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Illuminating Triangles, Quadrilaterals and Convex Polygons with Vertex Floodlights

by

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Abstract

This work presents the solution to three problems in Computational Geometry. First we introduce a theorem to illuminate every triangle with three $\frac{\pi}{6}$ vertex-floodlights and we provide three proofs. Secondly we show that three $\frac{\pi}{4}$ vertex-floodlights suffice to illuminate every quadrilateral. Finally we provide a theorem for illuminating a cocircular convex n -gon P of n vertices with a set of k vertex-floodlights, ($k \leq n$) whose total sum is π .

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Contents

| | | |
|----------|---|-----------|
| 1 | Introduction | 1 |
| 1.1 | Art Gallery History..... | 1 |
| 1.2 | Basic Terminology | 3 |
| 1.3 | Chvátal's Classical Art Gallery theorem..... | 10 |
| 1.4 | Orthogonal Art Galleries..... | 15 |
| 1.5 | Importance of Art Galleries..... | 16 |
| 1.6 | Variations on the Art Gallery Problem..... | 17 |
| 1.7 | Contribution..... | 18 |
| 2 | Illumination Problems | 20 |
| 2.1 | Introduction | 20 |
| 2.2 | Illuminating Families of Convex Sets..... | 26 |
| 2.3 | Illuminating Convex Sets using Line Segments | 35 |
| 2.4 | Illuminating Line Segments on the Plane | 36 |
| 2.5 | Protecting Line Segments..... | 42 |
| 2.6 | Guarding Rectangular Art Gallery | 44 |
| 3 | Floodlight Illumination Problems | 46 |
| 3.1 | Illuminating the Plane | 49 |
| 3.2 | Illumination of Stages..... | 51 |
| 3.2.1 | Optimal Illumination of Stages..... | 53 |
| 3.3 | Floodlight Illumination of Orthogonal Polygons..... | 59 |
| 3.4 | π -Floodlight Illumination of Polygons..... | 62 |

| | | |
|----------|--|------------|
| 4 | Illuminating Triangles by Vertex Lights | 68 |
| | 4.1 Introduction | 68 |
| | 4.2 Definitions and General Lemmas..... | 69 |
| | 4.3 Result | 78 |
| | 4.4 Conclusion | 84 |
| 5 | Illuminating Quadrilaterals by Vertex Lights | 85 |
| | 5.1 Introduction | 85 |
| | 5.2 Result | 85 |
| | 5.3 Conclusion | 105 |
| 6 | Illuminating Convex Polygons by Vertex Lights | 106 |
| | 6.1 Introduction | 106 |
| | 6.2 Result | 109 |
| | Conclusion | 117 |

List of Figures

| | |
|---|----|
| 1.1: A simple and an orthogonal polygon..... | 4 |
| 1.2: Two graphs with six vertices. | 4 |
| 1.3: A polygon and one of its triangulation | 5 |
| 1.4: A triangulation of a polygon together with its dual..... | 6 |
| 1.5: Point a can see b and c , but not d | 7 |
| 1.6: Graph G_1 has a matching with two edges, but no perfect matching. G_2 has a perfect matching..... | 8 |
| 1.7: A graph with no perfect matching..... | 9 |
| 1.8: Illustration of proof of Chvátal's Art Gallery Theorem..... | 11 |
| 1.9: P_1 has six vertices..... | 13 |
| 1.10: An orthogonal polygon that requires $\lfloor \frac{n}{3} \rfloor$ guards. | 16 |
| 2.1: Three lights suffice to illuminate a compact convex set..... | 21 |
| 2.2: Two-dimensional planar room..... | 22 |
| 2.3: Path $ABCD$ in a polygon and its unfolded one. | 23 |
| 2.4: Path $ABCDEF$ in a polygon and its unfolded one..... | 23 |
| 2.5: Isoceles right triangle ABC | 24 |

| | |
|--|----|
| 2.6: Lattice of mirror images of the triangular table..... | 25 |
| 2.7: Mirror images of the right angled isocoles triangle..... | 25 |
| 2.8: Arrangements requiring $4n-7$ lights..... | 28 |
| 2.9: A family of n disjoint circles requiring $2n$ light sources..... | 29 |
| 2.10: A family of n homothetic triangles that requires n point lights..... | 30 |
| 2.11: A polygon that requires $\lfloor \frac{(n+h)}{3} \rfloor$ vertex guards..... | 31 |
| 2.12: An example of family of $n = 3m - 3$ quadrilaterals that requires $4m - 4$ point lights to illuminate the free space they generate, $m= 6$ | 34 |
| 2.13: (a) A family of line segments, their extensions and their dual graph. (b) An example where n vertex guard are needed to illuminate the plane..... | 37 |
| 2.14: Illuminating line segments..... | 39 |
| 2.15: Placement of lights..... | 40 |
| 2.16: A set of orthogonal lines that requires $\lfloor \frac{n}{3} \rfloor$ lights..... | 41 |
| 2.17: A family F of eight line segments, and its corresponding graph $G(F)$ | 42 |
| 2.18: Generating a set of n line segments that require $\lfloor \frac{2n-3}{3} \rfloor$ points to protect them..... | 43 |
| 2.19: An example of an art gallery with eight rooms..... | 44 |
| 3.1: Illustration of proof of the Three-Floodlight Problem. | 47 |
| 3.2: Illustration of proof for the case where $u=w$ | 48 |
| 3.3: Four $\pi/2$ floodlights suffice to illuminate the plane..... | 50 |
| 3.4: The Stage Illumination Problem..... | 52 |
| 3.5: Illuminating the real line from two points..... | 54 |
| 3.6: Only points in the convex hull are useful..... | 55 |
| 3.7: Optimal illumination of the real line..... | 57 |

| | |
|--|----|
| 3.8: Top-left illumination rule | 60 |
| 3.9: Orthogonal polygons that need $\lfloor \frac{3n-4}{8} \rfloor$ orthogonal floodlights..... | 61 |
| 3.10: A polygon that cannot be illuminated by placing an orthogonal floodlight at each vertex. | 63 |
| 3.11: Vertex $(3\pi/4 - \varepsilon)$ lights do not suffice..... | 66 |
| 3.12: A family of polygons with $5n+1$ vertices that require $3n$ π -vertex floodlights..... | 66 |
| 4.1: l_1, l_2, l_3 the bisectors of triangle T | 68 |
| 4.2: Ceva Triangle..... | 70 |
| 4.3: Proof of Ceva Theorem..... | 70 |
| 4.4: Arithmetic-Geometric Mean proof..... | 74 |
| 4.5: Brocard points and Brocard angle..... | 74 |
| 4.6: Two circles intersecting at Ω | 75 |
| 4.7: Brocard Triangle..... | 76 |
| 4.8: A, Ω, C, D are cocircular..... | 76 |
| 4.9: Proof of Brocard Theorem. | 77 |
| 4.10: Isotomic Conjugate Q of a point P | 78 |
| 4.11: Maximum value of Brocard angle..... | 79 |
| 4.12: Illumination of triangle with three $\frac{\pi}{6}$ -vertex lights. | 80 |
| 4.13: AD, BF and CE are concurrent at point U | 80 |
| 4.14: Illuminating the triangle three $\frac{\pi}{6}$ -vertex lights. | 82 |
| 4.15: AA', BB' and CC' concur in a point U | 83 |
| 5.1: $\angle B_1BB_2 = \frac{\pi}{4}$ and $\angle B_2NR \leq \frac{\pi}{4}$ | 86 |
| 5.2: Proof of Lemma 5.1. | 87 |

| | |
|--|-----|
| 5.3: $D \in NB_1, \angle RDB_2 \leq \angle RNB_2$ | 87 |
| 5.4: Proof of Lemma 5.2 | 88 |
| 5.5: Two $\frac{\pi}{4}$ - vertex lights placed at A and B illuminates P | 89 |
| 5.6: AB is not an edge of P | 90 |
| 5.7: $AC \leq BC$ and $AD \leq BD$ | 91 |
| 5.8: (a) N is a vertex of P , (b) N is not a vertex of P | 92 |
| 5.9: Two $\frac{\pi}{4}$ -vertex lights placed on A and B suffice to illuminate P | 93 |
| 5.10: Arcs $2x$, $2y$ and $2z$ | 94 |
| 5.11: Two $\frac{\pi}{4}$ -vertex lights placed at A and B | 95 |
| 5.12: Vertex light of size less than $\frac{\pi}{6}$ placed at C illuminates the circular triangle $UA'B$ | 96 |
| 5.13: Coordinates of points A , A' , B , B' and R | 97 |
| 5.14: Three $\frac{\pi}{4}$ - vertex lights placed at A , B and C illumintes P | 98 |
| 5.15: (a) $AD \leq BD$ (b) $AD \geq BD$ | 99 |
| 5.16: $AD \leq BD, \angle QOB = \alpha < \angle ROC = \beta$ | 100 |
| 5.17: $AD \leq BD, \angle ROB = \alpha > \angle ROC = \beta$ | 100 |
| 5.18: Subcase 2b.2 where $AD \geq BD$ | 101 |
| 5.19: A , B , C and D all lie on the circle..... | 102 |
| 5.20: $AC \leq BC; AD \leq BD$ and $CR < DS$ | 103 |
| 5.21: $AC \leq BC; AD \leq BD$ and $CR < DS$ | 104 |
| 5.22: $AC \leq BC; AD \leq BD$ and $CR > DS$ | 105 |

| | |
|---|-----|
| 6.1: A polygon P with n vertices that cannot be covered by n equal $\alpha = \pi / n$ lights..... | 108 |
| 6.2 Partitioning of arc qp into arcs c_1, c_2, \dots, c_m | 110 |
| 6.3: Arc qp illuminated from vertices v_1, v_2, \dots, v_m | 110 |
| 6.4: Proof of Lemma 6.1. | 111 |
| 6.5: Illuminating wedge w_3 from vertex lights f_1, f_2, \dots, f_m located on the boundary of wedge w_1 in the clockwise direction. | 112 |
| 6.6: Illuminating arc p_2q_1 using f_i vertex lights located at vertex v_i on arc p_1q_2 | 113 |
| 6.7: Vertex floodlights located on the arc v_1v_j illuminate exactly wedge w_2 | 115 |
| 6.8: f_j vertex floodlight placed at v_j illuminates a wedge containing v_1 and satisfying Lemma 6. 2..... | 116 |

Chapter 1

Introduction

1.1 Art Gallery History

How many guards are necessary and how many are sufficient to patrol the paintings of any art gallery with n walls? This simple question of combinatorial geometry has since [V. Klee(1973)] been extended by mathematicians in several directions, and has stimulated many papers and surveys, most of them written in the last decade. The mathematical beginnings are found in the well-known theorem by V. Chvátal [15] according to which $\lfloor \frac{n}{3} \rfloor$ stationary guards are occasionally necessary and always sufficient. A guard or simply a light is a stationary point which can survey 360° about its fixed designated position. In this formulation, the gallery, having straight walls, is a polygon of n vertices and, for every point of the polygon and its interior, there is a light (guard) which illuminates (guards) it. Since Chvátal's result, numerous variations on the art gallery

problem have been studied, including floodlight illumination problems, mobile guards, guards with limited visibility, illumination of families of convex sets on the plane, guarding of orthogonal polygons and others.

In 1987, J. O'Rourke [59] published *Art Gallery Theorems and Algorithms*, the first book dedicated to the study of illumination problems of polygons in the plane. Two thorough survey papers have since been published on Art Gallery problems, one in 1992 by T. Shermer [68], and a second one in 1996 by J. Urrutia [78]. Since then a large number of papers in this area have appeared, and some important open problems have been solved. In this thesis we survey most of the results published to date in Art Gallery or illumination theorems and we focus our research on illumination problems using floodlights.

A *floodlight* is a light source that illuminates the area within a cone. Floodlight illumination problems were introduced in [8]. One of the motivations for the study of floodlight illumination problems is that many illumination or guarding devices cannot search or illuminate all around themselves in all directions. Floodlights illuminate only a circular cone. As it turns out, many communication devices used in the broadcasting of satellite and telecommunication signals can be idealized as floodlights. For example, satellites transmit signals using dishes that restrict the area of coverage to a cone.

1.2 Basic Terminology

A *polygon* P is generally defined as an ordered sequence of at least three points v_1, v_2, \dots, v_n in the plane, called vertices of P , and the n line segments $\overline{v_1v_2}, \overline{v_2v_3}, \dots, \overline{v_{n-1}v_n}$ and $\overline{v_nv_1}$, called its edges. A *simple* polygon is a polygon with the constraint that nonconsecutive edges do not intersect. A simple polygon is a Jordan curve, and thus divides the plane into three subsets: the polygon itself, the (bounded) interior and the (unbounded) exterior. However, we will henceforth use the term "polygon" to refer to the polygon together with its interior. Polygons are thus closed, bounded sets in the plane.

A polygon P is said to be covered by a collection of subsets of P if the union of these subsets is exactly P . The collection of subsets is called a cover of P . A cover of P is called a decomposition if the intersection of each pair of subsets in the cover has zero area.

A simple polygon is called *orthogonal* if all its edges alternate between horizontal (zero slope) and vertical (infinite slope). Orthogonal polygons have also been called isothetic and rectilinear. Orthogonal polygons are an important subclass of polygons which arise in many computing applications and in the design of many machines (such as image scanners and plotting devices) that are used in these applications. Restricting the polygons considered in the art gallery problem to orthogonal polygons creates an interesting subclass of problems, and has led to many results.

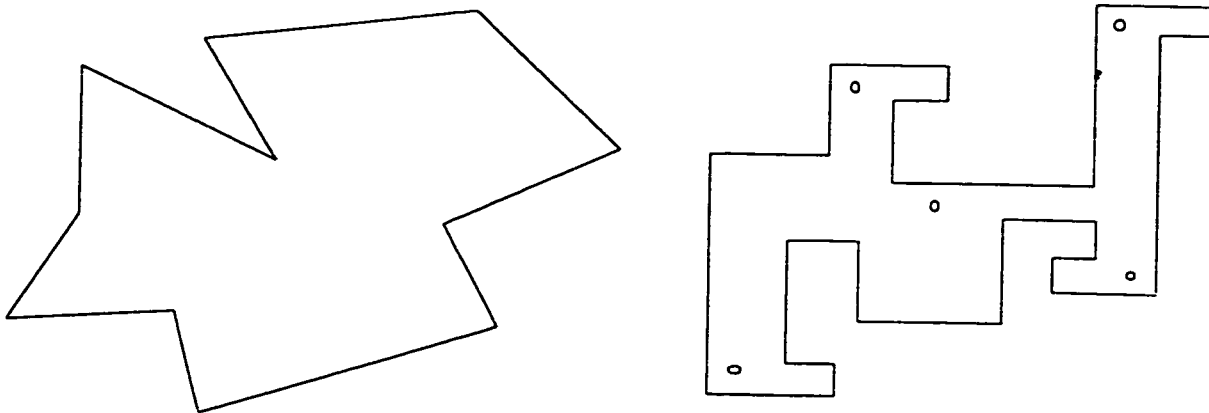


Figure 1.1: A simple and an orthogonal polygon.

A *graph* G consists of a set of elements V called the vertices of G , together with a set E of pairs of vertices of G called the *edges* of G . Two vertices u and v are called *adjacent* if the pair $\{u, v\}$ is an element of E . The *degree* of a vertex v of G is the number of vertices of G adjacent to v . A graph G is *planar* if it can be drawn on the plane in such a way that its vertices are represented by points on the plane, and each edge $\{u, v\}$ of G is represented by a simple curve joining points representing u and v ; edges of G may only intersect at their endpoints. In Figure 1.2 we show two graphs; the first one is planar, and the second is not.

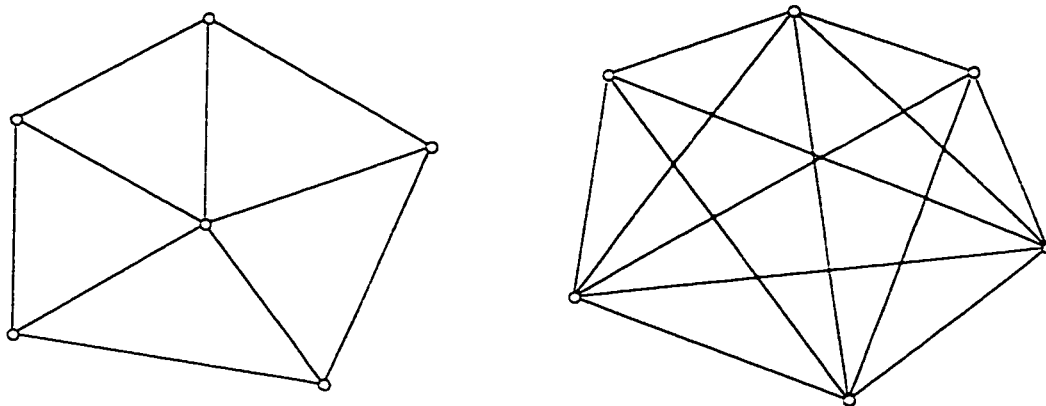


Figure 1.2: Two graphs with six vertices.

A *triangulation* T of a polygon P is a partitioning of the polygon into a set of triangles with disjoint interiors. This is done by cutting the polygon along *diagonals* joining pairs of nonadjacent vertices. A polygon and one of its triangulations are shown in Figure 1.3. It is easy to see that a triangulation of a polygon P with n vertices contains exactly $n-2$ triangles.

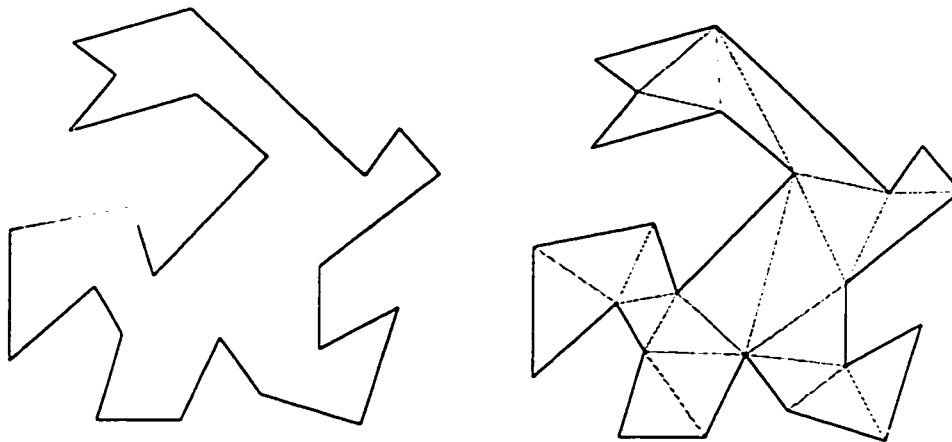


Figure 1.3: A polygon and one of its triangulations.

Many algorithms in computational geometry incorporate a polygon triangulation step. In 1978, Garey, Johnson, Preparata and Tarjan [37] obtained the first $O(n \ln n)$ time triangulation algorithm. This result was improved in 1988 by Trajan and van Wyk [74] to $O(n \ln \ln n)$. Finally in 1990 Chazelle [12] presented an algorithm that computes a triangulation of a polygon in $O(n)$ time.

Given a triangulation T of a polygon P , we can define a graph $GT(P)$ such that the vertices of $GT(P)$ are the vertices of P , and two vertices of $GT(P)$ are adjacent if they are connected by an edge of T . The dual of a

triangulation T of a polygon P is obtained by placing a vertex in the interior of each triangle of T , and connecting two vertices if their corresponding triangles share a common edge in T . It is easy to see that the dual graph of a triangulation of a polygon P is always a tree. In Figure 1.4 we show a polygon together with a triangulation T and its dual.

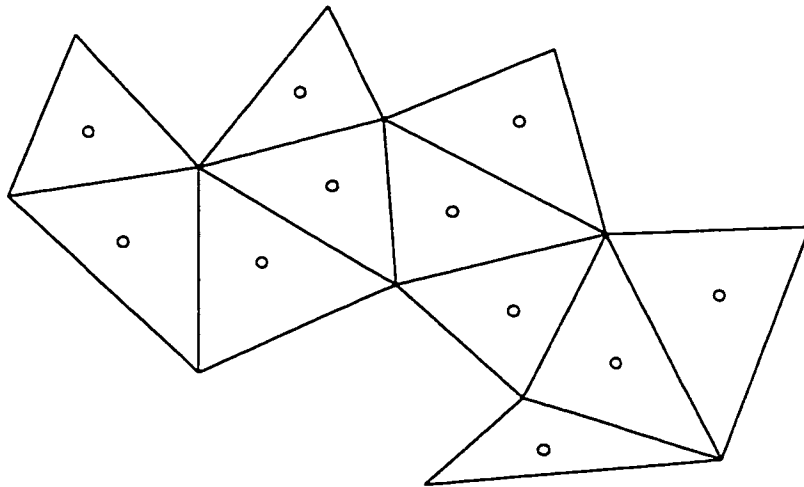


Figure 1.4: A triangulation of a polygon together with its dual.

A *coloring* of the vertices of a graph G is an assignment of colors to its vertices in such a way that adjacent vertices receive different colors. The chromatic number of a graph G is the smallest integer k for which a coloring of G exists. In this case, we say that G has chromatic number k . It can be easily proved that given a polygon P and T a triangulation of P , $GT(P)$ has chromatic number 3.

Let p and q be two points in a polygon P ; we will say that p and q are visible if the line segment \overline{pq} does not intersect the exterior of P . In Figure 1.5, the point a is visible from b and c , but not d . The set of all

points of P visible from p is a polygon, called the visibility polygon of p and is denoted by $V(p, P)$.

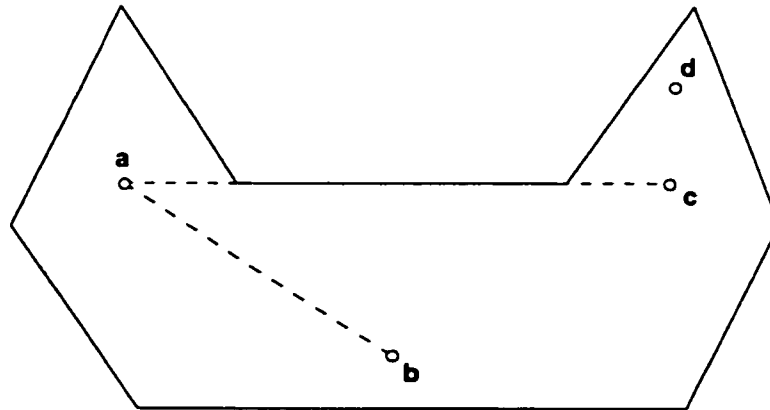


Figure 1.5: Point a can see b and c , but not d .

The visibility graph $VG(P)$ of P is the graph whose vertex set is the set of vertices of P , two vertices u and v being adjacent in $VG(P)$ if they are visible in P . Avis and ElGindy were the first to introduce the concept of visibility graphs [3].

Avis and ElGindy gave an optimal (worst case) $O(n^2)$ time algorithm to compute the visibility graph of a polygon. Their result was later improved by Hershberger [42], who gave an $O(|E|)$ time algorithm, where $|E|$ is the number of edges of the visibility graph of P . One important open problem in computational geometry is that of characterizing visibility graphs i.e. finding a set of graph-theoretic properties that exactly defines visibility graphs of polygons.

A collection H of points of P *illuminates* or *guards* P if every point u of P is visible from a point p in H . The term *illuminates* follows the

Theorem 1.2.1 (Tutte [76]): *A graph G has a perfect matching iff for every subset S of $V(G)$, $Odd(G - S) \leq |S|$.*

Necessity: We first observe that if G has a subset of vertices S such that $Odd(G - S) \geq |S| + 1$, then G cannot have a perfect matching. To see this, we simply observe that in any perfect matching of G , at least one element of an odd component of $G - S$ must be adjacent to a vertex in S , otherwise this component would have an even number of vertices, thus if we have more odd components in $G - S$ than elements in S , a perfect matching cannot exist; see Figure 1.7. For the sufficiency conditions, we refer the reader to the original paper [76].

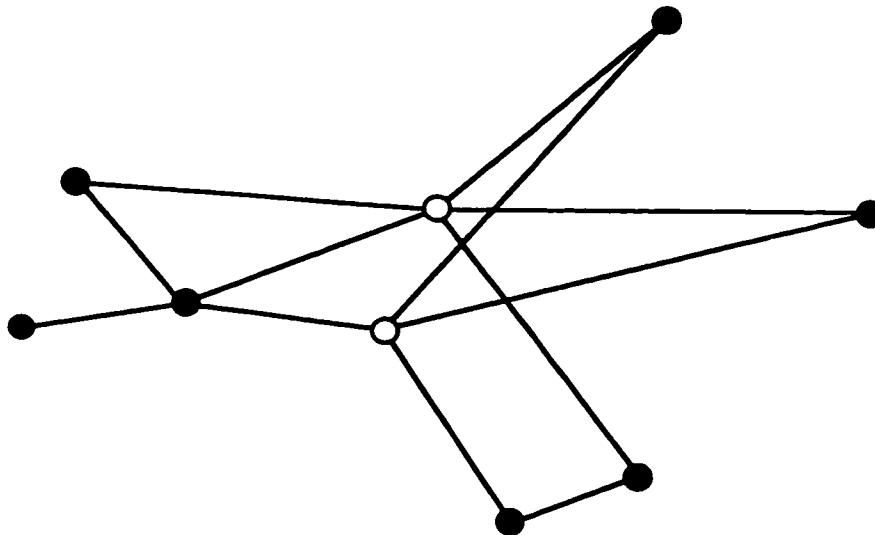


Figure 1.7: A graph with no perfect matching. The vertices of S are represented by small empty disks.

For the case of planar graphs, we will use the following result by T. Nishizeki:

Theorem 1.2.2 (Nishizeki [58]) *Any planar 2-connected graph G with $n \geq 14$ vertices and minimum vertex degree greater than or equal to 3 has a maximum matching with at least $\lfloor \frac{n+4}{3} \rfloor$ edges. When $n \leq 14$, the number of edges in such a matching is $\lfloor \frac{n}{2} \rfloor$.*

The study of Art Gallery problems deals mainly with illumination of polygons, families of convex sets and most recently, floodlight illumination problems; that is illumination problems using light sources with restricted angle of illumination.

1.3 Chvátal's classical Art Gallery theorem

We now show the proof of Chvátal's Art Gallery due to Fisk [36]. Fisk's proof came three years after Chvátal's original proof. We outline Fisk's proof:

Theorem 1.3.1 $\lfloor \frac{n}{3} \rfloor$ *stationary guards are always sufficient and occasionally necessary to illuminate a polygonal art gallery with n vertices.*

Proof. Consider an arbitrary simple polygon P with n vertices. Obtain a triangulation T of P by adding $n-2$ interior diagonals. (See Figure 1.8). Next three-color the vertices of the triangulation graph: assign each vertex one of three different colors, so that no two vertices which are adjacent in the graph have the same color. Each triangle of the graph, which corresponds to a triangle of the triangulation, will have one vertex of each color. Furthermore, every point of a triangle is visible to each vertex of that

triangle. Therefore, choosing any of the three color classes will result in a set of vertices from which every point of every triangle, and thus every point of the polygon, is visible. (i.e. each coloring class is a covering vertex guard set). By the pigeonhole principle, the smallest of these color classes will contain at most $\lfloor \frac{n}{3} \rfloor$ vertices. To see that $\lfloor \frac{n}{3} \rfloor$ guards are sometimes needed, consider the comb polygon $Comb_m$ with $n=3m$ vertices presented in Figure 1.8. It is easy to see that to guard P_m we need at least m guards. ■

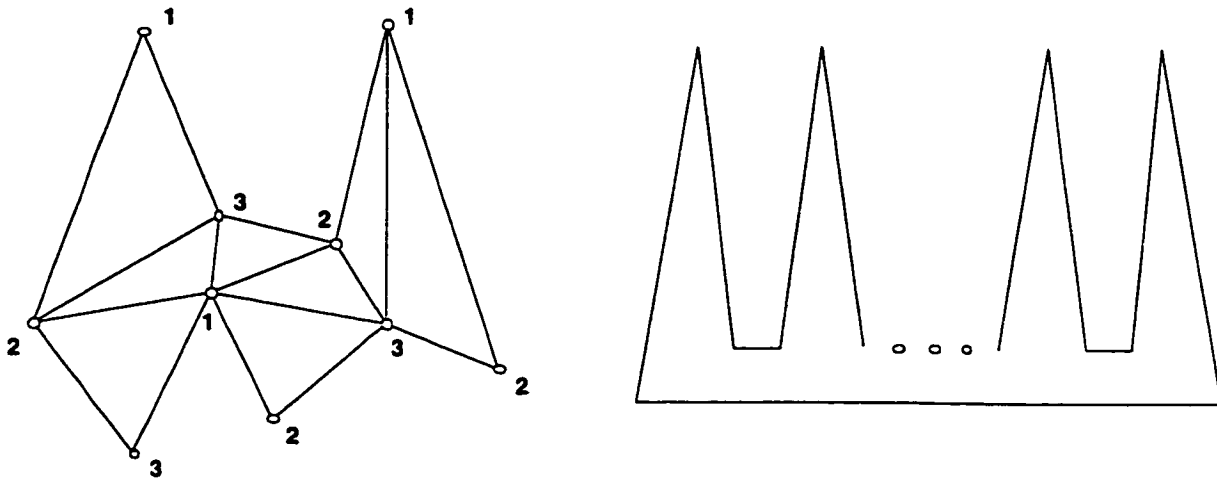


Figure 1.8: Illustration of proof of Chvátal's Art Gallery Theorem.

The first proof of Chvátal's Art Gallery Theorem was of course given by Chvátal, in 1975. His proof starts with a triangulation of the polygon, as does Fisk's, but does not use graph coloring. Rather the theorem is proven directly by induction. Although Chvátal's proof [15] is not as concise as Fisk's, it reveals aspects of the problem that are not brought to light by the coloring argument. Chvátal proves that there is always a diagonal that cuts

off 4 or 5 or 6 edges and then proceeds by induction on the number of vertices of P . Chvátal's proof was slightly more complicated than Fisk's proof. We now outline Chvátal's proof; the main idea behind it is the following result:

Lemma 1.3.1 *Any triangulation of a polygon with n vertices has a diagonal d that splits P into two polygons P_1 and P_2 , such that P_1 has 5, 6, or 7 vertices $n \geq 6$.*

Chvátal took as his induction hypothesis the following statement:

The triangles of any triangulation of an n -gon can be partitioned into at most $\lfloor \frac{n}{3} \rfloor$ subsets such that the triangles of each subset have a vertex in common.

Using an inductive argument and a case analysis, we can finish Chvátal's proof as follows:

Our statement is true if P has 3, 4, or 5 vertices; in any triangulation of P all its triangles have a common vertex. Suppose that P has at least six vertices, and let T be any triangulation of P .

Case 1: if P_1 has five vertices, then P_2 has $n-3$ vertices. By induction all the triangles of T in P_2 can be covered with $\lfloor \frac{n-3}{3} \rfloor$ vertices. Since P_1 has 5 vertices, one of its vertices belongs to all the triangles of T contained in P_1 .

This produces a total of $\lfloor \frac{n}{3} \rfloor$ vertices that cover all the triangles of T .

Case 2: if P_1 has six vertices, then P_2 has $n-4$ vertices. Consider the triangle T of P_1 supported by the diagonal d . See Figure 1.9. Now the quadrilateral (g,h,i,j) can be triangulated in 2 ways: if the diagonal (g,i) is

present then the vertex g is shared by all the triangles in P_1 . Then P_1 can be guarded with a single guard and P_2 with $\lfloor \frac{n-3}{3} \rfloor$ guards and we are done. ^s

If the diagonal (h,j) is present then let P_3 be the polygon formed by the union of P_2 and T . By the induction hypothesis, all the triangles of T' in P_3 can be covered with $\lfloor \frac{n}{3} \rfloor - 1$ vertices, one of which is a vertex of T .

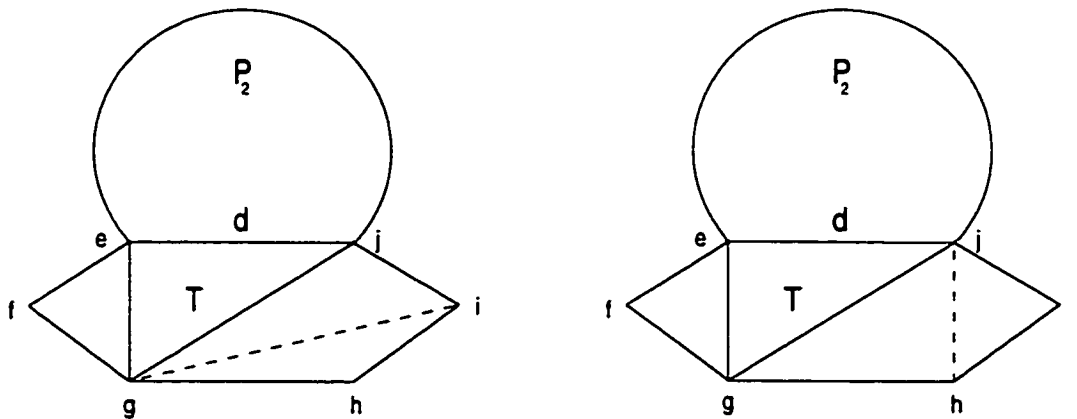


Figure 1.9: P_1 has six vertices.

If that vertex is g or e , then it covers the triangle with vertices efg . Selecting either one of h or j will produce a set with $\lfloor \frac{n}{3} \rfloor$ vertices that cover all the elements of T' .

The cases when P_1 has 7 vertices can be solved in a similar way.

Fisk's proof of Chvátal's theorem to find $\lfloor \frac{n}{3} \rfloor$ guards to cover a polygon leads in a natural way to an efficient algorithm. First triangulate the polygon P to be guarded, then 3-color the resulting graph and station a guard at every vertex of the smallest chromatic class. This was first noticed by Avis and Toussaint [5] who gave an $O(n \ln n)$ time algorithm to solve the Art Gallery Theorem. Their result was later improved when Tarjan and van

Wyk [74] obtained an $O(n \ln \ln n)$ time algorithm to triangulate polygons. The complexity of the Art Gallery Theorem was finally settled when Chazelle obtained a linear time triangulation algorithm [12].

Although Theorem 1.3.1 provides a general upper bound on the number of guards required to guard any polygon with n vertices, we can observe that most polygons can be guarded with fewer than $\lfloor \frac{n}{3} \rfloor$ guards. It is natural then to ask for the existence of an efficient algorithm to find the minimum number of vertex guards needed to guard a polygon. Unfortunately, the existence of such an algorithm is highly unlikely, due to the following result of Lee and Lin [51]:

Theorem 1.3.2 *The minimum vertex guard problem for polygons is NP-hard.*

Their proof is based on a reduction of the 3-satisfiability problem [51]. In the same paper, they show that the minimum edge and point guard problems are also NP-hard. This result gives rise to the study of algorithms that somehow take advantage of the structure of a polygon to obtain a better bound on the number of vertices required to guard them. For example, convex polygons can be guarded with a single vertex.

One approach to dealing with difficulty of covering polygons is to devise algorithms to find solutions that approximate the optimal ones. Ghosh [39] presented an $O(n^5 \ln n)$ time algorithm that finds a set of vertex guards that has at most $O(\ln n)$ times the minimum number of vertex guards needed to guard P .

For orthogonal polygons, Schuchardt and Hecker [67] recently proved:

Theorem 1.3.3 *The minimum vertex and point guard problems for orthogonal polygons are NP-hard.*

1.4 Orthogonal Art Galleries

Of particular interest is the study of guarding problems for orthogonal polygons; i.e. polygons whose edges are all parallel to the x - or y -axis. Perhaps one of the motivations for the study of these polygons is that most real life buildings are, after all orthogonal. From a mathematical point of view, their inherent structure allows us to obtain very interesting results. The first major result in this direction is due to Kahn, Klawe, and Kleitman [48]. They investigated the art gallery problem restricted to orthogonal polygons. They exhibit the orthogonal comb polygons of Figure 1.10, establishing that the maximum number of guards necessary to guard any orthogonal polygon of n vertices, $orth(n) \geq \lfloor \frac{n}{4} \rfloor$. They prove a matching upper bound, $orth(n) \leq \lfloor \frac{n}{4} \rfloor$, in the same manner as Fisk proved the original art gallery theorem, but they decompose the polygon into convex quadrilaterals rather than triangles, and then four-color the quadrilateralization graph, so that each quadrilateral has one vertex of each of the four colors. The bulk of their paper is devoted to proving that every orthogonal polygon has a decomposition into convex quadrilaterals. Edelsbrunner, O'Rourke, and Welzl gave an $O(n)$ point guard placement algorithm for orthogonal polygons, based on L-shaped partitioning [26].

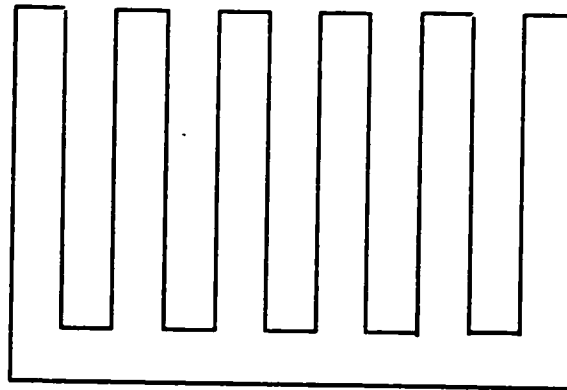


Figure 1.10: An orthogonal polygon that requires $\lfloor \frac{n}{4} \rfloor$ guards.

A different algorithm was later obtained by Lubiw [55]. Sack and Toussaint have presented also other linear placement algorithms based on quadrilateralization [66]. More recently, Estivill-Castro and Urrutia [30] proved that $\lfloor \frac{n}{4} \rfloor$ guards are also sufficient to guard an orthogonal polygon, but in this case using orthogonal floodlights; that is, guards which have an angle of vision of 90° degrees. Moreover, unlike the result of Edelsbrunner, O'Rourke, and Welzl [26], they locate the lights on the edges of the polygon, and give a linear time algorithm to solve this problem.

1.5 Importance of Art Galleries

Art gallery problems are studied by computer scientists because they are fundamental visibility problems, and visibility is a central issue in many computing applications. Application areas for visibility include robotics [49], [80], motion planning [52], [57], vision [70], [81], graphics [56],

[13], CAD/CAM [10], [28], computer-aided architecture [63], and pattern recognition [3], [74]. Other reasons that art gallery problems are studied are that they are a continuous form of classical facility-location problems, have a simple formulation, and require an interesting interplay of graph theory, geometry, and computer science in their solution.

1.6 Variations on the Art Gallery Problem

Let G be a finite set of points in a polygon P . We call G a guarding set if every point in P is visible from at least one point in G . The individual elements of a guarding set G are called guards. If all the points in a guarding set G are vertices of P , then G is called a vertex guard set, and the elements of G are called *vertex guards*. Otherwise G is called a point guard set, and its elements *point guards*. Other variations of the art gallery problem arise when specified subsets of the polygon, rather than just points, are allowed as elements of guarding sets. Typical types of subsets used as guards are edges of polygons. This branch of variations on the art gallery problem was started by Toussaint in 1981, when he asked how the art gallery theorem would change if guards were allowed to patrol individual edges of a polygon rather than having them standing at the same point all the time. He wanted to know the minimum number of *edge guards* necessary to cover any polygon of n vertices. Another type of subsets used as guards are *mobile guards* which are a slightly more general version of edge guards; each mobile guard can patrol either an edge or a diagonal of the polygon. Mobile guards were introduced by O'Rourke [60]. We also distinguish between vertex floodlights, located at vertices of a polygon, and

point floodlights, which can be located anywhere in the polygon to be guarded. Floodlights were introduced by J. Urrutia in 1990. The motivation for this type of guard is that many guarding and broadcasting devices have a limited range of visibility.

1.7 Contribution

This thesis is organized as follows:

1. In Chapters 2 and 3, we provide a comprehensive survey of most illumination problems, from the illumination of families of convex sets on the plane to the illumination of convex polygons using vertex-floodlights.
2. In Chapter 4, we prove that any triangle can be illuminated with three $\frac{\pi}{6}$ vertex-floodlights and we provide three proofs. It is known that three vertex-floodlights which sum to 180° degrees are required to illuminate any convex polygon. We pose the following question: Find a minimal angle α , such that any three vertex-floodlights whose apertures sum up to α , will illuminate any triangle. We conjecture that three vertex-floodlights whose total angle is at least equal to $2\frac{\pi}{3}$ suffice to illuminate any triangle.
3. In Chapter 5, we provide a theorem for illuminating a quadrilateral using three $\frac{\pi}{4}$ vertex-floodlights. This is an improvement over a result

by O'Rourke *et al.* In [62] they showed that four $\frac{\pi}{4}$ vertex-floodlights suffice to illuminate any quadrilateral. We also conjecture that any n -gon P may be illuminated by three vertex-floodlights, all of them on the edges of P , each one of them having angle $\frac{(n-2)\pi}{n}$ (half the average angle of P).

4. In Chapter 6, we provide a theorem for illuminating a cocircular convex n -gon P of n vertices with a set of k vertex-floodlights, ($k \leq n$) whose total sum is π . By establishing this theorem, we partially solve a question raised by O'Rourke in [62].

Some of our results in Chapter 4 and 5 were published in the proceedings of the 10th Canadian Conference on Computational Geometry.

Chapter 2

Illumination Problems

2.1 Introduction

In this chapter, we present most of the material known to us on illumination problems and leave floodlight illumination problems to be studied separately in the following chapters.

Illumination problems have always been a popular topic of study in mathematics. For example, it is well known that the boundary of any smooth compact convex set S on the plane can always be illuminated using three light sources located in the complement of S . An easy proof of this can be obtained by enclosing a smooth convex set within a triangle, then placing a light at each vertex of T ; see Figure 2.1.

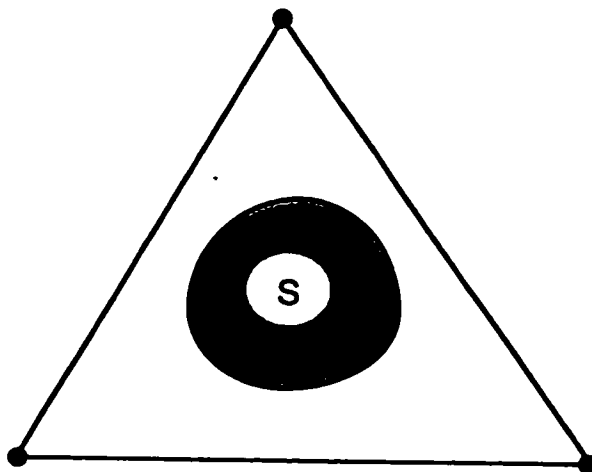


Figure 2.1: Three lights suffice to illuminate a compact convex set.

One famous—and until recently open—problem on illumination is attributed to Ernest Strauss (see E. G. Strauss and V. Klee [50]), who in the early fifties posed the following problem:

Suppose that we live in a two-dimensional room whose walls form a simple closed polygon P and each is a mirror.

1. *Is it true that if we place a light at any point of P , all of P will be illuminated using reflected rays as well as direct rays?*
2. *Is there necessarily a point from which a single light source will illuminate the entire room using reflected rays as well as direct rays?*

The first part of Strauss's problem can equivalently be posed in terms of a billiard ball bouncing around a pool table. Is there a pool shot between any two points on a polygonal pool table?

A *light ray* or *pool ball* reflects off the sides of the room in such a way that the angle of incidence equals the angle of reflection. A *light ray* or *pool ball* that strikes a vertex is considered to end or be absorbed there. All paths or pool shots will be taken to be of non-zero length.

The first part of Strauss's problem was recently proved to be false by G. Tokarsky [75]. He gave, in addition to the proof, elementary techniques for constructing rooms, both in the plane and in three-space, which are not illuminable from every point. Tokarsky's proof is surprisingly simple, using basic concepts of geometry that are easily understandable.

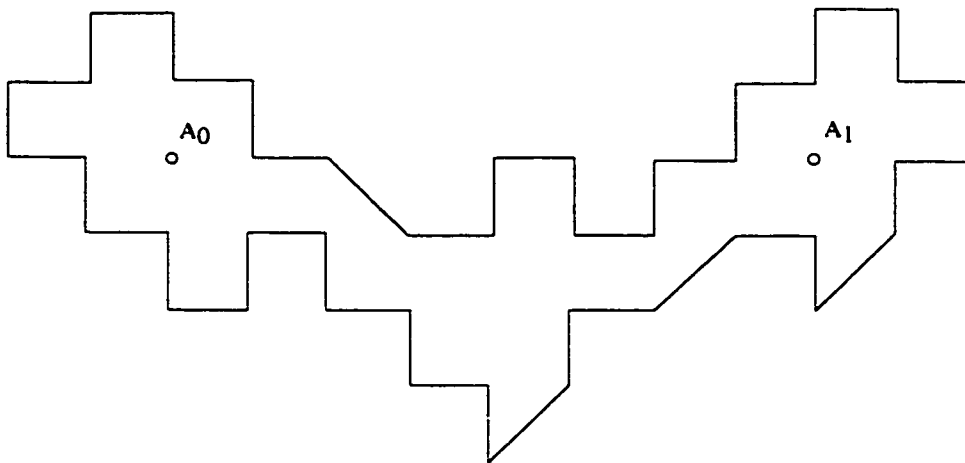


Figure 2.2: Two-dimensional planar room.

His main idea for solving this problem is that any path in a polygon unfolds to a path in another polygon constructed from mirror images of the first. Conversely, the second path can be considered to fold up to the first.

Example 1: Path $ABCD$ in Figure 2.3(a) corresponds to the straight line path $ABCD$ in 2.3(b).

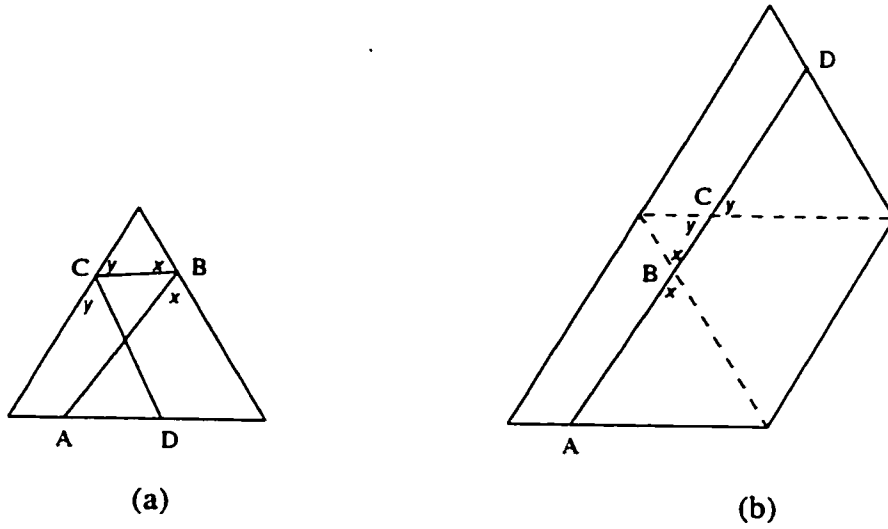


Figure 2.3: Path $ABCD$ in a polygon and its unfolded one.

Example 2: Path $ABCDEF$ in Figure 2.4(a) corresponds to path $ABCDEF$ in Figure 2.4(b).

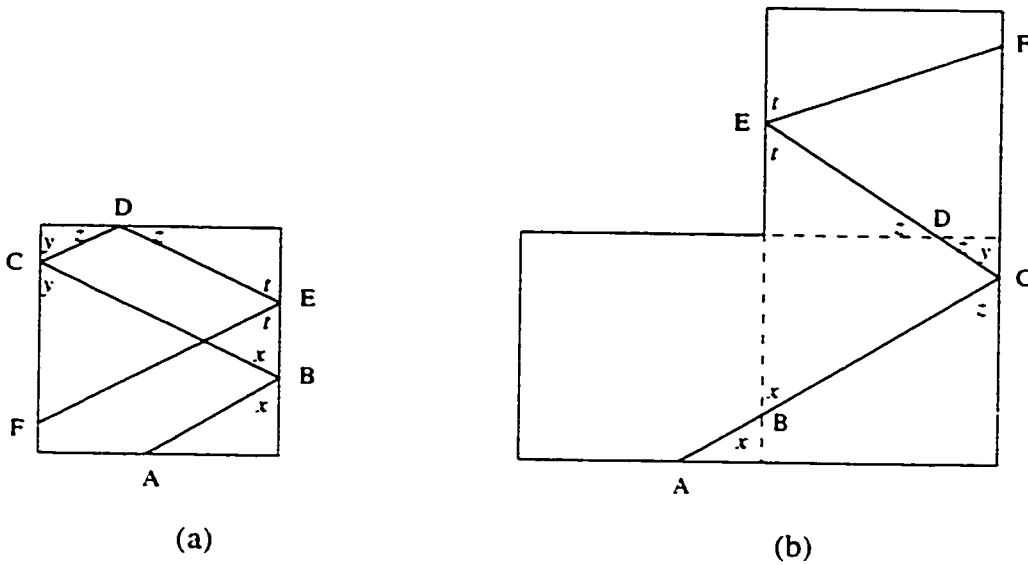


Figure 2.4: Path $ABCDEF$ in a polygon and its unfolded one.

His solution was based on the following lemma:

Lemma 2.1 *In any isosceles right triangle ABC (with right angle at C), there is no pool shot from A that returns back to A .*

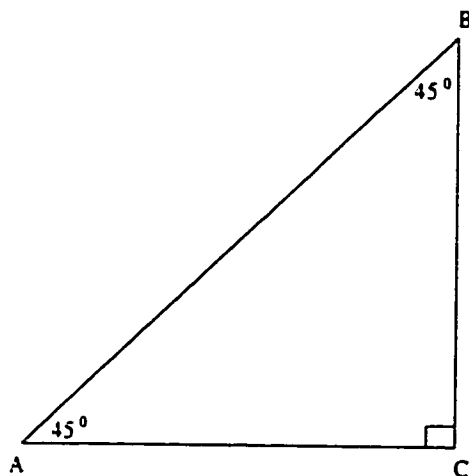


Figure 2.5: Isosceles right triangle ABC .

Proof. Take a lattice of mirror images of this triangular table and assign integer coordinates to the vertices as shown in Figure 2.6, with A the origin. Vertices labeled A have even coordinates $(2m, 2n)$ and vertices labeled B or C all have at least one odd coordinate. A pool shot from A to A on the original table would unfold or correspond to a straight line segment joining $A(0,0)$ to say $A(2m, 2n)$ in the lattice. This segment then must pass through the point (m,n) [or $(m/2, n/2)$ if both m and n are even, etc.] and thus must pass through a point labeled B or C . This means the pool shot would hit a vertex and be absorbed before returning to A .

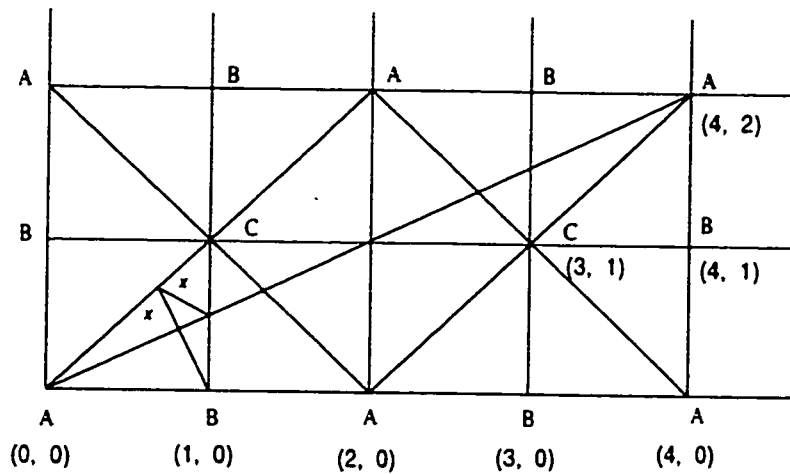


Figure 2.6: Lattice of mirror images of the triangular table.

Theorem 2.1 *There is no pool shot from A_0 to A_1 on the table shown in Figure 2.2*

Proof. This table is constructed by taking mirror images of a right angled isoceses triangle as shown in Figure 2.5. The key to the diagram and the proof is that any point labeled B or C must be a vertex of this table, while points labeled A do not have to be.

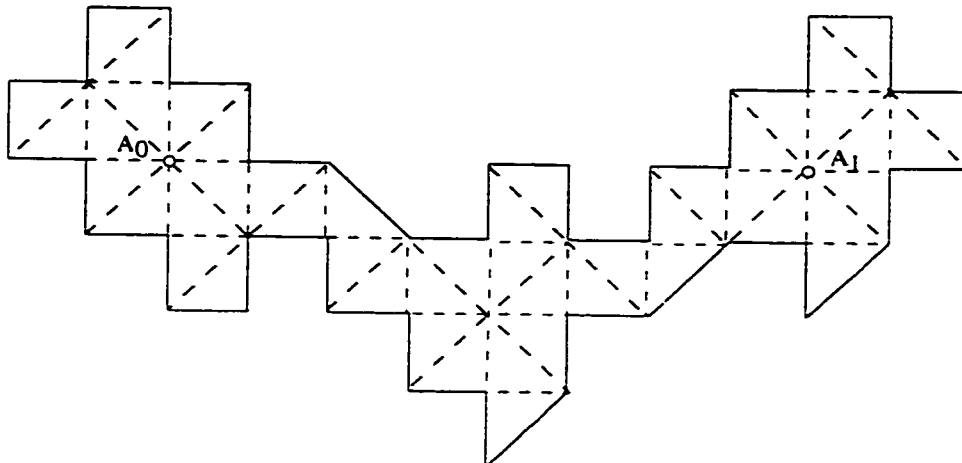


Figure 2.7: Mirror images of the right angled isoceses triangle.

If there were a pool shot from A_0 to A_1 , the initial path must pass through the interior of one of the eight triangles surrounding A_0 . Let us call this triangle T . As in the lemma, a pool shot from A_0 to A_1 would correspond or fold up to a pool shot from A_0 to A_0 in triangle T , which is impossible and the result follows. ■

Incidentally, it should be clear from the proof that there does not exist a pool shot between any two points labeled A on this table. The second part of Strauss's conjecture remains open.

2.2 Illuminating Families of Convex Sets

The following problem was first studied by Fejes Tóth [34]. Let $F = \{S_1, \dots, S_n\}$ be a family of n disjoint compact convex sets on the plane. How many light sources located in the complement of $S_1 \cup \dots \cup S_n$ are always sufficient to completely illuminate the boundaries of the elements of F ? Fejes Tóth proved in [34]:

Theorem 2.2.1 *For any family $F = \{S_1, \dots, S_n\}$ of n disjoint compact convex sets, $4n-7$ lights located in the complement of $S_1 \cup \dots \cup S_n$ are always sufficient and occasionally necessary to illuminate the boundaries of the elements of F .*

The proof is as follows:

Construct a family $T = \{T_1, \dots, T_n\}$ of n strictly convex sets such that:

1. S_i is contained in T_i , $i = 1, \dots, n$.
2. The interiors of T_i , $i = 1, \dots, n$ are disjoint.
3. The number of tangencies between the elements of T is maximized.

Suppose that T_i is tangent to $T_{s(j)}$, $j = 1, \dots, m$. Consider the lines $l_{i,s(j)}$ tangent to T_i at the points in which T_i intersects $T_{s(j)}$, $j = 1, \dots, m$. For each i , $T_{s(1)}, \dots, T_{s(m)}$ define a polygonal region P_i . Two cases arise:

1. P_i is a bounded polygonal region. In this case, place a light at each vertex of P_i .
2. P_i is an unbounded polygonal region. For this case, place a light at each vertex of P_i , and one more at each of the two semilines of P_i far enough from S_i .

It is easy to see that these lights illuminate S_i , $i = 1, \dots, n$ and that each of the lines defined above is assigned exactly two lights.

Construct a planar graph G with vertex set $\{T_1, \dots, T_n\}$ such that T_i and T_j are adjacent if they are tangent. It is clear that this graph is planar, and thus it has at most $3n-6$ edges. Notice that there is a one-to-one correspondence between the edges of G and the set of tangents generated by $T = \{T_1, \dots, T_n\}$. It now follows that the number of lights used so far is at most $2(3n-6)$. Let H be the complement of the union of P_i , $i = 1, \dots, n$. H consists of an unbounded face and a number of bounded ones. It is easy to see that the number of lights needed to illuminate S_1, \dots, S_n is maximized when all the faces of H , including the outer one, are bounded by exactly three elements of T . We notice that each face of H contains exactly 3 lights. Note however, that if for each face of H except the outer face, we remove any one of these three lights, the remaining lights still illuminate all the

elements of F . It is now easy to see that $4n - 7$ lights remain, and the result follows. ■

Examples of families of convex sets requiring $4n - 7$ lights are shown in Figure 2.8; each "triangular" region in the exterior of each arrangement is shaped so as to require two lights. Urrutia and Zacks rediscovered this result [77] and obtained bounds for this problem in higher dimensions. They proved:

Theorem 2.2.2 For any family F of n disjoint convex sets in E^d , $O(n[n + \lfloor \frac{d}{2} \rfloor]^{\lfloor \frac{d}{2} \rfloor})$ lights are always sufficient and sometimes necessary to illuminate the boundaries of the elements of F , $d \geq 3$.

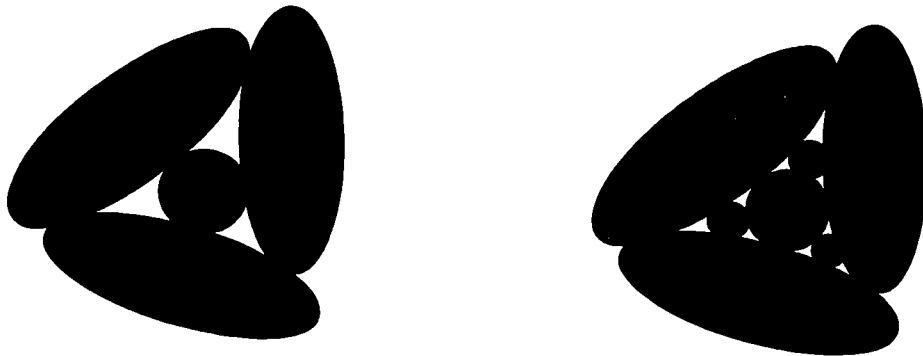


Figure 2.8: Arrangements requiring $4n - 7$ lights.

Fejes Tóth also considered illuminating families of disjoint circles. As any pair of circles is noncrossing, the $4n - 7$ upper bound on light sources holds. However, no families of disjoint circles require this many lights. In fact Fejes Tóth was able to show that no families of disjoint circles require more than $2n$ light sources, and that the only families of disjoint circles requiring this many are those in which the centers of the circles lie on a line, with consecutive circles just touching, as in Figure 2.9.

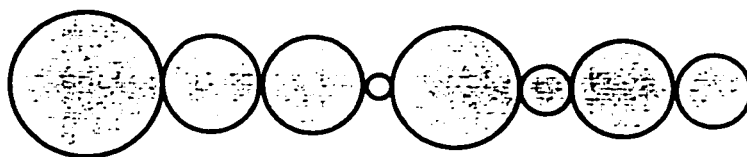


Figure 2.9 : A family of disjoint circles requiring $2n$ light sources.

For any other families of more than two disjoint circles, Fejes Tóth proved in [34] that:

Theorem 2.2.3 *$2n-2$ light sources are always sufficient and occasionally necessary to illuminate any family of n disjoint closed circles.*

This result was rediscovered (for pairwise disjoint circles) independently by Czyzowicz *et al.* [20] and Coullard *et al.* [16]. Although these results, and the ones for convex sets, are for guarding the boundary of the arrangement, the same proofs hold for guarding the entire exterior.

Czyzowicz, Rivera-Campo, and Urrutia [19] have also investigated the problem of illuminating families of triangles and rectangles. They show that $\lfloor \frac{4n+4}{3} \rfloor$ guards are sufficient to guard any family of n disjoint triangles.

However this bound does not seem to be tight, and they conjecture that there is some constant c for which every family of n disjoint triangles requires only $n+c$ guards. In the special case of disjoint homothetic triangles (triangles whose edges form three sets of parallel segments), they obtain such a bound, showing that $n+1$ guards suffice. They also exhibit a class of arrangements similar to Figure 2.8 but consisting of triangles that requires $n-1$ guards. Their work on illuminating disjoint isothetic rectangles also

yielded special case and general results; the general result is that $\lfloor \frac{4n+4}{3} \rfloor$ point guards are sufficient to guard any families of n disjoint isothetic rectangles. For the special case where the rectangles are restricted to all have the same width, they showed that $n+2$ guards suffice. They also exhibit a class of equal-width rectangle arrangements requiring $n-1$ guards. Everett and Toussaint have improved the $n+2$ bound to n for the case of equal sized squares and have given an $O(n \ln n)$ placement algorithm [33].

The following is an example of a family of n homothetic triangles that requires a lower bound of n point lights instead of $n-1$; see Figure 2.10.

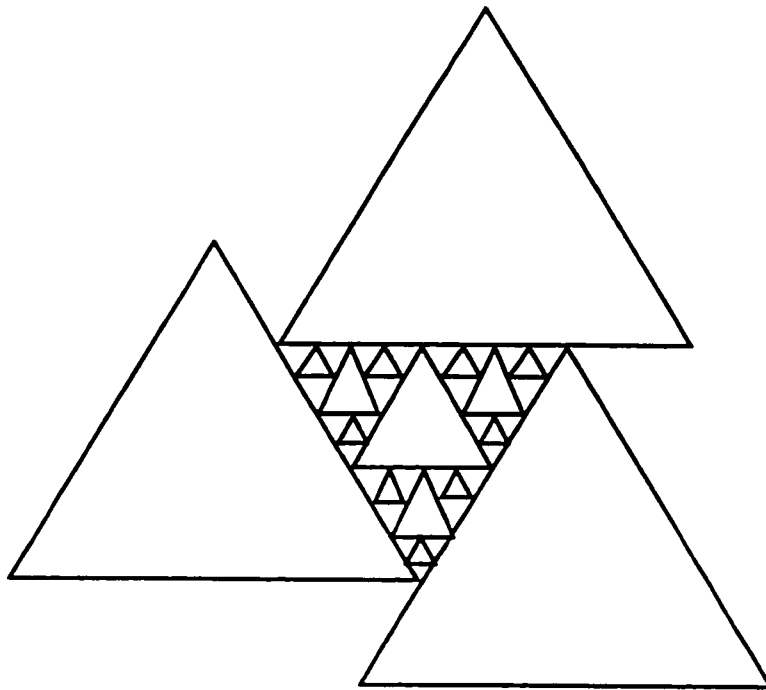


Figure 2.10: A family of n homothetic triangles that requires n point lights.

Polygons with holes

Given a polygon P and a set of m disjoint polygons P_1, \dots, P_m contained in the interior of P , we call the set $P - \{P_1 \cup \dots \cup P_m\}$ a polygon with holes. In this case, we say that P has m holes.

O'Rourke [59] was the first to prove the following result on guarding polygons with holes:

Theorem 2.2.4 *Any polygon with n vertices and h holes can always be guarded with $\lfloor \frac{(n+2h)}{3} \rfloor$ vertex guards.*

Shermer [68] conjectures:

Conjecture 2.2.1 [68] *Any polygon with n vertices and h holes can always be guarded with $\lfloor \frac{(n+h)}{3} \rfloor$ vertex guards.*

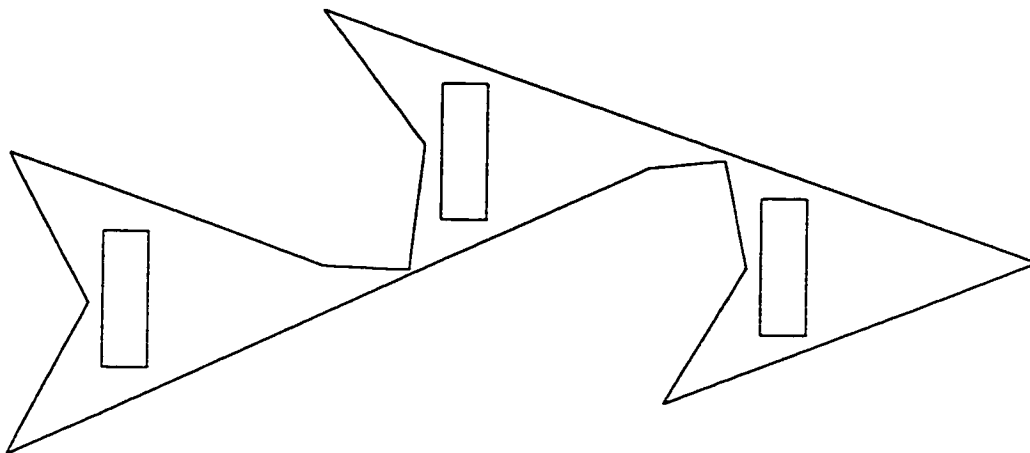


Figure 2.11: A polygon that requires $\lfloor \frac{(n+h)}{3} \rfloor$ vertex guards.

Shermer was able to prove his conjecture for $h=1$ and he believes that this bound is tight. However the conjecture remains open for $h > 1$. Bjorling-Sachs and Souvaine [6] and Hoffman, Kaufman and Kriegel [43] independently proved:

Theorem 2.2.5 $\lceil \frac{(n+h)}{3} \rceil$ point guards are always sufficient and occasionally necessary to guard any polygon with n vertices and h holes.

Bjorling-Sachs and Souvaine's [6] proof was based on connecting each hole of the polygon to the exterior by cutting away a quadrilateral channel such that one vertex is introduced for each channel, and there is a triangle T in the remaining polygon such that any point in it sees all of the channel. This triangle is then forced to be in a triangulation of the remaining polygon. Then they use Fisk's proof of Chvátal's theorem to place a guard at a vertex of T . This guard will cover the channel. They gave an $O(n^2)$ time algorithm to find the position of the $\lceil \frac{(n+h)}{3} \rceil$ guards.

O'Rourke, following his proof of Theorem 2.2.4, extended his method in 1982 to show that any orthogonal polygon with n vertices and h orthogonal holes can always be guarded with $\lfloor \frac{n+2h}{4} \rfloor$ vertex guards. He also conjectured that $\lfloor \frac{n}{4} \rfloor$ vertex guards are sufficient to guard them. Aggarwal proved O'Rourke's conjecture for the cases $h=1$ and $h=2$ [2]. Hoffman proved the full conjecture [43]. He proceeds by showing that any orthogonal polygon with holes can be divided into orthogonal star-shaped

polygons, each with at most 16 vertices. Hoffman also claims an $O(n^{1.5} \log^2 n \log \log n)$ guard placement algorithm.

García-López [38] studied the problem of illuminating the free space generated by a family of disjoint polygons, i.e. the complement of their union. Using Theorem 2.2.5, he showed:

Theorem 2.2.6 *The free space generated by any family of h disjoint polygons with a total of n vertices can be illuminated with at most $\lfloor \frac{n+h+3}{3} \rfloor$ point lights. There are families that require $\lfloor \frac{n+h-1}{3} \rfloor$ lights.*

This leads to an $O(n^2)$ time algorithm. For vertex lights, García-López proved [38]:

Theorem 2.2.7 $\lfloor \frac{5n}{9} \rfloor$ vertex lights are always sufficient and $\lceil \frac{n}{2} \rceil$ occasionally necessary to illuminate the free space generated by a family of disjoint polygons with n vertices.

The proof of Theorem 2.2.7 leads to an $O(n \ln n)$ time algorithm to find the $\lfloor \frac{5n}{9} \rfloor$ vertex lights. He also proved that to illuminate the free space generated by any family of m disjoint quadrilaterals, $2m$ vertex lights are always necessary and occasionally sufficient, and that $\lfloor \frac{5m+3}{3} \rfloor$ point guards are always sufficient. He also conjectured that the free space generated by m disjoint quadrilaterals can always be illuminated by $n+c$ point lights, c a

constant. This conjecture was proven to be false by Czyzowicz and Urrutia, see Urrutia [78]. They produced a family of $n = 3m-3$ quadrilaterals such that to illuminate the free space they generate requires $4m-4$ point lights, $m \geq 4$. See Figure 2.12. For illumination in higher dimensions it is shown in [22] that there are families of $O(n^2)$ boxes such that to illuminate the free space generated by them requires $O(n^3)$ light sources. On the other hand, Czyzowicz, Gaujal, Rivera-Campo, Urrutia and Zaks [25] proved:

Theorem 2.2.8 *For any compact convex set T in E^d , there is a constant $c_d(T)$ such that every family F consisting of n mutually disjoint congruent copies of T can be illuminated with $c_d(T) n$ lights.*

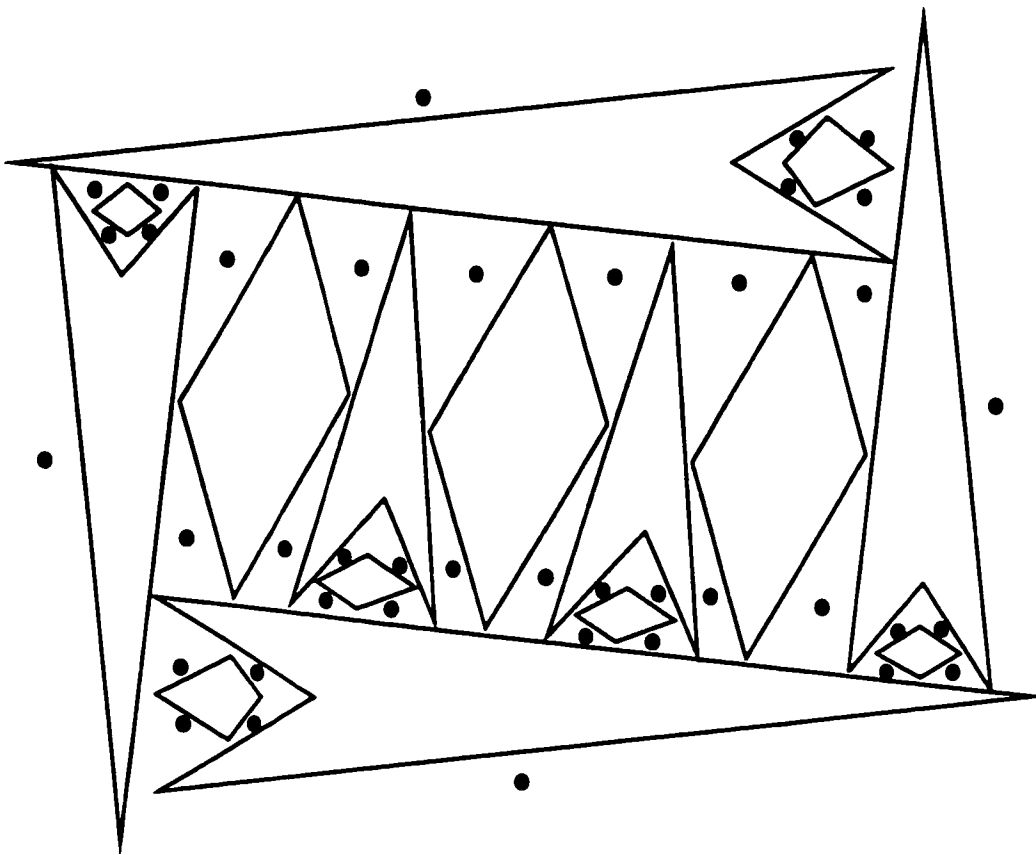


Figure 2.12: An example of a family of $n = 3m - 3$ quadrilaterals that require $4m-4$ point lights to illuminate the free space they generate, $m = 6$.

2.3 Illuminating Convex Sets using Line Segments

We now turn our attention to the problem of illuminating families of disjoint compact convex sets using line segment illuminators. These line segments are not allowed to intersect any of the compact sets. Urrutia [78] proves the following based on the idea used to prove Fejes Tóth's Theorem [34]:

Theorem 2.3.1 *Any family of n disjoint compact convex sets can always be illuminated with at most $n - 1$ line segments, $n > 2$.*

Proof: Consider a family $F = \{S_1, \dots, S_n\}$ of n disjoint compact convex sets. Find a family of sets $T = \{T_1, \dots, T_n\}$ such that:

1. S_i is contained in T_i , $i = 1, \dots, n$.
2. The interiors of T_i , $i = 1, \dots, n$ are disjoint.
3. The number of tangencies between the elements of T is maximized.

Let H be the complement of T_1, \dots, T_n . Again the worst case arises when H is partitioned into triangular faces. Construct a graph G whose vertex set is the faces of H , two of which are adjacent if they have two common elements of T on their boundaries. Note that G has $2n - 4$ vertices. The degree of each vertex of G is 3, and it is 3-connected, $n > 2$. Therefore by Tutte's Theorem, G has a perfect matching. Observe that if two triangular faces of H are adjacent in G , they can be illuminated by a common line segment not intersecting the interior of any T_i , $i = 1, \dots, n$. Using the perfect matching of G , we can choose $n - 2$ line segments to illuminate all the faces of H , except the outer one, which needs an extra line segment, i.e. we use

$n-1$ segments in all. These line segments also illuminate all the elements of S , and the result follows. ■

We close this section with the following conjecture due to Czyzowicz, Rival and Urrutia [18]:

Conjecture 2.3.1 [18] *Any family of disjoint line segments can always be illuminated with at most $\frac{n}{2} + c$ light sources, c a constant.*

2.4 Illuminating Line Segments on the Plane

Many people have studied art gallery problems on arrangements of disjoint line segments. A particularly interesting variation involving families of line segments was studied by O'Rourke [59]. Suppose we have a set of disjoint line segments $F = \{L_1, \dots, L_n\}$ representing obstacles on the plane, and we want to guard the whole plane. How many guards are always sufficient to guard the whole plane?

Some constraints were assumed by O'Rourke in his proof. A guard at a point p sees a point q on the plane if the line segment joining p to q does not cross any element of F . Moreover a line segment joining p to q is allowed to be collinear with any element of F . Under these assumptions, O'Rourke [59] proved:

Theorem 2.4.1 $\lfloor \frac{2n}{3} \rfloor$ *guards are always sufficient and occasionally necessary to guard the plane in the presence of n line segment obstacles.*

The main idea of O'Rourke's proof is as follows:

Consider F , a collection of n disjoint line segments in general position (i.e. no two of them are collinear). Extend the line segments one by one until

they hit another line, or an extension of another line of F . It is easy to see that we obtain a partitioning P of the plane into $n+1$ convex sets with disjoint interiors. Take the dual graph G of P ; see Figure 2.13(a). It is clear that the degree of every vertex is 3 and thus by Nishiseki's theorem, G has a matching of size at least $\lceil \frac{n+6}{3} \rceil$. Each pair of adjacent faces that are matched can be guarded with a single guard; the remaining ones are assigned a guard each. It now follows that at most $\lfloor \frac{2n}{3} \rfloor$ point guards are used. The case when parallel lines appear can be handled in a similar way. If the guards are required to be at endpoints, then Shermer and Boenke showed that n guards are sometimes needed and always sufficient [59]. ■

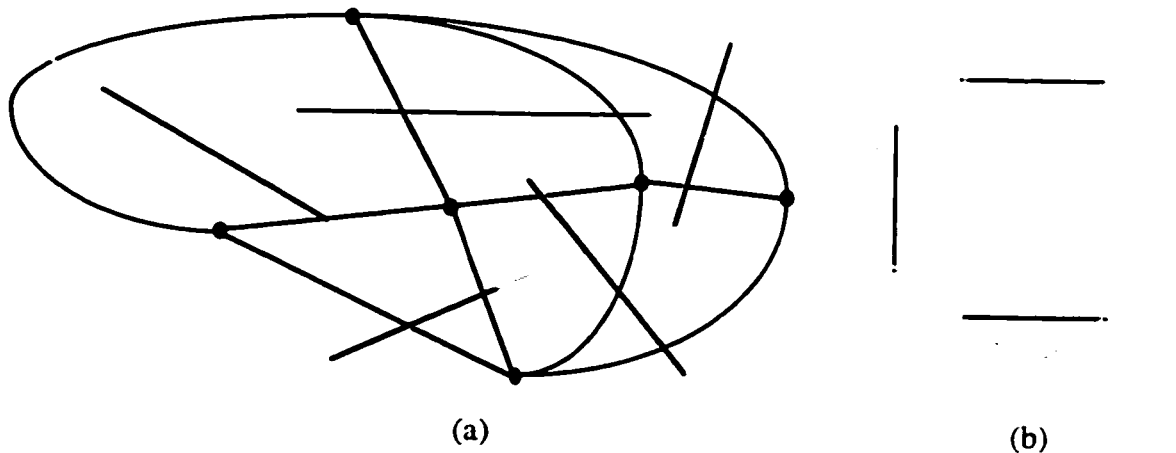


Figure 2.13: (a) A family of line segments, their extensions and their dual graph. (b) An example where n vertex guards are needed to illuminate the plane.

Recently, attention has been placed on seeing the "boundary" of families of line segments rather than the entire plane. In the previous model, the line segments were obstacles to the illumination of the plane; now they are objects to be illuminated. This leads to interesting variations, because

segments, unlike strictly convex sets, need only be seen from one side. For this type of guarding, Czyzowicz, Rival and Urrutia [18] proved that any collection of n line segments can always be illuminated using at most $\lfloor \frac{3n}{4} \rfloor$ line segments. This result was improved by Czyzowicz, Rivera-Campo, Urrutia and Zaks [21]. They proved:

Theorem 2.4.2 *Any collection F of n disjoint line segments can be illuminated with at most $\lfloor \frac{2n}{3} \rfloor - 3$ light sources. If the elements of F are all parallel to the x - or y -axis, $\lceil \frac{n+1}{2} \rceil$ suffice.*

Their proof was similar to O'Rourke's proof of Theorem 2.4.1; their main idea was to generate planar graphs which satisfy Nishiseki's and Tutte's theorems for matchings in planar graphs. Let $F = \{L_1, \dots, L_n\}$ be a family of n disjoint line segments. Choose a triangle T containing all of the elements of F in its interior and let F' be the family containing all the elements of F together with three line segments $L_{n+1}, L_{n+2}, L_{n+3}$ obtained from T by shortening the three sides of T by $e > 0$, e sufficiently small; see Figure 2.13. Construct a family $H = \{S_1, \dots, S_n, S_{n+1}, S_{n+2}, S_{n+3}\}$ of $n+3$ strictly convex compact sets with mutually disjoint interiors satisfying the following properties:

1. L_i is contained in S_i , $i = 1, \dots, n+3$.
2. The number of points at which pairs of elements of H are tangent is maximized.

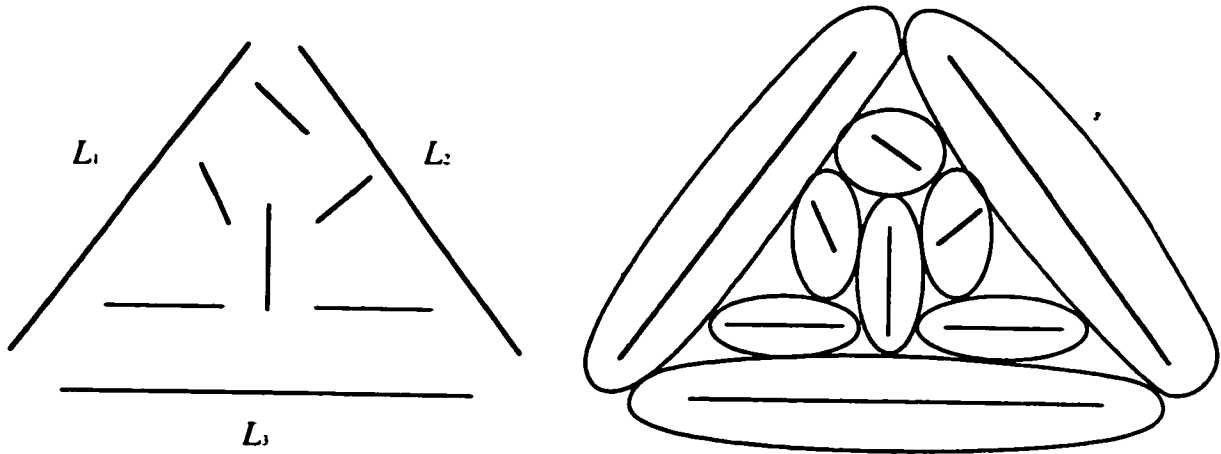


Figure 2.14: Illuminating line segments.

Now it is clear that every element in H is tangent to at least three element in H . Construct a graph G as follows: For each element of H insert a vertex in G . Two vertices are adjacent if their corresponding sets in G are tangent; see Figure 2.15. It is easy to see that G is planar and 2-connected. In addition, since each set in H is tangent to at least three other elements in H , the degree of each vertex in G is at least three. Then by Nishizeki's Theorem 1.2.2, G has a matching M of size at least $\lceil \frac{n+3+4}{3} \rceil = \lceil \frac{n+1}{3} \rceil + 2$. For each pair of elements S_i, S_j matched in M by an edge of G , place a light source at the point in which they intersect. This light source will completely illuminate the line segments L_i, L_j contained in S_i and S_j respectively. Since M has at least $\lceil \frac{n+1}{3} \rceil + 2$ elements, $2(\lceil \frac{n+1}{3} \rceil + 2)$ elements of F will be illuminated using $\lceil \frac{n+1}{3} \rceil + 2$ lights. For the remaining elements of F , an extra light source per element is needed. Then the total number of lights required with this technique is:

$$\left(\left\lceil \frac{n+1}{3} \right\rceil + 2 \right) + \left((n+3) - 2 \left\lceil \frac{n+1}{3} \right\rceil \right) = n+1 - \left\lceil \frac{n+1}{3} \right\rceil \leq \left\lceil \frac{2n}{3} \right\rceil - 3.$$

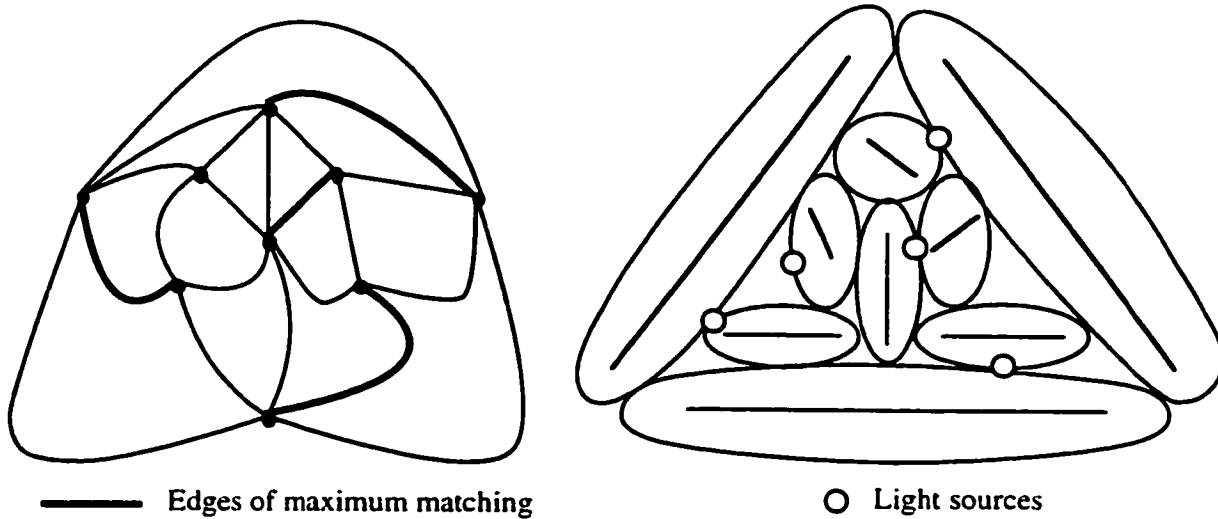


Figure 2.15: Placement of lights.

For the orthogonal case, the case where the elements of F are parallel to the x or the y -axis, they proceed by using Tutte's result [76] which guarantees the existence of a perfect matching in a graph. Their proof was constructed as follows: Let F be an orthogonal family of n line segments and let R be a rectangle containing all the elements of F in its interior. Suppose that F has an odd number of elements, otherwise add an extra line segment to F and the same result will follow. Extend all the elements of F on both sides, one at a time, until they hit the rectangle R or another element of F . It is easy to verify that after this step is done, R is subdivided into $n+1$ subrectangles. They then construct a graph G in which each subrectangle of R is represented by a vertex of G , and two vertices of G are adjacent if their corresponding subrectangles have a common corner point. Now they

proceed to show that the graph G satisfies Tutte's condition, thus G has a perfect matching M . So for every pair of regions matched in M , they place a light to illuminate both of them and the result follows. ■

In terms of bounds, they believe that neither bounds is tight and they think that, in the general case, the correct upper bound is $\lceil \frac{n}{2} \rceil + c$ for some constant c . For the lower bound of the orthogonal case, they were able to produce families of line segments that require $\lfloor \frac{n}{3} \rfloor$ lights. See Figure 2.16.

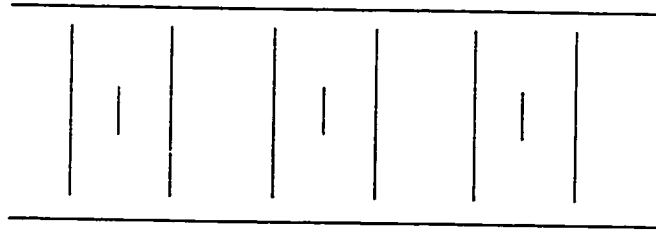


Figure 2.16: A set of orthogonal lines that requires $\lfloor \frac{n}{3} \rfloor$ lights

Another line illumination problem was studied by Jennings and Lenhart [46]. They were concerned with the following problem: Given a family F of n disjoint line segments, find a subset S of F such that every element of F is seen by at least one point on one of the elements of S . They proved that:

Theorem 2.4.3 *Given any set F of n pairwise disjoint line segments, it is possible to find a subset $S \in F$ with at most $\lfloor \frac{n}{2} \rfloor$ elements such that every element of F is seen by at least one point on one of the elements of S . The bound is tight.*

2.5 Protecting Line Segments

Czyzowicz *et al.* [23] have also studied another notion of guarding for families of line segments, suggested by Santoro. We will say that a family of sets is protected by a group of guards if each set contains a point that is seen by some guard. Thus if any set is removed, some guard would notice it. Czyzowicz *et al.* [23] have proved the following:

Theorem 2.5.1 *Any collection of n line segments can always be guarded using at most $\lfloor \frac{n}{2} \rfloor$ points; $\lfloor \frac{2n-3}{5} \rfloor$ points are occasionally necessary.*

Consider a collection $F = \{S_1, \dots, S_n\}$ of n disjoint line segments on the plane. Construct a graph $G(F)$ with n vertices v_1, \dots, v_n such that v_i is adjacent to v_j if and only if there is a point x on the plane that sees at least a point in the boundary of each of S_i and S_j , i.e. x can protect both line segments. In Figure 2.17 a collection F of five segments and its corresponding graph $G(F)$ is shown.

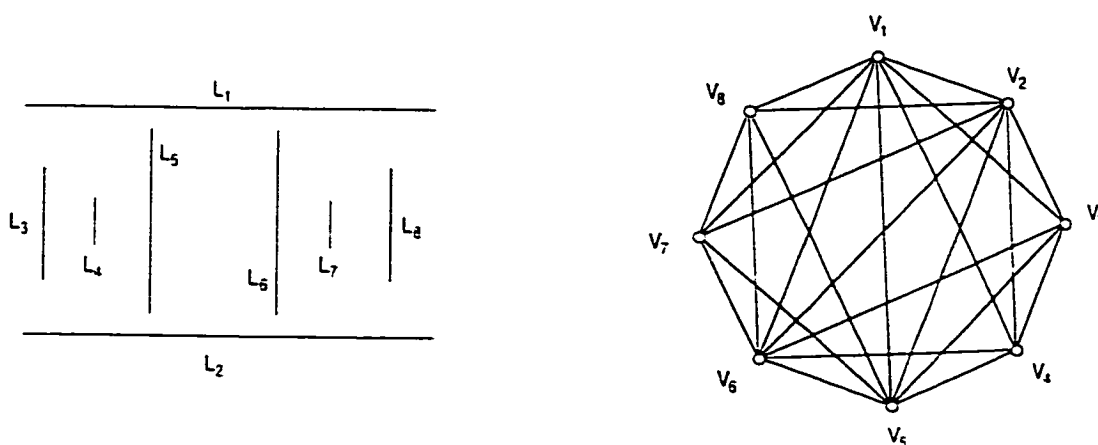


Figure 2.17: A family F of eight line segments, and its corresponding graph $G(F)$.

The main idea in their proof is to show that for any family F with an even number n of disjoint segments, the graph $G(F)$ satisfies Tutte's condition and thus has a perfect matching M . It follows that F can be illuminated by a set consisting of at most $\frac{n}{2}$ points, one for each edge of M . To obtain the lower bound, construct an example of a family F with n segments in which $\lfloor \frac{2n-3}{5} \rfloor$ points are required to guard F as follows: Let H be a cubic planar graph with a triangular outer face in which all the vertices, except the outer ones, are such that the three vectors emanating from the vertex along the edges positively span the plane. Let H have k vertices; it has $\frac{3k}{2}$ edges. Substitute the edges of H by segments such that at each of the $k-3$ inner vertices we obtain a triangular face in which we insert a small segment; see Figure 2.18.

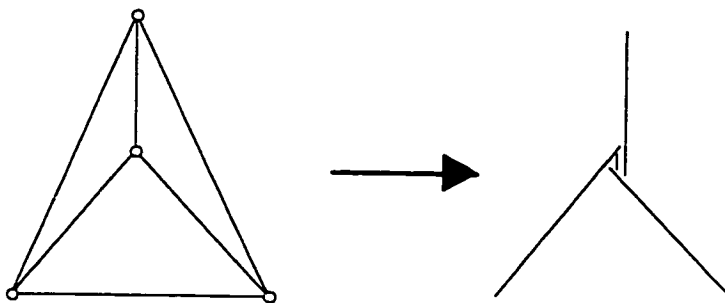


Figure 2.18: Generating a set of n line segments that require $\lfloor \frac{2n-3}{5} \rfloor$ points to protect them.

Discard the three edges of the outer face of H and disconnect each edge in a small neighborhood of its end vertices to form a collection of $n = (\frac{3k}{2}) - 3 + k - 3$ segments. No two of our $k-3$ small segments are visible from a single point, hence $k-3$ points are needed to guard the

collection of segments. It is easy to verify that $k-3$ points are also sufficient. $k-3 = \lfloor \frac{2n-3}{5} \rfloor$, completing the proof of Theorem 2.5.1. ■

2.6 Guarding Rectangular art galleries

Czyzowicz *et al.* consider the problem of guarding rectangular art galleries [22]. A rectangular art gallery is formed by subdividing a rectangle into any number of smaller rectangles (rooms). Any two rooms that share an edge have a door between them. A rectangular art gallery is shown in Figure 2.19.

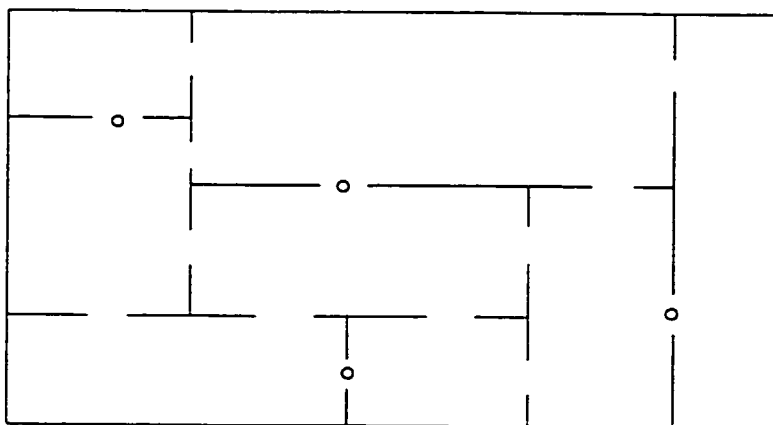


Figure 2.19: An example of an art gallery with eight rooms for which four guards suffice, their positions shown by with small circles \circ .

Guards are allowed either in a room or at a door between two rooms. Czyzowicz *et al.* [22] proved that, in rectangular art galleries with r rooms, the maximum number of guards needed is $\lceil \frac{r}{2} \rceil$. This number guards is needed by the rectangular gallery of r rooms constructed by dividing the outer rectangle by only vertical line segments. Their result extends to the

case where the outer shape is any orthogonal polygon of n vertices:

$\lceil \frac{(r+4)-2}{2} \rceil = \lceil \frac{(2r+n-4)}{4} \rceil$ guards are required. If the outer orthogonal polygon

has h holes and n vertices, $\lceil \frac{(2r+n-2h-4)}{2} \rceil$ guards are sometimes required.

Chapter 3

Floodlight Illumination Problems

In the previous chapters, the usual scenario is that we have some target objects in two or more dimensions that are to be illuminated, and some specified guards which are assumed to patrol around themselves in all directions, or some specified sites for lights which are assumed to shine light in every direction, i.e. with an angle of illumination of 360° in the planar case. In this chapter, we will consider a variation of these problems in which lights are constrained to shine in some specified angle of illumination. We call such light sources *floodlights*. Thus for the rest of this thesis, a floodlight f_i is a source of light located at a point p of the plane, called its apex; f_i illuminates only within a positive angle of illumination α_i , and can be rotated around its apex. In this chapter we survey results concerning illumination problems using floodlights and we start by studying the following problem due to J. Urrutia [78]:

The *three-floodlight illumination problem*: Let $\alpha_1 + \alpha_2 + \alpha_3 = \pi$ and consider any convex polygon P . Can we place three floodlights of sizes at most $\alpha_1, \alpha_2, \alpha_3$, at most one per vertex, in such a way that P is completely illuminated?

Urrutia established that any convex polygon may be illuminated by any three floodlights whose total angle is π . Let P be a convex polygon; if P has 3 vertices then clearly the result is true. Assume P has at least four vertices and suppose that $\alpha_1 \leq \alpha_2 \leq \alpha_3$. Notice first that $\alpha_2 < \pi/2$ and the sum of the internal angles of P is at least 2π . Now since P has at least four vertices, the interior angle at one vertex v of P is at least $\pi/2$. Find a triangle T with internal angles α_1, α_2 and α_3 such that the vertex of T with angle α_2 lies on v , and the other vertices of T lie on two points x and y on the boundary of P . Suppose that x and y lie on different edges, say e_1 and e_3 of P . See Figure 3.1.

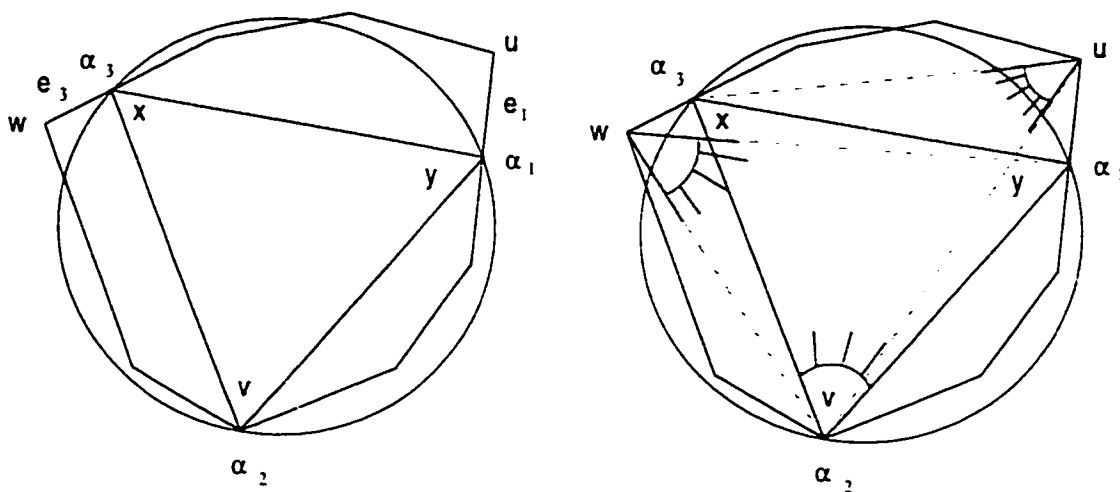


Figure 3.1: Illustration of proof of the Three-Floodlight Problem.

Place a floodlight f_2 with angle of illumination α_2 at v , illuminating T . Let C be the circle passing through the vertices of T . It is easy to see that at least one vertex of e_1 and e_3 is not contained in the interior of C . Let u and w be those endpoints outside C . Two cases arise:

1. $u = w$. Let t be the point of intersection of the tangents to C at x and y .

It is easy to verify that the angle β at t is $\pi - 2\alpha_2$ which is less than or equal to $\alpha_3 = \pi - (\alpha_1 + \alpha_2)$, therefore the angle α of P at u is at most β and placing a floodlight of size α at u illuminates P . The result follows.

See Figure 3.2.

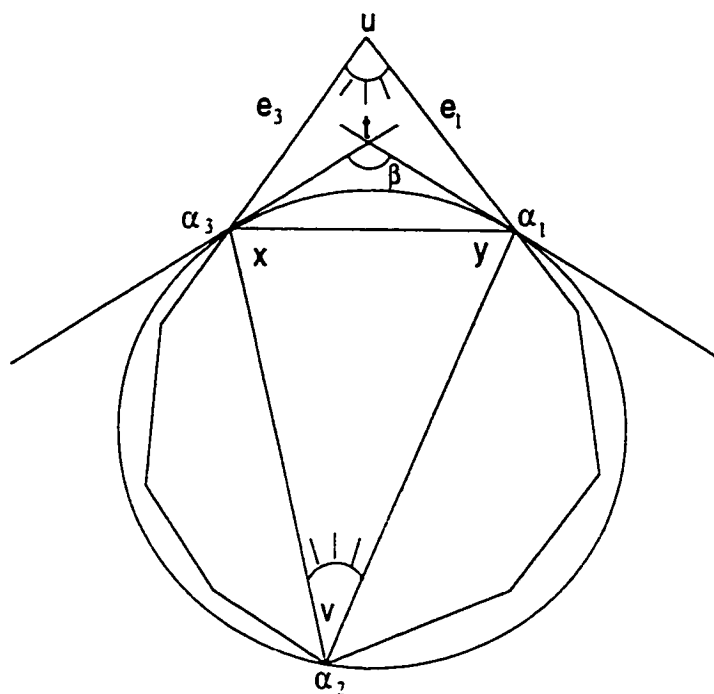


Figure 3.2: Illustration for the case where $u=w$, in which case two floodlights suffice to illuminate P .

2. $u \neq w$. In order to illuminate the angular region determined by v, u, \dot{x} and v, w, y , we have to place floodlights f_1 and f_3 at u and w respectively. Notice that u and w lie outside of the interior of C , therefore the angles of illumination of f_1 and f_3 are at most α_1 and α_3 respectively, and the result follows. See Figure 3.1. ■

3.1 Illuminating the Plane

We now examine the following floodlight illumination problem of the plane studied in [8].

Suppose we have four points p_1, \dots, p_4 on the plane and four $\pi/2$ floodlights, one at each p_i . Can we orient them in such a way that all of the plane is illuminated?

To see that under these conditions the plane can always be illuminated, consider a line that leaves two points on one side of it. Assume without loss of generality that the line is parallel to the x -axis. It is easy to see now that using the two floodlights on top of l , we can illuminate the region below it, and using the floodlights below l , we can illuminate the region above it, as in Figure 3.3. Observe that if we rotate the line l , we will obtain an infinite number of solutions to our plane illumination problem.

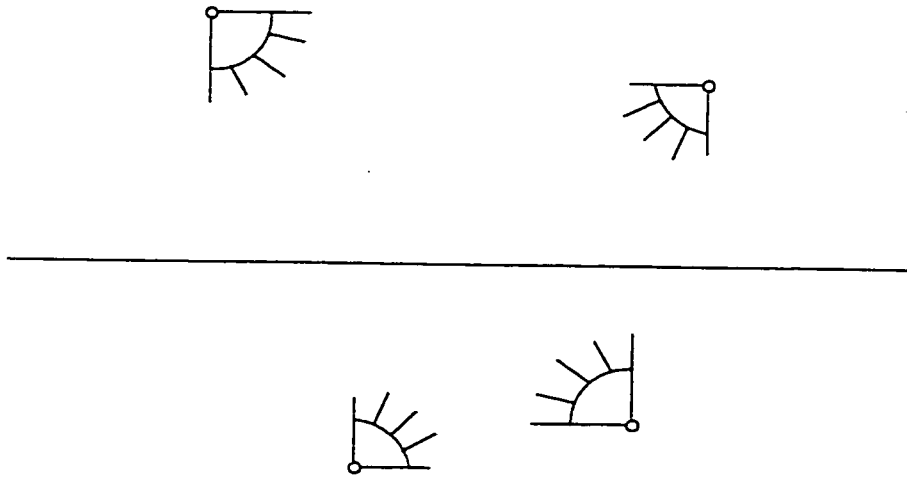


Figure 3.3: Four $\pi/2$ floodlights suffice to illuminate the plane.

A generalization of the previous result was proved in [8].

Let $\{\alpha_1, \dots, \alpha_n\}$ be a set of angles such that each of them is at most π and $\alpha_1 + \dots + \alpha_n = 2\pi$. Consider a set of floodlights f_1, \dots, f_n such that the size of f_i is α_i , $i = 1, \dots, n$.

Theorem 3.1.1 *Let P_n be a collection of n points on the plane and f_1, \dots, f_n a set of floodlight of sizes $\{\alpha_1, \dots, \alpha_n\}$ such that $\alpha_1 + \dots + \alpha_n = 2\pi$. Then we can locate all our floodlights, one per element of P_n and point them in such a way that the plane is completely illuminated.*

To prove this result, we split the floodlights into three disjoint sets $\{f_1, \dots, f_i\}$, $\{f_{i+1}, \dots, f_k\}$ and $\{f_{k+1}, \dots, f_n\}$ such that $\alpha_1 + \dots + \alpha_i = \beta_1$, $\alpha_{i+1} + \dots + \alpha_k = \beta_2$ and $\alpha_{k+1} + \dots + \alpha_n = \beta_3$ such that $\beta_i \leq \pi$, $i=1,2,3$. We then

subdivided the plane into three unbounded angular wedges w_i of sizes β_i , $i = 1, \dots, 3$ such that w_1, w_2 and w_3 contain exactly $i, j-i$ and $n-j$ elements of P_n respectively. Then we solve each of these wedge illumination problems separately to obtain a solution to the plane illumination problem. The solution leads to an $O(n \ln n)$ time algorithm.

Motivated by the proof of the Theorem 3.1.1, Steiger and Streinu have given a linear time algorithm for the tripartitioning problem, and a lower bound of $\Omega(n \ln n)$ for the floodlight illumination problem of the plane, in the case where no angle is greater than π . They have also shown that the tight floodlight wedge illumination problem with angles possibly greater than π is NP; see [72] [73] [71]. Rote [64] has devised an alternate proof of the floodlight theorem which he can generalize to higher dimensions. In the case that the cone of illumination arise from a polytope enclosing the origin, each cone is determined by the origin as apex, and by a facet of the polytope.

3.2 Illumination of Stages

We now study the *Stage Illumination problem* which is one of the most interesting problems in this area. This problem was presented by J. Urrutia in 1992. A stage is defined as a line segment on the x-axis of the plane. The problem asks whether or not a given set of floodlights with its apexes above a stage can be rotated around their apexes so as to obtain a final configuration such that the stage is completely illuminated. The problem for finding a polynomial time algorithm for this problem or proving NP-

hardness of this problem was open until most recently when Ito, Uehara and Yokoyama [45] were successful in proving that the *Stage Illumination problem* is NP-complete. They were able to show the NP-completeness even for the case where the following three restrictions hold:

- i) *The apexes of all the floodlights are on a single line.*
- ii) *The apexes of the floodlights lie on at most two points with the same x -coordinate.*
- iii) *The apexes of the floodlights lie on at most two points with the same y -coordinate.*

They showed their result by reduction from the partition problem, which is NP-complete. Thus, one of the remaining interesting problems is finding a significant subclass of the *Stage Illumination problem* that can be solved in polynomial time. They conjectured in [45] that it is also NP-complete even if all floodlights have the same size.

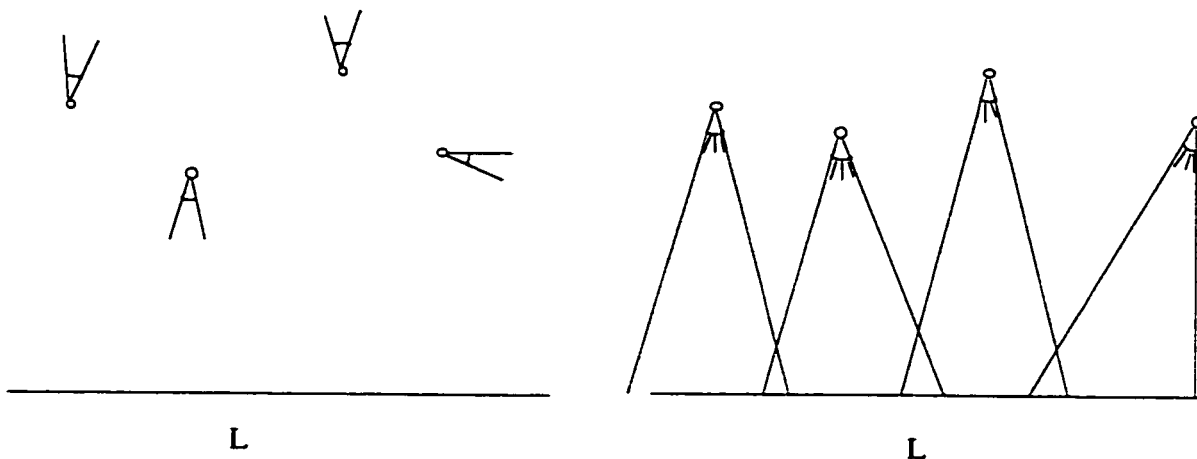


Figure 3.4: The Stage Illumination Problem.

But the following variation of the Stage Illumination Problem was solved efficiently in [24]. Their solution for the problem was also implemented in a video demonstration [35].

3.2.1 Optimal Illumination of Stages

Optimal floodlight illumination of stages: Consider a stage, represented by a line segment S and a set $P = \{p_1, \dots, p_n\}$ of n points. Determine a set of floodlights F that illuminates S such that the angular cost of F is minimized and the apex of each floodlight $f_i \in F$ is located at some point $p_j \in P$.

In their proof, they allow for more than one floodlight to be located at each point of P . Moreover, they assume that each floodlight has size strictly greater than 0. Their solution is such that only at one point in P they place two lights; at the other points they place one or no light at all.

To solve the floodlight illumination problem for line segments, they solve the problem of optimal floodlight illumination of the real line using floodlights placed on the elements of a point set $P = \{p_1, \dots, p_n\}$. To start, they consider the problem in which P has two points p_i and p_j . Assume that p_i is lower than p_j . Consider the two circles tangent to the real line and containing p_i and p_j , and let their intersection points with the real line be labeled $x_{i,j}$ in such a way that $x_{i,j} < y_{i,j}$; see Figure 3.5(a).

They prove:

Lemma 3.2.1 *In the optimal floodlight illumination of the real line L from $\{p_i, p_j\}$, all points in the interval $[x_{i,j}, y_{i,j}]$ are illuminated from p_j , and all points in the intervals $(-\infty, x_{i,j}]$ and $[y_{i,j}, \infty)$ are illuminated from p_i .*

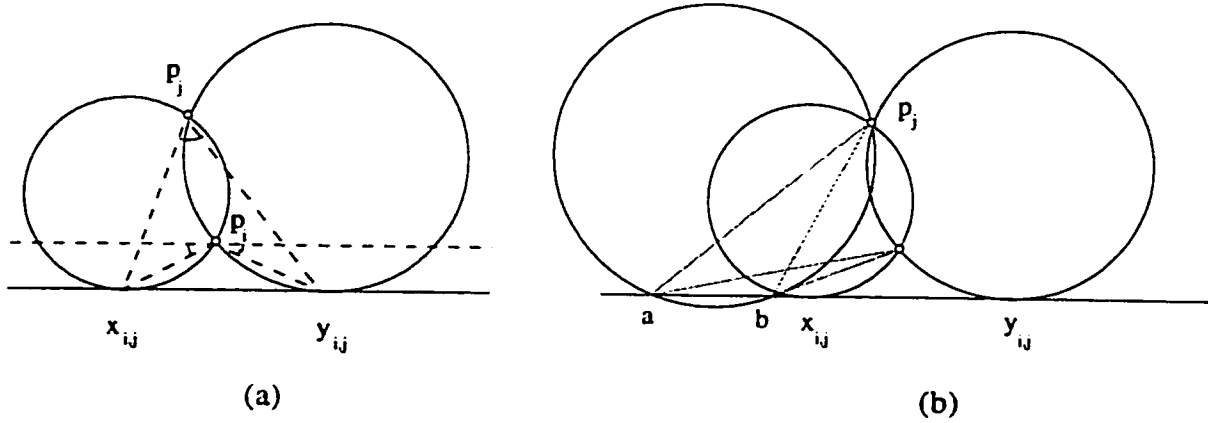


Figure 3.5: Illuminating the real line from two points.

Proof. Suppose that in an optimal illumination of the real line with a set of floodlights F , an interval with endpoints $a < b$ contained to the left of $x_{i,j}$ is illuminated by a small floodlight f at p_j . Consider the circle passing through a, b and p_j . It is easy to see that this circle leaves p_i outside, and thus the angle a, p_i, b is smaller than the angle a, p_j, b . Thus if we substitute f by a floodlight f' at p_i we obtain a set of lights that illuminate the real line with smaller weight than that of F , which is a contradiction. It now follows that all points to the left of $x_{i,j}$ are illuminated from p_j . Similarly we can conclude that the interval $[x_{i,j}, y_{i,j}]$ is illuminated from p_j and $[y_{i,j}, \infty)$ is illuminated from p_i ; see Figure 3.5. ■

Next they prove:

Lemma 3.2.2 *Let $P = \{p_1, \dots, p_n\}$ be a collection of points, and p_i a point in the interior of the convex hull of P . Then in any optimal illumination of the real line with floodlights at points of P , there is no floodlight located at p_i .*

Proof. Suppose that p_i is an interior point of the convex hull of P , and that there is an optimal illumination of the real line in which a floodlight f_i placed at p_i illuminated an interval, say $[x, y]$. Consider the smallest disk D containing x, y and all the elements of P , see Figure 3.6. Let p_j be a point of P located in the boundary of D . Since p_i is in the interior of the convex hull of P , $p_i \neq p_j$; moreover p_i belongs to the interior of D . Therefore angle x, p_i, y is greater than angle x, p_j, y .

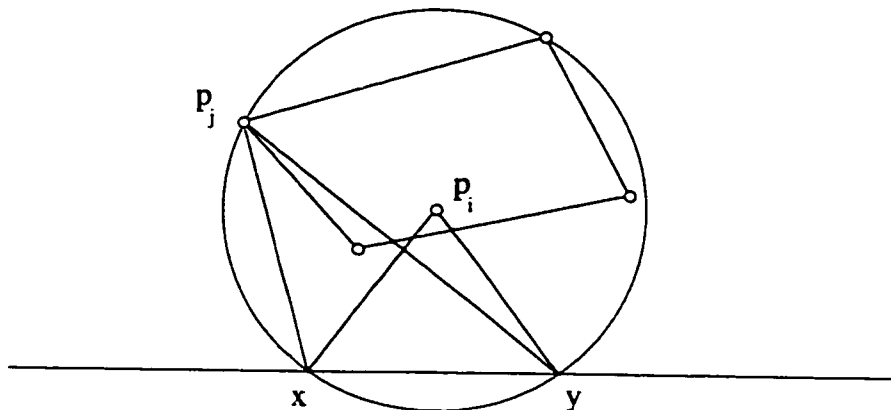


Figure 3.6: Only points in the convex hull are useful.

Thus one can substitute the floodlight at p_i that illuminates the interval $[x, y]$ by a smaller one placed at p_j that illuminates the same interval. This contradicts the assumption of optimality of F . ■

The following result is an easy consequence of Lemmas 3.2.1 and 3.2.2:

Lemma 3.2.3 *Consider an optimal floodlight illumination of the real line with a set of floodlights F on a set of points P . Then if a point x is illuminated by a floodlight of F located at a point p_i of P , the disk circle C tangent to the real line and containing p_i contains all of the elements of P .*

It now follows that the optimal solution to the floodlight illumination problem of the real line can be obtained as follows: Consider the leftmost point $y_{i,j}$ generated by all pairs of points in the convex hull of P , and obtain the circle tangent to the real line and passing through p_i and p_j . Slide to the left a point p initially located at $y_{i,j}$. While we move p to the left, maintain a circle C tangent to the real line at p , and tangent to the convex hull of P . For any position of p , the vertex p_k on the convex hull of P and C is the point where the floodlight that will illuminate p must be located, see Figure 3.7. Notice that for finite set of positions of p , C will contain two points in the convex hull of P . These points correspond to the beginning and end of the intervals to be illuminated by the elements of P , this situation is illustrated in Figure 3.7.

Using Lemma 3.2.3, Czyzowicz, Rivera-Campo and Urrutia [24] proved:

Theorem 3.2.1 *The Optimal Floodlight Illumination of Stages problem can be solved in $O(n \ln n)$ time.*

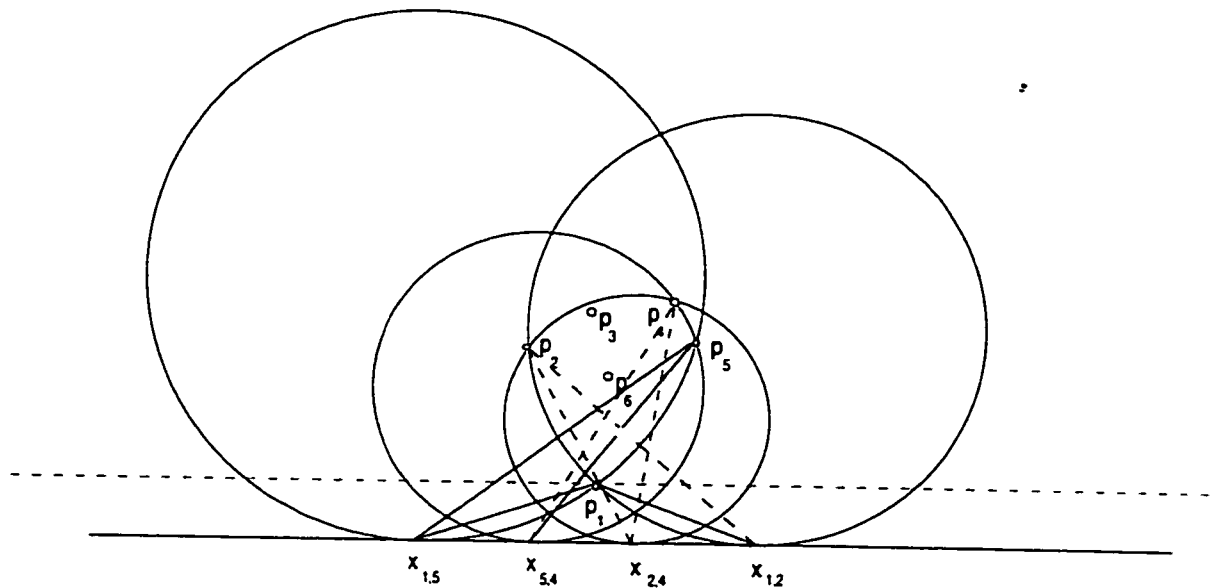


Figure 3.7: Optimal illumination of the real line.

Notice that the solutions obtained in [24] for Theorem 3.2.1 require two floodlights in at most one element of P_n . If we insist that at each point, we have exactly one floodlight, the problem remains open.

In [31], they study the *Two-Floodlight Illumination Problem* for convex polygons, that is the problem of finding two floodlights that illuminate a convex polygon with n vertices in such a way that the sum of their sizes is minimized. In [31] the following result is proved:

Theorem 3.2.2 *The Two-Floodlight Illumination Problem for convex polygons with n vertices can be solved in $O(n^2)$ time.*

It is not known if the complexity of the *Two Floodlight Illumination Problem* is optimal. The k -floodlights illumination problem, that is the problem of illuminating a convex polygon P with k floodlights, $k \geq 3$, such

that the sum of their sizes is minimized, is open. At this point they do not even know if the floodlights have to be located at vertices of the polygon to be illuminated. Moreover they do not even know if there is a constant l such that if $k > l$ then the optimal solution to the k -floodlight illumination problem uses at most l floodlights. They show, however, that if the vertices of P are cocircular, two floodlights suffice.

We close this section with a result of O'Rourke, Streinu and Shermer related to the *Three-Floodlight Illumination* problem presented in the first section of this chapter. As we showed in Problem 3.1, any convex polygon can always be illuminated with three vertex-floodlights such that the sum of their sizes is π . O'Rourke, Streinu and Shermer [62] showed that the generalization of this problem to k -floodlights is false, even when all lights have the same angle π/k .

An open problem is that of determining the smallest integer for which the result can be generalized. Recently, Ismailescu [44] showed that the result can be extended for the case $k = 4$; he proved that every convex n -gon P ($n \geq 4$) can be illuminated by four $\frac{\pi}{4}$ vertex-floodlights. We do not know if the result of Problem 3.1 can be extended, even for the case of $k=5$.

3.3 Floodlight Illumination of Orthogonal Polygons

We now turn our attention to study the illumination of orthogonal polygons with vertex floodlights. For orthogonal polygons, the first result was obtained by Estivill-Castro and Urrutia [30]. A floodlight of size $\frac{\pi}{2}$ will be called an *orthogonal floodlight*.

Theorem 3.3.1 *Any orthogonal polygon P with n vertices can be illuminated with at most $\lfloor \frac{3n-4}{8} \rfloor$ orthogonal vertex floodlights. If the floodlights are allowed to be anywhere on the boundary of the polygon, $\lfloor \frac{n}{4} \rfloor$ suffice.*

Let us partition the edges of P into four subsets. An edge e of P is called a *top edge* if there is an $\epsilon > 0$ such that any point directly below e and at distance at most ϵ from it lies in the interior of P . *Bottom, left and right edges* are defined in a similar way. We then choose a set of orthogonal floodlights according to four different rules: the *top-right, right-bottom, bottom-left* and *top-left* illumination rules. We now define the *top-left* illumination rule as follows. The other three rules can be defined in a similar way, each of which illuminates P .

The *top-left* illumination rule: At the top vertex of all *left* edges, and at the left end vertex of all *top* edges, put an orthogonal floodlight that illuminates the angular sector $3\pi/2$ to 2π . (see Figure 3.8) To see that the set of

floodlights chosen by the *top-left* illumination rule illuminates P , choose any point q in P . Consider the longest horizontal line segment h passing by q ; totally contained in P . Slide h upwards until it hits a top edge of P or it reaches the top vertex of the edge of P containing the left end point of h . In the former case, there is a floodlight at this vertex that illuminates q . Suppose then that h hits a top edge e of P . Notice that by the *top-left* illumination rule, we have a floodlight at the left vertex of e , and this floodlight illuminates q ; see Figure 3.8. A similar argument proves that the lights produced by the *right-bottom*, *bottom-left* and *top-right* illumination rules also illuminate P . A counting argument shows that one of these illumination rules uses at most $\lfloor \frac{3n-4}{8} \rfloor$ floodlights.

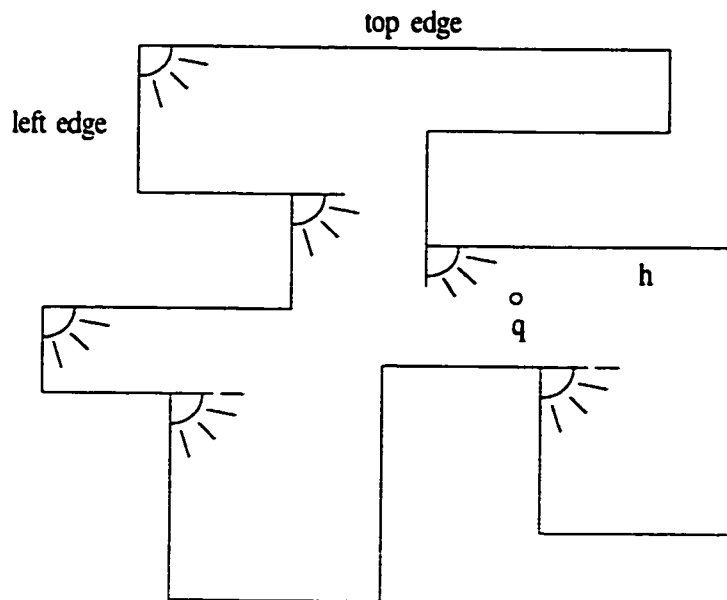


Figure 3.8: Top-left illumination rule

The polygon P_{12} shown in Figure 3.9 has 12 vertices and requires $4 = \frac{3(12)-4}{8}$ orthogonal vertex floodlights. If we now paste copies of this polygon as showing in the same figure, we can generate a family of polygons with $12 + 8k$ vertices, each of which requires $4 + 3k$ vertex floodlights. The result now follows. ■

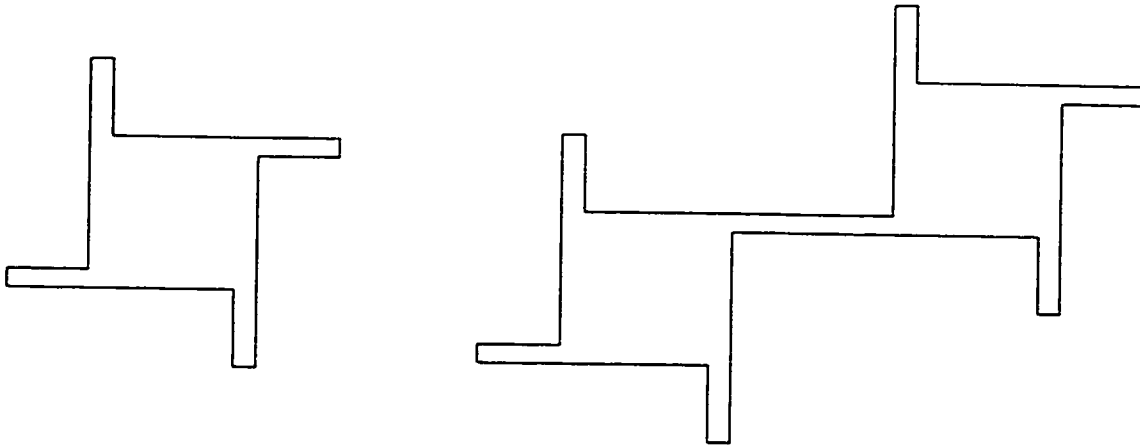


Figure 3.9: Orthogonal polygons that need $\lfloor \frac{3n-4}{8} \rfloor$ orthogonal floodlights.

For the $\lfloor \frac{n}{4} \rfloor$ bound, there are to our knowledge six proofs of this result. The first result is due to Kahn, Klawe, and Kleitman [48]. Kahn, Klawe, and Kleitman's proof was based on a similar technique to that used by Fisk in proving the original art gallery theorem. The main idea of their proof is to partition an orthogonal polygon into convex quadrilaterals. The internal diagonals of each of these quadrilaterals are then added, and the graph thus obtained is a four-vertex colored graph.

Kahn, Klawe, and Kleitman's result provided an incentive to study the problem of decomposing a rectilinear polygon into convex quadrilaterals.

The first $O(n)$ algorithm to achieve this was obtained by Sack [66]. A different algorithm was later obtained by Lubiw [55]. Sack and Toussaint have presented other linear placement algorithms based on quadrilateralization [66]. A linear time algorithm to guard orthogonal polygons using $\lfloor \frac{n}{4} \rfloor$ point guards, some of which may be located in the interior of the polygon, was then given by Edelsbrunner, O'Rourke, and Welzl [26]. Estivill-Castro and Urrutia [30] proved that $\lfloor \frac{n}{4} \rfloor$ guards are also sufficient to guard an orthogonal polygon, but in this case using orthogonal floodlights. Moreover, unlike the result of Edelsbrunner, O'Rourke, and Welzl, they locate the lights on the edges of the polygon, and also give a linear time algorithm to solve this problem. Recently, Urrutia [79] gave yet another proof, the sixth proof of the Orthogonal Art Gallery Theorem!

3.4 π -Floodlight Illumination of Polygons

The results of Theorem 3.3.1 raise the question: What is the smallest angle α such that we can illuminate any simple polygon by placing an α -vertex floodlight at each of its vertices? Let us consider a polygon P with n vertices. Observe that if we allowed to place $n-2$ floodlights of size $\frac{\pi}{3}$ at the vertices of P (allowing more than one light on each vertex), we can always do so in such a way that P is completely illuminated. This follows from the observation that any triangulation of P has $n-2$ triangles, each of which has an internal angle of size at most $\frac{\pi}{3}$. If we allow a floodlight of size at most π at each vertex, we can indeed illuminate all of P . If P is a triangle, a

floodlight of size at most π placed at any vertex suffices to cover it. Suppose then that P has more than 3 vertices. Take any triangulation T of \dot{P} and choose an ear e of T . Let v be the vertex of e of degree 2 in T . Place a π -floodlight at v , and cut e off from P . Proceed by induction on the number of vertices of P , and the result follows. A natural question now arises: Is there an $\alpha < \pi$ such that any polygon P can be illuminated by placing an α -floodlight at every vertex of P ? It was shown in [60] that $\alpha = \frac{\pi}{2}$ does not suffice. A polygon is shown in Figure 3.10 which cannot be illuminated by placing a $\frac{\pi}{2}$ -floodlight at each of its vertices. Shortly after, Estivill-Castro, O'Rourke, Urrutia and Xu [29] extended the result to prove the following:

Theorem 3.4.1 *For any $\epsilon > 0$ there is a polygon P that cannot be illuminated by placing at most one floodlight of size $\pi - \epsilon$ at each of its vertices.*

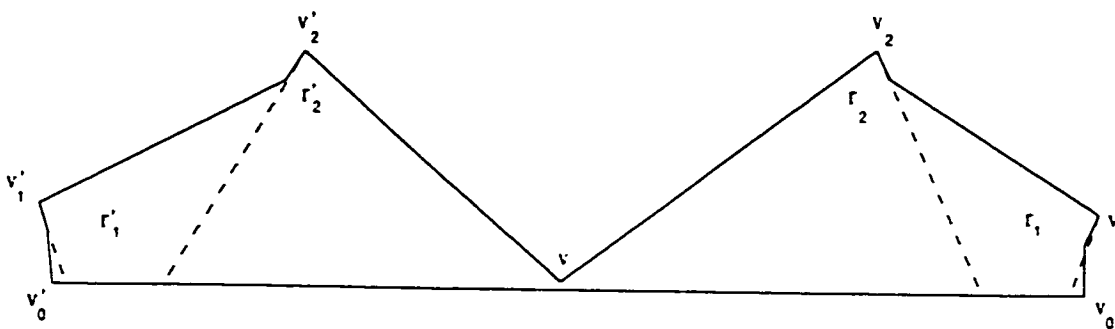


Figure 3.10: A polygon that cannot be illuminated by placing an orthogonal floodlight at each vertex.

Proof. The proof presented here is only for orthogonal floodlights and is due to O'Rourke and Xu [61]. Consider the symmetric polygon P shown in Figure 3.10. Divide P into a left and right section by cutting it along a vertical through vertex v . Let L be the half of P left of the vertical through v , and let R be the right half. Notice that all the convex internal angles at the vertices of P are slightly bigger than $\frac{\pi}{2}$. Label the vertices of P as shown in Figure 3.10 and notice that the extensions l_2, l_1 of the edges of P connecting v_2 to r_2 and v_1 to r_1 , respectively are such that l_2 hits the base of P to the left of the point at which l_1 hits it.

They show that P cannot be illuminated by placing an orthogonal floodlight at each of its vertices. The orthogonal floodlight at v illuminates vertices in L or R , but not both. Assume without loss of generality that this light illuminates vertices in L . Since the internal angle at v_2 is bigger than $\frac{\pi}{2}$, the only way to illuminate all the points in a small neighborhood of v_2 is by using the floodlights at v_2 and r_2 . Similarly to illuminate all the points in a small neighborhood of v_1 , the lights have to be at v_1 and r_1 . Notice, however, that this will prevent the floodlights at r_1 and r_2 from pointing towards v_0 . It now follows that all the points in a small neighborhood of v_0 have to be illuminated using only the floodlight at v_0 . Notice that this is impossible, since the internal angle of P at v_0 is greater than $\frac{\pi}{2}$ and the result follows. ■

We now generalize the previous result to show that any angle $\alpha < \pi$ does not suffice. Consider the polygon P shown in Figure 3.11. It is symmetric with respect to a vertical line through vertex v_m . The right section is composed of vertices $v_0, r_1, v_1, \dots, v_k, r_k$. Set all the convex internal angles at the vertices of P equal to α . Figure 3.11 shows the right section for $k = 5, \alpha = 3\pi/4$. We now argue that P cannot be covered by an angle $\alpha' < \alpha$. The proof is exactly analogous to the previous proof. Assume without loss of generality that the light at v_m illuminates vertices in L . Since all the vertices with indices $k-1, \dots, 0$ lie right of the extension of $r_k v_k$, the only way to illuminate all the points in a small neighborhood of v_k is by using the floodlights at v_k and r_k . Because the light at v_k cannot cover all points on either side of v_k , the light at r_k must be turned clockwise. Now the argument is repeated at v_{k-1}, \dots, v_1 , leading to the conclusion that the lights at r_1 must all be turned clockwise. It now follows that all the points in a small neighborhood of v_0 have to be illuminated using only the floodlight at v_0 . Notice that this is impossible, since the internal angle of P at v_0 is $\alpha < \pi$ and the result follows. ■

Now that we know that no angle less than π suffices, it becomes an interesting question to ask how many π -floodlights are always sufficient to illuminate any polygon with n vertices?

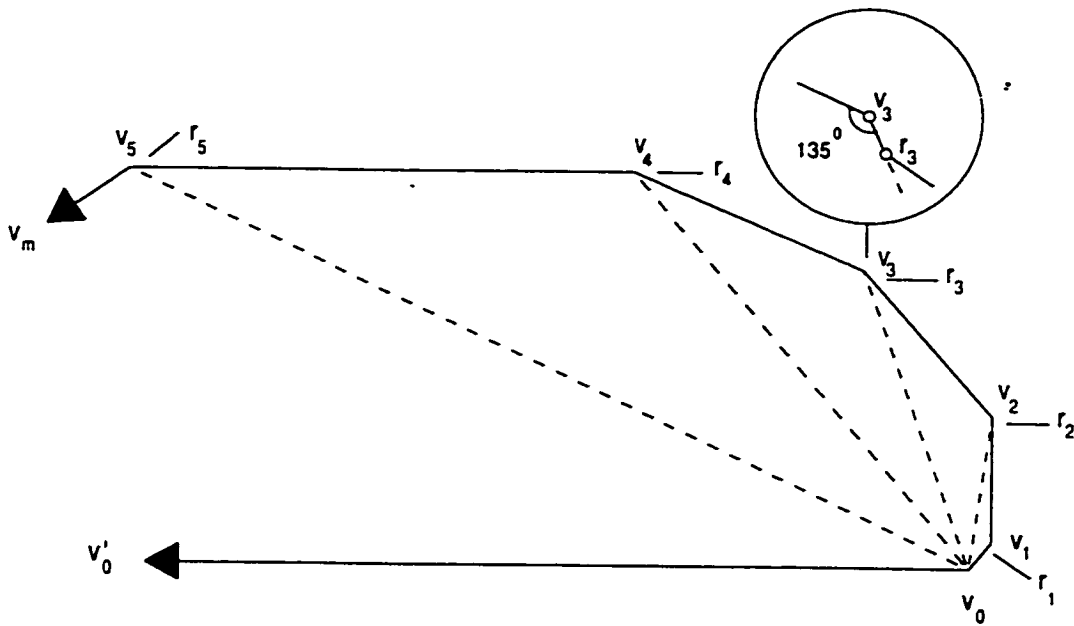


Figure 3.11: vertex $(3\pi/4 - \epsilon)$ lights do not suffice.

F. Santos has produced a family of polygons with $5n+1$ vertices that require $3n \pi$ -vertex floodlights to illuminate them; see Figure 3.12.

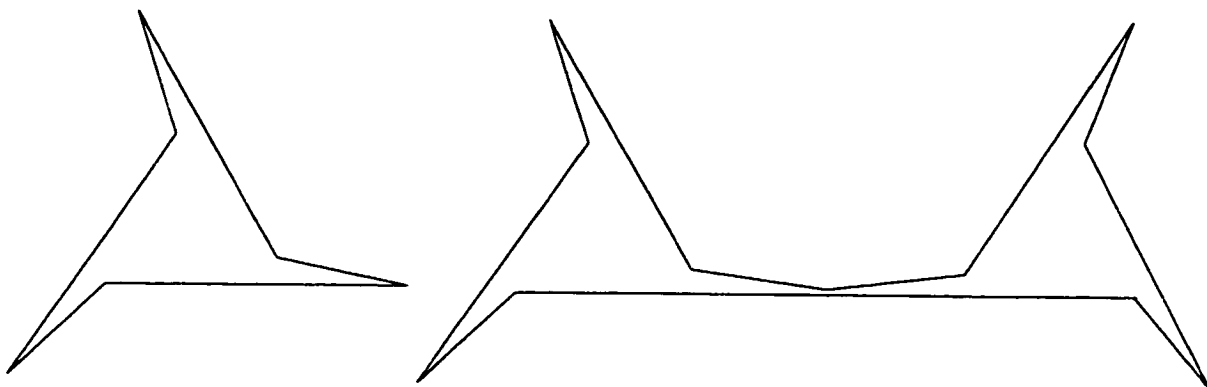


Figure 3.12: A family of polygons with $5n+1$ vertices that require $3n \pi$ -vertex floodlights.

For point floodlights, Bunting and Larman [11], and independently Csizmadia and Tóth proved in 1992 that $\lfloor \frac{9}{4}(n + \frac{1}{4}) \rfloor$ point π -floodlights suffice. Recently, Csizmadia and Tóth proved [17]:

Theorem 3.4.2 $\lceil \frac{2}{5}(n-3) \rceil$ point π -floodlights are always sufficient to illuminate any polygon P with n vertices, $n > 3$.

To prove Theorem 3.4.2, Csizmadia and Tóth first find a triangulation T of P . Then using the dual tree of T , they show that they can cut P along a diagonal of T chosen appropriately. They then proceed by induction on the number of vertices of P . From Theorem 3.4.1, it follows that $\lceil \frac{\pi}{\alpha} \rceil$ $\lceil \frac{2}{5}(n-3) \rceil$ α -point floodlights are always sufficient to illuminate an n -vertex polygon. The proof of Theorem 3.4.2 leads to a linear time algorithm.

Chapter 4

Illuminating Triangles by Vertex Lights

4.1 Introduction

It is easy to see that any triangle can be illuminated by placing a $\frac{\pi}{2}$ light at each of its vertices. This can be seen as follows: Consider a triangle T with vertices A , B , and C and let l_1 , l_2 , and l_3 be the bisectors of the angles of T . It is well known that l_1 , l_2 , and l_3 intersect at a point U in the interior of T .

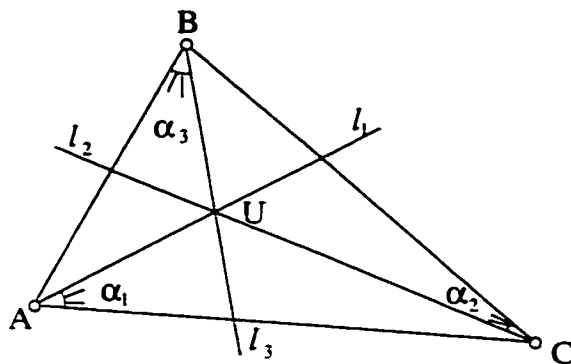


Figure 4.1: l_1, l_2, l_3 the bisectors of triangle T .

Let α_1 , α_2 and α_3 be the angles determined by UAC , UCB , and UBA respectively. Clearly $\alpha_1, \alpha_2, \alpha_3 \leq \frac{\pi}{2}$, from which our result follows. ■

This suggests the following problem:

Find the smallest α , $0 < \alpha \leq \frac{\pi}{2}$, such that any triangle T can be illuminated by placing an α -floodlight at each of its vertices.

Our goal here is to prove:

Theorem 4.1 *Any triangle can be illuminated by three $\frac{\pi}{6}$ -vertex floodlights.*

We start by observing that $\frac{\pi}{6}$ is best possible since we can easily verify that an equilateral triangle can not be illuminated with three α -floodlights, $\alpha < \frac{\pi}{6}$.

4.2 Definitions and General Lemmas

Some terminology and definitions will be given now which will be useful for theorems and proofs in this chapter. An α -floodlight is called α -vertex light if its apex is located at a vertex of a polygon.

We recall the following result:

Theorem 4.1 (Ceva[69]) *Three lines drawn from the vertices A , B and C of triangle ABC meeting the opposite sides in D , E and F respectively are concurrent if and only if:*

$$\frac{AF}{FB} \times \frac{BD}{DC} \times \frac{CE}{EA} = 1 \quad (1)$$

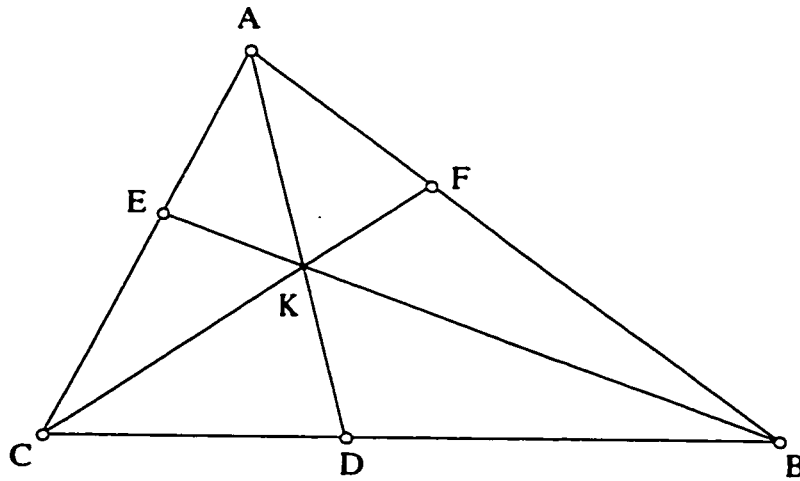


Figure 4.2: Ceva Triangle.

Proof. Extend the lines BE and CF beyond the triangle until they meet GH , the line through A parallel to BC . (See Figure 4.3). There are several pairs of similar triangles: AHF and BCF , AEG and BCE , AGK and BDK , and CDK and AHK . From these and in that order we derive the following proportions:

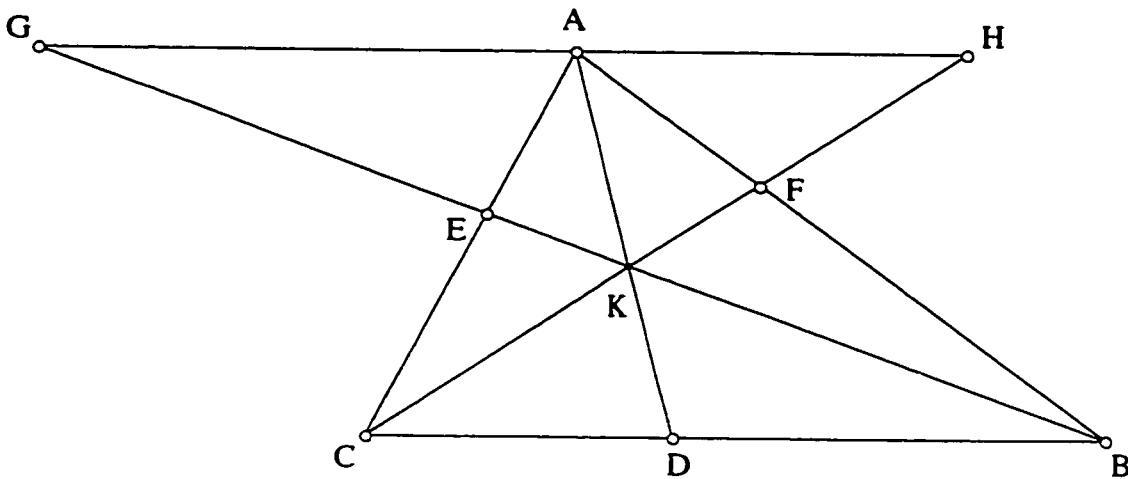


Figure 4.3: Proof of Ceva Theorem.

$$1. \frac{AF}{FB} = \frac{AH}{BC} \quad (*)$$

$$2. \frac{CE}{EA} = \frac{BC}{AG} \quad (*)$$

$$3. \frac{AG}{BD} = \frac{AK}{DK}$$

$$4. \frac{AH}{DC} = \frac{AK}{DK}$$

From the last two we conclude that $\frac{AG}{BD} = \frac{AH}{DC}$ and, hence, $\frac{BD}{DC} = \frac{AG}{AH}$ (*).

Multiplying the identities marked with (*) we get:

$$\frac{AF}{FB} \times \frac{BD}{DC} \times \frac{CE}{EA} = \frac{AH}{BC} \times \frac{BC}{AG} \times \frac{AG}{AH} = \frac{(AH \times BC \times AG)}{(BC \times AG \times AH)} = 1$$

which proves our result. We now prove that if (1) holds, then AD , BE , CF all intersect at a single point. Indeed, assume that K is the point of intersection of BE and CF and draw the line AK until it intersects BC at a point D' . Then, from the part of the theorem just proven it follows that:

$$\frac{AF}{FB} \times \frac{BD'}{D'C} \times \frac{CE}{EA} = 1$$

On the other hand, it is given that:

$$\frac{AF}{FB} \times \frac{BD}{DC} \times \frac{CE}{EA} = 1$$

Combining the two we get:

$$\frac{BD'}{D'C} = \frac{BD}{DC} \quad \text{or} \quad \frac{BD'}{D'C} + 1 = \frac{BD}{DC} + 1 \quad \text{or} \quad \frac{(BD' + D'C)}{D'C} = \frac{(BD + DC)}{DC}$$

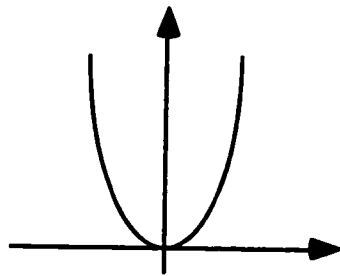
Finally

$$\frac{BC}{D'C} = \frac{BC}{DC}$$

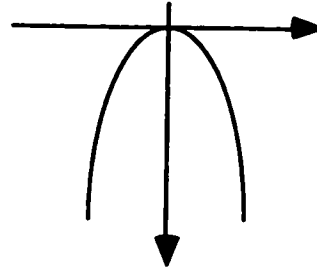
which immediately implies $D'C = DC$, that is D' and D are one and the same point. ■

Convex Function A function $f(x)$ is called *convex* on an interval $[a,b]$ if for any two points x_1 and x_2 in $[a,b]$,

$$f\left[\frac{1}{2}(x_1 + x_2)\right] \leq \frac{1}{2}[f(x_1) + f(x_2)].$$



convex



concave

Concave Function A function $f(x)$ is said to be *concave* on an interval $[a,b]$ if, for any points x_1 and x_2 in $[a,b]$, the function $-f(x)$ is *convex* on that interval.

Jensen's Inequality [40] For a Real Continuous Function

$$\frac{\sum f(x_i)}{n} \leq f\left(\frac{\sum x_i}{n}\right)$$

if f is *concave*,

$$\frac{\sum f(x_i)}{n} \geq f\left(\frac{\sum x_i}{n}\right)$$

if f is *convex*, and

$$\frac{\sum f(x_i)}{n} = f\left(\frac{\sum x_i}{n}\right)$$

if f is *linear*. A special case is

$$\sqrt[n]{x_1 x_2 \dots x_n} \leq \frac{x_1 + x_2 + \dots + x_n}{n},$$

with equality iff $x_1 = x_2 = \dots = x_n$.

The Arithmetic and Geometric mean Given a set of positive real numbers a_1, \dots, a_N , N a positive integer, the Arithmetic mean of the given

numbers is defined as:
$$\frac{(a_1 + \dots + a_N)}{N}$$

The Geometric mean of the given numbers is defined as:

$$(a_1 * \dots * a_N)^{1/N}$$

The two quantities are always related in the following way:

$$\frac{(a_1 + \dots + a_N)}{N} \geq (a_1 * \dots * a_N)^{1/N}$$

The Arithmetic-Geometric Mean inequality For positive a and b ,

$$\frac{(a+b)}{2} \geq \sqrt{(ab)} \text{ with equality iff } a = b.$$

Proof. Our result holds trivially if $a = b$. Assume then that $a \neq b$, and that $a > b$. Construct a right triangle with hypotenuse $\frac{(a+b)}{2}$ and one side equal to $\frac{(a-b)}{2}$ (See Figure 4.4). From the Pythagorean Theorem, the remaining side will equal $\sqrt{(ab)}$. However, in a right triangle, the hypotenuse is the largest side. Notice that the Pythagorean Theorem furnishes an intuitive geometric illustration by drawing two touching circles with radius a and b as in the following figure. ■

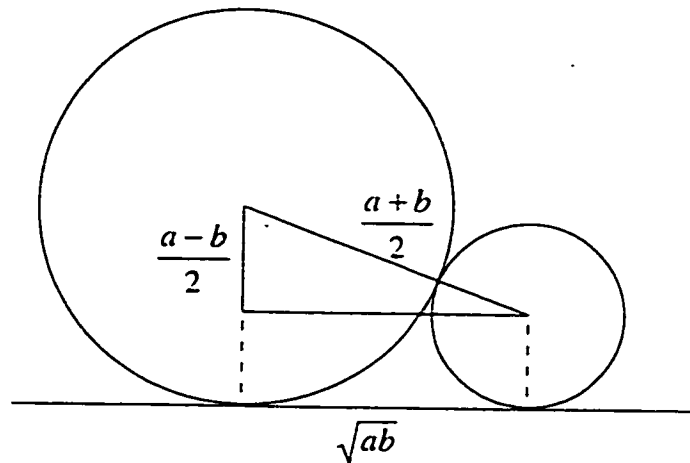


Figure 4.4: Arithmetic-Geometric Mean proof.

We now recall the following result:

Theorem 4.2 (Brocard[41]) *In any triangle ABC there is one and only one point Ω , such that $\angle BA\Omega = \angle CB\Omega = \angle AC\Omega = \omega$ and one point Ω' , such that $\angle AB\Omega' = \angle BC\Omega' = \angle CA\Omega' = \omega'$. These two points are called the Brocard points of the triangle and ω is called the Brocard angle. Moreover ω and ω' both satisfy: $\cot \omega = \cot \omega' = \cot A + \cot B + \cot C$.*

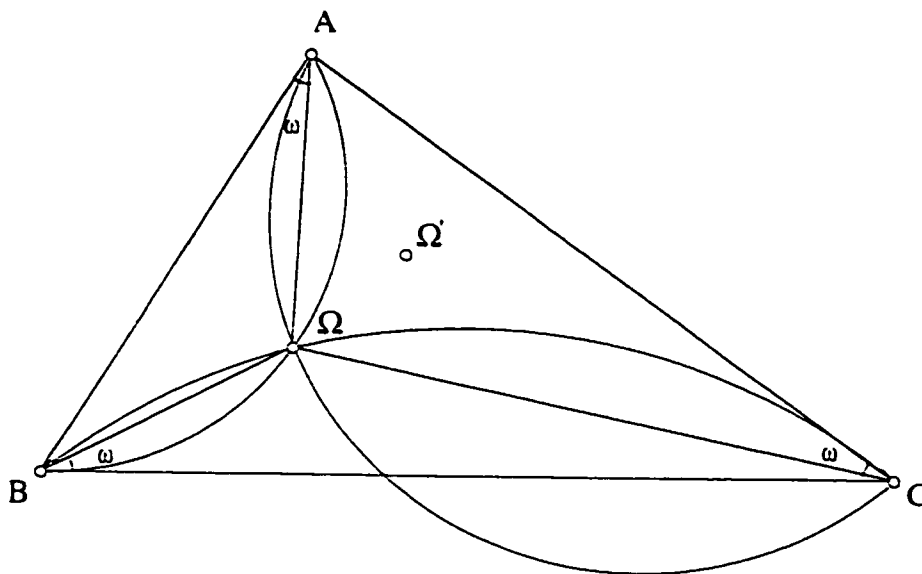


Figure 4.5: Brocard points and Brocard angle.

Proof. Given a triangle ABC , draw a circle tangent to BC at B passing through A , another tangent to CA at C passing through B and a third one tangent to AB at A passing through C . Let Ω be the second point of intersection between the first circle and the second one. (See Figure 4.6).

We have $\angle B\Omega C = 180^\circ - \angle C$, $\angle A\Omega B = 180^\circ - \angle B$. Combining these, we easily see that $\angle A\Omega C = 180^\circ - \angle A$ which shows that the third circle tangent to A passing through C also passes through Ω .

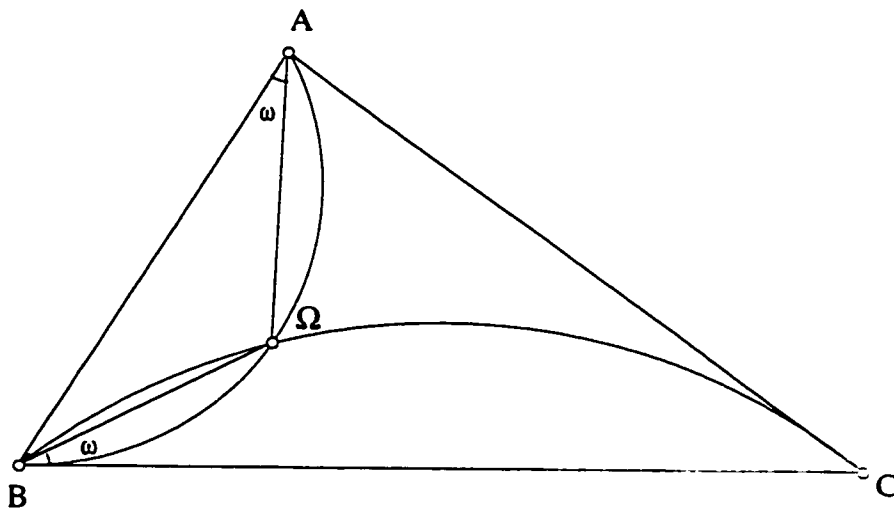


Figure 4.6: Two circles intersecting at Ω .

It now follows that $\angle BA\Omega = \angle CB\Omega = \angle AC\Omega$. (See Figure 4.7). Similarly, we can prove that the circle tangent to BC at C passing through A , the circle tangent to CA at A passing through B and the third circle tangent to AB at B passing through C , they all intersect at a point Ω' for which $\angle AB\Omega' = \angle BC\Omega' = \angle CA\Omega' = \omega'$. The points Ω and Ω' are the first and the second Brocard points of the triangle ABC .

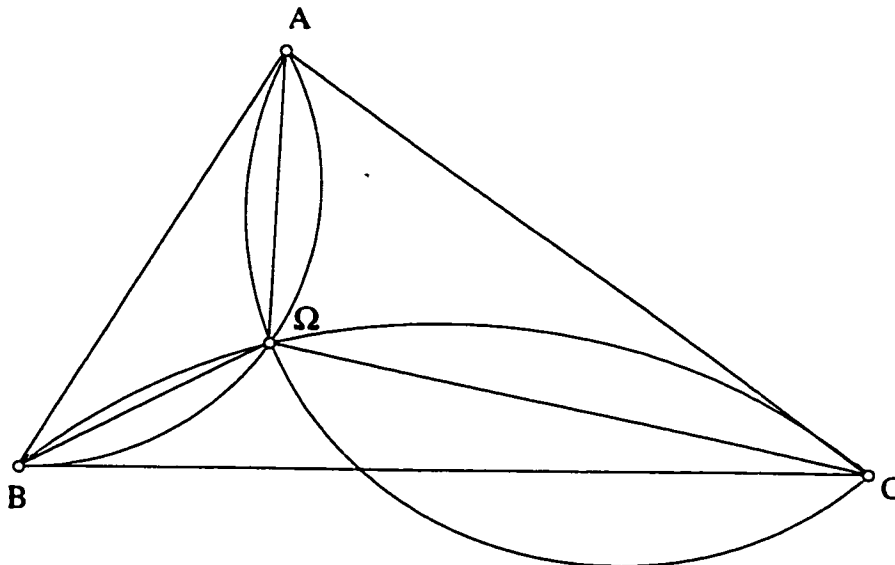


Figure 4.7: Brocard Triangle

We now prove the second part of Brocard's Theorem:

Extend the line segment $B\Omega$ to meet with the parallel to BC at D . Since $\angle\Omega CA = \angle\Omega BC$ and $\angle\Omega BC = \angle\Omega DA$ then $\angle\Omega CA = \angle\Omega DA$ and A, Ω, C, D are cocircular. (See Figure 4.8).

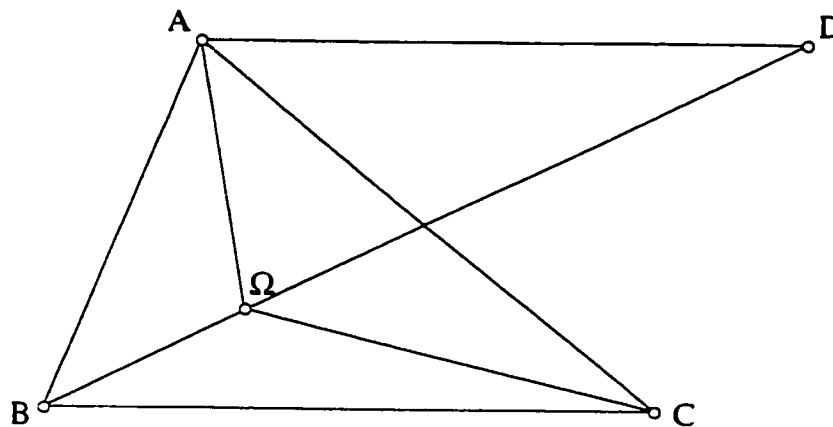


Figure 4.8: A, Ω, C, D are cocircular.

Denote $\angle CB\Omega = \omega$, we have that $\angle AC\Omega = \angle AD\Omega = \omega$. Furthermore $\angle C\Omega A = \angle B + \angle C$ and $\angle ADC = \angle A$ and $\angle DCA = \angle B$. Let E and F be the feet of the perpendiculars to AD from D and A respectively. Since $\angle C = \omega$, we have:

$$\cot \omega = \frac{BE}{ED} = \frac{BF}{FA} + \frac{FC}{FA} + \frac{CE}{ED} = \cot A + \cot B + \cot C. \quad \blacksquare$$

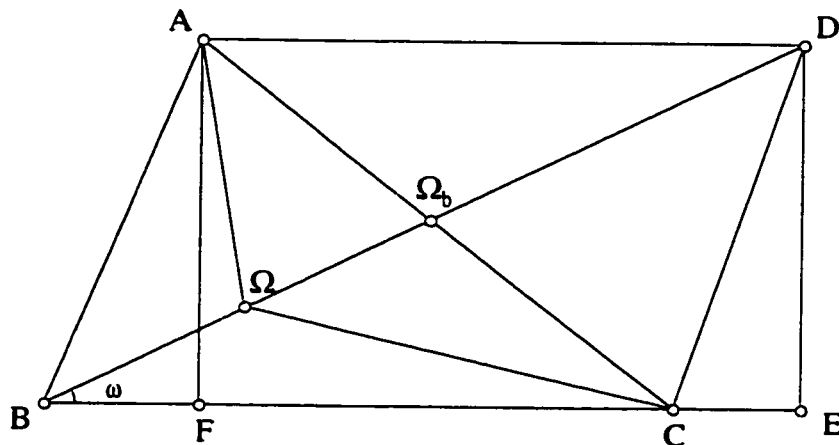


Figure 4.9: Proof of Brocard Theorem.

Isotomic Conjugate [47] If three lines from the vertices of a triangle, and concurrent at P , meet the opposite sides at P_1, P_2, P_3 respectively; and if we cut off A_2Q_1, A_3Q_2, A_1Q_3 equal respectively to P_1A_3, P_2A_1, P_3A_2 , then A_1Q_1, A_2Q_2, A_3Q_3 are concurrent at a point Q , called the *isotomic conjugate* of P .

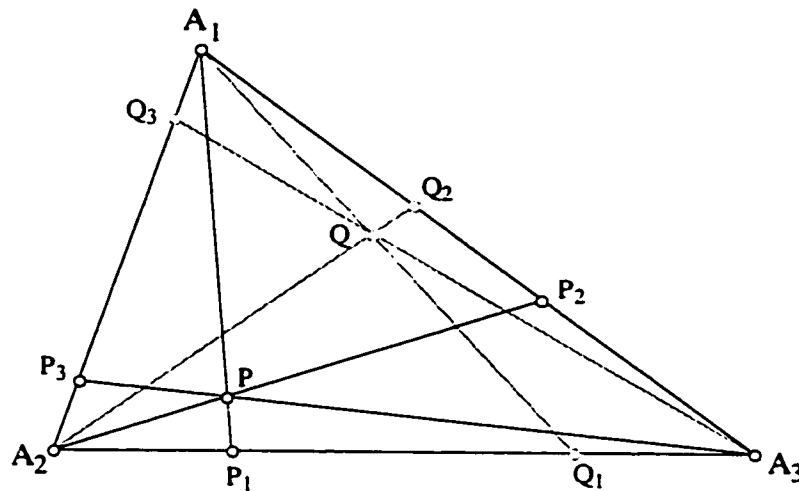


Figure 4.10: Isotomic Conjugate Q of a point P .

4.3 Result

We now prove the following result:

Theorem 4.3.1 *Every triangle can be illuminated by three $\frac{\pi}{6}$ -vertex lights.*

We will give three different proofs of Theorem 4.3.1.

First Proof. (Based on Brocard Theorem). Let Ω be the first Brocard point of the triangle ABC . Then by Brocard's Theorem :

$$\cot \omega = \frac{BE}{ED} = \frac{BF}{FA} + \frac{FC}{FA} + \frac{CE}{ED} = \cot A + \cot B + \cot C.$$

Let H be the feet of the perpendicular to BD from A and Ω_b the intersection point of $B\Omega$ and AC .

$$\frac{C\Omega_b}{\Omega_b A} = \frac{a \cdot \sin C B \Omega_b}{c \cdot \sin \Omega_b B A} = \frac{a^2}{b^2} \quad \text{where } BC = a; AB = c \text{ and } AC = b.$$

$$C\Omega_b + \Omega_b A = b \quad \text{and} \quad \frac{C\Omega_b}{\Omega_b A} = \frac{a^2}{b^2}$$

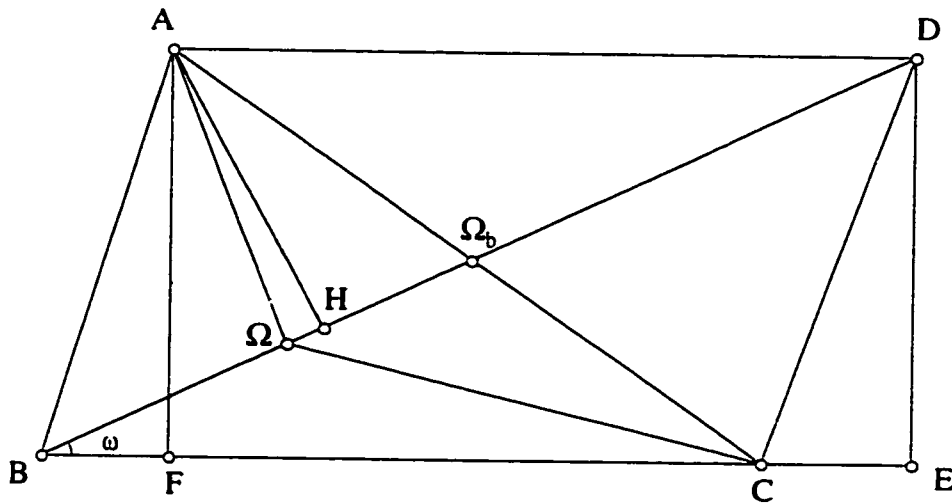


Figure 4.11: Maximum value of Brocard angle.

$$C\Omega_b = \frac{a^2 b}{a^2 + b^2} \quad ; \quad \Omega_b A = \frac{b^3}{a^2 + b^2}$$

$$\frac{\sin(C + \omega)}{\sin \omega} = \frac{a^2 + b^2}{ab} = \frac{a}{b} + \frac{a}{b} \geq 2.$$

As a result, $\sin \omega \leq \frac{1}{2}$ and the maximum value of $\omega \leq 30^\circ$, so now we place the vertex lights as shown in the following figure and the illumination of the triangle ABC is complete. ■

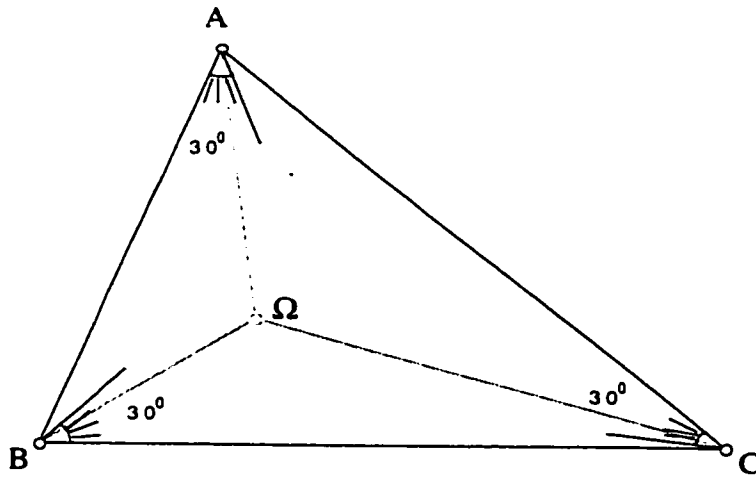


Figure 4.12: Illumination of triangle with three $\frac{\pi}{6}$ - vertex lights.

Second Proof. Let us consider the placing of three vertex lights, each $\frac{\pi}{6}$, in such a way as to illuminate the triangle ABC . Let θ be a small angle such that $\angle DAB = \angle FBC = \angle ECA = \theta$, as shown in Figure 4.13. By continuity, it is easy to see that incrementing θ in the clockwise direction makes the shaded triangle shown in Figure 4.13 converge to a point U , at which the lines AD , BF and CE are concurrent. Let $\angle UCA = \omega$.

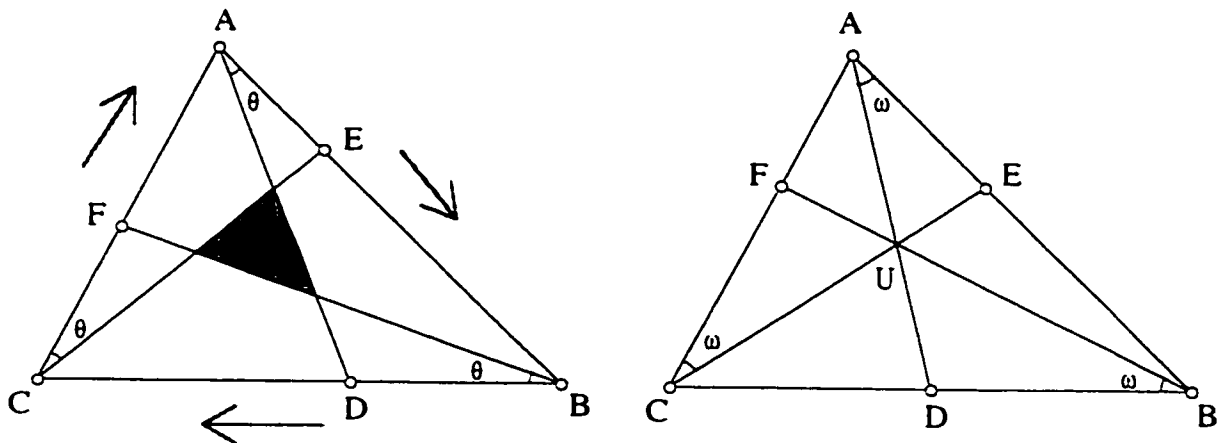


Figure 4.13: AD , BF and CE are concurrent at point U .

By Ceva's Theorem

$$\frac{CD}{DB} \times \frac{BE}{EA} \times \frac{AF}{FC} = 1$$

$$\frac{CD}{DB} = \frac{\text{area}\Delta CAD}{\text{area}\Delta DAB} = \frac{\frac{1}{2}(AD)(AC)\sin \angle CAD}{\frac{1}{2}(AD)(AB)\sin \angle DAB} = \frac{AC \sin \angle CAD}{AB \sin \angle DAB}$$

Similarly, $\frac{BE}{EA} = \frac{BC \sin \angle BCE}{AC \sin \angle ACE}$ and $\frac{AF}{FC} = \frac{AB \sin \angle ABF}{CB \sin \angle CBF}$

By multiplying

$$\frac{CD}{DB} \times \frac{BE}{EA} \times \frac{AF}{FC} = \frac{(AB)(BC)(AC)(\sin \angle CAD)(\sin \angle BCE)(\sin \angle ABF)}{(AB)(AC)(CB)(\sin \angle DAB)(\sin \angle ACE)(\sin \angle CBF)}$$

However, since by Ceva's theorem $\frac{CD}{DB} \times \frac{BE}{EA} \times \frac{AF}{FC} = 1$,

$$\frac{(\sin \angle CAD)(\sin \angle BCE)(\sin \angle ABF)}{(\sin \angle DAB)(\sin \angle ACE)(\sin \angle CBF)} = 1 \quad (\text{Trigonometric form of Ceva's$$

Theorem).

Let $\angle CAD = \alpha$, $\angle BCE = \beta$, and $\angle ABF = \gamma$, similarly

since $\angle DAB = \angle ACE = \angle CBF = \omega$ then

$$\frac{\sin \alpha \sin \beta \sin \gamma}{\sin \omega \sin \omega \sin \omega} = 1.$$

Hence $\sin^3 \omega = \sin \alpha \times \sin \beta \times \sin \gamma$

$\sin \omega = \sqrt[3]{\sin \alpha \times \sin \beta \times \sin \gamma}$ which is less than or equal to

$\frac{\sin \alpha + \sin \beta + \sin \gamma}{3}$ by the Arithmetic-Geometric mean.

By Jensen's inequality

$$\frac{\sin\alpha + \sin\beta + \sin\gamma}{3} \leq \sin\frac{\alpha + \beta + \gamma}{3}$$

Then

$$\sin\omega \leq \sin\frac{\alpha + \beta + \gamma}{3}$$

which implies that

$$\omega \leq \frac{\alpha + \beta + \gamma}{3}$$

and since $3\omega + \alpha + \beta + \gamma = \pi$, we have

$$\omega + \frac{\alpha + \beta + \gamma}{3} = \frac{\pi}{3}$$

Therefore

$$\omega \leq \frac{\alpha + \beta + \gamma}{3} \Rightarrow \omega \leq \frac{\pi}{6}$$

Now we place the three $\frac{\pi}{6}$ - vertex lights as shown in the following figure so that the interior of the triangle is completely illuminated.(See Figure 4.14).■

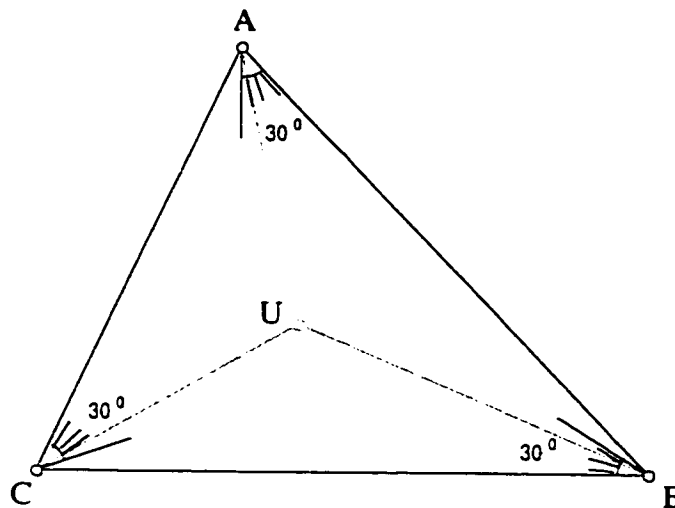


Figure 4.14: Illuminating the triangle with three $\frac{\pi}{6}$ - vertex lights.

Third Proof. Consider the triangle ABC , and let points A' , B' , C' be marked off some fixed distance x along each of the sides BC , CA , and AB : Let the length of the sides BC , CA , and AB be a , b and c respectively. Then the lines AA' , BB' , and CC' concur in a point U if :

$$x^3 = (a-x)(b-x)(c-x)$$

This equation has a single real root u , which can be obtained by solving the cubic equation:

$$f(x) = 2x^3 - px^2 + qx - r = 0,$$

Where

$$p = a + b + c,$$

$$q = ab + ac + bc,$$

$$r = abc.$$

if U' is the isotomic conjugate of U , the lines AU' , BU' and CU' meet the opposite sides at A'' , B'' and C'' respectively in such a way that $AB'' = BC'' = CA'' = x$. This follows readily from Ceva's Theorem.

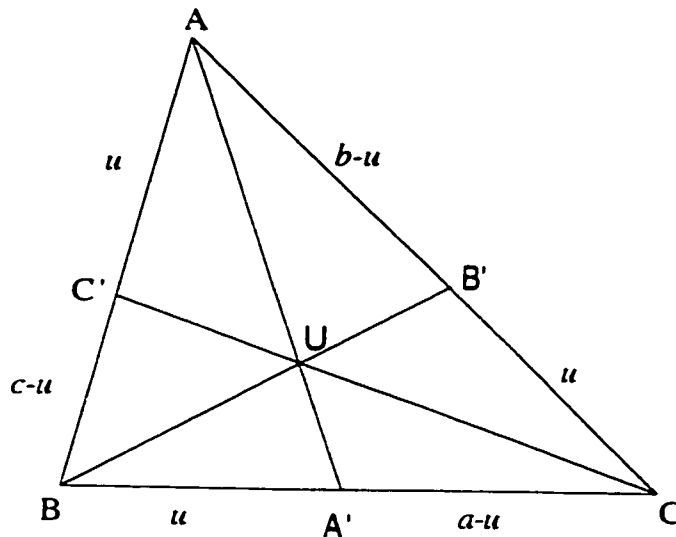


Figure 4.15: AA' , BB' , and CC' concur in a point U .

Now we have to show that $u \leq p/6$ is true. To prove this, consider:

$$f\left(\frac{p}{6}\right) = \frac{p^3}{108} - \frac{p^3}{36} + \frac{qp}{6} - r,$$

in which the last two terms constitute a function of two independent variables (e.g. a and b) if p is fixed. This function is found by ordinary methods to have $p/54$ as its minimum value. Hence $f(p/6) \geq 0$, and $u \leq p/6$.

Equality holds only in an equilateral triangle. ■

4.4 Conclusion

We close this chapter with another problem related to the vertex-light illumination of a triangle, the problem of finding a minimal angle α such that any three vertex lights whose apertures sum to α will illuminate any triangle. We have the following conjecture:

Conjecture 4.4.1 *Three vertex lights whose total angle is at least equal to $\frac{2\pi}{3}$ suffice to illuminate any triangle.*

It is easy to see that $\frac{2\pi}{3}$ is also the lower bound for this problem.

Chapter 5

Illuminating Quadrilaterals by Vertex Lights

5.1 Introduction

In this chapter, we study the quadrilateral illumination problem, that is the problem of finding an illumination for every quadrilateral with a minimum number of $\frac{\pi}{4}$ -vertex lights.

It has been proved that no more than four $\frac{\pi}{4}$ -vertex lights are needed to illuminate every quadrilateral [62] [44]. In this chapter, we prove that three $\frac{\pi}{4}$ -vertex lights suffice.

5.2 Result

Some definitions and general lemmas will be given now which will be useful to prove our result.

Let C_1 be a circular disk of radius 1, AB a horizontal diameter in C_1 , and let RS the perpendicular diameter to AB . Let B_1, B_2 be points on the arc AS and AR respectively. Denote by N the point of intersection between RS and B_1B_2 . We now prove:

Lemma 5.1 *If $\angle B_1BB_2 = \frac{\pi}{4}$, then $\angle B_2NR \leq \frac{\pi}{4}$.*

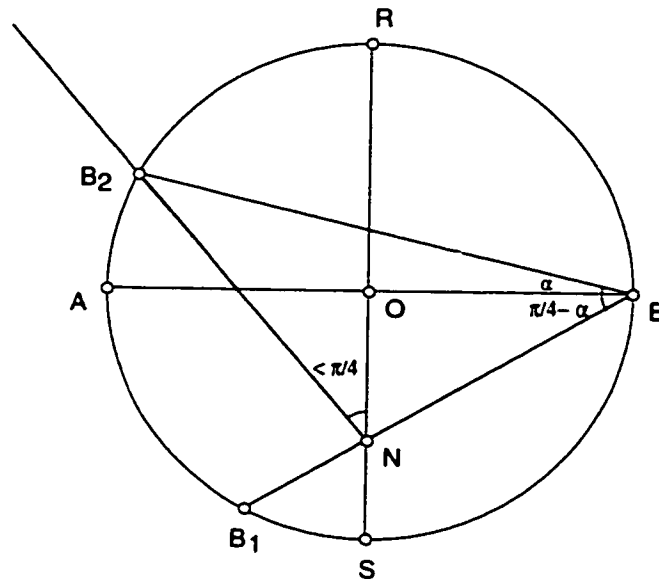


Figure 5.1: $\angle B_1BB_2 = \frac{\pi}{4}$ and $\angle B_2NR \leq \frac{\pi}{4}$.

Proof. We have $\angle B_1BB_2 = \frac{\pi}{4}$, now let $\angle B_1BA = \frac{\pi}{4} - \alpha$ and $\angle ABB_2 = \alpha$. It is easy to see that the triangle $\triangle BON$ is right-angled, therefore $\angle BON = \frac{\pi}{2}$ and $\angle BNO = \frac{\pi}{4} + \alpha$. Let A_1 be a point on the circle C symmetric to B_1 relative to line segment NS . It follows that $\angle A_1NS = \angle SNB_1$. Since $\angle B_2RS = \angle B_2RA + \angle ARS = \frac{\pi}{4} + \alpha$, it is easy to see that $RB_2 \parallel BN$ and $\angle BB_2N = \frac{\pi}{4} - \alpha + \beta$ where $\angle A_1NB_3 = \beta$. It follows that $\angle B_2NR = \frac{\pi}{4} - \beta$, and we are done. ■

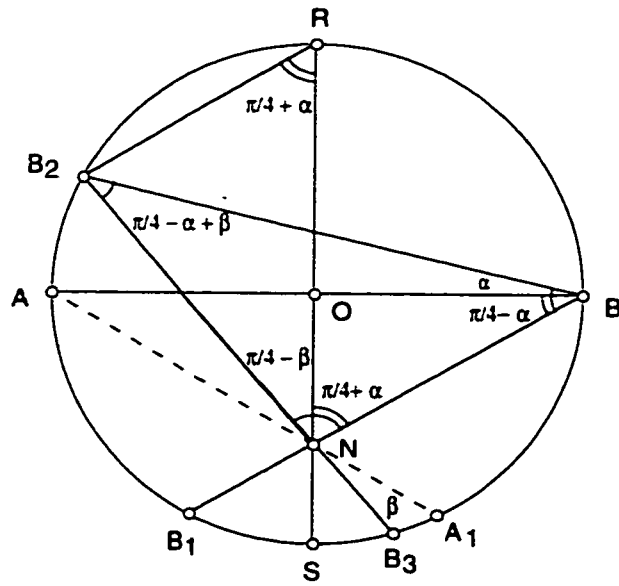


Figure 5.2: Proof of Lemma 5.1.

Lemma 5.2 Let D be a point which lies on line segment NB_1 . If $\angle RNB_2 \leq \frac{\pi}{4}$ then it follows that $\angle RDB_2 \leq \angle RNB_2 \leq \frac{\pi}{4}$.

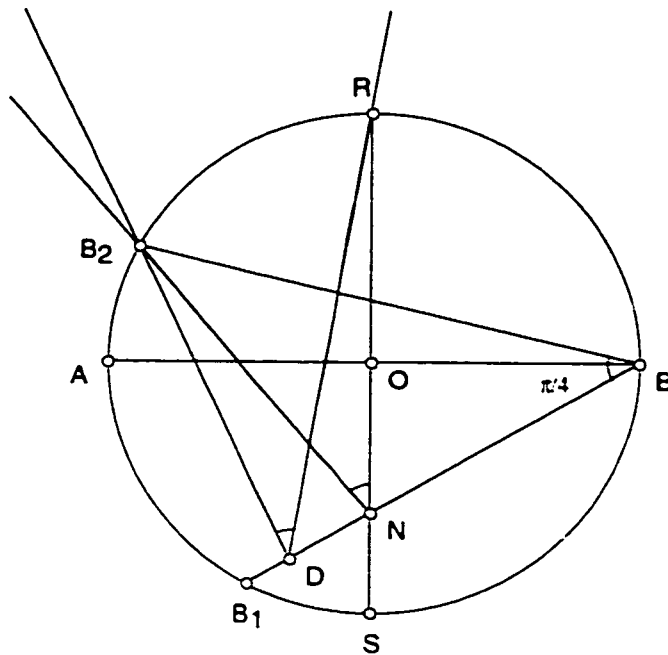


Figure 5.3: $D \in NB_1, \angle RDB_2 \leq \angle RNB_2$.

Proof. Consider a circle C_2 passing through R , N and B_2 ; we have to prove that the line segment NB_1 lies outside the circle C_2 , so that any point on that segment lies outside the circle C_2 .

In the proof of Lemma 5.1, we have showed that $BN \parallel RB_2$. Now let H be the mid-point of B_2R . The perpendicular to B_2R through H passes through the center of C_1 and C_2 and cuts the circle C_2 at K so HK is perpendicular to BN because $BN \parallel RB_2$. Let T be the tangent to C_2 at point K . Then T is perpendicular to HK and $T \parallel BN$. Therefore BN has to intersect the boundary of C_2 in two points in the bottom right quadrant which leaves the line segment NB_1 lies outside the circle C_2 . Since $\angle RNB_2 \leq \frac{\pi}{4}$, it follows that $\angle RDB_2 \leq \angle RNB_2 \leq \frac{\pi}{4}$ ■

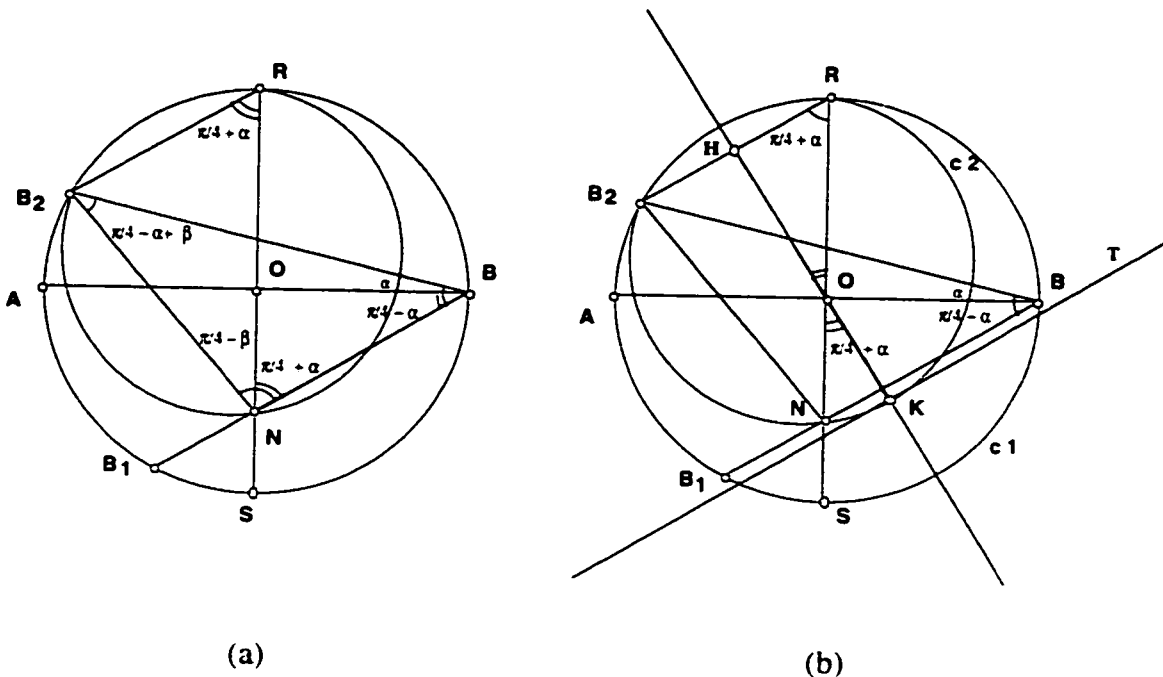


Figure 5.4: Proof of Lemma 5.2.

Theorem 5.1 Every convex quadrilateral can be illuminated by three $\frac{\pi}{4}$ -vertex lights.

Proof. Let P be a convex quadrilateral and let C_1 be the closed circular disk of minimal radius that contains P . We will analyze three cases:

Case1: $|C_1 \cap P| = \{2\}$.

$|C_1 \cap P| = \{A, B\}$ where A and B are antipodal points on the boundary of C_1 .

- If AB is an edge of P , then two $\frac{\pi}{4}$ -vertex lights placed at A and B suffice to illuminate P . (See Figure 5.5).

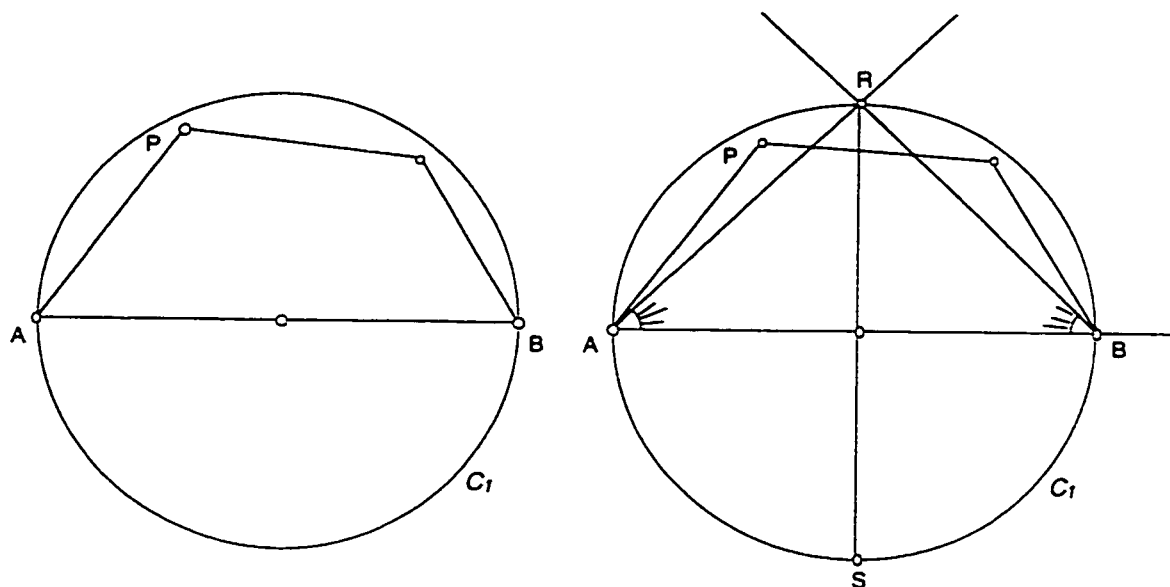


Figure 5.5: Two $\frac{\pi}{4}$ -vertex lights placed at A and B illuminate P .

- If AB is not an edge of P then there are 2 subcases to consider:

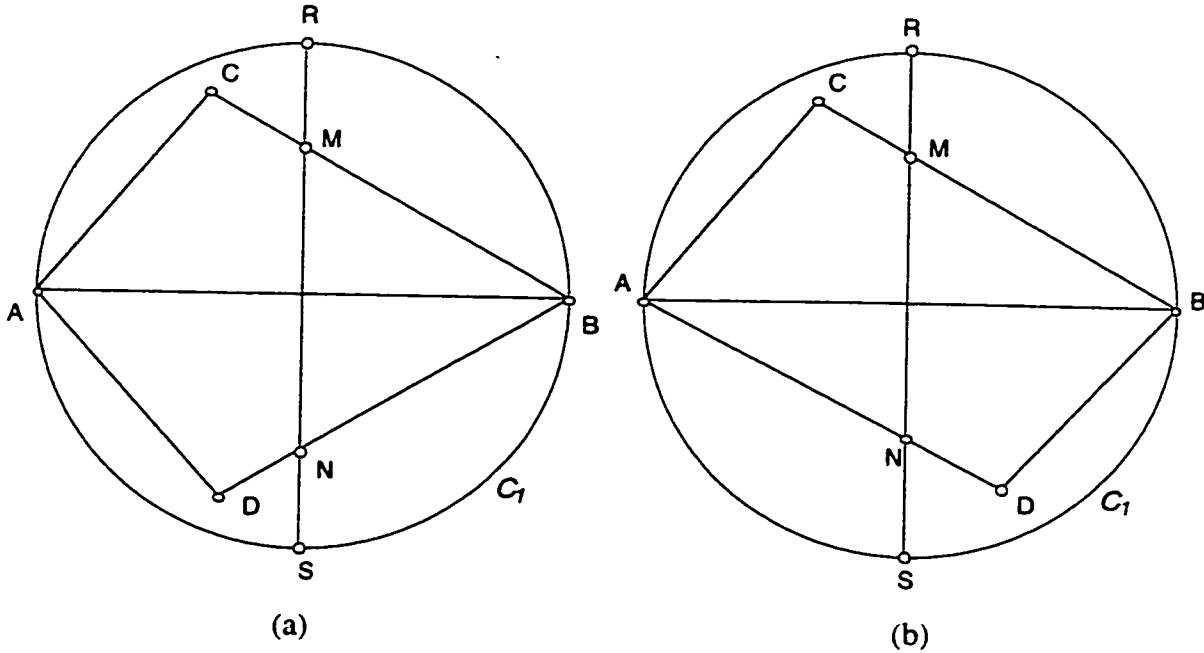


Figure 5.6: AB is not an edge of P .

Subcase 1.a: $AC \leq BC$
 $AD \leq BD$

Let RS be the perpendicular diameter on AB . Denote the points where RS intersects P by M and N . (See Figure 5.7a). Denote by B_1 the point where BN cuts the boundary of C_1 . (See Figure 5.7b).

Let T be the point of intersection of BB_1 and AR . In order to illuminate P , we place a $\frac{\pi}{4}$ -vertex light on B so that $\angle B_1BB_2 = \frac{\pi}{4}$ and we place another $\frac{\pi}{4}$ -vertex light on A so that $\angle RAB = \frac{\pi}{4}$.

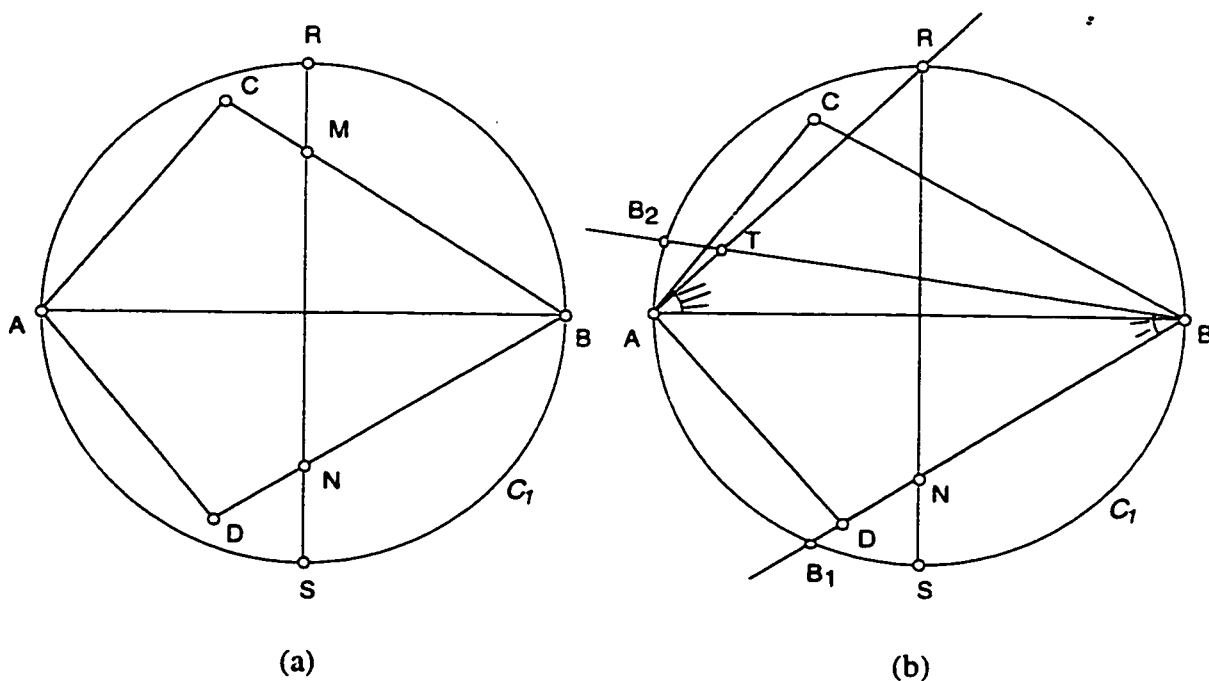


Figure 5.7: $AC \leq BC$ and $AD \leq BD$.

The circular triangle B_2TR remains not illuminated. Now it is easy to see that it would be sufficient to illuminate the circular triangle B_2TR in order to illuminate all of $\text{Int}(P)$.

Remember that $N \in P$. It is easy to see that $\angle RNB_2 \leq \frac{\pi}{4}$. If N is a vertex of P , then $\angle RNB_2$ illuminates the circular triangle B_2TR . (See Figure 5.8a). If N is not a vertex of P , it follows that there is an edge of P that contains N . Let BD be this edge. Since $\angle RDB_2 < \frac{\pi}{4}$ from Lemma 5.2, it follows that we can illuminate the circular triangle B_2TR with a vertex light of measure less than $\frac{\pi}{4}$ placed at D . (See Figure 5.8b).

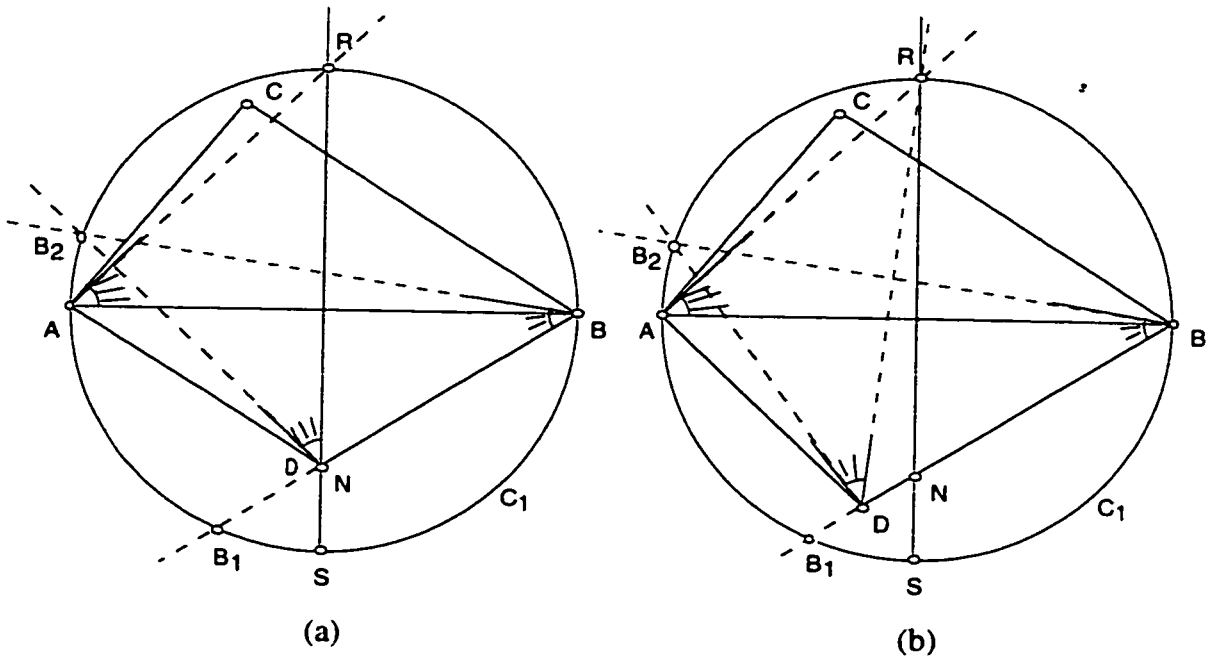


Figure 5.8: (a) N is a vertex of P , (b) N is not a vertex of P .

Subcase 1.b: $AC \leq BC$
 $AD \geq BD$

In this case, two vertex lights suffice to illuminate all of $\text{Int}(P)$. We place the first $\frac{\pi}{4}$ -vertex light on B so that $\angle RBA = \frac{\pi}{4}$, then we place the second $\frac{\pi}{4}$ -vertex light on A so that $\angle BAS = \frac{\pi}{4}$. (See Figure 5.9).

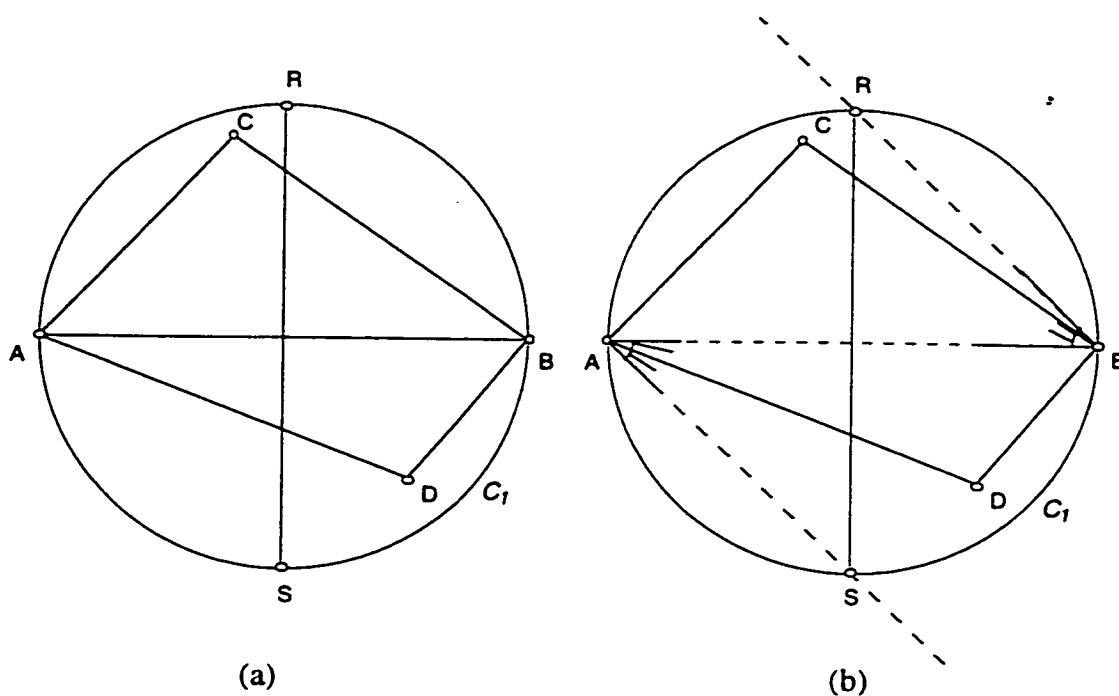


Figure 5.9: Two $\frac{\pi}{4}$ -vertex lights placed on A and B suffice to illuminate P .

Case2: $|C_1 \cap P| = \{3\}$

$|C_1 \cap P| = \{A, B, C\}$ where A, B and C are points on the boundary of C_1 .

Denote the measures of the circular arcs AC, BC and AB by $2x, 2y$ and $2z$ respectively. Without loss of generality, we can assume that $x \leq y \leq z$.

Clearly $z < \frac{\pi}{2}$.

Since $x + y + z = \pi$ it follows that $z \geq \pi/3$. Assume that AB is horizontal and that C belongs to the arc AR . (See Figure 5.10).

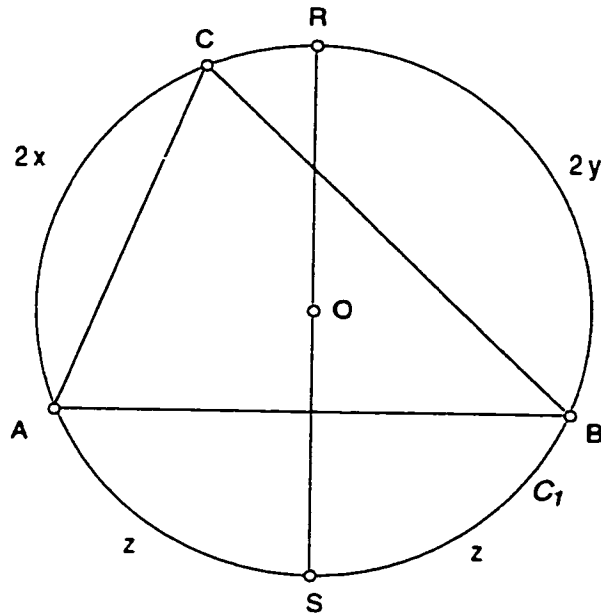


Figure 5.10: Arcs $2x$, $2y$ and $2z$.

We distinguish two subcases:

Subcase 2a: AB is an edge of the quadrilateral.

It will be sufficient to prove that the sector of C_1 above the horizontal through AB can be illuminated with three $\frac{\pi}{4}$ -vertex lights placed at points A , B and C . Let us consider the system of coordinates with origin O (the center of the circle C_1) and $yy' = RS$. Also denote the horizontal diameter by PQ . Obviously, $\angle AOP = \angle QOB = \frac{\pi}{2} - z$.

Consider the point B' on the arc PQ so that $\angle ABB' = \frac{\pi}{4}$; now it is easy to see that $\angle B'OR = \angle QOB = \frac{\pi}{2} - z$. Then take A' the antipodal point of A .

It can be easily seen that $\angle BAA' = \frac{\pi}{4}$ and $\angle A'OQ = \angle QOB$ because A' is symmetric to B relative to OQ .

Let U be the point of intersection of AA' and BB' . We place the first $\frac{\pi}{4}$ -vertex light on B , then the second $\frac{\pi}{4}$ -vertex light on A . What remains not illuminated is the circular triangle $UA'B$. (See Figure 5.11).

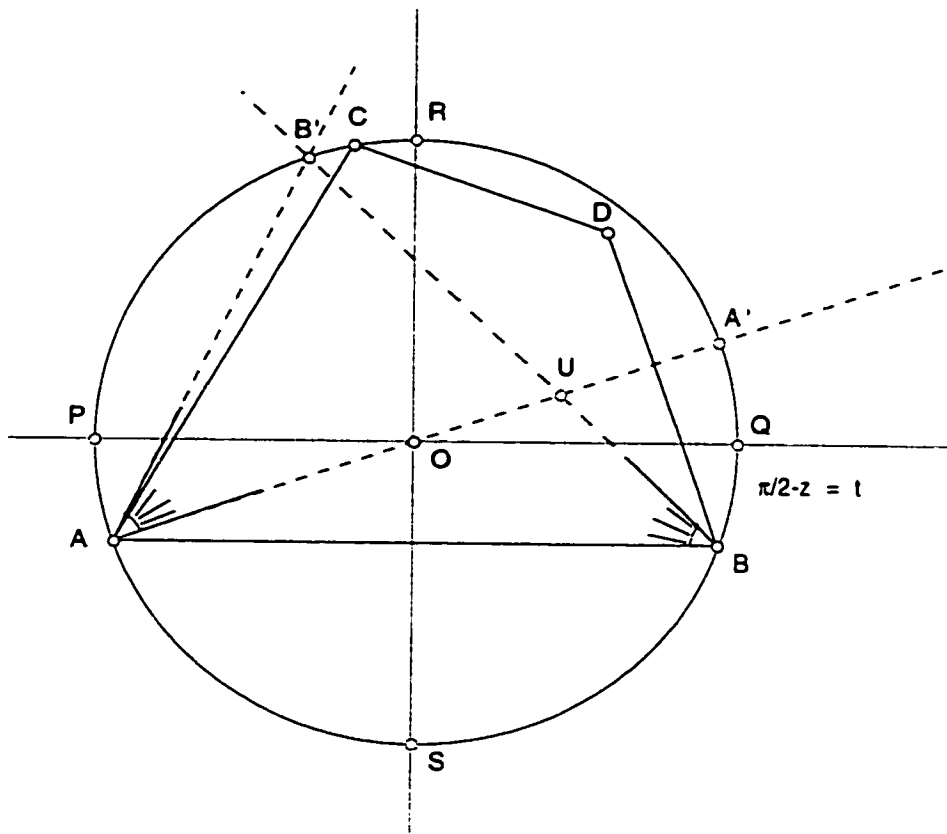


Figure 5.11: Two $\frac{\pi}{4}$ -vertex lights placed at A and B .

Recall that C belongs to the arc AR . If $C \in \text{arc } AB'$ then $\angle A'CB$ contains the circular triangle $UA'B$ and its measure is $\frac{\pi}{2} - z$. But since $z \geq \frac{\pi}{3}$, it

follows that $\angle A'CB \leq \frac{\pi}{6} < \frac{\pi}{4}$, so the illumination is complete. (See Figure 5.12).

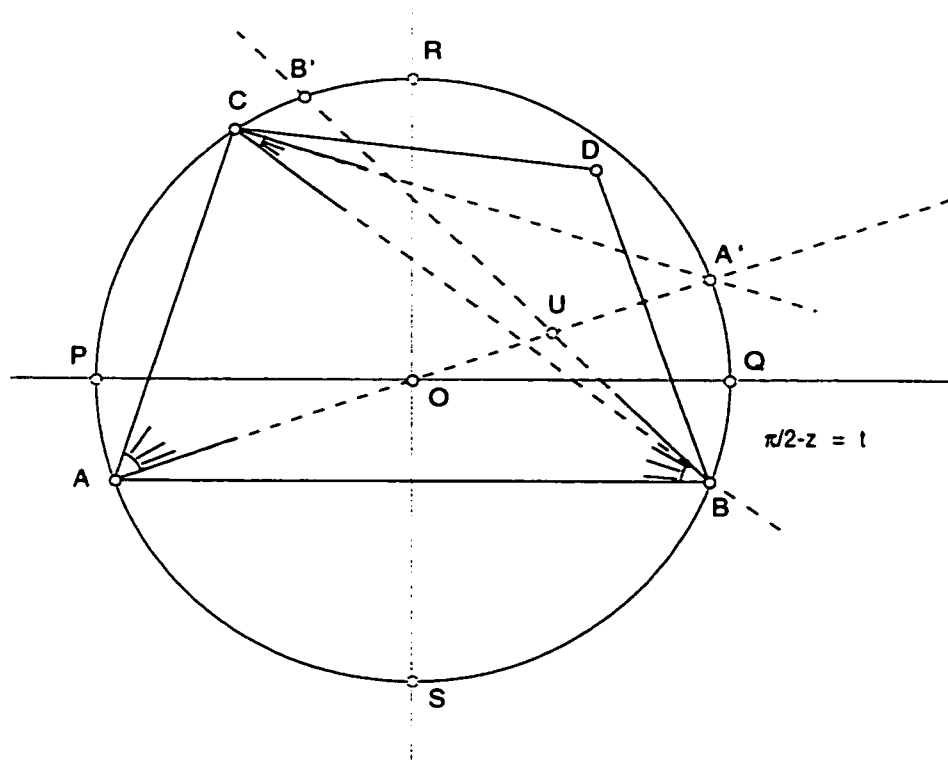


Figure 5.12: Vertex light of size less than $\frac{\pi}{6}$ placed at C illuminates the circular triangle $UA'B$.

If $C \in \text{arc } B'R$, it is easy to see that $A'CU \leq A'RU$, therefore it would be sufficient to prove that $A'RU \leq \frac{\pi}{4}$ in order to finish Subcase 2.a. Recall that the radius of C is 1 and let $\frac{\pi}{2} - z = t, t \in (0, \frac{\pi}{6}]$.

The points will have the coordinates shown in Figure 5.13. Some straightforward calculations give the coordinates of the point U .

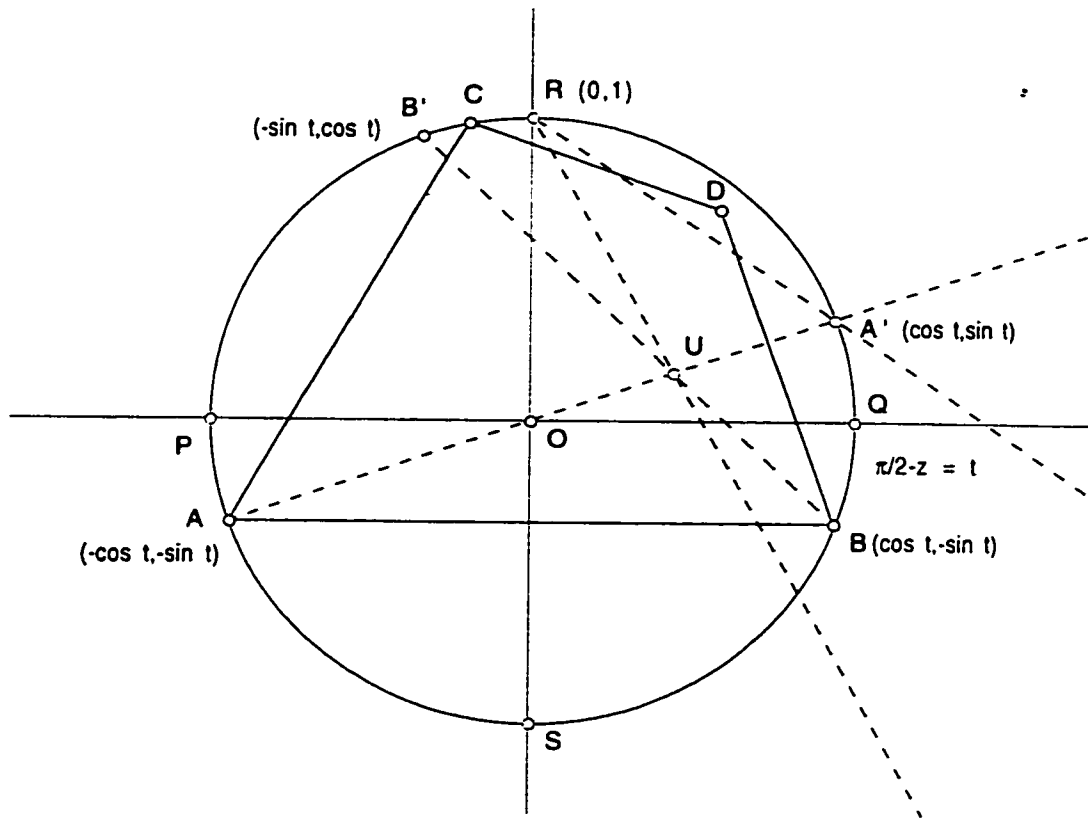


Figure 5.13: Coordinates of points A , A' , B , B' and R .

The coordinates of the point U are:

$$x_u = \frac{(\cos^2 t - \sin t \cos t)}{(\sin t + \cos t)}$$

$$y_u = \frac{(\sin t \cos t - \sin^2 t)}{(\sin t + \cos t)}$$

If m = the slope of AR and m' = the slope of RU it is known that:

$$\tan(\angle A'RU) = \frac{(m - m')}{1 + m \times m'}$$

$$m = \frac{(\sin t - 1)}{\cos t}$$

Subcase 2b: AB is not an edge of the quadrilateral:

There are two subcases to consider; the first when $AD \leq BD$, and the second when $AD \geq BD$ (See Figure 5.15).

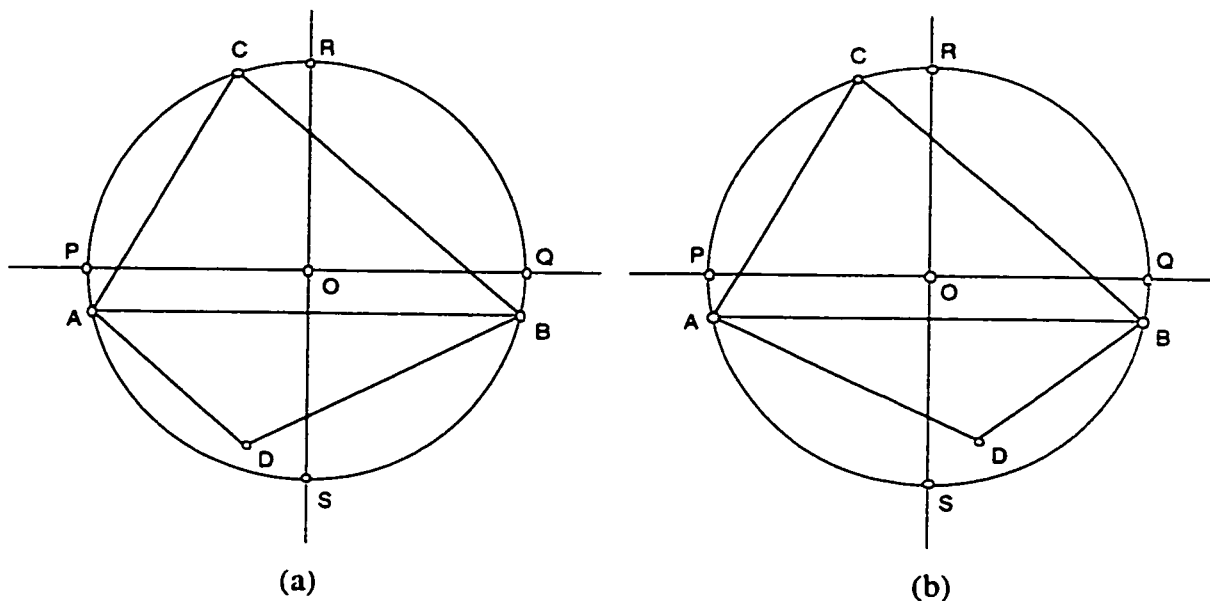


Figure 5.15: (a) $AD \leq BD$, (b) $AD \geq BD$

Subcase 2b.1: $AD \leq BD$

Let $\angle QOB = \alpha$ and $\angle ROC = \beta$; now we have the following 2 subcases:

1) $\alpha < \beta$

Let $\angle CBB_1 = \frac{\pi}{4}$. We place a $\frac{\pi}{4}$ - vertex light at B and the circular triangle BCB_1 is illuminated. Notice that $B_1 \in \text{arc } AS$ because $\alpha < \beta$. Let $\angle ACC_2 = \frac{\pi}{4}$; we place another $\frac{\pi}{4}$ - vertex light at C to illuminate the circular triangle CAC_2 . It is easy to see that $C_2 \in \text{arc } SB$, what is now left not illuminated is the circular area in the right bottom quadrant of the circle C_1 , but we have $\angle SAN = \frac{\pi}{4}$, so we can place a third $\frac{\pi}{4}$ - vertex light at A and the illumination is complete. (See Figure 5.16).

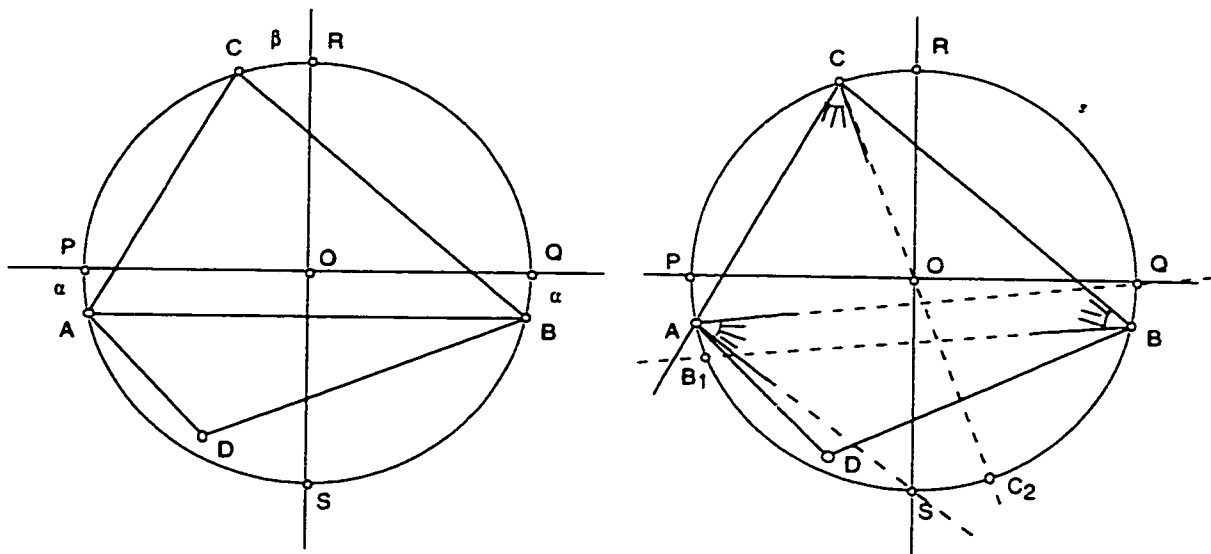


Figure 5.16: $AD \leq BD$, $\angle QOB = \alpha < \angle ROC = \beta$

2) $\alpha > \beta$

$\angle CBB_1 = \frac{\pi}{4}$; we place a $\frac{\pi}{4}$ - vertex light at B . $\angle ACC_2 = \frac{\pi}{4}$; we place another $\frac{\pi}{4}$ - vertex light at C and finally $\angle C_2AA_1 = \frac{\pi}{4}$; we place the third $\frac{\pi}{4}$ - vertex light at A and the illumination is complete. (See Figure 5.17).

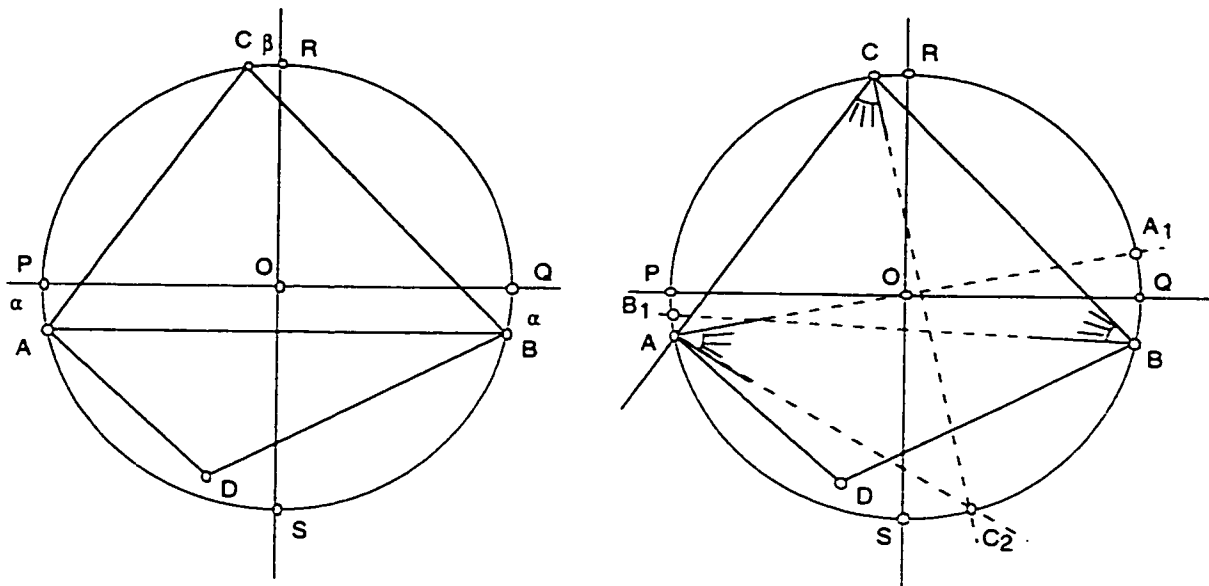


Figure 5.17: $AD \leq BD$, $\angle ROB = \alpha > \angle ROC = \beta$.

Subcase 2b.2: $AD \geq BD$

$\angle NBB_1 = \frac{\pi}{4}$; we place a vertex light at B .

$\angle RNB_1 \leq \frac{\pi}{4}$; (from Lemma 5.1) and $\angle RDB_1 \leq \frac{\pi}{4}$ (from Lemma 5.2); we

place a $\frac{\pi}{4}$ - vertex light at D to illuminate arc B_1R . $\angle SCQ = \frac{\pi}{4}$; we place

another $\frac{\pi}{4}$ vertex light at C to illuminate arc SQ and the illumination is complete.

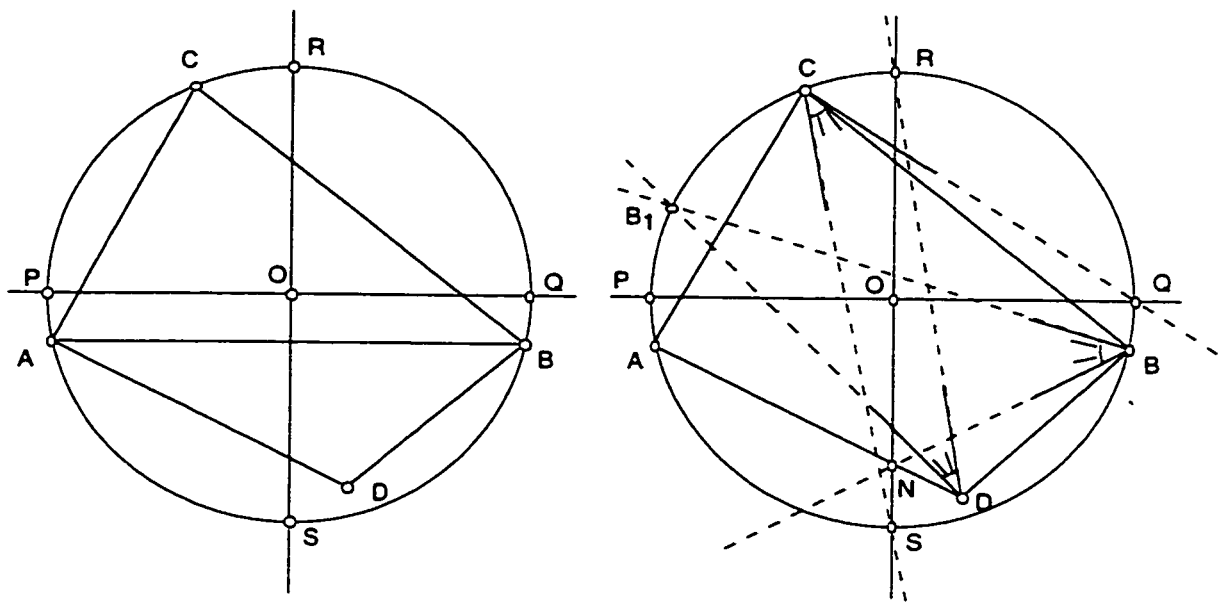


Figure 5.18: Subcase 2b.2 where $AD \geq BD$.

Case 3: $|C_1 \cap P| = \{4\}$

$|C_1 \cap P| = \{A, B, C, D\}$. All the points lie on the boundary of C_1 .

There are two cases to consider:

Subcase 3a: AB is an edge of the quadrilateral:

This case is already solved in **Case 2 Subcase 2.a.** the only difference is that D lies on the circle.

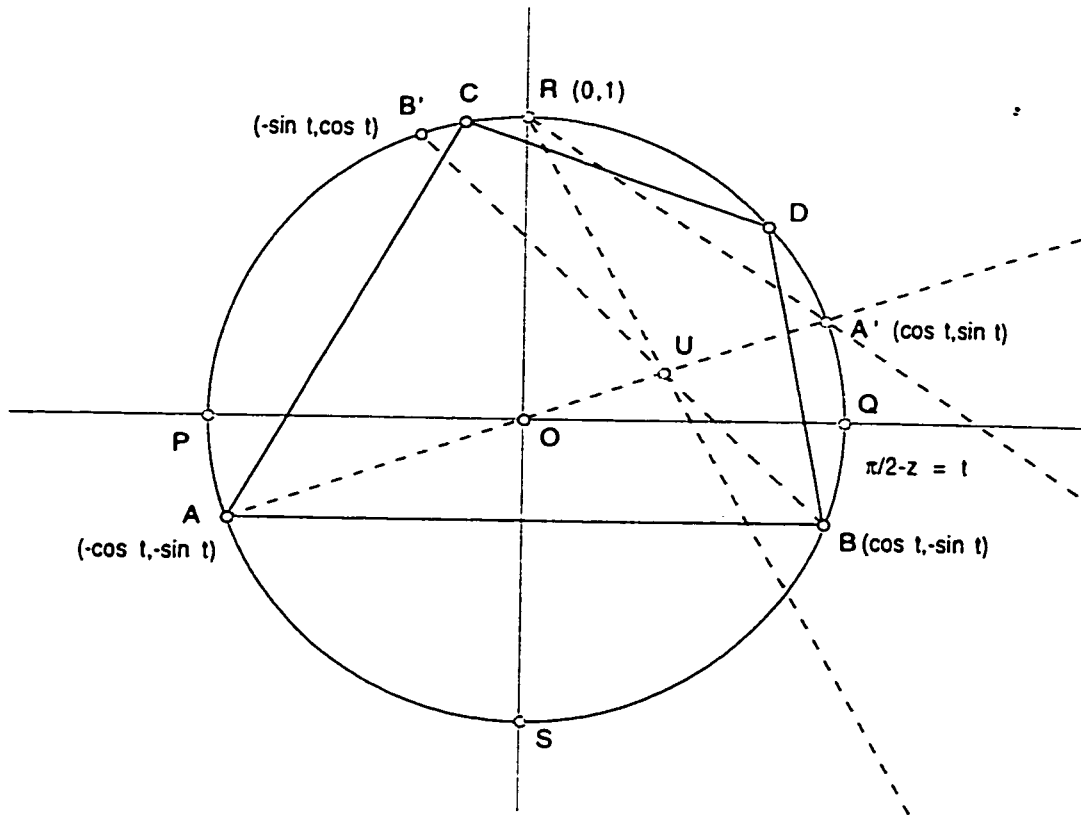


Figure 5.19: A, B, C and D all lies on the circle.

Subcase 3b: AB is not an edge of the quadrilateral:

There are two subcases to consider:

Subcase 3b.1: $AC \leq BC$

$$AD \geq BD$$

In this case we have the following:

$\angle MBP = \frac{\pi}{4}$: we place a $\frac{\pi}{4}$ - vertex light at B to illuminate the circular triangle ABM . Notice that $\angle PDR = \frac{\pi}{4}$: we then place a second $\frac{\pi}{4}$ - vertex light at D to illuminate the circular triangle PDR . Now because $C \in \text{arc}PR$,

$\angle QCS = \frac{\pi}{4}$, so in order to illuminate the circular triangle QCS , we place the third $\frac{\pi}{4}$ - vertex light at C and we are done.

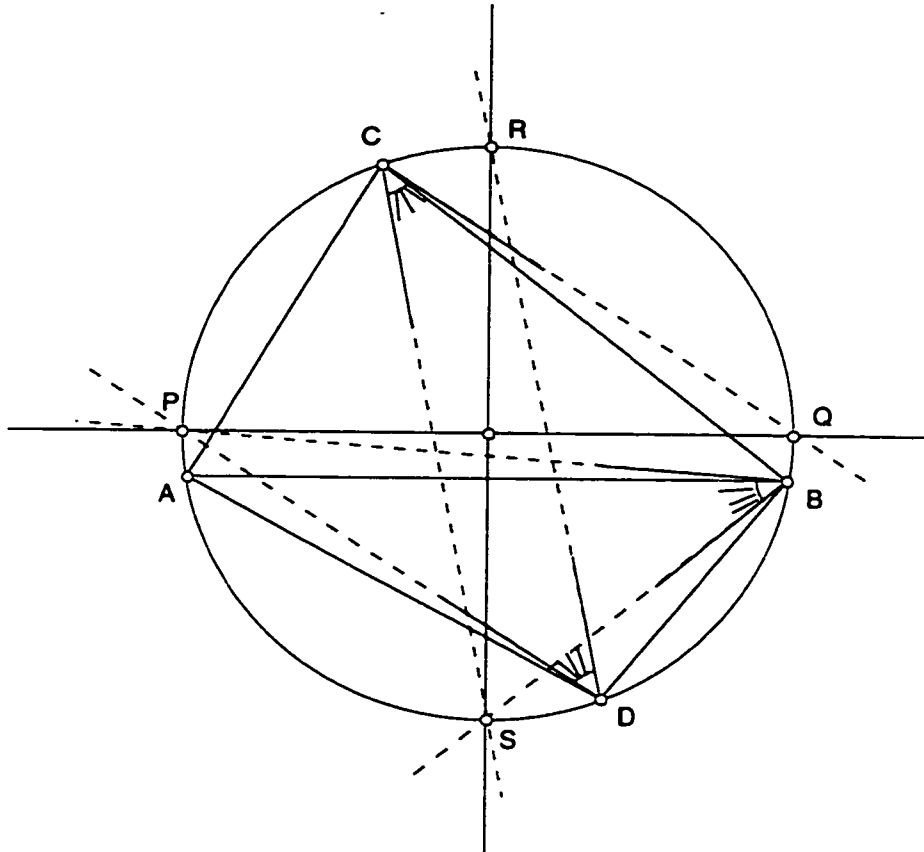


Figure 5.20: $AC \leq BC; AD \leq BD$ and $CR < DS$.

Subcase 3b.2: $AC \leq BC$
 $AD \leq BD$

In this case we have two subcases to consider:

Subcase 3b.2.1: $\text{arc}CR < \text{arc}DS$

We place a vertex light on B as shown to illuminate the circular triangle DBB' , $\angle DBB' = \frac{\pi}{4}$. BB' eats the center of the circle because $DM = AB'$, then we place the second $\frac{\pi}{4}$ - vertex light on D such that $\angle B'DD' = \frac{\pi}{4}$, so

DD' passes by the center of the circle because $\text{arc}DM = \text{arc}D'M$. Then the third $\frac{\pi}{4}$ -vertex light is placed at C to illuminate the circular triangle QCM and we are done.

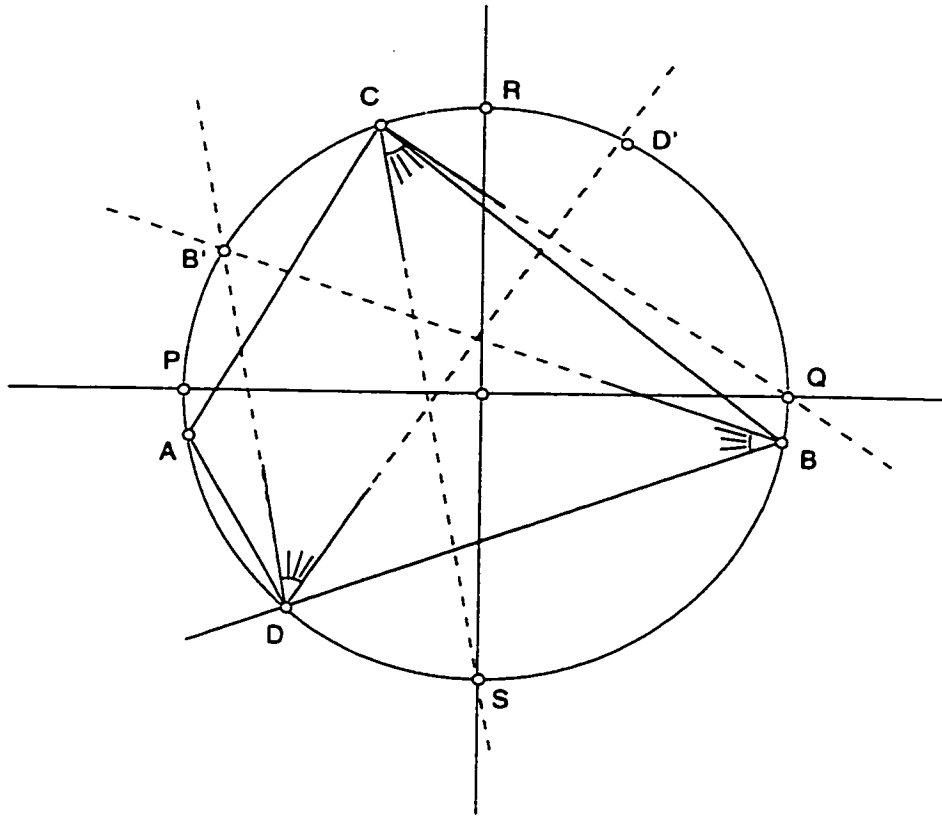


Figure 5.21: $AC \leq BC$; $AD \leq BD$ and $CR < DS$.

Subcase 3b.2.2: $\text{arc}CR > \text{arc}DS$

We place a vertex light on B as shown to illuminate the circular triangle CBB'' , $\angle CBB' = \frac{\pi}{4}$, BB' passes the center of the circle because $RC = AB'$, then we place the second vertex light on C such that $\angle B'CC' = \frac{\pi}{4}$, so CC' passes by the center of the circle because $\text{arc}RC = \text{arc}SC'$. Then the third vertex light is placed at A to illuminate the circular triangle QAS and we are done. This completes the proof of the theorem. ■

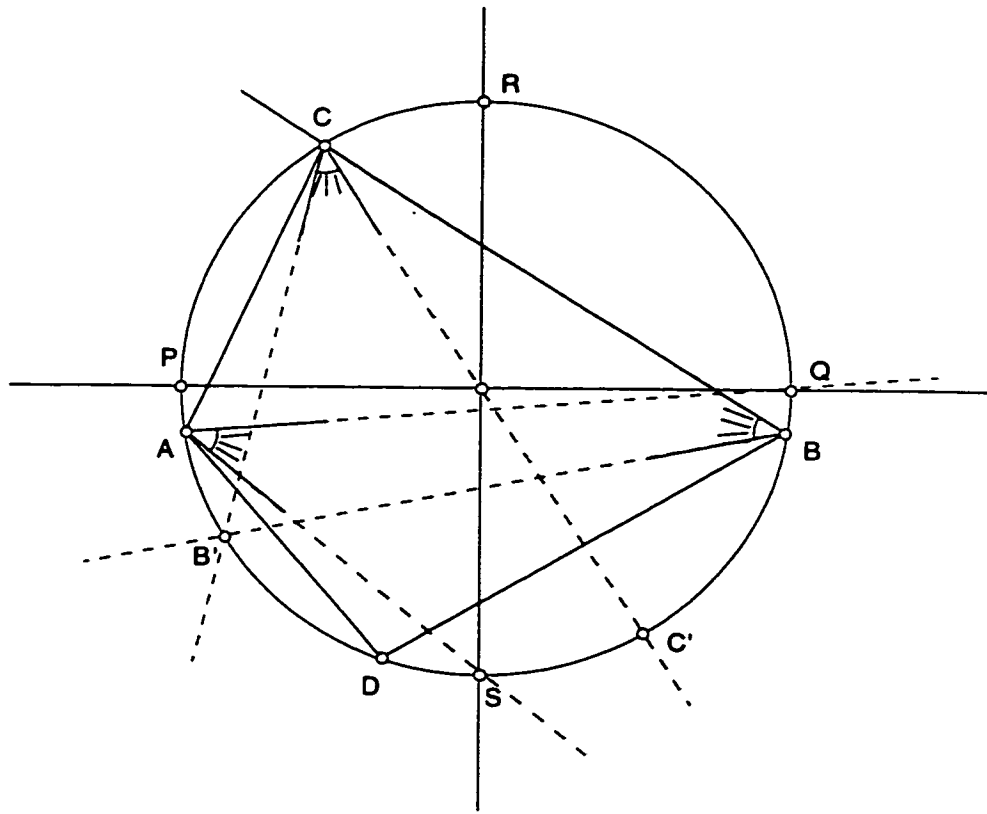


Figure 5.22: $AC \leq BC$; $AD \leq BD$ and $CR > DS$.

5.3 Conclusion

We close this chapter with the following conjecture:

Conjecture 5.3.1 *Any n -gon P may be illuminated by three vertex lights, all of them at the edges of P , and each one having angle $(n-2)\pi/n$ - {half the average angle of P }.*

Chapter 6

Illuminating Convex Polygons by Vertex Lights

6.1 Introduction

In this chapter, we explore which vertex floodlights suffice to cover any convex polygon with n vertices. Urrutia [78] established that any convex polygon may be illuminated by any three vertex floodlights whose total angle is π . Urrutia's result says that if we are given a convex polygon P , and three angles whose sum is π (or greater), then floodlights of those angles may be assigned to distinct vertices of P and oriented so that the interior of P is completely illuminated.

A point is illuminated if it lies in or on the boundary of some floodlight cone. i.e. the cones are closed.

It is natural to wonder if the following generalization holds:

Q0. *Given a convex polygon P of n vertices, a set of $k \leq n$ vertex floodlights whose total sum of angles is π , can the lights always be assigned to distinct vertices and oriented to fully illuminate P ?*

This question has been proved in [62] to have a negative answer. They establish their proof by showing a special case of this question when all floodlight angles are equal which has also a negative answer:

Q1. *Given a convex polygon P of n vertices, a set of $k \leq n$ vertex floodlights each with angle $\alpha = \pi / k$, can the lights always be assigned to distinct vertices and oriented to fully illuminate P ?*

First we should note that if the floodlight angles are not fixed in advance, the answer to Q0 is YES. More specifically, any convex polygon P may be covered by some set of vertex floodlights whose total angle is less than π : choose any vertex whose interior angle α is strictly less than π , and place one α -floodlight there. The same holds true for the analog of Q1 if the number of floodlights is fixed to $k < n$ but the angle distribution is not: again place an $\alpha < \pi$ light at some vertex, and distribute $k - 1$ lights each of angle $(\pi - \alpha) / (k - 1)$ at other vertices.

The more interesting problems involve lights whose apertures are fixed, and the case equal apertures is an especially intriguing case. The answer to Q1 is YES for $k = 2$: any convex polygon may be illuminated by placing two $\pi/2$ -lights facing one another at opposite ends of any edge. And Urrutia's result shows that the answer to Q1 is YES for $k = 3$, although they showed in [62] that the

answer to Q1 is NO for general k . They establish their result by showing once again a special case of question Q1, a case where they restrict each vertex to receive just one light, and at the same time fixing $k = n$. So in the following question, the notion of assignment has disappeared entirely:

Q3. *Given a convex polygon P of n vertices, a set of n vertex floodlights each with angle $\alpha = \pi / k$, placed one per vertex, can the lights always be oriented to fully illuminate P ?*

Given the constraints specified in Q3, they were able to find a polygon P with n vertices that cannot be covered by n equal $\alpha = \pi / n$ lights, one per vertex. Therefore the answer to Q3 is NO. (See Figure 6.1)

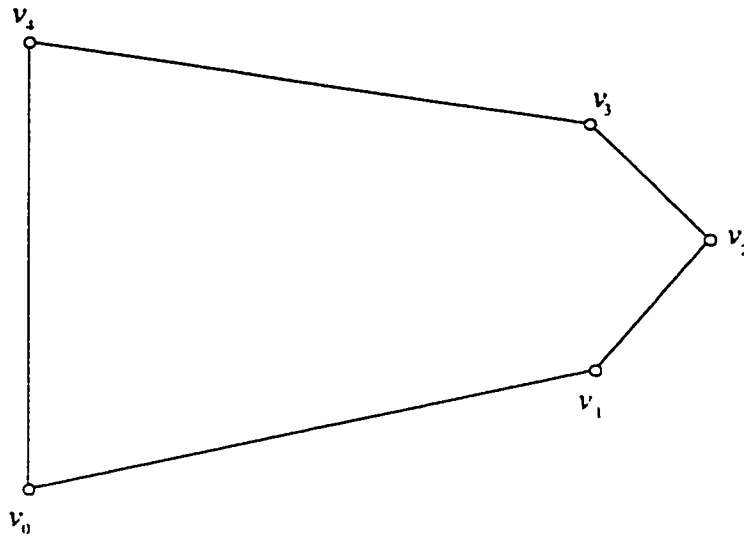


Figure 6.1: a polygon P with n vertices that cannot be covered by n equal $\alpha = \pi / n$ lights.

Q3 is a particular case of Q1, which in turn is a particular case of Q0. So the NO answer to Q3 implies NO answers to Q1 and to Q0.

In this chapter, we pursue this question in a special case where the convex polygon is cocircular, i.e. there is a circle C such that all the vertices of our polygon lie on C . We study the following question:

Q4. *Given a cocircular convex polygon P of n vertices and a set of $k \leq n$ vertex floodlights whose total angle sum is π , can the floodlights be aimed such that P is completely illuminated?*

We prove that this question has an affirmative answer.

6.2 Result

Let P be a cocircular convex n -gon and pq a line segment which partitions the circle C into two pieces C_1 and C_2 . Suppose that the boundary of C_1 is partitioned into m arcs c_1, c_2, \dots, c_m in clockwise order starting from q . Choose m points v_1, v_2, \dots, v_m on the boundary of C_2 in clockwise order. Suppose now we place an f_i floodlight at v_i that illuminates exactly the arc c_i of C_1 ; $i = 1, \dots, m$. (See Figure 6.2).

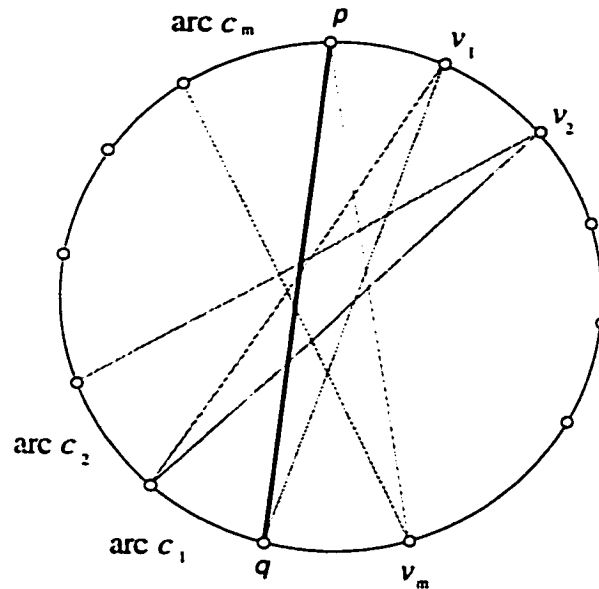


Figure 6.2 Partitioning of arc qp into arcs c_1, c_2, \dots, c_m .

We now prove:

Lemma 6.1 *The set of floodlights $\{f_i ; i = 1, \dots, m\}$ illuminates all of C_1 .*

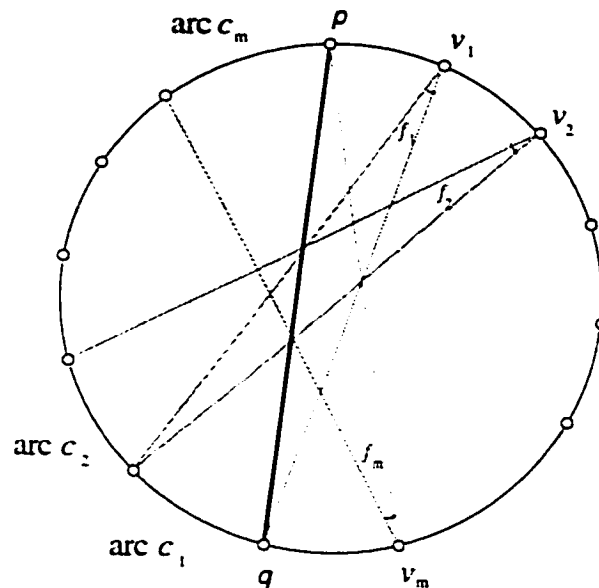


Figure 6.3: Arc qp illuminated from vertices v_1, v_2, \dots, v_m .

Proof. Let b_1, b_2, \dots, b_{m+1} be the endpoints of arcs c_1, c_2, \dots, c_m respectively; we have that $b_1 = q$ and $b_{m+1} = p$. Let aa' be a line joining v_1 to b_1 . Now alternatively displace first a' from b_i to b_{i+1} , then a from v_i to v_{i+1} , $i = 1, \dots, m$; until a' reaches b_{m+1} and a reaches b_1 . Notice that by displacing a' from b_i to b_{i+1} and a from v_i to v_{i+1} , we scan respectively the cone illuminated by the f_i -vertex floodlight placed at v_i , and a sector of C_1 which has already been illuminated by f_1, f_2, \dots, f_{i-1} . As a result, all of C_1 is completely illuminated. ■

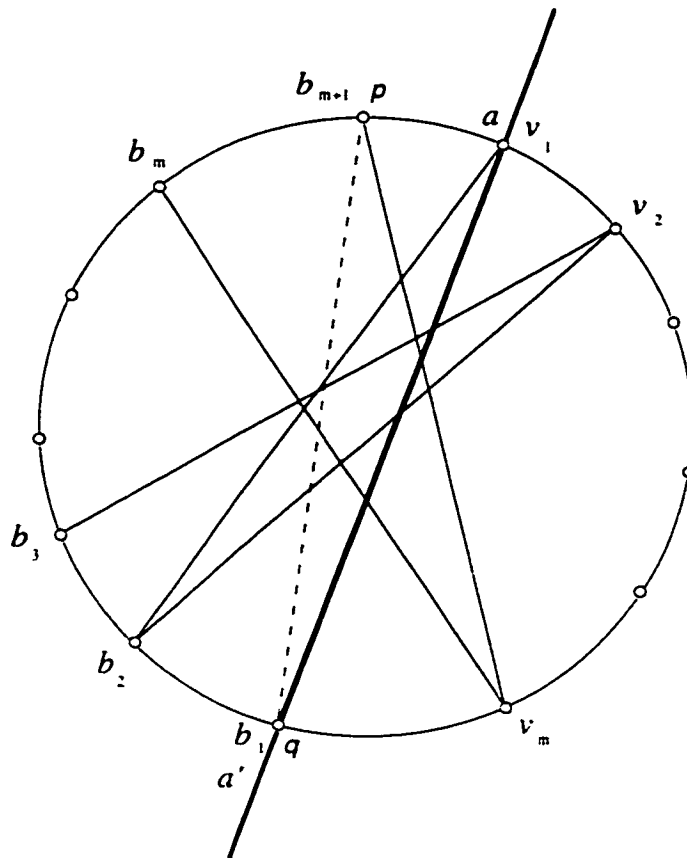


Figure 6.4: Proof of Lemma 6.1.

Given a circle C and two line segments p_1q_1 and p_2q_2 intersecting in the interior of C , these two lines partition the circle C into 4 wedges w_1 , w_2 , w_3 and w_4 . (See Figure 6.4). Let β be the size of arc p_1p_2 , the boundary of wedge w_4 . Choose m points on the boundary of wedge w_1 and now suppose that we place an f_i floodlight at v_i such that $\sum_{i=1}^m f_i = \alpha$; where α is the size of arc p_2q_1 ; the boundary of wedge w_3 .

We now prove:

Lemma 6.2 *The set of vertex floodlights f_1, f_2, \dots, f_m on the boundary of w_1 can be pointed such that w_3 is completely illuminated.*

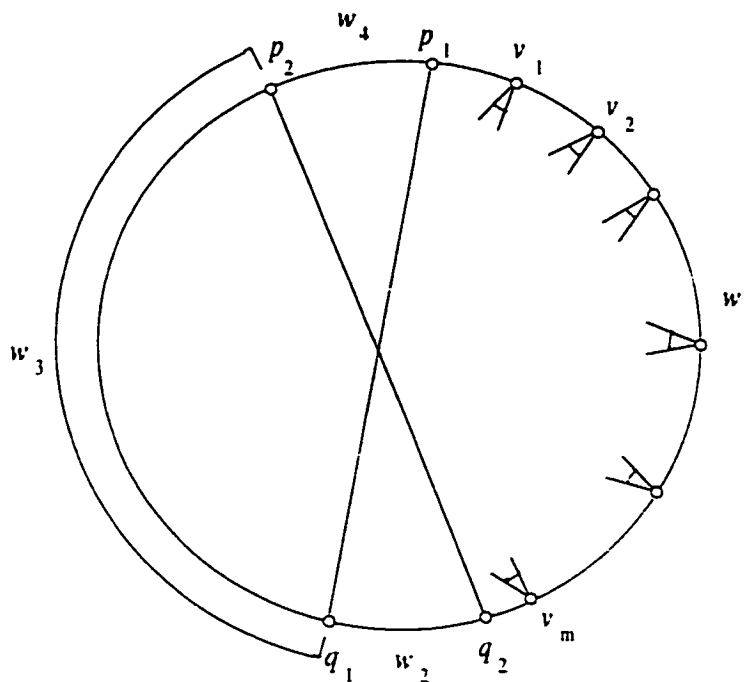


Figure 6.5: Illuminating wedge w_3 from vertex lights f_1, f_2, \dots, f_m located on the boundary of wedge w_1 in the clockwise direction.

Proof. Let f_1, f_2, \dots, f_m be the set of vertex lights placed at vertices v_1, v_2, \dots, v_m respectively, and suppose that the vertex light f_i placed at v_i illuminates exactly the arc c_i of C_1 ; $i = 1, \dots, m$. Now if we extend the vertex light f_m placed at v_m by β , we know from Lemma 6.1 that the lights f_1, f_2, \dots, f_m illuminates the area bounded by arc p_2q_1 of w_3 and the line L_1 passing through p_2 and q_1 , now reducing the vertex light f_m placed at v_m by β , would not affect the illumination, and the area bounded by arc p_2q_1 of wedge w_3 , p_1q_1 and p_2q_2 is still completely illuminated. (See Figure 6.6). ■

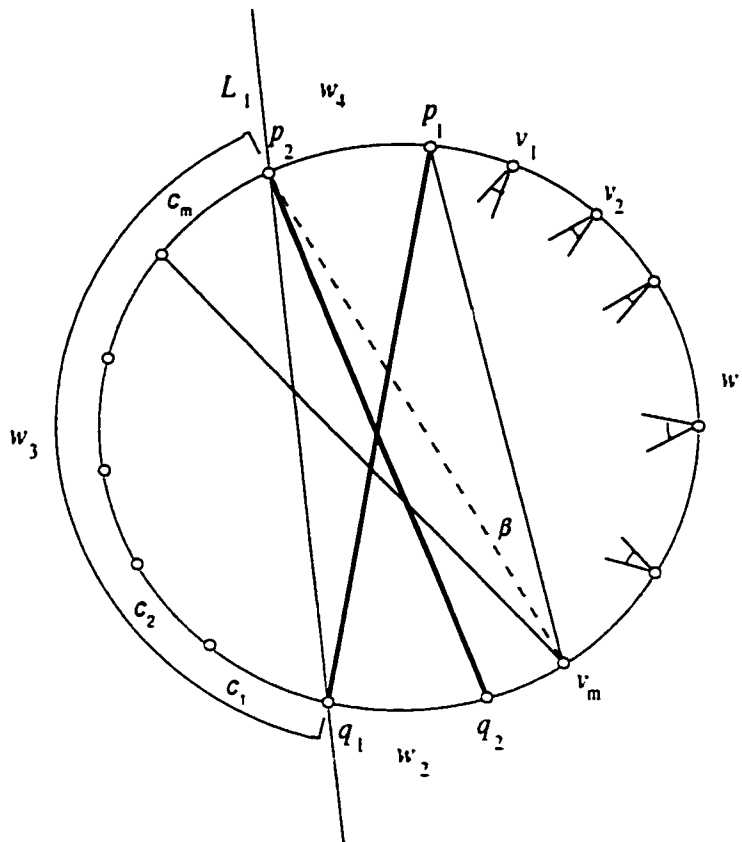


Figure 6.6: Illuminating arc p_2q_1 using f_i vertex lights located at vertex v_i on arc p_1q_2 .

We now prove the following theorem:

Theorem 6.1 *Every cocircular convex n -gon P with n vertices can be illuminated by a set of $k < n$ vertex lights whose total sum is π .*

Proof. Let f_1, f_2, \dots, f_k be the set of vertex lights placed at vertices v_1, v_2, \dots, v_k respectively. Suppose that the vertex light f_1 placed at v_1 can be pointed such that it illuminates an arc c_1 of circle C with endpoints v_j and q , $j = 2, \dots, k$. Let v_1v_j and v_1q be the boundaries of wedges w_1 and w_2 respectively. Start rotating the vertex light f_1 placed at v_1 in the clockwise direction up to the point where one of two cases arises:

Case 1: The length α of the arc joining v_1 to q equals the sum of the sizes of vertex floodlights located on arc v_1v_j .

$$\sum_{i=2}^j f_i = \alpha; \quad i = 2, \dots, j; \quad j = 2, \dots, k$$

Applying Lemma 6.1, the vertex floodlights located on the arc v_1v_j can be pointed such that wedge w_2 is completely illuminated and we are done. (See Figure 6.7).

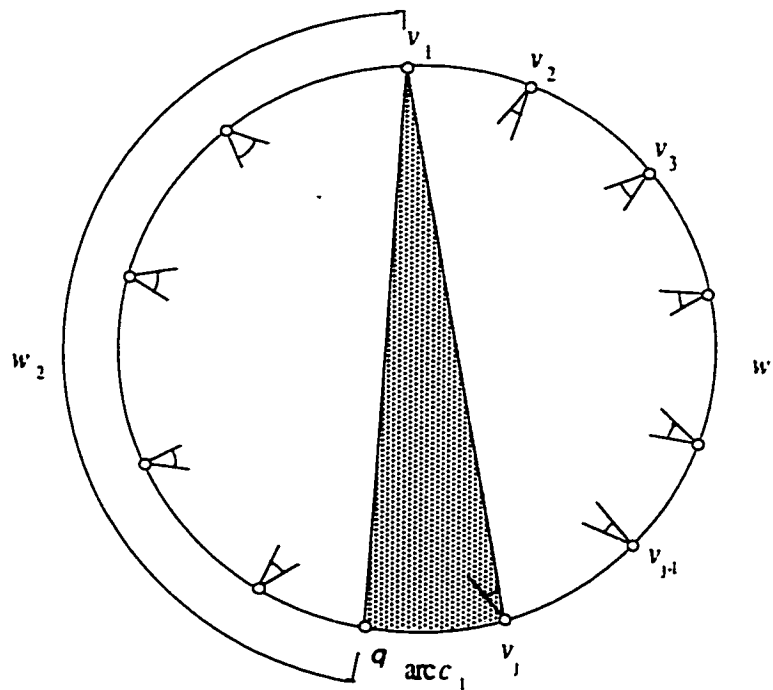


Figure 6.7: Vertex floodlights located on the arc v_1v_j illuminate exactly wedge w_2 .

Case 2: The sum of the sizes of vertex floodlights $f_i; i = 2, \dots, j$ located on arc v_1v_j is greater than α , the length of the arc joining v_1 to q ; (i.e. $\sum_{i=2}^j f_i > \alpha$), but $\sum_{i=2}^{j-1} f_i < \alpha$, so place a vertex floodlight f_j at v_j such that f_j can be pointed to illuminate a wedge containing point v_1 and satisfying Lemma 6.2. This completes the proof of the theorem. ■

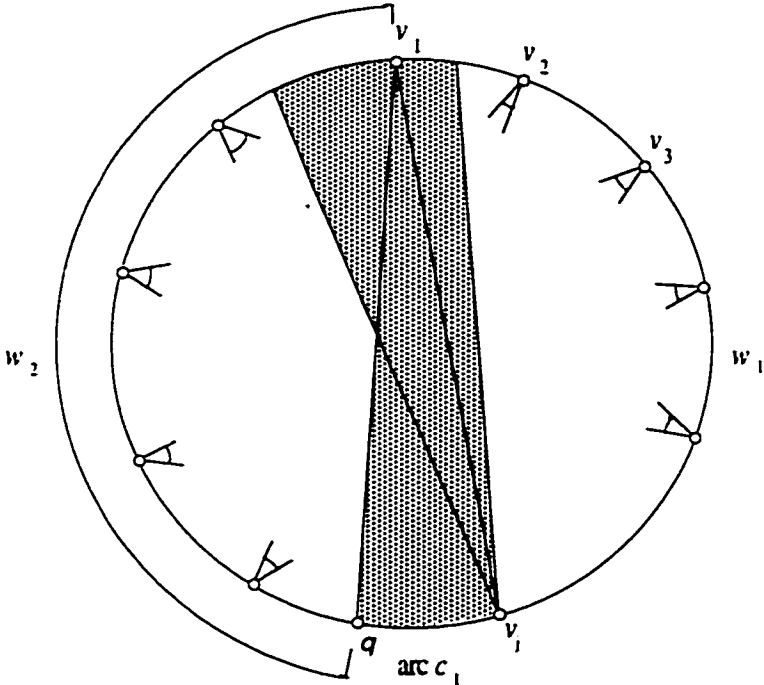


Figure 6.8: f_j vertex floodlight placed at v_j illuminates a wedge containing v_1 and satisfying Lemma 6.2.

Conclusion

In this thesis, we provide a comprehensive survey of most illumination problems from the illumination of families of convex sets on the plane to the illumination of convex polygons using vertex-floodlights. We have shown that any triangle can be illuminated with three $\frac{\pi}{6}$ vertex-floodlights. We also conjecture that three vertex-floodlights whose total angle is at least equal to $2\frac{\pi}{3}$ suffice to illuminate any triangle.

We also provide a theorem for illuminating a quadrilateral using three $\frac{\pi}{4}$ vertex-floodlights, and we conjecture that any n -gon P may be illuminated by three vertex-floodlights, all of them on the edges of P , each one of them having angle $\frac{(n-2)\pi}{n}$ - {half the average angle of P }.

Finally, we provide a theorem for illuminating a cocircular convex n -gon P of n vertices with a set of k vertex-floodlights, ($k \leq n$) whose total sum is π .

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