Polynomial Identities of Lie Subalgebras of M_n

by

Yuexiu Zhang

A Ph.D. thesis submitted to the School of Graduate Studies and Research
in partial fulfillment of the requirements for the degree of
Doctor of Philosophy in Mathematics*

University of Ottawa
Ottawa, Ontario
August, 1994

*The Ph.D. Program is a joint program with Carleton University, administered by the Ottawa-Carleton Institute of Mathematics and Statistics

ISBN 0-612-04892-6

UNIVERSITÉ D'OTTAWA
UNIVERSITY OF OTTAWA

To my parents

The undersigned hereby recommend to
The Faculty of Graduate Studies and Research
acceptance of the thesis,

Polynomial Identities of Lie Subalgebras of M_n

submitted by
Yuexiu Zhang
in partial fulfillment of the requirements
for the degree of Doctor of Philosophy

Dr. Ship-Fah, Wang, Chair,
Department of Mathematics

Dr. Michel Racine, Thesis Supervisor

Dr. John Faulkner, External Examiner

University of Ottawa
1994

Abstract

Let $K_n(F)$ denote the Lie algebra of skew symmetric matrices with coefficients in a field F. For $1 \leq k \leq m+1$, define

$$a_m(k)(x_1, \cdots, x_m; y) := \sum_{\sigma \in S_m} (-1)^\sigma x_{\sigma(1)} \cdots x_{\sigma(k-1)} y x_{\sigma(k)} \cdots x_{\sigma(m)},$$

and let a_m^+ (respectively a_m^-) be the sum of all $a_m(k)$ with k even (respectively odd).

This dissertation contains two parts. In the first part (Chapter 2), we prove that the almost standard polynomial

$$f(x_1, \cdots, x_m; y) := \alpha a_m^+ + \beta a_m^- + \sum_{\substack{k \leq m-2n+4 \\ \text{or } k \geq 2n-1}} \gamma_k a_m(k)$$

is a weak identity of K_n, for any $\alpha, \beta, \gamma \in F$ and any integer $n \geq 2$ if and only if $m \geq 2n-2$. As a consequence, the Lie standard polynomial identity of degree $4n-7$ on the Lie algebra K_n is obtained. The sharpness of the results is also examined in this part, in the case $n = 7$, a positive answer is given. In the second part (Chapter 3), the similar problem on $sp_{2n}(F)$, the Lie subalgebra of all skew-symmetric matrices with respect to the symplectic involution, is discussed.

Acknowledgements

I am grateful to my supervisor Dr. Michel Racine for his valuable guidance and advice which made this thesis possible. His effort and patience in helping me to prepare the thesis is highly appreciated.

I wish to thank the University of Ottawa and the Mathematics Department for providing me with financial support.

My special thanks go to my husband and my parents for their understanding, encouragement and support.

Contents

Chapter 1

Introduction

1.1 Standard Polynomials in Matrix Algebra

Let $X = \{x_1, x_2, \cdots, x_n, \cdots\}$ be a countable set which we call the set of variables and F be a field. $FA[X]$ denotes the free associative (non-commutative) algebra over F in countably many variables $x_1, x_2, \cdots$. Sometimes we will use other variables x, y, z, x_i, y_j, for notational simplicity. An associative polynomial over F is an element $f(x_1, x_2, \cdots, x_n) \in FA[X]$.

DEFINITION. Let A be an associative algebra over F. If there is a nonzero polynomial $f(x_1, x_2, \cdots, x_n) \in FA[X]$ such that $f(r_1, r_2, \cdots, r_n) = 0$ for all substitutions $r_1, r_2, \cdots, r_n \in A$, then $f(x_1, x_2, \cdots, x_n)$ is a *polynomial identity* of A.

The *standard polynomial* of degree k is defined to be the polynomial

$$S_k(x_1, \cdots, x_k) := \sum_{\sigma \in S_k} (-1)^\sigma x_{\sigma(1)} \cdots x_{\sigma(k)},$$

where S_k is the symmetric group on k letters and $(-1)^\sigma$ is the sign of the permutation σ.

The standard polynomial satisfies the following easily verified properties.

1. S_k is multilinear.

2. S_k is alternating.

3. It can be given recursively by

$$\begin{aligned} S_{k+1}(x_1, x_2, \cdots, x_{k+1}) &= \sum_{i=1}^{k+1}(-1)^{i+1}x_i S_k(x_1, \cdots, \hat{x}_i, \cdots, x_{k+1}) \\ &= \sum_{i=1}^{k+1}(-1)^{k+1-i} S_k(x_1, \cdots, \hat{x}_i, \cdots, x_{k+1})x_i. \end{aligned}$$

Let $M_n(F)$ denote the $n \times n$ matrix algebra with entries in F, $K_n(F)$ (respectively $H_n(F)$) denote the subspace of skew symmetric (respectively symmetric) matrices of $M_n(F)$(with respect to the transpose involution). In 1950, Amitsur and Levitzki [2] proved the following theorem.

Theorem 1.1.1 [*Amitsur, Levitzki*]

1. M_n has no identity of degree less than $2n$.

2. $S_{2n}(x_1, x_2, \cdots, x_{2n})$ is an identity of M_n.

3. If $|F| > 2$ or $n > 2$, any identity of M_n of degree $2n$ is a scalar multiple of the standard identity $S_{2n}(x_1, x_2, \cdots, x_{2n})$.

Razmyslov [16] introduced the concept of *weak identities* of the pair (A, W), namely, polynomials which vanish on some fixed subspace W of an algebra A. A weak identity is called *essential* if it is not a polynomial identity of A.

In 1958 and 1980, Kostant [13][14] proved that, if $char(F) \neq 2$ and n is even, $S_{2n-2}(x_1, \cdots, x_{2n-2})$ is a weak identity of $K_n(F)$ but $S_{2n-3}(x_1, \cdots, x_{2n-3})$ is not. This was extended to all n's by Rowen [17] in 1974. In the same paper, Rowen gave several other interesting results about standard polynomials.

Theorem 1.1.2 [*Rowen*]

1. $S_{2n-2}(x_1, \cdots, x_{2n-2})$ vanishes for all specializations of $x_1, \cdots, x_{2n-2}$ to elements of $K_n(F)$, for all n.

2. $S_{2n-1}(x_1, \cdots, x_{2n-1})$ *vanishes for all specializations of* $x_1, \cdots, x_{2n-2}$ *to elements of* $K_n(F)$ *and of* x_{2n-1} *to an element of* $H_n(F)$, *for all* n.

3. $S_{2n-2}(x_1, \cdots, x_{2n-2})$ *vanishes for all specializations of* $x_1, \cdots, x_{2n-3}$ *to elements of* $K_n(F)$ *and of* x_{2n-2} *to an element of* $H_n(F)$, *for* n *odd.*

These are the best possible results if F *has characteristic* 0.

But minimal polynomial identities need not resemble standard polynomials; this fact was pointed out in [15] by Ma and Racine who constructed a multilinear polynomial

$$T^i_{2n}(x_1, \cdots, x_{2n}) := \sum_{\substack{\sigma \in S_{2n} \\ 1 \le i \le 2n, \sigma^{-1}(i) \equiv 1,2 \bmod 4}} (-1)^\sigma x_{\sigma(1)} \cdots x_{\sigma(2n)}$$

such that any other homogeneous weak identity of $H_n(F)$ is a consequence of T^i_{2n} ($n \neq 3$) if the characteristic is 0 or larger than n.

1.2 Polynomial Identities in $sl_n(F)$

For any associative algebra A, one introduces a new multiplication

$$[x, y] := xy - yx$$

where $x, y \in A$ and obtains the Lie algebra A^-. If we let $yR_x = yx$ and $yL_x = xy$ then $[x, y] = x(R_y - L_y)$. We denote $sl_n(F)$ the Lie subalgebra of $M_n(F)^-$ of matrices with trace zero and $K_n(F)$ the Lie subalgebra of $M_n(F)^-$ of skew-symmetric matrices.

Recalling that $x\, ad_y := [x, y]$, we define the *Lie standard polynomial*

$$LS_n(x_1, \cdots, x_n; y) := yS_n(ad_{x_1}, ad_{x_2}, \cdots, ad_{x_n});$$

note that LS_n is of degree $n + 1$.

We denote the left justified Lie products $[\cdots[[x_1, x_2], x_3], \cdots, x_n]$ by $[x_1, x_2, \cdots, x_n]$.

Denote by $FL[X]$, the free Lie algebra generated by X. *A polynomial identity of a Lie algebra* L is a polynomial $f(x_1, \cdots, x_n) \in FL[X]$ such that $f(r_1, \cdots, r_n)$ vanishes for all substitutions $r_1, \cdots, r_n \in L$.

For $1 \leq k \leq m+1$, define the associative polynomials

$$a_m(k)(x_1, \cdots, x_m; y) := \sum_{\sigma \in S_m} (-1)^\sigma x_{\sigma(1)} \cdots, x_{\sigma(k-1)} y x_{\sigma(k)} \cdots x_{\sigma(m)},$$

and let a_m^+ (respectively a_m^-) be the sum of all $a_m(k)$ with k even (respectively odd). A multilinear polynomial f of degree $m+1$ over the field F is *almost standard* if $f = \sum_{k=1}^{m+1} \alpha_k a_m(k)$, where $\alpha_k \in F$. In particular, the standard polynomial

$$S_{n+1}(y, x_1, \cdots, x_n) = a_n^+ + a_n^- \tag{1.2.1}$$

is almost standard. By the Amitsur-Levitzki theorem, $S_{2n}(x_1, \cdots, x_{2n}) = a_{2n-1}^+ + a_{2n-1}^-$ is, up to scalar multiple, the only polynomial identity of degree $2n$ for $M_n(F)$. The Lie standard polynomial $LS_n(x_1, \cdots, x_n; y)$ is also almost standard and the relation between LS_n and the $a_n(k)$'s will be given in chapter 2.

By the Amitsur-Levitzki theorem, if $k \geq 2n+1$ or $m-(k-1) \geq 2n$ then $a_m(k)$ is an identity of $M_n(F)$.

In 1979, Benediktovic and Zalesski [5] proved the following theorem.

Theorem 1.2.1 [*Benediktovic, Zalesski*] *Let* $char(F) = 0$.

1. For $2n \leq m \leq 4n-4$, *an almost standard polynomial* f *of degree m+1 is a polynomial identity of* $M_n(F)$ *if and only if* f *is a linear combination of* a_m^+, a_m^- *and* $a_m(k)$ *with* $k \leq m-2n+1$ *or* $k \geq 2n+1$. *When* $m \geq 4n-3$, *every* $a_m(k)$ *is an identity of* $M_n(F)$.

2. The Lie standard polynomial LS_m *is an identity of* sl_n *if and only if* $m \geq 4n-4$.

Although this theorem does give a polynomial identity of minimal degree for $sl_2(F)$, In general, $4n-3$ is not the minimal degree of a polynomial identity in $sl_n(F)$. In fact, Drensky and Kasparian [6] constructed all polynomial identities of degree 8 for $sl_3(F)$ in 1983, in that paper, they found two identities of degree 8:

$$\begin{aligned} h_1(x_1,\cdots,x_5) &= \sum_\sigma (-1)^\sigma((-3/4)\left[\left[x_{\sigma(1)},x_{\sigma(2)}\right],\left[x_{\sigma(3)},x_1,x_1\right],\left[x_{\sigma(4)},x_{\sigma(5)},x_1\right]\right] \\ &\quad +(1/4)\left[\left[x_{\sigma(1)},x_{\sigma(2)}\right],\left[x_{\sigma(3)},x_{\sigma(4)},x_1,x_1\right],\left[x_{\sigma(5)},x_1\right]\right] \\ &\quad -(1/4)\left[\left[x_{\sigma(1)},x_{\sigma(2)},x_1,x_1\right],\left[x_{\sigma(3)},x_1\right],\left[x_{\sigma(4)},x_{\sigma(5)}\right]\right]) \end{aligned}$$

and

$$\begin{aligned} h_2(x_1,\cdots,x_6) &= \sum_\sigma (-1)^\sigma((1/4)\left[\left[x_{\sigma(1)},x_1\right],\left[x_{\sigma(2)},x_1\right],\left[x_{\sigma(3)},x_{\sigma(4)}\right],\left[x_{\sigma(5)},x_{\sigma(6)}\right]\right] \\ &\quad +(1/8)\left[\left[x_{\sigma(1)},x_{\sigma(2)},x_1\right],\left[x_{\sigma(3)},x_{\sigma(4)}\right],\left[x_{\sigma(5)},x_{\sigma(6)},x_1\right]\right] \\ &\quad +(1/2)\left[\left[x_{\sigma(1)},x_{\sigma(2)}\right],\left[x_{\sigma(3)},x_{\sigma(4)},x_1,x_1\right],\left[x_{\sigma(5)},x_{\sigma(6)}\right]\right] \end{aligned}$$

which generate all polynomial identities of degree 8 for sl_3.

1.3 The identities of algebras with involution

An *involution* * of an algebra A is an antiautomorphism of period 2. Amitsur [3] defined a **-polynomial* to be a polynomial where the variables can also appear with a *; such a polynomial can also be evaluated on associative algebras with involution in the obvious way. Let $F\{X,*\}$ be $F[x_1,x_1^*,x_2,x_2^*,\cdots,]$ the free algebra with a given involution * over F. The set $T((A,*))$ of all *-polynomial identities of A is a **-T-ideal* of $F\{X,*\}$, i.e., an ideal invariant under all endomorphisms of $F\{X,*\}$ commuting with the involution * [10]. In case $A=M_n(F)$, $n\geq 2$, two involutions play a very important role : the transpose involution, denoted (t), and the canonical symplectic involution (s), defined on $2n \times 2n$ matrices by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^s = \begin{pmatrix} D^t & -B^t \\ -C^t & A^t \end{pmatrix}$$

where A, B, C and D are $n \times n$ matrices. In fact, it can be proved (see [19], Theorem (3,1,62)) that if F is an infinite field and algebraically closed, then either $T((M_n(F), *)) = T((M_n(F), t))$ or $T((M_n(F), *)) = T((M_n(F), s))$.

We let $sp_{2n}(F)$ be the Lie subalgebra of all skew-symmetric matrices (with respect to involution s) of $M_{2n}(F)$, let $H_{2n}(F, s)$ be the Jordan subalgebra of all symmetric matrices (with respect to involution s) of $M_{2n}(F)$.

In this thesis, we follow the approach of Benediktovic and Zalesski's theorem to get the analogous results for Lie algebras $K_n(F)$ and $sp_{2n}(F)$. In chapter 2 we shall get that Lie standard polynomial of degree $4n-7$ is an identity of K_n for $n \geq 3$. In order to get this result, we shall first show that the almost standard polynomials

$$f(x_1, \cdots, x_m; y) := \alpha a_m^+ + \beta a_m^- + \sum_{\substack{k \leq m-2n+4 \\ \text{or } k \geq 2n-1}} \gamma_k a_m(k)$$

vanish for all specializations of $x_1, \cdots, x_m$ to elements of K_n and of y to an element of M_n, for any $\alpha, \beta, \gamma \in F$ and any integer $n \geq 2$ if and only if $m \geq 2n-2$. The sharpness of the results is examined in section 2.2. In the case $n = 7$, we will be able to get the following statements: $4n-7$ is the minimal degree of Lie standard polynomial identity for $K_n(F)$ and, when $m \geq 2n-2$, an almost standard polynomial $f(x_1, \cdots, x_m; y)$ is a weak identity of K_n if and only if $f(x_1, \cdots, x_m; y)$ is a linear combination of a_m^+, a_m^- and $a_m(k)$ with $k \leq m-2n+4$ or $k \geq 2n-1$. The content of chapter 3 is threefold: to give an analogue for the symplectic involution by proving that $4n$ is the minimal degree of the standard polynomial $S_m(x_1, \cdots, x_m)$ satisfied by $sp_{2n}(F)$, or satisfied by all elements $x_1, \cdots, x_{m-1} \in sp_{2n}(F)$ and $x_m \in H_{2n}(s, F)$, to show that the Jordan standard polynomial $JS_{8n-8}(x_1, \cdots, x_{8n-8}; y)$ (defined in section 3.3) of degree $8n-7$ is a polynomial identity of the Jordan algebra $H_{2n}(F, s)$ by the method of superalgebras we used in chapter 2, and finally to give a new basis of the T-ideal consisting of the polynomial identities of $sl_2(F)$. In this basis, every polynomial is Lie almost standard as defined in section 3.4.

Throughout the thesis F always denotes a field of characteristic 0 unless otherwise noted.

Chapter 2

The identities of $K_n(F)$

In this chapter, we will show that a multilinear polynomial

$$f(x_1, \cdots, x_m; y) := \alpha a_m^+ + \beta a_m^- + \sum_{\substack{k \leq m-2n+4 \\ \text{or } k \geq 2n-1}} \gamma_k a_m(k)$$

is an identity of $K_n(F)$ for any $\alpha, \beta, \gamma_k \in F$ and any integer $n \geq 2$ if and only if $m \geq 2n-2$. Furthermore, it will be proved that $f(x_1, \cdots, x_m; y)$ vanishes for all $x_1, \cdots, x_m \in K_n$ and $y \in M_n$ when $m \geq 2n-2$. As a consequence, we get that the Lie standard polynomial of degree $4n-7$ is an identity of the Lie algebra $K_n(F)$ where $n \geq 3$ and also, by a computational proof, we will find that this result is sharp for $n = 7$.

2.1 Weak Polynomial Identities For K_n

In order to carry out the proof, we shall first collect a few easily verified results ; most are observations of Kostant and the proofs can be found in [18, Lemma 3 and Lemma 4].

Lemma 2.1.1 *1. If $char(F) \neq 2$, then $K_n(F)$ and $H_n(F)$ are orthogonal relative to the trace bilinear form and the restriction of the trace form to $K_n(F)$ and to $H_n(F)$ is nondegenerate.*

2. Suppose $(*)$ is a given involution with respect to which $B_1,\cdots,B_k$ are symmetric and $A_{k+1},\cdots,A_d$ are skew; then

$$S_d(B_1,\cdots,B_k,A_{k+1},\cdots,A_d)^* = (-1)^{[d/2]+d-k} S_d(B_1,\cdots,B_k,A_{k+1},\cdots,A_d)$$

where $[r]$ is the greatest integer in r.

In the following Lemma, we consider some elementary properties of the trace on an almost standard polynomial.

Lemma 2.1.2 *Let $a_k(j) = \sum_{\sigma\in\mathcal{S}_k}(-1)^\sigma x_{\sigma(1)}\cdots x_{\sigma(j-1)} y x_{\sigma(j)}\cdots x_{\sigma(k)}$.*

For $X_i, Y \in M_n(F)$, $1\le i\le k$, we have

$$\begin{aligned}
1)\qquad Tr(a_k(j)(X_1,\cdots,X_k;Y)) &= (-1)^{(k+1)(j-1)}Tr(S_k(X_1,X_2,\cdots,X_k)Y).\\
2)\quad Tr(a_{2k+1}(j)(X_1,\cdots,X_{2k+1};Y)) &= Tr(X_1 \textstyle\sum_{i=1}^{2k+1} a_{2k}(i)(X_2,\cdots,X_{2k+1},Y)).\\
3)\qquad Tr(a_{2k}(j)(X_1,\cdots,X_{2k};Y)) &= (-1)^j Tr(X_1 \textstyle\sum_{i=1}^{2k}(-1)^{i-1}a_{2k-1}(i))\\
&= (-1)^j Tr(X_1 S_{2k}(Y,X_2,X_3,\cdots,X_{2k})).
\end{aligned}$$

Proof. 1) For any $\sigma\in\mathcal{S}_k$, $X_i, Y\in M_n(k)$, $1\le i\le k$, $1\le j\le k+1$,

$$Tr(X_{\sigma(1)}\cdots X_{\sigma(j-1)}YX_{\sigma(j)}\cdots X_{\sigma(k)}) = Tr(X_{\sigma(j)}\cdots X_{\sigma(k)}X_{\sigma(1)}\cdots X_{\sigma(j-1)}Y).$$

Therefore,

$$Tr(a_k(j)(X_1,\cdots,X_k;Y)) = Tr((\sum_{\sigma\in\mathcal{S}_k}(-1)^\sigma X_{\sigma(j)}\cdots X_{\sigma(k)}X_{\sigma(1)}\cdots X_{\sigma(j-1)})Y).$$

Moreover, the sign of the permutation

$$\begin{pmatrix} \sigma(1) & \cdots & \sigma(k-j) & \sigma(k-j+1) & \sigma(k-j+2) & \cdots & \sigma(k-j+j) \\ \sigma(j) & \cdots & \sigma(k-1) & \sigma(k) & \sigma(1) & \cdots & \sigma(j-1) \end{pmatrix}$$

is $(-1)^{(k-j+1)(j-1)} = (-1)^{(k+1)(j-1)}$.

Hence, $Tr(a_k(j)(X_1,\cdots,X_k;Y)) = (-1)^{(k+1)(j-1)}Tr(S_k(X_1,\cdots,X_k)Y)$.

2) $a_{2k+1}(j)$

$$= \sum_{\sigma\in S_{2k+1}} (-1)^{\sigma} x_{\sigma(1)}\cdots x_{\sigma(j-1)} y x_{\sigma(j)}\cdots x_{\sigma(2k+1)}$$

$$= \sum_{\tau\in S_{2k}} \sum_{i=2}^{j-1} (-1)^{\tau+i-1} x_{\tau(2)}\cdots x_{\tau(i)} x_1 \cdots x_{\tau(j-1)} y x_{\tau(j)}\cdots x_{\tau(2k+1)}$$

$$+ \sum_{i=j}^{2k+1} (-1)^{\tau+i-1} \Big(\sum_{\tau\in S_{2k}} x_{\tau(2)}\cdots x_{\tau(j-1)} y x_{\tau(j)}\cdots x_{\tau(i)} x_1 \cdots x_{\tau(2k+1)} \Big).$$

For any $\tau \in S_{2k}$, X_τ, $Y \in M_n(k)$, if $1 \le i \le j-1$,

$$Tr(X_{\tau(2)}\cdots X_{\tau(i)} X_1 X_{\tau(i+1)}\cdots Y\cdots X_{\tau(2k+1)})$$

$$= Tr(X_1 X_{\tau(i+1)}\cdots Y \cdots X_{\tau(2k+1)} X_{\tau(2)}\cdots X_{\tau(i)})$$

and if $j \le i \le 2k+1$,

$$Tr(X_{\tau(2)}\cdots Y\cdots X_{\tau(i)} X_1 X_{\tau(i+1)}\cdots X_{\tau(2k+1)})$$

$$= Tr(X_1 X_{\tau(i+1)}\cdots X_{\tau(2k+1)} X_{\tau(2)}\cdots Y\cdots X_{\tau(i)}).$$

Therefore

$$Tr(a_{2k+1}(j))$$

$$= Tr\Big(\sum_{i=1}^{j-1}(-1)^{i-1} \sum_{\sigma\in S_{2k}} (-1)^{\sigma} X_1 X_{\sigma(i+1)}\cdots Y\cdots X_{\sigma(2k+1)} X_{\sigma(2)}\cdots X_{\sigma(i)}\Big)$$

$$+ Tr\Big(\sum_{i=j}^{2k+1}(-1)^{i-1} \sum_{\sigma\in S_{2k}} (-1)^{\sigma} X_1 X_{\sigma(i+1)}\cdots X_{\sigma(2k+1)} X_{\sigma(2)}\cdots Y\cdots X_{\sigma(i)}\Big)$$

$$= Tr\Big(\sum_{i=1}^{j-1}(-1)^{i-1+(i-1)(2k+1-i)} X_1\, a_{2k}(j-i)(X_2,\cdots,X_{2k+1};Y)\Big)$$

$$+\,Tr(\sum_{i=j}^{2k+1}(-1)^{i-1+(i-1)(2k+1-i)}X_1\,a_{2k}(j+2k-i+1)(X_2,\cdots,X_{2k+1};Y))$$

$$=Tr(X_1\,(\sum_{i=1}^{j-1}a_{2k}(j-i)(X_2,\cdots,X_{2k+1};Y)$$

$$+\sum_{i=j}^{2k+1}a_{2k}(j+2k-i+1)(X_2,\cdots,X_{2k+1};Y))$$

$$=Tr(X_1\sum_{i=1}^{2k+1}a_{2k}(i)(X_2,\cdots,X_{2k+1};Y)).$$

3) Similarly,

$$\begin{aligned}
Tr(a_{2k}(j)) &= Tr(\sum_{i=1}^{j-1}(-1)^{i-1+(i-1)(2k-i)}X_1\,a_{2k-1}(j-i)) \\
&\quad +Tr(\sum_{i=j}^{2k}(-1)^{i-1+(i-1)(2k-i)}X_1\,a_{2k-1}(j+2k-i)) \\
&= Tr(X_1(\sum_{i=1}^{j-1}(-1)^{i-1}a_{2k-1}(j-i) \\
&\quad +\sum_{i=j}^{2k}(-1)^{i-1}a_{2k-1}(j+2k-i))) \\
&= (-1)^jTr(X_1\sum_{i=1}^{2k}(-1)^{i+1}a_{2k-1}(i)).
\end{aligned}$$

But,

$$S_{2k}(Y,X_2,\cdots,X_{2k})=\sum_{i=1}^{2k}(-1)^{i+1}a_{2k-1}(i)(X_2,\cdots,X_{2k};Y).$$

Hence,

$$Tr(a_{2k}(j))=(-1)^jTr(X_1\,S_{2k}(Y,X_2,\cdots,X_{2k})).$$

□

Theorem 2.1.3 *The almost standard polynomial*

$$f(x_1,\cdots,x_m;y)=\alpha a_m^+ + \beta a_m^- + \sum_{\substack{k\geq 2n-1 \\ \text{or } k\leq m-2n+4}}\gamma_k a_m(k), \tag{2.1.1}$$

where $m \geq 2n-2$, is a polynomial identity of K_n for all n and arbitrary $\alpha, \beta, \gamma_k \in F$.

Proof. First, we notice that If $k \geq 2n-1$ or $k \leq m-2n+4$ with $m \geq 2n-2$ then $a_m(k)$ is a sum of polynomial multiples of S_{2n-2}'s. Hence $\sum_{k \geq 2n-1 \text{ or } k \leq m-2n+4} \gamma_m a_m(k)$ is a polynomial identity of K_n.

It is easy to check that

$$a_{t+1}(k) = \sum_{j=1}^{t+1}(-1)^{t+j+1} a_t(k)(x_1, \cdots, \hat{x}_j, \cdots, x_{t+1}; y)x_j \tag{2.1.2}$$

and

$$a_{t+1}(k+1) = \sum_{j=1}^{t+1}(-1)^{j+1} x_j a_t(k)(x_1, \cdots, \hat{x}_j, \cdots, x_{t+1}; y). \tag{2.1.3}$$

Thus

$$a_{t+1}^{+} = \sum_{j=1}^{t+1}(-1)^{t+j+1} a_t^{+}(x_1, \cdots, \hat{x}_j, \cdots, x_{t+1}; y)x_j$$

and

$$a_{t+1}^{-} = \sum_{j=1}^{t+1}(-1)^{t+j+1} a_t^{-}(x_1, \cdots, \hat{x}_j, \cdots, x_{t+1}; y)x_j.$$

Since $S_{2n-1}(y, x_1, \cdots, x_{2n-2}) = a_{2n-2}^{+} + a_{2n-2}^{-}$ is a polynomial identity of K_n it is sufficient to show that a_{2n-2}^{-} is a weak identity of K_n.

By Lemma 2.1.2.1), we have

$$Tr(a_{2n-1}^{-}) = n\,Tr(S_{2n-1}(x_1, \cdots, x_{2n-1})y).$$

When n is odd, we assume that $X_1, \cdots, X_{2n-1}, Y \in K_n$. By Kostant's theorem, we get $S_{2n-1}(X_1, \cdots, X_{2n-1}) = 0$, so

$$Tr(a_{2n-1}^{-}(X_1, \cdots, X_{2n-1}; Y)) = 0. \tag{2.1.4}$$

On the other hand, by Lemma 2.1.2.2) and equation (1.2.1),

$$Tr(a_{2n-1}^{-}) = n\,Tr(X_1 \sum_{i=1}^{2n-1} a_{2n-2}(i)(X_2, \cdots, X_{2n-1}, Y)) \tag{2.1.5}$$

$$= n\,Tr(X_1(2a_{2n-2}^{-} - S_{2n-1})(X_2, \cdots, X_{2n-1}, Y)). \tag{2.1.6}$$

Since $S_{2n-1}(X_2,\cdots,X_{2n-1},Y)$ is an identity of K_n, $Tr(X_1 S_{2n-1}(X_2,\cdots,X_{2n-1},Y)) = 0$. By equations (2.1.4) and (2.1.5), $Tr(X_1\, a^-_{2n-2}) = 0$.

For any $1 \leq j \leq 2n-1$ and j odd,

$$\begin{aligned}(a_{2n-2}(j)(X_2,\cdots,X_{2n-1};Y))^t &= (-1)^n a_{2n-2}(2n-j)(X_2,\cdots,X_{2n-1};Y)\\ &= -a_{2n-2}(2n-j)(X_2,\cdots,X_{2n-1};Y).\end{aligned}$$

This implies that a^-_{2n-2} takes values in K_n. Hence $a^-_{2n-2} = 0$ by the nondegeneracy of the trace bilinear form on K_n.

For n even, we assume $X_2,\cdots,X_{2n-1},Y \in K_n$ and $X_1 \in H_n$. Rowen has proved that, for all n, $S_{2n-1}(x_1,\cdots,x_{2n-1})$ vanishes for all specializations of $x_1,\cdots,x_{2n-2}$ to elements of K_n and x_{2n-1} to an element of H_n, for all n. By Lemma 2.1.2.1),

$$Tr(a^-_{2n-1}(X_1,\cdots,X_{2n-1};Y)) = n\,Tr(S_{2n-1}(X_1,\cdots,X_{2n-1})Y) = 0.$$

Analogous arguments show $Tr(X_1\, a^-_{2n-2}) = 0$ and $a^-_{2n-2} \in H_n$. Again by the nondegeneracy of the trace bilinear form on H_n, we get $a^-_{2n-2}(X_2,\cdots,X_{2n-1};Y) = 0$.

□

In the rest of this section, we use the superalgebra method to give an alternate proof of above theorem, but only for the case n is odd. This supertrick which uses superalgebras to derive certain identities was first used, most probably, by Kemer in [12].

Let $\Xi = \{\xi_0,\xi_1,\cdots\}$ and $F_E[\Xi] = F[\Xi]/I$, where I is the ideal generated by all $\xi_i\xi_j + \xi_j\xi_i$ and ξ_i^2. We call $F_E[\Xi]$ the *exterior* or *Grassmann* algebra, and write also ξ_i for the image of ξ_i in $F_E[\Xi]$. Write $B = \{\xi_i | 0 \leq i < \infty\}$ and $B' = \{\xi_{i_1}\cdots\xi_{i_u} | 0 \leq u < \infty,\, i_1 < \cdots < i_u\}$. For the empty product $u = 0$ we write "1". Each element of $F_E[\Xi]$ can be written uniquely in the form $\sum \alpha_b b; \alpha_b \in F, b \in B'$, where all but a finite number of the α_b are 0. If $b = \xi_{i_1}\cdots\xi_{i_u}$, we write $deg(b) = u$. Let $F_E[\Xi]_0$

(resp. $F_E[\Xi]_1$) be the F−subspace of $F_E[\Xi]$ spanned by all monomials of even (resp. odd) degree, then $F_E[\Xi] = F_E[\Xi]_0 \oplus F_E[\Xi]_1$.

A *superalgebra* is a Z_2-graded algebra $A = A_0 + A_1$, $A_iA_j \subseteq A_{i+j}$ (modulo 2). An example of such an object is the Grassmann algebra $F_E[\Xi]$.

Let $X = x_1 \otimes \xi_1 + x_2 \otimes \xi_2 + \cdots + x_m \otimes \xi_m$. In this way we arrive at a superalgebra of the form $M_n \otimes F_E[\Xi]$. For every integer $k \geq 1$,

$$X^k = \sum_{i_1 < i_2 < \cdots < i_k} S_k(x_{i_1}, \cdots, x_{i_k}) \otimes \xi_{i_1} \cdots \xi_{i_k}.$$

In particular, $X^k = 0$ for $k > m$ and

$$X^m = S_m(x_1, \cdots, x_m) \otimes \xi_1 \cdots \xi_m, \tag{2.1.7}$$

which implies that X^m vanishes on M_n (respectively K_n) when $m \geq 2n$ (respectively $m \geq 2n-2$) by using Amitsur-Levitzki's theorem (respectively Kostant's theorem).

If we let $Y = y \otimes \xi_0 \in M_n(F) \otimes F_E[\Xi]$, then

$$X^kYX^{m-k} = (-1)^k a_m(k+1)(x_1, \cdots, x_m; y) \otimes \xi_0\xi_1 \cdots \xi_m; \tag{2.1.8}$$

or if we choose $Y = y \otimes 1 \in M_n(F) \otimes F_E[\Xi]$, then

$$X^kYX^{m-k} = a_m(k+1)(x_1, \cdots, x_m; y) \otimes \xi_1 \cdots \xi_m. \tag{2.1.9}$$

Now, we give another proof of theorem 2.1.3 in case n is odd.

Proof. Assuming n is odd and $m \geq 2n-2$, let $X = x_1 \otimes \xi_1 + x_2 \otimes \xi_2 + \cdots + x_m \otimes \xi_m \in K_n(F) \otimes F_E[\Xi]$ and $Y = y \otimes \xi_0 \in K_n(F) \otimes F_E[\Xi]$.

$$\begin{aligned}(X+Y)^{m+1} &= (y \otimes \xi_0 + x_1 \otimes \xi_1 + \cdots + x_m \otimes \xi_m)^{m+1} \\ &= S_{m+1}(y, x_1, \cdots, x_m) \otimes \xi_0\xi_1 \cdots \xi_m.\end{aligned}$$

By Kostant's theorem, we have $S_{m+1}(y, x_1, \cdots, x_m) = 0$, so $(X+Y)^{m+1} = 0$.

Expanding

$$(X+Y)^{m+1} = X^{m+1} + (YX^m + XYX^{m-1} + \cdots + X^{m-1}YX + X^mY) + \cdots + Y^{m+1},$$

since, over a field with enough elements, a homogenous component of an identity is also an identity and $X^m = 0$, we get

$$XYX^{m-1} + \cdots + X^{m-1}YX = 0. \tag{2.1.10}$$

Next let $Z = z_1 \otimes \xi_0\xi_1 + \cdots + z_m \otimes \xi_0\xi_m$. Then

$$\begin{aligned} ZX^{m-1} &= \sum_{i=1}^{m} a_{m-1}(1)(x_1, \cdots, x_{i-1}, \hat{x}_i, \cdots, x_m; z_i) \otimes \xi_0\xi_1 \cdots \xi_m, \\ XZX^{m-2} &= \sum_{i=1}^{m} a_{m-1}(2)(x_1, \cdots, x_{i-1}, \hat{x}_i, \cdots, x_m; z_i) \otimes \xi_0\xi_1 \cdots \xi_m, \\ &\vdots \qquad\qquad \vdots \\ X^{m-1}Z &= \sum_{i=1}^{m} a_{m-1}(m)(x_1, \cdots, x_{i-1}, \hat{x}_i, \cdots, x_m; z_i) \otimes \xi_0\xi_1 \cdots \xi_m. \end{aligned}$$

Therefore,

$$\begin{aligned} ZX^{m-1} &- XZX^{m-2} + \cdots + (-1)^{m-1}X^{m-1}Z \\ &= \sum_{i=1}^{m} S_m(z_i, x_1, \cdots, \hat{x}_i, \cdots, x_m) \otimes \xi_0\xi_1 \cdots \xi_m. \end{aligned} \tag{2.1.11}$$

Now, assume $Z = [Y, X] = \sum_{i=1}^{m} (y \circ x_i) \otimes \xi_0\xi_i$. Since $x_i, y \in K_n$, $z_i = y \circ x_i \in H_n$; by (2.1.11) and Rowen's result, we have

$$ZX^{m-1} - XZX^{m-2} + \cdots + (-1)^{m-1}X^{m-1}Z = 0 \tag{2.1.12}$$

when $z_i \in H_n$ and $x_i \in K_n$, $1 \le i \le m$.

Substituting $Z = YX - XY$ in equation (2.1.12), we have

$$YX^m - 2XYX^{m-1} + 2X^2YX^{m-2} - \cdots + (-1)^m X^mY = 0, \tag{2.1.13}$$

or

$$XYX^{m-1} - X^2YX^{m-2} + \cdots + (-1)^{m-1}X^{m-1}YX = 0, \tag{2.1.14}$$

since $char(F) \neq 2$ and $X^m = 0$. Summing equation (2.1.10) and equation (2.1.14), we get

$$XYX^{m-1} + X^3YX^{m-3} + \cdots = 0,$$

i.e.,

$$a_m^+ = a_m(2) + a_m(4) + \cdots = 0,$$

by equation (2.1.8). □

Furthermore, if we make certain modifications in the above proof, we can prove that theorem 2.1.3 holds for arbitrary y's. Since the characteristic is not 2 we only show that Theorem 2.1.3 holds with $y \in H_n$.

If we assume $m \geq 2n-2$ and let $X = x_1 \otimes \xi_1 + x_2 \otimes \xi_2 + \cdots + x_m \otimes \xi_m \in K_n(F) \otimes F_E[\Xi]$ and $Y = y \otimes 1 \in H_n(F) \otimes F_E[\Xi]$, then by Rowen's theorem,

$$\begin{aligned} YX^m - XYX^{m-1} + \cdots + (-1)^{m-1}X^mY \\ = S_{m+1}(y, x_1, \cdots, x_m) \otimes \xi_1 \cdots \xi_m = 0, \end{aligned} \tag{2.1.15}$$

since $m + 1 \geq 2n - 1$.

For any $Z = z_1 \otimes \xi_1 + \cdots + z_m \otimes \xi_m \in K_n \otimes F_E[\Xi]$, we have

$$\begin{aligned} (X+Z)^m &= ((x_1+z_1) \otimes \xi_1 + \cdots + (x_m + z_m) \otimes \xi_m)^m \\ &= S_m(x_1 + z_1, \cdots, x_m + z_m) \otimes \xi_1 \cdots \xi_m \\ &= 0. \end{aligned}$$

On the other hand, we expand

$$(X+Z)^m = X^m + (X^{m-1}Z + X^{m-2}ZX + \cdots + ZX^{m-1}) + \cdots + Z^m.$$

Since any homogeneous component of degree d of a polynomial identity is also an identity, when the field has more than d elements, we have

$$X^{m-1}Z + X^{m-2}ZX + \cdots + ZX^{m-1} = 0. \tag{2.1.16}$$

Since $x_i \in K_n$ and $y \in H_n$, $x_i \circ y \in K_n$ and we may substitute $Z = X \circ Y = x_1 \circ y \otimes \xi_1 + \cdots + x_m \circ y \otimes \xi_m$ in equation (2.1.16) to obtain

$$XYX^{m-1} + X^2YX^{m-2} + \cdots + X^{m-1}YX = 0, \tag{2.1.17}$$

since $char(F) \neq 2$. Summing equation (2.1.15) and equation (2.1.17), we get

$$XYX_{m-1} + X^3YX^{m-3} + \cdots = 0$$

i.e.,

$$a_m^+ = a_m(2) + a_m(4) + \cdots = 0.$$

This completes the proof of the following theorem.

Theorem 2.1.4 *The almost standard polynomial*

$$f(x_1, \cdots, x_m; y) = \alpha a_m^+ + \beta a_m^- + \sum_{\substack{k \geq 2n-1 \\ \text{or } k \leq m-2n+4}} \gamma_k a_m(k),$$

where $m \geq 2n-2$, vanishes for $x_1, \cdots, x_m \in K_n$ and $y \in M_n$ for all n and arbitrary $\alpha, \beta, \gamma_k \in F$.

2.2 Sharpness

Naturally one may ask the following two questions about theorem 2.1.3:

1. Is $2n-2$ the best result ? In other words, is $f_{2n-3}(x_1, \cdots, x_{2n-3}; y) = \alpha a_{2n-3}^+ + \beta a_{2n-3}^-$ (the third term in eqation (2.1.1) appears only if $m \geq 2n-1$) always an identity of K_n for arbitrary α and $\beta \in F$ too?

2. If $m \geq 2n-2$ and an almost standard polynomial $f(x_1, \cdots, x_m; y)$ is a weak identity of $K_n(F)$, does f have to be a linear combination of a_m^+, a_m^- and $a_m(k)$ with $k \geq 2n-1$ or $k \leq m-2n+2$?

In this section, we will discuss these two problems.

As a subspace of $M_n(F)$, $K_n(F)$ has a basis B_n^- of matrices $e_{ij} - e_{ji}$, $1 \leq i < j \leq n$, where e_{ij} is the matrix whose only nonzero entry is 1 in the (i,j)-position. Let $W = \{Y_1, \cdots, Y_t\}$, $t \geq 2$, be a subset of B_n^-. There is a directed graph G_W associated with W, constructed as follows: The vertices of G_W are the indices $1, \cdots, n$ and each element $Y_k = e_{i_k j_k} - e_{j_k i_k}$ of W is represented by an edge $\hat{Y}_k$ with an arrow, where $\{i_k, j_k\} = \hat{Y}_k$ and the arrowhead points to j_k. If $< \hat{Y}_{\pi 1}, \cdots, \hat{Y}_{\pi t} >$ is a sequence of edges in the graph G_W, π a permutation of $(1, \cdots, t)$, such that one may order the endpoints i_r and j_r of $\hat{Y}_{\pi r}$ in a way so that $i_2 = j_1, i_3 = j_2, \cdots, i_t = j_{t-1}$, then we call $\{< \hat{Y}_{\pi 1}, \cdots, \hat{Y}_{\pi t} >, i_1, j_t\}$ a path of length t with initial vertex i_1 and terminal vertex j_t. The *degree of a vertex* of G_W is the number of edges (in G_W) incident with it. It is easy to verify that $Y_{\pi 1} \cdots Y_{\pi t} = (-1)^l e_{i_1 j_t}$, where l is the number of edges which are traversed in the direction against the arrow when you go through all t edges from i_1 to j_t and we call $(-1)^l$ the *orientation coefficient* of this path.

DEFINITION. A *pseudo-Eulerian path* on a directed graph G is an undirected path which traverses every edge of G once and exactly once and which may travel in either direction on an edge e of G, regardless of the orientation of e. If there exists a pseudo-Eulerian path on G, then G is called *pseudo-Eulerian graph.* If the pseudo-Eulerian path forms a circuit, then it is an *Eulerian circuit.*

Since all intermediate vertices in a path are in pairs, only the end vertices can have odd degree. It follows that a pseudo-Eulerian graph always has exactly only zero or two vertices which have odd degree. Let $f(x_1, x_2, \cdots, x_t)$ be a multilinear polynomial and $W = \{Y_1, \cdots, Y_t\} \subset B_n^-$ then $f(Y_1, \cdots, Y_t) = \alpha e_{ij} + \beta e_{ji}$, if there are exactly two vertices i and j in G_W that have odd degree, or $f(Y_1, \cdots, Y_t) = \sum_{i=1}^{n} d_i e_{ii}$, for suitable $d_i \in Z$ if each vertex of G_W has even degree, or $f(Y_1, \cdots, Y_t) = 0$ if G_W is not a pseudo-Eulerian graph.

DEFINITION. The *sign of a pseudo-Eulerian path* P on a directed graph with labelled edges is the number $(-1)^\sigma(-1)^l$ where $(-1)^\sigma$ is the sign of the associated permutation of the edges of the graph and $(-1)^l$ is the orientation coefficient of the path. We denote it by $\xi(P)$.

For example, the sign of the path $\{< \hat{Y}_{\pi 1}, \cdots, \hat{Y}_{\pi t} >, i_1, j_t\}$ is $(-1)^\pi(-1)^l$.

A *positive* (respectively *negative*) pseudo-Eulerian path is a pseudo-Eulerian path whose sign is $+1$ (respectively -1).

We will show by example that $2n-2$ is the best result in theorem 2.1.3, i.e., for $n \geq 2$, $f_{2n-3}(x_1, \cdots, x_{2n-3}; y) = \alpha a_{2n-3}^+ + \beta a_{2n-3}^-$ is not always an identity of K_n for arbitrary α and β.

Example. Let $y = x_1 = e_{12} - e_{21}$, for $1 \leq r \leq n-2$ let $x_{2r} = e_{1,r+2} - e_{r+2,1}$ and $x_{2r+1} = e_{r+2,2} - e_{2,r+2}$. Then $y, x_1, \cdots, x_{2n-3} \in K_n$. In this example we deal first with case n odd and $n \geq 3$.

$$a_{2n-3}^+ = c_n(e_{12} - e_{21}),$$

where $c_n \in F$ can be calculated recursively:

$$c_3 = -3, \; c_n = -(n-2)(n-3)c_{n-2} + (-1)^{(n-1)/2}(2n-3)((n-2)!).$$

Proof. We prove this result by induction on n.

1) For $n = 3$, we have

$$a_3^+ = a_3(2) + a_3(4).$$

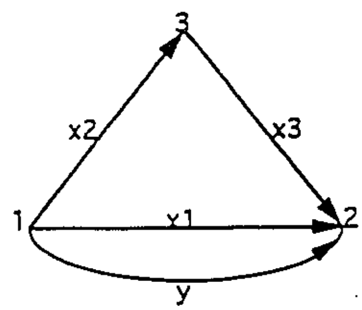
The substitution:

$$x_1 = e_{12} - e_{21} \quad , \quad x_2 = e_{13} - e_{31},$$

$$x_3 = e_{32} - e_{23} \quad , \quad y = e_{12} - e_{21};$$

gives the following diagram

Figure 1



From Figure 1 we can easily see that there are only two non-zero terms in $a_3(2)$:

$$\begin{aligned} x_1 y x_2 x_3 &= (e_{12} - e_{21})(e_{12} - e_{21})(e_{13} - e_{31})(e_{32} - e_{23}) \\ &= (-e_{11} - e_{22})e_{12} \\ &= -e_{12} \end{aligned}$$

and

$$\begin{aligned} -x_1 y x_3 x_2 &= -(e_{12} - e_{21})(e_{12} - e_{21})(e_{32} - e_{23})(e_{13} - e_{31}) \\ &= -(-e_{11} - e_{22})e_{21} \\ &= e_{21}. \end{aligned}$$

In $a_3(4)$, four terms are not zero:

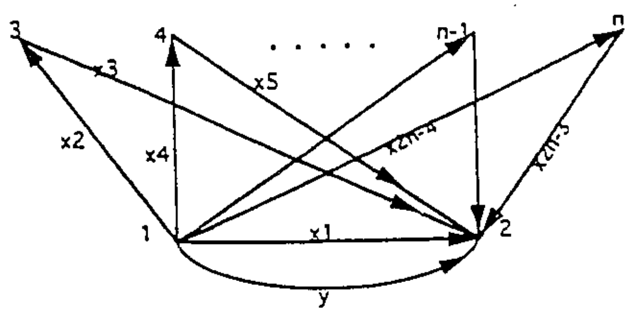
$$-x_1 x_3 x_2 y = -e_{12} \quad , \quad -x_3 x_2 x_1 y = e_{21},$$

$$x_2 x_3 x_1 y = -e_{12} \quad , \quad x_1 x_2 x_3 y = e_{21}.$$

So, $a_3^+ = -3(e_{12} - e_{21})$.

2) Assume the result is true for $n-2$. Considering the corresponding graph which has n vertices and $2n-2$ edges,

Figure 2



we get

$$\begin{aligned} a_{2n-3}(2) &= -\sum_{i\neq j} a_{2n-7}(2)(x_1, x_2, \cdots, \hat{x}_{2i}, \hat{x}_{2i+1}, \cdots, \\ &\qquad \hat{x}_{2j}, \hat{x}_{2j+1}, \cdots, x_{2n-4}, x_{2n-3}; y) \\ &\qquad \times(x_{2i+1}x_{2i}x_{2j}x_{2j+1} + x_{2j+1}x_{2j}x_{2i}x_{2i+1} \\ &\qquad +x_{2i}x_{2i+1}x_{2j+1}x_{2j} + x_{2j}x_{2j+1}x_{2i}x_{2i+1}) \\ &= -(n-2)\,(n-3)\,a_{2(n-2)-3}(2)(e_{11}+e_{22}), \end{aligned}$$

since the first two terms in the brackets are e_{22} and the other two terms are e_{11}.

By the same observation, we have

$$a_{2n-3}(2k) = -(n-2)\,(n-3)\,a_{2(n-2)-3}(2k)\,(e_{11}+e_{22})$$

for each $k \leq n-3$.

Since $\sum_{k=1}^{n-3} a_{2n-7}(2k) = a_{2n-7}^+ = c_{n-2}(e_{12}-e_{21})$, the following equality follows:

$$\begin{aligned} a_{2n-3}(2) + \cdots + a_{2n-3}(2n-6) &= -(n-2)(n-3)c_{n-2}(e_{12}-e_{21})(e_{11}+e_{22}) \\ &= -(n-2)(n-3)c_{n-2}(e_{12}-e_{21}). \end{aligned}$$

It remains to compute $a_{2n-3}(2n-4)$ and $a_{2n-3}(2n-2)$.

We notice that the general non-zero terms in $a_{2n-3}(2n-4)$ are

$$\pm x_{2\sigma(1)}x_{2\sigma(1)+1}x_{2\sigma(2)+1}x_{2\sigma(2)}\cdots x_1\cdots x_{2\sigma(n-3)}x_{2\sigma(n-3)+1}yx_{2\sigma(n-2)}x_{2\sigma(n-2)+1}$$

and

$$\mp x_{2\sigma(1)+1}x_{2\sigma(1)}x_{2\sigma(2)}x_{2\sigma(2)+1}\cdots x_1\cdots x_{2\sigma(n-3)+1}x_{2\sigma(n-3)}yx_{2\sigma(n-2)+1}x_{2\sigma(n-2)},$$

where x_1 is in the $(2k+1)th$ position and $k=0,1,\cdots,n-3$. Each term of the first type yields $(-1)^{(n-1)/2}e_{12}$ and each term of the second type yields $(-1)^{(n-3)/2}e_{21}$. So, $a_{2n-3}(2n-4)=(-1)^{(n-1)/2}(n-2)((n-2)!)(e_{12}-e_{21})$.

In $a_{2n-3}(2n-2)$, the general non-zero terms are

$$\pm x_{2\sigma(1)}x_{2\sigma(1)+1}x_{2\sigma(2)+1}x_{2\sigma(2)}\cdots x_1\cdots x_{2\sigma(n-2)+1}x_{2\sigma(n-2)}y$$

and

$$\mp x_{2\sigma(1)+1}x_{2\sigma(1)}x_{2\sigma(2)}x_{2\sigma(2)+1}\cdots x_1\cdots x_{2\sigma(n-2)}x_{2\sigma(n-2)+1}y,$$

where x_1 is in the $(2k+1)th$ position and $k=0,1,\cdots,n-3$. Each term of the first type yields $(-1)^{(n-1)/2}e_{12}$ and each term of the second type yields $(-1)^{(n-3)/2}e_{21}$. So, $a_{2n-3}(2n-2)=(-1)^{(n-1)/2}(n-1)!(e_{12}-e_{21})$. Hence, we have

$$\begin{aligned} a^+_{2n-3} &= (a_{2n-3}(2)+\cdots+a_{2n-3}(2n-6))+a_{2n-3}(2n-4)+a_{2n-3}(2n-2) \\ &= (-(n-2)(n-3)c_{n-2}+(-1)^{n-2/2}(2n-3)(n-2)!)(e_{12}-e_{21}). \end{aligned}$$

i.e. $c_n=-(n-2)(n-3)c_{n-2}+(-1)^{(n-1)/2}(2n-3)((n-2)!)$. □

Remarks:

1) Since $c_3=-3$, the two terms in c_n always have the same sign $(-1)^{(n-1)/2}$, which means that a^+_{2n-3} is not a weak identity of K_n, for n odd and $n\geq 3$.

2) For n even, if we assume that a^+_{2n-3} is a weak identity of K_n, then $Tr(a^+_{2n-3}) = 0 = (n-1)Tr(S_{2n-3}(X_1, \cdots, X_{2n-3})Y)$, by Lemma 2.1.2.1). But $S_{2n-3}(X_1, \cdots, X_{2n-3})$ is skew when n is even. By the nondegeneracy of the trace bilinear form on K_n, S_{2n-3} is an identity of K_n, which is impossible.

3) Since $S_{2n-2}(y, x_1, \cdots, x_{2n-3}) = (a^-_{2n-3} - a^+_{2n-3})(x_1, \cdots, x_{2n-3}; y)$ is an identity of K_n, the above calculation also gives us that $a^-_{2n-3} = c_n(e_{12} - e_{21})$, if we consider the same substitutions and the same constraints on n as above.

Now, we discuss the second question which is a much harder one.

In the case $n \leq 3$, the dimension of K_n is less than $2n-2$, which implies that every $a_m(k) = 0$, where $m \geq 2n-2$ and $1 \leq k \leq m+1$.

In the case $n \geq 4$ and $m \geq 4n-7$, we also have $a_m(k) = 0$ for each k. Since if $k-1 \geq 2n-2$ or $4n-7-k+1 \geq 2n-2$, then $a_m(k) = 0$, otherwise, it must be either $a_{4n-7}(2n-2)$ or $a_{4n-7}(2n-3)$.

For X and Y are the same as in the equation (2.1.9),

$$
\begin{aligned}
&X^{2n-3}YX^{2n-4} \\
&\quad = -a_{4n-7}(2n-2) \otimes \xi_1 \cdots \xi_{4n-7} \\
&\quad = -a^+_{4n-7} \otimes \xi_1 \cdots \xi_{4n-7} - \sum_{\substack{t \geq 2n-2 \\ \text{or } 4n-5-t \geq 2n}} X^t Y X^{4n-7-t} \\
&\quad = 0,
\end{aligned}
$$

for $m = 4n-7$. Similarly, $a_{4n-7}(2n-3) = 0$

So, in the two cases above, the theorem is not sharp in this sense. If we assume $2n-2 \leq m \leq 4n-8$ and $n \geq 4$ and we let

$$
g = \sum_{k=m-2n+4}^{2n-2} \alpha_k a_m(k)
$$

and

$$f = g + \sum_{\substack{k-1\geq 2n-2 \\ \text{or } m+1-k\geq 2n-2}} \beta_k a_m(k).$$

Using the same notation of X and Y as in equation(2.1.9), we may write g as

$$g = \cdots + \alpha_k X^{k-1} Y X^{m-k+1} + \alpha_{k+1} X^k Y X^{m-k} + \alpha_{k+2} X^{k+1} Y X^{m-k-1} + \cdots$$

where $k+1 \leq 2n-3$ and $m-k+1 \leq 2n-3$; here for notational simplicity, we replace $g \otimes \xi_1 \cdots \xi_{4n-7}$ simply by g.

We assume that f is a weak identity of K_n. As mentioned in the beginning of the proof of Theorem 2.1.3, $\sum_{\substack{k-1\geq 2n-2 \\ \text{or } m+1-k\geq 2n-2}} \beta_k a_m(k)$ is a weak identity of K_n, so is g.

In order to prove that $\alpha_k = \alpha_{k+2}$ for every k, we multiply g by $X^{2n-5-(k-1)}$ and $X^{2n-3-(m-k+1)}$ on the left and right respectively:

$$\begin{aligned} X^{2n-5-(k-1)} g X^{2n-3-(m-k+1)} &= \sum_{4n-8-i\geq 2n-2} X^i Y X^{4n-8-i} + \sum_{j\geq 2n-2} X^j Y X^{4n-8-j} \\ &+ \alpha_k X^{2n-5} Y X^{2n-3} + \alpha_{k+1} X^{2n-4} Y X^{2n-4} \\ &+ \alpha_{k+2} X^{2n-3} Y X^{2n-5}. \end{aligned}$$

Obviously, the two summations on the right-hand side of above equation are zero. Moreover,

$$X^{2n-4} Y X^{2n-4} = a^-_{4n-8} \otimes \xi_1 \cdots \xi_{4n-8} - X^{2n-2} Y X^{2n-6} - \cdots$$

and

$$X^{2n-5} Y X^{2n-3} + X^{2n-3} Y X^{2n-5} = -a^+_{4n-8} \otimes \xi_1 \cdots \xi_{4n-8} - X^{2n-1} Y X^{2n-7} - \cdots$$

vanish on K_n. Since g is an identity,

$$(\alpha_k - \alpha_{k+2}) X^{2n-5} Y X^{2n-3} = 0.$$

If we can prove that $X^{2n-5}YX^{2n-3}$ is not an identity of $K_n(F)$, then $\alpha_k = \alpha_{k+2}$ for every k, in another word, the result is sharp when $X^{2n-5}YX^{2n-3}$ is not an identity of $K_n(F)$. So, the sharpness is equivalent to whether $X^{2n-5}YX^{2n-3}$ is an identity of $K_n(F)$ or not. When $n \leq 6$, $4n-8$ is greater than the dimension of the space $K_n(F)$ which implies that $X^{2n-5}YX^{2n-3}$ is an identity of $K_n(F)$ for such n's.

When $n \geq 7$, one could conjecture that $X^{2n-5}YX^{2n-3}$ is not an identity of $K_n(F)$; i.e., an almost standard polynomial f of degree m is a weak identity of $K_n(F)$ if and if f is the form of equation (2.1.1). Unfortunately, we shall give a positive answer to this question by a computational proof only for the case $n = 7$. As will be seen, the quantity of computation is very large even for the case $n = 7$.

Assuming $n = 7$, the polynomial we will consider is X^9YX^{11} or

$$a_{20}(10) = \sum_{\sigma \in S_{20}} x_{\sigma(1)} \cdots x_{\sigma(9)} y x_{\sigma(10)} \cdots x_{\sigma(20)}.$$

In order to show that $a_{20}(10)$ is not an identity of K_7, we consider the following substitution:

$$x_1 = e_{12} - e_{21}, \quad x_2 = e_{14} - e_{41}, \quad x_3 = e_{15} - e_{51},$$

$$x_4 = e_{16} - e_{61}, \quad x_5 = e_{17} - e_{71}. \quad x_6 = e_{23} - e_{32},$$

$$x_7 = e_{24} - e_{42}, \quad x_8 = e_{25} - e_{52}, \quad x_9 = e_{26} - e_{62},$$

$$x_{10} = e_{27} - e_{72}, \quad x_{11} = e_{34} - e_{43}, \quad x_{12} = e_{35} - e_{53},$$

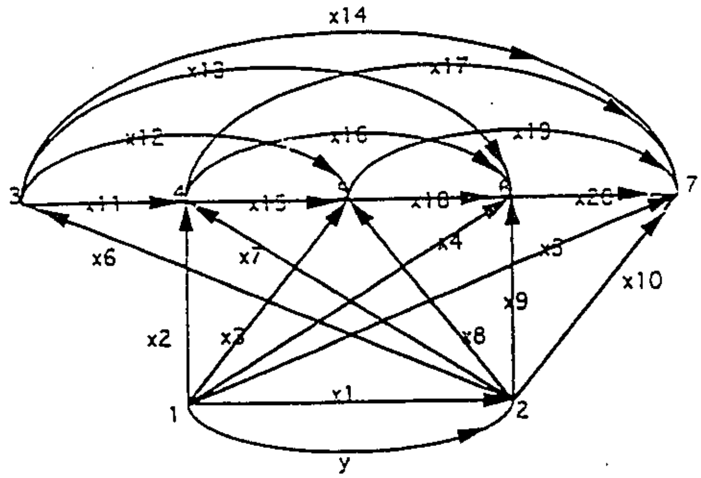
$$x_{13} = e_{36} - e_{63}, \quad x_{14} = e_{37} - e_{73}, \quad x_{15} = e_{45} - e_{54},$$

$$x_{16} = e_{46} - e_{64}, \quad x_{17} = e_{47} - e_{74}, \quad x_{18} = e_{56} - e_{65},$$

$$x_{19} = e_{57} - e_{75}, \quad x_{20} = e_{67} - e_{76}, \quad y = e_{12} - e_{21},$$

which yields the diagram of Figure 3.

Figure 3



Actually, since $a_{20}(10)$ is multilinear, we may choose $\{x_1, \cdots, x_{20}\}$ to be a subset of the basis $B_7^- = \{e_{ij} - e_{ji} | 1 \leq i < j \leq 7\}$; since $a_{20}(10)$ is alternating in all x_i, all the x_i's we choose must be distinct for $a_{20}(10)$ to be non-zero. So only one element in B_7^- is left and we assume that it is $e_{13} - e_{31}$. In order to have a non-zero substitution, we have to choose $y \in B_7^-$ such that the associated graph G_7 is a pseudo-Eulerian graph, i.e., G_7 must have only zero or two vertices which have odd degree. On the other hand, $a_{20}(10)(X_1, \cdots, X_{20}; Y)$ is skew symmetric for any X_i, Y in B_7^-, which tells us that G_7 has to have exactly two vertices which have odd degree. Hence, $y = e_{1i} - e_{i1}$ or $y = e_{3i} - e_{i3}$ where i can be $2, 4, 5, 6, 7$. No matter how y is chosen, we can always relabel the vertices of the graph G_7 to get above substitution. Therefore, this is the only non-zero substitution we could possibly have.

Now we consider the graph G_7 in which only vertices 3 and 2 have odd degree. since $a_{20}(10)$ is skew symmetric, $a_{20}(10)(x_1, \cdots, x_{20}; y) = d(e_{32} - e_{23})$ for some $d \in \mathbf{Z}$. By the discussion at the beginning of this section, we know that this d is the sum of the signs of all pseudo-Eulerian paths on G_7 whose initial vertex is 3, terminal vertex is 2 and tenth edge is $\hat{y}$. In this section, the value of $a_{20}(10)$ will be determined by counting the number of pseudo-Eulerian paths from 3 to 2 whose tenth edge goes either from 1 to 2 or from 2 to 1 which is called a *valid pseudo-Eulerian path* in G_7 in the following discussion, and by determining if each path contributes a positive or a negative value to the total sum.

We may denote a path by a sequence $p = \{p_1 p_2 \cdots p_{22}\}$ of vertices where $p_i = k$ and $p_{i+1} = l$ if the ith edge of this path is $e_{kl} - e_{lk}$ for any $1 \leq i \leq 22$. The sign of this path can be calculated as the product of the sign of the corresponding permutation and $\prod_{i=1}^{21} sign(p_{i+1} - p_i)$, since in this substitution we always choose $x_k = e_{i_k,j_k} - e_{j_k,i_k}$ such that $i_k < j_k$.

For example, the path

$$x_6 x_1 x_2 x_7 x_8 x_3 x_4 x_{20} x_5 y x_9 x_{13} x_{12} x_{15} x_{11} x_{14} x_{19} x_{18} x_{16} x_{17} x_{10}$$

in G_7 can be represented by

$$3\ 2\ 1\ 4\ 2\ 5\ 1\ 6\ 7\ \underline{1\ 2}\ 6\ 3\ 5\ 4\ 3\ 7\ 5\ 6\ 4\ 7\ 2.$$

The sign of this path is the product of the sign of the permutation

$$(1\ 6\ 3\ 2)(4\ 7)(5\ 8\ 20\ 10\ 9)(11\ 13\ 15\ 14)(12)(16\ 19\ 17\ 18)$$

and $(-1)^{11}$ which is $+1$.

We say that two vertices in G_7 are *symmetric* if after we exchange these two vertices in the graph the summation of signs of all valid pseudo-Eulerian paths is invariant.

Lemma 2.2.1 *In the graph G_7, vertices 4, 5, 6 and 7 are symmetric.*

Proof. We shall only show the symmetry of 4 and 5 since the proofs of symmetry of all other pairs of vertices are exactly the same.

Using the above notation, we get that every pseudo-Eulerian path in G_7 from 3 to 2 whose tenth edge is from 1 to 2 has the form

$$3 \cdots 4 \cdots 5 \cdots \underline{1\,2} \cdots 4 \cdots 5 \cdots 4 \cdots 5 \cdots 2$$

or

$$3 \cdots 4 \cdots 5 \cdots 4 \cdots 5 \cdots \underline{1\,2} \cdots 4 \cdots 5 \cdots 2$$

since the degree of vertex 4 or 5 is 6.

We notice that every vertex which is adjacent to 5 is adjacent to 4, so if we replace 4 by 5 and replace 5 by 4 in either the above cases we still get a valid pseudo-Eulerian path in G_7. Therefore it is sufficient to prove that the sign of the paths remain the same.

It also should be noticed first that either $\{4,5\}$ or $\{5,4\}$ must be an edge in every valid path.

The first change to be considered is the signatures of the corresponding permutations. Since this operation is equivalent to the operation of five edge exchanges which are exchanging $\{4,i\}$ (or $\{i,4\}$) and $\{5,i\}$ (or $\{i,5\}$) where $i = 1,2,3,6,7$, the signature part is changed after this operation.

Secondly, we have to consider the sign of the edge which is traversed. By the discussion preceding this lemma, the change of this part can be calculated by considering $\prod_{i=1}^{21} sign(p_{i+1} - p_i)$. All of $sign(p_{i+1} - p_i)$ are unchanged except the one with $p_{i+1} = 4$ and $p_i = 5$ (or $p_{i+1} = 5$ and $p_i = 4$). So this part is also changed.

Therefore, the sign of every valid pseudo-Eulerian path in G_7 is invariant under this operation, which means that 4 and 5 are symmetric. □

Let P be a valid pseudo-Eulerian path in G_7, then P contains three parts. The first part P_1 from the initial vertex of P, namely 3, to the initial vertex of y, 1 or 2, whose length is 9; the second part consists of only one edge y which is either from 1 to 2 or from 2 to 1 depending on whether the P_1 ends at 1 or 2 and the third part P_2 is the remaining part which has length 11.

If we assume that P_2 only traverses 6 or fewer vertices of G_7, the skew symmetric matrices corresponding to the edges of P_2 can be considered as matrices in K_6. Since we know that $S_{10}(x_1, \cdots, x_{10})$ is an identity of K_6, the sum of the signs of all valid pseudo-Eulerian paths of G_7, in which P_2 only traverses 6 or fewer vertices, is equal to 0. A similar discussion would lead to a similar conclusion about P_1.

Lemma 2.2.2 *The following hold for G_7:*

1. *The summation of the signs of all valid pseudo-Eulerian paths, in which P_2 only traverses 6 or fewer vertices, is equal to 0.*

2. *The summation of the signs of all valid pseudo-Eulerian paths, in which P_1 only traverses 5 or fewer vertices, is equal to 0.*

By Lemma 2.2.1, the summation of the signs of the paths which start with 3 2 1 4 2 5 is the same as that of those which start with 3 2 1 4 2 6, 3 2 1 4 2 7, 3 2 1 5 2 4, 3 2 1 5 2 6, 3 2 1 5 2 7, 3 2 1 6 2 4, 3 2 1 6 2 5, 3 2 1 6 2 7, 3 2 1 7 2 4, and 3 2 1 7 2 5 and which start with 3 2 1 7 2 6, so we need only calculate one of these summations and multiply it by 12. In the following computation, we say that this summation is with *multiplicity* of 12. A similar discussion can be applied to all other paths in G_7, this step will help us to reduce the quantity of computation by a factor of 12 . So, we only compute the summations of the signs of the paths which start with

3 2 1 4 2 5 with multiplicity of 12,

3 2 1 4 3 5 with multiplicity of 12,

3 2 1 4 5 1 with multiplicity of 12,

3 2 1 4 5 2 with multiplicity of 12,

3 2 1 4 5 3 with multiplicity of 12,

3 2 1 4 5 6 with multiplicity of 24,

3 2 4 1 2 5 with multiplicity of 12,

3 2 4 1 5 2 with multiplicity of 12,

3 2 4 1 5 3 with multiplicity of 12,

3 2 4 1 5 4 with multiplicity of 12,

3 2 4 1 5 6 with multiplicity of 24,

3 2 4 3 5 with multiplicity of 12,

3 2 4 5 1 2 with multiplicity of 12,

3 2 4 5 1 4 with multiplicity of 12,

3 2 4 5 1 6 with multiplicity of 24,

3 2 4 5 2 1 with multiplicity of 12,

3 2 4 5 2 6 with multiplicity of 24,

3 2 4 5 3 4 with multiplicity of 12,

3 2 4 5 3 6 with multiplicity of 24,

3 2 4 5 6 with multiplicity of 24,

3 4 1 2 3 5 with multiplicity of 12,

3 4 1 2 4 5 with multiplicity of 12,

3 4 1 2 5 with multiplicity of 12,

3 4 1 5 with multiplicity of 12,

3 4 2 1 4 5 with multiplicity of 12,

3 4 2 1 5 with multiplicity of 12,

3 4 2 3 5 with multiplicity of 12,

3 4 2 5 with multiplicity of 12,

3 4 5 1 with multiplicity of 12,

3 4 5 2 with multiplicity of 12,

3 4 5 3 with multiplicity of 12,

3 4 5 6 with multiplicity of 24,

We notice the following fact that can be used to reduce the number of calculations: Let P be a valid pseudo-Eulerian path in G_7. If its first part P_1 (or its third part P_2) contains a cycle C of length k at vertex v_0, then we can get another valid pseudo-Eulerian path P' in G_7 by only changing the direction of traverse in the cycle C into the reverse direction. It is easy to see that $\xi(P) = (-1)^{(k+[k/2])}\xi(P')$, i.e.

$$\xi(P) = \begin{cases} \xi(P') & \text{if } k = 4l \text{ or } k = 4l+3 \\ -\xi(P') & \text{otherwise} \end{cases}$$

So, the summation of the paths which start with 3 2 4 1 2 5 is equal to the summation of the paths which start with 3 2 1 4 2 5 and the same hold for all the following pairs: 324512 and 321542, 324514 and 324154, 341235 and 321435, 342145 and 341245, 34235 and 32435.

Since the calculation is very lengthy, we used a computer (the program is given in Appendix) to get the following numbers:

Paths starting with	Number of paths	Multiplicity	Sum of signs
3 2 1 4 2 5	4928	24	−1200
3 2 1 4 3 5	8424	24	−1920
3 2 1 4 5 1	2104	12	0
3 2 1 4 5 2	4112	24	−600
3 2 1 4 5 3	5080	12	−960
3 2 1 4 5 6	7092	24	−1560
3 2 4 1 5 2	1584	12	0
3 2 4 1 5 3	5424	12	960
3 2 4 1 5 4	5424	24	−960
3 2 4 1 5 6	5840	24	−360
3 2 4 3 5	36872	24	−6960
3 2 4 5 1 6	6036	24	−240
3 2 4 5 2 1	5064	12	1200
3 2 4 5 2 6	4732	24	840
3 2 4 5 3 4	6824	12	−1680
3 2 4 5 3 6	7744	24	−120
3 2 4 5 6	26636	24	−480
3 4 1 2 4 5	8424	24	−1920

Paths starting with	Number of paths	Multiplicity	Sum of signs
3 4 1 2 5	25472	12	3120
3 4 1 5	166736	12	36960
3 4 2 1 5	26720	12	−1080
3 4 2 5	126896	12	6120
3 4 5 1	129056	12	1680
3 4 5 2	122704	12	−1680
3 4 5 3	140544	12	−38880
3 4 5 6	167952	24	−840

Therefore, the summation of all valid pseudo-Eulerian paths (with their signatures) in G_7 is -322560.

The above calculation shows that the theorem is sharp in the case $n = 7$ and $2n - 2 \leq m \leq 4n - 8$. For the case $n \geq 8$, the problem is still open.

2.3 Lie identities of K_n

We have given the definitions of a Lie identity of a Lie algebra and of the Lie standard polynomial $LS_m(x_1, \cdots, x_m; y)$ in chapter 1. Following the approach of Smith in [20], it is not hard to find the relationship between LS_m and the almost standard polynomials $a_m(k)$.

First, we set $q(0,0) = 1$, $q(n,k) = 0$ for all $n < 0$ and $q(0,k) = 0$ for all $k \neq 0$. Then we recursively define

$$q(n+1,k) = q(n,k-1) - (-1)^n q(n,k) \text{ for all } n > 0.$$

The entries $q(n,k)$ thus obtained may be displayed in the usual fashion:

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & -1 & & 1 & & & & \\
 & & & -1 & & 0 & & 1 & & & \\
 & & 1 & & -1 & & -1 & & 1 & & \\
 & 1 & & 0 & & -2 & & 0 & & 1 & \\
-1 & & 1 & & 2 & & -2 & & -1 & & 1
\end{array}$$

$\cdots$etc.

The above pseudo-Pascal triangle yields the coefficients in the following theorem:

Theorem 2.3.1 $LS_m(x_1,\cdots,x_m;y)=\sum_{k=1}^{m+1} q(m,k-1)a_m(k)(x_1,\cdots,x_m;y)$

Proof. We proceed by induction.

$$\text{For } m=1,\ LS_1=[x_1,y]=-a_1(1)+a_1(2).$$

Suppose the result holds for m.

Since we have equation (2.1.2) and equation (2.1.3), by the induction hypothesis,

$$\begin{aligned}
LS_{m+1} &= \sum_{j=1}^{m+1}(-1)^{j+1}adx_j LS_m((x_1,\cdots,\hat{x}_j,\cdots,x_{m+1};y) \\
&= \sum_{j=1}^{m+1}(-1)^{j+1}adx_j \sum_{k=1}^{m+1} q(m,k-1)a_m(k)(x_1,\cdots,\hat{x}_j,\cdots,x_{m+1};y) \\
&= \sum_{k=1}^{m+1} q(m,k-1)\sum_{j=1}^{m+1}(-1)^{j+1}x_j a_m(k)(x_1,\cdots,\hat{x}_j,\cdots,x_{m+1};y) \\
&\quad -\sum_{k=1}^{m+1} q(m,k-1)\sum_{j=1}^{m+1}(-1)^{j+1}a_m(k)(x_1,\cdots,\hat{x}_j,\cdots,x_{m+1};y)x_j
\end{aligned}$$

$$\begin{aligned} &= \sum_{k=1}^{m+1} q(m,k-1)a_{m+1}(k+1)(x_1,\cdots,x_m;y) \\ &\quad - \sum_{k=1}^{m+1} q(m,k-1)(-1)^m a_m(k)(x_1,\cdots,x_m;y) \\ &= \sum_{k=1}^{m+2} (q(m,k-2)-q(m,k-1))a_{m+1}(k) \\ &= \sum_{k=1}^{m+2} q(m+1,k-1)a_{m+1}(k)(x_1,\cdots,x_{m+1};y) \end{aligned}$$

By induction, the result is true for all m. □

By the superalgebra method, we can get another expression of the relationship between LS_m and $a_m(k)$.

Define an operation on $K_n \otimes F_E[\Xi]$ as follows: for any $a \in K_n \otimes F_E[\Xi]_i$ and $b \in K_n \otimes F_E[\Xi]_j$, where $i, j = 0, 1$,

$$[a,b]_s = \begin{cases} ab+ba & \text{if i=j=1} \\ ab-ba & \text{otherwise} \end{cases}$$

and then extend it to $K_n \otimes F_E[\Xi]$ by linearity.

Let $X = x_1 \otimes \xi_1 + x_2 \otimes \xi_2 + \cdots + x_m \otimes \xi_m \in K_n \otimes F_E[\Xi]$ and $Y = y \otimes 1$, then we will show that

$$[Y, \overbrace{X, \cdots, X}^{m}]_s = LS_m(x_1, \cdots, x_m, y) \otimes \xi_1 \cdots \xi_m \tag{2.3.18}$$

holds for any m. In fact, when $m = 1$, we have $Y = y \otimes 1$ and $X = x_1 \otimes \xi_1$. Then

$$\begin{aligned} [Y,X]_s &= (yx_1 - x_1y) \otimes \xi_1 \\ &= LS_1(x_1;y) \otimes \xi_1. \end{aligned}$$

Assume equation (2.3.18) is true for $k-1$. Let $m = k$, then $Y = y \otimes 1$ and

$X = x_1 \otimes \xi_1 + \cdots + x_k \otimes \xi_k$. By the induction hypothesis,

$$[Y, \overbrace{X, \cdots, X}^{k-1}]_s = \sum_{i=1}^{k} y\, S_{k-1}(adx_1, \cdots, \hat{a}dx_i, \cdots, adx_k) \otimes \xi_1 \cdots \hat{\xi}_i \cdots \xi_k.$$

Therefore,

$$\begin{aligned}
&[Y, \overbrace{X, \cdots, X}^{k}]_s \\
&= [[Y, \overbrace{X, \cdots, X}^{k-1}], X]_s \\
&= \sum_{i=1}^{k} y\, S_{k-1}(adx_1, \cdots, \hat{a}dx_i, \cdots, adx_k) adx_i \otimes \xi_1 \cdots \hat{\xi}_i \cdots \xi_k \xi_i \\
&= \sum_{i=1}^{k} (-1)^i S_{k-1}(adx_1, \cdots, \hat{a}dx_i, \cdots, adx_k) adx_i \otimes \xi_1 \cdots \xi_k \\
&= y\, S_k(adx_1, \cdots, adx_k) \otimes \xi_1 \cdots \xi_k.
\end{aligned}$$

So, the equation (2.3.18) is true for any m.

On the other hand, we notice that for X, Y as above,

$$\begin{aligned}
[Y, X, X]_s &= [YX - XY, X]_s \\
&= (YX - XY)X + X(YX - XY) \\
&= YX^2 - XYX + XYX - X^2Y \\
&= [Y, X^2]_s \,.
\end{aligned}$$

In general, we have that

$$\begin{aligned}
[Y, \overbrace{X, \cdots, X}^{2k}]_s &= [Y, \overbrace{X^2, \cdots, X^2}^{k}]_s, \text{ if } m = 2k, k = 1, 2, \cdots \\
[Y, \overbrace{X, \cdots, X}^{2k+1}]_s &= [Y, \overbrace{X^2, \cdots, X^2}^{k}, X]_s, \text{ if } m = 2k+1, k = 1, 2, \cdots .
\end{aligned}$$

By the definition of bracket product, we have

$$[Y, \overbrace{X, \cdots, X}^{2k}]_s = Y(R_{X^2} - L_{X^2})^k.$$

Expanding $(R_{X^2} - L_{X^2})^k$ by the binomial formula, we get

$$[Y, \overbrace{X, \cdots, X}^{2k}]_s = \sum_{t=0}^{k} (-1)^{k-t} \begin{pmatrix} k \\ t \end{pmatrix} X^{2t} Y X^{2(k-t)}. \tag{2.3.19}$$

Similarly, we can get that

$$\begin{aligned}
[Y, \overbrace{X, \cdots, X}^{2k+1}]_s &= Y(R_{X^2} - L_{X^2})^k (R_X - L_X) \\
&= \sum_{t=0}^{k} (-1)^{k-t} \begin{pmatrix} k \\ t \end{pmatrix} X^{2t} Y X^{2(k-t)} (R_X - L_X) \\
&= \sum_{t=0}^{k} (-1)^{k-t} \begin{pmatrix} k \\ t \end{pmatrix} (X^{2t+1} Y X^{2(k-t)} - X^{2t} Y X^{2(k-t)+1}).
\end{aligned}$$

Since the following holds for any $0 \le t \le m$:

$$X^t Y X^{m-t} = a_m(t+1) \otimes \xi_1 \cdots \xi_m, \tag{2.3.20}$$

we have the following result:

$$\begin{aligned}
LS_m &= \sum_{t=0}^{k} (-1)^{k-t} \begin{pmatrix} k \\ t \end{pmatrix} a_m(2t+1), \text{ if } m = 2k, \\
LS_m &= \sum_{t=0}^{k} (-1)^{k-t} \begin{pmatrix} k \\ t \end{pmatrix} (a_m(2t+2) - a_m(2t+1)), \text{ if } m = 2k+1.
\end{aligned}$$

By this relation between LS_m and $a_m(k)$ and theorem 2.1.3, we can get the following result:

Theorem 2.3.2 *The Lie standard polynomial $LS_{4n-8}(x_1, \cdots, x_{4n-8}, y)$ is an identity of $K_n(F)$ for $n \ge 3$. When $n = 7$, $4n-7$ is the lowest degree Lie standard polynomial satisfied by $K_n(F)$.*

Proof. Letting $2k = 4n-8$ in equation (2.3.19), we get

$$[Y,\overbrace{X,\cdots,X}^{4n-8}]_s = \sum_{t=0}^{2n-4}(-1)^{2n-4-t}\begin{pmatrix} 2n-4 \\ t \end{pmatrix} X^{2t}YX^{2(2n-4-t)}$$

$$= (-1)^{n-2}\begin{pmatrix} 2n-4 \\ n-2 \end{pmatrix} X^{2n-4}YX^{2n-4}$$

since S_{2n-2} is an identity of K_n which implies $X^{2n-2}=0$. Moreover

$$a^{-}_{4n-8}\otimes\xi_1\cdots\xi_{4n-8} = YX^{4n-8}+\cdots+X^{2n-6}YX^{2n-2}$$
$$+X^{2n-4}YX^{2n-4}+X^{2n-2}YX^{2n-6}+\cdots+X^{4n-8}Y$$
$$= X^{2n-4}YX^{2n-4}$$

since $X^{2n-2}=0$.

But since a^{-}_{4n-8} is an identity of $K_n(F)$, $X^{2n-4}YX^{2n-4}=0$ and so is LS_{4n-8}.

When $n=7$, $4n-8$ is 20 and $2n-2$ is 12. We calculate LS_{19} in the following way:

$$[Y,\overbrace{X,\cdots,X}^{19}]_s = Y(R_{X^2}-L_{X^2})^9(R_X-L_X)$$

$$= \sum_{t=0}^{9}(-1)^{9-t}\begin{pmatrix} 9 \\ t \end{pmatrix} X^{2t}YX^{2(9-t)}(R_X-L_X)$$

$$= \sum_{t=0}^{9}(-1)^{9-t}\begin{pmatrix} 9 \\ t \end{pmatrix} (X^{2t+1}YX^{2(9-t)}-X^{2t}YX^{2(9-t)+1})$$

$$= \begin{pmatrix} 9 \\ 4 \end{pmatrix}(X^9YX^{10}-X^8YX^{11})-\begin{pmatrix} 9 \\ 5 \end{pmatrix}(X^{11}YX^8-X^{10}YX^9)$$

$$= \binom{9}{4}(-X^8YX^{11} + X^9YX^{10} + X^{10}YX^9 - X^{11}YX^8).$$

Since the coefficients of X^9YX^{10} and $X^{11}YX^8$ are distinct, by (2.3.20), LS_{19} cannot be of the form (2.1.1) in Theorem (2.1.3) and we have shown in section 2.2 that Theorem 2.1.3 is sharp for $n = 7$. Therefore LS_{19} is not an identity of K_7. □

Chapter 3

The identities of other Lie algebras

In the previous chapter, we discussed the almost standard polynomials and the Lie standard polynomial in $K_n(F)$; also the definition of the canonical symplectic involution (s) and the definition of $sp_{2n}(F)$ as the Lie algebra of all skew-symmetric matrices (with respect to the involution s) on $M_{2n}(F)$ is found in chapter 1. In the present chapter, we will study the almost standard polynomials and the Lie standard polynomial in $sp_{2n}(F)$ and discuss the polynomial identities of some other linear Lie algebras for small n's.

3.1 Weak Polynomial Identities For $sp_{2n}(F)$

The *symplectic involution* s is defined on $2n \times 2n$ matrices by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^s = \begin{pmatrix} D^t & -B^t \\ -C^t & A^t \end{pmatrix}$$

where A, B, C and D are $n \times n$ matrices, and t is the standard transpose involution. We denote $H_{2n}(F, s)$ and $sp_{2n}(F)$ as the space of symmetric matrices and the space of skew-symmetric matrices with respect to the symplectic involution respectively.

In 1982, Rowen [18] proved several results which are analogous to the results about the identities of matrices with the standard transpose involution mentioned in chapter 1:

Theorem 3.1.1 [*Rowen* [*18*]]

The following results hold:

1. $S_{4n-2}(x_1, \cdots, x_{4n-2})$ *vanishes for all specializations of* $x_1, \cdots, x_{4n-2}$ *to elements of* $H_{2n}(F, s)$.

2. $S_{4n-1}(x_1, \cdots, x_{4n-1})$ *vanishes for all specializations of* $x_1, \cdots, x_{4n-2}$ *to elements of* $H_{2n}(F, s)$ *and of* x_{4n-1} *to an element of* $sp_{2n}(F)$.

3. $S_{4n-2}(x_1, \cdots, x_{4n-2})$ *vanishes for all specializations of* $x_1, \cdots, x_{4n-3}$ *to elements of* $H_{2n}(F, s)$ *and of* x_{4n-2} *to an element of* $sp_{2n}(F)$.

The sharpness of Theorem 3.1.1 1) was proved only for $n = 1, 2$ by Rowen, then for $n = 3, 4$ by Jay M.H.Adamsson [1].

Now, we consider a matrix

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

in $sp_{2n}(F)$. If $M^s = -M$, we get

$$A^t = -D,\ B^t = B,\ C^t = C.$$

So, $sp_{2n}(F)$ has a basis $W = \{e_{ii} - e_{n+i,n+i},\ e_{ij} - e_{n+j,n+i}\ (i \neq j)\ , e_{i,n+i},\ e_{i,n+j} + e_{j,n+i}\ (i < j),\ e_{n+i,i},\ e_{n+i,j} + e_{n+j,i}\ (i < j)\ |1 \leq i, j \leq n\}$ and has dimension $2n^2 + n$. Each basis element in W can be represented on a graph with $2n$ vertices by a pair of labelled, directed edges or by a single directed edge in the following way. We label the vertices from 1 to $2n$ and draw a directed edge with positive label from point i to point j to represent e_{ij}. Similarly, we introduce an edge from point i to point j

with a negative label to represent $-e_{ij}$. In this manner, each of the basis elements in W is represented by a pair of labelled, directed edges or by a single directed edge on a graph.

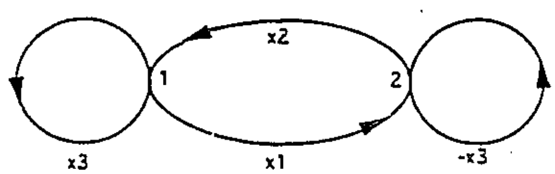
A pseudo-Eulerian path P on this graph is defined as a path in which every single edge and exactly one edge of each pair of edges is traversed exactly once. The *sign* of this pseudo-Eulerian path P, denoted by $sign(P)$, is determined by two factors which are the signature of the permutation of the edges in P and the signs of the edges which are traversed.

For example, in sp_2, define

$$x_1 = e_{12},$$
$$x_2 = e_{21},$$
$$x_3 = e_{11} - e_{22}$$

which corresponds to the following diagram.

Figure 4



The only three pseudo-Eulerian paths from point 1 which return to point 1 are $P_1 = x_1\, x_2\, x_3$, $P_2 = x_3\, x_1\, x_2$ and $P_3 = x_1\,(-x_3)\, x_2$. $sign(P_1)$ is the signature of the identity permutation on three elements which is $+1$. $sign(P_2)$ is the signature of the permutation $(3\,1\,2)$ which is $+1$ and $sign(P_3)$ is the signature of the permutation $(1\,3\,2)$ which is -1 times the sign of x_3 which is -1, so $sign(P_3) = 1$.

Theorem 3.1.2 *The standard polynomial $S_m(x_1, \cdots, x_m)$ is a weak identity of sp_{2n} if and only if $m \geq 4n$.*

Proof: Since sp_{2n} is a subset of M_{2n}, the standard polynomial of degree greater or equal $4n$ is an identity of sp_{2n}.

In the following, we will show that S_{4n-1} is not an identity of sp_{2n} by choosing a particular substitution. As in the case of K_n, we may determine the value of the (i,j)th entry of $S_m(x_1,\cdots,x_m)$ by counting the number of pseudo-Eulerian paths from vertex i to vertex j in the graph, and by determining if each path contributes a positive or a negative value to the total sum.

When $n=1$, we choose x_1, x_2, x_3 as in the above example and we know that there are only three pseudo-Eulerian paths from 1 to 1 and the sum of their signs is 3, which gives us that the $(1,1)$ entry in $S_3(x_1,x_2,x_3)$ is 3. So, S_{4n-1} is not an identity when $n=1$.

When $n=2$, we choose

$$\begin{aligned} x_1 &= e_{12}, & x_5 &= e_{31}-e_{24}, \\ x_2 &= e_{21}, & x_6 &= e_{34}, \\ x_3 &= e_{11}-e_{22}, & x_7 &= e_{43}, \\ x_4 &= e_{13}-e_{42}. \end{aligned}$$

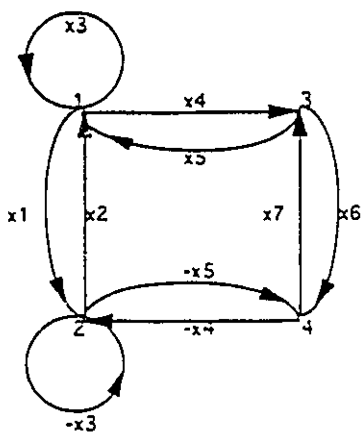
which yields the Figure 5.

By observation on the Figure 5, we can get that every pseudo-Eulerian path from 1 to 1 must have the form

$$\cdots \underline{x_4\, x_6\, x_7\, x_5} \cdots$$

$$\text{or} \quad \cdots \underline{(-x_5)\, x_7\, x_6\, (-x_4)} \cdots .$$

Figure 5



Since we know that the path $x_1x_2x_3$ in Figure 4 has positive sign and the signature of the even length permutations $(x_4\, x_6\, x_7\, x_5)$ or $(x_5\, x_7\, x_6\, x_4)$ is $+1$, every pseudo-Eulerian path from 1 to 1 in graph 5, which is given by adding $x_4\, x_6\, x_7\, x_5$ or $(-x_5)\, x_7\, x_6\, (-x_4)$ in $x_1x_2x_3$ at the front, at the end or in the middle of this path, is always positive.

Similarly, all other pseudo-Eulerian path from 1 to 1 in graph 5 are positive. So, S_{4n-1} is not an identity when $n = 2$.

In general, we choose

$$x_1 = e_{12}, \qquad x_2 = e_{21},$$

$$x_3 = e_{11} - e_{22},$$

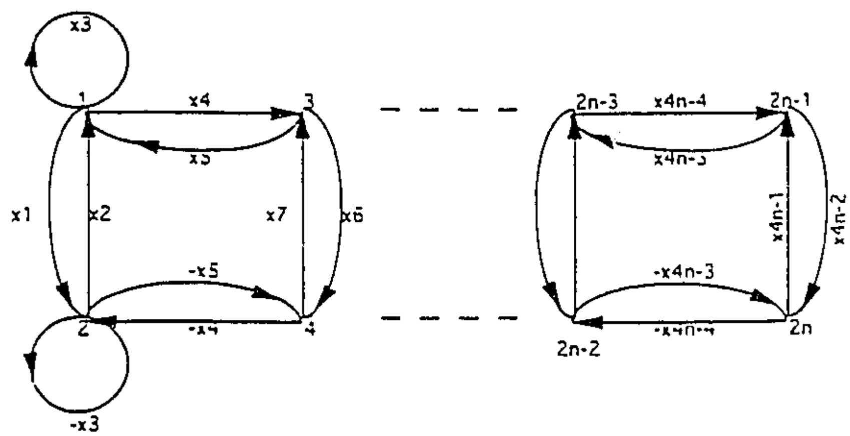
and for $2 \le r \le n$,

$$x_{4r-4} = e_{2r-3,2r-1} - e_{2r,2r-2}, \qquad x_{4r-2} = e_{2r-1,2r},$$

$$x_{4r-3} = e_{2r-1,2r-3} - e_{2r-2,2r}, \qquad x_{4r-1} = e_{2r,2r-1},$$

for which we have the following diagram

Figure 6



Arguing as in the case $n = 2$, we can get that every pseudo-Eulerian paths from 1 to 1 in graph 6 must be of the form

$$x_{i_1} \cdots x_{i_2} \underline{x_{4n-4}\, x_{4n-2}\, x_{4n-1}\, x_{4n-3}} x_{i_3} \cdots x_{i_4}$$

$$\text{or} \quad x_{j_1} \cdots x_{j_2} \underline{(-x_{4n-3})\, x_{4n-1}\, x_{4n-2}\, (-x_{4n-4})} x_{j_3} \cdots x_{j_4},$$

where $x_{i_1} \cdots x_{i_2} x_{i_3} \cdots x_{i_4}$ and $x_{j_1} \cdots x_{j_2} x_{j_3} \cdots x_{j_4}$ are the pseudo-Eulerian paths from 1 to 1 in the case $n - 1$.

By the same reasoning as in the case $n = 2$, we get that every pseudo-Eulerian paths from 1 to 1 in graph 6 has a positive sign.

Therefore, the $(1,1)$ entry in $S_{4n-1}(x_1, \cdots, x_{4n-1})$ is positive which implies that S_{4n-1} is not an identity of sp_{2n}. □

Note that while this result says that S_{4n} is the minimal standard identity satisfied by sp_{2n}, it does not say that sp_{2n} satisfies no other identity of degree $4n$ or less. In [9], Giambruno and Valenti constructed a polynomial identity of degree $4n - 1$ of

sp_{2n}:

$$T_{4n-1}(x_1, \cdots, x_{4n-2}; y) = a_{4n-2}^{-} + 2\sum_{i=1}^{n} a_{4n-2}(4i-2).$$

where $a_{4n-2}(k)$ and a_{4n-2}^{-} are the almost polynomials defined in chapter 1.

With a little modification of the above substitution, we can get another interesting result.

Theorem 3.1.3 *$S_{4n}(x_1, \cdots, x_{4n})$ is the standard polynomial of minimum degree which vanishes for all specializations of $x_1, \cdots, x_{4n-1}$ to elements of $sp_{2n}(F)$ and of x_{4n} to an element of $H_{2n}(F, s)$.*

Proof: By the Amitsur-Levitzki theorem, the standard polynomial of degree $4n$ is satisfied by all specializations of $x_1, \cdots, x_{4n-1}$ to elements of $sp_{2n}(F)$ and of x_{4n} to an element of $H_{2n}(F, s)$.

By the following substitution, we will get that S_{4n-1} is not satisfied by those elements. The whole calculation procedure is similar to the proof of Theorem 3.1.2.

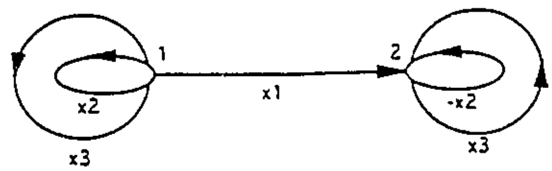
First, we notice that the space of $2n \times 2n$ symplectically symmetric matrices $H_{2n}(F, s)$ can be given a basis of $2n^2 - n$ matrices: $e_{ij} + e_{n+j,n+i}$, $(1 \leq i, j \leq n)$, $e_{i,n+j} - e_{j,n+i}$ $(1 \leq i < j \leq n)$, $e_{n+i,j} + e_{n+j,i}$ $(1 \leq i < j \leq n)$.

When $n = 1$, we choose

$$\begin{aligned} x_1 &= e_{12}, \\ x_2 &= e_{11} - e_{22}, \\ x_3 &= e_{11} + e_{22}, \end{aligned}$$

where $x_1,\ x_2 \in sp_2(F)$ and $x_3 \in H_2(F, s)$.

By checking the following graph,

Figure 7 

we find that there are only six pseudo-Eulerian paths from point 1 to point 2: $P_1 = x_1(-x_2)x_3$, $P_2 = x_1x_3(-x_2)$, $P_3 = x_2x_1x_3$, $P_4 = x_2x_3x_1$, $P_5 = x_3x_2x_1$ and $P_6 = x_3x_1(-x_2)$. An easy calculation gives us that $sign(P_1) = -1$, $sign(P_2) = +1$, $sign(P_3) = -1$, $sign(P_4) = +1$, $sign(P_5) = -1$ and $sign(P_6) = -1$. Hence the (1,2) entry in $S_3(x_1, x_2, x_3)$ is -2. This implies that S_{4n-1} does not vanish for all x_1, $x_2 \in sp_2(F)$ and $x_3 \in H_2(F, s)$.

When $n = 2$, we let

$$x_1 = e_{12}, \qquad x_5 = e_{31} - e_{24},$$

$$x_2 = e_{11} - e_{22}, \qquad x_6 = e_{34},$$

$$x_3 = e_{11} + e_{22}, \qquad x_7 = e_{43},$$

$$x_4 = e_{13} - e_{42},$$

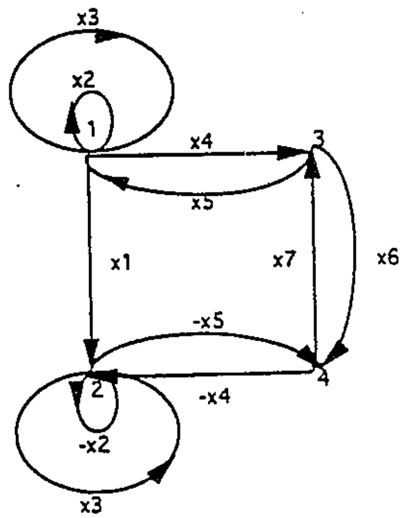
Then the associated diagram is as in Figure 8.

By observation on the Figure 8, we can get that every pseudo-Eulerian path from 1 to 2 must be of the form

$$\cdots \underline{x_4\, x_6\, x_7\, x_5} \cdots$$

$$\text{or} \quad \cdots \underline{(-x_5)\, x_7\, x_6\, (-x_4)} \cdots .$$

Figure 8



Since we know that the path $x_1(-x_2)x_3$ in Figure 7 has negative sign and the signature of the even length permutations $(x_4\,x_6\,x_7\,x_5)$ and $(x_5\,x_7\,x_6\,x_4)$ is $+1$, every pseudo-Eulerian path from 1 to 2 in graph 7, which is given by adding $x_4\,x_6\,x_7\,x_5$ or $(-x_5)\,x_7\,x_6\,(-x_4)$ in $x_1x_2x_3$ in the front of x_1, at the end of this path, in between of x_1 and x_2, and in between of x_2 and x_3, is always negative.

Similarly, we can get all the other pseudo-Eulerian paths from 1 to 2 in graph 8 and their signs. The summation of the signs of these paths is -8. So, S_{4n-1} is not an identity when $n = 2$.

For $n \geq 3$, we choose

$$x_1 = e_{12}, \qquad x_2 = e_{11} - e_{22},$$

$$x_3 = e_{11} + e_{22},$$

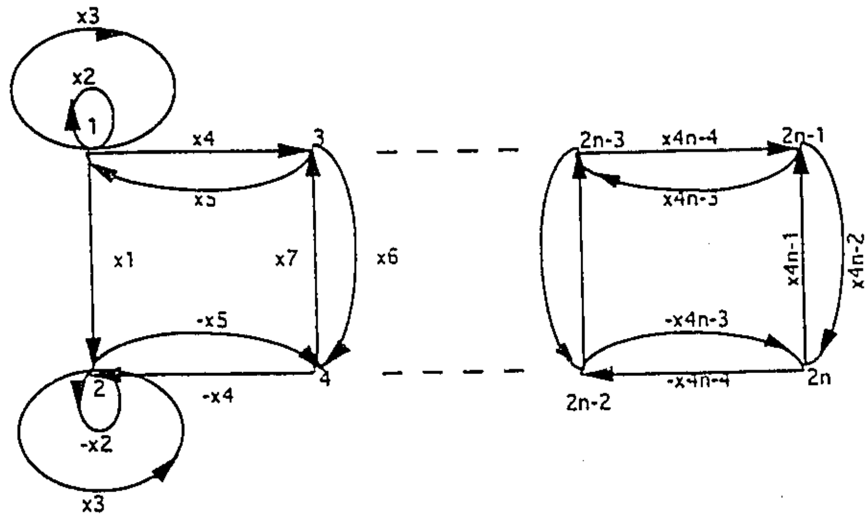
and for $2 \leq r \leq n$,

$$x_{4r-4} = e_{2r-3,2r-1} - e_{2r,2r-2}, \qquad x_{4r-2} = e_{2r-1,2r},$$

$$x_{4r-3} = e_{2r-1,2r-3} - e_{2r-2,2r}, \qquad x_{4r-1} = e_{2r,2r-1}.$$

for which we have the following diagram

Figure 9



Arguing as in the proof of Theorem 3.1.2, we can get that every pseudo-Eulerian paths from 1 to 2 in graph 8 must be of the form

(3.1.1) $$x_{i_1}\cdots x_{i_2}\underline{x_{4n-4}\,x_{4n-2}\,x_{4n-1}\,x_{4n-3}}x_{i_3}\cdots x_{i_4},$$

(3.1.2) $$\text{or}\quad x_{j_1}\cdots x_{j_2}\underline{(-x_{4n-3})\,x_{4n-1}\,x_{4n-2}\,(-x_{4n-4})}x_{j_3}\cdots x_{j_4},$$

where $x_{i_1}\cdots x_{i_2}x_{i_3}\cdots x_{i_4}$ and $x_{j_1}\cdots x_{j_2}x_{j_3}\cdots x_{j_4}$ are the pseudo-Eulerian paths from 1 to 2 in the case $n-1$.

By the same reasoning as in the case $n=2$, we get that every pseudo-Eulerian paths from 1 to 2 in graph 9 of the form (3.1.1) and the form (3.1.2) will have the same sign as the corresponding pseudo-Eulerian paths $x_{i_1}\cdots x_{i_2}x_{i_3}\cdots x_{i_4}$ or $x_{j_1}\cdots x_{j_2}x_{j_3}\cdots x_{j_4}$ have. Therefore, the summation of the signs of these paths is $-8\times 2^{n-2}$ which tells us that the $(1,2)$ entry in $S_{4n-1}(x_1,\cdots,x_{4n-1})$ is not zero, or S_{4n-1} is not an identity of sp_{2n}. □

Since sp_{2n} is a subset of $M_{2n}(F)$, Theorem 1.2.1 [5] by Benediktovic and Zalesski yields the following result.

Theorem 3.1.4 *For $n \geq 1$ and for $m \geq 4n$, the algebra sp_{2n} satisfies the almost standard polynomial*

$$f(x_1, \cdots, x_m; y) = \alpha a_m^+ + \beta a_m^- + \sum_{\substack{k \leq m-4n+1 \\ or\ k \geq 4n+1}} \gamma_k a_m(k)$$

for all α, β, $\gamma_k \in F$ and Lie standard polynomial $LS_{8n-4}(x_1, \cdots, x_{8n-4}; y)$.

In fact, we also can prove this theorem directly by the method we used to show Theorem 2.1.3. But since the minimum degree of the standard polynomial satisfied by sp_{2n} is $4n$, we cannot get a better result than Theorem 3.1.4.

3.2 The Mininum degree of the Lie standard identities of some low dimension Lie algebras

In the theory of Lie algebra, we have four infinite families of finite dimentional simple Lie algebras which are A_l, B_l, C_l and D_l.

The Lie algebra A_l is actually $sl_{(l+1)}$. A_l has a basis $\{e_{ij}(i \neq j), e_{ii} - e_{i+1,i+1}(1 \leq i \leq l)\}$ and its dimension is $l^2 + 2l$.

The Lie algebra B_l is a $2l^2 + l$ dimensional Lie algebra. It consists of all $(2l+1) \times (2l+1)$ matrices

$$\begin{pmatrix} a & b_1 & b_2 \\ c_1 & m & n \\ c_2 & p & q \end{pmatrix}$$

which satisfy that $a = 0$, $c_1 = -b_2^t$, $c_2 = -b_1^t$, $q = -m^t$, $n^t = -n$, $p^t = -p$.

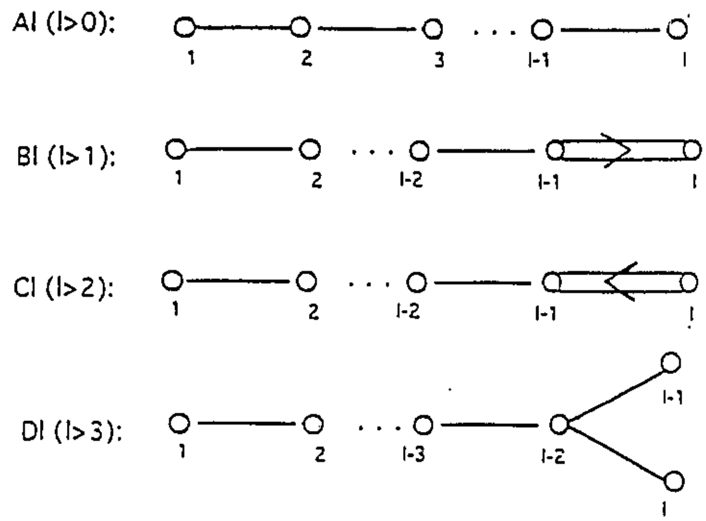
The Lie algebra C_l is the symplectic Lie algebra sp_{2l} which has the dimension $2l^2 + l$.

The Lie algebra D_l is another orthogonal algebra of $2l \times 2l$ matrices

$$\begin{pmatrix} m & n \\ p & q \end{pmatrix}$$

which satisfy $n^t = -n$, $p^t = -p$, $q = -m^t$.

The classification theorem gives us the Dynkin diagrams of their roots systems as the following:



By Benediktovic and Zalesski's theorem, we know that the minimun degree of a Lie standard polynomial identity of A_l (denoted by $MLI(A_l)$) is $4n-3$. i.e.,

$$MLI(A_1) = 5, \quad MLI(A_2) = 9, \quad MLI(A_3) = 13,$$

$$MLI(A_4) = 17, \quad MLI(A_5) = 21, \quad MLI(A_6) = 25.$$

Since $B_1 \cong A_1$ and $C_1 \cong A_1$ we have $MLI(B_1) = 5$ and $MLI(C_1) = 5$.

For B_2, $n = 5$, we calculate LS_8 first: let $X = x_1 \otimes \xi_1 + \cdots + x_8 \otimes \xi_8 \in B_2 \otimes F_E[\Xi]$ and $Y = y \otimes 1$, then $X^k = 0$ if $k \geq 8$. So

$$LS_8(x_1, \cdots, x_8; y) \otimes \xi_1 \cdots \xi_8$$

$$= Y(R_{X^2} - L_{X^2})^4$$

$$= YX^8 - 4X^2YX^6 + 6X^4YX^4 - 4X^6YX^2 + X^8Y$$

$$= -4X^2YX^6 + 6X^4YX^4 - 4X^6YX^2.$$

By theorem 2.1.3, $4X^2YX^6 + 6X^4YX^4 + 4X^6YX^2 = 0$. To show that LS_8 is an identity of B_2, it is enough to show that $X^4YX^4 = 0$.

Since the dimension of B_2 is 10 and LS_8 is multilinear and alternating in x_i's, we need only check the substitutions where $x_1, \cdots, x_8$ are chosen to be eight distinct elements of the basis $U = \{e_{ij} - e_{ji}, i \neq j, 1 \leq i, j \leq 5\}$ of B_2, letting the two missing basis elements be $e_{ij} - e_{ji}$ and $e_{kl} - e_{lk}$, then it is easy to see that there are only two cases: $i = k$ and $j \neq l$, or i, j, k, l are distinct. In the first case, we assume that $i = 1$, $j = 2$ and $l = 3$. Since X^4YX^4 is skew symmetric, in order to get a pseudo-Eulerian graph with exactly two vertices which have odd degree, we must choose that $y = e_{12} - e_{21}$ or $y = e_{24} - e_{42}$ (in the cases of $y = e_{13} - e_{31}$ or $y = e_{25} - e_{52}$, we can rearrange the indices of the graph to reduce them to the previous case). A similar discussion applied to the second case yields that there are only four subtitutions need to be checked:

1) $\{x_i\} = U \setminus \{ e_{12} - e_{21}, e_{13} - e_{31} \},\ y = e_{12} - e_{21}$;

2) $\{x_i\} = U \setminus \{ e_{12} - e_{21}, e_{13} - e_{31} \},\ y = e_{24} - e_{42}$;

3) $\{x_i\} = U \setminus \{ e_{12} - e_{21}, e_{34} - e_{43} \},\ y = e_{12} - e_{21}$;

4) $\{x_i\} = U \setminus \{ e_{12} - e_{21}, e_{34} - e_{43} \},\ y = e_{14} - e_{41}$.

It is not hard to get that the result of these substitutions into LS_8 are 0. So, LS_8 is an identity of B_2.

On the other hand, we compute the value of LS_7 in the same way. Let $X =$

$x_1 \otimes \xi_1 + \cdots + x_7 \otimes \xi_7 \in B_2 \otimes F_E[\Xi]$ and $Y = y \otimes 1$, then $X^k = 0$ if $k \geq 8$. So

$$\begin{aligned} &LS_7(x_1, \cdots, x_7; y) \otimes \xi_1 \cdots \xi_7 \\ &= Y(R_{X^2} - L_{X^2})^3(R_X - L_X) \\ &= YX^7 - 3X^2YX^5 + 3X^4YX^3 - X^6YX \\ &\quad -XYX^6 + 3X^3YX^4 - 3X^5YX^2 + X^7Y. \end{aligned}$$

If we choose $y = e_{12} - e_{21}$ and the x_i's to be the basis $U \backslash \{ e_{12} - e_{21}, e_{13} - e_{31}, e_{15} - e_{51} \}$, then we will get that the (3,5) entry of the matrix $LS_7(x_1, \cdots, x_7; y)$ is 24, which means that $LS_7(x_1, \cdots, x_7; y)$ is not an identity of B_2. Therefore, $MLI(B_2) = 9$.

Also $C_2 \cong B_2$, so $MLI(C_2) = 9$.

Theorem 2.3.2 tells us that $MLI(B_3) = 21$. Based on theorem 3.1.4, $MLI(C_3) \leq 21$. But the actual value of $MLI(C_3)$ is unkown.

Now we consider D_l. When $l = 1$, the dimension of D_1 is 1. Obviously $MLI(D_1) = 2$. When $l = 2, 3$, since $D_2 \cong A_1 \times A_1$ and $D_3 \cong A_3$ we have $MLI(D_2) = 5$ and $MLI(D_3) = 13$.

To make the results of above discussion clearer, we give the tabular version in the next page.

	A_l $(n = l+1)$			C_l $(n = 2l)$		B_l $(n = 2l+1)$			D_l $(n = 2l)$	
n	4n-3	MLI	dim	MLI	dim	4n-7	MLI	dim	MLI	dim
2	5	5	3	5	3	1	/	/	2	1
3	9	9	9	/	/	5	5	3	/	/
4	13	13	15	9	10	9	/	/	5	6
5	17	17	24	/	/	13	9	10	/	/
6	21	21	35	≤ 21	21	17	/	/	13	15
7	25	25	48	/	/	21	21	21	/	/

3.3 Identities of $H_{2n}(F, s)$

If A is an associative algebra over a field F then it inherits a quadratic Jordan algebra structure $A^{(q)}$ by

$$a\,U_b := aba;\ a^{\cdot 2} := a^2.$$

If A is unital we need only $a\,U_b$ since $b^{\cdot 2} = 1\,U_b$. A Jordan algebra J is said to be *special* if it embeds in an $A^{(q)}$ for some associative algebra A. Let $FJ[X]$ be the *free Jordan algebra* generated by X, $FSJ[X]$, the *free special Jordan algebra*, the Jordan subalgebra of $FA[X]^{[q]}$ generated by X. A *Jordan polynomial* is an element of the free Jordan algebra $FJ[X]$, a *special Jordan polynomial* an element of the free special Jordan algebra $FSJ[X]$.

In $A^{(q)}$

$$a\,V_b := a \circ b := ab + ba.$$

Smith [20] introduced the *Jordan standard polynomial*

$$JS_n(x_1, x_2, \cdots, x_n; y) := yS_n(V_{x_1}, V_{x_2}, \cdots, V_{x_n}).$$

Racine and Drensky [7, Proposition 4] .proved that the algebra $M_n(F)^{(q)}$ satifies JS_{4n-4} but not JS_{4n-5} by showing that

$$JS_{2n}(x_1, x_2, \cdots, x_{2n}; y) = LS_{2n}(x_1, x_2, \cdots, x_{2n}; y). \tag{3.3.3}$$

By Rowen's theorem 3.1.1 and the method we used to prove theorem 2.1.3, we can get results similar to theorem 2.1.3 and theorem 2.3.2 for Jordan algebra $H_{2n}(F, s)$ very easily.

Theorem 3.3.1 *The almost standard polynomial*

$$f(x_1, \cdots, x_m; y) = \alpha a_m^+ + \beta a_m^- + \sum_{\substack{k \geq 4n-1 \\ \text{or } k \leq m-4n+1}} \gamma_m a_m(k),$$

where $m \geq 4n-2$, *is a weak identity of* $H_{2n}(F, s)$ *for all* n *and arbitrary* $\alpha, \beta, \gamma \in F$.

Proof. Since the same reasoning as in the $K_n(F)$ case holds for $H_{2n}(F, s)$, it is sufficient to show that a_{4n-2}^- is a weak identity of $H_{2n}(F, s)$.

We assume $X_2, \cdots, X_{2n-1}, Y \in H_{2n}(F, s)$ and $X_1 \in sp_{2n}$. By Lemma 2.1.2.1), we have

$$Tr(a_{4n-1}^-(X_1, \cdots, X_{4n-1}; Y)) = n\,Tr(S_{4n-1}(X_1, \cdots, X_{4n-1})Y).$$

By Rowen's theorem 3.1.1.2), we get $S_{4n-1}(X_1, \cdots, X_{4n-1}) = 0$, so

$$Tr(a_{4n-1}^-(X_1, \cdots, X_{4n-1}; Y)) = 0.$$

Arguing as we did in theorem 2.1.3, by Lemma 2.1.2.2),

$$\begin{aligned} Tr(a_{4n-1}^-) &= n\,Tr(X_1 \sum_{i=1}^{4n-1} a_{4n-2}(i)(X_2, \cdots, X_{4n-1}, Y)) \\ &= n\,Tr(X_1(2a_{4n-2}^- - S_{4n-1})(X_2, \cdots, X_{4n-1}, Y)). \end{aligned}$$

By Rowen's theorem 3.1.1.1), $S_{4n-1}(X_2, \cdots, X_{4n-1}, Y)$ vanishes on $H_{2n}(F, s)$, we have $Tr(X_1 S_{4n-1}(X_2, \cdots, X_{4n-1}, Y)) = 0$. So, $Tr(X_1\, a^-_{4n-2}) = 0$ we may assume $char(F) = 0$.

For any $1 \leq j \leq 4n-1$ and j odd,

$$\begin{aligned}(a_{4n-2}(j)(X_2, \cdots, X_{4n-1}; y))^s &= (-1)^{2n-1} a_{4n-2}(4n-j)(X_2, \cdots, X_{4n-1}; y) \\ &= -a_{4n-2}(4n-j)(X_2, \cdots, X_{4n-1}; Y).\end{aligned}$$

This implies that a^-_{4n-2} takes values in sp_{2n}. Hence $a^-_{4n-2} = 0$ by the nondegeneracy of the trace bilinear form on sp_{2n}. □

Theorem 3.3.2 *For $n \geq 3$, the algebra $H_{2n}(F, s)$ satisfies JS_{8n-8}.*

Proof: Let $X = x_1 \otimes \xi_1 + \cdots + x_{8n-8} \otimes \xi_{8n-8} \in H_{2n}(F, s) \otimes F_E[\Xi]$ and $Y = y \otimes 1$, then exactly the same proof can give us that LS_{8n-8} vanishes on $H_{2n}(F, s)$. So by equation (3.3.3), JS_{8n-8} is an identity of $H_{2n}(F, s)$. □

3.4 Identities of Lie algebra $sl_2(F)$

We denote by $sl_k(F)$ the Lie subalgebra of $M_k(F)$ of matrices with trace zero. Let TL_k be the *T-ideal* of L, consisting of the polynomial identities of $sl_k(F)$, where a *T-ideal* is an ideal of $FA[X]$ which is closed under endomorphisms of $FA[X]$. The following lemma is obvious.

Lemma 3.4.1 *The T-ideal of identities of $sl_k(F)$ consists of those polynomials from the free Lie algebra FL[X], which are identities of $M_k(F)$.*

In 1973, Yu. P. Razmyslov [16, Theorem 2], found a finite basis of TL_2 in the following theorem:

Theorem 3.4.2 [*Razmyslov*]

The T-ideal of $sl_2(F)$ has a basis of the form

$$LS_4(x_1, x_2, x_3, x_4; y) = 0 \tag{3.4.4}$$

$$([y, z])(ad\, x)^3 = [(y)(ad\, x)^3, z] + [y, (z)(ad\, x)^3]. \tag{3.4.5}$$

Later V. T. Filippov [8, Corollary of Theorem 2], gave a basis consisting of one identity

$$[y, z, [t, x], x] + [y, x, [z, x], t] = 0.$$

In the present section, we will define Lie almost standard polynomials and then give a basis of identities of the Lie algebra $sl_2(F)$ consisting only of Lie almost standard identities.

DEFINITION: For $1 \leq k \leq n$, let

$$La_n(k)(x_1, \cdots, x_n; y) := \sum_{\sigma \in S_{n-1}} (-1)^\sigma [x_n, x_{\sigma(1)}, \cdots, x_{\sigma(k-1)}, y, x_{\sigma(k)}, \cdots, x_{\sigma(n-1)}],$$

or

$$La_n(k)(x_1, \cdots, x_n; y) := x_n a_{n-1}(k)(ad\, x_1, ad\, x_2, \cdots, ad\, x_{n-1}; ad\, y).$$

A multilinear polynomial f of degree $n+1$ over a field F is a *Lie almost standard* if $f(x_1, \cdots, x_n; y) = \sum_k \alpha_k La_n(k)$ where $\alpha_k \in F$. In particular, the Lie standard polynomial

$$LS_n(x_1, \cdots, x_n; y) = \sum_{k=1}^{n} (-1)^{k-1} La_n(k)(x_1, \cdots, x_{n-1}, y; x_n).$$

Lemma 3.4.3 *Let $F(x_1, x_2, \cdots, x_m; y)$ be a polynomial which is linear in each of $x_1, x_2, \cdots, x_m$ and homogeneous in y, then $F(x_1, x_2, \cdots, x_m; y)$ vanishes in $sl_n(F)$ if and only if $F(X_1, X_2, \cdots, X_m; Y)$ is identically zero whenever $X_1, X_2, \cdots, X_m$ are elements in the basis $B = \{e_{ij}(i \neq j, 1 \leq i, j \leq n), e_{ii} - e_{i+1,i+1}(1 \leq i \leq n-1)\}$ of $sl_n(F)$ and $Y = diag(u_1, \cdots, u_n) \in sl_n(F)$ such that $u_1, \cdots, u_n$ are distinct.*

Proof: Let $\Re$ be the set of all semi-simple regular elements in $sl_n(F)$, then, as proved in Humphreys's book [11, Appendix of section 23], $\Re$ is a non-empty open set in Zariski topology. So $\Re$ is a Zariski dense set in $sl_n(F)$. Therefore, $F(x_1, x_2, \cdots, x_m; y)$ is a weak identity of $sl_n(F)$ if and only if $F(X_1, X_2, \cdots, X_m; Y)$ is identically zero for $X_1, X_2, \cdots, X_m$ arbitrary matrices of $sl_n(F)$ and $Y \in \Re$.

Also we know that $Y \in \Re$ if and only if the eigenvalues of Y are distinct or, passing to a suitable field extension of F, there exists a matrix g such that $g^{-1}Yg = diag(u_1, \cdots, u_n) \in sl_n$ where $u_1, \cdots, u_n$ are distinct. Since

$$g^{-1}F(X_1, \cdots, X_m; Y)g = F(g^{-1}X_1g, \cdots, g^{-1}X_mg; g^{-1}Yg),$$

in order to prove $F(x_1, x_2, \cdots, x_m; y)$ vanishes in $sl_n(F)$ it suffices to show that $F(X_1, X_2, \cdots, X_m; Y)$ is identically zero for $X_1, X_2, \cdots, X_m$, arbitrary matrices of $sl_n(F)$ and Y is a diagonal matrix in the maximal toral subalgebra H of sl_n. Finally, since $F(x_1, x_2, \cdots, x_m; y)$ is linear in each of $x_1, \cdots, x_m$, it suffices to take $X_1, X_2, \cdots, X_m$ to be elements in the basis B, proving the lemma. □

Let $L = sl_2(F)$. The Cartan decomposition of L is

$$L = H + L_\alpha + L_{-\alpha},$$

where H is the maximal toral subalgebra generated by the element $h = e_{11} - e_{22}$, L_α is the subspace generated by $x = e_{12}$ and $L_{-\alpha}$, the subspace generated by $y = e_{21}$. We know that

$$[h, x] = 2x, \ [h, y] = -2y, \ [x, y] = h. \qquad (3.4.6)$$

Also $\phi(x) = y, \phi(y) = x, \phi(h) = -h$ yield an automorphism of sl_2.

Set

$$g_1(x_1, \cdots, x_4; y) \ = \ (La_4(1) - La_4(4))(x_1, \cdots, x_4; y), \qquad (3.4.7)$$

$$g_2(x_1,\cdots,x_4;y) = (La_4(2)-La_4(3))(x_1,\cdots,x_4;y), \tag{3.4.8}$$

$$g_3(x_1,\cdots,x_4;y) = (La_4(1)-2La_4(3))(x_1,\cdots,x_4;y). \tag{3.4.9}$$

We can prove the following lemma.

Lemma 3.4.4 *$g_1(x_1,\cdots,x_4;y)$, $g_2(x_1,\cdots,x_4;y)$ and $g_3(x_1,\cdots,x_4;y)$ are polynomial identities of sl_2.*

Proof: Since $La_4(1)-La_4(4)$ is multilinear, it is suffices to prove that $La_4(1)-La_4(4)$ is identically zero for $x_1,\cdots,x_4\in B$ and $y=h$. Also $La_4(1)-La_4(4)$ is alternating in x_1,x_2,x_3, so they must be distinct (otherwise $La_4(1)-La_4(4)$ will be zero already) which leads to $x_1=x, x_2=y, x_3=h$. Furthermore, since ϕ yields a automorphism of sl_2, if $(La_4(1)-La_4(4))(x,y,h,x;h)=0$ then

$$(La_4(1)-La_4(4))(y,x,-h,y;-h) = -(La_4(1)-La_4(4))(x,y,h,y;h)=0.$$

So only two substitutions have to be checked, namely,

$$x_1=x,\ x_2=y,\ x_3=h,\ x_4=x,\ y=h \tag{3.4.10}$$

and

$$x_1=x,\ x_2=y,\ x_3=h,\ x_4=h,\ y=h\ . \tag{3.4.11}$$

For the first substitution, we calculate $La_4(1)$ and $La_4(4)$ seperately. By the definition,

$$La_4(1)(x_1,x_2,x_3,x;h) = \sum_{\sigma\in S_3}(-1)^\sigma[x,h,x_{\sigma(1)},x_{\sigma(2)},x_{\sigma(3)}].$$

Since $[x,h,x]=0$, both of $[x,h,x_1,x_2,x_3]$ and $[x,h,x_1,x_3,x_2]$ are zero. Since $[x,h,h]\in L_\alpha$, $[x,h,x_3,x_1,x_2]=0$ and $[x,h,y]\in H$, $[x,h,x_2,x_3,x_1]=0$. The two remaining terms are

$$-[x,h,x_2,x_1,x_3] = [2x,y,x,h] = 2[h,x,h] = 4[x,h] = -8x$$

and

$$-[x, h, x_3, x_2, x_1] = [2x, h, y, x] = -4[x, y, x] = -4[h, x] = -8x.$$

So $La_4(1)(x, y, h, x; h) = -16x$.

For

$$La_4(4)(x_1, x_2, x_3, x; h) = \sum_{\sigma \in S_3} (-1)^\sigma [x, x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, h],$$

since $[x, x] = 0$, $[x, x_1, x_2, x_3, h]$ and $[x, x_1, x_3, x_2, h]$ are zero. By $[x, y, h] = 0$ and $[x, h, x] = 0$, we have $[x, x_2, x_3, x_1, h] = 0$ and $[x, x_3, x_1, x_2, h] = 0$. The rest of the terms are calculated in the following:

$$-[x, x_2, x_1, x_3, h] = -[h, x, h, h] = -8x$$

and

$$-[x, x_3, x_2, x_1, h] = [2x, y, x, h] = 2[h, x, h] = -8x.$$

So $La_4(4)(x, y, h, x; h) = -16x$. Hence $(La_4(1) - La_4(4))(x_1, \cdots, x_4; y)$ vanishes for the first substitution.

For the second substitution, the calculation is even simpler. Since $x_4 = y = h$, $La_4(1)(x_1, x_2, x_3, h; h) = 0$. For $La_4(4)$, since no matter which permutation σ is chosen, $[h, x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}]$ is always in H, so $[h, x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, h] = 0$. These suffice to establish the fact that $g_1(x_1, \cdots, x_4; y)$ is an identity. Applying the same arguments to $(La_4(2) - La_4(3))(x_1, \cdots, x_4; y)$ and $(La_4(1) - 2La_4(3))(x_1, \cdots, x_4; y)$ will complete the proof of the lemma. □

Theorem 3.4.5 *The polynomial identities $g_1(x_1, \cdots, x_4; y)$, $g_2(x_1, \cdots, x_4; y)$ and $g_3(x_1, \cdots, x_4; y)$ form a basis of TL_2.*

Proof: First, we will show that these three polynomials are linearly independent.

We know that $FL[x_1, x_2, \cdots, x_{n+1}]$ has a basis of $\{[x_{n+1}, x_{\sigma(1)}, \cdots, x_{\sigma(n)}] | \forall \sigma \in \mathcal{S}_n\}$[4, lemma 5 on page 126]. So $\{La_n(k)(x_1, \cdots, x_n; y) | 1 \leq k \leq n\}$ are linearly independent.

If $\alpha_1 g_1 + \alpha_2 g_2 + \alpha_3 g_3 = 0$, for any α_1, α_2, $\alpha_3 \in F$, then by the linear independence of $\{La_4(1), La_4(2), La_4(3), La_4(4)\}$ we have $\alpha_1 = \alpha_2 = \alpha_3 = 0$. So g_1, g_2 and g_3 are linearly independent.

Secondly, we prove that the polynomial identities (3.4.4) and (3.4.5) are consequences of g_1, g_2 and g_3. In fact, it is easy to see that $LS_4(x_1, x_2, x_3, x_4; y) = (g_1 - g_2)(x_1, x_2, x_3, y; x_4)$. Also by the following calculation we can see that the polynomial identity (3.4.5)

$$R_2(x, y, z) = [y, z, x, x, x] - [y, x, x, x, z] + [z, x, x, x, y]$$

is equivalent to the polynomial identity $g_3(x, y, z, x; x) = (La_4(1) - 2La_4(3))(x, y, z, x; x)$:

$$\begin{aligned} La_4(1)(x, y, z, x; x) &= 0, \\ La_4(3)(x, y, z, x; x) &= [x, y, z, x, x] - [x, z, y, x, x] \\ &\quad + [x, z, x, x, y] - [x, y, x, x, z]. \end{aligned}$$

By the Jacobi identity, $[x, y, z] - [x, z, y] = [z, y, x]$, so

$$\begin{aligned} La_4(3)(x, y, z, x; x) &= [z, y, x, x, x] + [x, z, x, x, y] - [x, y, x, x, z] \\ &= -R_2(x, y, z). \end{aligned}$$

This yields the result. □

Bibliography

[1] Adamsson, Jay, M. H., The standard polynomial as an identity on symplectic matrices, master's thesis, University of Ottawa, 1992.

[2] Amitsur, A. S. and Levitzki, J., Minimal identities for algebras, Proc. Amer. Math. Soc. **1** (1950), 449-463.

[3] Amitsur, A. S., Identities in rings with involution, Israel J. Math. **7** (1969), 63-68.

[4] Bahturin, Yu. A., Identical relations in Lie algebras, VUN Science Press, Utrecht, The Netherlands, 1987.

[5] Benediktovic, I. I., and Zalesski, A. E., Almost standard identities of matrix algebras, Dokl. Akad. Nauk BSSR **23** (1979), 201-204.

[6] Drensky, V. S. and Kasparian A., Polynomial identities of eighth degree for 3×3 matrices, C. R. Acad. Bulgare Sci. **77** (1983), 1-25.

[7] Drensky, V. S. and Racine, M. L., Distinguishing simple Jordan algebras by means of polynomial identities, Comm. Algebra **20** (2) (1992), 309-327.

[8] Filippov, A. T., Manifold of Mal'tsev algebras, Algebra i Logika **20**, No. 3, (1981), 300-314

[9] Giambruno, Antonio, On *-polynomial identities for $n \times n$ matrices, J. Algebra **133** (1990) 433-438.

[10] Giambruno, Antonio, and A. Regev, Wreath products and PI-algebras, J. Pure Appl. Algebra, **35** (1985), 133-149.

[11] Humphreys, J.E., Introduction to Lie algebras and representation theory, Springer-Verlag, New York. Heidelberg. Berlin-1972.

[12] Kemer, A. R., A remark on the standard identity, Mat. Zametki **23** (1978), 753-757; English transl. in Math. Notes.

[13] Kostant, B., A theorem of Frobenius, a theorem of Amisur- Levitzki, and cohomology theory, Indiana J. Math. **7** (1958), 237-264.

[14] Kostant, B., A Lie algebra generalization of the Amitsur- Levitzki theorem, Adv. in Math. **40** (1980) 155-175.

[15] Ma, Wenxin and Racine, M. L., Minimal identities of symmetric matrices, Trans. Amer. Math. Soc. **320** (1990), No.1, 171-192.

[16] Razmyslov, Y. P., Finite basing of the identities of a matrix algebra of second order over a field of characteristic 0, Algebra i Logika **12** (1973), 47-63.

[17] Rowen, L. H., Standard polynomials in matrix algebra, Trans. Amer. Math. Soc. **190** (1974), 253-284.

[18] Rowen, L. H., A simple proof of Kostant's theorem and an analogue for the symplectic involution, Comtemp. Math., vol.**13**, Amer. Math. Soc. (1982), 207-215.

[19] Rowen, L. H., Polynomial identities in ring theory, Academic Press, New York, 1980.

[20] Smith, B. D., A standard Jordan polynomial, Comm. Algebra **5**(2) 1977, 207-218.

[21] Zel'manov, E. I., Superalgebras and identities, Algebra and Analysis (Kemerovo, 1988) 39-46, Amer. Math. Soc. Transl. Ser. **2**, 148, Amer. Math. Soc., Providence, RI, 1991, 39-46

Appendix

The *Mathematica* program referenced in section 2.2 is presented in this appendix as follows:

```
*(This program will calculate the summation of the signatures of all valid pseudo-
Eulerian paths in the graph defined in section 2.2)*

graph={{1,2}, {1,4}, {1,5}, {1,6}, {1,7}, {2,3}, {2,4}, {2,5}, {2,6}, {2,7}, {3,4}, {3,5}, {3,6},
        {3,7}, {4,5}, {4,6}, {4,7}, {5,6}, {5,7}, {6,7}};
*(the graph is given by the set of arcs, where {i, j} denotes the arc from i to j)*

tepa[y_List]:=
   Block[{v, done, x, mdp1=1, mdp2=2, pegn1=9, pegn2=11, edp=2, ss, m=7,
          s={3}, edp=2, ss, j, m=7,  gi={}, p={}, sol=0, ind=Table[1,{7},{7}]},

       adjm[y]; d=0;
       h[s_List,v_]:=Count[s,v];

       While[ Length[s]>0,
         v=Last[s];sn=1;
         done=False;
        While[ind[[v,h[s,v]]]<=Length[e[[v]]] and !done,
             x=e[[v,ind[[v,h[s,v]]]++]];
             done=(pegn1+pegn2+2>Length[s]) and
                    (!MemberQ[gi,{v,x}] and !MemberQ[gi,{x,v}]);
         ];

         If[done,AppendTo[s,x];AppendTo[gi,{v,x}];
            If[MemberQ[y,{v,x}],AppendTo[p,Position[y,{v,x}]],
              AppendTo[p,Position[y,{x,v}]]
             ],
            If[(Length[s]==pegn1+2),
               ind[[v,h[s,v]]]=1;
               s=Drop[s,-2],
               ind[[v,h[s,v]]]=1;
               s=Drop[s,-1]
             ];
             If[Length[gi]>0, gi=Drop[gi,-1];p=Drop[p,-1]]
          ];

          If[(Length[s]==pegn1+1),
             If[(Last[s]==mdp1) or (Last[s]==mdp2),
                 ss=Union[s];
```

```
                 If[(Length[ss]==m),
                     AppendTo[s, Last[Complement[{mdp1,mdp2},{Last[s]}]]];
                     v=Last[s],
                      s=Drop[s,-1];
                      gi=Drop[gi,-1];
                      p=Drop[p,-1]
                   ],
                  s=Drop[s,-1];
                  gi=Drop[gi,-1];
                  p=Drop[p,-1]
               ]
             ];

          If[(Length[s]==pegn1+pegn2+2),
                If[(Last[s]==edp),
                     ss=Union[Drop[s,pegn1+1]];
                     If[(Length[ss]>m-2),
                         For[i=1,i<=Length[s]-1,i++,
                              If[s[[i]]<s[[i+1]],sn=sn,sn=-sn]
                         ];
                         sn=sn Signature[p];
                         sol=sol+sn;
                     ]
                 ];
                s=Drop[s,-1];
                gi=Drop[gi,-1];
                p=Drop[p,-1]
            ]
        ]
    ];
```

(The following subrotine will yield the adjacent matrix of the given graph)

```
adjm[g_List]:=
   Block[{a,b,k,t},t=Table[0,{i,7}];e=Table[0,{i,7},{j,7}];
          For[ k=1,k<=20,k++,a=g[[k,1]];b=g[[k,2]];
               t[[a]]=t[[a]]+1;t[[b]]=t[[b]]+1;
               e[[a,t[[a]]]]=b;e[[b,t[[b]]]]=a
          ];
          For[ i=1,i<=7,i++,e[[i]]=Complement[e[[i]],{0}]]
   ];
```