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LOCALLY NILPOTENT DERIVATIONS OF POLYNOMIAL RINGS

By

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Abstract

Let k be a field of characteristic 0. We classify locally nilpotent derivations $D : k[X, Y, Z] \to k[X, Y, Z]$ satisfying $D^2X = D^2Y = 0$; in particular, it is proved that every k-derivation D satisfying $D^2X = D^2Y = D^2Z = 0$ is essentially a partial derivative. Then we study three classes of k-derivations $D : k[X_1, \dots, X_n] \to k[X_1, \dots, X_n]$, namely the *elementary, constructible* and *nice* derivations. We note the simple fact that elementary $\Rightarrow$ constructible $\Rightarrow$ nice, and we investigate under which conditions the converses hold. We find that if $n = 3$ then all three notions are the same; in dimension 4, if D is irreducible then constructible is equivalent to elementary; in dimension 5, there is an example which is constructible, irreducible but not elementary. Another result states that rank $D \leq n - 2$ holds for all constructible derivations and all $n \geq 3$.

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Introduction

One of the basic objects of study in algebraic geometry is the affine n-space $\mathbf{A}^n$ over a field k. There are a number of basic problems in the study of $\mathbf{A}^n$. One of them is the *automorphism problem*, which can be stated as:

> Give an algebraic description of the group of (polynomial) automorphisms of $\mathbf{A}^n$.

Equivalently, one has to describe the automorphisms of $B = k[X_1, \ldots, X_n]$ as a k-algebra (recall that $\mathbf{A}^n = \operatorname{Spec} B$).

The question is trivial when $n = 1$, so the problem relates to a situation where $n \geq 2$. The group of all polynomial automorphisms of B is denoted by $\mathrm{GA}_n(k)$. Any element F of $\mathrm{GA}_n(k)$ can be represented as an n-tuple $F = (F_1, \ldots, F_n)$, where each $F_i \in k[X_1, \ldots, X_n]$; this means that $F : B \to B$ maps X_i to F_i. The *affine subgroup* of $\mathrm{GA}_n(k)$ is:

$$\mathrm{Af}_n(k) = \{F \in \mathrm{GA}_n(k) \mid \deg F_i = 1, \forall i\}.$$

The elements F of $\mathrm{Af}_n(k)$ with $F_i(0) = 0$ for each i form the *linear subgroup* $GL_n(k)$. The *triangular subgroup* $\mathrm{BA}_n(k)$ of $\mathrm{GA}_n(k)$ is:

$$\{F \in \mathrm{GA}_n(k) \mid F_i = a_i X_i + f_i,\ a_i \in k^*,\ f_i \in k[X_1, \ldots, X_{i-1}],\ \forall i\}.$$

The *tame subgroup* is that generated by $\mathrm{Af}_n(k)$ and $\mathrm{BA}_n(k)$, and is denoted $\mathrm{TA}_n(k)$.

In dimension 2, the structure of $\mathrm{GA}_2(k)$ is given by the following theorem, which goes back to van der Kulk [13]:

Theorem 1. The automorphism group $\mathrm{GA}_2(k)$ is the amalgamated product $\mathrm{Af}_2(k) *_I \mathrm{BA}_2(k)$ where $I = \mathrm{Af}_2(k) \cap \mathrm{BA}_2(k)$.

In particular, every element of $\mathrm{GA}_2(k)$ is tame. For $n \geq 3$, it is not known whether the inclusion $\mathrm{TA}_n(k) \subset \mathrm{GA}_n(k)$ is proper; it is, however, known that $\mathrm{TA}_n(k)$ is not the amalgamated free product of $\mathrm{Af}_n(k)$ and $\mathrm{BA}_n(k)$ along their intersection. An interesting fact is that, even in the dimension 2 case, the famous Jacobian Conjecture is unsolved. It can be stated as:

> If the Jacobian determinant of a polynomial morphism $\varphi : \mathbf{A}^n \to \mathbf{A}^n$ is a nonzero constant, then φ is an automorphism.

It is a common understanding that one way to approach the automorphism problem is to study the actions of the group G_a on $\mathbf{A}^n$, where $G_a = (k, +)$. An action $\rho : G_a \times \mathbf{A}^n \to \mathbf{A}^n$ corresponds to the one-parameter subgroup $\{\rho_t \mid t \in k\}$ of $\mathrm{GA}_n(k)$.

Note that $\mathbf{A}^n \cong \operatorname{Spec} B$ and $G_a \times \mathbf{A}^n \cong \operatorname{Spec}(B[T])$, where T is an indeterminate over B; since an action ρ is a morphism of varieties, it determines a homomorphism of k-algebras $\xi_T : B \to B[T]$. More precisely, one can check that a k-homomorphism $\xi_T : B \to B[T]$ corresponds to a G_a-action if and only if it satisfies the following two conditions:

1. The diagram

$$\begin{array}{ccc} B & \xrightarrow{\xi_T} & B[T] \\ & \searrow id & \downarrow ev_0 \\ & & B \end{array}$$

   is commutative, where ev_0 is evaluation at $T = 0$.

2. For all $f \in \mathrm{im}\xi_T$, $f(X+Y) = f^{(\xi_Y)}(X)$.

Given such a ξ_T, let $D : B \to B$ be the composition:

$$B \xrightarrow{\xi_T} B[T] \xrightarrow{\frac{d}{dT}} B[T] \xrightarrow{ev_0} B.$$

In other words, if $b \in B$ then $D(b) =$ coefficient of T in $\xi_T(b)$. One can verify that D is a locally nilpotent derivation of B and that $\exp(TD) = \xi_T$. This means that studying algebraic G_a-actions on $\mathbf{A}^n$ is equivalent to studying locally nilpotent derivations $D : B \to B$. In fact, the study of locally nilpotent derivations has proved to be an efficient way of investigating G_a-actions; substantial progress was made in recent years via this algebraic approach.

We will examine locally nilpotent derivations of polynomial rings. In Chapter 1, some well-known results are stated that will be referred to later, along with some useful terminology. At the end of the chapter, a brief description of the connection between derivations and Hilbert's fourteenth problem is provided. This is a very interesting aspect of derivations, and illustrates that some remarkable progress has been made recently.

Chapter 2 concerns homogeneous derivations. First, a technique on degree functions is developed, and some properties of homogeneous derivations are proved. The second part of Chapter 2 studies a method of associating a homogeneous derivation $\operatorname{gr} D$ to any given locally nilpotent derivation D of a polynomial ring. Subsequently one can apply the results for homogeneous derivations to $\operatorname{gr} D$ and then obtain some information about the original derivation D. In this way, a classification has been obtained for locally nilpotent derivations $D : k[X, Y, Z] \to k[X, Y, Z]$ satisfying $D^2X = D^2Y = 0$; in particular, it is proved that every k-derivation D satisfying $D^2X = D^2Y = D^2Z = 0$ is essentially a partial derivative.

Chapter 3 is an extension of the study of three classes of k-derivations D : $k[X_1, \ldots, X_n] \to k[X_1, \ldots, X_n]$, namely the *elementary, constructible* and *nice* derivations (these notions were introduced in papers of van den Essen, Janssen and Hubbers). It is known that the kernel of an elementary derivation can be non-finitely generated as a k-algebra. This increases the interest in elementary derivations.

We note the simple fact that elementary $\Rightarrow$ constructible $\Rightarrow$ nice, and we investigate under which conditions the converses hold. We find that if $n = 3$ then all three notions are the same; in dimension 4, if D is irreducible then constructible is equivalent to elementary; in dimension 5, there is an example which is constructible, irreducible but not elementary.

Chapter 3 also gives some results concerning the rank of constructible derivations, notably that $\operatorname{rank} D \leq n - 2$ holds for all constructible derivations and all $n \geq 3$.

The topic of locally nilpotent derivations of a polynomial ring is very interesting. Although some remarkable progress has been made recently, there are quite a few intriguing questions still unsolved. Little by little, we are discovering things about dimension 3, but dimension 4 is still a big mystery. This topic will undoubtedly continue to contribute to the development of algebraic geometry.

Contents

Chapter 1

Preliminaries

Throughout this thesis, k denotes a field of characteristic zero. All rings are commutative and have an identity. Let A be a ring. We use A^* to denote the group of units of A. The phrase "A is a domain" means that A is an integral domain, i.e., that A has no zero divisors; "A is a k-domain" means that A is a domain and a k-algebra. If A is a domain then $\operatorname{qt} A$ denotes the field of fractions of A. If $A \subset B$ are domains, then $\operatorname{tr.deg}_A(B)$ is an abbreviation for the transcendence degree $\operatorname{tr.deg}_{\operatorname{qt} A}(\operatorname{qt} B)$.

If A is a subring of a ring B and n is a positive integer, then $B = A^{[n]}$ means that B is A-isomorphic to the polynomial ring in n variables over A. Suppose that $B = A^{[n]}$. A *coordinate system of B over A* is an ordered n-tuple $(X_1, \dots, X_n)$ of elements of B satisfying $B = A[X_1, \dots, X_n]$; a *variable* of B over A is an element X of B such that $B = A[X, X_2, \dots, X_n]$ for some $X_2, \dots, X_n \in B$.

Let A be a subring of a domain B. An element b of B is *algebraic* over A if it is a root of a nonzero (but not necessarily monic) polynomial in $A[T]$. If every element of B which is algebraic over A belongs to A, we say that A is *algebraically closed* in B. The ring A is said to be *factorially closed in B* if, for all $x, y \in B$, we have $xy \in A \setminus \{0\} \Rightarrow x, y \in A$. Note:

Factorially closed implies algebraically closed.

To see this, suppose $x \in B$ is algebraic over A (we may assume that $x \neq 0$). Choose a nonzero polynomial $f(T) \in A[T]$ of minimal degree such that $f(x) = 0$, say,

$f(x) = a_0x^n + \cdots a_{n-1}x + a_n = 0$. Then

$$(a_0x^{n-1} + \cdots a_{n-1})x = -a_n \in A\backslash\{0\}.$$

If A is factorially closed, then $x \in A$, and thus A is algebraically closed. Note also that if A is factorially closed in B, then $A^* = B^*$.

1.1 Basic facts

In this section, some basic notations for derivations of a ring will be given, followed by a list of well-known and very important results concerning derivations.

Definition 1.1.1. Let B be a ring. We say a map $D : B \to B$ is a *derivation* of B if for any $x, y \in B$,

1. $D(x + y) = Dx + Dy$
2. $D(xy) = xDy + yDx$.

The *kernel* of D, defined by $\ker D = \{x \in B | Dx = 0\}$, is a subring of B. If R is a subring of B and $D(R) = 0$ then we say that D is an *R-derivation.*

Definition 1.1.2. A derivation D is *locally nilpotent* if $\forall x \in B$, $\exists n \in \mathbb{N}$ such that $D^n(x) = 0$. We say D is *irreducible* if the only principal ideal of B containing $D(B)$ is B (or equivalently, if D cannot be written as $D = \alpha D'$ with α representing a nonunit element of B, and D' representing a derivation of B).

1.1.3. Given a multiplicatively closed subset S of $B\backslash\{0\}$, a derivation D of B can be extended to a derivation of $S^{-1}B$ in the usual way, using $S^{-1}D$ to denote the extension. For all $b/s \in S^{-1}B$,

$$S^{-1}D(\frac{b}{s}) = \frac{sD(b) - bD(s)}{s^2}.$$

1.1.4. Let B be a domain containing $\mathbb{Q}$, let $D : B \to B$ be a locally nilpotent derivation, and let $B[T]$ be a polynomial ring of one variable. Define

$$\begin{aligned} \xi : \ B &\to B[T] \\ b &\mapsto \sum_{n=0}^{\infty} \frac{D^n b}{n!} T^n. \end{aligned}$$

Note that ξ is a homomorphism of A-algebras, where $A = \ker D$. For any $b \in B$ we can define the *exponent* $\nu_D(b)$ of b with respect to D as follows: Define $\nu_D : B \to \{-\infty\} \cup \mathbb{N}$ by

$$\nu_D(b) = \begin{cases} \max\{n \in \mathbb{N} | D^n b \neq 0\} & \text{if } b \neq 0; \\ -\infty & \text{if } b = 0. \end{cases}$$

When the context is clear we use ν instead of ν_D. Note the very useful formula:

$$\nu(ab) = \nu(a) + \nu(b) \tag{1}$$

for any $a, b \in B$. (Proof: ν is the composition $B \xrightarrow{\xi} B[T] \xrightarrow{\deg} \{-\infty\} \cup \mathbb{N}$.) Note that ν is a degree function (see 2.1.1).

1.1.5. Let B be an integral domain of characteristic zero, let $D : B \to B$ be a nonzero derivation of B, and let $A = \ker D$. Note the following well-known facts:

1. If D is locally nilpotent then A is a factorially closed subring of B. In particular, if D is locally nilpotent and B is a UFD then A is a UFD.

2. Let S be a multiplicatively closed subset of $B \setminus \{0\}$, and consider the derivation $S^{-1}D$ of $S^{-1}B$. Then

   (a) $S^{-1}D$ is locally nilpotent if and only if D is locally nilpotent and $S \subseteq A$.

   (b) If $S \subseteq A$ then $\ker S^{-1}D = S^{-1}A$ and $S^{-1}A \cap B = A$.

3. Assume that $\mathbb{Q} \subseteq B$. If D is locally nilpotent and $Db = 1$ for some $b \in B$, then $B = A[b] = A^{[1]}$.

4. Assume that $\mathbb{Q} \subseteq B$. If D is locally nilpotent, choose any $b \in B$ such that $Db \neq 0$ and $D^2 b = 0$, and let $S = \{1, Db, (Db)^2, \dots\} \subset A$. Then $S^{-1}D(b/Db) = 1$, so, by 3, $S^{-1}B = (S^{-1}A)[b] = (S^{-1}A)^{[1]}$.

5. If D is locally nilpotent, let $S = A \setminus \{0\}$. Then 4 implies $S^{-1}B = (\operatorname{qt} A)^{[1]}$, and 2(b) implies $\operatorname{qt} A \cap B = A$.

6. Let $b \in B\backslash\{0\}$. The derivation bD is locally nilpotent if and only if D is locally nilpotent and $b \in A$.

7. Let $f \in B[T] = B^{[1]}$ and let D' be the unique derivation of $B[T]$ which extends D and satisfies $D'T = f$. Then D' is locally nilpotent if and only if D is locally nilpotent and $f \in B$.

8. Suppose that B is a UFD. Then

   (a) $D = \alpha D_0$ where $\alpha \in B$ and D_0 is an irreducible derivation of B. Moreover, α and D_0 are unique, up to multiplication by units of B.

   (b) If D is locally nilpotent then so is D_0, and $\alpha \in \ker D$.

Proof. 1. Since $A = \{b \in B \mid \nu(b) \leq 0\}$, equation (1) implies that A is factorially closed in B.

Now if B is a UFD, for all $a \in A$, factor a in B, say $a = p_1 p_2 \cdots p_r$. Then $p_1, p_2, \ldots, p_r \in A$. Since B is a UFD, $p_1, p_2, \ldots, p_r$ are uniquely determined by a. So A is a UFD.

2. (a) If $S^{-1}D$ is locally nilpotent, then clearly D is locally nilpotent, and it can be shown that $S \subseteq A$. If we use ν to represent $\nu_{S^{-1}D}$, and let $s \in S$, then

$$\begin{aligned} 0 \geq -\nu(s) = \nu(1) - \nu(s) = \nu(1/s) \geq 0 \ &\Rightarrow \nu(s) = 0 \\ &\Rightarrow s \in \ker(S^{-1}D) \cap B = \ker D = A. \end{aligned}$$

Hence $S \subseteq A$.

Conversely suppose D is locally nilpotent and $S \subseteq A$. For all $b/s \in S^{-1}B$, we would have

$$S^{-1}D(\frac{b}{s}) = \frac{sD\,(b) - bD\,(s)}{s^2} = \frac{D\,(b)}{s}.$$

Inductively $(S^{-1}D)^n\,(b/s) = D^n\,(b)\,/s$, so $S^{-1}D$ is locally nilpotent.

(b) Suppose $S \subseteq A$. Let $a/s \in S^{-1}A$. Then $S^{-1}D\,(a/s) = D\,(a)\,/s = 0$. So $S^{-1}A \subseteq \ker S^{-1}D$. Conversely, let $b/s \in S^{-1}B$ and $S^{-1}D\,(b/s) = 0$. Since

$S^{-1}D\,(b/s) = D\,(b)\,/s$, it follows that $D\,(b) = 0$, hence $b \in A$ and $\ker S^{-1}D \subseteq S^{-1}A$. This proves that $\ker S^{-1}D = S^{-1}A$. So

$$S^{-1}A \cap B = \ker(S^{-1}D) \cap B = \ker D = A.$$

3. Given any $b \in B$ define

$$\begin{aligned} \varphi_{-b}: \ & B \to B \\ & x \mapsto \sum_{n=0}^{\infty} \frac{(-b)^n}{n!} D^n x. \end{aligned}$$

If $Db = 1$, one can verify that $\varphi_{-b}\,(x) \in A$ for any $x \in B$. Suppose $B \neq A[b]$, say, $\exists y \in B \backslash A[b]$. We can choose y such that $\nu(y)$ is minimal. Write

$$\varphi_{-b}(y) = y + \sum_{n=1}^{\infty} \frac{(-b)^n}{(n)!} D^n y.$$

Clearly $\nu(D^n y) < \nu(y)$ when $n \geq 1$. So $D^n y \in A[b]$ for all $n \geq 1$. Then

$$\sum_{n=1}^{\infty} \frac{D^n y}{n!} (-b)^n \in A[b].$$

It is known that $\varphi_{-b}(y) \in A$, and therefore $y \in A[b]$, which is a contradiction. Thus $B = A[b]$, and we know that A is algebraically closed in B. So $A[b] = A^{[1]}$.

4. Following these assumptions we have

$$S^{-1}D(\frac{b}{Db}) = \frac{DbDb - bD^2b}{(Db)^2} = 1.$$

From 2, $\ker S^{-1}D = S^{-1}A$. Using 3, we have $S^{-1}B = (S^{-1}A)[b/Db] = (S^{-1}A)^{[1]}$. So $S^{-1}B = (S^{-1}A)[b] = (S^{-1}A)^{[1]}$.

5. Choose any $b \in B\backslash A$. If $D^n b = 0$ but $D^{n-1}b \neq 0$, take $b_1 = D^{n-2}b$. Then

$$Db_1 = D^{n-1}b \neq 0, \quad \text{and} \quad D^2 b_1 = 0.$$

Let $S_1 = \{1, Db_1, (Db_1)^2, \ldots\}$. Then 4 implies that $S_1^{-1}B = (S_1^{-1}A)^{[1]}$. But $S_1 \subseteq S \subseteq A$. So $S^{-1}B = (S^{-1}A)^{[1]} = (\text{qt}\,A)^{[1]}$.

6. If D is locally nilpotent, and $b \in A$, it is clear that bD is locally nilpotent. Conversely if bD is locally nilpotent, write $D_1 = bD$. Choose $f \in B$ such that

$D_1 f \neq 0$ and $D_1^2 f = 0$; then $bDf = D_1 f \in \ker(D_1)\backslash\{0\}$ implies that $b \in \ker(D_1)$ (by part 1) so $b \in A$. Consequently, $D_1^n(x) = b^n D^n(x)$ for all $n \geq 1$ and all $x \in B$, and it follows that D is locally nilpotent.

7. First, suppose that D is locally nilpotent and $f \in B$. For any $h(T) \in B[T]$, say $h(T) = a_0 + a_1 T + \cdots + a_r T^r$, denote

$$h^D(T) = D(a_0) + D(a_1)T + \cdots + D(a_r)T^r.$$

Then

$$D'(h(T)) = h^D(T) + h'(T)D'(T) = h^D(T) + fh'(T).$$

Note that $\deg(D'h(T)) \leq r$ and coefficient of T^r in $D'(h(T))$ is $D(a_r)$, so we can reach some n such that $\deg(D'^n h(T)) < \deg h(T)$. Inductively we conclude that D' is locally nilpotent.

Now suppose that D' is locally nilpotent. Then D is locally nilpotent since D' extends D. We need to show that $f \in B$. We consider two cases:

Case 1. $\deg f > 1$:

For any $h(T) \in B[T]$, $D'(h(T)) = h^D(T) + fh'(T)$. Note that $\deg h^D(T) \leq \deg h(T)$ and $\deg(fh'(T)) \geq \deg(h(T)) + 1$. So $\deg D'(h(T)) > \deg h(T)$. This contradicts the assumption that D' is locally nilpotent.

Case 2. $\deg f = 1$:

Write $h(T) = \alpha T^n +$ lower terms, $f = aT + b$. Then

$$D'(h(T)) = h^D(T) + h'(T)(aT + b).$$

One can see that the coefficient of T^n in $D'(h(T))$ is $D\alpha + n\alpha a$. Note that

$$\nu(D\alpha) = \nu(\alpha) - 1, \qquad \nu(n\alpha a) = \nu(\alpha) + \nu(a).$$

So $D\alpha + n\alpha a \neq 0$. This implies that $\deg D'(h(T)) = \deg h(T)$. Once again, this contradicts the assumption that D' is locally nilpotent.

8. If D is reducible, then $D = \alpha_1 D_1$ with α_1 a nonunit element of B and D_1 a derivation of B. If D_1 is irreducible, we are finished. If not, then $D = \alpha_1 D_1 = \alpha_1\alpha_2 D_2$,

and so on... $D = \alpha_1\alpha_2\cdots\alpha_n D_n$. If we reach some n such that D_n is irreducible, again we are finished. If not, then

$$D(B) \subseteq \bigcap_{n=1}^{\infty} (\alpha_1\alpha_2\cdots\alpha_n)$$

Take $a \in D(B)$, $a \neq 0$, then $a = \alpha_1\alpha_2\cdots\alpha_n h_n$ for some $h_n \in B$. This is true for any n, and each α_n is a nonunit. This is impossible because B is a UFD. So we must have $D = \alpha D_0$ for some D_0 irreducible. For the uniqueness, let $D = \alpha D_0 = \beta D_1$. Then $D(B) \subseteq (\alpha)$, $D(B) \subseteq (\beta)$. Denote $d = \gcd(\alpha, \beta)$. Then

$$(\alpha/d)D = (\beta/d)D, \quad \gcd(\alpha/d, \beta/d) = 1.$$

So $D_0 B \subseteq (\beta/d)$, $D_1 B \subseteq (\alpha/d)$. But D_0 and D_1 are irreducible. So α/d and β/d are units, and hence α and β are associates. This proves (a), and (b) follows from 6. □

In connection with 3 and 4, we give the following:

Definition 1.1.6. Let B be a domain, and let $D : B \to B$ be a locally nilpotent derivation.

1. An element $s \in B$ is a *slice* for D if $Ds \in B^*$.
2. An element $b \in B$ is called a *local slice* for D if $Db \in \ker D \setminus \{0\}$.

A closely related concept is that of a fixed point.

Definition 1.1.7. Let B be an integral domain of characteristic zero and let $D : B \to B$ be a locally nilpotent derivation. Then $\mathrm{Fix}(D)$ denotes the closed subset $V(DB)$ of $\mathrm{Spec}\, B$, and the elements of $\mathrm{Fix}(D)$ are called the *fixed points* of D.

Remark 1. In 1.1.7, let α be the algebraic action of G_a on $\mathrm{Spec}\, B$ corresponding to D. Then one can see that $\mathrm{Fix}(D)$ is exactly the set of fixed points of α.

One notes that if D has a slice, i.e., $Ds \in B^*$ for some $s \in B$, then D is fixed point free (because $DB = (1)$ in this case, so that $V(DB) = \varphi$). However, the converse is not true: In 3.4.4, the derivation τ is fixed point free but does not have a slice.

The following related question is notoriously difficult: *If $B = k^{[3]}$ and D is a fixed point free locally nilpotent derivation of B, does it follow that D has a slice?*

1.2 Derivations of polynomial rings

Let k be a field of characteristic zero, and let B be a polynomial ring in n variables over k. One way to study k-derivations of B, is to classify them according to their ranks:

Definition 1.2.1. Let D be a k-derivation of $B = k^{[n]}$. The *rank* of D is the least integer $r \geq 0$ for which there exists a coordinate system $(X_1, \ldots, X_n)$ of B satisfying $k[X_{r+1}, \ldots, X_n] \subseteq \ker D$.

Consider locally nilpotent derivations of $B = k^{[n]}$. Derivations of low rank are easy to understand: $\operatorname{rank} D = 0$ means $D = 0$, and it is easy to see that if $\operatorname{rank} D = 1$ then D has the form $f(X_2, \ldots, X_n) \cdot \partial/\partial X_1$ for some coordinate system $(X_1, \ldots, X_n)$ of B. In other words, $\operatorname{rank} D = 1$ is equivalent to the two conditions:

$$\ker D = k^{[n-1]} \quad \text{and} \quad B = (\ker D)^{[1]}.$$

Let $A = \ker D$ (where $D : B \to B$ is locally nilpotent). By 3 of 1.1.5, we know that if D has a slice, then $B = A^{[1]}$. Since $B = k^{[n]}$, one is tempted to think that $A = k^{[n-1]}$, and hence $\operatorname{rank} D = 1$, but we do not know if this is true generally. In fact this is a special case of the following open problem:

Cancellation Problem: Given a k-algebra R, does $R^{[s]} \cong k^{[r+s]}$ imply that $R \cong k^{[r]}$?

Actually, our question is *equivalent* to the Cancellation Problem. So the Cancellation Problem can be formulated as: *Is it true that if $D : B \to B$ has a slice, where $B = k^{[n]}$, then* $\operatorname{rank} D = 1$*?* We do not know the answer in general, but in some special cases we have a positive answer:

Case 1: $n \leq 3$. By 1.2.2 and 1.2.10 below, $A \cong k^{[n-1]}$. So, if D has a slice then $\operatorname{rank} D = 1$.

Case 2: Let $B = k^{[n]}$. If D has a slice which happens to be a variable of B, then $\operatorname{rank} D = 1$. To see this, let s be a variable of B such that $Ds = 1$. By 3 of 1.1.5 we have $B = A[s] = A^{[1]}$. So

$$A = A[s]/sA[s] = B/sB = k^{[n-1]}.$$

This implies $\operatorname{rank} D = 1$.

Theorem 1.2.2. ***(Rentschler's Theorem)*** *[25] If L is a field of characteristic zero and D is a nonzero locally nilpotent derivation of $L[X,Y] = L^{[2]}$, then there exist P, Q such that $L[X,Y] = L[P, Q]$ and $D = \alpha\frac{\partial}{\partial Q}$ for some $\alpha \in L[P]$. Moreover,* $\ker D = L[P] = L^{[1]}$.

As to derivations of rank two, [7] gives an explicit description. Let us quote the main results from there for future reference. Let R be a UFD which contains $\mathbb{Q}$, $B = R[X,Y] = R^{[2]}$ and $K = \operatorname{qt} R$.

Definition 1.2.3. Given $P \in B$, define an R-derivation $\Delta_P : B \to B$ by

$$\Delta_P = -P_Y\frac{\partial}{\partial X} + P_X\frac{\partial}{\partial Y},$$

or equivalently

$$\Delta_P(h) = \begin{vmatrix} P_X & P_Y \\ h_X & h_Y \end{vmatrix}$$

for all $h \in B$.

Theorem 1.2.4. *(**Theorem 2.4** of [7]) Let R be a UFD containing $\mathbb{Q}$, let $B = R[X,Y] = R^{[2]}$ and let $K = \operatorname{qt} R$. For an R-derivation $D \neq 0$ of B, the following are equivalent:*

1. *D is locally nilpotent;*
2. *$D = \alpha\Delta_P$, for some $P \in B$ which is a variable of $K[X,Y]$ and satisfies $\gcd_B(P_X, P_Y) = 1$, and for some $\alpha \in R[P]\backslash\{0\}$.*

Moreover, if the above conditions are satisfied, then $\ker D = R[P] = R^{[1]}$.

Definition 1.2.5. Given a coordinate system $\gamma = (X_1, \ldots, X_n)$ of R_n and an element $P \in R_n = k[X_1, \ldots, X_n]$, define a derivation $\Delta_P^{\gamma} : R_n \to R_n$ by

$$\Delta_P^{\gamma} = -P_{X_n}\frac{\partial}{\partial X_{n-1}} + P_{X_{n-1}}\frac{\partial}{\partial X_n}.$$

Remark 2. A convenient notation is $\gamma = (X_1, \dots, X_{n-2}, Y, Z)$ and $\Delta_P^\gamma = -P_Z\partial/\partial Y + P_Y\partial/\partial Z$.

Corollary 1.2.6. ***(Corollary 3.2** of [7]) For a k-derivation $D \neq 0$ of $R_n = k^{[n]}$, the following are equivalent:*

1. *D is locally nilpotent and* $\operatorname{rank} D \leq 2$*;*
2. *$D = \alpha\Delta_P^\gamma$ for some γ, P and α satisfying*
   - *$\gamma = (X_1, \dots, X_{n-2}, Y, Z)$ is a coordinate system of R_n,*
   - *$P \in R_n$ is a variable of $k(X_1, \dots X_{n-2})[Y, Z]$ and* $\gcd_{R_n}(P_Y, P_Z) = 1$*,*
   - *α is a nonzero element of $k[X_1, \dots X_{n-2}, P]$.*

Moreover, if the above two conditions are satisfied then the following hold:

3. $\ker D = k[X_1, \dots X_{n-2}, P] = k^{[n-1]}$*;*
4. *Δ_P^γ is irreducible;*
5. *$\Delta_P^\gamma(R_n)$ contains a nonzero element of $k[X_1, \dots X_{n-2}]$.*

Given a k-derivation D of $B = k^{[n]}$, note that $\operatorname{rank} D = n$ if and only if there is no variable of B in $\ker D$. The existence of rank n locally nilpotent derivations of $k^{[n]}$ remained an open question for some time (such derivations obviously exist when $n = 1$ and, by Rentschler's Theorem, they do not exist when $n = 2$). Then Freudenburg [17] gave examples, for all $n \geq 3$, of rank n locally nilpotent derivations of $k^{[n]}$.

We now define some classes of derivations with special and interesting properties:

Definition 1.2.7. Let $B = k[X_1, \dots, X_n, Y_1, \dots, Y_m]$ be the polynomial ring in $n + m$ variables over k. A k-derivation D on B is called *elementary* if it is of the form

$$D = a_1(X_1, \dots, X_n)\,\partial/\partial Y_1 + \dots + a_m(X_1, \dots, X_n)\,\partial/\partial Y_m.$$

Observe that $DX_i = 0$ for all i and $D^2Y_j = 0$ for all j, and therefore every elementary derivation is locally nilpotent. Elementary derivations were studied in [12]. This class of derivations may appear to be trivial, but it is not. For instance, their kernels are not always finitely generated as k-algebras (see the next section).

Definition 1.2.8. Let $R = k^{[n]}$, let $D : R \to R$ be a derivation. We say D is *triangular* if there exists a coordinate system $(X_1, \dots, X_n)$ of R such that $D(X_1) \in k$ and $D(X_i) \in k[X_1, \dots, X_{i-1}]$ for $2 \le i \le n$.

Note that all triangular derivations are locally nilpotent. By Rentschler's Theorem, the restriction of D to $k[X_1, X_2]$ kills a variable, so the rank of a triangular derivation is less than n. The first example of a non-triangular locally nilpotent derivation of $k^{[n]}$ was given by Bass in [1]. Daigle in [2] gave a triangulability criterion for derivations of $k[X, Y, Z]$.

Definition 1.2.9. Let B be a $\mathbb{Z}$-graded ring, say, $B = \oplus_{i \in \mathbb{Z}} B_i$, let $D : B \to B$ be a locally nilpotent derivation. If there exists $d \in \mathbb{Z}$ such that $DB_i \subseteq B_{i+d}$ for all i, D is called *homogeneous* with respect to this grading.

The case of polynomial rings is of great interest. When $B = k^{[n]} = k[X_1, \dots, X_n]$, we usually make B a graded ring by assigning weights to the variables; say $w = (a_1, \dots, a_n) \in \mathbb{N}^n$ and $w(X_i) = a_i$. In this case, we call D *w-homogeneous*. The case $w = (1, \dots, 1)$ is just the standard grading. Note that Freudenburg's example of rank n derivation of $k^{[n]}$ $(n > 2)$ is homogeneous in standard grading.

There are some good properties for locally nilpotent derivations on a polynomial ring of three variables. The following theorem is due to Miyanishi [21]:

Theorem 1.2.10. *If D is any non-zero locally nilpotent derivation of $k^{[3]}$, then* $\ker D \cong k^{[2]}$.

Then, consider a theorem by Daigle [3] giving the jacobian formula for these derivations. First, some notation is required:

Let $B = k[X_1, \dots, X_n] = k^{[n]}$. Given $f = (f_1, \dots, f_{n-1}) \in B^{n-1}$, define Δ_f by

$$\Delta_f(g) = \frac{\partial(f_1, \dots, f_{n-1}, g)}{\partial(X_1, \dots, X_n)} \quad \text{(for each } g \in B\text{)}.$$

Theorem 1.2.11. *Let $B = k[X_1, \dots, X_n] = k^{[n]}$, and let $D : B \to B$ be a locally nilpotent k-derivation such that* $\ker D = k^{[n-1]}$. *If $f = (f_1, \dots, f_{n-1}) \in B^{n-1}$ is such that* $\ker D = k[f_1, \dots, f_{n-1}]$, *then Δ_f is irreducible, locally nilpotent, and satisfies $D = a\Delta_f$ for some $a \neq 0$ in* $\ker D$.

To make it clear, the two theorems can be combined:

Corollary 1.2.12. *Let $B = k^{[3]}$, let $D : B \to B$ be a nonzero locally nilpotent derivation, and let $A = \ker D = k[f, g]$. Then $D = a\Delta_{(f,g)}$ for some $a \in A$.*

Remark 3. If D is homogeneous, then f and g can be chosen to be homogeneous, by a theorem of Zurkowski [29]; it is also a consequence of:

Lemma 1.2.13. *Let $\mathbf{k}$ be a field, $A = \mathbf{k}^{[r]}$ ($r \geq 1$) and let $A = \oplus_{i \in \mathbf{N}} A_i$ be a grading such that $A_0 = \mathbf{k}$. If $f_1, \dots, f_n$ are homogeneous elements of A satisfying $\mathbf{k}[f_1, \dots, f_n] = A$, then there is a subset $\{g_1, \dots, g_r\}$ of $\{f_1, \dots, f_n\}$ satisfying $A = \mathbf{k}[g_1, \dots, g_r]$.*

Proof. Consider a subset $\{g_1, \dots, g_s\}$ of $\{f_1, \dots, f_n\}$ satisfying $A = \mathbf{k}[g_1, \dots, g_s]$ and minimal with respect to this property (in particular, $\deg(g_i) > 0$ for all i). Let $R = \mathbf{k}[T_1, \dots, T_s] = \mathbf{k}^{[s]}$, with grading $R = \oplus_{i \in \mathbf{N}} R_i$ determined by $R_0 = \mathbf{k}$ and $\deg(T_i) = \deg(g_i)$. Then the surjective $\mathbf{k}$-homomorphism $e : R \to A$, $\varphi \mapsto \varphi(g_1, \dots, g_s)$, is homogeneous of degree zero and it suffices to show that $\mathfrak{p} = \ker e$ is the zero ideal.

Assume the contrary. Note that $(T_1, \dots, T_s) \supseteq \mathfrak{p}$, i.e., the variety $V(\mathfrak{p}) \subseteq \mathbf{A}^s$ passes through the origin. Since the origin is a smooth point (A is smooth over $\mathbf{k}$), and since $\mathfrak{p}$ is generated by its homogeneous elements, the jacobian condition implies that some homogeneous $\varphi \in \mathfrak{p}$ contains a term λT_j ($\lambda \in \mathbf{k}^*$). Since φ is homogeneous and $\deg(T_i) > 0$ for all i, $\varphi - \lambda T_j \in \mathbf{k}[T_1, \dots, T_{j-1}, T_{j+1}, \dots T_s]$, so $g_j \in \mathbf{k}[g_1, \dots, g_{j-1}, g_{j+1}, \dots g_s]$, contradicting minimality of $\{g_1, \dots, g_s\}$. □

Lemma 1.2.14. *Let $B = k[X_1, \dots, X_n] = k^{[n]}$, and let $D : B \to B$ be a k-derivation. Suppose K is an extension field of k, let $\overline{B} = K[X_1, \dots, X_n]$ and extend D to a K-derivation $\overline{D} : \overline{B} \to \overline{B}$.*

1. *D is irreducible if and only if $\overline{D}$ is.*

2. *D has a slice if and only if* $\overline{D}$ *has a slice.*

3. *If* $n \leq 3$, $\operatorname{rank} D = 1 \iff \operatorname{rank} \overline{D} = 1$.

4. *Let* $\mathcal{M}$ *be a collection of monomials* $X_1^{e_1} \cdots X_n^{e_n}$ *(*$e_j \in \mathbb{N}$*), and let* $N \in \mathcal{M}$*. Then the following are equivalent:*

   (a) $\ker D$ *contains an element* $x = \sum_{M \in \mathcal{M}} \lambda_M M$ *(*$\lambda_M \in k$ *for all* M*) such that* $\lambda_N \neq 0$*;*

   (b) $\ker \overline{D}$ *contains an element* $x' = \sum_{M \in \mathcal{M}} \lambda'_M M$ *(*$\lambda'_M \in K$ *for all* M*) such that* $\lambda'_N \neq 0$.

Proof. (1) D is irreducible iff $\{DX_i\}_{i=1}^n$ are relatively prime in B, iff $\{DX_i\}_{i=1}^n$ are relatively prime in $\overline{B}$, iff $\overline{D}$ is irreducible.

(2) If D has a slice then, obviously, so does $\overline{D}$. Suppose that $\overline{D}$ has a slice: Let $x' \in \overline{B}$ be such that $\overline{D}x' = 1$. Choose a finite set $\mathcal{B}$ of monomials $X_1^{e_1} \cdots X_n^{e_n}$ ($e_i \in \mathbb{N}$) such that x' is a K-linear combination of $\mathcal{B}$. Let $V = \operatorname{Span}_k(\mathcal{B})$, $\overline{V} = \operatorname{Span}_K(\mathcal{B})$, $W = D(V)$ and $\overline{W} = \overline{D}(\overline{V})$, and let $T : V \to W$ and $\overline{T} : \overline{V} \to \overline{W}$ be the restrictions of D and $\overline{D}$ respectively. Note that T and $\overline{T}$ are linear maps of finite dimensional vector spaces. Then $T(x) = 1$ is a system of linear equations, and this system is consistent since $\overline{T}(x') = 1$ (consistency is independent of the field). So D has a slice.

(3) We claim that the following hold:

$$\operatorname{rank} D = 1 \overset{\forall n}{\Longrightarrow} \operatorname{rank} \overline{D} = 1 \overset{\forall n}{\Longrightarrow} B = (\ker D)^{[1]} \overset{n \leq 3}{\Longrightarrow} \operatorname{rank} D = 1.$$

The first implication is obvious and, to prove the second one, we may assume that D is irreducible. Then $\overline{D}$ is irreducible by (1) and, since $\operatorname{rank}(\overline{D}) = 1$, it follows that $\overline{D}$ has a slice; then D has a slice by (2), so $B = (\ker D)^{[1]}$ by 3 of 1.1.5. If $n \leq 3$ then $\ker D = k^{[2]}$; together with $B = (\ker D)^{[1]}$, this gives $\operatorname{rank}(D) = 1$.

(4) If $\ker D$ contains an element x of the form specified in (a), then $x \in \ker \overline{D}$, so (a) implies (b). Conversely, suppose $\ker \overline{D}$ contains $x' = \sum_{i=1}^r \lambda'_i M_i$ ($\lambda'_i \in K$, $M_i \in \mathcal{M}$), such that $\lambda'_1 \neq 0$. Starting from $\mathcal{B} = \{M_1, \ldots, M_r\}$, define V, $\overline{V}$, W, $\overline{W}$, T and $\overline{T}$ as in the proof of (2), and consider the system of linear equations $T(x) = 0$.

Then $\ker(T)$ and $\ker(\overline{T})$ have the same dimension (this property is independent of the field), so every k-basis of $\ker(T)$ is a K-basis of $\ker(\overline{T})$. So $\ker(T)$ must contain an element $x = \sum_{i=1}^{r} \lambda_i M_i$ ($\lambda_i \in k$) with $\lambda_1 \neq 0$, and this proves that (b) implies (a). □

Remark 4. As special cases of part (4) of 1.2.14, we obtain:

- Given a subset $\{X_{i_1}, \ldots, X_{i_r}\}$ of $\{X_1, \ldots, X_n\}$, $\ker D$ contains a nonzero linear form in $\{X_{i_1}, \ldots, X_{i_r}\}$ if and only if $\ker \overline{D}$ does.
- $\ker D$ contains an element of the form $X_i + f(X_1, \ldots, X_{i-1})$ if and only if $\ker \overline{D}$ does.

Lemma 1.2.15. *Let R be a UFD containing $\mathbb{Q}$, let $B = R[X,Y] = R^{[2]}$ and let $0 \neq D : B \to B$ be a locally nilpotent R-derivation. Assume that D is irreducible.*

1. *If $D^2X = 0$ then $\ker D = R[bY + f(X)]$, where $b \in R$ and $f(X) \in R[X]$. Moreover, $DX \in R$ and $DY \in R[X]$.*
2. *If $D^2X = 0 = D^2Y$, then $D = b\frac{\partial}{\partial X} - a\frac{\partial}{\partial Y}$ for some $a, b \in R$. Moreover, $\ker D = R[aX + bY]$ and $DX, DY \in R$.*
3. *If R is a PID and $D^2X = 0 = D^2Y$, then D has a slice.*

Proof. By 1.2.4, we know that for some $P \in R[X,Y]$

$$\ker D = R[P] \qquad \text{and} \qquad D = P_Y \frac{\partial}{\partial X} - P_X \frac{\partial}{\partial Y}.$$

Now $P_Y = DX \in \ker D = R[P]$ implies $P_Y \in R$, so $P = bY + f(X)$ and $DX = b$. Then $DY = -P_X = -f'(X) \in R[X]$, which proves (1).

If also $D^2Y = 0$ then, by (1), $DY \in R$; so $f(X) = aX + c$ ($a, c \in R$), $\ker D = R[P - c] = R[aX + bY]$, and $D = b\frac{\partial}{\partial X} - a\frac{\partial}{\partial Y}$. This proves (2). Note that $\gcd(a, b) = 1$ (because D is irreducible, $DX = b$ and $DY = -a$). If R is a PID then $ua + vb = 1$ for some $u, v \in R$, so $D(vX - uY) = 1$, and (3) holds. □

1.3 Hilbert's fourteenth problem

One interesting aspect of derivations is their connection with Hilbert's fourteenth problem, which can be stated as:

> Let k be a field and L a subfield of the field of rational functions $k(X_1, \dots, X_n)$ containing k. Is $L \cap k[X_1, \dots, X_n]$ a finitely generated k-algebra?

The first counterexample to Hilbert's 14th problem was found in 1958 by Nagata in dimension $n = 32$ [22]. Roberts in [26] gave a new counterexample to Hilbert's problem (with $n = 7$). This example was used by Deveney and Finston in [9] to show that there exists a locally nilpotent derivation of $k^{[7]}$ whose kernel is not a finitely generated k-algebra.

If one has a derivation of $k[X_1, \dots, X_n]$ whose kernel is not finitely generated then one obtains a counterexample to Hilbert's 14th problem by taking $L = \mathrm{qt}(\ker D)$. Furthermore, Derksen [8] and Nowicki [24] showed that essentially all counterexamples to Hilbert 14 can be realized as kernels of derivations. In particular, note the following interesting fact (Theorem 5.4 of [24]):

Theorem 1.3.1. *Let B be a domain and a finitely generated k-algebra. For a subalgebra A of B, the following are equivalent:*

1. *There exists a k-derivation d of B such that $A = \ker d$.*
2. *A is algebraically closed in B.*

It was proved by Zariski in [28] that Hilbert 14 is true if $\mathrm{tr.deg}_k(L) \le 2$, and this was used by Nagata and Nowicki to show in [23] that the kernel of any derivation of $k[X_1, \dots, X_n]$ is finitely generated if $n \le 3$.

The following derivation is essentially the example of Roberts, Deveney and Finston which we mentioned in the above:

Example 1.3.2. Let $B = k[x, y, z, s, t, u, v]$ be a polynomial ring of seven variables over a field k, and let $D : B \to B$ be the derivation defined by

$$D = x^3 \frac{\partial}{\partial s} + y^3 \frac{\partial}{\partial t} + z^3 \frac{\partial}{\partial u} + x^2 y^2 z^2 \frac{\partial}{\partial v}.$$

Then $\ker D$ is not finitely generated.

The derivation D in this example is elementary, and this shows that elementary derivations are not trivial. It is easy to construct similar examples (elementary, with non finitely generated kernel) in dimension higher than 7. Van den Essen and Janssen proved in [12] that elementary derivations of a polynomial ring of dimension less than 6 have finitely generated kernels. For elementary derivations of $k^{[6]}$, the question is still open. Recently, Freudenburg [14] constructed a triangular derivation of $k^{[6]}$ with a non finitely generated kernel:

Example 1.3.3. (Freudenburg's Example) Let $B = k[x, y, s, t, u, v]$, and let $D : B \to B$ be the derivation defined by

$$D = x^3\frac{\partial}{\partial s} + y^3 s\frac{\partial}{\partial t} + y^3 t\frac{\partial}{\partial u} + x^2y^2\frac{\partial}{\partial v}.$$

Then $\ker D$ is not finitely generated.

A few months later, Daigle and Freudenburg took a step further. They used the previous example to construct a counterexample in dimension 5, see [6].

Example 1.3.4. (Daigle and Freudenburg's Example) Let $B = k[a, b, x, y, z]$, and let $D : B \to B$ be the derivation defined by

$$D = a^2\frac{\partial}{\partial x} + (ax + b)\frac{\partial}{\partial y} + y\frac{\partial}{\partial z}$$

Then $\ker D$ is not finitely generated.

So now Hilbert 14 is open in dimension 4 only. When we look at the three examples above, we notice that they are all triangular and monomial (a derivation is *monomial* if it sends every variable to a monomial; in Example 1.3.4, this is the case after a change of variables). Maubach proved in [20] that in dimension 4, triangular monomial derivations have finitely generated kernels.

Theorem 1.3.5. ***(Maubach)*** *Let $A = k[X_1, X_2, X_3, X_4]$. Then the kernel of every monomial triangular derivation $D : A \to A$ is generated by at most 4 elements.*

Therefore if one wants to find a counterexample in dimension 4, one has to construct a more complicated example. On the other hand, one can always try to prove that every locally nilpotent derivation of $k^{[4]}$ has a finitely generated kernel (this would prove a special case of Hilbert 14 in dimension 4). In [5] some interesting results were given for triangular derivations of $k^{[4]}$. Using the notation $\#(R) = s$ for a finitely generated k-algebra R, indicating that R can be generated by s elements but not by $s - 1$, the following was proven:

Theorem 1.3.6. *Given any integer $n \geq 3$, there exists a triangular derivation Δ of the polynomial ring $k[X_1, X_2, X_3, X_4]$ whose kernel satisfies $n \leq \#(\ker \Delta) \leq n + 1$.*

Chapter 2

Homogeneous Derivations

We shall begin with degree functions, which are closely related to the notion of a graded ring. We shall then discuss homogeneous derivations on graded rings, and a method for obtaining a homogeneous derivation $\operatorname{gr} D$ from a derivation D of a graded ring. The method used in this chapter can be summarized as follows: given a derivation D of a polynomial ring B, we introduce a grading of B and the homogeneous derivation $\operatorname{gr} D$; in some cases, we hope to get information regarding D by studying the properties of $\operatorname{gr} D$.

2.1 Degree functions

Definition 2.1.1. Let B be an integral domain. By a *degree function* on B, we mean a map $w : B \to \mathbb{Z} \cup \{-\infty\}$ satisfying, for all $f, g \in B$,

1. $w(f) = -\infty$ if and only if $f = 0$,
2. $w(f + g) \leq \max(w(f), w(g))$ and
3. $w(fg) = w(f) + w(g)$.

If A is a subring of B and $w(\lambda) = 0$ for all $\lambda \in A \backslash \{0\}$, we can say that w is *over* A.

Every degree function $w : B \to \mathbb{Z} \cup \{-\infty\}$ determines a filtration $F = \{F_iB\}_{i\in\mathbb{Z}}$ of B, defined by $F_iB = \{f \in B \mid w(f) \leq i\}$. To be precise, each F_iB is a subgroup of $(B, +)$, with $F_iB \subseteq F_{i+1}B$ for all $i \in \mathbb{Z}$ and the following conditions hold:

1. $\bigcup_{i\in\mathbb{Z}} F_iB = B$,
2. $\bigcap_{i\in\mathbb{Z}} F_iB = \{0\}$,
3. $F_iBF_jB \subseteq F_{i+j}B$ (for all $i, j \in \mathbb{Z}$) and
4. the associated graded ring, $\operatorname{gr} B$, is an integral domain.

Here, the associated graded ring is $\operatorname{gr} B = \oplus_{i\in\mathbb{Z}} \operatorname{gr}_i B$, where $\operatorname{gr}_i B = F_iB/F_{i-1}B$. The *symbol map* $\operatorname{gr} : B \to \operatorname{gr} B$ can be defined by setting $\operatorname{gr}(0) = 0$ and, for $f \in B\backslash\{0\}$,

$$\operatorname{gr}(f) = f + F_{i-1}B \in \operatorname{gr}_i B,$$

where $f \in F_iB\backslash F_{i-1}B$. In order to stress that these objects are determined by w, one may write $F^w = \{F_i^w B\}_{i\in\mathbb{Z}}$, $\operatorname{gr}^w B = \oplus_{i\in\mathbb{Z}} \operatorname{gr}_i^w B$ and $\operatorname{gr}^w : B \to \operatorname{gr}^w B$.

Conversely, if $F = \{F_iB\}_{i\in\mathbb{Z}}$ is a filtration of B satisfying the above conditions, then the map $w : B \to \mathbb{Z} \cup \{-\infty\}$ defined by

$$w(f) = \inf\{i \in \mathbb{Z} \mid f \in F_iB\}$$

is a degree function on B.

By a *filtration* of B, we always mean the special type of filtration described in the above, meaning that it is always assumed that conditions (1-4) are satisfied. So filtrations are now equivalent to degree functions.

Consider the special case where B is already a graded ring:

2.1.2. Let $B = \oplus_{i\in\mathbb{Z}} B_i$ be a graded integral domain. For each i, let $h_i : B \to B_i$ denote the projection map (i.e., $f = \sum h_i(f)$ for all $f \in B$).

The grading determines the filtration $F_nB = \oplus_{i\leq n} B_i$ and the corresponding degree function on B is

$$w(f) = \sup\{i \in \mathbb{Z} \mid h_i(f) \neq 0\},$$

where $f \in B\backslash\{0\}$. The filtration F determines the associated graded ring $\operatorname{gr} B$ and, clearly, the maps $B_n \hookrightarrow F_nB \to \operatorname{gr}_n B$ give an isomorphism $B \to \operatorname{gr} B$ of graded rings. Note that the symbol map $\operatorname{gr} : B \to \operatorname{gr} B$ is not an isomorphism (not even an additive map), except in the trivial case $B = B_0$.

We will often need to consider two degree functions on a ring B. In this case we note:

2.1.3. Let B be an integral domain and $\nu, \nu' : B \to \mathbb{Z} \cup \{-\infty\}$ two degree functions satisfying $\nu \leq \nu'$ (i.e., $\nu(b) \leq \nu'(b)$ for all b). Then we have $F_i^{\nu'}B \subseteq F_i^{\nu}B$ for all i and this defines a ring homomorphism $\psi_{\nu,\nu'} : \operatorname{gr}^{\nu'} B \to \operatorname{gr}^{\nu} B$ which is homogeneous of degree zero. The kernel of $\psi_{\nu,\nu'}$ is a homogeneous prime ideal of $\operatorname{gr}^{\nu'} B$ and, given any $b \in B\backslash\{0\}$,

$$\psi_{\nu,\nu'}(\operatorname{gr}^{\nu'} b) = 0 \Leftrightarrow \nu(b) < \nu'(b).$$

In particular, $\psi_{\nu,\nu'}$ is injective if and only if $\nu = \nu'$.

The following special case of 2.1.3 is particularly useful:

2.1.4. Let B be an integral domain, $\nu : B \to \mathbb{Z} \cup \{-\infty\}$ a degree function, $F = \{F_iB\}_{i\in\mathbb{Z}}$ the filtration determined by ν and $R = \operatorname{gr}^{\nu} B$ the associated graded ring ($R_i = \operatorname{gr}_i^{\nu} B$). Let also $B = \oplus_{i\in\mathbb{Z}} B_i$ be a $\mathbb{Z}$-grading satisfying $B_i \subseteq F_iB$ for all $i \in \mathbb{Z}$; this is equivalent to assuming that $\nu \leq w$, where $w : B \to \mathbb{Z} \cup \{-\infty\}$ is the degree function corresponding to the grading.

We have $B_i \hookrightarrow F_iB \to \operatorname{gr}_i B = R_i$ for each i, and this defines a ring homomorphism $\varphi : B \to R$ which is homogeneous of degree zero. Note that φ is the composition of the isomorphism $B \to \operatorname{gr}^{w} B$ with the homomorphism $\psi_{\nu,w} : \operatorname{gr}^{w} B \to \operatorname{gr}^{\nu} B$. Denote $\mathfrak{p} = \ker \varphi$; then $\mathfrak{p}$ is a homogeneous prime ideal of B. If $h \in B$ is nonzero and homogeneous, then

$$h \in \mathfrak{p} \Leftrightarrow \nu(h) < w(h).$$

Also observe that:

- $\mathfrak{p} = 0$ if and only if $\nu = w$

and that, if we assume that ν is nonnegative (i.e., $\nu(B) \subseteq \mathbb{N} \cup \{-\infty\}$):

- $\mathfrak{p}$ is contained in the prime ideal $B_+ = \oplus_{i>0} B_i$;
- $\mathfrak{p}$ is maximal if and only if B_0 is a field and $\mathfrak{p} = B_+$.

2.2 Associated derivations

In the last section, we defined an associated graded ring $\operatorname{gr} B$ from a ring B and a degree function. We shall now define an associated derivation $\operatorname{gr} D$ from a given derivation D.

2.2.1. Let B be an integral domain, F a filtration of B and w the corresponding degree function. Recall the following notions.

1. Given a map $D : B \to B$, define $w(D) \in \mathbb{Z} \cup \{-\infty, \infty\}$ by

$$w(D) = \sup\{\, w(Dx) - w(x) \mid x \in B \setminus \{0\}\,\}.$$

   Note that $w(D) = -\infty$ if and only if D is identically zero. We call $w(D)$ the *degree* of D (with respect to w).

2. If $0 \neq D : B \to B$ is an additive map and $w(D) < \infty$, the *associated map* $\operatorname{gr} D : \operatorname{gr} B \to \operatorname{gr} B$ is defined as follows. Let $d = w(D) \in \mathbb{Z}$; then $D(F_i B) \subseteq F_{i+d} B$ holds for all $i \in \mathbb{Z}$, so D determines additive maps $\operatorname{gr}_i D : \operatorname{gr}_i B \to \operatorname{gr}_{i+d} B$; then the $\operatorname{gr}_i D$ extend uniquely to an additive map $\operatorname{gr} D : \operatorname{gr} B \to \operatorname{gr} B$. Note that $\operatorname{gr} D$ is nonzero and homogeneous of degree $w(D)$.

   If D is the zero map then we define $\operatorname{gr} D = 0$. Hence, $\operatorname{gr} D = 0$ if and only if $D = 0$. Note, also, that if D is a derivation then so is $\operatorname{gr} D$.

Remark 5. The degree of D might be infinite: Let $B = k[x_1, \dots, x_n, \dots]$, the polynomial ring in infinitely many variables, graded according to the standard total degree. Define a k-derivation $D : B \to B$ by $Dx_1 = 0$ and, for $n > 1$, $Dx_n = x_1^n$. Clearly D is locally nilpotent but $\deg D$ is infinite.

Definition 2.2.2. Let $B = \oplus_{i\in\mathbb{Z}} B_i$ be a graded integral domain and $w : B \to \mathbb{Z} \cup \{-\infty\}$ the degree function corresponding to the grading. Let $D : B \to B$ be an additive map satisfying $w(D) < \infty$. The *homogeneization of D* (with respect to the grading) is the map $\tilde{D} : B \to B$ defined by $\tilde{D} = \theta^{-1} \circ (\operatorname{gr} D) \circ \theta$, where $\operatorname{gr} D$ is defined in 2.2.1 and $\theta : B \to \operatorname{gr} B$ is the isomorphism of graded rings mentioned in 2.1.2.

Thus $\tilde{D} = 0$ if and only if $D = 0$ and, more generally, $w(\tilde{D}) = w(D)$. Also note that $\tilde{D}$ is an additive map, is homogeneous of degree $w(D)$ and is a derivation (resp. a locally nilpotent derivation) whenever D is one.

Remark 6. We will often identify the isomorphic graded rings B and $\operatorname{gr} B$. Consequently, we will often write "$\operatorname{gr} D$" when we mean "$\tilde{D}$", and use the terms "homogeneization of D" and "associated derivation" interchangeably.

We now give some explicit information about $\operatorname{gr} D$.

Lemma 2.2.3. *Let $B = \oplus_{i\in\mathbb{Z}} B_i$ be a graded integral domain of characteristic zero and $w : B \to \mathbb{Z} \cup \{-\infty\}$ the degree function corresponding to the grading. For each i, let $h_i : B \to B_i$ be the projection map. Let $0 \neq D : B \to B$ be a locally nilpotent derivation such that $w(D) < \infty$, let $\operatorname{gr} D : B \to B$ be its homogeneization and write $d = w(D)$.*

1. $(\operatorname{gr} D)(h_m f) = h_{m+d}(Df)$*, for all $f \in B$ and all $m \geq w(f)$.*

2. $\ker(\operatorname{gr} D) = \{f \in B \mid \forall i, w(Df_i) - w(f_i) < w(D)\}$*, where we write $f = \sum_i f_i$, $f_i \in B_i$.*

3. $\nu_{\operatorname{gr} D}(H) \leq \nu_D(H)$*, for all homogeneous elements H of B.*

Proof. Assertion (1) is obvious and (2) follows directly from (1). We prove (3). Let $m = w(H)$. Since H is w-homogeneous, we have $(\operatorname{gr} D)H = (\operatorname{gr} D)(h_m H)$. By (1), $(\operatorname{gr} D)(h_m H) = h_{m+d}(DH)$. Note that

$$w(DH) \leq w(H) + w(D) = m + d.$$

Refering to (1) again,

$$(\operatorname{gr} D)^2(H) = (\operatorname{gr} D)((\operatorname{gr} D)H) = (\operatorname{gr} D)(h_{m+d}(DH)) = h_{m+2d}(D^2 H).$$

Inductively, $(\operatorname{gr} D)^l(H) = h_{m+ld}(D^l H)$. So $D^l H = 0$ implies $(\operatorname{gr} D)^l H = 0$, which means $\nu_{\operatorname{gr} D}(H) \leq \nu_D(H)$. □

Remark 7. Without the assumption of H being w-homogeneous, assertion (3) would not be true. This is because $\ker D \not\subseteq \ker(\operatorname{gr} D)$ generally. So for $f \in B$ such that $Df = 0$ but $(\operatorname{gr} D)f \neq 0$ we obtain $\nu_{\operatorname{gr} D}(f) > \nu_D(f) = 0$.

Next, we would like to know some cases where $w(D)$ is finite.

2.2.4. Let $B = \oplus_{i \in \mathbb{Z}} B_i$ be a graded integral domain and $w : B \to \mathbb{Z} \cup \{-\infty\}$ the degree function corresponding to the grading. Let $D : B \to B$ be any map. It is convenient to introduce the auxiliary function $\delta = \delta_D : B \to \mathbb{Z} \cup \{-\infty\}$ defined by $\delta(0) = -\infty$ and, for $b \in B \setminus \{0\}$, $\delta(b) = w(Db) - w(b)$ (note that $w(Db) = \delta(b) + w(b)$ holds for *all* $b \in B$). We also define $\delta(S) \in \mathbb{Z} \cup \{-\infty, \infty\}$ for any subset S of B by setting $\delta(\emptyset) = -\infty$ and $\delta(S) = \sup\{\, \delta(s) \mid s \in S \,\}$ $(S \neq \emptyset)$. Observe that $\delta(S) < \infty$ whenever S is a finite set. Also, $w(D) = \delta(B)$.

Lemma 2.2.5. *Let $B = \oplus_{i \in \mathbb{Z}} B_i$ be a graded integral domain of characteristic zero and $w : B \to \mathbb{Z} \cup \{-\infty\}$ the degree function corresponding to the grading. Let $D : B \to B$ be a derivation and $\delta = \delta_D$. Then:*

1. $\delta(xy) \leq \max(\delta(x), \delta(y))$, *for all* $x, y \in B$;

2. $\delta(x + y) \leq \max(\delta(x), \delta(y))$, *for all* $x, y \in B_0$;

3. *if* $x_1, \dots, x_n \in B$ *satisfy* $w(x_1), \dots, w(x_{n-1}) < w(x_n)$, *then* $\delta(x_1 + \cdots + x_n) \leq \max(\delta(x_1), \dots, \delta(x_n))$;

4. *if E is a subring of B_0, $S \neq \emptyset$ a subset of E and E is algebraic over the subring $\overline{S}$ generated by S, then $\delta(E) = \delta(S)$;*

5. *if $H \subset B$ is a set of homogeneous elements such that $B = B_0[H]$, then $w(D) = \max(\delta(B_0), \delta(H))$.*

Proof. For assertion (1), we may assume that $x, y \neq 0$. Then

$$\begin{aligned}\delta(xy) &= w(D(xy)) - w(xy) = w(yDx + xDy) - w(xy)\\ &\leq \max(w(yDx), w(xDy)) - w(xy)\\ &= \max(w(Dx) + w(y), w(Dy) + w(x)) - w(xy) = \max(\delta(x), \delta(y)).\end{aligned}$$

For assertion (2), note that $\delta(t) = w(Dt)$ for all $t \in B_0$, so

$$\delta(x+y) = w(D(x+y)) = w(Dx + Dy) \leq \max(w(Dx), w(Dy)) = \max(\delta(x), \delta(y)).$$

In (3), we have $w(x_1 + \cdots + x_n) = w(x_n)$ and (for some j)

$$w(D(x_1 + \cdots + x_n)) \leq \max(w(Dx_1), \ldots, w(Dx_n)) = w(Dx_j),$$

so

$$\begin{aligned}\delta(x_1 + \cdots + x_n) &= w(D(x_1 + \cdots + x_n)) - w(x_1 + \cdots + x_n)\\ &\leq w(Dx_j) - w(x_n) = \delta(x_j) + w(x_j) - w(x_n) \leq \delta(x_j).\end{aligned}$$

Note that, in assertion (4), we have $\delta(\overline{S}) = \delta(S)$ by (1) and (2); so we may assume that S is a ring. Given $b \in E$, choose a polynomial $\Phi(T) \in S[T] \setminus \{0\}$ (where T is an indeterminate) of minimal degree such that $\Phi(b) = 0$. Then $0 = D\Phi(b) = \Phi^{(D)}(b) + \Phi'(b)Db$ and (using $\operatorname{char} B = 0$) $\Phi'(b) \in B_0 \setminus \{0\}$ imply $w(\Phi^{(D)}(b)) = w(Db) = \delta(b)$. Now $\Phi^{(D)}(b) = \sum_i D(s_i)b^i$ (where $s_i \in S$) and $w(D(s_i)b^i) = w(Ds_i) = \delta(s_i) \leq \delta(S)$ for each i, so $\delta(b) = w(\Phi^{(D)}(b)) \leq \delta(S)$.

For proving assertion (5), consider $b \in B$; choose a finite subset $\{x_1, \ldots, x_n\}$ of H such that $b \in B_0[x_1, \ldots, x_n]$. We will show that

$$\delta(b) \leq \max(\delta(B_0), \delta(x_1), \ldots, \delta(x_n)).$$

By a homogeneous subring of B, we mean a subring $R \subseteq B$ satisfying $R = \sum(B_i \cap R)$. Note that, for each $j = 0, \ldots, n-1$, $B_0[x_1, \ldots, x_j]$ is a homogeneous subring of B and x_{j+1} is homogeneous. So it suffices to prove that, if R is a homogeneous subring of B and $x \in B \setminus \{0\}$ is a homogeneous element, then $\delta(R[x]) \leq \max(\delta(R), \delta(x))$. Since $R[x]$ is a homogeneous subring of B, and in view of part (3) of Lemma 2.2.5, it's enough to check that $\delta(f) \le \max(\delta(R), \delta(x))$ holds for all *homogeneous* elements f of $R[x]$.

Let $0 \neq f \in R[x] \cap B_n$ and write $f = \Phi(x)$, where the polynomial $\Phi(X) = \sum_{i \in \mathbb{N}} a_i X^i$ ($a_i \in R$) satisfies $w(a_i) + iw(x) = n$ whenever $a_i \neq 0$. Then

$$w(a_i) + iw(x) \le n \quad \text{for all } i.$$

We have $w(D(a_i)x^i) = \delta(a_i) + w(a_i) + iw(x) \le \delta(R) + n$ for all $i \in \mathbb{N}$, so

$$w(\Phi^{(D)}(x)) \le \delta(R) + n.$$

Also, $w(\Phi'(x)) = w(\sum ia_i x^{i-1}) \le \max_i w(a_i x^{i-1}) \le n - w(x)$, so

$$w(\Phi'(x)Dx) \le n - w(x) + w(Dx) = n + \delta(x).$$

Since $Df = \Phi^{(D)}(x) + \Phi'(x)Dx$, we conclude that

$$\begin{aligned} w(Df) &\le \max(w(\Phi^{(D)}(x)), w(\Phi'(x)Dx)) \\ &\le \max(\delta(R) + n, \delta(x) + n) = \max(\delta(R), \delta(x)) + n, \end{aligned}$$

as desired. □

Corollary 2.2.6. *Let $B = \oplus_{i \in \mathbb{Z}} B_i$ be a graded integral domain of characteristic zero and $w : B \to \mathbb{Z} \cup \{-\infty\}$ the degree function corresponding to the grading. Let $H \subset B$ be a set of homogeneous elements satisfying $B = B_0[H]$. Suppose that B_0 contains a field k and let $T \subset B_0$ be a transcendence basis of $\operatorname{qt}(B_0)$ over k. Then every locally nilpotent derivation $D : B \to B$ satisfies $w(D) = \delta_D(T \cup H)$.*

Proof. Let $\delta = \delta_D$. Since D is locally nilpotent, all units of B are in $\ker D$ and in particular D is a k-derivation; so $\delta(k) = \{-\infty\}$. Observe that B_0 is algebraic over $k[T]$, which is the ring generated by the set $k \cup T$; so, by part (4) of Lemma 2.2.5, $\delta(B_0) = \delta(k \cup T) = \max(\delta(k), \delta(T)) = \delta(T)$. Then part (5) of Lemma 2.2.5 gives $w(D) = \max(\delta(B_0), \delta(H)) = \max(\delta(T), \delta(H)) = \delta(T \cup H)$. □

Corollary 2.2.7. *Let B be a finitely generated domain over a field k of characteristic zero and let $D : B \to B$ be a locally nilpotent derivation. If $w : B \to \mathbb{Z} \cup \{-\infty\}$ is* *the degree function of an arbitrary $\mathbb{Z}$-grading of B, then $w(D) < \infty$. More precisely, if $x_1, \ldots, x_n \in B$ are homogeneous and satisfy $B = k[x_1, \ldots, x_n]$, then $w(D) = \delta_D(\{x_1, \ldots, x_n\})$.*

Proof. Observe that each element of k^* is a unit in the graded integral domain B, and so is a homogeneous element of B. It follows that $k \subseteq B_0$, where $B = \oplus_{i \in \mathbb{Z}} B_i$ is the given grading (if $\lambda \in k \setminus B_0$ then one of λ, $1 + \lambda$ is non-homogeneous, which is absurd). Choose homogeneous elements $x_1, \ldots, x_n$ of B such that $B = k[x_1, \ldots, x_n]$; then $k \subseteq B_0$ and 2.2.6 gives $w(D) = \delta_D(\{x_1, \ldots, x_n\}) < \infty$. □

AN APPLICATION

It is desirable to give an example showing the use of $\operatorname{gr} D$. In [19], Makar-Limanov defined an invariant for any commutative ring R, and we use $\mathrm{ML}(R)$ to denote this invariant. The definition is:

$$\mathrm{ML}(R) = \bigcap \{\ker D \mid D \text{ is a locally nilpotent derivation of } R\}.$$

Clearly, $\mathrm{ML}(R)$ is a subring of R; it is called the ring of *absolute constants* of R.

Proposition 2.2.8. *Let R be a k-domain which has finite transcendence degree over k. If R satisfies $\mathrm{ML}(R) = R$, then $\mathrm{ML}(R^{[1]}) = R$.*

Proof. We want to prove that under the assumption of $\mathrm{ML}(R) = R$, every non-zero locally nilpotent derivation of $R^{[1]} = R[T]$ has kernel R. So, let D be such a derivation, and consider $R[T]$ as a graded ring, the grading being determined by the usual T-degree. Then, 2.2.6 shows that D has a finite degree, and the homogeneization $D_1 = \operatorname{gr} D$ of D is defined (and $D_1 \neq 0$). Let $d = \deg D = \deg D_1$, $E = \ker D$. Note that if $E \subset R$, then $E = R$, because otherwise the fact that E is algebraically closed in $R[T]$ would imply

$$\operatorname{tr.deg}_E R > 0, \text{ and then } \operatorname{tr.deg}_E R[T] > 1.$$

However by part (5) of 1.1.5,

$$\operatorname{tr.deg}_E R[T] = 1.$$

This is a contradiction. So, we may assume $E \not\subseteq R$, and choose $f(T) \in R[T]$ such that $\deg f(T) \geq 1$ and $D(f(T)) = 0$. Write

$$f(T) = a_m T^m + \cdots + a_1 T + a_0 \quad (a_i \in R \text{ and } a_m \neq 0),$$

then by 2.2.3

$$D_1(a_m T^m) = h_{m+d}(Df(T)) = 0,$$

which implies $D_1 T = 0$ because $\ker D_1$ is factorially closed in $R[T]$. We know that D_1 is homogeneous of degree d, so $D_1(RT^i) \subseteq RT^{d+i}$, and thus

$$\Delta = T^{-d} D_1$$

is a derivation of $R[T]$, homogeneous of degree zero. Since $T \in \ker D_1$, this Δ is also locally nilpotent. Note that $\Delta R \subseteq R$ and $\Delta R \neq 0$. For the second assertion, observe that $\Delta R = 0$ would imply $\Delta = 0$ (because $\Delta T = 0$), but $\Delta \neq 0$ because $D_1 \neq 0$. Restricting Δ to R, we get a non-zero locally nilpotent derivation of R. This contradicts $\mathrm{ML}(R) = R$, and we can conclude that $E = \ker D \subseteq R$, and thus $E = R$. So, we have proved that every non-zero locally nilpotent derivation of $R[T]$ has kernel R, and $\mathrm{ML}(R^{[1]}) = R$. □

2.3 The case $B = k[X_1, \ldots, X_n]$

In this section, B denotes the polynomial ring $k[X_1, \ldots, X_n]$ in $n \geq 1$ variables over a field k of characteristic zero. Note that every nonnegative degree function $w : B \to \mathbb{N} \cup \{-\infty\}$ is "over k", i.e., satisfies $w(\lambda) = 0$ for all $\lambda \in k$.

2.3.1. Let $B = k[X_1, \ldots, X_n]$. Then every degree function $\nu : B \to \mathbb{N} \cup \{-\infty\}$ determines an $\mathbb{N}$-grading of B and a degree function $\overline{\nu} : B \to \mathbb{N} \cup \{-\infty\}$. Namely, the grading $B = \oplus_{i \in \mathbb{N}} B_i$ is obtained by declaring that $k \subseteq B_0$ and that, for each j, X_j is homogeneous of degree $\nu(X_j)$; and we let $\overline{\nu}$ be the degree function corresponding to that grading.

Observe that $\overline{\nu}$ satisfies $\overline{\nu}(X_j) = \nu(X_j)$ for every j, so, keeping in mind that ν is over k, we get $\overline{\nu}(M) = \nu(M)$ for every monomial $M = \lambda X_1^{e_1} \cdots X_n^{e_n}$ ($\lambda \in k$, $e_j \in \mathbb{N}$). Moreover, if $f \in B\backslash\{0\}$ is written as a sum of monomials, $f = \sum M_j$, then

$$\overline{\nu}(f) = \max_j \overline{\nu}(M_j) = \max_j \nu(M_j) \geq \nu(f).$$

Hence, $\overline{\nu} \geq \nu$ and we are in the situation described in 2.1.4 (with $w = \overline{\nu}$). Note that, in this case, the following holds (notation as in 2.1.4):

- $\varphi : B \to \operatorname{gr}^{\nu} B$ is a homomorphism of k-algebras.
- No nonzero monomials belong to $\mathfrak{p} = \ker \varphi$.
- $\mathfrak{p}$ is not maximal. Indeed, if $\mathfrak{p}$ is maximal then, by 2.1.4, B_0 is a field (so $B_0 = k$) and $\mathfrak{p} = B_+$, so $\mathfrak{p} = (X_1, \ldots, X_n)$, resulting in $X_1 \in \mathfrak{p}$ being a contradiction.
- Given any monomial $M = \lambda X_1^{e_1} \cdots X_n^{e_n} (\lambda \in k,\ e_j \in \mathbb{N})$, $\mathfrak{p} \cap k[M] = 0$; in particular, $X_j - \lambda \notin \mathfrak{p}$, for all j and all $\lambda \in k$.

2.3.2. Let $B = k[X_1, \ldots, X_n]$ and $D : B \to B$ a locally nilpotent derivation. Then the following objects are determined by D:

1. The exponent $\nu = \nu_D : B \to \mathbb{N} \cup \{-\infty\}$ of D. Recall that ν is a degree function.
2. The $\mathbb{N}$-grading of B, $B = \oplus_{i=0}^{\infty} B_i$, determined by ν as in 2.3.1, and the degree function $\overline{\nu} : B \to \mathbb{N} \cup \{-\infty\}$ corresponding to this grading. Recall that $\overline{\nu} \geq \nu$.
3. A $\overline{\nu}$-homogeneous prime ideal $\mathfrak{p} \subset B$ (see 2.3.1). Note that the facts stated in 2.1.4 and 2.3.1 are valid here. In particular, if $h \in B \setminus \{0\}$ is $\overline{\nu}$-homogeneous,
$$h \in \mathfrak{p} \iff \nu(h) < \overline{\nu}(h).$$
4. The homogeneization $\operatorname{gr}^{\overline{\nu}} D : B \to B$ of D with respect to the grading (2).

From now-on, we will make use of these objects without further comments.

Lemma 2.3.3. *Let $B = k[X_1, \ldots, X_n]$, let $0 \neq D : B \to B$ be a locally nilpotent derivation and let $\overline{\nu} : B \to \mathbb{N} \cup \{-\infty\}$ be the degree function determined by D as in 2.3.2. Then $\overline{\nu}(D) \geq -1$ and if equality holds then D is triangular.*

Proof. Choose i such that $DX_i \neq 0$. Since $\overline{\nu}(X_i) = \nu_D(X_i)$ and $\overline{\nu}(DX_i) \geq \nu_D(DX_i)$,

$$\overline{\nu}(DX_i) - \overline{\nu}(X_i) \geq \nu_D(DX_i) - \nu_D(X_i) = -1,$$

so $\max_j(\overline{\nu}(DX_j) - \overline{\nu}(X_j)) \geq -1$. This implies that $\overline{\nu}(D) \geq -1$ by 2.2.7. For the second assertion, we may assume that $\nu_D(X_1) \leq \cdots \leq \nu_D(X_n)$. If $\overline{\nu}(D) = -1$ then $\overline{\nu}(DX_j) < \overline{\nu}(X_j) = \min\{\overline{\nu}(X_j), \ldots, \overline{\nu}(X_n)\}$, so $DX_j \in k[X_1, \ldots, X_{j-1}]$ (for each j), i.e., D is triangular. □

Lemma 2.3.4. *Let $B = k[X_1, \ldots, X_n]$ $(n \geq 2)$, let $0 \neq D : B \to B$ be a locally nilpotent derivation and let ν, $\overline{\nu}$ and $\mathfrak{p}$ be determined by D as in 2.3.2. Let $a_i = \nu(X_i) \in \mathbb{N}$ and assume that $\operatorname{ht} \mathfrak{p} = n - 1$. Then, for all i, j, $\exists \beta_{ij} \in k^*$ such that $X_i^{a_j} - \beta_{ij} X_j^{a_i} \in \mathfrak{p}$. In particular, $a_i > 0$ for all i.*

Proof. We first prove the Lemma under the assumption that k is algebraically closed. Let $C \subset \mathbf{A}^n$ be the zero set of $\mathfrak{p}$ and note that C is a curve. Since the monomial $X_1 \cdots X_n$ does not belong to $\mathfrak{p}$, we may choose $\alpha = (\alpha_1, \ldots, \alpha_n) \in C$ such that $\alpha_i \neq 0$ for all i. Define $t * \alpha = (t^{a_1}\alpha_1, \ldots, t^{a_n}\alpha_n)$ for each $t \in k$ and consider the subset $E = \{t * \alpha \mid t \in k\}$ of $\mathbf{A}^n$. Since each α_i is nonzero and some a_i is positive (because $D \neq 0$), E is an infinite set.

We claim that, given any $f \in B_i \setminus \{0\}$,

$$f \in \mathfrak{p} \Rightarrow f(\alpha) = 0 \Rightarrow f \text{ vanishes on } E \Rightarrow f \in \mathfrak{p}.$$

In fact, the first implication is trivial and the second one follows from $f(t*\alpha) = t^i f(\alpha)$. In particular, every homogeneous element of $\mathfrak{p}$ vanishes on E, so $E \subseteq C$, and since E is infinite, we can obtain the last implication.

In particular, let $f_{ij} = \alpha_j^{a_i} X_i^{a_j} - \alpha_i^{a_j} X_j^{a_i}$, then $f_{ij}(\alpha) = 0$, so $f_{ij} \in \mathfrak{p}$. Dividing f_{ij} by $\alpha_j^{a_i}$ gives $X_i^{a_j} - \beta_{ij} X_j^{a_i} \in \mathfrak{p}$, with $\beta_{ij} \in k^*$. If $a_i = 0$, then we may choose j such that $a_j > 0$. Then $X_i^{a_j} - \beta_{ij} \in \mathfrak{p}$, which contradicts an earlier observation. So $a_i > 0$ for all i.

We now drop the assumption that k is algebraically closed and let $\bar{k}$ be an algebraic closure of k. Write $\bar{B} = \bar{k}[X_1, \ldots, X_n]$ and let $\bar{D} : \bar{B} \to \bar{B}$ be the extension of D. Then, in addition to $\mathfrak{p} \subset B$, ν and $\overline{\nu}$, which are determined by D, we may also consider $\bar{\mathfrak{p}} \subset \bar{B}$, $\nu_{\bar{D}}$ and $\overline{\nu_{\bar{D}}}$, determined by $\bar{D}$. If $h \in B \setminus \{0\}$ is $\overline{\nu}$-homogeneous then

$$h \in \bar{\mathfrak{p}} \iff \nu_{\bar{D}}(h) < \overline{\nu_{\bar{D}}}(h) \iff \nu(h) < \overline{\nu}(h) \iff h \in \mathfrak{p},$$

so $\bar{\mathfrak{p}} \cap B = \mathfrak{p}$, and since $\bar{B}$ is an integral extension of B, $\bar{\mathfrak{p}}$ must have height $n-1$. By the special case proved in the above, we have $X_i^{a_j} - \beta_{ij} X_j^{a_i} \in \bar{\mathfrak{p}}$, with $\beta_{ij} \in \bar{k}^*$. This implies that $\nu_{\bar{D}}(X_i^{a_j} - \beta_{ij} X_j^{a_i}) < a_i a_j$, so

$$D^{a_i a_j}(X_i^{a_j}) - \beta_{ij} D^{a_i a_j}(X_j^{a_i}) = 0.$$

Since $D^{a_i a_j}(X_i^{a_j})$ and $D^{a_i a_j}(X_j^{a_i})$ are nonzero elements of B, it follows that $\beta_{ij} \in k^*$. □

HOMOGENEIZATION AND RANK

Lemma 2.3.5. *Let $R = k[x_1, \ldots, x_n]$, let $D : R \to R$ be a homogeneous locally nilpotent derivation with respect to the standard grading of R, and let $A = \ker D$. Use V_1 to denote the k-subspace of linear forms in R. If $\operatorname{rank} D = r$, then*

$$\dim_k(A \cap V_1) = n - r.$$

Moreover, if $r = 1$, then A is generated by $n-1$ linear forms.

Proof. Clearly $\dim_k(A \cap V_1) \leq n - r$. Let $(u_1, \ldots, u_n)$ be a coordinate system of R such that $k[u_{r+1}, \ldots, u_n] \subseteq A$. Then

$$u_i = a_{i0}(x) + a_{i1}(x) + \cdots + a_{im}(x),$$

where a_{ij} is the component of degree j of u_i. Since D is homogeneous, we have $a_{ij} \in A$ for $i > r$. By the chain rule, the jacobian determinant satisfies

$$\frac{\partial(u_1, \ldots, u_n)}{\partial(x_1, \ldots, x_n)} \in k^*$$

so the jacobian matrix evaluated at $x = 0$,

$$\|\frac{\partial(u_1, \ldots, u_n)}{\partial(x_1, \ldots, x_n)}(0)\|,$$

is invertible. This implies $\dim_k(a_{11}, \ldots, a_{n1}) = n$, and then $\dim_k(a_{r+1,1}, \ldots, a_{n1}) = n - r$. Since $a_{r+1,1}, \ldots, a_{n1} \in A \cap V_1$, $\dim_k(A \cap V_1) \geq n - r$. So $\dim_k(A \cap V_1) = n - r$. If $r = 1$, then , clearly, the subring $k[a_{2,1}, \ldots, a_{n,1}]$ of A is actually equal to A. □

Before we close this section, we make two observations concerning the relationship between $\operatorname{rank} D$ and $\operatorname{rank}(\operatorname{gr} D)$.

2.3.6. Let D be a locally nilpotent derivation of $B = k^{[n]}$. Then a coordinate system and a system of weights can be chosen such that $\operatorname{rank}(\operatorname{gr} D) \leq \operatorname{rank} D$. Indeed, let $r = \operatorname{rank} D$ and choose a coordinate system $(X_1, \ldots, X_n)$ of B such that $D(X_i) = 0$ for all $i > r$. Define $\operatorname{gr} D$ using the standard grading of $k[X_1, \ldots, X_n]$. Then $(\operatorname{gr} D)X_i = 0$ for all $i > r$, so $\operatorname{rank}(\operatorname{gr} D) \leq \operatorname{rank} D$.

The second observation is that replacing D by $\operatorname{gr} D$ may sometimes increase the rank, as is illustrated below:

Example 2.3.7. Let $B = k[X, Y, Z]$. Define $D : B \to B$ by $DX = 1, DY = X$ and $DZ = Y$. Clearly D is a locally nilpotent derivation of B, with a slice X , $\operatorname{rank} D = 1$, see §1.2. Consider $D_1 = \operatorname{gr} D$ with respect to the standard grading, then $D_1 X = 0, D_1 Y = X$ and $D_1 Z = Y$. We prove that $\operatorname{rank} D_1 = 2$. Clearly $\operatorname{rank} D_1 < 3$. We need only show that $\operatorname{rank} D_1 \neq 1$. Suppose $\operatorname{rank} D_1 = 1$, then $\ker D_1 = k[X, \alpha Y + \beta Z]$, by 2.3.5. But $D_1(\alpha Y + \beta Z) = \alpha X + \beta Y$ will never be 0 unless $\alpha = \beta = 0$, which is impossible. So $\operatorname{rank} D_1 = 2$.

2.4 Derivations of $k[X, Y, Z]$

Lemma 2.4.1. *Let $B = k[X; Y, Z]$, let $0 \neq D : B \to B$ be a locally nilpotent derivation and let $\overline{\nu}$ and $\mathfrak{p}$ be determined by D as in 2.3.2. Assume that DX, DY, DZ are nonzero and let $a = \overline{\nu}(X)$, $b = \overline{\nu}(Y)$ and $c = \overline{\nu}(Z)$.*

1. *If D is $\overline{\nu}$-homogeneous then* $\mathrm{ht}(\mathfrak{p}) = 2$ *and* $X^b - \alpha Y^a, X^c - \beta Z^a, Y^c - \gamma Z^b \in \mathfrak{p}$ *for some* $\alpha, \beta, \gamma \in k^*$.

2. *Suppose D is homogeneous with respect to some* $w_1 = (a_1, b_1, c_1) \in \mathbb{Z}^3$, $w_1 \neq (0,0,0)$. *If* $\mathrm{ht}\,\mathfrak{p} = 2$, *then* $nw_1 = m\overline{\nu}$ *for some* $m, n \in \mathbb{Z} \setminus \{0\}$. *It follows that D is $\overline{\nu}$-homogeneous.*

Proof. (1) Since D is homogeneous, we have $\ker D = k[f, g]$ with f and g homogeneous (see remark 3 in chapter 1). One can check that $f, g \in \mathfrak{p}$; since f and g are relatively prime, $\mathrm{ht}\,\mathfrak{p} > 1$. Since $\mathfrak{p}$ is not maximal, $\mathrm{ht}\,\mathfrak{p} = 2$. The second part follows from 2.3.4.

(2) If $\mathrm{ht}(\mathfrak{p}) = 2$ then, by 2.3.4, $X^b - \alpha Y^a \in \mathfrak{p}$ for some $\alpha \in k^*$, and

$$\nu(X^b - \alpha Y^a) < \overline{\nu}(X^b - \alpha Y^a) = ab.$$

It follows that $D^{ab}(X^b) - \alpha D^{ab}(Y^a) = 0$. So $w_1(X^b) = w_1(Y^a)$, i.e. $a_1 b = b_1 a$. Similarly, $a_1 c = c_1 a$ and $b_1 c = c_1 b$, so $\left(\begin{smallmatrix} a & b & c \\ a_1 & b_1 & c_1 \end{smallmatrix}\right)$ has rank one and we are done. □

Lemma 2.4.2. *Let $B = k[X, Y, Z]$ and let $0 \neq D : B \to B$ be a locally nilpotent derivation such that $D^2X = 0 = D^2Y$. Suppose that $DZ \neq 0$ and that D is homogeneous with respect to some weight $w = (1, 1, c)$, where $c \geq \nu_D(Z)$. Then* $\ker D$ *is either $k[X, Y]$ or $k[L, Z + h(X, Y)]$, where L is a linear form in X, Y. In particular,* $\mathrm{rank}\, D = 1$.

Proof. We may assume that D is irreducible. First, we claim that $\ker D$ contains a nonzero linear form in X, Y. To prove this claim, we may assume that k is algebraically closed (by 1.2.14); we may also assume that $DX \neq 0$ and $DY \neq 0$.

Let $\nu = \nu_D$, $\overline{\nu}$ and $\mathfrak{p}$ be as usual, and note that we have two gradings on B: The w-grading and the $\overline{\nu}$-grading. Write $c_1 = \nu(Z)$ and

$$(\overline{\nu}(X), \overline{\nu}(Y), \overline{\nu}(Z)) = (1, 1, c_1), \quad (c_1 \leq c).$$

Case 1: If $c_1 = c$, then D is homogeneous with respect to $\overline{\nu}$. By 2.4.1, $\mathfrak{p}$ contains $X + \alpha Y$ for some constant α. Then $D(X + \alpha Y) = 0$.

Case 2: If $c_1 < c$, write

$$DX = a_0 + a_1 Z + \cdots + a_r Z^r, \tag{2}$$

where $a_i \in k[X,Y]$. Note that DX is w-homogeneous. Let $d = w(DX)$. Then a_i is homogeneous of degree $(d - ci)$ (in $k[X,Y]$, homogeneous, w-homogeneous and $\overline{\nu}$-homogeneous are the same). If we compare the $\overline{\nu}$-weights of terms of DX, say

$$\begin{aligned}\overline{\nu}(a_iZ^i) - \overline{\nu}(a_jZ^j) &= \overline{\nu}(a_i) + ic_1 - \overline{\nu}(a_j) - jc_1 \\ &= (d - ci) + c_1 i - (d - cj) - c_1 j \\ &= c(j - i) - c_1(j - i) \\ &= (c - c_1)(j - i),\end{aligned}$$

it follows that, in the right hand side of (2), the first nonzero term, say a_iZ^i, has the highest $\overline{\nu}$-weight.

If $\nu(a_iZ^i) < \overline{\nu}(a_iZ^i)$, then $a_iZ^i \in \mathfrak{p}$; since $Z^i \notin \mathfrak{p}$, we must then have $a_i \in \mathfrak{p}$, so some linear factor L of a_i is in $\mathfrak{p}$. Then L satisfies $\nu(L) < \overline{\nu}(L) = 1$, so $L \in \ker D$.

If $\nu(a_iZ^i) = \overline{\nu}(a_iZ^i)$, then $\nu(a_iZ^i) > \overline{\nu}(a_jZ^j) \geq \nu(a_jZ^j)$ for all $j \neq i$, so $\nu(a_iZ^i) = \nu(DX) = 0$; it follows that $i = 0$, $a_0 \in \ker D$ and $DX = a_0$. If $a_0 \notin k$ then, as in the above paragraph, some factor L of a_0 is in $\ker D$. If $a_0 \in k$ then $w(D) = w(a_0) - w(X) < 0$, so $w(DY) < w(Y) = 1$, so $DY \in k$; then DX, DY are linearly dependent over k, so some nonzero linear form in X, Y belongs to $\ker D$.

This proves that $\ker D$ contains a linear form in X, Y. As we said at the beginning, this is valid without assuming that k is algebraically closed. (We now drop the assumption that $k = \bar{k}$.)

Now we prove the other assertions. Make a linear change in X, Y (keeping the same Z) and arrange that $DX = 0$. Note that D is still w-homogeneous, so, by 1.2.15, there is a w-homogeneous $Q \in B$ such that $\ker D = k[X,Q]$, $DY = Q_Z$ and $DZ = -Q_Y$ (see 1.2.12); moreover, Q is irreducible. Then $Q_Z = DY \in \ker D$, so $Q_Z = \lambda X^s$ ($\lambda \in k$ and $s \in \mathbb{N}$) and

$$Q = \lambda X^s Z + h(X,Y). \tag{3}$$

If $\lambda = 0$ then $\ker D = k[X,Y]$ and we are done; so assume that $\lambda \neq 0$. We claim that $s = 0$ (in which case we are done because $\lambda Z + h(X,Y) = Q \in \ker D$). To prove this, we assume the contrary: $s \geq 1$. Then $X \not| h(X,Y)$, so $\deg_Y(h) = s + c$, since h is homogeneous of degree $(s + c)$. It follows that, of all the nonzero monomials that occur in the right hand side of (3), Y^{s+c} is the one with highest ν-degree (because $\nu(X^iY^{s+c-i}) = s+c-i$ and $\nu(X^sZ) = c$). So $\nu(Q) = s+c > 0$, a contradiction. □

Proposition 2.4.3. *Let $B = k[X,Y,Z]$ and let $D : B \to B$ be a locally nilpotent derivation . If $D^2X = 0 = D^2Y$, then*

1. $\ker D$ *contains a non-zero linear form in X, Y or a polynomial of the form $Z + h(X,Y)$, and*

2. $\operatorname{rank} D < 3$.

Proof. It is clear that (2) is a direct consequence of (1). We need only to prove (1). By 1.2.14, we may assume that k is algebraically closed; we may also assume that $DX \neq 0$ and $DY \neq 0$. Let $c = \nu(Z)$ and let

$$\overline{\nu} = (\nu(X), \nu(Y), \nu(Z)) = (1,1,c).$$

If $\overline{\nu}(D) < 0$ then $\overline{\nu}(DX) = 0 = \overline{\nu}(DY)$, so $DX, DY \in k$, so DX, DY are linearly dependent over k and consequently $\ker D$ contains a nonzero linear form in X, Y. From now-on, we assume that $\overline{\nu}(D) \geq 0$.

We use induction on c, the case $c = 0$ being trivial.

Let $D_1 = \operatorname{gr} D$ with respect to $\overline{\nu}$. By 2.4.2, $\ker D_1$ is either equal to $k[X,Y]$ or to $k[X, Z + h(X,Y)]$.

If $\ker D_1 = k[X,Y]$ then $D_1 = P(X,Y)\frac{\partial}{\partial Z}$, for some $P(X,Y) \in k[X,Y]$; moreover, $P(X,Y) = D_1Z$ is $\overline{\nu}$-homogeneous. Then

$$DZ = P(X,Y) + \rho, \quad \text{where } \overline{\nu}(\rho) < \overline{\nu}(P). \tag{4}$$

Factor P into a product of linear forms in X, Y. We claim that at least one of these linear forms is in the kernel of D. Indeed, if this is not the case then $\nu(P) = \overline{\nu}(P)$. Note that $\overline{\nu}(DZ) = \overline{\nu}(P)$ by (4); since $\nu \leq \overline{\nu}$, we also have $\nu(DZ) = \nu(P)$ (because $\nu(P) = \overline{\nu}(P) > \overline{\nu}(\rho) \geq \nu(\rho)$). Since $X, Y \in \ker D_1$, part 2 of 2.2.3 gives $\overline{\nu}(DX) - \overline{\nu}(X) < \overline{\nu}(D)$ and similarly for Y; thus

$$\overline{\nu}(D) = \max\{\overline{\nu}(DT) - \overline{\nu}(T) \mid T \in \{X,Y,Z\}\} = \overline{\nu}(DZ) - \overline{\nu}(Z).$$

It follows that

$$\nu(DZ) = \nu(P) = \overline{\nu}(P) = \overline{\nu}(DZ) = \overline{\nu}(D) + \overline{\nu}(Z) = \overline{\nu}(D) + \nu(Z).$$

This is a contradiction, because we have assumed that $\overline{\nu}(D) \geq 0$. We can then conclude that some linear form of X and Y is in the kernel of D, so (1) holds in this case.

In the second case, $\ker D_1 = k[X, Z + h(X, Y)]$, denote

$$g = Z + h(X, Y).$$

Since $D_1 Y \in \ker D_1$, $D_1 Y = Q(X, g)$ is $\overline{\nu}$ -homogeneous. Then

$$DY = Q(X, g) + \rho, \quad \text{where } \overline{\nu}(\rho) < \overline{\nu}(Q). \tag{5}$$

Note that $D_1 Y \neq 0$, because otherwise $\ker D_1 = k[X, Y, Z + h(X, Y)] = B$, and this is impossible since $D_1 \neq 0$. Now $D_1 Y \neq 0$ implies that $\overline{\nu}(DY) - \overline{\nu}(Y) = \overline{\nu}(D)$, by 2.2.3. So

$$\overline{\nu}(Q(X, g)) = \overline{\nu}(DY) = \overline{\nu}(D) + \overline{\nu}(Y) = \overline{\nu}(D) + 1$$

and it follows that $\overline{\nu}(Q) \geq 1$. From this, equation (5) and $\nu(DY) = 0$, we obtain $\nu(Q) < \overline{\nu}(Q)$. Therefore some irreducible factor F of Q must satisfy $\nu(F) < \overline{\nu}(F)$. By $\overline{\nu}$-homogeneity and irreducibility, we have $F = X$ or $F = \alpha g + \beta X^c$ $(\alpha, \beta \in k)$, and $F = X$ is impossible because $\nu(X) = \overline{\nu}(X)$. Hence,

$$\nu(\alpha g + \beta X^c) < \overline{\nu}(\alpha g + \beta X^c) = \overline{\nu}(Z),$$

and we note that $\alpha \neq 0$ since $\nu(X^c) = \overline{\nu}(X^c)$. This results in

$$\nu(\alpha Z + \alpha h(X, Y) + \beta X^c) < \overline{\nu}(Z).$$

A change of coordinates of B can be made as follows:

$$Z_1 = \alpha Z + \alpha h(X, Y) + \beta X^c, \; Y_1 = Y, \; X_1 = X.$$

In this new coordinate system we still have $D^2 X_1 = 0 = D^2 Y_1$, but $\nu(Z_1) < \nu(Z) = c$. By the inductive hypothesis, we know that (1) holds for D. So (1) is true in both cases. $\square$

Proposition 2.4.4. *Let $B = k[X,Y,Z]$ and let D be a nonzero locally nilpotent derivation of B. If $D^2X = D^2Y = D^2Z = 0$, then*

1. $\ker D$ *contains a non-zero linear form in* $\{X,Y,Z\}$;
2. $\operatorname{rank} D = 1$;
3. *if D is irreducible then, for some coordinate system (X',Y',Z') of B related to (X,Y,Z) by a linear change of variables, we have $D = f(X')\frac{\partial}{\partial Y'} + g(X')\frac{\partial}{\partial Z'}$, where* $\gcd(f(X'),g(X')) = 1$.

Proof. We may assume that D is irreducible. If $\ker D$ contains a nonzero linear form in X,Y, or a nonzero linear form in Y,Z, or one in X,Z, then (1) holds. So assume the contrary. Then, by 2.4.3, $\ker D$ contains polynomials V_1,V_2,V_3 of the form

$$V_1 = X + f(Y,Z), \qquad V_2 = Y + g(X,Z), \qquad V_3 = Z + h(X,Y).$$

Let $R = k[V_3]$ and note that $B = R[X,Y]$; applying 1.2.15 to this situation gives $DX, DY \in k[V_3]$. By symmetry, $DY, DZ \in k[V_1]$ and $DX, DZ \in k[V_2]$.

Of the three polynomials DX, DY, DZ, at most one can belong to k (if, say, $DX, DY \in k$, then some nonzero linear form in X,Y is in $\ker D$, a contradiction). So we may assume that $DX, DY \notin k$, from which we obtain $k[V_2]\cap k[V_3] \not\subseteq k$ and $k[V_1]\cap k[V_3] \not\subseteq k$. It follows that $k[V_1] = k[V_2] = k[V_3]$. (Indeed, since $k[V_2]\cap k[V_3] \not\subseteq k$, we may write $\varphi(V_2) \in k[V_3]$ for some polynomial $\varphi(T) \in k[T]$ of positive degree and such that $\varphi(0) = 0$; then $\varphi(T) = T\varphi_1(T)$ ($\varphi_1(T) \in k[T]$), so $V_2\varphi_1(V_2) \in k[V_3]$. Since $B = k[V_3]^{[2]}$, $k[V_3]$ is factorially closed in B, so $V_2 \in k[V_3]$. By symmetry, $V_3 \in k[V_2]$ so $k[V_2] = k[V_3]$.) It follows that V_1, V_2, V_3 are actually linear forms in X,Y,Z. This proves (1).

We may choose a coordinate system (X',Y',Z') of B such that X',Y',Z' are linear forms in X,Y,Z and $X' \in \ker D$. Let $R = k[X']$ and write $B = R[Y',Z']$, where $D(R) = 0$, $D^2Y' = 0 = D^2Z'$. By 1.2.15, D has a slice, so $\operatorname{rank} D = 1$; also, $DY', DZ' \in R$, so assertion (3) holds. □

Remark 8. If we only assume $D^2X = 0 = D^2Y$, it may happen that $\operatorname{rank} D = 2$. In example 2.3.7, D_1 is just such a derivation.

Theorem 2.4.5 summarizes the results of this section concerning locally nilpotent derivations satisfying $D^2X = 0 = D^2Y$. We stress that this is a complete description of this class of derivations. Indeed, if D is such a derivation, then $D = \alpha D_0$ where $\alpha \in \ker D$ and D_0 is an irreducible derivation belonging to the same class ($D_0^2X = 0 = D_0^2Y$). So we may always assume that D is irreducible.

Theorem 2.4.5. *Let $B = k[X, Y, Z]$ and let $0 \neq D : B \to B$ be a locally nilpotent derivation. If D is irreducible and $D^2X = 0 = D^2Y$, then one of the following holds.*

1. *There exists a coordinate system (L_1, L_2, Z) of B, where L_1 and L_2 are linear forms in X, Y, such that $D(L_1) = 0$, $D(L_2) \in k[L_1]$ and $D(Z) \in k[L_1, L_2] = k[X, Y]$.*

2. *There exists a coordinate system (V, X, Y) of B such that $D(V) = 0$ and $DX, DY \in k[V]$.*

Conversely, if D is any k-derivation satisfying (1) or (2) then D is locally nilpotent and satisfies $D^2X = 0 = D^2Y$.

Remark 9. In case (1), the rank of D can be 1 or 2; in case (2), it must be 1 (by 1.2.15, D has a slice whenever (2) holds).

Proof of 2.4.5. By 2.4.3, one of the following holds: (i) $\ker D$ contains a nonzero linear form L_1 in X, Y; or (ii) $\ker D$ contains a polynomial of the form $Z + h(X, Y)$.

If (i) holds then choose a linear form L_2 in X, Y independent from L_1; then (L_1, L_2, Z) is a coordinate system of B, and $D(L_1) = 0$ and $D^2(L_2) = 0$. By 1.2.15 (with $R = k[L_1]$ and $B = R[L_2, Z]$), we obtain that (1) holds.

If (ii) holds then let $V = Z + h(X, Y)$ and $R = k[V]$, and note that $B = R[X, Y]$. Then 1.2.15 implies that (2) holds.

The last assertion, concerning the converse, is obvious. □

2.5 Further comments

We would naturally like to examine the same question in higher dimension. To be precise, let $B = k[X_1, \ldots, X_n]$, $n > 3$, and let $D : B \to B$ be a derivation. If

$D^2X_i = 0$ for all i, what can be said about $\operatorname{rank} D$? (We are particularly interested in the case where D is homogeneous with respect to standard weight.)

We first look at a very simple case. A derivation D of B is called *linear* if, for each i, DX_i is a linear form of $X_1, \dots, X_n$. Then $DV = MV$, where $V = (X_1, \dots, X_n)^t$ and M is a $n \times n$ matrix over k. Now the condition $D^2X_i = 0$ is equivalent to $M^2 = 0$. From a result of [16], the rank of D is simply the rank of M. But $M^2 = 0$ implies that $\operatorname{rank} M \le [n/2]$, so $\operatorname{rank} D \le [n/2]$ in this case.

Next, consider the following example:

Example 2.5.1. Let $B = k[x, y, t_0, t_1, \dots, t_n] = k^{[n+3]}$, and let $D : B \to B$ be the derivation defined by

$$D = x^n \frac{\partial}{\partial t_0} + x^{n-1}y\frac{\partial}{\partial t_1} + \cdots + x^{n-i}y^i\frac{\partial}{\partial t_i} + \cdots + y^n\frac{\partial}{\partial t_n}.$$

Thus, D is elementary, homogeneous in standard weight, and satisfies $D^2V = 0$ for each $V \in \{x, y, t_0, \dots, t_n\}$. Obviously, $\operatorname{rank} D \le n+1$. However we want to show that $\operatorname{rank} D = n+1$. Suppose $\operatorname{rank} D < n+1$. By 2.3.5, there is a linear form of $k[t_0, \dots, t_n]$ in $\ker D$. This means that we have $\sum_{i=0}^n a_i x^{n-i}y^i = 0$, for some $a_1, \dots, a_n \in k$ not all zero. This is impossible, so $\operatorname{rank} D = n+1$.

This example illustrates that there exists an elementary derivation of a polynomial ring of dimension n whose rank is $n-2$. Similarly, an elementary derivation can be constructed whose rank is i for $1 \le i \le n-2$.

Following the results of the last section and the example above we would tend to think that, for a derivation D of $B = k[X_1, \dots, X_n] = k^{[n]}$, if $D^2X_i = 0$ for all i then $\operatorname{rank} D \le n-2$. But we do not know if this is true, even in the dimension 4 case, as only a very weak result was obtained in dimension 4:

Lemma 2.5.2. *Let $B = k[T, X, Y, Z] = k^{[4]}$, and let D be a homogeneous derivation of B. If $D^2T = D^2X = D^2Y = D^2Z = 0$, then* $\operatorname{rank} D \ne 3$.

Proof. Suppose that $\operatorname{rank} D < 4$ and let us deduce that $\operatorname{rank} D \le 2$. Since D is homogeneous, we can make a linear change of variables such that one of the new variables is in the kernel of D. We can assume $DT = 0$ and we can extend D

to $\overline{D} : k(T)[X,Y,Z] \to k(T)[X,Y,Z]$, where $k(T) = \mathrm{qt}(k[T])$, such that $\overline{D}X = DX$, $\overline{D}Y = DY$, $\overline{D}Z = DZ$, so $\overline{D}^2X = \overline{D}^2Y = \overline{D}^2Z = 0$. Then, using 2.4.4, $\operatorname{rank}\overline{D} = 1$, and $\ker\overline{D}$ contains a linear form, that is some $aX + bY + cZ \in \ker\overline{D}$, where $a, b, c \in k(T)$ are not all zero. Eliminating denominators, we may assume $a, b, c \in k[T]$. Write

$$a = \sum_i a_iT^i,\ b = \sum_i b_iT^i,\ c = \sum_i c_iT^i, \quad a_i, b_i, c_i \in k.$$

Suppose $a \neq 0$, then some $a_j \neq 0$. Since D is homogeneous, $a_jT^jX + b_jT^jY + c_jT^jZ \in \ker D$. Then $a_jX + b_jY + c_jZ \in \ker D$, because $\ker D$ is factorially closed in B. This implies that $\operatorname{rank} D \leq 2$, as desired. □

To clarify this, see the following table, where $B = k[X_1, \dots, X_n] = k^{[n]}$ and $D : B \to B$ is a homogeneous derivation satisfying $D^2X_i = 0$ for all i. A "$\surd$" means that there exists such a derivation, a "×" means that there is no such derivation, and "?" means that we do not know.

Existence of a homogeneous k-derivation D of $k[X_1, \dots, X_n]$
satisfying $\operatorname{rank} D = r$ and $D^2X_i = 0$ for all $i = 1, \dots, n$.

	$r=1$	$r=2$	$r=3$	$r=4$	$r=5$	$r=6$	⋯
$n=1$	$\surd$	×	×	×	×	×	⋯
$n=2$	$\surd$	×	×	×	×	×	⋯
$n=3$	$\surd$	×	×	×	×	×	⋯
$n=4$	$\surd$	$\surd$	×	?	×	×	⋯
$n=5$	$\surd$	$\surd$	$\surd$	?	?	×	⋯
$n=6$	$\surd$	$\surd$	$\surd$	$\surd$	?	?	⋯
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

To be precise, if n is any integer greater than or equal to 5, the n-th row of the above table begins with $n-2$ "$\surd$"s, followed by two "?"s, and then "×"s for all $r > n$.

Chapter 3

Constructible Derivations

The notions of "constructible" and "nice" derivations presented here can be found in [11], but the reader should be warned that we modified some of the definitions. "Elementary" derivations were studied in [12], but refer to 3.3.1 for the definition.

3.1 Definitions and basic facts

Definition 3.1.1. Let B be a domain of characteristic zero and $\mathrm{Der}(B)$ the set of derivations $D : B \to B$. Recall that $\mathrm{Der}(B)$ is a B-module. Given a subset $\mathfrak{D}$ of $\mathrm{Der}(B)$, let $B^{\mathfrak{D}} = \{ b \in B \mid Db = 0 \ \forall D \in \mathfrak{D} \}$.

1. Given a subset $\mathfrak{D}$ of $\mathrm{Der}(B)$ and $D \in \mathrm{Der}(B)$, we say that D is *derived from* $\mathfrak{D}$ *in at most one step* if D has the form $D = \sum_i f_i D_i$, where $D_i \in \mathfrak{D}$ and $f_i \in B^{\mathfrak{D}}$. Let $\mathfrak{D}^+$ denote the set of all derivations D which are derived from $\mathfrak{D}$ in at most one step, and note that $\mathfrak{D}^+$ is simply the $B^{\mathfrak{D}}$-submodule of $\mathrm{Der}(B)$ spanned by $\mathfrak{D}$.

2. A *constructing sequence* is a sequence

$$\mathfrak{D}_0, \dots, \mathfrak{D}_m \quad (m \geq 0)$$

of finite subsets of $\mathrm{Der}(B)$ satisfying $\mathfrak{D}_i \subseteq (\mathfrak{D}_{i-1})^+$ for all i such that $0 < i \leq m$.

3. Given a finite subset $\mathfrak{D}$ of $\mathrm{Der}(B)$ and $D \in \mathrm{Der}(B)$, we say that D is *constructible from* $\mathfrak{D}$ if there exists a constructing sequence $\mathfrak{D}_0, \ldots, \mathfrak{D}_m$ satisfying $\mathfrak{D}_0 = \mathfrak{D}$ and $\mathfrak{D}_m \ni D$; such a sequence $\mathfrak{D}_0, \ldots, \mathfrak{D}_m$ is called a *constructing sequence from* $\mathfrak{D}$ *to* D. The set of all D which are constructible from $\mathfrak{D}$ is denoted $C(\mathfrak{D})$.

Remark 10. If $D \in C(\mathfrak{D})$ and $\alpha \in \ker(D)$ then $\alpha D \in C(\mathfrak{D})$. (Indeed, there exists a constructing sequence $\mathfrak{D}_0, \ldots, \mathfrak{D}_m$ such that $\mathfrak{D}_0 = \mathfrak{D}$ and $D \in \mathfrak{D}_m$. Then $\mathfrak{D}_0, \ldots, \mathfrak{D}_m, \{D\}, \{\alpha D\}$ is also a constructing sequence, so $\alpha D \in C(\mathfrak{D})$.) However, the converse does not hold, i.e., $\alpha D \in C(\mathfrak{D})$ does not imply $D \in C(\mathfrak{D})$.

Lemma 3.1.2. *Notation as for* 3.1.1.

1. *If* $\mathfrak{D}_0, \ldots, \mathfrak{D}_m$ *is a constructing sequence then* $C(\mathfrak{D}_0) \supset C(\mathfrak{D}_1) \supset \cdots \supset C(\mathfrak{D}_m)$.
2. *If* $\mathfrak{D}' \subset \mathfrak{D}$ *then* $C(\mathfrak{D}') \subset C(\mathfrak{D})$.
3. *If* $\mathfrak{D}_0, \ldots, \mathfrak{D}_m$ *is a constructing sequence then* $B^{\mathfrak{D}_0} \subset B^{\mathfrak{D}_1} \subset \cdots \subset B^{\mathfrak{D}_m}$.

Proof. 1. If $D \in C(\mathfrak{D}_i)$, where $i > 0$, then there exists a constructing sequence from $\mathfrak{D}_i$ to D:

$$\mathfrak{D}_i, \mathfrak{D}'_{i+1}, \ldots, \mathfrak{D}'_s.$$

Now $\mathfrak{D}_{i-1}, \mathfrak{D}_i, \mathfrak{D}'_{i+1}, \ldots, \mathfrak{D}'_s$ is a constructing sequence from $\mathfrak{D}_{i-1}$ to D, so $D \in C(\mathfrak{D}_{i-1})$, which implies that $C(\mathfrak{D}_i) \subset C(\mathfrak{D}_{i-1})$.

2. Let $\tau \in C(\mathfrak{D}')$. Then we have a constructing sequence from $\mathfrak{D}'$ to τ:

$$\mathfrak{D}', \mathfrak{D}'_1, \ldots, \mathfrak{D}'_r.$$

Note that every $D \in \mathfrak{D}' \subset \mathfrak{D}$ can be derived from $\mathfrak{D}$ in at most one step. So the sequence $\mathfrak{D}, \mathfrak{D}', \mathfrak{D}'_1, \ldots, \mathfrak{D}'_r$ is a constructing sequence from $\mathfrak{D}$ to τ. This means τ can be derived from $\mathfrak{D}$, or $\tau \in C(\mathfrak{D})$, and $C(\mathfrak{D}') \subset C(\mathfrak{D})$.

3. Let us show that $B^{\mathfrak{D}_i} \subset B^{\mathfrak{D}_{i+1}}$. Take $\tau \in \mathfrak{D}_{i+1}$, then $\tau = \sum_{D \in \mathfrak{D}_i} b_D D$, where $b_D \in B^{\mathfrak{D}_i}$. For any $x \in B^{\mathfrak{D}_i}$, we have

$$\tau(x) = \sum_{D \in \mathfrak{D}_i} b_D D x = 0,$$

because $Dx = 0$ for every $D \in \mathfrak{D}_i$. This means $B^{\mathfrak{D}_i} \subset \ker\tau$. Since τ is arbitrary, we have $B^{\mathfrak{D}_i} \subset B^{\mathfrak{D}_{i+1}}$, as desired. □

Lemma 3.1.3. *Let B be a domain of characteristic zero and let $\mathfrak{D} = \{D_1, \dots, D_m\} \subset$ Der(B). Given another subset $\mathfrak{D}' = \{D'_1, \dots, D'_m\}$ of* Der(B), *if we have*

$$\begin{pmatrix} D'_1 \\ \vdots \\ D'_m \end{pmatrix} = M \begin{pmatrix} D_1 \\ \vdots \\ D_m \end{pmatrix} \qquad (6)$$

such that M is an invertible matrix over $B^{\mathfrak{D}}$, then $B^{\mathfrak{D}'} = B^{\mathfrak{D}}$ and every derivation that can be derived from $\mathfrak{D}'$ in at most one step can also be derived from $\mathfrak{D}$ in at most one step, and vice versa. Consequently, $C(\mathfrak{D}) = C(\mathfrak{D}')$.

Proof. From (6) we see that $D'_1, \dots, D'_m$ can be derived from $\mathfrak{D}$ in at most one step; so $\mathfrak{D}, \mathfrak{D}'$ is a constructing sequence. By 3.1.2, $B^{\mathfrak{D}} \subset B^{\mathfrak{D}'}$ and $C(\mathfrak{D}) \supset C(\mathfrak{D}')$. Write

$$\begin{pmatrix} D_1 \\ \vdots \\ D_m \end{pmatrix} = M^{-1} \begin{pmatrix} D'_1 \\ \vdots \\ D'_m \end{pmatrix}$$

where M^{-1} has entries in $B^{\mathfrak{D}} \subset B^{\mathfrak{D}'}$, so M^{-1} is over $B^{\mathfrak{D}'}$. Thus $D_1, \dots, D_m$ can be derived from $\mathfrak{D}'$ in at most one step and consequently $\mathfrak{D}', \mathfrak{D}$ is a constructing sequence. Using 3.1.2 again, $B^{\mathfrak{D}} \supset B^{\mathfrak{D}'}$ and $C(\mathfrak{D}) \subset C(\mathfrak{D}')$. We conclude that $B^{\mathfrak{D}'} = B^{\mathfrak{D}}$ and $C(\mathfrak{D}) = C(\mathfrak{D}')$. □

Definition 3.1.4. Let $B = k^{[n]}$ and $D \in$ Der(B).

1. Let $\gamma = (X_1, \dots, X_n)$ be a coordinate system of B. We say that D is γ-*constructible* if it satisfies the following equivalent conditions:

   - $D \in C(\{\frac{\partial}{\partial X_1}, \dots, \frac{\partial}{\partial X_n}\})$,
   - $D \in C(\mathfrak{D})$ for some subset $\mathfrak{D}$ of $\{\frac{\partial}{\partial X_1}, \dots, \frac{\partial}{\partial X_n}\}$.

   The set of γ-constructible derivations is denoted $C(\gamma)$. Hence,

$$C(\gamma) = C(\{\frac{\partial}{\partial X_1}, \dots, \frac{\partial}{\partial X_n}\}).$$

2. We say that D is *constructible* if there exists a coordinate system γ of B such that D is γ-constructible.

Remark 11. It is known that constructible derivations are locally nilpotent (see 3.3.2), and this is why we are interested in them.

Remark 12. The set $C(\gamma)$ is not "closed under construction", i.e., if $\mathfrak{D} \subset C(\gamma)$ then it does not follow that $C(\mathfrak{D}) \subseteq C(\gamma)$. (Example: Let $B = k[X,Y] = k^{[2]}$, $\gamma = (X,Y)$, $D_1 = X\frac{\partial}{\partial Y}$ and $D_2 = Y\frac{\partial}{\partial X}$; then $\{D_1, D_2\} \subset C(\gamma)$, $D_1 + D_2 \in C(\{D_1, D_2\})$ but $D_1 + D_2 \notin C(\gamma)$ since $D_1 + D_2$ is not locally nilpotent—see the preceding remark.)

3.1.5. Let R be a domain and $B = R^{[n]}$. Two coordinate systems of B over R, $\gamma = (X_1, \ldots, X_n)$ and $\gamma' = (X_1', \ldots, X_n')$, are said to be *related by a linear change of variables* if there exists a matrix $A \in \mathrm{GL}_n(R)$ satisfying

$$\begin{pmatrix} X_1' \\ \vdots \\ X_n' \end{pmatrix} = A \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix}.$$

Lemma 3.1.6. *Given two coordinate systems of $B = k^{[n]}$, $\gamma = (X_1, \ldots, X_n)$ and $\gamma' = (X_1', \ldots, X_n')$, if the change between γ and γ' is linear, then $C(\gamma) = C(\gamma')$.*

Proof. Recall that $C(\gamma) = C(\{\frac{\partial}{\partial X_1}, \ldots, \frac{\partial}{\partial X_n}\})$. We know that the change from γ to γ' is linear:

$$\begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} X_1' \\ \vdots \\ X_n' \end{pmatrix},$$

where $a_{ij} \in k$ for all i, j. Use the notation $M = (a_{ij})$ to denote the above matrix. For any $f \in B$, we have

$$\frac{\partial f}{\partial X_1'} = \frac{\partial f}{\partial X_1}\frac{\partial X_1}{\partial X_1'} + \cdots + \frac{\partial f}{\partial X_n}\frac{\partial X_n}{\partial X_1'}$$

$$= (a_{11}\frac{\partial}{\partial X_1} + \cdots + a_{n1}\frac{\partial}{\partial X_n})(f).$$

Here f is any polynomial of B, and we conclude that

$$\frac{\partial}{\partial X_1'} = a_{11}\frac{\partial}{\partial X_1} + \cdots + a_{n1}\frac{\partial}{\partial X_n}.$$

Continuing with $\partial/\partial X_i'$ for each i, then

$$\begin{pmatrix} \frac{\partial}{\partial X_1'} \\ \vdots \\ \frac{\partial}{\partial X_n'} \end{pmatrix} = \begin{pmatrix} a_{11} & \cdots & a_{n1} \\ \vdots & \ddots & \vdots \\ a_{1n} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial X_1} \\ \vdots \\ \frac{\partial}{\partial X_n} \end{pmatrix}.$$

This matrix is M^t. We know that M is invertible over k, and so is M^t. Then according to Lemma 3.1.3, $C(\gamma) = C(\gamma')$. □

3.2 Rank of constructible derivations

Lemma 3.2.1. *Let R be a PID of characteristic zero and $B = R^{[n]}$. Let $\mathfrak{D}_0, \mathfrak{D}_1$ be a constructing sequence such that $\mathfrak{D}_0 = \{\frac{\partial}{\partial X_1}, \dots, \frac{\partial}{\partial X_n}\}$ for some coordinate system $(X_1, \dots, X_n)$ of B over R. Then there exists a coordinate system $(Y_1, \dots, Y_n)$ of B over R and a subset $\mathfrak{D}_1'$ of $\{\frac{\partial}{\partial Y_1}, \dots, \frac{\partial}{\partial Y_n}\}$ satisfying:*

1. *$\mathfrak{D}_0, \mathfrak{D}_1'$ is a constructing sequence and $(\mathfrak{D}_1')^+ \supseteq \mathfrak{D}_1^+$;*

2. *$(X_1, \dots, X_n)$ and $(Y_1, \dots, Y_n)$ are related by a linear change of variables.*

Proof. Let M be the R-submodule of $\operatorname{Der}(B)$ spanned by $\{\frac{\partial}{\partial X_1}, \dots, \frac{\partial}{\partial X_n}\}$, i.e., $M = \operatorname{Span}_R(\mathfrak{D}_0) = \mathfrak{D}_0^+$; let $N = \operatorname{Span}_R(\mathfrak{D}_1) \subseteq M$ and

$$\bar{N} = \{\, D \in M \mid rD \in N \text{ for some } r \in R \setminus \{0\} \,\}.$$

Note that $\bar{N}$ is a submodule of M and that $M/\bar{N}$ is torsion-free, hence free. So $M = \bar{N} \oplus Q$ for some submodule Q of M. Observe that $\bar{N}$ and Q are free, and choose a basis $\mathfrak{D}_1' = \{D_1, \dots, D_r\}$ of $\bar{N}$ and a basis $\{D_{r+1}, \dots, D_n\}$ of Q (where $0 \le r \le n$). Since $\{D_1, \dots, D_n\}$ and $\{\frac{\partial}{\partial X_1}, \dots, \frac{\partial}{\partial X_n}\}$ are two bases of $M \cong R^n$, there exists a matrix $A \in \mathrm{GL}_n(R)$ such that

$$\begin{pmatrix} \partial/\partial X_1 \\ \vdots \\ \partial/\partial X_n \end{pmatrix} = A \begin{pmatrix} D_1 \\ \vdots \\ D_n \end{pmatrix}.$$

Now define $Y_1, \dots, Y_n \in B$ by

$$\begin{pmatrix} Y_1 \\ \vdots \\ Y_n \end{pmatrix} = A^t \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix},$$

then $(Y_1, \dots, Y_n)$ is a coordinate system of B related to $(X_1, \dots, X_n)$ by a linear change of variables; also, $D_i = \frac{\partial}{\partial Y_i}$ for $1 \le i \le n$, so $\mathfrak{D}'_1 \subseteq \{\frac{\partial}{\partial Y_1}, \dots, \frac{\partial}{\partial Y_n}\}$. We have $\mathfrak{D}'_1 \subseteq \bar{N} \subseteq M = \mathfrak{D}_0^+$, so $\mathfrak{D}_0, \mathfrak{D}'_1$ is a constructing sequence. It is easy to see that

$$B^{\mathfrak{D}'_1} = B^{\bar{N}} = B^N = B^{\mathfrak{D}_1};$$

if we write $A = B^{\mathfrak{D}'_1} = B^{\mathfrak{D}_1}$, then $R \subseteq A$ and

$$\mathfrak{D}_1^+ = \mathrm{Span}_A(\mathfrak{D}_1) = \mathrm{Span}_A(N) \subseteq \mathrm{Span}_A(\bar{N}) = \mathrm{Span}_A(\mathfrak{D}'_1) = (\mathfrak{D}'_1)^+.$$

□

Proposition 3.2.2. *Let $B = R[X_1, \dots, X_n] = R^{[n]}$, where $n \ge 2$ and R is a PID of characteristic zero. If $D \in C(\{\frac{\partial}{\partial X_1}, \dots, \frac{\partial}{\partial X_n}\})$, then*

$$D \in C(\{\tfrac{\partial}{\partial Y_1}, \dots, \tfrac{\partial}{\partial Y_{n-1}}\}),$$

for some coordinate system $(Y_1, \dots, Y_n)$ of B related to $(X_1, \dots, X_n)$ by a linear change of variables.

Remark 13. The assumption that R is a PID is crucial. For instance, Example 2.5.1 shows that the result is false with $R = k^{[2]}$ (a very nice UFD).

Proof of 3.2.2. By assumption, there exist constructing sequences $\mathfrak{D}_0, \dots, \mathfrak{D}_m$ satisfying:

(i) for some coordinate system $(U_1, \dots, U_n)$ of B related to $(X_1, \dots, X_n)$ by a linear change of variables, and for some s such that $0 \le s \le n$, we have $\mathfrak{D}_0 = \{\frac{\partial}{\partial U_1}, \dots, \frac{\partial}{\partial U_s}\}$; and

(ii) $D \in \mathfrak{D}_m$;

choose such a sequence which makes m the smallest. We claim that some permutation $(Y_1, \dots, Y_n)$ of $(U_1, \dots, U_n)$ satisfies the requirement. Since this is clear if $m = 0$ or $s \leq n-1$, it suffices to show that the assumption

$$m \geq 1 \text{ and } s = n$$

leads to a contradiction. Since R is a PID, $B = R^{[n]}$ and $\mathfrak{D}_0 = \{\frac{\partial}{\partial U_1}, \dots, \frac{\partial}{\partial U_n}\}$, we may apply 3.2.1 to $\mathfrak{D}_0, \mathfrak{D}_1$; we obtain a set $\mathfrak{D}'_1$ such that $\mathfrak{D}'_1, \mathfrak{D}_2, \dots, \mathfrak{D}_m$ is a constructing sequence satisfying (i) and (ii). This contradicts the minimality of m. □

Corollary 3.2.3. *Let $B = k[X_1, \dots, X_n] = k^{[n]}$ and let $D \in C(\{\frac{\partial}{\partial X_1}, \dots, \frac{\partial}{\partial X_n}\})$ (where $n \geq 3$).*

1. *$D \in C(\{\frac{\partial}{\partial Y_1}, \dots, \frac{\partial}{\partial Y_{n-1}}\})$, for some coordinate system $(Y_1, \dots, Y_n)$ of B related to $(X_1, \dots, X_n)$ by a linear change of variables.*
2. *For some coordinate system $(Z_1, \dots, Z_n)$ of B, $D \in C(\{\frac{\partial}{\partial Z_1}, \dots, \frac{\partial}{\partial Z_{n-2}}\})$ and consequently $Z_{n-1}, Z_n \in \ker D$.*

In particular, $\operatorname{rank}(D) \leq n-2$.

Proof. Assertion (1) follows immediately from 3.2.2. To prove (2), fix $(Y_1, \dots, Y_n)$ satisfying (1), let $R = k[Y_n]$ (a PID) and note that $B = R[Y_1, \dots, Y_{n-1}] = R^{[n-1]}$. Applying 3.2.2 to $D \in C(\{\frac{\partial}{\partial Y_1}, \dots, \frac{\partial}{\partial Y_{n-1}}\})$ gives a coordinate system $(Z_1, \dots, Z_{n-1})$ of $B = k[Y_n]^{[n-1]}$ (over $k[Y_n]$), related to $(Y_1, \dots, Y_{n-1})$ by a linear change of variables and satisfying $D \in C(\{\frac{\partial}{\partial Z_1}, \dots, \frac{\partial}{\partial Z_{n-2}}\})$. Let $Z_n = Y_n$, then $(Z_1, \dots, Z_n)$ has the desired properties. The fact that $Z_{n-1}, Z_n \in \ker D$ follows from part (3) of 3.1.2. □

Remark 14. Corollary 3.2.3 cannot be improved, i.e., $\operatorname{rank}(D)$ may be equal to $n-2$ (see Example 2.5.1).

3.3 Relationships

Definition 3.3.1. Let $B = k^{[n]}$ and $D \in \operatorname{Der}(B)$.

1. Let $\gamma = (X_1, \ldots, X_n)$ be a coordinate system of B. We say that D is γ-*elementary* if there exists a subset $\mathfrak{D}$ of $\{\frac{\partial}{\partial X_1}, \ldots, \frac{\partial}{\partial X_n}\}$ such that $D \in \mathfrak{D}^+$. In other words, for a suitable permutation $(Y_1, \ldots, Y_n)$ of $(X_1, \ldots, X_n)$ and a suitable integer $m \in \{0, \ldots, n\}$, D is of the form
$$D = f_1(Y_{m+1}, \ldots, Y_n)\frac{\partial}{\partial Y_1} + \cdots + f_m(Y_{m+1}, \ldots, Y_n)\frac{\partial}{\partial Y_m}$$
(where $D = 0$ if $m = 0$).

2. Let $\gamma = (X_1, \ldots, X_n)$ be a coordinate system of B. We say that D is γ-*nice* if it is a k-derivation and satisfies $D^2(X_i) = 0$ for all $i = 1, \ldots, n$.

3. We say that D is *elementary* (resp. *nice*) if there exists a coordinate system γ of B such that D is γ-elementary (resp. γ-nice).

Remark 15. Every nice derivation is locally nilpotent.

Clearly, every γ-elementary derivation is γ-constructible. The following result of van den Essen and Hubbers implies that every γ-constructible derivation is γ-nice:

Lemma 3.3.2 (Proposition 2.2 of [11]). *Let $\gamma = (X_1, \ldots, X_n)$ be a coordinate system of $B = k^{[n]}$ and let $\mathfrak{D}_0, \ldots, \mathfrak{D}_m$ be a constructing sequence such that $D_1D_2(X_i) = 0$ for all $D_1, D_2 \in \mathfrak{D}_0$ and all i. Then $D_1D_2(X_i) = 0$ for all $D_1, D_2 \in \mathfrak{D}_m$ and all i. In particular, each $D \in C(\mathfrak{D}_0)$ is γ-nice.*

Proof. Clearly, we may assume that $m = 1$. Let $D_1, D_2 \in \mathfrak{D}_1$. Write $D_1 = \sum_r f_{1r}D_{1r}$, $D_2 = \sum_r f_{2r}D_{2r}$, where $D_{1r}, D_{2r} \in \mathfrak{D}_0$ for all r and $f_{1r}, f_{2r} \in B^{\mathfrak{D}_0}$. Then
$$D_1D_2(X_i) = \sum_{r,s} f_{1r}D_{1r}(f_{2s}D_{2s}X_i) = \sum_{r,s} f_{1r}f_{2s}D_{1r}D_{2s}(X_i) = 0.$$
□

Thus the following implications hold:

Corollary 3.3.3. *Let $B = k^{[n]}$ and $D \in \mathrm{Der}(B)$.*

1. *Given a coordinate system γ of B,*

$$D \text{ is } \gamma\text{-elementary} \implies D \text{ is } \gamma\text{-constructible} \implies D \text{ is } \gamma\text{-nice.}$$

2. *D is elementary $\implies$ D is constructible $\implies$ D is nice.*

We now consider the converses of the implications of 3.3.3. For $B = k^{[n]}$ and $D \in \mathrm{Der}(B)$, the situation is as follows:

n	D is γ-elem. $\Leftarrow$ D is γ-construc.	D is γ-construc. $\Leftarrow$ D is γ-nice
1	true (see 3.3.4)	true (see 3.3.4)
2	false (see 3.3.5)	true (see 3.3.6)
3	false (see 3.3.5)	true (see 3.3.6)
4	false (see 3.3.5)	?
> 4	false (see 3.3.5)	?

n	D is elem. $\Leftarrow$ D is construc.	D is construc. $\Leftarrow$ D is nice
1	true (see 3.3.4)	true (see 3.3.4)
2	true (see 3.3.7)	true (see 3.3.6)
3	true (see 3.3.7)	true (see 3.3.6)
4	essentially true (see 3.3.10, 3.3.9)	?
> 4	false (see 3.3.11)	?

The paragraphs below give justifications for these claims.

3.3.4. Suppose $n = 1$ and let $\gamma = (X)$ be any coordinate system of B ($B = k[X]$). Then every γ-nice derivation of B has the form $D = \lambda\frac{\partial}{\partial X}$ for some $\lambda \in k$, and hence is γ-elementary.

3.3.5. If $\gamma = (X_1, \ldots, X_n)$ (where $n \geq 2$) and $D = (X_1 - X_2)(\frac{\partial}{\partial X_1} + \frac{\partial}{\partial X_2})$, then D is γ-constructible but not γ-elementary. Hence, when $n \geq 2$,

$$D \text{ is } \gamma\text{-constructible} \not\Longleftarrow D \text{ is } \gamma\text{-elementary.}$$

3.3.6. *Let $n = 2$ or 3, let γ be a coordinate system and suppose that D is γ-nice. Then D is γ-constructible.*

Proof. By Remark 10 Remark 15, and 1.1.5 8(b), we may assume that D is irreducible. When $n = 2$, let $\gamma = (X, Y)$. Write $D = P\partial/\partial X + Q\partial/\partial Y$, where $P, Q \in B$ and $\gcd(P, Q) = 1$. Now $D^2X = D^2Y = 0$ implies that $PP_X = -QP_Y$, $PQ_X = -QQ_Y$. It follows that P and Q must be constants. So $D = a\partial/\partial X + b\partial/\partial Y$, $a, b \in k$, which is γ-constructible.

When $n = 3$, let $\gamma = (X, Y, Z)$. Since D is γ-nice, part (3) of 2.4.4 shows that we can make a linear change of variables, say $\gamma' = (X', Y', Z')$, such that

$$D = f(X')\frac{\partial}{\partial Y'} + g(X')\frac{\partial}{\partial Z'}.$$

Then D is γ'-elementary, so that $D \in C(\gamma')$. We know from 3.1.6 that $C(\gamma) = C(\gamma')$. So D is γ-constructible. □

3.3.7. *Suppose that $n = 2$ (resp. $n = 3$) and suppose that D is constructible. Then D is elementary.*

Proof. We have $\text{rank}(D) \leq 1$ by 1.2.2 (resp. 3.2.3), and it follows that D is elementary. □

In order to justify the above two tables, there remains to consider whether or not (when $n \geq 4$) constructible $\Longrightarrow$ elementary. We will see that this implication does not hold, but we will also see that the question is more interesting if we replace "elementary" by "pseudo-elementary" (definition below).

Definition 3.3.8. Let $B = k^{[n]}$, $D \in \text{Der}(B)$, and recall that we may write $D = \alpha D_0$ where $\alpha \in B$ and where $D_0 \in \text{Der}(B)$ is irreducible. We say that D is *pseudo-elementary* if D_0 is elementary and D is locally nilpotent.

3.3.9. Let $B = k[U, V, X, Y] = k^{[4]}$ and $D = (VX - UY)(U\frac{\partial}{\partial X} + V\frac{\partial}{\partial Y})$. Clearly, $D_0 = U\frac{\partial}{\partial X} + V\frac{\partial}{\partial Y}$ is elementary, so D is pseudo-elementary. We claim that D is not elementary. Indeed, assume that D is elementary and choose a coordinate system $\gamma = (X_1, X_2, X_3, X_4)$ such that, for some $s \in \{0, 1, 2, 3, 4\}$,

$$D = \sum_{i>s} f_i(X_1, \ldots, X_s)\frac{\partial}{\partial X_i}.$$

Then $VX - UY$ must be a common factor of the f_i; in particular, $s < 4$ (because $D \neq 0$) and $VX - UY \in k[X_1, X_2, X_3]$. So

$$B/(VX - UY) \cong R[X_4] = R^{[1]} \qquad \text{for some ring } R,$$

or equivalently the variety in $\mathbf{A}^4$ defined by "$VX - UY = 0$" is a product $S \times \mathbf{A}^1$ for some surface S. This is impossible, because the variety has an isolated singular point at the origin. Finally, note that

$$\{\tfrac{\partial}{\partial X}, \tfrac{\partial}{\partial Y}\},\ \{D_0\}, \{D\}$$

is a constructing sequence, so D is constructible (but not elementary).

Proposition 3.3.10. *Let $B = k^{[4]}$ and $D \in \mathrm{Der}(B)$. Then D is constructible if and only if it is pseudo-elementary.*

Proof. The implication from pseudo-elementary to constructible is easy. We only prove the converse. By part (2) of 3.2.3, we have a coordinate system $\gamma = (X, Y, U, V)$ of B such that $DU = DV = 0$ and D is γ-constructible. Since D is γ-constructible, we have $D^2X = D^2Y = 0$. Applying 1.2.15 to D_0 gives

$$D_0 = f(U,V)\frac{\partial}{\partial X} + g(U,V)\frac{\partial}{\partial Y},$$

so D_0 is γ-elementary and hence D is pseudo-elementary. □

3.3.11. Let $B = k[X_1, X_2, X_3, X_4, X_5] = k^{[5]}$ and

$$\tau = (1 + X_4X_2 - X_5X_3)\frac{\partial}{\partial X_1} + X_5\frac{\partial}{\partial X_2} + X_4\frac{\partial}{\partial X_3}.$$

Then τ is constructible but not pseudo-elementary. The proof is given in the next section.

3.4 An interesting example

We refer to the derivation τ of 3.3.11 as "Winkelmann's derivation"; our main goal, here, is to show that it is constructible but not pseudo-elementary, and we will also review some of its other properties. Winkelmann's derivation was given in [27] for other reasons, and it was noted in [11] that τ is constructible. The material presented in this section is not new, except maybe for the observation that τ is not pseudo-elementary, but we include it here for the reader's convenience.

Note that τ is irreducible; consequently, "elementary" and "pseudo-elementary" have the same meaning for τ. Our proof that τ is not elementary requires the following results:

3.4.1 (Theorem 2.5 of [7]). *Let D be a locally nilpotent derivation of $k^{[n]}$ of rank at most 2. If D is fixed point free, then* $\operatorname{rank} D = 1$.

3.4.2 (Theorem 2.1 of [12]). *Let D be an elementary derivation of $B = k^{[n]}$. If D is fixed point free, then* $\operatorname{rank} D = 1$.

Remark 16. See 1.1.7 for the definition of "fixed point". In 3.4.2, if instead of assuming that D is elementary we only assume that it is constructible, then the conclusion is false. (Indeed, this section shows that τ is constructible, fixed point free and of rank 3.)

We will need to find generators for $\ker\tau$. For this, we use an algorithm of van den Essen [10] (or [12] for a brief description without proofs). We now review this algorithm:

3.4.3. Let $D \neq 0$ be a locally nilpotent derivation of $B = k[X_1, \dots, X_n] = k^{[n]}$.

1. Choose an element $a \in B$ such that $d = Da \neq 0$ and $D^2 a = 0$. Let $s = a/d \in B[d^{-1}]$ and define

$$b_i = \sum_{j=0}^{\infty} \frac{1}{j!}(-s)^j D^j X_i.$$

   Since D is locally nilpotent, there exists $e_i \in \mathbb{N}$ such that

$$r_i = d^{e_i} b_i \in B.$$

   Define $R_0 = k[r_1, \dots, r_n, d]$. Then it is shown in [10] that

$$R_0 \subset \ker D \subset R_0[d^{-1}].$$

2. For each $m \geq 1$ define inductively R_m as the k-subalgebra of B generated by the elements $g \in B$ such that $dg \in R_{m-1}$. Then we get

$$R_0 \subset R_1 \subset \cdots \subset \ker D \quad \text{and} \quad \ker D = \bigcup_{m \geq 0} R_m.$$

3. Each R_m is a finitely generated k-algebra and can be computed as follows: suppose that $R_{m-1} = k[F_1, \dots, F_l]$ $(m \geq 1)$. Put

$$I(F_1, \dots, F_l) = \{P \in k[T_1, \dots, T_l] \mid P(F_1, \dots, F_l) \in dB\}.$$

Then $I(F_1, \dots, F_l)$ is an ideal in $k[T_1, \dots, T_l]$ generated by a finite number of elements, say $P_1(T), \dots, P_s(T)$. By definition this means that $P_i(F_1, \dots, F_l) = df_i$ for some $f_i \in B$, all i. It is shown in [12] that

$$R_m = k[F_1, \dots, F_l, f_1, \dots, f_s].$$

4. If $\ker D$ is a finitely generated k-algebra, then $\ker D = R_r$ for some $r \in \mathbb{N}$ (and $R_n = R_r$ for all $n \geq r$). Conversely, if there exists $r \in \mathbb{N}$ such that $R_r = R_{r+1}$, then $\ker D$ is finitely generated k-algebra and $\ker D = R_r$.

3.4.4. *The following hold:*

1. τ *is constructible.*

2. τ *is fixed point free.*

3. $\ker \tau = k[\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5]$, *where*

$$\begin{aligned} \alpha_1 &= X_1X_5 - X_2(1 + X_2X_4 - X_3X_5), \quad \alpha_2 = X_3X_5 - X_2X_4, \\ \alpha_3 &= X_4, \quad \alpha_4 = X_5, \quad \alpha_5 = X_1X_4 - X_3(1 + X_2X_4 - X_3X_5). \end{aligned}$$

4. $\operatorname{rank} \tau = 3$.

5. τ *is not elementary and* τ *has no slice.*

Proof. 1. Let $\mathfrak{D}_0 = \{\frac{\partial}{\partial X_1}, \frac{\partial}{\partial X_2}, \frac{\partial}{\partial X_3}\}$. Then define

$$\mathfrak{D}_1 = \{\frac{\partial}{\partial X_1}, \; X_5\frac{\partial}{\partial X_2} + X_4\frac{\partial}{\partial X_3}\}.$$

It is clear that the two derivations in $\mathfrak{D}_1$ can be derived from $\mathfrak{D}_0$ in at most one step. We claim that τ can be derived from $\mathfrak{D}_1$ in at most one step. To prove it we need only show that

$$1 + X_4X_2 - X_5X_3 \in \ker \frac{\partial}{\partial X_1} \cap \ker(X_5\frac{\partial}{\partial X_2} + X_4\frac{\partial}{\partial X_3}).$$

This is straightforward. If we take $\mathfrak{D}_2 = \{\tau\}$, then $\mathfrak{D}_0, \mathfrak{D}_1, \mathfrak{D}_2$ is a constructing sequence and consequenly τ is constructible.

2. This one is quite simple. Note that

$$\tau X_1 - X_2\tau X_3 + X_3\tau X_2 = 1.$$

So τ is fixed point free.

3. To find the generators of $\ker\tau$, we use van den Essen's algorithm 3.4.3. Take $a = X_2$, then $d = DX_2 = X_5$ and $s = X_2/X_5$. We have

$$\begin{aligned}
r_1 &= X_1X_5 - X_2(1 + X_2X_4 - X_3X_5) \\
r_2 &= 0 \\
r_3 &= X_3X_5 - X_2X_4 \\
r_4 &= X_4 \\
r_5 &= X_5
\end{aligned}$$

$$R_0 = k[X_1X_5 - X_2(1 + X_2X_4 - X_3X_5), \; X_3X_5 - X_2X_4, \; X_4, \; X_5].$$

Next we want to calculate R_1. According to step 3 of the algorithm, each $f \in R_1$ satisfies that $X_5 f = P(r_1, r_3, r_4)$ for some $P(T_1, T_3, T_4) \in k[T_1, T_3, T_4]$. So

$$P(r_1, r_3, r_4) = 0 \pmod{X_5},$$

and then $P(r_1 \pmod{X_5}, r_3 \pmod{X_5}, r_4 \pmod{X_5}) = 0$. That is

$$P(-X_2 - X_4X_2^2, \; -X_2X_4, \; X_4) = 0.$$

Now we consider a map

$$\begin{array}{ll} k[U_1,U_2,U_3] & \xrightarrow{\alpha} k[X_2,X_4] \\ U_1 & \mapsto -X_2 - X_4X_2^2 \\ U_2 & \mapsto -X_2X_4 \\ U_3 & \mapsto X_4 \end{array}$$

The image $\operatorname{Im}\alpha$ of α is a subring of $k[X_2,X_4]$, so it is an integral domain. What is $\dim(\operatorname{Im}\alpha)$? Clearly $\dim(\operatorname{Im}\alpha) \leq 2$. On the other hand,

$$(-X_2 - X_4X_2^2) - (-X_2X_4)X_2 + X_2 = 0,$$

which implies that X_2 is algebraic over $\operatorname{Im}\alpha$. So $k[X_2,X_4]$ is algebraic over $\operatorname{Im}\alpha$. Thus $\dim(\operatorname{Im}\alpha) = 2$. It follows that $\ker\alpha$ is a prime ideal of height 1, so a principal ideal. It is easy to see that

$$(-X_2 - X_4X_2^2)X_4 + (-X_2X_4)^2 - (-X_2X_4) = 0.$$

This means $U_1U_3 + U_2^2 - U_2 \in \ker\alpha$. So $\ker\alpha = (U_1U_3 + U_2^2 - U_2)$. We take

$$\begin{aligned} f &= X_5^{-1}(r_1r_4 + r_3^2 - r_3) \\ &= X_1X_4 - X_3(1 + X_2X_4 - X_3X_5), \end{aligned}$$

then we have

$$R_1 = k[r_1, r_3, r_4, r_5, f].$$

Then we must calculate R_2. Every $g \in R_2$ satisfies $X_5g = Q(r_1, r_3, r_4, f)$ for some $Q \in k[U_1,U_2,U_3,U_4]$ and such a Q must satisfy $Q(r_1 \pmod{X_5}, r_3 \pmod{X_5}, r_4 \pmod{X_5}, f \pmod{X_5}) = 0$, i.e.

$$Q(-X_2 - X_4X_2^2,\ -X_2X_4,\ X_4,\ X_1X_4 - X_3(1 + X_2X_4)) = 0.$$

It follows that, if we write $Q = \sum_i Q_i(U_1,U_2,U_3)U_4^i$, then

$$Q_i(-X_2 - X_4X_2^2, -X_2X_4, X_4) = 0,$$

so $Q_i \in \ker\alpha$ (for all i). Hence $R_2 = R_1$ and $\ker\tau = R_1$. This proves the third assertion.

4. Let's calculate the rank of τ. It's clear that $\operatorname{rank}\tau \leq 3$. To prove that $\operatorname{rank}\tau = 3$, let $A = \ker\tau$ and consider the morphism $\varphi : \operatorname{Spec} B \to \operatorname{Spec} A$ determined by the inclusion $A \hookrightarrow B$. Let $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ be as in the third assertion, let $R = k[Y_1, Y_2, Y_3, Y_4, Y_5]$ and let $\psi : \operatorname{Spec} A \to \operatorname{Spec} R$ be the closed immersion determined by the surjective k-homomorphism $R \to A$, $Y_i \mapsto \alpha_i$ $(1 \leq i \leq 5)$. Since $\dim A = 4$ and and $\dim R = 5$ the kernel of $R \to A$ is a prime ideal of height 1, denote it by p. Notice that

$$\alpha_4\alpha_5 - \alpha_1\alpha_3 - \alpha_2(1+\alpha_2) = 0, \text{ so } Y_4Y_5 - Y_1Y_3 - Y_2(1+Y_2) \in p,$$

and this polynomial is irreducible. Thus

$$p = (Y_4Y_5 - Y_1Y_3 - Y_2(1+Y_2)).$$

We know that $\operatorname{Im}\psi = V(p)$. Now consider the morphism

$$\psi \circ \varphi : \operatorname{Spec} B \to \operatorname{Spec} R.$$

This is a morphism from $\mathbf{A}^5$ to $\mathbf{A}^5$. One can check that for each point $x = (x_1, \ldots, x_5)$,

$$\psi \circ \varphi(x) = (\alpha_1(x), \ldots, \alpha_5(x)).$$

It is easy to see that the point $a = (0, -1, 0, 0, 0) \in \mathbf{A}^5$ belongs to $V(p)$, hence to the image of ψ. We claim that a is not in the image of $\psi \circ \varphi$. To see this, suppose that $a = \psi(\varphi(x))$ for some x. Then $x_4 = \alpha_3(x) = 0, x_5 = \alpha_4(x) = 0$. It follows that $\alpha_2(x) = 0$. but $\alpha_2(x) = a_2 = -1$, a contradiction. So we have a point

$$a \in \operatorname{Im}\psi \setminus \operatorname{Im}\psi \circ \varphi$$

and it follows that $\varphi : \operatorname{Spec} B \to \operatorname{Spec} A$ is not surjective. To prove that $\operatorname{rank}\tau = 3$, assume the contrary. Then $\operatorname{rank}\tau \leq 2$ and, since τ is fixed point free, we have $\operatorname{rank}\tau = 1$ by 3.4.1. Then $B = A^{[1]}$ and in particular φ is surjective, contradicting what we have just proved. So $\operatorname{rank}\tau = 3$.

5. From the proof above, we can conclude that τ has no slice. This is obvious because if there is a slice, then $B = A^{[1]}$; but we already know this is not the case. By 3.4.2, τ is not elementary. $\square$

Conclusion

This thesis studies locally nilpotent derivations of polynomial rings.

Chapter two classifies locally nilpotent derivations $D : k[X, Y, Z] \to k[X, Y, Z]$ satisfying $D^2X = D^2Y = 0$. In the special case where $D^2X = D^2Y = D^2Z = 0$, Proposition 2.4.4 shows that D is elementary.

This leads to the investigation of the relationships between elementary and nice derivations (where $D : k[X_1, \dots, X_n] \to k[X_1, \dots, X_n]$ is "nice" if $D^2X_i = 0$ for all i; note that all nice derivations are locally nilpotent). Chapter 3 is devoted to that question. It is clear that all elementary derivations are nice, and Proposition 2.4.4 essentially states that the converse holds in dimension three. Because this result does not hold in higher dimension, it is natural to introduce an intermediate class of derivations, namely, that of "constructible" derivations. The implications

$$D \text{ elementary} \Rightarrow D \text{ constructible} \Rightarrow D \text{ nice}$$

are fairly obvious, and we considered their converses. In dimension 3, as mentioned before, all converse implications are true. In dimension 4, a constructible derivation is "almost" elementary (pseudo-elementary). In dimension 5, it is shown that an example of Winkelmann gives a constructible derivation of $k[X_1, X_2, X_3, X_4, X_5]$ which is not elementary or even pseudo-elementary.

Also worth noting is Corollary 3.2.3, which states, in particular, that constructible derivations of $k[X_1, \dots, X_n]$ have rank $\leq n - 2$ (if $n > 2$).

The structure of locally nilpotent derivations of polynomial rings is very complicated. It is therefore good to restrict the study either to low dimension or to some special classes of locally nilpotent derivations. In dimension 3, as mentioned in Chapter 1, some properties of these derivations are known, but there are still many open questions. In dimension 4, very little is known;

for example, it is not known whether all "nice" derivations have finitely generated kernels, or whether all of them have rank strictly less than 4.

Another aspect of locally nilpotent derivations of polynomial rings is, as mentioned in Chapter 1, their connection with Hilbert's 14th Problem. By the recent work of Freudenburg and Daigle, Hilbert 14 is open in dimension 4 only. Another recent result of the same authors states that a triangular derivation of a polynomial ring of dimension 4 always has a finitely generated kernel.

In the last few years, some remarkable progress has been made in the study of locally nilpotent derivations, but there are many questions we still cannot answer. One very interesting question is certainly to determine whether or not there exists a locally nilpotent derivation of $k[X_1, X_2, X_3, X_4]$ with non finitely generated kernel; another one is to completely describe the locally nilpotent derivations of $k[X, Y, Z]$. We tend to believe that these problems will not be settled in the near future.

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