A FUNCTORIAL FORMULATION OF MODELS OF FIRST ORDER THEORIES

A thesis submitted

by

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Abstract

The aim of this thesis is to formulate models of first order theories in categorical terms. In the first chapter there is a brief description of Lawvere's theory and Volger's logical category. It is pointed out that in Volger's formulation only a certain class of model homomorphism is represented. In the second chapter the basic notion of this thesis, the theory category is defined. It is shown that a full subcategory of set valued functors from the theory category, which consists of functors corresponding to models (called model functors) is isomorphic to the category of models and H-homomorphisms. It is also shown that a certain factorization of a functor corresponding to a structure is necessary and sufficient for the structure to be a model. In the third chapter there is given a categorical characterization of functors isomorphic to model functors, for certain class of theory categories. There is shown via this characterization, that a category of models and certain homomorphism is closed under finite products and filtered colimits.

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Chapter I

Introduction

The categorical formulation of models for algebras which are definable by equations was given in [4], and it can be described as follows.

A skeletal category $\mathfrak{A}$ is called an *algebraic theory* if it satisfies the conditions:

1. $\mathfrak{A}$ has finite products and a terminal object.
2. There is an object A in $\mathfrak{A}$ such that for any object X in $\mathfrak{A}$ ~~which~~ is not terminal $X = \prod_{i=1}^{n} A$ for some n.

A functor F between algebraic theories $\mathfrak{A}$ and $\mathfrak{A}'$ is called an *algebraic functor* if:

1. F preserves finite products and the terminal object,
2. $F(A) = A'$.

The category of algebraic theories and algebraic functors will be denoted $\mathcal{A}$.

Let N be the discrete category of natural numbers. Any functor from N to the category of sets will be called an *algebraic type*. The category of algebraic types and natural transformations will be denoted O-type.

Define a functor $\mathcal{U}_0 \colon \mathcal{A} \to$ O-Type as follows:

$$\mathcal{U}_0(\mathfrak{A})(n) = \mathfrak{A}[A^n, A] \text{ for } \mathfrak{A} \in \mathrm{Obj}(\mathcal{A})$$

$$\mathcal{U}_0(F)_n(t) = F(t) \text{ for } F\colon \mathfrak{A} \to \mathfrak{A}' \in \mathrm{Mor}(\mathcal{A})$$

and $t \in \mathfrak{A}[A^n, A]$.

Then there is a functor V_0: O-Type $\to A$, which is left adjoint to U_0 (cf. [6]). For a given $\Sigma \in$ Obj(O-Type), $V_0(\Sigma)$ can be defined as follows:

$$\mathrm{Obj}(V_0(\Sigma)) = \{A^0, A, A^2, \ldots\}.$$
$$[A^n, A^m] = (T(n))^m,$$

where $T(n)$ is the set of all n-argument terms in the language of type Σ (i.e. the free word algebra on n generators).

The composition of morphisms in $V_0(\Sigma)$ is defined to be substitution of terms i.e. if $(t_1,\ldots,t_n) \in [A^m, A^n]$ and $(s_1,\ldots,s_m) \in [A^k, A^m]$, then $(t_1,\ldots,t_n) \circ (s_1,\ldots,s_m) = (t_1(s_1,\ldots,s_m),\ldots,t_n(s_1,\ldots,s_m))$.

In other words $V_0(\Sigma)$ is the free algebraic theory generated by the set of morphisms:

$$\{\sigma: A^n \to A \mid n \in N, \sigma \in \Sigma(n)\}.$$

Now, let $E = \{t = s \mid t, s \in T(n), n \in N\}$ be the set of all equations of terms in the language of type Σ. For $E \subset E$ define ~E to be the smallest congruence in the category $V_0(\Sigma)$ (an equivalence relation R is a <u>congruence</u> in a category C if $R \subset \bigcup_{X, Y \in \mathrm{Obj}(C)} C[X, Y]^2$ and R preserves composition of morphisms) which satisfies the conditions:

1. if $t = s \in E$, then $t \sim E\, s$
2. ~E preserves products, i.e. if $t_i \sim E\, s_i$ $i = 1\ldots m$, then $(t_1,\ldots,t_m) \sim E\, (s_1,\ldots,s_m)$.

Then the quotient category $V_0(\Sigma)/{\sim}E$ is also an algebraic theory, and the canonical projection π_E: $V_0(\Sigma) \to V_0(\Sigma)/{\sim}E$

is an algebraic functor. Every product preserving functor from $V_0(\Sigma)/{\sim}E$ to the category of sets is an algebra of type Σ in which all equations from E hold. Moreover, if Ψ is a natural transformation between two such functors, then Ψ_A is a homomorphism between the corresponding algebras.

Mathematical structures which are not definable by equations cannot be described in this way. The elementary theories defined in [8] are generalizations of algebraic theories and allow models of an arbitrary first order theory to be expressed in categorical terms. Below we present the main ideas.

A category $\mathcal{B}$ is called a <u>boolean category</u> if it satisfies the conditions:

1. $\mathcal{B}$ has finite products and a terminal object I. (The unique morphism from an object X to I will be denoted by $!_X$).
2. There is an object Ω and morphisms $I \xrightarrow{0} \Omega$, $I \xrightarrow{1} \Omega$ in $\mathcal{B}$ such that

$$\begin{tikzcd} I \arrow[dr, "1"] & \\ & \Omega \\ I \arrow[ur, "0"'] & \end{tikzcd} \quad \text{is a coproduct.}$$

3. For any object X in $\mathcal{B}$

$$\begin{tikzcd} X \arrow[dr, "{(id_X,\ 1_X)}"] & \\ & X \times \Omega \\ X \arrow[ur, "{(id_X,\ 0_X)}"'] & \end{tikzcd} \quad \text{is a coproduct.}$$

(where $1_X = 1 \circ\, !_X$, $0_X = 0 \circ\, !_X$).

A functor F between boolean categories $\mathcal{B}$ and $\mathcal{B}'$ is called a <u>boolean functor</u> if it preserves finite products and the terminal object and $F(\Omega) = \Omega'$, $F(1) = 1'$ and $F(0) = 0'$.

Define morphisms $\neg$, $\wedge$, $\vee$ in $\mathcal{B}$ to be the unique morphisms determined by the following diagrams:

1.
$$\begin{tikzcd}
I \arrow[dr, "1"] \arrow[drr, bend left, "0"] & & \\
& \Omega & \Omega \\
I \arrow[ur, "0"'] \arrow[urr, bend right, "1"'] & &
\end{tikzcd}$$

2.
$$\begin{tikzcd}
\Omega \arrow[dr, "(id_\Omega, 1_\Omega)"'] \arrow[drr, bend left, "id_\Omega"] & & \\
& \Omega \times \Omega \arrow[r, "\wedge"] & \Omega \\
\Omega \arrow[ur, "(id_\Omega, 0_\Omega)"] \arrow[urr, bend right, "0_\Omega"'] & &
\end{tikzcd}$$

3.
$$\begin{tikzcd}
\Omega \arrow[dr, "(id_\Omega, 1_\Omega)"'] \arrow[drr, bend left, "1_\Omega"] & & \\
& \Omega \times \Omega \arrow[r, "\vee"] & \Omega \\
\Omega \arrow[ur, "(id_\Omega, 0_\Omega)"] \arrow[urr, bend right, "id_\Omega"'] & &
\end{tikzcd}$$

It can be shown that Ω, with morphisms 0_Ω, 1_Ω, $\neg$, $\vee$, $\wedge$ is a boolean algebra object in $\mathcal{B}$. (For example: $\vee \circ (0_\Omega, 1_\Omega) = \vee \circ (id_\Omega, 1_\Omega) \circ 0_\Omega = 1_\Omega \circ 0_\Omega = 1_\Omega$). Thus, for any $X \in Obj(\mathcal{B})$ $[X, \Omega]$ is a boolean algebra, and therefore it can be regarded, as a partially ordered category.

A boolean category $\mathcal{B}$ is a <u>logical category</u> if for any $f: X \to Y \in Mor(\mathcal{B})$, there is a functor $\exists_f$ between the partially ordered categories $[X, \Omega]$ and $[Y, \Omega]$, such that

1. $\exists_f: [X, \Omega] \to [Y, \Omega]$ is left adjoint to the functor $\_ \circ f: [Y, \Omega] \to [X, \Omega]$
2. $\exists_{id_{Y_1} \times f_2}(g) \circ (f_1 \times id_{Y_2}) = \exists_{id_{X_1} \times f_2}(g \circ (f_1 \times id_{Y_2}))$ for $f_i: X_i \to Y_i$ $i = 1, 2$ and $g: Y_1 \times X_2 \to \Omega$ in $\mathcal{L}$.
3. $\exists_{id_Y \times id_Y}(1_Y) \circ (f \times id_Y) = \exists_{(id_X, f)}(1_Y \circ f)$ for $f: X \to Y$ in $\mathcal{L}$.

A boolean functor F between logical categories $\mathcal{L}, \mathcal{L}'$ is a <u>logical functor</u> if $F(\exists_f(g)) = \exists'_{F(f)}(F(g))$ for $f: X \to Y$, $g: X \to \Omega \in \mathrm{Mor}(\mathcal{L})$. A logical category $\mathcal{L}$ is called an <u>elementary theory</u> if

1. $\mathcal{L}$ is skeletal, and
2. there is a basic object A in $\mathcal{L}$, such that for any $X \in \mathrm{Obj}(\mathcal{L})$, $X \neq I$ there exist n, $m \in N$, such that $X = A^n \times \Omega^m$.

A logical functor between elementary theories is called a <u>theory map</u> if it preserves the basic object.

Any functor from the discrete category $N \times 2$ to the category of Sets is called a <u>relational type</u>. The category of relational types and natural transformations will be denoted Type.

Let $\mathcal{B}$ be the category of elementary theories and theory maps. Define a functor $\mathcal{U}: \mathcal{B} \to$ Type as follows:

$$\mathcal{U}(\mathcal{B})_0(n) = \mathcal{B}[A^n, A] \text{ for } \mathcal{B} \in Obj(\mathcal{B}), \; n \in N$$
$$\mathcal{U}(\mathcal{B})_1(n) = \mathcal{B}[A^n, \Omega]$$
$$\mathcal{U}(F)_0(n)(t) = F(t) \text{ for } F: \mathcal{B} \to \mathcal{L}' \in Mor(\mathcal{B}) \quad t \in \mathcal{B}[A^n, A]$$
$$\mathcal{U}(F)_1(n)(P) = F(P) \quad P \in \mathcal{B}[A^n, \Omega]$$

Then, there is a functor V: Type $\to \mathcal{B}$, which is left adjoint to $\mathcal{U}$ (cf. [8]).

Let Σ be a given relational type, then the category $V(\Sigma)$ is the free elementary theory generated by the set of morphisms $\{\sigma: A^n \to A \mid \sigma \in \Sigma_0(n), n \in N\} \cup \{P: A^n \to \Omega \mid P \in \Sigma_1(n), n \in N\}$.

The set of all morphisms of $V(\Sigma)$ can be regarded as a first order language of type Σ in the following sense. For $n \in N$, $t \in [A^n, A]$, $F \in [A^n, \Omega]$, t is a term of rank n (i.e. it is interpreted as an n-argument function) and F is a formula of rank n (i.e. it is interpreted as an n-argument predicate). This language differs from the usual one because

1. if t_1, t_2 are terms of rank n and F is a formula of the same rank, then the expression $\binom{t_1}{t_2} \circ (id_{A^n}, F)$ is a term of rank n. It is interpreted as the conditional term:

If F then t_1, else t_2.

2. if t is a term of rank n and F is a formula of the same rank, then the expression $\exists_t F$ is a formula, and it is interpreted as the image of F through t.

(Ordinary existential quantification can be expressed as an image along a projection from a product).

Any morphism $A^n \times \Omega^m \to A^p \times \Omega^q$ in $V(\Sigma)$ is uniquely specified by 2^m morphisms $A^n \to A^p \times \Omega^q$ (since $A^n \times \Omega^m = \coprod_{i=1}^{2^m} A^n$) and hence by $2^m \times p$ morphisms $A^n \to A$ and $2^m \times q$ morphisms $A^n \to \Omega$.

On the other hand, if there is given a first order language of type Σ, which contains conditional terms and extended quantification (i.e. the quantifier is interpreted as image) then morphisms in $V(\Sigma)$ can be defined as $2^m \times p + 2^m \times q$ tuples of terms and formulas, and the composition of morphisms is substitution in terms and formulas.

Now, let T be a set of closed formulas (i.e. formulas of rank 0) of this language. Define $\sim T$ to be the smallest congruence in the category $V(\Sigma)$, which satisfies the conditions:

1. if $F \in T$, then $F \sim T\ 1$.
2. $\sim T$ preserves finite products.
3. $\sim T$ preserves coproducts of the form $X + X$,

   i.e. if $t_i \sim T\ s_i$ $i = 1, 2$, then $\binom{t_1}{t_2} \sim T \binom{s_1}{s_2}$
4. if $t \sim T\ s$ and $F \sim T\ G$, then $\exists_t F \sim T\ \exists_s G$.

Then the quotient category $V(\Sigma)/\sim T$ is also an elementary theory (cf. [8]), and the canonical projection π_T: $V(\Sigma) \to V(\Sigma)/\sim T$ is a theory map. Any theory map from $V(\Sigma)/\sim T$ to an elementary theory, which is a full subcategory of Sets is a model of T.

Before discussing properties of natural transformations between model functors, consider the following definitions. Let $\mathfrak{M}$ and $\mathfrak{N}$ be two structures (see section 2) of type Σ and let H be a set of formulas. A function $h: |\mathfrak{M}| \to |\mathfrak{N}|$ is an <u>H-homomorphism</u> (cf. [2]) if it is a homomorphism with respect to all operations in Σ_0 (i.e. for a term t of rank n

$$h \circ \mathfrak{M}^{\ulcorner}t^{\urcorner} = \mathfrak{N}^{\ulcorner}t^{\urcorner} \circ h^n$$

and for any formula $F \in H$ of rank n

$$h^n(\mathfrak{M}^{\ulcorner}F^{\urcorner}) \subset \mathfrak{N}^{\ulcorner}F^{\urcorner}$$

If H is the set of all formulas, then it follows that h is a monomorphism and $\mathfrak{N}$ is an elementary extension of $h(\mathfrak{M})$ (a model $\mathfrak{N}$ is an elementary extension of a model $\mathfrak{N}'$ if $|\mathfrak{N}'| \subset |\mathfrak{N}|$ and for any formula F and a in the universe of $\mathfrak{N}'$, $a \in \mathfrak{N}^{\ulcorner}F^{\urcorner}$ iff $a \in \mathfrak{N}'^{\ulcorner}F^{\urcorner}$ (cf. [7])).

Now, let $\mathfrak{M}, \mathfrak{N}: V(\Sigma)/\sim T \to$ Sets be two model functors and let there be a natural transformation $\Psi: \mathfrak{M} \to \mathfrak{N}$. For any formula F of rank n the following diagram commutes

$$\begin{array}{ccc}
\mathfrak{M}(A^n) & \xrightarrow{\mathfrak{M}(F)} & \mathfrak{M}(\Omega) = \{1_{\mathfrak{M}}, 0_{\mathfrak{M}}\} \\
(\Psi_A)^n = \Psi_{A^n} \big\downarrow & & \big\downarrow \Psi_\Omega \\
\mathfrak{N}(A^n) & \xrightarrow{\mathfrak{N}(F)} & \mathfrak{N}(\Omega) = \{1_{\mathfrak{N}}, 0_{\mathfrak{N}}\}
\end{array}$$

where $\Psi_\Omega(1_{\mathfrak{M}}) = 1_{\mathfrak{N}}$, $\Psi_\Omega(0_{\mathfrak{M}}) = 0_{\mathfrak{N}}$ and, $\Psi_{A^n} = \Psi_A{}^n$ since

$$\begin{tikzcd}
\mathfrak{M}(I) \arrow[r, "\mathfrak{M}(1)"] \arrow[d, "\Psi_I"'] & \mathfrak{M}(\Omega) \arrow[d, "\Psi_\Omega"] \\
\mathfrak{N}(I) \arrow[r, "\mathfrak{N}(1)"'] & \mathfrak{N}(\Omega)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathfrak{M}(I) \arrow[r, "\mathfrak{M}(0)"] \arrow[d, "\Psi_I"'] & \mathfrak{M}(\Omega) \arrow[d, "\Psi_\Omega"] \\
\mathfrak{N}(I) \arrow[r, "\mathfrak{N}(0)"'] & \mathfrak{N}(\Omega)
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathfrak{M}(A^n) \cong (\mathfrak{M}(A))^n \arrow[r, "i^{th} \text{ proj}"] \arrow[d, "\Psi_{A^n}"'] & \mathfrak{M}(A) \arrow[d, "\Psi_A"] \\
\mathfrak{N}(A^n) \cong (\mathfrak{N}(A))^n \arrow[r, "i^{th} \text{ proj}"'] & \mathfrak{N}(A)
\end{tikzcd}$$

commute.

Thus $\mathfrak{N}(F) \circ (\Psi_A)^n = \Psi_\Omega \circ \mathfrak{M}(F)$. Now let $a \in \mathfrak{M}^{\ulcorner} F^{\urcorner}$, then $\mathfrak{M}(F)(a) = 1_{\mathfrak{M}}$, so $\Psi_\Omega \circ \mathfrak{M}(F)(a) = 1_{\mathfrak{N}}$, so $\mathfrak{N}(F) \circ (\Psi_A)^n(a) = 1_{\mathfrak{N}}$, so $(\Psi_A)^n(a) \in \mathfrak{N}^{\ulcorner} F^{\urcorner}$. Thus $(\Psi_A)^n(\mathfrak{M}^{\ulcorner} F^{\urcorner}) \subset \mathfrak{N}^{\ulcorner} F^{\urcorner}$ and this means that natural transformations between model functors are inclusions in elementary extensions.

It was the main aim of this work to describe a formulation in which a wider class of model homomorphisms is represented.

Chapter II

Theory Categories

In some respects, the first order language and the theory category are formulated in this chapter in a way similar to that of [8]. The main difference is in expressing formulas as subobjects not as characteristic functions. This latter idea comes from [1].

1. Relational types

Any function Σ from $N \times 2$ to sets, such that all sets $\Sigma_0(0), \Sigma_0(1), \ldots, \Sigma_1(0), \Sigma_1(1), \ldots$ are disjoint and $\Sigma_1(0) = \{1\}$ will be called a relational type. Elements of $\Sigma_0(n)$ are called n-ary function symbols, and elements of $\Sigma_1(n)$ are called n-ary predicate symbols. In the following it will be assumed that Σ is a fixed relational type.

2. Structures

Let A be a nonempty set. An interpretation $\ulcorner\;\urcorner$ is a function whose domain is the set $\bigcup_{n=0}^{\infty} \Sigma_0(n) \cup \bigcup_{n=0}^{\infty} \Sigma_1(n)$, such that if $f \in \Sigma_0(n)$ then $f : A^n \to A$ (i.e. $\ulcorner f\urcorner$ is an n-ary function), if $P \in \Sigma_1(n)$ and $n \neq 0$ then $\ulcorner P\urcorner \subset A^n$, and $\ulcorner 1\urcorner = A^0$.

The pair $\mathfrak{A} = (A, \ulcorner\;\urcorner)$ will be called a structure of type Σ (or simply a structure). The set A is the universe of the structure $\mathfrak{A}$, and will be denoted $|\mathfrak{A}|$.

3. The Alphabet

As usual the alphabet of a first order language is a disjoint union of a set V of individual variable symbols, a set F of function symbols, a set P of predicate symbols, a set L of logical symbols and a set A of auxiliary symbols. In the following alphabet will consist of the symbols:

1. $V = \bigcup_{n=0}^{\infty} V^*(n)$ where

$$V^*(n) = V(n) \cup \{p_0^n\}$$

$$V(n) = \begin{cases} \{p_1^n, \ldots, p_n^n\} & \text{if } n \neq 0 \\ \emptyset & \text{if } n = 0 \end{cases}$$

2. $F = \bigcup_{n=0}^{\infty} \Sigma_0(n)$

3. $P = \bigcup_{n=0}^{\infty} \Sigma_1(n)$

4. $L = \{\neg, \vee, \wedge, \exists\}$

5. $A = \{(,)\}$

The interpretation $\ulcorner\ \urcorner$ will be extended to the set V in the following way:

1. for $p_i^n \in V(n)$

   $\ulcorner p_i^n \urcorner : A^n \to A$ is the projection on the i^{th} axis, i.e.

   $\ulcorner p_i^n \urcorner (a_1, \ldots, a_n) = a_i$ for $a_1, \ldots, a_n \in A$

2. $\ulcorner p_0^n \urcorner$ is the unique function from A^n to A^0

4. Terms

Define the set T of proper terms as follows: $T = \bigcup_{n=0}^{\infty} T(n)$ where for each n, $T(n)$ is the smallest set satisfying the conditions:

1. $V(n) \cup \Sigma_0(n) \subset T(n)$

2. if $t_1, \ldots, t_m \in T(n)$ and $t \in T(m)$ then

$t(t_1 \ldots\ldots t_m) \in T(n)$.

3. if $t \in T(0)$ then $t(p_0^n) \in T(n)$.

Elements of the set $T^* = T \cup \bigcup_{n=0}^{\infty} \{p_0^n\}$ will be called <u>terms</u>.

The interpretation $\ulcorner\ \urcorner$ will be extended to the set of terms in the following way:

1. if $t \in T(n)$ and $t_1, \ldots, t_m \in T(n)$, then $\ulcorner t(t_1 \ldots\ldots t_m)\urcorner = \ulcorner t\urcorner(\ulcorner t_1\urcorner, \ldots, \ulcorner t_m\urcorner)$ (where $(\ulcorner t_1\urcorner, \ldots, \ulcorner t_m\urcorner)$ denotes the product function) i.e. $\ulcorner t(t_1 \ldots\ldots t_m)\urcorner(a) = \ulcorner t\urcorner(\ulcorner t_1\urcorner(a), \ldots, \ulcorner t_m\urcorner(a))$ for all $a \in A^n$.
2. if $t \in T(0)$ then $\ulcorner t(p_0^n)\urcorner$ is an n-argument constant function, such that $\ulcorner t(p_0^n)\urcorner(a_1, \ldots, a_n) = \ulcorner t\urcorner$.

5. <u>Formulas</u>

Define the set F of formulas as follows: $F = \bigcup_{n=0}^{\infty} F(n)$ where for each n, $F(n)$ is the smallest set satisfying the conditions:

1. $\Sigma_1(n) \subset F(n)$
2. if $t_1, \ldots, t_m \in T(n)$ and $F \in F(m)$, then $F(t_1 \ldots\ldots t_m) \in F(n)$,
3. if $F \in F(0)$ then $F(p_0^n) \in F(n)$,
4. if $F, G \in F(n)$, then $\neg F$, $\wedge FG$, $\vee FG \in F(n)$,
5. if $n \neq 0$, $t_1, \ldots, t_n \in T(m)$ and $F \in F(m)$ then $\exists_{t_1 \ldots\ldots t_n}(F) \in F(n)$,
6. if $n = 0$ and $F \in F(m)$, then $\exists_{p_0^m}(F) \in F(n)$.

Any subset of the set F will be called a <u>theory</u>.

The interpretation $\ulcorner\ \urcorner$ will be extended to the set of formulas in the following way:

1. if $t_1, \ldots, t_m \in T(n)$ and $F \in F(m)$, then $\ulcorner F(t_1 \ldots\ldots t_m)\urcorner \subset A^n$ and $a \in \ulcorner F(t_1 \ldots\ldots t_m)\urcorner$ iff $(\ulcorner t_1\urcorner(a), \ldots, \ulcorner t_m\urcorner(a)) \in \ulcorner F\urcorner$ (Thus $\ulcorner F(t_1 \ldots\ldots t_m)\urcorner$ is the inverse image of $\ulcorner F\urcorner$ by the product function $(\ulcorner t_1\urcorner, \ldots, \ulcorner t_m\urcorner)$).
2. if $F \in F(0)$ then $\ulcorner F(p_0^n)\urcorner = \begin{cases} A^n & \text{if } \ulcorner F\urcorner = A^0 \\ \emptyset & \text{if } \ulcorner F\urcorner = \emptyset \end{cases}$
3. if $F, G \in F(m)$, then $\ulcorner \neg F\urcorner = A^m - \ulcorner F\urcorner$
$$\ulcorner \vee FG\urcorner = \ulcorner F\urcorner \cup \ulcorner G\urcorner$$
$$\ulcorner \wedge FG\urcorner = \ulcorner F\urcorner \cap \ulcorner G\urcorner$$
4. if $F \in F(m)$ and $t_1, \ldots, t_n \in T(m)$, then $\ulcorner \exists t_1 \ldots\ldots t_n (F)\urcorner = (\ulcorner t_1\urcorner, \ldots, \ulcorner t_n\urcorner)(\ulcorner F\urcorner) = \{(\ulcorner t_1\urcorner(a), \ldots, \ulcorner t_n\urcorner(a)) \mid a \in \ulcorner F\urcorner\}$.
5. if $F \in F(m)$, then $\ulcorner \exists p_0^m (F)\urcorner = \begin{cases} A^0 & \text{if } \ulcorner F\urcorner \neq \emptyset \\ \emptyset & \text{if } \ulcorner F\urcorner = \emptyset \end{cases}$

6. Models

A structure $\mathfrak{A}$ is a model for the theory T if for all n: if $F \in T \cap F(n)$ then $\ulcorner F\urcorner = |\mathfrak{A}|^n$.

Let K_T denote the class of all models for the theory T. A formula $F \in F(n)$ is a semantical consequence of the theory T - in symbols $\models_T F$ - if for all $\mathfrak{A} \in K_T$, $\ulcorner F\urcorner = |\mathfrak{A}|^n$. The set of all formulas F such that $\models_T F$ will be denoted Cn(T).

7. Abbreviations

In the sequel the following symbols will be used as abbreviations of formulas or parts of formulas:

for $F, G \in F(n)$:

1. $F \vee G$ will denote $\vee FG$
2. $F \wedge G$ will denote $\wedge FG$
3. $F \rightarrow G$ will denote $\neg F \vee G$
4. $F \leftrightarrow G$ will denote $(F \rightarrow G) \wedge (G \rightarrow F)$

For $F_1, \ldots, F_n \in F(n)$:

5. $\bigcup_{i=1}^{n} F_i$ will denote $F_1 \vee \ldots \vee F_n$
6. $\bigcap_{i=1}^{n} F_i$ will denote $F_1 \wedge \ldots \wedge F_n$
7. 1_n will denote $1(p_0^n)$, where $1 \in \Sigma_1(0)$
8. 0 will denote $\neg 1$, 0_n will denote $\neg 1_n$

   For any structure $\mathfrak{A}$ (of type Σ)
   $\ulcorner 1_n \urcorner = \ulcorner 1(p_0^n) \urcorner = \{(a_1, \ldots, a_n) \in |\mathfrak{A}|^n \mid \ulcorner p_0^n \urcorner (a_1, \ldots, a_n) \in \ulcorner 1 \urcorner\} = |\mathfrak{A}|^n$,
   and $\ulcorner 0_n \urcorner = \emptyset$
9. for $t, s \in T(n)$

   $t \equiv s$ will denote $\exists_{p_1^1 p_1^1} 1(1_1)(t, s)$

   For any structure $\mathfrak{A}$, $\ulcorner t \equiv s \urcorner =$
   $\{a \in |\mathfrak{A}|^n \mid (\ulcorner t \urcorner, \ulcorner s \urcorner)(a) \in \ulcorner \exists_{p_1^1 p_1^1}(1_1) \urcorner\} =$
   $\{a \in |\mathfrak{A}|^n \mid (\ulcorner t \urcorner(a), \ulcorner s \urcorner(a)) \in (\ulcorner p_1^1 \urcorner, \ulcorner p_1^1 \urcorner)(|\mathfrak{A}|)\} = \{a \in |\mathfrak{A}|^n \mid \ulcorner t \urcorner(a) = \ulcorner s \urcorner(a)\}$
10. for $t_i, s_i \in T(n)$ $i = 1, \ldots, m$

    $t_1, \ldots, t_m \equiv s_1, \ldots, s_m$ will denote $\bigcap_{i=1}^{m} t_i \equiv s_i$.
11. for natural numbers $m_1, \ldots, m_k$, $1 \le i \le k$

    $(m_1, \ldots, \underline{m_i}, \ldots, m_k)$ will denote:

$$p^{m_1+\dots+m_k}_{m_1+\dots+m_{i-1}+1}, \dots, p^{m_1+\dots+m_k}_{m_1+\dots+m_i} \quad - \quad \text{if } m_i \neq 0$$

$$p^{m_1+\dots+m_k}_{0} \quad - \quad \text{if } m_i = 0,$$

thus for any structure $\mathfrak{A}$

$\ulcorner(m_1, \dots, \underline{m_i}, \dots, m_k)\urcorner$ is the projection:

$|\mathfrak{A}|^{m_1+\dots+m_k} \to |\mathfrak{A}|^{m_i}$.

a) For example, let α be the formula

$$\exists_{p^{m+n+k}_{m+1}, \dots, p^{m+n+k}_{m+n}} (F(p^{m+n+k}_1, \dots, p^{m+n+k}_{m+n})).$$

According to definition 11, α can be expressed as:

$$\exists_{(m,\underline{n},k)} (F(\underline{m}, \underline{n}, k)).$$

This last expression can be read as "There exist *m* and *k* such that F(*m*, *n*)" which agrees with the intuition attached to the formula α.

b) If $m \neq 0$ then the symbol $(k, \underline{m}, m) \equiv (k, m, \underline{m})$ makes sense and in any structure $\mathfrak{A}$

$\ulcorner(k, \underline{m}, m) \equiv (k, m, \underline{m})\urcorner = \{(a, b, c) \in |\mathfrak{A}|^{k+m+m} \mid b = c\}$

12. To make the symbol $(k, \underline{m}, m) \equiv (k, m, \underline{m})$ meaningful for any m, $p^k_0 \equiv p^k_0$ will denote 1_k.

13. $i(m)$ will denote $(\underline{m}, m) \equiv (m, \underline{m})$, thus for any structure $\mathfrak{A}^{\ulcorner i(m)\urcorner}$ is the diagonal in the set $(|\mathfrak{A}|^m)^2$.

14. For $F \in F(m)$ and $G \in F(n)$ $F \times G$ will denote $F(\underline{m}, n) \wedge G(m, \underline{n})$, thus for any structure $\mathfrak{A}$ $\ulcorner F \times G \urcorner = \{(a, b) \in |\mathfrak{A}|^{m+n} \mid a \in \ulcorner F \urcorner$ and $b \in \ulcorner G \urcorner\} = \ulcorner F \urcorner \times \ulcorner G \urcorner$.

8. H-homomorphism

Let H be a set of formulas

Let $\mathfrak{A} = (|\mathfrak{A}|, \mathfrak{A}^{\ulcorner\ \urcorner})$ and $\mathfrak{B} = (|\mathfrak{B}|, \mathfrak{B}^{\ulcorner\ \urcorner})$ be two structures of type Σ. For any function $h: |\mathfrak{A}| \to |\mathfrak{B}|$ define functions $h^n: |\mathfrak{A}|^n \to |\mathfrak{B}|^n$ as follows:

if $n \neq 0$ then

$h^n(a_1, \ldots, a_n) = (h(a_1), \ldots, h(a_n))$ for all $a_1, \ldots, a_n \in |\mathfrak{A}|$

if $n = 0$ then h^n is the unique function $|\mathfrak{A}|^0 \to |\mathfrak{B}|^0$. A function $h: |\mathfrak{A}| \to |\mathfrak{B}|$ is an $\underline{H\text{-homomorphism}}$ if for all n it satisfies the conditions:

1) $h \circ \mathfrak{A}^{\ulcorner t \urcorner} = \mathfrak{B}^{\ulcorner t \urcorner} \circ h^n$ for all $t \in T(n)$

2) $h^n(\mathfrak{A}^{\ulcorner F \urcorner}) \subset \mathfrak{B}^{\ulcorner F \urcorner}$ for all $F \in H(n) = H \cap F(n)$

8.1 Proposition

Let H be a set of formulas such that $\Sigma_1(n) \subset H(n)$ for all n, then $h: |\mathfrak{A}| \to |\mathfrak{B}|$ is an H-homomorphism iff h satisfies the condition 2) for all $F \in \overline{H}$, where $\overline{H}$ is the smallest set such that:

1) $H \subset \overline{H}$

2) $\overline{H}$ is closed under $\wedge$, $\exists$ and substitution (i.e. if $F, G \in \overline{H}(n)$, $t_1, \ldots, t_m \in T(n)$ and $s_1, \ldots, s_n \in T(k)$ then $F \wedge G$, $\exists_{t_1, \ldots, t_m}(F)$, $F(s_1, \ldots, s_n) \in \overline{H}$).

Proof

I Let h be an H-homomorphism, and $F \in \overline{H}(n)$.

1. if $F \in H$ then F satisfies the condition 2), because h is an H-homomorphism.
2. Let $F = G_1 \wedge G_2$ and suppose that G_1 and G_2 satisfy the condition 2), then $h^n(\mathfrak{A}\ulcorner G_1 \wedge G_2 \urcorner) = h^n(\mathfrak{A}\ulcorner G_1\urcorner \cap \mathfrak{A}\ulcorner G_2\urcorner) \subseteq h^n(\mathfrak{A}\ulcorner G_1\urcorner) \cap h^n(\mathfrak{A}\ulcorner G_2\urcorner) \subset \mathfrak{B}\ulcorner G_1\urcorner \cap \mathfrak{B}\ulcorner G_2\urcorner = \mathfrak{B}\ulcorner G_1 \wedge G_2\urcorner$, thus $h^n(\mathfrak{A}\ulcorner F\urcorner) \subset \mathfrak{B}\ulcorner F\urcorner$.
3. Let $F = \exists_{t_1, \ldots t_n}(G)$ for some $t_1, \ldots, t_n \in T(m)$, $G \in F(m)$ and suppose that G satisfies the condition 2), then $h^n(\mathfrak{A}\ulcorner\exists_{t_1, \ldots, t_n}(G)\urcorner) = h^n((\mathfrak{A}\ulcorner t_1\urcorner, \ldots, \mathfrak{A}\ulcorner t_n\urcorner)(\mathfrak{A}\ulcorner G\urcorner)) = (h \circ \mathfrak{A}\ulcorner t_1\urcorner, \ldots, h \circ \mathfrak{A}\ulcorner t_n\urcorner)(\mathfrak{A}\ulcorner G\urcorner) = (\mathfrak{B}\ulcorner t_1\urcorner \circ h^m, \ldots, \mathfrak{B}\ulcorner t_n\urcorner \circ h^m)(\mathfrak{A}\ulcorner G\urcorner) = (\mathfrak{B}\ulcorner t_1\urcorner, \ldots, \mathfrak{B}\ulcorner t_n\urcorner)(h^m(\mathfrak{A}\ulcorner G\urcorner)) \subset (\mathfrak{B}\ulcorner t_1\urcorner, \ldots, \mathfrak{B}\ulcorner t_n\urcorner)(\mathfrak{B}\ulcorner G\urcorner) = \mathfrak{B}\ulcorner\exists_{t_1, \ldots, t_n}(G)\urcorner$, thus $h^n(\mathfrak{A}\ulcorner F\urcorner) \subset \mathfrak{B}\ulcorner F\urcorner$.
4. Let $n = 0$ and $F = \exists_{p_0^m}(G)$ for some $G \in F(m)$. Suppose that G satisfies the condition 2), then $h^0(\mathfrak{A}\ulcorner\exists_{p_0^m}(G)\urcorner) = (h^0 \circ \mathfrak{A}\ulcorner p_0^m\urcorner)(\mathfrak{A}\ulcorner G\urcorner) = (\mathfrak{B}\ulcorner p_0^m\urcorner \circ h^m)(\mathfrak{A}\ulcorner G\urcorner) \subset \mathfrak{B}\ulcorner p_0^m\urcorner(\mathfrak{B}\ulcorner G\urcorner) = \mathfrak{B}\ulcorner\exists_{p_0^m}(G)\urcorner$, thus $h^n(\mathfrak{A}\ulcorner F\urcorner) \subset \mathfrak{B}\ulcorner F\urcorner$.
5. Let $F = G(t_1, \ldots, t_m)$ for some $t_1, \ldots, t_m \in T(n)$ and $G \in F(m)$, and suppose that G satisfies the condition 2). Let $b \in h^n(\mathfrak{A}\ulcorner G(t_1, \ldots, t_m)\urcorner)$ then $b = h^n(a)$ for some $a \in \mathfrak{A}\ulcorner G(t_1, \ldots, t_m)\urcorner$, so $(\mathfrak{A}\ulcorner t_1\urcorner, \ldots, \mathfrak{A}\ulcorner t_m\urcorner)(a) \in \mathfrak{A}\ulcorner G\urcorner$, so $(\mathfrak{B}\ulcorner t_1\urcorner, \ldots, \mathfrak{B}\ulcorner t_m\urcorner)(b) = (\mathfrak{B}\ulcorner t_1\urcorner, \ldots, \mathfrak{B}\ulcorner t_m\urcorner)(h^n(a)) =$

$(\mathfrak{B}\ulcorner t_1\urcorner \circ h^n, \ldots, \mathfrak{B}\ulcorner t_m\urcorner \circ h^n)(a) =$

$(h \circ \mathfrak{A}\ulcorner t_1\urcorner, \ldots, h \circ \mathfrak{A}\ulcorner t_m\urcorner)(a) =$

$h^m((\mathfrak{A}\ulcorner t_1\urcorner, \ldots, \mathfrak{A}\ulcorner t_m\urcorner)(a)) \in h^m(\mathfrak{A}\ulcorner G\urcorner) \subset$

$\mathfrak{B}\ulcorner G\urcorner$, so $(\mathfrak{B}\ulcorner t_1\urcorner, \ldots, \mathfrak{B}\ulcorner t_m\urcorner)(b) \in \mathfrak{B}\ulcorner G\urcorner$, so

$b \in \mathfrak{B}\ulcorner G(t_1 \ldots\ldots t_m)\urcorner$, thus $h^n(\mathfrak{A}\ulcorner F\urcorner) \subset \mathfrak{B}\ulcorner F\urcorner$.

6. Let $F = G(p_0^n)$ for some $G \in F(0)$ and suppose that G satisfies the condition 2). Let $b \in h^n(\mathfrak{A}\ulcorner G(p_0^n)\urcorner)$, then $b = h^n(a)$ for some $a \in \mathfrak{A}\ulcorner G(p_0^n)\urcorner$, so

$\mathfrak{A}\ulcorner p_0^n\urcorner(a) \in \mathfrak{A}\ulcorner G\urcorner$, so

$\mathfrak{B}\ulcorner p_0^n\urcorner(b) = (\mathfrak{B}\ulcorner p_0^n\urcorner \circ h^n)(a) = (h^0 \circ \mathfrak{A}\ulcorner p_0^n\urcorner)(a) \in$

$h^0(\mathfrak{A}\ulcorner G\urcorner) \subset \mathfrak{B}\ulcorner G\urcorner$, so $b \in \mathfrak{B}\ulcorner G(p_0^n)\urcorner$, thus

$h^n(\mathfrak{A}\ulcorner F\urcorner) \subset \mathfrak{B}\ulcorner F\urcorner$.

It follows from 1-6 that h satisfies the condition 2) for all $F \in \mathcal{H}$.

II Let h satisfy the condition 2) for all $F \in \mathcal{H}$. Let $t \in T(n)$, then $t(\underline{n}, 1) \equiv (n, \underline{1}) \in \mathcal{H}$, so

$h^{n+1}(\mathfrak{A}\ulcorner t(\underline{n}, 1) \equiv (n, \underline{1})\urcorner) \subset \mathfrak{B}\ulcorner t(\underline{n}, 1) \equiv (n, \underline{1})\urcorner$.

1. if $n \neq 0$ then let $(a_1, \ldots, a_n) \in |\mathfrak{A}|^n$, then

$(a_1, \ldots, a_n, \mathfrak{A}\ulcorner t\urcorner(a_1, \ldots, a_n)) \in \mathfrak{A}\ulcorner t(\underline{n}, 1) \equiv (n, \underline{1})\urcorner$,

so $(h(a_1), \ldots, h(a_n), h(\mathfrak{A}\ulcorner t\urcorner(a_1, \ldots, a_n))) \in$

$h^{n+1}(\mathfrak{A}\ulcorner t(\underline{n}, 1) \equiv (n, \underline{1})\urcorner) \subset \mathfrak{B}\ulcorner t(\underline{n}, 1) \equiv (n, \underline{1})\urcorner$, so

$\mathfrak{B}\ulcorner t\urcorner(h(a_1), \ldots, h(a_n)) = h(\mathfrak{A}\ulcorner t\urcorner(a_1 \ldots\ldots a_n))$

thus $h \circ \mathfrak{A}\ulcorner t\urcorner = \mathfrak{B}\ulcorner t\urcorner \circ h^n$.

2. if $n = 0$, then $\mathfrak{A}\ulcorner t\urcorner \in \mathfrak{A}\ulcorner t(\underline{n}, 1) \equiv (n, \underline{1})\urcorner$, so

$h(\mathfrak{A}\ulcorner t\urcorner) \in \mathfrak{B}\ulcorner t(\underline{n}, 1) \equiv (n, \underline{1})\urcorner$, so

$h(\mathfrak{A}\ulcorner t\urcorner) = \mathfrak{B}\ulcorner t\urcorner$, thus $h \circ \mathfrak{A}\ulcorner t\urcorner = \mathfrak{B}\ulcorner t\urcorner \circ h^n$.

Thus h is an $\mathcal{H}$-homomorphism $\square$

In the sequel we will always assume that the set H satisfies condition

(*) $\Sigma_1(n) \subset H(n)$ for all n, and $\overline{H} = H$.

Among all sets of formulas satisfying (*) we will distinguish the following:

0. H_0 - the smallest set satisfying (*).
1. H_1 - set of all formulas do which not involve $\neg$.
2. H_2 - any set of formulas which contains 0 and is closed under $\vee$, $\wedge$, $\exists$ and substitution and such that if $F \vee G$, $F \wedge G$, $\exists_{t_1 \ldots\ldots t_n}(F)$ or $F(t_1 \ldots\ldots t_n)$ is in H_2 then the constituants F, G are in H_2.

9. Theory categories

Let T be a theory for which there exists at least one model. Let K_T denote the class of all models for T.

9.1 The relation $\overset{T}{\sim}$

Define a relation $\overset{T}{\sim}$ on the set of formulas as follows:

$F \overset{T}{\sim} G$ iff

$\ulcorner F \urcorner = \ulcorner G \urcorner$ for all $\mathfrak{A} \in K_T$ and $F, G \in F(n)$ for some n.

Then $\overset{T}{\sim}$ is an equivalence relation. For $F \in F$ the equivalence class of F will be denoted by $\overline{F}$: (or if it is necessary to indicate that relation is relative to theory T, - by $\overline{F}_T$)

9.2 The relation $F \overset{T}{\sim} G$

For each $F \in F(m)$ and each $G \in F(n)$ define the relation $F \overset{T}{\sim} G$ on the set $F(m+n)$ as follows:

$\Phi \overset{T}{\underset{F,G}{\sim}} \Psi$ iff for all $\mathfrak{A} \in K_T$ $\ulcorner\Phi\urcorner$ and $\ulcorner\Psi\urcorner$ agree in the cartesian product of $\ulcorner F\urcorner$ and $\ulcorner G\urcorner$ (i.e. $\ulcorner F\urcorner \times \ulcorner G\urcorner \cap \ulcorner\Phi\urcorner = \ulcorner F\urcorner \times \ulcorner G\urcorner \cap \ulcorner\Psi\urcorner$). $\overset{T}{\underset{F,G}{\sim}}$ is an equivalence relation. Let $[\Phi]_{\overset{T}{\underset{F,G}{\sim}}}$ denote the equivalence class of Φ, then $F\ \Phi\ G$ (or $F\ \Phi\ G_T$) will denote the triple $(\overline{F}, [\Phi]_{\overset{T}{\underset{F,G}{\sim}}}, \overline{G})$. Clearly, if $F \overset{T}{\sim} F'$ and $G \overset{T}{\sim} G'$ then $F\ \Phi\ G = F'\ \Phi\ G'$. For any $\mathfrak{A} \in K_T$ $\ulcorner F\ \Phi\ G\urcorner$ will denote the set $\ulcorner F\urcorner_{\mathfrak{a}} \times \ulcorner G\urcorner \cap \ulcorner\Phi\urcorner$ regarded as a binary relation from $\ulcorner F\urcorner$ to $\ulcorner G\urcorner$.

An immediate consequence of the above definitions is the following:

9.2.1 Proposition

$F\ \Phi\ G = F'\ \Phi'\ G'$ iff $\ulcorner F\ \Phi\ G\urcorner = \ulcorner F'\ \Phi'\ G'\urcorner$ for all $\mathfrak{A} \in K_T$.

Proof

I. Let $F\ \Phi\ G = F'\ \Phi'\ G'$, then

$\overline{F} = \overline{F}'$, $\overline{G} = \overline{G}'$ and $[\Phi]_{F\tilde{,}G} = [\Phi']_{F'\tilde{,}G'}$, so

for any $\mathfrak{A} \in K_T$

$\ulcorner F\urcorner = \ulcorner F'\urcorner$ and $\ulcorner G\urcorner = \ulcorner G'\urcorner$, so

$\ulcorner F\urcorner \times \ulcorner G\urcorner \cap \ulcorner\Phi\urcorner = \ulcorner F'\urcorner \times \ulcorner G'\urcorner \cap \ulcorner\Phi\urcorner$, but

$\ulcorner F'\urcorner \times \ulcorner G'\urcorner \cap \ulcorner\Phi\urcorner = \ulcorner F'\urcorner \times \ulcorner G'\urcorner \cap \ulcorner\Phi'\urcorner$ because

$\Phi \in [\Phi']_{F'\tilde{,}G'}$ so

$\ulcorner F\urcorner \times \ulcorner G\urcorner \cap \ulcorner\Phi\urcorner = \ulcorner F'\urcorner \times \ulcorner G'\urcorner \cap \ulcorner\Phi'\urcorner$ and finally

$\ulcorner F\ \Phi\ G\urcorner = \ulcorner F'\ \Phi'\ G'\urcorner$

II If $\ulcorner F\ \Phi\ G\urcorner = \ulcorner F'\ \Phi'\ G'\urcorner$ for all $\mathfrak{A} \in K_T$ then

$\ulcorner F\urcorner = \ulcorner F'\urcorner$, $\ulcorner G\urcorner = \ulcorner G'\urcorner$ for all $\mathfrak{A} \in K_T$ so

$\overline{F} = \overline{F}'$, $\overline{G} = \overline{G}'$ and

$\ulcorner F \urcorner \times \ulcorner G \urcorner \cap \ulcorner \Phi \urcorner = \ulcorner F' \urcorner \times \ulcorner G' \urcorner \cap \ulcorner \Phi' \urcorner = \ulcorner F \urcorner \times \ulcorner G \urcorner \cap \ulcorner \Phi' \urcorner$,

so $[\Phi]_{\underset{F,G}{\tilde{T}}} = [\Phi']_{\underset{F,G}{\tilde{T}}} = [\Phi']_{\underset{F',G'}{\tilde{T}}}$

Thus $F \; \Phi \; G = F' \; \Phi' \; G'$ □.

9.3 $[\overline{F}, \overline{G}]$

For $F \in F(m)$ and $G \in F(n)$ define $[\overline{F}, \overline{G}] = \{F \; \Phi \; G \mid \Phi \in F(m$ and for all $\mathfrak{A} \in K_T$, $\ulcorner F \; \Phi \; G \urcorner$ is a function from $\ulcorner F \urcorner$ to $\ulcorner G \urcorner\}$.

9.4 The operation o

For $E \in F(k)$, $F \in F(m)$, $G \in F(n)$, $\Phi \in F(k+m)$ and $\Psi \in F(m+n)$ define $F \; \Psi \; G \circ E \; \Phi \; F = E(\Phi F \Psi)G$ where

$(\Phi F \Psi) = \exists(\underline{k}, m, \underline{n})(\Phi(\underline{k}, \underline{m}, n) \wedge F(k, \underline{m}, n) \wedge \Psi(k, \underline{m}, \underline{n}))$.

The intuitive sense of the formula $(\Phi F \Psi)$ can be explained in the following way:

For every $\mathfrak{A} \in K_T$ $\ulcorner(\Phi F \Psi)\urcorner = \{(k, n) \mid$ there exists $m \in \ulcorner F \urcorner$, such that $(k, m) \in \ulcorner \Phi \urcorner$ and $(m, n) \in \ulcorner \Psi \urcorner\}$, so

$\ulcorner E(\Phi F \Psi)G \urcorner = \{(k, n) \mid$ there exists $m \in \ulcorner F \urcorner$, such that $(k, m) \in \ulcorner \Phi \urcorner$, $(m, n) \in \ulcorner \Psi \urcorner$, $k \in \ulcorner E \urcorner$ and $n \in \ulcorner G \urcorner\} = \{(k, n) \mid$ there exists $m \in \ulcorner E \urcorner$ such that $(k, m) \in \ulcorner E \; \Phi \; F \urcorner$ and $(m, n) \in \ulcorner F \; \Psi \; G \urcorner\}$.

Thus $\ulcorner E(\Phi F \Psi)G \urcorner$ is the composition of $\ulcorner E \; \Phi \; F \urcorner$ and $\ulcorner F \; \Psi \; G \urcorner$.

9.4.1 Proposition

If $\overline{F} = \overline{F'}$, then $F \; \Psi_2 \; G \circ E \; \Psi_1 \; F' = E' \; \Phi \; G'$ iff $\ulcorner F \; \Psi_2 \; G \urcorner \circ \ulcorner E \; \Psi_1 \; F' \urcorner = \ulcorner E' \; \Phi \; G' \urcorner$ for all $\mathfrak{A} \in K_T$.

Proof

Let $\overline{F} = \overline{F'}$, then

$F \; \Psi_2 \; G \circ E \; \Psi_1 \; F' = F \; \Psi_2 \; G \circ E \; \Psi_1 \; F = E(\Psi_1 F \Psi_2)G$ and

$\ulcorner F \; \Psi_2 \; G \urcorner \circ \ulcorner E \; \Psi_1 \; F' \urcorner = \ulcorner F \; \Psi_2 \; G \urcorner \circ \ulcorner E \; \Psi_1 \; F \urcorner = \ulcorner E(\Psi_1 F \Psi_2)G \urcorner$

By Proposition 9.2.1

$E(\Psi_1 F \Psi_2)G = E' \; \Phi \; G'$ iff $\ulcorner E(\Psi_1 F \Psi_2)G \urcorner \doteq \ulcorner E' \; \Phi \; G' \urcorner$ for all $\mathfrak{A} \in K_T$ $\square$.

9.5 $[\overline{F}, \overline{G}]_H$

Let H be a set of formulas as in 8. Define

$[\overline{F}, \overline{G}]_H = \{F \; \Phi \; G \in [\overline{F}, \overline{G}] \mid \Phi \in H\}$.

9.6 Proposition

Let $Th(T, H) = \{Obj(Th(T, H)), Mor(Th(T, H)), o\}$

where

$Obj(Th(T, H)) = \{\overline{F} \mid F \in H\}$

$Mor(Th(T, H)) = \bigcup_{F, G \in H} [\overline{F}, \overline{G}]_H$, then

$Th(T, H)$ is a category.

Proof

I. Let $E \; \Phi \; F \in [\overline{E}, \overline{F}]_H$ and

$F' \; \Psi \; G \in [\overline{F}, \overline{G}]_H$, then

$F' \; \Psi \; G \; o \; E \; \Phi \; F = F \; \Psi \; G \; o \; E \; \Phi \; F = E(\Phi F \Psi)G$.

$\ulcorner E(\Phi F \Psi)G \urcorner = \ulcorner F \; \Psi \; G \urcorner \; o \; \ulcorner E \; \Phi \; F \urcorner$ for all $\mathfrak{A} \in K_T$.

For all $\mathfrak{A} \in K_T$:

$\ulcorner F \; \Psi \; G \urcorner$ is a function from $\ulcorner F \urcorner$ to $\ulcorner G \urcorner$

$\ulcorner E \; \Phi \; F \urcorner$ is a function from $\ulcorner E \urcorner$ to $\ulcorner F \urcorner$, so

$\ulcorner F \; \Psi \; G \urcorner \; o \; \ulcorner E \; \Phi \; F \urcorner$ is a function from $\ulcorner E \urcorner$ to $\ulcorner G \urcorner$, so

$\ulcorner E(\Phi F \Psi)G \urcorner$ is a function from $\ulcorner E \urcorner$ to $\ulcorner G \urcorner$, so

$E(\Phi F \Psi)G \in [\overline{E}, \overline{G}]$.

Now, there exist $E' \in \overline{E}$, $F'' \in \overline{F}$, $G' \in \overline{G}$, $\Phi' \in [\Phi]_{\underset{E,F}{\tilde{T}}}$ and

$\Psi' \in [\Psi]_{\underset{F,G}{\tilde{T}}}$ such that $E', F'', G', \Phi', \Psi' \in H$, so

$(\Phi' F'' \Psi') \in H$ and $(\Phi' F'' \Psi') \in [\Phi F \Psi]_{\underset{E,G}{\tilde{T}}}$, so

$E'(\Phi'F''\Psi')G' = E(\Phi F \Psi)G \in [\overline{E}, \overline{G}]_H$.

II For every $F \in H(m)$, $Fi(m)F$ is an identity morphism:

(1) $i(m) = (\underline{m}, m) \equiv (m, \underline{m}) = (p_1^{2m}, \ldots, p_m^{2m}) \equiv$

$(p_{m+1}^{2m}, \ldots, p_{2m}^{2m}) = \bigcap_{i=1}^{m} p_i^{2m} \equiv p_{m+i}^{2m} = \bigcap_{i=1}^{m} \exists_{p_1^1 p_1^1}(1(p_0^1))$

$(p_i^{2m}, p_{m+i}^{2m}) \in H$

(2) $\ulcorner Fi(m)F \urcorner = \{(m, m) \mid m \in \ulcorner F \urcorner\}$ is a function for all $\mathfrak{A} \in K_T$.

Thus from (1) and (2) it follows that $Fi(m)F \in [\overline{F}, \overline{F}]_H$

Now, let $E\ \Phi\ F \in [\overline{E}, \overline{F}]_H$ and $F\ \Psi\ G \in [\overline{F}, \overline{G}]_H$,

then $\ulcorner Fi(m)F \urcorner \circ \ulcorner E\ \Phi\ F \urcorner = \ulcorner E\ \Phi\ F \urcorner$

and

$$\ulcorner F\ \Psi\ G \urcorner \circ \ulcorner Fi(m)F \urcorner = \ulcorner F\ \Psi\ G \urcorner \text{ for all } \mathfrak{A} \in K_T$$

thus by Proposition 9.4.1

$Fi(m)F \circ E\ \Phi\ F = E\ \Phi\ F$ and

$F\ \Psi\ G \circ Fi(m)F = F\ \Psi\ G$

III The operation $\circ$ is associative:

Let $E\ \Phi_1\ F \in [\overline{E}, \overline{F}]_H$, $F\ \Phi_2\ G \in [\overline{F}, \overline{G}]_H$ and

$G\ \Phi_3\ H \in [\overline{G}, \overline{H}]_H$, then

$(G\ \Phi_3\ H \circ F\ \Phi_2\ G) \circ E\ \Phi_1\ F = F(\Phi_2 G \Phi_3)H \circ E\ \Phi_1\ F =$

$E(\Phi_1 F(\Phi_2 G \Phi_3))H$; and

$G\ \Phi_3\ H \circ (F\Phi_2 G \circ E\Phi_1 F) = G\ \Phi_3\ H \circ E(\Phi_1 F \Phi_2)G =$

$E((\Phi_1 F \Phi_2) G \Phi_3)H$

For any structure $\mathfrak{A} \in K_T$.

$\ulcorner E(\Phi_1 F(\Phi_2 G \Phi_3))H \urcorner = (\ulcorner G\Phi_3 H \urcorner \circ \ulcorner F\Phi_2 G \urcorner) \circ \ulcorner E\ \Phi_1\ F \urcorner =$

$\ulcorner G\ \Phi_3\ H \urcorner = (\ulcorner F\Phi_2 G \urcorner \circ \ulcorner E\Phi_1 F \urcorner) = \ulcorner E((\Phi_1 F \Phi_2) G \Phi_3)H \urcorner$,

so by Proposition 9.2.1

$E(\Phi_1 F(\Phi_2 G\Phi_3))H = E((\Phi_1 F\Phi_2)G\Phi_3)H$ □.

An immediate consequence of the previous definitions is the following:

9.7 Proposition

For any $\mathfrak{A} = (|\mathfrak{A}|, \ulcorner\urcorner) \in K_T$ the function $\ulcorner\urcorner$, defines a functor $\tilde{\mathfrak{A}}: Th(T, H) \to Sets$, such that $\tilde{\mathfrak{A}}(\bar{F}) = \ulcorner F\urcorner$ for $\bar{F} \in Obj(Th(T, H))$, and $\tilde{\mathfrak{A}}(F \Phi G) = \ulcorner F \Phi G\urcorner$ for $F \Phi G \in Mor(Th(T, H))$.

Proof

For any $F' \Phi G' \in [\bar{F}, \bar{G}]_H$, $\tilde{\mathfrak{A}}(F' \Phi G')$ is a function from $\tilde{\mathfrak{A}}(\bar{F}) = \ulcorner F\urcorner$ to $\tilde{\mathfrak{A}}(\bar{G}) = \ulcorner G\urcorner$. Moreover by Proposition 9.4.1 $\tilde{\mathfrak{A}}$ preserves the composition of morphisms and from the proof of Proposition 9.6 II it follows that $\tilde{\mathfrak{A}}$ preserves identity morphisms □.

Remark: The functor $\tilde{\mathfrak{A}}$ sometimes will be denoted $\tilde{\mathfrak{A}}^{(T, H)}$ to indicate the domain category. Functors of the form $\tilde{\mathfrak{A}}$ will be called model functors.

In the sequel we will use the following abbreviations:

1. For $t \in T(m)$

   $\underline{t}$ will denote the morphism $1_m\ t(\underline{m}, 1) \equiv p^{m+1}_{m+1}\ 1_1$,

2. For $t_1, \ldots, t_n \in T(m)$

   $(\underline{t}_1, \ldots, \underline{t}_n)$ will denote the morphism

   $1_m \bigcap_{i=1}^{n} t_i(\underline{m}, n) \equiv p^{m+n}_{m+i}\ 1_n$

Note, that since for any structure $\mathfrak{A} \in K_T$

1. $\ulcorner\underline{t}\urcorner = \ulcorner t\urcorner$ and
2. $\ulcorner(\underline{t}_1, \ldots, \underline{t}_n)\urcorner = (\ulcorner t_1\urcorner, \ldots, \ulcorner t_n\urcorner)$, hence

$\ulcorner\underline{t}\urcorner \circ (\underline{t}_1, \ldots, \underline{t}_n)^{\urcorner} = \ulcorner\underline{t}\urcorner \circ \ulcorner(\underline{t}_1, \ldots, \underline{t}_n)\urcorner =$
$\ulcorner t\urcorner \circ (\ulcorner t_1\urcorner, \ldots, \ulcorner t_n\urcorner) = \ulcorner t(t_1 \ldots\ldots t_n)\urcorner =$
$\underline{\ulcorner t(t_1 \ldots\ldots t_n)\urcorner}$, and
by Proposition 9.2.1
$\underline{t} \circ (\underline{t}_1, \ldots, \underline{t}_n) = \underline{t(t_1 \ldots\ldots t_n)}$

Let Mod(T, H) denote the full subcategory of $Sets^{Th(T,H)}$ consisting of all $\widetilde{\mathfrak{A}}$ where $\mathfrak{A} \in K_T$. Let $K_T(H)$ denote the category of models of T and H-homomorphisms, then

9.8 Theorem

The categories Mod(T, H) and $K_T(H)$ are isomorphic.

Proof

I If $\mathfrak{A}, \mathfrak{B} \in K_T$ and $\mathfrak{A} \neq \mathfrak{B}$ then $\mathfrak{A}^{\ulcorner P\urcorner} \neq \mathfrak{B}^{\ulcorner P\urcorner}$ for some $P \in \Sigma_1(n)$ or $\mathfrak{A}^{\ulcorner t\urcorner} \neq \mathfrak{B}^{\ulcorner t\urcorner}$ for some $t \in \Sigma_0(n)$, so since $P, t(\underline{n}, 1) \equiv (n, \underline{1}) \in H$, then $\widetilde{\mathfrak{A}}(\overline{P}) \neq \widetilde{\mathfrak{B}}(\overline{P})$ or $\widetilde{\mathfrak{A}}(t(\underline{n}, 1) \equiv (n, \underline{1})) \neq \widetilde{\mathfrak{B}}(t(\underline{n}, 1) \equiv (n, \underline{1}))$, thus $\widetilde{\mathfrak{A}} \neq \widetilde{\mathfrak{B}}$.

It follows from the above and the definition of Mod(T, H) that the map $(\widetilde{\ }): K_T \to$ Obj(Mod(T, H) is bijective.

II Now, let $\mathfrak{A}, \mathfrak{B} \in K_T$ and $h: |\mathfrak{A}| \to |\mathfrak{B}|$ be an H-homomorphism. For $F \in H(n)$ define

$\tilde{h}_{\overline{F}} = (h^n)_{\mathfrak{A}^{\ulcorner F\urcorner}}$ where $(h^n)_{\mathfrak{A}^{\ulcorner F\urcorner}}$ denotes the function h^n restricted to the set $\mathfrak{A}^{\ulcorner F\urcorner}$. We will show that $\tilde{h}$ is a natural transformation $\widetilde{\mathfrak{A}} \to \widetilde{\mathfrak{B}}$. Let $F \in H(n)$, $G \in H(m)$, $\Phi \in H(n+m)$ and $F \,\Phi\, G$ be a morphism in Th(T, H). Let $a \in \mathfrak{A}^{\ulcorner F\urcorner}$ and $a' = \mathfrak{A}^{\ulcorner F \Phi G\urcorner}(a)$, then $(a, a') \in \mathfrak{A}^{\ulcorner \Phi\urcorner}$ and $a' \in \mathfrak{A}^{\ulcorner G\urcorner}$, so $h^n(a) \in \mathfrak{B}^{\ulcorner F\urcorner}$,

$(h^n(a), h^m(a')) = h^{n+m}(a, a') \in \mathfrak{B}^{\ulcorner\Phi\urcorner}$ and

$h^m(a') \in \mathfrak{B}\ulcorner G\urcorner$, so $h^m(a') = \mathfrak{B}\ulcorner F \Phi G\urcorner(h^n(a))$, so $h^m(\mathfrak{A}\ulcorner F \Phi G\urcorner(a)) = \mathfrak{B}\ulcorner F \Phi G\urcorner(h^n(a))$, thus the following diagram commutes:

$$\begin{tikzcd}[column sep=huge, row sep=large]
\mathfrak{A}\ulcorner F\urcorner \arrow[r, "{\tilde{\mathfrak{A}}(F\,\Phi\,G) \,=\, \mathfrak{A}\ulcorner F\,\Phi\,G\urcorner}"] \arrow[d, "{\tilde{h}_{\overline{F}} \,=\, (h^n)_{\mathfrak{A}\ulcorner F\urcorner}}"'] & \mathfrak{A}\ulcorner G\urcorner \arrow[d, "{\tilde{h}_{\overline{G}} \,=\, (h^m)_{\mathfrak{A}\ulcorner G\urcorner}}"] \\
\mathfrak{B}\ulcorner F\urcorner \arrow[r, "{\mathfrak{B}\ulcorner F\,\Phi\,G\urcorner \,=\, \tilde{\mathfrak{B}}(F\,\Phi\,G)}"'] & \mathfrak{B}\ulcorner G\urcorner
\end{tikzcd}$$

Thus the map $(\tilde{\ })$ is a functor. It remains to show that $(\tilde{\ })$ is full and faithful.

III Let $h, h' : |\mathfrak{A}| \to |\mathfrak{B}|$ be two H-homomorphisms. If $h \neq h'$ then $\tilde{h}_{\overline{I}_1} \neq \tilde{h}'_{\overline{I}_1}$, so

$$\tilde{h} \neq \tilde{h}'.$$

Thus $(\tilde{\ })$ is faithful.

IV Let $\mathfrak{A}, \mathfrak{B} \in K_T$ and $\Psi : \tilde{\mathfrak{A}} \to \tilde{\mathfrak{B}}$ be a natural transformation. Define a function $\underset{\sim}{\Psi} : |\mathfrak{A}| \to |\mathfrak{B}|$ as $\underset{\sim}{\Psi} = \Psi_{\overline{I}_1}$, then $\underset{\sim}{\Psi}^n = \Psi_{\overline{I}_n}$, because the following diagram commutes for $i = 1 \ldots n$:

$$\begin{tikzcd}[column sep=huge, row sep=large]
|\mathfrak{A}|^n = \tilde{\mathfrak{A}}(\overline{I}_n) \arrow[r, "\Psi_{\overline{I}_n}"] \arrow[d, "{\mathfrak{A}\ulcorner p_i^n\urcorner \,=\, \tilde{\mathfrak{A}}(\underline{p_i^n})}"'] & \tilde{\mathfrak{B}}(\overline{I}_n) = |\mathfrak{B}|^n \arrow[d, "{\tilde{\mathfrak{B}}(\underline{p_i^n}) \,=\, \mathfrak{B}\ulcorner p_i^n\urcorner}"] \\
|\mathfrak{A}| = \tilde{\mathfrak{A}}(\overline{I}_1) \arrow[r, "\Psi_{\overline{I}_1}"'] & \tilde{\mathfrak{B}}(\overline{I}_1) = |\mathfrak{B}|
\end{tikzcd}$$

Let $F \in H(n)$, then the following diagram commutes:

$$\begin{tikzcd} \mathcal{O}\!\mathcal{L}^{\ulcorner F \urcorner} = \widetilde{\mathcal{O}\!\mathcal{L}}(\overline{F}) \arrow[r, "\Psi_{\overline{F}}"] \arrow[d, "\widetilde{\mathcal{O}\!\mathcal{L}}(Fi(n)1_n)"'] & \widetilde{\mathcal{L}}(\overline{F}) = \mathcal{L}^{\ulcorner F \urcorner} \arrow[d, "\widetilde{\mathcal{L}}(Fi(n)1_n)"] \\ |\mathcal{O}\!\mathcal{L}|^n = \widetilde{\mathcal{O}\!\mathcal{L}}(\overline{I}_n) \arrow[r, "\Psi_{I_n}"'] & \widetilde{\mathcal{L}}(\overline{I}_n) = |\mathcal{L}|^n \end{tikzcd}$$

If $a \in \widetilde{\mathcal{O}\!\mathcal{L}}(\overline{F})$, then $\underset{\sim}{\Psi}^n(a) = \Psi_{\overline{I}_n}(a) = \Psi_{\overline{I}_n}(\widetilde{\mathcal{O}\!\mathcal{L}}(Fi(n)1n)(a)) = (\widetilde{\mathcal{L}}(Fi(n)1_n) \circ \Psi_{\overline{F}})(a) = \Psi_{\overline{F}}(a) \in \widetilde{\mathcal{L}}(\overline{F}) = \mathcal{L}^{\ulcorner F \urcorner}$, so $\underset{\sim}{\Psi}^n(\mathcal{O}\!\mathcal{L}^{\ulcorner F \urcorner}) \subset \mathcal{L}^{\ulcorner F \urcorner}$, thus $\underset{\sim}{\Psi}$ is an H-homomorphism. Moreover $(\tilde{\underset{\sim}{\Psi}})_{\overline{F}} = (\underset{\sim}{\Psi})^n_{\mathcal{O}\!\mathcal{L}^{\ulcorner F \urcorner}} = (\Psi_{\overline{I}_n})_{\mathcal{O}\!\mathcal{L}^{\ulcorner F \urcorner}} = \Psi_{\overline{F}}$. Thus $(\tilde{\ })$ is full $\square$.

It remains to find an independent characterization of those functors $\mathfrak{M}: \mathrm{Th}(T, H) \to$ Sets which are isomorphic to functors $\widetilde{\mathcal{O}\!\mathcal{L}}$, $\mathcal{O}\!\mathcal{L} \in K_T$. An answer will be given in Chapter III with the aid of the following results.

Let $B(H)$ denote the set of all formulas $F \in F$ such that there are $n \in N$, $F_i, G_i \in H$ $i = 1, \ldots n$ and

$$\models_{\emptyset} F \leftrightarrow \bigcap_{i=1}^{n} (F_i \to G_i)$$

<u>Remark</u>: If $0 \in H$ then $B(H)$ consists of all formulas which are equivalent to formulas from the boolean closure of H (closure under $\vee$, $\wedge$, and $\neg$), because if a formula $F \in F(n)$ is in the boolean closure of H then there are formulas $F_{i,j} \in H(n)$, $i = 1 \ldots m$, $j = 1 \ldots k$ such that F is equivalent to

$$\bigcap_{j=1}^{k} \bigcup_{i=1}^{m} F^{*}_{i,j} \qquad \text{where } F^{*}_{i,j} \text{ is } F_{i,j} \text{ or } \neg F_{i,j}$$

(eg. conjunctive normal form).

For given j, $1 \leq j \leq k$ let

$$F_j = \bigcap_* F_{i,j} \quad \text{and}$$
$$F^*_{i,j} = \neg F_{i,j}$$
$$G_j = \bigcup_* F_{i,j} \quad \text{where}$$
$$F^*_{i,j} = F_{i,j}$$
$$\bigcap_{\emptyset} F^*_{i,j} = 1_n \text{ and } \bigcup_{\emptyset} F^*_{i,j} = 0_n, \text{ then}$$

F is equivalent to

$$\bigcap_{j=1}^{k} (\neg F_j \vee G_j) \quad \text{and} \quad F_j, G_j \in H(n)$$

Thus any formula equivalent to F is in $B(H)$ □.

Let u_T: $Th(\emptyset, H) \to Th(T, H)$, be the functor such that:
$u_T(\overline{F}_{\emptyset}) = \overline{F}_T$ for $\overline{F}_{\emptyset} \in Obj(Th(\emptyset, H))$, and
$u_T(F \Phi G_{\emptyset}) = F \widehat{\Phi} G_T$ for $F \Phi G_{\emptyset} \in Mor(Th(\emptyset, H))$.
Since $\emptyset \subset \underset{\sim}{T}$ and for any F, $G \in F$ $\;\overset{\emptyset}{F,G} \subset \overset{T}{\tilde{F,G}}$, hence it is easy to see that u_T is a well defined functor.

9.9 Proposition

For any structure $\mathfrak{A} \in K_{\emptyset}$, $\mathfrak{A}$ is a model of $Cn(T) \cap B(H)$ iff there is a factorization

$$\begin{tikzcd}
Th(\emptyset, H) \arrow[r, "\tilde{\mathfrak{A}}"] \arrow[d, "u_T"'] & Sets \\
Th(T, H) \arrow[ur, dashed] &
\end{tikzcd}$$

Proof

I Let T' denote the set $Cn(T) \cap B(H)$.

Let $F, G \in \mathcal{H}$ and $F \overset{T}{\sim} G$, then

$F \leftrightarrow G \in \mathcal{B}(\mathcal{H})$ and $F \leftrightarrow G \in Cn(T)$

Thus $F \leftrightarrow G \in T'$, so

$F \overset{T'}{\sim} G.$

It means that for formulas from $\mathcal{H}$ relations $\overset{T}{\sim}$ and $\overset{T'}{\sim}$ are equal and also relations $F \overset{T}{\sim} G$ and $F \overset{T'}{\sim} G$ are equal therefore

$$Th(T, \mathcal{H}) = Th(T', \mathcal{H})$$

and the following diagram commutes

$$\begin{tikzcd}
Th(\emptyset, \mathcal{H}) \arrow[r, "\tilde{\mathfrak{A}}^{(\emptyset, \mathcal{H})}"] \arrow[d, "u_T"'] & Sets \\
Th(T, \mathcal{H}) = Th(T', \mathcal{H}) \arrow[ur, "\tilde{\mathfrak{A}}^{(T', \mathcal{H})}"'] &
\end{tikzcd}$$

II Let $\mathfrak{A} \in K_{\emptyset}$ and let there be a functor $\mathfrak{M}$ such that the following diagram commutes:

$$\begin{tikzcd}
Th(\emptyset, \mathcal{H}) \arrow[r, "\tilde{\mathfrak{A}}"] \arrow[d, "u_T"'] & Sets \\
Th(T, \mathcal{H}) \arrow[ur, "\mathfrak{M}"'] &
\end{tikzcd}$$

We will show that for any $F \in T'(n)$

$$\mathfrak{A}^{\ulcorner F \urcorner} = |\mathfrak{A}|^n.$$

Let $F \in T'(n)$, then there are formulas $F_j, G_j \in \mathcal{H}$ $j = 1, \ldots, n$, such that

$$\models_{\emptyset} F \leftrightarrow \bigcap_{i=1}^{m} F_j \to G_j, \text{ so for } j = 1, \dots n$$

we have

$$\models_T F_j \to G_j, \text{ so}$$

$F_j\ i(n)\ G_{j_T} \in Mor(Th(T, H))$ and

$G_j\ i(n)\ 1_{n_T} \circ F_j\ i(n)\ G_{j_T} = F_j\ i(n)\ 1_{n_T}$ so

$\mathfrak{M}(G_j\ i(n) 1_{n_T}) \circ \mathfrak{M}(F_j\ i(n)\ G_{j_T}) = \mathfrak{M}(F_j\ i(n)\ 1_{n_T})$,

but $\mathfrak{M}(G_j\ i(n)\ 1_{n_T}) = \tilde{\mathfrak{A}}(G_j\ i(n)\ 1_{n_\emptyset})$ and

$\mathfrak{M}(F_j\ i(n)\ 1_{n_T}) = \tilde{\mathfrak{A}}(F_j\ i(n)\ 1_{n_\emptyset})$.

Now, let $a \in \tilde{\mathfrak{A}}(\bar{F}_{j_\emptyset}) = \mathfrak{M}(\bar{F}_{j_T})$, then

$a \in \tilde{\mathfrak{A}}(F_j\ i(n)\ 1_{n_\emptyset})(a) = \tilde{\mathfrak{A}}(G_j\ i(n)\ 1_{n_\emptyset}) \circ \mathfrak{M}(F_j\ i(n)\ G_{j_T})(a)$

$= \mathfrak{M}(F_j\ i(n)\ G_{j_T})(a) \in \mathfrak{M}(\bar{G}_{j_T}) = \tilde{\mathfrak{A}}(\bar{G}_{j_\emptyset})$

Thus $\tilde{\mathfrak{A}}(\bar{F}_{j_\emptyset}) \subset \tilde{\mathfrak{A}}(\bar{G}_{j_\emptyset})$, so

$$\mathfrak{A}\ulcorner F_j \urcorner \subset \mathfrak{A}\ulcorner G_j \urcorner, \text{ so}$$

$$\mathfrak{A}\ulcorner F_j \to G_j \urcorner = |\mathfrak{A}|^n \text{ and therefore}$$

$$\mathfrak{A}\ulcorner \bigcap_{j=1}^{m} F_j \to G_j \urcorner = |\mathfrak{A}|^n.$$

Since $\mathfrak{B}\ulcorner \bigcap_{j=1}^{m} F_j \to G_j \urcorner = \mathfrak{B}\ulcorner F \urcorner$ for all $\mathfrak{B} \in K_\emptyset$

therefore $\mathfrak{A}\ulcorner F \urcorner = |\mathfrak{A}|^n$. □.

Thus for a given T, if H is big enough, any functor $\mathfrak{M}$: Th(T, H) → Sets, such that $\mathfrak{M} \circ u_T$ corresponds to a structure will correspond to a model. On the other hand, for given T and $\mathcal{H}$ if T is not axiomatizable in $\mathcal{B}(\mathcal{H})$, it certainly is included in $\mathcal{F}$.

Let $\mathcal{V}_{\mathcal{H}}$ denote the inclusion functor from $\mathrm{Th}(T, \mathcal{H})$ to $\mathrm{Th}(T, \mathcal{F})$ (i.e. $\mathcal{V}_{\mathcal{H}}(\overline{F}) = \overline{F}$ for $F \in \mathcal{H}$ and $\mathcal{V}_{\mathcal{H}}(F \Phi G) = F \Phi G$ for $F \Phi G \in \mathrm{Mor}(\mathrm{Th}(T, \mathcal{H}))$.)

9.10 Proposition

For any functor $\mathfrak{M} : \mathrm{Th}(T, \mathcal{H}) \to \mathrm{Sets}$ $\mathfrak{M} \in \mathrm{Mod}(T, \mathcal{H})$ iff there is a factorization:

$$\begin{tikzcd}
\mathrm{Th}(T, \mathcal{H}) \arrow[r, "\mathfrak{M}"] \arrow[d, "\mathcal{V}_{\mathcal{H}}"'] & \mathrm{Sets} \\
\mathrm{Th}(T, \mathcal{F}) \arrow[ur, "\mathfrak{N}"'] &
\end{tikzcd}$$

for some $\mathfrak{N} \in \mathrm{Mod}(T, \mathcal{F})$.

Proof

It is an immediate consequence of the equality

$$\tilde{\mathfrak{A}}^{(T,\mathcal{H})} = \tilde{\mathfrak{A}}^{(T,\mathcal{F})} \circ \mathcal{V}_{\mathcal{H}} \quad \text{for all } \mathfrak{A} \in K_T \quad \square.$$

In the next chapter there will be given conditions under which any functor $\mathfrak{M} : \mathrm{Th}(T, \mathcal{H}_i) \to \mathrm{Sets}$ $i = 0, 1, 2$ is isomorphic to a functor $\tilde{\mathfrak{A}}$ for some $\mathfrak{A} \in K_{Cn(T)} \cap \mathcal{B}(\mathcal{H})$.

Finally we should remark that the semantically defined relations $\overset{T}{\sim}$ and $F \overset{T}{\sim} G$ have syntactical definitions; this follows from the Gödel completeness theorem.

Chapter III

Characterization of Model Functors

We shall begin this chapter with definitions of some categorical notions that will be used in the following.

Let I be a small category and A be a category, then a functor $D: I \to A$ is called a diagram. Usually we shall not distinguish between the functor D and its image, thus $Obj(D)$ will mean $\{D_i \mid i \in Obj(I)\}$ and $Mor(D)$ will mean $\{D_\alpha \mid \alpha \in Mor(I)\}$. A source (resp. a sink) in the category A is a diagram such that all its morphisms have common domain (resp. codomain). $(F, f_i, G_i)_{i \in J}$ will denote a source such that $f_i: F \to G_i$ for $i \in J$, and $(F_i, g_i, G)_{i \in J}$ will denote a sink such that $g_i: F_i \to G$ for $i \in J$.

A source $(F, f_i, D_i)_{i \in Obj(I)}$ will be called a source to the diagram D. A sink $(D_i, g_i, G)_{i \in Obj(I)}$ will be called a sink from the diagram D.

A source $(F, f_i, D_i)_{i \in Obj(I)}$ to a diagram D is a cone to the diagram D, if for any $\alpha: i \to j \in Mor(I)$, $D_\alpha \circ f_i = f_j$. A sink $(D_i, g_i, G)_{i \in Obj(I)}$ from a diagram D is a cocone from the diagram D, if for any $\alpha: i \to j \in Mor(I)$ $g_j \circ D_\alpha = g_i$. A cone $(F, f_i, D_i)_{i \in Obj(I)}$ to a diagram D is a limit of the diagram D if for any other cone $(F', f_i', D_i)_{i \in Obj(I)}$ to the diagram D there exists a unique morphism $h: F' \to F$, such that $f_i \circ h = f_i'$ for all $i \in Obj(I)$. A cocone $(D_i, g_i, G)_{i \in Obj(I)}$ from a diagram D is a colimit of the diagram D if for any other cocone

$(D_i, g_i', G')_{i \in Obj(I)}$ from the diagram D there exists a unique morphism $h: G \to G'$, such that $h \circ g_i = g_i'$ for all $i \in Obj(I)$. Let $\mathcal{O}_j$ be a functor from a category A to a category B_j for all $j \in J$. A family of functors $\{\mathcal{O}_j\}_{j \in J}$ is collectively faithful if for any F, G $\in$ Obj(A) and any f, g $\in$ [F, G], if $\mathcal{O}_j(f) = \mathcal{O}_j(g)$ for all $j \in J$, then f = g. A family of functors $\{\mathcal{O}_j\}_{j \in J}$ collectively reflects limits if for any diagram D in A and any cone C to D, if $\mathcal{O}_j(C)$ is a limit of $\mathcal{O}_j(D)$ for all $j \in J$ then C is a limit of D. A family $\{\mathcal{O}_j\}_{j \in J}$ collectively reflects colimits if for any diagram D in A and any cocone C from D, if $\mathcal{O}_j(C)$ is a colimit of $\mathcal{O}_j(D)$, for all $j \in J$, then C is a colimit of D. An immediate consequence of the above definitions is

10.1 Proposition

Let the family of functors $\{\mathcal{O}_j\}_{j \in J}$ be collectively faithful and collectively reflect limits (resp. colimits).

Let S be a source to (resp. sink from) a diagram $D: I \to A$. If $\mathcal{O}_j(S)$ is a limit (resp. colimit) of the diagram $\mathcal{O}_j(D)$ for all $j \in J$, then S is a limit (resp. colimit) of D.

Proof

Let $S = (F, f_i, D_i)_{i \in Obj(I)}$.

For all $\alpha: i \to k \in Mor(I)$

$$\mathcal{O}_j(D_\alpha) \circ \mathcal{O}_j(f_i) = \mathcal{O}_j(f_k) \quad \text{all } j \in J, \text{ so}$$

$$\mathcal{O}_j(D_\alpha \circ f_i) = \mathcal{O}_j(f_k), \text{ so}$$

$$D_\alpha \circ f_i = f_k, \text{ so}$$

S is a cone to the diagram D, so if $\mathfrak{A}_j(S)$ is a limit of the diagram $\mathfrak{A}_j(D)$ for all $j \in J$ then S is a limit of D. (For S a sink from D the proof is similar) □.

Let I and I^c be small categories such that

$$\begin{tikzcd} 1 \arrow[dr] & \\ & 3 \\ 2 \arrow[ur] & \end{tikzcd}$$ represents I, and

$$\begin{tikzcd} & 1 \\ 3 \arrow[ur] \arrow[dr] & \\ & 2 \end{tikzcd}$$ represents I^c.

Let $P, D : I \to A$ be diagrams in a category A such that $P(i) = D(i)$ $i = 1, 2$. The sink P is a <u>pushout of a pullback of the diagram</u> D if there exists a diagram $F: I^c \to A$, such that F is a pullback of D and P is a pushout of F.

<u>10.2 Example</u>

Let $f_i: A_i \to B$ $i = 1, 2$ be functions in Sets. Define functions $g_i: A_i \to C = f_1(A_1) \cup f_2(A_2)$ $i = 1, 2$, such that $g_i(x_i) = f_i(x_i)$, $x_i \in A_i$ $i = 1, 2$, then (g_1, g_2) is a pushout of a pullback of (f_1, f_2).

<u>Proof</u>

Let $F = \{(x_1, x_2) \in A_1 \times A_2 \mid f_1(x_1) = f_2(x_2)\}$ and $h_i: F \to A_i$ $i = 1, 2$ be functions such that $h_i(x_1, x_2) = x_i$ $i = 1, 2$ for $(x_1, x_2) \in F$, then (h_1, h_2) is a pullback of (f_1, f_2).

Now, we will show that (g_1, g_2) is a pushout of (h_1, h_2):

(1) $(g_1 \circ h_1)(x_1, x_2) = g_1(x_1) = f_1(x_1) = f_2(x_2) = g_2(x_2) = (g_2 \circ h_2)(x_1, x_2)$, for all $(x_1, x_2) \in F$, thus $g_1 \circ h_1 = g_2 \circ h_2$.

(2) Suppose that there are $g_i': A_i \to C'$ $i = 1, 2$, such that $g_1' \circ h_1 = g_2' \circ h_2$

$$\begin{tikzcd}
 & A_1 \arrow[dr, "g_1"] \arrow[drr, "g_1'"] \arrow[drrr, bend left, "f_1"] & & & \\
F \arrow[ur, "h_1"] \arrow[dr, "h_2"'] & & C = f_1(A_1) \cup f_2(A_2) \arrow[r, "u"] & C' & B \\
 & A_2 \arrow[ur, "g_2"] \arrow[urr, "g_2'"'] \arrow[urrr, bend right, "f_2"'] & & &
\end{tikzcd}$$

Define a function $u: C \to C'$ as follows

$$u(y) = \begin{cases} g_1'(x_1) & \text{if } y = f_1(x_1), \text{ for some } x_1 \in A_1 \\ g_2'(x_2) & \text{if } y = f_2(x_2), \text{ for some } x_2 \in A_2 \end{cases}$$

u is well defined, because if $y = f_1(x_1) = f_2(x_2)$, then $(x_1, x_2) \in F$ and by assumption

$$g_1'(x_1) = g_1' \circ h_1(x_1, x_2) = g_2' \circ h_2(x_1, x_2) = g_2'(x_2).$$

Now, for $x_i \in A_i$ $i = 1, 2$

$$u \circ g_i(x_i) = u \circ f_i(x_i) = g_i'(x_i) \quad i = 1, 2, \text{ thus}$$

$$u \circ g_i = g_i', \; i = 1, 2.$$

It remains to show that the function u is unique; suppose that there is $v: C \to C'$, such that $v \circ g_i = g_i'$ $i = 1, 2$. Let $y \in C$, then there is $x \in A_i$ such that $f_i(x) = y$, so $v(y) = v(f_i(x)) = v(g_i(x)) = g_i'(x) = u(y)$ □.

A functor $\mathcal{O}\!\mathcal{L}$ from the category A to a category B preserves pushouts of pullbacks, if for any diagrams P, D such that P is a pushout of a pullback of D, $\mathcal{A}(P)$ is a pushout of a pullback of $\mathcal{A}(D)$. A family of functors $\mathcal{A}_j: A \to B_j$ $j \in J$ <u>collectively reflects pushouts of pullbacks</u> if for any diagrams P, D: I $\to$ A, such that P(i) = D(i) i = 1, 2 (where I is the above category) if $\mathcal{A}_j(P)$ is a pushout of a pullback of $\mathcal{A}_j(D)$ for all $j \in J$ then P is a pushout of a pullback of D.

<u>10.4 Proposition</u>

The family of functors $\{\widetilde{\mathcal{A}}\}_{\mathcal{A} \in K_T}$ is collectively faithful.

<u>Proof</u>

This is a consequence of Proposition 9.2.1 □.

It follows from the above that the family of functors $\{\widetilde{\mathcal{A}}\}_{\mathcal{A} \in K_T}$ collectively reflects monomorphisms and epimorphisms (cf. [3], p. 70).

<u>10.5 Proposition</u>

The family $\{\widetilde{\mathcal{A}}\}_{\mathcal{A} \in K_T}$ collectively reflects finite limits.

<u>Proof</u>

Let $\{\overline{F}, F\Phi_i D_i, \overline{D}_i\}_{i=1}^n$ be a cone to a diagram D in Th(T, H), such that $\{\ulcorner F \urcorner, \ulcorner F\Phi_i D_i \urcorner, \ulcorner D_i \urcorner\}_{i=1}^n$ is a limit of $\ulcorner D \urcorner$ in Sets for any $\mathcal{A} \in K_T$. Let $\{\overline{F}', F'\Phi_i' D_i, \overline{D}_i\}_{i=1}^n$ be any other cone to D. Let m, m', $k_1, \ldots, k_n$ be numbers such that $F \in F(m)$, $F' \in F(m')$ and $D_i \in F(k_i)$ $i = 1, \ldots n$.

Define the formula Ψ to be

$$\exists(\underline{m}', \underline{m}, k_1, \ldots, k_n)\left(\bigcap_{i=1}^{n} (\Phi_i(m', \underline{m}, k_1, \ldots, \underline{k_i}, \ldots, k_n) \wedge\right.$$

$$D_i(m', m, k_1, \ldots, \underline{k_i}, \ldots, k_n) \wedge \Phi_i'(\underline{m}', m, k_1, \ldots, \underline{k_i}, \ldots, k_n)$$

Note that $\Psi \in H$, since we can assume that Φ_i, Φ'_i, D_i $i = 1 \ldots n$ are in H. Let $\mathfrak{A} \in K_T$ we will show that $\ulcorner F' \Psi F \urcorner$ is a function from $\ulcorner F' \urcorner$ to $\ulcorner F \urcorner$.

There exists a unique function $f\colon \ulcorner F' \urcorner \to \ulcorner F \urcorner$ such that $\ulcorner F\Phi_i D_i \urcorner \circ f = \ulcorner F'\Phi'_i D_i \urcorner$ $i = 1, \ldots, n$. Let $a \in \ulcorner F' \urcorner$, then $f(a) \in \ulcorner F \urcorner$ and

$$k_i = \ulcorner F\Phi_i D_i \urcorner (f(a)) = \ulcorner F'\Phi'_i D_i \urcorner (a) \quad i = 1 \ldots n,$$

so $\quad (f(a), k_i) \in \ulcorner \Phi_i \urcorner, \ (a, k_i) \in \ulcorner \Phi'_i \urcorner$ and $k_i \in \ulcorner D_i \urcorner$ $\quad i = 1 \ldots n,$

so $\quad (a, f(a)) \in \ulcorner \Psi \urcorner$, so

$$(a, f(a)) \in \ulcorner F'\Psi F \urcorner.$$

Now, suppose that $(a, b) \in \ulcorner F'\Psi F \urcorner$. Define a function $f'\colon \ulcorner F' \urcorner \to \ulcorner F \urcorner$, such that

$$f'(x) = \begin{cases} f(x) & \text{for } x \neq a \\ b & \text{for } x = a, \text{ then} \end{cases}$$

$$\ulcorner F\Phi_i D_i \urcorner \circ f' = \ulcorner F'\Phi'_i D_i \urcorner \quad i = 1 \ldots n, \text{ so}$$

by uniqueness of f, $f' = f$, so $f(a) = b$. Thus for any $\mathfrak{A} \in K_T$ $\ulcorner F'\Psi F \urcorner$ is a function from $\ulcorner F' \urcorner$ to $\ulcorner F \urcorner$ and it follows that $F'\Psi F$ is a morphism in $Th(T, H)$. Also $\ulcorner F\Phi_i D_i \urcorner \circ \ulcorner F'\Psi F \urcorner = \ulcorner F'\Phi_i D_i \urcorner$ $i = 1 \ldots n$ for all $\mathfrak{A} \in K_T$ so by Proposition 9.4.1, $F\Phi_i D_i \circ F'\Psi F = F'\Phi_i D_i$ $i = 1 \ldots n$.

Now, suppose that there is a morphism $F'\Psi' F\colon \overline{F}' \to \overline{F}$, such that $F\Phi_i D_i \circ F'\Psi' F = F'\Phi'_i D_i$ $i = 1 \ldots n$, then $\ulcorner F\Phi_i D_i \urcorner \circ \ulcorner F'\Psi' F \urcorner = \ulcorner F'\Phi_i D_i \urcorner$ $i = 1 \ldots n$ for all $\mathfrak{A} \in K_T$ but for any $\mathfrak{A} \in K_T$ $\ulcorner F'\Psi F \urcorner$ is unique, so $\ulcorner F'\Psi' F \urcorner = \ulcorner F'\Psi F \urcorner$ for all $\mathfrak{A} \in K_T$ so $F'\Psi' F = F'\Psi F$. Thus $\{\overline{F}, F\Phi_i D_i, \overline{D}_i\}_{i=1}^{n}$ is a limit of the diagram D $\square$.

10.6 Proposition

The family $\{\tilde{\mathcal{O}\!\mathcal{l}}\}_{\mathcal{O}\!\mathcal{l} \in K_T}$ collectively reflects finite colimits if H is closed under $\vee$.

Proof.

Let $\{\bar{D}_i, D_i\Phi_i G, \bar{G}\}_{i=1}^n$ be a cocone from a finite diagram D in Th(T, H), such that $\{\ulcorner D_i \urcorner, \ulcorner D_i\Phi_i G \urcorner, \ulcorner G \urcorner\}_{i=1}^n$ is a colimit of the diagram $\ulcorner D \urcorner$ in Sets for any $\mathcal{O}\!\mathcal{l} \in K_T$. Let $\{\bar{D}_i, D_i\Phi_i' G', \bar{G}'\}_{i=1}^n$ be another cocone from D. Let m, m', $k_1, \ldots, k_n$ be numbers such that $G \in F(m)$, $G' \in F(m')$ and $D_i \in F(k_i)$ $i = 1 \ldots n$. Define the formula Ψ to be

$$\exists(k_1, \ldots, k_n, \underline{m}, \underline{m}')\left(\bigcup_{i=1}^{n}(\Phi_i(k_1, \ldots, \underline{k}_i, \ldots, k_n, \underline{m}, m') \wedge D_i(k_1, \ldots, \underline{k}_i, \ldots, k_n, m, m') \wedge \Phi_i'(k_1, \ldots, \underline{k}_i, \ldots, k_n, m, \underline{m}'))\right)$$

Note that $\Psi \in H$, since we can assume that $\Phi_i, D_i, \Phi_i' \in H$ $i = 1 \ldots n$. Let $\mathcal{O}\!\mathcal{l} \in K_T$, we will show that $\ulcorner G\Psi G' \urcorner$ is a function from $\ulcorner G \urcorner$ to $\ulcorner G' \urcorner$. There exists a unique function $g: \ulcorner G \urcorner \to \ulcorner G' \urcorner$, such that $g \circ \ulcorner D_i\Phi_i G \urcorner = \ulcorner D_i\Phi_i' G' \urcorner$ $i = 1 \ldots n$. Let $a \in \ulcorner G \urcorner$, then exists i $1 \le i \le n$ and $k_i \in \ulcorner D_i \urcorner$ such that $\ulcorner D_i\Phi_i G \urcorner(k_i) = a$ (in Sets colimit sinks are collectively onto) $\ulcorner D_i\Phi_i' G' \urcorner(k_i) = (g \circ \ulcorner D_i\Phi_i G \urcorner)(k_i) = g(a)$, so $(k_i, g(a)) \in \ulcorner \Phi_i' \urcorner$, $(k_i, a) \in \ulcorner \Phi_i \urcorner$ and $k_i \in \ulcorner D_i \urcorner$, so $(a, g(a)) \in \ulcorner \Psi \urcorner$, so $(a, g(a)) \in \ulcorner G\Psi G' \urcorner$.

Now suppose that $(a, b) \in \ulcorner G\Psi G' \urcorner$ for some $b \in \ulcorner G' \urcorner$, then there exists j, $1 \le j \le n$ and $k_j \in \ulcorner D_j \urcorner$ such that $(k_j, a) \in \ulcorner \Phi \urcorner$ and $(k_j, b) \in \ulcorner \Phi' \urcorner$, so $b = \ulcorner D_j\Phi' G' \urcorner(k_j) = g \circ \ulcorner D_j\Phi_j G \urcorner(k_j) = g(a)$. Thus for any $\mathcal{O}\!\mathcal{l} \in K_T$ $\ulcorner G\Psi G' \urcorner$ is a function from $\ulcorner G \urcorner$ to $\ulcorner G' \urcorner$ and it follows that $G\Psi G'$ is a morphism in $\mathrm{Th}(T, H)$. Also for all $\mathfrak{A} \in K_T$ $\ulcorner G\Psi G' \urcorner \circ \ulcorner D_i \Phi_i G \urcorner = \ulcorner D_i \Phi_i' G' \urcorner$ $i = 1 \ldots n$, so by Proposition 9.4.1

$$G\Psi G' \circ D_i \Phi_i G = D_i \Phi_i' G'.$$

Now, suppose that there is a morphism $G\Psi' G' : \overline{G} \to \overline{G}'$, such that $G\Psi' G' \circ D_i \Phi_i G = D_i \Phi_i' G'$ $i = 1 \ldots n$, then $\ulcorner G\Psi' G' \urcorner \circ \ulcorner D_i \Phi_i G \urcorner = \ulcorner D_i \Phi_i' G' \urcorner$ $i = 1 \ldots n$ for all $\mathfrak{A} \in K_T$, but for any $\mathfrak{A} \in K_T$ $\ulcorner G\Psi G' \urcorner$ is unique, so

$$\ulcorner G\Psi' G' \urcorner = \ulcorner G\Psi G' \urcorner \quad \text{for all } \mathfrak{A} \in K_T$$

so $G\Psi' G' = G\Psi G'$.

Thus $\{\overline{D}_i, D_i \Phi_i G, \overline{G}\}_{i=1}^n$ is a colimit of the diagram D □.

Remark

In the above proof if the diagram D satisfies the condition:

there exists i, $1 \le i \le n$, such

for any j, $1 \le j \le n$ there exists α, such that

$D_\alpha : D_j \to D_i$ is a morphism in D,

then the formula Ψ can be defined as

$$\exists_{(k_i, \underline{m}, m')} (\Phi_i(\underline{k}_i, \underline{m}, m') \wedge D_i(\underline{k}_i, m, m') \wedge \Phi_i'(\underline{k}_i, m, \underline{m}'))$$

and the proof remains valid, so in this case the assumption that H is closed under $\vee$ is not necessary. In particular we have the following:

10.6.1 Corollary

The family $\{\tilde{\mathfrak{A}}\}_{\mathfrak{A} \in K_T}$ collectively reflects coequilizers and the initial object. □.

10.7 Proposition

The category $\mathrm{Th}(T, H)$ is finitely complete (i.e. every finite diagram has a limit).

Proof

It is sufficient to show that Th(T, H) has a terminal object and pullbacks [3]:

a) For any $\mathfrak{A} \in K_T$, $\ulcorner \bar{1} \urcorner = |\mathfrak{A}|^0$ is a terminal object in Sets, so by Proposition 10.5 $\bar{1}$ is a terminal object in Th(T, H).

b) Let $D_i \Phi_i G \in$ Mor(Th(T, H)) $i = 1, 2$, $D_1 \in F(k)$, $D_2 \in F(m)$ and $G \in F(n)$.

Define formulas F, Ψ_1, Ψ_2 as follows: F is

$D_1(\underline{k}, m) \wedge D_2(k, \underline{m}) \wedge \exists(\underline{k}, \underline{m}, n)(\Phi_1(\underline{k}, m, \underline{n}) \wedge G(k, m, \underline{n}) \wedge \Phi_2(k, \underline{m}, \underline{n}))$

Ψ_1 is $(\underline{k}, m, k) \equiv (k, m, \underline{k})$

Ψ_2 is $(k, \underline{m}, m) \equiv (k, m, \underline{m})$, then since we can assume that D_i, Φ_i, G, Ψ_i, F $i = 1, 2$ are in H, hence $F\Psi_1 D_1$ and $F\Psi_2 D_2$ are morphisms in Th(T, H), and for any $\mathfrak{A} \in K_T$

$\ulcorner F \urcorner = \{(k, m) \in \ulcorner D_1 \urcorner \times \ulcorner D_2 \urcorner \mid \ulcorner D_1 \Phi_1 G \urcorner (k) = \ulcorner D_2 \Phi_2 G \urcorner (m)\}$

$\ulcorner F\Psi_i D_i \urcorner : \ulcorner F \urcorner \to \ulcorner D_i \urcorner$ $i = 1, 2$ is a projection, so $(\ulcorner F \urcorner, \ulcorner F\Psi_i D_i \urcorner, \ulcorner D_i \urcorner)_{i=1}^2$ is a pullback of $\{\ulcorner D_1 \Phi_1 G \urcorner, \ulcorner D_2 \Phi_2 G \urcorner\}$, thus by Propositions 10.1, 10.4 and 10.5 $(\bar{F}, F\Psi_i D_i, \bar{D}_i)_{i=1}^2$ is a pullback of $\{D_1 \Phi_1 G, D_2 \Phi_2 G\}$ □. An immediate consequence of the above proof is

10.8 Proposition

For any $\mathfrak{A} \in K_T$, the functor $\tilde{\mathfrak{A}}$ preserves finite limits.

Proof

$\tilde{\mathfrak{A}}$ preserves the terminal object and pullbacks, so preserves finite limits [3] □.

10.9 Proposition

If H is closed under $\vee$, then the family of functors $\{\tilde{\mathfrak{A}}\}_{\mathfrak{A} \in K_T}$ collectively reflects pushouts of pullbacks.

Proof

Let P, D: I $\to$ Th(T, H) be diagrams such that for any $\mathfrak{A} \in K_T$, $\tilde{\mathfrak{A}}$(P) is a pushout of a pullback of $\tilde{\mathfrak{A}}$(D).

There exists a diagram F, such that F is a pullback of D, so $\tilde{\mathfrak{A}}$(F) is a pullback of $\tilde{\mathfrak{A}}$(D). There exists a diagram H in sets, such that H is a pullback of $\tilde{\mathfrak{A}}$(D) and $\tilde{\mathfrak{A}}$(P) is a pushout of H, thus H is isomorphic to $\tilde{\mathfrak{A}}$(F) (as pullbacks of $\tilde{\mathfrak{A}}$(D)), so $\tilde{\mathfrak{A}}$(P) is a pushout of $\tilde{\mathfrak{A}}$(F), so by Proposition 10.6, P is a pushout of F and finally P is a pushout of a pullback of D $\square$.

10.10 Proposition

The category Th(T, H) has an initial object $\overline{0}$, if $0 \in H$.

Proof

For any $\mathfrak{A} \in K_T$, $\tilde{\mathfrak{A}}(\overline{0})$ = $\ulcorner 0 \urcorner$ = $\emptyset$ is an initial object in ~~Sets~~, so by Corollary 10.6.1, $\overline{0}$ is an initial object in Th(T, H) $\square$.

10.11 Proposition

The category Th(T, H) has pushouts of pullbacks if H is closed under $\vee$.

Proof

Let $D_1\Phi_1G$, $D_2\Phi_2G \in$ Mor(Th(T, H)), $D_1 \in H(k)$, $D_2 \in H(m)$ and $G \in H(n)$. Define the formula H to be

$\exists_{(k,\ m,\ \underline{n})} (G(k,\ m,\ \underline{n}) \wedge ((D_1(\underline{k},\ m,\ n) \wedge \Phi_1(\underline{k},\ m,\ \underline{n})) \vee (D_2(k,\ \underline{m},\ n) \wedge \Phi_2(k,\ \underline{m},\ \underline{n}))))$, then for any $\mathfrak{A} \in K_T$

$\ulcorner H \urcorner = \{n \mid \ulcorner D_1\Phi_1 G\urcorner(k) = n$ for some $k \in \ulcorner D_1 \urcorner$ or $\ulcorner D_2\Phi_2 G\urcorner(m) = n$ for some $m \in \ulcorner D_2\urcorner\} = \ulcorner D_1\Phi_1 G\urcorner(\ulcorner D_1\urcorner) \cup \ulcorner D_2\Phi_2 G\urcorner(\ulcorner D_2\urcorner)$, and $\ulcorner D_1\Phi_1 H\urcorner(k) = \ulcorner D_1\Phi_1 G\urcorner(k)$ for all $k \in \ulcorner D_1\urcorner$ $\ulcorner D_2\Phi_2 H\urcorner(m) = \ulcorner D_2\Phi_2 G\urcorner(m)$ for all $m \in \ulcorner D_2\urcorner$, thus $D_1\Phi_1 H, D_2\Phi_2 H \in \mathrm{Mor}(\mathrm{Th}(T, H))$, since $H \in H$. For any $\mathfrak{A} \in K_T$ $(\ulcorner D_i\urcorner, \ulcorner D_i\Phi_i H\urcorner, \ulcorner H\urcorner)_{i=1}^{2}$ is a pushout of a pullback of $(\ulcorner D_i\urcorner, \ulcorner D_i\Phi_i G\urcorner, \ulcorner G\urcorner)_{i=1}^{2}$ (compare Example 10.2), so by Proposition 10.9 $(\overline{D}_i, D_i\Phi_i H, \overline{H})_{i=1}^{2}$ is a pushout of a pullback of $(\overline{D}_i, D_i\Phi_i G, \overline{G})_{i=1}^{2}$ □.

An immediate consequence of the above proof is

<u>10.12 Proposition</u>

For any $\mathfrak{A} \in K_T$ the functor $\tilde{\mathfrak{A}}$ preserves pushouts of pullbacks if H is closed under $\vee$ □.

<u>10.13 Proposition</u>

The category $\mathrm{Th}(T, H)$ has coequilizers of kernel pairs.

<u>Proof</u>

Let $F \in H(k)$, $G \in H(m)$, $H \in H(n)$ and $(F\Phi_1 G, F\Phi_2 G)$ be a kernel pair of $G\Psi H$, (By Proposition 10.7 for any morphism $G\Psi H$ such a pair exists).

Define the formula H' to be $\exists_{(m,\, \underline{n})}(G(\underline{m}, n) \wedge \Psi(\underline{m}, \underline{n}) \wedge H(m, \underline{n}))$, then $H' \in H$ and for any $\mathfrak{A} \in K_T$ $\ulcorner H'\urcorner = \{n \in \ulcorner H\urcorner \mid$ there exist $m \in \ulcorner G\urcorner$, such that $(n, \cancel{m) \in} \ulcorner\Psi\urcorner\}$.

Thus $\ulcorner H'\urcorner$ is an image of $\ulcorner H\urcorner$ through function $\ulcorner G\Psi H\urcorner$. Since $(\ulcorner F\Phi_1 G\urcorner, \ulcorner F\Phi_2 G\urcorner)$ is a kernel pair of $\ulcorner G\Psi H\urcorner$, it follows that $\ulcorner G\Psi H'\urcorner$ is a coequilizer of $(\ulcorner F\Phi_1 G\urcorner, \ulcorner F\Phi_2 G\urcorner)$. By Corollary 10.6.1 $G\Psi H'$ is a coequilizer of $(F\Phi_1 G, F\Phi_2 G)$ □.

An immediate consequence of the above proof is the following:

10.14 Proposition

For any $\mathfrak{A} \in K_T$, $\tilde{\mathfrak{A}}$ preserves coequilizers of kernel pairs $\square$.

11. The structure corresponding to a finite limit preserving functor.

Let $\mathfrak{M}$: Th(T, H) $\to$ Sets be a finite limit preserving functor. (Thus $\mathfrak{M}$ preserves monomorphisms [3], p. 168)

Let I_n be a discret category with n objects where $n \neq 0$. Let D: $I_n \to$ Th(T, H) be a diagram, such that $D_i = \bar{I}_1$ for $i = 1, \ldots, n$. For any $\mathfrak{A} \in K_T$ $(\ulcorner I_n^\urcorner, \ulcorner p_i^{n\urcorner}, \ulcorner I_1^\urcorner)_{i=1}^n$ is a product of the diagram $\tilde{\mathfrak{A}}(D)$, so by Proposition 10.5 $(\bar{I}_n, \underline{p}_i^n, \bar{I}_1)_{i=1}^n$ is a product of D and therefore $(\mathfrak{M}(\bar{I}_n), \mathfrak{M}(\underline{p}_i^n), \mathfrak{M}(\bar{I}_1))_{i=1}^n$ is a product of $\mathfrak{M}(D)$, so there exists a unique product isomorphism

$$j_n: \mathfrak{M}(\bar{I}_n) \to (\mathfrak{M}(\bar{I}_1))^n.$$

Clearly $j_n = \prod_{i=1}^{n} \mathfrak{M}(\underline{p}_i^n)$.

For $n = 0$, let j_n denote the unique isomorphism from the terminal object $\mathfrak{M}(\bar{I}_0)$ to the terminal object $(\mathfrak{M}(\bar{I}_1))^0$.

Define the structure $\underset{\sim}{\mathfrak{M}} = (|\underset{\sim}{\mathfrak{M}}|, \underset{\sim}{\mathfrak{M}}^{\ulcorner\urcorner})$ corresponding to the functor $\mathfrak{M}$, as follows:

1. $|\underset{\sim}{\mathfrak{M}}| = \mathfrak{M}(\bar{I}_1)$
2. if $f \in \Sigma_0(n)$, then $\underset{\sim}{\mathfrak{M}}^{\ulcorner f \urcorner} = \mathfrak{M}(\underline{f}) \circ j_n^{-1}$
3. if $P \in \Sigma_1(n)$, then

$$\underset{\sim}{\mathfrak{M}}^{\ulcorner P \urcorner} = (j_n \circ \mathfrak{M}(\underline{Pi}(n)1_n))(\mathfrak{M}(\bar{P})).$$

Note that since $\mathfrak{M}$ preserves monomorphisms, $\underset{\sim}{\mathfrak{M}}\ulcorner\overline{P}\urcorner \cong \mathfrak{M}(\overline{P})$.

11.1 Proposition

If $t \in T(n)$, then

(1) $\underset{\sim}{\mathfrak{M}}\ulcorner t\urcorner = \mathfrak{M}(\underline{t}) \circ j_n^{-1}$

Proof

1. if $t \in V(n)$, then $t = p_i^n$, for some i, $0 < i \le n$.

   $\underset{\sim}{\mathfrak{M}}\ulcorner p_i^n\urcorner : |\underset{\sim}{\mathfrak{M}}|^n \to |\underset{\sim}{\mathfrak{M}}|$ is the i-th projection,

   so

   $\underset{\sim}{\mathfrak{M}}\ulcorner p_i^n\urcorner \circ j_n = \underset{\sim}{\mathfrak{M}}\ulcorner p_i^n\urcorner \circ \prod_{i=1}^{n} \mathfrak{M}(\underline{p}_i^n) = \mathfrak{M}(\underline{p}_i^n)$, so

   $\underset{\sim}{\mathfrak{M}}\ulcorner p_i^n\urcorner = \mathfrak{M}(\underline{p}_i^n) \circ j_n^{-1}$.

2. if $t \in \Sigma_0(n)$, then (1) holds by the definition of $\underset{\sim}{\mathfrak{M}}$.

3. let $t = s(t_1 \ldots t_m)$, $s \in T(m)$, $t_i \in T(n)$ and suppose that (1) holds for $s, t_1, \ldots, t_m$.

   for $i = 1 \ldots m$ $\underline{p}_i^m \circ (\underline{t}_1, \ldots, \underline{t}_m) = \underline{t}_i$, so

   $\mathfrak{M}(p_i^m) \circ \mathfrak{M}(\underline{t}_1, \ldots, \underline{t}_m) = \mathfrak{M}(\underline{t}_i)$, so

   $j_m \circ \mathfrak{M}(\underline{t}_1, \ldots, \underline{t}_m) = \prod_{i=1}^{m} \mathfrak{M}(\underline{t}_i)$, thus

   $\underset{\sim}{\mathfrak{M}}\ulcorner s(t_1 \ldots t_m)\urcorner = \underset{\sim}{\mathfrak{M}}\ulcorner s\urcorner \circ (\underset{\sim}{\mathfrak{M}}\ulcorner t_1\urcorner, \ldots, \underset{\sim}{\mathfrak{M}}\ulcorner t_m\urcorner) =$

   $\mathfrak{M}(\underline{s}) \circ j_m^{-1} \circ \prod_{i=1}^{n} \mathfrak{M}(t_i) \circ j_n^{-1} =$

   $\mathfrak{M}(\underline{s}) \circ j_m^{-1} \circ j_m \circ \mathfrak{M}(\underline{t}_1, \ldots, \underline{t}_m) \circ j_n^{-1} =$

   $\mathfrak{M}(\underline{s}(\underline{t}_1, \ldots, \underline{t}_m)) \circ j_n^{-1} = \mathfrak{M}(\underline{s(t_1 \ldots t_m)}) \circ j_n^{-1}$.

4. let $t = s(p_0^n)$, $s \in T(0)$ and suppose that (1) holds for s.

   $\underset{\sim}{\mathfrak{M}}\ulcorner t\urcorner = \underset{\sim}{\mathfrak{M}}\ulcorner s\urcorner \circ \underset{\sim}{\mathfrak{M}}\ulcorner p_0^n\urcorner = \mathfrak{M}(\underline{s}) \circ j_0^{-1} \circ \underset{\sim}{\mathfrak{M}}\ulcorner p_0^n\urcorner$

   but

   $\underset{\sim}{\mathfrak{M}}\ulcorner p_0^n\urcorner \circ j_n = j_0 \circ \mathfrak{M}(1_n\, 1_n\, \Delta_0)$ as unique morphisms to $|\underset{\sim}{\mathfrak{M}}|^0$, so

$$\underset{\sim}{\mathfrak{M}}\ulcorner t \urcorner = \mathfrak{M}(\underline{s}) \circ j_0^{-1} \circ j_0 \circ \mathfrak{M}(1_n 1_n 1_0) \circ j_n^{-1} =$$
$$\mathfrak{M}(\underline{s} \circ 1_n 1_n 1_0) \circ j_n^{-1} = \mathfrak{M}(s(p_0^n)) \circ j_n^{-1} \quad \square.$$

11.2 Proposition

Let the functor $\mathfrak{M} : \mathrm{Th}(T, H_i) \to \mathrm{Sets}$ preserve finite limits, then

$$\underset{\sim}{\mathfrak{M}}\ulcorner F \urcorner = j_n \circ \mathfrak{M}(Fi(n)1_n)(\mathfrak{M}(\bar{F}))$$

for all $F \in H_i(n)$ provided that

(0) $i = 0$, $\mathfrak{M}$ preserves coequilizers of kernel pairs.

(1) $i = 1$, $\mathfrak{M}$ preserves pushouts of pullbacks.

(2) $i = 2$, $\mathfrak{M}$ preserves pushouts of pullbacks and the initial object.

Proof

In the proof we will use the following:

I

If $f_i: A_i \to B$, $g_i: C \to A_i$, $h: C \to B$ $i = 1, 2$ are morphisms in sets, such that

$$\begin{tikzcd} & A_1 \arrow[dr, "f_1"] & \\ C \arrow[ur, "g_1"] \arrow[rr, "h"] \arrow[dr, "g_2"'] & & B \\ & A_2 \arrow[ur, "f_2"'] & \end{tikzcd}$$

is a pullback, then

a) $h(C) = f_1(A_1) \cap f_2(A_2)$,

b) $g_2(C) = f_2^{-1}(f_1(A_1))$.

II

If $f_i: A_i \to B$, $g_i: A_i \to C$, $h: C \to B$ $i = 1, 2$ are morphisms in Sets, such that

$$\begin{tikzcd} A_1 \arrow[dr, "g_1"'] \arrow[drr, "f_1"] & & \\ & C \arrow[r, "h"] & B \\ A_2 \arrow[ur, "g_2"] \arrow[urr, "f_2"'] & & \end{tikzcd}$$

is a pushout of a pullback, then

$h(C) = f_1(A_1) \cup f_2(A_2)$. (compare Example 10.2).

III

If $g_i : A \to B$ $i = 1, 2$, $h: B \to D$, $f: B \to C$ and $u: D \to C$ are morphisms in Sets, such that h is a coequilizer of (g_1, g_2), $f \circ g_1 = f \circ g_2$ and u is the unique map such that $u \circ h = f$, then

$$f(B) = u(D)$$

$$\begin{tikzcd} A \arrow[r, shift left, "g_2"] \arrow[r, shift right, "g'"'] & B \arrow[r, "h"] \arrow[dr, "f"'] & D \arrow[d, "u"] \\ & & C \end{tikzcd}$$

Now, let $F \in \mathcal{H}_i(n)$

1. If $F \in \Sigma_1(n)$, then Proposition holds by the definition of the structure $\underset{\sim}{\mathfrak{M}}$.

2. Suppose that $F = G_1 \wedge G_2$, $G_1, G_2 \in \mathcal{H}_i(n)$ and $\underset{\sim}{\mathfrak{M}} \ulcorner G_k \urcorner = j_n \circ \mathfrak{M}(G_k i(n) 1_n)(\mathfrak{M}(\bar{G}_k))$ $k = 1, 2$.

For any $\mathfrak{A} \in K_T$

$$\begin{tikzcd} & \ulcorner G_1 \urcorner \arrow[dr, "\ulcorner G_1 i(n) 1_n \urcorner"] & \\ \ulcorner F \urcorner \arrow[ur, "\ulcorner Fi(n) G_1 \urcorner"] \arrow[rr, "\ulcorner Fi(n) 1_n \urcorner"] \arrow[dr, "\ulcorner Fi(n) G_2 \urcorner"'] & & \ulcorner 1_n \urcorner \\ & \ulcorner G_2 \urcorner \arrow[ur, "\ulcorner G_2 i(n) 1_n \urcorner"'] & \end{tikzcd}$$

is a pullback, so by Proposition 10.5

$$
\begin{tikzcd}
 & \mathfrak{M}(\overline{G}_1) \arrow[dr, "\mathfrak{M}(G_1 i(n) 1_n)"] & \\
\mathfrak{M}(\overline{F}) \arrow[ur, "\mathfrak{M}(Fi(n)G_1)"] \arrow[rr, "\mathfrak{M}(Fi(n)1_n)"] \arrow[dr, "\mathfrak{M}(Fi(n)G_2)"'] & & \mathfrak{M}(\overline{1}_n) \\
 & \mathfrak{M}(\overline{G}_2) \arrow[ur, "\mathfrak{M}(G_2 i(n) 1_n)"'] &
\end{tikzcd}
$$

is a pullback too, thus using Ia

$\mathfrak{M}(G_1 i(n) 1_n)(\mathfrak{M}(\overline{G}_1)) \cap \mathfrak{M}(G_2 i(n) 1_n)(\mathfrak{M}(\overline{G}_2)) =$
$\mathfrak{M}(Fi(n)1_n)(\mathfrak{M}(\overline{F}))$, so

$j_n \circ \mathfrak{M}(G_1 i(n) 1_n)(\mathfrak{M}(\overline{G}_1)) \cap j_n \circ \mathfrak{M}(G_2 i(n) 1_n)(\mathfrak{M}(\overline{G}_2)) =$
$j_n \circ \mathfrak{M}(Fi(n)1_n)(\mathfrak{M}(\overline{F}))$, so

$\underset{\sim}{\mathfrak{M}}\ulcorner F \urcorner = \underset{\sim}{\mathfrak{M}}\ulcorner G_1 \urcorner \cap \underset{\sim}{\mathfrak{M}}\ulcorner G_2 \urcorner = j_n \circ \mathfrak{M}(Fi(n)1_n)(\mathfrak{M}(\overline{F}))$.

3. For $i = 1$ or $i = 2$, suppose that $F = G_1 \vee G_2$,
$G_1, G_2 \in \mathcal{H}_i(n)$ and $\underset{\sim}{\mathfrak{M}}\ulcorner G_k \urcorner = j_n \circ \mathfrak{M}(G_k i(n) 1_n)(\mathfrak{M}(\overline{G}_k))$
$k = 1, 2$. For any $\mathfrak{A} \in K_T$

$$
\begin{tikzcd}
\ulcorner G_1 \urcorner \arrow[dr, "\ulcorner G_1 i(n) F \urcorner"'] \arrow[rr, bend left, "\ulcorner G_1 i(n) 1_n \urcorner"] & & \ulcorner 1_n \urcorner \\
 & \ulcorner F \urcorner \arrow[ur, "\ulcorner Fi(n) 1_n \urcorner"] & \\
\ulcorner G_2 \urcorner \arrow[ur, "\ulcorner G_2 i(n) F \urcorner"] \arrow[uurr, bend right, "\ulcorner G_2 i(n) 1_n \urcorner"'] & &
\end{tikzcd}
$$

is a pushout of a pullback, so by Proposition 10.9

$$\begin{tikzcd}
\mathfrak{M}(\bar{G}_1) \arrow[dr, "\mathfrak{M}(G_1 i(n) F)"'] \arrow[drr, "\mathfrak{M}(G_1 i(n) 1_n)"] & & \\
& \mathfrak{M}(\bar{F}) \arrow[r, "\mathfrak{M}(F i(n) 1_n)"] & \mathfrak{M}(\bar{I}_n) \\
\mathfrak{M}(\bar{G}_2) \arrow[ur, "\mathfrak{M}(G_2 i(n) F)"] \arrow[urr, "\mathfrak{M}(G_2 i(n) 1_n)"'] & &
\end{tikzcd}$$

is a pushout of a pullback too, thus using II

$\mathfrak{M}(Gi(n)1_n)(\mathfrak{M}(\bar{G}_1)) \cup \mathfrak{M}(G_2 i(n) 1_n)(\mathfrak{M}(\bar{G}_2)) =$
$\mathfrak{M}(Fi(n)1_n)(\mathfrak{M}(\bar{F}))$, so

$j_n \circ \mathfrak{M}(G_1 i(n) 1_n)(\mathfrak{M}(\bar{G}_1)) \cup j_n \circ \mathfrak{M}(G_2 i(n) 1_n)$
$(\mathfrak{M}(\bar{G}_2)) = j_n \circ \mathfrak{M}(Fi(n)1_n)(\mathfrak{M}(\bar{F}))$, so
$\underset{\sim}{\mathfrak{M}}\ulcorner F \urcorner = \underset{\sim}{\mathfrak{M}}\ulcorner G_1 \urcorner \cup \underset{\sim}{\mathfrak{M}}\ulcorner G_2 \urcorner = j_n \circ \mathfrak{M}(Fi(n)1_n)(\mathfrak{M}(\bar{F}))$.

4. For $i = 3$, suppose that $F = \neg G$, $G \in H_i(n)$ and
$\underset{\sim}{\mathfrak{M}}\ulcorner G \urcorner = j_n \circ \mathfrak{M}(Gi(n)1_n)(\mathfrak{M}(\bar{G}))$. For any $\mathfrak{A} \in K_T$

$$\begin{tikzcd}
& \ulcorner G \urcorner \arrow[dr, "\ulcorner Gi(n)1_n \urcorner"] & \\
\ulcorner 0_n \urcorner \arrow[ur, "\ulcorner 0_n i(n) G \urcorner"] \arrow[rr, "\ulcorner 0_n i(n) 1_n \urcorner"] \arrow[dr, "\ulcorner 0_n i(n) \neg G \urcorner"'] & & \ulcorner 1_n \urcorner \\
& \ulcorner \neg G \urcorner \arrow[ur, "\ulcorner \neg Gi(n)1_n \urcorner"'] &
\end{tikzcd}$$

is a pullback, so by Proposition 10.5

$$\begin{tikzcd}
& \mathfrak{M}(\bar{G}) \arrow[dr, "\mathfrak{M}(Gi(n)1_n)"] & \\
\emptyset = \mathfrak{M}(\bar{0}_n) \arrow[ur, "\mathfrak{M}(0_n i(n) G)"] \arrow[rr, "\mathfrak{M}(0_n i(n) 1_n)"] \arrow[dr, "\mathfrak{M}(0_n i(n) \neg G)"'] & & \mathfrak{M}(\bar{I}_n) \\
& \mathfrak{M}(\neg\bar{G}) \arrow[ur, "\mathfrak{M}(\neg Gi(n)1_n)"'] &
\end{tikzcd}$$

is a pullback too, thus

$\mathfrak{M}(Gi(n)1_n)(\mathfrak{M}(\bar{G})) \cap \mathfrak{M}(\neg Gi(n)1_n)(\mathfrak{M}(\neg\bar{G})) =$

$\mathfrak{M}(0_n i(n) 1_n)(\Phi) = \emptyset$, so

$j_n \circ \mathfrak{M}(Gi(n)1_n)(\mathfrak{M}(\bar{G})) \cap j_n \circ \mathfrak{M}(\neg Gi(n)1_n)$
$(\mathfrak{M}(\neg\bar{G})) = \emptyset$, so

(*) $\underset{\sim}{\mathfrak{M}}\ulcorner G\urcorner \cap j_n \circ \mathfrak{M}(\neg Gi(n)1_n)(\mathfrak{M}(\neg\bar{G})) = \emptyset$.

For any $\mathfrak{A} \in K_T$

$$\begin{tikzcd}
\ulcorner G\urcorner \arrow[dr, "\ulcorner Gi(n)1_n\urcorner"'] \arrow[drr, bend left, "\ulcorner Gi(n)1_n\urcorner"] & & \\
& \ulcorner \bar{1}_n\urcorner \arrow[r, "\ulcorner 1_n i(n)\bar{1}_n\urcorner"] & \ulcorner 1_n\urcorner \\
\ulcorner\neg G\urcorner \arrow[ur, "\ulcorner\neg Gi(n)1_n\urcorner"] \arrow[urr, "\ulcorner\neg Gi(n)1_n\urcorner"'] & &
\end{tikzcd}$$

is a pushout of a pullback, so by Proposition 10.9

$$\begin{tikzcd}
\mathfrak{M}(\bar{G}) \arrow[dr, "\mathfrak{M}(Gi(n)1_n)"'] \arrow[drr, "\mathfrak{M}(Gi(n)1_n)"] & & \\
& \mathfrak{M}(\bar{1}_n) \arrow[r, "\mathfrak{M}(1_n i(n)1_n)"] & \mathfrak{M}(1_n) \\
\mathfrak{M}(\neg\bar{G}) \arrow[ur, "\mathfrak{M}(\neg Gi(n)1_n)"] \arrow[urr, "\mathfrak{M}(\neg Gi(n)1_n)"'] & &
\end{tikzcd}$$

is a pushout of a pullback too, so

$\mathfrak{M}(Gi(n)1_n)(\mathfrak{M}(\bar{G})) \cup \mathfrak{M}(\neg Gi(n)1_n)(\mathfrak{M}(\neg\bar{G})) =$

$\mathfrak{M}(1_n i(n) 1_n)(\mathfrak{M}(\bar{1}_n)) = \mathfrak{M}(\bar{1}_n)$, so

$j_n \circ \mathfrak{M}(Gi(n)1_n)(\mathfrak{M}(\bar{G})) \cup j_n \circ \mathfrak{M}(\neg Gi(n)1_n)$
$(\mathfrak{M}(\neg\bar{G})) = j_n(\mathfrak{M}(\bar{1}_n)) = |\underset{\sim}{\mathfrak{M}}|^n$, so

(**) $\underset{\sim}{\mathfrak{M}}\ulcorner G\urcorner \cup j_n \circ \mathfrak{M}(\neg Gi(n)1_n)(\mathfrak{M}(\neg\bar{G})) = |\underset{\sim}{\mathfrak{M}}|^n$.

From (*) and (**) follows that

$j_n \circ \mathfrak{M}(\neg Gi(n)1_n) = |\underset{\sim}{\mathfrak{M}}|^n - \underset{\sim}{\mathfrak{M}}\ulcorner G\urcorner = \underset{\sim}{\mathfrak{M}}\ulcorner\neg G\urcorner$.

5. For $i = 0$, suppose that $F = \exists_{t_1 \ldots t_n}(G)$, $t_1, \ldots, t_n \in T(m)$, $G \in H_i(m)$ and

$$\mathfrak{M}\ulcorner G \urcorner = j_m \circ \mathfrak{M}(G.i(m)\ 1_m)(\mathfrak{M}(\bar{G}))$$

Define the formula Φ to be

$$\bigcap_{i=1}^{n} t_i(\underline{m},\ n) \equiv p^{m+n}_{m+i},$$

and the formula H to be

$$G \times G \wedge \bigcap_{i=1}^{n} t_i(\underline{m},\ m) \equiv t_i(m,\ \underline{m}),$$

let g_1 and g_2 denote respectively the morphisms $H(\underline{m},\ m,\ m) \equiv (m,\ m,\ \underline{m})G$ and $H(m,\ \underline{m},\ m) \equiv (m,\ m,\ \underline{m})G$, then for any $\mathfrak{A} \in K_T$ $(\ulcorner g_1 \urcorner,\ \ulcorner g_2 \urcorner)$ is a kernel pair of $\ulcorner G\Phi 1_n \urcorner$ and $\ulcorner G\Phi F \urcorner$ is a coequilizer of $(\ulcorner g_1 \urcorner,\ \ulcorner g_2 \urcorner)$. Thus by Proposition 10.5 and Corollary 10.6.1 $(g_1,\ g_2)$ is a kernel pair of $G\Phi 1_n$, and $G\Phi F$ is a coequilizer of $(g_1,\ g_2)$; so finally $\mathfrak{M}(G\Phi F)$ is a coequilizer of $(\mathfrak{M}(g_1),\ \mathfrak{M}(g_2))$ and the following diagram commutes

$$\begin{tikzcd}
\mathfrak{M}(\bar{H}) \arrow[r, shift left, "\mathfrak{M}(g_1)"] \arrow[r, shift right, "\mathfrak{M}(g_2)"'] & \mathfrak{M}(\bar{G}) \arrow[r, "\mathfrak{M}(G\Phi F)"] \arrow[dr, "\mathfrak{M}(G\Phi 1_n)"'] & \mathfrak{M}(\bar{F}) \arrow[d, "\mathfrak{M}(Fi(n)1_n)"] \\
 & & \mathfrak{M}(\bar{I}_n)
\end{tikzcd}$$

Thus by III:

$\mathfrak{M}(Fi(n)1_n)(\mathfrak{M}(\bar{F})) = \mathfrak{M}(G\Phi 1_n)(\mathfrak{M}(\bar{G})) =$

$\mathfrak{M}(1_m \Phi 1_n) \circ \mathfrak{M}(Gi(m)1_m)(\mathfrak{M}(\bar{G})) = \circ$

$\mathfrak{M}(\underline{t}_1,\ \ldots,\ \underline{t}_n)(\mathfrak{M}(G\,i(m)1_m)(\mathfrak{M}(\bar{G})))$, but for

$i = 1, \ldots, n \quad \underline{p}^n_i \circ (\underline{t}_1,\ \ldots,\ \underline{t}_n) = \underline{p^n_i(t_1 \ldots t_n)} \doteq \underline{t}_i,$

$\mathfrak{M}(p_i^n) \circ \mathfrak{M}(\underline{t}_1, \ldots, \underline{t}_n) = \mathfrak{M}(\underline{t}_i)$, so

$j_n \circ \mathfrak{M}(\underline{t}_1, \ldots, \underline{t}_n) = \prod_{i=1}^{n} \mathfrak{M}(\underline{t}_i) = \prod_{i=1}^{n} \underset{\sim}{\mathfrak{M}} \ulcorner t_i \urcorner \circ j_m$, so

$\mathfrak{M}(\underline{t}_1, \ldots, \underline{t}_n) = j_n^{-1} \circ \prod_{i=1}^{n} \underset{\sim}{\mathfrak{M}} \ulcorner t_i \urcorner \circ j_m$, thus

$\mathfrak{M}(Fi(n)1_n)(\mathfrak{M}(\bar{F})) = j_n^{-1} \circ \prod_{i=1}^{n} \underset{\sim}{\mathfrak{M}} \ulcorner t_i \urcorner \circ$

$j_m \circ \mathfrak{M}(Gi(m)1_m)(\mathfrak{M}(\bar{G})) = j_n^{-1} \circ \prod_{i=1}^{n} \underset{\sim}{\mathfrak{M}} \ulcorner t_i \urcorner (\underset{\sim}{\mathfrak{M}} \ulcorner G \urcorner) =$

$j_n^{-1}(\underset{\sim}{\mathfrak{M}} \ulcorner F \urcorner)$ so

$\underset{\sim}{\mathfrak{M}} \ulcorner F \urcorner = j_n \circ \mathfrak{M}(Fi(n)1_n)(\mathfrak{M}(\bar{F}))$.

6. If $i = 0$ and $F = \exists_{p_0^n}(G)$ for $G \in H_i(n)$ the proof is similar to case 5.

7. For $i = 1$ or $i = 2$, suppose that $F = \exists_{t_1 \ldots t_n}(G)$, $t_1, \ldots, t_n \in T(m)$, $G \in H_i(m)$ and $\underset{\sim}{\mathfrak{M}} \ulcorner G \urcorner = (j_m \circ \mathfrak{M}(Gi(m)1_m)(\mathfrak{M}(\bar{G}))$

Define the formula Φ to be

$\bigcap_{i=1}^{n} t_i(\underline{m}, n) \equiv p_{m+i}^{m+n}$, then for any $\mathfrak{A} \in K_T$

$$\begin{tikzcd}
\ulcorner G \urcorner \arrow[dr, "\ulcorner G\Phi F \urcorner"'] \arrow[drr, "\ulcorner G\Phi 1_n \urcorner"] & & \\
& \ulcorner F \urcorner \arrow[r, "\ulcorner Fi(n)1_n \urcorner"] & \ulcorner 1_n \urcorner \\
\ulcorner G \urcorner \arrow[ur, "\ulcorner G\Phi F \urcorner"] \arrow[urr, "\ulcorner G\Phi 1_n \urcorner"'] & &
\end{tikzcd}$$

is a pushout of a pullback, so by Proposition 10.9

$$\begin{tikzcd}
\mathfrak{M}(\bar{G}) \arrow[dr, "\mathfrak{M}(G\Phi F)"'] \arrow[drr, "\mathfrak{M}(G\Phi 1_n)"] & & \\
& \mathfrak{M}(\bar{F}) \arrow[r, "\mathfrak{M}(Fi(n)1_n)"] & \mathfrak{M}(\bar{I}_n) \\
\mathfrak{M}(\bar{G}) \arrow[ur, "\mathfrak{M}(G\Phi F)"] \arrow[urr, "\mathfrak{M}(G\Phi 1_n)"'] & &
\end{tikzcd}$$

is a pushout of a pullback too, thus

$\mathfrak{M}(G\Phi 1_n)(\mathfrak{M}(\bar{G})) = \mathfrak{M}(Fi(n)1_n)(\mathfrak{M}(\bar{F}))$, so

$$j_n \circ \mathfrak{M}(Fi(m)1_n)(\mathfrak{M}(\bar{F})) = j_n \circ \mathfrak{M}(G\Phi 1_n)(\mathfrak{M}(\bar{G})) =$$

$$j_n \circ \mathfrak{M}(1_m \Phi 1_n \circ Gi(m)1_m)(\mathfrak{M}(\bar{G})) =$$

$$j_n \circ \mathfrak{M}((\underline{t}_1, \ldots, \underline{t}_n) \circ Gi(m)1_m)(\mathfrak{M}(\bar{G})) =$$

$$j_n \circ \mathfrak{M}((\underline{t}_1, \ldots, \underline{t}_n)) \circ \mathfrak{M}(Gi(m)1_m)(\mathfrak{M}(\bar{G})) =$$

$$j_n \circ j_n^{-1} \circ \prod_{i=1}^{n} \underset{\sim}{\mathfrak{M}} \ulcorner t_i \urcorner \circ j_m \circ \mathfrak{M}(Gi(m)1_m)(\mathfrak{M}(\bar{G})) =$$

$$\prod_{i=1}^{n} \underset{\sim}{\mathfrak{M}} \ulcorner t_i \urcorner (\underset{\sim}{\mathfrak{M}} \ulcorner G \urcorner) = \underset{\sim}{\mathfrak{M}} \ulcorner F \urcorner.$$

8. For $i = 1$ or $i = 2$ suppose that

$F = \exists_{p_0}^{m}(G)$, $G \in H_i(m)$ and

$\underset{\sim}{\mathfrak{M}} \ulcorner G \urcorner = j_m \circ \mathfrak{M}(Gi(m)1_m)(\mathfrak{M}(\bar{G}))$

For any $\mathfrak{A} \in K_T$

$$\begin{tikzcd}
\ulcorner G \urcorner \arrow[dr, "\ulcorner G1_mF \urcorner"'] \arrow[drr, "\ulcorner G1_m1_0 \urcorner"] & & \\
& \ulcorner F \urcorner \arrow[r, "\ulcorner Fi(0)1_0 \urcorner"] & \ulcorner 1_0 \urcorner \\
\ulcorner G \urcorner \arrow[ur, "\ulcorner G1_mF \urcorner"] \arrow[urr, "\ulcorner G1_m1_0 \urcorner"'] & &
\end{tikzcd}$$

is a pushout of a pullback, so

$$\begin{tikzcd}[column sep=huge]
\mathfrak{M}(\bar{G}) \arrow[dr, "\mathfrak{M}(Gl_m F)"'] \arrow[drr, "\mathfrak{M}(Gl_m l_0)"] & & \\
& \mathfrak{M}(\bar{F}) \arrow[r, "\mathfrak{M}(Fi(0)l_0)"] & \mathfrak{M}(\bar{I}_0) \\
\mathfrak{M}(\bar{G}) \arrow[ur, "\mathfrak{M}(Gl_m l_0)"] \arrow[urr, "\mathfrak{M}(Gl_m l_0)"'] & &
\end{tikzcd}$$

is a pushout of a pullback too, so

$$\mathfrak{M}(Fi(0)l_0)(\mathfrak{M}(\bar{F})) = \mathfrak{M}(Gl_m l_0)(\mathfrak{M}(\bar{G})), \text{ so}$$

$$j_0 \circ \mathfrak{M}(Fi(0)l_0)(\mathfrak{M}(\bar{F})) = j_0 \circ \mathfrak{M}(Gl_m l_0)(\mathfrak{M}(\bar{G})) =$$

$$j_0 \circ \mathfrak{M}(l_m l_m l_0 \circ Gi(m)l_m)(\mathfrak{M}(\bar{G})) =$$

$$j_0 \circ \mathfrak{M}(l_m l_m l_0) \circ \mathfrak{M}(Gi(m)l_m)(\mathfrak{M}(\bar{G})) =$$

$$j_0 \circ j_0^{-1} \circ \underset{\sim}{\mathfrak{M}}\ulcorner p_0^m \urcorner \circ j_m \circ \mathfrak{M}(Gi(m)l_m)(\mathfrak{M}(\bar{G})) =$$

$$\underset{\sim}{\mathfrak{M}}\ulcorner p_0^m \urcorner(\underset{\sim}{\mathfrak{M}}\ulcorner G \urcorner) = \underset{\sim}{\mathfrak{M}}\ulcorner F \urcorner, \text{ thus}$$

$$j_0 \circ \mathfrak{M}(Fi(0)l_0)(\mathfrak{M}(\bar{F})) = \underset{\sim}{\mathfrak{M}}\ulcorner F \urcorner.$$

9. Let $F = G(t_1 \ldots t_m)$, $G \in H_i(m)$, $t, \ldots, t_m \in T(n)$ and

$$\underset{\sim}{\mathfrak{M}}\ulcorner G \urcorner = j_m \circ \mathfrak{M}(Gi(m)l_m)(\mathfrak{M}(\bar{G}))$$

Define the formula Φ to be

$$\bigcap_{i=1}^{m} t_i(\underline{n}, m) \equiv p_{n+i}^{n+m}$$

For any $\mathfrak{A} \in K_T$

$$\begin{tikzcd}[column sep=huge]
& \ulcorner G \urcorner \arrow[dr, "\ulcorner Gi(m)l_m \urcorner"] & \\
\ulcorner F \urcorner \arrow[ur, "\ulcorner F\Phi G \urcorner"] \arrow[rr, "\ulcorner F\Phi l_m \urcorner"] \arrow[dr, "\ulcorner Fi(n)l_n \urcorner"'] & & \ulcorner l_m \urcorner \\
& \ulcorner l_n \urcorner \arrow[ur, "\ulcorner l_n \Phi l_m \urcorner"'] &
\end{tikzcd}$$

is a pullback, thus

$$\begin{tikzcd}
 & \mathfrak{M}(\bar{G}) \arrow[dr, "\mathfrak{M}(Gi(m)1_m)"] & \\
\mathfrak{M}(\bar{F}) \arrow[ur, "\mathfrak{M}(F\Phi G)"] \arrow[rr, "\mathfrak{M}(F\Phi 1_m)"] \arrow[dr, "\mathfrak{M}(Fi(n)1_n)"'] & & \mathfrak{M}(I_m) \\
 & \mathfrak{M}(\bar{I}_n) \arrow[ur, "\mathfrak{M}(1_n\Phi 1_m)"'] &
\end{tikzcd}$$

is a pullback too, so using Ib

$$\mathfrak{M}(Fi(n)1_n)(\mathfrak{M}(\bar{F})) = (\mathfrak{M}(1_n\Phi 1_m))^{-1}(\mathfrak{M}(Gi(m)1_m)$$

$$(\mathfrak{M}(\bar{G})) = (\mathfrak{M}(\underline{t}_1, \ldots, \underline{t}_m))^{-1}(\mathfrak{M}(Gi(m)1_m)(\mathfrak{M}(\bar{G})) =$$

$$(j_m^{-1} \circ \prod_{i=1}^{m} \underset{\sim}{\mathfrak{M}}\ulcorner t_i \urcorner \circ j_n)^{-1}(\mathfrak{M}(Gi(m)1_m)(\mathfrak{M}(\bar{G})) =$$

$$j_n^{-1} \circ (\prod_{i=1}^{m} \underset{\sim}{\mathfrak{M}}\ulcorner t_i \urcorner)^{-1} \circ j_m \circ \mathfrak{M}(Gi(m)1_m)(\mathfrak{M}(\bar{G})) =$$

$$j_n^{-1} \circ (\prod_{i=1}^{m} \underset{\sim}{\mathfrak{M}}\ulcorner t_i \urcorner)^{-1} \circ \underset{\sim}{\mathfrak{M}}\ulcorner G \urcorner, \text{ so}$$

$$j_n \circ \mathfrak{M}(Fi(n)1_n)(\mathfrak{M}(\bar{F})) = (\prod_{i=1}^{m} \mathfrak{M}\ulcorner t_i \urcorner)^{-1}(\underset{\sim}{\mathfrak{M}}\ulcorner G \urcorner) =$$

$$\underset{\sim}{\mathfrak{M}}\ulcorner F \urcorner.$$

10. Let $F = G(p_0^n)$, $G \in H_i(0)$ and

$$\underset{\sim}{\mathfrak{M}}\ulcorner G \urcorner = j_0 \circ \mathfrak{M}(Gi(0)1_0)(\mathfrak{M}(\bar{G}))$$

For any $\mathfrak{O} \in K_T$

$$\begin{tikzcd}
 & \ulcorner G \urcorner \arrow[dr, "\ulcorner Gi(0)1_0 \urcorner"] & \\
\ulcorner F \urcorner \arrow[ur, "\ulcorner F1_nG \urcorner"] \arrow[rr, "\ulcorner F1_n1_0 \urcorner"] \arrow[dr, "\ulcorner Fi(n)1_n \urcorner"'] & & 1_0 \\
 & \ulcorner 1_n \urcorner \arrow[ur, "\ulcorner 1_n1_n1_0 \urcorner"'] &
\end{tikzcd}$$

is a pullback, thus

$$
\begin{tikzcd}
 & \mathfrak{M}(\bar{G}) \arrow[dr, "\mathfrak{M}(Gi(0)l_0)"] & \\
\mathfrak{M}(\bar{F}) \arrow[ur, "\mathfrak{M}(Fl_nG)"] \arrow[rr, "\mathfrak{M}(Fl_nl_0)"] \arrow[dr, "\mathfrak{M}(Fi(n)l_n)"'] & & \mathfrak{M}(\bar{I}_0) \\
 & \mathfrak{M}(\bar{I}_n) \arrow[ur, "\mathfrak{M}(l_nl_nl_0)"'] &
\end{tikzcd}
$$

is a pullback too, so

$\mathfrak{M}(Fi(n)l_n)(\mathfrak{M}(\bar{F})) = (\mathfrak{M}(l_nl_nl_0))^{-1}(\mathfrak{M}(Gi(0)l_0)$
$(\mathfrak{M}(\bar{G}))) = (j_0^{-1} \circ \underset{\sim}{\mathfrak{M}}\ulcorner p_0^n \urcorner \circ j_n)^{-1} \circ \mathfrak{M}(Gi(0)l_0)$
$(\mathfrak{M}(\bar{G})) = j_n^{-1} \circ (\underset{\sim}{\mathfrak{M}}\ulcorner p_0^n \urcorner)^{-1} \circ j_0 \circ \mathfrak{M}(Gi(0)l_0)$
$(\mathfrak{M}(\bar{G})) = j_n^{-1} \circ (\underset{\sim}{\mathfrak{M}}\ulcorner p_0^n \urcorner)^{-1}(\underset{\sim}{\mathfrak{M}}\ulcorner G \urcorner)$, so
$j_n \circ \mathfrak{M}(Fi(n)l_n)(\mathfrak{M}(\bar{F})) = \underset{\sim}{\mathfrak{M}}\ulcorner F \urcorner$ □.

11.3 Proposition

Let a functor $\mathfrak{M} : Th(T, \mathcal{H}_i) \to Sets$ preserve finite limits. If $\mathfrak{M}$ satisfies the condition (i) from Proposition 11.2, then

(a) $\underset{\sim}{\mathfrak{M}} \in K_T$, (where as before $T' = Cn(T) \cap B(\mathcal{H})$)

(b) $(\widetilde{\underset{\sim}{\mathfrak{M}}})^{(T', \mathcal{H})}$ is isomorphic to $\mathfrak{M}$

Proof

(a) It is sufficient to show that if $F, G \in \mathcal{H}(n)$ and $F \to G \in Cn(T)$ then $\underset{\sim}{\mathfrak{M}}\ulcorner F \urcorner \subset \underset{\sim}{\mathfrak{M}}\ulcorner G \urcorner$ (for the details compare with the proof of Proposition 9.9. II). Let $F, G \in \mathcal{H}(n)$ and $F \to G \in Cn(T)$, then $Fi(n)G_T \in Mor(Th(T, \mathcal{H}))$ and $Gi(n)l_{n_T} \circ Fi(n)G_T = Fi(n)l_{n_T}$ thus $\mathfrak{M}(Gi(n)l_{n_T}) \circ \mathfrak{M}(Fi(n)G_T) = \mathfrak{M}(Fi(n)l_{n_T})$ and by Proposotion 11.2

$\underset{\sim}{\mathfrak{M}}\ulcorner F\urcorner = \mathfrak{M}(F i(n) 1_{n_T})(\mathfrak{M}(\bar{F}_T)) =$
$\mathfrak{M}(G i(n) 1_{n_T}) \circ \mathfrak{M}(F i(n) G_T)(\mathfrak{M}(\bar{F}_T)) \subseteq$
$\mathfrak{M}(G i(n) 1_{n_T})(\mathfrak{M}(\bar{G}_T)) = \underset{\sim}{\mathfrak{M}}\ulcorner G\urcorner$, so
$\underset{\sim}{\mathfrak{M}}\ulcorner F\urcorner \subseteq \underset{\sim}{\mathfrak{M}}\ulcorner G\urcorner$.

(b) For any $F \in H(n)$ define
$\psi_{\bar{F}} = j_n \circ \mathfrak{M}(F\ i(n)\ 1_n)$, then
$\psi_{\bar{F}}$ is a bijection from
$\mathfrak{M}(\bar{F})$ to $(\widetilde{\underset{\sim}{\mathfrak{M}}})^{(T',H)}(\bar{F}) = \underset{\sim}{\mathfrak{M}}\ulcorner F\urcorner$.

We will show that the transformation ψ is natural:

Let $F \in H(n)$, $G \in H(m)$, $\Phi \in H(n+m)$ and $F\Phi G$ be a morphism in $Th(T, H)$. Let Ψ denote the formula $F \times G \cap \Phi$, then for any $\mathfrak{A} \in K_T$ $\ulcorner F\Phi G\urcorner = \ulcorner F\Psi G\urcorner$, thus $F\Phi G = F\Psi G$. Consider the diagram.

$$\begin{tikzcd}
\mathfrak{M}(\bar{F}) \arrow[r, "\mathfrak{M}(F\Psi G)"] \arrow[d, "\psi_{\bar{F}}"'] & \mathfrak{M}(\bar{G}) \arrow[d, "\psi_{\bar{G}}"] \\
\underset{\sim}{\mathfrak{M}}\ulcorner F\urcorner = (\widetilde{\underset{\sim}{\mathfrak{M}}})(\bar{F}) \arrow[r, "(\widetilde{\underset{\sim}{\mathfrak{M}}})(F\Psi G)"'] & (\widetilde{\underset{\sim}{\mathfrak{M}}})(\bar{G}) = \underset{\sim}{\mathfrak{M}}\ulcorner G\urcorner
\end{tikzcd}$$

$$(\widetilde{\underset{\sim}{\mathfrak{M}}})(F\Psi G) = \underset{\sim}{\mathfrak{M}}\ulcorner F\Psi G\urcorner = \underset{\sim}{\mathfrak{M}}\ulcorner F\urcorner \times \underset{\sim}{\mathfrak{M}}\ulcorner G\urcorner \cap \underset{\sim}{\mathfrak{M}}\ulcorner \Psi\urcorner$$

The above diagram commutes iff for any $a \in \mathfrak{M}(\bar{F})$ and $b = \mathfrak{M}(F\Psi G)(a)$, $(\psi_{\bar{F}}(a), \psi_{\bar{G}}(a)) \in \underset{\sim}{\mathfrak{M}}\ulcorner \Psi\urcorner$.

To show the last, notice that $\bar{F}$ and $\bar{\Psi}$ are isomorphic (since $\ulcorner F\urcorner$ and $\ulcorner \Psi\urcorner$ are isomorphic for any $\mathfrak{A} \in K_T$) and that the following diagram commutes in $Th(T, H)$ (since its image commutes for any $\mathfrak{A} \in K_T$)

$$\begin{tikzcd}
\overline{F} \arrow[rr, "F\Psi G"] \arrow[dd, bend right, "Fi(n)1_n"'] & & \overline{G} \arrow[dd, "Gi(m)1_m"] \\
\overline{\Psi} \arrow[u, "\Psi(\underline{n},m,n)\equiv(n,m,\underline{n})F"'] \arrow[d, "\Psi(\underline{n},m,n)\equiv(n,m,\underline{n})1_n"'] \arrow[drr, "\Psi(n,\underline{m},m)\equiv(n,m,\underline{m})1_m"] & & \\
\overline{1}_n & & \overline{1}_m
\end{tikzcd}$$

where $\Psi(\underline{n}, m, n) \equiv (n, m, \underline{n})F$ is an isomorphism. Therefore the following diagram commutes in sets:

$$\begin{tikzcd}
\mathfrak{M}(\overline{F}) \arrow[rr, "\mathfrak{M}(F\Psi G)"] \arrow[dd, bend right, "\mathfrak{M}(Fi(n)1_n)"'] & & \mathfrak{M}(\overline{G}) \arrow[dd, "\mathfrak{M}(Gi(m)1_m)"] \\
\mathfrak{M}(\overline{\Psi}) \arrow[u, "\mathfrak{M}(\Psi(\underline{n},m,n)\equiv(n,m,\underline{n})F)"'] \arrow[d, "\mathfrak{M}(\Psi(\underline{n},m,n)\equiv(n,m,\underline{n})1_n)"'] \arrow[drr, "\mathfrak{M}(\Psi(n,\underline{m},m)\equiv(n,m,\underline{m})1_m"] & & \\
\mathfrak{M}(\overline{1}_n) & & \mathfrak{M}(\overline{1}_m)
\end{tikzcd}$$

Let $a \in \mathfrak{M}(\overline{F})$ and $b = \mathfrak{M}(F\Psi G)(a)$, then there is $a' \in \mathfrak{M}(\overline{\Psi})$ such that

$a = \mathfrak{M}(\Psi(\underline{n}, m, n) \equiv (n, m, \underline{n})F)(a)$ and

$(\psi_{\overline{F}}(a), \psi_{\overline{G}}(b)) = (j_n' \circ \mathfrak{M}(Fi(n)1_n)(a), j_m \circ \mathfrak{M}(Gi(m)1_m)(b)) = (j_n \circ \mathfrak{M}(\Psi(\underline{n}, m, n) \equiv (n, m, \underline{n})1_n)(a'), j_m \circ \mathfrak{M}(\Psi(n, \underline{m}, m) \equiv (n, m, \underline{m})1_m)(a'))$.

But $\mathfrak{M}(\Psi(\underline{n}, m, n) \equiv (n, m, \underline{n})1_n) = \mathfrak{M}(\underline{p}_1^{n+m}, \ldots, \underline{p}_n^{n+m}) \circ \mathfrak{M}(\Psi\ i(n+m)\ 1_{n+m})$, and

$\mathfrak{M}(\Psi(n, \underline{m}, m) \equiv (n, m, \underline{m})1_m) =$

$\mathfrak{M}(\underline{p^{n+m}_{n+1}}, \ldots, \underline{p^{n+m}_{n+m}}) \circ \mathfrak{M}(\Psi\ i(n+m)1_{n+m})$, so

$(\psi_{\overline{F}}(a), \psi_{\overline{G}}(b)) = (j_n \circ \mathfrak{M}(\underline{p^{n+m}_1}, \ldots, \underline{p^{n+m}_n}),$

$j_m \circ \mathfrak{M}(\underline{p^{n+m}_{n+1}}, \ldots, \underline{p^{n+m}_{n+m}}))(\mathfrak{M}(\Psi\ i(n+m)1_{n+m})(a')) =$

$(j_{n+m} \circ \mathfrak{M}(\Psi\ i(n+m)1_{n+m}))(a') \in$

$j_{n+m} \circ \mathfrak{M}(\Psi\ i(n+m)1_{n+m})(\mathfrak{M}(\overline{\Psi})) = \underset{\sim}{\mathfrak{M}}\ulcorner\Psi\urcorner$

Thus

$(\psi_{\overline{F}}(a), \psi_{\overline{G}}(b)) \in \underset{\sim}{\mathfrak{M}}\ulcorner\Psi\urcorner$ □.

12. Theorem

Let $Sets^{(Th(T, H_i))}$ denote the full subcategory of $Sets^{Th(T, H_i)}$ consisting of those finite limit preserving functors which satisfy condition (i), then

$Sets^{(Th(T, H_i))}$ is equivalent to

$K_{T'}(H_i)$ where $T' = Cn(T) \cap B(H)$

Proof

1. By Proposition 9.8

$(\widetilde{\ \ }): K_{T'}(H_i) \to Mod(T', H_i)$

is an isomorphism, and $Mod(T', H_i)$ is a full subcategory of

$$Sets^{Th(T', H_i)} = Sets^{Th(T, H_i)}.$$

2. By Propositions 10.8, 10.10, 10.11 and 10.14, for any $\mathfrak{M} \in Mod(T', H_i)$; $\mathfrak{M} \in Sets^{(Th(T', H_i))}$ so

$Mod(T', H_i)$ is full subcategory of $Sets^{(Th(T, H_i))}$.

3. By Proposition 11.3(b) $Mod(T', H_i)$ is representative in the category $Sets^{(Th(T, H_i))}$. From 1, 2, and 3 follows that

$(\widetilde{\ \ }): K_{T'}(H_i) \to \mathrm{Sets}^{(Th(T,\ H_i))}$

is full, faithful and representative and therefore (cf. [3])

$K_{T'}(H_i)$ is equivalent to $\mathrm{Sets}^{(Th(T,\ H_i))}$ $\square$.

12.1 Corollary

Let $\mathrm{Sets}^{(Th(T,\ H))_{V_H}}$ denote the full subcategory of $\mathrm{Sets}^{Th(T,\ H)}$, whose objects are all $\mathfrak{M} \circ V_H$, where $\mathfrak{M} \in \mathrm{Sets}^{(Th(T,\ F))}$, then $\mathrm{Sets}^{(Th(T,\ H))_{V_H}}$ is equivalent to $K_T(H)$.

Proof

1. $\mathrm{Mod}(T,\ H)$ is a full subcategory of $\mathrm{Sets}^{(Th(T,\ H))_{V_H}}$, because:

   a) By Proposition 9.10 and Theorem 12 if $\mathfrak{M} \in \mathrm{Mod}(T,\ H)$ then

   $$\mathfrak{M} = \mathfrak{N} \circ V_H \text{ for some } \mathfrak{N} \in \mathrm{Sets}^{(Th(T,\ F))}$$

   b) $\mathrm{Mod}(T,\ H)$ is a full subcategory of $\mathrm{Sets}^{Th(T,\ H)}$.

2. $\mathrm{Mod}(T,\ H)$ is representative in $\mathrm{Sets}^{(Th(T,\ H))_{V_H}}$ because if $\mathfrak{M} \in \mathrm{Sets}^{(Th(T,\ F))}$, then $\mathfrak{M}$ is isomorphic to $(\widetilde{\mathfrak{M}})^{(T,\ F)}$ and since V_H is an inclusion $\mathfrak{M} \circ V_H$ is isomorphic to $(\widetilde{\mathfrak{M}})^{(T,\ F)} \circ V_H = (\widetilde{\mathfrak{M}})^{(T,\ H)} \in \mathrm{Mod}(T,\ H)$.

3. $(\widetilde{\ \ }): K_T(H) \to \mathrm{Mod}(T,\ H)$ is an isomorphism.

From 1, 2 and 3 follows that

$(\widetilde{\ \ }): K_T(H) \to \mathrm{Sets}^{(Th(T,\ H))_{V_H}}$

is a full, faithful and representative and therefore (cf. [3])

$$K_T \text{ is equivalent to } \mathrm{Sets}^{(Th(T,\ H))_{V_H}} \quad \square.$$

12.2 Corollary

The category $K_{T'}(H_0)$ has finite products which are preserved by the embeding

$$(\tilde{\ \ }): K_{T'}(H_0) \to \mathrm{Sets}^{Th(T,\ H_0)}$$

where as before $T' = Cn(T) \cap B(H_0)$.

Proof

Let $\mathfrak{M}_i \in \mathrm{Sets}^{(Th(T,\ H_0))}$ $\quad i = 1, 2.$

Define the functor

$$P: Th(T,\ H_0) \to \mathrm{Sets}, \text{ such that}$$

$P(\bar{F}) = \mathfrak{M}_1(\bar{F}) \times \mathfrak{M}_2(\bar{F})$ for $F \in H_0$, and

$P(F\Phi G) = \mathfrak{M}_1(F\Phi G) \times \mathfrak{M}_2(F\Phi G)$ for $F\Phi G \in \mathrm{Mor}(Th(T,\ H_0))$

then P is a product of functors $\mathfrak{M}_1$ and $\mathfrak{M}_2$ in $\mathrm{Sets}^{Th(T,\ H_0)}$.

We will show that

$$P \in \mathrm{Sets}^{(Th(T,\ H_0))}$$

1. P preserves finite limits. Suppose that $D: J \to Th(T,\ H_0)$ is a finite diagram in $Th(T,\ H_0)$, then

$$P(\lim_{j\in J} D_j) = (\lim_{i=1,2} \mathfrak{M}_i)(\lim_{j\in J} D_j) = \lim_{i=1,2} (\mathfrak{M}_i(\lim_{j\in J} D_j)) =$$

$$\lim_{i=1,2} (\lim_{j\in J} \mathfrak{M}_i(D_j)).$$

Since $\lim_{i=1,2} (\mathfrak{M}_i(D_j))$ exists for any $j \in \mathrm{Obj}(J)$ therefore

$$\lim_{i=1,2} (\lim_{j\in J} \mathfrak{M}_i(D_j)) = \lim_{j\in J}(\lim_{i=1,2} \mathfrak{M}_i(D_j)).$$ (cf. [5]), so

$$P(\lim_{j\in J} D_j) = \lim_{j\in J}(\lim_{i=1,2} \mathfrak{M}_i(D_j)) = \lim_{j\in J}(\lim_{i\in 1,2} \mathfrak{M}_i)(D_j) = \lim_{j\in J} P(D_j).$$

2. P preserves coequilizers of kernel pairs. In the category of sets a diagram

$$A \overset{f}{\underset{g}{\rightrightarrows}} B \xrightarrow{h} C$$

is a coequilizer of a kernel pair iff it satisfies the following:

1. $h(f(x)) = h(g(x))$ for all $x \in A$
2. if $h(y) = h(y')$ for some $y, y' \in B$, then there is $x \in A$, such that $f(x) = y$ and $g(x) = y'$.
3. if $f(x) = f(x')$ and $g(x) = g(x')$ for $x, x' \in A$ then $x = x'$
4. if $z \in C$ then there is $y \in B$, such that $z = h(y)$.

Now let

$$\bar{F} \overset{f}{\underset{g}{\rightrightarrows}} \bar{G} \xrightarrow{h} \bar{H} \quad \text{be}$$

a coequilizer of kernel pair in $Th(T, H_0)$, then

$$\mathfrak{M}_i(\bar{F}) \overset{\mathfrak{M}_i(f)}{\underset{\mathfrak{M}_i(g)}{\rightrightarrows}} \mathfrak{M}_i(\bar{G}) \xrightarrow{\mathfrak{M}_i(h)} \mathfrak{M}_i(\bar{H})$$

$i = 1, 2$

is a coequilizer of a kernel pair in Sets.

We will show that the diagram

$$(*) \qquad P(\overline{F}) \underset{P(g)}{\overset{P(f)}{\rightrightarrows}} P(\overline{G}) \xrightarrow{P(h)} P(\overline{H})$$

satisfies the conditions 1, 2, 3 and 4.

1. Let $(x_1, x_2) \in P(\overline{F})$, then

   $P(h)(P(f)(x_1, x_2)) = (\mathfrak{M}_1(h)(\mathfrak{M}_1(f)(x_1)),$

   $\mathfrak{M}_2(h)(\mathfrak{M}_2(f)(x_2))) = (\mathfrak{M}_1(h)(\mathfrak{M}_1(g)(x_1)),$

   $\mathfrak{M}_2(h)(\mathfrak{M}_2(g)(x_2))) = P(h)(P(g)(x_1, x_2)).$

2. Suppose that

   $P(h)(y_1, y_2) = P(h)(y_1', y_2')$ for some

   $(y_1, y_2), (y_1', y_2') \in P(\overline{G})$, then

   $\mathfrak{M}_1(h)(y_1) = \mathfrak{M}_1(h)(y_1')$ and $\mathfrak{M}_2(h)(y_2) = \mathfrak{M}_2(h)(y_2')$,

   so there are $x_1 \in \mathfrak{M}_1(\overline{F})$ and $x_2 \in \mathfrak{M}_2(\overline{F})$

   such that

   $\mathfrak{M}_1(f)(x_1) = y_1, \quad \mathfrak{M}_2(f)(x_2) = y_2$

   $\mathfrak{M}_1(g)(x_1) = y_1', \quad \mathfrak{M}_2(g)(x_2) = y_2'$, so

   $(x_1, x_2) \in P(\overline{F})$ and

   $P(f)(x_1, x_2) = (y_1, y_2)$

   $P(g)(x_1, x_2) = (y_1', y_2')$.

3. Suppose that $(z_1, z_2) \in P(\overline{H})$, then

   $z_1 \in \mathfrak{M}_1(\overline{H})$ and $z_2 \in \mathfrak{M}_2(\overline{H})$, so there are

   $y_1 \in \mathfrak{M}_1(\overline{G})$ and $y_2 \in \mathfrak{M}_2(\overline{G})$,

   such that

   $z_1 = \mathfrak{M}_1(h)(y_1)$ and $z_2 = \mathfrak{M}_2(h)(y_2)$, so

   $(y_1, y_2) \in P(\overline{G})$ and $(z_1, z_2) = P(h)(y_1, y_2)$.

4. Straightforward.

   Thus (*) is a coequilizer of kernel pair.

From 1. and 2. follows that $P \in \mathrm{Sets}^{(\mathrm{Th}(T,\ H_0))}$. Thus the category $\mathrm{Sets}^{(\mathrm{Th}(T,\ H_0))}$ has finite products which are preserved by inclusion in $\mathrm{Sets}^{(\mathrm{Th}(T,\ H_0)}$. Since $\mathrm{Sets}^{(\mathrm{Th}(T,\ H_0))}$ is equivalent to $K_{T'}(H_0)$, it follows that $K_{T'}(H_0)$ has finite products which are preserved by the embeding

$$(\widetilde{\ \ }): K_{T'}(H_0) \to \mathrm{Sets}^{\mathrm{Th}(T,\ H_0)} \qquad \square$$

12.3 Corollary

The category $K_{T'}(H_i)$ $i = 0, 1, 2$, has filtered colimits, which are preserved by the embeding

$$(\widetilde{\ \ }): K_{T'}(H_i) \to \mathrm{Sets}^{\mathrm{Th}(T, H_i)}$$

where as before $T' = Cn(T) \cap B(H_i)$.

Proof

Let $\{\mathfrak{M}_k\}_{k \in I}$ be a filtered diagram in $\mathrm{Sets}^{(\mathrm{Th}(T,\ H_i))}$ and let $C = \operatorname{colim}_{k \in I} \mathfrak{M}_k$ in $\mathrm{Sets}^{\mathrm{Th}(T,\ H_i)}$. Let $\{D_j\}_{j \in J}$ be a finite diagram in $\mathrm{Th}(T,\ H_i)$, then

I 1. $\lim_{j \in J} D_j$ exists

2. $\mathfrak{M}_k(\lim_{j \in J} D_j) = \lim_{j \in J} \mathfrak{M}_k(D_j)$ for all $k \in \mathrm{Obj}(I)$

Thus

$$C(\lim_{j \in J} D_j) = \operatorname{colim}_{k \in I} \mathfrak{M}_k(\lim_{j \in J} D_j) = \operatorname{colim}_{\epsilon I}(\mathfrak{M}_k(\lim_{j \in J} D_j)) =$$

$$\operatorname{colim}_{k \in I}(\lim_{j \in J} \mathfrak{M}_k(D_j)).$$

Since $\{D_j\}_{j \in J}$ is finite and $\{\mathfrak{M}_k\}_{k \in I}$ is filtered therefore

$$\operatorname{colim}_{k \in I}(\lim_{j \in J} \mathfrak{M}_k(D_j)) = \lim_{j \in J}(\operatorname{colim}_{k \in I} \mathfrak{M}_k(D_j)) \quad \text{(cf. [5]), so}$$

$C(\lim_{j\in J} D_j) = \lim_{j\in J}(\operatorname{colim}_{k\in I} \mathfrak{M}_k(D_j)) = \lim_{j\in J}(\operatorname{colim}_{k\in I} \mathfrak{M}_k)(D_j) =$

$\lim_{j\in J} C(D_j)$, so C preserves finite limits.

II 1. Suppose that $\operatorname{colim}_{j\in J} D_j$ exists

2. Suppose that $\mathfrak{M}_k(\operatorname{colim}_{j\in J} D_j) \cong \operatorname{colim}_{j\in J} \mathfrak{M}_k(D_j)$

for all $k \in \mathrm{Obj}(I)$

then

$C(\operatorname{colim}_{j\in J} D_j) = \operatorname{colim}_{k\in I} \mathfrak{M}_k(\operatorname{colim}_{j\in J} D_j) = \operatorname{colim}_{k\in I}(\mathfrak{M}_k(\operatorname{colim} D_j)) =$

$\operatorname{colim}_{k\in I}(\operatorname{colim}_{j\in J} \mathfrak{M}_k(D_j))$.

Since $\{D_j\}_{j\in J}$ is finite and $\{\mathfrak{M}_k\}_{k\in I}$ is filtered

therefore

$\operatorname{colim}_{k\in I}(\operatorname{colim}_{j\in J} \mathfrak{M}_k(D_j)) = \operatorname{colim}_{j\in J}(\operatorname{colim}_{k\in I} \mathfrak{M}_k(D_j))$

(cf. [5]), so

$C(\operatorname{colim}_{j\in J} D_j) = \operatorname{colim}_{j\in J}(\operatorname{colim}_{k\in I} \mathfrak{M}_k(D_j)) =$

$\operatorname{colim}_{j\in J}(\operatorname{colim} \mathfrak{M}_k)(D_j) = \operatorname{colim}_{j\in J} C(D_j)$, so

C preserves all finite colimites, which are preserved

by $\mathfrak{M}_k$ $k\in I$.

From I and II it follows that

$$C \in \mathrm{Sets}^{(\mathrm{Th}(T,\, H_i))}$$

Thus the category $\mathrm{Sets}^{(\mathrm{Th}(T,\, H_i))}$ has filtered coproducts, which are preserved by the inclusion in $\mathrm{Sets}^{\mathrm{Th}(T,\, H_i)}$. Since $\mathrm{Sets}^{(\mathrm{Th}(T,\, H_i))}$ is equivalent to $K_{T'}(H_i)$, it follows that $K_{T'}(H_i)$ has filtered colimits which are preserved by the embeding

$$(\widetilde{\ \ }) : K_{T'}(H_i) \to \mathrm{Sets}^{\mathrm{Th}(T', H_i)} \quad \square.$$

13. Example: Small categories

Let Σ^c be a relational type such that $\Sigma_0^c(1) = \{d, c\}$, $\Sigma_0^c(n) = \emptyset$ for $n \neq 1$, $\Sigma_1^c(0) = \{1\}$, $\Sigma_1^c(3) = P$ and $\Sigma_1^c(n) = \emptyset$ for $n \neq 0$ and $n \neq 3$. The symbols d and c interpreted as unary functions, such that for a morphism x, d(x) is the identity morphism on the domain of x and c(x) is the identity morphism on the codomain of x. The symbol P is interpreted as a three argument predicate, such that for morphisms x, y, z, P(x, y, z) holds iff the composition of x and y equals z.

There is a theory $T^c \subset B(H_0)$ such that $K_{T^c}(H_0)$ is the category of all small categories. For example T^c can consist of the following axioms (the formulas on the right hand side are rewritten in the usual first order language):

1. $c(p_1^2) \equiv d(p_2^2) \leftrightarrow \exists_{p_1^3 p_2^3}(P)$, $\quad c(x) \equiv d(y) \leftrightarrow \exists_z P(x, y, z)$
2. $P(\underline{3}, 1) \wedge P(\underline{2}, 1, \underline{1}) \to p_3^4 \equiv p_4^4$, $\quad P(x, y, z) \wedge P(x, y, w) \to z \equiv$
3. $P(d, p_1^1, p_1^1)$, $\quad P(d(x), x, x)$
4. $P(p_1^1, c, p_1^1)$, $\quad P(x, c(x), x)$
5. $c(c) \equiv c$, $\quad c(c(x)) \equiv c(x)$
6. $d(d) \equiv d$, $\quad d(d(x)) \equiv d(x)$

Thus $\mathrm{Th}(T^c, H_0)$ is the theory category for small categories, structures corresponding to set valued functors from

$Th(T^c, H_0)$ which preserve finite products and coequilizers of kernel pairs are small categories.

Now let R denote the following formula:

$$d(p_1^2) \equiv d(p_2^2) \wedge \neg\exists_{(\underline{2},\, 2)}(\neg(c(p_1^4) \equiv c(p_3^4) \wedge c(p_2^4) \equiv c(p_4^4) \wedge$$
$$d(p_3^4) \equiv d(p_4^4) \to \exists_{(\underline{4},\, 1)}(P(p_5^5, p_1^5, p_3^5) \wedge P(p_5^5, p_2^5, p_4^5)))) \wedge$$
$$\neg\exists_{(\underline{2},\, 4)}(\neg(P(p_5^6, p_1^6, p_3^6) \wedge (P(p_1^6, p_2^6, p_4^6) \wedge$$
$$P(p_6^6, p_1^6, p_3^6) \wedge P(p_6^6, p_2^6, p_4^6) \to p_5^6 \equiv p_6^6))$$

In the usual language $R(x, y)$ has the translation

$$d(x) \equiv d(y) \wedge \forall_{x',\, y'}[c(x) \equiv c(x') \wedge c(y) \equiv c(y') \wedge d(x') \equiv d(y')$$
$$\to \exists_u(P(u, x, x') \wedge P(u, y, y')] \wedge$$
$$\forall_{u,\, u',\, x',\, y'}[P(u, x, x') \wedge P(u, y, y') \wedge P(u', x, x') \wedge$$
$$P(u', y, y') \to u \equiv u']$$

(It is easy to see that the formula $R(x, y)$ has the interpretation: $d(x)$ is a product of $c(x)$ and $c(y)$ and x and y are projections.)

Let H be the closure of $H_0 \cup \{R\}$ under $\wedge$, $\exists$ and substitution, then $\mathrm{Sets}^{(Th(T^c,\, H))_{V_H}}$ is equivalent to the category of small categories and product preserving functors. This follows from Corollary 12.1 $\square$.

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