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RELIABILITY PREDICTION TECHNIQUES

by

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ABSTRACT

This thesis considers methods for predicting reliability and mean time to first failure of complex systems. The need for these methods, is a result of recent advances in pure materials technology and fabrication procedures, that potentially lead to more reliable systems. In this thesis, two types of complex systems are considered.

Chapter I, treats systems, with k independent parameters, each exhibiting degradation failure. A model is developed for predicting, the reliability and mean life of such systems, from data taken early in life tests. Therefore saving many hours of test time; and eliminating the need of testing to failure.

Chapter II, treats complex systems composed of independent systems, each exhibiting constant failure and repair rates. From the verbal description of a complex system, a Markov model is developed, from which predictions of reliability, availability and mean time to first failure, are made. This model also allows for the inclusion of a random waiting time between failure and beginning of repair.

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CHAPTER I

LIST OF SYMBOLS

$\{\quad\}_k$	Set of k elements
$\cap$	Set intersection operation
$\cup$	Set union operation
$\in$	Inclusion operation
$\subset$	Proper subset operation
$\forall$	Quantifier, meaning "for all".
$\{X^i\}_k^1$	Set of k independent system parameter A, i=1, 2, .. k.
$\{X^i(t)\}_k$	Set of k independent Brownian processes.
$x_2^i - x_1^i$	Region of acceptance of parameter X^i.
C^i	Drift coefficient for parameter X^i.
D^i	Diffusion coefficient for parameter X^i.
$R_i(t)$	Reliability function for parameter X^i.
$R(t) = \prod_{i=1}^{k} R_i(t)$	Reliability function for k parameter system
t_L^i	Mean time for which parameter X^i, has values in the range $x_2^i - x_1^i$.
$t_L - \inf \{t_L^i\}_k$	Mean life time for k parameter system

LIST OF SYMBOLS CONTINUED--

$p_{X^i}(x^i, t)$	Joint first order density function of $X^i(t)$ at time $t = t$.
$p_{X^i}(x_o^i / x^i; t)$	Conditional density function for $X^i(t)$, at time $t = t$.
$p_{\{X^i(t)\}}(x^1; x^2; .. x^k; t)$	Joint first order density function for $\{X^i(t)\}_k$, at time $t = t$.
$(\sigma_t^i)^2$	Variance about the mean m_t^i, of parameter X^i, at time t.
$(m_t)^i$	Expected, mean value of parameter X^i, at time t=t.
Erf (y)	Error function integral of argument y, That is $\text{Erf}(y) = \frac{1}{\sqrt{2\pi}} \int_o^y e^{-x^2} dx$.
L	Two sided Laplace transform operator
$\frac{\partial}{\partial x}$	Partial differential operator, with respect to x.

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[1] In the case where a single parameter system is being considered, the subscripts and superscripts in i; are dropped. That is, when k = 1, we write X^i as X, and $R_i(t)$ as R(t), etc. .

CHAPTER I

INTRODUCTION

In general system failures are classified either as random failures or degradation failures. This chapter treats systems, that have a set $\{X^i\}_k$, of k, independent parameters, with each parameter in this set, independently exhibiting degradation failure. The objective of this chapter, is to construct, for such systems a mathematical model, from which an overall reliability function and estimates of the mean lifetime, can be obtained.

In the development of the overall reliability function, R(t), it is shown that if the parameters of the system are independent, then $R(t) = \prod_{i=1}^{k} R_i(t)$, where $R_i(t)$, the reliability function for parameter X^i, gives a measure of the probability at time t, that parameter X^i has a value that as in some pre-assigned region, called the region of acceptance.

A method is developed, for computing for a given set $\{R(t_L^i) = R_i\}_k$, the set of times $\{t_L^i\}_k$, after which the parameters $\{X^i\}_k$, have values outside their respective acceptance regions $\{x_2^i - x_1^i\}_k$. The mean lifetime, that is, the mean time to first failure, of the complex system is then defined as the minimum time $t_L \in \{t_L^i\}_k$.

It is assumed that the fluctuations in each parameter's value is characterizable by a Brownian process. This assumption is reasonable, if one considers parameter fuctuations, as a "macroscopic" manifestation of internal "microscopic" activity. From a mathematical point of view, a Brownian process, is a type IV, stationary Markov process with Gaussian statistics. The transitional probability density function,

of such a process, completely characterizes it, and is obtained, as the unique solution of a linear partial differential equation of the Fokker - Planck type. As a consequence of stationarity, time and ensemble methods of averaging are equivalent, for an ensemble, (sample), of identical parameters having the same nominal values.

The above assumptions allow a closed form mathematical model to be constructed. From this model, predictions of reliability and mean lifetime, can then be made.

THEORETICAL DEVELOPMENT FOR SINGLE PARAMETER SYSTEMS

We begin by considering a system with a single parameter, denoted by X. The value of X, at any time, is some point on the real line. Let X(t) be the stochastic process characterizing the time behaviour of X. It is the Author's thesis, that X(t), can be treated as a stationary Markov process with Gaussian statistics.

The Joint second order density function $p_X(x_o : x; t)$, completely characterizes X(t), and can be written as the product of two density functions.

$$p_X(x_o : x;t) = p_X(x_o/x;t)\, p_X(x_o) \tag{1}$$

where $p_X(x_o/x;t)$ is the conditional density function of the random variable x(t), at t = t, and $p_X(x_o)$ is the first order density distribution function for the random variable X(t) at t = 0.

Since the random variables X(t), $\forall$ t, are assumed to be Gaussianly distributed, we can write for X(o) ,

$$p_X(x_o) = \sigma_o \sqrt{\frac{1}{2\pi}}\; e^{-\frac{(x_o - m_o)^2}{2\sigma_o^2}} \tag{2}$$

where m_o and σ_o^2 , are respectively the mean and variance of the random variable X(t), at time t = 0.

In Appendix A, it is shown that the conditional (transitional) density function, $p_X(x_o/x;t)$, is the unique solution of the linear partial differential equation, of the Fokker-Planck type,

$$D\, \partial^2 \frac{\{p_X(x_o/x;t)\}}{\partial x^2} + 2C\, \partial \frac{\{p_X(x_o/x;t)\}}{\partial x} = \partial \frac{\{p_X(x_o/x;t)\}}{\partial t}$$

and is given as,

$$p_X(x_o/x;t) = \frac{1}{\sqrt{2\pi 2Dt}} e^{-\frac{(x-x_o-2Ct)^2}{2\,2Dt}} \tag{3}$$

The first order density function $p_X(x;t)$, for the random variable X(t), at time t = t, is obtained from (1), by integration

$$p_X(x,t) = \int_{-\infty}^{\infty} p_X(x_o : x;t)\, dx_o$$

$$= \int_{\infty}^{\infty} p_X(x_o/x;t)\, p_X(x_o) dx_o \tag{4}$$

substituting (2) and (3) into (4), we obtain

$$p_X(x,t) = \frac{1}{\sqrt{2\pi(\sigma_o^2 + 2Dt)}} e^{-\frac{(x-m_o-2Ct)^2}{2(\sigma_o^2+2Dt)}} \tag{5}$$

therefore the random variable X(t), at time t = t, is again Gaussianly distributed. This follows from the fact that the transformation (4), is linear. This is shown schematically in Figure 1.

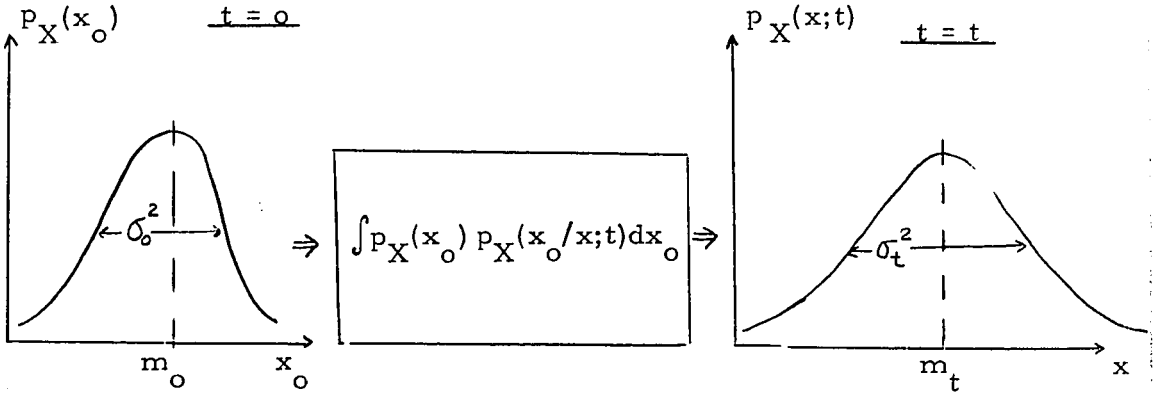


Figure 1.

Now (5), may be written as

$$p_X(x;t) = \frac{1}{\sigma_t\sqrt{2\pi}} e^{-\frac{(x-m_t)^2}{2\pi_t^2}} \tag{6}$$

where m_t and σ_t^2, are respectively, the mean and variance of the random variable X(t), at time t=t. and are given as

$$\begin{aligned} m_t &= m_o + 2Ct \\ \sigma_t^2 &= \sigma_o^2 + 2Dt \end{aligned} \tag{7}$$

C and D are constants, called the drift and diffusion coefficients respectively; as seen from (7), they give a measure of the rate at which the mean and variance change with time.

The probability R(t), that at time t = t, the parameter X, has a value that is in the acceptable region x_2-x_1, of the real line, is called the reliability function for the parameter X. That is

$$R(t) = P\{x_1 \leq X(t) \leq x_2\} \tag{8}$$

is called the reliability function of parameter X. Then from (5), we have

$$R(t) = \int_{x_1}^{x_2} p_X(x;t)dx \tag{9}$$

and as shown in Appendix A, on substituting (6) in (9), we obtain

$$R(t) = \text{Erf}\left(\frac{x_2-m_t}{\sigma_t}\right) - \text{Erf}\left(\frac{x_1-m_t}{\sigma_t}\right) \tag{10}$$

where Erf(y), is the error function integral of argument y, and m_t and σ_t are obtained from (7).

EXTENSION TO k - PARAMETER SYSTEMS

The following definitions now extend the techniques developed above, to systems with k-independent parameters.

Definition k-Brownian System. A k-Brownian system, is a system with a set $\{X^i\}_k$, of k independent parameters, where the time behaviour of each parameter in this set, is characterizable by a Brownian process. That is the k-Brownian system is characterized by a set $\{X^i(t)\}_k$, of k independent Brownian processes.

Since the processes are independent, the Joint first order density function, for the set of k random variables, $\{X^i(t)\}_k$, at time t=t, is

$$p_{\{X^i(t)\}_k}(x^1;x^2;\ldots;x^k;t) \equiv P\{\bigcap_{j=1}^{k}(x^j \le X^j(t) \le x^j + dx^j)\}$$

Because of the independence of the processes, the above eqn. can be written as

$$P\{\bigcap_{j=1}^{k}(x^j \le X^j(t) \le x^j+dx^j)\} = \bigcap_{j=1}^{k} P\{x^j \le X^j(t) \le x^h+dx^j\}$$

or

$$p_{\{X^i(t)\}_k}(x^1;x^2;\ldots;x^k;t) = \prod_{i=1}^{k} p_{X^i}(x^i;t). \qquad (11)$$

where each

$$p_{X^i}(x^i;t)dx^i \equiv P\{x^i \le X^i(t) \le x^i + dx^i\}, \quad 1 = 1,2,\ldots,k$$

is of the form given in (6), viz:,

$$p_{X^i}(x^i;t)dx^i = \frac{1}{\sigma_t^i\sqrt{2\pi}} e^{\frac{(x-m_t^i)^2}{2(\sigma_2^i)^2}} \qquad (12)$$

and m_t^i, $(\sigma_t^i)^2$, are respectively the mean and variance of the random variable $X^i(t)$, at time t = t.

Definition Acceptable region. The acceptable region for parameter X^i, i=1, 2, . . . , k; denoted by $[x_1^i, x_2^i]$, is some closed interval of the real line, $(x_2^i - x_1^i)$.

Definition Reliability. The reliability, R(t), of a k-Brownian system is the probability, that at time t = t, the set of k-random variables, $\{X^i(t)\}_k$ have values that lie respectively in $\{x_2^i - x_1^i\}_k$.

From the above definition of reliability, R(t), is expressed as

$$R(t) = P\{\bigcap_{j=1}^{k} (x_1^j \leq X^j(t) \leq x_2^j)\}$$

that is

$$R(t) = P\{x_1^1 \leq X^1(t) \leq x_2^1 : x_1^2 \leq X^2(t) \leq x_2^2 : \ldots : x_1^k \leq X^k(t) \leq X_2^k\}$$

or

$$R(t) = \int_{x_1^1}^{x_2^1} \int_{x_1^2}^{x_2^2} \cdots \int_{x_1^k}^{x_2^k} p_{\{X^i(t)\}_k}(x^1; x^2; , , , x^k; t)\, dx^1 dx^2 \ldots dx^k \ldots \quad (13)$$

by (11), we can write (13) as

$$R(t) = \prod_{i=1}^{k} \int_{x_1^i}^{x_2^i} p_{X^i}(x^i; t) dx^i \quad (14)$$

$$= \prod_{i=1}^{k} R_i(t) \quad (15)$$

where $R_i(t)$, is the reliability function for the single parameter X^i, which is given by (10) as

$$R_i(t) = \mathrm{Erf}\left(\frac{x_2^1 - m_t^i}{\sigma_t^i}\right) - \mathrm{Erf}\left(\frac{x_1^i - m_t^i}{\sigma_t^i}\right) \quad (16)$$

then substituting (16) into (15), yields,

$$R(t) = \prod_{i=1}^{k} \left\{ \text{Erf}: \left(\frac{x_2^i - m_t^i}{\sigma_t^i} \right) - \text{Erf} \left(\frac{x_1^i - m_t^i}{\sigma_t^i} \right) \right\} \tag{17}$$

or using equation (7),

$$R(t) = \prod_{i=1}^{k} \left\{ \text{Erf}: \left(\frac{x_2^i - m_o^i - 2C^i t}{\sqrt{(\sigma_o^i)^2 + 2D^i t}} \right) - \text{Erf} \left(\frac{x_1^i - m_o^i - 2C^i t}{\sqrt{(\sigma_o^i)^2 + 2D^i t}} \right) \right. \tag{18}$$

For example, when $\{x_1^i = -\infty\}_k$, equation (17) becomes

$$R(t) = \prod_{i=1}^{k} \left\{ \frac{1}{2} + \text{Erf} \left(\frac{x_2^i - m_t^i}{\sigma_t^i} \right) \right\} \tag{19}$$

Equation (18), is the reliability function for a k-Brownian system.

For given sets, $\{R_i(t_L^i) = R_i\}_k$ and $\{x_2^i - x_1^i\}_k$, (16), can be solved for a set of k times, $\{t_L^i\}_k$; thus enabling an estimate of the mean lifetime t_L, of the k-Brownian system to be made.

Definition Mean Lifetime of a k-Brownian System.

The mean lifetime t_L, of a k-Brownian system is the smallest time $t_L \in \{t_L^i\}_k$. where the set $\{t_L^i\}_k$, is obtained by solving the set of equations (16), for arbitrary, but specified $\{R_i(t_L^i)=R_i\}_k$ and $\{x_2^i - x_1^i\}_k$. That is

$$t_L = \inf \{t_L^i\}_k \tag{20}$$

is defined as the mean lifetime of a k-Brownian system.

PRACTICAL REALIZATION OF THE MODEL

In this section we wish to show how the reliability function equations (16) or (17), can be constructed from test data.

In order to obtain the reliability function of the form given in (17), the constants m_t^i and σ_t^i, $i = 1, 2, \ldots k$; have to be determined. It is seen in (7), that m_t^i and $(\sigma^i{}_t)^2$, are linear functions in t. That is, the set of equations, of the form of equation (7), are uniquely specified once m_t^i, $(\sigma_t^i)^2$, are known for $i = 1, 2, \ldots, k$, at two fixed points in time. Then the first step in the procedure, is to determine $m^i{}_t$ and $\sigma^i{}_t$, at two arbitrary faced instances of time; for example at $t = o$, and at $t = t_1$. Therefore, $(m^i{}_o, \sigma^i{}_o)$ and $(m_t^i, \sigma^i{}_t)$, can be obtained by making two cumulative plots for each parameter X^i, $i = 1, 2 \ldots, k$; one at $t = o$, the other at $t = t_1$.

To illustrate the technique of making these plots, let us assume, that we have a sample consisting of N, k-Brownian systems. Then select a parameter X^i, (where X^i could be, for example, a resistance, etc.). At any instant of time, X^i can assume values in some interval of the real line. Let x^i, be a sufficiently large interval, as to include the range of the random variables $X^i(t)$, $\forall$ t. Then make a plot at time t=o, of the fraction n/N, versus x^i, of the systems, whose parameter X^i, has a value that is less than or equal to x^i. By assumption, this data will be normally distributed, and well yield a graph as shown in Fig. (2).

Cumulative Plot at Time (t=o).

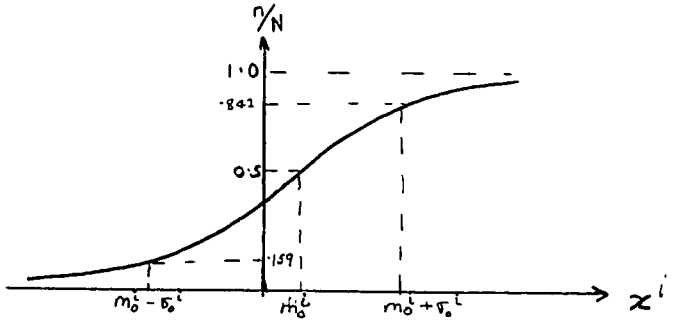
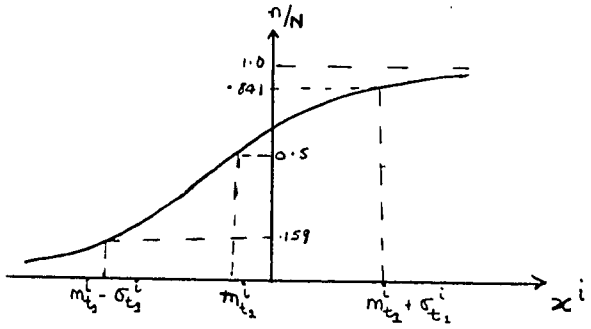


Figure 2.

From the theory of normal distributions, $(m_o \sigma_o)$ may be read of as shown in Fig. (2).

Similarly, $(m^i_{t_1}, \sigma^i_{t_1})$, are obtained from a cumulative plot, made exactly as described above, but made at $t = t_1$.



Note, that these two figures, represent $R_i(o)$ and $R_i(t_1)$ respectively.

The above procedure is then repeated for each of the k parameters, and the sets $\{m^i_o, \sigma^i_o\}_k$, and $\{m^i_{t_1}, \sigma^i_{t_1}, \}_k$, are obtained. Then from equation (7), the drift and diffusion coefficients C^i and D^i, $i = 1, 2, \ldots, k$; are computed. These coefficients are then substituted into (17) and R(t) is uniquely specified.

Then on specifying the sets $\{R_i(t^i_L) = R_i\}_k$ and $\{x^i_2 - x^i_1\}_k$,

the equations (16), are solved for the set of times, $\{t^i_L\}_k$, thus providing an estimate of the mean lifetime t_L, via (20).

Comment For any specified time $t_s > t_1$, each function, $R_i(t_s) \in \{R_i(t_s)\}_k$, yields for each parameter X^i, i=1,2,...,k; the cumulative distribution of its data, at time $t = t_s$. It should be noted, that these cumulative distributions are obtained a-priori, and hence are bona fide predictions.

EXAMPLE

To illustrate the theory presented in the Chapter, we now give an example of reliability prediction for a sample of resistors. As only one parameter, namely the resistance is of interest here, we treat each resistor as a 1-Brownian system. That is a typical resistor, whose resistance fluctuates with time, is treated as a 1-Brownian system.

In this example, the means and variances of a sample of 200 resistors, at times, t = 1000, 2000 and 5000 hrs, are predicted from data taken at times, t = o, and t = 250 hrs. The reliability function (19), is computed, and an estimate made, of mean life-time, t_L, based on a reliability, $R(t_L) = .75$, and a region of acceptance, $[x_1, x_2] \equiv [-\infty, 2433\Omega]$. These predictions are tabulated and compared with results obtained from test data.

Specifications

In this example, we are dealing with a 1-Brownian system, that is (k=1); Therefore all sets $\{\ \}_k$, have only one element. We drop the set and superscript notation and define.

a) Region of acceptance, $[x_1, x_2] \equiv [-\infty, x_2]$ with $x_2 = 2433\Omega$.

b) $\{R_i(t^i_L)\}_k \equiv R = .75$

c) Sample size, N = 200; sampling times, t=o, t=250 hrs.

d) The means and variances are to be predicted, for t = 1000, t = 2000, t= 5000 hrs, based on the times in (c).

The sample at time t = o, yielded the following cumulative data[2].

x	$\frac{n}{N} \leq x$
2436	1.0
2432	.85
2430	.75
2428	.65
2426	.50
2425	.35
2424	.25
2423	.20
2422	.10
2421	.05

Table 1.

[2] Using the Chi-Square test for goodness of fit; The hypotheses, that the data was drawn from a normal population, was established with a confidence of 95%.

The resulting mean and variance, (m_1, σ_o^2), at time t = o , computed digitally[2], are

$m_o = 2426.24\ \Omega$

$\sigma_o = 4.95\ \Omega$

At time t_1 = 250 hours, the following cumulative data, was obtained from the sample.

x	$\frac{n}{N} \leq x$
2437	1.0
2433	.85
2432	.80
2431	.75
2429	.60
2427	.55
2425	.35
2423	.20
2422	.10
2421	.05

Table 2.

The resulting mean and variance, $(m_{250}, \sigma_{250}^2)$, at time t= 250 hrs. computed digitally[3], are

$m_{250} = 2426.5\ \Omega$

$\sigma_{250} = 5.06\ \Omega$

By equation (7), the drift and diffusion coefficients are

$$C = \frac{m_{250} - m_o}{2t_1} = \frac{.25}{500}$$

[3] All calculations of mean and variance, in this example, were performed on an I. B. M. system 360 digital computer. The program used was called a "Data Reduction Program, and was taken from the "Scientic Subroutine Package", for the system 360.

$$D = \frac{\sigma_{250}^2 - \sigma_o^2}{2t_1} = \frac{1.1}{500}$$

Then the reliability function, equation (19),

$$R(t) = \frac{1}{2} + \text{Erf}\left(\frac{x_2 - m_o - 2Ct}{\sqrt{\sigma_o^2 + 2Dt}}\right) \qquad (19)$$

becomes

$$R(t) = \frac{1}{2} + \text{Erf}\left(\frac{2433 - 2426.25 - \frac{.5}{500}t}{\sqrt{24.5 + (\frac{2.2}{500})t}}\right) \qquad (21)$$

The mean lifetime t_L, equation (20), for $R(t_L) \equiv R = .75$, is obtained from (21), as the solution of

$$R(t_L) \equiv R = .75 = \frac{1}{2} + \text{Erf}\left(\frac{2433 - 2426.25 - \frac{.5}{500}t_L}{\sqrt{24.5 + \frac{2.2}{500}t_L}}\right)$$

solving,

$t_L \approx$ 2830 hrs.

Using equations (21) and (7), the reliability R(t), and the mean and variance, m_t, σ_t, were computed as predictions, at times t = 1000, t = 2000, t = 5000 hours. The results are given in Table 3. It should be appreciated, that all these results, are based on test data, for only the first 250 hrs. of test time.

t hours	Observed R(t)	Predicted R(t)	Observed m_t	Predicted m_t	Observed σ_t	Predicted σ_t
1000	.82	.85	2427.4	2427.25	5.35	5.38
2000	.78	.79	2428.2	2428.25	5.75	5.77
5000	*	.59	*	2431.2	*	6.82

TABLE III

* No 5000 hrs data was available at the time of writing this report. However, the test is still in progress.

Concluding Remarks

In chapter I, we have provided the first systematic approach for the reliability modelling of complex systems whose independent parameters exhibit non constant failure rates. It was shown, that a practical treatment of such systems is obtained by using elements of the theory of diffusion processes.

By using this approach, it was also shown, that computationally it is straightforward to obtain predictions of the mean life and the reliability function for such systems. Although too much should not be inferred from a single example, this method was illustrated by an example taken from industry, and a close agreement between prediction and experimental results was obtained. It is hoped in the near future, that this method will be applied to some further problems.

Since to the knowledge of this author, all existing methods for treating systems with non constant failure rates, are empirecal ones, there was no simple way of comparing them with the proposed method.

The major drawback of this method, is that it excludes the cases when periodicities or transients are present in the fluctuations of the parameters values. It is expected that this shortcoming can be overcome by using a more sophisticated diffusion model. However because of the complexity of the mathematics, this was beyond the present capabilities of this author.

CHAPTER II

LIST OF SYMBOLS

$\{N\}$	Set of N, states of complex system
$\cap$	Set intersection operation
$\cup$	Set union operation
$\subseteq$	Proper subset operation
$\forall$	Quantifier, meaning "for all"
$S(t)$	Stationary Markov Stochastic process of the third type.
$\overline{P(t)}$	N x 1, column vector of state probabilities
$\overline{P(o)}$	N x 1, column vector of initial state probabilities
$[A]$	N x N, constant matrix of transition rates
$\cdot = d/dt$	Total differential operator
$[\Phi(t)] \equiv e^{[A]t}$	N x N, state transition matrix
$R(t)$	Reliability function
$r(s)$	Laplace transform of R(t)
t_m	Mean time to first failure
$\overline{p(s)}$	Laplace transform of $\overline{P(t)}$
$\theta(t)$	Availability function
Ω	Steady state availability
L	One sided Laplace transform operator.

CHAPTER II

INTRODUCTION

This chapter treats complex systems, comprised of r, ($r \geq 1$), independent systems. The overall complex system is defined to have N, ($N \geq 1$), distinct states; where any particular state $i \in \{N\}$, indicates the subset of the, r, independent systems, that are operational at a given instant of time. The possible transitions between states, result because of the failure or repair of a particular system. It is assumed in this development, that each of the, r, independent systems, has known constant failure and repair rates. It is also assumed that the state transition behaviour of the complex system is characterizable, by a stationary Markov process of the third type, with poisson statistics. Then based on these assumptions, a mathematical model is developed, from which predictions of reliability availability and mean time to first failure are made.

There is associated, with each state $i \in \{N\}$, a state probability $P_i(t)$. Where $P_i(t)$, gives a measure of the probability, that at time $t = t$, the complex system is in state i. Then we can associate a N x 1, state probability vector, P(t), with the N, states of the system. Using the above assumptions of stationary Markov behaviour, and poisson statistics, it is shown that, $\overline{P(t)}$, satisfies the matrix differential equation, $\dot{\overline{P(t)}} = [A]\overline{P(t)}$; where $[A]$ is a constant N x N matrix, whose entries are the constant transition rates.

The solution of the above matrix differential equation, is of the form $\overline{P(t)} = [\Phi(t)]\ \overline{P(o)}$. Four well known methods of obtaining

explicity, $[\Phi(t)]$, are presented. One of these solution methods utilizes analog computer simulation, and it is shown, how using this method, the reliability function, and mean time to first failure may be simulated directly.

To consider the general reliability modelling of such a complex system, we characterize, its state transition behaviour, by a stationary, absorbing, Markov process. This process has (N-1), acceptable states, and a single failed state, labelled as N. Then the resulting reliability function $R(t) = \sum_{i=1}^{N-1} P_i(t)$; and an estimate of the mean time t_m, to first failure, is made by integrating R(t).

To consider the general availability modelling of such a complex system, we characterize, its state transition behaviour, by a stationary, nonabsorbing, Markov process. The resulting availability function $\theta(t)$, defined over some subset of states $\{k\} \subseteq \{N\}$, is $\theta(t) = \sum_{i=1}^{k} P_i(t)$. Then the steady state availability Ω, over the same set $\{k\}$, is obtained as, $\Omega = \lim_{t \to \infty} \theta(t)$.

It should be noted, that the state probability vectors, $\overline{P(t)}$, for the reliability and availability models, are not the same. Even-though, both satisfy the same general matrix differential equation, namely $\dot{\overline{P(t)}} = [A]\overline{P(t)}$; the [A] matrix, is different in the two cases, because of the different nature of the absorbing and the nonabsorbing Markov process.

A general procedure, in algorithmic for m, is given for obtaining the mathematical model of reliability or availability, directly from the verbal description, of the operational requirements of the complex system. Three examples are given to illustrate all aspects of the theory developed in this chapter. In example II, it is indicated, how the important case of a random waiting time, between failure and the beginning of repair, can be treated.

THEORETICAL DEVELOPMENT

Consider a complex system comprised of r, ($r \geq 1$), independent systems. It is assumed that these independent systems, all have constant, a priori known, failure and repair rates. The approach that will be taken in this chapter, is that of modelling this complex system, as a, {N} state stationary Markov process. Where a particular state, indicates what subset of the original r independent systems are operational at any given instant of time. Then transitions between states, are a result of the failure or repair of the independent systems.

Since the possible "next" states, through which the complex system will pass, depends only on its present state, and not on any previous states through which the complex system might have passed in attaining its present state; The assumption of a Markovian behaviour is quite plausible. Also because of the assumption of constant failure and repair, (transition), rates, we are led naturally to regard the behaviour of the complex system as characterizable by a stationary Markov process. Then assuming that all transitions have the poisson property; that is at most one transition, between states, can occur in any interval (t, t + dt), and that the probability of no transition in this interval is proportional to dt. we can model the complex system as a stationary Markov process with poisson statistics.[4]

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4. A number of authors, refs (10, 12, 16, 17, and 19), have utilized these assumptions in developing reliability models.

Accordingly then; let $S(t)$, be the stationary Markov process of the IIIrd type, which characterizes the state transition behaviour of the complex system. Then at any time $t = t$, $S(t)$, is a discrete random variable whose range is, $\{N\}$.

To characterize the state transition behaviour of such a complex system, we now develop method for obtaining the state probabilities, $P_i(t)$, defined as

$$P(S(t) = i \in \{N\}) \equiv P_i(t), \quad \forall i \in \{N\}.$$

The two assumptions below, provided the basis for our model

I. Markov Assumption. The next state of the complex system, is determined from considerations on its present state only; and not by any considerations on previous states through which the system passed in attaining its present state. Mathematically, let, $\{S(\tau)\}$, be the set of states, through which the complex system passed, $\forall\ \tau < t$. Let $S(t) = i$, be its state at time $t=t$. Then the Markov assumption, states with probability one, that

$$P(\{S(\tau)\}: S(t) = i \,/\, S(t+\tau) = j) = P(S(t) = i \,/\, S(t+dt) = j)$$

that is, the conditional probability of going from state i, to state j, in the interval $(t, t+dt)$, is independent of the previous state history $\{S(\tau)\}$, $\forall\ \tau < t$.

II. Poisson Assumption. Irrespective of what transitions occurred during the interval, $(0, t)$; if the system is in state, i, at time $t = t$, then the probability of a transition from state, i, to some state, $j \in \{N\}$, in the interval of time, $(t, t+dt)$, is

$$a_{ji}\, dt + \phi(dt)$$

and the probability of more than one transition occurring in in this interval, is

$\phi\,(dt)$

where $\phi(dt)$, has the property, that

$$\lim_{dt \to o} \frac{\phi(dt)}{dt} = 0$$

Then $P(S(t) = i \;/\; S(t+dt) = j) = a_{ji}dt$, is the conditional probability of a transition from state i to state j, in the interval of time $(t, t+dt)$; given that at time t, the state of the complex system was i. Thus a_{ji}, has limits of reciprocal time, and in the reliability literature, it is called a failure or repair rate, depending on the context in which it is used.

Accordingly then, let $P_i(t)$, be the probability that at time t = t, the complex system is in state i. Then at any time t = t, there is a N x 1, state probability vector $\overline{P(t)}$, associated with the {N} states of the complex system. By assumptions I and II, above, we can write the probability, $P_j(t+dt)$, that the complex system is in state j at time t = t +dt, as the sum of the probabilities of the two mutually exclusive events, (a) and (b), below. That is

$$P_j(t+dt) = P\{\text{Event (a)}\} \cup P\{\text{Event (b)}\} \qquad (1)$$

where,

Event (a):: the complex system was in state j at time t=t, and remained in state j, during the next interval of time, (t, t+dt). The probability of this event is

$$P\{\text{Event (a)}\} = \left(1 - \sum_{\substack{i \neq j \\ j=1}}^{N} a_{ij}\right) P_j(t) \qquad (2)$$

Event (b): The complex system was in some state i, $(i \neq j)$ at time t=t, then a transition from state i, to state j, occurred in the next interval of time, $(t, t+dt)$. The probability of this event is

$$P\{\text{Event (b)}\} = \left(\sum_{\substack{i=1 \\ i\neq j}}^{N} a_{ji}\,dt\right) P_i(t) \tag{3}$$

Adding, equation (2) to equation (3), we obtain (1),

$$P_j(t+dt) = \left(1 - \sum_{\substack{j=1 \\ i\neq j}}^{N} a_{ij}\,dt\right) P_j(t) + \left(\sum_{\substack{i=1 \\ i\neq j}}^{N} a_{ji}\,dt\right) P_i(t) \tag{4}$$

$$j = 1, 2, \ldots, N$$

putting,

$$a_{jj} = -\sum_{\substack{j=1 \\ i\neq j}}^{N} a_{ij} \tag{5}$$

and substituting (5), into (4), we obtain

$$P_j(t+dt) - P_j(t) = (a_{jj}\,P_j(t))dt + \left(\sum_{\substack{i=1 \\ i\neq j}}^{N} a_{ji} P_i(t)\right) dt \tag{6}$$

$$j = 1, 2, \ldots N.$$

In the limit $dt \to 0$, equation (5) becomes,

$$\frac{dP_j(t)}{dt} = \sum_{i=1}^{N} a_{ji}\,P_i(t) \tag{7}$$

$$j = 1, 2, \ldots N.$$

This set of equations, (7), can be conveniently written in matrix form as,

$$\dot{\overline{P}}(t) = [A]\ \overline{P(t)} \tag{8}$$

$[A]$, is an N x N, constant matrix of transition rates, and $\overline{P(t)}$, is the N x 1 column vector of state probabilities. Thus we have succeed in constructing the mathematical model for determining the state probabilities, that characterize the complex system.

The following theorem, plays an important role in the modelling of complex systems.

Theorem. Addition of Transition Rates.

Given a complex system, which is composed of r, $(r \geq 1)$, independent systems. The system has a finite set of sets, $\{N\}$. For any pair of states $i,j \in \{N\}$, let a_{ji}^{k}, $k = 1,2,\ldots,r$, be the transition rate of independent system k, from state i, to state j. Then at any instant of time t=t, when there are P, $(P \leq r)$, independent systems in state i; an overall transition rate from state i, to state j, for the complex system, in interval (t, t+dt), can be written as

$$a_{ji} = \sum_{k=1}^{P} a_{ji}^{k} \tag{9}$$

Proof:

Let $P_{ji}^{k}(t)$,[5] be the probability of a transition of independent system k, from state i, to state j. By assumption II, for t = dt, we have

$$P_{ji}^{k}(dt) = e^{-a_{ji}^{k}dt} \tag{10}$$

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5. $P_{ji}^{k}(t)$, should not be confused with the state probabilities $P_i(t)$ or $P_j(t)$.

since all of the r, systems are assumed to be independent, the Joint probability $P_{ji}(dt)$, that all P, ($P \leq r$), systems, will go from state i, to state j, in the interval of time, (t, t+dt), can be written as

$$P_{ji}(dt) = \prod_{k=1}^{P} P_{ji}^{k}(dt) \tag{11}$$

substituting (10) into (11), gives

$$P_{ji}(dt) = e^{-\sum_{k=1}^{P} a_{ji}^{k} dt}$$

This suggests that an overall transition rate, from state i, to state j, in interval (t, t + dt), given that at time t, that there are P, ($P \leq r$), independent systems in state i, can be defined as

$$a_{ji} = \sum_{k=1}^{P} a_{ji}^{k} \tag{9}$$

which establishes the theorem.

SOLUTION OF THE MATRIX DIFFERENTIAL EQUATION

In order to obtain an explicit expression for the state probability vector, $\overline{P(t)}$, we have to solve equation (8). That is we have to solve

$$\dot{\overline{P(t)}} = [A]\,\overline{P(t)} \tag{8}$$

It is a well known fact, that (8), has a unique solution, which varies continuously with given initial data, $\overline{P(o)}$.
This solution has the form,

$$\overline{P(t)} = [\Phi(t)]\overline{P(o)} \tag{12}$$

where $[\Phi(t)]$, is a N x N matrix, called the transition matrix, and is defined by

$$[\Phi(t)] \equiv e^{[A]t} = \sum_{i=o}^{\infty} \frac{[A]^i t^i}{i!} \tag{13}$$

Thus, the obtaining of an explicit expression for $\overline{P(t)}$, involves the compution of the matrix exponential function (13). For physically realizable situations, where $[A]$ is a constant square matrix, a closed form expression for $[\Phi(t)]$, always exists, and can be found by any of the following four methods.

Method I: Via Laplace transform techniques

Method II: Via Mason's rule and flow graph theory

Method III: Via the theory of functions of a matrix.

Method IV: Via Analog computer simulation.

Description of Method I

The one sided Laplace transform of (8), with $L(\overline{P(t)} \equiv p(s)$, yields the set of algebra equations in the complex variables, viz:

$$\overline{sp(s)} = \overline{P(o)} + [A]\overline{p(s)} \tag{14}$$

rearranging (14), gives

$$\overline{p(s)} = [\, sI - [A]]^{-1} \overline{P(o)} \tag{15}$$

where I is the identity matrix. Comparing (15) with (12), we have

$$[\Phi(t)] = L^{-1}\left([\, sI - [A]\,]^{-1}\right) \tag{16}$$

therefore the general solution of (8) is given as

$$P(t) = L^{-1}\left([\, sI - [A]\,]^{-1}\right) \overline{P(o)} \tag{17}$$

Description of Method II

The Laplace transform of equation (12) can be written as,

$$\overline{P(s)} = [\, \Phi(s)]\overline{P(o)} \tag{18}$$

since the elements of $\overline{P}(o)$ can be arbitrarily specified, the column vectors of $[\, \Phi(s)]$ are linearly independent. Then $\Phi_{ij}(s)$ is uniquely determined as,

$$\Phi_{ij}(s) = \frac{p_i(s)}{P_j(o)} \tag{19}$$

Equation (19) can be interpreted as the "gain" between two nodes , j, and i, of a Mason flow graph. Such a signal flow graph is a linear graph of the transformed state probabilities, equation(8);

The"gain", (19) can be computed directly from such a graph via the Mason rule[6], as

$$\Phi_{ij}(s) = \sum_{k} \left(\frac{\Delta_{ij}^{k}(s) G_{ij}^{k}(s)}{\Delta(s)} \right) \qquad (20)$$

$$\forall\, i, j, \in \{N\}$$

and

$$\Phi_{ij}(t) = L^{-1} \left[\Phi_{ij}(s) \right]$$

then equation (12) can be written as

$$P_i(t) = L^{-1} \left(\sum_{j=1}^{N} \sum_{k} \left(\frac{\Delta_{ij}^{k}(s) G_{ij}^{k}(s)}{\Delta(s)} \right) \right) \qquad (21)$$

$$i = 1, 2, \ldots, N.$$

To illustrate the method of constructing a Mason graph, consider the expression for the transformed state probability, $p_i(s)$, as obtained from (14)

$$p_i(s) = \frac{P_i(o)}{s} + \sum_{j=1}^{N} \frac{a_{ij} p_j(s)}{s} \qquad (22)$$

then regarding the transformed state variables as graph nodes, and the transition rates $\{a_{ij}\}$ as path transmittances, the Mason graph for (22); is shown in Fig. (4).

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[6] See reference (14). A description of this formula is given in Appendix B.

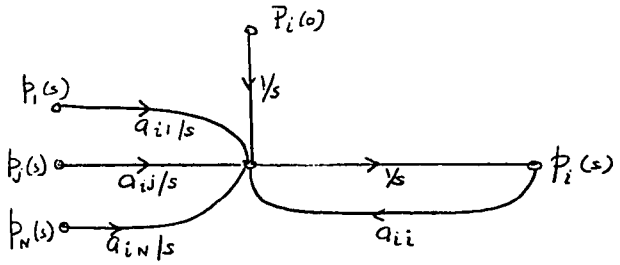


Figure 4

Description of Method III.

In this method, $[\Phi(t)]$, is computed directly from its series definition, (13). That is, from

$$[\Phi(t)] \equiv e^{[A]t} = \sum_{i=o}^{\infty} \frac{[A]^i t^i}{i!} \tag{13}$$

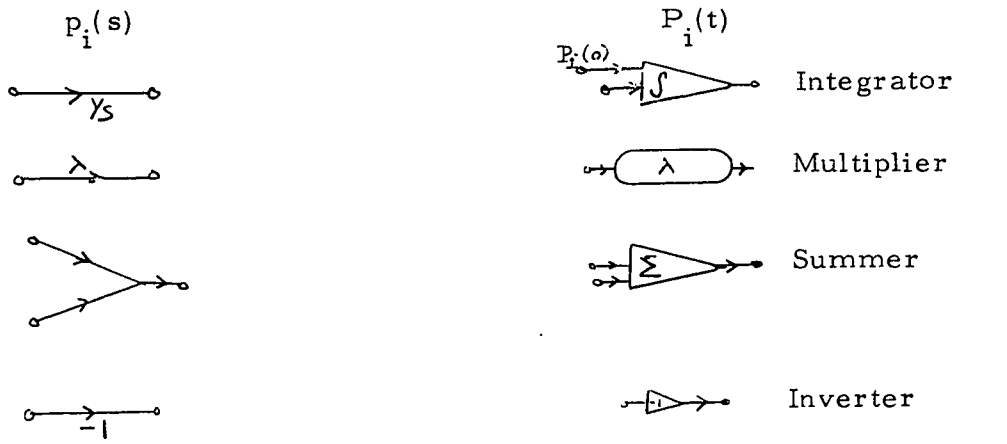
The theory of functions of a matrix, guarantees that for a matrix $[A]$, with a known spectrum, there exists an alternative finite expansion of (13). This theory, requires a knowledge of minimal polynomials and elementary divisors of a matrix[7]. To illustrate the theory, we present a method which is applicable to the case, when the matrix $[A]$, has, N distinct eigenvalues. That is when the spectrum of $[A]$, contains the set of eigenvalues; $B_i, i=1,2,\ldots,N$, which are all distinct. In this case.

$$[\Phi(t)] \equiv e^{[A]t} = \sum_{i=1}^{N} \left(e^{-\beta_i t} \frac{\prod_{\substack{j=1 \\ j\neq i}}^{N} ([A] - \beta_j I)}{\prod_{\substack{j=1 \\ j\neq i}}^{N} (\beta_i - \beta_j)} \right) \tag{23}$$

[7] This theory is covered in reference 15.

Description of Method IV.

To obtain an analog computer simulation of equation (8), a convenient approach is to construct from equation (14), the Mason graph for the transformed state variables, $\overline{p(s)}$. This Mason graph is then transformed directly into an analog computer schematic, via the scheme below



For example, using Method IV, the Mason graph figure 4, is transformed into the analog computer schematic as shown in Fig. 5.

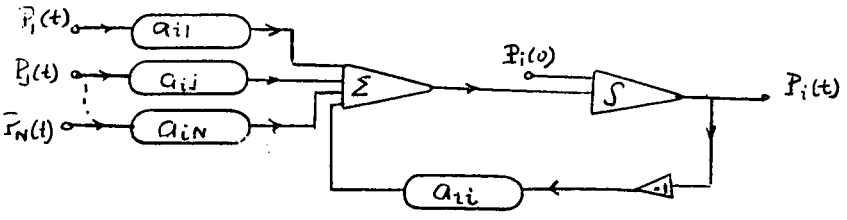


Figure 5

Having discussed methods for solving equation (8), we now develop the procedures for the reliability and availability modelling of complex systems, whose state probabilities $\overline{P(t)}$, satisfy (8).

RELIABILITY MODELLING

To construct a reliability model for the $\{N\}$, state complex system, we characterize its behaviour by an absorbing Markov process. Specifically by a process having (N-1), acceptable states, and a single absorbing state, which we label as N, and call the fail state. The specific characterization of the "fail" state N, is important, and accordingly then we give its properties in the following definition

Definition. Fail State N. A state N, of a stationary Markov process, is a fail, (absorbing) state, iff the following conditions are jointly true

a) It is a fail state

b) $\lim_{t \to \infty} P_N(t) = 1$

c) $A_{iN} = 0, \quad \forall\, i \in \{N\}$ (24)

d) $A_{Ni} \neq 0$, for some $i \in \{N\}$.

Physically, an absorbing state, expresses the fact, that after a sufficiently long time, any real system will fail.

Now the other (N-1), states, are acceptable states. Where in the model, by an acceptable state $i \in \{N-1\}$, is meant a state i, which satisfies all the conditions given in the definition below.

Definition. Acceptable State. A state $i \neq N$, $i \in \{N\}$, is an acceptable state in the context of a reliability model, iff, the following conditions are jointly satisfied

a) It is an acceptable state

b) The process S(t), is a stationary Markov process

$$\text{c)} \lim_{t \to \infty} P_i(t) = 0. \qquad \forall\, i \in \{N\},\ i \neq N$$

$$\text{d)} \ a_{ji} > 0 \qquad \text{for some } j \in \{N\} \tag{25}$$

$$\text{e)} \ a_{ij} > 0 \qquad \text{for some } j \in \{N-1\}.$$

That is, a state $i \in \{N\}$, is an acceptable state iff, there is at least one transition to, and one transition from state i.

Now, consider a $\{N\}$, state complex system, whose state probability vector $\overline{P}(t)$, satisfies (8), and can therefore be computed by any of the methods I, II, III, or IV. Then with equations (24) and (25), as constraints on the state probabilities $\overline{P}(t)$, we can define a reliability function for such a complex system, as,

Definition. Reliability Function R(t). For a $\{N\}$, state complex system, whose state probabilities $\overline{P}(t)$, satisfy (8), subject to constraints (24) and (25). The reliability function R(t) is the probability that a time t = t, the state S(t), of the complex system, will be contained in the set $\{N-1\}$, of acceptable states. That is,

$$R(t) = P(S(t) = \bigcup_{i=1}^{N-1} i)$$

or alternatively

$$R(t) = \sum_{i=1}^{N-1} P_i(t) \tag{26}$$

Having now defined the reliability function R(t); we can define the mean time t_m, to first failure of such a complex system. This definition applies only to a reliability function obtained as in (26).

Definition. Mean Time t_m, To First Failure.

The mean time t_m, to first, is defined to be the average time taken for the complex system to pass from the set $\{N-1\}$, of acceptable states, to the fail, (absorbing), state, N. That is, t_m = average of the first exit times from the set $\{N-1\}$. Alternatively

$$t_m = \int R(t)\, dt \tag{27}$$

then substituting (26) into (27).

$$t_m = \sum_{i=1}^{N-1} \int P_i(t)\, dt \tag{28}$$

and denoting the Laplace transform of R(t) as

$$L(R(t)) = r(s)$$

we can write (27), as

$$t_m = \lim_{s \to o} r(s) \tag{29}$$

or writing

$$L(P_i(t)) = p_i(s) \tag{30}$$

and

$$\lim_{s \to o} p_i(s) = \gamma_i \tag{31}$$

then substituting (31) into(28),

$$t_m = \sum_{i=1}^{N-1} \gamma_i \tag{32}$$

thus the problem of computing t_m, becomes a problem in obtaining $\bar{\gamma}$, that is γ_i, $i = 1, 2, \ldots, N-1$. as given by (31).

To compute $\bar{\gamma}$, we begin by Laplace transforming (8), and using the notation of (30),

$$s\,\bar{p}(s) - \bar{P}(o) = [A]\,\overline{p(s)} \tag{33}$$

and in the limit $s \to o$, from (24) and (25) and the final value Tauber theorem; equation (33), becomes

$$\bar{a} - \overline{P(o)} = [A]\,\bar{\gamma} \tag{34}$$

where

$$\bar{a} = \begin{bmatrix} o \\ \vdots \\ o \\ 1 \end{bmatrix} = \lim_{s \to o} s\,\bar{p}(s) = \lim_{t \to \infty} \bar{P}(t). \tag{35}$$

then multiplying both sides of (34), by $[A]^{-1}$, we obtain the desired result, viz:,

$$\bar{\gamma} = [A]^{-1}\,(\bar{a} - \bar{P}(o)) \tag{36}$$

$\bar{P}(o)$, is the constant vector of initial state probabilities and $\bar{a}$ is given by (35). Therefore the problem of computing $\bar{\gamma}$, and hence t_m, reduces to inverting matrix [A].

It can be shown[8], that the variance σ^2_m, about the mean time t_m given in (32), is

$$\sigma^2_m = -2 \lim_{s \to o} \left(\frac{d}{ds} \sum_{i=1}^{N-1} p_i(s)\right) - t^2_m \tag{37}$$

where t_m is obtained from (32), by first computing (36).

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[8] See refs (16, 19).

Comment. The reliability function, equation (26), is obtained for a specified complex system, by first computing $\overline{P(t)}$, by any of the methods, I, II, III, IV. Then $\overline{P(t)}$ is substituted in (26) to yield R(t). For example, using method IV, we would simulate the reliability function, (26), as shown in Figure 6.

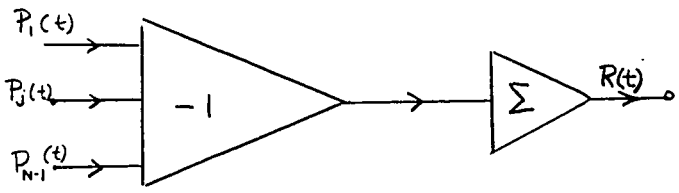

Figure 6

Then the mean time t_m, to first failure, given by equation (27), is simulated as shown in Figure 7.

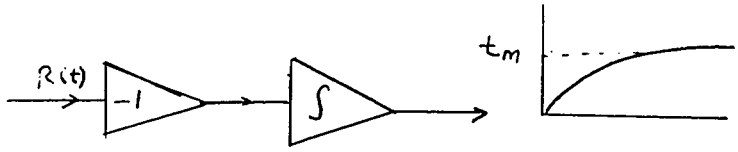

Figure 7.

AVAILABILITY MODELLING

To construct an availability model for a $\{N\}$, state complex system, we characterize its behaviour by a non absorbing, stationary, Markov process. Specifically, by a process, in which there are no fail, (absorbing), states, and all states are "acceptable" states, in the sense of equation (38) below.

Definition. Acceptable State For An Availability Model.
A state $i \in \{N\}$, is an acceptable state for an availability model, iff, the following conditions are jointly satisfied.

a) State i, is an acceptable state for an availability model.

b) The process S(t), is a nonabsorbing stationary Markov process.

c) $\lim_{t \to \infty} P_i(t) = \theta_i, \quad \forall i \in \{N\}$

where $(0 < \theta_i \leq 1)$ (38)

d) $a_{ij} > 0$, $a_{ji} > 0$, for some i, j, $\in \{N\}$

That is, in an availability model, transitions are possible, to and from each state, $i \in \{N\}$.

Then for a complex system, whose state probabilities $\overline{P}(t)$, satisfy (8), subject to the constraints (38); we define an availability function $\theta(t)$, as

Definition. Availability Function $\theta(t)$. For a $\{N\}$ state complex system, whose state probabilities $\overline{P(t)}$ satisfy (8) subject to constraints (38). The availability function $\theta(t)$, is the probability, that a time t = t, the state, S(t), of the complex system is contained in some proper subset $\{k\} \subseteq \{N\}$. That is

$$\theta(t) = P(S(t) = \bigcup_{i=1}^{k} i)$$

or alternatively

$$\theta(t) = \sum_{i=1}^{k} P_i(t) \qquad (39)$$

Now because of the existance, of steady state, non zero probabilities, in the case of the availability model, we also have a steady state availability Ω , defined as,

Definition. Steady state availability Ω. The steady state availability Ω , is the probability that after a sufficiently long time, the state $S(\infty)$ of the complex system will be in the subset, $\{k\} \subseteq \{N\}$. That is

$$\Omega = P(S(\infty) = \bigcup_{i=1}^{k} i)$$

or alternatively

$$\Omega = \lim_{t \to \infty} \theta(t) = \sum_{i=1}^{k} \lim_{t \to \infty} P_i(t) \tag{40}$$

$$= \sum_{i=1}^{k} \theta_i \tag{41}$$

To compute the steady state availability Ω , given by (40), consider

$$\theta = \lim_{t \to \infty} \overline{P(t)} \tag{40}$$

using equation (12), we can write (40) as

$$\theta = \lim_{t \to \infty} [\Phi(t)] \, \overline{P(o)} \tag{42}$$

Now it can be shown[9], that the transition matrix $[\Phi(t)]$, can be partioned as

$$[\Phi(t)] = [\Omega] + [M(t)] \tag{43}$$

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9 See reference (10) .

where $[Q]$ is a constant matrix characterizing the steady state behaviour of the system, and $[M(t)]$ is a matrix characterizing its transient behaviour. That is $[M(\infty)] = [0]$. On substituting (43) into (42)

$$\theta = [Q]\ \overline{P(o)} \tag{44}$$

or alternatively

$$\theta_i = \sum_{j=1}^{N} Q_{ij}\ P_j(o) \tag{45}$$

$$i = 1, 2, \ldots N.$$

then substituting (45) into (41), yields the required expression for Ω, namely

$$\Omega = \sum_{i=1}^{k} \sum_{j=1}^{N} Q_{ij}\ P_j(o) \tag{46}$$

Comments. a) The availability function (40), is obtained for a specified complex system, by using any of the methods, I, II, III, or IV. to compute the state probabilities $\overline{P(t)}$. For example using Method II, we would compute $\theta(t)$ as

$$\theta(t) = \sum_{i=1}^{k} \sum_{j=1}^{N} \Phi_{ij}(t)\ P_j(o) \tag{47}$$

where the $\Phi_{ij}(t)$ are first computed, using Masons' rule, equation (20).

b) It should be kept in mind, that eventhough in the reliability as well as the availability model, $\overline{P(t)}$, satisfies the same general form of the matrix differential equation (8). The explicit forms of (8) are different for the two cases, because of the different nature of absorbing and nonabsorbing Markov processes.

We now present a general algorithm for the reliability and availability modelling of complex systems.

ALGORITHM FOR THE RELIABILITY AND AVAILABILITY MODELLING OF COMPLEX SYSTEMS

This algorithm, which is divided into six sequential steps; permits the reliability and/or availability model to be constructed and analyzed, directly from the verbal description of the operational requirements of the complex system.

Step I. Verbal Description. Formulate a precise verbal description of the operational requirements of the complex system; that is, state precisely what requirements the complex system must satisfy, to be considered as successful; and what conditions produce failure. These requirements and conditions are always stated, in terms of the number of operational or non-operational independent systems, required to produce a given state of the complex system.

Step II. State Assignment. Choose as many states as are needed to characterize the complex system from its verbal description; including one fail state, when reliability is being considered.
N. B. The inclusion of extraneous states, increases the dimension of the problem, unnecessarily complicates it, and usually yields incorrect results .

Step III. Construction of the Transition Diagram. The transition diagram is the graph of the state space of the complex system. This diagram is constructed by enumerating the possible transitions between states, by a path, whose direction is the direction in which the transition occurs. The path transmittance is the transition rate between the two states in question.

Step IV. Construction of the Transition Graph. The transition graph, is important, because it yields on inspection the system matrix [A], for the particular model. This graph is constructed from the transition diagram by using the following rule, which is suggested by equation (5).

Rule. To each node, j, on the transition diagram, add a self loop of transmittance, a_{jj}, given as

a_{jj} = -(Σ of the transmittances of all paths emanating from node, j.)

Then matrix [A], is obtained on inspection of this graph, by having the self transmittances, a^{s}_{jj}, as the diagonal elements of [A]. The transmittances, (transition rates), a^{s}_{ij}, between states, are the off diagonal elements of [A]. Where the index i in a_{ij} refers to its row position in [A], and the, j, index refers to its column position in [A].

Comment. Steps I-IV, give the explicit form of equation (8), for the reliability or availability model, that was being considered. However to obtain predictions of reliability (26) or availability (39), equation (8) has to be solved by any of the techniques of methods I. II, III, or IV. An analog computer, usually gives the quickest and most flexible simulation of a model. Accordingly then, analysis and/or analog simulation of the model can be systematically realized, as outlined in steps V and VI, below.

Step 5. Construction of the Mason Signal Flow Graph. The Mason graph, is the graphical representation of equation (33), where the transformed state probabilities $p_i(s)$, and the initial conditions $P_i(o)$, are nodes on the graph. The Mason graph, may be obtained directly from the transition graph of step IV, by using the following rules.

Rules 1) Divide all path and loop transmittances, on the transition graph by the complex variable s.

2) Add to each node j, when appropriate, a node, $P_j(o)$, with transmittance to node j, given by $P_j(o)/s$. This is to take care of the initial conditions $\overline{P}(o)$. (see equation (33)).

Once the Mason graph has been constructed, we can compute $[\Phi(t)]$, equation (12), as the inverse Laplace transform of $[\Phi(s)]$, where $[\Phi(s)]$, is obtained from the Mason graph, by using the Mason rule, equation (20).

Step VI. Construction of the Analog Computer Schematic. The analog computer, provides flexible time domain simulation of a reliability or availability model. This simulation is readily achieved after the analog computer schematic has been obtained. The analog computer schematic is obtained directly from the Mason graph by using the transformation scheme given in method IV.

We now present three examples, to illustrate the theory covered in this chapter, and the use of the above algorithm.

Example I. Reliability and Availability Modelling of a Two State Complex System.

Step I. Verbal description

Reliability Model. The complex system consists of r, ($r \geq 1$), independent systems. Each independent system either works, that is, in an acceptable state, or fails, that is, is in the failed state. The complex system is considered as failed at the first time, that all of its r independent systems systems are simultaneously non-operational. Accordingly then, this complex system has only two states, an acceptable state and a failed state. The system operates in a non repair environment.

Availability Model. The description here is the same as that given above for the reliability model, except that there is no fail state and the system operates in a repair environment. Then the complex system is considered "available", at any time t = t, iff, all of its r, independent systems are operational at that time; eventhough any or all of these independent systems, may have failed, and been repaired prior to this time t.

Step II. State Assignment.

Reliability Model

State	Description
1 (Acceptable State)	All r, independent systems are operational
2 (Failed State)	All r, independent systems have failed and are not being repaired.

Availability Model.

State	Description
1	All r, independent systems are working
2	All r, independent systems have failed, but repairs begun immediately upon failure.

Then, the set of successful states $\{k\}$, consists of only one state, namely state 1, and therefore the availability function, equation (39), becomes $\theta(t) = P_1(t)$.

Step III. Construction of the Transition Diagram.

Let the failure and the repair rates of the r, independent systems be respectively a^i_{21}, a^i_{12}, $i = 1, 2, \ldots, r$. Now by equation (9), we can define the overall failure and repair rates, respectively

$$a_{21} = \sum_{i=1}^{r} a^i_{21}$$

and

$$a_{12} = \sum_{i=1}^{r} a^i_{12}$$

Then for the reliability model, the reliability transition diagram, is shown in Figure 8.

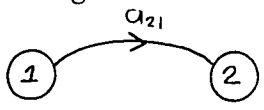


Figure 8.

and the availability transition diagram, is shown in Figure 9.

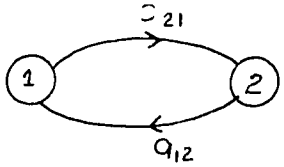


Figure 9.

Step IV. The transition graph, for the reliability and availability models, are shown respectively, in figures (10) and (11).

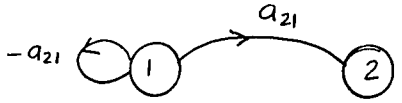


Figure 10

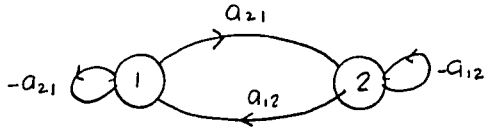


Figure 11

from the above graphs, the matrix, [A], for the reliability model, is

$$\begin{bmatrix} -a_{21} & 0 \\ a_{21} & 0 \end{bmatrix} \equiv [A]$$

and for the availability model,

$$\begin{bmatrix} -a_{21} & a_{12} \\ a_{21} & -a_{12} \end{bmatrix} \equiv [A]$$

Step V. The Mason graph, for the reliability and availability models, are shown respectively, in Figures (12) and (13)

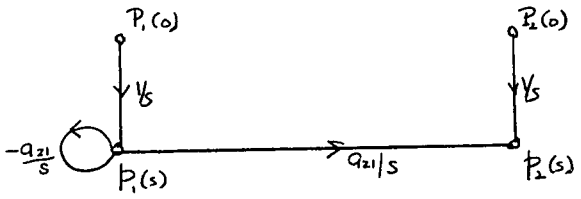
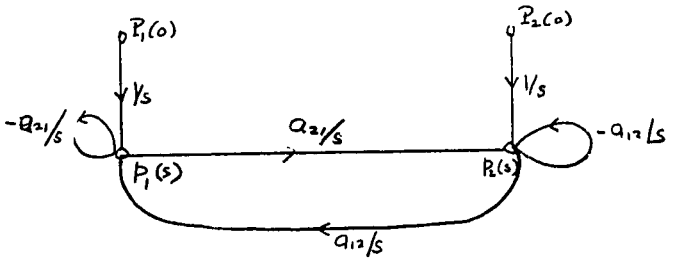


Figure 12

Figure 13



Step VI. The analog computer schematic, for simulating the reliability function and the mean time t_m, to first failure, is shown in Figure 14.

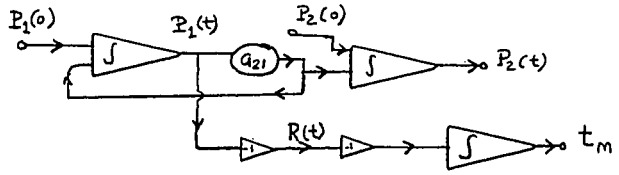


Figure 14

The analog computer schematic, for simulating the availability function $\theta(t)$, is shown in Figure 15.

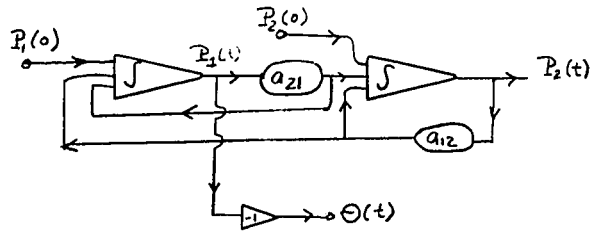


Figure 15.

To illustrate methods I, II, and III, for solving equation, we take a numerical example. Accordingly, let $a_{21} = 5$, $a_{12} = 4$. Then for the availability case, we have

$$[A] = \begin{bmatrix} -5 & 4 \\ 5 & -4 \end{bmatrix} \tag{48}$$

thus the $[\Phi(t)]$ matrix is a 2 x 2 matrix, which will now be computed by methods I, II, and III.

Method I. By equation (16),

$$[\Phi(t)] = L^{-1}\begin{bmatrix} s+5, & -4 \\ -5 & s+4 \end{bmatrix}^{-1} \tag{49}$$

$$= L^{-1}\begin{bmatrix} \frac{s+4}{s(s+9)} , & \frac{4}{s(s+9)} \\ \frac{5}{s(s+9)} , & \frac{s+5}{s(s+9)} \end{bmatrix} \tag{50}$$

taking the inverse Laplace transform of (50),

$$[\Phi(t)] = \begin{bmatrix} \frac{4}{9} + \frac{5e^{-9t}}{9} , & \frac{4}{9} - \frac{4}{9}e^{-9t} \\ \frac{5}{9} - \frac{5e^{-9t}}{9} , & \frac{5}{9} + \frac{4e^{-9t}}{9} \end{bmatrix} \tag{51}$$

then using equation (43), viz;

$$[\Phi(t)] = [Q] + [M(t)] \tag{43}$$

we can write (51), as

$$[\Phi(t)] = \frac{1}{9}\begin{bmatrix} 4 & 4 \\ 5 & 5 \end{bmatrix} + \frac{e^{-9t}}{9}\begin{bmatrix} 5 & -4 \\ -5 & 4 \end{bmatrix} \tag{52}$$

and replacing (52), in (12), we obtain

$$P_1(t) = \frac{1}{9}\left[4 + 5e^{-9t}\right] P_1(o) + \frac{1}{9}\left[4 - 4e^{-9t}\right] P_2(o) \tag{53}$$

then the availability function $\theta(t)$, given by (39); for this case is

$$\theta(t) = P_1(t) \tag{54}$$

and is therefore given by (53). The steady state availability Ω, may be obtained directly from (54), as

$$\Omega = \underset{t \to \infty}{\text{Limit}}\ \theta(t) \tag{40}$$

taking the limit, (53) gives

$$\Omega = \frac{4}{9} P_1(o) + 4/9\ P_2(o) \tag{55}$$

or alternatively by substituting (52) into (47), we find

$$\Omega = \sum_{i=1}^{1} \sum_{j=1}^{2} Q_{ij}\, P_j(o) \tag{56}$$

$$= \frac{4}{9} P_1(o) + \frac{4}{9} P_2(o) \tag{57}$$

as is given in (55).

Method II. From figure (13), using the Mason rule equation(20),[10] we find

$$\Phi_{11}(s) = \frac{p_1(s)}{P_1(o)} = \frac{G_{11}(s)\Delta_{11}(s)}{\Delta(s)} = \frac{s+4}{s(s+9)}$$

$$\Phi_{22}(s) = \frac{p_2(s)}{P_2(o)} = \frac{G_{22}(s)\Delta_{22}(s)}{\Delta(s)} = \frac{s+5}{s(s+9)}, \qquad (58)$$

$$\Phi_{12}(s) = \frac{p_1(s)}{P_2(o)} = \frac{G_{12}(s)\Delta_{12}(s)}{\Delta(s)} = \frac{4}{s(s+4)}$$

$$\Phi_{21}(s) = \frac{p_2(s)}{P_1(o)} = \frac{G_{21}(s)\Delta_{21}(s)}{\Delta(s)} = \frac{5}{s(s+9)}$$

Therefore, from (58),

$$[\Phi(s)] = \begin{bmatrix} \frac{s+4}{s(s+9)} & , & \frac{4}{s(s+9)} \\ \frac{5}{s(s+9)} & , & \frac{s+5}{s(s+9)} \end{bmatrix} \qquad (59)$$

comparing (59) and (50), we see that they are the same .

Method III. The eigenvalues $\{\beta_i\}$, of a matrix $[A]$, are the roots of the characteristic polynomial $C(\beta)$, where

$$C(\beta) = \det [\beta I - [A]] = 0 \qquad (60)$$

then for matrix $[A]$, given by (48); equation (60) yields the distinct eigenvalues β_1, and β_2 as

$$\beta_1 = 0, \quad \beta_2 = 9 \qquad (61)$$

---

10 See Appendix for a description of equation (20).

then equation (23) of method III, is applicable, and

$$[\Phi(t)] = e^{-\beta_1 t}\frac{[[A] - \beta_2 I]}{(\beta_1 - \beta_2)} - e^{-\beta_2 t}\frac{[[A] - \beta_1 I]}{(\beta_2 - \beta_1)} \tag{62}$$

on substituting (61), and (48), into (62), we find

$$[\Phi(t)] = \frac{1}{9}\begin{bmatrix} 4 & 4 \\ 5 & 5 \end{bmatrix} + \frac{e^{-9t}}{9}\begin{bmatrix} 5, & -4 \\ -5, & 4 \end{bmatrix} \tag{63}$$

which is the same result as was obtained by method I, equation (52).

Example II. Reliability Modelling of a Complex System with a Perfect Space and Random Delay Between Failure and Beginning of Repair[11]

The problem is to use the general algorith, to obtain the reliability function, and mean time t_m, to first failure, of the complex system shown in Figure (16). This is to be accomplished, by 1) Masons' rule and 2) analog computer simulation.

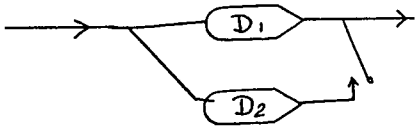


Figure 16

Step I. Verbal Description.

The complex system consists of an on line system, D_1, and a standby system, D_2. System D_2, acts as a perfect spare; that is D_2, cannot fail while it is on standby.

[11] This delay, is assumed to be a random variable, which is exponentially distributed, and has mean value $\equiv \tau$.

The operation of the system requires, that on failure of D_1, the system D_2, immediately becomes the on line system. Then after a time delay, of mean value τ, repairs to D_1, are begun. The complex system is considered to be in the failed state, when D_2, which is now operating on line, fails, before repairs have been completed to D_1.

The two systems, D_1, and D_2, are assumed to be identical and therefore have the same failure and repair rates, denoted by, λ and μ, respectively. We also introduce a waiting rate ρ, which is the reciprocal of the mean delay τ. That is $\rho = \frac{1}{\tau}$.

Step II. State Assignment [12].

State	Description
1	D_1 and D_2, are both operating
2	D_1 fails, and D_2, immediately operates on line
3	After a mean delay τ, service begins on the failed system D_1. System D_2, is now operating on line.
4	The on line system fails before repairs have been started on D_1 or before repairs have been completed on D_1

Step III Construction of the Transition Diagram.

The reliability transition diagram, constructed from the above state assignment, is shown in Figure (17).

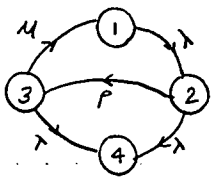


Figure 17

[12] In this state assignment, states 1, 2, and 3, are the acceptable states. (see equation (25)., of chapter II).

Step IV. Construction of the Transition Graph.

The reliability transition graph, constructed from the above transition diagram, is shown in Figure (18).

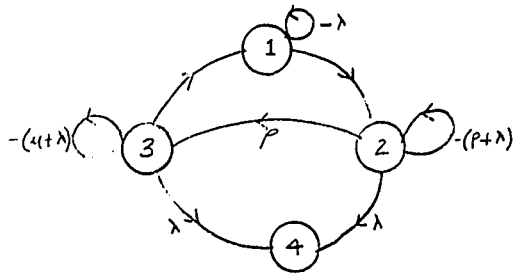


Figure 18

Then the matrix [A], and this equation (8), obtained by inspection of the above transition graph, is

$$\begin{bmatrix} \dot{P}_1(t) \\ \dot{P}_2(t) \\ \dot{P}_3(t) \\ \dot{P}_4(t) \end{bmatrix} = \begin{bmatrix} -\lambda & 0 & u & 0 \\ \lambda & -(\rho+\lambda) & 0 & 0 \\ 0 & \rho & -(\mu+\lambda) & 0 \\ 0 & \lambda & \lambda & 0 \end{bmatrix} X \begin{bmatrix} P_1(t) \\ P_2(t) \\ P_3(t) \\ P_4(t) \end{bmatrix} \qquad (64)$$

Step V. Construction of the Mason Graph.

The Mason graph, constructed from the above reliability transition graph, is shown in Figure (19).

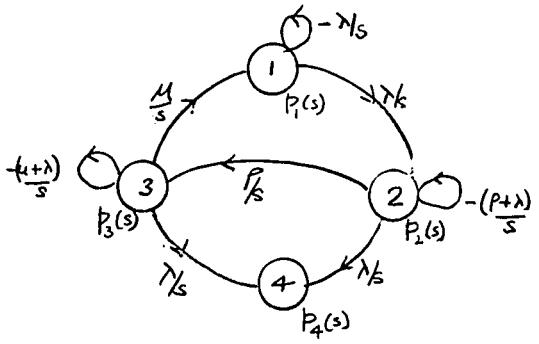


Figure 19

Now taking the Laplace transform of (26), we have

$$r(s) = \sum_{i=1}^{3} p_i(s) \tag{65}$$

and from (29),

$$t_m = \lim_{s=o} r(s) \tag{66}$$

On using the Mason rule, equation (20), we find for (65),

$$r(s) = \frac{s^2+\mu s+\lambda s+\rho s+\mu\rho+2\lambda s+2\lambda\mu+2\lambda^2+\lambda\rho}{(s+\rho+\lambda)(s+\lambda+\mu)(s+\lambda-\rho\lambda\mu)} \tag{67}$$

and by (66),

$$t_m = \frac{\mu\rho+\lambda\rho+2\lambda\mu+2\lambda^2+\lambda\rho}{(\rho+\lambda)(\mu+\lambda)(\lambda-\rho\lambda\mu)} \tag{68}$$

Then for the sample case, with

$$\begin{aligned} \rho &= 0.2 \text{ / hrs} \\ \lambda &= 0.001 \text{ / hrs} \\ \mu &= 0.04 \text{ / hrs} \end{aligned} \tag{69}$$

(68), gives, $t_m \approx 3.52 \times 10^4$ hrs. Another estimate of t_m, will now be made by the analog computer techniques of step VI.

Step VI. Analog Computer Simulation.

The analog computer schematic, obtained from the Mason graph, is shown in Figure (20). The reliability function R(t), and mean time to first failure, for the sample case (69), are shown in Figure (21).

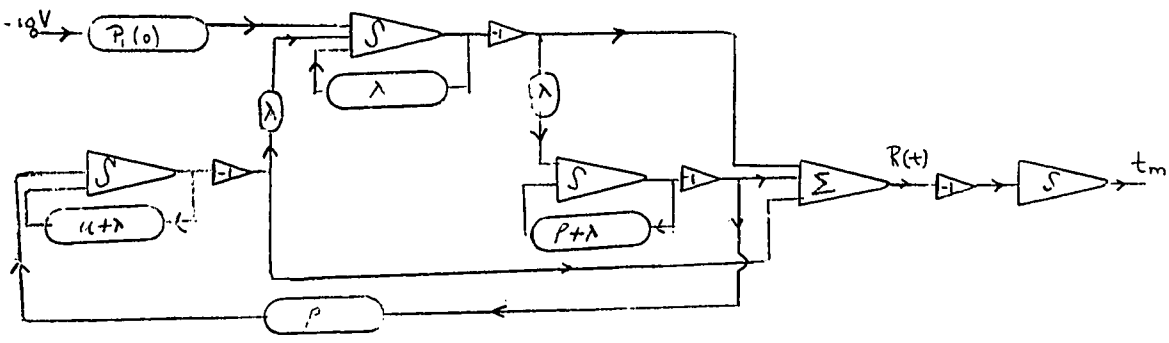


Figure 20

Analog Computer Graphs

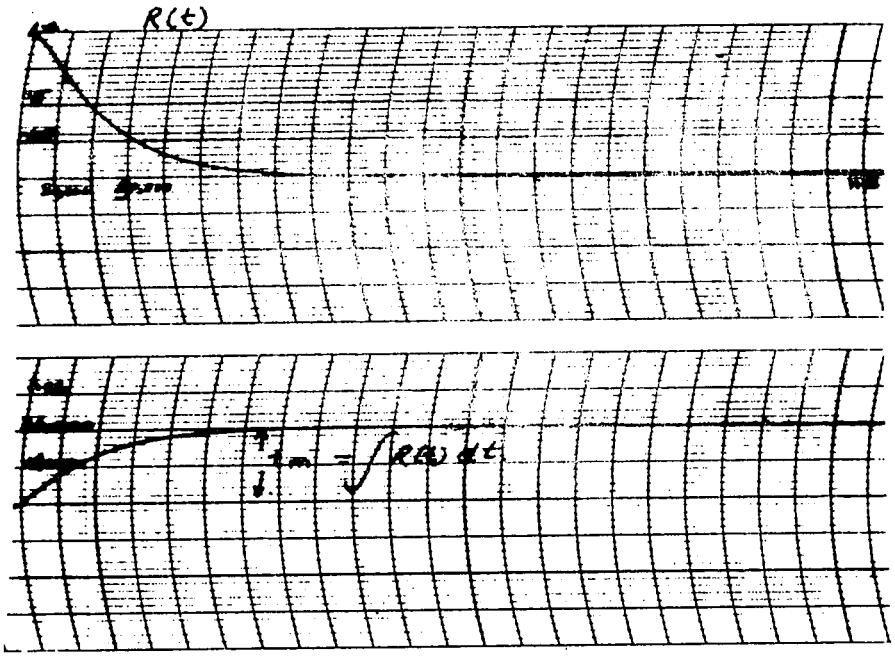


Figure 21.

Example III. Reliability Modelling of a Complex System with Hybrid Redundancy.

The problem is to use the general algorithm, to obtain by analog computer simulation the reliability function, and mean time t_m, to first failure, of the complex system shown in Figure (22).

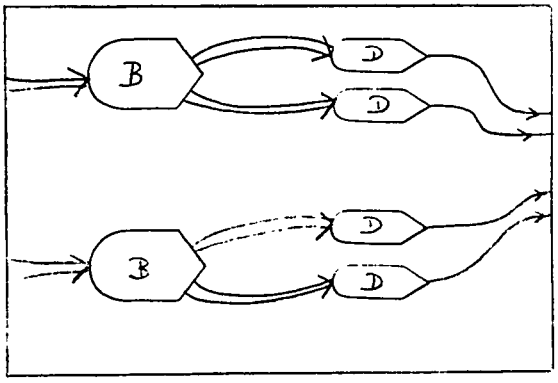


Figure 22

Step I. Verbal Description.

The complex system consists of four "identical" D-systems, and two "identical" B systems. The complex system is considered as successful provided that two or more of the D systems, survive for a specified period of time. The operation of the complex system, is such that failure of any of its B, systems, renders the associated D, system, inoperable. That is an operable D, system, rendered inoperable by the failure of its B system, is not considered as failed, and does not require servicing.

Let both B systems, have the same constant failure and repair rates, denoted by λ_1 and μ_1, respectively. Similarly denote

the identical constant, failure and repair rates of a D system by λ_2 and μ_2 respectively.
It is assumed that there are enough repair crews and service facilities, so that all queues are avoided.
Also repairs are begun immediately upon failure.

Step II State Assignment[13].

State	Description
1	Both B systems, and all four D systems, are operating.
2	One of the D systems, fail.
3	Two D systems, both associated with a single B system, fail.
4	Two D systems, one associated with each of the B systems fail.
5	One of the B systems fail.
6	One of the B systems, and one of its associated D systems, fail.
7	One of the B systems, and both of its D systems fail.
8	More than two D systems fail, or both B systems, fail.

Step III. Construction of the Transition Diagram.
The reliability transition diagram, constructed from the above

[13] States 1-7, are the "acceptable" states. State 8, is the fail state.

state assignment, is shown in Figure (23).

Step IV. Construction of the Transition Graph.
The reliability transition graph, and the matrix [A] are shown in Figures (24) and (25) respectively.

Step V. Construction of the Mason Graph.

The Mason graph, constructed from the transition graph, is shown in Figure (26).

Step VI. Analog Computer Simulation.

The analog computer schematic, obtained from the Mason graph, is shown in Figure (27). The plot of the reliability function R(t), and the mean time to first failure, for the sample case below, is shown in Figure (28).

Sample Case.

λ_1	μ_1	λ_2	μ_2	Mean time t_m to first failure.
0.05	0.5	1.0	2.0	5.8 hrs

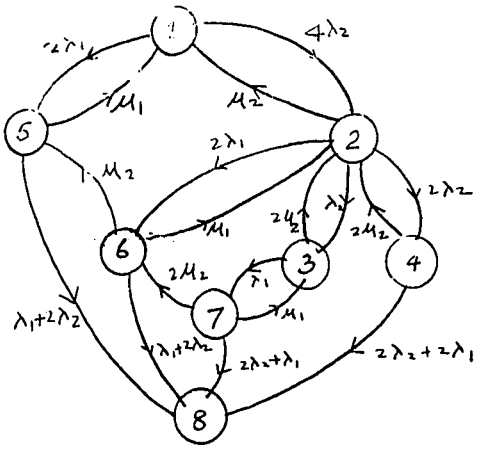


Figure 23.

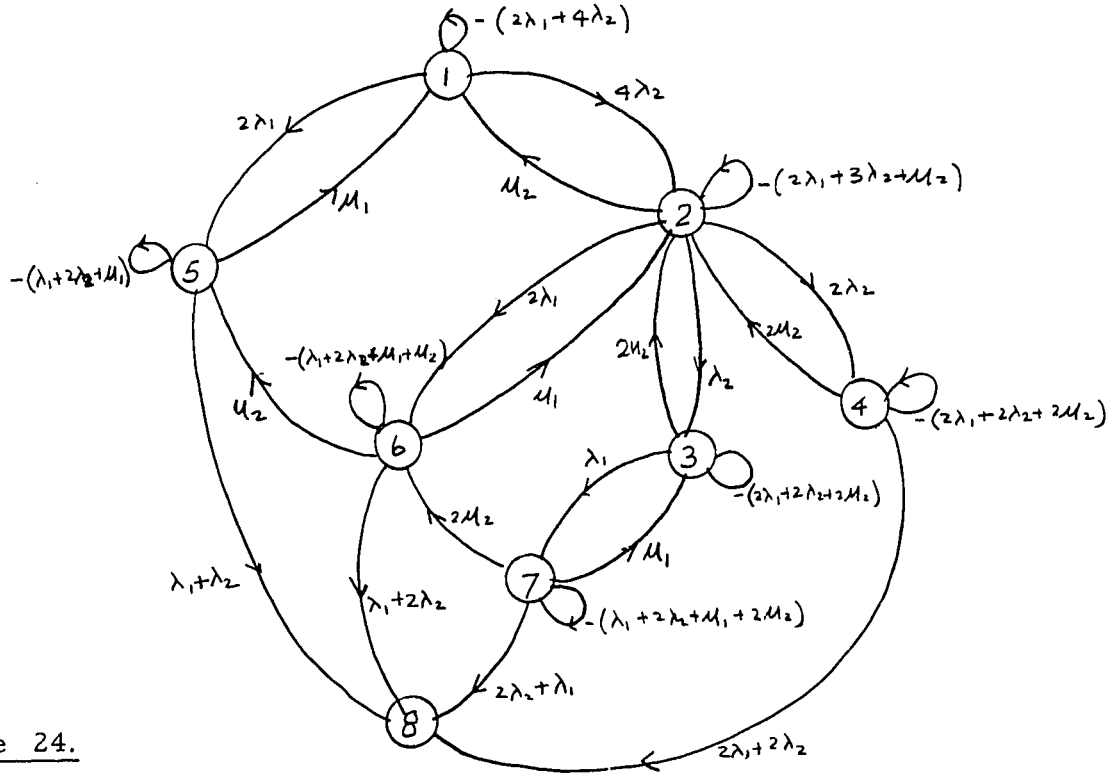


Figure 24.

$$
\begin{bmatrix} \dot{P}_1(t) \\ \dot{P}_2(t) \\ \dot{P}_3(t) \\ \dot{P}_4(t) \\ \dot{P}_5(t) \\ \dot{P}_6(t) \\ \dot{P}_7(t) \\ \dot{P}_8(t) \end{bmatrix} =
\begin{bmatrix}
-(2\lambda_1+4\lambda_2) & \mu_2 & 0 & 0 & \mu_1 & 0 & 0 & 0 \\
4\lambda_2 & -(2\lambda_1+3\lambda_2+\mu_2) & 2\mu_2 & 2\mu_2 & 0 & \mu_1 & 0 & 0 \\
0 & \lambda_2 & -(2\lambda_1+2\lambda_2+2\mu_2) & 0 & 0 & 0 & \mu_1 & 0 \\
0 & 2\lambda_2 & 0 & -2(\mu_2+2\lambda_2+2\mu_2) & 0 & 0 & 0 & 0 \\
2\lambda_1 & 0 & 0 & 0 & -(\lambda_1+2\lambda_2+\mu_1) & \mu_2 & 0 & 0 \\
0 & \lambda_1 & 0 & 0 & 0 & -(\lambda_1+2\lambda_2+\mu_1+2\mu_2) & 0 & 0 \\
0 & 0 & \lambda_1 & 0 & 0 & 0 & -(\lambda_1+2\lambda_2+\mu_1+2\mu_2) & 0 \\
0 & 0 & 0 & 2\lambda_2+2\lambda_1, & \lambda_1+2\lambda_2, & \lambda_1+2\lambda_2, & 2\lambda_2+\lambda_1, & 0
\end{bmatrix} \times
\begin{bmatrix} P_1(t) \\ P_2(t) \\ P_3(t) \\ P_4(t) \\ P_5(t) \\ P_6(t) \\ P_7(t) \\ P_8(t) \end{bmatrix}
$$

Figure 25.

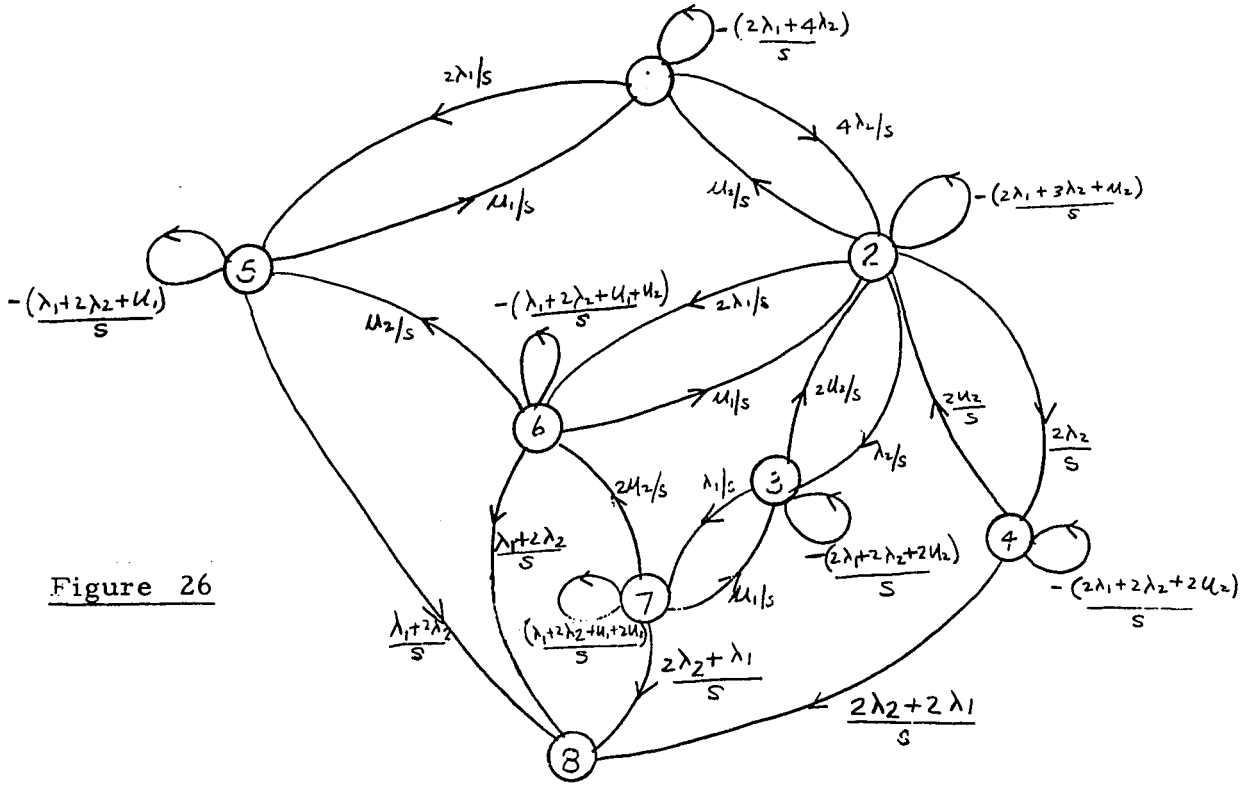


Figure 26

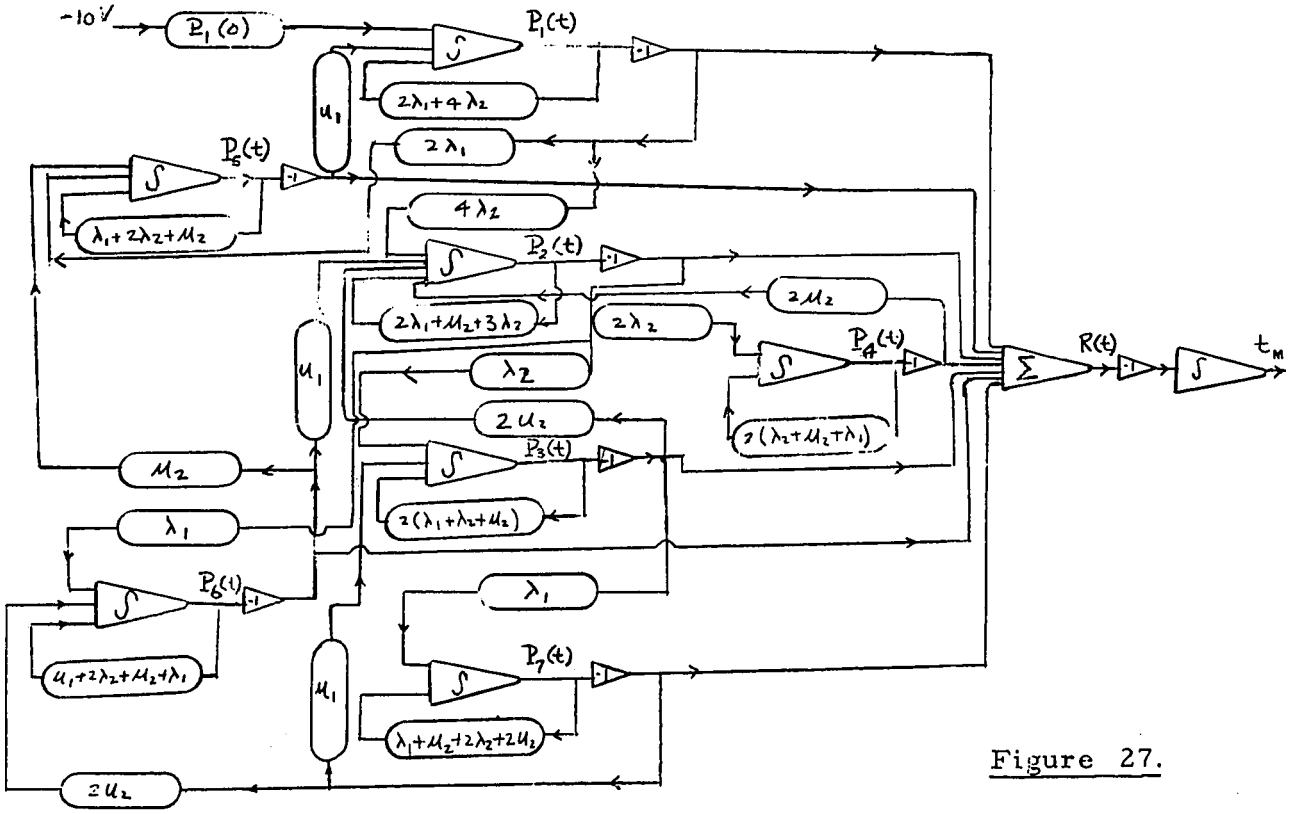


Figure 27.

Analog Computer Graphs

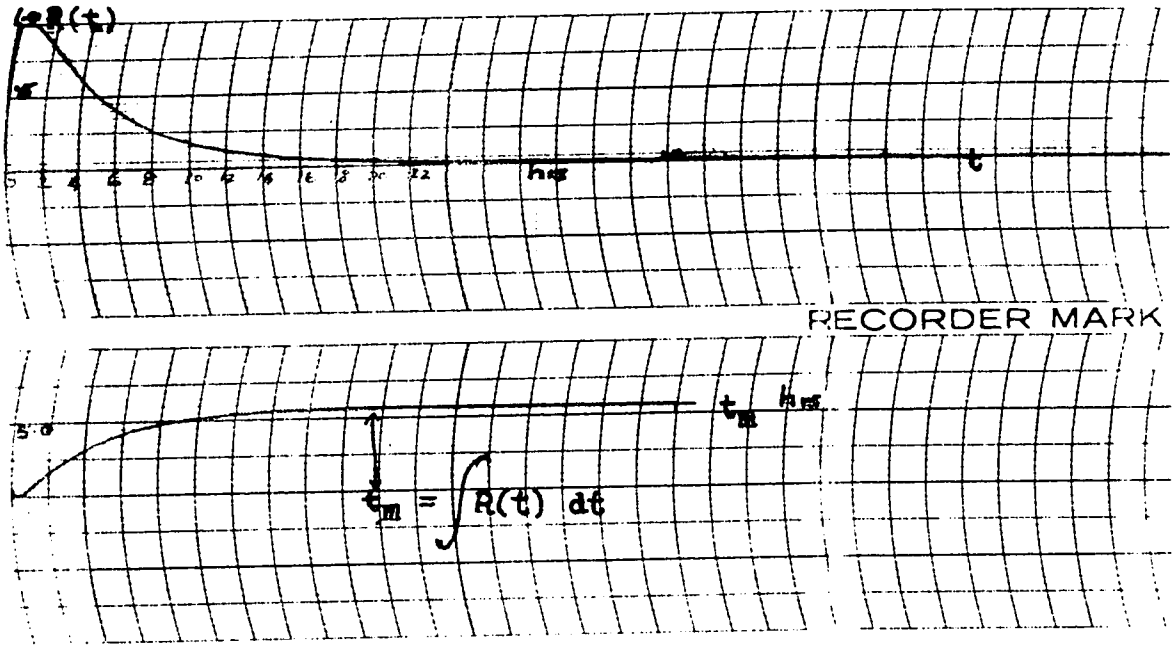


Figure 28.

CONCLUDING REMARKS.

In chapter II, we have presented a survey of the currently available techniques for the reliability and availability modelling of complex systems composed of independent systems, each of which exhibit constant failure and repair rates. This survey led to the development of a general algorithm for the systematic modelling and analysis of such types of complex systems. It was also indicated in example II, how a generally applicable technique of "state splitting", could be utilized, to take into account the important case of a random waiting time between failure and the beginning of repair.

SUGGESTIONS FOR FURTHER RESEARCH.

These suggestions are divided as indicated below, into those suggestions, which pertain to chapter I, and those that pertain to chapter II.

CHAPTER I

There exists extensive mathematical literature on the subject of diffusion processes[14]. The literature is the work of pure mathematicians, and therefore from an engineering point of view, the mathematics there is definately on an exalted plane. However there is to be found in this literature, many elegant results, which appear to have direct application in the reliability field. It is therefore this author's opinion, that any effort spent in translating these results into the reliability field, will make for significant and worthwhile contributions.

For example, hidden periodicities and transients in the fluctuations in a parameter's values, could be treated by well known methods from the theory of diffusion processes.

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14 See for example references (21 → 30).

CHAPTER II

It is this author's opinion, that researches interested in expanding the theory presented in this chapter, would find it profitable, to examine

1) The optimization techniques and methods of Control Theory, with a view to utilizing them in formulating a theory of optimum reliabile systems[15].

2) Since device operation is essentially binary, (that is, a device either works or fails, is repaired, or is not repaired), modelling problems in reliability and availability, starting with state assignments from word descriptions, could borrow from the theory of Switching Circuits, such concepts as minimal state assignments and realization schemes[16].

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[15] Since the completion of this work, there has been an attempt, (See Fan et al. IEEE Trans: Reliability Sept. 1967), to apply the discrete maximum principle to problems in reliability.

[16] See for example reference (20).

APPENDIX A

On Brownian Processes

In this appendix, we define mathematically, a Brownian process X(t). We then give, and inter relate, the statistics necessary to characterize X(t); and develop a linear partial differential equation, from the solution of which, explicit expressions for the statistics of X(t) are deduced. A theorem is then given, which states, that for certain initial data, (which correspond, with the initial data of our reliability model), this partial differential equation has an unique solution. This unique solution is then obtained, via the use of Laplace transform techniques, and the explicit expressions given in chapter I, for the conditional probability density function. $p_X(x_o/x;t)$, and first order density $p_X(x;t)$, established.

Accordingly then, let X, be a typical system parameter, and X(t), be the Brownian process which characterizes the time behaviour of X. We now give some basic definitions, and develop the pertinent statistics for X(t).

Definition. Stochastic Process. The family of time functions, generated, by assigning to sets, ξ, of outcomes of some experiment, according to some rule a time function, $X(t,\xi)$, is called a stochastic process. That is,

$$X(t,\xi) \equiv X(t)$$

is called, a stochastic process.

Comment At any specified time t = t, X(t) is a random variable, that is a function, whose range, we take, to be the real line.

Then, by the statistics of a stochastic process X(t), is meant, a set of probability density functions, necessary to characterize the random variables, X(t), $\forall$ t. Also, the statistics of X(t), are said to be Gaussian, when all the probability density functions, characterizing X(t), are of the Gaussian type. Such density functions, have the remarkable property, not possessed by any other type of density function, namely, they retain their Gaussian property, under all linear transformations. Also, from an experimental point of view, the occurrence of Gaussianly distributed random variables, is the rule, rather than the exception.

Definition. Stationarity. The stochastic process X(t), is said to be stationary in time, (in the "strict sense"), iff the random variables, X(t) and X(t+a), have the same statistics, $\forall$ t, and any arbitrary a .

Physically, by stationarity in time, is meant, that the underlying microscopic mechanisms, producing variations in the values of X, do not change with time.

Definition. Gaussianly Distributed Random Variable.
A random variable X(t), is said to be Gaussianly distributed $\forall$ t, iff

$$\langle X^n(t) \rangle = 0 \quad \forall\, n > 2, \ \forall\, t. \qquad \text{I-1}$$

$$= m_1 \quad \text{for } n = 1. \qquad \text{(first moment)}$$

$$= m_2 \quad \text{for } n = 2. \qquad \text{(second moment)}$$

That is, a Gaussianly distributed random variable, has all moments, except its first two, equal to zero.

Definition. Markov Process. A stochastic process X(t), is a markov process, iff, for every time sequence

$$t_o < t_1 < t_2 < \ldots\ldots < t_{n-1} < t_n$$

with probability one,

$$P\left\{ \bigcap_{j=o}^{n-1} X(t_j) = x_j \,/\, x_n \leq X(t_n) \leq x_n + dx_n \right\} =$$

$$P\{ \quad X(t_j) = x_j \ / \ x_n \leq X(t_n) \leq x_n + dx_n \} \qquad \text{I-2}$$

That is, X(t), is a Markov process, iff, $\forall\, t_i < t$, any condition on $X(t_i)$, has no influence on the statistics of X(t). In the case of a stationary Markov process, X(t), the time t_{n-1}, is arbitrary, and can be set equal to zero. Then t_n can be replaced by t. This brings as to the following definition

Definition Stationary Markov Process. The stochastic process X(t), is a stationary Markov process, iff $\forall\, t$, and any arbitrary a; with probability one,

$$P\{X(t_{n-1} + a) = x_{n-1} \ / \ x_n \leq X(t_n) \leq x_n + dx_n\}$$

$$= P\{X(t_{n-1}) = x_{n-1} \ / \ x_n \leq X(t_n) \leq x_n + dx_n\}$$

or alternatively

$$P\{X(t_{n-1}) = x_{n-1} \ / \ x_n \leq X(t_n) \leq x_n + dx_n\}$$

may be written as

$$P\{X(o) = x_o \ / \ x \leq X(t) \leq x + dx\} = p_X(x_o/x;t)dx \qquad \text{I-3}$$

Equation (I-3), is the conditional density function for the stationary Markov process X(t), defined in (3) of Chapter I.

We are now in a position, to define a Brownian process

Definition. Brownian Process. A stochastic process X(t), which has the properties given in equations (I-1) and (I-3), is called a Brownian process.

That is, a stochastic process X(t), is a Brownian process, iff, (1) it is a stationary Markov process and, (2), the random variables, X(t), $\forall\, t$, are Gaussianly distributed.

The Statistics of the Brownian Process X(t)

Since X(t), $\forall$ t, is a stationary Markov process, with Gaussian statistics, its first and second order statistics completely characterize it. We now define these statistics, and develop a linear partial differential equation, whose unique solution is $p_X(x_o/x;t)$.

Definition. First Order Density Function. The first order density function, $p_X(x;t)$, of the random variable X(t), is defined as

$$p_X(x;t)dx = P\{x \leq X(t) \leq x + dx\} \qquad \text{I-4}$$

and since X(t) has Gaussian statistics, $p_X(x,t)$, $\forall$ t, is of the form

$$p_X(x;t) = \frac{1}{\sqrt{2\pi\sigma_t^2}} e^{-\frac{(x-m_t)^2}{2\sigma_t^2}} \qquad \text{I-5}$$

in particular at t = o, the initial first order density of the random variable, X(t), is written as

$$p_X(x_o) = \frac{1}{\sqrt{2\pi\sigma_o^2}} e^{-\frac{(x_o-m_o)^2}{2\sigma_o^2}} \qquad \text{I-6}$$

Equation I-4, gives the probability that at time t, the random variable X(t), has a value that has in the interval (x,x+dx) that is $p_X(x;t)dx$, is the probability that at time t, parameter X, has a value that lies in the interval, (x, x+dx). Accordingly then, it appears natural to define a reliability function for the parameter X, as

Definition. Reliability Function For a Parameter. The reliability function R(t), called the reliability function of the parameter X, gives a measure of the probability, that at time t, the parameter X, has a value that lies in the interval (x_2-x_1), that is

$$R(t) = P\{x_1 \leq X(t) \leq x_2\}$$

$$= \int_{x_1}^{x_2} p_X(x;t)dx \qquad \text{I-7}$$

then on substituting (I-5) into (I-7), we obtain

$$R(t) = \frac{1}{\sigma_t\sqrt{2\pi}} \int_{x_1}^{x_2} e^{\frac{(x-m_t)^2}{2\sigma_t^2}} dx \qquad \text{I-8}$$

and putting $y = \frac{(x - m_t)}{\sigma_t}$, equation (I-8) becomes

$$R(t) = \frac{1}{\sqrt{2\pi}} \left(\int_o^{\frac{(x_2-m_t)}{\sigma_t}} e^{-y^2} dy + \int_{\frac{-(x_1-m_t)}{\sigma_t}}^{o} e^{-y^2} dy \right)$$

$$= \text{Erf}\left(\frac{x_2-m_t}{\sigma_t}\right) + \text{Erf}\left(\frac{x_1 - m_t}{\sigma_t}\right) \qquad \text{I-9}$$

It should be noted, that equation (I-9), is the same as equation (10), of Chapter I.

In the above development of the reliability function, R(t) it was assumed that $p_X(x,t)$ is of the form given in (I-5). The validity of this assumption is now established, by showing, that $p_X(x,t)$, is uniquely determined from $p_X(x_o/x,t)$, where $p_X(x_o/x,t)$, is the unique solution of a linear partial differential equation. Accordingly then we give the following definitions:

Definition. Conditional Probability Density Function. The conditional probability density $p_X(x_o/x, t)$, of the random variable X(t), $\forall$ t, is a second order statistic of X(t), defined as

$$p_X(x_o/x;t)dx = P\{X(o) = x_o / x \leq X(t) \leq x+dx\} \qquad \text{I-10}$$

that is, $p_X(x_o/x;t)dx$, gives a measure of the probability, that at any specified time t = t, the random variable X(t), has a value in the interval x, x+dx; given that a time t=o, the random variable, $X(o) = x_o$.

Definition. Joint Second Order Density Function. The joint second order density function $p_X(x_o:x;t)$ of the random variable X(t), $\forall$ t; is a second order statistic of X(t), defined as

$$p_X(x_o:x;t)\, dx_o dx = P\{x_o \leq X(o) \leq x_o + dx_o : x \leq X(t) \leq x+dx\} \qquad \text{I-11}$$

that is $p_X(x_o: x;t)\, dx_o\, dx$, gives a measure of the joint probability, that a time t = o, the random variable X(o) is in the interval (x_o, x_o+dx_o), and at time t = t, the random variable X(t), is in the interval x, x+dx.

Equation (I-11), is obtained from (I-6) and (I-5) by multiplication, as

$$p_X(x_o: x;t) = p_X(x_o/x;t)\; p_X(x_o) \qquad \text{I-12}$$

now $p_X(x_o;t)$, is obtained from $p_X(x_o: x;t)$ by integrating, that is,

$$p_X(x;t) = \int_{-\infty}^{\infty} p_X(x_o; x;t)\, dx_o \qquad \text{I-13}$$

and substituting (I-12), into (I-13), we obtain equation (4), of Chapter I,

$$p_X(x;t) = \int_{-\infty}^{\infty} p_X(x_o/x;t)\; p_X(x_o)dx_o \qquad \text{I-14}$$

then $p_X(x;t)$, is uniquely determined from $p_X(x_o/x;t)$. Thus it remains to be established that $p_X(x_o/x;t)$ is itself unique. We establish this, by developing a linear partial differential equation, which has as solution $p_X(x_o/x;t)$, and then give a theorem which states that this solution is unique. This unique solution for $p_X(x_o/x;t)$ is then replaced in (I-14), and equation (I-5), is obtained, thus proving the validity of the assumption of the form of (I-5); and establishing equation (5) of Chapter I.

The Partial Differential Equation For $p_X(x_o/x;t)$

The derivation presented below, is an outline of that given by Kac (ref 5).

We begin, by defining a discrete conditional density function

$$p_X(n\Delta x / m\Delta x; s\Delta\tau) = P\{X(o) = n\Delta x / x(s\Delta\tau) = m\Delta x\} \quad \text{I-15}$$

then (I-15), gives a measure of the probability, that parameter X has a value $m\Delta x$, at time $s\Delta\tau$, (that is, after s, "jumps", each of duration $\Delta\tau$), given that initially, at time $t = o$, it had the value $n\Delta x$. The transition from this discrete density, to the continuous density, is then effected by the limit,

$$p_X(x_o / x;t) = \underset{\substack{n\Delta x \to x_o \\ m\Delta x \to x_o \\ s\Delta\tau \to t}}{\text{Limit}} p_X(n\Delta x / m\Delta x; s\Delta\tau) \quad \text{I-16}$$

Now, by an inductive argument, we will show that $p_X(n\Delta x/m\Delta x;s\Delta\tau)$, satisfies the difference equation

$$p_X(n\Delta x / m\Delta x ; (s+1)\Delta\tau) = \alpha \{p_X(n\Delta x / (m+1)\Delta x; s\Delta\tau)\} + \beta \{p_X(n\Delta x / (m-1)\Delta x; s\Delta\tau)\} \quad \text{I-17}$$

where α , and β, are respectively, the probability of an increase or decrease, Δx, in the value of X, in time $\Delta\tau$; and

$$\alpha = \frac{1}{2} + \frac{C}{2D}\,\Delta x$$

$$\beta = \frac{1}{2} - \frac{C}{2D}\,\Delta x$$

C and D, are constants, which may be either positive or negative, and are called the drift and diffusion coefficients respectively.

Then equation (1-17), may be inductively established as follows. Since the process is Markov, the event, given by the L. H. S. of (I-17); namely the value $m\Delta x$ at time $(s+1)\,\Delta\tau$, from $n\Delta x$ at $t = o$. This event is the sum of the two mutually exclusive events, (a) and (b), below.

a) the value $(m+1)\Delta x$, was attained at time $s\Delta\tau$ from a value $n\Delta x$, at time $t = o$. Then a decrease of Δx, (with probability β), occurred, in the next interval of time $\Delta\tau$.

b) The value $(m-1)\Delta x$, was attained at time $s\Delta\tau$, from a value $n\Delta x$, at time $t = o$. Then an increase of Δx, (with probability α), occurred in the next interval of time $\Delta\tau$.

QED.

Then from Kac, ref (5); in the limit, equation (I-16), and

$$\Delta x \to dx, \quad \Delta\tau \to d\tau, ; \quad \frac{\Delta x^2}{2\Delta\tau} \equiv D$$

the difference equation (I-17), becomes the partial differential equation,

$$\frac{\partial\{p_X(x_o/x;t)\}}{\partial t} = \frac{D\,\partial^2\{p_X(x_o/x;t)\}}{\partial x^2} + \frac{2C\,\partial\{p_X(x_o/x;t)\}}{\partial x}$$

We now give a theorem, which states, that subject to certain intial conditions, (which are always satisfied in our reliability model), the solution of (I-18), is unique. Then we solve (I-18), via Laplace transform techniques, subject to these initial conditions, and obtain an explicit form for $p_X(x_o / x;t)$.

Theorem. If there exists a $p_X(x_o / x;t)$ such that the following conditions are satisfied,

a) $X(t)$, $\forall\, t$, is a stationary Markov process, with Gaussian statistics.

b) $\lim_{t \to o} p_X(x_o/x;t) = \delta(x-x_o)$

c) $\lim_{x \to o} p_X(x_o / x;t) = o$

then $p_X(x_o/x;t)$, subject to this initial data (b),(c), is the unique solution of (I-18).

Proof: See references (1), (2).

Using Laplace transform techniques, we now solve (I-18), subject to the initial data (b) and (c), given in the theorem. Accordingly then, putting x, as variable and denoting the two sided Laplace transform as

$$\mathcal{L}\{ p_X(x_o/x;t) \} = P(x_o / s;t)$$

equation (I-18), transforms into a first order, ordinary differential equation.

$$\frac{d(P_X(x_o/s;t))}{dt} = (Ds^2 + 2sC) P_X(x_o/s;t) \qquad \text{I-19}$$

and integrating (I-19), yields

$$P_X(x_o/s;t) = k e^{Ds^2 t + 2Cst} \qquad \text{I-20}$$

where the constant of integration k, is obtained from (b), of the theorem, as

$$k = \lim_{t \to o} P_X(x_o / s;t)$$

$$= \mathcal{L}\{\lim_{t \to o} p_X(x_o/x,t)\} = e^{sx_o} \qquad \text{I-21}$$

then substituting (I-21), into (I-20), we have a convolution,

$$P_X(x_o/s;t) = e^{sx_o} * e^{Ds^2t + 2sCt} \qquad \text{I-20}$$

and the inverse transform of (I-20), gives

$$p_X(x_o/x;t) = \frac{1}{\sqrt{2\pi.\,2tD}} \int_{-\infty}^{\infty} e^{-\frac{(x-\xi-2Ct)^2}{2.2Dt}} \delta(\xi - x_o)\,d\xi \qquad \text{I-22}$$

since, from ref (9),

$$\mathcal{L}^{-1}\{e^{Ds^2t+2Cst}\} = \frac{1}{\sqrt{2\pi.\,2tD}} e^{-\frac{(x-2Ct)^2}{2.2Dt}}$$

by the "sifting" property of the delta function, (I-22), gives, the explicit solution for $p_X(x_o/x;t)$. viz:,

$$p_X(x_o/x;t) = \frac{1}{\sqrt{2\pi.\,2tD}} e^{-\frac{(x-x_o-2Ct)^2}{2.2Dt}} \qquad \text{I-23}$$

which is equation (3) of Chapter I. Now from (I-4),

$$p_X(x;t) = \int_{-\infty}^{\infty} p_X(x_o/x;t)\, p_X(x_o)\,dx. \qquad \text{I-4}$$

on substituting (I-6) and (I-23), into the convolution integral (I-4), we obtain

$$p_X(x;t) = \frac{1}{\sqrt{2\pi(\sigma_o^2 + 2Dt)}} e^{\frac{(x-m_o-2Ct)^2}{2(\sigma_o^2+2Dt)}} \qquad \text{I-24}$$

and (I-24), maybe written as,

$$p_X(x;t) = \frac{1}{\sqrt{2\pi\sigma_t^2}} e^{-\frac{(x-m_t)^2}{2\sigma_t^2}}$$

which is equation (5) of Chapter I, and equation (I-5) of the appendix. where

$$m_t = m_o + 2Ct$$

$$\sigma_t^2 = \sigma_o^2 + 2Dt$$

as was given in equation (7) of Chapter I.

APPENDIX B

Description of Mason's formula.

In chapter II, it has been demonstrated, that the effecting of an explicit solution to a reliability or availability model reduces to solving, a set of equations of the form of equation (8). That is we have to obtain explicit solutions, to

$$\dot{\overline{P}}(t) = [A]\overline{P}(t) \tag{8}$$

where the general solution of (8) is of the form

$$\overline{P(t)} = [\Phi(t)]\ P(o) \tag{11}$$

taking the Laplace transform of (11), yields

$$p_i(s) = \sum_{j=1}^{n} \Phi_{ij}(s)\ P_j(o)$$

Since the $P_j(o)$ are arbitrary, the $\Phi_{ij}(s)$ can be made linearly independent, and uniquely determined, from the ratio

$$\Phi_{ij}(s) = \frac{p_i(s)}{P_j(o)} \tag{35}$$

This was pointed out in method IV, of chapter II. Equation (35), can be interpreted as the gain between modes j and i of a Mason signal flow graph. Then an alternative expression for (35), is

$$\Phi_{ij}(s) = \sum_{k} \frac{\Delta_{ij}^{k}(s)\ G_{ij}^{k}(s)}{\Delta(s)} \tag{36a}$$

where; a) $G_{ij}^{k}(s)$: is the gain k^{th} forward path from node j to node i.

b) $\Delta_{ij}^{k}(s)$: is that path of $\Delta(s)$, which remains, when all branches touching the k^{th} forward path between nodes i and j, and all branches having a node in common with it are removed.

c) $\Delta(s)$: is called the graph determinent; $\Delta(s) = 1 - \Sigma$ gains of the feedback paths taken one at a time and Σ product of the gains of non-touching feedback paths, taken feedback paths taken two at a time - Σ product of the gains of non-touching three at a time, etc. .

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