AN INTRODUCTION TO NON-STANDARD REAL ANALYSIS

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ABSTRACT

The non-standard analysis provides an alternate approach to real analysis including elementary calculus. To examine the validity and usefulness of an informal presentation proposed by H.J. Keisler a rigorous treatment of non-standard analysis is presented following W.A.J. Luxemburg. The pedagogical implications introducing infinitesimals are inconclusive due to the lack of sufficient experimental information.

CHAPTER 0

INTRODUCTION

Since the invention of differential and integral calculus by Newton and Leibniz at the end of the Seventeenth Century the question of infinitesimals has been an indispensable and often controversial topic in the foundation of analysis. The existence of "infinitely small" numbers which has been taken for granted by Leibniz and his followers has been disputed on mathematical as well as on philosophical grounds.

The developers of calculus in the Eighteenth Century (Bernoullis, Euler, Lagrange) made frequent, often ingenious, uses of the notions of "infinitely small" and "infinitely large". Occasionally, certain paradoxical situations (as in the convergence problems for infinite series) arose but these were handled skilfully if not rigorously and the end results were accepted as useful and correct.

During the Nineteenth Century with Cauchy, Balzano, Cantor, Dedekind and Weierstrass the infinitesimals were eliminated by using more precise notions as convergence, limits, completeness as applied to real number system. It is worth noting that Cantor whose work laid the foundations for classification of "infinities" (cardinalities of sets) categorically rejected the possible existence of infinitely small numbers. It appeared that this notion was now completely eliminated from mathematics.

However, in 1960 Abraham Robinson (1918-1974) introduced infinitesimals within the framework of model theory. He showed that there exists a new kind of extension of the real number system which properly includes the standard (classical) one. This extended system contains "infinitely large" and "infinitel small" elements and gives rise to so called "non-standard" analysis. Such non-standard models (first introduced by Skolem in 1934 for arithmetic) have been created for various mathematical structures.

The usefulness of non-standard analysis was demonstrated by several new proofs of old theorems and by solutions to old unsolved problems (invariant subspaces in Hilbert space theory, diophantine analysis in number theory, field theory in algebra) To quote the eminent logician Kurt Gödel who commenting on Robinson's talk at the Institute for Advanced Study at Princeton in March 1973 said: (in part)

"I would like to point out a fact that nonstandard analysis frequently simplifies substantially the proofs, not only of elementary theorems, but also of deep results. This is true, e.g. also for the proof of the existence of invariant subspaces for compact operators, disregarding the improvement of the result; and it is true in an even higher degree in other cases. This state of affairs should prevent a rather common misinterpretation of non-standard analysis, namely the idea that it is some kind of extravagance or fad of mathematical logicians. Nothing could be farther from the truth. Rather there are good reasons to believe that non-standard analysis, in some version or other, will be the analysis of the future.

One reason is the just mentioned simplification of proofs, since simplification facilitates discovery. Another, even more convincing reason, is the following: Arithmetic starts with the integers and proceeds by successively enlarging the number system by rational and negative numbers, irrational numbers, etc. But the next quite natural step after the reals, namely the introduction of infinitesimals, has simply been omitted. I think in coming centuries it will be considered a great oddity in the history of mathematics that the first exact theory of infinitesimals was developed 300 years after the invention of the differential calculus".*

The use of model theory as a foundation for non-standard analysis require a considerable amount of mathematical logic. An attempt has been made by W.A.J. Luxemburg to establish a minimal logical apparatus needed for a proper introduction of infinitesimals. His approach has been followed in this thesis.

In teaching elementary calculus two problems present themselves: 1) the intuitive concepts underlying the use of real number system and 2) the question of minimal rigor. Concerning the first, the notions of limits, completeness, continuity require some understanding of "small" and "large" which ultimatel lead to the use of "infinitely small" and "infinitely large".

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* Private communication from Mrs. A. Robinson.

As to the second points the classical ε-δ technique is usually beyond the ability of most students at that stage and could be often a self-defeating device. To circumvent these difficulties, hundreds of attempts have been made by text-book writers.

An interesting but by no means a definitive attempt has been made by H. Jerome Keisler to present introductory calculus based on infinitesimals. Since the present foundations of non-standard analysis require a considerable logical machinery and several rather subtle notions Keisler's approach is to establish an intuitive basis for the larger number system H (called hyper-reals) and assume certain axioms (rather loosely stated) in order to develop the usual techniques for limits, convergence, continuity derivatives and integrals.

In Chapters I - VI we describe the non-standard number system *R as an extension of the real number system R, in Chapter VII Keisler's approach to the elementary introduction is presented.

The references are given by a number in square brackets.

The Bibliography includes a list of general material used in the preparation of the thesis.

CHAPTER I

THE LOGICAL MACHINERY

§1 In real analysis quantification in the formal language is over real numbers (as "individuals" or elements) and over sets.

In example (1) the quantification is over individuals, - in examples (2) and (3), the quantification is over sets.

Examples

(1) The trichotomy law, in symbols:

$$(\forall x)(\forall y)\ [x \in R \wedge y \in R] \Rightarrow [x < y] \vee [x > y] \vee [x = y].$$

where the quantification of the variables x and y is over real numbers only.

(2) The Peano Induction Axiom, in symbols:

$$(\forall y)[[y \in N_1 \wedge 0 \in y \wedge n \in y] \Rightarrow [n + 1 \in y]] \Rightarrow (\forall n)[n + 1 \in y]$$

where N_1 is the power set of N.

The Dedekind upper bound property can be stated as follows:

(3) Every non-empty subset of R which is bounded above has a least upper bound. In symbols:

$$\begin{aligned}&(\forall x)(\forall y)(\exists z)[[[x \in R_1 \wedge x = \phi \wedge y \in R] \wedge [z \in x \Rightarrow z < y \vee z=y]] \Rightarrow \\ &\qquad (\exists w)[[w \in R] \wedge (\forall s)[s \in x \Rightarrow s < w \vee s = w] \wedge \\ &\qquad\qquad (\forall v)[(\forall r)[r \in x \Rightarrow r < v \vee r = v] \Rightarrow \\ &\qquad\qquad\qquad [w < v \vee w = v]]]\end{aligned}$$

where R_1 is the power set of R.

Our aim is to construct a language which should contain all statements of real analysis which deal with numbers, relations between numbers, relations between sets and numbers and so on. This language will be based on the axiomatic set theory and its constants will range over sets and numbers.

§2 The Language L

The atomic symbols of L are:

(1) The connectives: $\wedge$, $\vee$, $\Rightarrow$, $\Leftrightarrow$, $\neg$, which stand for "and", "or", "implies", "iff", "not", respectively.

(2) The variables: a countably infinite sequence denoted by x, y, z, ... with or without subscripts.

(3) The quantifiers: $(\exists\cdot)$ and $(\forall\cdot)$ for existential and universal quantifiers, respectively.

(4) Brackets: [ ], ( ).

(5) The basic predicates: $=$, $\in$ for "equality" and "membership", respectively.

(6) Extra logical constants (briefly, constants): this is a set of symbols which is large enough to be put in one-to-one correspondence with the entities of whatever structure may be under consideration. This set of constants is infinite but fixed. The constants will be denoted by Roman letters with or without subscripts and numerals 0, 1, 2, ...

The atomic formulas are $\alpha \in \beta$ and $\gamma = \rho$ where α, β, γ, ρ denote constants and variables. The wff (well-formed formulas) will be obtained in successive stages by applying connectives and quantifiers. If V is an atomic formula, then [V] is a wff If V, W are wff, then [V ∧ W], [V ∨ W], [V ⇒ W], [V ⇔ W] and [¬V] are wff. If V is a wff, then [(∀x)V] and [(∃x)V] are wff, where x denotes an arbitrary variable provided x does not already appear in V under the sign of a quantifier. In wff [(∀x)V] and [(∃x)V], V is called the <u>scope</u> of the quantifier.

A variable x is called <u>free</u> in a wff [V] if x is not in (∀x) or (∃x) or in the scope of a quantifier in V.

A wff is called a <u>sentence</u> if every variable is in the scope of a quantifier.

For real analysis we shall only consider those wff of L whose quantifiers are of the form:

"(∀x)[[x ∈ A] ⇒ ...]" and "(∃x)[[x ∈ A] ∧ ...]"

where A is an element in a certain structure. The wff of L wit this property are called the <u>admissible</u> wff.

The set of admissible wff of L will be denoted by K = K(L) and the subset of K of all <u>admissible sentences</u> which hold in a certain structure by $K_o = K_o(L)$.

§3 The Structure of $\hat{R}$ and Some of its Properties

Let R be the set of real numbers. We define inductively the sets

$R_0 = R$ and $R_{n+1} = P(\bigcup_{k=0}^{n} R_k)$ where P(X) denote the power set of

$\bigcup_{n\geq 0} R_n$ is called the <u>superstructure</u> on R and is denoted by $\hat{R}$.

The elements of $\hat{R}$ are called the entities, but the elements of R_o are referred to as the individuals.

Definition 3.1. The entities of $R_n - R_{n-1}$ $(n \geq 1)$ are called of rank n in $\hat{R}$. The individuals are given rank 0.

By this definition, the empty set gets assigned rank 1. (i.e. $\phi \in R_1 - R_o$.)

If $a \in \hat{R} \neq \phi$, then the rank of a is the smallest natura number n such that $a \in R_n$. In fact, if a is of rank n, then $a \in R_n - R_{n-1}$ which implies $a \in R_n$ and $a \notin R_{n-1}$; here n is t smallest natural number such that $a \in R_n$ is justified.

Lemma 3.1. (i) $R_p \subset R_n$ for all $n \geq p \geq 1$.

(ii) $\bigcup_{k=0}^{n} R_k = R_o \cup R_n$ for all $n \geq 1$.

(iii) $R_k \in R_{n+1}$ for all $0 \leq k \leq n$ and for all $n \geq 0$.

(iv) If $x \in y \in R_n$ $(n \geq 1)$, then $x \in R_o \cup R_{n-1}$.

(v) If $(x_1, \ldots, x_n) \in y \in R_p$ $(p \geq 1)$, then $x_1, \ldots, x_n \in R_o \cup R_{p-1}$. In particular, if an entity $\Phi \in \hat{R}$ is a binary relation, then its domain, dom $\Phi = \{x | (\exists y)(x,y) \in$ and its range, ran $\Phi = \{y | (\exists x)(x,y) \in \Phi\} \in \hat{R}$.

Proof. (i) If $x \in R_p$, then, by definition, $x \subset \bigcup_{k=0}^{p-1}$ and so $x \subset \bigcup_{k=0}^{q} R_k$ for all $q \geq p-1$. Hence $x \in P(\bigcup_{k=0}^{q} R_k) =$ for all $q+1 \geq p$.

(ii) For $n \geq 1$, $R_n \subset R_{n+1}$, and so

$\bigcup_{k=0}^{n} R_k = R_o \cup R_n$ for $R_o \cap R_n = \phi$, for $n \geq 1$.

(iii) From (ii) $R_k \subset R_o \cup R_n$ $(0 \leq k \leq n)$, then

$$R_k \in P(R_o \cup R_n) = R_{n+1}.$$

(iv) If $y \in R_n$, $(n \geq 1)$, then $y \subset R_o \cup R_{n-1}$, so $x \in R_o \cup R_{n-1}$.

(v) If $(x_1, \ldots, x_n) \in y \in R_p$ $(p \geq 1)$, then $(x_1, \ldots, x_n) \in R_o \cup R_{p-1}$.

Hence

$\{\{x_1\}, \{x_1, (x_2, \ldots, x_n)\}\} \in R_o \cup R_{p-1} = R_o \cup P(R_o \cup R_{p-2})$ implies $x_1 \in R_o \cup R_{p-2} \subset R_o \cup R_{p-1}$. Similarly, we can prove that the entities $x_2, \ldots, x_p \in R_o \cup R_{p-1}$ $(p \geq 1)$. In particular, dom $\Phi \in \hat{R}$ and ran $\Phi \in \hat{R} = \bigcup_{n \geq 0} R_n$.

° An ordered pair (a, b) is defined in the sense of Kuratowski by $(a, b) = \{\{a\}, \{a, b\}\}$.

The n-tuples $(a_1, \ldots, a_n)$ are defined inductively by $(a) = a$, $(a_1, \ldots, a_n) = ((a_1, \ldots, a_{n-1}), a_n)$. It follows that relations defined as sets of n-tuples $(n = 1, 2, \ldots)$ are all entities of $\hat{R}$.

For example, let $E = \{(x, y) | x, y \in R_n\}$ we shall prove that $E \in R_{n+2} \subset \hat{R}$.

The result can be obtained easily.

Indeed,

let

$$x,\ y \in R_n$$

then $$\{x\},\ \{x,\ y\} \in R_{n+1}$$

$$(x,\ y) = \{\{x\},\ \{x,\ y\}\} \in R_{n+2}$$

Hence $$E = \{(x,\ y)\} \subset \hat{R}$$

i.e. $$E \in \hat{R}.$$

In this manner, we can define the algebraic operations of R_o in terms of ternary relations as follows:

Let $a,\ b,\ c \in R_o$, then

$ab = c$ iff $(a,\ b,\ c) \in P \in \hat{R}$ (P is the relation which characterizes the multiplication)

$a + b = c$ iff $(a,\ b,\ c) \in S \in \hat{R}$ (S is the relation which characterizes the addition).

We shall now assume that the set of constants of L is brought in one-to-one correspondence with all the entities of the structure $\hat{R}$ and we shall from now on identify the constants of L with the entities of $\hat{R}$ so that $\hat{R}$ is part of L. If such an identification has been established, then we refer to $\hat{R}$ as an <u>L-structure</u>.

The interpretation of the basic predicate $\in$ of L in $\hat{R}$ will be the membership relation of axiomatic set theory.

CHAPTER II

ULTRAFILTERS AND MONOMORPHISMS

§1 Definition 2.1 (Filter).

By a filter over I (a non-empty set) we mean a non-empty set F of subsets of I such that

(i) the empty set $\phi \notin F$

(ii) if $A, B \in F$ then $A \cap B \in F$

(iii) if $A \in F$ and $B \supset A$ then $B \in F$.

A filter F_1 is called finer than a filter $F_2 (F_2 \leq F_1)$ whenever $A \in F_2$ implies $A \in F_1$.

The relation "$\leq$" orders the set of all filters over I.

Zorn's Lemma. If X is a partially ordered set (i.e. X is reflexive, antisymmetric and transitive) and every chain has an upper bound, then X has a maximal element.

Since the set of all filters over I is a partially ordered set and every chain of them has an upper bound, P(I), then it has a maximal element.

We call this maximal element an ultrafilter.

An important characterization of ultrafilters is the following:

A filter F is an ultrafilter iff for every $A \subset I$ either $A \in F$ or $I - A \in F$. (See [1] Theorem 2.2 on Page 5).

A filter is called free whenever $\cap (A : A \in F) = \phi$.

Definition 2.2 (δ-Incomplete Filter)

A filter $\mathcal{F}$ is called δ-incomplete, whenever there exists a sequence $F_n \in \mathcal{F}$ $(n = 1, 2, \ldots)$ such that $\bigcap_{n=1}^{\infty} F_n \notin \mathcal{F}$.

Lemma. Let $\mathcal{F}$, $\mathcal{E}$ be two filters in $P(I)$. Denote the filter generated by $\mathcal{F}$ and $\mathcal{E}$ by $\mathcal{F} \vee \mathcal{E}$. Then

$$\mathcal{F} \vee \mathcal{E} = \{F \cap E \mid F \in \mathcal{E}, E \in \mathcal{E}\}.$$

Proof. (a) R. S. is a filter.

Take $F_1 \; E_1$ and $F_2 \cap E_2$ in R.S. Then $F_1 \cap E_1 \cap F_2 \cap E_2 = F_1 \cap F_2 \cap E_1 \cap E_2 \in$ R.S. since $F_1 \cap F_2 \in \mathcal{F}$ and $E_1 \cap E_2 \in \mathcal{E}$. Take $F \cap E \in$ R. S. and let $F \cap E \in B$. Then $B = B \cup (F \cap E) = (B \cup F) \cap (B \cup E)$. Now $B \cup F \in \mathcal{F}$, $B \cup E \in \mathcal{E}$. Thus $B \in$ R. S. If $\mathcal{F} \neq \phi$, $\mathcal{E} \neq \phi$ then $\mathcal{F} \vee \mathcal{E} \neq \phi$.

(b) R. S. $\supseteq \mathcal{F}, \mathcal{E}$ since $F \cap I = F$, $I \cap E = E$.

(c) R. S. $\subseteq \mathcal{F} \vee \mathcal{E}$, therefore R. S. $= \mathcal{F} \vee \mathcal{E}$.

Lemma. Let $\{A_n\}_{n=1}^{\infty}$ be a family of sets and let $A = \bigcup_{n=1}^{\infty} A_n$ then $\{(A_n - \bigcup_{i=1}^{n-1} A_k)\}_{i=1}^{\infty}$ is a partition of A, that is $\bigcup_{i=1}^{1-1} A_i = \phi$ $\bigcup_{n=1}^{\infty} (A_n - \bigcup_{i=1}^{n-1} A_i) = A$ and if $n \neq m$

$$(A_n - \bigcup_{i=1}^{n-1} A_i) \cap (A_m - \bigcup_{i=1}^{m-1} A_i) = \phi.$$

Proof. Let $n \neq m$ suppose $n > m$ then $m \leq n - 1$.

$(A_n - \bigcup_{i=1}^{n-1} A_i) \cap (A_m - \bigcup_{i=1}^{m-1} A_i) = A_n \cap \bigcap_{n-1} CA_i \cap A_m \cap \bigcap_{m-1} CA_i$.

since $m \leq n - 1$. CA_m occurs here in $\bigcap_{n-1} CA_i$, but $CA_m \cap A_m = \phi$

therefore $(A_n - \bigcup_{i=1}^{n-1} A_i) \cap (A_m - \bigcup_{i=1}^{m-1} A_i) = \phi$. Clearly

$A_n - \bigcup_{i=1}^{n-1} A_i \subseteq A_n$ therefore $\bigcup_{n=1}^{\infty} (A_n - \bigcup_{i=1}^{n-1} A_i) \subseteq \bigcup_{n=1}^{\infty} A_n = A$.

Take $a \in A$. Then

Case 1 $a \in \bigcap_{i=1}^{\infty} A_i$. Then $a \in A_1$ and $a \in \bigcap_{n=1}^{\infty} A_i$).

Case 2 $a \notin \bigcap_{i=1}^{\infty} A_i$ then there exists n, $a \notin A_n$.

Let $S = \{n \in N | a \notin A_n\}$ and $S \neq \phi$. Therefore there exists the least n_o with $a \notin A_{n_o}$. Thus $a \in A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_{n_o-1}$.

If $a \in A_1$ the proof is complete.

If $a \notin A_1$. Case 1 $a \in A_2$, then $a \in A_2 - A_1$ the proof is complete

Case 2 $a \notin A_2$.

Case (i) $a \in A_3 \Rightarrow a \in A_3 - A_1 \cup A_2$, etc.

Eventually $a \in A_k - \bigcup_{i=1}^{k-1} A_i$. Thus it is a partition of A.

<u>Theorem</u>. Let U be an ultrafilter. Then U is δ-incomplete iff there exists a partition $\{I_n\}_{n=1}^{\infty}$ of I with $I_n \notin U$ for all n.

Proof. Let U be δ-incomplete. Then there exists $F_i \in U$, $i = 1, 2, \ldots$ with $\bigcap_{i=1}^{\infty} F_i \notin U$. Let $\overrightarrow{\bigcap_{i\geq 1} F_i}$ be the filter generated by $\bigcap_{i\geq 1} F_i$ and consider $\overrightarrow{\bigcap_{i\geq 1} F_i} \vee U$. Since U is an ultrafilter $\overrightarrow{\bigcap F_i} \vee U = \mathcal{P}(I)$. Thus $\phi \in \overrightarrow{\bigcap F_i} \vee U$. By Lemma there exists $A \in \overrightarrow{\bigcap F_i}$ and $F \in U$ such that $A \cap F = \phi$. Now $A \in \overrightarrow{\bigcap F_i}$ implies $A \supseteq \bigcap F_i$. Thus $A \cap F \supseteq F \cap \bigcap_i F_i$ since $A \cap F = \phi$ we have $F \cap \bigcap_i F_i = \bigcap_i F \cap F_i = \bigcap E_i$, $E_i = F \cap F_i \in U$. Let $J_i = CE_i$, $\bigcup_i J_i = \bigcup CE_i = C\bigcap E_i = C\phi = I$. To make J_i a partition let $I_n = J_n - \bigcup_{i=1}^{n-1} J_i$. Then $I = \bigcup_{n=1}^{\infty} I_n$, I_n are disjoint. Furthermore $I_n \notin U$.

In fact, suppose $I_n \in U$. Then $J_n - \bigcup_{i=1}^{n-1} J_i \in U$ since $J_n \supseteq J_n - \bigcup_{i=1}^{n-1} J_i$ we get $J_n \in U$ implies $CE_i \in U$. But $E_i \in U$. Conversely, let I_n be a partition of I with $I_n \notin U$ for all n. Then $CI_n \in U$, and $\bigcap CI_n = C\bigcup I_n = CI = \phi$, but $\phi \notin U$. Therefore U is δ-incomplete.

Another important property of δ-incomplete ultrafilter is:

<u>Lemma 2.2</u>. If U is a δ-incomplete ultrafilter on I and A, B are two subsets of I such that $A \cup B \in U$ and let $F = \{Y \subset I | A \cup Y \in U\}$. Then F is a filter. Indeed,

(i) $\phi \in F$ or else, $\phi \cup A = A \in U$ which contradicts the assumption that $A \notin U$.

(ii) $Y, Z \in F$ implies $Y \cup Z \in F$. Since $A \cup Y \in U$ and $A \cup Z \in U$ then $(A \cup Y) \cap (A \cup Z) \in U$. i.e., $A \cup (Y \cap Z) \in U$ therefore $Y \cap Z \in F$.

(iii) $Y \subset Z \subset I$ and $Y \in F$ implies $Z \in F$.

Since $A \cup Y \in U$ and $A \cup Z \supset A \cup Y$ then $A \cup Z \in U$ (since U is a δ-incomplete ultrafilter.) Therefore $Z \in F$. Thus F is a filter. Since $B \in F$, so F is strictly finer than U. This contradicts that U is a δ-incomplete ultrafilter. This completes the proof.

<u>Remark</u>. This above result can be extended to finite number of subsets of I such that $\bigcup_{k=1}^{n} I_k \in U$. Then there is one index k, $1 \leq k \leq n$ such that $I_k \in U$.

§2 The mapping from $\hat{R}$ into $\hat{R}^I$ can be treated in a more general way as follows: (with $\hat{R} = \hat{A}$ and $\hat{R}^I = \hat{B}$). (See [3]).

<u>Definition 2.3</u>. Let $\phi : \hat{A} \to \hat{B}$ be a map of $\hat{A}$ into $\hat{B}$ where $\hat{A}$ and $\hat{B}$ are superstructures over two sets of individuals A and B respectively. Given a wff α, then $^*\alpha$ is called a <u>Φ-transform</u> of α, where $^*\alpha$ is obtained from α by replacing each constant $c \in \hat{A}$ by $^*c$. e.g. ϕ-transform of "$(\forall x)[x \in A] \Rightarrow [x \in B]$" is "$(\forall x)[x \in {}^*A] \Rightarrow [x \in {}^*B]$.

Definition 2.4. $\phi : \hat{A} \to \hat{B}$ is a monomorphism iff $*\phi = \phi(\phi) = \phi$ and wfs (well-formed sentence)(it is simply called K-sentence defined similarly to K_o-sentence in §2 of Chapter I) α holds in $\hat{A}$ iff $*\alpha$ holds in $\hat{B}$.

If $x \in A_o$, i.e. x is an individual, we shall drop the $*$-notation and identify x with $*x$.

ϕ is assumed to be a monomorphism in the proof of the following Lemma.

Lemma 2.3. Given any constants $a, b, a_1, \ldots, a_n \in \hat{A}$ we have

(i) $a \subset b$ iff $*a \subset *b$

(ii) $a \in b$ iff $*a \in *b$

(iii) $*(a \times b) = *a \times *b$

(iv) Let $\alpha = \alpha(x_1, \ldots, x_n)$, its only free variables; if $E = \{(x_1, \ldots, x_n) \in C^n | \alpha\}$ then $*E = \{(x_1, \ldots, x_n) \in *C^n | *\alpha\}$, i.e. $*E$ consists of all members of $*C^n$ satisfying $*\alpha$, the ϕ-transform of α.

(v) $*\{a\} = \{*a\}$

(vi) $*(a - b) = *a - *b$

(vii) (a) $*(\bigcup_{i=1}^{n} a_i) = \bigcup_{i=1}^{n} *a_i$

(b) $*(\bigcap_{i=1}^{n} a_i) = \bigcap_{i=1}^{n} *a_i$

(c) $*\{a_1, \ldots, a_n\} = \{*a_1, \ldots, *a_n\}$

(d) $*(a_1, \ldots, a_n) = (*a_1, \ldots, *a_n)$

(viii) For any binary relation $R \in \hat{A}$ and any $Q \in \hat{A}$ we have[1])

$$\text{(a)} \quad {}^*(R[Q]) = ({}^*R)[{}^*Q]$$

$$\text{(b)} \quad {}^*(D_R) = D({}^*R)$$

$$\text{(c)} \quad {}^*(D_R') = D'({}^*R)$$

<u>Proof.</u>

(i) We translate "$a \subset b$" into "$(\forall x)[x \in a] \Rightarrow [x \in b]$" then by Φ we have "$(\forall x)[x \in {}^*a] \Rightarrow [x \in {}^*b]$. Since Φ is a monomorphism we have $(\forall x)[x \in {}^*a] \Rightarrow (\forall x)[x \in {}^*b]$ i.e. ${}^*a \subset {}^*b$.

(ii) This follows immediately from the definition of Φ-transform, that is to take α as "$a \in b$". The result follows.

(iii) $z \in a \times b$ can be translated into

$$(\forall z)(z \in (a \times b)] \Rightarrow (\exists x)(\exists y)[x \in a \wedge y \in b \wedge (x,y)=z]$$

and

$$(\forall z)[(\exists x)(\exists y)[x \in a \wedge y \in b \wedge (x,y) = z]] \Rightarrow [z \in a \times b]$$

By Φ-transform, in the same manner as above, they become

$$(\forall z)[z \in {}^*(a \times b)] \Rightarrow (\exists x)(\exists y)[x \in {}^*a \wedge y \in {}^*b \wedge (x,y) = z]$$

i.e. ${}^*(a \times b) \subset {}^*a \times {}^*b$ and

$$(\forall z)[(\exists x)(\exists y)[x \in {}^*a \wedge y \in {}^*b \wedge (x,y) = z]] \Rightarrow [z \in {}^*a \times {}^*b]$$

---

1) $R[Q] = \{y \mid (\ x \in Q)(x,y) \in R\}$, D_R is the domain of R and $D_R = D(R) = \{x \mid (\ y)(x,y) \in R\}$, $D_R' = D'(R) = D(R^{-1})$ is the range of R and $R^{-1} = \{(y,x) \mid (x,y) \in R\}$

i.e. $^*a \times {}^*b \subset {}^*(a \times b)$ therefore $^*(a \times b) = {}^*a \times {}^*b$. By induction, the result can be applied to Cartesian products of finite number of members of $\hat{A}$.

(iv) For $n = 2$, i.e. $\alpha = \alpha(x,y)$ use the K-sentences:

$(\forall z)(\exists x)(\exists y)[z \in E \wedge x \in C \wedge y \in C] \Rightarrow [(x,y)=z \wedge \alpha(x,y)]$

and

$$(\forall x)(\forall y)[x \in C \wedge y \in C] \Rightarrow$$
$$[\alpha(x,y) \Rightarrow (\exists z)[z \in E \wedge (x,y) = z]]$$

By Φ-transform they become $(\forall z)(\exists x)(\exists y)[z \in {}^*E \wedge x \in {}^*C \wedge y \in {}^*C] \Rightarrow [(x,y) = z \wedge {}^*\alpha(x,y)]$ i.e. $^*E \subset \{(x,y) \in {}^*C^2 | {}^*\alpha(x,y)]$ and $(\forall x)(\forall y)[x \in {}^*C \wedge y \in {}^*C] \Rightarrow [{}^*\alpha(x,y) \Rightarrow (\exists z)[z \in {}^*E \wedge (x,y) = z]]$ i.e. $^*E = \{(x,y) \in {}^*C^2 | {}^*\alpha\}$. The general result can be obtained by induction.

(v) By (iv), let $E = \{a\} = \{x \in A_n | x = a\}$ by Φ-transform, it gives

$^*E = {}^*\{a\} = \{x \in {}^*A_n | x = {}^*a\} = \{{}^*a\}$

(vi) Let $E = a - b = \{x \in a | x \notin b$, by (iv), with $n = 1$, $C = a$ and "$\alpha \notin b$" for "$\alpha(x)$" then,

$^*E = \{x \in {}^*a | x \notin {}^*b\} = {}^*a - {}^*b$.

(vii) (a) and (b) follow from (vi) by taking $a \cap b = a - (a - b)$ and $a \cup b = c - [(c - b) - a]$, with $c = a \cup b \in \hat{A}$.

(c) follows from (iv) and induction. Indeed, suppose $*\{a_1, \ldots, a_{n-1}\} = \{*a_1, \ldots, *a_{n-1}\}$.

Since $\{a_1, \ldots, a_n\} = \{a_1, \ldots, a_{n-1}\} \cup \{a_n\}$ then

$$*\{a_1, \ldots, a_n\} = *\{a_1, \ldots, a_{n-1}\} \cup \{*a_n\}$$

$$= \{*a_1, \ldots, *a_n\}$$

(d) follows from ordered pair and ordered n-tuple. Indeed, define

$$(a_1) = a_1,\ *(a_1) = *a_1$$

$$*(a_1, a_2) = *\{\{a_1\}, *\{a_1, a_2\}\}$$

$$= \{\{*a_1\}, \{*a_1, *a_2\}\}$$

$$= (*a_1, *a_2)$$

Suppose that

$$*(a_1, a_2, \ldots, a_{n-1}) = (*a_1, \ldots, *a_{n-1})$$

and $*((a_1, a_2, \ldots, a_{n-1}), a_n)$

$= *\{\{(a_1, a_2, \ldots, a_{n-1})\}, \{(a_1, \ldots, a_{n-1}), a_n\}\}$

$= \{*\{(a_1, a_2, \ldots, a_{n-1})\}, *\{(a_1, a_2, \ldots, a_{n-1}), a_n\}\}$

$= \{\{*(a_1, a_2, \ldots, a_{n-1})\}, \{*(a_1, a_2, \ldots, a_{n-1}), *a_n\}\}$

$= \{\{(*a_1, *a_2, \ldots, *a_{n-1})\}, \{(*a_1, *a_2, \ldots, *a_{n-1}), *a_n\}\}$

$= ((*a_1, *a_2, \ldots, *a_{n-1}), *a_n)$

$= (*a_1, *a_2, \ldots, *a_n)$

(viii) (a). If R is a binary relation in $\hat{A}$ then $R \subseteq (A_o \cup A_m)^2$. So $*R = *C^2$ where $C = A_o \subset A_m \in \hat{A}$.

So $(x,y) \in R$ implies $(x,y) \in *(C^2)$, i.e. $x, y \in C$

$(x,y) \in *R$ implies $(x,y) \in *(C^2)$, i.e. $x, y \in *C$.

The image of $R[Q] = \{y \in C \mid (\exists x \in Q)(\exists z \in R)(x,y) = z\}$

i.e. $*(R[Q]) = \{y \in *C \mid (\exists x \in *Q)(\exists z \in *R)(x,y) = z\}$

(by (iv)) which equals $*R[*Q]$.

(b) $D_R = \{x \mid (\exists y)(x,y) \in *R\}$. The image

$*(D_R) = \{x \mid (\exists y)(x,y) \in *R\} = D(*R)$

(c) $*(D_R') = \{(y,x) \mid (x,y) \in *R\} = D'(*R)$. q.e.d.

§3 $\hat{R}^I$ is the set of all mapping of I into $\hat{R}$. There exists a natural imbedding $a \to *a$ of $\hat{R}$ into $\hat{R}^I$ defined by $*a(i) = a$ for all $i \in I$, i.e. $\hat{R}$ is identified in $\hat{R}^I$ by the constant mappings.*

The undefined basic predicates "=" and "ϵ" of $\hat{R}$ can be extended to $\hat{R}^I$ by means of the following U-dependent definition, i.e. by the monomorphism ϕ_U which defines equivalence classes on $\hat{R}^I$.

<u>Definition 2.5 (U-Dependent Definition)</u>

If $a, b \in \hat{R}^I$ then $a =_U b \iff \{a \mid a(i) = b(a)\} \in U$ and $a \epsilon_U b \iff \{i \mid a(i) \in b(i)\} \in U$.

---

* The imbedding $\hat{R} \to \hat{R}^I$ is a monomorphism, the proof could be found on page 27 Theorem 2.

By Definition 2.1, we know that $I \in U$. It follows that if $a, b \in \hat{R}$, then $a = b$ iff $*a =_U *b$ and $a \in b$ iff $\{i | *a(i) \in *b(i)\} \in U$. Hence the relations "$=_U$" and "$\in_U$" are U extensions of "=" and "$\in$" of $\hat{R}$. We shall from now on drop the U-notation "$=_U$" and use the original notations "=" and "$\in$" respectively.

The above definition can be justified by the following:

For all $a, b \in \hat{R}^I$, either $(a \in b)$ or not $(a \in b)$ holds.

In fact, let

$$U_1 = \{i | a(i) \in b(i)\}$$

and

$$U_2 = \{i | a(i) \notin b(i)\}.$$

Since $U_1 \cup U_2 = I \in U$, it follows from the basic property (see Lemma 2.2) of an ultrafilter that either $U_1 \in U$ and $U_2 \notin U$ or $U_1 \notin U$ and $U_2 \in U$, i.e., either $a \in b$ or not $(a \in b)$ holds. Similarly we can prove that for all $a, b \in \hat{R}^I$ either $a = b$ or not $(a = b)$ holds. The justification of U-dependent definition is now completed.

By using Definition 2.5, we can prove Lemma 2.3 with respect to U. As an illustration, we prove Lemma 2.2 (i), (ii) and (v) as follows:

(i) Since $a \subset b$

then $\{i | c(i) \in a\} \subset \{i | c(i) \in b\}$, for all $i \in I$

$c \in *a$ implies $\{i | c(i) \in a\} \in U$, and U is an ultrafilter.

Therefore $\{i \mid c(i) \in b\} \in U$.

That is $c \in {}^*b$.

Hence ${}^*a \subset {}^*b$.

(ii) By Definition 2.5 we have

$$ {}^*a \in {}^*b \text{ iff } \{i \mid {}^*a(i) \in {}^*b(i)\} \in U. $$

Since ${}^*a(i) = a$, ${}^*b(i) = b$, for all $i \in I$,

thus ${}^*a(i) = a \in b = {}^*b(i)$, for all $i \in I$, implies

$$ {}^*a(i) \in {}^*b(i). $$

Therefore $a \in b$ iff ${}^*a \in {}^*b$.

For all $a \in \hat{R}$, let $x \in {}^*\{a\}$ then,

$$
\begin{aligned}
& \{i \mid x(i) \in \{a\}\} \in U \\
\iff & \{i \mid x(i) = a\} \in U \\
\iff & x = {}^*a \\
\iff & x \in \{{}^*a\} \\
& {}^*\{a\} = \{{}^*a\}.
\end{aligned}
$$

CHAPTER III

THE ULTRAPOWER OF $\hat{R}$ WITH RESPECT TO ULTRAFILTER U

§1 Now we are in a position to describe an <u>ultrapower</u> of $\hat{R}$.

Let I be a fixed infinite set, let U be a δ-incomplete ultrafilter, and let $\{I_n | n = 1, 2, \ldots\}$ be a countable partition of I satisfying $I_n \notin U$ $(n = 1, 2, \ldots)$.

<u>Definition 3.1</u>. An entity a of the *L-structure $\hat{R}^I$ is called <u>internal</u> whenever there exists a natural number $n \geq 0$ such that $a \in {}^*R_n$. An internal entity a is called a <u>standard</u> entity whenever there exists an entity $b \in \hat{R}$ such that $a = {}^*b$.

All entities which are not internal are called <u>external.</u>

The set $\bigcup_{n \geq 0} {}^*R_n$ of all internal entities is called the <u>ultrapower</u> of $\hat{R}$ w.r.t. the ultrafilter U and will be denoted by ${}^*(\hat{R})$. Observe that the mapping $a \to {}^*a$ of $\hat{R}$ into $\hat{R}^I$ imbeds $\hat{R}$ into the substructure ${}^*(\hat{R})$ of $\hat{R}^I$.

The notion of rank extends immediately to the internal entities.

<u>Definition 3.2.</u> An internal entity $a \in {}^*(\hat{R})$ is said to be of rank n $(n \geq 1)$ whenever $a \in {}^*R_n - {}^*R_{n-1}$, (by Lemma 2.3 (vi)) The entities of ${}^*R = {}^*R_0$ are said to be of rank 0. The entities of rank 0 are also referred to as the individual of ${}^*(\hat{R})$.

The empty set $*\phi$ has rank 1 (by Definition 2.4). By means of Lemma 2.2 the rank of an internal entity can also be specified in the following way. If $a \neq \phi$ and internal then $a \in *R_p$ for some $p \geq 0$ and by Definition 2.5, we have $\bigcup_{k=0}^{\infty} \{i|\ \text{rank}\ a(i) = k\} = U_2 \in U$. Let $I_k = \{i|\ \text{rank}\ a(i) = k\}$ and suppose that there exist a finite number of I_k, $1 \leq k \leq n$ such that

$$\{i|\ \text{rank}\ a(i) = k\} \subset U.$$

By definition of rank $a(i) = k$ where k is the smallest natural number such that $a(i) \in R_k$.

Therefore there exists exactly one index k, $1 \leq k \leq n$, such that $U_1 = \{i|\ \text{rank}\ a(i) = k\} \in U$ and for all $i \in U_1 \in U$. $a(i) \in R_k - R_{k-1}$ so $a \in *(R_k - R_{k-1})$, (by Lemma 2.3 (vi)(iii)) we have $a \in *R_k - *R_{k-1}$, i.e., rank $a = k$.

If $a = *b$, $b \in \hat{R}$ is a standard entity of $*(\hat{R})$ then its rank remains unchanged. In fact, if $b \in \hat{R}$ and b is of rank n, then $b \in R_n - R_{n-1}$ therefore $*b \in *R_n - *R_{n-1}$ and $a = *b$, therefore a is of rank n.

<u>Lemma 3.1</u>. There exist internal entities which are not standard.

<u>Proof</u>. In fact, let $a \in \hat{R}$ have infinitely many elements; then there exists a sequence $\{b_n : n = 1, 2, \ldots\}$ of elements of a such that $b_n \neq b_m$, $\forall\, n, m = 1, 2, \ldots$ and $n \neq m$.

Let $b : I \to a$ such that $b(i) = b_n$ for all $i \in I_n$ $(n = 1, 2, \ldots)$

$$b(i) \in a \text{ i.e. } b \in {}^*a$$

But there does not exist $c \in a \in \hat{R}$ such that $b = {}^*c$ i.e. $b(i) = c$ since $b(i)$ is not constant.

As a consequence of Definition 3.1, if a is an element of a standard entity, then a is internal that is, if $a \in {}^*b$, $b \in \hat{R}$ then a is internal. In fact, if $b \in \hat{R}$ then $b \in R_n$ for some n, so $b \subset R_o \cup R_{n-1}$. Thus, by Lemma 2.3 (ii)

$$a \in {}^*b \subset {}^*R_o \cup {}^*R_{n-1} \text{ implies } a \in {}^*R_o \cup {}^*R_{n-1}$$

i.e. either a is in ${}^*R_o$ or in ${}^*R_{n-1}$ which shows that a is internal by definition.

The following lemma also shows that the elements of an internal entity are internal.

<u>Lemma 3.2</u>. If $a \in b \in {}^*R_n$ $(n \geq 1)$, then

$$a \in {}^*R_o \cup {}^*R_{n-1}.$$

<u>Proof</u>. If $a \in b \in {}^*R_n$ $(n \geq 1)$ then

$U_1 = \{i | a(i) \in b(i)\} \in U$ and $U_1 = \{i | b(i) \in R_n\} \in U$

Thus

$U_2 = \{i | b(i) \subset R_o \cup R_{n-1}\}$ (since $b(i) \in R_n \Longleftrightarrow b(i) \subset R_o \cup R_{n-1}$)

To prove

$$a \in {}^*R_o \cup {}^*R_{n-1} = {}^*(R_o \cup R_{n-1})$$

It suffices to show that $U_3 = \{i | a(i) \in R_o \cup R_{n-1}\} \in U$ or $U_3 \supset U_1 \cap U_2$. In fact, take $i \in U_1 \cap U_2$ then $a(i) \in b(i)$ and $b(i) \in R_o \cup R_{n-1}$. Thus

$$a(i) \in R_o \cup R_{n-1} \text{ and } i \in U_3. \qquad \text{Q.E.D.}$$

Definition 3.3. A wff V in L is said to be in *prenex normal form*, if the connectives are in the scope of all quantifiers.

Lemma 3.3. Every admissible wff has a prenex normal form (for proof, see [7] 2.4.1 Theorem on page 11).

*Definition 3.4. (The Admissible wff of *L)*

A wff of *L is called admissible whenever all the quantifiers occurring in it are of the form "$(\forall x)[[x \in a] \Rightarrow \ldots]$" and "$(\exists x)[[x \in a] \wedge \ldots]$", where a is a constant denoting an entity of $\hat{R}^I$.

An admissible wff of *L is called internal whenever all the constants occurring in it denote internal entities. An admissible wff of *L is called standard whenever all constants occurring in it denote standard entities. Thus a standard wff is internal.

The set of all internal sentences of *L will be denoted by $*K = *K(*L)$, and the subset of all internal sentences which hold in $*(\hat{R})$ will be denoted by $*K_o = *K_o(*L)$.

If V is an admissible wff of L, then its *-transform *V is defined to be that *standard wff* of *L which is obtained from V by replacing in V all the constants, $*a_1, \ldots, *a_p$, but leaving the variables and bracketing unchanged.

<u>Theorem 1</u>. Let $V = V(x_1, \ldots, x_p)$ be an admissible L-wff with free variables $x_1, \ldots, x_p$ and let

$$A = \{(x_1, \ldots, x_p) | (x_1, \ldots, x_p) \in a \wedge V(x_1, \ldots, x_p)\},$$

where a is an arbitrary entity of $\hat{R}$. Then $A \in \hat{R}$ and

$$*A = \{(y_1, \ldots, y_p) | (y_1, \ldots, y_p) \in *a \wedge *V(y_1, \ldots, y_p)\}.$$

The proof of the above theorem is given in the Addenda [1].

§2 <u>Theorem 2.</u> $*(\hat{R})$ is a higher order nonstandard model of $\hat{R}$, that is, an admissible sentence V of K(L) holds in $\hat{R}$ if and only if $*V$ holds in $*(\hat{R})$, and $\hat{R}$ is properly imbedded in $*(\hat{R})$.

<u>Proof</u>. "$\hat{R}$ is properly imbedded in $*(\hat{R})$" is justified by Lemma 3.1.

Now we are going to show that if $V \in K(L)$, then $V \in K_o$ if and only if $*V \in *K_o$.

(i) If V has no quantifiers, then it follows immediately from Definition 2.5.

(ii) If V has quantifiers, assume that $V \in K$ has the prenex normal form $V = (qx_n) \ldots (qx_1)W$, where W has no quantifiers. Assume that (qx_n) is an existential quantifier $(\exists x_n)$. Then $V \in K_o(L)$ is equivalent to "the set $A = \{x_n | x_n \in a$ and $(qx_{n-1}) \ldots q(x_1)W\} \neq \phi$ where a is the domain of $(\exists x_n)$. Then by Theorem 1 and Definition 2.4 $\hat{A} \neq \phi$ is equivalent to $*A = \{x_n | x_n \in *a$ and $(qx_{n-1}) \ldots (qx_1)*W\} \neq *\phi$ which in turn is equivalent to $*V \in *K$. The proof is completed.

Theorem 3: Let $V = V(x_1, \ldots, x_n)$ be an internal wff with the free variables $x_1, \ldots, x_n$ and let $a \in *(\hat{R})$ be an internal entity. Then the set $\{(x_1, \ldots, x_n) | (x_1, \ldots, x_n) \epsilon a \wedge V(x_1, \ldots, x_n)\}$ is internal.

Proof. If V has no quantifiers, that is, $V = V(x_1, \ldots, x_n, a_1, \ldots, a_n)$ where $a_1, \ldots, a_p$ are the constants occurring in V, then $a_1, \ldots, a_p$ are internal entities.

Since a is internal (i.e. $a \in *(R) = \bigcup_{n \geq 0} *R_n$ therefore

$$a \in \bigcup_{n \geq 0} *R_n \text{ and so } a \in *R_n$$

for some n

$$\text{i.e. } a(i) \in R_n, \text{ for all } i \in I)$$

it follows that the mapping

$$i \to E(i) = \{(x_1, \ldots, x_n) | (x_1, \ldots, x_n) \in a(i) \wedge V(x_1, \ldots, x_n, a_1(i), \ldots, a_p(i))\}$$

is a mapping of I into R_n for some n, and so this mapping determines an internal entity denoted by E. Then $E = \{(x_1, \ldots, x_n) | (x_1, \ldots, x_n) \in a \wedge V\}$.

For the general internal wff, the proof will be by induction on the number of quantifiers. Assuming that the theorem holds for all internal wff with $\leq n$ quantifiers. Let $V = (qx_{n+1}) \ldots (qx_1)W$ be an internal wff with the free variables $y_1, \ldots, y_p$. There is no loss in generality to assume that $(qx_{n+1}) = (\exists x_{n+1})$ with domain $b \in {}^*\hat{(R)}$. Since b is internal, by the induction hypothesis, the binary relation

$$B = \{((y_1, \ldots, y_p), x_{n+1}) \mid ((y_1, \ldots, y_p), x_{n+1}) \in a \times b \text{ and } (qx_n) \ldots (qx_1)W(y_1, \ldots, y_p, x_{n+1})\}$$

is internal, thus by Lemma 3.1 (v) of Chapter I and F.T.[1], its domain

$$\{(y_1, \ldots, y_p) \mid (y_1, \ldots, y_p) \in a \text{ and } (\exists x_{n+1})(qx_n) \ldots (qx_1)W\}$$

is internal. The proof is completed.

---

[1] From now on whenever we quote Theorem 2, we refer to it as the Fundamental Theorem. (F.T.)

CHAPTER IV

A NONSTANDARD REAL NUMBER SYSTEM

§1 *R as a Totally Ordered Field

In Chapter III we have developed a model of $\hat{R}$ which is an U-ultrapower denoted by $*(\hat{R})$.

Let *R be the set of individuals of $*(\hat{R})$. By F.T. we know that *R is a totally ordered field. We shall not repeat the proof here, but for the sake of illustration, we are going to check some of the field axioms which hold in $\hat{R}$ and see how to interpret them by means of an ultrafilter U in the following examples.

(i) Addition.

$a + b = c$ can be expressed in the language L by specifying "+" as an extra logical constant. By F.T. we have

$$*a \; *\!+ \; *b = *c$$

in our new language *L.

In order to understand the new statement in the new language we construct an ultrafilter U. Through U, we can interpret *a as a(i), *b as b(i); *c as c(i); "+" as "$+_U$", "=" as "$=_U$", then

$$*a \; *\!+ \; *b = *c,$$

iff $$\{i \; a(i) + b(i) = c(i)\} \in U.$$

(ii) Commutativity of Multiplication.

$$[a \times b = c \wedge b \times a = d] \Rightarrow [c = d]$$

Taking "×" as an extra logical constant, the commutativity of multiplication can be expressed as a wfs in the language L. By F.T., we have

$$[{}^*a \times {}^*b = {}^*c \wedge {}^*b \overset{\frown}{\times} {}^*a = {}^*d] \Rightarrow [{}^*c = {}^*d]$$

By means of U, the commutativity of multiplication can be interpreted in *L as follows:

$$[a(i) \times_U b(i) =_U c(i), \wedge b(i) \times_U a(i) =_U d(i)] \Rightarrow [c(i) = d(i)]$$

$$\text{iff } \{i|a(i)b(i) = c(i)\} \wedge \{i|b(i)a(i) = d(i)\} \in U \Rightarrow \{i|c(i) = d(i)\} \in U$$

(iii) The Order Relation "<".

We define a set of all positive real numbers P, then $a < b$ can be defined as $b - a \in P \subset R \subset \hat{R}$. The statement $[b - a \in P]$ can be expressed in the language L by taking "-" as an extra logical constant; then by F.T., we can express the order relation "<", $a < b$ iff $[{}^*b - {}^*a \in P]$. Using the ultrafilter U we can interpret the corresponding new statement in *L as follows:

$$[{}^*b - {}^*a \in {}^*P]$$

$$\text{iff} \qquad \{i|b(i) - a(i) \in P\} \in U.$$

In the same manner we can check any one of the other field axioms of R which also hold in *R by the use of the ultrafilter U. Similarly other mathematical concepts can be interpreted in the new language *L.

Example (iv). The absolute value $|r|$ of a real number $r \in R$ can be considered as a mapping $|\cdot|$ of R into $R^+ = \{r|r \in R \text{ and } r \geq 0\}$ the set of all nonnegative real numbers. Let this mapping be denoted by a constant of L. Since the set of all constants of L is in one-to-one correspondence with all the entities of $\hat{R}$, this mapping can be extended from $\hat{R}$ to ${}^*(\hat{R})$ by a mapping ${}^*|\cdot|$ of ${}^*R$ into ${}^*(R^+)$. By F.T., we have ${}^*|a| = a$ for all $0 \leq a \in {}^*R$ and ${}^*|a| = -a$ for all $0 > a \in {}^*R$.

Example (v). Max (a, b) is a mapping of $R \times R$ into R such that if $\max(a, b) = x$ then it can be defined as a sentence of K_o:

$$(\exists x)[[x \in R \wedge x \in \{a, b\} \wedge a > b] \Rightarrow [x = a]] \vee$$
$$[[x \in R \wedge x \in \{a, b\} \wedge a = b] \Rightarrow [x = a]]]$$

By F.T., this statement can be interpreted as

$$(\exists x)[[[x \in {}^*R \wedge x \in \{a, b\} \wedge a > b] \Rightarrow [x = a]] \vee$$
$$[[x \in {}^*R \wedge x \in \{a, b\} \wedge a = b] \Rightarrow [x = a]]]$$

where $x \in {}^*R$ means $\{i \; x(i) \in R\} \in U$.

Similarly we can extend $\min(a, b)$ to ${}^*\min(a, b)$ of $R \times R$ into R.

Remark. The elements of ${}^*R$ which are identified with the elements of R will be called standard real numbers.

§2 The Archimedean Property

Express the Archimedean property of R as a sentence of K_o:

$$(\forall x)(\forall n)[x \in R] \wedge [n \in N]] \Rightarrow [[nx \leq 1] \Leftrightarrow [x \leq 0]]$$

By F.T., the following corresponding statement holds in ${}^*R$

$$(**) \quad (\forall x)(\forall n)[x \in {}^*R] \wedge [n \in {}^*N]] \Rightarrow [[nx \leq 1] \Leftrightarrow [x \leq 0]]$$

There are two possible interpretations of (**):

(i) If we interpret ${}^*N$, "the set of all natural numbers", which satisfies all Peano Axioms, as a proper extension of N (the set of all natural numbers in R), then *R is Archimedean.

(ii) If we interpret *R with respect to N then *R is non-Archimedean.

There does not exist an n such that the statement:

$$(\forall a)[0 < a \in {}^*R] \Rightarrow [na > 1],$$

where na means a + ... + a, n times +, holds.

§3 The Completeness Property of *R

In order to study the completeness of *R, we will need quantification over sets. For this purpose, we will study first some properties of internal sets.

Theorem 1. The union and intersection of two nonempty internal sets are internal.

Proof. Let R_p, R_q be two sets* where $R_{n+1} = P(\bigcup_{k=0}^{n} R_k)$ the power set of $\bigcup_{k=0}^{n} R_k$, since $R_p \subset R_n$ for all $n \geq p \geq 1$ $R_q \subset R_m$ for all $m \geq q \geq 1$. Let $s = \max\{m, n\}$. Then $R_p \subset R_q$ and $R_q \subset R_s$ therefore $R_p \subset R_q \subset R_s$ and $R_s \in R_{t+1}$ for all $0 \leq s \leq t$ and for all $t \geq 0$. Thus $R_s \in R_{t+1} \subset \bigcup_{n \geq 0} R_n$ therefore $R_s \in \bigcup_{n \geq 0} R_n$. So by F.T., ${}^*R_s \in \bigcup_{n \geq 0} {}^*R_n$ i.e. ${}^*R_p \cup {}^*R_q \in \bigcup^{*}_{n \geq 0} R_n$ so that ${}^*R_p \cup {}^*R_q$ is internal.

Similarly, we prove the intersection of two non-empty sets to be internal the same way.

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* R_p and R_q are nonempty sets.

<u>Theorem 2</u>. Every non-empty internal subset of *R which is bounded above has a least upper bound.

<u>Proof</u>. Since every non-empty subset of R which is bounded above has a least upper bound, (this is so-called the Dedekind completeness property), this statement can be expressed by a sentence of K_o. In symbols,

$$\begin{aligned}&(\forall y)(\forall x)(\exists z)[[x \in R_1] \wedge [x = \neg \phi] \wedge [y \in x] \wedge [y \in R] \wedge [z \in R] \wedge \\ &[z < y \vee z = y] \Rightarrow (\exists w)[[w \in R] \wedge (\forall s)[s \in x \Rightarrow \\ &s < w \vee s = w] \wedge (\forall q)[(\forall r)[r \in x \Rightarrow r < q \vee r = q] \Rightarrow [w < q \vee w=q].\end{aligned}$$

By F.T., the following *L-sentence holds.

$$\begin{aligned}&(\forall y)(\forall x)(\exists z)[[x \in {}^*R_1] \wedge [x = \neg \phi] \wedge [y \in x] \wedge [y \in {}^*R] \wedge [z \in R] \wedge \\ &[z > y \vee z = y]] \Rightarrow (\exists w)[[w \in {}^*R] \wedge (\forall s)[s \in x \Rightarrow s = w \vee s = w] \wedge \\ &\qquad (\forall q)[(\forall r)[r \in x \Rightarrow r < q \vee r = q] \Rightarrow [w < q \vee w = q]]\end{aligned}$$

which proves the theorem.

<u>Theorem 3</u>. Every non-empty internal subset of *N has a first element.

<u>Proof</u>. Since the principle of induction states that every non-empty set of natural numbers has a first element, this statement can be expressed by a sentence of K_o. In symbols,

$$\begin{aligned}&(\forall x)[[x \in N] \wedge [x = \neg \phi]] \Rightarrow (\exists y)[[y \in x] \wedge [(\forall z)[z \in x] \Rightarrow \\ &\qquad\qquad [y < z \vee y = z]]]\end{aligned}$$

By F.T. the following *L-sentence holds.

$$\begin{aligned}&(\forall x)[[x \in {}^*N] \wedge [x = \neg \phi]] \Rightarrow (\exists y)[[y \in x] \wedge [(\forall z)[z \in x] \Rightarrow \\ &\qquad\qquad [y < z \vee y = z]]]\end{aligned}$$

which proves the theorem.

<u>Theorem 4</u>. The least upper bound of a non-empty internal set of finite numbers is finite.

<u>Proof</u>. Let $B = \{y | y \in R \text{ and } y < t\}$ be a set of real numbers where t is a finite real number, then by the completeness theorem for the real number system R, the set B has a least upper bound, say, x.

Since B is an entity of $\hat{R}$ it can be expressed as a sentence of K_o. In symbols,

$$(\forall y)[[y \in R] \wedge [y < t] \wedge [t \in R]]$$
$$\Rightarrow (\exists x)[[x \in R] \wedge [y < x \vee y = x]]$$

By F.T. the following *L-setnence holds

$$(\forall y)[[y \in {}^*R] \wedge [y < t] \wedge [t \in {}^*R]]$$
$$\Rightarrow (\exists x)[[x \in {}^*R] \wedge [y < x \vee y = x]]$$

Q.E.D.

§4 <u>Finite and Infinite Numbers</u>

To prove $\omega > |r|$ for all $r \in R$.

It suffices to show that $\{i | \omega(i) > |r|\} \in U$, $\forall r$ or

$$\{i | \omega(i) \leq |r|\} \notin U, \forall r.$$

Claim $\{i | \omega(i) \leq |r|\} = \bigcup_{j=1}^{n} I_j$, where $n = [|r|]$.

<u>Proof</u>. Let $\omega(i) = m$ if $i \in I_m$, then take $i \in$ L.S. $\omega(i) \leq |r$ and let $[|r|] = n$, then $\omega(i) = 1 \vee \omega(i) = 2 \vee \ldots \vee \omega(i) = n$ i.e. $i \in \bigcup_{j=1}^{n} I_j$. Take $i \in$ R.S. $i \in I_j$ for some $j \leq n$ then $\omega(i) = j$ for some $j \leq n$ which implies $\omega(i) \leq n = [|r|] \leq |r|$, i.e $i \in$ L.S. Since $I_j \notin U$, it remains to show that $\bigcup_{j=1}^{n} I_j \notin U$, $I_j \notin$ then $I_j' \in U$ and $\bigcap^{n} I_j' \in U$, i.e. $(\bigcup^{n} I_j)' \in U$, thus $\bigcup^{n} I_j \notin U$. Q

Definition 4.1. A real number $a \in {}^*R$ is called finite whenever there exists a standard real number $0 < r \in R$ such that $|a| < r$. A real number $a \in {}^*R$ which is not finite will be called infinite.

Theorem 1. A natural number $n \in {}^*N$ is finite iff n is a standard natural number. In symbols, ${}^*N \cap M_o = N$.[1)]

Proof. Since $N \subset R \subset M_o$ i.e. if n is a standard natural number then n is finite and ${}^*N$ is a proper extension of N therefore $n \in {}^*N$ and is finite.

Conversely, if $n \in {}^*N$ is finite then there exists a standard real number $0 < r \in R$ such that $n < r$.

Since ${}^*K_o$ contains the sentence

$(\forall x)[x \in {}^*N] \Rightarrow [x \leq r] \Longleftrightarrow [x = 1] \vee [x = 2] \vee \ldots \vee [x = p]$,

where r and p are constants and $p = [r]$, the integral part of r. Thus by F.T. we get $n = 1$ or $n = 2$ or ... or $n = [r]$, i.e. n is a standard natural number which completes the proof.

As a consequence of the above theorem the set of all infinitely large natural numbers is given ${}^*N - N$, similarly, we can prove that ${}^*R - R$ is the set of all infinitely large real numbers.

Remarks. Integral Part Function

The mapping $]\ [: R^+ \to N \cup \{0\}$, where $]r[$ denotes the largest nonnegative integer less than or equal to r. This mapping can be denoted by a constant of L, then by passing from $\hat{R}$ to ${}^*(\hat{R})$ to a mapping ${}^*]\cdot[$ of ${}^*(R^+)$ into ${}^*N \cup \{0\}$, i.e., for all $0 \leq N \in {}^*(R^+)$, ${}^*]r[$ in the largest nonnegative integer $\leq r^*$.

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1) See Definition 1.1 Chapter V.

CHAPTER V

THE INFINITESIMALS

§1 Algebra of Infinitesimals

Definition 1.1. A real number $a \in {}^*R$ is called an infinitesimal or infinitely small whenever $|a| < r$ for all $0 < r \in R$.

The set of all finite real numbers of $^*R$ will be denoted by M_o and the set of all infinitesimals by M_1.

Observe that $R \subset M_o$, $M_1 \subset M_o$ and $R \cap M_1 = \{0\}$.

Lemma 1. A real number $0 \neq a \in {}^*R$ is an infinitesimal if and only if its reciprocal $1/a$ is infinite.

Proof. Indeed, $|a| < r$ for all $0 < r \in R$, i.e. a is an infinitesimal then $|\frac{1}{a}| > \frac{1}{r}$ and $|\frac{1}{a}|$ is infinite. Conversely, $|\frac{1}{a}| > \frac{1}{r}$ for all $0 < \frac{1}{r} \in R$ then $|a| < r$. The proof is completed.

Lemma 2. (i) The sum of two infinitesimals is an infinitesimal.

(ii) The product of two infinitesimals is an infinitesimal.

Proof. (i) Let ε_1, ε_2 be two infinitesimals, by Definition 2.1, we have $|\varepsilon_1| < r$, $|\varepsilon_2| < r$, for all $0 < r \in R$. Since $|\varepsilon_1 + \varepsilon_2| \leq |\varepsilon_1| + |\varepsilon_2| < 2r$. Therefore $\varepsilon_1 + \varepsilon_2$ is an infinitesimal.

(ii) Let ε_1, ε_2 be two infinitesimals, by Definition 2.1, we have $|\varepsilon_1| < r$, $|\varepsilon_2| < r$, for all $0 < r \in R$. Since

$$|\varepsilon_1\varepsilon_2| = |\varepsilon_1||\varepsilon_2| < r^2$$

Therefore $\varepsilon_1\varepsilon_2$ is an infinitesimal. The proof is completed.

Theorem 1. The set of all infinitesimals M_1 is a ring without unity.

Proof. Since *R is a proper extension of R, hence all field axioms, operations (i.e. addition and multiplication), order relation "<" and completeness property which hold in R, also hold in *R. Now we claim that M_1 is a ring without unity. First, we shall prove that M_1 is a ring.

(i) Closure. By Lemma 1 and 2 above we know that M_1 is closed under addition and multiplication.

(ii) The Additive Identity. The additive identity 0 in R, by F.T. we have *0 in *R which is an infinitesimal, by definition, so $^*0 \in M_1$.

(iii) The Additive Inverse. This follows immediately from Definition 2.1.

(iv) - (v). Commutativity and Distributivity. Since they hold in *R as a whole, so they also hold in M_1. Therefore M_1 is a ring. Since the unity *1 in *R is unique and by Definition 1.1, $^*1 \notin M_1$ then M_1 has no unity element. The proof is completed.

§2 - The Quotient Ring M_o/M_1.

Definition 2.1. If $a, b \in {}^*R$ and $a - b$ is infinitesimal, then we say that b is infinitely close to a and write $a \doteq_1 b$.

Lemma 1. M_1 is a maximal ideal in M_o.

Proof. (i) M_1 is an ideal in M_o. That is to prove (1) if $hk \in M_1$ then $h - k \in M_1$ (2) if $h \in M_1$ and $a \in M_o$ then $ah \in M_1$.

Proof of (1). By Definition 1.1

$$|h| < r, \ |k| < r \text{ for all } 0 < r \in R$$

$$|h - k| \leq |h| + |k| < 2r$$

therefore $\quad h - k \in M_1$

Proof of (2). By Definition 1.1

$|a| < s$, $|h| < r$ for some $0 < s \in R$ and for all $0 < r \in R$ so

$$|ah| < rs$$

therefore $\quad ah \in M_1$

therefore M_1 is an ideal in M_o.

(ii) M_1 is a maximal ideal in M_o. If not, let $a \in M_o$ and M_1' in M_o such that $a \notin M_1$, $M_1 \subseteq M_1'$ and $a \in M_1'$. Since $a \notin M_1$ then a has a multiplicative inverse and $a \in M_1'$. Then $aa^{-1} \in M_1'$ and so $M_1' = M_o$, then M_1 is a maximal ideal in M_o.

Lemma 2. M_o/M_1, where M_o is a commutative ring with unity of $*R$ and M_1 is a maximal ideal in M_o, is a field.

Proof. M_o is a subring of $*R$. This can be proved by checking step by step of the definition of a ring.

We shall prove that M_o is commutative, closed and with unity of $*R$.

M_o is closed, in fact, if $a, b \epsilon M_o$ then $|a| < r_1$, $|b| \epsilon r_2$ for some $0 < r_1, r_2 \epsilon R$

$$|ab| \leq |a||b| < r_1 r_2$$

therefore $ab \epsilon M_o$

$ab \epsilon M_o$ and $ba \epsilon M_o$ so $ba - ab$ (since $ab = ab$ holds in $*R$) therefore M_o is commutative.

We have, from Lemma 1, that $aa^{-1} = 1 \epsilon M_o$ therefore M_o is commutative, closed and with unity and M_1 is a maximal ideal in M_o. By a result from algebra we can conclude that M_o/M_1 is a field.

Theorem 2. The quotient ring M_o/M_1 is order isomorphic to the field R of the standard real numbers.

Proof. Let A be an equivalence class in M_o modulo M_1 then $A = a + M_1 = \{a + m_1 | m_1 \epsilon M_1\}$. Since $R \epsilon M_o$ then for every standard real number r in R, $r \epsilon$ some A. If r, s are two different standard real numbers which are in A then

$$r = a + m_1, \; s = a + m_2$$

$$r - s = a + m_1 - (a + m_2)$$

$$r - s = m_1 - m_2$$

is an infinitesimal. Therefore $|r - s| =_1 0$, by Definition 1.1, we can put $|r - s| < |r - s|$ which is impossible! Therefore for every $r \in R$, r is in only one equivalence class, say A, M_o modulo M_1. Consequently R is shown to be a subfield of M_o/M_1.

To complete the proof, we want to show that $a =_1 r$ where $a \in M_o$ and r is unique. Let $a \in M_o$ then $D = \{r : r \in R$ and $r \leq a\}$ and $D' = R - D$ define a Dedekind cut (D, D') in R. Let $r \in R$ be the real number in R which determine the same cut (D, D') The Dedekind cut is unique, if $a =_1 r$ then the proof of this theorem is completed. In fact, by indirect proof, i.e. if not $a =_1 r$ then by Definition 1.1 there exists a positive real number $0 < \varepsilon \in R$ such that $|a - r| \geq \varepsilon$.

Case I. If $a > r$, then $|a - r| \geq \varepsilon$ implies that $r + \frac{\varepsilon}{2} < a$.

Case II. If $a < r$ we have $r - \frac{\varepsilon}{2} > a$.

Both cases give the same contradiction to the fact that a and r determine the same cut. Therefore M_o/M_1 is order isomorphic to R.

Definition 2.2. The ring and order homomorphism of M_o onto R with kernel M_1 will be called the standard part homomorphism and will be denoted by st.

Definition 2.3. Let st: $M_o \rightarrow R$ be a ring and order homomorphism then $M_1 = \{r | r \in M_o$ and $st(r) = 0\}$ is called the kernel of st.

In the proof of Theorem 2, and the definitions above, we know that st(a) where $a \in M_o$ is the unique real number r which is infinitely close to a.

Theorem 3. (i) $st(a + b) = st(a) + st(b)$, $st(ab) = st(a)\,st(b)$ and $st(a - b) = st(a) - st(b)$ for all $a, b \in M_o$.

(ii) If $a, b \in M_o$, then $a \leq b$ implies $st(a) \leq st(b)$.

(iii) $st(|a|) = |st(a)|$, $st(\max(a, b)) = \max(st(a), st(b))$ and $st(\min(a, b)) = \min(st(a), st(b))$ for all $a, b \in M_o$.

(iv) $st(a) = 0$ if and only if $a \in M_1$.

(v) For all standard $r \in R$ we have $st(r) = r$.

(vi) If $a \in M_o$ and $st(a) \geq 0$, then $|a| =_1 st(a)$.

(vii) For all $a, b \in M_o$ we have $a =_1 b$ if and only if $st(a) = st(b)$.

(viii) If $b \in M_o$ and b is positive, then $st(\sqrt[n]{b}) = \sqrt[n]{st(b)}$, for all positive integer n.

Proof. (i) can be proved directly by the definition of homomorphism.

(ii) Let $a = st(a) + \varepsilon_1$, $b = st(b) + \varepsilon_2$ where $a, b \in M_o$, $\varepsilon_1, \varepsilon_2 \in M_1$ then $0 \leq b - a = (st(b) - st(a)) + (\varepsilon_2 - \varepsilon_1)$ since $-(\varepsilon_2 - \varepsilon_1) \leq st(b) - st(a) < 0$ is impossible. Therefore $st(b) - st(a) \geq 0$. i.e. $st(b) \geq st(a)$.

(iii)

$$st(|a|) = \begin{cases} st(a) & \text{if } a > 0 \\ -st(a) & \text{if } a < 0 \end{cases}$$

$$= |st(a)|.$$

$$st(\max(a,b)) = \begin{cases} st(a) & \text{if } a \geq b \\ st(b) & \text{if } b \geq a \end{cases}$$

From (ii), if $a \geq b$, $st(a) \geq st(b)$.

$$\max(st(a), st(b)) = st(a)$$

if $a \leq b$, $st(a) \leq st(b)$

$$\max(st(a), st(b)) = st(b).$$

In either case, $st(\max, (ab)) = \max(st(a), st(b))$. Similarly, we can prove that

$$st(\min(a,b)) = \min(st(a), st(b)).$$

(iv) $st(a) = 0$ iff $a \in M_1$

This can be obtained directly from Definition 2.3.

(v) For all standard $r \in R$ we have $st(r) = r$. This is asserted by Theorem 2 above.

(vi) If $a \in M_o$, $st(a) \geq 0$ then $|a| = {}_1st(a)$. By definition, $a = st(a) + \varepsilon$ where $a \in M_o$ and (is an infinitesimal) $\in M_1$

and $\mathrm{st}(a) \geq 0$ together imply that $a \geq 0$ i.e. $a = |a|$ therefore $|a| =_1 \mathrm{st}(a)$.

(vii) For all $a, b \in M_0$ we have $a =_1 b$ iff $\mathrm{st}(a) = \mathrm{st}(b)$.

If $a =_1 b$, i.e. $a - b = \varepsilon$ where $\varepsilon \in M_1$ $\mathrm{st}(a - b) = \mathrm{st}(\varepsilon)$. By (i) and (iv) we have

$$\mathrm{st}(a) - \mathrm{st}(b) = 0$$

i.e.

$$\mathrm{st}(a) = \mathrm{st}(b).$$

If

$$\mathrm{st}(a) = \mathrm{st}(b)$$

then

$$0 = \mathrm{st}(a) = \mathrm{st}(b)$$

$$= \mathrm{st}(a - b) \text{ by (i)}$$

thus $a - b$ is infinitesimal by (iv) therefore $a =_1 b$.

(viii) Since $\mathrm{st}(\sqrt[n]{b}) = \mathrm{st}[(\mathrm{st}(b) + \varepsilon)^{1/n}]$

$$= \mathrm{st}[(\mathrm{st}(b))^{\frac{1}{n}} + \frac{1}{n}\cdot \mathrm{st}(b)^{\frac{1}{n}-1} \cdot \varepsilon + \frac{1}{n}\,\frac{(\frac{1}{n}-1)}{2!}\,(\mathrm{st}(a))^{\frac{1}{n}-2} \cdot 2 + \ldots]$$

$$= \mathrm{st}(b)^{\frac{1}{n}} = \sqrt[n]{\mathrm{st}(b)}.$$

§3 Some External Sets and Their Properties

Theorem 1. The non-empty sets $^*N - N$, M_1, M_0 and the set of infinitely large real numbers $^*R_\infty = {}^*R - R$ are all external. ($M_0 \supset R$).

Proof. (i) Assume that $^*N - N$ is internal. Then since $^*N - N \neq \phi$ (by Lemma 2.1 of Chapter II). By Theorem 3 of Chapter IV on internal sets we know that $^*N - N$ has a first element, say,

ω_o. Thus $k + 1 < \omega_o$ for all $k \in N$ implies that $\omega_o - 1 \in {}^*N-N$ (Indeed, $n - 1 < n$ for every $n \in N$ which is true for R so by E.T. it is true for $^*R$ when interpreted in $^*R$, i.e. $\omega_o-1<\omega_o$) which shows that $^*N - N$ has no first element. Thus $^*N - N$ is external.

(ii) Assume that M_1 is internal. Since $0 \in M_1$, therefore $M_1 \neq \phi$ and $h \in M_1$ in particular $|h| < 1$, by Theorem 2 of Chapter IV, M_1 has a least upper bound, say, a_o. Furthermore $\{a_o\} \neq M_1$. By the definition of infinitesimal we have $a_o/2$ is also a least upper bound of M_1, so M_1 is external.

(iii) Assume that M_o is internal, by Theorem 2 of Chapter IV, M_o has a least upper bound, say, b_o, then $2b_o > b_o$ and $2b_o \in M_o$ which contradicts the fact that b_o is a least upper bound by M_o. Thus M_o is external.

(iv) If $^*R_\infty = {}^*R - M_o$ is internal, then also $M_o = {}^*R - {}^*R_\infty$ is internal which contradicts (iii). Thus $^*R_\infty = {}^*R - M_o$ is external.

§4 A Nonstandard Construction of the Real Number System R

Following the same pattern as in the proof of Theorem 2, we could construct R by using Q-infinitesimals instead of sequences of rational numbers.

Let Q be the set of all rational numbers, then $*(\hat{Q})$ is a higher order nonstandard model of the superstructure $\hat{Q}$. The set of individuals in $*Q \subset *R$ is a subfield of $*R$ which has the same properties as Q as far as they can be expressed by sentences of K_o. By Lemma 3.1 of Chapter III, we have $*Q \neq Q$, and in fact, we could prove that $*Q$ contains an element which is larger than any standard rational number. We define $*q \in *Q$ to be finite whenever $|*q| < q$ for some standard rational number, and $*q \in *Q$ to be infinitesimal whenever $|*q| < q$ all positive standard rationals q. Let Q_o denote the set of all finite rationals and let Q_1 denote the set of all infinitely small rationals, then it could be proved by the same techniques as before that Q_1 is a maximal ideal in the integral domain Q_o; thus the quotient ring Q_o/Q_1 is a field and order isomorphic to a field which is isomorphic to the field of Dedekind cuts of Q. Hence Q_o/Q_1 is isomorphic to R as a totally ordered field.

Remark. Since Q does not possess the completeness property, in order to make the construction of R starting from Q infinitesimals rigorous, one should prove that the completeness property holds in Q_o/Q_1.

CHAPTER VI

APPLICATIONS TO REAL ANALYSIS

§1 The Theory of Limits

Definition 1.1. A sequence $s = \{s_n | n = 1, 2, \ldots\}$ of real numbers is a mapping from N into R.

$s = \{s_n | n = 1, 2, \ldots\}$ is a subset of $N \times R$ and therefore an entity of $\hat{R}$. The entity s can be extended to an entity $*s$ in $*(\hat{R})$. By the F.T. and Lemma 2.3 (viii) of Chapter II $s \to *s$ is a mapping of $*N$ into $*R$.

Definition 1.2. The sequence $s = \{s_n | n = 1, 2, \ldots\}$ is bounded if the range of s is bounded, i.e. $s = \{s_n | n = 1, 2, \ldots\}$ is bounded if and only if there exists $r \in R$ such that

$$|s_n| \leq r \text{ for all } n \in N.$$

Theorem 1. A sequence $\{s_n | n = 1, 2, \ldots\}$ in R is bounded if and only if $*s_\omega$ is finite for all infinitely large natural numbers $\omega \in *N - N$.

Proof. If s in R is bounded then $|s_n| \leq r$, for some $r \in R$ and for all $n \in N$. By the F.T., we have

$$|*s_n| \leq r, \ \forall n \in *N \text{ and for some } r \in R.$$

Conversely, by Definition 1.2 if (ran $*s$) $\subset M_o$, then $|*s_n| \leq a$ for all $n \in *N$ and some $a \in M_o$ that is $|s_n| \leq st(a)$ for all $n \in N$ which shows that $\{s_n | n = 1, 2, \ldots\}$ is bounded.

<u>Definition 1.2.</u> A sequence $\{s_n | n = 1, 2, \ldots\}$ is said to be <u>convergent</u> with limit s if and only if

$$(*)\ [(\forall \varepsilon)[0 < \varepsilon \in R] \Rightarrow (\exists x)[x \in N] \wedge (\forall y)[y \in N \wedge x \leq y]] \Rightarrow [|s_y - s| < \varepsilon]$$

In nonstandard analysis this is expressed in a more intuitive fashion as follows.

<u>Theorem 2</u>. Let $\{s_n | n = 1, 2, \ldots\}$ be a sequence of numbers of R, and let $s \in R$. Then $\lim_{n\to\infty} s_n = s$ if and only if ${}^*s_\omega =_1 s$ for all $\omega \in {}^*N - N$.

<u>Proof</u>. To prove the condition is necessary, let $\lim_{n\to\infty} s_n = s$. Then (*) of Definition 1.2 above belongs to K_0 and also the following is a sentence of K_0.

$(\forall x)[x \in N \wedge x > n] \Rightarrow [|s_x - s| < \varepsilon]$, where $\varepsilon > 0$ and $n \in N$ are constants.

By F.T. the following *L-sentence holds:

$$(\forall x)[x \in {}^*N \wedge x > n] \Rightarrow [|{}^*s_x - s| < \varepsilon]$$

and so, for all $\omega \in {}^*N - N$ we have $|{}^*s_\omega - s| < \varepsilon$. Since ε is arbitrary, that is, ${}^*s_\omega =_1 s$ for all $\omega \in {}^*N - N$.

Next we are going to prove the condition is sufficient. Let $0 < \varepsilon \in R$, the following sentence holds in ${}^*(\hat{R})$.

$(\exists y)[y \in {}^*N] \wedge (\forall x)[x \in {}^*N \wedge y < x] \Rightarrow [|{}^*s_x - s| < \varepsilon]$.

By Theorem 2 the following L-sentence holds in $\hat{R}$.

$(\exists y)[y \in N] \wedge (\forall x)[x \in N \wedge y < x] \Rightarrow [|s_n - s| < \varepsilon]$.

The above statement means that there exist $n_0 \in N$ such that

$|s_n - s| < \varepsilon$ for all $n > n_o$, since ε is arbitrary therefore $\lim\limits_{n\to\infty} s_n = s$.

The condition $*s_\omega =_1 s$ for all $\omega \in *N - N$ is equivalent to $st(*s_\omega) = s$ for all $\omega \in *N - N$, for $*s_\omega$ in this case is finite.

Corollary 1. If the limit in Theorem 2 exists, it is unique. That is, let $\{s_n | n = 1, 2, \ldots\}$ be a sequence of numbers of R and let $s, t \in R$ and $\lim\limits_{n\to\infty} s_n = s$ and $\lim\limits_{n\to\infty} s_n = t$ then $s = t$.

Proof. The proof is indirect.

Let $s \neq t$ so $|s - t| > 0$ and let $\varepsilon = \frac{1}{2}|s - t|$. Since $(\forall x)[x \in N \wedge x > n] \Rightarrow |s_x - s| < \varepsilon$, where $\varepsilon > 0$ and $n \in N$ and $(\forall x)[x \in N \wedge x > n] \Rightarrow |s_x - t| < \varepsilon$ where $\varepsilon > 0$ and $n \in N$. By F.T. the following *L-sentence holds

$$(\forall x)[x \in *N \wedge x > n] \Rightarrow |*s_x - s| < \varepsilon$$

and $(\forall x)[x \in *N \wedge x > n] \Rightarrow |*s_x - t| < \varepsilon$

so

$$|s - t| \leq |(*s_x - s) - (*s_x - t)|$$

$$< 2\varepsilon = |s - t|$$

This contradicts the fact that $s \neq t$.

Corollary 2. If the sequence $\{s_n | n = 1, 2, \ldots\}$ of real numbers is convergent, then it is bounded.

Proof. Since $\{s_n | n = 1, 2, \ldots\}$ is convergent, we have $(\forall x)[x \in N \wedge x > n] \Rightarrow |s_x - s| < \varepsilon$ where $\varepsilon > 0$ and $n \in N$ are constants.

Then by the F.T., we have

$$(\forall x)[x \in {}^*N \wedge x > n] \Rightarrow |{}^*s_x - s| < \varepsilon$$

In particular for all $\omega \in {}^*N - N$ we have $|{}^*s_n - s| < \varepsilon$

i.e. $$|{}^*s_\omega| < \varepsilon + |s|$$

which shows that ${}^*s_\omega$ is finite by Theorem 1, we have shown that $\{s_n : n = 1, 2, \ldots\}$ is bounded.

Theorem 3. A sequence $\{s_n | n = 1, 2, \ldots\}$ of real numbers of R is convergent if and only if ${}^*s_\omega = {}_1{}^*s_\omega$ for all $\omega, \omega \in {}^*N - N$.

Proof. By Theorem 2, the necessary condition is justified.

To prove the condition is sufficient, we only have to show, by Theorem 2, that ${}^*s_\omega$ is finite for all $\omega \in {}^*N - N$. The proof of this part is indirect; suppose that there is an infinitely large number $\omega_0 \in {}^*N - N$ such that ${}^*s_{\omega_0}$ is finite. Define

$$A = \{n | n \in {}^*N \text{ and } |{}^*s_{\omega_0} - {}^*s_n| < 1\}$$

where A is a set of natural numbers. From Theorem 3 of Chapter III, A is therefore internal. By the definition of A, we have ${}^*N - N \subset A$. The inclusion here is improper. Indeed, if $n \in N$ is finite, then

$$|{}^*s_{\omega_0}| \le |{}^*s_{\omega_0} - {}^*s_n| + |{}^*s_n| \in M_0$$

which shows that $n \notin A$, and so $A = {}^*N - N$. Since ${}^*N - N$ is not internal, this contradiction shows the result.

§2 Algebra of Limits

Corollary 3. If $\{s_n | n = 1, 2, \ldots\}$, $\{t_n | n = 1, 2, \ldots\}$ are sequences of real numbers if $\lim\limits_{n\to\infty} s_n = s$, and $\lim\limits_{n\to\infty} t_n = t$ then

(i) $\lim\limits_{n\to\infty} (s_n + t_n) = s + t$

(ii) $\lim\limits_{n\to\infty} s_n t_n = st$

(iii) $\lim\limits_{n\to\infty} cs_n = cs$

(iv) $\lim\limits_{n\to\infty} (s_n/t_n) = s/t$ if $t \neq 0$

Proof. (i) By Theorem 2

$(\forall x)[x \in N \wedge x > n] \Longrightarrow |s_x - s| < \varepsilon/2$ where $\varepsilon > 0$ and $n \in N$

(and) $(\forall x)[x \in N \wedge x > n] \Longrightarrow |t_x - t| < \varepsilon/2$ where $\varepsilon > 0$ and $n \in N$ so $|(s_n + t_n) - s + t| \le |s_n - s| + |t_n - t| < \varepsilon$. By F.T. we have

$|({}^*s_\omega + {}^*t_\omega) - (s + t)| \le |{}^*s_\omega - s| + |{}^*t_\omega - t| < \varepsilon$ for all

$n \in {}^*N - N$ therefore ${}^*s_\omega + {}^*t_\omega = {}_1 s + t$

and $\quad {}^*(s + t)\omega = {}^*s_\omega + {}^*t_\omega$.

By Theorem 2, we have

$$\lim_{n\to\infty} (s_n + t_n) = s + t.$$

(ii) By the homomorphism st theorem

$st({}^*(st)_\omega) = st({}^*s_\omega\,{}^*t_\omega) = st({}^*s_\omega)st({}^*t_\omega) = st$ for all $\omega \in {}^*N - N$ i.e. ${}^*(st)_\omega = {}_1st$ therefore $\lim_{n\to\infty} s_n t_n = st$.

(iii) We only take $|s_\omega - s| < \frac{\varepsilon}{|c|}$ for all $\omega \in {}^*N - N$.

We get the desired result, $\lim_{n\to\infty} cs_n = cs$.

(iv) By (ii)

$$\lim_{n\to\infty} s_n \cdot \frac{1}{t_n} = s \cdot \frac{1}{t}$$

the result follows.

§3 Continuity and Differentiability

Definition 3.1. For every $\varepsilon > 0$, there exists a positive number $\delta(\varepsilon)$ such that if

$$0 < |x - a| < \delta, \text{ then } |f(x) - b| < \varepsilon$$

and $f(x)$ is said to have the limit b as $x \to a$ $[\lim_{x\to a} f(x) = b]$.

Definition 3.2. Let a function f be defined on a set S and $u \in S$, f is said to be continuous at u if and only if for every $\varepsilon > 0$, there exists a corresponding $\delta > 0$ such that $x \in S$ and $|x - u| < \delta$ imply that

$$|f(x) - f(u)| < \varepsilon.$$

Lemma 1. Let $u, \ell \in R$ and let x_0 be a limit point of the dom f then $\lim_{x\to x_0} f(x) = \ell$, iff $\lim_{n\to\infty} f(s_n) = \ell$ for every sequence $\{s_n : n = 1, 2, \ldots\}$ such that $s_n \in$ dom f for all n and $\lim_{n\to\infty} s_n = x_0$

Lemma 2. If a function f is defined on a set S and if u is a limit point of S belonging to S, then f is continuous at x_0 if and only if $\lim_{x\to x_0} f(x) = f(x_0)$.

Theorem 1. (i) $\lim_{x\to x_0} f(x) = \ell$ if and only if $*f(x_0+h) =_1 \ell$ for all $0 \neq h \in M_1$. (ii) In particular, f is continuous at x_0 if and only if $*f(x_0 + h) = {}_1 f(x)$ for all $h \in M_1$, that is, equivalently, $st(*f(a)) = f(st(a))$ for all $a \in *R$ such that $st(a)=x_0$.

Proof. (i) Assume that $\lim_{x\to x_0} f(x) = \ell$ then

$$(\forall x)[0 < \varepsilon \in R] \Rightarrow (\exists\delta)[0 < \delta \in R] \wedge (\forall x)[x \in R \wedge 0 < |x-x_0| < \delta] \Rightarrow [|f(x) - \ell| < \varepsilon]$$

is a L-sentence of K_0.

By F.T. the following *L-sentence holds.

$$(\forall\varepsilon)[0 < \varepsilon \in *R] \Rightarrow (\exists\delta)[0 < \delta \in *R] \wedge (\forall x)[x\in *R \wedge 0 < |x-x_0| < \delta] \Rightarrow [|*f(x) - \ell| < \varepsilon]$$

since ε is arbitrary, thus $*f(x) =_1 \ell$ where $x = x_0 + h$ for all $0 \neq h \in M_1$.

Conversely, just reverse the steps and use the F.T., the result follows:

(ii) By Lemma 2, we can use (i) to treat $\lim_{x\to x_0} f(x)=f(x_0)$

the result follows. That is, $\lim_{x \to x_o} f(x) = f(x_o)$ if and only if $*f(x_o + h) =_1 f(x_o)$, for all $h_1 \in M_1$.

Definition 3.3. Let f be a function, for each $x_o \in \operatorname{dom} f$, we define

$$f'(x_o) = \lim_{h \to 0} \frac{f(x_o+h) - f(x_o)}{h}.$$

If this limit exists, f(x) is said to be differentiable at $x=x_o$, and $f'(x_o)$ is called the derivative of f at x.

Remark. By the above definition and Theorem 1 f is differentiable at x_o if and only if there exists a constant $\ell \in R$. (Since $f'(x_o)$ exists means that $f'(x_o)$ is finite) such that

$$\frac{*f(x_o+h) - *f(x_o)}{h} =_1 \ell$$

for all $0 \neq h \in M_1$.

Corollary 1. If f is differentiable at x_o, then f is continuous at x_o.

Proof. From the above remark, we have

$$*f(x_o + h) - *f(x_o) =_1 hf'(x_o)$$

since $f'(x_o)$, by definition, is finite. So $hf'(x_o)$ is a infinitesimal therefore $*f(x_o + h) - *f(x_o) =_1 0$ for all $h \in M_1$.

Definition 3.4. Let f be a function continuous on dom f. f is said to be uniformly continuous on dom f iff for every $\varepsilon > 0$, there exists $\delta(\varepsilon)$ such that $|f(a) - f(b)| < \varepsilon$ for all a, b $\in$ dom f and $|a - b| < \delta$.

In symbol,

$$(**)\quad (\forall\varepsilon)(\exists\delta)[0 < \varepsilon \in R \wedge 0 < \delta \in R] \Rightarrow (\forall a)(\forall b)[|f(a) - f(b)| < \varepsilon \wedge |a - b| < \delta]$$

Theorem 2. Let f be a real function of a real variable. Then f is uniformly continuous if and only if $*f(a) =_1 *f(b)$ for all a, b $\in$ dom*f and $a =_1 b$.

Proof. The condition is necessary.

Assume that f is uniformly continuous. From the L-sentence (**) of K_o, the following L-sentence also belongs to K_o.

$(\forall a)(\forall b)[|f(a) - f(b)| < \varepsilon \wedge |a - b| < \delta]$ where $\varepsilon > 0$, $\delta > 0$ are constants. By the F.T. the following *L-sentence holds.

$$(\forall a)(\forall b)[|*f(a) - *f(b)| < \varepsilon \wedge |a - b| < \delta]$$

thus for all a, b $\in$ dom*f

$$*f(a) =_1 *f(b) \text{ and } a =_1 b.$$

For the proof of the sufficient condition is just to reverse the steps by using the F.T.

Theorem 3. (Heine). Let f be a real function of a real variable defined on the bounded and closed interval $x_1 \leq x \leq x_2$, $x_1, x_2 \in R$. If f is continuous, then f is uniformly continuous.

Proof. Let a, b ∈ *R satisfy $x_1 \le a, b \le x_2$ and $a = {}_1b$, then by definition of finite numbers of *R, a, b ∈ M_o and $x = st(a) = st(b)$ satisfies $x_1 \le x \le x_2$.

Since f is continuous by Theorem 1 we have $*f(a) = {}_1*f(b)$ where $a = st(a) + b_1$, $b = st(b) + b_2$, i.e. $a = x + b_1$, $b = x + b_2$, therefore $*f(a) = {}_1*f(b)$ again by the F.T. this result follows.

§4 Definite Integral

Definition 4.1. Given a function f and the closed interval [a, b] and $0 < \Delta x \in R$, we define $\sum_a^\ell f(x)\,\Delta x$ as the *Riemann Sum*.

$$\sum_a^b f(x)\,\Delta x = f(x_o)\Delta x + f(x_1)\Delta x + \dots + f(x_{n-1})\Delta x$$

where n is the greatest integer such that $a + n\Delta x \le b$ and x_k $(k = 0, \dots, n-1)$ where $x_{n-1} = a + (n-1)\Delta x$ are the partition points.

Definition 4.2. If $\lim\limits_{\Delta x \to 0} \sum_a^b f(x)\Delta x = \ell$ then

$$(\forall\varepsilon)(\exists\delta)[0 < \varepsilon\ R \wedge 0 < \delta \in R] \Rightarrow \left|\sum_a^b f(x)\Delta x - \ell\right| < \delta$$

where a, b are constants and a < b, $\Delta x = \frac{b-a}{n}$, n ∈ N which is a sentence of K_o. By the F.T. we have the following *L-sentence which holds in *R.

$(\forall\varepsilon)(\exists\delta)[0 < \varepsilon \in R \wedge 0 < \delta \in R] \Rightarrow |\sum_a^b f(x)dx - \ell| < \delta$ where $dx = \frac{b-a}{\omega}$, $\omega \in {}^*N$ we define $\sum_a^b f(x)dx$ as <u>an infinite Riemann sum</u>.

<u>Theorem 1</u>. If f is continuous on [a, b] and $dx > 0$ an infinitesimal then the infinite Riemann sum $\sum_a^b f(x)dx$ is a finite real number.

<u>Proof</u>. Since every $\sum_a^b f(x)\Delta x$ is bounded above and below, say

$$m(b - a) < \sum_a^b f(x)\Delta x < M(b - a)$$

where m, M are minimum and maximum of f, for f is continous on [a, b]. This statement can be expressed as a sentence of K_0. In symbols,

$$(\exists m)(\exists M)[0 < \Delta x \in R] \Rightarrow [[m(b - a) < \sum_a^b f(x)\Delta x] \wedge [\sum_a^b f(x)\Delta x < M(b - a)]]$$

where a, b are constants and $\Delta x = \frac{b-a}{n}$, $n \in N$.

By

$$(\exists m)(\exists M)[0 < dx \in {}^*R] \Rightarrow [[m(b - a) < \sum_a^b {}^*f(x)dx] \wedge [\sum_a^b {}^*f(x)dx < M(b - a)]]$$

where a, b are constant and $dx = \frac{b-a}{\omega}$, $\omega \in {}^*N$ which shows that $\sum_a^b f(x)dx$ is bounded below and above by $m(b - a)$ and $M(b - a)$ respectively, thus, by the definition of a finite number of *R, we know that $\sum_a^b f(x)dx$ is finite.

Remark. As we can define the derivative as the standard part of the quotient $\frac{\Delta y}{\Delta x}$ (denoted by $\frac{dy}{dx}$) the definite integral is now defined by the standard part of the infinite Riemann sum denoted by $\int_a^b f(x)dx = st(\sum_a^b f(x)dx)$.

CHAPTER VII

THE HYPERREAL NUMBERS

§1 A New Extension of the Real Number System

The real number system can be developed through a process of extension from N (the set of all natural numbers) to Z (the set of all integers) to Q (the set of all rational numbers) to R (the set of all real numbers).

The inclusion

$$N \subset Z \subset Q \subset R$$

is proper and the elements of the subset can be identified with special elements of the superset. Q and R are fields and Q is <u>dense</u> in R, i.e. between any two rational numbers there exists a non-rational real number, and R is a <u>closure</u> of Q with respect to limits of sequences. R is so constructed and is a set of equivalent classes of Cauchy sequences of rationals. Since a Cauchy sequence of real numbers converges to a real number the extension of R w.r. to Cauchy sequences is again R. We cannot obtain any non-real number system by following the same process as in the construction of R starting with Q.

§2 The Axioms for the Hyperreal Numbers

In order to avoid the heavy logical machinery, H.J. Keisler lists a set of axioms in such a way that H, the hyperreal number system would become an ordered field and also a proper extension of R.

I. Order Axioms.

(A) Every real number is a hyperreal number. That is, R is a subset of H.

(B) The Trichotomy Law holds for all hyperreal numbers, that is, exactly one of the following three relations holds:

$$a < b, \ a = b, \ a > b.$$

II. Axioms About Finite and Infinitesimal Numbers.

(A) Every finite hyperreal number is infinitely close to some real number.

(B) There exists a positive infinitesimal hyperreal number.

III. The Function Axioms.

(A) For every real function f of one or two variables, there is a corresponding hyperreal function f* of the same number of variables called the natural extension of f.

(B) If f is a real function of one variable and a is a real number, then:

When $f(a)$ is defined, $f^*(a) = f(a)$

When $f(a)$ is undefined, so is $f^*(a)$.

(C) If f is a real function of two variables and a, b are real numbers, then:

When $f(a,b)$ is defined, $f^*(a,b) = f(a, b)$.

IV. The Solution Axioms.

(A) Let S be any system of real formulas. If S is true of all real numbers, then S is true of all hyperreal numbers.

(B) Let S and T be two systems of real formulas. If every real solution of S is a solution of T, then every hyperreal solution of S is a hyperreal solution of T.

Remarks. With this set of axioms, H.J. Keisler proves that (i) for every finite hyperreal number b, there exists a unique (standard) real number c which is infinitely close to b; (ii) no real number is infinitely close to any infinite hyperreal number; (iii) every infinitesimal is infinitely close to zero; (iv) zero is the only infinitesimal which is also a standard real number; (v) every infinitesimal (except zero) is a reciprocal of an infinitely large hyperreal number.

The standard part of a finite hyperreal number, denoted by st(b), is defined as the unique real number c being infinitely close to b, i.e. $b = st(b) + \varepsilon$, where ε is an infinitesimal.

There is no standard part for any infinite hyperreal number. After establishing some rules for infinitesimals and standard part, the hyperreal numbers can be applied to elementary calculus. One shows that "standard part operation" commutes with algebraic operations (i.e. field operations, exponentiation, root extraction, order).

§3 Applications to Elementary Calculus

3.1 Derivative. In traditional elementary calculus course, as the definition of derivative of a function depends on the limit concept; so the concept of limit is usually treated before that of the derivative. By the concept of standard part, we can treat them the other way round in a more natural and simpler way.

<u>Definition</u>. Let f be a real function of one variable, the derivative of f is the new function f' whose value at x is the <u>slope</u> of f at x, denoted by f'(x), if

$$f'(x) = st\left(\frac{f(x + \Delta x) - f(x)}{\Delta x}\right)$$

for all $\Delta x \neq 0$.

<u>Remark</u>. Since every finite hyperreal number has a unique standard part, then the slope f'(x) exists if and only if

$$\frac{f(x + \Delta x) - f(x)}{\Delta x}$$

is finite and has the same standard part for all infinitesimal $\Delta x \neq 0$.

<u>Examples</u>.

(i) Find the derivative of the function

$$y = f(x) = x^2$$

$$y + \Delta y = (x + \Delta x)^2 = x^2 + 2x\ \Delta x + (\Delta x)^2$$

$$\Delta y = 2x\ \Delta x + (\Delta x)^2$$

$$\frac{\Delta y}{\Delta x} = 2x + \Delta x$$

$$\begin{aligned} st\left(\frac{\Delta y}{\Delta x}\right) &= st(2x + \Delta x) \\ &= 2x + 0 \\ &= 2x \end{aligned}$$

Therefore, $f'(x) = st(\frac{\Delta y}{\Delta x}) = 2x$, for all $x \in R$.

(ii) Find the derivative of the function

$$y = \sqrt[3]{x}$$

$$\frac{\Delta y}{\Delta x} = \frac{1}{(x+\Delta x)^{2/3} + x^{1/3}(x+\Delta x)^{1/3} + x^{2/3}}$$

$$f'(x) = st(\frac{\Delta y}{\Delta x})$$

$$= st(\frac{1}{(x+\Delta x)^{2/3} + x^{1/3}(x+\Delta x)^{1/3} + x^{2/3}})$$

$$= \frac{1}{[st(x+\Delta x)]^{2/3} + st(x^{1/3})[st(x+\Delta x)]^{1/3} + st(x^{2/3})}$$

$$= \frac{1}{(st(x))^{2/3} + st(x)^{1/3}(st(x))^{1/3} + st(x^{2/3})}$$

$$= \frac{1}{3(st(x))^{2/3}}$$

$$= \frac{1}{3x^{2/3}}, \text{ for all values of } x \in R.$$

The process of finding the derivative of f is called <u>differentiation</u>. f is said to be <u>differentiable</u> at x if f'(x) is defined, i.e., the slope of f exists.

3.2 Limits

<u>Definition</u>. L is said to be the limit of f(x) as x approac c whenever x is infinitely close to but not equal to c, f(x) is infinitely close to L. In symbols,

$$\lim_{x \to c} f(x) = L.$$

Theorem 1. The slope of f at x is given by the limit

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}.$$

The theorem follows immediately by comparing the definitions of slope and limit.

Examples.

(i) Find $\lim\limits_{x \to 1} \frac{x^2 - 3x + 6}{x^2 + x + 2}$.

Taking $x \neq 1$ and $st(x) = 1$

so $$st\left(\frac{x^2 - 3x + 6}{x^2 + x + 2}\right) = \frac{1 - 3 + 6}{1 + 1 + 2} = \frac{1}{2}.$$

(ii) Find $\lim\limits_{x \to 0} \sqrt{9 - x}$.

Since x approaches 0 at either side along the real line, $\sqrt{9-x}$ is defined therefore we can take the standard part of $\sqrt{9-x}$ as follows:

Taking $st(x) = 0$ but $x \neq 0$.

$$\begin{aligned} st\sqrt{9-x} &= [st(9 - x)]^{1/2} \\ &= [9 - st(x)]^{1/2} \\ &= (9)^{1/2} \\ &= 3. \end{aligned}$$

(iii) Find $\lim\limits_{x \to 0^+} \frac{1}{2 + e^{-(1/x)}}$

Taking the standard part of $\frac{1}{2 + e^{-(1/x)}}$ we have,

$$\mathrm{st}\left(\frac{1}{2+e^{-(1/x)}}\right) = \frac{1}{2+\mathrm{st}(e^{-(1/x)})} = \frac{1}{2+\mathrm{st}\left(\frac{1}{e^{1/x}}\right)}$$

$$= \frac{1}{2+\mathrm{st}\left(\frac{1}{e^{\omega}}\right)}$$

$$= \frac{1}{2+\mathrm{st}\left(\frac{1}{\omega}\right)},$$

$$= \frac{1}{2+\mathrm{st}(\eta)} \text{ (where } \eta \text{ is an infinitesimal)}$$

$$= \frac{1}{2+0}$$

$$= \frac{1}{2}$$

3.3 Continuity

Definition (A). f is said to be continuous at a point a iff

(i) f is defined at c

(ii) whenever x is infinitely close to c, f(x) is infinitely close to f(c).

Theorem 2. f is continuous at c iff $\lim_{x \to c} f(x) = f(c)$.

This theorem follows immediately from the definition.

Definition (B). We say that f is continuous on an open interval T if f is continuous at every point c in T. This is equivalent to saying that

$$f(\mathrm{st}(x)) = \mathrm{st}(f(x))$$

for every hyperreal number x whose standard part is in T.

3.4 The Definite Integral

Definition. f is said to be integrable over [a, b] if there exists a real number J such that

$$J \approx \sum_a^b f(x)\Delta x.$$

The number J~~, if~~ it exists, is denoted by $\int_a^b f$. ("$\approx$" means "is infinitely close to").

SUMMARY

The non-standard analysis provides an alternate approach to real analysis including elementary calculus. The present approach to non-standard analysis requires a considerable knowledge of mathematical logic. Although Luxemburg [2] has tried to reduce the logical machinery necessary for real analysis to a minimum it is still formidable. Another approach is presented by D. Laugwitz [13] who tries to minimize the logical apparatus but admits that his method lacks full rigor.

For the purpose of elementary introduction to non-standard analysis Keisler develops a hyperreal number system on intuitive basis. The advantage of such presentation appears to be some simplification in the formal manipulations dealing with sequences and limits. To a large extent, Keisler's method when dealing with derivatives and integrals differs insignificantly from most presentations in non-rigorous calculus text-books.

At present the teaching of non-standard calculus is in the preliminary stage. So far only one complete text-book (Keisler's) has appeared. Also the teaching experience has been very limited. Therefore no definitive evolution can be made at this time.

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ERRATA

Page 35, line 15, read $\omega \in {}^*R$.

ADDENDA

[1]

Theorem 1. Let $V = V(x_1, \ldots, x_p)$ be an admissible L-wff the free variables $x_1, \ldots, x_p$, and let

$A = \{(x_1, \ldots, x_p) | (x_1, \ldots, x_p) \in a$ and $V(x_1, \ldots, x_p)|$,

where a is an arbitrary entity of $\hat{R}$, then $A \in \hat{R}$ and

$*A = \{(y_1, \ldots, y_p) | (y_1, \ldots, y_p) \in *a$ and $*V(y_1, \ldots, y_p)\}$.

Proof. $A \in \hat{R}$ is trivial. In fact, A is a set of p-ary relation, then $A \in \hat{R}$.

If $V = V(x_1, \ldots, x_p, a_1, \ldots, a_q)$ is atomic, that is, V has the form $(x_1, \ldots, x_p, a_1, \ldots, a_{q-1}) \in a_q$ or $(x_1, \ldots, x_{p-1}, a_1, \ldots, a_q) \in x_p$ with possible permutation of the variables, then the result follows immediately from Definition 2.3 and Lemma 2.3 (iii). In order to show that the result holds for all wff V of L without quantifiers, we need only show that it holds for wff V and W then it also holds for $[V \wedge W]$ $[\neg V]$ (for $p \Rightarrow q$ can be replaced by $\neg p \vee q$, $p \Leftrightarrow q$ can be replaced by $[p \Rightarrow q] \wedge [q \Rightarrow p]$, and $p \vee q$ can be replaced by $\neg[[\neg p] \wedge [\neg q]]$). Assume that

$*A = \{(x_1, \ldots, x_p) | (x_1, \ldots, x_p) \in *a$ and $*V(x_1, \ldots, x_p)\}$

then we have to show that

$*B = \{(x_1, \ldots, x_p) | (x_1, \ldots, x_p) \in *a$ and $*V(x_1, \ldots, x_p)\}$

Where $B = a - A$. The result follows immediately. In fact, if

$(x_1, \ldots, x_p) \in {}^*a$ and $(x_1, \ldots, x_p) \notin {}^*A$ which in turn implies $(x_1, \ldots, x_p) \notin {}^*V(x_1, \ldots, x_p)$. That is $(x_1, \ldots, x_p) \in {}^*a$ and $(x_1, \ldots, x_p) \in \neg{}^*V(x_1, \ldots, x_p)$. Then $(x_1, \ldots, x_p) \in {}^*B$.

Assume now $V = V(x_1, \ldots, x_p, y_1, \ldots, y_q)$ and $W = W(x_1, \ldots, x_p, z_1, \ldots, z_r)$ be two L-wff without quantifiers, for which the result holds, and let

$$A = \{(x_1, \ldots, x_p, y_1, \ldots, \cancel{y_q}, x_1, \ldots, z_r) \mid (x_1, \ldots, z_r) \in a \text{ and } [V \wedge W]\}$$

then

$$A = \{(x_1, \ldots, z_r) \mid (x_1, \ldots, z_r) \in a \text{ and } V\} \cap \{(x_1, \ldots, z_r) \mid (x_1, \ldots, z_r) \in a \text{ and } W\}$$

implies by Lemma 2.3 (v) that

$${}^*A = {}^*\{\ldots\} \cap {}^*\{\ldots\} = \{(x_1, \ldots, z_r) \mid (x_1, \ldots, z_r) \in {}^*a \text{ and } [{}^*V \wedge {}^*W]\}$$

Next we are going to prove that admissible wff with quantifiers by induction on the number n of quantifiers. By Lemma 3.3, then V can be expressed as an admissible wff with (n+1) quantifiers of the form $q(x_{n+1}) \ldots q(x_1)W(x_1, \ldots, x_{n+1}, y_1, \ldots, y_q)$ where W has no quantifiers and $y_1, \ldots, y_q$ are without loss ~~of~~ generality, we may assume that q_{n+1} is the existential quantifier, otherwise consider not V.

For $n = 0$, the result holds as shown above for V without quantifiers.

Now let b denote the domain of $(\exists x_{n+1})$. Then, since V is admissible, $b \in \hat{R}$. Let

$$B = \{((y_1, \ldots, y_p), x_{n+1}) | ((y_1, \ldots, y_p), x_{n+1}) \in a \times b \text{ and } q(x_n) \ldots q(x_1)W\}$$

where $a \in \hat{R}$.

Then, by the induction hypothesis and Lemma 2.3 (v) we have

$$*B = \{((y_1, \ldots, y_p), x_{n+1}) | ((y_1, \ldots, y_p), x_{n+1}) \in *a \times *b \text{ and } (qx_n) \ldots (qx_1)*W\}$$

The domain of the binary relation B is the set

$$A = \{(y_1, \ldots, y_p) | (y_1, \ldots, y_p) \in a \text{ and } (\exists x_{n+1})(x_{n+1} \in b \cap (qx_n) \ldots (qx_1)W)\}$$
$$= \{(y_1, \ldots, y_p) | (y_1, \ldots, y_p) \in a \text{ and } V(y_1, \ldots, y_p)\}$$

The domain of the binary $*B$ is the set

$$\{(y_1, \ldots, y_p) | (y_1, \ldots, y_p) \in *a \text{ and } (\exists x_{n+1})(x_{n+1} \in *b \cap (qx_n) \ldots (qx_1)*W)\}$$
$$= \{(y_1, \ldots, y_p) | (y_1, \ldots, y_p) \in *a \text{ and } *V(y_1, \ldots, y_p)\}$$

Then by Lemma 2.3 (v), we have

$$*A = \{(y_1, \ldots, y_p) | (y_1, \ldots, y_p) \in *a \text{ and } *V(y_1, \ldots, y_p)\}.$$