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On simultaneous approximation to a real number and its cube by rational numbers

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Abstract

One of the fundamental problems in Diophantine approximation is approximation to real numbers by algebraic numbers of bounded degree. In 1969, H. Davenport and W. M. Schmidt developed a new method to approach the problem. This method combines a result on simultaneous approximation to successive powers of a real number ξ with geometry of numbers. For now, the only case where the estimates are optimal is the case of two consecutive powers. Davenport and Schmidt show that if a real number ξ is such that $1, \xi, \xi^2$ are linearly independent over $\mathbb{Q}$, then the exponent of simultaneous approximation to ξ and ξ^2 by rational numbers with the same denominator is at most $(\sqrt{5}-1)/2 = 0.618\ldots$, the inverse of the Golden ratio. In this thesis, we consider the case of a number and its cube. Our main result is that if a real number ξ is such that $1, \xi, \xi^3$ are linearly independent over $\mathbb{Q}$, then the exponent of simultaneous approximation to ξ and ξ^3 by rational numbers with the same denominator is at most $5/7 = 0.714\ldots$. As corollaries, we deduce a result on approximation by algebraic numbers and a version of Gel'fond's lemma for polynomials of the form $a + bT + cT^3$.

Résumé

Un des problèmes fondamentaux en approximation diophantienne est celui de l'approximation des nombres réels par des nombres algébriques de degré borné. En 1969, H. Davenport et W. M. Schmidt ont développé une nouvelle méthode pour approcher ce problème. Elle combine un résultat d'approximation simultané de puissances consécutives d'un nombre réel ξ avec des méthodes de géométrie des nombres. Pour l'instant le seul cas où les estimations sont optimales est celui de deux puissances consécutives. Davenport et Schmidt ont démontré que si un nombre réel ξ est tel que $1, \xi, \xi^2$ sont linérairement indépendants sur $\mathbb{Q}$, alors l'exposant d'approximation simultané de ξ et de ξ^2 par des nombres rationnels de même dénominateur est au plus $(\sqrt{5}-1)/2 = 0.618...$ soit l'inverse du nombre d'or. Dans ce mémoire, nous considérons le cas d'un nombre et de son cube. Notre résultat principal est que si un nombre réel ξ est tel que $1, \xi, \xi^3$ sont linéairement indépendants sur $\mathbb{Q}$, alors l'exposant d'approximation simultané de ξ et de ξ^3 par des nombres rationnels de même dénominateur est au plus $5/7 = 0.714....$ Nous en déduisons des corollaires sous la forme d'un résultat d'approximation par des nombres algébriques et d'une version du lemme de Gel'fond concernant les polynômes de la forme $a + bT + cT^3$.

Contents

Chapter 1

Introduction

1.1 History

In their 1969 paper [4], H. Davenport and W. M. Schmidt established the following result, where

$$\gamma = \frac{1+\sqrt{5}}{2} = 1 + \frac{1}{\gamma} = 1.618 \ .$$

denotes the golden ratio.

Theorem 1.1.1 (Davenport-Schmidt) *Suppose that $\xi \in \mathbb{R}$ is neither rational nor a quadratic irrational. Then there are arbitrarily large values of X such that the inequalities*

$$|x_0| \leq X, \quad |x_0\xi - x_1| \leq c_1 X^{-1/\gamma}, \quad |x_0\xi^2 - x_2| \leq c_1 X^{-1/\gamma}, \tag{1.1.1}$$

where c_1 is a suitable positive number depending only on ξ, have no solution in integers x_0, x_1, x_2 not all zero.

If $(x_0, x_1, x_2) \in \mathbb{Z}^3$ is a non-zero integer solution of the inequalities (1.1.1) of Theorem 1.1.1 for some large value of X, then we must have $x_0 \neq 0$ and we obtain

$$\max\left\{\left|\xi - \frac{x_1}{x_0}\right|, \left|\xi^2 - \frac{x_2}{x_0}\right|\right\} \leq c_1 \frac{X^{-1/\gamma}}{|x_0|} \leq c_1 |x_0|^{-\gamma}$$

which means that the pair $(x_1/x_0, x_2/x_0)$ is a good approximation to (ξ, ξ^2) by rational numbers with the same denominator.

By an argument based on geometry of numbers, Davenport and Schmidt deduce the following corollary where the *height* $H(\alpha)$ of an algebraic number α is defined as the height of its minimal polynomial in $\mathbb{Z}[T]$, the height $H(P)$ of a polynomial $P \in \mathbb{R}[T]$ being defined as the largest absolute value of its coefficients.

Corollary 1.1.2 (Davenport-Schmidt) *Let ξ be as in Theorem 1.1.1. Then there are infinitely many algebraic integers α of degree at most 3 over $\mathbb{Q}$ which satisfy*

$$|\xi - \alpha| \leq c_2 H(\alpha)^{-\gamma-1}$$

for a suitable constant $c_2 = c_2(\xi) > 0$.

The same method also leads to a similar result of approximation by algebraic numbers of degree at most 2:

Corollary 1.1.3 *Let ξ be as in Theorem 1.1.1. Then there are infinitely many algebraic numbers α of degree at most 2 over $\mathbb{Q}$ which satisfy*

$$|\xi - \alpha| \leq c_3 H(\alpha)^{-\gamma-1} \tag{1.1.2}$$

for a suitable constant $c_3 = c_3(\xi) > 0$.

However, although it is known that the estimates in Theorem 1.1.1 and in Corollary 1.1.2 are the best possible (see [7] and [8]), this is not the case for Corollary 1.1.3. Indeed, Davenport and Schmidt showed in [3] that in (1.1.2) the exponent $-\gamma - 1 \approx -2.618$ can be replaced by -3, in which case the estimate is optimal.

In another direction, Jarník's transference principle (Theorem 1 of [5]) also allows one to restate Theorem 1.1.1 in the following dual form:

Corollary 1.1.4 *Let ξ be as in Theorem 1.1.1. Then, there are arbitrarily large values of Y such that the inequalities*

$$|y_1| \leq Y, \quad |y_2| \leq Y, \quad |y_0 + y_1\xi + y_2\xi^2| \leq c_4 Y^{-\gamma-1},$$

where $c_4 = c_4(\xi) > 0$ *is a suitable positive number depending only on* ξ, *have no solution in integers* y_0, y_1, y_2 *not all zero.*

The latter corollary can be viewed as a version of Gel'fond's lemma in degree 2. A sharper result, with an explicit value for the constant c_4 is given in [1]:

Theorem 1.1.5 (Arbour-Roy) *Let* $\xi \in \mathbb{R}$. *Suppose that for each sufficiently large value of* Y *there exists a non-zero polynomial* $P \in \mathbb{Z}[T]$ *of degree* ≤ 2 *satisfying*

$$H(P) \leq Y \quad \text{and} \quad |P(\xi)| \leq \frac{1}{4} Y^{-\gamma-1}.$$

Then ξ *is algebraic over* $\mathbb{Q}$ *of degree* ≤ 2.

1.2 Main results

In this thesis, we are looking at simultaneous approximation to a real number and to its cube by rational numbers with the same denominator. Our main result is the following:

Theorem 1.2.1 *Let* $\xi \in \mathbb{R}$ *be such that* $1, \xi, \xi^3$ *are linearly independent over* $\mathbb{Q}$, *and let* $\lambda > 5/7$. *Then there are arbitrarily large values of* X *for which the inequalities*

$$|x_0| \leq X, \quad |x_0\xi - x_1| \leq X^{-\lambda}, \quad |x_0\xi^3 - x_2| \leq X^{-\lambda},$$

admit no non-zero solution $(x_0, x_1, x_2) \in \mathbb{Z}^3$.

We don't know if this statement remains true for $\lambda = 5/7$ even at the expense of replacing the upper bound $X^{-\lambda}$ by $c_5 X^{-\lambda}$ for a suitably small positive constant c_5.

Applying the method of Davenport and Schmidt to the above result, we deduce the following analog of Corollary 1.1.3:

Corollary 1.2.2 *Let* ξ *and* λ *be as in Theorem 1.2.1. Then there are infinitely many algebraic numbers* α *which are roots of polynomials of the form* $aT^3 + bT + c$ *with*

$a, b, c \in \mathbb{Z}$ and which satisfy

$$|\xi - \alpha| \leq X^{-1-1/\lambda}.$$

We do not present the details of the proof of this corollary in this thesis since there are abundant examples of such in the literature, starting from the original work of Davenport and Schmidt in Section 2 of [4] (see for example Section 2 of [9] or Chapter 2 of [11]).

Using Jarník's transference principle, Theorem 1.2.1 admits the following equivalent form, analogous to Theorem 1.1.5:

Corollary 1.2.3 *Let $\xi \in \mathbb{R}$ and let $\tau > 7/2$. Suppose that, for each sufficiently large Y, there exists a non-zero polynomial $P(T) \in \mathbb{Z}T^3 + \mathbb{Z}T + \mathbb{Z}$ satisfying*

$$H(P) \leq Y \quad \text{and} \quad |P(\xi)| \leq Y^{-\tau}.$$

Then ξ is a root of a non-zero polynomial in $\mathbb{Z}T^3 + \mathbb{Z}T + \mathbb{Z}$.

We give a direct proof a weaker result in Appendix B (corresponding to the measure found in Chapter 3).

The rest of the thesis is mainly concerned with the proof of Theorem 1.2.1. We proceed in several steps. In order to describe them, it is useful to make the following definitions:

Definition 1.2.4 *Let $\xi, \eta \in \mathbb{R}$. We say that a real number μ is a* measure of simultaneous approximation to $(1, \xi, \eta)$ *if there exists a constant $c > 0$, and arbitrarily large values of X, such that the inequalities*

$$|x_0| \leq X, \quad |x_0\xi - x_1| \leq cX^{-\mu}, \quad |x_0\eta - x_2| \leq cX^{-\mu},$$

have no non-zero solution $(x_0, x_1, x_2) \in \mathbb{Z}^3$. When this holds, we say that c is a constant associated to the measure μ.

Definition 1.2.5 *Let $\xi, \eta \in \mathbb{R}$. We say that a real number λ is an* exponent of simultaneous approximation to $(1, \xi, \eta)$ *if there exists a constant $c > 0$ such that the inequalities*

$$|x_0| \leq X, \quad |x_0\xi - x_1| \leq cX^{-\lambda}, \quad |x_0\eta - x_2| \leq cX^{-\lambda},$$

have a non-zero solution $(x_0, x_1, x_2) \in \mathbb{Z}^3$ for any sufficiently large X. In this case, we say that c is a constant associated to the exponent λ.

From these definitions, Theorem 1.1.1 can be restated by saying that $1/\gamma$ is a measure of approximation to $(1, \xi, \xi^2)$ for any $\xi \in \mathbb{R}$ such that $1, \xi, \xi^2$ are linearly independent over $\mathbb{Q}$. Similarly, Theorem 1.2.1 says that each real $\lambda > 5/7$ is a measure of approximation to $(1, \xi, \xi^3)$ for any $\xi \in \mathbb{R}$ such that $1, \xi, \xi^3$ are linearly independent over $\mathbb{Q}$.

Moreover, by Minkowski's Linear Forms Theorem [10, Chapter II, Theorem 2C], we have:

Theorem 1.2.6 *Let $\xi, \eta \in \mathbb{R}$. For every $X > 1$ the inequalities*

$$|x_0| \leq X, \quad |x_0\xi - x_1| < X^{-1/2}, \quad |x_0\eta - x_2| < X^{-1/2} \tag{1.2.1}$$

have a non-zero solution $(x_0, x_1, x_2) \in \mathbb{Z}^3$.

This shows that, for any $\xi, \eta \in \mathbb{R}$, the number $\lambda = 1/2$ is an exponent of simultaneous approximation to $(1, \xi, \eta)$ and $c = 1$ is a constant associated with this exponent. We also note the following properties which follow immediately from the definitions:

- If $\mu \in \mathbb{R}$ is a measure of approximation to a triple $(1, \xi, \eta)$, then any real number μ' with $\mu \leq \mu'$ is also a measure of approximation to the same triple.
- If $\lambda \in \mathbb{R}$ is an exponent of approximation to $(1, \xi, \eta)$, then any real number λ' with $\lambda' \leq \lambda$ is another such exponent of approximation.

- If μ is a measure of approximation to $(1, \xi, \eta)$ and if λ is an exponent of approximation to the same triple, then $\lambda \leq \mu$.

- If λ is not a measure of approximation to $(1, \xi, \eta)$, then λ is an exponent of approximation to the same triple and, as such, its associated constant can be taken to be arbitrarily small.

- If μ is not an exponent of approximation to $(1, \xi, \eta)$, then μ is a measure of approximation to the same triple and, as such, its associated constant can be taken to be arbitrarily large.

In view of these remarks, Theorem 1.2.1 can be restated by saying that, if λ is an exponent of approximation to $(1, \xi, \xi^3)$ for some $\xi \in \mathbb{R}$ such that $1, \xi, \xi^3$ are linearly independent over $\mathbb{Q}$, then $\lambda \leq 5/7$.

The reason why we restrict to triples $(1, \xi, \eta)$ where $1, \xi, \eta$ are linearly independent over $\mathbb{Q}$ is that, otherwise, Dirichlet's Theorem [10, Chapter I, Theorem 1A] implies that $\lambda = 1$ is an exponent of approximation to $(1, \xi, \eta)$.

1.3 Outline of the thesis and methodology

The chapters of this thesis represent the different stages which lead me finally to Theorem 1.2.1. In each one, new tools or new methods are introduced to improve the upper bound on any exponent of approximation λ to a triple $(1, \xi, \xi^3)$ with $\mathbb{Q}$-linearly

independent entries. I find successively:

$$\lambda \leq \sqrt{3} - 1 = 0.732... \qquad \text{(Chapter 3)}$$

$$\lambda \leq 0.7277... \qquad \text{(Chapter 5)}$$

$$\lambda \leq 8/11 = 0.7272... \qquad \text{(Chapter 6)}$$

$$\lambda \leq 0.7252... \qquad \text{(Chapter 7)}$$

$$\lambda \leq 0.7221... \qquad \text{(Chapter 8)}$$

$$\lambda \leq (9 + \sqrt{15})/18 = 0.7151... \qquad \text{(Chapter 9)}$$

$$\lambda \leq 5/7 = 0.7142... \qquad \text{(Chapter 10).}$$

All results rely on the construction of a so-called *sequence of minimal points* for $(1, \xi, \xi^3)$ following ideas of Davenport and Schmidt in Section 3 of [4]. This is a sequence of integer points $\mathbf{x}_i = (x_{i,0}, x_{i,1}, x_{i,2}) \in \mathbb{Z}^3$ which is canonically attached to the triple $(1, \xi, \xi^3)$ and from which one can deduce whether or not a given $\lambda > 0$ is an exponent of approximation to $(1, \xi, \xi^3)$. Upon putting

$$X_i = \max\left\{|x_{i,0}|, |x_{i,1}|, |x_{i,2}|\right\},$$
$$L_i = \max\left\{|x_{i,0}\xi - x_{i,1}|, |x_{i,0}\xi^3 - x_{i,2}|\right\},$$

this condition is simply that

$$L_i \leq cX_{i+1}^{-\lambda}$$

for some constant $c > 0$ (by construction, we also have $X_1 < X_2 < X_3 < \ldots$ and $L_1 > L_2 > L_3 > \ldots$).

The main tool is the construction of polynomials with specific properties. If

$$P(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_k) \in \mathbb{Z}[\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_k]$$

is a multi-homogeneous polynomial in triples of indeterminates $\mathbf{u}_j = (u_{j,0}, u_{j,1}, u_{j,2})$, upper bound estimates for

$$|P(\mathbf{x}_i, \mathbf{x}_{i+1}, \ldots, \mathbf{x}_{i+k})| \tag{1.3.1}$$

can be given in terms of $X_i, \ldots, X_{i+k}, L_i, \ldots, L_{i+k}$ (see Appendix A). When P is suitably constructed, then, depending on the value of λ, this upper bound tends to 0 as $i \to \infty$, and therefore the integers (1.3.1) are all zero for each sufficiently large i (see Proposition A.1).

Some of these polynomials are relatively easy to find, like the polynomial φ from Chapter 3. The polynomials or polynomial points w, f, g, S, k that appear in the subsequent chapters came from a computer search. Each new polynomial found in this way presents a challenge. First, one needs a compact way of writing it (some of them involve many monomials). Then, one has to understand how the polynomials are related with one another, and finally one has to understand what their vanishing means. When we can show that some polynomial P as above does not vanish at $(\mathbf{x}_i, \ldots, \mathbf{x}_{i+k})$ for infinitely many values of i, then we get an upper bound for λ. For example, the upper bound $\lambda \leq \sqrt{3} - 1 = 0.732...$ is obtained in Chapter 3 by showing that for $\lambda > 2/3$ we have $x_{i,0}^2 x_{i,2} - x_{i,1}^3 \neq 0$ for each i sufficiently large. The subsequent improvements are more challenging and we let the reader discover them as he goes through the thesis. We simply mention that the final estimate $\lambda \leq 5/7$ follows by observing first that if $\lambda > 5/7$ then f must vanish at certain pairs of points. From this, we deduce that k cannot vanish at related triples of points and this leads to a contradiction.

1.4 Notation

Notation 1.4.1 Throughout the text, for fixed real numbers ξ and η, we will use the notation $A \ll B$ or $B \gg A$, where A and B are non-negative variable real numbers, to mean $A \leq cB$ for some constant $c > 0$ depending only on ξ and η. We will also write $A \asymp B$ if $A \ll B \ll A$.

Chapter 2

Minimal points and sequences of minimal points

In their proof of Theorem 1.1.1, Davenport and Schmidt begin by defining a sequence of minimal points associated to a triple of the form $(1, \xi, \xi^2)$. In this chapter we work in greater generality by considering triples of the form $(1, \xi, \eta)$. The notion of minimal points that we attach to such a triple differs slightly from the definition of Davenport and Schmidt when $\eta = \xi^2$, but it is easy to adapt most of their arguments to our setting.

2.1 Minimal points

Definition 2.1.1 *Let $\xi, \eta \in \mathbb{R}$. For each point $\mathbf{x} = (x_0, x_1, x_2) \in \mathbb{R}^3$, we define the norm and proximity functions:*

$$\|\mathbf{x}\|_\infty := \max\{|x_0|, |x_1|, |x_2|\}, \quad L_{\xi,\eta}(\mathbf{x}) := \max\{|x_0\xi - x_1|, |x_0\eta - x_2|\}.$$

We usually write $\|\mathbf{x}\| := \|\mathbf{x}\|_\infty$ and $L(\mathbf{x}) := L_{\xi,\eta}(\mathbf{x})$.

Throughout this section, we fix $\xi, \eta \in \mathbb{R}\backslash\mathbb{Q}$. Then, for any non-zero point $\mathbf{x} \in \mathbb{Z}^3$, we have $L(\mathbf{x}) > 0$.

Definition 2.1.2 *For each real $X \geq 1$, consider the set of points $\mathbf{x} \in \mathbb{Z}^3$ with*

$$1 \leq \|\mathbf{x}\| \leq X, \quad L(\mathbf{x}) \leq 1. \tag{2.1.1}$$

A minimal point corresponding to X *is a point* $\mathbf{x}$ *of this set for which* $L(\mathbf{x})$ *is minimal.*

Note, that since the set defined by (2.1.1) is finite and non-empty (e.g. it contains (0,1,1)), for any real $X \geq 1$, there exists a minimal point corresponding to X.

Lemma 2.1.3 *If* $\mathbf{x} = (x_0, x_1, x_2)$ *is minimal point (for some real number* $X \geq 1$*), then* $\mathbf{x}$ *is* primitive, *i.e. we have* $\gcd(x_0, x_1, x_2) = 1$.

Proof: Indeed, if $a = \gcd(x_0, x_1, x_2) > 1$ then $\mathbf{x} = a\mathbf{x}'$ for some point $\mathbf{x}' \in \mathbb{Z}^3$. This implies that $\|\mathbf{x}'\| < \|\mathbf{x}\|$ and $L(\mathbf{x}') = \frac{1}{a}L(\mathbf{x}) < L(\mathbf{x})$, so $\mathbf{x}$ is not minimal. ∎

Lemma 2.1.4 *Let $\varepsilon > 0$. Then, for each X sufficiently large, a minimal point $\mathbf{x}$ corresponding to X satisfies $L(\mathbf{x}) \leq \varepsilon$.*

Proof: By Theorem 1.2.6, there exists a non-zero $\mathbf{y} \in \mathbb{Z}^3$ with $L(\mathbf{y}) \leq \min\{1, \varepsilon\}$. Choose $X \geq \|\mathbf{y}\|$ and let $\mathbf{x}$ be a minimal point corresponding to X, then since $\mathbf{y}$ is a solution of the inequalities (2.1.1), we have that $L(\mathbf{x}) \leq L(\mathbf{y}) \leq \varepsilon$. ∎

Lemma 2.1.5 *Let $\mathbf{x} = (x_0, x_1, x_2) \in \mathbb{Z}^3$ be a minimal point corresponding to some $X \geq 1$. If X is sufficiently large, then $\mathbf{x}$ has non-zero coordinates and satisfies*

$$\|\mathbf{x}\| \asymp |x_i|$$

for $i = 0, 1, 2$.

Proof: We first note that, for each $X \geq 1$, a minimal point $\mathbf{x} = (x_0, x_1, x_2)$ corresponding to X satisfies

$$|x_0\xi| - 1 \leq |x_1| \leq |x_0\xi| + 1,$$
$$|x_0\eta| - 1 \leq |x_2| \leq |x_0\eta| + 1.$$

If such a minimal point $\mathbf{x}$ satisfies $|x_0| < 2/\min\{1, |\xi|, |\eta|\}$, then, looking at the right-hand side of the above inequalities, we have $\|\mathbf{x}\| \ll 1$. Since there are a finite number of points which satisfy $\|\mathbf{x}\| \ll 1$, this means that $L(\mathbf{x}) \asymp 1$. But by Lemma 2.1.4, taking X sufficiently large we can always find a minimal point $\mathbf{x}'$ corresponding to X with $L(\mathbf{x}')$ as small as we want. So, for X sufficiently large, we have that $|x_0| \geq 2/\min\{1, |\xi|, |\eta|\}$. Then, from the above inequalities we find that the coordinates of $\mathbf{x}$ are non-zero with $|x_0| \asymp |x_1|$ and $|x_0| \asymp |x_2|$. ∎

2.2 Sequences of minimal points

From now on, we will suppose that $\xi, \eta \in \mathbb{R}$ are such that $1, \xi, \eta$ are linearly independent over $\mathbb{Q}$. We begin with a lemma which will be used to construct sequences of minimal points.

Lemma 2.2.1 *Let $\xi, \eta \in \mathbb{R}$ be such that $1, \xi, \eta$ are linearly independent over $\mathbb{Q}$. For any $X \geq 1 + \max\{|\xi|, |\eta|\}$, there are exactly two minimal points corresponding to X. If $\mathbf{x}$ is one of them, then the other is $-\mathbf{x}$.*

Proof: Fix some $X \geq 1 + \max\{|\xi|, |\eta|\}$, and a minimal point $\mathbf{x} = (x_0, x_1, x_2)$ corresponding to X.

We first show that $L(\mathbf{x}) < 1/2$ and deduce that $x_0 \neq 0$. Indeed, since $(1, [\xi], [\eta])$ is a solution of the inequalities (2.1.1) of Definition 2.1.2, where $[\xi]$ and $[\eta]$ denote the

integers nearest to ξ and η, respectively, we get $L(\mathbf{x}) \leq \max\{|\xi - [\xi]|, |\eta - [\eta]|\} < 1/2$. Now, suppose on the contrary that $x_0 = 0$. This gives $1 \leq \|\mathbf{x}\| = \max\{|x_1|, |x_2|\} = L(\mathbf{x})$, but since $L(\mathbf{x}) < 1/2$ this is impossible. So, we must have $x_0 \neq 0$.

Now suppose that $\mathbf{x}' = (x_0', x_1', x_2')$ is another minimal point for X. It satisfies $L(\mathbf{x}) = L(\mathbf{x}')$. Then we have 4 possible cases.

In the first case we have $L(\mathbf{x}) = |x_0\xi - x_1|$ and $L(\mathbf{x}') = |x_0'\eta - x_2'|$. This implies

$$x_0\xi - x_1 = \pm(x_0'\eta - x_2') \;\Rightarrow\; \mp x_0'\eta + x_0\xi - (x_1 \mp x_2') = 0 \;\Rightarrow\; x_0 = x_0' = 0,\; x_1 = \pm x_2'$$

which is impossible since $x_0 \neq 0$. The second case, when $L(\mathbf{x}) = |x_0\eta - x_2|$ and $L(\mathbf{x}') = |x_0'\xi - x_1'|$, leads similarly to a contradiction.

In the third case, $L(\mathbf{x}) = |x_0\xi - x_1|$ and $L(\mathbf{x}') = |x_0'\xi - x_1'|$, so we have

$$x_0\xi - x_1 = \pm(x_0'\xi - x_1') \;\Rightarrow\; (x_0 \mp x_0')\xi - (x_1 \mp x_1') = 0 \;\Rightarrow\; x_0 = \pm x_0',\; x_1 = \pm x_1'$$

and this implies

$$|-x_2 \pm x_2'| = |x_0\eta - x_2 \mp (x_0'\eta - x_2')| \leq |x_0\eta - x_2| + |x_0'\eta - x_2'| \leq 2L(\mathbf{x}) < 1 \;\Rightarrow\; x_2 = \pm x_2'.$$

Thus $\mathbf{x} = \pm\mathbf{x}'$. Finally, in the fourth case we have $L(\mathbf{x}) = |x_0\eta - x_2|$ and $L(\mathbf{x}') = |x_0'\eta - x_2'|$ and by a similar argument we again find that $\mathbf{x} = \pm\mathbf{x}'$. ∎

Note that if $\mathbf{x}$ is a minimal point (for some real number $X \geq 1$), then it is minimal for $\|\mathbf{x}\|$. So, on the basis of Lemma 2.2.1, we can construct recursively a sequence of minimal points $(\mathbf{x}_i)_{i\geq 1} \subseteq \mathbb{Z}^3$ which will satisfy the following definition.

Definition 2.2.2 *Let $\xi, \eta \in \mathbb{R}$ be such that $1, \xi, \eta$ are linearly independent over $\mathbb{Q}$. We define a* sequence of minimal points associated to $(1, \xi, \eta)$ *to be a sequence of minimal points $(\mathbf{x}_i)_{i\geq 1} \subseteq \mathbb{Z}^3$ of strictly increasing norms with the following property:*

if $\mathbf{x}$ is a minimal point satisfying $\|\mathbf{x}_i\| \leq \|\mathbf{x}\| < \|\mathbf{x}_{i+1}\|$ then $\mathbf{x} = \pm\mathbf{x}_i$.

Equivalently we could define such a sequence using the following property:

if $\mathbf{x}$ *is a minimal point satisfying* $L(\mathbf{x}_{i+1}) < L(\mathbf{x}) \leq L(\mathbf{x}_i)$, *then* $\mathbf{x} = \pm\mathbf{x}_i$.

Note that any two sequences of minimal points associated to $(1, \xi, \eta)$ differ only by the choice of their first element and a choice of sign factors. In particular, if $(\mathbf{x}_i)_{i\geq 1}$ is a sequence of minimal points, then $(\epsilon_i \mathbf{x}_i)_{i\geq r}$ is also a sequence of minimal points for any $r \geq 1$ and any choice of $\epsilon_i \in \{\pm 1\}$. Thus if a sequence of minimal points satisfies a certain property for each i sufficiently large, then by omitting its first elements we obtain a sequence of minimal points which obeys this property for each $i \geq 1$.

Notation 2.2.3 If $(\mathbf{x}_i)_{i\geq 1}$ is a sequence of minimal points associated to $(1, \xi, \eta)$, we usually write

$$\mathbf{x}_i = (x_{i,0}, x_{i,1}, x_{i,2}), \quad X_i := \|\mathbf{x}_i\| \quad \text{and} \quad L_i := L(\mathbf{x}_i).$$

We note that the X_i form a strictly increasing sequence of positive integers and that the L_i form a strictly decreasing sequence of positive real numbers.

Moreover, for any two points $\mathbf{x}_i, \mathbf{x}_j$ of the sequence with $i \neq j$, because they are primitive points with $\mathbf{x}_i \neq \mathbf{x}_j$ (see Lemma 2.1.4), we have that $\mathbf{x}_i$ and $\mathbf{x}_j$ are linearly independent.

Let $(\mathbf{x}_i)_{i\geq 1}$ be a sequence of minimal points associated to $(1, \xi, \eta)$. Note that if $x_{i,0} = 0$, then $(0, \pm 1, \pm 1)$ are four different points that are minimal points corresponding to $\|\mathbf{x}_i\|$, in contradiction with Definition 2.2.2. So, for all $i \geq 1$ we have $x_{i,0} \neq 0$. Because of this, we must have $L(\mathbf{x}_1) < 1$. Moreover, for an appropriate choice of $\epsilon_1, \epsilon_2 \in \{0, \pm 1\}$, the point $\mathbf{x} = (x_{1,0}, x_{1,1} + \epsilon_1, x_{1,2} + \epsilon_2)$ satisfies $L(\mathbf{x}) < 1/2$ and $\|\mathbf{x}\| \leq \|\mathbf{x}_1\| + 1 \leq \|\mathbf{x}_2\|$. So, for all $i \geq 2$ we have $L(\mathbf{x}_i) < 1/2$. Finally, we note that, for every i sufficiently large and appropriate choices of $\epsilon_i \in \{\pm 1\}$ the points $\epsilon_i \mathbf{x}_i$ remain in the same octant as $(1, \xi, \eta)$.

Lemma 2.2.4 *Let $\xi, \eta \in \mathbb{R}$ be such that $1, \xi, \eta$ are linearly independent over $\mathbb{Q}$. Let $(\mathbf{x}_i)_{i\geq 1}$ be a sequence of minimal points associated to $(1, \xi, \eta)$. Then for each choice of $j \geq i \geq 2$ we have $|x_{j,k}| \geq |x_{i,k}|$ for $k = 0, 1, 2$.*

Proof: For $i \geq 2$ we have $L(\mathbf{x}_j) \leq L(\mathbf{x}_i) < 1/2$. If $|x_{j,0}| \geq |x_{i,0}|$ then we have $|x_{j,1}| + 1/2 > |x_{j,0}\xi| \geq |x_{i,0}\xi| > |x_{i,1}| - 1/2$ which gives $|x_{j,1}| \geq |x_{i,1}|$ since $\mathbf{x}_i$ and $\mathbf{x}_j$ are integer points. Similarly we find $|x_{j,2}| \geq |x_{i,2}|$.

There are three possible cases: $\max\{1, |\xi|, |\eta|\} = 1$, $\max\{1, |\xi|, |\eta|\} = |\xi|$ or $\max\{1, |\xi|, |\eta|\} = |\eta|$.

In the first case we get $|x_{j,0}| > |x_{j,0}\xi| > |x_{j,1}| - 1/2$ which implies that $|x_{j,0}| \geq |x_{j,1}|$ and similarly we obtain $|x_{j,0}| \geq |x_{j,2}|$. Thus $\|\mathbf{x}_j\| = |x_{j,0}|$. Since $\|\mathbf{x}_j\| \geq \|\mathbf{x}_i\|$ we conclude that $|x_{j,0}| \geq |x_{i,0}|$ and we are done.

In the second case we get $|x_{j,1}| + 1/2 > |x_{j,0}\xi| > |x_{j,0}|$ and this gives $|x_{j,1}| \geq |x_{j,0}|$. We also have $|x_{j,1}| + 1/2 > |x_{j,0}\xi| > |x_{j,0}\eta| > |x_{j,2}| - 1/2$ and we get $|x_{j,1}| \geq |x_{j,2}|$. Thus $\|\mathbf{x}_j\| = |x_{j,1}|$. Since $\|\mathbf{x}_j\| \geq \|\mathbf{x}_i\|$ we now obtain $|x_{j,0}\xi| + 1/2 > |x_{j,1}| \geq |x_{i,1}| > |x_{i,0}\xi| - 1/2$ which implies $1 > 1/|\xi| > |x_{i,0}| - |x_{j,0}|$ and again we are done.

The third case leads similarly to $|x_{j,0}| \geq |x_{i,0}|$. ∎

2.3 Some key lemmas

The next series of Lemmas provide deeper results that we will use throughout this thesis. The first of these could be described as the main property of a sequence of minimal points. It lies at the basis of almost all further considerations.

Lemma 2.3.1 *Let $\xi, \eta \in \mathbb{R}$ be such that $1, \xi, \eta$ are linearly independent over $\mathbb{Q}$. Let $(\mathbf{x}_i)_{i\geq 1}$ be a sequence of minimal points associated to $(1, \xi, \eta)$. Suppose that $\lambda > 0$ is an exponent of approximation to $(1, \xi, \eta)$ and that $c > 0$ is a constant associated to*

this exponent. Then, for each i sufficiently large, we have

$$L_i \leq cc_0^{\lambda} X_{i+1}^{-\lambda}$$

where $c_0 = 2\max\{1, |\xi|, |\eta|\}$.

Proof: Fix some $i \geq 1$ and put $X = 2(X_{i+1} - 1)/c_0$. If i is sufficiently large, the hypotheses imply the existence of a non-zero solution $\mathbf{x} = (x_0, x_1, x_2) \in \mathbb{Z}^3$ of the inequalities of Definition 1.2.5. For such a point $\mathbf{x}$, we have

$$\begin{aligned} |x_0| &\leq X \leq X_{i+1} - 1 < X_{i+1}, \\ |x_1| &\leq |x_0\xi| + cX^{-\lambda} < |\xi|X + 1 \leq X_{i+1}, \\ |x_2| &\leq |x_0\eta| + cX^{-\lambda} < |\eta|X + 1 \leq X_{i+1}. \end{aligned}$$

Thus $\|\mathbf{x}\| < X_{i+1}$, and so we obtain

$$L_i = L(\mathbf{x}_i) \leq L(\mathbf{x}) \leq cX^{-\lambda} = cc_0^{\lambda}(2(X_{i+1} - 1))^{-\lambda} \leq cc_0^{\lambda} X_{i+1}^{-\lambda}$$

where the last estimate uses $X_{i-1} \geq X_2 \geq 2$ and so $2(X_{i+1} - 1) \geq X_{i+1}$. ∎

Lemma 2.3.2 *Let $\xi, \eta \in \mathbb{R}$ be such that $1, \xi, \eta$ are linearly independent over $\mathbb{Q}$. Let $(\mathbf{x}_i)_{i \geq 1}$ be a sequence of minimal points associated to $(1, \xi, \eta)$. Suppose that $\mathbf{x}_{i-1}, \mathbf{x}_i, \mathbf{x}_{i+1}$ are linearly independent. Then*

$$1 \ll X_{i+1} L_{i-1} L_i.$$

Proof: Since $\mathbf{x}_{i-1}, \mathbf{x}_i, \mathbf{x}_{i+1}$ are linearly independent, the determinant of these three points is a non-zero integer and so its absolute value is ≥ 1. We now find an upper bound for the determinant:

$$\left\| \begin{matrix} x_{i-1,0} & x_{i-1,1} & x_{i-1,2} \\ x_{i,0} & x_{i,1} & x_{i,2} \\ x_{i+1,0} & x_{i+1,1} & x_{i+1,2} \end{matrix} \right\| = \left\| \begin{matrix} x_{i-1,0} & x_{i-1,0}\xi - x_{i-1,1} & x_{i-1,0}\eta - x_{i-1,2} \\ x_{i,0} & x_{i,0}\xi - x_{i,1} & x_{i,0}\eta - x_{i,2} \\ x_{i+1,0} & x_{i+1,0}\xi - x_{i+1,1} & x_{i+1,0}\eta - x_{i+1,2} \end{matrix} \right\| \ll X_{i+1} L_{i-1} L_i.$$

∎

Lemma 2.3.3 *Let $\xi, \eta \in \mathbb{R}$ be such that $1, \xi, \eta$ are linearly independent over $\mathbb{Q}$. Let $(\mathbf{x}_i)_{i\geq 1}$ be a sequence of minimal points associated to $(1, \xi, \eta)$. Denote by $\mathbf{I}$ the set of all indices $i \geq 2$ for which $\mathbf{x}_{i-1}, \mathbf{x}_i, \mathbf{x}_{i+1}$ are linearly independent. Then $\mathbf{I}$ is an infinite set.*

Proof: For each $i \geq 2$, denote by W_i the subspace of $\mathbb{R}^3$ of dimension 2 generated by $\mathbf{x}_{i-1}$ and $\mathbf{x}_i$. Suppose $\mathbf{I}$ is finite, then it either has a maximal element i_0, or it is empty in which case we take $i_0 = 1$. Then, for all $i > i_0$, the points $\mathbf{x}_{i-1}, \mathbf{x}_i, \mathbf{x}_{i+1}$ are linearly dependent. This is equivalent to saying that for all $i > i_0$ we have $W_i = W_{i+1}$, which implies that there is a plane W such that $W = W_i$ for all $i > i_0$. Let $ax_0 + bx_1 + cx_2 = 0$ be an equation of W with $a, b, c \in \mathbb{Z}$ not all zero. This gives

$$ax_{i,0} + bx_{i,1} + cx_{i,2} = 0$$

for all $i > i_0$ and therefore

$$|x_{i,0}(a + b\xi + c\eta)| = |b(x_{i,0}\xi - x_{i,1}) + c(x_{i,0}\eta - x_{i,2})| \ll L(\mathbf{x}_i).$$

Since $a + b\xi + c\eta \neq 0$ by hypothesis and since $|x_{i,0}| \asymp \|\mathbf{x}_i\|$ by Lemma 2.1.5 we obtain

$$\|\mathbf{x}_i\| \asymp L(\mathbf{x}_i).$$

However, the left hand side tends to infinity and the right hand side tends to zero as $i \to 0$. This is a contradiction. ∎

Combining Lemmas 2.3.2 and 2.3.1, we obtain:

Lemma 2.3.4 *Let $\xi, \eta \in \mathbb{R}$ be such that $1, \xi, \eta$ are linearly independent over $\mathbb{Q}$. Let $(\mathbf{x}_i)_{i\geq 1}$ be a sequence of minimal points associated to $(1, \xi, \eta)$. Denote by $\mathbf{I}$ the set of*

all indices $i \geq 2$ for which $\mathbf{x}_{i-1}, \mathbf{x}_i, \mathbf{x}_{i+1}$ are linearly independent. Suppose that $\lambda > 0$ is an exponent of approximation to $(1, \xi, \eta)$. Then, for each $i \in \mathbf{I}$, we have

$$1 \ll X_i^{-\lambda} X_{i+1}^{1-\lambda}.$$

Theorem 2.3.5 *Let $\xi, \eta \in \mathbb{R}$ be such that $1, \xi, \eta$ are linearly independent over $\mathbb{Q}$. If λ is an exponent of approximation to $(1, \xi, \eta)$, then $\lambda < 1$.*

Proof: Suppose by contradiction that $\lambda \geq 1$. Let $(\mathbf{x}_i)_{i \geq 1}$ be a sequence of minimal points associated to $(1, \xi, \eta)$. Denote by $\mathbf{I}$ the set of all indices $i \geq 2$ for which $\mathbf{x}_{i-1}, \mathbf{x}_i, \mathbf{x}_{i+1}$ are linearly independent. By Lemma 2.3.3, the set $\mathbf{I}$ has infinitely many elements. This is impossible in view of Lemma 2.3.4. ∎

The next basic estimates will be useful throughout this thesis. They will allow us to simplify many computations.

Lemma 2.3.6 *Let $\xi, \eta \in \mathbb{R}$ be such that $1, \xi, \eta$ are linearly independent over $\mathbb{Q}$. Let $(\mathbf{x}_i)_{i \geq 1}$ be a sequence of minimal points associated to $(1, \xi, \eta)$. Denote by $\mathbf{I}$ the set of all indices $i \geq 2$ for which $\mathbf{x}_{i-1}, \mathbf{x}_i, \mathbf{x}_{i+1}$ are linearly independent. Suppose that $\lambda > 2/3$ is an exponent of approximation to $(1, \xi, \eta)$. Then for any $i \in \mathbf{I}$ we have*

$$X_i^2 \ll X_{i+1}, \quad L_i \ll L_{i-1}^2, \quad L_i \ll X_i^{-4/3}.$$

Proof: We note that $2(1-\lambda)/\lambda < 1$ since $\lambda > 2/3$. Then, by Lemma 2.3.4 we have

$$X_i^2 \ll X_{i+1}^{2(1-\lambda)/\lambda} \ll X_{i+1}.$$

We also find using Lemmas 2.3.1 and 2.3.2 that

$$L_i \ll L_i (X_{i+1} L_i L_{i-1})^2 \ll L_i^{1-2(1-\lambda)/\lambda} L_{i-1}^2 \leq L_{i-1}^2.$$

Finally, by Lemma 2.3.1

$$L_i \ll X_{i+1}^{-\lambda} \ll X_i^{-2\lambda} < X_i^{-4/3}.$$

∎

In particular, if $i < j$ are consecutive elements of $\mathbf{I}$, the first two estimates of Lemma 2.3.6 imply that $X_i^2 \ll X_j$ and $L_j \ll L_i^2$.

Chapter 3

The function φ and a first estimate

To prove Theorem 1.1.1, Davenport and Schmidt proceed by contradiction. They suppose that, for any $c > 0$, the inequalities of their theorem have a solution for each X sufficiently large. They then construct a sequence of minimal points $(\mathbf{x}_i)_{i\geq 1}$ associated to $(1, \xi, \xi^2)$ similar to our construction in Chapter 2. They first show that

$$\det(\mathbf{x}_i) = \begin{vmatrix} x_{i,0} & x_{i,1} \\ x_{i,1} & x_{i,2} \end{vmatrix} \neq 0$$

for all i sufficiently large. Using a result similar to Lemma 2.3.1, they conclude that

$$1 \leq |\det(\mathbf{x}_i)| \ll X_i L_i \ll c X_i X_{i+1}^{-1/\gamma}$$

where the lower bound comes from the fact that $\det(\mathbf{x}_i)$ is a non-zero integer. Then, they show as we do in Lemma 2.3.3 that, for infinitely many indices i, the points $\mathbf{x}_{i-1}, \mathbf{x}_i, \mathbf{x}_{i+1}$ are linearly independent. So, their determinant is a non-zero integer and arguing as we did in Lemma 2.3.4 they find

$$1 \ll |\det(\mathbf{x}_{i-1}, \mathbf{x}_i, \mathbf{x}_{i+1})| \ll X_{i+1} L_i L_{i-1} \ll c^2 X_{i+1}^{1-1/\gamma} X_i^{1/\gamma}.$$

Combining these results they derive a contradiction if c is sufficiently small.

Adapting the ideas of Davenport and Schmidt described above, we will show the following result:

Theorem 3.7 *Let $\xi \in \mathbb{R}$ be such that $1, \xi, \xi^3$ are linearly independent over $\mathbb{Q}$. Then $\lambda = \sqrt{3} - 1$ is a measure of approximation to $(1, \xi, \xi^3)$.*

We first introduce a polynomial which we will use throughout this paper, and which plays the role of the determinant in the work of Davenport and Schmidt:

Definition 3.8 *We set*

$$\varphi(\mathbf{x}, \mathbf{y}, \mathbf{z}) := \begin{vmatrix} x_0 y_0 & x_1 y_1 \\ z_1 & z_2 \end{vmatrix}, \qquad \varphi(\mathbf{x}) := \varphi(\mathbf{x}, \mathbf{x}, \mathbf{x})$$

where $\mathbf{x} = (x_0, x_1, x_2)$, $\mathbf{y} = (y_0, y_1, y_2)$, $\mathbf{z} = (z_0, z_1, z_2)$.

Lemma 3.9 *Let $\xi \in \mathbb{R}$ be such that $1, \xi, \xi^3$ are linearly independent over $\mathbb{Q}$. Let $(\mathbf{x}_i)_{i \geq 1}$ be a sequence of minimal points associated to $(1, \xi, \xi^3)$. Suppose that $\lambda > 2/3$ is an exponent of approximation to $(1, \xi, \xi^3)$. Then, for each sufficiently large i, we have*

$$\varphi(\mathbf{x}_i) = \begin{vmatrix} x_{i,0}^2 & x_{i,1}^2 \\ x_{i,1} & x_{i,2} \end{vmatrix} \neq 0.$$

Proof: If the result were false we would have $x_{i,1}^3 = x_{i,0}^2 x_{i,2}$. Since $x_{i,0} \neq 0$, this is equivalent to

$$\frac{x_{i,1}}{x_{i,0}} = \frac{n}{m} \quad \text{and} \quad \frac{x_{i,2}}{x_{i,0}} = \left(\frac{x_{i,1}}{x_{i,0}}\right)^3 = \frac{n^3}{m^3}$$

where m, n are integers with $m \neq 0$ and $\gcd(m, n) = 1$. It follows that

$$\left(1, \frac{x_{i,1}}{x_{i,0}}, \frac{x_{i,2}}{x_{i,0}}\right) = \left(1, \frac{n}{m}, \frac{n^3}{m^3}\right) \quad \Rightarrow \quad (x_{i,0}, x_{i,1}, x_{i,2}) \propto (m^3, m^2 n, n^3).$$

Since $x_{i,0}, x_{i,1}, x_{i,2}$ have no common factors, they then take the form

$$x_{i,0} = \epsilon m^3, \quad x_{i,1} = \epsilon m^2 n, \quad x_{i,2} = \epsilon n^3,$$

where $\epsilon \in \{\pm 1\}$. For i large enough, $|x_{i,j}| \asymp X_i$ for $j = 0, 1, 2$, and this gives

$$|m| \asymp |n| \asymp X_i^{1/3}.$$

Since $|\epsilon m^2(m\xi - n)| = |x_{i,0}\xi - x_{i,1}| \leq L_i$ we obtain

$$|m\xi - n| \ll X_i^{-2/3} L_i \tag{3.1}$$

and this implies

$$|m^2\xi^2 - n^2| = |m\xi - n||m\xi + n| \ll X_i^{-1/3} L_i. \tag{3.2}$$

The vectors $\mathbf{x}_{i-1}$, $\mathbf{x}_i$ are linearly independent so the matrix

$$\begin{pmatrix} x_{i-1,0} & x_{i-1,1} & x_{i-1,2} \\ x_{i,0} & x_{i,1} & x_{i,2} \end{pmatrix}$$

has rank 2 and therefore has a non-zero 2×2 minor. For i large enough, $x_{i-1,1}$ and $x_{i,1}$ are not zero, so we have

$$\begin{vmatrix} x_{i-1,0} & x_{i-1,1} \\ x_{i,0} & x_{i,1} \end{vmatrix} \neq 0 \quad \text{or} \quad \begin{vmatrix} x_{i-1,1} & x_{i-1,2} \\ x_{i,1} & x_{i,2} \end{vmatrix} \neq 0. \tag{3.3}$$

Suppose that the first inequality of (3.3) holds. We then have

$$\begin{vmatrix} x_{i-1,0} & x_{i-1,1} \\ m & n \end{vmatrix} \neq 0. \tag{3.4}$$

Using (3.1) and Lemma 2.3.1 we obtain

$$\left\| \begin{matrix} x_{i-1,0} & x_{i-1,0}\xi - x_{i-1,1} \\ m & m\xi - n \end{matrix} \right\| \ll X_{i-1} X_i^{-2/3} L_i + X_i^{1/3} L_{i-1} \ll X_i^{1/3} L_{i-1} \ll X_i^{1/3-\lambda}.$$

Since $\lambda > 1/3$, this contradicts (3.4) if i is sufficiently large.

Suppose that the second inequality of (3.3) holds. We then have

$$\begin{vmatrix} x_{i-1,1} & x_{i-1,2} \\ m^2 & n^2 \end{vmatrix} \neq 0. \tag{3.5}$$

Using (3.2) and Lemma 2.3.1 we obtain

$$\left\| \begin{matrix} x_{i-1,1} & x_{i-1,1}\xi^2 - x_{i-1,2} \\ m^2 & m^2\xi^2 - n^2 \end{matrix} \right\| \ll X_{i-1} X_i^{-1/3} L_i + X_i^{2/3} L_{i-1} \ll X_i^{2/3} L_{i-1} \ll X_i^{2/3-\lambda}.$$

Since $\lambda > 2/3$, this contradicts (3.5) if i is sufficiently large. ∎

Lemma 3.10 *Let $\xi \in \mathbb{R}$ be such that $1, \xi, \xi^3$ are linearly independent over $\mathbb{Q}$. Let $(\mathbf{x}_i)_{i\geq 1}$ be a sequence of minimal points associated to $(1, \xi, \xi^3)$. Suppose that $\lambda > 2/3$ is an exponent of approximation to $(1, \xi, \xi^3)$. Then, for each i, we have*

$$1 \ll X_i^2 L_i.$$

Proof: We find

$$|\varphi(\mathbf{x}_i)| = \left\| \begin{matrix} x_{i,0}^2 & x_{i,1}^2 \\ x_{i,1} & x_{i,2} \end{matrix} \right\| = \left\| \begin{matrix} x_{i,0}^2 & x_{i,0}^2\xi^2 - x_{i,1}^2 \\ x_{i,1} & x_{i,1}\xi^2 - x_{i,2} \end{matrix} \right\|$$
$$\ll X_i^2 L_i + X_i |x_{i,0}\xi - x_{i,1}||x_{i,0}\xi + x_{i,1}| \ll X_i^2 L_i.$$

But by Lemma 3.9, $\varphi(\mathbf{x}_i)$ is a non-zero integer. So its absolute value is ≥ 1. ∎

Proof of Theorem 3.7: Let $(\mathbf{x}_i)_{i\geq 1}$ be a sequence of minimal points associated to $(1, \xi, \eta)$. Denote by $\mathbf{I}$ the infinite set of all indices $i \geq 2$ for which $\mathbf{x}_{i-1}, \mathbf{x}_i, \mathbf{x}_{i+1}$ are linearly independent. Suppose on the contrary that λ is not a measure of approximation. Then for every $c > 0$, λ is an exponent of approximation.

Let $i \in \mathbf{I}$ with i arbitrarily large. We then find:

$$\begin{aligned} 1 &\ll (X_i^2 L_i)^\lambda && \text{(Lemma 3.10)} \\ &\ll (X_i^2 L_i)^\lambda (X_{i+1} L_{i-1} L_i)^2 && \text{(Lemma 2.3.2)} \\ &\ll c^{4+\lambda} X_{i+1}^{2-2\lambda-\lambda^2} && \text{(Lemma 2.3.1).} \end{aligned}$$

Since $\lambda = \sqrt{3} - 1$ is a root of $x^2 + 2x - 2$, we have $X_{i+1}^{2-2\lambda-\lambda^2} = 1$, and so the latter estimate becomes $1 \ll c^{4-\lambda}$. This is a contradiction. ∎

Chapter 4

A family of planes in $\mathbb{R}^3$

Define the height of a 2×3 matrix A, denoted $\|A\|$, to be the maximal absolute value of its minors of order 2, and define the height of a 2-dimensional subspace V of $\mathbb{Q}^3$, denoted $H(V)$, to be the height of any 2×3 matrix whose rows form a basis of $V \cap \mathbb{Z}^3$.

The next result generalizes Lemma 4.1 of [7]. The first part is essentially due to Davenport and Schmidt (see the proof of Lemma 2 of [3]).

Lemma 4.1 *Let $\xi, \eta \in \mathbb{R}$ be such that $1, \xi, \eta$ are linearly independent over $\mathbb{Q}$. Let $(\mathbf{x}_i)_{i \geq 1}$ be a sequence of minimal points associated to $(1, \xi, \eta)$. As in the proof of Lemma 2.3.3, we denote by W_i the subspace of $\mathbb{R}^3$ of dimension 2 generated by $\mathbf{x}_{i-1}$ and $\mathbf{x}_i$. For each $i \geq 2$, we have that $\{\mathbf{x}_{i-1}, \mathbf{x}_i\}$ is a basis of $W_i \cap \mathbb{Z}^3$ over $\mathbb{Z}$ and $H(W_i) \asymp \|\mathbf{x}_i\| L(\mathbf{x}_{i-1})$.*

Proof: Suppose on the contrary that $\{\mathbf{x}_{i-1}, \mathbf{x}_i\}$ is not a basis of $W_i \cap \mathbb{Z}^3$. Then there exists a non-zero point $\mathbf{x} = (x_0, x_1, x_2) \in W_i \cap \mathbb{Z}^3$ of the form $\mathbf{x} = r\mathbf{x}_{i-1} + s\mathbf{x}_i$ for rational numbers r and s with $\max\{|r|, |s|\} \leq 1/2$. We get

$$\|\mathbf{x}\| \leq |r|\|\mathbf{x}_{i-1}\| + |s|\|\mathbf{x}_i\| < \|\mathbf{x}_i\| \quad \text{and} \quad L(\mathbf{x}) \leq |r|L(\mathbf{x}_{i-1}) + |s|L(\mathbf{x}_i) < L(\mathbf{x}_{i-1})$$

which contradicts the fact that $\mathbf{x}_{i-1}$ and $\mathbf{x}_i$ are consecutive minimal points. Thus, $\{\mathbf{x}_{i-1}, \mathbf{x}_i\}$ is a basis of $W_i \cap \mathbb{Z}^3$.

We note that

$$\left\|\begin{matrix} x_{i-1,0} & x_{i-1,1} \\ x_{i,0} & x_{i,1} \end{matrix}\right\| = \left\|\begin{matrix} x_{i-1,0} & x_{i-1,0}\xi - x_{i-1,1} \\ x_{i,0} & x_{i,0}\xi - x_{i,1} \end{matrix}\right\| \ll \|\mathbf{x}_{i-1}\|L(\mathbf{x}_i) + \|\mathbf{x}_i\|L(\mathbf{x}_{i-1}),$$

$$\left\|\begin{matrix} x_{i-1,0} & x_{i-1,2} \\ x_{i,0} & x_{i,2} \end{matrix}\right\| = \left\|\begin{matrix} x_{i-1,0} & x_{i-1,0}\eta - x_{i-1,2} \\ x_{i,0} & x_{i,0}\eta - x_{i,2} \end{matrix}\right\| \ll \|\mathbf{x}_{i-1}\|L(\mathbf{x}_i) + \|\mathbf{x}_i\|L(\mathbf{x}_{i-1}),$$

$$\left\|\begin{matrix} x_{i-1,1} & x_{i-1,2} \\ x_{i,1} & x_{i,2} \end{matrix}\right\| = \left\|\begin{matrix} x_{i-1,1} & x_{i-1,1}\eta/\xi - x_{i-1,2} \\ x_{i,1} & x_{i,1}\eta/\xi - x_{i,2} \end{matrix}\right\| \ll \|\mathbf{x}_{i-1}\|L(\mathbf{x}_i) + \|\mathbf{x}_i\|L(\mathbf{x}_{i-1}).$$

Therefore,

$$H(W_i) = \left\|\begin{pmatrix} x_{i-1,0} & x_{i-1,1} & x_{i-1,2} \\ x_{i,0} & x_{i,1} & x_{i,2} \end{pmatrix}\right\| \ll \|\mathbf{x}_{i-1}\|L(\mathbf{x}_i) + \|\mathbf{x}_i\|L(\mathbf{x}_{i-1}) \ll \|\mathbf{x}_i\|L(\mathbf{x}_{i-1}).$$

Write $u_1 = x_{i-1,1} - x_{i-1,0}\xi$, $u_2 = x_{i-1,2} - x_{i-1,0}\eta$ and $v_1 = x_{i,1} - x_{i,0}\xi$, $v_2 = x_{i,2} - x_{i,0}\eta$. Choose an index j for which $|u_j| = L(\mathbf{x}_{i-1})$. This gives

$$\begin{aligned} H(W_i) &\geq \left\|\begin{matrix} x_{i-1,0} & x_{i-1,j} \\ x_{i,0} & x_{i,j} \end{matrix}\right\| = |x_{i-1,0}v_j - x_{i,0}u_j| \\ &\geq |x_{i,0}||u_j| - |x_{i-1,0}||v_j| \geq (|x_{i,0}| - |x_{i-1,0}|)L(\mathbf{x}_{i-1}). \end{aligned} \tag{4.1}$$

As we observed in Chapter 2, for i sufficiently large and for an appropriate choice of $\epsilon \in \{\pm 1\}$, the points $\mathbf{x}_{i-1}$ and $\epsilon\mathbf{x}_i$ lie in the same octant. Then the point $\mathbf{z} = \mathbf{x}_i - \epsilon\mathbf{x}_{i-1}$ satisfies $1 \leq \|\mathbf{z}\| < \|\mathbf{x}_i\|$. However, since $\mathbf{x}_{i-1}$ and $\mathbf{x}_i$ are consecutive minimal points, $\mathbf{x}_{i-1}$ is a minimal point corresponding to $\max\{\|\mathbf{x}_{i-1}\|, \|\mathbf{z}\|\}$. This means that $L(\mathbf{z}) \geq L(\mathbf{x}_{i-1})$. Choose an index k for which $|v_k - \epsilon u_k| = L(\mathbf{z}) \geq L(\mathbf{x}_{i-1})$. Then, we also find that

$$\begin{aligned} H(W_i) &\geq |x_{i-1,0}v_k - x_{i,0}u_k| = |x_{i-1,0}(v_k - \epsilon u_k) - (x_{i,0} - \epsilon x_{i-1,0})u_k| \\ &\geq |x_{i-1,0}|L(\mathbf{x}_{i-1}) - |x_{i,0} - \epsilon x_{i-1,0}|L(\mathbf{x}_{i-1}). \end{aligned}$$

Since from Lemma 2.2.4 we have $|x_{i,0}| \geq |x_{i-1,0}|$, it follows that $|x_{i,0} - \epsilon x_{i-1,0}| = |x_{i,0}| - |x_{i-1,0}|$ and we get

$$H(W_i) \geq |x_{i-1,0}|L(\mathbf{x}_{i-1}) - (|x_{i,0}| - |x_{i-1,0}|)L(\mathbf{x}_{i-1}). \tag{4.2}$$

Combining the inequalities (4.1) and (4.2) we get

$$3H(W_i) \geq |x_{i-1,0}|L(\mathbf{x}_{i-1}) + (|x_{i,0}| - |x_{i-1,0}|)L(\mathbf{x}_{i-1}) = |x_{i,0}|L(\mathbf{x}_{i-1}).$$

Therefore we have

$$\|\mathbf{x}_i\|L(\mathbf{x}_{i-1}) \gg 3H(W_i) \geq |x_{i,0}|L(\mathbf{x}_{i-1}) \asymp \|\mathbf{x}_i\|L(\mathbf{x}_{i-1})$$

which means that $H(W_i) \asymp \|\mathbf{x}_i\|L(\mathbf{x}_{i-1})$. ∎

From this lemma, we derive the following useful general estimates.

Corollary 4.2 *Let $\xi, \eta \in \mathbb{R}$ be such that $1, \xi, \eta$ are linearly independent over $\mathbb{Q}$. Let $(\mathbf{x}_i)_{i\geq 1}$ be a sequence of minimal points associated to $(1, \xi, \eta)$. Denote by $\mathbf{I}$ the set of all indices $i \geq 2$ for which $\mathbf{x}_{i-1}, \mathbf{x}_i, \mathbf{x}_{i+1}$ are linearly independent. Then for any pair $i < j$ consecutive elements of $\mathbf{I}$ we have*

$$X_{i+1}L_i \asymp X_j L_{j-1}.$$

Proof: Since $i < j$ are consecutive elements of $\mathbf{I}$, we have that $W_{i+1} = W_j$ and from the lemma we obtain

$$X_{i+1}L_i \asymp H(W_{i+1}) = H(W_j) \asymp X_j L_{j-1}.$$

∎

Corollary 4.3 *Let $\xi, \eta \in \mathbb{R}$ be such that $1, \xi, \eta$ are linearly independent over $\mathbb{Q}$. Let $(\mathbf{x}_i)_{i\geq 1}$ be a sequence of minimal points associated to $(1, \xi, \eta)$. Denote by $\mathbf{I}$ the set of all indices $i \geq 2$ for which $\mathbf{x}_{i-1}, \mathbf{x}_i, \mathbf{x}_{i+1}$ are linearly independent. Suppose that $\lambda > 0$ is an exponent of approximation to $(1, \xi, \eta)$. Then for any pair of consecutive elements $i < j$ of $\mathbf{I}$ we have*

$$X_j \ll (X_{i+1}X_{j+1})^{1-\lambda} \quad \textit{and} \quad L_i L_j \ll X_j^{-\lambda/(1-\lambda)}.$$

Proof: We first note that:

$$\begin{aligned} 1 &\ll X_{j+1} L_{j-1} L_j && \text{(Lemma 2.3.2)} \\ &\asymp X_{j+1}(X_{i+1} L_i X_j^{-1}) L_j && \text{(Corollary 4.2)} \\ &\ll X_{i+1}^{1-\lambda} X_j^{-1} X_{j+1}^{1-\lambda} && \text{(Lemma 2.3.1).} \end{aligned}$$

This proves the first estimate of the corollary. The second follows from Lemma 2.3.1 and the first estimate:

$$L_i L_j \ll (X_{i+1} X_{j+1})^{-\lambda} \ll X_j^{-\lambda/(1-\lambda)}.$$

∎

Chapter 5

The function Φ and the point ψ

We now introduce further tools which will be used throughout this thesis. In this chapter, we use them to improve on Theorem 3.7 by showing that, if $\xi \in \mathbb{R}$ is such that $1, \xi, \xi^3$ are linearly independent over $\mathbb{Q}$, then any exponent of approximation to $(1, \xi, \xi^3)$ is at most equal to the root of $7x^3 - 2x^2 - 5x + 2$ near 0.7277416.

Definition 5.1 *We set*

$$\Phi(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \varphi(\mathbf{x}, \mathbf{y}, \mathbf{z}) + \varphi(\mathbf{y}, \mathbf{z}, \mathbf{x}) + \varphi(\mathbf{z}, \mathbf{x}, \mathbf{y}),$$
$$\psi(\mathbf{x}, \mathbf{y}) = \Phi(\mathbf{x}, \mathbf{x}, \mathbf{y})\mathbf{x} - 2\varphi(\mathbf{x})\mathbf{y}.$$

The function ψ constructs new points of $\mathbb{R}^3$ out of each pair of points of $\mathbb{R}^3$. The following proposition provides upper bounds for the norm and the proximity function $L(\cdot)$ of these points.

Proposition 5.2 *For any* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$ *we have:*

$$\|\psi(\mathbf{x}, \mathbf{y})\| \ll \|\mathbf{x}\|^3 L(\mathbf{y}) + \|\mathbf{x}\|\|\mathbf{y}\| L(\mathbf{x})^2.$$
$$L(\psi(\mathbf{x}, \mathbf{y})) \ll \|\mathbf{x}\|^2 L(\mathbf{x}) L(\mathbf{y}) + \|\mathbf{x}\|\|\mathbf{y}\| L(\mathbf{x})^2.$$

Proof: Let $\mathbf{x} = (x_0, x_1, x_2)$ and $\mathbf{y} = (y_0, y_1, y_2)$ be points of $\mathbb{R}^3$. Write $\underline{\underline{\xi}} = (1, \xi, \xi^3)$ and define

$$\Delta\mathbf{x} = \mathbf{x} - x_0\underline{\underline{\xi}} = (0, \Delta x_1, \Delta x_2) \quad \text{and} \quad \delta(\mathbf{x}) := \Phi(\mathbf{x}, \underline{\underline{\xi}}, \underline{\underline{\xi}}) = \Delta x_2 - 3\xi^2\Delta x_1.$$

We find the following developments:

$$\begin{aligned}
\varphi(\mathbf{x}) &= x_0^2\delta(\mathbf{x}) - 3\xi\Delta x_1^2 x_0 - \Delta x_1^3, \\
\Phi(\mathbf{x}, \mathbf{x}, \mathbf{y}) &= x_0^2\delta(\mathbf{y}) + 2x_0y_0\delta(\mathbf{x}) - 3\xi y_0\Delta x_1^2 - 6\xi x_0\Delta x_1\Delta y_1 - 3\Delta x_1^2\Delta y_1.
\end{aligned}$$

Using these we get the following estimate for ψ:

$$\begin{aligned}
\psi(\mathbf{x}, \mathbf{y}) &= \Phi(\mathbf{x}, \mathbf{x}, \mathbf{y})(x_0\underline{\underline{\xi}} + \Delta\mathbf{x}) - 2\varphi(\mathbf{x})(y_0\underline{\underline{\xi}} + \Delta\mathbf{y}) \\
&= (x_0^2\delta(\mathbf{y}) + 2x_0y_0\delta(\mathbf{x}) - 3\xi y_0\Delta x_1^2 - 6\xi x_0\Delta x_1\Delta y_1 - 3\Delta x_1^2\Delta y_1)(x_0\underline{\underline{\xi}} + \Delta\mathbf{x}) \\
&\quad - 2(x_0^2\delta(\mathbf{x}) - 3\xi\Delta x_1^2 x_0 - \Delta x_1^3)(y_0\underline{\underline{\xi}} + \Delta\mathbf{y}) \\
&= (x_0^3\delta(\mathbf{y}) - 6\xi x_0^2\Delta x_1\Delta y_1 + 3\xi x_0y_0\Delta x_1^2 - 3x_0\Delta x_1^2\Delta y_1 + 2y_0\Delta x_1^3)\underline{\underline{\xi}} \\
&\quad + \mathcal{O}(\|\mathbf{x}\|^2 L(\mathbf{x})L(\mathbf{y}) + \|\mathbf{x}\|\|\mathbf{y}\|L(\mathbf{x})^2).
\end{aligned}$$

The conclusion follows upon noting that $L(\underline{\underline{\xi}}) = 0$. ∎

In our context, these estimates lead to:

Lemma 5.3 *Let $\xi \in \mathbb{R}$ be such that $1, \xi, \xi^3$ are linearly independent over $\mathbb{Q}$. Let $(\mathbf{x}_i)_{i\geq 1}$ be a sequence of minimal points associated to $(1, \xi, \xi^3)$. Denote by $\mathbf{I}$ the set of all indices $i \geq 2$ for which $\mathbf{x}_{i-1}, \mathbf{x}_i, \mathbf{x}_{i+1}$ are linearly independent. Suppose that $\lambda > 2/3$ is an exponent of approximation to $(1, \xi, \xi^3)$. Then for any pair of consecutive elements $i < j$ in $\mathbf{I}$ we have*

$$\|\psi(\mathbf{x}_i, \mathbf{x}_j)\| \ll X_iX_jL_i^2.$$

Proof: By Proposition 5.2, we have $\|\psi(\mathbf{x}_i, \mathbf{x}_j)\| \ll X_i^3 L_j + X_i X_j L_i^2$. So it suffices to show that $X_i^3 L_j \ll X_i X_j L_i^2$. Since i and j are in $\mathbf{I}$, Lemma 2.3.6 gives indeed

$$X_i^3 L_j \ll X_i X_{i+1} L_j \ll X_i X_{i+1} L_{j-1}^2 \leq X_i X_j L_i^2.$$

∎

We now observe that the function ψ has the following non-vanishing property.

Lemma 5.4 *Let $\xi \in \mathbb{R}$ be such that $1, \xi, \xi^3$ are linearly independent over $\mathbb{Q}$. Let $(\mathbf{x}_i)_{i \geq 1}$ be a sequence of minimal points associated to $(1, \xi, \xi^3)$. Suppose that $\lambda > 2/3$ is an exponent of approximation to $(1, \xi, \xi^3)$. Then for any $i, j \geq 1$ with i sufficiently large we have*

$$\psi(\mathbf{x}_i, \mathbf{x}_j) \neq 0.$$

Proof: By Lemma 3.9, we have $\varphi(\mathbf{x}_i) \neq 0$ for each i sufficiently large. If $i = j$ we find

$$\psi(\mathbf{x}_i, \mathbf{x}_i) = \varphi(\mathbf{x}_i)\mathbf{x}_i \neq 0.$$

Otherwise $i \neq j$, so $\mathbf{x}_i, \mathbf{x}_j$ are linearly independent and we obtain

$$\psi(\mathbf{x}_i, \mathbf{x}_j) = \Phi(\mathbf{x}_i, \mathbf{x}_i, \mathbf{x}_j)\mathbf{x}_i - 2\varphi(\mathbf{x}_i)\mathbf{x}_j \neq 0.$$

∎

Combining Lemmas 5.3 and 5.4 together with the fact that any non-zero integer point has norm at least 1, we obtain immediately:

Corollary 5.5 *Let $\xi \in \mathbb{R}$ be such that $1, \xi, \xi^3$ are linearly independent over $\mathbb{Q}$. Let $(\mathbf{x}_i)_{i \geq 1}$ be a sequence of minimal points associated to $(1, \xi, \xi^3)$. Denote by $\mathbf{I}$ the set of all indices $i \geq 2$ for which $\mathbf{x}_{i-1}, \mathbf{x}_i, \mathbf{x}_{i+1}$ are linearly independent. Suppose that*

$\lambda > 2/3$ is an exponent of approximation to $(1, \xi, \xi^3)$. Then for any pair of consecutive elements $i < j$ in $\mathbf{I}$ we have

$$1 \ll X_i X_j L_i^2.$$

We are now ready to prove the main result of this chapter:

Theorem 5.6 *Let $\xi \in \mathbb{R}$ be such that $1, \xi, \xi^3$ are linearly independent over $\mathbb{Q}$. If λ is an exponent of approximation to $(1, \xi, \xi^3)$, then $\lambda \leq \lambda_0$ where λ_0 is the largest real root of $7x^3 - 2x^2 - 5x + 2$, about 0.7277416.*

Proof: Suppose by contradiction that $\lambda > \lambda_0$. Let $(\mathbf{x}_i)_{i \geq 1}$ be a sequence of minimal points associated to $(1, \xi, \xi^3)$. Denote by $\mathbf{I}$ the set of all indices $i \geq 2$ for which $\mathbf{x}_{i-1}, \mathbf{x}_i, \mathbf{x}_{i+1}$ are linearly independent. Let $i < j$ be a pair of consecutive elements of $\mathbf{I}$ with i arbitrarily large.

We first note that

$$\begin{aligned} X_j^\lambda &\ll (X_{i+1} X_{j+1})^{\lambda(1-\lambda)} && \text{(Corollary 4.3)} \\ &\ll X_{i+1}^{\lambda(1-\lambda)} X_j^{2(1-\lambda)} && \text{(Lemma 3.10)} \end{aligned}$$

which gives us that

$$X_j^{3\lambda-2} \ll X_{i-1}^{\lambda(1-\lambda)}. \tag{5.1}$$

Combining this result with Corollary 5.5 we obtain

$$\begin{aligned} 1 &\ll (X_i X_j L_i^2)^{\lambda(3\lambda-2)} && \text{(Corollary 5.5)} \\ &\ll (X_{i+1}^{1-\lambda} X_j^\lambda L_i^{2\lambda})^{3\lambda-2} && \text{(Lemma 2.3.4)} \\ &\ll (X_{i+1}^{1-\lambda-2\lambda^2} X_j^\lambda)^{3\lambda-2} && \text{(Lemma 2.3.1)} \\ &\ll X_{i+1}^{(1-\lambda-2\lambda^2)(3\lambda-2)+\lambda^2(1-\lambda)} && \text{(equation (5.1))} \\ &= X_{i+1}^{-(7\lambda^3-2\lambda^2-5\lambda+2)}. \end{aligned}$$

However, since $\lambda > \lambda_0$ the exponent of X_{i+1} in the latter expression is negative and so that power of X_{i+1} tends to 0 as $i \to \infty$. This is a contradiction. ∎

Since $\lambda_0 < \sqrt{3}-1 \approx 0.732$, the theorem has the following immediate consequence.

Corollary 5.7 *Let $\xi \in \mathbb{R}$ be such that $1, \xi, \xi^3$ are linearly independent over $\mathbb{Q}$. Then $\lambda = \sqrt{3} - 1$ is not an exponent of approximation to $(1, \xi, \xi^3)$.*

Chapter 6

Intermediate points and large systems of inequalities

In this chapter we show that, if $\xi \in \mathbb{R}$ is such that $1, \xi, \xi^3$ are linearly independent over $\mathbb{Q}$, then any an exponent of approximation to $(1, \xi, \xi^3)$ is at most $8/11$. This improves on Theorem 5.6 since $8/11 = 0.\overline{72} < 0.7277$.

We begin by looking at a special case:

Proposition 6.1 *Let $\xi \in \mathbb{R}$ be such that $1, \xi, \xi^3$ are linearly independent over $\mathbb{Q}$. Let $(\mathbf{x}_i)_{i \geq 1}$ be a sequence of minimal points associated to $(1, \xi, \xi^3)$. Denote by $\mathbf{I}$ the set of all indices $i \geq 2$ for which $\mathbf{x}_{i-1}, \mathbf{x}_i, \mathbf{x}_{i+1}$ are linearly independent. Suppose that for infinitely many $i \in \mathbf{I}$, we have $i + 1 \in \mathbf{I}$. Then $\lambda = \sqrt{2}/2$ is a measure of approximation to $(1, \xi, \xi^3)$.*

Proof: Suppose on the contrary that λ is not a measure of approximation to $(1, \xi, \xi^3)$. Then for every $c > 0$, λ is an exponent of approximation to $(1, \xi, \xi^3)$.

Let $i \in \mathbf{I}$ be such that $i + 1 \in \mathbf{I}$. Lemma 2.3.4 shows that $1 \ll X_i^{-\lambda} X_{i+1}^{1-\lambda}$ and

Corollary 5.5 gives $1 \ll X_i X_{i+1} L_i^2$. For such an i we then have

$$\begin{aligned} 1 &\ll (X_i X_{i+1} L_i^2)^{(1-\lambda)} \ll c^{2(1-\lambda)} X_i^{(1-\lambda)} X_{i+1}^{(1-2\lambda)(1-\lambda)} \\ &\ll c^{2(1-\lambda)} X_i^{(1-\lambda)-\lambda(2\lambda-1)} = c^{2(1-\lambda)} X_i^{1-2\lambda^2}. \end{aligned}$$

Since $\lambda = \sqrt{2}/2$ is a root of $2x^2 - 1$ we have $X_i^{1-2\lambda^2} = 1$ and so for a choice of $c > 0$ sufficiently small this is a contradiction. ∎

From this result it is clear that, for any $\xi \in \mathbb{R}$ such that $1, \xi, \xi^3$ are linearly independent over $\mathbb{Q}$, if $\lambda > \sqrt{2}/2$ is an exponent of approximation to $(1, \xi, \xi^3)$ then for each $i \in \mathbf{I}$ sufficiently large we have $i + 1 \notin \mathbf{I}$.

At this point, we pause and make the following general observation about systems of linear inequalities:

Proposition 6.2 *Let $m, n \in \mathbb{Z}_{\geq 0}$ and $a_{i,j}, b_i \in \mathbb{R}$ for $i = 0, \ldots, m$ and $j = 0, \ldots, n$. Suppose the system of linear inequalities*

$$\sum_{j=0}^{n} a_{i,j} x_j + b_i \leq \varepsilon_i, \quad i = 0, \ldots, m$$

has a solution $(x_0, \ldots, x_n) \in \mathbb{R}^{n+1}$ for any choice of $\varepsilon_0, \ldots, \varepsilon_m > 0$. Then it has a solution for $\varepsilon_0 = \cdots = \varepsilon_m = 0$.

Proof: We note that for $0 \leq k \leq m$, the system of inequalities

$$\begin{cases} \displaystyle\sum_{j=0}^{n} -a_{i,j} x_j - b_i + \varepsilon_i \geq 0, & i = 0, \ldots, k-1 \\ \displaystyle\sum_{j=0}^{n} -a_{i,j} x_j - b_i \geq 0, & i = k+1, \ldots, m \end{cases} \tag{6.1}$$

together with the linear functional

$$\theta = \sum_{j=0}^{n} a_{k,j} x_j$$

is a linear program. According to Theorem 1 of §8.1 of [6], either the set of admissible solutions (i.e. solutions to the system of inequalities) is empty, or the set of admissible solutions has at least one minimal solution (i.e. a solution for which θ takes its minimum value), or there are admissible solutions for which θ is arbitrarily small (i.e. can be made smaller than any given real number, positive or negative).

We now show by reverse induction on k that the system

$$\begin{cases} \sum_{j=0}^{n} a_{i,j}x_j + b_i \leq \varepsilon_i, & i = 0, \ldots, k \\ \sum_{j=0}^{n} a_{i,j}x_j + b_i \leq 0, & i = k+1, \ldots, m \end{cases} \tag{6.2}$$

has a solution for every choice of $\varepsilon_0, \ldots, \varepsilon_k > 0$.

For $k = m$, this is true by hypothesis. Suppose now that $0 \leq k \leq m$ and that the system (6.2) has a solution for any choice of $\varepsilon_0, \ldots, \varepsilon_k > 0$. Fix $\varepsilon_0, \ldots, \varepsilon_{k-1} > 0$. Then, the set of admissible solutions to (6.1) is non-empty. If there are admissible solutions for which θ is arbitrarily small then there is a solution for which $\theta + b_k \leq 0$. We are left with the case where the set of admissible solutions has at least one minimal solution. Let θ_0 be the value of θ for a minimal solution. Since we have solutions such that $\theta + b_k \leq \epsilon_k$ for every $\epsilon_k > 0$, then we must have $\theta_0 + b_k \leq 0$ and so a minimal solution satisfies $\theta + b_k \leq 0$. ∎

Using this lemma we can deduce the following proposition.

Proposition 6.3 *Fix $n \in \mathbb{Z}_{\geq 0}$. Let $\xi \in \mathbb{R}$ be such that $1, \xi, \xi^3$ are linearly independent over $\mathbb{Q}$. Suppose that $\lambda > 2/3$ is an exponent of approximation to $(1, \xi, \xi^3)$.*

Then the following system of $4n+4$ *inequalities.*

$$B_0 + \lambda A_0 \leq 0 \tag{6.3}$$

$$0 \leq 2A_{2n} + B_{2n+1} \tag{6.4}$$

$$0 \leq A_{2n+1} + B_{2n} + B_{2n+1} \tag{6.5}$$

$$B_{2n+1} + \lambda A_{2n+1} \leq 0 \tag{6.6}$$

$$k = 0, \ldots, n-1 \begin{cases} A_{2k+1} + B_{2k+1} = A_{2k+2} + B_{2k+2} & (6.7) \\ 0 \leq A_{2k} + A_{2k+2} + 2B_{2k+1} & (6.8) \\ 0 \leq A_{2k+1} + B_{2k} + B_{2k+1} & (6.9) \\ B_{2k+1} + \lambda A_{2k+1} \leq 0 & (6.10) \end{cases}$$

has a solution $(A_0, \ldots, A_{2n+1}, B_0, \ldots, B_{2n+1})$ *in* $\mathbb{R}^{4n+4}$ *with* $A_0 = 1$.

Proof: Let $(\mathbf{x}_i)_{i \geq 1}$ be a sequence of minimal points associated to $(1, \xi, \xi^3)$. Denote by $\mathbf{I}$ the set of all indices $i \geq 2$ for which $\mathbf{x}_{i-1}, \mathbf{x}_i, \mathbf{x}_{i+1}$ are linearly independent. Let $i_0 < \cdots < i_n$ be $n+1$ consecutive elements of $\mathbf{I}$. Define real numbers $A_{i_0,0}, A_{i_0,1}, \ldots, A_{i_0,2n+1}$ and $B_{i_0,0}, B_{i_0,1}, \ldots, B_{i_0,2n+1}$ by the conditions

$$X_{i_0}^{A_{i_0,2k}} = X_{i_k},$$
$$X_{i_0}^{A_{i_0,2k+1}} = X_{i_k+1},$$
$$X_{i_0}^{B_{i_0,2k}} = L_{i_k-1},$$
$$X_{i_0}^{B_{i_0,2k+1}} = L_{i_k}$$

for $k = 0, \ldots, n$.

Since λ is an exponent of approximation to $(1, \xi, \xi^3)$, we have

$$L_{i_0-1} \ll X_{i_0}^{-\lambda} \qquad \text{by Lemma 2.3.1,}$$

$$1 \ll X_{i_n}^2 L_{i_n} \qquad \text{by Lemma 3.10,}$$

$$1 \ll X_{i_n+1} L_{i_n-1} L_{i_n} \qquad \text{by Lemma 2.3.2,}$$

$$L_{i_n} \ll X_{i_n+1}^{-\lambda} \qquad \text{by Lemma 2.3.1,}$$

$$k = 0, \dots, n-1 \begin{cases} X_{\iota_k+1} L_{\iota_k} \asymp X_{\iota_{k+1}} L_{\iota_{k+1}-1} & \text{by Corollary 4.2,} \\ 1 \ll X_{\iota_k} X_{\iota_k+1} L_{\iota_k}^2 & \text{by Corollary 5.5,} \\ 1 \ll X_{\iota_k+1} L_{\iota_k-1} L_{\iota_k} & \text{by Lemma 2.3.2,} \\ L_{\iota_k} \ll X_{\iota_k+1}^{-\lambda} & \text{by Lemma 2.3.1.} \end{cases}$$

We also note that the condition $X_{\iota_k+1} L_{\iota_k} \asymp X_{\iota_{k+1}} L_{\iota_{k+1}-1}$ is equivalent to

$$X_{\iota_k+1} L_{\iota_k} \ll X_{\iota_{k+1}} L_{\iota_{k+1}-1} \quad \text{and} \quad X_{\iota_k-1} L_{\iota_k} \gg X_{\iota_{k+1}} L_{\iota_{k+1}-1}.$$

Let $\varepsilon > 0$. Assuming that ι_0 is sufficiently large, we obtain the following inequalities by passing to the logarithm in base X_{ι_0}:

$$A_{\iota_0,0} = 1$$

$$B_{\iota_0,0} + \lambda A_{\iota_0,0} \le \varepsilon$$

$$-2A_{\iota_0,2n} - B_{\iota_0,2n+1} \le \varepsilon$$

$$-A_{\iota_0,2n+1} - B_{\iota_0,2n} - B_{\iota_0,2n+1} \le \varepsilon$$

$$B_{\iota_0,2n+1} + \lambda A_{\iota_0,2n+1} \le \varepsilon$$

$$\left.\begin{aligned} A_{\iota_0,2k+1} + B_{\iota_0,2k+1} - A_{\iota_0,2k+2} - B_{\iota_0,2k+2} &\le \varepsilon \\ -A_{\iota_0,2k+1} - B_{\iota_0,2k+1} + A_{\iota_0,2k+2} + B_{\iota_0,2k+2} &\le \varepsilon \\ -A_{\iota_0,2k} - A_{\iota_0,2k+2} - 2B_{\iota_0,2k+1} &\le \varepsilon \\ -A_{\iota_0,2k+1} - B_{\iota_0,2k} - B_{\iota_0,2k+1} &\le \varepsilon \\ B_{\iota_0,2k+1} + \lambda A_{\iota_0,2k+1} &\le \varepsilon \end{aligned}\right\} k = 0, \dots, n-1$$

For example, to get the third inequality, we use $1 \ll X_{\iota_n}^2 L_{\iota_n}$. It means that

$$1 \le c X_{\iota_n}^2 L_{\iota_n} = c X_{\iota_0}^{2A_{\iota_0,2n} + B_{\iota_0,2n+1}}$$

for some constant c depending only on ξ. Passing to the logarithm in base X_{ι_0} we get

$$0 \le \log c / \log X_{\iota_0} + 2A_{\iota_0,2n} + B_{\iota_0,2n+1}.$$

For i_0 sufficiently large, this gives

$$0 \leq \varepsilon + 2A_{i_0,2n} + B_{i_0,2n+1}.$$

Since for every $\varepsilon > 0$ this system of inequalities has a solution, Proposition 6.2 shows that it solution for $\varepsilon = 0$. ∎

The question now becomes to determine for which values of λ, the system of inequalities of Proposition 6.3 has a solution for each integer $n \geq 0$. To analyse this, we first make the following observation.

Lemma 6.4 *Let $n \in \mathbb{Z}_{\geq 1}$ and $c > 2$. Suppose that $a_0, a_1, \ldots, a_n$ are real numbers satisfying the inequalities*

$$a_{k+1} \geq ca_k - a_{k-1}, \quad k = 1, \ldots, n-1.$$

Then, for any choice of $k, l, m \in \mathbb{Z}_{\geq 0}$ with $k \leq n$, $m \leq k$ and $l + k \leq n$, we have

$$c_m a_{k+l} \geq c_k a_{l+m} - c_{k-m} a_l$$

where the the sequence $(c_i)_{i \geq 0}$ is the solution of the recurrence relation $c_{i+1} = cc_i - c_{i-1}$ with $c_1 = 1$ and $c_0 = 0$.

Proof: Set $\alpha = (c + \sqrt{c^2 - 4})/2$. The recurrence relation $c_{k+1} = cc_k - c_{k-1}$ has $x^2 - cx + 1$ as its companion polynomial with roots α and α^{-1} which are real and distinct since $c > 2$. The general solution of the recurrence relation is then $c_k = C\alpha^k + D\alpha^{-k}$. Solving for $c_0 = 0$ and $c_1 = 1$ we get

$$c_k = \frac{\alpha^k - \alpha^{-k}}{\alpha - \alpha^{-1}}.$$

We note that since $c > 2$ we have that $\alpha > 1$ and so $c_k \geq 0$ for all $k \geq 0$.

We begin by showing that for $k, l, m \in \mathbb{Z}_{\geq 0}$ with $m \leq k$ and $l + m \leq n - 1$ we have

$$c_k a_{l+m+1} - c_{k+1} a_{l+m} \geq c_{k-m} a_{l+1} - c_{k-m+1} a_l.$$

We prove this result by induction on k. The case $k = 0$ is clear. Suppose that for some $k \in \mathbb{Z}_{\geq 0}$ the result holds for each choice of $l, m \in \mathbb{Z}_{\geq 0}$ with $m \leq k$ and $l + m \leq n - 1$. Choose $l, m \in \mathbb{Z}_{\geq 0}$ with $m \leq k + 1$ and $l + m \leq n - 1$. If $m = 0$ there is nothing to prove. So we can assume $m > 0$. We find

$$\begin{aligned}
c_{(k+1)} a_{l+m+1} - c_{(k-1)+1} a_{l+m} &\geq c_{(k+1)}(c a_{l+m} - a_{l+m-1}) - c_{k+2} a_{l+m} && \text{(hypothesis)} \\
&= c_k a_{l+(m-1)+1} - c_{k+1} a_{l+(m-1)} && \text{(recurrence rel.)} \\
&\geq c_{k-(m-1)} a_{l+1} - c_{k-(m-1)+1} a_l && \text{(induction hyp.)} \\
&= c_{(k+1)-m} a_{l+1} - c_{(k+1)-m+1} a_l
\end{aligned}$$

We can now prove the lemma by induction on k. The case for $k = 0$ is clear. Suppose that for some $k \in \mathbb{Z}_{\geq 0}$ with $k < n$, the result holds for each choice of $l, m \in \mathbb{Z}_{\geq 0}$ with $m \leq k$ and $l + k \leq n$. Choose $l, m \in \mathbb{Z}_{\geq 0}$ with $m \leq k + 1$ and $l + k + 1 \leq n$. If $m = k + 1$ then the result is clear. So we can assume $m < k + 1$. We find

$$\begin{aligned}
c_m a_{(k+1)+l} &= c_m a_{k+(l+1)} \\
&\geq c_k a_{(l+1)+m} - c_{k-m} a_{l+1} && \text{(induction hypothesis)} \\
&\geq c_{(k+1)} a_{l+m} - c_{(k+1)-m} a_l && \text{(previous result)}
\end{aligned}$$

∎

Theorem 6.5 *Under the hypotheses of Proposition 6.3, the system*

$$A_2 \geq \frac{(\lambda+1)(2\lambda-1)}{1-\lambda}$$

$$A_{2n} \geq c_n A_2 - c_{n-1}$$

$$1 + c_{n-1}A_{2n} \geq c_n \frac{7\lambda-5}{1-\lambda} A_{2n}$$

has a solution $(A_2, A_{2n}) \in \mathbb{R}^2$ *with*

$$\alpha = \frac{2\lambda - 1 + \sqrt{\lambda(3\lambda-2)}}{1-\lambda} \quad \textit{and} \quad c_k = \frac{\alpha^k - \alpha^{-k}}{\alpha - \alpha^{-1}}. \tag{6.11}$$

Proof: Let $(A_0, \ldots, A_{2n+1}, B_0, \ldots, B_{2n+1}) \in \mathbb{R}^{4n+4}$ be a solution of the inequalities of Proposition 6.3, with $A_0 = 1$. We claim that A_2 and A_{2n} satisfy the conditions of the theorem. To show this we first note that, from (6.7) we have $A_{2k+1} = A_{2k+2} + B_{2k+2} - B_{2k+1}$ for $k = 0, \ldots, n-1$. Substituting this into (6.9) and (6.10) we get

$$A_{2k+2} + B_{2k} + B_{2k+2} \geq 0 \tag{6.12}$$

$$(1-\lambda)B_{2k+1} + \lambda(A_{2k+2} + B_{2k+2}) \leq 0 \tag{6.13}$$

for $k = 0, \ldots, n-1$.

From (6.8) we also have $B_{2k+1} \geq -(A_{2k} + A_{2k+2})/2$ for $k = 0, \ldots, n-1$. Since $\lambda < 1$, substituting this in (6.13) gives

$$(3\lambda - 1)A_{2k+2} + 2\lambda B_{2k+2} \leq (1-\lambda)A_{2k}$$

for $k = 0, \ldots, n-1$. This inequality can be rewritten in the form

$$B_{2k+2} \leq \frac{(1-\lambda)}{2\lambda} A_{2k} - \frac{(3\lambda-1)}{2\lambda} A_{2k+2} \tag{6.14}$$

for $k = 0, \ldots, n-1$.

To eliminate B_{2k} and B_{2k+2} from (6.12), we substitute these quantities by the upper bounds given by (6.14). After simplifications, this gives

$$A_{2k+2} - \frac{2(2\lambda-1)}{1-\lambda} A_{2k} + A_{2k-2} \geq 0 \tag{6.15}$$

for $k = 1, \ldots, n-1$. For $k = 0$, we only eliminate B_2 from (6.12) and get

$$A_2 \geq -\frac{(1-\lambda)A_0 + 2\lambda B_0}{1-\lambda}.$$

As (6.3) gives $B_0 \leq -\lambda A_0$, this leads to

$$A_2 \geq \frac{(\lambda+1)(2\lambda-1)}{1-\lambda} A_0.$$

Since $A_0 = 1$ this is the first inequality stated in the theorem.

Set $c = 2(2\lambda - 1)/(1-\lambda)$. Applying Lemma 6.4 to the inequalities (6.15) for $k = 1, \ldots, n-1$, with $l = 0$ and $m = 1$, we obtain

$$A_{2k} \geq c_k A_2 - c_{k-1} A_0$$

for $k = 1, \ldots, n$, where a and c_k are given by (6.11). In particular, for $k = n$, we obtain the second inequality stated in the theorem:

$$A_{2n} \geq c_n A_2 - c_{n-1} A_0.$$

Similarly, applying Lemma 6.4 with $l = 0$ and $m = k-1$, we get

$$A_0 + c_{k-1} A_{2k} \geq c_k A_{2k-2} \tag{6.16}$$

for $k = 1, \ldots, n$.

By (6.5), we have

$$A_{2n+1} \geq -B_{2n} - B_{2n+1}.$$

Substituting this in (6.6) we get

$$(1-\lambda)B_{2n+1} - \lambda B_{2n} \leq 0.$$

Combining this with the lower bound for B_{2n+1} provided by (6.4) and the upper bound for B_{2n} given by (6.14) for $k = n-1$, we obtain (since $\lambda < 1$):

$$A_{2n-2} \geq \frac{7\lambda - 5}{1-\lambda} A_{2n}.$$

By relation (6.16) with $k = n$, this finally gives

$$A_0 + c_{n-1}A_{2n} \geq c_n \frac{7\lambda - 5}{1 - \lambda} A_{2n}.$$

∎

Corollary 6.6 *Let $\xi \in \mathbb{R}$ be such that $1, \xi, \xi^3$ are linearly independent over $\mathbb{Q}$. If λ is an exponent of approximation to $(1, \xi, \xi^3)$, then we have $\lambda \leq 8/11$.*

Proof: Suppose on the contrary that $\lambda > 8/11$. For any $n \in \mathbb{Z}_{\geq 0}$ the theorem provides a solution $(A_2, A_{2n}) \in \mathbb{R}^2$ of the inequalities

$$A_2 \geq \frac{(\lambda + 1)(2\lambda - 1)}{1 - \lambda},$$

$$A_{2n} \geq c_n A_2 - c_{n-1},$$

$$1 + c_{n-1}A_{2n} \geq c_n \frac{7\lambda - 5}{1 - \lambda} A_{2n},$$

where α and $(c_k)_{k \geq 0}$ are given by (6.11).

Since α and $(\lambda + 1)(2\lambda - 1)/(1 - \lambda)$ are increasing functions of λ for $\lambda \in (2/3, 1)$, and since $\lambda > 8/11$, we find that $\alpha > 3$ and $(\lambda + 1)(2\lambda - 1)/(1 - \lambda) > 95/33$. Thus, the first displayed inequality implies that $A_2 > 95/33$. We also note that, since $\alpha > 3$, the sequence $(c_k)_{k \geq 1}$ tends to infinity with k and $c_k > 3c_{k-1}$ (since $c_k - \alpha c_{k-1} = \alpha^{1-k}$). In particular, if n is large enough, we obtain

$$A_{2n} \geq c_n A_2 - c_{n-1} > c_{n-1}(3A_2 - 1) > \frac{84}{33} c_{n-1}$$

while the last displayed inequality implies

$$1 \geq c_{n-1} \left(3 \frac{7\lambda - 5}{1 - \lambda} - 1 \right) A_{2n}.$$

Combining the two estimates, we get

$$1 \gg c_{n-1}^2 \frac{11\lambda - 8}{1 - \lambda}.$$

This is a contradiction since $11\lambda - 8 > 0$. ∎

Remark 6.7 Set $\lambda = 8/11$. Going back to the proof of Theorem 6.5. a more careful analysis shows that for this λ, we have for large values of k:

$$A_{2k+1} \approx \frac{11}{4} A_{2k}, \quad A_{2k+2} \approx 3A_{2k}. \quad B_{2k} \approx -\frac{3}{4} A_{2k}, \quad B_{2k+1} \approx -2A_{2k}.$$

For example, this is compatible with the existence of a sequence of minimal points $(\mathbf{x}_i)_{i\geq 1}$ satisfying the following conditions. For $i < j$ consecutive elements of $\mathbf{I}$, where, as usual $\mathbf{I}$ is set of all indices $i \geq 2$ for which $\mathbf{x}_{i-1}, \mathbf{x}_i, \mathbf{x}_{i+1}$ are linearly independent, we have:

$$\|\mathbf{x}_i\| = X_i, \quad \|\mathbf{x}_{i+1}\| \asymp X_i^{11/4}, \quad \|\mathbf{x}_j\| \asymp X_i^3.$$
$$L(\mathbf{x}_i) \asymp X_i^{-2}. \quad L(\mathbf{x}_{j-1}) \asymp X_i^{-9/4}, \quad L(\mathbf{x}_j) \asymp X_i^{-6}.$$

Chapter 7

The functions f and g

In this chapter, we improve the upper bound on the exponent of approximation from $8/11 = 0.\overline{72}$ to the root of $4x^3 - x^2 - 1$ near 0.7252700.

Definition 7.1 *We set*

$$g(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \Phi(\mathbf{x}, \mathbf{y}, \mathbf{z})\Phi(\mathbf{y}, \mathbf{y}, \mathbf{z}) - \Phi(\mathbf{x}, \mathbf{y}, \mathbf{y})\Phi(\mathbf{y}, \mathbf{z}, \mathbf{z}) - \varphi(\mathbf{y})\Phi(\mathbf{x}, \mathbf{z}, \mathbf{z}),$$
$$f(\mathbf{y}, \mathbf{z}) = g(\mathbf{y}, \mathbf{y}, \mathbf{z}) = \Phi(\mathbf{y}, \mathbf{y}, \mathbf{z})^2 - 4\varphi(\mathbf{y})\Phi(\mathbf{y}, \mathbf{z}, \mathbf{z}).$$

By expanding $g(\mathbf{x}, \mathbf{y}, \mathbf{z})$ in terms of $x_0, \Delta x_1, \ldots, \Delta z_2$ and collecting terms, a simple but tedious computation gives:

$$\begin{aligned} |g(\mathbf{x}, \mathbf{y}, \mathbf{z})| \ll \|\mathbf{y}\|^3\|\mathbf{z}\|L(\mathbf{x})L(\mathbf{z}) + \|\mathbf{z}\|^2 L(\mathbf{x})L(\mathbf{y})^3 \\ + \|\mathbf{x}\|\|\mathbf{y}\|^3 L(\mathbf{z})^2 + \|\mathbf{x}\|\|\mathbf{y}\|^2\|\mathbf{z}\|L(\mathbf{y})L(\mathbf{z}). \end{aligned} \tag{7.1}$$

In particular, for $f(\mathbf{y}, \mathbf{z})$ we have:

$$|f(\mathbf{y}, \mathbf{z})| \ll \|\mathbf{y}\|^3\|\mathbf{z}\|L(\mathbf{y})L(\mathbf{z}) + \|\mathbf{z}\|^2 L(\mathbf{y})^4 + \|\mathbf{y}\|^4 L(\mathbf{z})^2. \tag{7.2}$$

In the setting of Remark 6.7, this gives:

$$g(\mathbf{x}_{i+1}, \mathbf{x}_i, \mathbf{x}_j) = \mathcal{O}(X_i^{-1/4}) \quad \text{and} \quad f(\mathbf{x}_i, \mathbf{x}_j) = \mathcal{O}(X^{-2})$$

but since both values are integers we have:

$$g(\mathbf{x}_{i+1}, \mathbf{x}_i, \mathbf{x}_j) = 0 \quad \text{and} \quad f(\mathbf{x}_i, \mathbf{x}_j) = 0.$$

Since $\mathbf{x}_{i+1}$ is a linear combination of $\mathbf{x}_i$ and $\mathbf{x}_j$, but not a multiple of $\mathbf{x}_i$, the following proposition shows that this cannot happen.

Proposition 7.2 *Let $\xi \in \mathbb{R}$ be such that $1, \xi, \xi^3$ are linearly independent over $\mathbb{Q}$. Let $(\mathbf{x}_i)_{i\geq 1}$ be a sequence of minimal points associated to $(1, \xi, \xi^3)$. Suppose that $\lambda > 2/3$ is an exponent of approximation to $(1, \xi, \xi^3)$. Let $i, j \in \mathbb{Z}_{\geq 1}$ and $\mathbf{x} = a\mathbf{x}_i + b\mathbf{x}_j$ for some constants a, b with $b \neq 0$. If i and j are sufficiently large, we have $f(\mathbf{x}_i, \mathbf{x}_j) \neq 0$ or $g(\mathbf{x}, \mathbf{x}_i, \mathbf{x}_j) \neq 0$.*

Proof: Suppose $g(\mathbf{x}_i, \mathbf{x}_i, \mathbf{x}_j) = f(\mathbf{x}_i, \mathbf{x}_j) = 0$. Then, since g is linear in its first argument we obtain

$$\begin{aligned} g(\mathbf{x}, \mathbf{x}_i, \mathbf{x}_j) &= g(a\mathbf{x}_i + b\mathbf{x}_j, \mathbf{x}_i, \mathbf{x}_j) = ag(\mathbf{x}_i, \mathbf{x}_i, \mathbf{x}_j) + bg(\mathbf{x}_j, \mathbf{x}_i, \mathbf{x}_j) \\ &= bg(\mathbf{x}_j, \mathbf{x}_i, \mathbf{x}_j) = -3b\varphi(\mathbf{x}_i)\varphi(\mathbf{x}_j). \end{aligned}$$

By Lemma 3.9 this gives $g(\mathbf{x}, \mathbf{x}_i, \mathbf{x}_j) \neq 0$ if i and j are sufficiently large. ∎

Let $i < j$ be consecutive elements of $\mathbf{I}$. Our first goal is to show that, if $f(\mathbf{x}_i, \mathbf{x}_j) \neq 0$ for arbitrarily large values of i, then the exponents of approximation to $(1, \xi, \xi^3)$ are bounded above by $5/7$. As a consequence, for an exponent of approximation $\lambda > 5/7$ we must have $f(\mathbf{x}_i, \mathbf{x}_j) = 0$, for each sufficiently large i. In view of Proposition 7.2 this means that for each $h \in \mathbb{Z}$ with $i < h \leq j$, we have $g(\mathbf{x}_h, \mathbf{x}_i, \mathbf{x}_j) \neq 0$. This will lead to the main result of this chapter which tells us that exponents of approximation are bounded above by the root of $4x^3 - x^2 - 1$ near 0.7252700. We first establish a technical lemma.

Lemma 7.3 *Let $\xi \in \mathbb{R}$ be such that $1, \xi, \xi^3$ are linearly independent over $\mathbb{Q}$. Let $(\mathbf{x}_i)_{i\geq 1}$ be a sequence of minimal points associated to $(1, \xi, \xi^3)$. Denote by $\mathbf{I}$ the set of all indices $i \geq 2$ for which $\mathbf{x}_{i-1}, \mathbf{x}_i, \mathbf{x}_{i+1}$ are linearly independent. Suppose that*

$$1 \ll X_j L_i^2$$

for infinitely many pairs of consecutive elements $i < j$ of $\mathbf{I}$. If λ is an exponent of approximation to $(1, \xi, \xi^3)$, then $\lambda \leq 5/7$.

Proof: Suppose by contradiction that $\lambda > 5/7$. We find:

$$\begin{aligned} 1 &\ll (X_j L_i^2)^{1-\lambda} && \text{(hypothesis)} \\ &\ll X_j^{1-3\lambda} L_j^{-2(1-\lambda)} && \text{(Corollary 4.3)} \\ &\ll X_j^{1-3\lambda+4(1-\lambda)} && \text{(Lemma 3.10)} \\ &= X_j^{-(7\lambda-5)}. \end{aligned}$$

This is impossible if j is sufficiently large. ∎

We can now complete our first goal and show that $f(\mathbf{x}_i, \mathbf{x}_j) = 0$ when $\lambda > 5/7$.

Lemma 7.4 *Let $\xi \in \mathbb{R}$ be such that $1, \xi, \xi^3$ are linearly independent over $\mathbb{Q}$. Let $(\mathbf{x}_i)_{i\geq 1}$ be a sequence of minimal points associated to $(1, \xi, \xi^3)$. Denote by $\mathbf{I}$ the set of all indices $i \geq 2$ for which $\mathbf{x}_{i-1}, \mathbf{x}_i, \mathbf{x}_{i+1}$ are linearly independent. Suppose that $f(\mathbf{x}_i, \mathbf{x}_j) \neq 0$ for infinitely many pairs of consecutive elements $i < j$ of $\mathbf{I}$. If λ is an exponent of approximation to $(1, \xi, \xi^3)$, then $\lambda \leq 5/7$.*

Proof: Suppose by contradiction that $\lambda > 5/7$. Using the expansion of $f(\mathbf{x}, \mathbf{y})$ given by (7.2) we get

$$1 \leq |f(\mathbf{x}_i, \mathbf{x}_j)| \ll X_i^3 X_j L_i L_j + X_j^2 L_i^4.$$

If $1 \ll X_j^2 L_i^4$, then we also have $1 \ll X_j L_i^2$ in contradiction Lemma 7.3. Therefore we must have $1 \ll X_i^3 X_j L_i L_j$. We will show that this is impossible if $\lambda > (5-\sqrt{13})/2$. Indeed, we have:

$$\begin{aligned}
1 &\ll (X_i^3 X_j L_i L_j)^{\lambda(1-\lambda)} && \text{(hypothesis)}\\
&\ll X_i^{3\lambda(1-\lambda)} X_j^{-\lambda(2\lambda-1)} && \text{(Corollary 4.3)}\\
&\ll X_{i+1}^{3(1-\lambda)^2} X_j^{-\lambda(2\lambda-1)} && \text{(Lemma 2.3.4)}\\
&\leq X_j^{\lambda^2-5\lambda+3} && (X_{i+1} \leq X_j)
\end{aligned}$$

which is a contradiction for j sufficiently large since $\lambda^2 - 5\lambda + 3$ is negative for any λ in the interval from $(5-\sqrt{13})/2 \approx 0.697224$ to 1. ∎

Corollary 7.5 *Let $\xi \in \mathbb{R}$ be such that $1, \xi, \xi^3$ are linearly independent over $\mathbb{Q}$. Let $(\mathbf{x}_i)_{i\geq 1}$ be a sequence of minimal points associated to $(1, \xi, \xi^3)$. Denote by $\mathbf{I}$ the set of all indices $i \geq 2$ for which $\mathbf{x}_{i-1}, \mathbf{x}_i, \mathbf{x}_{i+1}$ are linearly independent. Suppose that $\lambda > 5/7$ is an exponent of approximation to $(1, \xi, \xi^3)$. Then for all pairs of consecutive elements $i < j$ of $\mathbf{I}$ with i sufficiently large we have $f(\mathbf{x}_i, \mathbf{x}_j) = 0$.*

We now look at what happens when $f(\mathbf{x}_i, \mathbf{x}_j) = 0$ for all sufficiently large i.

Lemma 7.6 *Let $\xi \in \mathbb{R}$ be such that $1, \xi, \xi^3$ are linearly independent over $\mathbb{Q}$. Let $(\mathbf{x}_i)_{i\geq 1}$ be a sequence of minimal points associated to $(1, \xi, \xi^3)$. Denote by $\mathbf{I}$ the set of all indices $i \geq 2$ for which $\mathbf{x}_{i-1}, \mathbf{x}_i, \mathbf{x}_{i+1}$ are linearly independent. Suppose that $\lambda > 2/3$ is an exponent of approximation to $(1, \xi, \xi^3)$. Then for any pair of consecutive elements $i < j$ of $\mathbf{I}$ with $f(\mathbf{x}_i, \mathbf{x}_j) = 0$ we have*

$$1 \ll X_j^2 L_i^4 + X_i^2 X_{i+1} X_j L_i L_j.$$

Proof: By Proposition 7.2 we have $g(\mathbf{x}_{i+1}, \mathbf{x}_i, \mathbf{x}_j) \neq 0$. Combining this with the expansion of $g(\mathbf{x}, \mathbf{y}, \mathbf{z})$ given by (7.1) we find that

$$1 \leq |g(\mathbf{x}_{i+1}, \mathbf{x}_i, \mathbf{x}_j)| \ll X_j^2 L_i^3 L_{i+1} + X_i^2 X_{i+1} X_j L_i L_j \leq X_j^2 L_i^4 + X_i^2 X_{i+1} X_j L_i L_j.$$

∎

Note that in the case where $\lambda > 5/7$ only the second term in the RHS of the inequality is of interest. Since $X_{i+1} \leq X_h$ for $h = i+1, \ldots, j$, the choice of $h = i+1$ in the preceding argument provides us with the smallest upper bound for $|g(\mathbf{x}_h, \mathbf{x}_i, \mathbf{x}_j)|$.

Corollary 7.7 *Let $\xi \in \mathbb{R}$ be such that $1, \xi, \xi^3$ are linearly independent over $\mathbb{Q}$. Let $(\mathbf{x}_i)_{i \geq 1}$ be a sequence of minimal points associated to $(1, \xi, \xi^3)$. Denote by $\mathbf{I}$ the set of all indices $i \geq 2$ for which $\mathbf{x}_{i-1}, \mathbf{x}_i, \mathbf{x}_{i+1}$ are linearly independent. Suppose that $\lambda > 5/7$ is an exponent of approximation to $(1, \xi, \xi^3)$. Then for any pair of consecutive elements $i < j$ of $\mathbf{I}$ we have*

$$1 \ll X_i^2 X_{i+1} X_j L_i L_j.$$

Proof: By Corollary 7.5 we have that $f(\mathbf{x}_i, \mathbf{x}_j) = 0$ and by Lemma 7.6 it follows that

$$1 \ll X_j^2 L_i^4 + X_i^2 X_{i+1} X_j L_i L_j.$$

The conclusion follows from Lemma 7.3 since the summand $X_j^2 L_i^4 \to 0$ as $i \to \infty$. ∎

Theorem 7.8 *Let $\xi \in \mathbb{R}$ be such that $1, \xi, \xi^3$ are linearly independent over $\mathbb{Q}$. If λ is an exponent of approximation to $(1, \xi, \xi^3)$, then $\lambda \leq \lambda_0$ where λ_0 is the root of $4x^3 - x^2 - 1$ near 0.7252700.*

Proof: Let $(\mathbf{x}_i)_{i\geq 1}$ be a sequence of minimal points associated to $(1, \xi, \xi^3)$. Denote by $\mathbf{I}$ the set of all indices $i \geq 2$ for which $\mathbf{x}_{i-1}, \mathbf{x}_i, \mathbf{x}_{i+1}$ are linearly independent. Let $i < j$ be a pair of consecutive elements of $\mathbf{I}$ with i arbitrarily large. Suppose by contradiction that $\lambda > \lambda_0$. We then get

$$\begin{aligned}
1 &\ll (X_i^2 X_{i+1} X_j L_i L_j)^{\lambda(1-\lambda)} && \text{(Corollary 7.7)}\\
&\ll (X_i^2 X_{i+1})^{\lambda(1-\lambda)} X_j^{-\lambda(2\lambda-1)} && \text{(Corollary 4.3)}\\
&\ll (X_i^2 X_{i+1})^{\lambda(1-\lambda)} (X_i L_i^2)^{\lambda(2\lambda-1)} && \text{(Corollary 5.5)}\\
&\ll X_{i+1}^{(1-\lambda)(1-\lambda)} L_i^{2\lambda(2\lambda-1)} && \text{(Lemma 2.3.4)}\\
&\ll X_{i+1}^{-(4\lambda^3-\lambda^2-1)} && \text{(Lemma 2.3.1)}
\end{aligned}$$

which is a impossible for i sufficiently large since $4\lambda^3 - \lambda^2 - 1 > 0$. ∎

Note that we can improve on this upper bound for λ using an argument similar to the one presented in Chapter 6 to prove $\lambda \leq 8/11$. To this end we replace the condition $0 \leq 2A_{2n} + B_{2n+1}$ with $0 \leq 2A_{2n-2} + A_{2n-1} + A_{2n} + B_{2n-1} + B_{2n+1}$ in Proposition 6.3. In this way we find that, if λ is an exponent of approximation to $(1, \xi, \xi^3)$, then $\lambda \leq \lambda_0$ where λ_0 is the root of $16x^3 - 7x - 1$ near 0.7237766.

Chapter 8

The point S and estimates for g

In this chapter, we first note that the upper bound for an exponent of approximation to $(1, \xi, \xi^3)$ found in Chapter 7 is not optimal. This is done by constructing various new points which must be zero for an exponent of approximation $\lambda > 0.72377...$, the value found at the end of Chapter 7. We also introduce a new point S which will play an important role in the sequel. Finally, we provide estimates for $g(\mathbf{x}_{i-1}, \mathbf{x}_i, \mathbf{x}_j)$ when $i < j$ are consecutive elements of $\mathbf{I}$ and conclude that these values are 0 when $(1, \xi, \xi^3)$ admits an exponent of approximation $\lambda > 5/7$.

8.1 Various new points

Definition 8.1.1 *We set*

$$\tilde{\psi}(\mathbf{x}, \mathbf{y}) := \Phi(\mathbf{x}, \mathbf{x}, \mathbf{y})\mathbf{y} - 2\Phi(\mathbf{x}, \mathbf{y}, \mathbf{y})\mathbf{x}$$

and let $\wedge$ *denote the usual cross product*

$$\mathbf{x} \wedge \mathbf{y} := \left(\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}, - \begin{vmatrix} x_0 & x_2 \\ y_0 & y_2 \end{vmatrix}, \begin{vmatrix} x_0 & x_1 \\ y_0 & y_1 \end{vmatrix} \right)$$

where $\mathbf{x} = (x_0, x_1, x_2)$ *and* $\mathbf{y} = (y_0, y_1, y_2)$.

Note that we have

$$\psi(\mathbf{x},\mathbf{y}) \wedge \tilde{\psi}(\mathbf{x},\mathbf{y}) = f(\mathbf{x},\mathbf{y})\mathbf{x} \wedge \mathbf{y}$$

which means that $\psi(\mathbf{x},\mathbf{y}) \propto \tilde{\psi}(\mathbf{x},\mathbf{y})$ when $f(\mathbf{x},\mathbf{y}) = 0$.

Definition 8.1.2 *For any* $\mathbf{x} = (x_0, x_1, x_2)$ *and* $\mathbf{y} = (y_0, y_1, y_2)$, *we also define*

$$\begin{aligned}
\theta_0(\mathbf{x},\mathbf{y}) &:= y_0\psi(\mathbf{x},\mathbf{y}) + x_0\tilde{\psi}(\mathbf{x},\mathbf{y}),\\
\theta_1(\mathbf{x},\mathbf{y}) &:= y_1\psi(\mathbf{x},\mathbf{y}) + x_1\tilde{\psi}(\mathbf{x},\mathbf{y}),\\
\theta_2(\mathbf{x},\mathbf{y}) &:= y_2\psi(\mathbf{x},\mathbf{y}) + x_2\tilde{\psi}(\mathbf{x},\mathbf{y}).
\end{aligned}$$

We can also write these points as

$$\begin{aligned}
\theta_0(\mathbf{x},\mathbf{y}) &= \psi(\mathbf{x},\mathbf{y})_0\mathbf{y} + \tilde{\psi}(\mathbf{x},\mathbf{y})_0\mathbf{x},\\
\theta_1(\mathbf{x},\mathbf{y}) &= \psi(\mathbf{x},\mathbf{y})_1\mathbf{y} + \tilde{\psi}(\mathbf{x},\mathbf{y})_1\mathbf{x},\\
\theta_2(\mathbf{x},\mathbf{y}) &= \psi(\mathbf{x},\mathbf{y})_2\mathbf{y} + \tilde{\psi}(\mathbf{x},\mathbf{y})_2\mathbf{x}.
\end{aligned}$$

It is interesting to note that

$$2g(\mathbf{x},\mathbf{y},\mathbf{z}) = \theta_0(\mathbf{y},\mathbf{z})_0x_2 - 3\theta_1(\mathbf{y},\mathbf{z})_1x_1 + 2\theta_0(\mathbf{y},\mathbf{z})_2x_0$$

which can also be written as

$$2g(\mathbf{x},\mathbf{y},\mathbf{z}) = \Phi(\mathbf{x},\mathbf{z}.\psi(\mathbf{y},\mathbf{z})) + \Phi(\mathbf{x},\mathbf{y},\tilde{\psi}(\mathbf{y},\mathbf{z})).$$

Using the points θ_i, we construct further new points:

$$\begin{gathered}
2x_ig(\mathbf{x},\mathbf{y},\mathbf{z})\mathbf{x} - \varphi(\mathbf{x})\theta_i(\mathbf{y},\mathbf{z}), \quad i = 0,1,2\\
4x_ig(\mathbf{x},\mathbf{y},\mathbf{z})\mathbf{x} \wedge \mathbf{y} + \theta_i(\mathbf{y},\mathbf{z}) \wedge \psi(\mathbf{x},\mathbf{y}), \quad i = 0,1,2\\
f(\mathbf{y},\mathbf{z})\psi(\mathbf{x},\mathbf{z}) + 4\varphi(\mathbf{y})\varphi(\mathbf{z})\psi(\mathbf{x},\mathbf{y}).
\end{gathered}$$

For the exponent $\lambda = 0.7237766$, it can be shown that all these points vanish when evaluated at $(\mathbf{x},\mathbf{y},\mathbf{z}) = (\mathbf{x}_i,\mathbf{x}_j,\mathbf{x}_k)$ where $i < j < k$ are consecutive elements of $\mathbf{I}$ with

i sufficiently large. Right away we note that since $\lambda > 5/7$ we have $f(\mathbf{x}_j, \mathbf{x}_k) = 0$ and so the last point is non-zero. This is a contradiction. More precisely, we find the following upper bound for the norm of this point:

$$\begin{aligned}
&\|\psi(\mathbf{x},\mathbf{z})f(\mathbf{y},\mathbf{z}) + 4\psi(\mathbf{x},\mathbf{y})\varphi(\mathbf{y})\varphi(\mathbf{z})\| \\
&\quad \ll \|\mathbf{x}\|^3\|\mathbf{y}\|^4 L(\mathbf{z})^3 + \|\mathbf{x}\|\|\mathbf{y}\|^4\|\mathbf{z}\| L(\mathbf{x})^2 L(\mathbf{z})^2 + \|\mathbf{x}\|^3\|\mathbf{y}\|^3\|\mathbf{z}\| L(\mathbf{y}) L(\mathbf{z})^2 \\
&\qquad + \|\mathbf{x}\|^3\|\mathbf{y}\|^2\|\mathbf{z}\|^2 L(\mathbf{y})^2 L(\mathbf{z}) + \|\mathbf{x}\|\|\mathbf{z}\|^3 L(\mathbf{x})^2 L(\mathbf{y})^4.
\end{aligned}$$

If $i < j < k$ are consecutive elements of $\mathbf{I}$ with $f(\mathbf{x}_j, \mathbf{x}_k) = 0$ and i sufficiently large then we have

$$f(\mathbf{x}_j, \mathbf{x}_k)\psi(\mathbf{x}_i, \mathbf{x}_k) + 4\varphi(\mathbf{x}_j)\varphi(\mathbf{x}_k)\psi(\mathbf{x}_i, \mathbf{x}_j) \neq 0$$

and so the above estimate applied to the norm of this non-zero integer point gives:

$$1 \ll X_i^3 X_j^2 X_k^2 L_j^2 L_k + X_i X_k^3 L_i^2 L_j^4.$$

Using this we can show that, if λ is an exponent of approximation, then $\lambda \leq 0.72211$. We will not provide a proof for this, but proceed to look at a point which will later be used to prove stronger results.

8.2 The point S

Definition 8.2.1 *We set*

$$\begin{aligned}
S(\mathbf{x},\mathbf{y},\mathbf{z}) = &(\Phi(\mathbf{x},\mathbf{x},\mathbf{y})\Phi(\mathbf{x},\mathbf{x},\mathbf{z}) - 4\varphi(\mathbf{x})\Phi(\mathbf{x},\mathbf{y},\mathbf{z}))\mathbf{x} \\
&- 2\varphi(\mathbf{x})\Phi(\mathbf{x},\mathbf{x},\mathbf{z})\mathbf{y} + 2\varphi(\mathbf{x})\Phi(\mathbf{x},\mathbf{x},\mathbf{y})\mathbf{z}.
\end{aligned}$$

Note that S satisfies

$$S(\mathbf{x},\mathbf{x},\mathbf{y}) = -3\varphi(\mathbf{x})\psi(\mathbf{x},\mathbf{y}) \quad \text{and} \quad S(\mathbf{x},\mathbf{y},\mathbf{y}) = f(\mathbf{x},\mathbf{y})\mathbf{x}.$$

By expanding $S(\mathbf{x},\mathbf{y},\mathbf{z})$ in terms of $x_0, \Delta x_1, \ldots, \Delta z_2$, we find the following upper bound for the norm of this point:

$$\begin{aligned}\|S(\mathbf{x},\mathbf{y},\mathbf{z})\| \ll \|\mathbf{x}\|^5 L(\mathbf{y})L(\mathbf{z}) + \|\mathbf{x}\|^4\|\mathbf{y}\|L(\mathbf{x})L(\mathbf{z}) \\ + \|\mathbf{x}\|^3\|\mathbf{z}\|L(\mathbf{x})^2L(\mathbf{y}) + \|\mathbf{x}\|\|\mathbf{y}\|\|\mathbf{z}\|L(\mathbf{x})^4.\end{aligned}$$

Proposition 8.2.2 *Let $\xi \in \mathbb{R}$ be such that $1, \xi, \xi^3$ are linearly independent over $\mathbb{Q}$. Let $(\mathbf{x}_i)_{i\geq 1}$ be a sequence of minimal points associated to $(1, \xi, \xi^3)$. Suppose that $\lambda > 2/3$ is an exponent of approximation to $(1, \xi, \xi^3)$. Let $i, j \in \mathbb{Z}_{\geq 1}$ and $\mathbf{x} = a\mathbf{x}_i + b\mathbf{x}_j$ for some constants a, b with $a \neq 0$. Then if i and j are sufficiently large we have $f(\mathbf{x}_i, \mathbf{x}_j) \neq 0$ or $S(\mathbf{x}_i, \mathbf{x}, \mathbf{x}_j) \neq 0$.*

Proof: Suppose $f(\mathbf{x}_i, \mathbf{x}_j) = 0$. Then, since S is linear in its second argument we obtain

$$\begin{aligned}S(\mathbf{x}_i, \mathbf{x}, \mathbf{x}_j) = S(\mathbf{x}_i, a\mathbf{x}_i + b\mathbf{x}_j, \mathbf{x}_j) &= -3a\varphi(\mathbf{x}_i)\psi(\mathbf{x}_i, \mathbf{x}_j) + bf(\mathbf{x}_i, \mathbf{x}_j)\mathbf{x}_i \\ &= -3a\varphi(\mathbf{x}_i)\psi(\mathbf{x}_i, \mathbf{x}_j).\end{aligned}$$

By Lemmas 3.9 and 5.4 this implies that $S(\mathbf{x}_i, \mathbf{x}, \mathbf{x}_j) \neq 0$ if i and j are sufficiently large. ∎

Lemma 8.2.3 *Let $\xi \in \mathbb{R}$ be such that $1, \xi, \xi^3$ are linearly independent over $\mathbb{Q}$. Let $(\mathbf{x}_i)_{i\geq 1}$ be a sequence of minimal points associated to $(1, \xi, \xi^3)$. Denote by $\mathbf{I}$ the set of all indices $i \geq 2$ for which $\mathbf{x}_{i-1}, \mathbf{x}_i, \mathbf{x}_{i+1}$ are linearly independent. Suppose that $\lambda > 2/3$ is an exponent of approximation to $(1, \xi, \xi^3)$. Let $i < j$ be a pair of elements of $\mathbf{I}$ for which $f(\mathbf{x}_i, \mathbf{x}_j) = 0$. Then we have*

$$1 \ll X_i^4 X_j L_i L_j + X_i^3 X_{i+1} L_i^3 + X_i X_j^2 L_i^4.$$

Proof: Since $\mathbf{x}_{j-1}$ is a linear combination of $\mathbf{x}_i$ and $\mathbf{x}_j$, but not a multiple of $\mathbf{x}_j$. Proposition 8.2.2 shows that $S(\mathbf{x}_i, \mathbf{x}_{j-1}, \mathbf{x}_j)$ is a non-zero point of $\mathbb{Z}^3$. So its norm is

bounded below by 1. On the other hand, using the general upper bound for the norm of S given above we find:

$$\begin{aligned}\|S(\mathbf{x}_i, \mathbf{x}_{j-1}, \mathbf{x}_j)\| &\ll X_i^5 L_{j-1} L_j + X_i^4 X_{j-1} L_i L_j + X_i^3 X_j L_i^2 L_{j-1} + X_i X_{j-1} X_j L_i^4 \\ &\ll X_i^4 X_{j-1} L_i L_j + X_i^3 X_j L_i^2 L_{j-1} + X_i X_{j-1} X_j L_i^4 \\ &\ll X_i^4 X_j L_i L_j + X_i^3 X_j L_i^2 L_{j-1} + X_i X_j^2 L_i^4 \\ &\asymp X_i^4 X_j L_i L_j + X_i^3 X_{i+1} L_i^3 + X_i X_j^2 L_i^4\end{aligned}$$

where the last equivalence comes from Corollary 4.3. ∎

8.3 New estimates for g

Since g is linear in its first argument, we can view it as a scalar product.

Definition 8.3.1 *For* $\mathbf{x} = (x_0, x_1, x_2)$ *and* $\mathbf{y} = (y_0, y_1, y_2)$ *we define the scalar product by*

$$\langle \mathbf{x}, \mathbf{y} \rangle = x_0 y_0 + x_1 y_1 + x_2 y_2.$$

Identifying polynomials $a + bT + cT^3 \in \mathbb{R}[T]$ *with points* $(a, b, c) \in \mathbb{R}^3$, *we also define a polynomial* $P_{\mathbf{yz}} \in \mathbb{R}[\mathbf{y}, \mathbf{z}][T]$ *by*

$$\langle \mathbf{x}, P_{\mathbf{yz}} \rangle = g(\mathbf{x}, \mathbf{y}, \mathbf{z}).$$

Lemma 8.3.2 *Let* $\mathbf{y}, \mathbf{z} \in \mathbb{Z}^3$ *with* $\varphi(\mathbf{y}) \neq 0$ *and* $f(\mathbf{y}, \mathbf{z}) = 0$. *Then we have*

$$\|P_{\mathbf{yz}}\| \ll \|\psi(\mathbf{y}, \mathbf{z})\|^2 \quad \textit{and} \quad |P_{\mathbf{yz}}(\xi)| \ll \|P_{\mathbf{yz}}\| \frac{L(\mathbf{y})}{\|\mathbf{y}\|}. \tag{8.3.1}$$

Moreover, for any $\mathbf{x} \in \mathbb{Z}^3$, *we also have*

$$|g(\mathbf{x}, \mathbf{y}, \mathbf{z})| \ll \left(\|\mathbf{x}\| \|\mathbf{y}\|^{-1} L(\mathbf{y}) + L(\mathbf{x})\right) \|\psi(\mathbf{y}, \mathbf{z})\|^2.$$

Proof: A computation shows the identity

$$\Phi(\mathbf{x}, \psi(\mathbf{y}, \mathbf{z}), \psi(\mathbf{y}, \mathbf{z})) = \Phi(\mathbf{x}, \mathbf{y}, \mathbf{y}) f(\mathbf{y}, \mathbf{z}) - 4\varphi(\mathbf{y}) g(\mathbf{x}, \mathbf{y}, \mathbf{z})$$

and also that

$$\Phi(\mathbf{x}, \mathbf{y}, \mathbf{y}) = 2x_0 y_0 y_2 - 3x_1 y_1^2 + x_2 y_0^2.$$

Since $f(\mathbf{y}, \mathbf{z}) = 0$, the first formula reduces to

$$-4\varphi(\mathbf{y}) g(\mathbf{x}, \mathbf{y}, \mathbf{z}) = 2x_0 \psi_0 \psi_2 - 3x_1 \psi_1^2 + x_2 \psi_0^2$$

where $\psi = \psi(\mathbf{y}, \mathbf{z})$. Since this holds for any $\mathbf{x}$, we conclude that

$$-4\varphi(\mathbf{y}) P_{\mathbf{yz}} = 2\psi_0 \psi_2 - 3T\psi_1^2 + T^3 \psi_0^2.$$

Since $\varphi(\mathbf{y})$ is a non-zero integer, we have $|\varphi(\mathbf{y})| \geq 1$ and so the above formula leads to

$$\|P_{\mathbf{yz}}\| \ll \frac{\|\psi(\mathbf{y}, \mathbf{z})\|^2}{|\varphi(\mathbf{y})|} \ll \|\psi(\mathbf{y}, \mathbf{z})\|^2.$$

We also note that

$$0 = f(\mathbf{y}, \mathbf{z}) = g(\mathbf{y}, \mathbf{y}, \mathbf{z}) = \langle \mathbf{y}, P_{\mathbf{yz}} \rangle = \langle y_0 \underline{\xi} + \Delta\mathbf{y}, P_{\mathbf{yz}} \rangle = y_0 P_{\mathbf{yz}}(\xi) + \langle \Delta\mathbf{y}, P_{\mathbf{yz}} \rangle$$

and so

$$|P_{\mathbf{yz}}(\xi)| = \frac{|\langle \Delta\mathbf{y}, P_{\mathbf{yz}} \rangle|}{|y_0|} \ll \|P_{\mathbf{yz}}\| \frac{L(\mathbf{y})}{\|\mathbf{y}\|}.$$

This proves (8.3.1). For $\mathbf{x} \in \mathbb{Z}^3$, we deduce that

$$\begin{aligned} |g(\mathbf{x}, \mathbf{y}, \mathbf{z})| &= |\langle \mathbf{x}, P_{\mathbf{yz}} \rangle| = |x_0 P_{\mathbf{yz}}(\xi) + \langle \Delta\mathbf{x}, P_{\mathbf{yz}} \rangle| \\ &\ll \|\mathbf{x}\| |P_{\mathbf{yz}}(\xi)| + \|P_{\mathbf{yz}}\| L(\mathbf{x}) \\ &\ll \left(\|\mathbf{x}\| \|\mathbf{y}\|^{-1} L(\mathbf{y}) + L(\mathbf{x}) \right) \|P_{\mathbf{yz}}\| \\ &\ll \left(\|\mathbf{x}\| \|\mathbf{y}\|^{-1} L(\mathbf{y}) + L(\mathbf{x}) \right) \|\psi(\mathbf{y}, \mathbf{z})\|^2. \end{aligned}$$

∎

We will need the following technical lemma which refines Lemma 7.3.

Lemma 8.3.3 *Let $\xi \in \mathbb{R}$ be such that $1, \xi, \xi^3$ are linearly independent over $\mathbb{Q}$. Let $(\mathbf{x}_i)_{i\geq 1}$ be a sequence of minimal points associated to $(1, \xi, \xi^3)$. Denote by $\mathbf{I}$ the set of all indices $i \geq 2$ for which $\mathbf{x}_{i-1}, \mathbf{x}_i, \mathbf{x}_{i+1}$ are linearly independent. Suppose that*

$$1 \ll X_i X_j^2 L_i^4$$

for infinitely many pairs of consecutive elements $i < j$ of $\mathbf{I}$. If λ is an exponent of approximation to $(1, \xi, \xi^3)$, then $\lambda \leq 5/7$.

Proof: Suppose by contradiction that $\lambda > 5/7$. We then find:

$$\begin{aligned} 1 &\ll (X_i X_j^2 L_i^4)^\lambda && \text{(hypothesis)} \\ &\ll (X_i X_j^2 X_{i+1}^{-4\lambda})^\lambda && \text{(Lemma 2.3.1)} \\ &\ll X_{i+1}^{1-\lambda-4\lambda^2} X_j^{2\lambda} && \text{(Lemma 2.3.4)} \end{aligned}$$

However we also have:

$$\begin{aligned} 1 &\ll (X_i^2 X_{i+1} X_j L_i L_j)^{\lambda(1-\lambda)} && \text{(Corollary 7.7)} \\ &\ll (X_i^2 X_{i+1})^{\lambda(1-\lambda)} X_j^{-\lambda(2\lambda-1)} && \text{(Corollary 4.3)} \\ &\ll X_{i+1}^{(2-\lambda)(1-\lambda)} X_j^{-\lambda(2\lambda-1)} && \text{(Lemma 2.3.4)} \end{aligned}$$

Combining these two results we obtain

$$1 \ll X_{i+1}^{-(8\lambda^3-4\lambda^2+3\lambda-3)}$$

which is a contradiction if i sufficiently large since $8\lambda^3 - 4\lambda^2 + 3\lambda - 3 > 0$ when $\lambda > \lambda_0$, where λ_0 is the real root of $8x^3 - 4x^2 + 3x - 3$ near 0.7124456. ∎

Lemma 8.3.4 *Let $\xi \in \mathbb{R}$ be such that $1, \xi, \xi^3$ are linearly independent over $\mathbb{Q}$. Let $(\mathbf{x}_i)_{i\geq 1}$ be a sequence of minimal points associated to $(1, \xi, \xi^3)$. Denote by $\mathbf{I}$ the set of all indices $i \geq 2$ for which $\mathbf{x}_{i-1}, \mathbf{x}_i, \mathbf{x}_{i+1}$ are linearly independent. Suppose that, for*

infinitely many pairs of consecutive elements $i < j$ of $\mathbf{I}$ we have $g(\mathbf{x}_{i-1}, \mathbf{x}_i, \mathbf{x}_j) \neq 0$. If λ is an exponent of approximation to $(1, \xi, \xi^3)$, then $\lambda \leq 5/7$.

Proof: Suppose by contradiction that $\lambda > 5/7$. Choose a triple $i < j < k$ of consecutive elements of $\mathbf{I}$ with i arbitrarily large such that $g(\mathbf{x}_{i-1}, \mathbf{x}_i, \mathbf{x}_j) \neq 0$. Then we find

$$\begin{aligned}
1 &\leq |g(\mathbf{x}_{i-1}, \mathbf{x}_i, \mathbf{x}_j)| && \text{(hypothesis)}\\
&\ll (X_{i-1} X_i^{-1} L_i + L_{i-1}) \|\psi(\mathbf{x}_i, \mathbf{x}_j)\|^2 && \text{(Lemma 8.3.2)}\\
&\ll L_{i-1} \|\psi(\mathbf{x}_i, \mathbf{x}_j)\|^2 && (X_{i-1} \leq X_i, L_i \leq L_{i-1})\\
&\ll X_i^2 X_j^2 L_{i-1} L_i^4 && \text{(Lemma 5.3).}
\end{aligned}$$

We first claim that $X_{i+1}^3 \ll X_{j+1}$. To show this, we observe that

$$\begin{aligned}
1 &\ll (X_i^2 X_j^2 L_{i-1} L_i^4)^\lambda && \text{(hypothesis)}\\
&\ll (X_i^{2-\lambda} X_j^2 X_{i+1}^{-4\lambda})^\lambda && \text{(Lemma 2.3.1)}\\
&\ll X_{i+1}^{(2-\lambda)(1-\lambda)} X_j^{2\lambda} X_{i+1}^{-4\lambda^2} && \text{(Lemma 2.3.4)}\\
&\ll X_{i+1}^{(2-\lambda)(1-\lambda)} (X_{i+1} X_{j+1})^{2\lambda(1-\lambda)} X_{i+1}^{-4\lambda^2} && \text{(Corollary 4.3)}\\
&= X_{i+1}^{-(5\lambda^2+\lambda-2)} X_{j+1}^{2\lambda(1-\lambda)}.
\end{aligned}$$

Since $5\lambda^2 + \lambda - 2$ and $2\lambda(1-\lambda)$ are respectively increasing and decreasing functions of λ for $\lambda > 1/2$, the above estimate combined with our assumption that $\lambda > 5/7$ leads to

$$X_{i+1}^{-62/49} \ll X_{j+1}^{20/49}$$

and therefore

$$X_{i+1}^3 \leq X_{i+1}^{31/10} \ll X_{j+1}.$$

It then follows from Corollary 4.3 that

$$X_j^3 \ll X_{j+1}^{4(1-\lambda)}. \tag{8.3.2}$$

By Corollary 7.5, we also have $f(\mathbf{x}_j, \mathbf{x}_k) = 0$ and Lemma 8.2.3 now tells us that

$$1 \ll X_j^4 X_k L_j L_k + X_j^3 X_{j+1} L_j^3 + X_j X_k^2 L_j^4.$$

This means that at least one of the summands must be $\gg 1$. We now look at each summand separately and find that this is impossible.

We first suppose that $1 \ll X_j^4 X_k L_j L_k$. This gives

$$\begin{aligned} 1 &\ll (X_j^4 X_k L_j L_k)^{3(1-\lambda)} && \text{(hypothesis)} \\ &\ll X_j^{12(1-\lambda)} X_k^{-3(2\lambda-1)} && \text{(Corollary 4.3)} \\ &\ll X_{j+1}^{16(1-\lambda)^2} X_k^{-3(2\lambda-1)} && \text{(equation (8.3.2))} \end{aligned}$$

but we also have

$$\begin{aligned} 1 &\ll (X_j X_k L_j^2)^3 && \text{(Corollary 5.5)} \\ &\ll (X_j X_k X_{j+1}^{-2\lambda})^3 && \text{(Lemma 2.3.1)} \\ &\ll X_{j+1}^{-2(5\lambda-2)} X_k^3 && \text{(equation (8.3.2)).} \end{aligned}$$

Combining these two results we obtain

$$1 \ll X_{j+1}^{-(2\lambda^2+7\lambda-6)}$$

which is a contradiction for j sufficiently large since $2\lambda^2 + 7\lambda - 6 > 0$ when $\lambda > (\sqrt{97} - 7)/4 \approx 0.712214$.

We now look at the second summand and suppose that $1 \ll X_j^3 X_{j+1} L_j^3$. We then find:

$$\begin{aligned} 1 &\ll X_j^3 X_{j+1} L_j^3 && \text{(hypothesis)} \\ &\ll X_j^3 X_{j+1}^{1-3\lambda} && \text{(Lemma 2.3.1)} \\ &\ll X_{j+1}^{-(7\lambda-5)} && \text{(equation (8.3.2))} \end{aligned}$$

which is a contradiction for j sufficiently large since $\lambda > 5/7$.

Therefore, we must have $1 \ll X_j X_k^2 L_j^4$. However, this is in contradiction with Lemma 8.3.3. ∎

Combining Corollary 7.5 and Lemma 8.3.4 we can now suppose that if $\lambda > 5/7$ is an exponent of approximation, then for all pairs of consecutive elements $i < j$ of $\mathbf{I}$ with i sufficiently large, we have $f(\mathbf{x}_i, \mathbf{x}_j) = 0$ and $g(\mathbf{x}_{i-1}, \mathbf{x}_i, \mathbf{x}_j) = 0$.

Chapter 9

Finer estimates

Let $\mathbf{T} = (1, T, T^3)$ and $\underline{\underline{\xi}} = (1, \xi, \xi^3)$. As in the proof of Proposition 5.2, for a triple $\mathbf{x} = (x_0, x_1, x_2)$, we define

$$\Delta\mathbf{x} = \mathbf{x} - x_0\underline{\underline{\xi}} = (0, \Delta x_1, \Delta x_2).$$

We also define the polynomial

$$P_{\mathbf{x}}(T) := \Phi(\mathbf{x}, \mathbf{T}, \mathbf{T}).$$

We then have

$$P_{\mathbf{x}}(\xi) = \Phi(\mathbf{x}, \underline{\underline{\xi}}, \underline{\underline{\xi}}) = \Phi(x_0\underline{\underline{\xi}} + \Delta\mathbf{x}, \underline{\underline{\xi}}, \underline{\underline{\xi}}) = \Phi(\Delta\mathbf{x}, \underline{\underline{\xi}}, \underline{\underline{\xi}}) = P_{\Delta\mathbf{x}}(\xi)$$

since Φ is multilinear and $\Phi(\mathbf{T}, \mathbf{T}, \mathbf{T}) = 0$. This gives the following upper bound:

$$|P_{\mathbf{x}}(\xi)| \ll L(\mathbf{x}). \tag{9.0.1}$$

A short computation shows that

$$P_{\mathbf{x}}(\xi) = \Delta x_2 - 3\xi^2\Delta x_1.$$

and therefore

$$L(\mathbf{x}) = \max\{|\Delta x_1|, |\Delta x_2|\} \asymp \max\{|\Delta x_1|, |P_{\mathbf{x}}(\xi)|\}. \tag{9.0.2}$$

For that reason, and for simplicity, we replace the polynomial notation by the notation used in the proof of Proposition 5.2:

$$\delta(\mathbf{x}) := \delta_\xi(\mathbf{x}) = P_{\mathbf{x}}(\xi)$$

and (9.0.2) then becomes

$$L(\mathbf{x}) \asymp \max\{|\Delta x_1|, |\delta(\mathbf{x})|\}.$$

In order to improve the upper bounds of the previous chapters, we will reexamine some results and refine them by taking expansions with respect to Δx_1 and $\delta(\mathbf{x})$ instead of Δx_1 and Δx_2.

9.1 Finer estimates for φ

For an arbitrary point $\mathbf{x} \in \mathbb{R}^3$, expanding $\varphi(\mathbf{x})$ as a polynomial in $x_0, \Delta x_1$ and $\delta(\mathbf{x})$, we find

$$\varphi(\mathbf{x}) = x_0^2\delta(\mathbf{x}) + \mathcal{O}(\|\mathbf{x}\|L(\mathbf{x})^2).$$

This leads to the following result which refines Lemma 3.10.

Lemma 9.1.1 *Let $\xi \in \mathbb{R}$ be such that $1, \xi, \xi^3$ are linearly independent over $\mathbb{Q}$. Let $(\mathbf{x}_i)_{i\geq 1}$ be a sequence of minimal points associated to $(1, \xi, \xi^3)$. Suppose that $\lambda > 2/3$ is an exponent of approximation to $(1, \xi, \xi^3)$. Then, for each sufficiently large i, we have*

$$1 \leq |\varphi(\mathbf{x}_i)| \asymp X_i^2|\delta(\mathbf{x}_i)|.$$

Proof: The preceding estimate gives

$$\varphi(\mathbf{x}_i) = x_{i,0}^2\delta(\mathbf{x}_i) + \mathcal{O}(X_iL_i^2).$$

Since $\lambda > 2/3$, we have

$$X_iL_i^2 \ll X_i^{1-2\lambda} < X_i^{-1/3} = o(1).$$

The conclusion follows since by Lemma 3.9 we also have $\varphi(\mathbf{x}_i) \neq 0$ for i sufficiently large which means that

$$1 \leq |\varphi(\mathbf{x}_i)|.$$

∎

The next lemma collects easy estimates that will be useful in the computations which follow.

Lemma 9.1.2 *Let $\xi \in \mathbb{R}$ be such that $1, \xi, \xi^3$ are linearly independent over $\mathbb{Q}$. Let $(\mathbf{x}_i)_{i\geq 1}$ be a sequence of minimal points associated to $(1, \xi, \xi^3)$. Denote by $\mathbf{I}$ the set of all indices $i \geq 2$ for which $\mathbf{x}_{i-1}, \mathbf{x}_i, \mathbf{x}_{i+1}$ are linearly independent. Suppose that $\lambda > 2/3$ is an exponent of approximation to $(1, \xi, \xi^3)$. Then for any pair of consecutive elements $i < j$ of $\mathbf{I}$ we have*

$$|\delta(\mathbf{x}_j)| \ll |\delta(\mathbf{x}_i)|, \quad L_i L_j \ll |\delta(\mathbf{x}_j)| \quad \textit{and} \quad L_i \ll |\delta(\mathbf{x}_i)| X_i.$$

Proof: By Lemmas 2.3.6 and 9.1.1, we have

$$|\delta(\mathbf{x}_j)| \ll L_j \ll X_j^{-4/3} \ll X_i^{-8/3} \ll |\delta(\mathbf{x}_i)|.$$

The second result follows from Corollary 4.3 and Lemma 9.1.1,

$$L_i L_j \ll X_j^{-\lambda/(1-\lambda)} \leq X_j^{-2} \ll |\delta(\mathbf{x}_j)|.$$

Finally, by Lemmas 2.3.6 and 9.1.1, we have

$$L_i \ll X_i^{-4/3} \leq X_i^{-2} X_i \ll |\delta(\mathbf{x}_i)| X_i.$$

∎

9.2 Finer estimates for g

We now turn our attention to g and begin by establishing a general upper bound for $|g(\mathbf{x}, \mathbf{x}_i, \mathbf{x}_j)|$.

Lemma 9.2.1 *Let $\xi \in \mathbb{R}$ be such that $1, \xi, \xi^3$ are linearly independent over $\mathbb{Q}$. Let $(\mathbf{x}_i)_{i\geq 1}$ be a sequence of minimal points associated to $(1, \xi, \xi^3)$. Denote by $\mathbf{I}$ the set of all indices $i \geq 2$ for which $\mathbf{x}_{i-1}, \mathbf{x}_i, \mathbf{x}_{i+1}$ are linearly independent. Suppose that $\lambda > 2/3$ is an exponent of approximation to $(1, \xi, \xi^3)$. Then for any point $\mathbf{x}$ and any pair of consecutive elements $i < j$ of $\mathbf{I}$ we have*

$$|g(\mathbf{x}, \mathbf{x}_i, \mathbf{x}_j)| \ll |\delta(\mathbf{x}_i)||\delta(\mathbf{x}_j)|\|\mathbf{x}\|X_i^2 X_j + |\delta(\mathbf{x})||\delta(\mathbf{x}_j)|X_i^3 X_j + X_j^2 L(\mathbf{x}) L_i^3.$$

Proof: Expanding $g(\mathbf{x}, \mathbf{y}, \mathbf{z})$ in terms of $x_0, \Delta x_1, \ldots, \delta(\mathbf{z})$, for arbitrary points $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^3$, we find

$$\begin{aligned} g(\mathbf{x}, \mathbf{y}, \mathbf{z}) = & -\delta(\mathbf{x})\delta(\mathbf{z}) y_0^3 z_0 - 3\delta(\mathbf{y})\delta(\mathbf{z}) x_0 y_0^2 z_0 + \delta(\mathbf{z})^2 x_0 y_0^3 \\ & + \mathcal{O}(\|\mathbf{z}\|^2 L(\mathbf{x}) L(\mathbf{y})^3 + \|\mathbf{y}\|^2 \|\mathbf{z}\| L(\mathbf{x}) L(\mathbf{y}) L(\mathbf{z}) + \|\mathbf{x}\|\|\mathbf{y}\|\|\mathbf{z}\| L(\mathbf{y})^2 L(\mathbf{z}) \\ & + \|\mathbf{y}\|^3 L(\mathbf{x}) L(\mathbf{z})^2 + \|\mathbf{x}\|\|\mathbf{y}\|^2 L(\mathbf{y}) L(\mathbf{z})^2). \end{aligned}$$

In particular, for i and j as in the statement of the lemma, we get:

$$\begin{aligned} |g(\mathbf{x}, \mathbf{x}_i, \mathbf{x}_j)| \ll & |\delta(\mathbf{x})||\delta(\mathbf{x}_j)| X_i^3 X_j + |\delta(\mathbf{x}_i)||\delta(\mathbf{x}_j)|\|\mathbf{x}\| X_i^2 X_j + |\delta(\mathbf{x}_j)|^2 \|\mathbf{x}\| X_i^3 \\ & + X_j^2 L(\mathbf{x}) L_i^3 + X_i^2 X_j L(\mathbf{x}) L_i L_j + \|\mathbf{x}\| X_i X_j L_i^2 L_j. \end{aligned}$$

The third term in this upper bound is superfluous because, by Lemma 9.1.2, we have:

$$|\delta(\mathbf{x}_j)|^2 \|\mathbf{x}\| X_i^3 \ll |\delta(\mathbf{x}_i)||\delta(\mathbf{x}_j)|\|\mathbf{x}\| X_i^2 X_j.$$

The fifth term can be dropped as well because, by Lemma 2.3.6, we have:

$$X_i^2 X_j L(\mathbf{x}) L_i L_j \ll X_j^2 L(\mathbf{x}) L_i^3.$$

Finally, we observe that the last term can also be discarded because, using Lemma 9.1.2, we find

$$\|\mathbf{x}\|X_iX_jL_i^2L_j \ll |\delta(\mathbf{x}_i)||\delta(\mathbf{x}_j)|\|\mathbf{x}\|X_i^2X_j.$$

∎

Corollary 9.2.2 *Let $\xi \in \mathbb{R}$ be such that $1, \xi, \xi^3$ are linearly independent over $\mathbb{Q}$. Let $(\mathbf{x}_i)_{i\geq 1}$ be a sequence of minimal points associated to $(1, \xi, \xi^3)$. Denote by $\mathbf{I}$ the set of all indices $i \geq 2$ for which $\mathbf{x}_{i-1}, \mathbf{x}_i, \mathbf{x}_{i+1}$ are linearly independent. Suppose that $\lambda > 2/3$ is an exponent of approximation to $(1, \xi, \xi^3)$. Then for any pair of consecutive elements $i < j$ of $\mathbf{I}$ with $f(\mathbf{x}_i, \mathbf{x}_j) = 0$ we have*

$$1 \ll |\delta(\mathbf{x}_i)||\delta(\mathbf{x}_j)|X_i^2X_{i+1}X_j + X_j^2L_i^4.$$

Proof: Since $f(\mathbf{x}_i, \mathbf{x}_j) = 0$, by Proposition 7.2 we have $g(\mathbf{x}_{i+1}, \mathbf{x}_i, \mathbf{x}_j) \neq 0$. By Lemma 9.2.1, we then get:

$$\begin{aligned} 1 &\ll |\delta(\mathbf{x}_i)||\delta(\mathbf{x}_j)|X_i^2X_{i+1}X_j + |\delta(\mathbf{x}_{i+1})||\delta(\mathbf{x}_j)|X_i^3X_j + X_j^2L_i^3L_{i+1} \\ &\ll |\delta(\mathbf{x}_i)||\delta(\mathbf{x}_j)|X_i^2X_{i+1}X_j + |\delta(\mathbf{x}_j)|X_i^3X_jL_i + X_j^2L_i^4. \end{aligned}$$

In the latter estimate, the second term can be dropped because using first Lemma 9.1.2 and then Lemma 2.3.6 we find

$$X_iL_i \ll X_i^2|\delta(\mathbf{x}_i)| \ll X_{i+1}|\delta(\mathbf{x}_i)|$$

which implies that $|\delta(\mathbf{x}_j)|X_i^3X_jL_i \ll |\delta(\mathbf{x}_i)||\delta(\mathbf{x}_j)|X_i^2X_{i+1}X_j$. ∎

This leads to the following refinement of Corollary 7.7.

Corollary 9.2.3 *Let $\xi \in \mathbb{R}$ be such that $1, \xi, \xi^3$ are linearly independent over $\mathbb{Q}$. Let $(\mathbf{x}_i)_{i\geq 1}$ be a sequence of minimal points associated to $(1, \xi, \xi^3)$. Denote by $\mathbf{I}$ the*

set of all indices $i \geq 2$ for which $\mathbf{x}_{i-1}, \mathbf{x}_i, \mathbf{x}_{i+1}$ are linearly independent. Suppose that $\lambda > 5/7$ is an exponent of approximation to $(1, \xi, \xi^3)$. Then for any pair of consecutive elements $i < j$ of $\mathbf{I}$ we have

$$1 \ll |\delta(\mathbf{x}_i)||\delta(\mathbf{x}_j)|X_i^2 X_{i+1} X_j.$$

Proof: Since $\lambda > 5/7$ we have $f(\mathbf{x}_i, \mathbf{x}_j) = 0$ by Corollary 7.5. From Corollary 9.2.2 we then get

$$1 \ll |\delta(\mathbf{x}_i)||\delta(\mathbf{x}_j)|X_i^2 X_{i+1} X_j + X_j^2 L_i^4.$$

The conclusion follows since by Lemma 7.3, we have $X_j^2 L_i^4 = o(1)$. ∎

9.3 A further consequence of the vanishing of f

In the next lemma, we follow essentially the same steps as in the proof of Lemma 9.2.1 to expand $f(\mathbf{x}_i, \mathbf{x}_j) = g(\mathbf{x}_i, \mathbf{x}_i, \mathbf{x}_j)$. When $f(\mathbf{x}_i, \mathbf{x}_j) = 0$ the main term of this expansion is bounded above in absolute value by the sum of the other terms. This gives:

Lemma 9.3.1 *Let $\xi \in \mathbb{R}$ be such that $1, \xi, \xi^3$ are linearly independent over $\mathbb{Q}$. Let $(\mathbf{x}_i)_{i \geq 1}$ be a sequence of minimal points associated to $(1, \xi, \xi^3)$. Denote by $\mathbf{I}$ the set of all indices $i \geq 2$ for which $\mathbf{x}_{i-1}, \mathbf{x}_i, \mathbf{x}_{i+1}$ are linearly independent. Suppose that $\lambda > 5/7$ is an exponent of approximation to $(1, \xi, \xi^3)$. Then for any pair of consecutive elements $i < j$ of $\mathbf{I}$ we have*

$$|\delta(\mathbf{x}_i)||\delta(\mathbf{x}_j)|X_i^3 \ll X_j L_i^4.$$

Proof: Expanding $f(\mathbf{y},\mathbf{z})$ in terms of $y_0, \Delta y_1, \ldots, \delta(\mathbf{z})$, for arbitrary points $\mathbf{x},\mathbf{y},\mathbf{z} \in \mathbb{R}^3$, we find

$$\begin{aligned} f(\mathbf{y},\mathbf{z}) = g(\mathbf{y},\mathbf{y},\mathbf{z}) &= -4\delta(\mathbf{y})\delta(\mathbf{z})y_0^3 z_0 + \delta(\mathbf{z})^2 y_0^4 \\ &\quad + \mathcal{O}(\|\mathbf{z}\|^2 L(\mathbf{y})^4 + \|\mathbf{y}\|^2\|\mathbf{z}\|L(\mathbf{y})^2 L(\mathbf{z}) + \|\mathbf{y}\|^3 L(\mathbf{y})L(\mathbf{z})^2). \end{aligned}$$

In particular for i and j as in the statement of the theorem, we have $f(\mathbf{x}_i,\mathbf{x}_j) = 0$. This now gives

$$|\delta(\mathbf{x}_i)||\delta(\mathbf{x}_j)|X_i^3 X_j \ll |\delta(\mathbf{x}_j)|^2 X_i^4 + X_j^2 L_i^4 + X_i^2 X_j L_i^2 L_j.$$

The conclusion follows because, by Lemma 2.3.6, we have

$$|\delta(\mathbf{x}_j)|^2 X_i^4 \ll X_i^4 L_j^2 \ll X_i^2 X_j L_i^2 L_j \ll X_j^2 L_i^4.$$

∎

9.4 Finer estimates for S

Expanding $S(\mathbf{x},\mathbf{y},\mathbf{z})$ in terms of $x_0, \Delta x_1, \ldots, \delta(\mathbf{z})$ we obtain:

$$\begin{aligned} S(\mathbf{x},\mathbf{y},\mathbf{z}) &= (\delta(\mathbf{y})\delta(\mathbf{z})x_0^5 - 4\delta(\mathbf{x})\delta(\mathbf{z})x_0^4 y_0)\underline{\underline{\xi}} \\ &\quad + \mathcal{O}(\|\mathbf{x}\|\|\mathbf{y}\|\|\mathbf{z}\|L(\mathbf{x})^4 + \|\mathbf{x}\|^3\|\mathbf{y}\|L(\mathbf{x})^2 L(\mathbf{z}) \\ &\quad + \|\mathbf{x}\|^3\|\mathbf{z}\|L(\mathbf{x})^2 L(\mathbf{y}) + \|\mathbf{x}\|^4 L(\mathbf{x})L(\mathbf{y})L(\mathbf{z})). \end{aligned}$$

Lemma 9.4.1 *Let $\xi \in \mathbb{R}$ be such that $1, \xi, \xi^3$ are linearly independent over $\mathbb{Q}$. Let $(\mathbf{x}_i)_{i\geq 1}$ be a sequence of minimal points associated to $(1,\xi,\xi^3)$. Denote by $\mathbf{I}$ the set of all indices $i \geq 2$ for which $\mathbf{x}_{i-1}, \mathbf{x}_i, \mathbf{x}_{i+1}$ are linearly independent. Suppose that $\lambda > 5/7$ is an exponent of approximation to $(1,\xi,\xi^3)$. Then for any $i \in \mathbf{I}$ we have*

$$1 \ll X_i^3 X_{i+1} L_i^3.$$

Proof: Let $i < j$ be a pair of consecutive elements of $\mathbf{I}$ with i arbitrarily large. By Corollary 7.5 we have $f(\mathbf{x}_i, \mathbf{x}_j) = 0$.

We first note that by Proposition 8.2.2 we have $S(\mathbf{x}_i, \mathbf{x}_{j-1}, \mathbf{x}_j) \neq 0$ since $\mathbf{x}_{j-1}$ is a linear combination of $\mathbf{x}_i$ and $\mathbf{x}_j$, but not a multiple of $\mathbf{x}_j$. Using the expansion of S given above, we then find

$$\begin{aligned}
1 &\leq \|S(\mathbf{x}_i, \mathbf{x}_{j-1}, \mathbf{x}_j)\| \\
&\ll |\delta(\mathbf{x}_{j-1})||\delta(\mathbf{x}_j)|X_i^5 + |\delta(\mathbf{x}_i)||\delta(\mathbf{x}_j)|X_i^4 X_{j-1} + X_i X_{j-1} X_j L_i^4 + X_i^3 X_j L_i^2 L_{j-1} \\
&\ll |\delta(\mathbf{x}_j)|X_i^5 L_{j-1} + |\delta(\mathbf{x}_i)||\delta(\mathbf{x}_j)|X_i^4 X_j + X_i X_j^2 L_i^4 + X_i^3 X_j L_i^2 L_{j-1} \\
&\asymp |\delta(\mathbf{x}_j)|X_i^5 L_{j-1} + |\delta(\mathbf{x}_i)||\delta(\mathbf{x}_j)|X_i^4 X_j + X_i X_j^2 L_i^4 + X_i^3 X_{i+1} L_i^3
\end{aligned}$$

where the last step is obtained by applying Corollary 4.3 to eliminate L_{j-1} from the last term in the RHS. We now show that the first and the second term in the RHS of this equation are superfluous. The first one is superfluous because

$$\begin{aligned}
X_i L_{j-1} &\leq X_i L_i && (L_{j-1} \leq L_i) \\
&\ll X_i^2 |\delta(\mathbf{x}_i)| && \text{(Lemma 9.1.2)} \\
&\ll X_{i+1} |\delta(\mathbf{x}_i)| && \text{(Lemma 2.3.6)} \\
&\leq X_j |\delta(\mathbf{x}_i)| && (X_{i+1} \leq X_j)
\end{aligned}$$

and so we find that $|\delta(\mathbf{x}_j)|X_i^5 L_{j-1} \ll |\delta(\mathbf{x}_i)||\delta(\mathbf{x}_j)|X_i^4 X_j$. For the second term, Lemma 9.3.1 immediately gives us that $|\delta(\mathbf{x}_i)||\delta(\mathbf{x}_j)|X_i^4 X_j \ll X_i X_j^2 L_i^4$. We are now left with

$$1 \ll X_i X_j^2 L_i^4 + X_i^3 X_{i+1} L_i^3.$$

The conclusion follows since by Lemma 8.3.3, we have $X_i X_j^2 L_i^4 = o(1)$. ∎

Theorem 9.4.2 *Let $\xi \in \mathbb{R}$ be such that $1, \xi, \xi^3$ are linearly independent over $\mathbb{Q}$. If λ is an exponent of approximation to $(1, \xi, \xi^3)$, then $\lambda \leq (9 + \sqrt{15})/18 \approx 0.7151657$*

Proof: Suppose by contradiction that $\lambda > (9+\sqrt{15})/18$. Let $(\mathbf{x}_i)_{i\geq 1}$ be a sequence of minimal points associated to $(1, \xi, \xi^3)$. Denote by $\mathbf{I}$ the set of all indices $i \geq 2$ for which $\mathbf{x}_{i-1}, \mathbf{x}_i, \mathbf{x}_{i+1}$ are linearly independent. Let $i < j$ be a pair of consecutive elements of $\mathbf{I}$ with i arbitrarily large.

Combining Corollary 9.2.3 and Lemma 9.3.1 we have

$$X_i \ll X_{i+1} X_j^2 L_i^4. \tag{9.4.1}$$

We then find that:

$$\begin{aligned}
1 &\ll X_i^3 X_{i+1} L_i^3 && \text{(Lemma 9.4.1)}\\
&\ll X_{i+1}^4 X_j^6 L_i^{15} && \text{(equation (9.4.1))}\\
&\ll X_{i+1}^{4-15\lambda} X_j^6 && \text{(Lemma 2.3.1)}\\
&\ll X_{i+1}^{4-15\lambda} (X_{i+1} X_{j+1})^{6(1-\lambda)} && \text{(Corollary 4.3)}\\
&= X_{i+1}^{10-21\lambda} X_{j+1}^{6(1-\lambda)}.
\end{aligned}$$

But we also have:

$$\begin{aligned}
1 &\ll X_j^3 X_{j+1} L_j^3 && \text{(Lemma 9.4.1)}\\
&\ll X_j^3 X_{j+1}^{1-3\lambda} && \text{(Lemma 2.3.1)}\\
&\ll (X_{i+1} X_{j+1})^{3(1-\lambda)} X_{j+1}^{1-3\lambda} && \text{(Corollary 4.3)}\\
&= X_{i+1}^{3(1-\lambda)} X_{j+1}^{2(2-3\lambda)}.
\end{aligned}$$

Combining these two results we obtain

$$1 \ll X_{i+1}^{-2(54\lambda^2-54\lambda+11)}$$

which would lead to a contradiction if $\lambda > (9+\sqrt{15})/18$ and i is sufficiently large. ∎

We end this chapter by noting that the upper bound provided by this theorem is not optimal. It can be slightly improved using the fact that $g(\mathbf{x}_{i-1}, \mathbf{x}_i, \mathbf{x}_j) = 0$

when $\lambda > 5/7$. In order to do this, one needs an analog of Lemma 9.3.1 where instead of decomposing $f(\mathbf{x}_i, \mathbf{x}_j)$, we decompose $g(\mathbf{x}_{i-1}, \mathbf{x}_i, \mathbf{x}_j)$ and observe that, since $g(\mathbf{x}_{i-1}, \mathbf{x}_i, \mathbf{x}_j) = 0$, the "main term" is bounded above in absolute value by the sum of the other terms.

Chapter 10

The function k and the last estimate

In this chapter we come to the main result of this thesis by showing that $5/7$ is an upper bound for any exponent of simultaneous approximation to $(1, \xi, \xi^3)$, when $1, \xi, \xi^3$ are linearly independent over $\mathbb{Q}$. This result relies on properties of the following function.

Definition 10.1 *We set*

$$k(\mathbf{x}, \mathbf{y}, \mathbf{z}) = g(\psi(\mathbf{x}, \mathbf{z}), \mathbf{x}, \mathbf{y}) + \varphi(\mathbf{x})^2 \Phi(\mathbf{y}, \mathbf{y}, \mathbf{z}).$$

Note that k satisfies the following remarkable property:

Proposition 10.2

$$k(\mathbf{x}, a\mathbf{x} + b\mathbf{y}, \mathbf{y}) = b^2(\Phi(\mathbf{x}, \mathbf{x}, \mathbf{y}) f(\mathbf{x}, \mathbf{y}) + 9\varphi(\mathbf{x})^2 \varphi(\mathbf{y})).$$

As an immediate consequence we get the following corollary.

Corollary 10.3 *Let $\xi \in \mathbb{R}$ be such that $1, \xi, \xi^3$ are linearly independent over $\mathbb{Q}$. Let $(\mathbf{x}_i)_{i \geq 1}$ be a sequence of minimal points associated to $(1, \xi, \xi^3)$. Denote by $\mathbf{I}$ the set*

of all indices $i \geq 2$ for which $\mathbf{x}_{i-1}, \mathbf{x}_i, \mathbf{x}_{i+1}$ are linearly independent. Suppose that $\lambda > 2/3$ is an exponent of approximation to $(1, \xi, \xi^3)$. Let $i < j$ be a pair consecutive elements of $\mathbf{I}$ and let $\mathbf{x} = a\mathbf{x}_i + b\mathbf{x}_j$ for some constants a, b with $b \neq 0$. Then, if i is sufficiently large we have $f(\mathbf{x}_i, \mathbf{x}_j) \neq 0$ or $k(\mathbf{x}_i, \mathbf{x}, \mathbf{x}_j) \neq 0$. In particular,

$$\text{if} \quad f(\mathbf{x}_i, \mathbf{x}_j) = 0 \quad \text{then} \quad k(\mathbf{x}_i, \mathbf{x}_{i+1}, \mathbf{x}_j) \neq 0.$$

Proof: Suppose $f(\mathbf{x}_i, \mathbf{x}_j) = 0$. Then, by Proposition 10.2 we find:

$$\begin{aligned} k(\mathbf{x}_i, \mathbf{x}, \mathbf{x}_j) = k(\mathbf{x}_i, a\mathbf{x}_i + b\mathbf{x}_j, \mathbf{x}_j) &= b^2 \left(\Phi(\mathbf{x}_i, \mathbf{x}_i, \mathbf{x}_j) f(\mathbf{x}_i, \mathbf{x}_j) + 9\varphi(\mathbf{x}_i)^2 \varphi(\mathbf{x}_j)\right) \\ &= 9b^2 \varphi(\mathbf{x}_i)^2 \varphi(\mathbf{x}_j). \end{aligned}$$

By Lemma 3.9 this gives $k(\mathbf{x}_i, \mathbf{x}, \mathbf{x}_j) \neq 0$ if i is sufficiently large.

The second assertion follows since $\mathbf{x}_{i+1}$ is a linear combination of $\mathbf{x}_i$ and $\mathbf{x}_j$ but not a multiple of $\mathbf{x}_i$. ∎

Combining this result with estimates for $k(\mathbf{x}_i, \mathbf{x}_{i+1}, \mathbf{x}_j)$, we get:

Lemma 10.4 *Let $\xi \in \mathbb{R}$ be such that $1, \xi, \xi^3$ are linearly independent over $\mathbb{Q}$. Let $(\mathbf{x}_i)_{i \geq 1}$ be a sequence of minimal points associated to $(1, \xi, \xi^3)$. Denote by $\mathbf{I}$ the set of all indices $i \geq 2$ for which $\mathbf{x}_{i-1}, \mathbf{x}_i, \mathbf{x}_{i+1}$ are linearly independent. Suppose that $\lambda > 2/3$ is an exponent of approximation to $(1, \xi, \xi^3)$. Then for any pair of consecutive elements $i < j$ of $\mathbf{I}$ with $f(\mathbf{x}_i, \mathbf{x}_j) = 0$ we have*

$$1 \ll |\delta(\mathbf{x}_i)||\delta(\mathbf{x}_j)| X_i^4 X_{i+1}^2 L_i + X_i X_{i+1}^2 X_j L_i^5 + X_i^5 X_{i+1} L_i^2 L_j + X_i^4 X_j L_i^4.$$

Proof: Upon expanding $k(\mathbf{x}, \mathbf{y}, \mathbf{z})$ in terms of $x_0, \Delta x_1, \ldots, \delta(\mathbf{z})$, for arbitrary points $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^3$, we get:

$$\begin{aligned} k(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \delta(\mathbf{x})^2 \delta(\mathbf{z}) x_0^4 y_0^2 \\ + \mathcal{O}(\|\mathbf{x}\|\|\mathbf{y}\|^2\|\mathbf{z}\| L(\mathbf{x})^5 + \|\mathbf{x}\|^2\|\mathbf{y}\|\|\mathbf{z}\| L(\mathbf{x})^4 L(\mathbf{y}) + \|\mathbf{x}\|^3\|\mathbf{y}\|^2 L(\mathbf{x})^3 L(\mathbf{z}) \\ + \|\mathbf{x}\|^4\|\mathbf{z}\| L(\mathbf{x})^2 L(\mathbf{y})^2 + \|\mathbf{x}\|^5\|\mathbf{y}\| L(\mathbf{x}) L(\mathbf{y}) L(\mathbf{z}) + \|\mathbf{x}\|^6 L(\mathbf{y})^2 L(\mathbf{z})). \end{aligned}$$

If $\mathbf{x}, \mathbf{y}, \mathbf{z}$ are minimal points with $\|\mathbf{x}\| \leq \|\mathbf{y}\| \leq \|\mathbf{z}\|$ then we have $L(\mathbf{x}) \geq L(\mathbf{y}) \geq L(\mathbf{z})$. So, the second summand in the error term is bounded above by the first, while the sixth summand is bounded above by the fifth, and the estimate becomes

$$k(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \delta(\mathbf{x})^2\delta(\mathbf{z})x_0^4y_0^2 + \mathcal{O}(\|\mathbf{x}\|\|\mathbf{y}\|^2\|\mathbf{z}\|L(\mathbf{x})^5 + \|\mathbf{x}\|^3\|\mathbf{y}\|^2L(\mathbf{x})^3L(\mathbf{z}) \\ + \|\mathbf{x}\|^4\|\mathbf{z}\|L(\mathbf{x})^2L(\mathbf{y})^2 + \|\mathbf{x}\|^5\|\mathbf{y}\|L(\mathbf{x})L(\mathbf{y})L(\mathbf{z})).$$

In particular, for i and j as in the statement of the lemma, we get

$$|k(\mathbf{x}_i, \mathbf{x}_{i+1}, \mathbf{x}_j)| \ll |\delta(\mathbf{x}_i)||\delta(\mathbf{x}_j)|X_i^4X_{i+1}^2L_i + X_iX_{i+1}^2X_jL_i^5 \\ + X_i^3X_{i+1}^2L_i^3L_j + X_i^4X_jL_i^4 + X_i^5X_{i+1}L_i^2L_j.$$

In this estimate, the third term is superfluous since Lemma 2.3.6 gives

$$X_i^3X_{i+1}^2L_i^3L_j \ll X_iX_{i+1}^2X_jL_i^5.$$

The conclusion follows since Corollary 10.3 shows that $k(\mathbf{x}_i, \mathbf{x}_{i+1}, \mathbf{x}_j)$ is a non-zero integer when i is sufficiently large. ∎

We can now prove our main result.

Theorem 10.5 *Let $\xi \in \mathbb{R}$ be such that $1, \xi, \xi^3$ are linearly independent over $\mathbb{Q}$. If λ is an exponent of approximation to $(1, \xi, \xi^3)$, then $\lambda \leq 5/7$.*

Proof: Suppose by contradiction that $\lambda > 5/7$. Let $(\mathbf{x}_i)_{i\geq 1}$ be a sequence of minimal points associated to $(1, \xi, \xi^3)$. Denote by $\mathbf{I}$ the set of all indices $i \geq 2$ for which $\mathbf{x}_{i-1}, \mathbf{x}_i, \mathbf{x}_{i+1}$ are linearly independent. Let $i < j < k$ be a triple of consecutive elements of $\mathbf{I}$ with i arbitrarily large. By Corollary 7.5 we have that $f(\mathbf{x}_i, \mathbf{x}_j) = f(\mathbf{x}_j, \mathbf{x}_k) = 0$. Then, Lemma 10.4 provides

$$1 \ll |\delta(\mathbf{x}_j)||\delta(\mathbf{x}_k)|X_j^4X_{j+1}^2L_j + X_jX_{j+1}^2X_kL_j^5 + X_j^5X_{j+1}L_j^2L_k + X_j^4X_kL_j^4.$$

We note that the last term in the RHS is superfluous since it is related to the second term by the inequality $X_j^4 X_k L_j^4 \ll X_j X_{j+1}^2 X_k L_j^5$. This follows from:

$$\begin{aligned} X_j^3 &\ll X_j^5 L_j && \text{(Lemma 3.10)} \\ &\ll X_{j+1}^{5(1-\lambda)/\lambda} L_j && \text{(Lemma 2.3.4)} \\ &\leq X_{j+1}^2 L_j && (\lambda > 5/7). \end{aligned}$$

Similarly, the third and the first terms satisfy $X_j^5 X_{j+1} L_j^2 L_k \ll X_j^4 X_{j+1}^2 L_j |\delta(\mathbf{x}_j)||\delta(\mathbf{x}_k)|$ because

$$\begin{aligned} X_j L_j L_k &\ll X_j X_k^{-\lambda/(1-\lambda)} && \text{(Corollary 4.3)} \\ &\ll X_j^3 X_k^{-(3\lambda-2)/(1-\lambda)} |\delta(\mathbf{x}_j)||\delta(\mathbf{x}_k)| && \text{(Lemma 9.1.1)} \\ &\ll X_{j+1}^{3/2} X_k^{-(3\lambda-2)/(1-\lambda)} |\delta(\mathbf{x}_j)||\delta(\mathbf{x}_k)| && \text{(Lemma 2.3.6)} \\ &\ll X_{j+1}^{1-(7\lambda-5)/(2-2\lambda)} |\delta(\mathbf{x}_j)||\delta(\mathbf{x}_k)| && (X_{j+1} \leq X_k) \\ &\leq X_{j+1} |\delta(\mathbf{x}_j)||\delta(\mathbf{x}_k)| && (\lambda > 5/7). \end{aligned}$$

We also note that the first term is also superfluous since by Lemma 9.3.1 we have:

$$|\delta(\mathbf{x}_j)||\delta(\mathbf{x}_k)| X_j^4 X_{j+1}^2 L_j \ll X_j X_{j+1}^2 X_k L_j^5.$$

Therefore, we must have $1 \ll X_j X_{j+1}^2 X_k L_j^5$. In order to analyse this fact, we first show that

$$X_j^{2(7\lambda-4)} \ll X_{j+1}^{(1-\lambda)(8\lambda-3)}. \tag{10.1}$$

Indeed, we have

$$\begin{aligned} 1 &\ll (X_i^2 X_{i+1} X_j |\delta(\mathbf{x}_i)||\delta(\mathbf{x}_j)|)^3 && \text{(Corollary 9.2.3)} \\ &\ll X_{i+1}^3 X_j^5 L_i^8 |\delta(\mathbf{x}_i)||\delta(\mathbf{x}_j)| && \text{(Lemma 9.3.1)} \\ &\ll X_{i+1}^{-(8\lambda-3)} X_j^5 L_i L_j && \text{(equation (9.0.1))} \\ &\ll (X_j^{-1/(1-\lambda)} X_{j+1})^{8\lambda-3} X_j^{5-\lambda/(1-\lambda)} && \text{(Corollary 4.3)} \\ &= X_j^{-2(7\lambda-4)/(1-\lambda)} X_{j+1}^{8\lambda-3}. \end{aligned}$$

This now gives:

$$\begin{aligned}
1 &\ll (X_j X_{j+1}^2 X_k L_j^5)^{2(7\lambda-4)} && \text{(hypothesis)}\\
&\ll X_{j+1}^{(1-\lambda)(8\lambda-3)+4(7\lambda-4)} (X_k L_j^5)^{2(7\lambda-4)} && \text{(equation (10.1))}\\
&\ll X_{j+1}^{(1-\lambda)(8\lambda-3)-2(5\lambda-2)(7\lambda-4)} X_k^{2(7\lambda-4)} && \text{(Lemma 2.3.1).}
\end{aligned}$$

But we also have:

$$\begin{aligned}
1 &\ll X_j^2 X_{j+1} X_k L_j L_k && \text{(Corollary 9.2.3)}\\
&\ll X_j^2 X_{j+1} X_k^{1-\lambda/(1-\lambda)} && \text{(Corollary 4.3)}\\
&\ll X_{j+1}^{(1-\lambda)(8\lambda-3)/(7\lambda-4)+1} X_k^{1-\lambda/(1-\lambda)} && \text{(equation (10.1)).}
\end{aligned}$$

Combining the two results we find

$$1 \ll X_{j+1}^{-\frac{(2\lambda-1)(10\lambda-1)(7\lambda-5)}{2(7\lambda-4)(1-\lambda)}}$$

which is a contradiction for $\lambda > 5/7$ and j sufficiently large. ∎

Appendix A

Finding new relations

The result presented in this appendix is the main tool that I used to find the various points and polynomials that appear throughout this thesis. It provides sufficient conditions for a polynomial to vanish when evaluated at a number of consecutive elements of a sequence of minimal points associated to a triple of the form $(1, \xi, \xi^3)$. The argument is adapted from Appendix A in [9] which deals with the case of $(1, \xi, \xi^2)$.

Let $\xi \in \mathbb{R}, \sigma_1 = 1, \sigma_2 = 3$ and let $(\mathbf{x}_i)_{i \geq 1}$ be a sequence of integer points of strictly increasing norms. We fix integers $k \geq 0$, $p \geq 0$ and a $(k+1)$-tuple

$$\mathbf{d} = (d_0, \ldots, d_k) \in \mathbb{Z}_{\geq 0}^{k+1}.$$

For each $j = 0, \ldots, k$, we also choose a triple of indeterminates $\mathbf{u}_j = (u_{j,0}, u_{j,1}, u_{j,2})$. We define the *weight* of a monomial

$$\mathbf{u}_0^{\mathbf{e}_0} \cdots \mathbf{u}_k^{\mathbf{e}_k} = \prod_{j=0}^{k} \prod_{l=0}^{2} u_{j,l}^{e_{j,l}} \in \mathbb{Q}[\mathbf{u}_0, \ldots, \mathbf{u}_k]$$

to be

$$\sum_{j=0}^{k} (\sigma_1 e_{j,1} + \sigma_2 e_{j,2}).$$

and its *multi-degree* to be

$$(|\mathbf{e}_0|, \ldots, |\mathbf{e}_k|) = (e_{0,0} + e_{0,1} + e_{0,2}, \ldots, e_{k,0} + e_{k,1} + e_{k,2}).$$

Finally, we denote by $E(\mathbf{d}, p)$ the subspace of $\mathbb{Q}[\mathbf{u}_0, \ldots \mathbf{u}_k]$ generated by all monomials of multi-degree $\mathbf{d}$ and weight p. We also write

$$\mathbf{1} = (1, \ldots, 1) \in \mathbb{Z}^{3k+3}.$$

Proposition A.1 *Suppose that there exist positive real numbers $\alpha_0, \ldots, \alpha_k, \beta_0, \ldots, \beta_k$ and an infinite set I^* of positive integers such that, for any $i \in I^*$, we have*

$$\|\mathbf{x}_{i+j}\| \ll \|\mathbf{x}_i\|^{\alpha_j} \quad \textit{and} \quad L(\mathbf{x}_{i+j}) := L_{\xi^{\sigma_1}, \xi^{\sigma_2}}(\mathbf{x}_{i+j}) \ll \|\mathbf{x}_i\|^{-\beta_j} \quad (j = 0, \ldots, k)$$

with implied constants not depending on i. Suppose also that a polynomial $P \in E(\mathbf{d}, p)$ satisfies

$$\left(\prod_{j=0}^{k} \prod_{l=0}^{2} \left(\frac{\partial}{\partial u_{j,l}} \right)^{f_{j,l}} P \right)(\mathbf{1}) = 0$$

for every choice of $f_{0,1}, f_{0,2}, \ldots, f_{k,1}, f_{k,2} \in \mathbb{Z}_{\geq 0}$ with

$$\sum_{j=0}^{k} (\alpha_j + \beta_j)(f_{j,1} + f_{j,2}) \leq \sum_{j=0}^{k} \alpha_j d_j \quad \textit{and} \quad f_{j,1} + f_{j,2} \leq d_j \quad (j = 0, \ldots, k).$$

Then we have

$$P(\mathbf{x}_i, \ldots, \mathbf{x}_{i+k}) = 0$$

for any sufficiently large $i \in I^*$.

Note that the conditions of the proposition provides a system of homogeneous linear equations with integer coefficients in the unknown coefficients of P. So, when this system has a non-zero solution (over $\mathbb{R}$), then it has a non-zero integer solution.

Proof: We note that

$$\begin{aligned} P(\mathbf{x}_i, \ldots, \mathbf{x}_{i+k}) &= \xi^p x_{i,0}^{d_0} \cdots x_{i+k,0}^{d_k} P\left(1, \frac{x_{i,1}}{x_{i,0}\xi^{\sigma_1}}, \frac{x_{i,2}}{x_{i,0}\xi^{\sigma_2}}, \ldots, 1, \frac{x_{i+k,1}}{x_{i+k,0}\xi^{\sigma_1}}, \frac{x_{i+k,2}}{x_{i+k,0}\xi^{\sigma_2}}\right) \\ &= \xi^p x_{i,0}^{d_0} \cdots x_{i+k,0}^{d_k} \sum \prod_{j=0}^{k} \prod_{l=1}^{2} \frac{1}{f_{j,l}!} \left(\left(\frac{x_{i+j,l}}{x_{i+j,0}\xi^{\sigma_l}} - 1 \right) \frac{\partial}{\partial u_{j,l}} \right)^{f_{j,l}} P(\mathbf{1}) \end{aligned}$$

where the sum ranges over all choices of $f_{0,1}, f_{0,2}, \ldots, f_{k,1}, f_{k,2} \in \mathbb{Z}_{\geq 0}$ with $f_{j,1} + f_{j,2} \leq d_j$ for $j = 0, \ldots, k$. Since we have

$$\begin{aligned}\left| x_{i,0}^{d_0} \cdots x_{i+k,0}^{d_k} \prod_{j=0}^{k} \prod_{l=1}^{2} \left(\frac{x_{i+j,l}}{x_{i+j,0}\xi^{\sigma_l}} - 1 \right)^{f_{j,l}} \right| &\leq \prod_{j=0}^{k} |x_{i+j,0}|^{d_j - f_{j,1} - f_{j,2}} \prod_{l=1}^{2} \left(\frac{L(\mathbf{x}_{i+j})}{|\xi^{\sigma_l}|} \right)^{f_{j,l}} \\ &\ll \prod_{j=0}^{k} \|\mathbf{x}_i\|^{\alpha_j d_j - (\alpha_j + \beta_j)(f_{j,1} + f_{j,2})} \\ &\ll \|\mathbf{x}_i\|^{\sum_{j=0}^{k} (\alpha_j d_j - (\alpha_j + \beta_j)(f_{j,1} + f_{j,2}))}\end{aligned}$$

the hypothesis implies

$$|P(\mathbf{x}_i, \ldots, \mathbf{x}_{i+k})| \ll \|\mathbf{x}_i\|^{-\varepsilon}$$

for some $\varepsilon > 0$ whenever $i \in I^*$. However, $P(\mathbf{x}_i, \ldots, \mathbf{x}_{i+k})$ is a rational number with bounded denominator. So, it must vanish for each sufficiently large $i \in I^*$. ∎

Appendix B

Polynomials

In this appendix, we state a result which, according to Jarník's transference principle, is dual to Theorem 3.7, and we give a direct independent proof of it, using resultants.

Definition B.1 *Let $\xi, \eta \in \mathbb{R}$. We say that a real number τ is a* measure of linear approximation to $(1, \xi, \eta)$ *if there exists a constant $c > 0$, and arbitrarily large values of X, such that the inequalities*

$$|x_0 + x_1\xi + x_2\eta| \leq cX^{-\tau}, \quad |x_1| \leq X, \quad |x_2| \leq X,$$

have no non-zero solution $(x_0, x_1, x_2) \in \mathbb{Z}^3$. When this holds, we say that c is a constant associated to the measure τ.

Definition B.2 *Let $\xi, \eta \in \mathbb{R}$. We say that a real number τ is a* exponent of linear approximation to $(1, \xi, \eta)$ *if there exists a constant $c > 0$ such that the inequalities*

$$|x_0 + x_1\xi + x_2\eta| \leq cX^{-\tau}, \quad |x_1| \leq X, \quad |x_2| \leq X,$$

have a non-zero solution $(x_0, x_1, x_2) \in \mathbb{Z}^3$ for any sufficiently large X. In this case, we say that c is a constant associated to the exponent τ.

According to Jarník's Transference Theorem, a number τ is an exponent (respectively a measure) of linear approximation to $(1, \xi, \eta)$ if and only if $\lambda = 1 - 1/\tau$ is

an exponent (respectively a measure) of simultaneous approximation to $(1, \xi, \eta)$ (see Theorem 1 of [5]).

By combining this result with our Theorem 3.7, we deduce:

Theorem B.3 *Let $\xi \in \mathbb{R}$ be such that $1, \xi, \xi^3$ are linearly independent over $\mathbb{Q}$. Then $\tau = \sqrt{3} + 2$ is a measure of linear approximation to $(1, \xi, \xi^3)$.*

Our proof below is largely inspired from [1]. We start with a definition and some preliminary results.

Definition B.4 *The* height *$H(P)$ of a polynomial P is the largest absolute value of its coefficients.*

Fix real numbers ξ and η such that $1, \xi, \eta$ are linearly independent over $\mathbb{Q}$. For each real $X \geq 1$, consider the finite set of non-zero polynomials $P(T, U) \in \mathbb{Z}T + \mathbb{Z}U + \mathbb{Z}$ with $H(P) \leq X$. Since $1, \xi, \eta$ are linearly independent over $\mathbb{Q}$, there exists a non-zero polynomial P from that set for which $|P(\xi, \eta)|$ is minimal. We call such a polynomial a *minimal polynomial corresponding to X*.

We note that minimal polynomials $P = aT + bU + c$ are primitive, in the sense that $\gcd(a, b, c) = 1$. We also note that for any $X \geq 1$ there are exactly two minimal polynomials corresponding to X. If P is one of them, then the other is $-P$.

If P is a minimal polynomial for some real number $X \geq 1$, then it is minimal for $H(P)$. We can recursively construct a sequence of minimal polynomials $(P_i)_{i \geq 1} \subseteq \mathbb{Z}T + \mathbb{Z}U + \mathbb{Z}$ with the following property:

if a minimal polynomial P satisfies $H(P_i) \leq H(P) < H(P_{i+1})$, then $P = \pm P_i$.

We say that such a sequence is a *sequence of minimal polynomials* associated to $(1, \xi, \eta)$.

We note that equivalently we could define such a sequence using the following property:

if a minimal polynomial P satisfies $|P_{i+1}(\xi,\eta)| < |P(\xi,\eta)| \leq |P_i(\xi,\eta)|$, then $P = \pm P_i$.

If $(P_i)_{i\geq 1}$ is a sequence of minimal polynomials associated to $(1,\xi,\eta)$, we usually write $Y_i := H(P_i)$ and $M_i := |P_i(\xi,\eta)|$. We note that the Y_i form a strictly increasing sequence of positive integers and that the M_i form a strictly decreasing sequence of positive real numbers. Moreover, for $i \neq j$, the polynomials P_i, P_j are linearly independent.

Lemma B.5 *Let $\xi,\eta \in \mathbb{R}$ such that $1,\xi,\eta$ are linearly independent over $\mathbb{Q}$. Let $(P_i)_{i\geq 1}$ be a sequence of minimal polynomials associated to $(1,\xi,\eta)$. Suppose that $\tau > 1$ is an exponent of approximation and that $c > 0$ is a constant associated to this exponent. Then, for each i sufficiently large, we have*

$$M_i \leq c(2c_0)^\tau Y_{i+1}^{-\tau}$$

where $c_0 = 2\max\{1,|\xi|,|\eta|\}$.

Proof: Fix some $i \geq 1$ and put $X = (Y_{i+1}-1)/c_0$. If i sufficiently large, the hypotheses imply the existence of a non-zero polynomial $P(T,U) = x_0 + x_1T + x_2U \in \mathbb{Z}T + \mathbb{Z}U + \mathbb{Z}$ such that

$$\max\{|x_1|,|x_2|\} \leq X \quad \text{and} \quad |x_0 + x_1\xi + x_2\eta| \leq cX^{-\tau},$$

and therefore:

$$\begin{aligned} |x_0| &\leq |x_1\xi + x_2\eta| + cX^{-\tau} < c_0X + 1 = Y_{i+1}, \\ |x_1| &\leq X = (Y_{i+1}-1)/c_0 < Y_{i+1}, \\ |x_2| &\leq X = (Y_{i+1}-1)/c_0 < Y_{i+1}. \end{aligned}$$

This means that $H(P) < Y_{i+1} = H(P_{i+1})$ and so we obtain

$$M_i \leq |P(\xi,\eta)| \leq cX^{-\tau} = c(2c_0)^\tau(2(Y_{i+1}-1))^{-\tau} \leq c(2c_0)^\tau Y_{i+1}^{-\tau}.$$

∎

The next two lemmas are very well known results (see [2]). In the second lemma, the notation $\mathrm{Res}(P, Q)$ stands for the Sylvester resultant of polynomials P and Q (in one variable).

Lemma B.6 *Let $P, Q \in \mathbb{C}[T]$ be polynomials of degree at most 3. Then*

$$H(PQ) \asymp H(P)H(Q).$$

Lemma B.7 *Let $P, Q \in \mathbb{Z}[T]$ be non-zero polynomials with $\deg(P) \leq m$ and $\deg(Q) \leq n$. Then, for any complex number ξ, we have*

$$|\mathrm{Res}(P, Q)| \ll H(P)^{n-1}H(Q)^{m-1}(|P(\xi)|H(Q) + |Q(\xi)|H(P)).$$

From the latter results, we deduce:

Lemma B.8 *Let $P, Q \in \mathbb{Z}[T]$ be non-zero polynomials of degree at most 3 with greatest common divisor $L \in \mathbb{Z}[T]$ of degree 1. Then, for any complex number ξ, we have*

$$H(L)^3|L(\xi)| \ll H(P)H(Q)(|P(\xi)|H(Q) + |Q(\xi)|H(P)).$$

Proof: If $L(\xi) = 0$ we are done. Suppose $L(\xi) \neq 0$. Since P/L and Q/L are relatively prime their resultant is a non-zero integer. From the previous lemmas, it follows that

$$\begin{aligned} 1 \leq |\mathrm{Res}(P/L, Q/L)| &\ll H(P/L)H(Q/L)(|(P/L)(\xi)|H(Q/L) + |(Q/L)(\xi)|H(P/L)) \\ &\asymp \frac{H(P)}{H(L)}\frac{H(Q)}{H(L)}\left(\frac{|P(\xi)|}{|L(\xi)|}\frac{H(Q)}{H(L)} + \frac{|Q(\xi)|}{|L(\xi)|}\frac{H(P)}{H(L)}\right). \end{aligned}$$

∎

Identifying polynomials in $\mathbb{Z}T^3 + \mathbb{Z}T + \mathbb{Z}$ with the triples of their coefficients in $\mathbb{Z}^3$, we also note that:

Lemma B.9 *Let $P, Q, R \in \mathbb{Z}T^3 + \mathbb{Z}T + \mathbb{Z}$. Then, for any $\xi \in \mathbb{C}$, we have*

$$|\det(P, Q, R)| \leq 2(|P(\xi)|H(Q)H(R) + |Q(\xi)|H(P)H(R) + |R(\xi)|H(P)H(Q)).$$

This follows by observing as in the proof of Lemma 4 of [3] that the determinant of the matrix (P, Q, R) does not change if, in this matrix, we replace the constant coefficients of the polynomials by their values at ξ.

Lemma B.10 *Let $\xi, \eta \in \mathbb{R}$ be such that $1, \xi, \eta$ are linearly independent over $\mathbb{Q}$. Let $(P_i)_{i\geq 1}$ be a sequence of minimal polynomials associated to $(1, \xi, \eta)$. Suppose that $\tau > 1$ is an exponent of approximation to $(1, \xi, \eta)$. Then there are infinitely many indices $i \geq 2$ for which P_{i-1}, P_i, P_{i+1} are linearly independent over $\mathbb{Q}$.*

Proof: For $i \geq 2$. Denote by W_i the subspace of $\mathbb{Q}[T]$ of dimension 2 generated by P_{i-1} and P_i. Suppose on the contrary that P_{i-1}, P_i, P_{i+1} are linearly dependent over $\mathbb{Q}$ for every $i \geq i_0$. Then $W_i = W_{i+1}$ for all $i > i_0$ and so there exists a subspace W of $\mathbb{Q}[T]$ such that $W = W_i$ for all $i \geq i_0$. Let $\{P, Q\}$ be a basis of $W \cap \mathbb{Z}[T]$. Then, for each $i \geq i_0$ we can write

$$P_i = a_i P + b_i Q$$

for some integers a_i and b_i with $|a_i| \ll Y_i$. Since P_i and P_{i+1} are linearly independent, we get

$$1 \leq \left\| \begin{matrix} a_i & b_i \\ a_{i+1} & b_{i+1} \end{matrix} \right\| = \frac{|a_i P_{i+1}(\xi) - a_{i+1} P_i(\xi)|}{|Q(\xi)|} \ll Y_{i+1}|P_i(\xi)| \ll Y_{i+1}^{1-\tau}$$

which is impossible for i sufficiently large. ∎

We now look at polynomials of a very specific form:

Proposition B.11 *Let $P \in \mathbb{Z}T^3 + \mathbb{Z}T + \mathbb{Z}$ be a non-zero primitive polynomial whose coefficient of X^2 is zero. Suppose that $P = (aT^2 + bT + c)(dT + e)$ for some $a, b, c, d, e \in \mathbb{Z}$. Then $a = \pm ds$ and $b = \mp es$ for some $s \in \mathbb{Z}$.*

Proof: If $a = b = 0$ then set $s = 0$. Otherwise, set $s = \gcd(a, b)$. We find

$$P = (aT^2 + bT + c)(dT + e) = adT^3 + (ae + bd)T^2 + (be + cd)T + ce$$

but $P \in \mathbb{Z}T^3 + \mathbb{Z}T + \mathbb{Z}$ so we must have $ae + bd = 0$ which implies $(a/s)e + (b/s)d = 0$. Note that $\gcd(a/s, b/s) = 1$ and that, since P is primitive, we also have $\gcd(d, e) = 1$. We conclude that $a = \pm ds$ and $b = \mp es$. ∎

Corollary B.12 *Let $P, Q \in \mathbb{Z}T^3 + \mathbb{Z}T + \mathbb{Z}$ be non-zero primitive polynomials and let $L = \gcd(P, Q)$. Then $P = \pm Q$ or $\deg(L) \leq 1$.*

Corollary B.13 *Let $P, Q, R \in \mathbb{Z}T^3 + \mathbb{Z}T + \mathbb{Z}$ be non-zero primitive polynomials and let $L = \gcd(P, Q, R)$. If $\deg(L) = 1$, then P, Q, R are linearly dependent.*

We conclude with:

Proof of Theorem B.3: Let $(P_i)_{i \geq 1}$ be a sequence of minimal polynomials associated to $(1, \xi, \xi^3)$. Suppose on the contrary that $\tau = \sqrt{3} + 2$ is not a measure of approximation. Then for every $c > 0$, τ is an exponent of approximation.

By the previous lemmas, for each i such that P_{i-1}, P_i, P_{i+1} are linearly independent, we have

$$\begin{aligned} 1 \leq |\det(P_{i-1}, P_i, P_{i+1})| &\ll |P_{i-1}(\xi)| H(P_i) H(P_{i+1}) \\ &\ll M_{i-1} Y_i Y_{i+1} \ll c Y_i^{1-\tau} Y_{i+1}. \end{aligned} \tag{B.1}$$

We also note that since $\tau > 3$ this gives

$$Y_i^2 \ll Y_{i+1}. \tag{B.2}$$

Let $L = \gcd(P_i, P_{i+1})$. Since P_i and P_{i+1} are linearly independent, by Corollary B.12 we have that $\deg(L) \in \{0, 1\}$. Suppose $\deg(L) = 1$. Then by Lemma B.8 we

have

$$\begin{aligned} H(L)^3|L(\xi)| &\ll H(P_i)H(P_{i+1})(|P_i(\xi)|H(P_{i+1}) + |P_{i+1}(\xi)|H(P_i)) \\ &\ll Y_iY_{i+1}(Y_iM_{i+1} + Y_{i+1}M_i) \\ &\ll Y_iY_{i+1}^{2-\tau}. \end{aligned} \tag{B.3}$$

We note that by Corollary B.13 we have $\mathrm{Res}(L, P_{i-1}) \neq 0$ since P_{i-1}, P_i, P_{i+1} are linearly independent. By Lemma B.7 we also have

$$\begin{aligned} 1 \leq |\mathrm{Res}(P_{i-1}, L)| &\ll H(L)^2(|P_{i-1}(\xi)|H(L) + |L(\xi)|H(P_{i-1})) \\ &\ll M_{i-1}H(L)^3 + Y_{i-1}H(L)^2|L(\xi)| \\ &\ll Y_i^{3-\tau} + Y_{i-1}H(L)^3|L(\xi)| && (H(L) \ll H(P_i) = Y_i) \\ &\ll Y_i^{3-\tau} + Y_{i-1}Y_iY_{i+1}^{2-\tau} && (\text{equation (B.3)}) \\ &\leq Y_i^{3-\tau} + Y_i^2Y_{i+1}^{2-\tau} && (Y_{i-1} \leq Y_i) \\ &\ll Y_i^{3-\tau} + Y_{i+1}^{3-\tau} && (\text{equation (B.2)}) \end{aligned}$$

which is a contradiction for i sufficiently large since $\tau > 3$.

Therefore, we may assume that $\deg(L) = 0$, and so, applying Lemma B.7, we get

$$\begin{aligned} 1 &\leq |\mathrm{Res}(P_i, P_{i+1})| \\ &\ll H(P_i)^2H(P_{i+1})^2(H(P_{i+1}|P_i(\xi)| + H(P_i)|P_{i+1}(\xi)|) \\ &\ll Y_i^2Y_{i+1}^2(Y_{i+1}M_i + Y_iM_{i+1}) \\ &\ll cY_i^2Y_{i+1}^{3-\tau}. \end{aligned} \tag{B.4}$$

Combining the estimates (B.1) and (B.4), we conclude that, for each i such that P_{i-1}, P_i, P_{i+1} are linearly independent, we have

$$1 \ll c^{\tau-1}Y_i^{2(\tau-1)}Y_{i+1}^{-\tau^2+4\tau-3} \ll c^{1+\tau}Y_{i+1}^{-\tau^2+4\tau-1}.$$

This occurs for infinitely many i (by Lemma B.10), but $\tau^2 - 4\tau + 1 = 0$ and so, for a c sufficiently small, this is impossible. ∎

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