

**Undergraduate Students' Connections Between the Embodied, Symbolic, and
Formal Mathematical Worlds of Limits and Derivatives:
A Qualitative Study Using Tall's Three Worlds of Mathematics**

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ABSTRACT

Calculus at the university level is taken by thousands of undergraduate students each year. However, a significant number of students struggle with the subject, resulting in poor problem solving, low achievement, and high failure rates in the calculus courses overall (e.g., Kaput, 1994; Szydlik, 2000; Tall, 1985; Tall & Ramos, 2004; White & Mitchelmore, 1996). This is cause for concern as the lack of success in university calculus creates further barriers for students who require the course for their programs of study. This study examines this issue from the perspective of Tall's Three Worlds of Mathematics (Tall, 2004a, 2004b, 2008), a theory of mathematics and mathematical cognitive development. A fundamental argument of Tall's theory suggests that connecting between the different mathematical worlds, named the Embodied-Conceptual, Symbolic-Proceptual, and Formal-Axiomatic worlds, is essential for full cognitive development and understanding of mathematical concepts. Working from this perspective, this research examined, through the use of calculus task questions and semi-structured interviews, how fifteen undergraduate calculus students made connections between the different mathematical worlds for the calculus topics of limits and derivatives. The analysis of the findings suggests that how the students make connections can be described by eight different *Response Categories*. The study also found that how the participants made connections between mathematical worlds might be influenced by the type of questions that are asked and their experience in calculus courses. I infer that these *Response Categories* have significance for this study and offer potential for further study and educational practice. I conclude by identifying areas of further research in regards to calculus achievement, the *Response Categories*, and other findings such as a more detailed study of the influence of experience.

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INTRODUCTION

This thesis presents a qualitative research study that examines undergraduate students' understanding of the calculus concepts of limits and derivatives through the lens of Tall's theory of Three Worlds of Mathematics. This study was specifically interested in identifying how students understand making connections between the Three Worlds of Mathematics.

Context of Study

Calculus, described as the mathematics of change or the study of rates of change, is not just used by mathematicians, but by individuals in any field where a problem can be mathematically modeled and an optimal solution desired. For example, engineers, economists, and medical scientists, to name a few, use calculus in their professions. Consequently, thousands of undergraduate students take calculus at the university level because they are required to for various programs. For instance, at one medium sized Canadian university at least 45.7% of undergraduate students in the 2008/2009 academic year were enrolled in a program that required them to take a calculus course (University of Ottawa, 2009). This statistic does not include students who may have taken calculus within a minor or as an option.

Unfortunately, studies that have examined university calculus have found that a considerable number of undergraduate students struggle with the subject, resulting in poor problem solving, low achievement, and high failure rates in the calculus courses overall (e.g., Kaput, 1994; Szydlik, 2000; Tall, 1985; Tall & Ramos, 2004; White & Mitchelmore, 1996). This is cause for concern for both students and university educators, as the lack of success in university calculus creates further barriers for

students who require the course for their programs of study.

To better understand and to ameliorate the issue of many students' weak understanding of calculus, which leads to low achievement and high failure rates, a variety of approaches and perspectives have been used to examine the problem. For example, some studies have looked at incorporating technology into the calculus classroom to improve problem solving (e.g., Porzio, 1995; Zaslavsky, Sela, & Leron, 2002), while others have examined specific calculus concepts in order to identify areas of concern (e.g., Hardy, 2009; Roh, 2008; Tall, 1985; White & Mitchelmore, 1996; Williams, 1991). There have also been studies that have employed teaching interventions with the goal of improving students' understanding (e.g., Thompson 1994; White & Micheltmore, 1996). However, despite the research that has already been conducted, educators still grapple with the reasons why calculus remains a subject that causes difficulties and with how to significantly improve calculus achievement.

As a result, it seems reasonable to examine the subject from a different perspective. The perspective that frames this study is Tall's Three Worlds of Mathematics (Tall, 2004a, 2004b, 2008), a theory of mathematics and mathematical cognitive development. According to Tall, cognitive development in mathematics is a process that takes place within three domains, or worlds, of mathematics, namely the embodied-conceptual, the symbolic-proceptual, and the formal-axiomatic (2004a, 2004b, 2008). These will be more fully described later in the thesis but, put simply, these worlds consist of a world of actions on images (embodied-conceptual), processes and concepts acting with symbols (symbolic-proceptual), and formal mathematical definitions and axioms (formal-axiomatic).

A fundamental feature of Tall's theory is the argument that connecting between the different mathematical worlds is essential for cognitive development and understanding of mathematical concepts (2008). Research conducted in a variety of mathematical areas supports Tall's argument by suggesting that a lack of understanding of a concept may result when students cannot connect between the different worlds of mathematical concepts (Nogueira de Lima & Tall, 2006; Smart, 2010; Stewart & Thomas, 2006, 2007; Watson, Spyrou, & Tall, 2003). Research also shows that students who can connect between at least the embodied-conceptual and symbolic-proceptual worlds of the studied concepts tend to be more successful at solving problems related to the concepts (Christou, Pitta-Pantazi, Souyoul, & Zachariades, 2005; Smart; Stewart & Thomas, 2006, 2007; Watson, Spyrou, & Tall). However, no research has yet described in detail what these connections look like or how students are making these connections. Thus, by examining students' connections between Tall's worlds I not only provide a meaningful way to look at their understanding of calculus concepts and to hopefully find ways to eventually address overall achievement in calculus, but I also contribute to the body of literature by providing more detail on how connections are made.

Research Question and Overview of Research Methods

As was mentioned, research using Tall's theory, on specific concepts at the secondary and post-secondary level, has shown that students who were not making connections between the symbolic-proceptual and embodied-conceptual worlds struggled with the concepts and problem solving related to the concepts (e.g., Nogueira de Lima & Tall, 2006; Stewart & Thomas, 2006, 2007). However, no research has yet

described these connections in order to address what students should aim for in their understanding of the concepts. In this thesis, I investigated *how* the participants were making connections in hopes that this increased level of detail may prove helpful in eventually influencing practice and consequently helping other students to make connections between different mathematics worlds as connecting seems to be a significant factor in developing an understanding of a mathematical concept. I specifically look at the concepts of limits and derivatives, as these concepts are fundamental to the development of an understanding of calculus and have been identified in the literature as often difficult for students (e.g., Hardy, 2009; Tall, 1985; White & Mitchelmore, 1996; Williams, 1991). These concepts had also not yet been studied from the perspective of Tall's Three Worlds. I initially decided that I would focus on the connections between the embodied-conceptual and symbolic-proceptual worlds for the concepts of limits of derivatives, as formal-axiomatic mathematics is not part of a university level introductory calculus course. However, as will be demonstrated, this was a misleading assumption since during the data collection process some participants did use representations that could be considered connections between the embodied-conceptual and formal-axiomatic as well as the symbolic-proceptual and formal-axiomatic. Therefore, I had to adjust the data analysis to incorporate the formal-axiomatic world.

Tall's theory also views cognitive development as an ongoing continual process. Since my study is not longitudinal in nature it does not look at the participants' cognitive development over time. Instead the study describes the connections students make between mathematical worlds at a given point in time. For this study, I use the

term “understanding” of a particular concept when referring to what I observed at the time of data collection as the participants’ cognitive development of the concept. In other words, a participant’s understanding of a concept is the participant’s point in the cognitive development of the concept at the time of the study.

Thus, my study uses the information provided from previous empirical research and theoretical literature to form the following research question of this thesis:

How do undergraduate calculus students make connections between the embodied-conceptual, symbolic-proceptual, and formal-axiomatic worlds of the calculus concepts of limits and derivatives?

In order to address this question, the study looked at the mathematical representations the participating students used on calculus task questions as well as the representations the students spoke about during an interview about the calculus tasks. Representations have been described as a means available for thinking and communicating in mathematics (Davis & Maher, 1990; Hiebert & Carpenter, 1992). For example, the number symbols of 0-9 are representations used to identify quantitative values. Other types of possible mathematical representations may be geometric, like shapes and angles, or graphic such as tables, Cartesian coordinates, diagrams, or figures, just to name a few. For this research, mathematical representations correspond to Tall’s different mathematical worlds and this correspondence is used to help examine the connections students make between the worlds.

The study included four task-based calculus questions on the concepts of limits and derivatives that were completed by undergraduate calculus student participants. The representations used by participants on the tasks were examined for connections

between mathematical worlds. Semi-structured interviews with the participants took place after the tasks. Through the interviews, I sought to understand the participants' impressions about their solutions to the task questions, identify their reasons for using particular representations, and describe how the students connected the mathematical worlds in relation to the mathematical concepts of limits and derivatives. These two sources of data were used to address the research question of this study.

Significance to the Field

Although there is research that has studied students' understanding of calculus at the post-secondary level (e.g., Sierpiska 1987; Tall and Vinner, 1981; White & Mitchelmore, 1996; Williams, 1991), to the best of my knowledge, none have explored the specific calculus concepts of limits and derivatives from the perspective of the Three Worlds of Mathematics. This study can offer significant findings in regards to how students understand these calculus topics as Tall's theory has previously demonstrated that the connections an individual makes between mathematical worlds of a concept provide valuable information about an individual's understanding of the concept and ability to successfully solve problems (e.g., Nogueira de Lima & Tall, 2006; Stewart & Thomas, 2007; Tall, 2008). This study contributes to the field by using the lens of the Three Worlds to describe students' understanding and the connections that students make between the worlds of the calculus concepts of limits and derivatives.

The research also contributes to the field through its focus on calculus at the university level. Insight into students' cognitive development of calculus concepts may lead to expanded research directions, provide valuable and relevant feedback to the university community about the teaching and learning of calculus, and potentially

inform teachers and policy makers. This could ultimately provide information that could be used to help students be more successful in university calculus courses.

Researcher's Role and Qualifications

My educational background and research experience helped situate me to be able to successfully conduct this study. As someone who has undergraduate and graduate degrees in mathematics, has worked as an instructor and teaching assistant for calculus courses, and has tutored numerous individuals in calculus courses, I have developed the ability to recognize the different representations of calculus concepts as well as the connections between them and among them. This experience was invaluable while conducting this study as it provided me with in-depth knowledge about calculus concepts and in particular the calculus concepts that are studied in first year university calculus.

Prior to conducting this study, I undertook a single-case pilot study (Smart, 2010) that helped to shape this research. From the pilot study, I gained experience using the framework of the Three Worlds of Mathematics, linking representations with their corresponding world(s), and analyzing the representations for connections and students' understanding of the concepts. My calculus-based task questions, semi-structured interview questions, and consent form from the pilot study served as templates for this research. Overall the skills developed in the pilot study assisted with the full research study.

Overview of Thesis Organization

In the remainder of this chapter I present an overview of the organization of this thesis. In Chapter 2, I describe the theoretical perspectives and position that informed

this study, which include Tall's Three Worlds of Mathematics combined with a theory of mathematical representations and connections between representations. I begin with an outline of the Three Worlds of Mathematics, which is the foundational theory of the research. I then discuss the theory of mathematical representation with specific focus on connections between mathematical representations. I conclude the chapter by defining the overall perspective of this study with a summary of how I used Tall's theory combined with a theory of mathematical representations.

Chapter 3 provides an overview of the literature that informed this study. I analyze and synthesize the previous studies that have also used the Three Worlds as their fundamental theoretical foundation and explore how empirical research that used Tall's theory informed my research in terms of research design, implementation, findings, and analysis. I then review empirical works that focused on connections between multiple representations. Lastly, because the research question of this study focuses on the specific calculus topics of limits and derivatives, I provide a review of research on these topics that demonstrates (a) that students struggle with these topics, (b) there have been many studies conducted on these topics using a variety of different methodologies, and (c) that the topics of limits and derivatives can be situated within multiple worlds, according to Tall's theory, thus making them appropriate subjects to analyze according to this perspective.

In Chapter 4, I describe the research methods employed in the study. This chapter begins with a review of the research question that structured the study. I then expand on the research and instrument designs and describe the data collection. I justify how the research method established an appropriate means by which to address

the research question. The majority of the chapter finishes up with an in-depth description of the six-step data analysis method that was undertaken and how this method worked within the chosen theoretical perspective of this study.

In Chapter 5, the findings of the study are presented. The observations from the data analysis are communicated in detail, which include the descriptions of the participants, and the findings from the six data analysis steps that were undertaken as the data analysis method.

In Chapter 6, I review and discuss the findings with regard to how they address the research question. I describe the significance of the Response Categories that emerged from my investigation and their implications for future research. As well, the findings are linked to other literature in order to place the findings in the context of current empirical studies in mathematics education. Lastly, I talk about the main claims that can be made from the findings of this study.

In chapter 7, the final chapter of this thesis, I talk about the implications of the research as a whole and the overall purpose of the study. I elaborate further on the significance of the eight Response Categories that emerged from my investigation and discuss the significance of the findings about the influence of prior experience and the types of questions asked. I also address the implications the Response Categories may have on practice. This chapter then describes a portion of the further research questions that were prompted by this study. I conclude by addressing the limitations of the study in order to ensure that the results and findings are appropriately interpreted and generalized.

CHAPTER TWO: THEORETICAL PERSPECTIVES

This study uses Tall's (2004a, 2004b, 2008) theory of Three Worlds of Mathematics because it provides a theory of both mathematics and mathematical cognitive development. Tall's theory (a) provides clear categories and descriptions of development and understanding of mathematics and mathematical objects, (b) clearly describes the interrelationships between the categories of development, and (c) allows for the incorporation of mathematical representations as visual objects of study in order to recognize students' understanding according to the different categories in Tall's theory. These features are important as they allow for a well-rounded theory that provides a framework to investigate undergraduate calculus students' understanding.

This chapter begins by outlining Tall's theory and describing the components of the theory that frame the perspective of this study. I then provide an overview of the literature on a theory of mathematical representations, a definition for mathematical representations, and their correspondence with the Three Worlds. Aspects of representations that are specific to this research will be described, which include connections between representations, connections between representations and mathematical worlds, and the differences between internal and external representations. This chapter concludes with a summary of the theoretical ideas presented and how they will be used throughout the research study.

Three Worlds of Mathematics

Tall's (2004a, 2004b, 2008) theory was developed from a reflection of what he observed to be three fundamentally different ways of operating on mathematical objects. These objects include: 1) geometrically formed mathematical objects, like lines

or shapes, 2) symbolic mathematical objects, such as formulas, and equations, and 3) the objects of advanced mathematics, such as definitions, axioms, theorems, and proofs. The idea of three different types of mathematical objects was initially presented by Gray and Tall (2001). They speculated that the different types of mathematical objects, along with the concepts that correspond to the objects, are each cognitively developed in their own way with their own language. For example, Tall suggests that students' development of geometric objects and their corresponding concepts are found to be similar to the development described by van Hiele (Burger & Shaughnessy, 1986), where students build an understanding of geometric objects in a hierarchical manner moving from one level to the next. Similarly, Tall proposes that symbolic number objects and the corresponding arithmetic concepts are developed in a manner described by the Actions, Processes, Objects, Schemas (APOS) theory of Dubinsky and McDonald (2001). In the APOS theory, students' development is based on first constructing mental actions with symbolic numbers and arithmetic concepts. As these mental actions on numbers and arithmetic are developed, they become processes. Once students have fully developed these processes they can be characterized as their own objects, as defined by the APOS theory. APOS objects are not to be confused with Tall's description of mathematical objects, despite the two theories' use of the same word. According to the APOS theory, objects are then organized into schemas in order to make sense of mathematical situations.

Gray and Tall's (2001) initial work has since been expanded and developed by Tall into a theory that is centered on the idea that all mathematical cognitive development fits into three domains, or worlds (Tall, 2004a, 2004b, 2008). These

worlds are named the embodied-conceptual, symbolic-proceptual, and formal-axiomatic worlds. Each of the three worlds focuses on different qualities of concepts and objects of mathematics (Watson, Spyrou, & Tall, 2003). These worlds will be described in detail hereafter, but in brief the embodied-conceptual world focuses on the qualities of physical objects and their properties, the symbolic-proceptual world focuses on the combination of the processes and concepts that are represented by symbolic objects, and the formal-axiomatic world focuses on the properties of physical and symbolic objects and builds a logical structure that identifies the relationships between the objects.

Embodied-Conceptual World

The embodied-conceptual world of mathematics is a combination of Bruner's Enactive and Iconic modes of representation (1966). According to Bruner, the Enactive mode of representation signifies an action or actions such as putting together or breaking apart, while the Iconic mode of representation is attributed to all visual or image organizations; for example, common shapes like triangles or quadrilaterals are organizations of images. Tall's embodied-conceptual world draws on these modes of representation and is made up of actions on visual or image representations of the mathematics (2004a). These action-image representations take the form of objects that are seen or sensed in the real world or imagined in the mind. For example, the amount of pebbles held in a hand can be conceptually embodied in the real world by pointing and counting each rock. The image can also be counted in the mind's eye, that is, I can imagine I have ten pebbles in my hand and mentally count them.

Students' development within the embodied-conceptual world of mathematics is

described by Tall as beginning with embodiment, which grows out of their perceptions, and the way they think about things that they perceive and sense. As students reflect on their sensory experiences, the reflections can be built upon to grow into conceptions that do not exist in the real world, such as a perfectly round circle. Thus, the development includes how students build more sophisticated notions from their sensory experiences. Overall, the embodied-conceptual world develops to include not only mental perceptions of real-world objects, but also internal conceptions that involve visuospatial imagery.

Validity in the embodied-conceptual world takes the form of visually being able to see something is correct, or predicting something because of an observable fact. For example, students can see it is correct that two and four make the same total as four and two because they can observe this with physical objects.

Examples of mathematical activities that take place in the embodied-conceptual world include the activity of pointing or organizing and counting objects into a physically quantifiable amount, or the activity of imaging in our minds a line that is perfectly straight which does not exist in reality. An embodied activity could also include recognizing and naming shapes of objects, such as doors being rectangles and clocks being circles, or an instructor sweeping his or her arms wide to represent the infinite nature of a function on a graph. Even in higher mathematics graph theory uses embodied representations to facilitate the exploration of the axioms and propositions. These examples make use of Bruner's (1966) action-image representations that form mathematical objects.

Symbolic-Proceptual World

The symbolic-proceptual world is made up of mathematical symbols that are acted upon by the combination of mathematical processes and concepts. The idea of the combination of processes and concepts is termed procepts and was developed by Gray and Tall (1994) to describe how processes and concepts can be thought of as acting in conjunction with each other in mathematics. More specifically, procepts are described as single mental objects made up of the encapsulation of a mathematical process that uses symbols and the related concept. For example, the symbols of $3 + 8$ are a case of both a process of addition, represented by the number symbols for 3 and 8 and the operator symbol for addition, as well as a representation of the concept of (the number) 11. Thus, $3 + 8$ can be flexibly viewed as an amalgam of a process and concept, or procept (Gray & Tall, 1994). Overall, proceptual objects of the symbolic-proceptual world of mathematics are grounded in the idea that flexible thinking of mathematical processes and concepts is necessary for the development of mathematical knowledge.

Tall (2004a, 2004b, 2008) describes students' development within the symbolic-proceptual world as beginning with the symbols that are used for calculations and manipulation, such as number symbols and operator symbols. In this world, the actions are taken on mathematical symbols. Development takes place as a student focuses on the properties of the symbols and the relationships between them, rather than the physical or embodied meaning of the symbols. Thus, through this encapsulation, a student's development moves towards symbolic activities, like arithmetic or algebra. This development of symbolic mathematical activities is pivotal, as it is through using symbols in mathematics that students can effortlessly switch between the processes of

doing mathematics to concepts for thinking about mathematics. In other words, as students develop an understanding of how to use mathematical symbols, these symbols can actually represent mathematical concepts.

Validity and truth in the symbolic-proceptual world can be tested by computations and manipulation. For example, $x + y = y + x$ can be perceived as true because a student can test it through trial and error computation. Similarly, $(a - b)(a + b) = a^2 - b^2$ can be shown to be true by carrying out algebraic manipulations or substituting numerical values into the equation.

Examples of symbolic mathematical systems include Arabic numerals, Roman numerals, or in the case of this proposed study, the symbols used in symbolic calculus, such as $\frac{dx}{dy}$ or $\int$. On their own, these symbols are meaningless. However, once combined with mathematical processes and concepts, they result in symbolic-proceptual objects. For example, the concepts of arithmetic can be combined with the concepts of algebra and symbols of Arabic numerals to produce an infinite number of symbolic-proceptual objects. Other objects of the symbolic-proceptual world are formed in the similar manner of combining symbols with mathematical processes and concepts.

Formal-Axiomatic World

The formal-axiomatic world is developed from the properties of mathematical objects and their concepts that are used to specify mathematical structure. This world works not with the familiar objects of experience, such as embodied objects or symbols. Instead, the formal-axiomatic world uses the properties of the embodied and symbolic objects as well as logical deductions to create formal definitions and axioms and deduce

theorems that construct axiomatic mathematical systems. These definitions and axioms are carefully formulated to define mathematical structures in terms of specific properties.

Students' development of the formal-axiomatic world begins with generalizing. Generalizing is considered the forming of conclusions from specific instances (Tall, 1991). For example, $(a + b) = (b + a)$ is a generalization of $2 + 3 = 3 + 2$ and $\mathbf{R}^n$, which is a real space of n -dimensions, is a generalization of $\mathbf{R}^2$, a 2-dimensional space or the Cartesian plane. In mathematics, definitions are developed from experience with objects whose properties can be generalized (Tall, 1991). Some of the property-based definitions are used to form axioms, which are considered self-evident truths that do not need to be proved because of their self-evident nature (Tall, 1991). Once definitions and axioms are set, further properties can be proved with logical deduction, which become theorems. As students engage in the formal-axiomatic world they deduce sets of theorems that describe the structure of formal mathematical systems (Tall, 2004a, 2004b).

One example of the formal-axiomatic world is Euclidean Geometry. The objects of Euclidean Geometry, which consist of objects such as shapes and angles, exist in the embodied-conceptual world. However, within Euclidean Geometry there also exist definitions that address the structure of the geometry objects and axioms that describe relationships between the objects. Combining the definitions and axioms with the embodied conceptual objects can lead to the development of the propositions or theorems of Euclidean Geometry. While Euclidean Geometry develops out of embodiment, changing the structural definitions or axioms can then lead to the

development of Non-Euclidean geometries, such as Spherical and Hyperbolic Geometry. This demonstrates how development within the formal-axiomatic world of mathematics can continue to progress.

Validity and truth within the formal-axiomatic world exists based on logical proof. When moving to the formal-axiomatic world, the truths from the embodied-conceptual world and symbolic-proceptual world are no longer considered true simply because they are seen to be true or manipulation and computation show them to be true. Instead, these embodied or symbolic *truths* often develop into the axioms that are applied logically to prove theorems of mathematical systems.

Some examples of formal axiomatic mathematical systems are within the structures of analysis, vector algebra, or field theory. While these examples all employ the use of symbolic mathematics, the carefully formulated definitions and axioms set them apart from merely symbolic mathematics and the symbolic-proceptual world. Similarly, Non-Euclidean Geometries, as mentioned above, do employ properties of conceptual embodiment but are considered formal-axiomatic because of the necessary use of axioms to describe the structure of these geometries. For example, Spherical Geometry alters Euclid's 5th postulate into an axiom that results in no parallel lines existing within the mathematical structure, thus describing a unique geometric space (Henderson & Taimina, 2005). However, these examples emphasize the strong interrelationships that exist between the three mathematical worlds.

Connecting Between Mathematical Worlds

One essential aspect of Tall's theory is the emphasis on the interrelationships and connections between the mathematical worlds, which hereafter will be referred to

as the embodied, symbolic, and formal worlds for the sake of simplicity. While many of Tall's earlier works have discussed this, his 2008 paper specifically elaborates on the connections between the worlds and provides a coherent and integrated framework. According to Tall (2008), mathematical cognitive development begins with the embodiment of physical concepts and actions: for example, pointing and counting objects or putting shapes into collections. As we develop within the embodied world, these actions become routine. These routines can be represented by symbols, such as numbers, on which mathematical operations can be performed. Many symbolic concepts arise from natural embodiments, and lead on to more sophisticated symbolism.

However, this transition and development from the embodied world to the symbolic world does not necessarily follow a linear path (Tall, 2008). Instead, the transition can move back and forth between the two worlds a number of times, developing a network of connections between the embodied and symbolic worlds. Tall describes these processes of development as symbolizing embodiment and embodying symbolism. An example of these relationships occurs with analytical geometry. We embody symbolism when symbolic equations and functions are represented as curves on a coordinate graph. Conversely, we are symbolizing embodiment when we represent circles or ellipses by algebraic equations.

Tall (2008) also describes how developing an understanding of the embodied and symbolic worlds and developing the connections between these worlds leads to the formal world of mathematics being developed. This takes the form of formal definitions and theorems being formulated and proved about embodied and symbolic

mathematical concepts. Again, this development is not linear, but dynamic in nature with understanding moving back and forth between the different worlds.

It is important to note that while it would seem that the overall goal of mathematical development is to reach the formal world, the theory is not hierarchical. Tall's (2008) theory actually emphasizes that cognitive development is strengthened as individuals develop the flexibility to move back and forth between the worlds, rather than "up" towards another world. In Figure 1 the three worlds and the many connections between them are modeled. It is these connections that this research is aiming to identify.

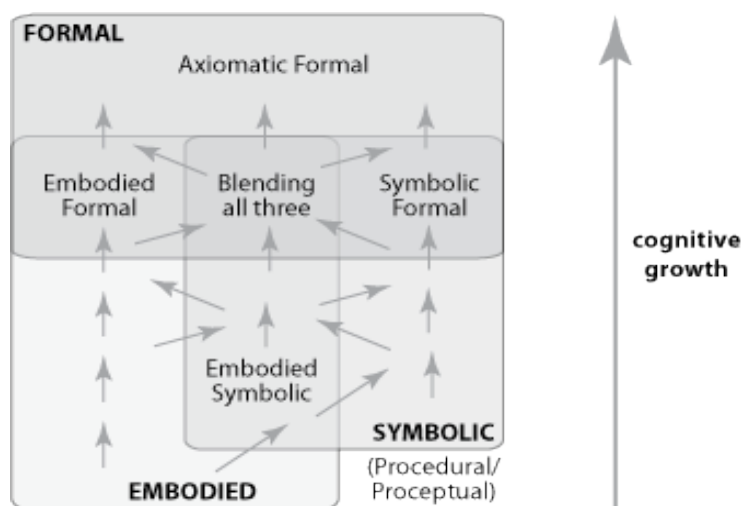


Figure 1. Development through Tall's Three Worlds of Mathematics (Tall, 2009).

Mathematical Representations

It has been suggested that to communicate and think about mathematics we use representations (Davis & Maher, 1990; Hiebert & Carpenter, 1992). All levels of mathematics, from kindergarten to advanced graduate work, can be seen as employing numerous and varied representations of concepts. Examples of mathematical representations include the number symbols 0-9, but symbols are just one form of

representation. Other examples of mathematical representations are geometric like angles or shapes. Some mathematical representations are graphical like curves in the Cartesian plane or polar coordinate plane. Mathematical representations can also include tabular representations, such as representing a function in a table, or technological representations, in the form of images created using dynamic geometry software or scatter plots created on graphing calculators, to name just a few.

Different representations allow for different properties to be depicted or conveyed. For example, a function can be represented as an equation, a curve on a Cartesian graph, or a table of values. While each is a representation of a function, each representation presents a different perspective of the concept (Goldin, 2003; Sierpinska, 1992). Mathematical representations are the language of mathematics with different representations “saying” different information. Overall, representations play a significant role in modern mathematics and mathematics education and it could be argued that mathematics as it is taught, practiced and used would not be possible without representations (Hiebert & Carpenter, 1992; Vergnaud, 1987).

There are a number of theoretical research papers on mathematical representations that provide an explicit definition and/or description for mathematical representations (e.g. Goldin and Shteingold, 2001; Hiebert and Carpenter, 1992; Janvier, 1987; Kaput, 1985). Most definitions and descriptions talk about representations as being something that “can stand for” (Goldin and Shteingold) something else. Representations are described by some authors as taking many forms, such as “configurations of signs, characters, icons, or objects” (Goldin, 2002, 2003), “spoken language” (Hiebert and Carpenter), “written symbols” (Hiebert and Carpenter;

Janvier), or “ideas in one’s mind” (Janvier). Many authors also describe representations as involving a type of “relationship” (Kaput) between the representation and what is being represented. Goldin (2003) describes the representing relationship as relationships that “correspond to, denote, depict, embody, encode, evoke label, mean, produce, refer to, suggest or symbolize” (p. 276), while the NCTM (2000) describes it as “the act of capturing a mathematical concept” (p. 67). From these descriptions of representations, several key words and ideas stand out. For example one idea is that a representation involves a relationship between some sorts of configurations. Thus, from summarizing the above descriptions, in this research I will define mathematical representations as: *configurations, designs, or arrangements that relate to mathematical concepts and depict those concepts into thinkable and communicable forms.*

Several authors also describe how representations can be organized according to external and internal representations (e.g. Hiebert & Carpenter, 1992; Janvier, 1987; NCTM, 2000). In terms of external and internal representations, many authors discuss a difference between the representations that we physically sense or work with and the representations that only exist in our minds. Dreyfus (1981) offers a comprehensive description of external and mental representations, which is another term for internal representations:

A symbolic representation is externally written or spoken, usually with the aim of making communication about the concept easier... A mental representation refers to internal schemata or frames of references which a person uses to interact with the external world. It is what occurs in the mind when thinking of that particular part of the external world and may differ from person to person

(p. 31).

A great deal of importance is placed on the distinction between external and internal representations, as it is only possible to communicate with external representations (Dreyfus, 1981). External representations can also be described as individually or socially constructed to allow for communication of mathematical concepts (Goldin & Shteingold, 2001; Hiebert & Carpenter). Thus, it is important to understand the distinction between internal and external representations when undertaking an analysis of mathematical representations particularly because internal representations differ for each individual and external representations are the only representations that can be visually analyzed through research.

Connections and translations between mathematical representations also need to be considered. Connections and translations from or between mathematical representations have been described as “the psychological processes involved in going from one mode of representation to another” (Janvier, 1987, p. 27). Connections and translations have been examined between external and internal representations (e.g. Janvier, 1987; Kaput, 1987; Lesh, Post, & Behr, 1987) between external and external (e.g. Dufour-Janvier, Bednarz, & Belanger, 1987; Hiebert and Carpenter, 1992), and between internal and internal representations (e.g. Hiebert and Carpenter; Kaput), suggesting an infinite network of ways to connect representations (Hiebert & Carpenter). In these works external and internal representations are described in a similar manner to Dreyfus’ above description (1981). Goldin and Shteingold (2001) state that representation connections have a two-way nature and that translation can take place back and forth between related representations. Pimm (1995) likewise

suggests that the connection and translation process is not only a one-to-one mapping of representations but also one-to-many, many-to-one, and many-to-many. Overall, the connections between representations and the translation processes used to move and connect between representations are an essential feature of mathematical representations.

Taking into account the division of external and internal representations as well as the connections that exist between representations, the above definition of representations can now be expanded and described as: *configurations, designs, or arrangements that relate to mathematical concepts and depict those concepts into thinkable and communicable forms. Thinkable mathematical representations are considered internal representations while communicable mathematical representations are considered external representations. There are also connections and translations between mathematical representations and mathematical concepts as well as between different mathematical representations, both internal and external.*

Mathematical Representations Within the Three Worlds of Mathematics

Incorporating a theory of mathematical representations with the Three Worlds of Mathematics provided this research with a comprehensive theoretical perspective. According to Tall, mathematical representations are the mathematical objects within the Three Worlds (2004a, 2008). The external mathematical representations or objects provide a means of communication to describe the concepts within the different worlds. This communication can be achieved by categorizing mathematical representations according to the descriptions and definitions of the embodied, symbolic, and formal worlds and allows for the study and analysis of the cognitive

development of the concepts, In other words, there exist representations that could be considered as the objects of the embodied world, representations that are the objects of the symbolic world, and representations that are the objects of the formal world. For example, in calculus, representations from the embodied world may take the form of a physical drawing of a tangent line to a curve, or a professor sweeping his or her arms in a particular way to demonstrate exponential rate of change. A calculus representation from the symbolic world may be in the form of the symbols that make up the language of calculus, such as dy/dx for derivatives or $\int$ for integration. A calculus representation from the formal world could be the definitions of limits, derivatives, or integrals.

It is also the case, as has been previously discussed and illustrated in Figure 1, that the different mathematical worlds have connections between them resulting in domains of two or more worlds. Since representations are the objects that correspond to each world and representations can be connected to one another, as was discussed above (e.g. Goldin and Shteingold, 2001; Hiebert and Carpenter, 1992; Pimm, 1995), it follows that these connected representations are the objects that exist in the domains of the connected mathematical worlds. These connected representations are demonstrated in Figure 2.

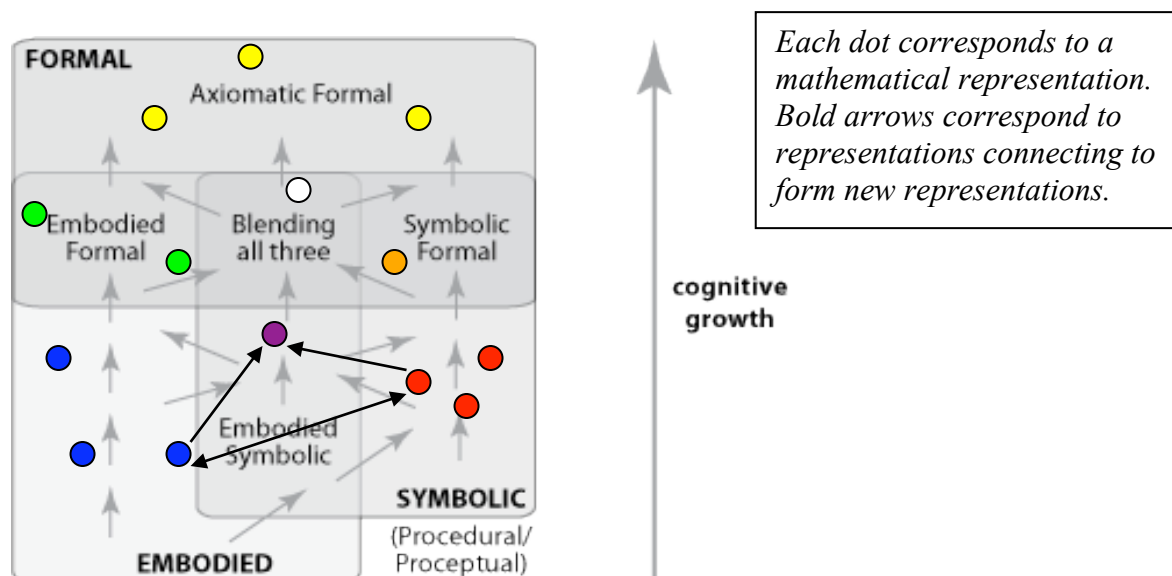


Figure 2. Representations within Tall's Three Worlds of Mathematics (adapted from Tall, 2009).

Examples of representations that correspond to a connected domain between mathematical worlds include Cartesian graphs. Cartesian graphs are used to embody or physically represent how a function changes as its values change. While a Cartesian graph incorporates features of embodiment, it also utilizes proper mathematical symbolism in the way the graph is constructed, thus a Cartesian graph is a connected embodied-symbolic representation. Other examples of representations that correspond to a connected domain between mathematical worlds are Euclidean geometric proofs. While Euclidean proofs incorporate formal definitions and axioms, pictures of geometric objects are used within the argument of the proofs. Thus, Euclidean proofs are examples of representations that are situated in the connected domain between the embodied and formal worlds.

Overall, an individual's ability to work simultaneously with representations from different worlds demonstrates that the individual is actually drawing some connections between the two worlds and working within the united domain of the

worlds (Goldin and Shteingold, 2001; Hiebert and Carpenter, 1992; Pimm, 1995; Tall, 2008). Thus, it is possible that representation connections can demonstrate an individual's development of concepts and the connection between the concepts.

Summary and Perspective

Tall's theory claims that mathematical cognitive development of concepts occurs in embodied, symbolic, and formal mathematical worlds and particularly as connections are formed between the mathematical worlds. In order to investigate an individual's understanding of mathematical concepts in terms of the Three Worlds of Mathematics and the connections between them, I chose to look at the representations the individual uses. Different representations are the objects of the different mathematical worlds or the connected domains between mathematical worlds and all representations provide different assumptions and information about mathematics.

Connecting representation use to the mathematical worlds has previously been described in Tall's theoretical work where representations are mathematical objects (Tall, 2004a, 2004b, 2008). Connecting representation use to the worlds has also been used in several studies (e.g. Nogueira de Lima & Tall, 2006; Stewart & Thomas, 2007) as well as in my pilot study (Smart, 2010), which will be explained in more detail in the next chapter. In these studies, representations were described as belonging to different mathematical worlds based on an interpretation of Tall's definitions of the objects of the worlds. This helped to describe students' connections between the worlds as well as their understanding of mathematical concepts. Combining the theory of Three Worlds of Mathematics with representations thus appears to be an appropriate way to investigate how students approach mathematical concepts, make connections, and

develop an understanding of the concepts. The models in Figure 1 and Figure 2 also support the perspective of combining Tall's theory with a theory of mathematical representations as they illustrate how the three worlds do not act in isolation but instead the process of cognitive development includes connecting concepts and their representations from the three worlds.

This research focuses on the connections that students make between the embodied, symbolic, and formal worlds. This research will also concentrate only on external representations. While both internal and external representations can be situated within the Three Worlds, this research only examines external representations because they are the only representations visible to the researcher. In my pilot study (Smart, 2010), I looked at external representations that were written down in some form, spoken about, or physical objects such as manipulatives or calculators, which is consistent with how the external representations have been described within the theoretical literature (Dreyfus, 1991; Dufour-Janvier, Bednarz, and Belanger, 1987; Janvier, 1987). I focus on similar forms of external representations in this study.

CHAPTER THREE: LITERATURE REVIEW

In this review I examine three themes from the body of educational literature:

(a) research which has used the theoretical perspective of Tall's Three Worlds of Mathematics, (b) research on representation use in mathematics with a particular focus on connecting between representations, and (c) research that focused on calculus teaching and learning of limit and derivative concepts. These three themes were examined as they help describe and inform the major underlying ideas of this thesis. Since this thesis focuses on Tall's theory of cognitive development, I analyze and synthesize the previous studies that have also used Tall's theory as their primary theoretical perspective. By reviewing these studies I was able to explore how empirical research that used Tall's theory could inform my research in terms of research design, implementation, findings, and analysis. I also identify how the studies that used Tall's theory have not addressed the topic that this thesis studies, emphasizing the significance of this research. I then review empirical works that focused on connections between multiple representations. These studies were reviewed because a perspective of mathematical representation and connections between representations was layered with Tall's theory to form the theoretical perspective of this thesis. These studies were also able to inform my research in a similar manner to the studies that used Tall's theory. Lastly, because the research question of this thesis focuses on the specific calculus topics of limits and derivatives, I provide a review of research on students' understanding of these topics. The review of research on limits and derivatives focused on studies that showed that students struggle with these topics regardless of the variety of different methodologies that have been used, and that the topics of limits and

derivatives can be situated within multiple mathematical worlds, according to Tall's theory, thus making them appropriate subjects to analyze according to Tall's perspective.

Tall's Three Worlds of Mathematics

Tall's theory of Three Worlds of Mathematics has been used as a framework for studies in mathematics education with respect to a variety of mathematical concepts. For example, Tall's theory has been used to examine the concepts of equations, vectors, and functions, to name a few. These studies have also used different research methods to explore areas like students' understanding of a concept, connections students make between worlds, and whether students' procedures are predominately embodied or symbolic. Of the studies that have used Tall's theory, they have all been conducted with participants at either the secondary school level or the university undergraduate level. In this section, I begin by describing and summarizing the results of the empirical studies. Each of the studies was influential to my thesis research and I take the opportunity to address how my research question, research design, and analysis method were all shaped in some way by this literature. I conclude by summarizing what I learned from the literature and provide some thoughts about how Tall's theory will be used in my thesis.

My review included a pair of companion studies that used Tall's Three Worlds of Mathematics as their primary theoretical framework (Stewart & Thomas, 2006, 2007). These studies looked at specific concepts within Linear Algebra at the university level. Stewart and Thomas (2006) began by looking at the way ten undergraduate students understood the concepts of eigenvalues and eigenvectors. They used Tall's theory to

study the embodied, symbolic, and formal aspects (Tall, 2004a, 2004b) of mathematics task questions on eigenvalues and eigenvectors completed by the students. According to the researchers, the task questions were not designed to assess course progress, but rather, the students' understanding of eigenvectors and eigenvalues as it could be situated in Tall's Three Worlds. In order to analyze students' understanding, the solutions to the task questions were corresponded to one or more of Tall's world(s). This analysis was then used to suggest that a student's understanding of a task question was developing within the corresponded world(s). It was concluded by the researchers, among other results, that the students who understood the concepts of eigenvalues and eigenvectors from both an embodied and symbolic perspective were better able to conduct the procedural calculations in the symbolic world. The researchers attribute this to the embodied world allowing students to help build their thinking, while those who only have a symbolic understanding of the concepts tend to show a weakness in the procedural calculations. These findings support Tall's theory, which suggests that a full understanding of a mathematical concept begins with the embodied world and that those individuals with only a symbolic understanding will struggle with applying the mathematical concepts appropriately (Tall 2004a, 2004b, 2008). This study influenced my research by providing an example of research at the undergraduate level that employed a research method using task questions. However, while Stewart and Thomas (2006) designed their study to look at where students' understanding could be situated in Tall's Three Worlds, my study design also intended to look at connections students made. This differs in that Stewart and Thomas (2006) situated the students' understanding in one or more worlds, while my study planned to describe the students'

understanding based on the connections the students made between the mathematical worlds.

Employing a slightly modified methodological design, Stewart and Thomas (2007) later used Tall's theory to examine undergraduate students' thinking and understanding of the concepts of linear combinations, span, and subspaces within linear algebra. In this study, ten students first attended supplementary linear algebra tutorials at the end of the course, which aimed to provide the students with an embodied, symbolic, and formal explanation for the concepts of linear combinations, span, and subspaces. The students were given a set of task questions during the tutorials designed to assess their existing thinking on the topics in terms of the embodied, symbolic, and formal worlds (Tall, 2004a, 2004b), similar to Stewart and Thomas's previous study (2006). The researchers also conducted interviews with two of the students. The task questions and interviews were analyzed to determine in which worlds of mathematics the students' thinking was situated and how this affected performance on task questions. The researchers found that those students whose answers to the task question and interview responses could be situated in the embodied world showed better abilities at connecting between the embodied and symbolic world as well as providing correct answers to the task questions. Again, this result supports Tall's theory (Tall 2004a, 2004b, 2008) that claims that being able to connect between the embodied and symbolic worlds of a concept contributes to successfully being able to solve problems related to the concept. The results also align with the researchers' findings from their previous study (2006). In this study the researchers adapted their previous research method to include both tutorials and some

student interviews along with the task questions. For my thesis research I also included the use of interviews with task questions. However I chose not to include tutorials of any kind. This was because I wanted to examine the connections students made between mathematical worlds without any supplementary instruction or teaching on my part, so as not to influence the participants' understanding. Stewart and Thomas' research (2006, 2007) also differs from my research in that they specifically looked at correctness of the task question. While correctness of solutions has been suggested by Tall as being related to the connections individuals make between worlds, for my study I chose to focus more on detailing how connections are made between the worlds. Looking at correctness was not necessary in order to address the research question of my study. However, I did look at correctness as one step of the data analysis, in order to examine if it did offer more information about how connections were being made between mathematical worlds.

Another pair of companion studies that employed Tall's theory also intrigued me. Different from the Stewart and Thomas's research, these studies examined the concept of equations with students at the secondary school level. One study focused on how secondary school students interpret the concept of equation and its solution (Nogueira de Lima & Tall, 2006). The researchers examined 77 15-16 year old students' development of the concept of equations in terms of the connections students form between the embodied and symbolic worlds of mathematics (Tall 2004a, 2004b). It was theorized before the study took place that individuals' understandings of equations develop from an embodied perspective and symbolic perspective in parallel, although in different ways for different students. The data were collected from questionnaires

concerning the concept of equations, written equation-solving tasks, and interviews with selected students. In the analysis of the data, the information from the questionnaires or equation-solving tasks and the interview were linked to Tall's Three Worlds according to Tall's definitions (Tall 2004a, 2004b). The researchers found that the students in this study exhibited very little or no embodied development, as well as poor connections from the symbolic to embodied world. They also found that the students significantly struggled with the concept of equivalence between two sides of an equation. This is significant, as Tall's theory suggests that to completely develop an understanding of a concept it is necessary to develop both an embodied and symbolic understanding of the concept. In this study, the researchers concluded that the students lacked an embodied understanding because it did not feature in the experiences of the students. In other words, it was speculated that the students did not study or use the concept of equations from the embodied perspective. As a result the students exhibited limited procedural knowledge that was "fragile and prone to error" (Nogueira de Lima & Tall, 2006). This study was influential to my research as it again provided an example of research that used both task questions and interviews. This study also used questionnaires, which I decided were not necessary for my work as I could use the interview to ask questions that would have been included in a questionnaire. As well, the suggestion by the researchers that the students' experience or lack of experience played a role in the findings was also influential in directing the design of my research.

The companion study by the same researchers was conducted to understand secondary school students' procedures as they worked through solving algebraic linear equations. The goal of the research was to see if the students solved the equations from

an embodied perspective or from a symbolic perspective, which was predominately taught to the students (Nogueira de Lima & Tall, 2008). This study involved 68 15-16 year old students from the same demographic as the companion study (Nogueira de Lima & Tall, 2006). For this research Tall's theoretical framework of Three Worlds of Mathematics (Tall 2004a, 2004b) was used to analyze what the students wrote in response to task questions involving algebraic problems, and spoke about in follow-up interviews. The study found that a significant number of students did not speak about the procedures of solving algebraic equations in terms of the algebraic principles from a symbolic-proceptual perspective, but instead talked about what the researchers termed as procedural embodiment of the process. For example, the students would often speak about moving symbols or picking up numbers and putting them on the other side. The researcher found that the students would often conform to a method of solving linear equations, instead of looking for a value of the unknown to satisfy the given equations. From this, the researchers concluded that the students were working through procedural embodiment, where they had developed ways of solving using a method they were confident with, without needing a reason about why it works. It was determined that the students were not yet developing connections but in a process of transition between embodiment and symbolism in terms of algebraic equations. Comparable to its companion study, this research also used task questions and follow-up interviews similar to what I planned to use for my research. The results of this study also helped me design my interview questions and carry out my data analysis. In particular, the finding that students conformed to a specific method for solving, whether it was the most appropriate procedure to take, prompted me to ensure I asked

students during the interviews why they chose to solve problems in certain ways and ensured that I paid attention to the procedures used during my analysis.

Tall's theory has also been used as the primary theoretical framework for single studies that explored mathematical concepts with strong embodied, symbolic, and formal perspectives. At the secondary school level, a group of researchers conducted a study with two sets of students on their development of an understanding of the concept of vectors in terms of Tall's Three Worlds of Mathematics (Watson, Spyrou, & Tall, 2003). The participants consisted of 23 16-17 years old and 26 17-18 year old high school students, respectively. The choice of the two groups differing by one school year was to see if students developed different understandings based on their previous experience with vectors. In particular, the 17-18 year old group had more experience with vectors than the 16-17 year old group. The researchers' interest in vectors was based on the fact that the concept offers distinct aspects from the embodied world and the symbolic world, depending on the context. For example, a vector is embodied as a quantity having a magnitude and direction, with real-world applications to physics and mechanics as a force, a velocity, or an acceleration. Alternatively, a vector situated in the symbolic world begins as a translation in a plane or three dimensions and represented as column matrices. The data collected in this study consisted of lesson observations, standard class tests to assess progress and ability, conceptual questionnaires on the concept of vectors, and clinical interviews with students and teacher. The researchers "triangulated" the data by analyzing the textbooks the students used, interviewing the teachers on their expectations and the students on their knowledge, and particularly emphasizing the results on the conceptual

questionnaire. Overall, the results from these many data sources showed that having a physical embodiment of the mathematical concept of vectors, such as visually interpreting vectors when adding or translating them, was beneficial to both developing an understanding of the concept and successfully solving problems involving vectors. Of significance was the finding that this result did not differ considerably based on the school year of the students. It was also concluded that a strongly developed understanding of the concept of vectors resides in the intersection between the embodied and symbolic worlds of mathematics. This research was undertaken with more data sources than my thesis research. This was appropriate for the goals of the research, which wanted to take into consideration the participants' previous experience with the concept of vectors. My thesis takes into account the participants' experience from their own perspective and as a result, I felt it was only necessary to include data sources directly from the participants and not from class observations, teacher interviews, or textbooks, as this study has.

At the university level, researchers have examined how first year university students think about the concept of functions as situated in Tall's Three Worlds (Christou, Pitta-Pantazi, Souyoul, & Zachariades, 2005). For this research 236 students who were registered in calculus were administered a test that included ten task questions on functions. Three of the tasks were designed to investigate the students' abilities in the embodied world, three others aimed to investigate the students' abilities in the symbolic world, and four aimed to investigate the students' abilities in the formal world (Tall, 2004a, 2004b). The researchers hypothesized that because the questions each related to a particular mathematical world, the solutions would present distinct

representations. This was confirmed in the analysis and it was concluded that these particular mathematical tasks and potentially similar tasks could be categorized based on Tall's worlds. It was also found that students who could successfully complete the embodied and symbolic tasks were 50% more successful at completing the formal tasks, thus suggesting that the students may have been making some connections that led them to be more successful on the formal tasks. This result supports Tall's theory that claims that development of a better and more complete understanding of a mathematical concept occurs when connections are made between the mathematical worlds. This finding also supports the research question of my thesis as an appropriate topic to investigate further. Overall, this study was particularly influential in how I designed the task questions for my thesis research, as it demonstrated that task questions could be written to prompt solutions from different mathematical worlds. The results of this research were taken into consideration when designing the questions for this study, which will be addressed in more detail in the following chapter.

The pilot study to this research (Smart, 2010) also used Tall's theory as the main theoretical framework to examine a student's development of calculus concepts and connections in terms of the Three Worlds of Mathematics. The study included a single participant completing ten task-based calculus questions and participating in a semi-structured interview. The participant had recently completed a university level introductory calculus course. The combination of the two data sources was used to see if both would be beneficial in addressing the research question. During the data collection, the participant first completed the ten task-based calculus questions, which

included the calculus concepts of derivative, limits, and integrals. An interview was then conducted with the participant, which focused on the participant describing his answers to the task questions. The task questions were initially analyzed according to which mathematical worlds the representations fit into. The interview was analyzed according to the comments made about how the participant solved the task questions and were then compared to the results from the task questions. The results of this study suggested that the participant had developed the calculus concepts symbolically and that there was very little embodied development. The participant also demonstrated hesitation and lack of confidence in solving calculus questions in general. Overall, the results of this study warranted further investigation. The pilot study also significantly influenced the methodology of this thesis as it specifically helped develop the research question, which asked about the connections between the embodied and symbolic worlds of only two calculus concepts. The pilot study also demonstrated that ten calculus based task-questions could take a participant a long time to complete and that it was probably more appropriate to have fewer task questions that focused on the two calculus concepts that I wanted to examine in this thesis. From the pilot study I was also able to gauge that the design of collecting data from task-questions and an interview would be appropriate in addressing the research question.

Summary

In general, the studies helped reviewed above inform my research as they all made use of Tall's Three Worlds of Mathematics as a framework for situating the findings of the research. Tall's theory appears to be a valuable framework to investigate mathematical development and understanding (Smart, 2010; Stewart & Thomas, 2006,

2007; Watson, Spyrou, & Tall, 2003), how individuals makes connections between different worlds (Nogueira de Lima & Tall, 2006; Smart 2010), what worlds the students' procedures are situated in (Nogueira de Lima & Tall, 2008), and how students perform on task questions situated in different worlds (Christou, Pitta-Pantazi, Souyoul, & Zachariades, 2005). Therefore, since Tall's theory was successfully used in this variety of research scenarios, it seemed appropriate that Tall's theory is the theory employed in this thesis research, which planned to look at how students make connections between the mathematical worlds of the calculus concepts of limits and derivatives.

These studies also informed my research in terms of what data I wanted to collect, and how the data would be analyzed. Each study used some form of task questions as either their only part (Christou, Pitta-Pantazi, Souyoul, & Zachariades, 2005; Stewart & Thomas, 2006) or one part of their data collection. Other studies included interviews with participants (Nogueira de Lima & Tall, 2006, 2008; Smart, 2010; Stewart & Thomas, 2007; Watson, Spyrou, & Tall, 2003) and in one case even interviewed teachers and analyzed the textbooks as well (Watson, Spyrou, & Tall, 2003). By reviewing these studies and considering the goals of my research, it became apparent that data in the forms of task questions and interviews with participants would most likely be appropriate to address my research question. The above studies also influenced my data analysis method. In each case the researchers, in their analysis, examined the external representations students wrote on task questions (Christou, Pitta-Pantazi, Souyoul, & Zachariades, 2005; Nogueira de Lima & Tall, 2006, 2008; Smart, 2010; Stewart & Thomas, 2006, 2007) or spoke about during an interview

(Nogueira de Lima & Tall, 2006, 2008; Smart, 2010; Stewart & Thomas, 2008; Watson, Spyrou, & Tall, 2003) and made judgments as to what world(s) the external representations corresponded to, based on Tall's own descriptions of the worlds (2004a, 2004b, 2008). This analysis method informed my research and I also chose to use a similar method of examining external representations and corresponding them to Tall's world in my thesis study. Consequently, literature on mathematical representations and in particular connections between external representations has been analyzed and is reported on in the following section of this chapter.

These studies also helped situate this research by demonstrating that the concepts of limits and derivatives have not yet, as far as I know, been examined from the perspective of Tall's Three Worlds of Mathematics. As well, the concepts of limits and derivatives can be described similar to how Watson, Spyrou, & Tall (2003) described the concept of vectors in their study, which is a concept that offers distinct aspects from the embodied world or the symbolic world, depending on the context. Thus, the concepts of limits and derivatives also have distinct aspects in the different worlds, depending on the context, and are concepts that can readily be analyzed according to the framework of the Three Worlds.

Last and most significantly, the studies that looked at students' understanding of different mathematical concepts (Christou, Pitta-Pantazi, Souyoul, & Zachariades, 2005; Smart, 2010; Stewart & Thomas, 2006, 2007; Watson, Spyrou, & Tall, 2003) all found that individuals who could connect the concepts between the embodied and symbolic worlds were more successful problem solvers. This result helped to inform this thesis research by supporting the need for further investigation on the topic of connections

between mathematical worlds. However, while these studies all demonstrated that connecting between the embodied and symbolic worlds was beneficial, none of the studies described in detail how the participants were making the connections. This is a significant disadvantage as there is no way for educators to know how to influence or replicate the connections students make if there is no information on how the connecting takes place. As was mentioned in the Introduction, I investigated specifically *how* the participants were making connections. This increased level of detail, which no other research has yet to consider, may prove helpful in eventually influencing practice and consequently helping other students to make connections between different mathematics worlds as connecting seems to be a significant factor in developing an understanding of a mathematical concept.

Mathematical Representations and Connections

The topic of mathematical representations has been covered quite extensively in both theoretical and empirical research. In this section, I focus on empirical studies that pay particular attention to connections between mathematical representations. Specifically, I review studies that look at mathematical concepts that play a significant role in calculus. As well, the mathematical concepts covered are concepts that offer a variety of mathematical representations, which correspond to Tall's different worlds, as described in Chapter 2. Reviewing studies that deal with the connections between mathematical representations not only helps to clarify what work has been done in this area but also helps to guide and support the methodology of this study.

Several studies focus on the multiple representations and the connections between the representations of the concept of functions, which is fundamental to the

study of calculus (Even, 1998; Hitt, 1998; Knuth, 2000). As well, the concept of function allows for multiple representations that can be corresponded to different mathematical worlds (Tall 2004a, 2004b). Hitt (1998) investigated how 30 mathematics teachers at the secondary level connected between different types of representations of functions. This research utilized task questions designed to explore what difficulties the teachers had with function representations and connecting between them. Data was collected in the form of 14 task questionnaires over 7 consecutive weeks, with two task questionnaires per week, which were analyzed qualitatively for trends that appeared in the results. This study found that one of the biggest challenges the mathematics teachers had was with preserving the meaning of the function representations that were connected. More specifically, the teachers struggled with going from graphical or physical contexts to constructing the algebraic function. It was also found that some forms of function representations were more difficult to articulate and thus, less likely to be shown, such as the physical context where a function describes a natural event like the velocity of a car over the course of a trip. I speculate that if the representations in this study were situated within Tall's Three Worlds of Mathematics, it would demonstrate that these mathematics teachers struggled with connections between the embodied and symbolic worlds, which may likely be a result of this thesis research as well. This study was also informative to my research as it was an example of a study that successfully employed the use of task questions as a means to collect data on how the participants made connections between multiple representations of functions. This provides further support to the use of task questions in my thesis.

At the postsecondary level, one study on functions looked at the abilities of 152

college mathematics students to connect between different representations within the same problem (Even, 1998). Even argued that the ability to connect and move between representations allows for better conceptual understanding and stronger problem solving skills, which has also been argued by Tall's theory (2008). Even employed the use of open-ended questionnaires that used non-standard mathematics problems, with the goal of addressing several features of the participants' function understanding. It was assumed that non-standard questions would better demonstrate what Even termed the students' "conceptions' boundaries", or the boundaries of the students' understanding of the concepts. The study also included follow-up interviews with 10 of the participating students, with the goal to allow those 10 students to speak about how they answered the questionnaire. It was found that the students in the study had difficulties linking different representations of functions, such as point-wise functions and understanding the role of the parameters in different representations of quadratic functions. The researcher speculated that this might be because many of the students demonstrated that they think about functions in a point-wise manner and do not focus on how functions behave over intervals or globally. This study supports the conclusion that connecting between function representations can be a challenging task, which could effect how successful students would be on calculus tasks. As well, the use of qualitative interviews as part of the study helped the research develop more detail about the questionnaire responses and is an example of the type of qualitative interview I planned to use. This research demonstrated the depth of detail a qualitative interview can provide and thus supported my use of a similar interview method for this thesis.

The concept of functions as it is used in calculus is often introduced at the secondary school level. Knuth (2000) looked at multiple representations of functions and the abilities of secondary school students to connect between the algebraic and geometric representations of functions. Data was collected from 178 secondary school students through the use of problem questionnaires administered in class, which required the students to connect between the algebraic and graphical representations of functions. The students' responses were classified as either algebraic solutions or graphical solutions according to Knuth's criteria. Knuth found that the students exhibited an overwhelming reliance on algebraic representations of functions, even for tasks that would have been more appropriately solved geometrically. It was speculated that this might be a result of algebraic representations being emphasized more strongly in the classroom setting. Despite the reasons for the reliance on algebraic function representations, the students in this study also demonstrated that it was challenging to connect multiple representations of functions. This study offered many informative aspects to my research. Knuth's use of problem questions is similar to a portion of my data collection instruments. Knuth's analysis of the representations that categorizes them as either algebraic or geometric is comparable to the intentions of my study to categorize the representations according to Tall's Three Worlds of Mathematics. Lastly, one of Knuth's findings that suggest students may use one type of representation over another depending on what is emphasized in the classroom setting provides support for questions I included in the interview portion of the data collection, which asks the participants what representations were emphasized in their calculus classes.

Overall, these three studies were beneficial in informing my research, as they are

examples of different approaches to examining connections between multiple representations of the concept of functions, which as was stated, is a fundamental concept of calculus. They all concluded that the development of connections between function representations is a very challenging task, regardless of level of education (Even, 1998; Hitt, 1998; Knuth, 2000). They also provide support to my use of tasks and interviews as similar methods were used in these studies.

I also examined studies that looked at multiple representations of mathematical concepts that could be situated within Tall's Three Worlds, in order to see if there were similar results as those studies that specifically examined functions (Herbert, 2009; Mesquita, 1998). Overall, it was concluded that connecting between multiple representations can be problematic for students, regardless of the mathematical concept (Even, 1998; Herbert, 2009; Hitt, 1998; Knuth, 2000; Mesquita, 1998). For instance, Mesquita (1998) investigated 14 year-old students' connections between the geometric representations of diagrams and figures to mathematical problems, and whether these connections contributed to students' geometric reasoning and problem solving. The research looked at the results of approximately 300 students' answers to a problem with a misleading illustration. It was found that without proper knowledge or understanding of the connections between the problem and the geometric figure, the geometric representations can act as a cognitive obstacle to the students' problem solving. This study suggests that it is necessary to use correct diagrams or figures when connecting to a problem. Diagrams and figures are used frequently in calculus to describe what is taking place in the problem and can be situated in Tall's embodied world of mathematics. Thus, I propose that diagrams and figures offer a way for

students to connect calculus concepts between the embodied and symbolic worlds. The findings of this research are informative as they suggest that inaccurate diagrams and figures can actually be a hindrance to the students' problem solving. Thus for my own thesis research it is important to observe whether the participants are using correct diagrams or figures and how the connections are made between them and the representations situated in the symbolic world.

In another study, Herbert (2009) investigated students' connections between multiple representations of the calculus topic of rate of change. I found this study of particular interest as it reinforces the idea that calculus is a topic that allows for multiple representation use. In this case, 25 students in secondary school were interviewed about the concept of rate of change. It was apparent from the results of this study that the students had difficulties with multiple representations and struggled with connecting between algebraic and geometric representations of the concept of rate. This was concluded from the interview evidence that found that the participants did not have an understanding of the connection between the rules and the rate the area was changing at. Herbert's use of interviews further informs and supports my use of interviews as part of my data collection, as Herbert demonstrated that interviews with appropriate questioning can likely ascertain the details about how a student understands a problem, not just how they solved the problem, something I hoped to study. As well, the researcher's results that students struggle connecting between algebraic and geometric representations may be situated in Tall's Three Worlds to demonstrate students' difficulty connecting between the embodied and symbolic worlds.

In summary, the results of these studies support the idea that students are able to make some connections between representations on familiar or routine problems, but that for many students these connections seem to be underdeveloped (Herbert, 2009), or superficial (Knuth, 2000). These studies all support the claim that connecting between representations can be a challenging process for learners, which further supports the value of my research. As well, these studies all demonstrate difficulty connecting between representations that may be situated in Tall's embodied and symbolic worlds, if Tall's theory were to be applied to the results. For example, Hitt's (1998) results that suggest mathematics teachers struggle articulating and connecting to the physical context of the concept of functions could demonstrate a weak embodied understanding of the concept. Even's (1998) findings that demonstrate the students in her study struggled to connect the algebraic representation of functions to the appropriate graphical representation could be situated in Tall's theory to suggest that the students are struggling to connect between the embodied and symbolic worlds in terms of the concept of functions. Knuth (2000) specifically looked at how students connected between geometric and algebraic representation of functions and analyzed the results by categorizing the representations, which could also be situated in Tall's theory as examining the connections between representations from the embodied and symbolic worlds. Knuth's results of students' reliance on algebraic representations could be the result of poor connections between the symbolic world and other mathematical worlds. Thus, from my interpretation, if Tall's theory were to be applied to the results of any of the above-mentioned studies, they demonstrate difficulty connecting between the embodied and symbolic worlds. These results offer support for

the theoretical perspective of my research, which layers Tall's theory of Three Worlds of Mathematics with a theory of multiple representations.

Even though the findings discussed above suggest that developing connections is a challenging task, some studies I reviewed on multiple representation have found that those who can connect between different representations gain more knowledge and understanding about concepts (e.g. Ainsworth, Bibby, & Wood, 2002; Davis, 2007; Benner, Herman, Ho, & Zimmer, 1999) and develop improved problem solving skills (e.g. Benner, Herman, Ho, & Zimmer; Panaoura, Gagatsis, Deliyianni, & Elia, 2009). These findings are consistent with the studies using Tall's theory, in that individuals that could connect representations between Tall's worlds demonstrated better problem solving and a deeper understanding of the concepts (e.g. Stewart & Thomas, 2006, 2007). Tall's theory also stresses the importance of developing connections between the mathematical worlds with respect to different concepts. This is important to my research as it provides support for the theoretical perspective of this study. It also offers backing for this study as it confirms that individuals who can make connections are better problem solvers and hence, it would be logical to argue that students who could make connections between calculus concepts might be more successful in calculus.

These studies also use research methods, which, in part, are similar to the research method of this thesis and therefore offer support to the methods used in my study. Many of the studies used tasks that students were asked to complete (Even, 1998; Hitt, 1998; Knuth, 2000; Mesquita, 1998). Some used interviews (Even, 1998; Herbert, 2009), which helped to provide more detail of why the students solved

problems in certain ways and how they understood the topics, not just the processes they used. My use of combining tasks and interviews, similar to Even (1998), helped to provide nice detail of the processes the students used, which helps to answer the research question of this thesis. Overall, all of these studies inform my research as they reinforce the theory that connections between and within mathematical concepts are important for cognitive development of the concepts and that the methodology my research employed would be appropriate to look at these connections.

Mathematics Education and Calculus

There have been many studies in mathematics education on the topic of students' understanding of calculus and how students struggle to understand calculus concepts. In reviewing literature on students' understanding of calculus the concepts of limits and derivatives emerge as essential concepts of calculus and areas that students tend to have difficulty with (e.g., Prezenioslo, 2004; Sierpinska, 1987; Szydlik 2000). The concepts of limits and derivatives also appear to be worth a closer examination from the perspective of Tall's Three Worlds as they are both concepts that can be situated in multiple mathematical worlds with strong connections between the worlds. In this section I present a review of studies that specifically examined students' difficulties understanding the concepts of limits of derivatives with the goal of demonstrating how this literature informed my research. This literature informed my study by providing information when I developed the task questions for the data collection, and reinforced the value of including clinical interviews in the data collection. It also demonstrated the roles that multiple representations played in the calculus concepts as well as the students' understanding. Lastly, it provided

information on why and how students struggle with the calculus concepts of limits and derivatives.

Limit Concept

In this section I pay particular attention to studies that focused on how students understand the calculus concept of limits and common misconceptions that students have with their understanding. This identified areas of difficulty and differences in understanding and informed my design of task questions on limits and derivatives and eventually assisted in the categorization of the representations students use on limit task questions according to Tall's Three Worlds. Studies were examined that looked at students' understanding at both the high school and undergraduate level.

A study that influenced my work was Williams (1991), which involved administering a questionnaire about limits to 341 undergraduate university students in the hopes of assessing how the students understood the concept of limits and the different ways these students modeled this understanding. The primary purpose of the questionnaire was to determine a way to classify students' views about limits. Williams developed categories of classifications from the results of the questionnaire and hence, the students' understanding, according to whether limits were understood and modeled as (a) dynamic-theoretical, (b) acting as a boundary, (c) formal, (d) unreachable, (e) an approximation, and (f) dynamic-practical. Williams found, among other results, that the majority of students who participated in the questionnaire held an informal understanding of limits and were resistant to changing their understanding to a more formal view. Williams also found that there was significant confusion about whether limits are dynamic processes or static objects. For instance, there was

confusion about whether a limit is the process of moving infinitely closer to a point but never reaching it, which was considered a dynamic understanding, or whether a limit had a specific real number value similar to functions, which was considered a static understanding. This difference is significant since viewing limits as dynamic processes could be associated with Tall's embodied world, while limits as static objects could be associated with the symbolic world. These findings also influenced the design of my task questions on limits. More specifically, I designed one task question where the limits had specific real number answers and one task question where the limits were infinity. This was purposely done to give participants the opportunity to undertake limit task questions that could be associated with both the embodied and the symbolic worlds of mathematics. It is also the case that Williams' process of developing categories for his findings influenced and validated my process that used a similar method.

Prior to developing the theory of Three Worlds of Mathematics, Tall along with a partner researcher Vinner, undertook a study that looked at how 70 undergraduate university students solved limits and their definitions of limits (Tall & Vinner, 1981). After collecting and analyzing data from a questionnaire, Tall and Vinner (1981) found that only four students were able to give a correct formal definition of a limit, while the majority of students described a definition of a limit as a dynamic process of moving infinitely closer to a point but never reaching it. This is similar to Williams' (1991) findings and as was stated above, is important as it demonstrates that the students' understanding of limits could be situated within Tall's (2004a, 2004b) embodied world. Tall and Vinner also found that more students could correctly explain how to solve a

limit than give a definition of a limit. However, 54 out of 70 students used and described how to solve a limit with a dynamic process. Again, this could be situated in Tall's (2004a, 2004b) theory to imply that the students' understanding of the procedure of how to solve limits was most likely situated in the embodied world. The fact that students were not able to provide the proper definition may demonstrate incomplete or no understanding of limits in the formal world. Overall, Tall and Vinner concluded that the majority of students had developed a *concept image* of limits as a dynamic process, where *concept image* is defined as all of the cognitive structures in a student's mind that is associated with a given concept. Therefore, it could be described that the majority of students held cognitive structures of limits as dynamic processes. Overall, this study had very similar conclusions to Williams' study (1991) and is informative in a similar way as it reflects how the limit concept can be situated in Tall's (2004a, 2004b) three worlds and that the students' understanding can be positioned in the embodied world of mathematics.

Tall and Vinner's (1981) theory of concept images of limits was also used in a study by Przenioslo (2004), who collected data from 238 university students over 9 years, to determine how they understood the concept of limits. In other words, by using Tall and Vinner's definition of *concept image*, Przenioslo's research described students' understanding as their cognitive structures of concept of limits. Data was collected through written tests, observations of groups of students, and interviews with individual students. From analyzing all of this data, which included a categorization process similar to Williams (1991), it was concluded that there were five different classes of concept images of limits. This research also found many of the conceptions

the students had about limits had been formed in secondary school and thus students brought this previous understanding with them as they tried to develop a more formal concept image. As a result, while students could correctly state the official definition of a limit, they were often unable to recognize contradictions between the definition and their own personal concept images. This study is another demonstration of the difficulty students have with developing proper cognitive structures of limits, or a proper understanding. This study was also another example of an analysis process that categorized the data, similar to the process I undertook in this thesis, and which demonstrates that categorizing data can be an appropriate method for developing an account of the information provided from these types of data sources. As well, the finding that students' understanding is influenced by their previous experience with the concept of limits from secondary school influenced the interview protocol of this thesis that asked students specific questions about their calculus courses and how calculus was taught. This was in order to see how the students in my study were influenced by what they experienced.

Two studies reviewed expanded the focus beyond students' understanding to also include students' beliefs about mathematics and the concept of limits. In one case, through previous work analyzing historical and experimental research, Sierpiska (1987) identified four epistemological or knowledge-based obstacles related to how individuals develop a proper understanding of limits. These obstacles were: scientific knowledge or the belief about what mathematics is and what it involves, the concept of infinity or the belief that infinity exists only potentially, the understanding that functions are reduced to a set of values, and a lack of a uniform concept of real

numbers. Using the list of epistemological obstacles, the researcher conducted five study sessions with 17 volunteer students, aged 16 and 17 with the aim to help the students overcome these obstacles. It was concluded from the sessions that while most students started to overcome the epistemological obstacles of their conceptions of infinity, functions, and real numbers, the hardest obstacle to overcome was the idea of scientific knowledge and the students' beliefs of validity and truth in mathematics. Sierpinska found that mathematics was viewed as objective, impersonal knowledge, not something that could be used to validate or prove mathematical truths.

Szydluk (2000) also examined students' beliefs in mathematics and how those beliefs related to their understanding of limits. The researcher used questionnaires and interviews to investigate how 27 university students' beliefs about real numbers, infinity, functions, and validity and truth in mathematics related to their beliefs about limit concepts and how this affected their ability to solve limit problems. Szydluk also paid attention to what the students relied on when developing their beliefs about limits. It was found that some students relied on an authority like an instructor or the textbook, termed an external source of conviction, while other students relied on empirical evidence and logic, termed an internal source of conviction. Overall, the majority of students in this study held incoherent or misplaced beliefs and misconceptions about limits while at the same time relying on external sources of conviction for their beliefs. These beliefs and external sources of conviction in turn negatively affected their ability to solve limit problems. This research is informative in that it suggests that students' beliefs about mathematics and what they rely on to develop those beliefs can affect their problem solving ability on limit calculus problems.

These studies that looked at beliefs were informative as they prompted me to want to investigate *why* a participant might answer a limit task question in a particular way, as the answer may be influenced by what he or she believes about limits, or mathematics overall. As such Sierpinska's (1987) and Szydlik's (2000) studies supported the importance of including interviews with participants as part of the data collection so that a student's beliefs would come forward. This helps to fully investigate the research question of this thesis, and specifically, to accurately describe how the participants were making connections between the mathematical worlds as they solved the task questions.

In summary, these studies agree that students have difficulty with the limit concept in calculus; regardless of whether the level of calculus they have studied is high school or undergraduate at university (Przenioslo, 2004; Sierpinska, 1987; Szydlik, 2000; Tall & Vinner, 1981; Williams, 1991). Overall, the difficulties with the limit concept can be summarized to suggest that students develop the concept erroneously due to prior experience (Prezenioslo, 2004; Sierpinska, 1987; Szydlik 2000), or due to misconceptions about the concept being a dynamic process or static object (Prezenioslo, 2004; Szydlik, 2000; Williams 1991). As well, difficulties in understanding the limit concept could be due to the epistemological obstacles or beliefs that result from concepts related to limits, such as real numbers, infinity, and functions (Sierpinska, 1987; Szydlik, 2000). Regardless of the reasons, the research on limits suggests that students often develop inconsistent or incomplete understanding of the limit concept and its definition (Sierpinska, 1987; Tall & Vinner, 1981). William and Prezenioslo also demonstrate that the concept of limits can be situated within Tall's

different worlds, with an understanding of limits as a dynamic process or a static object being situated in the embodied and symbolic worlds respectively. This research lends support to the use of Tall's theory when examining the concept of Limits.

Derivative Concept

In this last section I describe research that has examined the calculus concept of derivatives and identified it as a concept students struggle to understand. A derivative can be defined as the slope of a tangent line to a curve at a single point. As this point on the curve changes, the slope of the tangent line also changes, and is termed the *rate of change*. Consequently, derivatives, rate of change, and slope are all mathematical concepts that can be represented visually or graphically, as well as symbolically and formally.

Because these concepts naturally allow for visual representations in tandem with symbolic representations, studies have been conducted that have explored how students understand the concepts of derivatives, slope, and rate of change from a visual and symbolic perspective. In one case a group of researchers looked at how a student's understanding of the imagery of the concept of slope and rate of change, and derivatives, can be a hindrance to understanding the concept symbolically if the mental images are incorrect (Aspinwall, Shaw, & Presmeg, 1997). In this study a single participant undertook 20 derivative based tasks in a clinical interview setting. The tasks consisted of problems that required both a procedural understanding and graphical and image understanding of derivatives to solve. It was found that while the participant was able to apply differentiation rules to solve derivatives, he had misconceptions of the graphical and visual representations of functions, derivative

functions, and tangent lines. In particular, the participant struggled throughout the tasks with a misunderstanding of the nature of asymptotes, which hindered his ability to explore derivatives and rate of change in graphical representations. This study could apply Tall's theory of Three Worlds of Mathematics with image task questions being considered embodied, the task question requiring only differentiation rules being symbolic, and the graphical task question being considered in the connected domain between the embodied and symbolic worlds. This was helpful as I was designing the derivative task questions for my study in that I paid particular attention to whether the task questions could be situated in one mathematical world or another, or if they were in the connected domain between worlds, which was my goal.

A similar study also looked at students' visual understanding of rate of change and how it related to their usage of the derivative rules (Herbert, 2009). This research involved 25 high school students with data collected through the use of individual clinical interviews. The main focus of the interviews was to investigate the participants' understanding of rate of change on two questions that were developed in Geometer's Sketch Pad™. The significant difference between the two questions was that one represented a situation with a constant rate of change and the other represented a situation with a variable rate of change. By using Geometers Sketch Pad™ to model the two scenarios Herbert was able to explore the participants' understanding of rate of change using the multiple representations of tables, graphs, and images, as well as symbolic procedures. In this case it was also found that while many students had procedural competence with problems involving rate of change such as using the derivative rules, they tended to exhibit poor understanding of the rules' connections

with image and visual representations of rate of change on the derivatives. The researcher concluded that this was because the students in this study did not have an understanding of derivatives beyond the algebraic rules and that to develop a more complete understanding attention needed to be paid to familiarizing students with multiple representations of the concept, such as graphs or images, as well as connecting the representations. This study is significant as it demonstrates that the concept of derivatives does allow for multiple representation use, which can be influenced by the students' understanding of the concept. It also shows that poor understanding of the multiple representations could be a factor in whether students use multiple representations when solving derivative problems, which provides support for my theoretical framework that links representation use with Tall's Three Worlds and students' understanding.

Exploring students' understanding from a visual and symbolic perspective, as is done in these two studies (Aspinwall, Shaw, & Presmeg, 1997; Herbert, 2009), is of particular interest to my thesis research as the two perspectives provide verification for the argument that derivatives can have multiple representations from different mathematical worlds. These works were also influential as they provided examples of research methods that used clinical interviews to gather detailed information about how the participants solved the task questions. This provided support for my research method as I hoped to collect comprehensive information from my participants in order to address the thesis research question.

There has also been research that has employed the use of teaching interventions to explore and improve students' understanding of the concepts of rate of

change and derivatives. This demonstrates that researchers and educators have recognized a deficit in students' understanding of this concept and have explored improving understanding of derivatives through teaching interventions. In one such study, Thompson (1994) explored how 19 senior university and university graduated mathematics students understood concept of derivatives with particular attention paid to rate of change and how the students understand how this concept applies to the Fundamental Theorem of Calculus (Thompson, 1994). This highlights that an understanding of the concept of rate of change is essential to understanding the Fundamental Theorem of Calculus, a necessary theorem for university senior mathematics students and university mathematics graduates. The data was collected through videotaping the students participating in 11 1.5-hour teaching interventions over five weeks. The teaching intervention was divided into four phases, each designed to stimulate conceptual development. The videotapes were analyzed for how the students' conceptual development progressed throughout the teaching interventions. It was found, among other results, that the students' difficulties with the Fundamental Theorem of Calculus stemmed from weak conceptions and multiple representations of rate of change and how it relates to derivative rules. In particular, the students struggled with poorly developed and poorly connected image representations of functional variation. This research impressed upon me that even mathematics students at the highest undergraduate level still have difficulty with the concept of derivatives, which seems to be influenced by their understanding of the multiple representations of the concept.

Another study used a teaching intervention, periodic testing and interviews, to

examine first year university calculus students' understanding of rate of change and derivatives (White & Mitchelmore, 1996). Unlike Thompson (1994), this study was conducted with students who had not yet taken university calculus and had only experienced some pre-calculus topics in secondary school, such as functions and slope. The researchers used the setting of a university calculus course to conduct their study in which the students were given calculus-based task questions four times throughout the course in order to examine their understanding of derivatives and rate of change and how their understanding developed as a result of the particular teaching intervention that took place during the course. The researchers found that between the second, third, and fourth tests the students' ability to correctly model the task questions and receive correct solutions greatly improved, thus concluding that the teaching was a positive factor. However, overall, the researchers found that almost 50% of the students still struggled to properly model and correctly answer the derivative and rate of change questions. It was determined that this may be the result of an underdevelopment of the general relationship between functions, derivatives, and rate of change. I found this result of 50% of students still struggling with the derivative concept as supportive of my argument that despite the research and teaching interventions that have previously taken place, derivatives are still a difficult concept for students.

In a similar study, although again with different age group, Bloch (2003) employed a teaching intervention, with 35 17-year old secondary school students, with the aim to help develop a better understanding of calculus and particularly, derivatives and rate of change. This particular intervention took place in a regular mathematics

classroom setting and focused on presenting the subjects of functions, derivatives, rate of change, and slope, within a graphical setting as well as with symbolic representations. The goal was to observe whether emphasizing these concepts from a graphical perspective would be beneficial in leading the students to link the knowledge to a more formal understanding of calculus. At the end of the teaching intervention, the students were administered a test along with 105 students of the same school year who had not participated in the teaching intervention, which acted as a control group. The most significant finding of the study was that the students who participated in the teaching intervention showed more mathematical knowledge about the concepts of derivatives, rate of change, and slope, and were able to use graphical and image representations in their calculus problem solving, as well as differentiation rules, thus offering more varied ways of investigation.

These studies that employed teaching interventions (Bloch, 2003; Thompson, 1994; White & Mitchelmore, 1996) are all informative to my research as examples of students who struggle because of limited understanding of the multiple representations of derivatives as well as examples of students who were more successful because of the intervention that provided a better understanding of the multiple representations of the concept. In particular, these studies demonstrated that students who can connect image representations to algebraic and symbolic representations of derivatives were more successful at solving derivative problems and had a better understanding of the concept. This is informative as it supports the theoretical perspective of this study that looks at the use of multiple representations, situated in Tall's different worlds of mathematics, as well as for connections between the representations, to describe the

participants' understanding of the derivative concept. As well, because of the importance that making connections between multiple representations appears to have in these research studies, I chose to design my thesis to look at the connections students make in order to develop a more complete comprehension on how connections influence understanding.

Overall, this research on the understanding of the concept of derivatives has found that students tend to struggle with conceptions of slope and rate of change as they relate to derivatives (e.g. Herbert, 2009; White & Mitchelmore, 1996). These poorly formed concepts of rate of change and slope result in their limited ability to successfully model or solve derivative calculus problems (Herbert, 2009; Thompson, 1994; White & Mitchelmore, 1996). The research also suggests that an understanding and knowledge of slope and rate of change can contribute significantly to student success with the concept of derivative (e.g. Bloch, 2003; Thompson, 1994). However, while some studies concluded that connecting the concepts to graphical or image representations did improve understanding, this was limited (Bloch, 2003) and in some cases how the students visualized the derivative functions, or rate of change, was actually a "hindrance" to the students' development of the concepts (Aspinwall, Shaw, & Presmeg, 1997). Thus, it is indicated that the concept of derivatives, and particularly rate of change and slope, is an area that causes students significant difficulty.

CHAPTER FOUR: RESEARCH METHOD

The purpose of this research is to investigate how undergraduate students make connections between the embodied, symbolic, and formal mathematical worlds with respect to the calculus concepts of limits and derivatives. In this chapter I begin by presenting the research question that guided this study, and then discuss the research design, instrument design, and data collection methods, that helped to address the research question. This chapter then concludes with a detailed outline of the data analysis methods.

Research Question

The design of this thesis was initially led by the following question:

How do undergraduate calculus students make connections between the embodied-conceptual and symbolic-proceptual worlds of the calculus concepts of limits and

derivatives? However, as was mentioned in the Introduction and as will be demonstrated in the Data Analysis described in this chapter, it became apparent that some participants were actually connecting between the embodied and formal worlds and the symbolic and formal worlds. As such, the research question that was actually answered from this design and thesis research was: *How do undergraduate calculus students make connections between the embodied-conceptual, symbolic-proceptual, and formal-axiomatic worlds of the calculus concepts of limits and derivatives?*

Research Design

To explore the research question, the design of the study used two types of data collection: (a) four task-based calculus questions and (b) a semi-structured interview.

The four calculus-based task questions include two questions on limits and two

questions on derivatives. The tasks were designed such that solving the questions might involve multiple representation use from different mathematical worlds and connections between different mathematical worlds. Prior to the data collection taking place, I wrote potential solutions to the task questions. This was done to demonstrate and confirm that the questions did allow for multiple representations from different mathematical worlds as well as connections between the worlds. The calculus courses from which participants were recruited all cover the concepts of limits and derivatives that were included in the task questions and thus, participants would have experience with these concepts (personal communication with program coordinator, April 20, 2010).

The semi-structured interview included some introductory general questions about the participants' past and current educational experience and status. There were then seven open-ended questions that covered the participants' impressions about taking calculus in general and about the calculus task questions they had just completed, as well as explored participants' answers to and reasoning about the calculus task questions. The interview concluded with some questions about the participants' background in order to better describe the participant population set.

The research design proposed inviting at least 15 and up to 20 participants who were currently enrolled in or had just completed a first year calculus course at university. The goal was that every participant would complete both the four task-based calculus questions and a semi-structured interview. This would result in 60-80 written samples and 15-20 interviews for an in-depth analysis that would provide a large enough sample, while still manageable, for meaningful analysis.

Participants were to be recruited from six courses, which were all considered first year university calculus courses (personal communication with program coordinator, April 20, 2010). The differences between the courses were that two were for math/engineering/physics majors, two were for business commerce and business finance majors, and two were for life sciences majors. The content of the courses only differed in terms of the context of the real world examples (personal communication, April 20, 2010). The concepts of limits and derivatives were all to be covered in a similar manner in the first year calculus courses.

Overall, the research design was influenced by some of the previous literature that also included calculus task questions and a semi-structured interview (e.g. Nogueira de Lima & Tall, 2006, 2008; Smart, 2010; Stewart & Thomas, 2007; Watson, Spyrou, & Tall, 2003) and was modeled particularly after my pilot study (Smart, 2010). The results of the pilot study helped me to refine the design of this study to ensure that the research question would be appropriately addressed. From the task questions in the pilot study I was able to observe what representations the participant chose to use on limit and derivative questions. From the semi-structured interview in the pilot study I was able to observe how the participant spoke about his understanding of the calculus concepts of limits and derivatives and how that understanding may have influenced the representations used and connections made on the task questions. The interview in the pilot study also provided more explanation about the participant's thinking as he discussed his solutions to the problems. Taken together, the two data sources allowed me to draw conclusions with regards to the research question in the pilot study. The design of the research presented in this thesis is very similar to the

pilot study as it also uses task questions and interviews. The differences between the two studies in terms of the research design are that some of the task and semi-structured interview questions were modified, some were eliminated, and new ones were added to the research study reported here. I will discuss some of the similarities and differences in the next section.

Instrument Design

As was mentioned above, the instruments designed and used in this research included four calculus-based task questions (Appendix A) and nine semi-structured interview questions (Appendix B). In this section I present the four calculus-based task questions and demonstrate how each of the task questions aligns with the goals of this research by allowing for multiple representation in the solutions. I then describe how these task questions compare to the ones used in my pilot study, and reflect on the previous literature that influenced the task question design. This section also provides an overview of the nine semi-structured interview questions, details about why the questions were included in the interview, and how they help address the research the question.

Calculus-Based Task Questions

The four calculus-based task questions used in this study are presented below in the order they are posed to the participants during the data collection. However, it should be noted that the participants could choose the order to answer the task questions. Question #1 is as follows:

Question 1.

At what point(s) of $\mathbf{x^2 + 2y^2 = 1}$ does the tangent line have a slope of 1? Show your work and explain your answer.

Prior to the data collection taking place, potential solutions to all of the task questions were written up (Appendix C) to confirm that the questions allowed for multiple representations from different mathematical worlds as well as connections between the worlds. From these potential solutions it can be seen that Question #1 may be solved by first recognizing the equation as an ellipse on a Cartesian graph, with a centre at (0, 0), an x-axis centre to edge distance equal to 1, and a y-axis centre to edge distance equal to $1/\sqrt{2}$. With this information, it is possible to sketch the graph of the equation on graphing paper, which provides a graphical representation. The value of this representation is that from the sketched graph and by understanding that the slope of a curve at a point is the slope of a tangent to the curve at that point, it is possible to see that there are two points on the ellipse that have a slope = 1; one in the second quadrant and one in the fourth quadrant. Recognizing this information from the graphical representation is valuable as it demonstrates that there are two correct answers to Question #1.

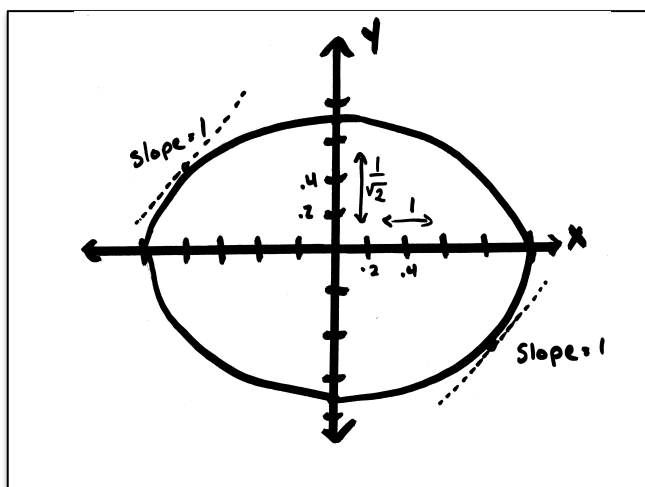


Figure 3. Hand drawn picture of equation and approximate points where slope of tangent = 1.

Another way to begin this question could involve rearranging the equation to equal y and plugging the arrangement into a graphing calculator. This also presents a graphical representation but is an alternative to graphing the equation on paper. Similar to the hand drawn graph, this representation on the graphing calculator could also allow for recognition that there are two points that have a solution to this problem. The graphing calculator representation would look like:

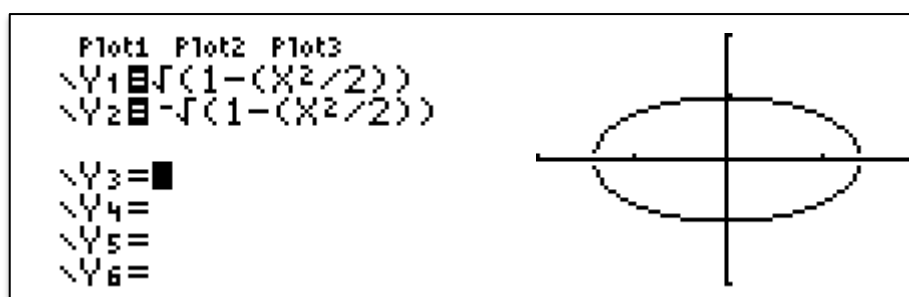


Figure 4. Graphing calculator input and graphed screen on a TI-83.

For Question #1, graphing the equation, either on paper or on a graphing calculator, is not a necessary step for solving the problem. The solution may in fact be

found by taking the derivative, requiring numeral and calculus symbolic representations. The derivative can be found by either using implicit differentiation on the equation as it is presented in the question, or rearranging the equation to equal y and solving with the use of the differentiation rules and algebraic manipulation. Overall, Question #1 allows for solutions that include graphical representations, symbolic representations, or a combination of both.

Question #2 is as follows:

Question 2.

Given the piecewise function

$$g(x) = \begin{cases} x & \text{if } x < 1 \\ 3 & \text{if } x = 1 \\ 2 - x^2 & \text{if } 1 < x \leq 2 \\ x - 3 & \text{if } x > 2 \end{cases}$$

Find the limits:

$$\lim_{x \rightarrow 1^-} g(x)$$

$$\lim_{x \rightarrow 1} g(x)$$

$$\lim_{x \rightarrow 2^-} g(x)$$

$$\lim_{x \rightarrow 2^+} g(x)$$

$$\lim_{x \rightarrow 2} g(x)$$

Show your work and explain your process and thinking.

For Question #2, the potential solutions (Appendix C) demonstrate that a

solution to this problem may be found by graphing the piecewise function on graphing paper in order to have a visual/graphical representation of the function.

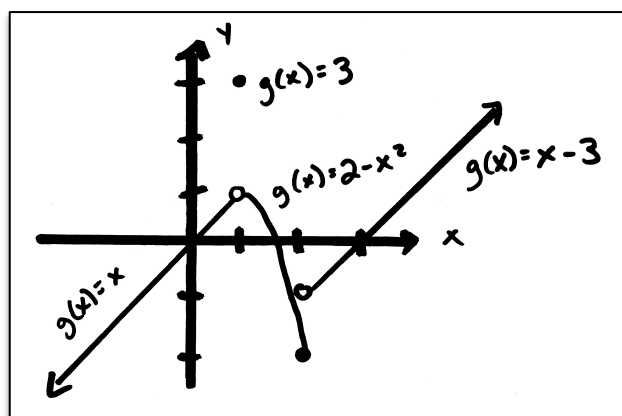


Figure 5. Hand drawn picture of piecewise function.

If a clear and accurate graphical representation was produced then the desired limits could be determined by examining the graph on the appropriate intervals or at the appropriate points. Therefore, using only a graphical representation, the answers to the limits could be found. Alternatively, this question could be solved by assigning $g(x)$ in each limit to be the part of the piecewise function for the region that the limit is to be found. For example, $\lim_{x \rightarrow 1^-} g(x) = \lim_{x \rightarrow 1^-} x = 1$ since $g(x) = x$, if $x < 1$. The other parts of the piecewise function could then be disregarded when solving the particular limit. This second method would involve only the use of symbolic representations and an understanding of limits and inequality statements. It is also possible that a combination of both of these methods could be used, resulting in a solution that uses a combination of the graphical and symbolic representations.

Question #3 is as follow:

Question 3.

Given the function $f(x) = 3x^5 - 5x^3$

On what interval(s) is the function increasing?

On what interval(s) is the function decreasing?

What is(are) the local maximum(s)?

What is(are) the local minimum(s)?

Show your work and explain your process and thinking.

For Question #3, this question could be solved by first using a graphing calculator to visualize the graph. Entering the function into a graphing calculator would produce the following images.

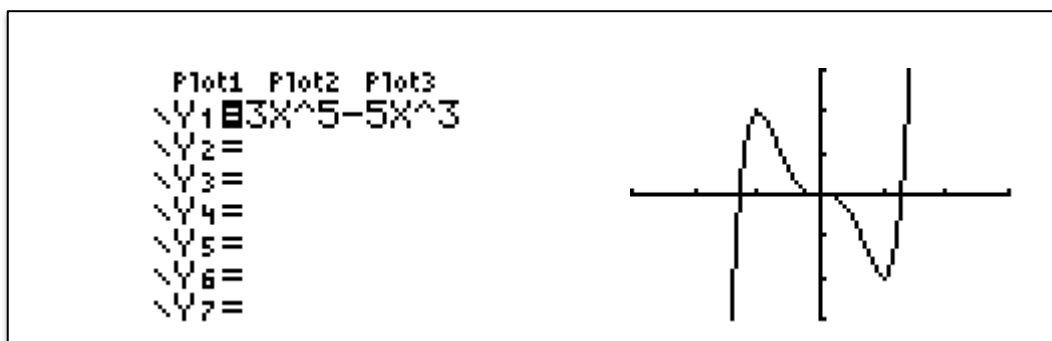


Figure 6. Graphing calculator input and graphed screen on a TI-83.

From the graphical representation it could be seen that there is definitely one local maximum, one local minimum, and different intervals of increasing and decreasing. A series of steps could then be undertaken with the graphing calculator to determine what these exact minimum and maximum values and intervals of increasing and decreasing are. This solution method does not actually require any calculus knowledge and only the use of a graphing calculator and the representations it

presents.

Another way to solve Question #3 is to use algebraic symbolic representations of the function and find the first derivative in order to determine the intervals and maximum and minimum values. The first derivative of a function represents the slope of the function at a particular point. By setting the derivative of the function equal to 0 and then solving for the x-values, it can be determined where the function has a slope of 0, or where the function levels off. This is significant as a function has to level off or have a slope of 0 when it changes from increasing to decreasing and thus, there would be a maximum or minimum value at that point.

Question #4 is as follows:

Question 4.

Given the function $f(x) = x - \frac{1}{(x^2)(x+2)}$

Find:

$$\lim_{x \rightarrow -2^+} f(x)$$

$$\lim_{x \rightarrow -2^-} f(x)$$

$$\lim_{x \rightarrow 0} f(x)$$

Show your work and explain your process and thinking.

Similar to the other questions, a solution for Question #4 may also begin by graphing the function, in this case on a graphing calculator, in order to have a visual or

graphical representation. The figure below shows the function input and graph screen.

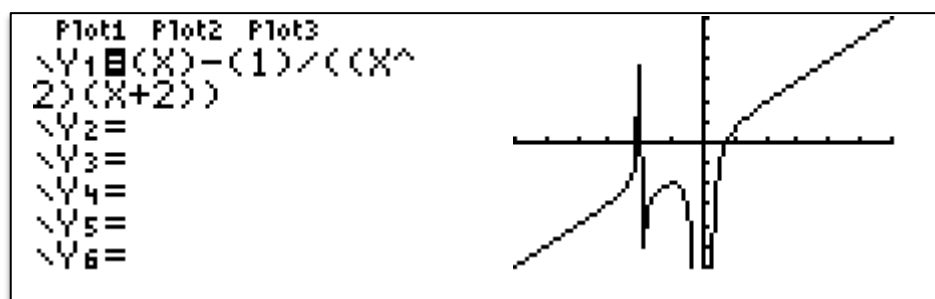


Figure 7. Taken from graphing TI 83 calculator of input screen and graph.

To use this graphical representation to solve the problem, familiarity with how the graphing calculator draws asymptotes for discontinuous functions would be needed. If this were the case then the desired limits would be able to be found by examining the graph. For example, the straight vertical line at $x = -2$ is how the graphing calculator draws an asymptote, indicating that the function does not exist at this point. As the function gets close to the point $x = -2$ on the right, the curve of the graph proceeds down to negative infinity. Similarly, as the function gets close to the point $x = -2$ on the left, the curve of the graph proceeds up to positive infinity. A similar method can be used by examining the asymptote on the graphical representation at $x = 0$.

Alternatively, a solution could proceed without a graphical representation and instead use symbolic representations and limit theorems to break up the function into two limits. For example, the first limit would result in one limit that is $\lim_{x \rightarrow -2+} x = -2$ and

one limit is $\lim_{x \rightarrow -2+} \frac{1}{(x^2)(x+2)} = \text{positive infinity}$. The process used to determine this

second limit is to imagine a number infinitely close to -2, but still slightly larger,

because the limit is approaching -2 from the right. In that case, $\frac{1}{(x^2)(x+2)}$ would result

in a fraction of one over an infinitely small positive number. Thus, $\lim_{x \rightarrow -2+} x - \lim_{x \rightarrow -2+} \frac{1}{(x^2)(x+2)} = -2 - \text{positive infinity} = \text{negative infinity}$. The other two limits can be found through a similar process. Overall, this second suggested solution requires only symbolic representations and an understanding of the concept of limits, limit theorems, and the nature of infinity.

It is also possible that solutions to these questions could use a combination of methods and apply multiple representations when solving. For example, a solution may begin with a graphical representation but then continue to find the answers by using symbolic representations. Once an answer is found it could be referred back to the graphical representation for accuracy. Alternatively, a solution could begin with symbolic representations and apply a graphical representation during the process or at the end of the process as a means to check correctness. These are just two examples of then many ways these questions may be solved with the possible use of multiple representations.

When designing and deciding which task questions to use for the study, I choose Questions #1 and #3, which required derivatives, as they were used in my pilot study (Smart, 2010) and were shown to be able to be solved with a variety of representations from multiple mathematical worlds. I then designed task Questions #2 and #4, with the help of reviewing numerous Calculus textbooks, as questions that required the use of limits to solve and in a similar manner to Questions #1 and #3 in that they could also be solved with a variety of representations from multiple mathematical worlds. Both the task questions (Appendix A) and the potential solution (Appendix C) were reviewed for

accuracy prior to the data collection taking place by another researcher with their own experience teaching calculus.

When designing and choosing tasks to use I also kept in mind the study by Christou, Pitta-Pantazi, Souyoul, and Zachariades (2005), which demonstrated that questions can be written in such a way to promote being solved with representations from one world over another, as I wanted to try to ensure my task questions avoided this restriction as much as possible. The choice of task questions was also influenced by my review of the literature on how students understand the concepts of limits and derivatives. Some studies that examined limits found that students think of limits as dynamic processes and static objects (Prezenioslo, 2004; Szydlik, 2000; Williams 1991). This influenced me to include two limit questions, one that has answers that are real number values like a static object, and one that has answers that are infinite like a dynamic object. In terms of the derivative questions, I made sure to include one question that required an understanding of slope and rate of change, Question #1, since this is identified in the literature as a derivative concept that was difficult for students but also requiring multiple representations (Aspinwall, Shaw, & Presmeg, 1997; Herbert, 2009). The other derivative question, Question #3, could involve modeling the function or using imagery representations and was influenced by the findings in Aspinwall, Shaw, and Presmeg (1997) and Bloch (2003).

Semi-Structured Interview

In regards to the semi-structured interview (Appendix B), the general questions (1a-1d) and the concluding questions (9a-9d) were designed to learn about the participants' background and current experiences in university to better describe the

individual participants and the overall population I was collecting data from. The seven open-ended questions were designed to encourage the participants to reconstruct and reflect on their experiences, since reconstruction “is based partially on memory and partially on what the participant now senses is important about the past event” (Seidman, 2006, p. 88). Specifically, questions 2-4 were designed to gain the participants’ impressions about the task questions, how well they felt they understood the questions, to determine if the task questions are similar to calculus tasks the participants had experienced, and to understand better the participants’ analysis and reasoning they used in solving the task questions. Questions 5-8 were designed to better uncover details about the participants’ understanding of calculus in general based on their personal perspectives and what they sense was important about calculus. Questions 2-8 assisted in addressing the research question by allowing me to see if a participant’s responses related to the representations a participant used and the connections made.

Data Collection

Data collection began with recruitment of participants through a variety of methods. Initially, I approached the Director of the Department of Mathematics and Statistics at the university where the research was to take place in order to gain access to calculus classes, to talk to potential participants about my study and to seek permission to display posters in the Department’s building. With these recruitment methods, I provided my contact information to potential participants and requested that they contact me if they were interested in participating. Once contacted by potential participants, I provided more information about the study. If they were

interested in participating, we arranged a mutually agreeable time and location for the interviews and tasks. I was confident that a sufficient number of participants would be recruited through these methods because of the volume of students who enrol in calculus courses each semester. For example, in the 2009-2010 school year there were 4779 undergraduate students enrolled in calculus courses at the university where the research was to take place (personal communication, April 20, 2010).

As was previously mentioned, the research was designed to have at least 15 participants, allowing for upwards of 20 participants, should more volunteer. This number was chosen in order to develop a sample large enough to offer some breadth to see similarities and differences, while at the same time limiting the data to a manageable size. The participants for this study were drawn from students who were near completion, or who had just completed a first year calculus course at the university where the research took place. Since they were nearing completion or had just completed a calculus course, if they were successful in calculus they should have had the skills required to attempt the limit and derivative task questions and be able to reflect on their problem solving. Further descriptions and information about the participants is covered in the Findings chapter of the thesis.

Through the above-mentioned recruitment methods 16 calculus students volunteered and I was able to arrange times to meet with each of them. Of the 16 calculus students who volunteered, 15 showed up at the arranged times and participated fully in the study. One participant did not show up at the arranged time and did not respond to my email invitations to reschedule the interview. Thus, data collection was completed with 15 participants.

Prior to meeting with each participant, I set up a workspace on the table in the interview room of the mathematics education research lab so that the students would have a space to complete the tasks. The workspace included a task question booklet, two copies of the consent form, a variety of pencils, pens, and erasers, some extra blank, lined, and graph paper, and two calculators, one graphing and one scientific. The variety of supplies was meant to be comparative to what calculus students may normally have at their disposal when completing calculus questions. There were also two video cameras set up in the room that would be used during the interviews. One video camera faced the participant and me; the other was set up facing the task question booklet. The decision to have one camera focused on the participant was because when speaking about mathematics some people make hand gestures to explain what they mean, and I wanted to record these gestures or movements. The reason to have one camera focused on the participant's booklet was in order to have a record of what each person pointed to on the page or commented about on the page while they are speaking. The cameras remained off while the participants completed the tasks and were only turned on for the interview portion of the data collection. This was because I was not evaluating how the participants finished the questions and I did not want to cause any anxieties. The participants were also provided with a variety of snacks and drinks in the interview room that they could help themselves to.

Upon entering the research lab, each participant was shown the interview room that we would be working in. Each participant had been sent an information document by email prior to the scheduled interview informing him or her of the study and what he or she would be doing if they chose to participate. At this time I repeated a summary

of this information and asked the participant if he or she had any questions about the research process. After I answered all questions, we signed and dated two copies of the consent form.

The data collection process then began with the participants individually completing the four task-based calculus questions. I informed each participant that they had as much time as they needed. Each participant was then left alone in the interview room to complete the four task-based calculus questions, using any of the aids that had been set up in the workspace, or any aids he or she had brought, like a personal calculator. During this time I was in another room of the research lab and waited for the participant to come and get me when they were finished. No participants came to ask me questions while they completed the work. The tasks were designed to take approximately 30-40 minutes to complete all four, based on the experience of similarly designed questions used in my pilot study (Smart, 2010). The amount of time each participant used varied with a range of about 30-70 minutes, with an average of about 40 minutes.

Once a participant had completed the task questions, he or she was asked if they would like to take a break before continuing on with the interview portion of the study. Each participant declined the need for a break although a few did take the opportunity to help themselves to snacks at this point.

After the task questions were complete, I joined the participant and conducted the interview portion of the study (Appendix B). Each interview was video recorded with the participant's approval. The semi-structured interview was also designed to take approximately 30 minutes, again based on the experience of the similarly designed

semi-structured interview from my pilot study (Smart, 2010). According to Creswell (2005) a semi-structured interview is appropriate for an in-depth analysis of a particular issue, and as such was suitable for the goals of this study. Each interview followed the interview protocol without a lot of deviation. The only time a question in the interview protocol was skipped was when the participant had already answered what it was asking without prompting. If the order of the interview questions was changed it was because the discussion had led to a place in the conversation where it seemed appropriate to ask a specific question at that time. Each participant also answered questions on their background. Overall, the interviews took between 30-35 minutes for each participant.

After each interview, I thanked the participant for coming and asked them if they had any further questions. It was my impression that each participant left after the data collection without feeling any discomfort. At the end of each interview I placed the participant's booklets with the calculus questions, my copy of their consent form, and the interview protocol sheet I used during each interview, in a large envelope. On the envelope I put the participant's assigned identification number as well as the date and time of the interview. These envelopes were stored in a locked cabinet in the research lab in anticipation to the data analysis process.

During the data collection I strove to be aware and receptive to the participants' comfort and feelings. From my experience as a mathematics teacher and tutor I have become sensitive to recognizing when individuals feel uncomfortable with mathematics. If any of the participants showed evidence of this, I tried to reassure and comfort them by reminding them of the confidentiality of their participation. Finally, I

also offered each participant the opportunity to do member checks and review the transcripts from the interview in order to ensure they agree with that has been recorded (Creswell, 2005). No participants were interested in reviewing the interview transcripts.

Description of Participants

This section provides details about the participant population. It begins with a summary table of the background, and educational status of the participants who took part in this research study. I then describe and elaborate on the participant population.

Summary Table of Participants

Table 1

Summary of Participants Background, and Education Status

Participant	Age	Gender	Country of birth	Languages spoken	High school	Took intro to calculus in...	University Calculus course	Year of study
#1	24	Male	Canada	English French	Ontario	High School	Calculus for Business/ Finance	2nd
#2	18	Female	Canada	English French	Ontario	High School	Calculus for Commerce	1st
#3	18	Male	Canada	English French	Ontario	High School	Calculus for Life Science	1 st
#4	19	Female	China	Mandarin English	Ontario	High School	Calculus for Math/ Engineering	1st
#5	18	Male	Columbia	Spanish English French	Ontario	High school	Calculus for Math/ Engineering	1st

Table 1 (*continued*)

Summary of Participants Background, and Education Status

Participant	Age	Gender	Country of birth	Languages spoken	High school	Took Intro Calculus in...	University Calculus course	Year of study
#6	18	Male	Canada	English French	Ontario	High school	Calculus for Math/ Engineering	1st
#7	18	Female	Canada	French English	Quebec	University	Calculus for Math/ Engineering	1st
#8	19	Male	Canada	English	Ontario	University	Calculus for Commerce students	1st
#9	19	Female	Canada	English	Ontario	High School	Calculus for Life Science	1st
#10	18	Male	Romania	Romanian French English	Quebec	University	Calculus for Math/ Engineering	1st
#11	19	Female	Canada	French English	Ontario	High school	Calculus for Life Science	1st

Table 1 (*continued*)

Summary of Participants Background, and Education Status

Participant	Age	Gender	Country of birth	Languages spoken	High school	Took Intro Calculus in...	University Calculus course	Year of study
#12	23	Male	Rwanda	Swahili French Chichewa English	Malawi	University	Calculus for Math/ Engineering	1st
#13	19	Male	Canada	Farsi English	Ontario	High School	Calculus for Math/ Engineering	1st
#14	19	Female	Canada	English French German	Ontario	High School	Calculus for Math/ Engineering	2nd
#15	19	Female	Canada	English French	Ontario	High School	Calculus for Commerce	1st

From the table we can see that of the 15 participants, 8 were male and 7 were female. The participants came from a variety of backgrounds, with 12 participants reporting being Canadian by birth and 3 participants being born outside of Canada. English was the first language for only nine of the participants although all of the participants appeared to speak English fluently.

Participant #12 was the only participant to not have studied in Canada prior to university and was the only designated international student that took part in the study. The remaining 14 students had all completed high school in Canada, with 12 studying in Ontario and 2 studying in Quebec. Of the 12 participants who had completed high school in Ontario, 10 had taken the Ontario high school calculus course, Calculus and Vectors (Ministry of Education, 2007). The two participants who had studied in Quebec had not attended Collège d'enseignement général et professionnel (CEGEP), which is postsecondary preparatory college attended by the majority of individuals in Quebec who plan to continue in postsecondary studies.

The participants had also taken, or were taking at the time of data collection, a variety of different first year calculus courses, which included 4 participants from the calculus courses for business finance and business commerce majors, 9 participants from the calculus courses for mathematics/engineering/physics majors, and 2 participants from the calculus courses for life science majors. Overall, the participants were a diverse group.

Data Analysis Methods

The data analysis process for this research can be broken down into six distinct steps. Step #1 began with the organization of the data that was collected. Step #2 was

an analysis of the organized data according to each participant. Step #3 was an analysis of the organized data according to each task question. Step #4 was a further analysis designed to review the accuracy of the findings from Step #3. Step #5 was an analysis of each participant's solutions and answers for correctness and sophistication. Lastly, Step #6 was an analysis of other interesting findings that became evident during the previous five steps. These steps are hereafter detailed.

Step #1: Organization of Data

Organizing the data was undertaken with the goal of eventually developing a layered analysis of the information from the task question and the material from the interviews. This organizing began with transcribing the video-recorded interviews to have the information from the interviews in written documents. Each interview transcript was then summarized according to the main ideas. The organizing of the data then proceeded to summarize the written work according to what representations were used and the sequences they were used in while solving. Once this was completed, the two data sources were connected to develop a layered analysis. Full details of the processes undertaken at each of these organizing stages are described in the following sections.

Transcribing and summarizing the video-recorded interviews.

After completing the data collection with 15 participants the first step in my data analysis included transcribing the video-recorded interviews in order to have the interviews in written format for review and analysis. Each interview was transcribed verbatim by watching the recorded interviews.

I then read through the transcripts as I reviewed the video-recorded interview

and added comments to the interview transcripts, in italics, about hand gestures the participants used during the interviews. Gestures were included as they are a method of communication that can relate to mathematical thinking about different concepts. In determining how I would deal with gestures displayed during the interviews I used the categories developed by McNeill (2005). McNeill classifies gestures as: (a) *deixis*, (b) *metaphoricity*, (c) *iconicity* (d) *temporal highlighting*, and (e) *social interactivity*. Below I have included a brief definition and examples from the interview transcripts of instances of the first four classifications of gestures.

- Deixis – includes the gestures of pointing to objects, whether existing or imaginary, such as pointing while counting, or pointing to the object that is being referred to.

P4: Well, I can see, for example, from this one, when x ...when the limit of this function...when x approaches 1 from the left side, so I just go to 1 and approach from the left side (*points to the graph she had drawn*) and see that the function is just going to a value of 1 and the limit is 1 at that point.

- Metaphoricity – includes the gestures that present an abstract idea, such as, holding the index and thumb fingers closely together when describing a “short” temper, or sweeping your arm to the right when discussing increasing numbers, as though on a number line.

P4: And for the regular limit is where...well I kind of think of it this way. If you have a left limit going to some a and some right limit going to some number b and they are not equal so the limit does not exist at that point. Because the limit on the left side and the right side are not equal to each other. So the limit does not exist

at that point. (*moves one hand from the left and one hand from the right not meeting up*)

- Iconicity – which includes the gestures that are directly related to the semantic content of the speech, such as pretending to hold and move a steering wheel when speaking about driving.

P15: So like, if it is a circle it will be round (*traces the outline of a circle with finger*). And then on the line you can pick a point where it will...where you can find a tangent to.

- Temporal highlighting – which includes gestures that are used for emphasis or significance, such as when someone gestures time in the past by sweeping their hand behind them, or sweeps their arms wide when speaking about something large.

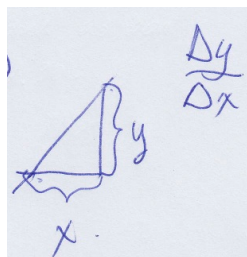
P1: I remember there was a question like this on one of the midterms and I didn't do so well on limits, so...(*waves hands in front of the paper to try to block what is seen*), I always struggle with limits.

The fifth category of gestures, or *social interactivity* gestures, which includes all gestures that mediate and modulate interactions during speech, was not recorded as I focused on the gestures that were communicating about the mathematics. The role gestures played in the analysis of the interviews will be discussed shortly.

Some participants also drew diagrams or solved equations in the task question booklets during the interview. This was to better explain what they were verbally describing. When this occurred during the interview it was also noted in the transcripts in italics. This was in order to be clearer about what the participants were speaking

about. For example, Participant 7 made the following statement and drew the included diagram during her interview:

P7: Cause...y' is the difference between...is dy over dx, which is the difference between y and x (*grabs a pen and write on page*) as ...this is the way I look at it...Is this what you want to know from me?



When the transcripts for the 15 interviews were finished, I reviewed them by reading the transcripts and watching/listening to the videos again to check for accuracy and to make sure there were no obvious typos, or spelling mistakes. I also hoped to begin familiarizing myself with the information in the transcripts. During this review, I decided to add in picture images of the participants' written work from the task questions so that when I came back to the transcripts later I would be able to quickly see what parts of his or her written work a participant was talking about. These images included the participants' written work completed in the first portion of the data collection and any diagrams drawn during the interview portion. I was able to accurately add the picture images of the written work because of the design of the interview protocol, in which I reviewed a participant's task question solutions with them one after another. As well, in situations where it was unclear from the interview videos, I was able to use the video recordings from the camera that was focused on the participant's booklet to correlate what a participant was saying to what he or she was referring to in the task question booklet. An example of what the transcripts with the

gestures in italics and these image additions looked like is included in Figure 8. Further examples are included in Appendix D.

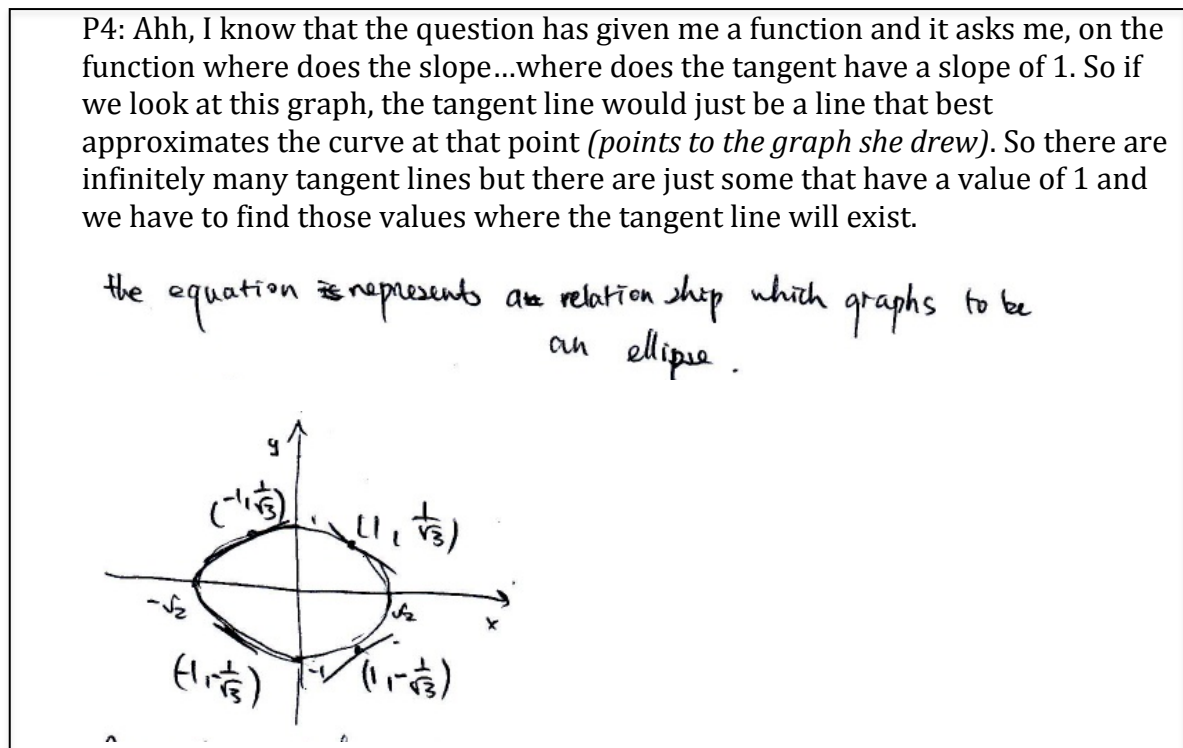


Figure 8. An example of a participant referring to her written work during the semi-structured interview.

Once all of the interviews were transcribed, gestures added in italics, and images from the written work were included, I summarized each participant's interview transcripts according to the main ideas they provided. In order to write the summaries of the transcripts, I began by reporting a participant's personal and descriptive information using the responses to questions 1 and question 9 from the interview protocol. Similarly, I was able to write summary paragraphs based on the other interview questions. As was mentioned in the instrument design, the interview questions were designed to address specific themes. For example, questions 2-3 and their potential follow-up questions were designed to better understand a participant's

impressions and how they felt about the task questions, as well as to inquire if he or she had seen similar task questions in his or her calculus class. In summarizing these responses I also included excerpts from the transcripts to support the summary. I used the responses from interview question 4, which asked a participant to describe his or her answers to each task question, to summarize the participant's responses to the four task questions. At this point, an analysis of the written work was not undertaken, as the summaries of the transcripts only included how the participants spoke about their task solutions. Different images of the participant's written work were also included in this portion of the summary, similar to how they were included in the interview transcripts. Lastly, questions 5-8 were designed to gain details about a participant's understanding of calculus, in general, based on his or her reflection of their personal experience of taking a calculus class. The responses to questions 5-8 were summarized into a paragraph describing whether the participant noticed any specific types of teaching techniques from his or her calculus professor, what the participant felt he or she did best and worst on in the course, and the participant's description of how graphs and diagrams were used in course, as well as the balance in the course between graphs and diagrams, algebraic manipulation, and technology. Excerpts from the transcripts were used to support the summary.

The purpose of summarizing the transcripts was to consolidate the main ideas from the interviews into easily accessible documents so that the same information from the 15 interviews would be available during a further analysis to compare the information from the interviews to the written work. A portion of an interview summary is provided in Figure 9 with further examples in Appendix E.

The interview then moved onto the discussing Participant #3's solutions to the task questions. With the first question, I asked him what he understood the question was asking him to do:

P3: It is asking me to find the slope of the tangent when the slope is 1. So you want the point coordinates of the original function when the tangent has a slope of 1.

A: And so what does it mean for the tangent line to have a slope of 1

P3: A tangent with a slope of 1 would be the line that touches the function once, like a very, very tiny point (*makes a little point between thumb and first finger*), would have a slope of 1, like increasing by 1 (*motions increasing with finger*).

He then discussed how he went about solving the question. He began by isolating for y so that he could take the derivative. He then set the derivative equal to 1. When asked why he set the derivative equal to 1 he said:

P3: Why do I set the derivative equal to 1? Because, the slope is 1 on the tangent line (*makes line with finger*), on our derivative function (*makes motion of a function*) the x coordinate will still be 1 so we are finding the "new y" coordinate, which is the slope of the tangent of the function (*makes curve motion with both hand*)...if that makes any sense.

Figure 9. An example of a portion of a summarized transcript.

Review and analysis of the written work.

The data analysis moved onto an analysis of the written work from the task questions. Up to this point, the written work had only been used to add image documents to the interview transcripts and the transcript summaries. For the analysis of the written work I began by describing the solution, from what I saw in the written work, for each question for each participant. Thus, there were 60 description/summaries completed. The descriptions were brief accounts of the different types of representations that appeared to have been used in the question, for

example, whether graphs were used or numerical and operator symbols. How the representations were described and defined will be described in more detail in the sections to come. The descriptions also included the length of the solution to a question, and generally what the solution looked like; for example, did the work look linear and organized. The summaries of the solutions included details about the perceived steps a participant followed when solving a question. Initially, this was done without consulting the interview, as a further analysis comparing the two data sources was planned for later. The purpose of describing and summarizing the written work was to have the information in a condensed document and to ensure I reviewed each question for each participant in a consistent manner, again providing me with parallel information for each participant. By describing and summarizing the written work I was able to ensure that I made similar observations to every question in this portion of the analysis. It also gave me the opportunity to become more familiar with the written work data.

Throughout the process of describing and summarizing the written work, and in keeping with the goals of the research and the research question, I paid particular attention to the representations that were used in the solutions. In order to organize the representations I was seeing, I began to make a working document that categorized representations used in the written responses with the help of Tall's Three Worlds of Mathematics. This document would later be named the *Categorization of Representations* document. Overall, the purpose of the *Categorization of Representations* document was to keep track of decisions I made about specific representations so that I could refer to it later and more importantly to ensure consistency in categorizing

representations.

The development of the *Categorization of Representations* document began by listing the different mathematical worlds that included the Embodied, Symbolic, and Formal worlds, as well as the connections between the worlds of Embodying-Symbolism, Symbolizing-Embodiment, Embodying-Formalism, Formalizing-Embodiment, Symbolizing- Formalism, and Formalizing-Symbolism.

For each category listed above, I added a description of the world and connections between worlds by reviewing the theoretical works on Tall's Three Worlds of Mathematics (2004a, 2004b, 2008). For example, for the Embodied world I added the description:

This world is defined as actions on visual or image representations of the mathematics. These action-image representations take the form of objects that are seen or sensed in the real world or imagined in the mind.

Examples:

- Shapes, tallies, diagram of problem, pointing up for increasing
- Curves not on graphical axes

Figure 10. Example of Embodied World descriptions in the Categorization of Representations document.

Similarly, the Symbolic world had the description:

This world is a combination of processes and concepts, which are termed procepts, that act together with symbolic systems of mathematics. (Procepts are described as single mental object made up of the encapsulation of a mathematical process that uses symbols and the relating concept.)

Examples:

- Any equations, functions, formulas, or inequalities, whether using algebra or only numbers and operators. Also includes any manipulation of equations, functions, formulas, or inequalities, from one form to another.
- Step-by-step procedures that only use symbolic algebra or numbers.

Figure 11. Example of Symbolic World descriptions in the Categorization of Representations document.

I then used these listed and described categories of the worlds as a guide in writing the summaries and descriptions of the students' written work where I identify the worlds, and connections between worlds, that different representations belonged to. I made note on the *Categorization of Representations* document of the allocation of representations as well as a brief justifications as to why a representation was allocated to a specific category. For example, under the Symbolizing-Embodiment heading was included:

Symbolizing-Embodiment

- Taking a curve and determining the function/formula/equation of the curve or solving the question from the graph

just looking at graph.

- representing circles or ellipses by algebraic equations

(knowing it's a sample figure, ellipse.)

- Number lines (represent the numbers in a structured order that reflects their quantified amounts)

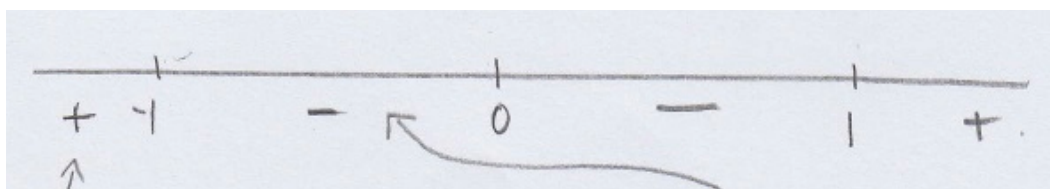


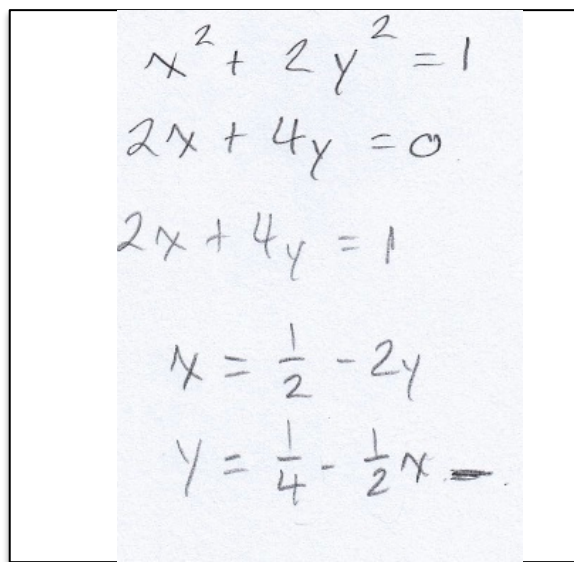
Figure 12. Example of Symbolizing-Embodiment in the *Categorization of Representations* document.

Overall, the development of the *Categorization of Representations* document and the written work summaries were completed in tandem to each other.

As was mentioned, in the original research design it was not anticipated that I would find any representations or indications of connections involving the formal world. This is because the formal world is not covered in a first year calculus course at university, but is in other more advanced calculus courses. However, during this process of describing and summarizing the written work while using the *Categorization*

of *Representations* document it became apparent that there were some instances of connections with the formal world, and hence the adaptation of the research question to include the formal world in the analysis.

The process of analyzing the written work also led to a realization that representations were used in different ways throughout a solution. For example, some representations or types of representations were used within the primary solution method, which means that the primary way of solving the problem used one type of representations more predominately than another. For instance, the example of written work below uses only symbolic representations.



The image shows a piece of paper with handwritten mathematical work. The equations are written in black ink on a light blue background. The equations are:

$$x^2 + 2y^2 = 1$$

$$2x + 4y = 0$$

$$2x + 4y = 1$$

$$x = \frac{1}{2} - 2y$$

$$y = \frac{1}{4} - \frac{1}{2}x =$$

Figure 13. Written work with only symbolic representations.

Alternatively, a solution could have also been made up of a number of different unique parts, with each part comprising of a different type of representation.

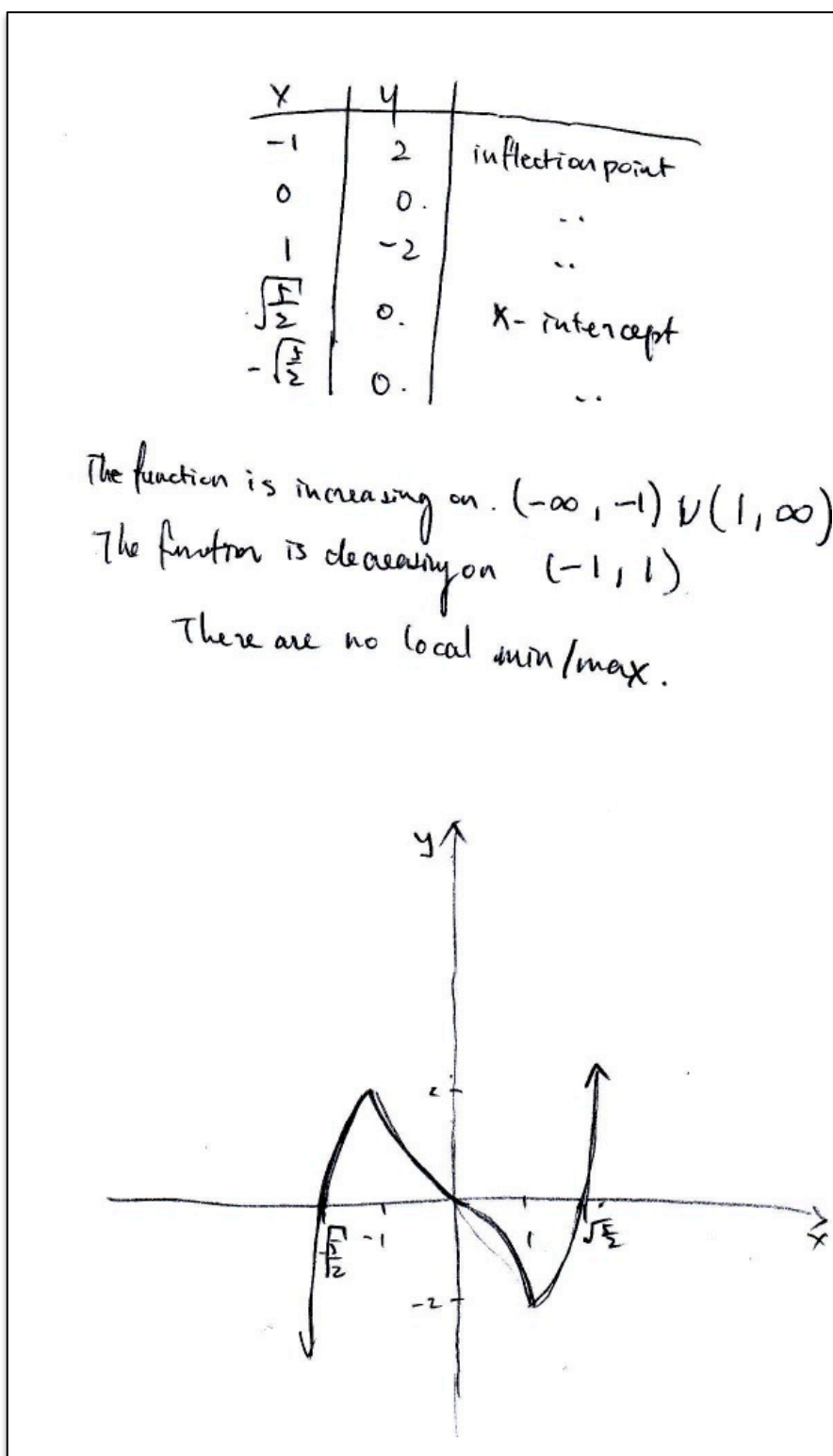


Figure 14. Written work with multiple representations.

From these observations I began a working document that eventually became the document entitled *Roles of Representations*, which comprised of five different roles

the representations played within the written work solutions. These roles included representations that were used as the primary part of a solution, representations that were used as a unique part within a solution with multiple types of representations, representations that do not appear to affect the solution, representations used as supplementary to check steps or solution, and representations drawn during the interview. A portion of the *Roles of Representations* document is included in Figure 15. In this example it shows what is meant by a representation that is used as a primary part of a solution. The entire document can be found in Appendix G.

Used as the primary part of the solution:

The designation of *primary part of the solution* will be given when a specific representation or a group of the same type of representations is used as the major method for the solution to the problem. For example, a solution that involves only algebraic manipulation of numbers and variables would be classified as using a Symbolic representation as the primary part of the solution. Another example is one where a student uses a graphical representation to find the answers as the primary part of the solution. This would be categorized as Embodied as the primary part. This was sometimes the case for question #2. A participant would draw a graph and then state that the solutions to the limits were found from looking at the graph. Although the answer statements are written with symbolic algebra, the primary part of the solution is the graphical representation.

Figure 15. An example of a category from the *Roles of Representations* document.

From the observations made while describing and summarizing the written work and the creation of the working documents *Categorizations of Representations* and *Roles of Representations*, I developed a table that was used to summarize a participant's answers to each of the four calculus task question (Figure 16). This table, referred to as the *Representation Table*, was created for each task question for each participant, resulting in 60 tables in total. The *Representation Table* considers both the representations that were used in the solutions as well as *how* the representations

were used. Both of these areas were important to address the research question, which asked *how* students made connections between representations, not just what representations were used. In the few situations where a participant's representations demonstrated connections with the formal world, a row was added to the bottom of the table to include the Formalizing-Embodiment or Formalizing-Symbolism representations, depending on what was needed. As this was a rare occurrence, this row was not regularly included in the *Representation Table*.

Representations	Used as the primary part of the solution	Used as a unique part within a solution	Does not appear to affect solution	Used as supplementary to check steps or solution	Drawn during the interview
Embodied					
Embodying-Symbolism					
Symbolizing-Embodiment					
Symbolic					

Figure 16. The *Representation Table* used in the organization step of the data analysis.

At this point I have referred to a number of documents and which I am including a table (Table 2) for the sake of simplicity. Table 2 names the documents, describes each document and also presents the purpose of the document. I will use these document names throughout the rest of my discussion of the analysis of the data.

Table 2

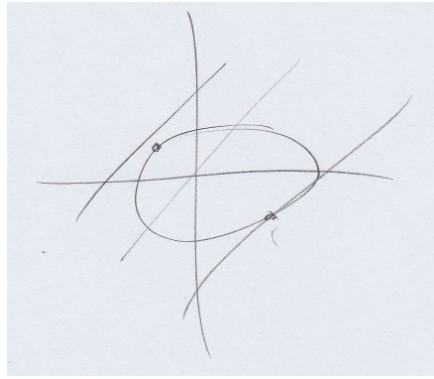
Documents Developed from the Initial Steps of the Data Analysis

Document	Description	Purpose
Interview Transcripts (Appendix D)	Interview typed verbatim with Italic notes for hand gestures used during the interview and information added to the written work during the interview. Also includes image documents of the participant's written work when referred to in the interview.	Initial step to organizing information from interview into workable documents.
Interview Transcript Summaries (Appendix E)	Summary of main ideas from interview into condensed paragraph form. Also includes image documents of the participant's written work when referred to in the interview.	Consolidate the main ideas from the interview transcripts into easily accessible documents.
Written Work Summaries and Analysis (Appendix H)	A description and summary of the solution for each question for each participant. The descriptions include the different types of representations that appeared to have been used in the question, the length of the solution to a question, and generally what the solution	To have the information in easy to use documents and to ensure a review of each question for each participant in a consistent manner.

	looked like. The summaries include more detail about the steps a participant followed when solving a question.	
Categorizations of Representations (Appendix F)	List of each of Tall's worlds and connections between worlds with definitions for each world and connected category. Examples of images of representations from written work added to the different categories.	To keep track of decisions made about specific representations so that it could be refer to later as well as to ensure consistency in categorizing the representations.
Roles of Representations (Appendix G)	A list of the different roles the representations played within the written work solutions as well as a detailed described of what each of the roles looks like so they could be identified from the written work.	To identify the different ways representations were being used in the written work, which will help address the research question of <i>how</i> students make connections.
Representation Table (Figure 16)	A table with the representations that were used in the written work solution in the first column as well as <i>how</i> the representations were used in the first row. The rest of the table is blank and allows for comments to be placed in the boxes.	To provide a visual depiction of both what representations are used and how they are used in the same space. One representation table was created for each participant's response to each task, resulting in 60 tables.

Development of Layered Analysis.

The next step of the analysis was to use the interview summaries to further analyze the written work. Comparing a participant's interview to his/her solutions to the task questions provides more information and aids in understanding of a participant's solution. For example, during my pilot study (Smart, 2010), the participant had a written solution with a correct graphical representation of the problem, suggesting a connection with an embodied understanding. However, when questioned during the interview about why he drew the graphical representation he responded, "I don't really know, that's what my teacher always did." This example demonstrates that the written response could have been misleading as his response in the interview demonstrated that the graphical representation was merely a procedure that he had learned and he did not use the graph as part of his solution and thus was not connecting between the Symbolic and Embodied Worlds. In an example from this research, a participant drew a diagram-graph at the beginning of the solution. When asked why, he gave an explanation of how the diagram-graph helped him solve the problem.



P10: I simply, well at the beginning I imagined the shape so I just knew there was two equations and I knew what it was going to look like. And then I just forgot about it and did all of the math. (*talks with hands, makes gestures while speaking*)

Figure 17. Example of how the participant used the diagram while solving the problem.

In this example it is interesting how the participant states that he “forgot about it” or forgot about the information he obtained from the diagram-graph, even though the information was valuable for finding an answer to the question. The goal of this layered analysis of the written work and interview summaries was to accurately fill in the *Representation Table* for each question according to the representations a participant used when solving a problem.

To conduct each of the layered analyses, I began with the *Written Work Summaries and Analyzes*. From these documents I paid particular attention to what mathematical worlds the representations in the written work corresponded to and the role of the representations within a solution. I then reviewed the *Interview Transcript Summaries*, and sometimes the *Interview Transcripts* if I needed more details, to see if any new information was provided with regards to what mathematical worlds the representations in the task corresponded to and the role of the representations within the solution. During this review of the *Interview Summaries* and *Interview Transcripts* I

also paid attention the gestures that were recorded and layered selected ones with the information from the interview. In my selection, I only focused on Iconicity and Metaphoricity gestures, as defined by McNeill (2005), as these were the gestures that I perceived as sometimes demonstrating embodied-conceptual representations. However, I limited the analysis even further and only included gestures as representations that demonstrated connections between mathematical worlds if the accompanying verbal communication provided support for this conclusion. For example, some participants, when discussing the maximum and minimum values to be found in task Question #3, would gesture upwards and then downwards when saying maximum and minimum. While this can be considered a representation of how the concepts of maximum and minimum were understood, in my analysis I only included it as an embodied representation that demonstrated connections if a participant's interview was also able to provide support for this. Overall, it can be summarized that certain hand gestures were identified as embodied-conceptual representations and that a portion of these could also be identified from the verbal communication in the interview and demonstrating connections between different mathematical worlds.

From this layering of these documents a *Representation Table* was filled in for each question for each participant (making 60 tables in total). Lastly, I added to each *Representation Table* a summary of the information about what mathematical worlds the representations in the written work corresponded to and the role of the representations within a solution. Altogether, these three steps made up the layered analysis and produced 60 documents, which will be referred to as the *Layered Analysis* documents. A few examples of *Layered Analysis* documents are provided below in

Figure 18 and Figure 19.

Question #1: Analysis of Worlds

For question #1 the participant's written work appears to be all symbolic-proceptual. The solution follows only step-by-step procedures, using symbolic algebra. There is no evidence of any connections between worlds in the written work. However, during the interview the participant demonstrates that he knew that the derivative of an equation is the slope of a tangent line (lines 85-86). The participant also made hand gestures that suggest he has some knowledge of the relationship between the derivative of an equation and the slope of a tangent line (lines 88-89). This knowledge could involve some connecting between the Symbolic and Embodied worlds as tangent lines and slopes concepts that can be embodied as well as symbolized. However, in lines 86-87 the participant states that he was not sure how to solve this question and that he approached the solving of the question with manipulation of algebraic symbols, which are part of the Symbolic world. Thus, it appears from the written work and the interview that the solution to this question only took place in the Symbolic world.

Representations	Used as the primary part of the solution	Used as a unique part within a solution	Does not appear to affect solution	Used as supplementary to check steps or solution	Drawn during the interview
Embodied					
Embodying-Symbolism					
Symbolizing-Embodiment					
Symbolic	Step-by-step procedures using symbolic algebra				

Figure 18. An example of a *Layered Analysis* for Question 1.

Question #3 Analysis of Worlds

This written work to this question appears to use mostly symbolic representations although it is combined with some embodied representations. The algebraic work of finding the derivative, setting it to zero, and solving for x , is all symbolic work. During the interview the participant was able to describe in an embodied manner, as well as with hand motions, why he would set the derivative equal to zero to find a tangent line with a slope of zero. The table representation appears to use mostly symbolic characters, however, there are arrows used to represent increasing and decreasing. These arrows can be considered embodying-symbolism (since they embody the idea of increasing and decreasing). The arrows are also used elsewhere within the question to represent increasing and decreasing. There is also a curve sketched, which was done during the interview, that would be embodied since the curve is a model of the function in the question. The curve is not embodying-symbolism since it is not placed on axes or has any other type of graphical notation associated with it. Thus, overall it seems as though the participant was drawing from both and making connections between the symbolic and embodied worlds to solve this question.

Representations	Used as the primary part of the solution	Used as a unique part within a solution	Does not appear to affect solution	Used as supplementary to check steps or solution	Drawn during the interview
Embodied					-curve
Embodying-Symbolism		Arrows used for increasing and decreasing -interview comments			
Symbolizing-Embodiment					
Symbolic	Finding derivative and setting to 0				

Figure 19. An example of a *Layered Analysis* for Question 3.

The example in Figure 18 was chosen because it demonstrates an answer that was entirely Symbolic and provided no indications of any connections taking place

between mathematical worlds. In contrast, the example in Figure 19 is a *Layered Analysis* of an answer that did use multiple representations from different mathematical worlds, in both the written work and the interview. These two examples also show how the *Representation Tables* were completed to address these differences in the answers given. Thus, the *Layered Analysis* documents can be added to Table 2 of documents used throughout my analysis. A complete copy of Table 2 is in Appendix I.

Step #2: Analysis by Participant

I then used the documents in Table 2, and particularly the *Layered Analysis* documents, to try to address the research question of this study by analyzing the results according to participant. This was done in order to see how the participants made connections between mathematical worlds for the calculus topics of limits and derivatives. The first action I took was to organize my *Layered Analysis* documents according to participants. Thus, I had 15 groups, one group for each participant, with 4 *Layered Analysis* documents, one for each task question, in each group. This was in order to observe whether each participant had a particular way of representing and connecting between mathematical worlds. I then reviewed each participant's *Layered Analysis* documents to see if any patterns or trends emerged in the representations that were used and in the connections that were or were not made while solving the task questions. I did this by thoroughly reading each *Layered Analysis* for a participant and documenting how a participant was making connections between mathematical worlds.

It was through this review that it became evident that the participants demonstrated little consistency in how they connected between worlds when solving from task question to task question. For example, the Participant who was identified as

#3 answered Question #1 with algebraic written work but then proceeded to give an embodied description during the interview about how he understood the question, suggesting connections between the worlds in terms of the participant's understanding. Question #2 is done primarily embodied in both the written work and interview explanation. Question #3 and Question #4 are both primarily symbolic in written work. The interview suggests some connections to the embodied world (Question #3) and the Formal world (Question #4) in understanding. However, the connections in understanding that did appear during the interview were not necessarily part of the problem solving. This information showed that there was little consistency between how Participant#3 solved the various questions. Overall, there was little consistency between how the majority of the participants solved the various questions. From this analysis it suggested that how a participant made or did not make connections might actually depend more on the type of question he or she was solving. Thus, I chose to move on from my review of the *Layered Analysis* by participant and look at the data from a different perspective.

Step #3: Analysis by Task Question

The previous step demonstrated to me that the research question was not going to be addressed by comparing the data as 15 separate data sets by participant. It did however, bring to light that there may be a relationship between how a participant made connections between mathematical worlds and the type of task question that was asked. In this next step of the analysis process I began by using the *Layered Analysis* documents and organizing the documents according to the task questions. This was in order to review if how the question was asked influenced how a participant made

connections between worlds.

I then carefully read through each *Layered Analysis* by task question. While the *Layered Analysis* documents were first divided into four groups, one for each task question, with fifteen documents each, one for each participant, this analysis led me to be able to divide the fifteen documents in each of the four task question groups into further subgroups based on how the participants made connections and used representations. For example, one subgroup was made for solutions that were only symbolic. These subgroups were organized based on two analyzes of the *Layered Analysis* documents. The participants' responses, as described in the *Layered Analyses*, were first analyzed and put into categories based on how much or how little connecting between worlds was taking place, ranging from no connections to making connections throughout the entire solution for each task. The *Layered Analyses* were then analyzed and grouped based on the representations that were used, such as symbolic or embodied, and how they were used. For example, in the subgroup Category (A), all of the written and verbal representations that a participant used in a response could be considered symbolic. Alternatively, in the subgroup Category (B) the majority of written and verbal representations that a participant used in a response are also symbolic like in Category (A), but there are a few representations that correspond to either the embodied or formal worlds. Regardless of these embodied or formal representations, there is no connecting taking place in the responses in both Category (A) and Category (B). These are just two examples. The *Layered Analyses* were divided into eight sub-groups in total. It was through this review that I was able to start to develop a good description of how the participants were making connections between

mathematical worlds. The use of categories to describe how the participants were making connections was also influenced by the some of the reviewed literature, which used similar analysis/categorization methods (e.g. Przenioslo, 2004; Williams, 1991). Table 3 lists the categories as well as their descriptions and more detail about these categories follows the table.

Table 3.

Categories of Layered Analysis Based on How Connections Were Made and Representations Used

Response Categories	Description
(A) No Connections/ All Symbolic	A response categorized in this group appears to have only symbolic-proceptual written work. As well, the information provided in the interview suggests that the participant was only solving the question in a symbolic-proceptual manner and that there was no connecting going on.
(B) No Connections/ Multiple Representations	A response categorized in this group appears to be mostly symbolic in both written work and information from the interview. However, these responses also have one or two other representations that are not symbolic but these other representations are not necessarily demonstrating connections or affecting the way the problem was solved.

Table 3 (continued)

*Categories of Layered Analysis Based on How Connections Were Made and**Representations Used*

Response Categories	Description
(C) No Connections/ Disconnect between Tasks and Interview	A response categorized in this group appears to have written work that is primarily symbolic and the information from the interview is primarily embodied. However, there is a disconnect between how the participant described how the question was solved in the interview and how the question was actually solved with the written work. There are no connections demonstrated between the worlds.
(D) Mostly Symbolic with some Connections	A response categorized in this group appears to have mostly symbolic written work. However, the responses also include one or more written representations from a different world. From the information provided in the interview, it is suggested that the participant was connecting the representations from different worlds.
(E) Tasks Symbolic/ Interview Embodied with some Connections	A response categorized in this group appears to have only symbolic written work. The information from the interview suggests that the participant was thinking about the question and the response with some embodied representations and making some connections between the two worlds. The difference between Category (E) and Category (C) is that responses in this category do demonstrate some connections being made.

Table 3 (continued)

Categories of Layered Analysis Based on How Connections Were Made and Representations Used

Response Categories	Description
(F) Multiple Representations with Connections/ Memorized Process	A response categorized in this group has multiple representations from different worlds. It also has connections between these worlds. However, the information from the interview suggests that the process to solve this question was memorized and that although the participant uses multiple representations and appears to make connections, there is actually little understanding as to why.
(G) Only Graphical Solution	A response categorized in this group appears to be solved only with a graph, making it embodied-symbolic and symbolic-embodied. The answer may be written with mathematical symbols, but the problem is still only solved with a graphical representation.
(H) Strong Connections/ Multiple Representations	A response categorized in this group appears to have strong connections between the symbolic and embodied worlds in both the written work and the interview information. There are also a few cases where connections are made to the formal world.

Further Description of Categories

The categories in Table 3 are organized by the categories where connections are not made and the categories where connections are made and by the types of representations. Categories (A), (B), and (C) can all be considered categories where

there are no connections taking place between mathematical worlds. The only difference between these three categories is the representations that are being used come from different mathematical worlds. Category (A) does not demonstrate any representations other than symbolic, while Category (B) does include more than just symbolic representations. Category (C) on the other hand describes the situation of a participant having written a solution with a certain type of representations and then discusses what he or she did by describing another type of representations, while at the same time no connections are taking place. Examples of participants' work that falls into Categories (A), (B), and (C) are provided in the three figures below respectively.

(Line 84-89) P1: My first line, I just wrote the equations and then I tried deriving because the derivative of the equation is the slope of a tangent line. I knew we had to find where it equals 1 and that's where I wasn't sure what to do. So I isolated the variables, so when x equals this and y equals that we will have a slope with tangent equal to 1. *(points to the pages and talked with his fingers. Makes a backwards and forward motion with his finger when talking about the relationship between derivatives and tangents)*

$$\begin{aligned}x^2 + 2y^2 &= 1 \\2x + 4y &= 0 \\2x + 4y &= 1 \\x &= \frac{1}{2} - 2y \\y &= \frac{1}{4} - \frac{1}{2}x\end{aligned}$$

Figure 20. Example of participant work in Category (A).

P9: This one confused me. Actually I should have known this one very well because I'm not bad at this

A: Yeah

P9: So I found where the tangent was zero and I subbed it...I remember from this um calculus course, you know when it is unstable it is greater than zero and when it is stable it is less than zero, but that wouldn't help me...oh, actually it would help me with stable and unstable, which would be increasing and decreasing *(talks with hands, makes gestures while speaking and points to different work on the page)*

$$f(x) = 3x^5 - 5x^3$$

$$= 15x^4 - 15x^2$$

$$0 = 15x^2(x^2 - 1)$$

$$0 = x^2(x^2 - 1)$$

Tangent is zero
when $x=0$
and at $x=1, -1$

Figure 21. Example of participant work in Category (B).

A: Okay. So how about this second part? What does it mean for this tangent line to have a slope of 1.

P15: So like, if it is a circle it will be round (*makes a circle with hand*). And then on the line you can pick a point where it will...where you can find a tangent to. And the tangent to the circle of the curve, or whatever you have, should be 1 (*makes a slope of 1 with finger*).

The participant then described what she did to solve the problem:

P15: So I just solved for y. Like I put it in line form (*points to work on page*)

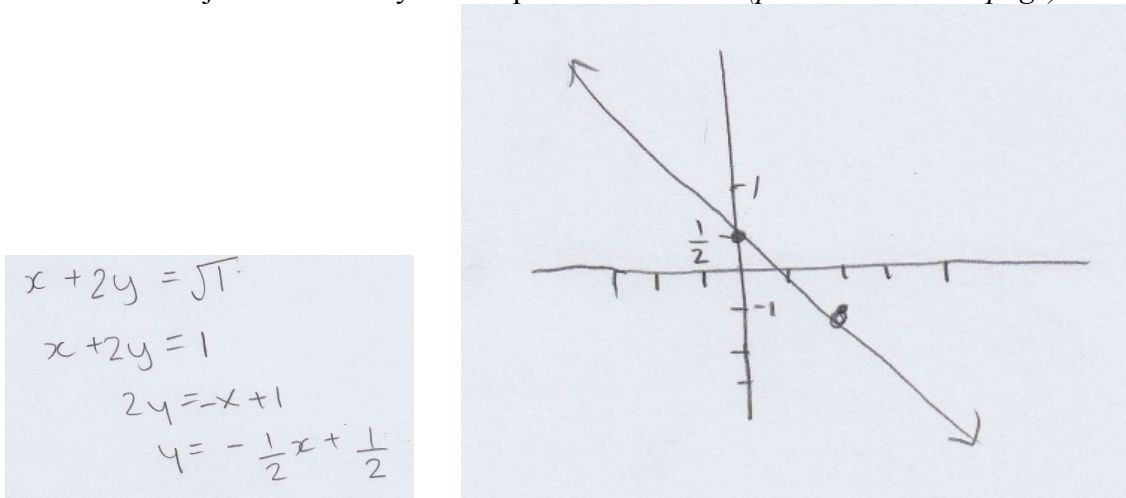


Figure 22. Example of participant work in Category (C).

The remaining categories in Table 3, Categories (D), (E), (F), (G), and (H), can all be considered categories where at least some connections up to a lot of connections are being made between mathematical worlds. The differences between these categories depend on the degree of mathematical connections being made and the uses of different mathematical representations. For instance, Category (D) is the first category that describes at least some connections being made between different mathematical worlds using different representations. Responses grouped in this category still seem to be mostly symbolic however they also include at least one example of a non-symbolic representation that is being connected to the symbolic work. Category (H) on the other

hand, is the category that is used for responses that demonstrate several connections being made between mathematical worlds during more than one step of the problem solving in a solution. Thus, if Categories (D) and (H) were placed on a spectrum between having few connections and having many connections, they would be on opposite ends. Examples of participants work from Categories (D) and (H) are provided in the figures below respectively. These examples were chosen as they demonstrate the difference between participant work from the two categories.

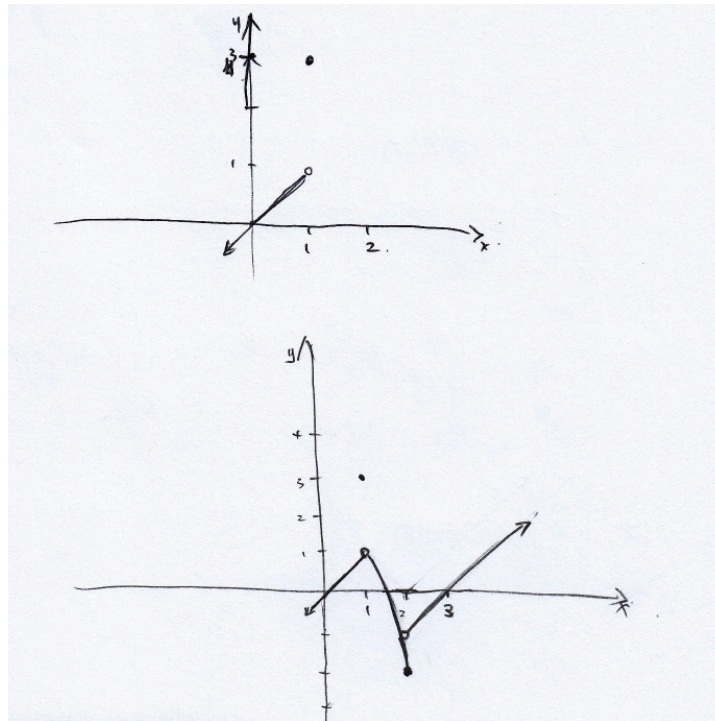
P5: Yeah, so the first one was taking the limit as x approaches 1 from the left (*points to work on page*). And when you evaluate a piecewise function if it is just from one side you just have to check which side it is going on (*points to two different sides*). If it is going from the left then it is the piece that is going from the left (*sweeps hand in from the left*). So that was the first 1 where x is smaller than 1.

$$\begin{aligned} \lim_{x \rightarrow 1^-} g(x) &= 1 \quad (\text{1st statement of piecewise}) \\ \lim_{x \rightarrow 1} g(x) &= \text{DNE} \quad \text{b/c} \quad \lim_{x \rightarrow 1^-} g(x) \neq \lim_{x \rightarrow 1^+} g(x) \\ \lim_{x \rightarrow 2^-} g(x) &= -2 \rightarrow (2-x^2) = (2-2^2) = -2 \\ \lim_{x \rightarrow 2^+} g(x) &= -1 \rightarrow (2-3) = (2-3) = -1 \\ \lim_{x \rightarrow 2} g(x) &= \text{DNE} \quad \text{b/c} \quad \lim_{x \rightarrow 2^-} g(x) \neq \lim_{x \rightarrow 2^+} g(x) \end{aligned}$$

He stated that he solved every left and right limit in this question using similar logic that he described above. In regards to the limits that weren't left or right he stated:

P5: yeah, so these are the left and right parts and this one is just the limit as a whole, not the left or the right (*points to different limits on the page*). So for this one I put that it doesn't exist because the limit ...like it is not approaching the same thing from the left and the right (*moves hand from the left and from the right*). Because from the right it is 3 ...as it is approaching 1 from the right it is 3 and as it is approaching from the left it is 1. So there is a gap (*moves hand up and down to demonstrate a gap*). There is no limit.

Figure 23. Example of participant work in Category (D).



When asked how the graph helped her solve the question she said:

P4: Well, I can see, for example, from this one, when $x \dots$ when the limit of this function when x approaches 1 from the left side so I just go to 1 and approach from the left side (*points to the graph*) and see that the function is just going to a value of 1 and the limit is 1 at that point.

Figure 24. Example of participant work in Category (H).

The other three categories with connections, namely (E), (F), and (G), all have responses that show some varying degree of connections being made with the major differences between the categories depending more on the representations that are used and how they are used. For Category (E), the written work appears to use only symbolic representations. However, the interview has embodied representations in the description that are connected to the symbolic representations in the written work. An example of Category E is presented in the figure below.

P8: So I pictured the graph in my head...cause I knew it wasn't just a full graph (*moves hand in a curve motion*) because this is piecewise. So what I saw was when x is approaching 1 from the left, so I kind of looked at which one it would be. And since this is less than 1 it would just be approaching from the left side and it would be like this part of it (*moves hand from the left*). And same with the next one, like x or the limit of x as it goes to 1 is close to 3 cause x equals 1. And then 2 from the left, then I just saw it here, it was 2 to the left. And then 2 to the right, is greater than 2 so it is coming that way. And then when x equals 2 it is that one because 2 is less than or equal to 2. (*points to the different parts of the question on the page*)

Show your work and explain your process and thinking.

$\lim_{x \rightarrow 1^-} g(x) = x$ $\lim_{x \rightarrow 1^-} g(x) = 1$	$\lim_{x \rightarrow 1} g(x) = 3$	$\lim_{x \rightarrow 2^-} g(x) = 2 - x^2$ $\lim_{x \rightarrow 2^-} g(x) = 2 - (2)^2$ $\lim_{x \rightarrow 2^-} g(x) = -2$	$\lim_{x \rightarrow 2^+} g(x) = x - 3$ $\lim_{x \rightarrow 2^+} g(x) = (2) - 3$ $\lim_{x \rightarrow 2^+} g(x) = -1$
$\lim_{x \rightarrow 2} g(x) = 2 - x^2$ $\lim_{x \rightarrow 2} g(x) = -2$			

Figure 25. Example of participant work in Category (E).

Category (F) uses both symbolic and embodied representations in the written work, which are connected. In this category, the interview describes a memorized process and very little understanding of why the question uses multiple representations that are connected. An example of a participant's interview comments placing the work in Category (F) is presented in the figure below.

A: Why did you choose to draw these pictures along here, or solve the problem this way.

P8: I have just been taught that way

Figure 26. Example of participant work in Category (F).

Responses in Category (G) are solved and described with only graphical

representations. Since graphical representations result when connections are made between certain concepts from the symbolic and embodied worlds, graphical responses do demonstrate connections being made even if they only use one type of representation. An example of a participant's work from Category (G) is provided in the figure below.

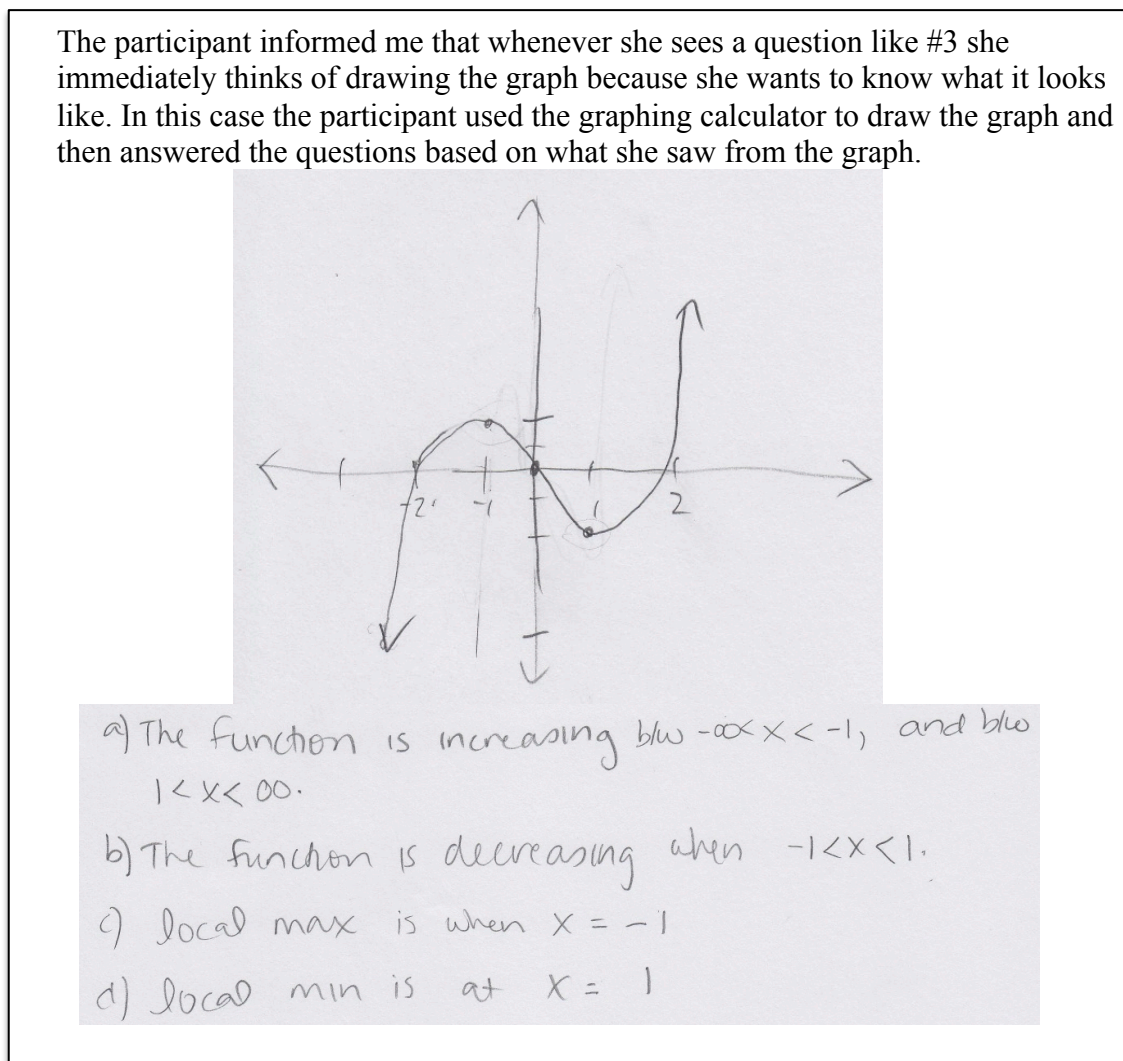


Figure 27. Example of participant work in Category (G).

Step #4: Further Analysis of Findings from Step #3

The analysis that took place in Step #3, which looked at the *Layered Analyses* based on the task questions, led to some further ideas of how to look at the data. This

further analysis was done in order to develop an even clearer picture of how participants were making connections between mathematical worlds and to evaluate and substantiate the *Response Categories*, developed in Step #3, for accuracy in classifying participants work according to the *Response Categories*. In this analysis step, I decided to review the written work summaries and interview summaries to see whether participants who connect between worlds were starting and finishing their answers to their question with particular representations. I also decided to look at whether the connecting was taking place in a particular order, for example, from the symbolic world to the embodied world. Both of these actions were undertaken in order to see if how an answer was started or finished might have contributed to how a participant made connections or whether the connecting was taking place in a particular order.

I began by making a list of more detailed questions about how a participant was making connections. These questions included:

- How is the participant starting the question and with what representations?
- How is the participant finishing the question and with what representations?
- Are the connections taking place in a particular direction (i.e. from symbolic to embodied)?
- What types of representations is the participant connecting?

With these questions I made tables to record my observations. Each table included the four further analysis questions above. A table was made for each of the different *Response Categories* from Step #3 where connections are made, which were Categories (D), (E), (F), (G), and (H). These tables were then duplicated for each of the task

questions. The purpose of only asking these further analysis questions of the participants in Categories (D), (E), (F), (G), and (H) was because it was unnecessary to ask further questions about how a participant was making connections for those responses where no connections were made at all. The last step of this analysis was to record in each table the answers to the questions after reviewing the participant's *Layered Analysis*, *Written Work Summary*, and *Transcript Summary*.

After conducting a further analysis on the *Response Categories* with connections, I looked at the *Response Categories* without connections. I developed two questions to ask about the solutions that fit into Categories (A), (B), and (C). The questions are:

- What representations is the participant using when solving?
- In what way is the participant using (different) representations, if any?

I asked these two questions in order to see if some further detail about representation use could be provided for the participants' responses in these *Response Categories* where no connections were made. I used the same method as I did to analyze the *Response Categories* with connections by making tables for each *Response Category* without connections to record my observations. I then duplicated the tables for each task question and recorded the answers to the two above questions after reviewing the participant's *Layered Analysis*, *Written Work Summary*, and *Transcript Summary*.

Overall, it was felt that these further analysis questions helped to draw a more detailed picture about how participants were or were not making connections between mathematical worlds and provided support to the finding from Step #3 and consequently, helped to address the research question. This is discussed in more detail in the Findings chapter.

Step #5: Analysis of Correctness of Answers and Solutions

The purpose of analyzing correctness within the participants' answers was to see if participants who connected in particular ways were more or less likely to use a correct method for the solution and/or get a correct answer on the calculus based task question. It was also to see if the findings from this research could be related back to the literature, which suggests that individuals who connect between mathematics worlds (e.g. Nogueira de Lima and Tall, 2008; Stewart and Thomas, 2007) and between multiple representations (e.g. Herbert 2009; Knuth, 2000) are more successful in obtaining the correct answer on mathematical problems.

In order to make these observations, the written work solutions were analyzed for two types of correctness, namely method correctness and answer correctness. It was determined that both areas needed to be analyzed to draw appropriate conclusions. For instance, it was observed from the analysis of the written work that many participants used a method to produce the solution for a question that could have provided a correct answer, but unfortunately made one or more arithmetic mistakes that prevented them from getting the correct answer. If the solution was only analyzed for correctness of the answer it would be misleading since a participant whose solution used an appropriate method but made an arithmetic mistake would be grouped with someone that used an inappropriate method and also received an incorrect answer. An example of a participant's work with a correct method but an incorrect solution because of an arithmetic mistake is included below. The arithmetic mistake in this case happens while taking the derivative.

Isolating for y

$$2y^2 = 1 - x^2$$

$$y^2 = \frac{1 - x^2}{2}$$

$$y = \sqrt{\frac{1 - x^2}{2}}$$

$$y = \left(\frac{1 - x^2}{2}\right)^{\frac{1}{2}}$$

Finding formula of derivative

$$y' = \frac{1}{2} \left(\frac{1 - x^2}{2}\right)^{-\frac{1}{2}} (-x)$$

$$f'(x) = \frac{1}{2} \left(\frac{1 - x^2}{2}\right)^{-\frac{1}{2}} (-x)$$

$$1 = \frac{1}{2} \left(\frac{1 - x^2}{2}\right)^{-\frac{1}{2}} (-x)$$

$$2 = -x \left(\frac{1 - x^2}{2}\right)^{-\frac{1}{2}}$$

$$2 = \frac{-x}{\left(\frac{1 - x^2}{2}\right)^{\frac{1}{2}}}$$

$$4 = \frac{-x^2}{\left(\frac{1 - x^2}{2}\right)}$$

using x value to find y coordinate

$$y = \left(\frac{1 - x^2}{2}\right)^{\frac{1}{2}}$$

$$y = \left(\frac{1 - (\sqrt{2})^2}{2}\right)^{\frac{1}{2}}$$

$$y = \left(\frac{1 - 2}{2}\right)^{\frac{1}{2}}$$

$$y = \left(-\frac{1}{2}\right)^{\frac{1}{2}}$$

$$y = \sqrt{-\frac{1}{2}}$$

Incorrect solution (arithmetic mistake):

$$m = \frac{1}{2} - \frac{x^2}{2}$$

$$m' = 0 - x$$

$$4 = \frac{-2x^2}{1 - x^2}$$

$$4 - 4x^2 = -2x^2$$

$$4 - 2x^2 = 0$$

$$2(2 - x^2) = 0$$

$$2 - x^2 = 0$$

$$-x^2 = -2$$

$$x^2 = 2$$

$$x = \sqrt{2}$$

Figure 28. Example of participant work with a correct method and incorrect solution because of arithmetic mistake.

For this analysis the term “correct” can be described as something that is true,

accurate, or right. The term “method” is used to describe the procedure, technique, or way of answering the question. The “solution” refers to a participant’s entire written work on a question, while the “answer” refers only to the final part of the solution that addresses what the question was asking.

Method Correctness

To analyze the correctness of the method for a solution, categories were initially developed and described, based on my previous experience of reviewing and analyzing the written work. I divided the analysis of method correctness into two steps, an analysis that looked at how the solutions were started and then an analysis that looked at how the solutions were continued and finished. I was able to accurately tell how the solutions were started and finished based on the combination of the written work and the transcripts from the interviews.

First I considered the method that the solutions began with and developed two categories:

- Solutions started with an appropriate method: A solution placed in this category would have been started with a method that, if continued, could lead to a correct answer.
- Solution started with an inappropriate method or an unnecessary first step: A solution placed in this category would have been started with a method that, if continued, would not lead to a correct answer. This could take the form of either an inappropriate method or an unnecessary first step of the method.

Second I considered the method that the solution continued and finished with and developed three categories:

- Continued and finished solution with an appropriate method: A solution placed in this category would continue and finish the question, after the first step of the method, with a method that could lead to a correct answer.
- Continued and finished solution with an inappropriate method: A solution placed in this category would continue and finish the question, after the first step of the method, with a method that would not lead to a correct answer.
- Solution was left incomplete: A solution was placed in this category if the participant began a question but then left it incomplete.

Since how a participant started a solution and then continued a solution are not mutually exclusive, it seemed appropriate to place the categories on different axes of a chart and fill in a chart in this manner for each of the four calculus based task questions.

The Method Correctness chart is displayed below:

Method Correctness	Started solution with an appropriate method	Solution started with an inappropriate method/or an unnecessary step
Continued and finished answer with an appropriate method		
Continued and finished answer with an inappropriate method		
Solution was left incomplete		

Figure 29. Method Correctness chart for Analysis Step #5.

Answer Correctness

To analyze the correctness of the answer four categories were developed and described.

- Completely correct answer: An answer in this category is completely correct for all parts of the question.
- Partially correct answer: An answer in this category is correct for a portion of the question and incorrect for another portion. This category was possible since some of the questions have multiple parts. For example, Question #2 asked for five different limits to be found from the same function. It is the case that a participant could answer three of the limits correctly and two incorrectly, thus placing them in this category.
- Incorrect answer because of an arithmetic mistake: An answer in this category was on track to be correct in terms of the method used, but because of one or more arithmetic mistakes the final answer is not correct.
- Incorrect answer or incomplete: An answer in this category is assigned to questions that are incorrect because of more than an arithmetic mistake, or are left incomplete by the participant.

These four categories were placed in a chart and the participants' numbers were added to the appropriate cells based on the responses. The Answer Correctness chart is provided below:

Answer Correctness

Completely correct answer	Partially correct answer (partially incorrect)	Incorrect answer because of arithmetic mistake	Incorrect answer/or incomplete

Figure 30. Answer Correctness chart for Analysis Step #5.

Thus, for the analysis, participants' numbers were placed into the appropriate category in these two charts for each of the four task questions. This resulted in four Method Correctness charts and four Answer Correctness charts. Examples of the Method Correctness chart and Answer Correctness chart for task Question #1 are included below in Figure 31, with all of the charts presented in the Findings chapter of this thesis.

Method Correctness: Question #1	Started solution with an appropriate method	Solution started with an inappropriate method/or an unnecessary step
Continued and finished answer with an appropriate method	#3, #4, #6, #7, #10, #13, #14	
Continued and finished answer with an inappropriate method	#8, #12	#15
Solution was left incomplete	#1, #2, #5, #9, #11	

Answer Correctness: Question 1

Completely correct answer	Partially correct answer (partially incorrect)	Incorrect answer because of arithmetic mistake	Incorrect answer/or incomplete
#6, #14	#13	#3, #4, #7, #10	#1, #2, #5, #8, #9, #11, #12, #15

Figure 31. Examples of Method Correctness and Answer Correctness Charts with Participant Numbers.

These charts were then compared to the finding from the Analysis by Question in Step #3 in order to see if there was a relationship between how a participant made connections between mathematical worlds and the correctness of a participant's answer. The results of this comparison will be discussed in more detail in the Findings chapter of this thesis.

Step #6: Observations of other Interesting Findings

Throughout the previous five analysis steps, I made notes about observations

that I thought might warrant some further investigation. These observations came up mostly during Step #1 as I was organizing the data for a more in-depth analysis. These observations include: (a) the way many participants used a table when solving task Question #3 to find the properties of the function, (b) the way some participants used the word “math” to only refer to algebraic manipulation, as opposed to drawing graphs or diagrams, and (c) the frequency with which participants commented that first year university calculus was mostly a “repeat” of what they learned in high school calculus. These three observations emerged from the study and were examined as they may have contributed to describing how the participants made connections between mathematical worlds. Alternatively, they may just have pointed to areas that deserved further investigation in future research.

In order to analyze observation (a) I began by making note of the participants who used a table in task Question #3. I then grouped the tables according to similarities in structure in order to better describe the tables that were used. Lastly, I made note of whether the participants who used tables also took Calculus and Vectors (Ministry of Education, 2007) in high school or the equivalent course in university. This was to observe whether students who took Calculus and Vectors in high school tended to use tables for Question #3 more than students who took the equivalent course in university, or vice versa.

To analyze observation (b) I reviewed the participants’ semi-structured interviews in detail and recorded the participants who used the word “math” and the meaning they gave to it. For example, some participants used the word “math” to refer to the task questions and the problem solving they undertook to solve them. However,

many participants also used the word “math” only to refer to algebraic manipulation. In this research, I defined algebra as the branch of mathematics in which arithmetic operations and relationships are generalized by using alphabetic symbols to represent unknown numbers or members of a specified set of numbers. Thus, algebraic manipulation was any instance where the student manipulated or used arithmetic operations and relationships with alphabetic symbols to represent unknown numbers. I then took the entire statement from the different participants’ transcripts where they referenced the word “math” and grouped them into a single document. From this I was able to observe the similarities and difference between the statements while paying particular attention to those statements that used the word “math” to mean algebraic manipulation.

Lastly, to analyze observation (c) I referred to my participant information in order to know exactly which participants took calculus in high school prior to university and which participants took the equivalent to high school calculus in university. The province of Ontario high school course Calculus and Vectors (Ministry of Education, 2007), or an equivalent for those students who come from other provinces, is a prerequisite for first year university calculus at Ontario universities. For this situation I reviewed the interview transcripts and made note of which participants felt that first year university calculus repeated a lot from high school calculus or the equivalent course. The statements that the participants made were organized into a single document and reviewed for content.

The results from these three observations are described in the Findings chapter of this thesis. In the Discussion chapter I elaborate further on these findings and how

they contributed to the results of the study.

Summary of Data Analysis Steps

Overall, a summary of the data analysis steps that I undertook are represented in the flow chart included below (Figure 32). The steps in the flow chart have all been explained in detail in the previous sections, however a brief description of these steps follows the chart.

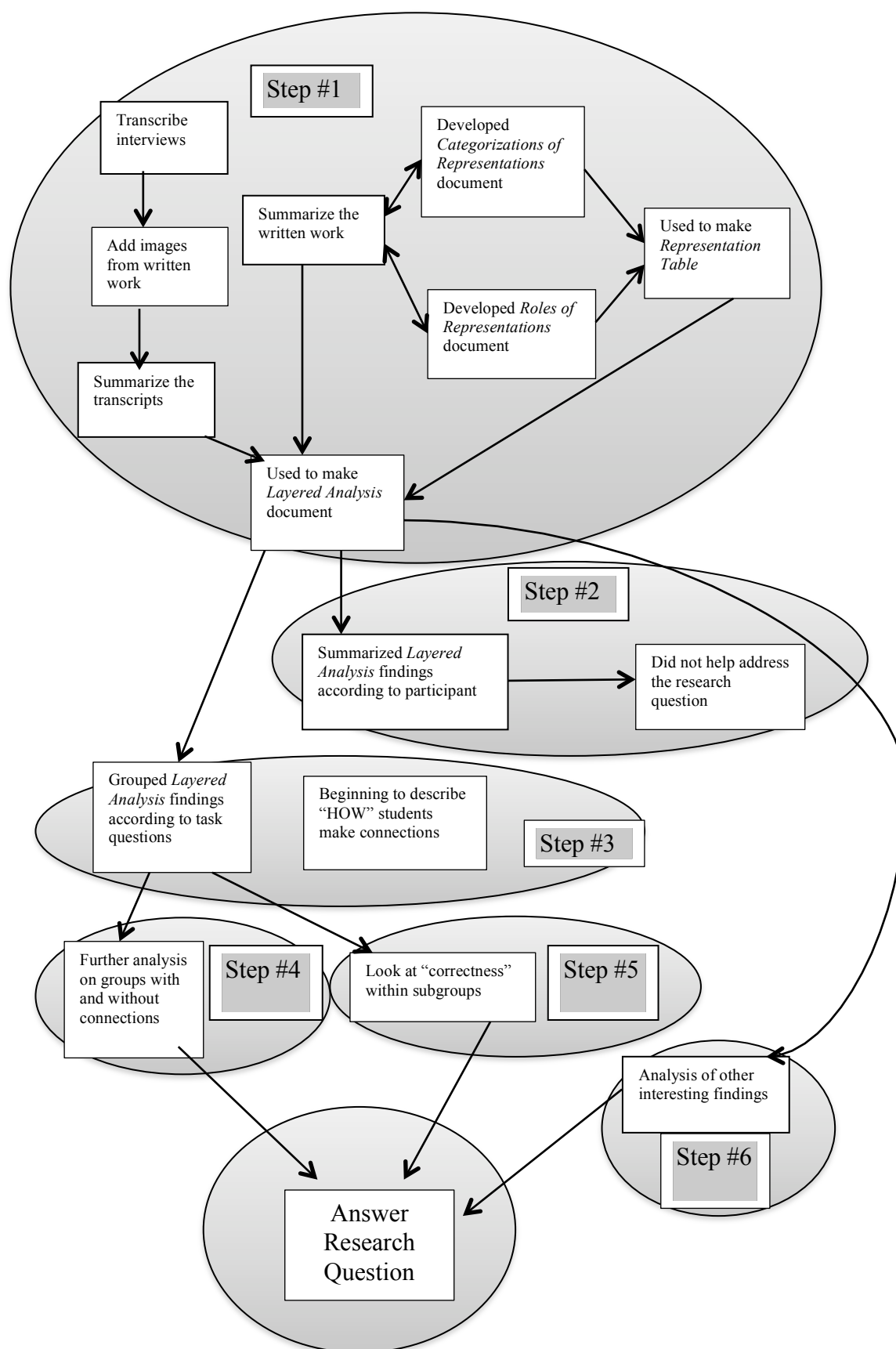


Figure 32. Flow chart of data analysis steps.

Each of the steps taken in the data analysis helped me address the research question of this study. Step #1, which consisted of many sub-steps where I organized the data into category documents and developed the *Layered Analysis* documents, allowed me to easily refer back to specific information throughout the other steps of the data analysis. It also helped me familiarize myself with the data, which allowed me to determine what further steps needed to be taken in the data analysis. In Step #2 I grouped the *Layered Analysis* documents according to participant and reviewed them to see if there was anything significant about how each participant made connections on the task question. This step demonstrated to me that the research question was not going to be addressed by comparing the data as 15 separate data sets by participant. It did however, bring to light that there may be a relationship between how a participant made connections between mathematical worlds and the type of task question that was asked. Thus, Step #2 helped by providing the idea for another way to analyze the data. In Step #3 I began by grouping the *Layered Analysis* documents according to the four calculus task questions. I then reviewed, sub-grouped, and documented similarities between how participants made connections when answering the task questions and speaking about them during the interview. The analysis from Step #3 helped me by making it possible to begin to recognize the similarities and differences in how participants were connecting between mathematical worlds, which would become of the *Response Categories*. In Step #4 I used what I had found in Step #3 and further analyzed the similarities and differences in how the participants were connecting between mathematical worlds. This step helped me to be able to develop a well-rounded picture of how connections were made and support my findings from Step #3,

and particularly support the accuracy of the *Response Categories*. In Step #5 I analyzed the correctness of each answer for method correctness and answer correctness. This step helped me to relate my research to previous literature that used a similar methodology in order to compare my findings. This step also added to the information from Step #3 and Step #4, which in turn provided more detailed findings that helped address the research question. Lastly, in Step #6 I examined particular emergent ideas that had come to my attention through the data analysis process with the goal of seeing if some of these findings added information to the findings from Step #3 and Step #4 and helped address the research question.

CHAPTER FIVE: FINDINGS

The findings reported in this chapter reflect what was learned from the data analysis steps, as described in the Research Method chapter. This chapter also reports on any relationships between these findings, such as whether the way a participant made connections affected the correctness of his or her solutions to the task questions. This chapter does not cover in detail the findings from Step #2, the Analysis by Participant but begins with the findings from Step #3, the Analysis by Question. This is because it was through the Analysis by Question that that *Response Categories* began to be developed and describe how the participants were making connections. This step also led to a Further Analysis, Step #4, which provided a more detailed description of how participants made connections within the different *Response Categories*. With regards to Step #2, it is sufficient to say that from that analysis there did not appear to be any clear trends or relationships on how a participant made connections between mathematical worlds and did not provide a means by which to address the research question of the study. It did, however, offer insight into an alternative analysis idea, namely an analysis by task question. The findings from the Analysis by Participant are located in Appendix K.

Following the findings from the Analysis by Question and the Further Analysis of the *Response Categories*, the chapter describes the findings from Step #5 of the data analysis that looked at correctness of solutions and answers. Following this, I provide a summary of the participant responses to the interview questions that did not ask the participants about their task questions. The last section of this chapter discusses the other interesting findings, or analysis Step #6. These interesting findings included the

use of a consistent problem solving method in task Question #3, the meaning assigned to the word “math” by the participants, and the participants’ comments about the similarities between high school calculus and university calculus. Overall, the findings are presented in this order to reflect the significance and verification of the *Response Categories*. I then present the correctness analysis as correctness is described in the literature as a significant benefit to being able to make connection between mathematical worlds. I conclude this chapter with the findings that may have also contributed to how participants made connections.

Findings from Analysis by Question

The findings for this analysis were reported in the *Response Categories* table (Table 3). As a reminder, Table 3 includes the eight *Response Categories* that the participants’ *Layered Analyses* were grouped into, as well as a description of each category. The *Response Categories* were what resulted from the analysis by question. Participants’ numbers were added to the appropriate category in the table based on information from his or her *Layered Analysis* for each of the task questions. This table is now identified as Table 4.

Table 4.

Categories of Layered Analysis Based on How Connections Were Made and Representations Used with Participant Responses

Response Categories		Description			
(A) No Connections/ All Symbolic		A response categorized in this group appears to have only symbolic-proceptual written work. As well, the information provided in the interview suggests that the participant was only solving the question in a symbolic-proceptual manner and that there was no connecting going on.			
	Question #1	Question #2	Question #3	Question #4	
Participant	#1, #9, #11	#2		#1, #12, #14	
(B) No Connections/ Multiple Representations		A response categorized in this group appears to be mostly symbolic in both written work and information from the interview. However, these responses also have one or two other representations that are not symbolic but these other representations are not necessarily demonstrating connections or affecting the way the problem was solved.			
	Question #1	Question #2	Question #3	Question #4	
Participant	#7, #8		#3, #9	#2, #3, #6, #8	

Table 4. *(continued)*

Response Categories		Description			
(C) No Connections/ Disconnect between Tasks and Interview		A response categorized in this group appears to have written work that is primarily symbolic and the information from the interview is primarily embodied. However, there is a disconnect between how the participant described how the question was solved in the interview and how the question was actually solved with the written work. There are no connections demonstrated between the worlds.			
	Question #1	Question #2	Question #3	Question#4	
Participant	#12, #15				
(D) Mostly Symbolic with some Connections		A response categorized in this group appears to have mostly symbolic written work. However, the responses also include one or more written representations from a different world. From the information provided in the interview, it is suggested that the participant was connecting the representations from different worlds.			
	Question #1	Question #2	Question #3	Question #4	
Participant	#2, #3, #5, #6, #10, #13, #14	#1, #5, #7, #12, #14	#1, #6, #7, #12, #15	#4, #7, #9, #10, #11, #15	

Table 4. (continued)

Response Categories		Description		
(E) Tasks Symbolic/ Interview Embodied with some Connections		A response categorized in this group appears to have only symbolic written work. The information from the interview suggests that the participant was thinking about the question and the response with some embodied representations and making some connections between the two worlds. The difference between Category (E) and Category (C) is that responses in this category do demonstrate some connections being made.		
	Question #1	Question #2	Question #3	Question #4
Participant		#3, #8, #10		
(F) Multiple Representations with Connections/ Memorized Process		A response categorized in this group has multiple representations from different worlds. It also has connections between these worlds. However, the information from the interview suggests that the process to solve this question was memorized and that although the participant uses multiple representations and appears to make connections, there is actually little understanding as to why.		
	Question #1	Question #2	Question #3	Question #4
Participant			#8	
(G) Only Graphical Solution		A response categorized in this group appears to be solved only with a graph, making it embodied-symbolic and symbolic-embodied. The answer may be written with mathematical symbols, but the problem is still only solved with a graphical representation.		
	Question #1	Question #2	Question #3	Question #4
Participant			#11	

Table 4. *(continued)*

Response Categories		Description			
(H) Strong Connections/ Multiple Representations		A response categorized in this group appears to have strong connections between the symbolic and embodied worlds in both the written work and the interview information. There are also a few cases where connections are made to the formal world.			
	Question #1	Question #2	Question #3	Question #4	
Participant	#4	#4, #6, #9, #11, #13, #15	#2, #4, #5, #10, #13, #14	#5, #13	

Findings from Table 4

There are many significant findings that can be seen from the results in Table 4. For instance, while there are eight different *Response Categories*, there are only three categories that have responses situated in them for all four of the calculus task questions, namely categories (A) No Connections, All Symbolic, (D) Mostly Symbolic with Some Connections, and (H) Strong Connections, Multiple Representations. In total 44/60 responses were placed into these three categories. This suggests, among other things, that the other five *Response Categories* may actually be more unique to the individual responses or task question and in fact may not reflect how connections are or are not generally made between mathematical worlds. In other words, the five categories that do not have responses for every calculus task question may not appear in similar situations. This can be supported by the fact that four of the five categories have a responses for only one question each, with a total of 7 responses for all four categories ((C) No Connections/ Disconnect between Tasks and Interview, (E) Tasks Symbolic/ Interview Embodied with some Connections, (F) Multiple Representations

with Connections/ Memorized Process, and (G) Only Graphical Solution).

The table also shows that the category with the most responses for all four questions is (D) Mostly Symbolic with Some Connections, with 23/60 total responses. This is a significant result since, as was described in the Data Analysis section, responses placed in category (D) can be considered a first step of connections being made between mathematical worlds. The number of responses per question for category (D) is also relatively consistent with between 5-7 responses for each question. This suggests that approximately one third of the participants were at least able to begin to make connections between the mathematical worlds for each of the different task questions.

Each task question had a category with the most responses and of importance is that they are all in categories where connections between mathematical worlds are being made to some varying degree. Specifically, Question #1 and Question #4 both have the most responses in category (D) while Question #2 and Question #3 also share the category with the most responses, namely (H).

For the categories with no connections ((A) No Connections, All Symbolic, (B) No Connections, Multiple Representations, and (C) No Connections/ Disconnect between Tasks and Interview), there were a total of 17/60 responses. Thus, for the categories with some degree of connections between mathematical worlds, there were a total of 43/60 responses.

Only two participants had responses that fit into category (C) No Connections/ Disconnect between Tasks and Interview. As was mentioned in the description of the categories, while (C) is a category with no connections, it also seems to describe the

unique situation of a participant's written work being done with one type of representation and his or her interview describing the written work referring to different types of representations. The results shown in Table 4 seem to confirm the suggestion that category (C) describes a unique situation, since the two participants that had solutions that fit into this category were both for task Question #1. The fact that category (C) only has responses from task Question #1 also suggests that there may be a relationship between this type of response and the type of question that was asked.

Three categories only have responses for one task question each, ((E) Tasks Symbolic/ Interview Embodied with some Connections, (F) Multiple Representations with Connections/ Memorized Process, and (G) Only Graphical Solution, only have responses for one task question each). Category (E) has three responses for Question #2 while categories (F) and (G) have one response each for Question #3. This suggests that there again may actually be a relationship between categories (E), (F), and (G) and the task questions for which they have responses.

Comparison of Findings Between the Calculus Task Questions

Table 4 also presents some interesting findings in terms of the calculus task questions. In particular, participant responses are not evenly distributed across the *Response Categories* for the four different task questions. For instance, Questions #1 and #4 have the highest number of responses where no connections are being made, with Question #1 having seven responses and Question #4 having eight responses. In comparison, Question #2 has only one response and Question #3 only two responses where no connections are being made. Similarly, Questions #1 and #4 have the lowest

number of responses with strong connections and multiple representations, while Questions #2 and #3 have the highest. This could suggest that certain types of calculus questions may be more likely to have connections made and multiple representations used.

Another interesting finding from Table 4, which was already touched upon briefly above, is that certain *Response Categories* are unique to particular task questions. For instance, Category (C) No Connections/ Disconnect between Tasks and Interview has two responses both for only Question #1. Likewise, Category (E) Tasks Symbolic/ Interview Embodied with some Connections, has three responses for only Question #2 and Category (F) Multiple Representations with Connections/ Memorized Process, and (G) Only Graphical Solution both have one response each for only Question #3. This again tells us that there may be a link between the types of task questions that were asked and these unique *Response Categories* being used.

Further Analysis Findings

In order to better describe how the participants were making connections between the mathematical worlds, a further analysis was conducted on the findings from Table 4 with the use of the written work and interview data, not just the *Layered Analyses*. This further analysis is described in more detail in Step #4 in the Research Methods Chapter. To summarize, the further analysis reviewed the written work summaries and interview summaries to see whether participants who connected between worlds were starting and finishing their solutions to their questions with particular representations. It also looked at whether the connecting was taking place in a particular order, for example, from the symbolic world to the embodied world. The

further analysis also looked at the categories without connections to see what types of representations were being used and how they were being used. This further analysis extended the *Analysis by Question* (Step #3 of the data analysis). The significant results of this further analysis are divided up by question and then category. These results are described below and include examples from participants' work to help demonstrate the findings. It should be noted that the interview comments are excerpts that provide a summary of the most relevant portions of the transcripts.

Question 1:

At what point(s) of $x^2 + 2y^2 = 1$ does the tangent line have a slope of 1? Show your work and explain your answer.

Category A (No Connections/All Symbolic) - Further Analysis:

There were three participant responses for this category. These three responses answered question #1 by writing down and speaking only about symbolic representations. They all used the symbolic representations in step-by-step procedures in the written work and then spoke of the procedures in the interview. Also, the three participant responses in this category all left the question incomplete.

The image shows handwritten mathematical work on a light blue background. The equations are written in black ink:

$$2x + 4yy' = 1$$

$$y'' = \frac{1 - 2x}{4y}$$

$$4y = 1 - 2x$$

$$4y - 2x = 1$$

Figure 33. Example for Question #1 in Category (A).

The example in Figure 33 demonstrates one of the responses to Question #1 that was considered to be Category (A). As this example shows, the participant did not use any representations other than symbolic. No interview comments were included because the interview did not provide any information to suggest that the participant was using other representations or making any connections.

Category B (No Connections/Multiple Representations) - Further Analysis:

There were two participant responses for this category. Each participant response in this category answered Question #1 by writing down and speaking about mostly symbolic representations. They all used the symbolic representations in step-by-step procedures in the written work and then spoke of the procedures in the interview. Each participant also had one representation that was not symbolic. Two participants had graphical diagrams and one participant had a formalizing-symbolism statement. However, from the interview it was apparent that these representations were not used in the solving of the question and were not connected to the other representations of the solution. It is not clear why these representations are there.

$$x^2 + 2y^2 = 1$$

Set $f(x) = 0$

$$x^2 + 2y^2 - 1 = 0$$

Derivative

$$2x + 4y = 0$$

$$4y = -2x + n$$

$$y = \frac{-2x + n}{4}$$

slope = $-\frac{1}{2}$ $\therefore$ tangent line does not have a slope of 1

$y = mx + n \leftarrow$ tangent line
slope of 1
 $y = 1x + n$

P8: I was kind of trying to remember what we did in class. Because I knew the professor kind of wrote it off to the side, like the y equals mx plus n form and then he would substitute the numbers in (*points to the $y=mx+n$ on the page*)

A: Yeah

P8: But I didn't know how to put it together to do it like myself

Figure 34. Example for Question #1 in Category (B).

In this figure we have an example of the participant making a formalizing-symbolic statement by writing out the generalized equation of a tangent line. However, in the interview the participant stated that he did not know why he wrote this equation, only that it was used in class. This shows that the participant was not making connections to this representation, only that he remembered it was used in his class. Therefore, this demonstrates how the participant was using more than just symbolic representations, but that there were no connections being made.

Category C (No Connections/Disconnect between Tasks and Interview) - Further

Analysis:

There were two participant responses for this category. The participant responses in this category used mostly symbolic representations in the written work. They also both include diagrams/graphs in the written work, but the majority of representations are still symbolic. During the interview both participants described their steps by using primarily embodied representations. It is the case that each participant's description does not correlate with what was actually written down in the written work. Hence there is a disconnect between the written work and interview and any connections between mathematical worlds are not apparent.

$$\begin{array}{l}
 x + 2y = \sqrt{1} \\
 x + 2y = 1 \\
 2y = -x + 1 \\
 y = -\frac{1}{2}x + \frac{1}{2}
 \end{array}
 \qquad
 \begin{array}{l}
 m=1 \\
 y = -\frac{1}{2}(1) + \frac{1}{2} \\
 y = -\frac{1}{2} + \frac{1}{2} \\
 y = 0 \\
 0 = -\frac{1}{2}x + \frac{1}{2} \\
 -\frac{1}{2} = -\frac{1}{2}x \\
 x = 1 \\
 \boxed{(0, 1)}
 \end{array}$$

P15: Okay, when I look at that I think of like a circle (*points to work on page*)...because there are two squares equal to like whatever...So like, if it is a circle it will be round (*makes a circle with hand*). And then on the line you can pick a point where it will...where you can find a tangent to. And the tangent to the circle of the curve, or whatever you have, should be 1 (*makes a slope of 1 with finger*)...So I just solved for y. Like I put it in line form (*points to work on page*)

Figure 35. Example of Question #1 for Category (C).

In this example, the participant demonstrated that she knew that the equation in Question #1 represented a circle, although it would be more accurate to consider the equation an ellipse. This participant then stated that she found the tangent and made its equation equal to y . This was done by solving for y and writing the equation of a line. However, there is some confusion in that the participant understood that the original equation represented a curved shape in that she stated it was a circle, but then rearranged the equation to be the equation of a line without questioning how it is illogical to go from a circle to a line by rearranging a single equation. The participant also at one point states that $x=1$ and $y=0$ but then writes the ordered pair as $(0, 1)$. Overall, what the participant expressed she understood and what she wrote down are disconnected and are taken to demonstrate that she has not been able to develop connections between these representations.

Category D (Mostly Symbolic with some Connections) - Further Analysis:

There were seven participant responses for this category. Each participant response began Question #1 by either re-arranging the equation to isolate for y and then differentiating or by implicitly differentiating the original equation. In all cases symbolic representations are used. Of the six participants in this category, three finished the question by sketching of graph of their findings, while the other three finished the question by stating the answer symbolically. Each participant made some connections between the symbolic and embodied worlds by either sketching a graph in the written work or describing their solutions in terms of a graphical representation during the interview. One participant even made some connections between the symbolic and formal worlds with a description of the slope of a tangent line. Overall,

while the majority of representations used by these participants are symbolic, each participant demonstrates at least one connection between the symbolic and embodied or formal worlds.

P5: So first I did the implicit differentiating (*pointing to page*)

Implicit differentiation

$$2x + 4y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1-2x}{4y}$$

$y = 1$ slope

$$\frac{dy}{dx} = 1 = \frac{1-2x}{4y}$$

Sub x = $4y = 1-2x$

$$4y + 2x = 1$$

P5: The derivative is the slope of this implicit function. To find where that happens you have to set it equal to 1. (*points to work on page*)

P5: ...x squared plus 2y squared equals 1 is an ellipse. So I was thinking of an ellipse with a ...would it be a radius of 1 (*moves fingers in an ellipse motion*). It should have multiple points where the slope is 1 (*points to different places*)

A: Would you like to draw it, you keep waving your fingers around?

P5: yeah, sorry (*laughs and takes pencil*). So if it is an ellipse...I forget exactly what the ellipse equations are...then it would be...there could be more than 1 point with the slope of 1, depending on where you are on the ellipse.

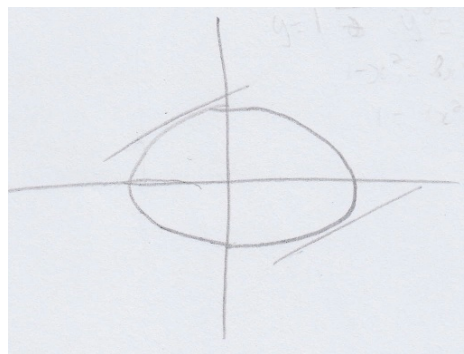


Figure 36. Example of Question #1 for Category (D).

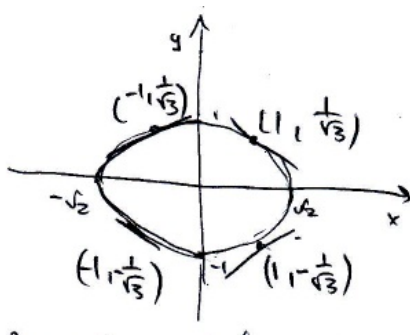
This figure provides an example of a participant who used symbolic representations when completing the written work and then spoke about the problem in terms of graphical representations. The participant even drew an informal graphical representation during the interview to better illustrate what he was talking about. Thus, this example clearly fits Category (D) as it demonstrates that connections were being made to at least one representations that was not symbolic

Category H (Strong Connections/Multiple Representations) - Further Analysis:

Only one participant had a solution that fit into this category for Question #1. This solution began by drawing the graph of the equation and finished with the participant plotting the solution on the graph to check if the solution seems correct. The written work and interview is a mix of symbolic and embodied representations with connections between the two worlds being made at almost every step of the problem solving. An example of this participant's work is included in the figure below. As it can be seen, the participant moved between and connected embodied and symbolic representations while solving Question #1.

P4: Ahh, I know that the question has given me a function and it asks me, on the function where does the slope...where does the tangent have a slope of 1. So if we look at this graph, the tangent line would just be a line that best approximates the curve at that point *(points to the graph she drew)*. So there are infinitely many tangent lines but there are just some that have a value of 1 and we have to find those values where the tangent line will exist.

the equation ~~is~~ represents a relationship which graphs to be an ellipse.



P4: ...well we know that $x^2 + y^2 = r^2$, which is a circle. And this is just a variation of a circle, it is an ellipse [circle] with...I don't know how to really call this...a radii of root 2. So I drew the graph like this and then I got the value of plus or minus 1 over root 3 for my x values. And for both of these x values I get the y value of 1 over plus or minus root 3. So I just plot all of these points and I drew tangent lines at these points and from the graph I can know...I can see only at these two points are the slopes positive. At these two the slopes are negative. So these two points are the answers that I want. *(points to the page to show the steps she took)*

The points,
 $(-1, \frac{1}{\sqrt{3}})$ & $(1, -\frac{1}{\sqrt{3}})$
 has a tangent of 1

Figure 37. Example of Question #1 for Category (H).

Question 2:

Given the piecewise function

$$g(x) = \begin{cases} x & \text{if } x < 1 \\ 3 & \text{if } x = 1 \\ 2 - x^2 & \text{if } 1 < x \leq 2 \\ x - 3 & \text{if } x > 2 \end{cases}$$

Find the limits

$$\lim_{x \rightarrow 1^-} g(x)$$

$$\lim_{x \rightarrow 1} g(x)$$

$$\lim_{x \rightarrow 2^-} g(x)$$

$$\lim_{x \rightarrow 2^+} g(x)$$

$$\lim_{x \rightarrow 2} g(x)$$

Show your work and explain your process and thinking.

Category A (No Connections/All Symbolic) - Further Analysis:

Only one response fits into this category for Question #2. This participant used symbolic representations for her written work and also described her understanding with only symbolic representations during the interview. There was nothing in her explanation on how she solved the problem to indicate any other representations from or connections with other worlds. An example of her work is included in the figure below.

P2: Well, I knew I had to choose which piece to use for each limit. (*points to the pieces and limits*)

A: Okay.

P2: Yeah, then just basically sub in the values because there is no denominator or anything so I know there was probably not going to be any limits, I mean any holes or asymptotes or things like that. (*talks with hand gestures for emphasis*)

$$g(x) = x \quad \lim_{x \rightarrow 1^-} x = 1$$

$$\lim_{x \rightarrow 1} \text{g(x)} = 3$$

$$\lim_{x \rightarrow 2^-} 2 - x^2 = -2$$

$$\lim_{x \rightarrow 2^+} x - 3 = -1$$

$$\lim_{x \rightarrow 2} 2 - x^2 = -2$$

Figure 38. Example of Question #2 for Category (A).

Category D (Mostly Symbolic with some Connections)- Further Analysis:

There were five participant responses that fit in this category, which all began by using symbolic representations. Four of the participants just stated the answers to the limits and included explanation statements next to the answers as to why the limits equaled what they did. One participant did this as well but also included the intervals that the limits exist in. However, the way this question was finished differed between participants. For example two participants made symbolizing-formalism statements about their answers, one participant drew a graph, one participant drew a number line diagram of left and right limits, and one participant left this question incomplete. Four

of the participants in this category were making connections between the symbolic and embodied worlds, with one also making connections to the formal world. One participant only connected between the symbolic and formal worlds. Overall, even though the questions were finished with different methods and representations, they all essentially started this question with the same method, demonstrating that they may have had similar problem solving methods in mind.

P5: Yeah, so the first one was taking the limit as x approaches 1 from the left (*points to work on page*). And when you evaluate a piecewise function if it is just from one side you just have to check which side it is going on (*points to two different sides*). If it is going from the left then it is the piece that is going from the left (*sweeps hand in from the left*). So that was the first 1 where x is smaller than 1.

Handwritten work showing limit calculations for a piecewise function $g(x)$:

- $\lim_{x \rightarrow 1^-} g(x) = 1$ (1st statement of piecewise)
- $\lim_{x \rightarrow 1} g(x) = \text{DNE}$ b/c $\lim_{x \rightarrow 1^-} g(x) \neq \lim_{x \rightarrow 1^+} g(x)$
- $\lim_{x \rightarrow 2^-} g(x) = -2 \rightarrow (2 - x^2) = (2 - 2^2) = -2$
- $\lim_{x \rightarrow 2^+} g(x) = -1 \rightarrow (x - 3) = (2 - 3) = -1$
- $\lim_{x \rightarrow 2} g(x) = \text{DNE}$ b/c $\lim_{x \rightarrow 2^-} g(x) \neq \lim_{x \rightarrow 2^+} g(x)$

A: Okay

P5: So if we are going from 1 to the left it is x and so it is going to equal 1 (*points to work on page*).

A: So why is it going to equal 1?

P5: Because it is going to the left (*sweeps hand towards the left*)...going towards 1 on the left and at that point the piecewise function is x . So as it approaches from the left it is x so it is approaching 1.

Figure 39. Example of Question #2 for Category (D).

In the example above the participant demonstrated in the interview that he was thinking about the limits in terms of what they were doing physically on a graph. From

this it shows that he was connecting to graphical or embodied representations while using the symbolic representations. The participant also wrote a formalizing-symbolism statement for his second and fifth limit, which shows that he was starting to connect to the formal world.

Category E (Tasks Symbolic/Interview Embodied with some Connections)- Further Analysis:

Category E is unique to Question #2 and has only three responses that fit into this category. I speculate that this might have something to do with the type of question that was being asked, which I will elaborate on further in the Discussion chapter. The three participants in this category began their solutions by mentally picturing the graph of the function in their heads. They then continued to write down only symbolic representations and explanation statements for their written work. This mental picturing of the graph became apparent during the interview portion of the data collection and demonstrated that the participants were making connections between the embodied and symbolic worlds, even if only symbolic representations were used in the written work. An example is included below.

P8: So I pictured the graph in my head...cause I knew it wasn't just a full graph (*moves hand in a curve motion*) because this is piecewise. So what I saw was when x is approaching 1 from the left, so I kind of looked at which one it would be. And since this is less than 1 it would just be approaching from the left side and it would be like this part of it (*moves hand from the left*). And same with the next one, like x or the limit of x as it goes to 1 is close to 3 cause x equals 1. And then 2 from the left, then I just saw it here, it was 2 to the left. And then 2 to the right, is greater than 2 so it is coming that way. And then when x equals 2 it is that one because 2 is less than or equal to 2. (*points to the different parts of the question on the page*)

Show your work and explain your process and thinking.

$\lim_{x \rightarrow 1^-} g(x) = x$ $\lim_{x \rightarrow 1^-} g(x) = 1$	$\lim_{x \rightarrow 1} g(x) = 3$	$\lim_{x \rightarrow 2^-} g(x) = 2 - x^2$ $\lim_{x \rightarrow 2^-} g(x) = 2 - (2)^2$ $\lim_{x \rightarrow 2^-} g(x) = -2$	$\lim_{x \rightarrow 2^+} g(x) = x - 3$ $\lim_{x \rightarrow 2^+} g(x) = (2) - 3$ $\lim_{x \rightarrow 2^+} g(x) = -1$
$\lim_{x \rightarrow 2} g(x) = 2 - x^2$ $\lim_{x \rightarrow 2} g(x) = -2$			

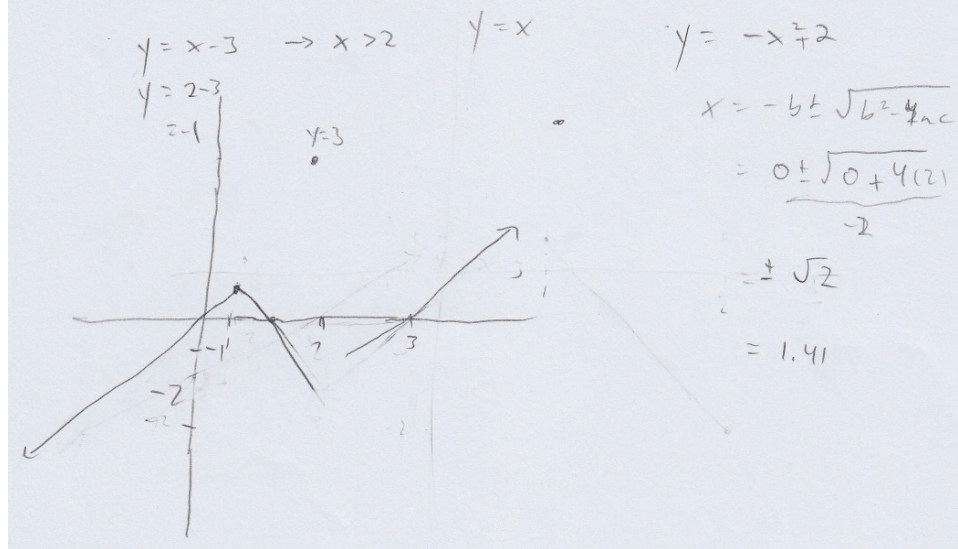
Figure 40. Example of Question #2 for Category (E).

Category H (Strong Connections/Multiple Representations) - Further Analysis:

The six participant responses placed in this category all began the question the same way, which was to graph the function using embodying symbolism representations. Of the six responses, four of them finished the question by finding the answers to the limits in the question by looking at the graph that was drawn. One participant also did this and then continued to state a formal rule about why a specific limit does not exist. One participant stopped after drawing the graph and stated that he could not remember how to solve the question. Overall, every participant was connecting between the embodied and symbolic worlds, with embodying-symbolism, when they drew the graph of the function. The participants who used the graph to find the limits were symbolizing-embodiment. The participant who made the formal

explanation statement was also formalizing-symbolism. Overall, the responses in this category demonstrated a lot of connections taking place between mathematical worlds.

P9: I took the individual equations and subbed in the limit like where the limit is and then I graphed them. So this would be the y equals x, this is where the y equals 1, so it would be the parabola one, and this would be where x minus 3 is. *(talks with hands, makes gestures while speaking and also points to the different work on the page)*



$$\begin{aligned} \lim_{x \rightarrow 1^-} g(x) &= 1 \\ \lim_{x \rightarrow 1} g(x) &\in \text{systems } \pm 1 \\ \lim_{x \rightarrow 2^-} g(x) &= -2 \\ \lim_{x \rightarrow 2^+} g(x) &= -1 \\ \lim_{x \rightarrow 2} g(x) &= \text{DNE} \end{aligned}$$

Figure 41. Example of Question #2 for Category (H).

From the example above it demonstrates that the participant was clearly connecting and moving between the embodied and symbolic worlds as she describes using the pieces of the function to find the limit and graph what the piece would look like as part of her problem solving process.

Question 3:

Given the function $f(x) = 3x^5 - 5x^3$

On what interval(s) is the function increasing?

On what interval(s) is the function decreasing?

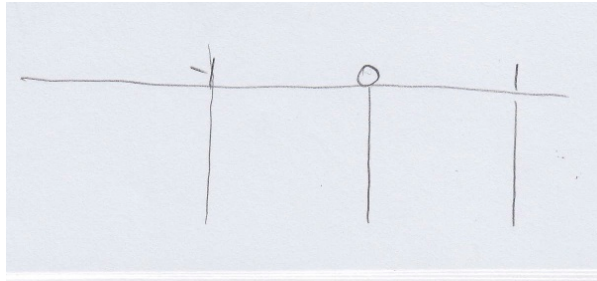
What is(are) the local maximum(s)?

What is(are) the local minimum(s)?

Show your work and explain your process and thinking.

Category B (No Connections/Multiple Representations)- Further Analysis:

There were two participants who had a response that fit into this category for Question #3. The responses had written work that consisted of mostly symbolic representations. One response also had a graph (embodying-symbolism) and a number line (symbolizing-embodiment), and the other referred to the graph of the function during the interview. However, during the interview the participants were questioned about these graphs and the number line. For both participants the interviews stated that they did not use them to solve the question and was only putting them there as a rote action remembered from class. Thus, the graphs and number line representations were not connected to the symbolic representations. An example from one participant's work is included below.



P9: yeah. Then I forgot what it meant

A: Okay, but you remembered it from class

P9: yeah, cause it is like the intervals right. Where you are increasing and decreasing, but I don't remember which values to put in there. *(makes intervals with hands and then moves hand up for increasing and does for decreasing)*

Figure 42. Example of Question #3 for Category (B).

Category D (Mostly Symbolic with some Connections) - Further Analysis:

There were five participant responses for this category. Four of the five responses began the question by taking the derivative of the function. One participant started by factoring the function, realized the mistake in this step and then differentiated. All of the derivative work involved symbolic representations. The participants also all finished the question by stating the answer with statements that used some symbolic representations. One participant also stated the inflection points of the function, even though this was not asked for. Every participant connected between the embodied and symbolic worlds in the form of a number line (symbolizing-embodiment) used in the first row of a table. Three participants also used graphical representations of the function, and two used an embodied representation of arrows for increasing and decreasing. Overall, the participants' solutions that fit into this category all essentially used the same method and representation connections when

solving.

	$15x^2$	$x+1$	$x-1$	$f'(x)$
$(-\infty, -1)$	+	-	-	+ /
$(-1, 0)$	+	+	-	- /
$(0, 1)$	+	+	+	- /
$(1, \infty)$	+	+	+	+ /

P15: I just put the roots, like you always want to make it equal to zero. So if you put zero there, and then negative 1 and 1 would be zero so I just put these in

A: Okay

P15: And then to find these I picked a number between the intervals (*points to the cells in the table and describes how she found the entries*). So like -1000 and you plug it into there and you plug it into there and you plug it into there. And you would just sum up these signs, like a negative and negative makes positive, plus positive is positive

Figure 43. Example of Question #3 for Category (D).

In the example in Figure 43 the participant used upwards dashes and downwards dashes next to the + and - signs in the last column. These dashes denote whether the function is increasing or decreasing on the graph. This shows that the participant was thinking about what the increasing and decreasing would mean to the function in a graphical representation.

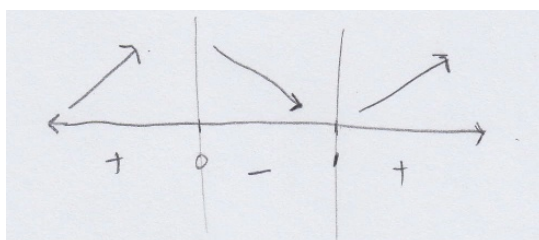
Category F (Multiple Representations with Connections/Memorized Process)-

Further Analysis:

This category is unique to Question #3 and also only has one response that fits into it. This suggests that this category may be the result of the type of question that was asked for Question #3 or the result of how the participant understands questions like #3. The participant's written work was very similar to the written work that fits

into Category H for the same question. The difference is that during the interview the participant was not able to describe why he was using multiple representations from different mathematical worlds and instead described his method as strictly memorized. The example below includes one of the quotes that demonstrate that the participant was using a memorized process that he learned in class.

P8: And then I just plugged in values less than zero so then I saw that it was positive so it was increasing (*moves hand up*) and then between zero and 1 it was decreasing (*moves hand down*) and the same with the 1.



A: Okay, Can you explain to me, um, what, why positive means increasing?

P8: I don't, but I just knew, I just knew in class that it was told to us that if it is positive it is increasing and if it is negative it is decreasing.

Figure 44. Example of Question #3 for Category (F).

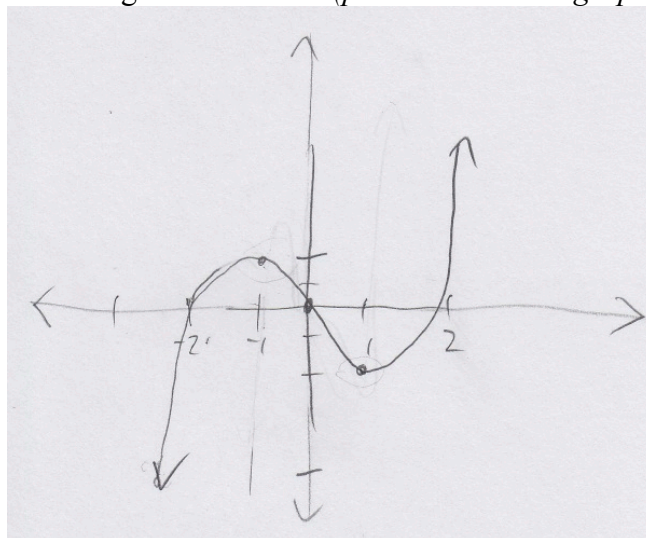
Category G (Only Graphical Solution) - Further Analysis:

This category is unique to Question #3 and also only has one response that fits into it. This suggests that this category may be the result of the type of question that was asked for #3. I speculate that this is the case since Category G describes a solution that I had included in Appendix C of potential solutions. Overall, the participant solved this question without using any calculus at all but instead used the graphing calculator to produce a graph of the function and finding the answers from that graph. This does show the ability to connect between the symbolic and embodied worlds with a graphical representation. This solution and category are interesting as none of the

participants reported using technology in their classes. An excerpt from this participant's written work and interview is included below.

A: So now did you graph this using a graphing calculator?

P11: yeah. So then I graphed it and then wherever it was increasing I wrote that and where it was decreasing I wrote that... *(points to work on graph)*



- a) The function is increasing b/w $-\infty < x < -1$, and b/w $1 < x < \infty$.
- b) The function is decreasing when $-1 < x < 1$.
- c) local max is when $x = -1$
- d) local min is at $x = 1$

Figure 45. Example of Question #3 for Category (G).

Category H (Strong Connections/Multiple Representations)- Further Analysis:

Each of the five participant responses in this category began by differentiating the function with symbolic representations. However, one participant also stated that she wanted to sketch the curve and another stated what the derivative function would be doing when it is increasing or decreasing. The responses however, finished differently. Two participants sketched the graph of the function (although this was not

required for the question). Three participants stated the answers to the question using symbolic representations and one used a combination of symbolic and embodied representations. Overall, every participant used a number line in his or her solution, which is symbolizing-embodiment. As well each participant described during the interview how he or she understood the connections between the symbolic and embodied world with regards to this question.

P14: Um, first of all, at first I was going to answer it giving a whole bunch of extra information. But I realized that there is no need for me to necessarily say certain details. So I decided to just go with finding the derivative, well, first derivative, in order to get the increasing and decreasing for the function (*points to the work on the page and gestures with hands while talking*)

$$\begin{aligned}
 f(x) &= 3x^5 - 5x^3 \\
 f'(x) &= 15x^4 - 15x^2 \\
 0 &= 15x^2(x^2 - 1) \\
 15x^2 &= 0 \quad \leftarrow \quad x^2 - 1 = 0 \\
 x^2 &= 0 \quad \quad \quad x^2 = 1 \\
 & \quad \quad \quad x = \pm 1 \\
 & \therefore x = \pm 1
 \end{aligned}$$

P14: And over here, I was just reminding myself that local max is going to be the highest point (*makes a curve up and points to highest point*) of the if there is a curve and local min is obviously the lowest (*makes a curve down and points to the lowest point*). And that means it is the point where, I actually explain it over here, (*points to work on page*) it is when the intervals change signs. So when it goes from increasing to decreasing it is going to be the local max and when it goes from decreasing to increasing it is local min. (*moves hand up and down and then down and up*)

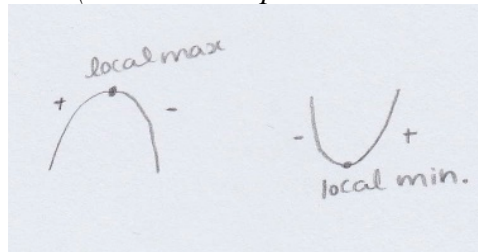


Figure 46. Example for Question #3 for Category (H).

As can be seen in this example, the participant used embodied representations to serve as a reminder about how to interpret the results from the symbolic representations. Her answer to this question continued in this manner, by using both symbolic and embodied representations in the written work and the interview. This demonstrated a lot of connections being made.

Question 4:

Given the function $f(x) = x - \frac{1}{(x^2)(x+2)}$

Find:

$$\lim_{x \rightarrow -2^+} f(x)$$

$$\lim_{x \rightarrow -2^-} f(x)$$

$$\lim_{x \rightarrow 0} f(x)$$

Show your work and explain your process and thinking.

Category A (No Connections/All Symbolic) - Further Analysis:

Question #4 had three participants with responses for this category. This may be connected with the type of question that is asked in Question #4. Regardless, all of the responses in this category used only symbolic representations for the written work. As well, all of the representations talked about in the interview were also symbolic. An example of a response in this category is included below.

P14: I guess I am someone who likes to put everything as one and not try to deal with...I mean like for me its... *(points to the work on the page while talking)*

*I vaguely remember
the formula for a limit...
I think?*

$$\lim_{x \rightarrow 2^+} f(x) = \frac{f(x+h) - f(x)}{h}$$

$$= \frac{\left(\frac{(x+h)(x+h)^2(x+h+2) - 1}{(x+h)^2(x+h+2)} \right) - \left(\frac{x(x^2)(x+2) - 1}{(x^2)(x+2)} \right)}{h}$$

Figure 47. Example of Question #4 for Category (A).

Category B (No Connections/Multiple Representations)- Further Analysis:

The four participant responses in this category are mostly completed with symbolic representations. Three of the participants did use some representations that could be considered symbolizing-formalism. However, during the interview it was apparent that the participants who wrote symbolizing-formalism representations did not actually use these representations to solve the question and were not connecting them to the solutions. As well, one participant mentioned picturing a number line during the interview, but also stated that this did not affect the way he solved the question. The example below shows how the participant made some formalizing-symbolism statements but solved the problem by using symbolic manipulation.

P6: Kay, so I saw that as x approaches these numbers, these...the x was not going to have anything to do with the actual limit, so you could just disregard it. So if you plug in a value for x as it approaches -2 from the right, like -1.9999 or whatever, you will get a really big number. But there is a negative here, so it will be negative infinity. (*points to question and work on page while answering*).

Handwritten work showing limit calculations and reasoning:

Plug in values as $x \rightarrow -2^+$, and as $x \rightarrow -2^-$, and as $x \rightarrow 0$.

Since: the x out front will not make a difference, we can ignore it for all of the limits because just by looking at them, they will either be positive or negative infinity.

$\lim_{x \rightarrow -2^+} f(x) = -\infty$
 $\lim_{x \rightarrow -2^-} f(x) = \infty$
 $\lim_{x \rightarrow 0} f(x) = -\infty$

$\lim_{x \rightarrow -2^+} f(x) = -\infty$ $\lim_{x \rightarrow -2^-} f(x) = \infty$
 and $\lim_{x \rightarrow 0} f(x) = -\infty$

P6: ...you get a small number in the denominator, so that makes it infinity

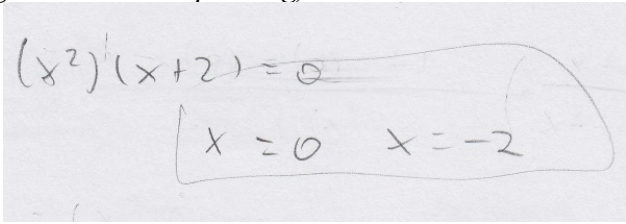
Figure 48. Example of Question #4 for Category (B).

Category D (Mostly Symbolic with some Connections)- Further Analysis:

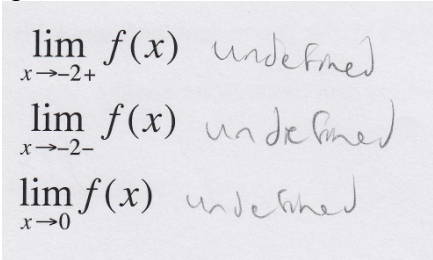
The six participant responses in this category started this question with a variety of methods and representations. One participant tried to sketch the graph with the help of the graphing calculator. Two participants made statements about the limit values being asymptotes using embodying-symbolism representations, two participants began by solving the first limit with symbolic representations, and one participant put the function under a common denominator. However, there was more consistency between how the participants finished the question. They all concluded by stating what they found to be the answers to the limits. Two participants found the limits to be undefined, while the others did not. One participant made a symbolizing-

formalism statement and another drew a graph of the limits as they approach the asymptote points. Thus, despite starting this question with different methods, the participants in this category all concluded by stating the answers to the limits and some even offered further explanations as to why. Half of the participants made some connections to the formal world, and all but one made connections between the symbolic and embodied worlds when answering this question.

P9: Because that's where the bottom is zero and they all are undefined (*talks with hands, makes gestures while speaking*)



A: so you found that all of these limits were undefined because you were able to find that they were all asymptotes.



P9: Yep

Figure 49. Example of Question #4 for Category (D).

The example in Figure 50 shows how the participant was able to connect to a formal understanding of fractions with zero in the denominator. While the limits could be solved with real numbers, the solutions given are correct according to the participant's interpretation of the question.

Category H (Strong Connections/Multiple Representations)- Further Analysis:

Only two participants had responses that fit into this category for Question #4. However, the two responses were very similar to each other. They both started the question by thinking about what the graph of the function looked like, which is embodying-symbolism. They also both finished the question by plugging in values to the limits. Both participants diagramed their understanding of the asymptotes of the function. Overall, the responses in this category showed a lot of connections between the embodied and symbolic worlds, and even some connections to the formal world.

A: So here we are given a function and we are asked these three limits. So it says, x does not exist when x is equal to 0 and x is equal to -2.

$f(x)$ does not exist for $x = 0$ and $x = -2$

P13: Yeah. These are asymptotes, or asymptotes are discontinuities. So then I used the calculator to plug in values on either side of the asymptotes to find out the behaviour roughly. So I see here that around 1 on one side...oh, sorry about 0, right.

There are asymptotes/discontinuities

Use Calculator plug in vldes of x on either side of asymptotes/discontinuities to see fcn behavior (roughly)

$$F(0.1) = -47.519$$

$$F(-0.1) = -52.73$$

$$F(2.1) = 2.0446$$

$$F(1.9) = 1.8289$$

$$F(2.001) = 1.93857$$

$$F(1.999) = 1.93642 \quad \left. \vphantom{F(1.999)} \right\} \text{close to } 2$$



A: You were finding they were both negative on both sides. So how did that help you answer this question?

P13: I just saw this in my head so I decided to draw this down. (points to work on page)

Figure 50. Example of Question #4 for Category (H).

This above example demonstrates how the participant used both embodied and symbolic representations when solving this problem. It also demonstrates how these embodied and symbolic representations were connected to each other. Thus, this example is clearly situated in Category (H).

Summary of Further Analysis Findings

Overall, the findings from the further analysis provide a means by which to describe in more detail *how* the participants are making connections between

mathematical worlds, which helps provide a detailed answer to the research question of how undergraduate students make connections. The results from this part of the analysis also demonstrated that participant responses categorized together in analysis Step #3 were done so appropriately based on the representations that were used, how they were used, and how connections were made between the different representations. The further analysis also demonstrates that there is a lot of consistency between participants' responses within the same *Response Categories* and therefore, the *Response Categories* are likely an accurate way to classify the participants' responses.

A Note on the Trustworthiness of the Response Categories

Creswell (2007) describes validation or trustworthiness of a qualitative study as a process that can take many different forms and provides examples of a number of strategies that he recommends researchers employ to demonstrate the trustworthiness of their findings. During the data analysis I undertook a number of steps and used some of the recommended strategies to ensure the trustworthiness of the *Response Categories*, as well as the classification of the participants' responses according to the categories. Specifically, the research was designed to use multiple sources of data, in the forms of the written work from the task questions and the interviews, which probed the written responses and gathered other relevant data. The multiple sources and intense probing provided the opportunity to more fully understand how students were approaching the problem so that it was clear where the representations lie. Creswell states that the benefit of multiple data sources is that it allows for corroborating evidence to bring to light themes from data. As the analysis progressed I

readily refined my working hypotheses when needed, and allowing me to ensure that “all cases fit” (Creswell, 2007, p. 208). In other words, I refined my research question and my data analysis to include the formal world of mathematics when it became apparent that some of the participant responses were demonstrating connections being made to the formal world. I then used a peer review process, another strategy described by Creswell, with my thesis supervisor who reviewed my development of the *Response Categories* at every stage. This was to help ensure the trustworthiness of my findings, interpretations, and conclusions and confirm that they were supported by the data. Once I was comfortable with the analysis I had completed and the development of the *Response Categories*, I made sure to include rich and thick descriptions throughout the thesis that described the data analysis process, the *Response Categories*, and the classifying of the participants’ *Layered Analyses* according to the *Response Categories* so that the classification was convincing to the reader. I also included multiple examples of the participants’ work in my thesis to substantiate the classifications. Creswell comments that rich and thick descriptions, like the ones I included, allow for transferability of the research. Lastly, once the rich descriptions were prepared and the examples included, my thesis committee had the opportunity to review the *Response Categories* and the classifications. This again offered another form of peer review that helped ensure the trustworthiness of this part of the data analysis.

Findings from the Correctness Analysis

The data analysis that took place in Step #5 looked at a participant’s correctness of method and answer. This was in order to further describe how participants made connections and if making connections was related to the correct use of a problem

solving method or answer. Analyzing correctness would also provide a good means by which to relate the findings from the research back to previous research using similar theoretical perspectives. In particular it was to see if the findings from this research could be related back to the literature that suggests that individuals who connect between mathematics worlds (e.g. Nogueira de Lima and Tall, 2008; Stewart and Thomas, 2007) and between multiple representations (e.g. Davis, 2007; Herbert 2009; Knuth, 2000) are more successful in receiving the correct answer on mathematical problems.

The findings from the correctness analysis were recorded in two tables for each task question. One table recorded the analysis of method correctness, while the other recorded the analysis of answer correctness. Method correctness analyzed the procedure, technique, or way of solving the question. Answer correctness analyzed only the final part of the solution that addresses what the question was asking. These tables are presented below with some of the findings discussed for each task question.

Correctness Analysis: Question #1

Method Correctness	Started solution with an appropriate method	Solution started with an inappropriate method/or an unnecessary step
Continued and finished solution with an appropriate method	#3, #4, #6, #7, #10, #13, #14	
Continued and finished solution with an inappropriate method	#8, #12	#15
Solution was left incomplete	#1, #2, #5, #9, #11	

Figure 51. Task Question #1 method correctness chart.

Answer Correctness

Completely correct answer	Partially correct answer (partially incorrect)	Incorrect answer because of arithmetic mistake	Incorrect answer/or incomplete
#6, #14	#13	#3, #4, #7, #10	#1, #2, #5, #8, #9, #11, #12, #15

Figure 52. Task Question #1 answer correctness chart.

For task Question #1, only seven of the fifteen participants were able to use an appropriate method for their entire solution to the question. Of these seven participants who used an appropriate method for the entire solution, only two participants were able to come to the correct answer. However, one participant was able to find one of the correct points and four participants made small arithmetic errors that made their answer incorrect. For this question, if a participant did not both start with and continue to use a correct method, the solution was then either incorrect or left incomplete. This question actually had the highest number of participants who left their solution incomplete or got an incorrect answer (8/15).

Correctness Analysis: Question #2

Method Correctness	Started solution with an appropriate method	Solution started with an inappropriate method/or an unnecessary step
Continued and finished solution with an appropriate method	#2, #3, #4, #5, #6, #7, #8, #9, #10, #11, #12, #15	
Continued and finished solution with an inappropriate method		#1
Solution was left incomplete	#13, #14	

Figure 53. Task Question #2 method correctness chart.

Answer Correctness

Completely correct answer	Partially correct answer (partially incorrect)	Incorrect answer because of arithmetic mistake	Incorrect answer/or incomplete
#2, #8	#3, #4, #5, #6, #7, #9, #10, #12	#11, #15	#1, #13, #14

Figure 54. Task Question #2 answer correctness chart.

Question #2 had the highest number of individuals who started and continued with an appropriate method (12). While only two participants got the answer completely correct, eight participants had the answer partially correct. I believe this is due to the question asking for five different limits to be found from one function, which allowed for the possibility to get some correct and some not. These eight participants were able to find some of the correct answers, while making mistakes on others. As well, two participants made arithmetic mistakes that led to incorrect answers, but still used appropriate methods. Overall, the majority of participants were using a correct method and finding at least partially correct answers.

Correctness Analysis: Question #3

Method Correctness	Started solution with an appropriate method	Solution started with an inappropriate method/or an unnecessary step
Continued and finished solution with an appropriate method	#2, #3, #4, #5, #6, #7, #10, #11, #13, #14	#15
Continued and finished solution with an inappropriate method	#9, #12	#1, #8

Figure 55. Task Question #3 method correctness chart.

Answer Correctness

Completely correct answer	Partially correct answer (partially incorrect)	Incorrect answer because of arithmetic mistake	Incorrect answer/or incomplete
#5, #10, #13, #14	#2, #4, #6, #7, #11, #12	#3	#1, #8, #9, #15

Figure 56. Task Question #3 answer correctness chart.

Most participants started and completed Question #3 with a correct method (10). However, only four participants were able to get a completely correct answer. Of the six participants who provided a partially correct answer, four of them made the same mistake, which was to state that the function was decreasing from -1 to 1, including the point 0. However, the function is not decreasing at 0, it is an inflection point. The two other participants with partially correct answers made mistakes calculating the maximum and minimum values.

It is also interesting that no participants left this question incomplete. Why this

is the case is unclear and requires some further investigation, although I would hypothesize it might have something to do with Question #3 being very similar to questions the participants would have seen in their calculus courses. Overall, it appeared that the majority of the participants had a correct method for which to solve this question and were able to get partially or completely correct answer.

Correctness Analysis: Question #4

Method Correctness	Started solution with an appropriate method	Solution started with an inappropriate method/or an unnecessary step
Continued and finished solution with an appropriate method	#2, #3, #5, #6, #8, #9, #10, #13, #15	#4, #7
Continued and finished solution with an inappropriate method	#11	#1, #12
Solution was left incomplete		#14

Figure 57. Task Question #4 method correctness chart.

Answer Correctness

Completely correct answer	Partially correct answer (partially incorrect)	Incorrect answer because of arithmetic mistake	Incorrect answer/or incomplete
#3, #6, #7	#2, #4, #5, #9, #10, #15	#8, #13	#1, #11, #12, #14

Figure 58. Task Question #4 answer correctness chart.

Many participants attempted this question by beginning and continuing with a correct method (9/15). However, similar to Question #2, only three participants

provided a completely correct answer. The six participants with partially correct answers either got one or two of the limits correct but not all three. In some cases participants wrote the answer as undefined or DNE, instead of with positive or negative infinity. Writing undefined or DNE is partially correct, as it is the case for real numbers. As well, only one participant left the question incomplete, while three others used inappropriate methods and produced incorrect answers.

Comparison of Analysis by Question to Correctness Analysis.

A comparison of the participants' responses based on the *Response Categories* to *Method Correctness* and *Answer Correctness* provided a few interesting findings but nothing that appears remarkably significant in terms of addressing the research question of this study. The findings from this comparison are presented below.

The comparison was done for each task question. It first linked the participants in different *Response Categories* to their *Method Correctness*. Once this was completed, a participant's *Answer Correctness* was also added. This information was summarized into four tables, one for each task question, for easy review. In the tables, acronyms were used for the purpose of simplifying. In particular, CC stands for Complete Correct, PC stands for Partially Correct, AM stands for Arithmetic Mistake, IN stands for Incorrect, and IC stands for Incomplete. Table 5, Table 6, Table 7, and Table 8 were made for task Question #1-#4 respectively.

Table 5.

Question #1 Comparison of Response Categories with Correctness

Started and finished solution with an appropriate method							
Participant	#3	#4	#6	#7	#10	#13	#14
Response Category	D	H	D	B	D	D	D
Answer Correctness	AM	AM	CC	AM	AM	PC	CC
Started solution with an appropriate method and finished in an inappropriate method							
Participant	#8				#12		
Response Category	B				C		
Answer Correctness	IN				IN		
Started and finished solution with an inappropriate method							
Participant	#15						
Response Category	C						
Answer Correctness	IN						
Started solution with an appropriate method and left incomplete							
Participant	#1	#2	#5	#9	#11		
Response Category	A	D	D	A	A		
Answer Correctness	IC	IC	IC	IC	IC		

Table 6.

Question #2 Comparison of Response Categories with Correctness

Started and finished solution with an appropriate method												
Participant	#2	#3	#4	#5	#6	#7	#8	#9	#10	#11	#12	#15
Response Category	A	E	H	D	H	D	E	H	E	H	D	H
Answer Correctness	CC	PC	PC	PC	PC	PC	CC	PC	PC	AM	PC	AM
Started and finished solution with an inappropriate method												
Participant	#1											
Response Category	D											
Answer Correctness	IN											
Started solution with an appropriate method and left incomplete												
Participant	#13						#14					
Response Category	H						D					
Answer Correctness	IC						IC					

Table 8.

Question #4 Comparison of Response Categories with Correctness

Started and finished solution with an appropriate method									
Participant	#2	#3	#5	#6	#8	#9	#10	#13	#15
Response Category	B	B	H	B	B	D	D	H	D
Answer Correctness	PC	CC	PC	CC	AM	PC	PC	AM	PC
Started solution with an appropriate method and finished in an inappropriate method									
Participant	#11								
Response Category	D								
Answer Correctness	IN								
Started and finished solution with an inappropriate method									
Participant	#1				#12				
Response Category	A				A				
Answer Correctness	IN				IN				
Started with an inappropriate method and finished with an appropriate method									
Participant	#4				#7				
Response Category	D				D				
Answer Correctness	PC				CC				
Started solution with an appropriate method and left incomplete									
Participant	#14								
Response Category	A								
Answer Correctness	IC								

From Table 5 we can observe that for Question #1 the three incorrect answers were from participants who had responses in categories without connections between mathematical worlds. As well, 3/5 of the incomplete solutions were from *Response Category* (A): No Connections/All Symbolic, which includes responses that are all symbolic. However, 2/5 of the incomplete solutions were from *Response Category* (D):

Mostly Symbolic with some Connections, which are responses that are beginning to make connections between mathematical worlds. Lastly, it can be observed that 6/7 of the responses that started and finished with an appropriate method were from *Response Categories* (D) or (H): Strong Connection with Multiple Representations, which are responses that are making connections between mathematical worlds.

From Table 6 we can observe that for Question #2 the majority of responses had partially correct answers and that all of the partially correct answers were from *Response Categories* where connections were made. Question #2 actually had only one participant response in a *Response Category* with no connections. However, this response also had a Complete Correct Answer. Table 6 also shows that answers that were Completely Incorrect or Incomplete did not start and finish the solution with an appropriate method.

From Table 7 we can observe that for Question #3 there were no responses left Incomplete. For Question #3, all of the Completely Correct responses were from *Response Category* (H): Strong Connections with Multiple Representations. Question #3 had the highest amount of Incorrect answers, at 4/15, however, it also had no Incomplete solutions.

From Table 8 we can observe for that for Question #4 there do not appear to be any relationships between method correctness, answer correctness, and the *Response Category* that the participants' responses fit into. For example, of the 9/15 responses that started and finished the solution with a correct method, 4 were from *Response Categories* with no connections being made and 5 were from *Response Categories* with connections being made. Similarly, 2/3 of the Incorrect Answers were from *Response*

Category (A): No Connections/All Symbolic. However, the other 1 Incorrect Answer was from *Response Category (D): Mostly Symbolic with some Connections.*

An observation from every table is that participants who started and finished with an appropriate method had a Completely Correct, Partially Correct, or incorrect answer because of an Arithmetic Mistake. There were no Completely Incorrect or Incomplete solutions from participants who started and finished the answer with an appropriate method. However, overall it appears that the sample size of 15 participants for each task question was not big enough to uncover any significant findings in terms of the relationship between correctness and how connections are made. As well, how method and answer correctness related to the connections participants made was not originally a part of the research design of this study and was only looked at as a source of further reflection on the findings.

Summary of Participant Interview Responses

In this section I summarize information obtained during the interview related to the participants' experience taking calculus. This information was collected in order to observe if there was a relationship between a participant's responses to the interview questions and how a participant made connections in the written work. I begin with the responses to how each participant described calculus was taught. I then include the calculus topics they said they did the best and or worst in, if they shared this information. I next cover the participants' descriptions of how graphs and diagrams were used in the course. Lastly, I finish with a summary of how the participants described the balance between algebra, graphs and diagrams, and technology in their calculus courses.

How Calculus Was Taught: Summary of Participant Responses

When asked how calculus was taught, a common theme that appeared between the majority of participants was that the teaching focused on “solving problems”, “doing examples on the board”, or teaching the topics with “step-by-step” algorithms (Participant #1, #2, #3, #4, #5, #6, #8, #9, #10, #12, #15). For instance, one participant described the examples he saw in calculus class as being more “quantity than quality” (Participant #8). Only two participants, of the fifteen, mentioned that calculus was taught with a visual or graphical component. Participant #6 stated that his professor would draw graphs and diagrams at every opportunity. Participant #13 stated that his professor was good at demonstrating how to solve a question in more than one way, with a visual technique and an algebraic technique. Three participants mentioned the calculus was taught by beginning with some theory or rules, prior to the examples or solving problems (Participant #5, #10, #15). Two participants stated that they could not comment on how calculus was taught, only to say they struggled in class (Participant #11, #14).

Calculus Topics Participants Did Best and Worst In and Why: Summary of Participant Responses

Not every participant could comment on topics in calculus that they felt they did best or worst in. In some cases, this was attributed to the fact that they did not learn anything new, as their calculus class was just a review from high school (Participant #2, #3, #4, #5, #9, #15). For those who did comment on a topic they did best or worst in, there was no consistency between participants. For example, Participant #1 and Participant #11 stated that they did best on curve sketching questions because they just

had to memorize the steps to solve the questions correctly, while Participant #8 said the complete opposite in that curve sketching was his worst topic as there were too many steps to memorize. Three participants stated that they struggled with the topic of Integrals (Participants #3, #10, #15), with two of them claiming it was because it was the only new topic that was no review from high school (Participants #3, #15). However, Participant #6 stated that he did best in Integrals because it was the only new topic. Overall, there were no common themes that appeared when the participants were asked this question.

Description of How Graphs and Diagrams Were Used In Calculus: Summary

Participant Responses

The participants were specifically asked how graphs and diagrams were used in the course. This was in order to gather information about whether they were exposed to representations from the embodied world or representations that connected between the symbolic and embodied worlds. A number of participants mentioned that graphs and diagrams were mostly used when studying the topics of curve sketching and finding properties of a function (Participant #1, #2, #5, #8, #10, #11). Other topics that were mentioned were area under the curve (Participant #5) and finding equilibrium points (Participant #9). Another theme that appeared in the responses is that many participants described the way graphs and diagrams were used as supplementary to the algebraic manipulation portion of the solutions, or for explanation or visual aid purposes only (Participant #1, #2, #3, #6, #7, #8, #10, #13, #14, #15). One participant did describe how his professor would often begin with a graph or diagram and refer to it throughout the explanation of the algebraic steps,

which allowed the participant to connect between the two representations (Participant #12). Overall, only one participant stated that graphs and diagrams were used as both a primary and supplementary way to solve problems (Participant #4). These findings might suggest that not many participants were exposed to graphs and diagrams as a primary way to solve calculus problems, or were even exposed to the use of graphs and diagrams as a portion of a primary solution, which in turn could affect the connections made to these representations from the symbolic world.

Balance Between Algebraic Manipulation, Graphs and Diagrams, and Technology

In Calculus: Summary of Participant Responses

Since the majority of participants described how, in their experience, graphs and diagrams were used only for a supplementary or explanation purpose, it is no surprise that the majority of participants also reported that their courses were more heavily weighted towards algebraic manipulation (Participant #1, #2, #3, #4, #6, #8, #9, #10, #11, #12, #13, #14, #15). Two participants speculated that their calculus courses were taught with primarily symbolic representations since that is what is tested (Participant #12, 13). Only one participant felt that there was an even balance of algebraic manipulation and graphs and diagrams in her calculus course (Participant #5). The participants were also asked what the balance of technology was in their courses. This was in order to see if the participants were exposed to graphs or diagrams in technological representations. All of the participants reported that there was no technology used in the classroom. However, certain participants sought out their own technology outside of the classroom. For example, some participants used graphing software, either from a graphing calculator or on a computer (Participant #5, 6, 8, 11,

13). One participant also stated that he used the computer quite frequently to look up the calculus topics for his own personal studying (Participant #10).

Other Interesting Findings

As was mentioned in the Data Analysis addressed in the previous chapter, some investigation was conducted on three observations that appeared during the analysis process of the data. These observations were: (a) the use of a table in task Question #3 to find the properties of the function, (b) using the word “math” when referring to algebraic manipulation, as opposed to drawing graphs or diagrams, and (c) the frequency with which participants commented that first year calculus was mostly a “repeat” of what they learned in high school. The findings from investigating these three observations are described below.

The Use of a Table in Task Question #3

When investigating the observation of the use of a table representation in task Question #3, I began by making note of the number of participants who did use a table. From this I found that 11/15 of the participants used some form of table in their problem solving. There were 4 participants who did not use a table, however, 2 of the 4 did use a number line.

In terms of the 11 participants who used a table in Question #3, these tables can be divided into 3 distinct structures, which I have termed the Number-line Table, the Interval-Row Table, and the Interval-Column Table. The Number-line Table structure has numbers placed in the first row as though it is a number line. The numbers placed in the first row are the critical points found from taking the derivative of the function in Question #3 and solving the derivative when it is set equal to 0. The table is then

divided into boxes with the critical points on the lines. There are 3 tables that follow this structure. Of the 3 tables, 2 then have $f'(x)$ written as the first entry of the second row and then positive or negative values assigned in the other entry places. The one table that does not have $f'(x)$ in the second row is left incomplete and not filled in. An example of a Number-line Table is given below in Figure 60.

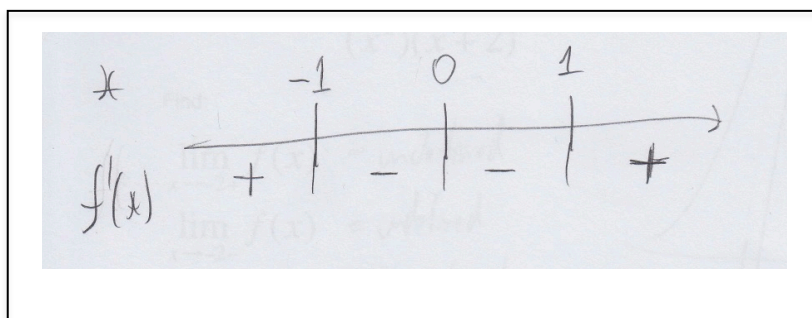


Figure 59. Example of a Number-line Table from task Question #3.

The Interval-Row Table was the most frequently used table structure, with a total of 6 out of 11. This type of table has intervals placed in the ascending order in the first row of the table. The intervals are made up of the values that lay less than, greater than, and in between the critical points found from solving the derivative of the function for 0. Boxes are then made for the remainder of the table with lines drawn between the different intervals. The first column of the Interval-Row Table all include $f'(x)$. Similar to the Number-line Table, the boxes are then filled in with positive or negative signs. Some of the Interval-Row Tables also have other column entries. For example, 4 of the tables also include a row that begins with $f(x)$. An example of an Interval-Row Table is given in Figure 61.

	$x < -1$	$-1 < x < 0$	$0 < x < 1$	$x > 1$
$f''(x)$	-	+	-	+
$f'(x)$	+	-	-	+
$f(x)$	∪	∩	∪	∩

$\wedge$ $\wedge$ $\wedge$
 inflection point inflection point inflection point

Figure 60. Example of an Interval-Row Table from task Question #3.

The Interval-Column Table is very similar to the Interval-Row Table, except that the intervals are placed in the first column instead of the first row. Again, these intervals are made up of the values that lay less than, greater than, and in between the critical points found from solving the derivative of the function for 0. There are only 2 tables that follow an Interval-Column Table structure. In both of these cases, the first row of the table is made up of the factored pieces of the derivative function with the last column labeled as $f(x)$. The tables are filled in with positive and negative signs. An example of an Interval-Column Table is given in Figure 62. Overall, the three table structures do not diverge very much in terms of the content they include and the mathematical context in which they are used

	$15x^2$	$x+1$	$x-1$	$f(x)$
$(-\infty, -1)$	+	-	-	+
$(-1, 0)$	+	+	-	-
$(0, 1)$	+	+	+	+
$(1, \infty)$	+	+	+	+

Figure 61. Example of an Interval-Column Table from task Question #3.

The participants' transcripts were also reviewed to see how different participants spoke about the tables they had drawn. As was mentioned, the observation of the repeated use of a table in Question #3 was not noticed until the Data Analysis steps. Thus, during the Data Collection, I did not tailor any questions during the interview that discussed the use of a table. However, 10 participants still made comments about the table. Of the 10 participants that mentioned the table during their interviews, 6 of the participants described the table as just part of the steps that needed to be taken to solve the question. Similarly, 6 participants also mentioned that using the table is how they were taught in class to answer these types of questions. Only 1 participant actually described how the information provided in the table is connected to the problem in the question and gave an explanation for why the table was used. However, I am inclined to believe that this is not representative of the number of participants who could have given a similar description if they had been asked. Again, specific questions about the table representation were not included in the interview. Overall, I speculate that this specific use of a tabular representation was most likely the result of instruction methods within the calculus courses, and not explicitly an attempt

to make connections between mathematical worlds. If this is the case, then using a table for questions like Question #3 is an example of how instruction methods can have a strong influence on representations that are used and connections that are made between mathematical worlds.

The Use of the Word “math”

During the interviews I also began to notice a pattern in that some of the participants would use the word “math” to describe the algebraic work or algebraic manipulation they had done, and not the graphing, diagraming, or drawing of figures. I wanted to investigate this further to observe it in more detail and see if there was some sort of relationship between what the participants were saying. In order to investigate this observation of participants using the word “math” when referring to algebraic manipulation, I first reviewed the participants’ interview transcripts and organized into a document all of the statements that used the word math. Three examples of participant statements that referred to algebraic manipulation as “math” are included below in Figure 63.

Participant #3

A: So now would you describe the graphs and diagrams in a primary role, secondary role, supplementary role in terms of the teaching?

P3: Secondary role. It was writing the equation and doing the algebra and math on the board (*mimics writing*) and the graphs were to supplement that with the teaching (*mimics drawing a graph*).

Participant #10

A: So was there anything visual going on in your mind while you solved?

P10: Nope, I simply, well at the beginning I imagined the shape so I just knew there was two equations and I knew what it was going to look like. And then I just forgot about it and did all of the math. (*talks with hands, makes gestures while speaking*)

Participant #15

A: Okay, so what would you say, from the lectures and stuff, what would you say the balance is between algebra and graphing and diagrams is

P15: The difference

A: The balance. Like how much is he putting...or how much is he doing one than the other

P15: well, there is like hardly any graphing, it is mostly like math stuff. And lately, it is not even numbers it is like letters (laughs)

Figure 62. Examples of statement with “math” for algebraic manipulation.

In total, 8/15 participants made at least one statement during their interview that referred to algebraic manipulation as math. As well, each of the 8 participants also made some explicit or implicit suggestion that either graphing, diagrams, or picturing while solving, either in their heads or on paper, is not actually math. I speculate that there might be some links between what certain participants understand is appropriate

mathematical problem solving (ie. only algebraic manipulation) and whether they make connections between the symbolic and embodied worlds when problem solving. As well, these findings from the participant statements also suggest that instructors may not be connecting between mathematical worlds either, at least not in the methods they are using when instructing students on how to solve calculus problems. This suggests the need for possible classroom observation in future studies to fully understand these implications.

The Repeat of Material from High School Calculus in University Calculus

To observe how many participants commented that university calculus was a repeat or a review of high school calculus, I began by making note of whether the participant took calculus in high school or not. Ten of fifteen participants took the Calculus and Vectors course in high school (Ministry of Education, 2007). The other 5/15 participants took a course equivalent to the high school Calculus and Vectors course in university, prior to taking their first year university calculus.

Of the 10 participants who took Calculus and Vectors in high school, 9 of them made some sort of statement that they felt first year university calculus was a repeat, a review, or the same, as high school calculus. Figure 64 has an example of three different participant statements.

Participant #2

P2: Umm, well the first two thirds of it was actually pretty much review from high school

Participant #3

P3: yeah, but I had already learned it in grade 12 so it was kind of review

Participant #6

P6: Yeah, because everything else was pretty much review.

Figure 63. Participant statements that claim university calculus was a review.

Of the 5 other participants who took the Calculus and Vectors equivalence course in university, 1 of them also made a comment that some of the material covered in the university calculus course was similar to what was covered in the high school equivalence course. In particular, this participant stated:

P8: I didn't take calculus in high school so I took the equivalent here. So it seemed pretty much like those questions, it is just that those were a little simpler. And same with the [university calculus] course, like the beginning of the course, the questions are like almost the same. So I didn't feel overwhelmed by it, or I didn't like not know what to do.

Overall, 10/15 of the participants stated that the university calculus course that they took was a repeat, a review, or the same, as the high school calculus course, or the equivalent course at university. It is interesting to note that a majority of the participants felt this way and might suggest that how a participant makes connections between mathematical worlds on calculus questions may be influenced by their experience in the high school calculus course, and not just the university calculus course.

CHAPTER SIX: DISCUSSION

The research question of this study began by asking: *How do undergraduate calculus students make connections between the embodied-conceptual and symbolic-proceptual worlds of the calculus concepts of limits and derivatives?* As mentioned earlier in this thesis, after the data was collected and analysis began it became apparent that the findings were not limited to connections being made only between the embodied-conceptual and symbolic-proceptual worlds. Thus, the research question was adapted from the original to be: *How do undergraduate calculus students make connections between the embodied-conceptual, symbolic-proceptual, and formal-axiomatic worlds of the calculus concepts of limits and derivatives?*

The findings from the research describe a series of results that support a number of conclusions and provide a strong argument for addressing this research question. A very significant result from this research is the creation of the eight *Response Categories*, detailed in Table 3 and again in Table 4. These *Response Categories* are important as they provide a way to describe the types of connections students make between the mathematical worlds, and how they make them. Another significant result, with many supportive findings, suggests that how students make connections, and thus the *Response Category* their work is classified into, may be related to the type of question that is asked. A third result, from the comparison of the analysis of correctness to how participants made connections, is that there is not enough evidence from this study to support whether or not there is a relationship between correctness and how connections are made. As a fourth result, further analysis of unintended yet relevant observations suggests that how students make connections is most likely

linked to their previous experiences in calculus courses as well as their personal understanding of what they consider “math”. This chapter discusses these results and describes their implications towards answering the research question in more detail.

The Creation of the Eight Response Categories

The eight *Response Categories* are a significant result of this study as they provide a means by which to concisely and consistently describe the types of connections the participants made. The creation of these categories was a result of a review of the participants’ *Layered Analysis*, which is a combination of the analysis from the written work layered with an analysis from the *Transcript Summaries* and a filled in *Representation Table*. While the creation of the *Response Categories* was a result of this particular study, these categories are noteworthy as a framework that could potentially be used to describe how connections are made and representations used in other similar scenarios. It is also a substantial finding that these categories are not limited to only the Embodied and Symbolic worlds, but also allow for connections between the Formal and Embodied or Symbolic worlds as well. Thus, it may be possible to use the framework of the *Response Categories* to analyze how students are making connections on other topics in university calculus or other areas of university mathematics, such as linear algebra. While it might be the case that not every situation will provide results in all eight *Response Categories* since this work found that some of the categories were unique to particular questions, this framework would nonetheless provide a way to categorize different findings.

As mentioned, one of the noteworthy characteristics of the *Response Categories* is that they describe connections to be made between the Formal world of

mathematics, as well as the Embodied and Symbolic worlds. Initially, the research question of this study was based on the assumption that the Formal world of calculus was not included in first year university calculus. While it is the case that the representations purely situated in the Formal world are still not addressed in first year university calculus, some courses do begin to make connections between representations from the Embodied and Formal or Symbolic and Formal worlds, which some students reflected in their task question solutions and interview. For example, when solving task Question #2, some participants wrote a variation of the formal proposition for the existence of right and left limits, namely: If $\lim_{x \rightarrow a-} f(x) = \lim_{x \rightarrow a+} f(x)$ then $\lim_{x \rightarrow a} f(x)$ exists. Similarly, for task Question #4 one participant drew a graphical diagram to describe the asymptotes of the function. These are examples of Formalizing-Symbolism and Formalizing-Embodiment respectively. Thus, this framework of *Response Categories* offers a means to classify and identify the connections that are made between all three of the mathematical worlds.

How Students Make Connections: The Response Categories

The strongest findings from this research are that the *Response Categories* and the added details from the Further Analysis of the *Response Categories* describe *how* the participants in this study made connections. As well, how the connections are made can include the formal world, not just the embodied or symbolic world. These findings are supported by the data analysis that looked at the participants' *Layered Analysis* and analyzed the participant responses by question, which resulted in the *Response Categories*. As well, the Further Analysis as mentioned above, supports these findings as it looked not only at the *Layered Analysis*, but more deeply at the written work

summaries and interview summaries, in order to examine the trustworthiness of the *Response Categories*. The results from the Further Analysis demonstrated that participant responses that were categorized together were done so appropriately based on the representations that were used, *how* they were used, and *how* connections were made between the different representations. For example, Category (D): Mostly Symbolic with some Connections is described in Table 3 and Table 4 as “A response categorized in this group appears to have mostly symbolic written work. However, the responses also include one or more written representations from a different world. From the information provided in the interview, it is suggested that the participant was connecting the representations from different worlds” (Table 3 pp. 109-111, Table 4 pp. 136-139). However, if we examine the Further Analysis for Category (D)- Question #1 it states:

There were six participant responses for this category. Each participant response began Question #1 by either re-arranging the equation to isolate for y and then differentiating or by implicitly differentiating the original equation. In all cases symbolic representations are used. Of the six participants in this category, three finished the question by sketching a graph of their findings, while the other three finished the question by stating the answer symbolically. Each participant made some connections between the symbolic and embodied worlds by either sketching a graph in the written work or describing their solutions in terms of a graphical representation during the interview. One participant even made some connections between the symbolic and formal worlds with a description of the slope of a tangent line. Overall, while the

majority of representations used by these participants are symbolic, each participant demonstrates at least one connection between the symbolic and embodied or formal worlds. (p. 147)

From this description we can better understand *how* the participants whose responses were categorized as (D) for Question #1 were making connections. Examples of two participants' work and interviews for Category (D) –Question #1 are included in the figures below.

P6: It makes sense based on the oval cause there is one right here and right here (points to page)

our answer.

$$\begin{aligned}
 y &= \pm \sqrt{\frac{1-x^2}{2}} \\
 y' &= \frac{1}{2} \left(\frac{1-x^2}{2} \right)^{-\frac{1}{2}} \left(-2x \right) \\
 &= \frac{1}{2} \left(\frac{1-x^2}{2} \right)^{-\frac{1}{2}} (-x) \\
 1 &= \frac{-x}{2 \sqrt{\frac{1-x^2}{2}}} \\
 2 \sqrt{\frac{1-x^2}{2}} &= -x \\
 2 \sqrt{\frac{1-x^2}{2}} &= x^2 \\
 1-x^2 &= \frac{x^2}{2} \\
 1 &= \frac{3}{2} x^2 \\
 \sqrt{\frac{2}{3}} &= \sqrt{x^2} \\
 x &= \pm \frac{\sqrt{6}}{3} \\
 \left(\frac{\sqrt{6}}{3} \right)^2 + 2y^2 &= 1 \\
 2y^2 &= 1 - \frac{6}{9} \\
 y &= \sqrt{\frac{1}{6}} = \frac{\sqrt{6}}{6}
 \end{aligned}$$

Check: Since $x^2 + 2y^2 = 1$ is an oval, then the tangent lines when the slope is 1, will look something like this

The points where the tangent line has a slope of 1 are $\left(-\frac{\sqrt{6}}{3}, \frac{\sqrt{6}}{6}\right)$ and $\left(\frac{\sqrt{6}}{3}, -\frac{\sqrt{6}}{6}\right)$

P6: I knew what it was going to look like from the beginning. But I didn't really picture it until I got the two points, here? So that's when I drew it

Figure 64. Example a. of Category (D) for Question #1.

P14: So I decided that would mean that the slope of the tangent, as is stated in the question, is 1. And I thought I was going to be working with that, but I wasn't really sure so I just...*(points to the work on the page while talking)*

$$\rightarrow \text{tangent line: } f'(a)(x-a) + f(a).$$

$$f'(a) = \text{slope of tangent}$$

$$\therefore f'(a) = 1.$$

Take first derivative of $x^2 + 2y^2 = 1$. (Implicit differentiation).

$$x^2 + 2y^2 = 1.$$

$$2x + 4y \cdot y' = 0.$$

$$4y \cdot y' = -2x$$

$$y' = \frac{-2x}{4y}$$

$$\therefore y' = \frac{-x}{2y}.$$

Now that I have my first derivative, I can substitute the slope of the tangent line into $f'(x)$ or y' .

$$y' = \frac{-x}{2y}$$

$$1 = \frac{-x}{2y}$$

$$2y = -x$$

$$y = \frac{-x}{2}$$

Figure 65. Example b. of Category (D) for Question #1.

In the Figure 64 we can observe that the participant choose to draw an informal graphical representation as he completed the problem, which is how he was making connections between the embodied and symbolic worlds. The example in Figure 65 however, shows an excerpt from an answer in which the particular participant used

some formalizing-symbolism statements with regards to tangent lines and the slope of the tangent, which demonstrate how she was making connections between the formal and symbolic worlds. In both examples, the participants did use symbolic representations for the majority of the written work. The example in Figure 65 also demonstrates how the Further Analysis also supported that some participants used representations that were connected with the formal world. Overall, while this Further Analysis might seem redundant from the analysis that took place in the creation of the *Response Categories*, the Further Analysis in fact looked at more than just the *Layered Analysis* documents to draw conclusions and thus provided further support for the reliability of the *Response Categories*.

It is also noteworthy that while the data analysis steps did provide eight *Response Categories* to describe how the participants made connections, the majority of participants responses were categorized within only three *Response Categories* namely, (A): No Connection/All Symbolic, (D): Mostly Symbolic with Some Connections, and (H): Strong Connections/Multiple Representations. As was suggested in the Findings Chapter, the other five *Response Categories* may actually be specific to the individual responses or task question and in fact may not reflect how connections are or are not generally made between mathematical worlds. Thus, with this result it seems reasonable to assert that *how* most undergraduate calculus students would likely be to make connections as described by *Response Categories* (A), (D), or (H). This assertion is made from the fact that 44/60 participant responses, almost 75%, fell into these three categories. This information tells us that there appears to be a scale of how connections are made and representations used. However, this scale would only place *Response*

Category (A) on one end of the scale with no connections being made and only symbolic representations being used. On the other end of the scale would only be *Response Category (H)* with multiple representations from the different worlds being used and a lot of connections being made between the representations. Last in the middle of the scale would be *Response Category (D)*. This is because responses classified in *Category (D)* still use a majority of symbolic representations, but are beginning to use and make connections to representations from other worlds. However, while these three categories appeared for every task question, the other *Response Categories*, (B), (C), (E), (F), and (G), may not appear in similar situations. This uniqueness of five categories is another fact in support that a student's response would most likely be categorized into either *Response Categories (A), (D), and (H)*.

Another finding is that of the 44/60 responses that were categorized into *Response Categories (A), (D), and (H)*, only 7 of the responses were in category (A). This is significant since categories (D) and (H) are both *Response Categories* where connections are being made, although of a varying degree. The results from Table 4 actually show that the majority of participant responses were classified in a *Response Category* where connections are being made, namely *Categories (D)-(H)*, with 43/60 responses, or about 72%, falling into these categories. Thus, it may be the case that the majority of undergraduate calculus students are able to make some connections between mathematical worlds when solving derivative and limit calculus questions, and that potentially more may be able to given the right support or depending on the question, as will be discussed in the following section.

Connections Are Made Depending On the Questions Asked

While the *Response Categories* can describe how the participants in this study made connections between mathematical worlds, the analysis appears to suggest that how connections are made is also connected to the type of calculus question that is asked. There are a number of findings that support this claim. For example, the initial analysis by participant (Step #2 of data analysis), which grouped the responses by participant and examined them to see if each participant had a specific way of making connections, did not demonstrate any trends or relationships. Alternatively, the analysis by question (Step #3 of data analysis), which grouped the responses by the four task questions and examined them to see if there was a specific way that connections were made according to the question, did demonstrate trends and relationships. These trends and relationships between the responses based on the task questions became the *Response Categories* and helped to assist with further analysis. Another supportive finding is that each participant's responses were categorized into at least two different *Response Categories*, with many of the participants having responses that spanned three or four categories. Again, this supports the findings from the analysis by participant (Step#2 of the data analysis) that there was no specific way each participant made connections and provides support to the findings from the analysis by question (Step #3 of the data analysis), which also supports the conclusion that the way undergraduate students make connections may have a relationship the question that was asked.

Another finding that supports this conclusion, is the fact that the responses are not evenly distributed across the eight *Response Categories* for each of the calculus task

questions with some questions having more of one type of response than another. For example, it was found that Question #1, which asked:

At what point(s) of $x^2 + 2y^2 = 1$ does the tangent line have a slope of 1? Show your work and explain your answer.

and Question #4, which asked :

Given the function $f(x) = x - \frac{1}{(x^2)(x+2)}$

Find:

$$\lim_{x \rightarrow -2^+} f(x)$$

$$\lim_{x \rightarrow -2^-} f(x)$$

$$\lim_{x \rightarrow 0} f(x)$$

Show your work and explain your process and thinking.

had the majority of responses classified in Category (D): Mostly Symbolic with some Connections.

Alternatively, Question #2, which asked:

Given the piecewise function

$$g(x) = \begin{cases} x & \text{if } x < 1 \\ 3 & \text{if } x = 1 \\ 2 - x^2 & \text{if } 1 < x \leq 2 \\ x - 3 & \text{if } x > 2 \end{cases}$$

Find the limits

$$\lim_{x \rightarrow 1^-} g(x)$$

$$\lim_{x \rightarrow 1} g(x)$$

$$\lim_{x \rightarrow 2^-} g(x)$$

$$\lim_{x \rightarrow 2^+} g(x)$$

$$\lim_{x \rightarrow 2} g(x)$$

Show your work and explain your process and thinking.

and Question #3, which asked:

Given the function $f(x) = 3x^5 - 5x^3$

On what interval(s) is the function increasing?

On what interval(s) is the function decreasing?

What is(are) the local maximum(s)?

What is(are) the local minimum(s)?

Show your work and explain your process and thinking.

had the majority of responses classified in Category (H): Strong Connections/Multiple Representations.

I believe that Questions #2 and #3 prompted more connections and multiple

representation use because of what these questions were asking and how it was being asked. For example, Question #2 asks about limits, including left and right limits, of a piecewise function. These two factors may prompt or encourage the students to want to visualize the function graphically or place some of the values on a number line, in order to ensure there is an understanding of the discontinuities or jumps that take place in the piecewise function. Graphing the function or placing values on a number line can also help when solving left and right limits, as they provide a visual reference to refer to. Similarly, Question #3 asks about characteristics of the function that all have visual or physical meanings as well as numerical, such as function intervals of increasing or decreasing, or maximum and minimum values. Therefore, I speculate that by asking about these characteristics of the function students feel inclined to use a representation of the function that has embodied connections to offer some insight in the visual or physical meanings. These take the form of a graph, number line, or diagram for example.

Conversely, Questions #1 and #4 may not necessarily be the types of questions that prompt strong connections between mathematics worlds or multiple representation use from different worlds. For example, Question #1 asks about an implicit function of an ellipse. Some of the participants may not have known how to visualize or graphically represent an ellipse. Participant #2 for example, stated that she wanted to graph the function to visualize it, but that she found it too difficult. Similarly, Question #4 includes a rational function with asymptotes that may also cause some difficulty with visualizing or graphing. I make this assertion from the fact that only one participant drew the graph of the function in Question #4, and this participant stated

that she had done it by looking at the graphing calculator. As well, the connections that were made between the mathematical worlds for Question #4 tended to be discussed and described in the interview and were not necessarily evident from the written work. Question #1 and Question #4 were also presented purely with algebraic symbols and this may have then prompted some of the participants to move in an algebraic direction when solving. Therefore, it would seem as though Question #1 and Question #4 may not be the type of questions that promoted strong connections between mathematical worlds or multiple representation use from the different worlds. Thus, as was previously asserted, it may be the case that participants made more connections and used multiple representations from different mathematical worlds based on the type of question that was asked.

An additional fact in support of this claim is that half of the *Response Categories* are unique to a particular task question. Specifically, categories (C): No Connections/ Disconnect between Tasks and Interview, (E): Tasks Symbolic/ Interview Embodied with some Connections, (F): Multiple Representations with Connections/ Memorized Process, and (G): Only Graphical Solution, are unique to only one of the calculus task questions. Category (C) was unique to Question #1, Category (E) was unique to Question #2, and Categories (F) and (G) were unique to Question #3. This suggests that these categories may thus be the result of the questions that were asked. For example, the response in Category (G) uses only a graphical representation to find the solution for Question #3. It would have been very difficult to answer the other questions, especially Question #1 and Question #4, using only a graphical representation. Likewise, Category (E), in which the written work on the task is all symbolic and the

interview description of the task is then embodied, is most likely unique to Question #2 because the function in the question is not too difficult to mentally visualize. In fact, a few participants stated that they went about answering this question by picturing the function in their head in a visual and graphical form and then solved the limits from what they visualized. This processes of just visualizing the function internally could have been too difficult for the functions and equations in the other task question, thus making Category (E) unique to Question #2 because of what was asked in the question. These findings can also be compared to the study by Christou, Pitta-Pantazi, Souyoul, and Zachariades (2005). In this study the researchers hypothesized that calculus questions can be related to a particular mathematical world and as such the solutions would present distinct representations. This seems to be the case for the task questions in this thesis as well, although it was demonstrated prior to data collection that the questions could be answered with multiple representations from different mathematical worlds (Appendix C).

A Comparison of the Responses with Correctness

As was previously suggested in the Findings Chapter from the results of this comparison, this study did not find any conclusive evidence for or against the suggestion that there is a relationship between how connections are made and the correctness of an answer or solution when solving a mathematical problem. One of the only findings that seems to potentially have significance is the fact that for every question participants who started and finished with an appropriate method for solving the problem had no Completely Incorrect or Incomplete solutions. Thus, it seems suitable to assert that starting and finishing a solution with an appropriate method will

make it more likely for the answer to be completely or partially correct. Although, starting and finishing a solution with an appropriate method does not require the use of multiple representations or connections being made between mathematical worlds. Overall, the only conclusion from this comparison is that it seems that the sample size of 15 participants for each task question was not big enough to uncover any other significant findings. As well, this comparison was not part of the research question so the study was not necessarily designed to look at the relationship between correctness and multiple representation use with connections. Examining the findings to see if there was a relationship was something that I decided to look at in addition to the research findings as it had been suggested in some of the literature that there is a relationship between how students make connections from different mathematical worlds, and their success at problem solving (e.g. Nogueira de Lima and Tall, 2008; Stewart and Thomas, 2007). However, I cannot make conclusions about it. In particular, no findings address the relationship between making connections and method or answer correctness.

Previous Experience and Met-Befores

The participants' prior experience with questions similar to the task questions may have also played a role in how the connections were made. In particular, the participants may have been taught to answer particular types of calculus questions in a certain way. This assertion comes from information received during the interview portion of the data collection. As discussed in the findings, 9 of the participants who had taken the high school calculus course prior to university mentioned that their first year university calculus course repeated a lot of the same material. As well, 1 participant

who had taken the equivalence to the high school calculus course in university also made a similar statement. Thus, 10/15 of the participants felt that their first year university calculus course was repeating at least a portion of the material they had already been taught. Therefore, the participants in this study all had prior experience from their high school calculus course as well as their university calculus courses that may have influenced how they made connections on the task questions.

Tall (2004b) describes the influence of previous experiences on current learning and development as a “met-before”. The notion of a met-before is what takes place when students build from experiences that they already have had. Nogueira de Lima and Tall (2008) define a met-before as “a mental construct that an individual uses at a given time based on experiences they have met before” (p. 4). Thus, it may be the case that the participants in this study were influenced by their previous experiences in high school calculus as well as in their university calculus courses to answer certain calculus questions in particular ways. For example, the majority of participants (10/15) used a table in task Question #3. As the findings report in the previous chapter, 6 of the 10 participants who used a table stated that they used a table for Question #3 because that is how they were taught in class to answer questions like Question #3. Therefore, it appears that the use of a table to answer Question #3 may be the result of a met-before and not necessarily an attempt to make connections between mathematical worlds or use alternative representations. This may also be true for other solutions to the calculus task questions, although the use of the table in Question #3 is the most obvious.

What is “math”?

Another finding that may have influenced how participants made connections

between mathematical worlds is the understanding of what counts as “math” and what does not. As was reported, 8/15 participants used the word “math” at least once during the interview portion of the data collection to mean the use of symbolic algebra and algebraic manipulation when solving problems, rather than a broader definition that might have included embodied mathematics, such as graphs and diagrams. For example, when Participant #10 was asked:

A: So was there anything visual going on in your mind while you solved?

The participant responded with:

P10: Nope, I simply, well at the beginning I imagined the shape so I just knew there was two equations and I knew what it was going to look like. And then I just forgot about it and did all of the math.

These 8/15 participants also either implicitly or explicitly stated that graphing, diagrams, or picturing while solving problems was not “math”. In the Theoretical Perspective, graphing, diagraming, or picturing is described as solving techniques where connections are being made between the symbolic and embodied worlds of mathematics. Thus, the participants who used the word “math” to mean the algebra and algebraic manipulation may not have been considering graphing, diagrams, or pictures as “math”. This perspective of what “math” is may influence how the participants decided to solve the task questions, maybe assuming that a “math” or symbolic solution was more appropriate. This finding may play a role in how undergraduate students make connections between mathematical worlds. Overall, I believe this finding warrants further investigation.

Summary

From the findings of the study I feel comfortable making the following concluding claims.

(1) *The connections that undergraduate calculus students make on derivative and limit questions can be classified using the eight **Response Categories**. As well:*

- *The eight **Response Categories** could provide an accurate way to describe how connections are made between all three mathematical worlds in other similar situations.*
- *Out of the eight **Response Categories**, categories (A), (D), and (H) are the most likely way to describe how students are making connections.*
- *Students who do not have responses that fall into the **Response Categories** (A), (D), and (H), fall into categories that may be unique to this study and may not appear in similar situations.*
- *The majority of the responses are in categories where connections are made, namely (D)-(H) and thus, the majority of undergraduate calculus students are most likely able to make some connections between mathematical worlds when solving derivative and limit calculus questions.*

(2) *How undergraduate students make connections may be dependent on the question that is asked. This is supported by the following findings:*

- *The analysis by participant did not provide any results suggesting any trends or relationship in regards to how each participant was making connections.*

- *The analysis by question did provide results that demonstrated strong relationships between how connections were made based on the question that was asked.*

*This analysis resulted in the eight **Response Categories**.*

- *Participant responses are not evenly distributed across the eight **Response Categories** for each of the calculus task questions with some questions having more of one type of response than another. This may be due to the nature of the question that was asked.*

- *Questions #2 and #3 promoted connections being made and the use of multiple representations.*
- *Questions #1 and #4 did not promote as many connections or multiple representation use.*

- *Half of the **Response Categories** are unique to a particular task question and this may be due to what the question was asking.*

(3) There was no evidence in support for or against the suggestion that there is a relationship between how participants make connections and the correctness of an answer or solution. However:

- *Participants who started and finished an answer with an appropriate method seem to be more likely to achieve a completely correct or partially correct answer to the question.*

(4) The participant's prior experience with questions similar to the task questions may have influenced how the connections were made.

(5) What undergraduate calculus students think is considered a mathematical solution may influence the way the calculus questions are answered and the connections that are

made in the answers. In particular:

- *More than half of the participants used the word “math” to describe solving with algebraic manipulation.*
- *Graphing, drawing diagrams, or using pictures, was not considered part of the “math” when solving a question.*
- *This result requires further investigation to fully comprehend the influence this perspective may have on how undergraduate calculus students make connections between mathematical worlds.*

Overall, these above claims help to address the research question by providing evidence to support that how undergraduate calculus students make connections between mathematical worlds on derivative and limit questions can be described by the eight *Response Categories*. As well, it is most likely that undergraduate calculus students are making connections similar to those described in categories (A): No connections/all symbolic, (D): Mostly symbolic with some connections, and (H): Strong connections/multiple representations. How connections are made is probably influenced by the nature of the question that is asked and previous experience, or met-
before, with similar questions. It may also be the case that how an undergraduate calculus students understands what “math” is could influence how a question is answered and the connections that are made in an answer, but this result requires further investigation to determine the extend of its influence.

CHAPTER SEVEN: CONCLUSIONS

This research used Tall's theory of three worlds of mathematics combined with a theory of mathematical representations to investigate how undergraduate students make connections between the embodied, symbolic, and formal mathematical worlds on limit and derivative task questions. The analysis results in the identification of the ways the participating students make connections, and in the development of the *Response Categories*, which describe what representations are used, how they are used, and what connections are made between representations from different mathematical worlds. This research also suggests that the way students make connections is dependent upon the type of question that is asked as well as prior experience with similar content.

The theoretical perspective of Tall's Three Worlds of Mathematics, and the research method of collecting both written work and interview data for a layered analysis, allowed me to develop this in-depth description of how the participants were making connections. In particular, collecting both written work and interview responses allowed for a thorough analysis of the participants' understanding to enable categorizing the findings with Tall's Three Worlds of mathematics. Thus, this suggests that the study used an appropriate methodology for investigating and analyzing how students make connections between mathematical worlds for other mathematical concepts.

In this chapter I address some of the implications of this research on both future research as well as on practice. I then address some of the further questions that this research has prompted. Some of these questions are directly related to Tall's Three

Worlds of mathematics and how connections are made between them. However, some of the further questions were prompted by other interesting findings that appeared from the data analysis, such as how prior experience influences the way students solve problems and what students understand as mathematics. I touch on some of the limitations of this study in order to ensure that the findings and results are appropriately interpreted. Lastly, I conclude by summarizing the most significant results of this research.

Implications for Research

This study is a first step in the investigation of the connections students make between Tall's Three Worlds on calculus task based questions that focused on limits and derivatives. Although this was a small qualitative study it produced a particular contribution to this field of research in the development of what I termed the eight *Response Categories*. The study also found that how students make connections might depend on the types of questions that were asked and prior experience. These findings offer significant implications for future research, which are addressed below.

Response Categories

The eight *Response Categories* were developed in the third step of the data analysis through a process of layering the findings from each participant's written work and interview. The *Response Categories* arose as a way to describe how the participants were making connections between the mathematical worlds when they solved limit and derivative task questions. While the *Response Categories* helped me effectively organize and illustrate how the participants were making connections, they also became a tool of information that I continually referred back to while addressing the

research question of this study.

I believe there is potential for the *Response Categories* to accurately describe how students make connections based on other mathematical concepts and thus, the *Response Categories* could become a valuable research instrument in similar investigations. This is a result of the usefulness of the *Response Categories* in describing how the participants made connections between mathematical worlds. The *Response Categories* allowed me to: (a) identify the mathematical worlds that the participants' representations were situated in; (b) identify how the different representations were used within the solution; and (c) identify the connections that were made between mathematical worlds. Being able to identify these themes allowed me to accurately describe *how* participants were making connections on the task problems. In particular, I was able to combine the findings of what representations were used with how the representations were used in order to identify the connections that were or were not being made between mathematical worlds. Overall, the *Response Categories* can allow other researchers to also identify representation use and connections between mathematical worlds.

The Influence of the Task Questions

The research also found that the type of question that is asked could influence how participants make connections. This finding is supported by the uneven distribution of the participants' responses among the *Response Categories* as demonstrated in Table 4. It is also supported by the fact that four of the *Response Categories* have responses from only one task question, namely categories (C): No Connections/ Disconnect between Tasks and Interview, (E): Tasks Symbolic/ Interview

Embodied with some Connections, (F): Multiple Representations with Connections/Memorized Process, and (G): Only Graphical Solution. As well, support is offered from the fact that the analysis by participant did not produce any relationships, while the analysis by question did. Lastly, the works of Christou, et al. (2005) demonstrated that task questions could be written to promote being solved with representations from one world over another. It would be beneficial for future research that use task questions or looks to describe students' connections between mathematical worlds to acknowledge all of this support for the influence of the types of questions on how connections are made between mathematical worlds.

The Influence of Prior Experience

The research found that participants' prior experience could also be an influencing factor in how connections are made between mathematical worlds. This finding is supported by the fact that 10/15 participants commented that their first year university level calculus course repeated a significant amount of what was taught in their high school level calculus course. For another example, 6/10 of the participants who used a table while solving task Question #3 stated that they did so because that is how they were taught to solve questions like Question #3 in their calculus classes. Lastly, many participants spoke about undertaking certain problem solving steps in their solutions to the task questions because of their experience in calculus classes, which demonstrated solving certain questions with particular problem solving processes. The participants did not necessarily undertake the procedures because they were the most efficient way to solve the problem or because they fully understood the steps, but because they perceived the question had to be solved that way. Future

research that uses task questions should consider the influence that prior experience with what is emphasized in mathematics classes and what is demonstrated as the “correct” way to solve certain types of problems has on how questions are solved and as a result how connections are made between mathematical worlds.

Implications for Practice

While the design of this study did not seek ways to change current practices in university calculus, only to better describe what was going on, some of the findings could still have implications for how calculus is taught. In particular, the finding that 10/15 of the participants stated that their first year university level calculus course repeated a lot of material from the high school level calculus course, or Calculus and Vectors (Ministry of Education, 2007), suggests a need to investigate these accounts. Universities need to consider what is taught in the first year calculus courses and compare it with what is taught in high school and potentially adjust course content if they find that significant portions of topics are being unnecessarily repeated. Another finding that could influence practice is the result that the type of questions that are asked can impact how students make connections. Instructors could consider assigning problems that promote connections being made between mathematical worlds with the goal of expanding understanding of the concepts. Instructors could also consider the representations that they use, and the way that they themselves make connections between mathematical worlds in the presentation of material in the classroom, as this could impact the way students make connections and develop an understanding of the topic. Many mathematical concepts allow for very rapid efficient calculations with the use of only symbolic representation or objects from the symbolic world. However, if

these are the only representations that are taught, students may develop a gap in their understanding of the concepts, as according to Tall (2004a, 2004b, 2008), full development takes places as students learn to understand the concepts in other mathematical worlds, and make connections between them. Thus, instructors could take the opportunity to show single problems solved numerous times, each time with different of multiple representations. An example of how this could be done with a single problem can be seen in Appendix C where I solve the task questions used in this study multiple time with different representations for each scenario.

Further Questions

This thesis has contributed to the bodies of research concerned with the understanding and development of the calculus concepts of limits and derivatives, while demonstrating that Tall's theory of Three Worlds of Mathematics is a way of describing cognitive development of mathematical concepts, and specifically with how students make connections between mathematical worlds. However, along with these contributions are many possibilities for further study. Some possible questions for further research are concerned with investigating in more detail how students make connections. For example, some questions that could be pursued are:

1. How are students making connections between mathematical worlds for other calculus concepts? Are students making connections similar to or the same as the eight response categories that were found in this study?
2. How are instructors making connections between mathematical worlds when teaching calculus? Do these align with the eight *Response Categories* that were identified in this study?

3. Within what mathematical worlds are the representations used in calculus textbooks situated? How do the representations in calculus textbooks make connections between mathematical worlds?

Some of the results of the study also prompt questions that are not directly related to Tall's Three Worlds of mathematics of the *Response Categories*. Some of these research questions could be:

4. What do undergraduate students understand to be doing "math" when solving problems? What do they understand is not math? How does this understanding influence solving problem?
5. How does previous experience, or "met-befores", in mathematics courses effect problem solving?

The result of the majority of participants expressing that first year university calculus is a repeat from high school calculus, or the Ontario Calculus and Vectors course (Ministry of Education, 2007) also prompts some policy type questions, such as:

6. What material is repeated from the high school Calculus and Vectors course in first year university calculus? And if material is repeated, is it represented in similar ways? Do the variety of ways the material is presented help or hinder building stronger connections?

Limitations of the Study

One limitation of the study could be that the volunteers may not be representative of the total population who takes calculus. The participants of this study were recruited as volunteers. Quite often students volunteer because they are comfortable with mathematics, particularly calculus, and may not be representative of

the population taking calculus. However, the participants that volunteered for this study still offered a diverse population and while most reported being comfortable with mathematics, two stated that they were uncomfortable with calculus and struggled. The participants also reported receiving a variety of grades in their calculus courses.

The other two areas that I will discuss are actually topics that could prompt further investigation rather than actual limitations of the study. The focus of the study was on how these students made connections between mathematical worlds on selected calculus based task questions, which were focused only on the concepts of limits and derivatives. Therefore the findings may be limited to these concepts, which prompts further investigation for other calculus concepts.

The findings of the study also pointed to a number of external variables that may have played a role in why the participants made connections the way they did during the study. In particular, some participants referenced the way they were taught, either in university or in high school. However, this study did not make any observations of university or high school classes. This study also did not explore how the participants' experience in high school calculus may have influenced them. It also might be the case that the textbooks and other materials the participants have access to while taking university calculus may influence the way they make connections. Overall, this study is limited to only examining the participants' written work and interviews. Although this was sufficient for addressing the research question, by exploring some of these other potential influencing factors it might be possible to address *why* participants made connections in particular ways, not just *how* they made connections.

Summary

At the conclusion of this research it is evident that this study has produced a valuable framework, in the form of the *Response Categories*. The *Response Categories* provide an exhaustive and comprehensive way of describing how the participants were making connections between Tall's Three Worlds of mathematics. This research also implies that the *Response Categories*, or some portion of them, could accurately describe how individuals would make connections between mathematical worlds on other task questions. It can also be concluded from this research that how connections are made depends upon the type of questions that are asked. Some mathematical task questions may be worded in ways that elicit responses with more or fewer connections between mathematical worlds than other types of task questions. Thus, when describing how participants make connections it should be considered whether the way the task question was worded was an influencing factor. Overall, while the *Response Categories* are a very detailed way to describe how connections are made, the findings from the study prompt further investigation into the effect the type of task question has on how connections are made.

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APPENDICES

Appendix A: Calculus-Based Task Questions

Calculus-Based Task Questions

Instructions: Please read and complete all of the questions. For each question be sure to show your work and explain your process and thinking. Feel free to use any of the supplies provided including extra paper. Please be assured that your solutions will not be marked or graded.

Date:

Time:

Location:

Interviewer: Angela Smart

Found out:

Participant:

Student Status:

Email Address:

Phone Number:

Age:

1.

At what point(s) of $\mathbf{x^2 + 2y^2 = 1}$ does the tangent line have a slope of 1? Show your work and explain your answer.

2.

Given the piecewise function

$$g(x) = \begin{cases} x & \text{if } x < 1 \\ 3 & \text{if } x = 1 \\ 2 - x^2 & \text{if } 1 < x \leq 2 \\ x - 3 & \text{if } x > 2 \end{cases}$$

Find the limits

$$\lim_{x \rightarrow 1^-} g(x)$$

$$\lim_{x \rightarrow 1} g(x)$$

$$\lim_{x \rightarrow 2^-} g(x)$$

$$\lim_{x \rightarrow 2^+} g(x)$$

$$\lim_{x \rightarrow 2} g(x)$$

Show your work and explain your process and thinking.

3.

Given the function $f(x) = 3x^5 - 5x^3$

On what interval(s) is the function increasing?

On what interval(s) is the function decreasing?

What is(are) the local maximum(s)?

What is(are) the local minimum(s)?

Show your work and explain your process and thinking.

4.

Given the function $f(x) = x - \frac{1}{(x^2)(x+2)}$

Find:

$$\lim_{x \rightarrow -2^+} f(x)$$

$$\lim_{x \rightarrow -2^-} f(x)$$

$$\lim_{x \rightarrow 0} f(x)$$

Show your work and explain your process and thinking.

Appendix B: Interview Protocol

Task Based Calculus Questions and Semi-Structured Reflection Interview Protocol

Project: PhD Thesis – Undergraduate Calculus Students’ Connections Between the Embodied and Symbolic Mathematical Worlds of Limits and Derivatives: A Qualitative Study using Tall’s Three Worlds of Mathematics

Date:
Time:
Location:
Interviewer: Angela Smart

Participant:
Email Address:
Phone Number:
Age:

[Greet the participant and thank him or her for participating. Give a brief description of the study, task question process, and interview process, including audio or video recording. Assure the participant of confidentiality and anonymity of responses. Review the consent form and provide time for the participant to read through the consent form, and if they consent, to sign, and date the consent form.]

Task Based Calculus Questions:

Note: *The Calculus Questions will be provided in a booklet with one question per page. Participant will solve the calculus questions in this booklet, with the options to use extra paper where needed. The booklet and any potential extra work sheets will be attached to this document on completion on the interview.*

Break: *Participants will be offered a ten-minute break between completion of the calculus questions and the semi-structured interview. During this time the interviewer will review the participant’s responses in order to determine the solutions she would like to ask about during the interview (see interview question #8).*

Questions:

Note: *Potential follow-up questions are listed in bullet points under the main questions.*

1) Education background information questions:

- a) *What year of study are you in?*
 - b) *What are you studying/What is your major?*
 - c) *Is calculus a required course for your current program?*
 - d) *Have you always studied at the University of Ottawa?*
-

2) Please describe your impressions of the tasks you just completed.

- *How do you feel about the tasks you just completed?*
- *How well did you understand what the tasks were asking you to do?*

3) Had you seen questions like these before?

- *Where did you see the question(s)?*
- *Were they the same or just similar?*

4) Can you tell me more about your solution to number _____?

(Follow up with as many explanations about solutions as there is time, but ask about at least two.)

5) Can you tell me what a typical calculus class was like?

- *What types of topics were emphasized?*
- *What techniques were used in solving calculus problems?*

6) What topic(s) did you learn the best?

- *Why do you think you learned this (these) topic (s) best?*

7) What topic(s), if any, did you have the most trouble with?

- *Do you know why you struggled with this topic/these topics?*

8) Can you describe how:

- i. *graphs or diagrams were used in the course?*
- ii. *Technology was used in the course, if any?*
 - *What was the balance between algebra/graphing/diagrams/technology?*

9) Population descriptive questions (invite participants to share this information as long as they feel comfortable):

- a) *What nationality do you identify yourself with?*
- b) *Have you always studied in Canada, including pre-university education?*
- i. *(If participant has always studied in Canada) Which province(s) have you attended school in?*
- c) *Do you identify yourself with any minority, either visible or invisible? If so, what?*
- d) *Is English your first language? What other languages do you speak (if any)?*

[Ask the participant if he/she has any further questions or comments. Thank him/her for participating in the study.]

Appendix C: Potential Solutions to Calculus-Based Task Questions and Discussions

1.

At what point(s) of $x^2 + 2y^2 = 1$ does the tangent line have a slope of 1? Show your work and explain your answer.

Solution and Discussion

First suggested solution to graphing the equation: To solve this question the student may begin by recognizing the equation as an ellipse. This ellipse has a centre at (0, 0), an x-axis centre to edge distance equal to 1, and a y-axis centre to edge distance equal to $\frac{1}{\sqrt{2}}$. This information can be found by using the general formula for an ellipse:

$$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1$$

Where (h, k) is the centre of the ellipse, 'a' represents the horizontal distance from the centre to the edge of the ellipse, and 'b' represents the vertical distance from the centre to the edge of the ellipse.

With this information a student can write the equation in this question as:

$$x^2 + 2y^2 = \frac{(x - 0)^2}{1^2} + \frac{(y - 0)^2}{\left(\frac{1}{\sqrt{2}}\right)^2} = 1$$

Where the centre of the ellipse (h, k) is at (0, 0), a=1, and $b = \frac{1}{\sqrt{2}}$.

With this information, the student would be able to sketch the graph on graphing paper (Figure 66). By sketching on graphing paper, and by understanding that the slope of a

curve at a point is the slope of a tangent to the curve at a point, the student would be able to visually see that there are two points on the ellipse that have a slope = 1; one in the second quadrant and one in the fourth quadrant (Figure 66).

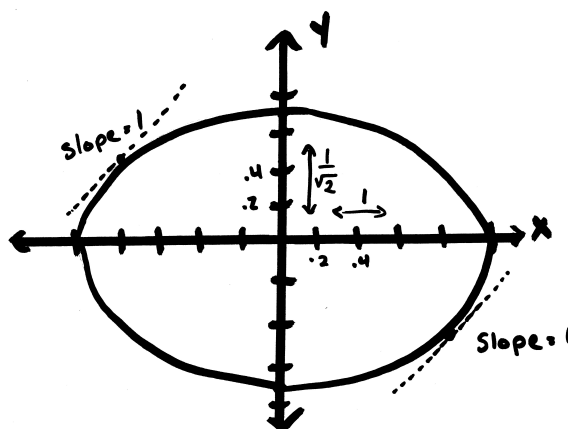


Figure 66. Hand drawn picture of equation and approximate points where slope of tangent = 1.

Second suggested solution to graphing the equation: Alternatively, the student may begin by rearranging the equation such that it is in the form of y equal to a function of x. This rearranging would result in a function that looks like:

$$y = \pm \sqrt{1 - \frac{x^2}{2}} \text{ or}$$

$$y = \pm \left(1 - \frac{x^2}{2}\right)^{1/2}$$

At this point the student could proceed by graphing the equation using a graphing calculator. The equation would need to be inputted as two separate equations to ensure both the positive and the negative side are graphed (Figure 67).

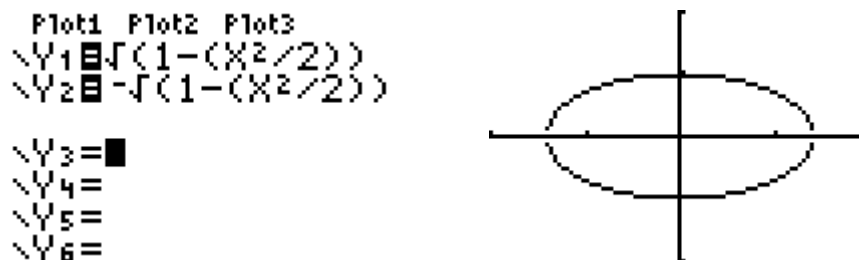


Figure 67. Positive and negative equation input and result graph on TI-83.

It can be seen from this graphing calculator image that there would be two points with a slope = 1, just as was demonstrated with the first graphing method.

However, it is important to note that it is not necessary for the student to graph the equation using either of the above graphing methods in order to solve the question.

Whether the student graphed the equation (either on graphing paper or with a graphing calculator) or not, the student would need to use the knowledge that the first derivative of an equation is equal to the slope of the tangent line to the equation and calculate the first derivative using symbolic manipulation rules of differentiation.

Solution to equations equal to y: The first derivative of the equations equal to y

(namely: $y = \pm \sqrt{1 - \frac{x^2}{2}}$ or $y = \pm \left(1 - \frac{x^2}{2}\right)^{1/2}$) is equal to:

$$y' = \frac{x}{2\left(1 - \frac{x^2}{2}\right)^{1/2}}$$

To determine where the slope of the tangent is equal to 1 the student would set $y' = 1$ and solve for x:

$$1 = \frac{x}{2\left(\frac{1-x^2}{2}\right)^{1/2}}$$

$$\left(\frac{1-x^2}{2}\right)^{1/2} = \frac{1}{2}x$$

$$\left(\frac{1-x^2}{2}\right) = \left(\frac{1}{2}x\right)^2$$

$$\left(\frac{1}{2} - \frac{x^2}{2}\right) = \frac{1}{4}x^2$$

$$\frac{1}{2} - \frac{3}{4}x^2 = 0$$

Using the quadratic formula:

$$x = \pm \frac{\sqrt{6}}{-3}$$

Plugging these x values into the original equation $x^2 + 2y^2 = 1$ and find the points:

$$\left(\frac{\sqrt{6}}{-3}, \sqrt{\frac{1}{6}}\right) \text{ and } \left(-\frac{\sqrt{6}}{-3}, -\sqrt{\frac{1}{6}}\right)$$

Therefore, the equation has a tangent with a slope = 1 at the two points found above,

namely: $\left(\frac{\sqrt{6}}{-3}, \sqrt{\frac{1}{6}}\right) \text{ and } \left(-\frac{\sqrt{6}}{-3}, -\sqrt{\frac{1}{6}}\right).$

Solution to original equation ($x^2 + 2y^2 = 1$): Alternatively, it is also the case that the

student may use a different procedure for finding the point(s), namely implicit

differentiation. Implicit differentiation would proceed with the following steps:

$$x^2 + 2y^2 = 1$$

$$x^2 \frac{dy}{dx} + 2y^2 \frac{dy}{dx} = 1 \frac{dy}{dx}$$

$$2x + 4y \frac{dy}{dx} = 0$$

$$2x = -4y \frac{dy}{dx}$$

$$\frac{-2x}{4y} = \frac{-x}{2y} = \frac{dy}{dx}$$

Since we can solve the equation for y as a function of x to equal $\pm \sqrt{1 - \frac{x^2}{2}}$, we have:

$$\frac{dy}{dx} = - \left(\frac{x}{2(\pm \sqrt{1 - \frac{x^2}{2}})} \right)$$

Which, when $\frac{dy}{dx}$ is set equal to 1, the same points are found as in the above solution,

which are: $\left(\frac{\sqrt{6}}{-3}, \sqrt{\frac{1}{6}} \right)$ and $\left(-\left(\frac{\sqrt{6}}{-3} \right), -\sqrt{\frac{1}{6}} \right)$.

Therefore, to answer the question, the equation has a tangent with a slope =1 at the two

points found above: $\left(\frac{\sqrt{6}}{-3}, \sqrt{\frac{1}{6}} \right)$ and $\left(-\left(\frac{\sqrt{6}}{-3} \right), -\sqrt{\frac{1}{6}} \right)$.

2.

Given the piecewise function

$$g(x) = \begin{cases} x & \text{if } x < 1 \\ 3 & \text{if } x = 1 \\ 2-x^2 & \text{if } 1 < x \leq 2 \\ x-3 & \text{if } x > 2 \end{cases}$$

Find the limits

$$\lim_{x \rightarrow 1^-} g(x)$$

$$\lim_{x \rightarrow 1} g(x)$$

$$\lim_{x \rightarrow 2^-} g(x)$$

$$\lim_{x \rightarrow 2^+} g(x)$$

$$\lim_{x \rightarrow 2} g(x)$$

Show your work and explain your process and thinking.

Solutions and Discussion

First suggested solution: To solve this question a student may begin by graphing the piecewise function on graphing paper in order to have a visual representation of the function (Figure 68).

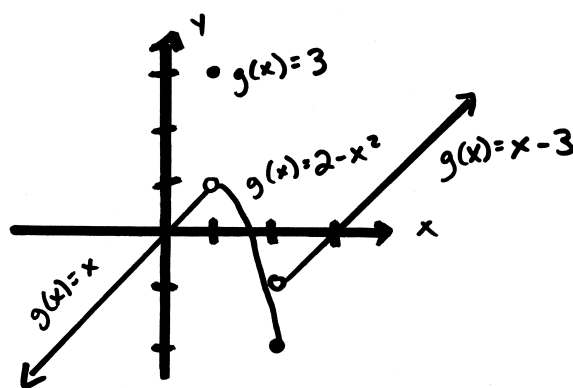


Figure 68. Hand drawn picture of piecewise function.

If the student were able to produce a clear and accurate graph image then the desired

limits would be able to be found by examining the graph. The student would be able to find that the limit values are:

$$\lim_{x \rightarrow 1^-} g(x) = 1$$

$$\lim_{x \rightarrow 1} g(x) = 3$$

$$\lim_{x \rightarrow 2^-} g(x) = -2$$

$$\lim_{x \rightarrow 2^+} g(x) = -1$$

$$\lim_{x \rightarrow 2} g(x) = DNE,$$

since $\lim_{x \rightarrow 2^-} g(x) = -2$ does not equal $\lim_{x \rightarrow 2^+} g(x) = -1$.

Second suggested solution: Alternatively, the student could proceed by assigning $g(x)$ in each limit to be the part of the piecewise function where the limit is to be found. The other parts of the piecewise function could then be disregarded. This would produce the limits of:

$$\lim_{x \rightarrow 1^-} g(x) = \lim_{x \rightarrow 1^-} x = 1$$

$$\lim_{x \rightarrow 1} g(x) = \lim_{x \rightarrow 1} 3 = 3$$

$$\lim_{x \rightarrow 2^-} g(x) = \lim_{x \rightarrow 2^-} 2 - x^2 = -2$$

$$\lim_{x \rightarrow 2^+} g(x) = \lim_{x \rightarrow 2^+} x - 3 = -1$$

$$\lim_{x \rightarrow 2} g(x) = DNE.$$

Because $\lim_{x \rightarrow 2^-} g(x) = \lim_{x \rightarrow 2^-} 2 - x^2 = -2 \neq \lim_{x \rightarrow 2^+} g(x) = \lim_{x \rightarrow 2^+} x - 3 = -1$.

3.

Given the equation $y = 3x^5 - 5x^3$

On what interval(s) is the equation increasing?

On what interval(s) is the equation decreasing?

What is(are) the local maximum(s)?

What is(are) the local minimum(s)?

Show your work and explain your process and thinking.

Solution and Discussion

First suggested solution (note: this solution does not actually require the use of calculus, only an understanding of polynomial equation properties and graphing calculators):

One way a student might solve this question is to first use a graphing calculator to visualize the graph. By entering the equation into a graphing calculator a student would get the image below (Figure 69).

```

Plot1 Plot2 Plot3
Y1=3X^5-5X^3
Y2=
Y3=
Y4=
Y5=
Y6=
Y7=

```

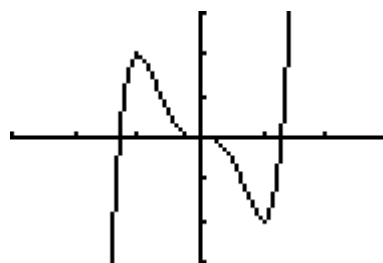


Figure 69. Graphing calculator input and graphed screen on a TI-83.

This image of the graph demonstrates that there is definitely one local maximum, one local minimum, and different intervals of increasing and decreasing. A student could now proceed with determining what these actual values and intervals are, in order to answer the question. Although it appears that the maximum and minimum values may

just be read straight from the graph, the graphing calculator could then be used to calculate the exact minimum and maximum values using the following steps:

For minimum value:

2nd CALC

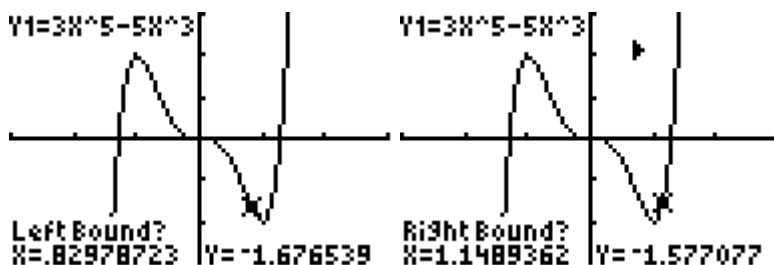
select 3: minimum

```

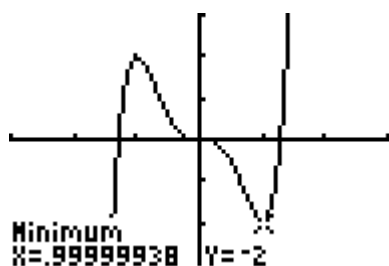
MATH>
1:value
2:zero
3:minimum
4:maximum
5:intersect
6:dy/dx
7:∫f(x)dx

```

Select left and right bound



Press enter to receive answer:



For maximum value:

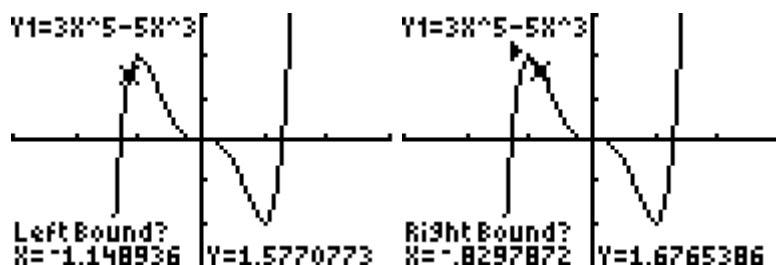
2nd CALC

select 4: maximum

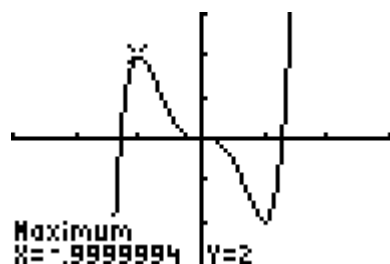
CHOOSE

1:value
2:zero
3:minimum
4:maximum
5:intersect
6:dy/dx
7: $\int f(x)dx$

Select left and right bounds:



Press enter to receive answer:



Using the graphing calculator, a student would thus find that the maximum value is at the point $(-1, 2)$ and the minimum value is at the point $(1, -2)$. (Note the infinite values of $-.99999999$ and $.99999999$ equal -1 and 1 respectively). With these values and the visual representation of the graph, a student could now state that the equation is increasing on the interval $(-\infty, -1)$ and $(1, \infty)$. The interval between $(-1, 1)$ may need further investigation because from a visual examination it may not be clear whether the equation is continually decreasing. For a function to be decreasing it must be the case that as x increases in value, y decreases in value. One way to determine if the equation

is continually decreasing or not is to examine a table of values of the equation to see if y is always decreasing as x is increasing. It is also possible to draw a table of values with a graphing calculator. Figure 70 is an example of what such a table would look like with table intervals set of 0.1:

X	Y ₁
-3	.12771
-2	.03904
-1	.00497
0	0
.1	-.005
.2	-.039
.3	-.1277

X=.3

Figure 70. Table of values of equation with interval set of 0.1 on TI-83.

The intervals on the table are chosen to be sufficiently small so as to examine a number of values between $x = -1$ and $x = 1$. A student could pick different intervals, which would provide a different table of values. Regardless, a table shows that the equation values are not only decreasing but also reflective about the origin, which is consistent with the symmetrical nature polynomial equations.

Therefore, from the information received using a graphing calculator representation of both a graph and a table, a student would be able to answer that the equation is increasing between $(-\infty, -1)$ and $(1, \infty)$, decreasing between $(-1, 1)$, has a local maximum at $(-1, 2)$ and a local minimum at $(1, -2)$.

Second suggested solution (this solution uses derivatives):

One way a student might solve this question is to use algebraic symbolic representations of the equation and find the first derivative in order to determine the

intervals and max/min values. The first derivative of an equation represents the slope of the equation at a particular point. By setting the derivative of the equation equal to 0 and then solving for the x-values, we can determine where the equation has a slope of 0, or where the equation levels off (that is, where the equation would have a maximum or a minimum). Using this method, a student may start with the equation,

$$y = 3x^5 - 5x^3$$

and, using symbolic derivative rules, find that the first derivative is,

$$y' = 15x^4 - 15x^2.$$

If a student then sets $y' = 0$ and solve for x he/she would get,

$$x = 0, x = 1, \text{ and } x = -1.$$

The student would thus be left with three points that may possibly be maximum or minimum values. According to the properties of equations and first derivatives, it is the case that if an equation increases at values less than a point and decreases at values greater than a point, then the point is a maximum. The reverse is true for a minimum point. In situations where the slope of the equation at a point is 0 but the equation does not change direction from increasing to decreasing or decreasing to increasing, then the point is an inflection point and not a maximum or minimum.

To determine if a equation is increasing or decreasing on the intervals between the x-values found above, a student may plug a value from the intervals into the derivative equation y' . A student could have proceeded from here in the solution using a graphing calculator as well after finding the maximum and minimum values. According to the properties of a derivative, if the derivative values are negative on an interval than the equation is decreasing and if the derivative values are positive on an interval than the

equation is increasing.

One representation a student might use to test derivative values between intervals is a number line/tabular representation (Figure 71).

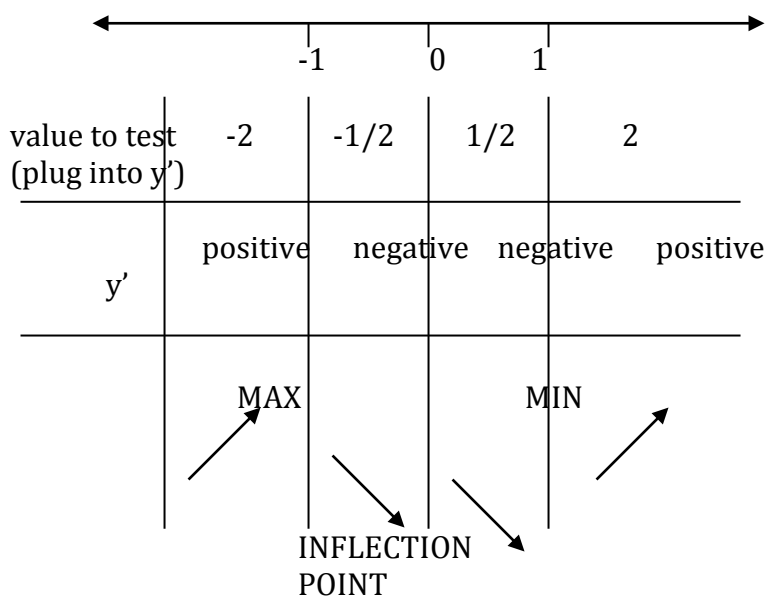


Figure 71. Number line/table representations of first derivative interval test.

Therefore, this representation provides the information that the equation y has a local maximum at $x = -1$, a local minimum at $x = 1$, is increasing between $(-\infty, -1)$ and $(1, \infty)$ and is decreasing between $(-1, 1)$. By plugging $x = -1$ and $x = 1$ into the original equation we can determine that the points $(-1, 2)$ and $(1, -2)$ are the maximum and minimum points respectively.

It is important to note that a student might use a combination of the two above solutions. For example, a student may begin by graphing the equation on a graphing

calculator in order to have a visual representation, but then may proceed with answering the question by using the first derivative of the equation. While using the first derivative of the equation, a student may refer back to the graphical representation to confirm that the results from the derivative are consistent with how the equation looks.

4.

Given the function $f(x) = x - \frac{1}{(x^2)(x+2)}$

Find:

$$\lim_{x \rightarrow -2^+} f(x)$$

$$\lim_{x \rightarrow -2^-} f(x)$$

$$\lim_{x \rightarrow 0} f(x)$$

Show your work and explain your process and thinking.

Solution and Discussion

First suggested solution: To solve this question a student may begin by graphing the function on a graphing calculator in order to have a visual representation on the function. Figure 72 shows the function input screen and the graph.

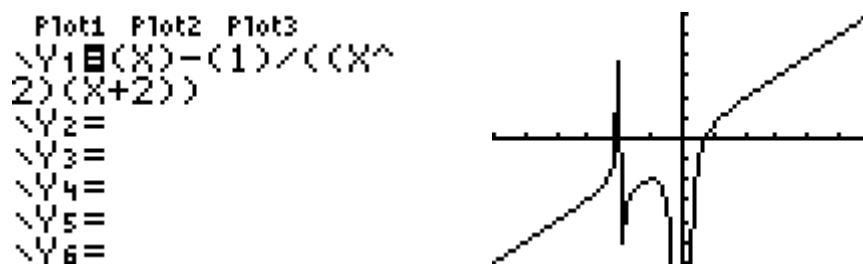


Figure 72. Taken from graphing TI 83 calculator of input screen and graph.

If the student was familiar with the way graphing calculators draw functions that go to infinity or negative infinity, then the desired limits would be able to be found by examining the graph. For example, the straight vertical line at $x = -2$ is how the graphing calculator draws an asymptote, indicating that the function does not exist at this point. As the function gets close to the point $x = -2$ on the right, the curve of the

graph proceeds down to negative infinity. Similarly, as the function gets close to the point $x = -2$ on the left, the curve of the graph proceeds up to positive infinity. Thus, from this examination, a student could conclude that:

$$\lim_{x \rightarrow -2^+} f(x) = \text{negative infinity}$$

$$\lim_{x \rightarrow -2^-} f(x) = \text{positive infinity}$$

A similar method can be used to determine the $\lim_{x \rightarrow 0} f(x)$. First, the graphing calculator can be used to determine if there is a real number solution at $x = 0$ by tracing the graph at $x = 0$, show in Figure 73.

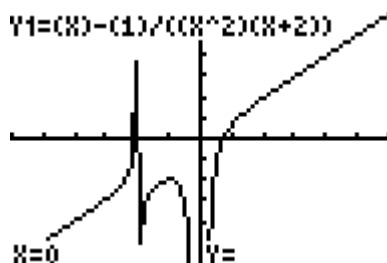


Figure 73. Graph from TI-83 calculator of answer of function at $x=0$.

The bottom of the graph screen shows that when $x = 0$, there is no real value solution for y . Thus, it is the case that we need to see if the left and right limits of $\lim_{x \rightarrow 0} f(x)$ are the same in order to say that $\lim_{x \rightarrow 0} f(x)$ has a solution. An examination of the graphing calculator image shows that as the function approaches $x = 0$ on the left and on the right, the function proceeds to negative infinity. Thus, since the left and right limits are equal, we can say that:

$$\lim_{x \rightarrow 0} f(x) = \text{negative infinity}$$

Second suggested solution: Alternatively, the student could proceed by using limit

theorems to break up the function into two limits and proceed without a graphical representation. For example, the limits could be written as:

$$\lim_{x \rightarrow -2+} f(x) = \lim_{x \rightarrow -2+} x - \lim_{x \rightarrow -2+} \frac{1}{(x^2)(x+2)}$$

$$\lim_{x \rightarrow -2-} f(x) = \lim_{x \rightarrow -2-} x - \lim_{x \rightarrow -2-} \frac{1}{(x^2)(x+2)}$$

$$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} x - \lim_{x \rightarrow 0} \frac{1}{(x^2)(x+2)}$$

For the first limit, $\lim_{x \rightarrow -2+} x = -2$. The second half of the limit is equal to positive infinity. The process used to determine this limit is to imagine a number infinitely close to -2, but still slight larger, as we are taking the limit as we approach -2 from the right.

In that case, $\frac{1}{(x^2)(x+2)}$ would result in a fraction of one over an infinitely small

positive number. Thus, $\lim_{x \rightarrow -2+} \frac{1}{(x^2)(x+2)} = \text{positive infinity}$ and $\lim_{x \rightarrow -2+} x - \lim_{x \rightarrow -2+}$

$\frac{1}{(x^2)(x+2)} = -2 - \text{positive infinity} = \text{negative infinity}$.

An identical process is used to solve the second limit. Again, $\lim_{x \rightarrow -2-} x = -2$. The second half of the limit is equal to negative infinity this time. The process used to determine this limit is to imagine a number infinitely close to -2, but still slight smaller,

as we are taking the limit as we approach -2 from the left. In that case, $\frac{1}{(x^2)(x+2)}$

would result in a fraction of one over an infinitely small negative number. Thus,

$$\lim_{x \rightarrow -2^-} \frac{1}{(x^2)(x+2)} = \text{negative infinity} \text{ and } \lim_{x \rightarrow -2^-} x - \lim_{x \rightarrow -2^-} \frac{1}{(x^2)(x+2)} = -2 -$$

negative infinity = positive infinity.

To solve the $\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} x - \lim_{x \rightarrow 0} \frac{1}{(x^2)(x+2)}$, we first find that

$\lim_{x \rightarrow 0} x = 0$. Then $\lim_{x \rightarrow 0} \frac{1}{(x^2)(x+2)}$, is equal to positive infinity whether it is from the

left or the right, because $(x^2)(x+2)$ will always equal a positive number infinity close to

zero. Thus, $\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} x - \lim_{x \rightarrow 0} \frac{1}{(x^2)(x+2)} = 0 - \text{positive infinity} = \text{negative}$

infinity.

Overall, this second suggested solution requires a strong understanding of the concept of limits and the nature of infinity.

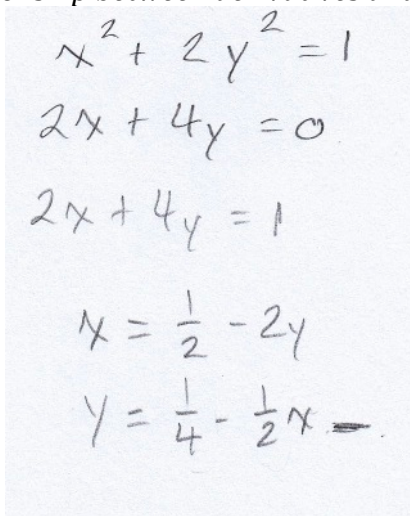
Appendix D: Examples of Transcripts with Written Work Images

Example of a portion of a transcript from Participant #1

A: Let's open the page then and just look at these questions, okay. *(opens booklet)* So this asks us "on what point on this equation does the tangent have a slope of 1 and to show your work and explain your answer." So can you tell me what you started to do here?

At what point(s) of $x^2 + 2y^2 = 1$ does the tangent line have a slope of 1? Show your work and explain your answer.

P1: My first line, I just wrote the equations and then I tried deriving because the derivative of the equation is the slope of a tangent line. I knew we had to find where it equals 1 and that's where I wasn't sure what to do. So I isolated the variables, so when x equals this and y equals that we will have a slope with tangent equal to 1. *(points to the pages and talked with his fingers. Makes a backwards and forward motion with his finger when talking about the relationship between derivatives and tangents)*



Handwritten work showing the derivation of the tangent line slope for the equation $x^2 + 2y^2 = 1$:

$$x^2 + 2y^2 = 1$$

$$2x + 4y = 0$$

$$2x + 4y = 1$$

$$x = \frac{1}{2} - 2y$$

$$y = \frac{1}{4} - \frac{1}{2}x$$

A: Okay, so for this question, what type of equation is this?

P1: *(silence)*

A: I am not sure if my question is making sense. If we have a y equal a value and then another value in front of x . So do you recognize what type of equations those are? *(points to the page with pencil to show the equation)*

P1: I remember a specific name...

A: This is an equation of a line, right? Remember y equals mx plus b .

P1: Okay?

A: So you were on the right track because you were talking about tangent lines, okay.

P1: Okay.

A: So now what type of equation is this *(points to equation in question)*?

P1: It would be a curve because that x has to the power of 2.

A: *(nods)* okay. So I think we are going to move on from this one because you didn't do a lot of written work for this one. But is there anything else you would like to say about this question?

P1: Not really.

A: Do you want to ask me any questions about this?

P1: I would like to know the answer, but...

A: Okay, we can do it after if you would like to know. ***(Turns pages to question #2)***. So here is number 2. Now, we mentioned that you had seen equations, sorry, not equations, questions like number 2 before.

P1: Yeah.

A: So first off we are given a piecewise function and asked to find limits but we are actually also asked to find left and right limits with it. Okay. *(points to the function and question on the paper for a pencil)* So what did you end up doing here? *(gestures to the paper)*

Given the piecewise function

$$g(x) = \begin{cases} x & \text{if } x < 1 \\ 3 & \text{if } x = 1 \\ 2 - x^2 & \text{if } 1 < x \leq 2 \\ x - 3 & \text{if } x > 2 \end{cases}$$

Find the limits

$$\lim_{x \rightarrow 1^-} g(x)$$

$$\lim_{x \rightarrow 1} g(x)$$

$$\lim_{x \rightarrow 2^-} g(x)$$

$$\lim_{x \rightarrow 2^+} g(x)$$

$$\lim_{x \rightarrow 2} g(x)$$

P1: I remember there was a question like this on one of the midterms and I didn't do so well on limits, so, *(looks a little embarrassed)* I always struggled with limits. *(Hands waving in front of the paper)*.

A: (I offer him a pencil) Would you like to point out what you did, or whatever works?

P1: *(Doesn't take pencil but does point with finger. Points to the function and then the limit question and then answer)*. What I had done was so the limit when x approaches 1 from the left, I found where x is less than 1, which is equal to 1, so I said the limit of the expression at negative 1 to the left, $g(x)$, is x , according to what I was given.

$$\begin{aligned} \lim_{x \rightarrow 1^-} g(x) &= x \\ \lim_{x \rightarrow 1} g(x) &= 3 \\ \lim_{x \rightarrow 2^+} g(x) &= x - 3 \\ x &= 3 \\ \lim_{x \rightarrow 2} g(x) &= 2 - x^2 \\ x^2 &= 2 \\ x &= \sqrt{2} \end{aligned}$$

A: Okay.

P1: And then I did the same for x equals 2, approaches 1. So I found where x approaches 1 from the right, so I have $x-3$, so I did $x-3$. And then I did for when x approaches 2 from the right, so have the piece where 2 is included in, which is ... *(pointing to function, limits and answer)*

A: It looks like 2 is included right here *(points at the piece of the function)*.

P1: Yeah, that one. And then when it is equal to 2.

A: Oh, that one is when it is equal to 2 *(realizes we pointed to the wrong piece)*.

P1: Yeah, and that one *(points to a different piece)* is where we approach 2 from the right *(sweeps his hand like it is coming from the right)*.

A: *(mimics the gesture)*

P1: So then I just solved it so x equals 3 and x equals the square root of 2.

A: Okay, so, umm, you did understand the piecewise portion of the function. How do you picture this function?

P1: It seems very broken. Like there are a lot of *(uses hands to demonstrate pieces)* I guess you would say asymptotes. Oh wait no that's not the word...

A: No, don't worry about the word. Just explain to me how you see it.

P1: *(silence)*.

A: Like if I gave you the pencil and asked you to try to sketch it and it doesn't even have to be accurate... *(gestures to hand over pencil)*

P1: I couldn't draw it... *(seems very uncomfortable)*

A: you wouldn't be able to draw it.

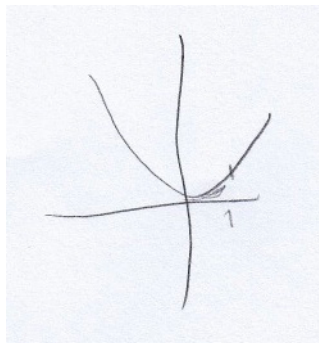
P1: yeah, I can't picture this... *(seems nervous)*

A: Okay. So yeah, you understood very well what it means to go from left to right on the points on the number line *(points to the left and right limits)*. Do you, umm, understand what it means from the visual perspective of the function... *(makes a gesture with hands for a whole)*

P1: *(silence, looks at page)*

A: From the graphical perspective?

P1: Umm, I guess *(takes pencil from me)*. So lets say it was like this *(he draws an xy-plane and puts a random curve on it)*. If you are approaching, the function approaches 2, or let's say 1 in this case, from the left, the value would approach 1 as it gets closer...*(looks at me)*, right?



A: Yep, that's the correct interpretation. Interpreted as approaching from only one direction, not from both. *(sweeps hand slowly in one direction first and then sweeps both hands towards the same point)*

P1: *(nods)*.

Example of a portion of a transcript from Participant #6

A: Alright, thank you. *(turns page)*. So here we were given a function...

Given the function $f(x) = 3x^5 - 5x^3$

On what interval(s) is the function increasing?

On what interval(s) is the function decreasing?

What is(are) the local maximum(s)?

What is(are) the local minimum(s)?

Show your work and explain your process and thinking.

P6: Yeah

A: ...and we were asked to find the properties of the function. So where did you start? Did you start right here?

P6: yeah, right here.

A: and you took the derivative and set it equal to zero. So why did you do that?

P6: Umm, I don't know what you call them... because you have to find the points where the slope is zero, to find where the function changes its intervals of increasing to decreasing

A: Is that where the slope is zero?

P6: well, not always, but sometimes. Like here, the slope was zero, but here it stayed decreasing, like *(points to work on page)* something like that *(moves figure along in a curve)*

$$\begin{aligned}
 f(x) &= 3x^5 - 5x^3 \\
 f'(x) &= 15x^4 - 15x^2 \\
 0 &= 15(x^4 - x^2) \\
 0 &= x^4 - x^2 \\
 0 &= x^2(x^2 - 1) \\
 &= x^2(x+1)(x-1) \\
 x &= -1, 0, 1
 \end{aligned}$$

A: Yep, okay. So you set it equal to zero and found these points. So then what did you do with them?

P6: You put the points, like you set them in intervals. So you do x is less than -1 , x is less than -1 and greater than 0 , x is less than 1 greater than 0 and x is greater than 1 . And then you have here and then you have x squared is always positive so that is why you have positive for the first one, and a negative times and negative is always positive so that's why it is increasing, positive times positive times negatives is negative so it is decreasing, etc. *(throughout the explanation he points to the work he wrote on the page)*

	$x < -1$	$-1 < x < 0$	$0 < x < 1$	$x > 1$
$f'(x)$	$(+)(-)(-)$	$(+)(+)(-)$	$(+)(+)(-)$	$(+)(+)(+)$
	+ increasing	- decreasing	- decreasing	+ increasing
	local max.		local min.	

A: Okay, and you found these by plugging in values from the interval.

P6: Like if you plug in a value that is less than -1 you get -1 squared which is positive, and *(mumbles through the other parts)* ...and so that's why you get positive, negative, negative, which is positive

A: And so I notice you have this table. How does this table help you answer this question?

P6: Because if you know that is increasing up to -1 and decreasing after -1 then you know that is like a local max. *(moves finger up for increasing and down for decreasing)*

A: Okay

P6: and then x is...it tells you if it is increasing, right. Like if it is increasing to decreasing then it is a local max and then if it is decreasing to increasing that is a local minimum. *(moves finger around for increasing and decreasing and points to questions on the page)*

A: Okay. So um, from the answer you got does that feel...do you feel this is the correct answer based on the function you were given.

' the function is increasing when $x < -1$ and when $x > 1$.
 ↳ the function is decreasing when $-1 < x < 1$.
 ↳ There is a local max. when $x = -1$, and this point is $(-1, 2)$
 ↳ There is a local min. when $x = 1$, and this point is $(1, -2)$.

P6: Do I feel it is correct? Yeah *(sounds confused that I would ask that)*.

A: Okay **(turns pages)**. Alright, question four. So again for question four we are given a function and we are asked to find three different limits. So can you explain to me what you did here?

$$f(x) = x - \frac{1}{(x^2)(x+2)}$$

Find:

$$\lim_{x \rightarrow -2^+} f(x)$$

$$\lim_{x \rightarrow -2^-} f(x)$$

$$\lim_{x \rightarrow 0} f(x)$$

P6: Kay, so I saw that as x approaches these numbers, these...the x was not going to

have anything to do with the actual limit, so you could just disregard it. So if you plug in a value for x as it approaches -2 from the right, like -1.9999 or whatever, you will get a really big number. But there is a negative here, so it will be negative infinity. (points to question and work on page while answering).

Plug in values as $x \rightarrow -2^+$, and as $x \rightarrow -2^-$, and as $x \rightarrow 0$.

Since: the x out front will not make a difference, we can ignore it for all of the limits because just by looking at them, they will either be positive or negative infinity.

$\lim_{x \rightarrow -2^+} f(x) = -\infty$ $\lim_{x \rightarrow -2^-} f(x) = \infty$ $\lim_{x \rightarrow 0} f(x) = -\infty$	$\lim_{x \rightarrow -2^+} f(x) = -\infty$ $\lim_{x \rightarrow -2^-} f(x) = \infty$ and $\lim_{x \rightarrow 0} f(x) = -\infty$
--	---

A: So you get the big number in the denominator, right so that makes it infinity?

P6: Well, you get a small number in the denominator, so that makes it infinity

A: Oh, yeah, right. And so did you use the same logic to answer the other ...

P6: yeah, that's the exact same

A: Was there anything visually going on in your mind as you solved this one?

P6: No. There didn't really need to be for this one?

A: Well, whether there needs to be or not, sometimes people picture things.

P6: yeah, I know. But I didn't.

A: But you didn't for this one. You were able to solve it strictly with algebraic manipulation.

P6: yeah.

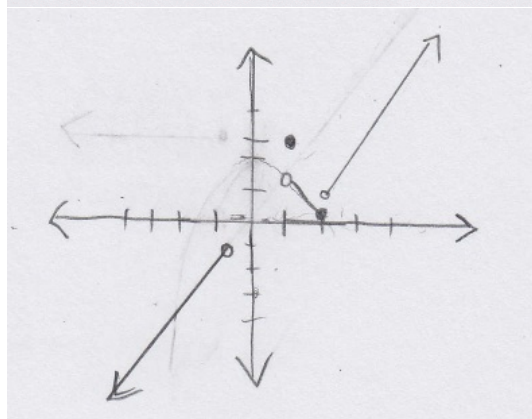
Example of a portion of a transcript from Participant #11

P11: yeah, that one, I am pretty visual. But again I hadn't seen this....

A: Piecewise function?

P11:in awhile. So I was like okay. And I did them all and I was erasing what wasn't good anymore. And I did that quick and I feel that might be wrong. But what I did basically is I graphed it and wherever 1, like it approaches 1 from the left, and I guess it was going to 3, that point. *(talks with hands, makes gestures while speaking and points to work on page)*

- graphed the functions first to see
 - described what was happening when $g(x)$ was approaching 1 or 2 from left or right.
 - if 2^- and 2^+ are equal, limit exists, if not equal, the limit does not exist.



A: Okay

P11: And at 1, it doesn't exist because

A: because of the gap

P11: yeah, because on the other side it would be that open dot point

A: yeah

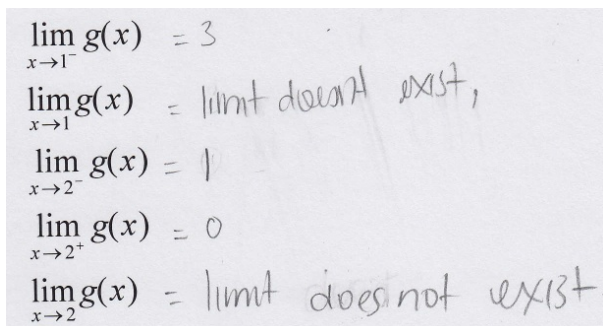
P11: And then at 2^+ , 2 from the right, okay I think that is supposed to be, ah,.... I made a mistake on my picture *(points to work on page)*

A: I think those are supposed to be*(talking over can't understand)*...and then 1 is what you have drawn here.

P11: yeah

A: But then again, interpreted

P11: yeah, cause they don't match. And then I wrote, graphed the function and then figured out what $g(x)$ is approaching when it approaches 1 or 2 from left or right. And if they are equal at the 2's and 1's there is a limit and if they are not equal than it doesn't exist. *(talks with hands, makes gestures while speaking)*



Handwritten mathematical limits on a piece of paper:

$$\lim_{x \rightarrow 1^-} g(x) = 3$$

$$\lim_{x \rightarrow 1} g(x) = \text{limit doesn't exist,}$$

$$\lim_{x \rightarrow 2^-} g(x) = 1$$

$$\lim_{x \rightarrow 2^+} g(x) = 0$$

$$\lim_{x \rightarrow 2} g(x) = \text{limit does not exist.}$$

A: Okay, so you said that graphed the function first. Did you find that by graphing the function it helped you to solve it?

P11: Oh, I don't think I could solve that without seeing it.

A: Okay

P11: I am so visual

A: So do you find that like, along that train of thought, if you could visualize what is going on in the math...

P11:...you can understand better, yeah

A: You find that for yourself

P11: Yeah, and I find teachers don't really always consider that

A: they don't teach the visual

P11: yeah, and I am like someone who really, really needs it. But if they don't I will like manage to do it, but I won't understand. But I can still do it. *(talks with hands, makes gestures while speaking)*

A: yeah

P11: it just means that I'll forget because it is not stuck anywhere

A: Okay, so let's think about it this way. Let's just compare these two questions. So up here you have some steps and they are the right steps, but you got stuck somewhere

P11: yeah

A: Here you visualized and you drew it and were able to get all of the answers.

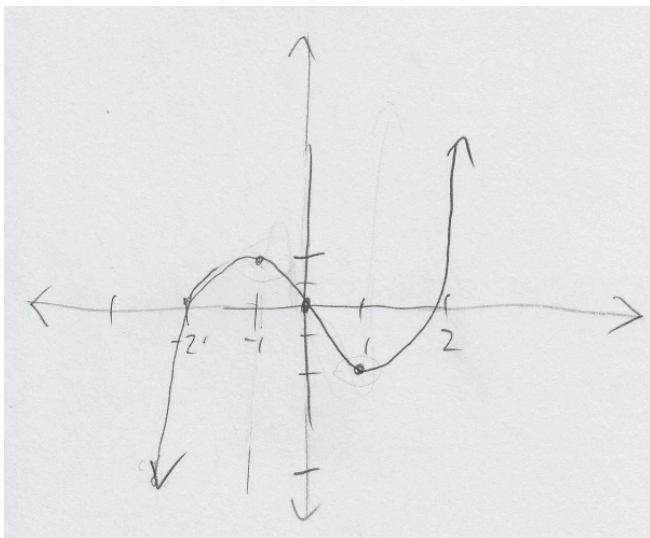
P11: yeah

A: So is that kind of a good contrast, a good way of describing

P11: I guess. With this I think it is just like a basic calculus question, I think that's why I didn't know it. But yeah, I think you are right. It definitely helped to visualize

A: Alright **(turns page)**. Here's another one and I see another graph. So here we are given a function and we were asked to find certain properties about this function. So what came to your mind as you saw this?

P11: Graph the function



A: Graph the function

P11: Yeah

A: So did it come to your mind that I should graph the function first, or did it come to your mind that I should find these properties and then draw the graph?

P11: Mmmm, I wanted to see it first and then I could see what was happening

A: Okay

P11: But again, I know the local max and min, is this, I think it is. I don't know, we did this stuff in high school again so I don't remember. But *(talks with hands, makes gestures while speaking)*

A: So now did you graph this using a graphing calculator?

P11: yeah, but I mean, I just wanted to make sure it was right

A: yeah

P11: So then I graphed it and then wherever it was increasing I wrote that and where it was decreasing I wrote that... *(points to work on graph)*

a) The function is increasing b/w $-\infty < x < -1$, and b/w $1 < x < \infty$.

b) The function is decreasing when $-1 < x < 1$.

c) local max is when $x = -1$

d) local min is at $x = 1$

A: So where did you find these max min points. Did you use the calculator or did you just...

P11: No I, yeah, I used that calculator and I just saw that at -1 it was the highest, so then I saw what y would be...I don't know... *(points to work on page)*

A: Yeah, you saw that as the highest so you put it as the max

P11: yeah

A: so you used the calculator then to recognize these two point

P11: I mean I know I could have derived it, but the calculator was given to me.

A: No, that's fine. I just wanted to know what tools you used. The tools were given to

you, you can use whatever you want

P11: That's what I figured so I used the calculator

A: no, you were allowed to use the resources in front of you. Alright, so I guess this is kind of self explanatory, but when you see a question like this you have something visual come to your mind first.

P11: Yeah yeah

A: So now, do you understand these from a calculus perspective or has that slipped your mind?

P11: Probably slipped my mind

A: okay

P11: Yeah, again, these are questions I haven't seen in a long time. So on what intervals is the function increasing. I just kind of guessed about what made sense, when it has a positive slope. I don't know if that's right *(moves hand up for positive slope)*

A: So you just used the statement positive slope, why would you say positive slope for increasing.

P11: ahhh, because a positive slope usually means it is a positive number if it is increasing, negative number if it is always downwards, decreasing. *(moves hand down for decreasing)*

A: So that statement you made, is that something you understand or is that something you know

P11: Ah, no I think I understand that

A: Okay

P11: A little bit of both, it makes sense whenever it is taught

Appendix E: Examples of Summarized Interviews

Example of Interview Summary from Participant #2

Participant #2 was an 18 year old female in her first year of study at the University of Ottawa. She was Canadian by birth, did not identify herself with any minorities, and had always studied in the province of Ontario for her pre-university education. English was her first language and she reported speaking a decent amount of French, although poorly. At the time of the interview, Participant #2 was majoring in a Bachelors of Commerce with a specialization in Accounting. Calculus was required for her program of study and she reported taking Math 1300 in the Fall 2010 semester and received an A+ grade upon completion. Prior to attending university she took all three of her Ontario grade 12 university pre-requisite mathematics courses, which included Calculus and Vectors. She was a little shy during the interview and had to be encouraged to answer questions. However, she warmed up as the interview went on and felt more comfortable sharing information.

When asked about her impressions about the calculus tasks, Participant #2 stated that she didn't get the first question and that it was difficult. However, she stated that the rest of the questions were okay but that she was concerned that she wasn't going to pass. This is interesting as it was made very clear to Participant #2 that the calculus tasks would not be marked. She stated that she understood what the questions were asking her to do and that she had seen questions like the tasks in her class, on assignments, and in DGD lectures in her calculus class.

The interview then moved onto the task questions. Participant #2 was asked how she understood the questions. She said she knew that she had to find a derivative because the tangent line is the derivative, although she made this statement very hesitantly. She began by rearranging the equation for y and then took the derivative.

A: Okay. So you knew you had to find the derivative, so I see here you started by rearranging the equation for y and then you went and you took the derivative from there... (*pointing to page*)

$$\begin{aligned}
 2y &= -x^2 + 1 \\
 y &= \frac{-x^2 + 1}{2} \\
 &= \frac{1}{2} (-x^2 + 1)^{\frac{1}{2}} \\
 y' &= \frac{1}{4} (-x^2 + 1)^{-\frac{1}{2}} (-2x) \\
 y' &= \frac{-\frac{1}{2}x}{\sqrt{-x^2 + 1}}
 \end{aligned}$$

Afterwards Participant #2 set the derivative equation to 1. However, when asked what to do to find the slope, she said she her memory failed.

$$y = 1 = \frac{-\frac{1}{2}x}{x^2+1}$$

$$1 \left(\frac{1}{\sqrt{x^2+1}} \right) = -\frac{1}{2}x$$

$$\frac{1}{\sqrt{x^2+1}} = -\frac{1}{2}x$$

I asked Participant #2 if there was anything visually going on in her mind while she solved this question. She said that she considered graphing it but found it too hard. I asked her if she considered using a graphing calculator and she informed me that she had never used one before and didn't know how to. However, when I hypothetically asked her what she would have looked for from the graph she responded:

P2: I would have looked to see where...I would have just liked to visualize it to see where the tangent was approximately 1, I don't know.... *(holds up hand in a straight/flat manner when mentions tangent)*

A: Okay, so you would have been looking for the tangent to see how the tangent changed around the equation? *(makes curving motion with finger when mentioning changing equation)*

P2: Yeah, yeah.

A: Now a tangent that has a slope of 1, what does that tangent look like?

P2: *(demonstrates with her hand)* It would be like going upwards.

A: Yeah, it is pretty upwards, at y equals x. *(mimics hand motion)*

P2: *(nods)* yeah.

Example of Interview Summary Participant #8

Participant #8 was a 19 year old male student studying full time at the university of Ottawa. He was born in Canadian and identifies himself as Chinese Canadian. He reported always studying in the province of Ontario, including all of his pre-university education. He did not identify himself with any minorities, and only spoke English. At the time of the interview, his program of study at the University of Ottawa was Commerce with a specialization in Accounting. Calculus was required for his program of study, only Math 1300. He had taken Math 1339 in the Fall 2010 semester and reported getting an A+ and was currently in Math 1300 at the time of the interview. He had not taken Calculus and Vectors in his grade 12 year of high school in Ontario. When asked about his impressions of the task questions he had completed and how he felt about them, he informed me that he felt that they weren't too bad and that he had definitely seen questions like the task question during his calculus courses. He said he had seen questions like these in mostly his Math 1339 course on examples, in the textbook and also on tests.

The interview then moved onto a discussion of the participant's solutions to the task questions. For question #1 I asked the participant what he understood the question wanted him to do.

P8: Okay, now that I've read it I think I did it, like the question wrong. Like I answered it totally wrong. Like the slope of the tangent line is the derivative. So I need to find the equation of the derivative and then once I have found that I need to work it out...I have to find like where the slope is equal to 1 (*talks with hand throughout explanation*)

$$x^2 + 2y^2 = 1$$

Set $f(x) = 0$

$$x^2 + 2y^2 - 1 = 0$$

Derivative

$$2x + 4y = 0$$

$$4y = -2x + n$$

$$y = \frac{-2x + n}{4}$$

slope = $-\frac{1}{2}$ $\therefore$ tangent line does not have a slope of 1

$y = mx + n \leftarrow$ tangent line
slope of 1
 $y = 1x + n$

This statement demonstrates that although the participant initially answered the question completely wrong, once he read it through a second time he knew how to solve it correctly. When I asked the participant if there was anything visually going on in his mind while he solved this question:

P8: But for like drawing it, like I didn't need to draw it out, I didn't think that would help or anything. I just was thinking about what we did in lectures and stuff

He also said that he didn't feel like he needed to draw the graph for this problem and that if he had drawn the graph it just would have been an extra step.

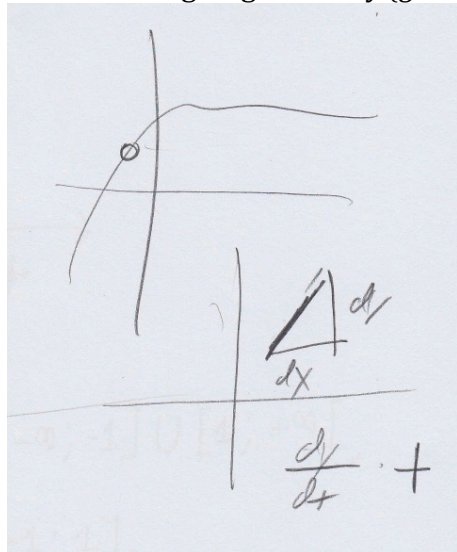
Example of Interview Summary Participant #10

For question #3, the participant stated that the first thing that came to his mind when he read this question was that he needed to take the derivative. When asked why he needed to take the derivative he said:

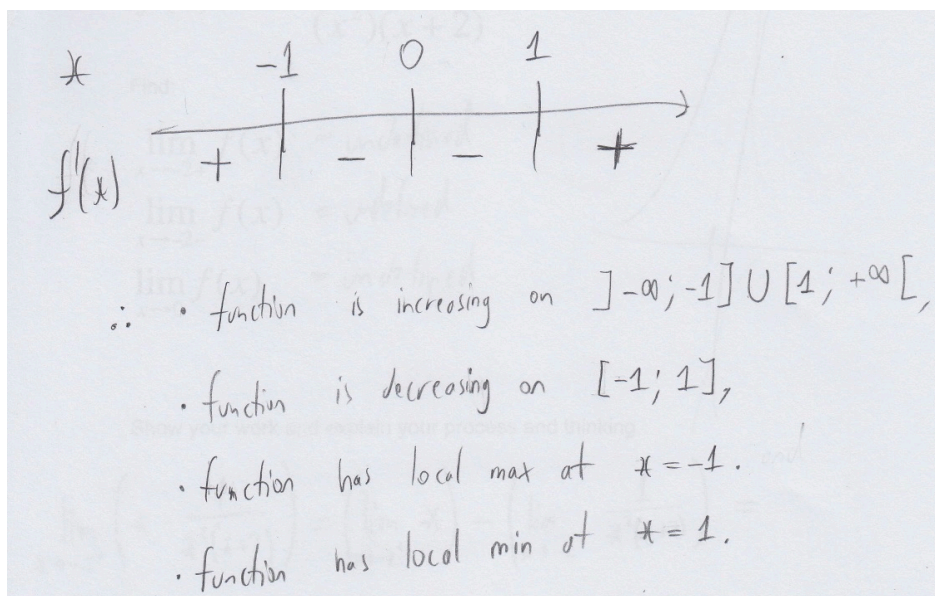
P10: Well, if the derivative is positive it is increasing, if it is negative it is decreasing. Then if you have a change from...I don't remember exactly, from plus to minus, we have a minimum, I mean maximum. And this if it is minus to plus it is the other way. *(points to work on page and make up and down motion with hand for min and max)*

When asked why a positive derivative is increasing he told me:

P10: Because, if you take...the derivative is basically the limit if we zoom in the function. So if we got here for example and we zoom this in we have something here that is straight. Since this, if we take this to be dy/dx , we know that dy/dx is going positive because it is going that way *(grabs pencil and starts writing his explanation)*



I then moved onto to asking Participant #10 to explain to me the steps he took to solve the question. He began by taking the derivative and setting it equal 0. He said this is because 0 is where he would find sign changes. He found three points where the derivative equals 0 and set intervals between these points. Within these intervals the participant tested to see if the derivative was positive or negative, in order to determine if the derivative was increasing or decreasing. The participant's answer concluded with a number line chart and some answer statements.



I also asked the participant if he could describe what was happening at the point 0 on the function. With some discussion he was able to tell me that it was a point where the concavity changed, but not the sign of the derivative. I concluded the discussion of this question by asking him if there was anything else going on in his mind while he solved this problem. He told me that he never memorized what happens when the sign changes direction and instead he always just pictures what is happening, in order to say whether it is a maximum or minimum. He said that in situations like these he never memorizes the facts.

We then turned the page and moved onto discussing the last task question. I began by asking the participant how he solved for the limits in this problem.

P10: Well, we know that since this value is finite it is a constant multiplied by a limit so we can simple sub in $-1/4$ becauseand here we can substitute in 2 or -2 plus h or something like that equals x plus 2, or -2, whatever (*points to work on page*)

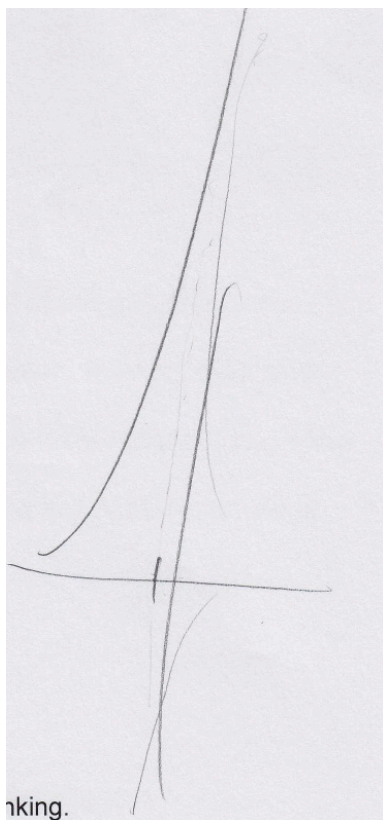
$$\begin{aligned}
 \lim_{x \rightarrow -2^+} \left(x - \frac{1}{x^2(x+2)} \right) &= \left(\lim_{x \rightarrow -2^+} x \right) - \left(\lim_{x \rightarrow -2^+} \frac{1}{x^2(x+2)} \right) = \\
 &= -2 - \left(\lim_{x \rightarrow -2^+} \frac{1}{x^2} \right) \left(\lim_{x \rightarrow -2^+} \frac{1}{x+2} \right) = \\
 &\quad \text{we can do this because it is finite.} \\
 &= -2 - \frac{1}{4} \left(\lim_{h \rightarrow 0^+} \frac{1}{h} \right) = \text{undefined.} \quad \text{---} \infty
 \end{aligned}$$

Similar argument applies for $\lim_{x \rightarrow -2^-} f(x)$ and $\lim_{x \rightarrow 0} f(x)$.

$$\begin{aligned}
 \lim_{x \rightarrow -2^+} f(x) &= \text{undefined} \\
 \lim_{x \rightarrow -2^-} f(x) &= \text{undefined} \\
 \lim_{x \rightarrow 0} f(x) &= \text{undefined.}
 \end{aligned}$$

He stated that he used similar logic to determine that all three of the limits were at infinity or negative infinity so that made him conclude that the limits were undefined. However, upon finishing his explanation he told me that he felt he had screwed up and that the answers should have been positive infinity or negative infinity. I reassured him that based on the way the question was asked that his answer was correct as well. I then asked the participant if he could describe for me what he understood was going on at the limits.

P10: Well, it just goes really up.... (first makes hand sweep up really high and then grabs the pencil and draws what he is thinking)



This was his way of describing what a limit at infinity looks like. I concluded by asking the participant if there was anything visually going on while he solved this question. He told me that there wasn't, but that now he thinks that he should have pictured it more, because then he would have understood that the questions should have been positive or negative infinity. This demonstrated that he still felt that his answers were wrong, despite my comment that they were not.

At this point we closed the booklet with the task questions and moved onto questions about the participants experience of taking calculus in general. I began by asking the participant what a typical class was like. He described it as the professor giving theory and then doing a few examples and then moving on. Participant #10 informed me that he attended class and took notes on the theory and then "googled" the different topics. He said he hardly learned anything in class, but instead did most of his learning from the internet, particularly the site Wolfram-Alpha. The participant told me that the professor did not seem to have any specific solving techniques, other than just going through examples in a step-by-step algebraic manner.

When I enquired what he had learned the best in calculus, he said that there was no specific topic. In terms of topic(s) he struggled with, he told me that at first he was very confused by integrals because the teacher had never told the class that not every integral has a closed form answer. He said he got stuck on some unsolvable integrals for awhile until he did some reading on the topic.

I then asked the participant how graphs and diagrams were used in the course. He said that apart from the actual chapter on graphing, that the teacher would sometimes use them in his lecturing. When I asked if he could describe how he told me:

P10: well there is more than one way to learn anything. For some people it might just be the derivation, for some people it might just be applying the theory, for example you don't go through the entire derivation when you want to solve a quadratic. You just do the derivation to obtain $-b$ or $-\text{square root whatever}$. *(talks with hands, makes gestures while speaking)*

A: yeah

P10: So you just use it. Someone else might trace a graph and imagine what point it is. Or just start solving it, using the regular formula. So there is more than one way of obtaining an answer. I think he should use at least two of them to capture as many people as possible. *(talks with hands, makes gestures while speaking)*

A: Would he often use at least two of them?

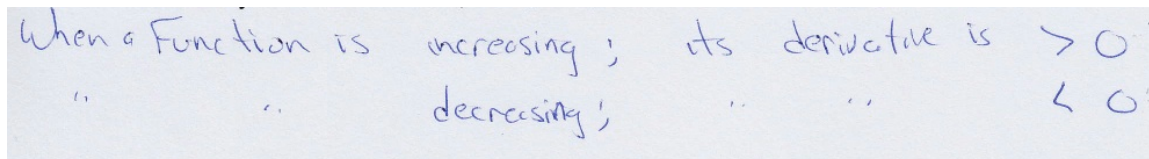
P10: *(nods)*

He concluded by telling me that the actual class used no technology or calculators. However, the participant himself used the internet as almost a constant resource. He informed me that he didn't even buy the textbook and that he just used his notes from class to look up the topics online for studying. He also told me that the class lectures were very heavily weighted towards algebra versus graphing and diagrams. He said that graphing and diagrams sometimes showed up in the theory. If the professor used them on an example, then it was only on the first example or on the hardest examples.

Example of Interview Summary for Participant #13

The interview then went onto the third question in the task booklet. I began by asking the participant how he began his solution to his question:

P13: Once again, I've done lots of these questions, so more like an algorithm, like a mechanical process, like you said (*gestures with hands while talking*). So it asked me to explain my thinking, so this is kind of just proving, or justifying why I am using this. So I would write like these facts. (*pointing to work while talking*)



When a Function is increasing ; its derivative is > 0
 " " decreasing ; " " < 0

I asked the participant if he was able to explain to me why the facts he included throughout his solution were correct.

A: So can you just explain...let's start with these two facts here....how do you understand these facts. Even just start with the first one "when a function is increasing the derivative is greater than 0"

P13: So the function is going up, let's say it is increasing, its y value is increasing, (*makes finger goes up*) so the derivative would be positive because at every point on this function, this original function since it is going higher and higher (*moves hand up for higher and higher*), the slope is going to positive, it is going to be a positive slope. So the derivative is just I guess a collection of all of the slopes

The participant was able to explain all of the facts he had written down and why they were correct in terms of the mathematics and what he wanted to solve for the question. He also described how he solved the question by taking the derivative of the function, setting the derivative equal to 0 and then finding the roots of the derivative function. From here he placed the intervals between the roots into a table, which he said he learned from high school. When asked how he used the table he said:

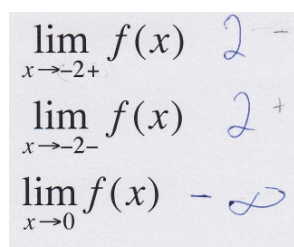
P13: And then determine in this interval whether the value would be positive or negative. Fill out the interval and multiply it through. Determine...and then when you do this on the bottom, this is where you find out your function itself if it is positive or negative on that interval. (*point to work on page while talking*)

When asked if there was anything visual going on while he solved the question he stated that there was nothing visual about how he solved the problem, but more visual in how he understood the problem. He repeated that he understood this question in graphical terms but solved it mechanically.

The interview then moved onto the fourth and final task question. The participants

solution described the points $x=0$ and $x=2$ as asymptotes and then used the calculator to plug in values on either side of the asymptotes to find out the behaviour roughly. For example, he described his answer to the third limit as:

P13: Um, since it was 0 from both sides I just decided to follow it and I was like yeah, as x gets 0...as x approaches 0 this function keeps going down and that doesn't reach a single finite value. It just keeps going down forever (*points down for the function, and points to work on page*)



Handwritten mathematical limits on a piece of paper:

$$\lim_{x \rightarrow -2^+} f(x) \quad 2^-$$

$$\lim_{x \rightarrow -2^-} f(x) \quad 2^+$$

$$\lim_{x \rightarrow 0} f(x) \quad -\infty$$

It is interesting however that he was not able to apply this logic to the first two limits and instead found that they go to finite answer.

When asked if there was anything going on visually the participant stated that he wanted to see the graph of the function but that he wasn't able to picture it in his head. When asked why he didn't use the graphing calculator he said that if it had been a real test he wouldn't have been able to so he choose not to use it here.

At this point we closed the booklet and moved onto to discussing calculus in general. When asked what a typical calculus class is like the participant described them as very structured classes. He stated that he always would work ahead so that he didn't fall behind before class started. He also said that he takes math very serious so he would always do all of the homework and assignments. When asked why he always took math very serious he said that his parents and teachers instructed him to.

He said that there was no specific topic emphasized in calculus. In terms of teaching techniques, he said that his professors were good at describing more than one way to solve a question. The examples the participant gave were an algebraic technique and a visual technique. He also said that while he might use the algebraic technique more, he understands better with the visual technique.

In terms of how graphs or diagrams were used in calculus, the participant told me that his high school lessons were done almost 100% on a Smart Board and that he found it a really good way to learn. In university he said that the graphing or diagram component of calculus doesn't stand out in his mind. He explained that professors will usually start with the algebra portion of how to solve a problem and that a graph or diagram might be added as supplementary at the end of the question.

He stated that the only technology he used in university calculus courses was graphing software on his own computer at home. He said that there was no technology used in

class and although he felt that 100% of what he was taught could be represented by graphs or diagrams, that it was mostly done algebraically and class and that the algebra component was definitely what was tested.

Appendix F: Categorizations of Representations Document

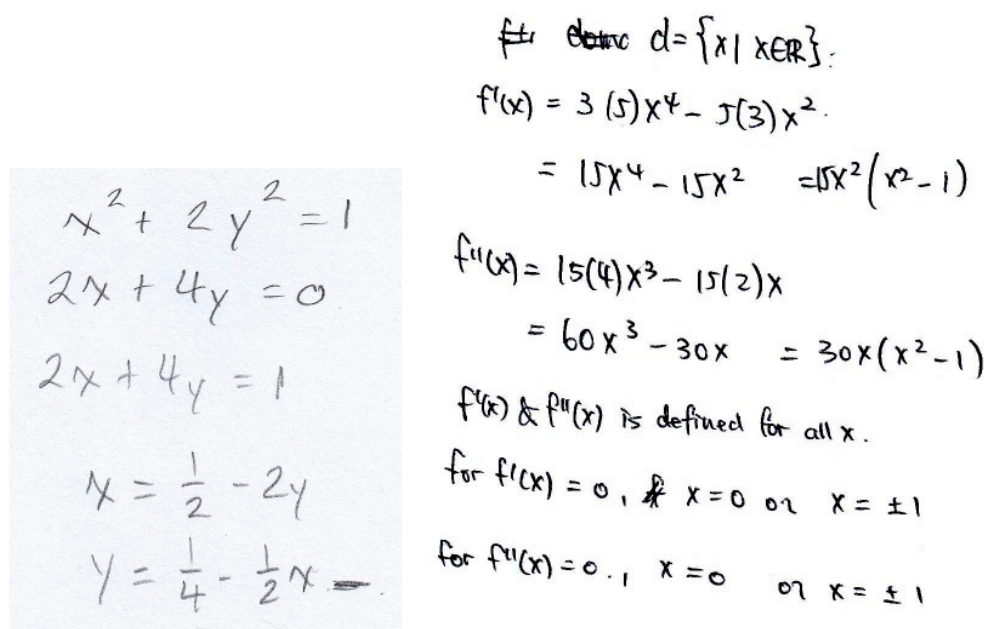
Categorizations of Representations according to the Three Worlds

Symbolic-Proceptual (Symbolic)

This world is a combination of processes and concepts, which are termed procepts, that act together with symbolic systems of mathematics. (Procepts are described as single mental object made up of the encapsulation of a mathematical process that uses symbols and the relating concept.)

Examples:

- Any equations, functions, formulas, or inequalities, whether using algebra or only numbers and operators. Also includes any manipulation of equations, functions, formulas, or inequalities, from one form to another.
- Step-by-step procedures that only use symbolic algebra or numbers.



Handwritten mathematical work:

$$x^2 + 2y^2 = 1$$

$$2x + 4y = 0$$

$$2x + 4y = 1$$

$$x = \frac{1}{2} - 2y$$

$$y = \frac{1}{4} - \frac{1}{2}x$$

$$f(x) = 3(5)x^4 - 5(3)x^2$$

$$= 15x^4 - 15x^2 = 15x^2(x^2 - 1)$$

$$f'(x) = 15(4)x^3 - 15(2)x$$

$$= 60x^3 - 30x = 30x(x^2 - 1)$$

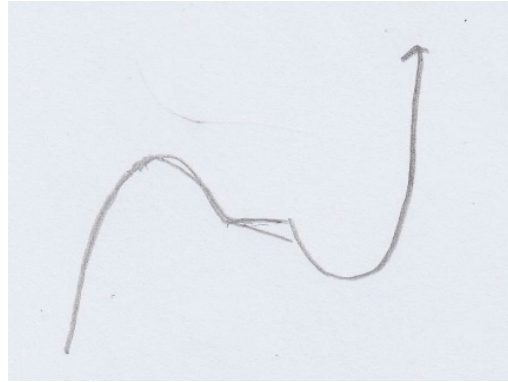
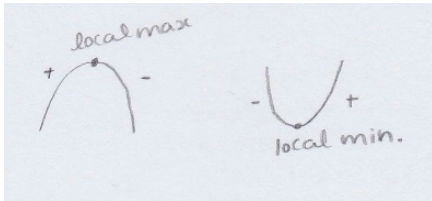
$f'(x)$ & $f''(x)$ is defined for all x .
 for $f'(x) = 0$, $x = 0$ or $x = \pm 1$
 for $f''(x) = 0$, $x = 0$ or $x = \pm 1$

Embodied-Conceptual (Embodied)

This world is defined as actions on visual or image representations of the mathematics. These action-image representations take the form of objects that are seen or sensed in the real world or imagined in the mind.

Examples:

- Shapes, tallies, diagram of problem, pointing up for increasing
- Curves not on graphical axes



Connecting between Embodied and Symbolic worlds:

Symbolizing-Embodiment

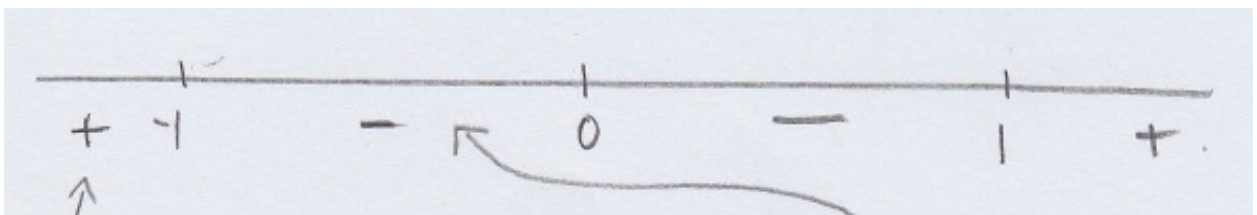
- Taking a curve and determining the function/formula/equation of the curve or solving the question from the graph

just looking at graph.

- representing circles or ellipses by algebraic equations

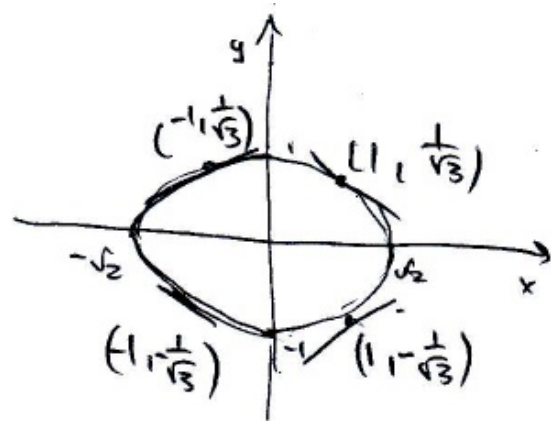
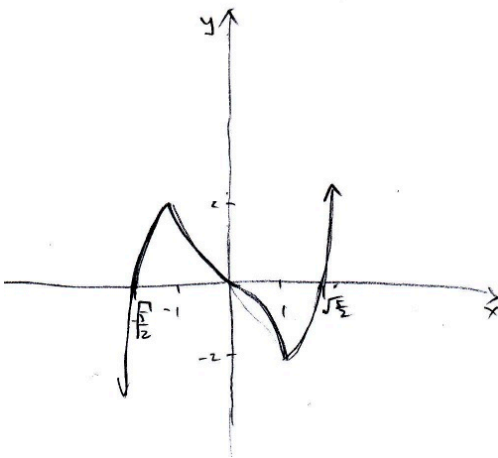
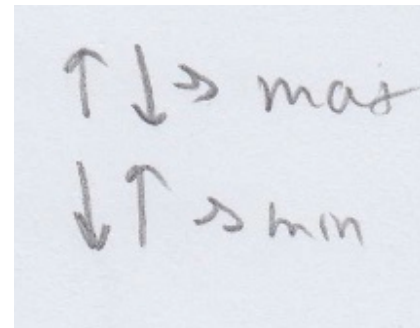
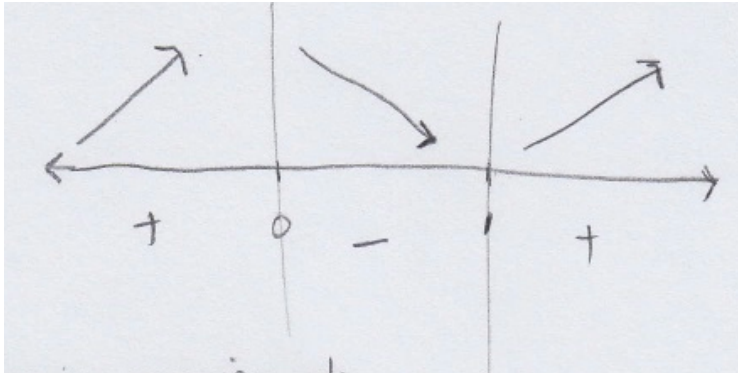
(knowing it's a sample figure, ellipse.)

- Number lines (represent the numbers in a structured order that reflects their quantified amounts)



Embodying-Symbolism

- Taking symbolic equations and functions and representing them as curves on a coordinate graph
- graphical representations that do not represent any equation or function in particular but arbitrary curves are more embodied than symbolic
- arrows pointing up to represent positive or increasing, arrows pointing down to represent negative or decreasing



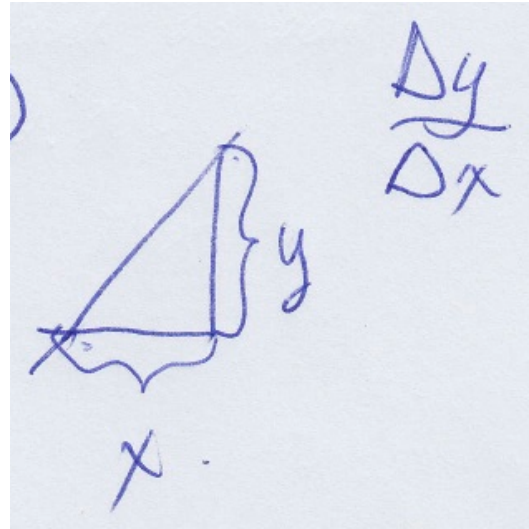
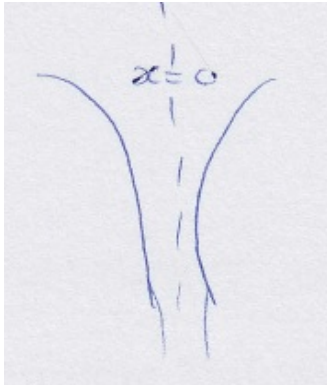
The Formal-Axiomatic (Formal) world and combination

Formal-Axiomatic (Formal)

This world uses the properties of the embodied and symbolic objects as well as logical deductions to create formal definitions and deduce axioms and theorems that construct axiomatic mathematical systems.

Embodying-Formalism

- Diagrams that represent a formal idea...discontinuity, left and right limit concepts, asymptotes



Symbolizing-Formalism

- Writing out a portion of the theorem of limits (left and write limit not equal, limit DNE, etc.)
- Explanation statements that refer to some formal property: limits, derivatives, etc.
- Setting $1/0$ to infinity

$$e) \lim_{x \rightarrow 1^-} g(x) \neq \lim_{x \rightarrow 1^+} g(x) \text{ D.N.E.}$$

Recalling that ~~limit~~ $\lim_{x \rightarrow a} f(x)$ exist if $\lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^+} f(x)$.

Appendix G: Roles of Representations Document

Roles of Representation Use Within a Question

Used as the primary part of the solution:

The designation of *primary part of the solution* will be given when a specific representation or a group of the same type of representations is used as the major method for the solution to the problem. For example, a solution that involves only algebraic manipulation of numbers and variables would be classified as using a Symbolic representation as the primary part of the solution. Another example is one where a student uses a graphical representation to find the answers as the primary part of the solution. This would be categorized as Embodied as the primary part. This was sometimes the case for question #2. A participant would draw a graph and then state that the solutions to the limits were found from looking at the graph. Although the answer statements are written with symbolic algebra, the primary part of the solution is the graphical representation.

Used as a unique part within a solution with multiple types of representations:

The designation of *unique part within a solution* will be given to a representation when it is part of a solution to a question that is made up of a number of distinct steps that use a variety of different types of representations. For example, a solution may begin with an equation or function from the question being drawn on a graph for in visual representation. The solution then may have some algebraic manipulation of numbers and variables, or may use a table, within its next steps. These representations would include Embodied and Symbolic representations. For this solution, the graph, algebraic manipulation and the representations in the table would all be considered a unique part within a single solution.

This designation may also be assigned if a solution uses one type of representation as the primary part of solution, while at the same time including another representation as a unique and separate part to the primary solution. This often takes the appearance of graphs or diagrams. In this case the solution would have both a primary part (above category) and a representations that is unique part within that solution.

Does not appear to affect solution:

The designation of *does not appear to affect solution* is assigned when a representation does not appear to affect the solution to a question, or sometimes even necessarily relate to the question. These types of representations often appear in a participant's written solution as "side work" or are identified during the interview. For example, some solutions have work written on the sides that use functions or equations different from those in the actual question. Often these are identified during the interview as the participant's way to remind them of the process. Another example is that some solutions to the first question have the equation of a line written out. Although the question does relate to the slope of an equation of a line, having the equation of a line may not affect the solution to the question in anyway.

Used as supplementary to check steps or solution:

The designation of *supplementary* is given to representations that are used to check

either unique steps within a solution, or the solution overall. It is the case that the representations that are used to check steps or solutions may be the same type of representation as what is used for the primary part of the solution. However, these representations still fall within a unique category, as they are not a necessary part needed to answer the question, but used for validity purposes. For example, a participant may have taken the derivative of the equation in the first question in two ways, implicitly and after isolating y , in order to make sure he/she gets the same answer. Supplementary representations are identified either from the participant's written work or during the interview.

Drawn during the interview:

The designation of *drawn during the interview* is assigned to representation that a participant wrote out or drew within the booklet while they were explaining their solutions during the interview. Representations drawn during the interview fall within this category because they were not initially part of the written work, but were only added because of the interview. These representations will be analyzed within the interview portion of the study. For example, when asked why a specific value was an asymptote, a student may have drawn a graph/diagram to demonstrate why.

Appendix H: Example of Written Work Summaries and Analysis

Example of Written Task Summary: Participant #5

Question 1

Description: The solution to this question includes five lines of algebraic work, one written statement, and an informal graphical representation. There are no tables, diagrams or other forms or representations.

Summary: The solution to this question begins with the statement “Implicit differentiation”. I have made the assumption that this statement is the participants’ way of describing what he was doing for this question.

The solution then has the five lines of algebraic work. The first line has the left hand side of the equation implicitly differentiate. However, the right hand side is not differentiated. The next line then solves the equation for dy/dx . The third line then sets dy/dx equal to 1. The last two lines are the equation rearranged and solved for 1 again. This is where the algebraic work stops. It should be mentioned that there is an error in that the right hand side of the equation is not differentiated.

The informal graphical representation consists of horizontal and vertical axes, which are not labeled. There is then an ellipse drawn on the axis with two straight lines, like tangents. This informal graphical representation is very similar to what the equation in the question would look like graphed with tangents that have a slope of 1.

Question 2

Description: This question consists of only algebraic or written statements. There are no graphs, diagrams, tables or other types of presentations. The algebraic answers do include the formal limit symbol for calculus.

Summary: Next to the limit question the participant has drawn arrows and has written left, limit, left, right, limit, respectively. I assume these words are there to identify whether the limit is a left or right limit, or neither.

There are then five limits written, namely the five limits that are asked in the question. Next to these are answers and statements. For the first limit the answer is written as 1 and the statement “1st statement of piecewise”. I assume this means that the answer was found from the first part of the piecewise function. The second limits has DNE for the answer and the statement “b/c $\lim_{x \rightarrow 1^-} g(x) \neq \lim_{x \rightarrow 1^+} g(x)$ ”. The third limits is set equal to -2 and the statement “ $(2-x^2)=(2-2^2)=-2$ ”. The forth limit is equal to -1 and has the statement “ $(x-3)=(2-3)=-1$ ”. The last limit has DNE written next to the limit and the statement “b/c $\lim_{x \rightarrow 2^+} g(x) \neq \lim_{x \rightarrow 2^-} g(x)$ ”.

Question 3

Description: The solution of this question includes algebraic work, a table, some diagrams, and an explanation statement. This question is done one page.

Summary: There is some algebraic work right written right next to the function in the question. It includes the function factored as well as the function with the values (-1) and (1) plugged in.

In the solution area of the question there is first written the derivative of the function as well as the derivative of the function factored. The factored derivative is then set equal to 0 and then x is set equal to 0, 1 and -1.

There is then a 5x5 table. The upper left hand cell is blank. The first row then has written $15x^2$, x^2-1 , y' , y , respectively for the next four columns. In the first column then have the intervals between negative infinity to -1, -1 to 0, 0 to 1, and 1 to positive infinity in the next four rows. The column under $15x^2$ has all + signs in it. The column under x^2-1 and y' both have +, -, -, + in the four rows. The last column has arrows, namely an upwards arrow, two downwards arrows, and an upwards arrow respectively for the four rows.

Under the table is the statement "double decreasing happens at a point b/c it is multiplied by x^2 ." I have made the assumption that this statement is referring to the derivative function and the fact that x^2 is factored out in front giving $x=0$ as one of the solutions. Under this statement is drawn a curve that first increases then decreases, plateaus, then decreases some more.

There are two answer statements for increasing and decreasing intervals where upwards and downwards arrows are used to identify increasing and decreasing respectively. There are also two answer statements for max and min points that also use upwards and downwards arrows.

Lastly, there is another curve diagram drawn, which is very similar to what the function in the question actually looks like.

Question 4

Description: The solution to this question includes a combination of algebraic manipulation, written statements, and an informal Cartesian graph. There are no other tables, diagrams, or other forms of representations.

Summary: The solution to this question first has the asymptotes identified as $y=x$, $x=0$, and $x=-2$. There is then an informal Cartesian graph with the asymptotes identified by dotted lines and curves drawn between the asymptotes. The axes of the graph are not labeled but are assumed to be x/y axes.

There are then two statements. The first statement is “behaves like $f(x)=x$ @ large x ” and the second statement is “behaves like $f(x) = -1/(x^2)(x + 2)$ @ x 's close to 0, 2.” I have assumed that these statements are talking about the function $f(x)$.

Next is written the three limits with answers and explanations statements next to the answers. The first limit is set next to negative infinity with the statement “by the graph, -2 is an asymptote when it approaches -2 ($-1/(x^2)(x + 2) \rightarrow \infty$).” Next to the second limit is the answer positive infinity and the statement “b/c same as above”. The last limit is set equal to -5 and has the statement “plug into calc.” next to it.

Back page

Description: On the back page there is a drawn a Cartesian graph (axes not labeled) and some algebraic work. This work was added during the interview portion and was not original written down after the completion of the task questions.

Summary: The graph has the line $y=x$ drawn in the first quadrant. The x and y sides of the axis are then labeled to make it look like a triangle. Next to graph are five lines of algebraic work. The first line is the area equation of a triangle. The next two lines is the area equation with the x and y values plugged in. The last two lines are the integral for x and the solution to the integral.

Example of Written Task Summary: Participant #7

Question 1

Description: The work on this question seems very scattered and is a little confusing about the directions the participant undertook. The solution includes algebraic work, some diagrams, and graphical representations. There are no explanation statements or tables.

Summary: It appears that at first the participant began by differentiating implicitly and then rearranging the solution for y' . It also appears that the equation is rearranged for y and this rearranged value for y is plugged in for y in the y' equation.

It next appears that the participant set $y' = 1$. Once y' is set equal to 1 it seems as though the participant solved for x . There is however a lot of scribbled out work so it isn't really clear how the participant went about arriving at the final answer for x .

There is then a portion of diagrams drawn near the bottom of the page. There are two ellipses, one vertical and one horizontal. There is also a right triangle with x and y identified as the vertical and horizontal sides. Lastly, there appears to be an informal Cartesian graph with a circle on it, a radius identified and the x and y parts labeled. There are also two tangent lines drawn on the circle however, these tangent lines both have negative slopes. So it is unclear how the participant used this information to solve the answer.

Question 2

Description: The solution to this question is made up mostly of explanation statements as to how the solutions to the limits were arrived at. There are five of these solutions labeled a)-e). There are also two diagrams drawn on the page. There are no other graphs or tables.

Summary: It appears that the participant began by labeling the parts of the piecewise function as 1-5 and labeling the limits as a)-e). The answer then has statements a)-e), which I have assumed are the explanations to how the answers to a)-e) are found.

Limit a) is set equal to 1. The solution statement beside a) is "This is when $x < 1$, Use eq. (1). $g(x) = x = 1$. Limit b) is set equal to 1 and the statement beside b) is "This is when $\lim_{x \rightarrow 1^-} = \lim_{x \rightarrow 1^+} = eq(1) = eq(2) = 1$." Limit c) is set equal to -2 and the statement beside c) is "use eq. (3) $g(x) = 2 - x^2 = 2 - 4 = -2$." Limit d) is set equal to -1 and the statement beside d) is "use eq. (4) $g(x) = x - 3 = 2 - 3 = -1$." Limit e) is set equal to DNE and the statement for e) is: " $\lim_{x \rightarrow 1^-} g(x) = \lim_{x \rightarrow 1^+} g(x) = DNE$."

One of the diagrams seems to be modeling discontinuity. The other diagram appears to be modeling the idea of left and right limits.

Question 3

Description: The solution to this question is also very scattered. There does appear to be some algebraic work, a table, and four explanation statements. There are no other graphs or diagrams.

Summary: It appears that the solution begins with the participant taking the derivative of the function and then factoring the derivative. There is then a statement that says " x^2-1 will be + iff $x < -1$ $x > 1$ ". It appears as though the participant was making a general statement on her observation. This statement may have helped her with answering the question. There is also the statement " $15x^2$ is always positive". Again this appears to be an observation statement but it may have helped the participant with the final solution.

There is then a 2×5 table. The first row is x , $x < -1$, $-1 < x < 0$, $0 < x < 1$, $1 < x$, respectively for the five cells. The second row is $f'(x)$, +, -, -, +, respectively for each cell.

There are four answer statements that answer the questions asked in the problem. These statements do incorporate a portion of symbolism in their answers. There is also an explanation statement in regards to the one of the x values being equal to 0. The statement is "the 0 doesn't make a diff. it is just a funky curve in the graph." While the statement is not really mathematical in nature, it does demonstrate that the participant was connecting the x values in the function to a graphical representation.

Question 4

Description: The solution for this question is scattered. It is difficult to interpret exactly what the participant did in what order. The solution does however have some algebraic work and two diagrams. There are a few short written statements, but they don't give a lot of explanation as to what the participant was doing.

Summary: On the page the participant expanded the denominator and wrote the answer underneath the function in the question. The three limits in the question are then labeled (A), (B) and (C). Beside limit (B) is an arrow and the inequality $x^3 < 2x^2$. Beside limit (C) is an arrow and the inequality $x^3 > 2x^2$.

There is then some written work. It begins by writing (A) and then $f'(x)$ and the derivative being taken. The derivative is then set equal to 0 and two lines of algebraic work follow. However, the participant stopped and wrote the statement "This didn't work". I assume that the participant was attempting to find the critical points of the function in the question and realized that it was not the optimum way to go about solving the question. There is then a #2 and the statement "try again". Again it appears that the participant began by trying to find the critical points of the derivative. However the participant only wrote down one line before stopping.

The solution then has a number 3 and the statement "Attempt #3" written next to it. There is then an (A), (B) and (C) with the limits and their solutions. Limit (A) is found to be negative infinity, limit (B) positive infinity and limit (C) negative infinity. Beside the

answer to limit (C) is the statement “similar to 4(A)”. I assume that this mean that the participant found limit (C) in a similar way to how she found limit (A).

One of the diagrams is written near the top of the page. It appears to be a Cartesian graph, however, the axes aren't labeled. There is then a vertical line drawn and labeled $x=2$. Two arrows are drawn, one from the left and one from the right approaching the line $x=2$. I assume that the participant was demonstrating limit (A) and limit (B) in a graphical form, maybe to help with finding the solution to the question. The second diagram is similar, yet smaller. This time there is no vertical line drawn, but the $x=0$ axis is labeled. There are also two arrows, one from the right and one from the left approaching $x=0$. Again, I assume that the participant was drawing a graphical representation of limit (C).

Back page

Description: During the interview the participant wrote some work on the back page to describe what she was talking about. It consists of five lines of algebraic work.

Summary: The first line is the fractions $0/0$ and infinity/infinity . The second line is a fraction with brackets on the top and bottom and a derivative symbol. This fraction is not set equal to anything. The third line is $f(x) \cdot g(x) = g(x)/1/f(x)$ which is set equal to $g'(x)/f'(x)/1$. The next line is $f(x)=\ln x$, then the line $(1/f(x))' = \text{infinity}$ and then set equal to $1/x \ln x^2$.

Example of Written Task Summary: Participant #9

Question 1

Description: The solution to this question includes four lines of work, which are all algebraic. There are no graphs, diagrams, tables, or other forms of representations. It does appear that the participant may have written other work on the page that was erased. However, this work is not legible and I am unable to see what might have been written.

Summary: The first line of work is the implicit differentiation of the equation. The left hand side of the equation is implicitly differentiated. The right hand side is not. The second line of work is the implicitly differentiated equation rearranged for y' . The third line of work is the equation rearranged, this time with y' set equal to 1. The last line of work is also the equation rearranged. This is where the solution stops. It is not complete and there is no answer given for the question.

Question 2

Description: The solution to this question includes algebraic work and a graphical representation. There are no other forms of representations included with the question.

Summary: This question asks for five limits. The participant has written the answer to each limit next to the limits in the question. Below this are a graphical representation and some algebraic work. The graphical representation is an informal Cartesian graph. The axes are not labeled and the scale is only labeled on the positive horizontal and negative vertical. The curve on the graph appears to be the four parts of the piecewise function from the question. The algebraic work around the graph appears to be the participants attempt to identify specific points in order to facilitate drawing the graph. There does not appear to be any algebraic work that identifies how the participant found the solutions to the limits.

Question 3

Description: The solution to this question includes mostly algebraic work, as well as one graphical representation and what appears to be the beginning of a tabular/number line representation. This is also an explanation statement within the written work.

Summary: The first four lines of algebraic work consist of the participant re-writing the function from the question, taking the derivative of the function, setting the derivative equal to 0 and factoring the derivative to find the values of x when the derivative is 0. There is then the explanation statement "tangent is zero when $x=0$ and at $x=1, -1$ ". This statement is interesting as it refers to the tangent being zero, which is not mentioned in the question at all.

Next to this work the original function is also set equal to zero and solved for x . It is

found the $x=0$ and x equal the positive and negative square root of $5/3$. It is not clear why the participant found these values, as they are not used anywhere else in the solution for the problem.

The participant also took the second derivative of the function and set it equal to zero. She then plugged in the x values found from the first derivative being set equal to 0. She found that $f''(0) = 0$, $f''(1) > 0$, $f''(-1) < 0$. She also labeled $f''(1)$ as unstable and $f''(-1)$ as stable. There is no other information as to how this contributed to answer the questions in the problem.

Below this work is what appears to be the start of a table/number line. There is a line drawn and the values -1, 0, and 1 placed on the line. Underneath these values, intervals are sectioned off with vertical lines. However, there is no other work on this table.

There is then a graphical representation. It appears to be an informal Cartesian graph. The axes are not labeled and only the point -1 and 1 are labeled on the horizontal axis. There is then a curve drawn on the graph and two minimum points labeled on the curve.

Lastly, there is a statement. However, this statement is not a sentence and it is hard to conclude what the participant's intentions were by the statement. It may be the case that the participant was trying to articulate that $x=0$, -1, 1 are minimum values.

Question 4

Description: There is not a lot of work written down for this question. Next to each limit are answers. There is then one explanation statement and two lines of algebraic work. There are no diagrams, graphs, tables or other forms of representations.

Summary: Next to each limit function is written the word "undefined". It is the case that this is not necessarily incorrect, if an answer is wanted in terms of real numbers. There is one explanation statement, which says "All asymptotes". I assume that the participant identified the points that the limits were looking at as asymptotes for the function in the question. It seems safe to assume that the participant concludes that asymptotes have undefined answers. The algebraic work consists of the denominator of the function being set equal to zero and the x values found to be 0 and -2. I assume this is the work that the participant did in order to conclude the statement "All asymptotes".

Appendix I: Table 2

Table 2.

Documents Developed from the Initial Steps of the Data Analysis

Document	Description	Purpose
Interview Transcripts (Appendix D)	Interview typed verbatim with Italic notes for hand gestures used during the interview and information added to the written work during the interview. Also include image documents of the participant's written work when referred to in the interview.	Initial step to organizing information from interview into workable documents.
Interview Transcript Summaries (Appendix E)	Summary of main ideas of from interview into condensed paragraph form. Also include image documents of the participant's written work when referred to in the interview.	Consolidate the main ideas that from the interview transcripts into easily accessible documents.
Written Work Summaries and Analysis (Appendix H)	A description and summary of the solution for each question for each participant. The descriptions include the different types of representations that appeared to have been used in the question, the length of the solution to a question, and generally what the solution looked like. The summaries include more detail about the steps a participant followed when solving a question.	To have the information in easy to use documents and to ensure a review of each question for each participant in a consistent manner.

Table 2 (*continued*)

Documents Developed from the Initial Steps of the Data Analysis

Document	Description	Purpose
Categorizations of Representations (Appendix F)	List of each of Tall's worlds and categorizes between worlds with definitions for each world and category. Examples of images of representations from written work added to the different categories.	To keep track of decisions made about specific representations so that could be refer to later as well as to ensure consistency in categorizing the representations.
Roles of Representations (Appendix G)	A list of the different roles the representations played within the written work solutions as well as a detailed described of what each of the roles looks like so they could be identified from the written work.	To identify the different ways representations were being used in the written work, which will help address the research question of <i>how</i> students make connections.
Representation Table (Figure 16)	A table with the representations that were used in the question answers in the first column as well as <i>how</i> the representations were used in the first row. The rest of the table is blank and allows for comments to be placed in the boxes.	To provide a visual representation that addresses both what representations are used and how they are used in the same space. This is to help address the research question. On representation table was created for each participants' response to each task resulting in 60 tables.

Table 2 (continued)

Documents Developed from the Initial Steps of the Data Analysis

Document	Description	Purpose
Layered Analysis (Appendix J)	A combination of the analysis from the written work layered with an analysis from the transcript summaries as well as a filled in connection/representation table.	To have a complete analysis of each question from each participant in an easily accessible document with the goal to help address the research question.

Appendix J: Examples of Layered Analysis

Participant #3: Question #1 Layered Analysis

The written work with steps outlined in this question is all symbolic-proceptual. However, the statement “using x value to find y coordinate” could suggest that the participant may have been thinking of the ordered pair in a graphical sense, or embodying-symbolism. There is also more evidence of embodying-symbolism from the participant’s interview statements. For example, he describes why he took certain steps or followed a certain procedure because of what he was “visualizing in his mind”. For example, he was visualizing that he wanted to find the point on a function where the tangent that touches has a slope of 1. This example could demonstrate that he was thinking about the question in embodied-symbolic manner and connecting between the two worlds while he was solving.

Representations	Used as the primary part of the solution	Used as a unique part within a solution	Does not appear to affect solution	Used as supplementary to check steps or solution	Drawn during the interview
Embodied					
Embodying-Symbolism		-Statement of finding y coordinate -interview statement of “visualizing”			
Symbolizing-Embodiment					
Symbolic	Step-by-step procedures of algebra symbols				

Participant #7 Question #2 Layered Analysis

For the written work, the answers to the limits and their accompanying explanation statements appear to be mostly symbolic. There are two symbolic statements that have some formal aspects (namely the statement for limit b) and limit e)). In the case of these two statements, the participant mentions a formal proposition of limits. This could be an indication of some connections being made to the formal world. The two diagrams, which were drawn during the interview for explanation purposes, seem to be embodying formalism as they are modeling concepts of functions and limits. During the interview the participant also made the statement that she tries not to visualize while solving problems as it causes a larger margin for error. This statement may suggest that the participant was trying to solve this question with mainly a symbolic method.

Representations	Used as the primary part of the solution	Used as a unique part within a solution	Does not appear to affect solution	Used as supplementary to check steps or solution	Drawn during the interview
Embodied					
Embodying-Symbolism					
Symbolizing-Embodiment					
Symbolic		-Limit answers			
Symbolizing-Formalism		-limit statements			
Embodying-Formalism					-limit concept diagrams

Participant #11 Question #3 Layered Analysis

From the written work and the interview this question appears to be solved only with a graphical representation, thus embodying-symbolism. The answers to this question were found from looking at the graphical representation. In the interview the participant did state that she would have been able to solve this question with a derivative, but that she felt it was unnecessary since she had a graphing calculator. Overall, it appears that this question was primarily solved by embodying symbolism.

Representations	Used as the primary part of the solution	Used as a unique part within a solution	Does not appear to affect solution	Used as supplementary to check steps or solution	Drawn during the interview
Embodied					
Embodying-Symbolism		-graphical			
Symbolizing-Embodiment					
Symbolic					

Participant #14 Question #4 Layered Analysis

The written solution to this question as well as the comments made during the interview suggests that the question was solved with symbolic representations. For example, the participant put the function under a single denominator. As well, she spoke about wanting to remember a formula to help her solve the question. Overall, there does not appear to be any connections made between worlds.

Representations	Used as the primary part of the solution	Used as a unique part within a solution	Does not appear to affect solution	Used as supplementary to check steps or solution	Drawn during the interview
Embodied					
Embodying-Symbolism					
Symbolizing-Embodiment					
Symbolic	-symbolic representations				

Appendix K: Examples of Overall Summary of Analysis by Participant

Participant #1: All of the written work for each question is primarily symbolic with step-by-step procedures using mathematical symbols. There were some comments during the interview and a few representations on some question that could suggest an embodied or connected understanding. However, the participant's primary solving technique or problem solving process remained symbolic.

From the interview it was also learned that the participant's considered graphing questions his best, because he just followed the steps. No comments about being able to picture anything. However, he did comment that limit questions were his worse because he couldn't picture them. Did elaborate on what that meant.

Participant #2: All of the written work for each question, except question #3, is primarily symbolic with step-by-step procedures using mathematical symbols. Question #3 exhibits a lot of examples of connecting between the embodied and symbolic worlds in both the written work and the interview. The participant did state that she wanted to graph the function in question #1 but couldn't. She wanted to graph the question to visualize the problem, suggesting that she understood the connections between the graph and the equation. There were also some embodied references in the interview but nothing that suggested that it had an affect on the problem solving process.

From the interview it was also learned that the participant felt that the teaching style she experienced in calculus was step-by-step with algorithms. She commented that the teacher only used graphs for a supplementary purpose. Interestingly, this participant had never used a graphing calculator before in her life as they were not used at her privately run high school.

Participant #3: Question #1 has algebraic written work but an embodied description from the interview, suggesting connections between the worlds in terms of the participant's understanding. Question #2 is done primarily embodied in both the written work and interview explanation. Question #3 and Question #4 are both primarily symbolic in written work. The interview suggests some connections to the embodied world (Question #3) and the Formal world (Question #4) in understanding. However, the connections in understanding that did appear during the interview were not necessarily part of the problem solving.

From the interview it was also learned that the participant found his professors teaching technique as mostly algorithmic. He stated that diagrams and graphs were used in the course as mostly supplementary to the teaching of the algebra and "math".

Participant #4: All of the written work for each question demonstrated strong connecting between the symbolic-embodied worlds. Question #4 does have some symbolic than embodied written work as compared to the other questions, but during the interview it was apparent that there were still many connections being made, even to the formal world.

From the interview it was also learned that the participant found that the teacher used a lot of visual representations, such as graph and diagrams, when teaching. She also stated that the teacher used graphs and diagrams as both primary and secondary in problem solving.

Participant #5: For question #1 and question #2 the written work is all symbolic. During the interview some comments were made suggesting that the participant was connecting to the embodied world while solving question #1. For question #2 some statements were made during the interview that connect to the formal world. However, in both these cases, these connects were only made during the interview and were not a major part of the problem solving process. Question #3 and question #4 were done with mostly symbolic-embodied connections that are apparent from both the written work and the interview comments.

From the interview it was also learned that the participant found that the teacher used whatever method was better or easier for solving a question, whether it was primarily symbolic or primarily embodied. The DGD instructor also used a lot of diagrams and graphs for explanations, which the participant liked.

Participant #6: For question #1 the written work was mostly symbolic but the explanation during the interview demonstrated that there were a lot of connections being made between the symbolic and the embodied. For question #2 it was the opposite. While the written work was primarily embodied, although with some symbolic representations, the process described during the interview was primary symbolic. For question #3 and question #4 the written work is primarily symbolic. The interview did suggest some connections to the embodied (question #3) and the formal (question #4) but these connections to the other worlds did not necessarily affect the problem solving process.

From the interview it was also learned that the professors teaching style included “drawing” while solving the problems. The participant stated he found this helpful. He described these drawings as mostly visual aids for students.

Participant #7: Most of the written work on all of the questions is symbolic. However, statements made during the interview do suggest that the participant’s understanding has some connections to the embodied worlds and in some cases the formal world. Despite this, the problem solving is mostly algorithmic using symbolic representations.

From the interview it was also learned that graphs and diagrams were used in the participants class for explanation purposes while teaching and not for problem solving purposes. The participant also personally stated that she thinks people make more mistakes when they try to visual the problems.

Participant #8: For question #1 and question #4 the majority of written work and interview comments are symbolic. For question #2 and question #3 the written work is a combination of both embodied and symbolic representations. For question #2 the

interview also demonstrates that the problem solving method was primarily embodied. For question #3 the interview comments suggest that the embodied written work was done to follow a process that the participant was taught in class, although he did successfully know how to follow the process and use the embodied representations.

From the interview it was also learned that the teacher only used graphs or diagrams for graphing question and application problems in the course. The participant stated that he struggled with graphing questions because of the steps that he needed to memorize. He however, was very good at the application question because he could visualize and picture the problems.

Participant #9: All of the questions, except question #2, were primarily solved with symbolic representation in the written work, as well as in the interview comments. In some cases there were comments made during the interview that could be considered embodied, but they did not affect the problem solving or understanding. For question #2 the participant used a combination of embodied and symbolic representations to answer the question. There was no information provided as to why the participant made connections only on this question.

From the interview it was also learned that the teacher used a mostly step-by-step teaching technique using symbolic representations. The participant did say that she liked using visual representations but that she didn't see many in her calculus course.

Participant #10: The participant's first question has written work that is all symbolic however, the interview comments make it clear that he was connecting to the embodied world in his problem solving and understanding while completing the question. For Question #2 and Question #3 the written work is a combination of symbolic and embodied representations. The interview confirms that the participant was connecting between the two worlds when solving these questions. For Question #4 the written work is symbolic with one graphical representation of an asymptote that has some formal connections. The interview suggests that the participants understanding of this question does have some embodied understanding.

From the interview it was also learned that the participant felt that he should visualize questions as much as possible as it helps him solve. He stated that the teacher's teaching style was step-by-step with algebra. During his course graphs and diagrams only showed up sometimes in theory or on harder examples.