Weighted Limits in Categories Graded by Monoidal Categories

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Abstract

Categories graded by a monoidal category $\mathcal{V}$ generalize both $\mathcal{V}$-actegories and $\mathcal{V}$-enriched categories without requiring additional properties of $\mathcal{V}$. However, $\mathcal{V}$-graded categories are themselves enriched in the monoidal category $\hat{\mathcal{V}}$ of presheaves on $\mathcal{V}$. In this text, we define a notion of weighted limit for $\mathcal{V}$-graded categories, and show that $\mathcal{V}$-graded weighted limits are precisely the $\hat{\mathcal{V}}$-enriched weighted limits whose weights take on representable values. When $\mathcal{V}$ is biclosed and the $\mathcal{V}$-graded categories involved are $\mathcal{V}$-enriched, we recover precisely the familiar notion of $\mathcal{V}$-enriched weighted limit. We use $\mathcal{V}$-graded structure to define weighted limits in $\mathcal{V}$-actegories and in $\mathcal{V}$-categories for a non-biclosed monoidal $\mathcal{V}$. We develop both a convenient concrete formulation and an equivalent abstract description as $\mathcal{V}$-graded representations, and explore examples including $\mathcal{V}$-graded powers and conical limits.

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Contents

Introduction

Categories graded by a monoidal category $\mathcal{V}$, or $\mathcal{V}$-graded categories, were introduced by Richard Wood in his 1976 PhD thesis [19] under the name *large $\mathcal{V}$-categories*. They were later renamed *locally $\mathcal{V}$-graded categories* in 2019 by Levy [13], and then abbreviated to *$\mathcal{V}$-graded categories* in 2025 by Lucyshyn-Wright [15]. Categories graded by a monoidal category $\mathcal{V}$ are categories enriched in the presheaf category $\hat{\mathcal{V}} := [\mathcal{V}^{op}, SET]$ equipped with Day convolution monoidal structure. We can unpack this definition to obtain an elementary definition. A $\mathcal{V}$-graded category $\mathcal{C}$ then simply consists of a set $ob(\mathcal{C})$ of objects and a set of graded morphisms of the form $f : X, A \to B$ with a grade X in $ob(\mathcal{V})$ and $A, B \in ob(\mathcal{C})$, including identity graded morphisms $1_A : I, A \to A$ for $A \in ob(\mathcal{C})$. The composite of the graded morphisms $f : X, A \to B$ and $g : Y, B \to C$ in $\mathcal{C}$ is a graded morphism $g \circ f : Y \otimes X, A \to C$ in $\mathcal{C}$, and $\mathcal{C}$ is equipped with a *reindexing* operation which takes $f : X, A \to B$ to a graded morphism $\alpha^* f : Y, A \to B$ for $\alpha : Y \to X$ in $\mathcal{V}$. These data is required to satisfy certain axioms that we will recall in Chapter 2.

One of the advantages of working with $\mathcal{V}$-graded categories is that these generalise both $\mathcal{V}$-categories and $\mathcal{V}$-actegories. $\mathcal{V}$-graded structure reveals a common architecture underlying enrichment and action, enabling constructions that apply uniformly to both. The concept of interest in this thesis is that of weighted limits. There is an existing notion of weighted limit in $\mathcal{V}$-categories that has been widely known for some time. A $\mathcal{V}$-enriched weighted limit is given by a representation of a particular $\mathcal{V}$-functor which we recall in Chapter 2. Here, we define a notion of $\mathcal{V}$-graded weighted limit which specializes to recover this familiar notion of $\mathcal{V}$-enriched weighted limits.

We describe a weighted limit in the $\mathcal{V}$-graded category $\mathcal{C}$ on a diagram $D : \mathcal{K} \to \mathcal{C}$ with weight $W : \mathcal{K} \to \mathcal{V}$ in elementary terms as an object $L \in \mathcal{C}$ with a family of graded morphisms $\lambda_K : WK, L \to DK$ $(K \in \mathcal{K})$ called a *weighted cone* satisfying a certain naturality condition, and this weighted cone must further satisfy a *graded universal property* as described in Chapter 3. When the $\mathcal{V}$-graded categories involved are $\mathcal{V}$-enriched and $\mathcal{V}$ is biclosed, the data we describe above gives us equivalently the familiar notion of $\mathcal{V}$-enriched weighted limits. Note that while $\mathcal{V}$-graded categories are $\hat{\mathcal{V}}$-categories, the weight W of a $\mathcal{V}$-graded weighted limit has codomain $\mathcal{V}$ and not $\hat{\mathcal{V}}$, as one allows for general $\hat{\mathcal{V}}$-enriched weighted limits. In fact, $\mathcal{V}$-graded weighted limits are precisely the $\hat{\mathcal{V}}$-enriched limits whose weights take on representable values.

Additionally, we may use $\mathcal{V}$-graded structure to extend the existing notion of $\mathcal{V}$-enriched weighed limits to the case where $\mathcal{V}$ is simply monoidal and not necessarily biclosed. We may also use $\mathcal{V}$-graded weighted limits to form weighted limits in $\mathcal{V}$-actegories, finding that these admit a particularly simple description where rather than satisfying a graded universal property, it is sufficient for the weighted cone (L, λ) to satisfy a weaker version of the universal property.

We now provide an overview of the thesis structure. In Chapter 1, we review some background material on monoidal categories, including Day Convolution monoidal structure on a presheaf category. We then recall the concept of *actegories* since these are an interesting example of $\mathcal{V}$-graded categories which we will continue to encounter throughout the text. Afterwards, we discuss briefly the theory of enriched categories and describe weighted (co)limits in enriched categories, giving the foundational examples of (co)powers and conical (co)limits. However, unlike the more widely-known treatment popularized by Kelly in [11], we opt for an approach given by Street [18] and reformulated by Gordon and Power [9] that does not require symmetry on the base of enrichment. In Chapter 2, we provide an overview of $\mathcal{V}$-graded categories, highlighting examples and concepts that will become relevant in the next chapter. We also describe $\mathcal{V}$-graded presheaf categories, which arise as the $\hat{\mathcal{V}}$-enriched case of Street's enriched presheaf categories [18] and are a special case of the general $\mathcal{V}$-graded functor categories of [15]. We discuss representability of $\mathcal{V}$-graded functors, and briefly outline a method for constructing commutative diagrams with graded morphisms which Lucyshyn-Wright calls *envelope diagrams* [15]. The final chapter of the thesis is dedicated to the presentation of our original contributions. In Chapter 3, we establish the concrete definitions of the above notions of weighted limit and colimit in $\mathcal{V}$-graded categories in terms of *weighted cones* satisfying a certain universal property, and we establish equivalent but more abstract formulations as representations of certain $\mathcal{V}$-graded functors. We explore the relationship between enriched weighted limits and $\mathcal{V}$-graded weighted limits. Namely, we prove that $\mathcal{V}$-graded weighted limits are the $\hat{\mathcal{V}}$-enriched weighted limits for a particular class of weights, and we prove that every $\mathcal{V}$-enriched weighted limit is equivalently a $\mathcal{V}$-graded weighted limit. We also use $\mathcal{V}$-graded weighted limits to define weighted limits in $\mathcal{V}$-actegories, which we discover to admit a simpler definition. Finally, we discuss $\mathcal{V}$-graded powers and $\mathcal{V}$-graded conical limits, which are examples of $\mathcal{V}$-graded limits, and we use $\mathcal{V}$-graded structure to define $\mathcal{V}$-enriched weighted limits where $\mathcal{V}$ is monoidal but not necessarily biclosed or complete.

Author's Note. This thesis is based on joint work with Rick Blute and Rory Lucyshyn-Wright. It presents definitions and results which were developed collaboratively, which will form the basis for a forthcoming shorter journal paper coauthored by the three of us.

Chapter 1

Preliminaries

1.1 Monoidal Categories

In this section, we recall concepts from monoidal category theory that will be required throughout the thesis. For a general reference, we recommend [16] and [4].

Definition 1.1.1. A **monoidal category** $\mathcal{V}$ consists of

- a category $\mathcal{V}$;
- a functor $\otimes : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$, called the *tensor product* – we write $X \otimes Y$ for the image under $\otimes$ of the pair (X, Y);
- an object $I \in \mathcal{V}$, called the *unit*;
- an *associator* isomorphism $a_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z)$ natural in $X, Y, Z \in \mathcal{V}$;
- a *left unitor* and a *right unitor* isomorphism $\ell_X : I \otimes X \xrightarrow{\sim} X$ and $r_X : X \otimes I \xrightarrow{\sim} X$ natural in $X \in \mathcal{V}$;

The data must satisfy the associativity (pentagon) and unit (triangle) coherence conditions; i.e., the following diagrams must commute for all $X, Y, Z, U \in ob(\mathcal{V})$.

$$\begin{tikzcd}
((X \otimes Y) \otimes Z) \otimes U \arrow[r, "a_{X,Y,Z} \otimes U"] \arrow[d, "a_{X \otimes Y,Z,U}"'] & (X \otimes (Y \otimes Z)) \otimes U \arrow[r, "a_{X,Y \otimes Z,U}"] & X \otimes ((Y \otimes Z) \otimes U) \arrow[d, "X \otimes a_{Y,Z,U}"] \\
(X \otimes Y) \otimes (Z \otimes U) \arrow[rr, "a_{X,Y,Z \otimes U}"] & & X \otimes (Y \otimes (Z \otimes U))
\end{tikzcd} \tag{1.1.1}$$

$$\begin{tikzcd}
(X \otimes I) \otimes Y \arrow[rr, "a_{X,I,Y}"] \arrow[dr, "r_X \otimes id_Y"'] & & X \otimes (I \otimes Y) \arrow[dl, "id_X \otimes \ell_Y"] \\
& X \otimes Y &
\end{tikzcd} \tag{1.1.2}$$

We say $\mathcal{V}$ is **strict monoidal** if a, ℓ, and r are identities.

The reader is very likely to be already familiar with tensor products in their studies of linear algebra. For this reason, the canonical example of a monoidal category is $Vect_k$, the category of vector spaces and linear maps over a field k. In this case, our tensor product functor maps two vector spaces V and W to their traditional tensor product $V \otimes W$ and two linear maps $f : V \to V'$ and $g : W \to W'$ to $f \otimes g : V \otimes W \to V' \otimes W'$. The monoidal unit in $Vect$ is the 1-dimensional vector space k.

There are many other interesting monoidal categories. For instance, monoidal structure can be given by category-theoretic product with a terminal object as unit. This type of monoidal category is called **cartesian monoidal**. Another interesting point to consider is that a category can carry more than one monoidal structure.

Example 1.1.2. *Set*, the category of sets and functions, can be made into a monoidal category with cartesian product $\times : Set \times Set \to Set$. Alternatively, we can determine our tensor product to be the disjoint union $+ : Set \times Set \to Set$.

Remark 1.1.3. Throughout this thesis, we adopt the same set-theoretic framework of (Grothendieck) universes as in [15]. For convenience, we do not distinguish between a universe and the category of sets it determines. We fix a universe SET, which we refer to as the *category of large sets*. We say that a set is SET-**small** if it is an object of SET, and we say that a category is SET-small if it a category internal to SET. We refer to the category Set from the previous example as the *category of small sets* and we note that it is a SET-small universe; i.e., Set itself is an object of SET.

Definition 1.1.4. A **symmetric monoidal category** is a monoidal category $\mathcal{V}$ equipped with an additional symmetry isomorphism $\sigma_{X,Y} : X \otimes Y \to Y \otimes X$ natural in $X, Y \in \mathcal{V}$. In addition to the coherence conditions given by Diagram (1.1) and Diagram (1.2), $\mathcal{V}$ is subject to the commutativity of the following diagrams for all $X, Y, Z \in ob(\mathcal{V})$.

$$\begin{tikzcd}
(X \otimes Y) \otimes Z \arrow[r, "a_{X,Y,Z}"] \arrow[d, "\sigma_{X,Y} \otimes Z"'] & X \otimes (Y \otimes Z) \arrow[r, "\sigma_{X,Y \otimes Z}"] & (Y \otimes Z) \otimes X \arrow[d, "a_{Y,Z,X}"] \\
(Y \otimes X) \otimes Z \arrow[r, "a_{Y,X,Z}"'] & Y \otimes (X \otimes Z) \arrow[r, "Y \otimes \sigma_{X,Z}"'] & Y \otimes (Z \otimes X)
\end{tikzcd} \tag{1.1.3}$$

$$\begin{tikzcd}
X \otimes Y \arrow[r, "\sigma_{X,Y}"] \arrow[dr, "id_{X \otimes Y}"'] & Y \otimes X \arrow[d, "\sigma_{Y,X}"] \\
& X \otimes Y
\end{tikzcd} \tag{1.1.4}$$

A more intuitive interpretation of the latter diagram is that applying symmetry on $X \otimes Y$ and then again on $Y \otimes X$ is equal to the original.

For any monoidal category $(\mathcal{V}, \otimes, I)$, we can construct a monoidal category $(\mathcal{V}^{rev}, \bar{\otimes}, I)$ with the same objects but whose tensor product is such that $A \bar{\otimes} B$ is $B \otimes A$. We can say that if $\mathcal{V}$ is symmetric monoidal then $\mathcal{V}$ is monoidally isomorphic to $\mathcal{V}^{rev}$.

Definition 1.1.5 (Yoneda Embedding). Let $\mathcal{V}$ be a category and let $[\mathcal{V}^{op}, SET]$ be the functor category whose objects are presheaves $F : \mathcal{V}^{op} \to SET$. There is a functor denoted

$\mathsf{Y} : \mathcal{V} \to [\mathcal{V}^{op}, SET]$ called the **Yoneda embedding** which maps an object $X \in ob(\mathcal{V})$ to its contravariant representable presheaf $\mathcal{V}(-, X) : \mathcal{V}^{op} \to SET$ and maps a morphism $f : X \to Y$ in $\mathcal{V}$ to the natural transformation $\mathcal{V}(-, f) : \mathcal{V}(-, X) \Rightarrow \mathcal{V}(-, Y)$ whose components are given by post-composition with f.

Theorem 1.1.6 (Yoneda Lemma)**.** Let $\mathcal{V}$ be a category and let $F : \mathcal{V}^{op} \to SET$. There is a bijection between elements $x \in F(X)$ for some $X \in ob(\mathcal{V})$ and natural transformations of the form $\mathsf{Y}X \Rightarrow F$; i.e.,

$$[\mathcal{V}^{op}, SET](\mathsf{Y}X, F) \cong FX$$

Additionally, this isomorphism is natural in X and F.

Remark 1.1.7. The bijection above is given in the following way. Let $x \in FX$. We can construct a natural transformation $\beta : \mathsf{Y}X \Rightarrow F$ such that $\beta_Z(f) := (Ff)(x)$ for every $f : Z \to X$ in $\mathcal{V}$. Conversely, let $\alpha : \mathsf{Y}X \Rightarrow F$ be a natural transformation. We can obtain an element of FX by evaluating the component $\alpha_X : \mathcal{V}(X, X) \to FX$ at $id_X : X \to X$.

Remark 1.1.8. The Yoneda embedding is a fully faithful functor. In fact, this follows from an application of the Yoneda Lemma.

Definition 1.1.9. A monoidal category $\mathcal{V}$ is **left closed** if for every $X \in ob(\mathcal{V})$ the functor $X \otimes (-) : \mathcal{V} \to \mathcal{V}$ has a right adjoint denoted $(-)^X : \mathcal{V} \to \mathcal{V}$. Equivalently, there is a bijection

$$\mathcal{V}(X \otimes Y, Z) \cong \mathcal{V}(Y, Z^X) \tag{1.1.5}$$

natural in $X, Y, Z \in \mathcal{V}$. For a morphism $f : Y \to Z^X$, we call the corresponding morphism under the adjunction $\bar{f} : X \otimes Y \to Z$ the exponential transpose of f and we call $Z^X \in ob(\mathcal{V})$ the left-closed hom. The counit of the adjunction is a family of morphisms $e_Y : X \otimes Y^X \to Y$ $(Y \in \mathcal{V})$ whose component at Y is called the *evaluation* at Y.

Similarly, $\mathcal{V}$ is **right closed** if for every $X \in ob(\mathcal{V})$ the functor $(-) \otimes X : \mathcal{V} \to \mathcal{V}$ has a right adjoint denoted ${}^X(-) : \mathcal{V} \to \mathcal{V}$. Equivalently, there is a bijection

$$\mathcal{V}(Y \otimes X, Z) \cong \mathcal{V}(Y, {}^X Z) \tag{1.1.6}$$

natural in $X, Y, Z \in \mathcal{V}$. For a morphism $f : Y \to {}^X Z$, we call the corresponding morphism under the adjunction $\bar{f} : Y \otimes X \to Z$ the exponential transpose of f and we call ${}^X Z \in ob(\mathcal{V})$ the right-closed hom. The counit of the adjunction is a family $e_Y : {}^X Y \otimes X \to Y$ $(Y \in \mathcal{V})$ whose component at Y is called the *evaluation* at Y.

If $\mathcal{V}$ is both left and right closed, we call it **biclosed**. When $\mathcal{V}$ is symmetric monoidal, there is a natural isomorphism between the functors $(-) \otimes X$ and $X \otimes (-)$, so being closed on either side guarantees that it is biclosed. In this case, we simply say that $\mathcal{V}$ is symmetric monoidal closed.

Lemma 1.1.10. The left-closed and right-closed homs satisfy the following statements:

- $Z^{X \otimes Y} \cong (Z^X)^Y$ and ${}^{X \otimes Y}Z \cong {}^X({}^Y Z)$
- $Z^I \cong Z \cong {}^I Z$

- ${}^X(Z^Y) \cong ({}^XZ)^Y$

naturally in $X, Y, Z \in \mathcal{V}$ for a monoidal biclosed $\mathcal{V}$.

Proof. For any U in $\mathcal{V}$, we have

$$\begin{aligned} \mathcal{V}(U, Z^{X\otimes Y}) &\cong \mathcal{V}((X \otimes Y) \otimes U, Z) \\ &\cong \mathcal{V}((X \otimes (Y \otimes U), Z) \\ &\cong \mathcal{V}(Y \otimes U, Z^X) \\ &\cong \mathcal{V}(U, (Z^X)^Y) \end{aligned}$$

$$\begin{aligned} \mathcal{V}(U, {}^{X\otimes Y}Z) &\cong \mathcal{V}(U \otimes (X \otimes Y), Z) \\ &\cong \mathcal{V}((U \otimes X) \otimes Y, Z) \\ &\cong \mathcal{V}(U \otimes X, {}^YZ) \\ &\cong \mathcal{V}(U, {}^X({}^YZ)) \end{aligned}$$

natural in every variable. By the fullness and faithfulness of the Yoneda embedding, it follows that $Z^{X\otimes Y} \cong (Z^X)^Y$ and ${}^{X\otimes Y}Z \cong {}^X({}^YZ)$.

We have for any U in $\mathcal{V}$,

$$\mathcal{V}(U, Z^I) \cong \mathcal{V}(I \otimes U, Z) \cong \mathcal{V}(U, Z) \cong \mathcal{V}(U \otimes I, Z) \cong \mathcal{V}(U, {}^IZ)$$

natural in every variable. By the fullness and faithfulness of the Yoneda embedding, it follows that $Z^I \cong Z \cong {}^IZ$.

We have for any U in $\mathcal{V}$,

$$\begin{aligned} \mathcal{V}(U, {}^X(Z^Y)) &\cong \mathcal{V}(U \otimes X, Z^Y) \\ &\cong \mathcal{V}(Y \otimes (U \otimes X), Z) \\ &\cong \mathcal{V}((Y \otimes U) \otimes X, Z) \\ &\cong \mathcal{V}(Y \otimes U, {}^XZ) \\ &\cong \mathcal{V}(U, ({}^XZ)^Y) \end{aligned}$$

natural in every variable. By the fullness and faithfulness of the Yoneda embedding, it follows that ${}^X(Z^Y) \cong ({}^XZ)^Y$. ∎

While $[\mathcal{V}^{op}, SET]$ can be regarded as cartesian monoidal, we often prefer to equip it with a different monoidal structure that in many ways is more compatible with the monoidal structure of $\mathcal{V}$.

Example 1.1.11 (Day Convolution)**.** In the following discussion, we summarize some standard material on Day Convolution in a manner similar to the exposition in [15].

Let $\mathcal{V}$ be a SET-small monoidal category. From this point forward, we will use $\hat{\mathcal{V}}$ to denote $[\mathcal{V}^{op}, SET]$. The category $\hat{\mathcal{V}}$ is biclosed monoidal under Day Convolution [7], a monoidal product given by the coend

$$(P \otimes Q)(X) = \int^{Y,Z \in \mathcal{V}} \mathcal{V}(X, Y \otimes Z) \times PY \times QZ \qquad (P, Q \in \hat{\mathcal{V}},\ X \in \mathcal{V})$$

the monoidal unit is given by $\mathsf{Y}I = \mathcal{V}(-, I)$. There is a bijective correspondence between morphisms $\Phi : P \otimes Q \Rightarrow R$ in $\hat{\mathcal{V}}$ and families of functions $\phi_{X,Y} : PX \times QY \to R(X \otimes Y)$ natural in $X, Y \in \mathcal{V}$. To construct the left-closed hom Q^P we consider the requirement by (1.1.5) and the Yoneda Lemma that

$$\hat{\mathcal{V}}(P \otimes \mathsf{Y}X, Q) \cong \hat{\mathcal{V}}(\mathsf{Y}X, Q^P) \cong Q^P(X)$$

so we define $Q^P(X) := \hat{\mathcal{V}}(P \otimes \mathsf{Y}X, Q)$. There is a bijective correspondence between elements of $\hat{\mathcal{V}}(P \otimes \mathsf{Y}X, Q)$ and families $PY \times \mathcal{V}(Z, X) \to Q(Y \otimes Z)$ natural in $Y, Z \in \mathcal{V}$, which themselves correspond to families $PY \to Q(Y \otimes X)$ natural in $Y \in \mathcal{V}$. Thus, the left-closed hom is equivalently given by $Q^P(X) = \hat{\mathcal{V}}(P, Q(- \otimes X))$. A similar process gives ${}^{P}Q(X) = \hat{\mathcal{V}}(P, Q(X \otimes -))$. When $P = \mathsf{Y}X$, we have that $Q^{\mathsf{Y}X} \cong Q(X \otimes -)$ and ${}^{\mathsf{Y}X}Q \cong Q(- \otimes X)$ by the Yoneda Lemma.

When $\hat{\mathcal{V}}$ has Day convolution monoidal structure, the Yoneda embedding $\mathsf{Y} : \mathcal{V} \to \hat{\mathcal{V}}$ is a strong monoidal functor [10]. In particular, $\mathsf{Y}(X \otimes Z) \cong \mathsf{Y}X \otimes \mathsf{Y}Z$.

For any $\alpha : Y \to X$ in $\mathcal{V}$ and $P : \mathcal{V}^{op} \to SET$, we often use the notation α^* for $P\alpha : PX \to PY$, and for $p \in PX$ we call $\alpha^*(p) \in PY$ the *reindexing* of p along α. By the Yoneda Lemma, every $p \in PX$ is equivalently given by a corresponding $\tilde{p} : \mathsf{Y}X \Rightarrow P$. We have that $\widetilde{\alpha^*(p)} : \mathsf{Y}Y \Rightarrow P$ is given by

$$\mathsf{Y}Y \xRightarrow{\mathsf{Y}\alpha} \mathsf{Y}X \xRightarrow{\tilde{p}} P$$

For any $\Phi : P \otimes Q \Rightarrow R$ in $\hat{\mathcal{V}}$ with its corresponding family of $\phi_{X,Y} : PX \times QY \to R(X \otimes Y)$, the element $\phi_{X,Y}(p, q) \in R(X \otimes Y)$ for $p \in PX$ and $q \in QY$ corresponds under the Yoneda Lemma to the natural transformation

$$\mathsf{Y}(X \otimes Y) \xRightarrow{\sim} \mathsf{Y}X \otimes \mathsf{Y}Y \xRightarrow{\tilde{p} \otimes \tilde{q}} P \otimes Q \xRightarrow{\Phi} R$$

Lemma 1.1.12. Let $\mathcal{V}$ be a biclosed monoidal category. The Yoneda embedding preserves left-closed and right-closed homs up to isomorphism; i.e., $\mathsf{Y}(B^A) \cong \mathsf{Y}B^{\mathsf{Y}A}$ and $\mathsf{Y}({}^{A}B) \cong {}^{\mathsf{Y}A}\mathsf{Y}B$.

Proof. By an application of the Yoneda Lemma, the strong monoidal property of the Yoneda embedding, and the adjunction (1.6), we have

$$\begin{aligned} \mathsf{Y}B^{\mathsf{Y}A}(X) &\cong \hat{\mathcal{V}}(\mathsf{Y}X, \mathsf{Y}B^{\mathsf{Y}A}) \\ &\cong \hat{\mathcal{V}}(\mathsf{Y}A \otimes \mathsf{Y}X, \mathsf{Y}B) \\ &\cong \hat{\mathcal{V}}(\mathsf{Y}(A \otimes X), \mathsf{Y}B) \end{aligned}$$

$$\begin{aligned} &\cong \mathcal{V}(A \otimes X, B) \\ &\cong \mathcal{V}(X, B^A) \\ &= \mathsf{Y}(B^A)(X) \end{aligned}$$

natural in $X \in \mathcal{V}$. Thus, $\mathsf{Y}(B^A) \cong \mathsf{Y}B^{\mathsf{Y}A}$. A similar argument gives $\mathsf{Y}({}^{A}B) \cong {}^{\mathsf{Y}A}\mathsf{Y}B$. ∎

1.2 Actegories

Much like how a monoid can act on a set, a monoidal category $\mathcal{V}$ can act on an ordinary category $\mathcal{C}$. The notion of a category equipped with an action of $\mathcal{V}$ was introduced by Bénabou [3] in 1967, and the term *actegory* first appears in print years later, in McCrudden's work [17]. For a general reference on actegories, we recommend [6], whose notation we adopt in this thesis.

Definition 1.2.1. Let $\mathcal{V}$ be a monoidal category. A **left $\mathcal{V}$-actegory** $\mathcal{C}$ consists of:

- an ordinary category $\mathcal{C}$;
- a functor $- \bullet = : \mathcal{V} \times \mathcal{C} \to \mathcal{C}$ often called the *left action* of $\mathcal{V}$ on $\mathcal{C}$;
- a *multiplicator* isomorphism $\mu_{X,Y,A} : X \bullet (Y \bullet A) \xrightarrow{\sim} (X \otimes Y) \bullet A$ natural in $X, Y \in \mathcal{V}$, $A \in \mathcal{C}$;
- a *unitor* isomorphism $\eta_A : A \xrightarrow{\sim} I \bullet A$ natural in $A \in \mathcal{C}$

The data must satisfy the following coherence conditions; i.e., the following diagrams must commute for all $X, Y, Z \in ob(\mathcal{V})$ and $A \in ob(\mathcal{C})$.

$$\begin{tikzcd}
X \bullet (Y \bullet (Z \bullet A)) \arrow[r, "\mu_{X,Y,Z\bullet A}"] \arrow[d, "X\bullet\mu_{Y,Z,A}"'] & (X \otimes Y) \bullet (Z \bullet A) \arrow[r, "\mu_{X\otimes Y,Z,A}"] & ((X \otimes Y) \otimes Z) \bullet A \arrow[d, "a_{X,Y,Z}\bullet A"] \\
X \bullet ((Y \otimes Z) \bullet A) \arrow[rr, "\mu_{X,Y\otimes Z,A}"] & & (X \otimes (Y \otimes Z)) \bullet A
\end{tikzcd} \tag{1.2.1}$$

$$\begin{tikzcd}
X \bullet A \arrow[rr, "\eta_{X\bullet A}"] \arrow[dr, "\ell_X^{-1}\bullet A"'] & & I \bullet (X \bullet A) \arrow[dl, "\mu_{I,X,A}"] & X \bullet A \arrow[rr, "X\bullet\eta_A"] \arrow[dr, "r_X^{-1}\bullet A"'] & & X \bullet (I \bullet A) \arrow[dl, "\mu_{X,I,A}"] \\
& (I \otimes X) \bullet A & & & (X \otimes I) \bullet A &
\end{tikzcd} \tag{1.2.2}$$

There is also a notion of **right $\mathcal{V}$-actegory** where the action of $\mathcal{V}$ on $\mathcal{C}$ is given on the right. In this case, the multiplicator and unitor are such that $(A \bullet Y) \bullet X \cong A \bullet (Y \otimes X)$ and $A \cong A \bullet I$.

We note that a right $\mathcal{V}$-actegory is equivalently a left $\mathcal{V}^{rev}$-actegory. In this text, we will use the term $\mathcal{V}$-actegory to mean left $\mathcal{V}$-actegory.

Example 1.2.2. Let $\mathcal{V}$ be a monoidal category. We can consider $\mathcal{V}$ as both a left $\mathcal{V}$-actegory and a right $\mathcal{V}$-actegory using the tensor product in $\mathcal{V}$ as the action.

Definition 1.2.3. Let $(\mathcal{C}, \bullet, \mu, \eta)$ and $(\mathcal{D}, \bar{\bullet}, \bar{\mu}, \bar{\eta})$ be $\mathcal{V}$-actegories. A **lax morphism of actegories** between them is a functor $F : \mathcal{C} \to \mathcal{D}$ equipped with a lineator morphism $\ell_{X,A} : X \mathbin{\bar{\bullet}} FA \to F(X \bullet A)$ natural in $X \in \mathcal{V}$ and $A \in \mathcal{C}$ such that the following diagrams commute for all $X, Y \in \mathcal{V}$ and $A \in \mathcal{C}$.

$$\begin{tikzcd}
X \mathbin{\bar{\bullet}} (Y \mathbin{\bar{\bullet}} FA) \arrow[r, "X \mathbin{\bar{\bullet}} \ell_{Y,A}"] \arrow[d, "\bar{\mu}_{X,Y,FA}"'] & X \mathbin{\bar{\bullet}} F(Y \bullet A) \arrow[r, "\ell_{X,Y \bullet A}"] & F(X \bullet (Y \bullet A)) \arrow[d, "F(\mu_{X,Y,A})"] \\
(X \otimes Y) \mathbin{\bar{\bullet}} FA \arrow[rr, "\ell_{X \otimes Y,A}"] & & F((X \otimes Y) \bullet A)
\end{tikzcd} \tag{1.2.3}$$

$$\begin{tikzcd}
I \mathbin{\bar{\bullet}} FA \arrow[rr, "\ell_{I,A}"] & & F(I \bullet A) \\
& FA \arrow[ul, "\bar{\eta}_{FA}"] \arrow[ur, "F(\eta_A)"'] &
\end{tikzcd} \tag{1.2.4}$$

If ℓ is a natural isomorphism, we call F a **strong morphism of actegories**.

Definition 1.2.4. Let $F, G : \mathcal{C} \to \mathcal{D}$ be lax morphisms of actegories with lineators ℓ and ν respectively. A **natural transformation of actegories** between them is a natural transformation $\beta : F \Rightarrow G$ such that the following diagram commutes for every $X, Y \in ob(\mathcal{V})$ and $A \in ob(\mathcal{C})$.

$$\begin{tikzcd}
X \mathbin{\bar{\bullet}} FA \arrow[r, "X \mathbin{\bar{\bullet}} \beta_A"] \arrow[d, "\ell_{X,A}"'] & X \mathbin{\bar{\bullet}} GA \arrow[d, "\nu_{X,A}"] \\
F(X \bullet A) \arrow[r, "\beta_{X \bullet A}"'] & G(X \bullet A)
\end{tikzcd} \tag{1.2.5}$$

Remark 1.2.5. There is a 2-category ${}_{\mathcal{V}}ACT$ whose objects are $\mathcal{V}$-actegories, 1-cells are lax morphisms, and 2-cells are natural transformations. There is also a 2-category ${}_{\mathcal{V}}ACT^{Strong}$ whose objects are $\mathcal{V}$-actegories, 1-cells are strong morphisms, and 2-cells are natural transformations. Similarly, $ACT_{\mathcal{V}}$ and $ACT_{\mathcal{V}}^{Strong}$ are the analogous 2-categories whose objects are right $\mathcal{V}$-actegories.

1.3 Enriched Category Theory

In working with categories, we often encounter situations where the hom-sets $\mathcal{C}(A, B)$, whose elements are morphisms from A to B, carry additional structure beyond that of a set. For instance, a hom-set might be equipped with operations of addition and scalar multiplication, forming a vector space of morphisms. Enriched category theory provides a framework to handle cases like these by allowing the hom-sets to underlie *hom-objects* in a monoidal category. Enrichment provides methods to capture and work coherently with the extra structure of the morphisms.

In this section, we recall some standard material from enriched category theory. For a general reference on $\mathcal{V}$-enriched categories, we recommend [4] and [11], from which we also borrow notation and terminology from these. We additionally follow the convention of [15] in distinguishing between left and right $\mathcal{V}$-categories.

Let $(\mathcal{V}, \otimes, I)$ be a monoidal category.

Definition 1.3.1. A left $\mathcal{V}$-enriched category or simply **left $\mathcal{V}$-category** $\mathcal{A}$ consists of:

- a set of objects $ob(\mathcal{A})$;
- a *hom-object* $\mathcal{A}(A, B) \in ob(\mathcal{V})$ for each pair of objects $A, B \in \mathcal{A}$;
- a composition morphism $m_{A,B,C} : \mathcal{A}(B, C) \otimes \mathcal{A}(A, B) \to \mathcal{A}(A, C)$ in $\mathcal{V}$ for every $A, B, C \in ob(\mathcal{A})$;
- an identity morphism $j_A : I \to \mathcal{A}(A, A)$ for every $A \in ob(\mathcal{A})$;

subject to the associativity and unit axioms expressed by the commutativity of the following diagrams.

$$\begin{tikzcd}
(\mathcal{A}(C,D) \otimes \mathcal{A}(B,C)) \otimes \mathcal{A}(A,B) \arrow[rr, "a"] \arrow[d, "m_{B,C,D} \otimes \mathcal{A}(A,B)"'] & & \mathcal{A}(C,D) \otimes (\mathcal{A}(B,C) \otimes \mathcal{A}(A,B)) \arrow[d, "\mathcal{A}(C,D) \otimes m_{A,B,C}"] \\
\mathcal{A}(B,D) \otimes \mathcal{A}(A,B) \arrow[dr, "m_{A,B,D}"'] & & \mathcal{A}(C,D) \otimes \mathcal{A}(A,C) \arrow[dl, "m_{A,C,D}"] \\
& \mathcal{A}(A,D) &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{A}(B,B) \otimes \mathcal{A}(A,B) \arrow[r, "m_{A,B,B}"] & \mathcal{A}(A,B) & \mathcal{A}(A,B) \otimes \mathcal{A}(A,A) \arrow[l, "m_{A,A,B}"'] \\
I \otimes \mathcal{A}(A,B) \arrow[u, "j_B \otimes \mathcal{A}(A,B)"] \arrow[ur, "\ell"'] & & \mathcal{A}(A,B) \otimes I \arrow[u, "\mathcal{A}(A,B) \otimes j_A"'] \arrow[ul, "r"]
\end{tikzcd}$$

A **right $\mathcal{V}$-category** is precisely a left $\mathcal{V}^{rev}$-category. It consists of the same data as a left $\mathcal{V}$-category with the only difference being that its composition morphisms have the form

$$m_{A,B,C} : \mathcal{A}(A, B) \otimes \mathcal{A}(B, C) \to \mathcal{A}(A, C)$$

In this text, we will use the term $\mathcal{V}$-category to mean left $\mathcal{V}$-category, unless otherwise specified.

When $\mathcal{V}$ is symmetric, a left $\mathcal{V}$-category is equivalently a right $\mathcal{V}$-category since we can apply symmetry to the tensor product in the composition morphisms. It is then unnecessary to distinguish between left and right enriched categories over a symmetric base, and we only need to choose one convention (and stick to it), as in [11] where Kelly works with left $\mathcal{V}$-categories or in [2] where Bénabou works with right $\mathcal{V}$-categories. In both cases, these are are simply referred to as enriched categories.

It is possible for a category to be enriched both on the left and right.

Example 1.3.2 ($\mathcal{V}$ as a $\mathcal{V}$-category)**.** Let $\mathcal{V}$ be a biclosed monoidal category. $\mathcal{V}$ carries both a left and right $\mathcal{V}$-enriched structure. As a left $\mathcal{V}$-category, $\mathcal{V}$ has hom-objects $\mathcal{V}(X,Y)$ given by the right-closed hom ${}^XY$ in $ob(\mathcal{V})$. The composition morphism $m_{X,Y,Z} : {}^YZ \otimes {}^XY \to {}^XZ$ corresponds under the adjunction (1.1.6) to the composite

$$({}^YZ \otimes {}^XY) \otimes X \xrightarrow{\sim} {}^YZ \otimes ({}^XY \otimes X) \xrightarrow{{}^YZ \otimes e_{X,Y}} {}^YZ \otimes Y \xrightarrow{e_{Y,Z}} Z$$

and the identity morphism $j_X : I \to {}^XX$ corresponds under the adjunction (1.1.6) to the left unitor $\ell_X : I \otimes X \to X$.

As a right $\mathcal{V}$-category, $\mathcal{V}$ has hom-objects $\mathcal{V}(X,Y)$ given by the left-closed hom Y^X in $ob(\mathcal{V})$. The composition morphism $m_{X,Y,Z} : Y^X \otimes Z^Y \to Z^X$ corresponds under the adjunction (1.1.5) to the composite

$$X \otimes (Y^X \otimes Z^Y) \xrightarrow{\sim} (X \otimes Y^X) \otimes Z^Y \xrightarrow{e_{X,Y} \otimes Z^Y} Y \otimes Z^Y \xrightarrow{e_{Y,Z}} Z$$

and the identity morphism $j_X : I \to X^X$ corresponds under the adjunction (1.1.5) to the right unitor $r_X : X \otimes I \to X$.

Example 1.3.3. A *Set*-category is a (locally small) ordinary category.

A *Cat*-category is called a 2-category. The objects and morphisms of the hom-categories are called 1-cells and 2-cells respectively. For a general reference on 2-categories, see [16].

Since $Vect$ is a symmetric monoidal closed category, it is enriched in itself. Its hom-objects $Vect(U,V)$ are the spaces of linear transformations from U to V.

Example 1.3.4. The unit $\mathcal{V}$-category $\mathcal{I}$ is the $\mathcal{V}$-category with one object $*$ and with $\mathcal{I}(*,*) = I$. We note that $\mathcal{I}$ is both a left $\mathcal{V}$-category and a right $\mathcal{V}$-category.

Definition 1.3.5. Let $\mathcal{A}$ be a left $\mathcal{V}$-category for an arbitrary (potentially non-symmetric) monoidal $\mathcal{V}$. The **formal opposite** $\mathcal{A}^\circ$ of $\mathcal{A}$ is a right $\mathcal{V}$-category with the same objects as $\mathcal{A}$ but whose hom-objects are given by $\mathcal{A}^\circ(A,B) := \mathcal{A}(B,A)$ and whose composition morphisms $\mathcal{A}^\circ(A,B) \otimes \mathcal{A}^\circ(B,C) \to \mathcal{A}^\circ(A,C)$ are given by the $m_{C,B,A} : \mathcal{A}(B,A) \otimes \mathcal{A}(C,B) \to \mathcal{A}(C,A)$ from $\mathcal{A}$. Similarly, for a right $\mathcal{V}$-category, the formal opposite is a left $\mathcal{V}$-category.

When $\mathcal{V}$ is symmetric, we do not need to concern ourselves with whether the formal opposite of $\mathcal{A}$ is left or right enriched. In [11], Kelly uses $\mathcal{A}^{op}$ for the left $\mathcal{V}$-category isomorphic to the right $\mathcal{V}$-category $\mathcal{A}^\circ$.

Given a monoidal category $\mathcal{V}$, we say that a morphism $f : X \to Y$ is a **generalized element** of Y with shape X. A **global element** of Y is a generalized element whose shape is the monoidal unit; i.e., $f : I \to Y$. When $\mathcal{V} = Set$, a global element of A is a morphism $f : \{*\} \to A$, which picks out an element $a \in A$. When $\mathcal{V} = Vect_k$, a global element of V is a morphism $f : k \to V$, which corresponds to a vector $v \in V$ since $Hom_k(k,V) \cong V$.

Definition 1.3.6. Let $\mathcal{A}$ be a $\mathcal{V}$-category. The **underlying ordinary category** $\mathcal{A}_0$ of $\mathcal{A}$ has the same objects as $\mathcal{A}$ and a morphism $f : A \to B$ in $\mathcal{A}_0$ is given by a global element of $\mathcal{A}(A, B)$; i.e., a morphism $f : I \to \mathcal{A}(A, B)$ in $\mathcal{V}$. The composite $g \cdot f : A \to C$ of $f : A \to B$ and $g : B \to C$ in $\mathcal{A}_0$ is given by

$$I \cong I \otimes I \xrightarrow{g \otimes f} \mathcal{A}(B, C) \otimes \mathcal{A}(A, B) \xrightarrow{m_{A,B,C}} \mathcal{A}(A, C)$$

in $\mathcal{V}$ and the identities $id_A : A \to A$ in $\mathcal{A}_0$ are the $j_A : I \to \mathcal{A}(A, A)$ in $\mathcal{V}$.

Remark 1.3.7. Taking $\mathcal{V}$ enriched in itself, its underlying ordinary category $\mathcal{V}_0$ is isomorphic to $\mathcal{V}$. We may identify a morphism $f : I \to {}^X Z$ in $\mathcal{V}_0$ with its corresponding morphism $\bar{f} \cdot \ell^{-1} : X \to Z$ in $\mathcal{V}$.

Definition 1.3.8. For $\mathcal{V}$-categories $\mathcal{A}$ and $\mathcal{B}$, a (left) **$\mathcal{V}$-functor** $T : \mathcal{A} \to \mathcal{B}$ consists of a function $T : ob(\mathcal{A}) \to ob(\mathcal{B})$ and a morphism, called a *structural morphism*,

$$T_{A,A'} : \mathcal{A}(A, A') \to \mathcal{B}(TA, TA')$$

in $\mathcal{V}$ for every $A, A' \in ob(\mathcal{A})$ such that the following diagrams commute.

$$\begin{tikzcd}
\mathcal{A}(B,C) \otimes \mathcal{A}(A,B) \arrow[r, "m_{A,B,C}"] \arrow[d, "T_{B,C} \otimes T_{A,B}"'] & \mathcal{A}(A,C) \arrow[d, "T_{A,C}"] \\
\mathcal{B}(TB,TC) \otimes \mathcal{B}(TA,TB) \arrow[r, "m_{TA,TB,TC}"'] & \mathcal{B}(TA,TC)
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathcal{A}(A,A) \arrow[dd, "T_{A,A}"] \\
I \arrow[ur, "j_A"] \arrow[dr, "j_{TA}"'] & \\
 & \mathcal{B}(TA,TA)
\end{tikzcd}$$

A right $\mathcal{V}$-functor is precisely a left $\mathcal{V}^{rev}$-functor. These consist of the same data as a left $\mathcal{V}$-functor satisfying analogous axioms, however we write $F^{C,C'}$ for the structural morphisms of $F : \mathcal{C} \to \mathcal{D}$. The $\mathcal{V}$-functor T is **fully faithful** if every structural morphism $T_{A,A'}$ is an isomorphism in $\mathcal{V}$.

Note that when $\mathcal{B} = \mathcal{V}$, the structural morphisms $T_{A,A'} : \mathcal{A}(A, A') \to {}^{TA}TA'$ are equivalently given by their transposes $\overline{T}_{A,A'} : \mathcal{A}(A, A') \otimes TA \to TA'$. Similarly, when $\mathcal{D} = \mathcal{V}$, the structural morphisms $F^{C,C'} : \mathcal{C}(C, C') \to FC'^{FC}$ are equivalently given by their transposes $\overline{F}^{C,C'} : FC \otimes \mathcal{C}(C, C') \to FC'$.

Definition 1.3.9. Let $T, S : \mathcal{A} \to \mathcal{B}$ be $\mathcal{V}$-functors. A (left) **$\mathcal{V}$-natural transformation** $\alpha : T \Rightarrow S$ is a family $\alpha_A : TA \to SA$ $(A \in \mathcal{A})$ in $\mathcal{B}_0$ such that the diagram

$$\begin{tikzcd}
TA \arrow[r, "Tf"] \arrow[d, "\alpha_A"'] & TA' \arrow[d, "\alpha_{A'}"] \\
SA \arrow[r, "Sf"'] & SA'
\end{tikzcd}$$

commutes for every $f : A \to A'$ in $\mathcal{A}_0(A, A')$.

Remark 1.3.10. There is a 2-category denoted ${}_{\mathcal{V}}CAT$ whose objects are $\mathcal{V}$-categories, 1-cells are $\mathcal{V}$-functors, and 2-cells are $\mathcal{V}$-natural transformations. Similarly, $CAT_{\mathcal{V}}$ is the analogous 2-category whose objects are right $\mathcal{V}$-categories.

1.3.1 Weighted Limits in Enriched Categories

Weighted limits were introduced in the 1970s, under the names *mean cotensor products* [5] and *formal Homs* [1], as a generalization of limits in ordinary categories. A decade later, Kelly referred to them as *indexed limits* in [11], a book which to this day is a popular reference on the topic of weighted limits.

The treatment of weighted limits in [11] relies on a notion of $\mathcal{V}$-enriched presheaf categories that requires $\mathcal{V}$ to be symmetric monoidal closed. However, Street later established the concept of enriched presheaf categories over a non-symmetric base. These were introduced in [18] using $\mathcal{V}$-modules between $\mathcal{V}$-categories. Here, we opt to use an equivalent formulation given by Gordon and Power [9] that makes no reference to $\mathcal{V}$-modules.

For this section only, we let $\mathcal{V}$ be a monoidal biclosed and complete category.

Definition 1.3.11. Let $\mathcal{A}$ be a $\mathcal{V}$-category let $A \in ob(\mathcal{A})$. The covariant hom-$\mathcal{V}$-functor $\mathcal{A}(A,-) : \mathcal{A} \to \mathcal{V}$ maps an object $B \in ob(\mathcal{A})$ to the hom-object $\mathcal{A}(A,B)$ in $\mathcal{V}$, and its structural morphisms

$$\mathcal{A}(A,-)_{B,C} : \mathcal{A}(B,C) \to {}^{\mathcal{A}(A,B)}\mathcal{A}(A,C) \qquad (B,C \in \mathcal{A})$$

correspond under the adjunction (1.1.6) to the $m_{A,B,C} : \mathcal{A}(B,C) \otimes \mathcal{A}(A,B) \to \mathcal{A}(A,C)$ in $\mathcal{V}$. Similarly, the contravariant hom-$\mathcal{V}$-functor $\mathcal{A}(-,A) : \mathcal{A}^\circ \to \mathcal{V}$ is the right $\mathcal{V}$-functor whose structural morphisms

$$\mathcal{A}(-,A)_{B,C} : \mathcal{A}(C,B) \to \mathcal{A}(C,A)^{\mathcal{A}(B,A)} \qquad (B,C \in \mathcal{A})$$

correspond under the adjunction (1.1.5) to the $m_{C,B,A} : \mathcal{A}(B,A) \otimes \mathcal{A}(C,B) \to \mathcal{A}(C,A)$ in $\mathcal{V}$.

When $\mathcal{A} = \mathcal{V}$ as a left $\mathcal{V}$-category, the covariant and contravariant hom-$\mathcal{V}$-functors may be denoted ${}^X(-) : \mathcal{V} \to \mathcal{V}$ and ${}^{(-)}X : \mathcal{V}^\circ \to \mathcal{V}$.

Definition 1.3.12 (Street's presheaf $\mathcal{V}$-category)**.** Let $\mathcal{K}$ be a small left $\mathcal{V}$-category. There is a left $\mathcal{V}$-category $\mathcal{P}_{\mathcal{V}}\mathcal{K}$ whose objects are right $\mathcal{V}$-functors $\mathcal{K}^\circ \to \mathcal{V}$. For $F, G : \mathcal{K}^\circ \to \mathcal{V}$ in $\mathcal{P}_{\mathcal{V}}\mathcal{K}$, we denote the hom-object $\mathcal{P}_{\mathcal{V}}\mathcal{K}(F,G)$ by ${}^FG \in ob(\mathcal{V})$ and we define it to be a limit in $\mathcal{V}$ over the diagram

$${}^{FK}GK \xrightarrow{R_{K,K'}} {}^{FK\otimes\mathcal{K}(K',K)}GK' \xleftarrow{S_{K,K'}} {}^{FK'}GK' \qquad (K,K' \in \mathcal{K})$$

where R is the transpose of

$${}^{FK}GK \otimes FK \otimes \mathcal{K}(K',K) \xrightarrow{ev_{FK}\otimes\mathcal{K}(K',K)} GK \otimes \mathcal{K}(K',K) \xrightarrow{\overline{G}^{KK'}} GK'$$

and S is the transpose of

$${}^{FK'}GK' \otimes FK \otimes \mathcal{K}(K',K) \xrightarrow{{}^{FK'}GK'\otimes\overline{F}^{K,K'}} {}^{FK'}GK' \otimes FK' \xrightarrow{ev_{FK'}} GK'$$

There is also a right $\mathcal{V}$-category $\mathcal{P}^{\dagger}_{\mathcal{V}}\mathcal{K}$ whose objects are left $\mathcal{V}$-functors $\mathcal{K} \to \mathcal{V}$. For $H, L : \mathcal{K} \to \mathcal{V}$ in $\mathcal{P}^{\dagger}_{\mathcal{V}}\mathcal{K}$, we denote the hom-object $\mathcal{P}^{\dagger}_{\mathcal{V}}\mathcal{K}(H, L)$ by $L^H \in ob(\mathcal{V})$ and we define it to be a limit in $\mathcal{V}$ over the diagram

$$LK^{HK} \xrightarrow{T_{K,K'}} LK'^{\,\mathcal{K}(K,K')\otimes HK} \xleftarrow{U_{K,K'}} LK'^{HK'} \qquad (K, K' \in \mathcal{K})$$

where T is the transpose of

$$\mathcal{K}(K,K')\otimes HK\otimes LK^{HK} \xrightarrow{\mathcal{K}(K,K')\otimes ev_{HK}} \mathcal{K}(K,K')\otimes LK \xrightarrow{\overline{L}_{K,K'}} LK'$$

and U is the transpose of

$$\mathcal{K}(K,K')\otimes HK\otimes LK'^{HK'} \xrightarrow{\overline{H}_{K,K'}\otimes LK'^{HK'}} HK'\otimes LK'^{HK'} \xrightarrow{ev_{HK'}} LK'$$

We call $\mathcal{P}^{\dagger}_{\mathcal{V}}\mathcal{K}$ **Street's covariant presheaf $\mathcal{V}$-category** and we call $\mathcal{P}_{\mathcal{V}}\mathcal{K}$ **Street's contravariant presheaf $\mathcal{V}$-category**. We note that, while it is convenient to have distinct notations for both cases, it is not necessary. We can recover one of these structures as a special case of the other. In fact, $\mathcal{P}^{\dagger}_{\mathcal{V}}\mathcal{K}$ is exactly $\mathcal{P}_{\mathcal{V}^{rev}}\mathcal{K}^{\circ}$.

When $\mathcal{V}$ is symmetric monoidal closed, $\mathcal{P}_{\mathcal{V}}\mathcal{K}$ and $\mathcal{P}^{\dagger}_{\mathcal{V}}\mathcal{K}$ are isomorphic to the enriched presheaf categories $[\mathcal{K}^{op}, \mathcal{V}]$ and $[\mathcal{K}, \mathcal{V}]$ respectively, as seen as [11].

Remark 1.3.13. We can identify the underlying ordinary category $(\mathcal{P}_{\mathcal{V}}\mathcal{K})_0$ with the hom-category $CAT_{\mathcal{V}}(\mathcal{K}^{\circ}, \mathcal{V})$ where $CAT_{\mathcal{V}}$ is the 2-category of right $\mathcal{V}$-categories.

Theorem 1.3.14 ([18], Strong Yoneda Lemma)**.** Let $\mathcal{A}$ be a $\mathcal{V}$-category and let $F : \mathcal{A}^{\circ} \to \mathcal{V}$ be a right $\mathcal{V}$-functor. There is an isomorphism

$$FA \cong \mathcal{P}_{\mathcal{V}}\mathcal{A}(\mathcal{A}(-, A), F)$$

$\mathcal{V}$-natural in $A \in \mathcal{A}$ and $F \in \mathcal{P}_{\mathcal{V}}\mathcal{A}$. This isomorphism is constructed in the following way.

The structural morphisms $F_{A,B}$ are equivalently given by the transposes $\bar{F}_{A,B} : FA \to {}^{\mathcal{A}(B,A)}FB$. Since the hom-object ${}^{\mathcal{A}(-,A)}F$ is a limit in $\mathcal{V}$, there exists a unique morphism $u : FA \to {}^{\mathcal{A}(-,A)}F$ such that $\bar{F}_{A,B} = p_{A,B} \cdot u$ for every $A, B \in \mathcal{A}$ where $p_{A,B} : {}^{\mathcal{A}(-,A)}F \to {}^{\mathcal{A}(B,A)}FB$ is the component of the limiting cone. The inverse morphism $u^{-1} : {}^{\mathcal{A}(-,A)}F \to FA$ is given by the composite

$${}^{\mathcal{A}(-,A)}F \xrightarrow{p_{A,A}} {}^{\mathcal{A}(A,A)}FA \xrightarrow{r^{-1}} ({}^{\mathcal{A}(A,A)}FA)\otimes I \xrightarrow{1\otimes j_A} ({}^{\mathcal{A}(A,A)}FA)\otimes \mathcal{A}(A,A) \xrightarrow{ev} FA$$

Theorem 1.3.15 (Weak Yoneda Lemma)**.** Let $F : \mathcal{A}^{\circ} \to \mathcal{V}$ be a right $\mathcal{V}$-functor. There is a bijection between the set of $\mathcal{V}$-natural transformations of the form $\alpha : \mathcal{A}(-, A) \Rightarrow F$ and the set of global elements of FA.

Definition 1.3.16 ([18], Yoneda Embedding)**.** Let $\mathcal{A}$ be a $\mathcal{V}$-category. There is a $\mathcal{V}$-functor $\mathsf{Y} : \mathcal{A} \to \mathcal{P}_{\mathcal{V}}\mathcal{A}$ called the enriched Yoneda embedding which maps an object $A \in ob(\mathcal{A})$ to the contravariant hom-$\mathcal{V}$-functor $\mathcal{A}(-, A) : \mathcal{A}^{\circ} \to \mathcal{V}$. The structural morphisms

$$\mathsf{Y}_{A,B} : \mathcal{A}(A, B) \to {}^{\mathcal{A}(-,A)}\mathcal{A}(-, B)$$

are isomorphisms by an application of the Strong Yoneda Lemma with $F = \mathcal{A}(-, B)$. Thus, the enriched Yoneda embedding is a fully faithful $\mathcal{V}$-functor.

Taking $\mathcal{V} = Set$, the enriched Yoneda embedding reduces to the ordinary Yoneda embedding of Definition 1.1.5.

Definition 1.3.17. Let $F : \mathcal{A} \to \mathcal{V}$ be a $\mathcal{V}$-functor. A **representation** of F is given by an object $A \in \mathcal{A}$ and a $\mathcal{V}$-natural isomorphism $\theta : \mathcal{A}(A, -) \Rightarrow F$. We say F is **representable** if it admits a representation (A, θ). The element $\eta : I \to FA$ corresponding to θ by the Weak Yoneda Lemma is called the unit of the representation.

Similarly, for a right $\mathcal{V}$-functor $G : \mathcal{A}^\circ \to \mathcal{V}$, a representation is given by an object $A \in ob(\mathcal{A})$ and a right $\mathcal{V}$-natural isomorphism $\theta : \mathcal{A}(-, A) \Rightarrow G$. The element $\epsilon : I \to FA$ corresponding to θ by the Weak Yoneda Lemma is called the counit of the representation.

Now we have all the necessary structures required to define enriched weighted limits. To motivate the idea of weighted limits, we begin by noting that limits in ordinary categories can be formulated in the following way:

Let $D : \mathcal{K} \to \mathcal{C}$ be a functor. A cone to D is given by a natural transformation $\gamma : \Delta_A \Rightarrow D$ where Δ_A is the constant functor at $A \in ob(\mathcal{C})$, and we have that

$$[\mathcal{K}, \mathcal{C}](\Delta_A, D) \cong [\mathcal{K}, Set](\Delta_{\{*\}}, \mathcal{C}(A, D-))$$

We can then describe a limit of D as a representation of $[\mathcal{K}, Set](\Delta_{\{*\}}, \mathcal{C}(-, D?)) : \mathcal{C}^{op} \to Set$, i.e., an object $L \in \mathcal{C}$ equipped with isomorphisms

$$\mathcal{C}(A, L) \cong [\mathcal{K}, Set](\Delta_{\{*\}}, \mathcal{C}(A, D-))$$

natural in $A \in \mathcal{C}$. The counit of the representation corresponds to the limiting cone $\pi_K : L \to DK$ $(K \in \mathcal{K})$.

We note that ordinary categories are $\mathcal{V}$-categories where $\mathcal{V} = Set$. From this starting point, we can hope to achieve two things: (1) find a way to replace $\Delta_{\{*\}}$ with an arbitrary functor $W : \mathcal{K} \to Set$; (2) find a way to formulate this for $\mathcal{V}$-categories with a given biclosed monoidal and complete $\mathcal{V}$ rather than just Set.

Definition 1.3.18. Let $\mathcal{K}$ be a small $\mathcal{V}$-category and let $\mathcal{A}$ be another (not necessarily small) $\mathcal{V}$-category. Let $\mathcal{W} : \mathcal{K} \to \mathcal{V}$ and $D : \mathcal{K} \to \mathcal{A}$ be $\mathcal{V}$-functors, which we call the *weight* and *diagram* respectively. For all $A \in ob(\mathcal{A})$, we can form the composite $\mathcal{V}$-functor $\mathcal{A}(A, D?) : \mathcal{K} \to \mathcal{V}$ where we use "?" rather than "$-$" as the placeholder for the input variable. $\mathcal{A}(A, D?)$ is an object in $\mathcal{P}^\dagger_\mathcal{V}\mathcal{K}$. By varying over all $A \in \mathcal{A}$, we can further form the right $\mathcal{V}$-functor $\mathcal{P}^\dagger_\mathcal{V}\mathcal{K}(W, \mathcal{A}(-, D?)) : \mathcal{A}^\circ \to \mathcal{V}$ which acts on objects by mapping $A \in \mathcal{A}$ to the object $\mathcal{P}^\dagger_\mathcal{V}\mathcal{K}(W, \mathcal{A}(A, D?))$ of $\mathcal{V}$. Note that we opted initially to use "?" rather than "$-$" because in the latter right $\mathcal{V}$-functor we wanted to avoid writing $\mathcal{A}(-, D-)$ which could mislead the reader.

An enriched **weighted limit** of D with weight W is a representation of this right $\mathcal{V}$-functor; i.e., an object $L \in \mathcal{A}$ equipped with a family of isomorphisms

$$\mathcal{A}(A, L) \cong \mathcal{P}^\dagger_\mathcal{V}\mathcal{K}(W, \mathcal{A}(A, D-))$$

right $\mathcal{V}$-natural in $A \in \mathcal{A}$. The counit of the representation is a $\mathcal{V}$-natural transformation $\lambda : W \Rightarrow \mathcal{A}(L, D?)$ called the *limit cylinder.*

For the same diagram $D : \mathcal{K} \to \mathcal{A}$ and a different weight given by a right $\mathcal{V}$-functor $W' : \mathcal{K}^\circ \to \mathcal{V}$, an enriched **weighted colimit** of D with weight W' is a representation given by $C \in ob(\mathcal{A})$ and isomorphisms

$$\mathcal{A}(C, A) \cong \mathcal{P}_\mathcal{V}\mathcal{K}(W', \mathcal{A}(D-, A)) \qquad (A \in \mathcal{A})$$

$\mathcal{V}$-natural in $A \in \mathcal{A}$. The unit of the representation is a right $\mathcal{V}$-natural transformation $\kappa : W' \Rightarrow \mathcal{A}(D-, C)$ called the *colimit cylinder.*

We now discuss a few notable examples of weighted (co)limits, which serve as building blocks for small weighted (co)limits in (co)complete enriched categories.

Example 1.3.19 ((Co)powers)**.** Let the shape category $\mathcal{K}$ be $\mathcal{I}$, the unit $\mathcal{V}$-category. A weight $\mathcal{V}$-functor $W : \mathcal{I} \to \mathcal{V}$ is equivalently given by an object $X \in \mathcal{V}$ and a diagram $\mathcal{V}$-functor $D : \mathcal{I} \to \mathcal{A}$ is equivalently given by an object $A \in \mathcal{A}$. A weighted limit for this weight and diagram is given by an object A^X in $\mathcal{A}$ equipped with isomorphisms

$$\mathcal{A}(B, A^X) \cong \mathcal{P}^\dagger_\mathcal{V}\mathcal{I}(W, \mathcal{A}(B, D-))$$

$\mathcal{V}$-natural in $B \in \mathcal{A}$. We can simplify this by noting that $\mathcal{A}(B, D-)^W$ is isomorphic to $\mathcal{A}(B, A)^X$. Thus, this weighted limit is given by an object A^X equipped with isomorphisms

$$\mathcal{A}(B, A^X) \cong \mathcal{A}(B, A)^X$$

$\mathcal{V}$-natural in $B \in \mathcal{A}$. The counit of the representation is the morphism $\epsilon : X \to \mathcal{A}(A^X, A)$. We call A^X a **power** of A by X.

Consider the same diagram $D : \mathcal{I} \to \mathcal{A}$ but take instead the weight right $\mathcal{V}$-functor $W' : \mathcal{I}^\circ \to \mathcal{V}$ determined by $W'(*) = X$. A weighted colimit for this weight and diagram is given by an object $X \cdot A$ in $\mathcal{A}$ equipped with isomorphisms

$$\mathcal{A}(X \cdot A, B) \cong {}^X\mathcal{A}(A, B)$$

$\mathcal{V}$-natural in $B \in \mathcal{A}$. We call $X \cdot A$ a **copower** of A by X.

When $\mathcal{A} = \mathcal{V}$, the power A^X is simply the left-closed hom since ${}^B(A^X) \cong ({}^BA)^X$ and the copower $X \cdot A$ is simply the tensor product since ${}^{X \otimes A}B \cong {}^X({}^AB)$. This aligns well with alternate terminology where copowers are called tensors and powers called cotensors. However, we opt instead for the terms powers and copowers since when $\mathcal{V} = Set$, we recover the familiar notions of power and copower in ordinary categories.

Let $\mathcal{C}$ be a right $\mathcal{V}$-category. Taking instead the weight and diagram to be the right $\mathcal{V}$-functors $W' : \mathcal{I} \to \mathcal{V}$ and $D' : \mathcal{I} \to \mathcal{C}$ determined by $W'(*) = X$ and $D'(*) = C$, a power of C by X is an object ${}^XC$ in $\mathcal{C}$ equipped with isomorphisms

$$\mathcal{C}(B, {}^XC) \cong {}^X\mathcal{C}(B, C)$$

$\mathcal{V}$-natural in $B \in \mathcal{C}$. A copower of C by X is an object $C \cdot X$ in $\mathcal{C}$ equipped with isomorphisms

$$\mathcal{C}(C \cdot X, B) \cong \mathcal{C}(C, B)^X$$

$\mathcal{V}$-natural in $B \in \mathcal{C}$. When $\mathcal{C} = \mathcal{V}$ regarded as a right $\mathcal{V}$-category, a power is a right-closed hom.

Example 1.3.20 (Conical (Co)Limits)**.** Let $\mathcal{A}$ be a $\mathcal{V}$-category and let $\mathcal{L}$ be an ordinary category. Consider a functor $D : \mathcal{L} \to \mathcal{A}_0$. A **conical limit** in $\mathcal{A}$ over D is a cone $p_K : A \to DK$ $(K \in \mathcal{L})$ in $\mathcal{A}_0$ that is mapped to a limit cone in $\mathcal{V}$ by the covariant hom-$\mathcal{V}$-functors $\mathcal{A}(B, -) : \mathcal{A} \to \mathcal{V}$ for all $B \in ob(\mathcal{A})$. In other words, A in $\mathcal{A}$ is a conical limit if, for all $B \in \mathcal{A}$,

$$\mathcal{A}(B, p_k) : \mathcal{A}(B, A) \to \mathcal{A}(B, DK) \qquad\qquad (K \in \mathcal{L})$$

is a limit cone of the ordinary functor $\mathcal{A}(B, D-) : \mathcal{L} \to \mathcal{V}$.

Similarly, a **conical colimit** in $\mathcal{A}$ over D is a cocone $c_K : DK \to C$ $(K \in \mathcal{L})$ in $\mathcal{A}_0$ that is mapped to a limit cone in $\mathcal{V}$ by the contravariant hom-$\mathcal{V}$-functors $\mathcal{A}(-, B) : \mathcal{A}^\circ \to \mathcal{V}$ for all $B \in ob(\mathcal{A})$. In other words, C in $\mathcal{A}$ is a conical colimit if

$$\mathcal{A}(c_K, B) : \mathcal{A}(C, B) \to \mathcal{A}(DK, B) \qquad\qquad (K \in \mathcal{L})$$

is a limit cone of the ordinary diagram $\mathcal{A}(D-, B) : \mathcal{L}^{op} \to \mathcal{V}$.

We mention in passing that, if $\mathcal{L}$ is small and $\mathcal{V}$ has small coproducts, a conical limit is a weighted limit in the sense of Definition 1.3.18 where the shape category $\mathcal{K}$ is the free $\mathcal{V}$-category of $\mathcal{L}$, the diagram is the $\mathcal{V}$-functor $\mathcal{K} \to \mathcal{A}$ induced by $D : \mathcal{L} \to \mathcal{A}_0$, and the weight is the $\mathcal{V}$-functor $\mathcal{K} \to \mathcal{V}$ induced by the functor $\Delta_I : \mathcal{L} \to \mathcal{V}_0$ constant at the tensor unit I, as outlined in [11, §3.8].

Proposition 1.3.21. A $\mathcal{V}$-category $\mathcal{A}$ admits all small weighted limits if and only if it admits all small conical limits and all powers. Similarly, $\mathcal{A}$ admits all small weighted colimits if and only if it admits all small conical colimits and all copowers.

Chapter 2

$\mathcal{V}$-Graded Categories

The structures introduced by Wood in [19] as *large $\mathcal{V}$-categories* were later reformulated by Lucyshyn-Wright in [15]. He adopted the terminology *$\mathcal{V}$-graded categories* and developed the notation and methods we use in this chapter. Except where noted otherwise, we work in Lucyshyn-Wright's setting.

From this point forward, let $(\mathcal{V}, \otimes, I)$ be a SET-small monoidal category, where SET is the category of large sets.

Definition 2.0.1. A **left $\mathcal{V}$-graded category** $\mathcal{C}$ is a left $\widehat{\mathcal{V}}$-category. That is, $\mathcal{C}$ is given by:

- a set $ob(\mathcal{C})$;
- for all $A, B \in ob(\mathcal{C})$, a functor $\mathcal{C}(-, A; B) : \mathcal{V}^{op} \to SET$;
- for all $A, B, C \in ob(\mathcal{C})$, a family of maps
$$\circ^{Y,X}_{A,B,C} : \mathcal{C}(Y, B; C) \times \mathcal{C}(X, A; B) \to \mathcal{C}(Y \otimes X, A; C);$$
- for all $A \in ob(\mathcal{C})$, an element $1_A \in \mathcal{C}(I, A; A)$;

subject to associativity and unit conditions derived from those for $\hat{\mathcal{V}}$-categories. We will describe these concretely below.

The functor $\mathcal{C}(-, A; B) : \mathcal{V}^{op} \to SET$ maps an object $X \in \mathcal{V}$ to a SET-small set $\mathcal{C}(X, A; B)$ whose elements we call **graded morphisms** $g : X, A \to B$ with grade X. For $\alpha : Y \to X$ in $\mathcal{V}$, we get

$$\mathcal{C}(\alpha, A; B) : \mathcal{C}(X, A; B) \to \mathcal{C}(Y, A; B)$$

mapping an X-graded morphism $g : X, A \to B$ to a Y-graded morphism $\alpha^* g : Y, A \to B$ called the **reindexing of g along α**.

Using this notation and terminology, we obtain a concrete definition of $\mathcal{V}$-graded categories. A left $\mathcal{V}$-graded category $\mathcal{C}$ consists of

- set $ob(\mathcal{C})$;
- for all $A, B \in ob(\mathcal{C})$ and all $X \in ob(\mathcal{V})$, a SET-small set $\mathcal{C}(X, A; B)$ whose elements are called graded morphisms $g : X, A \to B$;
- for any $f : X, A \to B$ and $g : Y, B \to C$, graded morphism $g \circ f : Y \otimes X, A \to C$ called the composite of f and g;
- for all $A \in ob(\mathcal{C})$, a graded morphism $1_A : I, A \to A$ called the identity at A;
- for all $f : X, A \to B$ in $\mathcal{C}$ and $\alpha : Y \to X$ in $\mathcal{V}$, a graded morphism $\alpha^* f : Y, A \to B$ called the reindexing of f along α

subject to the following four axioms:

Functoriality of reindexing For all $f : X, A \to B$ in $\mathcal{C}$ and all $\alpha : Z \to Y$, $\beta : Y \to X$ in $\mathcal{V}$, we have that $id_X^* f = f$ and that $(\beta \cdot \alpha)^* f = \alpha^*(\beta^* f) : Z, A \to B$.

Naturality of composition For all $f : X, A \to B$, $g : Y, B \to C$ in $\mathcal{C}$ and all $\alpha : X' \to X$, $\beta : Y' \to Y$ in $\mathcal{V}$, we have that $\beta^* g \circ \alpha^* f = (\beta \otimes \alpha)^*(g \circ f) : Y' \otimes X', A \to C$.

Essential associativity For all $f : X, A \to B$, $g : Y, B \to C$, $h : Z, C \to D$ in $\mathcal{C}$, the composite graded morphism $(h \circ g) \circ f : (Z \otimes Y) \otimes X, A \to D$ is the reindexing of $h \circ (g \circ f) : Z \otimes (Y \otimes X), A \to D$ along the associator $\alpha_{Z,Y,X} : (Z \otimes Y) \otimes X \xrightarrow{\sim} Z \otimes (Y \otimes X)$.

Essential identity For all $f : X, A \to B$, the composite graded morphism $f \circ 1_A : X \otimes I, A \to B$ is the reindexing of f along the right unitor $r_X : X \otimes I \xrightarrow{\sim} X$. Similarly, the composite $1_B \circ f : I \otimes X, A \to B$ is the reindexing of f along the left unitor $\ell_X : I \otimes X \xrightarrow{\sim} X$.

When $\mathcal{V}$ is strict monoidal, the essential associativity and identity axioms reduce to strict associativity and identity axioms. We require simply the equations $(h \circ g) \circ f = h \circ (g \circ f)$ and $f \circ 1_A = f = 1_B \circ f$.

Definition 2.0.2. A **right V-graded category** $\mathcal{C}$ is a right $\hat{\mathcal{V}}$-category. For $A, B \in ob(\mathcal{C})$, the hom-presheaf $\mathcal{C}(A, B)$ is denoted $\mathcal{C}(A, -; B)$. A graded morphism $f \in \mathcal{C}(A, X; B)$ is written $f : A, X \to B$ and the composite of $f : A, X \to B$ and $g : B, Y \to C$ is a graded morphism $g \circ f : A, X \otimes Y \to C$ with grade $X \otimes Y$.

In this text, we will use the term $\mathcal{V}$-graded category to mean left $\mathcal{V}$-graded category.

When $\mathcal{V}$ is symmetric monoidal, then $\hat{\mathcal{V}}$ is also symmetric monoidal, and we have that a left $\mathcal{V}$-graded category is equivalently given by a right $\mathcal{V}$-graded category.

Remark 2.0.3. Recall that a right $\hat{\mathcal{V}}$-category is a left $\hat{\mathcal{V}}^{rev}$-category. Since $\hat{\mathcal{V}}^{rev} \cong \widehat{\mathcal{V}^{rev}}$, we have that a right $\mathcal{V}$-graded category is a left $\mathcal{V}^{rev}$-graded category. Identity graded morphisms and reindexing in a right $\mathcal{V}$-graded category are analogous to those of a left $\mathcal{V}$-graded category, and the data of a right $\mathcal{V}$-graded category are subject to the same four axioms as for left $\mathcal{V}$-graded categories.

Remark 2.0.4. There is a 2-category ${}_{\mathcal{V}}GCAT = {}_{\hat{\mathcal{V}}}CAT$ whose objects are $\mathcal{V}$-graded categories, 1-cells are $\mathcal{V}$-graded functors, and 2-cells are $\mathcal{V}$-graded natural transformations. There is also a 2-category $GCAT_{\mathcal{V}} = CAT_{\hat{\mathcal{V}}}$ of right $\mathcal{V}$-graded categories.

Example 2.0.5 ($\mathcal{V}$-actegories are $\mathcal{V}$-graded)**.** Every $\mathcal{V}$-actegory $\mathcal{C}$ has the structure of a $\mathcal{V}$-graded category where

$$\mathcal{C}(X, A; B) := \mathcal{C}(X \bullet A, B)$$

for $X \in ob(\mathcal{V})$ so a graded morphism $f : X, A \to B$ is given by a morphism $f : X \bullet A \to B$ in the actegory $\mathcal{C}$. The identity graded morphisms $1_A : I, A \to A$ are given by the unitors $\eta_A^{-1} : I \bullet A \xrightarrow{\sim} A$ in the actegory. The reindexing $\alpha^*(f) : Y, A \to B$ of f along $\alpha : Y \to X$ is

$$Y \bullet A \xrightarrow{\alpha \bullet A} X \bullet A \xrightarrow{f} B$$

Given $f : X, A \to B$ and $g : Y, B \to C$ in $\mathcal{C}$, the composite graded morphism $g \circ f : (Y \otimes X), A \to C$ is

$$(Y \otimes X) \bullet A \xrightarrow{\mu^{-1}_{Y,X,A}} Y \bullet (X \bullet A) \xrightarrow{Y \bullet f} Y \bullet B \xrightarrow{g} C$$

Similarly, a right $\mathcal{V}$-actegory has the structure of a right $\mathcal{V}$-graded category. This construction extends to an embedding of 2-categories ${}_{\mathcal{V}}ACT \hookrightarrow {}_{\mathcal{V}}GCAT$.

Example 2.0.6 ([19], [15], $\mathcal{V}$-categories are $\mathcal{V}$-graded)**.** Since $\hat{\mathcal{V}}$ is equipped with Day Convolution monoidal structure, we know that the Yoneda embedding $\mathsf{Y} : \mathcal{V} \to \hat{\mathcal{V}}$ is a fully faithful strong monoidal functor. Applying the change of base of enrichment process given in [8], we obtain a fully faithful 2-functor $\mathsf{Y}_* : {}_{\mathcal{V}}CAT \to {}_{\mathcal{V}}GCAT$ that sends every $\mathcal{V}$-category $\mathcal{C}$ to a $\mathcal{V}$-graded category $\mathsf{Y}_*(\mathcal{C})$. This $\mathcal{V}$-graded category has the same objects as $\mathcal{C}$ and its hom-objects are

$$\mathsf{Y}_*(\mathcal{C})(-, A; B) := \mathsf{Y}(\mathcal{C}(A, B)) = \mathcal{V}(-, \mathcal{C}(A, B)) : \mathcal{V}^{op} \to SET$$

for $A, B \in ob(\mathcal{C})$. Therefore, a graded morphism $f : X, A \to B$ in $\mathsf{Y}_*(\mathcal{C})$ is given by $f : X \to \mathcal{C}(A, B)$ in $\mathcal{V}$. The identity graded morphisms $1_A : I, A \to A$ are given by the identities $j_A : I \to \mathcal{C}(A, A)$. The reindexing $\alpha^*(f) : Y, A \to B$ of f along $\alpha : Y \to X$ is

$$Y \xrightarrow{\alpha} X \xrightarrow{f} \mathcal{C}(A, B)$$

Given $f : X, A \to B$ and $g : Y, B \to C$ in $\mathcal{C}$, the composite $g \circ f : Y \otimes X, A \to C$ in $\mathcal{C}$ is

$$Y \otimes X \xrightarrow{g \otimes f} \mathcal{C}(B, C) \otimes \mathcal{C}(A, B) \xrightarrow{m_{A,B,C}} \mathcal{C}(A, C)$$

Similarly, a right $\mathcal{V}$-category gives rise to a right $\mathcal{V}$-graded category.

Example 2.0.7 ($\mathcal{V}$ is a $\mathcal{V}$-graded category)**.** From Example 2.1, $\mathcal{V}$ has the structure of a $\mathcal{V}$-actegory. Applying the concepts from Example 2.0.5, we then have that $\mathcal{V}$ is graded in itself with the hom-presheaves $\mathcal{V}(-, Y; Z) := \mathcal{V}(- \otimes Y, Z)$, meaning that a graded morphism $f : X, Y \to Z$ is given by $f : X \otimes Y \to Z$ in $\mathcal{V}$. Recall that $\mathcal{V}$ is enriched in itself when it is biclosed monoidal. In this case, by Example 2.0.6, $\mathcal{V}$ has an additional $\mathcal{V}$-graded structure

$\mathsf{Y}_*(\mathcal{V})$ where a graded morphism $f : X, Y \to Z$ is given by $f : X \to {}^{Y}Z$ in $\mathcal{V}$. Both graded structures on $\mathcal{V}$ are equivalent; i.e., $\mathcal{V} \cong \mathsf{Y}_*(\mathcal{V})$ in ${}_{\mathcal{V}}GCAT$. This follows from the property (1.1.6) which gives us isomorphisms between sets of graded morphisms

$$\mathcal{V}(X, Y; Z) = \mathcal{V}(X \otimes Y, Z) \cong \mathcal{V}(X, {}^{Y}Z) = \mathsf{Y}_*(\mathcal{V})(X, Y; Z)$$

Since $\mathcal{V}$ also has the structure of a right $\mathcal{V}$-actegory, it is also a right $\mathcal{V}$-graded category with the hom-presheaves $\mathcal{V}(Y, -; Z) := \mathcal{V}(Y \otimes X, Z)$. When $\mathcal{V}$ is biclosed monoidal, the right $\mathcal{V}$-graded category $\mathsf{Y}_*(\mathcal{V})$ has graded morphisms $f : Y, X \to Z$ given by $f : X \to Z^Y$ in $\mathcal{V}$. By a similar argument to the one above, $\mathcal{V} \cong \mathsf{Y}_*(\mathcal{V})$ in $GCAT_{\mathcal{V}}$.

Example 2.0.8 ($\hat{\mathcal{V}}$ as a $\mathcal{V}$-graded category)**.** By Example 1.1.11, $\hat{\mathcal{V}}$ is biclosed under Day Convolution monoidal structure. We can then regard it as a left (resp. right) $\hat{\mathcal{V}}$-category whose hom-objects are the internal homs ${}^{P}Q$ (resp. Q^P) for $P, Q \in ob(\hat{\mathcal{V}})$. This gives us a left and right $\mathcal{V}$-graded structure for $\hat{\mathcal{V}}$. As a left $\mathcal{V}$-graded category, $\hat{\mathcal{V}}$ has graded morphisms $\phi : X, P \to Q$ given by $\phi \in {}^{P}Q(X)$, which we know to be natural transformations $\phi : P \Rightarrow Q(X \otimes -) \cong Q^{\mathsf{Y}(X)}$. As a right $\mathcal{V}$-graded category, $\hat{\mathcal{V}}$ has graded morphisms $\psi : P, X \to Q$ given by $\psi \in Q^P(X)$, a natural transformations $\psi : P \Rightarrow Q(- \otimes X) \cong {}^{\mathsf{Y}(X)}Q$.

Definition 2.0.9. Let $\mathcal{C}$ be a $\mathcal{V}$-graded category and let $\mathcal{G} := (\mathcal{G}_0, \mathcal{G}_1)$ be a pair consisting of a collection of objects $\mathcal{G}_0 \subseteq ob(\mathcal{C})$ and a collection of graded morphisms $\mathcal{G}_1 \subseteq \coprod_{X \in \mathcal{V},\, A, B \in \mathcal{G}_0} \mathcal{C}(X, A; B)$. We say $\mathcal{G}$ is a **$\mathcal{V}$-graded subcategory** of $\mathcal{C}$ if $\mathcal{G}$ contains the identities 1_A for each $A \in \mathcal{G}_0$ and is closed under composition and reindexing. We denote $\langle \mathcal{G} \rangle$ the smallest $\mathcal{V}$-graded subcategory of $\mathcal{C}$ that contains $\mathcal{G}$. We say that $\mathcal{G}$ is a **generating collection** of $\mathcal{C}$ if $\langle \mathcal{G} \rangle = \mathcal{C}$.

It is possible for $\mathcal{G}_0$ to contain an object A that is not a domain or codomain of any generating graded morphism in $\mathcal{G}_1$ since its identity 1_A is added to $\langle \mathcal{G} \rangle$ by definition. We may call these **isolated generating objects**. Note that it is also possible to have a generating collection consisting exclusively of isolated generating objects.

It should be noted that [15] has a slightly different notion of *generating collection* which is defined entirely using generating graded morphisms.

Example 2.0.10. Let $\mathcal{V}$ be a monoidal category. The **unit $\mathcal{V}$-graded category**, $\mathcal{I}$, is the unit $\hat{\mathcal{V}}$-category. It is generated by the collection containing only one object $* \in \mathcal{I}$. Its only graded morphisms are reindexings of the identity morphism $1_* : I, * \to *$.

Example 2.0.11. Let $\mathcal{L}$ be a *SET*-small category. The **free $\mathcal{V}$-graded category on $\mathcal{L}$**, denoted $\langle \mathcal{L} \rangle$, is precisely the free $\hat{\mathcal{V}}$-category on $\mathcal{L}$ as in [11]. We have that $ob(\langle \mathcal{L} \rangle) = ob(\mathcal{L})$ and the hom-presheaves

$$\langle \mathcal{L} \rangle(-, A; B) := \mathcal{L}(A, B) \cdot \mathcal{V}(-, I)$$

formed by copowers in $\hat{\mathcal{V}}$. Thus, we have

$$\langle \mathcal{L} \rangle(X, A; B) = \coprod_{\mathcal{A}(A,B)} \mathcal{V}(X, I)$$

for all $X \in ob(\mathcal{V})$ and $A, B \in ob(\mathcal{L})$. For any $f : A \to B$ in $\mathcal{L}$, we may identify the pair (f, id_I) with the associated graded morphism $f : I, A \to B$ in $\langle \mathcal{L} \rangle$. Any graded morphism

$g : X, A \to B$ may then be identified with the pair (f, α) for some $\alpha \in \mathcal{V}(X, I)$. Thus, we can simply describe $\langle \mathcal{L} \rangle$ as a $\mathcal{V}$-graded category with the same objects as $\mathcal{L}$ and whose graded morphisms are all reindexings of morphisms in $\mathcal{L}$.

Definition 2.0.12. Let $\mathcal{C}$ be a $\mathcal{V}$-graded category. Its **underlying ordinary category** $\mathcal{C}_0$ is the usual underlying ordinary category of the $\hat{\mathcal{V}}$-category $\mathcal{C}$. An ordinary morphism from A to B in $\mathcal{C}_0$ is given by a graded morphism from A to B with grade I.

Definition 2.0.13. We have a method for composing graded morphisms in $\mathcal{C}$ with ordinary morphisms in $\mathcal{C}_0$. Let $p : A' \to A$ in $\mathcal{C}_0$ and $f : X, A \to B$ in $\mathcal{C}$. We define

$$f \star p := (r_X^{-1})^*(f \circ p) : X, A' \to B$$

where we take p in its I-graded form $p : I, A' \to A$ in $\mathcal{C}$ and $f \circ p$ to be the graded composite.

Similarly, let $q : B \to B'$ in $\mathcal{C}_0$ and $f : X, A \to B$ in $\mathcal{C}$. We define

$$q \star f := (\ell_X^{-1})^*(q \circ f) : X, A \to B'$$

where we take q in its I-graded form $q : I, B \to B'$ in $\mathcal{C}$ and $q \circ f$ to be the graded composite.

Definition 2.0.14. Since $\mathcal{V}$-graded categories are simply $\hat{\mathcal{V}}$-categories, we can apply Definition 1.3.5 to obtain a **$\mathcal{V}$-graded formal opposite**. Given a $\mathcal{V}$-graded category $\mathcal{C}$, there is a right $\mathcal{V}$-graded category $\mathcal{C}^\circ$ with the same objects as $\mathcal{C}$ and whose graded morphisms $f : B, X \to A$ in $\mathcal{C}^\circ$ are precisely graded morphisms $f : X, A \to B$ in $\mathcal{C}$. The composite $f \circ g : C, Y \otimes X \to A$ in $\mathcal{C}^\circ$ of the graded morphisms $f : B, X \to A$ and $g : C, Y \to B$ in $\mathcal{C}^\circ$ is the composite $g \circ f : Y \otimes X, A \to C$ in $\mathcal{C}$ of the graded morphisms $f : X, A \to B$ and $g : Y, B \to C$ in $\mathcal{C}$. Similarly, the formal opposite of a right $\mathcal{V}$-graded category is a left $\mathcal{V}$-graded category.

Definition 2.0.15. Let $\mathcal{C}$ and $\mathcal{D}$ be $\mathcal{V}$-graded categories. A left **V-graded functor** $F : \mathcal{C} \to \mathcal{D}$ is a left $\hat{\mathcal{V}}$-functor. It consists of the following:

- for each $A \in ob(\mathcal{C})$, an object $FA \in \mathcal{D}$;
- for all $A, B \in ob(\mathcal{C})$, a family of morphisms $F^X_{A,B} : \mathcal{C}(X, A; B) \to \mathcal{D}(X, FA; FB)$ natural in $X \in \mathcal{V}$;

satisfying the property that for every $f : X, A \to B$ and $g : Y, B \to C$ in $\mathcal{C}$, we have $F(g \circ f) = Fg \circ Ff$, and the property that $F1_A = 1_{FA}$. We note that the $F^X_{A,B}$ map graded morphisms $f : X, A \to B$ in $\mathcal{C}$ to graded morphisms $Ff : X, FA \to FB$ in $\mathcal{D}$, and the naturality of these $F_{A,B}$ is precisely the condition that F preserves reindexing; i.e., $\alpha^*(Ff) = F(\alpha^* f)$ for any $\alpha : Y \to X$ in $\mathcal{V}$.

Similarly, a right $\mathcal{V}$-graded functor $G : \mathcal{A} \to \mathcal{B}$ is such that the $G^X_{A,B} : \mathcal{A}(A, X; B) \to \mathcal{B}(GA, X; GB)$ preserve composition, identities, and reindexing.

Remark 2.0.16. A $\mathcal{V}$-graded functor $F : \mathcal{C} \to \mathcal{D}$ is **fully faithful** if it is fully faithful as a $\hat{\mathcal{V}}$-functor. Concretely, F is fully faithful if $F_{A,B}$ is a natural isomorphism; i.e., the $F^X_{A,B}$ are isomorphisms in SET.

Definition 2.0.17 ($\mathcal{V}$-graded hom-functors)**.** Let $\mathcal{C}$ be a left $\mathcal{V}$-graded category. Since $\mathcal{V}$-graded categories are precisely $\hat{\mathcal{V}}$-categories, the **$\mathcal{V}$-graded hom-functors** are simply the covariant and contravariant hom-$\hat{\mathcal{V}}$-functors as in Definition 1.3.11; $\mathcal{C}(A,-) : \mathcal{C} \to \hat{\mathcal{V}}$ and $\mathcal{C}(-,B) : \mathcal{C}^{\circ} \to \hat{\mathcal{V}}$ respectively. We now unpack the data of these $\hat{\mathcal{V}}$-functors and formulate them again using $\mathcal{V}$-graded notation.

The $\mathcal{V}$-graded functor $\mathcal{C}(A,-)$ maps an object $B \in ob(\mathcal{C})$ to the presheaf $\mathcal{C}(-,A;B) : \mathcal{V}^{op} \to SET$. On graded morphisms, $\mathcal{C}(A,-)$ takes $f : X, B \to B'$ in $\mathcal{C}$ to the graded morphism $\mathcal{C}(A,f) : X, \mathcal{C}(-,A;B) \Rightarrow \mathcal{C}(-,A;B')$ in $\hat{\mathcal{V}}$ given by the natural transformation with components

$$\mathcal{C}(Y,A;f) : \mathcal{C}(Y,A;B) \to \mathcal{C}(X \otimes Y, A; B') \qquad (Y \in \mathcal{V})$$

mapping $g : Y, A \to B$ to $f \circ g : X \otimes Y, A \to B'$.

The right $\mathcal{V}$-graded functor $\mathcal{C}(-,B)$ acts on objects much like $\mathcal{C}(A,-)$ does. On graded morphisms, $\mathcal{C}(-,B)$ takes $h : X, A \to A'$ in $\mathcal{C}$ to the graded morphism $\mathcal{C}(h,B) : \mathcal{C}(-,A';B), X \Rightarrow \mathcal{C}(-,A;B)$ in $\hat{\mathcal{V}}$ given by the natural transformation with components

$$\mathcal{C}(Y,h;B) : \mathcal{C}(Y,A';B) \to \mathcal{C}(Y \otimes X, A; B) \qquad (Y \in \mathcal{V})$$

mapping $k : Y, A' \to B$ to $k \circ h : Y \otimes X, A \to B$.

Example 2.0.18 (Canonical Embedding)**.** We can define a left $\mathcal{V}$-graded functor $\mathsf{Y} : \mathcal{V} \to \hat{\mathcal{V}}$ which we have named the **canonical embedding** since its underlying ordinary functor Y_0 is the Yoneda embedding. This $\mathcal{V}$-graded functor maps an object $X \in \mathcal{V}$ to its contravariant representable functor $\mathcal{V}(-,X)$ in $\hat{\mathcal{V}}$. It maps a graded morphism $\phi : U, X \to Y$ in $\mathcal{V}$ to a graded morphism $\mathsf{Y}(\phi) : U, \mathcal{V}(-,X) \to \mathcal{V}(-,Y)$ in $\hat{\mathcal{V}}$. By Example 2.0.8, this graded morphism is a natural transformation $\mathsf{Y}(\phi) : \mathcal{V}(-,X) \Rightarrow \mathcal{V}(U \otimes -, Y)$. Each component is defined as

$$\begin{aligned} \mathsf{Y}(\phi)_Z : \mathcal{V}(Z,X) &\to \mathcal{V}(U \otimes Z, Y) \\ g : Z \to X \quad &\mapsto \quad \phi \cdot (U \otimes g) \end{aligned}$$

where $\phi \cdot (U \otimes g)$ is the composite $U \otimes Z \xrightarrow{U \otimes g} U \otimes X \xrightarrow{\phi} Y$ in $\mathcal{V}$. Note that we may equivalently consider the components of this natural transformation to be

$$\begin{aligned} \mathsf{Y}(\phi)_Z : \mathcal{V}(Z,X) &\to \mathcal{V}(U,Z;Y) \\ g : Z \to X \quad &\mapsto \quad \phi \star g : U, Z \to Y \end{aligned}$$

Therefore, the structural morphisms $\mathsf{Y}^U_{X,Y} : \mathcal{V}(U,X;Y) \to \hat{\mathcal{V}}(U, \mathsf{Y}X; \mathsf{Y}Y)$ are precisely the Yoneda isomorphisms between the set $\mathcal{V}(U,X;Y)$ and the set of natural transformations of the form $\mathsf{Y}X \Rightarrow \mathcal{V}(U,-;Y)$.

There is also a right $\mathcal{V}$-graded canonical embedding of $\mathcal{V}$ into $\hat{\mathcal{V}}$ where $\mathcal{V}$ and $\hat{\mathcal{V}}$ are regarded as right $\mathcal{V}$-graded categories.

Lemma 2.0.19. The canonical embedding is a fully faithful $\mathcal{V}$-graded functor.

Proof. For Y to be fully faithful, we require $\mathsf{Y}^U_{X,Y} : \mathcal{V}(U, X; Y) \to \hat{\mathcal{V}}(U, \mathsf{Y}X; \mathsf{Y}Y)$ to be an isomorphism for every $U, X, Y \in \mathcal{V}$. We identified above that the $\mathsf{Y}^U_{X,Y}$ are precisely the isomorphisms that constitute the Yoneda Lemma. Thus, Y must be a fully faithful $\mathcal{V}$-graded functor. ∎

As the reader may have already expected, for an enriched category $\mathcal{C}$, there is a relationship between its hom-$\mathcal{V}$-functors and the graded hom-functors for its $\mathcal{V}$-graded structure.

Lemma 2.0.20. Let $\mathcal{V}$ be biclosed monoidal and let $\mathcal{C}$ be a left $\mathcal{V}$-category regarded as a $\mathcal{V}$-graded category $\mathsf{Y}_*(\mathcal{C})$. The $\mathcal{V}$-graded hom-functor $\mathsf{Y}_*(\mathcal{C})(-, B)$ is equivalently given by postcomposing the enriched hom-functor $\mathcal{C}(-, B) : \mathcal{C}^\circ \to \mathcal{V}$ (regarded as a $\mathcal{V}$-graded functor) with the canonical embedding Y.

Proof. To regard $\mathcal{C}(-, B)$ as a right $\mathcal{V}$-graded functor, we apply the 2-functor $\mathsf{Y}_* : CAT_{\mathcal{V}} \to CAT_{\hat{\mathcal{V}}}$, which gives us a right $\mathcal{V}$-graded functor $\mathsf{Y}_*(\mathcal{C}(-, B)) : \mathsf{Y}_*\mathcal{C}^\circ \to \mathsf{Y}_*\mathcal{V}$ mapping a morphism $g : Z \to \mathcal{C}(A, A')$ to the morphism $\mathcal{C}(-, B)_{AA'} \cdot g : Z \to \mathcal{C}(A, B)^{\mathcal{C}(A', B)}$. By Example 2.0.7, we can identify $\mathsf{Y}_*\mathcal{V}$ with the $\mathcal{V}$-graded category $\mathcal{V}$, so we have the $\mathcal{V}$-graded functor

$$\mathsf{Y}_*(\mathcal{C}(-, B)) : \mathsf{Y}_*\mathcal{C}^\circ \to \mathcal{V}$$

which acts on graded morphisms by

$$g : Z \to \mathcal{C}(A, A') \qquad \mapsto \qquad m_{A,A',B} \cdot (\mathcal{C}(A', B) \otimes g) : \mathcal{C}(A', B) \otimes Z \to \mathcal{C}(A, B)$$

Note that $m_{A,A',B} \cdot (\mathcal{C}(A', B) \otimes g)$ is the transpose of $\mathcal{C}(-, B)_{AA'} \cdot g$. Postcomposing with Y, we obtain

$$\mathsf{Y} \circ \mathsf{Y}_*(\mathcal{C}(-, B)) : \mathsf{Y}_*\mathcal{C}^\circ \to \hat{\mathcal{V}}$$

mapping g to the natural transformation

$$\mathsf{Y}(m_{A,A',B} \cdot (\mathcal{C}(A', B) \otimes g)) : \mathcal{C}(-, A'; B) \Rightarrow \mathcal{C}(- \otimes Z, A; B)$$

We claim that the latter natural transformation is equal to

$$\mathsf{Y}_*(\mathcal{C})(g, B) : Z, \mathsf{Y}_*(\mathcal{C})(A', B) \to \mathsf{Y}_*(\mathcal{C})(A, B)$$

which we may also write as $\mathsf{Y}_*(\mathcal{C})(-, g; B) : \mathsf{Y}_*(\mathcal{C})(-, A'; B) \to \mathsf{Y}_*(\mathcal{C})(-, A; B)$. Indeed, for every $h : U \to \mathcal{C}(A', B)$ in $\mathcal{C}(U, A'; B)$. Then

$$\begin{aligned} \mathsf{Y}(m_{A,A',B} \cdot (\mathcal{C}(A', B) \otimes g))(h) &= (m_{A,A',B} \cdot (\mathcal{C}(A', B) \otimes g)) \cdot (h \otimes Z) \\ &= m_{A,A',B} \cdot (h \otimes g) \\ &= h \circ g \end{aligned}$$

Thus, $\mathsf{Y} \circ \mathsf{Y}_*(\mathcal{C}(-, B))$ is the $\mathcal{V}$-graded hom-functor $\mathcal{C}(-, B)$. ∎

Definition 2.0.21. Let $F, G : \mathcal{C} \to \mathcal{D}$ be $\mathcal{V}$-graded functors. A (left) **$\mathcal{V}$-graded natural transformation** $\phi : F \Rightarrow G$ is a (left) $\hat{\mathcal{V}}$-natural transformation. Concretely, ϕ consists of a family of morphisms $\phi_A : FA \to GA$ $(A \in \mathcal{C})$ in $\mathcal{C}_0$ satisfying the property that for any $f : X, A \to B$ in $\mathcal{C}$, $\phi_B \star Ff = Gf \star \phi_A : X, FA \to GB$.

Similarly, a right $\mathcal{V}$-graded natural transformation is a right $\hat{\mathcal{V}}$-natural transformation.

2.1 Graded Presheaf Categories

The graded presheaf categories in this text are an example of the more general concept of graded functor categories introduced in [15]. Graded functor categories must be valued in $\mathcal{V}$-$\mathcal{W}$-bigraded categories: each of which is simultaneously left $\mathcal{V}$-graded and right $\mathcal{W}$-graded. We can start to understand why this is by noting that our graded presheaf categories are constructed using both the left and right $\mathcal{V}$-graded structure of $\hat{\mathcal{V}}$, however, we will not dwell on the $\mathcal{V}$-$\mathcal{V}$-bigraded structure of $\hat{\mathcal{V}}$. When describing $\mathcal{V}$-graded presheaf categories, we will observe that these are equivalently Street's presheaf categories in $\hat{\mathcal{V}}$.

Definition 2.1.1. Let $\mathcal{C}$ be a $\mathcal{V}$-graded category. There is a right $\mathcal{V}$-graded category, denoted $[\mathcal{C}, \hat{\mathcal{V}}]$, whose objects are left $\mathcal{V}$-graded functors $F : \mathcal{C} \to \hat{\mathcal{V}}$. Given objects $F, G : \mathcal{C} \to \hat{\mathcal{V}}$, a graded morphism $\phi : F, X' \Rightarrow G$ in $[\mathcal{C}, \hat{\mathcal{V}}]$ is called a **graded transformation** and is given by a family $\phi_A : FA, X' \to GA$ $(A \in \mathcal{C})$ in $\hat{\mathcal{V}}$, equivalently by Example 2.0.8, a family of natural transformations $\phi_A : FA \Rightarrow GA(- \otimes X')$ $(A \in \mathcal{C})$, satisfying the property that

$$\begin{tikzcd}
FA \arrow[r, Rightarrow, "\phi_A"] \arrow[d, Rightarrow, "Ff"'] & GA(- \otimes X') \arrow[r, Rightarrow, "(Gf)(-\otimes X')"] & GB(X \otimes (- \otimes X')) \arrow[d, Rightarrow, "(GB)(a_{X,-,X'})"] \\
FB(X \otimes -) \arrow[rr, Rightarrow, "(\phi_B)(X\otimes -)"] & & GB((X \otimes -) \otimes X')
\end{tikzcd} \tag{2.1.1}$$

commutes for all $f : X, A \to B$ in $\mathcal{C}$. Note that $(Gf)(-\otimes X')$ is the whiskering of the natural transformation $Gf : GA \Rightarrow GB(X \otimes -)$ by the functor $- \otimes X' : \mathcal{V} \to \mathcal{V}$ and $(\phi_B)(X \otimes -)$ is the whiskering of the natural transformation $\phi_B : FB \Rightarrow GB(- \otimes X')$ by the functor $X \otimes - : \mathcal{V} \to \mathcal{V}$.

A graded transformation $\phi : F, X' \Rightarrow G$ is equivalently given by a family of natural transformations $\tilde{\phi}_A : FA \otimes \mathsf{Y}(X') \Rightarrow GA$ $(A \in \mathcal{C})$ satisfying the property that

$$\begin{tikzcd}
\mathsf{Y}(X) \otimes FA \otimes \mathsf{Y}(X') \arrow[r, Rightarrow, "\mathsf{Y}(X)\otimes\tilde{\phi}_A"] \arrow[d, Rightarrow, "\overline{Ff}\otimes\mathsf{Y}(X')"'] & \mathsf{Y}(X) \otimes GA \arrow[d, Rightarrow, "\overline{Gf}"] \\
FB \otimes \mathsf{Y}(X') \arrow[r, Rightarrow, "\tilde{\phi}_B"] & GB
\end{tikzcd} \tag{2.1.2}$$

commutes for all $f : X, A \to B$ in $\mathcal{C}$, where $\overline{Ff}$ is the transpose of $Ff : FA \Rightarrow FB^{\mathsf{Y}(X)}$ and $\overline{Gf}$ is the transpose of $Gf : GA \Rightarrow GB^{\mathsf{Y}(X)}$.

We may call $[\mathcal{C}, \hat{\mathcal{V}}]$ the **covariant graded presheaf category**. There also exsits a **contravariant graded presheaf category**, denoted $[\mathcal{C}^\circ, \hat{\mathcal{V}}]$. It is a left $\mathcal{V}$-graded category whose objects are right $\mathcal{V}$-graded functors $T : \mathcal{C}^\circ \to \hat{\mathcal{V}}$. Given $T, S : \mathcal{C}^\circ \to \hat{\mathcal{V}}$, a graded transformation in $[\mathcal{C}^\circ, \hat{\mathcal{V}}]$, $\psi : X', T \Rightarrow S$, is given by a family $\psi_A : X', TA \to SA$ $(A \in \mathcal{C})$ in $\hat{\mathcal{V}}$; i.e., a family of natural transformations $\psi_A : TA \Rightarrow SA(X' \otimes -)$ $(A \in \mathcal{C})$, satisfying the property that

$$\begin{tikzcd}
TB \arrow[r, Rightarrow, "\psi_B"] \arrow[d, Rightarrow, "Tf"'] & SB(X' \otimes -) \arrow[r, Rightarrow, "(Sf)(X'\otimes -)"] & SA((X' \otimes -) \otimes X) \arrow[d, Rightarrow, "(SA)(a^{-1}_{X',-,X})"] \\
TA(- \otimes X) \arrow[rr, Rightarrow, "(\psi_A)(-\otimes X)"] & & SA(X' \otimes (- \otimes X))
\end{tikzcd} \tag{2.1.3}$$

commutes for all $f : X, A \to B$ in $\mathcal{C}$.

A graded transformation $\psi : X', T \Rightarrow S$ is equivalently given by a family of natural transformations $\tilde{\psi}_A : \mathsf{Y}(X') \otimes TA \Rightarrow SA$ ($A \in \mathcal{C}$) satisfying the property that

$$\begin{tikzcd}
\mathsf{Y}(X') \otimes TB \otimes \mathsf{Y}(X) \arrow[r, Rightarrow, "\tilde{\psi}_B \otimes \mathsf{Y}(X)"] \arrow[d, Rightarrow, "\mathsf{Y}(X') \otimes \overline{Tf}"'] & SB \otimes \mathsf{Y}(X) \arrow[d, Rightarrow, "\overline{Sf}"] \\
\mathsf{Y}(X') \otimes TA \arrow[r, Rightarrow, "\tilde{\psi}_A"] & SA
\end{tikzcd} \tag{2.1.4}$$

commutes for all $f : X, A \to B$ in $\mathcal{C}$.

Definition 2.1.2. Let $\mathcal{D}$ be a right $\mathcal{V}$-graded category. The covariant graded presheaf category $[\mathcal{D}, \hat{\mathcal{V}}]$ has right $\mathcal{V}$-graded functors $F : \mathcal{D} \to \hat{\mathcal{V}}$ as objects. Given $F, G : \mathcal{D} \to \hat{\mathcal{V}}$, a graded transformation $\phi : X', F \Rightarrow G$ is given by a family of natural transformations $\phi_U : FU \Rightarrow GU(X' \otimes -)$ ($U \in \mathcal{D}$) satisfying the property that

$$\begin{tikzcd}
FU \arrow[r, Rightarrow, "\phi_U"] \arrow[d, Rightarrow, "Fh"'] & GU(X' \otimes -) \arrow[r, Rightarrow, "(Gh)(X' \otimes -)"] & GV((X' \otimes -) \otimes X) \arrow[d, Rightarrow, "(GV)(a^{-1}_{X',-,X})"] \\
FV(- \otimes X) \arrow[rr, Rightarrow, "(\phi_V)(- \otimes X)"] & & GV(X' \otimes (- \otimes X))
\end{tikzcd} \tag{2.1.5}$$

commutes for all $h : U, X \to V$ in $\mathcal{D}$.

A graded transformation $\phi : X', F \Rightarrow G$ is equivalently given by a family of natural transformations $\tilde{\phi}_U : \mathsf{Y}(X') \otimes FU \Rightarrow GU$ ($U \in \mathcal{D}$) satisfying the property that

$$\begin{tikzcd}
\mathsf{Y}(X') \otimes FU \otimes \mathsf{Y}(X) \arrow[r, Rightarrow, "\tilde{\phi}_U \otimes \mathsf{Y}(X)"] \arrow[d, Rightarrow, "\mathsf{Y}(X') \otimes \overline{Fh}"'] & GU \otimes \mathsf{Y}(X) \arrow[d, Rightarrow, "\overline{Gh}"] \\
\mathsf{Y}(X') \otimes FV \arrow[r, Rightarrow, "\tilde{\phi}_V"] & GV
\end{tikzcd} \tag{2.1.6}$$

commutes for all $h : U, X \to V$ in $\mathcal{D}$.

The contravariant graded presheaf category $[\mathcal{D}^\circ, \hat{\mathcal{V}}]$ is a right $\mathcal{V}$-graded category whose objects are $\mathcal{V}$-graded functors $T : \mathcal{D}^\circ \to \hat{\mathcal{V}}$. Given $T, S : \mathcal{D}^\circ \to \hat{\mathcal{V}}$, a graded transformation $\psi : T, X' \Rightarrow S$ is given by a family of natural transformations $\psi_U : TU \Rightarrow SU(- \otimes X')$ ($U \in \mathcal{D}$) satisfying the property that

$$\begin{tikzcd}
TV \arrow[r, Rightarrow, "\psi_V"] \arrow[d, Rightarrow, "Th"'] & SV(- \otimes X') \arrow[r, Rightarrow, "(Sh)(- \otimes X')"] & SU(X \otimes (- \otimes X')) \arrow[d, Rightarrow, "(SU)(a_{X,-,X'})"] \\
TU(X \otimes -) \arrow[rr, Rightarrow, "(\psi_U)(X \otimes -)"] & & SU((X \otimes -) \otimes X')
\end{tikzcd} \tag{2.1.7}$$

commutes for all $h : U, X \to V$ in $\mathcal{D}$.

A graded transformation $\psi : T, X' \Rightarrow S$ is equivalently given by a family of natural transformations $\tilde{\psi}_U : TU \otimes \mathsf{Y}(X') \Rightarrow SU$ ($U \in \mathcal{D}$) satisfying the property that

$$\begin{tikzcd}
\mathsf{Y}(X) \otimes TU \otimes \mathsf{Y}(X') \arrow[r, Rightarrow, "\mathsf{Y}(X)\otimes\tilde{\psi}_U"] \arrow[d, Rightarrow, "\overline{Th} \otimes \mathsf{Y}(X')"'] & \mathsf{Y}(X) \otimes SU \arrow[d, Rightarrow, "\overline{Sh}"] \\
TV \otimes \mathsf{Y}(X') \arrow[r, Rightarrow, "\tilde{\psi}_V"] & SV
\end{tikzcd} \tag{2.1.8}$$

commutes for all $h : U, X \to V$ in $\mathcal{D}$

Proposition 2.1.3. If $\mathcal{C}$ has a generating collection $\mathcal{G}$, then a family of graded morphisms $\phi_A : FA, X' \to GA$ ($A \in \mathcal{C}$) from Example 2.1.1 form a graded transformation if and only if Diagram 2.1.1 commutes for all generating morphisms $f \in \mathcal{G}$. It is therefore sufficient to verify this commutativity condition only for the generating morphisms.

Proof. The case where $\mathcal{G}$ does not contains any isolated generating objects is proven in [15, Proposition 8.2]. For the general case where $\mathcal{G}$ may contain isolated generating objects, we define an alternative generating collection $\mathcal{G}'$ where $\mathcal{G}'_0 = \mathcal{G}_0$ and

$$\mathcal{G}'_1 = \mathcal{G}_1 \cup \{1_A \mid \text{ A is an isolated generating object in } \mathcal{G}\}$$

Then $\mathcal{G}'$ is a generating collection for $\mathcal{C}$ and $\mathcal{G}'$ does not contain any isolated generating objects. Given any family of graded morphisms $\phi_A : FA, X' \to GA$ ($A \in \mathcal{C}$), if Diagram 2.1.1 commutes for all graded morphisms $f \in \mathcal{G}$ then it commutes for all graded morphisms $f \in \mathcal{G}'$, so by [15, Proposition 8.2], ϕ is a graded transformation. ∎

Remark 2.1.4. Note that in the situation of Proposition 2.1.3, if $\mathcal{G}$ only contains objects, then any such ϕ is vacuously a graded transformation.

As one might expect, there is a relationship between Street's presheaf category and our graded presheaf category. The right $\hat{\mathcal{V}}$-category $\mathcal{P}^{\dagger}_{\hat{\mathcal{V}}}\mathcal{K}$ has $\hat{\mathcal{V}}$-functors $F : \mathcal{K} \to \hat{\mathcal{V}}$ as objects. For $F, G : \mathcal{K} \to \hat{\mathcal{V}}$, the hom-object G^F in $\hat{\mathcal{V}}$ is a limit for the diagram

$$GK^{FK} \xRightarrow{T} GK'^{\,\mathcal{K}(-,K;K')\otimes FK} \xLeftarrow{U} GK'^{FK'} \qquad (K, K' \in \mathcal{K})$$

$\mathcal{P}^{\dagger}_{\hat{\mathcal{V}}}\mathcal{K}$ is then a right $\mathcal{V}$-graded category whose graded morphisms of grade $X \in \mathcal{V}$ are elements of $G^F(X)$.

By the Yoneda Lemma, an element of $G^F(X)$ corresponds to a morphism $\eta : \mathsf{Y}X \Rightarrow G^F$. Since G^F is a limit in $\hat{\mathcal{V}}$, η corresponds to a cone with vertex $\mathsf{Y}X$ over the above diagram. We can simplify the data required for this cone; that is, the cone corresponds to a family $\eta_K : \mathsf{Y}X \Rightarrow GK^{FK}$ ($K \in \mathcal{K}$) such that

$$\begin{tikzcd}
\mathsf{Y}X \arrow[r, Rightarrow, "\eta_K"] \arrow[d, Rightarrow, "\eta_{K'}"'] & GK^{FK} \arrow[d, Rightarrow, "T"] \\
GK'^{FK'} \arrow[r, Rightarrow, "U"'] & GK'^{\,\mathcal{K}(K,K')\otimes FK}
\end{tikzcd}$$

commutes for all K in $\mathcal{K}$. We call such a family a **strong wedge** with vertex $\mathsf{Y}X$. Equivalently, a strong wedge with vertex $\mathsf{Y}X$ is given by $\eta_K : \mathsf{Y}X \Rightarrow GK^{FK}$ $(K \in \mathcal{K})$ such that

$$\begin{tikzcd}
\mathcal{K}(K,K') \otimes FK \otimes \mathsf{Y}X \arrow[rr, Rightarrow, "\mathcal{K}(K,K')\otimes FK\otimes\eta_K"] \arrow[d, Rightarrow, "\overline{F}_{K,K'}\otimes\mathsf{Y}X"'] & & \mathcal{K}(K,K') \otimes FK \otimes GK^{FK} \arrow[d, Rightarrow, "\mathcal{K}(K,K')\otimes ev_{FK,GK}"] \\
FK' \otimes \mathsf{Y}X \arrow[d, Rightarrow, "FK'\otimes\eta_{K'}"'] & & \mathcal{K}(K,K') \otimes GK \arrow[d, Rightarrow, "\overline{G}_{K,K'}"] \\
FK' \otimes GK'^{FK'} \arrow[rr, Rightarrow, "ev_{FK',GK'}"'] & & GK'
\end{tikzcd}$$

commutes for all K in $\mathcal{K}$. By transposition„ such a strong wedge is also given by a family of morphisms $\overline{\eta}_K : FK \otimes \mathsf{Y}X \to GK$ $(K \in \mathcal{K})$ such that

$$\begin{tikzcd}
\mathcal{K}(K,K') \otimes FK \otimes \mathsf{Y}X \arrow[r, Rightarrow, "\mathcal{K}(K,K')\otimes\overline{\eta}_K"] \arrow[d, Rightarrow, "\overline{F}_{K,K'}\otimes\mathsf{Y}X"'] & \mathcal{K}(K,K') \otimes GK \arrow[d, Rightarrow, "\overline{G}_{K,K'}"] \\
FK' \otimes \mathsf{Y}X \arrow[r, Rightarrow, "\overline{\eta}_{K'}"'] & GK'
\end{tikzcd}$$

commutes for all K in $\mathcal{K}$. Thus, a graded morphism in $\mathcal{P}^{\dagger}_{\hat{\mathcal{V}}}\mathcal{K}$ is equivalently given by such a family $\overline{\eta}_K : FK \otimes \mathsf{Y}X \to GK$ $(K \in \mathcal{K})$.

Since $\mathcal{P}^{\dagger}_{\hat{\mathcal{V}}}\mathcal{K}$ is a right $\hat{\mathcal{V}}$-category, the composition morphism for $F, G, H \in \mathcal{P}^{\dagger}_{\hat{\mathcal{V}}}\mathcal{K}$ is given by the unique morphism $\circ_{F,G,H} : G^F \otimes H^G \Rightarrow H^F$ in $\hat{\mathcal{V}}$ such that

$$\begin{tikzcd}
G^F \otimes H^G \arrow[r, Rightarrow, "\circ_{F,G,H}"] \arrow[d, Rightarrow] & H^F \arrow[d, Rightarrow] \\
GK^{FK} \otimes HK^{GK} \arrow[r, Rightarrow, "m_{FK,GK,HK}"] & HK^{FK}
\end{tikzcd} \tag{2.1.9}$$

commutes for every $K \in ob(\mathcal{K})$. The vertical maps are given by the components at K of the limiting cones out of G^F, H^G and H^F, and $m_{FK,GK,HK}$ is the composition in $\hat{\mathcal{V}}$ enriched over itself.

Lemma 2.1.5. There is an isomorphism $\mathcal{P}^{\dagger}_{\hat{\mathcal{V}}}\mathcal{K} \cong [\mathcal{K}, \hat{\mathcal{V}}]$ of right $\mathcal{V}$-graded categories.

Proof. For a concise abstract proof of this result, we recommend [15, 12.4]. However, here we choose to construct the $\mathcal{V}$-graded isomorphism directly to highlight the relationship between graded morphisms in both $\mathcal{V}$-graded categories.

Note that objects in both these right $\mathcal{V}$-graded categories are $\mathcal{V}$-graded functors $F : \mathcal{K} \to \hat{\mathcal{V}}$. Therefore, we wish to construct an identity-on-objects isomorphism. For $F, G : \mathcal{K} \to \hat{\mathcal{V}}$, we wish to find a bijective correspondence between elements of $G^F(X)$ and elements of $[\mathcal{K}, \hat{\mathcal{V}}](F, X; G)$.

Recall that $\eta : F, X \Rightarrow G$ in $[\mathcal{K}, \hat{\mathcal{V}}]$ is given by a family $\overline{\eta}_K : FK \otimes \mathsf{Y}X \Rightarrow GK$ ($K \in \mathcal{K}$) such that for all $\phi : Z, K \to K'$ in $\mathcal{K}$, Diagram 2.1.2 commutes. By the Yoneda Lemma, ϕ corresponds to a morphism $\tilde{\phi}_Z : \mathsf{Y}Z \Rightarrow \mathcal{K}(-, K; K')$. The graded morphisms $\overline{F\phi} : \mathsf{Y}Z \otimes FK \to FK'$ and $\overline{G\phi} : \mathsf{Y}Z \otimes GK \to GK'$ are equivalently the composites $\overline{F}_{K,K'} \cdot (\tilde{\phi}_Z \otimes FK)$ and $\overline{G}_{K,K'} \cdot (\tilde{\phi}_Z \otimes GK)$ respectively, with $F_{K,K'} : \mathbb{K}(K, K') \otimes FK \to FK'$ and $G_{K,K'} : \mathcal{K}(K, K') \otimes GK \to GK'$. We can then consider the diagram

$$\begin{tikzcd}
\mathsf{Y}Z \otimes FK \otimes \mathsf{Y}X \arrow[r, Rightarrow, "\mathsf{Y}Z \otimes \overline{\eta}_K"] \arrow[d, Rightarrow, "\tilde{\phi}_Z \otimes FK \otimes \mathsf{Y}X"'] \arrow[dd, Rightarrow, bend right=60, "\overline{F\phi} \otimes \mathsf{Y}X"'] & \mathsf{Y}Z \otimes GK \arrow[d, Rightarrow, "\tilde{\phi}_Z \otimes GK"] \arrow[dd, Rightarrow, bend left=60, "\overline{G\phi}"] \\
\mathcal{K}(K, K') \otimes FK \otimes \mathsf{Y}X \arrow[r, Rightarrow, "\mathcal{K}(K,K') \otimes \overline{\eta}_K"] \arrow[d, Rightarrow, "\overline{F}_{K,K'} \otimes \mathsf{Y}X"'] & \mathcal{K}(K, K') \otimes GK \arrow[d, Rightarrow, "\overline{G}_{K,K'}"] \\
FK' \otimes \mathsf{Y}X \arrow[r, Rightarrow, "\overline{\eta}_{K'}"'] & GK'
\end{tikzcd}$$

We note that $\left(\tilde{\phi}_Z\right)_{Z \in \mathcal{V}}$ is a jointly epimorphic family since it is a colimit cocone for the presheaf $\mathcal{K}(-, K; K')$ (a colimit of representable functors). By the jointly epimorphic property of $\left(\tilde{\phi}_Z\right)_{Z \in \mathcal{V}}$, the outer diagram commutes if and only if the lower square commutes. In other words, the family $(\overline{\eta}_K)_{K \in \mathcal{K}}$ is a graded transformation if and only if it is also a strong wedge with vertex $\mathsf{Y}X$; i.e., an element of $G^F(X)$ as described above. Therefore, we have constructed a fully faithful identity-on-objects functor, meaning that we have an isomorphism.

It remains to show that this isomorphism is a $\mathcal{V}$-graded functor by verifying that it is preserves composition, reindexing, and identities.

The identities case is immediately satisfied from the work above.

Taking elements $\eta \in G^F(U)$ and $\gamma \in H^G(X)$, the composition in $\mathcal{P}^{\dagger}_{\mathcal{V}}\mathcal{K}$ is given by

$$\circ_{F,G,H} : G^F \otimes H^G \Longrightarrow H^F\ , \qquad \circ^{U,X}_{F,G,H} : G^F(U) \times H^G(X) \to H^F(U \otimes X)$$

We know from Example 1.1.11 that $\gamma \circ \eta \in H^F(U \otimes X)$ corresponds by the Yoneda Lemma to

$$\widetilde{\gamma \circ \eta} = \left(\mathsf{Y}(U \otimes X) \cong \mathsf{Y}U \otimes \mathsf{Y}X \xRightarrow{\tilde{\eta} \otimes \tilde{\gamma}} G^F \otimes H^G \xRightarrow{\circ_{F,G,H}} H^F\right)$$

Given that H^F is a limit in $\hat{\mathcal{V}}$, then $\gamma \circ \eta$ corresponds to the strong wedge whose components $(\widetilde{\gamma \circ \eta})_K$ are given by

$$\mathsf{Y}(U \otimes X) \cong \mathsf{Y}U \otimes \mathsf{Y}X \xRightarrow{\tilde{\eta} \otimes \tilde{\gamma}} G^F \otimes H^G \xRightarrow{\circ_{F,G,H}} H^F \Longrightarrow HK^{FK} \qquad (K \in \mathcal{K})$$

By the commutativity of Diagram 2.1.9, the component $(\widetilde{\gamma \circ \eta})_K$ is equally given by

$$\mathsf{Y}(U \otimes X) \cong \mathsf{Y}U \otimes \mathsf{Y}X \xRightarrow{\tilde{\eta}_K \otimes \tilde{\gamma}_K} GK^{FK} \otimes HK^{GK} \xRightarrow{m_{FK,GK,HK}} HK^{FK} \qquad (K \in \mathcal{K})$$

Taking the transposes, we have a family whose components $\overline{(\gamma \circ \eta)}_K$ are given by

$$FK \otimes \mathsf{Y}(U \otimes X) \cong FK \otimes \mathsf{Y}U \otimes \mathsf{Y}X \xRightarrow{\bar{\eta}_K \otimes \mathsf{Y}X} GK \otimes \mathsf{Y}X \xRightarrow{\bar{\gamma}_K} HK \qquad (K \in \mathcal{K})$$

We recall that $\gamma \circ \eta$ is a graded transformation if and only if it is a strong wedge, and we note that the composite of η and γ as graded transformation is precisely the same as the composite given above.

Let $\eta \in G^F(U)$ and $\alpha : X \to U$ in $\mathcal{V}$. The reindexing $\alpha^*(\eta) \in G^F(X)$ is given by $G^F(\alpha)(\eta)$ where $G^F(\alpha)$ is the induced function between the limit sets $G^F(U)$ and $G^F(X)$; i.e., $G^F(\alpha) : G^F(U) \to G^F(X)$ is the function such that

$$\begin{tikzcd}
G^F(U) \arrow[r, Rightarrow, "G^F(\alpha)"] \arrow[d, Rightarrow] & G^F(X) \arrow[d, Rightarrow] \\
GK^{FK}(U) \arrow[r, Rightarrow, "GK^{FK}(\alpha)"] & GK^{FK}(X)
\end{tikzcd}$$

commutes for all K in $\mathcal{K}$. Chasing $\eta \in G^F(U)$ through the bottom left of the diagram, we take the strong wedge $\eta = (\bar{\eta}_K : FK \otimes \mathsf{Y}U \to GK)_{K \in \mathcal{K}}$, we pick out the component $\bar{\eta}_K : FK \otimes \mathsf{Y}U \to GK$, and we map this component to

$$FK \otimes \mathsf{Y}X \xRightarrow{FK \otimes \mathsf{Y}\alpha} FK \otimes \mathsf{Y}U \xRightarrow{\bar{\eta}_K} GK$$

For this diagram to commute, we require $G^F(\alpha)(\eta)$ to be the family

$$FK \otimes \mathsf{Y}X \xRightarrow{FK \otimes \mathsf{Y}\alpha} FK \otimes \mathsf{Y}U \xRightarrow{\bar{\eta}_K} GK \qquad (K \in \mathcal{K})$$

which we know to be a strong wedge since the outer diagram

$$\begin{tikzcd}
\mathcal{K}(K,K') \otimes FK \otimes \mathsf{Y}X \arrow[r, Rightarrow, "{\mathcal{K}(K,K') \otimes FK \otimes \mathsf{Y}\alpha}"] \arrow[d, Rightarrow, "\overline{F}_{K,K'} \otimes \mathsf{Y}X"'] & \mathcal{K}(K,K') \otimes FK \otimes \mathsf{Y}U \arrow[r, Rightarrow, "{\mathcal{K}(K,K') \otimes \bar{\eta}_K}"] \arrow[d, Rightarrow, "\overline{F}_{K,K'} \otimes \mathsf{Y}U"] & \mathcal{K}(K,K') \otimes GK \arrow[d, Rightarrow, "\overline{G}_{K,K'}"] \\
FK' \otimes \mathsf{Y}X \arrow[r, Rightarrow, "FK' \otimes \mathsf{Y}\alpha"] & FK' \otimes \mathsf{Y}U \arrow[r, Rightarrow, "\bar{\eta}_{K'}"] & GK'
\end{tikzcd}$$

commutes by the commutativity of the two inner squares. Finally, we recall that $\alpha^*(\eta)$ is a graded transformation if and only if it is a strong wedge, and we note that the reindexing along α of η as a graded transformation is precisely the same as the $G^F(\alpha)(\eta)$ we defined above. Thus, the isomorphism $\mathcal{P}^{\dagger}_{\mathcal{V}}\mathcal{K} \cong [\mathcal{K}, \hat{\mathcal{V}}]$ preserves reindexings. ∎

Lemma 2.1.6. Let $\mathcal{K}$ be a small $\mathcal{V}$-category for a monoidal biclosed and complete category $\mathcal{V}$, and let $F, G : \mathcal{K} \to \mathcal{V}$ be left $\mathcal{V}$-functors. Then there is an isomorphism $\mathsf{Y}(\mathcal{P}^{\dagger}_{\mathcal{V}}\mathcal{K}(F,G)) \cong [\mathcal{K}, \hat{\mathcal{V}}](YF, YG)$.

Proof. Note that $\mathsf{Y} : \mathcal{V} \to \hat{\mathcal{V}}$ preserves limits so $\mathsf{Y}(G^F)$ is a limit in $\hat{\mathcal{V}}$ for the diagram

$$\begin{tikzcd}
\mathsf{Y}(GK^{FK}) \arrow[r] & \mathsf{Y}(GK'^{\mathcal{K}(K,K') \otimes FK}) & \mathsf{Y}(GK'^{FK'}) \arrow[l]
\end{tikzcd}$$

By Lemma 1.1.12, Y preserves homs up to isomorphism, so $\mathsf{Y}(G^F)$ is a limit for the diagram

$$\begin{tikzcd}
\mathsf{Y}GK^{\mathsf{Y}FK} \arrow[r] & \mathsf{Y}GK'^{\mathcal{K}(-,K;K')\otimes \mathsf{Y}FK} & \mathsf{Y}GK'^{\mathsf{Y}FK'} \arrow[l]
\end{tikzcd}$$

Thus, $\mathsf{Y}(G^F)$ is isomorphic to $\mathcal{P}^{\dagger}_{\hat{\mathcal{V}}}\mathcal{K}(\mathsf{Y}F, \mathsf{Y}G)$. By Lemma 2.1.5, we have

$$\mathsf{Y}(\mathcal{P}^{\dagger}_{\mathcal{V}}\mathcal{K}(F,G)) \cong \mathcal{P}^{\dagger}_{\hat{\mathcal{V}}}\mathcal{K}(\mathsf{Y}F, \mathsf{Y}G) \cong [\mathcal{K}, \hat{\mathcal{V}}](YF, YG)$$

■

2.2 Representability of $\mathcal{V}$-Graded Functors

Definition 2.2.1. Let $\mathcal{C}$ be a left $\mathcal{V}$-graded category. The **$\mathcal{V}$-graded Yoneda embedding** for $\mathcal{C}$ is a left $\mathcal{V}$-graded functor

$$y : \mathcal{C} \to [\mathcal{C}^{\circ}, \hat{\mathcal{V}}]$$

mapping an object $A \in \mathcal{C}$ to the contravariant right $\mathcal{V}$-graded hom-functor $\mathcal{C}(-, A) : \mathcal{C}^{\circ} \to \hat{\mathcal{V}}$ from Definition 2.0.17, and a graded morphism $f : X, A \to B$ in $\mathcal{C}$ to the graded transformation $\mathcal{C}(-, f) : X, \mathcal{C}(-, A) \to \mathcal{C}(-, B)$ given by the family of graded morphisms $\mathcal{C}(D, f) : X, \mathcal{C}(-, D; A) \to \mathcal{C}(-, D; B)$ $(D \in \mathcal{C})$ in $\hat{\mathcal{V}}$.

Theorem 2.2.2 (Strong Graded Yoneda Lemma)**.** Let $\mathcal{C}$ be a $\mathcal{V}$-graded category and let $F : \mathcal{C}^{\circ} \to \hat{\mathcal{V}}$ be a right $\mathcal{V}$-graded functor. There is an isomorphism

$$[\mathcal{C}^{\circ}, \hat{\mathcal{V}}](\mathcal{C}(-, A), F) \cong FA$$

$\mathcal{V}$-graded natural in $A \in \mathcal{C}$ and $F \in [\mathcal{C}^{\circ}, \hat{\mathcal{V}}]$. This gives us a bijective correspondence between elements $\delta \in (FA)X$ and graded transformations $\tilde{\delta} : X, \mathcal{C}(-, A) \Rightarrow F$.

Proof. By the $\hat{\mathcal{V}}$-enriched Strong Yoneda Lemma, we have

$$\mathcal{P}_{\hat{\mathcal{V}}}\mathcal{C}(\mathcal{C}(-, A), F) \cong FA$$

$\mathcal{V}$-graded natural in $A \in \mathcal{C}$ and $F \in \mathcal{P}_{\hat{\mathcal{V}}}\mathcal{C}$. By Lemma 2.1.6, the above isomorphism is equivalently given by

$$[\mathcal{C}^{\circ}, \hat{\mathcal{V}}](\mathcal{C}(-, A), F) \cong FA$$

$\mathcal{V}$-graded natural in $A \in \mathcal{C}$ and $F \in [\mathcal{C}^{\circ}, \hat{\mathcal{V}}]$.

There is also an excellent proof of this result in ([15], 12.4) using instead the $\mathcal{V}$-modules formulation of Street's presheaf category. ■

Remark 2.2.3. The correspondence of the $\mathcal{V}$-graded Yoneda Lemma is given in the following way:

Let $\delta \in (FA)X$. We may obtain a graded transformation $\tilde{\delta} : X, \mathcal{C}(-, A) \Rightarrow F$ with components

$$(\tilde{\delta}_B)_Y : \mathcal{C}(Y, B; A) \to FB(X \otimes Y) \qquad (B \in \mathcal{C}, Y \in \mathcal{V})$$

given by $(\tilde{\delta}_B)_Y(g) := (Fg)_X\delta$ for $g : Y, B \to A$ in $\mathcal{C}$. Conversely, let $\mu : X, \mathcal{C}(-, A) \Rightarrow F$ be a graded transformation. We may obtain an element

$$FA(\ell_X^{-1})((\mu_A)_I 1_A) \in (FA)X$$

Theorem 2.2.4 (Weak Yoneda Lemma)**.** Let $\mathcal{C}$ be a $\mathcal{V}$-graded category and let $F : \mathcal{C}^\circ \to \hat{\mathcal{V}}$ be a right $\mathcal{V}$-graded functor. There is a bijective correspondence between elements $\delta \in (FA)I$ and $\mathcal{V}$-graded natural transformations $\tilde{\delta} : \mathcal{C}(-, A) \Rightarrow F$.

Proof. By the Strong Graded Yoneda Lemma, there is an isomorphism

$$[\mathcal{C}^\circ, \hat{\mathcal{V}}](I, \mathcal{C}(-, A); F) \cong (FA)I$$

giving a bijective correspondence between elements $\delta \in (FA)I$ and graded transformations of the form $\tilde{\mu} : I, \mathcal{C}(-, A) \Rightarrow F$. Taking the components $(\tilde{\mu}_B)_Y : \mathcal{C}(Y, B; A) \Rightarrow FB(I \otimes Y)$ $(B \in \mathcal{C}, Y \in \mathcal{V})$ and postcomposing

$$\mathcal{C}(Y, B; A) \Longrightarrow FB(I \otimes Y) \Longrightarrow (FB)Y$$

with $FB(\ell_Y^{-1})$ gives us components of a $\mathcal{V}$-graded natural transformation $\tilde{\delta} : \mathcal{C}(-, A) \Rightarrow F$ such that

$$(\tilde{\delta}_B)_Y(g) := FB(\ell_Y^{-1})(\tilde{\mu}_B)_Y(g) = FB(\ell_Y^{-1})(Fg)_I\delta$$

for $g : Y, B \to A$ in $\mathcal{C}$. For the inverse, a $\mathcal{V}$-graded natural transformation $\tilde{\delta} : \mathcal{C}(-, A) \Rightarrow F$ corresponds to the element $(\tilde{\delta}_A)_I(1_A) \in (FA)I$. ∎

Definition 2.2.5. A **representation** of a right $\mathcal{V}$-graded functor $F : \mathcal{C}^\circ \to \hat{\mathcal{V}}$ is given by an object $A \in \mathcal{C}$ and a graded natural isomorphism $\theta : \mathcal{C}(-, A) \Rightarrow F$. We say F is representable if it admits a $\mathcal{V}$-graded representation (A, θ).

By the (weak) Graded Yoneda Lemma, a representation is equivalently given by an element $\delta \in (FA)I$ whose corresponding graded natural transformation $\tilde{\delta} : \mathcal{C}(-, A) \Rightarrow F$ is invertible. We call δ a counit of the representation.

Given that $\mathcal{V}$-graded categories are $\hat{\mathcal{V}}$-categories, $\mathcal{V}$-graded functors are $\hat{\mathcal{V}}$-functors, and $\mathcal{C}(-, A)$ is the contravariant representable $\hat{\mathcal{V}}$-functor, we have that $\mathcal{V}$-graded representations are enriched representations in $\hat{\mathcal{V}}$.

Proposition 2.2.6. Let $\mathcal{V}$ be monoidal biclosed and let $\mathcal{C}$ be a left $\mathcal{V}$-category. A $\mathcal{V}$-enriched representation of the right $\mathcal{V}$-functor $P : \mathcal{C}^\circ \to \mathcal{V}$ is equivalently a $\mathcal{V}$-graded representation of the composite $\mathsf{Y}P$ where P is regarded as a $\mathcal{V}$-graded functor and $\mathsf{Y} : \mathcal{V} \to \hat{\mathcal{V}}$ is the canonical embedding of right $\mathcal{V}$-graded categories.

Proof. We can regard the right $\mathcal{V}$-category $\mathcal{C}^\circ$ as a right $\mathcal{V}$-graded category $\mathsf{Y}_*\mathcal{C}^\circ$, and P as a $\hat{\mathcal{V}}$-functor Y_*P. Recall that $\mathsf{Y}_* : CAT_{\mathcal{V}} \to CAT_{\hat{\mathcal{V}}}$ is a fully faithful 2-functor. Therefore, it preserves and reflects isomorphisms between 1-cells. We then have for $\mathcal{C}(-, A) \cong P$ as $\mathcal{V}$-functors an associated $\mathsf{Y}_*\left(\mathcal{C}(-, A)\right) \cong \mathsf{Y}_*P$ as $\hat{\mathcal{V}}$-functors.

Since, as shown in Lemma 2.0.19, $\mathsf{Y} : \mathcal{V} \to \hat{\mathcal{V}}$ is a fully faithful $\hat{\mathcal{V}}$-functor, it is a representably fully faithful as a 1-cell in $\hat{\mathcal{V}}$-CAT, in the sense of [12]. This gives us that whiskering on the left by Y preserves and reflects isomorphisms. Therefore, for $\mathsf{Y}_* \left(\mathcal{C}(-, A)\right) \cong \mathsf{Y}_* P$ there is a corresponding $\mathsf{Y}\left(\mathsf{Y}_* \left(\mathcal{C}(-, A)\right)\right) \cong \mathsf{Y}\left(\mathsf{Y}_* P\right)$. Finally, by Lemma 2.0.20, $\mathsf{Y}\left(\mathsf{Y}_* \left(\mathcal{C}(-, A)\right)\right)$ is the contravariant $\mathcal{V}$-graded hom-functor at A.

Thus, every enriched representation $\mathcal{C}(-, A) \cong P$ has an associated $\mathcal{V}$-graded representation $\mathcal{C}(?, -; A) \cong \mathsf{Y}\left(\mathsf{Y}_* P\right)$. ∎

2.3 Envelope Diagrams

In working with ordinary categories, we often express certain properties or provide proofs by constructing commutative diagrams. With $\mathcal{V}$-graded categories, there is the analogous notion of envelope diagrams developed by Lucyshyn-Wright in [15]. In this section, we briefly introduce the concept of envelope diagrams as they will serve in this text almost exclusively as a tool for representing data rather than a proof method. For a deeper look into envelope diagrams and to investigate the extent of their power, see [15, §6].

Given a $\mathcal{V}$-graded category $\mathcal{C}$, an **actegorical environment** of $\mathcal{C}$ is an actegory $\mathcal{E}$ with action $\cdot\,, \cdot : \mathcal{V} \times \mathcal{E} \to \mathcal{E}$ equipped with a $\mathcal{V}$-graded functor $E : \mathcal{C} \hookrightarrow \mathcal{E}$ that is fully faithful and injective on objects. Concretely, we require there to be a bijective correspondence between graded morphisms $f : X, A \to B$ in $\mathcal{C}$ and morphisms $Ef : X, EA \to EB$ in the actegory $\mathcal{E}$, and we require that $EA = EB$ if and only if $A = B$. Note that the domain of $Ef : X, A \to B$ is the object $X, A \in \mathcal{E}$ given by the action of X on A.

In [15], Lucyshyn-Wright constructs what he calls the **enveloping actegory** of $\mathcal{C}$ which is an actegorical environment $(\mathcal{V}{,}\mathcal{C}\,, E : \mathcal{C} \hookrightarrow \mathcal{V}{,}\mathcal{C})$ with the property that for any actegory $\mathcal{D}$, the functor

$$ {}_{\mathcal{V}}ACT^{strong}(\mathcal{V}{,}\mathcal{C}\,, \mathcal{D}) \;\to\; {}_{\mathcal{V}}GCAT(\mathcal{C}, \mathcal{D}) $$

mapping a strong morphism of $\mathcal{V}$-actegories $F : \mathcal{V}{,}\mathcal{C} \to \mathcal{D}$ to the $\mathcal{V}$-graded functor $F \circ E : \mathcal{C} \to \mathcal{D}$, is an equivalence of categories.

Via the embedding $E : \mathcal{C} \to \mathcal{V}{,}\mathcal{C}$, we regard $\mathcal{C}$ as the full $\mathcal{V}$-graded subcategory of $\mathcal{V}{,}\mathcal{C}$ spanned by the objects EA for all $A \in \mathcal{C}$. We may then choose to adopt the convention of simply using A to denote the EA in $\mathcal{V}{,}\mathcal{C}$ and $f : X, A \to B$ for the $Ef : X, EA \to EB$. It is important to note that $\mathcal{V}{,}\mathcal{C}$ is generally significantly larger than $\mathcal{C}$.

In this text, it is sufficient to work with any actegorical environment of $\mathcal{C}$ rather than its enveloping actegory. For a deeper look into the structure of the enveloping actegory, we refer the reader to ([15], 6.1, 6.2, 6.3).

Lucyshyn-Wright define an **envelope diagram** to be a commutative diagram in the enveloping actegory of $\mathcal{C}$. In this way, we may construct diagrams with graded morphisms. When an envelope diagram commutes in $\mathcal{V}{,}\mathcal{C}$, the equality it represents also holds in the $\mathcal{V}$-graded category $\mathcal{C}$. The following are foundational examples from which we can build any commutative diagrams to represent equalities in $\mathcal{C}$.

Example 2.3.1. Let $f : X, A \to B$ and $g : Y, B \to C$ in a $\mathcal{V}$-graded category $\mathcal{C}$. The following commutative envelope diagram in $\mathcal{V}{,}\mathcal{C}$ represents the composition in $\mathcal{C}$ of f and g.

$$\begin{tikzcd}
Y \otimes X, A \arrow[r, "\sim"] \arrow[rrd, "g\circ f"'] & Y, X, A \arrow[r, "{Y,f}"] & Y, B \arrow[d, "g"] \\
& & C
\end{tikzcd} \tag{2.3.1}$$

Note that we use $\sim$ to mean the multiplicator isomorphism in $\mathcal{V}{,}\mathcal{C}$ (i.e., $(Y \otimes X), A \cong Y, (X, A))$, and the f, g, and $g \circ f$ in the diagram are exactly the Ef, Eg, and $E(g \circ f)$ corresponding to the graded morphisms.

Consider a morphism $\alpha : Z \to X$ in $\mathcal{V}$. The reindexing of f along α is represented by the following commutative envelope diagram.

$$\begin{tikzcd}
Z, A \arrow[r, "{\alpha\ ,\ A}"] \arrow[rd, "\alpha^* f"'] & X, A \arrow[d, "f"] \\
& B
\end{tikzcd} \tag{2.3.2}$$

We recall from Definition 2.0.17 the composition of a graded morphism and an ordinary morphism. For $p : A \to B$ and $m : B \to C$ in $\mathcal{C}_0$, the following diagrams in $\mathcal{V}{,}\mathcal{C}$ represent $g \star p$ and $m \star f$ in $\mathcal{C}$ respectively.

$$\begin{tikzcd}
X, A \arrow[r, "{X,p}"] \arrow[rd, "g\star p"'] & X, B \arrow[d, "g"] & & X, A \arrow[r, "f"] \arrow[rd, "m\star f"'] & B \arrow[d, "m"] \\
& C & & & C
\end{tikzcd} \tag{2.3.3}$$

Since a right $\mathcal{V}$-graded category $\mathcal{D}$ is a $\mathcal{V}^{rev}$-graded category, we can consider an actegorical environment $\mathcal{D}{,}\mathcal{V}$ with respect to $\mathcal{V}^{rev}$. One of these $\mathcal{D}{,}\mathcal{V}$ is then a right $\mathcal{V}$-actegory with action $\cdot, \cdot : \mathcal{D}{,}\mathcal{V} \times \mathcal{V} \to \mathcal{D}{,}\mathcal{V}$ whose embedding $E : \mathcal{D} \hookrightarrow \mathcal{D}{,}\mathcal{V}$ is instead a right $\mathcal{V}$-graded functor. We then have the following examples similar to the ones above.

Example 2.3.2. Let $h : A, X \to B$ and $k : B, Y \to C$ in a right $\mathcal{V}$-graded category $\mathcal{D}$. The following commutative envelope diagram represents the composition in $\mathcal{D}$ of h and k.

$$\begin{tikzcd}
A, X \otimes Y \arrow[r, "\sim"] \arrow[rrd, "k\circ h"'] & A, X, Y \arrow[r, "{h,Y}"] & B, Y \arrow[d, "k"] \\
& & C
\end{tikzcd} \tag{2.3.4}$$

We note here that $\sim$ refers to the multiplicator isomorphism in $\mathcal{D}{,}\mathcal{V}$ (i.e., $A, (X \otimes Y) \cong (A, X), Y)$.

Consider a morphism $\alpha : Z \to X$ in $\mathcal{V}$. The reindexing of h along α is represented by the following commutative envelope diagram.

$$\begin{tikzcd}
A, Z \arrow[r, "A\,,\,\alpha"] \arrow[dr, "\alpha^* h"'] & A, X \arrow[d, "h"] \\
 & B
\end{tikzcd} \tag{2.3.5}$$

We recall from Definition 2.0.17 the composition of a graded morphism and an ordinary morphism. For $q : A \to B$ and $n : B \to C$ in $\mathcal{D}_0$, we have

$$\begin{tikzcd}
A, X \arrow[r, "q{,}X"] \arrow[dr, "k\star q"'] & B, X \arrow[d, "k"] \\
 & C
\end{tikzcd}
\qquad
\begin{tikzcd}
A, X \arrow[r, "h"] \arrow[dr, "n\star h"'] & B \arrow[d, "n"] \\
 & C
\end{tikzcd} \tag{2.3.6}$$

Chapter 3

Weighted Limits in $\mathcal{V}$-Graded Categories

Let $(\mathcal{V}, \otimes, I)$ be a monoidal category. Further let $\mathcal{C}$ and $\mathcal{K}$ be left $\mathcal{V}$-graded categories and $\mathcal{E}$ and $\mathcal{J}$ be right $\mathcal{V}$-graded categories.

Let us recall that a limit for a diagram of ordinary categories is a terminal cone over the diagram. Applying this idea in the $\mathcal{V}$-graded context, we may start by asking what a "cone" in $\mathcal{C}$ over D might be.

Definition 3.0.1. Let $W : \mathcal{K} \to \mathcal{V}$ and $D : \mathcal{K} \to \mathcal{C}$ be $\mathcal{V}$-graded functors. A W**-weighted cone** is a pair (L, λ) consisting of an object $L \in \mathcal{C}$ and a family of graded morphisms

$$\lambda_K : WK,\ L \to DK \qquad (K \in \mathcal{K})$$

in $\mathcal{C}$ such that for every $\phi : Y, K \to K'$ in $\mathcal{K}$, the following envelope diagram (naturality of weighted cones) commutes

$$\begin{tikzcd}
Y \otimes WK, L \arrow[r, "\sim"] \arrow[d, "W\phi{,}L"'] & Y, WK, L \arrow[r, "Y{,}\lambda_K"] & Y, DK \arrow[d, "D\phi"] \\
WK', L \arrow[rr, "\lambda_{K'}"] & & DK'
\end{tikzcd} \tag{3.0.1}$$

i.e., $D\phi \circ \lambda_K = (W\phi)^*(\lambda_{K'})$.

A **graded** W**-weighted cone** with grade $X \in ob(\mathcal{V})$ is a pair (A, γ) consisting of an object $A \in \mathcal{C}$ and a family of graded morphisms

$$\gamma_K : WK \otimes X, A \to DK \qquad (K \in \mathcal{K})$$

in $\mathcal{C}$ such that for every $\phi : Y, K \to K'$ in $\mathcal{K}$, the following envelope diagram (naturality of graded weighted cones) commutes

$$\begin{tikzcd}
Y \otimes (WK \otimes X), A \arrow[r, "\sim"] \arrow[d, "a^{-1}_{Y,WK,X}{,}A"'] & Y, WK \otimes X, A \arrow[r, "Y{,}\gamma_K"] & Y, DK \arrow[d, "D\phi"] \\
(Y \otimes WK) \otimes X, A \arrow[r, "W\phi \otimes X{,}A"] & WK' \otimes X, A \arrow[r, "\gamma_{K'}"] & DK'
\end{tikzcd} \tag{3.0.2}$$

i.e., $\left[(W\phi \otimes X) \cdot a^{-1}_{Y,WK,X}\right]^*(\gamma_{K'}) = D\phi \circ \gamma_K$.

Let $U : \mathcal{J} \to \mathcal{V}$ and $F : \mathcal{J} \to \mathcal{E}$ be right $\mathcal{V}$-graded functors. A U-weighted cone is a pair (L, λ) consisting of an object $L \in \mathcal{E}$ and a family of graded morphisms $\lambda_J : L, UJ \to FJ$ $(J \in \mathcal{J})$ such that for every $\psi : J, Y \to J'$ in $\mathcal{J}$, the following envelope diagram commutes

$$\begin{tikzcd}
L,\, UJ \otimes Y \arrow[r, "\sim"] \arrow[d, "L,\, U\psi"'] & L,\, UJ,\, Y \arrow[r, "\lambda_J,\, Y"] & FJ,\, Y \arrow[d, "F\psi"] \\
L,\, UJ' \arrow[rr, "\lambda_{J'}"] & & FJ'
\end{tikzcd} \tag{3.0.3}$$

i.e., $F\psi \circ \lambda_J = (W\phi)^*(\lambda_{J'})$.

A graded U-weighted cone with grade $X \in ob(\mathcal{V})$ is (A, γ) consisting of an object $A \in \mathcal{E}$ and a family of graded morphisms $\gamma_J : A, X \otimes UJ \to FJ$ $(J \in \mathcal{J})$ such that for every $\psi : J, Y \to J'$ in $\mathcal{J}$, the following envelope diagram commutes

$$\begin{tikzcd}
A,\, (X \otimes UJ) \otimes Y \arrow[r, "\sim"] \arrow[d, "A,\, a_{X,UJ,Y}"'] & A,\, X \otimes UJ,\, Y \arrow[r, "\gamma_J,\, Y"] & FJ,\, Y \arrow[d, "F\psi"] \\
A,\, X \otimes (UJ \otimes Y) \arrow[r, "A,\, X \otimes U\psi"] & A,\, X \otimes UJ' \arrow[r, "\gamma_{J'}"] & FJ'
\end{tikzcd} \tag{3.0.4}$$

i.e., $\left[(X \otimes U\psi) \cdot a_{X,UJ,Y}\right]^*(\gamma_{J'}) = F\psi \circ \gamma_J$.

Remark 3.0.2. When $\mathcal{C}$ is an actegory, a W-weighted cone is a pair (L, λ) where $\lambda_K : WK \bullet L \to DK$ $(K \in \mathcal{K})$ such that for every $\phi : Y, K \to K'$ in $\mathcal{K}$, the following diagram in $\mathcal{C}$ commutes

$$\begin{tikzcd}
Y \otimes WK \bullet L \arrow[r, "\sim"] \arrow[d, "W\phi \bullet L"'] & Y \bullet WK \bullet L \arrow[r, "Y \bullet \lambda_K"] & Y \bullet DK \arrow[d, "D\phi"] \\
WK' \bullet L \arrow[rr, "\lambda_{K'}"] & & DK'
\end{tikzcd} \tag{3.0.5}$$

Similarly, a graded W-weighted cone is a pair (A, γ) where $\gamma_A : WK \otimes X \bullet A \to DK$ $(K \in \mathcal{K})$ such that for every $\phi : Y, K \to K'$ in $\mathcal{K}$, the following diagram in $\mathcal{C}$ commutes

$$\begin{tikzcd}
Y \otimes (WK \otimes X) \bullet A \arrow[r, "\sim"] \arrow[d, "a^{-1}_{Y,WK,X} \bullet A"'] & Y \bullet WK \otimes X \bullet A \arrow[r, "Y \bullet \gamma_K"] & Y \bullet DK \arrow[d, "D\phi"] \\
(Y \otimes WK) \otimes X \bullet A \arrow[r, "W\phi \otimes X \bullet A"] & WK' \otimes X \bullet A \arrow[r, "\gamma_{K'}"] & DK'
\end{tikzcd} \tag{3.0.6}$$

Lemma 3.0.3. A W-weighted cone is equivalently given by an I-graded W-weighted cone.

Proof. Let $(B, \beta_K : WK, B \to DK)$ $(K \in \mathcal{K})$ be a family of graded morphisms. We consider the pair (B, b) where we have $b_K := (r_{WK})^*(\beta_K) : WK \otimes I, B \to DK$. For every $\phi : Y, K \to K'$ in $\mathcal{K}$, we can construct the following envelope diagram

$$\begin{tikzcd}
Y \otimes (WK \otimes I), B \arrow[rr, "\sim"] \arrow[dddd, "\alpha^{-1}_{Y,WK,I}{,}B"'] \arrow[ddr, "Y\otimes r_{WK}{,}B"] & & Y, WK \otimes I, B \arrow[rr, "Y{,}b_K"] \arrow[dr, "Y{,}r_{WK}{,}B"'] & & Y, DK \arrow[dddd, "D\phi"] \\
 & & & Y, WK, B \arrow[ur, "Y{,}\beta_K"'] & \\
 & Y \otimes WK, B \arrow[urr, "\sim"'] \arrow[drr, "W\phi{,}B"] & & & \\
 & & & WK', B \arrow[dr, "\beta_{K'}"] & \\
(Y \otimes WK) \otimes I, B \arrow[uur, "r_{Y\otimes WK}{,}B"'] \arrow[rr, "W\phi \otimes I{,}B"'] & & WK' \otimes I, B \arrow[ur, "r_{WK'}{,}B"] \arrow[rr, "b_{K'}"'] & & DK'
\end{tikzcd}$$

Note that the pentagon is exactly Diagram 3.0.1. The two triangles on the top and bottom right commute by definition of b. The triangle on the left commutes by the monoidal property of $\mathcal{V}$ as in [16, §7.1]. The squares on the top and bottom commute by the naturality of the multiplicator $\sim$ and right unitor r respectively. Since all these inner diagrams commute, we can conclude that the outer diagram commutes if and only if the pentagon commutes. Thus, (B, b) is an I-graded W-weighted cone if and only if (B, β) is a W-weighted cone. ∎

Definition 3.0.4. We say the W-weighted cone (L, λ) given by $L \in ob(\mathcal{C})$ and $\lambda_K : WK, L \to DK$ $(K \in \mathcal{K})$ is a W-**weighted limit of** D if it satisfies the following graded universal property:

> For every graded W-weighted cone (A, γ) on D given by $A \in ob(\mathcal{C})$ and the family $\gamma_K : WK \otimes X, A \to DK$ $(K \in \mathcal{K})$, there is a unique graded morphism $g : X, A \to L$ in $\mathcal{C}$ such that the following envelope diagram commutes
>
> $$\begin{tikzcd}
WK \otimes X, A \arrow[r, "\sim"] \arrow[dr, "\gamma_K"'] & WK, X, A \arrow[r, "WK{,}g"] & WK, L \arrow[dl, "\lambda_K"] \\
 & DK &
\end{tikzcd}$$
>
> i.e.,$\gamma_K = \lambda_K \circ g$ for all K in $\mathcal{K}$.

Lemma 3.0.5. We can equivalently say that (L, λ) is a W-weighted limit of D if it satisfies the property that the function

$$\begin{array}{ccc}
\mathcal{C}(X, A; L) & \to & WCone(W, A, D)(X) \\
g : X, A \to L & \mapsto & (\lambda_K \circ g)_{K \in \mathcal{K}}
\end{array}$$

is bijective for all $A \in \mathcal{C}$ and $X \in \mathcal{V}$. Here we take $WCone(W, A, D)(X)$ to be the set of X-graded weighted cones with vertex A on D.

Proof. This is immediate once we verify that the given function is well-defined; i.e., that $(\lambda_K \circ g)_{K \in \mathcal{K}}$ is an X-graded weighted cone. This follows conveniently from

$$D\phi \circ (\lambda_K \circ g) = (a^{-1}_{Y,WK,X})^*[(D\phi \circ \lambda_K) \circ g]$$

$$
\begin{aligned}
&= (a^{-1}_{Y,WK,X})^*[(W\phi)^*(\lambda_{K'}) \circ g] \\
&= (a^{-1}_{Y,WK,X})^*[(W\phi \otimes X)^*(\lambda_{K'} \circ g)] \\
&= [(W\phi \otimes X) \cdot (a^{-1}_{Y,WK,X})]^*(\lambda_{K'} \circ g)
\end{aligned}
$$

Now we recall what it means for this function to be bijective:

∎

Remark 3.0.6. When $\mathcal{C}$ is an actegory, the graded universal property of the limiting weighted cone is satisfied when the following diagram in $\mathcal{C}$ commutes

$$\begin{tikzcd}
WK \otimes X \bullet A \arrow[r, "\sim"] \arrow[dr, "\gamma_K"'] & WK \bullet X \bullet A \arrow[r, "WK \bullet g"] & WK \bullet L \arrow[dl, "\lambda_K"] \\
& DK &
\end{tikzcd}$$

for all K in $\mathcal{K}$. The combination of this and the previous remark allows us to define weighted limits over an actegory using only its actegorical structure rather than its $\mathcal{V}$-graded structure.

There is a weaker version of the universal property of the limiting weighted cone which one can contrast with the above graded universal property. Given a W-weighted cone (L, λ) on $D : \mathcal{K} \to \mathcal{C}$, the ordinary universal property of the limiting weighted cone is the following:

> For every W-weighted cone $\gamma_K : WK, A \to DK)$ $(K \in \mathcal{K})$ on D, there is a unique ordinary morphism $g : A \to L$ in $\mathcal{C}_0$ such that the following envelope diagram commutes
>
> $$\begin{tikzcd}
WK, A \arrow[rr, "{WK,g}"] \arrow[dr, "\gamma_K"'] & & WK, L \arrow[dl, "\lambda_K"] \\
& DK &
\end{tikzcd}$$
>
> i.e., $\gamma_K = \lambda_K \star g$ for all K in $\mathcal{K}$.

It is important to note that while satisfying the ordinary universal property of the limiting weighted cone is necessary, it is not in general sufficient to conclude that (L, λ) is a weighted limit. However, it is sufficient in a $\mathcal{V}$-actegory $\mathcal{C}$.

Proposition 3.0.7. Let $\mathcal{C}$ be an actegory. If a W-weighted cone (L, λ) on $D : \mathcal{K} \to \mathcal{C}$ satisfies the ordinary universal property, then it further satisfies the graded universal property, making (L, λ) a W-weighted limit.

Proof. Suppose we have $L \in ob(\mathcal{C})$ and $\lambda_K : WK \bullet L \to DK$ for all $K \in ob(\mathcal{K})$ such that (L, λ) is a W-weighted cone satisfying the ordinary universal property of the limiting weighted cone.

Let $(A, \gamma_K : (WK \otimes X) \bullet A \to DK)$ be a graded W-weighted cone on D. Composing γ_K and the multiplicator $\mu_{WK,X,A}$, we obtain $\bar{\gamma_K} : WK \bullet (X \bullet A) \to DK$ in $\mathcal{C}$. Notice that for every $\phi : Y, K \to K'$ in $\mathcal{K}$, we have the following diagram

$$\begin{tikzcd}
(Y \otimes WK) \bullet X \bullet A \arrow[rr, "\sim"] \arrow[ddr, "\sim"] \arrow[dddd, "W\phi \bullet X \bullet A"'] & & Y \bullet WK \bullet X \bullet A \arrow[r, "Y \bullet \bar{\gamma}_K"] \arrow[d, "\sim"] & Y \bullet DK \arrow[dddd, "D\phi"] \\
 & & Y \bullet (WK \otimes X) \bullet A \arrow[ur, "Y \bullet \gamma_L"'] & \\
 & (Y \otimes WK) \otimes X \bullet A \arrow[r, "a_{Y,WK,X} \bullet A"] \arrow[d, "W\phi \otimes X \bullet A"] & Y \otimes (WK \otimes X) \bullet A \arrow[u, "\sim"'] & \\
 & WK \otimes X \bullet A \arrow[drr, "\gamma_{K'}"] & & \\
WK' \bullet X \bullet A \arrow[ur, "\sim"] \arrow[rrr, "\bar{\gamma}_{K'}"] & & & DK'
\end{tikzcd}$$

The two triangles commute by definition. The left pentagon is precisely Diagram 1.2.1, the coherence of the multiplicator. The square and remaining diagram commute by naturality of the multiplicator $\sim$ and naturality of graded weighted cones (Diagram 3.0.6) respectively. It follows that the outer diagram commutes, making $(A, \bar{\gamma}_K : WK \bullet (X \bullet A) \to DK)$ a W-weighted cone. By the ordinary universal property, each of these $\bar{\gamma}$ has an associated unique $\bar{g} : X \bullet A \to L$ in $\mathcal{C}$ such that

$$\begin{tikzcd}
WK \bullet X \bullet A \arrow[r, "WK \bullet \bar{g}"] \arrow[d, "\bar{\gamma_K}"'] & WK \bullet L \arrow[dl, "\lambda_K"] \\
DK &
\end{tikzcd}$$

commutes. It follows that the diagram

$$\begin{tikzcd}
WK \otimes X \bullet A \arrow[r, "\sim"] \arrow[dr, "\gamma_K"'] & WK \bullet X \bullet A \arrow[r, "WK \bullet g"] \arrow[d, "\bar{\gamma_K}"] & WK \bullet L \arrow[dl, "\lambda_K"] \\
 & DK &
\end{tikzcd}$$

commutes. Thus, (L, λ) satisfies the graded universal property of the limiting weighted cone. ∎

Definition 3.0.8 (Weighted Colimits)**.** Let $W' : \mathcal{K}^{\circ} \to \mathcal{V}$ be a right $\mathcal{V}$-graded functor and let $D : \mathcal{K} \to \mathcal{C}$ be a $\mathcal{V}$-graded functor. A **W'-weighted cocone** is a pair (C, κ) where $C \in ob(\mathcal{C})$ and

$$\kappa_K : W'K, DK \to C \qquad (K \in \mathcal{K})$$

such that for every $\phi : Y, K \to K'$ in $\mathcal{K}$, the following envelope diagram commutes

$$\begin{tikzcd}
W'K' \otimes Y,\, DK \arrow[r, "\sim"] \arrow[d, "W'\phi,\, DK"'] & W'K',\, Y,\, DK \arrow[r, "W'K',\, D\phi"] & W'K',\, DK' \arrow[d, "\kappa_{K'}"] \\
W'K,\, DK \arrow[rr, "\kappa_K"] & & C
\end{tikzcd} \tag{3.0.7}$$

i.e., $\kappa_{K'} \circ D\phi = (W'\phi)^*(\kappa_K)$ for all K in $\mathcal{K}$.

A **graded W'-weighted cocone** is a pair (B, β) where $B \in ob(\mathcal{C})$ and

$$\beta_K : X \otimes W'K, DK \to B \qquad (K \in \mathcal{K})$$

such that for every $\phi : Y, K \to K'$ in $\mathcal{K}$, the following envelope diagram commutes

$$\begin{tikzcd}
(X \otimes W'K') \otimes Y,\, DK \arrow[r, "\sim"] \arrow[d, "a_{X,W'K',Y},\, DK"'] & X \otimes W'K',\, Y,\, DK \arrow[r, "X\otimes W'K',\, D\phi"] & X \otimes W'K', DK' \arrow[d, "\beta_{K'}"] \\
X \otimes (W'K' \otimes Y),\, DK \arrow[r, "X\otimes W'\phi,\, DK"] & X \otimes W'K,\, DK \arrow[r, "\beta_K"] & B
\end{tikzcd} \tag{3.0.8}$$

i.e., $[(X \otimes W'\phi) \cdot a_{X,W'K',Y}]^*(\beta_K) = \beta_{K'} \circ D\phi$ for all K in $\mathcal{K}$.

Definition 3.0.9. We say the W'-weighted cocone (C, κ) is a **W'-weighted colimit of** D if it satisfies the following graded universal property:

> For every graded W'-weighted cocone $\beta_K : X \otimes W'K, DK \to B$ $(K \in \mathcal{K})$ on D, there exists a unique graded morphism $b : X, C \to B$ such that the following envelope diagram commutes
>
> $$\begin{tikzcd}
X \otimes \tilde{W}K,\, DK \arrow[r, "\sim"] \arrow[dr, "\beta_K"'] & X,\, \tilde{W}K,\, DK \arrow[r, "X,\, c_K"] & X,\, C \arrow[dl, "b"] \\
& B &
\end{tikzcd}$$
>
> i.e., $\beta_K = b \circ c_K$ for all K in $\mathcal{K}$.

Lemma 3.0.10. A W'-weighted colimit in $\mathcal{C}$ of $D : \mathcal{K} \to \mathcal{C}$ is equivalently given by a W'-weighted limit in the right $\mathcal{V}$-graded category $\mathcal{C}^\circ$ of the diagram $D^\circ : \mathcal{K}^\circ \to \mathcal{C}^\circ$.

Proof. Let $A \in ob(\mathcal{C})$. We have

$$\alpha_K : W'K, DK \to A \text{ in } \mathcal{C} \qquad (K \in \mathcal{K})$$

$$\alpha_K : A, W'K \to DK \text{ in } \mathcal{C}^\circ \qquad (K \in \mathcal{K})$$

with the former being a weighted cocone in $\mathcal{C}$ if and only if the latter is a weighted cone in $\mathcal{C}^\circ$. We know this since $\alpha_{K'} \circ D\phi = (W'\phi)^*(\alpha_K)$ in $\mathcal{C}$ if and only if $D\phi \circ \alpha_{K'} = (W'\phi)^*(\alpha_K)$ in $\mathcal{C}^\circ$ Similarly, we have

$$\gamma_K : X \otimes W'K, DK \to B \text{ in } \mathcal{C} \qquad (K \in \mathcal{K})$$

$$\gamma_K : B, X \otimes W'K \to DK \text{ in } \mathcal{C}^\circ \qquad (K \in \mathcal{K})$$

with the former being a graded weighted cocone in $\mathcal{C}$ if and only if the latter is a graded weighted cone in $\mathcal{C}^\circ$. We know this since

$$\left[(X \otimes \tilde{W}\phi) \cdot a_{X,\tilde{W}K',Y}\right]^*(\beta_K) = \beta_{K'} \circ D\phi$$

in $\mathcal{C}$ if and only if

$$\left[(X \otimes \tilde{W}\phi) \cdot a_{X,\tilde{W}K',Y}\right]^*(\beta_K) = D\phi \circ \beta_{K'}$$

in $\mathcal{C}^\circ$. Finally, (A, α) satisfies the graded universal property of the limiting cocone in $\mathcal{C}$ if and only if it satisfies the graded universal property of the limiting cone in $\mathcal{C}^\circ$. This follows from the fact that for every family $(\gamma_K)_{K\in\mathcal{K}}$ as given above, there exists a unique $g : X, A \to B$ in $\mathcal{C}$, equivalently $g : B, X \to A$ in $\mathcal{C}^\circ$, such that $\gamma_K = g \circ \alpha_K$ in $\mathcal{C}$ if and only if $\gamma_K = \alpha_K \circ g$ in $\mathcal{C}^\circ$. ∎

3.1 A More Abstract Perspective

Let $W : \mathcal{K} \to \mathcal{V}$ and $D : \mathcal{K} \to \mathcal{C}$ be $\mathcal{V}$-graded functors. There is a more abstract way of defining a graded W-weighted limit of D as a representation of a certain $\mathcal{V}$-graded functor. To do this, we first introduce the right $\mathcal{V}$-graded functor of graded weighted cones

$$WCone(W, -, D) : \mathcal{C}^\circ \to \hat{\mathcal{V}}$$

which acts on objects by taking $A \in \mathcal{C}$ to the presheaf $WCone(W, A, D) : \mathcal{V}^{op} \to SET$. This presheaf maps an object $X \in \mathcal{V}$ to the set $WCone(W, A, D)(X)$ of X-graded W-weighted cones over D with domain A. It maps a morphism $\alpha : X \to Y$ in $\mathcal{V}$ to the function

$$WCone(W, A, D)(\alpha) : WCone(W, A, D)(Y) \to WCone(W, A, D)(X)$$

which takes a Y-graded weighted cone $\gamma_K : WK \otimes Y, A \to DK$ $(K \in \mathcal{K})$ to the X-graded weighted cone given by reindexing along all the $WK \otimes \alpha$ $(K \in \mathcal{K})$.

The $\mathcal{V}$-graded weighted cone functor acts on graded morphisms by taking $f : X, A \to B$ in $\mathcal{C}$ to the graded morphism $WCone(W, f, D) : WCone(W, B, D), X \to WCone(W, A, D)$ in $\hat{\mathcal{V}}$; that is, the natural transformation

$$WCone(W, f, D)_Y : WCone(W, B, D)(Y) \to WCone(W, A, D)(Y \otimes X) \qquad (Y \in \mathcal{V})$$

mapping the Y-graded weighted cone $(\gamma_K : WK \otimes Y, B \to DK)_{K\in\mathcal{K}}$ to the $(Y \otimes X)$-graded weighted cone $\left({a^{-1}_{WK,Y,X}}^*(\gamma_K \circ f)\right)_{K\in\mathcal{K}}$.

Theorem 3.1.1. A W-weighted limit of D is equivalently given by a representation of the right $\mathcal{V}$-graded functor $WCone(W, -, D) : \mathcal{C}^\circ \to \hat{\mathcal{V}}$.

Proof. A representation of $WCone(W, -, D)$ is an object $L \in \mathcal{C}$ and an element $\lambda \in WCone(W, L, D)(I)$ such that its corresponding $\tilde{\lambda} : \mathcal{C}(-, L) \Rightarrow WCone(W, -, D)$ under Yoneda is invertible. In other words, for every graded weighted cone $\gamma_K : WK \otimes X, A \to DK$ $(K \in \mathcal{K})$, there exists a unique graded morphism $g : X, A \to L$ in $\mathcal{C}$ such that

$$\begin{aligned}\gamma_K &= (\tilde{\lambda}_K)_X(g)\\ &= WCone(W, A, D)(\ell_X^{-1})(WCone(W, g, D)_I(\lambda_K))\\ &= WCone(W, A, D)(\ell_X^{-1})({a^{-1}_{WK,I,X}}^*(\lambda_K \circ g))\end{aligned}$$

$$
\begin{aligned}
&= (WK \otimes \ell_X^{-1})^*({a_{WK,I,X}^{-1}}^*(\lambda_K \circ g)) \\
&= (a_{WK,I,X}^{-1} \cdot (WK \otimes \ell_X^{-1}))^*(\lambda_K \circ g) \\
&= (r_{WK} \otimes X)^*(\lambda_K \circ g) \\
&= {r_{WK}^{-1}}^*(\lambda_K) \circ g
\end{aligned}
$$

By Lemma 3.0.3, the I-graded weighted cone (L, λ) corresponds to the weighted cone $(L, {r_{WK}^{-1}}^*\lambda_K)$. Applying Lemma 3.0.5 to this weighted cone, we get that $(L, {r_{WK}^{-1}}^*\lambda_K)$ is a W-weighted limit of D in the sense of Definition 3.0.4. Thus, a $\mathcal{V}$-graded representation of $WCone(W, -, D)$ is equivalently given by a graded weighted limit. ∎

Lemma 3.1.2. Let $W : \mathcal{K} \to \mathcal{V}$ and $D : \mathcal{K} \to \mathcal{C}$ be $\mathcal{V}$-graded functors. There is an isomorphism $WCone(W, A, D) \cong [\mathcal{K}, \hat{\mathcal{V}}](\mathsf{Y}W, \mathcal{C}(A, D-))$ in $\hat{\mathcal{V}}$ that is V-graded natural in $A \in \mathcal{C}$, where $\mathsf{Y}W$ is the composite of W and the canonical embedding Y.

Proof. To prove that these two presheaves are isomorphic, we construct the components of a natural transformation then prove naturality. Let $X \in ob(\mathcal{V})$. We construct an isomorphism

$$WCone(W, A, D)(X) \cong [\mathcal{K}, \hat{\mathcal{V}}](\mathsf{Y}W, X; \mathcal{C}(A, D-))$$

Let $\eta : \mathsf{Y}W, X \Rightarrow \mathcal{C}(A, D-)$ be a graded transformation in $[\mathcal{K}, \hat{\mathcal{V}}]$. By Example 2.0.8, η is a family of natural transformations $\eta_K : \mathsf{Y}(WK) \Longrightarrow \mathcal{C}(- \otimes X, A; DK)$ $(K \in \mathcal{K})$ such that for all $\phi : U, K \to K'$ the diagram

$$\begin{tikzcd}
\mathcal{V}(-, WK) \arrow[r, Rightarrow, "\eta_K"] \arrow[d, Rightarrow, "\mathsf{Y}(W\phi)"] & \mathcal{C}(- \otimes X, A; DK) \arrow[r, Rightarrow, "\mathcal{C}(- \otimes X, A; D\phi)"] & \mathcal{C}(U \otimes (- \otimes X), A; DK') \\
\mathcal{V}(U \otimes -, WK') \arrow[rr, Rightarrow, "(\eta_{K'})(U \otimes -)"'] & & \mathcal{C}((U \otimes -) \otimes X, A; DK') \arrow[u, Rightarrow, "\mathcal{C}(a_{U,-,X}^{-1}, A; DK')"']
\end{tikzcd}$$

commutes, where $\mathsf{Y}(W\phi) : U, \mathsf{Y}(WK) \to \mathsf{Y}(WK')$ is the graded transformation obtained by applying the composite $\mathsf{Y}W$ at ϕ. Each natural transformation η_K corresponds under the Yoneda Lemma to the element $\gamma_K := (\eta_K)_{WK}(id_{WK})$ of $\mathcal{C}(WK \otimes X, A; DK)$. The Yoneda Lemma also tells us that $(\eta_K)_Z : \mathcal{V}(Z, WK) \to \mathcal{C}(Z \otimes X, A; DK)$ is given by $(\eta_K)_Z(g) = (g \otimes X)^*(\gamma_K)$ for $g : Z \to WK$. Taking the WK component of the above diagram gives

$$\begin{tikzcd}
\mathcal{V}(WK, WK) \arrow[r, "(\eta_K)_{WK}"] \arrow[d, "(\mathsf{Y}(W\phi))_{WK}"] & \mathcal{C}(WK \otimes X, A; DK) \arrow[r, "\mathcal{C}(WK \otimes X, A; D\phi)"] & \mathcal{C}(U \otimes (WK \otimes X), A; DK') \\
\mathcal{V}(U \otimes WK, WK') \arrow[rr, "(\eta_{K'})_{U \otimes WK}"] & & \mathcal{C}((U \otimes WK) \otimes X, A; DK') \arrow[u, "\mathcal{C}(a_{U,WK,X}^{-1}, A; DK')"']
\end{tikzcd}$$

Chasing the element $id_{WK} : WK \to WK$ through the top of the diagram gives

$$D\phi \circ (\eta_K)_{WK}(id_{WK}) = D\phi \circ \gamma_K$$

Chasing id_{WK} through the bottom of the diagram is a bit more involved. Firstly, we apply $(\mathsf{Y}(W\phi))_{WK}$ and obtain $(\mathsf{Y}(W\phi))_{WK}(id_{WK}) = W\phi \cdot (U \otimes id_{WK}) = W\phi$ by Example 2.0.18. Then, applying $(\eta_{K'})_{Y \otimes WK}$ gives $(\eta_{K'})_{Y\otimes WK}(W\phi) = (W\phi \otimes X)^*(\gamma_{K'})$. Finally, we have

$$a_{Y,WK,X}^{-1}{}^*((W\phi \otimes X)^*(\gamma_{K'})) = [(W\phi \otimes X) \cdot a_{Y,WK,X}^{-1}]^*(\gamma_{K'})$$

By commutativity of the diagram, we get

$$[(W\phi \otimes X) \cdot a_{Y,WK,X}^{-1}]^*(\gamma_{K'}) = D\phi \circ \gamma_K$$

which is exactly the condition required for $(\gamma_K)_{K\in\mathcal{K}}$ to be a graded weighted cone.

Great! We have a mapping from $[\mathcal{K}, \hat{\mathcal{V}}](\mathsf{Y}W, X; \mathcal{C}(A, D-))$ to $WCone(W, A, D)(X)$. In the inverse direction, we take a graded weighted cone $(\gamma_K : WK \otimes X, A \to DK)_{K\in\mathcal{K}}$ to the family of natural transformations $\eta_K : \mathsf{Y}(WK) \Rightarrow \mathcal{C}(- \otimes X, A; DK)$ $(K \in \mathcal{K})$ given by $(\eta_K)_Z(g) = (g \otimes X)^*(\gamma_K)$ for $g : Z \to WK$. We claim that the following diagram commutes

$$\begin{tikzcd}
\mathcal{V}(Z, WK) \arrow[r, "(\eta_K)_Z"] \arrow[d, "(\mathsf{Y}(W\phi))_Z"] & \mathcal{C}(Z \otimes X, A; DK) \arrow[r, "D\phi\circ-"] & \mathcal{C}(U \otimes (Z \otimes X)), A; DK') \\
\mathcal{V}(U \otimes Z, WK) \arrow[rr, "(\eta_{K'})_{U\otimes Z}"] & & \mathcal{C}((U \otimes Z) \otimes X, A; DK') \arrow[u, "(a_{U,Z,X}^{-1})^*(-)"]
\end{tikzcd}$$

for all $\phi : U, K \to K'$ in $\mathcal{K}$ and all $Z \in ob(\mathcal{V})$. Chasing an arbitrary morphism $g : Z \to WK$ through this diagram, we note that for the diagram to commute, we require

$$\left[a_{U,Z,X}^{-1}\right]^* \left[(W\phi \cdot (U \otimes g)) \otimes X\right]^* (\gamma_K') = D\phi \circ ((g \otimes X)^*(\gamma_K)) \tag{3.1.1}$$

Consider the envelope diagram

$$\begin{tikzcd}
U \otimes (Z \otimes X), A \arrow[r, "\sim"] \arrow[dr, "U\otimes(g\otimes X),A"'] \arrow[dd, "a_{U,Z,X}^{-1},A"'] & U, Z \otimes X, A \arrow[dr, "U,g\otimes X,A"] & & \\
& U \otimes (WK \otimes X), A \arrow[r, "\sim"] \arrow[d, "a_{U,WK,X}^{-1},A"'] & U, WK \otimes X, A \arrow[r, "U,\gamma_K"] & U, DK \arrow[d, "D\phi"] \\
(U \otimes Z) \otimes X, A \arrow[r, "(U\otimes g)\otimes X,A"] \arrow[rr, bend right, "(W\phi\cdot(U\otimes g))\otimes X,A"'] & (U \otimes WK) \otimes X, A \arrow[r, "W\phi\otimes X,A"] & WK' \otimes X, A \arrow[r, "\gamma_{K'}"] & DK'
\end{tikzcd}$$

The two squares commute by the naturality of the associator a and the multiplicator $\sim$ respectively. The remaining inner diagram commutes by the naturality of the graded weighted cone γ. Therefore, the outer diagram commutes; i.e., the required Equation 3.1.1 holds. Thus, the family $(\eta_K)_{K\in\mathcal{K}}$ is a graded transformation in $[\mathcal{K}, \hat{\mathcal{V}}](\mathsf{Y}W, X; \mathcal{C}(A, D-))$. Having established that in both directions, the functions are well-defined, we can easily conclude that they are inverses of each other since γ and η are obtained from each other by Yoneda isomorphisms.

Now, we claim that this bijection is natural in $X \in \mathcal{V}$. Let $f : Y \to X$ in $\mathcal{V}$. We consider the following naturality diagram

$$\begin{tikzcd}
WCone(W,A,D)(X) \arrow[r, "WCone(W{,}A{,}D)(f)"] \arrow[d, "\cong_X"'] & WCone(W,A,D)(Y) \\
{[\mathcal{K},\hat{\mathcal{V}}](\mathsf{Y}W, X; \mathcal{C}(A,D-))} \arrow[r, "{[\mathcal{K},\hat{\mathcal{V}}](\mathsf{Y}W,f;\mathcal{C}(A,D-))}"] & {[\mathcal{K},\hat{\mathcal{V}}](\mathsf{Y}W, Y; \mathcal{C}(A,D-))} \arrow[u, "\cong_Y"']
\end{tikzcd}$$

Along the top of the diagram, a graded weighted cone $\gamma_K : WK \otimes X, A \to DK$ $(K \in \mathcal{K})$ maps to the graded weighted cone $(WK \otimes f)^*(\gamma_K) : WK \otimes Y, A \to DK$ $(K \in \mathcal{K})$ and then to it's corresponding graded transformation $\nu : \mathsf{Y}W, Y \Rightarrow \mathcal{C}(A, D-)$ given by the family of natural transformation $\nu_{WK} : \mathsf{Y}(WK) \Rightarrow \mathcal{C}(- \otimes Y, A; DK)$ $(K \in \mathcal{K})$ defined by

$$(\nu_{WK})_Z(g) := (g \otimes Y)^*(WK \otimes f)^*(\gamma_K) = (g \otimes f)^*(\gamma_K)$$

for $g : Z \to WK$ in $\mathcal{V}$. Along the bottom of the diagram, γ maps first to the graded transformation $\eta : \mathsf{Y}W, X \to \mathcal{C}(A, D-)$ given by the family of natural transformations $\eta_K : \mathcal{V}(-, WK) \Rightarrow \mathcal{C}(- \otimes X, A; DK)$ $(K \in \mathcal{K})$ defined by $(\eta_K)_Z(g) := (g \otimes X)^*(\gamma_K)$ for $g : Z \to WK$ in $\mathcal{V}$. Then, $[\mathcal{K}, \hat{\mathcal{V}}](\mathsf{Y}W, f; \mathcal{C}(A, D-))(\eta) = \psi$ for $\psi : \mathsf{Y}W, Y \to \mathcal{C}(A, D-)$ a graded transformation given by a family of natural transformations $\psi_K : \mathcal{V}(-, WK) \Rightarrow \mathcal{C}(- \otimes Y, A; DK)$ $(K \in \mathcal{K})$ defined component-wise by $(\psi_K)_Z(g) := (g \otimes f)^*(\gamma_K)$ for $g : Z \to WK$. Thus, the diagram commutes and the naturality condition holds.

It remains only to prove that $WCone(W, A, D) \cong [\mathcal{K}, \hat{\mathcal{V}}](\mathsf{Y}W, \mathcal{C}(A, D))$ is $\mathcal{V}$-graded natural in $A \in \mathcal{C}$. We construct a $\mathcal{V}$-graded natural transformation $\delta : WCone(W, -, D) \Rightarrow [\mathcal{K}, \hat{\mathcal{V}}](\mathsf{Y}W, \mathcal{C}(-, D))$ given by a family of morphisms in $\hat{\mathcal{V}}_0$,

$$\delta_A : WCone(W, A, D) \Longrightarrow [\mathcal{K}, \hat{\mathcal{V}}](\mathsf{Y}W, \mathcal{C}(A, D)) \quad (A \in \mathcal{C})$$

natural transformations whose components $(\delta_A)_X$ are the

$$WCone(W, A, D)(X) \cong [\mathcal{K}, \hat{\mathcal{V}}](\mathsf{Y}W, X; \mathcal{C}(A, D-))$$

we defined above. Let $f : X, A \to B$ in $\mathcal{C}$. We consider the following diagram.

$$\begin{tikzcd}
WCone(W,B,D) \arrow[r, Rightarrow, "WCone(W{,}f{,}D)"] \arrow[d, Rightarrow, "\delta_B"'] & WCone(W,A,D)(- \otimes X) \arrow[d, Rightarrow, "\delta_A(-\otimes X)"] \\
{[\mathcal{K},\hat{\mathcal{V}}](\mathsf{Y}W, -; \mathcal{C}(B,D))} \arrow[r, Rightarrow, "{[\mathcal{K},\hat{\mathcal{V}}](\mathsf{Y}W,-;\mathcal{C}(f,D))}"] & {[\mathcal{K},\hat{\mathcal{V}}](\mathsf{Y}W, - \otimes X; \mathcal{C}(A,D))}
\end{tikzcd}$$

Let $Y \in ob(\mathcal{V})$ and take the Y component of the above diagram.

$$\begin{tikzcd}
WCone(W,B,D)(Y) \arrow[r, "WCone(W,f,D)(Y)"] \arrow[d, "(\delta_B)_Y"'] & WCone(W,A,D)(Y \otimes X) \arrow[d, "\delta_A(Y\otimes X)"] \\
{[\mathcal{K},\hat{\mathcal{V}}](\mathsf{Y}W, Y; \mathcal{C}(B,D))} \arrow[r, "{[\mathcal{K},\hat{\mathcal{V}}](\mathsf{Y}W,Y;\mathcal{C}(f,D))}"] & {[\mathcal{K},\hat{\mathcal{V}}](\mathsf{Y}W, Y \otimes X; \mathcal{C}(A,D))}
\end{tikzcd}$$

Along the top and right of the diagram, a graded cone $\gamma_K : WK \otimes Y, B \to DK$ $(K \in \mathcal{K})$ maps first to $(a_{WK,Y,X})^*(\gamma_K \circ f) : WK \otimes (Y \otimes X), A \to DK$ $(K \in \mathcal{K})$ then to $\mu_K : \mathcal{V}(-, WK) \Rightarrow \mathcal{C}(- \otimes (Y \otimes X), A; DK)$ $(K \in \mathcal{K})$ where $(\mu_K)_Z(g) = (g \otimes (Y \otimes X))^*(a_{WK,Y,X})^*(\gamma_K \circ f)$ for $g : Z \to WK$. Along the left and bottom of the diagram, the same γ_K maps to $\eta_K : \mathcal{V}(-, WK) \Rightarrow \mathcal{C}(- \otimes Y, B; DK)$ $(K \in \mathcal{K})$ where $(\eta_K)_Z(g) = (g \otimes Y)^*(\gamma_K)$ then to $\rho_K : \mathcal{V}(-, WK) \Rightarrow \mathcal{C}(- \otimes (Y \otimes X), A; DK)$ $(K \in \mathcal{K})$ where $(\rho_K)_Z(g) = (a_{Z,Y,X})^*[(g \otimes Y)^*(\gamma_K) \circ f]$. We have that

$$(\mu_K)_Z(g) = (g \otimes (Y \otimes X))^*(a_{WK,Y,X})^*(\gamma_K \circ f) = [a_{WK,Y,X} \cdot g \otimes (Y \otimes X)]^*(\gamma_K \circ f)$$

$$(\rho_K)_Z(g) = (a_{Z,Y,X})^*[(g \otimes Y)^*(\gamma_K) \circ f] = [(g \otimes Y) \otimes X \cdot a_{Z,Y,X}]^*(\gamma_K \circ f)$$

so $\mu = \rho$ by naturality of the associator in $\mathcal{V}$. Thus, the diagram commutes and we can conclude that $WCone(W, A, D) \cong [\mathcal{K}, \hat{\mathcal{V}}](\mathsf{Y}W, \mathcal{C}(A, D))$ is $\mathcal{V}$-graded natural in $A \in \mathcal{C}$. ∎

Corollary 3.1.3. A W-weighted limit over D is a representation of the $\mathcal{V}$-graded functor $[\mathcal{K}, \hat{\mathcal{V}}](\mathsf{Y}W?, \mathcal{C}(-, D?)) : \mathcal{C}^\circ \to \hat{\mathcal{V}}$.

Proof. By Lemma 3.1.1 and Lemma 3.1.2, there is an object $L \in ob(\mathcal{C})$ and an isomorphism

$$\mathcal{C}(-, L) \cong WCone(W, -, D) \cong [\mathcal{K}, \hat{\mathcal{V}}](\mathsf{Y}W?, \mathcal{C}(-, D?))$$

∎

Given that $\mathcal{V}$-graded structure is exactly $\hat{\mathcal{V}}$-enriched structure, the reader may be inclined to believe that $\mathcal{V}$-graded weighted limits and $\hat{\mathcal{V}}$-enriched weighted limits are equivalent. However, the weight W of a $\mathcal{V}$-graded weighted limit must be of the form $W : \mathcal{K} \to \mathcal{V}$ while $\hat{\mathcal{V}}$-enriched weighted limits in general take on any $W' : \mathcal{K} \to \hat{\mathcal{V}}$ as a weight. In fact, $\mathcal{V}$-graded weighted limits are only the $\hat{\mathcal{V}}$-enriched weighted limits for a class of weights we call *$\mathcal{V}$-valued weights*. These are the weight $\mathcal{V}$-graded functors given by composites $\mathsf{Y}W : \mathcal{K} \to \hat{\mathcal{V}}$ where $W : \mathcal{K} \to \mathcal{V}$ and $\mathsf{Y} : \mathcal{V} \to \hat{\mathcal{V}}$ is the canonical embedding.

Theorem 3.1.4. Let $\mathcal{K}$ and $\mathcal{C}$ be $\mathcal{V}$-graded categories and let $W : \mathcal{K} \to \mathcal{V}$ and $D : \mathcal{K} \to \mathcal{C}$ be $\mathcal{V}$-graded functors. A graded weighted limit of D with weight W is equivalently a $\hat{\mathcal{V}}$-enriched weighted limit with weight $\mathsf{Y}W : \mathcal{K} \to \hat{\mathcal{V}}$.

Proof. A graded weighted limit is given by an object $L \in ob(\mathcal{C})$ and an isomorphism

$$\mathcal{C}(A, L) \cong [\mathcal{K}, \hat{\mathcal{V}}](\mathsf{Y}W, \mathcal{C}(A, D-))$$

$\mathcal{V}$-graded natural in $A \in \mathcal{C}$. By Lemma 2.1.5, there is an isomorphism

$$[\mathcal{K}, \hat{\mathcal{V}}](\mathsf{Y}W, \mathcal{C}(A, D-)) \cong \mathcal{P}^{\dagger}_{\mathcal{V}}\mathcal{K}(\mathsf{Y}W, \mathcal{C}(A, D-))$$

Therefore, the representation above is equivalently given by

$$\mathcal{C}(A, L) \cong \mathcal{P}^{\dagger}_{\mathcal{V}}\mathcal{K}(\mathsf{Y}W, \mathcal{C}(A, D-))$$

$\mathcal{V}$-graded natural in $A \in \mathcal{C}$. ∎

3.2 The Enriched-Graded Relationship

At this point in the thesis, the reader is likely already convinced that $\mathcal{V}$-graded category theory and $\mathcal{V}$-enriched category theory are interrelated topics. They generalise each other in interesting and meaningful ways. We take the opportunity here to establish what this means for their respective weighted limits.

When the $\mathcal{V}$-graded categories involved are $\mathcal{V}$-enriched categories, $\mathcal{V}$-graded weighted limits specialise to the case of enriched weighted limits from Chapter 1.

Proposition 3.2.1. Let $\mathcal{V}$ be monoidal biclosed and complete. Let $\mathcal{K}$ and $\mathcal{C}$ be $\mathcal{V}$-categories with $\mathcal{K}$ small. Let $D : \mathcal{K} \to \mathcal{C}$ and $W : \mathcal{K} \to \mathcal{V}$ be $\mathcal{V}$-functors. A $\mathcal{V}$-enriched weighted limit of D is equivalently a $\mathcal{V}$-graded weighted limit where we regard W and D as $\mathcal{V}$-graded functors.

Proof. First, we recall from Chapter 1 that an enriched weighted limit is a representation of the $\mathcal{V}$-functor $\mathcal{P}^{\dagger}_{\mathcal{V}}\mathcal{K}(W?, \mathcal{C}(-, D?)) : \mathcal{C}^{\circ} \to \mathcal{V}$. By Proposition 2.2.6, a $\mathcal{V}$-enriched representation of $\mathcal{P}^{\dagger}_{\mathcal{V}}\mathcal{K}(W?, \mathcal{C}(-, D?))$ is equivalently given by a $\mathcal{V}$-graded representation of $\mathsf{Y} \circ \mathcal{P}^{\dagger}_{\mathcal{V}}\mathcal{K}(W?, \mathcal{C}(-, D?))$ where $\mathsf{Y} : \mathcal{V} \to \hat{\mathcal{V}}$ is the right $\mathcal{V}$-graded canonical embedding; i.e., an object $L \in \mathcal{C}$ such that

$$\mathcal{C}(-, A; L) \cong \mathsf{Y}\mathcal{P}^{\dagger}_{\mathcal{V}}\mathcal{K}(W, \mathcal{C}(A, D?))$$

$\mathcal{V}$-graded natural in $A \in \mathcal{C}$. Applying Lemma 2.1.6, the above representation is equivalently given by

$$\mathcal{C}(-, A; L) \cong [\mathcal{K}, \hat{\mathcal{V}}](\mathsf{Y}W?, -; \mathcal{C}(=, A; D?))$$

$\mathcal{V}$-graded natural in $A \in \mathcal{C}$, making L a representation of $[\mathcal{K}, \hat{\mathcal{V}}](\mathsf{Y}W?, \mathcal{C}(-, D?)) : \mathcal{C}^{\circ} \to \hat{\mathcal{V}}$. Thus, an enriched W-weighted limit over D is equivalently a $\mathcal{V}$-graded W-weighted limit over D. ∎

We note that in the above proposition we must presuppose that $\mathcal{V}$ is blclosed and complete and that $\mathcal{K}$ is small. However, these assumptions are not required for graded weighted limits. We can then use these to form weighted limits in categories enriched in an arbitrary monoidal base.

Example 3.2.2. Let $\mathcal{K}$ and $\mathcal{C}$ be $\mathcal{V}$-categories for some monoidal category $\mathcal{V}$. Let $D : \mathcal{K} \to \mathcal{C}$ be a $\mathcal{V}$-functor. Since $\mathcal{V}$ is not necessarily biclosed, we cannot take a weight $\mathcal{V}$-functor. However, we can consider a $\mathcal{V}$-graded functor $W : \mathcal{K} \to \mathcal{V}$ where $\mathcal{V}$ has the $\mathcal{V}$-graded structure described in Example 2.0.7. A W-weighted limit of D is an object L of $\mathcal{C}$ equipped with a family of morphisms

$$\lambda_K : WK \to \mathcal{C}(L, DK) \qquad (K \in \mathcal{K})$$

in $\mathcal{V}$ such that for every $\phi : Y \to \mathcal{K}(K, K')$ in $\mathcal{V}$

$$\begin{tikzcd}
Y \otimes WK \arrow[r, "\phi \otimes WK"] \arrow[d, "W\phi"'] & \mathcal{K}(K, K') \otimes WK \arrow[r, "D_{K,K'} \otimes \lambda_K"] & \mathcal{C}(DK, DK') \otimes \mathcal{C}(L, DK) \arrow[d, "m_{L,DK,DK'}"] \\
WK' \arrow[rr, "\lambda_{K'}"] & & \mathcal{C}(L, DK')
\end{tikzcd}$$

commutes, and such that λ satisfies the following graded universal property: for every $\gamma_K : WK \otimes X \to \mathcal{C}(A, DK)$ $(K \in \mathcal{K})$ such that the diagram

$$\begin{tikzcd}
Y \otimes (WK \otimes X) \arrow[r, "\phi \otimes (WK \otimes X)"] \arrow[d, "a^{-1}_{Y,WK,X}"'] & \mathcal{K}(K, K') \otimes (WK \otimes X) \arrow[r, "D_{K,K'} \otimes \gamma_K"] & \mathcal{C}(DK, DK') \otimes \mathcal{C}(A, DK) \arrow[d, "m_{A,DK,DK'}"] \\
(Y \otimes WK) \otimes X \arrow[r, "W\phi \otimes X"] & WK' \otimes X \arrow[r, "\gamma_{K'}"] & \mathcal{C}(A, DK')
\end{tikzcd}$$

commutes for all $\phi : Y \to \mathcal{K}(K, K')$ in $\mathcal{V}$, there exists a unique morphism $g : X \to \mathcal{C}(A, L)$ in $\mathcal{V}$ such that the diagram

$$\begin{tikzcd}
WK \otimes X \arrow[rr, "\lambda_K \otimes g"] \arrow[dr, "\gamma_K"'] & & \mathcal{C}(L, DK) \otimes \mathcal{C}(A, L) \arrow[dl, "m_{A,L,DK}"] \\
& \mathcal{C}(A, DK) &
\end{tikzcd}$$

commutes for all K in $\mathcal{K}$.

3.3 Foundational Examples of Graded Weighted Limits

Let $\mathcal{C}$ and $\mathcal{K}$ be $\mathcal{V}$-graded categories and let $W : \mathcal{K} \to \mathcal{V}$ and $D : \mathcal{K} \to \mathcal{C}$ be $\mathcal{V}$-graded functors.

Lemma 3.3.1. If $\mathcal{K}$ has a generating collection $\mathcal{G}$, then a family of graded morphisms $\gamma_K : WK \otimes X, A \to DK$ $(K \in \mathcal{K})$ form a graded weighted cone if and only if Diagram 3.0.2 commutes for all graded morphisms $\phi \in \mathcal{G}$. It is therefore sufficient to verify the naturality of graded weighted cones only for the generating morphisms.

Proof. By the Yoneda Lemma, the family $\gamma_K : WK \otimes X, A \to DK$ ($K \in \mathcal{K}$) corresponds to a family of natural transformations $\eta_K : \mathsf{Y}WK \Rightarrow \mathcal{C}(- \otimes X, A; DK)$ ($K \in \mathcal{K}$). We observed in the proof of 3.1.2 that for a graded morphism $\phi \in \mathcal{K}$, Diagram 3.0.2 for γ commutes if and only if Diagram 2.1.1 for η commutes. Suppose that Diagram 3.0.2 commutes for all generating graded morphisms $\phi \in \mathcal{G}$. Then, Diagram 2.1.1 commutes also for every generating graded morphism $\phi \in \mathcal{G}$. By Proposition 2.1.3, η is a graded transformation. Thus, γ must be a graded weighted cone. ∎

Revisiting the concrete definition (Definition 3.0.4) of graded weighted limits, we note now that when the shape category $\mathcal{K}$ has a generating collection, we only need to verify the naturality conditions over the generating graded morphisms. The following few examples illustrate the convenience of this concept.

Example 3.3.2 (Powers)**.** Let $\mathcal{K} = \mathcal{I}$ the unit $\mathcal{V}$-graded category given in Example 2.0.10 and let $\mathcal{C}$ be any $\mathcal{V}$-graded category. $\mathcal{V}$-graded functors $W : \mathcal{I} \to \mathcal{V}$ and $D : \mathcal{I} \to \mathcal{C}$ are in effect objects $X \in \mathcal{V}$ and $C \in \mathcal{C}$, respectively. A $\mathcal{V}$-graded W-weighted limit of D is an object C^X in $\mathcal{C}$ equipped with graded morphism

$$\lambda : X, C^X \to C$$

satisfying the property that for every graded morphism $\gamma : X \otimes Y, A \to C$ in $\mathcal{C}$ there exists a unique graded morphism $g : Y, A \to C^X$ in $\mathcal{C}$ such that

$$\begin{tikzcd}
X \otimes Y, A \arrow[r, "\sim"] \arrow[dr, "\gamma"'] & X, Y, A \arrow[r, "{X,g}"] & X, C^X \arrow[dl, "\lambda"] \\
& C &
\end{tikzcd}$$

i.e., $\gamma = \lambda \circ g$. We call the limit C^X a **power** of C by X in the $\mathcal{V}$-graded category $\mathcal{C}$. We note that λ and γ vacuously satisfy the naturality of weighted cones and graded weighted cones respectively since $\mathcal{I}$ has no generating graded morphisms.

We next verify directly that a power C^X is a $\mathcal{V}$-graded representation

$$\mathcal{C}(-, A; C^X) \cong \mathcal{C}(-, A; C)^{\mathsf{Y}X} \cong \mathcal{C}(X \otimes -, A; C) \qquad (A \in \mathcal{C})$$

In counit form, this means a $\mathcal{V}$-graded power is given by an object C^X in $\mathcal{C}$ and a graded morphism $\lambda \in \mathcal{C}(X \otimes I, C^X; C) \cong \mathcal{C}(X, C^X; C)$, such that its corresponding $\mathcal{V}$-graded natural transformation under the Weak Graded Yoneda Lemma, whose components are given by

$$(\tilde{\lambda}_A)_Z : \mathcal{C}(Z, A; C^X) \to \mathcal{C}(X \otimes Z, A; C) , \qquad g : Z, A \to C^X \mapsto \mathcal{C}(X, g; C)(\lambda) = \lambda \circ g ,$$

is invertible. To be invertible is exactly the property that for every $\gamma \in \mathcal{C}(X \otimes Z, A; C)$ there exists a unique $g \in \mathcal{C}(Z, A; C^X)$ such that $\gamma = \lambda \circ g$.

Let $W' : \mathcal{I}^\circ \to \mathcal{V}$ be a right $\mathcal{V}$-graded functor given by $W'(*) = X$. A $\mathcal{V}$-graded W-weighted colimit of D is an object $X \cdot C$ in $\mathcal{C}$ equipped with graded morphism

$$c : X, C \to X \cdot C$$

satisfying the property that for every graded morphism $\beta : Z \otimes X, C \to A$ in $\mathcal{C}$ there exists a unique graded morphism $b : Z, X \cdot C \to A$ in $\mathcal{C}$ such that

$$\begin{tikzcd}
Z \otimes X, C \arrow[r, "\sim"] \arrow[dr, "\beta"'] & Z, X, C \arrow[r, "{Z,c}"] & Z, X \cdot C \arrow[dl, "b"] \\
 & A &
\end{tikzcd}$$

i.e., $\beta = b \circ c$. We call $X \cdot C$ a **copower** of C by X in the $\mathcal{V}$-graded category $\mathcal{C}$.

Definition 3.3.3. Let $\mathcal{A}$ be an ordinary category, let $\mathcal{C}$ be a $\mathcal{V}$-graded category, and let $D : \mathcal{A} \to \mathcal{C}_0$ be an ordinary functor. A **conical limit** in $\mathcal{C}$ of the ordinary diagram D is an object L of $\mathcal{C}$ equipped with an ordinary cone $\lambda_A : L \to DA$ $(A \in \mathcal{A})$ satisfying the property that:

for every family of graded morphisms $\gamma_A : X, B \to DA$ $(A \in \mathcal{A})$ such that the envelope diagram

$$\begin{tikzcd}
X, B \arrow[d, "\gamma_A"'] \arrow[dr, "\gamma_{A'}"] & \\
DA \arrow[r, "D\phi"'] & DA'
\end{tikzcd}$$

commutes for all $\phi : A \to A'$ in $\mathcal{A}$, there exists a unique graded morphism $g : X, B \to L$ in $\mathcal{C}$ such that the envelope diagram

$$\begin{tikzcd}
X, B \arrow[r, "g"] \arrow[dr, "\gamma_A"'] & L \arrow[d, "\lambda_A"] \\
 & DA
\end{tikzcd}$$

commutes.

Similarly, a **conical colimit** of D is an object $C \in \mathcal{C}$ equipped with an ordinary cocone $c_A : DA \to C$ $(A \in \mathcal{A})$ satisfying the property that for every family of graded morphisms $\beta_A : X, DA \to B$ $(A \in \mathcal{A})$ such that

$$\begin{tikzcd}
X, DA \arrow[d, "{X,D\phi}"'] \arrow[dr, "\beta_A"] & \\
X, DA' \arrow[r, "\beta_{A'}"'] & B
\end{tikzcd}$$

commutes for all $\phi : A \to A'$ in $\mathcal{A}$, there exists a unique graded morphism $b : X, C \to B$ such that

$$\begin{tikzcd}
X, DA \arrow[r, "{X,c_A}"] \arrow[dr, "\beta_A"'] & X, C \arrow[d, "b"] \\
 & B
\end{tikzcd}$$

commutes.

Example 3.3.4 (Conical Limits)**.** Let $\mathcal{A}$ be an ordinary category. Let $\mathcal{K} = \langle \mathcal{A} \rangle$ the free $\mathcal{V}$-graded category on $\mathcal{A}$ as in Example 2.0.11. Let $\mathcal{C}$ be any $\mathcal{V}$-graded category and $D : \mathcal{A} \to \mathcal{C}_0$ be an ordinary functor landing in the underlying ordinary category of $\mathcal{C}$. In an effort to effectively erase the weight, we will take our weight be the $\mathcal{V}$-graded functor determined by the constant functor on the monoidal unit I; i.e., we take $W = \Delta I$.

A W-weighted limit of the ordinary diagram D is an object $L \in \mathcal{C}$ equipped with a family of morphisms

$$\lambda_A : L \to DA \qquad (A \in \mathcal{A})$$

in $\mathcal{C}_0$ satisfying the naturality of weighted cones; i.e., for every $\phi : A \to A'$ in $\mathcal{A}$, a generating morphism in $\langle \mathcal{A} \rangle$, the diagram

$$\begin{tikzcd} L \arrow[r, "\lambda_A"] \arrow[dr, "\lambda_{A'}"'] & DA \arrow[d, "D\phi"] \\ & DA' \end{tikzcd}$$

commutes. This is precisely the property that λ is an ordinary cone of D. We further require that the cone (L, λ) satisfy the property that for every family $\gamma_A : I \otimes X, B \to DA$ $(A \in \mathcal{A})$ (equivalently given by $\gamma_A : X, B \to DA$) such that

$$\begin{tikzcd} X, B \arrow[d, "\gamma_A"'] \arrow[dr, "\gamma_{A'}"] & \\ DA \arrow[r, "D\phi"'] & DA' \end{tikzcd}$$

commutes for all $\phi : A \to A'$ in $\mathcal{A}$; i.e., $\gamma_{A'} = D\phi \star \gamma_A$, there exists a unique graded morphism $g : X, B \to L$ in $\mathcal{C}$ such that

$$\begin{tikzcd} X, B \arrow[r, "g"] \arrow[dr, "\gamma_A"'] & L \arrow[d, "\lambda_A"] \\ & DA \end{tikzcd}$$

commutes; i.e., $\gamma_A = \lambda_A \star g$.

Lemma 3.3.5. A $\mathcal{V}$-graded conical limit is equivalently a $\hat{\mathcal{V}}$-enriched conical limit.

Proof. We claim that a $\mathcal{V}$-graded conical limit is an ordinary cone (L, λ) in $\mathcal{C}_0$ that is mapped to a limit cone in $\hat{\mathcal{V}}$ by the covariant $\mathcal{V}$-graded hom-functor $\mathcal{C}(B, -) : \mathcal{C} \to \hat{\mathcal{V}}$ for all $B \in ob(\mathcal{C})$.

Let $\lambda_K : L \to DA$ $(A \in \mathcal{A})$ be a cone in $\mathcal{C}_0$. For any $B \in ob(\mathcal{C})$, consider the family

$$\mathcal{C}(-, B; \lambda_A) : \mathcal{C}(-, B; L) \Rightarrow \mathcal{C}(-, B; DA) \qquad (A \in \mathcal{A})$$

we claim this is a limit cone in $\hat{\mathcal{V}}$. Since limits of presheaves are computed componentwise, we consider the family

$$\mathcal{C}(X, B; \lambda_A) : \mathcal{C}(X, B; L) \to \mathcal{C}(X, B; DA) \qquad (A \in \mathcal{A})$$

for some $X \in ob(\mathcal{V})$. Such a family is a limit cone in SET if and only if for every compatible family $(\gamma_A)_{A \in \mathcal{A}}$ of elements $\gamma_A \in \mathcal{C}(X, B; DA)$, there exists a unique element $g \in \mathcal{C}(X, B; L)$ such that

$$\mathcal{C}(X, B; \lambda_A)(g) = \lambda_A \star g = \gamma_A$$

for all $A \in \mathcal{A}$. Unpacking the compatibility condition, we have that the family $\mathcal{C}(X, B; \lambda_A) : \mathcal{C}(X, B; L) \to \mathcal{C}(X, B; DA)$ $(A \in \mathcal{A})$ is a limit cone in SET if and only if for every family of graded morphisms $\gamma_A : X, B \to DA$ $(A \in \mathcal{A})$ such that

$$\begin{tikzcd}
X, B \arrow[d, "\gamma_A"'] \arrow[dr, "\gamma_{A'}"] & \\
DA \arrow[r, "D\phi"'] & DA'
\end{tikzcd}$$

commutes for all $\phi : A \to A'$ in $\mathcal{A}$, there exists a unique graded morphism $g : X, B \to L$ such that

$$\begin{tikzcd}
X, B \arrow[r, "g"] \arrow[dr, "\gamma_A"'] & L \arrow[d, "\lambda_A"] \\
& DA
\end{tikzcd}$$

commutes. Thus, a $\mathcal{V}$-graded conical limit is equivalently a $\hat{\mathcal{V}}$-enriched conical limit. ∎

Definition 3.3.6 ([14], Weighted pullbacks)**.** Let $\mathcal{C}$ be a $\mathcal{V}$-graded category. An **X-weighted pullback** of a morphism $q : Q \to B$ in $\mathcal{C}_0$ along a graded morphism $f : X, A \to B$ in $\mathcal{C}$ is an object P of $\mathcal{C}$ equipped with

$$p : P \to A\ , \qquad f' : X, P \to Q$$

in $\mathcal{C}_0$ and $\mathcal{C}$ respectively, such that the following envelope diagram

$$\begin{tikzcd}
X, P \arrow[r, "f'"] \arrow[d, "{X,p}"'] & Q \arrow[d, "q"] \\
X, A \arrow[r, "f"'] & B
\end{tikzcd}$$

commutes. The data (P, p, f') must further satisfy the property that for every $r : Y, R \to A$ and $g : X \otimes Y, R \to Q$ in $\mathcal{C}$ for which the envelope diagram

$$\begin{tikzcd}
X, Y, R \arrow[r, "\sim"] \arrow[d, "{X,r}"'] & X \otimes Y, R \arrow[r, "g"] & Q \arrow[d, "q"] \\
X, A \arrow[rr, "f"'] & & B
\end{tikzcd}$$

commutes, there exists a unique graded morphism $t : Y, R \to P$ such that the envelope diagrams

$$\begin{tikzcd}
Y, R \arrow[r, "t"] \arrow[dr, "r"'] & P \arrow[d, "p"] \\
& A
\end{tikzcd}
\qquad
\begin{tikzcd}
X, Y, R \arrow[r, "\sim"] \arrow[d, "{X,t}"'] & X \otimes Y, R \arrow[d, "g"] \\
X, P \arrow[r, "f'"'] & Q
\end{tikzcd}$$

commute.

Example 3.3.7. Let $\mathcal{C}$ be a $\mathcal{V}$-graded category and $\mathcal{K}$ be the free $\mathcal{V}$-graded category generated by the graded morphism ϕ_1 and the ordinary morphism ϕ_2

$$\begin{tikzcd}
& K_2 \arrow[d, "\phi_2"] \\
X, K_1 \arrow[r, "\phi_1"'] & K_3
\end{tikzcd}$$

Concretely, $\mathcal{K}$ is the $\mathcal{V}$-graded category with $ob(\mathcal{K}) = \{K_1, K_2, K_3\}$ and the hom-presheaves

$$\mathcal{K}(-, K_1; K_1) = \mathcal{V}(-, I)\ , \quad \mathcal{K}(-, K_2; K_2) = \mathcal{V}(-, I)\ , \quad \mathcal{K}(-, K_3; K_3) = \mathcal{V}(-, I)$$

$$\mathcal{K}(-, K_1; K_2) = \mathcal{K}(-, K_2; K_1) = \mathcal{K}(-, K_3; K_1) = \mathcal{K}(-, K_3; K_2) = \Delta_\emptyset$$
$$\mathcal{K}(-, K_2; K_3) = \mathcal{V}(-, I)\ , \qquad \mathcal{K}(-, K_1; K_3) = \mathcal{V}(-, X)$$

We have that every graded morphism $h : Y, K_1 \to K_3$ in $\mathcal{K}$ is given by a reindexing $\alpha^*(\phi_1)$ for some $\alpha : Y \to X$. Every graded morphism $k : Y, K_2 \to K_3$ in $\mathcal{K}$ is given by a reindexing $\beta^*(\phi_2)$ for some $\beta : Y \to I$. The remaining graded morphisms in $\mathcal{K}$ are reindexings of the identities 1_{K_1}, 1_{K_2}, and 1_{K_3}.

Let $D : \mathcal{K} \to \mathcal{C}$ be the $\mathcal{V}$-graded functor determined by

$$D(K_1) = A\ , \quad D(K_2) = Q\ , \quad D(K_3) = B$$

$$D(\phi_1) = f\ , \quad D(\phi_2) = q$$

Let $W : \mathcal{K} \to \mathcal{V}$ be the $\mathcal{V}$-graded functor determined by

$$W(K_2) = X\ , \qquad W(K_1) = W(K_3) = I$$

Then, $W\phi_1 : X \otimes I \to I$ and $W\phi_2 : I \otimes X \to I$ correspond to the same $w : X \to I$ in $\mathcal{V}$ by precomposition with the unitors in $\mathcal{V}$.

A W-weighted limit of D is given by an object $P \in \mathcal{C}$ equipped with

$$p : P \to A\ , \qquad f' : X, P \to Q\ , \qquad \lambda : P \to B$$

satisfying the naturality of weighted cones for ϕ_1 and ϕ_2, the generating morphisms of $\mathcal{K}$; i.e.,

$$\begin{tikzcd}
X, P \arrow[r, "X{,}p"] \arrow[d, "w{,}P"'] & X, A \arrow[d, "f"] \\
P \arrow[r, "\lambda"] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
X, P \arrow[r, "f'"] \arrow[d, "w{,}P"'] & Q \arrow[d, "q"] \\
P \arrow[r, "\lambda"] & B
\end{tikzcd}$$

commute. We can combine these to obtain a single diagram, making λ redundant. Therefore, a W-weighted limit of D is an object $P \in \mathcal{C}$ equipped with $p : P \to A$ in $\mathcal{C}_0$ and $f' : X, P \to Q$ in $\mathcal{C}$ such that

$$\begin{tikzcd}
X, P \arrow[r, "f'"] \arrow[d, "X{,}p"'] & Q \arrow[d, "q"] \\
X, A \arrow[r, "f"'] & B
\end{tikzcd}$$

commutes; i.e., $f \star p = q \star f'$. The data (P, p, f') must further satisfy the property that for any

$$r : Y, R \to A\,, \qquad g : X \otimes Y, R \to Q$$

such that

$$\begin{tikzcd}
X, Y, R \arrow[r, "\sim"] \arrow[d, "X{,}r"'] & X \otimes Y, R \arrow[r, "g"] & Q \arrow[d, "q"] \\
X, A \arrow[rr, "f"'] & & B
\end{tikzcd}$$

commutes; i.e., $q \star g = f \circ r$, there exists a unique graded morphism $t : Y, R \to P$ such that

$$\begin{tikzcd}
Y, R \arrow[r, "t"] \arrow[dr, "r"'] & P \arrow[d, "p"] \\
& A
\end{tikzcd}
\qquad
\begin{tikzcd}
X, Y, R \arrow[r, "X{,}t"] \arrow[d, "\sim"'] & X, P \arrow[d, "f'"] \\
X \otimes Y, R \arrow[r, "g"'] & Q
\end{tikzcd}$$

both commute; i.e., $p \star t = r$ and $f' \circ t = g$.

Note that the graded weighted cone also contains a redundant $\gamma : Y, R \to B$ enabling us to simplify the naturality of graded weighted cones down to the one diagram above, much like the process for the weighted cone.

This weighted limit is an X-weighted pullback of $q : Q \to B$ along $f : X, A \to B$.

Remark 3.3.8. In this final chapter of the thesis, we have explored the relationship between $\mathcal{V}$-graded weighted limits and the familiar notion of enriched weighted limits. This relationship is twofold: graded weighted limits themselves form a special class of $\hat{\mathcal{V}}$-enriched weighted limits, and $\mathcal{V}$-enriched weighted limits are a special case of $\mathcal{V}$-graded weighted limits. Given these findings, we strongly believe that, analogously to Lemma 1.3.21, a $\mathcal{V}$-graded category admits all small $\mathcal{V}$-graded weighted limits if and only if it admits all small $\mathcal{V}$-graded conical limits and $\mathcal{V}$-graded powers. Since $\hat{\mathcal{V}}$ is not locally small, we cannot define a *small* $\mathcal{V}$-graded category as a "small $\hat{\mathcal{V}}$-category". We defer the task of defining the concept of a *small* $\mathcal{V}$-graded category and we leave the proof of this theorem for future publications.

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