} p_5(t) &= \frac{(D_{106} + D_{107} - 2D_{109})\lambda_b\lambda_a e^{-D_{109}t}}{(D_{104} - D_{109})(D_{107} - D_{109})(D_{106} - D_{109})} \\ &\quad \frac{\lambda_b\lambda_a e^{-D_{107}t}}{(D_{107} - D_{104})(D_{107} - D_{109})} + \\ &\quad \frac{\lambda_b\lambda_a e^{-D_{106}t}}{(D_{106} - D_{104})(D_{106} - D_{109})} + \\ &\quad \frac{(2D_{104} - D_{107} - D_{106})\lambda_b\lambda_a e^{-D_{104}t}}{(D_{104} - D_{109})(D_{104} - D_{106})(D_{104} - D_{107})} \end{aligned} \tag{2.204}$$

$$
\begin{aligned}
p_6(t) \quad = \quad & \frac{(D_{107} - 2D_{110} + D_{105})\lambda_c\lambda_q e^{-D_{110}t}}{(D_{104} - D_{110})(D_{107} - D_{110})(D_{105} - D_{110})} + \\
& \frac{\lambda_c\lambda_q e^{-D_{107}t}}{(D_{107} - D_{104})(D_{107} - D_{110})} + \\
& \frac{\lambda_c\lambda_q e^{-D_{105}t}}{(D_{105} - D_{104})(D_{105} - D_{110})} + \\
& \frac{(2D_{104} - D_{105} - D_{107})\lambda_c\lambda_q e^{-D_{104}t}}{(D_{104} - D_{110})(D_{104} - D_{107})(D_{104} - D_{105})} \qquad (2.205)
\end{aligned}
$$

The system reliability is given by

$$
R(t) = p_0(t) + p_1(t) + p_2(t) + p_2(t) + p_3(t)p_4(t) + p_5(t) + p_6(t) \qquad (2.206)
$$

The plots of Equations 2.206 are shown in Figure 2.30. The plots clearly indicate a decrease in system reliability with an increase in common-cause failures.

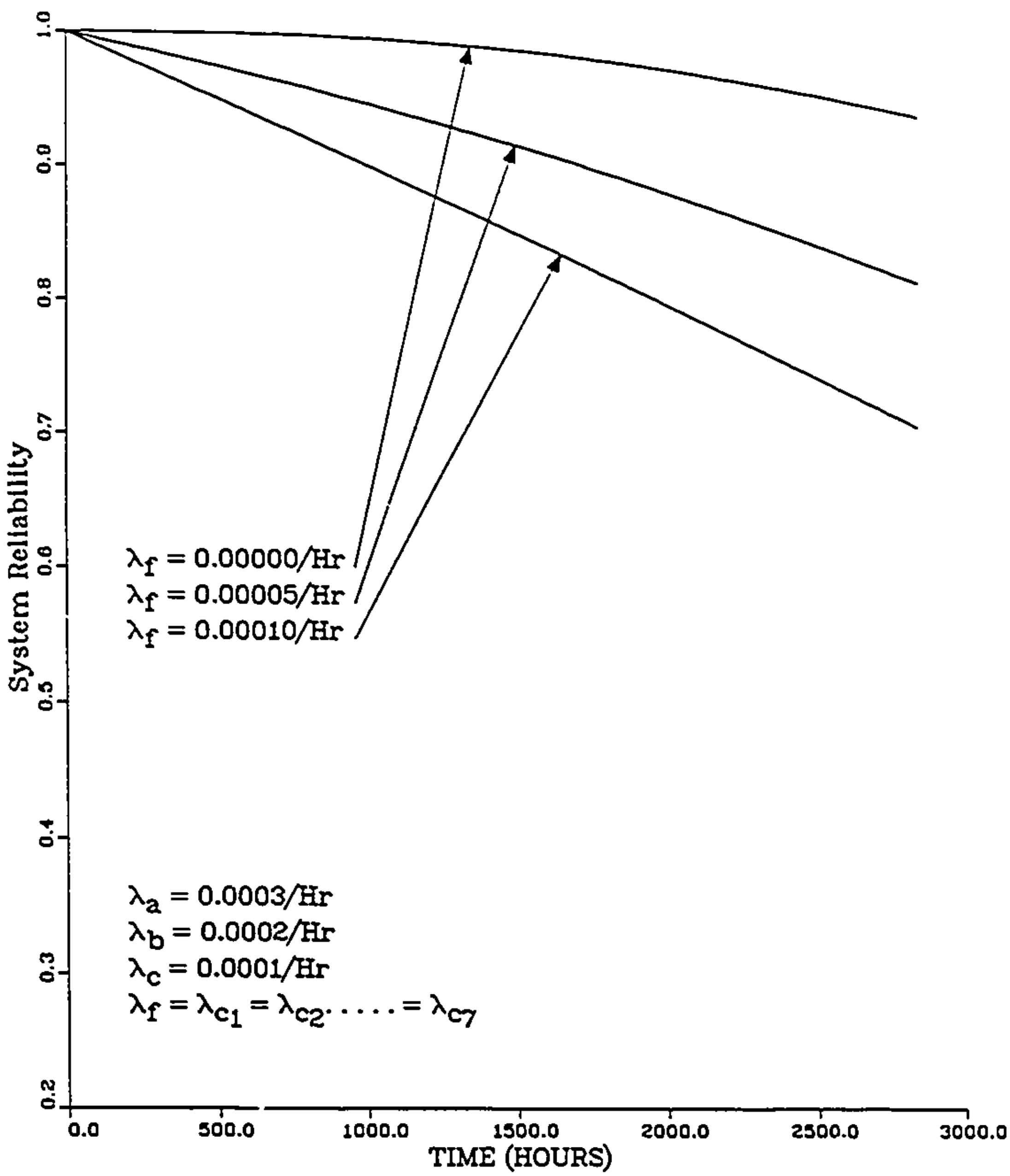

Figure 2.30: System Reliability plots for Three Non-identical Unit Parallel System without Repair

Chapter 3

k-out-of-n System

This is another type of redundancy. It is used where a specified number of units must be good for the system success. The system consists of n number of active units out of which k units must be operative to ensure that the system functions normally. The parallel system analyzed in the preceding chapter can be considered as a special case of this configuration (with k = 1).

Sections 3.1 and 3.2 present the reliability analysis of such a system with identical and non-identical units respectively, taking into consideration the occurrence of common-cause failures.

3.1 Identical Unit k-out-of-n System

This section is concerned with newly developed Markov models for both repairable and non-repairable identical unit k-out-of-n system. At time $t = 0$, all the units start working simultaneously. At least k units must operate normally for the system to function successfully. The failure rate of each unit is independent of time. The occurrence of a common-cause failure can cause the total system failure. Furthermore, a common-cause failure can occur when the system is in any one of its operating states (e.g. $0, 1, 2, \ldots, n-k$).

The parallel system discussed in Chapter 2 can be considered as a special case of this configuration, with $k = 1$. Two types of repair policies, similar to those discussed in the identical unit parallel system, are considered in the analysis of identical unit standby system.

1. Type I Repair : Under this repair policy, the partially failed system (1 unit failed, $n-1$ units operating; 2 units failed, $n-2$ units operating; ...; $n-k$ units failed, k units operating) is repaired to its previous state and to state 0. In addition, the completely failed system (either due to a common-cause failure or due to other types of failures) is also repaired back to state 0. Furthermore, if a common-cause failure occurs at state n, resulting in a total system failure, the system is repaired back to its state n ($n = 1, 2, 3, \ldots, n-k$).
2. Type II Repair : In this type of repair policy, the completely failed system is not repaired. However, the partially failed system (1 unit failed, $n-1$ units operating; 2 units failed, $n-2$ units operating, ..., $n-k$ units failed, k units operating) is repaired back to its previous state as well as to state 0 (where all units are operating normally).

Assumptions

The following assumptions are associated with all the models under this configuration :

1. The system has n identical and active units.
2. A system can fail either due to a common-cause failure or other failures.
3. Common-cause and other failures are statistically independent.
4. A common-cause failure can occur when the system is in any one its good states.
5. Common-cause and other failure rates are constant.
6. The system is said to be in a failed state if more than $n-k$ units become non-operative.
7. Repair rates are constant.
8. A repaired unit is as good as new.

Notation

The symbols associated with all the models under this configuration are as follows:

n	Number of units in the system.
t	time.
λ_i	Constant failure rate from the $(i-1)^{th}$ system state; for $i = 1, 2, 3, \ldots, r+1$.
μ_i	Constant repair rate from the i^{th} system state; for $i = 1, 2, 3, \ldots, r+1$.
μ_{io}	Constant repair rate from the i^{th} system state to system state 0; for $i = 2, 3, 4, \ldots, r+1$.
$\mu_{c_{i+1}}$	Constant repair rate of the failed system from state $r+2$ to state i; for $i = 0, 1, 2, \ldots, r$.
$\lambda_{c_{i+1}}$	Constant common-cause failure rate of the system from state i; for $i = 0, 1, 2, 3, \ldots, r$.
j	j^{th} state of the system for $j = 0, 1, 2, 3 \ldots, r+2$.
$p_j(t)$	Probability that the system is in state j at time t; for $j = 0, 1, 2, 3, \ldots, r+2$.
$R(t)$	Reliability of the system in $[0, t]$.
s	Laplace transform variable.
CCF	Common-cause failure.
$MTTF$	System mean time to failure.
AV_{ss}	Steady state system availability.
UV_{ss}	Steady state system unavailability.
$\dot{p}_j(t)$	Derivative of $p_j(t)$ with respect to time t; for $j = 0, 1, 2, \ldots, r+2$.
p_i	Steady state probability, that the system is in state i; for $i = 0, 1, 2, \ldots, r+2$.

3.1.1 k-out-of-n System With Type I Repair

The state space diagram for a k-out-of-n unit system is shown in Figure 3.1. The numerals plus a letter in the boxes of the figure denote the system state numbers.

If k number of units are required to be operative for the system to function

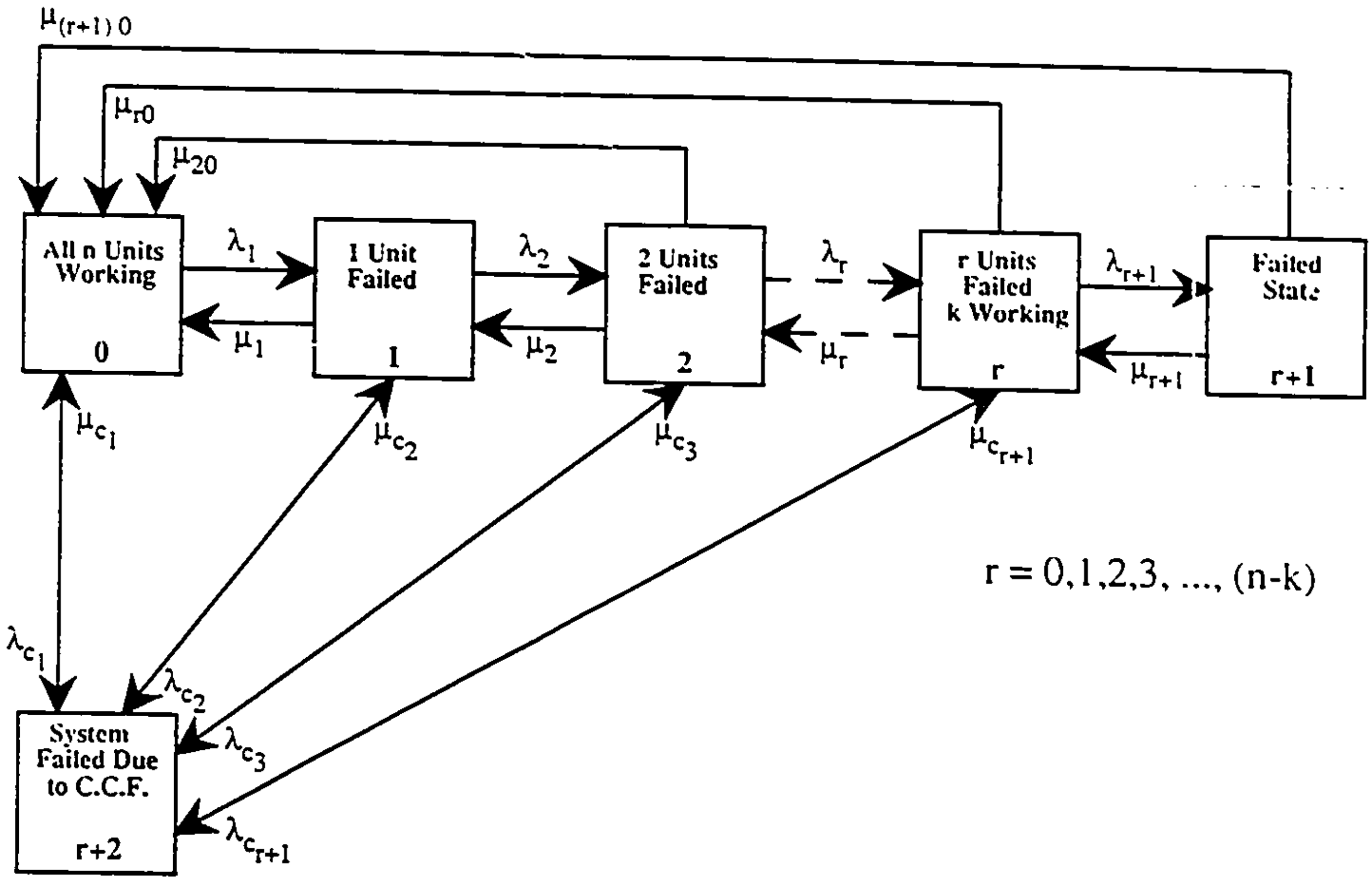

Figure 3.1: State Space Diagram for an Identical Unit k-out-of-n System

successfully, out of n number of units, then r is defined as :

$$r = n - k \tag{3.1}$$

The system of differential equations for the model shown in Figure 3.1 is

$$\dot{p}_0(t) = -(\lambda_1 + \lambda_{c_1})p_0(t) + \mu_1 p_1(t) + \sum_{i=2}^{r+1} \mu_{io} p_i(t) + \mu_{c_1} p_{r+2}(t) \tag{3.2}$$

$$\dot{p}_1(t) = \lambda_1 p_0(t) - (\mu_1 + \lambda_2 + \lambda_{c_2})p_1(t) + \mu_2 p_2(t) + \mu_{c_2} p_{r+2}(t) \tag{3.3}$$

$$\begin{aligned} \dot{p}_i(t) &= \lambda_i p_{i-1}(t) - \{\mu_{io} + \mu_i + \lambda_{i+1} + \lambda_{c_{i+1}}\}p_i(t) + \\ & (\mu_{i+1})p_{i+1}(t) + (\mu_{c_{i+1}})p_{r+2}(t) \end{aligned} \tag{3.4}$$

for $i = 2, 3, 4, \ldots, n - k$

$$\dot{p}_{r+1}(t) = \lambda_{r+1} p_r(t) - \{\mu_{r+1} + \mu_{(r+1)o}\}p_{r+1}(t) \tag{3.5}$$

$$\dot{p}_{r+2}(t) = \sum_{i=0}^{r} \lambda_{c_{i+1}} p_i(t) - \{\sum_{i=1}^{r+1} \mu_{c_i}\}p_{r+2}(t) \tag{3.6}$$

At time $t = 0$, $p_0(0) = 1$ and all other initial condition probabilities are equal to zero.

Since the number of terms in the steady state availability expression become too large, for the purpose of generalization, $\mu_{c_{i+1}}$; for $i = 1, 2, 3, \ldots, r$ is set to zero and the system of differential equations reduces to

$$\dot{p}_0(t) = -(\lambda_1 + \lambda_{c_1})p_0(t) + \mu_1 p_1(t) + \sum_{i=2}^{r+1} \mu_{io} p_i(t) + \mu_{c_1} p_{r+2}(t) \tag{3.7}$$

$$\dot{p}_1(t) = \lambda_1 p_0(t) - (\mu_1 + \lambda_2 + \lambda_{c_2})p_1(t) + \mu_2 p_2(t) \tag{3.8}$$

$$\dot{p}_i(t) = \lambda_i p_{i-1}(t) - \{\mu_{io} + \mu_i + \lambda_{i+1} + \lambda_{c_{i+1}}\}p_i(t) + (\mu_{i+1})p_{i+1}(t) \tag{3.9}$$

for $i = 2, 3, 4, \ldots, n - k$

$$\dot{p}_{r+1}(t) = \lambda_{r+1} p_r(t) - \{\mu_{r+1} + \mu_{(r+1)o}\}p_{r+1}(t) \tag{3.10}$$

$$\dot{p}_{r+2}(t) = \sum_{i=0}^{r} \lambda_{c_{i+1}} p_i(t) - \mu_{c_1} p_{r+2}(t) \tag{3.11}$$

By setting the derivatives of Equations (3.7) – (3.11) to zero and using the relationship $\sum_{i=0}^{r+2} p_i = 1$, the following steady state probability expression in terms of p_0 is obtained :

$$p_i = \frac{\{\prod_{j=1}^{i} \lambda_j A_{r-i}\}\, p_0}{(\mu_1 + \lambda_2 + \lambda_{c_2})A_{r-1} - \mu_2 \lambda_2 A_{r-2}} \tag{3.12}$$

For $i = 1, 2, 3, \ldots, n-k$

where

$$\begin{aligned}
A_0 &= \{\mu_{r+1} + \mu_{(r+1)o}\} \\
For\ j &= 0, 1, 2, 3, \ldots, r-2 \\
A_{j+1} &= \{\mu_{(r-j)o} + \mu_{r-j} + \lambda_{(r+1)-j} + \lambda_{c_{(r+1)-j}}\} A_j - (\mu_{(r+1)-j})(\lambda_{(r+1)-j}) A_{j-1} \\
A_{j-1} &= 1\ for\ (j-1) < 0
\end{aligned}$$

$$p_0 = \frac{\mu_1 p_1 + \sum_{i=2}^{r+1} \mu_{io} p_i + \mu_{c_1}\{1 - \sum_{i=1}^{r+1} p_i\}}{\lambda_1 + \lambda_{c_1} + \mu_{c_1}} \tag{3.13}$$

The steady state probabilities of a parallel system for given values of n and k, can be determined using Equations (3.12) and (3.13). The steady state system availability and steady state system unavailability, respectively, are

$$\begin{aligned}
AV_{ss} &= \sum_{i=0}^{r} p_i \\
UV_{ss} &= \sum_{i=r+1}^{r+2} p_i
\end{aligned}$$

Figures 3.2 and 3.3 show the plots of steady state system availability and steady state system unavailability, respectively, for different values of n and specified values of model parameters. It may be noted that the repair rate $\mu_{c_{i+1}}$; for $i = 1, 2, 3, \ldots, r$ is also considered for these plots. From these plots it can be clearly seen that the steady state system availability decreases with an increase in the number of common-cause failures, and conversely the steady state system unavailability increases with an increase in common-cause failures. In addition, the plots also show that, for a given value of k, the steady state system availability increases with an increase in the total number of units n in the system.

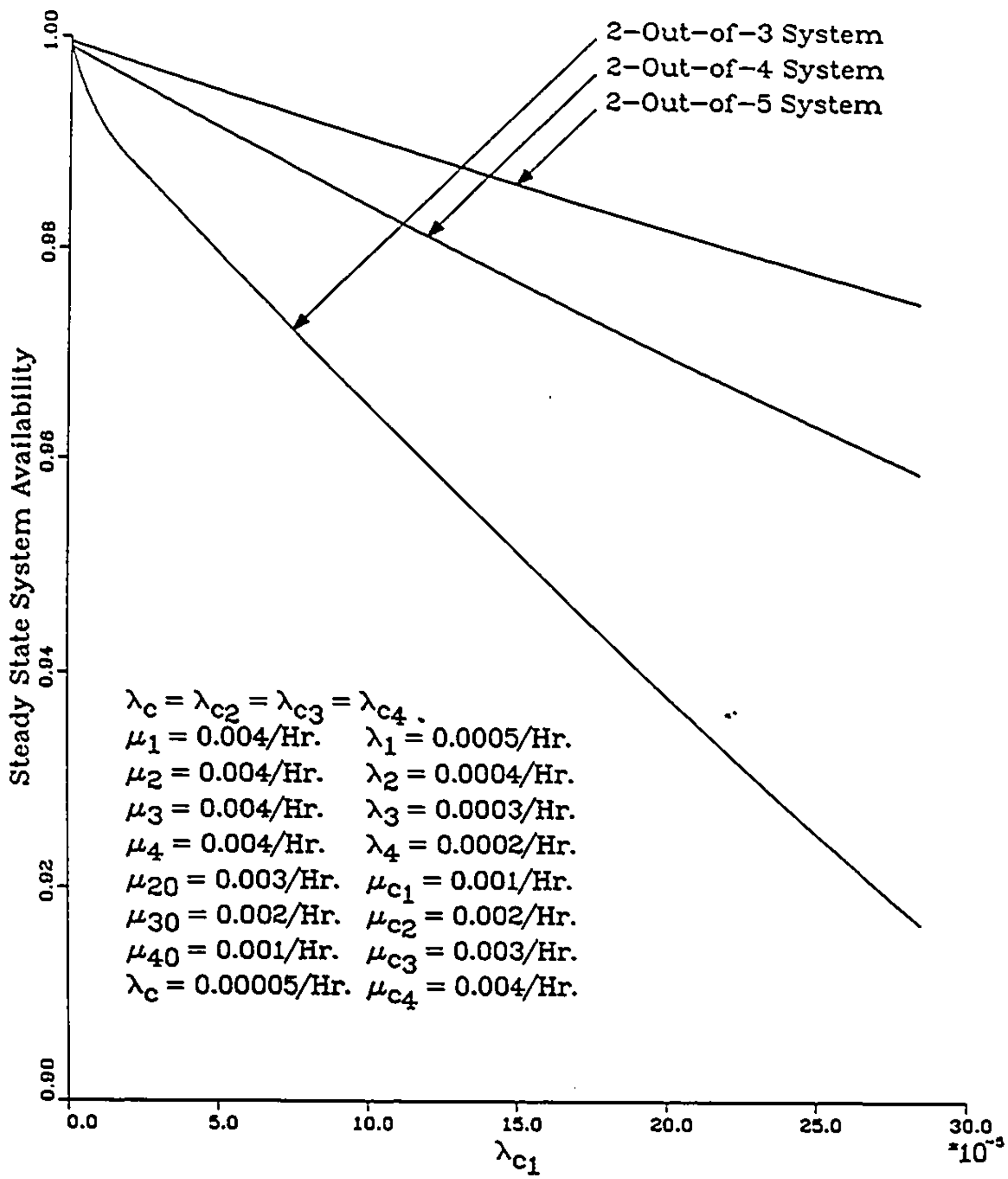

Figure 3.2: Steady State System Availability Plots for an Identical Unit k-out-of-n System with Type I Repair

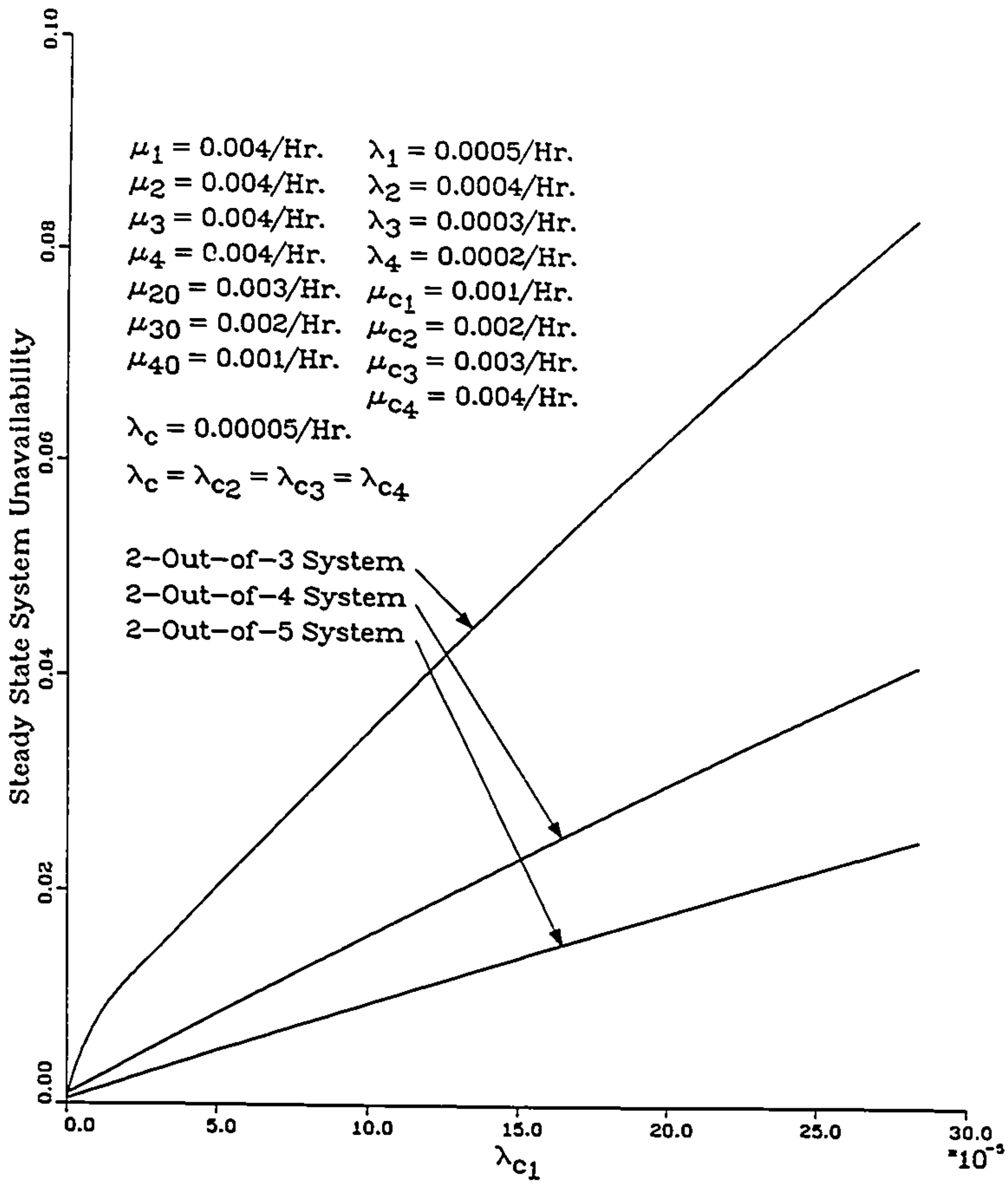

Figure 3.3: Steady State System Unavailability Plots for an Identical Unit k-out-of-n System with Type I Repair

Special Case Model I

For a system where $(n-2)$ units are required to be operative out of n units to ensure system success, setting $k = n-2$ in Equations $(3.2)-(3.6)$ yields the following system of differential equations :

$$\begin{aligned}
\dot{p}_0(t) &= -(\lambda_1 + \lambda_{c_1})p_0(t) + \mu_1 p_1(t) + \mu_{20}p_2(t) + \mu_{30}p_3(t) + \mu_{c_1}p_4(t) && (3.14)\\
\dot{p}_1(t) &= \lambda_1 p_0(t) - (\mu_1 + \lambda_2 + \lambda_{c_2})p_1(t) + \mu_2 p_2(t) + \mu_{c_2}p_4(t) && (3.15)\\
\dot{p}_2(t) &= \lambda_2 p_1(t) - (\mu_2 + \mu_{20} + \lambda_3 + \lambda_{c_3})p_2(t) + \mu_3 p_3(t) + \mu_{c_3}p_4(t) && (3.16)\\
\dot{p}_3(t) &= \lambda_3 p_2(t) - (\mu_3 + \mu_{30})p_3(t) && (3.17)\\
\dot{p}_4(t) &= \lambda_{c_1}p_0(t) + \lambda_{c_2}p_1(t) + \lambda_{c_3}p_2(t) - (\mu_{c_1} + \mu_{c_2} + \mu_{c_3})p_4(t) && (3.18)
\end{aligned}$$

At time $t = 0$, $p_0(0) = 1$ and all other initial condition probabilities are equal to zero.

By setting the derivatives of the above equations to zero and using the relationship $\sum_{i=0}^{4} p_i = 1$, we get the following steady state probabilities :

$$\begin{aligned}
p_0 &= \frac{E_1}{E_6} && (3.19)\\
p_1 &= \frac{E_2}{E_6} && (3.20)\\
p_2 &= \frac{E_3}{E_6} && (3.21)\\
p_3 &= \frac{E_4}{E_6} && (3.22)\\
p_4 &= \frac{E_5}{E_6} && (3.23)
\end{aligned}$$

where the constants E_1, E_2, E_3, E_4, E_5 *and* E_6 are defined in Appendix B. The steady state system availability and unavailability are given by

$$\begin{aligned}
AV_{ss} &= p_0 + p_1 + p_2 && (3.24)\\
UV_{ss} &= p_3 + p_4 && (3.25)
\end{aligned}$$

The plots for the Equations (3.24) and (3.25) for specified values of model parameters are given in Figure 3.4 and 3.5. It can be clearly seen from the plots that

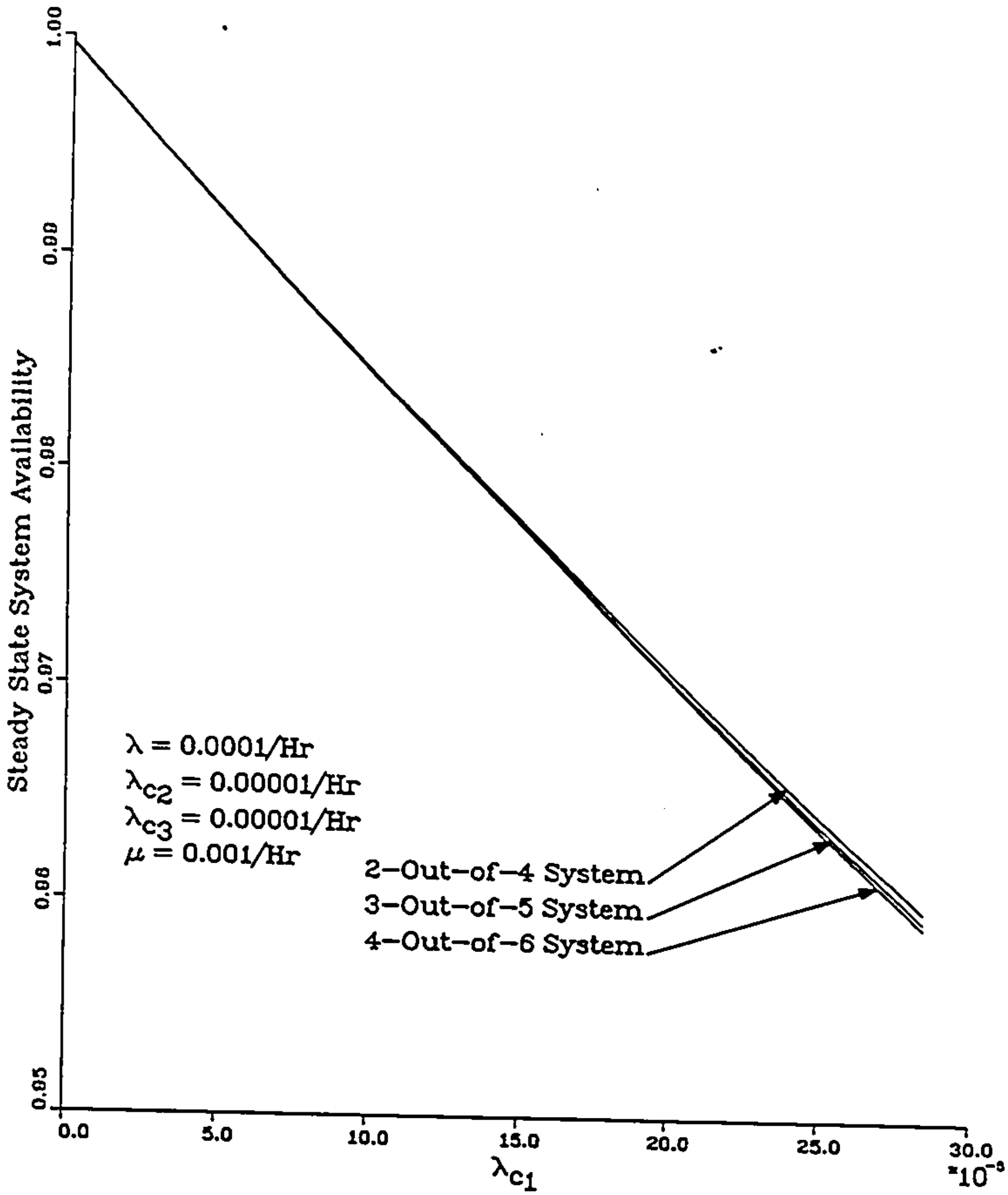

Figure 3.4: Steady State System Availability Plots for an Identical Unit (n-2)-out-of-n System with Type I Repair

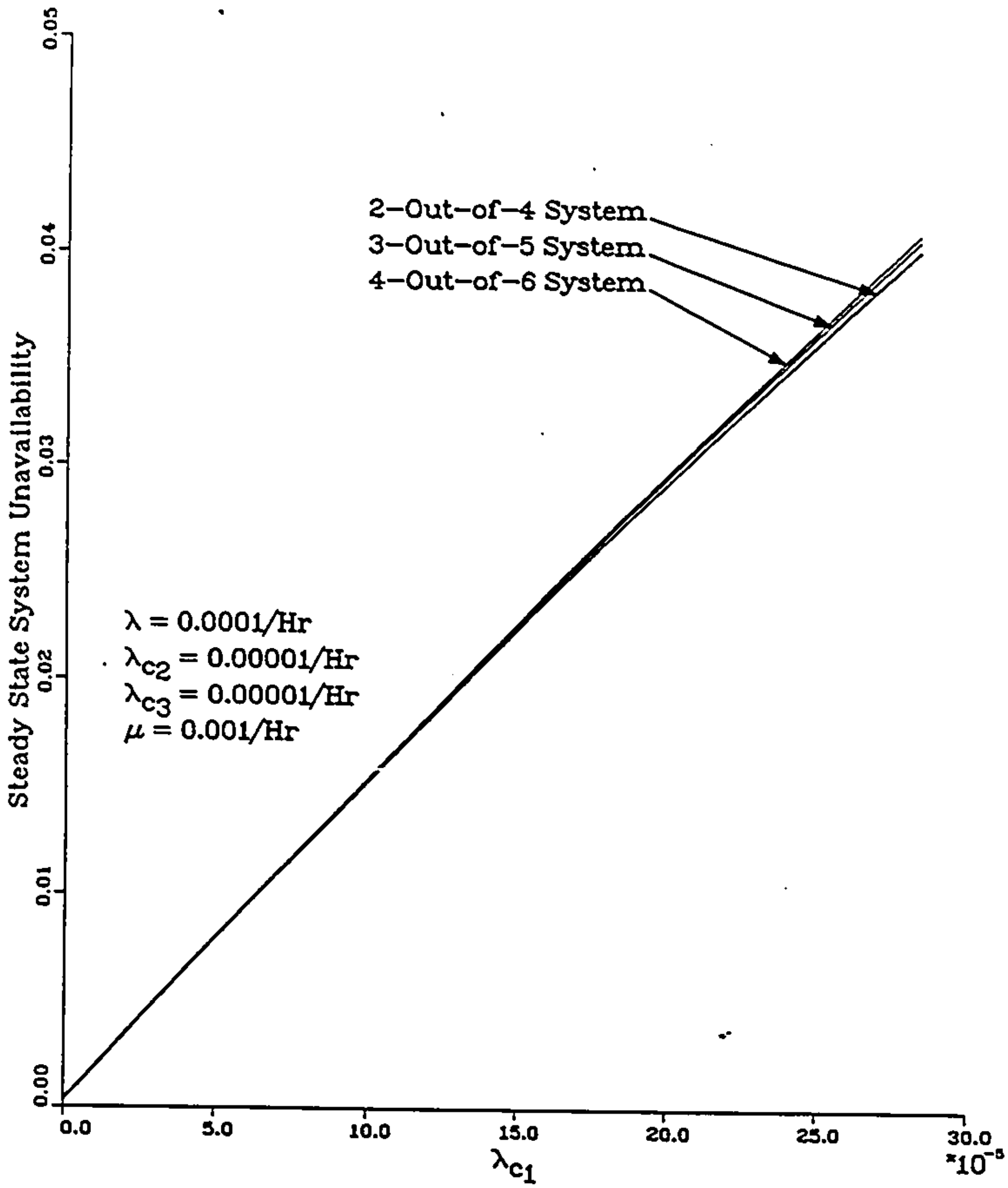

Figure 3.5: Steady State System Unavailability Plots for an Identical Unit (n-2)-out-of-n System with Type I Repair

the steady state system availability decreases with increasing values of common-cause failures. The converse is true for steady state system unavailability.

Special Case Model II

When $n-1$ units out of n units are required to be operative for the system to function normally, the system of differential equations can be obtained by setting $k = n-1$ in Equations (3.2) – (3.6).

$$\dot{p}_0(t) = -(\lambda_1 + \lambda_{c_1})p_0(t) + \mu_1 p_1(t) + \mu_{20}p_2(t) + \mu_{c_1}p_3(t) \quad (3.26)$$

$$\dot{p}_1(t) = \lambda_1 p_0(t) - (\mu_1 + \lambda_2 + \lambda_{c_2})p_1(t) + \mu_2 p_2(t) + \mu_{c_2}p_3(t) \quad (3.27)$$

$$\dot{p}_2(t) = \lambda_2 p_1(t) - (\mu_2 + \mu_{20})p_2(t) \quad (3.28)$$

$$\dot{p}_3(t) = \lambda_{c_1}p_0(t) + \lambda_{c_2}p_1(t) - (\mu_{c_1} + \mu_{c_2})p_3(t) \quad (3.29)$$

At time $t = 0$, $p_0(0) = 1$, $p_1(0) = 0$, $p_2(0) = 0$ and $p_3(0) = 0$.

By setting the derivatives of the above equations to zero and using the relationship $\sum_{i=0}^{3} p_i = 1$, we get the following steady state probabilities :

$$p_0 = \frac{E_7}{E_{11}} \quad (3.30)$$

$$p_1 = \frac{E_8}{E_{11}} \quad (3.31)$$

$$p_2 = \frac{E_9}{E_{11}} \quad (3.32)$$

$$p_3 = \frac{E_{10}}{E_{11}} \quad (3.33)$$

where the constants E_7, E_8, E_9, E_{10} *and* E_{11} are defined in Appendix B. The steady state system availability and steady state system unavailability expressions are as follows :

$$AV_{ss} = p_0 + p_1 \quad (3.34)$$

$$UV_{ss} = p_2 + p_3 \quad (3.35)$$

The plots of Equations (3.34) and (3.35) are given in Figures 3.6 and 3.7 respectively for different values of n. The plots clearly depict the negative effect that the common-cause failures have on steady state system availability.

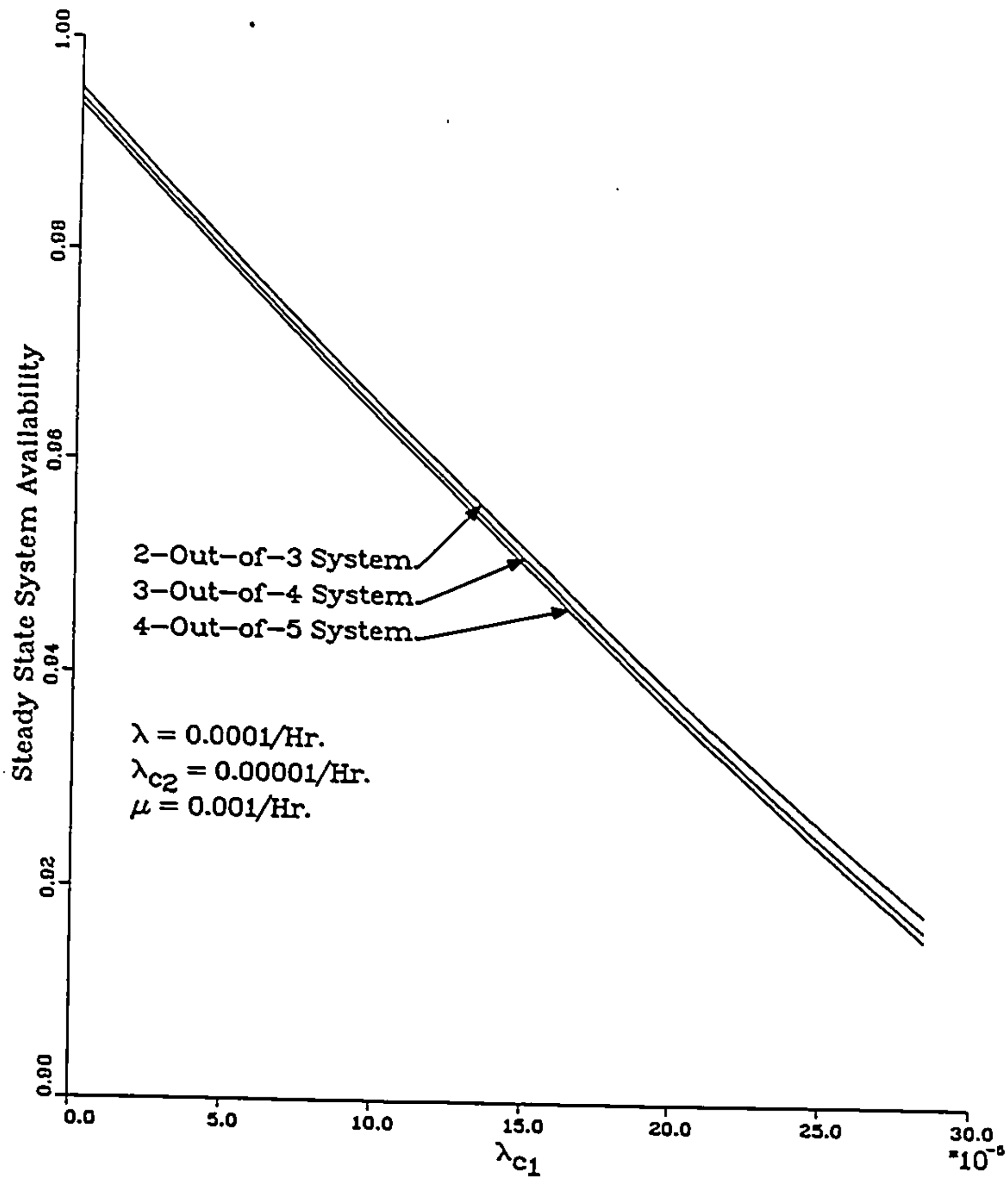

Figure 3.6: Steady State System Availability Plots for an Identical Unit (n-1)-out-of-n System with Type I Repair

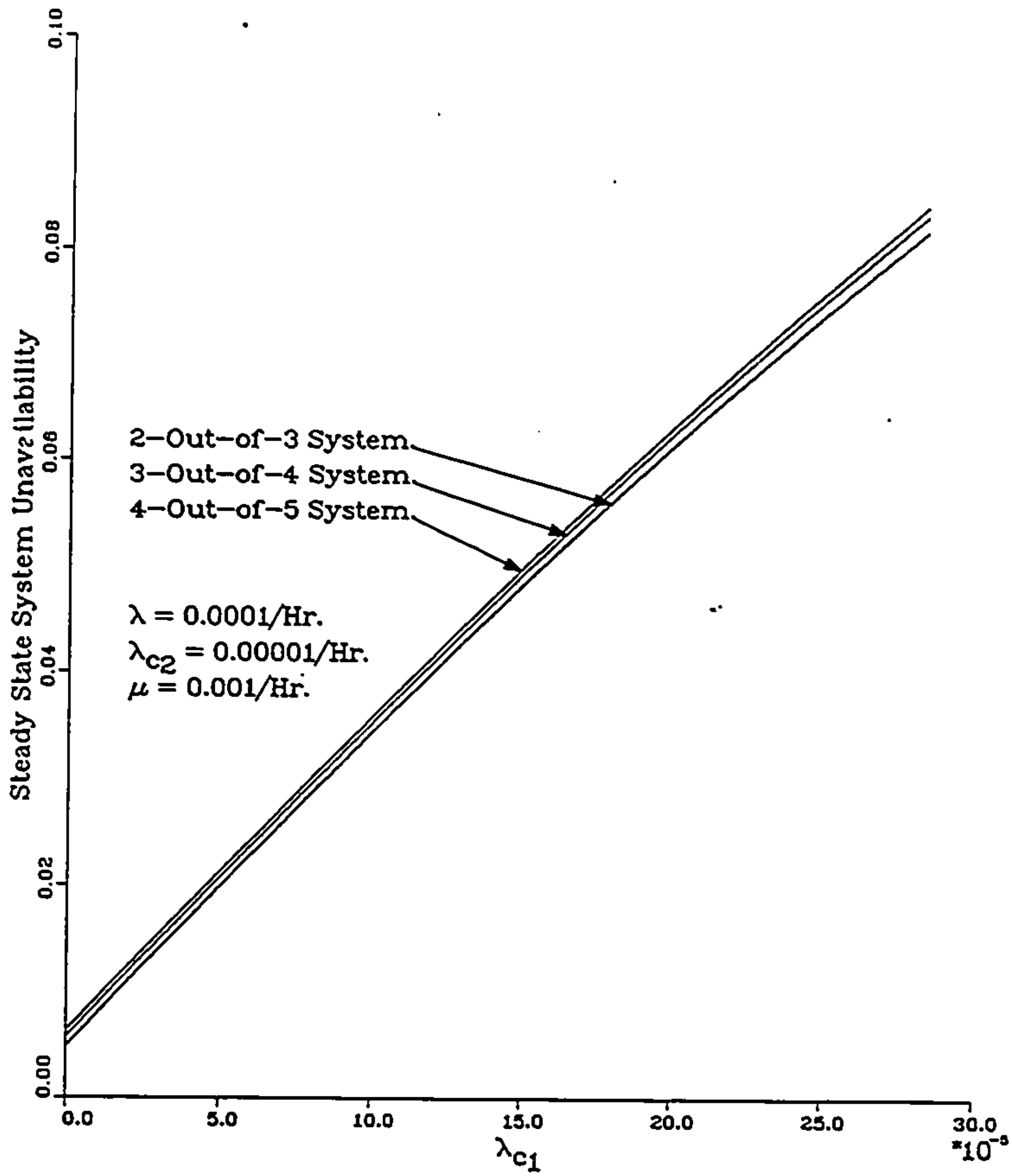

Figure 3.7: Steady State System Unavailability Plots for an Identical Unit (n-1)-out-of-n System with Type I Repair

3.1.2 k-out-of-n System With Type II Repair

By setting the repair rates $\mu_{c_{i+1}}$; for $i = 0, 1, 2, \ldots, r$, $\mu_{(r+1)o}$ and μ_{r+1} equal to zero, in Figure 3.1, yields the following system of differential equations :

$$\dot{p}_0(t) = -(\lambda_1 + \lambda_{c_1})p_0(t) + \mu_1 p_1(t) + \sum_{i=2}^{r} \mu_{io} p_i(t) \tag{3.36}$$

$$\dot{p}_1(t) = \lambda_1 p_0(t) - (\mu_1 + \lambda_2 + \lambda_{c_2})p_1(t) + \mu_2 p_2(t) \tag{3.37}$$

$$\dot{p}_{j-1}(t) = (\lambda_{j-1})p_{j-2}(t) - \{\mu_{(j-1)o} + \mu_{j-1} + \lambda_j + \lambda_{c_j}\}p_{j-1}(t) + \mu_j p_j(t) \tag{3.38}$$

for $j = 3, 4, 5 \ldots, n-k$

$$\dot{p}_r(t) = \lambda_r p_{r-1}(t) - \{\mu_{ro} + \mu_r + \lambda_{r+1} + \lambda_{c_{r+1}}\}p_r(t) \tag{3.39}$$

for $r = n - k$

$$\dot{p}_{r+1}(t) = \lambda_{r+1} p_r(t) \tag{3.40}$$

$$\dot{p}_{r+2}(t) = \sum_{i=0}^{r} \lambda_{c_{i+1}} p_i(t) \tag{3.41}$$

At time $t = 0$, $p_0(0) = 1$ and all other initial condition probabilities are equal to zero.

Solving Equations (3.36) – (3.41) with the aid of Laplace transforms, we get the following Laplace transforms of the state probabilities :

$$p_0(s) = \frac{1 + \mu_1 p_1(s) + \sum_{i=2}^{r} \mu_{io} p_i(s)}{s + \lambda_1 + \lambda_{c_1}} \tag{3.42}$$

$$p_1(s) = \frac{\lambda_1 p_0(s) + \mu_2 p_2(s)}{s + \mu_1 + \lambda_2 + \lambda_{c_2}} \tag{3.43}$$

$$p_{j-1}(s) = \frac{\lambda_{j-1} p_{j-2}(s) + \mu_j p_j(s)}{s + \mu_{(j-1)o} + \mu_{j-1} + \lambda_j + \lambda_{c_j}} \tag{3.44}$$

for $j = 3, 4, 5, \ldots, n-k$

$$p_r(s) = \frac{\lambda_r p_{r-1}(s)}{s + \mu_{ro} + \mu_r + \lambda_{r+1} + \lambda_{c_{r+1}}} \tag{3.45}$$

for $r = n - k$

$$p_{r+1}(s) = \frac{\lambda_{r+1} p_r(s)}{s} \tag{3.46}$$

$$p_{r+2}(s) = \frac{\sum_{i=0}^{r} \lambda_{c_{i+1}} p_i(s)}{s} \tag{3.47}$$

The k-out-of-n unit system mean time to failure ($MTTF_{k/n}$) is given by

$$MTTF_{k/n} = \lim_{s \to 0} [p_0(s) + p_1(s) + \ldots + p_{r-1}(s) + p_r(s)] \tag{3.48}$$

For $r = 0$ and $r = 1$, we get the following expressions for the system mean time to failure :

$$\begin{aligned} MTTF_0 &= \frac{1}{\lambda_1 + \lambda_{c_1}} \\ MTTF_1 &= \frac{\mu_1 + \lambda_2 + \lambda_{c_2} + \lambda_1}{(\lambda_2\lambda_{c_1} + \mu_1\lambda_{c1} + \lambda_{c_2}\lambda_{c_1} + \lambda_2\lambda_1 + \lambda_{c_2}\lambda_1)} \end{aligned}$$

Thus, for a k-out-of-n system,

$$MTTF_{k/n} = \frac{A_{r-1}\{E_{12}\} - \mu_r\lambda_r\{E_{13}\} + \prod_{i=1}^{r} \lambda_i}{A_{r-1}\{E_{14}\} - \mu_r\lambda_r\{E_{15}\} - \mu_{ro}\prod_{i=1}^{r} \lambda_i} \tag{3.49}$$

where

$$\begin{aligned} A_i &= \{\mu_{i+1} + \mu_{(i+1)o} + \lambda_{i+2} + \lambda_{c_{i+2}}\} \\ E_{12} &= Numerator\ of\ MTTF_{r-1} \\ E_{13} &= Numerator\ of\ MTTF_{r-2} \\ E_{14} &= Denominator\ of\ MTTF_{r-1} \\ E_{15} &= Denominator\ of\ MTTF_{r-2} \end{aligned}$$

Figure 3.8 shows the plots of Equation (3.49) for specified values of model parameters. These plots show that the system mean time to failure decreases with an increase in the number of common-cause failures. Furthermore, the increase in system mean time to failure with increasing values of $(n - k)$ can be easily seen.

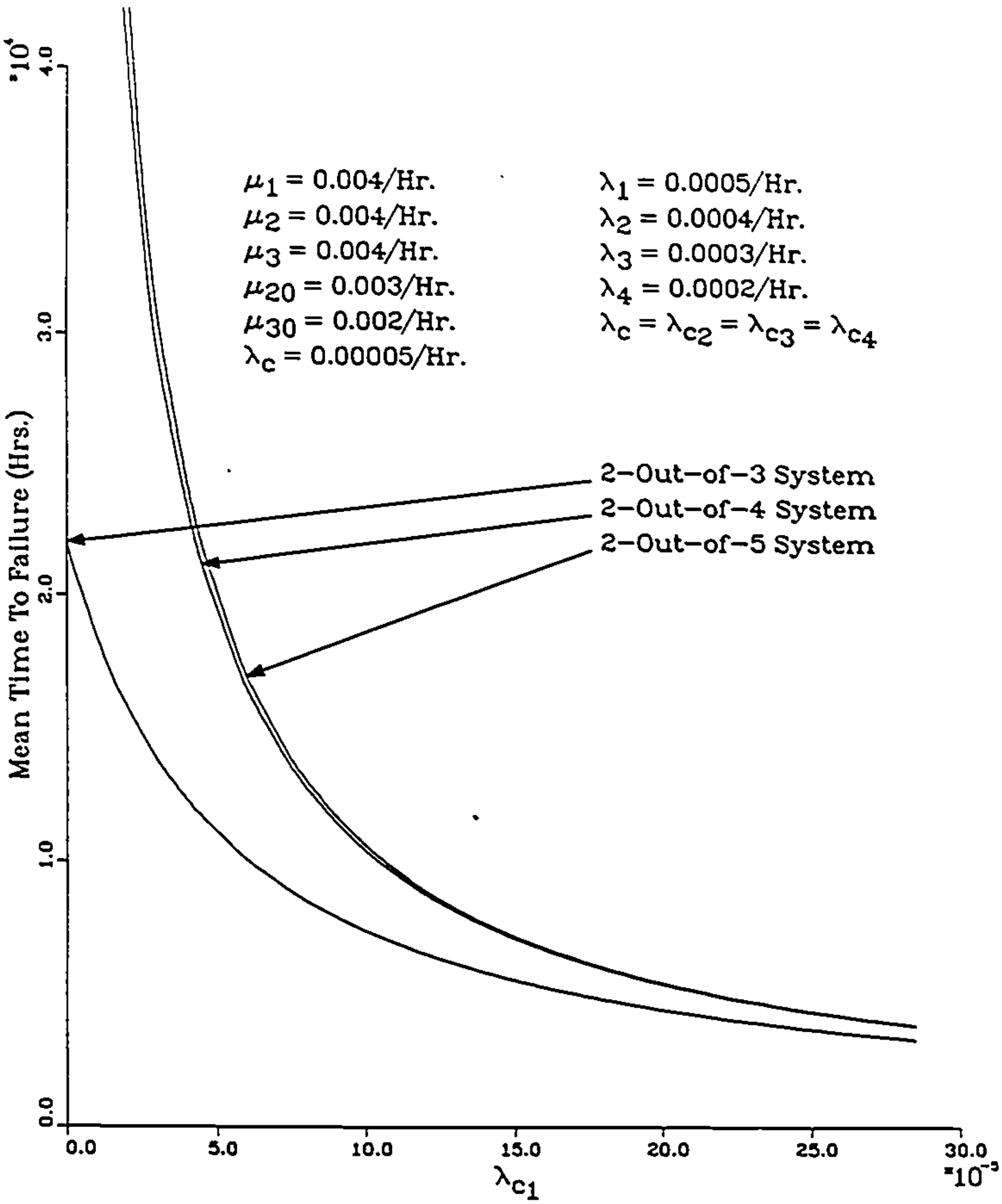

Figure 3.8: System Mean Time to Failure Plots for an Identical Unit k-out-of-n System with Type II Repair

Special Case Model I

By setting $k = (n-2)$ in Equation (3.36) – (3.41), the following system of differential equations are obtained :

$$\dot{p}_0(t) = -(\lambda_1 + \lambda_{c_1})p_0(t) + \mu_1 p_1(t) + \mu_{20}p_2(t) \tag{3.50}$$

$$\dot{p}_1(t) = \lambda_1 p_0(t) - (\mu_1 + \lambda_2 + \lambda_{c_2})p_1(t) + \mu_2 p_2(t) \tag{3.51}$$

$$\dot{p}_2(t) = \lambda_2 p_1(t) - (\mu_2 + \mu_{20} + \lambda_3 + \lambda_{c_3})p_2(t) \tag{3.52}$$

$$\dot{p}_3(t) = \lambda_3 p_2(t) \tag{3.53}$$

$$\dot{p}_4(t) = \lambda_{c_1}p_0(t) + \lambda_{c_2}p_1(t) + \lambda_{c_3}p_2(t) \tag{3.54}$$

At time $t = 0$, $p_0(0) = 1$, $p_1(0) = 0$, $p_2(0) = 0$, $p_3(0) = 0$ *and* $p_4(0) = 0$.

Solving Equations (3.50) – (3.54) using Laplace transforms yields the following state probability expressions :

$$\begin{aligned} p_0(t) &= \frac{(s_1^2 + E_{20}s_1 + E_{17})e^{s_1 t}}{(s_1 - s_2)(s_1 - s_3)} + \frac{(s_2^2 + E_{20}s_2 + E_{17})e^{s_2 t}}{(s_3 - s_2)(s_1 - s_2)} - \\ &\quad \frac{(s_3^2 + E_{20}s_3 + E_{17})e^{s_3 t}}{(s_3 - s_2)(s_1 - s_3)} \qquad (3.55) \\ p_1(t) &= \frac{\lambda_1(s_1 + \mu_2 + \lambda_{c_3} + \lambda_3 + \mu_{20})e^{s_1 t}}{(s_3 - s_1)(s_2 - s_1)} + \frac{\lambda_1(s_2 + \mu_2 + \lambda_{c_3} + \lambda_3 + \mu_{20})e^{s_2 t}}{(s_2 - s_1)(s_2 - s_3)} + \\ &\quad \frac{\lambda_1(s_3 + \mu_2 + \lambda_{c_3} + \lambda_3 + \mu_{20})e^{s_3 t}}{(s_3 - s_1)(s_3 - s_2)} \qquad (3.56) \\ p_2(t) &= \frac{\lambda_1\lambda_2 e^{(s_1 t)}}{(s_1 - s_2)(s_1 - s_3)} + \frac{\lambda_1\lambda_2 e^{(s_2 t)}}{(s_1 - s_2)(s_3 - s_2)} + \\ &\quad \frac{\lambda_1\lambda_2 e^{(s_3 t)}}{(s_3 - s_2)(s_3 - s_1)} + \qquad (3.57) \end{aligned}$$

where s_1, s_2, *and* s_3 are the roots of the cubic equation and are determined as follows:

$$s^3 + E_{18}s^2 + E_{19}s + E_{20} = 0$$

$$
\begin{aligned}
Let\ \alpha &= \frac{3E_{19} - (E_{18})^2}{9} \\
\beta &= \frac{9E_{18}E_{19} - 27E_{20} - 2(E_{18})^3}{54} \\
\Phi &= \sqrt[3]{\beta + \sqrt{\alpha^3 + \beta^2}} \\
\Omega &= \sqrt[3]{\beta - \sqrt{\alpha^3 + \beta^2}} \\
Thus,\ s_1 &= \Phi + \Omega - \frac{1}{3}E_{18} \\
s_2 &= -\frac{1}{2}(\Phi + \Omega) - \frac{1}{3}E_{18} + \frac{1}{2}i\sqrt{3}(\Phi - \Omega) \\
s_3 &= -\frac{1}{2}(\Phi + \Omega) - \frac{1}{3}E_{18} - \frac{1}{2}i\sqrt{3}(\Phi - \Omega)
\end{aligned}
$$

The above i is associated with complex numbers.

$$
\begin{aligned}
-E_{18} &= s_1 + s_2 + s_3 \\
E_{19} &= s_1 s_2 + s_2 s_3 + s_3 s_1 \\
-E_{20} &= s_1 s_2 s_3
\end{aligned}
$$

where the constants E_{16}, E_{17}, E_{18}, E_{19} *and* E_{20} are defined in Appendix B.

The system reliability can be obtained from

$$
R(t) = p_0(t) + p_1(t) + p_2(t) \tag{3.58}
$$

The plots of the Equation (3.58) are shown in Figure 3.9. It can be clearly seen from the plots that the system reliability decreases with an increase in the number of common-cause failures.

Figures 3.10 and 3.11 show the system reliability and system unreliability for a 3-out-of-5 and a 4-out-of-6 system, respectively, for specified values of model parameters.

From Equation (3.49) the system mean time to failure can be expressed as

$$
MTTF_2 = \frac{A_1\{\mu_1 + \lambda_2 + \lambda_{c_2} + \lambda_1\} - \mu_2\lambda_2\{1\} + \lambda_1\lambda_2}{[A_1\{\lambda_2\lambda_{c_1} + \mu_1\lambda_{c_1} + \lambda_{c_2}\lambda_{c_1} + \lambda_2\lambda_1 + \lambda_{c_2}\lambda_1\} - \mu_2\lambda_2\{\lambda_1 + \lambda_{c_1}\} - \mu_{20}\lambda_1\lambda_2]} \tag{3.59}
$$

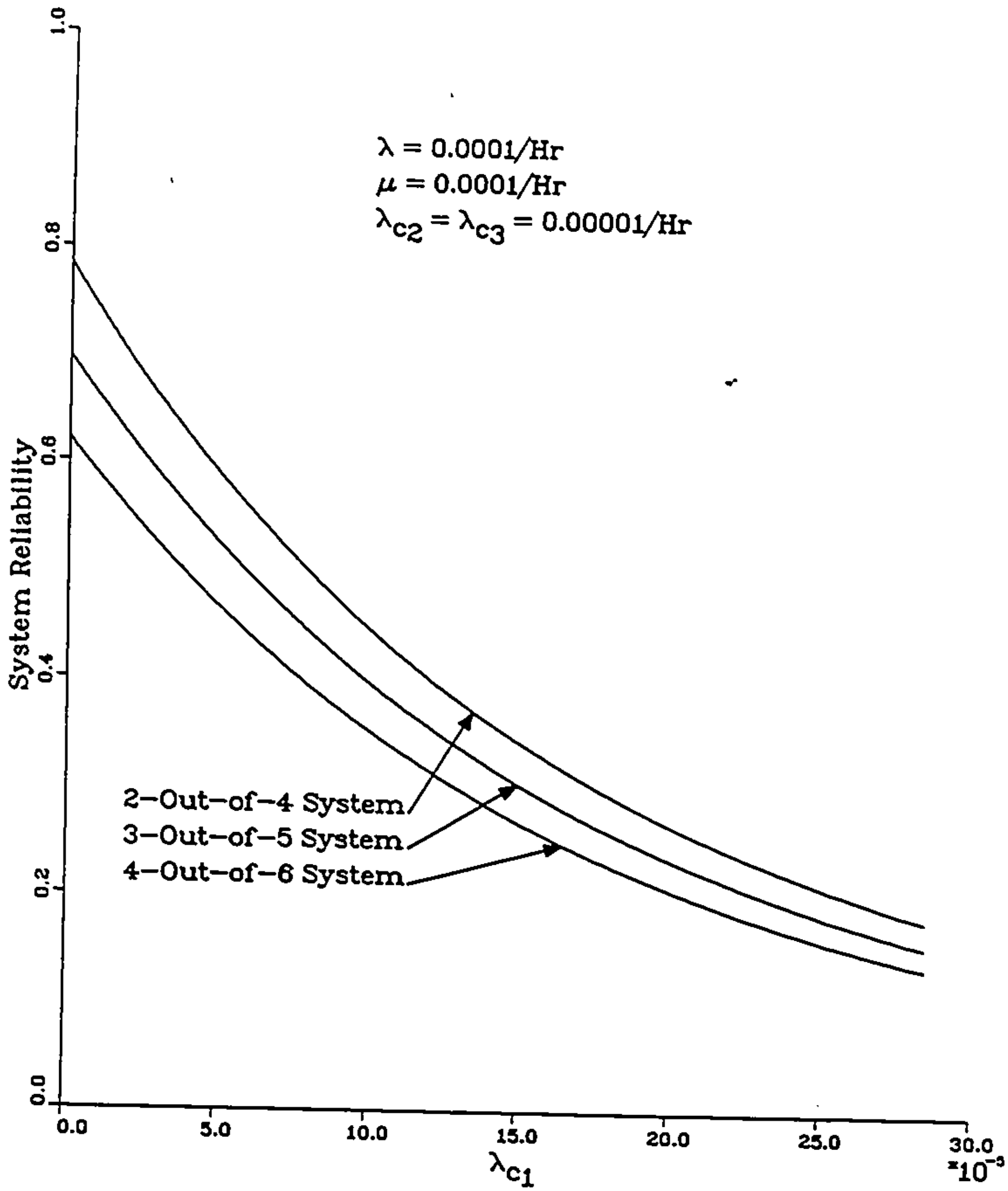

Figure 3.9: System Reliability Plots for an Identical Unit (n-2)-out-of-n System with Type II Repair

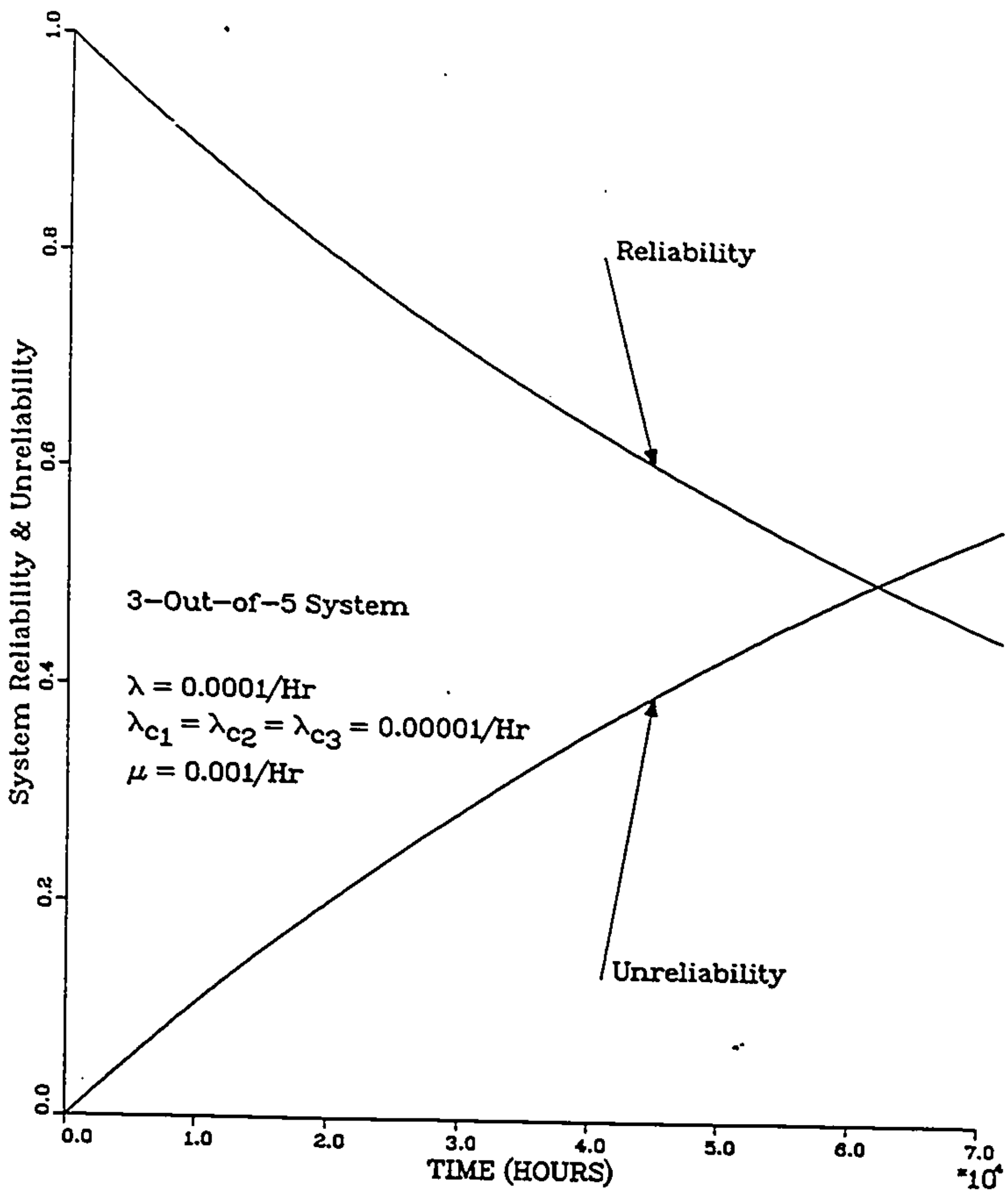

Figure 3.10: System Reliability and System Unreliability Plots for an Identical Unit 3-out-of-5 System with Type II Repair

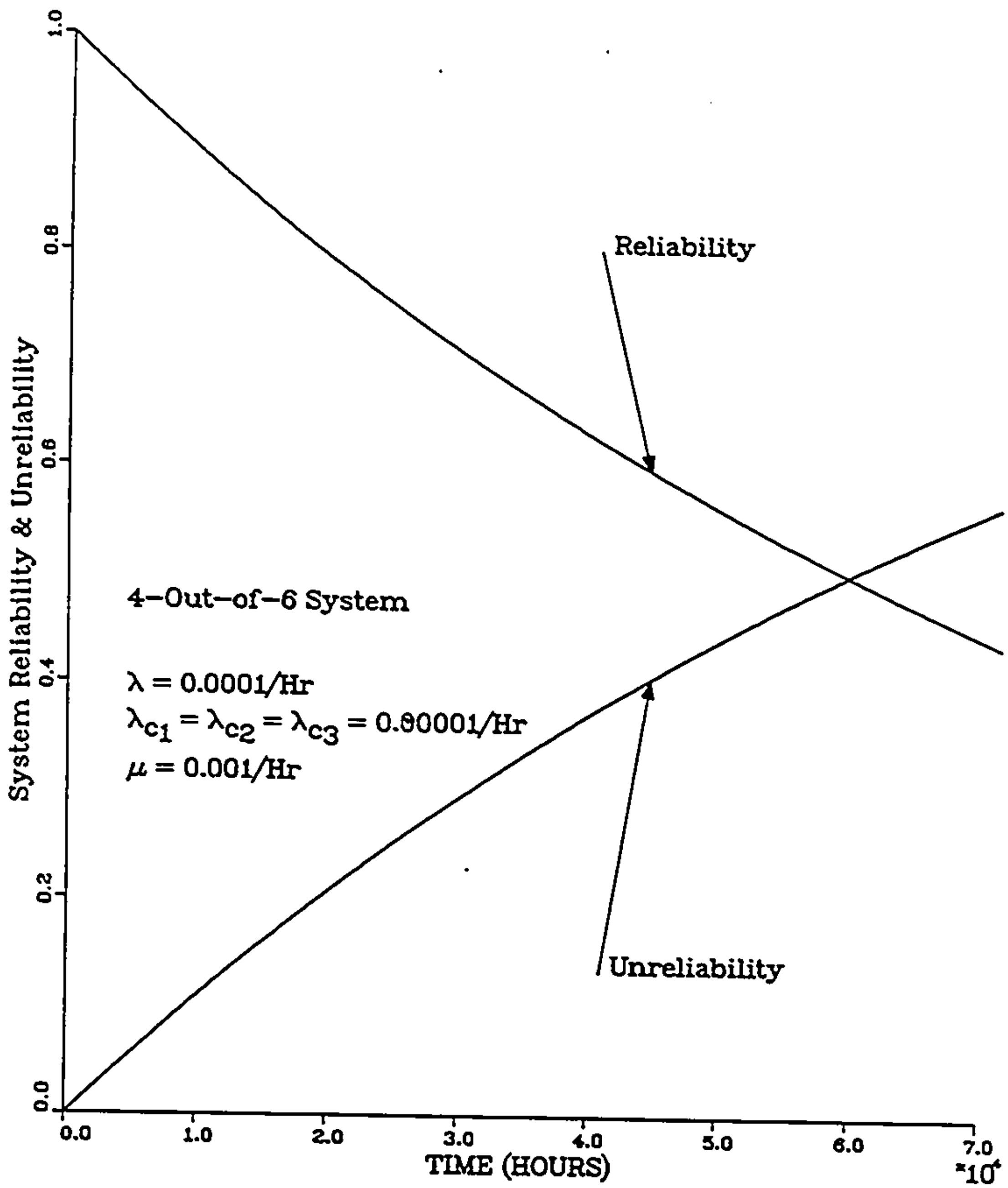

Figure 3.11: System Reliability and System Unreliability Plots for an Identical Unit 4-out-of-6 System with Type II Repair

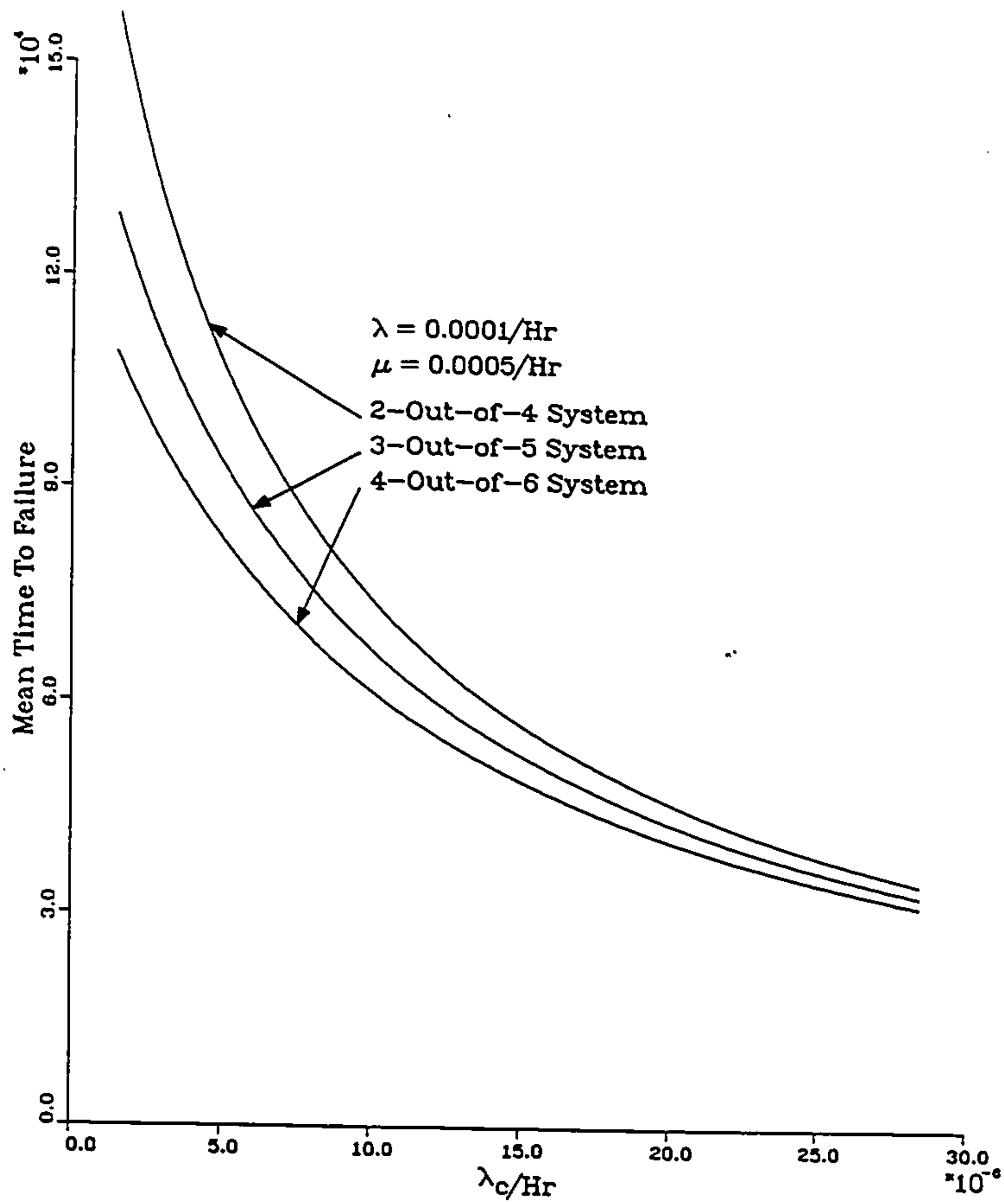

Figure 3.12: System Mean Time to Failure Plots for an Identical Unit (n-2)-out-of-n System With Type II Repair

where

$$A_1 = \{\mu_2 + \mu_{20} + \lambda_3 + \lambda_c\}$$

The plots of Equation (3.30) for specified values of model parameters are shown in Figure 3.12. The plots clearly indicate the decreasing trend in system mean time to failure with increasing values of common-cause failures.

Special Case Model II

For a system which requires $(n-1)$ units to be operative out of n units to ensure system success, the system of differential equations is obtained by setting $k = n - 1$ in Equations (3.36) − (3.41).

$$\dot{p}_0(t) = -(\lambda_1 + \lambda_{c_1})p_0(t) + \mu_1 p_1(t) \tag{3.60}$$

$$\dot{p}_1(t) = \lambda_1 p_0(t) - (\mu_1 + \lambda_2 + \lambda_{c_2})p_1(t) \tag{3.61}$$

$$\dot{p}_2(t) = \lambda_2 p_1(t) \tag{3.62}$$

$$\dot{p}_3(t) = \lambda_{c_1} p_0(t) + \lambda_{c_2} p_1(t) \tag{3.63}$$

The initial conditions are, at time $t = 0$, $p_0(0) = 1$, $p_1(0) = 0$, $p_2(0) = 0$, *and* $p_3(0) = 0$. Solving Equations (3.60)−(3.63) with the aid of Laplace transforms results in the following state probability expressions :

$$\begin{aligned} p_0(t) \;=\; & \frac{(\mu_1 + \lambda_2 - \lambda_1 - \lambda_{c_1} + \lambda_{c_2} + E_{21})e^{-\frac{1}{2}(\mu_1+\lambda_2+\lambda_1+\lambda_{c_1}+\lambda_{c_2}-E_{21})t}}{2E_{21}} + \\ & \frac{(\lambda_1 + \lambda_{c_1} - \mu_1 - \lambda_2 - \lambda_{c_2} + E_{21})e^{-\frac{1}{2}(\mu_1+\lambda_2+\lambda_1+\lambda_{c_1}+\lambda_{c_2}+E_{21})t}}{2E_{21}} \end{aligned} \tag{3.64}$$

$$p_1(t) = \frac{\lambda_1\{e^{E_{21}t} - 1\}e^{-\frac{1}{2}(\mu_1+\lambda_2+\lambda_1+\lambda_{c_1}+\lambda_{c_2}+E_{21})t}}{E_{21}} \tag{3.65}$$

The constant E_{21} is defined in Appendix B.

The system reliability is given by

$$R(t) \;=\; p_0(t) + p_1(t) \tag{3.66}$$

The plots of system reliability expressed in Equation (3.66) are given in Figure 3.13 from which it can be easily concluded that the system reliability decreases with increase in common-cause failures. Figure 3.14 and Figure 3.15 show the system reliability and unreliability for specified values of the model parameters for a 3-out-of-4 and a 4-out-of-5 system respectively.

The system mean time to failure of a (n-1)-out-of-n system is given by the expression

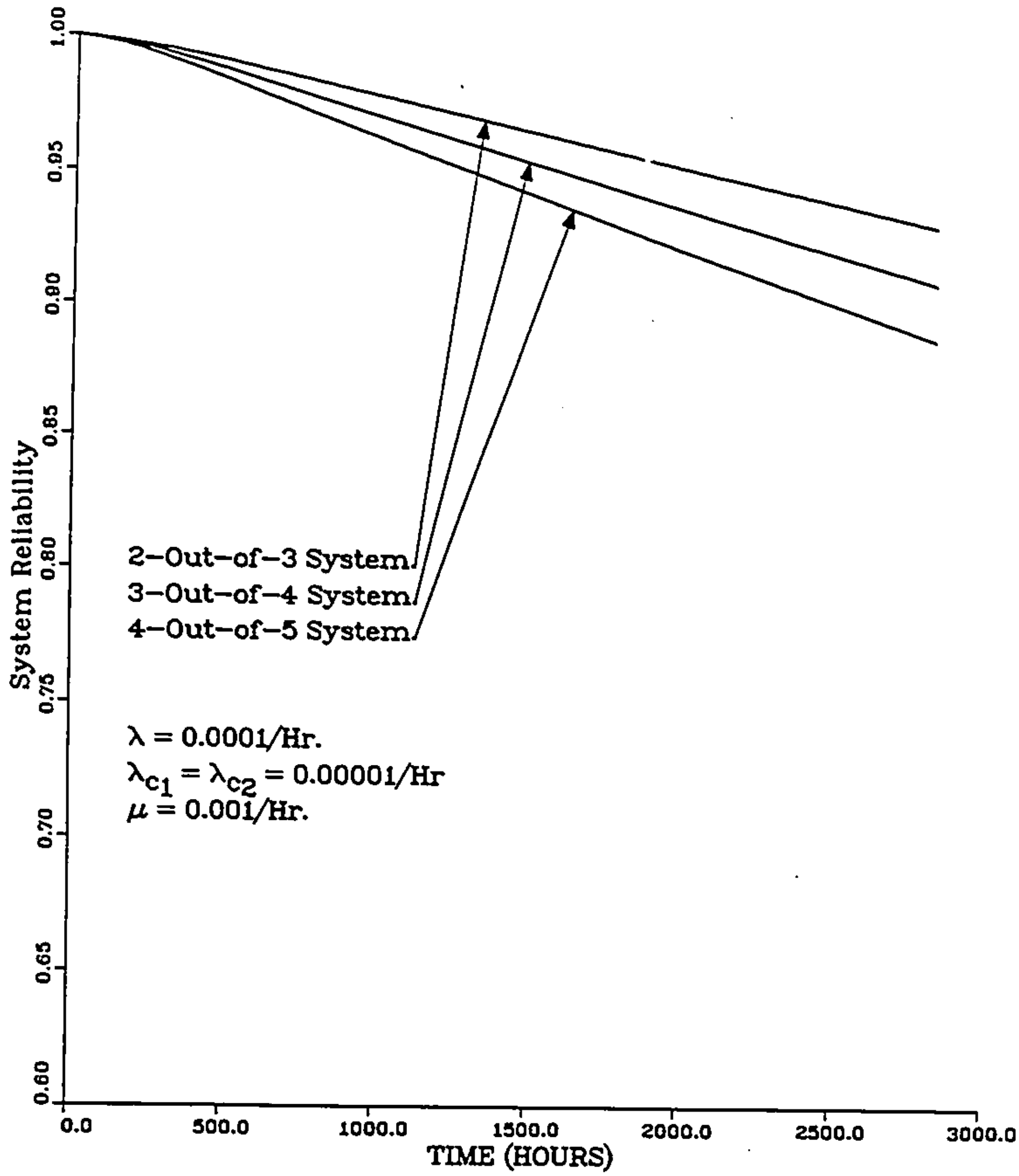

Figure 3.13: System Reliability Plots for an Identical Unit (n-1)-out-of-n System With Type II Repair

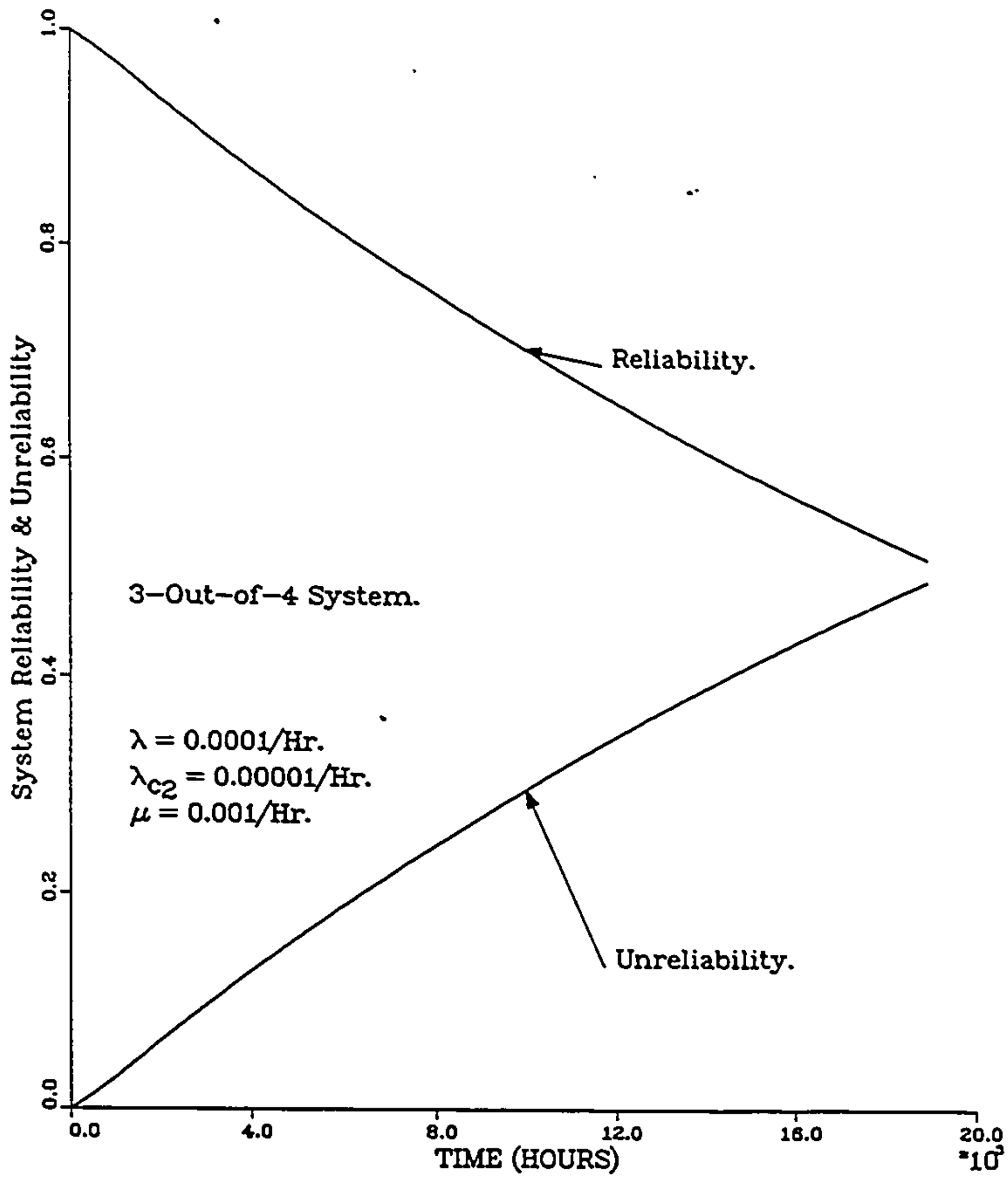

Figure 3.14: System Reliability and System unreliability Plots for an Identical Unit 3-out-of-4 System With Type II Repair

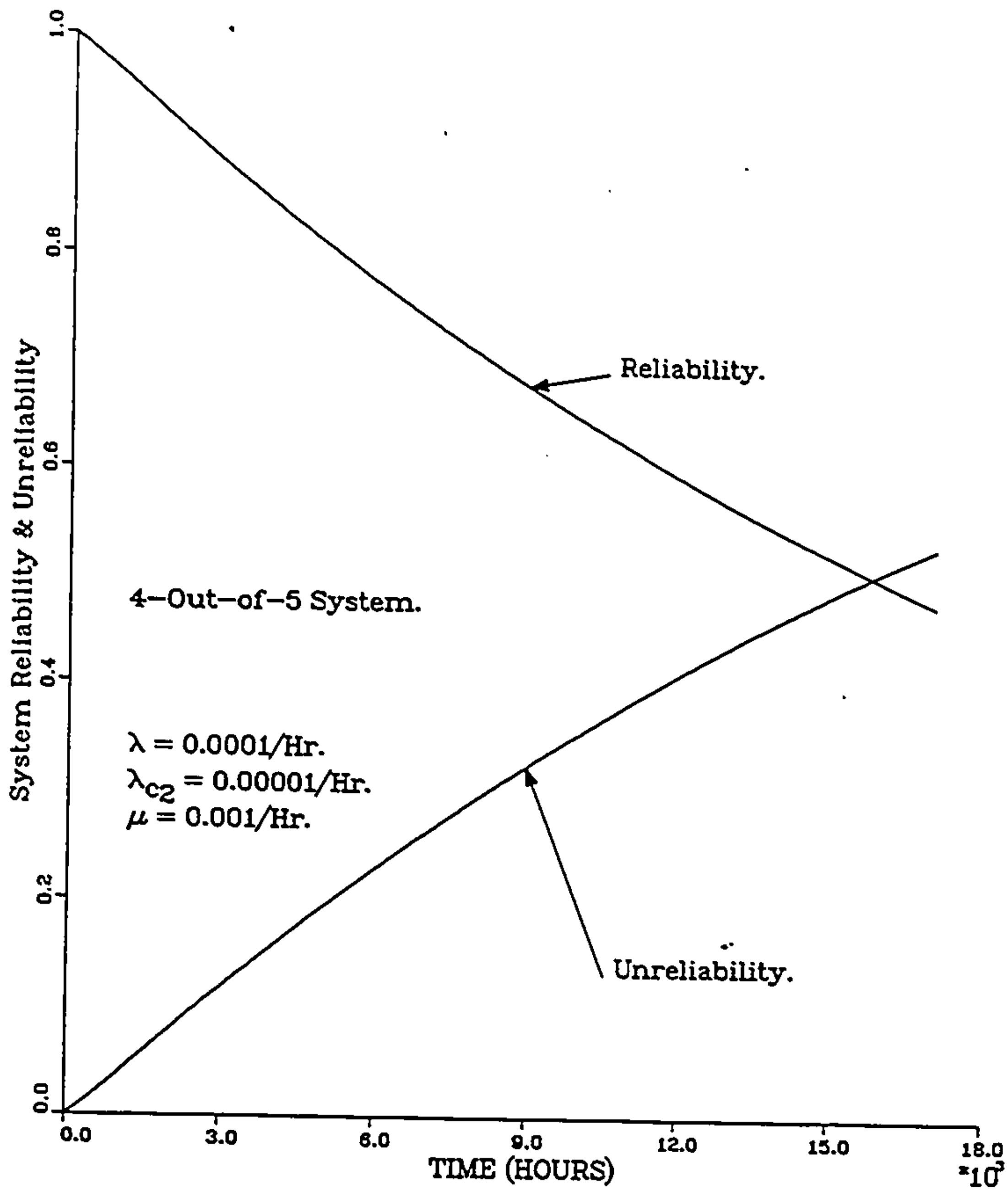

Figure 3.15: System Reliability and System unreliability Plots for an Identical Unit 4-out-of-5 System With Type II Repair

$$MTTF = \frac{\mu_1 + \lambda_2 + \lambda_{c_2} + \lambda_1}{\lambda_2\lambda_{c_1} + \mu_1\lambda_{c_1} + \lambda_{c_1}\lambda_{c_2} + \lambda_1\lambda_2 + \lambda_{c_2}\lambda_1} \qquad (3.67)$$

The system mean time to failure for a 2-out-of-3, 3-out-of-4 and for a 4-out-of-5 system are plotted in Figure 3.16. The plots clearly show that the system mean time to failure decreases with an increase in the number of common-cause failures.

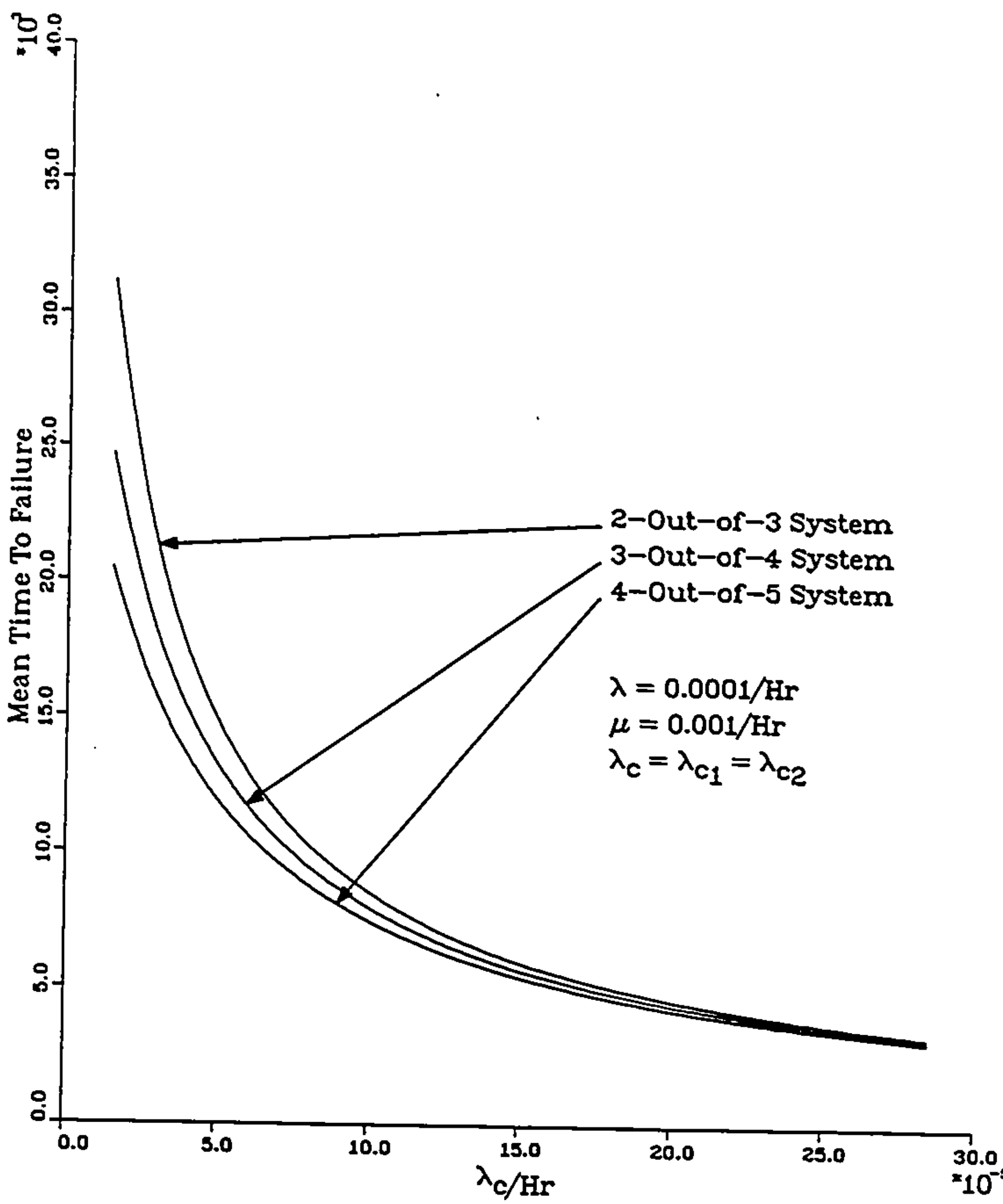

Figure 3.16: System Mean Time to Failure Plots for an Identical Unit (n-1)-out-of-n System with Type II Repair

3.1.3 k-out-of-n System Without Repair

Setting the repair rates μ_i; for $i = 1, 2, 3, \ldots, r+1$, μ_{ω_i}; for $i = 2, 3, 4, \ldots, r+1$ and $\mu_{c_{i+1}}$; for $i = 0, 1, 2, \ldots, r$ in Figure 3.1 results in a k-out-of-n system without repair. The system of differential equations for such a system is as follows :

$$\dot{p}_0(t) = -(\lambda_1 + \lambda_{c_1})p_0(t) \tag{3.68}$$

$$\dot{p}_j(t) = \lambda_j p_{j-1}(t) - (\lambda_{j+1} + \lambda_{c_{j+1}})p_j(t) \tag{3.69}$$

for $j = 1, 2, 3, \ldots, r$

$$\dot{p}_{r+1}(t) = \lambda_{r+1}p_r(t) \tag{3.70}$$

$$\dot{p}_{r+2}(t) = \sum_{i=0}^{r} \lambda_{c_{i+1}} p_i(t) \tag{3.71}$$

The initial conditions are, at time $t = 0$, $p_0(0) = 1$ and all other initial condition probabilities are equal to zero.

The Laplace transforms of the state probability expressions can be obtained by solving Equations (3.68) – (3.71).

$$p_0(s) = \frac{1}{s + \lambda_1 + \lambda_{c_1}} \tag{3.72}$$

$$p_j(s) = \frac{\prod_{i=1}^{j} \lambda_i}{\prod_{i=1}^{j+1}(s + \lambda_i + \lambda_{c_i})} \tag{3.73}$$

for $j = 1, 2, 3, \ldots, r$

The k-out-of-n unit system mean time to failure is given by

$$MTTF_{k/n} = \lim_{s \to 0}[p_0(s) + p_1(s) + \ldots + p_{r-1}(s) + p_r(s)]$$

Taking inverse Laplace transforms of the above expressions (i.e., Equations (3.72)–3.73)), we get the following time dependent state probability expressions :

$$p_0(t) = e^{-(\lambda_1 + \lambda_{c_1})t} \tag{3.74}$$

$$p_1(t) = \lambda_1 e^{-\frac{1}{2}(\lambda_2 + \lambda_{c_2} + \lambda_{c_1} + \lambda_1)t} \times \frac{\left\{e^{\frac{1}{2}(\lambda_2 + \lambda_{c_2} - \lambda_{c_1} - \lambda_1)t} - e^{-\frac{1}{2}(\lambda_2 + \lambda_{c_2} - \lambda_{c_1} - \lambda_1)t}\right\}}{(\lambda_2 + \lambda_{c_2} - \lambda_{c_1} - \lambda_1)} \tag{3.75}$$

$$p_j(t) = \sum_{k=1}^{j+1} \frac{\prod_{i=1}^{j} \lambda_i e^{-(\lambda_k + \lambda_{c_k})t}}{\prod_{i=1, i \neq k}^{j+1} (\lambda_i + \lambda_{c_i} - \lambda_k - \lambda_{c_k})} \tag{3.76}$$

for $j = 2, 3, 4, \ldots, n - k$

The system reliability is given by

$$R(t) = p_o(t) + p_1(t) + \sum_{i=2}^{r} p_i(t) \tag{3.77}$$

The system reliability plots of Equation (3.77) are given in Figure 3.17 for specified values of model parameters and it is evident from the plots that an increase in the number of common-cause failures decreases the system reliability.

The system mean time to failure ($MTTF$) is given by

$$\begin{aligned} MTTF &= \lim_{s \to 0} R(s) \\ &= \lim_{s \to 0} \{p_0(s) + \sum_{i=1}^{r} p_i(s)\} \\ &= \{\frac{1}{(\lambda_1 + \lambda_{c_1})} + \sum_{j=1}^{r} \frac{\prod_{i=1}^{j} \lambda_i}{\prod_{i=1}^{j+1} (\lambda_i + \lambda_{c_i})}\} \end{aligned} \tag{3.78}$$

The system mean time to failure plots, for specified values of parameters, are shown in Figure 3.18. The plots indicate a decreasing trend in system mean time to failure with an increase in common-cause failures.

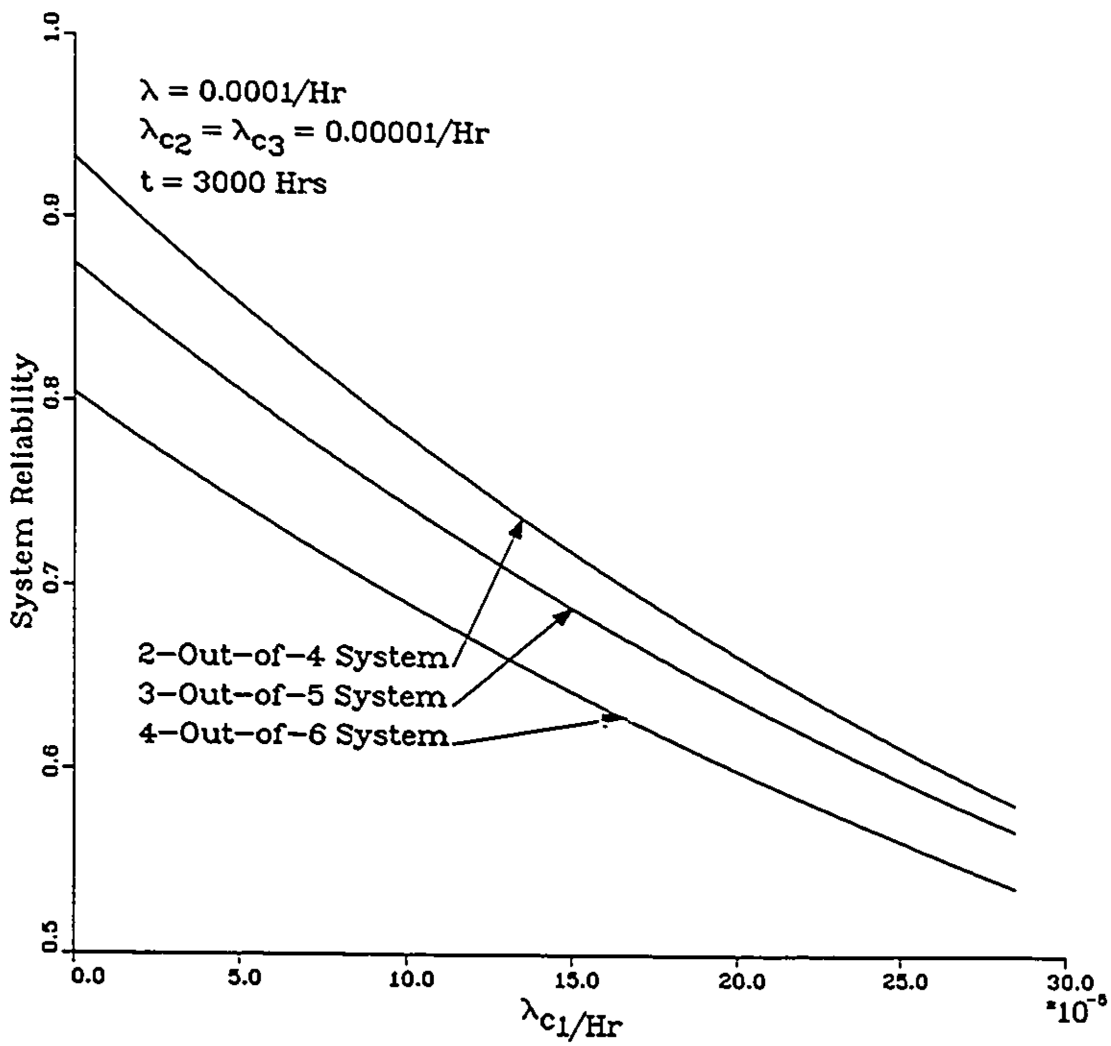

Figure 3.17: System Reliability Plots for an Identical Unit k-out-of-n System Without Repair

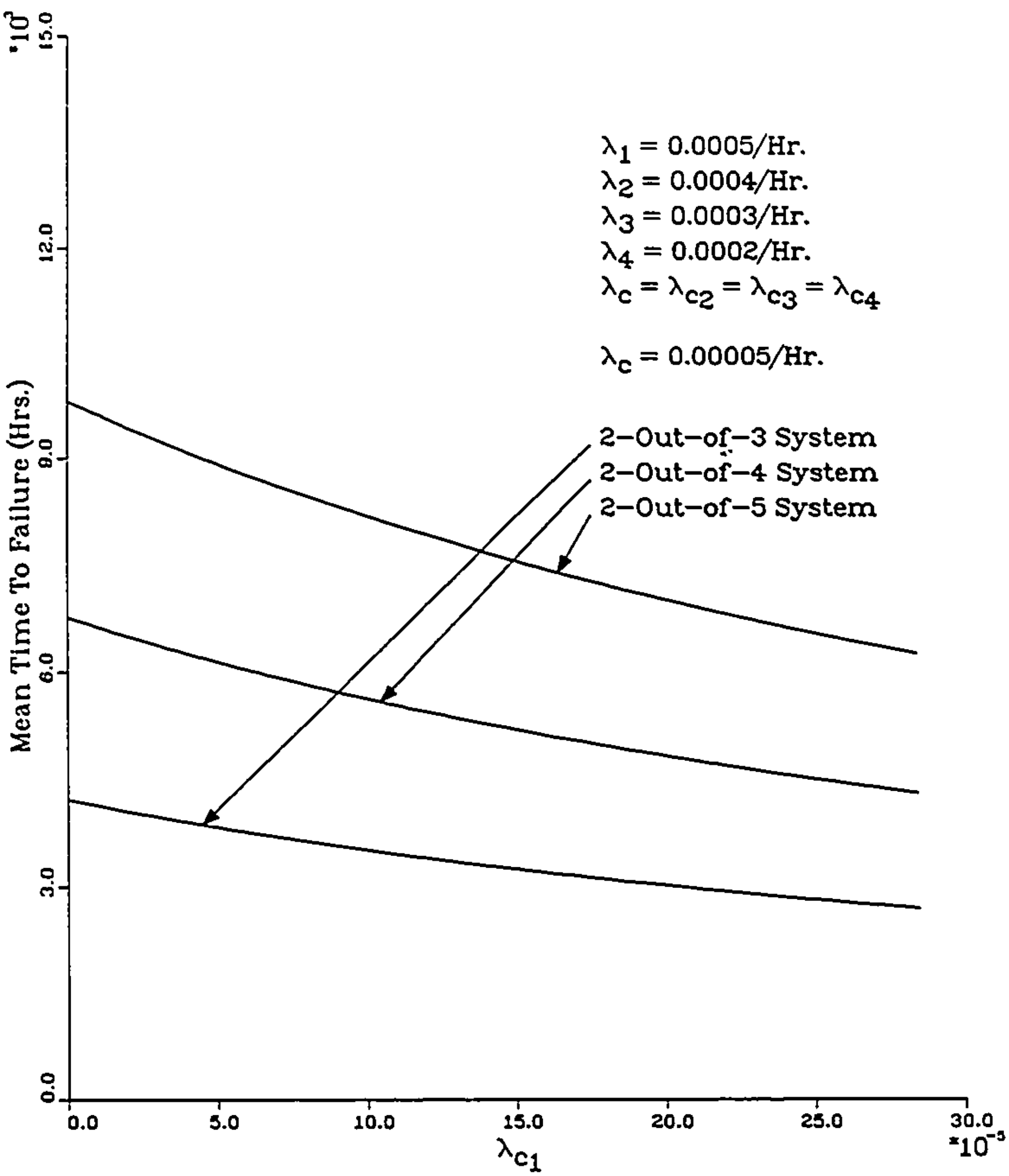

Figure 3.18: System Mean Time to Failure Plots for an Identical Unit k-out-of-n System Without Repair

Special Case Model I

Setting $k = n - 2$ in Equations (3.68) – (3.71), yields the following system of differential equations for a (n-2)-out-of-n system :

$$\dot{p}_0(t) = -(\lambda_1 + \lambda_{c_1})p_0(t) \quad (3.79)$$

$$\dot{p}_1(t) = \lambda_1 p_0(t) - (\lambda_2 + \lambda_{c_2})p_1(t) \quad (3.80)$$

$$\dot{p}_2(t) = \lambda_2 p_1(t) - (\lambda_3 + \lambda_{c_3})p_2(t) \quad (3.81)$$

$$\dot{p}_3(t) = \lambda_3 p_2(t) \quad (3.82)$$

$$\dot{p}_4(t) = \lambda_{c_1} p_0(t) + \lambda_{c_2} p_1(t) + \lambda_{c_3} p_2(t) \quad (3.83)$$

At time $t = 0$, $p_0(0) = 1$, $p_1(0) = 0$, $p_2(0) = 0$, $p_3(0) = 0$ *and* $p_4(0) = 0$.

Solving the above equations with the aid of Laplace transforms, the following time dependent state probability expressions are obtained :

$$p_0(t) = \epsilon^{-(\lambda_1+\lambda_{c_1})t} \quad (3.84)$$

$$p_1(t) = \lambda_1 \epsilon^{-\frac{1}{2}(\lambda_2+\lambda_{c_2}+\lambda_{c_1}+\lambda_1)t} \times \frac{\{\epsilon^{\frac{1}{2}(\lambda_2+\lambda_{c_2}-\lambda_{c_1}-\lambda_1)t} - \epsilon^{-\frac{1}{2}(\lambda_2+\lambda_{c_2}-\lambda_{c_1}-\lambda_1)t}\}}{(\lambda_2 + \lambda_{c_2} - \lambda_{c_1} - \lambda_1)} \quad (3.85)$$

$$p_2(t) = \frac{\lambda_1\lambda_2\epsilon^{-(\lambda_1+\lambda_{c_1})t}}{(\lambda_2 + \lambda_{c_2} - \lambda_1 - \lambda_{c_1})(\lambda_3 + \lambda_{c_3} - \lambda_1 - \lambda_{c_1})} + \frac{\lambda_1\lambda_2\epsilon^{-(\lambda_2+\lambda_{c_2})t}}{(\lambda_1 + \lambda_{c_1} - \lambda_2 - \lambda_{c_2})(\lambda_3 + \lambda_{c_3} - \lambda_2 - \lambda_{c_2})} + \frac{\lambda_1\lambda_2\epsilon^{-(\lambda_3+\lambda_{c_3})t}}{(\lambda_1 + \lambda_{c_1} - \lambda_3 - \lambda_{c_3})(\lambda_2 + \lambda_{c_2} - \lambda_3 - \lambda_{c_3})} \quad (3.86)$$

The system reliability, given by $p_0(t) + p_1(t) + p_2(t)$, is plotted in Figure 3.19. Furthermore, Figures 3.20 and 3.21 show the system reliability and system unreliability for a 3-out-of-5 and 4-out-of-6 system respectively.

The mean time to failure of the system can be expressed as follows :

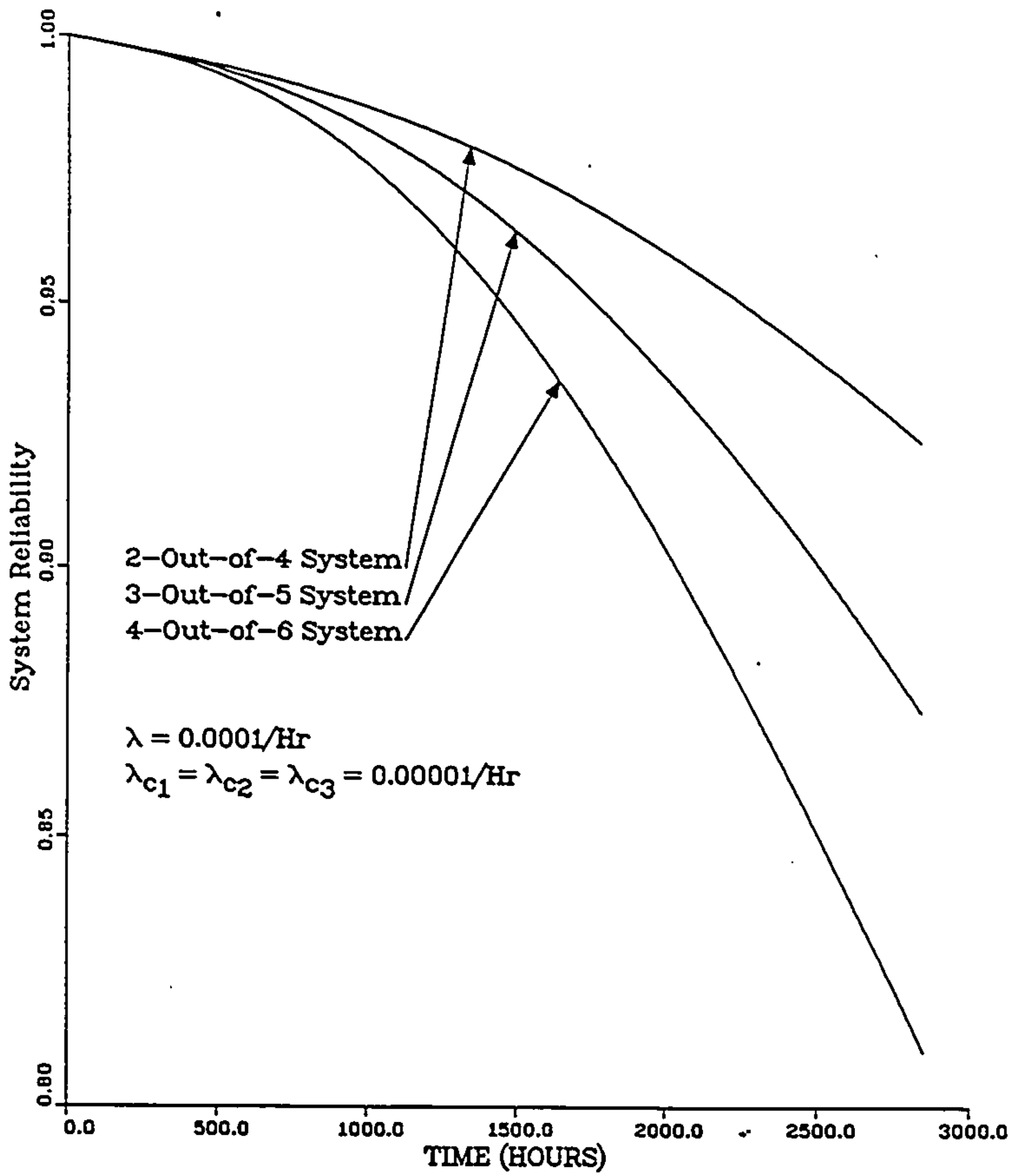

Figure 3.19: System Reliability Plots for an Identical Unit (n-2)-out-of-n System Without Repair

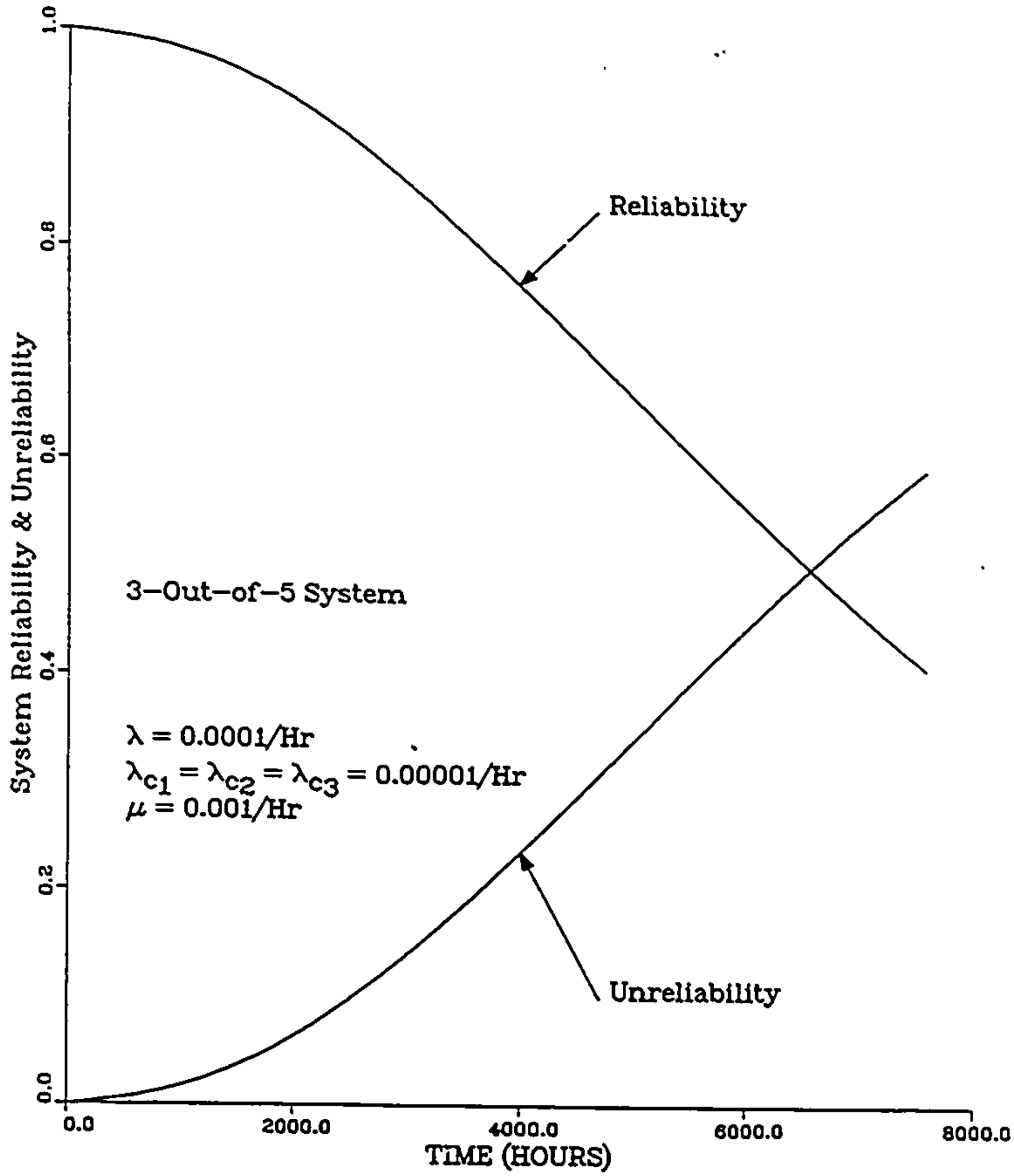

Figure 3.20: System Reliability and System Unreliability Plots for an Identical Unit 3-out-of-5 System Without Repair

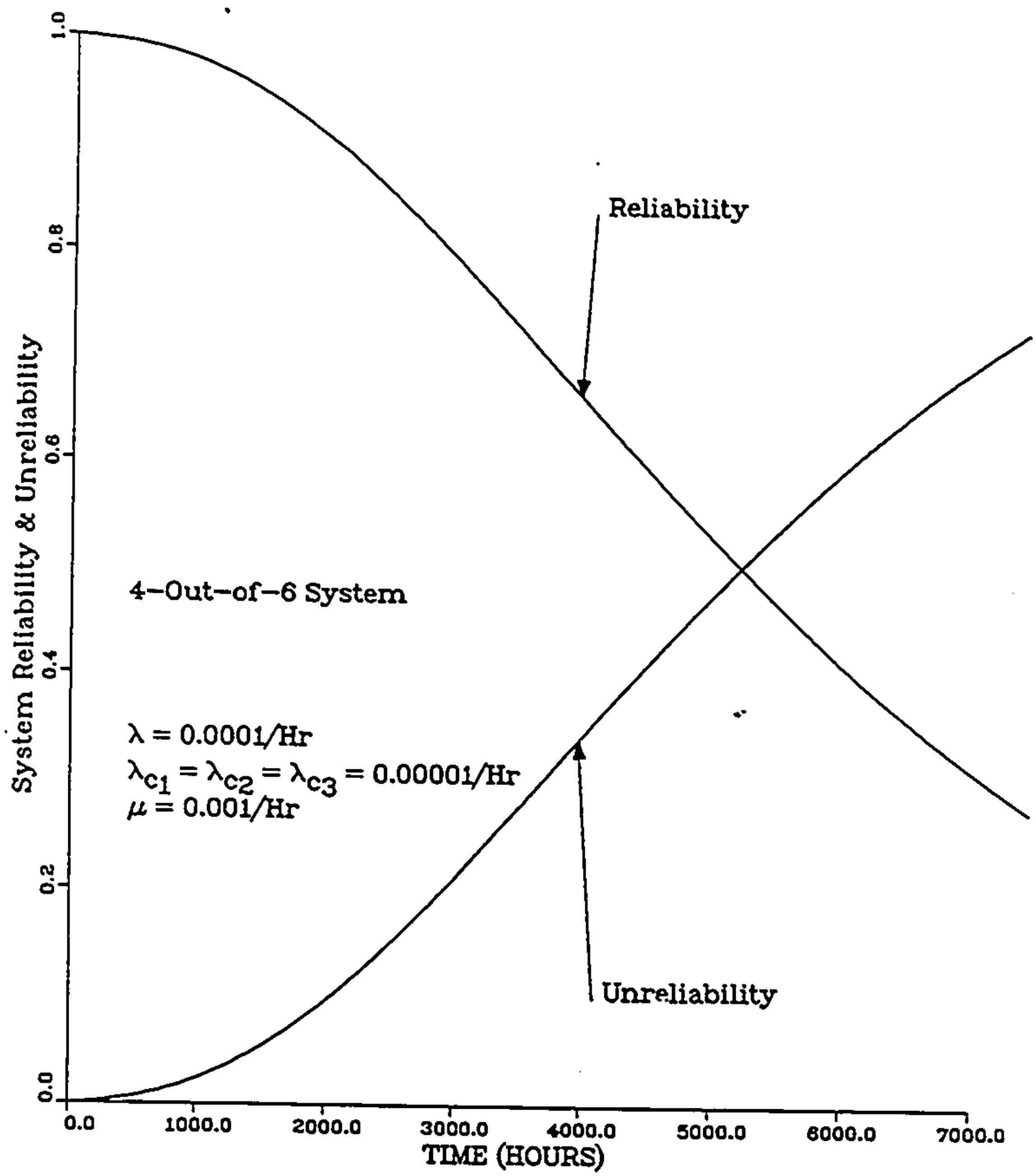

Figure 3.21: System Reliability and System Unreliability Plots for an Identical Unit 4-out-of-6 System Without Repair

$$MTTF = \frac{1}{\lambda_1 + \lambda_{c_1}} + \frac{\lambda_1}{(\lambda_1 + \lambda_{c_1})(\lambda_2 + \lambda_{c_2})} + \frac{\lambda_1 \lambda_2}{(\lambda_1 + \lambda_{c_1})(\lambda_2 + \lambda_{c_2})(\lambda_3 + \lambda_{c_3})} \tag{3.87}$$

The plots in Figure 3.22 show the system mean time to failure for specified values of the model parameters. These plots indicate a decreasing trend in system mean time to failure with an increasing number of common-cause failures.

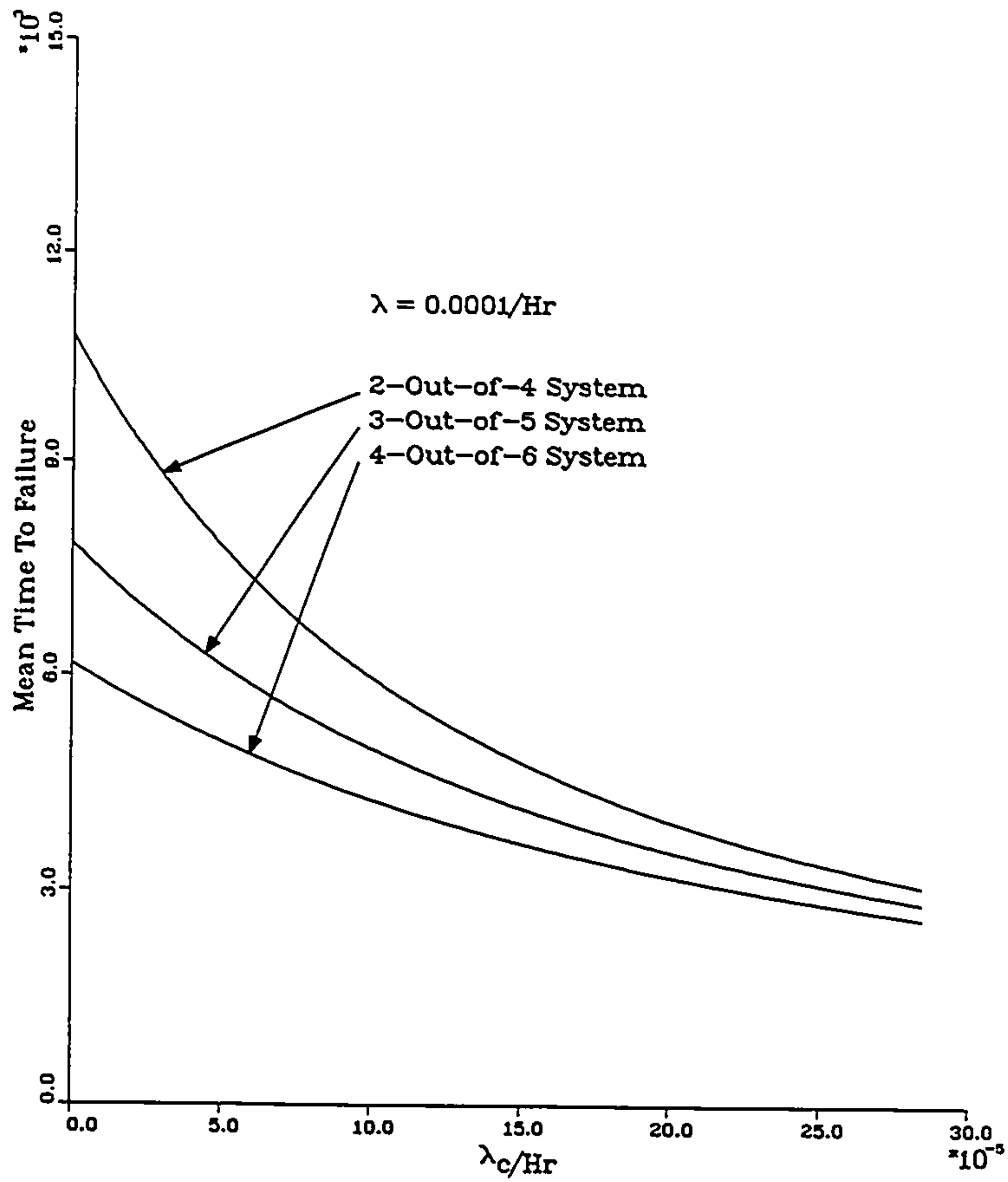

Figure 3.22: System Mean Time to Failure for an Identical Unit (n-2)-out-of-n System Without Repair

Special Case Model II

When $n-1$ units out of n units are required to be operative for the system to function normally (3-out-of-4), the system of differential equations can be obtained by setting $k = n-1$ in Equations (3.68) – (3.71).

$$\dot{p}_0(t) = -(\lambda_1 + \lambda_{c_1})p_0(t) \tag{3.88}$$

$$\dot{p}_1(t) = \lambda_1 p_0(t) - (\lambda_2 + \lambda_{c_2})p_1(t) \tag{3.89}$$

$$\dot{p}_2(t) = \lambda_2 p_1(t) \tag{3.90}$$

$$\dot{p}_3(t) = \lambda_{c_1} p_0(t) + \lambda_{c_2} p_1(t) \tag{3.91}$$

At time $t = 0, p_0(0) = 1, \ p_1(0) = 0, \ p_2(0) = 0, \ and \ p_3(0) = 0.$

Solving the above Equations with the aid of Laplace transforms results in the following state probability expressions :

$$p_0(t) = e^{-(\lambda_1 + \lambda_{c_1})t} \tag{3.92}$$

$$p_1(t) = \lambda_1 e^{-\frac{1}{2}(\lambda_2+\lambda_{c_2}+\lambda_{c_1}+\lambda_1)t} \times \frac{\{e^{\frac{1}{2}(\lambda_2+\lambda_{c_2}-\lambda_{c_1}-\lambda_1)t} - e^{-\frac{1}{2}(\lambda_2+\lambda_{c_2}-\lambda_{c_1}-\lambda_1)t}\}}{(\lambda_2 + \lambda_{c_2} - \lambda_{c_1} - \lambda_1)} \tag{3.93}$$

The system reliability is given by

$$R(t) = p_0(t) + p_1(t) \tag{3.94}$$

Figure 3.23 shows some typical plots of Equation (3.94). The inverse relationship between the system reliability and failure rate is clearly shown in these plots. Furthermore, Figures 3.24 and 3.25 show the system reliability and system unreliability for a 3-out-of-4 and 4-out-of-5 system respectively for specified values of the model parameters.

The mean time to failure for a (n-1)-out-of-n system is given by

$$MTTF = \frac{1}{\lambda_1 + \lambda_{c_1}} + \frac{\lambda_1}{(\lambda_1 + \lambda_{c_1})(\lambda_2 + \lambda_{c_2})} \tag{3.95}$$

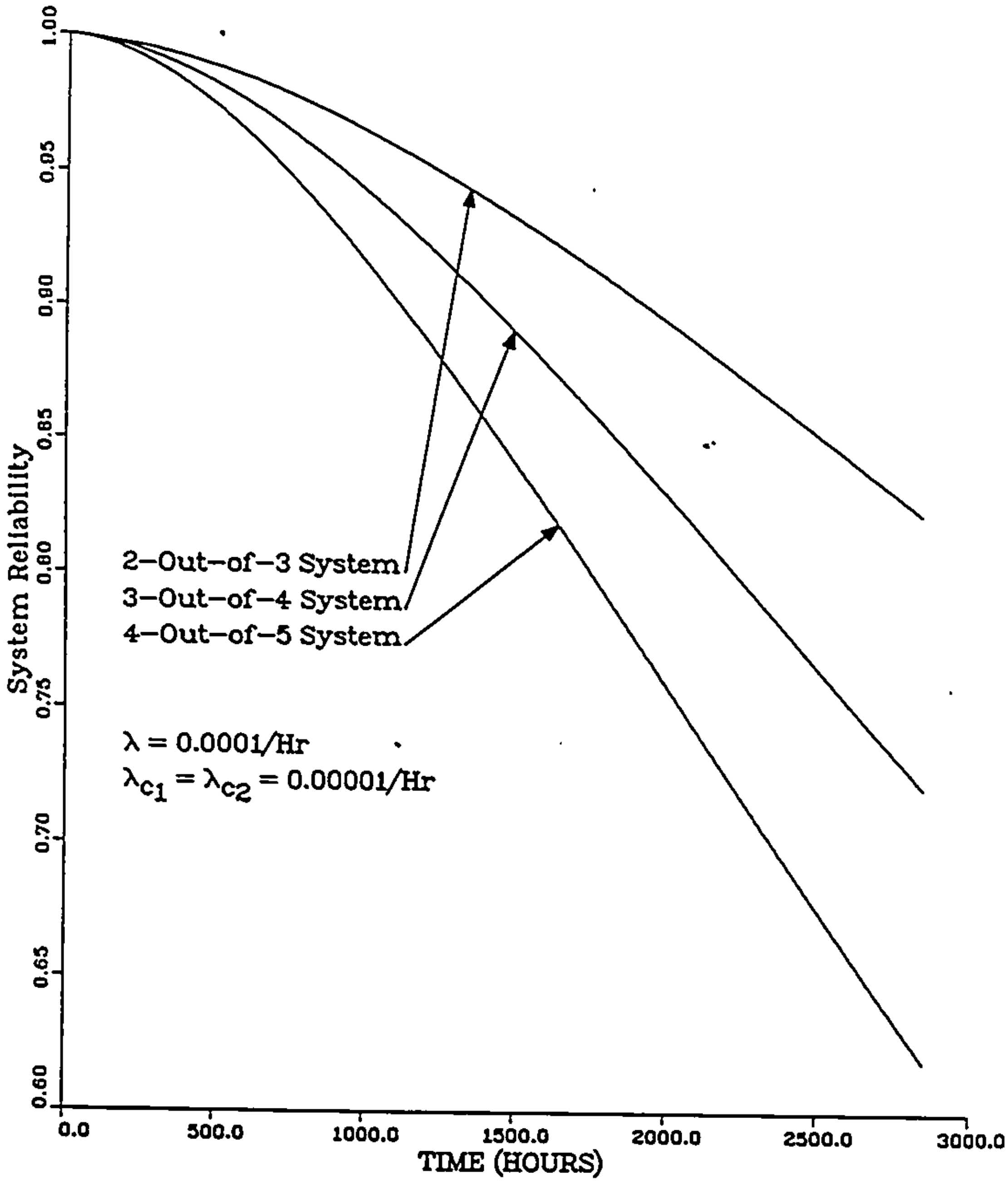

Figure 3.23: System Reliability Plots for an Identical Unit (n-1)-out-of-n System Without Repair

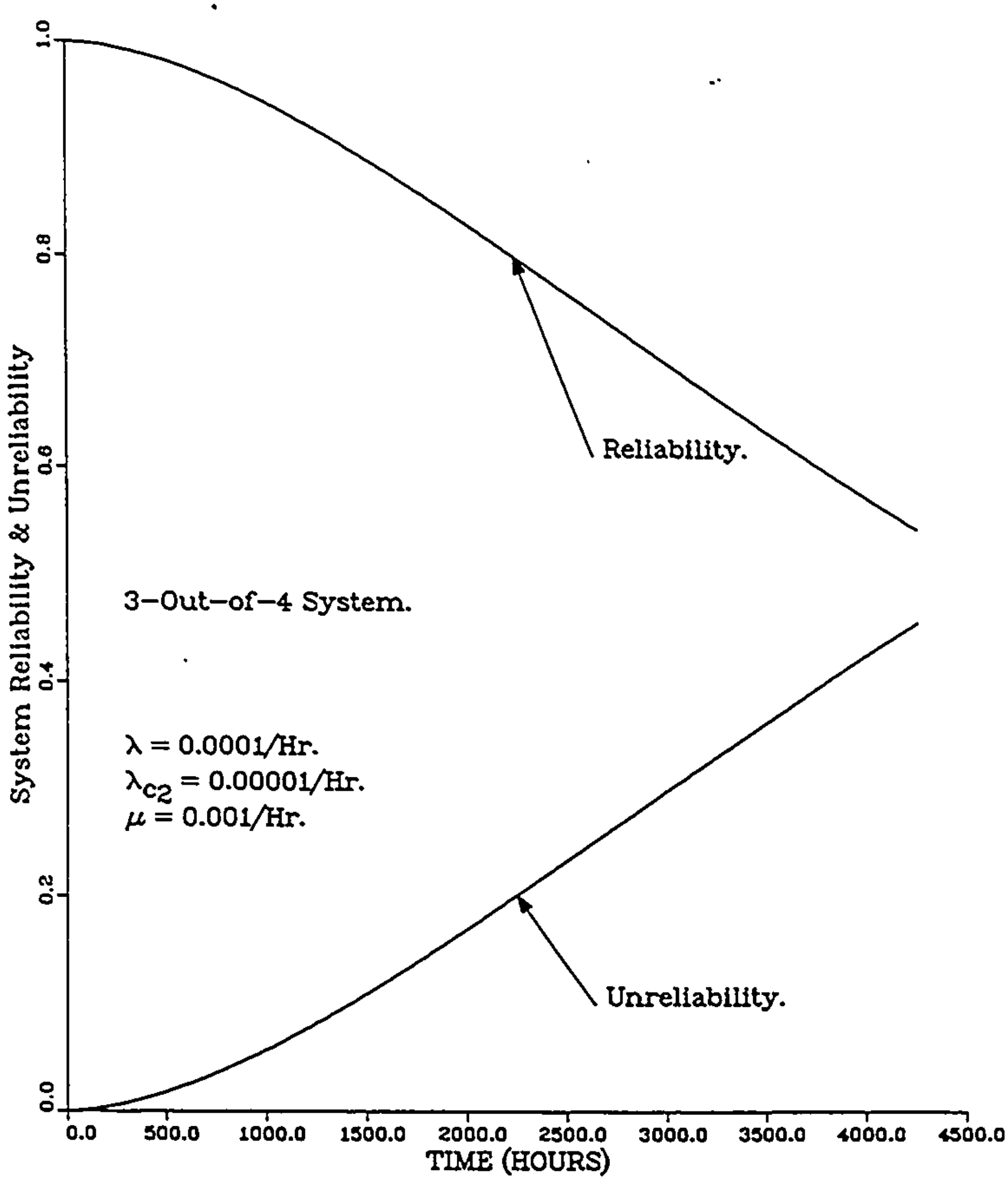

Figure 3.24: System Reliability Plots and System Unreliability Plots for an Identical Unit 3-out-of-4 System Without Repair

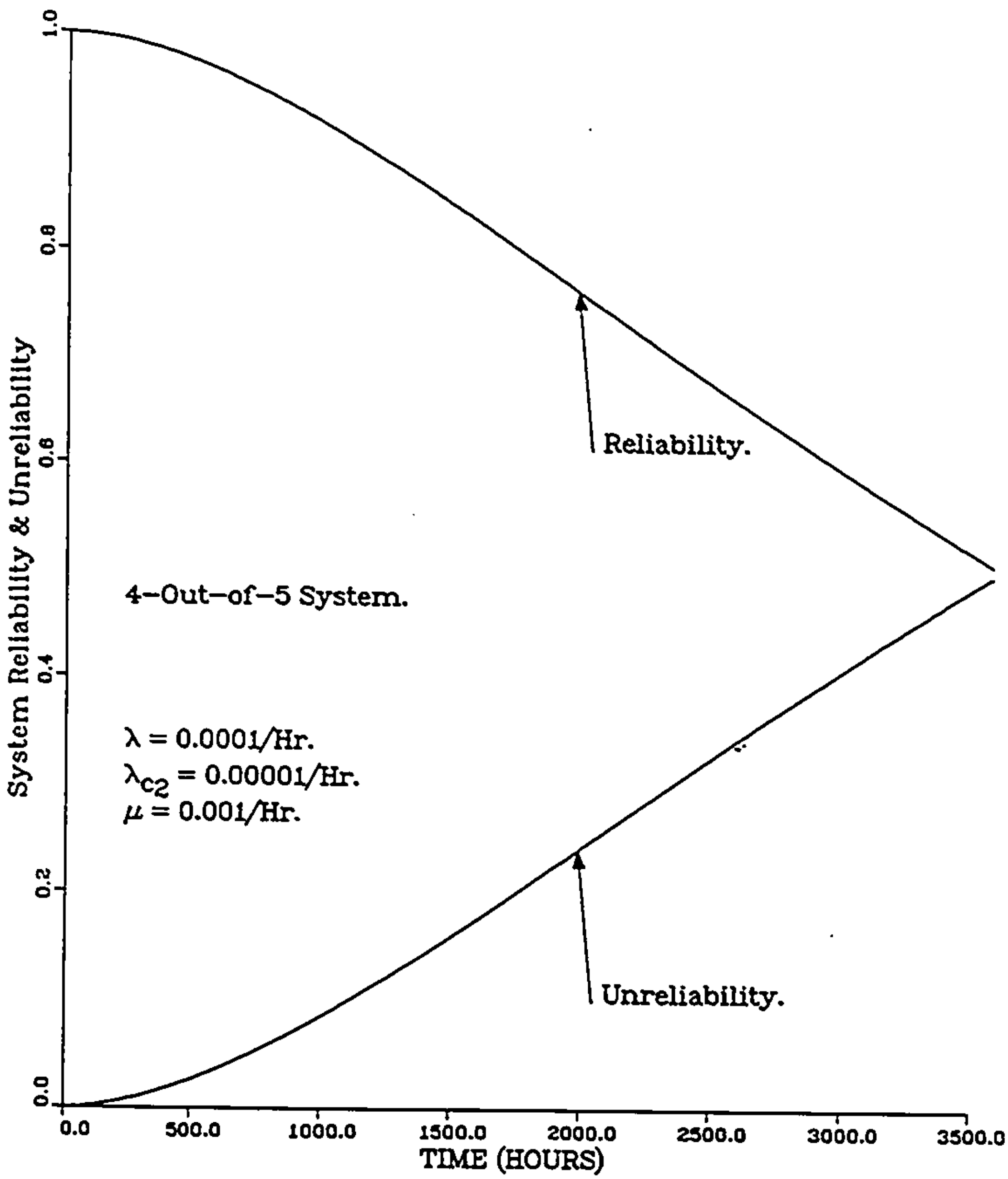

Figure 3.25: System Reliability Plots and System Unreliability Plots for an Identical Unit 4-out-of-5 System Without Repair

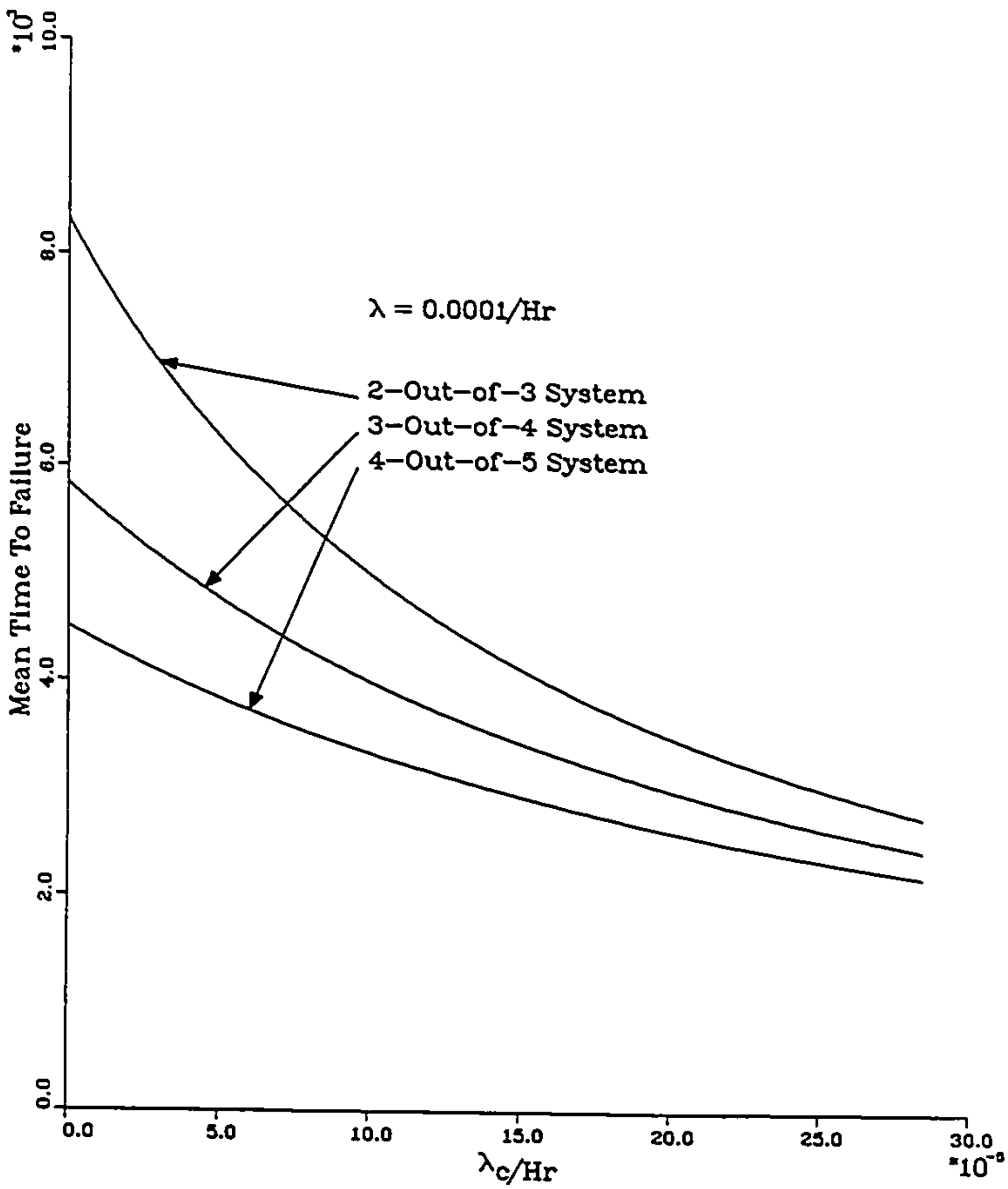

Figure 3.26: System Mean Time to Failure Plots for an Identical Unit (n-1)-out-of-n System Without Repair

Figure 3.26 shows typical plots of the system mean time to failure using Equation (3.93). It can be seen that the system mean time to failure decreases with an increase in the number of common-cause failures.

3.1.4 Comparison of Markov Method Results with that of Block Diagram Method

We can compare the results obtained through Markov state space technique by considering a modified k-out-of-n network [73] shown in Figure 3.27 (Block diagram method). The k-out-of-n stage (labelled 1) of Figure 3.27 represents all the independent failures for any k-out-of-n unit system. The series unit stage (labelled 2) in Figure 3.27 represents all common-cause failures of the system.

Let λ be the constant failure rate of each unit and λ_c be the constant common-cause failure rate. The common-cause failure probability hypothetical unit is connected in series with the independent failure mode units. A failure of the hypothetical series unit will cause a total system failure.

The system reliability can be expressed as

$$R_p(t) \;=\; R_1(t)R_2(t)$$

where $R_1(t)$ is the reliability of the k-out-of-n system and $R_2(t)$ is the reliability of the hypothetical common-cause failure unit.

Let us consider a 2-out-of-6 system, that is, at least 2 units out of 6 units must function normally for system success. From the block diagram we have,

$$R(t) \;=\; \left\{\sum_{i=2}^{6}\binom{6}{i} R^i(1-R)^{6-i}\right\} R_2(t) \tag{3.96}$$

$$\begin{aligned} MTTF \;&=\; \int_0^\infty R(t) \\ &=\; \frac{1044\lambda^4 + 580\lambda^3\lambda_c + 155\lambda^2\lambda_c^2 + 20\lambda\lambda_c^3 + \lambda_c^4}{(6\lambda+\lambda_c)(5\lambda+\lambda_c)(4\lambda+\lambda_c)(3\lambda+\lambda_c)(2\lambda+\lambda_c)} \end{aligned} \tag{3.97}$$

By substituting λ_1 with 6λ, λ_2 with 5λ, λ_3 with 4λ, λ_4 with 3λ and λ_5 with 2λ, and by letting $\lambda_{c_{i+1}} = \lambda_c$; for $i = 0, 1, 2, \ldots, r$, the results of system mean time to failure for a 2-out-of-6 system obtained by using Equation (3.78) is the same as given in Equation (3.97).

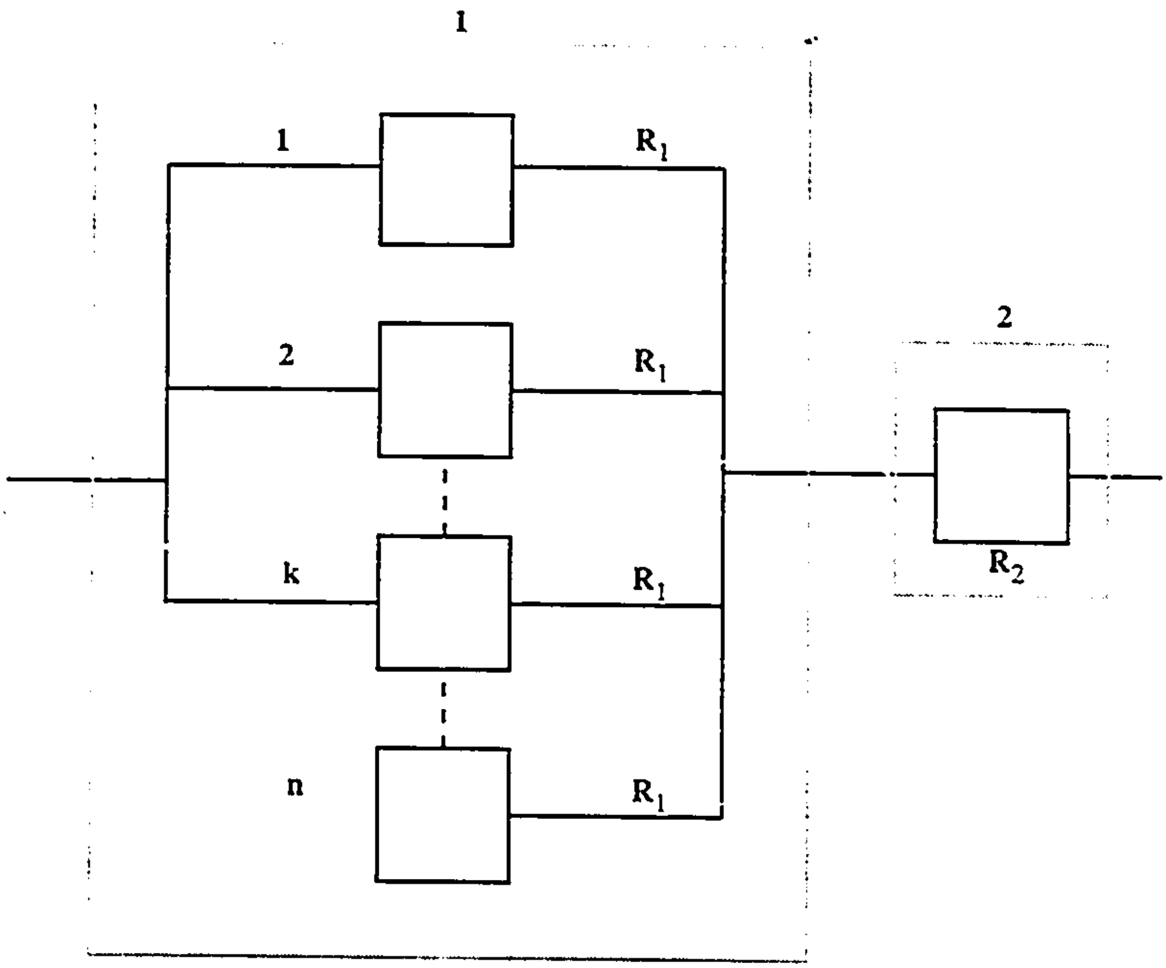

Figure 3.27: Block Diagram of a Modified k-out-of-n System with CCF

3.2 A Non-identical Unit k-out-of-n System

This section presents the reliability analysis of a repairable and non-repairable, non-identical unit 2-out-of-3 system with common-cause failures. At time $t = 0$, all the three units start working simultaneously. At least two units must function properly to ensure that the system operates successfully. The occurrence of a common-cause failure leads to total system failure. Furthermore, a common-cause failure can occur when all the units are operating normally or when 1 unit has failed and the other two operating.

Two types of repair policies are considered in the analysis of such a system.

- Type I Repair : In this case, in addition to the partially failed system (1 unit failed and the other two working) being repaired to state 0, the completely failed system is also repaired to its normal operating state 0 (where all the three units are operating normally).
- Type II Repair : The completely failed system is not repaired under this repair policy. However, the partially failed system is restored back to its normal operating state.

The assumptions associated with the identical unit k-out-of-n system hold good in this case as well, except that the system has a specified number of non-identical units.

Notation

The symbols listed below are associated with a non-identical unit 2-out-of-3 system.

t	time.
λ_a	Constant failure rate of unit A.
λ_b	Constant failure rate of unit B.
λ_c	Constant failure rate of unit C.
μ_a	Constant repair rate of the unit A.

μ_b	Constant repair rate of the unit B.
μ_c	Constant repair rate of the unit C.
μ_{40}	Constant repair rate of the failed system from state 4 to state 0.
μ_{c_1}	Constant repair rate of the failed system from state 5 to state 0.
$\lambda_{c_{i+1}}$	Constant common-cause failure rate of the system from state i; for $i = 0, 1, 2, 3$.
j	j^{th} state of the system; for $j = 0, 1, 2, \ldots, 5$.
$p_j(t)$	Probability that the system is in state j at time t; for $j = 0, 1, 2, \ldots, 5$.
$R(t)$	Reliability of the system in $[0, t]$.
s	Laplace transform variable.
CCF	Common-cause failures.
$MTTF$	System mean time to failure.
AV_{ss}	Steady state system availability.
UV_{ss}	Steady state system unavailability.
$\dot{p}_j(t)$	Derivative of $p_j(t)$ with respect to time t; for $j = 0, 1, 2, \ldots, 5$.
p_i	Steady state probability, that the system is in state i; for $i = 0, 1, 2, \ldots, 5$.

2-out-of-3 System With Type I Repair

The state space diagram for a 2-out-of-3 non-identical unit system is shown in Figure 3.28. The numerals in the boxes of the figure denote the system state numbers.

The system of differential equations for the model shown in Figure 3.28 is

$$\dot{p}_0(t) = -(\lambda_a + \lambda_b + \lambda_c + \lambda_{c_1})p_0(t) + \mu_c p_1(t) + \mu_b p_2(t) + \mu_a p_3(t) + \mu_{40} p_4(t) + \mu_{c_1} p_5(t) \quad (3.98)$$

$$\dot{p}_1(t) = \lambda_c p_0(t) - (\mu_c + \lambda_a + \lambda_b + \lambda_{c_2})p_1(t) \quad (3.99)$$

$$\dot{p}_2(t) = \lambda_b p_0(t) - (\mu_b + \lambda_a + \lambda_c + \lambda_{c_3})p_2(t) \quad (3.100)$$

$$\dot{p}_3(t) = \lambda_a p_0(t) - (\mu_a + \lambda_b + \lambda_c + \lambda_{c_4})p_3(t) \quad (3.101)$$

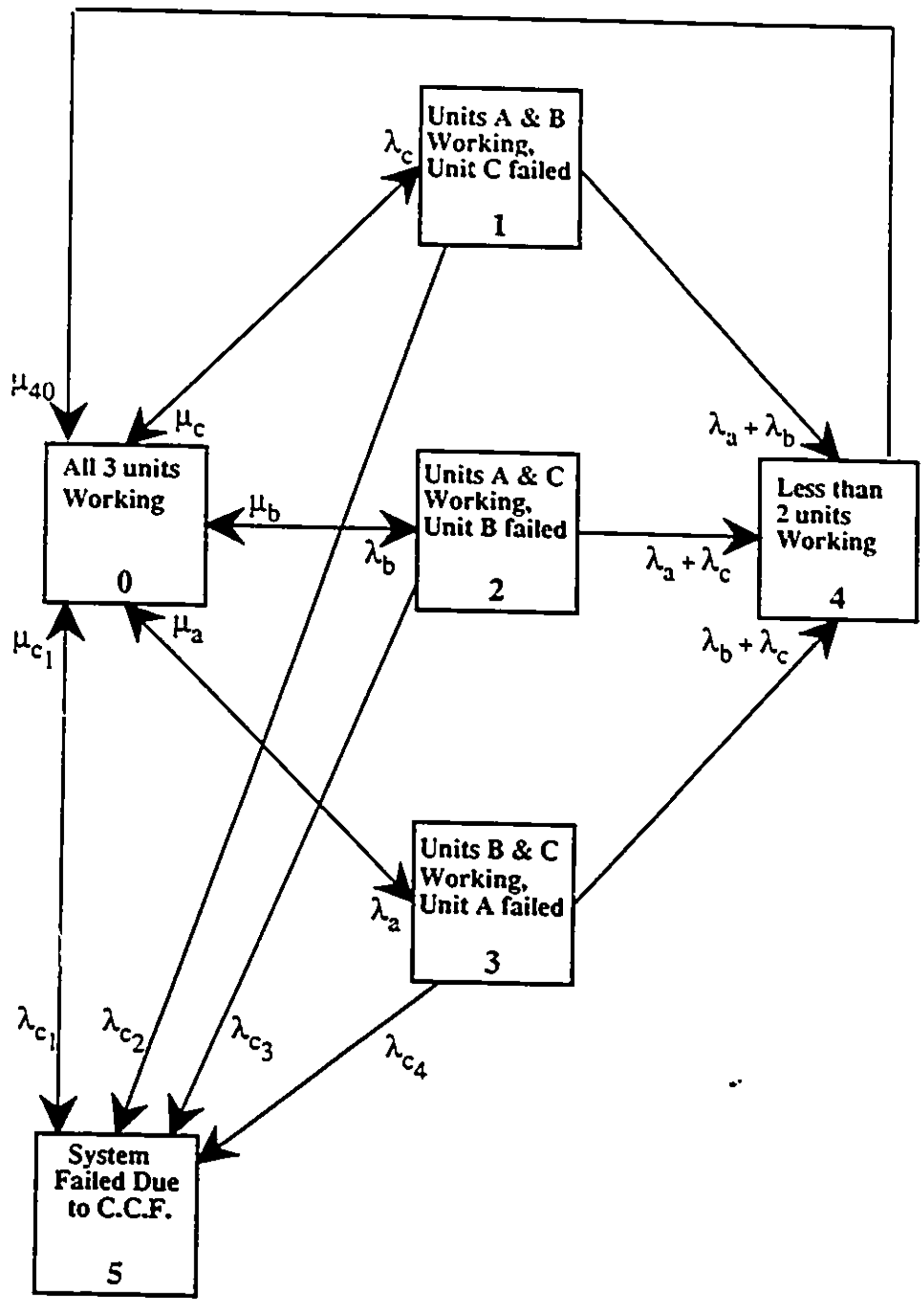

Figure 3.28: State Space Diagram for a Non-identical Unit 2-out-of-3 System

$$\dot{p}_4(t) = (\lambda_a + \lambda_b)p_1(t) + (\lambda_a + \lambda_c)p_2(t) + (\lambda_b + \lambda_c)p_3(t) - \mu_{40}p_4(t) \quad (3.102)$$

$$\dot{p}_5(t) = \lambda_{c_1}p_0(t) + \lambda_{c_2}p_1(t) + \lambda_{c_3}p_2(t) + \lambda_{c_4}p_3(t) - \mu_{c_1}p_5(t) \quad (3.103)$$

At time $t = 0$, $p_0(0) = 1$ and all other initial condition probabilities are equal to zero.

Solving the above equations, by setting their derivatives equal to zero and utilizing the relationship $\sum_{i=0}^{5} p_i = 1$, leads to the following steady state probabilities :

$$p_0 = \frac{E_{22}}{E_{28}} \quad (3.104)$$

$$p_1 = \frac{E_{23}}{E_{28}} \quad (3.105)$$

$$p_2 = \frac{E_{24}}{E_{28}} \quad (3.106)$$

$$p_3 = \frac{E_{25}}{E_{28}} \quad (3.107)$$

$$p_4 = \frac{E_{26}}{E_{28}} \quad (3.108)$$

$$p_5 = \frac{E_{27}}{E_{28}} \quad (3.109)$$

Where the constants E_{22}, $E_{23}, \ldots, E_{28}$ are defined in Appendix B.

The system steady state system availability and steady state system unavailability are given by

$$AV_{ss} = p_0 + p_1 + p_2 + p_3 \quad (3.110)$$

$$UV_{ss} = p_4 + p_5 \quad (3.111)$$

Figures 3.29 and 3.30 show the plots for steady state system availability and steady state system unavailability, respectively, for specified values of model parameters. It may be noted from the plots that the steady state system availability decreases with increase in the number of common-cause failures, and conversely that the steady state system unavailability increases with an increase in common-cause failures.

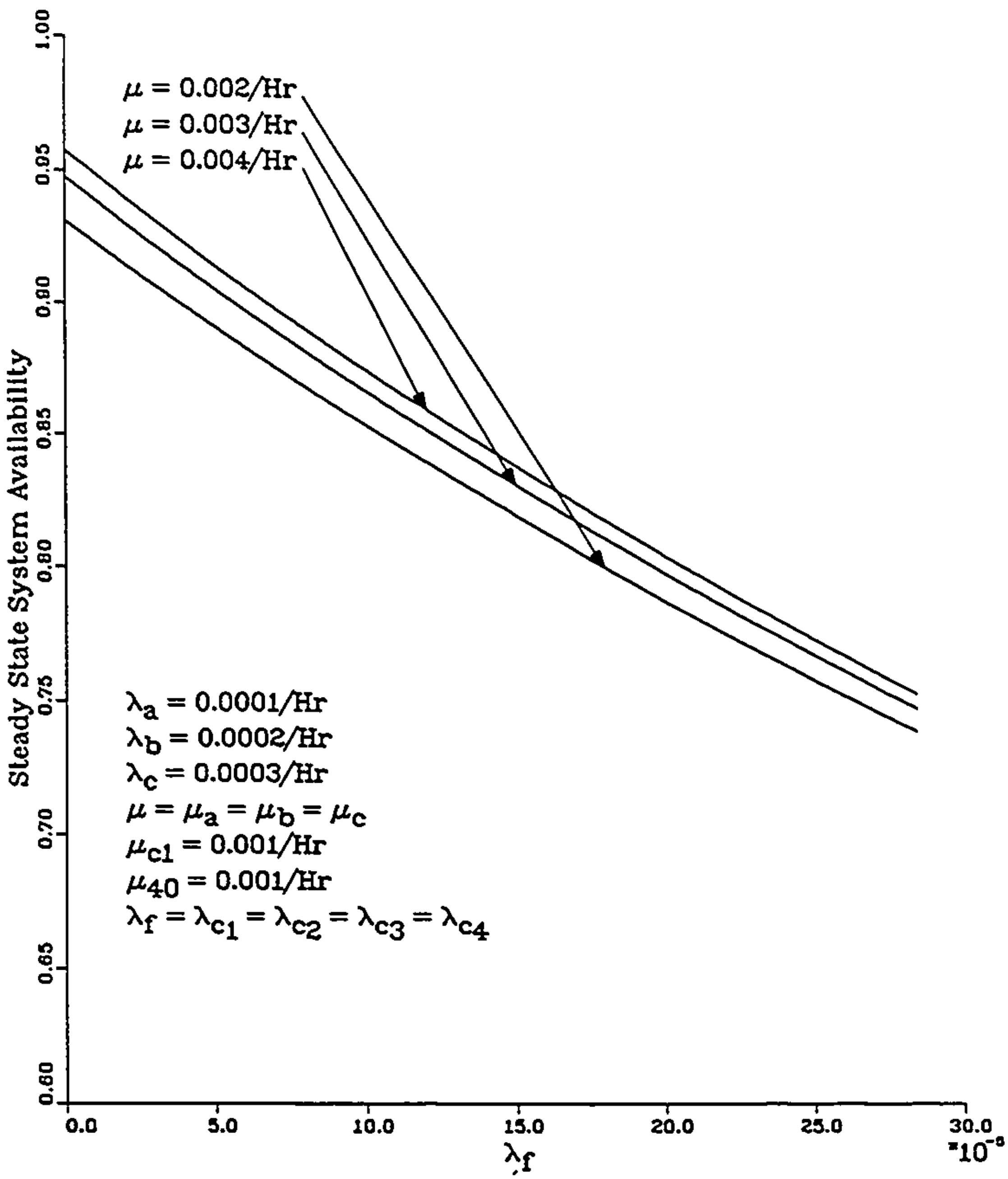

Figure 3.29: Steady State System Availability Plots for a Non-identical Unit 2-out-of-3 System With Type I Repair

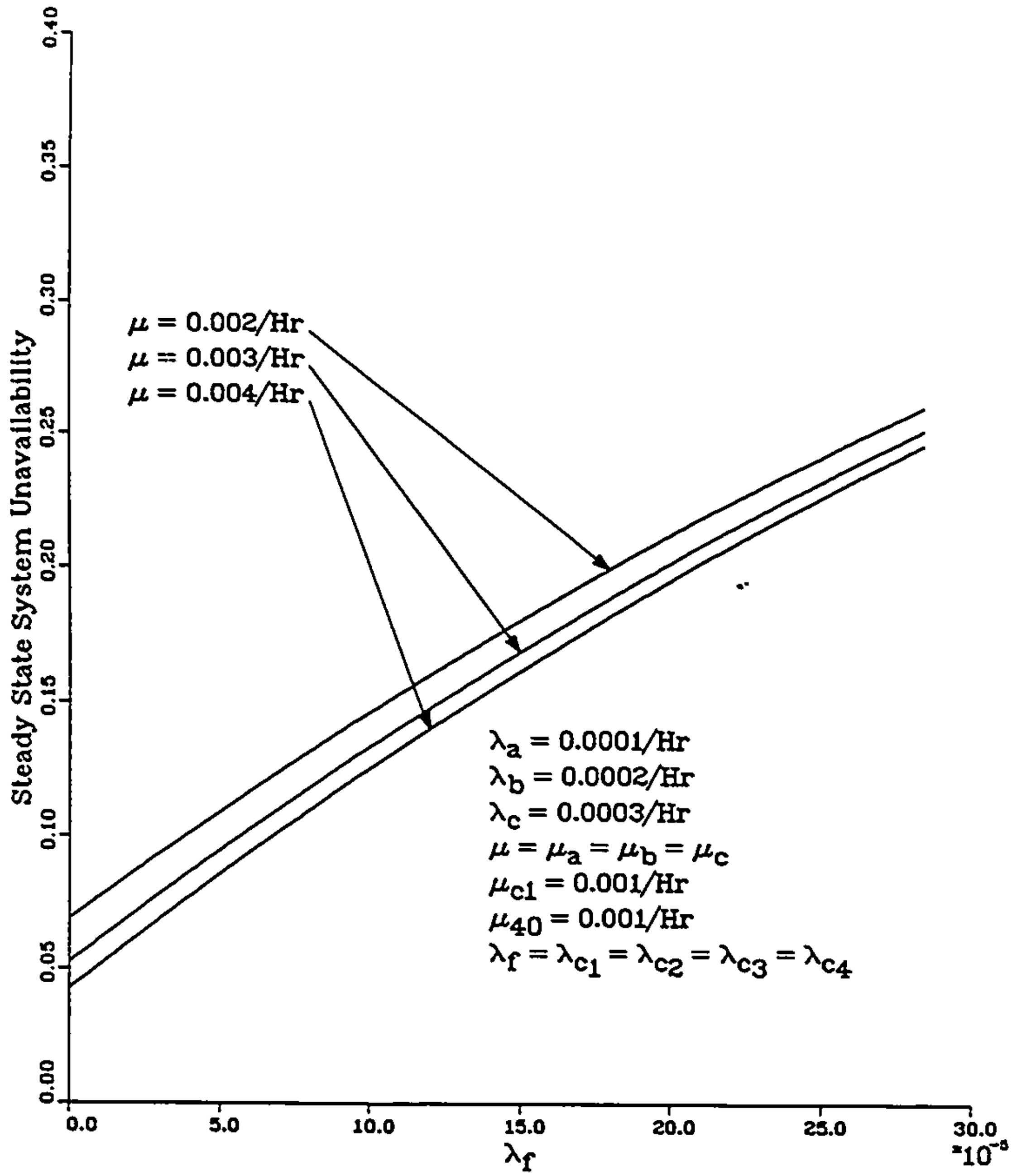

Figure 3.30: Steady State System Unavailability Plots for a Non-identical Unit 2-out-of-3 System With Type I Repair

2-out-of-3 System With Type II Repair

By setting the repair rates μ_{c_1} and μ_{40} equal to zero in Figure 3.28 we get the following system of differential equations :

$$\dot{p}_0(t) = -(\lambda_a + \lambda_b + \lambda_c + \lambda_{c_1})p_0(t) + \mu_c p_1(t) + \mu_b p_2(t) + \mu_a p_3(t) \tag{3.112}$$

$$\dot{p}_1(t) = \lambda_c p_0(t) - (\mu_c + \lambda_a + \lambda_b + \lambda_{c_2})p_1(t) \tag{3.113}$$

$$\dot{p}_2(t) = \lambda_b p_0(t) - (\mu_b + \lambda_a + \lambda_c + \lambda_{c_3})p_2(t) \tag{3.114}$$

$$\dot{p}_3(t) = \lambda_a p_0(t) - (\mu_a + \lambda_b + \lambda_c + \lambda_{c_4})p_3(t) \tag{3.115}$$

$$\dot{p}_4(t) = (\lambda_a + \lambda_b)p_1(t) + (\lambda_a + \lambda_c)p_2(t) + (\lambda_b + \lambda_c)p_3(t) \tag{3.116}$$

$$\dot{p}_5(t) = \lambda_{c_1}p_0(t) + \lambda_{c_2}p_1(t) + \lambda_{c_3}p_2(t) + \lambda_{c_4}p_3(t) \tag{3.117}$$

At time $t = 0$, $p_0(0) = 1$ and all other initial condition probabilities are equal to zero.

Solving Equations (3.112) – (3.117) with the aid of Laplace transforms yields the following Laplace transforms of the state probabilities :

$$p_0(s) = \frac{(s + \lambda_{c_4} + \mu_a + \lambda_b + \lambda_c)(\lambda_{c_3} + \mu_b + s + \lambda_c + \lambda_a)}{(s - s_1)(s - s_2)(s - s_3)(s - s_4)} \times (s + \lambda_a + \lambda_b + \lambda_{c_2} + \mu_c) \tag{3.118}$$

$$p_1(s) = \frac{(s + \lambda_{c_4} + \mu_a + \lambda_b + \lambda_c)(\lambda_{c_3} + \mu_b + s + \lambda_c + \lambda_a)\lambda_c}{(s - s_1)(s - s_2)(s - s_3)(s - s_4)} \tag{3.119}$$

$$p_2(s) = \frac{(s + \lambda_{c_4} + \mu_a + \lambda_b + \lambda_c)(s + \lambda_a + \lambda_b + \lambda_{c_2} + \mu_c)\lambda_b}{(s - s_1)(s - s_2)(s - s_3)(s - s_4)} \tag{3.120}$$

$$p_3(s) = \frac{\lambda_a(s + \lambda_a + \lambda_b + \lambda_{c_2} + \mu_c)(\lambda_{c_3} + \mu_b + s + \lambda_c + \lambda_a)}{(s - s_4)(s - s_3)(s - s_2)(s - s_1)} \tag{3.121}$$

where s_1, s_2, s_3 *and* s_4 are the roots of the quartic equation and can be determined as explained below.

$$s^4 + E_{29}s^3 + E_{30}s^2 + E_{31}s + E_{32} = 0$$

Let y_1 be a real root of the cubic equation

$$y^3 - E_{30}y^2 + (E_{29}E_{31} - 4E_{32})y + (4E_{30}E_{32} - E_{31}^2 - E_{29}^2E_{32}) = 0$$

s_1, s_2, s_3 *and* s_4 are the roots of

$$z^2 + \frac{1}{2}\{E_{29} \pm \sqrt{E_{29}^2 - 4E_{30} + 4y_1}\}z + \frac{1}{2}\{y_1 \pm \sqrt{y_1^2 - 4E_{32}}\} = 0$$

$$\begin{aligned}
-E_{29} &= s_1 + s_2 + s_3 + s_4 \\
E_{30} &= s_1s_2 + s_2s_3 + s_3s_4 + s_4s_1 + s_1s_3 + s_2s_4 \\
-E_{31} &= s_1s_2s_3 + s_2s_3s_4 + s_1s_2s_4 + s_1s_3s_4 \\
E_{32} &= s_1s_2s_3s_4
\end{aligned}$$

where the constants E_{29}, E_{30}, E_{31}, *and* E_{32} are defined in the Appendix B.

Taking inverse Laplace transforms of Equations (3.118) – (3.121) the following time dependent state probabilities are obtained :

$$\begin{aligned}
p_0(t) = {} & \frac{(s_1 + \lambda_a + \lambda_b + \lambda_{c_2} + \mu_c)(\lambda_c + \mu_b + \lambda_{c_3} + \lambda_a + s_1)}{(s_1 - s_2)(s_3 - s_1)(s_4 - s_1)} \times \\
& (\lambda_{c_4} + \mu_a + \lambda_c + s_1 + \lambda_b)e^{s_1 t} + \\
& \frac{(s_2 + \lambda_a + \lambda_b + \lambda_{c_2} + \mu_c)(\lambda_c + \mu_b + \lambda_{c_3} + \lambda_a + s_2)}{(s_2 - s_1)(s_3 - s_2)(s_4 - s_2)} \times \\
& (\lambda_{c_4} + \mu_a + \lambda_c + s_2 + \lambda_b)e^{s_2 t} + \\
& \frac{(s_3 + \lambda_a + \lambda_b + \lambda_{c_2} + \mu_c)(\lambda_c + \mu_b + \lambda_{c_3} + \lambda_a + s_3)}{(s_2 - s_3)(s_3 - s_1)(s_4 - s_3)} \times \\
& (\lambda_{c_4} + \mu_a + \lambda_c + s_3 + \lambda_b)e^{s_3 t} + \\
& \frac{(s_4 + \lambda_a + \lambda_b + \lambda_{c_2} + \mu_c)(\lambda_c + \mu_b + \lambda_{c_3} + \lambda_a + s_4)}{(s_4 - s_2)(s_4 - s_1)(s_4 - s_3)} \times \\
& (\lambda_{c_4} + \mu_a + \lambda_c + s_4 + \lambda_b)e^{s_4 t}
\end{aligned} \tag{3.122}$$

$$p_1(t) = \frac{\lambda_c(\lambda_{c_4} + \mu_a + \lambda_c + s_1 + \lambda_b)(\lambda_c + \mu_b + \lambda_{c_3} + \lambda_a + s_1)e^{s_1 t}}{(s_1 - s_2)(s_3 - s_1)(s_4 - s_1)} +$$

$$\frac{\lambda_c(\lambda_{c_4} + \mu_a + \lambda_c + s_2 + \lambda_b)(\lambda_c + \mu_b + \lambda_{c_3} + \lambda_a + s_2)e^{s_2 t}}{(s_2 - s_1)(s_3 - s_2)(s_4 - s_2)} +$$
$$\frac{\lambda_c(\lambda_{c_4} + \mu_a + \lambda_c + s_3 + \lambda_b)(\lambda_c + \mu_b + \lambda_{c_3} + \lambda_a + s_3)e^{s_3 t}}{(s_2 - s_3)(s_3 - s_1)(s_4 - s_3)} +$$
$$\frac{\lambda_c(\lambda_{c_4} + \mu_a + \lambda_c + s_4 + \lambda_b)(\lambda_c + \mu_b + \lambda_{c_3} + \lambda_a + s_4)e^{s_4 t}}{(s_4 - s_2)(s_4 - s_1)(s_4 - s_3)} \quad (3.123)$$

$$\begin{aligned} p_2(t) = & \frac{\lambda_b(s_1 + \lambda_a + \lambda_b + \lambda_{c_2} + \mu_c)(\lambda_{c_4} + \mu_a + \lambda_c + s_1 + \lambda_b)e^{s_1 t}}{(s_1 - s_2)(s_3 - s_1)(s_4 - s_1)} + \\ & \frac{\lambda_b(s_2 + \lambda_a + \lambda_b + \lambda_{c_2} + \mu_c)(\lambda_{c_4} + \mu_a + \lambda_c + s_2 + \lambda_b)e^{s_2 t}}{(s_1 - s_2)(s_3 - s_2)(s_4 - s_2)} + \\ & \frac{\lambda_b(s_3 + \lambda_a + \lambda_b + \lambda_{c_2} + \mu_c)(\lambda_{c_4} + \mu_a + \lambda_c + s_3 + \lambda_b)e^{s_3 t}}{(s_2 - s_3)(s_3 - s_1)(s_4 - s_3)} + \\ & \frac{\lambda_b(s_4 + \lambda_a + \lambda_b + \lambda_{c_2} + \mu_c)(\lambda_{c_4} + \mu_a + \lambda_c + s_4 + \lambda_b)e^{s_4 t}}{(s_4 - s_2)(s_4 - s_1)(s_4 - s_3)} \end{aligned} \quad (3.124)$$

$$\begin{aligned} p_3(t) = & \frac{\lambda_a(\lambda_a + s_1 + \lambda_{c_2} + \mu_c + \lambda_b)(\lambda_c + \mu_b + \lambda_{c_3} + \lambda_a + s_1)e^{s_1 t}}{(s_1 - s_2)(s_3 - s_1)(s_4 - s_1)} + \\ & \frac{\lambda_a(\lambda_a + s_2 + \lambda_{c_2} + \mu_c + \lambda_b)(\lambda_c + \mu_b + \lambda_{c_3} + \lambda_a + s_2)e^{s_2 t}}{(s_2 - s_1)(s_3 - s_2)(s_4 - s_2)} + \\ & \frac{\lambda_a(\lambda_a + s_3 + \lambda_{c_2} + \mu_c + \lambda_b)(\lambda_c + \mu_b + \lambda_{c_3} + \lambda_a + s_3)e^{s_3 t}}{(s_2 - s_3)(s_3 - s_1)(s_4 - s_3)} + \\ & \frac{\lambda_a(\lambda_a + s_4 + \lambda_{c_2} + \mu_c + \lambda_b)(\lambda_c + \mu_b + \lambda_{c_3} + \lambda_a + s_4)e^{s_4 t}}{(s_4 - s_2)(s_4 - s_1)(s_4 - s_3)} \end{aligned} \quad (3.125)$$

The system reliability is given by

$$R(t) = p_0(t) + p_1(t) + p_2(t) + p_3(t) \quad (3.126)$$

The system reliability given by Equation (3.126) is plotted in Figure 3.31. The plots clearly show that the system reliability decreases with an increase in the number of common-cause failures.

The system mean time to failure can be expressed as

$$\begin{aligned} MTTF &= \lim_{s \to 0} R(s) \\ &= \lim_{s \to 0} [p_0(s) + p_1(s) + p_2(s) + p_3(s)] \end{aligned} \quad (3.127)$$

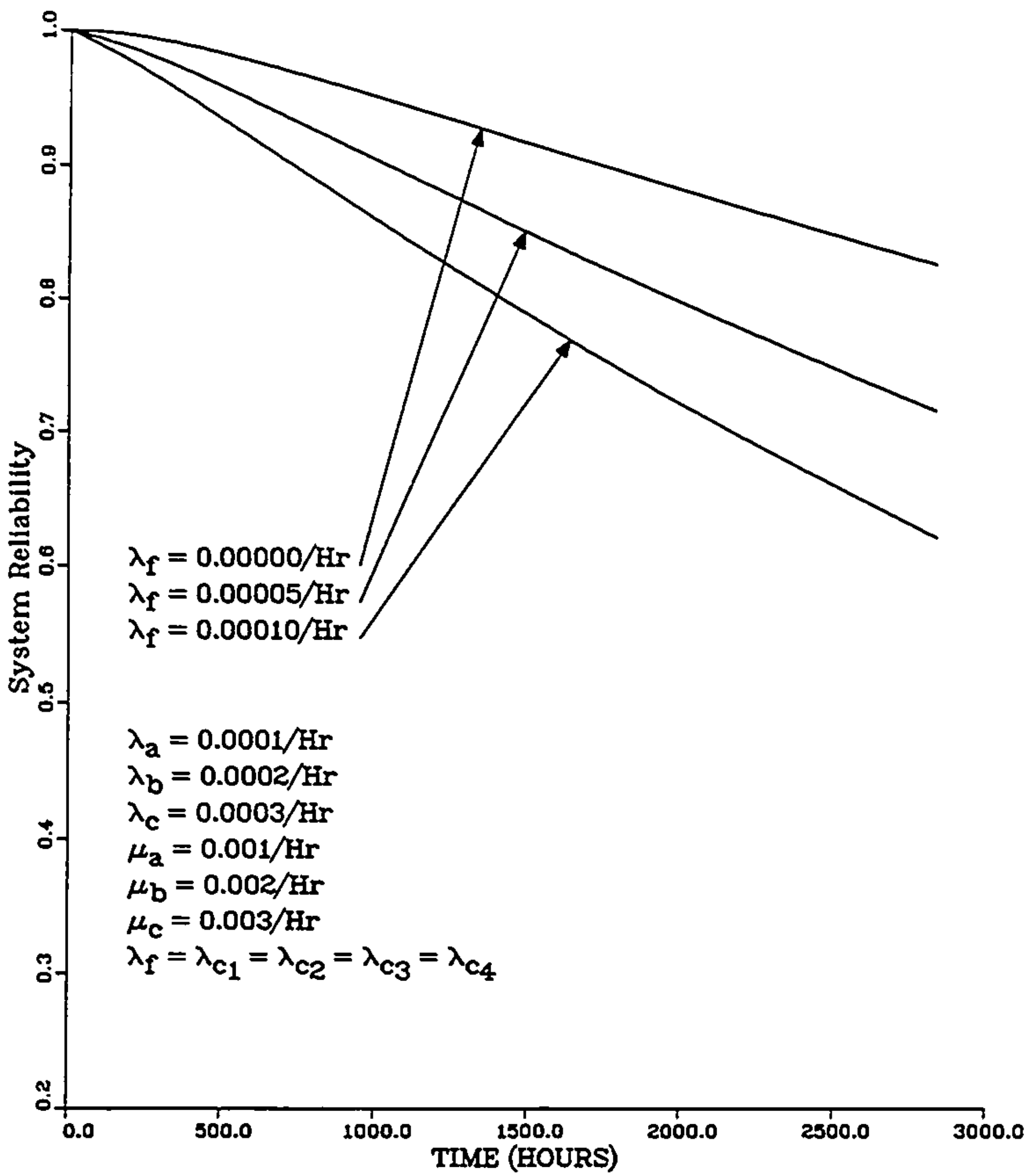

Figure 3.31: System Reliability Plots for a Non-identical Unit 2-out-of-3 System With Type II Repair

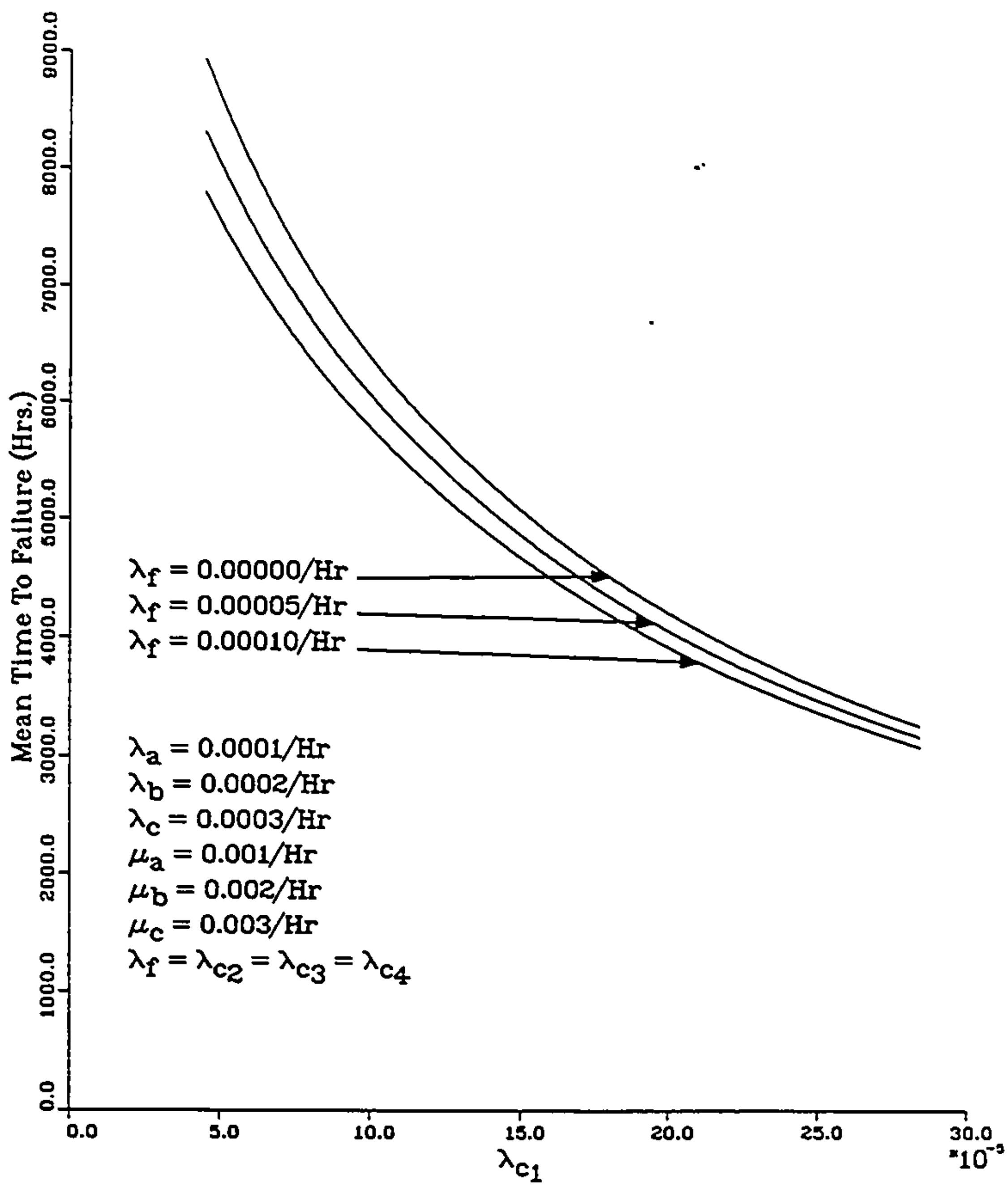

Figure 3.32: System Mean Time to Failure Plots for a Non-identical Unit 2-out-of-3 System With Type II Repair

where $R(s)$ is the Laplace transform of the system reliability. The system mean time to failure plots are shown in Figure 3.32. From the plots it can be seen that the system mean time to failure decreases with an increase in common-cause failures.

2-out-of-3 System Without Repair

The following system of differential equations is obtained by setting the repair rates μ_a, μ_b, μ_c, μ_{40} *and* μ_{c_1} equal to zero in Figure 3.28 :

$$\dot{p}_0(t) = -(\lambda_a + \lambda_b + \lambda_c + \lambda_{c_1})p_0(t) \tag{3.128}$$

$$\dot{p}_1(t) = \lambda_c p_0(t) - (\lambda_a + \lambda_b + \lambda_{c_2})p_1(t) \tag{3.129}$$

$$\dot{p}_2(t) = \lambda_b p_0(t) - (\lambda_a + \lambda_c + \lambda_{c_3})p_2(t) \tag{3.130}$$

$$\dot{p}_3(t) = \lambda_a p_0(t) - (\lambda_b + \lambda_c + \lambda_{c_4})p_3(t) \tag{3.131}$$

$$\dot{p}_4(t) = (\lambda_a + \lambda_b)p_1(t) + (\lambda_a + \lambda_c)p_2(t) + (\lambda_b + \lambda_c)p_3(t) \tag{3.132}$$

$$\dot{p}_5(t) = \lambda_{c_1}p_0(t) + \lambda_{c_2}p_1(t) + \lambda_{c_3}p_2(t) + \lambda_{c_4}p_3(t) \tag{3.133}$$

At time $t = 0$, $p_0(0) = 1$. All other initial condition probabilities are equal to zero.

The following Laplace transforms of state probabilities are obtained by solving Equations (3.128) – (3.133) with the aid of Laplace transforms :

$$p_0(s) = \frac{1}{s + \lambda_a + \lambda_c + \lambda_b + \lambda_{c_1}} \tag{3.134}$$

$$p_1(s) = \frac{\lambda_c}{(s + \lambda_a + \lambda_c + \lambda_b + \lambda_{c_1})(s + \lambda_a + \lambda_b + \lambda_{c_2})} \tag{3.135}$$

$$p_2(s) = \frac{\lambda_b}{(s + \lambda_a + \lambda_c + \lambda_b + \lambda_{c_1})(s + \lambda_a + \lambda_c + \lambda_{c_3})} \tag{3.136}$$

$$p_3(s) = \frac{\lambda_a}{(s + \lambda_a + \lambda_c + \lambda_b + \lambda_{c_1})(s + \lambda_{c_4} + \lambda_b + \lambda_c)} \tag{3.137}$$

Taking inverse Laplace transforms of Equations (3.134) – (3.137) the following state probabilities expressions are obtained :

$$p_0(t) = e^{-(\lambda_c + \lambda_b + \lambda_a + \lambda_{c_1})t} \tag{3.138}$$

$$p_1(t) = \frac{-\lambda_c \epsilon^{-\frac{1}{2}(2\lambda_a+2\lambda_b+\lambda_{c_2}+\lambda_c+\lambda_{c_1})t}\{\epsilon^{-\frac{1}{2}(\lambda_{c_1}+\lambda_c-\lambda_{c_2})t} - \epsilon^{\frac{1}{2}(\lambda_{c_1}+\lambda_c-\lambda_{c_2})t}\}}{\lambda_{c_1}+\lambda_c-\lambda_{c_2}} \quad (3.139)$$

$$p_2(t) = \frac{-\lambda_b \epsilon^{-\frac{1}{2}(2\lambda_a+2\lambda_c+\lambda_{c_3}+\lambda_b+\lambda_{c_1})t}\{\epsilon^{-\frac{1}{2}(\lambda_{c_1}+\lambda_b-\lambda_{c_3})t} - \epsilon^{\frac{1}{2}(\lambda_{c_1}+\lambda_b-\lambda_{c_3})t}\}}{\lambda_{c_1}+\lambda_b-\lambda_{c_3}} \quad (3.140)$$

$$p_3(t) = \frac{-\lambda_a \epsilon^{-\frac{1}{2}(2\lambda_b+2\lambda_c+\lambda_{c_4}+\lambda_a+\lambda_{c_1})t}\{\epsilon^{-\frac{1}{2}(\lambda_{c_1}+\lambda_a-\lambda_{c_4})t} - \epsilon^{\frac{1}{2}(\lambda_{c_1}+\lambda_a-\lambda_{c_4})t}\}}{\lambda_{c_1}+\lambda_a-\lambda_{c_4}} \quad (3.141)$$

The system reliability is given by

$$R(t) = p_0(t) + p_1(t) + p_2(t) + p_3(t) \quad (3.142)$$

Figure 3.33 show the plots of system reliability as expressed by Equation (3.142). These plots clearly indicate that the system reliability decreases with an increase in common-cause failures.

The system mean time to failure is given by

$$\begin{aligned} MTTF &= \lim_{s \to 0} R(s) \\ &= \lim_{s \to 0} [p_0(s) + p_1(s) + p_2(s) + p_3(s)] \end{aligned} \quad (3.143)$$

where $R(s)$ is the Laplace transform of the system reliability. The plots of system mean time to failure shown in Figure 3.34 clearly indicate the decreasing effect that the common-cause failure has on the system mean time to failure.

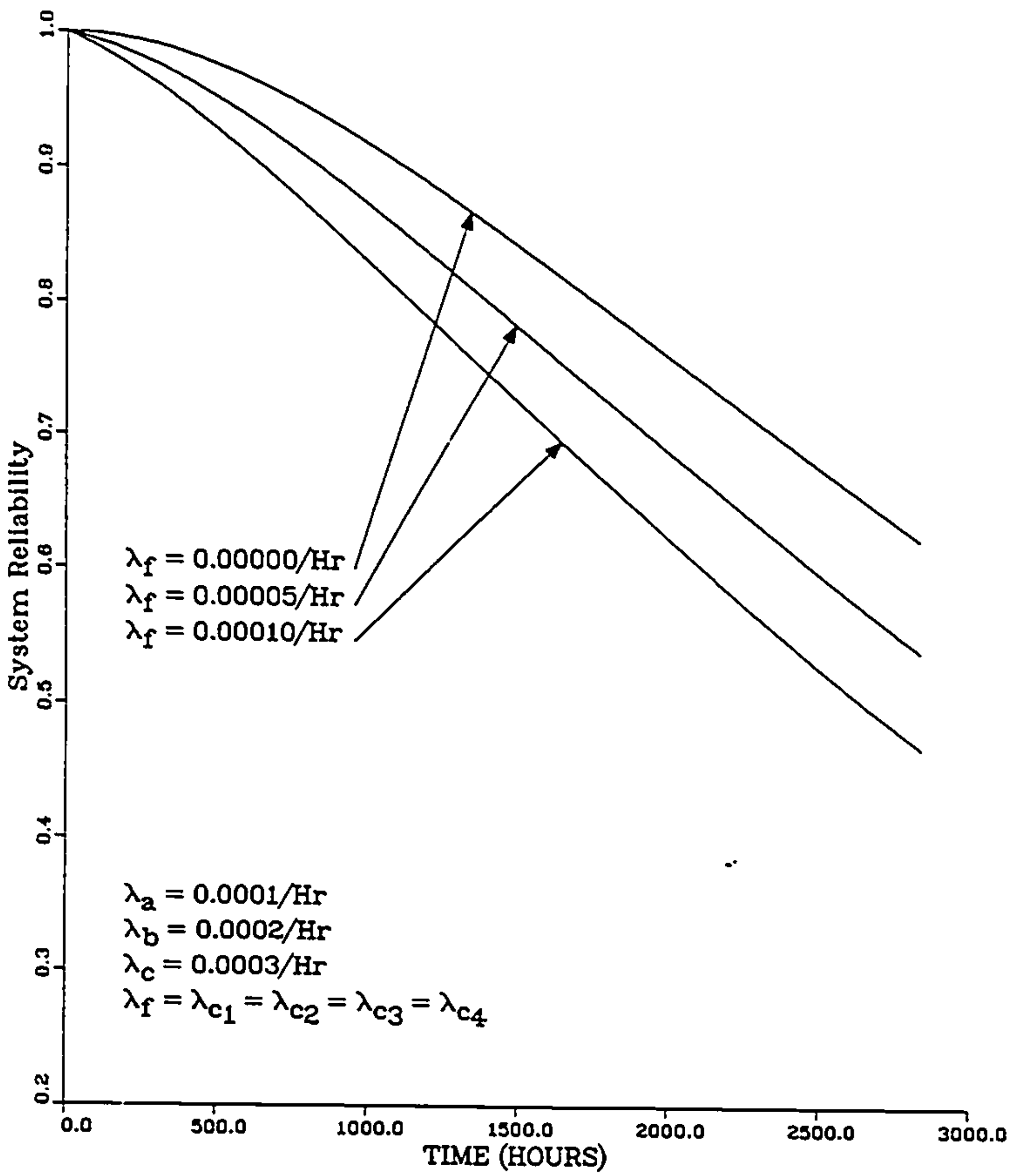

Figure 3.33: System Reliability Plots for a Non-identical Unit 2-out-of-3 System Without Repair

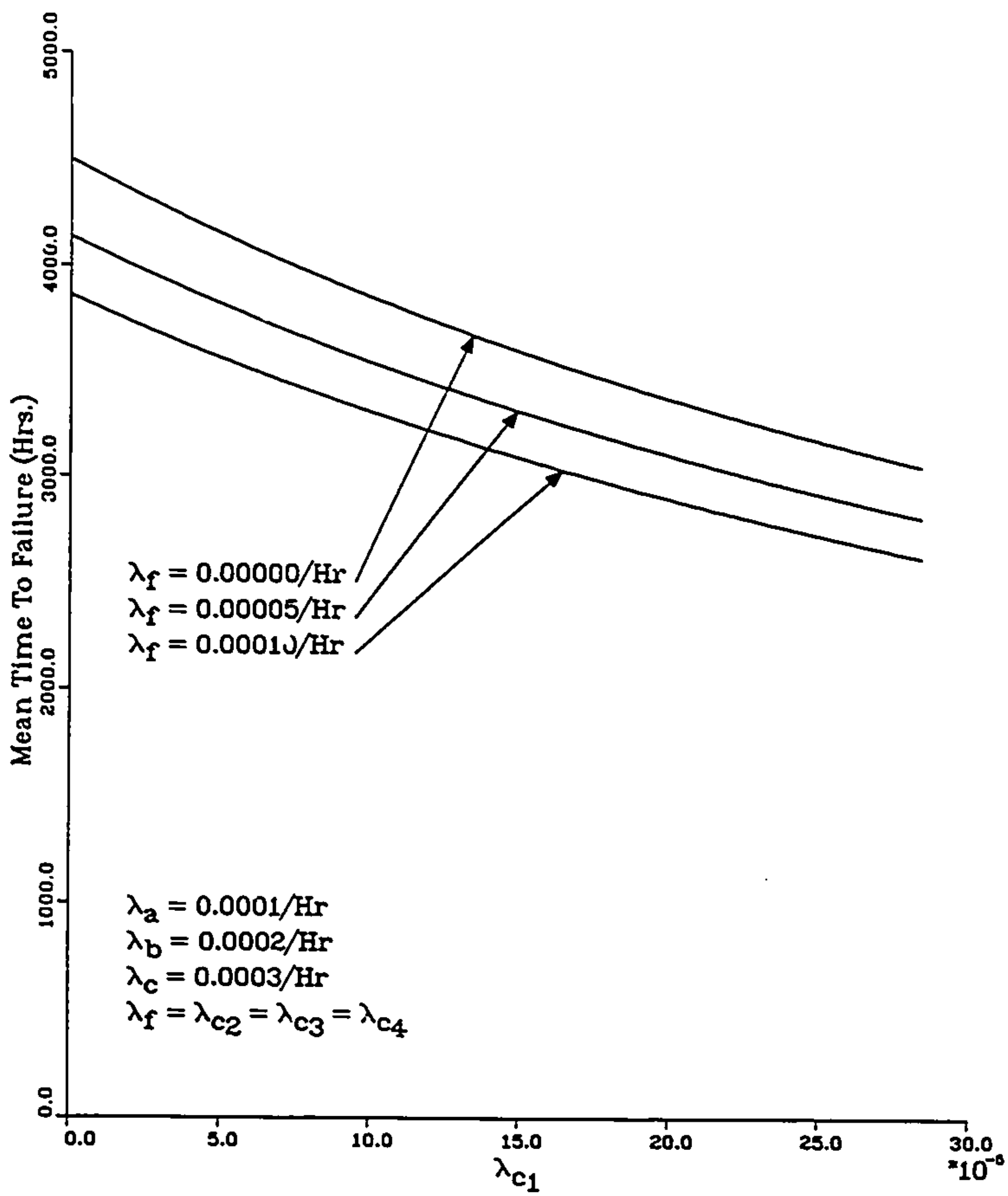

Figure 3.34: System Mean Time to Failure Plots for a Non-identical Unit 2-out-of-3 System

Chapter 4

Standby System

This type of redundancy represents a situation where one unit is active (operating) and the remaining on standby. As soon as the operating unit fails, it is replaced by one of the standby units. The system fails only when the operating unit as well as all the standby units fail. More specifically, only one unit is required to operate for the system success. In addition, unlike a parallel network where all the units are active, the standby units are not active.

The reliability analysis of such a system with identical units is given in section 4.1 and section 4.2 presents the reliability analysis of a standby system comprised of non-identical units. In both cases the occurrence of common-cause failures is taken into consideration.

4.1 Identical Unit Standby System

The reliability analysis of a repairable and non-repairable identical unit standby system is presented in this section. At time $t = 0$, only one unit starts operating and the remaining n units are in standby mode. One of the standby units takes over the operating unit's function as soon as it fails. The failure rate of each unit is constant. As in the case of parallel and k-out-of-n systems, a common-cause failure can cause the system failure. Furthermore, a common-cause failure can occur when the system is in any one of its operating states, such as, $0, 1, 2, \ldots, n$, even though a demand on a particular standby unit is not made.

The reliability analysis of such a system is performed by taking into consideration two types of repair policies similar to those discussed in chapter 2.

1. Type I Repair : This repair policy deals with repair of not only the partially failed system (1 unit failed, 1^{st} standby unit working, $(n-1)$ units on standby; 2 units failed, 2^{nd} standby unit working, $(n-2)$ units on standby; and so on.) to its previous state and to state 0, but also of the completely failed system (either due to a common-cause failure or due to other types of failures) to state 0. In addition, the completely failed system (due to a common-cause failure occurring at state n) is restored back to state n; for $n = 1, 2, 3, \ldots, n$.
2. Type II Repair : The basic distinction between this type of repair policy and that of Type I Repair policy is that, unlike the Type I Repair policy, the completely failed system either due to a common-cause failure or due to other failures is not repaired. However, the partially failed system is restored back to both its previous state as well as to state 0.

Assumptions

The assumptions associated with the models in this configuration are as follows :

1. The system has one working unit and n identical units in standby.
2. A unit can fail either due to a common-cause failure or other failures.
3. Common-cause and other failures are statistically independent.
4. A common-cause failure can occur irrespective of the number of units acting as standby in the system.
5. All the units in standby may fail due to a common-cause failure, even though only one particular unit is active.
6. The system is said to be down when all the system units become non-operative.
7. Common-cause and other failure rates are constant.
8. The switching arrangement of the standby system is perfect.
9. Repair rates are constant.
10. A repaired unit is as good as new.

Notation

The following symbols are associated with all the models considered under this configuration:

n	number of standby units.
t	time.
λ	Constant failure rate of a unit.
μ	Constant repair rate of a unit.
μ_i	Constant repair rate from the i^{th} standby system state to state 0; for $i = 2, 3, 4, \ldots, n$.
μ_c	Constant repair rate of the system from the system failed state to state 0.
μ_{c_i}	Constant repair rate of the failed system from system state $n+2$ to state i; for $i = 1, 2, 3, \ldots, n$.
λ_c	Constant common-cause failure rate of the system.
j	j^{th} state of the system; for $j = 0, 1, 2, 3, \ldots, n+2$.
$p_j(t)$	Probability that the system is in state j at time t; for $j = 0, 1, 2, 3, \ldots, n+2$.
$R(t)$	Reliability of the system in $[0, t]$.
s	Laplace transform variable.
CCF	Common-cause failure.
$MTTF$	System mean time to failure.
AV_{ss}	Steady state system availability.
UV_{ss}	Steady state system unavailability.
$\dot{p}_j(t)$	Derivative of $p_j(t)$ with respect to time t; for $j = 0, 1, 2, \ldots, n+2$.
p_i	Steady state probability that the system is in state i; for $i = 0, 1, 2, \ldots, n+2$.

4.1.1 Standby System With Type I Repair

The state space diagram for a n-Identical unit standby system is given in Figure 4.1. The system state numbers are represented by the numerals plus a letter in the boxes of the figure.

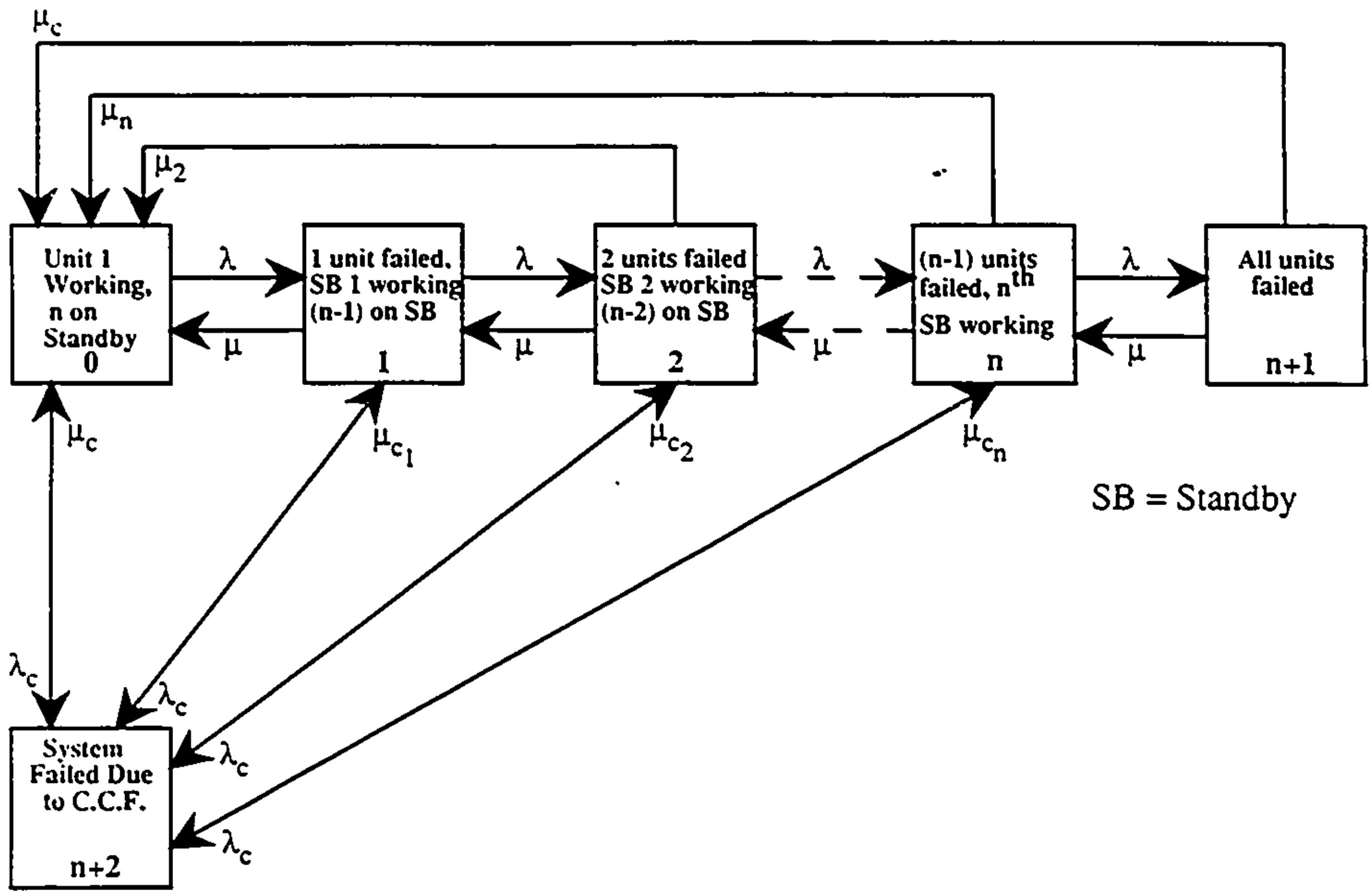

Figure 4.1: State Space Diagram for an Identical Unit Standby System

The system of differential equations for the model represented in Figure 4.1 is as follows:

$$\dot{p}_0(t) = -(\lambda + \lambda_c)p_0(t) + \mu p_1(t) + \sum_{i=2}^{n} \mu_i p_i(t) + \sum_{j=n+1}^{n+2} \mu_c p_j(t) \tag{4.1}$$

$$\dot{p}_1(t) = \lambda p_0(t) - (\mu + \lambda + \lambda_c)p_1(t) + \mu p_2(t) + \mu_{c_1} p_{n+2}(t) \tag{4.2}$$

$$\dot{p}_r(t) = \lambda p_{r-1}(t) - \{\mu_r + \mu + \lambda + \lambda_c\}p_r(t) + \mu p_{r+1}(t) + \mu_{c_r} p_{n+2}(t) \tag{4.3}$$

for $r = 2, 3, 4, \ldots, n$

$$\dot{p}_{n+1}(t) = \lambda p_n(t) - (\mu + \mu_c)p_{n+1}(t) \tag{4.4}$$

$$\dot{p}_{n+2}(t) = \sum_{i=0}^{n} \lambda_c p_i(t) - \{\mu_c + \sum_{j=1}^{n} \mu_{c_j}\}p_{n+2}(t) \tag{4.5}$$

At time $t = 0$, $p_0(0) = 1$ and all other initial condition probabilities are equal to zero.

For the purpose of generalization, the repair rate μ_{c_i}; for $i = 1, 2, 3, \ldots, n$ is set to zero (since, with this repair rate the number of terms in the steady state system availability expression become too many). The differential equations under such a condition are as follows :

$$\dot{p}_0(t) = -(\lambda + \lambda_c)p_0(t) + \mu p_1(t) + \sum_{i=2}^{n} \mu_i p_i(t) + \sum_{j=n+1}^{n+2} \mu_c p_j(t) \tag{4.6}$$

$$\dot{p}_1(t) = \lambda p_0(t) - (\mu + \lambda + \lambda_c)p_1(t) + \mu p_2(t) \tag{4.7}$$

$$\dot{p}_r(t) = \lambda p_{r-1}(t) - \{\mu_r + \mu + \lambda + \lambda_c\}p_r(t) + \mu p_{r+1}(t) \tag{4.8}$$

for $r = 2, 3, 4, \ldots, n$

$$\dot{p}_{n+1}(t) = \lambda p_n(t) - (\mu + \mu_c)p_{n+1}(t) \tag{4.9}$$

$$\dot{p}_{n+2}(t) = \sum_{i=0}^{n} \lambda_c p_i(t) - \mu_c p_{n+2}(t) \tag{4.10}$$

At time $t = 0$, $p_0(0) = 1$ and all other initial condition probabilities are equal to zero.

By setting the derivatives of Equations (4.6) – (4.10) equal to zero and by making use of the relationship $\sum_{i=0}^{n+2} p_i = 1$, the following steady state probability expression in terms of p_0 is obtained :

$$p_r = \frac{\{\lambda^r A_{n-r}\} p_0}{(\mu + \lambda + \lambda_c)A_{n-1} - \mu\lambda A_{n-2}} \tag{4.11}$$

For $r = 1, 2, 3, \ldots, n$
where

$$\begin{aligned} A_0 &= (\mu + \mu_c) \\ A_{i-1} &= 1 \; for \; (i-1) < 0 \\ A_{i+1} &= \{\mu_{n-i} + \mu + \lambda + \lambda_c\}A_i - \mu\lambda A_{i-1} \\ For \; i &= 0, 1, 2, 3, \ldots, n-2 \end{aligned}$$

$$p_0 = \frac{\mu p_1 + \sum_{i=2}^{n} \mu_i p_i + \mu_c\{1 - \sum_{i=1}^{n} p_i\}}{\lambda + \lambda_c + \mu_c} \tag{4.12}$$

The steady state probabilities for a standby system with known number units in standby can be obtained by using Equations (4.11) and (4.12). The steady state system availability and unavailabilities can be determined by using the expressions

$$\begin{aligned} AV_{ss} &= \sum_{i=0}^{n} p_i \\ UV_{ss} &= \sum_{i=n+1}^{n+2} p_i \end{aligned}$$

Figures 4.2 and 4.3 show the plots for steady state system availability and steady state system unavailability, respectively, for specified values of model parameters. It may be noted that the repair rate μ_{c_i}; for $i = 1, 2, 3, \ldots, n$ is also considered for these plots. It is clear from the plots that steady state system availability decreases with an increase in the number of common-cause failures. Conversely the steady state system unavailability increases with an increase in the number of common-cause failures.

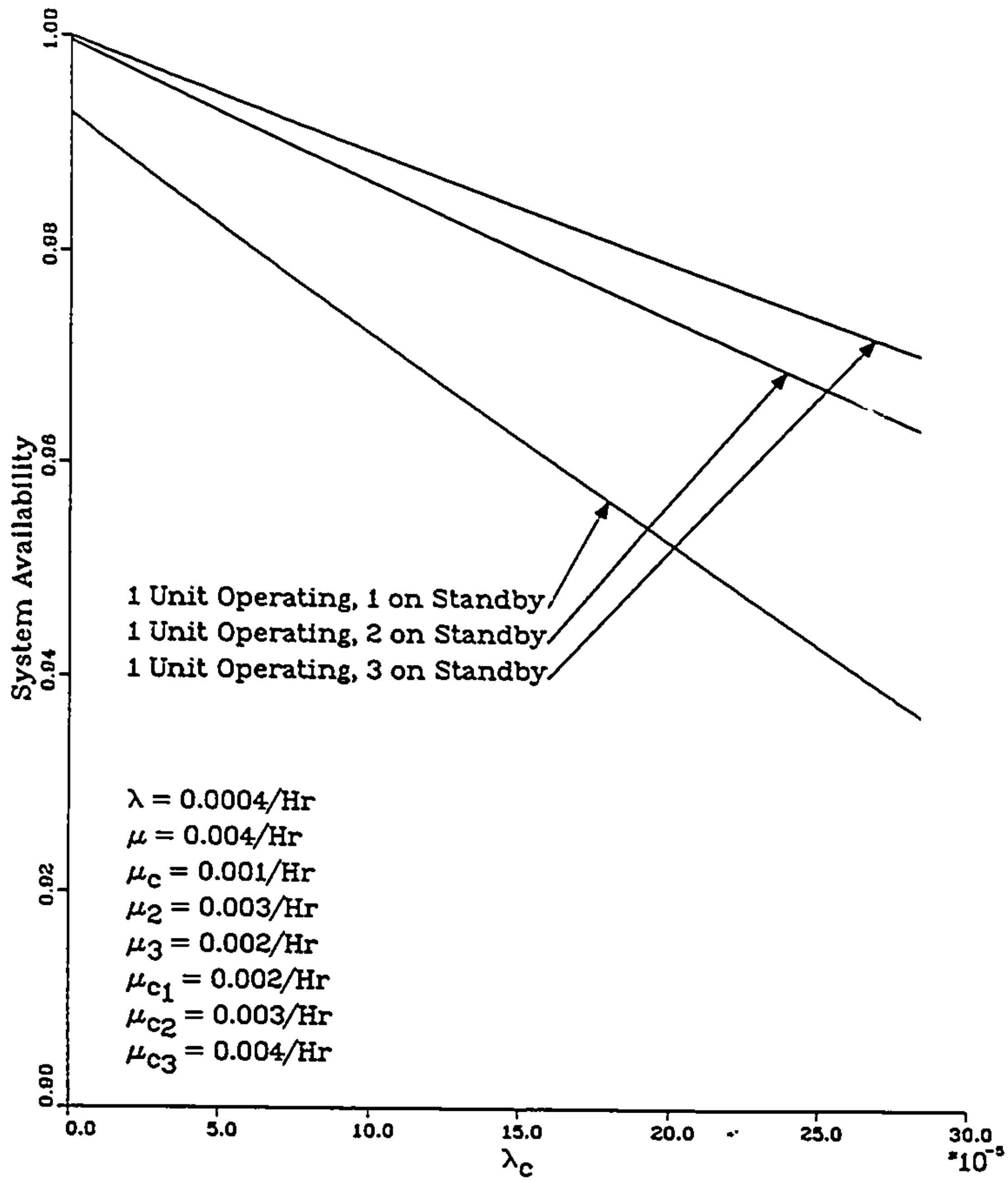

Figure 4.2: Steady State System Availability Plots for an Identical Unit Standby System with Type I Repair

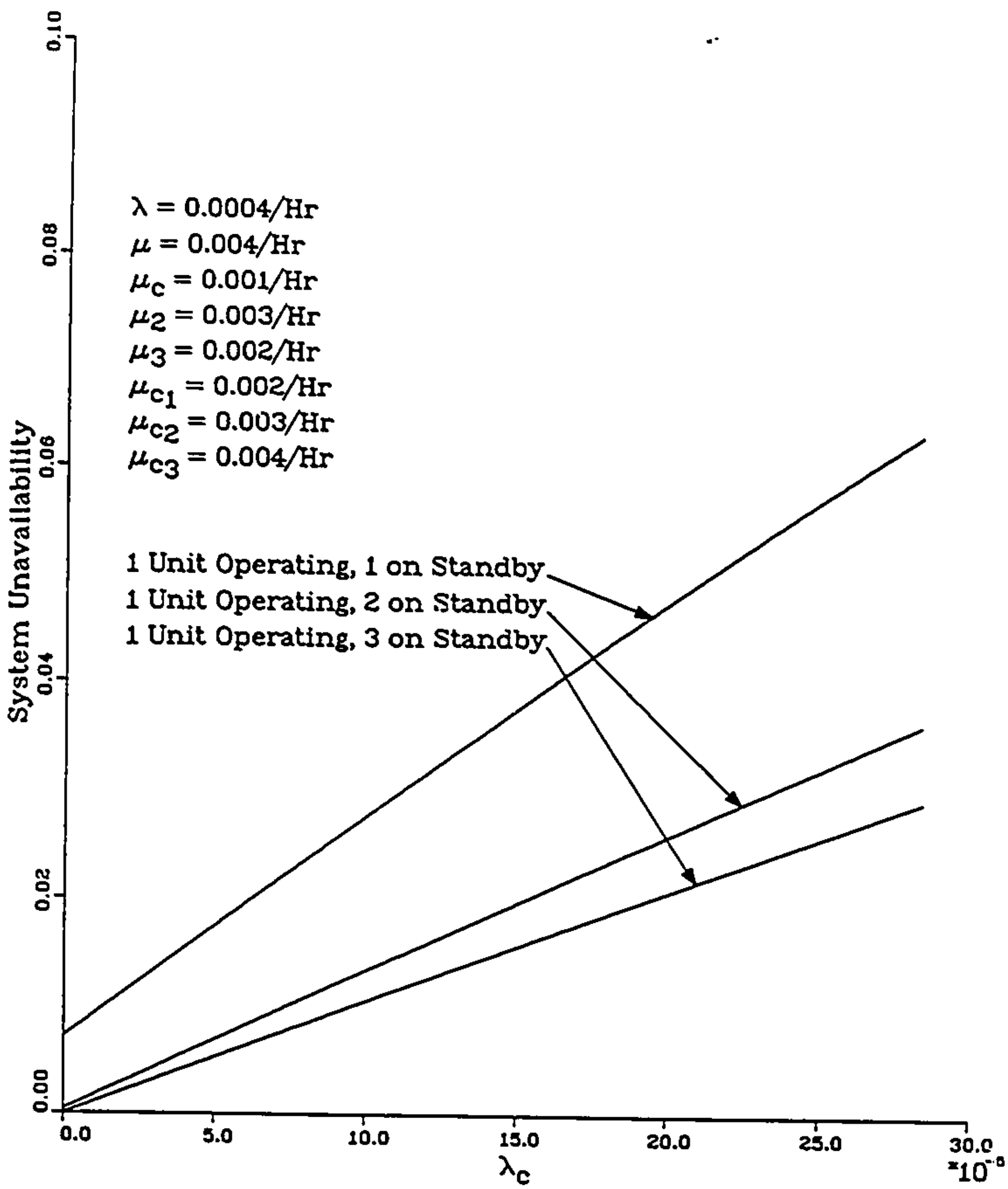

Figure 4.3: Steady State System Unavailability Plots for an Identical Unit Standby System with Type I Repair

Special Case Model I

The following system of differential equations for a single unit standby system is obtained by setting $n = 1$ in Equations $(4.1) - (4.5)$:

$$\dot{p}_0(t) = -(\lambda + \lambda_c)p_0(t) + \mu p_1(t) + \mu_c p_2(t) + \mu_c p_3(t) \quad (4.13)$$

$$\dot{p}_1(t) = \lambda p_0(t) - (\mu + \lambda + \lambda_c)p_1(t) + \mu p_2(t) + \mu_{c_1} p_3(t) \quad (4.14)$$

$$\dot{p}_2(t) = \lambda p_1(t) - (\mu_c + \mu)p_2(t) \quad (4.15)$$

$$\dot{p}_3(t) = \lambda_c p_0(t) + \lambda_c p_1(t) - (\mu_c + \mu_{c_1})p_3(t) \quad (4.16)$$

At time $t = 0$, $p_0(0) = 1$, $p_1(0) = 0$, $p_2(0) = 0$ and $p_3(0) = 0$.

By setting the derivatives of the above expressions equal to zero and by making use of the relationship $\sum_{i=0}^{3} p_i = 1$, results in the following steady state probabilities :

$$p_0 = \frac{F_1}{F_5} \quad (4.17)$$

$$p_1 = \frac{F_2}{F_5} \quad (4.18)$$

$$p_2 = \frac{F_3}{F_5} \quad (4.19)$$

$$p_3 = \frac{F_4}{F_5} \quad (4.20)$$

The constants $F_1, F_2, F_3, F_4, \; and \; F_5$ are defined in Appendix C.

The system steady state availability and unavailability are given by

$$AV_{ss} = p_0 + p_1 \quad (4.21)$$

$$UV_{ss} = p_2 + p_3 \quad (4.22)$$

Figure 4.4 and 4.5 show the plots of steady state system availability and steady state system unavailability respectively for specified values of model parameters. The plots clearly show that the steady state system availability decreases with increasing values of common-cause failures. The plots also show that as the repair rate increases the steady state system availability also increases.

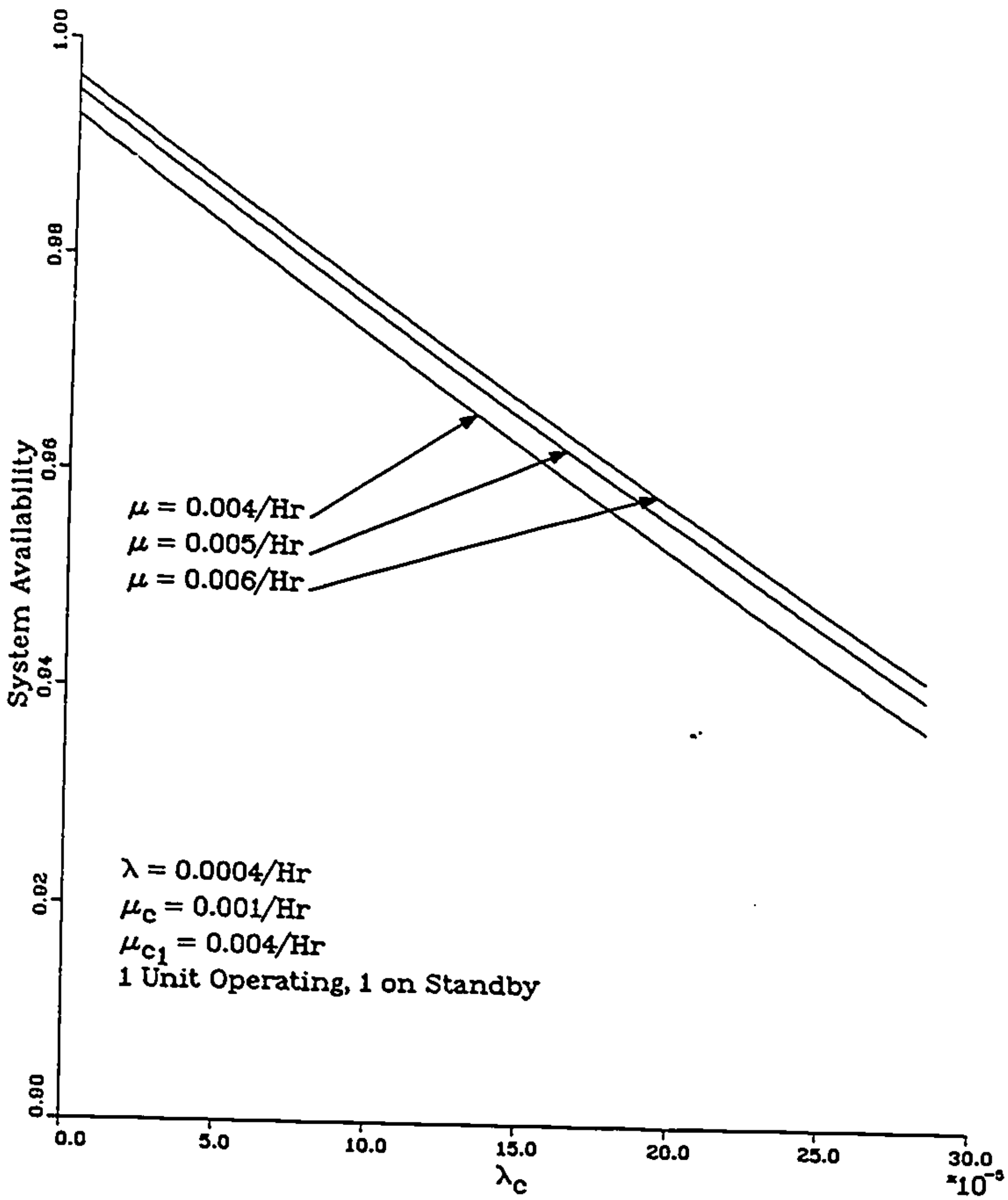

Figure 4.4: Steady State System Availability Plots for a Single Identical Unit Standby System with Type I Repair

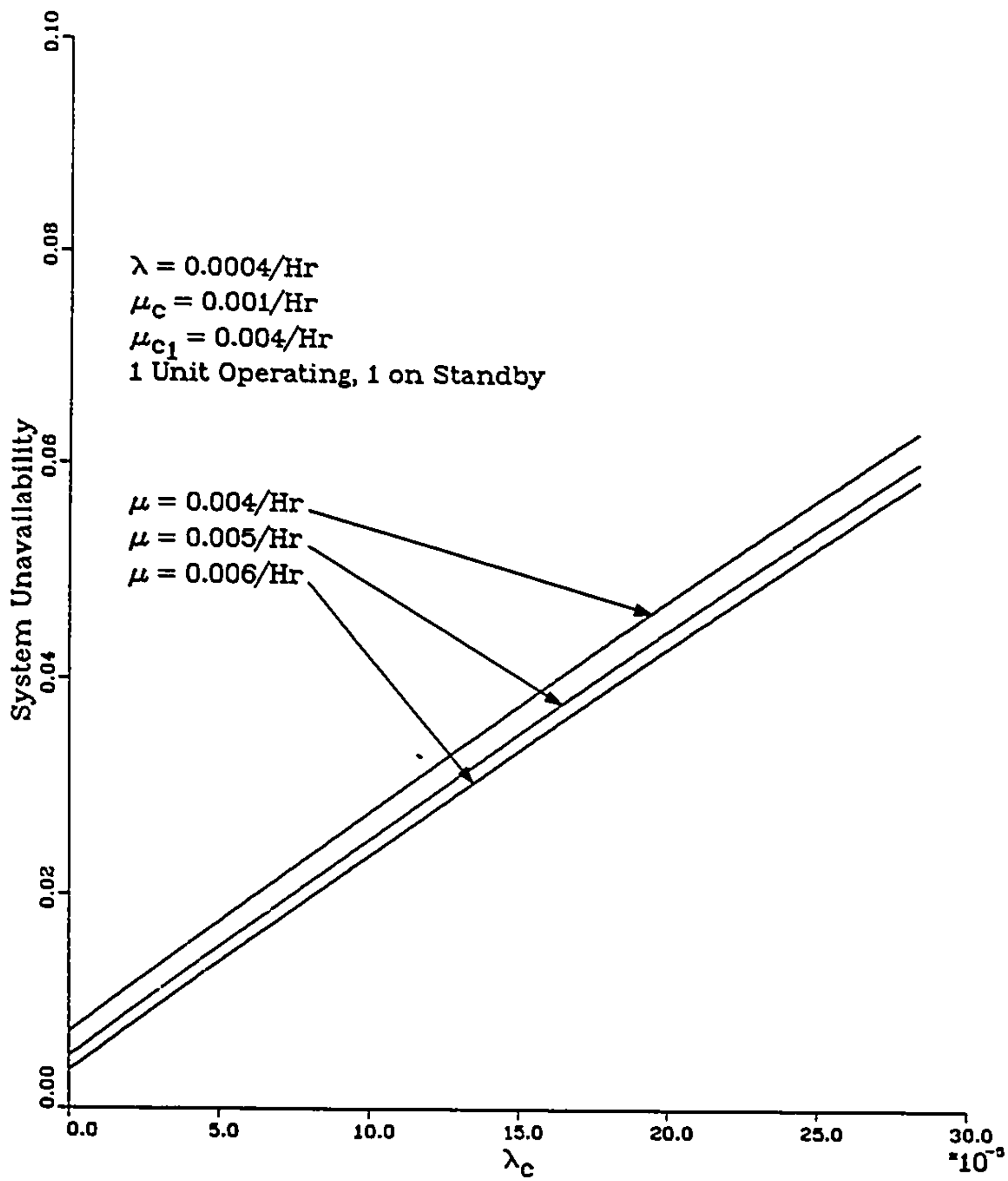

Figure 4.5: Steady State System Unavailability Plots for a Single Identical Unit Standby System with Type I Repair

Special Case Model II

Setting $n = 2$ in Equations (4.1) – (4.5) results in the following system of differential equations :

$$\begin{aligned}
\dot{p}_0(t) &= -(\lambda + \lambda_c)p_0(t) + \mu p_1(t) + \mu_2 p_2(t) + \mu_c p_3(t) + \mu_c p_4(t) && (4.23)\\
\dot{p}_1(t) &= \lambda p_0(t) - (\mu + \lambda + \lambda_c)p_1(t) + \mu p_2(t) + \mu_{c_1} p_4(t) && (4.24)\\
\dot{p}_2(t) &= \lambda p_1(t) - (\mu_2 + \mu + \lambda + \lambda_c)p_2(t) + \mu p_3(t) + \mu_{c_2} p_4(t) && (4.25)\\
\dot{p}_3(t) &= \lambda p_2(t) - (\mu_c + \mu)p_3(t) && (4.26)\\
\dot{p}_4(t) &= \lambda_c p_0(t) + \lambda_c p_1(t) + \lambda_c p_2(t) - (\mu_c + \mu_{c_1} + \mu_{c_2})p_4(t) && (4.27)
\end{aligned}$$

At time $t = 0$, $p_0(0) = 1$, $p_1(0) = 0$, $p_2(0) = 0$, $p_3(0) = 0$ and $p_4(0) = 0$.

The steady state probabilities are obtained by solving the above equations after setting their derivatives equal to zero and making use of the relationship $\sum_{i=0}^{4} p_i = 1$.

$$\begin{aligned}
p_0 &= \frac{F_6}{F_{11}} && (4.28)\\
p_1 &= \frac{F_7}{F_{11}} && (4.29)\\
p_2 &= \frac{F_8}{F_{11}} && (4.30)\\
p_3 &= \frac{F_9}{F_{11}} && (4.31)\\
p_4 &= \frac{F_{10}}{F_{11}} && (4.32)
\end{aligned}$$

The constants $F_6, F_7, F_8, F_9, F_{10}$ *and* F_{11} are defined in Appendix C.

The system steady state availability and unavailability are given by

$$\begin{aligned}
AV_{ss} &= p_0 + p_1 + p_2 && (4.33)\\
UV_{ss} &= p_3 + p_4 && (4.34)
\end{aligned}$$

Figure 4.6 and 4.7 show the plots of steady state system availability and steady state system unavailability respectively for specified values of model parameters. It can be clearly seen from the plots that the steady state system availability decreases

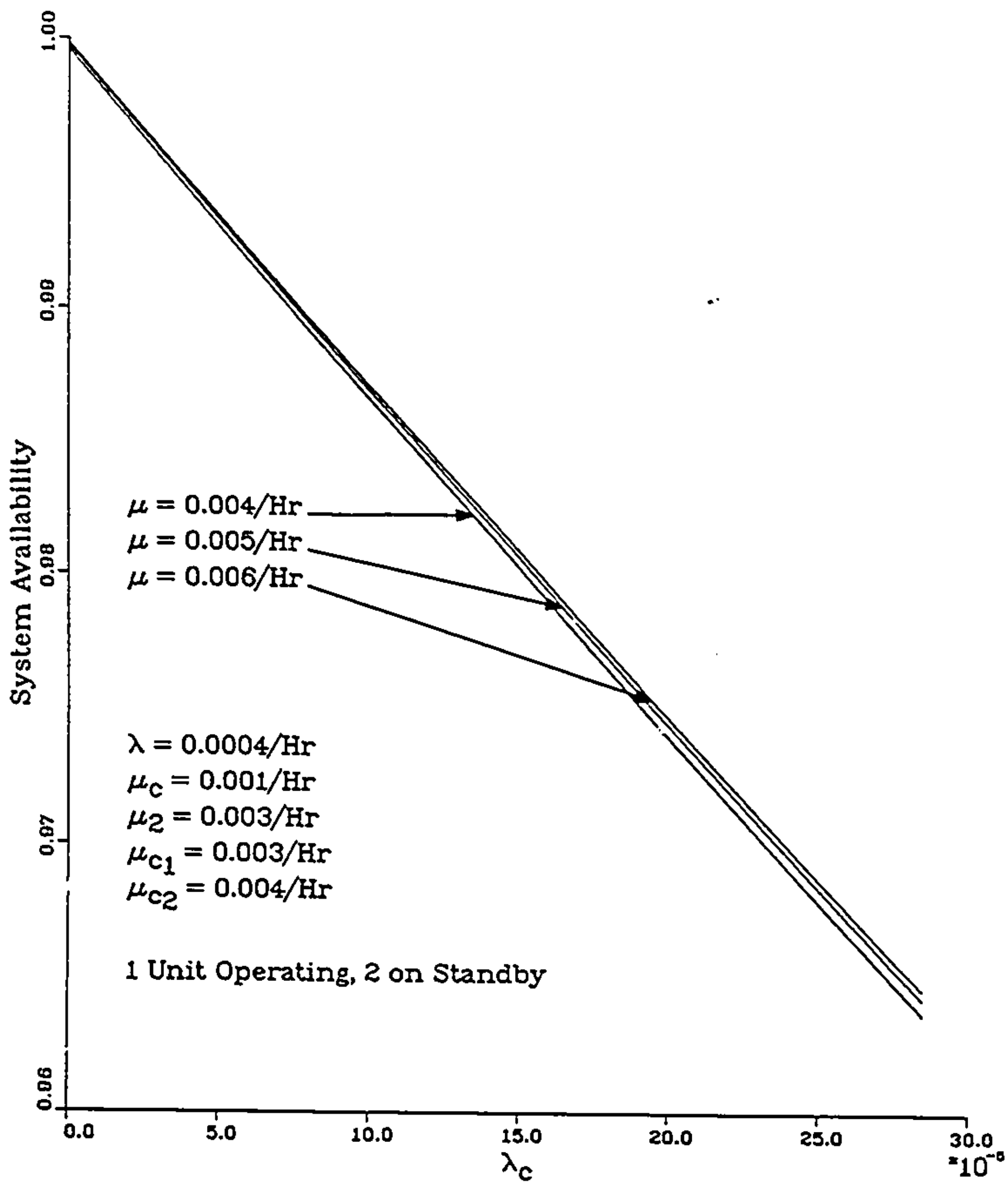

Figure 4.6: Steady State System Availability Plots for a two Identical Unit Standby System

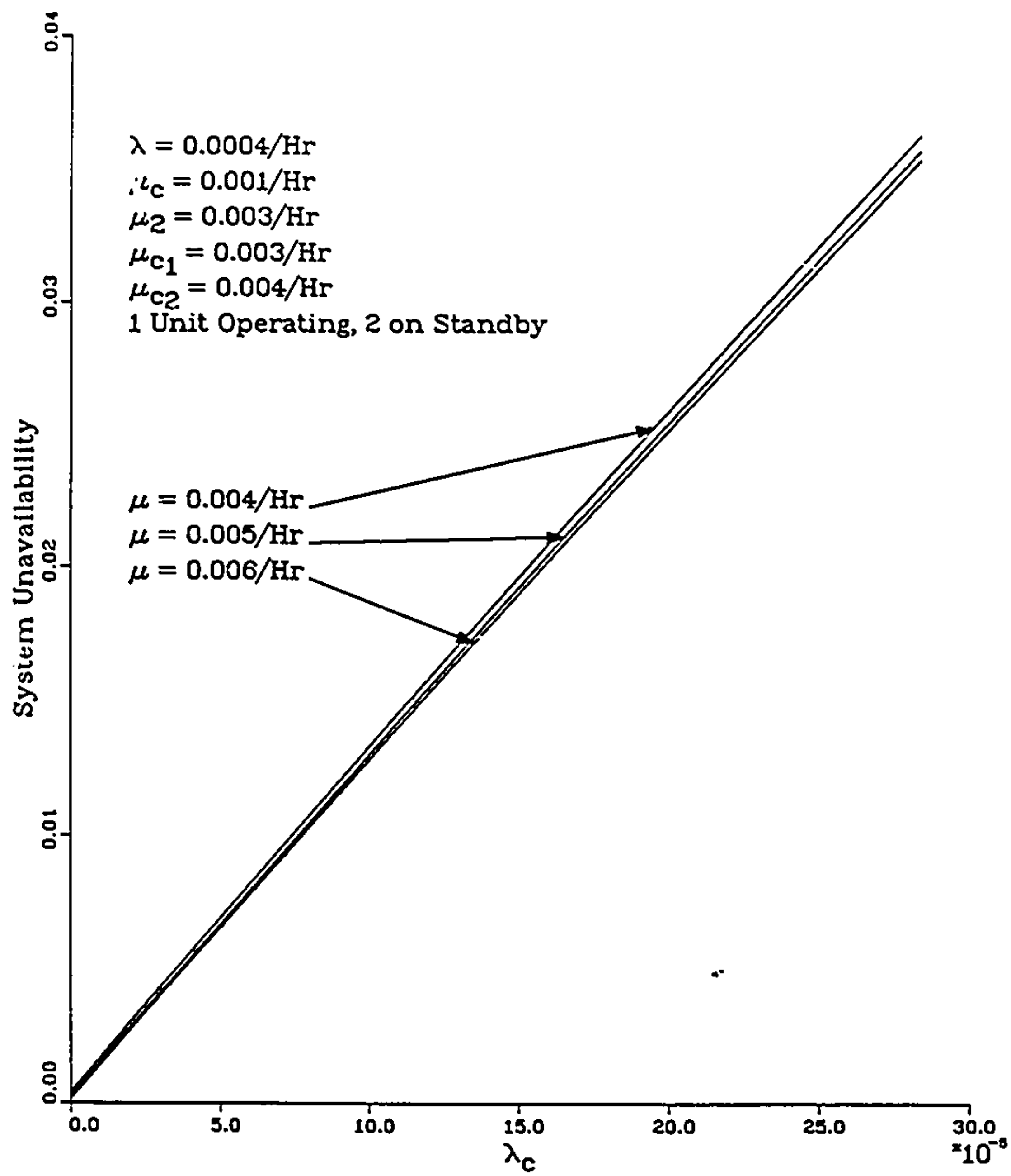

Figure 4.7: Steady State System Unavailability Plots for a two Identical Unit Standby System

with increasing values of common-cause failures. The plots also show an increase in steady state system availability with an increase in the repair rate.

Special Case Model III

The following differential equations for a three unit standby system are obtained by setting $n = 3$ in Equations (4.1) – (4.5) :

$$\dot{p}_0(t) = -(\lambda + \lambda_c)p_0(t) + \mu p_1(t) + \mu_2 p_2(t) + \mu_3 p_3(t) + \mu_c p_4(t) + \mu_c p_5(t) \quad (4.35)$$

$$\dot{p}_1(t) = \lambda p_0(t) - (\mu + \lambda + \lambda_c)p_1(t) + \mu p_2(t) + \mu_{c_1} p_5(t) \quad (4.36)$$

$$\dot{p}_2(t) = \lambda p_1(t) - (\mu_2 + \mu + \lambda + \lambda_c)p_2(t) + \mu p_3(t) + \mu_{c_2} p_5(t) \quad (4.37)$$

$$\dot{p}_3(t) = \lambda p_2(t) - (\mu_3 + \mu + \lambda + \lambda_c)p_3(t) + \mu p_4(t) + \mu_{c_3} p_5(t) \quad (4.38)$$

$$\dot{p}_4(t) = \lambda p_3(t) - (\mu_c + \mu)p_4(t) \quad (4.39)$$

$$\begin{aligned} \dot{p}_5(t) &= \lambda_c p_0(t) + \lambda_c p_1(t) + \lambda_c p_2(t) + \lambda_c p_3(t) - \\ &\quad (\mu_c + \mu_{c_1} + \mu_{c_2} + \mu_{c_3})p_5(t) \end{aligned} \quad (4.40)$$

At time $t = 0$, $p_0(0) = 1$ and all other initial condition probabilities are equal to zero.

By applying the same method as in the earlier special case models, the following steady state probability expressions are obtained :

$$p_0 = \frac{F_{12}}{F_{18}} \quad (4.41)$$

$$p_1 = \frac{F_{13}}{F_{18}} \quad (4.42)$$

$$p_2 = \frac{F_{14}}{F_{18}} \quad (4.43)$$

$$p_3 = \frac{F_{15}}{F_{18}} \quad (4.44)$$

$$p_4 = \frac{F_{16}}{F_{18}} \quad (4.45)$$

$$p_5 = \frac{F_{17}}{F_{18}} \quad (4.46)$$

The constants $F_{12}, F_{13}, F_{14}, F_{15}, F_{16}, F_{17}$ *and* F_{18} are defined in Appendix C.

The system steady state availability and unavailability are given by

$$AV_{ss} = p_0 + p_1 + p_2 + p_3 \tag{4.47}$$

$$UV_{ss} = p_4 + p_5 \tag{4.48}$$

Figures 4.8 and 4.9 show the steady state system availability and steady state system unavailability, respectively, for specified values of model parameters. It can be clearly seen from these plots that the steady state system availability decreases with an increase in the number of common-cause failures.

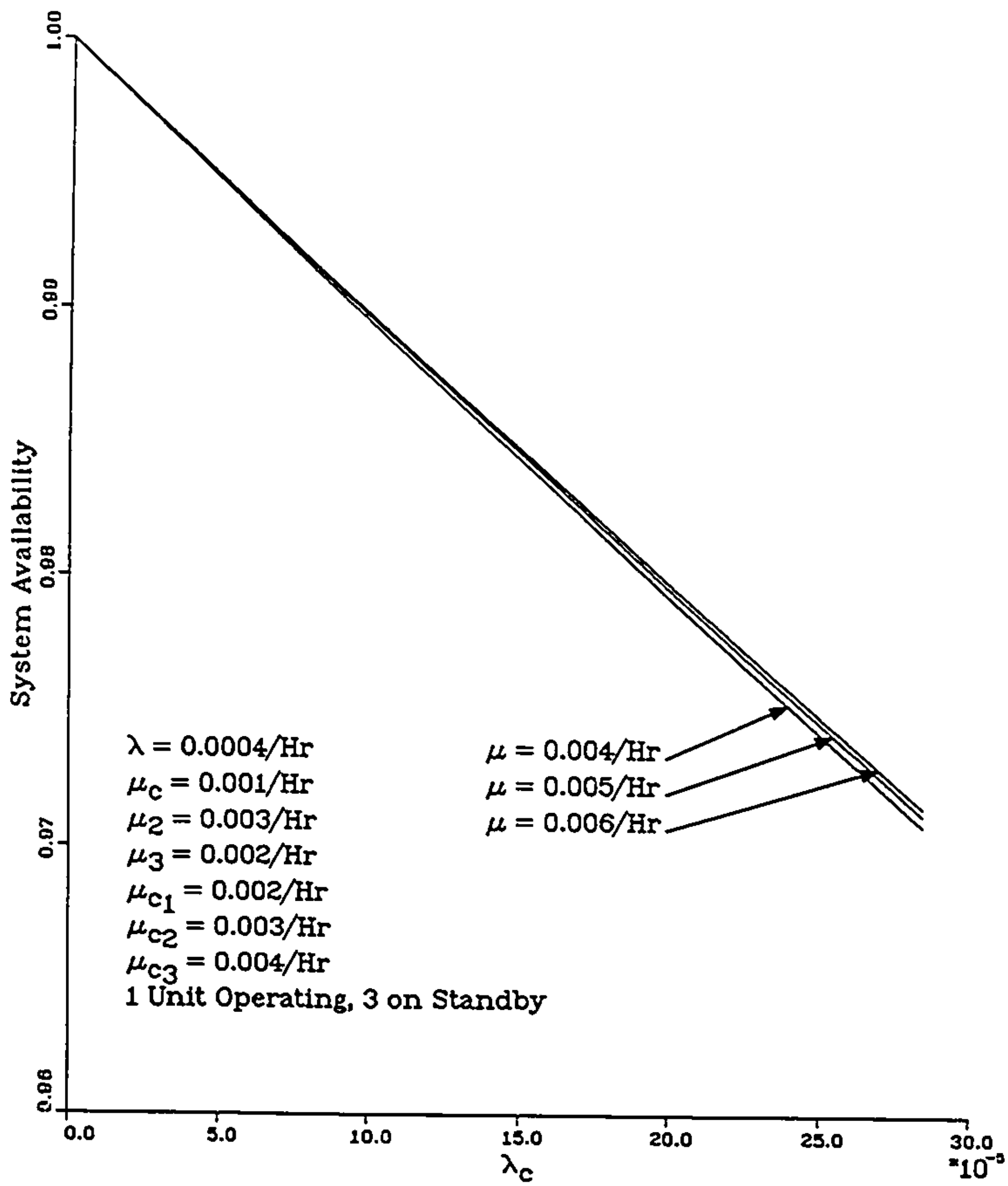

Figure 4.8: Steady State System Availability Plots for a three Identical Unit Standby System with Type I Repair

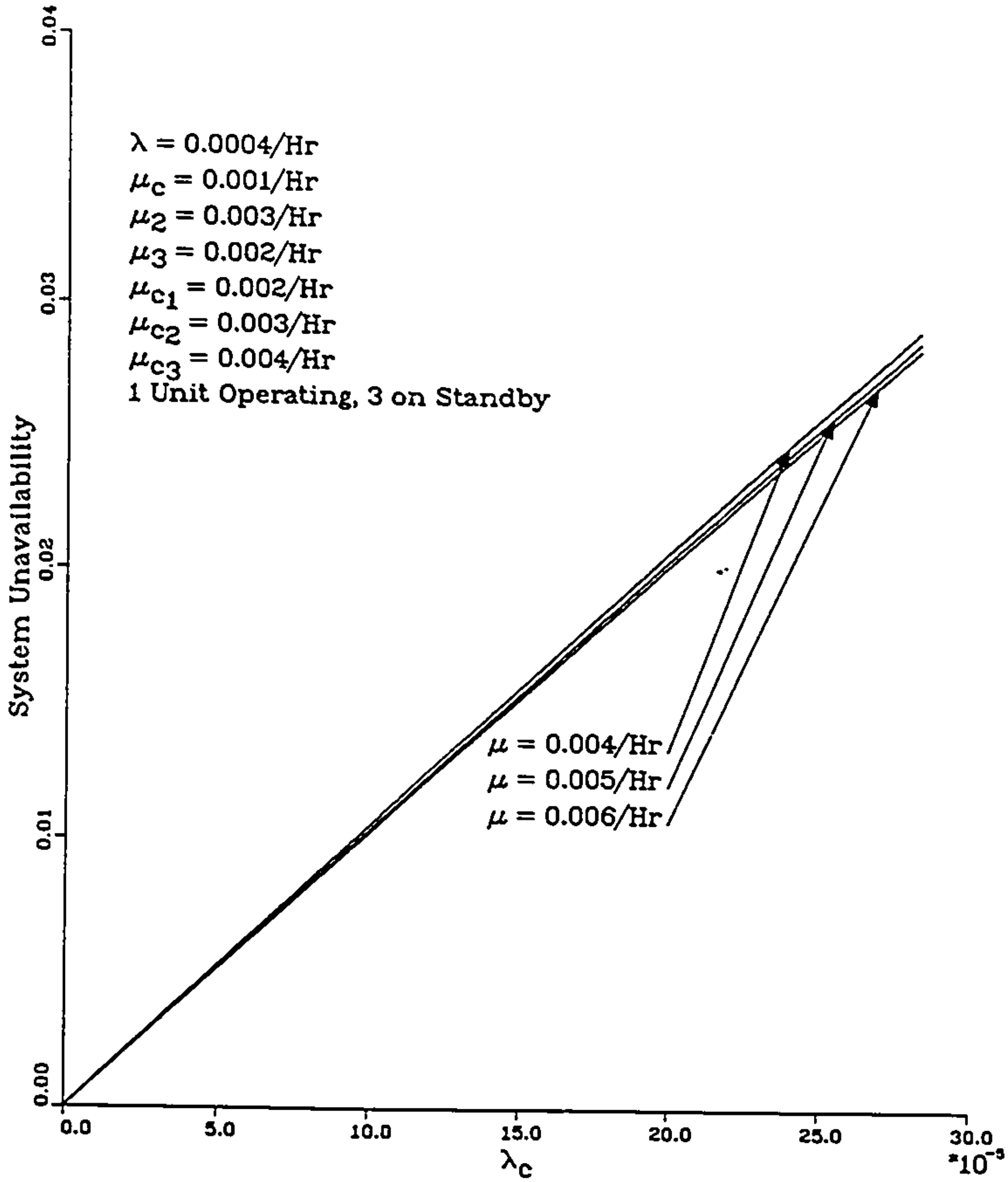

Figure 4.9: Steady State System Unavailability Plots for a three Identical Unit Standby System with Type I Repair

4.1.2 Standby System With Type II Repair

A n unit standby system with Type II repair can be represented as shown in Figure 4.1 with repair rates μ_c and μ_{c_i}; for $i = 1, 2, 3, \ldots, n$ set to zero. The system of differential equations governing such a system is as follows :

$$\dot{p}_0(t) = -(\lambda + \lambda_c)p_0(t) + \mu p_1(t) + \sum_{i=2}^{n} \mu_i p_i(t) \tag{4.49}$$

$$\dot{p}_1(t) = \lambda p_0(t) - (\mu + \lambda + \lambda_c)p_1(t) + \mu p_2(t) \tag{4.50}$$

$$\dot{p}_r(t) = \lambda p_{r-1}(t) - \{\mu + \mu_r + \lambda + \lambda_c\}p_r(t) + \mu p_{r+1}(t) \tag{4.51}$$

for $r = 2, 3, 4, \ldots, n-1$

$$\dot{p}_n(t) = \lambda p_{n-1}(t) - \{\mu + \mu_n + \lambda + \lambda_c\}p_n(t) \tag{4.52}$$

$$\dot{p}_{n+1}(t) = \lambda p_n(t) \tag{4.53}$$

$$\dot{p}_{n+2}(t) = \sum_{i=0}^{n} \lambda_c p_i(t) \tag{4.54}$$

At time $t = 0$, $p_0(0) = 1$ and all other initial condition probabilities are equal to zero.

Solving Equations (4.49) – (4.54) yields the following Laplace transforms of the state probabilities :

$$p_0(s) = \frac{1 + \mu p_1(s) + \sum_{i=2}^{n} \mu_i p_i(s)}{s + \lambda + \lambda_c} \tag{4.55}$$

$$p_1(s) = \frac{\lambda p_0(s) + \mu p_2(s)}{s + \mu + \lambda + \lambda_c} \tag{4.56}$$

$$p_{r-1}(s) = \frac{\lambda p_{r-2}(s) + \mu p_r(s)}{s + \mu_{(r-1)} + \mu + \lambda + \lambda_c} \tag{4.57}$$

for $r = 3, 4, 5, \ldots, n$

$$p_n(s) = \frac{\lambda p_{n-1}(s)}{s + \mu_n + \mu + \lambda + \lambda_c} \tag{4.58}$$

$$p_{n+1}(s) = \frac{\lambda p_n(s)}{s} \tag{4.59}$$

$$p_{n+2}(s) = \frac{\sum_{i=0}^{n} \lambda_c p_i(s)}{s} \tag{4.60}$$

The standby system mean time to failure is given by

$$MTTF = \lim_{s \to 0} [p_0(s) + p_1(s) + \ldots + p_{n-1}(s) + p_n(s)] \tag{4.61}$$

For zero and one unit standby, the mean time to failure, respectively, are

$$MTTF_0 = \frac{1}{\lambda + \lambda_c}$$

and

$$MTTF_1 = \frac{\mu + 2\lambda + \lambda_c}{\mu\lambda_c + \lambda_c^2 + \lambda^2 + 2\lambda\lambda_c}$$

For n number of standbys ($n > 1$), the general formula for system mean time to failure is

$$MTTF_n = \frac{A_{n-1}\{F_a\} - \mu\lambda\{F_b\} + \lambda^n}{A_{n-1}\{F_c\} - \mu\lambda\{F_d\} - \mu_n\lambda^n} \tag{4.62}$$

where

$$\begin{aligned}
A_i &= \{\mu + \mu_{i+1} + \lambda + \lambda_c\} \\
F_a &= Numerator\ of\ MTTF_{n-1} \\
F_b &= Numerator\ of\ MTTF_{n-2} \\
F_c &= Denominator\ of\ MTTF_{n-1} \\
F_d &= Denominator\ of\ MTTF_{n-2}
\end{aligned}$$

The system mean time to failure for specified values of model parameters using Equation (4.62) are plotted in Figure 4.10. The plots clearly indicate the decrease in system mean time to failure with an increase in common-cause failures. It can also be noted from the plots that the greater the number of standby units in the system, the higher the system mean time to failure.

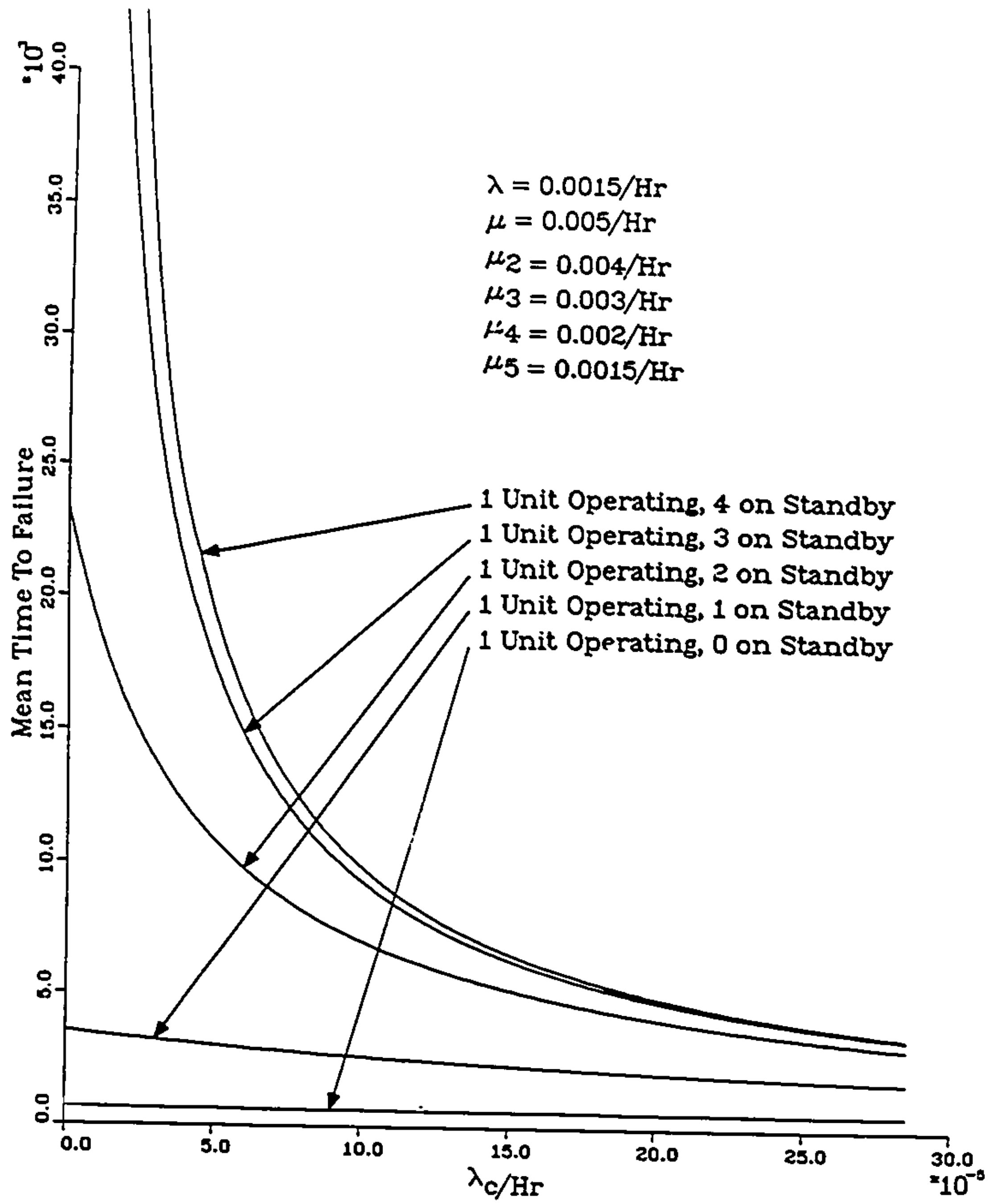

Figure 4.10: System Mean Time to Failure Plots for an Identical unit Standby System with Type II Repair

Special Case Model I

Consider a system with one unit as standby. By setting $n = 1$ in Equations (4.49) – (4.54) the differential equations for such a system can be expressed as follows :

$$\dot{p}_0(t) = -(\lambda + \lambda_c)p_0(t) + \mu p_1(t) \tag{4.63}$$

$$\dot{p}_1(t) = \lambda p_0(t) - (\mu + \lambda + \lambda_c)p_1(t) \tag{4.64}$$

$$\dot{p}_2(t) = \lambda p_1(t) \tag{4.65}$$

$$\dot{p}_3(t) = \lambda_c p_0(t) + \lambda_c p_1(t) \tag{4.66}$$

At time $t = 0$, $p_0(0) = 1$, $p_1(0) = 0$, $p_2(0) = 0$, *and* $p_3(0) = 0$ are the initial conditions.

Solving Equations (4.63) – (4.66), yields the following time dependent state probabilities :

$$p_0(t) = \frac{[(F_{19} + \frac{1}{2}F_{20})e^{(2F_{19}F_{20})t} - F_{19} + \frac{1}{2}F_{20}]e^{-\frac{1}{2}(2F_{19}F_{20}+2\lambda_c+\mu+2\lambda)t}}{F_{20}} \tag{4.67}$$

$$p_1(t) = \frac{\lambda[e^{(2F_{19}F_{20})t} - 1]e^{[-\frac{1}{2}(2\lambda+2\lambda_c+\mu)-F_{19}F_{20}]t}}{2F_{19}F_{20}} \tag{4.68}$$

where

$$F_{19} = \frac{1}{2}\mu^{\frac{1}{2}}$$

$$F_{20} = [4\lambda + \mu]^{\frac{1}{2}}$$

The system reliability is given by

$$R(t) = p_0(t) + p_1(t) \tag{4.69}$$

The plots of the Equation (4.69) are shown in Figure 4.11, from which it can be clearly seen that the system reliability decreases with an increase in the number of common-cause failures.

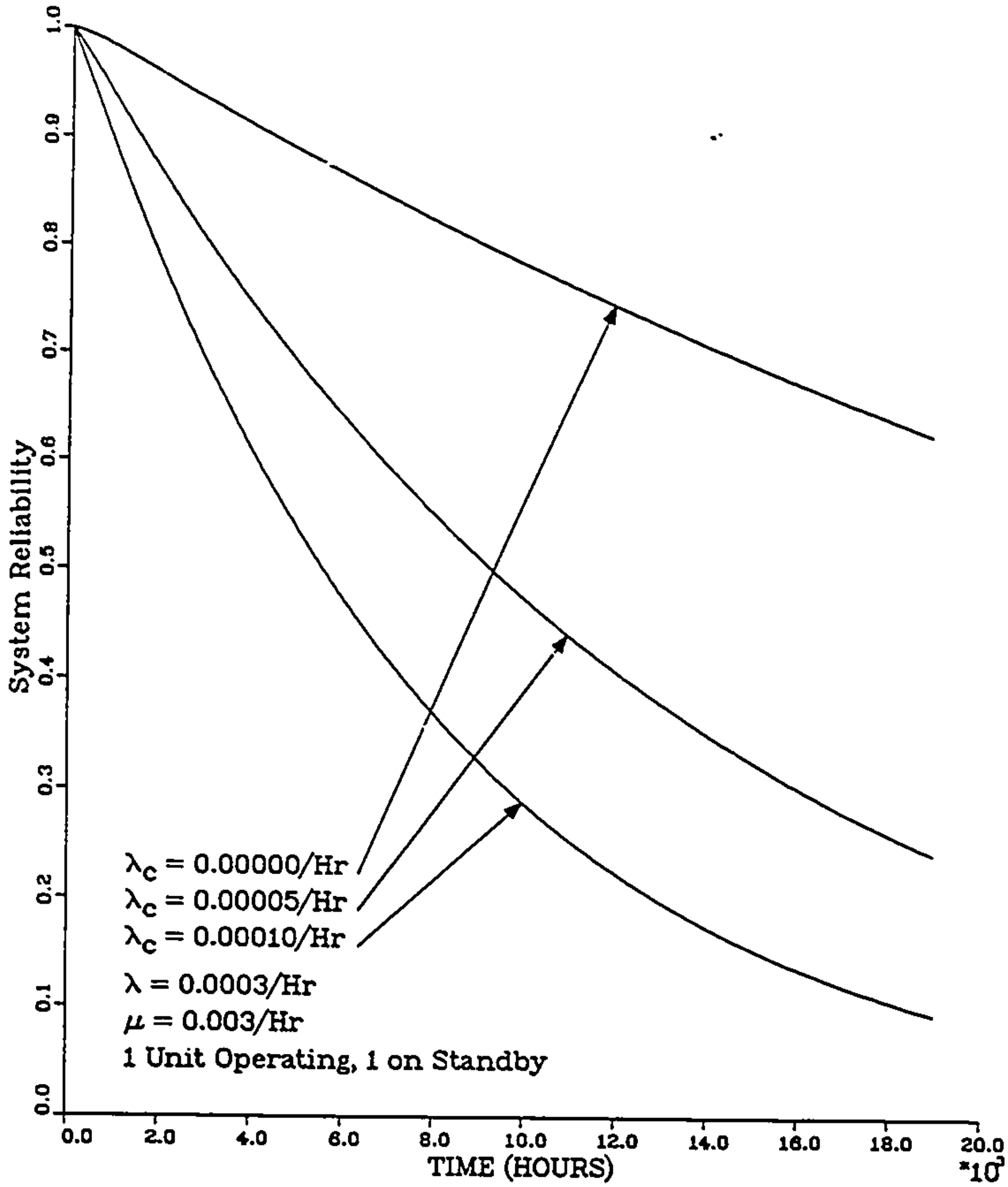

Figure 4.11: System Reliability Plots for One Identical Unit Standby System with Type II Repair

Special Case Model II

The system of differential equations for a system with two units in standby can be obtained by setting $n = 2$ in Equations (4.49) – (4.54).

$$\dot{p}_0(t) = -(\lambda + \lambda_c)p_0(t) + \mu p_1(t) + \mu_2 p_2(t) \tag{4.70}$$

$$\dot{p}_1(t) = \lambda p_0(t) - (\mu + \lambda + \lambda_c)p_1(t) + \mu p_2(t) \tag{4.71}$$

$$\dot{p}_2(t) = \lambda p_1(t) - (\mu + \mu_2 + \lambda + \lambda_c)p_2(t) \tag{4.72}$$

$$\dot{p}_3(t) = \lambda p_2(t) \tag{4.73}$$

$$\dot{p}_4(t) = \lambda_c p_0(t) + \lambda_c p_1(t) + \lambda_c p_2(t) \tag{4.74}$$

The initial conditions of the system are, at time $t = 0$, $p_0(0) = 1$, $p_1(0) = 0$, $p_2(0) = 0$, $p_3(0) = 0$ *and* $p_4(0) = 0$.

The results of the Laplace transforms of the state probabilities obtained by solving Equations (4.70) – (4.74) are given below.

$$p_0(s) = \frac{s^2 + F_{24}s + F_{25}}{s^3 + F_{21}s^2 + F_{22}s + F_{23}} \tag{4.75}$$

$$p_1(s) = \frac{\lambda(s + \mu + \lambda + \mu_2 + \lambda_c)}{s^3 + F_{21}s^2 + F_{22}s + F_{23}} \tag{4.76}$$

$$p_2(s) = \frac{\lambda^2}{s^3 + F_{21}s^2 + F_{22}s + F_{23}} \tag{4.77}$$

$$p_3(s) = \frac{\lambda^3}{s(s^3 + F_{21}s^2 + F_{22}s + F_{23})} \tag{4.78}$$

$$p_4(s) = \frac{\lambda_c s^2 + F_{26}s + F_{27}}{s(s^3 + F_{21}s^2 + F_{22}s + F_{23})} \tag{4.79}$$

The constants $F_{21}, F_{22}, F_{23}, F_{24}, F_{25}, F_{26}$ *and* F_{27} are defined in Appendix C.

Taking the inverse Laplace transforms of the Equations (4.75) – (4.79), we get the following time dependent state probabilities :

$$p_0(t) = \frac{(s_1^2 + F_{24}s_1 + F_{25})e^{(s_1 t)}}{(s_1 - s_2)(s_1 - s_3)} + \frac{(s_2^2 + F_{24}s_2 + F_{25})e^{(s_2 t)}}{(s_3 - s_2)(s_1 - s_2)} - \frac{(s_3^2 + F_{24}s_3 + F_{25})e^{(s_3 t)}}{(s_3 - s_2)(s_1 - s_3)} \quad (4.80)$$

$$p_1(t) = \frac{\lambda(s_1 + \lambda_c + \mu_2 + \mu + \lambda)e^{s_1 t}}{(s_2 - s_1)(s_3 - s_1)} + \frac{\lambda(s_2 + \lambda_c + \mu_2 + \mu + \lambda)e^{s_2 t}}{(s_2 - s_1)(s_2 - s_3)} - \frac{\lambda(s_3 + \lambda_c + \mu_2 + \mu + \lambda)e^{s_3 t}}{(s_3 - s_1)(s_2 - s_3)} \quad (4.81)$$

$$p_2(t) = \frac{\lambda^2 e^{s_1 t}}{(s_2 - s_1)(s_3 - s_1)} + \frac{\lambda^2 e^{s_2 t}}{(s_1 - s_2)(s_3 - s_2)} - \frac{\lambda^2 e^{s_3 t}}{(s_3 - s_2)(s_1 - s_3)} \quad (4.82)$$

where s_1, s_2 *and* s_3 are the roots of the cubic equation and can be determined as explained below

$$s^3 + F_{21}s^2 + F_{22}s + F_{23} = 0$$

$$Let\ \alpha = \frac{3F_{22} - (F_{21})^2}{9}$$

$$\beta = \frac{9F_{21}F_{22} - 27F_{23} - 2(F_{21})^3}{54}$$

$$\Phi = \sqrt[3]{\beta + \sqrt{\alpha^3 + \beta^2}}$$

$$\Omega = \sqrt[3]{\beta - \sqrt{\alpha^3 + \beta^2}}$$

$$Thus,\ s_1 = \Phi + \Omega - \frac{1}{3}F_{21}$$

$$s_2 = -\frac{1}{2}(\Phi + \Omega) - \frac{1}{3}F_{21} + \frac{1}{2}i\sqrt{3}(\Phi - \Omega)$$

$$s_3 = -\frac{1}{2}(\Phi + \Omega) - \frac{1}{3}F_{21} - \frac{1}{2}i\sqrt{3}(\Phi - \Omega)$$

The above i is associated with complex numbers.

$$-F_{21} = s_1 + s_2 + s_3$$

$$F_{22} = s_1 s_2 + s_2 s_3 + s_3 s_1$$

$$-F_{23} = s_1 s_2 s_3$$

The system reliability is given by

$$R(t) = p_0(t) + p_1(t) + p_2(t) \tag{4.83}$$

The plots of the Equation (4.83) are shown in Figure 4.12. It can be clearly seen from the plots that the system reliability decreases with an increase in the number of common-cause failures.

The system mean time to failure can be obtained by substituting $n = 2$ in Equation (4.62).

$$MTTF_2 = \frac{A_1\{\mu + 2\lambda + \lambda_c\} - \mu\lambda\{1\} + \lambda^2}{A_1\{\mu\lambda_c + \lambda_c^2 + \lambda^2 + 2\lambda\lambda_c\} - \mu\lambda\{\lambda + \lambda_c\} - \lambda^2\mu_2} \tag{4.84}$$

where

$$A_1 = \{\mu + \mu_2 + \lambda + \lambda_c\}$$

Figure 4.10 shows the plots of Equation (4.62) for different values of n. The plots clearly demonstrate the decrease in system mean time to failure with an increase in the number of common-cause failures. Furthermore, an increase in system mean time to failure with an increase in the number of units in the standby system can also be seen.

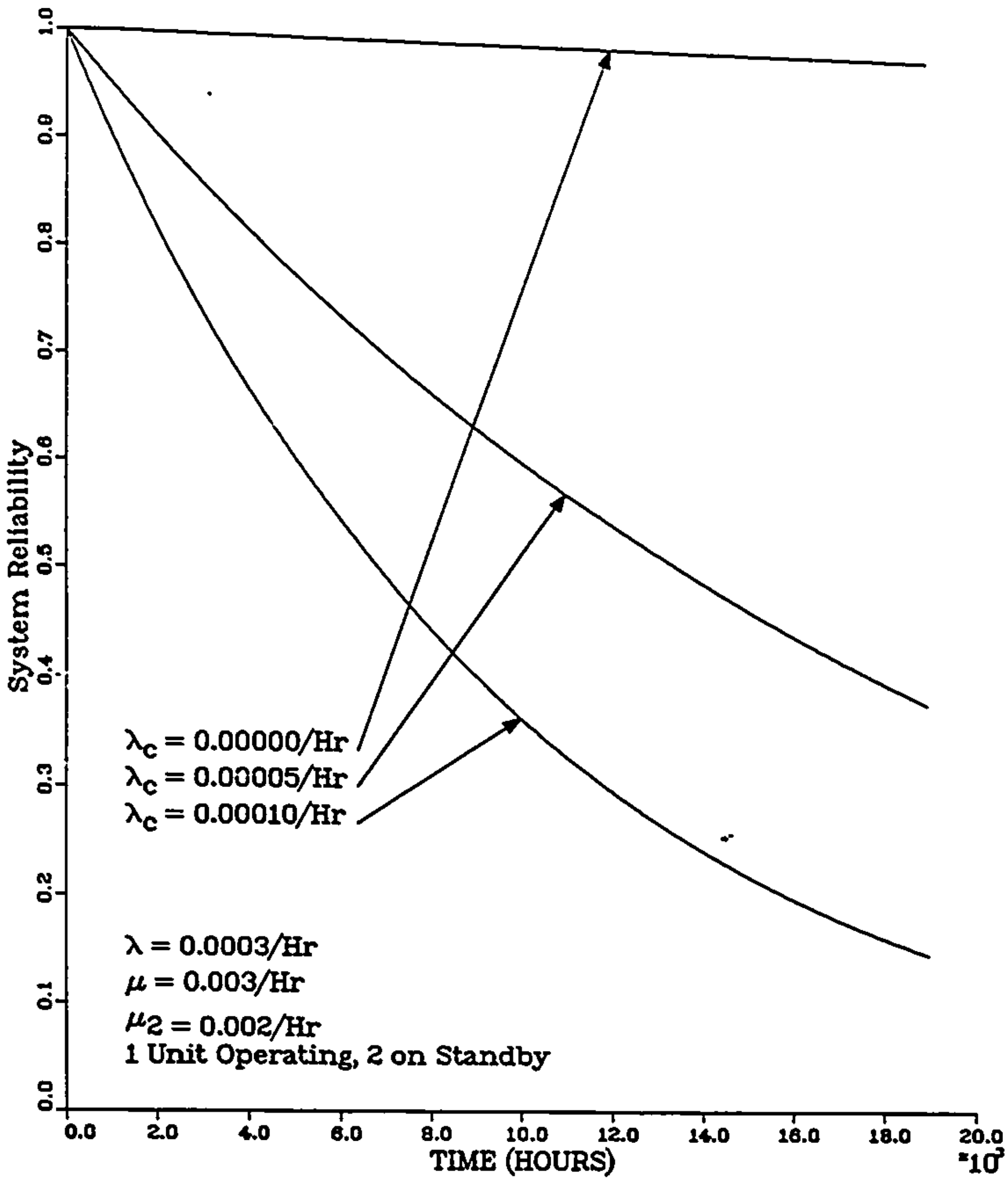

Figure 4.12: System Reliability Plots for a Two Identical Unit Standby System with Type II Repair

Special Case Model III

For a system with 3 standby units, the following differential equations are obtained by setting $n = 3$ in Equations (4.49) – (4.54) :

$$\dot{p}_0(t) = -(\lambda + \lambda_c)p_0(t) + \mu p_1(t) + \mu_2 p_2(t) + \mu_3 p_3(t) \tag{4.85}$$

$$\dot{p}_1(t) = \lambda p_0(t) - (\mu + \lambda + \lambda_c)p_1(t) + \mu p_2(t) \tag{4.86}$$

$$\dot{p}_2(t) = \lambda p_1(t) - (\mu + \mu_2 + \lambda + \lambda_c)p_2(t) + \mu p_3(t) \tag{4.87}$$

$$\dot{p}_3(t) = \lambda p_2(t) - (\mu + \mu_3 + \lambda + \lambda_c)p_3(t) \tag{4.88}$$

$$\dot{p}_4(t) = \lambda p_3(t) \tag{4.89}$$

$$\dot{p}_5(t) = \lambda_c p_0(t) + \lambda_c p_1(t) + \lambda_c p_2(t) + \lambda_c p_3(t) \tag{4.90}$$

At time $t = 0$, $p_0(0) = 1$, $p_1(0) = 0$, $p_2(0) = 0$, $p_3(0) = 0$, $p_4 = 0$, *and* $p_5(0) = 0$.

Solving Equations (4.85) – (4.90) using Laplace transforms result in the following Laplace transforms of the state probabilities :

$$p_0(s) = \frac{s^3 + F_{28}s^2 + F_{29}s + F_{30}}{s^4 + F_{31}s^3 + F_{32}s^2 + F_{33}s + F_{34}} \tag{4.91}$$

$$p_1(s) = \frac{\lambda s^2 + F_{35}s + F_{36}}{s^4 + F_{31}s^3 + F_{32}s^2 + F_{33}s + F_{34}} \tag{4.92}$$

$$p_2(s) = \frac{\lambda^2[s + \mu + \mu_3 + \lambda + \lambda_c]}{s^4 + F_{31}s^3 + F_{32}s^2 + F_{33}s + F_{34}} \tag{4.93}$$

$$p_3(s) = \frac{\lambda^3}{s^4 + F_{31}s^3 + F_{32}s^2 + F_{33}s + F_{34}} \tag{4.94}$$

$$p_4(s) = \frac{\lambda^4}{s\{s^4 + F_{31}s^3 + F_{32}s^2 + F_{33}s + F_{34}\}} \tag{4.95}$$

$$p_5(s) = \frac{\lambda_c s^3 + F_{37}s^2 + F_{38}s + F_{39}}{s\{s^4 + F_{31}s^3 + F_{32}s^2 + F_{33}s + F_{34}\}} \tag{4.96}$$

The constants $F_{28}, F_{29}, \ldots, F_{36}$ are defined in Appendix C.

The following time dependent state probabilities are obtained by taking inverse Laplace transforms of Equations (4.91) – (4.96) :

$$
\begin{aligned}
p_0(t) \;=\; & \frac{[s_1^3 + F_{28}s_1^2 + F_{29}s_1 + F_{30}]e^{s_1 t}}{(s_1 - s_2)(s_1 - s_3)(s_1 - s_4)} + \frac{[s_2^3 + F_{28}s_2^2 + F_{29}s_2 + F_{30}]e^{s_2 t}}{(s_2 - s_3)(s_2 - s_4)(s_2 - s_1)} + \\
& \frac{[s_3^3 + F_{28}s_3^2 + F_{29}s_3 + F_{30}]e^{s_3 t}}{(s_3 - s_2)(s_4 - s_3)(s_1 - s_3)} + \\
& \frac{[s_4^3 + F_{28}s_4^2 + F_{29}s_4 + F_{30}]e^{s_4 t}}{(s_2 - s_4)(s_3 - s_4)(s_4 - s_1)}
\end{aligned} \qquad (4.97)
$$

$$
\begin{aligned}
p_1(t) \;=\; & \frac{[\lambda s_1^2 + F_{35}s_1 + F_{36}]e^{s_1 t}}{(s_1 - s_3)(s_2 - s_1)(s_4 - s_1)} + \frac{[\lambda s_2^2 + F_{35}s_2 + F_{36}]e^{s_2 t}}{(s_2 - s_1)(s_3 - s_2)(s_4 - s_2)} + \\
& \frac{[\lambda s_3^2 + F_{35}s_3 + F_{36}]e^{s_3 t}}{(s_3 - s_1)(s_2 - s_3)(s_4 - s_3)} + \\
& \frac{[\lambda s_4^2 + F_{35}s_4 + F_{36}]e^{s_4 t}}{(s_4 - s_1)(s_4 - s_3)(s_4 - s_2)}
\end{aligned} \qquad (4.98)
$$

$$
\begin{aligned}
p_2(t) \;=\; & \frac{\lambda^2[s_1 + \lambda + \mu + \mu_3 + \lambda_c]e^{s_1 t}}{(s_4 - s_1)(s_1 - s_3)(s_2 - s_1)} + \frac{\lambda^2[s_2 + \lambda + \mu + \mu_3 + \lambda_c]e^{s_2 t}}{(s_2 - s_4)(s_2 - s_3)(s_2 - s_1)} + \\
& \frac{\lambda^2[s_3 + \lambda + \mu + \mu_3 + \lambda_c]e^{s_3 t}}{(s_3 - s_4)(s_1 - s_3)(s_2 - s_3)} + \\
& \frac{\lambda^2[s_4 + \lambda + \mu + \mu_3 + \lambda_c]e^{s_4 t}}{(s_4 - s_3)(s_1 - s_4)(s_2 - s_4)}
\end{aligned} \qquad (4.99)
$$

$$
\begin{aligned}
p_3(t) \;=\; & \frac{\lambda^3 e^{s_1 t}}{(s_1 - s_3)(s_4 - s_1)(s_2 - s_1)} + \frac{\lambda^3 e^{s_2 t}}{(s_2 - s_1)(s_2 - s_3)(s_2 - s_4)} + \\
& \frac{\lambda^3 e^{s_3 t}}{(s_3 - s_1)(s_4 - s_3)(s_2 - s_3)} + \frac{\lambda^3 e^{s_4 t}}{(s_1 - s_4)(s_4 - s_3)(s_2 - s_4)}
\end{aligned} \qquad (4.100)
$$

where s_1, s_2, s_3 *and* s_4 are the roots of the quartic equation and can be determined as explained below.

$$s^4 + F_{31}s^3 + F_{32}s^2 + F_{33}s + F_{34} = 0$$

Let y_1 be a real root of the cubic equation

$$y^3 - F_{32}y^2 + (F_{31}F_{33} - 4F_{34})y + (4F_{32}F_{34} - F_{33}^2 - F_{31}^2F_{34}) = 0$$

s_1, s_2, s_3 *and* s_4 are the roots of

$$z^2 + \frac{1}{2}\{F_{31} \pm \sqrt{F_{31}^2 - 4F_{32} + 4y_1}\}z + \frac{1}{2}\{y_1 \pm \sqrt{y_1^2 - 4F_{34}}\} = 0$$

$$-F_{31} = s_1 + s_2 + s_3 + s_4$$

$$F_{32} = s_1 s_2 + s_2 s_3 + s_3 s_4 + s_4 s_1 + s_1 s_3 + s_2 s_4$$

$$-F_{33} = s_1 s_2 s_3 + s_2 s_3 s_4 + s_1 s_2 s_4 + s_1 s_3 s_4$$

$$F_{34} = s_1 s_2 s_3 s_4$$

The system reliability is given by

$$R(t) = p_0(t) + p_1(t) + p_2(t) \tag{4.101}$$

The plots of the Equation (4.101) are shown in Figure 4.13. It can be clearly seen from the plots that the system reliability decreases with an increase in the number of common-cause failures.

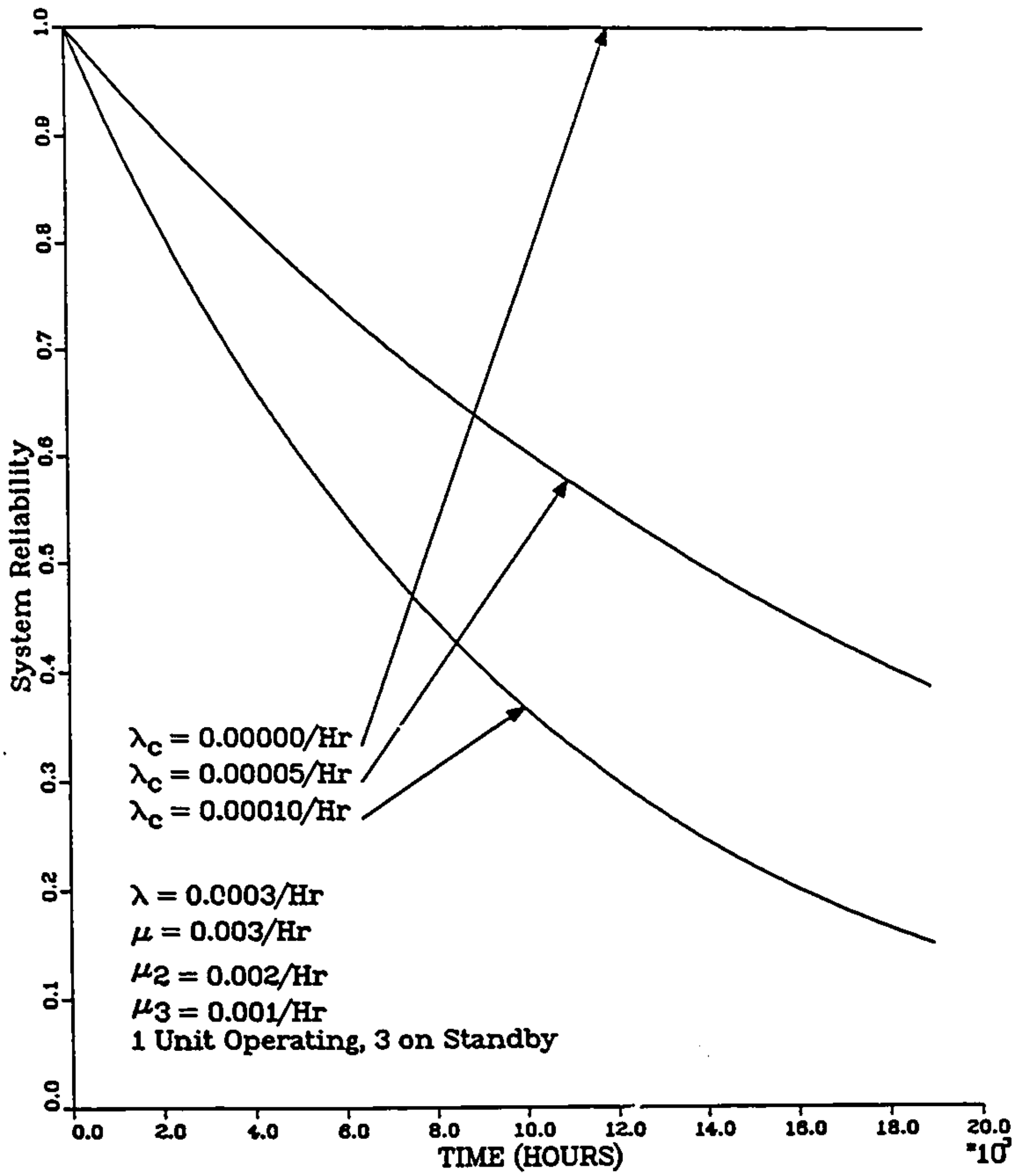

Figure 4.13: System Reliability Plots for a Three Identical Unit Standby System with Type II Repair

4.1.3 Standby System Without Repair

Setting the repair rates μ_{c_i}; for $i = 1, 2, 3, \ldots, n$, μ_c and μ equal to zero in Figure 4.1, yields the following differential equations :

$$\dot{p}_0(t) = -(\lambda + \lambda_c)p_0(t) \tag{4.102}$$

$$\dot{p}_i(t) = \lambda p_{i-1}(t) - (\lambda + \lambda_c p_i(t) \tag{4.103}$$

for $i = 1, 2, 3, \ldots, n$

$$\dot{p}_j(t) = \lambda p_{j-1}(t) \tag{4.104}$$

for $j = n + 1$

$$\dot{p}_{n+2}(t) = \sum_{i=0}^{n} \lambda_c p_i(t) \tag{4.105}$$

At time $t = 0$, $p_0(0) = 1$ and all other initial condition probabilities are equal to zero. The Laplace transforms of the state probability expressions obtained by solving Equations (4.102) – (4.105) are

$$p_0(s) = \frac{1}{s + \lambda + \lambda_c} \tag{4.106}$$

$$p_i(s) = \frac{\lambda^i}{(s + \lambda + \lambda_c)^{i+1}} \tag{4.107}$$

for $i = 1, 2, 3, \ldots, n$

The system mean time to failure ($MTTF$) is given by

$$\begin{aligned} MTTF_n &= \lim_{s \to 0} R(s) \\ &= \lim_{s \to 0} \{p_0(s) + \sum_{i=1}^{n} p_i(s)\} \\ &= \left\{ \frac{1}{(\lambda + \lambda_c)} + \sum_{i=1}^{n} \frac{\lambda^i}{(\lambda + \lambda_c)^{i+1}} \right\} \end{aligned} \tag{4.108}$$

Using the above expression, the system mean time to failure for specified values of model parameters are plotted in Figure 4.14. The plots show a decreasing trend in the system mean time to failure with an increase in common-cause failures, as well as

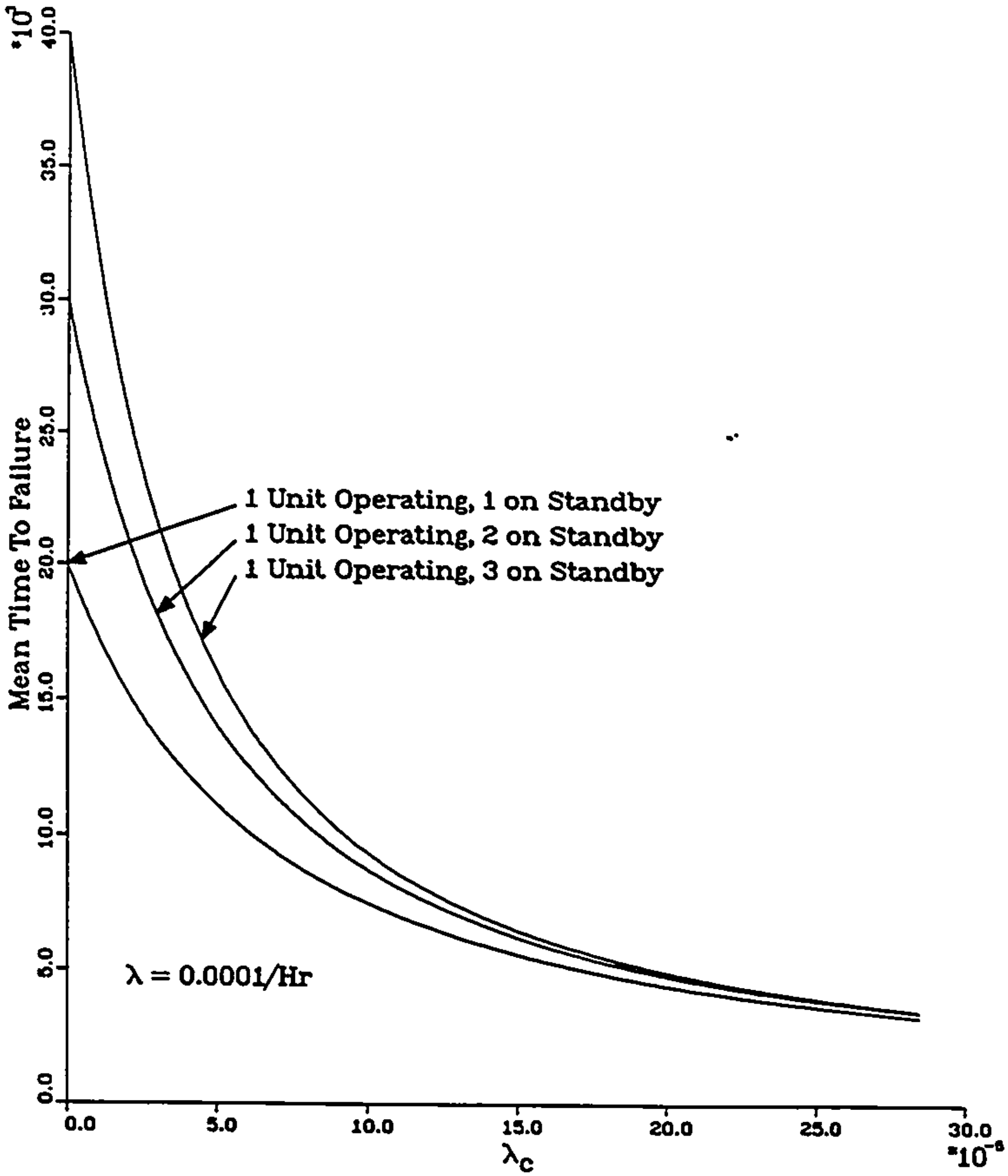

Figure 4.14: System Mean Time to Failure Plots for an Identical Unit Standby System Without Repair

with a decrease in the number of units in the standby system.

By taking inverse Laplace transforms of Equations (4.106) and (4.107), the following time dependent state probability expressions are obtained :

$$p_0(t) = e^{-(\lambda+\lambda_c)t} \quad (4.109)$$

$$p_i(t) = \frac{\lambda^i t^i e^{-(\lambda+\lambda_c)t}}{i!} \quad (4.110)$$

for $i = 1, 2, 3, \ldots, n$

The system reliability is given by

$$R(t) = p_o(t) + \sum_{i=1}^{n} p_i(t) \quad (4.111)$$

The plots of system reliability given by Equation (4.111) are shown in Figure 4.15 for specified values of model parameters. From these plots it is apparent that the increase in the number of common-cause failures results in a decrease in the system reliability.

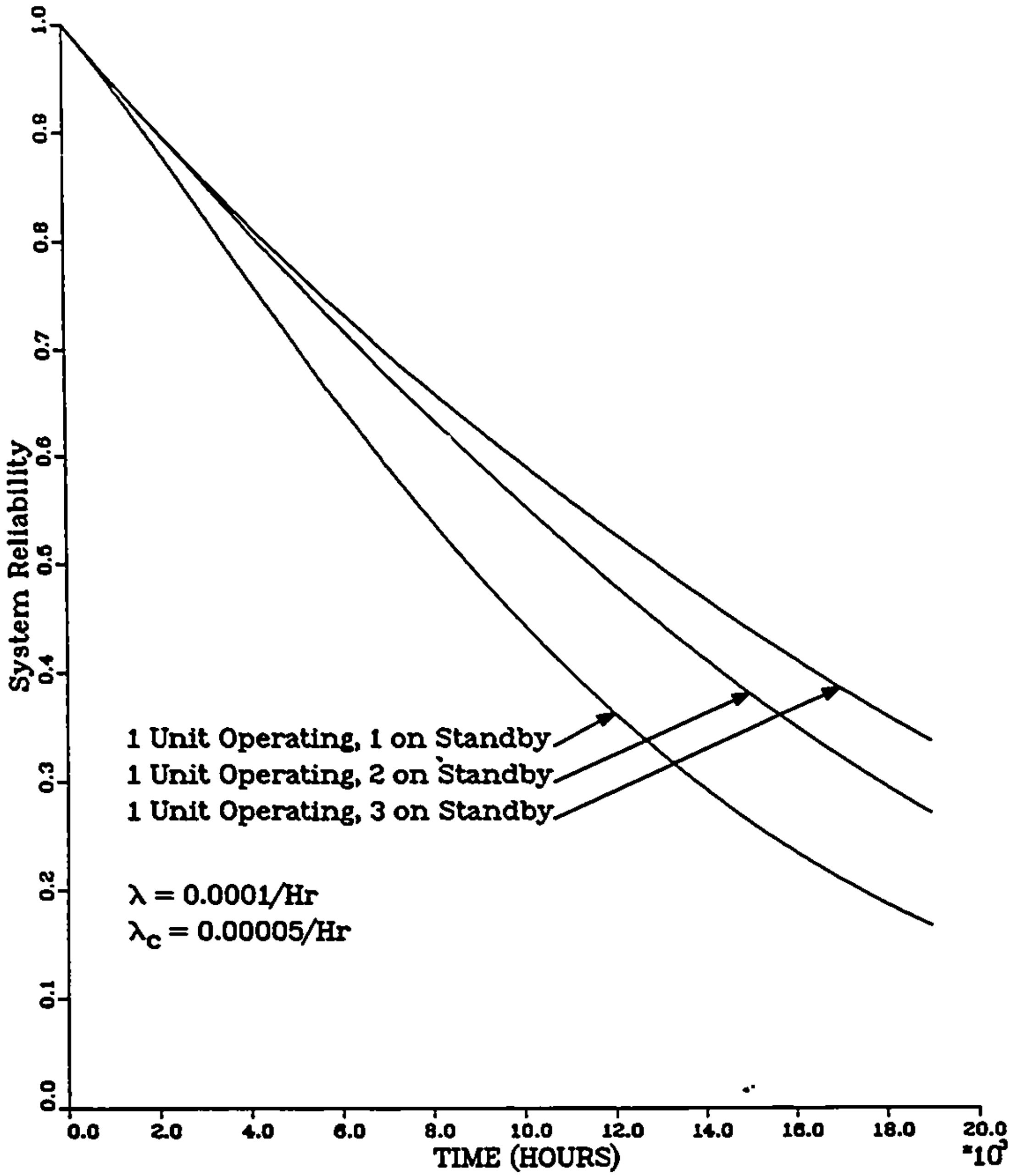

Figure 4.15: System Reliability Plots for an Identical Unit Standby System Without Repair

Special Case Model I

Consider a system with 1 unit as standby. By substituting $n = 1$ in Equations (4.102) – (4.105), the following system of differential equations is obtained :

$$\dot{p}_0(t) = -(\lambda + \lambda_c)p_0(t) \tag{4.112}$$

$$\dot{p}_1(t) = \lambda p_0(t) - (\lambda + \lambda_c)p_1(t) \tag{4.113}$$

$$\dot{p}_2(t) = \lambda p_1(t) \tag{4.114}$$

$$\dot{p}_3(t) = \lambda_c p_0(t) + \lambda_c p_1(t) \tag{4.115}$$

At time $t = 0$, $p_0(0) = 1$, $p_1(0) = 0$, $p_2(0) = 0$ and $p_3(0) = 0$.

Solving the above Equations with the aid of Laplace transforms, we get the following state probability expressions :

$$p_0(t) = e^{-(\lambda+\lambda_c)t} \tag{4.116}$$

$$p_1(t) = \lambda t e^{-(\lambda+\lambda_c)t} \tag{4.117}$$

The system reliability is given by

$$R(t) = p_0(t) + p_1(t) \tag{4.118}$$

The system reliability plotted in Figure 4.16 shows a inverse relationship between system reliability and the number of common-cause failures.

The system mean time to failure of a single unit standby system is given by

$$MTTF = \frac{1}{\lambda + \lambda_c} + \frac{\lambda}{(\lambda + \lambda_c)^2} \tag{4.119}$$

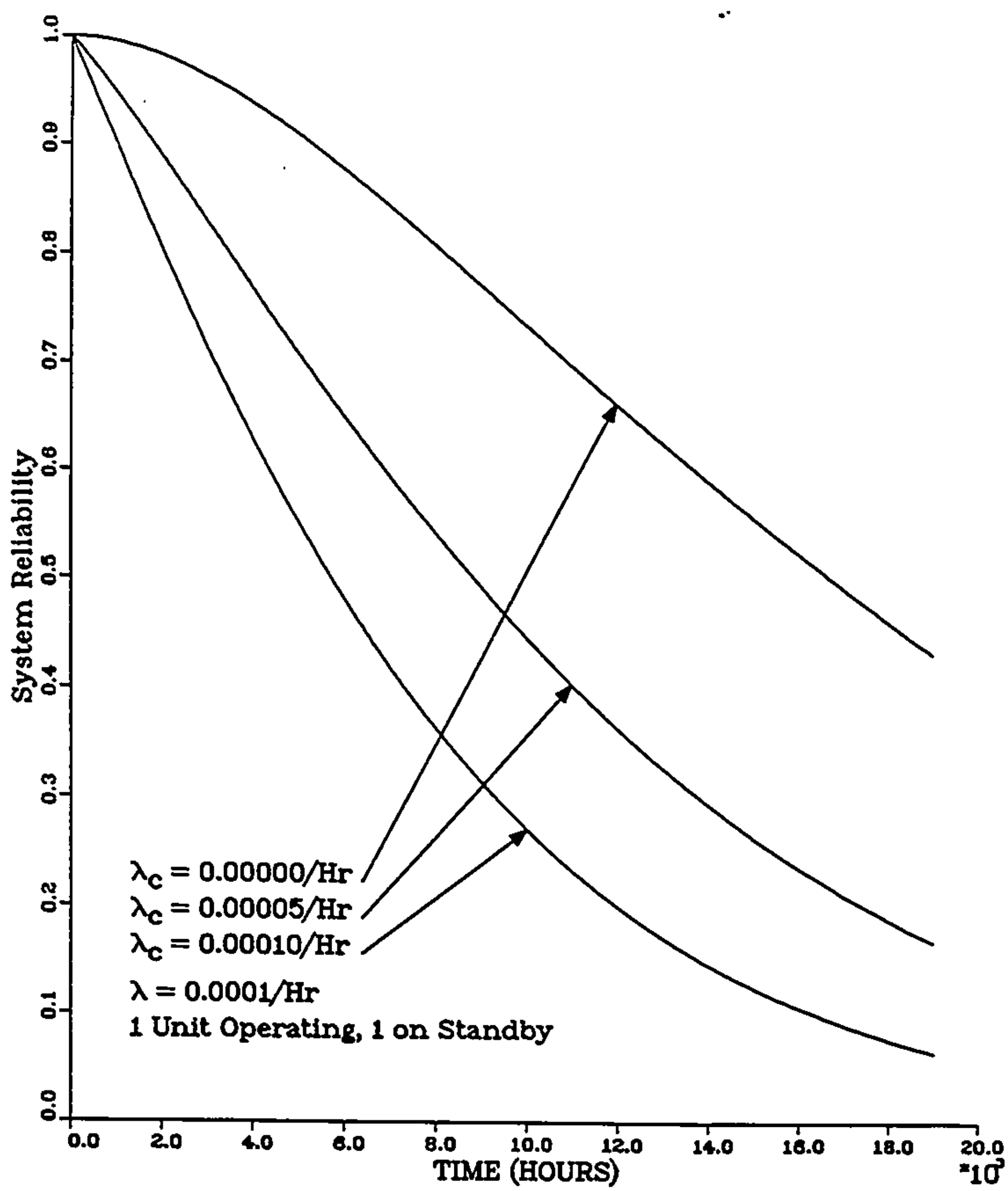

Figure 4.16: System Reliability Plots for a Single Identical Unit Standby System Without Repair

Special Case Model II

For a system with two units as standby the differential equations are as follows :

$$\dot{p}_0(t) = -(\lambda + \lambda_c)p_0(t) \quad (4.120)$$

$$\dot{p}_1(t) = \lambda p_0(t) - (\lambda + \lambda_c)p_1(t) \quad (4.121)$$

$$\dot{p}_2(t) = \lambda p_1(t) - (\lambda + \lambda_c)p_2(t) \quad (4.122)$$

$$\dot{p}_3(t) = \lambda p_2(t) \quad (4.123)$$

$$\dot{p}_4(t) = \lambda_c p_0(t) + \lambda_c p_1(t) + \lambda_c p_2(t) \quad (4.124)$$

At time $t = 0$, $p_0(0) = 1$, $p_1(0) = 0$, $p_2(0) = 0$, $p_3(0) = 0$ and $p_4(0) = 0$.

The following state probabilities are obtained by solving the above equations with the aid of Laplace transforms :

$$p_0(t) = e^{-(\lambda+\lambda_c)t} \quad (4.125)$$

$$p_1(t) = \lambda t e^{-(\lambda+\lambda_c)t} \quad (4.126)$$

$$p_2(t) = \frac{1}{2}\lambda^2 t^2 e^{-(\lambda+\lambda_c)t} \quad (4.127)$$

The system reliability is given by

$$R(t) = p_0(t) + p_1(t) + p_2(t) \quad (4.128)$$

Figure 4.17 shows the plots of system reliability. The plots clearly show the decreasing trend of system reliability with an increase in the number of common-cause failures.

The system mean time to failure of a two unit standby system is

$$MTTF = \frac{1}{\lambda + \lambda_c} + \frac{\lambda}{(\lambda + \lambda_c)^2} + \frac{\lambda^2}{(\lambda + \lambda_c)^3} \quad (4.129)$$

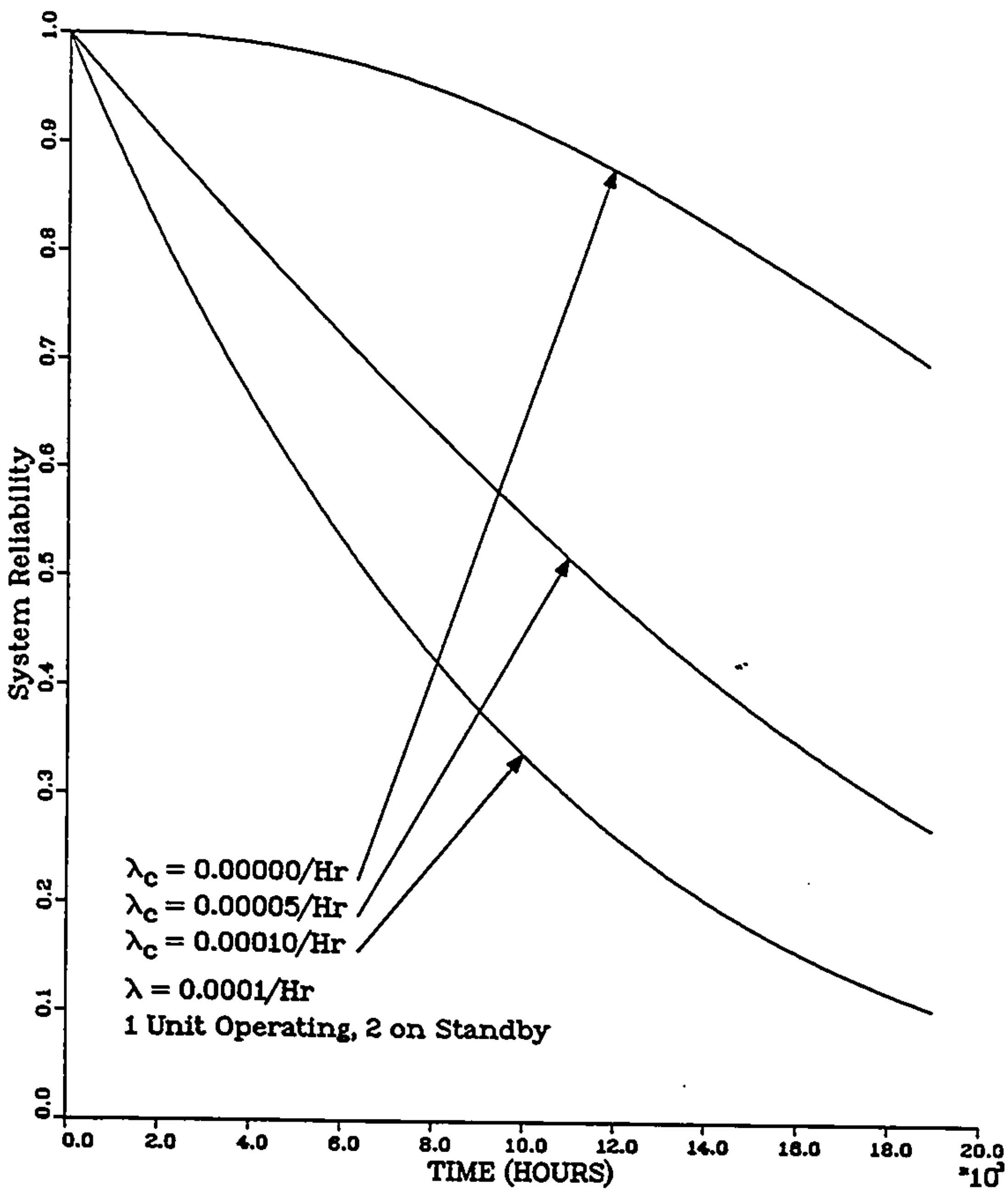

Figure 4.17: System Reliability Plots for Two Identical Unit Standby System Without Repair

Special Case Model III

Setting $n = 3$ in Equations (4.102)–(4.105) results in a system with three identical units in standby. The differential equations for this arrangement are as follows :

$$\dot{p}_0(t) = -(\lambda + \lambda_c)p_0(t) \quad (4.130)$$

$$\dot{p}_1(t) = \lambda p_0(t) - (\lambda + \lambda_c)p_1(t) \quad (4.131)$$

$$\dot{p}_2(t) = \lambda p_1(t) - (\lambda + \lambda_c)p_2(t) \quad (4.132)$$

$$\dot{p}_3(t) = \lambda p_2(t) - (\lambda + \lambda_c)p_3(t) \quad (4.133)$$

$$\dot{p}_4(t) = \lambda p_3(t) \quad (4.134)$$

$$\dot{p}_5(t) = \lambda_c p_0(t) + \lambda_c p_1(t) + \lambda_c p_2(t) + \lambda_c p_3(t) \quad (4.135)$$

At time $t = 0$, $p_0(0) = 1$ and all other initial condition probabilities are equal to zero.

By solving the above equations with the aid of Laplace transforms the following state probability expressions are obtained :

$$p_0(t) = e^{-(\lambda+\lambda_c)t} \quad (4.136)$$

$$p_1(t) = \lambda t e^{-(\lambda+\lambda_c)t} \quad (4.137)$$

$$p_2(t) = \frac{1}{2}\lambda^2 t^2 e^{-(\lambda+\lambda_c)t} \quad (4.138)$$

$$p_3(t) = \frac{1}{6}\lambda^3 t^3 e^{-(\lambda+\lambda_c)t} \quad (4.139)$$

The system reliability is given by

$$R(t) = p_0(t) + p_1(t) + p_2(t) + p_3(t) \quad (4.140)$$

The system reliability is plotted in Figure 4.18. These plots indicate an inverse relationship between system reliability and common-cause failures.

The system mean time to failure of a three unit standby system can be obtained by substituting $n = 3$ in Equation (4.108).

$$MTTF = \frac{1}{\lambda + \lambda_c} + \frac{\lambda}{(\lambda + \lambda_c)^2} + \frac{\lambda^2}{(\lambda + \lambda_c)^3} + \frac{\lambda^3}{(\lambda + \lambda_c)^4} \quad (4.141)$$

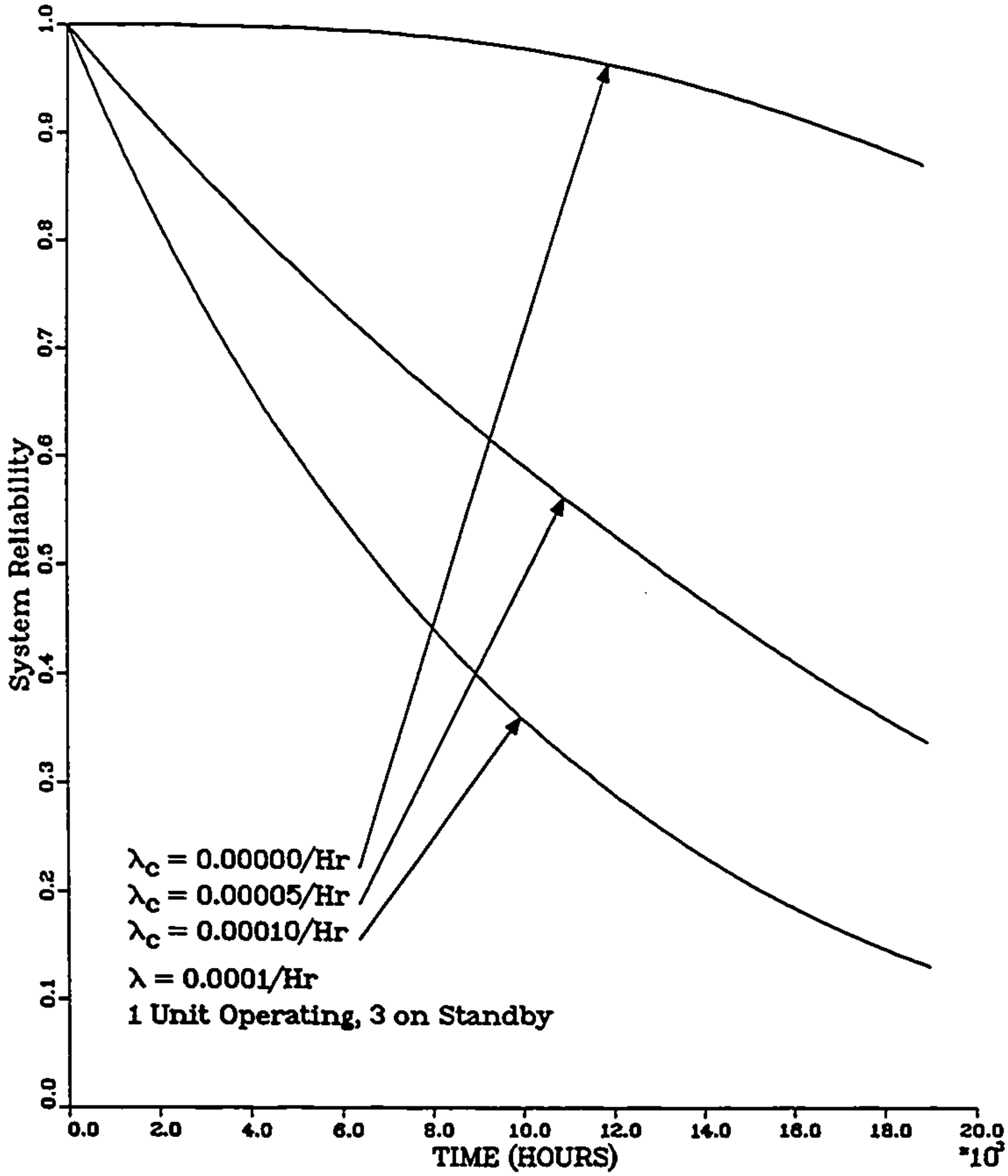

Figure 4.18: System Reliability Plots for a Three Identical Unit Standby System Without Repair

4.1.4 Comparison of Markov Method Results with that of Block Diagram Method

In order to compare the results obtained through Markov state space approach with those obtained by the standard block diagram approach with modified network shown in Figure 4.19. The standby stage, labelled 1 in Figure 4.19, represents all the independent failures. The series unit, labelled 2 in Figure 4.19, represents all common-cause failures of the system.

The common-cause failure probability hypothetical unit is connected in series with the independent failure mode units. A failure of this hypothetical series unit (that is, the occurrence of a common-cause failure) will cause a total system failure.

The system reliability can be expressed as

$$R_{st}(t) \quad = \quad R_1(t)R_2(t)$$

In the case of $(n+1)$ non-identical units whose failure time density functions are different, the standby redundant system failure density is expressed in [64] as

$$f_{st}(t) = \int_{y_n=0}^{t} \int_{y_{n-1}=0}^{y_n} \cdots \int_{y_1=0}^{y_2} f_1(y_1)f_2(y_2-y_1)\ldots f_{n+1}(t-y_n)dy_1dy_2\ldots dy_n$$

Thus, the system reliability can be obtained by integrating $f_{st}(t)$ over the interval $[t, \infty]$ as follows :

$$R_1(t) = \int_t^{\infty} f_{st}(t)dt \qquad (4.142)$$

If λ is the constant failure rate of each unit in the standby system, the system reliability of the $(n+1)$ unit, in which one unit is operating and n units are on standby, is given by

$$R_1(t) = \sum_{i=0}^{n} \frac{(\lambda t)^i e^{-\lambda t}}{i!} \qquad (4.143)$$

The reliability of the hypothetical common-cause failure unit is given by

$$R_2(t) = e^{-\lambda_c t} \qquad (4.144)$$

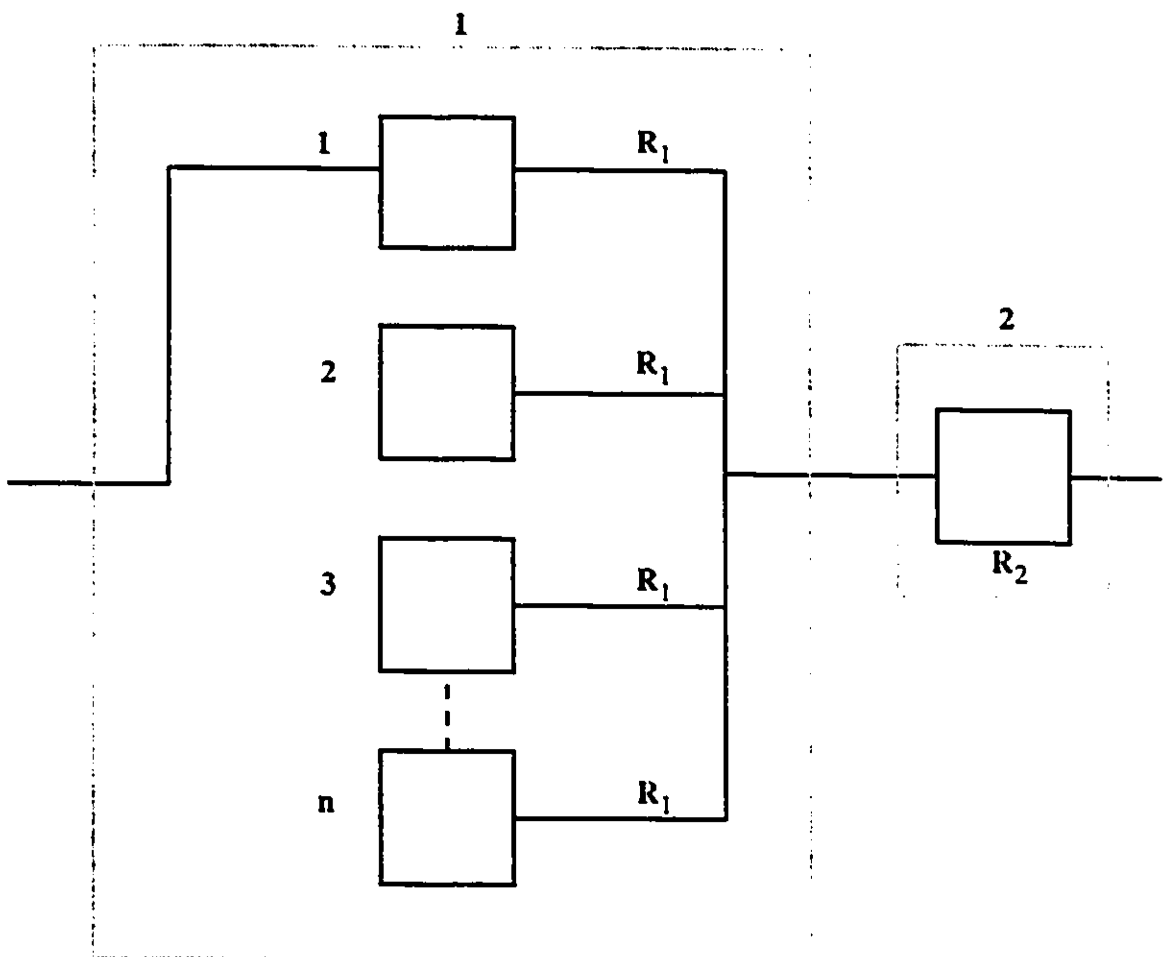

Figure 4.19: Block Diagram of a Standby System in Series with a Hypothetical CCF Unit

where λ_c is the constant common-cause failure rate of the hypothetical unit.

For a two identical unit standby system

$$R_{st}(t) = \left\{\sum_{i=0}^{2} \frac{(\lambda t)^i e^{-\lambda t}}{i!}\right\} R_2(t) \tag{4.145}$$

$$= \frac{1}{2} e^{-(\lambda+\lambda_c)t}\{2 + 2\lambda t + \lambda_2 t^2\} \tag{4.146}$$

and

$$MTTF = \int_0^\infty R(t)$$

$$= \frac{3\lambda^2 + 3\lambda_c\lambda + \lambda_c^2}{(\lambda + \lambda_c)^3} \tag{4.147}$$

Substituting $n = 2$ in Equations (4.108) and (4.111) the expressions obtained for system mean time to failure as well as for system reliability are found to be the same those expressed by Equations (4.146) and (4.147).

4.2 Non-identical Unit Standby System

This section deals with the reliability analysis of both repairable and non-repairable non-identical unit standby system. At $t = 0$, only one unit starts operating and the remaining n units are in standby mode. One of the standby units takes over the function of the operating unit as soon as it fails. Furthermore, the occurrence of a common-cause failure in any of the system's operating state $(0, 1, 2, \ldots, n)$ can result in a system failure.

Two types of repair policies, namely, Type I Repair and Type II Repair, similar to those discussed in Section 4.1, are taken into consideration in the analysis of non-identical unit standby system.

Notation

The following symbols are associated with non-identical unit standby system :

n	number of units in standby.
t	time.
λ_i	Constant failure rate form the $(i-1)^{th}$ system state; for $i = 1, 2, 3, \ldots, n+1$.
μ_i	Constant repair rate from the i^{th} system state; for $i = 1, 2, 3, \ldots, n+1$.
μ_{io}	Constant repair rate from the i^{th} system state to system state 0; for $i = 2, 3, 4, \ldots, n+1$.
$\mu_{c_{i+1}}$	Constant repair rate of the failed system from state $n+2$ to state i; for $i = 0, 1, 2, \ldots, n$.
$\lambda_{c_{i+1}}$	Constant common-cause failure rate of the system from state i; for $i = 0, 1, 2, 3, \ldots, n$.
j	j^{th} state of the system; for $j = 0, 1, 2, \ldots, n+2$.
$p_j(t)$	Probability that the system is in state j at time t; for $j = 0, 1, 2, \ldots, n+2$.
$R(t)$	Reliability of the system in $[0, t]$.
s	Laplace transform variable.

CCF	Common-cause failure.
$MTTF$	System mean time to failure.
AV_{ss}	Steady state system availability.
UV_{ss}	Steady state system unavailability.
$\dot{p}_j(t)$	Derivative of $p_j(t)$ with respect to time t; for $j = 0, 1, 2, \ldots, n+2$.
p_i	Steady state probability, that the system is in state i; for $i = 0, 1, 2, \ldots, n+2$.

4.2.1 Standby System with Type I Repair

The state space diagram for a n non-identical unit standby system is shown in Figure 4.20. The system state numbers are denoted by the numerals plus a letter in the boxes of the figure.

The system of differential equations for the model shown in Figure 4.20 is

$$\dot{p}_0(t) = -(\lambda_1 + \lambda_{c_1})p_0(t) + \mu_1 p_1(t) + \sum_{i=2}^{n+1} \mu_{io} p_i(t) + \mu_{c_1} p_{n+2}(t) \tag{4.148}$$

$$\dot{p}_1(t) = \lambda_1 p_0(t) - (\mu_1 + \lambda_2 + \lambda_{c_2})p_1(t) + \mu_2 p_2(t) + \mu_{c_2} p_{n+2}(t) \tag{4.149}$$

$$\begin{aligned} \dot{p}_r(t) &= \lambda_r p_{r-1}(t) - \{\mu_{ro} + \mu_r + \lambda_{r+1} + \lambda_{c_{r+1}}\}p_r(t) + \\ &\quad (\mu_{r+1})p_{r+1}(t) + (\mu_{c_{r+1}})p_{n+2}(t) \end{aligned} \tag{4.150}$$

for $r = 2, 3, 4, \ldots, n$

$$\dot{p}_{n+1}(t) = \lambda_{n+1} p_n(t) - \{\mu_{n+1} + \mu_{(n+1)o}\}p_{n+1}(t) \tag{4.151}$$

$$\dot{p}_{n+2}(t) = \sum_{i=0}^{n} \lambda_{c_{i+1}} p_i(t) - \sum_{i=1}^{n+1} \mu_{c_i} \{p_{n+2}(t) \tag{4.152}$$

At time $t = 0$, $p_0(0) = 1$ and all other initial condition probabilities are equal to zero.

The repair rate $\mu_{c_{i+1}}$; for $i = 1, 2, 3, \ldots, n$ was set equal to zero for the purpose of generalizing the steady state probability expression, because it was found that with

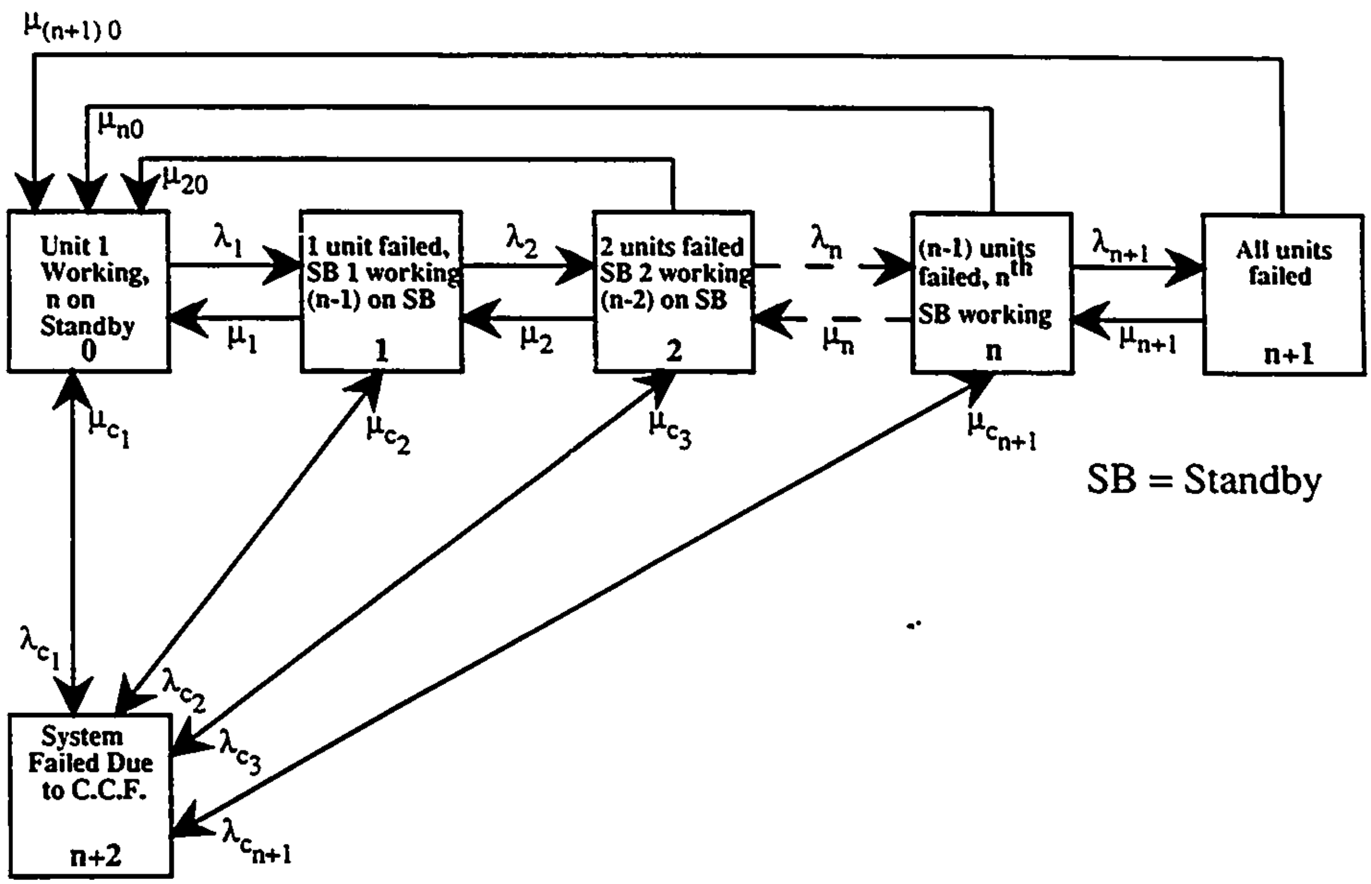

Figure 4.20: State Space Diagram for a Non-identical Unit Standby System

this repair rate the number of terms in the steady state system availability expression become too large. Hence generalization was not possible even though a trend in the expressions could be seen.

The following differential equations are obtained when $\mu_{c_{i+1}}$; for $i = 1, 2, 3, \ldots, n$ is set to zero :

$$\dot{p}_0(t) = -(\lambda_1 + \lambda_{c_1})p_0(t) + \mu_1 p_1(t) + \sum_{i=2}^{n+1} \mu_{io} p_i(t) + \mu_{c_1} p_{n+2}(t) \tag{4.153}$$

$$\dot{p}_1(t) = \lambda_1 p_0(t) - (\mu_1 + \lambda_2 + \lambda_{c_2})p_1(t) + \mu_2 p_2(t) \tag{4.154}$$

$$\dot{p}_r(t) = \lambda_r p_{r-1}(t) - \{\mu_{ro} + \mu_r + \lambda_{r+1} + \lambda_{c_{r+1}}\}p_r(t) + \mu_{r+1} p_{r+1}(t) \tag{4.155}$$

for $r = 2, 3, 4, \ldots, n$

$$\dot{p}_{n+1}(t) = \lambda_{n+1} p_n(t) - \{\mu_{n+1} + \mu_{(n+1)o}\}p_{n+1}(t) \tag{4.156}$$

$$\dot{p}_{n+2}(t) = \sum_{i=0}^{n} \lambda_{c_{i+1}} p_i(t) - \mu_{c_1} p_{n+2}(t) \tag{4.157}$$

At time $t = 0$, $p_0(0) = 1$ and all other initial condition probabilities are equal to zero.

The following steady state probability expression in terms of p_0 is obtained by setting the derivatives of the above equations equal to zero and by making use of the relationship $\sum_{i=0}^{n+2} p_i = 1$:

$$p_r = \frac{\{\prod_{i=1}^{r} \lambda_i A_{n-r}\}\, p_0}{(\mu_1 + \lambda_2 + \lambda_{c_2})A_{n-1} - \mu_2 \lambda_2 A_{n-2}} \tag{4.158}$$

For $r = 1, 2, 3, \ldots, n$
where

$$\begin{aligned} A_0 &= \{\mu_{n+1} + \mu_{(n+1)o}\} \\ A_{i-1} &= 1 \; for \; (i-1) < 0 \\ A_{i+1} &= \{\mu_{(n-i)o} + \mu_{n-i} + \lambda_{(n+1)-i} + \lambda_{c_{(n+1)-i}}\}A_i - (\mu_{(n+1)-i})(\lambda_{(n+1)-i})A_{i-1} \end{aligned}$$

for $i = 0, 1, 2, 3, \ldots, n-2$

$$p_0 = \frac{\mu_1 p_1 + \sum_{i=2}^{n+1} \mu_{c_i} p_i + \mu_{c_1}\{1 - \sum_{i=1}^{n+1} p_i\}}{\lambda_1 + \lambda_{c_1} + \mu_{c_1}} \tag{4.159}$$

The steady state probabilities of a non-identical unit standby system for a given value of n can determined using Equations (4.158) and (4.159). The steady state system availability and unavailability, respectively, are

$$AV_{ss} = \sum_{i=0}^{n} p_i$$
$$UV_{ss} = \sum_{i=n+1}^{n+2} p_i$$

For different values of n the plots of steady state system availability and steady state system unavailability are shown in Figures 4.21 and 4.22, respectively. The repair rate $\mu_{c_{i+1}}$; for $i = 1, 2, 3, \ldots, n$ is also taken into consideration in these plots. It is clear from these figures that the steady state system availability decreases with increasing number of common-cause failures, and conversely that the steady state system unavailability increases with an increase in common-cause failures.

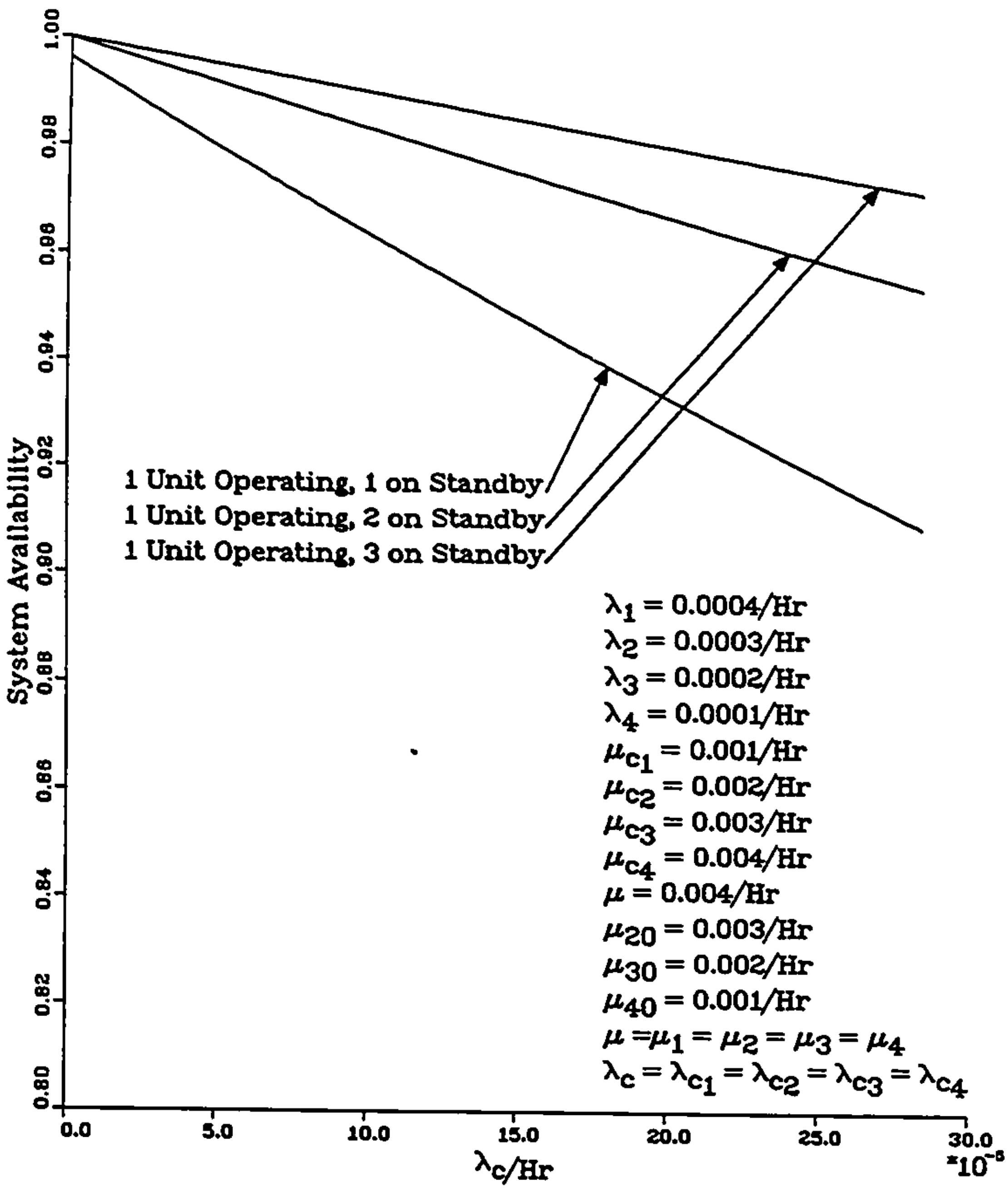

Figure 4.21: Steady State System Availability Plots for a Non-identical Unit Standby System with Type I Repair

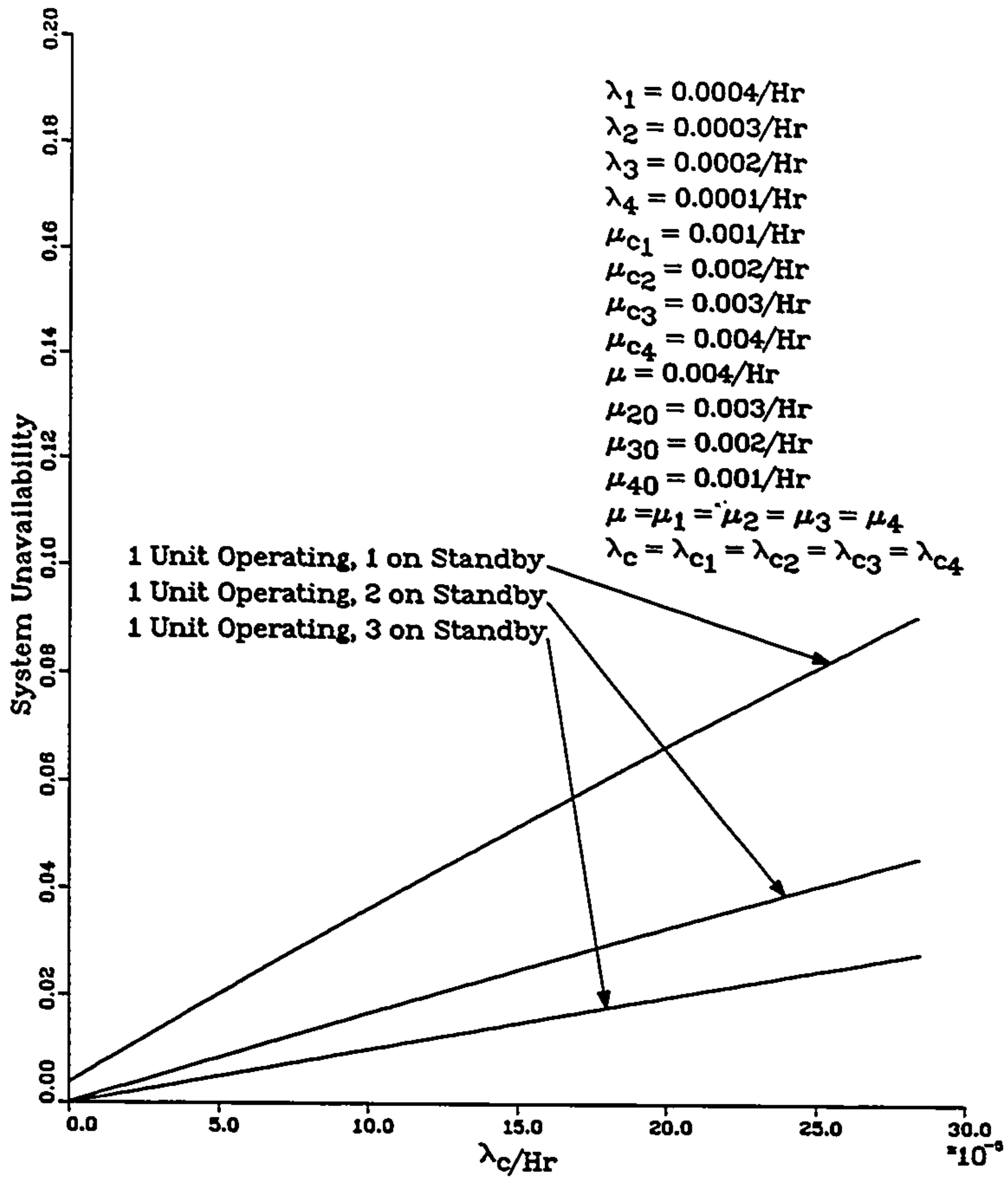

Figure 4.22: Steady State System Unavailability Plots for a Non-identical Unit Standby System with Type I Repair

Special Case Model I

The differential equations for a system with single unit as standby can be obtained by setting $n = 1$ in Equations (4.148) – (4.152).

$$\dot{p}_0(t) = -(\lambda_1 + \lambda_{c_1})p_0(t) + \mu_1 p_1(t) + \mu_{20} p_2(t) + \mu_{c_1} p_3(t) \quad (4.160)$$

$$\dot{p}_1(t) = \lambda_1 p_0(t) - (\mu_1 + \lambda_2 + \lambda_{c_2})p_1(t) + \mu_2 p_2(t) + \mu_{c_2} p_3(t) \quad (4.161)$$

$$\dot{p}_2(t) = \lambda_2 p_1(t) - (\mu_2 + \mu_{20})p_2(t) \quad (4.162)$$

$$\dot{p}_3(t) = \lambda_{c_1} p_0(t) + \lambda_{c_2} p_1(t) - (\mu_{c_1} + \mu_{c_2})p_3(t) \quad (4.163)$$

At time $t = 0$, $p_0(0) = 1$, $p_1(0) = 0$, $p_2(0) = 0$, and $p_3(0) = 0$.

By setting the derivatives of the above equations equal to zero and by using the relationship $\sum_{i=0}^{3} p_i = 1$, results in the following steady state probabilities :

$$p_0 = \frac{F_{40}}{F_{42}} \quad (4.164)$$

$$p_1 = \frac{F_{41}}{F_{42}} \quad (4.165)$$

The constants F_{40}, F_{41} *and* F_{42} are given in Appendix C.

The steady state system availability is given by

$$AV_{ss} = p_0 + p_1 \quad (4.166)$$

The plots of Equation (4.166) are shown in Figure 4.23. The plots clearly show that an increase in the number of common-cause failures decreases the steady state system availability. The plots also indicate that an increase in the repair rate increases the steady state system availability.

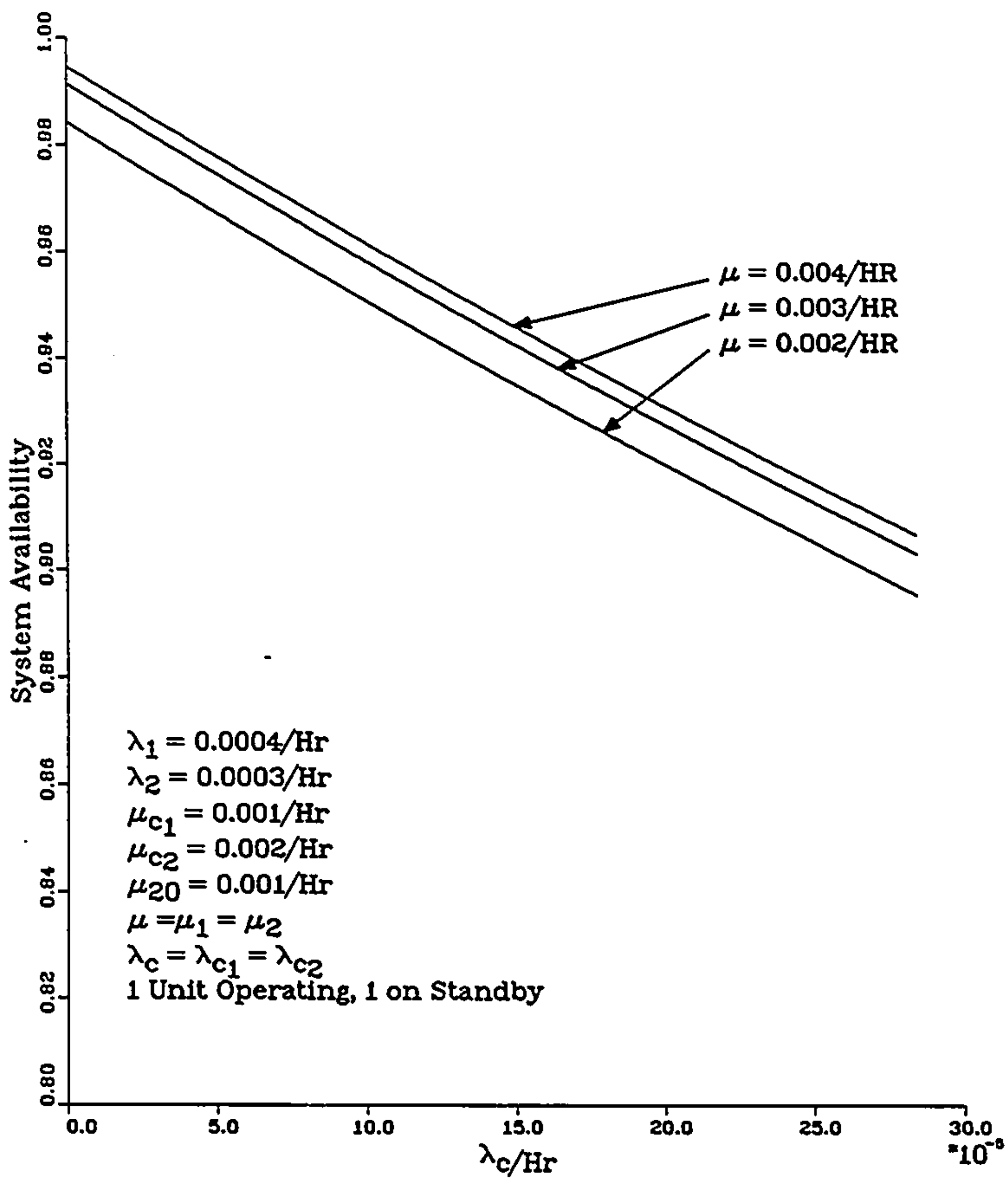

Figure 4.23: Steady State System Availability Plots for a Single Non-identical Unit Standby System with Type I Repair

Special Case Model II

Setting $n = 2$ in Equations (4.148) – (4.152) the following differential equations for a 2 unit standby system are obtained :

$$\begin{aligned}
\dot{p}_0(t) &= -(\lambda_1 + \lambda_{c_1})p_0(t) + \mu_1 p_1(t) + \mu_{20}p_2(t) + \mu_{30}p_3(t) + \mu_{c_1}p_4(t) && (4.167)\\
\dot{p}_1(t) &= \lambda_1 p_0(t) - (\mu_1 + \lambda_2 + \lambda_{c_2})p_1(t) + \mu_2 p_2(t) + \mu_{c_2}p_4(t) && (4.168)\\
\dot{p}_2(t) &= \lambda_2 p_1(t) - (\mu_2 + \mu_{20} + \lambda_3 + \lambda_{c_3})p_2(t) + \mu_3 p_3(t) + \mu_{c_3}p_4(t) && (4.169)\\
\dot{p}_3(t) &= \lambda_3 p_2(t) - (\mu_3 + \mu_{30})p_3(t) && (4.170)\\
\dot{p}_4(t) &= \lambda_{c_1}p_0(t) + \lambda_{c_2}p_1(t) + \lambda_{c_3}p_3(t) - (\mu_{c_1} + \mu_{c_2} + \mu_{c_3})p_4(t) && (4.171)
\end{aligned}$$

At time $t = 0$, $p_0(0) = 1$ and all other initial condition probabilities are equal to zero.

Setting the derivatives of the above equations equal to zero and using the relationship $\sum_{i=0}^{4} p_i = 1$, results in the following steady state probabilities :

$$\begin{aligned}
p_0 &= \frac{F_{43}}{F_{48}} && (4.172)\\
p_1 &= \frac{F_{44}}{F_{48}} && (4.173)\\
p_2 &= \frac{F_{45}}{F_{48}} && (4.174)\\
p_3 &= \frac{F_{46}}{F_{48}} && (4.175)\\
p_4 &= \frac{F_{47}}{F_{48}} && (4.176)
\end{aligned}$$

The constants $F_{43}, F_{44}, F_{45}, F_{46}, F_{47},\ and\ F_{48}$ are given in Appendix C.

The steady state system availability and steady state system unavailability expressions are given by

$$\begin{aligned}
AV_{ss} &= p_0 + p_1 + p_2 && (4.177)\\
UV_{ss} &= p_3 + p_4 && (4.178)
\end{aligned}$$

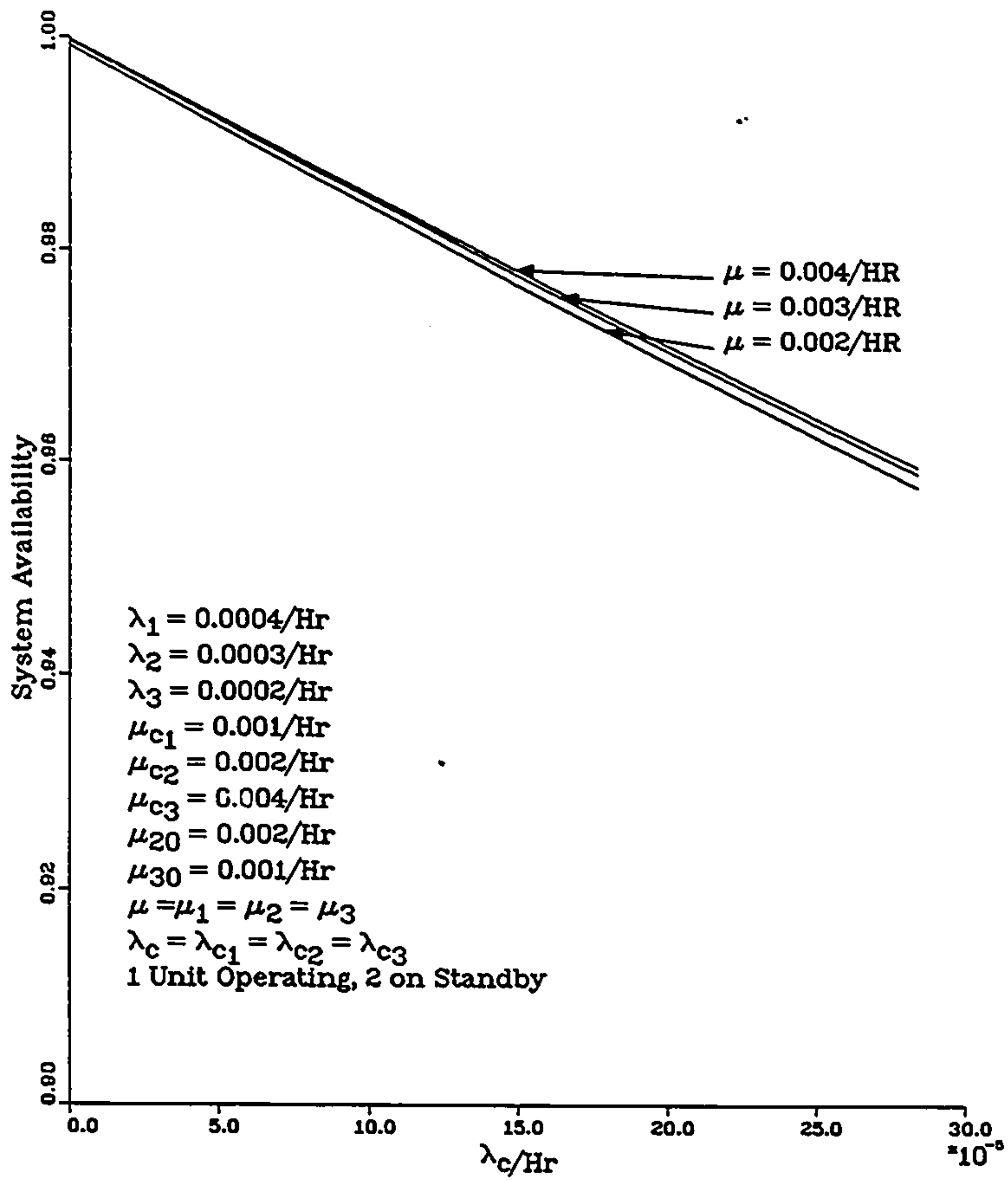

Figure 4.24: Steady State System Availability Plots for a Two Non-identical Unit Standby System with Type I Repair

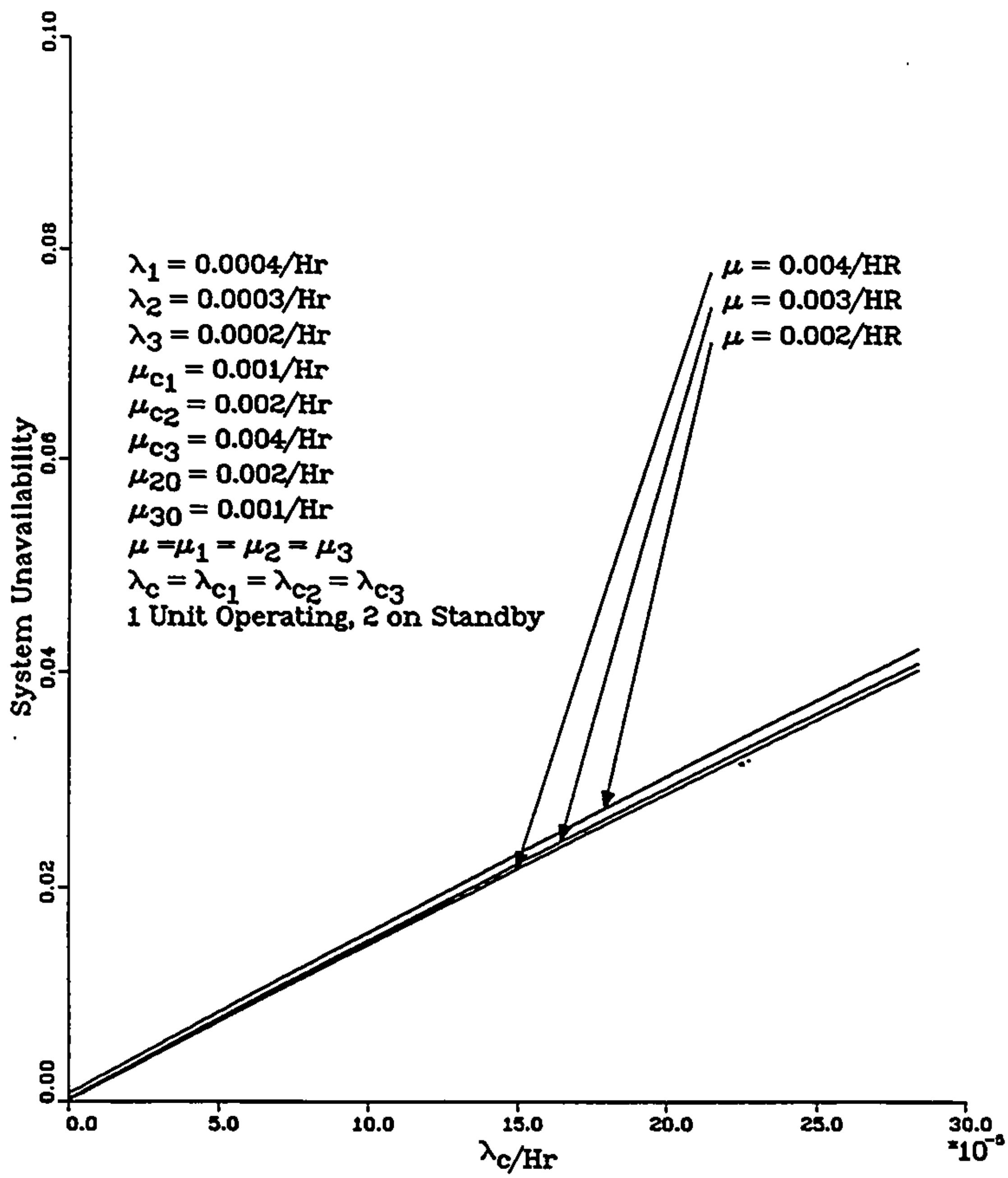

Figure 4.25: Steady State System Unavailability Plots for a Two Non-identical Unit Standby System with Type I Repair

Figures 4.24 and 4.25 show the plots of steady state system availability and steady state system unavailability, respectively, for varying repair rates. The plots indicate an increase in steady state system availability with increase in repair rate. In addition, the plots also show that the steady state system availability decreases with increase in common-cause failures.

4.2.2 Standby System With Type II Repair

Setting the repair rates $\mu_{c_{i+1}}$; for $i = 0, 1, 2, \ldots, n$, $\mu_{(n+1)o}$ and μ_{n+1} equal to zero, in Figure 4.20, results in a n non-identical unit standby system with Type II Repair. The differential equations for such a system can be represented as follows :

$$\dot{p}_0(t) = -(\lambda_1 + \lambda_{c_1})p_0(t) + \mu_1 p_1(t) + \sum_{i=2}^{n} \mu_{io} p_i(t) \tag{4.179}$$

$$\dot{p}_1(t) = \lambda_1 p_0(t) - (\mu_1 + \lambda_2 + \lambda_{c_2})p_1(t) + \mu_2 p_2(t) \tag{4.180}$$

$$\dot{p}_{k-1}(t) = \lambda_{k-1} p_{k-2}(t) - \{\mu_{(k-1)o} + \mu_{k-1} + \lambda_k + \lambda_{c_k}\} p_{k-1}(t) + \mu_k p_k(t) \tag{4.181}$$

for $k = 3, 4, 5, \ldots, n$

$$\dot{p}_n(t) = \lambda_n p_{n-1}(t) - \{\mu_{no} + \mu_n + \lambda_{n+1} + \lambda_{c_{n+1}}\} p_n(t) \tag{4.182}$$

$$\dot{p}_{n+1}(t) = \lambda_{n+1} p_n(t) \tag{4.183}$$

$$\dot{p}_{n+2}(t) = \sum_{i=0}^{n} \lambda_{c_{i+1}} p_i(t) \tag{4.184}$$

The initial conditions are, at time $t = 0$, $p_0(0) = 1$ and all other probabilities are equal to zero.

The state probability expressions can be obtained by solving Equations (4.179) – (4.184) with the aid of Laplace transforms. Listed below are the Laplace transforms of the state probabilities.

$$p_0(s) = \frac{1 + \mu_1 p_1(s) + \sum_{i=2}^{n} \mu_{io} p_i(s)}{s + \lambda_1 + \lambda_{c_1}} \tag{4.185}$$

$$p_1(s) = \frac{\lambda_1 p_0(s) + \mu_2 p_2(s)}{s + \mu_1 + \lambda_2 + \lambda_{c_2}} \tag{4.186}$$

$$p_{r-1}(s) = \frac{\lambda_{r-1} p_{r-2}(s) + \mu_r p_r(s)}{s + \mu_{(r-1)o} + \mu_{r-1} + \lambda_r + \lambda_{c_r}} \tag{4.187}$$

for $r = 3, 4, 5, \ldots, n$

$$p_n(s) = \frac{\lambda_n p_{n-1}(s)}{s + \mu_{no} + \mu_n + \lambda_{n+1} + \lambda_{c_{n+1}}} \tag{4.188}$$

$$p_{n+1}(s) = \frac{\lambda_{n+1} p_n(s)}{s} \tag{4.189}$$

$$p_{n+2}(s) = \frac{\sum_{i=0}^{n} \lambda_{c_{i+1}} p_i(s)}{s} \tag{4.190}$$

The standby system (with Type II Repair) mean time to failure is given by

$$MTTF = \lim_{s \to 0} [p_0(s) + p_1(s) + \ldots + p_{n-1}(s) + p_n(s)] \tag{4.191}$$

For 0 and 1 unit standby system, the mean time to failure, respectively, are

$$MTTF_0 = \frac{1}{\lambda_1 + \lambda_{c_1}}$$

and

$$MTTF_1 = \frac{\mu_1 + \lambda_2 + \lambda_{c_2} + \lambda_1}{(\lambda_2 \lambda_{c_1} + \mu_1 \lambda_{c1} + \lambda_{c_2} \lambda_{c_1} + \lambda_2 \lambda_1 + \lambda_{c_2} \lambda_1)}$$

for n number of standbys, the general formula for system mean time to failure is

$$MTTF_n = \frac{A_{n-1}\{F_{49}\} - \mu_n \lambda_n \{F_{50}\} + \prod_{i=1}^{n} \lambda_i}{A_{n-1}\{F_{51}\} - \mu_n \lambda_n \{F_{52}\} - \mu_{no} \prod_{i=1}^{n} \lambda_i} \tag{4.192}$$

where

$$\begin{aligned}
A_i &= \{\mu_{i+1} + \mu_{(i+1)o} + \lambda_{i+2} + \lambda_{c_{i+2}}\} \\
F_{49} &= Numerator\ of\ MTTF_{n-1} \\
F_{50} &= Numerator\ of\ MTTF_{n-2} \\
F_{51} &= Denominator\ of\ MTTF_{n-1} \\
F_{52} &= Denominator\ of\ MTTF_{n-2}
\end{aligned}$$

Figure 4.26 shows a number of plots of Equation (4.192) for specified values of model parameters. The plots clearly show that the system mean time to failure decreases with an increase in the number of common-cause failures.

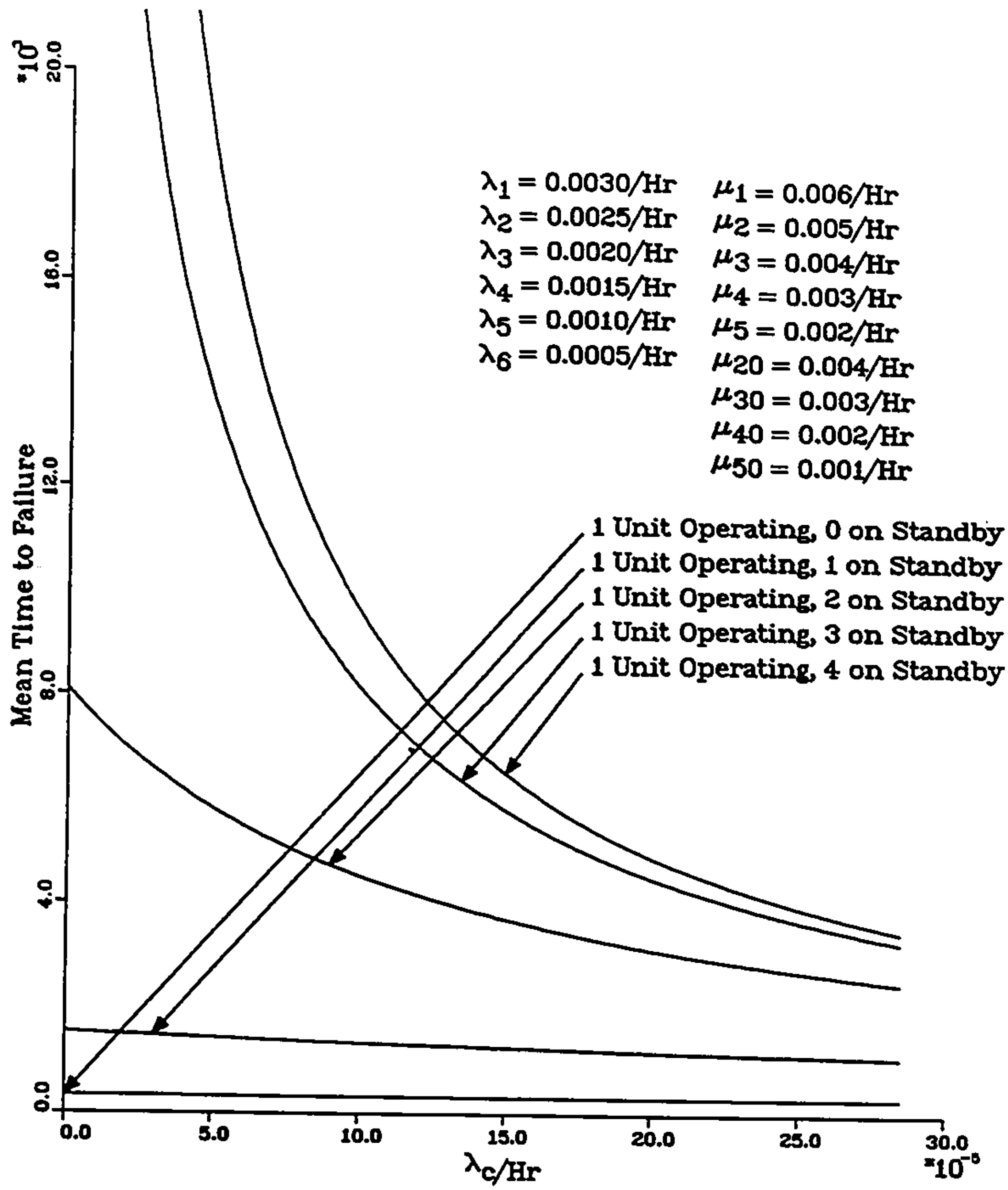

Figure 4.26: Typical System Mean Time to Failure Plots for a Non-identical Unit Standby System with Type II Repair

Special Case Model I

The system of differential equations for a single unit standby system can be obtained by setting $n = 1$ in Equations (4.179) – (4.184). These are as follows :

$$\dot{p}_0(t) = -(\lambda_1 + \lambda_{c_1})p_0(t) + \mu_1 p_1(t) \tag{4.193}$$

$$\dot{p}_1(t) = \lambda_1 p_0(t) - (\mu_1 + \lambda_2 + \lambda_{c_2})p_1(t) \tag{4.194}$$

$$\dot{p}_2(t) = \lambda_2 p_1(t) \tag{4.195}$$

$$\dot{p}_3(t) = \lambda_{c_1} p_0(t) + \lambda_{c_2} p_1(t) \tag{4.196}$$

At time $t = 0$, $p_0(0) = 1$, $p_1(0) = 0$, $p_2(0) = 0$, *and* $p_3(0) = 0$. Solving the above equations with the aid of Laplace transforms the following time dependent state probabilities are obtained :

$$p_0(t) = \frac{\frac{1}{2}(\mu_1 + \lambda_2 - \lambda_1 - \lambda_{c_1} + \lambda_{c_2} + F_{53})e^{\{-\frac{1}{2}[F_{53}+\lambda_2+\lambda_{c_2}+\mu_1+\lambda_1+\lambda_{c_1}]+F_{53}\}t}}{F_{53}} - \frac{\frac{1}{2}(\mu_1 + \lambda_2 - \lambda_1 - \lambda_{c_1} + \lambda_{c_2} - F_{53})e^{-\frac{1}{2}[F_{53}+\lambda_2+\lambda_{c_2}+\mu_1+\lambda_1+\lambda_{c_1}]t}}{F_{53}} \tag{4.197}$$

$$p_1(t) = \frac{\lambda_1[e^{F_{53}t} - 1]e^{-\frac{1}{2}[\lambda_1+\lambda_{c_1}+\lambda_2+\mu_1+\lambda_{c_2}+F_{53}]t}}{F_{53}} \tag{4.198}$$

where the constant F_{53} is defined in Appendix C.

The system reliability is given by

$$R(t) = p_0(t) + p_1(t) \tag{4.199}$$

The plots of the Equation (4.199) are shown in Figure 4.27. From the plots it is evident that the system reliability decreases with an increase in the number of common-cause failures.

The system mean time to failure of a single unit standby system is given by

$$MTTF_1 = \frac{\mu_1 + \lambda_2 + \lambda_{c_2} + \lambda_1}{(\lambda_2\lambda_{c_1} + \mu_1\lambda_{c1} + \lambda_{c_2}\lambda_{c_1} + \lambda_2\lambda_1 + \lambda_{c_2}\lambda_1)} \tag{4.200}$$

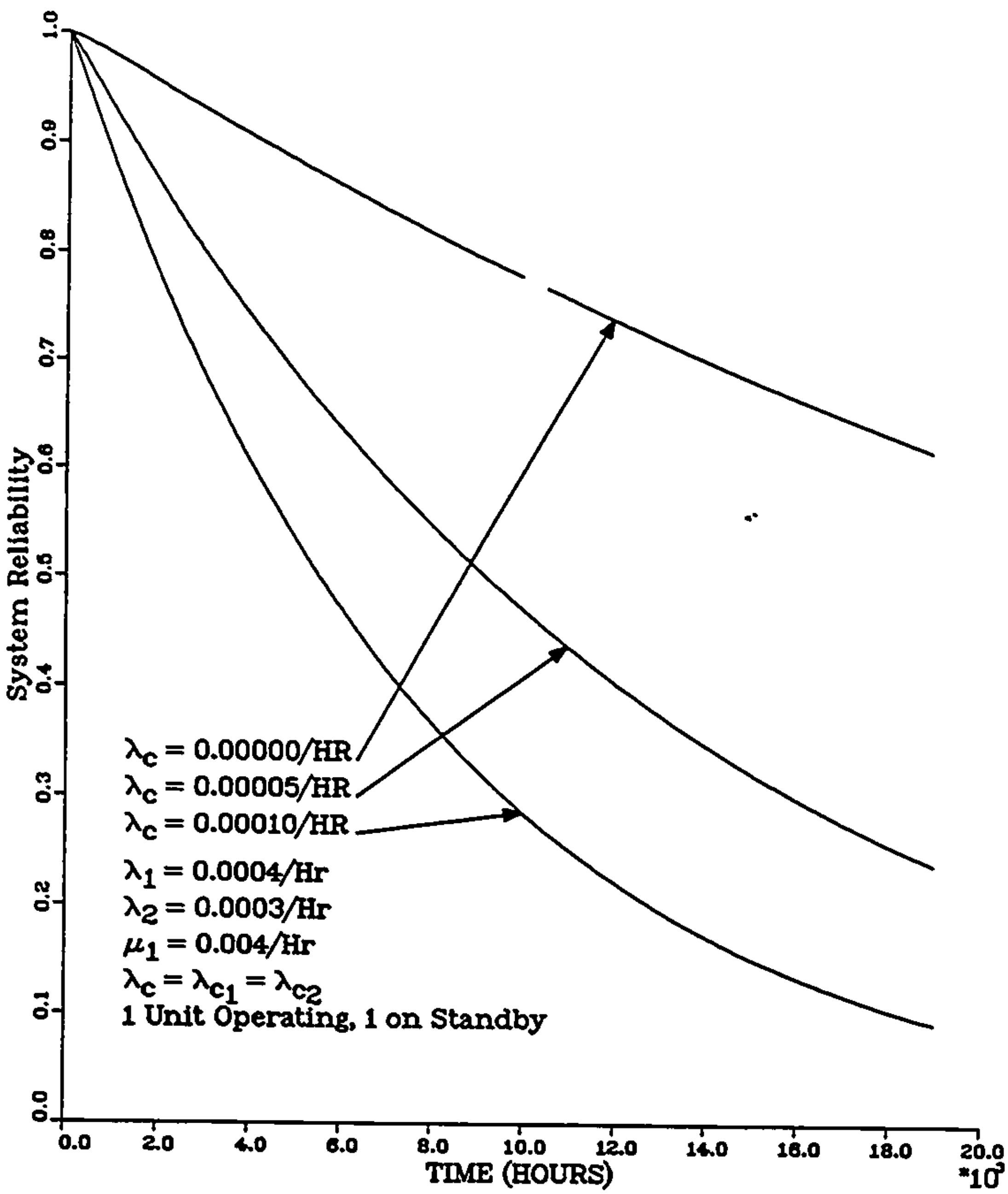

Figure 4.27: System Reliability Plots for a Single Non-identical Unit Standby System with Type II Repair

Special Case Model II

Setting $n = 2$ in Equations (4.179) – (4.184), the following system of differential equations is obtained :

$$\dot{p}_0(t) = -(\lambda_1 + \lambda_{c_1})p_0(t) + \mu_1 p_1(t) + \mu_{20} p_2(t) \tag{4.201}$$

$$\dot{p}_1(t) = \lambda_1 p_0(t) - (\mu_1 + \lambda_2 + \lambda_{c_2})p_1(t) + \mu_2 p_2(t) \tag{4.202}$$

$$\dot{p}_2(t) = \lambda_2 p_1(t) - (\mu_2 + \mu_{20} + \lambda_3 + \lambda_{c_3})p_2(t) \tag{4.203}$$

$$\dot{p}_3(t) = \lambda_3 p_2(t) \tag{4.204}$$

$$\dot{p}_4(t) = \lambda_{c_1} p_0(t) + \lambda_{c_2} p_1(t) + \lambda_{c_3} p_2(t) \tag{4.205}$$

At time $t = 0$, $p_0(0) = 1$, $p_1(0) = 0$, $p_2(0) = 0$, $p_3(0) = 0$ *and* $p_4(0) = 0$.

Solving Equations (4.201) – (4.205) using Laplace transforms yields the following state probability expressions :

$$p_0(t) = \frac{(s_1^2 + F_{54}s_1 + F_{55})e^{s_1 t}}{(s_1 - s_2)(s_1 - s_3)} + \frac{(s_2^2 + F_{54}s_2 + F_{55})e^{s_2 t}}{(s_3 - s_2)(s_1 - s_2)} - \frac{(s_3^2 + F_{54}s_3 + F_{55})e^{s_3 t}}{(s_3 - s_2)(s_1 - s_3)} \tag{4.206}$$

$$p_1(t) = \frac{\lambda_1(s_1 + \mu_2 + \lambda_{c_3} + \lambda_3 + \mu_{20})e^{s_1 t}}{(s_3 - s_1)(s_2 - s_1)} + \frac{\lambda_1(s_2 + \mu_2 + \lambda_{c_3} + \lambda_3 + \mu_{20})e^{s_2 t}}{(s_2 - s_1)(s_2 - s_3)} + \frac{\lambda_1(s_3 + \mu_2 + \lambda_{c_3} + \lambda_3 + \mu_{20})e^{s_3 t}}{(s_3 - s_1)(s_2 - s_3)} \tag{4.207}$$

$$p_2(t) = \frac{\lambda_1\lambda_2 e^{(s_1 t)}}{(s_2 - s_1)(s_3 - s_1)} + \frac{\lambda_1\lambda_2 e^{(s_2 t)}}{(s_1 - s_2)(s_3 - s_2)} + \frac{\lambda_1\lambda_2 e^{(s_3 t)}}{(s_3 - s_2)(s_3 - s_1)} \tag{4.208}$$

where s_1, s_2, *and* s_3 are the roots of the cubic equation and determined as follows :

$$s^3 + F_{56}s^2 + F_{57}s + F_{58} = 0$$

$$\begin{aligned}
Let\ \alpha &= \frac{3F_{57} - (F_{56})^2}{9} \\
\beta &= \frac{9F_{56}F_{57} - 27F_{58} - 2(F_{56})^3}{54} \\
\Phi &= \sqrt[3]{\beta + \sqrt{\alpha^3 + \beta^2}} \\
\Omega &= \sqrt[3]{\beta - \sqrt{\alpha^3 + \beta^2}} \\
Thus,\ s_1 &= \Phi + \Omega - \frac{1}{3}F_{56} \\
s_2 &= -\frac{1}{2}(\Phi + \Omega) - \frac{1}{3}F_{56} + \frac{1}{2}i\sqrt{3}(\Phi - \Omega) \\
s_3 &= -\frac{1}{2}(\Phi + \Omega) - \frac{1}{3}F_{56} - \frac{1}{2}i\sqrt{3}(\Phi - \Omega)
\end{aligned}$$

The above i is associated with complex numbers.

$$\begin{aligned}
-F_{56} &= s_1 + s_2 + s_3 \\
F_{57} &= s_1 s_2 + s_2 s_3 + s_3 s_1 \\
-F_{58} &= s_1 s_2 s_3
\end{aligned}$$

and the constants $F_{54}, F_{55}, F_{56}, F_{57}$ *and* F_{58} are given in Appendix C.

The system reliability is given by

$$R(t) = p_0(t) + p_1(t) + p_2(t) \tag{4.209}$$

The plots of the Equation (4.209) are shown in Figure 4.28. From these plots it is evident that the system reliability decreases with an increase in the number of common-cause failures.

From Equation (4.192) the system mean time to failure for a two non-identical unit standby system can be expressed as

$$MTTF_2 = \frac{A_1\{\mu_1 + \lambda_2 + \lambda_{c_2} + \lambda_1\} - \mu_2\lambda_2\{1\} + \lambda_1\lambda_2}{[A_1\{\lambda_2\lambda_{c_1} + \mu_1\lambda_{c_1} + \lambda_{c_2}\lambda_{c_1} + \lambda_2\lambda_1 + \lambda_{c_2}\lambda_1\} - \mu_2\lambda_2\{\lambda_1 + \lambda_{c_1}\} - \mu_{20}\lambda_1\lambda_2]} \tag{4.210}$$

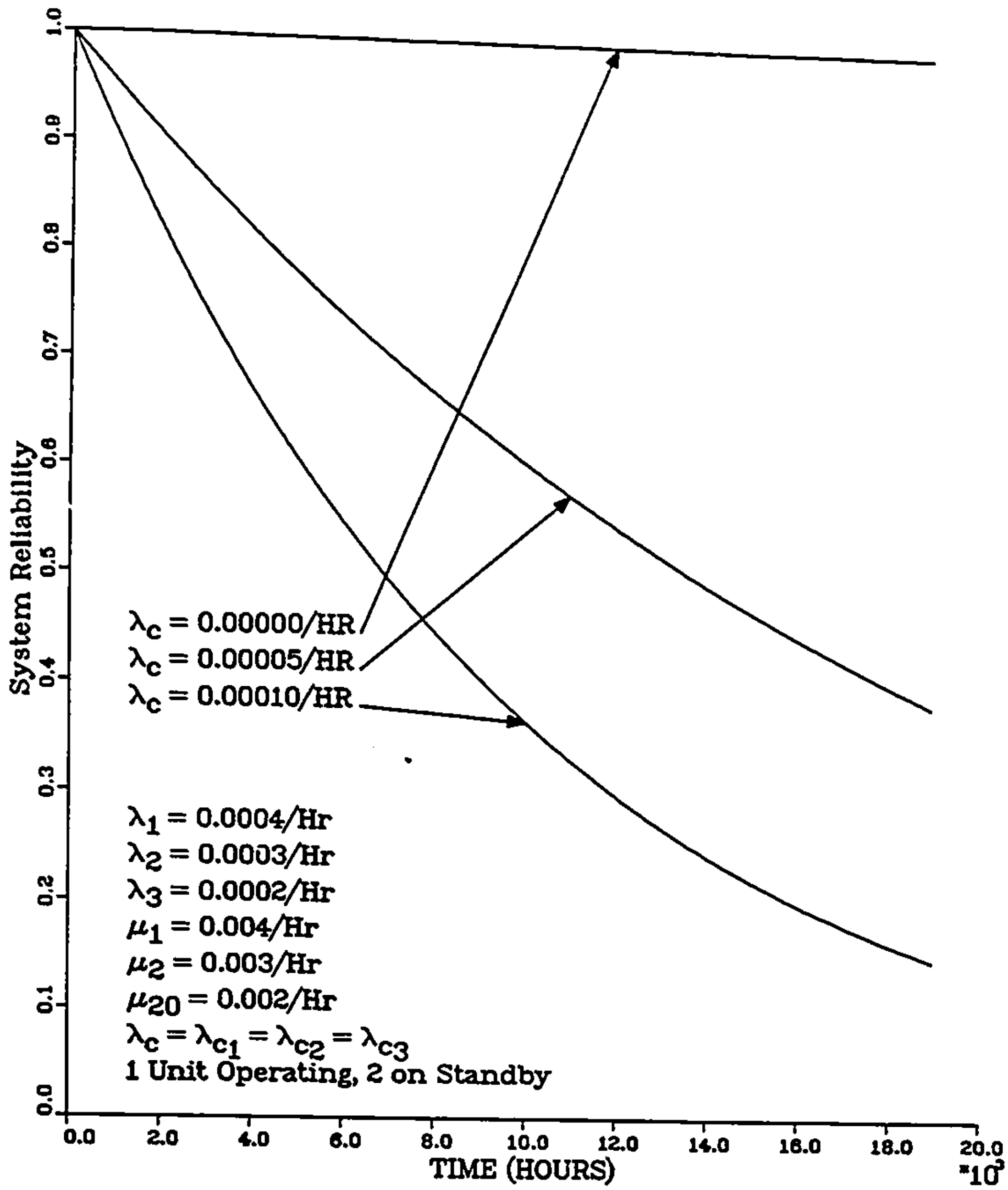

Figure 4.28: System Reliability Plots for a Two Non-identical Unit Standby System with Type II Repair

Special Case Model III

If we set $n = 3$ in Equations (4.170) – (4.184), the following system of differential equations for a three unit standby system is obtained :

$$\dot{p}_0(t) = -(\lambda_1 + \lambda_{c_1})p_0(t) + \mu_1 p_1(t) + \mu_{20}p_2(t) + \mu_{30}p_3(t) \quad (4.211)$$

$$\dot{p}_1(t) = \lambda_1 p_0(t) - (\mu_1 + \lambda_2 + \lambda_{c_2})p_1(t) + \mu_2 p_2(t) \quad (4.212)$$

$$\dot{p}_2(t) = \lambda_2 p_1(t) - (\mu_2 + \mu_{20} + \lambda_3 + \lambda_{c_3})p_2(t) + \mu_3 p_3(t) \quad (4.213)$$

$$\dot{p}_3(t) = \lambda_3 p_2(t) - (\mu_3 + \mu_{30} + \lambda_4 + \lambda_{c_4})p_3(t) \quad (4.214)$$

$$\dot{p}_4(t) = \lambda_4 p_3(t) \quad (4.215)$$

$$\dot{p}_5(t) = \lambda_{c_1}p_0(t) + \lambda_{c_2}p_1(t) + \lambda_{c_3}p_2(t) + \lambda_{c_4}p_3(t) \quad (4.216)$$

At time $t = 0$, $p_0(0) = 1$, $p_1(0) = 0$, $p_2(0) = 0$, $p_3(0) = 0$, $p_4(0)$ *and* $p_5(0) = 0$, are the initial conditions.

Solving Equations (4.211) – (4.216) using Laplace transforms yields the following state probability expressions :

$$p_0(t) = \frac{[s_1^3 + F_{59}s_1^2 + F_{60}s_1 + F_{61}]e^{s_1 t}}{(s_1 - s_2)(s_1 - s_3)(s_1 - s_4)} + \frac{[s_2^3 + F_{59}s_2^2 + F_{60}s_2 + F_{61}]e^{s_2 t}}{(s_3 - s_2)(s_4 - s_2)(s_1 - s_2)} + \frac{[s_3^3 + F_{59}s_3^2 + F_{60}s_3 + F_{61}]e^{s_3 t}}{(s_3 - s_2)(s_4 - s_3)(s_1 - s_3)} + \frac{[s_4^3 + F_{59}s_4^2 + F_{60}s_4 + F_{61}]e^{s_4 t}}{(s_2 - s_4)(s_4 - s_3)(s_1 - s_4)} \quad (4.217)$$

$$p_1(t) = \frac{[\lambda_1 s_1^2 + F_{66}s_1 + F_{67}]e^{s_1 t}}{(s_1 - s_4)(s_3 - s_1)(s_2 - s_1)} + \frac{[\lambda_1 s_2^2 + F_{66}s_2 + F_{67}]e^{s_2 t}}{(s_2 - s_1)(s_2 - s_4)(s_2 - s_3)} + \frac{[\lambda_1 s_3^2 + F_{66}s_3 + F_{67}]e^{s_3 t}}{(s_1 - s_3)(s_3 - s_4)(s_2 - s_3)} + \frac{[\lambda_1 s_4^2 + F_{66}s_4 + F_{67}]e^{s_4 t}}{(s_4 - s_1)(s_3 - s_4)(s_2 - s_4)} \quad (4.218)$$

$$p_2(t) = \frac{\lambda_1\lambda_2[s_1 + \mu_3 + \mu_{30} + \lambda_4 + \lambda_{c_4}]e^{s_1 t}}{(s_3 - s_1)(s_1 - s_4)(s_2 - s_1)} + \frac{\lambda_1\lambda_2[s_2 + \mu_3 + \mu_{30} + \lambda_4 + \lambda_{c_4}]e^{s_2 t}}{(s_2 - s_3)(s_2 - s_4)(s_2 - s_1)} + \frac{\lambda_1\lambda_2[s_3 + \mu_3 + \mu_{30} + \lambda_4 + \lambda_{c_4}]e^{s_3 t}}{(s_3 - s_4)(s_1 - s_3)(s_2 - s_3)} + \frac{\lambda_1\lambda_2[s_4 + \mu_3 + \mu_{30} + \lambda_4 + \lambda_{c_4}]e^{s_4 t}}{(s_4 - s_3)(s_1 - s_4)(s_2 - s_4)} \tag{4.219}$$

$$p_3(t) = \frac{\lambda_3\lambda_2\lambda_1 e^{s_1 t}}{(s_1 - s_4)(s_3 - s_1)(s_2 - s_1)} + \frac{\lambda_3\lambda_2\lambda_1 e^{s_2 t}}{(s_2 - s_1)(s_2 - s_4)(s_2 - s_3)} + \frac{\lambda_3\lambda_2\lambda_1 e^{s_3 t}}{(s_1 - s_3)(s_3 - s_4)(s_2 - s_3)} + \frac{\lambda_3\lambda_2\lambda_1 e^{s_4 t}}{(s_4 - s_1)(s_3 - s_4)(s_2 - s_4)} \tag{4.220}$$

where s_1, s_2, s_3 *and* s_4 are the roots of the quartic equation and can be determined as explained below.

$$s^4 + F_{62}s^3 + F_{63}s^2 + F_{64}s + F_{65} = 0$$

Let y_1 be a real root of the cubic equation

$$y^3 - F_{63}y^2 + (F_{62}F_{64} - 4F_{65})y + (4F_{63}F_{65} - F_{64}^2 - F_{62}^2F_{65}) = 0$$

s_1, s_2, s_3 *and* s_4 are the roots of

$$z^2 + \frac{1}{2}\{F_{62} \pm \sqrt{F_{62}^2 - 4F_{63} + 4y_1}\}z + \frac{1}{2}\{y_1 \pm \sqrt{y_1^2 - 4F_{65}}\} = 0$$

$$-F_{62} = s_1 + s_2 + s_3 + s_4$$

$$F_{63} = s_1s_2 + s_2s_3 + s_3s_4 + s_4s_1 + s_1s_3 + s_2s_4$$

$$-F_{64} = s_1s_2s_3 + s_2s_3s_4 + s_1s_2s_4 + s_1s_3s_4$$

$$F_{65} = s_1s_2s_3s_4$$

where the constants F_{59}, F_{60}, F_{61}, F_{62}, F_{62}, F_{63}, F_{64}, F_{65}, F_{66}, *and* F_{67} are defined in the Appendix C.

The system reliability is given by

$$R(t) = p_0(t) + p_1(t) + p_2(t) + p_3(t) \tag{4.221}$$

The plots of the Equation (4.221) are shown in Figure 4.29. From these plots it is evident that the system reliability decreases with an increase in the number of common-cause failures.

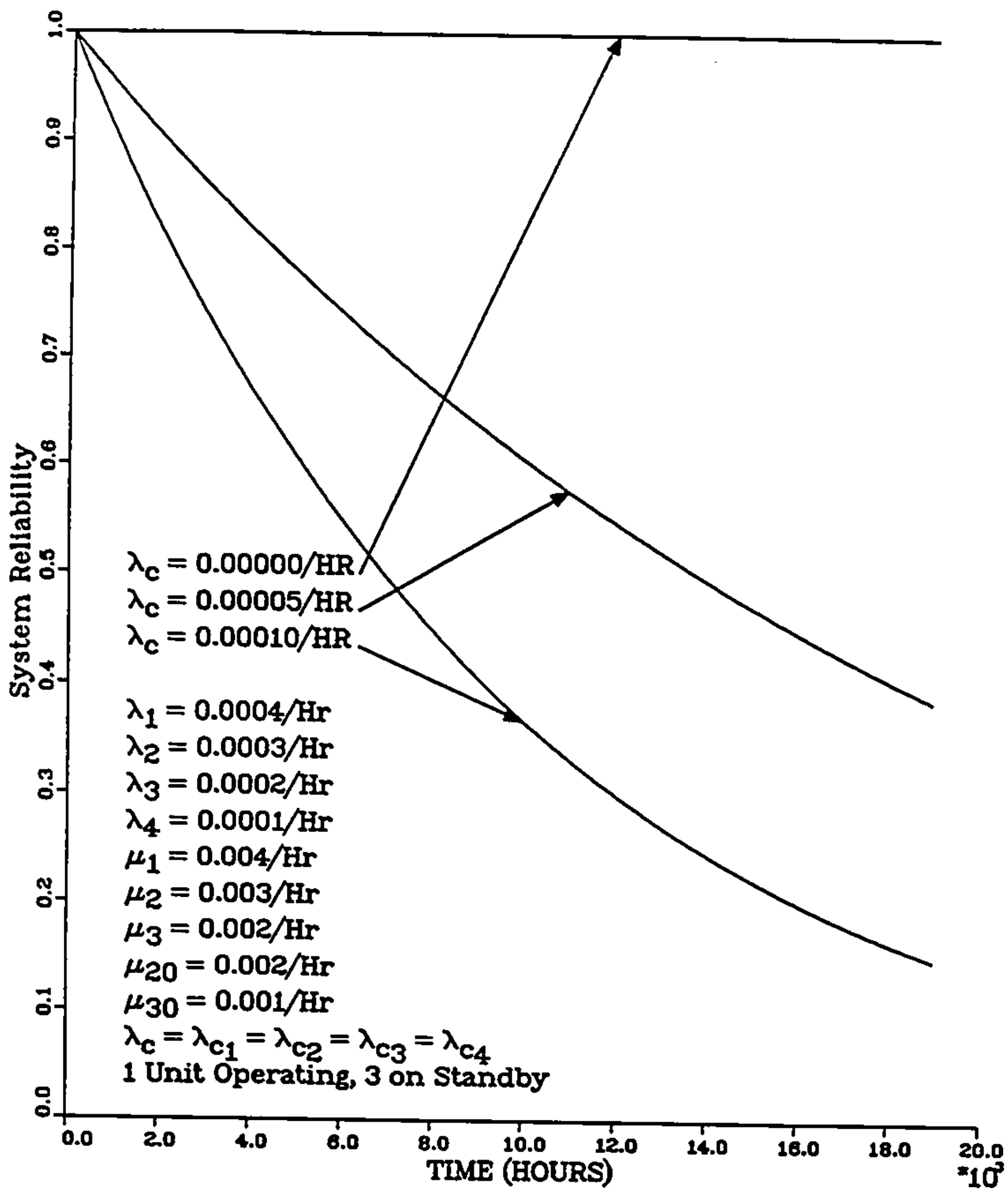

Figure 4.29: System Reliability Plots for a Three Non-identical Unit Standby System with Type II Repair

4.2.3 Standby System Without Repair

Setting the repair rates μ_i; for $i = 1, 2, 3, \ldots, n+1$, $\mu_{i\omega}$; for $i = 2, 3, 4, \ldots, n+1$ and $\mu_{c_{i+1}}$; for $i = 0, 1, 2, \ldots, n$ in Figure 4.20, results in the following system of differential equations :

$$\dot{p}_0(t) = -(\lambda_1 + \lambda_{c_1})p_0(t) \tag{4.222}$$

$$\dot{p}_i(t) = \lambda_i p_{i-1}(t) - (\lambda_{i+1} + \lambda_{c_{i+1}})p_i(t) \tag{4.223}$$

for $i = 1, 2, 3, \ldots, n$

$$\dot{p}_{n+1}(t) = \lambda_{n+1} p_n(t) \tag{4.224}$$

$$\dot{p}_{n+2}(t) = \sum_{i=0}^{n} \lambda_{c_{i+1}} p_i(t) \tag{4.225}$$

At time $t = 0$, $p_0(0) = 1$. All other initial condition probabilities are equal to zero.

By solving Equations (4.222) – (4.225) with the aid of Laplace transforms the following Laplace transforms of the state probabilities are obtained :

$$p_0(s) = \frac{1}{s + \lambda_1 + \lambda_{c_1}} \tag{4.226}$$

$$p_i(s) = \frac{\prod_{j=1}^{i} \lambda_j}{\prod_{j=1}^{i+1}(s + \lambda_j + \lambda_{c_j})} \tag{4.227}$$

for $i = 1, 2, 3, \ldots, n$

Taking inverse Laplace transforms of the above expressions, we get the following time dependent state probability expressions :

$$p_0(t) = e^{-(\lambda_1 + \lambda_{c_1})t} \tag{4.228}$$

$$p_1(t) = \lambda_1 e^{-\frac{1}{2}(\lambda_2 + \lambda_{c_2} + \lambda_{c_1} + \lambda_1)t} \times \frac{\{e^{\frac{1}{2}(\lambda_2 + \lambda_{c_2} - \lambda_{c_1} - \lambda_1)t} - e^{-\frac{1}{2}(\lambda_2 + \lambda_{c_2} - \lambda_{c_1} - \lambda_1)t}\}}{(\lambda_2 + \lambda_{c_2} - \lambda_{c_1} - \lambda_1)} \tag{4.229}$$

$$p_r(t) = \sum_{k=1}^{r+1} \frac{\prod_{j=1}^{r} \lambda_j e^{-(\lambda_k+\lambda_{c_k})t}}{\prod_{i=1,\neq k}^{r+1}(\lambda_i + \lambda_{c_i} - \lambda_k - \lambda_{c_k})} \tag{4.230}$$

for $r = 2, 3, 4, \ldots, n$

The system reliability is given by

$$R(t) \;=\; p_o(t) + p_1(t) + \sum_{i=2}^{n} p_i(t) \tag{4.231}$$

The system reliability plots of Equation (4.231) are given in Figure 4.30 for specified values of model parameters. It is evident from the plots that an increase in the number of common-cause failures decreases the system reliability.

The system mean time to failure ($MTTF$) is given by

$$\begin{aligned} MTTF &= \lim_{s \to 0} R(s) \\ &= \lim_{s \to 0} \{p_0(s) + \sum_{i=1}^{n} p_i(s)\} \\ &= \left\{ \frac{1}{(\lambda_1 + \lambda_{c_1})} + \sum_{j=1}^{n} \frac{\prod_{i=1}^{j} \lambda_i}{\prod_{i=1}^{j+1}(\lambda_i + \lambda_{c_i})} \right\} \end{aligned} \tag{4.232}$$

The plots of the system mean time to failure for specified values of the model parameters are given in Figure 4.31. The plots indicate that an increase in common-cause failures decreases the system mean time to failure.

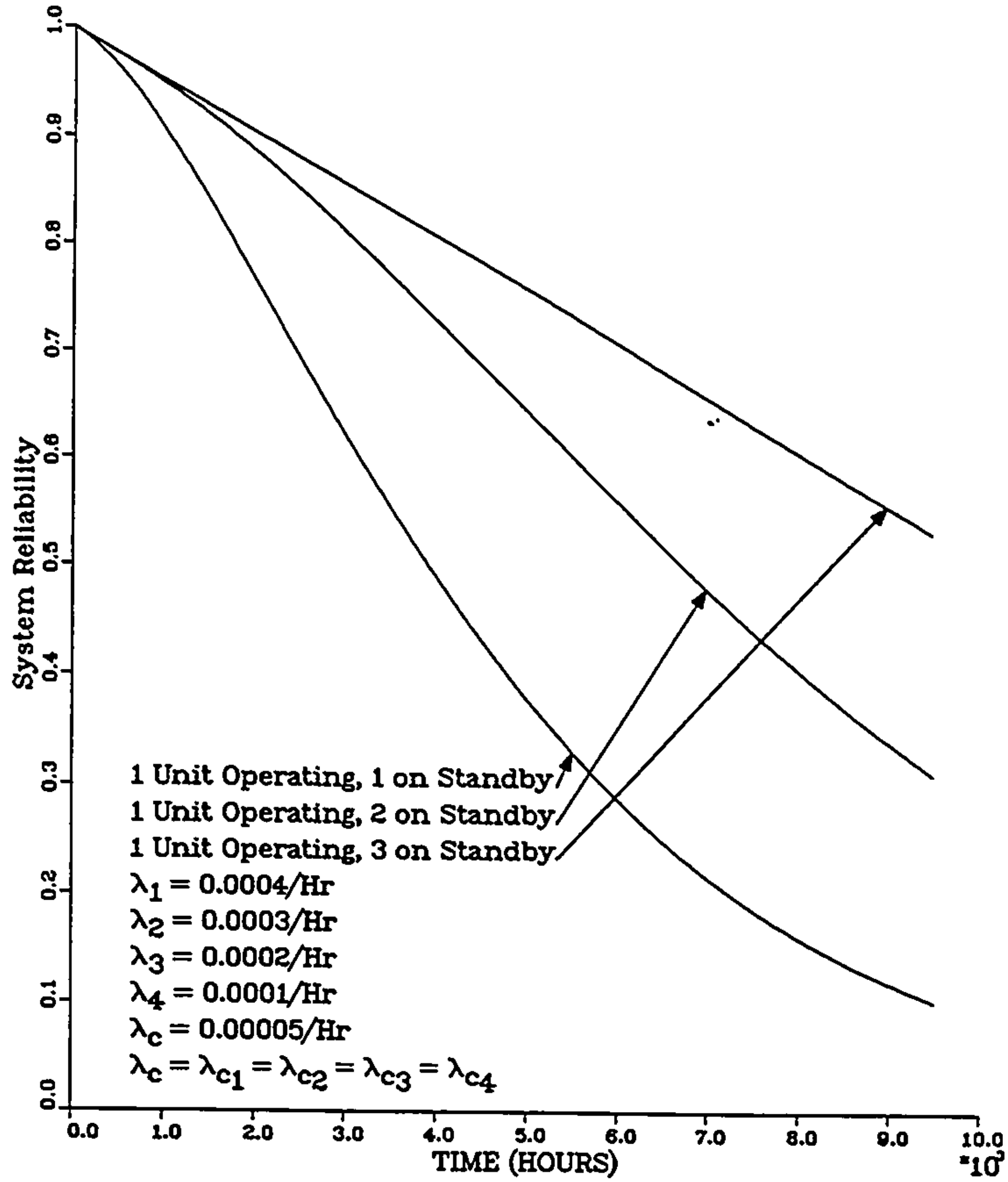

Figure 4.30: Typical System Reliability Plots for a Non-identical Unit Standby System Without Repair

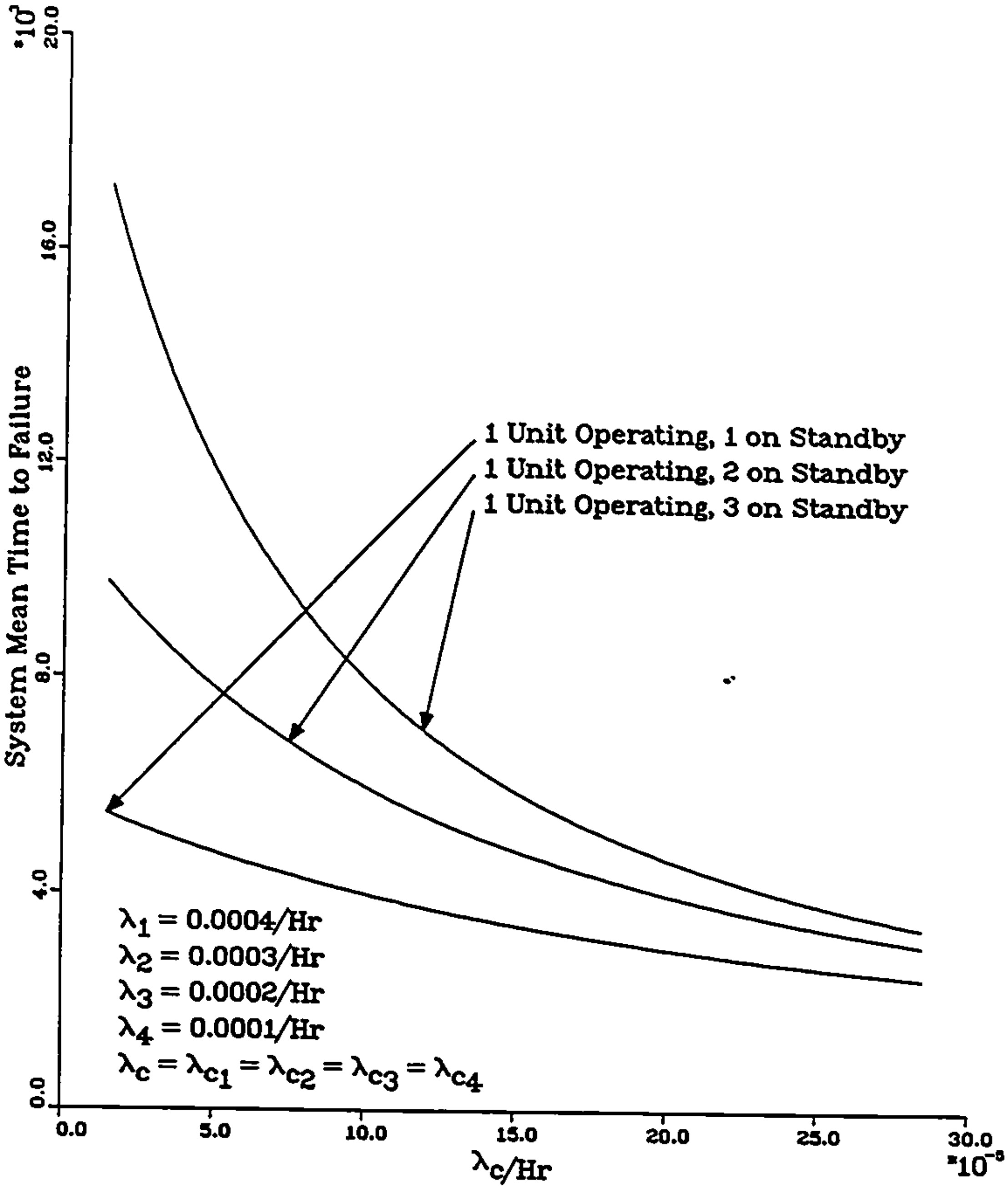

Figure 4.31: Typical System Mean Time to Failure Plots for a Non-identical Unit Standby System Without Repair

Special Case Model I

The following system of differential equations is obtained by setting $n = 1$ in Equations (4.222) – (4.225) :

$$\dot{p}_0(t) = -(\lambda_1 + \lambda_{c_1})p_0(t) \quad (4.233)$$

$$\dot{p}_1(t) = \lambda_1 p_0(t) - (\lambda_2 + \lambda_{c_2})p_1(t) \quad (4.234)$$

$$\dot{p}_2(t) = \lambda_2 p_1(t) \quad (4.235)$$

$$\dot{p}_3(t) = \lambda_{c_1} p_0(t) + \lambda_{c_2} p_1(t) \quad (4.236)$$

By solving the above equations with the aid of Laplace transforms, the following time dependent state probability expressions are obtained :

$$p_0(t) = e^{-(\lambda_1+\lambda_{c_1})t} \quad (4.237)$$

$$p_1(t) = \lambda_1 e^{-\frac{1}{2}(\lambda_2+\lambda_{c_2}+\lambda_{c_1}+\lambda_1)t} \times \frac{\{e^{\frac{1}{2}(\lambda_2+\lambda_{c_2}-\lambda_{c_1}-\lambda_1)t} - e^{-\frac{1}{2}(\lambda_2+\lambda_{c_2}-\lambda_{c_1}-\lambda_1)t}\}}{(\lambda_2 + \lambda_{c_2} - \lambda_{c_1} - \lambda_1)} \quad (4.238)$$

The system reliability given by $p_0(t) + p_1(t)$, is plotted in Figure 4.32 and the decreasing effect of common-cause failures on system reliability can easily be noted from these plots.

The mean time to failure of a single non-identical unit standby system can be expressed as

$$MTTF = \frac{1}{\lambda_1 + \lambda_{c_1}} + \frac{\lambda_1}{(\lambda_1 + \lambda_{c_1})(\lambda_2 + \lambda_{c_2})} \quad (4.239)$$

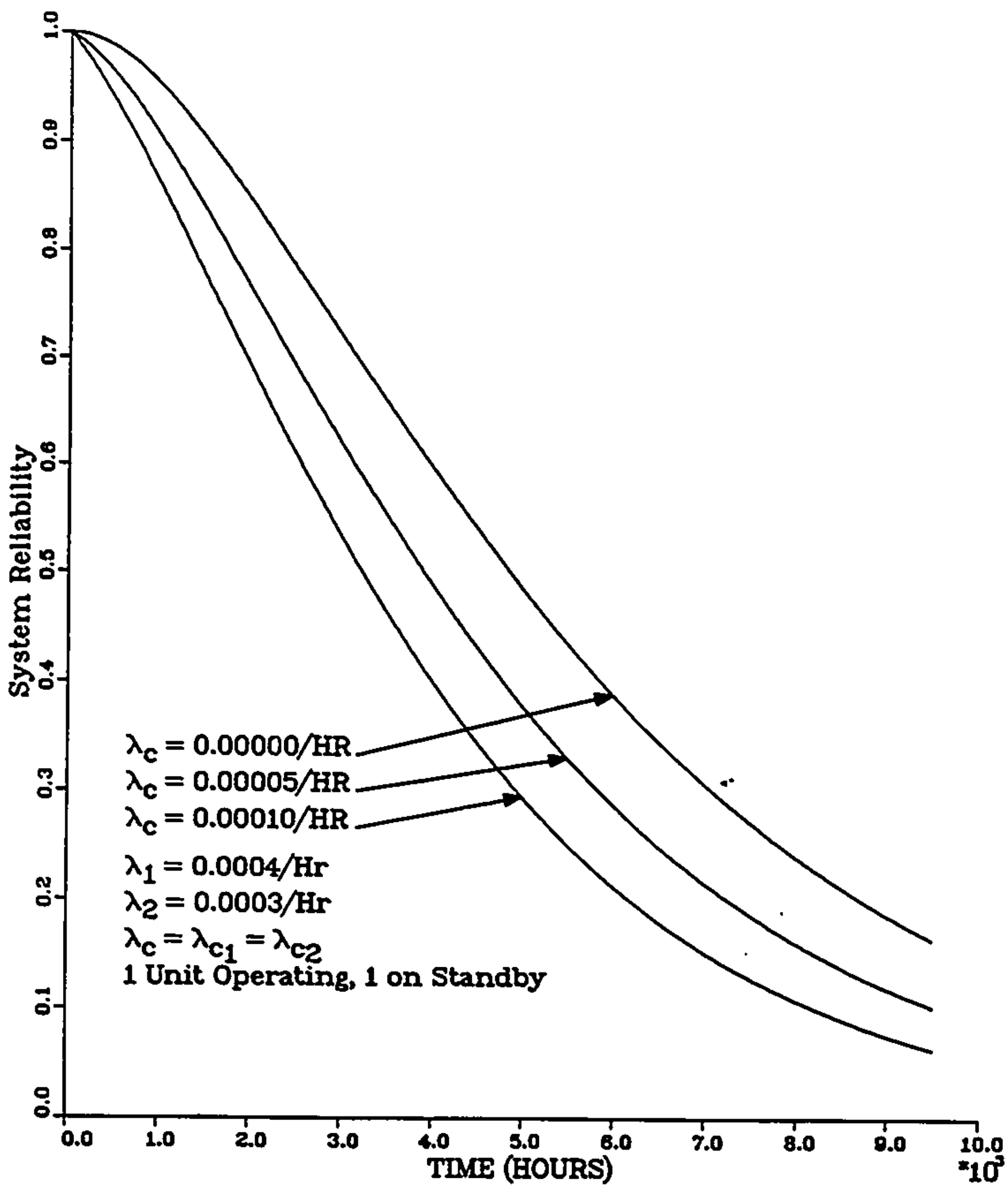

Figure 4.32: System Reliability Plots for a Single Non-identical Unit Standby System Without Repair

Special Case Model II

For a system with two non-identical units, the system of differential equations can be represented as

$$\dot{p}_0(t) = -(\lambda_1 + \lambda_{c_1})p_0(t) \quad (4.240)$$

$$\dot{p}_1(t) = \lambda_1 p_0(t) - (\lambda_2 + \lambda_{c_2})p_1(t) \quad (4.241)$$

$$\dot{p}_2(t) = \lambda_2 p_1(t) - (\lambda_3 + \lambda_{c_3})p_2(t) \quad (4.242)$$

$$\dot{p}_3(t) = \lambda_3 p_2(t) \quad (4.243)$$

$$\dot{p}_4(t) = \lambda_{c_1} p_0(t) + \lambda_{c_2} p_1(t) + \lambda_{c_3} p_2(t) \quad (4.244)$$

Solving the above equations with the aid of Laplace transforms, we get the following time dependent state probability expressions :

$$p_0(t) = e^{-(\lambda_1+\lambda_{c_1})t} \quad (4.245)$$

$$p_1(t) = \lambda_1 e^{-\frac{1}{2}(\lambda_2+\lambda_{c_2}+\lambda_{c_1}+\lambda_1)t} \times \frac{\{e^{\frac{1}{2}(\lambda_2+\lambda_{c_2}-\lambda_{c_1}-\lambda_1)t} - e^{-\frac{1}{2}(\lambda_2+\lambda_{c_2}-\lambda_{c_1}-\lambda_1)t}\}}{(\lambda_2 + \lambda_{c_2} - \lambda_{c_1} - \lambda_1)} \quad (4.246)$$

$$p_2(t) = \frac{\lambda_1\lambda_2 e^{-(\lambda_1+\lambda_{c_1})t}}{(\lambda_2 + \lambda_{c_2} - \lambda_1 - \lambda_{c_1})(\lambda_3 + \lambda_{c_3} - \lambda_1 - \lambda_{c_1})} + \frac{\lambda_1\lambda_2 e^{-(\lambda_2+\lambda_{c_2})t}}{(\lambda_1 + \lambda_{c_1} - \lambda_2 - \lambda_{c_2})(\lambda_3 + \lambda_{c_3} - \lambda_2 - \lambda_{c_2})} + \frac{\lambda_1\lambda_2 e^{-(\lambda_3+\lambda_{c_3})t}}{(\lambda_1 + \lambda_{c_1} - \lambda_3 - \lambda_{c_3})(\lambda_2 + \lambda_{c_2} - \lambda_3 - \lambda_{c_3})} \quad (4.247)$$

The system reliability given by $p_0(t) + p_1(t) + p_2(t)$, is shown in Figure 4.33. The decreasing effect of common-cause failures on system reliability can be easily noted from these plots.

The mean time to failure of a two non-identical is

$$MTTF = \frac{1}{\lambda_1 + \lambda_{c_1}} + \frac{\lambda_1}{(\lambda_1 + \lambda_{c_1})(\lambda_2 + \lambda_{c_2})} + \frac{\lambda_1\lambda_2}{(\lambda_1 + \lambda_{c_1})(\lambda_2 + \lambda_{c_2})(\lambda_3 + \lambda_{c_3})} \quad (4.248)$$

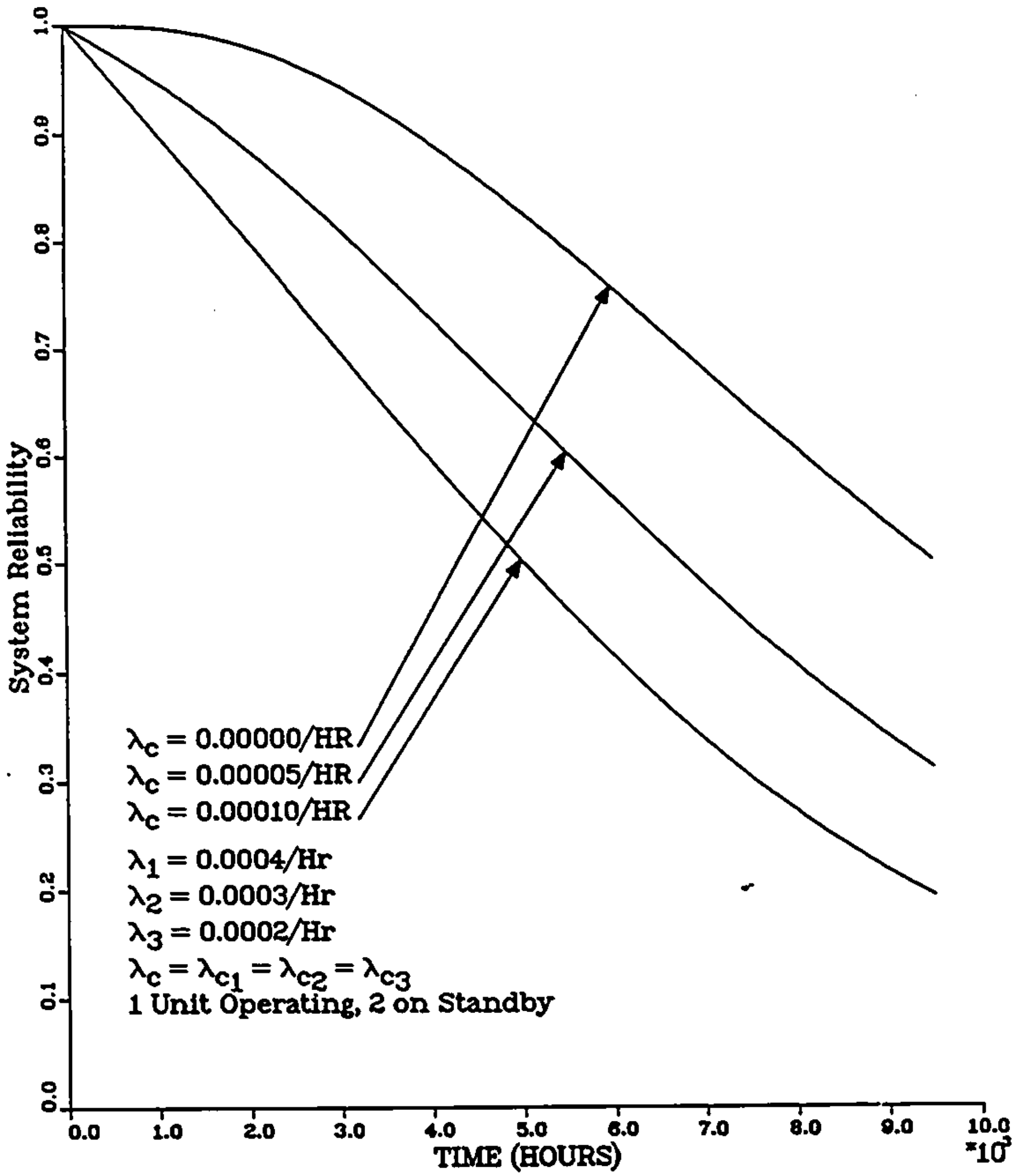

Figure 4.33: System Reliability Plots for a Two Non-identical Unit Standby System Without Repair

Special Case Model III

Similar to the above Special Case models, by setting $n = 3$ in Equations (4.222)–(4.225) results in the following differential equations :

$$\begin{aligned}
\dot{p}_0(t) &= -(\lambda_1 + \lambda_{c_1})p_0(t) && (4.249)\\
\dot{p}_1(t) &= \lambda_1 p_0(t) - (\lambda_2 + \lambda_{c_2})p_1(t) && (4.250)\\
\dot{p}_2(t) &= \lambda_2 p_1(t) - (\lambda_3 + \lambda_{c_3})p_2(t) && (4.251)\\
\dot{p}_3(t) &= \lambda_3 p_2(t) - (\lambda_4 + \lambda_{c_4})p_3(t) && (4.252)\\
\dot{p}_4(t) &= \lambda_4 p_3(t) && (4.253)\\
\dot{p}_5(t) &= \lambda_{c_1} p_0(t) + \lambda_{c_2} p_1(t) + \lambda_{c_3} p_2(t) + \lambda_{c_4} p_3(t) && (4.254)
\end{aligned}$$

At time $t = 0$, $p_0(0) = 1$ and all other initial condition probabilities are equal to zero.

The following state probabilities are obtained by solving the above equations with the aid of Laplace transforms :

$$p_0(t) = e^{-(\lambda_1+\lambda_{c_1})t} \qquad (4.255)$$

$$p_1(t) = \lambda_1 e^{-\frac{1}{2}(\lambda_2+\lambda_{c_2}+\lambda_{c_1}+\lambda_1)t} \times \frac{\{e^{\frac{1}{2}(\lambda_2+\lambda_{c_2}-\lambda_{c_1}-\lambda_1)t} - e^{-\frac{1}{2}(\lambda_2+\lambda_{c_2}-\lambda_{c_1}-\lambda_1)t}\}}{(\lambda_2 + \lambda_{c_2} - \lambda_{c_1} - \lambda_1)} \qquad (4.256)$$

$$\begin{aligned}
p_2(t) = {} & \frac{\lambda_1\lambda_2 e^{-(\lambda_1+\lambda_{c_1})t}}{(\lambda_2 + \lambda_{c_2} - \lambda_1 - \lambda_{c_1})(\lambda_3 + \lambda_{c_3} - \lambda_1 - \lambda_{c_1})} + \\
& \frac{\lambda_1\lambda_2 e^{-(\lambda_2+\lambda_{c_2})t}}{(\lambda_1 + \lambda_{c_1} - \lambda_2 - \lambda_{c_2})(\lambda_3 + \lambda_{c_3} - \lambda_2 - \lambda_{c_2})} + \\
& \frac{\lambda_1\lambda_2 e^{-(\lambda_3+\lambda_{c_3})t}}{(\lambda_1 + \lambda_{c_1} - \lambda_3 - \lambda_{c_3})(\lambda_2 + \lambda_{c_2} - \lambda_3 - \lambda_{c_3})} \qquad (4.257)
\end{aligned}$$

$$\begin{aligned}
p_3(t) = {} & \frac{\lambda_3\lambda_2\lambda_1 e^{-(\lambda_1+\lambda_{c_1})t}}{(\lambda_2 + \lambda_{c_2} - \lambda_1 - \lambda_{c_1})(\lambda_4 + \lambda_{c_4} - \lambda_1 - \lambda_{c_1})(\lambda_3 + \lambda_{c_3} - \lambda_1 - \lambda_{c_1})} + \\
& \frac{\lambda_3\lambda_2\lambda_1 e^{-(\lambda_2+\lambda_{c_2})t}}{(\lambda_2 + \lambda_{c_2} - \lambda_1 - \lambda_{c_1})(\lambda_4 + \lambda_{c_4} - \lambda_2 - \lambda_{c_2})(\lambda_2 + \lambda_{c_2} - \lambda_3 - \lambda_{c_3})} +
\end{aligned}$$

$$\frac{\lambda_3\lambda_2\lambda_1 e^{-(\lambda_3+\lambda_{c_3})t}}{(\lambda_3+\lambda_{c_3}-\lambda_1-\lambda_{c_1})(\lambda_2+\lambda_{c_2}-\lambda_3-\lambda_{c_3})(\lambda_4+\lambda_{c_4}-\lambda_3-\lambda_{c_3})}+$$
$$\frac{\lambda_3\lambda_2\lambda_1 e^{-(\lambda_4+\lambda_{c_4})t}}{(\lambda_1+\lambda_{c_4}-\lambda_1-\lambda_{c_1})(\lambda_4+\lambda_{c_4}-\lambda_2-\lambda_{c_2})(\lambda_3+\lambda_{c_3}-\lambda_4-\lambda_{c_4})} \tag{4.258}$$

The system reliability is given by

$$R(t) = p_0(t) + p_1(t) + p_2(t) + p_3(t) \tag{4.259}$$

The system reliability given by Equation (4.259) is plotted in Figure 4.34 and the plots clearly show the inverse relationship between system reliability and common-cause failures.

The mean time to failure of a three non-identical unit standby system is

$$\begin{aligned} MTTF &= \frac{1}{\lambda_1+\lambda_{c_1}} + \frac{\lambda_1}{(\lambda_1+\lambda_{c_1})(\lambda_2+\lambda_{c_2})} + \\ &\quad \frac{\lambda_1\lambda_2}{(\lambda_1+\lambda_{c_1})(\lambda_2+\lambda_{c_2})(\lambda_3+\lambda_{c_3})} + \\ &\quad \frac{\lambda_1\lambda_2\lambda_3}{(\lambda_1+\lambda_{c_1})(\lambda_2+\lambda_{c_2})(\lambda_3+\lambda_{c_3})(\lambda_4+\lambda_{c_4})} \end{aligned} \tag{4.260}$$

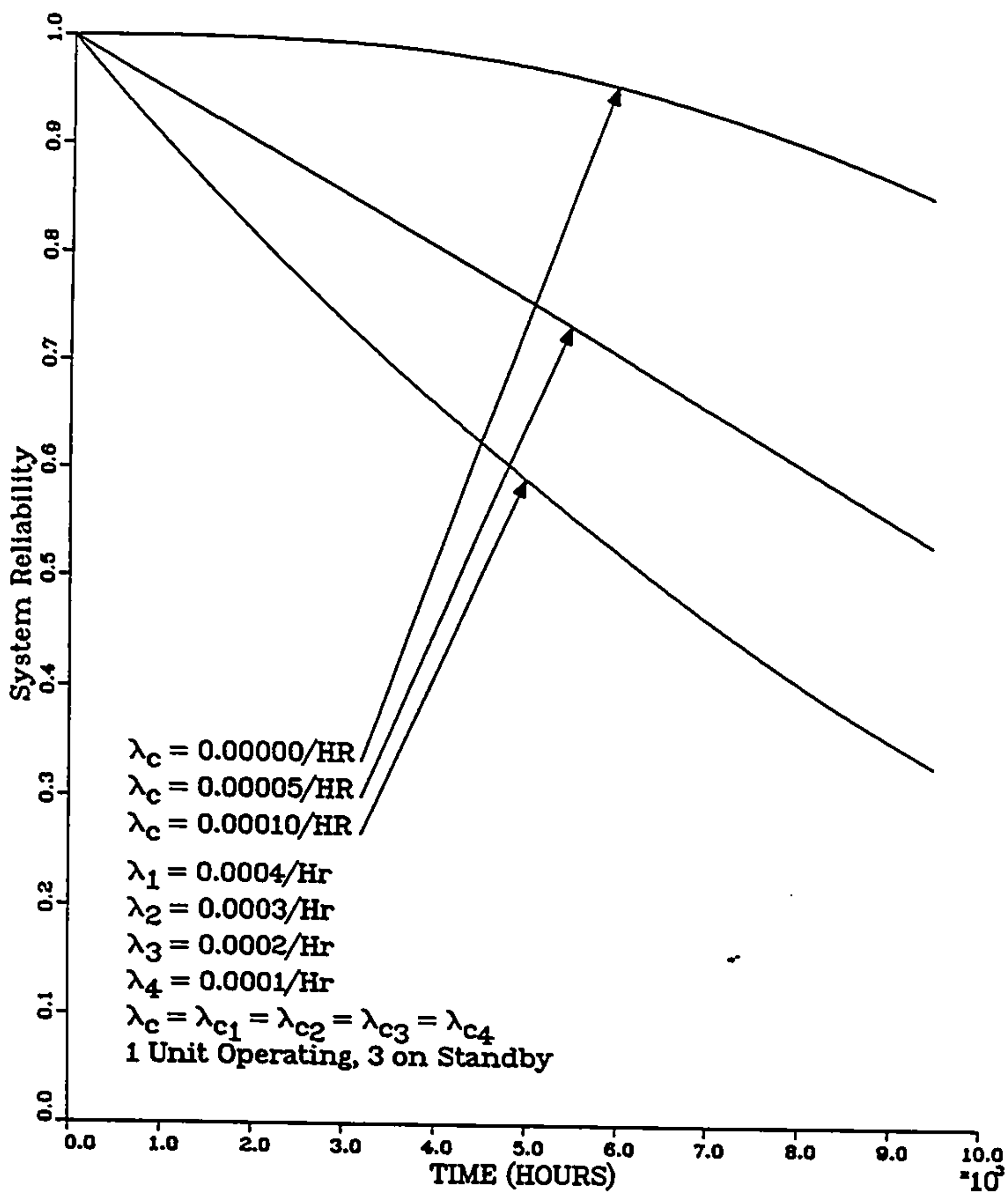

Figure 4.34: System Reliability Plots for a Three Non-identical Unit Standby System Without Repair

Chapter 5

Conclusions and Recommendations

5.1 Conclusions

In this study the common-cause failure analysis of three of the most commonly used repairable and non-repairable redundant systems were presented. These systems were parallel, k-out-of-n and standby with identical as well as non-identical units. Generalized expressions for system reliability and mean time to failure for identical unit parallel, k-out-of-n and standby systems were developed with constant failure and repair rates.

In the case of parallel system, with identical as well as non-identical units, it was noted that the steady state system availability increases with an increase in the number of units in the system. In addition, the steady state system availability plots indicated a decrease in steady state system availability with an increase in the number of common-cause failures. An increase in steady state system availability with an increase in the repair rates was observed in the special case models. The system reliability plots showed a decrease in system reliability with an increase in time as well as with an increase in common-cause failures. It was also seen that the system mean time to failure increased with increase in number of units in the parallel system. However, a decreasing effect of common-cause failures on system mean time to failure was observed.

The reliability analysis of identical as well as non-identical unit k-out-of-n system yielded the following results :

- For a given number of units k, required to be operative to ensure system success, the steady state system availability increases with an increase in the total number of units n in the system.
- The steady state system availability decreases with an increase in the number of common-cause failures.
- The system reliability decreases with an increase in time as well as increase in the number of common-cause failures.
- The system mean time to failure decreases with an increase in the number of common-cause failures.

In addition, the system reliability and system mean time to failure for a 2-out-of-6 identical unit system without repair, obtained by Markov method, was compared with those obtained by considering common-cause failures to be acting in series with a 2-out-of-6 system (the block diagram method). The results from both the methods were found to be identical. In a similar exercise for the parallel and standby systems, the system reliability and system mean time to failure obtained through Markov method were found to be the same as those obtained by considering the common-cause failure to be acting in series with a corresponding identical unit block diagram network.

Similar to the parallel and k-out-of-n systems, the reliability analysis of a standby system, with identical as well as non-identical units, indicated a decrease in steady state system availability with an increase in the number of common-cause failures. In addition, an increase in number of units in standby was seen to increase the steady state system availability. The effects of common-cause failures on system reliability and system mean time to failure were no different than those observed for the parallel and k-out-of-n systems.

Thus it is concluded that the reliability analysis of redundant systems without taking into consideration the occurrence of common-cause failures would lead to the prediction of optimistic values of reliability parameters.

5.2 Recommendations for Further Study

The reliability analysis performed in this study assumes that both the failure rates and repair rates are independent of time. This study can be extended to include time dependent failure and repair rates. Generalized expressions for system reliability and mean time to failure for various failure and repair rate distributions could then be obtained. In addition, since the number of terms in steady state system availability expression were too large, the generalized expression for steady state system availability could not be developed, even though a trend could be seen. A more efficient method of handling the terms in these expressions may lead to generalized steady state probability expressions.

Chapter 6

Bibliography

1. "A Study of Common-Cause Failures Phase I: A Classification System", Electrical Power Research Institute, Report No. EPRI, NP 3383, Research Project 2169-1, Prepared by Los Alamos Technical Associates Inc., New Mexico, U.S.A., Jan 1984.
2. "Advanced Seminar on Common-cause Failure Analysis in Probabilistic Safety Assessment", Proc. of the ISPRA Course Held at the Joint Research Center, Ispra, Italy, Edited by Aniello Amendola, Kluwer Publishers, Dordrecht, The Netherlands, Nov. 1987.
3. Allan, R.N., Dialynas, E.N., Homer, I.R., "Modeling Common-Mode Failures in the Reliability Evaluation of Power System Networks", IEEE Power Engineering Society Discussions and Closures from the Winter Meeting, New York, N.Y., Feb. 1979, pp. DISC. A 79 040-7.
4. Allan, R.N., Rondiris, I.L., Fryer, D.M., "An Efficient Computational Technique for Evaluating the Cut/Tie sets and Common-Cause Failures", IEEE Transactions on Reliability, Vol. 30, No. 2, June 1981, pp. 101-109.
5. Amendola, A., "Classification of Multiple Related Failures", Advanced Seminar of Common-cause Failure Analysis in Probabilistic Safety Assessment, Edited by Aniello Amendola, Kluwer Publications, Dordrecht, The Netherlands, 1989, pp. 31-46.
6. Apostolakis, G.E., "Effect of a Certain Class of Potential Common-Cause Mode Analysis on Reliability of Redundant Systems", Nuclear Engineering and Design, Vol. 36, 1976, pp. 123-133.

7. Apostolakis, G.E., "On a Certain Class of Potential Common-Mode Failures", Transactions of American Nuclear Society on Safety System Reliability Methods and Applications, Vol. 22, 1975, pp. 476-477.

8. Apostolakis, G.E., "On the Reliability of Redundant Systems", Transactions of American Nuclear Society on Safety Systems Reliability Methods and Applications, Vol. 22, 1975, pp. 477-478.

9. Apostalakis, G.E., Moieni, P., "On the Correlation of Failure Rates", Reliability Data Collection and Use in Risk and Availability Assessment, Proc. of the Fifth EUREDATA Conference, Heidelberg, Germany, April 1986.

10. Atwood, C.L., "Common-Cause and Individual Failure and Fault Rates for Licensee Event Reports of Pumps at U.S. Commercial Nuclear Power Plants", Report No. EGG EA 5289, EG & G Idaho Falls, Idaho, U.S. Nuclear Regulatory Commission, Washington D.C., 1980.

11. Atwood, C.L., "The Binomial Failure Rate Common-Cause Model", Technometrics, Vol. 28, 1986, pp. 139-148.

12. Atwood, C.L., "Common-Cause Fault Rates for Pumps", Report No. NUREG/CR-2098, EGG-EA-5289, U.S. Nuclear Regulatory Commission, Washington D.C., August 1982.

13. Atwood, C.L., "Estimators for the Binomial Failure Rate Common-Cause Model", Report No. NUREG/CR-1401, EGG-EA-5112, U.S. Nuclear Regulatory Commission, Washington D.C., April 1981.

14. Atwood, C.L., "Data Analysis Using the Binomial Failure Rate Common-cause Model", NUREG/CR-3737, U.S. Nuclear Regulatory Commission, Washington, D.C., Sept. 1983.

15. Atwood, C.L., Steverson, J.A., "Common-cause Rates for Diesel Generators", Report No. NUREG/CR-2099, EGG-EA-5359, Rev. 1, U.S. Nuclear Regulatory Commission, Washington D.C., June 1982.

16. Atwood, C.L., Steverson, J.A., "Common-cause Fault Rates for Valves : Estimates Based on Licensee Event Reports at U.S. Nuclear Regulatory Commission, EG & G, Idaho Inc., NUREG/CR-2770, EGA-EA-5485, Feb. 1983.

17. Atwood, C.L., Suitt, W.J., "User's Guide to BFR: A Computer Code Based on the Binomial Failure Rate Common-Cause Model", Report No. NUREG/CR-

2729, EGG-EA-5502, U.S. Nuclear Regulatory Commission Report, Washington D.C., Feb. 1983.

18. Atwood, C.L., Teresa, R.M., "Common-Cause Failure Rates for Instrumentation and Control Assemblies", Report No. NUREG/CR- 3289, EGG-EA-2258, U.S. Nuclear Regulatory Commission, Washington D.C., 1983.

19. Bank, J.V., "Catastrophic Failure Modes Limit Redundancy Effectiveness", IEEE Transactions on Reliability, Vol. 32, Dec. 1983, pp. 409-411.

20. Bengtz, M., Bjore, S., Hirschberg, S., "NKA Project Risk Analysis (RAS-470), Identification of Common-cause Failure Events for Motor Operated Valves in Swedish Boiling Water Reactor Plants, Final Report, RAS-470(84) 4, Jan. 1986.

21. Bickel, J.H., Caivano, J.D., "Generation of Common-Cause Failure Rate Distributions for Monte Carlo Analysis Using Beta Factor Method", Proc. of the International Topical Meeting on Probabilistic Safety Methods and Applications, EPRI NP-3912, Vol. 3, Sessions 17-23, California, Feb. 1985, pp. 179/1-179/6.

22. Billington, R., Kumar, Y., "Transmission Line Reliability Models Including Common-Mode and Adverse Weather Effects", IEEE Power Engineering Society Discussions and Closures of Abstracted Papers from the Winter Meeting, New York, N.Y., Paper No. A80 080, Feb. 1980.

23. Billington, R., Kumar, Y., "Transmission Line Reliability Models Including Common-Mode and Adverse Weather Effects", IEEE Transactions on Power Apparatus and Systems, Vol. 100, No. 8, Aug. 1981, pp. 3899-3910.

24. Billington, R., Medicherla, T.K.P., Sachdev, M.S., "Application of Common-Cause Outage Models in Composite System Reliability Evaluation", IEEE Transactions on Power Apparatus and Systems, Vol. 100, No. 7, July 1981, pp. 3648-3657.

25. Billington, R., Medicherla, T.K.P., Sachdev, M.S., "Common-Cause Outages in Multiple Circuit Transmission Lines", IEEE Transactions on Reliability, Vol. 27, 1978, pp. 128-131.

26. Bourne, A.J., Edwards, G.T., Hunns, D.M., Poulter, D.R., Watson, I.A., "Defences Against Common-Mode Failures in Redundancy Systems", Report No. SRD R 196, A Guide for Management, Designers and Operators, Safety and

Reliability Directorate, United Kingdom Atomic Energy Authority, Warrington, U.K.

27. Burdick, G.R., "COMCAN - A Computer Code for Common-Cause Analysis", IEEE Transactions on Reliability, Vol. 26, No. 2, June 1977, pp. 100-102.

28. Canterella, S., "Treatment of Multiple Related Failures by Markov Method", Advanced Seminar of Common-cause Failure Analysis in Probabilistic Safety Assessment, Edited by Aniello Amendola, Kluwer Publications, Dordrecht, The Netherlands, 1989, pp. 145-157.

29. Carnino, A., "Views on the Problems of 'Common-Modes' in CEA/Department of Nuclear Safety", Proc. of the Second National Reliability Conference, U.K., March 1979, pp. 5A/3/1-5A/3/3.

30. Cate, C.L., Fussel, J.B., "BACFIRE - A Computer Code for Common-Cause Failure Analysis", Chemical Engineering Dept., University of Tennessee, Knoxville, Tennessee 37916, Feb. 1977.

31. Cate, C.L., Wagner, D.P., Fussell, J.B., "A Computer Aided Approach to Qualitative and Quantitative Common-Cause Analysis", Proc. of the 8th Annual Pittsburg Conference on Modeling and Simulation, Vol. 1, Part 8, 1977, pp. 25-29.

32. Chae, K.C., Clark, G.M., "System Reliability in the Presence of Common-Cause Failures", IEEE Transactions on Reliability, Vol. 35, No. 1, April 1986, pp. 32-35.

33. Chari, A.A.R., "A Stochastic Model for Availability Measure with Common-Cause Failures", IEEE Transactions on Reliability, Vol. 35, No. 5, Dec. 1986, pp. 570.

34. Chu, B.B., Gaver, D.P., "Stochastic Models for Repairable Redundant Systems Susceptible to Common-Mode Failures", Proc. of the International Conference on Nuclear Systems Reliability Engineering And Risk Assessment, Gatlinburg, Tennessee, 1977, pp. 342-367.

35. Chung, W.K., "A K-out-of-N:G Redundant System with Cold Standby Units and Common-Cause Failures", Microelectronics and Reliability, Vol. 24, No. 4, 1984, pp. 691-695.

36. Chung, W.K., "A N-Unit Redundant System with Common-Cause Failures", Microelectronics and Reliability, Vol. 19, 1979, pp. 377-378.

37. Chung, W.K., "A Two Non-Identical Three-State Units Redundant System with Common-Cause Failure and One Standby Unit", Microelectronics and Reliability, Vol. 21, No. 5, 1981, pp. 707- 709.

38. Chung, W.K., "A K-out-of-N:G Three-State Unit Redundant Systems with Common-Cause Failures and Replacements", Microelectronics and Reliability, Vol. 21, 1981, pp. 589-591.

39. Chung, W.K., "A K-Out-Of-N Redundant System With Common-Cause Failures", IEEE Transactions on Reliability, Vol. 29, 1980, pp. 344.

40. Chung, W.K., "Reliability Analysis of a Repairable Parallel System with Standby Involving Human Error and Common-Cause Failures", Microelectronics and Reliability, Vol. 27, No. 2, 1987, pp. 269-271.

41. Chung, W.K., "Reliability Analysis of Repairable and Non-Repairable Systems with Common-Cause Failures", Microelectronics and Reliability, Vol. 29, No. 4, 1989, pp. 345-347.

42. Chung, W.K., "A Reliability Model for a K-Out-Of-N:G Redundant System with Multiple Failure Modes and Common-Cause Failures", Microelectronics and Reliability, Vol. 27, No. 4, 1987, pp. 621- 623.

43. Chung, W.K., "An Availability Calculation for K-out-of-N Redundant Systems with Common-Cause Failures and Replacement", Microelectronics and Reliability, Vol. 20, 1980, pp. 517.

44. Chung, W.K., "A K-Out-Of-N:G Redundant System with Dependent Failure Rates and Common-Cause Failures", Microelectronics and Reliability, Vol. 28, No. 2, 1988, pp. 201-203.

45. Chung, W.K., "An Availability Analysis of K-Out-Of-N:G Redundant System with Dependant Failure Rates and Common-Cause Failures", Microelectronics and Reliability, Vol. 28, No. 3, 1988, pp. 391- 393.

46. "Common-Mode Failures Bounding Techniques and Special Techniques", Report No. WASH-1400, Reactor Safety Study, Appendix IV, NUREG-75/1400, US Nuclear Regulatory Commission, Washington D.C., 1975.

47. "Common-Mode Failure of Incore Instrumentation, Reactor Operating Experiences", Report No. ROE-73-7, Oak Ridge National Laboratory, Oak Ridge, Tennessee, U.S.A., 1975.

48. "Common-Mode Failure Of Incore Instrumentation: Reactor Operating Experiences No. ROE-73-7", USAEC Report No. ORNL/NSIC-64, United States Atomic Energy Council, Oak Ridge National Laboratory, Oak Ridge, Tennessee, 1972, pp. 51-54.

49. Contini, S, "Algorithms for Common-cause Analysis - a preliminary Report", TN I.06.01.81.16, CEC-JRC, Ispra, Italy, 1981.

50. Contini, S., "Dependent Failure Modelling by Fault Tree Technique", Advanced Seminar of Common-cause Failure Analysis in Probabilistic Safety Assessment, Edited by Aniello Amendola, Kluwer Publications, Dordrecht, The Netherlands, 1989, pp. 145-157.

51. Crellin, et al. (LATA), "A Study of Common-cause Phase II : A Comprehensive Classification System for Component Fault Analysis, EPRI-NP-3837, Electrical Power Research Institute, May 1985.

52. David, S.J., "Reliability and Maintainability in Perspective: Technical, Management and Commercial Aspects", Macmillan Press, London, 1981, pp. 158-163.

53. Dhillon, B.S., "On Common-Cause Failures - Bibliography", Microelectronics and Reliability, Vol. 18, 1979, pp. 533-534.

54. Dhillon, B.S., "A K-Out-Of-N Three-State Device System with Common-Cause Failures", Microelectronics and Reliability, Vol. 18, 1978, pp. 447-448.

55. Dhillon, B.S., "Effects of Weibull Hazard Rate on Common-Cause Failure Analysis of Reliability Networks", Microelectronics and Reliability, Vol. 17, 1978, pp. 59-66.

56. Dhillon, B.S., "A Common-Cause Failure Availability Model", Microelectronics and Reliability, Vol. 17, 1978, pp. 583-584.

57. Dhillon, B.S., "A 4-Unit Redundant System with Common-Cause Failures", IEEE Transactions on Reliability, Vol. 28, No. 3, August 1979, pp. 267.

58. Dhillon, B.S., "Unified Availability Model: A Redundant System with Mechanical, Electrical, Software, Human and Common-Cause Failures", Microelectronics and Reliability, Vol. 21, No. 5, 1981, pp. 653-659.

59. Dhillon, B.S., "A 1-out-of-N:G System with Duplex Elements", IEEE Transactions on Reliability, 1979, pp. 169.

60. Dhillon, B.S., "Multi-State Device Redundant Systems with Common- Cause Failure and One Standby Unit", Microelectronics and Reliability, Vol. 20, 1980, pp. 411-417.

61. Dhillon, B.S., "Optimal Maintenance Policy for Systems with Common-Cause Failures", Proc. of the 9th Annual Pittsburgh Conference on Modeling and Simulation, Pittsburgh, Pennsylvania, 1978, pp. 1121-1215.

62. Dhillon, B.S., "A System with Two Kinds of 3-State Elements", IEEE Transactions on Reliability, Vol. 29, No. 4, Oct. 1979, pp. 345.

63. Dhillon, B.S., "Reliability Engineering in Systems Design and Operation", Van Nostrand Reinhold Company, New York, 1983.

64. Dhillon, B.S., Singh, C., "Engineering Reliability: New Techniques and Applications", Wiley & Sons, New York, 1981, pp. 93-111, 167, 260.

65. Dhillon, B.S., "Mechanical Reliability: Theory, Models and Applications", American Institute of Aeronautics and Astronautics, Washington D.C., 1988, pp. 127-129.

66. Dhillon, B.S., "Power System Reliability, Safety and Management", Ann Arbor Science, The Butterworth Group, Borough Green, Sevenoaks, England, 1983.

67. Dhillon, B.S., "Quality Control, Reliability and Engineering Design", Marcel Dekker Inc., New York, 1985.

68. Dhillon, B.S., "Reliability in Computer System Design", Ablex Publishing Corporation, Norwood, New Jersey, 1987, pp. 149-156.

69. Dhillon, B.S., "A 4-Unit Redundant System with Common-Cause Failures", IEEE Transactions on Reliability, Vol. 20, 1977, pp. 373-377.

70. Dhillon, B.S., "Stochastic Analysis of a Parallel System with Common-Cause Failures and Critical Human Errors", Microelectronics and Reliability, Vol. 29, No. 4, 1989, pp. 627- 627.

71. Dhillon, B.S., "Mathematical Modeling of Common-Cause Failures and Human Errors in Engineering Systems", Proc. of the 7th International Conference of the Israel Society for Quality Assurance, 1988, pp. 2.1.1.1-2.1.1.5.

72. Dhillon, B.S., Natesan, J., "Probabilistic Analysis of a Pulverizer System with Common-Cause Failures", Microelectronics and Reliability, Vol. 22, 1982, pp. 1121.

73. Dhillon, B.S., Proctor, C.L., "Common-Mode Failure Analysis of Reliability Networks", Proc. of the Annual Reliability and Maintainability Symposium, Jan. 18-20, 1977, pp. 404-408.

74. Dhillon, B.S., Rayapati, S.N., "Reliability and Availability Analysis of a Parallel System with Common-Cause Failures and Human Errors", Proc. of the International Conference on Nuclear Power Plant Aging, Availability Factor and Reliability Analysis, San Diego, California, U.S.A, July 1985, pp. 459-465.

75. Dhillon, B.S., Rayapati, S.N., "Human Error and Common-Cause Failure Modelling of Standby-Systems", Maintenance Management International, Vol. 7, 1988, pp. 93-110.

76. Dhillon, B.S., Rayapati, S.N., "Common-Cause Failure and Human Error Modelling of Redundant Systems with Partially Energized Units", Reliability Engineering, Vol. 19, 1987, pp. 1-14.

77. Dhillon, B.S., Rayapati, S.N., "Human Error and Common-Cause Failure Modelling of Redundant Systems", Microelectronics and Reliability, Vol. 26, No. 6, 1986, pp. 1139-1162.

78. Dhillon, B.S., Rayapati, S.N., "Common-Cause Failures in Repairable Systems", Proc. of the Annual Reliability and Maintainability Symposium, 1988, pp. 283-289.

79. Dhillon, B.S., Rayapati, S.N., "Probabilistic Analysis of Redundant Systems with Human Errors and Common-Cause Failures", Stochastic Analysis and Applications, Vol. 4, 1986, pp. 367-398.

80. Dhillon, B.S., Rayapati, S.N., "Reliability Modeling of Redundant Computer Systems with Common-Cause Failures", Computers and Electrical Engineering, Vol. 14, 1988, pp. 125-136.

81. Dhillon, B.S., Rayapati, S.N., "Analysis of Systems with Human Errors and Common-Cause Failures", International Journal of Modeling and Simulation, Vol. 9, 1989, pp. 124-128.

82. Dhillon, B.S., Rayapati, S.N., "Common-Cause Failure and Human Error Analysis of Redundant Systems", Proc. of the Inter-Ram Conference for Electric Power Industry, 1986, pp. 398-405.

83. Dhillon, B.S., Rayapati, S.N., "Reliability and Availability Analysis of Redundant Systems with Common-Cause Failures and Human Errors", Proc. of the International Conference on Factory of Future, 1986, pp. 3-10.

84. Dhillon, B.S., Sambi, A., Khan, M.R., "Common-Cause Failure Analysis of a Three-State Device System", Microelectronics and Reliability, Vol. 19, 1979, pp. 345-348.

85. Dhillon, B.S., Viswanath, H.C., "On Common-cause Failures - Bibliography", Microelectronics and Reliability, (to appear).

86. Dhillon, B.S., Viswanath, H.C., "Reliability Modeling of a Two Identical Unit Redundant System with Common-cause Failures", Submitted for Publication in International Journal of Quality and Reliability Management.

87. Dhillon, B.S., Viswanath, H.C., "Reliability Analysis of a Non-Identical Unit Parallel System with Common-cause Failures", Microelectronics and Reliability, (to appear).

88. Dhillon, B.S., Viswanath, H.C., "Stochastic Analysis of Common Systems with Common-cause Failures", Submitted for Publication in Stochastic Analysis and Applications Journal.

89. Dichirico, C., Singh, C., "Reliability Analysis of Transmission Lines with Common-Mode Failures When Repair Times are Random", IEEE Transactions on Power Systems, Vol. 3, No. 3, Aug. 1988, pp. 1012-1019.

90. Dichirico, C., Singh, C., "Availability Analysis of Two Unit Repairable Parallel Redundant System with Common-Cause Failures", Microelectronics and Reliability, Vol. 26, No. 6, 1986, pp. 1183- 1188.

91. Dorre, P., "Pitfalls in Common-cause Failure Data Evaluation", Advanced Seminar of Common-cause Failure Analysis in Probabilistic Safety Assessment, Edited by Aniello Amendola, Kluwer Publications, 1989, Dordrecht, The Netherlands, pp. 205-219.

92. Dorre, P., "Possible Pitfalls in the Process of CCF Event Data Evaluation", Proc. of the International Topical Conference on Probabilistic Safety Assessment and

Risk Management, Zurich, Switzerland, Vol. I, Aug.-Sept. 1987, pp. 74-79.

93. Dorre, P., Schiling, R., "Design Defences Against Common-cause/Multiple Related Failures", Advanced Seminar of Common-cause Failure Analysis in Probabilistic Safety Assessment, Edited by Aniello Amendola, Kluwer Publications, Dordrecht, The Netherlands, 1989, pp. 101-106.

94. Easterling, R.G., "Probabilistic Analysis of Common-Mode Failures", Proc. of the Topical Meeting on Probabilistic Analysis of Nuclear Reactor Safety, Newport Beach, California, Vol. 3, May 1978, pp. X.7-1-X.7-12.

95. Edison, G.E., "Common-Cause Failure Experience for Reliability Analysis and Design Guidelines in Large Plants", Proc. of the Topical Meeting on Probabilistic Analysis of Nuclear Reactor Safety, New Port, California, May 1978, pp. X.6/1-X.6/14.

96. Edwards, G.T., "Alleviation of CMF Problems in Protective Systems", Proc. of the 2nd National Reliability Conference, Birmingham, U.K., March 1979, pp. 5A/2/1-5A/7/1.

97. Edwards, G.T., Watson, I.A., "A Study of Common-Mode Failures", Report No. SRD R 146, Safety and Reliability Directorate, United Kingdom Atomic Energy Authority, Warrington, U.K., July 1976.

98. Endrenyi, J., "Reliability Modeling in Electric Power Systems", John Wiley & sons, New York, 1978.

99. Epler, E.P., "Diversity and Periodic Testing in Defense Against Common-Mode Failure", Proc. of the International Conference on Nuclear Systems Reliability Engineering and Risk Assessment, Gatlinburg, Tennessee, June 1977, pp. 269-287.

100. Epler, E.P., "Common-Mode Failure Considerations in the Design of Systems for Protection and Control", Nuclear Safety, Vol. 10, No. 1, Jan./Feb. 1969, pp. 38-45.

101. Evans, M.G.K., Parry, G.W., Wreathall, J., "On the Treatment of Common-Cause Failures in System Analysis", Reliability Engineering, Vol. 9, 1984, pp. 107-115.

102. Evans, R.A., "Statistical Independence and Common-Mode Failures", IEEE Transactions on Reliability, Vol. 24, Dec. 1985, pp. 289.

103. Fleming, K.N., "A Reliability Model for Common-Mode Failures in Redundant Safety Systems", Proc. of the 6th Pittsburgh Conference on Modeling and Simulation, Part 1, Vol. 6, Pittsburgh, 1975, pp. 579-581.

104. Fleming, K.N., "Parametric Models for Common-cause Failure Analysis", Advanced Seminar of Common-cause Failure Analysis in Probabilistic Safety Assessment, Edited by Aniello Amendola, Kluwer Publications, Dordrecht, The Netherlands, 1989, pp. 159-174.

105. Fleming, K.N., Hannaman, G.W., "Common-Cause Failure Considerations in Predicting HTGR Cooling System Reliability", IEEE Transactions on Reliability, Vol. 25, No. 3, August 1976, pp. 171-177.

106. Fleming, K.N., Mosleh, A, Deremer, R.K., "A Systematic Procedure for the Incorporation of Common-Cause Events Into Risk and Reliability Models", Nuclear Engineering and Design, Vol. 93, 1986, pp. 245-273.

107. Fleming, K.N., Mosleh, A., "Common-Cause Data Analysis and Implications in System Modeling", Proc. of the International Topical Meeting on Probabilistic Safety Methods and Applications, EPRI NP-3912-SR, Vol. 1, Sessions 1-8, California, Feb. 1975, pp. 3/1-3/12.

108. Fleming, K.N., Mosleh, A., Acey, A.L., Worledge, D.H., "Event Classification and Systems Modelling of Common-Cause Failures", Transactions of American Nuclear Society Annual Meeting on Probabilistic Risk Analysis: Methods Development, New Orleans, June 1984, pp. 519-521.

109. Fleming, K.N., Raabe, P.H., "A Comparison of Three Methods for the Quantitative Analysis of Common-Cause Failures", Proc. of the Topical Meeting on Probabilistic Analysis of Nuclear Safety, Vol. III, Newport Beach, California, May 1978, pp. X.3/1-X.3-12.

110. Fragola, J.R., "Common-Mode/Common-Cause Failure Comments for Discussion", Proc. of the 2nd National Reliability Conference, Birmingham, U.K., March 1978.

111. Frederick, L.G., "An Analysis of Functional Common-Mode Failures in GE BWR Protection and Control Instrumentation", Report No. NEDO-10189, General Electric Company, U.S.A., July 1970.

112. Gachot, B., "A Probabilistic Approach Design for the ECCS of PWR", Proc. of the Annual Reliability and Maintainability Symposium, 1977, pp. 332-342.

113. Games, A.M., "Some Aspects of Common-cause Failure Analysis in Engineering Systems", PhD Thesis, University of Liverpool, U.K., Oct. 1986.

114. Games, A.M., Amendola, A., Martin, P., "Exploitation of a Component Event Data Bank for Common-Cause Failure Analysis", Proc. of the International Topical Meeting on Probabilistic Safety Methods and Applications, EPRI NP-3912-SR, Vol. 1, Sessions 1-8, California, Feb. 1975, pp. 1/1-1/11.

115. Games, A.M., Breewood, M., Amendola, A., Martin, P., Keller, A.Z., "Common-Cause Failure Investigation Using the European Reliability Data System", Reliability Engineering, Vol. 13, 1985, pp. 33-44.

116. Games, A.M., Martin, P., Amendola, A., "Multiple Related Component Failure Events", Proceedings of Reliability '85, NCSR, Birmingham, U.K., July 1985.

117. Gangloff, W.C., "Common-Mode Failure Analysis is 'in"', Electrical World, Oct. 1972, pp. 30-33.

118. Gangloff, W.C., "Common-Mode Failure Analysis", IEEE Transactions on Power Apparatus and Systems, Vol. 94, No. 1, Jan./Feb. 1975, pp. 27-30.

119. Garg, R., Goel, L.R., "Cost Analysis of a System with Common-Cause Failure and Two Types of Repair Facilities", Microelectronics and Reliability, Vol. 25, No. 2, 1985, pp. 281- 284.

120. Goel, L.R., Gupta, R., Singh, S.K., "A Two (Multi-component) Unit Parallel System with Standby and Common-Cause Failure", Microelectronics and Reliability, Vol. 24, No. 3, 1984, pp. 415- 418.

121. Goldberg, H., "Extending the Limits of Reliability Theory", Wiley & Sons, New York, 1981, pp. 110-114.

122. Govil, A.K., "Availability of a Complex System Having Shelf-Life of the Components and Common-Cause Failures", Microelectronics and Reliability, Vol. 22, No. 4, 1982, pp. 685-688.

123. Govil, A.K., "Reliability of a Stand-By System with Common-Cause Failures and Scheduled Maintenance", Microelectronics and Reliability, Vol. 21, No. 2, 1981, pp. 269-271.

124. Grant Ireson, W., Clyde, F.C., Editors, "Handbook of Reliability Engineering and Management", McGraw-Hill, New York, 1988, pp. 13.23.

125. Green, A.E., "Safety Systems Reliability", John Wiley & Sons, New York, 1983.

126. Guey, C.N., Heising, C.D., "A Method for Estimating Common-Cause Failure Probability and Model Parameters", Proc. of the International Topical Meeting on Probabilistic Safety Methods and Applications, EPRI NP-3912-SR, Palo Alto, CA, Vol. 3, Sessions 17-23, Feb. 1985, pp. 182/1-182/13.

127. Hagen, E.W., "Common-Mode/Common-Cause Failure: A Review", Nuclear Engineering and Design, Vol. 59, 1980, pp. 423-431.

128. Hagen, E.W., "Technical Note: The Kahl Relay Common-Mode Failure", Nuclear Safety, Vol. 20, No. 5, Sept. 1979, pp. 579-581.

129. Han, S.G., Yoon, W.H., Chang, S.H., "The Trinomial Failure Rate Model for Treating Common-Mode Failures", Reliability Engineering and System Safety, Vol. 25, No. 2, 1989, pp. 131-146.

130. Harris, B., "Stochastic Models for Common-Cause Failures", Proc. of the International Conference on Reliability and Quality Control, 1986, pp. 185-200.

131. Hartung, J., "Common-Cause Failure Theory", Report No. N00IT1000 145, Atomics International Division, Rockwell International, U.S.A., July 1981.

132. Hartung, J., "A Statistical Correlation Model and Proposed General Statement of Theory for Common-Cause Failures", Proc. of the International Meeting on Probabilistic Risk Assessment, Sept. 1981, Port Chester, New York, pp. 239-246.

133. Hayden, K.C., "Common-Mode Failure Mechanisms in Redundant Systems Important to Reactor Safety", Nuclear Safety, Vol. 17, No. 5, 1976, pp. 686-693.

134. Hayden, K.C., "Common-Mode Failure Mechanisms in Nuclear Plants Protection Systems", ERDA Report No. ORNL/TM-4984, Oak Ridge National Laboratory, Oak Ridge, Tennessee, U.S.A., Dec. 1975.

135. Heikkila, M., "A Model for Common-Mode Failures", Proc. of the 2nd National Reliability Conference, Birmingham, U.K., March 1979, pp. 6C/6/1-6C/6/5.

136. Heising, C.D., "Development of Unavailability Expressions for One and Two Component Systems with Periodic Testing and Common-Cause Failures", Reliability Engineering, Vol. 6, No. 4, 1983, pp. 229-254.

137. Heising, C.D., Guey, C.N., "A Comparison of Methods for Calculating System Unavailability Due to Common-Cause Failures", Reliability Engineering, Vol. 8, 1984, pp. 101-116.

138. Heising, C.D., Luciani, D.M., "Application of a Computerized Methodology for Performing Common-Cause Failure Analysis (MOBB) Code", Reliability Engineering, Vol. 17, 1987, pp. 193-210.

139. Heising, C.D., Rasmussen, N.C., Mak, C., "Common-Cause Failure Analysis", Energy Laboratory Report No. MIT-E182-038, MIT, Cambridge, Massachusetts, October 1982.

140. Henley, E.J., Hiromitsu, K., "Reliability Engineering and Risk Management", Prentice-Hall, Englewood Cliffs, New Jersey, 1981, pp. 120-127, 385-390.

141. Henley, E.J., Hiromitsu, K., "Designing for Reliability and Safety Control", Prentice-Hall, Englewood Cliffs, New Jersey, 1985, pp. 18, 401-402.

142. Heylmun, N.F., et al., "Common-Cause Failure Analyses for the Primary Heat Transport System and Reactor Vessel", Report No. WARD-B69101-2, Westinghouse Electric Corporation, Madison, Pennsylvania, July 1976.

143. Hirschberg, S., "Comparison of Methods for Quantitative Analysis of Common-Cause Failures - A Case Study", Proc. of the International Topical Meeting on Probabilistic Safety Methods and Applications, EPRI NP-3912-SR, Vol. 3, Sessions 17-23, California, Feb. 1985, pp. 183/1-183/10.

144. Hirschberg, S., "Treatment of Common-cause Failures, The Nordic Perspective", Advanced Seminar of Common-cause Failure Analysis in Probabilistic Safety Assessment, Edited by Aniello Amendola, Kluwer Publications, 1989, Dordrecht, The Netherlands, pp. 9-29.

145. Hirschberg, S., Pulkkinen, U., "Common-Cause Failure Data: Experience from Diesel Generator Studies", Nuclear Safety, Vol. 26, No. 3, 1985, pp. 305-313.

146. Hirschberg, S., Tiren, L.I., "Design-Related Defensive Measures Against Dependent Failures, ABB Atom's Approach", Advanced Seminar of Common-cause Failure Analysis in Probabilistic Safety Assessment, Edited by Aniello Amendola, Kluwer Publications, 1989, Dordrecht, The Netherlands, pp. 71-100

147. Hirschmann, H., Nicolescu, Weber, R., "Steady State Unavailability of a K-out-of-N Systems with Total Repair", Reliability Engineering, Vol. 4, 1983, pp.

181-197.

148. Hokstad, P., "A Shock Model for Common-Cause Failures", Reliability Engineering and System Safety, Vol. 23, 1988, pp. 127-145.

149. Hudson, J.M. Jean, G., "Common-Mode Failure in Nuclear Power Plants", Proc. of the Annual Reliability and Maintainability Symposium, Jan. 1981, pp. 149-155.

150. Hughes, R.P., "A New Approach to Common-Cause Failures", Reliability Engineering, Vol. 17, 1987, pp. 211-236.

151. Hughes, R.P., "The Relationship Between Common-Cause Failure and Data Uncertainties", International Atomic Energy Authority Seminar on Implications of Probabilistic Risk Analysis, Edited by M.C. Cullingford, S.M. Shah, J.H. Gittus, IAEA-SR-111/10, 1985, pp. 239-252.

152. Humphreys, R.A., "Assigning Numerical Value to the Beta Factor Common-Cause Evaluation", Proc. of National Reliability Conference, Birmingham, U.K., Vol. 1, 1987, pp. 2C/5/1-2C/5/8.

153. Humphreys, P., "Design Defences Against Multiple Related Failures", Advanced Seminar of Common-cause Failure Analysis in Probabilistic Safety Assessment, Edited by Aniello Amendola, Kluwer Publications, 1989, Dordrecht, The Netherlands, pp. 47-70

154. Jacobs, I.M., "The Common-Mode Failure Study Discipline", IEEE Transactions on Nuclear Science, Vol. 17, 1970, pp. 594-598.

155. Jacobs, I.M., "The Nature, Analysis and Impact of Common-Cause Failures on the Design and Licensing of Future LBR's", Report No. REM 80-22, General Electric Co., U.S.A., August 1980.

156. Jerwood, D., Georgiakodia, F.A., "Application of Multivariate Techniques to Monitor System Reliability and Detect Common-Mode Failures", Proc. of the 2nd National Reliability Conference, Birmingham, U.K., Vol. 1, March 1979, pp. 2B/5/1-2B/5/7.

157. John, D., Editor, "The Reliability of Mechanical Systems", Published by Mechanical Engineering Publications Ltd., London, 1988, pp. 81-85.

158. Johnson, J.W., Vesely, W.E., "Common-Mode Analysis of Valve Leakages", Proc. of the Topical Meeting on Probabilistic Analysis of Nuclear Safety, Newport Beach, California, May 1978, Vol. 3, pp. X.5/1-X.5/8.

159. Jolly, M.E., Wreathall, J., "Common-Mode Failures in Reactor Safety Systems", Nuclear Safety, Vol. 18, No. 5, 1977, pp. 624- 632.

160. Jorgensen, G.E., Patton, A.D., Piede, D.L., Ringlee, R.J., "Common-Mode Forced Outages of Overhead Transmission Lines", IEEE Transactions on Power Apparatus and Systems, Vol. 95, No. 3, May 1976, pp. 859-863.

161. Lakner, A.A., "Reliability Engineering for Nuclear and Other High Technology Systems: A Practical Guide", Elsevier Applied Science Publishers, New York, 1985, pp. 85, 129-130.

162. Lewis, E.E., "Introduction to Reliability Engineering", John Wiley & Sons, New York, 1987.

163. Lisboa, J.J., "Quantification of Human Error and Common-Mode Failures in Man-Machine Systems", IEEE Transactions on Nuclear Science, Vol. 35, No. 1, Feb. 1988, pp. 907-911.

164. Lisoba, J.J., "Quantification of Human Error and Common-Mode Failure in Man-Machine Systems", IEEE Transactions on Energy Conversion, Vol. 3, No. 2, June 1988, pp. 292-300. Los Alamos Technical Associates Inc., "Data Benchmark Test of a Classification Procedure for Common-cause Failures", Prepared for EPRI, LATA-EPR-02-02, U.S.A., June 1983.

165. Mahmoud, M, Mokhles, M.A., Saleh, E.H., "Availability Analysis of a Repairable System with Common-Cause Failure and Standby Unit", Microelectronics and Reliability, Vol. 27, No. 4, 1987, pp. 741-754.

166. Mankamo, T., "Common-Load Model, A Tool for Common-Cause Failure Analysis", Finnish Report No. 31, Technical Research Center, Electrical Engineering Laboratory, Espoo, Finland, Dec. 1977.

167. Mankamo, T., "Common-Cause Failures of Reactor Pressure Components", International Atomic Energy Agency, Report No. IAEA-SM-218/5, Vienna, Austria.

168. Mankamo, T., "Common-Mode Failures", Finnish Report No. 18, Technical Research Center, Electrical Engineering Laboratory, Espoo, Finland, May 1976.

169. Mankamo, T., Pulkkinen, U., "Dependent Failures of Diesel Generators", Nuclear Safety, Vol. 23, No. 1, Jan./Feb. 1982, pp. 32-40.

170. Martin, B.R., Wright, R.I., "A Practical Method of Common-Cause Failure Modeling", Reliability Engineering, Vol. 19, 1987, pp. 185-199.

171. Matsuoka, T., "Component Failure Model Dependent on Time and Causes", Nuclear Engineering and Design, Vol. 75, 1982, pp. 109-116.

172. Meachum, T.R., Atwood, C.L., "Common-Cause Fault Rates for Instrumentation and Control Assemblies", Report No. NUREG/CR - 2700, EGG-EA-5485, U.S. Nuclear Regulatory Commission, Washington D.C., 1983.

173. Meslin, T., "Measures Taken at Design Level to Counter Common-cause Failures, A Few Comments Concerning the Approach to EDF, Advanced Seminar of Common-cause Failure Analysis in Probabilistic Safety Assessment, Edited by Aniello Amendola, Kluwer Publications, 1989, Dordrecht, The Netherlands, pp. 107-111.

174. Meslin, T., "Analysis of Common-cause Failures Based on Operating Experience: Possible Approaches and Results", Advanced Seminar of Common-cause Failure Analysis in Probabilistic Safety Assessment, Edited by Aniello Amendola, Kluwer Publications, 1989, Dordrecht, The Netherlands, pp. 257-276.

175. Moody, J.H., Follen, S.M., "Common-Cause Modeling of Reactor Trip Breaker Configuration", Proc. of the International Topical Meeting on Probabilistic Safety Methods and Applications, EPRI NP-3912-SR, Vol. 3, Sessions 17-23, California, Feb. 1985, pp. 180/1-180/10.

176. Mosleh, A., "Estimation of Parameter of Common-cause Failure Models", Advanced Seminar of Common-cause Failure Analysis in Probabilistic Safety Assessment, Edited by Aniello Amendola, Kluwer Publications, 1989, Dordrecht, The Netherlands, pp. 175-203.

177. Mosleh, A., Fleming, K.N., Perry, G.W., Paula, H.M., Worledge, D.H., Dasmuson, D.M., "Procedures for Testing Common-cause Failure Events in Safety and Reliability Studies, Vols. I and II, EPRI NP-5613 and NUREG/CR-4780, Electrical Power Research Institute, 1988.

178. Mosleh, A., Siu, N.O., "A Multi-Parameter, Event Based Common-cause Failure Model", Paper M 7/3, Proc. of the Ninth International Conference on Structural

Mechanics in Reactor Technology, Lausanne, Switzerland, Aug. 1987.

179. Mosleh, A., Siu, N.O., "On the Use of Uncertain Data in Common-cause Failure Analysis," Proc. PSA-87, International Topical Conference on Probabilistic Safety Assessment and Risk Management, Aug.-Sept. 1987.

180. Moret, B.M.E., Thomson, M.G., "Boolean Difference Techniques for Time-Sequence and Common-Cause Analysis of Fault Trees", IEEE Transactions on Reliability, Vol. 33, No. 5, Dec. 84, pp. 399-405.

181. Natesan, J., Jardine, A.K.S., "Stochastic Behaviour of a Single Server n-Unit Pulverizer System with Common-Cause Failures", Microelectronics and Reliability, Vol. 24, No. 6, 1984, pp. 1045-1055.

182. Page, R.J., Kamis, G.J., Marbach, R.A., Mueller, C.J., "A Common- Cause Analysis of the TREAT Upgrade Reactor Protection System", Proc. of the International Topical Meeting on Probabilistic Safety Methods and Applications, EPRI NP-3912-SR, Vol. 1, Sessions 1-8, California, Feb. 1975, pp. 64/1-64/10.

183. Pages, A., Gondran, M., Translated By Griffin, E., "System Reliability: Evaluation and Prediction in Engineering", Springer-Verlag, New York, 1986, pp. 282-289.

184. Paul, F., "Reliability Assessment of Technical Systems by Reference to Examples from the Field of Process Automation", Siemens Power Engineering, Vol. 6, No.2, 1984, pp. 86-91.

185. Peterson, K.E., "Analysis of Common-cause Failure Data - Identification, The Experience from the Nordic Benchmark," Advanced Seminar of Common-cause Failure Analysis in Probabilistic Safety Assessment, Edited by Aniello Amendola, Kluwer Publications, Dordrecht, The Netherlands, 1989, pp. 235-241.

186. Plastiras, J.K., "Inter System Common-Cause Analysis of a Diesel Generator Failure", Risk Analysis, Vol. 6, No. 4, 1986, pp. 463-476.

187. Platz, O., "A Markov Model for Common-Cause Failures", Reliability Engineering, Vol. 9, 1984, pp. 25-31.

188. Porn, K., "Some Comments on CCF Quantification - the Experience from the Nordic Benchmark", Advanced Seminar of Common-cause Failure Analysis in Probabilistic Safety Assessment, Edited by Aniello Amendola, Kluwer Publications, 1989, Dordrecht, The Netherlands, pp. 243-256.

189. Poucet, A., "Experience and Results of the Common-cause Failure - Reliability Benchmark Exercise", Advanced Seminar of Common-cause Failure Analysis in Probabilistic Safety Assessment, Edited by Aniello Amendola, Kluwer Publications, 1989, Dordrecht, The Netherlands, pp. 221-234.

190. Poucet, A., Amendola, A., Cacciabue, P.C., "Common-cause Failure Reliability Benchmark Exercise", Final Report, CEC-JRC, Ispra, EUR 11054 EN, Italy.

191. Poucet, A., Amendola, A., Cacciabue, P.C., "Summary of the Common-cause Failure Reliability Benchmark Exercise", Joint Research Center Report, PER 1133/86, Ispra, Italy, April 1986.

192. Rankin, J.P., "Identification of Common-Cause Failures in Instrumentation and Control Systems", IEEE Transactions on Nuclear Science, Vol. 29, No. 1, Feb. 1982, pp. 979-986.

193. Rankin, J.P., "Common-Cause Hazard Analysis for Random Glitches", Proc. of the Annual Reliability and Maintainability Symposium, 1982, pp. 1-4.

194. Rankin, J.P., "Common-Cause Failure Analysis of Instrumentation and Control Systems", Proc. of the 8th Annual Reliability Engineering Conference for the Electrical Power Industry, Portland, Oregon, April 1981, pp. 195-204.

195. Rankin, J.P., "Common-Cause Failure Analysis – Why Interlocked Redundant Systems Fail", Society of Automotive Engineers, Inc., SAE Technical Paper Series on Turbine Powered Executive Aircraft Meeting, Arizona, U.S.A., Paper No. 800631, April 1980, pp. 1-6.

196. Rasmuson, D.M., Burdick, G., Wilson, J.R., "Common-Cause Failure Analysis Techniques: A review and Comparative Evaluation", Report No. TREE-1349, Idaho National Electric Laboratory, Idaho Falls, Idaho, Sept. 1979.

197. Rasmuson, D.M., et al., "COMCAN IIA - A Computer Program for Automated Common-Cause Failure Analysis", Report No. TREE-1361, EG & G Inc., Idaho National Electric Laboratory, Idaho Falls, Idaho.

198. Sambi, A., Dhillon, B.S., Khan, M.R., "Common-Cause Failure Analysis of Multi-Failure Mode Systems", Proc. of the 10th Annual Pittsburg Conference on Modeling and Simulation, Part 2, Vol. 10, April 1979, pp. 511-515.

199. Sharma, G.C., Goel, L.R., Gupta, P., "Stochastic Analysis of a Parallel System with Common-Cause Failure, Preventive Maintenance", Microelectronics and

Reliability, Vol. 25, No. 6, 1985, pp. 1035-1039.

200. Singh, C., "Reliability Modeling of TMR Computer Systems with Repair and Common-Mode Failures", Microelectronics and Reliability, Vol. 21, No. 2, 1981, pp. 259-262.

201. Singh, C., Reza Ebrahimian, M., "Non-Markovian Models for Common- Mode Failures in Transmission Systems", IEEE Transactions on Power Apparatus and Systems, Vol. 101, No. 6, June, 1982, pp. 1545-1550.

202. Singh, C., Reza Ebrahimian, M., Patton, A.D., "Modeling Common-Mode Failures in Transmission Systems", Proc. of 11th Annual Pittsburg Conference on Modeling and Simulation, Vol. 11, 1980, pp. 863-867.

203. Singh, J., "A Warm Standby Redundant System with Common-Cause Failures", Reliability Engineering and System Safety, Vol. 26, No. 2, 1989, pp. 135-141.

204. Singh, H.R., Singh, S.K., Shukla, S., "Common-Mode Failure Consideration in a Cold Standby Duplex System", Microelectronics and Reliability, Vol. 29, No. 5, 1989, pp. 723-728.

205. Smith, A.M., "Common-Cause Failure - A Mountain or Molehill", Proc. of the 2nd National Reliability Conference, Birmingham, U.K., March 1979, pp. 5A/5/1-5A/5/3.

206. Smith, A.M., Watson, I.A., "Common-Cause Failures - A Dilemma in Perspective", Proc. of the Annual Reliability and Maintainability Symposium, Jan. 1980, pp. 332-339.

207. Stamatelatos, M.G., "Improved Method for Evaluating Common-Cause Failure Probabilities", Transactions of the American Nuclear Society on Probabilistic Risk Assessment, Vol. 43, 1982, pp. 474- 481.

208. Stecher, K., "Fault Tree Analysis, Taking into Account Causes of Common-Mode Failures", Siemens Forschungs-und Entwicklungs-berichte, R & D Reports, Vol. 13, No. 4, 1984, pp. 184-191.

209. Steverson, J.A., Atwood, C.L., "Common-cause Failure Rate Estimate for Diesel Generators in Nuclear Power Plants", International American Nuclear Society/European Nuclear Society Topical Meeting on Probabilistic Risk Assessment, Portchester, Sept. 1981.

210. Taniguchi, T., et al., "Common-Cause Evaluations in Applied Risk Analysis of Nuclear Power Plants", Report No. ORNL/TM-8297, Oak Ridge National Laboratory, Oak Ridge, Tennessee, April 1983.

211. Taylor, J.R., "A Study of Failure Causes Based on U.S. Power Reactor Abnormal Occurrence Reports", Proc. of the Symposium on Reliability of Nuclear Power Plants, Innsbruck, Report No. IAEA-SM-195/16, 1975, pp. 119-130.

212. Taylor, J.R., "Common-Mode and Coupled Failure", Report No. Riso-M-1826, Danish Atomic Energy Commission, Research Establishment RISO, Electronics Department, Oct. 1975, pp. 1-60.

213. Taylor, J.R., "Comments on the Subject of Common-Mode Failures", Proc. of the 2nd National Reliability Conference, Birmingham, U.K., March 1979, pp. 5A/7/1.

214. Unione, A.J., Ritzman, R.L., "Examination of Common-Mode Failure Analysis Methodology", Report No. SAI/SR-141-PA, Science Applications, Inc., U.S.A., Feb. 1976.

215. Vaurio, J., "Availability of Redundant Safety Systems with Common-Mode and Undetected Failures", Nuclear Engineering and Design, Vol. 58, 1980, pp. 415-424.

216. Vaurio, J.K., "Structures for Common-Cause Failure Analysis", International American Nuclear Society/European Nuclear Society Topical Meeting on Probabilistic Risk Assessment, Portchester, Sept., 1981.

217. Vavrek, K.J., Andre, G.R., "Sensitivity Study of Common-Mode Failure Rates for Sizewall B", Proc. of the International Topical Meeting On Probabilistic Safety Methods and Applications, EPRI NP-3912-SR, Vol. 2, Sessions 9-16, California, Feb. 1985, pp. 102/1-102/18.

218. Vesely, W.E., "Estimating Common-Cause Failure Probabilities in Reliability and Risk Analysis", Proc. of the International Conference on Nuclear Systems, Reliability Engineering and Risk Assessment, Gatlinburg, Tennessee, June 1977, pp. 314-341.

219. Victor, B., Joyce, F., "Understanding Systems Failures", Manchester University Press, Manchester, 1984, pp. 187-188.

220. Virolainen, R., "On Common-Cause Failures, Statistical Dependence and Calculation of Uncertainty", Nuclear Engineering and Design, Vol. 77, 1984, pp. 103-108.

221. Virolainen, R., "On Common-Cause Failure Methods Dealing with Dependent Failures; A Comparative Applications", Proc. of the International Topical Meeting on Probabilistic Safety Methods and Applications, EPRI NP-3912-SR, Vol. 1, Sessions 1-8, California, Feb. 1985, pp. 4/1-4/11.

222. Volta, G., "The Common-Mode Failure Analysis", Report No. CEC ISPRA, SR 76, No. 8, Italy, May 1976.

223. Wagner, D.P., "A Procedure for Qualitative Common-Cause Failure Analysis of Complex Systems", M.Sc Thesis, University of Tennessee, Knoxville, Tennessee, 1977.

224. Wagner, D.P., Cate, L.L., Fussel, J.B., "Common-Cause Failure Analysis Methodology for Complex Systems", Proc. of the International Conference on Nuclear Systems Reliability Engineering and Risk Assessment, Gatlinburg, Tennessee, June 1977, pp. 289-341.

225. Watson, I.A., "Analysis of Dependent Events and Multiple Unavailabilities with Particular Reference to Common-Cause", Nuclear Engineering and Design, Vol. 93, 1986, pp. 227-244.

226. Watson, I.A., "Common-Mode/Cause Failures - an Overall Review", in Mechanical Reliability, Science and Technology Press Ltd., U.K., 1980, pp. 136-162.

227. Watson, I.A., "The Rare Event Dilemma and Common-Cause Failures", Proc. of Annual Reliability and Maintainability Symposium, 1982, pp. 5-10.

228. Watson, I.A., "Introductory Paper: Common-Mode Failures", Proc. of the 2nd National Reliability Conference, Birmingham, U.K., March 1979, pp. 5A/1/1-5A/1/7.

229. Watson, I.A., "Review of Common-Cause Failures", Report No. NCSR R 27, United Kingdom Atomic Energy Authority, National Center of Systems Reliability, Warrington, U.K., July 1981, pp. 1-31.

230. Watson, I.A., Edwards, G.T., "Common-Mode Failures in Redundancy Systems", Nuclear Technology, Vol. 46, Dec. 1979, pp. 183-191.

231. Williams, D.W., "Common-Mode Failures in U.S. Commercial Power Reactors", Thesis for M.S. Degree, University of Tennessee, Knoxville, Tennessee, June 1972.

232. Wilson, J.R., Crump, R.J., "Computer-Aided Common-Cause Analysis of an LMFBR System", Transactions of the American Nuclear Society on Safety Systems Reliability Methods and Applications, Vol. 22, 1975, pp. 474-475.

233. Worledge, D.H., Chu, B.B., "Common-Cause Failure and Systems Interactions Issues – An Overview", Proc. of the International Topical Meeting on Probabilistic Safety Methods and Applications, EPRI NP-3912-SR, Vol. 2, Sessions 9-16, California, Feb. 1985, pp. 97/1-97/10.

234. Worledge, D., Wall, I.B., "What Has Been Learned About Common-cause Failures in the Last 5 Years ?", International Topical Conference on Probabilistic Safety Assessment and Risk Management, Zurich, Switzerland, Aug.- Sept. 1987.

235. Worrell, R.B., Burdick, G.R., "Qualitative Analysis in Reliability and Safety Studies", IEEE Transactions on Reliability, Vol. 25, No. 3, Aug. 1976, pp. 164-170.

236. Worrell, R.B., Stack, D.W., "Common-Cause Analysis Using SETS", Report No. SAND 77 - 1832, Sandia Laboratory, Albuquerque, New Mexico, U.S.A., Dec. 1977.

237. Worrell, R.B., Stack, D.W., "A Boolean Approach to Common-Cause Analysis", Proc. of the Annual Reliability and Maintainability Symposium, 1980, pp. 363-366.

238. Wreathall, J., "Limits to Reliability Technology Results", Proc. of the 2nd National Reliability Conference, Birmingham, U.K., March 1979, pp. 5A/4/1-5A/4/4.

239. Wright, R.I., "Some Data On Common-Cause Failures in Redundancy Industrial Computer Systems", The Nuclear Engineer, Vol. 26, No. 3, May 1985, pp. 72-79.

240. Yuan, J., "Pivotal Decomposition to find Availability and Failure-Frequency of Systems with Common-Cause Failures", IEEE Transactions on Reliability, Vol. 36, No. 1, April 1987, pp. 48-53.

241. Yuan, J., "A Conditional Probability Approach to Reliability with Common-Cause Failures", IEEE Transactions on Reliability, Vol. 34, No. 1, April 1985, pp. 38-42.

242. Yuan, J., Lai, M., Kai-Li Ko, "Evaluation of System Reliability with Common-Cause Failures by Pseudo-environments Model", IEEE Transactions on Reliability, Vol. 38, No. 3, Aug. 1989, pp. 328-332.

243. Yun, W.Y., Bai, D.S., "Optimal Numbers of Redundant Units for Parallel Systems with Common-Mode Failures", Reliability Engineering, Vol. 16, 1986, pp. 201-206.

Appendix A

The following constants are associated with the models considered under parallel system :

$$
\begin{aligned}
D_1 \;=\;& (\lambda_1\mu_{c_1}\mu_{20} + \mu_{c_2}\lambda_2\mu_{20} + \lambda_1\mu_{c_2}\mu_2 + \lambda_1\mu_{c_2}\mu_{20} + \mu_{c_2}\mu_1\mu_2 + \\
& \mu_{c_2}\mu_1\mu_{20} + \lambda_{c_1}\mu_1\mu_{20} + \lambda_{c_1}\lambda_2\mu_{20} + \lambda_{c_1}\lambda_2\mu_{c_2} + \lambda_{c_1}\mu_1\mu_2 + \\
& \lambda_1\mu_{c_1}\mu_2 + \lambda_{c_1}\mu_{c_2}\mu_{20} + \mu_{c_1}\mu_1\mu_2 + \mu_{c_1}\mu_1\mu_{20} + \mu_{c_1}\lambda_2\mu_{20} + \mu_{c_1}\lambda_{c_2}\mu_2 + \\
& \mu_{c_1}\lambda_{c_2}\mu_{20} + \lambda_1\lambda_2\mu_{c_1} + \lambda_1\lambda_{c_2}\mu_2 + \lambda_1\lambda_{c_2}\mu_{20} + \lambda_1\lambda_2\mu_{c_2} + \\
& \lambda_{c_1}\lambda_{c_2}\mu_2 + \lambda_{c_1}\lambda_{c_2}\mu_{20} + \lambda_{c_1}\mu_{c_2}\mu_2)
\end{aligned}
$$

$$
\begin{aligned}
D_2 \;=\;& (\mu_{c_3}\mu_2\mu_{30}\mu_1 + \mu_{c_3}\mu_3\lambda_2\mu_{20} + \lambda_{c_2}\mu_{c_1}\mu_{30}\mu_{20} + \mu_{c_3}\lambda_{c_2}\mu_{30}\mu_{20} + \\
& \mu_1\mu_{c_2}\mu_2\mu_{30} + \mu_3\lambda_{c_3}\lambda_{c_2}\mu_{c_1} + \mu_3\lambda_{c_3}\mu_{c_1}\mu_1 + \mu_2\mu_3\mu_{c_1}\mu_1 + \\
& \lambda_3\mu_{c_3}\mu_{30}\mu_1 + \mu_{c_3}\mu_3\lambda_{c_2}\mu_{20} + \mu_{c_3}\lambda_2\mu_{30}\mu_{20} + \mu_{c_1}\mu_{30}\mu_1\mu_{20} + \\
& \mu_1\mu_{c_2}\mu_2\mu_3 + \mu_1\mu_{c_2}\lambda_3\mu_{30} + \mu_1\mu_{c_2}\mu_{20}\mu_{30} + \mu_1\mu_{c_2}\lambda_{c_3}\mu_3 + \\
& \mu_2\mu_3\lambda_{c_2}\mu_{c_1} + \mu_3\lambda_{c_3}\lambda_2\mu_{c_1} + \lambda_3\mu_{c_1}\mu_{30}\mu_1 + \mu_3\lambda_{c_2}\mu_{c_1}\mu_{20} + \\
& \lambda_2\mu_{c_1}\mu_{30}\mu_{20} + \mu_1\mu_{c_2}\mu_{20}\mu_3 + \mu_1\mu_{c_2}\lambda_{c_3}\mu_{30} + \mu_{c_3}\lambda_3\lambda_{c_2}\mu_{30} + \\
& \mu_{c_1}\lambda_3\lambda_{c_2}\mu_{30} + \lambda_{c_3}\lambda_2\mu_{c_1}\mu_{30} + \mu_{c_3}\mu_2\mu_3\mu_1 + \lambda_{c_3}\mu_{c_1}\mu_{30}\mu_1 + \\
& \mu_3\lambda_2\mu_{c_1}\mu_{20} + \mu_{c_3}\mu_3\mu_1\mu_{20} + \mu_3\mu_{c_1}\mu_1\mu_{20} + \mu_{c_3}\mu_{30}\mu_1\mu_{20} + \\
& \lambda_2\mu_{c_2}\lambda_3\mu_{30} + \mu_2\mu_{c_1}\mu_{30}\mu_1 + \mu_{c_3}\lambda_3\lambda_2\mu_{30} + \lambda_{c_3}\lambda_{c_2}\mu_{c_1}\mu_{30} + \\
& \mu_2\lambda_{c_2}\mu_{c_1}\mu_{30} + \mu_{c_1}\lambda_3\lambda_2\mu_{30} + \lambda_2\mu_{c_2}\mu_{20}\mu_3 + \lambda_2\mu_{c_2}\mu_{20}\mu_{30})
\end{aligned}
$$

$$
\begin{aligned}
D_3 \;=\;& (\mu_{c_2}\lambda_1\lambda_{c_3}\mu_{30} + \mu_{c_2}\lambda_1\mu_2\mu_{30} + \mu_{c_2}\lambda_1\lambda_3\mu_{30} + \mu_{c_2}\lambda_1\lambda_{c_3}\mu_3 + \\
& \mu_{c_2}\lambda_1\mu_{20}\mu_{30} + \mu_{c_2}\lambda_1\mu_2\mu_3 + \mu_{c_3}\lambda_1\mu_2\mu_3 + \mu_{c_3}\lambda_1\mu_2\mu_{30} +
\end{aligned}
$$

$$\lambda_1\mu_{c_3}\mu_3\mu_{20} + \mu_{c_1}\lambda_1\lambda_{c_3}\mu_{30} + \mu_{c_1}\lambda_1\lambda_{c_3}\mu_3 + \mu_{c_1}\lambda_1\mu_2\mu_3 + \mu_{c_1}\lambda_1\mu_2\mu_{30} + \mu_{c_1}\lambda_1\lambda_3\mu_{30} + \mu_{c_3}\lambda_1\lambda_3\mu_{30} + \mu_{c_3}\mu_2\mu_3\lambda_{c_1} + \mu_{c_2}\mu_{20}\mu_3\lambda_{c_1} + \mu_{c_2}\mu_2\mu_{30}\lambda_{c_1} + \mu_{c_2}\mu_2\mu_3\lambda_{c_1} + \mu_{c_1}\lambda_1\mu_{30}\mu_{20} + \lambda_1\mu_{c_3}\mu_{30}\mu_{20} + \mu_{c_1}\lambda_1\mu_3\mu_{20} + \mu_{c_2}\mu_{20}\mu_{30}\lambda_{c_1} + \mu_{c_2}\lambda_{c_3}\mu_{30}\lambda_{c_1} + \mu_{c_2}\lambda_3\mu_{30}\lambda_{c_1} + \mu_{c_2}\lambda_{c_3}\mu_3\lambda_{c_1} + \mu_{c_2}\lambda_1\mu_{20}\mu_3 + \mu_{c_3}\mu_2\mu_{30}\lambda_{c_1})$$

$$D_4 = (\mu_{30} + \mu_3)(\mu_{c_3}\mu_1\lambda_{c_1} + \mu_{c_3}\lambda_2\lambda_{c_1} + \mu_{c_2}\lambda_2\lambda_{c_1} + \mu_{c_3}\lambda_{c_2}\lambda_{c_1} + \lambda_2\mu_{c_1}\lambda_1 + \lambda_2\lambda_1\mu_{c_3} + \lambda_2\lambda_1\mu_{c_2} + \lambda_{c_2}\lambda_1\mu_{c_3})$$

$$D_5 = \lambda_3(\mu_{c_3}\mu_1\lambda_{c_1} + \mu_{c_3}\lambda_2\lambda_{c_1} + \mu_{c_2}\lambda_2\lambda_{c_1} + \mu_{c_3}\lambda_{c_2}\lambda_{c_1} + \lambda_2\mu_{c_1}\lambda_1 + \lambda_2\lambda_1\mu_{c_3} + \lambda_2\lambda_1\mu_{c_2} + \lambda_{c_2}\lambda_1\mu_{c_3})$$

$$D_6 = (\lambda_{c_2}\lambda_1\mu_2\mu_{30} + \lambda_{c_2}\lambda_1\lambda_3\mu_{30} + \lambda_2\lambda_1\lambda_{c_3}\mu_3 + \lambda_2\lambda_1\lambda_{c_3}\mu_{30} + \lambda_{c_2}\lambda_1\mu_2\mu_3 + \lambda_{c_2}\lambda_1\mu_{20}\mu_3 + \lambda_{c_2}\lambda_1\lambda_{c_3}\mu_{30} + \lambda_{c_2}\lambda_1\lambda_{c_3}\mu_3 + \lambda_{c_2}\lambda_1\mu_{20}\mu_{30} + \lambda_{c_3}\mu_{30}\mu_1\lambda_{c_1} + \mu_2\mu_3\mu_1\lambda_{c_1} + \mu_3\lambda_{c_3}\mu_1\lambda_{c_1} + \mu_3\lambda_2\mu_{20}\lambda_{c_1} + \mu_3\lambda_{c_2}\mu_{20}\lambda_{c_1} + \mu_3\mu_1\mu_{20}\lambda_{c_1} + \mu_3\lambda_{c_3}\lambda_2\lambda_{c_1} + \mu_{30}\mu_1\mu_{20}\lambda_{c_1} + \mu_2\mu_3\lambda_{c_2}\lambda_{c_1} + \lambda_{c_2}\mu_{30}\mu_{20}\lambda_{c_1} + \lambda_3\lambda_2\mu_{30}\lambda_{c_1} + \mu_3\lambda_{c_3}\lambda_{c_2}\lambda_{c_1} + \lambda_3\mu_{30}\mu_1\lambda_{c_1} + \lambda_2\mu_{30}\mu_{20}\lambda_{c_1} + \lambda_{c_3}\lambda_2\mu_{30}\lambda_{c_1} + \mu_2\mu_{30}\mu_1\lambda_{c_1} + \mu_2\lambda_{c_2}\mu_{30}\lambda_{c_1} + \lambda_3\lambda_{c_2}\mu_{30}\lambda_{c_1} + \lambda_{c_3}\lambda_{c_2}\mu_{30}\lambda_{c_1})$$

$$D_7 = (\lambda_2\lambda_1\lambda_3\mu_{c_3} + \lambda_{c_2}\lambda_1\lambda_3\mu_{c_3} + \lambda_{c_2}\lambda_1\mu_2\mu_{30} + \lambda_{c_2}\lambda_1\lambda_3\mu_{30} + \lambda_2\lambda_1\lambda_{c_3}\mu_3 + \lambda_2\lambda_1\lambda_{c_3}\mu_{30} + \lambda_2\lambda_1\mu_{c_3}\mu_3 + \mu_{c_2}\lambda_1\lambda_{c_3}\mu_{30} + \mu_{c_2}\lambda_1\mu_2\mu_{30} + \mu_{c_2}\lambda_1\lambda_3\mu_{30} + \mu_{c_2}\lambda_1\lambda_{c_3}\mu_3 + \mu_{c_2}\lambda_1\mu_{20}\mu_{30} + \mu_{c_2}\lambda_1\mu_2\mu_3 + \lambda_{c_2}\lambda_1\mu_2\mu_3 + \lambda_{c_2}\lambda_1\mu_{20}\mu_3 + \lambda_{c_2}\lambda_1\lambda_{c_3}\mu_{30} + \lambda_{c_2}\lambda_1\mu_{c_3}\mu_3 + \lambda_{c_2}\lambda_1\mu_{c_3}\mu_{30} + \lambda_{c_2}\lambda_1\lambda_{c_3}\mu_3 + \lambda_{c_2}\lambda_1\mu_{20}\mu_{30} + \mu_{c_3}\lambda_1\mu_2\mu_3 + \mu_{c_3}\lambda_1\mu_2\mu_{30} + \mu_{c_3}\mu_2\mu_{30}\mu_1 + \mu_{c_3}\mu_3\lambda_2\mu_{20} + \lambda_{c_2}\mu_{c_1}\mu_{30}\mu_{20} + \mu_{c_3}\lambda_{c_2}\mu_{30}\mu_{20} + \mu_1\mu_{c_2}\mu_2\mu_{30} + \lambda_1\mu_{c_3}\mu_3\mu_{20} + \lambda_2\mu_{c_1}\lambda_1\lambda_3 + \mu_{c_1}\lambda_1\lambda_{c_3}\mu_{30} + \mu_{c_1}\lambda_1\lambda_{c_3}\mu_3 + \mu_{c_1}\lambda_1\mu_2\mu_3 + \mu_{c_1}\lambda_1\mu_2\mu_{30} + \mu_{c_1}\lambda_1\lambda_3\mu_{30} + \mu_{c_3}\lambda_1\lambda_3\mu_{30} + \lambda_2\mu_{c_1}\lambda_1\mu_3 +$$

$$
\begin{aligned}
&\lambda_2\mu_{c_1}\lambda_1\mu_{30} + \lambda_{c_3}\mu_{30}\mu_1\lambda_{c_1} + \mu_{c_3}\mu_{30}\mu_1\lambda_{c_1} + \mu_2\mu_3\mu_1\lambda_{c_1} + \\
&\mu_3\lambda_{c_3}\mu_1\lambda_{c_1} + \mu_3\lambda_2\mu_{20}\lambda_{c_1} + \mu_3\lambda_{c_2}\mu_{20}\lambda_{c_1} + \mu_{c_3}\lambda_3\lambda_2\lambda_{c_1} + \\
&\mu_3\mu_1\mu_{20}\lambda_{c_1} + \mu_{c_3}\mu_2\mu_3\lambda_{c_1} + \mu_{c_2}\mu_{20}\mu_3\lambda_{c_1} + \mu_{c_2}\mu_2\mu_{30}\lambda_{c_1} + \\
&\mu_{c_2}\mu_2\mu_3\lambda_{c_1} + \mu_{c_3}\lambda_3\lambda_{c_2}\lambda_{c_1} + \mu_3\lambda_{c_3}\lambda_2\lambda_{c_1} + \mu_{c_1}\lambda_1\mu_{30}\mu_{20} + \\
&\lambda_1\mu_{c_3}\mu_{30}\mu_{20} + \mu_{c_1}\lambda_1\mu_3\mu_{20} + \lambda_2\lambda_1\lambda_3\mu_{c_2} + \mu_{30}\mu_1\mu_{20}\lambda_{c_1} + \\
&\mu_2\mu_3\lambda_{c_2}\lambda_{c_1} + \lambda_2\mu_{c_2}\mu_3\lambda_{c_1} + \mu_{c_3}\mu_3\mu_1\lambda_{c_1} + \lambda_2\mu_{c_2}\mu_{30}\lambda_{c_1} + \\
&\lambda_{c_2}\mu_{30}\mu_{20}\lambda_{c_1} + \lambda_3\lambda_2\mu_{30}\lambda_{c_1} + \mu_{c_2}\mu_{20}\mu_{30}\lambda_{c_1} + \mu_{c_2}\lambda_3\lambda_2\lambda_{c_1} + \\
&\mu_3\lambda_{c_3}\lambda_{c_2}\lambda_{c_1} + \mu_{c_3}\mu_3\lambda_{c_2}\lambda_{c_1} + \mu_{c_3}\mu_3\lambda_2\lambda_{c_1} + \mu_{c_2}\lambda_{c_3}\mu_{30}\lambda_{c_1} + \\
&\mu_{c_2}\lambda_3\mu_{30}\lambda_{c_1} + \lambda_3\mu_{30}\mu_1\lambda_{c_1} + \lambda_2\mu_{30}\mu_{20}\lambda_{c_1} + \lambda_{c_3}\lambda_2\mu_{30}\lambda_{c_1} + \\
&\mu_2\mu_{30}\mu_1\lambda_{c_1} + \mu_{c_2}\lambda_{c_3}\mu_3\lambda_{c_1} + \mu_{c_3}\lambda_3\mu_1\lambda_{c_1} + \mu_3\lambda_{c_3}\lambda_{c_2}\mu_{c_1} + \\
&\mu_3\lambda_{c_3}\mu_{c_1}\mu_1 + \mu_2\mu_3\mu_{c_1}\mu_1 + \lambda_3\mu_{c_3}\mu_{30}\mu_1 + \mu_{c_3}\mu_3\lambda_{c_2}\mu_{20} + \\
&\mu_{c_3}\lambda_2\mu_{30}\mu_{20} + \mu_{c_1}\mu_{30}\mu_1\mu_{20} + \mu_1\mu_{c_2}\mu_2\mu_3 + \mu_1\mu_{c_2}\lambda_3\mu_{30} + \\
&\mu_1\mu_{c_2}\mu_{20}\mu_{30} + \mu_1\mu_{c_2}\lambda_{c_3}\mu_3 + \lambda_2\lambda_1\mu_{c_3}\mu_{30} + \lambda_2\lambda_1\mu_{c_2}\mu_3 + \\
&\lambda_2\lambda_1\mu_{c_2}\mu_{30} + \mu_{c_2}\lambda_1\mu_{20}\mu_3 + \mu_{c_3}\mu_2\mu_{30}\lambda_{c_1} + \mu_{c_3}\lambda_2\mu_{30}\lambda_{c_1} + \\
&\mu_2\lambda_{c_2}\mu_{30}\lambda_{c_1} + \lambda_3\lambda_{c_2}\mu_{30}\lambda_{c_1} + \lambda_{c_3}\lambda_{c_2}\mu_{30}\lambda_{c_1} + \mu_{c_3}\lambda_{c_2}\mu_{30}\lambda_{c_1} + \\
&\mu_2\mu_3\lambda_{c_2}\mu_{c_1} + \mu_3\lambda_{c_3}\lambda_2\mu_{c_1} + \lambda_3\mu_{c_1}\mu_{30}\mu_1 + \mu_3\lambda_{c_2}\mu_{c_1}\mu_{20} + \\
&\lambda_2\mu_{c_1}\mu_{30}\mu_{20} + \mu_1\mu_{c_2}\mu_{20}\mu_3 + \mu_1\mu_{c_2}\lambda_{c_3}\mu_{30} + \mu_{c_3}\lambda_3\lambda_{c_2}\mu_{30} + \\
&\mu_{c_1}\lambda_3\lambda_{c_2}\mu_{30} + \lambda_{c_3}\lambda_2\mu_{c_1}\mu_{30} + \mu_{c_3}\mu_2\mu_3\mu_1 + \lambda_{c_3}\mu_{c_1}\mu_{30}\mu_1 + \\
&\mu_3\lambda_2\mu_{c_1}\mu_{20} + \mu_{c_3}\mu_3\mu_1\mu_{20} + \mu_3\mu_{c_1}\mu_1\mu_{20} + \mu_{c_3}\mu_{30}\mu_1\mu_{20} + \\
&\lambda_2\mu_{c_2}\lambda_3\mu_{30} + \mu_2\mu_{c_1}\mu_{30}\mu_1 + \mu_{c_3}\lambda_3\lambda_2\mu_{30} + \lambda_{c_3}\lambda_{c_2}\mu_{c_1}\mu_{30} + \\
&\mu_2\lambda_{c_2}\mu_{c_1}\mu_{30} + \mu_{c_1}\lambda_3\lambda_2\mu_{30} + \lambda_2\mu_{c_2}\mu_{20}\mu_3 + \lambda_2\mu_{c_2}\mu_{20}\mu_{30})
\end{aligned}
$$

$$
\begin{aligned}
D_{12} \;=\;& \{2\lambda_1\lambda_{c_1} - 2\mu_1\lambda_{c_1} - 2\lambda_2\lambda_1 - 2\lambda_2\lambda_{c_1} - 2\lambda_{c_2}\lambda_1 - 2\lambda_{c_2}\lambda_{c_1} + \lambda_1^2 + \lambda_{c_2}^2 + \\
& 2\mu_1\lambda_{c_2} + \mu_1^2 + 2\lambda_2\lambda_{c_2} + 2\lambda_2\mu_1 + \lambda_2^2 + \lambda_{c_1}^2 + 2\lambda_1\mu_1\}^{\frac{1}{2}}
\end{aligned}
$$

$$
\begin{aligned}
D_{13} \;=\;& (\mu_1 + \mu_{20} + \lambda_3 + \lambda_{c_2} + \mu_2 + \lambda_2 + \lambda_{c_3}) \\
D_{14} \;=\;& (\lambda_{c_3}\mu_1 + \mu_2\lambda_{c_2} + \lambda_2\mu_{20} + \lambda_3\lambda_2 + \lambda_3\lambda_{c_2} + \mu_2\mu_1 + \mu_{20}\lambda_{c_2} \\
& \lambda_{c_3}\lambda_2 + \lambda_3\mu_1 + \mu_1\mu_{20} + \lambda_{c_3}\lambda_{c_2}) \\
D_{15} \;=\;& (\lambda_2 + \mu_1 + \lambda_{c_3} + \mu_2 + \lambda_1 + \lambda_3 + \mu_{20} + \lambda_{c_1} + \lambda_{c_2}) \\
D_{16} \;=\;& (\lambda_{c_1}\lambda_{c_3} + \lambda_1\mu_{20} + \mu_1\mu_{20} + \mu_2\lambda_{c_2} + \lambda_{c_3}\mu_1 + \lambda_1\lambda_{c_2} + \lambda_{c_1}\lambda_{c_2} + \lambda_{c_3}\lambda_{c_2} +
\end{aligned}
$$

$$\lambda_1\mu_2 + \lambda_{c_3}\lambda_2 + \lambda_2\mu_{20} + \lambda_3\lambda_{c_2} + \lambda_{c_1}\lambda_3 + \lambda_{c_1}\mu_2 + \lambda_{c_1}\mu_1 + \lambda_3\lambda_2 +$$
$$\mu_{20}\lambda_{c_2} + \lambda_1\lambda_{c_3} + \mu_{20}\lambda_{c_1} + \lambda_3\mu_1 + \mu_2\mu_1 + \lambda_2\lambda_{c_1} + \lambda_2\lambda_1 + \lambda_1\lambda_3)$$

$$
\begin{aligned}
D_{17} &= (\lambda_1\lambda_{c_3}\lambda_{c_2} + \mu_{20}\lambda_{c_1}\lambda_{c_2} + \lambda_1\mu_{20}\lambda_{c_2} + \lambda_{c_1}\lambda_3\mu_1 + \lambda_{c_1}\lambda_3\lambda_2 + \lambda_{c_1}\lambda_3\lambda_{c_2} + \\
&\quad \lambda_1\lambda_3\lambda_{c_2} + \lambda_1\lambda_{c_3}\lambda_2 + \lambda_{c_1}\lambda_{c_3}\mu_1 + \mu_{20}\lambda_{c_1}\lambda_2 + \lambda_{c_1}\lambda_{c_3}\lambda_{c_2} + \mu_1\mu_{20}\lambda_{c_1} + \\
&\quad \lambda_{c_1}\mu_2\mu_1 + \lambda_{c_1}\lambda_{c_3}\lambda_2 + \lambda_1\lambda_3\lambda_2 + \lambda_1\mu_2\lambda_{c_2} + \lambda_{c_1}\mu_2\lambda_{c_2})
\end{aligned}
$$

$$
\begin{aligned}
D_{18} &= (\lambda_4 + \lambda_{c_4} + \lambda_{c_3} + \mu_{20} + \mu_2 + \lambda_2 + \mu_{30} + \mu_1 + \lambda_3 + \lambda_{c_2} + \mu_3) \\
D_{19} &= (\lambda_2\lambda_{c_4} + \mu_1\lambda_3 + \lambda_{c_2}\mu_{30} + \lambda_2\lambda_3 + \lambda_{c_2}\lambda_{c_4} + \lambda_2\mu_3 + \mu_{20}\mu_3 + \lambda_4\lambda_3 + \\
&\quad \lambda_{c_3}\mu_3 + \mu_{30}\mu_2 + \mu_{30}\lambda_3 + \lambda_4\mu_2 + \lambda_{c_4}\mu_2 + \lambda_{c_2}\mu_{20} + \lambda_{c_2}\lambda_{c_3} + \lambda_{c_2}\mu_3 + \\
&\quad \lambda_{c_2}\mu_2 + \mu_1\mu_2 + \lambda_2\lambda_{c_3} + \mu_1\lambda_4 + \lambda_{c_2}\lambda_3 + \mu_1\lambda_{c_4} + \lambda_2\lambda_4 + \mu_1\mu_3 + \\
&\quad \lambda_2\mu_{20} + \lambda_{c_2}\lambda_4 + \mu_1\mu_{30} + \mu_{20}\lambda_4 + \lambda_{c_3}\lambda_4 + \mu_3\mu_2 + \mu_{20}\lambda_{c_4} + \lambda_{c_3}\lambda_{c_4} + \\
&\quad \lambda_2\mu_{30} + \mu_1\mu_{20} + \mu_1\lambda_{c_3} + \mu_{20}\mu_{30} + \lambda_{c_3}\mu_{30} + \lambda_{c_4}\lambda_3) \\
D_{20} &= (\lambda_2\mu_{20}\mu_{30} + \lambda_2\lambda_{c_3}\mu_{30} + \lambda_2\lambda_{c_3}\lambda_4 + \lambda_{c_2}\mu_{30}\mu_2 + \\
&\quad \mu_1\lambda_{c_3}\mu_3 + \lambda_{c_2}\lambda_{c_3}\mu_{30} + \lambda_2\lambda_4\lambda_3 + \lambda_2\lambda_{c_3}\mu_3 + \\
&\quad \mu_1\lambda_4\lambda_3 + \lambda_{c_2}\lambda_4\mu_2 + \mu_1\lambda_4\mu_2 + \lambda_2\mu_{20}\lambda_4 + \\
&\quad \mu_1\mu_{20}\mu_{30} + \lambda_{c_2}\mu_{30}\lambda_3 + \lambda_{c_2}\lambda_{c_3}\mu_3 + \mu_1\mu_{30}\lambda_3 + \\
&\quad \lambda_{c_2}\mu_3\mu_2 + \lambda_{c_2}\lambda_4\lambda_3 + \lambda_{c_2}\lambda_{c_4}\mu_2 + \mu_1\lambda_{c_3}\lambda_4 + \\
&\quad \mu_1\mu_{20}\lambda_4 + \lambda_{c_2}\lambda_{c_3}\lambda_4 + \lambda_{c_2}\mu_{20}\lambda_4 + \lambda_{c_2}\mu_{20}\lambda_{c_4} + \\
&\quad \mu_1\mu_3\mu_2 + \mu_1\lambda_{c_3}\lambda_{c_4} + \lambda_{c_2}\lambda_{c_3}\lambda_{c_4} + \lambda_{c_2}\mu_{20}\mu_3 + \\
&\quad \mu_1\mu_{30}\mu_2 + \lambda_{c_2}\lambda_{c_4}\lambda_3 + \lambda_2\lambda_{c_4}\lambda_3 + \lambda_2\mu_{20}\lambda_{c_4} + \\
&\quad \mu_1\mu_{20}\lambda_{c_4} + \mu_1\mu_{20}\mu_3 + \mu_1\lambda_{c_3}\mu_{30} + \lambda_2\lambda_{c_3}\lambda_{c_4} + \\
&\quad \lambda_{c_2}\mu_{20}\mu_{30} + \lambda_2\mu_{20}\mu_3 + \lambda_2\mu_{30}\lambda_3 + \mu_1\lambda_{c_4}\mu_2 + \mu_1\lambda_{c_4}\lambda_3) \\
D_{21} &= (\lambda_4 + \mu_3 + \lambda_{c_4} + \mu_{20} + \lambda_{c_1} + \lambda_1 + \mu_2 + \lambda_2 + \mu_1 + \lambda_{c_3} + \mu_{30} + \lambda_3 + \lambda_{c_2}) \\
D_{22} &= (\mu_1\lambda_{c_1} + \lambda_2\lambda_{c_4} + \mu_1\lambda_3 + \lambda_{c_2}\mu_{30} + \lambda_2\lambda_3 + \lambda_{c_2}\lambda_{c_4} + \\
&\quad \lambda_2\mu_3 + \lambda_1\mu_3 + \lambda_{c_1}\mu_3 + \lambda_{c_1}\mu_{20} + \lambda_1\lambda_{c_3} + \mu_{20}\mu_3 + \lambda_4\lambda_3 + \lambda_{c_3}\mu_3 + \\
&\quad \mu_{30}\mu_2 + \mu_{30}\lambda_3 + \lambda_4\mu_2 + \lambda_{c_4}\mu_2 + \lambda_{c_2}\mu_{20} + \lambda_{c_2}\lambda_{c_3} + \lambda_{c_2}\mu_3 + \lambda_{c_2}\mu_2 + \\
&\quad \mu_1\mu_2 + \lambda_2\lambda_{c_3} + \mu_1\lambda_4 + \lambda_{c_2}\lambda_3 + \mu_1\lambda_{c_4} + \lambda_2\lambda_4 + \mu_1\mu_3 + \lambda_2\mu_{20} + \\
&\quad \lambda_{c_2}\lambda_4 + \mu_1\mu_{30} + \mu_{20}\lambda_4 + \lambda_{c_3}\lambda_4 + \mu_3\mu_2 + \mu_{20}\lambda_{c_4} + \lambda_{c_3}\lambda_{c_4} + \lambda_2\mu_{30} + \\
&\quad \mu_1\mu_{20} + \mu_1\lambda_{c_3} + \mu_{20}\mu_{30} + \lambda_{c_3}\mu_{30} + \lambda_{c_4}\lambda_3 + \lambda_{c_1}\lambda_3 + \lambda_1\mu_{30} + \lambda_{c_1}\lambda_4 + \\
&\quad \lambda_{c_1}\lambda_{c_4} + \lambda_1\mu_2 + \lambda_{c_1}\mu_2 + \lambda_1\mu_{20} + \lambda_{c_1}\mu_{30} + \lambda_1\lambda_{c_4} + \lambda_1\lambda_4 +
\end{aligned}
$$

$$
\begin{aligned}
& \lambda_{c_1}\lambda_2 + \lambda_1\lambda_2 + \lambda_1\lambda_{c_2} + \lambda_{c_1}\lambda_{c_2} + \lambda_1\lambda_3 + \lambda_{c_1}\lambda_{c_3}) \\
D_{23} \;=\; & (\lambda_2\mu_{20}\mu_{30} + \lambda_1\lambda_{c_2}\mu_2 + \lambda_{c_1}\lambda_2\lambda_3 + \lambda_1\lambda_{c_2}\lambda_4 + \lambda_1\lambda_4\mu_2 + \\
& \lambda_1\lambda_{c_3}\mu_{30} + \lambda_{c_1}\mu_{20}\mu_{30} + \lambda_{c_1}\lambda_{c_2}\mu_{30} + \lambda_1\mu_3\mu_2 + \lambda_{c_1}\lambda_2\mu_3 + \\
& \lambda_1\lambda_2\lambda_3 + \lambda_{c_1}\lambda_2\lambda_{c_4} + \lambda_2\lambda_{c_3}\mu_{30} + \lambda_2\lambda_{c_3}\lambda_4 + \lambda_{c_2}\mu_{30}\mu_2 + \\
& \lambda_{c_1}\lambda_4\lambda_3 + \mu_1\lambda_{c_1}\lambda_{c_3} + \lambda_1\lambda_{c_3}\mu_3 + \lambda_1\mu_{20}\mu_3 + \lambda_1\lambda_{c_3}\lambda_{c_4} + \\
& \lambda_1\lambda_{c_2}\lambda_{c_4} + \lambda_1\lambda_{c_2}\lambda_3 + \lambda_1\lambda_2\lambda_{c_3} + \lambda_1\lambda_2\lambda_{c_4} + \lambda_{c_1}\mu_{20}\lambda_4 + \\
& \lambda_1\lambda_2\mu_{30} + \lambda_{c_1}\lambda_{c_2}\lambda_3 + \lambda_1\lambda_{c_2}\mu_{30} + \lambda_1\mu_{30}\lambda_3 + \lambda_{c_1}\lambda_{c_4}\mu_2 + \\
& \lambda_1\lambda_2\lambda_4 + \lambda_{c_1}\lambda_{c_3}\lambda_4 + \lambda_{c_1}\mu_{30}\mu_2 + \lambda_{c_1}\mu_{20}\lambda_{c_4} + \lambda_1\lambda_{c_2}\mu_3 + \\
& \lambda_1\lambda_{c_2}\lambda_{c_3} + \lambda_{c_1}\lambda_2\mu_{30} + \lambda_{c_1}\lambda_4\mu_2 + \lambda_{c_1}\lambda_{c_3}\lambda_{c_4} + \\
& \lambda_{c_1}\mu_{30}\lambda_3 + \mu_1\lambda_{c_1}\mu_2 + \lambda_1\lambda_{c_2}\mu_{20} + \mu_1\lambda_{c_1}\lambda_3 + \lambda_1\lambda_2\mu_3 + \\
& \lambda_{c_1}\lambda_{c_3}\mu_{30} + \mu_1\lambda_{c_3}\mu_3 + \lambda_{c_1}\lambda_2\lambda_{c_3} + \lambda_{c_1}\lambda_2\lambda_4 + \\
& \mu_1\lambda_{c_1}\lambda_{c_4} + \lambda_{c_1}\lambda_{c_3}\mu_3 + \lambda_{c_1}\mu_3\mu_2 + \mu_1\lambda_{c_1}\mu_{30} + \lambda_1\lambda_4\lambda_3 + \\
& \lambda_{c_2}\lambda_{c_3}\mu_{30} + \lambda_2\lambda_4\lambda_3 + \lambda_2\lambda_{c_3}\mu_3 + \mu_1\lambda_4\lambda_3 + \lambda_{c_2}\lambda_4\mu_2 + \\
& \mu_1\lambda_4\mu_2 + \lambda_2\mu_{20}\lambda_4 + \mu_1\mu_{20}\mu_{30} + \lambda_{c_2}\mu_{30}\lambda_3 + \lambda_1\mu_{20}\lambda_4 + \\
& \lambda_{c_1}\lambda_2\mu_{20} + \lambda_1\lambda_{c_3}\lambda_4 + \lambda_{c_1}\lambda_{c_2}\mu_{20} + \mu_1\lambda_{c_1}\mu_3 + \lambda_1\mu_{20}\lambda_{c_4} + \\
& \lambda_1\mu_{20}\mu_{30} + \lambda_1\lambda_{c_4}\lambda_3 + \lambda_{c_2}\lambda_{c_3}\mu_3 + \lambda_{c_1}\lambda_{c_2}\mu_3 + \mu_1\mu_{30}\lambda_3 + \\
& \lambda_{c_2}\mu_3\mu_2 + \lambda_{c_2}\lambda_4\lambda_3 + \lambda_{c_2}\lambda_{c_4}\mu_2 + \mu_1\lambda_{c_3}\lambda_4 + \mu_1\mu_{20}\lambda_4 + \\
& \lambda_{c_2}\lambda_{c_3}\lambda_4 + \lambda_1\mu_{30}\mu_2 + \lambda_{c_2}\mu_{20}\lambda_4 + \lambda_{c_2}\mu_{20}\lambda_{c_4} + \\
& \mu_1\mu_3\mu_2 + \mu_1\lambda_{c_3}\lambda_{c_4} + \lambda_{c_2}\lambda_{c_3}\lambda_{c_4} + \lambda_{c_2}\mu_{20}\mu_3 + \\
& \mu_1\lambda_{c_1}\mu_{20} + \lambda_{c_1}\lambda_{c_2}\lambda_{c_4} + \lambda_{c_1}\mu_{20}\mu_3 + \mu_1\mu_{30}\mu_2 + \\
& \mu_1\lambda_{c_1}\lambda_4 + \lambda_{c_2}\lambda_{c_4}\lambda_3 + \lambda_2\lambda_{c_4}\lambda_3 + \lambda_2\mu_{20}\lambda_{c_4} + \\
& \mu_1\mu_{20}\lambda_{c_4} + \mu_1\mu_{20}\mu_3 + \lambda_{c_1}\lambda_{c_2}\lambda_{c_3} + \lambda_{c_1}\lambda_{c_4}\lambda_3 + \\
& \mu_1\lambda_{c_3}\mu_{30} + \lambda_{c_1}\lambda_{c_2}\mu_2 + \lambda_2\lambda_{c_3}\lambda_{c_4} + \lambda_{c_2}\mu_{20}\mu_{30} + \\
& \lambda_1\lambda_{c_4}\mu_2 + \lambda_2\mu_{20}\mu_3 + \lambda_{c_1}\lambda_{c_2}\lambda_4 + \lambda_2\mu_{30}\lambda_3 + \mu_1\lambda_{c_4}\mu_2 + \mu_1\lambda_{c_4}\lambda_3) \\
D_{24} \;=\; & (\lambda_{c_1}\lambda_{c_2}\lambda_{c_3}\lambda_{c_4} + \lambda_{c_1}\lambda_2\mu_{20}\mu_3 + \lambda_{c_1}\lambda_2\lambda_{c_3}\lambda_4 + \\
& \lambda_{c_1}\lambda_2\mu_{30}\lambda_3 + \lambda_{c_1}\lambda_2\mu_{20}\lambda_{c_4} + \lambda_1\lambda_{c_2}\lambda_{c_4}\lambda_3 + \lambda_1\lambda_{c_2}\lambda_4\lambda_3 + \\
& \mu_1\lambda_{c_1}\lambda_{c_3}\lambda_4 + \lambda_1\lambda_{c_2}\mu_{30}\lambda_3 + \lambda_1\lambda_{c_2}\mu_3\mu_2 + \mu_1\lambda_{c_1}\lambda_4\mu_2 + \\
& \lambda_{c_1}\lambda_{c_2}\lambda_{c_3}\lambda_4 + \lambda_1\lambda_{c_2}\mu_{20}\mu_{30} + \mu_1\lambda_{c_1}\mu_{30}\mu_2 + \mu_1\lambda_{c_1}\mu_{30}\lambda_3 + \\
& \mu_1\lambda_{c_1}\mu_3\mu_2 + \lambda_1\lambda_{c_2}\lambda_{c_3}\lambda_{c_4} + \lambda_1\lambda_2\lambda_4\lambda_3 + \mu_1\lambda_{c_1}\lambda_{c_4}\lambda_3 +
\end{aligned}
$$

$$
\begin{aligned}
& \mu_1\lambda_{c_1}\mu_{20}\lambda_4 + \mu_1\lambda_{c_1}\mu_{20}\lambda_{c_4} + \mu_1\lambda_{c_1}\mu_{20}\mu_3 + \mu_1\lambda_{c_1}\lambda_4\lambda_3 + \\
& \mu_1\lambda_{c_1}\lambda_{c_3}\mu_{30} + \lambda_{c_1}\lambda_{c_2}\mu_{20}\mu_3 + \lambda_1\lambda_{c_2}\mu_{30}\mu_2 + \lambda_{c_1}\lambda_{c_2}\mu_{20}\lambda_4 + \\
& \lambda_{c_1}\lambda_2\lambda_{c_3}\lambda_{c_4} + \lambda_1\lambda_2\lambda_{c_3}\mu_{30} + \lambda_1\lambda_2\lambda_{c_3}\lambda_4 + \lambda_1\lambda_2\lambda_{c_4}\lambda_3 + \\
& \lambda_{c_1}\lambda_{c_2}\mu_{20}\lambda_{c_4} + \lambda_{c_1}\lambda_2\mu_{20}\lambda_4 + \lambda_1\lambda_{c_2}\lambda_4\mu_2 + \lambda_{c_1}\lambda_2\lambda_{c_3}\mu_{30} + \\
& \lambda_{c_1}\lambda_2\lambda_4\lambda_3 + \lambda_{c_1}\lambda_2\lambda_{c_3}\mu_3 + \lambda_{c_1}\lambda_2\lambda_{c_4}\lambda_3 + \lambda_1\lambda_{c_2}\mu_{20}\mu_3 + \\
& \lambda_{c_1}\lambda_2\mu_{20}\mu_{30} + \mu_1\lambda_{c_1}\mu_{20}\mu_{30} + \lambda_{c_1}\lambda_{c_2}\mu_{30}\lambda_3 + \lambda_{c_1}\lambda_{c_2}\lambda_4\lambda_3 + \\
& \mu_1\lambda_{c_1}\lambda_{c_3}\mu_3 + \lambda_{c_1}\lambda_{c_2}\lambda_{c_3}\mu_{30} + \lambda_1\lambda_2\lambda_{c_3}\lambda_{c_4} + \lambda_1\lambda_{c_2}\mu_{20}\lambda_4 + \\
& \lambda_1\lambda_{c_2}\mu_{20}\lambda_{c_4} + \lambda_{c_1}\lambda_{c_2}\mu_3\mu_2 + \lambda_{c_1}\lambda_{c_2}\mu_{30}\mu_2 + \lambda_{c_1}\lambda_{c_2}\lambda_4\mu_2 + \\
& \lambda_1\lambda_{c_2}\lambda_{c_4}\mu_2 + \lambda_1\lambda_2\lambda_{c_3}\mu_3 + \mu_1\lambda_{c_1}\lambda_{c_4}\mu_2 + \lambda_{c_1}\lambda_{c_2}\lambda_{c_4}\mu_2 + \\
& \lambda_1\lambda_{c_2}\lambda_{c_3}\mu_{30} + \lambda_{c_1}\lambda_{c_2}\lambda_{c_3}\mu_3 + \lambda_{c_1}\lambda_{c_2}\mu_{20}\mu_{30} + \mu_1\lambda_{c_1}\lambda_{c_3}\lambda_{c_4} + \\
& \lambda_1\lambda_{c_2}\lambda_{c_3}\mu_3 + \lambda_{c_1}\lambda_{c_2}\lambda_{c_4}\lambda_3 + \lambda_1\lambda_{c_2}\lambda_{c_3}\lambda_4)
\end{aligned}
$$

$$
\begin{aligned}
D_{25} &= (\mu_3 + \lambda_{c_4} + \lambda_3 + \mu_{30} + \mu_{20} + \lambda_{c_3} + \mu_2 + \lambda_4)\lambda_1 \\
D_{26} &= (\mu_{20}\mu_{30} + \lambda_{c_3}\mu_{30} + \lambda_{c_4}\lambda_3 + \mu_{20}\lambda_4 + \lambda_{c_3}\lambda_4 + \mu_3\mu_2 + \mu_{20}\lambda_{c_4} + \lambda_{c_3}\lambda_{c_4} + \\
&\quad \mu_{20}\mu_3 + \lambda_4\lambda_3 + \lambda_{c_3}\mu_3 + \mu_{30}\mu_2 + \mu_{30}\lambda_3 + \lambda_4\mu_2 + \lambda_{c_4}\mu_2)\lambda_1)
\end{aligned}
$$

$$
\begin{aligned}
D_{27} &= [s_4^4 + (D_{31} + D_{32})s_4^3 + \\
&\quad (D_{31}D_{32} + D_{33})s_4^2 + (D_{31}D_{33} + D_{34})s_4 + D_{31}D_{34}] \\
D_{28} &= [s_3^4 + (D_{31} + D_{32})s_3^3 + \\
&\quad (D_{31}D_{32} + D_{33})s_3^2 + (D_{31}D_{33} + D_{34})s_3 + D_{31}D_{34}] \\
D_{29} &= [s_2^4 + (D_{31} + D_{32})s_2^3 + \\
&\quad (D_{31}D_{32} + D_{33})s_2^2 + (D_{31}D_{33} + D_{34})s_2 + D_{31}D_{34}] \\
D_{30} &= [s_1^4 + (D_{31} + D_{32})s_1^3 + \\
&\quad (D_{31}D_{32} + D_{33})s_1^2 + (D_{31}D_{33} + D_{34})s_1 + D_{31}D_{34}] \\
D_{31} &= (\mu_b + \lambda_a + \lambda_{c_2}) \\
D_{32} &= (\lambda_b + \lambda_{c_3} + \mu_a + \mu_c + \mu_c) \\
D_{33} &= (\mu_a\mu_c + \mu_c\lambda_b + \lambda_{c_3}\mu_c + \mu_a\mu_c + \lambda_{c_3}\mu_c + \mu_c\mu_c + \mu_c\lambda_b) \\
D_{34} &= (\mu_c\lambda_{c_3}\mu_c + \mu_a\mu_c\mu_c + \mu_c\lambda_b\mu_c) \\
D_{35} &= (2\lambda_b + \lambda_{c_3} + \lambda_{c_1} + \mu_c + \mu_a + 2\lambda_a + \mu_b + \lambda_{c_2} + \mu_c) \\
D_{36} &= (\lambda_{c_1}\mu_a + \mu_b\mu_c + \lambda_{c_2}\lambda_{c_3} + \lambda_b\mu_b + \lambda_a^2 + \lambda_{c_2}\mu_a + 2\lambda_a\mu_c + \lambda_b^2 + \mu_a\lambda_b + \\
&\quad 2\lambda_a\lambda_{c_3} + \lambda_{c_1}\lambda_{c_3} + \lambda_b\lambda_{c_3} + \mu_c\mu_b + \mu_c\lambda_{c_2} + \mu_a\mu_b + \lambda_{c_2}\mu_c + 3\lambda_b\lambda_a + \lambda_{c_1}\mu_c +
\end{aligned}
$$

$$2\lambda_a\mu_c + \lambda_{c_1}\lambda_b + 2\mu_c\lambda_b + \lambda_a\mu_b + \lambda_a\mu_a + \lambda_{c_1}\lambda_{c_2} + \lambda_{c_3}\mu_c + \mu_c\mu_c + 2\mu_c\lambda_b +$$
$$\mu_a\mu_c + \mu_a\mu_c + \lambda_{c_3}\mu_c + 2\lambda_b\lambda_{c_2} + \lambda_a\lambda_{c_2} + \lambda_{c_1}\lambda_a + \lambda_{c_1}\mu_b + \lambda_{c_3}\mu_b)$$

$$D_{37} = (2\lambda_a\mu_c\mu_c + \mu_c\lambda_{c_3}\mu_c + \mu_a\mu_c\mu_c + \lambda_b^2\lambda_a + \mu_c\mu_c\mu_b + \mu_c\mu_c\lambda_{c_2} +$$
$$\lambda_b^2\mu_c + \lambda_{c_1}\lambda_{c_2}\lambda_{c_3} + \lambda_b\lambda_{c_1}\lambda_a + \lambda_a\mu_b\lambda_{c_3} + \mu_c\lambda_b^2 + \lambda_a^2\mu_c + \mu_a\lambda_{c_2}\lambda_b +$$
$$\lambda_a^2\lambda_{c_3} + \lambda_b^2\lambda_{c_2} + \lambda_{c_1}\mu_b\lambda_{c_3} + \lambda_a\lambda_{c_2}\lambda_{c_3} + \lambda_b\lambda_{c_1}\lambda_{c_2} + \mu_a\lambda_{c_1}\lambda_a + \lambda_a\mu_c\mu_a +$$
$$\lambda_{c_1}\lambda_a\lambda_{c_3} + \lambda_a\mu_c\lambda_{c_2} + \mu_a\lambda_{c_1}\mu_b + \mu_c\lambda_b\lambda_{c_2} + \mu_c\lambda_{c_2}\mu_a + \mu_a\mu_c\mu_b + \lambda_a\mu_c\mu_b +$$
$$\mu_c\lambda_b\mu_b + \mu_c\lambda_{c_3}\mu_b + 3\lambda_b\lambda_a\mu_c + \lambda_a\lambda_{c_3}\mu_c + \lambda_{c_2}\lambda_{c_3}\mu_c + \lambda_{c_1}\mu_a\mu_c +$$
$$\lambda_a\mu_b\mu_c + \lambda_b\lambda_a\mu_b + \lambda_b\lambda_a\lambda_{c_2} + \lambda_a\mu_a\mu_c + \lambda_b\lambda_a\mu_a + \lambda_{c_1}\mu_b\mu_c + \lambda_{c_1}\lambda_a\mu_c +$$
$$2\lambda_b\lambda_{c_2}\mu_c + \lambda_{c_1}\lambda_{c_2}\mu_c + \lambda_{c_3}\mu_b\mu_c + \lambda_a\lambda_{c_2}\mu_c + \lambda_{c_2}\lambda_b\lambda_{c_3} + \mu_c\lambda_b\lambda_{c_3} +$$
$$\lambda_b\lambda_a\mu_c + \lambda_{c_2}\mu_a\mu_c + \mu_a\mu_b\mu_c + \lambda_b\mu_b\mu_c + \mu_a\mu_c\lambda_b + \mu_a\mu_c\lambda_b +$$
$$2\lambda_a\lambda_{c_3}\mu_c + \lambda_{c_2}\lambda_{c_3}\mu_c + \lambda_b\lambda_{c_3}\mu_c + \mu_c\lambda_{c_1}\lambda_{c_3} + \lambda_{c_1}\lambda_b\mu_c + \lambda_a\lambda_{c_3}\lambda_b +$$
$$\lambda_a\lambda_{c_1}\mu_b + \mu_a\lambda_{c_1}\lambda_{c_2} + 2\mu_c\lambda_b\mu_c + \lambda_a^2\mu_c + \lambda_b\lambda_a^2)$$

$$D_{38} = (\lambda_a\lambda_{c_3}\mu_c\mu_c + \lambda_a\lambda_{c_3}\mu_c\lambda_b + \lambda_a\lambda_{c_3}\mu_b\mu_c + \lambda_b\lambda_a\mu_c\mu_b + \lambda_{c_1}\lambda_a\lambda_{c_3}\mu_c +$$
$$\lambda_{c_1}\lambda_{c_2}\lambda_{c_3}\mu_c + \lambda_{c_2}\lambda_b\lambda_{c_3}\mu_c + \lambda_b\lambda_{c_1}\lambda_a\mu_c + \lambda_b\lambda_a\mu_c\mu_c + \lambda_b\lambda_a^2\mu_c +$$
$$\lambda_b^2\lambda_a\mu_c + \lambda_b\lambda_a\mu_c\lambda_{c_2} + \lambda_{c_1}\lambda_{c_3}\mu_b\mu_c + \lambda_b\lambda_a\mu_c\mu_a + \mu_c\lambda_{c_2}\mu_a\mu_c +$$
$$\mu_a\mu_c\mu_b\mu_c + \mu_a\lambda_{c_1}\mu_b\mu_c + \lambda_a\mu_c\mu_b\mu_c + \mu_c\lambda_b\mu_b\mu_c + \mu_c\lambda_{c_3}\mu_b\mu_c +$$
$$\lambda_b\lambda_{c_1}\mu_b\mu_c + \lambda_a^2\mu_c\mu_c + \lambda_b^2\lambda_{c_2}\mu_c + \lambda_{c_2}\lambda_b\mu_a\mu_c + \lambda_{c_1}\lambda_{c_2}\mu_a\mu_c +$$
$$\lambda_a\mu_c\mu_a\mu_c + \lambda_{c_1}\lambda_a\mu_a\mu_c + \lambda_a\mu_c\lambda_{c_2}\mu_c + \mu_c\lambda_b\lambda_{c_2}\mu_c + \lambda_a\lambda_{c_3}\lambda_{c_2}\mu_c +$$
$$\lambda_b\lambda_{c_1}\lambda_{c_2}\mu_c + \mu_c\lambda_b\lambda_{c_3}\mu_c + \mu_a\mu_c\mu_c\lambda_b + \lambda_{c_2}\lambda_{c_3}\mu_c\mu_c + \lambda_a^2\lambda_{c_3}\mu_c + \mu_c\lambda_b^2\mu_c)$$

$$D_{39} = (\mu_c + \lambda_a + \mu_b + \lambda_{c_2} + \mu_c)$$

$$D_{40} = (\mu_c\mu_b + \mu_c\lambda_{c_2} + \lambda_a\mu_c + \mu_b\mu_c + \lambda_a\mu_c + \mu_c\mu_c + \lambda_{c_2}\mu_c)$$

$$D_{41} = (\mu_c\mu_b\mu_c + \lambda_a\mu_c\mu_c + \mu_c\lambda_{c_2}\mu_c)$$

$$D_{42} = (\lambda_b + \lambda_{c_3} + \mu_a + \mu_c + \mu_c)$$

$$D_{43} = (\mu_a\mu_c + \mu_c\lambda_b + \lambda_{c_3}\mu_c + \mu_a\mu_c + \lambda_{c_3}\mu_c + \mu_c\mu_c + \mu_c\lambda_b)$$

$$D_{44} = (\mu_c\lambda_{c_3}\mu_c + \mu_a\mu_c\mu_c + \mu_c\lambda_b\mu_c)$$

$$D_{45} = (\lambda_a + 2\mu_c + \mu_b + \mu_a + \lambda_b + \lambda_{c_3} + \lambda_{c_2})$$

$$D_{46} = (\lambda_a\mu_c + \mu_a\mu_c + \mu_c\mu_b + \mu_c\lambda_{c_2} + \lambda_{c_3}\mu_c + \mu_c\lambda_b)$$

$$D_{47} = (\lambda_{c_1}\mu_a + \lambda_{c_1}\mu_b + \lambda_{c_1}\lambda_a + \lambda_{c_1}\lambda_{c_3} + \lambda_a\lambda_{c_3} + \lambda_{c_1}\lambda_{c_2} + \lambda_{c_1}\lambda_b + \lambda_b\lambda_{c_2})$$

$$D_{48} = (\lambda_{c_2}\lambda_b\lambda_{c_3} + \lambda_{c_1}\lambda_{c_2}\lambda_{c_3} + \lambda_{c_1}\mu_b\lambda_{c_3} + \lambda_b\lambda_{c_1}\lambda_a + \lambda_b\lambda_{c_1}\lambda_{c_2} + \lambda_b\lambda_{c_1}\mu_b + \mu_a\lambda_{c_1}\lambda_a +$$

$$\mu_a\lambda_{c_1}\lambda_{c_2} + \lambda_a\mu_b\lambda_{c_1} + \mu_a\lambda_{c_1}\mu_b + \lambda_b^2\lambda_{c_2} + \lambda_a\lambda_{c_3} + \mu_a\lambda_{c_2}\lambda_b + \lambda_a\lambda_{c_2}\lambda_{c_3} + \lambda_a^2\lambda_{c_3})$$

$$\begin{aligned}
D_{49} = {} & (\lambda_a\mu_c\mu_c\lambda_b + \lambda_{c_2}\mu_c\mu_c\lambda_{c_3} + \lambda_a\mu_c\mu_c\lambda_{c_3} + \lambda_a\mu_c\mu_c\mu_a + \mu_b\lambda_a\mu_c\mu_c + \\
& \lambda_{c_2}\lambda_a\mu_c\mu_c + \mu_b\lambda_a\mu_c\lambda_b + \lambda_{c_2}\lambda_a\mu_c\lambda_b + \lambda_a\mu_c\lambda_b\mu_a + \lambda_a\mu_c\lambda_b\lambda_{c_3} + \\
& \lambda_{c_2}\mu_c\lambda_{c_1}\mu_a + \mu_b\mu_c\lambda_{c_1}\lambda_{c_3} + \mu_b\mu_c\lambda_{c_1}\mu_a + \mu_b\lambda_a\mu_c\lambda_{c_3} + \mu_b\mu_c\lambda_{c_1}\lambda_b + \\
& \lambda_{c_2}\mu_c\lambda_{c_1}\lambda_{c_3} + \lambda_{c_2}\mu_c\lambda_b\mu_a + \lambda_a\mu_c\lambda_{c_1}\lambda_{c_3} + \lambda_a\mu_c\lambda_{c_1}\lambda_b + \lambda_a\mu_c\lambda_{c_1}\mu_a + \\
& \lambda_{c_2}\mu_c\lambda_b\lambda_{c_3} + \lambda_{c_2}\lambda_a\mu_c\lambda_{c_3} + \lambda_{c_2}\mu_c\lambda_{c_1}\lambda_b + \mu_c\mu_c\lambda_b\mu_a + \mu_c\mu_c\lambda_b\lambda_{c_3} + \\
& \lambda_{c_2}\mu_c\mu_c\mu_a + \lambda_{c_2}\mu_c\mu_c\lambda_b + \mu_b\mu_c\mu_c\lambda_{c_3} + \mu_b\mu_c\mu_c\mu_a + \lambda_a^2\mu_c\mu_c + \\
& \mu_c\mu_c\lambda_b^2 + \mu_b\mu_c\mu_c\lambda_b + \lambda_a^2\mu_c\lambda_b + \lambda_a\mu_c\lambda_b^2 + \lambda_{c_2}\mu_c\lambda_b^2 + \lambda_a^2\mu_c\lambda_{c_3})
\end{aligned}$$

$$\begin{aligned}
D_{50} = {} & (\mu_b + \lambda_a + \lambda_{c_2} + \mu_a + \lambda_b + \lambda_{c_3}) \\
D_{51} = {} & (\mu_a\mu_b + \mu_a\lambda_a + \mu_a\lambda_{c_2} + \lambda_b\mu_b + \lambda_a\lambda_b + \lambda_b\lambda_{c_2} + \lambda_{c_3}\mu_b + \lambda_{c_3}\lambda_a + \lambda_{c_3}\lambda_{c_2}) \\
D_{52} = {} & (\lambda_{c_3} + 2\lambda_b + \lambda_{c_1} + \mu_b + \lambda_{c_2} + \mu_a + 2\lambda_a) \\
D_{53} = {} & (\lambda_{c_3}\mu_b + \lambda_b^2 + \lambda_{c_3}\lambda_{c_2} + \lambda_b\mu_a + \lambda_a\lambda_{c_2} + 3\lambda_a\lambda_b + \mu_a\mu_b + \lambda_a^2 + \\
& \mu_a\lambda_a + \lambda_{c_1}\lambda_b + \lambda_a\mu_b + \mu_a\lambda_{c_2} + \lambda_{c_1}\mu_a + \lambda_b\mu_b + \lambda_{c_1}\lambda_{c_3} + \lambda_{c_1}\mu_b + \\
& 2\lambda_b\lambda_{c_2} + \lambda_{c_1}\lambda_{c_2} + \lambda_b\lambda_{c_3} + \lambda_{c_1}\lambda_a + 2\lambda_{c_3}\lambda_a) \\
D_{54} = {} & (\lambda_a\lambda_{c_3}\lambda_{c_2} + \lambda_b^2\lambda_{c_2} + \lambda_{c_1}\lambda_b\lambda_{c_2} + \lambda_{c_1}\lambda_a\mu_a + \lambda_{c_1}\lambda_{c_3}\lambda_{c_2} + \lambda_{c_1}\lambda_{c_3}\lambda_a + \\
& \lambda_{c_1}\mu_b\lambda_{c_3} + \lambda_b\lambda_{c_3}\lambda_{c_2} + \lambda_{c_1}\lambda_b\mu_b + \lambda_a\lambda_b\mu_b + \mu_b\lambda_a\lambda_{c_3} + \lambda_a\lambda_b^2 + \\
& \lambda_b\lambda_a^2 + \lambda_b\lambda_{c_2}\lambda_a + \lambda_a\lambda_b\mu_a + \lambda_a\lambda_b\lambda_{c_3} + \lambda_a^2\lambda_{c_3} + \lambda_{c_1}\lambda_b\lambda_a + \\
& \lambda_{c_1}\lambda_{c_2}\mu_a + \lambda_b\lambda_{c_2}\mu_a + \lambda_{c_1}\mu_b\mu_a) \\
D_{55} = {} & \lambda_a(\mu_b + \lambda_a + \lambda_{c_2}) \\
D_{56} = {} & \lambda_b(\mu_a + \lambda_b + \lambda_{c_3}) \\
D_{57} = {} & \lambda_a\lambda_b(\mu_b + \lambda_a + \lambda_{c_2} + \mu_a + \lambda_b + \lambda_{c_3}) \\
D_{58} = {} & (\lambda_{c_1}\mu_b + \lambda_{c_1}\lambda_a + \lambda_{c_1}\lambda_{c_2} + \lambda_{c_1}\mu_a + \lambda_b\lambda_{c_2} + \lambda_{c_1}\lambda_b + \lambda_{c_1}\lambda_{c_3} + \lambda_{c_3}\lambda_a) \\
D_{59} = {} & (\lambda_{c_1}\mu_b\mu_a + \lambda_{c_1}\lambda_a\mu_a + \lambda_{c_1}\lambda_b\mu_b + \lambda_{c_1}\lambda_b\lambda_a + \lambda_{c_1}\lambda_b\lambda_{c_2} + \lambda_{c_1}\mu_b\lambda_{c_3} + \\
& \lambda_{c_1}\lambda_{c_3}\lambda_a + \lambda_{c_1}\lambda_{c_3}\lambda_{c_2} + \mu_b\lambda_a\lambda_{c_3} + \lambda_a^2\lambda_{c_3} + \lambda_a\lambda_{c_3}\lambda_{c_2} + \lambda_{c_1}\lambda_{c_2}\mu_a + \\
& \lambda_b\lambda_{c_2}\mu_a + \lambda_b^2\lambda_{c_2} + \lambda_b\lambda_{c_3}\lambda_{c_2})
\end{aligned}$$

$$\begin{aligned}
D_{60} = {} & (-\lambda_b\lambda_{c_2}^2\lambda_{c_1}\lambda_a - \lambda_{c_1}\lambda_{c_2}\lambda_b\lambda_a^2 - \lambda_{c_3}^2\lambda_{c_1}\lambda_b\lambda_a - \lambda_{c_3}\lambda_{c_1}\lambda_b^2\lambda_a - \\
& \lambda_{c_1}\lambda_{c_2}\lambda_{c_3}\lambda_a - \lambda_{c_1}\lambda_b^2\lambda_{c_3}\lambda_{c_2} - \lambda_{c_1}\lambda_{c_2}^2\lambda_{c_3}\lambda_a - \lambda_{c_3}^2\lambda_{c_2}\lambda_b\lambda_{c_1} +
\end{aligned}$$

$$
\begin{aligned}
& \lambda_{c_1}^2\lambda_{c_3}\lambda_a^2 - \lambda_b^2\lambda_{c_2}\lambda_a^2 - \lambda_b^2\lambda_{c_2}^2\lambda_{c_1} - \lambda_{c_3}^2\lambda_b\lambda_a^2 + \\
& \lambda_{c_1}^3\lambda_{c_2}\lambda_{c_3} + \lambda_{c_3}^2\lambda_{c_2}^2\lambda_b + \lambda_b\lambda_{c_1}^3\lambda_a + \lambda_{c_1}^2\lambda_b\lambda_a^2 - \\
& \lambda_{c_1}^2\lambda_{c_2}^2\lambda_b - \lambda_b^2\lambda_{c_3}\lambda_a^2 + \lambda_{c_3}\lambda_{c_1}^3\lambda_a + \lambda_{c_3}^2\lambda_{c_2}^2\lambda_a + \\
& \lambda_b\lambda_{c_2}\lambda_{c_1}^3 - \lambda_{c_3}^2\lambda_{c_1}^2\lambda_a - \lambda_{c_3}^2\lambda_{c_1}\lambda_a^2 - \lambda_{c_1}^2\lambda_{c_2}^2\lambda_{c_3} + \\
& \lambda_b^2\lambda_{c_1}\lambda_a^2 - \lambda_b^2\lambda_{c_2}^2\lambda_a - \lambda_{c_3}^2\lambda_{c_1}^2\lambda_{c_2} + \lambda_b^2\lambda_{c_1}^2\lambda_a + \\
& \lambda_{c_2}\lambda_{c_3}^2\lambda_a^2 + \lambda_{c_3}\lambda_{c_2}^2\lambda_b^2 + \lambda_{c_3}^2\lambda_{c_1}\lambda_{c_2}^2 + \lambda_b^2\lambda_{c_2}\lambda_{c_1}^2)
\end{aligned}
$$

$$
\begin{aligned}
D_{61} &= (\lambda_{c_2}\lambda_b\lambda_{c_3}\lambda_a^2 + 2\lambda_{c_3}\lambda_{c_1}\lambda_b\lambda_{c_2}\lambda_a + \lambda_b^3\lambda_{c_2}\lambda_a - \lambda_{c_2}\lambda_b^2\lambda_{c_3}^2 - \\
&\quad \lambda_{c_3}\lambda_b^3\lambda_{c_2} + \lambda_b^2\lambda_{c_2}\lambda_a^2 + \lambda_{c_1}\lambda_b^3\lambda_a - \lambda_{c_2}^2\lambda_b\lambda_{c_2}\lambda_a + \\
&\quad \lambda_b^2\lambda_{c_2}\lambda_{c_1}^2 + \lambda_{c_1}^2\lambda_b\lambda_{c_2}\lambda_{c_3} - \lambda_{c_3}^2\lambda_{c_2}\lambda_b\lambda_{c_1} + 2\lambda_b^2\lambda_{c_2}\lambda_{c_1}\lambda_a) \\
D_{62} &= (-\lambda_{c_3}\lambda_{c_2}\lambda_a^3 - \lambda_{c_3}\lambda_{c_2}^2\lambda_a^2 + \lambda_{c_3}\lambda_{c_2}\lambda_b^2\lambda_a + \lambda_{c_1}^2\lambda_{c_2}\lambda_{c_3}\lambda_a + \\
&\quad \lambda_{c_1}^2\lambda_{c_3}\lambda_a^2 + \lambda_b^2\lambda_{c_3}\lambda_a^2 + 2\lambda_{c_3}\lambda_{c_1}\lambda_b\lambda_{c_2}\lambda_a - \lambda_{c_3}\lambda_{c_2}^2\lambda_b\lambda_a - \\
&\quad \lambda_{c_1}\lambda_{c_2}^2\lambda_{c_3}\lambda_a + \lambda_{c_1}\lambda_{c_3}\lambda_a^3 + 2\lambda_b\lambda_{c_3}\lambda_{c_1}\lambda_a^2 + \lambda_b\lambda_{c_3}\lambda_a^3) \\
D_{63} &= (\lambda_b^2\lambda_{c_2} + \lambda_{c_2}\lambda_b\lambda_{c_3} + \lambda_{c_1}\lambda_{c_2}\lambda_b + \lambda_{c_1}\lambda_a\lambda_b + \lambda_{c_2}\lambda_{c_3}\lambda_a + \lambda_{c_1}\lambda_{c_2}\lambda_{c_3} + \lambda_{c_3}\lambda_a^2 + \\
&\quad \lambda_{c_1}\lambda_{c_3}\lambda_a)
\end{aligned}
$$

$$
\begin{aligned}
D_{64} &= (D_{80}D_{79}D_{78}\mu_b D_{75}D_{77}D_{76}\mu_a) \\
D_{65} &= (D_{80}D_{79}D_{78}\mu_b D_{77}D_{76}\mu_a\lambda_c) \\
D_{66} &= (D_{80}D_{79}D_{78}\mu_b D_{75}D_{77}\mu_a\lambda_b) \\
D_{67} &= (D_{80}D_{79}D_{78}\mu_b D_{75}D_{76}\mu_a\lambda_a) \\
D_{68} &= (\lambda_c\lambda_b(D_{75} + D_{76})D_{80}D_{79}\mu_b D_{77}\mu_a) \\
D_{69} &= (\lambda_a\lambda_b(D_{77} + D_{76})D_{80}D_{78}\mu_b D_{75}\mu_a) \\
D_{70} &= (\lambda_c\lambda_a(D_{75} + D_{77})D_{79}D_{78}\mu_b D_{76}\mu_a) \\
D_{71} &= \lambda_a\lambda_c\lambda_b\mu_a(D_{75}D_{80}D_{79}D_{77} + D_{76}D_{80}D_{79}D_{77} + D_{77}D_{80}D_{78}D_{75} + \\
&\quad D_{76}D_{80}D_{78}D_{75} + D_{75}D_{79}D_{78}D_{76} + D_{77}D_{79}D_{78}D_{76})
\end{aligned}
$$

$$
\begin{aligned}
D_{72} &= -\mu_b(D_{80}D_{78}D_{75}\lambda_c D_{77}\lambda_a\lambda_b + D_{80}D_{78}D_{75}\lambda_c D_{76}\lambda_b\lambda_a - \\
&\quad D_{80}D_{79}D_{78}D_{75}D_{77}D_{76}D_{74} + D_{79}D_{78}D_{76}\lambda_b D_{77}\lambda_a\lambda_c + \\
&\quad D_{79}D_{78}D_{76}\lambda_b D_{75}\lambda_c\lambda_a + D_{80}D_{79}D_{77}\lambda_a D_{75}\lambda_c\lambda_b + \\
&\quad D_{80}D_{79}D_{77}\lambda_a D_{76}\lambda_b\lambda_c)
\end{aligned}
$$

$$
D_{73} = (D_{80}D_{79}D_{78}\mu_b D_{75}D_{77}D_{76}D_{74} + D_{80}D_{79}D_{78}\mu_b D_{75}D_{77}D_{76}\mu_a +
$$

$$
\begin{aligned}
&D_{80}D_{79}D_{78}\mu_b D_{75}D_{77}\mu_a\lambda_b + D_{80}D_{79}D_{78}\mu_b D_{75}D_{76}\mu_a\lambda_a + \\
&D_{80}D_{79}D_{78}\mu_b D_{77}D_{76}\mu_a\lambda_c - D_{80}D_{79}D_{77}\mu_b\lambda_a D_{75}\lambda_c\lambda_b - \\
&D_{80}D_{79}D_{77}\mu_b\lambda_a D_{76}\lambda_b\lambda_c + \\
&D_{80}D_{79}\mu_b D_{77}\mu_a D_{75}\lambda_c\lambda_b + D_{80}D_{79}\mu_b D_{77}\mu_a D_{76}\lambda_b\lambda_c + \\
&D_{80}D_{79}D_{77}\mu_a\lambda_a D_{75}\lambda_c\lambda_b + D_{80}D_{79}D_{77}\mu_a\lambda_a D_{76}\lambda_b\lambda_c + \\
&D_{80}D_{78}\mu_b D_{75}\mu_a D_{77}\lambda_a\lambda_b + D_{80}D_{78}\mu_b D_{75}\mu_a D_{76}\lambda_b\lambda_a + \\
&D_{80}D_{78}D_{75}\lambda_c\mu_a D_{77}\lambda_a\lambda_b + D_{80}D_{78}D_{75}\lambda_c\mu_a D_{76}\lambda_b\lambda_a - \\
&D_{80}D_{78}D_{75}\lambda_c\mu_b D_{77}\lambda_a\lambda_b - D_{80}D_{78}D_{75}\lambda_c\mu_b D_{76}\lambda_b\lambda_a + \\
&D_{79}D_{78}D_{76}\mu_b\mu_a D_{75}\lambda_c\lambda_a + D_{79}D_{78}D_{76}\mu_b\mu_a D_{77}\lambda_a\lambda_c + \\
&D_{79}D_{78}D_{76}\lambda_b\mu_a D_{75}\lambda_c\lambda_a + D_{79}D_{78}D_{76}\lambda_b\mu_a D_{77}\lambda_a\lambda_c - \\
&D_{79}D_{78}D_{76}\lambda_b\mu_b D_{75}\lambda_c\lambda_a - D_{79}D_{78}D_{76}\lambda_b\mu_b D_{77}\lambda_a\lambda_c)
\end{aligned}
$$

$$
\begin{aligned}
D_{74} &= (\lambda_a + \lambda_b + \lambda_c + \lambda_{c_1}) \\
D_{75} &= (\lambda_a + \lambda_b + \lambda_{c_2}) \\
D_{76} &= (\lambda_a + \lambda_c + \lambda_{c_3}) \\
D_{77} &= (\lambda_b + \lambda_c + \lambda_{c_4}) \\
D_{78} &= (\lambda_a + \lambda_{c_5}) \\
D_{79} &= (\lambda_c + \lambda_{c_6}) \\
D_{80} &= (\lambda_b + \lambda_{c_7})
\end{aligned}
$$

$$
\begin{aligned}
D_{81} &= (\lambda_a + \lambda_b + \lambda_c + \lambda_{c_1}) \\
D_{82} &= (\lambda_a + \lambda_b + \mu_c + \lambda_{c_2}) \\
D_{83} &= (\lambda_a + \lambda_c + \mu_b + \lambda_{c_3}) \\
D_{84} &= (\lambda_b + \lambda_c + \mu_a + \lambda_{c_4}) \\
D_{85} &= (\lambda_a + \lambda_{c_5}) \\
D_{86} &= (\lambda_c + \lambda_{c_6}) \\
D_{87} &= (\lambda_b + \lambda_{c_7}) \\
D_{88} &= D_{83} + D_{82} + D_{84} \\
D_{89} &= D_{82}D_{83} + D_{84}D_{83} + D_{84}D_{82} \\
D_{90} &= D_{84}D_{82}D_{83}
\end{aligned}
$$

$$
\begin{aligned}
D_{91} &= D_{S4} + D_{S1} + D_{S2} + D_{S3} \\
D_{92} &= (D_{S4}D_{S2} - \mu_b\lambda_b + D_{S4}D_{S3} + D_{S2}D_{S1} + D_{S2}D_{S3} + \\
&\quad D_{S1}D_{S3} + D_{S4}D_{S1} - \lambda_a\mu_a - \mu_c\lambda_c) \\
D_{93} &= (-\mu_b D_{S4}\lambda_b - \mu_a\lambda_a D_{S3} - \mu_c\lambda_c D_{S3} + D_{S1}D_{S4}D_{S3} + D_{S4}D_{S2}D_{S3} + \\
&\quad D_{S1}D_{S2}D_{S3} - \mu_c D_{S4}\lambda_c + D_{S1}D_{S4}D_{S2} - D_{S2}\mu_b\lambda_b - \mu_a\lambda_a D_{S2}) \\
D_{94} &= (-\mu_c D_{S4}\lambda_c D_{S3} - D_{S4}D_{S2}\mu_b\lambda_b - \mu_a\lambda_a D_{S2}D_{S3} + D_{S1}D_{S4}D_{S2}D_{S3}) \\
D_{95} &= D_{S3} + D_{S4} \\
D_{96} &= D_{S2} + D_{S4} \\
D_{97} &= D_{S3} + D_{S2} \\
D_{98} &= D_{S2} + 2D_{S4} + D_{S3} \\
D_{99} &= D_{S4}D_{S2} + D_{S4}D_{S3} \\
D_{100} &= 2D_{S2} + D_{S4} + D_{S3} \\
D_{101} &= D_{S4}D_{S2} + D_{S2}D_{S3} \\
D_{102} &= 2D_{S3} + D_{S4} + D_{S2} \\
D_{103} &= D_{S4}D_{S3} + D_{S2}D_{S3} \\
D_{104} &= (\lambda_a + \lambda_b + \lambda_c + \lambda_{c_1}) \\
D_{105} &= (\lambda_a + \lambda_b + \lambda_{c_2}) \\
D_{106} &= (\lambda_a + \lambda_c + \lambda_{c_3}) \\
D_{107} &= (\lambda_b + \lambda_c + \lambda_{c_4}) \\
D_{108} &= (\lambda_a + \lambda_{c_5}) \\
D_{109} &= (\lambda_c + \lambda_{c_6}) \\
D_{110} &= (\lambda_b + \lambda_{c_7})
\end{aligned}
$$

Appendix B

The following constants are associated with the models considered under the k-out-of-n category :

$$
\begin{aligned}
E_1 = {} & (\lambda_2\mu_{c_2}\mu_{20}\mu_{30} + \lambda_2\mu_{c_2}\lambda_3\mu_{30} + \lambda_3\mu_{c_1}\mu_{30}\mu_1 + \mu_2\mu_{c_1}\mu_{30}\mu_1 + \\
& \mu_3\lambda_{c_2}\mu_{c_1}\mu_{20} + \mu_3\lambda_{c_3}\lambda_{c_2}\mu_{c_1} + \mu_2\mu_3\lambda_{c_2}\mu_{c_1} + \mu_3\lambda_{c_3}\lambda_2\mu_{c_1} + \\
& \mu_{c_1}\lambda_3\lambda_{c_2}\mu_{30} + \mu_3\lambda_{c_3}\mu_{c_1}\mu_1 + \mu_1\mu_{c_2}\mu_{20}\mu_3 + \mu_1\mu_{c_2}\mu_2\mu_{30} + \\
& \mu_1\mu_{c_2}\lambda_3\mu_{30} + \mu_1\mu_{c_2}\lambda_{c_3}\mu_{30} + \mu_{c_3}\lambda_{c_2}\mu_{30}\mu_{20} + \lambda_{c_2}\mu_{c_1}\mu_{30}\mu_{20} + \\
& \mu_{c_3}\mu_3\mu_1\mu_{20} + \mu_3\mu_{c_1}\mu_1\mu_{20} + \mu_{c_1}\lambda_3\lambda_2\mu_{30} + \mu_{c_3}\lambda_3\lambda_2\mu_{30} + \\
& \mu_{c_1}\mu_{30}\mu_1\mu_{20} + \lambda_{c_3}\lambda_{c_2}\mu_{c_1}\mu_{30} + \mu_1\mu_{c_2}\mu_2\mu_3 + \mu_1\mu_{c_2}\lambda_{c_3}\mu_3 + \\
& \mu_{c_3}\lambda_2\mu_{30}\mu_{20} + \mu_3\lambda_2\mu_{c_1}\mu_{20} + \lambda_2\mu_{c_1}\mu_{30}\mu_{20} + \mu_{c_3}\mu_2\mu_3\mu_1 + \\
& \mu_{c_3}\mu_2\mu_{30}\mu_1 + \lambda_{c_3}\mu_{c_1}\mu_{30}\mu_1 + \mu_2\mu_3\mu_{c_1}\mu_1 + \mu_{c_3}\mu_3\lambda_{c_2}\mu_{20} + \\
& \mu_{c_3}\mu_3\lambda_2\mu_{20} + \mu_{c_3}\mu_{30}\mu_1\mu_{20} + \lambda_{c_3}\lambda_2\mu_{c_1}\mu_{30} + \mu_{c_3}\lambda_3\mu_{30}\mu_1 + \\
& \mu_{c_3}\lambda_3\lambda_{c_2}\mu_{30} + \mu_2\lambda_{c_2}\mu_{c_1}\mu_{30} + \mu_1\mu_{c_2}\mu_{20}\mu_{30} + \lambda_2\mu_{c_2}\mu_{20}\mu_3)
\end{aligned}
$$

$$
\begin{aligned}
E_2 = {} & (\mu_{c_2}\lambda_3\mu_{30}\lambda_{c_1} + \mu_{c_3}\mu_2\mu_{30}\lambda_{c_1} + \mu_{c_3}\mu_2\mu_3\lambda_{c_1} + \mu_{c_2}\mu_2\mu_3\lambda_{c_1} + \\
& \mu_{c_2}\mu_2\mu_{30}\lambda_{c_1} + \mu_{c_2}\lambda_{c_3}\mu_3\lambda_{c_1} + \mu_{c_2}\lambda_1\mu_{20}\mu_3 + \mu_{c_2}\lambda_1\lambda_{c_3}\mu_{30} + \\
& \mu_{c_2}\lambda_1\lambda_{c_3}\mu_3 + \mu_{c_2}\lambda_1\lambda_3\mu_{30} + \mu_{c_2}\lambda_1\mu_{20}\mu_{30} + \lambda_1\mu_{c_3}\mu_{30}\mu_{20} + \\
& \mu_{c_1}\lambda_1\mu_3\mu_{20} + \lambda_1\mu_{c_3}\mu_3\mu_{20} + \mu_{c_1}\lambda_1\mu_{30}\mu_{20} + \mu_{c_1}\lambda_1\mu_2\mu_3 + \\
& \mu_{c_1}\lambda_1\mu_2\mu_{30} + \mu_{c_1}\lambda_1\lambda_{c_3}\mu_3 + \mu_{c_1}\lambda_1\lambda_3\mu_{30} + \mu_{c_3}\lambda_1\lambda_3\mu_{30} + \\
& \mu_{c_1}\lambda_1\lambda_{c_3}\mu_{30} + \mu_{c_2}\lambda_1\mu_2\mu_{30} + \mu_{c_2}\lambda_1\mu_2\mu_3 + \mu_{c_3}\lambda_1\mu_2\mu_{30} + \\
& \mu_{c_3}\lambda_1\mu_2\mu_3 + \mu_{c_2}\lambda_{c_3}\mu_{30}\lambda_{c_1} + \mu_{c_2}\mu_{20}\mu_{30}\lambda_{c_1} + \mu_{c_2}\mu_{20}\mu_3\lambda_{c_1}) \\
E_3 = {} & (\mu_{30} + \mu_3)(\mu_{c_3}\lambda_2\lambda_{c_1} + \lambda_2\mu_{c_1}\lambda_1 + \mu_{c_3}\mu_1\lambda_{c_1} + \\
& \mu_{c_3}\lambda_{c_2}\lambda_{c_1} + \lambda_{c_2}\lambda_1\mu_{c_3} + \lambda_2\lambda_1\mu_{c_2} + \mu_{c_2}\lambda_2\lambda_{c_1} + \lambda_2\lambda_1\mu_{c_3})
\end{aligned}
$$

$$
\begin{aligned}
E_4 &= \lambda_3(\mu_{c_3}\lambda_2\lambda_{c_1} + \lambda_2\mu_{c_1}\lambda_1 + \mu_{c_3}\mu_1\lambda_{c_1} + \mu_{c_3}\lambda_{c_2}\lambda_{c_1} + \lambda_{c_2}\lambda_1\mu_{c_3} + \\
&\quad \lambda_2\lambda_1\mu_{c_2} + \mu_{c_2}\lambda_2\lambda_{c_1} + \lambda_2\lambda_1\mu_{c_3}) \\
E_5 &= (\mu_3\lambda_{c_3}\lambda_{c_2}\lambda_{c_1} + \mu_3\lambda_{c_3}\lambda_2\lambda_{c_1} + \lambda_3\lambda_2\mu_{30}\lambda_{c_1} + \lambda_{c_2}\lambda_1\mu_2\mu_{30} + \\
&\quad \lambda_{c_2}\lambda_1\mu_{20}\mu_{30} + \lambda_{c_2}\lambda_1\lambda_{c_3}\mu_3 + \lambda_{c_2}\lambda_1\mu_2\mu_3 + \mu_2\lambda_{c_2}\mu_{30}\lambda_{c_1} + \\
&\quad \lambda_3\lambda_{c_2}\mu_{30}\lambda_{c_1} + \lambda_{c_3}\lambda_{c_2}\mu_{30}\lambda_{c_1} + \mu_2\mu_3\lambda_{c_2}\lambda_{c_1} + \lambda_2\lambda_1\lambda_{c_3}\mu_{30} + \\
&\quad \lambda_2\lambda_1\lambda_{c_3}\mu_3 + \lambda_{c_2}\lambda_1\mu_{20}\mu_3 + \lambda_{c_2}\lambda_1\lambda_3\mu_{30} + \lambda_{c_2}\lambda_1\lambda_{c_3}\mu_{30} + \\
&\quad \mu_3\mu_1\mu_{20}\lambda_{c_1} + \lambda_2\mu_{30}\mu_{20}\lambda_{c_1} + \lambda_{c_2}\mu_{30}\mu_{20}\lambda_{c_1} + \mu_3\lambda_2\mu_{20}\lambda_{c_1} + \\
&\quad \mu_3\lambda_{c_2}\mu_{20}\lambda_{c_1} + \mu_{30}\mu_1\mu_{20}\lambda_{c_1} + \lambda_{c_3}\mu_{30}\mu_1\lambda_{c_1} + \mu_2\mu_3\mu_1\lambda_{c_1} + \\
&\quad \mu_3\lambda_{c_3}\mu_1\lambda_{c_1} + \mu_2\mu_{30}\mu_1\lambda_{c_1} + \lambda_3\mu_{30}\mu_1\lambda_{c_1} + \lambda_{c_3}\lambda_2\mu_{30}\lambda_{c_1})
\end{aligned}
$$

$$
\begin{aligned}
E_6 &= (\mu_{c_2}\lambda_3\lambda_2\lambda_{c_1} + \lambda_2\mu_{c_1}\lambda_1\lambda_3 + \mu_3\lambda_{c_3}\lambda_{c_2}\lambda_{c_1} + \mu_{c_3}\mu_3\lambda_{c_2}\lambda_{c_1} + \\
&\quad \mu_3\lambda_{c_3}\lambda_2\lambda_{c_1} + \mu_{c_3}\mu_3\lambda_2\lambda_{c_1} + \lambda_3\lambda_2\mu_{30}\lambda_{c_1} + \mu_{c_2}\lambda_3\mu_{30}\lambda_{c_1} + \\
&\quad \mu_{c_3}\mu_2\mu_{30}\lambda_{c_1} + \mu_{c_3}\mu_2\mu_3\lambda_{c_1} + \mu_{c_2}\mu_2\mu_3\lambda_{c_1} + \mu_{c_2}\mu_2\mu_{30}\lambda_{c_1} + \\
&\quad \mu_{c_2}\lambda_{c_3}\mu_3\lambda_{c_1} + \lambda_2\mu_{c_2}\mu_{20}\mu_{30} + \lambda_2\mu_{c_2}\lambda_3\mu_{30} + \lambda_{c_2}\lambda_1\mu_{c_3}\mu_{30} + \\
&\quad \lambda_{c_2}\lambda_1\mu_2\mu_{30} + \lambda_{c_2}\lambda_1\mu_{20}\mu_{30} + \mu_{c_2}\lambda_1\mu_{20}\mu_3 + \lambda_{c_2}\lambda_1\lambda_{c_3}\mu_3 + \\
&\quad \lambda_{c_2}\lambda_1\mu_2\mu_3 + \mu_{c_2}\lambda_1\lambda_{c_3}\mu_{30} + \mu_{c_2}\lambda_1\lambda_{c_3}\mu_3 + \mu_{c_2}\lambda_1\lambda_3\mu_{30} + \\
&\quad \mu_{c_2}\lambda_1\mu_{20}\mu_{30} + \mu_2\lambda_{c_2}\mu_{30}\lambda_{c_1} + \lambda_3\lambda_{c_2}\mu_{30}\lambda_{c_1} + \lambda_{c_3}\lambda_{c_2}\mu_{30}\lambda_{c_1} + \\
&\quad \mu_{c_3}\lambda_{c_2}\mu_{30}\lambda_{c_1} + \mu_{c_3}\mu_3\mu_1\lambda_{c_1} + \mu_2\mu_3\lambda_{c_2}\lambda_{c_1} + \lambda_1\mu_{c_3}\mu_{30}\mu_{20} + \\
&\quad \mu_{c_1}\lambda_1\mu_3\mu_{20} + \lambda_1\mu_{c_3}\mu_3\mu_{20} + \mu_{c_1}\lambda_1\mu_{30}\mu_{20} + \lambda_2\mu_{c_1}\lambda_1\mu_{30} + \\
&\quad \lambda_2\mu_{c_1}\lambda_1\mu_3 + \mu_{c_1}\lambda_1\mu_2\mu_3 + \mu_{c_1}\lambda_1\mu_2\mu_{30} + \mu_{c_1}\lambda_1\lambda_{c_3}\mu_3 + \\
&\quad \mu_{c_1}\lambda_1\lambda_3\mu_{30} + \mu_{c_3}\lambda_1\lambda_3\mu_{30} + \mu_{c_1}\lambda_1\lambda_{c_3}\mu_{30} + \lambda_2\lambda_1\lambda_3\mu_{c_3} + \\
&\quad \lambda_{c_2}\lambda_1\lambda_3\mu_{c_3} + \mu_{c_2}\lambda_1\mu_2\mu_{30} + \mu_{c_2}\lambda_1\mu_2\mu_3 + \lambda_2\lambda_1\mu_{c_2}\mu_3 + \\
&\quad \lambda_2\lambda_1\mu_{c_2}\mu_{30} + \lambda_2\lambda_1\mu_{c_3}\mu_3 + \lambda_2\lambda_1\mu_{c_3}\mu_3 + \lambda_2\lambda_1\lambda_{c_3}\mu_{30} + \\
&\quad \lambda_2\lambda_1\lambda_{c_3}\mu_3 + \mu_{c_3}\lambda_1\mu_2\mu_{30} + \mu_{c_3}\lambda_1\mu_2\mu_3 + \mu_{c_3}\lambda_3\mu_1\lambda_{c_1} + \\
&\quad \mu_{c_3}\lambda_3\lambda_{c_2}\lambda_{c_1} + \mu_{c_3}\lambda_3\lambda_2\lambda_{c_1} + \lambda_3\mu_{c_1}\mu_{30}\mu_1 + \mu_2\mu_{c_1}\mu_{30}\mu_1 + \\
&\quad \mu_3\lambda_{c_2}\mu_{c_1}\mu_{20} + \mu_3\lambda_{c_3}\lambda_{c_2}\mu_{c_1} + \mu_2\mu_3\lambda_{c_2}\mu_{c_1} + \mu_3\lambda_{c_3}\lambda_2\mu_{c_1} + \\
&\quad \mu_{c_1}\lambda_3\lambda_{c_2}\mu_{30} + \mu_3\lambda_{c_3}\mu_{c_1}\mu_1 + \mu_1\mu_{c_2}\mu_{20}\mu_3 + \mu_{c_3}\lambda_2\mu_{30}\lambda_{c_1} + \\
&\quad \lambda_2\lambda_1\lambda_3\mu_{c_2} + \lambda_{c_2}\lambda_1\mu_{20}\mu_3 + \lambda_{c_2}\lambda_1\lambda_3\mu_{30} + \lambda_{c_2}\lambda_1\lambda_{c_3}\mu_{30} + \\
&\quad \lambda_{c_2}\lambda_1\mu_{c_3}\mu_3 + \mu_1\mu_{c_2}\mu_2\mu_{30} + \mu_1\mu_{c_2}\lambda_3\mu_{30} + \mu_1\mu_{c_2}\lambda_{c_3}\mu_{30} + \\
&\quad \mu_{c_3}\lambda_{c_2}\mu_{30}\mu_{20} + \lambda_{c_2}\mu_{c_1}\mu_{30}\mu_{20} + \mu_{c_3}\mu_3\mu_1\mu_{20} + \mu_3\mu_{c_1}\mu_1\mu_{20} +
\end{aligned}
$$

$$
\begin{aligned}
&\mu_{c_1}\lambda_3\lambda_2\mu_{30} + \mu_{c_1}\lambda_3\lambda_2\mu_{30} + \mu_{c_1}\mu_{30}\mu_1\mu_{20} + \lambda_{c_3}\lambda_{c_2}\mu_{c_1}\mu_{30} + \\
&\mu_1\mu_{c_2}\mu_2\mu_3 + \mu_1\mu_{c_2}\lambda_{c_2}\mu_3 + \mu_{c_3}\lambda_2\mu_{30}\mu_{20} + \mu_3\lambda_2\mu_{c_1}\mu_{20} + \\
&\lambda_2\mu_{c_1}\mu_{30}\mu_{20} + \mu_{c_3}\mu_2\mu_3\mu_1 + \mu_{c_3}\mu_2\mu_{30}\mu_1 + \lambda_{c_3}\mu_{c_1}\mu_{30}\mu_1 + \\
&\mu_2\mu_3\mu_{c_1}\mu_1 + \mu_{c_3}\mu_3\lambda_{c_2}\mu_{20} + \mu_{c_3}\mu_3\lambda_2\mu_{20} + \mu_{c_3}\mu_{30}\mu_1\mu_{20} + \\
&\lambda_{c_3}\lambda_2\mu_{c_1}\mu_{30} + \mu_{c_3}\lambda_3\mu_{30}\mu_1 + \mu_{c_3}\lambda_3\lambda_{c_2}\mu_{30} + \mu_2\lambda_{c_2}\mu_{c_1}\mu_{30} + \\
&\mu_{c_2}\lambda_{c_3}\mu_{30}\lambda_{c_1} + \lambda_2\mu_{c_2}\mu_{30}\lambda_{c_1} + \lambda_2\mu_{c_2}\mu_3\lambda_{c_1} + \mu_{c_2}\mu_{20}\mu_{30}\lambda_{c_1} + \\
&\mu_{c_2}\mu_{20}\mu_3\lambda_{c_1} + \mu_3\mu_1\mu_{20}\lambda_{c_1} + \lambda_2\mu_{30}\mu_{20}\lambda_{c_1} + \lambda_{c_2}\mu_{30}\mu_{20}\lambda_{c_1} + \\
&\mu_3\lambda_2\mu_{20}\lambda_{c_1} + \mu_3\lambda_{c_2}\mu_{20}\lambda_{c_1} + \mu_{30}\mu_1\mu_{20}\lambda_{c_1} + \lambda_{c_3}\mu_{30}\mu_1\lambda_{c_1} + \\
&\mu_{c_3}\mu_{30}\mu_1\lambda_{c_1} + \mu_2\mu_3\mu_1\lambda_{c_1} + \mu_3\lambda_{c_3}\mu_1\lambda_{c_1} + \mu_2\mu_{30}\mu_1\lambda_{c_1} + \\
&\lambda_3\mu_{30}\mu_1\lambda_{c_1} + \lambda_{c_3}\lambda_2\mu_{30}\lambda_{c_1} + \mu_1\mu_{c_2}\mu_{20}\mu_{30} + \lambda_2\mu_{c_2}\mu_{20}\mu_3)
\end{aligned}
$$

$$
\begin{aligned}
E_7 &= (\mu_{c_1}\lambda_{c_2}\mu_2 + \mu_{c_1}\mu_1\mu_{20} + \mu_{c_1}\mu_1\mu_2 + \mu_{c_2}\lambda_2\mu_{20} + \mu_{c_1}\lambda_2\mu_{20} + \mu_{c_2}\mu_1\mu_2 + \\
&\quad \mu_{c_1}\lambda_{c_2}\mu_{20} + \mu_{c_2}\mu_1\mu_{20}) \\
E_8 &= (\mu_{20} + \mu_2)(\lambda_{c_1}\mu_{c_2} + \lambda_1\mu_{c_1} + \lambda_1\mu_{c_2}) \\
E_9 &= \lambda_2(\lambda_{c_1}\mu_{c_2} + \lambda_1\mu_{c_1} + \lambda_1\mu_{c_2}) \\
E_{10} &= (\lambda_1\lambda_{c_2}\mu_2 + \lambda_{c_1}\lambda_2\mu_{20} + \lambda_{c_1}\lambda_{c_2}\mu_{20} + \lambda_{c_1}\lambda_{c_2}\mu_2 + \lambda_1\lambda_{c_2}\mu_{20} + \\
&\quad \lambda_{c_1}\mu_1\mu_{20} + \lambda_{c_1}\mu_1\mu_2) \\
E_{11} &= (\lambda_{c_1}\mu_1\mu_{20} + \lambda_{c_1}\mu_1\mu_2 + \mu_{c_1}\mu_1\mu_2 + \mu_{c_2}\lambda_2\mu_{20} + \mu_{c_1}\lambda_2\mu_{20} + \mu_{c_1}\mu_1\mu_{20} + \\
&\quad \lambda_1\lambda_{c_2}\mu_{20} + \lambda_1\lambda_{c_2}\mu_2 + \mu_{c_2}\mu_1\mu_{20} + \mu_{c_2}\mu_1\mu_2 + \lambda_{c_1}\lambda_2\mu_{20} + \lambda_1\mu_{c_2}\mu_2 + \\
&\quad \lambda_{c_1}\lambda_2\mu_{c_2} + \lambda_{c_1}\lambda_{c_2}\mu_{20} + \lambda_{c_1}\lambda_{c_2}\mu_2 + \lambda_{c_1}\mu_{c_2}\mu_{20} + \lambda_{c_1}\mu_{c_2}\mu_2 + \mu_{c_1}\lambda_{c_2}\mu_{20} + \\
&\quad \mu_{c_1}\lambda_{c_2}\mu_2 + \lambda_1\lambda_2\mu_{c_2} + \lambda_1\lambda_2\mu_{c_1} + \lambda_1\mu_{c_1}\mu_{20} + \lambda_1\mu_{c_1}\mu_2 + \lambda_1\mu_{c_2}\mu_{20}) \\
E_{16} &= (\mu_1 + \mu_{20} + \lambda_3 + \lambda_{c_2} + \mu_2 + \lambda_2 + \lambda_{c_3}) \\
E_{17} &= (\lambda_{c_3}\mu_1 + \mu_2\lambda_{c_2} + \lambda_2\mu_{20} + \lambda_3\lambda_2 + \lambda_3\lambda_{c_2} + \mu_2\mu_1 + \mu_{20}\lambda_{c_2} \\
&\quad \lambda_{c_3}\lambda_2 + \lambda_3\mu_1 + \mu_1\mu_{20} + \lambda_{c_3}\lambda_{c_2}) \\
E_{18} &= (\lambda_2 + \mu_1 + \lambda_{c_3} + \mu_2 + \lambda_1 + \lambda_3 + \mu_{20} + \lambda_{c_1} + \lambda_{c_2}) \\
E_{19} &= (\lambda_{c_1}\lambda_{c_3} + \lambda_1\mu_{20} + \mu_1\mu_{20} + \mu_2\lambda_{c_2} + \lambda_{c_3}\mu_1 + \lambda_1\lambda_{c_2} + \lambda_{c_1}\lambda_{c_2} + \lambda_{c_3}\lambda_{c_2} + \\
&\quad \lambda_1\mu_2 + \lambda_{c_3}\lambda_2 + \lambda_2\mu_{20} + \lambda_3\lambda_{c_2} + \lambda_{c_1}\lambda_3 + \lambda_{c_1}\mu_2 + \lambda_{c_1}\mu_1 + \lambda_3\lambda_2 + \\
&\quad \mu_{20}\lambda_{c_2} + \lambda_1\lambda_{c_3} + \mu_{20}\lambda_{c_1} + \lambda_3\mu_1 + \mu_2\mu_1 + \lambda_2\lambda_{c_1} + \lambda_2\lambda_1 + \lambda_1\lambda_3) \\
E_{20} &= (\lambda_1\lambda_{c_3}\lambda_{c_2} \quad \mu_{20}\lambda_{c_1}\lambda_{c_2} + \lambda_1\mu_{20}\lambda_{c_2} + \lambda_{c_1}\lambda_3\mu_1 + \lambda_{c_1}\lambda_3\lambda_2 + \lambda_{c_1}\lambda_3\lambda_{c_2} + \\
&\quad \lambda_1\lambda_3\lambda_{c_2} + \lambda_1\lambda_{c_3}\lambda_2 + \lambda_{c_1}\lambda_{c_3}\mu_1 + \mu_{20}\lambda_{c_1}\lambda_2 + \lambda_{c_1}\lambda_{c_3}\lambda_{c_2} + \mu_1\mu_{20}\lambda_{c_1} +
\end{aligned}
$$

$$
\begin{aligned}
& \lambda_{c_1}\mu_2\mu_1 + \lambda_{c_1}\lambda_{c_2}\lambda_2 + \lambda_1\lambda_3\lambda_2 + \lambda_1\mu_2\lambda_{c_2} + \lambda_{c_1}\mu_2\lambda_{c_2})
\end{aligned}
$$

$$
\begin{aligned}
E_{21} &= (-2\mu_1\lambda_{c_1} - 2\lambda_2\lambda_1 - 2\lambda_2\lambda_{c_1} - 2\lambda_{c_2}\lambda_1 - 2\lambda_{c_2}\lambda_{c_1} + \lambda_1^2 + \lambda_{c_2}^2 + 2\mu_1\lambda_{c_2} + \mu_1^2 + \\
&\quad 2\lambda_2\lambda_{c_2} + 2\lambda_2\mu_1 + \lambda_2^2 + \lambda_{c_1}^2 + 2\lambda_1\mu_1 + 2\lambda_1\lambda_{c_1})^{\frac{1}{2}} \\
E_{22} &= (\mu_b + \lambda_a + \lambda_c + \lambda_{c_3})\mu_{40}\mu_{c_1}(\lambda_a + \mu_c + \lambda_b + \lambda_{c_2})(\lambda_{c_4} + \mu_a + \lambda_c + \lambda_b) \\
E_{23} &= (\lambda_{c_4} + \mu_a + \lambda_c + \lambda_b)\mu_{c_1}\mu_{40}\lambda_c(\mu_b + \lambda_a + \lambda_c + \lambda_{c_3}) \\
E_{24} &= \mu_{40}\lambda_b\mu_{c_1}(\lambda_a + \mu_c + \lambda_b + \lambda_{c_2})(\lambda_{c_4} + \mu_a + \lambda_c + \lambda_b) \\
E_{25} &= (\lambda_a + \mu_c + \lambda_b + \lambda_{c_2})\mu_{c_1}\mu_{40}\lambda_a(\mu_b + \lambda_a + \lambda_c + \lambda_{c_3})
\end{aligned}
$$

$$
\begin{aligned}
E_{26} &= \mu_{c_1}(\mu_a\lambda_a\lambda_c\lambda_{c_3} + \lambda_c\lambda_a^2\lambda_{c_3} + \lambda_b\mu_a\lambda_c^2 + \lambda_b\mu_c\lambda_a u_b + \\
&\quad \lambda_b^2\lambda_a\lambda_{c_3} + 4\lambda_b\lambda_c^2\lambda_a + 2\lambda_b\mu_c\lambda_a\lambda_c + \lambda_b\lambda_{c_4}\lambda_c\mu_b + \\
&\quad \lambda_b^2\lambda_c\mu_c + \lambda_c^2\lambda_a\mu_b + 2\lambda_b\lambda_{c_4}\lambda_c\lambda_a + 2\lambda_b\lambda_{c_2}\lambda_a\lambda_c + \\
&\quad \lambda_b\lambda_{c_2}\lambda_a\mu_b + \lambda_c\mu_c\lambda_a\mu_b + 2\lambda_b\lambda_c\lambda_a\lambda_{c_3} + \lambda_b\lambda_c^2\mu_b + \\
&\quad 4\lambda_b^2\lambda_a\lambda_c + \lambda_c^2\mu_3\lambda_a + \lambda_{c_4}\lambda_a\lambda_c^2 + \lambda_b^2\lambda_c\lambda_{c_3} + \\
&\quad \lambda_b\lambda_a^2\lambda_{c_4} + \lambda_b\lambda_a^2\mu_a + \lambda_b\lambda_a\mu_a\mu_c + \lambda_c\mu_c\lambda_a\lambda_{c_3} + \\
&\quad \lambda_c\lambda_{c_2}\lambda_a\lambda_{c_3} + \mu_a\lambda_a\lambda_c^2 + \lambda_b\lambda_c^2\lambda_{c_3} + \\
&\quad \lambda_b\mu_c\lambda_a^2 + \lambda_b\lambda_c\lambda_{c_4}\mu_c + \lambda_b\lambda_a\lambda_{c_4}\lambda_{c_2} + \lambda_b\lambda_c\mu_a\lambda_{c_2} + \\
&\quad 2\lambda_b\lambda_c\lambda_a\mu_b + \mu_a\lambda_a\lambda_c\mu_b + \lambda_{c_4}\lambda_a\lambda_c\mu_b + \lambda_b\lambda_a\lambda_{c_4}\mu_c + \\
&\quad \lambda_b\mu_1\lambda_c\lambda_{c_3} + \lambda_b\lambda_a\mu_a\lambda_{c_2} + \lambda_b\lambda_c\lambda_{c_4}\lambda_{c_2} + \lambda_b\lambda_c\mu_a\mu_c + \\
&\quad \lambda_c\lambda_{c_2}\lambda_a^2 + \lambda_b^2\lambda_c\lambda_{c_2} + \lambda_c^2\lambda_a\lambda_{c_3} + \lambda_b^2\lambda_c\mu_b + \\
&\quad \lambda_b\lambda_c^2\mu_c + \lambda_b\mu_c\lambda_a\lambda_{c_3} + \lambda_c\lambda_{c_2}\lambda_a\mu_b + \lambda_b\mu_a\lambda_c\mu_b + \\
&\quad \lambda_{c_4}\lambda_a\lambda_c\lambda_{c_3} + \lambda_{c_4}\lambda_a^2\lambda_c + \lambda_b\lambda_a^2\lambda_{c_3} + \\
&\quad 4\lambda_b\lambda_a^2\lambda_3 + \lambda_b^2\lambda_c\lambda_{c_4} + \lambda_b^2\lambda_a\mu_b + \lambda_b\lambda_{c_2}\lambda_a^2 + \\
&\quad \mu_a\lambda_a^2\lambda_c + \lambda_c\mu_c\lambda_a^2 + \lambda_b\lambda_{c_4}\lambda_c^2 + \lambda_b^2\lambda_c\mu_a + \\
&\quad \lambda_b\lambda_{c_4}\lambda_c\lambda_{c_3} + 2\lambda_b\mu_a\lambda_c\lambda_a + \lambda_b\lambda_{c_2}\lambda_a\lambda_{c_3} + \lambda_c^2\lambda_{c_2}\lambda_a + \\
&\quad \lambda_b\lambda_c^2\lambda_{c_2} + \lambda_b\lambda_a^2\mu_b + \lambda_b^2\lambda_a\mu_3 + \lambda_c\lambda_a^2\mu_b + \lambda_b^2\lambda_a\mu_a + \\
&\quad 2\lambda_b^2\lambda_a^2 + \lambda_b\lambda_a^3 + \lambda_c^3\lambda_a + 2\lambda_b^2\lambda_c^2 + \\
&\quad \lambda_b\lambda_c^3 + \lambda_b^3\lambda_c + \lambda_b^3\lambda_1 + \lambda_c\lambda_a^3 + \lambda_b^2\lambda_a\lambda_{c_4} + \lambda_b^2\lambda_1\lambda_{c_2} + 2\lambda_c^2\lambda_a^2)
\end{aligned}
$$

$$
\begin{aligned}
E_{27} &= \mu_{40}(\lambda_b^3\lambda_{c_3} + \lambda_b\lambda_{c_2}\lambda_{c_1}\lambda_c + \lambda_c\lambda_{c_2}\lambda_{c_1}\mu_b + \lambda_{c_4}\lambda_b\lambda_{c_1}\lambda_a + \\
&\quad \mu_a\lambda_a\lambda_{c_1}\lambda_{c_3} + \mu_a\mu_c\lambda_{c_1}\mu_b + \lambda_{c_2}\lambda_b^2\lambda_{c_3} + \mu_c\lambda_b^2\lambda_{c_3} +
\end{aligned}
$$

$$
\begin{aligned}
&\lambda_{c_4}\lambda_{c_2}\lambda_{c_1}\lambda_{c_3} + \lambda_c\mu_c\lambda_{c_1}\lambda_{c_3} + \lambda_c\lambda_b\lambda_{c_1}\lambda_{c_3} + \lambda_{c_4}\lambda_a\lambda_{c_1}\mu_b + \\
&\mu_a\lambda_b\lambda_{c_1}\mu_b + \lambda_b\lambda_{c_1}\lambda_a^2 + \lambda_{c_4}\mu_c\lambda_{c_1}\lambda_{c_3} + \lambda_c\lambda_b\lambda_{c_1}\mu_b + \\
&\lambda_{c_4}\lambda_{c_2}\lambda_{c_1}\lambda_a + \mu_a\lambda_{c_2}\lambda_c^2 + \lambda_b^2\lambda_{c_1}\lambda_{c_3} + \lambda_b^2\lambda_{c_1}\lambda_c + \\
&\lambda_b\mu_c\lambda_{c_1}\lambda_c + \mu_a\lambda_a\lambda_b\lambda_{c_3} + \lambda_{c_4}\mu_c\lambda_a\lambda_{c_3} + \lambda_{c_4}\lambda_{c_2}\lambda_b\lambda_{c_3} + \\
&\mu_a\lambda_{c_2}\lambda_{c_1}\mu_b + \mu_a\mu_c\lambda_{c_1}\lambda_{c_3} + \lambda_b\lambda_{c_2}\lambda_c\mu_b + \lambda_{c_4}\lambda_a\lambda_{c_1}\lambda_c + \\
&\lambda_b\mu_c\lambda_{c_1}\lambda_{c_3} + \lambda_{c_4}\mu_c\lambda_a\mu_b + \mu_a\lambda_{c_2}\lambda_{c_1}\lambda_c + \mu_a\mu_c\lambda_b\lambda_{c_3} + \\
&\lambda_{c_4}\lambda_a^2\mu_b + \lambda_{c_4}\mu_c\lambda_{c_1}\lambda_c + \lambda_b\mu_c\lambda_{c_1}\lambda_a + \mu_a\lambda_{c_2}\lambda_{c_1}\lambda_{c_3} + \\
&\lambda_{c_4}\lambda_{c_2}\lambda_c^2 + \lambda_{c_4}\lambda_{c_2}\lambda_a\mu_b + \lambda_{c_4}\lambda_{c_2}\lambda_a^2 + \lambda_c\lambda_{c_2}\lambda_{c_1}\lambda_{c_3} + \\
&\lambda_2\lambda_{c_2}\lambda_{c_1}\lambda_{c_3} + \lambda_c\mu_c\lambda_b\lambda_{c_3} + \lambda_{c_4}\lambda_a^2\lambda_{c_3} + \mu_a\mu_c\lambda_{c_1}\lambda_a + \\
&\lambda_{c_2}\lambda_3^2\lambda_{c_3} + \lambda_{c_4}\lambda_{c_2}\lambda_{c_1}\lambda_c + \lambda_c\lambda_{c_2}\lambda_{c_1}\lambda_a + \mu_a\lambda_{c_2}\lambda_c\mu_b + \\
&\lambda_{c_4}\lambda_{c_2}\lambda_{c_1}\mu_b + \lambda_b^2\lambda_{c_1}\lambda_a + \lambda_c\mu_c\lambda_{c_1}\lambda_a + \lambda_c\lambda_a\lambda_{c_1}\lambda_{c_3} + \\
&\lambda_c\lambda_a\lambda_{c_1}\mu_b + \mu_a\mu_c\lambda_{c_1}\lambda_c + \lambda_b\lambda_a\lambda_{c_1}\mu_b + \lambda_{c_4}\mu_c\lambda_{c_1}\mu_b + \\
&2\lambda_{c_4}\lambda_{c_2}\lambda_c\lambda_a + \lambda_{c_2}\lambda_{c_1}\lambda_c^2 + \lambda_{c_4}\lambda_{c_2}\lambda_c\lambda_{c_3} + \mu_c\lambda_{c_1}\lambda_c^2 + \\
&\lambda_{c_4}\mu_c\lambda_c\lambda_a + \lambda_{c_4}\lambda_a^3 + \lambda_{c_4}\lambda_b\lambda_a\mu_b + \lambda_{c_2}\lambda_c^3 + \\
&\mu_a\lambda_{c_1}\lambda_a^2 + \lambda_c\lambda_{c_1}\lambda_a^2 + \mu_a\lambda_b^2\lambda_{c_3} + \lambda_{c_2}\lambda_c^2\mu_b + \\
&\lambda_b\lambda_{c_2}\lambda_{c_1}\lambda_1 + \lambda_{c_4}\lambda_{c_2}\lambda_c\mu_b + \lambda_a\lambda_{c_1}\lambda_c^2 + \lambda_b\lambda_a\lambda_{c_1}\lambda_{c_3} + \\
&\lambda_{c_4}\mu_c\lambda_{c_1}\lambda_a + \lambda_{c_4}\lambda_b\lambda_{c_1}\mu_b + 2\lambda_b\lambda_{c_2}\lambda_c\lambda_{c_3} + \mu_a\lambda_a\lambda_{c_1}\mu_b + \\
&\lambda_{c_4}\lambda_{c_1}\lambda_a^2 + \mu_a\lambda_{c_2}\lambda_c\lambda_{c_3} + \mu_a\lambda_a\lambda_{c_1}\lambda_c + 2\lambda_{c_4}\lambda_a\lambda_b\lambda_{c_3} + \\
&\lambda_c\mu_c\lambda_{c_1}\mu_b + \lambda_{c_4}\lambda_a\lambda_{c_1}\lambda_{c_3} + \lambda_{c_4}\lambda_b\lambda_{c_1}\lambda_c + \mu_a\lambda_b\lambda_{c_1}\lambda_c + \\
&\lambda_b\lambda_{c_2}\lambda_{c_1}\mu_b + 2\lambda_2\lambda_a\lambda_{c_1}\lambda_c + \mu_a\lambda_b\lambda_{c_1}\lambda_{c_3} + \lambda_{c_4}\mu_c\lambda_a^2 + \\
&\lambda_b^2\lambda_a\lambda_{c_3} + \lambda_b\lambda_{c_4}\lambda_c\lambda_a + \lambda_b\lambda_{c_2}\lambda_a\lambda_c + \lambda_b\lambda_3\lambda_a\lambda_{c_3} + \\
&\lambda_b^2\lambda_c\lambda_{c_3} + \lambda_b\lambda_a^2\lambda_{c_4} + \lambda_{c_4}\lambda_a^2\lambda_c + \lambda_c^2\lambda_{c_2}\lambda_a + \\
&\lambda_b\lambda_c^2\lambda_{c_2} + \lambda_{c_4}\lambda_b^2\lambda_{c_3} + \mu_a\lambda_{c_2}\lambda_{c_1}\lambda_a + \lambda_{c_4}\lambda_b\lambda_{c_1}\lambda_{c_3} + \\
&\mu_1\lambda_{c_2}\lambda_c\lambda_a + \lambda_b\mu_c\lambda_{c_1}\mu_b + \lambda_b\lambda_{c_1}\lambda_c^2 + \lambda_{c_4}\mu_c\lambda_b\lambda_{c_3} + \\
&\lambda_b^2\lambda_{c_1}\mu_b + \mu_a\lambda_{c_2}\lambda_b\lambda_{c_3} + \mu_a\lambda_b\lambda_{c_1}\lambda_a + \lambda_{c_4}\lambda_{c_2}\lambda_a\lambda_{c_3})
\end{aligned}
$$

$$
\begin{aligned}
E_{28_a} \;=\; &(\mu_a\lambda_{c_2}\mu_{40}\lambda_b\mu_{c_1} + \lambda_{c_4}\lambda_{c_2}\mu_{40}\lambda_b\lambda_{c_3} + 2\lambda_{c_4}\lambda_a\mu_{40}\lambda_b\lambda_{c_3} + \\
&\lambda_{c_4}\lambda_a\mu_{40}\lambda_{c_1}\lambda_c + \lambda_{c_4}\lambda_{c_2}\mu_{40}\lambda_1\mu_b + \mu_c\mu_{40}\mu_{c_1}\lambda_c^2 + \\
&\lambda_b\lambda_{c_2}\mu_{40}\mu_{c_1}\mu_2 + \lambda_c\mu_{40}\lambda_{c_1}\lambda_a^2 + 2\lambda_b\mu_{c_1}\lambda_a\lambda_3\lambda_{c_3} + \\
&\lambda_b\mu_c\mu_{40}\mu_{c_1}\lambda_a + 2\lambda_b\mu_{40}\mu_{c_1}\lambda_c\lambda_{c_3} + 2\lambda_b\mu_{c_1}\lambda_a\lambda_c\mu_b +
\end{aligned}
$$

$$\lambda_{c_4}\lambda_a\mu_{40}\lambda_{c_1}\lambda_{c_3} + \lambda_b\lambda_{c_2}\mu_{40}\lambda_{c_1}\lambda_a + \lambda_b\lambda_{c_2}\mu_{40}\lambda_{c_1}\lambda_{c_3} +$$
$$2\lambda_b\lambda_1\mu_{40}\mu_{c_1}\mu_b + \lambda_b\lambda_{c_2}\mu_{40}\lambda_{c_1}\mu_b + \mu_a\lambda_a\mu_{40}\mu_{c_1}\lambda_{c_3} +$$
$$\lambda_b\mu_c\mu_{40}\mu_{c_1}\lambda_{c_3} + \lambda_b\mu_c\mu_{40}\lambda_{c_1}\lambda_c + \lambda_b\mu_c\mu_{40}\mu_{c_1}\mu_b +$$
$$2\lambda_{c_4}\lambda_a\lambda_b\mu_{c_1}\lambda_c + \lambda_c\lambda_{c_2}\mu_{40}\lambda_{c_1}\mu_b + \lambda_b\mu_{c_1}\lambda_c^2\lambda_{c_3} +$$
$$\lambda_c\lambda_a\mu_{40}\mu_{c_1}\mu_b + \lambda_c\lambda_a\mu_{40}\lambda_{c_1}\lambda_{c_3} + 2\lambda_2\lambda_{c_2}\mu_{40}\lambda_c\lambda_{c_3} +$$
$$\lambda_c\lambda_a\mu_{40}\lambda_{c_1}\mu_b + 2\lambda_c\lambda_{c_2}\lambda_b\mu_{c_1}\lambda_a + \mu_{c_1}\lambda_a\lambda_3^2\mu_b +$$
$$\lambda_c\lambda_{c_2}\mu_{40}\lambda_{c_1}\lambda_{c_3} + \mu_a\lambda_{c_2}\mu_{40}\lambda_{c_1}\lambda_{c_3} + \mu_{40}\mu_{c_1}\lambda_c^2\mu_b +$$
$$\lambda_c\mu_c\mu_{40}\lambda_{c_1}\lambda_a + \lambda_{c_4}\lambda_{c_2}\mu_{40}\lambda_a\lambda_{c_3} + \mu_a\lambda_b^2\mu_{c_1}\lambda_c +$$
$$\mu_a\mu_{c_1}\lambda_a\lambda_c^2 + 2\lambda_b\lambda_a\mu_{40}\lambda_{c_1}\lambda_c + \mu_a\mu_{c_1}\lambda_a^2\lambda_c +$$
$$\lambda_b^2\mu_{c_1}\lambda_c\mu_b + \lambda_{c_4}\lambda_b\mu_{c_1}\lambda_a^2 + \mu_a\mu_c\mu_{40}\mu_{c_1}\mu_b +$$
$$\mu_a\mu_{c_1}\lambda_a\lambda_c\mu_b + \mu_a\mu_c\mu_{40}\mu_{c_1}\lambda_c + \mu_a\mu_{c_1}\lambda_a\lambda_c\lambda_{c_3} +$$
$$\mu_a\mu_{40}\lambda_b^2\lambda_{c_3} + \mu_c\mu_{c_1}\lambda_c\lambda_1\lambda_{c_3} + \lambda_{c_4}\mu_{c_1}\lambda_a\lambda_c^2 +$$
$$\lambda_{c_2}\mu_{40}\lambda_3^2\mu_b + \mu_a\mu_c\mu_{40}\lambda_b\lambda_{c_3} + \mu_c\lambda_b\mu_{c_1}\lambda_a\mu_b +$$
$$\mu_a\lambda_b^2\mu_{c_1}\lambda_a + \mu_c\mu_{40}\mu_{c_1}\lambda_1\lambda_{c_3} + \mu_c\mu_{40}\mu_{c_1}\lambda_a\mu_b +$$
$$\mu_a\lambda_b\mu_{c_1}\lambda_3\lambda_{c_3} + \mu_a\lambda_{c_2}\mu_{40}\lambda_{c_1}\lambda_a + \lambda_b\mu_{40}\lambda_{c_1}\lambda_c^2 +$$
$$\lambda_{c_2}\mu_{40}\mu_{c_1}\lambda_c^2 + \mu_c\mu_{c_1}\lambda_c\lambda_a^2 + 2\lambda_c\mu_{40}\mu_{c_1}\lambda_a^2 +$$
$$\lambda_{c_4}\lambda_1\mu_{40}\mu_{c_1}\mu_b + \mu_a\lambda_b\mu_{40}\lambda_{c_1}\lambda_a + \mu_a\lambda_{c_2}\mu_{40}\mu_{c_1}\lambda_a +$$
$$\lambda_b\mu_c\mu_{40}\lambda_{c_1}\mu_b + 2\lambda_b\mu_{40}\mu_{c_1}\lambda_a^2 + \lambda_b\lambda_{c_2}\mu_{40}\lambda_c\lambda_a +$$
$$\mu_a\lambda_{c_2}\mu_{40}\mu_{c_1}\lambda_c + \lambda_{c_4}\lambda_b\mu_{40}\lambda_{c_1}\lambda_c + \lambda_b\mu_c\mu_{40}\lambda_{c_1}\lambda_{c_3} +$$
$$\mu_a\lambda_{c_2}\mu_{40}\mu_{c_1}\lambda_{c_3} + \lambda_a\mu_{40}\lambda_{c_1}\lambda_c^2 + \mu_c\mu_{40}\lambda_{c_1}\lambda_c^2 +$$
$$\lambda_{c_2}\mu_{40}\lambda_{c_1}\lambda_c^2 + 2\lambda_1\mu_{40}\mu_{c_1}\lambda_c^2 + \lambda_{c_4}\mu_c\mu_{40}\mu_{c_1}\lambda_{c_3} +$$
$$\mu_c\lambda_b\mu_{c_1}\lambda_c^2 + \mu_a\lambda_{c_2}\mu_{40}\mu_{c_1}\mu_b + 4\lambda_b\mu_{c_1}\lambda_a^2\lambda_c +$$
$$\lambda_{c_4}\lambda_b\mu_{40}\lambda_{c_1}\mu_b + 2\lambda_{c_4}\lambda_{c_2}\mu_{40}\lambda_c\lambda_a + \lambda_c\mu_c\mu_{40}\mu_{c_1}\mu_b +$$
$$\lambda_b\mu_c\mu_{40}\lambda_{c_1}\lambda_a + \lambda_c\mu_c\mu_{40}\mu_{c_1}\lambda_{c_3} + \lambda_c\mu_c\mu_{40}\lambda_b\lambda_{c_3} +$$
$$2\lambda_b\mu_{40}\mu_{c_1}\lambda_c^2 + 2\lambda_c\mu_c\mu_{40}\mu_{c_1}\lambda_a + 2\lambda_b\lambda_1\mu_{40}\mu_{c_1}\lambda_{c_3} +$$
$$\lambda_{c_4}\lambda_b\mu_{40}\lambda_a\mu_b + \mu_a\mu_{40}\mu_{c_1}\lambda_a^2 + \lambda_b^2\mu_{c_1}\lambda_a\lambda_{c_3} +$$
$$\lambda_{c_4}\mu_3\lambda_b\mu_{c_1}\lambda_a + \lambda_{c_4}\mu_c\lambda_b\mu_{c_1}\lambda_c + \lambda_{c_4}\mu_c\mu_{40}\lambda_{c_1}\lambda_{c_3} +$$
$$\mu_a\lambda_a\mu_{40}\mu_{c_1}\mu_2 + \mu_a\lambda_{c_2}\mu_{40}\lambda_{c_1}\mu_b + \mu_c\mu_{c_1}\lambda_c^2\lambda_a +$$
$$\lambda_{c_4}\mu_c\mu_{40}\lambda_{c_1}\lambda_a + \lambda_{c_4}\lambda_{c_2}\mu_{40}\lambda_c\mu_b + \lambda_{c_4}\lambda_{c_2}\mu_{40}\lambda_{c_1}\lambda_c +$$

$$
\begin{aligned}
&\lambda_{c_4}\lambda_{c_2}\mu_{40}\mu_{c_1}\mu_b + \lambda_b\lambda_{c_2}\mu_{40}\mu_{c_1}\lambda_a + \mu_1\lambda_a\mu_{40}\lambda_b\lambda_{c_3} + \\
&\lambda_{c_4}\lambda_{c_2}\mu_{40}\mu_{c_1}\lambda_{c_3} + \mu_{c_1}\lambda_a\lambda_c^2\lambda_{c_3} + \lambda_b\lambda_{c_2}\mu_{40}\mu_{c_1}\lambda_{c_3} + \\
&\lambda_{c_4}\lambda_{c_2}\mu_{40}\mu_{c_1}\lambda_c + \mu_a\lambda_{c_2}\mu_{40}\lambda_b\lambda_{c_3} + \mu_a\lambda_a\mu_{40}\lambda_{c_1}\lambda_c + \\
&4\lambda_b\mu_{c_1}\lambda_a\lambda_c^2 + \lambda_{c_4}\mu_c\mu_{40}\lambda_a\lambda_{c_3} + \lambda_{c_4}\mu_c\mu_{40}\lambda_{c_1}\mu_b + \\
&\lambda_{c_4}\lambda_b\mu_{40}\lambda_c\lambda_a + \lambda_{c_4}\lambda_b\mu_{40}\lambda_{c_1}\lambda_{c_3} + \mu_a\lambda_{c_2}\mu_{40}\lambda_{c_1}\lambda_c + \\
&\lambda_{c_4}\lambda_b\mu_{40}\lambda_{c_1}\lambda_a + \lambda_b\lambda_a\mu_{40}\lambda_{c_1}\lambda_{c_3} + \lambda_b\lambda_{c_2}\mu_{40}\lambda_{c_1}\lambda_c + \\
&\lambda_b\mu_{40}\lambda_{c_1}\lambda_a^2 + \lambda_{c_4}\lambda_a\mu_{40}\lambda_{c_1}\mu_b + \lambda_b\lambda_a\mu_{40}\lambda_{c_1}\mu_b + \\
&\mu_c\lambda_b\mu_{c_1}\lambda_a\lambda_{c_3} + \lambda_{c_2}\mu_{40}\mu_{c_1}\lambda_a\mu_b + \lambda_c\lambda_{c_2}\mu_{40}\mu_{c_1}\lambda_{c_3} + \\
&4\lambda_a\lambda_b^2\mu_{c_1}\lambda_c + 2\lambda_c\lambda_{c_2}\mu_{40}\mu_{c_1}\lambda_a + \lambda_{c_2}\mu_{c_1}\lambda_c\lambda_a\lambda_{c_3} + \\
&\lambda_c\lambda_b\mu_{40}\lambda_{c_1}\lambda_{c_3} + \lambda_{c_2}\mu_{40}\mu_{c_1}\lambda_a\lambda_{c_3} + \lambda_{c_4}\lambda_b^2\mu_{c_1}\lambda_c + \\
&\lambda_c\lambda_{c_2}\mu_{40}\mu_{c_1}\mu_b + 2\lambda_c\mu_c\lambda_2\mu_{c_1}\lambda_a + \lambda_c\lambda_a\mu_{40}\lambda_b\lambda_{c_3} + \\
&\lambda_c\lambda_b\mu_{40}\lambda_{c_1}\mu_b + \lambda_{c_4}\mu_{40}\lambda_b^2\lambda_{c_3} + \lambda_{c_2}\mu_{c_1}\lambda_c\lambda_a\mu_b + \\
&\mu_a\mu_c\mu_{40}\lambda_{c_1}\lambda_c + \mu_a\mu_{40}\mu_{c_1}\lambda_c\lambda_{c_3} + \mu_a\mu_{40}\mu_{c_1}\lambda_3^2 + \\
&\mu_a\lambda_b\mu_{c_1}\lambda_c^2 + \mu_a\mu_{40}\mu_{c_1}\lambda_c\mu_b + \lambda_{c_4}\mu_c\mu_{40}\mu_{c_1}\mu_b + \\
&\lambda_c\mu_{40}\lambda_b^2\lambda_{c_3} + \lambda_{c_4}\mu_c\mu_{40}\lambda_a\mu_b + \lambda_{c_4}\mu_c\mu_{40}\lambda_{c_1}\lambda_c + \\
&\lambda_{c_4}\mu_c\mu_{40}\lambda_a^2 + \lambda_{c_4}\mu_{40}\lambda_c\lambda_a^2 + \lambda_{c_2}\lambda_b\mu_{c_1}\lambda_a\lambda_{c_3} + \\
&\lambda_b\lambda_{c_2}\mu_{40}\lambda_c\mu_b + \lambda_b^2\mu_{40}\mu_{c_1}\lambda_{c_3} + \lambda_{c_4}\lambda_b\mu_{40}\mu_{c_1}\lambda_{c_3} + \\
&\lambda_{c_4}\lambda_b\mu_{40}\mu_{c_1}\lambda_c + \lambda_{c_4}\lambda_b\mu_{40}\mu_{c_1}\mu_b + \lambda_{c_4}\mu_c\mu_{40}\lambda_b\lambda_{c_3} + \\
&\lambda_{c_2}\lambda_b\mu_{c_1}\lambda_a\mu_b + \mu_a\lambda_b\mu_{c_1}\lambda_c\mu_b + \lambda_{c_4}\mu_c\mu_{40}\mu_{c_1}\lambda_c + \\
&\lambda_{c_4}\lambda_b^2\mu_{c_1}\lambda_a + \lambda_{c_4}\mu_{c_1}\lambda_a\lambda_c\lambda_{c_3} + \lambda_{c_4}\mu_c\mu_{40}\mu_{c_1}\lambda_a + \\
&\lambda_{c_2}\lambda_b\mu_{c_1}\lambda_c^2 + \lambda_2^2\mu_{40}\mu_{c_1}\mu_b + \mu_a\lambda_{c_2}\lambda_b\mu_{c_1}\lambda_c + \\
&\lambda_a^3\mu_{c_1}\lambda_c + 2\lambda_b^2\mu_{c_1}\lambda_a^2 + \lambda_{c_2}\mu_{40}\lambda_c^2\lambda_a + \\
&\mu_a\lambda_b\mu_{40}\lambda_{c_1}\lambda_{c_3} + \lambda_{c_4}\mu_{40}\mu_{c_1}\lambda_a^2 + \lambda_{c_2}\lambda_b\mu_{c_1}\lambda_a^2 + \\
&\lambda_a^2\mu_{40}\mu_{c_1}\mu_b + \lambda_{c_4}\lambda_a\mu_{40}\mu_{c_1}\lambda_{c_3} + 2\lambda_{c_4}\lambda_a\mu_{40}\mu_{c_1}\lambda_c + \\
&\lambda_{c_2}\mu_{40}\mu_{c_1}\lambda_a^2 + \lambda_{c_2}\mu_{c_1}\lambda_c^2\lambda_a + \lambda_{c_2}\mu_{c_1}\lambda_c\lambda_a^2 + \\
&\mu_c\lambda_b\mu_{c_1}\lambda_a^2 + \mu_c\mu_{40}\mu_{c_1}\lambda_a^2 + \lambda_{c_4}\mu_c\mu_{40}\lambda_c\lambda_a + \\
&\lambda_{c_4}\lambda_b\mu_{40}\lambda_a^2 + \lambda_b^2\mu_{c_1}\lambda_a\mu_b + \mu_a\lambda_b\mu_{40}\mu_{c_1}\lambda_c)
\end{aligned}
$$

$$
\begin{aligned}
E_{28_b} \;=\; &(\lambda_{c_4}\mu_{c_1}\lambda_a\lambda_c\mu_b + \lambda_a^2\mu_{c_1}\lambda_c\lambda_{c_3} + \lambda_a^2\mu_{c_1}\lambda_c\mu_b + \\
&\lambda_{c_4}\lambda_{c_2}\mu_{40}\mu_{c_1}\lambda_a + \mu_a\lambda_b\mu_{40}\mu_{c_1}\lambda_{c_3} + \lambda_{c_4}\lambda_b\mu_{c_1}\lambda_c\lambda_{c_3} +
\end{aligned}
$$

$$
\begin{aligned}
&\lambda_{c_4}\lambda_b\mu_{c_1}\lambda_c\mu_b + \lambda_{c_4}\lambda_b\mu_{c_1}\lambda_c^2 + \lambda_{c_4}\mu_{40}\mu_{c_1}\lambda_c\lambda_{c_3} + \\
&\mu_1\mu_c\mu_{40}\lambda_{c_1}\lambda_a + \lambda_a^2\lambda_b\mu_{c_1}\lambda_{c_3} + \lambda_c\lambda_a\mu_{40}\mu_{c_1}\lambda_{c_3} + \\
&\mu_c\mu_{c_1}\lambda_c\lambda_a\mu_b + \lambda_a^2\lambda_b\mu_{c_1}\mu_b + \lambda_a^2\mu_{40}\mu_{c_1}\lambda_{c_3} + \\
&\lambda_{c_4}\mu_{40}\mu_{c_1}\lambda_c\mu_b + \mu_a\mu_c\mu_{40}\mu_{c_1}\lambda_a + \mu_a\lambda_b\mu_{40}\lambda_{c_1}\lambda_c + \\
&\mu_a\lambda_b\mu_{40}\lambda_{c_1}\mu_b + \lambda_{c_4}\mu_{40}\lambda_{c_1}\lambda_a^2 + \lambda_{c_2}\mu_{40}\lambda_c^2\lambda_{c_3} + \\
&\lambda_{c_4}\mu_{c_1}\lambda_a^2\lambda_c + \lambda_{c_4}\mu_{40}\mu_{c_1}\lambda_c^2 + \lambda_{c_4}\lambda_{c_2}\mu_{40}\lambda_c^2 + \\
&\mu_a\mu_c\mu_{40}\mu_{c_1}\lambda_{c_3} + \lambda_{c_4}\lambda_{c_2}\mu_{40}\lambda_a^2 + \lambda_{c_4}\mu_{40}\lambda_1^2\lambda_{c_3} + \\
&\mu_a\lambda_b\mu_{40}\mu_{c_1}\mu_b + \lambda_{c_4}\mu_{40}\lambda_1^2\mu_b + 2\lambda_b^2\mu_{c_1}\lambda_c^2 + \\
&\mu_a\mu_c\mu_{40}\lambda_{c_1}\lambda_{c_3} + \lambda_{c_2}\lambda_b^2\mu_{c_1}\lambda_c + \mu_c\lambda_b^2\mu_{c_1}\lambda_a + \\
&2\mu_a\lambda_a\lambda_b\mu_{c_1}\lambda_c + \mu_a\mu_{40}\lambda_{c_1}\lambda_a^2 + \mu_a\mu_c\mu_{40}\lambda_{c_1}\mu_b + \\
&\lambda_b^2\mu_{40}\lambda_{c_1}\mu_b + \mu_c\mu_{40}\lambda_b^2\lambda_{c_3} + \lambda_b^2\mu_{40}\lambda_{c_1}\lambda_c + \\
&\mu_a\lambda_{c_2}\mu_{40}\lambda_c^2 + 2\lambda_c\mu_{40}\lambda_b^2\mu_{c_1} + 2\lambda_c\mu_c\mu_{40}\lambda_b\mu_{c_1} + \\
&\lambda_b^2\mu_{40}\lambda_{c_1}\lambda_a + \lambda_b^2\mu_{c_1}\lambda_c\lambda_{c_3} + \lambda_{c_4}\lambda_{c_2}\mu_{40}\lambda_{c_1}\lambda_a + \\
&2\lambda_c\lambda_{c_2}\mu_{40}\lambda_b\mu_{c_1} + \lambda_b^2\mu_{40}\lambda_{c_1}\lambda_{c_3} + \lambda_c\mu_c\mu_{40}\lambda_{c_1}\mu_b + \\
&\lambda_{c_4}\mu_c\mu_{40}\lambda_b\mu_{c_1} + \mu_a\mu_c\mu_{40}\lambda_b\mu_{c_1} + \mu_a\mu_{40}\lambda_b^2\mu_{c_1} + \\
&\lambda_{c_4}\lambda_{c_2}\mu_{40}\lambda_{c_1}\mu_b + \mu_{40}\mu_{c_1}\lambda_c^2\lambda_{c_3} + \mu_a\lambda_{c_2}\mu_{40}\lambda_c\mu_b + \\
&2\lambda_{c_4}\lambda_a\mu_{40}\lambda_b\mu_{c_1} + 5\lambda_c\lambda_a\mu_{40}\lambda_b\mu_{c_1} + \lambda_{c_2}\mu_{40}\lambda_c^3 + \\
&\lambda_c\mu_c\mu_{40}\lambda_{c_1}\lambda_{c_3} + \lambda_{c_4}\lambda_{c_2}\mu_{40}\lambda_b\mu_{c_1} + \lambda_{c_2}\mu_{40}\lambda_b^2\lambda_{c_3} + \\
&\lambda_b\mu_{c_1}\lambda_c^3 + \lambda_a^3\lambda_b\mu_{c_1} + 2\mu_{c_1}\lambda_a^2\lambda_c^2 + \lambda_{c_4}\mu_{40}\lambda_b^2\mu_{c_1} + \\
&\mu_a\mu_c\lambda_b\mu_{c_1}\lambda_c + \mu_a\lambda_{c_2}\mu_{40}\lambda_c\lambda_{c_3} + \mu_a\lambda_{c_2}\mu_{40}\lambda_c\lambda_a + \\
&\lambda_c\lambda_{c_2}\mu_{40}\lambda_{c_1}\lambda_a + \lambda_a^3\mu_{40}\mu_{c_1} + \mu_{40}\mu_{c_1}\lambda_c^3 + \lambda_b^3\mu_{c_1}\lambda_a)
\end{aligned}
$$

$$
\begin{aligned}
E_{28} \;=\;& E_{28_a} + E_{28_b} + (\lambda_{c_2}\lambda_b^2\mu_{c_1}\lambda_a + \mu_c\lambda_b^2\mu_{c_1}\lambda_c + \mu_{40}\lambda_b^3\lambda_{c_3} + \mu_{40}\lambda_b^3\mu_{c_1} + \\
&\mu_a\mu_c\lambda_b\mu_{c_1}\lambda_a + \lambda_{c_4}\lambda_{c_2}\lambda_b\mu_{c_1}\lambda_3 + \lambda_a\mu_{40}\lambda_b^2\lambda_{c_3} + \\
&\mu_a\lambda_a\mu_{40}\lambda_{c_1}\lambda_{c_3} + \lambda_{c_4}\lambda_{c_2}\mu_{40}\lambda_c\lambda_{c_3} + \lambda_{c_4}\lambda_{c_2}\lambda_b\mu_{c_1}\lambda_a + \\
&\mu_a\lambda_a\mu_{40}\lambda_{c_1}\mu_b + \mu_a\lambda_b\mu_{c_1}\lambda_a^2 + 2\mu_a\lambda_a\mu_{40}\mu_{c_1}\lambda_c + \\
&\lambda_{c_4}\lambda_{c_2}\mu_{40}\lambda_{c_1}\lambda_{c_3} + 2\lambda_a\mu_{40}\lambda_b^2\mu_{c_1} + 2\lambda_b\mu_{40}\mu_{c_1}\lambda_c\mu_b + \\
&\lambda_b\mu_{c_1}\lambda_c^2\mu_2 + 2\mu_a\lambda_a\mu_{40}\lambda_b\mu_{c_1} + \lambda_{c_2}\mu_{40}\lambda_b^2\mu_{c_1} + \mu_c\mu_{40}\lambda_b^2\mu_{c_1} + \\
&\lambda_b\lambda_{c_2}\mu_{40}\lambda_c^2 + \lambda_{c_4}\mu_{40}\lambda_a^3 + \lambda_b^3\mu_{c_1}\lambda_c + \mu_{c_1}\lambda_c\lambda_c^3 + \mu_a\lambda_{c_2}\lambda_b\mu_{c_1}\lambda_a)
\end{aligned}
$$

$$
\begin{aligned}
E_{29} &= (\lambda_{c_4} + \lambda_{c_3} + 3\lambda_3 + \mu_3 + 3\lambda_1 + \lambda_{c_2} + \mu_1 + \mu_2 + \lambda_{c_1} + 3\lambda_2) \\
E_{30} &= (\lambda_{c_1}\mu_1 + 2\lambda_2\lambda_{c_2} + 2\lambda_2\mu_3 + \mu_3\mu_1 + 3\lambda_3\lambda_{c_2} + 2\lambda_3\mu_3 + 3\lambda_{c_4}\lambda_1 + 2\lambda_1\mu_1 + \\
&\quad \lambda_{c_2}\mu_1 + \lambda_{c_4}\lambda_{c_2} + 7\lambda_3\lambda_1 + \mu_3\lambda_{c_4} + 2\lambda_2\lambda_{c_4} + \lambda_{c_1}\lambda_{c_3} + \lambda_{c_1}\mu_3 + 2\lambda_{c_1}\lambda_2 + \\
&\quad \mu_3\mu_2 + 2\lambda_3\lambda_{c_3} + 7\lambda_3\lambda_2 + 2\lambda_2\mu_1 + 7\lambda_2\lambda_1 + 3\lambda_1^2 + 2\lambda_1\mu_2 + 2\lambda_3\mu_2 + \\
&\quad \lambda_{c_1}\mu_2 + 2\lambda_{c_1}\lambda_3 + \mu_2\lambda_{c_2} + 2\mu_1\lambda_3 + 2\lambda_2\mu_2 + 3\lambda_2^2 + 2\lambda_{c_1}\lambda_1 + 2\lambda_1\lambda_{c_2} + \\
&\quad \mu_3\lambda_{c_3} + \lambda_{c_3}\mu_1 + 2\lambda_{c_4}\lambda_3 + \lambda_{c_4}\lambda_{c_3} + 2\lambda_1\lambda_{c_3} + 3\lambda_2\lambda_{c_3} + 3\lambda_3^2 + 2\lambda_1\mu_3 + \\
&\quad \mu_2\mu_1 + \lambda_{c_2}\lambda_{c_3} + \lambda_{c_1}\lambda_{c_2} + \mu_2\lambda_{c_4} + \lambda_{c_1}\lambda_{c_4})
\end{aligned}
$$

$$
\begin{aligned}
E_{31} &= (3\lambda_{c_1}\lambda_1\lambda_3 + 2\lambda_{c_2}\lambda_3\mu_2 + \mu_2\lambda_{c_2}\mu_1 + \lambda_{c_2}\lambda_{c_3}\mu_1 + \mu_3\lambda_{c_3}\mu_1 + \\
&\quad \lambda_3\mu_2\mu_1 + \lambda_{c_1}\lambda_3\mu_2 + \lambda_{c_1}\mu_2\mu_1 + 3\lambda_2\lambda_1\mu_3 + \lambda_{c_1}\lambda_{c_4} \\
&\quad \lambda_{c_3} + \lambda_1^2\lambda_{c_3} + \mu_3\lambda_1\mu_2 + 3\lambda_2\lambda_1\lambda_{c_2} + \lambda_2\lambda_{c_4}\lambda_{c_2} + \lambda_1^2\mu_1 + \lambda_2\mu_3\lambda_{c_4} + \\
&\quad \mu_2\lambda_{c_4}\lambda_{c_2} + \lambda_{c_1}\lambda_3\lambda_{c_2} + 3\lambda_2\lambda_3\mu_2 + \lambda_{c_4}\lambda_3\mu_2 + \mu_3\mu_2\mu_1 + \\
&\quad 2\lambda_{c_1}\lambda_2\lambda_{c_3} + 3\lambda_1\lambda_3\lambda_{c_3} + \lambda_1^2\lambda_{c_2} + \mu_3\lambda_{c_4}\lambda_{c_3} + \lambda_{c_1}\lambda_{c_4}\lambda_3 + \\
&\quad 11\lambda_3\lambda_2\lambda_1 + 2\lambda_2\lambda_{c_4}\lambda_{c_3} + \lambda_{c_2}\lambda_1\lambda_{c_3} + 2\lambda_{c_1}\lambda_2\mu_2 + \lambda_{c_1}\lambda_2\lambda_{c_4} + \\
&\quad 2\lambda_{c_1}\lambda_1\lambda_{c_4} + \lambda_2^2\mu_3 + \lambda_2\mu_3\mu_1 + \lambda_3\lambda_{c_3}\mu_1 + \lambda_{c_1}\lambda_1\mu_3 + \lambda_{c_1}\lambda_2\mu_1 + \\
&\quad 4\lambda_3\lambda_2\lambda_{c_2} + \lambda_{c_1}\lambda_1\mu_2 + \mu_3\lambda_3\mu_2 + \lambda_2\mu_3\mu_2 + \lambda_1^2\mu_2 + \lambda_2^2\lambda_{c_2} + 3\lambda_3^2\lambda_{c_2} + \\
&\quad \lambda_{c_1}\lambda_3^2 + 5\lambda_1\lambda_2^2 + \lambda_3^2\lambda_{c_3} + \lambda_{c_1}\lambda_1^2 + \lambda_2^2\mu_2 + \lambda_3^2\mu_1 + 3\lambda_2^2\lambda_{c_3} + 5\lambda_1^2\lambda_3 + \\
&\quad \lambda_{c_1}\lambda_2\mu_3 + 2\mu_3\lambda_1\lambda_{c_4} + 2\lambda_{c_1}\mu_3\lambda_3 + \mu_3\lambda_{c_4}\lambda_3 + \lambda_2\mu_2\mu_1 + 3\lambda_2\lambda_1\mu_1 + \\
&\quad 2\lambda_2\lambda_{c_2}\lambda_{c_3} + 3\lambda_{c_1}\lambda_1\lambda_2 + 2\lambda_{c_4}\lambda_{c_2}\lambda_3 + 2\lambda_{c_1}\lambda_1\mu_1 + \mu_3\lambda_1\mu_1 + \\
&\quad \lambda_{c_1}\mu_3\mu_1 + 3\lambda_3\mu_1\lambda_2 + 3\mu_3\lambda_1\lambda_3 + 2\lambda_{c_2}\mu_1\lambda_3 + \lambda_{c_1}\lambda_{c_2}\lambda_{c_3} + \\
&\quad \lambda_1\mu_2\mu_1 + \lambda_2^3 + 2\lambda_{c_4}\lambda_1\lambda_{c_2} + 2\lambda_{c_2}\lambda_3\lambda_{c_3} + 4\lambda_{c_2}\lambda_1\lambda_3 + \lambda_{c_1}\lambda_{c_2}\mu_1 + \\
&\quad \lambda_2\lambda_{c_2}\mu_1 + \lambda_2\mu_2\lambda_{c_4} + \lambda_{c_1}\lambda_2\lambda_{c_2} + 3\lambda_3\lambda_2\mu_3 + \lambda_2^2\lambda_{c_4} + 2\lambda_1\mu_2\lambda_{c_4} + \\
&\quad \lambda_{c_1}\mu_2\lambda_{c_2} + \mu_3\mu_1\lambda_3 + \lambda_1\mu_2\lambda_{c_2} + 4\lambda_2\lambda_1\lambda_{c_4} + \lambda_{c_1}\mu_2\lambda_{c_4} + \lambda_{c_1}\mu_3\lambda_{c_4} + \\
&\quad \mu_3\mu_2\lambda_{c_4} + \lambda_3^2\mu_2 + \lambda_{c_2}\lambda_1\mu_1 + 2\lambda_{c_4}\lambda_1\lambda_{c_3} + 3\lambda_2\lambda_1\mu_2 + \mu_3\lambda_3\lambda_{c_3} + \\
&\quad 4\lambda_{c_4}\lambda_1\lambda_3 + \lambda_{c_1}\lambda_1\lambda_{c_2} + 4\lambda_2\lambda_3\lambda_{c_3} + \lambda_{c_1}\mu_1\lambda_3 + \lambda_{c_1}\lambda_{c_3}\mu_1 + \\
&\quad \lambda_{c_1}\lambda_1\lambda_{c_3} + \mu_3\lambda_1\lambda_{c_3} + 4\lambda_2\lambda_1\lambda_{c_3} + \lambda_{c_4}\lambda_{c_2}\lambda_{c_3} + 3\lambda_1\mu_1\lambda_3 + \\
&\quad \lambda_3\lambda_{c_4}\lambda_{c_3} + \lambda_1\lambda_{c_3}\mu_1 + \lambda_{c_1}\mu_3\lambda_{c_3} + \lambda_3^3 + \lambda_2\mu_2\lambda_{c_2} + \\
&\quad \lambda_{c_1}\lambda_{c_4}\lambda_{c_2} + \lambda_{c_1}\mu_3\mu_2 + 3\lambda_1\lambda_3\mu_2 + 3\lambda_{c_1}\lambda_2\lambda_3 + 2\lambda_2\mu_3\lambda_{c_3} + \\
&\quad 2\lambda_{c_1}\lambda_{c_2}\lambda_3 + 3\lambda_3\lambda_{c_4}\lambda_2 + \lambda_{c_1}\lambda_2^2 + 5\lambda_3\lambda_2^2 + \lambda_3^2\mu_3 + \lambda_1^2\mu_3 + 5\lambda_3^2\lambda_2 + \\
&\quad \lambda_2^2\mu_1 + 5\lambda_1^2\lambda_2 + 3\lambda_1^2\lambda_{c_4} + \lambda_3^2\lambda_{c_4} + 5\lambda_1\lambda_3^2 + 2\lambda_2\lambda_{c_3}\mu_1 + \lambda_1^3)
\end{aligned}
$$

$$
\begin{aligned}
E_{32} \;=\; & (\lambda_1^2\lambda_{c_4}\lambda_{c_3} + 2\lambda_3^2\lambda_2\lambda_{c_2} + \lambda_2\lambda_1^2\lambda_{c_3} + \lambda_{c_1}\lambda_3^2\mu_3 + \\
& \lambda_2^2\lambda_{c_3}\mu_1 + \lambda_3^2\lambda_2\mu_3 + \lambda_1^2\lambda_{c_4}\lambda_{c_2} + \lambda_3\lambda_{c_4}\lambda_2^2 + \\
& \lambda_{c_1}\lambda_2^2\lambda_3 + \lambda_{c_1}\lambda_1\lambda_2^2 + 2\lambda_1^2\lambda_{c_4}\lambda_3 + \lambda_3^2\lambda_2\lambda_{c_3} + \\
& \lambda_{c_1}\lambda_2^2\lambda_{c_3} + \lambda_3^2\lambda_{c_4}\lambda_{c_2} + \lambda_3^2\lambda_{c_2}\lambda_{c_3} + \lambda_{c_1}\mu_3\mu_1\lambda_3 + \\
& \lambda_2^3\lambda_{c_3} + \lambda_1\lambda_3^3 + \lambda_3\lambda_2^3 + \lambda_1^3\lambda_2 + 2\lambda_3^2\lambda_2^2 + 2\lambda_1^2\lambda_2^2 + 2\lambda_3^2\lambda_1^2 + \\
& \lambda_1\lambda_2^3 + \lambda_3^3\lambda_2 + \lambda_1^3\lambda_3 + \lambda_2\mu_3\lambda_1\lambda_{c_3} + 2\lambda_3\mu_1\lambda_2\lambda_1 + \\
& \lambda_3\lambda_{c_2}\lambda_1\lambda_{c_3} + \lambda_1\lambda_3\lambda_{c_4}\lambda_{c_3} + \lambda_1\lambda_3\mu_3\lambda_{c_3} + \\
& 2\lambda_2\mu_3\lambda_1\lambda_3 + 4\lambda_2\lambda_1^2\lambda_3 + \lambda_2\lambda_1^2\lambda_{c_2} + \lambda_3\mu_1\lambda_2^2 + \lambda_{c_1}\lambda_2^2\mu_2 + \\
& \lambda_3\lambda_1^2\lambda_{c_3} + \lambda_2^2\mu_3\lambda_{c_3} + \lambda_1^2\mu_3\lambda_{c_4} + \lambda_2^2\lambda_{c_2}\lambda_{c_3} + \\
& \lambda_2^2\lambda_{c_4}\lambda_{c_3} + \lambda_3^2\lambda_2\mu_2 + \lambda_{c_1}\lambda_1^2\lambda_3 + \lambda_{c_1}\lambda_3^2\lambda_{c_2} + \\
& \lambda_3^2\lambda_{c_2}\mu_1 + 2\lambda_2^2\lambda_3\lambda_{c_3} + \lambda_{c_1}\lambda_3^2\lambda_1 + 2\lambda_2^2\lambda_1\lambda_{c_3} + \\
& 2\lambda_2\lambda_1^2\lambda_{c_4} + \lambda_1^2\mu_2\lambda_{c_4} + \lambda_1\mu_2\lambda_{c_4}\lambda_{c_2} + 2\lambda_{c_1}\lambda_2\lambda_1\lambda_3 + \\
& \lambda_{c_1}\lambda_2\mu_1\lambda_3 + \lambda_{c_1}\mu_3\lambda_3\mu_2 + \lambda_{c_1}\lambda_1\mu_2\lambda_{c_4} + \\
& \lambda_{c_1}\lambda_1\mu_3\lambda_{c_4} + \lambda_{c_1}\mu_3\lambda_3\lambda_1 + \lambda_{c_1}\lambda_2\lambda_3\mu_2 + \lambda_{c_1}\lambda_2\lambda_1\mu_2 + \\
& 3\lambda_3\lambda_{c_4}\lambda_2\lambda_1 + \lambda_2\lambda_1^2\mu_2 + \lambda_{c_1}\lambda_1^2\lambda_{c_4} + \lambda_3^3\lambda_{c_2} + \\
& \lambda_{c_1}\lambda_{c_2}\mu_1\lambda_3 + \lambda_2\lambda_{c_4}\lambda_1\lambda_{c_2} + \lambda_{c_1}\lambda_2\lambda_1\lambda_{c_2} + \\
& \lambda_{c_1}\lambda_{c_4}\lambda_1\lambda_{c_2} + \lambda_{c_1}\lambda_2\mu_2\lambda_{c_4} + \lambda_{c_1}\mu_3\mu_2\lambda_{c_4} + \lambda_2\lambda_1\mu_2\lambda_{c_4} + \\
& 2\lambda_2\lambda_{c_2}\lambda_3\lambda_{c_3} + 3\lambda_2\lambda_1\lambda_3\lambda_{c_3} + \lambda_2\mu_3\lambda_3\lambda_{c_3} + \mu_3\lambda_1\mu_2\lambda_{c_4} + \\
& 2\lambda_3^2\lambda_{c_2}\lambda_1 + \lambda_2\mu_3\lambda_{c_4}\lambda_{c_3} + \mu_3\lambda_1\lambda_{c_4}\lambda_{c_3} + \lambda_{c_1}\lambda_2\mu_2\lambda_{c_2} + \\
& \lambda_3\lambda_2\mu_2\lambda_{c_2} + \lambda_{c_1}\lambda_2\lambda_{c_4}\lambda_{c_3} + \lambda_3\mu_2\lambda_{c_4}\lambda_{c_2} + \lambda_{c_1}\lambda_2\mu_3\mu_2 + \\
& \lambda_{c_1}\mu_3\lambda_3\lambda_{c_3} + \lambda_{c_1}\lambda_1\mu_1\lambda_3 + \lambda_{c_1}\mu_3\lambda_{c_4}\lambda_3 + \lambda_{c_1}\lambda_2\mu_3\lambda_3 + \\
& \lambda_2\lambda_1^2\mu_3 + \mu_3\lambda_1\lambda_{c_4}\lambda_3 + \lambda_{c_1}\lambda_2\mu_3\lambda_{c_3} + \lambda_{c_1}\lambda_{c_4}\lambda_{c_2}\lambda_{c_3} + \\
& \lambda_2\lambda_{c_4}\lambda_{c_2}\lambda_{c_3} + \lambda_{c_1}\lambda_{c_4}\lambda_1\lambda_3 + \lambda_{c_1}\lambda_2\lambda_{c_2}\lambda_{c_3} + \lambda_{c_1}\mu_3\lambda_{c_3}\mu_1 + \\
& \lambda_{c_1}\lambda_2\lambda_1\lambda_{c_3} + \lambda_2^2\lambda_1\mu_3 + \lambda_{c_1}\mu_2\lambda_{c_2}\mu_1 + \lambda_{c_1}\lambda_{c_2}\lambda_3\lambda_1 + \\
& \lambda_{c_1}\lambda_{c_2}\lambda_3\mu_2 + \lambda_{c_1}\lambda_{c_2}\lambda_3\lambda_{c_3} + \lambda_{c_1}\lambda_1\mu_2\mu_1 + \lambda_{c_1}\lambda_1\lambda_{c_3}\mu_1 + \\
& \lambda_2\lambda_1\lambda_{c_3}\mu_1 + \lambda_{c_1}\lambda_2\lambda_{c_4}\lambda_3 + \lambda_3^2\lambda_{c_2}\mu_2 + \lambda_{c_1}\lambda_2\lambda_3\lambda_{c_3} + \\
& \lambda_{c_1}\mu_3\mu_2\mu_1 + \lambda_{c_1}\lambda_{c_2}\lambda_{c_3}\mu_1 + \lambda_2\mu_3\lambda_{c_3}\mu_1 + \lambda_{c_1}\lambda_1\lambda_{c_2}\mu_1 + \\
& \lambda_2\lambda_{c_2}\lambda_{c_3}\mu_1 + \lambda_3\mu_2\lambda_{c_2}\mu_1 + \lambda_{c_1}\lambda_2\lambda_{c_3}\mu_1 + \lambda_3\lambda_2^2\mu_2 + \lambda_{c_1}\lambda_1\mu_3\mu_1 + \\
& \lambda_{c_1}\mu_2\lambda_{c_4}\lambda_{c_2} + \lambda_{c_1}\lambda_1\lambda_3\mu_2 + \lambda_{c_1}\lambda_2\lambda_1\mu_3 + \lambda_3\lambda_{c_2}\lambda_{c_3}\mu_1 +
\end{aligned}
$$

$$2\lambda_2\lambda_1\lambda_3\mu_2 + \lambda_{c_1}\lambda_1\lambda_3\lambda_{c_3} + 3\lambda_2\lambda_{c_2}\lambda_1\lambda_3 + \lambda_{c_1}\lambda_3^2\lambda_2 +$$

$$\lambda_3\lambda_{c_4}\lambda_2\mu_3 + \lambda_3\lambda_{c_4}\lambda_2\lambda_{c_2} + \lambda_3\mu_1\lambda_2\lambda_{c_2} + \lambda_3\mu_1\lambda_2\mu_3 +$$

$$\lambda_3\lambda_1\lambda_{c_3}\mu_1 + \lambda_2\mu_3\lambda_1\mu_2 + \lambda_1\lambda_{c_4}\lambda_3\mu_2 + \mu_3\lambda_1\lambda_3\mu_2 +$$

$$\lambda_{c_1}\lambda_1^2\mu_1 + \lambda_{c_1}\lambda_2\lambda_1\mu_1 + \lambda_3\lambda_{c_2}\lambda_1\mu_1 + \lambda_{c_1}\lambda_{c_4}\lambda_{c_2}\lambda_3 +$$

$$\lambda_3\lambda_{c_4}\lambda_{c_2}\lambda_{c_3} + \lambda_3\lambda_1\mu_2\lambda_{c_2} + \lambda_3\lambda_2\mu_2\lambda_{c_4} + \lambda_2\lambda_1\mu_2\lambda_{c_2} +$$

$$\lambda_2\mu_3\lambda_1\lambda_{c_4} + \lambda_{c_1}\lambda_1^2\lambda_2 + 2\lambda_1\lambda_{c_4}\lambda_{c_2}\lambda_3 + \lambda_{c_1}\lambda_2\lambda_{c_2}\lambda_3 +$$

$$\lambda_{c_1}\lambda_2\lambda_1\lambda_{c_4} + \lambda_{c_1}\lambda_2\mu_2\mu_1 + \lambda_1\lambda_{c_4}\lambda_{c_2}\lambda_{c_3} + \lambda_{c_1}\lambda_{c_4}\lambda_1\lambda_{c_3} +$$

$$2\lambda_2\lambda_{c_4}\lambda_1\lambda_{c_3} + \lambda_{c_1}\mu_3\lambda_{c_4}\lambda_{c_3} + 4\lambda_3^2\lambda_2\lambda_1 + \lambda_3\lambda_2^2\lambda_{c_2} + \lambda_3\lambda_2^2\mu_3 +$$

$$4\lambda_3\lambda_2^2\lambda_1 + \lambda_3^2\lambda_2\mu_1 + \lambda_2\lambda_1^2\mu_1 + \lambda_3\lambda_1^2\mu_1 + \lambda_2^2\lambda_1\mu_1 + \lambda_1\lambda_3^2\mu_2 + \lambda_1\lambda_3^2\lambda_{c_3} +$$

$$\lambda_1\lambda_3^2\mu_3 + \lambda_2^2\lambda_1\lambda_{c_2} + \lambda_3\lambda_2\mu_2\mu_1 + \lambda_3\lambda_1\mu_2\mu_1 + \lambda_3\lambda_2\lambda_{c_3}\mu_1 + \lambda_3\lambda_2\lambda_{c_4}\lambda_{c_3} +$$

$$\lambda_2\lambda_{c_2}\lambda_1\lambda_{c_3} + \lambda_2\lambda_{c_2}\lambda_1\mu_1 + \mu_3\lambda_1^2\lambda_3 +$$

$$\lambda_3\lambda_1^2\lambda_{c_2} + \lambda_1^2\lambda_3\mu_2 + \lambda_1\lambda_3^2\lambda_{c_4} + \lambda_1\lambda_3^2\mu_1 + \lambda_3^2\lambda_2\lambda_{c_4} +$$

$$\lambda_2^2\lambda_1\lambda_{c_4} + \lambda_2\mu_3\lambda_1\mu_1 + \lambda_1^3\lambda_{c_4} + \lambda_2^2\lambda_1\mu_2)$$

Appendix C

The constants defined below are associated with the models considered under the category of Standby Systems.

$$
\begin{aligned}
F_1 &= (\mu_c^2\lambda_c + \mu\mu_c\lambda_c + \mu\mu_c^2 + \mu_c^2\lambda + \mu^2\mu_c + \mu_{c_1}\mu\mu_c + \mu_{c_1}\mu_c\lambda + \mu_{c_1}\mu^2) \\
F_2 &= (\mu + \mu_c)(\lambda\mu_c + \lambda\mu_{c_1} + \mu_{c_1}\lambda_c) \\
F_3 &= (\lambda\mu_c + \lambda\mu_{c_1} + \mu_{c_1}\lambda_c)\lambda \\
F_4 &= \lambda_c(\mu^2 + 2\lambda\mu_c + \mu\lambda + \mu\mu_c + \mu\lambda_c + \mu_c\lambda_c) \\
F_5 &= (\mu\lambda_c^2 + \mu_c\lambda_c^2 + \mu_c^2\lambda_c + 2\mu\mu_c\lambda_c + \mu_{c_1}\mu_c\lambda_c + 2\lambda\mu_c\lambda_c + \mu^2\lambda_c + \\
&\quad \mu_{c_1}\mu\lambda_c + \mu\lambda\lambda_c + \lambda\mu_{c_1}\lambda_c + \mu\mu_c^2 + 2\mu_c^2\lambda + \mu^2\mu_c + \mu_{c_1}\mu\mu_c + \\
&\quad \mu\lambda\mu_c + 2\mu_{c_1}\mu_c\lambda + \lambda^2\mu_c + \mu_{c_1}\mu^2 + \mu_{c_1}\mu\lambda + \lambda^2\mu_{c_1})
\end{aligned}
$$

$$
\begin{aligned}
F_6 &= (\mu_{c_2}\mu_2\mu_c\lambda_c + \mu\mu_2\mu_c\lambda + \mu_{c_2}\mu_2\mu\lambda_c + \mu_{c_2}\mu_2\mu_c\lambda + \\
&\quad \mu_{c_1}\mu_2\mu\lambda + \mu\mu_{c_2}\mu_c\mu_2 + \mu_{c_2}\mu_2\mu\lambda + \mu_{c_1}\mu_2\mu_c\lambda + \\
&\quad \mu\mu_2\mu_c\lambda_c + \mu\mu_{c_1}\mu_c\mu_2 + \mu_{c_1}\mu^2\mu_2 + \mu_{c_2}\mu^2\mu_2 + \\
&\quad \mu_c\mu_2\mu^2 + \mu_c^2\lambda_c\mu_2 + \mu_2\mu_c^2\lambda + \mu\mu_2\mu_c^2 + \\
&\quad \mu\mu_{c_1}\mu_c\lambda + \mu\mu_{c_2}\mu_c\lambda + \mu_{c_2}\lambda_c\mu_c\lambda + \lambda_c\mu\mu_{c_1}\mu_c + \\
&\quad \mu\lambda_c\mu_c\lambda + \mu_{c_1}\mu^3 + \lambda_c\mu_{c_1}\mu^2 + \lambda_c^2\mu\mu_c + \\
&\quad 2\lambda_c\mu_c\mu^2 + \mu_c\mu_{c_1}\mu^2 + 2\lambda_c\mu_c^2\mu + \mu_{c_2}\mu^2\mu_c + \\
&\quad \mu_c^2\lambda^2 + \mu_{c_2}\mu_c\lambda^2 + \mu_{c_1}\mu_c\lambda^2 + \mu_c\mu^3 + \mu_c^2\mu + \\
&\quad \lambda_c^2\mu_c^2 + \mu_c^2\mu\lambda + 2\lambda_c\mu_c^2\lambda + \mu_{c_2}\mu^3)
\end{aligned}
$$

$$
\begin{aligned}
F_7 &= (\mu\mu_2\mu_c\lambda + \mu_{c_1}\mu_2\mu\lambda_c + \mu_{c_2}\mu_2\mu_c\lambda + \mu_{c_1}\mu_2\mu\lambda + \\
&\quad 2\mu_{c_1}\lambda_c\mu_c\lambda + \mu\mu_{c_2}\mu_c\lambda_c + \lambda_c\mu\mu_{c_1}\lambda + \mu^2\lambda_c\mu_{c_2} +
\end{aligned}
$$

$$\mu_{c_2}\mu_2\mu\lambda + \mu_{c_1}\mu_2\mu_c\lambda + \mu_{c_1}\mu_2\mu_c\lambda_c + \mu_c\mu^2\lambda +$$
$$\mu_{c_1}\mu\lambda_c + \mu_{c_1}\mu_c\lambda_c^2 + \mu_{c_1}\mu^2\lambda + \mu_{c_2}\mu^2\lambda + \mu_2\mu_c^2\lambda +$$
$$\mu\mu_{c_1}\mu_c\lambda + \mu\mu_{c_2}\mu_c\lambda + \lambda_c\mu\mu_{c_1}\mu_c + \mu\lambda_c\mu_c\lambda +$$
$$\lambda_c\mu_{c_1}\mu^2 + \mu_c^2\lambda^2 + \mu_{c_2}\mu_c\lambda^2 + \mu_{c_1}\mu_c\lambda^2 + \mu_c^2\mu\lambda + \lambda_c\mu_c^2\lambda)$$

$$F_8 = (\mu + \mu_c)(\lambda^2\mu_c + \mu_{c_1}\lambda^2 + \mu_{c_2}\lambda^2 + \mu_{c_1}\lambda\lambda_c + 2\mu_{c_2}\lambda\lambda_c + \mu_{c_2}\lambda_c^2 + \lambda_c\mu\mu_{c_2})$$

$$F_9 = \lambda(\lambda^2\mu_c + \mu_{c_1}\lambda^2 + \mu_{c_2}\lambda^2 + \mu_{c_1}\lambda\lambda_c + 2\mu_{c_2}\lambda\lambda_c + \mu_{c_2}\lambda_c^2 + \lambda_c\mu\mu_{c_2})$$

$$\begin{aligned}F_{10} = \ & \lambda_c(2\mu\mu_c\lambda + \mu\mu_2\mu_c + 2\mu_2\mu\lambda + 2\mu_2\mu_c\lambda + \mu\lambda_c^2 + \mu_c\lambda_c^2 + \\ & 2\mu\lambda_c\lambda + 3\mu_c\lambda_c\lambda + \mu\lambda^2 + 3\lambda^2\mu_c + \mu^3 + 2\lambda_c\mu^2 + \mu_2\mu\lambda_c + \\ & \mu_2\mu^2 + 2\lambda_c\mu\mu_c + \mu^2\lambda + \mu_2\mu_c\lambda_c + \mu_c\mu^2)\end{aligned}$$

$$\begin{aligned}F_{11} = \ & (\mu_{c_2}\mu_2\mu_c\lambda_c + 2\mu\mu_2\mu_c\lambda + 2\lambda_c\mu_2\mu_c\lambda + \mu_{c_1}\mu_2\mu\lambda_c + \\ & \mu_{c_2}\mu_2\mu\lambda_c + 2\mu_{c_2}\mu_2\mu_c\lambda + 2\mu_{c_1}\mu_2\mu\lambda + 3\mu_{c_1}\lambda_c\mu_c\lambda + \\ & 3\lambda_c\mu\mu_{c_2}\lambda + 2\mu\mu_{c_2}\mu_c\lambda_c + 2\lambda_c\mu\mu_{c_1}\lambda + \mu\lambda_c^3 + \\ & 2\mu_{c_2}\lambda_c\lambda + 2\mu^2\lambda_c\mu_{c_2} + \mu_{c_2}\lambda_c^2\mu_c + \mu\mu_{c_2}\mu_c\mu_2 + \\ & 2\mu_{c_2}\mu_2\mu\lambda + 2\mu_{c_1}\mu_2\mu_c\lambda + 2\mu\mu_2\mu_c\lambda_c + \mu\mu_{c_1}\mu_c\mu_2 + \\ & 2\lambda_c\mu_2\mu\lambda + \mu_{c_1}\mu_2\mu_c\lambda_c + \mu_2\mu_c\lambda_c^2 + \mu_{c_1}\mu^2\mu_2 + \\ & \mu_{c_2}\mu^2\mu_2 + \lambda_c\mu_2\mu^2 + \mu_c\mu^2\lambda + \lambda_c\mu^2\lambda + \\ & \mu_{c_1}\mu\lambda_c^2 + 2\mu\lambda_c^2\lambda + \mu_{c_1}\mu_c\lambda_c + \mu_{c_2}\lambda_c^2\lambda + \\ & \mu_{c_1}\mu^2\lambda + \mu_{c_1}\lambda_c\lambda + \mu_{c_2}\mu^2\lambda + \mu_c\lambda_c^3 + \mu_c\mu_2\mu^2 + \mu_c^2\lambda_c\mu_2 + \\ & 2\mu_2\mu_c^2\lambda + \mu\mu_2\mu_c^2 + \mu_2\mu\lambda_c^2 + \mu_c\mu\lambda^2 + \lambda_c^2\mu\mu_{c_2} + \\ & 2\mu\mu_{c_1}\mu_c\lambda + 2\mu\mu_{c_2}\mu_c\lambda + 3\mu_{c_2}\lambda_c\mu_c\lambda + \mu\lambda_c\lambda^2 + \\ & 3\mu_c\lambda_c\lambda^2 + \mu_{c_1}\mu\lambda^2 + \mu_{c_2}\mu\lambda^2 + 2\lambda_c\mu\mu_{c_1}\mu_c + 4\mu\lambda_c\mu_c\lambda + \\ & 2\lambda_c^2\mu^2 + \lambda_c\mu^3 + \mu_c\lambda^3 + \mu_{c_2}\lambda^3 + \mu_{c_1}\lambda^3 + \mu_{c_1}\mu^3 + 3\mu_c\lambda_c^2\lambda + \\ & 2\lambda_c\mu_{c_1}\mu^2 + 3\lambda_c^2\mu\mu_c + 3\lambda_c\mu_c\mu + \mu_c\mu_{c_1}\mu^2 + 2\lambda_c\mu_c^2\mu + \\ & \mu_{c_2}\mu^2\mu_c + 3\mu_c^2\lambda^2 + 3\mu_{c_2}\mu_c\lambda^2 + 3\mu_{c_1}\mu_c\lambda^2 + \mu_c\mu^3 + \mu_c^2\mu^2 + \\ & \lambda_c^2\mu_c^2 + 2\mu_c^2\mu\lambda + 3\lambda_c\mu_c^2\lambda + \mu_{c_2}\mu^3)\end{aligned}$$

$$
\begin{aligned}
F_{12} \;=\;& (\mu_{c_2}\mu^2\lambda\mu_c + 4\mu\lambda\lambda_c\mu_c^2 + \mu^2\mu_{c_1}\mu_c\mu_3 + \mu^2\lambda_c^2\mu_{c_1} + \mu_{c_2}\mu^3\mu_3 + \\
& \mu^3\mu_{c_1}\mu_3 + 3\mu^2\lambda_c^2\mu_c + 2\mu^3\lambda_c\mu_{c_1} + 3\mu^3\lambda_c\mu_c + \mu_{c_3}\mu^4 + \mu_{c_3}\lambda\mu_c\mu^2 + \\
& \mu_{c_3}\lambda^2\mu\mu_c + \lambda_c\mu^2\lambda\mu_{c_1} + 2\mu_{c_3}\lambda_c\mu^2\mu_3 + \mu_{c_2}\mu^2\lambda\mu_2 + \mu^2\lambda\mu_{c_1}\mu_2 + \\
& \mu_{c_2}\mu^2\lambda\mu_3 + 2\lambda_c\mu^2\lambda\mu_c + \mu_{c_2}\mu^2\mu_c\mu_3 + \mu\lambda\mu_{c_1}\mu_3\mu_2 + \\
& 2\mu_{c_2}\lambda_c\mu\mu_c\mu_2 + \mu^4\mu_{c_1} + \mu^4\mu_c + \lambda^3\mu_c^2 + \lambda_c^3\mu_c^2 + \mu^3\mu_c^2 + \mu^4\mu_{c_2} + \\
& 2\mu_{c_3}\lambda^2\mu_c\lambda_c + \mu_{c_2}\mu_c\lambda^2\mu_3 + 2\lambda_c\mu\mu_{c_3}\lambda\mu_3 + \mu_{c_3}\lambda_c^2\mu\mu_3 + \mu_{c_3}\mu^2\lambda\mu_3 + \\
& \mu^2\mu_{c_1}\mu_c\mu_2 + \mu\lambda_c^2\mu_c\mu_2 + \mu_{c_1}\lambda^2\mu_c\mu_2 + 2\lambda_c\mu^2\mu_c\mu_2 + \mu^2\mu_{c_1}\mu_3\mu_2 + \\
& \mu\lambda_c^2\mu_{c_1}\mu_c + \mu_{c_2}\lambda_c\mu^2\mu_c + \mu\mu_{c_1}\lambda^2\mu_c + \mu\mu_{c_3}\lambda^2\mu_3 + \mu^3\mu_{c_3}\mu_3 + \\
& \mu_{c_2}\lambda_c\lambda^2\mu_c + \lambda_c\mu\lambda^2\mu_c + \mu_{c_3}\lambda_c\mu_c\mu\mu_2 + \mu_{c_2}\mu\lambda_c^2\mu_2 + \\
& \mu_{c_2}\mu_c\lambda_c^2\mu_2 + \mu_{c_2}\mu_c\lambda^2\mu_2 + \mu_{c_2}\mu^2\mu_c\mu_2 + \mu_{c_2}\mu_3\mu^2\mu_2 + \\
& 2\mu_{c_2}\lambda_c\mu^2\mu_2 + \mu_{c_3}\mu^2\lambda\mu_2 + \mu\lambda_c\mu_{c_1}\mu_c\mu_3 + \mu\lambda\mu_{c_1}\mu_c\mu_3 + \\
& 2\lambda_c\mu^2\mu_{c_1}\mu_c + 2\mu_{c_3}\lambda\mu\mu_c\lambda_c + \mu_{c_2}\mu\lambda^2\mu_c + \mu^3\mu_{c_1}\mu_c + \\
& 3\lambda^2\lambda_c\mu_c^2 + \mu^3\mu_{c_1}\mu_2 + \lambda_c^2\mu_c^2\mu_2 + \mu^3\mu_c\mu_3 + \mu^2\mu_c^2\mu_3 + \\
& 3\lambda_c\mu^2\mu_c^2 + 3\mu\lambda_c^2\mu_c^2 + \mu^2\lambda\mu_c^2 + \lambda_c\mu^2\mu_{c_1}\mu_3 + \\
& 2\mu\lambda_c\mu_c^2\mu_3 + \mu^2\mu_{c_1}\mu_c\mu_3 + \mu\lambda_c^2\mu_c\mu_3 + \mu_{c_1}\lambda^2\mu_c\mu_3 + \\
& 2\lambda_c\mu^2\mu_c\mu_3 + \mu_{c_3}\lambda_c^2\mu_c\mu_3 + \mu\mu_{c_1}\lambda^2\mu_3 + \mu_{c_2}\lambda_c\mu\lambda\mu_3 + \\
& 2\mu_{c_3}\lambda_c\mu_c\lambda\mu_3 + 2\lambda_c\mu\mu_c\lambda\mu_3 + \mu_{c_2}\mu\mu_c\lambda\mu_3 + \mu\mu_{c_3}\mu_c\lambda\mu_3 + \\
& \mu_{c_3}\mu_c\lambda^2\mu_3 + \mu\mu_c\lambda^2\mu_3 + 2\lambda_c\mu_c^2\lambda\mu_3 + \mu^2\mu_c\lambda\mu_3 + \\
& \mu\mu_c^2\lambda\mu_3 + 2\mu_{c_3}\lambda_c\mu_c\mu\mu_3 + \mu_{c_2}\mu_c\mu_3\lambda\mu_2 + \mu_{c_3}\mu_3\mu\lambda\mu_2 + \\
& \mu_{c_3}\mu_3\mu_c\lambda\mu_2 + \mu\mu_c\mu_3\lambda\mu_2 + 2\mu_{c_2}\mu\mu_c\lambda\mu_2 + 2\mu\mu_{c_3}\mu_c\lambda\mu_2 + \\
& \mu_{c_2}\mu\mu_c\mu_3\mu_2 + \mu_{c_2}\mu\mu_3\lambda_c\mu_2 + \mu_{c_2}\mu_c\mu_3\lambda_c\mu_2 + \mu_{c_3}\mu_c\lambda^2\mu_2 + \\
& 2\lambda_c\mu_c^2\lambda\mu_2 + \mu_3\mu_c^2\lambda\mu_2 + \mu^2\mu_c\lambda\mu_2 + 2\mu\lambda\lambda_c\mu_{c_1}\mu_c + \\
& \mu_{c_3}\lambda\mu_c\lambda_c^2 + \mu_{c_2}\mu\mu_3\lambda\mu_2 + 2\mu\mu_c^2\lambda\mu_2 + \mu^2\lambda\mu_{c_1}\mu_3 + \\
& \mu^2\mu_c\mu_3\mu_2 + \mu_{c_3}\mu^2\mu_3\mu_2 + \lambda_c\mu_c^2\mu_3\mu_2 + \mu\mu_c^2\mu_3\mu_2 + \\
& \mu_{c_2}\lambda_c\mu\lambda\mu_2 + 2\mu_{c_2}\lambda_c\mu_c\lambda\mu_2 + \mu_{c_3}\lambda_c\mu_c\lambda\mu_2 + \lambda_c\mu\mu_c\lambda\mu_2 + \\
& \mu_{c_3}\lambda_c\mu^2\mu_2 + \mu^2\mu_{c_3}\mu_c\mu_2 + \mu\mu_{c_1}\mu_c\mu_3\mu_2 + \lambda\mu_{c_1}\mu_c\mu_3\mu_2 + \\
& \lambda_c\mu\mu_{c_3}\mu_3\mu_2 + \mu\lambda_c\mu_{c_1}\mu_c\mu_2 + 2\mu\lambda\mu_{c_1}\mu_c\mu_2 + \mu_{c_1}\lambda\lambda_c\mu_c\mu_2 + \\
& \mu\lambda_c\mu_c\mu_3\mu_2 + \mu_{c_3}\lambda_c\mu_c\mu_3\mu_2 + \mu\lambda\lambda_c\mu_{c_1}\mu_2 + \lambda_c\mu^2\mu_{c_1}\mu_2 + \\
& 2\mu\lambda_c\mu_c^2\mu_2 + \mu_{c_2}\lambda^3\mu_c + 3\lambda\lambda_c^2\mu_c^2 + \mu^2\mu_c^2\mu_2 +
\end{aligned}
$$

$$
\begin{aligned}
& \mu_c^2\lambda^2\mu_2 + \mu^3\lambda_c\mu_{c_2} + \mu_{c_2}\mu^3\mu_c + \mu_{c_2}\mu^3\mu_2 + \\
& \mu\lambda_c^3\mu_c + \mu\lambda^2\mu_c^2 + \lambda_c^2\mu_c^2\mu_3 + \mu_{c_1}\lambda^3\mu_c + \\
& \mu^3\mu_{c_3}\mu_2 + \mu_{c_3}\lambda^3\mu_c + \mu_c^2\lambda^2\mu_3 + \mu_{c_3}\mu^3\mu_c + \\
& \mu_{c_2}\lambda_c\mu_c\lambda\mu_3 + \mu\mu_{c_3}\mu_c\mu_3\mu_2 + \mu^3\mu_c\mu_2 + 2\mu\lambda_c^2\lambda\mu_c + \\
& \mu^2\lambda\mu_{c_1}\mu_c + \mu_{c_2}\mu\lambda^2\mu_3)
\end{aligned}
$$

$$
\begin{aligned}
F_{13} \;=\; & (\mu_{c_2}\mu^2\lambda\mu_c + 2\mu\lambda\lambda_c\mu_c^2 + 2\mu^2\lambda_c^2\mu_{c_1} + \mu^2\lambda_c^2\mu_{c_2} + \\
& \lambda_c^3\mu_{c_1}\mu_c + \mu^3\lambda\mu_{c_2} + \mu^3\lambda\mu_c + \mu\lambda_c^3\mu_{c_1} + \\
& \mu^3\lambda_c\mu_{c_1} + \mu^3\lambda\mu_{c_1} + \mu_{c_3}\lambda\mu_c\mu^2 + \mu_{c_3}\lambda^2\mu\mu_c + \\
& 2\lambda_c\mu^2\lambda\mu_{c_1} + \mu_{c_2}\mu^2\lambda\mu_2 + \mu^2\lambda\mu_{c_1}\mu_2 + \mu_{c_2}\mu^2\lambda\mu_3 + \\
& 2\lambda_c\mu^2\lambda\mu_c + \mu\lambda\mu_{c_1}\mu_3\mu_2 + \lambda^3\mu_c^2 + \mu_{c_3}\lambda^2\mu_c\lambda_c + \\
& \mu_{c_2}\mu\lambda_c^2\mu_c + \mu_{c_2}\mu_c\lambda^2\mu_3 + \mu_{c_2}\lambda_c\mu\mu_c\mu_3 + \lambda_c\mu\mu_{c_3}\lambda\mu_3 + \\
& \mu_{c_2}\lambda_c\mu^2\mu_3 + \mu_{c_3}\mu^2\lambda\mu_3 + \lambda_c^2\mu_{c_1}\mu_c\mu_2 + \mu_{c_1}\lambda^2\mu_c\mu_2 + \\
& 3\lambda\lambda_c^2\mu_{c_1}\mu_c + 3\mu_{c_1}\lambda_c\lambda^2\mu_c + 2\mu_{c_2}\lambda_c\mu\lambda\mu_c + 2\mu\lambda_c^2\mu_{c_1}\mu_c + \\
& \mu_{c_2}\lambda_c\mu^2\mu_c + \mu\mu_{c_1}\lambda^2\mu_c + \mu\mu_{c_3}\lambda^2\mu_3 + \lambda_c\mu\lambda^2\mu_c + \\
& \mu_{c_2}\mu_c\lambda^2\mu_2 + \mu_{c_3}\lambda_c\mu_c\mu^2 + \mu_{c_3}\mu^2\lambda\mu_2 + \mu\lambda_c\mu_{c_1}\mu_c\mu_3 + \\
& \mu\lambda\mu_{c_1}\mu_c\mu_3 + 2\mu_{c_1}\lambda\lambda_c\mu_c\mu_3 + 2\mu\lambda\lambda_c\mu_{c_1}\mu_3 + \lambda_c\mu^2\mu_{c_1}\mu_c + \\
& 2\mu\lambda_c^2\lambda\mu_{c_1} + \lambda_c\mu^2\lambda\mu_{c_2} + \mu_{c_2}\mu\lambda^2\mu_c + 2\lambda^2\lambda_c\mu_c^2 + \\
& \mu_{c_3}\mu^3\lambda_c + \mu^2\lambda\mu_c^2 + \mu\lambda_c^2\mu_{c_1}\mu_3 + \lambda_c\mu^2\mu_{c_1}\mu_3 + \\
& \lambda_c^2\mu_{c_1}\mu_c\mu_3 + \mu_{c_1}\lambda^2\mu_c\mu_3 + \mu\mu_{c_1}\lambda^2\mu_3 + \mu_{c_3}\lambda_c\mu_c\lambda\mu_3 + \\
& \lambda_c\mu\mu_c\lambda\mu_3 + \mu_{c_2}\mu\mu_c\lambda\mu_3 + \mu\mu_{c_3}\mu_c\lambda\mu_3 + \mu_{c_3}\mu_c\lambda^2\mu_3 + \\
& \mu\mu_c\lambda^2\mu_3 + \lambda_c\mu_c^2\lambda\mu_3 + \mu^2\mu_c\lambda\mu_3 + \mu\mu_c^2\lambda\mu_3 + \\
& \mu_{c_2}\mu_c\mu_3\lambda\mu_2 + \mu_{c_3}\mu_3\mu\lambda\mu_2 + \mu_{c_3}\mu_3\mu_c\lambda\mu_2 + \mu\mu_c\mu_3\lambda\mu_2 + \\
& \mu_{c_2}\mu\mu_c\lambda\mu_2 + \mu\mu_{c_3}\mu_c\lambda\mu_2 + \mu_{c_3}\mu_c\lambda^2\mu_2 + \lambda_c\mu_c^2\lambda\mu_2 + \\
& \mu_3\mu_c^2\lambda\mu_2 + \mu^2\mu_c\lambda\mu_2 + 3\mu\lambda\lambda_c\mu_{c_1}\mu_c + \mu_{c_2}\mu\mu_3\lambda\mu_2 + \\
& \mu\mu_c^2\lambda\mu_2 + \mu^2\lambda\mu_{c_1}\mu_3 + \mu_{c_2}\lambda_c\mu\lambda\mu_2 + \mu_{c_2}\lambda_c\mu_c\lambda\mu_2 + \\
& \lambda_c\mu\mu_c\lambda\mu_2 + \lambda\mu_{c_1}\mu_c\mu_3\mu_2 + \mu\lambda_c\mu_{c_1}\mu_c\mu_2 + \mu\lambda\mu_{c_1}\mu_c\mu_2 + \\
& 2\mu_{c_1}\lambda\lambda_c\mu_c\mu_2 + \mu\lambda_c\mu_{c_1}\mu_3\mu_2 + \lambda_c\mu_{c_1}\mu_c\mu_3\mu_2 + \mu\lambda\lambda_c\mu_{c_1}\mu_2 + \\
& \mu\lambda_c^2\mu_{c_1}\mu_2 + \lambda_c\mu^2\mu_{c_1}\mu_2 + \mu_{c_2}\lambda^3\mu_c + \lambda\lambda_c^2\mu_c^2 + \\
& \mu_c^2\lambda^2\mu_2 + \mu^3\lambda_c\mu_{c_2} + \mu\lambda^2\mu_c^2 + \mu_{c_1}\lambda^3\mu_c +
\end{aligned}
$$

$$
\begin{aligned}
& \mu_{c_3}\lambda\mu^3 + \mu_{c_2}\lambda^3\mu_c + \mu_c^2\lambda^2\mu_3 + \mu\lambda_c^2\lambda\mu_c + \\
& \mu^2\lambda\mu_{c_1}\mu_c + \lambda_c\mu\lambda^2\mu_{c_1} + \mu_{c_2}\mu\lambda^2\mu_3) \\
\\
F_{14} \;=\; & (\lambda_c\mu\lambda^2\mu_{c_2} + 2\mu^2\lambda_c^2\mu_{c_2} + \lambda_c^3\mu_{c_2}\mu_c + \mu^2\lambda^2\mu_{c_2} + \\
& \mu_{c_3}\lambda^2\mu\mu_c + \lambda_c\mu^2\lambda\mu_{c_1} + \lambda^3\mu_c^2 + 2\mu_{c_2}\mu\lambda_c^2\mu_c + \\
& 3\lambda\lambda_c^2\mu_{c_2}\mu_c + \mu_{c_2}\mu_c\lambda^2\mu_3 + \mu_{c_2}\lambda_c\mu\mu_c\mu_3 + \mu_{c_2}\lambda_c\mu^2\mu_3 + \\
& \lambda\lambda_c^2\mu_{c_1}\mu_c + 2\mu_{c_1}\lambda_c\lambda^2\mu_c + 3\mu_{c_2}\lambda_c\mu\lambda\mu_c + 2\mu\lambda_c^2\lambda\mu_{c_2} + \\
& \mu_{c_2}\lambda_c\mu^2\mu_c + \mu\mu_{c_1}\lambda^2\mu_c + \mu\mu_{c_3}\lambda^2\mu_3 + 3\mu_{c_2}\lambda_c\lambda^2\mu_c + \\
& \lambda_c\mu\lambda^2\mu_c + \mu_{c_3}\lambda_c\mu_c\mu^2 + \mu_{c_1}\lambda\lambda_c\mu_c\mu_3 + \mu\lambda\lambda_c\mu_{c_1}\mu_3 + \\
& \mu\lambda_c^2\lambda\mu_{c_1} + 2\mu_{c_3}\lambda\lambda_c\mu^2 + 2\mu_{c_3}\lambda\mu\mu_c\lambda_c + 2\lambda_c\mu^2\lambda\mu_{c_2} + \\
& \mu_{c_2}\mu\lambda^2\mu_c + \lambda^2\lambda_c\mu_c^2 + \mu^2\lambda^2\mu_{c_1} + \mu_{c_3}\lambda_c^2\mu^2 + \\
& \mu_{c_3}\mu^3\lambda_c + \mu_{c_1}\lambda^2\mu_c\mu_3 + \mu\mu_{c_1}\lambda^2\mu_3 + 2\mu_{c_2}\lambda_c\mu\lambda\mu_3 + \\
& \mu_{c_3}\mu_c\lambda^2\mu_3 + \mu\mu_c\lambda^2\mu_3 + \mu_{c_2}\mu\lambda_c^2\mu_3 + \mu_{c_3}\lambda^2\mu^2 + \\
& \mu_{c_3}\lambda_c^2\mu\mu_c + \mu\lambda\lambda_c\mu_{c_1}\mu_c + \mu_{c_2}\mu_c\lambda_c^2\mu_3 + \mu_{c_2}\lambda^3\mu_c + \\
& \mu\lambda_c^3\mu_{c_2} + \mu^3\lambda_c\mu_{c_2} + \mu\lambda^2\mu_c^2 + \mu_{c_1}\lambda^3\mu_c + \\
& \mu_{c_3}\lambda^3\mu_c + \mu^2\lambda^2\mu_c + \mu_c^2\lambda^2\mu_3 + 2\mu_{c_1}\lambda_c\mu_c\lambda\mu_3 + \lambda_c\mu\lambda^2\mu_{c_1} + \mu_{c_2}\mu\lambda^2\mu_3) \\
\\
F_{15} \;=\; & (\mu + \mu_c)(2\mu_{c_3}\mu_2\lambda_c\lambda + \mu_{c_3}\lambda_c^2\mu_2 + \lambda_c\mu\mu_{c_3}\mu_2 + \\
& \mu_c\lambda^3 + \mu_{c_1}\lambda^3 + \mu_{c_1}\lambda_c\lambda^2 + \mu_{c_2}\lambda^3 + \\
& 2\mu_{c_2}\lambda_c\lambda^2 + \lambda\lambda_c^2\mu_{c_2} + \mu_{c_2}\lambda_c\mu\lambda + \mu_{c_3}\lambda^3 + \\
& 3\mu_{c_3}\lambda_c\lambda^2 + 3\mu_{c_3}\lambda_c^2\lambda + 2\lambda_c\mu\mu_{c_3}\lambda + \mu_{c_3}\lambda_c^3 + 2\mu_{c_3}\lambda_c^2\mu + \mu_{c_3}\lambda_c\mu^2) \\
\\
F_{16} \;=\; & \lambda(2\mu_{c_3}\mu_2\lambda_c\lambda + \mu_{c_3}\lambda_c^2\mu_2 + \lambda_c\mu\mu_{c_3}\mu_2 + \mu_c\lambda^3 + \\
& \mu_{c_1}\lambda^3 + \mu_{c_1}\lambda_c\lambda^2 + \mu_{c_2}\lambda^3 + 2\mu_{c_2}\lambda_c\lambda^2 + \\
& \lambda\lambda_c^2\mu_{c_2} + \mu_{c_2}\lambda_c\mu\lambda + \mu_{c_3}\lambda^3 + 3\mu_{c_3}\lambda_c\lambda^2 + \\
& 3\mu_{c_3}\lambda_c^2\lambda + 2\lambda_c\mu\mu_{c_3}\lambda + \mu_{c_3}\lambda_c^3 + 2\mu_{c_3}\lambda_c^2\mu + \mu_{c_3}\lambda_c\mu^2) \\
\\
F_{17} \;=\; & \lambda_c(\mu\lambda_c\mu_3\mu_2 + 2\mu\lambda_c\lambda\mu_2 + 3\mu\lambda_c\lambda\mu_3 + \\
& 3\mu_c\lambda_c\lambda\mu_3 + 2\mu\mu_c\lambda\mu_3 + \mu^3\lambda + \mu^2\lambda^2 + \lambda_c^2\mu_c\mu_3 + \\
& 2\mu\lambda_c\mu_c\mu_2 + \lambda_c\mu_c\mu_3\mu_2 + 2\mu\lambda_c\mu_c\mu_3 + 3\mu_c\lambda_c\lambda\mu_2 + \\
& \mu\mu_c\mu_3\mu_2 + 3\mu^3\lambda_c + 3\mu^2\lambda_c^2 + \mu\lambda_c^2\mu_2 +
\end{aligned}
$$

$$
\begin{aligned}
& 6\mu\lambda_c\lambda\mu_c + \mu\lambda^3 + \mu\lambda_c^3 + 3\mu\lambda_c\lambda^2 + 2\mu_3\mu\lambda\mu_2 + \\
& 4\mu_c\lambda^3 + 2\mu_c\lambda^2\mu_2 + 4\lambda_c^2\lambda\mu_c + 2\mu^2\lambda\mu_3 + \\
& \mu^2\mu_c\mu_2 + 2\mu^2\lambda_c\mu_2 + 3\mu\lambda_c^2\mu_c + \mu^3\mu_2 + \mu^3\mu_3 + \\
& \mu^3\mu_c + \mu\lambda_c^2\mu_3 + 3\mu\mu_c\lambda\mu_2 + \mu^2\mu_c\mu_3 + \\
& 3\mu_c\lambda^2\mu_3 + 2\mu^2\lambda\mu_c + 2\mu_3\mu_c\lambda\mu_2 + 3\mu\lambda^2\mu_3 + \\
& 2\mu^2\lambda_c\mu_3 + \mu^2\mu_3\mu_2 + 3\mu\lambda_c^2\lambda + \lambda_c^2\mu_c\mu_2 + \\
& 3\mu^2\lambda_c\mu_c + 4\mu^2\lambda_c\lambda + 2\mu^2\lambda\mu_2 + 3\mu\lambda^2\mu_c + 6\lambda_c\lambda^2\mu_c + \mu^4 + \lambda_c^3\mu_c)
\end{aligned}
$$

$$
\begin{aligned}
F_{18} \;=\; & (4\lambda_c\mu\lambda^2\mu_{c_2} + 2\mu_{c_2}\mu^2\lambda\mu_c + 6\mu\lambda\lambda_c\mu_c^2 + \mu^2\mu_{c_3}\mu_c\mu_3 + \\
& 2\mu^2\lambda_c^2\mu_3 + \lambda^3\mu\mu_{c_3} + 3\mu^2\lambda_c^2\mu_{c_1} + 4\lambda_c^3\lambda\mu_c + \\
& \mu_{c_2}\mu^3\mu_3 + 3\mu_{c_3}\lambda^2\lambda_c^2 + \mu^3\mu_{c_1}\mu_3 + 3\mu^2\lambda_c^2\mu_{c_2} + \\
& \lambda_c^3\mu_{c_1}\mu_c + \mu^3\lambda\mu_{c_2} + 4\mu^2\lambda_c^2\lambda + \mu_{c_3}\lambda_c^3\mu + \\
& \lambda_c^3\mu_{c_2}\mu_c + \mu^2\lambda^2\mu_{c_2} + 6\mu^2\lambda_c^2\mu_c + \mu^3\lambda\mu_c + \\
& \mu\lambda_c^3\mu_{c_1} + \mu^3\lambda_c\lambda + 3\mu^3\lambda_c\mu_{c_1} + 4\mu^3\lambda_c\mu_c + \\
& \mu^3\lambda\mu_{c_1} + \mu\lambda^3\mu_c + \lambda_c^3\mu_c\mu_2 + \mu_{c_3}\mu^4 + \\
& 2\mu_{c_3}\lambda\mu_c\mu^2 + 3\mu_{c_3}\lambda^2\mu\mu_c + 4\lambda_c\mu^2\lambda\mu_{c_1} + 2\mu_{c_3}\lambda_c\mu^2\mu_3 + \\
& 2\mu_{c_2}\mu^2\lambda\mu_2 + 2\mu^2\lambda\mu_{c_1}\mu_2 + 2\mu_{c_2}\mu^2\lambda\mu_3 + 6\lambda_c\mu^2\lambda\mu_c + \\
& \mu_{c_2}\mu^2\mu_c\mu_3 + 2\mu\lambda\mu_{c_1}\mu_3\mu_2 + 2\mu_{c_2}\lambda_c\mu\mu_c\mu_2 + \lambda^4\mu_{c_1} + \\
& \mu^4\mu_{c_1} + \lambda_c^4\mu_c + \mu^4\mu_c + \lambda^4\mu_c + 4\lambda^3\mu_c^2 + \\
& \lambda_c^3\mu_c^2 + \mu^3\mu_c^2 + \mu^4\mu_{c_2} + \mu^4\lambda_c + \lambda^4\mu_{c_2} + \\
& 3\mu^2\lambda_c^3 + 3\mu^3\lambda_c^2 + \mu\lambda_c^4 + \mu_{c_3}\lambda^4 + \\
& 6\mu_{c_3}\lambda^2\mu_c\lambda_c + 3\mu_{c_2}\mu\lambda_c^2\mu_c + 4\lambda\lambda_c^2\mu_{c_2}\mu_c + 5\mu_{c_3}\lambda^2\mu\lambda_c + \\
& 3\lambda_c\mu\lambda^2\mu_3 + 3\lambda_c\mu_c\lambda^2\mu_3 + 3\mu_{c_2}\mu_c\lambda^2\mu_3 + 2\mu_{c_2}\lambda_c\mu\mu_c\mu_3 + \\
& 3\lambda_c\mu\mu_{c_3}\lambda\mu_3 + 2\mu_{c_2}\lambda_c\mu^2\mu_3 + \mu_{c_3}\lambda_c^2\mu\mu_3 + 3\mu\lambda_c^2\lambda\mu_3 + \\
& 2\mu_{c_3}\mu^2\lambda\mu_3 + 3\mu_c\lambda_c^2\lambda\mu_3 + \lambda_c^2\mu_{c_1}\mu_c\mu_2 + \mu^2\mu_{c_1}\mu_c\mu_2 + \\
& 3\mu\lambda_c^2\mu_c\mu_2 + 2\mu_{c_1}\lambda^2\mu_c\mu_2 + 3\lambda_c\mu^2\mu_c\mu_2 + \mu_{c_3}\lambda_c^2\mu_c\mu_2 + \\
& \mu^2\mu_{c_1}\mu_3\mu_2 + \lambda_c^2\mu_c\mu_3\mu_2 + \mu\lambda_c^2\mu_3\mu_2 + 4\lambda\lambda_c^2\mu_{c_1}\mu_c + \\
& 6\mu_{c_1}\lambda_c\lambda^2\mu_c + 6\mu_{c_2}\lambda_c\mu\lambda\mu_c + 3\mu\lambda_c^2\mu_{c_1}\mu_c + 3\mu\lambda_c^2\lambda\mu_{c_2} + \\
& 3\mu_{c_2}\lambda_c\mu^2\mu_c + 3\mu\mu_{c_1}\lambda^2\mu_c + 3\mu\mu_{c_3}\lambda^2\mu_3 + \mu^3\mu_{c_3}\mu_3 + \\
& 6\mu_{c_2}\lambda_c\lambda^2\mu_c + 6\lambda_c\mu\lambda^2\mu_c + 2\mu_{c_3}\lambda_c\mu_c\mu\mu_2 + \mu_{c_2}\mu\lambda_c^2\mu_2 +
\end{aligned}
$$

$$\mu_{c_2}\mu_c\lambda_c^2\mu_2 + 2\lambda_c\mu_c\lambda^2\mu_2 + 2\mu_{c_2}\mu_c\lambda^2\mu_2 + 3\lambda_c\mu\mu_{c_3}\lambda\mu_2 +$$
$$\mu_{c_2}\mu^2\mu_c\mu_2 + \mu_{c_2}\mu_3\mu^2\mu_2 + 2\mu_{c_2}\lambda_c\mu^2\mu_2 + \mu_{c_3}\lambda_c^2\mu\mu_2 +$$
$$3\mu_{c_3}\lambda_c\mu_c\mu^2 + 2\mu\lambda_c^2\lambda\mu_2 + 2\mu_{c_3}\mu^2\lambda\mu_2 + 3\mu_c\lambda_c^2\lambda\mu_2 +$$
$$\mu_{c_3}\lambda_c^2\lambda\mu_2 + 2\mu\lambda_c\mu_{c_1}\mu_c\mu_3 + 2\mu\lambda\mu_{c_1}\mu_c\mu_3 + 3\mu_{c_1}\lambda\lambda_c\mu_c\mu_3 +$$
$$3\mu\lambda\lambda_c\mu_{c_1}\mu_3 + 3\lambda_c\mu^2\mu_{c_1}\mu_c + 3\mu\lambda_c^2\lambda\mu_{c_1} + 5\mu_{c_3}\lambda\lambda_c\mu^2 +$$
$$6\mu_{c_3}\lambda\mu\mu_c\lambda_c + 4\lambda_c\mu^2\lambda\mu_{c_2} + 3\mu_{c_2}\mu\lambda^2\mu_c + \mu^3\lambda_c\mu_3 +$$
$$\mu^3\mu_{c_1}\mu_c + \mu_{c_3}\lambda\lambda_c^3 + 6\lambda^2\lambda_c\mu_c^2 + 3\mu\lambda_c^2\lambda^2 +$$
$$\mu\lambda_c^3\mu_3 + \mu^2\lambda^2\mu_{c_1} + \lambda_c^2\lambda^2\mu_{c_2} + \mu\lambda^3\mu_{c_2} +$$
$$2\lambda_c\lambda^3\mu_{c_2} + \mu^3\mu_{c_1}\mu_2 + \lambda_c\lambda^3\mu_{c_1} + \mu\lambda^3\mu_{c_1} +$$
$$\lambda_c^2\mu_c^2\mu_2 + 4\lambda_c\lambda^3\mu_c + \lambda_c\mu^2\lambda^2 + \mu^3\mu_c\mu_3 +$$
$$3\mu_{c_3}\lambda_c^2\mu^2 + \mu^2\mu_c^2\mu_3 + 3\mu_{c_3}\mu^3\lambda_c + 3\lambda_c\mu^2\mu_c^2 +$$
$$3\mu\lambda_c^2\mu_c^2 + 2\mu^2\lambda\mu_c^2 + \mu\lambda_c^2\mu_{c_1}\mu_3 + 2\lambda_c\mu^2\mu_{c_1}\mu_3 +$$
$$2\mu\lambda_c\mu_c^2\mu_3 + \lambda_c^2\mu_{c_1}\mu_c\mu_3 + \mu^2\mu_{c_1}\mu_c\mu_3 + 3\mu\lambda_c^2\mu_c\mu_3 +$$
$$3\mu_{c_1}\lambda^2\mu_c\mu_3 + 3\lambda_c\mu^2\mu_c\mu_3 + \mu_{c_3}\lambda_c^2\mu_c\mu_3 + 3\mu\mu_{c_1}\lambda^2\mu_3 +$$
$$3\mu_{c_2}\lambda_c\mu\lambda\mu_3 + 3\mu_{c_3}\lambda_c\mu_c\lambda\mu_3 + 5\lambda_c\mu\mu_c\lambda\mu_3 + 2\mu_{c_2}\mu\mu_c\lambda\mu_3 +$$
$$2\mu\mu_{c_3}\mu_c\lambda\mu_3 + 2\lambda_c\mu^2\lambda\mu_3 + 3\mu_{c_3}\mu_c\lambda^2\mu_3 + 3\mu\mu_c\lambda^2\mu_3 +$$
$$3\lambda_c\mu_c^2\lambda\mu_3 + 2\mu^2\mu_c\lambda\mu_3 + 2\mu\mu_c^2\lambda\mu_3 + 2\mu_{c_3}\lambda_c\mu_c\mu\mu_3 +$$
$$\mu_{c_2}\mu\lambda_c^2\mu_3 + \mu_{c_3}\lambda^2\mu^2 + 2\mu_{c_2}\mu_c\mu_3\lambda\mu_2 + 2\mu_{c_3}\mu_3\mu\lambda\mu_2 +$$
$$2\mu_{c_3}\mu_3\mu_c\lambda\mu_2 + 2\mu\mu_c\mu_3\lambda\mu_2 + 3\mu_{c_2}\mu\mu_c\lambda\mu_2 + 3\mu\mu_{c_3}\mu_c\lambda\mu_2 +$$
$$2\mu_3\mu\lambda_c\lambda\mu_2 + 2\mu_3\mu_c\lambda_c\lambda\mu_2 + 2\lambda_c\mu^2\lambda\mu_2 + \mu_{c_2}\mu\mu_c\mu_3\mu_2 +$$
$$\mu_{c_2}\mu\mu_3\lambda_c\mu_2 + \mu_{c_2}\mu_c\mu_3\lambda_c\mu_2 + 2\mu_{c_3}\mu_c\lambda^2\mu_2 + 3\lambda_c\mu_c^2\lambda\mu_2 +$$
$$2\mu_3\mu_c^2\lambda\mu_2 + 2\mu^2\mu_c\lambda\mu_2 + 3\mu_{c_3}\lambda_c^2\mu\mu_c + 6\mu\lambda\lambda_c\mu_{c_1}\mu_c +$$
$$5\mu_{c_3}\lambda\mu\lambda_c^2 + 4\mu_{c_3}\lambda\mu_c\lambda_c^2 + 2\mu_{c_2}\mu\mu_3\lambda\mu_2 + 3\mu\mu_c^2\lambda\mu_2 +$$
$$2\mu^2\lambda\mu_{c_1}\mu_3 + \mu_{c_2}\mu_c\lambda_c^2\mu_3 + \mu^2\mu_c\mu_3\mu_2 + \mu_{c_3}\mu^2\mu_3\mu_2 +$$
$$\lambda_c\mu^2\mu_3\mu_2 + \lambda_c\mu_c^2\mu_3\mu_2 + \mu\mu_c^2\mu_3\mu_2 + 2\mu_{c_2}\lambda_c\mu\lambda\mu_2 +$$
$$3\mu_{c_2}\lambda_c\mu_c\lambda\mu_2 + 3\mu_{c_3}\lambda_c\mu_c\lambda\mu_2 + 5\lambda_c\mu\mu_c\lambda\mu_2 + 2\mu_{c_3}\lambda_c\lambda^2\mu_2 +$$
$$2\mu_{c_3}\lambda_c\mu^2\mu_2 + \mu^2\mu_{c_3}\mu_c\mu_2 + \mu\mu_{c_1}\mu_c\mu_3\mu_2 + 2\lambda\mu_{c_1}\mu_c\mu_3\mu_2 +$$
$$\lambda_c\mu\mu_{c_3}\mu_3\mu_2 + 2\mu\lambda_c\mu_{c_1}\mu_c\mu_2 + 3\mu\lambda\mu_{c_1}\mu_c\mu_2 + 3\mu_{c_1}\lambda\lambda_c\mu_c\mu_2 +$$
$$2\mu\lambda_c\mu_c\mu_3\mu_2 + \mu_{c_3}\lambda_c\mu_c\mu_3\mu_2 + \mu\lambda_c\mu_{c_1}\mu_3\mu_2 + \lambda_c\mu_{c_1}\mu_c\mu_3\mu_2 +$$

$$
\begin{aligned}
& 2\mu\lambda\lambda_c\mu_{c_1}\mu_2 + \mu\lambda_c^2\mu_{c_1}\mu_2 + 2\lambda_c\mu^2\mu_{c_1}\mu_2 + 2\mu\lambda_c\mu_c^2\mu_2 + \\
& \mu^3\lambda_c\mu_2 + 4\mu_{c_2}\lambda^3\mu_c + \mu\lambda_c^3\mu_{c_2} + 4\lambda\lambda_c^2\mu_c^2 + \\
& \mu^2\mu_c^2\mu_2 + 2\mu_c^2\lambda^2\mu_2 + \lambda_c^3\mu_c\mu_3 + 3\mu^3\lambda_c\mu_{c_2} + \\
& \mu_{c_2}\mu^3\mu_c + \mu_{c_3}\lambda_c^3\mu_c + \mu_{c_2}\mu^3\mu_2 + 4\mu\lambda_c^3\mu_c + \\
& 3\mu\lambda^2\mu_c^2 + \lambda_c^2\mu_c^2\mu_3 + 4\mu_{c_1}\lambda^3\mu_c + 6\lambda_c^2\lambda^2\mu_c + \\
& \mu^3\mu_{c_3}\mu_2 + 2\mu^2\lambda_c^2\mu_2 + \mu_{c_3}\lambda\mu^3 + \mu\lambda_c^3\mu_2 + \\
& 4\mu_{c_3}\lambda^3\mu_c + \mu\lambda_c\lambda^3 + 3\mu_{c_3}\lambda^3\lambda_c + \mu^2\lambda^2\mu_c + \\
& 3\mu_c^2\lambda^2\mu_3 + 3\mu\lambda_c^3\lambda + \mu_{c_3}\mu^3\mu_c + 3\mu_{c_2}\lambda_c\mu_c\lambda\mu_3 + \\
& \mu\mu_{c_3}\mu_c\mu_3\mu_2 + \mu^3\mu_c\mu_2 + 9\mu\lambda_c^2\lambda\mu_c + 2\mu^2\lambda\mu_{c_1}\mu_c + \\
& 3\lambda_c\mu\lambda^2\mu_{c_1} + 3\mu_{c_2}\mu\lambda^2\mu_3)
\end{aligned}
$$

$$
\begin{aligned}
F_{21} &= (3\lambda_c + 2\mu + \mu_2 + 3\lambda) \\
F_{22} &= (6\lambda\lambda_c + 2\lambda_c\mu_2 + 2\lambda\mu + \mu\mu_2 + 3\lambda_c + 2\lambda\mu_2 + \mu^2 + 4\lambda_c\mu + 3\lambda^2) \\
F_{23} &= (\lambda_c^3 + 2\lambda\mu_2\lambda_c + \mu\mu_2\lambda_c + 2\mu\lambda\lambda_c + 3\lambda^2\lambda_c + \mu_2\lambda_c^2 + \lambda^3 + \\
&\quad \mu^2\lambda_c + 2\mu\lambda_c^2 + 3\lambda\lambda_c^2) \\
F_{24} &= (2\lambda_c + 2\mu + \mu_2 + 2\lambda) \\
F_{25} &= (\lambda\mu_2 + \lambda_c\mu_2 + \lambda\mu + \mu^2 + 2\lambda_c\mu + \lambda_c^2 + 2\lambda\lambda_c + \lambda^2 + \mu\mu_2) \\
F_{26} &= (2\lambda_c^2 + 2\lambda_c\mu + \lambda_c\mu_2 + 3\lambda\lambda_c) \\
F_{27} &= (2\lambda\mu_2\lambda_c + \mu_2\lambda_c^2 + 2\mu\lambda\lambda_c + \mu^2\lambda_c + 2\mu\lambda_c^2 + \lambda_c^3 + 3\lambda\lambda_c^2 + \\
&\quad 3\lambda^2\lambda_c + \mu\mu_2\lambda_c)
\end{aligned}
$$

$$
\begin{aligned}
F_{28} &= (\mu_2 + \mu_3 + 3\mu + 3\lambda + 3\lambda_c) \\
F_{29} &= (\mu_2\mu_3 + 3\mu^2 + 6\lambda_c\mu + 6\lambda\lambda_c + 3\lambda_c^2 + 2\mu_2\mu + 2\mu_2\lambda + 2\lambda_c\mu_2 + 2\lambda_c\mu_3 + \\
&\quad 4\lambda\mu + 2\mu_3\mu + 3\lambda^2 + 2\lambda\mu_3) \\
F_{30} &= (\lambda\mu^2 + \mu\lambda^2 + \mu_2\mu^2 + \mu_3\mu^2 + \lambda_c^2\mu_3 + 3\lambda^2\lambda_c + 3\lambda_c\mu^2 + \\
&\quad \lambda_c^2\mu_2 + 3\lambda_c^2\mu + 3\lambda\lambda_c^2 + \lambda^2\mu_3 + \mu_2\lambda^2 + \lambda\mu_2\mu_3 + 2\mu\mu_2\lambda + 2\mu\lambda_c\mu_3 + \\
&\quad \mu\lambda\mu_3 + \mu\mu_2\mu_3 + 2\lambda_c\mu\mu_2 + 2\lambda\lambda_c\mu_2 + 2\lambda\lambda_c\mu_3 + \\
&\quad 4\lambda_c\mu\lambda + \mu^3 + \lambda_c^3 + \lambda^3 + \lambda_c\mu_2\mu_3) \\
F_{31} &= (\mu_2 + \mu_3 + 3\mu + 4\lambda_c + 4\lambda) \\
F_{32} &= (\mu_2\mu_3 + 2\mu_2\mu + 6\lambda^2 + 3\mu_2\lambda + 3\lambda_c\mu_2 + 3\lambda_c\mu_3 + 6\lambda_c^2 + 2\mu_3\mu + 3\mu^2 + 3\lambda\mu_3 +
\end{aligned}
$$

$$6\lambda\mu + 12\lambda\lambda_c + 9\lambda_c\mu)$$

$$\begin{aligned}
F_{33} &= (2\lambda\mu^2 + 3\mu\lambda^2 + \mu_2\mu^2 + \mu_3\mu^2 + 3\lambda_c^2\mu_3 + 12\lambda^2\lambda_c + 6\lambda_c\mu^2 + \\
&\quad 3\lambda_c^2\mu_2 + 9\lambda_c^2\mu + 12\lambda\lambda_c^2 + 3\lambda^2\mu_3 + 2\mu_2\lambda^2 + 2\lambda\mu_2\mu_3 + 3\mu\mu_2\lambda + 4\mu\lambda_c\mu_3 + \\
&\quad 2\mu\lambda\mu_3 + \mu\mu_2\mu_3 + 4\lambda_c\mu\mu_2 + 6\lambda\lambda_c\mu_2 + 6\lambda\lambda_c\mu_3 + \\
&\quad 12\lambda_c\mu\lambda + \mu^3 + 4\lambda_c^3 + 4\lambda^3 + 2\lambda_c\mu_2\mu_3) \\
F_{34} &= (2\lambda_c\mu\mu_3\lambda + 3\mu\lambda^2\lambda_c + 2\lambda^2\lambda_c\mu_2 + 2\lambda_c^2\mu\mu_2 + \\
&\quad \lambda_c\mu^2\mu_2 + 2\lambda_c\mu^2\lambda + 2\lambda_c^2\mu\mu_3 + 3\lambda\lambda_c^2\mu_3 + \\
&\quad 3\lambda_c\mu\mu_2\lambda + \lambda_c\mu\mu_2\mu_3 + 2\lambda\lambda_c\mu_2\mu_3 + 6\lambda^2\lambda_c^2 + \\
&\quad 4\lambda\lambda_c^3 + 3\lambda_c^3\mu + 3\lambda_c^2\mu^2 + \lambda_c^3\mu_2 + \lambda_c\mu^3 + \\
&\quad 4\lambda^3\lambda_c + \lambda_c^3\mu_3 + 3\lambda\lambda_c^2\mu_2 + 6\lambda_c^2\mu\lambda + \\
&\quad 3\lambda^2\lambda_c\mu_3 + \lambda^4 + \lambda_c^4 + \lambda_c^2\mu_2\mu_3 + \lambda_c\mu^2\mu_3) \\
F_{35} &= (2\lambda\mu + 2\lambda^2 + 2\lambda\lambda_c + \lambda\mu_3 + \mu_2\lambda) \\
F_{36} &= (\lambda\mu_2\mu_3 + \mu\lambda\mu_3 + \lambda^2\mu_3 + \lambda\lambda_c\mu_3 + \mu\mu_2\lambda + \mu_2\lambda^2 + \lambda\lambda_c\mu_2 + \lambda^3 + \\
&\quad 2\lambda^2\lambda_c + \mu\lambda^2 + \lambda\lambda_c^2 + 2\lambda_c\mu\lambda + \lambda\mu^2) \\
F_{37} &= (\lambda_c\mu_3 + 3\lambda_c^2 + 4\lambda\lambda_c + \lambda_c\mu_2 + 3\lambda_c\mu) \\
F_{38} &= (3\lambda\lambda_c\mu_2 + 3\lambda_c^3 + 3\lambda\lambda_c\mu_3 + 2\mu\lambda_c\mu_3 + 2\lambda_c\mu\mu_2 + \\
&\quad 3\lambda_c\mu^2 + 2\lambda_c^2\mu_2 + 6\lambda_c\mu\lambda + 2\lambda_c^2\mu_3 + 8\lambda\lambda_c^2 + \lambda_c\mu_2\mu_3 + 6\lambda_c^2\mu + 6\lambda^2\lambda_c) \\
F_{39} &= (2\lambda_c\mu\mu_3\lambda + 3\mu\lambda^2\lambda_c + 2\lambda^2\lambda_c\mu_2 + 2\lambda_c^2\mu\mu_2 + \\
&\quad \lambda_c\mu^2\mu_2 + 2\lambda_c\mu^2\lambda + 2\lambda_c^2\mu\mu_3 + 3\lambda\lambda_c^2\mu_3 + \\
&\quad 3\lambda_c\mu\mu_2\lambda + \lambda_c\mu\mu_2\mu_3 + 2\lambda\lambda_c\mu_2\mu_3 + 6\lambda^2\lambda_c^2 + \\
&\quad 4\lambda\lambda_c^3 + 3\lambda_c^3\mu + 3\lambda_c^2\mu^2 + \lambda_c^3\mu_2 + \lambda_c\mu^3 + \\
&\quad 4\lambda^3\lambda_c + \lambda_c^3\mu_3 + 3\lambda\lambda_c^2\mu_2 + 6\lambda_c^2\mu\lambda + \\
&\quad 3\lambda^2\lambda_c\mu_3 + \lambda_c^4 + \lambda_c^2\mu_2\mu_3 + \lambda_c\mu^2\mu_3)
\end{aligned}$$

$$\begin{aligned}
F_{40} &= (\mu_{c_1}\lambda_{c_2}\mu_2 + \mu_{c_1}\mu_1\mu_{20} + \mu_{c_1}\mu_1\mu_2 + \mu_{c_2}\lambda_2\mu_{20} + \mu_{c_1}\lambda_2\mu_{20} + \mu_{c_2}\mu_1\mu_2 + \\
&\quad \mu_{c_1}\lambda_{c_2}\mu_{20} + \mu_{c_2}\mu_1\mu_{20})
\end{aligned}$$

$$F_{41} = (\mu_{20} + \mu_2)(\lambda_{c_1}\mu_{c_2} + \lambda_1\mu_{c_1} + \lambda_1\mu_{c_2})$$

$$F_{42} = (\lambda_{c_1}\mu_1\mu_{20} + \lambda_{c_1}\mu_1\mu_2 + \mu_{c_1}\mu_1\mu_2 + \mu_{c_2}\lambda_2\mu_{20} + \mu_{c_1}\lambda_2\mu_{20} + \mu_{c_1}\mu_1\mu_{20} +$$

$$\lambda_1\lambda_{c_2}\mu_{20} + \lambda_1\lambda_{c_2}\mu_2 + \mu_{c_2}\mu_1\mu_{20} + \mu_{c_2}\mu_1\mu_2 + \lambda_{c_1}\lambda_2\mu_{20} + \lambda_1\mu_{c_2}\mu_2 +$$
$$\lambda_{c_1}\lambda_2\mu_{c_2} + \lambda_{c_1}\lambda_{c_2}\mu_{20} + \lambda_{c_1}\lambda_{c_2}\mu_2 + \lambda_{c_1}\mu_{c_2}\mu_{20} + \lambda_{c_1}\mu_{c_2}\mu_2 + \mu_{c_1}\lambda_{c_2}\mu_{20} +$$
$$\mu_{c_1}\lambda_{c_2}\mu_2 + \lambda_1\lambda_2\mu_{c_2} + \lambda_1\lambda_2\mu_{c_1} + \lambda_1\mu_{c_1}\mu_{20} + \lambda_1\mu_{c_1}\mu_2 + \lambda_1\mu_{c_2}\mu_{20})$$

$$\begin{aligned}
F_{43} \;=\; & (\mu_{c_3}\mu_2\mu_{30}\mu_1 + \mu_{c_3}\mu_3\lambda_2\mu_{20} + \lambda_{c_2}\mu_{c_1}\mu_{30}\mu_{20} + \mu_{c_3}\lambda_{c_2}\mu_{30}\mu_{20} + \\
& \mu_1\mu_{c_2}\mu_2\mu_{30} + \mu_3\lambda_{c_3}\lambda_{c_2}\mu_{c_1} + \mu_3\lambda_{c_3}\mu_{c_1}\mu_1 + \mu_2\mu_3\mu_{c_1}\mu_1 + \\
& \lambda_3\mu_{c_3}\mu_{30}\mu_1 + \mu_{c_3}\mu_3\lambda_{c_2}\mu_{20} + \mu_{c_3}\lambda_2\mu_{30}\mu_{20} + \mu_{c_1}\mu_{30}\mu_1\mu_{20} + \\
& \mu_1\mu_{c_2}\mu_2\mu_3 + \mu_1\mu_{c_2}\lambda_3\mu_{30} + \mu_1\mu_{c_2}\mu_{20}\mu_{30} + \mu_1\mu_{c_2}\lambda_{c_3}\mu_3 + \\
& \mu_2\mu_3\lambda_{c_2}\mu_{c_1} + \mu_3\lambda_{c_3}\lambda_2\mu_{c_1} + \lambda_3\mu_{c_1}\mu_{30}\mu_1 + \mu_3\lambda_{c_2}\mu_{c_1}\mu_{20} + \\
& \lambda_2\mu_{c_1}\mu_{30}\mu_{20} + \mu_1\mu_{c_2}\mu_{20}\mu_3 + \mu_1\mu_{c_2}\lambda_{c_3}\mu_{30} + \mu_{c_3}\lambda_3\lambda_{c_2}\mu_{30} + \\
& \mu_{c_1}\lambda_3\lambda_{c_2}\mu_{30} + \lambda_{c_3}\lambda_2\mu_{c_1}\mu_{30} + \mu_{c_3}\mu_2\mu_3\mu_1 + \lambda_{c_3}\mu_{c_1}\mu_{30}\mu_1 + \\
& \mu_3\lambda_2\mu_{c_1}\mu_{20} + \mu_{c_3}\mu_3\mu_1\mu_{20} + \mu_3\mu_{c_1}\mu_1\mu_{20} + \mu_{c_3}\mu_{30}\mu_1\mu_{20} + \\
& \lambda_2\mu_{c_2}\lambda_3\mu_{30} + \mu_2\mu_{c_1}\mu_{30}\mu_1 + \mu_{c_3}\lambda_3\lambda_2\mu_{30} + \lambda_{c_3}\lambda_{c_2}\mu_{c_1}\mu_{30} + \\
& \mu_2\lambda_{c_2}\mu_{c_1}\mu_{30} + \mu_{c_1}\lambda_3\lambda_2\mu_{30} + \lambda_2\mu_{c_2}\mu_{20}\mu_3 + \lambda_2\mu_{c_2}\mu_{20}\mu_{30})
\end{aligned}$$

$$\begin{aligned}
F_{44} \;=\; & (\mu_{c_2}\lambda_1\lambda_{c_3}\mu_{30} + \mu_{c_2}\lambda_1\mu_2\mu_{30} + \mu_{c_2}\lambda_1\lambda_3\mu_{30} + \mu_{c_2}\lambda_1\lambda_{c_3}\mu_3 + \\
& \mu_{c_2}\lambda_1\mu_{20}\mu_{30} + \mu_{c_2}\lambda_1\mu_2\mu_3 + \mu_{c_3}\lambda_1\mu_2\mu_3 + \mu_{c_3}\lambda_1\mu_2\mu_{30} + \\
& \lambda_1\mu_{c_3}\mu_3\mu_{20} + \mu_{c_1}\lambda_1\lambda_{c_3}\mu_{30} + \mu_{c_1}\lambda_1\lambda_{c_3}\mu_3 + \mu_{c_1}\lambda_1\mu_2\mu_3 + \\
& \mu_{c_1}\lambda_1\mu_2\mu_{30} + \mu_{c_1}\lambda_1\lambda_3\mu_{30} + \mu_{c_3}\lambda_1\lambda_3\mu_{30} + \mu_{c_3}\mu_2\mu_3\lambda_{c_1} + \\
& \mu_{c_2}\mu_{20}\mu_3\lambda_{c_1} + \mu_{c_2}\mu_2\mu_{30}\lambda_{c_1} + \mu_{c_2}\mu_2\mu_3\lambda_{c_1} + \mu_{c_1}\lambda_1\mu_{30}\mu_{20} + \\
& \lambda_1\mu_{c_3}\mu_{30}\mu_{20} + \mu_{c_1}\lambda_1\mu_3\mu_{20} + \mu_{c_2}\mu_{20}\mu_{30}\lambda_{c_1} + \mu_{c_2}\lambda_{c_3}\mu_{30}\lambda_{c_1} + \\
& \mu_{c_2}\lambda_3\mu_{30}\lambda_{c_1} + \mu_{c_2}\lambda_{c_3}\mu_3\lambda_{c_1} + \mu_{c_2}\lambda_1\mu_{20}\mu_3 + \mu_{c_3}\mu_2\mu_{30}\lambda_{c_1})
\end{aligned}$$

$$\begin{aligned}
F_{45} \;=\; & (\mu_{30} + \mu_3)(\mu_{c_3}\mu_1\lambda_{c_1} + \mu_{c_3}\lambda_2\lambda_{c_1} + \mu_{c_2}\lambda_2\lambda_{c_1} + \\
& \mu_{c_3}\lambda_{c_2}\lambda_{c_1} + \lambda_2\mu_{c_1}\lambda_1 + \lambda_2\lambda_1\mu_{c_3} + \lambda_2\lambda_1\mu_{c_2} + \lambda_{c_2}\lambda_1\mu_{c_3})
\end{aligned}$$

$$\begin{aligned}
F_{46} \;=\; & \lambda_3(\mu_{c_3}\mu_1\lambda_{c_1} + \mu_{c_3}\lambda_2\lambda_{c_1} + \mu_{c_2}\lambda_2\lambda_{c_1} + \mu_{c_3}\lambda_{c_2}\lambda_{c_1} + \\
& \lambda_2\mu_{c_1}\lambda_1 + \lambda_2\lambda_1\mu_{c_3} + \lambda_2\lambda_1\mu_{c_2} + \lambda_{c_2}\lambda_1\mu_{c_3})
\end{aligned}$$

$$\begin{aligned}
F_{47} \;=\; & (\lambda_{c_2}\lambda_1\mu_2\mu_{30} + \lambda_{c_2}\lambda_1\lambda_3\mu_{30} + \lambda_2\lambda_1\lambda_{c_3}\mu_3 + \lambda_2\lambda_1\lambda_{c_3}\mu_{30} + \\
& \lambda_{c_2}\lambda_1\mu_2\mu_3 + \lambda_{c_2}\lambda_1\mu_{20}\mu_3 + \lambda_{c_2}\lambda_1\lambda_{c_3}\mu_{30} + \lambda_{c_2}\lambda_1\lambda_{c_3}\mu_3 +
\end{aligned}$$

$$
\begin{aligned}
& \lambda_{c_2}\lambda_1\mu_{20}\mu_{30} + \lambda_{c_3}\mu_{30}\mu_1\lambda_{c_1} + \mu_2\mu_3\mu_1\lambda_{c_1} + \mu_3\lambda_{c_3}\mu_1\lambda_{c_1} + \\
& \mu_3\lambda_2\mu_{20}\lambda_{c_1} + \mu_3\lambda_{c_2}\mu_{20}\lambda_{c_1} + \mu_3\mu_1\mu_{20}\lambda_{c_1} + \mu_3\lambda_{c_3}\lambda_2\lambda_{c_1} + \\
& \mu_{30}\mu_1\mu_{20}\lambda_{c_1} + \mu_2\mu_3\lambda_{c_2}\lambda_{c_1} + \lambda_{c_2}\mu_{30}\mu_{20}\lambda_{c_1} + \lambda_3\lambda_2\mu_{30}\lambda_{c_1} + \\
& \mu_3\lambda_{c_3}\lambda_{c_2}\lambda_{c_1} + \lambda_3\mu_{30}\mu_1\lambda_{c_1} + \lambda_2\mu_{30}\mu_{20}\lambda_{c_1} + \lambda_{c_3}\lambda_2\mu_{30}\lambda_{c_1} + \\
& \mu_2\mu_{30}\mu_1\lambda_{c_1} + \mu_2\lambda_{c_2}\mu_{30}\lambda_{c_1} + \lambda_3\lambda_{c_2}\mu_{30}\lambda_{c_1} + \lambda_{c_3}\lambda_{c_2}\mu_{30}\lambda_{c_1})
\end{aligned}
$$

$$
\begin{aligned}
F_{48} = {} & (\lambda_2\lambda_1\lambda_3\mu_{c_3} + \lambda_{c_2}\lambda_1\lambda_3\mu_{c_3} + \lambda_{c_2}\lambda_1\mu_2\mu_{30} + \lambda_{c_2}\lambda_1\lambda_3\mu_{30} + \\
& \lambda_2\lambda_1\lambda_{c_3}\mu_3 + \lambda_2\lambda_1\lambda_{c_3}\mu_{30} + \lambda_2\lambda_1\mu_{c_3}\mu_3 + \mu_{c_2}\lambda_1\lambda_{c_3}\mu_{30} + \\
& \mu_{c_2}\lambda_1\mu_2\mu_{30} + \mu_{c_2}\lambda_1\lambda_3\mu_{30} + \mu_{c_2}\lambda_1\lambda_{c_3}\mu_3 + \mu_{c_2}\lambda_1\mu_{20}\mu_{30} + \\
& \mu_{c_2}\lambda_1\mu_2\mu_3 + \lambda_{c_2}\lambda_1\mu_2\mu_3 + \lambda_{c_2}\lambda_1\mu_{20}\mu_3 + \lambda_{c_2}\lambda_1\lambda_{c_3}\mu_{30} + \\
& \lambda_{c_2}\lambda_1\mu_{c_3}\mu_3 + \lambda_{c_2}\lambda_1\mu_{c_3}\mu_{30} + \lambda_{c_2}\lambda_1\lambda_{c_3}\mu_3 + \lambda_{c_2}\lambda_1\mu_{20}\mu_{30} + \\
& \mu_{c_3}\lambda_1\mu_2\mu_3 + \mu_{c_3}\lambda_1\mu_2\mu_{30} + \mu_{c_3}\mu_2\mu_{30}\mu_1 + \mu_{c_3}\mu_3\lambda_2\mu_{20} + \\
& \lambda_{c_2}\mu_{c_1}\mu_{30}\mu_{20} + \mu_{c_3}\lambda_{c_2}\mu_{30}\mu_{20} + \mu_1\mu_{c_2}\mu_2\mu_{30} + \lambda_1\mu_{c_3}\mu_3\mu_{20} + \\
& \lambda_2\mu_{c_1}\lambda_1\lambda_3 + \mu_{c_1}\lambda_1\lambda_{c_3}\mu_{30} + \mu_{c_1}\lambda_1\lambda_{c_3}\mu_3 + \mu_{c_1}\lambda_1\mu_2\mu_3 + \\
& \mu_{c_1}\lambda_1\mu_2\mu_{30} + \mu_{c_1}\lambda_1\lambda_3\mu_{30} + \mu_{c_3}\lambda_1\lambda_3\mu_{30} + \lambda_2\mu_{c_1}\lambda_1\mu_3 + \\
& \lambda_2\mu_{c_1}\lambda_1\mu_{30} + \lambda_{c_3}\mu_{30}\mu_1\lambda_{c_1} + \mu_{c_3}\mu_{30}\mu_1\lambda_{c_1} + \mu_2\mu_3\mu_1\lambda_{c_1} + \\
& \mu_3\lambda_{c_3}\mu_1\lambda_{c_1} + \mu_3\lambda_2\mu_{20}\lambda_{c_1} + \mu_3\lambda_{c_2}\mu_{20}\lambda_{c_1} + \mu_{c_3}\lambda_3\lambda_2\lambda_{c_1} + \\
& \mu_3\mu_1\mu_{20}\lambda_{c_1} + \mu_{c_3}\mu_2\mu_3\lambda_{c_1} + \mu_{c_2}\mu_{20}\mu_3\lambda_{c_1} + \mu_{c_2}\mu_2\mu_{30}\lambda_{c_1} + \\
& \mu_{c_2}\mu_2\mu_3\lambda_{c_1} + \mu_{c_3}\lambda_3\lambda_{c_2}\lambda_{c_1} + \mu_3\lambda_{c_3}\lambda_2\lambda_{c_1} + \mu_{c_1}\lambda_1\mu_{30}\mu_{20} + \\
& \lambda_1\mu_{c_3}\mu_{30}\mu_{20} + \mu_{c_1}\lambda_1\mu_3\mu_{20} + \lambda_2\lambda_1\lambda_3\mu_{c_2} + \mu_{30}\mu_1\mu_{20}\lambda_{c_1} + \\
& \mu_2\mu_3\lambda_{c_2}\lambda_{c_1} + \lambda_2\mu_{c_2}\mu_3\lambda_{c_1} + \mu_{c_3}\mu_3\mu_1\lambda_{c_1} + \lambda_2\mu_{c_2}\mu_{30}\lambda_{c_1} + \\
& \lambda_{c_2}\mu_{30}\mu_{20}\lambda_{c_1} + \lambda_3\lambda_2\mu_{30}\lambda_{c_1} + \mu_{c_2}\mu_{20}\mu_{30}\lambda_{c_1} + \mu_{c_2}\lambda_3\lambda_2\lambda_{c_1} + \\
& \mu_3\lambda_{c_3}\lambda_{c_2}\lambda_{c_1} + \mu_{c_3}\mu_3\lambda_{c_2}\lambda_{c_1} + \mu_{c_3}\mu_3\lambda_2\lambda_{c_1} + \mu_{c_2}\lambda_{c_3}\mu_{30}\lambda_{c_1} + \\
& \mu_{c_2}\lambda_3\mu_{30}\lambda_{c_1} + \lambda_3\mu_{30}\mu_1\lambda_{c_1} + \lambda_2\mu_{30}\mu_{20}\lambda_{c_1} + \lambda_{c_3}\lambda_2\mu_{30}\lambda_{c_1} + \\
& \mu_2\mu_{30}\mu_1\lambda_{c_1} + \mu_{c_2}\lambda_{c_3}\mu_3\lambda_{c_1} + \mu_{c_3}\lambda_3\mu_1\lambda_{c_1} + \mu_3\lambda_{c_3}\lambda_{c_2}\mu_{c_1} + \\
& \mu_3\lambda_{c_3}\mu_{c_1}\mu_1 + \mu_2\mu_3\mu_{c_1}\mu_1 + \lambda_3\mu_{c_3}\mu_{30}\mu_1 + \mu_{c_3}\mu_3\lambda_{c_2}\mu_{20} + \\
& \mu_{c_3}\lambda_2\mu_{30}\mu_{20} + \mu_{c_1}\mu_{30}\mu_1\mu_{20} + \mu_1\mu_{c_2}\mu_2\mu_3 + \mu_1\mu_{c_2}\lambda_3\mu_{30} + \\
& \mu_1\mu_{c_2}\mu_{20}\mu_{30} + \mu_1\mu_{c_2}\lambda_{c_3}\mu_3 + \lambda_2\lambda_1\mu_{c_3}\mu_{30} + \lambda_2\lambda_1\mu_{c_2}\mu_3 + \\
& \lambda_2\lambda_1\mu_{c_2}\mu_{30} + \mu_{c_2}\lambda_1\mu_{20}\mu_3 + \mu_{c_3}\mu_2\mu_{30}\lambda_{c_1} + \mu_{c_3}\lambda_2\mu_{30}\lambda_{c_1} + \\
& \mu_2\lambda_{c_2}\mu_{30}\lambda_{c_1} + \lambda_3\lambda_{c_2}\mu_{30}\lambda_{c_1} + \lambda_{c_3}\lambda_{c_2}\mu_{30}\lambda_{c_1} + \mu_{c_3}\lambda_{c_2}\mu_{30}\lambda_{c_1} +
\end{aligned}
$$

$$
\begin{aligned}
& \mu_2\mu_3\lambda_{c_2}\mu_{c_1} + \mu_3\lambda_{c_3}\lambda_2\mu_{c_1} + \lambda_3\mu_{c_1}\mu_{30}\mu_1 + \mu_3\lambda_{c_2}\mu_{c_1}\mu_{20} + \\
& \lambda_2\mu_{c_1}\mu_{30}\mu_{20} + \mu_1\mu_{c_2}\mu_{20}\mu_3 + \mu_1\mu_{c_2}\lambda_{c_3}\mu_{30} + \mu_{c_3}\lambda_3\lambda_{c_2}\mu_{30} + \\
& \mu_{c_1}\lambda_3\lambda_{c_2}\mu_{30} + \lambda_{c_3}\lambda_2\mu_{c_1}\mu_{30} + \mu_{c_3}\mu_2\mu_3\mu_1 + \lambda_{c_3}\mu_{c_1}\mu_{30}\mu_1 + \\
& \mu_3\lambda_2\mu_{c_1}\mu_{20} + \mu_{c_3}\mu_3\mu_1\mu_{20} + \mu_3\mu_{c_1}\mu_1\mu_{20} + \mu_{c_3}\mu_{30}\mu_1\mu_{20} + \\
& \lambda_2\mu_{c_2}\lambda_3\mu_{30} + \mu_2\mu_{c_1}\mu_{30}\mu_1 + \mu_{c_3}\lambda_3\lambda_2\mu_{30} + \lambda_{c_3}\lambda_{c_2}\mu_{c_1}\mu_{30} + \\
& \mu_2\lambda_{c_2}\mu_{c_1}\mu_{30} + \mu_{c_1}\lambda_3\lambda_2\mu_{30} + \lambda_2\mu_{c_2}\mu_{20}\mu_3 + \lambda_2\mu_{c_2}\mu_{20}\mu_{30})
\end{aligned}
$$

$$
\begin{aligned}
F_{53} &= \{2\lambda_1\lambda_{c_1} - 2\mu_1\lambda_{c_1} - 2\lambda_2\lambda_1 - 2\lambda_2\lambda_{c_1} - 2\lambda_{c_2}\lambda_1 - 2\lambda_{c_2}\lambda_{c_1} + \lambda_1^2 + \lambda_{c_2}^2 + \\
&\quad 2\mu_1\lambda_{c_2} + \mu_1^2 + 2\lambda_2\lambda_{c_2} + 2\lambda_2\mu_1 + \lambda_2^2 + \lambda_{c_1}^2 + 2\lambda_1\mu_1\}^{\frac{1}{2}}
\end{aligned}
$$

$$
\begin{aligned}
F_{54} &= (\mu_1 + \mu_{20} + \lambda_3 + \lambda_{c_2} + \mu_2 + \lambda_2 + \lambda_{c_3}) \\
F_{55} &= (\lambda_{c_3}\mu_1 + \mu_2\lambda_{c_2} + \lambda_2\mu_{20} + \lambda_3\lambda_2 + \lambda_3\lambda_{c_2} + \mu_2\mu_1 + \mu_{20}\lambda_{c_2} \\
&\quad \lambda_{c_3}\lambda_2 + \lambda_3\mu_1 + \mu_1\mu_{20} + \lambda_{c_3}\lambda_{c_2}) \\
F_{56} &= (\lambda_2 + \mu_1 + \lambda_{c_3} + \mu_2 + \lambda_1 + \lambda_3 + \mu_{20} + \lambda_{c_1} + \lambda_{c_2}) \\
F_{57} &= (\lambda_{c_1}\lambda_{c_3} + \lambda_1\mu_{20} + \mu_1\mu_{20} + \mu_2\lambda_{c_2} + \lambda_{c_3}\mu_1 + \lambda(1)\lambda_{c_2} + \lambda_{c_1}\lambda_{c_2} + \lambda_{c_3}\lambda_{c_2} + \\
&\quad \lambda_1\mu_2 + \lambda_{c_3}\lambda_2 + \lambda_2\mu_{20} + \lambda_3\lambda_{c_2} + \lambda_{c_1}\lambda_3 + \lambda_{c_1}\mu_2 + \lambda_{c_1}\mu_1 + \lambda_3\lambda_2 + \\
&\quad \mu_{20}\lambda_{c_2} + \lambda_1\lambda_{c_3} + \mu_{20}\lambda_{c_1} + \lambda_3\mu_1 + \mu_2\mu_1 + \lambda_2\lambda_{c_1} + \lambda_2\lambda_1 + \lambda(1)\lambda_3) \\
F_{58} &= (\lambda_1\lambda_{c_3}\lambda_{c_2} + \mu_{20}\lambda_{c_1}\lambda_{c_2} + \lambda_1\mu_{20}\lambda_{c_2} + \lambda_{c_1}\lambda_3\mu_1 + \lambda_{c_1}\lambda_3\lambda_2 + \lambda_{c_1}\lambda_3\lambda_{c_2} + \\
&\quad \lambda_1\lambda_3\lambda_{c_2} + \lambda_1\lambda_{c_3}\lambda_2 + \lambda_{c_1}\lambda_{c_3}\mu_1 + \mu_{20}\lambda_{c_1}\lambda_2 + \lambda_{c_1}\lambda_{c_3}\lambda_{c_2} + \mu_1\mu_{20}\lambda_{c_1} + \\
&\quad \lambda_{c_1}\mu_2\mu_1 + \lambda_{c_1}\lambda_{c_3}\lambda_2 + \lambda_1\lambda_3\lambda_2 + \lambda_1\mu_2\lambda_{c_2} + \lambda_{c_1}\mu_2\lambda_{c_2})
\end{aligned}
$$

$$
\begin{aligned}
F_{59} &= (\lambda_4 + \lambda_{c_4} + \lambda_{c_3} + \mu_{20} + \mu_2 + \lambda_2 + \mu_{30} + \mu_1 + \lambda_3 + \lambda_{c_2} + \mu_3) \\
F_{60} &= (\lambda_2\lambda_{c_4} + \mu_1\lambda_3 + \lambda_{c_2}\mu_{30} + \lambda_2\lambda_3 + \lambda_{c_2}\lambda_{c_4} + \lambda_2\mu_3 + \mu_{20}\mu_3 + \lambda_4\lambda_3 + \\
&\quad \lambda_{c_3}\mu_3 + \mu_{30}\mu_2 + \mu_{30}\lambda_3 + \lambda_4\mu_2 + \lambda_{c_4}\mu_2 + \lambda_{c_2}\mu_{20} + \lambda_{c_2}\lambda_{c_3} + \lambda_{c_2}\mu_3 + \\
&\quad \lambda_{c_2}\mu_2 + \mu_1\mu_2 + \lambda_2\lambda_{c_3} + \mu_1\lambda_4 + \lambda_{c_2}\lambda_3 + \mu_1\lambda_{c_4} + \lambda_2\lambda_4 + \mu_1\mu_3 + \\
&\quad \lambda_2\mu_{20} + \lambda_{c_2}\lambda_4 + \mu_1\mu_{30} + \mu_{20}\lambda_4 + \lambda_{c_3}\lambda_4 + \mu_3\mu_2 + \mu_{20}\lambda_{c_4} + \lambda_{c_3}\lambda_{c_4} + \\
&\quad \lambda_2\mu_{30} + \mu_1\mu_{20} + \mu_1\lambda_{c_3} + \mu_{20}\mu_{30} + \lambda_{c_3}\mu_{30} + \lambda_{c_4}\lambda_3) \\
F_{61} &= (\lambda_2\mu_{20}\mu_{30} + \lambda_2\lambda_{c_3}\mu_{30} + \lambda_2\lambda_{c_3}\lambda_4 + \lambda_{c_2}\mu_{30}\mu_2 + \\
&\quad \mu_1\lambda_{c_3}\mu_3 + \lambda_{c_2}\lambda_{c_3}\mu_{30} + \lambda_2\lambda_4\lambda_3 + \lambda_2\lambda_{c_3}\mu_3 + \\
&\quad \mu_1\lambda_4\lambda_3 + \lambda_{c_2}\lambda_4\mu_2 + \mu_1\lambda_4\mu_2 + \lambda_2\mu_{20}\lambda_4 + \\
&\quad \mu_1\mu_{20}\mu_{30} + \lambda_{c_2}\mu_{30}\lambda_3 + \lambda_{c_2}\lambda_{c_3}\mu_3 + \mu_1\mu_{30}\lambda_3 +
\end{aligned}
$$

$$\lambda_{c_2}\mu_3\mu_2 + \lambda_{c_2}\lambda_4\lambda_3 + \lambda_{c_2}\lambda_{c_4}\mu_2 + \mu_1\lambda_{c_3}\lambda_4 +$$
$$\mu_1\mu_{20}\lambda_4 + \lambda_{c_2}\lambda_{c_3}\lambda_4 + \lambda_{c_2}\mu_{20}\lambda_4 + \lambda_{c_2}\mu_{20}\lambda_{c_4} +$$
$$\mu_1\mu_3\mu_2 + \mu_1\lambda_{c_3}\lambda_{c_4} + \lambda_{c_2}\lambda_{c_3}\lambda_{c_4} + \lambda_{c_2}\mu_{20}\mu_3 +$$
$$\mu_1\mu_{30}\mu_2 + \lambda_{c_2}\lambda_{c_4}\lambda_3 + \lambda_2\lambda_{c_4}\lambda_3 + \lambda_2\mu_{20}\lambda_{c_4} +$$
$$\mu_1\mu_{20}\lambda_{c_4} + \mu_1\mu_{20}\mu_3 + \mu_1\lambda_{c_3}\mu_{30} + \lambda_2\lambda_{c_3}\lambda_{c_4} +$$
$$\lambda_{c_2}\mu_{20}\mu_{30} + \lambda_2\mu_{20}\mu_3 + \lambda_2\mu_{30}\lambda_3 + \mu_1\lambda_{c_4}\mu_2 + \mu_1\lambda_{c_4}\lambda_3)$$

$$F_{62} = (\lambda_4 + \mu_3 + \lambda_{c_4} + \mu_{20} + \lambda_{c_1} + \lambda_1 + \mu_2 + \lambda_2 + \mu_1 + \lambda_{c_3} + \mu_{30} + \lambda_3 + \lambda_{c_2})$$

$$F_{63} = (\mu_1\lambda_{c_1} + \lambda_2\lambda_{c_4} + \mu_1\lambda_3 + \lambda_{c_2}\mu_{30} + \lambda_2\lambda_3 + \lambda_{c_2}\lambda_{c_4} +$$
$$\lambda_2\mu_3 + \lambda_1\mu_3 + \lambda_{c_1}\mu_3 + \lambda_{c_1}\mu_{20} + \lambda_1\lambda_{c_3} + \mu_{20}\mu_3 + \lambda_4\lambda_3 + \lambda_{c_3}\mu_3 +$$
$$\mu_{30}\mu_2 + \mu_{30}\lambda_3 + \lambda_4\mu_2 + \lambda_{c_4}\mu_2 + \lambda_{c_2}\mu_{20} + \lambda_{c_2}\lambda_{c_3} + \lambda_{c_2}\mu_3 + \lambda_{c_2}\mu_2 +$$
$$\mu_1\mu_2 + \lambda_2\lambda_{c_3} + \mu_1\lambda_4 + \lambda_{c_2}\lambda_3 + \mu_1\lambda_{c_4} + \lambda_2\lambda_4 + \mu_1\mu_3 + \lambda_2\mu_{20} +$$
$$\lambda_{c_2}\lambda_4 + \mu_1\mu_{30} + \mu_{20}\lambda_4 + \lambda_{c_3}\lambda_4 + \mu_3\mu_2 + \mu_{20}\lambda_{c_4} + \lambda_{c_3}\lambda_{c_4} + \lambda_2\mu_{30} +$$
$$\mu_1\mu_{20} + \mu_1\lambda_{c_3} + \mu_{20}\mu_{30} + \lambda_{c_3}\mu_{30} + \lambda_{c_4}\lambda_3 + \lambda_{c_1}\lambda_3 + \lambda_1\mu_{30} + \lambda_{c_1}\lambda_4 +$$
$$\lambda_{c_1}\lambda_{c_4} + \lambda_1\mu_2 + \lambda_{c_1}\mu_2 + \lambda_1\mu_{20} + \lambda_{c_1}\mu_{30} + \lambda_1\lambda_{c_4} + \lambda_1\lambda_4 +$$
$$\lambda_{c_1}\lambda_2 + \lambda_1\lambda_2 + \lambda_1\lambda_{c_2} + \lambda_{c_1}\lambda_{c_2} + \lambda_1\lambda_3 + \lambda_{c_1}\lambda_{c_3})$$

$$F_{64} = (\lambda_2\mu_{20}\mu_{30} + \lambda_1\lambda_{c_2}\mu_2 + \lambda_{c_1}\lambda_2\lambda_3 + \lambda_1\lambda_{c_2}\lambda_4 + \lambda_1\lambda_4\mu_2 +$$
$$\lambda_1\lambda_{c_3}\mu_{30} + \lambda_{c_1}\mu_{20}\mu_{30} + \lambda_{c_1}\lambda_{c_2}\mu_{30} + \lambda_1\mu_3\mu_2 + \lambda_{c_1}\lambda_2\mu_3 +$$
$$\lambda_1\lambda_2\lambda_3 + \lambda_{c_1}\lambda_2\lambda_{c_4} + \lambda_2\lambda_{c_3}\mu_{30} + \lambda_2\lambda_{c_3}\lambda_4 + \lambda_{c_2}\mu_{30}\mu_2 +$$
$$\lambda_{c_1}\lambda_4\lambda_3 + \mu_1\lambda_{c_1}\lambda_{c_3} + \lambda_1\lambda_{c_3}\mu_3 + \lambda_1\mu_{20}\mu_3 + \lambda_1\lambda_{c_3}\lambda_{c_4} +$$
$$\lambda_1\lambda_{c_2}\lambda_{c_4} + \lambda_1\lambda_{c_2}\lambda_3 + \lambda_1\lambda_2\lambda_{c_3} + \lambda_1\lambda_2\lambda_{c_4} + \lambda_{c_1}\mu_{20}\lambda_4 +$$
$$\lambda_1\lambda_2\mu_{30} + \lambda_{c_1}\lambda_{c_2}\lambda_3 + \lambda_1\lambda_{c_2}\mu_{30} + \lambda_1\mu_{30}\lambda_3 + \lambda_{c_1}\lambda_{c_4}\mu_2 +$$
$$\lambda_1\lambda_2\lambda_4 + \lambda_{c_1}\lambda_{c_3}\lambda_4 + \lambda_{c_1}\mu_{30}\mu_2 + \lambda_{c_1}\mu_{20}\lambda_{c_4} + \lambda_1\lambda_{c_2}\mu_3 +$$
$$\lambda_1\lambda_{c_2}\lambda_{c_3} + \lambda_{c_1}\lambda_2\mu_{30} + \lambda_{c_1}\lambda_4\mu_2 + \lambda_{c_1}\lambda_{c_3}\lambda_{c_4} +$$
$$\lambda_{c_1}\mu_{30}\lambda_3 + \mu_1\lambda_{c_1}\mu_2 + \lambda_1\lambda_{c_2}\mu_{20} + \mu_1\lambda_{c_1}\lambda_3 + \lambda_1\lambda_2\mu_3 +$$
$$\lambda_{c_1}\lambda_{c_3}\mu_{30} + \mu_1\lambda_{c_3}\mu_3 + \lambda_{c_1}\lambda_2\lambda_{c_3} + \lambda_{c_1}\lambda_2\lambda_4 +$$
$$\mu_1\lambda_{c_1}\lambda_{c_4} + \lambda_{c_1}\lambda_{c_3}\mu_3 + \lambda_{c_1}\mu_3\mu_2 + \mu_1\lambda_{c_1}\mu_{30} + \lambda_1\lambda_4\lambda_3 +$$
$$\lambda_{c_2}\lambda_{c_3}\mu_{30} + \lambda_2\lambda_4\lambda_3 + \lambda_2\lambda_{c_3}\mu_3 + \mu_1\lambda_4\lambda_3 + \lambda_{c_2}\lambda_4\mu_2 +$$
$$\mu_1\lambda_4\mu_2 + \lambda_2\mu_{20}\lambda_4 + \mu_1\mu_{20}\mu_{30} + \lambda_{c_2}\mu_{30}\lambda_3 + \lambda_1\mu_{20}\lambda_4 +$$
$$\lambda_{c_1}\lambda_2\mu_{20} + \lambda_1\lambda_{c_3}\lambda_4 + \lambda_{c_1}\lambda_{c_2}\mu_{20} + \mu_1\lambda_{c_1}\mu_3 + \lambda_1\mu_{20}\lambda_{c_4} +$$

$$
\begin{aligned}
& \lambda_1\mu_{20}\mu_{30} + \lambda_1\lambda_{c_4}\lambda_3 + \lambda_{c_2}\lambda_{c_3}\mu_3 + \lambda_{c_1}\lambda_{c_2}\mu_3 + \mu_1\mu_{30}\lambda_3 + \\
& \lambda_{c_2}\mu_3\mu_2 + \lambda_{c_2}\lambda_4\lambda_3 + \lambda_{c_2}\lambda_{c_4}\mu_2 + \mu_1\lambda_{c_3}\lambda_4 + \mu_1\mu_{20}\lambda_4 + \\
& \lambda_{c_2}\lambda_{c_3}\lambda_4 + \lambda_1\mu_{30}\mu_2 + \lambda_{c_2}\mu_{20}\lambda_4 + \lambda_{c_2}\mu_{20}\lambda_{c_4} + \\
& \mu_1\mu_3\mu_2 + \mu_1\lambda_{c_3}\lambda_{c_4} + \lambda_{c_2}\lambda_{c_3}\lambda_{c_4} + \lambda_{c_2}\mu_{20}\mu_3 + \\
& \mu_1\lambda_{c_1}\mu_{20} + \lambda_{c_1}\lambda_{c_2}\lambda_{c_4} + \lambda_{c_1}\mu_{20}\mu_3 + \mu_1\mu_{30}\mu_2 + \\
& \mu_1\lambda_{c_1}\lambda_4 + \lambda_{c_2}\lambda_{c_4}\lambda_3 + \lambda_2\lambda_{c_4}\lambda_3 + \lambda_2\mu_{20}\lambda_{c_4} + \\
& \mu_1\mu_{20}\lambda_{c_4} + \mu_1\mu_{20}\mu_3 + \lambda_{c_1}\lambda_{c_2}\lambda_{c_3} + \lambda_{c_1}\lambda_{c_4}\lambda_3 + \\
& \mu_1\lambda_{c_3}\mu_{30} + \lambda_{c_1}\lambda_{c_2}\mu_2 + \lambda_2\lambda_{c_3}\lambda_{c_4} + \lambda_{c_2}\mu_{20}\mu_{30} + \\
& \lambda_1\lambda_{c_4}\mu_2 + \lambda_2\mu_{20}\mu_3 + \lambda_{c_1}\lambda_{c_2}\lambda_4 + \lambda_2\mu_{30}\lambda_3 + \mu_1\lambda_{c_4}\mu_2 + \mu_1\lambda_{c_4}\lambda_3) \\
F_{65} \;=\; & (\lambda_{c_1}\lambda_{c_2}\lambda_{c_3}\lambda_{c_4} + \lambda_{c_1}\lambda_2\mu_{20}\mu_3 + \lambda_{c_1}\lambda_2\lambda_{c_3}\lambda_4 + \\
& \lambda_{c_1}\lambda_2\mu_{30}\lambda_3 + \lambda_{c_1}\lambda_2\mu_{20}\lambda_{c_4} + \lambda_1\lambda_{c_2}\lambda_{c_4}\lambda_3 + \lambda_1\lambda_{c_2}\lambda_4\lambda_3 + \\
& \mu_1\lambda_{c_1}\lambda_{c_3}\lambda_4 + \lambda_1\lambda_{c_2}\mu_{30}\lambda_3 + \lambda_1\lambda_{c_2}\mu_3\mu_2 + \mu_1\lambda_{c_1}\lambda_4\mu_2 + \\
& \lambda_{c_1}\lambda_{c_2}\lambda_{c_3}\lambda_4 + \lambda_1\lambda_{c_2}\mu_{20}\mu_{30} + \mu_1\lambda_{c_1}\mu_{30}\mu_2 + \mu_1\lambda_{c_1}\mu_{30}\lambda_3 + \\
& \mu_1\lambda_{c_1}\mu_3\mu_2 + \lambda_1\lambda_{c_2}\lambda_{c_3}\lambda_{c_4} + \lambda_1\lambda_2\lambda_4\lambda_3 + \mu_1\lambda_{c_1}\lambda_{c_4}\lambda_3 + \\
& \mu_1\lambda_{c_1}\mu_{20}\lambda_4 + \mu_1\lambda_{c_1}\mu_{20}\lambda_{c_4} + \mu_1\lambda_{c_1}\mu_{20}\mu_3 + \mu_1\lambda_{c_1}\lambda_4\lambda_3 + \\
& \mu_1\lambda_{c_1}\lambda_{c_3}\mu_{30} + \lambda_{c_1}\lambda_{c_2}\mu_{20}\mu_3 + \lambda_1\lambda_{c_2}\mu_{30}\mu_2 + \lambda_{c_1}\lambda_{c_2}\mu_{20}\lambda_4 + \\
& \lambda_{c_1}\lambda_2\lambda_{c_3}\lambda_{c_4} + \lambda_1\lambda_2\lambda_{c_3}\mu_{30} + \lambda_1\lambda_2\lambda_{c_3}\lambda_4 + \lambda_1\lambda_2\lambda_{c_4}\lambda_3 + \\
& \lambda_{c_1}\lambda_{c_2}\mu_{20}\lambda_{c_4} + \lambda_{c_1}\lambda_2\mu_{20}\lambda_4 + \lambda_1\lambda_{c_2}\lambda_4\mu_2 + \lambda_{c_1}\lambda_2\lambda_{c_3}\mu_{30} + \\
& \lambda_{c_1}\lambda_2\lambda_4\lambda_3 + \lambda_{c_1}\lambda_2\lambda_{c_3}\mu_3 + \lambda_{c_1}\lambda_2\lambda_{c_4}\lambda_3 + \lambda_1\lambda_{c_2}\mu_{20}\mu_3 + \\
& \lambda_{c_1}\lambda_2\mu_{20}\mu_{30} + \mu_1\lambda_{c_1}\mu_{20}\mu_{30} + \lambda_{c_1}\lambda_{c_2}\mu_{30}\lambda_3 + \lambda_{c_1}\lambda_{c_2}\lambda_4\lambda_3 + \\
& \mu_1\lambda_{c_1}\lambda_{c_3}\mu_3 + \lambda_{c_1}\lambda_{c_2}\lambda_{c_3}\mu_{30} + \lambda_1\lambda_2\lambda_{c_3}\lambda_{c_4} + \lambda_1\lambda_{c_2}\mu_{20}\lambda_4 + \\
& \lambda_1\lambda_{c_2}\mu_{20}\lambda_{c_4} + \lambda_{c_1}\lambda_{c_2}\mu_3\mu_2 + \lambda_{c_1}\lambda_{c_2}\mu_{30}\mu_2 + \lambda_{c_1}\lambda_{c_2}\lambda_4\mu_2 + \\
& \lambda_1\lambda_{c_2}\lambda_{c_4}\mu_2 + \lambda_1\lambda_2\lambda_{c_3}\mu_3 + \mu_1\lambda_{c_1}\lambda_{c_4}\mu_2 + \lambda_{c_1}\lambda_{c_2}\lambda_{c_4}\mu_2 + \\
& \lambda_1\lambda_{c_2}\lambda_{c_3}\mu_{30} + \lambda_{c_1}\lambda_{c_2}\lambda_{c_3}\mu_3 + \lambda_{c_1}\lambda_{c_2}\mu_{20}\mu_{30} + \mu_1\lambda_{c_1}\lambda_{c_3}\lambda_{c_4} + \\
& \lambda_1\lambda_{c_2}\lambda_{c_3}\mu_3 + \lambda_{c_1}\lambda_{c_2}\lambda_{c_4}\lambda_3 + \lambda_1\lambda_{c_2}\lambda_{c_3}\lambda_4) \\
F_{66} \;=\; & (\mu_3 + \lambda_{c_4} + \lambda_3 + \mu_{30} + \mu_{20} + \lambda_{c_3} + \mu_2 + \lambda_4)\lambda_1 \\
F_{67} \;=\; & (\mu_{20}\mu_{30} + \lambda_{c_3}\mu_{30} + \lambda_{c_4}\lambda_3 + \mu_{20}\lambda_4 + \lambda_{c_3}\lambda_4 + \mu_3\mu_2 + \mu_{20}\lambda_{c_4} + \lambda_{c_3}\lambda_{c_4} + \\
& \mu_{20}\mu_3 + \lambda_4\lambda_3 + \lambda_{c_3}\mu_3 + \mu_{30}\mu_2 + \mu_{30}\lambda_3 + \lambda_4\mu_2 + \lambda_{c_4}\mu_2)\lambda_1
\end{aligned}
$$