Properties of Arithmetic Universes and Locoi

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Abstract

In this thesis, we study two kinds of categories: locoi, which are lextensive categories with list objects, and arithmetic universes, which are pretoposes with list objects. We show three main results: first, if $\mathcal{C}$ is a locos, then the list object functor $L : \mathcal{C} \to \mathcal{C}$ is a polynomial functor. Second, if $\mathcal{C}$ is a locos, then the full subcategory $\mathbf{Fin}(\mathcal{C})$ of finite objects is a Boolean topos. Third, if $\mathcal{S}$ is an arithmetic universe, then the free extension of $\mathcal{S}$ by an object is the category $[\mathbf{Fin}_{\mathcal{S}}, \mathbb{S}]$ of indexed copresheaves on the internal category $\mathbf{Fin}_{\mathcal{S}}$ of finite sets in $\mathcal{S}$.

Résumé

Dans cette thèse, nous étudions deux types de catégories: les locos, qui sont des catégories lextensives avec des "objets des listes" (*list objects*), et les univers arithmétiques, qui sont des prétopos avec des objets des listes. Nous démontrons trois résultats principaux: d'abord, si $\mathcal{C}$ est un locos, alors le foncteur des listes $L : \mathcal{C} \to \mathcal{C}$ est un foncteur polynomial. Ensuite, si $\mathcal{C}$ est un locos, alors la sous-catégorie pleine $\mathbf{Fin}(\mathcal{C})$ des objects finis est un topos booléen. Enfin, si $\mathcal{S}$ est un univers arithmétique, alors l'extension libre de $\mathcal{S}$ par un objet est la catégorie $[\mathbf{Fin}_{\mathcal{S}}, \mathbb{S}]$ des copréfaisceaux indexés sur la catégorie interne $\mathbf{Fin}_{\mathcal{S}}$ des objets finis dans $\mathcal{S}$.

Dedications

I first want to dedicate this thesis to Phil Scott. Phil was my co-supervisor for the first two years of my PhD, before passing away in 2023. I wish he had been able to see the completion of this thesis. Phil talked about math with such enthusiasm, and he was always welcoming and kind. I am proud to call myself one of his students.

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Chapter 1

Introduction

Contents

1.1 Background and related work

As suggested by the title, this thesis focuses on locoi and arithmetic universes. These are types of categories which feature *list objects*.

When working with sets, if we have a set X, then we can form the set $L(X)$ of finite lists with entries in X. List objects, a concept introduced by Cockett [Cock 90], are a way to generalize this notion to any category $\mathcal{C}$ with finite products. Given an object $X \in \mathcal{C}$, the *list object associated to* X is an object $L(X)$ which is characterized by a certain universal property; we think of $L(X)$ as the set of lists with elements in X. Roughly, the universal property of $L(X)$ states that an arrow $f : L(X) \to Y$ can be defined inductively as follows (see also section 4.1).

- For the base case, we specify the value of $f(\varnothing)$.
- For the inductive step, we require that $f(x :: \ell)$ is defined as a function of x, ℓ, and $f(\ell)$. That is, $f(x :: \ell) = h(x, \ell, f(\ell))$ for some function h.

(Here, $x :: \ell$ represents an element x appended to a list ℓ.) In general, we say a category $\mathcal{C}$ with finite products *has list objects* if for every $X \in \mathcal{C}$, there is an object $L(X)$ satisfying this universal property.

As mentioned above, this thesis focuses on locoi and arithmetic universes. A *locos* is a category with finite limits, list objects, and finite coproducts which are well-behaved in a certain sense. In short, it is a lextensive category with (parametrized) list objects [Cock 90].

Because of their connection to data types, list objects and locoi have been studied mostly in the context of computer science (in addition to [Cock 90], see [Jay 93a], [Jay 93b], [Jay 94]). Some notable results include the fact that locoi are local – meaning that if $\mathcal{C}$ is a locos, then every slice category $\mathcal{C}/I$ is too [Cock 90, proposition 6.7], and the pullback functors preserve the locos structure [Maie 05, proposition 2.13] – and that the functor $L : \mathcal{C} \to \mathcal{C}$ mapping X to $L(X)$ is part of a Cartesian monad when $\mathcal{C}$ is a locos [Jay 94, corollary 2.7]. However, list objects gained some relevance in the context of topos theory when Maietti [Maie 10] included them in an axiomatization of *arithmetic universes*.

According to [Maie 10], arithmetic universes were introduced in the seventies by André Joyal in some unpublished lectures. Joyal used arithmetic universes to provide a categorical proof of Gödel's incompleteness theorems (which has now been reproduced [Dijk 20]), but at the time, he did not provide a general definition of the notion. Today, the standard definition is due to Maietti [Maie 10]: an arithmetic universe is a pretopos with (parametrized) list objects. Spelled out, this means that an arithmetic universe is a category with finite limits, list objects, finite coproducts, and quotients of equivalence relations, and such that the coproducts and quotients are well-behaved in a certain sense.

Examples of arithmetic universes include all elementary toposes with a natural numbers object; this includes categories of sheaves on a space (or, more generally, Grothendieck toposes). The initial arithmetic universe is also fairly well understood: it was originally described by Joyal as the exact completion of the category $\mathrm{Pred}(\mathcal{S}_{in})$ of *primitive recursive predicates*. That is, $\mathrm{Pred}(\mathcal{S}_{in})$ (which is also the initial locos) is the category of subobjects of N defined by Δ_0 formulas, and the initial arithmetic universe is obtained by freely adding quotients of equivalence relations. Maietti discusses this characterization at length in [Maie 10], and gives an alternative characterization in [Maie 24].

The category $\mathrm{Pred}(\mathcal{S}_{in})$ is notable because the arrows $N \to N$ are primitive recursive functions up to provable equality in PRA, and because all arrows $\mathbb{1} \to N$ are standard numerals (arrows of the form $s^n 0$ for $n \in \mathbb{N}$). However, we can vary this construction of an arithmetic universe by using a stronger theory of arithmetic (like peano arithmetic); we can even get arithmetic universes with non-standard arrows $\mathbb{1} \to N$ by using non-standard models of PRA.

In addition to Maietti's work, arithmetic universes have recently been investigated by Vickers, who suggested in 1999 [Vick 99] that they could be used to provide a "base-free" way to study Grothendieck toposes. Traditionally, one studies a Grothendieck topos with respect to an elementary topos $\mathcal{S}$ called the *base*: a Grothendieck topos with respect to this base consists of another elementary topos with a bounded geometric morphism to $\mathcal{S}$. This approach has its drawbacks, since geometric morphisms are not the most natural notion of morphism between elementary toposes.

On the other hand, morphisms of arithmetic universes are much closer to geometric morphisms, and Vickers' idea led to a research program exploring whether it is possible to develop the theory of Grothendieck toposes using arithmetic universes. This has seen some success ([Maie 12], [Vick 19], [Hazr 20]); for instance, Vickers recently developed an analogue for classifying toposes using arithmetic universes [Vick 19], which supplements the examples above with a more general construction of arithmetic universes by generators, albeit without explicit descriptions. As we'll discuss below, one of the main results of this thesis is a step in the direction of this research program.

1.2 The main results of this thesis

In this thesis, we essentially prove three important results.

1. If $\mathcal{C}$ is a locos, then the list object functor $L : \mathcal{C} \to \mathcal{C}$ is a *polynomial functor* (corollary 5.4.2). Specifically, the map $\pi_2^E : E \to N$ (where E is the *collection of finite cardinals*, see section 3.4) is exponentiable, and L is the polynomial functor associated to this exponentiable arrow. Informally, this means that we can write $L(X) = \sum_{n \in N} X^n$.

2. If $\mathcal{C}$ is a locos, then the full subcategory $\mathbf{Fin}(\mathcal{C})$ of finite objects in $\mathcal{C}$ (where finite means isomorphic to a finite cardinal; see section 13) is a Boolean topos (theorem 15.5.1). Moreover, the topos structure of $\mathbf{Fin}(\mathcal{C})$ is preserved by the inclusion $\mathbf{Fin}(\mathcal{C}) \hookrightarrow \mathcal{C}$. Notably, result 1 is what allows us to take exponentials of finite objects.

3. If $\mathcal{S}$ is an arithmetic universe, then the arithmetic universe $\mathcal{S}[O]$ – obtained by freely adjoining an object O – can be described explicitly (theorem 17.2.1). The category $\mathcal{S}[O]$ is the category of indexed functors $[\mathbf{Fin}_{\mathcal{S}}, \mathbb{S}]$, where $\mathbb{S}$ is the canonical self-indexing of $\mathcal{S}$, and $\mathbf{Fin}_{\mathcal{S}}$ is the internal category of finite sets in $\mathcal{S}$ (which is closely related to $\mathbf{Fin}(\mathcal{S})$; see section 14). It is equipped with the object $O = \iota : \mathbf{Fin}_{\mathcal{S}} \hookrightarrow \mathbb{S}$ (the canonical inclusion), and the functor $\Delta : \mathcal{S} \to [\mathbf{Fin}_{\mathcal{S}}, \mathbb{S}]$ which maps an object I to the constant indexed functor at I.

Below, we give an overview of each of these results separately, and then comment on the overall structure of the thesis.

The list object functor is polynomial

Given a set X, it is possible to form the set $L(X)$ of finite lists with elements in X. This mapping $X \mapsto L(X)$ extends to a functor $L : \mathbf{Set} \to \mathbf{Set}$, where $f : X \to Y$ becomes the map $L(f) : L(X) \to L(Y)$ given by $L(f)([x_1, ..., x_n]) = [f(x_1), ..., f(x_n)]$.

It is well known that this functor $L : \mathbf{Set} \to \mathbf{Set}$ is an example of a *polynomial functor*. (A good reference for this notion is [Gamb 13], but we also review the definition in section 2.1.) Indeed, if we partition lists by length, we can see that this functor is given by

$$L(X) = \coprod_{n \in \mathbb{N}} X^n.$$

This expression lets us to conclude that L is a polynomial functor. Specifically, it is represented by the polynomial

$$\mathbb{1} \longleftarrow E \xrightarrow{\pi_2^E} \mathbb{N} \longrightarrow \mathbb{1} \ ,$$

where $E = \{(m,n) \in \mathbb{N} \times \mathbb{N} \mid m < n\}$, and $\pi_2^E : E \to \mathbb{N}$ is the second projection. (See also example 1.9 in [Gamb 13].)

If $\mathcal{C}$ is a category with finite limits and a (parametrized) natural numbers object N, it is possible to construct the analogue of π_2^E as an arrow $\pi_2^E : E \to N$ in $\mathcal{C}$ (see section 3.4). Therefore, if $\mathcal{C}$ also has list objects, then we might expect that the list object functor $L : \mathcal{C} \to \mathcal{C}$ is also a polynomial functor, represented by the above polynomial. In order to prove this result (corollary 5.4.2), we make a slightly stronger assumption: we assume $\mathcal{C}$ is a locos. This part of the thesis (proving that L is polynomial when $\mathcal{C}$ is a locos) has already been published to the mathematical arXiv ([Desr 25]).

As mentioned previously, if $\mathcal{C}$ is a locos, then the functor L is part of a Cartesian monad [Jay 94, corollary 2.7]. Therefore, with this result, we conclude that L is part of a *polynomial monad* (see [Gamb 13] for the definition).

It is worth discussing why we assume $\mathcal{C}$ is a locos. Firstly, though we have not constructed a counter-example, we suspect that L is not necessarily a polynomial functor if we only assume $\mathcal{C}$ has finite limits and list objects (and hence a natural numbers object, namely $L(\mathbb{1})$). On the other hand, the assumption that $\mathcal{C}$ is a locos can be weakened (as is shown in [Desr 25]). Indeed, we don't need to assume $\mathcal{C}$ has all binary coproducts, only that certain coproducts (related to the list objects) are well behaved. However, we stick to locoi because extensive categories are well studied, and because when list objects were introduced in [Cock 90], they were mostly studied in a locos. As Cockett puts it in that paper, "it might reasonably be argued that the type of mathematics determined by [a locos] is precisely what is traditionally called 'discrete mathematics'".

One of the difficulties in proving that L is polynomial is that usually, polynomial functors are discussed for locally cartesian closed categories (as in [Gamb 13]). Indeed, given an arrow $f : B \to A$ in $\mathcal{C}$, the induced polynomial functor $P_f : \mathcal{C} \to \mathcal{C}$ is defined using the right adjoint Π_f of $f^* : \mathcal{C}/A \to \mathcal{C}/B$. Since locoi are generally not cartesian closed, in order to show that the list object functor L is induced by the arrow $\pi_2^E : E \to N$, we will need to prove that π_2^E is exponentiable (which is equivalent to saying $\Pi_{\pi_2^E}$ exists; see fact 2.1.1). This fact is established as a consequence of our proof strategy, which we explain below (see also the proof of corollary 5.4.2).

The proof of this first main result is wholly contained in part II of this thesis, and is mostly limited to chapter 5, though it does require some of the preliminary tools of chapters 3 and 4, and it uses the internal language of section 2.2. The idea of the proof is the following: it suffices to prove that the functor

$$L_N : \mathcal{C} \to \mathcal{C}/N : X \mapsto (\text{len} : L(X) \to N)$$

(where len is the length arrow) is a right adjoint of the functor

$$\mathcal{C}/N \to \mathcal{C} : (l_A : A \to N) \mapsto E \times_N A.$$

(This result is theorem 5.4.1.) The counit of this adjunction is $(\text{nth}_X : E \times_N L(X) \to X)_{X \in \mathcal{C}}$, where nth is the "$n^{th}$ element" arrow defined in section 4.4. So, proving the result can be rephrased as establishing a universal property, which we do in chapter 5, culminating in corollary 5.4.2, which proves the first main result.

The category of finite objects is a Boolean topos

The category **FinSet** of finite sets has many interesting properties. It is an (elementary) topos, it is the free category with finite coproducts on one object, and it is used to build the classifying topos for the theory of objects (see section D3.2 in [John 02]).

If we are in a category $\mathcal{C}$ with finite limits and a natural numbers object N, then we can talk about *finite cardinals*. For any arrow $m : \mathbb{1} \to N$, the finite cardinal E_m is defined as the pullback of $\pi_2^E : E \to N$ along m. We think of E_m as "the set with m elements".

$$\begin{array}{ccc} E_m & \longrightarrow & E \\ \big\downarrow & \lrcorner & \big\downarrow \pi_2^E \\ \mathbb{1} & \xrightarrow{m} & N \end{array}$$

We may then say that an object of $\mathcal{C}$ is *finite* if it is isomorphic to a finite cardinal. This is not the only definition of "finiteness" in a category, but it is the notion that will be most relevant to us.

We can then consider the full subcategory $\mathbf{Fin}(\mathcal{C})$ of finite objects in $\mathcal{C}$, and ask if it has any nice properties. If $\mathcal{C}$ is a topos with a natural numbers object, then it is known that $\mathbf{Fin}(\mathcal{C})$ is a Boolean topos, and that the finite limits, finite colimits, and exponentials are preserved by the inclusion $\mathbf{Fin}(\mathcal{C}) \hookrightarrow \mathcal{C}$ (see theorem D5.2.7 of [John 02]). The second main result of this thesis states that this remains true if $\mathcal{C}$ is instead only a locos (theorem 15.5.1). Given the first main result, this should not be so surprising – the arrow π_2^E being exponentiable essentially tells us that exponentials by finite objects exist in a locos.

This second main result is established in part IV of the thesis. The first thing we show is that subobjects of finite objects are finite if and only if they are complemented (theorem 13.2.3). The proof is not easy: it relies on some technical tools for locoi that we develop in appendix B. These tools, in turn, rely heavily on the first main result. However, once this has been established, we can also show that $\mathbf{Fin}(\mathcal{C})$ has finite limits and finite coproducts; this is done in chapter 13.

In order to proceed, we need some tools from the theory of internal categories (which is why this part of the thesis comes after part III, where we develop some results for indexed and internal categories). In chapter 14, we introduce $\mathbf{Fin}_{\mathcal{S}}$, the *internal category of finite sets* in a locos $\mathcal{S}$. By considering this internal category as an $\mathcal{S}$-indexed category, it turns out that $\mathbf{Fin}_{\mathcal{S}}(I) \simeq \mathbf{Fin}(\mathcal{S}/I)$ (proposition 14.1.1). Our results about $\mathbf{Fin}(\mathcal{C})$ let us conclude that $\mathbf{Fin}_{\mathcal{S}}$ has finite products, and this implies (by theorem 12.4.2, a technical result for internal categories) that it has "internally finite products": products indexed by finite cardinals. This is what allows us to conclude that the exponential of finite objects is finite (theorem 15.1.1).

In chapter 15, we finish proving the second main result (theorem 15.5.1); what remains is showing that $\mathbf{Fin}(\mathcal{C})$ has coequalizers which are preserved by the inclusion. The proof is rather technical, and requires us to establish an internal version of the pigeonhole principle (theorem 15.2.6). The proof strategy is based on Maietti's construction of coequalizers in an arithmetic universe (proposition 3.10 in [Maie 10]).

Adjoining an object to an arithmetic universe

As we discussed above, there has been some recent interest in developing the theory of Grothendieck toposes using arithmetic universes, and doing so would involve a theory of "classifying arithmetic universes".

Perhaps the simplest example of a classifying topos is the one for the theory of objects $\mathbb{O}$: the theory with a single type, no function symbols, and no axioms. For toposes over a base $\mathcal{S}$, this classifying topos $\mathcal{S}[\mathbb{O}]$ is equipped with an object $O \in \mathcal{S}[\mathbb{O}]$ and a geometric morphism $\mathcal{S} \hookrightarrow \mathcal{S}[\mathbb{O}]$ (in the direction of the left adjoint functor), and it is defined by the following universal property. For any topos $\mathcal{T}$, any $X \in \mathcal{T}$, and any geometric morphism $\mathcal{S} \to \mathcal{T}$, there exists a unique $\mathcal{S}[\mathbb{O}] \to \mathcal{T}$ such that O is mapped to X and the following diagram commutes.

$$\begin{array}{ccc} O & \longmapsto & X \\ \mathcal{S}[\mathbb{O}] & \dashrightarrow & \mathcal{T} \\ \uparrow & \nearrow & \\ \mathcal{S} & & \end{array}$$

We can also describe $\mathcal{S}[\mathbb{O}]$ as freely adjoining an object O to the topos $\mathcal{S}$.

If we replace toposes by arithmetic universes and geometric morphisms by AU morphisms, we get a description of the "classifying arithmetic universe" for the theory of objects. The third result of this thesis is an explicit characterization of this category. In short, given an arithmetic universe $\mathcal{S}$, we construct an arithmetic universe $\mathcal{S}[\mathbb{O}]$ which satisfies the above universal property.

To give an explicit characterization of $\mathcal{S}[\mathbb{O}]$ for an arithmetic universe $\mathcal{S}$, we are inspired by the classifying topos $\mathbf{Set}[\mathbb{O}]$, which is given by $[\mathbf{FinSet}, \mathbf{Set}]$ (see section D3.2 in [John 02]). As discussed above, we can develop notions of finiteness in an arithmetic universe, and it turns out that the category $\mathcal{S}[\mathbb{O}]$ we're looking for is

$$\mathcal{S}[\mathbb{O}] = [\mathbf{Fin}_{\mathcal{S}}, \mathbb{S}].$$

That is, it is the category of *$\mathcal{S}$-indexed functors* from the *internal category of finite sets* $\mathbf{Fin}_{\mathcal{S}}$ to the canonical self-indexing $\mathbb{S}$ of $\mathcal{S}$. (In other words, it is the category of internal copresheaves on $\mathbf{Fin}_{\mathcal{S}}$.)

We only discuss and prove this result (theorem 17.2.1) in chapter 17, but most of the thesis is dedicated to establishing results in preparation for this proof, which mostly comes down to establishing two equivalences of categories. The first equivalence is corollary 16.2.8:

$$\mathcal{T} \simeq [\mathbf{Fin}_{\mathcal{S}}^{op}, F^*\mathbb{T}]_{flex}.$$

Roughly, it says that a functor on $\mathbf{Fin}^{op}$ which preserves finite limits is uniquely determined by its action on 1. Indeed, this implies preserving (internally) finite coproducts on $\mathbf{Fin}$, and each $n \in \mathbf{Fin}$ is an (internally) finite coproduct of 1. To prove this result, we need to show that $\mathbf{Fin}$ has finite colimits, and that these are preserved by the inclusion into $\mathcal{S}$: this is essentially achieved by the second main result. We also need some results about functors on **Fin**, which are proved in chapters 8 and 16 (the results of chapter 8 are developed in a more general context).

The second equivalence is corollary 11.3.1:

$$[[\![C^{op}, \mathbb{S}]\!], F^*\mathbb{T}]_{flexcc} \simeq [C, F^*\mathbb{T}]_{flex}.$$

Roughly, it says that a lex functor on C extends uniquely to a lex cocontinuous functor on the copresheaf category $[\![C^{op}, \mathbb{S}]\!]$. This is a well-known result in ordinary topos theory, but it is not easy to adapt to the context of indexed categories and internal categories. An extra wrinkle is that we must consider *internally* finite products – that is, products indexed by the natural numbers object. Most of part III is dedicated to this task: after some preliminaries in chapters 6 and 7, we take chapters 9 and 10 to develop the key technical tool, which is that filtered colimits of lex functors are lex. We arrive at a first version of the equivalence of categories with corollary 10.5.2, and chapter 11 slightly generalizes it to get the equivalence above.

1.3 The structure of this thesis

In the previous section, we outlined the proofs of the various main results and how they fit into the structure of the thesis. Now, we explain the structure from a more global perspective.

Part I contains the introduction and preliminaries. Most notably, section 2.2 establishes the *internal language* that we will use throughout the thesis. It also contains a review of certain categorical notions (section 2.1) and a brief discussion of subobjects and relations (section 2.3).

Part II is focused on natural numbers objects and list objects. The first main result of this thesis, corollary 5.4.2, is proved in chapter 5. Moreover, chapters 3 and 4 develop many technical tools for working with natural numbers objects and list objects, respectively.

Part III develops many tools for working with indexed categories and internal categories. Some basics are covered in chapters 6 and 7, including the properties of indexed categories that come from arithmetic universes. In chapter 8, we discuss the construction of a certain kind of internal category, which we will later use (in chapter 14) to define the internal category of finite sets.

Next, chapters 9, 10, and 11 together establish an equivalence of categories (corollary 11.3.1) that is key for proving the third main result of the thesis. Chapter 9 discusses the indexed colimit and filtered colimits of internally lex (flex) functors; chapter 10 describes a specific Kan extension and establishes some of its properties; and chapter 11 shows how to perform a "change of base" operation with these tools.

Finally, chapter 12 discusses a certain class of indexed categories, and shows how finite products are linked to internally finite products for this class.

Part IV mostly discusses finite objects and the internal category of finite sets in a locos: chapters 13 through 16 work to establish properties of these constructions, including the second main result of the thesis (theorem 15.5.1). Finally, chapter 17 uses all the previous results to establish the third main result of the thesis (corollary 17.1.3). We finish with a conclusion in chapter 18.

Part V contains the appendices and end matter. Appendix A contains detailed proofs of some basic results about indexed categories (discussed in section 6.3), while appendix B establishes technical tools that are used in the calculations of chapter 13.

Chapter 2

Preliminaries

Contents

This chapter contains some preliminaries for the rest of the thesis. Section 2.1 reviews some standard definitions and facts from the literature, and establishes some notation. Section 2.2 builds the internal language that will be used throughout the thesis. Section 2.3 proves some basic facts about subobjects and relations.

2.1 Review of standard notions

In this chapter, we review important definitions and results from the literature that we will reference in the rest of the thesis.

Finite limits

We will not review the definitions of finite limits, but we will establish some notation.

A category which has all finite products is called *Cartesian*. For a binary product $A \times B$, we denote the projection maps $\pi_1^{A,B}$ and $\pi_2^{A,B}$, omitting superscripts when they can be inferred from the context. The pairing of $f : C \to A$ and $g : C \to B$ is denoted $\langle f, g \rangle : C \to A \times B$, and given $\alpha : X \to A$ and $\beta : Y \to B$, we write $\alpha \times \beta$ for $\langle \alpha \circ \pi_1, \beta \circ \pi_2 \rangle : X \times Y \to A \times B$. This notation extends to the n-fold product $A_1 \times ... \times A_n$, which has projections $\pi_1, ..., \pi_n$ and pairing $\langle f_i \rangle_{i=1}^n$. In particular, for the terminal object $\mathbb{1}$, the unique arrow $X \to \mathbb{1}$ is denoted $\langle \rangle_X$.

For pullbacks, the notation is similar. The pullback of $f : A \to C$ and $g : B \to C$ is denoted $A \times_C B$, and the projections are denoted $\pi_1^{A,B}$ and $\pi_2^{A,B}$. We will also use the notation $\langle p, q \rangle : X \to A \times_C B$ and $\alpha \times_C \beta : X \times_C Y \to A \times_C B$ whenever these constructions are well-defined.

A category with all finite limits, or a functor which preserves finite limits, may be called *lex*. We note that if a category $\mathcal{C}$ has all finite limits, then so do all its slice categories $\mathcal{C}/I$.

Throughout this thesis, we will sometimes make reference to the *pasting law* or *gluing law* for pullbacks. This refers to the standard fact that, if we have a commutative diagram as below where the right square is a pullback, then the left square is a pullback if and only if the outer rectangle is a pullback.

$$\begin{array}{ccccc} \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet \\ \downarrow & & \downarrow & & \downarrow \\ \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet \end{array}$$

Finally, we may sometimes write $\mathrm{eq}(f, g)$ for the equalizer of arrows f, g.

Functors between slice categories

Given an arrow $f : A \to B$ in a category $\mathcal{C}$, we write Σ_f for the composition functor $\mathcal{C}/A \to \mathcal{C}/B$. Its right adjoint is the pullback functor $f^* : \mathcal{C}/B \to \mathcal{C}/A$ (sometimes denoted Δ_f), assuming it exists. In the special case $B = \mathbb{1}$, these functors become $\Sigma_A : \mathcal{C}/A \to \mathcal{C}$ (the forgetful functor) and $A^* : \mathcal{C} \to \mathcal{C}/A$ (or Δ_A).

If f^* has a further right adjoint, we call it the *dependent product functor* along f and denote it $\Pi_f : \mathcal{C}/A \to \mathcal{C}/B$. We record the following fact from [Nief 82] (corollary 1.2) for later use.

Fact 2.1.1. Let $\mathcal{C}$ be a category with finite limits, and let $f : A \to B$ be an arrow. Then the following are equivalent:

- f^* has a right adjoint (i.e. Π_f exists);
- $- \times f$ has a right adjoint (i.e. f is exponentiable, per p.8 of [Joya 95]);
- $\Sigma_A \circ f^*$ has a right adjoint.

Polynomials

Let $\mathcal{C}$ be a category with finite limits. A *polynomial* over a category $\mathcal{C}$ is a diagram P of the following form.

$$I \xleftarrow{\ s\ } A \xrightarrow{\ f\ } B \xrightarrow{\ t\ } J$$

If the functor Π_f exists (i.e. if f is exponentiable, see fact 2.1.1), then the functor $F_P : \mathcal{C}/I \to \mathcal{C}/J$ given by

$$\mathcal{C}/I \xrightarrow{\Delta_s} \mathcal{C}/A \xrightarrow{\Pi_f} \mathcal{C}/B \xrightarrow{\Sigma_t} \mathcal{C}/J$$

is called the *polynomial functor associated to* P.

A functor $F : \mathcal{C}/I \to \mathcal{C}/J$ is called a *polynomial functor* if there exists a polynomial P such that F_P exists and $F \cong F_P$. In this case, we say P *represents* F. If $F : \mathcal{C} \to \mathcal{C}$, then we say it is a polynomial functor if this is the case when using the usual identification between $\mathcal{C}$ and $\mathcal{C}/\mathbb{1}$.

These definitions are from [Gamb 13]; however, in that paper, they assume that $\mathcal{C}$ is locally Cartesian closed. We avoid this assumption, at the cost of not all polynomial functors existing.

Extensivity

We refer to [Carb 93] for the general definition of an extensive category: a category $\mathcal{C}$ with finite coproducts is *extensive* if for any $X, Y \in \mathcal{C}$, the canonical functor $\mathcal{C}/X \times \mathcal{C}/Y \to \mathcal{C}/(X+Y)$ is an equivalence. (So, whenever we say a category is extensive, we implicitly assume it has finite coproducts.) Since we will always assume our categories have finite limits, we use the following characterization, drawn from proposition 2.2 of [Carb 93].

Fact 2.1.2. Let $\mathcal{C}$ be a category with finite limits and finite coproducts. Then $\mathcal{C}$ is extensive if and only if for every coproduct $X_1 \longrightarrow X_1 + X_2 \longleftarrow X_2$ and any commutative diagram as below,

$$\begin{array}{ccccc} A_1 & \longrightarrow & A & \longleftarrow & A_2 \\ \downarrow & & \downarrow & & \downarrow \\ X_1 & \longrightarrow & X_1 + X_2 & \longleftarrow & X_2 \end{array}$$

the squares are pullbacks if and only if the top row is a coproduct.

In particular, in an extensive category, every binary coproduct is *universal*, which means that its pullback along any arrow is still a coproduct. An extensive category which has finite limits is called *lextensive*.

Finally, note that we may write $\mathbb{O}$ to refer to the initial object of a category. (We reserve the symbol $\varnothing$ for list objects.)

Regular and exact categories

For the definition of regular category, we refer to section 2 of [Borc 94b]. First, recall that the *kernel pair* of an arrow $f : X \to Y$ is a pair of arrows $g, h : R \to X$ such that the following square is a pullback.

$$\begin{array}{ccc} R & \xrightarrow{h} & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ X & \xrightarrow[f]{} & Y \end{array}$$

Recall also that a *regular epimorphism* is an arrow which is the coequalizer of two others. Then a *regular category* is one in which

1. every arrow has a kernel pair;

2. every kernel pair has a coequalizer; and

3. the pullback of a regular epimorphism along any morphism exists and is again a regular epimorphism.

(This is definition 2.1.1 in [Borc 94b].)

We also refer to section 2.6 of [Borc 94b] for the definition of exact categories. Recall that an equivalence relation $r_1, r_2 : E \rightrightarrows A$ on an object A is *effective* if the coequalizer q of r_1, r_2 (also called the *quotient*) exists, and if (r_1, r_2) is the kernel pair of q. A category is *exact* if it is regular and if every equivalence relation is effective.

Finally, per the nLab definition[1], a *pretopos* is a category which is both exact and extensive. In [Maie 10], the definition is spelled out as follows: a pretopos is a category with finite limits, stable disjoint finite coproducts, and stable effective quotients of equivalence relations.

We take this opportunity to recall the definition of an (elementary) *topos*: it is a category with finite limits, exponentials, and a subobject classifier (definition A2.1.1 in [John 02]). We refer to section A1.6 of [John 02] for the definition of a subobject classifier.

Natural numbers objects

Let $\mathcal{C}$ be a Cartesian category. A *(parametrized) natural numbers object* (NNO) is a triple $(N, 0, s)$ consisting of an object N and arrows $0 : \mathbb{1} \to N$, $s : N \to N$ which satisfy the following property.

> For any objects A, B and arrows $g : A \to B$, $h : A \times N \times B \to B$, there exists a unique arrow $f : A \times N \to B$ which makes the following diagrams commute.

$$\begin{array}{ccc} & & A \times N \\ & \nearrow^{\langle \mathrm{Id}_A, 0_A \rangle} & \downarrow f \\ A & \xrightarrow[g]{} & B \end{array} \qquad \begin{array}{ccc} A \times N & \xrightarrow{\mathrm{Id}_A \times s} & A \times N \\ \downarrow \langle \pi_1, \pi_2, f \rangle & & \downarrow f \\ A \times N \times B & \xrightarrow[h]{} & B \end{array}$$

This definition comes from [Roma 89]. Following the convention of [Roma 89], when we say “natural numbers object”, we will always refer to the parametrized version. We will discuss natural numbers objects in much more detail in chapter 3, where we'll use the internal language of section 2.2.

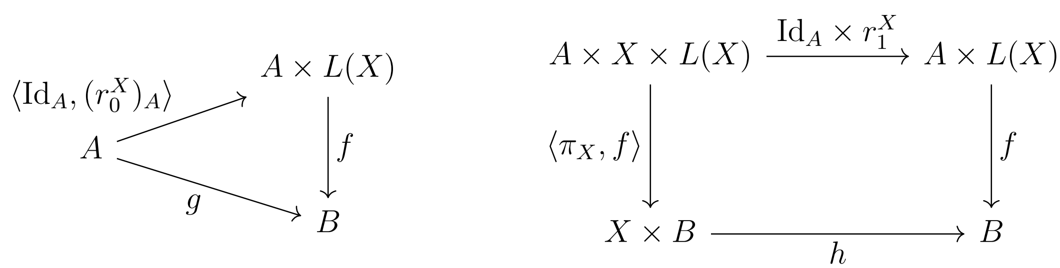
[1] https://ncatlab.org/nlab/show/pretopos

A category which is lextensive and which has an NNO is called *arithmetic lextensive* (we take this terminology from [Maie 10]).

List objects

Let $\mathcal{C}$ be a Cartesian category, and let $X \in \mathcal{C}$ be an object. A *(parametrized) list object* on X is a triple $(L(X), r_0^X, r_1^X)$ consisting of an object $L(X)$ and arrows $r_0^X : \mathbb{1} \to L(X)$, $r_1^X : X \times L(X) \to L(X)$ which satisfy the following property.

> For any objects A, B and arrows $g : A \to B$, $h : X \times B \to B$, there exists a unique arrow $f : A \times L(X) \to B$ which makes the following diagrams commute.



This definition of list objects comes from [Cock 90]. As with natural numbers objects, we will always talk about parametrized list objects.

Note that in the second diagram, we are abusing notation by writing $\langle \pi_X, f \rangle$. Indeed, to be precise, we should write $\langle \pi_2, f \circ \langle \pi_1, \pi_3 \rangle \rangle$. Moreover, the domain of $\mathrm{Id}_A \times r_1^X$ should be $A \times (X \times L(X))$ (with brackets). The definition will become much easier to parse when, in chapter 4, we state it with the internal language of section 2.2.

A category which is lextensive and which has list objects is called a *locos*. (This terminology is due to [Cock 90].) In the same paper, Cockett also showed that locoi are local (proposition 6.7 of [Cock 90]) in the sense of each slice being local. It was also shown in proposition 2.13 of [Maie 05] that the locos structure is preserved by pullbacks, which we consider to be part of the definition of "local" (per [Maie 10]).

Fact 2.1.3. The property of being a locos is local. That is, if $\mathcal{C}$ is a locos, then every slice $\mathcal{C}/I$ is a locos, and the functor $f^* : \mathcal{C}/A \to \mathcal{C}/B$ is a locos morphism for every arrow $f : B \to A$ of $\mathcal{C}$.

Arithmetic universes

We refer to [Maie 10] for the definition of arithmetic universes, though (as noted in [Maie 10]) the name dates back to an idea of Joyal from the seventies.

Definition. An *arithmetic universe* (AU), also called a *list-arithmetic pretopos*, is a pretopos with (parametrized) list objects.

A *morphism of arithmetic universes*, also called an *AU morphism* or an *AU functor*, is a functor which preserves finite limits, finite coproducts, quotients of equivalence relations, and list objects.

In [Maie 10], there is the following important result.

Fact 2.1.4 (Proposition 3.10 in [Maie 10])**.** Arithmetic universes have all coequalizers. (And therefore, all finite colimits.)

With this result, we can give a simpler and more useful characterization of AU functors.

Proposition 2.1.5. Let $\mathcal{A}, \mathcal{B}$ be arithmetic universes. Then $F : \mathcal{A} \to \mathcal{B}$ is an AU functor if and only if it preserves finite limits, finite colimits, and list objects.

Proof. We just need to show that if F is an AU functor, then it preserves arbitrary coequalizers. In section 3.10 of [Maie 10], arbitrary coequalizers are constructed explicitly using pretopos structures and list objects. Since AU functors preserve these, they also preserve the coequalizers. ∎

Maietti also proved the following fact.

Fact 2.1.6 (Proposition 2.7 of [Maie 10])**.** The property of being an arithmetic universe is local. That is, if $\mathcal{A}$ is an AU, then every slice $\mathcal{A}/I$ is an AU, and the functor $f^* : \mathcal{A}/A \to \mathcal{A}/B$ is an AU morphism for every arrow $f : B \to A$ of $\mathcal{A}$.

In particular, this means that the pullback functors for arithmetic universes preserve finite limits, finite colimits, and list objects.

Functors between slices of different categories

Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Then for any $I \in \mathcal{C}$, there is a functor

$$F/I : \mathcal{C}/I \to \mathcal{D}/F(I)$$

(usually just denoted F) which maps an object $(p : X \to I) \in \mathcal{C}/I$ to $F(p) : F(X) \to F(I)$. Similarly, an arrow f is mapped to $F(f)$.

We will often need to know that these functors preserve limits, colimits, and list objects. This is summarized by the following result.

Proposition 2.1.7. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor.

1. If $\mathcal{C}, \mathcal{D}$ have finite limits and F preserves them, then so does F/I for every I.
2. If $\mathcal{C}, \mathcal{D}$ have finite colimits of a given type and F preserves them, then so does F/I for every I.
3. If $\mathcal{C}, \mathcal{D}$ are locoi and F is a locos morphism, then so is F/I for every I.
4. If $\mathcal{C}, \mathcal{D}$ are AUs and F is an AU morphism, then so is F/I for every I.

Proof. The first point is easy. The terminal object in $\mathcal{C}/I$ is $\mathrm{Id}_I : I \to I$, which is mapped by F to $\mathrm{Id}_{F(I)}$, the terminal object in $\mathcal{D}/F(I)$. So, F/I preserves terminal objects. Moreover, pullbacks in slice categories are computed as in the base category, so F/I preserves pullbacks. Therefore, F/I preserves all finite limits.

The second point is even easier: all finite colimits are computed in the slice as in the base, so they are preserved by every F/I.

The third point is more nuanced. The first two points show that F/I preserves finite limits and finite coproducts, but the description of a list object in $\mathcal{C}/I$ is more complicated. However, it can be constructed explicitly using finite limits, finite coproducts, and list objects in $\mathcal{C}$ (see the proof of proposition 6.7 in [Cock 90]). Since F preserves all these, we find that F/I does indeed preserve list objects.

The fourth point is a direct consequence of the first three. ∎

2.2 Internal language

In any Cartesian category, there is a way to express equalities between arrows in a "variable form", so that an equality $f = g$ is instead written as $f(x) =_x g(x)$. This improves readability: to define some $h : X \times Y \times Z \to A$, it is much clearer to write

$$h(x, y, z) =_{x,y,z} f(x, g(y, z)) \qquad \text{than} \qquad h = f \circ \langle \pi_1,\ g \circ \langle \pi_2, \pi_3 \rangle \rangle.$$

We will use this notation in the remainder of the thesis.

The way to make this precise is to develop an *internal language* for our ambient category $\mathcal{C}$, a process which is explained in great detail in sections D1.1 - D1.3 of [John 02], particularly on p.837. In this section, we will briefly review this process, partly because the language we'll use in this thesis (which we take from [Roma 89]) is even weaker than what is done in [John 02], and partly because we also introduce some shorthands for dealing with equalizers and pullbacks.

We should note that, in addition to what is presented in this section, we introduce internal language notation for arithmetic and list-arithmetic operations in sections 3.2, 3.3, and 4.1, and we use infix notation when appropriate.

Setting up the language

First, the internal language has a collection of types, which are the objects of $\mathcal{C}$. It has function symbols of the form $f : A_1 \times ... \times A_n \to B$, which come from the arrows of $\mathcal{C}$. Arrows with domain $\mathbb{1}$ are treated as constant symbols. We inductively build *typed terms* using variables and these function symbols in the usual way.

Next, a *context* is a list of variables and their types, such as $C = (x : X, y : Y, z : Z)$, and we write $[C]$ for $X \times Y \times Z$. A context is *valid* for a term t if all the variables in t occur in C, and the *interpretation* of a term $t : B$ in a valid context C is an arrow $[t]_C : [C] \to B$. This interpretation is defined inductively: for a variable x_i in a context $C = (x_1 : X_1, ..., x_n : X_n)$, $[x_i]_C$ is the projection map $\pi_i : [C] \to X_i$, and for a term $f(t_1, ..., t_n)$, its interpretation is $f \circ \langle [t_1], ..., [t_n] \rangle$.

Finally, if t_1, t_2 are terms in a context C, we write $t_1 =_C t_2$ to mean that $[t_1]_C = [t_2]_C$ (as arrows in $\mathcal{C}$). Since this is enough to make sense of the example above with $h(x, y, z)$, this completes the formal setup of the language (note that we don't use relations). Note that the types of the variables in C are omitted if clear from context.

Below, we state some basic facts about this language (the only non-trivial rule is the second one, which requires lemma D1.2.4 in [John 02]).

- The equality $=_C$ is an equivalence relation.
- The equality $=_C$ is preserved by substitution: if $a =_C b$, then $a[\vec{t}/\vec{y}] =_C b[\vec{t}/\vec{y}]$, and if $t_i =_C s_i$ for each i, then $a[\vec{t}/\vec{y}] = a[\vec{s}/\vec{y}]$.
- Given arrows $f, g : X_1 \times ... \times X_n \to Y$ and the context $\vec{x} = (x_1 : X_1, ..., x_n : X_n)$, we have $f = g$ if and only if $f(x_1, ..., x_n) =_{\vec{x}} g(x_1, ..., x_n)$.
- For any appropriate arrows $f, (g_i)_{i=1}^m, (h_j)_{j=1}^n$, terms $(t_i)_{i=1}^m$, and context C, we have

$$f\big(g_1(t_1, ..., t_n), ..., g_m(t_1, ..., t_n)\big) =_C (f \circ \langle g_1, ..., g_m \rangle)(t_1, ..., t_n)$$

  and

$$f(h_1(t_1), ..., h_n(t_n)) =_C (f \circ (h_1 \times ... \times h_n))(t_1, ..., t_n).$$

  In particular, $f(g(t)) =_C (f \circ g)(t)$.

Additional notation

Terms of product types

Given two terms $a : A$, $b : B$ in a context C, we can let (a, b) be shorthand for the term $\mathrm{Id}_{A \times B}(a, b)$ (and similarly for products of any length). Since $(a, b) : A \times B$, this notation lets us mimic products of types without having to build them into the language. Remark that $[(a, b)]_C = \langle [a]_C, [b]_C \rangle$, as we'd expect. In particular, if we have an arrow $g : A \times B \to X$, then $g((a, b)) =_C g(a, b)$, and so we will not bother to distinguish between these terms.

Equalizers

Suppose we have a context $D = (y_1 : Y_1, ..., y_n : Y_n)$ and terms $s, t : Z$ in the context D. Then we can form the following equalizer diagram.

$$[D'] = E \overset{i}{\hookrightarrow} [D] \underset{[t]_D}{\overset{[s]_D}{\rightrightarrows}} Z$$

We will write $\{y_1 : Y_1, ..., y_n : Y_n \mid s = t\}$ as notation for the object E, and we will write $(y_1 : Y_1, ..., y_n : Y_n \mid s = t)$ or $(D \mid s = t)$ as notation for the context $D' = (e : E)$. This makes it easier to refer to equalizers without using new variable names; for example, the context for a pullback of $f : U \to W$, $g : V \to W$ is $(u : U,\ v : V \mid f(u) = g(v))$.

We want to be talk about "terms of type E" and "terms of type $[D]$ which equalize s and t" interchangeably. This will require us to abuse notation in the following ways. First, if we have a term $e : E$, then we can also think of it as the term $i(e) : [D]$. So, we may write $e : [D]$ to refer to this term.

Second, if we have terms $u_1 : Y_1, ..., u_n : Y_n$ in a context $C = (\vec{x} : \vec{X})$, then the term $(u_1, ..., u_n)$ is formally of type $[D]$. However, if $s[\vec{u}/\vec{y}] =_C t[\vec{u}/\vec{y}]$, we can say $(u_1, ..., u_n)$ is of type E by considering it as shorthand for $\psi(\vec{x}) : E$, where ψ comes from the universal property of the equalizer (see the diagram below).

$$\begin{array}{ccccc} [D'] = E & \overset{i}{\hookrightarrow} & [D] & \underset{[t]_D}{\overset{[s]_D}{\rightrightarrows}} & Z \\ {\scriptstyle\psi}\uparrow & \nearrow{\scriptstyle \langle [u_1]_C, ..., [u_n]_C \rangle} & & & \\ [C] & & & & \end{array}$$

Remark that $[(u_1, ..., u_n)]_C = \psi$.

It is important to note that these two ways of abusing notation are compatible with each other. On one hand, if $(u_1, ..., u_n) : [D]$ equalizes s and t, then $i(u_1, ..., u_n) =_C (u_1, ..., u_n)$ as terms of type $[D]$; on the other hand, if $e : E$, then $i(e) =_C e$ as terms of type E (these are easy to check).

Finally, we remark that the latter notation behaves well with respect to substitution. The proof is straightforward, though it does require lemma D1.2.4 from [John 02].

Remark 2.2.1. With the above setup, let $w_1 : X_1, ..., w_m : X_m$ be terms in another context B. Then

$$(u_1, ..., u_n)[\vec{w}/\vec{x}] =_B (u_1[\vec{w}/\vec{x}], ..., u_n[\vec{w}/\vec{x}]) \quad \text{(as terms of type } E)$$

Substitution for equalizers

Let $D = (y_1 : Y_1, ..., y_n : Y_n)$, let $s, t : Z$ be terms in the context D, and let $D' = (y_1 : Y_1, ..., y_n : Y_n \mid s = t) = (e : E)$, as in the previous section. Remark that terms in the context

D can be also considered to be terms in the context D', simply by "pulling back" along the inclusion map $i : [D'] \to [D]$. Notably, if we view s, t as terms in D', then we have $s =_{D'} t$.

Now, consider a term like $g(y_1, ..., y_n)$ in the context D. If we consider this term in the context D', can we still substitute terms for the variables y_i? The y_i are no longer variables in the context D', so this requires special consideration.

Formally, let $r : R$ be a term in the context D', and let $u_1 : Y_1, ..., u_n : Y_n$ be terms in a context C. If we assume that $s[\vec{u}/\vec{y}] =_C t[\vec{u}/\vec{y}]$, then we can say that

$$r[\vec{u}/\vec{y}] \text{ is shorthand for } r[e/\psi(\vec{x})],$$

where ψ once again comes from the universal property (see the previous subsection). This definition ensures that our substitution notation enjoys many of the same properties of regular substitution, and we can check that it behaves the way we want with the following remark (the proof is straightforward).

Remark 2.2.2. With the above setup, $y_k[\vec{u}/\vec{y}] =_C u_k$.

Example usage

To show how this internal language may be used, we use it to express some universal properties.

Products

In a category $\mathcal{C}$, let A, B, P be objects, and let $\pi_1 : P \to A$, $\pi_2 : P \to B$ be arrows. Then (P, π_1, π_2) is a product of A and B if and only if the following condition holds.

> For any context C and any terms $a : A$, $b : B$ in the context C, there exists a unique term $p : P$ in the context C such that $\pi_1(p) =_C a$ and $\pi_2(p) =_C b$.

Remark 2.2.3. In this example, when we say "there exists a unique term $p : P$ such that...", what we mean is uniqueness up to equality. This means that for any two terms $p_1, p_2 : P$ (in the context C) satisfying the condition, we have $p_1 =_C p_2$. Going forward, when talking about uniqueness of terms in the internal language, we always mean uniqueness up to equality.

Equalizers

In a category $\mathcal{C}$, let A, B, E be objects, and let $f, g : A \to B$ and $i : E \to A$ be arrows such that $f \circ i = g \circ i$. Then (E, i) is an equalizer of f, g if and only if the following condition holds.

> For any context C and any term $a : A$ in C such that $f(a) =_C g(a)$, there exists a unique term $e : E$ in C such that $i(e) =_C a$.

Pullbacks

In a category $\mathcal{C}$, suppose the following square commutes.

$$\begin{array}{ccc} P & \xrightarrow{\pi_2} & B \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle g} \\ A & \xrightarrow[f]{} & C \end{array}$$

Then (P, π_1, π_2) is a pullback of f, g if and only if the following condition holds.

> For any context C and any terms $a : A$ and $b : B$ in C such that $f(a) =_C g(b)$, there exists a unique term $p : P$ in C such that $\pi_1(p) =_C a$ and $\pi_2(p) =_C b$.

2.3 Subobjects and relations

In this section, we establish some basic results about subobjects and equivalence relations. These are not new results, but we could not find them stated in the literature in the form we wanted.

Given an object A in a category $\mathcal{C}$, recall that a *subobject* of A is a monomorphism $B \hookrightarrow A$. The term "subobject" may also mean an equivalence class of monomorphisms into A, where $f : B \hookrightarrow A$ and $f' : B' \hookrightarrow A$ are equivalent if there is an isomorphism $g : B \to B'$ such that $f'g = f$. (See, for instance, definition 4.1.1 of [Borc 94a] or section A1.3 of [John 02].)

In a category $\mathcal{C}$ with pullbacks, we say that a subobject $f : B \hookrightarrow A$ is *complemented* if there is another subobject $f' : B' \hookrightarrow A$ such that the diagram

$$B \xhookrightarrow{f} A \xhookleftarrow{f'} B'$$

is a coproduct and the pullback $B \times_A B'$ is an initial object. We say $\mathcal{C}$ is *Boolean* if every subobject is complemented. The usual definition of complemented subobjects (and Boolean categories) is for a coherent category (see, e.g. the text above lemma A1.4.10 in [John 02]), but this definition is clearly equivalent in that case. Remark that, in a lextensive category, we only need to check the coproduct condition, since sums are disjoint in extensive categories (see proposition 2.6 in [Carb 93]).

It turns out that subobjects are complemented if and only if they can be expressed as a certain kind of equalizer. To express this fact, we will write $\{0, 1\}$ for the coproduct $\mathbb{1} + \mathbb{1}$, and the inclusion maps will be denoted $0, 1 : \mathbb{1} \to \{0, 1\}$.

Theorem 2.3.1. Let $\mathcal{C}$ be a lextensive category, and let $Y \in \mathcal{C}$. For any map $d : Y \to \{0, 1\}$, the equalizer $R \hookrightarrow Y$ of d and 0 is a complemented subobject: its complement is the equalizer of d and 1.

Moreover, the map

$$\left\{ \begin{matrix} \text{arrows} \\ d : Y \to \{0,1\} \end{matrix} \right\} \xrightarrow{d \mapsto \mathrm{eq}(d,0)} \left\{ \begin{matrix} \text{complemented} \\ \text{subobjects } R \hookrightarrow Y \end{matrix} \right\}$$

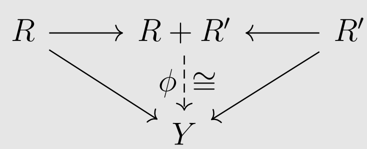
is a bijection (where subobjects are consider up to isomorphism).

Proof. Let $R_0 = \mathrm{eq}(d, 0)$ and $R_1 = \mathrm{eq}(d, 1)$. We start by showing that these are complementing subobjects of Y. To this end, consider the following diagram.

$$\begin{CD} R_0 @>>> Y @<<< R_1 \\ @VVV @VdVV @VVV \\ \mathbb{1} @>0>> \{0,1\} @<1<< \mathbb{1} \end{CD} \tag{2.3.1}$$

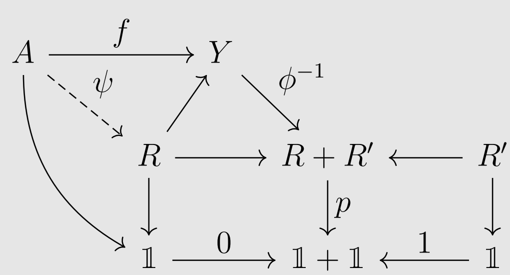
The squares are pullbacks (they correspond to R_0, R_1 being defined as equalizers), so by extensivity, the top row is a coproduct. (Also, in an extensive category, finite sums are disjoint.) Thus, we've shown that $d \mapsto \mathrm{eq}(d, 0)$ is a well-defined function between the above sets.

Now, we show that $d \mapsto \mathrm{eq}(d, 0)$ is surjective. So, let $R \hookrightarrow Y$ be a complemented subobject; that means it has a complement $R' \hookrightarrow Y$ such that the induced map $\phi : R + R' \to Y$ is an isomorphism.



Let $p : R + R' \to \mathbb{1} + \mathbb{1} = \{0, 1\}$ be the natural map, and let $d = p \circ \phi^{-1}$. We claim that R is the equalizer of d and 0 (this will show surjectivity).

Let $f : A \to Y$ be such that $d \circ f = 0$. Consider the following diagram.



Since the top row is a coproduct, extensivity implies the squares must be pullbacks. So, there exists a unique $\psi : A \to R$ such that the diagram commutes, and this gives the equalizer property.

Finally, we show that $d \mapsto \mathrm{eq}(d, 0)$ is injective. Let d, d' be two arrows, and assume $R \hookrightarrow Y$ is both $\mathrm{eq}(d, 0)$ and $\mathrm{eq}(d', 0)$. We will show that $d = p \circ \phi^{-1}$, where p and ϕ are as in the previous part of the proof (we assume $i : R \to Y$ is complemented by $i' : R' \to Y$). This is sufficient to conclude injectivity: we can argue that $d' = p \circ \phi^{-1}$ in the same way, so we get $d = d'$.

To show $d = p \circ \phi^{-1}$, we will show that $d \circ \phi = p$. Since the domain of these maps is $R + R'$, it suffices to show that $d \circ \phi \circ i = p \circ i$ and $d \circ \phi \circ i' = p \circ i'$. Note that $p \circ i = 0$ and $p \circ i' = 1$, so we need to show the following diagram commutes.

$$\begin{array}{ccccc} R & \xrightarrow{\;i\;} & R + R' & \xleftarrow{\;i'\;} & R' \\ & & \downarrow\phi & & \\ & & Y & & \\ & {\scriptstyle 0}\searrow & \downarrow d & \swarrow{\scriptstyle 1} & \\ & & \mathbb{1} + \mathbb{1} & & \end{array}$$

This is easy: on the left, $\phi \circ i$ is the map $R \to Y$, and this map is by definition the equalizer of d and 0, so we get the commutativity. The right side is similar, so we're done. ∎

Remark 2.3.2. In an arithmetic lextensive category, it can be shown that $\{0, 1\} \cong \{x \in N \mid x \leq 1\}$ (see proposition 3.3.9). Therefore, a map $d : Y \to \{0, 1\}$ can be defined by giving a map $d' : Y \to N$ such that $d' \leq 1$. In other words, given $d' : Y \to N$ such that $d' \leq 1$, the equalizer $\mathrm{eq}(d', 0)$ is a complemented subobject of Y.

This theorem gives the following immediate corollary.

Corollary 2.3.3. In a lextensive category where every subobject is complemented, $\{0, 1\}$ is a subobject classifier.

Proof. Let $A \hookrightarrow B$ be a monomorphism, so that A is a subobject of B. Since A is complemented by assumption, theorem 2.3.1 tells us there exists a unique arrow $d : B \to \{0, 1\}$ such that $A = \mathrm{eq}(d, 0)$. But saying that $A = \mathrm{eq}(d, 0)$ is equivalent to saying that

$$\begin{array}{ccc} A & \longrightarrow & \mathbb{1} \\ \big\downarrow & & \big\downarrow{\scriptstyle 0} \\ B & \xrightarrow{\;d\;} & \{0, 1\} \end{array}$$

is a pullback square. Thus, $\{0, 1\}$ is a subobject classifier. ∎

Next, we recall the definition of relations and equivalence relations. These can be defined for arbitrary categories, but since we will only deal with categories with finite limits, we restrict our attention to this case, which makes the definitions simpler. Notably, the definition we give below comes from proposition 2.5.5 of [Borc 94b].

Definition. A *relation* on an object X is a subobject (i.e. a monomorphism) $r = \langle r_1, r_2 \rangle : R \hookrightarrow X \times X$. Moreover, we say the relation is...

- ...*reflexive* if the diagonal $\Delta_X : X \to X \times X$ factors through r.
- ...*symmetric* if $\sigma \circ r : R \to X \times X$ factors through r, where $\sigma = \langle \pi_1, \pi_2 \rangle : X \times X \to X \times X$ swaps the entries.
- ...*transitive* if, when we consider the pullback square

$$
\begin{array}{ccccc}
R \times_X R & \xrightarrow{\pi_2} & R & \xrightarrow{r_2} & X \\
\downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle r_1} & & \\
R & \xrightarrow{r_2} & X & & \\
\downarrow{\scriptstyle r_1} & & & & \\
X & & & &
\end{array},
$$

  the arrow $\langle r_1\pi_1, r_2\pi_2 \rangle : R \times_X R \to X \times X$ factors through r.
- ...*an equivalence relation* if it is reflexive, symmetric, and transitive.

If we have a relation $R \hookrightarrow X \times X$ which is a complemented subobject, then theorem 2.3.1 tells us it is associated to a map $d : X \times X \to \{0, 1\}$. We would like to express the fact that R is an equivalence relation by rephrasing the conditions in terms of the map d. This is what we do in the following theorem, but first, we need a lemma.

Lemma 2.3.4. In a lextensive category, consider a relation $R \hookrightarrow X \times X$ which is complemented. If R is symmetric, then its complement is symmetric too.

Proof. A proof of this result can be found on the Mathematics Stack Exchange website [favi]. We reproduce the proof for completeness.

Let $r' : R' \to X \times X$ be the complement of $r : R \to X \times X$. If R is symmetric, then there is a map $\phi : R \to R$ such that $r\phi = \sigma r$, where $\sigma : X \times X \to X \times X$ is the arrow swapping the components. We must show there is an arrow $\psi : R' \to R'$ such that $r'\psi = \sigma r'$. To this end, consider the following diagram.

$$
\begin{array}{ccccc}
\mathbb{O} & \longrightarrow & R' & \xleftarrow{\pi_1} & R' \times_{X \times X} R' \\
\downarrow & & \downarrow{\scriptstyle r'} & & \downarrow{\scriptstyle \pi_2} \\
R & \xrightarrow{r} & X \times X & & \\
{\scriptstyle \phi}\downarrow & & \downarrow{\scriptstyle \sigma} & & \\
R & \xrightarrow[r]{} & X \times X & \xleftarrow[r']{} & R'
\end{array}
$$

It's clear that σ is an involution, and this implies ϕ is an involution too (notably because

r is a monomorphism). Therefore, the bottom left square is a pullback. The top left square is also a pullback because sums are disjoint in an extensive category ($\mathbb{O}$ is the initial object). Finally, the right square is a pullback by definition.

Since we're in an extensive category, all this implies that the top row is a coproduct; therefore, the arrow π_1 is invertible. We can therefore set $\psi = \pi_2 \circ (\pi_1)^{-1}$, and the right square gives the desired equation. ∎

Theorem 2.3.5. In a lextensive category, let $i : R \hookrightarrow X \times X$ be a complemented relation, and let $d : X \times X \to \{0, 1\}$ be the associated map by theorem 2.3.1. Then:

- R is reflexive if and only if $d(x, x) =_x 0$;
- R is symmetric if and only if $d(x, y) =_{x,y} d(y, x)$;
- R is transitive if and only if for any context C and any terms $x, y, z : X$ in C, if $d(x, y) =_C 0$ and $d(y, z) =_C 0$, then $d(x, z) =_C 0$.

Proof. We know R is reflexive if and only if the diagonal $\Delta_X : X \to X \times X$ factors through R; that is, there is a map $f : X \to R$ such that $i \circ f = \langle \mathrm{Id}, \mathrm{Id}\rangle$. In that case, clearly $d(x, x) = d(i(f(x))) = 0$ (because $i = \mathrm{eq}(d, 0)$). On the other hand, if $d(x, x) = 0$, then we just use the equalizer property:

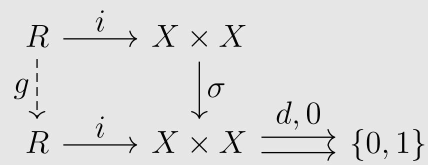
$$\begin{array}{ccccc} R & \overset{i}{\hookrightarrow} & X \times X & \overset{d,0}{\rightrightarrows} & \{0,1\} \\ f \uparrow & \nearrow_{\langle \mathrm{Id}, \mathrm{Id}\rangle} & & & \\ X & & & & \end{array}$$

Next, by unravelling the definition, we find that R is symmetric if there is a map $g : R \to R$ such that the following square commutes (where $\sigma = \langle \pi_2, \pi_1\rangle$).

$$\begin{array}{ccc} R & \xrightarrow{i} & X \times X \\ g \downarrow & & \downarrow \sigma \\ R & \xrightarrow{i} & X \times X \end{array}$$

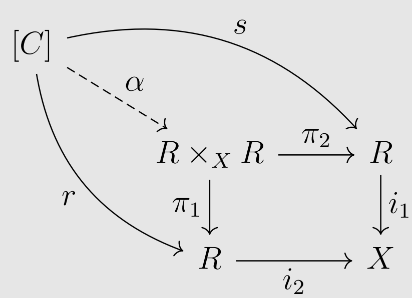
Suppose this is true. We must show $d = d \circ \sigma$; we do this by decomposing $X \times X = R + R'$, so that it suffices to show $d \circ i = d \circ \sigma \circ i$ and $d \circ i' = d \circ \sigma \circ i'$. For the first of these, we have $d \circ i = 0$ (because $i = \mathrm{eq}(d, 0)$) and $d \circ \sigma \circ i = d \circ i \circ g = 0 \circ g = 0$. For the second, we similarly get $d \circ i' = 1$ (because $R' = \mathrm{eq}(d, 1)$) and $d \circ \sigma \circ i' = 1$ (because R' is also symmetric by lemma 2.3.4).

On the other hand, suppose $d(x, y) = d(y, x)$, i.e. $d = d \circ \sigma$. Then $d \circ \sigma \circ i = d \circ i = 0$, so by the equalizer, we get the desired g.



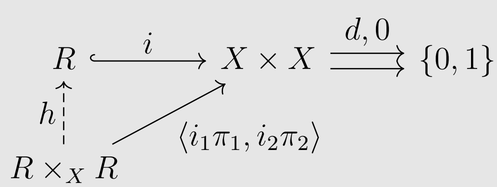
Finally, we tackle transitivity. Write $i = \langle i_1, i_2 \rangle$, and let $R \times_X R$ be the pullback defined in the definition of a transitive relation. By definition, $i_2\pi_1 = i_1\pi_2$, and R is transitive if $\langle i_1\pi_1, i_2\pi_2 \rangle$ factors through R.

Suppose R is transitive, so $\langle r_1\pi_1, r_2\pi_2 \rangle = i \circ h$ for some h. Then, let $x, y, z : X$ be terms in a context C such that $d(x, y) = 0$ and $d(y, z) = 0$. Since the terms $(x, y), (y, z) : X \times X$ equalize d and 0, the equalizer property tells us that there are terms $r, s : R$ such that $i(r) = (x, y)$, $i(s) = (y, z)$. But then we can use r, s with the pullback square $R \times_X R$ to get a term $\alpha : R \times_X R$ such that $\pi_1\alpha = r$ and $\pi_2\alpha = s$. (Notably, because $i_2(r) = y = i_1(s)$.)



Then $x = i_1(r) = i_1\pi_1(\alpha)$ and $z = i_2(s) = i_2\pi_2(\alpha)$, so $(x, z) = \langle i_1\pi_1, i_2\pi_2 \rangle(\alpha) = i(h(\alpha))$. Thus $d(x, z) = d(i(h(\alpha))) = 0$ (because i equalizes $d, 0$), as desired.

On the other hand, suppose that for any terms x, y, z, if $d(x, y) = 0$ and $d(y, z) = 0$, then $d(x, z) = 0$. To get h, it suffices to show that $d \circ \langle i_1\pi_1, i_2\pi_2 \rangle = 0$ (see the diagram below).



So, let $\alpha : R \times_X R$. Note that $d(i_1\pi_1\alpha, i_2\pi_1\alpha) = d(i(\pi_1\alpha)) = 0$ and $d(i_1\pi_2\alpha, i_2\pi_2\alpha) = d(i(\pi_2\alpha)) = 0$. Since $i_2\pi_1\alpha = i_1\pi_2\alpha$ (by definition of the pullback), transitivity of d lets us conclude that $d(i_1\pi_1\alpha, i_2\pi_2\alpha) = 0$. Thus, we indeed have $d \circ \langle i_1\pi_1, i_2\pi_2 \rangle = 0$. ∎

Part II

Locoi and the polynomial list functor

Chapter 3

Natural numbers objects

Contents

In this chapter, we establish some important theory and results about natural numbers objects (NNOs). Section 3.1 revisits the definition using the internal language of section 2.2, and sections 3.2, 3.3, 3.4 set up essential concepts that we use throughout the thesis: order and inequalities, definitions and proofs by cases, and the collection of finite cardinals. Sections 3.5, 3.6, and 3.7 present a hodgepodge of tools that will be used in various places throughout the thesis.

3.1 NNOs with the internal language

We start by expressing the definition of natural numbers objects using the internal language.

Definition. Let $\mathcal{C}$ be a Cartesian category. A *(parametrized) natural numbers object* (NNO) is a triple $(N, 0, s)$ consisting of an object N and arrows $0 : \mathbb{1} \to N$, $s : N \to N$ which satisfy the following property. For any objects A, B and arrows $g : A \to B$, $h : A \times N \times B \to B$, there exists a unique arrow $f : A \times N \to B$ such that $f(a, 0) =_a g(a)$ and $f(a, sn) =_{a,n} h(a, n, f(a, n))$.

In fact, we can formulate the universal property in a way that uses the internal language even more. This formulation, given below, avoids the need to think of the terms f, f_1, f_2, g, h as coming from arrows.

Remark 3.1.1. Let N be a natural numbers object. Let A, B be objects, let g be a term in the context $(a : A)$, and let h be a term in the context $(a : A, n : N, b : B)$. Then there exists a term f in the context $(a : A, n : N)$ such that $f[0/n] =_a g$ and $f[s(n)/n] =_{a,n} h[f/b]$. Moreover, if f_1, f_2 are two such terms, then $f_1 =_{a,n} f_2$.

Following the convention of [Roma 89], the term "natural numbers object" will always refer to *parametrized* natural numbers objects; we will not discuss unparametrized NNOs (where we omit the parameter A in the definition).

The NNO property lets us define the operations of addition $+ : N \times N \to N$, multiplication $\bullet : N \times N \to N$, predecessor $P : N \to N$, and (truncated) subtraction $\dot{-} : N \times N \to N$ in the usual inductive way.

$$\begin{array}{cccc} x + 0 =_x x & x \bullet 0 =_x 0 & P(0) =_\varnothing 0 & x \dot{-} 0 =_x x \\ x + sy =_{x,y} s(x+y) & x \bullet sy =_{x,y} (x \bullet y) + x & P(sy) =_y y & x \dot{-} sy =_{x,y} P(x \dot{-} y) \end{array}$$

The operations $+$ and $\bullet$, along with 0 and $1 = s0$, make N into an (internal) commutative semiring. We will prove some facts about these operations in this chapter, but we will also frequently use results from [Roma 89] (themselves inspired by [Good 57]).

For instance, we will borrow the function $|x, y| =_{x,y} (x \dot{-} y) + (y \dot{-} x)$ from [Roma 89], which has the property that for $f, g : X \to N$, $f = g$ if and only if $|f, g| = 0_X$ (corollary 1.4 in [Roma 89]). Equivalently, given any two terms $x, y : N$ in a context C, we have $x =_C y$ if and only if $|x, y| =_C 0$. We will also use Román's min and max functions, defined as $\min(x, y) =_{x,y} x \dot{-} (x \dot{-} y) =_{x,y} y \dot{-} (y \dot{-} x)$ and $\max(x, y) =_{x,y} x + (y \dot{-} x) =_{x,y} y + (x \dot{-} y)$ (see his proposition 1.3).

We finish this section by proving the following preliminary result.

Proposition 3.1.2. We have the following equalities.

1. $a \dot{-} (b + c) =_{a,b,c} (a \dot{-} b) \dot{-} c$
2. $|\max(a, b), b| =_{a,b} a \dot{-} b$
3. $(a \dot{-} b) \bullet (b \dot{-} a) =_{a,b} 0$

Proof. For part 1, we go by induction on c. In the base case $c = 0$, both sides clearly equal $a \dot{-} b$. For the induction step, we compute

$$\begin{aligned} a \dot{-} (b + s(c)) =_{a,b,c} a \dot{-} s(b + c) &=_{a,b,c} P(a \dot{-} (b + c)), \\ (a \dot{-} b) \dot{-} s(c) &=_{a,b,c} P((a \dot{-} b) \dot{-} c). \end{aligned}$$

So, the equality follows by induction.

For part 2, since $|\max(a, b), b| =_{a,b} (\max(a, b) \dot{-} b) + (b \dot{-} \max(a, b))$, it suffices to show $\max(a, b) \dot{-} b =_{a,b} a \dot{-} b$ and $b \dot{-} \max(a, b) =_{a,b} 0$. The latter equality is an easy computation with part 1:

$$b \dot{-} \max(a,b) =_{a,b} b \dot{-} (a + (b \dot{-} a)) =_{a,b} (b \dot{-} a) \dot{-} (b \dot{-} a) =_{a,b} 0.$$

For the former equality, we use corollary 2.2(e) from Román's paper [Roma 89]:

$$\begin{aligned} \max(a,b) \dot{-} b =_{a,b} (a + (b \dot{-} a)) \dot{-} b &=_{a,b} (a \dot{-} b) + ((b \dot{-} a) \dot{-} (b \dot{-} a)) \\ &=_{a,b} (a \dot{-} b) + 0 =_{a,b} a \dot{-} b. \end{aligned}$$

For part 3, we will use the induction principle provided by proposition 3.1(c) in Román's paper [Roma 89]. This principle states that, given $f : N^2 \to N$, if $f(a + sb, b) =_{a,b} 0$, then $(a \dot{-} b) \bullet f(a,b) =_{a,b} 0$. This conclusion is what we want with $f(a,b) =_{a,b} b \dot{-} a$, so we compute

$$f(a + sb, b) =_{a,b} b \dot{-} (a + sb) =_{a,b} (b \dot{-} sb) \dot{-} a =_{a,b} 0 \dot{-} a =_{a,b} 0.$$

This is all we needed to show. ∎

3.2 Order in NNOs

In this section, we develop various facts about the order relations on a natural numbers object.

First, the definition: given terms $m, n : N$ in a context C, we write $m \leq_C n$ for $m \dot{-} n =_C 0$, and we write $m <_C n$ for $s(m) \dot{-} n =_C 0$.

When working in a category with finite limits, we can use inequalities in our notation for defining objects as equalizers. For instance, we will write $\{m, n : N \mid m < n\}$ for $\{m, n : N \mid sm \dot{-} n = 0\}$, and similarly for $\{m, n : N \mid m \leq n\}$.

We will now prove some basic facts about inequality.

Proposition 3.2.1. Let $a, b : N$ be terms in a context C. Then the following are equivalent:

1. $a \leq_C b$
2. $\max(a,b) =_C b$
3. $\min(a,b) =_C a$
4. There exists a term $x : N$ in the context C such that $a + x =_C b$
5. There exists a term $x : N$ in the context C such that $a =_C b \dot{-} x$.

Proof. For this proof, we show that $1 \Rightarrow 2 \Rightarrow 4 \Rightarrow 1$ and $1 \Rightarrow 3 \Rightarrow 5 \Rightarrow 1$. All the equalities here will take place in the context C, so we will omit the subscripts.

($1 \Rightarrow 2$). If $a \leq b$, then $a \dot{-} b = 0$. Using proposition 3.1.2(2), we find that $|\max(a,b), b| = a \dot{-} b = 0$, which implies $\max(a,b) = b$, as desired.

($2 \Rightarrow 4$). Suppose $\max(a,b) = b$, and set $x = b \dot{-} a$. Then $a + x = \max(a,b) = b$.

($4 \Rightarrow 1$). Assume $a + x = b$. Then, using 3.1.2(1), $a \dot{-} b = a \dot{-} (a + x) = (a \dot{-} a) \dot{-} x = 0 \dot{-} x = 0$, which is equivalent to $a \leq b$.

($1 \Rightarrow 3$). Suppose $a \leq b$, i.e. $a \dot{-} b = 0$. By definition, $\min(a,b) = a \dot{-} (a \dot{-} b) = a \dot{-} 0 = a$.

($3 \Rightarrow 5$). Suppose $\min(a,b) = a$, and set $x = b \dot{-} a$. Then $b \dot{-} x = \min(b,a) = \min(a,b) = a$.

($5 \Rightarrow 1$). Assume $a = b \dot{-} x$. By Román's proposition 1.3(f), $a \dot{-} b = (b \dot{-} x) \dot{-} b = (b \dot{-} b) \dot{-} x = 0 \dot{-} x = 0$, which is equivalent to $a \leq b$. ∎

Proposition 3.2.2. For any context C, the relation $\leq_C$ is a partial order on terms of type N in the context C. That is:

1. $x \leq_C x$;
2. If $x \leq_C y$ and $y \leq_C x$, then $x =_C y$;
3. If $x \leq_C y$ and $y \leq_C z$, then $x \leq_C z$.

Moreover, if $x \leq_C y$, then

- $x + z \leq_C y + z$,
- $x \bullet z \leq_C y \bullet z$,
- $x \dot{-} z \leq_C y \dot{-} z$, and
- $z \dot{-} y \leq_C z \dot{-} x$.

Proof. We will omit the C subscript for clarity. For 1, see Román, proposition 1.3(b). For 2, note that $x \leq y$ and $y \leq x$ imply $|x,y| = (x \dot{-} y) + (y \dot{-} x) = 0 + 0 = 0$. By Román, corollary 1.4, we get $x = y$.

For 3, we use proposition 3.2.1 to conclude that there exist $a, b : N$ such that $x + a = y$ and $y + b = z$. Then $x + (a + b) = (x + a) + b = y + b = z$, so $x \leq z$, as desired.

For the last part, assume that $x \leq y$, so $y = x + k$ for some k (by proposition 3.2.1). For the addition inequality, $y + z = (x + k) + z = (x + z) + k$, so $x + z \leq y + z$. For the multiplication, $y \bullet z = (x + k) \bullet z = x \bullet z + k \bullet z$, so $x \bullet z \leq y \bullet z$. For the first subtraction, we apply Román's corollary 2.2(e) to find

$$y \dot{-} z = (x + k) \dot{-} z = (x \dot{-} z) + (k \dot{-} (z \dot{-} x)),$$

so $x \dot{-} z \leq y \dot{-} z$. For the second subtraction, we just compute

$$\begin{aligned}(z \dot{-} y) \dot{-} (z \dot{-} x) &= (z \dot{-} (x + k)) \dot{-} (z \dot{-} x) \\ &= ((z \dot{-} x) \dot{-} k) \dot{-} (z \dot{-} x) \\ &= ((z \dot{-} x) \dot{-} (z \dot{-} x)) \dot{-} k = 0.\end{aligned}$$

Here, we just used proposition 3.1.2 and commutativity of addition. ∎

Now that we've established the basic facts for inequalities, we present some results that we will use a few times throughout the paper.

Proposition 3.2.3. Let $\mathcal{C}$ be a Cartesian category with NNO, and let $n : N$ be a term in a context C. If $n >_C 0$, then $n =_C s(P(n))$.

Proof. If $n >_C 0$, then $n \geq_C s(0) = 1$, so $n =_C m + 1 =_C s(m)$ for some term $m : N$ (by proposition 3.2.1). Thus $s(P(n)) =_C s(P(s(m))) =_C s(m) =_C n$ (since $P(s(m)) =_C m$ by definition of P). ∎

Proposition 3.2.4. Let $a, b : N$ be terms in a context C. Then $a <_C b$ if and only if $b \dot{-} a >_C 0$.

Proof. We omit the C subscript for brevity. Remark that the proposition is the same as saying $b \geq a + 1$ if and only if $b \dot{-} a \geq 1$.

First, suppose $b \geq a + 1$. Then $b \dot{-} a \geq (a + 1) \dot{-} a = 1$ by proposition 3.2.2 and Román's proposition 1.3(c).

On the other hand, suppose $b \dot{-} a \geq 1$. Then, using propositions 3.1.2(3) and 3.2.2,

$$a \dot{-} b = (a \dot{-} b) \bullet 1 \leq (a \dot{-} b) \bullet (b \dot{-} a) = 0,$$

so $a \dot{-} b = 0$. Thus,

$$b = b + (a \dot{-} b) = a + (b \dot{-} a) \geq a + 1.$$

(The second equality is the definition of max; see Román's proposition 1.3(e).) ∎

Proposition 3.2.5. We have $sx \dot{-} (x \dot{-} y) =_{x,y} s(x \dot{-} (x \dot{-} y))$.

Proof. By Román's corollary 2.2(e),

$$\begin{aligned} sx \dot{-} (x \dot{-} y) &= (x+1) \dot{-} (x \dot{-} y) \\ &= (x \dot{-} (x \dot{-} y)) + (1 \dot{-} ((x \dot{-} y) \dot{-} x)) \\ &= (x \dot{-} (x \dot{-} y)) + (1 \dot{-} 0) \\ &= s(x \dot{-} (x \dot{-} y)). \end{aligned}$$

Note that $(x \dot{-} y) \dot{-} x = (x \dot{-} x) \dot{-} y = 0 \dot{-} y = 0$ by Román's proposition 1.3(e). ∎

Proposition 3.2.6. Let $a : N$ be a term in a context C. If $\min(a, 1) = 0$, then $a = 0$.

Proof. By definition, $\min(a, 1) = 1 \dot{-} (1 \dot{-} a)$, so $\min(a, 1) = 0$ implies $1 \leq 1 \dot{-} a$. This inequality can be written $0 < 1 \dot{-} a$; by proposition 3.2.4, this implies $a < 1$, i.e. $sa \leq s0$. Thus $a \leq 0$, and so $a = 0$, as desired. ∎

3.3 Arithmetic lextensive categories

In this section, we'll show how working in an extensive category provides additional tools for dealing with natural numbers objects. This is all based around the following simple fact (see, e.g., [John 02, lemma A2.5.5]).

Fact 3.3.1. In a Cartesian category with a parametrized NNO, the diagram $\mathbb{1} \xrightarrow{0} N \xleftarrow{s} N$ is a coproduct.

Standard definition by cases

By combining fact 3.3.1 with the characterization of extensivity of fact 2.1.2, we obtain the following result.

Proposition 3.3.2. Let $t : N$ be a term in a context $C = (x_1 : X_1, ..., x_n : X_n)$, and let C_0, $C_{>0}$ be the contexts $C_0 = (C \mid t = 0)$ and $C_{>0} = (C \mid t > 0)$.

Then, in an arithmetic lextensive category, $[C_0] \hookrightarrow [C] \hookleftarrow [C_{>0}]$ is a coproduct.

Proof. Write $i_0, i_{>0}$ for the inclusion maps of $[C_0]$, $[C_{>0}]$ into $[C]$, respectively, and consider the following diagram.

$$\begin{array}{ccccc} [C_0] & \xrightarrow{i_0} & [C] & \xleftarrow{i_{>0}} & [C_{>0}] \\ \downarrow & & \downarrow [t]_C & & \downarrow P \circ [t]_C \circ i_{>0} \\ \mathbb{1} & \xrightarrow{0} & N & \xleftarrow{s} & N \end{array}$$

We claim that this diagram commutes and that the squares are pullbacks. If this is true, then extensivity tells us that the top row is a coproduct, as desired (see fact 2.1.2).

First, we check that the squares commute. The left square states that, in the context C_0, we have $t =_{C_0} 0$, which is true by definition. The right square states that, in the context $C_{>0}$, we have $t =_{C_{>0}} s(P(t))$. We know $t > 0$ in this context, so the equality follows by proposition 3.2.3.

Second, we claim that the left square is a pullback because of how $[C_0]$ is defined as an equalizer, and the right square is a pullback because of how $[C_{>0}]$ is defined as an equalizer. This is mostly a routine check, so we omit the details. We just note that for the latter case, if $f : B \to [C]$ and $g : B \to N$ are such that $[t]_C \circ f = s \circ g$, then $[t]_C \circ f > 0$ and $g = P \circ [t]_C \circ f$. ∎

The conclusion of this proposition can be reformulated using the internal language so that it is easier to use.

Remark 3.3.3. Assume we are in an arithmetic lextensive category. Let $t, C, C_0, C_{>0}$ be as in proposition 3.3.2, let $a_0 : Y$ be a term in the context C_0, and let $a_{>0} : Y$ be a term in the context $C_{>0}$. Then there exists a unique arrow $h : [C] \to Y$ such that $h(x_1, ..., x_n) =_{C_0} a_0$ and $h(x_1, ..., x_n) =_{C_{>0}} a_{>0}$. Moreover, for any terms $p, q : Y$ in the context C, if $p =_{C_0} q$ and $p =_{C_{>0}} q$, then $p =_C q$.

We use the following special notation for the term $h(x_1, ..., x_n)$ of the above remark:

$$h(x_1, ..., x_n) =_C \begin{cases} a_0 & \text{if } t = 0 \\ a_{>0} & \text{else} \end{cases}.$$

In other words, working in an arithmetic lextensive category allows us to define a function "by cases", conditional on a term of type N. The second part of the above remark tells us that we can also prove two terms are equal "by cases".

For legibility, we will introduce some notation for when we want to consider more than one case at a time. We'll write

$$\left\{\begin{matrix} x & \text{if } m = 0 \\ y & \text{else if } n = 0 \\ z & \text{else} \end{matrix}\right\} = \left\{\begin{matrix} x & \text{if } m = 0 \\ \left\{\begin{matrix} y & \text{if } n = 0 \\ z & \text{else} \end{matrix}\right\} & \text{else} \end{matrix}\right\}.$$

Finally, we have the following result, that lets us convert between two different "conditions".

Proposition 3.3.4. Let $f, g : X \to B$ and $h_1, h_2 : X \to N$. If we're working in an arithmetic lextensive category, then the statement

> "for any term $x : X$ in any context C, if $h_1(x) =_C 0$ then $h_2(x) =_C 0$, and if $h_1(x) >_C 0$ then $h_2(x) >_C 0$"

implies that

$$\begin{Bmatrix} f(x) & \text{if } h_1(x) = 0 \\ g(x) & \text{else} \end{Bmatrix} = \begin{Bmatrix} f(x) & \text{if } h_2(x) = 0 \\ g(x) & \text{else} \end{Bmatrix}.$$

Proof. It suffices to show that the right hand side satisfies the defining property of the left hand side. So, let $X_0 = \{x : X \mid h_1(x) = 0\}$ and $X_{>0} = \{x : X \mid h_1(x) > 0\}$. Given $x : X_0$, we have $h_1(x) =_x 0$, so $h_2(x) =_x 0$ too by assumption. Thus, plugging x into the right hand side yields $f(x)$. Similarly, plugging some $x : X_{>0}$ into the right hand side yields $g(x)$. These two equalities are what uniquely define the left hand side, so we get equality. ∎

Definition by cases without extensivity

It is worth noting that certain functions can be defined by cases without requiring extensivity. In a category with finite limits and NNO, we can define a function $ITE_Y : Y \times Y \times N \to B$ (short for "if-then-else") by induction as follows:

$$ITE_Y(a, b, 0) =_{a,b} a, \qquad ITE_Y(a, b, sk) =_{a,b,k} b.$$

Then, if we have terms $t : N$ and $a_0, a_{>0} : Y$ in a context $C = (x_1, ..., x_n)$, the function $h(x_1, ..., x_n) = ITE_Y(a_0, a_{>0}, t)$ satisfies $h(x_1, ..., x_n) =_{C_0} a_0$ and $h(x_1, ..., x_n) =_{C_{>0}} a_{>0}$. Thus, if it turns out that we are in an extensive category, we have

$$ITE_Y(a_0, a_{>0}, t) =_C \begin{cases} a_0 & \text{if } t = 0 \\ a_{>0} & \text{else} \end{cases}.$$

So, we will use this "by cases" notation for $ITE_Y(a, b, k)$ even when we don't assume the ambient category is extensive.

There is, of course, a drawback to working without extensivity: the terms $a_0, a_{>0}$ must both exist in the context C, not in the contexts C_0 and $C_{>0}$ (respectively). However, this turns out to be an acceptable restriction in many cases.

Definition by cases with inequality

We will sometimes want to split into cases $m \leq n$ and $m > n$ instead of $t = 0$ and $t > 0$. The idea is to use $m \leq n$ to represent $m \dot{-} n = 0$, and to use $m > n$ to represent $sn \dot{-} n = 0$. To do this formally, we will need the following result.

Corollary 3.3.5. Let $u, w : N$ be terms in a context $C = (x_1 : X_1, ..., x_n : X_n)$, and let C_1, C_2 be the contexts $C_1 = (C \mid u < w)$ and $C_2 = (C \mid u \geq w)$.

Then, in an arithmetic lextensive category, $[C_1] \hookrightarrow [C] \hookleftarrow [C_2]$ is a coproduct.

Proof. Remark that $C_2 = (C \mid w \mathbin{\dot{-}} u = 0)$. So, if we let $C_{>0} = (C \mid w \mathbin{\dot{-}} u > 0)$, then by proposition 3.3.2, the following diagram is a coproduct.

$$[C_2] \hookrightarrow [C] \hookleftarrow [C_{>0}]$$

So, to show the result we want, we need to show that there is an isomorphism $[C_1] \cong [C_{>0}]$ which respects the inclusion into $[C]$.

The only choice for an isomorphism between the objects $[C_1] = \{C \mid u < w\}$ and $[C_{>0}] = \{C \mid w \mathbin{\dot{-}} u > 0\}$ which respects the inclusion into $[C]$ is the "identity map" $(x_1, ..., x_n) \mapsto (x_1, ..., x_n)$. Therefore, we just need to show that this map is well-defined in both directions. This just involves showing that $w \mathbin{\dot{-}} u >_{C_1} 0$ (knowing that $u <_{C_1} w$) and that $u <_{C_{>0}} w$ (knowing that $w \mathbin{\dot{-}} u >_{C_{>0}} 0$). This is true by proposition 3.2.4. ∎

Now, let C be a context, let $m, n : N$ be terms in C, let $C_1 = (C \mid m \leq n)$, let $C_2 = (C \mid m > n)$, let $a_1 : Y$ be a term in C_1, and let $a_2 : Y$ be a term in C_2. Then we write

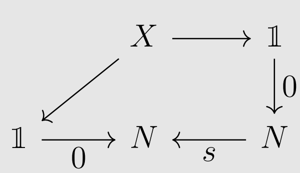
$$\begin{cases} a_1 & \text{if } m \leq n \\ a_2 & \text{else} \end{cases} \quad \text{or} \quad \begin{cases} a_2 & \text{if } n < m \\ a_1 & \text{else} \end{cases} \quad \text{for} \quad \begin{cases} a_1 & \text{if } m \mathbin{\dot{-}} n = 0 \\ a_2 & \text{else} \end{cases}.$$

Proof by contradiction

Working in an extensive category lets us do a sort of proof by contradiction.

Proposition 3.3.6. In an arithmetic lextensive category, if X is an object such that $0 =_{x:X} 1$, then X is initial.

Proof. The equation $0 =_{x:X} 1$ means that the following diagram commutes.



Therefore, there is an arrow from X into the pullback of the arrows $0, s$. But 0 and s form a coproduct, and in an extensive category, coproducts are disjoint, so this pullback is an initial object. Thus we have an arrow $X \to \mathbb{O}$, but in an extensive category, initials are strict, so X is an initial object too. ∎

Corollary 3.3.7. Suppose we are working in an arithmetic lextensive category.

- Let $t : N$ be a term in a context C, and write $C_0 = (C \mid t = 0)$, $C_{>0} = (C \mid t > 0)$. If $0 =_{C_0} 1$, then $t >_C 0$, and if $0 =_{C_{>0}} 1$, then $t =_C 0$.
- Let $m, n : N$ be terms in a context C, and write $C_1 = (C \mid m \leq n)$, $C_2 = (C \mid m > n)$. If $0 =_{C_1} 1$, then $m >_C n$, and if $0 =_{C_2} 1$, then $m \leq_C n$.

Proof. We only prove one case; the rest are similar. So, let $t : N$ be a term in a context C, and assume that $0 =_{C_{>0}} 1$. By proposition 3.3.6, $[C_{>0}]$ is an initial object, and since $[C] = [C_0] + [C_{>0}]$, this implies that the inclusion $i : [C_0] \hookrightarrow [C]$ is an isomorphism. By definition, i is the equalizer of $[t]_C$ and 0, so $[t]_C \circ i = 0 \circ i$; applying i^{-1} to each side gives $[t]_C = 0$, which is equivalent to saying $t =_C 0$. ∎

Some technical facts

Here, we prove two small technical facts.

Proposition 3.3.8. In an arithmetic lextensive category, let $f : A \to N$ and $g : B \to N$ be arrows, and let $[f, g] : A + B \to N$ be the map induced by the coproduct property.

Then, for any $n : A + B \to N$, if $f \leq n$ and $g \leq n$, then $[f, g] \leq n$. Similarly, if $f \geq n$ and $g \geq n$, then $[f, g] \geq n$.

Proof. We must show that $[f, g] \dot{-} n = 0$. But since the domain is a coproduct, it suffices to check on each component. On the A component, $[f, g] \dot{-} n$ becomes $f \dot{-} n$, which is zero because $f \leq n$. The B component is similar, so we get the desired inequality. The reverse case is proved the same way. ∎

Proposition 3.3.9. In an arithmetic lextensive category, let $\{0, 1\}$ denote the coproduct $\mathbb{1} + \mathbb{1}$ (with inclusion maps $0, 1 : \mathbb{1} \to \{0, 1\}$), and let $\{x \leq 1\}$ denote the object $\{x \in N \mid x \leq 1\}$ (with inclusion map $i : \{x \leq 1\} \hookrightarrow N$). Then $\{0, 1\} \cong \{x \leq 1\}$.

Proof. First, define $f : N \to \{0, 1\}$ by $f(0) = 0$, $f(sn) = 1$, and define $g : \{0, 1\} \to N$ by $g(0) = 0$, $g(1) = 1$. It is clear that $fg = \mathrm{Id}_{\{0,1\}}$ by the coproduct property.

Then, we have $fi : \{x \leq 1\} \to \{0, 1\}$, and the map $g : \{0, 1\} \to N$ induces a map $h : \{0, 1\} \to \{x \leq 1\}$ such that $ih = g$ (by the equalizer property, since $g \leq 1$ by proposition 3.3.8). We claim that fi and h are inverses of each other.

On one hand, we easily compute $fih = fg = \mathrm{Id}_{\{0,1\}}$. On the other hand, to show $hfi = \mathrm{Id}_{\{x \leq 1\}}$, it suffices to show $ihfi = i$, since i is a monomorphism. Note that $ihfi = gfi$, so we must show $gfi = i$.

So, let $x \leq 1$. By proposition 3.3.2, we can split into the cases $x = 0$ and $x > 0$. In the former case, we have $gfi(x) = gf(0) = g(0) = 0 = i(x)$. In the latter case, $x > 0$ means $x \geq 1$, but since $x \leq 1$, we get $x = 1$. Thus $gfi(x) = gf(1) = g(1) = 1 = i(x)$, and we're done. ∎

3.4 The collection of finite cardinals

A key concept for all the results in this thesis is *finiteness*. In our case, we examine finiteness by using a map which has "finite fibers". We want an analogue of the set $E = \{(m, n) \in \mathbb{N} \times \mathbb{N} : m < n\}$, which can be easily achieved in a category with finite limits and a natural numbers object N. We set

$$E = \{m : N,\, n : N \mid m < n\}.$$

(Here, we use notation from section 3.2; explicitly, E is the equalizer of the arrows $N \times N \to N$ given by $(m, n) \mapsto s(m) \dot{-} n$ and $(m, n) \mapsto 0$.) Now, if we write D for the context $(m : N,\, n : N \mid m < n)$ (remark that $[D] = E$), then we can define the map $\pi_2^E : E \to N$ by $\pi_2^E(m, n) =_D n$. Going forward, we'll use the notation E, π_2^E to refer to this object, which we call *the collection of finite cardinals*, without mentioning it explicitly. In [John 02], (E, π_2^E) is referred to as the *generic finite cardinal* (see the discussion above lemma 2.5.14).

Remark 3.4.1. It is not hard to convince yourself that the above equalizer exists even if we only assume the existence of finite products (instead of finite limits): E is a countably infinite set, and is therefore isomorphic to N. However, checking this formally is technical and not relevant to the rest of this paper, so we omit the proof of this.

An important construction with the object E, which will be used many times throughout the thesis, is the pullback of $\pi_2^E : E \to N$ and another arrow $l_A : A \to N$.

$$\begin{array}{ccc} E \times_N A & \longrightarrow & A \\ \downarrow & & \downarrow l_A \\ E & \xrightarrow{\pi_2^E} & N \end{array}$$

We use this notation because the arrow l_A will often be understood from the context.

Using the internal language, we have $E \times_N A = \{e : E,\, a : A \mid \pi_2^E(e) = l_A(a)\}$. Since E is the object $\{m : N, n : N \mid m < n\}$, we could also write this as $\{(m, n) : E,\, a : A \mid n = l_A(a)\}$. However, this shows us that the "n" parameter is redundant, and that we can just think of this object as $\{m : N,\, a : A \mid m < l_A(a)\}$. Specifically, we'd like to use the context $(m : N, a : A \mid m < l_A(a))$ when referencing the object $E \times_N A$. The following proposition assures us that this is valid.

Proposition 3.4.2. Let $l_A : A \to N$. Then the following maps are well-defined and provide an isomorphism $E \times_N A \cong \{m : N,\, a : A \mid m < l_A(a)\}$.

$$E \times_N A \rightleftarrows \{m : N,\, a : A \mid m < l_A(a)\}$$

$$((m,n),a) \mapsto (m,a)$$

$$((m, l_A(a)), a) \mapsfrom (m,a)$$

Proof. Straightforward check. ∎

Note that we do need to be careful when writing $\{m : N, a : A \mid m < l_A(a)\}$ for $E \times_N A$, because applying the projection maps does not behave the way we expect. For instance, we have the map $\pi_1^{E,A} : E \times_N A \to E$, and if we consider $(m,a) : E \times_N A$ with $m : N, a : A$, then $\pi_1^{E,A}(m,a)$ is not equal to m, as we might think if we're not being careful.

Fortunately, this issue does not come up very often. We'll only need the following fact to assure us that we aren't making mistakes (the proof is a straightforward check, so we omit it).

Fact 3.4.3. Suppose we have arrows $l_A : A \to N$, $l_B : B \to N$ and $f : A \to B$. Assume $l_B \circ f = l_A$, so that we can consider the following arrow.

$$E \times_N A \xrightarrow{\mathrm{Id}_E \times_N f} E \times_N B$$

Then, in the context $C = (m : N, a : A \mid m < l_A(a))$, we have

$$(\mathrm{Id}_E \times_N f)(m,a) =_C (m, f(a)).$$

3.5 New forms of NNO induction

In this section, we develop some new inductive principles for natural numbers objects that make it easier to prove certain results. We start with an easy result about induction for inequalities.

Theorem 3.5.1. In a Cartesian category $\mathcal{C}$ with NNO, let $f : A \times N \to N$. If $f(a,0) =_a 0$ and $f(a, sn) \leq_{a,n} f(a,n)$, then $f(a,n) =_{a,n} 0$.

Moreover, suppose $\mathcal{C}$ has list objects (see chapter 4). Then, similarly, given $f : A \times L(X) \to N$, if $f(a, \varnothing) =_a 0$ and $f(a, x :: \ell) \leq_{a,x,\ell} f(a,\ell)$, then $f(a,\ell) =_{a,\ell} 0$.

Proof. We start with the first statement. Since $f(a, sn) \leq f(a, n)$, there exists a term $k(a, n)$ such that

$$f(a, sn) = f(a, n) \dot{-} k(a, n).$$

Thus, if we set $g(a) = 0$ and $h(a, n, x) = x \dot{-} k(a, n)$, we find that $f(a, 0) =_a g(a)$ and $f(a, sn) =_{a,n} h(a, n, f(a, n))$. But the constant zero map also satisfies these equations, so these must be equal.

The proof of the second statement is identical: since $f(a, x :: \ell) \leq f(a, \ell)$, we can write $f(a, x :: \ell) = f(a, \ell) \dot{-} k(a, x, \ell)$, which is a recursion also satisfied by the zero map. Since f and the zero map are also equal (to 0) in the base case, we get equality by list object induction. ∎

Next, we prove a statement for "finite induction".

Theorem 3.5.2. Let $\mathcal{C}$ be an arithmetic lextensive category. Let $f : A \to N$, and let $g_1, g_2 : N \times A \to B$. Consider the following statement:

> For any terms $n : N$, $a : A$ in a context C, if $n \leq_C f(a)$, then $g_1(n, a) =_C g_2(n, a)$.

To prove this statement, it suffices to show the following.

1. $g_1(0, a) =_a g_2(0, a)$;
2. $g_1(f(a), a) =_a g_2(f(a), a)$;
3. There is an arrow $r : N \times A \times B \to B$ such that, for any terms $n : N$, $a : A$ in a context C satisfying $n <_C f(a)$, we have $g_i(sn, a) =_C r(n, a, g_i(n, a))$ for $i = 1, 2$.

Proof. Note that if $n \leq_C f(a)$, then $\min(n, f(a)) =_C n$. Therefore, it suffices to prove that

$$g_1\Big(\min(n, f(a)),\, a\Big) =_{n,a} g_2\Big(\min(n, f(a)),\, a\Big).$$

Call these terms $h_1(n, a)$ and $h_2(n, a)$. We'll prove they're equal by induction on n. In the base case, we use assumption 1:

$$h_1(0, a) = g_1\Big(\min(0, f(a)),\, a\Big) = g_1(0, a) = g_2(0, a) = g_2\Big(\min(0, f(a)),\, a\Big) = h_2(0, a).$$

For the inductive step, we show that both functions satisfy the recurrence

$$h_i(sn, a) =_{n,a} \begin{cases} r(n, a, h_i(n, a)) & \text{if } n < f(a) \\ g(f(a), a) & \end{cases}.$$

Here, $g(f(a), a)$ just represents either of $g_1(f(a), a)$ or $g_2(f(a), a)$ (they are equal by assumption 2), and r comes from assumption 3.

To show this equality, we split into the cases $n < f(a)$ and $n \geq f(a)$ (which we can do because we're in an extensive category). In the former case, we get $\min(sn, f(a)) = sn$, so we have

$$h_i(sn, a) = g_i\Big(\min(sn, f(a)),\ a\Big) = g_i(sn, a) = r(n, a, g_i(n, a)) = r(n, a, h_i(n, a)).$$

Note that we used the equality from assumption 3, and $g_i(n, a) = h_i(n, a)$ because $n < f(a)$ implies $\min(n, f(a)) = n$. Now, in the latter case, we have $\min(sn, f(a)) = f(a)$, so

$$h_i(sn, a) = g_i(f(a), a) = g(f(a), a),$$

which is what we wanted. Thus, we've proven the desired equality. ∎

Finally, we show that we can define an arrow $f(a, n)$ inductively even if $f(a, sn)$ depends on $f(p(a), n)$ instead of just $f(a, n)$.

Theorem 3.5.3. Let N be an NNO in an arithmetic lextensive category. For any arrows $g : A \to B$, $h : A \times N \times B \to B$, and $p : A \to A$, there exists a unique arrow $f : A \times N \to B$ such that

$$f(a, 0) =_a g(a),$$
$$f(a, sn) =_{a,n} h(a, n, f(p(a), n)).$$

Note: for this proof, we use the following notation. If $r : X \to X$, we write $(x, n) \mapsto r^n(x)$ for the map $X \times N \to X$ defined recursively by $r^0(x) = x$, $r^{sn}(x) = r(r^n(x))$. It is easy to show (by induction on n) that $r^{sn}(x) = r^n(r(x))$.

Proof. Consider the function $\hat{f} : N \times A \times N \to B$ defined inductively by

$$\hat{f}(m, a, 0) = g(p^m(a)) \quad \text{and} \quad \hat{f}(m, a, sn) = \begin{cases} h\Big(p^{m \dot{-} sn}(a), n, \hat{f}(m, a, n)\Big) & \text{if } n < m \\ \hat{f}(m, a, n) & \text{else} \end{cases}.$$

We claim that setting $f(a, n) = \hat{f}(n, a, n)$ satisfies the equations of the theorem, and moreover if f is any function satisfying those equations, then $f(a, n) = \hat{f}(n, a, n)$. Proving these two facts will give us the desired existence and uniqueness.

Existence.

For existence, we must show $f(a, n) = \hat{f}(n, a, n)$ satisfies the equations of the theorem. The first equation is easy:

$$f(a, 0) = \hat{f}(0, a, 0) = g(p^0(a)) = g(a).$$

For the second equation, we must prove an intermediate result: we claim that

$$\hat{f}(sm, a, \min(m, n)) = \hat{f}(m, p(a), \min(m, n)).$$

We prove this by induction on n. In the base case, we have $\min(m, 0) = 0$, so

$$\begin{aligned} \hat{f}(sm, a, \min(m, 0)) &= \hat{f}(sm, a, 0) = g(p^{sm}(a)) \\ \text{and } \hat{f}(m, p(a), \min(m, 0)) &= \hat{f}(m, p(a), 0) = g(p^m(p(a))) = g(p^{sm}(a)). \end{aligned}$$

For the inductive step, we claim that both sides satisfy

$$k(m, a, sn) = \begin{cases} h\Big(p^{m \dot{-} n}(a),\ n,\ k(m, a, n)\Big) & \text{if } n < m \\ k(m, a, n) & \text{else} \end{cases}.$$

We start with the left side. If $n < m$, then $\min(m, sn) = sn$ and $\min(m, n) = n$. Therefore,

$$\begin{aligned} \hat{f}(sm, a, \min(m, sn)) &= \hat{f}(sm, a, sn) = h\Big(p^{sm \dot{-} sn}(a), n, \hat{f}(sm, a, n)\Big) \\ &= h\Big(p^{m \dot{-} n}(a), n, \hat{f}(sm, a, n)\Big) = h\Big(p^{m \dot{-} n}(a), n, \hat{f}(sm, a, \min(m, n))\Big), \end{aligned}$$

as desired. If $n \geq m$, then $\min(m, sn) = m = \min(m, n)$, so

$$\hat{f}(sm, a, \min(m, sn)) = \hat{f}(sm, a, \min(m, n)),$$

as desired. Now, we show the right hand side satisfies the recurrence. If $n < m$, then $\min(m, sn) = sn$ and $\min(m, n) = n$, so

$$\begin{aligned} \hat{f}(m, p(a), \min(m, sn)) = \hat{f}(m, p(a), sn) &= h\Big(p^{m \dot{-} sn}(p(a)), n, \hat{f}(m, a, n)\Big) \\ &= h\Big(p^{m \dot{-} n}(a), n, \hat{f}(m, a, \min(m, n)). \end{aligned}$$

(For the last step, since $n < m$, we have $m \dot{-} n > 0$ by proposition 3.2.4, and so $s(m \dot{-} sn) = s(P(m \dot{-} n)) = m \dot{-} n$.) If $n \geq m$, then $\min(m, sn) = \min(m, n)$, so

$$\hat{f}(m, p(a), \min(m, sn)) = \hat{f}(m, p(a), \min(m, n)).$$

Thus both sides satisfy the recurrence, so we have the desired intermediate result.

Finally, with the intermediate result established, we plug in $m = n$ to get the equality $\hat{f}(sn, a, n) = \hat{f}(n, p(a), n)$. Using that, we compute

$$\begin{aligned} f(a, sn) = \hat{f}(sn, a, sn) &= h\Big(p^{sn \dot{-} sn}(a), n, \hat{f}(sn, a, n)\Big) \\ &= h(a, n, \hat{f}(n, p(a), n)) = h(a, n, f(p(a), n)). \end{aligned}$$

Thus, this f does indeed satisfy the second equation of the theorem, and we have finished the existence proof.

Uniqueness.

For uniqueness, suppose f is some function satisfying the equations of the theorems. Then set

$$f'(m,a,n) = f\Big(p^{m \dot{-} \min(m,n)}(a),\, \min(m,n)\Big);$$

we claim f' satisfies the defining equations of $\hat{f}$. If that's the case, then clearly we have $\hat{f}(n,a,n) = f'(n,a,n) = f(a,n)$, as desired.

So, we check that f' satisfies the defining equations of $\hat{f}$. For the base case, we compute

$$f'(m,a,0) = f(p^{m \dot{-} 0}(a), 0) = f(p^m(a), 0) = g(p^m(a)).$$

For the inductive step, we split into two cases: $n < m$ and $n \geq m$. In the case $n < m$, we note that $\min(m, sn) = sn$ and $\min(m,n) = n$. Then we compute

$$\begin{aligned} f'(m,a,sn) = f(p^{m \dot{-} sn}(a),\, sn) &= h\Big(p^{m \dot{-} sn}(a), n, f(p(p^{m \dot{-} sn}(a)), n)\Big) \\ &= h\Big(p^{m \dot{-} sn}(a), n, f(p^{m \dot{-} n}(a), n)\Big) \\ &= h\Big(p^{m \dot{-} sn}(a), n, f(p^{m \dot{-} \min(m,n)}(a), \min(m,n))\Big) \\ &= h\Big(p^{m \dot{-} sn}(a), n, f'(m,a,n)\Big). \end{aligned}$$

Note that, to pass to the second line, we used $s(m \dot{-} sn) = m \dot{-} n$ (as before) since we're in the case $n < m$. Next, in the case $n \geq m$, we have $\min(m, sn) = m = \min(m,n)$. Therefore,

$$\begin{aligned} f'(m,a,sn) &= f\Big(p^{m \dot{-} \min(m,sn)}(a),\, \min(m,sn)\Big) \\ &= f\Big(p^{m \dot{-} \min(m,n)}(a),\, \min(m,n)\Big) \\ &= f'(m,a,n). \end{aligned}$$

So, f' does satisfy the same recurrence as $\hat{f}$, and this finishes the proof of uniqueness. ∎

Corollary 3.5.4. In an arithmetic lextensive category, let $f : A \times N \to N$ and $p : A \to A$. If $f(a,0) =_a 0$ and $f(a,sn) \leq_{a,n} f(p(a),n)$, then $f(a,n) =_{a,n} 0$.

Proof. Since $f(a, sn) \leq f(p(a), n)$, there exists a term $k(a, n)$ such that

$$f(a, sn) = f(p(a), n) \dot{-} k(a, n).$$

Thus, if we set $g(a) = 0$ and $h(a, n, x) = x \dot{-} k(a, n)$, we find that $f(a, 0) =_a g(a)$ and $f(a, sn) =_{a,n} h(a, n, f(p(a), n))$. But the constant zero map also satisfies these equations, so by theorem 3.5.3, these must be equal. ∎

3.6 Increasing functions

Definition. In a Cartesian category with NNO, let $f : N \to N$. We say f is *increasing* if $f(m) \leq_m f(sm)$, and we say f is *strictly increasing* if $f(m) <_m f(sm)$.

Given $f : I \times N \to N$, we say $f(i, -)$ is increasing if $f(i, m) \leq_{i,m} f(i, sm)$. We say $f(i, -)$ is strictly increasing if $f(i, m) <_{i,m} f(i, sm)$.

Proposition 3.6.1. In a Cartesian category with NNO, let $f : N \to N$.

- f is increasing if and only if for any context C and any terms $m, n : N$ in C such that $m \leq_C n$, we have $f(m) \leq_C f(n)$.
- f is strictly increasing if and only if for any context C and any terms $m, n : N$ in C such that $m <_C n$, we have $f(m) <_C f(n)$.

For an arrow $f : I \times N \to N$, we have analogous statements.

Proof. We start with increasing functions. It's clear that the stated condition implies f is increasing, so we prove the other implication. We claim that, if f is increasing, then $f(m) \leq_{m,k} f(m + k)$. In other words, we must show that

$$f(m) \dot{-} f(m + k) =_{m,k} 0.$$

To prove this, we appeal to theorem 3.5.1. The base case is clear, and for the inductive step, we use proposition 3.2.2 and the assumption $f(a) \leq_a f(sa)$ to get $f(m) \dot{-} f(m + sk) = f(m) \dot{-} f(s(m + k)) \leq f(m) \dot{-} f(m + k)$.

So, if we have terms m, n such that $m \leq n$, we can write $n = m + k$ for some k, and use this result to conclude $f(m) \leq f(m + k) = f(n)$.

For f strictly increasing, we can argue similarly, showing that $f(m) <_{m,k} f(m+sk)$. That is, we must show that

$$s(f(m)) \dot{-} f(m + sk) =_{m,k} 0.$$

We again appeal to theorem 3.5.1. The base case is just the assumption that f is strictly increasing; for the inductive step, we note that $f(m+sk) \leq f(s(m+sk))$ (we have a strict inequality by assumption, so in particular we get this inequality) and compute

$$s(f(m)) \dot{-} f(m+ssk) = s(f(m)) \dot{-} f(s(m+sk)) \leq s(f(m)) \dot{-} f(m+sk).$$

This gives the desired result.

For arrows of the form $f : I \times N \to N$, we can use exactly the same proof and just add the parameter $i : I$ everywhere. ∎

Theorem 3.6.2. In an arithmetic lextensive category, if $f : N \to N$ is strictly increasing, then f is monic.

Proof. Let $m, n : X \to N$ be such that $f(m(x)) =_x f(n(x))$. Román showed (proposition 3.2(b) [Roma 89]) that

$$N \times N = \{a, b \in N \mid a < b\} + \{a, b \in N \mid a = b\} + \{a, b \in N \mid a > b\},$$

and since we're in an extensive category, we can pull back this coproduct along $\langle m, n\rangle : X \to N \times N$ to get

$$X = \{x \in X \mid m(x) < n(x)\} + \{x \in X \mid m(x) = n(x)\} + \{x \in X \mid m(x) > n(x)\}.$$

Write this coproduct as $X = X_< + X_= + X_>$. Now, if we have $x : X_<$, then $m(x) < n(x)$, so by proposition 3.6.1 and the assumption on m, n, we have

$$f(n(x)) =_x f(m(x)) <_x f(n(x)),$$

and by subtracting $f(n(x))$ from each side, we get $0 <_x 0$. That is, $0 =_x s0 \dot{-} 0 = 1$. By proposition 3.3.6, we conclude that $X_<$ is initial.

A similar argument shows that $X_>$ is initial, and therefore the inclusion $X_= \hookrightarrow X$ is an isomorphism. If we write this map as i, then for $x : X$ we have

$$m(x) = m(i(i^{-1}(x))) = n(i(i^{-1}(x))) = n(x),$$

as desired (note that $m \circ i = n \circ i$ because of how $X_=$ is defined). ∎

Corollary 3.6.3. In an arithmetic lextensive category, let $f : N \to N$, and let $n : \mathbb{1} \to N$. If for any term $m : N$ in a context C, $m <_C n$ implies $f(m) <_C f(sm)$, then $f \circ i$ is monic, where $i : E_n \hookrightarrow N$.

Proof. Define $g : N \to N$ by

$$g(m) = \begin{cases} f(m) & \text{if } m < n \\ f(n) + (m \dot{-} n) & \text{else} \end{cases}.$$

Note that $f \circ i = g \circ i$, because $g(m) = f(m)$ for $m < n$. Note also that i is monic (it's an equalizer), so to prove the result, it suffices to show g is monic; by theorem 3.6.2, it suffices to show g is strictly increasing.

Before continuing, note that $g(m) = f(m)$ for $m \leq n$. To show this, we split into the cases $m < n$ and $m = n$; these are each easy to check.

So, we now show $g(m) <_m g(sm)$. We split into the cases $m < n$ and $m \geq n$. If $m < n$, then $g(m) = f(m)$ and $g(sm) = f(sm)$ (because $sm \leq n$). Therefore, $g(m) < g(sm)$ by the assumption on f. On the other hand, if $m \geq n$, then

$$\begin{aligned} s(g(m)) \dot{-} g(sm) = s\Big[f(n) + (m \dot{-} n)\Big] \dot{-} \Big[f(n) + (sm \dot{-} n)\Big] &= s(m \dot{-} n) \dot{-} (sm \dot{-} n) \\ &= s(m \dot{-} n) \dot{-} s(m \dot{-} n) \\ &= 0. \end{aligned}$$

Note: for this last line, we have $sm \dot{-} n = s(m \dot{-} n)$ because $m \geq n$. Indeed, $m \geq n$ means $m = n + k$ for some k, and then $m \dot{-} n = k$ and $sm \dot{-} n = (1 + n + k) \dot{-} n = 1 + k$.

So, in each case, we have shown that $g(m) < g(sm)$, so we're done. ∎

3.7 Maximums

In this section, we construct a function which finds the maximum value in a bounded set for which a given condition is satisfied.

Definition. Given arrows $n : I \to N$ and $t : I \times N \to N$, we construct an arrow $g : I \times N \to N$ by induction as follows.

$$g(i, 0) = 0$$
$$g(i, sk) = \begin{cases} sk & \text{if } t(i, k) = 0 \\ g(i, k) & \text{else} \end{cases}$$

Then, set $\text{smax}(i) = g(i, n(i))$. We also define $\max(i) = P(\text{smax}(i))$.

What is this function $\text{smax} : I \to N$? Well, we'd like to have a function $I \to N$ which, given $i \in I$, picks out the value $\max(x < n(i) \mid t(i, x) = 0)$. The easiest way to do this is to start with a default value which indicates that there is no maximum, and then recursively define $g(i, n)$ to be n if $t(i, n) = 0$ and $g(i, n-1)$ otherwise. The problem is that we don't have easy access to a default value; the easiest one is 0. But 0 could just as easily be the maximum we're looking for. So, we define smax to be zero if there is no maximum, and one more than the maximum if it does exist.

Note: this means that it is not meaningful to consider the value $\max(i)$ if the maximum does not exist, i.e. if $\text{smax}(i) = 0$. In other words, we should only use $\max(i)$ if we have already checked that $\text{smax}(i) > 0$. On the other hand, if we do have $\text{smax}(i) > 0$, then from proposition 3.2.3 it is clear that $s(\max(i)) = \text{smax}(i)$, and we use the following notation:

$$\max(i) = \max(x < n(i) \mid t(i, x) = 0).$$

We now show some properties of this smax function.

Proposition 3.7.1. In an arithmetic lextensive category, let $n : I \to N$ and $t : I \times N \to N$. In a context C, let $i : I$. Then we have the following properties.

- If there is a term $k : N$ in C such that $k <_C n(i)$ and $t(i, k) =_C 0$, then $\text{smax}(i) \geq_C sk$.
- If $\text{smax}(i) >_C 0$, then $\max(i) <_C n(i)$ and $t(i, \max(i)) =_C 0$.
- If $t(i, n(i)) >_C 0$, then $t(i, \text{smax}(i)) >_C 0$.

In particular, if $n(i) >_C 0$ and $t(i, 0) =_C 0$, then $\max(i) <_C n(i)$ and $t(i, \max(i)) =_C 0$; and if moreover $t(i, n(i)) >_C 0$, then $t(i, s(\max(i))) >_C 0$.

Proof. First, note that $g(i, k) \leq_{i,k} k$; this is easy to show by induction. On the other hand, we can also show by induction (using this first inequality) that $g(i, sk) \geq_{i,k} g(i, k)$, so $g(i, -)$ is increasing. With these two facts, we can show that if there is a term k such that $k < n(i)$ and $t(i, k) = 0$, then $\text{smax}(i) \geq sk$. Indeed, we can compute

$$\text{smax}(i) = g(i, n(i)) \geq g(i, sk) = sk.$$

The last equality is because we assumed $t(i, k) = 0$, and the inequality is because $g(i, -)$ is increasing (and we assumed $k < n(i)$), using proposition 3.6.1.

Next, we can easily show by induction that $g(i, g(i, k)) =_{i,k} g(i, k)$. We also claim that if k is such that $g(i, sk) = sk$, then $t(i, k) = 0$. Indeed, if $t(i, k) > 0$, then the definition of g tells us that $sk = g(i, sk) = g(i, k) \leq k$. This is a contradiction (see corollary 3.3.7), so $t(i, k) = 0$.

Now, suppose $\text{smax}(i) > 0$; this means we can write $\text{smax}(i)$ as the successor of $\max(i)$. Then, by using the inequality $g(i, k) \leq k$, we get

$$\max(i) < \text{smax}(i) = g(i, n(i)) \leq n(i).$$

Moreover, note that

$$\text{smax}(i) = g(i, n(i)) = g(i, g(i, n(i))) = g(i, \text{smax}(i))$$

(using what we proved above), and since $\operatorname{smax}(i) = s(\max(i))$, we conclude (again, by above) that $t(i, \max(i)) = 0$.

Finally, suppose $t(i, n(i)) > 0$; we want to show that $t(i, \operatorname{smax}(i)) > 0$. To do so, we split into the cases $\operatorname{smax}(i) < n(i)$ and $\operatorname{smax}(i) \geq n(i)$ (see corollary 3.3.5). In the latter case, we already know that $\operatorname{smax}(i) \leq n(i)$ (we proved it first), so $\operatorname{smax}(i) = n(i)$ and we get $t(i, \operatorname{smax}(i)) = t(i, n(i)) > 0$ by assumption. In the former case, we assume $\operatorname{smax}(i) < n(i)$. Then, assume $t(i, \operatorname{smax}(i)) = 0$; these two assumptions tell us (by the first bullet point in the proposition) that $\operatorname{smax}(i) \geq s(\operatorname{smax}(i))$, a contradiction. Thus (by corollary 3.3.7) we get $t(i, \operatorname{smax}(i)) > 0$. ∎

Chapter 4

List objects

Contents

In this chapter, we present various facts and constructions for list objects. In section 4.1, we revisit the definition using the internal language of section 2.2, and in section 4.2, we make a note about what it means to preserve list objects. In sections 4.3 and 4.4, we define some important functions that we use for proving things about list objects; notably, in section 4.4, we define the "n^{th} element of a list" arrow, which requires us to assume we're working in a locos. In sections 4.5 and 4.6, we define and prove some technical facts about two more list functions: the "concatenation" arrow and the "head" arrow.

4.1 List objects with the internal language

We start by restating the definition of list objects using the internal language.

Definition. Let $\mathcal{C}$ be a Cartesian category, and let $X \in \mathcal{C}$ be an object. A *(parametrized) list object* on X is a triple $(L(X), r_0^X, r_1^X)$ consisting of an object $L(X)$ and arrows $r_0^X : \mathbb{1} \to L(X)$, $r_1^X : X \times L(X) \to L(X)$ which satisfy the following property. In what follows, we write $\varnothing$ for r_0^X and $x :: \ell$ for $r_1^X(x, \ell)$.

> For any objects A, B and arrows $g : A \to B$, $h : X \times B \to B$, there exists a unique arrow $f : A \times L(X) \to B$ such that
>
> $$f(a, \varnothing) =_a g(a) \qquad \text{and} \qquad f(a, x :: \ell) =_{a,x,\ell} h(x, f(a, \ell)).$$

A version of the universal property with terms can also be formulated, like for natural numbers objects (see remark 3.1.1).

Given a term $x : X$, we will write $[x]$ for the list $x :: \varnothing$ in $L(X)$.

The list object property can actually be generalized slightly to the following proposition.

Proposition 4.1.1. Let $\mathcal{C}$ be a Cartesian category which has all list objects, and let $X \in \mathcal{C}$. For any objects A, B and arrows $g : A \to B$, $h : A \times X \times L(X) \times B \to B$, there exists a unique arrow $f : A \times L(X) \to B$ such that

$$f(a, \varnothing) =_a g(a) \qquad \text{and} \qquad f(a, x :: \ell) =_{a,x,\ell} h(a, x, \ell, f(a, \ell)).$$

Proof. In the definition of list objects, replace B with $A \times L(X) \times B$. With some small straightforward adjustments, we get the desired property. ∎

From this proposition, it becomes clear that $L(\mathbb{1})$ is a natural numbers object. This means that it is redundant to assume that a category has list objects *and* a natural numbers object.

Finally, if a Cartesian category $\mathcal{C}$ has all list objects, then we can form a functor $L : \mathcal{C} \to \mathcal{C}$ which maps X to the list object $L(X)$. Its action on arrows is defined as follows: given $f : X \to Y$, we define $L(f) : L(X) \to L(Y)$ inductively by

$$L(f)(\varnothing) = \varnothing \qquad \text{and} \qquad L(f)(x :: \ell) =_{x,\ell} f(x) :: L(f)(\ell).$$

We omit the easy check that this is a functor.

4.2 Preserving list objects

Definition. Let $\mathcal{C}, \mathcal{D}$ be Cartesian categories with list objects, and let $F : \mathcal{C} \to \mathcal{D}$ be a functor preserving finite products. Then we say F *preserves list objects* if for any $X \in \mathcal{C}$, the tuple $(F(LX), F(r_0^X), F(r_1^X))$ is a list object on FX in $\mathcal{D}$.

Remark 4.2.1. In the above definition, we implicitly mean that $(F(LX), F(r_0^X), F(r_1^X))$ is a *parametrized* list object. However, it suffices to check that it is an unparametrized list object: indeed, in a category with all parametrized list objects, an unparametrized list object is automatically parametrized (because it has a canonical isomorphism to the parametrized list object).

Proposition 4.2.2. Let $\mathcal{C}, \mathcal{D}$ be Cartesian categories with list objects. Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors forming an adjunction $F \dashv G$, and assume that the counit $\epsilon : FG \Rightarrow \mathrm{Id}_{\mathcal{D}}$ is a natural isomorphism.

If F preserves finite products, then it also preserves list objects.

Proof. We start with a technical remark: for any $X \in \mathcal{C}$ and $A \in \mathcal{D}$, there is a bijection $\bar{\cdot} : \mathrm{Hom}(FX \times A, A) \to \mathrm{Hom}(X \times GA, GA)$ such that for any $b : FX \times A \to A$, the following diagram commutes.

$$\begin{array}{ccc} FX \times FGA & \xrightarrow{F(\bar{b})} & FGA \\ {\scriptstyle \mathrm{Id} \times \epsilon_A}\big\downarrow & & \big\downarrow{\scriptstyle \epsilon_A} \\ FX \times A & \xrightarrow{\quad b \quad} & A \end{array}$$

Indeed, the adjunction $F \dashv G$ means that the map

$$\begin{aligned} \mathrm{Hom}(X \times GA, GA) &\to \mathrm{Hom}(FX \times FGA, A) \\ f &\mapsto \varepsilon_A \circ Ff \end{aligned}$$

is a bijection. Since ε is an isomorphism, we have a bijection $\mathrm{Hom}(FX \times FGA, A) \to \mathrm{Hom}(FX \times A, A)$ given by precomposition with $\mathrm{Id} \times \epsilon_A^{-1}$. Combining these two bijections gives (the inverse of) the desired bijection.

Now, we show that F preserves list objects. Let $X \in \mathcal{C}$; we must show that $(F(LX), F(r_0^X), F(r_1^X))$ is an unparametrized list object on FX (per remark 4.2.1). So, let $A \in \mathcal{D}$, let $a : \mathbb{1} \to A$, and let $b : FX \times A \to A$. We claim there is a unique arrow $h : F(LX) \to A$ such that the following diagrams commute.

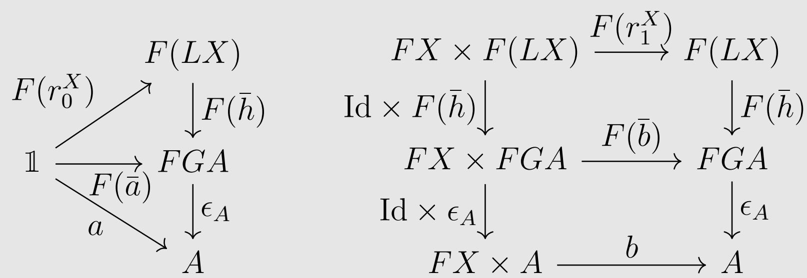
$$\begin{array}{ccc} & \overset{F(r_0^X)}{\nearrow} & F(LX) \\ \mathbb{1} & & \big\downarrow{\scriptstyle h} \\ & \underset{a}{\searrow} & A \end{array} \qquad \begin{array}{ccc} FX \times F(LX) & \xrightarrow{F(r_1^X)} & F(LX) \\ {\scriptstyle \mathrm{Id} \times h}\big\downarrow & & \big\downarrow{\scriptstyle h} \\ FX \times A & \xrightarrow{\quad b \quad} & A \end{array} \tag{4.2.1}$$

Using the isomorphism $\mathrm{Hom}(\mathbb{1}, A) = \mathrm{Hom}(F\mathbb{1}, A) \cong \mathrm{Hom}(\mathbb{1}, GA)$ given by the adjunction (and F preserving finite products), we get $\bar{a} : \mathbb{1} \to GA$. Then, using the technical remark, we get some $\bar{b} : X \times GA \to GA$. Since LX is a list object, there is a unique arrow $\bar{h}$ such that the following diagrams commute.

$$\begin{array}{ccc} & \overset{r_0^X}{\nearrow} & LX \\ \mathbb{1} & & \big\downarrow{\scriptstyle \bar{h}} \\ & \underset{\bar{a}}{\searrow} & GA \end{array} \qquad \begin{array}{ccc} X \times LX & \xrightarrow{r_1^X} & LX \\ {\scriptstyle \mathrm{Id} \times \bar{h}}\big\downarrow & & \big\downarrow{\scriptstyle \bar{h}} \\ X \times GA & \xrightarrow{\quad \bar{b} \quad} & GA \end{array} \tag{4.2.2}$$

Since $F \dashv G$, we have a bijection $\phi : \mathrm{Hom}(LX, GA) \to \mathrm{Hom}(F(LX), A)$ given by $\phi(\bar{h}) = \epsilon_A \circ F(\bar{h})$. We claim that $\bar{h}$ makes diagrams 4.2.2 commute if and only if $\phi(\bar{h})$ makes diagrams 4.2.1 commute; since there is a unique such $\bar{h}$, this implies the list object property for $F(LX)$.

So, we first assume that $\bar{h}$ makes diagrams 4.2.2 commute. If we apply F to these diagrams and post-compose with ϵ_A, we get the following.



The right diagram commutes by the technical remark, and the left diagram commutes because the isomorphism $\mathrm{Hom}(\mathbb{1}, GA) \to \mathrm{Hom}(\mathbb{1}, A)$ is given by $f \mapsto \epsilon_A \circ Ff$. This shows that $\phi(\bar{h}) = \epsilon_A \circ F(\bar{h})$ makes diagrams 4.2.1 commute, as desired.

On the other hand, suppose $\phi(\bar{h}) = \epsilon_A \circ F(\bar{h})$ makes diagrams 4.2.1 commute; we must show diagrams 4.2.2 commute. For the first one, recall that we have the isomorphism $\mathrm{Hom}(\mathbb{1}, GA) \to \mathrm{Hom}(\mathbb{1}, A)$ given by $f \mapsto \epsilon_A \circ Ff$. So, to check that $\bar{a} = \bar{h} \circ r_0^X$, it suffices to show that their images under this functor are equal. But, as we just checked above, their images under this functor give a and $\epsilon_A \circ F(\bar{h}) \circ F(r_0^X)$, which are equal because we assumed $\phi(\bar{h})$ satisfies equations 4.2.1. Thus, $\bar{a} = \bar{h} \circ r_0^X$, as desired. The same argument applies for the second diagram. ∎

4.3 Important list arrows

In this section, we define some important arrows related to list objects and prove some properties about them. Throughout, we assume we are working in a Cartesian category $\mathcal{C}$ with parametrized list objects (and therefore a parametrized NNO).

We start with the definitions. These operators are all indexed by an object X, but we remove this index if it is clear from context.

Definition. In a Cartesian category with parametrized list objects, we can define the following arrows.

- The *length* arrow, $\mathrm{len}_X : L(X) \to N$, is defined inductively by

$$\mathrm{len}_X(\varnothing) = 0, \qquad \mathrm{len}_X(x :: \ell) =_{x,\ell} s(\mathrm{len}_X(\ell)).$$

- The *truncate* arrow, $\mathrm{tr}_X : L(X) \to L(X)$, is the arrow which removes the leading element from a list. It is defined inductively by

$$\mathrm{tr}_X(\varnothing) = \varnothing, \qquad \mathrm{tr}_X(x :: \ell) =_{x,\ell} \ell.$$

- The *tail* arrow, $\mathrm{tail}_X : N \times L(X) \to L(X)$, is an arrow which iterates truncation: $\mathrm{tail}_X(n, \ell)$ returns the list ℓ with the n leading elements removed. It is defined inductively by

$$\mathrm{tail}_X(0, \ell) =_\ell \ell, \qquad \mathrm{tail}_X(s(n), \ell) =_{n,\ell} \mathrm{tr}_X(\mathrm{tail}_X(n, \ell)).$$

- The *zeroth-or-default* arrow, $\mathrm{zerothDef}_X : X \times L(X) \to X$, is an arrow which takes a pair (x, ℓ) as input. If ℓ is a non-empty list, it returns its leading element; however, if ℓ is empty, it returns x, the "default" element. This arrow is defined inductively by

$$\mathrm{zerothDef}_X(x, \varnothing) =_x x, \qquad \mathrm{zerothDef}_X(x, y :: \ell) =_{x,y,\ell} y.$$

- The *nth-or-default* arrow, $\mathrm{nthDef}_X : X \times N \times L(X) \to X$, is an arrow which takes a tuple (x, n, ℓ) as input. If $n < \mathrm{len}(\ell)$, then it returns the n^{th} element of ℓ (by removing the first n elements of ℓ and returning the leading element of the result). Otherwise, it returns x, the default element. This arrow is defined by

$$\mathrm{nthDef}_X(x, n, \ell) =_{x,n,\ell} \mathrm{zerothDef}_X\Big(x,\ \mathrm{tail}_X\big(n, \ell\big)\Big).$$

Note that we say "the zeroth element" for the leading element of a list instead of "the first element". This is to avoid confusion when we say "the n^{th} element", since "the n^{th} element" with $n = 0$ refers to the leading element.

Proposition 4.3.1. Let $f : X \to Y$ be an arrow. Then

- $\mathrm{len}_Y\big(L(f)(\ell)\big) =_\ell \mathrm{len}_X(\ell)$;
- $\mathrm{tr}_Y\big(L(f)(\ell)\big) =_\ell L(f)\big(\mathrm{tr}_X(\ell)\big)$;
- $\mathrm{tail}_Y\big(m,\ L(f)(\ell)\big) =_{m,\ell} L(f)\big(\mathrm{tail}_X(m,\ \ell)\big)$;
- $\mathrm{zerothDef}_Y\big(f(x),\ L(f)(\ell)\big) =_{x,\ell} f\big(\mathrm{zerothDef}_X(x,\ \ell)\big)$.
- $\mathrm{nthDef}_Y(f(x), m, L(f)(\ell)) =_{x,m,\ell} f(\mathrm{nthDef}_X(x, m, \ell))$.

Proof. Each of these equalities is proved by induction – except for the last, which is straightforward. For the first, we show $\mathrm{len}_Y \circ L(f)$ satisfies the equations defining len_X. For the third, we show that both terms satisfy $h(0, \ell) =_\ell L(f)(\ell)$ and $h(s(n), \ell) =_{n,\ell} \mathrm{tr}_Y(h(n, \ell))$, using the previous result about tr. The other two are simple. ∎

The next proposition shows how to decompose list objects based on their length. We draw particular attention to the second case: we would like to decompose a list $t : L(X)$ of positive length into $t = x :: t'$, but this is not possible unless we are working in an extensive category. The key limitation is that we cannot extract a term of type X from the context $(\ell : L(X) \mid \mathrm{len}(\ell) > 0)$ without additional assumptions (see also proposition 4.4.2), so in order to perform the decomposition, we need to be given an arbitrary term $\mathrm{def} : X$.

For most of the thesis, we will work in an extensive category, so this restriction will not be important. Nevertheless, we will attempt to be parsimonious with our assumptions when possible.

Proposition 4.3.2. Let $t : L(X)$ be a term in a context C.

1. If $\mathrm{len}(t) =_C 0$, then $t =_C \varnothing$.
2. If $\mathrm{len}(t) >_C 0$, then for any term $\mathrm{def} : X$ in the context C, we have

$$t =_C \mathrm{zerothDef}(\mathrm{def}, t) :: \mathrm{tr}(t).$$

Proof. For the first point, consider the map $Z : N \times L(X) \to L(X)$ defined inductively by $Z(0, \ell) =_\ell \varnothing$, $Z(sn, \ell) =_{n,\ell} \ell$. By induction on ℓ, it is easy to show that $Z(\mathrm{len}(\ell), \ell) =_\ell \ell$. We conclude by using the hypothesis $\mathrm{len}(t) =_C 0$ to compute

$$t =_C Z(\mathrm{len}(t), t) =_C Z(0, t) =_C \varnothing,$$

as desired.

For the second point, we similarly consider an arrow $Z : X \times N \times L(X) \to L(X)$ defined by $Z(x, 0, \ell) =_{x,\ell} \varnothing$ and $Z(x, sn, \ell) =_{x,n,\ell} \mathrm{zerothDef}(x, \ell) :: \mathrm{tr}(\ell)$. As before, we can easily show by induction that $Z(x, \mathrm{len}(\ell), \ell) =_{x,\ell} \ell$. Now, if $\mathrm{len}(t) >_C 0$, then $\mathrm{len}(t) =_C s(n)$ for some $n : N$ (by proposition 3.2.3). If moreover we have some $\mathrm{def} : X$, then

$$t =_C Z(\mathrm{def}, \mathrm{len}(t), t) =_C Z(\mathrm{def}, s(n), t) =_C \mathrm{zerothDef}(\mathrm{def}, t) :: \mathrm{tr}(t),$$

as desired. ∎

Next, we check that the tail operation has the length we expect. Combined with the previous result, this gives us a way to tell when $\mathrm{tail}(n, \ell)$ is the empty list.

Proposition 4.3.3. We have $\text{len}(\text{tail}(n, \ell)) =_{n,\ell} \text{len}(\ell) \dot{-} n$. In particular, for any terms $n : N$ and $\ell : L(X)$ in a context C, if $n \geq_C \text{len}(\ell)$, then $\text{tail}(n, \ell) =_C \varnothing$.

Proof. First, we note that $\text{len}(\text{tr}(\ell)) =_\ell P(\text{len}(\ell))$; this is easily checked by induction on ℓ. Using this fact, it is easy to check that the terms $\text{len}(\text{tail}(n, \ell))$ and $\text{len}(\ell) \dot{-} n$ both satisfy $h(0, \ell) =_\ell \text{len}(\ell)$ and $h(sn, \ell) =_{n,\ell} P(h(n, \ell))$, so we get the desired equality by induction.

For the second part, let $n : N$ and $\ell : L(X)$ be terms in a context C. If $n \geq_C \text{len}(\ell)$, then $\text{len}(\text{tail}(n, \ell)) =_C \text{len}(\ell) \dot{-} n =_C 0$, so $\text{tail}(n, \ell) =_C \varnothing$ by proposition 4.3.2. ∎

The next proposition tells us how $\text{nthDef}(x, n, \ell)$ behaves when $n \geq \text{len}(\ell)$: there is no n^{th} element and we should return the default x.

Proposition 4.3.4. Let $x : X$, $n : N$, $\ell : L(X)$ be terms in a context C. If $n \geq_C \text{len}(\ell)$, then

$$\text{nthDef}(x, n, \ell) =_C x.$$

Proof. Since $n \geq_C \text{len}(\ell)$ by assumption, we have $\text{tail}(n, \ell) =_C \varnothing$ by proposition 4.3.3. So, by definition of nthDef, we have

$$\text{nthDef}(x, n, \ell) =_C \text{zerothDef}(x,\ \text{tail}(n, \ell)) =_C \text{zerothDef}(x, \varnothing) =_C x.$$

This is what we wanted. ∎

This proposition handles the case $n \geq \text{len}(\ell)$, but what about the case $n < \text{len}(\ell)$? In that case, we expect to get the n^{th} element of ℓ, so the default x should have no impact on the outcome. Perhaps surprisingly, to obtain this result, we must assume that we are working in an extensive category. Therefore, we defer that result (proposition 4.4.3) to the next section.

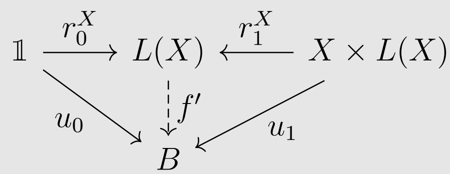
4.4 Lists in extensive categories

In this section, we assume that we are working in a *locos*, i.e. a category with finite limits and list objects which is extensive. Just like with natural numbers objects, adding the assumption of extensivity provides us with many additional tools. It is possible to be even more parsimonious with our assumptions here (as is done in my paper [Desr 25]), but for simplicity, we will assume full extensivity.

Similarly to natural numbers objects, list objects can be expressed as a coproduct. While this fact can be found in [Cock 90], it is stated in a much more abstract way, so we provide a simple proof.

Fact 4.4.1. In a Cartesian category with list objects, the diagram $\mathbb{1} \xrightarrow{r_0^X} L(X) \xleftarrow{r_1^X} X \times L(X)$ is a coproduct.

Proof. With the generalized version of the list object property, if we set $A = \mathbb{1}$, $g = u_0$, and $h = u_1 \circ \langle \pi_2, \pi_3 \rangle$, the diagrams become the following.

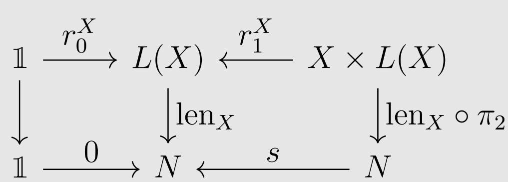


This is the usual coproduct diagram. ∎

The above fact is what makes it so advantageous to work in an extensive category. Most importantly, we get the following consequence.

Proposition 4.4.2. Let $\mathcal{C}$ be a locos, and let $X \in \mathcal{C}$. If $\ell : L(X)$ is a term in a context C such that $\mathrm{len}(\ell) >_C 0$, then there exist unique terms $x : X$, $\ell' : L(X)$ such that $\ell =_C x :: \ell'$.

Proof. Consider the following diagram.



It's clear that this diagram commutes, and both rows are coproducts (by facts 3.3.1 and 4.4.1). By extensivity (see fact 2.1.2), both squares are pullbacks.

Now, if $\mathrm{len}(\ell) >_C 0$, we can write $\mathrm{len}(\ell) =_C s(n)$ for some term $n : N$ (by proposition 3.2.3). Since the right square is a pullback, this means there exists a unique term $(x, \ell') : X \times L(X)$ such that $\ell =_C x :: \ell'$ (and $\mathrm{len}(\ell') = n$, but this is automatic since $\mathrm{len}(\ell') = \mathrm{len}(\mathrm{tr}(\ell)) = P(\mathrm{len}(\ell)) = P(s(n)) = n$). ∎

Immediately, we get the complement of proposition 4.3.4.

Proposition 4.4.3. Assume we are working in a locos. Let $x : X$, $n : N$, $\ell : L(X)$ be terms in a context C. If $n <_C \mathrm{len}(\ell)$, then for any other term $x' : X$ in the context C,

$$\mathrm{nthDef}(x, n, \ell) =_C \mathrm{nthDef}(x', n, \ell).$$

Proof. By proposition 3.2.4, $n <_C \mathrm{len}(\ell)$ implies $\mathrm{len}(\ell) \dot{-} n >_C 0$, so

$$\mathrm{len}(\mathrm{tail}(n, \ell)) =_C \mathrm{len}(\ell) \dot{-} n >_C 0$$

by proposition 4.3.3. By proposition 4.4.2, this inequality implies that $\mathrm{tail}(n, \ell) =_C y :: \ell'$ for some terms y, ℓ' in the context C. We conclude that

$$\mathrm{nthDef}(x, n, \ell) =_C \mathrm{zerothDef}(x, \mathrm{tail}(n, \ell)) =_C \mathrm{zerothDef}(x, y :: \ell') =_C y,$$

and $\mathrm{nthDef}(x', n, \ell) =_C y$ by the same argument. This gives the desired equality. ∎

In fact, in a locos, we can define maps "zeroth" and "nth" that do not require us to have a default element. The proper domain of zeroth is $L(X)_{>0} = \{\ell : L(X) \mid \mathrm{len}(\ell) > 0\}$, and the proper domain of nth is $\{m : N, \ell : L(X) \mid m < \mathrm{len}(\ell)\}$, i.e. the object $E \times_N L(X)$ (see proposition 3.4.2; note that we implicitly assume $L(X)$ is equipped with the arrow len_X). This will be key for some later results.

Definition. Let $\mathcal{C}$ be a locos, and let $X \in \mathcal{C}$. We define the arrows $\mathrm{zeroth} : L(X)_{>0} \to X$ and $\mathrm{nth}_X : E \times_N L(X) \to X$ as follows.

- Given $\ell : L(X)_{>0}$, proposition 4.4.2 tells us there exist unique terms $x : X$, $\ell' : L(X)$ such that $\ell =_\ell x :: \ell'$. Then we set

$$\mathrm{zeroth}(\ell) =_\ell x.$$

- Let $C = (m : N, \ell : L(X) \mid m < \mathrm{len}(\ell))$, so that $[C] = E \times_N L(X)$. Given $(m, \ell) : C$, we have $\mathrm{len}(\mathrm{tail}(n, \ell)) =_0 \mathrm{len}(\ell) \dot{-} n >_C 0$ (by propositions 4.3.3 and 3.2.4), so $\mathrm{tail}(m, \ell) : L(X)_{>0}$, and we set

$$\mathrm{nth}(m, \ell) =_C \mathrm{zeroth}(\mathrm{tail}(m, \ell)).$$

The following proposition shows how these are related to their default counterparts.

Proposition 4.4.4. Let $\mathcal{C}$ be a locos, let $X \in \mathcal{C}$, and let C be a context.

- If $\ell : L(X)$ is a term in the context C such that $\mathrm{len}(\ell) > 0$, then there is a term def : X in the context C such that

$$\text{zeroth}(\ell) =_C \text{zerothDef}(\text{def}, \ell).$$

In fact, for any $x : X$ in the context C, we have $\text{zeroth}(\ell) =_C \text{zerothDef}(x, \ell)$.

- If $m : N$, $\ell : L(X)$ are terms in the context C such that $m < \text{len}(\ell)$, then there is a term def : X in the context C such that

$$\text{nth}(m, \ell) =_C \text{nthDef}(\text{def}, m, \ell).$$

In fact, for any $x : X$ in the context C, we have $\text{nth}(m, \ell) =_C \text{nthDef}(x, m, \ell)$.

Proof. We start with the first point. As seen in the definition of zeroth, there exist unique terms $x : X$, $\ell' : L(X)$ such that $\ell =_C x :: \ell'$, and $\text{zeroth}(\ell) =_C x$. Note that $\text{def} = x$ is a term of type X, and

$$\text{zerothDef}(\text{def}, \ell) = \text{zerothDef}(\text{def}, x :: \ell') = x = \text{zeroth}(\ell).$$

In fact, for any term $x' : X$, we have $\text{zerothDef}(x', \ell) = \text{zeroth}(\ell)$ by the same calculation.

For the second point, we know by definition of nth that $\text{len}(\text{tail}(m, \ell)) > 0$, and so $\text{tail}(m, \ell) = x :: \ell'$ for some unique x, ℓ'. Again, $\text{def} = x$ is a term of type X in C, and

$$\begin{aligned}\text{nthDef}(\text{def}, m, \ell) &= \text{zerothDef}(\text{def}, \text{tail}(m, \ell)) \\ &= \text{zerothDef}(\text{def}, x :: \ell') \\ &= x = \text{zeroth}(\text{tail}(m, \ell)) = \text{nth}(m, \ell).\end{aligned}$$

For any other $x' : X$, we have $\text{nthDef}(x', m, \ell) = \text{nth}(m, \ell)$ by the same calculation. ∎

We record an easy fact about the tail arrow.

Proposition 4.4.5. Let $\mathcal{C}$ be a Cartesian category with list objects. Then

$$\text{tail}(a + b, \ell) =_{a,b,\ell} \text{tail}(a, \text{tail}(\ell, b)).$$

Therefore, $\text{nthDef}(x, a + b, \ell) =_{x,a,b,\ell} \text{nthDef}(x, a, \text{tail}(\ell, b))$. If $\mathcal{C}$ is a locos, then for $a + b < \text{len}(\ell)$, we have $\text{nth}(a + b, \ell) = \text{nth}(a, \text{tail}(b, \ell))$.

Proof. The first equality is easy to prove by induction on a, and the two consequences follow directly. ∎

We end this section with a technical calculation that we will use later on.

Proposition 4.4.6. Let $\mathcal{C}$ be a Cartesian category with list objects. Then we have

$$\mathrm{tail}(m, \ell) =_{x,m,\ell} \begin{cases} \varnothing & \text{if } \mathrm{len}(\ell) \leq m \\ \mathrm{nthDef}(x, m, \ell) :: \mathrm{tail}(sm, \ell) & \text{else} \end{cases}.$$

Moreover, if $\mathcal{C}$ is a locos, then

$$\mathrm{tail}(m, \ell) =_{m,\ell} \begin{cases} \varnothing & \text{if } \mathrm{len}(\ell) \leq m \\ \mathrm{nth}(m, \ell) :: \mathrm{tail}(sm, \ell) & \text{else} \end{cases}.$$

Proof. We start with the first equality. We first prove by induction on ℓ that

$$\ell =_{x,\ell} \begin{cases} \varnothing & \text{if } \mathrm{len}(\ell) = 0 \\ \mathrm{zerothDef}(x, \ell) :: \mathrm{tr}(\ell) & \text{else} \end{cases}.$$

If $\ell = \varnothing$, then $\mathrm{len}(\ell) = 0$, so the right side reduces to $\varnothing$. If $\ell = y :: \ell'$, then $\mathrm{len}(\ell) = s(\mathrm{len}(\ell'))$, and the right side reduces to $\mathrm{zerothDef}(x, y :: \ell') :: \mathrm{tr}(y :: \ell') = y :: \ell' = \ell$.

Now, in the above equality, we replace ℓ by $\mathrm{tail}(m, \ell)$. The else case simplifies (using the definition of nthDef) to

$$\mathrm{zerothDef}(x, \mathrm{tail}(m, \ell)) :: \mathrm{tr}(\mathrm{tail}(m, \ell)) = \mathrm{nthDef}(x, m, \ell) :: \mathrm{tail}(sm, \ell).$$

The term $\mathrm{len}(\mathrm{tail}(m, \ell))$ in the "if" condition simplifies (by proposition 4.3.3) to $\mathrm{len}(\ell) \dot{-} m$, so the condition is equivalent to $\mathrm{len}(\ell) \leq m$. Thus, we get the desired equality.

Now, suppose that $\mathcal{C}$ is a locos. It's clear (from previous results) that

$$\mathrm{tail}(m, \ell) =_{m,\ell} \begin{cases} \varnothing & \text{if } \mathrm{len}(\ell) \leq m \\ \mathrm{tail}(m, \ell) & \text{else} \end{cases}.$$

But now, the "else" case is the context $C = (m : N, \ell : L(X) \mid m < \mathrm{len}(\ell))$; note that $\mathrm{nth}(m, \ell) =_C \mathrm{nthDef}(\mathrm{def}, m, \ell)$ for some term $\mathrm{def} : X$ by proposition 4.4.4. Then, if we take the first part of this proposition and substitute $\mathrm{def} \mapsto x$, $m \mapsto m$, $\ell \mapsto \ell$, we find that

$$\mathrm{tail}(m, \ell) =_C \mathrm{nthDef}(\mathrm{def}, m, \ell) :: \mathrm{tail}(sm, \ell) =_C \mathrm{nth}(m, \ell) :: \mathrm{tail}(sm, \ell).$$

Substituting this into the previous equality gives the desired result. ∎

4.5 The concatenation arrow

In this section, we define concatenation of lists and prove some of its properties.

Definition. Let $\mathcal{C}$ be a Cartesian category with list objects. We define the *concatenation* function $L(X) \times L(X) \to L(X)$, which we denote by $\ell_1 +\!\!+ \ell_2$, by induction as follows:

$$\varnothing +\!\!+ \ell_2 =_{\ell_2} \ell_2, \qquad\qquad (x :: \ell_1) +\!\!+ \ell_2 =_{x,\ell_1,\ell_2} x :: (\ell_1 +\!\!+ \ell_2).$$

Proposition 4.5.1. We have the following facts about concatenation.

- $\mathrm{len}(\ell_1 +\!\!+ \ell_2) =_{\ell_1,\ell_2} \mathrm{len}(\ell_1) + \mathrm{len}(\ell_2)$
- $\ell_1 +\!\!+ (\ell_2 +\!\!+ \ell_3) =_{\ell_1,\ell_2,\ell_3} (\ell_1 +\!\!+ \ell_2) +\!\!+ \ell_3$

Proof. In both cases, this is an easy induction on ℓ_1. ∎

The next results show how concatenation interacts with taking the tail of a list.

Proposition 4.5.2. In a locos, we have:

1. If $\mathrm{len}(\ell_1) > 0$, then $\mathrm{tr}(\ell_1) +\!\!+ \ell_2 = \mathrm{tr}(\ell_1 +\!\!+ \ell_2)$;
2. $\mathrm{tail}(\mathrm{len}(\ell_1), \ell_1 +\!\!+ \ell_2) = \ell_2$;
3. If $k \leq \mathrm{len}(\ell_1)$, then $\mathrm{tail}(k, \ell_1 +\!\!+ \ell_2) = \mathrm{tail}(k, \ell_1) +\!\!+ \ell_2$.

Proof. For part 1, we note that $\mathrm{len}(\ell_1) > 0$ implies that we can write $\ell_1 = x :: \ell$ for some x, ℓ (by proposition 4.4.2). Then

$$\begin{aligned} \mathrm{tr}(\ell_1) +\!\!+ \ell_2 &= \mathrm{tr}(x :: \ell) +\!\!+ \ell_2 = \ell +\!\!+ \ell_2 \\ \text{and } \mathrm{tr}(\ell_1 +\!\!+ \ell_2) &= \mathrm{tr}(x :: (\ell +\!\!+ \ell_2)) = \ell +\!\!+ \ell_2, \end{aligned}$$

so we have the desired equality.

For part 2, we go by induction on ℓ_1. The base case $\ell_1 = \varnothing$ is trivial. For the inductive step, we compute

$$\begin{aligned} \mathrm{tail}\Big(\mathrm{len}(x :: \ell_1),\ (x :: \ell_1) +\!\!+ \ell_2\Big) &= \mathrm{tail}\Big(1 + \mathrm{len}(\ell_1),\ x :: (\ell_1 +\!\!+ \ell_2)\Big) \\ &= \mathrm{tail}\Big(\mathrm{len}(\ell_1),\ \mathrm{tr}(x :: (\ell_1 +\!\!+ \ell_2))\Big) \\ &= \mathrm{tail}(\mathrm{len}(\ell_1),\ \ell_1 +\!\!+ \ell_2). \end{aligned}$$

(Note: $\mathrm{tail}(sk, \ell) = \mathrm{tail}(k, \mathrm{tr}(\ell))$ is easy to prove by induction on k.) Since $(\ell_1, \ell_2) \mapsto \ell_2$ satisfies the same recurrence, we have the desired equality.

Finally, for part 3, we use theorem 3.5.2 to prove this by finite induction. For the case $k = 0$, both sides clearly just reduce to $\ell_1 +\!\!+ \ell_2$. For the case $k = \mathrm{len}(\ell_1)$, we note (using proposition 4.3.3) that

$$\mathrm{tail}(\mathrm{len}(\ell_1),\ \ell_1) +\!\!+\ \ell_2 = \varnothing +\!\!+\ \ell_2 = \ell_2,$$

which is equal to $\mathrm{tail}(\mathrm{len}(\ell_1), \ell_1 +\!\!+\ \ell_2)$ by part 2.

For the inductive step, let $k < \mathrm{len}(\ell_1)$. We compute

$$\mathrm{tail}(sk, \ell_1 +\!\!+\ \ell_2) = \mathrm{tr}(\mathrm{tail}(k, \ell_1 +\!\!+\ \ell_2))$$

and

$$\mathrm{tail}(sk, \ell_1) +\!\!+\ \ell_2) = \mathrm{tr}(\mathrm{tail}(k, \ell_1)) +\!\!+\ \ell_2 = \mathrm{tr}(\mathrm{tail}(k, \ell_1) +\!\!+\ \ell_2).$$

This last equality is by part 1, which we can apply because $k < \mathrm{len}(\ell_1)$ implies that $\mathrm{len}(\mathrm{tail}(k, \ell_1)) > 0$. Thus, we see that both sides satisfy the same recurrence, and we're done. ∎

Corollary 4.5.3. In a locos, let $\ell_1, \ell_2 : L(X)$.

- If $m < \mathrm{len}(\ell_1)$, then $\mathrm{nth}(m, \ell_1 +\!\!+\ \ell_2) = \mathrm{nth}(m, \ell_1)$.
- If $m < \mathrm{len}(\ell_2)$, then $\mathrm{nth}(\mathrm{len}(\ell_1) + m, \ell_1 +\!\!+\ \ell_2) = \mathrm{nth}(m, \ell_2)$.

Proof. For the first point, since $m < \mathrm{len}(\ell_1)$, we use proposition 4.5.2 to compute

$$\mathrm{nth}(m, \ell_1 +\!\!+\ \ell_2) = \mathrm{zeroth}(\mathrm{tail}(m, \ell_1 +\!\!+\ \ell_2)) = \mathrm{zeroth}(\mathrm{tail}(m, \ell_1) +\!\!+\ \ell_2).$$

Now, we have $\mathrm{len}(\mathrm{tail}(m, \ell_1)) = \mathrm{len}(\ell_1) \dot{-} m > 0$ (by propositions 4.3.3 and 3.2.4), so by proposition 4.4.2, we can write $\mathrm{tail}(m, \ell_1) = x :: \ell'$. By definition, we have $\mathrm{nth}(m, \ell_1) = \mathrm{zeroth}(\mathrm{tail}(m, \ell_1)) = x$. However, we also have

$$\mathrm{nth}(m, \ell_1 +\!\!+\ \ell_2) = \mathrm{zeroth}(\mathrm{tail}(m, \ell_1) +\!\!+\ \ell_2) = \mathrm{zeroth}(x :: (\ell' +\!\!+\ \ell_2)) = x,$$

so we get $\mathrm{nth}(m, \ell_1) = \mathrm{nth}(m, \ell_1 +\!\!+\ \ell_2)$, as desired.

For the second point, we first use propositions 4.5.2 and 4.4.5 to compute

$$\operatorname{tail}(\operatorname{len}(\ell_1) + m, \ell_1 +\!\!+ \ell_2) = \operatorname{tail}\Big(m,\ \operatorname{tail}(\operatorname{len}(\ell_1), \ell_1 +\!\!+ \ell_2)\Big) = \operatorname{tail}(m, \ell_2).$$

Therefore,

$$\begin{aligned}\operatorname{nth}(\operatorname{len}(\ell_1) + m, \ell_1 +\!\!+ \ell_2) &= \operatorname{zeroth}\Big(\operatorname{tail}(\operatorname{len}(\ell_1) + m, \ell_1 +\!\!+ \ell_2)\Big)\\ &= \operatorname{zeroth}(\operatorname{tail}(m, \ell_2)) = \operatorname{nth}(m, \ell_2),\end{aligned}$$

as desired. ∎

4.6 The head of a list

In this section, we define a function that gives the head of a list (its first n elements). Throughout this section, we will be working in a locos.

First, we define a preliminary function that will be needed for the head function. Note that we need extensivity for this definition by cases, since the term $\operatorname{nth}(n, \ell)$ only exists in the context $(n, \ell \mid n < \operatorname{len}(\ell))$.

Definition. Let $\mathcal{C}$ be a locos, and let $X \in \mathcal{C}$. We define $\operatorname{nthSingleton} : N \times L(X) \to L(X)$ by

$$\operatorname{nthSingleton}(n, \ell) =_{n,\ell} \begin{cases} [\operatorname{nth}(n, \ell)] & \text{if } n < \operatorname{len}(\ell) \\ \varnothing & \text{if } n \geq \operatorname{len}(\ell) \end{cases}.$$

Proposition 4.6.1. In a locos, $\operatorname{nthSingleton}(sn, x :: \ell) = \operatorname{nthSingleton}(n, \ell)$.

Proof. Note that $sn < \operatorname{len}(x :: \ell)$ is the same as $n < \operatorname{len}(\ell)$, and if $n < \operatorname{len}(\ell)$, then

$$\operatorname{nth}(sn, x :: \ell) = \operatorname{nth}(n, \operatorname{tr}(x :: \ell)) = \operatorname{nth}(n, \ell).$$

Hence the two terms are the same. ∎

Finally, we are ready to define the head function. Intuitively, $\operatorname{head}(n, \ell)$ should give the first n elements of ℓ.

Definition. We define $\operatorname{head} : N \times L(X) \to L(X)$ inductively by

$$\begin{aligned}\operatorname{head}(0, \ell) &= \varnothing,\\ \operatorname{head}(sn, \ell) &= \operatorname{head}(n, \ell) +\!\!+ \operatorname{nthSingleton}(n, \ell).\end{aligned}$$

Proposition 4.6.2. In a locos, we have:

1. $\mathrm{head}(sm, x :: \ell) = x :: \mathrm{head}(m, \ell)$
2. $\mathrm{head}(\mathrm{len}(\ell), \ell) = \ell$
3. If $n \geq \mathrm{len}(\ell)$, then $\mathrm{head}(n, \ell) = \ell$.

Proof. Part 1. By induction on m. Base case:

$$\begin{aligned} \mathrm{head}(1, x :: \ell) &= \mathrm{head}(0, x :: \ell) +\!\!+ \mathrm{nthSingleton}(0, x :: \ell) = \varnothing +\!\!+ [\mathrm{nth}(0, x :: \ell)] \\ &= \varnothing +\!\!+ [x] = [x], \\ x :: \mathrm{head}(0, \ell) &= x :: \varnothing = [x]. \end{aligned}$$

Inductive step (using proposition 4.6.1):

$$\begin{aligned} \mathrm{head}(ssm, x :: \ell) &= \mathrm{head}(sm, x :: \ell) +\!\!+ \mathrm{nthSingleton}(sm, x :: \ell) \\ &= \mathrm{head}(sm, x :: \ell) +\!\!+ \mathrm{nthSingleton}(m, \ell), \\ x :: \mathrm{head}(sm, \ell) &= x :: \Big(\mathrm{head}(m, \ell) +\!\!+ \mathrm{nthSingleton}(m, \ell)\Big) \\ &= \Big(x :: \mathrm{head}(m, \ell)\Big) +\!\!+ \mathrm{nthSingleton}(m, \ell). \end{aligned}$$

So, they satisfy the same recursion.

Part 2. By induction on ℓ. Base case:

$$\mathrm{head}(\mathrm{len}(\varnothing), \varnothing) = \mathrm{head}(0, \varnothing) = \varnothing.$$

Inductive step (using part 1):

$$\mathrm{head}(\mathrm{len}(x :: \ell), x :: \ell) = \mathrm{head}(1 + \mathrm{len}(\ell), x :: \ell) = x :: \mathrm{head}(\mathrm{len}(\ell), \ell).$$

This recursion is satisfied by the identity.

Part 3. This is equivalent to saying $\mathrm{head}(\mathrm{len}(\ell) + m, \ell) = \ell$ for any m; we prove this by induction on m. The base case is just part 2; for the inductive case, we compute

$$\begin{aligned} \mathrm{head}(\mathrm{len}(\ell) + sm, \ell) &= \mathrm{head}(\mathrm{len}(\ell) + m, \ell) +\!\!+ \mathrm{nthSingleton}(\mathrm{len}(\ell) + m, \ell) \\ &= \mathrm{head}(\mathrm{len}(\ell) + m, \ell) +\!\!+ \varnothing \\ &= \mathrm{head}(\mathrm{len}(\ell) + m, \ell). \end{aligned}$$

Note that the second line just uses the definition of nthSingleton. ∎

Proposition 4.6.3. In a locos, $\ell =_{n,\ell} \mathrm{head}(n, \ell) +\!\!+ \mathrm{tail}(n, \ell)$

Proof. We go by induction on n. In the zero case,

$$\mathrm{head}(0, \ell) +\!\!+ \mathrm{tail}(0, \ell) = \varnothing +\!\!+ \ell = \ell.$$

For the inductive step, we split into the cases $n \geq \mathrm{len}(\ell)$ and $n < \mathrm{len}(\ell)$. In the first case, we we have $sn \geq \mathrm{len}(\ell)$, so $\mathrm{head}(sn, \ell) = \ell$ (by proposition 4.6.2) and $\mathrm{tail}(sn, \ell) = \varnothing$ (proposition 4.3.3), so

$$\mathrm{head}(sn, \ell) +\!\!+ \mathrm{tail}(sn, \ell) = \ell.$$

In the case $n < \mathrm{len}(\ell)$, we compute

$$\begin{aligned} \mathrm{head}(sn, \ell) +\!\!+ \mathrm{tail}(sn, \ell) &= \mathrm{head}(n, \ell) +\!\!+ [\mathrm{nth}(n, \ell)] +\!\!+ \mathrm{tail}(sn, \ell) \\ &= \mathrm{head}(n, \ell) +\!\!+ \mathrm{tail}(n, \ell) \end{aligned}$$

by the definition of head and by proposition 4.4.6. (Note also that associativity is used implicitly.)

This recursion is also satisfied by $f(n, \ell) = \ell$, so we're done. ∎

Corollary 4.6.4. In a locos, $\mathrm{len}(\mathrm{head}(n, \ell)) =_{n,\ell} \min(n, \mathrm{len}(\ell))$. In particular, if $n \leq \mathrm{len}(\ell)$, then we have $\mathrm{len}(\mathrm{head}(n, \ell)) = n$.

Proof. By propositions 4.5.1 and 4.6.3,

$$\mathrm{len}(\ell) = \mathrm{len}(\mathrm{head}(n, \ell)) + \mathrm{len}(\mathrm{tail}(n, \ell)).$$

Then, by proposition 4.3.3,

$$\mathrm{len}(\mathrm{head}(n, \ell)) = \mathrm{len}(\ell) \dot{-} \mathrm{len}(\mathrm{tail}(n, \ell)) = \mathrm{len}(\ell) \dot{-} (\mathrm{len}(\ell) \dot{-} n) = \min(n, \mathrm{len}(\ell)),$$

as desired. ∎

Proposition 4.6.5. In a locos, we have the following:

1. $\mathrm{tr}(\mathrm{head}(n, x :: \ell)) = \mathrm{head}(Pn, \ell)$
2. $\mathrm{tr}(\mathrm{head}(n, \ell)) = \mathrm{head}(Pn, \mathrm{tr}(\ell))$
3. $\mathrm{tail}(m, \mathrm{head}(n, \ell)) = \mathrm{head}(n \dot{-} m, \mathrm{tail}(m, \ell))$

Proof. For 1, we go by cases on n. The $n = 0$ case is trivial (both reduce to $\varnothing$ by definition of head). For the successor case, we use proposition 4.6.2 to compute

$$\mathrm{tr}(\mathrm{head}(sn, x :: \ell)) = tr(x :: \mathrm{head}(n, \ell)) = \mathrm{head}(n, \ell) = \mathrm{head}(Psn, \ell).$$

For 2, we go by cases on ℓ. If $\ell = \varnothing$, both sides equal $\varnothing$ (we can apply part 3 of proposition 4.6.2 to note that $\mathrm{head}(m, \varnothing) = \varnothing$). In the $x :: \ell$ case, we use part 1 to compute

$$\mathrm{tr}(\mathrm{head}(n, x :: \ell)) = \mathrm{head}(Pn, \ell) = \mathrm{head}(Pn, \mathrm{tr}(x :: \ell)).$$

For 3, we go by induction on m. For the $m = 0$ case, both terms clearly reduce to $\mathrm{head}(n, \ell)$. For the inductive step, we show that both terms satisfy $f(sm, n, \ell) = \mathrm{tr}(f(m, n, \ell))$. This is clear for the left hand side; for the right hand side, we use part 2 to compute

$$\begin{aligned}\mathrm{head}(n \dot{-} sm, \mathrm{tail}(sm, \ell)) &= \mathrm{head}(P(n \dot{-} m), \mathrm{tr}(\mathrm{tail}(m, \ell))) \\ &= \mathrm{tr}(n \dot{-} m, \mathrm{head}(\mathrm{tail}(m, \ell)).\end{aligned}$$

This is all we needed to show. ∎

Chapter 5

The list adjunction

Contents

This chapter is mostly dedicated to proving corollary 5.4.2, which is the first main result of this thesis. Section 5.1 outlines the proof of this result, and explains how the remaining chapters fit into the proof. Briefly, sections 5.2 and 5.3 perform some technical calculations; then, section 5.4 puts the proof together and proves a few easy corollaries.

5.1 Goal

In a locos $\mathcal{C}$, we can define a functor $L_N : \mathcal{C} \to \mathcal{C}/N$ by mapping $X \in \mathcal{C}$ to $\mathrm{len}_X : L(X) \to N$. If $f : X \to Y$ is an arrow, then $L_N(f) = L(f)$; this is well-defined because $\mathrm{len}_Y \circ L(f) = \mathrm{len}_X$ by proposition 4.3.1.

In this chapter, we aim to prove that L_N is the right adjoint of the functor $\Sigma_E \circ \Delta_{\pi_2^E} : \mathcal{C}/N \to \mathcal{C}$ (which maps $l_A : A \to N$ to $E \times_N A$). An important consequence of this adjunction is the first main result of this thesis: that the arrow $\pi_2^E : E \to N$ is exponentiable, and that the list object functor L is a polynomial functor. We will see these consequences in section 5.4.

Before we can prove L_N is the right adjoint of $\Sigma_E \circ \Delta_{\pi_2^E}$, we need to specify the counit of the adjunction, which will be $(\mathrm{nth}_X)_X$; recall that $\mathrm{nth}_X : E \times_N L(X) \to X$ was defined in section 4.4. Indeed, the collection $(\mathrm{nth}_X)_X$ forms a natural transformation.

Proposition 5.1.1. Assume we are working in a locos. The collection of arrows $(\mathrm{nth}_X)_X$ is a natural transformation $\Sigma_E \circ \Delta_{\pi_2^E} \circ L_N \Rightarrow \mathrm{Id}_{\mathcal{C}}$.

Proof. Let $f : X \to Y$ be an arrow of $\mathcal{C}$. We must show that the following square commutes.

$$\begin{array}{ccc} E \times_N L(X) & \xrightarrow{\mathrm{nth}_X} & X \\ {\scriptstyle \mathrm{Id}_E \times_N L(f)}\downarrow & & \downarrow{\scriptstyle f} \\ E \times_N L(Y) & \xrightarrow[\mathrm{nth}_Y]{} & Y \end{array}$$

Let C be the context $(m : N,\ \ell : L(X) \mid m < \mathrm{len}(\ell))$. Note that, in the context C, $\mathrm{def} = \mathrm{nth}(m, \ell)$ is a term of type X, and so $f(\mathrm{def})$ is a term of type Y. Then we compute:

$$\begin{aligned} \mathrm{nth}_Y(m, L(f)(\ell)) &=_C \mathrm{nthDef}_Y\big(f(\mathrm{def}),\ m,\ L(f)(\ell)\big) \\ &=_C f(\mathrm{nthDef}_X(\mathrm{def}, m, \ell)) \\ &=_C f(\mathrm{nth}_X(m, \ell)). \end{aligned}$$

For the first and last equalities, we used proposition 4.4.4. This is allowed because $m < \mathrm{len}(\ell) = \mathrm{len}(L(f)(\ell))$ (using proposition 4.3.1). Finally, the second equality also uses proposition 4.3.1. ∎

Then, to prove that L_N is a right adjoint of $\Sigma_E \circ \Delta_{\pi_2^E}$ (with counit $(\mathrm{nth}_X)_X$), it will suffice to show that for each X, $(L_N(X), \mathrm{nth}_X)$ is a universal morphism from $\Sigma_E \circ \Delta_{\pi_2^E}$ to X. Unravelling the definitions, this means that we must prove the following universal property for each X:

For any $l_A : A \to N$ and $g : E \times_N A \to X$, there exists a unique $h : A \to L(X)$ such that $l_A = \mathrm{len}_X \circ h$ and $g = \mathrm{nth}_X \circ (\mathrm{Id} \times_N h)$.

We will show existence in section 5.2 and uniqueness in section 5.3. We finish the proof of the adjunction in section 5.4, and establish some easy corollaries, including the fact that the list object functor L is polynomial.

5.2 Constructing lists

Throughout this section, we assume we're working in a category $\mathcal{C}$ with finite limits and parametrized list objects (and therefore a parametrized natural numbers object).

As noted in section 5.1, the goal of this section is to show the existence part of the statement that $(L_N(X), \mathrm{nth}_X)$ is a universal morphism. That is, given arrows $l_A : A \to N$ and $g : E \times_N A \to X$, we must construct an arrow $h : A \to L(X)$ such that $l_A = \mathrm{len}_X \circ h$ and $g = \mathrm{nth}_X \circ (\mathrm{Id} \times_N h)$.

It is particularly difficult to construct maps into a list object because its universal property only provides inductive constructions of maps out of it. Therefore, we start this section by developing a general technique for constructing such maps. Then, we establish some properties of this technique, before finally applying it to this particular case.

5.2.1 Constructing maps into a list object

The key idea for constructing maps into $L(X)$ is that we should use the inductive property of NNOs. Indeed, suppose we have a map $f : N \times A \to X$, which we think of as a collection of infinite sequences $(x_{m,a})_m$ indexed by $a \in A$. We can use this map to construct the map $\mathrm{Seq}[f] : N \times N \times A \to L(X)$, where $\mathrm{Seq}[f](m, n, a)$ represents $[x_{m,a}, ..., x_{m+n-1,a}]$. This is done by induction on n, the length of this list, as follows.

$$\mathrm{Seq}[f](m, 0, a) =_{m,a} \varnothing \qquad \mathrm{Seq}[f](m, s(n), a) =_{m,n,a} \mathrm{Seq}[f](m, n, a) +\!\!+ [f(m + n, a)]$$

If we are also given a map $p : A \to N$ which represents the length of a list associated to a, then we can define $\mathrm{List}[f, p] : A \to L(X)$ by setting

$$\mathrm{List}[f, p](a) =_a \mathrm{Seq}[f](0, p(a), a).$$

Intuitively, we have $\mathrm{List}[f, p](a) = [x_{0,a}, ..., x_{p(a)-1,a}]$. The drawback of this technique is that it requires an infinite sequence $(x_{n,a})_n$ to specify this finite list; we address this issue later with lemma 5.2.6.

We end this section with some very basic facts about this construction.

Proposition 5.2.1. $\mathrm{len} \circ \mathrm{List}[f, p] = p$

Proof. To obtain this equality, it suffices to show that that $\mathrm{len}(\mathrm{Seq}[f](m, n, a)) =_{m,n,a} n$. This is done by induction on n: using proposition 4.5.1 and noting that $\mathrm{len}([x]) =_x 1$, it is easy to show that these terms both satisfy the introductory equations $h(m, 0, a) =_{m,a} 0$ and $h(m, sn, a) =_{m,n,a} h(m, n, a) + 1$. ∎

Proposition 5.2.2. $\mathrm{Seq}[f](m, sn, a) =_{m,n,a} f(m, a) :: \mathrm{Seq}[f](sm, n, a)$.

Proof. This is proved by induction on n: it is straightforward to check that both terms satisfy $h(m, 0, a) =_{m,a} [f(m, a)]$ and $h(m, sn, a) =_{m,n,a} h(m, n, a) +\!\!+ [f(s(m + n), a)]$. ∎

5.2.2 The nth elements of constructed lists

In this section, we prove an important result about the interaction between nth and $\text{List}[f, p]$: specifically, we show with theorem 5.2.5 that the m^{th} element of $\text{List}[f, p](a)$ is $f(m, a)$, as long as $m < p(a)$.

We start with some intermediate results. The first thing we need to do is investigate the interaction between truncation and $\text{Seq}[f]$, because nthDef is built using tr, and List is built using Seq. Recall that P represents the predecessor function.

Proposition 5.2.3. Let $f : N \times A \to X$. Then

$$\text{tr}(\text{Seq}[f](m, n, a)) =_{m,n,a} \text{Seq}[f](sm, Pn, a).$$

Proof. By induction on n, it suffices to show that the arrows agree on 0 and s. For the base case, it's clear that both terms reduce to $\varnothing$; for the inductive case, we use proposition 5.2.2:

$$\begin{aligned} \text{tr}(\text{Seq}[f](m, sn, a)) =_{m,n,a} \text{tr}\Big(f(m, a) :: \text{Seq}[f](sm, n, a)\Big) &=_{m,n,a} \text{Seq}[f](sm, n, a), \\ \text{Seq}[f](sm, Psn, a) &=_{m,n,a} \text{Seq}[f](sm, n, a). \end{aligned}$$

This is all we need to show. ∎

From this fact, we expect that iterating the truncation map just turns into iterating the successor s and predecessor P. However, iterating the successor just becomes addition, and iterating the predecessor just becomes subtraction. This observation gives us the following important lemma.

Lemma 5.2.4. Let $f : N \times A \to X$. Then

$$\text{tail}\Big(k,\ \text{Seq}[f](m, n, a)\Big) =_{k,m,n,a} \text{Seq}[f]\Big(m + k,\ n \dot{-} k,\ a\Big).$$

In particular, given $p : A \to N$, we have

$$\text{tail}\Big(k,\ \text{List}[f, p](a)\Big) =_{k,a} \text{Seq}[f]\Big(k,\ p(a) \dot{-} k,\ a\Big).$$

Proof. We start with the first equality, which we prove by induction on k. To this end, we show that both arrows satisfy the equalities $h(0, m, n, a) =_{m,n,a} \text{Seq}[f](m, n, a)$ and $h(sk, m, n, a) =_{k,m,n,a} \text{tr}(h(k, m, n, a))$. The first equality is trivial, since $\text{tail}(0, \ell) =_\ell \ell$, $m + 0 =_m m$, and $n \dot{-} 0 =_n n$. The second equality is immediate for the left-hand side function (it's the definition of tail); for the right-hand side function, we use proposition

5.2.3.

$$\begin{aligned}\mathrm{Seq}[f]\Big(m+sk,\ n\dot{-}sk,\ a\Big) &=_{k,m,n,a} \mathrm{Seq}[f]\Big(s(m+k),\ P(n\dot{-}k),\ a\Big)\Big) \\ &=_{k,m,n,a} \mathrm{tr}\Big(\mathrm{Seq}[f]\Big(m+k,\ n\dot{-}k,\ a\Big)\Big)\end{aligned}$$

This is all we needed to show for the first part. For the second part, we simply replace m by 0 and n by $p(a)$, then use the definition of $\mathrm{List}[f,p]$. ∎

Finally, we arrive at the desired result. Note that we need to work in a locos in order to talk about the nth arrow.

Theorem 5.2.5. In a locos, let $f : N \times A \to X$ and $p : A \to N$, and let $m : N$, $a : A$ be terms in a context C. If $m <_C p(a)$, then

$$\mathrm{nth}(m, \mathrm{List}[f,p](a)) =_C f(m,a).$$

Proof. First, note that the term on the left is well-defined, because $\mathrm{len}(\mathrm{List}[f,p](a)) =_C p(a)$ (by proposition 5.2.1) and $m <_C p(a)$ be assumption. Then, by lemma 5.2.4 and proposition 4.4.4,

$$\begin{aligned}\mathrm{nth}(m, \mathrm{List}[f,p](a)) &=_C \mathrm{nthDef}(\mathrm{def}, m, \mathrm{List}[f,p](a)) \\ &=_C \mathrm{zerothDef}\Big(\mathrm{def},\ \mathrm{tail}\Big(m,\ \mathrm{List}[f,p](a)\Big)\Big) \\ &=_C \mathrm{zerothDef}\Big(\mathrm{def},\ \mathrm{Seq}[f]\Big(m, p(a)\dot{-}m, a\Big)\Big).\end{aligned}$$

Since $m <_C p(a)$ by assumption, we have $p(a)\dot{-}m >_C 0$ (by proposition 3.2.4). Therefore, by proposition 3.2.3, $p(a)\dot{-}m =_C s(k)$ for some term k. We substitute this into the above equality, apply proposition 5.2.2, and use the definition of zerothDef. We get:

$$\begin{aligned}\mathrm{nth}(m, \mathrm{List}[f,p](a)) &=_C \mathrm{zerothDef}\Big(\mathrm{def},\ \mathrm{Seq}[f](m, sk, a)\Big) \\ &=_C \mathrm{zerothDef}\Big(\mathrm{def},\ f(m,a) :: \mathrm{Seq}[f](sm, k, a)\Big) =_C f(m,a).\end{aligned}$$

This is what we wanted to show. ∎

5.2.3 The universal property arrow

The construction $\mathrm{List}[f,p]$ allows us to construct an arrow $A \to L(X)$ based on two arrows $f : N \times A \to X$ and $p : A \to N$. However, the goal stated at the beginning of this chapter is to construct $h : A \to L(X)$ from $g : E \times_N A \to X$ and $l_A : A \to N$, so we need to adjust this technique.

Recall from proposition 3.4.2 that $E \times_N A = \{m : N,\, a : A \mid m < l_A(a)\}$. This means that the arrow $g : E \times_N A \to X$ only gives us finite sequences $(x_{0,a}, ..., x_{l_A(a)-1,a})$ instead of the infinite sequences used in List$[f, p]$. We account for this discrepancy by using two tricks.

- First, if $l_A(a) > 0$, then we can extend the finite sequence to an infinite one simply by repeating the last element; we will use lemma 5.2.6, below, to do this formally.
- Second, we note that the first trick doesn't work if $l_A(a) = 0$, because then g doesn't give us any elements of X. However, $l_A(a) = 0$ just means that we should set $h(a) = \varnothing$. So, we have to use extensivity to split into two cases: $l_A(a) = 0$ and $l_A(a) > 0$.

We employ these techniques in the proof of theorem 5.2.7, below. Note that extensivity is also required to talk about nth_X.

Lemma 5.2.6. In a locos, let $l_A : A \to N$ be an arrow, let $C = (m : N,\, a : A \mid m < l_A(a))$, and let $A_{>0} = (a : A \mid l_A(a) > 0)$. Then, for any $g : [C] \to X$, there is an arrow $g' : N \times [A_{>0}] \to X$ such that

$$g'(m, a) =_C g(m, a).$$

Proof. We define g' by

$$g'(m, a) =_{m,a} \begin{cases} g(m, a) & \text{if } m < l_A(a) \\ g(0, a) & \text{else} \end{cases}.$$

We must first check that this is well-defined. If $m < l_A(a)$, then of course $(m, a) : C$ and we can apply g to this pair. On the other hand, the domain of g' tells us that $l_A(a) > 0$, so $g(0, a)$ is still well-defined even if $m \geq l_A(a)$. (Note that extensivity is required for this definition.)

Now, if we take $(m, a) : C$, then $l_A(a) > m \geq 0$, so we can say that $a : A_{>0}$. Thus $g'(m, a)$ is well-defined, and since we are in the case $m < l_A(a)$, we find that $g'(m, a) =_C g(m, a)$ by definition. ∎

Theorem 5.2.7. Let $\mathcal{C}$ be a locos. Given objects $X, A \in \mathcal{C}$ and arrows $l_A : A \to N$ and $g : E \times_N A \to X$, there exists an arrow $h : A \to L(X)$ such that $l_A = \mathrm{len}_X \circ h$ and $g = \mathrm{nth}_X \circ (\mathrm{Id}_E \times_N h)$.

Proof. Proposition 3.3.2 tells us that we have the coproduct $A = [A_0] + [A_{>0}]$, where $A_0 = (a : A \mid l_A(a) = 0)$ and $A_{>0} = (a : A \mid l_A(a) > 0)$. This lets us define $h : A \to L(X)$ "by cases" on each part of this coproduct.

In the case $l_A(a) = 0$, we simply set $h(a) =_{A_0} \varnothing$. For the case $l_A(a) > 0$, note that the domain of g is $[C]$, where $C = (m : N,\, a : A \mid m < l_A(a))$. Then lemma 5.2.6 tells us that there is an arrow $g' : N \times [A_{>0}] \to X$ such that $g'(m, a) =_C g(m, a)$, and we can define h in the $A_{>0}$ case to be

$$h(a) =_{A_{>0}} \text{List}[g', l'_A](a),$$

where $l'_A : [A_{>0}] \to N$ is just given by $l'_A(a) =_{A_{>0}} l_A(a)$. This completes the definition of h; we must now check that it satisfies the required equalities.

First, we check that $\text{len}(h(a)) =_a l_A(a)$. This can be done by checking equality on both parts of the coproduct. In the A_0 case, we have $\text{len}(h(a)) =_{A_0} \text{len}(\varnothing) =_{A_0} 0 =_{A_0} l_A(a)$, and in the $A_{>0}$ case, we have

$$\text{len}(h(a)) =_{A_{>0}} \text{len}(\text{List}[g', l'_A](a)) =_{A_{>0}} l'_A(a) =_{A_{>0}} l_A(a)$$

by using proposition 5.2.1.

Second, we check that $\text{nth}_X(m, h(a)) =_C g(m, a)$. We could check on both parts of the coproduct, but this would be redundant: the condition $m < l_A(a)$ in C implies that we are already in the $A_{>0}$ case. Formally, performing the substitution $a \mapsto a$ in the equation $h(a) =_{A_{>0}} \text{List}[g', l'_A](a)$ gives us the equality $h(a) =_C \text{List}[g', l'_A](a)$, and this substitution is valid because $l_A(a) > 0$ in C. Using this equality and theorem 5.2.5, we compute

$$\text{nth}_X(m, h(a)) =_C \text{nth}_X(m, \text{List}[g', l'_A](a)) =_C g'(m, a) =_C g(m, a),$$

which is what we wanted. ∎

5.3 Equality of lists

Throughout this section, we assume we're working in a category $\mathcal{C}$ with finite limits and parametrized list objects (and therefore a parametrized natural numbers object).

As noted in section 5.1, the goal of this section is to show the uniqueness part of the statement that $(L_N(X), \text{nth}_X)$ is a universal morphism. That is, if we have two arrows $h_1, h_2 : A \to L(X)$ such that $l_A = \text{len}_X \circ h_i$ and $g = \text{nth}_X \circ (\text{Id} \times_N h_i)$ for $i = 1, 2$, then $h_1 = h_2$ (here, $l_A : A \to N$ and $g : E \times_N A \to X$). Intuitively, what we need to show is that if two lists have the same length and the same elements, then they are equal.

To simplify this problem, we can start by using the arrow nthDef instead of nth. So, given two arrows $h_1, h_2 : A \to L(X)$, we'd like to show the following statement:

$$\text{If } \text{len}(h_1(a)) =_a \text{len}(h_2(a)) \text{ and } \text{nthDef}(x, n, h_1(a)) =_{x,n,a} \text{nthDef}(x, n, h_2(a)),$$
$$\text{then } h_1(a) =_a h_2(a).$$

It turns out that we need to work in a locos to prove this statement. Without that assumption, the best we can do is show that $h_1(a) =_{x,a} h_2(a)$, where $x : X$; this is the content of theorem 5.3.3.

This section will therefore be divided into two subsections. In the first, we prove theorem 5.3.3, which doesn't require us to work in a locos. In the second, we work in a locos to improve this result: first, we change the conclusion to be $h_1 = h_2$, and then we adjust our hypotheses so they involve nth instead of nthDef.

5.3.1 Equality without extra hypotheses

Our goal for this section is to prove theorem 5.3.3, below. Essentially, we want to prove that two lists $h_1, h_2 : A \to L(X)$ are equal if they have the same length and the same elements. The idea of the proof is to go by induction on their length, and show that they are "built up in the same way".

More specifically, the strategy is as follows. Given arrows $h_1, h_2 : A \to L(X)$, we form arrows $H_1, H_2 : N \times A \to L(X)$ where $H_i(k, a)$ is the list $h_i(a)$ with all but the last k elements removed. We then want to show that $H_1 = H_2$; if this is true, then by setting $k = \mathrm{len}(h_i(a))$, we get $h_1(a) = h_2(a)$, as desired. To show that $H_1 = H_2$, we go by induction on k: the base case is clear since $H_1(0, a) = H_2(0, a) = \varnothing$, but the inductive step is more tricky.

For the inductive step, we must analyse the transition from having the last k elements of h_i to having the last $k+1$ elements of h_i. What we're doing is "adding back" an element to h_i; specifically, the one in position $\mathrm{len}(h_i) \dot{-} k$. So, we'd expect to have

$$H_i(k+1, a) = \mathrm{nthDef}\big(\mathrm{len}(h_i(a)) \dot{-} k, h_i(a)\big) :: H_i(k, a).$$

If we're assuming that nthDef and len agree for h_1 and h_2, then this gives us a recurrence relation that both H_1 and H_2 satisfy, and we're done. However, there's a problem: nthDef must include a default parameter. For this reason, H_i must also include a default parameter, making it of the form $H_i(x, k, a)$. This is why we can only conclude $h_1(a) =_{x,a} h_2(a)$.

Formalizing this proof idea requires several steps. First, we must have a better understanding of the arrows H_i. These arrows will involve the mapping $(x, k, \ell) \mapsto \mathrm{tail}(\mathrm{len}(\ell) \dot{-} k, \ell)$, which we want to show interacts nicely with appending an n^{th} element of ℓ. This first proposition gives us more information about this idea.

Proposition 5.3.1. We have

$$\mathrm{tail}(Pm, \ell) =_{x,m,\ell} \begin{cases} \mathrm{tail}(m, \ell) & \text{if } m = 0 \\ \varnothing & \text{else if } \mathrm{len}(\ell) < m \\ \mathrm{nthDef}(x, Pm, \ell) :: \mathrm{tail}(m, \ell) & \text{else} \end{cases}.$$

Proof. We prove this by induction on m. If $m = 0$, then the left side is just $\mathrm{tail}(0, \ell)$. On the right side, the condition $m = 0$ is true, so it reduces to $\mathrm{tail}(0, \ell)$, as desired.

On the other hand, suppose $m = sn$. On the right hand side, the outer condition becomes $sn = 0$, which is false, so it reduces to the else-if. So, the equality we want to prove becomes

$$\mathrm{tail}(Psn, \ell) =_{x,n,\ell} \begin{cases} \varnothing & \text{if } \mathrm{len}(\ell) < sn \\ \mathrm{nthDef}(x, Psn, \ell) :: \mathrm{tail}(sn, \ell) & \text{else} \end{cases}.$$

Clearly $Psn = n$, and the condition $\mathrm{len}(\ell) < sn$ is by definition equivalent to $s(\mathrm{len}(\ell)) \dot{-} sn = 0$. But $sa \dot{-} sb =_{a,b} a \dot{-} b$, so this condition is just $\mathrm{len}(\ell) \dot{-} n = 0$, i.e. $\mathrm{len}(\ell) \leq n$. Thus, the equality we want to prove is precisely given by proposition 4.4.6. ∎

In the next lemma, we show that an arrow H (which will act like the arrows H_1, H_2 we mentioned) satisfies a recurrence relation like the one we described in the proof outline.

Lemma 5.3.2. Consider the arrow $A : X \times N \times L(X) \times L(X) \to L(X)$ given by

$$A(x, k, \ell, L) =_{x,k,\ell,L} \begin{cases} L & \text{if } \mathrm{len}(\ell) \leq k \\ \mathrm{nthDef}\big(x, P(\mathrm{len}(\ell) \dot{-} k), \ell\big) :: L & \text{else} \end{cases}.$$

Next, consider $H : X \times N \times L(X) \to L(X)$ given by

$$H(x, k, \ell) =_{x,k,\ell} \mathrm{tail}(\mathrm{len}(\ell) \dot{-} k, \ell).$$

Intuitively, H takes ℓ and removes all but the last k elements. Then,

$$H(x, s(k), \ell) =_{x,k,\ell} A(x, k, \ell, H(x, k, \ell)).$$

Proof. Using proposition 5.3.1, we compute

$$\begin{aligned} & H(x, sk, \ell) \\ &=_{x,k,\ell} \mathrm{tail}(\mathrm{len}(\ell) \dot{-} sk, \ell) \\ &=_{x,k,\ell} \mathrm{tail}(P(\mathrm{len}(\ell) \dot{-} k), \ell) \\ &=_{x,k,\ell} \begin{cases} \mathrm{tail}(\mathrm{len}(\ell) \dot{-} k, \ell) & \text{if } \mathrm{len}(\ell) \dot{-} k = 0 \\ \varnothing & \text{else if } \mathrm{len}(\ell) < \mathrm{len}(\ell) \dot{-} k \\ \mathrm{nthDef}(x, P(\mathrm{len}(\ell) \dot{-} k), \ell) :: \mathrm{tail}(\mathrm{len}(\ell) \dot{-} k, \ell) & \text{else} \end{cases}. \end{aligned}$$

Note that the top condition, $\mathrm{len}(\ell) \dot{-} k = 0$, is the same as $\mathrm{len}(\ell) \leq k$. Moreover, the else-if condition is just false, since $s(\mathrm{len}(\ell)) \dot{-} (\mathrm{len}(\ell) \dot{-} k)$ is greater than zero by proposition 3.2.5. Thus, the else-if-else reduces to just the else case. Finally, we can just replace the term $\mathrm{tail}(\mathrm{len}(\ell) \dot{-} k, \ell)$ by $H(x, k, \ell)$. With all this consideration, the above expression reduces to

$$H(x, sk, \ell) = \begin{cases} H(x, k, \ell) & \text{if } \mathrm{len}(\ell) \leq k \\ \mathrm{nthDef}(x, P(\mathrm{len}(\ell) \dot{-} k), \ell) :: H(x, k, \ell) & \text{else} \end{cases},$$

which is just $A(x, k, \ell, H(x, k, \ell))$, as desired. ∎

Finally, we reach the desired result. As noted previously, this theorem does not allow us to conclude $h_1(a) =_a h_2(a)$, i.e. $h_1 = h_2$. It says that $h_1 \circ \pi_A = h_2 \circ \pi_A$ as maps $X \times A \to L(X)$; we can't yet get rid of the parameter X.

Theorem 5.3.3. Let $h_1, h_2 : A \to L(X)$. Suppose that

$$\operatorname{len}(h_1(a)) =_a \operatorname{len}(h_2(a)),$$
$$\operatorname{nthDef}(x, m, h_1(a)) =_{x,m,a} \operatorname{nthDef}(x, m, h_2(a)).$$

Then $h_1(a) =_{x,a} h_2(a)$, where x is a variable of type X.

Proof. For $i = 1, 2$, we define the map $A_i : X \times N \times A \times L(X) \to L(X)$ by setting $A_i(x, k, a, L) =_{x,k,a,L} A(x, k, h_i(a), L)$, where A is as in lemma 5.3.2. For $i = 1, 2$, define $H_i : X \times N \times A \to L(X)$ by $H_i(x, k, a) =_{x,k,a} H(x, k, h_i(a))$, where H is as in lemma 5.3.2. By this lemma, we know that $H_i(x, sk, a) =_{x,k,a} A_i(x, k, a, H_i(x, k, a))$.

We claim that $H_1 = H_2$; we'll prove this by induction on k. In the base case, we have $H_1(x, 0, a) =_{x,a} \varnothing =_{x,a} H_2(x, 0, a)$; indeed, for $i = 1, 2$, we compute

$$\begin{aligned} H_i(x, 0, a) =_{x,a} H(x, 0, h_i(a)) &=_{x,a} \operatorname{tail}(\operatorname{len}(h_i(a)) \dot{-} 0, h_i(a)) \\ &=_{x,a} \operatorname{tail}(\operatorname{len}(h_i(a)), h_i(a)) =_{x,a} \varnothing. \end{aligned}$$

The last equality follows from combining propositions 4.3.3 and 4.3.2. (The length of this list is $\operatorname{len}(h_i(a)) \dot{-} \operatorname{len}(h_i(a)) =_{x,a} 0$, so the list is empty.)

For the inductive step, we note that $A_1 = A_2$. Indeed, we have

$$A_i(x, k, a, L) =_{x,k,a,L} \begin{cases} L & \text{if } \operatorname{len}(h_i(a)) \leq k \\ \operatorname{nthDef}(x, P(\operatorname{len}(h_i(a)) \dot{-} k), h_i(a)) :: L & \text{else} \end{cases},$$

so the equalities assumed for this theorem give us $A_1 = A_2$. Writing $A_\times$ for $A_1 = A_2$, we therefore have $H_i(x, sk, a) = A_\times(x, k, a, H_i(x, a, k))$ for $i = 1, 2$.

So, H_1, H_2 satisfy the same defining equations, and we conclude by induction that $H_1 = H_2$. Now, note that

$$H_i(x, \operatorname{len}(h_i(a)), a) =_{x,a} \operatorname{tail}(0, h_i(a)) =_{x,a} h_i(a),$$

so $H_1 = H_2$ and $\operatorname{len}(h_1(a)) =_a \operatorname{len}(h_2(a))$ let us conclude $h_1(a) =_{x,a} h_2(a)$, as desired. ∎

5.3.2 Equality with extensivity hypotheses

In this section, we work in a locos to strengthen theorem 5.3.3. First, we show that the conclusion can be changed from $h_1(a) =_{x,a} h_2(a)$ to $h_1 = h_2$.

Lemma 5.3.4. Assume we are working in a locos. If $h_1, h_2 : A \to L(X)$ are arrows which satisfy the equations $\operatorname{len}(h_1(a)) =_a \operatorname{len}(h_2(a))$ and $h_1(a) =_{x,a} h_2(a)$ (x is a variable of type X), then $h_1 = h_2$.

Proof. Let $l_A = \text{len}_X \circ h_1 = \text{len}_X \circ h_2$. Let D_0 be the context $(a : A \mid l_A(a) = 0)$ and let $D_{>0}$ be the context $(a : A \mid l_A(a) > 0)$. By proposition 3.3.2, we have $A = [D_0] + [D_{>0}]$. Therefore, to show $h_1 = h_2$, it suffices to show that

$$h_1(a) =_{D_0} h_2(a) \qquad \text{and} \qquad h_1(a) =_{D_{>0}} h_2(a).$$

For the first equality, we claim that both terms equal $\varnothing$. Indeed, in the context D_0, we have $\text{len}(h_1(a)) =_{D_0} l_A(a) =_{D_0} 0$, and this implies that $h_1(a) =_{D_0} \varnothing$ by proposition 4.3.2. The same goes for h_2.

For the second equality, we need the existence of a term $\text{def} : X$ in the context $D_{>0}$. Indeed, if such a term exists, then by substituting $x \mapsto \text{def}$, $a \mapsto a$ in the equation $h_1(a) =_{x,a} h_2(a)$, we get the desired equality

$$h_1(a) =_{D_{>0}} h_2(a).$$

To obtain def, we note that $0 <_{D_{>0}} l_A(a) =_{D_{>0}} \text{len}(h_1(a))$, and so $\text{def} = \text{nth}(0, h_1(a))$ is a well-defined term of type X in $D_{>0}$. ∎

Next, we show that the hypotheses of theorem 5.3.3 can be changed so that we use nth instead of nthDef (recall that the arrow nth_X is only defined when working in a locos).

Lemma 5.3.5. In a locos, let $h_1, h_2 : A \to L(X)$. Suppose that

$$\text{len} \circ h_1 = \text{len} \circ h_2,$$

and denote this arrow l_A. Then, in the context $C = (m : N,\ a : A \mid m < l_A(a))$, suppose that

$$\text{nth}(m, h_1(a)) =_C \text{nth}(m, h_2(a)).$$

Then $\text{nthDef}(x, m, h_1(a)) =_{x,m,a} \text{nthDef}(x, m, h_2(a))$.

Proof. Let $D = (x : X, m : N, a : A)$, and let $D_0 = (D \mid m < l_A(a))$, $D_1 = (D \mid m \geq l_A(a))$. Since we're in a locos, we can use corollary 3.3.5 to conclude that $[D] = [D_0] + [D_1]$. Therefore, to show that $\text{nthDef}(x, m, h_1(a)) =_D \text{nthDef}(x, m, h_2(a))$, it suffices to show the following.

$$\text{nthDef}(x, m, h_1(a)) =_{D_0} \text{nthDef}(x, m, h_2(a))$$
$$\text{nthDef}(x, m, h_1(a)) =_{D_1} \text{nthDef}(x, m, h_2(a))$$

We start with the second equality, since it is the easiest. Indeed, since we have $m \geq_{D_1} l_A(a) =_{D_1} \text{len}(h_1(a))$, proposition 4.3.4 tells us that

$$\text{nthDef}(x, m, h_1(a)) =_{D_1} x.$$

The same is true if we replace h_1 by h_2, so the equality follows.

For the first equality, we note that $\text{nth}(m, h_1(a)) =_{D_0} \text{nth}(m, h_2(a))$ by assumption (we can "add $x : X$ to the context" by performing a substitution). To establish the desired equality, then, it suffices to show that

$$\text{nthDef}(x, m, h_1(a)) =_{D_0} \text{nth}(m, h_1(a)),$$

and the same for h_2 (which is done the same way). But this equality is simply true by proposition 4.4.4, using the hypothesis $m < l_A(a) = \text{len}(h_1(a))$ from the context D_0. So, we're done. ∎

Putting this all together, we get the following result, which is the uniqueness part of showing that $(L_N(X), \text{nth}_X)$ is a universal morphism.

Corollary 5.3.6. In a locos, if $h_1, h_2 : A \to L(X)$ are such that $\text{len}_X \circ h_1 = \text{len}_X \circ h_2$ and $\text{nth}_X \circ (\text{Id}_E \times_N h_1) = \text{nth}_X \circ (\text{Id}_E \times_N h_2)$, then $h_1 = h_2$.

Proof. Apply lemma 5.3.5, then theorem 5.3.3, then lemma 5.3.4. Remark that, if we set $l_A = \text{len}_X \circ h_1 = \text{len}_X \circ h_2$, then in the context $C = (m : N, a : A \mid m < l_A(a))$, we have $\big(\text{nth}_X \circ (\text{Id}_E \times_N h)\big)(m, \ell) =_C \text{nth}_X(n, h(a))$ by fact 3.4.3. ∎

5.4 Putting it together

We are now ready to prove the result that was promised in section 5.1.

Theorem 5.4.1. In a locos, L_N is a right adjoint to $\Sigma_E \circ \Delta_{\pi_2^E}$, with counit $(\text{nth}_X)_X$.

In particular, for any $l_A : A \to N$ and $g : E \times_N A \to X$, there exists a unique $h : A \to L(X)$ such that $l_A = \text{len}_X \circ h$ and $g = \text{nth}_X \circ (\text{Id} \times_N h)$.

Proof. Proposition 5.1.1 tells us that $\text{nth} : \Sigma_E \circ \Delta_{\pi_2^E} \circ L_N \Rightarrow \text{Id}$ is a natural transformation. Moreover, theorem 5.2.7 and corollary 5.3.6 together tell us that for each X, $(L_N(X), \text{nth}_X)$ is a universal morphism from $\Sigma_E \circ \Delta_{\pi_2^E}$ to X (this is what the second part of the theorem says). Thus, L_N is right adjoint to $\Sigma_E \circ \Delta_{\pi_2^E}$ (see e.g. [Borc 94a, dfn. 3.1.4]). ∎

Corollary 5.4.2. In a locos $\mathcal{C}$, the arrow $\pi_2^E : E \to N$ is exponentiable, and the list object functor $L : \mathcal{C} \to \mathcal{C}$ is represented by the following polynomial.

$$\mathbb{1} \longleftarrow E \xrightarrow{\pi_2^E} N \longrightarrow \mathbb{1}$$

Proof. By theorem 5.4.1, L_N is right adjoint to $\Sigma_E \circ \Delta_{\pi_2^E}$. By fact 2.1.1 (due to [Nief 82]), this right adjoint existing implies that $\Pi_{\pi_2^E}$, the right adjoint of $\Delta_{\pi_2^E}$, also exists, and so π_2^E is exponentiable.

To show that L is represented by the above polynomial, we must show that

$$\Sigma_N \circ \Pi_{\pi_2^E} \circ \Delta_E \cong L.$$

We note that $L = \Sigma_N \circ L_N$, so we just need to show that $\Pi_{\pi_2^E} \circ \Delta_E \cong L_N$. For this, we note that $\Pi_{\pi_2^E}$ and Δ_E are the right adjoints of $\Delta_{\pi_2^E}$ and Σ_E, respectively, and so $\Pi_{\pi_2^E} \circ \Delta_E$ is right adjoint to $\Sigma_E \circ \Delta_{\pi_2^E}$ (see [Borc 94a, prop. 3.2.1]). However, L_N is already a right adjoint of $\Sigma_E \circ \Delta_{\pi_2^E}$, so by uniqueness of adjoints, we get $\Pi_{\pi_2^E} \circ \Delta_E \cong L_N$, as desired. ∎

Corollary 5.4.3. In a locos $\mathcal{C}$, let $X \in \mathcal{C}$. Then the exponential

$$(\Delta_N X)^{\pi_2^E}$$

(in $\mathcal{C}/N$) exists and is given by $\mathrm{len} : L(X) \to N$.

Proof. It is easy to check that the following diagram of functors commutes.

$$\begin{array}{ccc} \mathcal{C}/N & \xrightarrow{- \times \pi_2^E} & \mathcal{C}/N \\ {\scriptstyle \Delta_{\pi_2^E}}\downarrow & & \downarrow{\scriptstyle \Sigma_N} \\ \mathcal{C}/E & \xrightarrow[\Sigma_E]{} & \mathcal{C} \end{array}$$

By theorem 5.4.1, L_N is a right adjoint to $\Sigma_E \circ \Delta_{\pi_2^E}$, so it is also a right adjoint to $\Sigma_N \circ (- \times \pi_2^E)$. However, note that the right adjoints of Σ_N and $- \times \pi_2^E$ are Δ_N and $(-)^{\pi_2^E}$, respectively (the latter exists because, as noted in corollary 5.4.2, the arrow π_2^E is exponentiable – see fact 2.1.1). Therefore, L_N is isomorphic to $(-)^{\pi_2^E} \circ \Delta_N$, which is all we needed to show. ∎

We end with the following result, which will be useful later.

Proposition 5.4.4. Let $\mathcal{A}, \mathcal{B}$ be locoi, and let $F : \mathcal{A} \to \mathcal{B}$ be a functor which preserves finite limits and NNO. Then the following are equivalent.

- F preserves list objects
- The canonical natural transformation

$$\begin{array}{ccc} \mathcal{A}/E & \xrightarrow{\Pi_{\pi_2^E}} & \mathcal{A}/N \\ {\scriptstyle F}\big\downarrow & \swarrow & \big\downarrow{\scriptstyle F} \\ \mathcal{B}/E & \xrightarrow[\Pi_{\pi_2^E}]{} & \mathcal{B}/N \end{array}$$

is an isomorphism.

Proof. Since F preserves finite limits and NNO, it preserves N and E (so we just write N and E instead of FN and FE), and it maps π_2^E in $\mathcal{A}$ to π_2^E in $\mathcal{B}$. Moreover, since $\mathcal{A}$ and $\mathcal{B}$ are locoi, we know by corollary 5.4.2 that π_2^E is exponentiable in both $\mathcal{A}$ and $\mathcal{B}$, and the polynomial functor associated to π_2^E is the list object functor. Therefore, we can apply proposition 2.12 in [Uemu 22], which says that the canonical natural transformation $F\Pi_{\pi_2^E} \Rightarrow \Pi_{\pi_2^E}F$ is an isomorphism if and only if the canonical natural transformation $FL \Rightarrow LF$ is an isomorphism (see the diagram below).

$$\begin{array}{ccccccc} \mathcal{A} & \xrightarrow{E^*} & \mathcal{A}/E & \xrightarrow{\Pi_{\pi_2^E}} & \mathcal{A}/N & \xrightarrow{\Sigma_N} & \mathcal{A} \\ \big\downarrow{\scriptstyle F} & & \big\downarrow{\scriptstyle F} & & \big\downarrow{\scriptstyle F} & & \big\downarrow{\scriptstyle F} \\ \mathcal{B} & \xrightarrow[E^*]{} & \mathcal{B}/E & \xrightarrow[\Pi_{\pi_2^E}]{} & \mathcal{B}/N & \xrightarrow[\Sigma_N]{} & \mathcal{B} \end{array}$$

It is straightforward to check that F preserves list objects if and only if the canonical natural transformation $FL \Rightarrow LF$ is an isomorphism. ∎

Part III

Indexed categories

Chapter 6

Indexed categories

Contents

This chapter introduces indexed categories, a key tool for the rest of the thesis. In general set theory, we often discuss families of things which are indexed by some set I. Indexed categories provide a way to discuss indexed families when the index I is taken to be an object of a "base" category $\mathcal{S}$, rather than a set. Specifically, an $\mathcal{S}$-indexed category $\mathbb{C}$ consists of a category $\mathcal{C}^I$ for each object $I \in \mathcal{S}$, and we think of $\mathcal{C}^I$ as "the category of I-indexed families of things in $\mathcal{C}$" (we give the precise definition in section 6.1). These categories are related by "re-indexation" maps (also called transition maps): for each arrow $x : I \to J$ of $\mathcal{S}$, we have a functor $x^* : \mathcal{C}^J \to \mathcal{C}^I$ which we think of as mapping a family $(A_j \mid j \in J)$ to $(A_{x(i)} \mid i \in I)$. A more detailed justification of this definition can be found in section B1.2 of [John 02].

In this chapter, we cover some basic definitions and results about indexed categories, many of which come from section B1.2 of [John 02]. Section 6.1 spells out the basic definitions in detail. Section 6.2 introduces limits and colimits: what it means for an indexed category to have them, and for an indexed functor to preserve them; we also introduce a new kind of indexed products which does not appear in [John 02].

In section 6.3, we cover an essential basic definition: the canonical self-indexing $\mathbb{S}$ of a category $\mathcal{S}$. We discuss its limits and colimits, including in the special case where $\mathcal{S}$ is an arithmetic universe. We also study functors associated with canonical indexings and their properties.

Finally, in section 6.4, we establish some properties of the category $[\mathbb{C}, \mathbb{D}]$ of indexed functors $\mathbb{C} \to \mathbb{D}$ (and the diagonal functor $\Delta : \mathcal{D} \to [\mathbb{C}, \mathbb{D}]$), and in section 6.5, we study the *indexed* category $[\![\mathbb{C}, \mathbb{D}]\!]$.

6.1 Basics

We start with the definitions of indexed category, indexed functor, and indexed natural transformation, which we take from definition B1.2.1 in [John 02]. These definitions can be given concisely as follows.

Definition. Let $\mathcal{S}$ be a category. An $\mathcal{S}$*-indexed category* is a pseudofunctor $\mathcal{S}^{op} \to \mathbf{CAT}$ (the large category of categories); an $\mathcal{S}$*-indexed functor* is a pseudo-natural transformation; and an $\mathcal{S}$*-indexed natural transformation* is a modification.

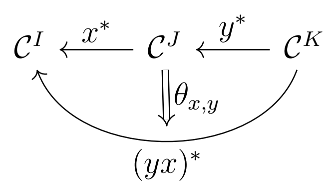
However, we will also spell out these definitions in more detail for easier reference. We make the following notes.

- Most of the notation here also comes from definition B1.2.1 of [John 02].
- We will avoid discussing issues of size, since these never come into play in a meaningful way.
- We choose to assume that indexed categories are strictly unital, meaning that the identity arrows are preserved strictly (and the corresponding coherence isomorphisms behave nicely; see the definition below), since they can always be easily modified to fit this criterion.

We are now ready for the full definitions.

Definition. Let $\mathcal{S}$ be a category. An $\mathcal{S}$*-indexed category* $\mathbb{C}$ consists of

- For each object $I \in \mathcal{S}$, a category $\mathcal{C}^I$ (sometimes called the *fibers*);
- For each morphism $x : I \to J$ in $\mathcal{S}$, a functor $x^* : \mathcal{C}^J \to \mathcal{C}^I$ (sometimes called the *transition functors*), with $(\mathrm{Id}_I)^* = \mathrm{Id}_{\mathcal{C}^I}$;
- For any $x : I \to J$ and $y : J \to K$ in $\mathcal{S}$, a natural isomorphism $\theta_{x,y} : x^* y^* \Rightarrow (yx)^*$,

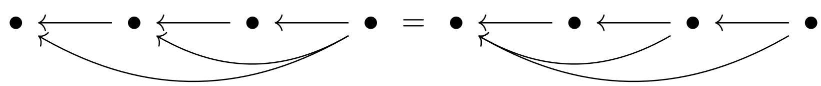
  

  such that the following equations hold for $I \xrightarrow{x} J \xrightarrow{y} K \xrightarrow{z} L$.

$$\theta_{\mathrm{Id}_I,x} = \mathrm{Id}_{x^*} = \theta_{x,\mathrm{Id}_J} \qquad \theta_{x,zy}(x^* \circ \theta_{y,z}) = \theta_{yx,z}(\theta_{x,y} \circ z^*)$$

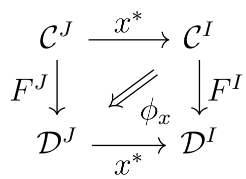
If $\mathcal{S}$ has a terminal object $\mathbb{1}$, then we call $\mathcal{C}^{\mathbb{1}}$ the *underlying ordinary category* of $\mathbb{C}$, and we denote it $\mathcal{C}$.

The second equality for the natural isomorphisms can be understood with the following diagram.

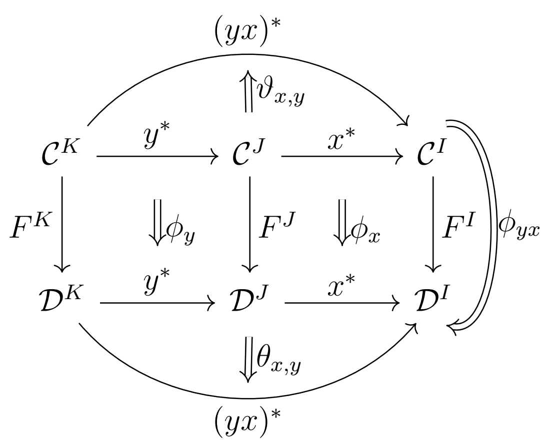


Definition. Let $\mathcal{S}$ be a category, and let $\mathbb{C}, \mathbb{D}$ be $\mathcal{S}$-indexed categories, whose natural isomorphisms are denoted ϑ and θ, respectively. An *$\mathcal{S}$-indexed functor* $F : \mathbb{C} \to \mathbb{D}$ consists of

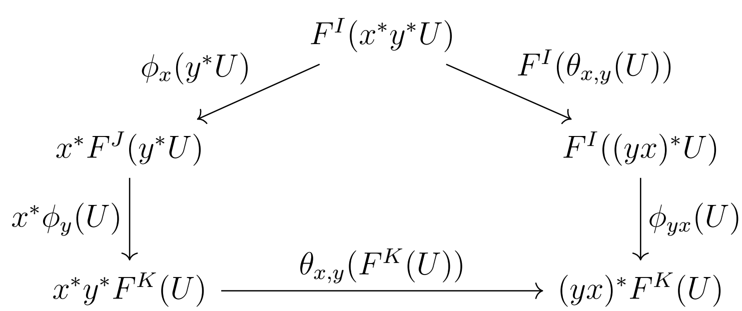
- For each object $I \in \mathcal{S}$, a functor $F^I : \mathcal{C}^I \to \mathcal{D}^I$;
- For each morphism $x : I \to J$ in $\mathcal{S}$, a natural isomorphism $\phi_x : F^I x^* \Rightarrow x^* F^J$,



such that $\phi_{\mathrm{Id}_I} = \mathrm{Id}_{F^I}$ (recall that $\mathrm{Id}^* = \mathrm{Id}$ by assumption). Moreover, we have to have coherence between ϕ and the isomorphisms ϑ, θ, based on the following diagram for any arrows $I \xrightarrow{x} J \xrightarrow{y} K$ in $\mathcal{S}$.



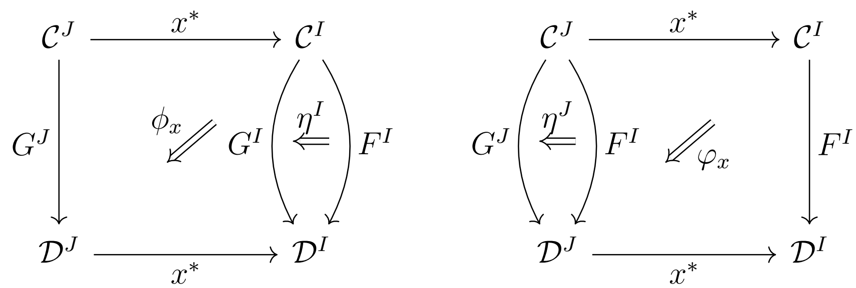
What this means is that for any such arrows x, y, and for any object $U \in \mathcal{C}^K$, the following diagram commutes.



Definition. Let $\mathcal{S}$ be a category, let $\mathbb{C}, \mathbb{D}$ be $\mathcal{S}$-indexed categories (equipped with ϑ and θ), and let $F, G : \mathbb{C} \to \mathbb{D}$ be $\mathcal{S}$-indexed functors (equipped with φ and ϕ). An *$\mathcal{S}$-indexed natural transformation* $\eta : F \Rightarrow G$ consists of

- For each object $I \in \mathcal{S}$, a natural transformation $\eta^I : F^I \Rightarrow G^I$

which satisfies the following condition. For any arrow $x : I \to J$ of $\mathcal{S}$, the following natural transformations are equal.



In other words, for any arrow $x : I \to J$ and any object $U \in \mathcal{C}^J$, the following diagram commutes.

$$\begin{array}{ccc}
F^I(x^*U) & \xrightarrow{\varphi_x(U)} & x^*F^J(U) \\
{\scriptstyle \eta^I_{x^*U}} \downarrow & & \downarrow {\scriptstyle x^*(\eta^J_U)} \\
G^I(x^*U) & \xrightarrow[\phi_x(U)]{} & x^*G^J(U)
\end{array}$$

Most of the time, as in [John 02], we will avoid mentioning the coherence isomorphisms to alleviate the text. We will only do a few proofs in full detail (in appendix A) to show the machinery in action.

We end this section with one more definition, taken from example B1.2.2(e) of [John 02].

Definition. Let $\mathcal{S}, \mathcal{T}$ be categories, let $F : \mathcal{S} \to \mathcal{T}$ be a functor, and let $\mathbb{C}$ be a $\mathcal{T}$-indexed category. Then we form a $\mathcal{S}$-indexed category $F^*\mathbb{C}$ as follows. (We call this operation *change of base* or *pullback* along F.)

- For each $I \in \mathcal{S}$, we set $(F^*\mathbb{C})^I = \mathcal{C}^{F(I)}$;
- For each $x : I \to J$ in $\mathcal{S}$, the functor $x^* : \mathcal{C}^{F(J)} \to \mathcal{C}^{F(I)}$ is just the functor $F(x)^*$ coming from the structure of $\mathbb{C}$;
- The isomorphism $x^*y^* \Rightarrow (yx)^*$ is the isomorphism $F(x)^*F(y)^* \Rightarrow (F(y)F(x))^* = F(yx)^*$ from $\mathbb{C}$.

We omit the trivial check that this forms a well-defined $\mathcal{S}$-indexed category. Note that if $\mathcal{S}$ has a terminal object, and it is preserved by F, then the underlying ordinary category of $F^*\mathbb{C}$ is $\mathcal{C}$.

6.2 Limits and colimits

Indexed categories having (co)limits

An important aspect of indexed categories is limits and colimits. In this section, we recall the relevant definitions from [John 02], and give an additional definition that will be important for our purposes.

We start with the definition of finite limits and colimits in an indexed category, which is given at the beginning of section B1.4 in [John 02].

Definition. An indexed category $\mathbb{C}$ *has finite (co)limits of a particular shape* (binary products, terminal objects, equalizers, binary coproducts, etc.) if each $\mathcal{C}^I$ has them and they are preserved by the transition functors. We also say that $\mathbb{C}$ *has finite (co)limits* if each $\mathcal{C}^I$ has finite (co)limits and the transition functors preserve them.

As with ordinary categories, we say an indexed category is *lex* if it has finite limits.

Proposition 6.2.1. Let $F : \mathcal{S} \to \mathcal{T}$ be a functor, and let $\mathbb{C}$ be a $\mathcal{T}$-indexed category. If $\mathbb{C}$ has finite (co)limits of a particular shape, or all finite (co)limits, then so does $F^*\mathbb{C}$.

Proof. We must show that for any $J \in \mathcal{S}$, the category $\mathcal{C}^{F(J)}$ has finite (co)limits (of a given shape), and they are preserved by $F(x)^* : \mathcal{C}^{F(J)} \to \mathcal{C}^{F(I)}$ (for $x : I \to J$). But this is just true by the assumption on $\mathbb{C}$. ∎

Next, we look at infinite products and coproducts; the following is definition B1.4.2 from [John 02].

Definition. Let $\mathcal{S}$ be a category with pullbacks, and let $\mathbb{C}$ be an $\mathcal{S}$-indexed category. We say that $\mathbb{C}$ *has $\mathcal{S}$-indexed products* if the following conditions hold.

- For each $x : I \to J$ in $\mathcal{S}$, there is a right adjoint Π_x to $x^* : \mathcal{C}^J \to \mathcal{C}^I$.
- (Beck-Chevalley condition) For each pullback square in $\mathcal{S}$ (as below on the left), the canonical natural transformation ψ in the diagram below on the right is an isomorphism.

$$\begin{array}{ccc} I & \xrightarrow{x} & J \\ {\scriptstyle y}\downarrow & & \downarrow{\scriptstyle z} \\ K & \xrightarrow[w]{} & L \end{array} \qquad\qquad \begin{array}{ccc} \mathcal{C}^I & \xleftarrow{x^*} & \mathcal{C}^J \\ {\scriptstyle \Pi_y}\downarrow & \overset{\psi}{\nwarrow} & \downarrow{\scriptstyle \Pi_z} \\ \mathcal{C}^K & \xleftarrow[w^*]{} & \mathcal{C}^L \end{array}$$

We say $\mathbb{C}$ is *$\mathcal{S}$-complete* if it has finite limits and $\mathcal{S}$-indexed products.

Similarly, we say $\mathbb{C}$ *has $\mathcal{S}$-indexed coproducts* if each x has a left adjoint Σ_x, and the Beck-Chevalley condition holds. We say $\mathbb{C}$ is *$\mathcal{S}$-cocomplete* if it has finite colimits and $\mathcal{S}$-indexed coproducts.

The above definition is too strong for our purposes: we will want to consider indexed categories that have only some infinite products, not all of them. To that end, we use the following definition.

Definition. Let $\mathcal{S}$ be a category with finite limits, and let $\mathbb{C}$ be an $\mathcal{S}$-indexed category. Given an arrow $f : I \to J$ in $\mathcal{S}$, we say that $\mathbb{C}$ *has f-indexed products* if the following conditions hold.

- For any pullback $g : A \to B$ of f, the functor $g^* : \mathcal{C}^B \to \mathcal{C}^A$ has a right adjoint $\Pi_g : \mathcal{C}^A \to \mathcal{C}^B$. (In particular, f^* has a right adjoint Π_f.)
- (Beck-Chevalley condition) For any pullback $g : A \to B$ of f, as witnessed by the square below on the left, the canonical natural transformation ψ in the diagram below on the right is an isomorphism.

$$\begin{array}{ccc} A & \xrightarrow{y} & I \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ B & \xrightarrow[z]{} & J \end{array} \qquad\qquad \begin{array}{ccc} \mathcal{C}^A & \xleftarrow{y^*} & \mathcal{C}^I \\ {\scriptstyle \Pi_g}\downarrow & \overset{\psi}{\Nwarrow} & \downarrow{\scriptstyle \Pi_f} \\ \mathcal{C}^B & \xleftarrow[z^*]{} & \mathcal{C}^J \end{array}$$

Similarly, we say that $\mathbb{C}$ has f-indexed coproducts if each pullback g of f has a left adjoint, and the Beck-Chevalley condition holds.

Remark 6.2.2. If $\mathcal{S}$ is a category with finite limits and $\mathbb{C}$ is an $\mathcal{S}$-indexed category, then $\mathbb{C}$ has $\mathcal{S}$-indexed (co)products if and only if $\mathbb{C}$ has f-indexed (co)products for every arrow f of $\mathcal{S}$.

Next, we prove some basic results about this last definition.

Proposition 6.2.3. Let $\mathcal{S}$ be a category with finite limits, and let $\mathbb{C}$ be an $\mathcal{S}$-indexed category. If $\mathbb{C}$ has f-indexed products, then it has g-indexed products for any pullback g of f.

Proof. There are two conditions to show. The first is easy: if h is a pullback of g, then it is also a pullback of f (by the pasting law), so Π_h exists.

For the second condition, let the squares below on the left be pullbacks, showing that h is a pullback of g and g is a pullback of f.

$$\begin{array}{ccccc} X & \longrightarrow & A & \longrightarrow & I \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ Y & \longrightarrow & B & \longrightarrow & J \end{array} \qquad\qquad \begin{array}{ccccc} \mathcal{C}^X & \longleftarrow & \mathcal{C}^A & \longleftarrow & \mathcal{C}^I \\ \downarrow{\scriptstyle \Pi_h} & \overset{\psi_{hg}}{\Nwarrow} & \downarrow{\scriptstyle \Pi_g} & \overset{\psi_{gf}}{\Nwarrow} & \downarrow{\scriptstyle \Pi_f} \\ \mathcal{C}^Y & \longleftarrow & \mathcal{C}^B & \longleftarrow & \mathcal{C}^J \end{array}$$

Then, in the above diagram on the right, we have canonical natural transformations ψ_{gf}, ψ_{hf}, ψ_{hg}. We have, essentially, $\psi_{hg} \circ \psi_{gf} = \psi_{hf}$. Because $\mathbb{C}$ has f-indexed products, we know that ψ_{gf} and ψ_{hf} are invertible. This implies that ψ_{hg} is invertible too, as desired. ∎

Proposition 6.2.4. Let $\mathcal{S}$, $\mathcal{T}$ be categories with finite limits, let $F : \mathcal{S} \to \mathcal{T}$ be a functor which preserves pullbacks, and let $\mathbb{C}$ be a $\mathcal{T}$-indexed category.

If f is an arrow in $\mathcal{S}$ and $\mathbb{C}$ has $F(f)$-indexed (co)products, then $F^*\mathbb{C}$ has f-indexed (co)products.

Hence, if $\mathbb{C}$ has $\mathcal{T}$-indexed (co)products, then $F^*\mathbb{C}$ has $\mathcal{S}$-indexed (co)products, and if $\mathbb{C}$ is $\mathcal{T}$-(co)complete, then $F^*\mathbb{C}$ is $\mathcal{S}$-(co)complete. (See also lemma B1.4.9 of [John 02].)

Proof. Let g be a pullback of f. Since F preserves pullbacks, we know that $F(g)$ is a pullback of $F(f)$. The functor g^* in $F^*\mathbb{C}$ is the functor $F(g)^*$ in $\mathbb{C}$, which must have a right adjoint because $\mathbb{C}$ has $F(f)$-indexed products.

For Beck-Chevalley, if we have a square witnessing a pullback g of f in $\mathcal{S}$, it remains such a square in $\mathcal{T}$ since F preserves pullbacks. Thus, because $\mathbb{C}$ has $F(f)$-indexed products, the canonical map is an isomorphism, as desired.

For the final remark about (co)completeness, we also use proposition 6.2.1. ∎

Finally, we give a specific case of the previous definition, which we will study extensively in chapters 12, 14, and 16.

Definition. Let $\mathcal{S}$ be a category with finite limits and a natural numbers object N, and let $\mathbb{C}$ be an $\mathcal{S}$-indexed category. We say $\mathbb{C}$ *has N-finite products* if it has π_2^E-indexed products, where $\pi_2^E : E \to N$ is the arrow defined in section 3.4.

We also use the following terminology.

Definition. Let $\mathcal{S}$ be a category with finite limits and NNO, and let $\mathbb{C}$ be an $\mathcal{S}$-indexed category. We say $\mathbb{C}$ is *flex* if it has finite limits and N-finite products, and we say $\mathbb{C}$ is *flexcc* if it has finite limits, it has N-finite products, and it is $\mathcal{S}$-cocomplete.

We use the term "N-finite products" because we think of this condition as meaning that $\mathbb{C}$ has finite products which are indexed by the natural numbers object. Indeed, the products come from the fibers of π_2^E, which we think of as sets that are finite from the internal perspective of N. We'll see later that having N-finite products is related to having list objects (propositions 6.3.6 and 6.3.7), and preserving N-finite products (which we'll discuss below) is related to preserving list objects (propositions 6.3.8 and 6.3.9). These connections will be key to proving the third main result of this thesis (theorem 17.2.1), as we'll see in its proof.

Indexed functors preserving (co)limits

We now discuss indexed functors which preserve (co)limits. The definition given in [John 02] (definition B1.4.13) is again too general for us; we present some more specific definitions.

Definition. Let $\mathcal{S}$ be a category with finite limits, let $\mathbb{C}$ and $\mathbb{D}$ be $\mathcal{S}$-indexed categories, and let $F : \mathbb{C} \to \mathbb{D}$ be an $\mathcal{S}$-indexed functor.

- If $\mathbb{C}$ and $\mathbb{D}$ have finite (co)limits (of a particular shape), then we say F *preserves finite (co)limits* (of a particular shape) if each $F^I : \mathcal{C}^I \to \mathcal{D}^I$ preserves them. As with ordinary functors, we say F is *lex* if it preserves finite limits.
- Let $f : I \to J$ be an arrow in $\mathcal{S}$. If $\mathbb{C}$ and $\mathbb{D}$ have f-indexed (co)products, then we say F *preserves f-indexed (co)products* if for any pullback $g : A \to B$ of f, the canonical natural transformation $F^B \Pi_g \Rightarrow \Pi_g F^A$ ($F^B \Sigma_g \Rightarrow \Sigma_g F^B$ for coproducts), which is the "mate" of the isomorphism $g^* F^B \cong F^A g^*$, is an isomorphism.
- If $\mathbb{C}$ and $\mathbb{D}$ have $\mathcal{S}$-indexed (co)products, then we say that F *preserves $\mathcal{S}$-indexed (co)products* if it preserves f-indexed (co)products for each arrow f of $\mathcal{S}$.
- If $\mathbb{C}$ and $\mathbb{D}$ are (co)complete, we say F is *(co)continuous* if it preserves finite (co)limits and $\mathcal{S}$-indexed (co)products.

Remark 6.2.5. If F preserves f-indexed (co)products, then it also preserves g-indexed (co)products for any pullback g of f. (This is trivial from the definition.)

We also use the following terminology.

Definition. Let $\mathcal{S}$ be a category with finite limits and NNO, let $\mathbb{C}$, $\mathbb{D}$ be $\mathcal{S}$-indexed categories, and let $F : \mathbb{C} \to \mathbb{D}$ be an $\mathcal{S}$-indexed functor.

- Assume $\mathbb{C}$ and $\mathbb{D}$ are flex. We say F is a *flex morphism* if it preserves finite limits and N-finite products.
- Assume $\mathbb{C}$ and $\mathbb{D}$ are flexcc. We say F is a *flexcc morphism* if it preserves finite limits, it preserves N-finite products, and it is cocontinuous.

Proposition 6.2.6. Let $\mathcal{S}, \mathcal{T}$ be categories with finite limits, let $F : \mathcal{S} \to \mathcal{T}$ preserve finite limits, let $\mathbb{C}, \mathbb{D}$ be $\mathcal{T}$-indexed categories, and let $G : \mathbb{C} \to \mathbb{D}$ be a $\mathcal{T}$-indexed functor.

- Suppose $\mathbb{C}$ and $\mathbb{D}$ have finite limits. If G preserves finite limits, then so does $F^*G : F^*\mathbb{C} \to F^*\mathbb{D}$.
- Let f be an arrow in $\mathcal{S}$, and suppose $\mathbb{C}, \mathbb{D}$ have $F(f)$-indexed (co)products. If G preserves $F(f)$-indexed (co)products, then $F^*G : F^*\mathbb{C} \to F^*\mathbb{D}$ preserves f-indexed (co)products.

Proof. The first part is easy: for any $I \in \mathcal{S}$, $(F^*G)^I = G^{F(I)}$, which preserves finite limits by assumption.

For the second part, let $g : A \to B$ be a pullback of f. Then Fg is a pullback of Ff (since F preserves pullbacks), and we have to show that the following diagram commutes.

$$\begin{array}{ccc} \mathcal{C}^{FA} & \xrightarrow{\Pi_g} & \mathcal{C}^{FB} \\ {\scriptstyle G^{FA}}\downarrow & & \downarrow{\scriptstyle G^{FB}} \\ \mathcal{D}^{FA} & \xrightarrow{\Pi_g} & \mathcal{D}^{FB} \end{array}$$

But Π_g here (in both $\mathbb{C}$ and $\mathbb{D}$) is just the right adjoint of $F(g)^*$, so this diagram commutes because G preserves $F(f)$-indexed products. ∎

6.3 The canonical indexing

In this section, we describe the canonical indexing of a category over itself, and prove many properties of it. This construction is of fundamental importance for the rest of this thesis.

Definitions

We start with the definition, which comes from example B1.2.2(c) of [John 02].

Definition. Let $\mathcal{S}$ be a category with finite limits. The *canonical indexing* of $\mathcal{S}$ over itself is an $\mathcal{S}$-indexed category $\mathbb{S}$ defined as follows.

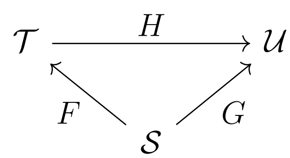
- For $I \in \mathcal{S}$, we set $\mathcal{S}^I = \mathcal{S}/I$;
- For $x : I \to J$, the functor $\mathcal{S}/J \to \mathcal{S}/I$ is the "pullback along x" functor;
- The isomorphism $x^*y^* \cong (yx)^*$ is the one given by the pasting law.

There is also a way to define functors between the canonical indexings of various categories; this comes from example B1.2.2(d) of [John 02].

Definition. Let $\mathcal{S}, \mathcal{T}$ be categories with finite limits, and let $F : \mathcal{S} \to \mathcal{T}$ be a functor which preserves finite limits. Then F extends to an $\mathcal{S}$-indexed functor $\mathcal{F} : \mathbb{S} \to F^*\mathbb{T}$ as follows (where $\mathbb{S}, \mathbb{T}$ are the canonical indexings of $\mathcal{S}, \mathcal{T}$, respectively, and F^* is the change of base operation described in section 6.1).

- For $I \in \mathcal{S}$, the functor $F^I : \mathcal{S}/I \to \mathcal{T}/F(I)$ maps $p : A \to I$ to $F(p) : F(A) \to F(I)$;
- For $x : I \to J$, the isomorphism $F^I x^* \cong x^* F^J$ comes from the fact that F preserves pullbacks.

Moreover, suppose $\mathcal{U}$ is another category with finite limits, $G : \mathcal{S} \to \mathcal{U}$ and $H : \mathcal{T} \to \mathcal{U}$ are functors which preserve finite limits, and $HF \cong G$.



Then H extends to an $\mathcal{S}$-indexed functor $\mathcal{H} : F^*\mathbb{T} \to G^*\mathbb{U}$ as follows.

- For $I \in \mathcal{S}$, the functor $H^I : \mathcal{T}/F(I) \to \mathcal{U}/G(I)$ maps $p : A \to F(I)$ to $H(p) : H(A) \to H(F(I)) \cong G(I)$;
- For $x : I \to J$, the isomorphism $H^I x^* \cong x^* H^J$ comes from the fact that H preserves pullbacks.

In section A.1 of the appendix, we check that this construction is well-defined.

The extension of H to $\mathcal{H}$ is not described explicitly in [John 02], but it is referenced in example B1.2.2(d): it's what Johnstone describes as the 2-functor $\mathcal{S}\backslash\mathfrak{CART} \to \mathfrak{CAT}_{\mathcal{S}}$. (Here, $\mathfrak{CART}$ is the 2-category of lex categories – referred to as Cartesian categories in [John 02] – and $\mathfrak{CAT}_{\mathcal{S}}$ is the 2-category of $\mathcal{S}$-indexed categories.)

Remark 6.3.1. The extension of $F : \mathcal{S} \to \mathcal{T}$ to $\mathcal{F} : \mathbb{S} \to F^*\mathbb{T}$ is a special case of the extension of $H : \mathcal{T} \to \mathcal{U}$ to $\mathcal{H} : F^*\mathbb{T} \to G^*\mathbb{U}$. Indeed, if we consider F as an arrow from $\mathrm{Id} : \mathcal{S} \to \mathcal{S}$ to $F : \mathcal{S} \to \mathcal{T}$, then applying the latter construction is the same as applying the former. Therefore, results about the latter construction also apply to the former.

General properties

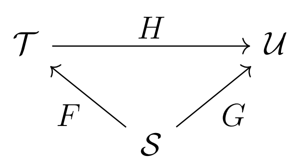
We now state some properties that the canonical indexing, and its associated functors, will always have.

Proposition 6.3.2. Let $\mathcal{S}$ be a category with finite limits. Then $\mathbb{S}$ has finite limits and $\mathcal{S}$-indexed coproducts. In fact, if $\mathcal{S}$, $\mathcal{T}$ have finite limits and $F : \mathcal{S} \to \mathcal{T}$ preserves them, then $F^*\mathbb{T}$ has finite limits and $\mathcal{S}$-indexed coproducts.

Proof. Since $\mathcal{S}$ has finite limits, so does each slice category, and the transition functors preserve them since pullbacks preserve finite limits. Thus, $\mathbb{S}$ has finite limits. Next, $\mathbb{S}$ has $\mathcal{S}$-indexed coproducts because the left adjoint of the pullback functor $x^* : \mathcal{S}/J \to \mathcal{S}/I$ is just the "composition with x" functor $\Sigma_x : \mathcal{S}/I \to \mathcal{S}/J$. It is also well-known that the Beck-Chevalley condition holds for these squares (see also lemma B1.4.7 in [John 02]).

Finally, since $\mathbb{T}$ has finite limits and $\mathcal{T}$-indexed coproducts, we know $F^*\mathbb{T}$ has finite limits and $\mathcal{S}$-indexed coproducts by propositions 6.2.1 and 6.2.4. ∎

Proposition 6.3.3. In the diagram below, assume that $\mathcal{S}$, $\mathcal{T}, \mathcal{U}$ have finite limits, that F, G, H preserve finite limits, and $HF \cong G$.



Then the extension $\mathcal{H} : F^*\mathbb{T} \to G^*\mathbb{U}$ of H preserves finite limits and $\mathcal{S}$-indexed coproducts. (Proposition 6.3.2 guarantees that this is meaningful.)

Proof. We leave the proof that $\mathcal{H}$ preserves coproducts to section A.2 of the appendix. For finite limits, we must show that each functor $\mathcal{H}^I : \mathcal{T}/F(I) \to \mathcal{U}/G(I)$ preserves finite limits. But, by examining the definition of $\mathcal{H}^I$, we find that it is just the functor

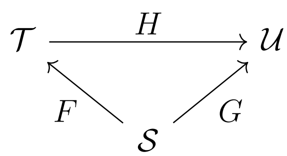
$$H/F(I) : \mathcal{T}/F(I) \to \mathcal{U}/H(F(I))$$

composed with the equivalence of categories $\mathcal{U}/H(F(I)) \simeq \mathcal{U}/G(I)$ induced by the isomorphism $HF \cong G$. The functor $H/F(I)$ preserves finite limits by proposition 2.1.7, so we're done. ∎

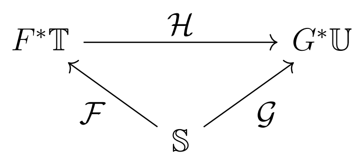
Proposition 6.3.4. Let $\mathcal{S}$ be a category with finite limits, and let $\mathbb{C}$ be an $\mathcal{S}$-indexed category with finite limits and $\mathcal{S}$-indexed coproducts. Up to isomorphism, there is at most one $\mathcal{S}$-indexed functor $\mathbb{S} \to \mathbb{C}$ which preserves finite limits and $\mathcal{S}$-indexed coproducts.

Proof. This proof is done in section A.4 of the appendix. ∎

Remark 6.3.5. Suppose we have a diagram as below, consisting of categories with finite limits and functors which preserve finite limits, which commutes up to isomorphism.



We know that H extends to $\mathcal{H} : F^*\mathbb{T} \to G^*\mathbb{U}$, but we also know that F, G extend to indexed functors, giving us the following diagram of $\mathcal{S}$-indexed categories and $\mathcal{S}$-indexed functors.



As one might expect, this diagram commutes up to isomorphism. We can prove this directly, but we can instead appeal to the machinery we've developed above. Indeed, proposition 6.3.2 tells us that these indexed categories all have finite limits and $\mathcal{S}$-indexed coproducts, and proposition 6.3.3 tells us that these indexed functors all preserve this structure. Therefore, $\mathcal{HF}$ and $\mathcal{G}$ are indexed functors $\mathbb{S} \to G^*\mathbb{U}$ which preserve finite limits and $\mathcal{S}$-indexed coproducts; by proposition 6.3.4, they must be isomorphic.

Additional structure

We now discuss properties of the canonical indexing of categories with additional structure. In particular, we establish properties of $\mathbb{S}$ when $\mathcal{S}$ is an arithmetic universe. However, we first need an intermediate result.

Proposition 6.3.6. Let $\mathcal{S}$ be a category with finite limits, let $\mathbb{S}$ be the canonical indexing, and let f be an arrow of $\mathcal{S}$. Then $\mathbb{S}$ has f-indexed products iff f is exponentiable.

Proof. The forward implication is trivial, because f being exponentiable just means Π_f exists in the canonical indexing $\mathbb{S}$. So, for the converse, assume f is exponentiable. Since pullbacks of exponentiable arrows are exponentiable (Niefield, corol 1.4), we know that Π_g exists for any pullback g of f. As is well-known, the Beck-Chevalley conditions hold for such squares. (See e.g. the proof of lemma B1.4.6 in [John 02]; this result is a version of lemma B1.4.7 in [John 02].) ∎

Now, we prove some results about flexcc indexed categories and functors.

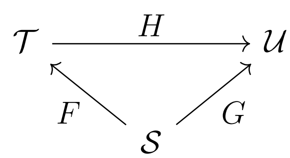
Proposition 6.3.7. If $\mathcal{S}$ is an AU, then the canonical indexing $\mathbb{S}$ is flexcc.

Moreover, if $\mathcal{S}, \mathcal{T}$ are AUs and $F : \mathcal{S} \to \mathcal{T}$ is an AU morphism, then $F^*\mathbb{T}$ is flexcc.

Proof. Fact 2.1.6 (and proposition 2.1.5) tell us that $\mathbb{S}$ has all finite limits and finite colimits (i.e. each slice has them and they are preserved by the transition functors). We know that $\mathbb{S}$ has all indexed coproducts by proposition 6.3.2, and it has N-finite products because π_2^E is exponentiable in an AU (by corollary 5.4.2); proposition 6.3.6 assures us this is sufficent. Thus, $\mathbb{S}$ is flexcc.

For the second part, since $\mathbb{T}$ is flexcc, the fact that $F^*\mathbb{T}$ is flexcc follows from propositions 6.2.1 and 6.2.4. ∎

Proposition 6.3.8. Suppose the diagram below commutes up to isomorphism.



Suppose $\mathcal{S}, \mathcal{T}, \mathcal{U}$ are AUs and F, G are AU functors. If H preserves finite limits and list objects, then the extension $\mathcal{H} : F^*\mathbb{T} \to G^*\mathbb{U}$ is flex; if H additionally preserves finite colimits (i.e. H is an AU functor), then $\mathcal{H}$ is flexcc.

Proof. Suppose H preserves finite limits and list objects. By proposition 6.3.3, $\mathcal{H}$ preserves finite limits, and $\mathcal{H}$ preserves N-finite products as a direct consequence of proposition 5.4.4.

Now, suppose H also preserves finite colimits. Then each H^I preserves finite colimits too. (As in the proof of proposition 6.3.3, H^I is just $H/F(I)$, modulo an equivalence of categories; this functor preserves finite colimits by proposition 2.1.7.) Moreover, by proposition 6.3.3, the extension of H preserves indexed coproducts, and we conclude that $\mathcal{H}$ is cocontinuous. ∎

Proposition 6.3.9. Suppose $\mathcal{S}, \mathcal{T}, \mathcal{U}$ are AUs and $F : \mathcal{S} \to \mathcal{T}$, $G : \mathcal{S} \to \mathcal{U}$ are AU functors. If $\mathcal{H} : F^*\mathbb{T} \to G^*\mathbb{U}$ is a flexcc indexed functor, then the underlying functor $H : \mathcal{T} \to \mathcal{U}$ is an AU functor.

Proof. By definition, each fiber of an indexed flexcc morphism must preserve finite limits and finite colimits, so all we need to check is that H preserves list objects. That is, we must show H commutes with the list object functor.

We know that the list object functor is polynomial, so in $G^*\mathbb{U}$, it is given by the composite $L = \Sigma_N \circ \Pi_{\pi_2^E} \circ \langle\rangle_E^*$. The fibers of $\mathcal{H}$ commute with $\langle\rangle_E^*$ by axioms of indexed functors, they commute with $\Pi_{\pi_2^E}$ because $\mathcal{H}$ preserves N-finite products, and they commute with Σ_N because $\mathcal{H}$ is cocontinuous. All together, this implies H commutes with L, as desired. ∎

6.4 The category of indexed functors

Definition. Let $\mathcal{S}$ be a category with finite limits, and let $\mathbb{C}, \mathbb{D}$ be $\mathcal{S}$-indexed categories. Then we write $[\mathbb{C}, \mathbb{D}]_{\mathcal{S}}$ for the category of $\mathcal{S}$-indexed functors $\mathbb{C} \to \mathbb{D}$ and $\mathcal{S}$-indexed natural transformations. (We drop the subscript if it is clear from the context.)

In section 6.5, we will make this into an indexed category, but for now, we just consider it as an ordinary category. We write $[\mathbb{C}, \mathbb{D}]_{cc}$ for the category of cocontinuous indexed functors, and we use similar notation for flex or flexcc indexed functors.

It will be important to know when $[\mathbb{C}, \mathbb{D}]$ has limits, colimits, and other properties.

Proposition 6.4.1. Let $\mathcal{S}$ be a category with finite limits, and let $\mathbb{C}, \mathbb{D}$ be $\mathcal{S}$-indexed categories. If $\mathbb{D}$ has finite limits or finite colimits, then so does $[\mathbb{C}, \mathbb{D}]$; these are computed pointwise.

Proof. We just need to check that computing finite (co)limits pointwise does indeed give finite (co)limits for $[\mathbb{C}, \mathbb{D}]$. This is rather straightforward; we do the case of binary products as an example.

Let $F, G : \mathbb{C} \to \mathbb{D}$. We define $F \times G : \mathbb{C} \to \mathbb{D}$ by setting $(F \times G)^I = F^I \times G^I$ for each $I \in \mathcal{S}$. For this to be well-defined, we must check coherence with the transition maps. That is, for each $x : I \to J$, the following diagram should commute.

$$\begin{array}{ccc} \mathcal{C}^J & \xrightarrow{F^J \times G^J} & \mathcal{D}^J \\ {\scriptstyle x^*}\downarrow & & \downarrow{\scriptstyle x^*} \\ \mathcal{C}^I & \xrightarrow{F^I \times G^I} & \mathcal{D}^I \end{array}$$

Since $\mathbb{D}$ has finite limits, we know that x^* preserves binary products in $\mathbb{D}$. Therefore, for $u \in \mathcal{C}^J$,

$$\begin{aligned} x^*((F^J \times G^J)(u)) = x^*(F^J(u) \times G^J(u)) &\cong (x^* F^J(u)) \times (x^* G^J(u)) \\ &\cong F^I(x^*u) \times G^I(x^*u) = (F^I \times G^I)(x^*u), \end{aligned}$$

as desired.

We can similarly define $\pi_1 : F \times G \to F$ and $\pi_2 : F \times G \to G$. Checking the universal property is similarly done pointwise: if we have maps $\alpha : H \to F$, $\beta : H \to G$, then we can form $\langle \alpha, \beta \rangle : H \to F \times G$ by setting $\langle \alpha, \beta \rangle^I_u = \langle \alpha^I_u, \beta^I_u \rangle$. Indeed, we have

$$\langle \alpha^I_u, \beta^I_u \rangle : H^I(u) \to F^I(u) \times G^I(u) = (F \times G)^I(u).$$

So, $F \times G$ is indeed the desired product. ∎

The most important result for us will be the following.

Proposition 6.4.2. If $\mathcal{S}$ is an arithmetic universe, then so is $[\mathbb{C}, \mathbb{S}]_{\mathcal{S}}$ for any $\mathcal{S}$-indexed category $\mathbb{C}$. Moreover, the finite limits, finite colimits, and list objects in $[\mathbb{C}, \mathbb{S}]$ are computed pointwise.

Proof. First, recall from proposition 6.3.7 that $\mathbb{S}$ has finite limits and is cocomplete, so proposition 6.4.1 tells us $[\mathbb{C}, \mathbb{S}]$ has finite limits and finite colimits.

Next, since AUs are local (fact 2.1.6), we know that each slice $\mathcal{S}/I$ has list objects, and these are preserved by the pullback functors (i.e., the transition maps of $\mathbb{S}$). Therefore, we find that the list objects in $[\mathbb{C}, \mathbb{S}]$ can be computed pointwise; this is done just as in the proof of proposition 6.4.1.

Finally, we must check the AU coherence axioms between limits and colimits (i.e. coproducts are stable and disjoint, quotients of equivalence relations are stable and effective). Again, this can all be done pointwise, since each fiber $\mathcal{S}/I$ is an AU and each transition map is an AU functor.

For example, we show that coproducts are stable. Let $F, G, H \in [C, \mathbb{S}]$, and let $\alpha : H \to F + G$. If we pull back α along the inclusions into $F + G$, we get the following diagram; we must show that $H_F + H_G = H$.

$$\begin{array}{ccccc} H_F & \longrightarrow & H & \longleftarrow & H_G \\ \downarrow & & \downarrow \alpha & & \downarrow \\ F & \longrightarrow & F + G & \longleftarrow & G \end{array}$$

For each $I \in \mathcal{S}$ and $u \in C(I)$, we get a corresponding diagram in $\mathcal{S}/I$, where the bottom row is a coproduct and the squares are pullbacks since these are computed pointwise. But then, since $\mathcal{S}/I$ is an AU, the top row $H^I_F(u) \to H^I(u) \leftarrow H^I_G(u)$ must be a coproduct. Since coproducts in $[C, \mathbb{S}]$ are computed pointwise, this tells us that $H_F \to H \leftarrow H_G$ is a coproduct too, as desired. ∎

There is an important kind of indexed functors which generalize the usual notion of constant functors.

Definition. Let $\mathcal{S}$ be a category with finite limits, and let $\mathbb{C}, \mathbb{D}$ be $\mathcal{S}$-indexed categories. Then there is a functor $\Delta_{\mathbb{C}} : \mathcal{D} \to [\mathbb{C}, \mathbb{D}]$, sometimes called the *diagonal functor*, defined as follows.

- For $X \in \mathcal{D}$, the indexed functor $\Delta_{\mathbb{C}} X : \mathbb{C} \to \mathbb{D}$ is given by $(\Delta_{\mathbb{C}} X)^I(c) = I^* X$. That is, for each $I \in \mathcal{S}$, $(\Delta_{\mathbb{C}} X)^I$ is the constant functor $\mathcal{C}^I \to \mathcal{D}^I$ taking the value $I^* X$. (Note that we write I^* for the transition functor coming from $I \to \mathbb{1}$.)
- For $f : X \to Y$ in $\mathcal{D}$, the indexed natural transformation $\Delta_{\mathbb{C}} f : \Delta_{\mathbb{C}} X \Rightarrow \Delta_{\mathbb{C}} Y$ is given by $(\Delta_{\mathbb{C}} f)^I(c) = I^* f : I^* X \to I^* Y$.

We omit the easy check that this is well-defined. We may drop the subscript $\mathbb{C}$ if it is clear from context; we may also call this functor CF (for "constant functor").

Proposition 6.4.3. If $\mathcal{S}$ is an AU, then for any $\mathcal{S}$-indexed category $\mathbb{C}$, the functor $\Delta_{\mathbb{C}} : \mathcal{S} \to [\mathbb{C}, \mathbb{S}]$ is an AU morphism. (This makes sense by proposition 6.4.2.)

Proof. We must check that $\Delta_{\mathbb{C}}$ preserves finite limits, finite colimits, and list objects. Since these are computed pointwise in $[C, \mathbb{S}]$ (by proposition 6.4.2), this becomes straightforward. To illustrate, we do binary products as an example.

Given $X, Y \in \mathcal{S}$, the indexed functor $\Delta_{\mathbb{C}}(X \times Y)$ is given by

$$\begin{aligned} \Delta_{\mathbb{C}}(X \times Y)^I(u) = I^*(X \times Y) \cong I^* X \times I^* Y &= \Delta_{\mathbb{C}}(X)^I(u) \times \Delta_{\mathbb{C}}(Y)^I(u) \\ &= \big(\Delta_{\mathbb{C}}(X) \times \Delta_{\mathbb{C}}(Y)\big)^I(u). \end{aligned}$$

Thus, $\Delta_{\mathbb{C}}(X \times Y) \cong \Delta_{\mathbb{C}}(X) \times \Delta_{\mathbb{C}}(Y)$. This works because the product is computed pointwise in the functor category, and because the functor $I^* : \mathcal{S} \to \mathcal{S}/I$ (the pullback functor associated to the map $\langle\rangle_I : I \to \mathbb{1}$) is an AU morphism (hence preserve limits, colimits, and list objects). ∎

6.5 The indexed category of indexed functors

In this section, we see how to make $[\mathbb{C}, \mathbb{D}]$ into an indexed category. There are many ways of doing this; we use the following definition because it is the most useful for us.

Definition. Let $\mathcal{S}$ be a category with finite limits, and let $\mathbb{C}, \mathbb{D}$ be $\mathcal{S}$-indexed categories. We define the $\mathcal{S}$-indexed category $[\![\mathbb{C}, \mathbb{D}]\!]$ as follows: for any $I \in \mathcal{S}$, we set

$$[\![\mathbb{C}, \mathbb{D}]\!]^I = [\Sigma_I^* \mathbb{C}, \Sigma_I^* \mathbb{D}]_{\mathcal{S}/I},$$

where $\Sigma_I : \mathcal{S}/I \to \mathcal{S}$ is the forgetful functor. That is, we take the pullback of $\mathbb{C}$ and $\mathbb{D}$ along Σ_I to get $\mathcal{S}/I$-indexed categories, and we consider the category of $\mathcal{S}/I$-indexed functors between them.

The transition functors for $[\![\mathbb{C}, \mathbb{D}]\!]$ are defined as follows. Given $f : I \to J$ in $\mathcal{S}$, the functor f^* maps $(G^y)_{y:Y \to J} \in [\Sigma_J^* \mathbb{C}, \Sigma_J^* \mathbb{D}]$ to $(G^{fx})_{x:X \to I} \in [\Sigma_I^* \mathbb{C}, \Sigma_I^* \mathbb{D}]$.

Remark 6.5.1. In [John 02], the indexed category $[\![\mathbb{C}, \mathbb{D}]\!]$ is defined differently. The definition is given in the discussion above example B1.2.5, and more equivalent characterizations are given in example B2.3.12 for when $\mathbb{C}$ is an internal category. It is not hard to check that these definitions are equivalent to the one we gave above.

However, there is one additional equivalent characterization that will be important to us.

Proposition 6.5.2. Let $\mathcal{S}$ be a category with finite limits, and let $\mathbb{C}$ be an $\mathcal{S}$-indexed category. Then, for each $I \in \mathcal{S}$, we have

$$[\mathbb{C}, \mathbb{S}]^I \simeq [\mathbb{C}, \mathbb{S}]/\Delta(I),$$

where $\Delta : \mathcal{S} \to [\mathbb{C}, \mathbb{S}]$ is the diagonal functor (see section 6.4).

In other words, if we take the canonical self-indexing of $[\mathbb{C}, \mathbb{S}]$, then the $\mathcal{S}$-indexed category obtained by pulling back along Δ is equivalent to $[\![\mathbb{C}, \mathbb{S}]\!]$.

Proof. We omit the straightforward check. ∎

We will also need the following observations.

Remark 6.5.3. Let $\mathcal{S}$ be a category with finite limits. First, given $I \in \mathcal{S}$, the $\mathcal{S}/I$-indexed category $\Sigma_I^* \mathbb{S}$ is equivalent to the canonical self-indexing of $\mathcal{S}/I$ (which we may denote $\mathbb{S}/I$).

Next, let $\mathbb{C}$ be an internal category in $\mathcal{S}$ (see chapter 7). Given $I \in \mathcal{S}$, the $\mathcal{S}/I$-indexed category $\Sigma_I^* \mathbb{C}$ is isomorphic to an internal category in $\mathcal{S}/I$, which we'll denote $I^*\mathbb{C}$, given by $\pi_1 : I \times C_0 \to I$ and $\pi_1 : I \times C_1 \to I$.

We now show that $[\![\mathbb{C}, \mathbb{D}]\!]$ behaves like an exponential.

Proposition 6.5.4. Let $\mathcal{S}$ be a category with finite limits, and let $\mathbb{C}, \mathbb{D}$ be $\mathcal{S}$-indexed categories. Then $[\![\mathbb{C}, \mathbb{D}]\!]$ is the exponential of $\mathbb{C}$ and $\mathbb{D}$ (in the 2-category of $\mathcal{S}$-indexed categories).

This means that $[\![\mathbb{C}, \mathbb{D}]\!]$ is equipped with an indexed functor $\mathrm{ev} : \mathbb{C} \times [\![\mathbb{C}, \mathbb{D}]\!] \to \mathbb{D}$, and for any $\mathbb{E}$, the following functor is an equivalence of categories.

$$\begin{aligned}[][\mathbb{E}, [\![\mathbb{C}, \mathbb{D}]\!]] &\to [\mathbb{C} \times \mathbb{E}, \mathbb{D}] \\ F &\mapsto \mathrm{ev} \circ (\mathrm{Id} \times F)\end{aligned}$$

Note: in following sections, we will often need to use the explicit constructions of ev and of the inverse in this equivalence of categories. They are given in the proof of this proposition.

Proof. First, we define ev. For any $I \in \mathcal{S}$, the functor $\mathrm{ev}^I : \mathcal{C}^I \times [\Sigma_I^*\mathbb{C}, \Sigma_I^*\mathbb{D}]_{\mathcal{A}/I} \to \mathcal{D}^I$ takes a pair (c, F) and returns $F^{T_I}(c)$, where T_I is $\mathrm{Id} : I \to I$, the terminal object of $\mathcal{A}/I$. Note that $\Sigma_I(T_I) = I$, so we indeed have $F^{T_I} : \mathcal{C}^I \to \mathcal{D}^I$.

Next, given an indexed functor $G : \mathbb{C} \times \mathbb{E} \to \mathbb{D}$, we must construct an indexed functor $F_G : \mathbb{E} \to [\![\mathbb{C}, \mathbb{D}]\!]$ such that $\mathrm{ev} \circ (\mathrm{Id} \times F_G) \cong G$. So, for each $I \in \mathcal{S}$, we must define $F_G^I : \mathcal{E}^I \to [\Sigma_I^*\mathbb{C}, \Sigma_I^*\mathbb{D}]_{\mathcal{A}/I}$, which means that for $e \in \mathcal{E}^I$, we must define the indexed functor $F_G^I(e) : \Sigma_I^*\mathbb{C} \to \Sigma_I^*\mathbb{D}$. So, let $\alpha : J \to I$ be an object of $\mathcal{A}/I$. We must define the functor

$$\left(F_G^I(e)\right)^\alpha : \mathcal{C}^J \to \mathcal{D}^J,$$

and we set

$$\left(F_G^I(e)\right)^\alpha(c) = G^J(c, \alpha^* e).$$

This finishes the definition of F_G; now, note that for $I \in \mathcal{S}$, $c \in \mathcal{C}^I$, and $e \in \mathcal{E}^I$, we have

$$\mathrm{ev}^I(c, F_G^I(e)) = \left(F_G^I(e)\right)^{T_I}(c) = G^I(c, T_I^* e) = G^I(c, e)$$

(because $T_I = \mathrm{Id}_I$, so $T_I^* = \mathrm{Id}$). Thus this F_G was constructed to have the desired property. ∎

It will be useful to know when $[\![\mathbb{C}, \mathbb{D}]\!]$ has various limits. We therefore end this section with the following characterization.

Proposition 6.5.5. Let $\mathcal{S}$ be a category with finite limits, and let $\mathbb{C}, \mathbb{D}$ be $\mathcal{S}$-indexed categories. If $\mathbb{D}$ has finite (co)limits, then so does $[\![\mathbb{C}, \mathbb{D}]\!]$. Moreover, given an arrow f in $\mathcal{S}$, if $\mathbb{D}$ has f-indexed (co)products, then so does $[\![\mathbb{C}, \mathbb{D}]\!]$.

In particular, let $\mathcal{S}$ be a category with finite limits and NNO. If $\mathbb{D}$ is flex, then so is $[\![\mathbb{C}, \mathbb{D}]\!]$; and if $\mathbb{D}$ is flexcc, then so is $[\![\mathbb{C}, \mathbb{D}]\!]$.

Proof. First, we check that all the fibers have finite (co)limits. For each $I \in \mathcal{S}$, we know that $\Sigma_I^*\mathbb{D}$ has finite (co)limits (by proposition 6.2.1), so $[\Sigma_I^*\mathbb{C}, \Sigma_I^*\mathbb{D}]$ does too (by proposition 6.4.1).

The transition maps preserve finite (co)limits since these are computed pointwise (as noted in proposition 6.4.1). For example, consider the case of binary products. Let $x : I \to J$, and let $F, G : \Sigma_J^*\mathbb{C} \to \Sigma_J^*\mathbb{D}$; then, for $g : U \to I$ in $\mathcal{S}/I$ and $A \in \mathcal{C}^U$, we have

$$\begin{aligned} ((x^*F) \times (x^*G))^g(A) = (x^*F)^g(A) \times (x^*G)^g(A) &= F^{xg}(A) \times G^{xg}(A) \\ &= (F \times G)^{xg}(A) = (x^*(F \times G))(A). \end{aligned}$$

This shows that $x^*F \times x^*G \cong x^*(F \times G)$, as desired. Therefore, we know that $[\![\mathbb{C}, \mathbb{D}]\!]$

has finite (co)limits.

Next, let f be an arrow in $\mathcal{S}$, and assume $\mathbb{D}$ has f-indexed products (coproducts are done in the same way). We will show that the functor $f^* : [\mathbb{C}, \mathbb{D}]^A \to [\mathbb{C}, \mathbb{D}]^B$ has a right adjoint and omit the other details. Recall that f^* maps an indexed functor $(F^x)_{x:I\to A}$ to $(F^{fy})_{y:I\to B}$.

Given an indexed functor $G \in [\mathbb{C}, \mathbb{D}]^B$, we define $\Pi_f G \in [\mathbb{C}, \mathbb{D}]^A$ as follows. For $x : I \to A$ in $\mathcal{S}/A$, form the pullback square below.

$$\begin{array}{ccc} I \times_x B & \xrightarrow{\pi_2^x} & B \\ {\scriptstyle \pi_1^x}\downarrow & & \downarrow{\scriptstyle f} \\ I & \xrightarrow{x} & A \end{array}$$

Then, we define $(\Pi_f G)^x : \mathcal{C}^I \to \mathcal{D}^I$ as the following composite.

$$\mathcal{C}^I \xrightarrow{(\pi_1^x)^*} \mathcal{C}^{I\times_x B} \xrightarrow{G^{\pi_2^x}} \mathcal{D}^{I\times_x B} \xrightarrow{\Pi_{\pi_1^x}} \mathcal{D}^I$$

To show that Π_f is the right adjoint of f^*, we claim that for any $F \in [\mathbb{C}, \mathbb{D}]^A$ and $G \in [\mathbb{C}, \mathbb{D}]^B$, we have the isomorphism $\mathrm{Hom}(f^*F, G) \cong \mathrm{Hom}(F, \Pi_f G)$. To make sense of this, we make the following remarks.

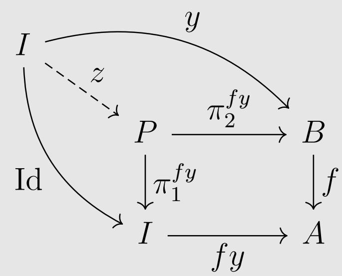
- An arrow $\eta : f^*F \to G$ consists of a family of arrows $\eta_u^y : F^{fy}(u) \to G^y(u)$ in $\mathcal{D}^I$, indexed by $(y : I \to B) \in \mathcal{S}/B$ and $u \in \mathcal{C}^I$ (satisfying some conditions).
- An arrow $\mu : F \to \Pi_f G$ consists of a family of arrows $\mu_u^x : F^x(u) \to \Pi_{\pi_1^x} \circ G^{\pi_2^x} \circ (\pi_1^x)^*(u)$, indexed by $(x : I \to A) \in \mathcal{S}/A$ and $u \in \mathcal{C}^I$.

Now, by the adjunction property of $\Pi_{\pi_1^x}$ (which comes from $\mathbb{D}$ having f-indexed products), we know that an arrow μ_u^x corresponds to an arrow $(\pi_1^x)^* F^x(u) \to G^{\pi_2^x}(\pi_1^x)^* u$. Moreover, since F commutes with the transition maps, this is equivalent to

$$\tilde{\mu}_u^x : F^{f\pi_2^x}\big((\pi_1^x)^* u\big) \to G^{\pi_2^x}\big((\pi_1^x)^* u\big).$$

(Note that π_1^x is an arrow from $f\pi_2^x$ to x in $\mathcal{S}/A$.) So, from η, we can define μ by setting $\tilde{\mu}_u^x = \eta^{\pi_2^x}_{(\pi_1^x)^* u}$.

Conversely, suppose we are given μ. Then, for any $(y : I \to B) \in \mathcal{S}/B$ and $u \in \mathcal{C}^I$, we can define η_u^y as follows. First, consider the following diagram, where P is a pullback.



Note that $z^*(\pi_1^{fy})^* = \mathrm{Id}$. Moreover, note that z is an arrow from fy to $f\pi_2^{fy}$ in $\mathcal{S}/A$, so

$$z^* F^{f\pi_2^{fy}}\big((\pi_1^{fy})^* u\big) \cong F^{fy}\big(z^*(\pi_1^{fy})^* u\big) \cong F^{fy}(u).$$

Similarly, z is an arrow from y to π_2^{fy} in $\mathcal{S}/B$, so

$$z^* G^{\pi_2^{fy}}\big((\pi_1^{fy})^* u\big) \cong G^{y}\big(z^*(\pi_1^{fy})^* u\big) \cong G^{y}(u).$$

Therefore, we can take η_u^y to be $z^*\mu_u^{fy}$ with its domain and codomain adjusted by the above isomorphisms.

This correspondence between μ and η gives the desired hom-set isomorphism. ∎

Chapter 7

Internal categories

Contents

In this chapter, we review some basic definitions and facts about internal categories. Moreover, we explain how to handle the following concepts with our internal language: internal categories and functors (section 7.1), copresheaves on internal categories (section 7.2, and finite limits in internal categories (section 7.3). Finally, in section 7.4, we discuss filtered internal categories and the category of elements for a copresheaf.

A particularly important concept for internal categories is how they relate to indexed categories. Indeed, each internal category in $\mathcal{S}$ can be viewed as an $\mathcal{S}$-indexed category (see the definition at the end of section 7.1). Broadly, if we think about $\mathcal{S}$-indexed categories as a way of doing category theory with respect to a base $\mathcal{S}$, then internal categories correspond to *small* categories, and we think of general $\mathcal{S}$-indexed categories as potentially large. Indeed, Johnstone uses the term *essentially small* for indexed categories which are equivalent to an internal category (see the discussion above B2.3.5 in [John 02]). We will see this come into play (somewhat indirectly) starting in chapter 9, where we start taking colimits of indexed functors whose domain is an internal category.

7.1 Definition and description via internal language

The definitions of an internal category, internal functor, and internal natural transformation are somewhat technical. We defer to [John 02] (definition B2.3.1) for the full definitions, but we give a summarized version here.

Definition. Let $\mathcal{S}$ be a category with finite limits. An *internal category* C in $\mathcal{S}$ consists of the following:

- Objects $C_0, C_1 \in \mathcal{S}$ with arrows $\mathrm{dom}, \mathrm{cod} : C_1 \rightrightarrows C_0$;
- An arrow $\iota : C_0 \to C_1$ such that $\mathrm{dom}\iota = \mathrm{cod}\iota = \mathrm{Id}_{C_0}$;
- An arrow $\gamma : C_1 \times_{C_0} C_1 \to C_1$ (where the pullback is as below) satisfying $\mathrm{dom}\gamma = \mathrm{dom}\pi_1$, $\mathrm{cod}\gamma = \mathrm{cod}\pi_1$, the identity axioms (with respect to ι), and the associativity axiom.

$$\begin{array}{ccc} C_1 \times_{C_0} C_1 & \xrightarrow{\pi_2} & C_1 \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \mathrm{dom}} \\ C_1 & \xrightarrow{\mathrm{cod}} & C_0 \end{array}$$

Remark 7.1.1. In order to discuss internal categories more easily, we would like to use our internal language. For instance, we will want to talk about "objects" and "arrows" of the internal category C, but this will always have to be with respect to a given context. So, let T be a context. We may refer to a term $c : C_0$ as an "object" of C in the context T, and we may refer to $f : C_1$ as an "arrow" of C in the context T. If, in a context T, we say that we are considering an arrow $f : c \to d$ of C, what we mean is that $f : C_1$ and we are denoting $c =_T \mathrm{dom}(f)$, $d =_T \mathrm{cod}(f)$. We may also write $f : \mathrm{Hom}(c, d)$. Furthermore, given $c : C_0$, we will denote $\iota(c) : C_1$ by Id_c.

Next, we consider composition. The domain of the composition arrow γ is $C_1 \times_{C_0} C_1 = \{f, g : C_1 \mid \mathrm{cod}(f) = \mathrm{dom}(g)\}$. If we have two terms $f, g : C_1$ in a context T such that $\mathrm{cod}(f) =_T \mathrm{dom}(g)$, then (by the abuse of notation in our internal language) we can say that $(f, g) : C_1 \times_{C_0} C_1$, and therefore $\gamma(f, g)$ is well-defined. In such a case, we denote $\gamma(f, g)$ by $g \circ f$. Note that the condition $\mathrm{cod}(f) =_T \mathrm{dom}(g)$ is implicitly fulfilled if we write something like $f : b \to c$ and $g : c \to d$.

Finally, we can describe the axioms that must be satisfied (throughout, we work in a fixed context T without mentioning it). First, we have requirements for the domain and codomain of identities and composites. Given $c : C_0$, we must have $\mathrm{dom}(\mathrm{Id}_c) = \mathrm{cod}(\mathrm{Id}_c) = c$; this can be summarized by saying that $\mathrm{Id}_c : c \to c$. Moreover, given $f : b \to c$ and $g : c \to d$, we must have $g \circ f : b \to d$. Next, the identities must behave like identities. Given $f : c \to d$, we must have $f \circ \mathrm{Id}_c = f$ and $\mathrm{Id}_d \circ f = f$. (Note that the composites are well-defined because of the requirements on the domain and codomain of Id_c. Moreover, note that the equality here is between terms of type C_1, but there is no conflict between the domains and codomains of these arrows - that is, we have $\mathrm{dom}(f \circ \mathrm{Id}_c) = \mathrm{dom}(f)$ and other similar equalities even before we impose the equality $f \circ \mathrm{Id}_c = f$.) Finally, we must have associativity of composition. Given terms $f : a \to b$, $g : b \to c$, $h : c \to d$, we must have $h \circ (g \circ f) = (h \circ g) \circ f$. (Again, it is easy to check that all these composites are well-defined based on the axioms of domain and codomain for composition.)

This finishes the discussion of how we will talk about internal categories using the internal language. It may be interesting to note that we are **not** using dependent types to discuss these concepts, as one might usually do. At least, we are not baking this into our internal language; we do use notation that looks like dependent types, but it is just abuse of notation for a simply typed system. This is to avoid using complicated type theories that may require the underlying category to have more structure.

Remark 7.1.2. Now that we've discussed internal categories, we must also discuss internal functors between internal categories. So, let C, D be internal categories in $\mathcal{S}$. An internal functor $F : C \to D$ consists of arrows $F_0 : C_0 \to D_0$ and $F_1 : C_1 \to D_1$ in $\mathcal{S}$ such that the following properties hold. In the internal language, we may write F for either of F_0 or F_1, as we do with usual functors.

- First, the action on arrows must respect the domain and codomain. We can write this equationally as $\mathrm{dom}(F_1(f)) =_{f:C_1} F_0(\mathrm{dom}(f))$ and $\mathrm{cod}(F_1(f)) =_{f:C_1} F_0(\mathrm{cod}(f))$. Internally, we can say that in any context T, if we have an arrow $f : c \to c'$ in C, then $Ff : Fc \to Fc'$ in D.
- Second, the functor must respect identities. This means that, in any context T, for any term $c : C_0$, we have $F(\mathrm{Id}_c) = \mathrm{Id}_{Fc}$.
- Third, the functor must respect composition. This means that, in any context T, if we have $f : c \to c'$ and $g : c' \to c''$, then $F(g \circ f) = Fg \circ Ff$. (The identities for the domain and codomain ensure that everything here is well-defined.)

Internal categories play an important role in the theory of indexed categories, as each internal category can be viewed as an indexed category. This definition comes from [John 02], in the discussion before lemma B2.3.3.

Definition. Let $\mathcal{S}$ be a category with finite limits, and let C be an internal category in $\mathcal{S}$. The *$\mathcal{S}$-indexed category corresponding to C*, which we also denote C, is defined as follows.

- For $I \in \mathcal{S}$, the category $C(I)$ is defined as follows. The objects of $C(I)$ are arrows $I \to C_0$ in $\mathcal{S}$, and the arrows of $C(I)$ are arrows $I \to C_1$ in $\mathcal{S}$. The domain, codomain, identity, and composition operations are induced by those in C.
- For $x : I \to J$ in $\mathcal{S}$, the functor $x^* : C(J) \to C(I)$ is given by pre-composition with x. (That is, an object $p : J \to C_0$ is mapped to px, and an arrow $q : J \to C_1$ is mapped to qx.)

Remark 7.1.3. Given $I \in \mathcal{S}$, the objects of $C(I)$ are terms of type C_0 in the context $(i : I)$ (up to equality), and the arrows of $C(I)$ are terms of type C_1 in the context $(i : I)$ (up to equality). This shows that our internal language is well-suited to discuss these concepts.

7.2 Copresheaves on internal categories

Definition. Let $\mathcal{S}$ be a category with finite limits, and let C be an internal category in $\mathcal{S}$. An *(internal) copresheaf* on C is an $\mathcal{S}$-indexed functor $C \to \mathbb{S}$ (when considering C as an $\mathcal{S}$-indexed category).

Note that the category of copresheaves on C is simply the category $[C, \mathbb{S}]$ of indexed functors.

As it turns out, a copresheaf on an internal category C can be expressed internally quite nicely, using its corresponding *diagram*. We take this concept from definition B2.3.11 in [John 02]. (Again, we do not give the full details.)

Definition. Let $\mathcal{S}$ be a category with finite limits, let C be an internal category in $\mathcal{S}$, and let $\mathbb{D}$ be an $\mathcal{S}$-indexed category. A *diagram F of shape C in $\mathbb{D}$* consists of a pair $(F_\bullet, \phi_F)$, where

- $F_\bullet$ is an object of $\mathcal{D}^{C_0}$, and
- ϕ_F is an arrow $\mathrm{dom}^* F_\bullet \to \mathrm{cod}^* F_\bullet$ in $\mathcal{D}^{C_1}$ which is compatible with the identity and composition of C.

A *morphism of diagrams* $(F_\bullet, \phi_F) \to (G_\bullet, \phi_G)$ is an arrow $F_\bullet \to G_\bullet$ which commutes with the arrows ϕ_F, ϕ_G in the obvious way.

The motivation for this definition is the following fact, which we take from lemma B2.3.13 in [John 02].

Fact 7.2.1. Let $\mathcal{S}$ be a category with finite limits, let C be an internal category in $\mathcal{S}$, and let $\mathbb{D}$ be an $\mathcal{S}$-indexed category. Then the category of diagrams of shape C in $\mathbb{D}$ is equivalent to the category $[C, \mathbb{D}]$.

Specifically, an indexed functor $F : C \to \mathbb{D}$ is mapped to the diagram $(F_\bullet, \phi_F)$ where $F_\bullet = F^{C_0}(\mathrm{Id}_{C_0})$ and $\phi_F = F^{C_1}(\mathrm{Id}_{C_1})$ (modulo some coherence isomorphisms). Conversely, given a diagram $(F_\bullet, \phi_F)$, we form $F^I : C(I) \to \mathcal{D}^I$ by setting $F^I(p : I \to C_0) = p^* F_\bullet$ and $F^I(q : I \to C_1) = q^* \phi_F$ (again, modulo some coherence isomorphisms).

Now, let's see how this applies to copresheaves (the special case where $\mathbb{D} = \mathbb{S}$).

Remark 7.2.2. A diagram of shape C in $\mathbb{S}$ is a pair $(F_\bullet, \phi)$. First, $F_\bullet$ is an object of $\mathcal{S}/C_0$, so we may as well write it as $\beta : F_0 \to C_0$. Second, ϕ is an arrow $\mathrm{dom}^* F_\bullet \to \mathrm{cod}^* F_\bullet$. Note that ϕ is an arrow in $\mathcal{S}/C_1$ between objects formed using the following pullback squares.

$$
\begin{array}{ccc}
C_1 \times_{\mathrm{dom}} F_0 & \longrightarrow & F_0 \\
\downarrow & & \downarrow{\scriptstyle \beta} \\
C_1 & \xrightarrow{\mathrm{dom}} & C_0
\end{array}
\qquad
\begin{array}{ccc}
C_1 \times_{\mathrm{cod}} F_0 & \longrightarrow & F_0 \\
\downarrow & & \downarrow{\scriptstyle \beta} \\
C_1 & \xrightarrow{\mathrm{cod}} & C_0
\end{array}
\qquad
\begin{array}{ccc}
C_1 \times_{\mathrm{dom}} F_0 & \xrightarrow{\phi} & C_1 \times_{\mathrm{cod}} F_0 \\
& \searrow \quad \swarrow & \\
& C_1 &
\end{array}
$$

The arrow ϕ must satisfy some requirements, but before that we will first describe the notation we use to talk about $F_\bullet$ and ϕ.

We think of a diagram as a copresheaf, in the sense that it maps each $c \in C_0$ to a "set" $F(c)$. This is represented by the arrow $\beta : F_0 \to C_0$; we think of it as the collection $(F(c) \mid c \in C_0)$. So, in a context T, if we write $x \in F(c)$, what we mean is that $x : F_0$ and we denote $c =_T \beta(x)$. Note that we have chosen the letter β here to stand for "basepoint".

Next, we know that a copresheaf must also act on arrows, which is done indirectly with ϕ. Using the above notation, ϕ takes a pair (f, x) with $f : c \to d$ and $x \in F(c)$ and maps it to (f, y) with $y \in F(d)$. Indeed, note that the constraint on ϕ means that we can write it as $\phi = \langle \pi_1, F_1 \rangle$, where F_1 is an arrow $C_1 \times_{\mathrm{dom}} F_0 \to F_0$ such that $\mathrm{cod} \circ \pi_1 = \beta \circ F_1$. Thus, ϕ maps (f, x) to (f, y) with $y = F_1(f, x)$; we will denote $F_1(f, x)$ by $Ff(x)$. To summarize: if we have terms $f : c \to d$ and $x \in F(c)$, then we get a term $Ff(x) \in F(d)$.

Finally, we can express the axioms of a copresheaf. The first is action on the identity: for any terms $c : C_0$ and $x \in F(c)$, we have $F\mathrm{Id}_c(x) = x$. The second is action on composition: for any terms $f : b \to c$, $g : c \to d$, and $x \in F(b)$, we have $F(g \circ f)(x) = Fg(Ff(x))$. (Note that all of this is well-defined.)

We will also need an internal description of morphisms of diagrams, in order to talk about arrows between copresheaves.

Remark 7.2.3. We have the notion of indexed natural transformations between indexed functors, and an indexed natural transformation between copresheaves $F, G : C \to \mathbb{S}$ corresponds to a morphism between the corresponding diagrams of shape C in $\mathbb{S}$. Indeed, $\eta : F \Rightarrow G$ gives us $\eta^{C_0}_{\mathrm{Id}_{C_0}} : F^{C_0}(\mathrm{Id}_{C_0}) \to G^{C_0}(\mathrm{Id}_{C_0})$, which we abbreviate as η_0.

$$\begin{array}{ccc} F_0 & \xrightarrow{\eta_0} & G_0 \\ & {\scriptstyle\beta}\searrow \quad \swarrow{\scriptstyle\beta} & \\ & C_0 & \end{array}$$

This tells us that if $x : F_0$, then $\eta(x) : G_0$, and if $x \in F(c)$, then $\eta(x) \in G(c)$. We also have the naturality property: given $f : c \to d$ and $x \in F(c)$, we have $Gf(\eta(x)) = \eta(Ff(x))$. This is obtained by applying π_2 to the following square, which comes from naturality of η (applied to the arrow $\mathrm{Id}_{C_1} : \mathrm{dom} \to \mathrm{cod}$ in $C(C_1)$).

$$\begin{array}{ccc} C_1 \times_{\mathrm{dom}} F_0 & \xrightarrow{\mathrm{Id} \times_{\mathrm{dom}} \eta_0} & C_1 \times_{\mathrm{dom}} G_0 \\ \big\downarrow{\scriptstyle \phi_F} & & \big\downarrow{\scriptstyle \phi_G} \\ C_1 \times_{\mathrm{cod}} F_0 & \xrightarrow{\mathrm{Id} \times_{\mathrm{cod}} \eta_0} & C_1 \times_{\mathrm{cod}} F_0 \end{array}$$

Conversely, if we have an arrow η_0 which satisfies this naturality property, then it is an arrow between copresheaves.

Finally, we saw in section 6.4 that for any $\mathcal{S}$-indexed categories $\mathbb{C}$, $\mathbb{D}$, we can form the functor $\Delta : \mathcal{D} \to [\mathbb{C}, \mathbb{D}]$ which gives us the "constant functors" $\Delta X : \mathbb{C} \to \mathbb{D}$ for $X \in \mathcal{D}$. Note that in the case $\mathbb{D} = \mathbb{S}$, the functor $(\Delta X)^I : \mathcal{C}^I \to \mathcal{S}/I$ is just the constant functor taking the value $I^*X = (\pi_1 : I \times X \to I) \in \mathcal{S}/I$. When studying copresheaves, these constant functors can be described internally.

Remark 7.2.4. Let X be an object of $\mathcal{S}$. The diagram of the constant copresheaf $\Delta X : C \to \mathbb{S}$ (see section 6.4) is given as follows. First, the arrow $\beta : F_0 \to C_0$ is simply

$$\pi_1 : C_0 \times X \to C_0.$$

With this choice, we find that the pullbacks $C_1 \times_{\text{dom}} F_0$ and $C_1 \times_{\text{cod}} F_0$ become

$$\begin{array}{ccc} C_1 \times X & \xrightarrow{\text{dom}\times\text{Id}} & C_0 \times X \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_1} \\ C_1 & \xrightarrow{\text{dom}} & C_0 \end{array} \qquad \begin{array}{ccc} C_1 \times X & \xrightarrow{\text{cod}\times\text{Id}} & C_0 \times X \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_1} \\ C_1 & \xrightarrow{\text{cod}} & C_0 \end{array} \qquad \begin{array}{ccc} C_1 \times X & \xrightarrow{\phi} & C_1 \times X \\ & {\scriptstyle \pi_1}\searrow \quad \swarrow{\scriptstyle \pi_1} & \\ & C_1 & \end{array}$$

and so we just have $\phi = \text{Id}$. This corresponds to the constant copresheaf mapping arrows to the identity. Indeed, let $f : c \to d$, and let $(c, x) \in (\Delta X)(c)$ (i.e., we have $(c', x) : C_0 \times X$ with $\pi_1(c', x) = c$, i.e. $c' = c$). If we unravel the notations, we find that $(\Delta X f)(c, x) = (d, x)$.

There is one more thing we will write here for simplicity. Let $F \in [C, \mathbb{S}]$ and let $X \in \mathcal{S}$. Suppose we have a natural transformation $\eta : F \Rightarrow \Delta X$. Then the naturality condition on η can be expressed much more simply. The underlying arrow of η is η_0, as below.

$$\begin{array}{ccc} F_0 & \xrightarrow{\eta_0} & C_0 \times X \\ & {\scriptstyle \beta}\searrow \quad \swarrow{\scriptstyle \pi_1} & \\ & C_0 & \end{array}$$

This diagram implies that $\eta_0 = \langle \pi_1, \eta_0' \rangle$ for $\eta_0' : F_0 \to X$. Then, if we have $f : c \to d$ and $x \in F(c)$, then the naturality condition says that $\eta_0(Ff(x)) = (\Delta X f)(\eta_0(x))$. Using the definition of η_0, we get

$$(d, \eta_0'(Ff(x))) = (\Delta X f)(c, \eta_0'(x)) = (d, \eta_0'(x)).$$

So, the natuality here tells us that $\eta_0'(Ff(x)) = \eta_0'(x)$.

7.3 Finite limits in internal categories

We choose to define finite limits and colimits in internal categories via their interpretation as indexed categories.

Definition. Let $\mathcal{S}$ be a category with finite limits, and let C be an internal category in $\mathcal{S}$. We say that C *has finite (co)limits (of a particular shape)* if the corresponding $\mathcal{S}$-indexed category C has them.

Specifically, this means that each $C(I)$ has finite (co)limits (of a particular shape), and that they are preserved by the transition functors $x^* : C(J) \to C(I)$. However, since we can interpret $C(I)$ as a category of terms in a context (see remark 7.1.3), this gives us a more familiar characterization. (Recall that, in the internal language, "unique" terms means up to equality. See remark 2.2.3.)

Proposition 7.3.1. Let $\mathcal{S}$ be a category with finite limits, and let C be an internal category in $\mathcal{S}$.

- If C has a terminal object, then for any context T, there is a term $t : C_0$ in the context T such that for any $t' : C_0$, there is a unique arrow $t' \to t$.
- If C has products, then for any context T and any terms $c, c' : C_0$ in T, there is a term $c \times c' : C_0$ in T and arrows $\pi : c \times c' \to c$, $\pi' : c \times c' \to c'$ which satisfy the universal property of products.
- If C has equalizers, then for any context T and for any terms $c, d : C_0$ and $f, g : c \to d$ in T, there is a term $b : C_0$ and an arrow $i : b \to c$ which satisfy the universal property of equalizers.

The converse of each of these statements holds if we additionally assume that these structures are preserved (up to isomorphism) by substitution.

Proof. If C has finite limits of a particular shape, then so does $C([T])$ for any context T. Since we can interpret $C([T])$ as a category of terms in the context T (by remark 7.1.3), each of the above statements is just saying that the corresponding finite limit exists in $C([T])$.

For the converse statements, we know that for C to have finite limits of a given shape, we additionally need the transition functors to preserve them. But the transition functor $C(J) \to C(I)$ corresponding to $x : I \to J$ is just pre-composition with x, and this corresponds to substitution in the internal language. ∎

We could give similar characterizations of finite colimits, but we will not need them. Next, we use the internal language to characterize functors which preserve these finite limits.

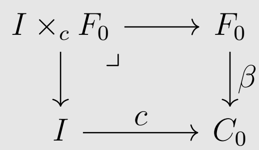
Proposition 7.3.2. Let $\mathcal{S}$ be a category with finite limits, let C be an internal category in $\mathcal{S}$, and let $F : C \to \mathbb{S}$. Then:

- Suppose C has a terminal object (i.e. each $C(I)$ has a terminal object t_I, and these are preserved up to isomorphism by the transition maps). Then F preserves terminal objects if and only if for any context I, there is a unique term $* \in F(t_I)$.
- Suppose C has binary products. Then F preserves binary products if and only if for any context I, for any terms $c, c' : C_0$, $x \in F(c)$, and $x' \in F(c')$, there exists a unique term $(x, x') \in F(c \times c')$ such that $F\pi_1(x, x') =_I x$ and $F\pi_2(x, x') =_I x'$.

- Suppose C has equalizers. Then F preserves equalizers if and only if for any context I, for any $g, h : c \to d$ with equalizer $f : b \to c$, and any term $x \in F(c)$ such that $Fg(x) =_I Fh(x)$, there exists a unique term $w \in F(b)$ such that $Ff(w) =_I x$.

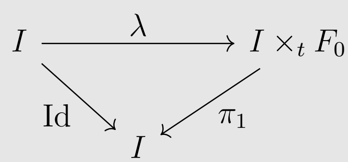
Proof. We do the case of terminal objects; the other cases are laborious but similar. Recall that F preserves terminal objects if for each I, the functor $F^I : C(I) \to \mathcal{S}/I$ preserves terminal objects. Write $(F_\bullet, \phi)$ for the diagram corresponding to F (see remark 7.2.2).

Let I be a context. The functor $F^I : C(I) \to \mathcal{S}/I$ maps a term $c : C_0$ to $c^* F_\bullet$; see the diagram below.

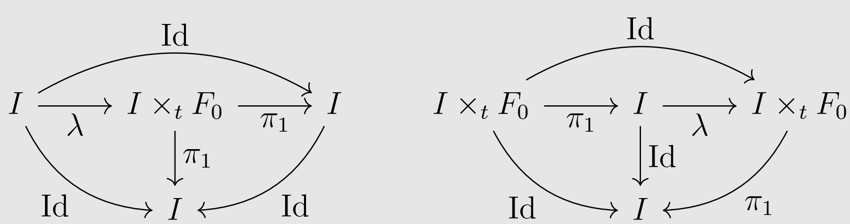


Thus, if t is the terminal object of $C(I)$, it is mapped by F^I to $\pi_1 : I \times_t F_0 \to I$. But $\mathcal{S}/I$ already has a terminal object, $\mathrm{Id}_I : I \to I$, and so there is a canonical map $F^I(t) \to \mathrm{Id}_I$. By definition, F^I preserves terminal objects if and only if this canonical map is an isomorphism.

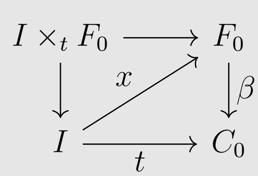
So, suppose $F^I(t)$ preserves terminal objects. Then there is a map



which is an inverse of the canonical map $F^I(t) \to \mathrm{Id}_I$ - i.e., if the following diagrams commute.



The first diagram tells us that we can write $\lambda = \langle \mathrm{Id}, x \rangle$ – which, in particular, tells us that $\beta \circ x = t$ – and the second tells us that $x \circ \pi_1 = \pi_2$. Thus, all together, we know that there exists a map $x : I \to F_0$ that makes the following diagram commute.



So, in the context I, there exists a term $x \in F(t)$, and for any other term $y \in F(t)$, we have $x =_I y$.

Conversely, if there exists a unique term x in the context $F(t)$, then we can run this argument in reverse and find that $\lambda = \langle \mathrm{Id}, x \rangle$ is an inverse of the canonical map $F^I(t) \to \mathrm{Id}_I$. ∎

Finally, it turns out that an internal category having finite limits – in the way we defined above – is stronger than we might expect. For instance, we'll consider binary products: even though our definition only required each $C(I)$ to have binary products separately (and for the transition functors to preserve them), it turns out that the binary products are "global". That is, they can all be determined by an arrow $C_0 \times C_0 \to C_0$, a sort of "product map on objects". We make this precise with the following result.

Proposition 7.3.3. Let $\mathcal{S}$ be a category with finite limits, and let C be an internal category in $\mathcal{S}$. If C has binary products, then these binary products are *global* in the following sense. There exist arrows

$$\begin{aligned} \mathrm{prodOb} &: C_0 \times C_0 \to C_0, & \mathrm{prodAr} &: C_1 \times C_1 \to C_1, \\ \mathrm{proj}_1, \mathrm{proj}_2 &: C_0 \times C_0 \to C_1, & \mathrm{pair} &: C_1 \times_{\mathrm{dom}} C_1 \to C_1 \end{aligned}$$

with the following properties. (For legibility, we will use the $\times$ symbol to replace prodOb and prodAr when used in the internal language.)

First, the domains and codomains of these arrows are connected by the following facts. These could be expressed diagrammatically (or equationally), but we opt for a more legible presentation.

- If $f : c_1 \to c_2$ and $g : d_1 \to d_2$, then $f \times g : c_1 \times c_2 \to d_1 \times d_2$.
- We have $\mathrm{proj}_1(c, d) : c \times d \to c$ and $\mathrm{proj}_2(c, d) : c \times d \to d$.
- If $f : c \to d$ and $g : c \to e$, then $\mathrm{pair}(f, g) : c \to d \times e$.

Second, for each $I \in \mathcal{S}$, the functor $\Pi^I : C(I) \times C(I) \to C(I)$ given by

$$\begin{aligned} \big(p_1, p_2 : I \to C_0\big) &\mapsto \big(\mathrm{prodOb} \circ \langle p_1, p_2 \rangle : I \to C_0\big), \\ \big(q_1, q_2 : I \to C_1\big) &\mapsto \big(\mathrm{prodAr} \circ \langle q_1, q_2 \rangle : I \to C_1\big). \end{aligned}$$

is right adjoint to the diagonal Δ^I. (Hence, it defines the product.) Third, the natural transformation $\epsilon^I : \Delta^I \Pi^I \Rightarrow \mathrm{Id}$ given by

$$\epsilon^I_{(p_1,p_2)} = \big(\mathrm{proj}_1 \circ \langle p_1, p_2 \rangle,\ \mathrm{proj}_2 \circ \langle p_1, p_2 \rangle\big)$$

is the counit of the adjunction $\Delta^I \dashv \Pi^I$. Finally, the map $\mathrm{Hom}(\Delta^I q, (p_1, p_2)) \to \mathrm{Hom}(q, \Pi^I(p_1, p_2))$ given by

$$(f, g) \mapsto \mathrm{pair} \circ \langle f, g \rangle$$

is the inverse of $h \mapsto \epsilon^I_{(p_1,p_2)} \circ \Delta^I(h)$, and this is the isomorphism giving the adjunction $\Delta^I \dashv \Pi^I$.

Proof. For each $I \in \mathcal{S}$, let $\tilde{\Pi}^I : C(I) \times C(I) \to C(I)$ denote the binary product functor on $C(I)$, which exists because C is assumed to have binary products. (We will omit the superscript when it can be inferred from context.) In the category $C(C_0 \times C_0)$, remark that $\pi_1, \pi_2 : C_0 \times C_0 \to C_0$ are objects. Therefore, $\tilde{\Pi}(\pi_1, \pi_2)$ is an object of $C(C_0 \times C_0)$, which is to say an arrow $C_0 \times C_0 \to C_0$ in $\mathcal{S}$. Therefore, we set

$$\mathrm{prodOb} = \tilde{\Pi}^{C_0 \times C_0}(\pi_1, \pi_2).$$

Next, in the category $C(C_1 \times C_1)$, remark that $\pi_1, \pi_2 : C_1 \times C_1 \to C_1$ are arrows: the former is an arrow $\mathrm{dom}\pi_1 \to \mathrm{cod}\pi_1$, and the latter is an arrow $\mathrm{dom}\pi_2 \to \mathrm{cod}\pi_2$. Therefore, we can consider

$$\tilde{\Pi}(\pi_1, \pi_2) : \tilde{\Pi}(\mathrm{dom}\pi_1, \mathrm{dom}\pi_2) \to \tilde{\Pi}(\mathrm{cod}\pi_1, \mathrm{cod}\pi_2),$$

an arrow in $C(C_1 \times C_1)$.

Moreover, C having binary products means that these products are preserved by the transition maps. Thus, the following diagram commutes up to a canonical isomorphism.

$$\begin{array}{ccc} C(C_0 \times C_0) \times C(C_0 \times C_0) & \xrightarrow{(\mathrm{dom}\times\mathrm{dom})^* \times (\mathrm{dom}\times\mathrm{dom})^*} & C(C_1 \times C_1) \times C(C_1 \times C_1) \\ \tilde{\Pi}\downarrow & & \downarrow\tilde{\Pi} \\ C(C_0 \times C_0) & \xrightarrow{(\mathrm{dom} \times \mathrm{dom})^*} & C(C_1 \times C_1) \end{array}$$

In particular, if we start with (π_1, π_2) in the top left, then following the bottom path yields $\mathrm{prodOb} \circ (\mathrm{dom} \times \mathrm{dom})$, while following the top path yields $\tilde{\Pi}(\mathrm{dom}\pi_1, \mathrm{dom}\pi_2)$ (because $\pi_1 \circ (\mathrm{dom} \times \mathrm{dom}) = \mathrm{dom} \circ \pi_1$). Therefore, there is a canonical isomorphism

$$\mathrm{prodOb} \circ (\mathrm{dom} \times \mathrm{dom}) \cong \tilde{\Pi}(\mathrm{dom}\pi_1, \mathrm{dom}\pi_2)$$

as objects of $C(C_1 \times C_1)$. Therefore, we can define prodAr as the arrow in $C(C_1 \times C_1)$ given by the following composition.

$$\begin{array}{ccc} \mathrm{prodOb} \circ (\mathrm{dom} \times \mathrm{dom}) & \xrightarrow{\mathrm{prodAr}} & \mathrm{prodOb} \circ (\mathrm{cod} \times \mathrm{cod}) \\ \cong & & \cong \\ \tilde{\Pi}(\mathrm{dom}\pi_1, \mathrm{dom}\pi_2) & \xrightarrow{\tilde{\Pi}(\pi_1, \pi_2)} & \tilde{\Pi}(\mathrm{cod}\pi_1, \mathrm{cod}\pi_2) \end{array}$$

This means that prodAr is an arrow $C_1 \times C_1 \to C_1$ in $\mathcal{S}$, and because of how we chose its domain and codomain (as an arrow in $C(C_1 \times C_1)$), they are related to those of prodOb in the desired way.

So, we have now defined prodOb and prodAr and shown their (co)domains are related in the desired way. Therefore, we can define the functors $\Pi^I : C(I) \times C(I) \to C(I)$ in the statement of the proposition. We claim that Π^I is a right adjoint of Δ^I; to show this, we will just show that $\Pi^I \cong \tilde{\Pi}^I$. First, let $I \in \mathcal{S}$, and let $p_1, p_2 : I \to C_0$ be objects of $C(I)$. Then we get an arrow $\langle p_1, p_2 \rangle : I \to C_0 \times C_0$, and the fact that $\tilde{\Pi}$ commutes with transition maps means we get the following diagram which commutes up to isomorphism.

$$\begin{array}{ccc} C(C_0 \times C_0) \times C(C_0 \times C_0) & \xrightarrow{\langle p_1, p_2 \rangle^* \times \langle p_1, p_2 \rangle^*} & C(I) \times C(I) \\ \tilde{\Pi} \big\downarrow & & \big\downarrow \tilde{\Pi} \\ C(C_0) & \xrightarrow{\qquad \langle p_1, p_2 \rangle^* \qquad} & C(I) \end{array}$$

If we start with (π_1, π_2) in the top left, then following the bottom path yields $\text{prodOb} \circ \langle p_1, p_2 \rangle = \Pi^I(p_1, p_2)$, while following the top path yields $\tilde{\Pi}(p_1, p_2)$. Thus, the two are isomorphic.

A similar computation shows that, for arrows $q_1, q_2 : I \to C_1$ in $C(I)$, the arrow $\tilde{\Pi}(q_1, q_2)$ is the same as $\Pi^I(q_1, q_2) = \text{prodAr} \circ \langle q_1, q_2 \rangle$, when we account for the modifications in the domain and codomain. It's also not hard to show that Π commutes with the transition maps since $\tilde{\Pi}$ does.

Now, we examine the projection maps. Since Π^I is a right adjoint of Δ^I, we know the adjunction has a counit $\tilde{\epsilon}^I : \Delta^I \Pi^I \Rightarrow \text{Id}$. If we take $I = C_0 \times C_0$ and apply this to (π_1, π_2), we get

$$\tilde{\epsilon}^{C_0 \times C_0}_{(\pi_1, \pi_2)} : (\text{prodOb}, \text{prodOb}) \to (\pi_1, \pi_2).$$

That is, we get two arrows $\text{prodOb} \to \pi_1$ and $\text{prodOb} \to \pi_2$ in $C(C_0 \times C_0)$; these are the desired arrows proj_1 and proj_2. Using these, we can define ϵ^I for each I as in the proposition, and easily check that this yields a natural transformation (the fact that it is the counit will follow from showing the isomorphism with the pair map).

Next, we define the pairing arrow. Remark that, in $C(C_1 \times_{\text{dom}} C_1)$, the map $\pi_1 : C_1 \times_{\text{dom}} C_1 \to C_1$ is an arrow from $\text{dom}\pi_1 \to \text{cod}\pi_1$, and similarly π_2 is an arrow from $\text{dom}\pi_2 \to \text{cod}\pi_2$. But by definition of the pullback, we have $\text{dom}\pi_1 = \text{dom}\pi_2$ (which we'll write as $\text{dom}\pi_\bullet$), and so we have

$$(\pi_1, \pi_2) : \Delta(\text{dom}\pi_\bullet) \to (\text{cod}\pi_1, \text{cod}\pi_2),$$

an arrow in $C(C_1 \times_{\text{dom}} C_1) \times C(C_1 \times_{\text{dom}} C_1)$. But now, because $\Delta \dashv \Pi$, there exists a unique arrow

$$\phi : \text{dom}\pi_\bullet \to \text{prodOb}(\text{cod}\pi_1, \text{cod}\pi_2)$$

such that

$$(\pi_1, \pi_2) = (\text{proj}_1, \text{proj}_2) \circ \Delta\phi.$$

This arrow $\phi : C_1 \times_{\text{dom}} C_1 \to C_1$ is the desired map pair. It's now easy to check that pair and the projections form the isomorphisms as described in the proposition. ∎

7.4 Filtered categories and the category of elements

In this section, we discuss filtered internal categories, which will play an important role later on.

It is tempting to say that an internal category C is filtered if each $C(I)$ is filtered. However, the definition given in [John 02] (definition B2.6.2) is weaker than this criterion. Therefore, we leave the definition of a filtered internal category to [John 02], and introduce some new terminology.

Definition. Let $\mathcal{S}$ be a category with finite limits, and let C be an internal category in $\mathcal{S}$. We say that C is *strongly filtered* if $C(I)$ is filtered for each I. We say C is *strongly cofiltered* if C^{op} is strongly filtered, i.e. if $C(I)$ is cofiltered for each I.

Remark 7.4.1. Filtered internal categories in [John 02] are only discussed when the ambient category $\mathcal{S}$ is a regular category. Therefore, suppose $\mathcal{S}$ is a regular category and C is an internal category in $\mathcal{S}$. If C is strongly filtered, then C is filtered; this is easy to check.

Now, we discuss the category of elements of a copresheaf. Intuitively, this is the same as for a regular copresheaf F: the objects of Elt$F$ are pairs (A, a) with $A \in C$ and $a \in F(A)$, while an arrow $(A, a) \to (B, b)$ is an arrow $f : A \to B$ in C such that $F(f)(a) = b$.

Definition. Let $\mathcal{S}$ be a category with finite limits, let C be an internal category in $\mathcal{S}$, and let $F : C \to \mathbb{S}$ be a copresheaf on C. The *category of elements* of F is an internal category Elt$F$ in $\mathcal{S}$, equipped with an internal functor $\pi : \text{Elt}F \to C$, defined as follows.

The object of objects of Elt$F$ is $(\text{Elt}F)_0 = F_0$, and the object of arrows is $(\text{Elt}F)_1 = C_1 \times_{\text{dom}} F_0$. The domain and codomain arrows are

$$\text{dom} = \pi_2 : C_1 \times_{\text{dom}} F_0 \to F_0, \qquad \text{cod} = F_1 : C_1 \times_{\text{dom}} F_0 \to F_0.$$

The identity arrow is $\langle \iota \circ \beta, \text{Id} \rangle : F_0 \to C_1 \times_{\text{dom}} F_0$, and composition is given as follows. Note that the object of composable pairs is given by

$$\{f, g : C_1, x, y : F_0 \mid f : c \to d, g : d \to e, x \in F(c), y \in F(d), Ff(x) = y\},$$

and the composition just maps this to $(g \circ f, x)$.

Finally, the internal functor $\pi : \mathrm{Elt}F \to C$ acts as follows. The arrow on objects is $\beta : F_0 \to C_0$, and the arrow on arrows is $\pi_1 : C_1 \times_{\mathrm{dom}} F_0 \to C_1$.

Remark 7.4.2. If instead of a functor $F : C \to \mathbb{S}$, we have a functor $E : C^{op} \to \mathbb{S}$, then we can consider E to be a copresheaf on C^{op}. The above definition tells us we have an internal category $\mathrm{Elt}E$ and an internal functor $\mathrm{Elt}E \to C^{op}$. However, we are often interested in the opposite of this category and functor, i.e. $(\mathrm{Elt}E)^{op} \to (C^{op})^{op} = C$. We will denote this as $\pi : \mathrm{Elt}^{op}E \to C$ to emphasize that the focus is on the category C, not C^{op}.

Remark 7.4.3. If $x, y : F_0$ are objects of $\mathrm{Elt}F$ such that $x \in F(c)$ and $y \in F(d)$, then an arrow $x \to y$ consists of a pair (f, x), where $f : c \to d$ is an arrow in C such that $Ff(x) = y$.

In [John 02], the category of elements of $F : C \to \mathbb{S}$ is constructed as part of an equivalence of categories $[C, \mathbb{S}] \simeq \mathbf{doFib}(\mathcal{S})/C$; the mapping $F \mapsto (\pi : \mathrm{Elt}F \to C)$ is one direction of this equivalence. Here, $\mathbf{doFib}(\mathcal{S})$ refers to the category of internal categories in $\mathcal{S}$ and *discrete opfibrations* (see definition B2.5.1 of [John 02]) between them. We will briefly revisit this notion in section 11, but we will not spend much time on it.

Finally, we show that the category of elements is strongly cofiltered. This is the internal analogue of a standard fact, which can be found (e.g.) as proposition 6.1.2 in [Borc 94a]. There is also a similar result in [John 02] (lemma B2.6.7).

Proposition 7.4.4. Let $\mathcal{S}$ be a category with finite limits, and let C be an internal category with finite limits. If $F : C \to \mathbb{S}$ preserves finite limits, then $\mathrm{Elt}F$ is strongly cofiltered. (And therefore, $(\mathrm{Elt}F)^{op}$ is strongly filtered.)

Proof. Let $I \in \mathcal{S}$; we claim $(\mathrm{Elt}F)(I)$ is cofiltered (as an ordinary category). Per definition 2.13.1 of [Borc 94a], it suffices to show three conditions.

First, $(\mathrm{Elt}F)(I)$ is not empty. Since F is lex, proposition 7.3.2 tells us that (in the context I) there is a term $x \in F(t)$, where $t : C_0$ is the terminal object of $C(I)$. Thus $x : F_0$, so x is an object of $(\mathrm{Elt}F)(I)$.

Second, let $x, x' : I \to F_0$ be two objects of $(\mathrm{Elt}F)(I)$. Write $c = \beta(x)$ and $c' = \beta(x')$ (i.e., $x \in F(c)$ and $x' \in F(c')$). Then $c, c' : I \to C_0$ are two objects of $C(I)$, so since C has finite limits, they have a product $c \times c' \in C(I)$ with projection maps $\pi : c \times c' \to c$, $\pi' : c \times c' \to c'$. By proposition 7.3.2, there is a term $(x, x') \in F(c \times c')$ such that $F\pi(x, x') = x$ and $F\pi'(x, x') = x'$.

So, we have $(x, x') : (\mathrm{Elt}F)_0$, and we have $(\pi, (x, x')), (\pi', (x, x')) \in (\mathrm{Elt}F)_1$ (indeed, $\mathrm{dom}(\pi) = \mathrm{dom}(\pi') = c \times c' = \beta(x, x')$). We claim that $(\pi, (x, x'))$ is an arrow $(x, x') \to x$ and that $(\pi', (x, x'))$ is an arrow $(x, x') \to x'$. The domains are clear; for the codomains, we recall that $Ff(y)$ is notation for $F_1(f, y)$, so $F_1(\pi, (x, x')) = F\pi(x, x') = x$ and $F_1(\pi', (x, x')) = F\pi'(x, x') = x'$, as desired.

Third, let y, z be two objects of $(\mathrm{Elt}F)(I)$; say that $y \in F(d)$ and $z \in F(e)$. Consider two arrows $y \to z$ in Elt$F$; they are given by pairs $(g, y), (h, y)$ such that $g, h : d \to e$ are arrows in C and $Fg(y) = z = Fh(y)$. Since C has equalizers, we get an arrow $f : c \to d$ such that $h \circ f = g \circ f$. Since F preserves equalizers, proposition 7.3.2 tells us there is a term $x \in F(c)$ such that $Ff(x) = y$.

Thus, x is an object of $(\text{Elt}F)(I)$, and $(f, x) : x \to y$ is an arrow of $(\text{Elt}F)(I)$. Moreover, $(g, y) \circ (f, x) = (g \circ f, x) = (h \circ f, x) = (h, y) \circ (f, x)$, which is what we wanted. ∎

Chapter 8

Internal category generated by an exponentiable arrow

Contents

In this chapter, we develop an important example of an internal category: the internal category generated by an exponentiable arrow. The reason for including this chapter is that we will later (in chapter 14) use this construction to build the internal category of finite sets **Fin**, and we will need the results proved here.

In definition B2.3.5 in [John 02], Johnstone describes the notion of an internal full subcategory. If $\mathbb{C}$ is a locally small $\mathcal{S}$-indexed category, then for any object X of $\mathcal{C}^I$ (for some I), we can form the internal full subcategory of $\mathbb{C}$ generated by X. In the case of $\mathbb{C} = \mathbb{S}$ (i.e., the canonical indexing of $\mathcal{S}$ over itself), we know that $\mathbb{S}$ is locally small if and only if $\mathcal{S}$ is locally cartesian closed (see lemma B2.2.3 in [John 02]), and X will be some arrow of $\mathcal{S}$.

However, given an arrow $f : B \to A$ in a category $\mathcal{S}$ which is *not* cartesian closed, the construction of the internal full subcategory (which we'll denote $\mathbb{S}[f]$) still works, as long as we assume that f is an exponentiable arrow. Assuming that $\mathcal{S}$ is locally cartesian closed – which is to say that all arrows are exponentiable – is not necessary. In section 8.1, we'll describe the construction of this internal category.

Next, in section 8.2, we define two families of indexed functors on the internal category $\mathbb{S}[f]^{op}$, which we'll denote $\mathbb{P}_f X$ and $\mathbb{E}_f X$. Broadly, these correspond to taking f-indexed products and taking exponentials by f. As one might expect, these turn out to be the same (theorem 8.2.3), but the different descriptions allow us to prove different things about them.

In sections 8.3 and 8.4, we show that the functors $\mathbb{P}_f X$ and $\mathbb{E}_f X$ preserve certain limits, subject to some assumptions on $\mathcal{S}$ and f. Finally, in section 8.5, we prove an easy result about these functors having a left inverse.

8.1 Definition

In this section, we construct the internal full subcategory generated by an exponentiable arrow $f : B \to A$ in a category $\mathcal{S}$ with finite limits. We will denote this internal category by $\mathbb{S}[f]$ in order to be consistent with Johnstone's notation.

Before getting into it, it'll be important to have some intuition. The idea is that the "objects" of this internal category are $f^{-1}(a)$ for $a \in A$, which we'll denote $[a]$. Then, since f is exponentiable, it'll make sense to form $[b]^{[a]}$ for $a, b \in A$, and elements of this exponential will be the arrows from $[a]$ to $[b]$.

More precisely, the object of objects of $\mathbb{S}[f]$ will be A, and the object of arrows will be based on $([b]^{[a]} \mid a, b \in A)$, which is the exponential of the two objects $([a] \mid a, b \in A)$ and $([b] \mid a, b \in A)$ of $\mathcal{S}/(A \times A)$, which are shorthand for $\pi_1^*(f) = f \times \mathrm{Id}_A$ and $\pi_2^*(f) = \mathrm{Id}_A \times f$, respectively.

Remark 8.1.1. Let $\mathcal{S}$ be a category with finite limits, and let $f : B \to A$ be an exponentiable arrow. Then for any arrow $p : X \to A$, the arrow $p^* f$ is also exponentiable. (See corollary 1.4 of [Nief 82].)

In particular, if π_1, π_2 are the projections $A \times A \to A$, then the exponential $(\pi_2^* f)^{(\pi_1^* f)}$ exists (in the category $\mathcal{S}/A \times A$).

In order to construct $\mathbb{S}[f]$, we will need a convenient description of the arrows into $(\pi_2^* f)^{(\pi_1^* f)}$. This is done with the following proposition.

Proposition 8.1.2. Let $\mathcal{S}$ be a category with finite limits, and let $f : B \to A$ be an exponentiable arrow in $\mathcal{S}$. Let π_1, π_2 denote the projections $A \times A \to A$.

For any $I \in \mathcal{S}$ and arrows $m, n : I \to A$, there is a bijection

$$\Phi^f_{m,n} : \mathrm{Hom}_{\mathcal{S}/A \times A}\left(\langle m, n\rangle, (\pi_2^* f)^{(\pi_1^* f)}\right) \to \mathrm{Hom}_{\mathcal{S}/I}\left(m^* f, n^* f\right).$$

Proof. First, we use the universal property of exponentials to get the following bijection.

$$\mathrm{Hom}_{\mathcal{S}/A \times A}\left(\langle m, n\rangle, (\pi_2^* f)^{(\pi_1^* f)}\right) \cong \mathrm{Hom}_{\mathcal{S}/A \times A}\Big(\langle m, n\rangle \times (\pi_1^* f), (\pi_2^* f)\Big)$$

Then, we note that products in a slice category are the same as taking a pullback, then applying the composition functor. So, the above is equal to:

$$= \mathrm{Hom}_{\mathcal{S}/A\times A}\Big(\Sigma_{\langle m,n\rangle}\langle m,n\rangle^*(\pi_1^* f), (\pi_2^* f)\Big)$$

But, we know that $\Sigma_{\langle m,n\rangle}$ is left adjoint to $\langle m,n\rangle^*$; so, we can use the adjunction to say that the above is isomorphic to:

$$\cong \mathrm{Hom}_{\mathcal{S}/I}\Big(\langle m,n\rangle^*(\pi_1^* f), \langle m,n\rangle^*(\pi_2^* f)\Big)$$

Finally, we know that $x^*y^* \cong (yx)^*$, so we can combine the projections with $\langle m,n\rangle$ and get the following.

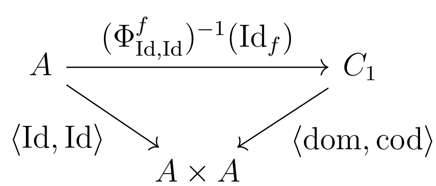
$$\cong \mathrm{Hom}_{\mathcal{S}/I}(m^* f, n^* f)$$

This is what we wanted to show. ∎

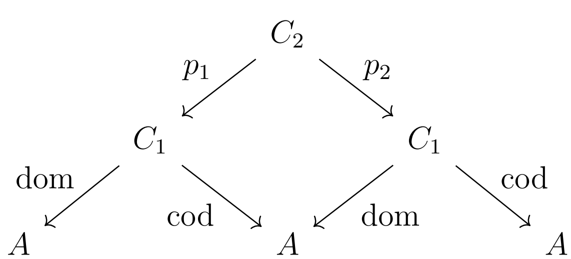
We are now ready to define the internal category $\mathbb{S}[f]$.

Definition. Let $\mathcal{S}$ be a category with finite limits, and let $f : B \to A$ be an exponentiable arrow in $\mathcal{S}$. The internal category $\mathbb{S}[f]$ in $\mathcal{S}$ is defined as follows.

- The object of objects is A.
- Write the exponential $(\pi_2^* f)^{(\pi_1^* f)}$ (in $\mathcal{S}/(A\times A)$) as $\langle \mathrm{dom}, \mathrm{cod}\rangle : C_1 \to A \times A$. Then C_1 is the object of arrows, and $\mathrm{dom}, \mathrm{cod}$ are the domain and codomain arrows.
- The arrow representing identities in this internal category is $(\Phi^f_{\mathrm{Id},\mathrm{Id}})^{-1}(\mathrm{Id}_f)$, where Φ is as in proposition 8.1.2; see the diagram below.



- The arrow determining composition is defined as follows. First, we form the object C_2 of composable pairs of arrows in the usual way.



Then, note that in $\mathcal{S}/A \times A$, we have arrows $p_1 : \langle \mathrm{dom}p_1, \mathrm{cod}p_1 \rangle \to (\pi_2^* f)^{(\pi_1^* f)}$ and $p_2 : \langle \mathrm{dom}p_2, \mathrm{cod}p_2 \rangle \to (\pi_2^* f)^{(\pi_1^* f)}$, so we can apply Φ and get arrows as follows.

$$(\mathrm{dom}p_1)^* f \xrightarrow{\Phi^f_{dp_1,cp_1}(p_1)} (\mathrm{cod}p_1)^* f = (\mathrm{dom}p_2)^* f \xrightarrow{\Phi^f_{dp_2,cp_2}(p_2)} (\mathrm{cod}p_2)^* f$$

So, by applying $(\Phi^f_{dp_1,cp_2})^{-1}$ to this composite, we get an arrow as follows.

$$\begin{array}{ccc} C_2 & \xrightarrow{(\Phi^f_{dp_1,cp_2})^{-1}\left(\Phi^f_{dp_2,cp_2}(p_2) \circ \Phi^f_{dp_1,cp_1}(p_1)\right)} & C_1 \\ & {\scriptstyle \langle \mathrm{dom}p_1, \mathrm{cod}p_2 \rangle} \searrow \quad \swarrow {\scriptstyle \langle \mathrm{dom}, \mathrm{cod} \rangle} & \\ & A \times A & \end{array}$$

This is the desired composition arrow.

We omit the straightforward proof that this produces a valid internal category.

In [John 02], Johnstone notes (immediately after definition B2.3.5) that an internal full subcategory comes equipped with a canonical indexed functor to the original category. This indexed functor is full and faithful, which justifies the terminology. The next definition describes this indexed functor in our case, which we call the "inclusion functor".

Definition. For each $I \in \mathcal{S}$, we define a functor $\iota^I : \mathbb{S}[f](I) \to \mathcal{S}/I$ as follows.

- An object of $\mathbb{S}[f](I)$, which is an arrow $m : I \to A$, is mapped to the object $m^* f$ of $\mathcal{A}/I$.
- Let $p : m \to n$ be an arrow in $\mathbb{S}[f](I)$. That is, we have $m, n : I \to A$, and $p : I \to C_1$ is such that $\mathrm{dom}p = m$ and $\mathrm{cod}p = n$. In other words, p is an arrow $\langle m, n \rangle \to (\pi_2^* f)^{(\pi_1^* f)}$ in $\mathcal{S}/A \times A$; thus, we have $\Phi^f_{m,n}(p) : m^* f \to n^* f$ in $\mathcal{S}/I$ (where Φ is from proposition 8.1.2). This is the image of p under ι^I.

We omit the straightforward check that this is a valid functor.

Proposition 8.1.3. For each $I \in \mathcal{S}$, the functor ι^I is full and faithful. Moreover, $(\iota^I)_I$ forms an indexed functor, which we denote $\iota : \mathbb{S}[f] \hookrightarrow \mathbb{S}$.

Proof. The fact that ι^I is full and faithful follows immediately from $\Phi^f_{m,n}$ being a bijection. We omit the straightforward check that this defines a valid indexed functor. ∎

The takeaway of this result is the following. An object of $\mathbb{S}[f](I)$ is an arrow $n : I \to A$ of $\mathcal{S}$, but we associate it with the object n^*f of $\mathcal{S}/I$, which we also write as $\pi_1^n : I \times_n B \to I$ (see the pullback square below).

$$\begin{array}{ccc} I \times_n B & \xrightarrow{\pi_2^n} & B \\ {\scriptstyle \pi_1^n}\downarrow & & \downarrow{\scriptstyle f} \\ I & \xrightarrow{n} & A \end{array}$$

Then, if $m, n : I \to A$ are objects of $\mathbb{S}[f](I)$, then an arrow $p : m \to n$ is an arrow $m^*f \to n^*f$ of $\mathcal{S}$, which we also denote p. That is, it's an arrow $p : I \times_m B \to I \times_n B$ which commutes with the first projection (see the diagram below).

$$\begin{array}{ccc} I \times_m B & \xrightarrow{\quad p \quad} & I \times_n B \\ & {\scriptstyle \pi_1^m}\searrow \quad \swarrow{\scriptstyle \pi_1^n} & \\ & I & \end{array}$$

8.2 Functors

In this section, we define two indexed functors out of $\mathbb{S}[f]$, and show they are equivalent. The reason for this is that the two different definitions are useful for different purposes.

8.2.1 Products

Let $\mathcal{S}$ be a category with finite limits, and let $f : B \to A$ be an exponentiable arrow in $\mathcal{S}$. If $\mathbb{C}$ is an $\mathcal{S}$-indexed category with f-indexed products (see section 6.2), then for any $X \in \mathcal{C}$, we can define an indexed functor

$$\mathbb{P}_f X : \mathbb{S}[f]^{op} \to \mathbb{C}.$$

For each $I \in \mathcal{S}$, we define the functor $(\mathbb{P}_f X)^I : \mathbb{S}[f]^{op}(I) \to \mathcal{C}^I$ as follows.

Objects

Let n be an object of $\mathbb{S}[f](I)$, which is to say it's an arrow $n : I \to A$, and, when considering $\mathbb{S}[f] \subseteq \mathbb{S}$, it corresponds to n^*f, which we will write as $\pi_1^n : I \times_n B \to I$ (see the diagram below).

$$\begin{array}{ccc} I \times_n B & \xrightarrow{\pi_2^n} & B \\ {\scriptstyle \pi_1^n}\downarrow & & \downarrow{\scriptstyle f} \\ I & \xrightarrow{n} & A \end{array}$$

Then, consider the following functors.

$$\mathcal{C} \xrightarrow{\langle\rangle_n^*} \mathcal{C}^{I \times_n B} \xrightarrow{\Pi_n} \mathcal{C}^I$$

Here, $\langle\rangle_n : I \times_n B \to \mathbb{1}$, and Π_n is right adjoint to $(\pi_1^n)^*$, which exists because $\mathbb{C}$ has f-indexed products and π_1^n is a pullback of f. Then, we just set

$$(\mathbb{P}_f X)^I(n) = \prod_n \langle\rangle_n^* X.$$

Arrows

Let $p : m \to n$ be an arrow of $\mathbb{S}[f](I)$; it corresponds to an arrow $p : m^* f \to n^* f$, i.e. an arrow $p : I \times_m B \to I \times_n B$ which commutes with the first projections. Then, we define $(\mathbb{P}_f X)^I(p) : (\mathbb{P}_f X)^I(n) \to (\mathbb{P}_f X)^I(m)$ to be the unique arrow which makes the following diagram commute.

$$\begin{array}{ccccc}
(\pi_1^m)^* \prod_n \langle\rangle_n^* X & & \xrightarrow{(\pi_1^m)^*(\mathbb{P}_f X)^I(p)} & & (\pi_1^m)^* \prod_m \langle\rangle_m^* X \\
\cong\downarrow & & & & \downarrow\varepsilon_m \\
p^*(\pi_1^n)^* \prod_n \langle\rangle_n^* X & \xrightarrow{p^*\varepsilon_n} & p^* \langle\rangle_n^* X & \xrightarrow{\cong} & \langle\rangle_m^* X
\end{array}$$

Here, ε_n and ε_m are the counits of the adjunctions defining Π_n and Π_m. Note also that we are using the equalities $\pi_1^n \circ p = \pi_1^m$ and $\langle\rangle_n \circ p = \langle\rangle_m$ to obtain the isomorphisms in the diagram. Finally, note that the existence and uniqueness of $(\mathbb{P}_f X)^I(p)$ is given by the universal property of Π_m as a right adjoint.

Proposition 8.2.1. The above defines a valid indexed functor $\mathbb{P}_f X : \mathbb{S}[f]^{op} \to \mathbb{C}$. Moreover, the mapping $X \mapsto \mathbb{P}_f X$ extends to a functor $\mathcal{C} \to [\mathbb{S}[f]^{op}, \mathbb{C}]$.

Proof. This is a straightforward check. ∎

8.2.2 Exponentials

Let $\mathcal{S}$, $\mathcal{T}$ be categories with finite limits, let $F : \mathcal{S} \to \mathcal{T}$ be a functor which preserves finite limits, and let $f : B \to A$ be an arrow in $\mathcal{S}$ such that both f and $F(f)$ are exponentiable. For any $X \in \mathcal{T}$, we can define an indexed functor

$$\mathbb{E}_f X : \mathbb{S}[f]^{op} \to F^* \mathbb{T}.$$

For each $I \in \mathcal{S}$, we define the functor $(\mathbb{E}_f X)^I : \mathbb{S}[f]^{op}(I) \to \mathcal{T}/F(I)$ as follows.

Objects

Let n be an object of $\mathbb{S}[f](I)$, which is to say it's an arrow $n : I \to A$, and, when considering $\mathbb{S}[f] \subseteq \mathbb{S}$, it corresponds to $n^* f$, which we write as $\pi_1^n : I \times_n B \to I$. Note that $F(\pi_1^n)$ is exponentiable as an object of $\mathcal{T}/F(I)$, since $F(\pi_1^n) = F(n^* f) \cong n^* F(f)$ and $F(f)$ is exponentiable (see corollary 1.4 of [Nief 82]).

Note that $\pi_1 : F(I) \times X \to F(I)$ is the same as $\langle\rangle_I^* X$. Then, we let

$$(\mathbb{E}_f X)^I(n) = (\langle\rangle_I^* X)^{F(\pi_1^n)}.$$

If we write n for π_1^n (which is hardly even abusing notation) and X for $\langle\rangle_I^* X$, we can say that

$$(\mathbb{E}_f X)^I(n) = X^{F(n)}.$$

Arrows

Let $p : m \to n$ be an arrow of $\mathbb{S}[f](I)$, and identify m, n with their corresponding objects of $\mathcal{S}/I$ (i.e., π_1^m and π_1^n). Then define $(\mathbb{E}_f X)^I(p)$ to be the unique arrow which makes the following diagram commute.

$$\begin{array}{ccc} F(m) \times X^{F(m)} & \xrightarrow{\mathrm{ev}_{F(m)}} & X \\ {\scriptstyle \mathrm{Id} \times (\mathbb{E}_f X)^I(p)}\uparrow & & \uparrow{\scriptstyle \mathrm{ev}_{F(n)}} \\ F(m) \times X^{F(n)} & \xrightarrow{F(p) \times \mathrm{Id}} & F(n) \times X^{F(n)} \end{array}$$

In this diagram, X is considered as an object of $\mathcal{T}/F(I)$, as before. Moreover, $\mathrm{ev}_{F(m)}$ and $\mathrm{ev}_{F(n)}$ are the evaluation arrows that come with the definition of an exponential, and the existence and uniqueness of $(\mathbb{E}_f X)^I(p)$ is guaranteed by the universal property of the exponential $X^{F(m)}$.

Proposition 8.2.2. The above defines a valid indexed functor $\mathbb{E}_f X : \mathbb{S}[f]^{op} \to F^*\mathbb{T}$. Moreover, the mapping $X \mapsto \mathbb{E}_f X$ extends to a functor $\mathcal{T} \to [\mathbb{S}[f]^{op}, F^*\mathbb{T}]$.

Proof. This is a straightforward check. ∎

8.2.3 Equivalence

We will show that that the indexed functors $\mathbb{P}_f X$ and $\mathbb{E}_f X$ are equivalent. For this to make sense, we set the codomain of $\mathbb{P}_f X$ to be $\mathbb{C} = F^*\mathbb{T}$; note that $F^*\mathbb{T}$ has f-indexed products if $F(f)$ is exponentiable (see propositions 6.3.6 and 6.2.4), so defining $\mathbb{P}_f X$ is valid.

Theorem 8.2.3. Let $\mathcal{S}$, $\mathcal{T}$ be categories with finite limits, let $F : \mathcal{S} \to \mathcal{T}$ be a functor which preserves finite limits, and let $f : B \to A$ be an arrow in $\mathcal{S}$ such that both f and $F(f)$ are exponentiable. Then the functors

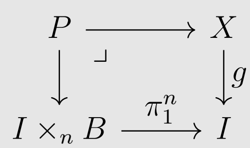
$$\mathbb{P}_f, \mathbb{E}_f : \mathcal{T} \to [\mathbb{S}[f]^{op}, F^*\mathbb{T}]$$

are isomorphic.

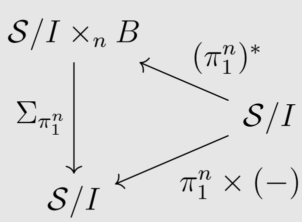
Proof. For each $X \in \mathcal{T}$ and $I \in \mathcal{S}$, we will construct a natural isomorphism $\alpha_X^I : (\mathbb{P}_f X)^I \to (\mathbb{E}_f X)^I$.

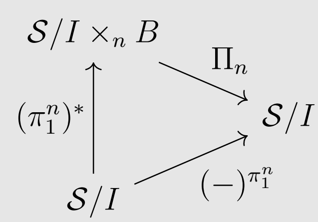
Let $n : I \to A$ be an object of $\mathbb{S}[f]$; we will construct an isomorphism $(\alpha_X^I)_n : (\mathbb{P}_f X)^I(n) \to (\mathbb{E}_f X)^I(n)$ and show it is natural in n. Recall that n corresponds to the arrow $\pi_1^n : I \times_n B \to I$ (via the inclusion $\mathbb{S}[f] \hookrightarrow \mathbb{S}$).

Then note that the functor $\pi_1^n \times (-) : \mathcal{S}/I \to \mathcal{S}/I$ is the same as $\Sigma_{\pi_1^n} \circ (\pi_1^n)^*$. Indeed, given $g : X \to I$, consider the pullback square below: $(\pi_1^n)^*(g)$ is the left vertical arrow, so $\Sigma_{\pi_1^n}((\pi_1^n)^*(g))$ is the diagonal, which is the same as $\pi_1^n \times g$.



In short, we find that the diagram below on the left commutes.





If we take right adjoints along the outside of the above left diagram, we get the above right diagram. By uniqueness of right adjoints, we find that these functors are isomorphic. Spelled out, this means that for any $Y \in \mathcal{S}/I$, we have an isomorphism

$$\beta_n^Y : \Pi_n(\pi_1^n)^* Y \xrightarrow{\sim} Y^{\pi_1^n},$$

and this isomorphism is natural in Y. In fact, there is a unique such natural isomorphism β_n which commutes with the counits of the adjunctions. If ε_n is the counit of $\Sigma_{\pi_1^n}(\pi_1^n)^* \dashv \Pi_n(\pi_1^n)^*$ and ev_n is the counit of $\pi_1^n \times (-) \dashv (-)^{\pi_1^n}$, then β_n^Y is the unique map which makes the following diagram commute.

$$\begin{array}{ccc}
\pi_1^n \times \prod_n ((\pi_1^n)^* Y) & \xrightarrow{\pi_1^n \times \beta_n^Y} & \pi_1^n \times Y^{\pi_1^n} \\
= & & \downarrow \mathrm{ev}_n^Y \\
\Sigma_{\pi_1^n} (\pi_1^n)^* \prod_n (\pi_1^n)^* Y & \xrightarrow{\varepsilon_n^Y} & Y
\end{array}$$

Now, note that $\langle\rangle_n = \langle\rangle_I \circ \pi_1^n$, and so $(\pi_1^n)^* \langle\rangle_I^* \cong \langle\rangle_n^*$. So, if $X \in \mathcal{S}$, then we can form the following isomorphism.

$$(\alpha_X^I)_n : \Pi_n \langle\rangle_n^* X \xrightarrow{\cong} \Pi_n (\pi_1^n)^* \langle\rangle_I^* X \xrightarrow{\beta_n^{\langle\rangle_I^* X}} (\langle\rangle_I^* X)^{\pi_1^n}$$

This is the isomorphism $(\mathbb{P}_f X)^I(n) \to (\mathbb{E}_f X)^I(n)$ we were looking for. Note that, by the unique characterization of β_n, we can say that $(\alpha_X^I)_n$ is the unique arrow that makes the following diagram commute.

$$\begin{array}{ccccc}
\pi_1^n \times \Pi_n \langle\rangle_n^* X & & \xrightarrow{\mathrm{Id} \times (\alpha_X^I)_n} & & (\langle\rangle_I^* X)^{\pi_1^n} \\
\cong \downarrow & & & & \downarrow \mathrm{ev}_n^{\langle\rangle_I^* X} \\
\pi_1^n \times \Pi_n (\pi_1^n)^* \langle\rangle_I^* X & = & \Sigma_n (\pi_1^n)^* \Pi_n (\pi_1^n)^* \langle\rangle_I^* X & \xrightarrow{\varepsilon_n^{\langle\rangle_I^* X}} & \langle\rangle_I^* X
\end{array}$$

We must now show that the isomorphism $(\alpha_X^I)_n$ is natural in n. So, let $p : m \to n$ be an arrow in $\mathbb{S}[f](I)$; we need to show that the following diagram commutes.

$$\begin{array}{ccc}
\prod_n \langle\rangle_n^* X & \xrightarrow{(\mathbb{P}X)^I(p)} & \prod_m \langle\rangle_m^* X \\
(\alpha_X^I)_n \downarrow & & \downarrow (\alpha_X^I)_m \\
(\langle\rangle_I^* X)^{\pi_1^n} & \xrightarrow{(\mathbb{E}X)^I(p)} & (\langle\rangle_I^* X)^{\pi_1^m}
\end{array}$$

Since these are two arrows into an exponential, to check they are equal, it suffices to show they are equal when we take the product with (π_1^m) and compose with the counit; this is straightforward. ∎

8.3 Preserving indexed products

In this section, we prove that the functor $\mathbb{E}_f X$ preserves f-indexed products, which is to say it turns f-indexed sums in $\mathbb{S}[f]$ into f-indexed products in $F^*\mathbb{T}$, subject to some conditions on the categories involved. The idea is simple: we think of the functor $(\mathbb{E}_f X)^I$ as being the composite of the following functors:

$$(\iota^I)^{op} : \mathbb{S}[f](I)^{op} \hookrightarrow (\mathcal{S}/I)^{op},$$

$$F^{op} : (\mathcal{S}/I)^{op} \to (\mathcal{T}/F(I))^{op},$$
$$(\langle\rangle_I^* X)^{(-)} : (\mathcal{T}/F(I))^{op} \to \mathcal{T}/F(I).$$

The second and third functors always preserve f-indexed products, so we just need to assume that the first one does. However, the problem is that the exponential $(\langle\rangle_I^* X)^{(-)}$ is not defined on all of $\mathcal{T}/F(I)$, so we can't use this argument. We must therefore recreate this argument while restricting our attention to objects of $\mathbb{S}[f](I)$.

Before we get the desired result, we need some intermediate facts.

Proposition 8.3.1. Let $\mathcal{S}$ be a category with finite limits, let $f : B \to A$ be an arrow in $\mathcal{S}$, let $Y \in \mathcal{S}/A$, and let $Z \in \mathcal{S}/B$. Let ε denote the counit of the adjunction $\Sigma_f \vdash f^*$. Then we have $Y \times \Sigma_f(Z) \cong \Sigma_f(f^*Y \times Z)$ via the following map.

$$\Sigma_f(f^*Y \times Z) \xrightarrow{\langle \Sigma_f(\pi_1),\, \Sigma_f(\pi_2)\rangle} \Sigma_f f^*Y \times \Sigma_f Z \xrightarrow{\varepsilon \times \mathrm{Id}} Y \times \Sigma_f Z$$

Proof. Say that Y is short for $g : Y \to A$, and Z is short for $h : Z \to B$. Then consider the following diagrams.

$$\begin{array}{ccc} f^*Y \times Z & \longrightarrow & Z \\ \downarrow & & \downarrow h \\ f^*Y & \longrightarrow & B \\ \downarrow & & \downarrow f \\ Y & \xrightarrow{g} & A \end{array} \qquad\qquad \begin{array}{ccc} Y \times \Sigma_f Z & \longrightarrow & Z \\ \Big\downarrow & & \downarrow h \\ & & B \\ \Big\downarrow & & \downarrow f \\ Y & \xrightarrow{g} & A \end{array}$$

Let's start by examining the left diagram. The lower square is a pullback defining f^*Y, which is an arrow we write as $f^*Y \to B$. Then the product (in $\mathcal{S}/B$) of f^*Y and Z is given by the top square, which is also a pullback. Thus, the overall diagram is a pullback, and the arrow from top left to bottom right is $\Sigma_f(f^*Y \times Z)$.

Moreover, in the left diagram, the top arrow is the second projection π_2, but if we think of it as an arrow from $\Sigma_f(f^*Y \times Z)$, it becomes $\Sigma_f(\pi_2)$. Similarly, the left column can be viewed as an arrow from $\Sigma_f(f^*Y \times Z) \to A$ to $Y \to A$, and we can easily check that it is precisely the composition of $\Sigma_f(\pi_1) : \Sigma_f(f^*Y \times Z) \to \Sigma_f(f^*Y)$ and $\varepsilon : \Sigma_f(f^*Y) \to Y$.

Next, the right diagram is a pullback, because it is the definition of $Y \times \Sigma_f Z$ (the product in $\mathcal{S}/A$). Moreover, the arrows $\varepsilon \circ \Sigma_f(\pi_1)$ and $\Sigma_f(\pi_2)$ provide arrows from the left diagram to the right diagram, inducing a map $\Sigma_f(f^*Y \times Z) \to Y \times \Sigma_f Z$. But since both squares are pullbacks of the same maps (g and hf), this induced map is an isomorphism. ∎

Proposition 8.3.2. Let $\mathcal{S}$ be a category with finite limits. Let $f : B \to A$ be an arrow in $\mathcal{S}$, let $X, Y \in \mathcal{S}/A$, and assume the exponential X^Y exists, with the evaluation map $\mathrm{ev} : Y \times X^Y \to X$.

Then $f^*(X^Y)$, together with the evaluation map ev' given by

$$f^*Y \times f^*(X^Y) \cong f^*(Y \times X^Y) \xrightarrow{f^*(\mathrm{ev})} f^*X,$$

is an exponential $(f^*X)^{f^*Y}$.

Proof. Given $Z \in \mathcal{S}/B$, we must show that the following map is a bijection.

$$\mathrm{Hom}\left(Z,\ f^*(X^Y)\right) \xrightarrow{g \mapsto \mathrm{ev}' \circ (\mathrm{Id} \times g)} \mathrm{Hom}\left(f^*Y \times Z,\ f^*X\right)$$

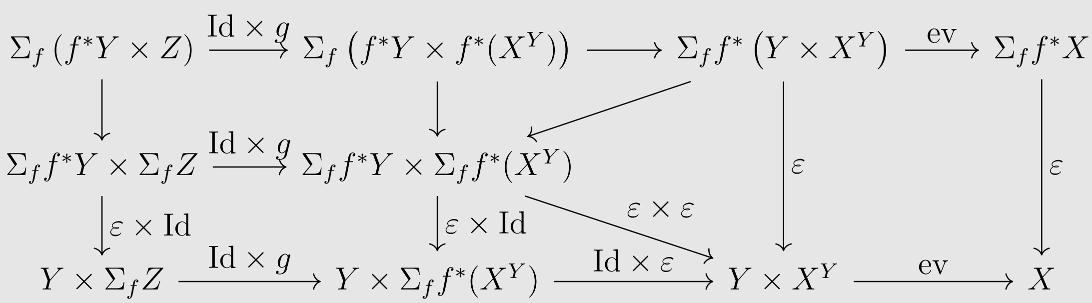
Note that $\mathrm{Hom}(f^*Y \times Z, f^*X) \cong \mathrm{Hom}(\Sigma_f(f^*Y \times Z), X)$, so in fact it suffices to show that the following map is a bijection.

$$\phi : \mathrm{Hom}(Z, f^*(X^Y)) \to \mathrm{Hom}(f^*Y \times Z, f^*X) \cong \mathrm{Hom}(\Sigma_f(f^*Y \times Z), X)$$

To prove this, we show that this map is the same as the following chain of bijections (the third bijection comes from proposition 8.3.1).

$$\begin{aligned}\psi : \mathrm{Hom}\left(Z,\ f^*(X^Y)\right) \cong \mathrm{Hom}\left(\Sigma_f Z,\ X^Y\right) &\cong \mathrm{Hom}\left(Y \times \Sigma_f Z,\ X\right)\\ &\cong \mathrm{Hom}\left(\Sigma_f(f^*Y \times Z),\ X\right)\end{aligned}$$

So, let $g : Z \to f^*(X^Y)$. We can follow the two maps ϕ, ψ above to get two corresponding arrows $\phi(g), \psi(g) : \Sigma_f(f^*Y \times Z) \to X$; we just need to check that they are equal. This is done with the following diagram: the top path is the arrow $\phi(g)$ and the bottom path is the arrow $\psi(g)$, so we just need to show that the diagram commutes.



The fact that this diagram commutes is straightforward (it is mostly by naturality of the counits), so we will not elaborate further. ∎

Finally, we arrive at the promised result of this section. Note that $\mathbb{S}$ has f-indexed products since f is exponentiable (see proposition 6.3.6).

Theorem 8.3.3. Let $\mathcal{S}$, $\mathcal{T}$ be categories with finite limits, let $F : \mathcal{S} \to \mathcal{T}$ be a functor which preserves finite limits, and let $f : B \to A$ be an arrow in $\mathcal{S}$ such that both f and $F(f)$ are exponentiable.

Assume that $\mathbb{S}[f]$ has f-indexed sums, and that the inclusion $\mathbb{S}[f] \hookrightarrow \mathbb{S}$ preserves f-indexed sums.

Then, for any $X \in \mathcal{T}$, the functor $\mathbb{E}_f X : \mathbb{S}[f]^{op} \to F^*\mathbb{T}$ preserves f-indexed products. (That is, it maps f-indexed sums in $\mathbb{S}[f]$ to f-indexed products in $F^*\mathbb{T}$.)

Proof. A pullback of f is an arrow of the form $\pi_1^n : I \times_n B \to I$ for some $n : I \to A$. So, for any $X \in \mathcal{S}$, any $n : I \to A$, and any $Z \in \mathbb{S}[f](I \times_n B)$, we must show that the canonical arrow

$$(\mathbb{E}_f X)^I \left(\sum_n Z \right) \to \prod_n (\mathbb{E}_f X)^{I \times_n B}(Z)$$

is an isomorphism. Note that, if write out the definition of $\mathbb{E}_f X$, this arrow is

$$(\langle\rangle_I^* X)^{F(\Sigma_n Z)} \to \prod_n (\langle\rangle_n^* X)^{F(Z)} .$$

(On the right side, we view Z as an object of $\mathcal{S}/I \times_n B$.)

To start, we explain how we use the assumption that the inclusion $\mathbb{S}[f] \hookrightarrow \mathbb{S}$ preserves f-indexed sums. The functor $\Sigma_n : \mathbb{S}[f](I \times_n B) \to \mathbb{S}[f](I)$ is, by definition, the left adjoint of $(\pi_1^n)^*$. In $\mathbb{S}$, the left adjoint of $(\pi_1^n)^*$ is $\Sigma_{\pi_1^n}$, the functor given by composition with π_1^n. So, if we take $Z \in \mathbb{S}[f](I \times_n B)$ and view $\Sigma_n(Z)$ as an object of $\mathcal{S}/I$, then this is isomorphic to $\Sigma_{\pi_1^n}(Z)$ (while viewing Z as an object of $\mathcal{S}/I \times_n B$). So, when we write $\Sigma_n(Z)$, we can just treat Σ_n as being the post-composition functor.

Moreover, we note that the functor $F : \mathbb{S} \to F^*\mathbb{T}$ preserves f-indexed sums (see proposition 6.3.3 and remark 6.3.1), so we have an isomorphism $\Sigma_n(FZ) \cong F(\Sigma_n Z)$.

Now, before we show the canonical arrow is an isomorphism, note that for any $Y \in \mathcal{T}/F(I)$, we have the following chain of isomorphisms. (We use the isomorphism of proposition 8.3.1.)

$$\begin{aligned}
\mathrm{Hom}\left(Y,\ (\langle\rangle_I^* X)^{F(\Sigma_n Z)}\right) &\cong \mathrm{Hom}\left((\Sigma_n FZ) \times Y,\ \langle\rangle_I^* X\right) \\
&\cong \mathrm{Hom}\left(\sum_n (FZ \times (\pi_1^n)^* Y),\ \langle\rangle_I^* X\right) \\
&\cong \mathrm{Hom}\left(FZ \times (\pi_1^n)^* Y,\ (\pi_1^n)^* \langle\rangle_I^* X\right) \\
&\cong \mathrm{Hom}\left(FZ \times (\pi_1^n)^* Y,\ \langle\rangle_n^* X\right) \\
&\cong \mathrm{Hom}\left((\pi_1^n)^* Y,\ (\langle\rangle_n^* X)^{FZ}\right) \\
&\cong \mathrm{Hom}\left(Y,\ \prod_n (\langle\rangle_n^* X)^{FZ}\right)
\end{aligned}$$

By the Yoneda embedding, we know these two objects are isomorphic, but we need to show that the canonical arrow is an isomorphism. So, we must show that the arrow given by this isomorphism is indeed the canonical one. This is a very technical calculation, since there are many coherences to keep track of, but it is straightforward with proposition 8.3.2, so we omit the proof. ∎

8.4 Preserving equalizers

In this section, we prove that the functor $\mathbb{E}_f X$ preserves equalizers, which is to say it turns coequalizers in $\mathbb{S}[f]$ into equalizers in $F^*\mathbb{T}$. We need some assumptions on the categories involved, but the spirit of the proof is the same as in section 8.3.

We start with a simple fact.

Proposition 8.4.1. Let $\mathcal{C}$ be a category with finite limits. Let A, B, X, Y be objects of $\mathcal{C}$, and assume that the exponentials X^A and X^B exist. Finally, let $f : A \to B$, and let $\overline{f} : X^B \to X^A$ be the usual "pre-composition with f" arrow induced by the universal property. Then the following diagram commutes.

$$\begin{array}{ccc} \mathrm{Hom}(Y, X^B) & \xrightarrow{\ \overline{f} \circ -\ } & \mathrm{Hom}(Y, X^A) \\ {\scriptstyle\cong}\downarrow & & \downarrow{\scriptstyle\cong} \\ \mathrm{Hom}(B \times Y, X) & \xrightarrow{\ - \circ (f \times \mathrm{Id})\ } & \mathrm{Hom}(A \times Y, X) \end{array}$$

Proof. Let $\phi : Y \to X^B$, and consider the following diagram.

$$\begin{array}{ccccc} A \times Y & \xrightarrow{\mathrm{Id} \times \phi} & A \times X^B & \xrightarrow{\mathrm{Id} \times \overline{f}} & A \times X^A \\ \downarrow{\scriptstyle f \times \mathrm{Id}} & & \downarrow{\scriptstyle f \times \mathrm{Id}} & & \downarrow{\scriptstyle ev_A} \\ B \times Y & \xrightarrow[\mathrm{Id} \times \phi]{} & B \times X^B & \xrightarrow[ev_B]{} & X \end{array}$$

The left square commutes trivially, and the right square commutes by definition of $\overline{f}$. The top path of this diagram is the arrow obtained by following the top path of the diagram in the theorem (applied to ϕ), and the bottom path here comes from the bottom path in the theorem. So, the two paths are equal. ∎

We are now ready for the desired result.

Theorem 8.4.2. Let $\mathcal{S}$, $\mathcal{T}$ be categories with finite limits, let $F : \mathcal{S} \to \mathcal{T}$ be a functor which preserves finite limits, and let $f : B \to A$ be an arrow in $\mathcal{S}$ such that both f and $F(f)$ are exponentiable.

Assume that the indexed categories $\mathbb{S}[f]$, $\mathbb{S}$, and $\mathbb{T}$ have coequalizers, and that the indexed functors $\mathbb{S}[f] \hookrightarrow \mathbb{S}$ and $F : \mathbb{S} \to F^*\mathbb{T}$ preserve them.

Then for any $X \in \mathcal{S}$, the functor $\mathbb{E}_f X : \mathbb{S}[f]^{op} \to F^*\mathbb{T}$ preserves equalizers. That is, it maps coequalizers in $\mathbb{S}[f]$ to equalizers in $F^*\mathbb{T}$.

Proof. First, we note that for any $I \in \mathcal{S}$ and any $Y \in \mathcal{T}/F(I)$, the functor $- \times Y : \mathcal{T}/F(I) \to \mathcal{T}/F(I)$ preserves coequalizers. Indeed, Y is an arrow of $\mathcal{T}$, which we'll denote $g : Y \to F(I)$, and $- \times Y$ is the same as $\Sigma_g \circ g^*$. Since $\mathbb{T}$ has coequalizers, we know that they are preserved by the transition map g^*, and Σ_g preserves coequalizers since it is a left adjoint (of g^*). Thus, the composite $- \times Y$ must also preserve them.

Now, let $I \in \mathcal{S}$, and consider a coequalizer diagram in $\mathbb{S}[f](I)$ as below.

$$k \underset{q}{\overset{p}{\rightrightarrows}} m \xrightarrow{r} n$$

We claim that the image of this diagram under $\mathbb{E}_f X$ is an equalizer. We'll use denote this image as follows.

$$X^{F(n)} \xrightarrow{\overline{F(r)}} X^{F(m)} \underset{\overline{F(q)}}{\overset{\overline{F(p)}}{\rightrightarrows}} X^{F(k)}$$

So, let $Y \in \mathcal{T}/F(I)$, and let $\phi : Y \to X^{F(m)}$ be such that $\overline{F(p)} \circ \phi = \overline{F(q)} \circ \phi$. By the universal property of exponentials, ϕ corresponds to a map $\psi : F(m) \times Y \to X$, and we find that $\psi \circ (F(p) \times \mathrm{Id}) = \psi \circ (F(q) \times \mathrm{Id})$ (see proposition 8.4.1).

$$\begin{array}{ccccc} F(k) \times Y & \underset{F(q)\times \mathrm{Id}}{\overset{F(p)\times \mathrm{Id}}{\rightrightarrows}} & F(m) \times Y & \xrightarrow{F(r) \times \mathrm{Id}} & F(n) \times Y \\ & & & \searrow^{\psi} & \downarrow{\scriptstyle \beta} \\ & & & & X \end{array}$$

Now, by assumption (and by the discussion at the start of the proof), the composite functor

$$\mathbb{S}[f](I) \hookrightarrow \mathcal{S}/I \xrightarrow{F} \mathcal{T}/F(I) \xrightarrow{- \times Y} \mathcal{T}/F(I)$$

preserve coequalizers, so the above is a coequalizer diagram. This tells us that there is a unique $\beta : F(n) \times Y \to X$ such that $\alpha \circ (F(r) \times \mathrm{Id}) = \psi$. But then β corresponds to $\alpha : Y \to X^{F(n)}$, and this is a unique arrow such that $\overline{F(r)} \circ \alpha = \phi$ (again, see proposition 8.4.1).

$$\begin{array}{ccccc} X^{F(n)} & \xrightarrow{\overline{F(r)}} & X^{F(m)} & \underset{\overline{F(q)}}{\overset{\overline{F(p)}}{\rightrightarrows}} & X^{F(k)} \\ \uparrow{\scriptstyle \alpha} & \nearrow_{\phi} & & & \\ Y & & & & \end{array}$$

This completes the proof. ∎

8.5 A left inverse

We start with a simple remark.

Remark 8.5.1. Let $\mathcal{S}$ be a category with finite limits, and let $\mathbb{C}, \mathbb{D}$ be $\mathcal{S}$-indexed categories. Then for any $u \in \mathcal{C}$, the following is a well-defined functor.

$$\begin{aligned} [\mathbb{C}, \mathbb{D}] &\longrightarrow \mathcal{D} \\ F &\longmapsto F^{\mathbb{1}}(u) \\ (\alpha : F \Rightarrow G) &\longmapsto (\alpha^{\mathbb{1}}_u : F^{\mathbb{1}}(u) \to G^{\mathbb{1}}(u)) \end{aligned}$$

In fact, this is still true if $[\mathbb{C}, \mathbb{D}]$ is replaced by any of its subcategories.

We use this to establish the following isomorphism of functors.

Proposition 8.5.2. Let $\mathcal{S}, \mathcal{T}$ be categories with finite limits, let $F : \mathcal{S} \to T$ be a functor which preserves finite limits, and let $f : B \to A$ be an arrow in $\mathcal{S}$ such that f and $F(f)$ are both exponentiable.

Let u be an object of $\mathbb{S}[f](\mathbb{1})$, and consider the composition of $\mathbb{E}_f$ and the "evaluation at u" functor of remark 8.5.1.

$$\mathcal{T} \xrightarrow{X \mapsto \mathbb{E}_f X} [\mathbb{S}[f]^{op}, F^*\mathbb{T}]_{\mathcal{S}} \xrightarrow{G \mapsto G^{\mathbb{1}}(u)} \mathcal{T}, \qquad \mathcal{T} \xrightarrow{X \mapsto (\mathbb{E}_f X)^{\mathbb{1}}(u)} \mathcal{T}$$

If $\iota^{\mathbb{1}}(u)$ is a terminal object of $\mathcal{S}$, where $\iota : \mathbb{S}[f] \hookrightarrow \mathbb{S}$ is the usual inclusion, then the functor $X \mapsto (\mathbb{E}_f X)^{\mathbb{1}}(u)$ is naturally isomorphic to the identity.

Proof. Note that $\iota^{\mathbb{1}}(u)$ is $u^* f$, so this is a terminal object of $\mathcal{S}$ by assumption. Since F preserves finite products, $F(u^* f)$ is a terminal object of $\mathcal{T}$.

Moreover, $(\mathbb{E}_f X)^{\mathbb{1}}(u)$ is just $X^{F(u^* f)}$, by definition. But X exponentiated with a terminal object is just X, so $(\mathbb{E}_f X)^{\mathbb{1}}(u) \cong X$, as desired. ∎

Chapter 9

Indexed colimits

Contents

In this chapter, we take the first concrete steps towards proving the third main result of this thesis (theorem 17.2.1), which states that freely adding an object to an arithmetic universe $\mathcal{S}$ gives us $[\mathbf{Fin}_{\mathcal{S}}, \mathbb{S}]$. Roughly, chapters 9, 10, and 11 are strung together to prove the following equivalence of categories (corollary 11.3.1); as discussed in the introduction (section 1.2), this is a big part of proving the third main result.

$$[[\![C^{op}, \mathbb{S}]\!], F^*\mathbb{T}]_{flexcc} \simeq [C, F^*\mathbb{T}]_{flex}.$$

Here, $\mathcal{S}$ and $\mathcal{T}$ are AUs, $F : \mathcal{S} \to \mathcal{T}$ is an AU morphism, and C is a flex internal category. We also recall the terms flex and flexcc from section 6.2: an indexed functor is flex if it preserves finite limits and N-finite products, and it is flexcc if it is also cocontinuous. Informally, we think of flexcc indexed functors as coming from AU functors (see the results of section 6.3).

Where does this equivalence come from? It starts with corollary B2.5.8 of [John 02] (which we recall in fact 10.2.1). It states that

$$[[\![C^{op}, \mathbb{S}]\!], \mathbb{D}]_{cc} \simeq [C, \mathbb{D}],$$

where $\mathbb{D}$ is any cocomplete $\mathcal{S}$-indexed category. One direction of this equivalence is given by precomposition with the Yoneda functor Y, and the other direction maps G to $\mathrm{Lan}_Y\, G$, the left Kan extension of G along Y (we describe these constructions in chapter 10). To obtain the equivalence we want, we need to show that these maps both preserve flex functors, and since $\mathrm{Lan}_Y\, G$ is defined using colimits, this requires us to have a good understanding of colimits in indexed categories. That is the goal of this chapter.

In section 9.1, we define the colimit functor colim : $[\mathbb{C}, \mathbb{D}] \to \mathcal{D}$, and see how it extends to an indexed functor $[\![\mathbb{C}, \mathbb{D}]\!] \to \mathbb{D}$. Sections 9.2 and 9.3 each show a basic result: the fact that colimits are computed pointwise, and the colimit interchange law. Section 9.4 shows that the colimit functor preserves finite limits and N-finite products, and section 9.5 concludes that taking a filtered colimit of flex functors yields a flex functor.

9.1 Defining the colimit functor

Our definition of colimits in indexed categories is based on the following fact, which comes from proposition B2.3.20 of [John 02]. We also use the notation from the paragraph underneath that proposition.

Fact 9.1.1. Let $\mathcal{S}$ be a category with finite limits, and let $\mathbb{D}$ be a cocomplete $\mathcal{S}$-indexed category. Then for any internal functor $f : \mathbb{C} \to \mathbb{C}'$, the functor $f^* : [\mathbb{C}', \mathbb{D}] \to [\mathbb{C}, \mathbb{D}]$ has a left adjoint colim_f. If f is the functor $\mathbb{C} \to \mathbb{1}$, then f^* is the diagonal functor $\Delta_{\mathbb{C}} : \mathcal{D} \to [\mathbb{C}, \mathbb{D}]$ (see section 6.4), and we denote its left adjoint by colim : $[\mathbb{C}, \mathbb{D}] \to \mathcal{D}$.

Based on this fact, we extend the colimit functor to an indexed functor.

Definition. Let $\mathcal{S}$ be a category with finite limits, let $\mathbb{D}$ be a cocomplete $\mathcal{S}$-indexed category, and let $\mathbb{C}$ be an internal category in $\mathcal{S}$. We define an $\mathcal{S}$-indexed functor

$$\text{colim} : [\![\mathbb{C}, \mathbb{D}]\!] \to \mathbb{D}$$

as follows. For any $I \in \mathcal{S}$, we must define $\text{colim}^I : [\Sigma_I^* \mathbb{C}, \Sigma_I^* \mathbb{D}]_{\mathcal{S}/I} \to \mathcal{D}^I$. We can view $\Sigma_I^* \mathbb{C}$ as an internal category in $\mathcal{S}/I$ (see remark 6.5.3), and $\Sigma_I^* \mathbb{D}$ is a cocomplete $\mathcal{S}/I$-indexed category (by proposition 6.2.4, since Σ_I preserves pullbacks) whose ordinary underlying category is $\mathcal{D}^I$. So, we can just view this as a special case of the ordinary colimit functor given in fact 9.1.1.

In the special case when $\mathbb{D} = \mathbb{S}$, the canonical indexing of $\mathcal{S}$, we can give an equivalent definition of the colimit. Since $[\![\mathbb{C}, \mathbb{S}]\!]$ can be viewed as the pullback by $\Delta_{\mathbb{C}}$ of the canonical indexing of $[\mathbb{C}, \mathbb{S}]$ (see proposition 6.5.2), we expect that colim can be defined as an extension like we saw in section 6.3. However, for this to make sense, we first need the following lemma.

Lemma 9.1.2. Let $\mathcal{S}$ be a category with finite limits such that $\mathbb{S}$ is cocomplete, and let C be an internal category in $\mathcal{S}$ with a terminal object. Then the following diagram commutes (up to isomorphism).

$$\begin{array}{ccc} [C, \mathbb{S}] & \xrightarrow{\text{colim}} & \mathcal{S} \\ {\scriptstyle \Delta_C} \uparrow & \nearrow_{\text{Id}} & \\ \mathcal{S} & & \end{array}$$

In fact, the counit $\operatorname{colim} \circ \Delta \Rightarrow \mathrm{Id}$ is an isomorphism. (Recall that colim is defined as the left adjoint of Δ.)

Proof. Let $X \in \mathcal{S}$. To show that $\operatorname{colim}(\Delta X) \cong X$, it suffices to show that X, equipped with the arrow $\mathrm{Id} : \Delta X \to \Delta X$, is a universal morphism from ΔX to Δ. In fact, this also shows that the counit $\operatorname{colim} \circ \Delta \Rightarrow \mathrm{Id}$ is an isomorphism: since both $(\operatorname{colim}(\Delta X), \eta_{\Delta X})$ and $(X, \mathrm{Id}_{\Delta X})$ are universal morphisms, there is an isomorphism $\phi : \operatorname{colim}(\Delta X) \to X$ such that $\Delta\phi \circ \eta_{\Delta X} = \mathrm{Id}_{\Delta X}$. But note that the adjunction bijection $\mathrm{Hom}(\operatorname{colim}(\Delta X), X) \cong \mathrm{Hom}(\Delta X, \Delta X)$ is given by the maps $g \mapsto \Delta g \circ \eta_{\Delta X}$ and $h \mapsto \epsilon_X \circ \operatorname{colim}(h)$, so we find that $\phi = \epsilon_X \circ \operatorname{colim}(\mathrm{Id}_{\Delta X}) = \epsilon_X$. Thus, each ϵ_X is an isomorphism.

So, we now show that $(X, \mathrm{Id}_{\Delta X})$ is a universal morphism from ΔX to Δ. That is, we show that for every $Z \in \mathcal{S}$ and $g : \Delta X \to \Delta Z$, there exists a unique $h : X \to Z$ such that $\Delta h = g$.

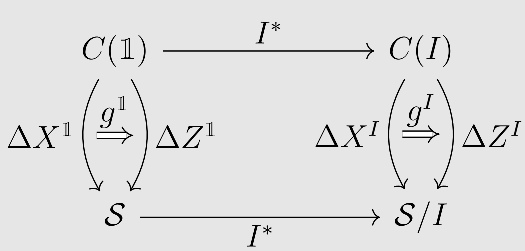
We start with uniqueness: suppose $h : X \to Z$ satisfies $\Delta h = g$. Since we assumed C has terminal objects, we know in particular that $C(\mathbb{1})$ has a terminal object T. Then, note that $(\Delta X)^{\mathbb{1}}(T) = X$, $(\Delta Z)^{\mathbb{1}}(T) = Z$ and $(\Delta h)^{\mathbb{1}}_T = h : X \to Z$. Therefore, we have $h = (\Delta h)^{\mathbb{1}}_T = g^{\mathbb{1}}_T$; this shows uniqueness.

For existence, it remains to show that if we take $h = g^{\mathbb{1}}_T$, then $\Delta(g^{\mathbb{1}}_T) = g$. Let $I \in \mathcal{S}$ and $p \in C(I)$. Then there is an arrow $p \to T_I$ (where T_I is the terminal object of $C(I)$), so naturality of g^I gives a commuting square as below.

$$\begin{array}{ccc} I^*X & \xrightarrow{\mathrm{Id}} & I^*X \\ {\scriptstyle g^I_p}\big\downarrow & & \big\downarrow{\scriptstyle g^I_{T_I}} \\ I^*Z & \xrightarrow{\mathrm{Id}} & I^*Z \end{array}$$

Indeed, we know that $(\Delta X)^I(p) = I^*X$ and $(\Delta X)^I$ applied to an arrow is Id (and similarly for Z). Therefore, we have $g^I_p = g^I_{T_I}$.

Now, we assumed C has a terminal object, which means in particular that the terminal objects are preserved by the transition maps. So, the functor $I^* : C(\mathbb{1}) \to C(I)$ maps T to T_I. Then, by the coherence axioms of g, we find that $g^I_{T_I} = I^*(g^{\mathbb{1}}_T)$ (see the diagram below).



So, $g^I_p = g^I_{T_I} = I^*(g^{\mathbb{1}}_T)$, but this is precisely the definition of $(\Delta g^{\mathbb{1}}_T)^I_p$, so we have the desired equality. ∎

Proposition 9.1.3. Let $\mathcal{S}$ be a category with finite limits such that $\mathbb{S}$ is cocomplete, and let C be an internal category in $\mathcal{S}$ with a terminal object. The commuting diagram of lemma 9.1.2 induces (using the definition of section 6.3) an indexed functor

$$\operatorname{colim} : [\![C, \mathbb{S}]\!] \to \mathbb{S},$$

where we use proposition 6.5.2 to note that $[\![C, \mathbb{S}]\!]$ is the pullback along Δ of the canonical indexing of $[C, \mathbb{S}]$. This indexed functor is the same as the one defined at the beginning of this section.

Proof. This is a straightforward check. ∎

9.2 Pointwise colimits

In this section, we show that colimits of functor categories can be computed pointwise. However, in order to state this result, we first need to explain what "computing pointwise" means. For this, we need the following remark.

Remark 9.2.1. Recall that proposition 6.5.4 shows that $[\![\mathbb{C}, \mathbb{D}]\!]$ is an exponential for indexed categories. If we have a third indexed category $\mathbb{E}$, then this property (along with symmetry of the cartesian product) gives us the following equivalences of categories.

$$[\mathbb{E}, [\![\mathbb{C}, \mathbb{D}]\!]] \simeq [\mathbb{C} \times \mathbb{E}, \mathbb{D}] \simeq [\mathbb{E} \times \mathbb{C}, \mathbb{D}] \simeq [\mathbb{C}, [\![\mathbb{E}, \mathbb{D}]\!]]$$

We will use the following pieces of notation for passing between these categories.

- If we are given an indexed functor $F : \mathbb{C} \times \mathbb{E} \to \mathbb{D}$, then we may write $\widehat{F}$ for the corresponding functor $\mathbb{E} \to [\![\mathbb{C}, \mathbb{D}]\!]$ and $\widehat{F'}$ for the corresponding functor $\mathbb{C} \to [\![\mathbb{E}, \mathbb{D}]\!]$.
- If we are given an indexed functor $H : \mathbb{E} \to [\![\mathbb{C}, \mathbb{D}]\!]$, then we may write $\tilde{H}$ for the corresponding functor $\mathbb{C} \to [\![\mathbb{E}, \mathbb{D}]\!]$.

It will be useful to give an explicit description of the mapping $H \mapsto \tilde{H}$, which we can do using the construction given in the proof of proposition 6.5.4. So, given $H : \mathbb{E} \to [\![\mathbb{C}, \mathbb{D}]\!]$, the functor $\tilde{H} : \mathbb{C} \to [\![\mathbb{E}, \mathbb{D}]\!]$ is defined as follows. For $I \in \mathcal{S}$, the functor

$$\tilde{H}^I : \mathcal{C}^I \to [\Sigma_I^* \mathbb{E}, \Sigma_I^* \mathbb{D}]_{\mathcal{S}/I}$$

maps $c \in \mathcal{C}^I$ to the indexed functor

$$\tilde{H}^I(c) : \Sigma_I^* \mathbb{E} \to \Sigma_I^* \mathbb{D}$$

which, for any $(\alpha : J \to I) \in \mathcal{S}/I$, is given by

$$\begin{aligned} \left(\tilde{H}^I(c)\right)^\alpha &: \mathcal{E}^J \to \mathcal{D}^J \\ a &\mapsto \left(H^J(a)\right)^{T_J}(\alpha^* c). \end{aligned}$$

(Recall that T_J is $\mathrm{Id}_J : J \to J$, viewed as an object of $\mathcal{S}/J$.)

We now use this remark to formulate the result about pointwise colimits.

Proposition 9.2.2 (Colimits are computed pointwise)**.** Let S be a category with finite limits. Let $\mathbb{C}, \mathbb{D}$ be $\mathcal{S}$-indexed categories, with $\mathbb{D}$ cocomplete. Then $[\![\mathbb{C}, \mathbb{D}]\!]$ is cocomplete, and its colimits are computed pointwise. That is, for any internal category B in $\mathcal{S}$, the following diagram commutes up to isomorphism.

$$\begin{array}{ccc} [B, [\![\mathbb{C}, \mathbb{D}]\!]] & \xrightarrow{\text{colim}} & \\ \simeq\downarrow & & [\mathbb{C}, \mathbb{D}] \\ [\mathbb{C}, [\![B, \mathbb{D}]\!]] & \xrightarrow{\text{colim}\circ -} & \end{array}$$

(The equivalence comes from remark 9.2.1.)

Using the notation of remark 9.2.1, if we have a functor $F : \mathbb{C} \times B \to \mathbb{D}$, then we have $\text{colim}(\widehat{F}) \cong \text{colim} \circ \widehat{F'}$. In other words, if we have $G : B \times \mathbb{C} \to \mathbb{D}$, then $\text{colim}(\widehat{G'}) \cong \text{colim} \circ \widehat{G}$.

Proof. We must show that the pointwise colimit provides a left adjoint to the diagonal. So, let $F \in [B, [\![\mathbb{C}, \mathbb{D}]\!]]$ and $G \in [\mathbb{C}, \mathbb{D}]$; we claim that

$$\text{Hom}(\text{colim} \circ \tilde{F}, G) \cong \text{Hom}(F, \Delta G).$$

Let $\theta \in \text{Hom}(F, \Delta G)$. By unpacking all the definitions, we get the following: for each $I \in \mathcal{S}$, $(\alpha : J \to I) \in \mathcal{S}/I$, $b \in B(I)$, and $c \in \mathcal{C}^J$, we have an arrow

$$(\theta^I_b)^\alpha_c : (F^I(b))^\alpha(c) \to G^J(c)$$

in $\mathcal{D}^J$. Now, remark that $b : I \to C_0$, so this is equivalent to saying

$$(\theta^I_b)^\alpha_c : (F^{C_0}(\text{Id}_{C_0})^{b\alpha}(c) \to G^J(c).$$

On the other hand, let $\eta \in \text{Hom}(\text{colim} \circ \tilde{F}, G)$. Again, by unpacking all the definitions (and using the adjunction for the colimit), we get: for each $I \in \mathcal{S}$, $(\alpha : J \to I) \in \mathcal{S}/I$, $c \in \mathcal{C}^I$, and $b \in C(J)$, an arrow

$$(\widetilde{\eta^I_c})^\alpha_b : (F^J(b))^{T_J}(\alpha^* c) \to G^J(\alpha^* c)$$

in $\mathcal{D}^J$. Again, using the fact that $b : J \to C_0$, we can consider

$$(\widetilde{\eta^I_c})^\alpha_b : (F^{C_0}(\text{Id}_{C_0}))^b(c) \to G^J(\alpha^* c).$$

So, if we're given η, then we can define θ as follows: given $\alpha : J \to I$, $b \in B(I)$, and $c \in \mathcal{C}^J$, we can set

$$(\theta^I_b)^\alpha_c = (\widetilde{\eta^J_c})^{\mathrm{Id}}_{b\alpha}.$$

Conversely, if we're given θ, we can define η as follows: given $\alpha : J \to I$, $c \in \mathcal{C}^I$, and $b \in B(J)$, we can set

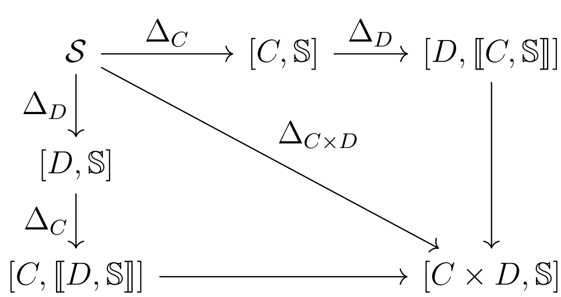
$$(\widetilde{\eta^I_c})^\alpha_b = (\theta^J_b)^{\mathrm{Id}}_{\alpha^* c}.$$

This gives the desired correspondence. ∎

9.3 The interchange law for colimits

In this section, we show that colimits satisfy an interchange law. In order to show this, we need the following lemma.

Lemma 9.3.1. Let $\mathcal{S}$ be a category with finite limits, and let C, D be internal categories in $\mathcal{S}$. Then the following diagram commutes.



Proof. Direct computation. ∎

We can now state and prove the interchange law.

Theorem 9.3.2. Let $\mathcal{S}$ be a category with finite limits such that $\mathbb{S}$ is cocomplete. Let C, D be internal categories in $\mathcal{S}$, and let

$$F : C \times D \to \mathbb{S}$$

be an indexed functor. Using the notation of remark 9.2.1, F induces two indexed functors $\widehat{F}$, $\widehat{F'}$ as below.

$$\widehat{F} : D \to [\![C, \mathbb{S}]\!] \qquad\qquad \widehat{F'} : C \to [\![D, \mathbb{S}]\!]$$

Then $\mathrm{colim}_C(\mathrm{colim}_D(\widehat{F})) \cong \mathrm{colim}_D(\mathrm{colim}_C(\widehat{F'}))$.

In the above theorem, note that $\mathrm{colim}_D(\widehat{F})$ is an indexed functor $C \to \mathbb{S}$, so we can take its colimit again. Similarly, $\mathrm{colim}_C(\widehat{F'})$ is an indexed functor $D \to \mathbb{S}$.

Proof. Since $\mathrm{colim}_D : [D, [\![C, \mathbb{S}]\!]] \to [C, \mathbb{S}]$ is defined as the left adjoint of Δ_D, we have an arrow

$$t : \widehat{F} \to \Delta_D\left(\mathrm{colim}_D \widehat{F}\right),$$

and for the same reason, we have

$$s : \mathrm{colim}_D \widehat{F} \to \Delta_C\left(\mathrm{colim}_C\left(\mathrm{colim}_D \widehat{F}\right)\right).$$

Therefore, we get the following composite.

$$\widehat{F} \xrightarrow{\ t\ } \Delta_D\left(\mathrm{colim}_D \widehat{F}\right) \xrightarrow{\Delta_D(s)} \Delta_D\left(\Delta_C\left(\mathrm{colim}_C\left(\mathrm{colim}_D \widehat{F}\right)\right)\right)$$

Now, the above arrow is in the category $[D, [\![C, \mathbb{S}]\!]]$, so there is a corresponding arrow in $[C, [\![D, \mathbb{S}]\!]]$, which we denote r. We would like to know the domain and codomain of r; its domain is $\widehat{F'}$ by definition of $\widehat{F}$ and $\widehat{F'}$, and its codomain is as below by lemma 9.3.1.

$$\widehat{F'} \xrightarrow{\ r\ } \Delta_C\left(\Delta_D\left(\mathrm{colim}_C\left(\mathrm{colim}_D \widehat{F}\right)\right)\right)$$

Then, by the adjoint property, this r induces a map

$$\mathrm{colim}_C \widehat{F'} \xrightarrow{\ q\ } \Delta_D\left(\mathrm{colim}_C\left(\mathrm{colim}_D \widehat{F}\right)\right),$$

and finally, again by the adjoint property, we get a map

$$\mathrm{colim}_D\left(\mathrm{colim}_C \widehat{F'}\right) \xrightarrow{\ \lambda\ } \mathrm{colim}_C\left(\mathrm{colim}_D \widehat{F}\right).$$

By the same process, we get a map $\lambda' : \mathrm{colim}_C(\mathrm{colim}_D\widehat{F}) \to \mathrm{colim}_D(\mathrm{colim}_C\widehat{F'})$.

It remains to show that λ and λ' are inverses of each other. To do this, we just appeal to the uniqueness part of the adjoint property. ∎

9.4 The colimit preserving limits

We now turn our attention to the question of the colimit functor preserving limits. This question is mostly answered by the following fact from [John 02].

Fact 9.4.1 (Theorem B2.6.8 in [John 02]). Let $\mathcal{S}$ be a pretopos, and let C be an internal category in $\mathcal{S}$. If C is filtered, then $\mathrm{colim} : [C, \mathbb{S}] \to \mathcal{S}$ preserves finite limits.

We give a slight strengthening of this fact.

Corollary 9.4.2. Let $\mathcal{S}$ be an arithmetic universe, and let C be an internal filtered category in $\mathcal{S}$ with a terminal object. Then $\mathrm{colim} : [C, \mathbb{S}] \to \mathcal{S}$ is an AU functor. Moreover, the indexed functor $\mathrm{colim} : [\![C, \mathbb{S}]\!] \to \mathbb{S}$ is flexcc.

Proof. First, note that the colimit functor exists (without any additional assumptions) because $\mathbb{S}$ is cocomplete (see proposition 6.3.7).

We know colim preserves finite limits by fact 9.4.1, and it preserves finite colimits because it is a left adjoint. It remains to show that colim preserves list objects; however, since it is a left adjoint and its counit is an isomorphism (by lemma 9.1.2), proposition 4.2.2 tells us that this follows from it preserving finite limits. Thus, we know that colim is an AU functor.

For the second part, we just recall that $\mathrm{colim} : [\![C, \mathbb{S}]\!] \to \mathbb{S}$ is a canonical extension (see proposition 9.1.3), so the underlying functor being an AU functor implies that its extension is flexcc (by proposition 6.3.8). ∎

9.5 Filtered colimits of flex indexed functors

The goal of this section is to prove that colimits of flex functors are flex (under certain hypotheses, including C being filtered). Roughly, what we want is to show that taking colimits gives us a functor

$$[C, [\![\mathbb{D}, \mathbb{S}]\!]_{flex}] \to [\mathbb{D}, \mathbb{S}]_{flex}.$$

However, to even state this result, we need to know what we mean by $[\![\mathbb{D}, \mathbb{S}]\!]_{flex}$. To define this indexed category, we first make a remark.

Remark 9.5.1. Let $\mathcal{S}$ be a category with finite limits and NNO, and let $\mathbb{C}$ be a flex $\mathcal{S}$-indexed category. We claim that $\Sigma_I^*\mathbb{C}$ is also flex for any $I \in \mathcal{S}$. It's clear that $\Sigma_I^*\mathbb{C}$ has finite limits since $\mathbb{C}$ does (see proposition 6.2.1). But what about N-finite products?

Since $\mathcal{S}$ is a locos, we know that $\mathcal{S}/I$ is a locos too, and its version of π_2^E is as below.

$$\begin{array}{ccc} I \times E & \xrightarrow{\ \mathrm{Id} \times \pi_2^E\ } & I \times N \\ & {\scriptstyle \pi_1} \searrow \quad \swarrow {\scriptstyle \pi_1} & \\ & I & \end{array}$$

If we apply Σ_I to this arrow, we get the arrow $\mathrm{Id} \times \pi_2^E : I \times E \to I \times N$ of $\mathcal{S}$. Note that this is a pullback of π_2^E (along $\pi_2 : I \times N \to N$), so $\mathbb{C}$ has $(\mathrm{Id} \times \pi_2^E)$-indexed products (by proposition 6.2.3). But then we can apply proposition 6.2.4 to conclude that $\Sigma_I^*\mathbb{C}$ has N-finite products (since Σ_I preserves pullbacks).

With this remark in mind, we give the following definition.

Definition. Let $\mathcal{S}$ be a category with finite limits and NNO, and let $\mathbb{C}, \mathbb{D}$ be flex $\mathcal{S}$-indexed categories. We write $[\![\mathbb{C}, \mathbb{D}]\!]_{flex}$ for the indexed category where $[\mathbb{C}, \mathbb{D}]^I_{flex} = [\Sigma_I^*\mathbb{C}, \Sigma_I^*\mathbb{D}]_{flex}$. (We showed that $\Sigma_I^*\mathbb{C}$ and $\Sigma_I^*\mathbb{D}$ are flex for any I in remark 9.5.1.) Note that there is an obvious inclusion $[\![\mathbb{C}, \mathbb{D}]\!]_{flex} \hookrightarrow [\![\mathbb{C}, \mathbb{D}]\!]$.

Before we prove the result promised at the beginning of this section, we need an intermediate result.

Proposition 9.5.2. Let $\mathcal{S}$ be a category with finite limits and NNO, and let $\mathbb{A}, \mathbb{C}, \mathbb{D}$ be flex $\mathcal{S}$-indexed categories. If $H : \mathbb{A} \to [\![\mathbb{C}, \mathbb{D}]\!]_{flex}$, then the corresponding indexed functor $\tilde{H} : \mathbb{C} \to [\![\mathbb{A}, \mathbb{D}]\!]$ (see remark 9.2.1) is flex. (This is well-defined: $[\![\mathbb{A}, \mathbb{D}]\!]$ is flex by proposition 6.5.5.)

Proof. Note that the explicit definition of $\tilde{H}$ is found in remark 9.2.1. We start by showing that $\tilde{H}$ preserves finite limits. So, let $I \in \mathcal{S}$; for any $c \in \mathcal{C}^I$, $\tilde{H}^I(c)$ is an indexed functor $\Sigma_I^*\mathbb{A} \to \Sigma_I^*\mathbb{D}$ such that for each $(\alpha : J \to I) \in \mathcal{S}/I$, we have

$$\begin{aligned} \left(\tilde{H}^I(c)\right)^\alpha &: \mathcal{A}^J \to \mathcal{D}^J \\ a &\mapsto \left(H^J(a)\right)^{T_J}(\alpha^* c). \end{aligned}$$

Note that α^* is lex (because $\mathbb{C}$ is lex), and $(H^J(a))^{T_J}$ is lex by the assumption on H. Therefore, since finite limits are computed pointwise for functor categories, we find that $\tilde{H}^I$ is lex.

Next, we show that $\tilde{H}$ preserves N-finite products. Let $\pi_1^n : I \times_n E \to I$ be a pullback of π_2^E along $n : I \to N$; we must show that the following diagram commutes up to isomorphism.

$$\begin{array}{ccc} \mathcal{C}^{I\times_n E} & \xrightarrow{\tilde{H}^{I\times_n E}} & [\mathbb{A},\mathbb{D}]^{I\times_n E} \\ \Pi_n\big\downarrow & & \big\downarrow\Pi_n \\ \mathcal{C}^{I} & \xrightarrow{\quad\tilde{H}^{I}\quad} & [\mathbb{A},\mathbb{D}]^{I} \end{array}$$

So, let $x \in \mathcal{C}^{I\times_n E}$. Following the two paths of this diagram gives us two $\mathcal{S}/I$-indexed functors

$$\tilde{H}^I\left(\Pi_n x\right),\ \Pi_n\tilde{H}^{I\times_n E}(x) : \Sigma_I^*\mathbb{A} \to \Sigma_I^*\mathbb{D}.$$

We must show that for each $(\alpha : J \to I) \in \mathcal{S}/I$ and $a \in \mathcal{A}^J$, we have

$$\left(\tilde{H}^I\left(\Pi_n x\right)\right)^{\alpha}(a) \cong \left(\Pi_n\tilde{H}^{I\times_n E}(x)\right)^{\alpha}(a).$$

We do this in two steps.

First, we can easily check that the squares in the following diagrams are pullbacks.

$$\begin{array}{ccccc} J\times_{n\alpha} E & \xrightarrow{\beta=\langle\alpha\pi_1,\pi_2\rangle} & I\times_n E & \xrightarrow{\pi_2} & E \\ \pi_1^{n\alpha}\big\downarrow & & \pi_1^{n}\big\downarrow & & \pi_2^{E}\big\downarrow \\ J & \xrightarrow[\alpha]{\qquad\qquad} & I & \xrightarrow[n]{\quad} & N \end{array} \qquad \begin{array}{ccc} J\times_{n\alpha} E & \xrightarrow{\langle\pi_1,\pi_2\rangle} & J\times E \\ \pi_1^{n\alpha}\big\downarrow & & \big\downarrow \mathrm{Id}\times\pi_2^E \\ J & \xrightarrow{\langle \mathrm{Id}, n\alpha\rangle} & J\times N \end{array}$$

Since $\mathbb{C}$ has N-finite products, the left square in the left diagram above gives us the commutative square below on the left. Moreover, the right diagram above can be considered to be a pullback in $\mathcal{S}/J$ (if we add in all the obvious arrows to J). In this way, $\pi_1^{n\alpha}$ is an arrow in $\mathcal{S}/J$ from itself to $T_J = \mathrm{Id} : J \to J$. Therefore, since $\Sigma_J^*\mathbb{D}$ has N-finite products and $H^J(a)$ preserves them (by assumption), we get the commutative square below on the right.

$$\begin{array}{ccc} \mathcal{C}^{I\times_n E} & \xrightarrow{\beta^*} & \mathcal{C}^{J\times_{n\alpha} E} \\ \Pi_{\pi_1^n}\big\downarrow & & \big\downarrow\Pi_{\pi_1^{n\alpha}} \\ \mathcal{C}^{I} & \xrightarrow{\quad\alpha^*\quad} & \mathcal{C}^{J} \end{array} \qquad \begin{array}{ccc} \mathcal{C}^{J\times_{n\alpha} E} & \xrightarrow{\Pi_{\pi_1^{n\alpha}}} & \mathcal{C}^{J} \\ \left(H^J(a)\right)^{\pi_1^{n\alpha}}\big\downarrow & & \big\downarrow\left(H^J(a)\right)^{T_J} \\ \mathcal{D}^{J\times_{n\alpha} E} & \xrightarrow[\Pi_{\pi_1^{n\alpha}}]{} & \mathcal{D}^{J} \end{array}$$

Now, using these squares and the definition of $\tilde{H}$ in remark 9.2.1, we compute

$$\begin{aligned} \left(\tilde{H}^I\left(\Pi_n x\right)\right)^{\alpha}(a) &= \left(H^J(a)\right)^{T_J}\left(\alpha^*\Pi_n x\right) \\ &\cong \left(H^J(a)\right)^{T_J}\left(\Pi_{n\alpha}(\beta^* x)\right) \\ &\cong \Pi_{n\alpha}\Big((H^J(a))^{\pi_1^{n\alpha}}(\beta^* x)\Big) \end{aligned}$$

$$\cong \Pi_{n\alpha}\Big(\Big(H^{J\times_{n\alpha}E}\big((\pi_1^{n\alpha})^* a\big)\Big)^{T_{J\times_{n\alpha}E}}(\beta^* x)\Big).$$

This last isomorphism uses the coherence of H with the pullback $(\pi_1^{n\alpha})^*$.

For the second step, we apply the construction of N-finite products in $[\![\mathbb{A},\mathbb{D}]\!]$ that was given in the proof of proposition 6.5.5. We also apply the definition of $\tilde{H}$ from remark 9.2.1. We get

$$\begin{aligned}\Big(\Pi_n \tilde{H}^{I\times_n E}(x)\Big)^{\alpha}(a) &= \Pi_{n\alpha}\Big(\big(\tilde{H}^{I\times_n E}(x)\big)^{\beta}\big((\pi_1^{n\alpha})^* a\big)\Big)\\ &\cong \Pi_{n\alpha}\Big(\Big(H^{J\times_{n\alpha}E}\big((\pi_1^{n\alpha})^* a\big)\Big)^{T_{J\times_{n\alpha}E}}(\beta^* x)\Big).\end{aligned}$$

This is what we wanted. ∎

We are now ready to show that (most) filtered colimits of flex functors are flex. The following theorem can be viewed as an analogue of proposition 6.2.2 in [Borc 94a].

Theorem 9.5.3. Let $\mathcal{S}$ be an arithmetic universe, let C be a filtered internal category in $\mathcal{S}$ with a terminal object, and let $\mathbb{D}$ be a flex $\mathcal{S}$-indexed category.

Then, for every indexed functor $G : C \to [\![\mathbb{D},\mathbb{S}]\!]_{flex}$, the indexed functor $\mathrm{colim}(\iota\circ G) : \mathbb{D} \to \mathbb{S}$ is flex. (Here, ι is the inclusion $[\![\mathbb{D},\mathbb{S}]\!]_{flex} \hookrightarrow [\![\mathbb{D},\mathbb{S}]\!]$.)

Proof. By calculating colimits pointwise (proposition 9.2.2), we find that $\mathrm{colim}(\iota \circ G)$ is isomorphic to $\mathrm{colim} \circ \tilde{G}$, where $\mathrm{colim} : [\![C,\mathbb{S}]\!] \to \mathbb{S}$ and $\tilde{G} : \mathbb{D} \to [\![C,\mathbb{S}]\!]$ is the indexed functor corresponding to G via remark 9.2.1. Corollary 9.4.2 and proposition 9.5.2 show (respectively) that colim and $\tilde{G}$ are both flex, so their composite is too. ∎

Chapter 10

The Kan extension of indexed functors

Contents

In this chapter, we continue the progress of chapter 9 towards proving the equivalence of categories

$$[[\![C^{op}, \mathbb{S}]\!], F^*\mathbb{T}]_{flexcc} \simeq [C, F^*\mathbb{T}]_{flex},$$

which is an important part of proving the third main result of this thesis (theorem 17.2.1). In this chapter, we ignore F and $\mathcal{T}$ (this is left to chapter 11), and we prove corollary 10.5.2, which states that

$$[[\![C^{op}, \mathbb{S}]\!], \mathbb{S}]_{flexcc} \simeq [C, \mathbb{S}]_{flex}.$$

We start by defining the Yoneda functor Y (in section 10.1) and the Kan extension $\mathrm{Lan}_Y G$ (in section 10.2), which allow us to define the functors that make up this equivalence. Then, the only difficult step to obtaining this equivalence is showing that G being flex implies $\mathrm{Lan}_Y G$ is flex (theorem 10.5.1). This step requires two intermediate results: lemma 10.3.5 (proven in section 10.3), and lemma 10.4.3 (proven in section 10.4). We put everything together in section 10.5.

10.1 The Yoneda functor

We start this section by defining the Hom functor.

Definition. Let C be an internal category in $\mathcal{S}$. Then there is an indexed functor

$$\mathrm{Hom} : C^{op} \times C \to \mathbb{S}$$

defined as follows. For any $I \in \mathcal{S}$ and $c, c' : I \to C_0$, the object $\mathrm{Hom}^I(c, c') \in \mathcal{S}/I$ is defined by the following pullback.

$$\begin{array}{ccc} \mathrm{Hom}^I(c,c') & \longrightarrow & C_1 \\ \big\downarrow & \lrcorner & \big\downarrow \langle \mathrm{dom}, \mathrm{cod} \rangle \\ I & \xrightarrow[\langle c, c' \rangle]{} & C_0 \times C_0 \end{array}$$

We use this functor to define the Yoneda embeddings.

Definition. Since indexed functor categories are exponentials (proposition 6.5.4), the indexed functor Hom induces two indexed functors Y, Y^* as below.

$$Y : C \to [\![C^{op}, \mathbb{S}]\!] \qquad\qquad Y^* : C^{op} \to [\![C, \mathbb{S}]\!]$$

We call these the *Yoneda embeddings.* Using the notation of remark 9.2.1, we have $Y = \widehat{\mathrm{Hom}}$ and $Y^* = \widehat{\mathrm{Hom}'}$. Note in particular that $\mathrm{ev} \circ (\mathrm{Id} \times Y) = \mathrm{Hom}$ and $\mathrm{ev} \circ (\mathrm{Id} \times Y^*) = \mathrm{Hom} \circ \sigma$ (where $\sigma : C \times C^{op} \to C^{op} \times C$ is the swap arrow).

Proposition 10.1.1. Let $\mathcal{S}$ be a category with finite limits, and let C be an internal category in $\mathcal{S}$.

- If C has finite limits, then $Y : C \to [\![C^{op}, \mathbb{S}]\!]$ preserves them.
- If f is an exponentiable arrow in $\mathcal{S}$ and C has f-indexed products, then $Y : C \to [\![C^{op}, \mathbb{S}]\!]$ preserves f-indexed products.

Note: in the second point, f being exponentiable in $\mathcal{S}$ implies $\mathbb{S}$ has f-indexed products (proposition 6.3.6), and so $[\![C^{op}, \mathbb{S}]\!]$ does too (proposition 6.4.1); thus, the statement that Y preserves these products is meaningful.

Proof. For finite limits, we do the example of binary products. Given $I \in \mathcal{S}$ and $c_1, c_2 \in C(I)$, we must show that $Y^I(c_1 \times c_2) \cong Y^I(c_1) \times Y^I(c_2)$. These are indexed functors $\Sigma_I^* C^{op} \to \Sigma_I^* \mathbb{S}$, so we will show that, for each $(\alpha : J \to I) \in \mathcal{S}/I$ and each $d \in C^{op}(J)$, we have

$$\Big(Y^I(c_1 \times c_2)\Big)^{\alpha}(d) \cong \Big(Y^I(c_1) \times Y^I(c_2)\Big)^{\alpha}(d).$$

Since $Y = \widehat{\mathrm{Hom}}$, we can use the proof of proposition 6.5.4 to obtain an explicit description of Y based on Hom. Moreover, we know that products in functor categories are computed pointwise (by proposition 6.4.1). Finally, we know α preserves binary products by the assumption on C. So, we compute

$$\begin{aligned}\Big(Y^I(c_1 \times c_2)\Big)^\alpha(d) &= \mathrm{Hom}^J(d, \alpha^*(c_1 \times c_2)) \cong \mathrm{Hom}^J(d, \alpha^* c_1 \times \alpha^* c_2),\\ \Big(Y^I(c_1) \times Y^I(c_2)\Big)^\alpha(d) &\cong \Big(Y^I(c_1)\Big)^\alpha(d) \times \Big(Y^I(c_2)\Big)^\alpha(d)\\ &\cong \mathrm{Hom}^J(d, \alpha^* c_1) \times \mathrm{Hom}^J(d, \alpha^* c_2).\end{aligned}$$

(Note that in the last part, the product is in $\mathcal{S}/J$.) So, to establish the desired isomorphism, we essentially need to show that for any $d, c_1, c_2 \in C(I)$, we have

$$\mathrm{Hom}^I(d, c_1 \times c_2) \cong \mathrm{Hom}^I(d, c_1) \times_I \mathrm{Hom}^I(d, c_2).$$

These are objects of $\mathcal{S}/I$, so this is not quite the isomorphism defining the product $c_1 \times c_2$ (because that isomorphism is between external hom-sets). However, the situation is closely related.

To analyze this situation, we'll use our internal language; specifically, we'll use the characterizations of finite limits given in proposition 7.3.1. A context for the object on the left is $T = (i : I,\ f, g : C_1 \mid f : d(i) \to c_1(i),\ g : d(i) \to c_2(i))$; by the universal property of products, there exists a unique term $h : C_1$ in the context T such that $h : d(i) \to (c_1 \times c_2)(i)$ and $\pi_1 h = f, \pi_2 h = g$. But then (i, h) is a term whose type is the object on the left, and this gives us an arrow

$$\mathrm{Hom}^I(d, c_1) \times_I \mathrm{Hom}^I(d, c_2) \to \mathrm{Hom}^I(d, c_1 \times c_2).$$

We can construct the reverse arrow in the same way, and we see that these objects are isomorphic because of the universal property of $c_1 \times c_2$.

For f-indexed products, a similar technique applies. Using the construction of products in functor categories given by proposition 6.5.5, we eventually just need to show an isomorphism of the following form. (Here, $c \in C(B)$, $\alpha : I \to A$, $d \in C(I)$, and $\pi_1^\alpha, \pi_2^\alpha$ come from forming the pullback $I \times_A B$ of α and f.)

$$\mathrm{Hom}^I\left(d,\ \Pi_{\pi_1^\alpha}\big((\pi_2^\alpha)^* c\big)\right) \cong \Pi_{\pi_1^\alpha}\Big(\mathrm{Hom}^{I \times_A B}\big((\pi_1^\alpha)^* d,\ c\big)\Big)$$

This is just an internal version of the external isomorphism defining the indexed product. ∎

We end this section with the following fact, which states that (co)presheaves can be expressed as a colimit of representables. This fact comes from lemma B2.5.7 in [John 02]; there, it is expressed with different notation, but it's not hard to check that its statement implies the fact below.

Fact 10.1.2. Let $\mathcal{S}$ be a pretopos, and let C be an internal category in $\mathcal{S}$. Let $F : C \to \mathbb{S}$, and consider the morphism $\pi^{op} : (\mathrm{Elt}F)^{op} \to C^{op}$. Then $F \cong \mathrm{colim}(Y^* \circ \pi^{op})$.

Dually, if $E : C^{op} \to \mathbb{S}$, then $E \cong \mathrm{colim}(Y \circ \pi)$, where $\pi : \mathrm{Elt}^{op}E \to C$.

10.2 The Left Yoneda Kan Extension

In this section, we define an indexed functor $\mathrm{Lan}_Y\, F$ based on an indexed functor F, which is the analogue of the left Kan extension of F along the Yoneda functor Y. This construction is inspired by the one given in the proof of corollary B2.5.8 in [John 02]. The definition uses the category of elements of a copresheaf, which we recall from section 7.4.

Definition. Let $\mathcal{S}$ be a category with finite limits, let $\mathbb{D}$ be a cocomplete $\mathcal{S}$-indexed category, let C be an internal category in $\mathcal{S}$, and let $F : C \to \mathbb{D}$. Then we can construct an indexed functor $\mathrm{Lan}_Y\, F : [\![C^{op}, \mathbb{S}]\!] \to \mathbb{D}$ as follows.

First, we define the ordinary underlying functor $[C^{op}, \mathbb{S}] \to \mathcal{D}$. An object $E \in [C^{op}, \mathbb{S}]$ is mapped to

$$\mathrm{colim}\left(\mathrm{Elt}^{op}E \xrightarrow{\pi} C \xrightarrow{F} \mathbb{D} \right).$$

Next, for some $I \in \mathcal{S}$, we want to define $(\mathrm{Lan}_Y\, F)^I : [\Sigma_I^* C^{op}, \Sigma_I^* \mathbb{S}] \to \mathcal{D}^I$. By remark 6.5.3, we know that $\Sigma_I^* C^{op}$ is the internal category $I^*(C^{op}) \cong (I^*C)^{op}$ in $\mathcal{S}/I$, and $\Sigma_I^*\mathbb{S}$ is the canonical self-indexing of $\mathcal{S}/I$. Therefore, a functor $E \in [\Sigma_I^* C^{op}, \Sigma_I^*\mathbb{S}]$ is a presheaf on I^*C, so we can form

$$\mathrm{Elt}^{op}E \xrightarrow{\pi} \Sigma_I^* C.$$

Then, we have $\Sigma_I^* F : \Sigma_I^* C \to \Sigma_I^* \mathbb{D}$, and the underlying ordinary category of $\Sigma_I^*\mathbb{D}$ is $\mathcal{D}^I$. So, we map E to

$$\mathrm{colim}\left(\mathrm{Elt}^{op}E \xrightarrow{\pi} I^*C \xrightarrow{\Sigma_I^* F} \Sigma_I^*\mathbb{D} \right),$$

and this is indeed an object of $\mathcal{D}^I$.

The motivation for introducing the functor $\mathrm{Lan}_Y\, F$ is the following fact, which comes from corollary B2.5.8 in [John 02]. (Note that the *cc* subscript indicates that we are referring to cocontinuous indexed functors.)

Fact 10.2.1. Let $\mathcal{S}$ be a pretopos, let C be an internal category in $\mathcal{S}$, and let $\mathbb{D}$ be a cocomplete $\mathcal{S}$-indexed category. Then the functor

$$- \circ Y : [[\![C^{op}, \mathbb{S}]\!], \mathbb{D}]_{cc} \to [C, \mathbb{D}]$$

is part of an equivalence of categories whose inverse is given by $F \mapsto \mathrm{Lan}_Y F$.

We must comment on the this fact since, in [John 02], the functors establishing the equivalence of categories are not included in the statement. However, they are given in the proof of this statement, and although they are expressed slightly differently, it's not hard to check that they are the same as the ones we have given.

In section 10.5 and chapter 11, we will strengthen this equivalence of categories.

10.3 An alternative characterization

The goal of this section is to prove a technical result: lemma 10.3.5. This lemma provides an alternate description of the indexed functor $\mathrm{Lan}_Y F$ for a copresheaf F, and is crucial for the proof of theorem 10.5.1. To arrive at the proof of this lemma, we start with a very basic fact.

Proposition 10.3.1. Using the notation of remark 9.2.1, if $F = G \circ (\mathrm{Id} \times f)$, then $\widehat{F} \cong \widehat{G} \circ f$. This is illustrated by the diagrams below.

$$\mathbb{B} \times \mathbb{C} \xrightarrow{\mathrm{Id} \times f} \mathbb{B} \times \mathbb{D} \xrightarrow{G} \mathbb{E}, \quad F : \mathbb{B} \times \mathbb{C} \to \mathbb{E} \qquad \mathbb{C} \xrightarrow{f} \mathbb{D} \xrightarrow{\widehat{G}} [\![\mathbb{B}, \mathbb{E}]\!], \quad \widehat{F} : \mathbb{C} \to [\![\mathbb{B}, \mathbb{E}]\!]$$

Similarly, if $F = G \circ (f \times \mathrm{Id})$, then $\widehat{F'} \cong \widehat{G'} \circ f$.

$$\mathbb{B} \times \mathbb{D} \xrightarrow{f \times \mathrm{Id}} \mathbb{C} \times \mathbb{D} \xrightarrow{G} \mathbb{E}, \quad F : \mathbb{B} \times \mathbb{D} \to \mathbb{E} \qquad \mathbb{B} \xrightarrow{f} \mathbb{C} \xrightarrow{\widehat{G'}} [\![\mathbb{D}, \mathbb{E}]\!], \quad \widehat{F'} : \mathbb{B} \to [\![\mathbb{D}, \mathbb{E}]\!]$$

Proof. To show $\widehat{G} \circ f \cong \widehat{F}$, it suffices to show that their images are isomorphic under the equivalence $[\mathbb{C}, [\![\mathbb{B}, \mathbb{E}]\!]] \to [\mathbb{B} \times \mathbb{C}, \mathbb{E}]$, which is given by $H \mapsto \mathrm{ev} \circ (\mathrm{Id}_{\mathbb{B}} \times H)$. Applying this map to the diagram on the right gives the diagram on the left, because $\widehat{F}$ is defined such that $\mathrm{ev} \circ (\mathrm{Id} \times \widehat{F}) = F$ (and similarly for G). The second case is done similarly. ∎

The following result is the key step for proving lemma 10.3.5. It is a version of proposition 3.8.1 in [Borc 94a].

Proposition 10.3.2. Let $\mathcal{S}$ be a pretopos such that $\mathbb{S}$ is cocomplete. The two following functors $[C, \mathbb{S}] \times [C^{op}, \mathbb{S}] \to \mathcal{S}$ are isomorphic.

$$(F, E) \mapsto \operatorname{colim}\left((\mathrm{Elt}F)^{op} \xrightarrow{\pi^{op}} C^{op} \xrightarrow{E} \mathbb{S}\right)$$
$$(F, E) \mapsto \operatorname{colim}\left(\mathrm{Elt}^{op}E \xrightarrow{\pi} C \xrightarrow{F} \mathbb{S}\right)$$

Proof. Fix F and E, and consider the following indexed functor, which we'll denote H.

$$(\mathrm{Elt}F)^{op} \times \mathrm{Elt}^{op}E \xrightarrow{\pi^{op} \times \pi} C^{op} \times C \xrightarrow{\mathrm{Hom}} \mathbb{S}$$

We claim that $\operatorname{colim}(\operatorname{colim}(\widehat{H}))$ is isomorphic to the image of the first functor, and that $\operatorname{colim}(\operatorname{colim}(\widehat{H'}))$ is isomorphic to the image of the second functor. If we can show this, then we're done by theorem 9.3.2.

First, write $K = \mathrm{Hom} \circ (\pi^{op} \times \mathrm{Id})$, so that $H = K \circ (\mathrm{Id} \times \pi)$. Proposition 10.3.1 tells us that $\widehat{H} = \widehat{K} \circ \pi$, and proposition 9.2.2 tells us that

$$\operatorname{colim}(\widehat{H'}) = \operatorname{colim} \circ \widehat{H} = \operatorname{colim} \circ \widehat{K} \circ \pi = \operatorname{colim}(\widehat{K'}) \circ \pi.$$

Applying proposition 10.3.1 again gives us $\widehat{K'} = \widehat{\mathrm{Hom}'} \circ \pi^{op} = Y^* \circ \pi^{op}$ (see the definition of Y^*), and so we can apply fact 10.1.2 to find that $\operatorname{colim}(\widehat{K'}) = \operatorname{colim}(Y^* \circ \pi^{op}) = F$. Thus, all together,

$$\operatorname{colim}(\operatorname{colim}(\widehat{H'})) = \operatorname{colim}(\operatorname{colim}(\widehat{K'}) \circ \pi) = \operatorname{colim}(F \circ \pi),$$

which is precisely the image of the second functor.

The argument for the first functor works in exactly the same way. ∎

We now just make note of two short facts before arriving at the proof of lemma 10.3.5.

Remark 10.3.3. Let $\mathcal{S}$ be a category with finite limits, and let $\mathbb{C}, \mathbb{D}, \mathbb{E}$ be $\mathcal{S}$-indexed categories. Let $f : \mathbb{C} \to \mathbb{D}$ be an indexed functor. Then there is an indexed functor $\phi : [\![\mathbb{D}, \mathbb{E}]\!] \to [\![\mathbb{C}, \mathbb{E}]\!]$ defined by the universal property of the exponential (see the diagram below), which we denote $- \circ f$.

$$\begin{array}{ccc}
\mathbb{C} \times [\![\mathbb{C}, \mathbb{E}]\!] & \xrightarrow{\mathrm{ev}} & \mathbb{E} \\
{\scriptstyle \mathrm{Id} \times (- \circ f)} \uparrow & & \uparrow {\scriptstyle \mathrm{ev}} \\
\mathbb{C} \times [\![\mathbb{D}, \mathbb{E}]\!] & \xrightarrow[f \times \mathrm{Id}]{} & \mathbb{D} \times [\![\mathbb{D}, \mathbb{E}]\!]
\end{array}$$

The functor $\phi = (- \circ f)$ is given by the following formula.

$$\phi^I : [\Sigma_I^* \mathbb{D}, \Sigma_I^* \mathbb{E}] \to [\Sigma_I^* \mathbb{C}, \Sigma_I^* \mathbb{E}]$$

$$g \mapsto g \circ \Sigma_I^* f$$

The ordinary functor $[\mathbb{D}, \mathbb{E}] \to [\mathbb{C}, \mathbb{E}]$ underlying ϕ is just the usual $- \circ f$.

Proposition 10.3.4. Let $\mathcal{S}$ be a category with finite limits, let C be an internal category in $\mathcal{S}$, and let $F : C \to \mathbb{S}$. Then we have an isomorphism as follows.

$$\begin{array}{ccc} I^*(\mathrm{Elt}F)^{op} & \cong & (\mathrm{Elt}(\Sigma_I^* F))^{op} \\ & {\scriptstyle I^*\pi^{op}} \searrow \quad \swarrow {\scriptstyle \pi^{op}} & \\ & I^*C^{op} \cong (I^*C)^{op} & \end{array}$$

Proof. We give a sketch of the proof. First, note that the isomorphism $I^*C^{op} \cong (I^*C)^{op}$ means we can drop the "ops" in this proof. So, we need to show that $I^*(\mathrm{Elt}F)$ and $\mathrm{Elt}(\Sigma_I^* F)$ are the same, in a way compatible with the projection to I^*C.

Recall (see section 7.2) that a copresheaf F corresponds to its diagram, which includes $\beta : F_0 \to C_0$. The object of objects of $\mathrm{Elt}F$ is F_0, and the functor π acts on objects by the map β. That means that $I^*\pi : I^*(\mathrm{Elt}F) \to I^*C$ is the following map.

$$\begin{array}{ccc} I \times F_0 & \xrightarrow{\mathrm{Id} \times \beta} & I \times C_0 \\ & {\scriptstyle \pi_1} \searrow \quad \swarrow {\scriptstyle \pi_1} & \\ & I & \end{array}$$

Next, recall that $\beta : F_0 \to C_0$ is given by $F^{C_0}(\mathrm{Id}_{C_0})$. So, if we consider $\Sigma_I^* F : I^*C \to \mathbb{S}/I$, the part of its diagram corresponding to β is

$$(\Sigma_I^* F)^{I \times C_0 \to I}(\mathrm{Id}) \qquad = \qquad \mathrm{Id} \times \beta : I \times F_0 \to I \times C_0.$$

Thus, the functor $\pi : \mathrm{Elt}(\Sigma_I^* F) \to I^*C$ is just determined by $\mathrm{Id} \times \beta$. It's therefore clear that $\mathrm{Elt}(\Sigma_I^* F)$ is the same as $I^*(\mathrm{Elt}F)$ (they are both just $I \times F_0$) in a way that respects the projection to I^*C (both projections are just $\mathrm{Id} \times \beta$). ∎

Finally, we arrive at the main result of this section.

Lemma 10.3.5. Let $\mathcal{S}$ be a pretopos such that $\mathbb{S}$ is cocomplete, let C be an internal category in $\mathcal{S}$, and let $F : C \to \mathbb{S}$. Consider the indexed functor $\mathrm{Lop}(F) : [\![C^{op}, \mathbb{S}]\!] \to \mathbb{S}$ defined as the composite below. (The functor $- \circ \pi^{op}$ is as in remark 10.3.3.)

$$[\![C^{op}, \mathbb{S}]\!] \xrightarrow{- \circ \pi^{op}} [\![(\mathrm{Elt}F)^{op}, \mathbb{S}]\!] \xrightarrow{\mathrm{colim}} \mathbb{S}.$$

Then $\mathrm{Lop}(F) \cong \mathrm{Lan}_Y F$.

Proof. By definition, and by recalling remark 6.5.3, the I slice of $\mathrm{Lop}(F)$ is given by

$$\begin{aligned} &[I^*C^{op}, \mathbb{S}/I] \to \mathcal{S}/I \\ &\qquad E \mapsto \mathrm{colim}\left(I^*(\mathrm{Elt}F)^{op} \xrightarrow{I^*\pi^{op}} I^*C^{op} \xrightarrow{E} \mathbb{S}/I\right). \end{aligned}$$

By proposition 10.3.4, this is just

$$E \mapsto \mathrm{colim}\left((\mathrm{Elt}(\Sigma_I^* F))^{op} \xrightarrow{\pi^{op}} (I^*C)^{op} \xrightarrow{E} \mathbb{S}/I\right).$$

On the other hand, the I slice of $\mathrm{Lan}_Y F$ is, by definition,

$$E \mapsto \mathrm{colim}\left(\mathrm{Elt}^{op} E \xrightarrow{\pi} I^*C \xrightarrow{\Sigma_I^* F} \mathbb{S}/I\right).$$

If we let $F_I = \Sigma_I^* F$, note that $F_I \in [I^*C, \mathbb{S}/I]$ (it's a copresheaf). So the fact that these two things are isomorphic is just an application of proposition 10.3.2. ∎

10.4 Revisiting the evaluation map

In this section, we prove another technical result in order to arrive at theorem 10.5.1. Essentially, we show that for any $c \in \mathbb{C}$, the evaluation map $\mathrm{ev}_c : [\![\mathbb{C}, \mathbb{D}]\!] \to \mathbb{D}$ is flex. However, this statement is not very formal, since $\mathbb{C}$ here should be an indexed category. Therefore, we first need the following remarks to formalize this idea.

Remark 10.4.1. As a special case of proposition 6.5.4, we have an equivalence of categories

$$[\mathbb{C},\ [\![[\![\mathbb{C}, \mathbb{D}]\!], \mathbb{D}]\!]] \simeq [[\![\mathbb{C}, \mathbb{D}]\!] \times \mathbb{C},\ \mathbb{D}].$$

Note that $\mathrm{ev} : \mathbb{C} \times [\![\mathbb{C}, \mathbb{D}]\!] \to \mathbb{D}$ is an object of the category on the right (modulo symmetry of $\times$), so it has a corresponding functor on the left, that we denote

$$\mathrm{ev}_{(-)} : \mathbb{C} \to [\![[\![\mathbb{C}, \mathbb{D}]\!], \mathbb{D}]\!].$$

(In the non-indexed case, this functor takes $c \in C$ and maps it to the "evaluation functor" $\mathrm{ev}_c : [C, D] \to D : F \mapsto F(c)$.)

Explicitly, this functor can be described as follows. For any $I \in \mathcal{S}$, the functor $\mathrm{ev}^I_{(-)} : \mathcal{C}^I \to [\Sigma^*_I[\![\mathbb{C},\mathbb{D}]\!],\ \Sigma^*_I\mathbb{D}]$ maps $x \in \mathcal{C}^I$ to an $\mathcal{S}/I$-indexed functor $\mathrm{ev}^I_x : \Sigma^*_I[\![\mathbb{C},\mathbb{D}]\!] \to \Sigma^*_I\mathbb{D}$. For any $(\alpha : J \to I) \in \mathcal{S}/I$, we get a fiber $(\mathrm{ev}^I_x)^\alpha : [\Sigma^*_J\mathbb{C}, \Sigma^*_J\mathbb{D}] \to \mathcal{D}^J$. Then,

$$(\mathrm{ev}^I_x)^\alpha = \mathrm{ev}^J(\alpha^* x, -).$$

Remark 10.4.2. Given a functor $F : \mathbb{C} \to [\![\mathbb{D},\mathbb{E}]\!]$, what does it mean to say "for each $c \in \mathbb{C}$, $F(c)$ is flex"? Since there is an inclusion $i : [\![\mathbb{D},\mathbb{E}]\!]_{flex} \hookrightarrow [\![\mathbb{D},\mathbb{E}]\!]$, we should interpret this statement to mean that F factors through $[\![\mathbb{D},\mathbb{E}]\!]_{flex}$.

What does this mean concretely? The indexed functor F consists of, for each $I \in \mathcal{S}$, a functor $F^I : \mathcal{C}^I \to [\mathbb{D},\mathbb{E}]^I$. We'd like to check that the image of F is actually in $[\mathbb{D},\mathbb{E}]^I_{flex}$. Since this is a full subcategory of $[\mathbb{D},\mathbb{E}]$, it just suffices to show that $F^I(x) \in [\![\mathbb{D},\mathbb{E}]\!]^I_{flex}$ for each x.

Therefore, to check that F factors through $[\![\mathbb{D},\mathbb{E}]\!]_{flex}$, we need to show that for each $I \in \mathcal{S}$ and $x \in \mathcal{C}^I$, the functor $F^I(x) \in [\mathbb{D},\mathbb{E}]^I$ is flex.

Finally, we arrive at the main result. Note that in the statement of this lemma, it makes sense to write $[\![[\![\mathbb{C},\mathbb{S}]\!],\mathbb{S}]\!]_{flex}$, because both $\mathbb{S}$ and $[\![\mathbb{C},\mathbb{S}]\!]$ are flex. Indeed, $\mathbb{S}$ is flex by proposition 6.3.7, and $[\![\mathbb{C},\mathbb{S}]\!]$ is flex by proposition 6.5.5.

Lemma 10.4.3. Let $\mathcal{S}$ be an arithmetic universe, and let C be an internal category. The indexed functor $\mathrm{ev}_{(-)} : C \to [\![[\![C,\mathbb{S}]\!],\mathbb{S}]\!]$ of remark 10.4.1 factors through $[\![[\![C,\mathbb{S}]\!],\mathbb{S}]\!]_{flex}$.

Proof. By remark 10.4.2, we need to show that for each $I \in \mathcal{S}$ and each $x \in C(I)$, the indexed functor $\mathrm{ev}^I_x : \Sigma^*_I[\![C,\mathbb{S}]\!] \to \Sigma^*_I\mathbb{S}$ is flex.

We start by checking that this functor preserves finite limits. So, for each $(\alpha : J \to I) \in \mathcal{S}/I$, we must check that $(\mathrm{ev}^I_x)^\alpha : [C,\mathbb{S}]^J \to \mathcal{S}/J$ preserves finite limits. By remark 10.4.1, this functor is simply

$$\begin{aligned}(\mathrm{ev}^I_x)^\alpha = \mathrm{ev}^J(\alpha^* x, -) : [\Sigma^*_J C, \Sigma^*_J\mathbb{S}] &\to \mathcal{S}/J \\ F &\mapsto F^{T_J}(\alpha^* x).\end{aligned}$$

This functor preserves finite limits because they are computed pointwise in $[\Sigma^*_J C, \Sigma^*_J\mathbb{S}]$.

Next, we show that $\mathrm{ev}^I_x : \Sigma^*_I[\![C,\mathbb{S}]\!] \to \Sigma^*_I\mathbb{S}$ preserves N-finite products. This is an $\mathcal{S}/I$-indexed functor, so recall that the version of π^E_2 in $\mathcal{S}/I$ is $\mathrm{Id} \times \pi^E_2 : I \times E \to I \times N$ (an arrow between the first projections). Now, let $(\alpha : J \to I) \in \mathcal{S}/I$; an arrow from α to $(\pi_1 : I \times N \to I)$ in $\mathcal{S}/I$ is of the form $\langle \alpha, n\rangle : J \to I \times N$, and a pullback along this arrow in $\mathcal{S}/I$ is still a pullback of π^E_2 in $\mathcal{S}$ (see remark 9.5.1). In short, we have the following diagram.

$$\begin{tikzcd} J \times_n E \arrow[rr] \arrow[dd, "\pi_1^n"'] \arrow[dr] & & I \times E \arrow[dl, "\pi_1"'] \arrow[dd, "\mathrm{Id} \times \pi_2^E"] \\ & I & \\ J \arrow[ur, "\alpha"] \arrow[rr, "\langle \alpha, n \rangle"'] & & I \times N \arrow[ul, "\pi_1"] \end{tikzcd}$$

So, π_1^n is an arrow in $\mathcal{S}/I$ from $\alpha\pi_1^n$ to α, and it is a pullback of $\mathrm{Id} \times \pi_2^E$. Therefore, we must show that the following diagram commutes.

$$\begin{tikzcd} {[C, \mathbb{S}]^{J \times_n E}} \arrow[r, "(\mathrm{ev}_x^I)^{\alpha\pi_1^n}"] \arrow[d, "\Pi_{\pi_1^n}"'] & \mathcal{S}/J \times_n E \arrow[d, "\Pi_{\pi_1^n}"] \\ {[C, \mathbb{S}]^J} \arrow[r, "(\mathrm{ev}_x^I)^{\alpha}"] & \mathcal{S}/J \end{tikzcd}$$

Let $F \in [C, \mathbb{S}]^{J \times_n E}$. For the top path, we just use the definition of the evaluation functor to compute

$$\Pi_{\pi_1^n}\Big((\mathrm{ev}_x^I)^{\alpha\pi_1^n}(F)\Big) = \Pi_{\pi_1^n}\Big(\mathrm{ev}^{J \times_n E}\big((\alpha\pi_1^n)^* x, F\big)\Big) = \Pi_{\pi_1^n}\Big(F^{T_{J \times_n E}}\big((\pi_1^n)^* \alpha^* x\big)\Big).$$

For the bottom path, we again use the definition of the evaluation functor to compute

$$(\mathrm{ev}_x^I)^{\alpha}(\Pi_{\pi_1^n} F) = \mathrm{ev}^J(\alpha^* x, \Pi_{\pi_1^n} F) = \big(\Pi_{\pi_1^n} F\big)^{T_J}(\alpha^* x).$$

To proceed, we must use the construction of products in functor categories that we gave in the proof of proposition 6.5.5. This involves taking the pullback of π_1^n along T_J. But T_J is just the identity on J, so the corresponding pullback square is just the one below.

$$\begin{tikzcd} J \times_n E \arrow[r, "\mathrm{Id} = T_{J \times_n E}"] \arrow[d, "\pi_1^n"'] & J \times_n E \arrow[d, "\pi_1^n"] \\ J \arrow[r, "\mathrm{Id} = T_J"] & J \end{tikzcd}$$

Therefore, when we apply the construction, we end up with the same result as the top path, as desired. ∎

10.5 Kan extension preserves limits

In this section, we show that if a copresheaf F is flex, then so is $\mathrm{Lan}_Y F$. We use this fact to strengthen the equivalence of categories of fact 10.2.1.

Theorem 10.5.1. Let $\mathcal{S}$ be an arithmetic universe. Let C be an internal category in $\mathcal{S}$, and let $F : C \to \mathbb{S}$. If C and F are flex, then so is $\mathrm{Lan}_Y F : [\![C^{op}, \mathbb{S}]\!] \to \mathbb{S}$.

Proof. Note that $\mathbb{S}$ is cocomplete by proposition 6.3.7. Consider the following indexed functor (since F is a copresheaf, we can form its category of elements).

$$(\mathrm{Elt}F)^{op} \xrightarrow{\pi^{op}} C^{op} \xrightarrow{\mathrm{ev}_{(-)}} [\![[\![C^{op}, \mathbb{S}]\!], \mathbb{S}]\!] \qquad (*)$$

Since $(\mathrm{Elt}F)^{op}$ is an internal category, we can take the colimit of this indexed functor. We claim its colimit is $\mathrm{Lan}_Y F$ and that it is flex.

Step 1: We claim the colimit of $(*)$ is $\mathrm{Lan}_Y F$.

By proposition 9.2.2, colimits in functor categories are computed pointwise, so let's examine the pointwise colimit of $(*)$. That is, we must see what functor $[\![C^{op}, \mathbb{S}]\!] \to [\![(\mathrm{Elt}F)^{op}, \mathbb{S}]\!]$ it corresponds to. By applying the definitions of $\mathrm{ev}_{(-)}$ (remark 10.4.1) and $- \circ \pi$ (remark 10.3.3), it's easy to check that $(*)$ corresponds to the indexed functor $- \circ \pi^{op}$, and so its colimit corresponds (by 9.2.2) to the functor

$$[\![C^{op}, \mathbb{S}]\!] \xrightarrow{- \circ \pi^{op}} [\![(\mathrm{Elt}F)^{op}, \mathbb{S}]\!] \xrightarrow{\mathrm{colim}} \mathbb{S}.$$

By lemma 10.3.5, this functor is isomorphic to $\mathrm{Lan}_Y F$, so we have shown that $\mathrm{Lan}_Y F$ is isomorphic to the colimit of $(*)$.

Step 2: We claim the colimit of $(*)$ is flex.

By lemma 10.4.3, the image of $(*)$ lands in $[\![[\![C^{op}, \mathbb{S}]\!], \mathbb{S}]\!]_{flex}$ Moreover, since C has and F preserves finite limits, proposition 7.4.4 tells us that $(\mathrm{Elt}F)^{op}$ is strongly filtered (hence filtered; see remark 7.4.1). So, the colimit of $(*)$ is a filtered colimit of flex functors. Thus, theorem 9.5.3 tells us that the colimit is flex too. Since the colimit is $\mathrm{Lan}_Y F$, we conclude that $\mathrm{Lan}_Y F$ is flex. ∎

Corollary 10.5.2. Let $\mathcal{S}$ be an arithmetic universe, and let C be a flex internal category in $\mathcal{S}$. Then the functor

$$- \circ Y : [[\![C^{op}, \mathbb{S}]\!], \mathbb{S}]_{flexcc} \to [C, \mathbb{S}]_{flex}$$

is part of an equivalence of categories whose inverse is given by $F \mapsto \mathrm{Lan}_Y F$.

Proof. Fact 10.2.1 already tells us that $- \circ Y$ and $F \mapsto \mathrm{Lan}_Y F$ form an equivalence of categories between $[[\![C^{op}, \mathbb{S}]\!], \mathbb{S}]_{cc}$ and $[C, \mathbb{S}]$. So, it just remains to show that, when C is flex, both of these maps preserve flex indexed functors. For $- \circ Y$, this is just a result of

Y preserving limits (see proposition 10.1.1). For $F \mapsto \operatorname{Lan}_Y F$, this is done in theorem 10.5.1. ∎

Chapter 11

Base changes

Contents

The goal of this section is to strengthen the equivalence of categories of corollary 10.5.2. Specifically, we want to adjust the codomain: instead of an equivalence

$$[[\![C^{op}, \mathbb{S}]\!], \mathbb{S}]_{flexcc} \simeq [C, \mathbb{S}]_{flex},$$

we want an equivalence

$$[[\![C^{op}, \mathbb{S}]\!], F^*\mathbb{T}]_{flexcc} \simeq [C, F^*\mathbb{T}]_{flex},$$

where $\mathbb{T}$ is the canonical indexing of some AU $\mathcal{T}$, and we have an AU functor $F : \mathcal{S} \to \mathcal{T}$. In order to make this adjustment, we will need some technical results about applying the "change of base" F^*, and how it interacts with the other parts of this equivalence.

In section 11.1, we present the equivalence of categories $[C, F^*\mathbb{D}] \simeq [F(C), \mathbb{D}]$; in section 11.2, we establish a naturality result for the mapping $G \mapsto \mathrm{Lan}_Y\, G$; and in section 11.3, we put these results together to get the desired equivalence.

11.1 An adjunction-like result

In this section, we'll look at an equivalence of categories of the form $[C, F^*\mathbb{D}] \simeq [F(C), \mathbb{D}]$. In order to state this precisely, we first need to consider the following proposition.

Proposition 11.1.1. Let $\mathcal{S}$ and $\mathcal{T}$ be categories with finite limits, let $F : \mathcal{S} \to \mathcal{T}$ preserve finite limits, and let C be an internal category in $\mathcal{S}$. Then there is an indexed functor

$$F_C : C \to F^*(F(C))$$

where $F_C^I : C(I) \to F(C)(F(I))$ maps $p : I \to C_0$ to $F(p) : F(I) \to F(C_0)$.

Moreover, F_C defines a natural transformation between the functors

$$\mathrm{Id}, F^*F : \mathbf{Cat}(\mathcal{S}) \to \mathfrak{CAT}_{\mathcal{S}}.$$

Here, $\mathbf{Cat}(\mathcal{S})$ is the category of internal categories in $\mathcal{S}$, and $\mathfrak{CAT}_{\mathcal{S}}$ is the category of $\mathcal{S}$-indexed categories.

Proof. The fact that each F_C is a well-defined indexed functor is easy to check. Naturality is also easy. ∎

We are now ready for the equivalence of categories. We present it as the following fact, which comes from [John 02] (corollary B2.3.14).

Fact 11.1.2. Let $\mathcal{S}$ and $\mathcal{T}$ be categories with finite limits, let $F : \mathcal{S} \to \mathcal{T}$ preserve finite limits, let C be an internal category in $\mathcal{S}$, and let $\mathbb{D}$ be a $\mathcal{T}$-indexed category.

Then diagrams of shape $F(C)$ in $\mathbb{D}$ are the same as diagrams of shape C in $F^*\mathbb{D}$. In particular, we get an equivalence of categories

$$[C, F^*\mathbb{D}]_{\mathcal{S}} \simeq [F(C), \mathbb{D}]_{\mathcal{T}}.$$

The functor $[F(C), \mathbb{D}]_{\mathcal{T}} \to [C, F^*\mathbb{D}]_{\mathcal{S}}$ in this equivalence is given by $G \mapsto F^*G \circ F_C$, where $F_C : C \to F^*F(C)$ is as in proposition 11.1.1.

Note: the latter part of this fact is not included in [John 02], but it is easy to check, using the correspondence between functors and diagrams given in lemma B2.3.13 of [John 02].

For our purposes, we need to know that this equivalence preserves limits in a special case.

Proposition 11.1.3. Let $\mathcal{S}, \mathcal{T}$ be arithmetic universes, let $F : \mathcal{S} \to \mathcal{T}$ be an AU functor, and let C be a flex internal category in $\mathcal{S}$. Then $F(C)$ is flex, and the equivalence

$$[C, F^*\mathbb{T}]_{\mathcal{S}} \simeq [F(C), \mathbb{T}]_{\mathcal{T}}$$

of fact 11.1.2 preserves flex functors both ways. That is, it restricts to an equivalence

$$[C, F^*\mathbb{T}]_{\mathcal{S},flex} \simeq [F(C), \mathbb{T}]_{\mathcal{T},flex}.$$

Proof. We give a sketch of the proof. Roughly, a functor $G : C \to F^*\mathbb{T}$ preserves finite limits if for any finite category K, the canonical natural transformation α_1 in the square below is an isomorphism.

$$
\begin{array}{ccc}
C^K & \xrightarrow{G^K} & (F^*\mathbb{T})^K \\
{\scriptstyle \lim}\downarrow & \overset{\alpha_1}{\Nearrow} & \downarrow{\scriptstyle \lim} \\
C & \xrightarrow[G]{} & F^*\mathbb{T}
\end{array}
$$

However, if $G' : F(C) \to \mathbb{T}$ is the indexed functor corresponding to G, it's clear that α_1 induces the transformation α_2 in the following diagram.

$$
\begin{array}{ccc}
F(C^K) & \xrightarrow{(G')^K} & D^K \\
{\scriptstyle \lim}\downarrow & \Nearrow_{\alpha_2} & \downarrow{\scriptstyle \lim} \\
F(C) & \xrightarrow[G']{} & D
\end{array}
$$

Therefore, α_1 is an isomorphism if and only if α_2 is an isomorphism, which yields the desired equivalence.

For N-finite products, the argument is similar. Indeed, having N-finite products is equivalent to a similar condition as above, but where K is replaced by an *internally finite* object (which we'll see in chapter 13). Otherwise, the argument works the same. ∎

Finally, we apply the equivalence of categories fact to the computation of colimits.

Proposition 11.1.4. Let $\mathcal{S}$ and $\mathcal{T}$ be categories with finite limits, let $F : \mathcal{S} \to \mathcal{T}$ preserve finite limits, let C be an internal category in $\mathcal{S}$, and let $\mathbb{D}$ be a cocomplete $\mathcal{T}$-indexed category.

Let $G \in [C, F^*\mathbb{D}]$, and let $G' \in [F(C), \mathbb{D}]$ be the functor corresponding to G via the equivalence of fact 11.1.2. Then $\operatorname{colim}(G) \cong \operatorname{colim}(G')$ as objects of $\mathcal{D}$.

Proof. Colimits are defined at the level of diagrams (see the definition in [John 02], below proposition B2.3.20), and fact 11.1.2 tells us the diagrams corresponding to G and G' are the same. ∎

11.2 Kan extension naturality

In this section, we show a kind of naturality property of the Kan extension $F \mapsto \operatorname{Lan}_Y F$. Recall from section 10.2 that the Kan extension allows us to extend $F \in [C, \mathbb{D}]$ to $\operatorname{Lan}_Y F \in [[\![C^{op}, \mathbb{S}]\!], \mathbb{D}]$ for any cocomplete $\mathcal{S}$-indexed category $\mathbb{D}$. The goal of this section is to relate these extensions when $\mathbb{D}$ changes from $\mathbb{S}$ to $F^*\mathbb{T}$. In order to do this, we need to define a functor $F_+ : [\![C, \mathbb{S}]\!] \to F^*[\![F(C), \mathbb{T}]\!]$, and in order to do that, we need to talk about discrete opfibrations.

Per definition B2.5.1 of [John 02], given a category $\mathcal{S}$ with finite limits and an internal category C in $\mathcal{S}$, a *discrete opfibration* on C is an internal functor $f : D \to C$ such that the following square is a pullback.

$$\begin{array}{ccc} D_1 & \xrightarrow{\text{dom}} & D_0 \\ \downarrow{f_1} & & \downarrow{f_0} \\ C_1 & \xrightarrow{\text{dom}} & C_0 \end{array}$$

We write $\mathbf{doFib}(\mathcal{S})$ for the category of internal categories in $\mathcal{S}$ and discrete opfibrations between them. Discrete opfibrations are important because they correspond to copresheaves via the category of elements.

Fact 11.2.1. Let $\mathcal{S}$ be a category with finite limits, and let C be an internal category in $\mathcal{S}$. Then $[C, \mathbb{S}] \simeq \mathbf{doFib}(\mathcal{S})/C$ via the map $E \mapsto (\mathrm{Elt}E \xrightarrow{\pi} C)$.

The above fact is proposition B2.5.3 in [John 02]. There, the category of elements isn't referred to by that name, but the construction of the functor $[C, \mathbb{S}] \to \mathbf{doFib}(\mathcal{S})/C$ is the same as the definition we gave in section 7.4. With this fact, we can construct the functor F_+.

Remark 11.2.2. Let $\mathcal{S}, \mathcal{T}$ be categories with finite limits, and let $F : \mathcal{S} \to \mathcal{T}$ preserve finite limits. If $f : D \to C$ is a discrete opfibration in $\mathcal{S}$, then $F(f) : F(D) \to F(C)$ is a discrete opfibration in $\mathcal{T}$ – indeed, the pullback square is preserved by F.

Therefore, we have a functor $F : \mathbf{doFib}(\mathcal{S})/C \to \mathbf{doFib}(\mathcal{T})/F(C)$. By fact 11.2.1, this gives us a functor $F_+ : [C, \mathbb{S}]_{\mathcal{S}} \to [F(C), \mathbb{T}]_{\mathcal{T}}$. By construction, this means that given $E \in [C, \mathbb{S}]$, we have an isomorphism $\mathrm{Elt}(F_+(E)) \cong F(\mathrm{Elt}E)$ that commutes with the projections (see below).

$$\begin{array}{ccc} \mathrm{Elt}(F_+(E)) & \xrightarrow{\pi} & \\ \cong & & F(C) \\ F(\mathrm{Elt}E) & \xrightarrow{F(\pi)} & \end{array}$$

Moreover, F_+ extends to an $\mathcal{S}$-indexed functor $F_+ : [\![C, \mathbb{S}]\!] \to F^*[\![F(C), \mathbb{T}]\!]$. Indeed, on a fiber I, we should have a functor

$$F_+^I : [I^*C, \mathbb{S}/I]_{\mathcal{S}} \to [F(I)^*F(C), \mathbb{T}/F(I)]_{\mathcal{T}}.$$

Note that I^*C is an internal category in $\mathcal{S}/I$, and $\mathbb{S}/I$ is the canonical indexing of $\mathcal{S}/I$ (see remark 6.5.3). It's also easy to check that $F(I^*C) \cong F(I)^*F(C)$, so we're just in a special case of the context that lets us define F_+ (coming from the functor $F/I : \mathcal{S}/I \to \mathcal{T}/F(I)$, which preserves finite limits by proposition 2.1.7).

We now arrive at the naturality result we promised.

Proposition 11.2.3. Let $\mathcal{S}, \mathcal{T}$ be categories with finite limits, let $F : \mathcal{S} \to \mathcal{T}$ preserve finite limits, let C be an internal category in $\mathcal{S}$, and let $\mathbb{D}$ be a cocomplete $\mathcal{T}$-indexed category. Then the following diagram commutes (up to isomorphism), where the left side is the equivalence from fact 11.1.2, and the indexed functor F_+ is from remark 11.2.2.

$$\begin{array}{ccc} [F(C), \mathbb{T}]_{\mathcal{T}} & \xrightarrow{G \mapsto \operatorname{Lan}_Y G} & \big[[\![F(C)^{op}, \mathbb{T}]\!],\ \mathbb{T} \big]_{\mathcal{T}} \\ \simeq \big\downarrow & & \big\downarrow G \mapsto F^* G \circ F_+ \\ [C, F^*\mathbb{T}]_{\mathcal{S}} & \xrightarrow[G \mapsto \operatorname{Lan}_Y G]{} & \big[[\![C^{op}, \mathbb{S}]\!],\ F^*\mathbb{T} \big]_{\mathcal{S}} \end{array}$$

Proof. Let $G \in [F(C), \mathbb{T}]_{\mathcal{T}}$. Recall from fact 11.1.2 that the equivalence here is given by $G \mapsto F^*G \circ F_C$, so the bottom path yields $\operatorname{Lan}_Y(F^*G \circ F_C)$, while the top path yields $F^*(\operatorname{Lan}_Y G) \circ F_+$. We claim these indexed functors are isomorphic. We will show this is true for the underlying ordinary functors; the other fibers are done similarly.

So, we check that the two underlying functors $[C^{op}, \mathbb{S}] \to \mathcal{T}$ are the same. For the bottom path, we have $(\operatorname{Lan}_Y(F^*G \circ F_C))^{\mathbb{1}}$; by definition of Lan_Y, this maps $E \in [C^{op}, \mathbb{S}]$ to

$$\operatorname{colim}\Big(\mathrm{Elt}^{op} E \xrightarrow{\pi} C \xrightarrow{F_C} F^*F(C) \xrightarrow{F^*G} F^*\mathbb{T}\Big).$$

For the top path, we have $(F^* \operatorname{Lan}_Y G)^{\mathbb{1}} \circ F_+$. So, this functor maps E to

$$\operatorname{colim}\Big(\mathrm{Elt}^{op}(F_+E) \xrightarrow{\pi} F(C) \xrightarrow{G} \mathbb{B}\Big).$$

Now, as observed in remark 11.2.2, we can replace $\mathrm{Elt}^{op}(F_+E)$ in the above formula by $F(\mathrm{Elt}^{op} E)$.

$$\operatorname{colim}\Big(F(\mathrm{Elt}^{op} E) \xrightarrow{F(\pi)} F(C) \xrightarrow{G} \mathbb{D}\Big).$$

But now, we are taking the colimit of an object in $[F(\mathrm{Elt}^{op} E), \mathbb{D}]$. By proposition 11.1.4, the colimit of this functor is the same as the colimit of its partner in $[C, F^*\mathbb{D}]$, which we obtain by applying F^* and pre-composing with F_C (see fact 11.1.2). In short, the above colimit is isomorphic to

$$\operatorname{colim}\Big(\mathrm{Elt}^{op} E \xrightarrow{F_{\mathrm{Elt}^{op} E}} F^*F(\mathrm{Elt}^{op} E) \xrightarrow{F^*F(\pi)} F^*F(C) \xrightarrow{F^*G} F^*\mathbb{D}\Big).$$

But now, it's clear that to show this matches the result of the bottom path, we just need to show that the following diagram commutes.

$$\begin{array}{ccc} F^*F(\mathrm{Elt}^{op}E) & \xrightarrow{F^*F(\pi)} & F^*F(C) \\ {\scriptstyle F_{\mathrm{Elt}^{op}E}}\uparrow & & \uparrow{\scriptstyle F_C} \\ \mathrm{Elt}^{op}E & \xrightarrow{\quad\pi\quad} & C \end{array}$$

This is just naturality of F_C (proposition 11.1.1), so we're done. ∎

To finish off this section, we establish the following fact about the functor F_+.

Proposition 11.2.4. Let $\mathcal{S}, \mathcal{T}$ be arithmetic universes, let $F : \mathcal{S} \to \mathcal{T}$ be an AU morphism, and let C be an internal category in $\mathcal{S}$. Then the indexed functor $F_+ : [\![C, \mathbb{S}]\!] \to F^*[\![F(C), \mathbb{T}]\!]$ from remark 11.2.2 is flex.

Proof. First, it's easy to check that the following diagram commutes. .

$$\begin{array}{ccc} [C, \mathbb{S}]_{\mathcal{S}} & \xrightarrow{F_+} & [F(C), \mathbb{T}]_{\mathcal{T}} \\ \Delta\uparrow & & \Delta\uparrow \\ \mathcal{S} & \xrightarrow{\quad F\quad} & \mathcal{T} \end{array}$$

Therefore, F_+ extends to an indexed functor $\Delta^*[C, \mathbb{S}] \to F^*\Delta^*[F(C), \mathbb{T}]$, and we find that this is the same as the indexed functor F_+ we defined in remark 11.2.2 (using the equivalence $\Delta^*[\mathbb{C}, \mathbb{S}] \simeq [\![\mathbb{C}, \mathbb{S}]\!]$ of proposition 6.5.2). By proposition 6.3.8, to show F_+ is flex, it suffices to show that the underlying functor F_+ preserves finite limits and list objects.

To show that F_+ preserves finite limits and list objects, we first re-examine its definition. If we look at the proof of fact 11.2.1 in [John 02] (proposition B2.5.3), we find that F_+ acts on objects as follows. Let $E : C \to \mathbb{S}$ be a copresheaf; it gives us an object $E^{C_0}(\mathrm{Id}_{C_0}) = (\beta : E_0 \to C_0) \in \mathcal{S}/C_0$. Applying F gives $F(\beta) : F(E_0) \to F(C_0)$, and then we define $(F_+E)^J : F(C)(J) \to \mathcal{T}/J$ for $J \in \mathcal{T}$ as follows.

$$q : J \to F(C_0) \qquad \mapsto \qquad \begin{array}{ccc} J \times_q F(E_0) & \longrightarrow & F(E_0) \\ \downarrow & & \downarrow{\scriptstyle F(\beta)} \\ J & \xrightarrow{\quad q\quad} & F(C_0) \end{array}$$

Using this description, it's easy (if tedious) to show that F_+ preserves finite limits and list objects. We show the case of binary products as an example.

Let $D, E : C \to \mathbb{S}$ be copresheaves. Their product $D \times E$ is computed pointwise (see proposition 6.4.1), so

$$(D \times E)^{C_0}(\mathrm{Id}_{C_0}) = D^{C_0}(\mathrm{Id}_{C_0}) \times E^{C_0}(\mathrm{Id}_{C_0}) = (\beta : D_0 \times_{C_0} E_0 \to C_0).$$

Since F preserves pullbacks, applying F gives $F(D_0) \times_{F(C_0)} F(E_0) \to F(C_0)$, which is just the product $F(D^{C_0}(\mathrm{Id}_{C_0})) \times F(E^{C_0}(\mathrm{Id}_{C_0}))$ in $\mathcal{T}/C_0$. Now, for $J \in \mathcal{T}$ and $q \in F(C)(J)$, the pullback along q^* preserves products, so

$$\begin{aligned}(F_+(D \times E))^J(q) &= q^*\Big(F(D^{C_0}(\mathrm{Id}_{C_0})) \times F(E^{C_0}(\mathrm{Id}_{C_0}))\Big) \\ &= q^*\Big(F(D^{C_0}(\mathrm{Id}_{C_0}))\Big) \times q^*\Big(F(E^{C_0}(\mathrm{Id}_{C_0}))\Big) \\ &= (F_+D)^J(q) \times (F_+E)^J(q) \\ &= (F_+D \times F_+E)^J(q).\end{aligned}$$

Thus, $F_+(D \times E) \cong F_+D \times F_+E$, as desired. ∎

11.3 Another equivalence of categories

Finally, we obtain another equivalence of categories.

Corollary 11.3.1. Let $\mathcal{S}, \mathcal{T}$ be arithmetic universes, let $F : \mathcal{S} \to \mathcal{T}$ be an AU functor, and let C be a flex internal category in $\mathcal{S}$. Then the functor

$$- \circ Y : [[\![C^{op}, \mathbb{S}]\!], F^*\mathbb{T}]_{flexcc} \to [C, F^*\mathbb{T}]_{flex}$$

is part of an equivalence of categories whose inverse is given by $G \mapsto \mathrm{Lan}_Y G$.

Proof. Fact 10.2.1 already tells us that $- \circ Y$ and $G \mapsto \mathrm{Lan}_Y G$ form an equivalence of categories between $[[\![C^{op}, \mathbb{S}]\!], F^*\mathbb{T}]_{cc}$ and $[C, F^*\mathbb{T}]$. So, it just remains to show that, when C is flex, both of these maps preserve flex indexed functors. For $- \circ Y$, this is just a result of Y preserving limits (see proposition 10.1.1).

So, it remains to show that if $G : C \to F^*\mathbb{T}$ is flex, then so is the functor $\mathrm{Lan}_Y G : [\![C^{op}, \mathbb{S}]\!] \to F^*\mathbb{T}$. By proposition 11.2.3, we know that $\mathrm{Lan}_Y G$ is isomorphic to $F^*(\mathrm{Lan}_Y(G')) \circ F_+$, where $G' : F(C) \to \mathbb{T}$ is the indexed functor corresponding to G. If G is flex, then so is G' (by proposition 11.1.3). Then, theorem 10.5.1 tells us that $\mathrm{Lan}_Y G'$ is flex; we can apply this theorem because we're considering the mapping

$$[F(C), \mathbb{T}]_{\mathcal{T}} \xrightarrow{G' \mapsto \mathrm{Lan}_Y G'} \big[[\![F(C)^{op}, \mathbb{T}]\!],\ \mathbb{T}\big]_{\mathcal{T}}.$$

Finally, $F^*(\mathrm{Lan}_Y(G'))$ is flex by proposition 6.2.6, and proposition 11.2.4 tells us F_+ is flex. Therefore, the composite $F^*(\mathrm{Lan}_Y(G')) \circ F_+ \cong \mathrm{Lan}_Y G$ is also flex, as desired. ∎

Chapter 12

ES indexed categories

Contents

Let $\mathbb{C}$ be an $\mathcal{S}$-indexed category. In remark B1.4.3 of [John 02], Johnstone notes that "the two conditions '$\mathbb{C}$ has $\mathcal{S}$-indexed products' and '$\mathbb{C}$ has finite limits' are not entirely independent". Indeed, even if $\mathbb{C}$ has all $\mathcal{S}$-indexed products, it is not guaranteed that $\mathbb{C}$ has finite products. However, there are some assumptions we can place on $\mathcal{S}$ and $\mathbb{C}$ so that having $\mathcal{S}$-indexed products implies having finite products. This is described briefly in remark B1.4.3 of [John 02]; in this chapter, we elaborate on this topic.

For this chapter, we will assume $\mathcal{S}$ is lextensive. In order to obtain a link between $\mathcal{S}$-indexed products and finite products, we will assume that $\mathbb{C}$ satisfies the *extensive stack conditions* (ES conditions), which we define in section 12.2 (after proving some simple facts in section 12.1). We use this terminology because the assumptions we use are related to the assumption that $\mathbb{C}$ is a stack for the extensive topology on $\mathcal{S}$, but we will not discuss stacks directly.

In section 12.3, we establish the link between $\mathcal{S}$-indexed products and finite products for indexed categories satisfying the ES conditions. Then, in section 12.4, we show that having N-finite products is a sufficient condition for having finite products (proposition 12.4.1, one of the main results of this chapter). Perhaps surprisingly, it turns out that the converse of this fact is true for internal categories; this is the content of theorem 12.4.2. Finally, in section 12.5, we establish a similar link between preserving indexed products and preserving finite products.

The goal of this chapter is to simplify discussions of flex categories and functors, and to use theorem 12.4.2 to show that exponentials of finite objects are finite (theorem 15.1.1).

12.1 Facts about lextensive categories

Before we introduce the notion of "extensive stack conditions", we will prove some preliminary results about extensive categories. Recall that a category $\mathcal{C}$ is extensive if, for any $I, J \in \mathcal{C}$, the canonical functor $\mathcal{C}/I \times \mathcal{C}/J \to \mathcal{C}/(I+J)$ is an equivalence. The first result shows that, in a lextensive category, we can explicitly describe the inverse of the canonical functor.

Proposition 12.1.1. If $\mathcal{C}$ a lextensive category, then for any objects I, J, the functors $+ : \mathcal{C}/I \times \mathcal{C}/J \to \mathcal{C}/(I+J)$ and $\langle i_I^*, i_J^* \rangle : \mathcal{C}/(I+J) \to \mathcal{C}/I \times \mathcal{C}/J$ form an equivalence.

Proof. We must show that the two composites are naturally isomorphic to the identity. On one hand, consider the functor $+ \circ \langle i_I^*, i_J^* \rangle$. Given $f : X \to I + J$, we pull back along the inclusion maps to get the diagram below.

$$\begin{array}{ccccc} X_I & \longrightarrow & X & \longleftarrow & X_J \\ \big\downarrow f_I & & \big\downarrow f & & \big\downarrow f_J \\ I & \longrightarrow & I+J & \longleftarrow & J \end{array}$$

According to fact 2.1.2, extensivity implies the top row is a coproduct, so X "is" $X_I + X_J$, which means that f "is" $f_I + f_J$. Thus $\langle i_I^*, i_J^* \rangle \circ +$ is "equal" (isomorphic) to the identity functor.

On the other hand, consider $\langle i_I^*, i_J^* \rangle \circ +$. Given $f : X \to I$ and $g : Y \to J$, we form the coproduct diagram below.

$$\begin{array}{ccccc} X & \longrightarrow & X+Y & \longleftarrow & Y \\ \big\downarrow f & & \big\downarrow f+g & & \big\downarrow g \\ I & \longrightarrow & I+J & \longleftarrow & J \end{array}$$

Extensivity now implies that the two squares are pullbacks. Thus f "is" $i_I^*(f+g)$ and g "is" $i_J^*(f+g)$, so as before $+ \circ \langle i_I^*, i_J^* \rangle$ is isomorphic to the identity. ∎

The following is a technical result we'll use later.

Proposition 12.1.2. In a lextensive category, for any $x : I \to J$, the following diagram is a pullback.

$$\begin{array}{ccc} I+I & \xrightarrow{x+x} & J+J \\ \downarrow{\scriptstyle \nabla} & & \downarrow{\scriptstyle \nabla} \\ I & \xrightarrow{\quad x \quad} & J \end{array}$$

(Here, ∇ is the codiagonal.)

Proof. We first note that it suffices to prove this in the case $J = \mathbb{1}$. Indeed, if we have $x : I \to J$, then we can form the following commutative diagram.

$$\begin{array}{ccccc} I+I & \xrightarrow{x+x} & J+J & \xrightarrow{\langle\rangle+\langle\rangle} & \mathbb{1}+\mathbb{1} \\ \downarrow{\scriptstyle \nabla} & & \downarrow{\scriptstyle \nabla} & & \downarrow{\scriptstyle \nabla} \\ I & \xrightarrow{\quad x \quad} & J & \xrightarrow{\quad \langle\rangle \quad} & \mathbb{1} \end{array}$$

The right square is the diagram where x is $\langle\rangle : J \to \mathbb{1}$, and the whole rectangle is the diagram where x is $\langle\rangle : I \to \mathbb{1}$. If these two squares are pullbacks, then the left square is too, by the pasting law.

So, we focus our attention on the case $x = \langle\rangle : I \to \mathbb{1}$. But now that $J = \mathbb{1}$, this is equivalent to saying that the following diagram is a product.

$$I \xleftarrow{\nabla} I+I \xrightarrow{\langle\rangle+\langle\rangle} \mathbb{1}+\mathbb{1}$$

Moreover, note that $\big\langle \nabla,\ \langle\rangle+\langle\rangle \big\rangle : I+I \to I \times (\mathbb{1}+\mathbb{1})$ is the same as the canonical map $(I \times \mathbb{1}) + (I \times \mathbb{1}) \to I \times (\mathbb{1}+\mathbb{1})$. Since lextensive categories are distributive, this map is an isomorphism, which implies that the above diagram is indeed a product. ∎

12.2 Extensive stack conditions

Definition. Let $\mathcal{S}$ be a lextensive category, and let $\mathbb{C}$ be an $\mathcal{S}$-indexed category. We say $\mathbb{C}$ *satisfies the extensive stack conditions* (ES conditions) if the two following conditions hold.

- $\mathcal{C}^{\mathbb{O}} \simeq \mathbb{1}$ (where $\mathbb{O}$ is the initial object of $\mathcal{S}$)
- For any $I, J \in \mathcal{S}$, the functor $\langle i_1^*, i_2^* \rangle : \mathcal{C}^{I+J} \to \mathcal{C}^I \times \mathcal{C}^J$ is an equivalence of categories.

We also call $\mathbb{C}$ an ES $\mathcal{S}$-indexed category (ES for "extensive stack").

The first important result characterizes the important classes of ES indexed categories.

Proposition 12.2.1. Let $\mathcal{S}$ be a lextensive category.

- The canonical indexing $\mathbb{S}$ is an ES $\mathcal{S}$-indexed category. Moreover, if $\mathcal{T}$ is a lextensive category and $F : \mathcal{S} \to \mathcal{T}$ preserves finite limits and finite coproducts, then $F^*\mathbb{T}$ is an ES $\mathcal{S}$-indexed category.
- Any internal category of $\mathcal{S}$ is an ES $\mathcal{S}$-indexed category.

Proof. We start with the first point. We only show that $F^*\mathbb{T}$ satisfies the ES conditions; the result then follows for $\mathbb{S}$.

We have two ES conditions to check. First, we must show that $(F^*\mathbb{T})^{\mathbb{O}} \simeq \mathbb{1}$. Since F preserves finite coproducts, we have $(F^*\mathbb{T})^{\mathbb{O}} = \mathcal{T}/F(\mathbb{O}) = \mathcal{T}/\mathbb{O}$. But since $\mathcal{T}$ is lextensive, $\mathcal{T}/\mathbb{O} \simeq \mathbb{1}$ (see corollary 2.9 of [Carb 93]).

Second, we must show that $\langle i_1^*, i_2^* \rangle : \mathcal{T}/F(I+J) \to \mathcal{T}/F(I) \times \mathcal{T}/F(J)$ is an equivalence of categories. Since F preserves finite coproducts, this is just asking whether $\langle i_1^*, i_2^* \rangle : \mathcal{T}/(A+B) \to \mathcal{T}/A \times \mathcal{T}/B$ is an equivalence, where $A = F(I)$ and $B = F(J)$. This is true because $\mathcal{T}$ is lextensive (see proposition 12.1.1). This finishes the first point.

Now, we prove the second point. Let C be an internal category of $\mathcal{S}$. Again, we have two ES conditions to check. First, we note that $C(\mathbb{O}) \simeq \mathbb{1}$. Indeed, $C(\mathbb{O})$ has precisely one object and one arrow, because the arrows $\mathbb{O} \to C_0$ and $\mathbb{O} \to C_1$ in $\mathcal{S}$ are unique by definition.

Second, given I, J, we must show that $\langle i_1^*, i_2^* \rangle : C(I+J) \to C(I) \times C(J)$ is an equivalence; we must construct its inverse. This is easy: given objects $f : I \to C_0$ and $g : J \to C_0$ in $C(I)$ and $C(J)$, we map the pair (f, g) to $[f, g] : I + J \to C_0$, an object of $C(I+J)$. We define the action on arrows in the same way; it's easy to see that this preserves domain and codomain, and that the assignment is functorial. Moreover, it's clear that these functors are strict inverses of each other (not just up to isomorphism). So, we're done. ∎

12.3 Products when assuming ES conditions

We are now ready to see how indexed products correspond to finite products when assuming the ES conditions. As noted earlier, the following propositions are inspired by remark B1.4.3 in [John 02].

Proposition 12.3.1. Let $\mathcal{S}$ be a lextensive category, and let $\mathbb{C}$ be an ES $\mathcal{S}$-indexed category. Then the following statements are equivalent:

- $\mathbb{C}$ has $\nabla_{\mathbb{1}}$-indexed products, where $\nabla_{\mathbb{1}} : \mathbb{1} + \mathbb{1} \to \mathbb{1}$ is the codiagonal;

- $\mathbb{C}$ has binary products.

In particular, if $\mathbb{C}$ has all $\mathcal{S}$-indexed products, then it has binary products.

Proof. We start by proving the equivalence between the following two statements.

- For any pullback $g : A \to B$ of $\nabla_{\mathbb{1}}$, the functor g^* has a right adjoint Π_g.
- For each $I \in \mathcal{S}$, the category $\mathcal{C}^I$ has binary products.

By proposition 12.1.2, we know that the pullback of $\nabla_{\mathbb{1}}$ along $I \to \mathbb{1}$ is the codiagonal $\nabla_I : I + I \to I$. So, the first condition is true if and only if each ∇_I^* has a right adjoint. It remains to show that ∇_I^* has a right adjoint if and only if $\mathcal{C}^I$ has binary products. Since $\mathbb{C}$ is ES, we know that $\langle i_1^*, i_2^* \rangle : \mathcal{C}^{I+I} \to \mathcal{C}^I \times \mathcal{C}^I$ is an equivalence of categories; this implies that ∇_I^* has a right adjoint iff $\langle i_1^*, i_2^* \rangle \circ \nabla_I^*$ has one. Now, note that

$$\langle i_1^*, i_2^* \rangle \circ \nabla^* \cong \langle i_1^* \nabla^*,\ i_2^* \nabla^* \rangle \cong \langle (\nabla i_1)^*, (\nabla i_2)^* \rangle = \langle (\mathrm{Id}_I)^*, (\mathrm{Id}_I)^* \rangle \cong \langle \mathrm{Id}, \mathrm{Id} \rangle = \Delta,$$

where $\Delta : \mathcal{C}^I \to \mathcal{C}^I \times \mathcal{C}^I$ is the diagonal functor. Therefore, ∇_I^* has a right adjoint if and only if Δ has one, but this is equivalent to saying $\mathcal{C}^I$ has binary products.

Now, we check the equivalence of the proposition. To do so, we must show that, under the assumption that the pair of statements above are true, we can prove the equivalence of the following two statements:

- The Beck-Chevalley condition holds for pullbacks of $\nabla_{\mathbb{1}}$;
- The transition maps preserve products.

Let $x : I \to J$ in $\mathcal{S}$, and consider the following commutative square, which is a pullback by proposition 12.1.2.

$$\begin{array}{ccc}
I + I & \xrightarrow{x + x} & J + J \\
\downarrow{\scriptstyle \nabla_I} & & \downarrow{\scriptstyle \nabla_J} \\
I & \xrightarrow{x} & J
\end{array}$$

The arrows ∇_I and ∇_J are pullbacks of $\nabla_{\mathbb{1}}$, so by assumption the functors ∇_I^* and ∇_J^* have right adjoints. Therefore, we can form the following square, which has a canonical natural transformation in it.

$$\begin{array}{ccc}
\mathcal{C}^{I+I} & \xleftarrow{(x + x)^*} & \mathcal{C}^{J+J} \\
{\scriptstyle \Pi_\nabla}\downarrow & \Leftarrow & \downarrow{\scriptstyle \Pi_\nabla} \\
\mathcal{C}^I & \xleftarrow[x^*]{} & \mathcal{C}^J
\end{array}$$

If we throw in the equivalence $\mathcal{C}^{I+I} \simeq \mathcal{C}^I \times \mathcal{C}^I$ provided by the ES conditions, the diagram changes to the following, where Δ^* is the right adjoint of the diagonal Δ.

$$\begin{array}{ccc} \mathcal{C}^I \times \mathcal{C}^I & \xleftarrow{x^* \times x^*} & \mathcal{C}^J \times \mathcal{C}^J \\ {\scriptstyle \Delta^*}\downarrow & \overset{\psi}{\Leftarrow} & \downarrow{\scriptstyle \Delta^*} \\ \mathcal{C}^I & \xleftarrow[x^*]{} & \mathcal{C}^J \end{array}$$

Now, we remark that the canonical map ψ is an isomorphism if and only if x^* preserves products. So, if the transition maps preserve products, then the Beck-Chevalley condition holds.

Conversely, if the Beck-Chevalley condition holds for pullbacks of $\nabla_{\mathbb{1}}$, then $\mathbb{C}$ has $\nabla_{\mathbb{1}}$-indexed products. But then it also has ∇_J-indexed products because ∇_J is a pullback of $\nabla_{\mathbb{1}}$ (by proposition 12.1.2). By the same proposition, this square is a pullback. Therefore, by the Beck-Chevalley conditions (from having ∇_J-indexed products), the canonical natural transformation in the diagram above is an isomorphism. ∎

Proposition 12.3.2. Let $\mathcal{S}$ be a lextensive category, and let $\mathbb{C}$ be an ES $\mathcal{S}$-indexed category. Then the following statements are equivalent:

- $\mathbb{C}$ has Z-indexed products, where Z is the unique arrow $Z : \mathbb{O} \to \mathbb{1}$;
- $\mathbb{C}$ has terminal objects.

In particular, if $\mathbb{C}$ has all $\mathcal{S}$-indexed products, then it has terminal objects.

Proof. Before starting the proof, note that every arrow $[]_I : \mathbb{O} \to I$ is a pullback of $\mathbb{O} \to \mathbb{1}$; this is a consequence of extensivity. (We can rephrase this fact as saying that $I \times \mathbb{O} \cong \mathbb{O}$; this is a consequence of distributivity per proposition 3.2 in [Carb 93].)

First, we note that $[]_I^* : \mathcal{C}^I \to \mathcal{C}^{\mathbb{O}}$ (where $[]_I$ is a pullback of Z) has a right adjoint if and only if $\mathcal{C}^I$ has a terminal object. Indeed, by the ES property, we know that $\mathcal{C}^{\mathbb{O}} \simeq \mathbb{1}$, so $[]_I^*$ is just the unique functor $\mathcal{C}^I \to \mathbb{1}$, and for this functor to have a right adjoint is equivalent to saying $\mathcal{C}^I$ has terminal objects.

Now, suppose that each $\mathcal{C}^I$ has a terminal object, i.e. each $[]_I^*$ has a right adjoint. Given $x : I \to J$, remark that the following diagram is a pullback. Indeed, if we have A with arrows $A \to I$ and $A \to \mathbb{O}$, then extensivity implies $A \cong \mathbb{O}$.

$$\begin{array}{ccc} \mathbb{O} & \longrightarrow & \mathbb{O} \\ \downarrow & & \downarrow \\ I & \xrightarrow{x} & J \end{array}$$

If $\mathbb{C}$ has Z-indexed products, then it also has $[]_J$-indexed products (by proposition 6.2.3), so in the diagram below (where we replace $\mathcal{C}^{\mathbb{O}}$ with $\mathbb{1}$ thanks to the ES condition), the natural map ψ is an isomorphism.

$$\begin{array}{ccc} \mathbb{1} & \longleftarrow & \mathbb{1} \\ {\scriptstyle T}\downarrow & \overset{\psi}{\nwarrow} & \downarrow{\scriptstyle T} \\ \mathcal{C}^I & \underset{x^*}{\longleftarrow} & \mathcal{C}^J \end{array}$$

This implies that the terminal object is preserved by x^*.

Conversely, if $\mathbb{C}$ has terminal objects, then we can still form the above square, and ψ is an isomorphism since x^* preserves terminal objects. Therefore, the Beck-Chevalley condition holds for pullbacks of Z, so $\mathbb{C}$ has Z-indexed products. ∎

Corollary 12.3.3. Let $\mathcal{S}$ be a lextensive category, and let $\mathbb{C}$ be an $\mathcal{S}$-indexed category. If $\mathbb{C}$ is ES, then it has finite products if and only if it has $\nabla_{\mathbb{1}}$-indexed products and Z-indexed products (where $\nabla_{\mathbb{1}}$ and Z are as in propositions 12.3.1 and 12.3.2).

Proof. Immediate from propositions 12.3.1 and 12.3.2. ∎

12.4 N-finite products

Let $\mathcal{S}$ be an arithmetic lextensive category. Recall from section 6.2 that an $\mathcal{S}$-indexed category $\mathbb{C}$ has N-finite products if it has π_2^E-indexed products, where $\pi_2^E : E \to N$. We use this terminology to suggest that $\mathbb{C}$ has finite products, but in a way where "finiteness" is indexed by the natural numbers object N of $\mathcal{S}$. In this section, we show the relationship between N-finite products and finite products in the ordinary sense.

Proposition 12.4.1. Let $\mathcal{S}$ be an arithmetic lextensive category, and let $\mathbb{C}$ be an ES $\mathcal{S}$-indexed category. If $\mathbb{C}$ has N-finite products, then it has finite products.

Proof. It suffices to show that the arrows $\nabla_{\mathbb{1}} : \mathbb{1} + \mathbb{1} \to \mathbb{1}$ and $Z : \mathbb{O} \to \mathbb{1}$ are pullbacks of π_2^E. Indeed, if this is true, then $\mathbb{C}$ has both $\nabla_{\mathbb{1}}$-indexed products and Z-indexed products (by proposition 6.2.3), and so it has finite products by corollary 12.3.3.

We start with $\nabla_{\mathbb{1}}$. We claim that the following diagram is a pullback.

$$\begin{array}{ccc} \mathbb{1} + \mathbb{1} & \xrightarrow{\langle [0,1], 2 \rangle} & E \\ {\scriptstyle \nabla_{\mathbb{1}}}\downarrow & & \downarrow{\scriptstyle \pi_2^E} \\ \mathbb{1} & \xrightarrow{\quad 2 \quad} & N \end{array}$$

Briefly, we need to show that if $f : I \to N$ satisfies $f < 2$, then there is a map $g : I \to \mathbb{1}+\mathbb{1}$ such that $f = [0, 1] \circ g$. Since we're working in an extensive category, we can decompose $I = I_0+I_{>0}$ (see proposition 3.3.4), and let g be the obvious arrow $I_0+I_{>0} \to \mathbb{1}+\mathbb{1}$. Then, the equality $f = [0, 1] \circ g$ is equivalent to saying $f = 0$ in the context $(i : I \mid f(i) = 0)$ (which is obviously true) and $f = 1$ in the context $(i : I \mid f(i) > 0)$ (which is true since $f(i) > 0$ means $f(i) \geq 1$, and we assumed $f < 2$, i.e. $f \leq 1$, so we get $f(i) = 1$).

For Z, we claim that the following square is a pullback.

$$\begin{array}{ccc} \mathbb{O} & \longrightarrow & E \\ {\scriptstyle Z}\downarrow & & \downarrow{\scriptstyle \pi_2^E} \\ \mathbb{1} & \xrightarrow{0} & N \end{array}$$

This is shown in the proof of proposition 13.4.1. ∎

The above proposition is helpful, but not too surprising. More interesting would be a converse: a case where having finite products implies having N-finite products.

This is not true in general. We can form an example similar to $\mathbf{Fin}_{\mathcal{S}}$, the internal category of finite sets in $\mathcal{S}$, which we'll see in chapter 14. We modify its definition slightly to define an $\mathcal{S}$-indexed category **BFin** (B for bounded) as follows. For each $I \in \mathcal{S}$, we set

$$\mathbf{BFin}(I) = \{p : I \to N \mid \text{there exists } n : \mathbb{1} \to N \text{ such that } p < n\}.$$

This is a subcategory of $\mathbf{Fin}(I)$, and its finite products are computed the same way as those in $\mathbf{Fin}(I)$. However, **BFin** does not have N-finite products. If it did, then it would have to be equal to **Fin**, which is of course not true (for instance, $\mathrm{Id}_N \in \mathbf{Fin}(N)$ while $\mathrm{Id}_N \notin \mathbf{BFin}(N)$).

So, in general, having finite products doesn't imply having N-finite products. However, it turns out that this is true for internal categories.

Theorem 12.4.2. Let $\mathcal{S}$ be a locos, and let C be an internal category in $\mathcal{S}$. If C has finite products, then it has N-finite products.

Proof. If C has finite products, then $C(\mathbb{1})$ has a terminal object $t : \mathbb{1} \to C_0$, which gives us a term $t : C_0$ in any context; we also get the term $\mathrm{Id}_t : C_1$ in any context. Moreover, proposition 7.3.3 tells us that the binary products of C are given by arrows $\mathrm{bProdOb} : C_0 \times C_0 \to C_0$ and $\mathrm{bProdAr} : C_1 \times C_1 \to C_1$. So, we can define an arrow $\mathrm{prodOb} : L(C_0) \to C_0$ inductively by setting

$$\mathrm{prodOb}(\varnothing) = t, \qquad \mathrm{prodOb}(c :: \ell) = \mathrm{bProdOb}(c, \mathrm{prodOb}(\ell)).$$

In the same way, we get $\mathrm{prodAr} : L(C_1) \to C_1$ (with $\mathrm{prodAr}(\varnothing) = \mathrm{Id}_t$).

To show C has N-finite products, let $\pi_1^n : I \times_n E \to I$ be a pullback of $\pi_2^E : E \to N$ via some arrow $n : I \to N$. We must show that $(\pi_1^n)^*$ has a right adjoint

$\Pi_n : C(I \times_n E) \to C(I)$.

Defining the right adjoint. We define Π_n as follows: given an object $g : I \times_n E \to C_0$ of $C(I \times_n E)$, theorem 5.4.1 gives us an arrow $\overline{g} : I \to L(C_0)$, and we let $\Pi_n(g) = \mathrm{prodOb} \circ \overline{g}$. For an arrow $h : I \times_n E \to C_1$ of $C(I \times_n E)$, we let $\Pi_n(h) = \mathrm{prodAr} \circ \overline{h}$. We must show that this is well-defined: notably, we must show $\mathrm{dom}(\Pi_n(h)) = \Pi_n(\mathrm{dom} h)$ and $\mathrm{cod}(\Pi_n(h)) = \Pi_n(\mathrm{cod} h)$. To show this, we first claim that

$$\mathrm{dom}(\mathrm{prodAr}(\ell)) =_\ell \mathrm{prodOb}(L(\mathrm{dom})(\ell)).$$

This is easy to show by induction on ℓ, using the properties of bProdOb and bProdAr from proposition 7.3.3. Then, we simply calculate

$$\begin{aligned} \mathrm{dom} \circ \Pi_n(h) = \mathrm{dom} \circ \mathrm{prodAr} \circ \overline{h} = \mathrm{prodOb} \circ L(\mathrm{dom}) \circ \overline{h} &= \mathrm{prodOb} \circ \overline{\mathrm{dom} \circ h} \\ &= \Pi_n(\mathrm{dom} h). \end{aligned}$$

(The penultimate equality is a general fact; it's easy to prove with theorem 5.4.1.) The corresponding equality for the codomain is proved in the same way, so Π_n is well-defined.

Defining the counit. To prove Π_n is a right adjoint, we first define the counit. For $\ell : L(C_1)$, write $\mathrm{codProd}(\ell) = \mathrm{prodOb}(L(\mathrm{cod})(\ell))$. Then, let

$$L^* = \{\ell : L(C_1) \mid L(\mathrm{dom})(\ell) = \mathrm{const}(\mathrm{codProd}(\ell), \mathrm{len}(\ell))\},$$

the collection of lists $[f_1, ..., f_n]$ where $f_i : p \to c_i$ and $p = \Pi_i c_i$. Next, define $f : C_0 \times L^* \to L(C_1)$ using theorem 5.4.1 by setting

$$\mathrm{len}(f(c, \ell)) = \mathrm{len}(\ell), \qquad \mathrm{nth}(k, f(c, \ell)) = \mathrm{nth}(k, \ell) \circ \mathrm{proj}_2(c \times \mathrm{codProd}(\ell)).$$

This is well-defined: the arrow $\mathrm{proj}_2(c, \mathrm{codProd}(\ell))$ has codomain $\mathrm{codProd}(\ell)$, which is the domain of $\mathrm{nth}(k, \ell)$ by the definition of L^*.

Now, define $f' : C_0 \times L^* \to L(C_1)$ by $f'(c, \ell) = \mathrm{proj}_1(c, \mathrm{codProd}(\ell)) :: f(c, \ell)$. We claim that f' is an arrow $C_0 \times L^* \to L^*$. To see this, we must show that for any $(c, \ell) : C_0 \times L^*$, we have

$$L(\mathrm{dom})(f'(c, \ell)) = \mathrm{const}\big(\mathrm{codProd}(f'(c, \ell)), \mathrm{len}(f'(c, \ell))\big). \tag{12.4.1}$$

To show this, we first note that $\mathrm{codProd}(f'(c, \ell)) = c \times \mathrm{codProd}(\ell)$. Indeed, it's easy to compute $\mathrm{codProd}(f'(c, \ell)) = c \times \mathrm{codProd}(f(c, \ell))$, and we have $\mathrm{codProd}(f(c, \ell)) = \mathrm{codProd}(\ell)$ because $L(\mathrm{cod})(f(c, \ell)) = L(\mathrm{cod})(\ell)$. (This last equality can be shown with the uniqueness of theorem 5.4.1: we have $\mathrm{cod}(\mathrm{nth}(k, f(c, \ell))) = \mathrm{cod}(\mathrm{nth}(k, \ell))$.)

We can now show equation 12.4.1 by appealing to the uniqueness of theorem 5.4.1. These lists clearly have the same length, which is $\mathrm{len}(f'(c, \ell)) = 1 + \mathrm{len}(\ell)$. We must show these lists have the same k^{th} element for $k < 1 + \mathrm{len}(\ell)$; this amounts to saying the

k^{th} element of the list on the left is $\mathrm{codProd}(f'(c,\ell)) = c \times \mathrm{codProd}(\ell)$ for each k. We split this into two cases: $k = 0$ and $k > 0$. For the case $k = 0$, we have

$$\mathrm{nth}(0, L(\mathrm{dom})(f'(c,\ell))) = \mathrm{dom}(\mathrm{proj}_1(c, \mathrm{codProd}(\ell))) = c \times \mathrm{codProd}(\ell),$$

as desired. For the case $k > 0$, we have

$$\begin{aligned}\mathrm{nth}(sk, L(\mathrm{dom})(f'(c,\ell))) = \mathrm{nth}(k, L(\mathrm{dom})(f(c,\ell))) &= \mathrm{dom}(\mathrm{nth}(k, f(c,\ell))) \\ &= c \times \mathrm{codProd}(\ell),\end{aligned}$$

as desired. So, f' is indeed a map $C_0 \times L^* \to L^*$.

Finally, we define a map $\mathrm{proj} : L(C_0) \to L^*$ inductively by setting $\mathrm{proj}(\varnothing) = \varnothing$ and $\mathrm{proj}(c :: \ell) = f'(c, \mathrm{proj}(\ell))$. We note that $L(\mathrm{cod})(\mathrm{proj}(\ell)) = \ell$; we can easily show this by induction on ℓ (using $L(\mathrm{cod})(f(c,\ell)) = L(\mathrm{cod})(\ell)$, which we proved above). This also implies that $\mathrm{codProd}(\mathrm{proj}(\ell)) = \mathrm{prodOb}(\ell)$.

We can now define the counit. For any object $g \in C(I \times_n E)$, we know that $g : I \times_n E \to C_0$. This induces a map $\overline{g} : I \to L(C_0)$, and we define $\epsilon_g : I \times_n E \to C_1$ by

$$\epsilon_g(i,k) = \mathrm{nth}(k, \mathrm{proj}(\overline{g}(i))).$$

We claim this is an arrow $(\pi_1^n)^*\Pi_n g \to g$ in $C(I \times_n E)$. For the codomain, we compute

$$\begin{aligned}\mathrm{cod}(\epsilon_g(i,k)) = \mathrm{cod}(\mathrm{nth}(k, \mathrm{proj}(\overline{g}(i)))) &= \mathrm{nth}(k, L(\mathrm{cod})(\mathrm{proj}(\overline{g}(i)))) \\ &= \mathrm{nth}(k, \overline{g}(i)) = g(i,k).\end{aligned}$$

For the domain, we use the fact that proj lands in L^* to compute

$$\mathrm{dom}(\epsilon_g(i,k)) = \mathrm{nth}(k, L(\mathrm{dom})(\mathrm{proj}(\overline{g}(i)))) = \mathrm{codProd}(\mathrm{proj}(\overline{g}(i))) = \mathrm{prodOb}(\overline{g}(i)).$$

Since $\Pi_n(g) = \mathrm{prodOb} \circ \overline{g}$, and $(\pi_1^n)^*$ is just pre-composition with π_1^n, this is indeed the desired domain. So, this arrow is well-defined.

Establishing the adjunction. We now must show that Π_n is indeed a right adjoint of $(\pi_1^n)^*$ with counit $(\epsilon_g)_g$. So, let $f \in C(I)$ and $g \in C(I \times_n E)$; we must show there is a map

$$\mathrm{Hom}((\pi_1^n)^* f, g) \to \mathrm{Hom}(f, \Pi_n g)$$

which is the inverse of the map going the other way given by the counit. So, let ϕ be an arrow $(\pi_1^n)^* f \to g$ in $C(I \times_n E)$. That means that $\phi : I \times_n E \to C_1$, and this induces a map $\overline{\phi} : I \to L(C_1)$. Moreover, since $\mathrm{dom} \circ \phi = f \circ \pi_1^n$, the domains of the elements in the list $\overline{\phi}$ are all $f(i)$. So, we can say $(f(i), \overline{\phi}(i)) \in P^*$, where

$$P^* = \{c : C_0, \ell : L(C_1) \mid L(\mathrm{cod})(\ell) = \mathrm{const}(c, \mathrm{len}(\ell))\}.$$

Now, proposition 7.3.3 gives us a pairing function $C_1 \times_{\text{dom}} C_1 \to C_1$. By using induction (as we did for proj), we can extend it to a map pair : $P^* \to C_1$ which satisfies $\text{dom}(\text{pair}(c, \ell)) = c$ and $\text{cod}(\text{pair}(c, \ell)) = \text{codProd}(\ell)$. Then, we define $\psi : I \to C_1$ by

$$\psi(i) = \text{pair}(f(i), \overline{\phi}).$$

We claim that ψ is the desired arrow $f \to \Pi_n g$ in $C(I)$ which gets mapped to ϕ. First, we must check that the domain and codomain match. For the codomain, we compute

$$\text{cod}(\text{pair}(f(i), \overline{\phi}(i))) = \text{codProd}(\overline{\phi}(i)) = \text{prodOb}(\overline{g}),$$

which is what we want. For the domain, we immediately get $\text{dom}(\text{pair}(f(i), \overline{\phi}(i))) = f(i)$, as desired.

So, we have defined a map $\text{Hom}((\pi_1^n)^* f, g) \to \text{Hom}(f, \Pi_n g)$ given by $\phi \mapsto \text{pair} \circ \langle f, \overline{\phi} \rangle$; we must show it is the inverse of the map $\psi \mapsto \epsilon_g \circ (\pi_1^n)^* \psi$. If we write $\gamma : C_1 \times_{C_0} C_1 \to C_1$ for the composition arrow, then for one direction, we need to show

$$\gamma\big(\text{pair}(f(i), \overline{\phi}(i)),\ \epsilon_g(i, k)\big) = \phi(i, k).$$

By making a couple substitutions, this is equivalent to

$$\gamma\big(\text{pair}(f(i), \overline{\phi}(i),\ \text{nth}(k, \text{proj}(L(\text{cod})(\overline{\phi}(i))))\big) = \text{nth}(k, \overline{\phi}(i)).$$

Basically, this says that if we take a list $[f_1, ..., f_n]$ of arrows with common codomain, take their pairing $\langle f_i \rangle_{i=1}^n$, and then compose with the k^{th} projection, we get f_i. We will omit this proof, but it boils down to using induction combined with the similar equality of proposition 7.3.3. A similar argument works for the reverse direction. ∎

12.5 Functors preserving finite products

Now that we've discussed ES indexed categories which have finite products, we consider functors on such categories which preserve these products.

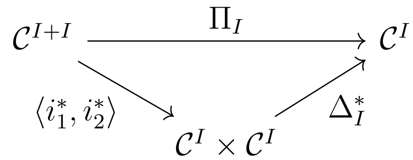
Remark 12.5.1. Let $\mathcal{S}$ be a lextensive category, let $\mathbb{C}, \mathbb{D}$ be $\mathcal{S}$-indexed categories, and let $F : \mathbb{C} \to \mathbb{D}$ be an $\mathcal{S}$-indexed functor. Then the following diagram commutes (up to isomorphism).

$$\begin{array}{ccc} \mathcal{C}^{I+I} & \xrightarrow{\langle i_1^*, i_2^* \rangle} & \mathcal{C}^I \times \mathcal{C}^I \\ {\scriptstyle F^{I+I}}\downarrow & & \downarrow{\scriptstyle F^I \times F^I} \\ \mathcal{D}^{I+I} & \xrightarrow[\langle i_1^*, i_2^* \rangle]{} & \mathcal{D}^I \times \mathcal{D}^I \end{array}$$

Indeed, this is trivial by noting that $F^I \circ i_k^* \cong i_k^* \circ F^{I+I}$ (by the axioms of an indexed functor).

Remark 12.5.2. Let $\mathcal{S}$ be a lextensive category, and let $\mathbb{C}$ be an ES $\mathcal{S}$-indexed category. For any $I \in \mathcal{S}$, let $\nabla_I : I + I \to I$ be the codiagonal, and let $\Delta_I : \mathcal{C}^I \to \mathcal{C}^I \times \mathcal{C}^I$ be the diagonal functor.

If the right adjoints Π_I, Δ_I^* of ∇_I^* and Δ_I (respectively) exist, then the following diagram commutes (up to isomorphism).

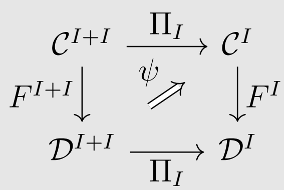


Indeed, it's easy to check that $\langle i_1^*, i_2^* \rangle \circ \nabla_I^* \cong \Delta_I$. Taking right adjoints gives $\Pi_I \circ F \cong \Delta_I^*$, where F is the right adjoint of $\langle i_1^*, i_2^* \rangle$. But since $\mathbb{C}$ is ES, we know $\langle i_1^*, i_2^* \rangle$ is an equivalence, so its right adjoint is its inverse. Therefore, we have $\Pi_I \cong \Delta_I^* \circ \langle i_1^*, i_2^* \rangle$, as desired.

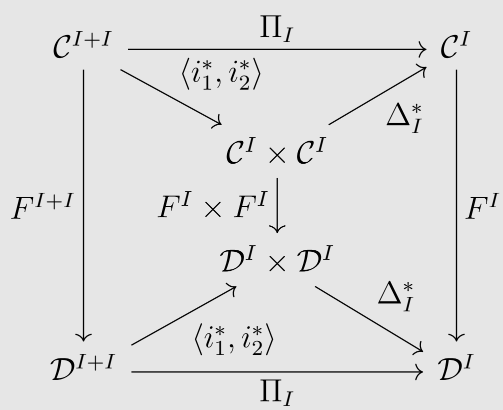
Lemma 12.5.3. Let $\mathcal{S}$ be a lextensive category, let $\mathbb{C}, \mathbb{D}$ be ES $\mathcal{S}$-indexed categories with binary products, and let $F : \mathbb{C} \to \mathbb{D}$ be an $\mathcal{S}$-indexed functor.

Then F preserves binary products if and only if it preserves $\nabla_{\mathbb{1}}$-indexed products.

Proof. By proposition 12.1.2, any pullback of $\nabla_{\mathbb{1}}$ is of the form $\nabla_I : I + I \to I$. So, F preserves $\nabla_{\mathbb{1}}$-indexed products iff for any I, the canonical natural transformation ψ in the following square is an isomorphism.



(Note that we use proposition 12.3.1, along the assumption that $\mathbb{C}, \mathbb{D}$ are ES with binary products, to get the Π_I functors.) But now let's enlarge this square and add some components.



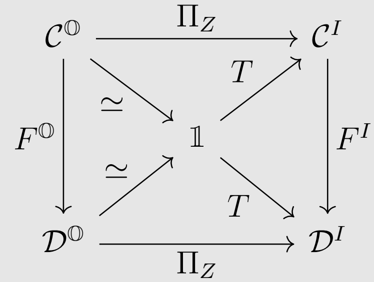
Here, $\Delta_I^* : \mathcal{C}^I \times \mathcal{C}^I \to \mathcal{C}^I$ is the right adjoint of the diagonal; it's the functor creating binary products. Remarks 12.5.1 and 12.5.2 tell us that there are natural isomorphisms in all the cells except the right square. In the right square, there is a canonical natural transformation $\phi : \Delta_I^* \circ (F^I \times F^I) \Rightarrow F^I \circ \Delta_I^*$, and its composition with the rest of the elements gives the canonical transformation of the outer square.

Moreover (by remarks 12.5.1 and 12.5.2), since the other elements of the square are isomorphisms, ψ is an isomorphism if and only if ϕ is. Therefore, F preserves $\nabla_{\mathbb{1}}$-indexed products if and only if for each I, the natural transformation ϕ is an isomorphism. But this latter condition is equivalent to saying F^I preserves binary products for each I, i.e. F preserves binary products. ∎

Lemma 12.5.4. Let $\mathcal{S}$ be a lextensive category, let $\mathbb{C}, \mathbb{D}$ be ES $\mathcal{S}$-indexed categories with terminal objects, and let $F : \mathbb{C} \to \mathbb{D}$ be an $\mathcal{S}$-indexed functor.

Then F preserves terminal objects if and only if it preserves Z-indexed products (where Z is the unique map $\mathbb{O} \to \mathbb{1}$).

Proof. A pullback of Z is $[]_I : \mathbb{O} \to I$. Then we can argue similarly to the previous lemma, with the following diagram.



The details are straightforward. ∎

Theorem 12.5.5. Let $\mathcal{S}$ be an arithmetic lextensive category, let $\mathbb{C}, \mathbb{D}$ be ES $\mathcal{S}$-indexed categories with N-finite products, and let $F : \mathbb{C} \to \mathbb{D}$ be an $\mathcal{S}$-indexed functor.

If F preserves N-finite products, then it preserves finite products.

Proof. In the proof of proposition 12.4.1, we showed that $\nabla_{\mathbb{1}}$ and Z are pullbacks of π_2^E. The theorem follows from this and lemmas 12.5.3, 12.5.4. ∎

One might wonder if there are situations where a converse to this theorem holds, analogous to theorem 12.4.2. That is, if C is an internal category in $\mathcal{S}$ with (N-)finite products, and $F : C \to \mathbb{D}$ preserves finite products, does F preserve N-finite products? We are not yet able to answer this question. We expect that it does not hold in general, but perhaps it holds in the special case where $\mathbb{D} = F^*\mathbb{T}$, for $F : \mathcal{S} \to \mathcal{T}$ a morphism of lextensive categories.

Part IV

Finite objects and extensions of arithmetic universes

Chapter 13

Finite objects in a locos, part 1

Contents

In this chapter and chapter 15, we prove some facts about the category of *finite objects* in a locos. In general, there are many notions of finite objects in a category, such as Dedekind finiteness or Kuratowski-finiteness (see e.g. section D5.4 of [John 02]). However, the notion that will be most relevant to us is the notion a *finite cardinal*: a pullback of $\pi_2^E : E \to N$ along some arrow $\mathbb{1} \to N$ (see also section D5.2 of [John 02]). Therefore, we say an object is finite if it is isomorphic to a finite cardinal (we state this as a definition in section 13.1, below).

Our goal for this chapter and chapter 15 is to show that, in a locos $\mathcal{C}$, the full subcategory of finite objects is a Boolean topos, and that the inclusion of this subcategory preserves all the relevant structure (see theorem 15.5.1); this is the second main result of the thesis. This is well-known when $\mathcal{C}$ is a topos (see theorem D5.2.7 of [John 02]), but we show that this is still true in the weaker setting of $\mathcal{C}$ being a locos.

In this chapter, we start by presenting the definition and some basic facts (in section 13.1). The first big result, in section 13.2, is that subobjects of finite objects are finite if and only if they are complemented (theorem 13.2.3). To prove this result, we rely heavily on the tools developed in appendix B.

Finally, we discuss finite limits and coproducts of finite objects in sections 13.3 and 13.4, and summarize the results so far in section 13.5. We defer more complicated constructions (exponentials and quotients) to chapter 15, since we will need the additional tools that we develop in chapter 14.

13.1 Definitions

Definition. In an arithmetic lextensive category $\mathcal{C}$, for any arrow $m : \mathbb{1} \to N$, we write E_m for the object given by the following pullback square. We call E_m the *finite cardinal* associated to m.

$$\begin{array}{ccc} E_m & \longrightarrow & E \\ \downarrow & \lrcorner & \downarrow \pi_2^E \\ \mathbb{1} & \xrightarrow{m} & N \end{array}$$

Equivalently, we have $E_m = \{k : N \mid k < m\}$, so each E_m has a canonical inclusion into N. We say an object $X \in \mathcal{C}$ is *finite* if $X \cong E_m$ for some $m : \mathbb{1} \to N$, and we write $\mathbf{Fin}(\mathcal{C})$ for the full subcategory of finite objects in $\mathcal{C}$.

Later on, we will talk about finite objects in slice categories. The following proposition gives a characterization of these objects, but first, we establish some notation. Given an arrow $m : I \to N$, we write $I \times_m E$ for the following pullback.

$$\begin{array}{ccc} I \times_m E & \xrightarrow{\pi_2^m} & E \\ \pi_1^m \downarrow & \lrcorner & \downarrow \pi_2^E \\ I & \xrightarrow{m} & N \end{array}$$

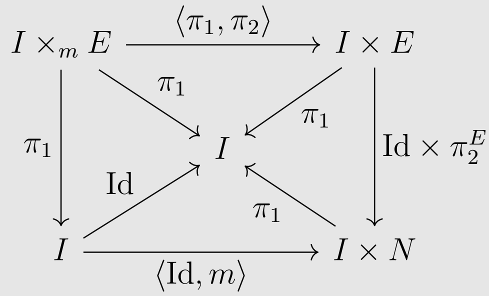
Note that we write π_1^m and π_2^m for the projections of this particular pullback.

Proposition 13.1.1. Let $\mathcal{C}$ be an arithmetic lextensive category, and let $I \in \mathcal{C}$. An object $X \in \mathcal{C}/I$ is finite if and only if X is isomorphic to $\pi_1^m : I \times_m E \to I$ for some $m : I \to N$.

Proof. Arithmetic lextensive categories are local (see proposition 2.7 in [Maie 10]), so the functor $I^* : \mathcal{C} \to \mathcal{C}/I$ preserves the collection of finite cardinals. (Also, the terminal object in $\mathcal{C}/I$ is $\mathrm{Id}_I : I \to I$.) Therefore, a finite object in $\mathcal{C}/I$ is the pullback of $\mathrm{Id} \times \pi_2^E$ and some n; see below.

$$\begin{array}{ccccc} I & \xrightarrow{n} & I \times N & \xleftarrow{\mathrm{Id} \times \pi_2^E} & I \times E \\ & \underset{\mathrm{Id}}{\searrow} & \downarrow \pi_1 & \underset{\pi_1}{\swarrow} & \\ & & I & & \end{array}$$

It's clear that we can write $n = \langle \mathrm{Id}, m \rangle$ for some $m : I \to N$. So, since pullbacks in slice categories are computed as in the base, the pullback of these arrows gives the following.



This is precisely what we wanted. ∎

A special case of the notation defined above is when the arrow $I \to N$ is of the form $I \to \mathbb{1} \xrightarrow{m} N$ for some $m : \mathbb{1} \to N$. Since there is no risk of confusion if we specify the domain of m, we also denote this pullback by $I \times_m E$. Then we have the following proposition, which will be useful for converting maps into lists.

Proposition 13.1.2. Let $\mathcal{C}$ be an arithmetic lextensive category, let $I \in \mathcal{C}$, and let $m : \mathbb{1} \to E$. Then $I \times_m E \cong I \times E_m$. Specifically, the map $I \times_m E \to I \times E_m$ given by $(i, k) \mapsto (i, k)$ is an isomorphism.

In particular, if $\mathcal{C}$ is a locos, then for any arrow $g : I \times E_m \to X$, there exists a unique arrow $h : I \to L(X)$ such that $\mathrm{len}(h(i)) = m$ and $\mathrm{nth}(k, h(i)) = g(i, k)$.

Proof. By proposition 3.4.2, we know that $I \times_m E = \{i : I, k : E \mid k < m\}$. So, the map $I \times_m E \to I \times E_m$ given by $(i, k) \mapsto (i, k)$ is well-defined, and so is its clear inverse. Thus, we have the desired isomorphism.

The second part of the proposition is just a special case of theorem 5.4.1. ∎

One might note that we have changed the order of the pullback, writing $I \times_m E$ instead of $E \times_m I$ (as we did in chapter 5). This is simply because it will be easier to draw diagrams in this chapter with I as the first component.

13.2 Subobjects of finite objects

In this section, we show that if F is a subobject of E_m, then F is finite if and only if F is complemented. These results rely heavily on the machinery developed in appendix B, and we make use of the maps cZ, kZ, and more from that appendix.

Lemma 13.2.1. In a locos, let $n : \mathbb{1} \to N$, and let $F \hookrightarrow E_n$ be a subobject. If F is complemented, then F is finite (meaning $F \cong E_m$ for some $m : \mathbb{1} \to N$).

Proof. Since F is complemented, then by theorem 2.3.1, there is arrow $d : E_n \to \{0,1\}$ such that $F \hookrightarrow E_n$ is the equalizer of d and 0. To such a map d, we can associate a map $\ell_d : \mathbb{1} \to L(\{0,1\})$ such that

$$\operatorname{len}(\ell_d) = n \qquad \text{and} \qquad \operatorname{nth}(\ell_d, k) = d(k) \text{ for } k < n.$$

(See proposition 13.1.2.) Now, set $m = cZ(\ell_d)$, and define $f_1 : N \to N$ by

$$f_1(k) = kZ(\ell_d, k).$$

Remark that, if $k < cZ(\ell_d) = m$, then by lemma B.7.3, we have

$$f_1(k) = kZ(\ell_d, k) < kZ(\ell_d, sk) = f_1(sk).$$

Therefore, if we let i be the inclusion $E_m \hookrightarrow N$, then corollary 3.6.3 tells us that the composite $f_2 = f_1 \circ i$ is monic.

Next, note that for any $k < m = cZ(\ell_d)$, we have

$$f_2(k) = kZ(\ell_d, k) < \operatorname{len}(\ell_d) = n$$

by proposition B.6.3. This implies that $f_2 : E_m \to N$ induces a map $f_3 : E_m \to E_n$. Note that f_3 is monic because $i' \circ f_3 = f_2$, which is monic (where $i' : E_n \hookrightarrow N$).

Now, we claim that $d \circ f_3 = 0$. Indeed, if $k < m = cZ(\ell_d)$, then

$$d(f_3(k)) = \operatorname{nth}(\ell_d, f_3(k)) = \operatorname{nth}(\ell_d, kZ(\ell_d, k)) = 0$$

by proposition B.8.1. So, we now plan to show that the following diagram is an equalizer.

$$E_m \xrightarrow{f_3} E_n \overset{d}{\underset{0}{\rightrightarrows}} \{0,1\}$$

We need to show that for any $g : X \to E_n$ such that $d \circ g = 0$, there exists a map $t : X \to E_m$ such that $g = f_3 \circ t$; the uniqueness is guaranteed since f_3 is monic.

So, let $g : X \to E_n$ satisfy $d(g(x)) =_x 0$. In particular, we have

$$g(x) < n = \operatorname{len}(\ell_d) \qquad \text{and} \qquad \operatorname{nth}(\ell_d, g(x)) = d(g(x)) = 0.$$

Therefore, by proposition B.8.2, if we set $t(x) = chZ(\ell_d, g(x))$, then $t(x) < cZ(\ell_d) = m$ and $g(x) = kZ(\ell_d, t(x))$. The former condition tells us that t is a valid arrow $X \to E_m$, and the latter tells us that $g(x) = f_3(t(x))$. This is all we needed to show. ∎

Lemma 13.2.2. In a locos, let $m, n : \mathbb{1} \to N$, and let $i : E_m \hookrightarrow E_n$ be a subobject. Then this subobject is complemented.

Proof. Define $g : E_n \times E_m \to \{0,1\}$ by $g(b,a) = \min\{|b, i(a)|, 1\}$. By proposition 13.1.2, there is a map $h : E_n \to L(\{0,1\})$ such that

$$\mathrm{len}(h(b)) =_b m \qquad \text{and} \qquad \mathrm{nth}(h(b), a) =_{b,a} \min\{|b, i(a)|, 1\} \text{ (for } a < m\text{)}.$$

Then, define $d : E_n \to \{0,1\}$ by $d(b) = 1 \dot{-} cZ(h(b))$. We claim that the following is an equalizer diagram (if so, we conclude by theorem 2.3.1 that i is complemented).

$$E_m \xhookrightarrow{\;i\;} E_n \underset{}{\overset{d,0}{\rightrightarrows}} \{0,1\}$$

First, we must show that $d \circ i = 0$. To show this, first note that for any $a < m$, we have

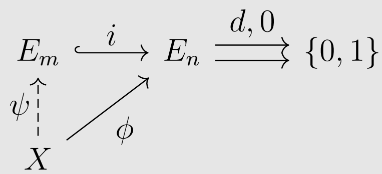
$$\mathrm{nth}(h(i(a)), a) = \min\{|i(a), i(a)|, 1\} = \min\{0, 1\} = 0.$$

By proposition B.8.2, the existence of $a < m = \mathrm{len}(h(i(a)))$ such that $\mathrm{nth}(h(i(a)), a) = 0$ implies in particular that $cZ(h(i(a))) > 0$. Thus,

$$d(i(a)) = 1 \dot{-} cZ(h(i(a))) = 0,$$

as desired.

Now, let $\phi : X \to E_n$ be such that $d \circ \phi = 0$. We claim there exists a unique $\psi : X \to E_m$ such that $i \circ \psi = \phi$. Uniqueness is guaranteed by the fact that i is monic, so we just check existence.



Given $x : X$, since $0 = d(\phi(x)) = 1 \dot{-} cZ(h(\phi(x)))$, we have $cZ(h(\phi(x))) \geq 1$. Therefore, if we set $\psi(x) = kZ(h(\phi(x)), 0)$, then proposition B.8.1 tells us that $\mathrm{nth}(h(\phi(x)), \psi(x)) = 0$. By definition of h, we get

$$0 = \mathrm{nth}(h(\phi(x)), \psi(x)) = \min\{|\phi(x), i(\psi(x))|, 1\},$$

and therefore $0 = |\phi(x), i(\psi(x))|$ (by proposition 3.2.6), which implies $\phi(x) = i(\psi(x))$. Thus $\phi = i \circ \psi$, as desired. ∎

Combining the above lemmas, we immediately get the following result.

Theorem 13.2.3. In a locos, let $n : \mathbb{1} \to N$, and let $F \hookrightarrow E_n$ be a subobject. Then F is complemented if and only if F is finite.

Proof. Immediate by lemmas 13.2.1 and 13.2.2. ∎

13.3 Finite limits

In this section, we show that finite limits of finite objects are finite.

Proposition 13.3.1. In an arithmetic lextensive category $\mathcal{C}$, the terminal object $\mathbb{1} \in \mathcal{C}$ is finite. In particular, $\mathbf{Fin}(\mathcal{C})$ has a terminal object, and it is preserved by the inclusion $\mathbf{Fin}(\mathcal{C}) \hookrightarrow \mathcal{C}$.

Proof. We claim $\mathbb{1} \cong E_1$, so we just need to show that the following square is a pullback.

$$\begin{array}{ccc} \mathbb{1} & \xrightarrow{\langle 0,1\rangle} & E \\ \downarrow & & \downarrow \pi_2^E \\ \mathbb{1} & \xrightarrow{1} & N \end{array}$$

This is not hard: if we have a term $e = (x, y) \in E$ with $1 = \pi_2^E(e) = y$, then $x < y = 1$, so $sx = s0$, so $x = 0$. ∎

Proposition 13.3.2. Let $\mathcal{C}$ be a locos, and let $f, g : A \to B$ be arrows in $\mathcal{C}$. If A and B are finite, then so is the equalizer $\text{eq}(f, g)$. In particular, $\mathbf{Fin}(\mathcal{C})$ has equalizers, and these are preserved by the inclusion $\mathbf{Fin}(\mathcal{C}) \hookrightarrow \mathcal{C}$.

Proof. Let $X \hookrightarrow A$ be the equalizer of arrows f, g. Note that we can consider $B \subseteq N$, and since $f = g$ iff $|f, g| = 0$, we find that $X = \text{eq}(d, 0)$, where $d : A \to \{0, 1\}$ is given by $d(a) = \min\{|f(a), g(a)|, 1\}$. Thus X is a complemented subobject of A (by theorem 2.3.1), and since A is finite, X must be finite too by theorem 13.2.3. ∎

Proposition 13.3.3. Let $\mathcal{C}$ be a locos. If $A, B \in \mathcal{C}$ are finite, then so is $A \times B$. In particular, $\mathbf{Fin}(\mathcal{C})$ has binary products, and these are preserved by the inclusion $\mathbf{Fin}(\mathcal{C}) \hookrightarrow \mathcal{C}$.

Proof. Let $m, n : \mathbb{1} \to N$; we claim that $E_m \times E_n$ is isomorphic to $E_{m \bullet n}$. To show this, we will use the maximum function defined in section 3.7. We define maps $f : E_m \times E_n \to E_{m \bullet n}$ and $g : E_{m \bullet n} \to E_m \times E_n$ as follows.

$$f(a, b) = a \bullet n + b$$

$$g(c) = \Big(\max(x < m \mid x \bullet n \le c),\ c \dot{-} \max(x < m \mid x \bullet n \le c) \bullet n\Big)$$

First, we check that these maps are well-defined. We start with f. If $a < m$ and $b < n$, then $f(a,b) < a \bullet n + n = (sa) \bullet n \leq m \bullet n$, so this is well-defined.

Next, we consider g. Let $c < m \bullet n$; since $m \bullet n > c \geq 0$, $0 \bullet n \leq c$, and $m \bullet n > c$, proposition 3.7.1 tells us that there is a term max $< m$ such that max $\bullet\, n \leq c$ and $s(\max) \bullet n > c$. This max is what we wrote as $\max(x < m \mid x \bullet n \leq c)$ in the definition of g. Now, since max $\bullet\, n \leq c$, there is a term k such that max $\bullet\, n + k = c$; note that this implies $c \dot{-} \max \bullet\, n = k$. Thus, $g(c) = (\max, k)$; we've noted that max $< m$, so it remains to check that $k < n$. To see this, note that

$$\max \bullet\, n + k = c < s(\max) \bullet n = \max \bullet\, n + n;$$

subtracting max $\bullet\, n$ from both sides yields $k < n$. This completes the check that f and g are well-defined.

Finally, we claim these maps are inverses of each other. On one hand, if we start with $c < m \bullet n$, then we've shown that $g(c) = (\max, k)$ where max $\bullet\, n + k = c$; hence, $f(g(c)) = c$. On the other hand, let $a < m$ and $b < n$. Denote $\max(x < m \mid x \bullet n \leq a \bullet n + b)$ by max. We must show that max $= a$; if so, then $(a \bullet n + b) \dot{-} (\max \bullet\, n) = b$, and we get $g(f(a,b)) = g(a \bullet n + b) = (a,b)$.

So, it remains to show that max $= a$, where max $= \max(x < m \mid x \bullet n \leq a \bullet n + b)$ and $a < m$, $b < n$. On one hand, we know that $a < m$ and $a \bullet n \leq a \bullet n + b$, so proposition 3.7.1 tells us that max $\geq a$. On the other hand, well-definedness of g means that max $\bullet n \leq a \bullet n + b$, so if we had max $> a$, then max $\bullet n \geq sa \bullet n = a \bullet n + n > a \bullet n + b$, a contradiction. So, by corollary 3.3.7, we get max $\leq a$, and we conclude max $= a$. ∎

Corollary 13.3.4. If $\mathcal{C}$ is a locos, then $\mathbf{Fin}(\mathcal{C})$ has finite limits, and the inclusion $\mathbf{Fin}(\mathcal{C}) \hookrightarrow \mathcal{C}$ preserves them.

Proof. Immediate from propositions 13.3.1, 13.3.2, and 13.3.3. ∎

13.4 Finite coproducts

Proposition 13.4.1. In a locos $\mathcal{C}$, the initial object $\mathbb{O} \in \mathcal{C}$ is finite. In particular, $\mathbf{Fin}(\mathcal{C})$ has an initial object, and it is preserved by the inclusion map $\mathbf{Fin}(\mathcal{C}) \hookrightarrow \mathcal{C}$.

Proof. We claim $\mathbb{O} \cong E_0$. Note that $\pi_2^E > 0$: indeed, if $(m,n) : E$, then $\pi_2^E(m,n) = n > m \geq 0$. So, by proposition 3.2.3, we have $\pi_2^E = s \circ P \circ \pi_2^E$, and hence the following diagram commutes.

$$\begin{array}{ccccc} & & E_0 & \longrightarrow & E \\ & \swarrow & & & \big\downarrow P \circ \pi_2^E \\ \mathbb{1} & \xrightarrow{\;0\;} & N & \xleftarrow{\;s\;} & N \end{array}$$

Therefore, there is a map from E_0 into the pullback of $0, s$. But since $0, s$ form a coproduct, and we're in an extensive category, their pullback is the initial object. Thus we have a map $E_0 \to \mathbb{O}$; since initials are strict in an extensive category, this is an isomorphism. ∎

Proposition 13.4.2. Let $\mathcal{C}$ be a locos. If $A, B \in \mathcal{C}$ are finite, then so is $A + B$. In particular, $\mathbf{Fin}(\mathcal{C})$ has binary coproducts, and these are preserved by the inclusion $\mathbf{Fin}(\mathcal{C}) \hookrightarrow \mathcal{C}$.

Proof. Let $m, n : \mathbb{1} \to N$; we claim that $E_m + E_n$ is isomorphic to E_{m+n}. We'll denote the inclusion maps into $E_m + E_n$ by i_m and i_n, and we define maps $f : E_m + E_n \to E_{m+n}$ and $g : E_{m+n} \to E_m + E_n$ as follows.

$$f(k) = \begin{cases} k & \text{if } k \in E_m \\ m + k & \text{if } k \in E_n \end{cases} \qquad\qquad g(k) = \begin{cases} i_m(k) & \text{if } k < m \\ i_n(k \mathbin{\dot{-}} m) & \text{else} \end{cases}$$

First, we check that these maps are well-defined. We start with f: if $k \in E_m$, then $f(k) = k < m \leq m + n$, and if $k \in E_n$, then $f(k) = m + k < m + n$. Either way, we have $f(k) < m + n$, so we can say $f(k) \in E_{m+n}$. For g, let $k : E_{m+n}$ (i.e. $k < m + n$). If $k < m$, then $k : E_m$, so $i_m(k) : E_m + E_n$ is well-defined. On the other hand, if $k \geq m$, then since $k < m + n$, we have $k \mathbin{\dot{-}} m < n$: indeed, $k \geq m$ means $k = m + k'$, with $k' = k \mathbin{\dot{-}} m$, and so

$$\begin{aligned} s(k \mathbin{\dot{-}} m) \mathbin{\dot{-}} n = s(k') \mathbin{\dot{-}} n &= ((sk' + m) \mathbin{\dot{-}} m) \mathbin{\dot{-}} n \\ &= s(k' + m) \mathbin{\dot{-}} (m + n) = sk \mathbin{\dot{-}} (m + n) = 0. \end{aligned}$$

So, $k : E_n$, and $i_n(k)$ is well-defined.

Now, we check that f and g are inverses are of each other. First, we show that $g \circ f$ is the identity on $E_m + E_n$. We split into cases: if $k \in E_m$, then $g(f(k)) = g(k) = i_m(k)$ (because $k < m$), and if $k \in E_n$, then $g(f(k)) = g(m + k) = i_n((m + k) \mathbin{\dot{-}} m) = i_n(k)$ (because $m + k \geq m$). So, $gf(i_m(k)) = i_m(k)$ and $gf(i_n(k)) = i_n(k)$; this shows that $gf = \mathrm{Id}$.

Finally, we show that $f \circ g$ is the identity on E_{m+n}. Again, we split into cases. If $k < m$, then $f(g(k)) = f(i_m(k)) = k$, and if $k \geq m$, then $f(g(k)) = f(i_n(k \mathbin{\dot{-}} m)) = m + (k \mathbin{\dot{-}} m) = k$ (because $k \geq m$). Thus, we get $fg = \mathrm{Id}$. ∎

Corollary 13.4.3. If $\mathcal{C}$ is a locos, then $\mathbf{Fin}(\mathcal{C})$ has finite coproducts, and the inclusion $\mathbf{Fin}(\mathcal{C}) \hookrightarrow \mathcal{C}$ preserves them.

Proof. Immediate from propositions 13.4.1 and 13.4.2. ∎

13.5 Summary

We summarize the results so far in the following theorem.

Theorem 13.5.1. If $\mathcal{C}$ is a locos, then $\mathbf{Fin}(\mathcal{C})$ is a lextensive category, and the inclusion $\mathbf{Fin}(\mathcal{C}) \hookrightarrow \mathcal{C}$ preserves finite limits and finite coproducts. Moreover, if $B \in \mathcal{C}$ is finite, then a subobject $A \hookrightarrow B$ is finite if and only if it is complemented.

Proof. In corollaries 13.3.4 and 13.4.3, we showed that $\mathbf{Fin}(\mathcal{C})$ has all finite limits and finite products, and that these are preserved by the inclusion $\mathbf{Fin}(\mathcal{C}) \hookrightarrow \mathcal{C}$. To show $\mathbf{Fin}(\mathcal{C})$ is lextensive, it just remains to show that coproducts are stable and disjoint, but this is clear since it is true in $\mathcal{C}$.

Finally, we showed in theorem 13.2.3 that subobjects of finite objects are finite if and only if they are complemented. ∎

Chapter 14

The internal category of finite sets, part 1

Contents

In this chapter, we define the internal category of finite sets, $\mathbf{Fin}_{\mathcal{S}}$, and explore its properties.

We start with the definition in section 14.1: it is a special case of the construction discussed in chapter 8. We also explain its connection to the category of finite objects discussed in chapter 13. Next, in section 14.2, we show that this internal category has N-finite sums, and that they are preserved by the inclusion $\mathbf{Fin}_{\mathcal{S}} \hookrightarrow \mathbb{S}$. Finally, in section 14.3, we see how the results of chapter 13 apply to give us results about $\mathbf{Fin}_{\mathcal{S}}$.

We include this chapter here so that, in chapter 15, we can use $\mathbf{Fin}_{\mathcal{S}}$ to prove results about the category of finite objects in a locos. Specifically, we will use theorem 12.4.2 to conclude that $\mathbf{Fin}_{\mathcal{S}}$ has N-finite products, and this will let us show that the category of finite objects has exponentials (theorem 15.1.1). Later, in chapter 16, we will explore more properties specific to $\mathbf{Fin}_{\mathcal{S}}$.

14.1 Basics

Definition. Let $\mathcal{S}$ be a locos. Since the arrow $\pi_2^E : E \to N$ in $\mathcal{S}$ is exponentiable (see corollary 5.4.2), we can form the internal category $\mathbb{S}[\pi_2^E]$ in $\mathcal{S}$ (see section 8.1). We call this the *internal category of finite sets* in $\mathcal{S}$, and denote it $\mathbf{Fin}_{\mathcal{S}}$. (We drop the subscript if it is clear from the context.)

Of course, since $\mathbf{Fin}_{\mathcal{S}}$ is defined from an exponentiable arrow, all the results from chapter 8 apply to it. For one, we have a full and faithful inclusion $\mathbf{Fin}_{\mathcal{S}} \hookrightarrow \mathbb{S}$ (see proposition 8.1.3). Moreover, if $F : \mathcal{S} \to \mathcal{T}$ is a morphism of locoi, then $\mathbb{P}_{\pi_2^E}$ and $\mathbb{E}_{\pi_2^E}$ are well-defined. Indeed, a locos morphism F preserves the arrow $\pi_2^E : E \to N$, so π_2^E and $F(\pi_2^E)$ are both exponentiable. Therefore, we also introduce the following notation.

Definition. Let $\mathcal{S}, \mathcal{T}$ be locoi, and let $F : \mathcal{S} \to \mathcal{T}$ be a locos morphism. We will write $\mathbb{L} : \mathcal{T} \to [\mathbf{Fin}_{\mathcal{S}}^{op}, F^*\mathbb{T}]$ for either of the functors $\mathbb{P}_{\pi_2^E}, \mathbb{E}_{\pi_2^E} : \mathcal{T} \to [\mathbf{Fin}_{\mathcal{S}}^{op}, F^*\mathbb{T}]$ (which are isomorphic by theorem 8.2.3).

In chapter 16, we will establish some results about the functors $\mathbb{L}X$.

The remarkable thing about **Fin** is how it relates to the concepts of finite objects that we discussed in section 13. (Indeed, the abuse of notation is no coincidence!)

Proposition 14.1.1. If $\mathcal{S}$ is a locos, then for any $I \in \mathcal{S}$, we have $\mathbf{Fin}_{\mathcal{S}}(I) \simeq \mathbf{Fin}(\mathcal{S}/I)$.

(More specifically, the inclusion $\mathbf{Fin}_{\mathcal{S}}(I) \hookrightarrow \mathcal{S}/I$ lands in $\mathbf{Fin}(\mathcal{S}/I)$, and this forms an equivalence of categories.)

Proof. We know that there is a full and faithful inclusion $\mathbf{Fin}_{\mathcal{S}}(I) \hookrightarrow \mathcal{S}/I$ which maps an object $n : I \to N$ of $\mathbf{Fin}_{\mathcal{S}}(I)$ to $n^*(\pi_2^E) = (\pi_1^n : I \times_n E \to I)$. Therefore, $\mathbf{Fin}_{\mathcal{S}}(I)$ is (isomorphic to) the full subcategory of $\mathcal{S}/I$ spanned by pullbacks of $\pi_2^E : E \to N$.

On the other hand, $\mathbf{Fin}(\mathcal{S}/I)$ is defined to be the full subcategory of $\mathcal{S}/I$ spanned by finite objects of $\mathcal{S}/I$. But by proposition 13.1.1, finite objects of $\mathcal{S}/I$ are precisely those which are isomorphic to $\pi_1^n : I \times_n E \to I$ for some $n : I \to N$. Therefore, the inclusion $\mathbf{Fin}_{\mathcal{S}}(I) \hookrightarrow \mathcal{S}/I$ lands in $\mathbf{Fin}(\mathcal{S}/I)$, and it is full, faithful, and essentially surjective, so we have the desired equivalence. ∎

We will use this characterization in the next section to obtain several facts about **Fin**.

14.2 Finite sums

In this section, we want to show that $\mathbf{Fin}_{\mathcal{S}}$ has N-finite sums (when considered as an indexed category). In order to do this, we need a technical tool about maps into list objects.

The technical result is based on the fact that, by the adjunction of theorem 5.4.1, arrows $m : I \to N$ and $f : I \times_n E \to X$ uniquely induce a map $\overline{f} : I \to L(X)$. We think of $\bar{f}(i)$ as being the list $[f(i, 0), ..., f(i, m(i) - 1)]$, so we should be able to build this list by adding the values $f(i, k)$ to the list step by step. This step-by-step construction would be captured by a map $\bar{f}_2 : I \times_{sm} E \to L(X)$ with properties given in the following proposition.

Proposition 14.2.1. In a locos, let $m : I \to N$ and $f : I \times_m E \to X$, and let $\bar{f} : I \to L(X)$ be the arrow corresponding to f via the adjunction of theorem 5.4.1. Then there exists an arrow $\bar{f}_2 : I \times_{sm} E \to L(X)$ that satisfies the following properties.

1. $\bar{f}_2(i, 0) = \varnothing$.
2. $\bar{f}_2(i, m(i)) = \bar{f}(i)$.
3. For $a < m(i)$, $\bar{f}_2(i, sa) = \bar{f}_2(i, a) +\!\!+ [f(i, a)]$.

Proof. We start by constructing $\bar{f}_2$. First, let $p_2 : I \times_{sm} E \to N$ be the arrow $\pi_1^E \circ \pi_2$; that is, given a pair (i, k) with $k < sm(i)$, we have $p_2(i, k) = k$. Then, we can form the following pullback square.

$$\begin{array}{ccc} (I \times_{sm} E) \times_{p_2} E & \longrightarrow & E \\ \downarrow & & \downarrow \pi_2^E \\ I \times_{sm} E & \xrightarrow{p_2} & N \end{array}$$

Note that we can write $(I \times_{sm} E) \times_{p_2} E = \{i \in I, a, b \in N \mid b < a < sm(i)\}$. Then, we can define $f_2 : (I \times_{sm} E) \times_{p_2} E \to X$ by setting $f_2(i, a, b) = f(i, b)$ (note that this is well-defined because $b < a \leq m(i)$). The list object correspondence gives us an arrow

$$\bar{f}_2 : I \times_{sm} E \to L(X)$$

which satisfies $\text{len}(\bar{f}_2(i, a)) = a$ and $\text{nth}(\bar{f}_2(i, a), b) = f_2(i, a, b) = f(i, b)$ (for $b < a \leq m(i)$). We now check that $\bar{f}_2$ satisfies the three claimed properties of the proposition.

For the first part, we note that $\text{len}(\bar{f}_2(i, 0)) = 0$, so the equality follows by proposition 4.3.2.

For the second part, we appeal to uniqueness. Indeed, we have $\text{len}(\bar{f}_2(i, m(i))) = m(i)$, and for $a < m(i)$, we have $\text{nth}(\bar{f}_2(i, m(i)), a) = f(i, b)$. So we get the equality by the defining equations of $\bar{f}(i)$.

For the third part, fix $i \in I$ and $a < m(i)$; we will show that the two lists are equal by again invoking uniqueness. First, it is clear that these lists have the same length, sa. Next, we must show that for any $b < sa$, i.e. $b \leq a$, we have

$$\text{nth}\Big(\bar{f}_2(i, sa),\ b\Big) = \text{nth}\Big(\bar{f}_2(i, a) +\!\!+ [f(i, a)],\ b\Big).$$

We know the left hand side is equal to $f(i, b)$. For the right hand side, we split into the cases $b < a$ and $b \geq a$.

- If $b < a$, then since $\text{len}(\bar{f}_2(i,a)) = a$, we can use corollary 4.5.3 to get

$$\text{nth}\Big(\bar{f}_2(i,a) \mathbin{+\!\!+} [f(i,a)],\ b\Big) = \text{nth}(\bar{f}_2(i,a), b) = f(i,b).$$

- If $b \geq a$, then $b = a$ since we already have $b \leq a$. Then, again using the fact that $\text{len}(\bar{f}_2(i,a)) = a = b$ and corollary 4.5.3, we have

$$\text{nth}\Big(\bar{f}_2(i,a) \mathbin{+\!\!+} [f(i,a)],\ b\Big) = \text{nth}([f(i,a)], 0) = f(i,a) = f(i,b).$$

Thus, we have the desired equality. ∎

With this tool under our belt, we can show that **Fin** has N-finite sums. We will put the bulk of the proof in the following lemma, and obtain the final conclusion afterwards.

Lemma 14.2.2. In a locos $\mathcal{S}$, let $m : I \to N$ and let $n : I \times_m E \to N$. Then there exist arrows $s : I \to N$ and $p : (I \times_m E) \times_n E \to E$ such that the following square is a pullback.

$$\begin{array}{ccc}
(I \times_m E) \times_n E & \xrightarrow{\ p\ } & E \\
\pi_1^n \big\downarrow & & \\
I \times_m E & & \big\downarrow \pi_2^E \\
\pi_1^m \big\downarrow & & \\
I & \xrightarrow{\quad s \quad} & N
\end{array}$$

Proof. Let $m : I \to N$, let $n : I \times_m E \to N$, and let $\bar{n} : I \to L(N)$ denote the arrow corresponding to n (by theorem 5.4.1). We define $s = S[n] : I \to N$ to be $S[n] = \text{sum} \circ \bar{n}$, where $\text{sum} : L(N) \to N$ is the arrow define inductively by

$$\text{sum}(\varnothing) = 0, \qquad\qquad \text{sum}(n :: \ell) = n + \text{sum}(\ell).$$

Note that $\text{sum}(\ell_1 \mathbin{+\!\!+} \ell_2) = \text{sum}(\ell_1) + \text{sum}(\ell_2)$; this is easy to show by induction on ℓ_1.

Next, let $\bar{n}_2 : I \times_{sm} E \to L(N)$ be the arrow given by proposition 14.2.1, and define $PS[n] : I \times_{sm} E \to N$ to be $PS[n] = \text{sum} \circ \bar{n}_2$. Then, by viewing the object $(I \times_m E) \times_n E$ as

$$(I \times_m E) \times_n E = \{i \in I, a, b \in N \mid a < m(i), b < n(i,a)\},$$

we can define $p : (I \times_m E) \times_n E \to E$ to be

$$p(i,a,b) = \Big(PS[n](i,a) + b,\ S[n](i)\Big).$$

We claim that these choices of s and p make the square into a pullback.

First, we must check that p is well-defined. To start, let (i, a, b) be in the domain of p. Then $a < m(i)$, so $sa < sm(i)$, so $PS[n](i, sa)$ is well-defined. Using proposition 14.2.1, we compute

$$\begin{aligned} PS[n](i, sa) &= \operatorname{sum}\Big(\bar{n_2}(i, sa)\Big) = \operatorname{sum}\Big(\bar{n_2}(i, a) \mathbin{+\!\!+} [n(i, a)]\Big) \\ &= \operatorname{sum}(\bar{n_2}(i, a)) + n(i, a) = PS[n](i, a) + n(i, a) > PS[n](i, a) + b. \end{aligned}$$

It is easy to check that $PS[n](i, -)$ is increasing (using the above inequality), so since $sa \leq m(i)$, we have (using propositions 14.2.1 and 3.6.1)

$$PS[n](i, sa) \leq PS[n](i, m(i)) = \operatorname{sum}(\bar{n_2}(i, m(i))) = \operatorname{sum}(\bar{n}(i)) = S[n](i).$$

Thus, we have checked that $PS[n](i, a) + b < S[n](i)$, which shows that p is well-defined.

It is clear by definition that s and p make the square commute. So, we need to check it is a pullback square. Unravelling the definitions, this is equivalent to checking the following statement.

> In a context C, let $i : I$ and $k : N$ be such that $k <_C S[n](i)$. Then there exist unique terms $a, b : N$ such that $a < m(i)$, $b < n(i, a)$, and $k = PS[n](i, a) + b$.

We claim that this should be accomplished by setting

$$a = \max\big(x < m(i) \mid PS[n](i, x) \leq k\big) \qquad \text{and} \qquad b = k \mathbin{\dot{-}} PS[n](i, a).$$

First, we check that this a is a valid maximum, i.e. $a < m(i)$ and $PS[n](i, a) \leq k$. By proposition 3.7.1, it suffices to check that $m(i) > 0$ and $PS[n](i, 0) \leq k$. For the latter, we simply use proposition 14.2.1 to compute

$$PS[n](i, 0) = \operatorname{sum}(\bar{n_2}(i, 0)) = \operatorname{sum}(\varnothing) = 0.$$

For the condition $m(i) > 0$, we note that if $m(i) = 0$, then proposition 14.2.1 tells us that $S[n](i) = \operatorname{sum}(\bar{n}(i)) = \operatorname{sum}(\bar{n_2}(i, m(i))) = \operatorname{sum}(\bar{n_2}(i, 0)) = \operatorname{sum}(\varnothing) = 0$. This contradicts the condition $k < S[n](i)$, so (by corollary 3.3.7) we must have $m(i) > 0$, as desired.

Therefore, we know that $a < m(i)$ and $PS[n](i, a) \leq k$. The latter inequality tells us that there is some b such that $PS[n](i, a) + b = k$, and subtracting $PS[n](i, a)$ from both sides gives the desired expression of b. So, we have chosen our a and b, and we have shown so far that $a < m(i)$ and $k = PS[n](i, a) + b$. It remains to show that $b < n(i, a)$.

To show this last condition, we note that $PS[n](i, m(i)) = S[n](i)$ (using proposition 14.2.1), and since $k < S[n](i)$, we must have $PS[n](i, m(i)) > k$. Then proposition 3.7.1

tells us that $PS[n](i, sa) > k$. As noted earlier, $PS[n](i, sa) = PS[n](i, a) + n(i, a)$, so we get

$$PS[n](i, a) + b = k < PS[n](i, a) + n(i, a).$$

Subtracting $PS[n](i, a)$ from both sides gives $b < n(i, a)$, as desired. ∎

Finally, we are ready for the statement that **Fin** has N-finite sums.

Proposition 14.2.3. Let $\mathcal{S}$ be a locos. Then the internal category $\mathbf{Fin}_{\mathcal{S}}$ has N-finite sums, and these are preserved by the inclusion $\mathbf{Fin}_{\mathcal{S}} \hookrightarrow \mathbb{S}$.

Proof. Recall that having N-finite sums means having π_2^E-indexed sums. So, consider a pullback of π_2^E, i.e. an arrow of the form $\pi_1^m : I \times_m E \to I$ for some $m : I \to N$. We must show that $(\pi_1^m)^* : \mathbf{Fin}_{\mathcal{S}}(I \times_m E) \to \mathbf{Fin}_{\mathcal{S}}(I)$ has a left adjoint Σ_n, and that this left adjoint commutes with the inclusion.

Now, we know that $(\pi_1^m)^* : \mathcal{S}/I \times_m E \to \mathcal{S}/I$ has a left adjoint, $\Sigma_{\pi_1^m}$, which is simply given by composition with π_1^m. Since $\mathbf{Fin}_{\mathcal{S}}$ can be viewed as a full subcategory of $\mathbb{S}$, it suffices to show that $\Sigma_{\pi_1^m}$ restricts to a functor $\mathbf{Fin}_{\mathcal{S}}(I \times_m E) \to \mathbf{Fin}_{\mathcal{S}}(I)$. Or, equivalently, we'll show that $\Sigma_{\pi_1^m}$ restricts to a functor $\mathbf{Fin}(\mathcal{S}/I \times_m E) \to \mathbf{Fin}(\mathcal{S}/I)$. (This will also imply that the Beck-Chevalley condition holds, since it holds in $\mathbb{S}$.)

So, let n be an object of $\mathbf{Fin}(\mathcal{S}/I \times_m E)$. We will express this by saying that $n : I \times_m E \to N$, and the corresponding object of $\mathbf{Fin}(\mathcal{S}/I \times_m E)$ is $\pi_1^n : (I \times_m E) \times_n E \to I \times_m E$ (by definition, every object of $\mathbf{Fin}(\mathcal{S}/I \times_m E)$ is of this form, up to isomorphism). We must show that $\Sigma_{\pi_1^m}(\pi_1^n)$ is an object of $\mathbf{Fin}(\mathcal{S}/I)$; that is, we must show it is the pullback of some $s : I \to N$.

$$\begin{array}{ccc}
(I \times_m E) \times_n E & \xrightarrow{\;p\;} & E \\
\pi_1^n \downarrow & & \\
I \times_m E & & \downarrow \pi_2^E \\
\pi_1^m \downarrow & & \\
I & \xrightarrow{\;s\;} & N
\end{array}$$

In short, given m and n, we must construct s and p such that the above square is a pullback (since $\Sigma_{\pi_1^m}$ is just post-composition with π_1^m). This is done in lemma 14.2.2. ∎

14.3 Key facts

With the following proposition, we take the results from chapter 13 and apply them to the internal category **Fin**.

Proposition 14.3.1. Let $\mathcal{S}$ be a locos. Then the internal category $\mathbf{Fin}_{\mathcal{S}}$ has finite limits, finite coproducts, and N-finite sums, and these are preserved by the inclusion $\mathbf{Fin}_{\mathcal{S}} \hookrightarrow \mathbb{S}$.

Proof. The N-finite sums are covered by proposition 14.2.3. Let $I \in \mathcal{S}$; note that $\mathcal{S}/I$ is a locos since locoi are local (see proposition 2.13 in [Maie 05]). By proposition 14.1.1 and theorem 13.5.1, we know that $\mathbf{Fin}_{\mathcal{S}}(I)$ has finite limits and finite coproducts, and that these are preserved by the inclusion $\mathbf{Fin}_{\mathcal{S}}(I) \hookrightarrow \mathcal{S}/I$. It just remains to check that the finite limits and finite coproducts are preserved by the transition maps, but this is true since the transition maps in $\mathbb{S}$ are pullbacks, which preserve finite limits and finite coproducts since $\mathcal{S}$ is a locos. ∎

Chapter 15

Finite objects in a locos, part 2

Contents

In this chapter, we continue the work of chapter 13 and obtain more results about the category $\mathbf{Fin}(\mathcal{C})$ when $\mathcal{C}$ is a locos, culminating in theorem 15.5.1, the second main result of this thesis.

In section 15.1, we show that $\mathbf{Fin}(\mathcal{C})$ has exponentials (theorem 15.1.1), thanks to the connection between $\mathbf{Fin}(\mathcal{C})$ and $\mathbf{Fin}_{\mathcal{C}}$ that we developed in chapter 14.

The goal for the remainder of the chapter is to show that $\mathbf{Fin}(\mathcal{C})$ has coequalizers, and that they are preserved by the inclusion $\mathbf{Fin}(\mathcal{C}) \hookrightarrow \mathcal{C}$. In section 15.3, we show this for quotients of equivalence relations, and in section 15.4, we show it for general coequalizers. However, the latter result requires some technical tools. So, we first show that collections of finite lists are finite (theorem 15.1.8 in section 15.1), and we show an internal version of the pigeonhole principle (theorem 15.2.6 in section 15.2).

We end this chapter with section 15.5, which summarizes the results of chapters 13 and 15.

15.1 Exponentials and finite lists

In this section, we show that the exponential of two finite objects is finite. We expect the exponential $(E_n)^{E_m}$ to be given by E_{n^m}; however, since we don't need this fact specifically, and since such a proof seems very technical, we prefer the following indirect proof.

Theorem 15.1.1. Let $\mathcal{C}$ be a locos. If $A, B \in \mathcal{C}$ are finite, then the exponential B^A exists and is finite. In particular, $\mathbf{Fin}(\mathcal{C})$ has exponentials, and these are preserved by the inclusion $\mathbf{Fin}(\mathcal{C}) \hookrightarrow \mathcal{C}$.

Proof. As we noted in proposition 14.3.1, the internal category $\mathbf{Fin}_{\mathcal{C}}$ has finite products. By theorem 12.4.2, $\mathbf{Fin}_{\mathcal{C}}$ therefore has N-finite products.

Now, let $A, B \in \mathbf{Fin}(\mathcal{C}) \simeq \mathbf{Fin}_{\mathcal{C}}(\mathbb{1})$. By definition, we have $A = E_m$ for some $m : \mathbb{1} \to N$; this means that $A \to \mathbb{1}$ is the pullback along m of $\pi_2^E : E \to N$. Since $\mathbf{Fin}_{\mathcal{C}}$ has N-finite products, the functor $A^* : \mathbf{Fin}_{\mathcal{C}}(\mathbb{1}) \to \mathbf{Fin}_{\mathcal{C}}(A)$ has a right adjoint $\Pi_A : \mathbf{Fin}_{\mathcal{C}}(A) \to \mathbf{Fin}_{\mathcal{C}}(\mathbb{1})$, and we can set

$$X = \Pi_A(A^*B) \in \mathbf{Fin}_{\mathcal{C}}(\mathbb{1}) \simeq \mathbf{Fin}(\mathcal{C}).$$

We claim that X is the exponential B^A in $\mathcal{C}$.

To show that X is the exponential B^A in $\mathcal{C}$, it suffices to show that

$$\mathrm{Hom}_{\mathcal{C}/I}(I^*A, I^*B) \cong \mathrm{Hom}_{\mathcal{C}/I}(\mathbb{1}, I^*X).$$

But now, we know that A, B, X are finite, the pullback I^* preserves finite objects, and $\mathbb{1} \in \mathcal{C}/I$ (by proposition 13.3.1). Therefore, these hom-sets are between objects of $\mathbf{Fin}(\mathcal{C}/I) \simeq \mathbf{Fin}_{\mathcal{C}}(I)$, and since this is a full subcategory, what we actually need to show is

$$\mathrm{Hom}_{\mathbf{Fin}_{\mathcal{C}}(I)}(I^*A, I^*B) \cong \mathrm{Hom}_{\mathbf{Fin}_{\mathcal{C}}(I)}(\mathbb{1}, I^*X).$$

To show this, we remark that the diagram on the left is a pullback square, which gives us the commutative diagram on the right (since $\mathbf{Fin}$ has N-finite products).

$$\begin{array}{ccc} I \times A & \xrightarrow{\pi_2} & A \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \langle\rangle_A} \\ I & \xrightarrow[\langle\rangle_I]{} & \mathbb{1} \end{array} \qquad\qquad \begin{array}{ccc} \mathbf{Fin}(I \times A) & \xleftarrow{\pi_2^*} & \mathbf{Fin}(A) \\ {\scriptstyle \Pi_{\pi_1}}\downarrow & & \downarrow{\scriptstyle \Pi_A} \\ \mathbf{Fin}(I) & \xleftarrow[I^*]{} & \mathbf{Fin}(\mathbb{1}) \end{array}$$

Then, we compute

$$\begin{aligned} \mathrm{Hom}_{\mathbf{Fin}(I)}(\mathbb{1}, I^*X) = \mathrm{Hom}_{\mathbf{Fin}(I)}(\mathbb{1}, I^*\Pi_A A^*B) &\cong \mathrm{Hom}_{\mathbf{Fin}(I)}(\mathbb{1}, \Pi_{\pi_1}\pi_2^*A^*B) \\ \cong \mathrm{Hom}_{\mathbf{Fin}(I\times A)}(\pi_1^*\mathbb{1}, \pi_2^*A^*B) &\cong \mathrm{Hom}_{\mathbf{Fin}(I\times A)}(\pi_1^*\mathbb{1}, \pi_1^*I^*B). \end{aligned}$$

But now, we can think of $\mathbf{Fin}(I \times A)$ as a full subcategory of $\mathcal{C}/I \times A$ again, and we have

$$\mathrm{Hom}_{\mathbf{Fin}(I\times A)}(\pi_1^*\mathbb{1}, \pi_1^*I^*B) = \mathrm{Hom}_{\mathcal{C}/I\times A}(\pi_1^*\mathbb{1}, \pi_1^*I^*B) \cong \mathrm{Hom}_{\mathcal{C}/I}(\Sigma_{\pi_1}\pi_1^*\mathbb{1}, I^*B).$$

We can directly compute in $\mathcal{C}$ that $\Sigma_{\pi_1}\pi_1^*\mathbb{1} = I^*A$, so this gives us the desired isomorphism. ∎

With this result, we can already say that $\mathbf{Fin}(\mathcal{C})$ is a Boolean topos.

Corollary 15.1.2. If $\mathcal{C}$ is a locos, then $\mathbf{Fin}(\mathcal{C})$ is a Boolean topos.

Proof. In theorem 13.5.1, we showed that $\mathbf{Fin}(\mathcal{C})$ is a lextensive category and that all its subobjects are complemented. By corollary 2.3.3, this implies that $\{0, 1\}$ is a subobject classifier for $\mathbf{Fin}(\mathcal{C})$. Finally, since $\mathbf{Fin}(\mathcal{C})$ is cartesian closed by theorem 15.1.1, we conclude that it is a Boolean topos. ∎

However, we will need to know that the colimits of $\mathbf{Fin}(\mathcal{C})$ are preserved by the inclusion into $\mathcal{C}$, which is not yet guaranteed.

Now that we have shown exponentials of finite objects are finite, we see how this has implications for collections of finite lists in a locos.

Proposition 15.1.3. Let $\mathcal{C}$ be a locos. Given $X \in \mathcal{C}$ and $m : \mathbb{1} \to N$, the exponential X^{E_m} exists and is given by $L(X)_m = \{\ell : L(X) \mid \mathrm{len}(\ell) = m\}$.

In particular, given $m, n : \mathbb{1} \to N$, the object $L(E_n)_m$ is finite.

Proof. We claim that $L(X)_m$ is the exponential X^{E_m}, with the evaluation map $E_m \times L(X)_m \to X$ given by $(k, \ell) \mapsto \mathrm{nth}(k, \ell)$. So, we must show the universal property of the exponential, which says that for any $g : E_m \times I \to X$, there exists a unique map $h : I \to L(X)_m$ such that $\mathrm{nth}(k, h(i)) = g(k, i)$. But having $h : I \to L(X)_m$ is the same as having $h : I \to L(X)$ with $\mathrm{len}(h(i)) = m$. Therefore, this is just the property proven in theorem 5.4.1 (see also proposition 13.1.2).

For any $m, n : \mathbb{1} \to N$, the above argument shows that $L(E_n)_m$ is the exponential $(E_n)^{E_m}$, and we showed in theorem 15.1.1 that this exponential is finite. ∎

Next, we would like to show that the object $L(X)_{\leq m} = \{\ell : L(X) \mid \mathrm{len}(\ell) \leq m\}$ is finite when X is finite. We show this in theorem 15.1.8, below; our technique is to show that $L(E_n)_{\leq m}$ is a complemented subobject of $L(E_{sn})_m$. Specifically, a list $[a_1, ..., a_k]$ in $L(E_n)_{\leq m}$ should correspond to the list $[0, ..., 0, 1 + a_1, ..., 1 + a_k]$ in $L(E_{sn})_m$ (where we add enough zeroes to make the length be m). However, doing this argument formally requires some technical tools, which we establish below.

Tools for finite lists

We start by defining some functions.

- We define $fnz : L(N) \to N$ inductively below. Intuitively, $fnz(\ell)$ gives the position of the first non-zero element of ℓ.

$$fnz(\varnothing) = 0, \qquad fnz(k :: \ell) = \begin{cases} 1 + fnz(\ell) & \text{if } k = 0 \\ 0 & \text{else} \end{cases}.$$

- We define $rZ : L(N) \to L(N)$ by $rz(\ell) = \text{tail}(\ell, fnz(\ell))$.
- We define $gz : L(N) \to N$ inductively below. Intuitively, $gz(\ell) = 0$ means all elements of ℓ are greater than zero.

$$gz(\varnothing) = 0, \qquad gz(k :: \ell) = \begin{cases} 1 & \text{if } k = 0 \\ gz(\ell) & \text{else} \end{cases}.$$

- We define $ohz : L(N) \to N$ by $ohz(\ell) = gz(rz(\ell))$. Intuitively, $ohz(\ell) = 0$ means ℓ is of the form $[0, ..., 0, x_1, ..., x_n]$ with $x_1, ..., x_n > 0$.

For the following propositions, we assume we're working in a locos.

Proposition 15.1.4. We have:

1. $fnz(\ell) \leq \text{len}(\ell)$
2. $fnz(\ell_1 +\!\!+ \ell_2) \leq \text{len}(\ell_1) + fnz(\ell_2)$
3. If $a < \text{len}(\ell)$ and $\text{nth}(a, \ell) > 0$, then $fnZ(\ell) \leq a$.
4. If $a < fnz(\ell)$, then $\text{nth}(a, \ell) = 0$.

Proof. The first part is an easy induction on ℓ, and the second part is an easy induction on ℓ_1. (See theorem 3.5.1 for doing induction on inequalities.)

For the third part, write $\ell = \text{head}(a, \ell) +\!\!+ \text{tail}(a, \ell)$. Since $a < \text{len}(\ell)$, we can write $\text{tail}(a, \ell) = \text{nth}(a, \ell) :: \text{tail}(sa, \ell)$ (proposition 4.4.6). Then by the second part,

$$\begin{aligned} fnz(\ell) &= fnz\Big(\text{head}(a, \ell) +\!\!+ (\text{nth}(a, \ell) :: \text{tail}(sa, \ell))\Big) \\ &\leq a + fnz(\text{nth}(a, \ell) :: \text{tail}(sa, \ell)) \\ &= a + 0. \end{aligned}$$

For the fourth part, if $a < fnz(\ell)$, then $a < \text{len}(\ell)$ by the first part. So $\text{nth}(a, \ell)$ is well-defined, and we can split into the cases $\text{nth}(a, \ell) = 0$ and $\text{nth}(a, \ell) > 0$. The latter case implies $fnz(\ell) \leq a$, which is a contradiction (see corollary 3.3.7), so we get $\text{nth}(a, \ell) = 0$, as desired. ∎

For the next proposition, we use the const arrow, which we define formally in section 15.2. Essentially, $\text{const}(x, n)$ is the list of length n whose entries are all equal to x.

Proposition 15.1.5. We have $\ell = \text{const}(0, fnz(\ell)) +\!\!+\, rz(\ell)$.

Proof. First, note that these lists have the same length. Indeed, the length of the right side is $fnz(\ell) + (\text{len}(\ell) \dot{-} fnz(\ell))$, which is equal to $\text{len}(\ell)$ since $fnz(\ell) \leq \text{len}(\ell)$. To show these lists are equal, then, it remains to show that their k^{th} elements are the same for all $k < \text{len}(\ell)$.

So, let $k < \text{len}(\ell)$. We split into two cases. If $k < fnz(\ell)$, then we have

$$\text{nth}\Big(\text{const}(0, fnz(\ell)) +\!\!+\, rz(\ell),\ k\Big) = \text{nth}(\text{const}(0, fnz(\ell)),\ k) = 0 = \text{nth}(\ell, k)$$

by proposition 15.1.4 and corollary 4.5.3. On the other hand, if $k \geq fnz(\ell)$, then

$$\begin{aligned}\text{nth}\Big(\text{const}(0, fnz(\ell)) +\!\!+\, rz(\ell),\ k\Big) &= \text{nth}(rz(\ell), k \dot{-} fnz(\ell)) \\ &= \text{nth}(\text{tail}(fnz(\ell), \ell),\ k \dot{-} fnz(\ell)) \\ &= \text{nth}(fnz(\ell) + (k \dot{-} fnz(\ell)),\ \ell) = \text{nth}(k, \ell).\end{aligned}$$

That's all we needed to show. ∎

Proposition 15.1.6. We have $gz(\ell) \leq 1$ and $gz(\ell_1 +\!\!+\, \ell_2) \geq gz(\ell_2)$.

Proof. Easy by induction. ∎

Proposition 15.1.7. If $gz(\ell) = 0$, then we have $\text{nth}(a, \ell) > 0$ for each $a < \text{len}(\ell)$, and moreover there is a unique ℓ', given by $\ell' = L(P)(\ell)$, such that $L(s)(\ell') = \ell$.

Proof. Since $a < \text{len}(\ell)$, we have $\ell = \text{head}(a, \ell) +\!\!+\, (\text{nth}(a, \ell) :: \text{tail}(sa, \ell))$. By proposition 15.1.6, we have

$$0 = gz(\ell) \geq gz(\text{nth}(a, \ell) :: \text{tail}(sa, \ell)).$$

So, $gz(\text{nth}(a, \ell) :: \text{tail}(sa, \ell)) = 0$, which implies (by definition of gz) that we cannot have $\text{nth}(a, \ell) = 0$. Thus $\text{nth}(a, \ell) > 0$, as desired.

For the second part, set $\ell' = L(P)(\ell)$. Then for any $a < \text{len}(\ell)$, we have

$$\text{nth}(L(s)(\ell'), a) = s(\text{nth}(\ell', a)) = s(\text{nth}(L(P)(\ell), a)) = s(P(\text{nth}(\ell, a))),$$

and this equals $\text{nth}(\ell, a)$ since $\text{nth}(\ell, a) > 0$. This tells us that $L(s)(\ell') = \ell$. For uniqueness, note that we must have $\text{len}(\ell') = \text{len}(\ell)$, and $\text{nth}(\ell, a) = s(\text{nth}(\ell', a))$, so $\text{nth}(\ell', a) = P(\text{nth}(\ell, a)) = \text{nth}(L(P)(\ell), a)$. This forces us to have $\ell' = L(P)(\ell)$. ∎

We can now prove the main result.

Theorem 15.1.8. Let $\mathcal{C}$ be a locos. Given $m : \mathbb{1} \to N$, if X is finite, then so is $L(X)_{\leq m}$.

Proof. Say that $X = E_n$. Note that the successor map $s : E_n \to E_{sn}$ is well-defined, so we get $L(s) : L(E_n) \to L(E_{sn})$. Then we define the following map.

$$\begin{aligned} L(E_n) &\xrightarrow{\alpha} L(E_{sn}) \\ \ell &\mapsto \text{const}(0, m \mathbin{\dot{-}} \text{len}(\ell)) \mathbin{+\!\!+} L(s)(\ell). \end{aligned}$$

Note that if $\text{len}(\ell) \leq m$, then $\text{len}(\ell) = m$, so this induces a map

$$\alpha : L(E_n)_{\leq m} \to L(E_{sn})_m.$$

Next, note that the inclusion $E_{sn} \hookrightarrow N$ induces $L(E_{sn}) \hookrightarrow L(N)$, and so we can apply the arrow ohz to terms of type $L(E_{sn})$.

We now claim that the following is an equalizer diagram.

$$L(E_n)_{\leq m} \xrightarrow{\alpha} L(E_{sn})_m \underset{0}{\overset{ohz}{\rightrightarrows}} N$$

We showed that $\mathbf{Fin}(\mathcal{C})$ is closed under equalizers (proposition 13.3.2), so this implies the desired result (together with theorem 15.1.1, which says that $L(X)_m$ is finite when X is finite). The rest of the proof is dedicated to showing that this is indeed an equalizer.

First, we note that $ohz \circ \alpha = 0$. It's easy to show by induction that $ohz(\text{const}(0, a) \mathbin{+\!\!+} \ell') = ohz(\ell')$ and $ohz(\ell') = gz(\ell') = 0$ when $\ell' = L(s)(\ell)$. Thus, given some $\ell : L(E_n)_{\leq m}$, applying ohz to $\alpha(\ell)$ does indeed give 0.

Now, we show the equalizer property. Let $\ell : L(E_{sn})_m$ be a term in some context, and assume that $ohz(\ell) = 0$. We must show that there is a unique $\ell' : L(E_n)_{\leq m}$ such that $\alpha(\ell') = \ell$. We claim that $\ell' = L(P)(rz(\ell))$ satisfies this condition.

First, we must show that $L(P)(rz(\ell))$ is a valid element of $L(E_n)_{\leq m}$. It's clear that $\text{len}(L(P)(rz(\ell))) = \text{len}(rz(\ell)) \leq \text{len}(\ell) = m$. Moreover, it's easy to check that $rz(\ell) : L(E_{sn})$ since $\ell : L(E_{sn})$, so applying P to each component gives us something in E_n. Thus, this ℓ' is a valid candidate.

Next, since we assume that $0 = ohz(\ell) = gz(rz(\ell))$, proposition 15.1.7 tells us that $\ell' = L(P)(rz(\ell))$ is the unique term such that $L(s)(\ell') = rz(\ell)$. Moreover, we know by proposition 15.1.5 that

$$\ell = \text{const}(0, fnz(\ell)) \mathbin{+\!\!+} rz(\ell),$$

and $\mathrm{len}(\ell') = \mathrm{len}(rz(\ell)) = \mathrm{len}(\ell) \dot{-} fnz(\ell) = m \dot{-} fnz(\ell)$, so

$$\alpha(\ell') = \mathrm{const}(0, m \dot{-} \mathrm{len}(\ell')) +\!\!+ L(s)(\ell') = \mathrm{const}(0, fnz(\ell)) +\!\!+ rz(\ell) = \ell.$$

Thus, ℓ' is the desired term.

Finally, we must show uniqueness. Suppose that $\ell' : L(E_n)_{\leq m}$ satisfies $\alpha(\ell') = \ell$. We claim that we must have $fnz(\ell) = m \dot{-} \mathrm{len}(\ell')$. We split into two cases: if $\ell' = \varnothing$, then $\ell = \mathrm{const}(0, m)$ and we find that $fnz(\ell) = m$, so we get the equality. On the other hand, if $\ell' = k :: \ell''$, then

$$\begin{aligned} fnz(\ell) &= fnz\Big(\mathrm{const}(0, m \dot{-} \mathrm{len}(\ell')) +\!\!+ (sk :: L(s)(\ell''))\Big) \\ &= (m \dot{-} \mathrm{len}(\ell')) + fnz(sk :: L(s)(\ell'')) \\ &= m \dot{-} \mathrm{len}(\ell') + 0. \end{aligned}$$

Thus, $\ell = \mathrm{const}(0, fnz(\ell)) +\!\!+ L(s)(\ell')$. But we also know that $\ell = \mathrm{const}(0, fnz(\ell)) +\!\!+ rz(\ell)$; by taking the tail of both of these, we find that $L(s)(\ell') = rz(\ell)$. By proposition 15.1.7, this tells us that $\ell' = L(P)(rz(\ell))$, as desired. ∎

15.2 The pigeonhole principle

In this section, we prove an internal version of the pigeonhole principle (theorem 15.2.6), which will be a key tool for discussing general coequalizers of finite objects. (Indeed, we only use this result in section 15.4, while working towards the proof of theorem 15.4.5.)

Proving this internal pigeonhole principle requires some more technical tools, which we establish below. Throughout this section, we work in a locos.

Pointwise sum of lists

First, let $\mathrm{sum} : L(N) \to N$ be the map inductively by $\mathrm{sum}(\varnothing) = 0$ and $\mathrm{sum}(x :: \ell) = x + \mathrm{sum}(\ell)$. It's easy to show that $\mathrm{sum}(\ell_1 +\!\!+ \ell_2) = \mathrm{sum}(\ell_1) + \mathrm{sum}(\ell_2)$ by induction on ℓ_1.

Next, we want a map that can add two lists in $L(N)$ component wise. We start with a map $\mathrm{headsum} : N \times L(N) \times L(N) \to L(N)$ defined inductively by

$$\mathrm{headsum}(0, \ell_1, \ell_2) = \varnothing,$$
$$\mathrm{headsum}(sn, \ell_1, \ell_2) = (\mathrm{zerothDef}(\ell_1, 0) + \mathrm{zerothDef}(\ell_2, 0)) :: \mathrm{headsum}(n, \mathrm{tr}(\ell_1), \mathrm{tr}(\ell_2)).$$

Note that we require the induction scheme of theorem 3.5.3 for this definition. Then, consider $L(N) \times_N L(N)$, the collection of pairs of lists of equal length. We define the arrow $+ : L(N) \times_N L(N) \to L(N)$ by

$$\ell_1 + \ell_2 = \mathrm{headsum}(\mathrm{len}(\ell_1), \ell_1, \ell_2).$$

Note that this is different from the concatenation of lists, which is denoted $\ell_1 +\!\!+ \ell_2$. This arrow has the following properties.

Proposition 15.2.1. In a locos, the following are true.

1. $\mathrm{len}(\ell_1 + \ell_2) = \mathrm{len}(\ell_1) = \mathrm{len}(\ell_2)$;
2. $\mathrm{nth}(k, \ell_1 + \ell_2) = \mathrm{nth}(k, \ell_1) + \mathrm{nth}(k, \ell_2)$ (for $k < \mathrm{len}(\ell_1) = \mathrm{len}(\ell_2)$);
3. $\mathrm{sum}(\ell_1 + \ell_2) = \mathrm{sum}(\ell_1) + \mathrm{sum}(\ell_2)$.

Proof. For the first item, it's easy to show that $\mathrm{len}(\mathrm{headsum}(n, \ell_1, \ell_2)) = n$ (go by induction on n), so the result follows easily.

For the second item, it's easy to show (by cases on n) that $\mathrm{tr}(\mathrm{headsum}(n, \ell_1, \ell_2)) = \mathrm{headsum}(Pn, \mathrm{tr}(\ell_1), \mathrm{tr}(\ell_2))$, and it follows that

$$\mathrm{tail}(m,\ \mathrm{headsum}(m, \ell_1, \ell_2)) = \mathrm{headsum}(n \dot{-} m,\ \mathrm{tail}(m, \ell_1),\ \mathrm{tail}(m, \ell_2)).$$

Therefore, if $k < \mathrm{len}(\ell_1)$, then we can write $\mathrm{len}(\ell_1) \dot{-} k = sn$ for some n, and we have

$$\begin{aligned} \mathrm{tail}(k, \mathrm{headsum}(\mathrm{len}(\ell_1), \ell_1, \ell_2)) &= \mathrm{headsum}(sn, \mathrm{tail}(k, \ell_1), \mathrm{tail}(k, \ell_2)) \\ = \big(zDef(\mathrm{tail}(k, \ell_1), 0) + zDef(\mathrm{tail}(k, \ell_2), 0)\big) &:: \mathrm{headsum}(n, \mathrm{tail}(sk, \ell_1), \mathrm{tail}(sk, \ell_2)) \\ &= (\mathrm{nth}(k, \ell_1) + \mathrm{nth}(k, \ell_2)) :: \mathrm{headsum}(n, \mathrm{tail}(sk, \ell_1), \mathrm{tail}(sk, \ell_2)). \end{aligned}$$

(For the last equality, we used the assumption $k < \mathrm{len}(\ell_1) = \mathrm{len}(\ell_2)$.) Therefore,

$$\begin{aligned} \mathrm{nth}(k, \ell_1 + \ell_2) &= \mathrm{zeroth}(\mathrm{tail}(k, \mathrm{headsum}(\mathrm{len}(\ell_1), \ell_1, \ell_2))) \\ &= \mathrm{nth}(k, \ell_1) + \mathrm{nth}(k, \ell_2). \end{aligned}$$

Finally, for the third point, proposition 4.6.2 tells us that it suffices to show

$$\mathrm{headsum}(n, \ell_1, \ell_2) = \mathrm{sum}(\mathrm{head}(n, \ell_1)) + \mathrm{sum}(\mathrm{head}(n, \ell_2)).$$

We show that the left hand side satisfies the defining equations of headsum. The case $n = 0$ is easy; for the induction step, it suffices to show that

$$\mathrm{sum}(\mathrm{head}(sn, \ell)) = \mathrm{zerothDef}(\ell, 0) + \mathrm{sum}(\mathrm{head}(n, \mathrm{tr}(\ell))).$$

To see this, we split into cases with ℓ. If $\ell = \varnothing$, then clearly both sides equal zero. If $\ell = x :: \ell'$, then the left side is

$$\mathrm{sum}(\mathrm{head}(sn, x :: \ell)) = \mathrm{sum}(x :: \mathrm{head}(n, \ell)) = x + \mathrm{sum}(\mathrm{head}(n, \ell))$$

(using proposition 4.6.2), while the right side is

$$\text{zerothDef}(x :: \ell, 0) + \text{sum}(\text{head}(n, \text{tr}(x :: \ell))) = x + \text{sum}(\text{head}(n, \ell)),$$

as desired. ∎

Proposition 15.2.2. In a locos, let $\ell_1, \ell_2 : L(N)$ be two terms in a context C such that $\text{len}(\ell_1) = \text{len}(\ell_2)$. If $\text{nth}(k, \ell_1) \leq \text{nth}(k, \ell_2)$ holds in the context $(C, k : N \mid k < \text{len}(\ell_1))$, then $\text{sum}(\ell_1) \leq \text{sum}(\ell_2)$.

Proof. Denote $n = \text{len}(\ell_1) = \text{len}(\ell_2)$, and for simplicity, let C be $(i : I)$. Then $n : I \to N$, and we have a map $I \times_n E \to N$ given by $(i, k) \mapsto \text{nth}(k, \ell_2) \dot{-} \text{nth}(k, \ell_1)$. Then there is a list $\ell_3 : I \to L(N)$ such that $\text{len}(\ell_3) = n$ and $\text{nth}(k, \ell_3) = \text{nth}(k, \ell_2) \dot{-} \text{nth}(k, \ell_1)$ (by theorem 5.4.1).

Now, since $\text{nth}(k, \ell_1) \leq \text{nth}(k, \ell_2)$, we have $\text{nth}(k, \ell_1) + \text{nth}(k, \ell_3) = \text{nth}(k, \ell_2)$. But this implies $\ell_2 = \ell_1 + \ell_3$: these lists have the same length and elements (by proposition 15.2.1), so we get equality by theorem 5.4.1. Using proposition 15.2.1 one more time, we have $\text{sum}(\ell_2) = \text{sum}(\ell_1) + \text{sum}(\ell_3)$, so we get $\text{sum}(\ell_1) \leq \text{sum}(\ell_2)$, as desired. ∎

Constant lists

We can define a map $\text{const} : X \times N \to L(X)$ which maps (x, n) to the list of length n which has the value x in each position. There are two equivalent ways of defining this map. First, we can define this inductively: $\text{const}(x, 0) = \varnothing$ and $\text{const}(x, sn) = x :: \text{const}(x, n)$. Second, we can define this via the list adjunction (theorem 5.4.1): the length of $\text{const}(x, n)$ is n, and $\text{nth}(k, \text{const}(x, n)) = x$. (It's not hard to show that these definitions are the same.) It's easy to show that, when dealing with lists in $L(N)$, we have $\text{sum}(\text{const}(x, n)) = x \bullet n$.

Next, we can define a map $\text{single} : N \times N \to L(N)$ given by

$$\text{single}(a, b) = \text{const}(0, a) +\!\!+ [1] +\!\!+ \text{const}(0, b).$$

It's clear that $\text{len}(\text{single}(a, b)) = a + b + 1$ and $\text{sum}(\text{single}(a, b)) = 1$. Moreover, we can use corollary 4.5.3 to show that:

- if $k < a$, then $\text{nth}(k, \text{single}(a, b)) = 0$;
- $\text{nth}(a, \text{single}(a, b)) = 1$;
- if $k < b$, then $\text{nth}(a + 1 + k, \text{single}(a, b)) = 0$.

In short, for $k < a + b + 1$, we have

$$\text{nth}(k, \text{single}(a, b)) = \begin{cases} 1 & \text{if } |k, a| = 0 \\ 0 & \text{else} \end{cases}.$$

(To prove this formally, we can use a trichotomy argument like in the proof of theorem 3.6.2.)

Counting occurrences

First, we define a map $\mathrm{eqTests} : N \times L(N) \to L(N)$ by induction as follows:

$$\mathrm{eqTests}(x, \varnothing) = \varnothing, \qquad \mathrm{eqTests}(x, y :: \ell) = |x, y| :: \mathrm{eqTests}(x, \ell).$$

Proposition 15.2.3. Let $\mathcal{C}$ be a locos. Then we have $\mathrm{len}(\mathrm{eqTests}(x, \ell)) = \mathrm{len}(\ell)$, and for $k < \mathrm{len}(\ell)$, we have $\mathrm{nth}(k, \mathrm{eqTests}(x, \ell)) = |x, \mathrm{nth}(k, \ell)|$.

Proof. The first equality is an easy proof by induction on ℓ. For the second, we first note that $\mathrm{tr}(\mathrm{eqTests}(x, \ell)) = \mathrm{eqTests}(x, \mathrm{tr}(\ell))$ (by induction on ℓ), and therefore $\mathrm{tail}(k, \mathrm{eqTests}(x, \ell)) = \mathrm{eqTests}(x, \mathrm{tail}(k, \ell))$ (by induction on k).

Now, suppose that $k < \mathrm{len}(\ell)$. Then we can write $\mathrm{tail}(k, \ell) = y :: \ell'$, and $\mathrm{nth}(k, \ell) = y$. But then we compute

$$\begin{aligned}
\mathrm{nth}(k, \mathrm{eqTests}(x, \ell)) &= \mathrm{zeroth}(\mathrm{tail}(k, \mathrm{eqTests}(x, \ell))) \\
&= \mathrm{zeroth}(\mathrm{eqTests}(x, \mathrm{tail}(k, \ell))) \\
&= \mathrm{zeroth}(\mathrm{eqTests}(x, y :: \ell')) \\
&= \mathrm{zeroth}(|x, y| :: \mathrm{eqTests}(x, \ell')) \\
&= |x, y| = |x, \mathrm{nth}(k, \ell)|,
\end{aligned}$$

which is what we wanted. ∎

Next, we fix some $m : \mathbb{1} \to N$, and define a map $\mathrm{countOc}_m : L(E_m) \to L(N)$ which takes a list $\ell : L(E_m)$ and outputs a list of length m whose k^{th} entry tells us how many times k occurs in the list ℓ. This is defined as follows: we have a map $E_m \times L(E_m) \to N$ given by $(k, \ell) \mapsto cZ(\mathrm{eqTests}(k, \ell))$, so by proposition 13.1.2, we can define $\mathrm{countOc}_m$ as the unique map such that $\mathrm{len}(\mathrm{countOc}_m(\ell)) = m$ and $\mathrm{nth}(k, \mathrm{countOc}_m(\ell)) = cZ(\mathrm{eqTests}(k, \ell))$.

Proposition 15.2.4. In a locos, let $m : \mathbb{1} \to N$. Then, for any $\ell : L(E_m)$, we have $\mathrm{sum}(\mathrm{countOc}_m(\ell)) = \mathrm{len}(\ell)$.

Proof. We will show this by induction on ℓ. First, consider the base case $\ell = \varnothing$. For any $k < m$, we have

$$\mathrm{nth}(k, \mathrm{countOc}_m(\varnothing)) = cZ(\mathrm{eqTests}(k, \varnothing)) = cZ(\varnothing) = 0,$$

and so we conclude by uniqueness that $\mathrm{countOc}_m(\varnothing) = \mathrm{const}(0, m)$. Therefore,

$$\mathrm{sum}(\mathrm{countOc}_m(\varnothing)) = \mathrm{sum}(\mathrm{const}(0, m)) = 0.$$

For the inductive step, we claim that

$$\mathrm{countOc}_m(y :: \ell) = \mathrm{countOc}_m(\ell) + \mathrm{single}(y, m \dot{-} sy).$$

First, we check that all the lists involved have the same length. We know by definition that $\mathrm{countOc}_m(y :: \ell)$ and $\mathrm{countOc}_m(\ell)$ have length m. For the third term, since $y < m$ (because $y : E_m$), we have $sy \leq m$, and so $1 + y + (m \dot{-} sy) = sy + (m \dot{-} sy) = m$. Hence, the lists have the same length.

Next, for $k < m$, we compute

$$\begin{aligned}
\mathrm{nth}(k, \mathrm{countOc}_m(y :: \ell)) &= cZ(\mathrm{eqTests}(k, y :: \ell)) = cZ(|k, y| :: \mathrm{eqTests}(k, \ell)) \\
&= \begin{cases} 1 + cZ(\mathrm{eqTests}(k, \ell)) & \text{if } |k, y| = 0 \\ cZ(\mathrm{eqTests}(k, \ell)) & \text{else} \end{cases} \\
&= cZ(\mathrm{eqTests}(k, \ell)) + \begin{cases} 1 & \text{if } |k, y| = 0 \\ 0 & \text{else} \end{cases} \\
&= \mathrm{nth}(k, \mathrm{countOc}_m(\ell)) + \mathrm{nth}(k, \mathrm{single}(y, m \dot{-} sy)).
\end{aligned}$$

By uniqueness and proposition 15.2.1, we conclude $\mathrm{countOc}_m(y :: \ell) = \mathrm{countOc}_m(\ell) + \mathrm{single}(y, m \dot{-} sy)$, as desired.

Finally, with this equality, we can check the inductive step. Using proposition 15.2.1, we compute

$$\begin{aligned}
\mathrm{sum}(\mathrm{countOc}_m(y :: \ell)) &= \mathrm{sum}(\mathrm{countOc}_m(\ell)) + \mathrm{sum}(\mathrm{single}(y, m \dot{-} sy)) \\
&= \mathrm{sum}(\mathrm{countOc}_m(\ell)) + 1,
\end{aligned}$$

which is what we wanted. ∎

Final result

Before we prove the final result, we need a fact about the "count zeroes" operation from appendix B.

Proposition 15.2.5. In a locos, let $\ell : L(\{0, 1\})$ be a term in a context C. If $cZ(\ell) = 0$, then $\mathrm{nth}(k, \ell) > 0$ in the context $(C, k : N \mid k < \mathrm{len}(\ell))$.

Proof. In the context $(C, k : N \mid k < \mathrm{len}(\ell))$, consider the term k. Since $k < \mathrm{len}(\ell)$, we know that $\mathrm{tail}(k, \ell) = \mathrm{nth}(k, \ell) :: \mathrm{tail}(sk, \ell)$ (see proposition 4.4.6). Now, suppose $\mathrm{nth}(k, \ell) = 0$, so that $\mathrm{tail}(k, \ell) = 0 :: \mathrm{tail}(sk, \ell)$. Then, by propositions 4.6.3 and B.5.2, we have

$$\begin{aligned}
cZ(\ell) &= cZ(\mathrm{head}(k, \ell)) + cZ(\mathrm{tail}(k, \ell)) \\
&\geq cZ(\mathrm{tail}(k, \ell))
\end{aligned}$$

$$= cZ(0 :: \text{tail}(sk, \ell)) = 1 + cZ(\text{tail}(sk, \ell)) \geq 1.$$

But this contradicts the assumption $cZ(\ell) = 0$, so we must have $\text{nth}(k, \ell) > 0$ (using corollary 3.3.7). ∎

Theorem 15.2.6 (Internal Pigeonhole Principle)**.** Let $m : \mathbb{1} \to N$, and let $\ell : L(E_m)$ be a term in a context C such that $\text{len}(\ell) > m$. Then there are terms $0 \leq a < b < \text{len}(\ell)$ such that $\text{nth}(a, \ell) = \text{nth}(b, \ell)$.

Proof. Define $f : N \to N$ by $f(x) = 1 \dot{-} (x \dot{-} 1)$, and let $\ell' = L(f)(\text{countOc}_m(\ell))$. We claim that $cZ(\ell') > 0$.

For a contradiction (using corollary 3.3.7), suppose that $cZ(\ell') = 0$. By proposition 15.2.5, we have $\text{nth}(k, \ell') > 0$ in the context $(C, k : N \mid k < \text{len}(\ell'))$. That is, in the context $(C, k : N \mid k < \text{len}(\ell'))$, we have

$$\begin{aligned} 0 &< \text{nth}(k, \ell') \\ &= \text{nth}\big(k, L(f)(\text{countOc}_m(\ell))\big) \\ &= f\big(\text{nth}(k, \text{countOc}_m(\ell))\big). \end{aligned}$$

(For the second equality, we used proposition 5.1.1.) Note that $1 \dot{-} a > 0$ implies $a = 0$ (see proposition 3.2.6), so this inequality implies $0 = \text{nth}(k, \text{countOc}_m(\ell)) \dot{-} 1$, i.e.

$$\text{nth}(k, \text{countOc}_m(\ell)) \leq 1 = \text{nth}(k, \text{const}(1, m)).$$

Using propositions 15.2.2 and 15.2.4, this implies that

$$\text{len}(\ell) = \text{sum}(\text{countOc}_m(\ell)) \leq \text{sum}(\text{const}(1, m)) = m,$$

a contradiction to our assumption $\text{len}(\ell) > m$.

So, we have $cZ(\ell') > 0$. By proposition B.8.1, we have $\text{nth}(v, \ell') = 0$ for some $v < \text{len}(\ell')$ (specifically, for $v = kZ(\ell', 0)$). Then, as above, we compute

$$\begin{aligned} 0 &= \text{nth}(v, \ell') \\ &= \text{nth}\big(v, L(f)(\text{countOc}_m(\ell))\big) \\ &= f\big(\text{nth}(v, \text{countOc}_m(\ell))\big) \\ &= f\big(cZ(\text{eqTests}(v, \ell))\big). \end{aligned}$$

Note that $1 \dot{-} a = 0$ implies $a \geq 1$ (by definition), so the above inequality implies

$$cZ(\text{eqTests}(v, \ell)) \dot{-} 1 \geq 1.$$

Using proposition 3.2.4, this gives us $cZ(\text{eqTests}(v, \ell)) \geq 2$.

Now, let $a = kZ(\text{eqTests}(v,\ell),0)$ and $b = kZ(\text{eqTests}(v,\ell),1)$. Since we showed $cZ(\text{eqTests}(v,\ell)) \geq 2$, proposition B.6.3 and lemma B.7.3 tell us that

$$0 \leq a < b < \text{len}(\text{eqTests}(v,\ell)) = \text{len}(\ell).$$

(We also use proposition 15.2.3 for the equality). Proposition B.8.1 tells us that

$$\text{nth}(a, \text{eqTests}(v,\ell)) = 0 = \text{nth}(b, \text{eqTests}(v,\ell)).$$

And, proposition 15.2.3 tells us that

$$\text{nth}(a, \text{eqTests}(v,\ell)) = |v, \text{nth}(a,\ell)|, \qquad \text{nth}(b, \text{eqTests}(v,\ell)) = |v, \text{nth}(b,\ell)|,$$

so we get $\text{nth}(a,\ell) = v = \text{nth}(b,\ell)$. Thus, we have found values of a and b that satisfy the conditions we want. ∎

15.3 Quotients of equivalence relations

In this section, we show that quotients of finite objects by equivalence relations exist and are finite. In fact, we show that these quotients are split, and thus absolute.

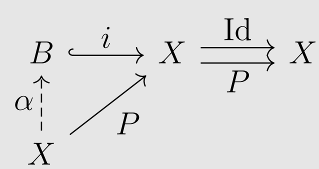
We start by proving some intermediate results.

Proposition 15.3.1. Let $\mathcal{C}$ be a lextensive category, and let $f, g : R \rightrightarrows X$. Suppose there exists a map $P : X \to X$ such that

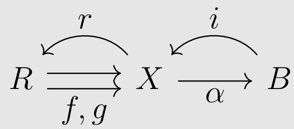
- $P^2 = P$;
- $Pf = Pg$; and
- $\langle \text{Id}, P \rangle : X \to X \times X$ factors through $\langle f, g \rangle : R \to X \times X$.

Then $\{x \in X \mid x = Px\}$ is the coequalizer of f, g. In fact, this object is part of a split coequalizer.

Proof. Let $B = \{x \in X \mid x = Px\}$. It is equipped with the inclusion $i : B \hookrightarrow X$ which makes it the equalizer of P and Id_X. Using the assumption $P^2 = P$, we can construct $\alpha : X \to B$ with the universal property as follows.



Finally, let $r : X \to R$ be the arrow witnessing that $\langle \text{Id}, P \rangle$ factors through R. Then consider the following diagram.



To prove the result, we will show that this diagram is a split coequalizer. To do this, we need to prove four equalities:

1. $\alpha \circ f = \alpha \circ g$;
2. $\alpha \circ i = \mathrm{Id}_B$;
3. $f \circ r = \mathrm{Id}_X$;
4. $g \circ r = i \circ \alpha$.

For the first one, we use the assumption $Pf = Pg$ and the definition of α to compute

$$i \circ \alpha \circ f = P \circ f = P \circ g = i \circ \alpha \circ g,$$

and so $\alpha \circ f = \alpha \circ g$ since i is a monomorphism.

For the second, we have $i \circ \alpha \circ i = P \circ i = i$ (by definition of α and because i equalizes P, Id), so $\alpha \circ i = \mathrm{Id}$ because i is a monomorphism.

For the third and fourth, we note that $\langle f, g \rangle \circ r = \langle \mathrm{Id}, P \rangle$ by definition of r. Taking the first projection gives $f \circ r = \mathrm{Id}$, and taking the second projection gives $g \circ r = P = i \circ \alpha$ (by definition of α). ∎

Proposition 15.3.2. Let $\mathcal{C}$ be a locos, let $n : \mathbb{1} \to N$, let $\langle f, g \rangle : R \hookrightarrow E_n \times E_n$ be a complemented equivalence relation on a finite object E_n, and let $d : E_n \times E_n \to \{0, 1\}$ be the associated map (see theorems 2.3.1 and 2.3.5).

Suppose there exists a map $P : E_n \to E_n$ such that

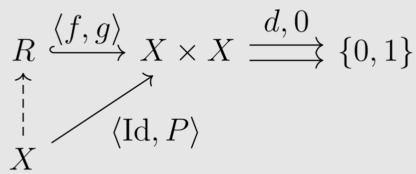
- $d(x, Px) =_x 0$;
- if $x, z : E_n$ are terms such that $d(x, z) = 0$, then $Px \leq z$. (Note that $\leq$ is applied by noting that $E_n \hookrightarrow N$.)

Then $P^2 = P$, $Pf = Pg$, and $\langle \mathrm{Id}, P \rangle$ factors through R. Thus, by proposition 15.3.1, $\{x \in E_n \mid x = Px\}$ is the quotient E_n/R (in particular, it is a split coequalizer).

Proof. First, we show $P^2 = P$. Let $x \in E_n$, i.e. $x \in N$ with $x < n$. Then $d(x, Px) = 0$ and $d(Px, P(Px)) = 0$ by assumption. By symmetry and transitivity of d, we have $d(x, P^2x) = 0$. By the second assumption, this tells us that $Px \leq P^2x$. On the other hand, $d(Px, Px) = 0$ by reflexivity, so the second assumption tells us $P^2x \leq Px$. Thus $Px = P^2x$.

Next, we show $Pf = Pg$. Let $r : R$. By definition of d, we have $d(f(r), g(r)) = 0$; and by the first assumption, we have $d(f(r), Pf(r)) = 0$. Using symmetry and transitivity of d, we get $d(g(r), Pf(r)) = 0$, and so the second assumption implies that $Pg(r) \leq Pf(r)$. But we can obtain $Pf(r) \leq Pg(r)$ in the same way, so we conclude $Pf(r) = Pg(r)$.

Finally, we claim that $\langle \mathrm{Id}, P \rangle$ factors through R. Since $\langle f, g \rangle$ is complemented, it is the equalizer of $d, 0$.



Therefore, by the equalizer property, to show $\langle \mathrm{Id}, P \rangle$ factors through R, it suffices to show that $d \circ \langle \mathrm{Id}, P \rangle = 0$. But this is just saying $d(x, Px) = 0$, which is true by assumption. ∎

Proposition 15.3.3. Let $\mathcal{C}$ be a locos, and let $n : \mathbb{1} \to N$. For any $d : E_n \times E_n \to \{0, 1\}$ which satisfies the equivalence relation axioms (see theorem 2.3.5), there exists a map $P : E_n \to E_n$ such that

1. $d(x, Px) =_x 0$;
2. if $x, z : E_n$ are terms in a context C such that $d(x, z) =_C 0$, then $Px \leq_C z$.

Proof. Since we have $d : E_n \times E_n \to \{0, 1\}$, proposition 13.1.2 tells us that there exists a (unique) arrow $g : E_n \to L(\{0, 1\})$ such that $\mathrm{len}(g(x)) =_x n$ and $\mathrm{nth}(g(x), k) =_{x,k} d(x, k)$. Then, let

$$P(x) =_x zZ(g(x)).$$

(The arrow zZ is from appendix B.) Note that $P : E_n \to N$. However, since d is reflexive, we have $0 =_x d(x, x) =_x \mathrm{nth}(g(x), x)$, so proposition B.2.3 tells us that

$$P(x) =_x zZ(g(x)) \leq_x x <_x n.$$

Thus, we can say that $P : E_n \to E_n$. We now check that P satisfies the desired properties.

For (1), note that

$$d(x, Px) = \mathrm{nth}(g(x), Px) = \mathrm{nth}\big(g(x), zZ(g(x))\big).$$

Since $zZ(g(x)) < n = \mathrm{len}(g(x))$ (as noted above), we can apply proposition B.2.1 to conclude that $\mathrm{nth}(g(x), zZ(g(x))) = 0$, as desired.

For (2), if $d(x, z) = 0$, then $\mathrm{nth}(g(x), z) = 0$, so by proposition B.2.3 we conclude that

$$P(x) = zZ(g(x)) \leq z,$$

as desired. ∎

Corollary 15.3.4. Let $\mathcal{C}$ be a locos, let $B \in \mathcal{C}$, and let $A \hookrightarrow B \times B$ be an equivalence relation on B. If A and B are finite, then the quotient B/A exists and is finite, and $A \rightrightarrows B \to B/A$ is a split coequalizer.

In particular, $\mathbf{Fin}(\mathcal{C})$ has split quotients of equivalence relations.

Proof. Since B is finite, so is $B \times B$ (by proposition 13.3.3). Therefore, the subobject $A \hookrightarrow B \times B$ is complemented (by theorem 13.2.3), so we get an associated equivalence relation map $d : B \times B \to \{0, 1\}$ (by theorems 2.3.1 and 2.3.5).

Now, since B is finite, say that $B = E_n$ for some $n : \mathbb{1} \to N$. Since $d : E_n \times E_n \to \{0, 1\}$ is an equivalence relation map, proposition 15.3.3 tells us there exists a map $P : E_n \to E_n$ such that (1) $d(x, Px) = 0$, and (2) $d(x, z) = 0$ implies $Px \leq z$. By proposition 15.3.2, this implies that $\{x \in E_n \mid x = Px\}$ is the desired quotient, and it is a split coequalizer.

It remains to show that this quotient is finite; since it is defined as the equalizer of the maps $\mathrm{Id}, P : E_n \to E_n$, this is true by proposition 13.3.2. ∎

15.4 General coequalizers

Finally, in this section, we show that $\mathbf{Fin}(\mathcal{C})$ has arbitrary coequalizers, and that these are preserved by the inclusion into $\mathcal{C}$. This is stated as theorem 15.4.5; as we'll see in its proof, the strategy is based on Maietti's construction of coequalizers in an arithmetic universe (proposition 3.10 of [Maie 10]).

To achieve this proof, we'll need some technical tools, which we establish before arriving at the main result.

Cutting out repeats from a list

We define a map cutRep : $L(N) \to L(N)$ by induction as follows:

$$\begin{aligned} \operatorname{cutRep}(\varnothing) &= \varnothing, \\ \operatorname{cutRep}(x :: \ell) &= \begin{cases} \operatorname{tail}\big(zz(\operatorname{eqTests}(x,\ell)),\ \ell\big) & \text{if } zz(\operatorname{eqTests}(x,\ell)) < \operatorname{len}(\ell) \\ x :: \operatorname{cutRep}(\ell) & \text{else} \end{cases}. \end{aligned}$$

Proposition 15.4.1. In a locos $\mathcal{C}$, we have $\operatorname{len}(\operatorname{cutRep}(\ell)) \leq \operatorname{len}(\ell)$ and

$$\operatorname{len}(\operatorname{cutRep}(\ell)) \leq \max\Big\{P(\operatorname{len}(\ell)),\ k + \operatorname{len}(\operatorname{cutRep}(\operatorname{tail}(k,\ell)))\Big\}.$$

Proof. The first part is an easy induction on ℓ: it's clear that $\operatorname{len}(\operatorname{tail}(k,\ell)) \leq \operatorname{len}(\ell)$ for any k, so we have

$$\operatorname{len}(\operatorname{cutRep}(x :: \ell)) \leq \begin{cases} \operatorname{len}(\ell) & \text{if } zz(\operatorname{eqTests}(x,\ell)) < \operatorname{len}(\ell) \\ 1 + \operatorname{len}(\operatorname{cutRep}(\ell)) & \text{else} \end{cases}.$$

The inequality then follows easily by induction.

For the second part, if we use induction on k, it suffices to show that

$$\operatorname{len}(\operatorname{cutRep}(\ell)) \leq \max\Big\{P(\operatorname{len}(\ell)),\ 1 + \operatorname{len}(\operatorname{cutRep}(\operatorname{tr}(\ell)))\Big\}.$$

To show this, we go by cases on ℓ. The case $\ell = \varnothing$ is easy; for the case $\ell = x :: \ell'$, we start by noting as before that

$$\operatorname{len}(\operatorname{cutRep}(x :: \ell')) \leq \begin{cases} \operatorname{len}(\ell') & \text{if } zz(\operatorname{eqTests}(x,\ell')) < \operatorname{len}(\ell') \\ 1 + \operatorname{len}(\operatorname{cutRep}(\ell')) & \text{else} \end{cases}.$$

Then, since

$$P(\operatorname{len}(x :: \ell')) = \operatorname{len}(\ell') \quad \text{and} \quad 1 + \operatorname{len}(\operatorname{cutRep}(\operatorname{tr}(x :: \ell'))) = 1 + \operatorname{len}(\operatorname{cutRep}(\ell')),$$

the desired inequality is clear. ∎

Proposition 15.4.2. Let $\mathcal{C}$ be a locos, and let $m : \mathbb{1} \to N$. If $\ell : L(E_m)$ is a term in a context C with $\operatorname{len}(\ell) > m$, then $\operatorname{len}(\operatorname{cutRep}(\ell)) \leq P(\operatorname{len}(\ell))$.

Proof. If $\operatorname{len}(\ell) > m$, then $\operatorname{len}(\ell) \geq sm$, so $\ell : L(E_m)_{sm}$. By the internal pigeonhole principle (theorem 15.2.6), there are terms $0 \leq a < b < \operatorname{len}(\ell)$ such that $\operatorname{nth}(a,\ell) = \operatorname{nth}(b,\ell)$. Write x for this common value.

We start by using proposition 15.2.3 to note that

$$\begin{aligned}\operatorname{nth}\big(b \mathbin{\dot{-}} sa,\ \operatorname{eqTests}(x, \operatorname{tail}(sa,\ell))\big) &= |x, \operatorname{nth}(b \mathbin{\dot{-}} sa,\ \operatorname{tail}(sa,\ell))| \\ &= |x, \operatorname{nth}((b \mathbin{\dot{-}} sa) + sa, \ell)| \\ &= |x, \operatorname{nth}(b,\ell)| = |x,x| = 0.\end{aligned}$$

Therefore, by proposition B.2.3, we have

$$zz(\operatorname{eqTests}(x, \operatorname{tail}(sa,\ell))) \leq b \mathbin{\dot{-}} sa < \operatorname{len}(\operatorname{tail}(sa,\ell)).$$

Now, note that $\operatorname{tail}(a,\ell) = \operatorname{nth}(a,\ell) :: \operatorname{tail}(sa,\ell) = x :: \operatorname{tail}(sa,\ell)$ by proposition 4.4.6, and so by applying the definition of cutRep with the above inequality, we have

$$\begin{aligned}\operatorname{cutRep}(\operatorname{tail}(a,\ell)) &= \operatorname{cutRep}\big(x :: \operatorname{tail}(sa,\ell)\big) \\ &= \operatorname{tail}\Big(zz(\operatorname{eqTests}(x, \operatorname{tail}(sa,\ell))),\ \operatorname{tail}(sa,\ell)\Big).\end{aligned}$$

Therefore,

$$\operatorname{len}(\operatorname{cutRep}(\operatorname{tail}(a,\ell))) \leq \operatorname{len}(\operatorname{tail}(sa,\ell)) \leq \operatorname{len}(\ell) \mathbin{\dot{-}} sa.$$

With the above equality, we find that $a+\operatorname{len}(\operatorname{cutRep}(\operatorname{tail}(a,\ell))) \leq a+(\operatorname{len}(\ell)\mathbin{\dot{-}}sa) = P(\operatorname{len}(\ell))$ (because $\operatorname{len}(\ell) > b \geq sa$). Combining this with proposition 15.4.1, we have

$$\begin{aligned}\operatorname{len}(\operatorname{cutRep}(\ell)) &\leq \max\Big\{P(\operatorname{len}(\ell)),\ a + \operatorname{len}(\operatorname{cutRep}(\operatorname{tail}(a,\ell)))\Big\} \\ &\leq \max\Big\{P(\operatorname{len}(\ell)),\ P(\operatorname{len}(\ell))\Big\} \\ &= P(\operatorname{len}(\ell)),\end{aligned}$$

which is what we wanted to show. ∎

Corollary 15.4.3. Let $\mathcal{C}$ be a locos, and let $m : \mathbb{1} \to N$. Then for any $\ell : L(E_m)$, we have

$$\operatorname{len}(\operatorname{cutRep}^k(\ell)) \leq \max\{m, \operatorname{len}(\ell) \mathbin{\dot{-}} k\}.$$

Proof. First, we show that $\operatorname{len}(\operatorname{cutRep}(\ell)) \leq \max\{m, P(\operatorname{len}(\ell))\}$. This is easy by splitting into cases: if $\operatorname{len}(\ell) \leq m$, then by proposition 15.4.1 we have $\operatorname{len}(\operatorname{cutRep}(\ell)) \leq \operatorname{len}(\ell) \leq m$. On the other hand, if $\operatorname{len}(\ell) > m$, then proposition 15.4.2 tells us that $\operatorname{len}(\operatorname{cutRep}(\ell)) \leq P(\operatorname{len}(\ell))$.

The desired inequality follows by induction on k. ∎

Definition. Let $m : \mathbb{1} \to N$. We define the arrow $\operatorname{cut}_m : L(E_m) \to L(E_m)$ by setting $\operatorname{cut}_m(\ell) = \operatorname{cutRep}^{\operatorname{len}(\ell)}(\ell)$.

Corollary 15.4.4. Let $\ell : L(E_m)$. Then $\text{len}(\text{cut}_m(\ell)) \leq m$.

Proof. This is immediate by corollary 15.4.3. ∎

Main result

Theorem 15.4.5. Let $\mathcal{C}$ be a locos, let $A, B \in \mathcal{C}$, and let $f, g : A \rightrightarrows B$. If A and B are finite, then the coequalizer of f, g exists and is finite. In particular, $\mathbf{Fin}(\mathcal{C})$ has coequalizers, and these are preserved by the inclusion $\mathbf{Fin}(\mathcal{C}) \hookrightarrow \mathcal{C}$.

Proof. Since we have already shown that quotients of finite equivalence relations are finite (corollary 15.3.4), we just need to construct the equivalence relation $E \hookrightarrow B \times B$ generated by f and g and show that E is finite. (The quotient of this equivalence relation will be the coequalizer of f and g.)

This is done as in the proof of proposition 3.10 in [Maie 10]: we start by creating the reflexive symmetric relation $\langle r_1, r_2 \rangle : R \hookrightarrow B \times B$ generated by f, g. This construction only uses pretopos tools, so we already know that R is finite since $\mathbf{Fin}(\mathcal{C})$ is a pretopos (by corollary 15.1.2).

Now, we need to construct the transitive closure of R. As in [Maie 10], we want to say that two elements $b, b' : B$ are related if they are connected by a finite list x_1, x_2, x_3 such that all pairs $(b, x_1), (x_1, x_2), (x_2, x_3), (x_3, b)$ are in R. Formally, we set

$$U = \{b, b' : B, \ell : L(R)_{>0} \mid L(r_1)(back(\ell)) = L(r_2)(front(\ell)), \\ b = r_1(\text{zeroth}(\ell)),\ b' = r_2(last(\ell))\}.$$

(Here, $front$ removes the last element of ℓ, $back$ removes the first element, and $last$ picks out the last element.) Then, we have two obvious projections $p_1, p_2 : U \to B$, and we let $E \hookrightarrow B \times B$ be the image of $\langle p_1, p_2 \rangle : U \to B \times B$. This is how the equivalence relation is constructed in [Maie 10]; she notes that it follows easily that E is an equivalence relation, and that the quotient gives the desired coequalizer.

However, it is not clear that this construction produces a finite object E, because the object U is not finite - it involves the infinite object $L(R)_{>0}$. In order to fix this, we would like to restrict ourselves to a collection of finite lists. The idea is that we shouldn't need a list which is longer than the size of R (remember, we noted that R is finite). Indeed, if a list connecting b to b' is longer than $|R|$, it will have a repetition (by the internal pigeonhole principle, theorem 15.2.6). If we cut out the elements in between the repetition, we still get a list connecting b and b'.

So, let U_{fin} be defined just as U, but with the restriction that $\ell : L(R)_{\leq |R|}$; theorem 15.1.8 tells us that this object is finite. Then its image E_{fin} will also be finite. We claim that taking the quotient B/E_{fin} will give the desired coequalizer, following the same proof strategy as in [Maie 10]. There is only one problem: the proof of transitivity doesn't work the same.

With the original equivalence relation, if we have a list ℓ showing that $b \sim b'$ (i.e., $(b, b', \ell) : U$) and another list ℓ' showing that $b' \sim b''$, then the list $\ell +\!\!+ \ell'$ shows that $b \sim b''$ (i.e., $(b, b'', \ell +\!\!+ \ell') : U$). Now, if we have $(b, b', \ell), (b', b'', \ell') : U_{fin}$, it's not guaranteed that $(b, b'', \ell +\!\!+ \ell') : U_{fin}$ because the list $\ell +\!\!+ \ell'$ may be too big. So, we have to formalize the idea that we can cut out a part of the list and still have it be of the right format.

In short, given $(b, b', \ell), (b', b'', \ell') : U_{fin}$, we must show that $(b, b'', \mathrm{cut}_{|R|}(\ell +\!\!+ \ell')) : U_{fin}$. We showed with corollary 15.4.4 that $\mathrm{cut}_{|R|}(\ell +\!\!+ \ell')$ is of the right length. So, to get the desired conclusion, it suffices to show the following (since cut is just defined by repeated application of cutRep).

If $(b, b', \ell) : U$, then $(b, b', \mathrm{cutRep}(\ell)) : U$.

We sketch the proof as follows. If ℓ has no repetitions, then $\mathrm{cutRep}(\ell) = \ell$, and we're done. If ℓ has repetitions, then there is a minimal $x < \mathrm{len}(\ell)$ such that there is some $x < y < \mathrm{len}(\ell)$ with $\mathrm{nth}(x, \ell) = \mathrm{nth}(y, \ell)$ (we can test for this using eqTests). In this case, if we set $r = \mathrm{nth}(x, \ell) = \mathrm{nth}(y, \ell)$, then we have

$$\mathrm{cutRep}(\ell) = \mathrm{head}(x, \ell) +\!\!+ [r] +\!\!+ \mathrm{tail}(sy, \ell).$$

But note also that $\mathrm{head}(x, \ell) +\!\!+ [r] = \mathrm{head}(sx, \ell)$ and $[r] +\!\!+ \mathrm{tail}(sy, \ell) = \mathrm{tail}(y, \ell)$. Therefore, we can check the conditions of U on each part of this list, and we conclude that $(b, b', \mathrm{cutRep}(\ell)) : U$. ∎

15.5 Summary

All in all, we get the following result. (The is essentially theorem D5.2.7 from [John 02], but for a locos.)

Theorem 15.5.1. If $\mathcal{C}$ is a locos, then:

- $\mathbf{Fin}(\mathcal{C})$ is a Boolean topos;
- all coequalizers of equivalence relations in $\mathbf{Fin}(\mathcal{C})$ are split; and
- all finite limits, finite colimits, and exponentials are preserved by the inclusion $\mathbf{Fin}(\mathcal{C}) \hookrightarrow \mathcal{C}$.

Proof. In corollary 15.1.2, we noted that $\mathbf{Fin}(\mathcal{C})$ is a Boolean topos. In corollary 15.3.4, we showed that the quotients of equivalence relations in $\mathbf{Fin}(\mathcal{C})$ are split. Finally, in theorems 13.5.1, 15.1.1, and 15.4.5, we showed that the finite limits, finite colimits, and exponentials of $\mathbf{Fin}(\mathcal{C})$ are preserved by the inclusion $\mathbf{Fin}(\mathcal{C}) \hookrightarrow \mathcal{C}$. ∎

Chapter 16

The internal category of finite sets, part 2

Contents

In this chapter, we elaborate on the properties of the internal category $\mathbf{Fin}_{\mathcal{S}}$. We start in section 16.1 by applying previous facts to this internal category. First, applying the results of chapter 15 tells us that $\mathbf{Fin}_{\mathcal{S}}^{op}$ is flex (theorem 16.1.1). Second, applying the results of chapter 8 tells us that the functors $\mathbb{L}X$ have some nice properties; notably, the mapping $X \mapsto \mathbb{L}X$ has a left inverse (corollary 16.1.3).

As it turns out that, this left inverse is a two-sided inverse, giving rise to an equivalence of categories:

$$\mathcal{T} \simeq [\mathbf{Fin}_{\mathcal{S}}^{op}, F^*\mathbb{T}]_{flex}.$$

This result (corollary 16.2.8) is a big part of establishing the third main result of the thesis (theorem 17.2.1), as discussed in section 1.2 of the introduction. Its proof is rather technical, and it takes up section 16.2.

16.1 Applying known results

To start with, we apply the results of theorem 15.5.1 to the internal category $\mathbf{Fin}_{\mathcal{S}}$ and the inclusion $\mathbf{Fin}_{\mathcal{S}} \hookrightarrow \mathbb{S}$. However, in order to conclude that the finite colimits are preserved by the transition functors, we must assume that $\mathcal{S}$ is an arithmetic universe.

Theorem 16.1.1. Let $\mathcal{S}$ be an arithmetic universe. Then the internal category $\mathbf{Fin}_{\mathcal{S}}$ has finite limits, finite colimits, and N-finite sums, and these are preserved by the inclusion

$\mathbf{Fin}_{\mathcal{S}} \hookrightarrow \mathbb{S}$. In particular, $\mathbf{Fin}^{op}_{\mathcal{S}}$ is flex.

Proof. We already saw in proposition 14.3.1 that $\mathbf{Fin}_{\mathcal{S}}$ has finite limits, finite coproducts, and N-finite sums, and that they are preserved by the inclusion $\mathbf{Fin}_{\mathcal{S}} \hookrightarrow \mathbb{S}$. Since $\mathcal{S}$ is an arithmetic universe, the argument in the proof of proposition 14.3.1 applies to coequalizers: theorem 15.5.1 tells us $\mathbf{Fin}_{\mathcal{S}}(I)$ has coequalizers and that they are preserved by the inclusion, and the pullback functors for arithmetic universes preserve coequalizers (see fact 2.1.6). ∎

Next, we prove some results about the indexed functor $\mathbb{L}X$, which are easy consequences of previous theorems.

Theorem 16.1.2. Let $\mathcal{S}$, $\mathcal{T}$ be arithmetic universes, and let $F : \mathcal{S} \to \mathcal{T}$ be an AU morphism. Then, for any $X \in \mathcal{T}$, the indexed functor $\mathbb{L}X : \mathbf{Fin}^{op}_{\mathcal{S}} \to F^*\mathbb{T}$ preserves finite limits and N-finite products. That is, $\mathbb{L}X$ is a flex morphism.

In particular, this means that the mapping $X \mapsto \mathbb{L}X$ is a functor

$$\mathbb{L} : \mathcal{T} \to [\mathbf{Fin}^{op}_{\mathcal{S}}, F^*\mathbb{T}]_{flex}.$$

Proof. By propositions 6.3.7 and 6.3.8, $\mathbb{S}$ and $\mathbb{T}$ have all finite colimits, and $F : \mathbb{S} \to F^*\mathbb{T}$ preserves them. Moreover, by theorem 16.1.1, $\mathbf{Fin}_{\mathcal{S}}$ has finite colimits and N-finite sums, and these are preserved by the inclusion $\mathbf{Fin}_{\mathcal{S}} \hookrightarrow \mathbb{S}$.

Thus, all the hypotheses of theorems 8.3.3 and 8.4.2 are satisfied, and they let us conclude that $\mathbb{L}X$ preserves equalizers and N-finite products. The latter condition implies that $\mathbb{L}X$ also preserves finite products (by theorem 12.5.5, which we can apply because the domain and codomain of $\mathbb{L}X$ are ES by proposition 12.2.1), so we're done. ∎

Corollary 16.1.3. Let $\mathcal{S}$, $\mathcal{T}$ be arithmetic universes, and let $F : \mathcal{S} \to \mathcal{T}$ be an AU morphism. Consider the following composite (the first functor is $\mathbb{L}$, the second is from remark 8.5.1, and 1 refers to $1 : \mathbb{1} \to N$).

$$\mathcal{T} \xrightarrow{X \mapsto \mathbb{L}X} [\mathbf{Fin}^{op}_{\mathcal{S}}, F^*\mathbb{T}]_{flex} \xrightarrow{G \mapsto G^{\mathbb{1}}(1)} \mathcal{T}$$

This composite is naturally isomorphic to the identity.

Proof. Note that this composite functor is the same as if we had omitted the "flex" condition from the middle category. Moreover, note that the object $1 \in \mathbf{Fin}(\mathbb{1})$ is a terminal object, and so is $\iota^{\mathbb{1}}(1) \in \mathcal{S}$ (see proposition 13.3.1). Therefore, by proposition 8.5.2, the composite is naturally isomorphic to the identity. ∎

We claim that the functors in the above corollary form an equivalence of categories. To show this, we must show that the other way of composing the functors also produces the identity. However, this proof is rather difficult, so we leave it to the next section.

16.2 Uniqueness of functors on finite sets

In this section, we build up to showing the fact that if G is an indexed functor on $\mathbf{Fin}^{op}$ which preserves N-finite products, then G is isomorphic to $\mathbb{L}X$, where $X = G^{\mathbb{1}}(1)$.

We start with an explicit computation of certain N-finite sums in **Fin**.

Remark 16.2.1. Let $\mathcal{S}$ be a locos, let $I \in \mathcal{S}$, and let $m \in \mathbf{Fin}(I)$. Recall that, via the inclusion $\mathbf{Fin} \hookrightarrow \mathbb{S}$, m corresponds to $\pi_1^m : I \times_m E \to I$.

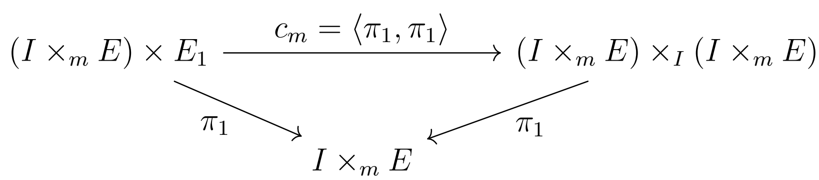
Write 1_m for the arrow $1 \circ \langle\rangle_m : I \times_m E \to N$ (where $\langle\rangle_m$ is the arrow from $I \times_m E$ to the terminal object). Remark that we can view 1_m as an object of $\mathbf{Fin}(I \times_m E)$, and in fact, it is given by $\langle\rangle_m^*(1)$. Now, we know the object $1 : \mathbb{1} \to N$ of $\mathbf{Fin}(\mathbb{1})$ corresponds to the object $E_1 \to \mathbb{1}$ of $\mathcal{S}/\mathbb{1}$ (by definition; see section 13.1). Therefore, the object of $\mathcal{S}/I \times_m E$ corresponding to $1_m = \langle\rangle_m^*(1)$ is

$$\pi_1 : (I \times_m E) \times E_1 \to (I \times_m E).$$

Next, remark that $(\pi_1^m)^* : \mathbf{Fin}(I) \to \mathbf{Fin}(I \times_m E)$, and since $m \in \mathbf{Fin}(I)$, we can consider $(\pi_1^m)^* m$. As an object of $\mathcal{S}/I \times_m E$, this corresponds to

$$\pi_1 : (I \times_m E) \times_I (I \times_m E).$$

We can therefore form an arrow $c_m : 1_m \to (\pi_1^m)^* m$ in $\mathbf{Fin}(I \times_m E)$ which is given by the following arrow in $\mathcal{S}/I \times_m E$.



Proposition 16.2.2. Let $\mathcal{S}$ be a locos, let $I \in \mathcal{S}$, let $m \in \mathbf{Fin}(I)$, and let $c_m : 1_m \to (\pi_1^m)^* m$ be the arrow in $\mathbf{Fin}(I \times_m E)$ from remark 16.2.1. Then (m, c_m) is a universal morphism from 1_m to $(\pi_1^m)^*$.

In particular, recall that **Fin** has N-finite sums, so π_1^m has a left adjoint Σ_m; let η_m denote the unit of this adjunction. Then there exists an isomorphism $\chi_m : m \to \Sigma_m(1_m)$ such that $(\pi_1^m)^* \chi_m \circ c_m = \eta_m$.

Proof. First, note that $E_1 \cong \mathbb{1}$; see the proof of proposition 13.3.1. To prove the first part of this proposition, we must show that for any object n and any arrow $f : 1_m \to (\pi_1^m)^* n$, there is a unique $p : m \to n$ such that $f = (\pi_1^m)^* p \circ c_m$. Note that such an arrow f becomes the following in $\mathcal{S}/(I \times_m E)$.

$$\begin{array}{ccc} (I \times_m E) \times E_1 & \xrightarrow{\quad f \quad} & (I \times_m E) \times_I (I \times_n E) \\ & {\scriptstyle \pi_1} \searrow \quad \swarrow {\scriptstyle \pi_1} & \\ & I \times_m E & \end{array}$$

Since this diagram commutes, we must have $f = \langle \pi_1, f_2 \rangle$ for some $f_2 : (I \times_m E) \times E_1 \to I \times_n E$. But since $E_1 \cong \mathbb{1}$, we know that $\pi_1 : (I \times_m E) \times E_1 \to I \times_m E$ is an isomorphism, and so there exists a unique $p : I \times_m E \to I \times_n E$ such that $f_2 = p \circ \pi_1$. Finally, note that $(\pi_1^m)^* p$ is $(\mathrm{Id}_{I \times_m E}) \times_I p$, and so

$$(\pi_1^m)^* p \circ c_m = (\mathrm{Id}_{I \times_m E} \times_I p) \circ \langle \pi_1, \pi_1 \rangle = \langle \pi_1, p \circ \pi_1 \rangle = \langle \pi_1, f_2 \rangle = f,$$

as desired. The uniqueness is also clear from the construction.

The second part is standard from the theory of adjunctions. ∎

Next, we need a general tool for defining maps when indexed products and sums are involved.

Remark 16.2.3. Let $\mathcal{S}$ be a category with finite limits, and let $\mathbb{C}$ be an $\mathcal{S}$-indexed category. Consider a commutative diagram in $\mathcal{S}$ as follows.

$$\begin{array}{ccc} E_m & \xrightarrow{\quad p \quad} & E_n \\ & {\scriptstyle \pi_m} \searrow \quad \swarrow {\scriptstyle \pi_n} & \\ & I & \end{array}$$

Assume that $(\pi_m)^* : \mathcal{C}^I \to \mathcal{C}^{E_m}$ and $(\pi_n)^* : \mathcal{C}^I \to \mathcal{C}^{E_n}$ have right adjoints Π_m and Π_n (we can do this construction for left adjoints too). Then, for any $A \in \mathcal{C}^I$ and $T \in \mathcal{C}^{E_n}$, we can define a map

$$\phi_p^{A,T} : \mathrm{Hom}_{\mathcal{C}^I}\left(A, \prod_n T\right) \to \mathrm{Hom}_{\mathcal{C}^I}\left(A, \prod_m p^* T\right)$$

as the following composite.

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{C}^I}\left(A, \prod_n T\right) & \cong & \mathrm{Hom}_{\mathcal{C}^{E_n}}\left((\pi_n)^* A, T\right) \\
 & & \downarrow p^* \\
\phi_p^{A,T} \Big\downarrow & & \mathrm{Hom}_{\mathcal{C}^{E_m}}\left(p^*(\pi_n)^* A, p^* T\right) \\
 & & \cong \\
\mathrm{Hom}_{\mathcal{C}^I}\left(A, \prod_m p^* T\right) & \cong & \mathrm{Hom}_{\mathcal{C}^{E_m}}\left((\pi_m)^* A, p^* T\right)
\end{array}$$

It's clear that $\phi_p^{A,T}$ is natural in A and T.

Now, remark that if we set $A = \Pi_n T$, then we can apply $\phi_p^{A,T}$ to the identity map to get an arrow

$$\phi_p^{\Pi_n T, T}(\mathrm{Id}_{\Pi_n T}) : \prod_n T \to \prod_m p^* T.$$

We simply denote this arrow by ϕ_p^T. Note that, if ε_m and ε_n are the counits of the adjunctions for Π_m and Π_n, then ϕ_p^T is the unique map which makes the following diagram commute.

$$\begin{array}{ccc}
(\pi_m)^* \prod_n T & \xrightarrow{\cong} & p^*(\pi_n)^* \prod_n T \\
(\pi_m)^* \phi_p^T \Big\downarrow & & \Big\downarrow p^* \varepsilon_n^T \\
(\pi_m)^* \prod_m p^* T & \xrightarrow{\varepsilon_m^{p^* T}} & p^* T
\end{array}$$

Moreover, we note that $(\phi_p^T)_T$ is a natural transformation.

Proposition 16.2.4. Let $\mathcal{S}$ be a locos, let $I \in \mathcal{S}$, and let $p : m \to n$ be an arrow in $\mathbf{Fin}(I)$. Let χ_m, χ_n be as in proposition 16.2.2, and let $\phi_p^{1_n}$ be as in remark 16.2.3. Then the following diagram commutes.

$$\begin{array}{ccc}
m & \xrightarrow{\chi_m} & \sum_m (1_m) \\
 & & \cong \\
p \Big\downarrow & & \sum_m (p^* 1_n) \\
 & & \Big\downarrow \phi_p^{1_n} \\
n & \xrightarrow{\chi_n} & \sum_n (1_n)
\end{array}$$

Proof. Because (m, c_m) is a universal morphism from 1_m to $(\pi_1^m)^*$ (proposition 16.2.2), to show the two arrows are equal, it suffices to show they're equal when we apply $(\pi_1^m)^*$ and pre-compose with c_m.

First, applying $(\pi_1^m)^*$ to $\chi_n \circ p$ and precomposing with c_m lets us consider the following diagram.

$$\begin{array}{ccccc}
1_m & \xrightarrow{\cong} & p^*1_n & \overset{p^*\eta_n^{1_n}}{\searrow} & \\
 & & \downarrow{\scriptstyle p^*c_n} & & \\
\downarrow{\scriptstyle c_m} & & p^*(\pi_1^n)^*n & \xrightarrow[p^*(\pi_1^n)^*\chi_n]{} & p^*(\pi_1^n)^*\sum_n(1_n) \\
 & & \downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} \\
(\pi_1^m)^*m & \xrightarrow[(\pi_1^m)^*p]{} & (\pi_1^m)^*n & \xrightarrow[(\pi_1^m)^*\chi_n]{} & (\pi_1^m)^*\sum_n(1_n)
\end{array}$$

We can check that this diagram commutes. The triangle commutes by definition of χ_n, and the bottom right square commutes by naturality of $x^*y^* \cong (yx)^*$. For the left square, we must consider the corresponding square in $\mathcal{S}/I \times_m E$ and check that it commutes. This is technical but straightforward, so we omit it.

Second, applying $(\pi_1^m)^*$ to $\phi_p^{1_n} \circ \cong \circ \chi_m$ and precomposing with c_m lets us consider the following diagram.

$$\begin{array}{ccccccc}
1_m & \xrightarrow{\cong} & p^*1_n & & \xrightarrow{p^*\eta_n^{1_n}} & & p^*(\pi_1^n)^*\sum_n(1_n) \\
\downarrow{\scriptstyle c_m} & \overset{\eta_m^{1_m}}{\searrow} & & \overset{\eta_m^{p^*1_n}}{\searrow} & & & \downarrow{\scriptstyle\cong} \\
(\pi_1^m)^*m & \xrightarrow[(\pi_1^m)^*\chi_m]{} & (\pi_1^m)^*\sum_m(1_m) & \xrightarrow[\cong]{} & (\pi_1^m)^*\sum_m(p^*1_n) & \xrightarrow[(\pi_1^m)^*\phi_p^{1_n}]{} & (\pi_1^m)^*\sum_n(1_n)
\end{array}$$

Again, it's easy to check that this diagram commutes. The left triangle commutes by definition of χ_m. The middle square commutes by naturality of η_m. The right square commutes by definition of ϕ_p.

Note that the paths along the top of both diagrams are the same; thus, we get the desired equality. ∎

We have one more technical fact to check before proving the desired result.

Proposition 16.2.5. Let $\mathcal{S}$ be a locos, let $\mathbb{C}, \mathbb{D}$ be two $\mathcal{S}$-indexed categories with N-finite products, and let $F : \mathbb{C} \to \mathbb{D}$. Let $\psi_n : F^I\Pi_n \Rightarrow \Pi_n F^n$ and $\psi_m : F^I\Pi_m \Rightarrow \Pi_m F^m$ be the canonical natural transformations, and let ϕ_p be as in remark 16.2.3.

Then, for any $m, n : I \to N$ in $\mathcal{S}$, any $p : m^*(\pi_2^E) \to n^*(\pi_2^E)$ in $\mathcal{S}/I$, and any

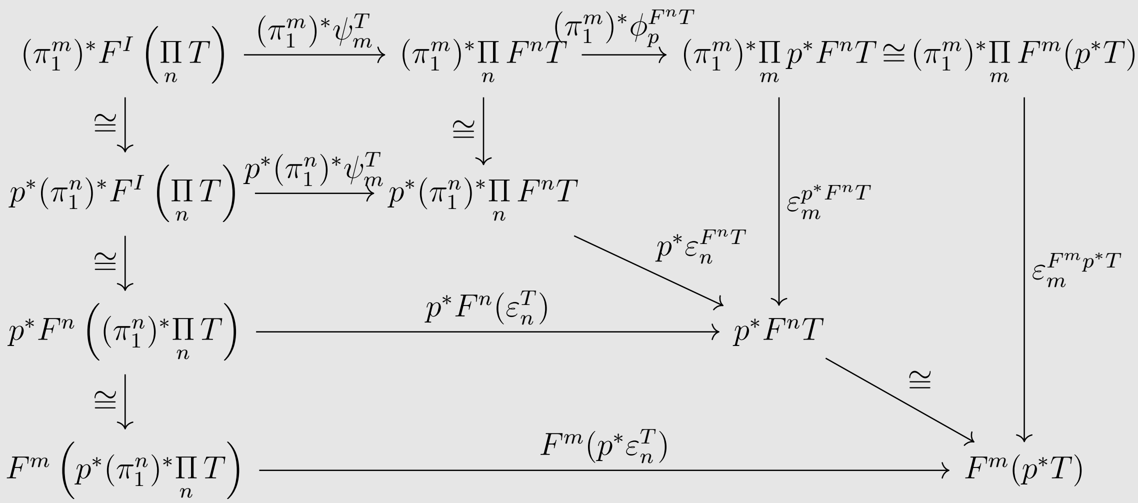
$T \in \mathcal{C}^{I \times_n E}$, the following diagram commutes.

$$\begin{array}{ccc}
F^I\left(\prod_n T\right) & \xrightarrow{\psi_n^T} & \prod_n F^n(T) \\
\Big\downarrow F^I(\phi_p^T) & & \Big\downarrow \phi_p^{F^n T} \\
 & & \prod_m p^* F^n(T) \\
 & & \Big\downarrow \Pi_m(\cong) \\
F^I\left(\prod_m p^* T\right) & \xrightarrow{\psi_m^{p^*T}} & \prod_m F^m(p^*T)
\end{array}$$

Proof. We have two arrows going into Π_m applied to something. So, by adjoint properties, it suffices to check that they are equal when we apply $(\pi_1^m)^*$ and post-compose with the counit ε_m. When we do this, we get the two following diagrams.

$$\begin{array}{ccccc}
(\pi_1^m)^* F^I\left(\prod_n T\right) & \xrightarrow{(\pi_1^m)^* F^I(\phi_p^T)} & (\pi_1^m)^* F^I\left(\prod_m p^* T\right) & \xrightarrow{(\pi_1^m)^* \psi_m^{p^*T}} & (\pi_1^m)^* \prod_m F^m(p^*T) \\
\cong\Big\downarrow & & \cong\Big\downarrow & & \\
F^m\left((\pi_1^m)^* \prod_n T\right) & \xrightarrow{F^m\left((\pi_1^m)^* \phi_p^T\right)} & F^m\left((\pi_1^m)^* \prod_m p^* T\right) & & \Big\downarrow \varepsilon_m^{F^m(p^*T)} \\
\cong\Big\downarrow & & & \searrow F^m(\varepsilon_m^{p^*T}) & \\
F^m\left(p^* (\pi_1^n)^* \prod_n T\right) & & \xrightarrow{F^m(p^* \varepsilon_n^T)} & & F^m(p^*T)
\end{array}$$

It's easy to check that this diagram commutes. The top right square commutes by naturality of the isomorphism $(\pi_1^m)^* F^I \cong F^m (\pi_1^m)^*$; the right square commutes by definition of ψ_m; and the bottom square commutes by definition of ϕ_p (and then applying F^m).



It's also easy to check that this diagram commutes. The top right square commutes by naturality of $x^*y^* \cong (yx)^*$; the upper middle square commutes by definition of ϕ_p; the left middle square commutes by definition of ψ_m (and applying p^*); the left square commutes by naturality of ε_m; and the bottom square commutes by naturality of $p^*F^n \cong F^m p^*$.

Finally, we notice that we can glue these two diagrams together along the bottom and right sides. We can also glue together the left sides; these two compositions are equal by the coherence of F with the structure of the indexed categories. Thus, the two ways of going along the top and the right must be equal; this is what we wanted to show. ∎

Proposition 16.2.6. Let $\mathcal{S}$ be a locos, let $\mathbb{C}$ be an $\mathcal{S}$-indexed category with N-finite products, and let $G : \mathbf{Fin}_{\mathcal{S}}^{op} \to \mathbb{C}$ preserve N-finite products. Then $G \cong \mathbb{P}_{\pi_2^E}(G^{\mathbb{1}}(1))$.

Proof. Let $X = G^{\mathbb{1}}(1)$. For $I \in \mathcal{S}$ and $n \in \mathbf{Fin}(I)$, define $\alpha_n^I : G^I(n) \to (\mathbb{P}_{\pi_2^E} X)^I(n)$ to be the following chain of maps.

$$G^I(n) \xrightarrow{G^I(\chi_n^{-1})} G^I\left(\sum_n \langle\rangle_n^* 1\right) \xrightarrow{\cong} \prod_n G^n(\langle\rangle_n^* 1) \xrightarrow{\cong} \prod_n \langle\rangle_n^* G^{\mathbb{1}}(1)$$

Note that χ_n is the map from proposition 16.2.2, the second map is the canonical isomorphism $G^I\Sigma_n \cong \Pi_n G^n$ (because G preserves N-finite products), and the third map is the isomorphism that makes G commute with the change of base functors. It is clear α_n^I is an isomorphism.

We must check that $(\alpha_n^I)_n$ is natural. So, given some $p : m \to n$ in $\mathbf{Fin}(I)$, we consider the following diagram.

$$
\begin{array}{ccccccc}
G^I(n) & \xrightarrow{G^I(\chi_n^{-1})} & G^I\left(\sum_n \langle\rangle_n^* 1\right) & \xrightarrow{\cong} & \prod_n G^n(\langle\rangle_n^* 1) & \xrightarrow{\cong} & \prod_n \langle\rangle_n^* G^{\mathbb{1}}(1) \\
 & & & & \big\downarrow \phi_p^{G(-)} & & \big\downarrow \phi_p^{\langle\rangle G} \\
 & & \big\downarrow G^I(\phi_p^{\langle\rangle_n^* 1}) & & \prod_m p^* G^n(\langle\rangle_n^* 1) & \xrightarrow{\cong} & \prod_m p^* \langle\rangle_n^* G^{\mathbb{1}}(1) \\
\big\downarrow G^I(p) & & & & \big\downarrow \cong & & \\
 & & G^I\left(\sum_m p^* \langle\rangle_n^* 1\right) & \xrightarrow{\cong} & \prod_m G^m(p^* \langle\rangle_n^* 1) & & \big\downarrow \cong \\
 & & \big\downarrow \cong & & \big\downarrow \cong & & \\
G^I(m) & \xrightarrow{G^I(\chi_m^{-1})} & G^I\left(\sum_m \langle\rangle_m^* 1\right) & \xrightarrow{\cong} & \prod_m G^m(\langle\rangle_m^* 1) & \xrightarrow{\cong} & \prod_m \langle\rangle_m^* G^{\mathbb{1}}(1)
\end{array}
$$

We note that the left side of this diagram is $(\mathbb{P}_{\pi_2^E} X)^I(p)$, by definition; so, it remains to show that this diagram commutes.

- The left square commutes because it's just G^I applied to the square of proposition 16.2.4.
- The upper middle square commutes by proposition 16.2.5.
- The lower middle square commutes by naturality of $G^I \Sigma_m \cong \Pi_m G^m$.
- The upper right square commutes by naturality of ϕ_p.
- The lower right square commutes since it's Π_m applied to the coherence diagram between G and the structure of the indexed categories.

So, we've constructed a natural isomorphism $\alpha^I : G^I \to (\mathbb{P}_{\pi_2^E} X)^I$ for each I. ∎

Theorem 16.2.7. Let $\mathcal{S}$, $\mathcal{T}$ be arithmetic universes, and let $F : \mathcal{S} \to \mathcal{T}$ be an AU morphism. For any flex indexed functor $G : \mathbf{Fin}^{op} \to F^*\mathbb{T}$, if we set $X = G^{\mathbb{1}}(1)$, then $G \cong \mathbb{L}X$ naturally in G. That is, the following composite (these are the same functors from corollary 16.1.3, but reordered) is naturally isomorphic to the identity.

$$
[\mathbf{Fin}_{\mathcal{S}}^{op}, F^*\mathbb{T}]_{flex} \xrightarrow{G \mapsto G^{\mathbb{1}}(1)} \mathcal{T} \xrightarrow{X \mapsto \mathbb{L}X} [\mathbf{Fin}_{\mathcal{S}}^{op}, F^*\mathbb{T}]_{flex}
$$

Proof. Recall that $\mathbb{L}$ is the same functor as $\mathbb{P}_{\pi_2^E}$ (see section 14.1). The result follows from proposition 16.2.6. ∎

Corollary 16.2.8. Let $\mathcal{S}$, $\mathcal{T}$ be arithmetic universes, and let $F : \mathcal{S} \to \mathcal{T}$ be an AU morphism. We have an equivalence of categories $\mathcal{T} \simeq [\mathbf{Fin}_{\mathcal{S}}^{op}, F^*\mathbb{T}]_{flex}$ given by $X \mapsto \mathbb{L}X$ and $G \mapsto G^{\mathbb{1}}(1)$.

Proof. Immediate from corollary 16.1.3 and theorem 16.2.7. ∎

Chapter 17

Extending an arithmetic universe by an object

Contents

In this section, we finally prove the third main result of this thesis: an explicit characterization of $\mathcal{S}[\mathbb{O}]$, the free extension of an arithmetic universe $\mathcal{S}$ by an object (theorem 17.2.1). We need a minor technical result, which we prove in section 17.1, before proving the main result in section 17.2, building on many previous results.

17.1 Equivalence of categories for finite sets

Our goal in this section is to slightly adapt the equivalence of categories of corollary 11.3.1. Specifically, we want to consider this equivalence for the internal category $\mathbf{Fin}^{op}$. It turns out that, in this case, the indexed functor Y has a particular form, as we see with the proposition below.

Remark 17.1.1. Let $\mathcal{S}$ be an arithmetic universe, and let ι denote the inclusion $\mathbf{Fin}_{\mathcal{S}} \hookrightarrow \mathbb{S}$. Note that $\iota \in [\mathbf{Fin}_{\mathcal{S}}, \mathbb{S}]$, and $[\![\mathbf{Fin}_{\mathcal{S}}, \mathbb{S}]\!]$ is the pullback along CF of the canonical indexing of $[\mathbf{Fin}_{\mathcal{S}}, \mathbb{S}]$ (see proposition 6.5.2). So, we can form the indexed functor

$$\mathbb{L}\iota : \mathbf{Fin}_{\mathcal{S}}^{op} \to [\![\mathbf{Fin}_{\mathcal{S}}, \mathbb{S}]\!].$$

Proposition 17.1.2. Let $\mathcal{S}$ be an arithmetic universe. Then the indexed functors

$$\mathbb{L}\iota, Y : \mathbf{Fin}_{\mathcal{S}}^{op} \to [\![\mathbf{Fin}_{\mathcal{S}}, \mathbb{S}]\!]$$

are isomorphic.

Proof. These are indexed functors on an internal category, so it suffices to check that their corresponding diagrams are isomorphic. This is a rather tedious calculation, so we just sketch the proof.

The functor $\mathbb{L}\iota$ corresponds to a diagram whose object is

$$(\mathbb{E}_{\pi_2^E}\iota)^{C_0}(\mathrm{Id}_{C_0}) = (\langle\rangle_N^*\iota)^{CF(\pi_2^E)}.$$

Since CF preserves finite limits (proposition 6.4.3), we can just replace $CF(\pi_2^E)$ with "the π_2^E of $[\mathbf{Fin}, \mathbb{S}]$". Moreover, since CF preserves NNO, we know that $\langle\rangle_N^*\iota$ is just $\pi_1 : N \times \iota \to N$ (where here, we write N for the NNO in $[\mathbf{Fin}, \mathbb{S}]$). Therefore, by corollary 5.4.3, this exponential (which is happening in $[\mathbf{Fin}, \mathbb{S}]/N$) is just $\mathrm{len} : L(\iota) \to N$.

Now, $(\mathrm{len} : L(\iota) \to N) \in [\mathbf{Fin}, \mathbb{S}]/N$ is the object in the diagram corresponding to $\mathbb{L}\iota$. To understand it better, we must understand $L(\iota)$, which is an object of $[\mathbf{Fin}, \mathbb{S}]$. In particular, it is an indexed functor on an internal category – therefore, we can consider its corresponding diagram.

Now, the diagram associated with this indexed functor is obtained by considering the fiber at $C_0 = N$ and evaluating at Id_N. List objects are computed pointwise for functors, so

$$L(\iota)^N(\mathrm{Id}_N) = L(\iota^N(\mathrm{Id}_N)),$$

the list object on $\iota^N(\mathrm{Id}_N)$. The inclusion functor maps n to $n^*(\pi_2^E)$, so this is just $\pi_2^E = ([n] \mid n \in N)$. List objects are computed fiberwise for slice categories, so we have

$$L(\iota^N(\mathrm{Id}_N)) = (L([n]) \mid n \in N).$$

But finally, we are also indexing by the length arrow, so throwing that in gives us

$$([n]^{[m]} \mid m, n \in N)$$

as an object of $\mathcal{S}/N \times N$. This is the object of the diagram representing this whole indexed functor, and it's just $\langle \mathrm{dom}, \mathrm{cod}\rangle : C_1 \to C_0$ specified to $\mathbf{Fin}$. So, this diagram matches the one from the Yoneda embedding. ∎

With this proposition, we get a refinement of corollary 11.3.1.

Corollary 17.1.3. Let $\mathcal{S}$, $\mathcal{T}$ be arithmetic universes, and let $F : \mathcal{S} \to \mathcal{T}$ be an AU functor. Then the functor

$$- \circ \mathbb{L}\iota : [[\![\mathbf{Fin}_{\mathcal{S}}, \mathbb{S}]\!], F^*\mathbb{T}]_{flexcc} \to [\mathbf{Fin}_{\mathcal{S}}^{op}, F^*\mathbb{T}]_{flex}$$

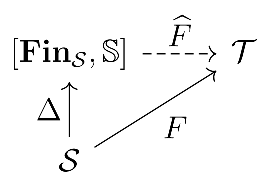
is part of an equivalence of categories.

Proof. Since $\mathbb{L}\iota \cong Y$ by proposition 17.1.2, this is precisely the statement of corollary 11.3.1 applied to $\mathbf{Fin}_{\mathcal{S}}^{op}$, which is valid because $\mathbf{Fin}_{\mathcal{S}}^{op}$ is flex by theorem 16.1.1. ∎

17.2 Final result

Theorem 17.2.1. Let $\mathcal{S}$ be an arithmetic universe. Then the category $[\mathbf{Fin}_{\mathcal{S}}, \mathbb{S}]$, together with the object $\iota \in [\mathbf{Fin}_{\mathcal{S}}, \mathbb{S}]$ and the functor $\Delta : \mathcal{S} \to [\mathbf{Fin}_{\mathcal{S}}, \mathbb{S}]$, satisfy the following property.

For any arithmetic universe $\mathcal{T}$, any AU functor $F : \mathcal{S} \to \mathcal{T}$, and any object $X \in \mathcal{T}$, there exists a unique (up to isomorphism) AU functor $\widehat{F} : [\mathbf{Fin}_{\mathcal{S}}, \mathbb{S}] \to \mathcal{T}$ such $\widehat{F}(\iota) \cong X$ and the following diagram commutes up to isomorphism.



Proof. By corollaries 16.2.8 and 17.1.3, the following functor is part of an equivalence of categories.

$$[[\![\mathbf{Fin}_{\mathcal{S}}, \mathbb{S}]\!], F^*\mathbb{T}]_{flexcc} \longrightarrow \mathcal{T}$$
$$G \longmapsto (G \circ \mathbb{L}\iota)^{\mathbb{1}}(1)$$

Note that $(G \circ \mathbb{L}\iota)^{\mathbb{1}}(1) \cong G^{\mathbb{1}}(\mathbb{L}\iota^{\mathbb{1}}(1)) \cong G^{\mathbb{1}}(\iota)$ (by corollary 16.2.8), so there exists a unique (up to isomorphism) flexcc indexed functor

$$\mathcal{H} : [\![\mathbf{Fin}_{\mathcal{S}}, \mathbb{S}]\!] \to F^*\mathbb{T}$$

such that $\mathcal{H}^{\mathbb{1}}(\iota) \cong X$. We claim that $H = \mathcal{H}^{\mathbb{1}}$ is the functor we want.

We start with existence. The functor $H : [\mathbf{Fin}_{\mathcal{S}}, \mathbb{S}] \to \mathcal{T}$ that we've constructed satisfies $H(\iota) \cong X$. Moreover, since $\mathcal{H}$ is flexcc, we know that H is an AU morphism by proposition 6.3.9. (Note that, by proposition 6.5.2, $[\![\mathbf{Fin}_{\mathcal{S}}, \mathbb{S}]\!]$ is the same as the pullback along Δ of the canonical indexing of $[\mathbf{Fin}_{\mathcal{S}}, \mathbb{S}]$.) So, it just remains to show that H makes the diagram commute up to isomorphism. To this end, we use propositions 6.3.7 and

6.3.8 to get the following diagram.

$$\begin{array}{ccc} [\![\mathbf{Fin}_{\mathcal{S}}, \mathbb{S}]\!] & \xrightarrow{\mathcal{H}} & F^*\mathbb{T} \\ \Delta \uparrow & \nearrow_{F} & \\ \mathbb{S} & & \end{array}$$

Since F and Δ are AU morphisms (see proposition 6.4.3), they extend to flexcc indexed functors. Thus $\mathcal{H} \circ \Delta$ and F are flexcc functors $\mathbb{S} \to F^*\mathbb{T}$, so by proposition 6.3.4, they must be isomorphic. Their underlying functors must therefore be isomorphic too, so we get $H \circ \Delta \cong F$, as desired.

Next, we show uniqueness. Suppose that $H_1, H_2 : [\mathbf{Fin}_{\mathcal{S}}, \mathbb{S}] \to \mathcal{T}$ are AU morphisms such that $H_i \circ \Delta \cong F$ and $H_i(\iota) \cong X$ for $i = 1, 2$. Then, again by propositions 6.3.7, 6.3.8, and 6.3.4, they extend to flexcc morphisms $\mathcal{H}_1, \mathcal{H}_2$ which make the following diagram commute.

$$\begin{array}{ccc} [\![\mathbf{Fin}_{\mathcal{S}}, \mathbb{S}]\!] & \xrightarrow{\mathcal{H}_1, \mathcal{H}_2} & F^*\mathbb{T} \\ \Delta \uparrow & \nearrow_{F} & \\ \mathbb{S} & & \end{array}$$

But since $\mathcal{H}_i^{\mathbb{1}}(\iota) = H_i(\iota) \cong X$, the uniqueness we obtained from the equivalence of categories tells us that $\mathcal{H}_1 \cong \mathcal{H}_2$, and so $H_1 \cong H_2$, as desired. ∎

Chapter 18

Conclusion

As promised in the introduction, in this thesis, we showed three main results. First, we showed that if $\mathcal{C}$ is a locos, then the list object functor $L : \mathcal{C} \to \mathcal{C}$ is a polynomial functor, induced by the arrow $\pi_2^E : E \to N$ (corollary 5.4.2). The proof mostly amounted to proving (in section 5) the adjunction of theorem 5.4.1, which turned out to be an important technical tool for future results.

Second, we showed that, if $\mathcal{C}$ is a locos, then the full subcategory $\mathbf{Fin}(\mathcal{C})$ of finite objects is a Boolean topos (theorem 15.5.1). The proof, which mostly occupied chapters 13 and 15, relied heavily on the first main result and the technical tools of appendix B. It also used some technical tools of internal categories (notably, theorem 12.4.2), and used the link between $\mathbf{Fin}(\mathcal{C})$ and the internal category $\mathbf{Fin}_{\mathcal{C}}$ of chapter 14 (proposition 14.1.1).

Finally, we showed that if $\mathcal{S}$ is an arithmetic universe, then the extension $\mathcal{S}[O]$ by an object O – equivalently, the classifying AU $\mathcal{S}[\mathbb{O}]$ for the theory of objects $\mathbb{O}$ – can be explicitly described as the category $[\mathbf{Fin}_{\mathcal{S}}, \mathbb{S}]$ of indexed copresheaves on $\mathbf{Fin}_{\mathcal{S}}$. This result was based on two equivalences of categories (corollaries 11.3.1 and 16.2.8), and required the development of tools for indexed categories (in part III) and tools for the internal category of finite sets (in part IV). Notably, we used the concepts of flex and flexcc indexed functors to provide a link between AU functors and indexed functors.

We can view these three results as interesting in their own right. However, we can also view the first two as stepping stones for the third, which fits into the larger story we discussed in the introduction: using arithmetic universes to develop the theory of Grothendieck toposes.

Where does the story go from here? We hope to mimic the classical theory, and show that Grothendieck toposes with respect to an arithmetic universe $\mathcal{S}$ can be identified with finitely presented extensions of $\mathcal{S}$.

For toposes, it is well known that all geometric theories can be built from two operations: "adding a collection of objects" (that is, an internally indexed collection, i.e. adding an object $X \to B$ for some fixed $B \in \mathcal{S}$), which corresponds to specifying types in a signature, and "localization", which corresponds to adding an axiom. (Indeed, function symbols can be built from a combination of these two.) These two operations both give us finitely presented extensions of a category, and one might suspect that all finitely presented extensions of arithmetic universes can be obtained in this way; thus, they are what we should consider in adapting the theory of Grothendieck toposes to arithmetic universes. In short, to continue this line of research, we should study extensions of AUs by collections of objects (which should not be too different from the case of a single object) and localizations of AUs (which will be more complex).

We expect that results for arithmetic universes should be similar to the classical results for toposes. In this thesis, while we only study the extension by a single object instead of a family of objects, we find a first confirmation of this hypothesis. It is an encouraging step in the direction of this greater goal.

Part V

Appendices and end matter

Appendix A

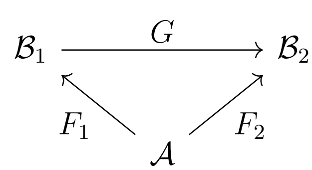
Basic properties of indexed categories

Contents

In this appendix, we give detailed proofs of some of the facts about indexed categories in section 6.3. The details of indexed categories make these proofs quite involved (albeit relatively straightforward), which is why we avoid this level of detail in the rest of the thesis.

A.1 Canonical functors are well defined

Recall the following fact from section 6.3. Suppose $\mathcal{A}, \mathcal{B}_1, \mathcal{B}_2$ are categories with finite limits, and suppose F_1, F_2, G are functors as in the diagram below which preserve finite limits and satisfy $GF_1 \cong F_2$.



Then G extends to an $\mathcal{A}$-indexed functor $\mathcal{G} : F_1^*\mathbb{B}_1 \to F_2^*\mathbb{B}_2$.

In this section, we give the definition of $\mathcal{G}$ more formally and check that it is well-defined.

Notation

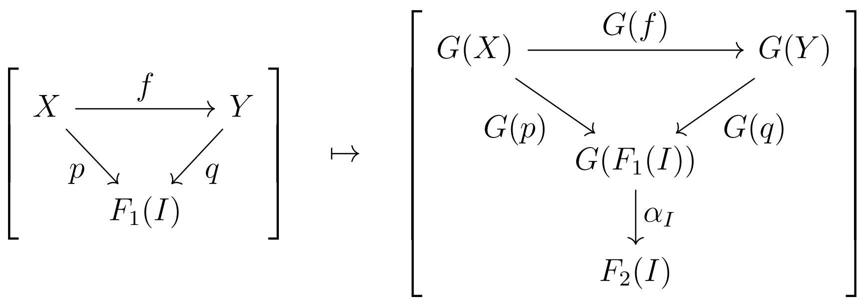
Let $\alpha : G \circ F_1 \to F_2$ denote the isomorphism assumed to exist in the proposition statement.

Defining the functors

For each $I \in \mathcal{A}$, we must define a functor $\mathcal{G}^I : \mathcal{B}_1/F_1(I) \to \mathcal{B}_2/F_2(I)$. An object $X \xrightarrow{p} F_1(I)$ is mapped to the object

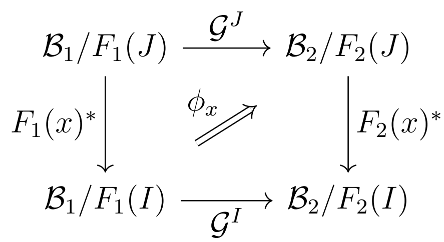
$$G(X) \xrightarrow{G(p)} G(F_1(I)) \xrightarrow{\alpha_I} F_2(I),$$

and the mapping of arrows is depicted below.

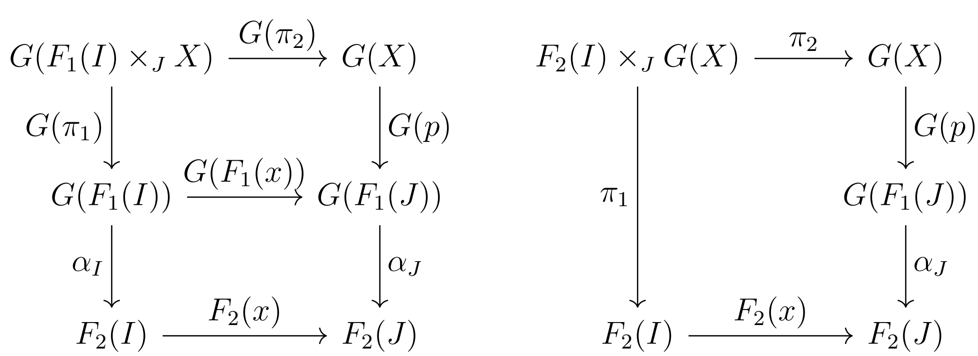


Defining the natural isomorphisms

For each $x : I \to J$ in $\mathcal{A}$, we must define a natural isomorphism $\phi_x : \mathcal{G}^I x^* \Rightarrow x^* \mathcal{G}^J$, as in the square below.



Given an object $X \xrightarrow{p} F_1(J)$ in $\mathcal{B}_1/F_1(J)$, the arrow $\phi_x(X, p)$ is defined by considering the following two diagrams.



Remark that the left column of the left diagram is $\mathcal{G}^I F_1(x)^*(X, p)$, and the left column of the right diagram is $F_2(x)^* \mathcal{G}^J(X, p)$. Moreover, the left diagram consists of two pullback squares glued together (because G preserves pullbacks and α is an isomorphism); thus, the left and right diagrams are both pullbacks of $F_2(x)$ and $\alpha_J \circ G(p)$. We will therefore let $\phi_x(X, p)$

be the unique isomorphism $G(F_1(I) \times_J X) \to F_2(I) \times_J G(X)$ that respects the projection maps.

It remains to check that this collection of isomorphisms is natural in (X, p). Given an arrow $f : (X, p) \to (Y, q)$, we must check that the naturality square commutes.

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & F_1(J) & \end{array} \qquad \begin{array}{ccc} G(F_1(I) \times_J X) & \xrightarrow{\phi_x(X,p)} & F_2(I) \times_J G(X) \\ {\scriptstyle G(\mathrm{Id} \times_J f)}\downarrow & & \downarrow{\scriptstyle \mathrm{Id} \times_J G(f)} \\ G(F_1(I) \times_J Y) & \xrightarrow{\phi_x(Y,q)} & F_2(I) \times_J G(Y) \end{array}$$

That is, we must show $(\mathrm{Id} \times_J G(f)) \circ \phi_x(X, p) = \phi_x(Y, q) \circ G(\mathrm{Id} \times_J f)$. Since these are arrows into a pullback, it suffices to show that they are equal when the projection maps are applied. Since the ϕ_x are chosen to commute with the projections, this becomes a trivial calculation; we omit the details.

Coherence conditions

We have defined all the parts of the indexed functor $\mathcal{G}$. However, we must check that some coherence conditions are satisfied.

First, we note that if $x = \mathrm{Id}_I$, then $\mathcal{G}^I x^*(X, p)$ and $x^*\mathcal{G}^I(X, p)$ coincide: they are precisely the arrow $G(X) \xrightarrow{G(p)} G(F_1(J)) \xrightarrow{\alpha_I} F_2(I)$, viewed as the pullback of itself via the identity map. Then $\phi_x(X, p)$ is the canonical map from this pullback to itself, which is the identity. Thus, $\phi_{\mathrm{Id}_I} = \mathrm{Id}_{\mathcal{G}^I}$, as desired.

Next, we must establish the coherence condition between the ϕ_x arrows and the isomorphisms $x^*y^* \cong (yx)^*$ of the $\mathcal{A}$-indexed categories $F_1^*\mathbb{B}_1$ and $F_2^*\mathbb{B}_2$. For this calculation, the precise definitions of these indexed categories are important: the isomorphisms come from the pasting laws for pullbacks.

Without getting too much into the details, the important idea is that we have two diagrams of the following form, where all the squares are pullbacks.

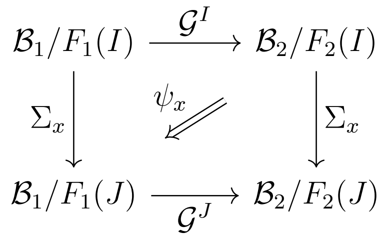
$$\begin{array}{ccccc} P' & \longrightarrow & P & \longrightarrow & G(X) \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle G(p)} \\ G(F_1(I)) & \xrightarrow{x} & G(F_1(J)) & \xrightarrow{y} & G(F_1(K)) \\ \downarrow{\scriptstyle \alpha_I} & & \downarrow{\scriptstyle \alpha_J} & & \downarrow{\scriptstyle \alpha_K} \\ F_2(I) & \xrightarrow{x} & F_2(J) & \xrightarrow{y} & F_2(K) \end{array} \qquad \begin{array}{ccccc} P'' & & \longrightarrow & & G(X) \\ \Big\downarrow & & & & \downarrow{\scriptstyle G(p)} \\ & & & & G(F_1(K)) \\ \Big\downarrow & & & & \downarrow{\scriptstyle \alpha_K} \\ F_2(I) & \xrightarrow{x} & F_2(J) & \xrightarrow{y} & F_2(K) \end{array}$$

We need to establish that two arrows $P' \to P''$ are equal, but P' and P'' turn out to both be pullbacks of the arrows $F_2(y) \circ F_2(x)$ and $\alpha_K \circ G(p)$. Moreover, it turns out that the arrows $P' \to P''$ each commute with the projections of these pullbacks (because of how the arrows ϕ_x and the isomorphisms $x^*y^* \cong (yx)^*$ are defined), so they must be equal.

A.2 Canonical functors preserve coproducts

In section A.1, we showed how to extend a functor G to $\mathcal{G} : F_1^*\mathbb{B}_1 \to F_2^*\mathbb{B}_2$. In this section, we show that this indexed functor always preserves indexed coproducts. (This provides part of the proof of proposition 6.3.3.)

Let $x : I \to J$ be an arrow of $\mathcal{A}$. Recall that $\phi_x : \mathcal{G}^I x^* \Rightarrow x^* \mathcal{G}^J$ is the natural isomorphism defined previously. Recall that $F_1^*\mathbb{B}_1$ and $F_2^*\mathbb{B}_2$ have arbitrary internal coproducts, so they each have an adjunction $(\Sigma_x, x^*, \eta_x, \epsilon_x)$, where Σ_x is the usual functor (post-composition with x). Then we can form a natural transformation ψ_x as in the following diagram.

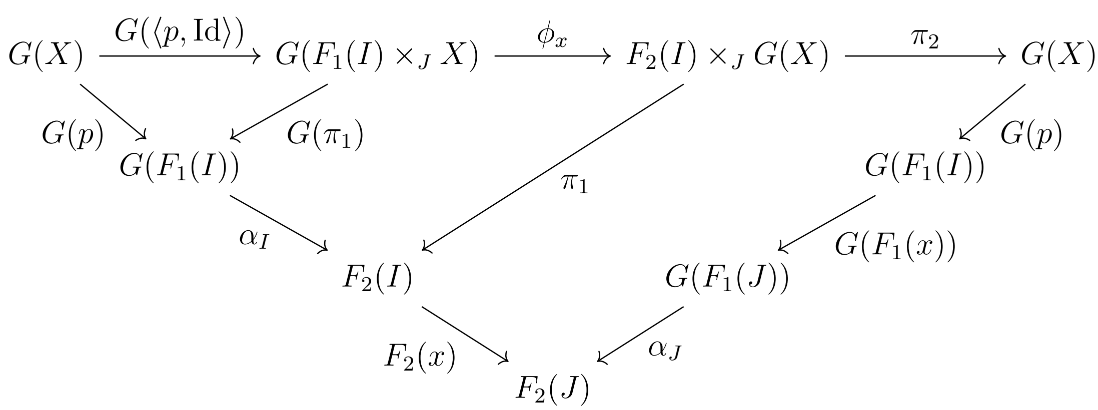


Specifically, ψ_x is given by the following composition.

$$\Sigma_x\mathcal{G}^I \xrightarrow{\Sigma_x\mathcal{G}^I\eta_x} \Sigma_x\mathcal{G}^I x^*\Sigma_x \xrightarrow{\Sigma_x\phi_x\Sigma_x} \Sigma_x x^*\mathcal{G}^J\Sigma_x \xrightarrow{\varepsilon_x\mathcal{G}^J\Sigma_x} \mathcal{G}^J\Sigma_x$$

To show $\mathcal{G}$ preserves arbitrary internal coproducts, we must show this ψ_x is an isomorphism.

Let $X \xrightarrow{p} F_1(I)$ be an object of $\mathcal{B}_1/F_1(I)$; we will show that $\psi_x(X,p)$ is an isomorphism. By unravelling the definitions of the functors Σ_x (composition) and x^* (pullback) and the transformations $\eta_x, \varepsilon_x, \phi_x$, we find that $\psi_x(X,p)$ is the following arrow in $\mathcal{B}_2/F_2(J)$.



Note that $F_2(x) \circ \alpha_I = \alpha_J \circ G(F_1(x))$ because α is natural, so the domain and codomain of the arrow $\psi_x(X,p) = \pi_2 \circ \phi_x \circ G(\langle p, \mathrm{Id}\rangle)$ are equal. We will therefore actually claim that $\psi_x(X,p)$ is the identity map. This is true because ϕ_x commutes with the pullback projection maps and G preserves pullbacks, so we can make the following calculation.

$$\psi_x(X,p) = \pi_2 \circ \phi_x \circ G(\langle p, \mathrm{Id}\rangle) = \pi_2 \circ \langle G(p), \mathrm{Id}\rangle = \mathrm{Id}$$

This is all we needed to show.

A.3 Indexed categories with coproducts

In this section, we develop some results that will be used in section A.4

First, let $\mathbb{C}$ be an $\mathcal{S}$-indexed category with all $\mathcal{S}$-indexed coproducts. Let $I \xrightarrow{x} J \xrightarrow{y} K$ be a diagram in $\mathcal{S}$. For any $A \in \mathcal{C}^I$ and $B \in \mathcal{C}^K$, consider the following series of isomorphisms (which come from the adjunctions and the indexed category):

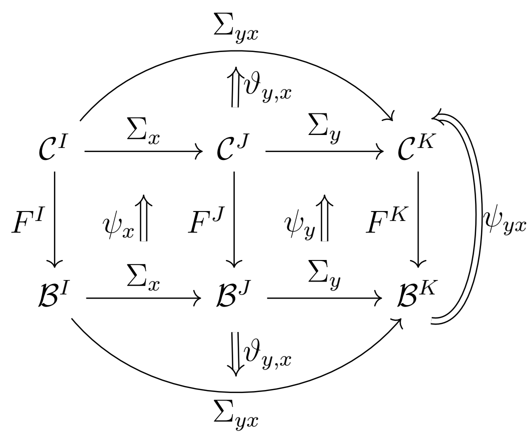
$$\begin{aligned}\mathrm{Hom}(\Sigma_y\Sigma_x A, B) \cong \mathrm{Hom}(\Sigma_x A, y^*B) &\cong \mathrm{Hom}(A, x^*y^*B)\\ &\cong \mathrm{Hom}(A, (yx)^*B) \cong \mathrm{Hom}(\Sigma_{yx}A, B).\end{aligned}$$

This induces a canonical isomorphism $\vartheta_{y,x} : \Sigma_y\Sigma_x \Rightarrow \Sigma_{yx}$.

Now, recall that an $\mathcal{S}$-indexed category $\mathbb{C}$ is just a pseudofunctor $\mathcal{S}^{op} \to \mathbf{CAT}$. But, if $\mathbb{C}$ has all $\mathcal{S}$-indexed coproducts, we can form another pseudofunctor $\mathbb{C}^L : \mathcal{S} \to \mathbf{CAT}$ with the categories $\mathcal{C}^I$, the functors Σ_x, and the canonical isomorphisms $\vartheta_{y,x} : \Sigma_y\Sigma_x \Rightarrow \Sigma_{yx}$. In particular, we have

$$\vartheta_{\mathrm{Id},x} = \mathrm{Id}_{\Sigma_x} = \vartheta_{x,\mathrm{Id}} \qquad \text{and} \qquad \vartheta_{z,yx} \circ (z^*\vartheta_{y,x}) = \vartheta_{zy,x} \circ (\vartheta_{z,y}x^*).$$

Furthermore, suppose $\mathbb{C}, \mathbb{D}$ are $\mathcal{S}$-indexed categories with all $\mathcal{S}$-indexed coproducts, and let $F : \mathbb{C} \to \mathbb{D}$ be an indexed functor which preserves coproducts. Then F, equipped with the canonical isomorphisms $\psi_x : \Sigma_x F^I \to F^J\Sigma_x$, is also a pseudonatural transformation $\mathbb{C}^L \to \mathbb{D}^L$. In particular, $\psi_{\mathrm{Id}_I} = \mathrm{Id}_{F^I}$, and the following diagram commutes for any x, y.



This diagram is the analogue of the one given in section 6.1.

A.4 Unique functor preserving coproducts

In this section, we prove proposition 6.3.4. It states that if $\mathcal{S}$ is a category with finite limits, and $\mathbb{C}$ is an $\mathcal{S}$-indexed category with finite limits and $\mathcal{S}$-indexed coproducts, then there is at most one $\mathcal{S}$-indexed functor $\mathbb{S} \to \mathbb{C}$ (up to isomorphism) which preserves finite limits and $\mathcal{S}$-indexed coproducts. In order to prove this result, we rely on the facts from section A.3.

So, let $F_1, F_2 : \mathbb{S} \to \mathbb{C}$ be two $\mathcal{S}$-indexed functors which preserve finite limits and arbitrary coproducts. They are equipped with natural isomorphisms $\phi_{i,x} : F_i^I x^* \Rightarrow x^* F_i^J$ and $\psi_{i,x} : \Sigma_x F_i^I \Rightarrow F_i^J \Sigma_x$. Our goal is to construct an $\mathcal{S}$-indexed natural isomorphism $\alpha : F_1 \Rightarrow F_2$.

We start by fixing some notation. For any object $A \in \mathcal{S}$, remark that Id_A is a terminal object in $\mathcal{S}/A$. Since F_1, F_2 preserve finite limits, we find that $F_1^A(\mathrm{Id}_A)$ and $F_2^A(\mathrm{Id}_A)$ are terminal objects, and so there exists a unique arrow $F_1^A(\mathrm{Id}_A) \to F_2^A(\mathrm{Id}_A)$ (which is an isomorphism). We call this arrow T_A.

Now, let $B \in \mathcal{S}$; we will define a natural isomorphism $\alpha^B : F_1^B \Rightarrow F_2^B$ between these two functors $\mathcal{S}/B \to \mathcal{C}^B$. Given an object $f : A \to B$ in $\mathcal{S}/B$, note that $f = \Sigma_f(\mathrm{Id}_A)$. We can therefore define $\alpha^B(f)$ to be the following composite.

$$\begin{array}{ccl} F_1^B(f) & = & F_1^B(\Sigma_f(\mathrm{Id}_A)) \\ & & \quad \downarrow \psi_{1,f}^{-1}(\mathrm{Id}_A) \\ & & \Sigma_f(F_1^A(\mathrm{Id}_A)) \\ & & \quad \downarrow \Sigma_f(T_A) \\ & & \Sigma_f(F_2^A(\mathrm{Id}_A)) \\ & & \quad \downarrow \psi_{2,f}(\mathrm{Id}_A) \\ & & F_2^B(\Sigma_f(\mathrm{Id}_A)) \quad = \quad F_2^B(f) \end{array}$$

We must now check naturality. Consider an arrow of $\mathcal{S}/B$ as follows.

$$\begin{array}{ccc} X & \xrightarrow{g} & A \\ & h \searrow \quad \swarrow f & \\ & B & \end{array}$$

Of course, $h = f \circ g$, and we can moreover note the following.

$$\begin{array}{ccc} X & \xrightarrow{g} & A \\ & h \searrow \quad \swarrow f & \\ & B & \end{array} \quad = \quad \Sigma_f \left(\begin{array}{ccc} X & \xrightarrow{g} & A \\ & g \searrow \quad \swarrow \mathrm{Id}_A & \\ & A & \end{array} \right) \quad = \quad \Sigma_f\Big(\Sigma_g(\mathrm{Id}_X) \xrightarrow{g} \mathrm{Id}_A\Big)$$

We will focus our attention on the arrow $\Sigma_g(\mathrm{Id}_X) \xrightarrow{g} \mathrm{Id}_A$ in $\mathcal{S}/A$. If we apply F_1^A to it, we can consider the following diagram in $\mathcal{C}^A$.

$$\begin{array}{ccc} F_1^A(\Sigma_g(\mathrm{Id}_X)) & \xrightarrow{F_1^A(g)} & F_1^A(\mathrm{Id}_A) \\ {\scriptstyle \alpha^A(g)} \downarrow & & \downarrow {\scriptstyle T_A} \\ F_2^A(\Sigma_g(\mathrm{Id}_X)) & \xrightarrow{F_2^A(g)} & F_2^A(\mathrm{Id}_A) \end{array}$$

This diagram commutes because $F_2^A(\mathrm{Id}_A)$ is a terminal object. Then, we apply Σ_f to this diagram, and add commutative squares on the top and bottom based on the natural isomorphisms $\Sigma_f F_i^A \cong F_i^B \Sigma_f$. (Remark that $\Sigma_f(\mathrm{Id}_A) = f$ and $\Sigma_f \Sigma_g(\mathrm{Id}_X) = f \circ g = h$.)

$$\begin{array}{ccc}
F_1^B(h) & \xrightarrow{\;F_1^B(g)\;} & F_1^B(f) \\
{\scriptstyle \psi_{1,f}^{-1}(\Sigma_g(\mathrm{Id}_X))}\big\downarrow & & \big\downarrow{\scriptstyle \psi_{1,f}^{-1}(\mathrm{Id}_A)} \\
\Sigma_f F_1^A \Sigma_g(\mathrm{Id}_X) & \xrightarrow{\;\Sigma_f F_1^A(g)\;} & \Sigma_f F_1^A(\mathrm{Id}_A) \\
{\scriptstyle \Sigma_f(\alpha^A(g))}\big\downarrow & & \big\downarrow{\scriptstyle \Sigma_f(T_A)} \\
\Sigma_f F_2^A \Sigma_g(\mathrm{Id}_X) & \xrightarrow{\;\Sigma_f F_2^A(g)\;} & \Sigma_f F_2^A(\mathrm{Id}_A) \\
{\scriptstyle \psi_{2,f}(\Sigma_g(\mathrm{Id}_X))}\big\downarrow & & \big\downarrow{\scriptstyle \psi_{2,f}(\mathrm{Id}_A)} \\
F_2^B(h) & \xrightarrow{\;F_2^B(g)\;} & F_2^B(f)
\end{array}$$

We now notice that the right column is precisely $\alpha^B(f)$. To finish our proof of naturality, it now suffices to show that the left column is $\alpha^B(h)$. Put another way, we want to show that the following square commutes.

$$\begin{array}{ccc}
\Sigma_f F_1^A \Sigma_g(\mathrm{Id}_X) & \xrightarrow{\;\psi_{1,f}(\Sigma_g(\mathrm{Id}_X))\;} & F_1^B(h) \\
{\scriptstyle \Sigma_f(\alpha^A(g))}\big\downarrow & & \big\downarrow{\scriptstyle \alpha^B(h)} \\
\Sigma_f F_2^A \Sigma_g(\mathrm{Id}_X) & \xrightarrow{\;\psi_{2,f}(\Sigma_g(\mathrm{Id}_X))\;} & F_2^B(h)
\end{array}$$

In the above diagram, we can expand the definition of α, yielding the following.

$$\begin{array}{ccc}
\Sigma_f F_1^A \Sigma_g(\mathrm{Id}_X) & \xrightarrow{\;\psi_{1,f}(\Sigma_g(\mathrm{Id}_X))\;} & F_1^B \Sigma_h(\mathrm{Id}_X) \\
{\scriptstyle \Sigma_f(\psi_{1,g}^{-1}(\mathrm{Id}_X))}\big\downarrow & & \big\downarrow{\scriptstyle \psi_{1,h}^{-1}(\mathrm{Id}_X)} \\
\Sigma_f \Sigma_g F_1^X(\mathrm{Id}_X) & & \Sigma_h F_1^X(\mathrm{Id}_X) \\
{\scriptstyle \Sigma_f(\Sigma_g(T_X))}\big\downarrow & & \big\downarrow{\scriptstyle \Sigma_h(T_X)} \\
\Sigma_f \Sigma_g F_2^X(\mathrm{Id}_X) & & \Sigma_h F_2^X(\mathrm{Id}_X) \\
{\scriptstyle \Sigma_f(\psi_{2,g}(\mathrm{Id}_X))}\big\downarrow & & \big\downarrow{\scriptstyle \psi_{2,h}(\mathrm{Id}_X)} \\
\Sigma_f F_2^A \Sigma_g(\mathrm{Id}_X) & \xrightarrow{\;\psi_{2,f}(\Sigma_g(\mathrm{Id}_X))\;} & F_2^B \Sigma_h(\mathrm{Id}_X)
\end{array}$$

To see that this commutes, we break it down into three squares, where the vertical arrows in the top square have been reversed (replacing $\psi_{1,g}^{-1}$ and $\psi_{1,h}^{-1}$ with $\psi_{1,g}$ and $\psi_{1,h}$).

$$
\begin{array}{ccc}
\Sigma_f F_1^A \Sigma_g(\mathrm{Id}_X) & \xrightarrow{\psi_{1,f}(\Sigma_g(\mathrm{Id}_X))} & F_1^B \Sigma_h(\mathrm{Id}_X) \\
{\scriptstyle \Sigma_f(\psi_{1,g}(\mathrm{Id}_X))}\uparrow & & \uparrow{\scriptstyle \psi_{1,h}(\mathrm{Id}_X)} \\
\Sigma_f \Sigma_g F_1^X(\mathrm{Id}_X) & \xrightarrow{\vartheta_{f,g}(F_1^X(\mathrm{Id}_X))} & \Sigma_h F_1^X(\mathrm{Id}_X) \\
{\scriptstyle \Sigma_f(\Sigma_g(T_X))}\downarrow & & \downarrow{\scriptstyle \Sigma_h(T_X)} \\
\Sigma_f \Sigma_g F_2^X(\mathrm{Id}_X) & \xrightarrow{\vartheta_{f,g}(F_2^X(\mathrm{Id}_X))} & \Sigma_h F_2^X(\mathrm{Id}_X) \\
{\scriptstyle \Sigma_f(\psi_{2,g}(\mathrm{Id}_X))}\downarrow & & \downarrow{\scriptstyle \psi_{2,h}(\mathrm{Id}_X)} \\
\Sigma_f F_2^A \Sigma_g(\mathrm{Id}_X) & \xrightarrow{\psi_{2,f}(\Sigma_g(\mathrm{Id}_X))} & F_2^B \Sigma_h(\mathrm{Id}_X)
\end{array}
$$

To finish off the proof, we claim that each square here commutes. The top and bottom squares commute by the diagram in section A.3, and the middle square commutes by naturality of ϑ, so we're done.

Appendix B

Calculations for binary lists

Contents

In chapters 13 and 15, we will prove results about finite objects in a locos. However, we will need some tools to do this, which we will develop in this appendix. Specifically, we will define some functions on binary lists (elements of $L(\{0,1\})$) and prove facts about them. Throughout, we work in a locos.

We attempt to summarize these results as follows: they are used to identify the zeroes in a binary list. We define a function $cZ : L(\{0,1\}) \to N$ which counts the number of zeroes in a binary list, and we define a function $kZ : L(\{0,1\}) \times N \to N$ where $kZ(\ell, k)$ gives the position of the k^{th} zero in the list ℓ (if it exists). We want to show that, if $k < cZ(\ell)$, then $kZ(\ell, k)$ does indeed give us the k^{th} zero of the list ℓ. But how can we express this fact?

First, the condition $k < cZ(\ell)$ should imply that $kz(\ell, k)$ does indeed give the position of a zero of the list ℓ. That is, we should have $kZ(\ell, k) < \mathrm{len}(\ell)$ (proposition B.6.3) and $\mathrm{nth}(\ell, kZ(\ell, k)) = 0$ (proposition B.8.1).

Second, the function $kZ(\ell, k)$ should not pick out the same zero twice. We formalize this by showing that $kZ(\ell, k) < kZ(\ell, sk)$ as long as $k < cZ(\ell)$ (lemma B.7.3).

Finally, we should know that all zeroes of ℓ are identified by the kZ function. So, if there is some $n < \mathrm{len}(\ell)$ such that $\mathrm{nth}(\ell, n) = 0$, then there should be some $k < cZ(\ell)$ such that $n = kZ(\ell, n)$ (proposition B.8.1).

This appendix is dedicated to proving the above results. Unfortunately, the proof strategies are quite ad hoc, and it is difficult to organize them into a sensible narrative.

B.1 Key definitions

In this section, we define the functions on binary lists that will allow us to do our calculations. We preceed these definitions with the following remark, which will justify the form of these definitions.

Remark B.1.1. Since a locos is extensive and hence distributive, we have

$$\{0,1\} \times L(\{0,1\}) \cong \Big[\{0\} \times L(\{0,1\})\Big] + \Big[\{1\} \times L(\{0,1\})\Big].$$

So, when defining functions on $L(\{0,1\})$ (and proving facts about them), the inductive step – which involves considering the list $b :: \ell$ – can be split into two cases: $0 :: \ell$ and $1 :: \ell$.

We first define the arrows tZ ("trim to zero") and tpZ ("trim past zero"). Intuitively, $tZ(\ell)$ trims ℓ up to the zeroth zero and leaves it, while $tpZ(\ell)$ trims ℓ up to the zeroth zero and removes it. Formally, we have the following definition.

Definition. We define $tZ, tpZ : L(\{0,1\}) \to L(\{0,1\})$ inductively as follows.

$$\begin{aligned} tZ(\varnothing) &= \varnothing \\ tZ(0 :: \ell) &= 0 :: \ell \\ tZ(1 :: \ell) &= tZ(\ell) \end{aligned} \qquad\qquad \begin{aligned} tpZ(\varnothing) &= \varnothing \\ tpZ(0 :: \ell) &= \ell \\ tpZ(1 :: \ell) &= tpZ(\ell) \end{aligned}$$

Next, we define zZ ("zeroth zero") and pzZ ("past the zeroth zero"). Intuitively, $zZ(\ell)$ gives the position of the zeroth zero in the list ℓ; it also counts the number of trims necessary to turn ℓ into $tZ(\ell)$. Moreover, $pzZ(\ell)$ gives the position after the zeroth zero, i.e. it counts the number of trims necessary to turn ℓ into $tpZ(\ell)$.

Note that we expect $pzZ(\ell) = 1 + zZ(\ell)$, unless ℓ doesn't have any zeroes, in which case $pzZ(\ell) = zZ(\ell) = \text{len}(\ell)$, but we don't offer a formal proof yet.

Definition. We define $zZ, pzZ : L(\{0,1\}) \to N$ inductively as follows.

$$\begin{aligned} zZ(\varnothing) &= 0 \\ zZ(0 :: \ell) &= 0 \\ zZ(1 :: \ell) &= 1 + zz(\ell) \end{aligned} \qquad\qquad \begin{aligned} pzZ(\varnothing) &= 0 \\ pzZ(0 :: \ell) &= 1 \\ pzZ(1 :: \ell) &= 1 + pzZ(\ell) \end{aligned}$$

Next, we define arrows that iterate these definitions. We start with tkZ ("trim to the k^{th} zero") and $tpkZ$ ("trim past the k^{th} zero"). These descriptions are pretty self-explanatory.

Note that for these definitions, we need to use the new induction scheme of theorem 3.5.3. Also, note that in both cases, for the inductive step, we need to *trim past* the current zero.

Definition. We define $tkZ, tpkZ : L(\{0,1\}) \times N \to L(\{0,1\})$ inductively as follows.

$$tkZ(\ell, 0) = tZ(\ell) \qquad tpkZ(\ell, 0) = tpZ(\ell)$$
$$tkZ(\ell, sn) = tkZ(tpZ(\ell), n) \qquad tpkZ(\ell, sn) = tpkZ(tpZ(\ell), n)$$

Finally, we define kZ ("k^{th} zero") and pkZ ("past the k^{th} zero"). Intuitively, $kZ(\ell, k)$ gives the position of the k^{th} zero; equivalently, it gives the number of trims necessary to turn ℓ into $tkZ(\ell, k)$. The arrow pkZ is analogous for trimming just past the k^{th} zero.

Definition. We define $kZ, pkZ : L(\{0,1\}) \times N \to N$ inductively as follows.

$$kZ(\ell, 0) = zZ(\ell) \qquad pkZ(\ell, 0) = pzZ(\ell)$$
$$kZ(\ell, sn) = pzZ(\ell) + kZ(tpZ(\ell), n) \qquad pkZ(\ell, sn) = pzZ(\ell) + pkZ(tpZ(\ell), n)$$

We define an arrow cZ ("count zeroes") which counts the number of zeroes in a binary list.

Definition. We define $cZ : L(\{0,1\}) \to N$ inductively as follows.

$$cZ(\varnothing) = 0$$
$$cZ(0 :: \ell) = 1 + cZ(\ell)$$
$$cZ(1 :: \ell) = cZ(\ell)$$

Note that the induction relation can also be written as $cZ(b :: \ell) = \overline{b} + cZ(\ell)$, where $\overline{b} = 1 \dot{-} b$.

We also define chZ ("count head zeroes") and ctZ ("count tail zeroes") via the following abbreviations.

Definition. We write

$$chZ(\ell, n) = cZ(\text{head}(\ell, n)) \qquad \text{and} \qquad ctZ(\ell, n) = cZ(\text{tail}(\ell, n)).$$

B.2 First results - zZ

Proposition B.2.1. Let $\ell : L(\{0,1\})$ be a term in a context C. If $zZ(\ell) <_C \text{len}(\ell)$, then $\text{nth}(\ell, zZ(\ell)) =_C 0$.

Proof. It suffices to show that $\text{nthDef}(\ell, zZ(\ell), 0) =_\ell 0$. Indeed, if $zZ(\ell) <_C \text{len}(\ell)$, then proposition 4.4.4 tells us that

$$\text{nth}(\ell, zZ(\ell)) =_C \text{nthDef}(\ell, zZ(\ell), 0) =_C 0,$$

as desired.

So, we show $\text{nthDef}(\ell, zZ(\ell), 0) =_\ell 0$ by induction on ℓ. In the base case, we have

$$\text{nthDef}(\varnothing, zZ(\varnothing), 0) = \text{nthDef}(\varnothing, 1, 0) = \text{zerothDef}(\varnothing, 0) = 0.$$

In the induction case, we must evaluate $\mathrm{nthDef}(b :: \ell, zZ(b :: \ell), 0)$. By splitting b into the cases $b = 0$, $b = 1$, we can easily check that

$$\mathrm{nthDef}(b :: \ell, zZ(b :: \ell), 0) =_{b,\ell} \begin{cases} 0 & \text{if } b = 0 \\ \mathrm{nthDef}(\ell, zZ(\ell), 0) & \text{if } b = 1 \end{cases}.$$

This inductive characterization is shared by the zero map, so the two must be equal, as desired. ∎

Proposition B.2.2. In a locos, we have the following.

1. $zZ(\ell) \leq_\ell 1 + zZ(\mathrm{tr}(\ell))$
2. $zZ(\ell) \leq_{\ell,k} k + zZ(\mathrm{tail}(\ell, k))$

Proof. For part (1), we simply split the list into cases. In the empty case, we trivially have

$$zZ(\varnothing) = 1 \leq 2 = 1 + zZ(\mathrm{tr}(\varnothing)).$$

In the non-empty case, we split into cases for $b \in \{0, 1\}$:

$$zZ(b :: \ell) =_{b,\ell} \begin{cases} 0 & \text{if } b = 0 \\ 1 + zZ(\ell) & \text{if } b = 1 \end{cases}.$$

In either case, the term is less or equal to $1 + zZ(\ell) = 1 + zZ(\mathrm{tr}(b :: \ell))$.

For part (2), we go by induction on k. The base case is trivial; for the induction step, we start by using part (1) to note that

$$\begin{aligned} sk + zZ(\mathrm{tail}(\ell, sk)) &= k + 1 + zZ(\mathrm{tr}(\mathrm{tail}(\ell, k))) \\ &\geq zZ(\mathrm{tail}(\ell, k)). \end{aligned}$$

Thus, there exists a term $\phi(\ell, k)$ such that $sk + zZ(\mathrm{tail}(\ell, sk)) = zZ(\mathrm{tail}(\ell, k)) + \phi(\ell, k)$. Then we compute:

$$zZ(\ell) \dot{-} \big(sk + zZ(\mathrm{tail}(\ell, sk))\big) = \big(zZ(\ell) \dot{-} zZ(\mathrm{tail}(\ell, k))\big) \dot{-} \phi(\ell, k).$$

This recursion is also satisfied by zero, so we're done. ∎

Proposition B.2.3. Let $k : N$, $\ell : L(\{0, 1\})$ be terms in a context C such that $k < \mathrm{len}(\ell)$. If $\mathrm{nth}(\ell, k) =_C 0$, then $zZ(\ell) \leq_C k$.

Proof. Note that $\mathrm{len}(\mathrm{tail}(\ell, k)) = \mathrm{len}(\ell) \dot{-} k > 0$, so we can write $\mathrm{tail}(\ell, k) = b :: \ell'$. Then we compute

$$0 = \mathrm{nth}(\ell, k) = \mathrm{zeroth}(\mathrm{tail}(\ell, k)) = \mathrm{zeroth}(b :: \ell') = b,$$

which tells us that $zZ(\mathrm{tail}(\ell, k)) = 0$. Using proposition B.2.2, we conclude that

$$zZ(\ell) \leq k + zZ(\mathrm{tail}(\ell, k)) = k + 0 = k,$$

as desired. ∎

B.3 First results - tK

Proposition B.3.1. For every $\ell \in L(\{0, 1\})$ and $k \in N$, we have

1. $tZ(\ell) = \mathrm{tail}(\ell, zZ(\ell))$
2. $tpZ(\ell) = \mathrm{tail}(\ell, pzZ(\ell))$
3. $tkZ(\ell, k) = \mathrm{tail}(\ell, kZ(\ell, k))$
4. $tpkZ(\ell, k) = \mathrm{tail}(\ell, pkZ(\ell, k))$

Proof. Part 1. We check that $\mathrm{tail}(\ell, zZ(\ell))$ satisfies the equations defining $tZ(\ell)$.

$$\begin{aligned} \mathrm{tail}(\varnothing, zZ(\varnothing)) &= \varnothing \\ \mathrm{tail}(0 :: \ell, zZ(0 :: \ell)) &= \mathrm{tail}(0 :: \ell, 0) = 0 :: \ell \\ \mathrm{tail}(1 :: \ell, zZ(1 :: \ell)) &= \mathrm{tail}(1 :: \ell), 1 + zZ(\ell)) = \mathrm{tail}(\ell, zZ(\ell)) \end{aligned}$$

Part 2. We check that $\mathrm{tail}(\ell, pzZ(\ell))$ satisfies the equations defining $tpZ(\ell)$.

$$\begin{aligned} \mathrm{tail}(\varnothing, pzZ(\varnothing)) &= \varnothing \\ \mathrm{tail}(0 :: \ell, pzZ(0 :: \ell)) &= \mathrm{tail}(0 :: \ell, 1) = \ell \\ \mathrm{tail}(1 :: \ell, pzZ(1 :: \ell)) &= \mathrm{tail}(1 :: \ell, 1 + pzZ(\ell)) = \mathrm{tail}(\ell, pzZ(\ell)) \end{aligned}$$

Part 3. We check that $\mathrm{tail}(\ell, kZ(\ell, k))$ satisfies the equations defining $tkZ(\ell, k)$.

$$\begin{aligned} \mathrm{tail}(\ell, kZ(\ell, 0)) &= \mathrm{tail}(\ell, zZ(\ell)) = tZ(\ell) \\ \mathrm{tail}(\ell, kZ(\ell, sk)) &= \mathrm{tail}(\ell, pzZ(\ell) + kZ(tpZ(\ell), k)) \\ &= \mathrm{tail}\Big(\mathrm{tail}(\ell, pzZ(\ell)),\ kZ(tpZ(\ell), k)\Big) \end{aligned}$$

$$= \text{tail}\Big(tpZ(\ell),\ kZ(tpZ(\ell), k)\Big)$$

Note that we used parts 1 and 2 for the computation (along with proposition 4.4.5).

Part 4. We check that $\text{tail}(\ell, pkZ(\ell, k))$ satisfies the equations defining $tpkZ(\ell, k)$.

$$\begin{aligned} \text{tail}(\ell, pkZ(\ell, 0)) &= \text{tail}(\ell, pzZ(\ell)) = tpZ(\ell) \\ \text{tail}(\ell, pkZ(\ell, sk)) &= \text{tail}(\ell,\ pzZ(\ell) + pkZ(tpZ(\ell), k)) \\ &= \text{tail}\Big(\text{tail}(\ell, pzZ(\ell)),\ pkZ(tpZ(\ell), k)\Big) \\ &= \text{tail}\Big(tpZ(\ell),\ pkZ(tpZ(\ell), k)\Big) \end{aligned}$$

Again, we used part 2 for the computation. ∎

Proposition B.3.2. We have $kZ(\varnothing, k) = 0$.

Proof. We go by induction on k. For $k = 0$, we have $kZ(\varnothing, 0) = zZ(\varnothing) = 0$. For the successor case, we have

$$kZ(\varnothing, sk) = pzZ(\varnothing) + kZ(tpZ(\varnothing), k) = 0 + kZ(\varnothing, k) = kZ(\varnothing, k),$$

a recurrence which is also satisfied by the constant zero map. ∎

Proposition B.3.3. $kZ(1 :: \ell, k) = 1 + kZ(\ell, k)$

Proof. We split into the zero and successor case for k. For the base case,

$$\begin{aligned} kZ(1 :: \ell, 0) &= zZ(1 :: \ell) = 1 + zZ(\ell), \\ 1 + kZ(\ell, 0) &= 1 + zZ(\ell). \end{aligned}$$

For the successor case,

$$\begin{aligned} kZ(1 :: \ell, sk) = pzZ(1 :: \ell) + kZ(tpZ(1 :: \ell), k) &= 1 + pzZ(\ell) + kZ(tpZ(\ell), k), \\ 1 + kZ(\ell, sk) &= 1 + pzZ(\ell) + kZ(tpZ(\ell), k). \end{aligned}$$

That's all we needed to show. ∎

Proposition B.3.4. We have $tpkZ(\ell, k) = tpZ^{k+1}(\ell)$. In particular, $tpkZ(\ell, sk) = tpZ(tpkZ(\ell, k))$.

Proof. This is a special case of a more general fact. Given $f : X \to X$, we write $f^k(x)$ for the function defined inductively by $f^0(x) = x$ and $f^{sk}(x) = f(f^k(x))$. However, using the induction scheme of theorem 3.5.3, we can also define $g(x, k)$ by $g(x, 0) = f(x)$ and $g(x, sk) = g(f(x), k)$. This proposition is the special case of proving $g(x, k) = f^{sk}(x)$, where f is replaced by tpZ ($tpkZ$ is the same as the corresponding g).

So, let's prove $g(x, k) = f^{sk}(x)$. First, we show that $f^{sk}(x) = f^k(f(x))$ by induction; the base case is trivial, and for the induction step, it's easy to check that both satisfy $h(x, sk) = f(h(x, k))$. Then, we note that $f^{ssk}(x) = f^{sk}(f(x))$ is the same recursion as the one defining $g(x, k)$. ∎

Proposition B.3.5. $tkZ(\ell, sk) = tZ(tpkZ(\ell, k))$

Proof. We go by induction on k. In the base case, we have

$$tkZ(\ell, 1) = tkZ(tpZ(\ell), 0) = tZ(tpZ(\ell)) = tZ(tpkZ(\ell, 0)).$$

For the inductive step, we compute

$$tkZ(\ell, ssk) = tkZ(tpZ(\ell), sk) \quad \text{and} \quad tZ(tpkZ(\ell, k)) = tZ(tpkZ(tpZ(\ell), k)).$$

Thus both terms satisfy the recursion $f(\ell, sk) = f(tpZ(\ell), k)$. ∎

Proposition B.3.6. $kZ(\ell, sk) = pkZ(\ell, k) + zZ(tpkZ(\ell, k))$

Proof. We go by induction on k. If $k = 0$, both sides easily reduce to $pzZ(\ell) + zZ(tpZ(\ell))$. For the inductive step, we compute

$$pkZ(\ell, sk) + zZ(tpkZ(\ell, sk)) = pzZ(\ell) + pkZ(tpZ(\ell), k) + zZ(tpkZ(tpZ(\ell), k)),$$

which is the same recurrence as kZ, as desired. ∎

Proposition B.3.7. We have

1. $zZ(\ell) \leq pzZ(\ell)$
2. $zZ(\ell) \leq kZ(\ell, sk)$

Proof. For the first part, go by induction on ℓ. For $\ell = \varnothing$, both are zero. For $0 :: \ell$, the subtraction is zero; for $1 :: \ell$, we have

$$zZ(1 :: \ell) \dot{-} pzZ(1 :: \ell) = zZ(\ell) \dot{-} pzZ(\ell).$$

So the recurrence is clear.

For the second part, just apply part 1 with the definition of kZ:

$$kZ(\ell, sk) = pzZ(\ell) + kZ(tpZ(\ell), k) \geq pzZ(\ell) \geq zZ(\ell).$$

This is all we needed. ∎

Corollary B.3.8. If $pzZ(\ell) \geq \operatorname{len}(\ell)$, then $tpZ(\ell) = \varnothing$.

Proof. By proposition B.3.1, $tpZ(\ell) = \operatorname{tail}(\ell, pzZ(\ell))$. Since $pzZ(\ell) \geq \operatorname{len}(\ell)$ by assumption, we have $\operatorname{tail}(\ell, pzZ(\ell)) = \varnothing$ (by proposition 4.3.3), as desired. ∎

B.4 Intermediate results - zeroes and heads

Proposition B.4.1. We have $zZ(\ell_1) \leq zZ(\ell_1 +\!\!+ \ell_2)$, and also

$$zZ(\ell_1 +\!\!+ \ell_2) = \begin{cases} zZ(\ell_1) & \text{if } zz(\ell_1 +\!\!+ \ell_2) < \operatorname{len}(\ell_1) \\ zZ(\ell_1 +\!\!+ \ell_2) & \text{else} \end{cases}$$

Proof. We start with the inequality. We prove it by induction on ℓ_1, using theorem 3.5.1 to show that $zZ(\ell_1) \dot{-} zZ(\ell_1 +\!\!+ \ell_2) = 0$. The base case $\ell_1 = \varnothing$ is trivial. For the induction step, we must show that

$$zZ(x :: \ell) \dot{-} zZ((x :: \ell) +\!\!+ \ell_2) \leq_{x,\ell,\ell_2} zZ(\ell) \dot{-} zZ(\ell +\!\!+ \ell_2).$$

We split into the cases $x = 0$ and $x = 1$. In the case $x = 0$, the left hand side reduces to 0, so the inequality holds. In the case $x = 1$, we have

$$zz(1 :: \ell) \dot{-} zZ((1 :: \ell) +\!\!+ \ell_2) = [1 + zZ(\ell)] \dot{-} [1 + zZ(\ell +\!\!+ \ell_2)] = zZ(\ell) \dot{-} zZ(\ell +\!\!+ \ell_2),$$

so we also have the inequality. This finishes the induction.

Next, we show the equality. Again, we go by induction on ℓ_1. In the base case $\ell_1 = \varnothing$, the if-else reduces to the else case (because $\operatorname{len}(\ell_1) = 0$, and $x < 0$ is impossible), so the equality is clear. For the inductive step, we show that both terms satisfy

$$f(x :: \ell, \ell_2) = \begin{cases} 0 & \text{if } x = 0 \\ 1 + f(\ell, \ell_2) & \text{else} \end{cases}.$$

For the left hand side, this is clear. For the right hand side, we must check that $f(0 :: \ell, \ell_2) = 0$ and $f(1 :: \ell, \ell_2) = 1 + f(\ell, \ell_2)$. With $x = 0$, it's clear that both branches of the if-else reduce to zero, so the whole function is indeed zero. With $x = 1$, we apply the definitions of zZ and len to find that it equals

$$\begin{cases} 1 + zZ(\ell) & \text{if } 1 + zZ(\ell +\!\!+ \ell_2) < 1 + \text{len}(\ell) \\ 1 + zZ(\ell +\!\!+ \ell_2) & \text{else} \end{cases}.$$

The if condition simplifies to $zZ(\ell +\!\!+ \ell_2) < \text{len}(\ell)$, and we can pull the $+1$ out of the if-then, so we get the recursion we wanted. ∎

Corollary B.4.2. We have $zZ(\text{head}(\ell, n)) \leq zZ(\ell)$. Moreover, if $n \leq \text{len}(\ell)$, then

$$zZ(\ell) = \begin{cases} zZ(\text{head}(\ell, n)) & \text{if } zz(\ell) < n \\ zZ(\ell) & \text{else} \end{cases}.$$

Proof. By proposition 4.6.3, we can write $\ell = \text{head}(\ell, n) +\!\!+ \text{tail}(\ell, n)$. The inequality follows directly by applying proposition B.4.1; for the equality, we also need to note that $n \leq \text{len}(\ell)$ implies $\text{len}(\text{head}(\ell, n)) = n$ (by corollary 4.6.4). ∎

Proposition B.4.3. We have $pzZ(\ell_1) \leq pzZ(\ell_1 +\!\!+ \ell_2)$, and also

$$pzZ(\ell_1 +\!\!+ \ell_2) = \begin{cases} pzZ(\ell_1) & \text{if } zz(\ell_1 +\!\!+ \ell_2) < \text{len}(\ell_1) \\ pzZ(\ell_1 +\!\!+ \ell_2) & \text{else} \end{cases}$$

and

$$pzZ(\ell_1 +\!\!+ \ell_2) = \begin{cases} pzZ(\ell_1) & \text{if } pzZ(\ell_1) < \text{len}(\ell_1) \\ pzZ(\ell_1 +\!\!+ \ell_2) & \text{else} \end{cases}$$

Proof. We start with the inequality. We prove it by induction on ℓ_1, using theorem 3.5.1 to show that $pzZ(\ell_1) \dot{-} pzZ(\ell_1 +\!\!+ \ell_2) = 0$. The base case $\ell_1 = \varnothing$ is trivial. For the induction step, we must show that

$$zZ(x :: \ell) \dot{-} zZ((x :: \ell) +\!\!+ \ell_2) \leq_{x,\ell,\ell_2} zZ(\ell) \dot{-} zZ(\ell +\!\!+ \ell_2).$$

We split into the cases $x = 0$ and $x = 1$. In the case $x = 0$, the left hand side reduces to

$1 \dot{-} 1 = 0$, so the inequality holds. In the case $x = 1$, we have

$$\begin{aligned} pzz(1 :: \ell) \dot{-} pzZ((1 :: \ell) +\!\!+ \ell_2) &= [1 + pzZ(\ell)] \dot{-} [1 + pzZ(\ell +\!\!+ \ell_2)] \\ &= pzZ(\ell) \dot{-} pzZ(\ell +\!\!+ \ell_2), \end{aligned}$$

so we also have the inequality. This finishes the induction.

Next, we show the first equality. Again, we go by induction on ℓ_1. In the base case $\ell_1 = \varnothing$, the if-else reduces to the else case (because $\mathrm{len}(\ell_1) = 0$, and $x < 0$ is impossible), so the equality is clear. For the inductive step, we show that both terms satisfy

$$f(x :: \ell, \ell_2) = \begin{cases} 1 & \text{if } x = 0 \\ 1 + f(\ell, \ell_2) & \text{else} \end{cases}.$$

For the left hand side, this is clear. For the right hand side, we must check that $f(0 :: \ell, \ell_2) = 1$ and $f(1 :: \ell, \ell_2) = 1 + f(\ell, \ell_2)$. With $x = 0$, it's clear that both branches of the if-else reduce to 1, so the whole function is indeed 1. With $x = 1$, we apply the definitions of pzZ, zZ, and len to find that it equals

$$\begin{cases} 1 + pzZ(\ell) & \text{if } 1 + zZ(\ell +\!\!+ \ell_2) < 1 + \mathrm{len}(\ell) \\ 1 + pzZ(\ell +\!\!+ \ell_2) & \text{else} \end{cases}.$$

The "if" condition simplifies to $zZ(\ell +\!\!+ \ell_2) < \mathrm{len}(\ell)$, and we can pull the $+1$ out of the if-then, so we get the recursion we wanted.

Finally, we prove the second equality, by showing that the right hand side satisfies the same recurrence as the first equality. Once again, the base case $\ell = \varnothing$ is trivial, and in the inductive $x = 0$ case, both branches reduce to 1. Finally, in the inductive $x = 1$ case, the expression reduces to

$$\begin{cases} 1 + pzZ(\ell) & \text{if } 1 + pzZ(\ell) < 1 + \mathrm{len}(\ell) \\ 1 + pzZ(\ell +\!\!+ \ell_2) & \text{else} \end{cases},$$

which simplifies in the same way as before to get the desired recurrence. ∎

Corollary B.4.4. We have $pzZ(\mathrm{head}(\ell, n)) \leq pzZ(\ell)$. Moreover, if $n \leq \mathrm{len}(\ell)$, then

$$pzZ(\ell) = \begin{cases} pzZ(\mathrm{head}(\ell, n)) & \text{if } zz(\ell) < n \\ pzZ(\ell) & \text{else} \end{cases}.$$

Proof. By proposition 4.6.3, we can write $\ell = \mathrm{head}(\ell, n) +\!\!+ \mathrm{tail}(\ell, n)$. The inequality follows directly by applying proposition B.4.3; for the equality, we also need to note that $n \leq \mathrm{len}(\ell)$ implies $\mathrm{len}(\mathrm{head}(\ell, n)) = n$ (by corollary 4.6.4). ∎

Corollary B.4.5. If $pzZ(\ell_1) < \text{len}(\ell_1)$, then $tpZ(\ell_1 +\!\!+ \ell_2) = tpZ(\ell_1) +\!\!+ \ell_2$.

Proof. First, by proposition B.3.1, we have

$$tpZ(\ell_1 +\!\!+ \ell_2) = \text{tail}\Big(\ell_1 +\!\!+ \ell_2,\ pzZ(\ell_1 +\!\!+ \ell_2)\Big).$$

Moreover, since $pzZ(\ell_1) < \text{len}(\ell_1)$, we can use proposition B.4.3 to note that $pzZ(\ell_1 +\!\!+ \ell_2) = pzZ(\ell_1)$. Therefore,

$$tpZ(\ell_1 +\!\!+ \ell_2) = \text{tail}\Big(\ell_1 +\!\!+ \ell_2,\ pzZ(\ell_1)\Big).$$

Proposition B.3.1 also tells us that

$$tpZ(\ell_1) +\!\!+ \ell_2 = \text{tail}(\ell_1, pzZ(\ell_1)) +\!\!+ \ell_2.$$

Since $pzZ(\ell_1) < \text{len}(\ell_1)$, proposition 4.5.2 tells us that these two terms are equal. ∎

Proposition B.4.6. We have $kZ(\ell_1, k) \leq kZ(\ell_1 +\!\!+ \ell_2, k)$.

Proof. We go by induction on k, using corollary 3.5.4 to show that the function given by $f(\ell_1, \ell_2, k) = kZ(\ell_1, k) \dot{-} kZ(\ell_1 +\!\!+ \ell_2, k)$ is equal to zero. The base case $k = 0$ is covered by proposition B.4.1. For the induction step, we compute

$$\begin{aligned}
&f(\ell_1, \ell_2, sk) \\
&= kZ(\ell_1, sk) \dot{-} kZ(\ell_1 +\!\!+ \ell_2, sk) \\
&= \Big[pzZ(\ell_1) + kZ(tpZ(\ell_1), k)\Big] \dot{-} \Big[pzZ(\ell_1 +\!\!+ \ell_2) + kZ(tpZ(\ell_1 +\!\!+ \ell_2), k)\Big] \\
&\leq \Big[pzZ(\ell_1 +\!\!+ \ell_2) + kZ(tpZ(\ell_1), k)\Big] \dot{-} \Big[pzZ(\ell_1 +\!\!+ \ell_2) + kZ(tpZ(\ell_1 +\!\!+ \ell_2), k)\Big] \\
&= kZ(tpZ(\ell_1), k) \dot{-} kZ(tpZ(\ell_1 +\!\!+ \ell_2), k).
\end{aligned}$$

Note that, for the third step, we used proposition B.4.3 to get the inequality.

We now split into two cases: $pzZ(\ell_1) < \text{len}(\ell_1)$, and $pzZ(\ell_1) \geq \text{len}(\ell_1)$. In the first case, we use corollary B.4.5 to get

$$\begin{aligned}
f(\ell_1, \ell_2, sk) &\leq kZ(tpZ(\ell_1), k) \dot{-} kZ(tpZ(\ell_1 +\!\!+ \ell_2), k) \\
&= kZ(tpZ(\ell_1), k) \dot{-} kZ(tpZ(\ell_1) +\!\!+ \ell_2, k) \\
&= f(tpZ(\ell_1), \ell_2, k).
\end{aligned}$$

In the second case, we use corollary B.3.8 and proposition B.3.2 to get

$$f(\ell_1, \ell_2, sk) \leq kZ(tpZ(\ell_1), k) \dot{-} kZ(tpZ(\ell_1 +\!\!+ \ell_2), k)$$

$$\begin{aligned}
&= kZ(\varnothing, k) \dot{-} kZ(tpZ(\ell_1 +\!\!+ \ell_2), k) \\
&= 0 \dot{-} kZ(tpZ(\ell_1 +\!\!+ \ell_2), k) \\
&= 0 \leq f(tpZ(\ell_1), \ell_2, k).
\end{aligned}$$

So, in both cases, we get $f(\ell_1, \ell_2, sk) \leq f(tpZ(\ell_1), \ell_2, k)$, which is what we needed to complete the inductive step. ∎

Corollary B.4.7. $tpZ(\text{head}(\ell, n)) = \text{head}\Big(tpZ(\ell),\ n \dot{-} pzZ(\ell)\Big)$

Proof. By propositions B.3.1 and 4.6.5,

$$\begin{aligned}
tpZ(\text{head}(\ell, n)) = \text{tail}(\text{head}(\ell, n),\ pzZ(\ell)) &= \text{head}(\text{tail}(\ell, pzZ(\ell)),\ n \dot{-} pzZ(\ell)) \\
&= \text{head}(tpZ(\ell),\ n \dot{-} pzZ(\ell)),
\end{aligned}$$

as desired. ∎

Proposition B.4.8. If $n \leq \text{len}(\ell)$, then

$$kZ(\ell, k) = \begin{cases} kZ(\text{head}(\ell, n), k) & \text{if } kZ(\ell, k) < n \\ kZ(\ell, k) & \text{else} \end{cases}$$

Proof. We go by induction on k. In the base case, this reduces to corollary B.4.2. For the inductive step, we claim both sides satisfy the recurrence

$$f(\ell, n, sk) = pzZ(\ell) + f\Big(tpZ(\ell),\ n \dot{-} pzZ(\ell),\ k\Big).$$

This is clear for the left hand side by the definition of kZ, so we just need to compute this for the right hand side.

Let's look at the top branch first. We have

$$\begin{aligned}
kZ(\text{head}(\ell, n), sk) &= pzZ(\text{head}(\ell, n)) + kZ(tpZ(\text{head}(\ell, n)), k) \\
&= pzZ(\ell) + kZ\Big(\text{head}(tpZ(\ell), n \dot{-} pzZ(\ell)),\ k\Big).
\end{aligned}$$

There are two key steps in this second equality. First, we use corollary B.4.7 to transform $tpZ(\text{head}(\ell, n))$. Second, we claim that $pzZ(\text{head}(\ell, n)) = pzZ(\ell)$; to justify this, we can appeal to corollary B.4.4, but for this result to apply, we must have $zZ(\ell) < n \leq \text{len}(\ell)$. We have $n \leq \text{len}(\ell)$ by assumption; moreover, we're in the case $kZ(\ell, sk) < n$, so by proposition B.3.7,

$$zZ(\ell) \leq kZ(\ell, sk) < n,$$

as desired.

Next, let's consider the condition, which is $kZ(\ell, sk) < n$. Expanding the definition of kZ, we get $pzZ(\ell)+kZ(tpZ(\ell), k) < n$. We claim this is equivalent to $kZ(tpZ(\ell), k) < n \dot{-} pzZ(\ell)$; in fact, we claim that $a + b < c$ is equivalent to $b < c \dot{-} a$ in general. To show this, we invoke proposition 3.3.4:

- If $a + b < c$, then subtracting a from both sides gives $b < c \dot{-} a$;
- If $a + b \geq c$, then (again) subtracting a from both sides gives $b \geq c \dot{-} a$.

Thus, the condition for the successor case is indeed $kZ(tpZ(\ell), k) < pzZ(\ell)$.

So, all together, replacing k by sk in the right hand side gives us the following:

$$\begin{cases} pzZ(\ell) + kZ\Big(\text{head}(tpZ(\ell), n \dot{-} pzZ(\ell)),\ k\Big) & \text{if } kZ(tpZ(\ell), k) < n \dot{-} pzZ(\ell) \\ pzZ(\ell) + kZ(tpZ(\ell), k) & \text{else} \end{cases}$$
$$= pzZ(\ell) + \begin{cases} kZ\Big(\text{head}(tpZ(\ell), n \dot{-} pzZ(\ell)),\ k\Big) & \text{if } kZ(tpZ(\ell), k) < n \dot{-} pzZ(\ell) \\ kZ(tpZ(\ell), k) & \text{else} \end{cases}$$

So, the claimed recurrence holds. ∎

Corollary B.4.9. Let $\ell \in L(\{0, 1\})$ and $k, n \in N$. Then

$$kZ(\text{head}(\ell, n), k) \leq kZ(\ell, k).$$

Moreover, if $kZ(\ell, k) < n \leq \text{len}(\ell)$, then

$$kZ(\text{head}(\ell, n), k) = kZ(\ell, k).$$

Proof. For the first part, first note that $\ell = \text{head}(\ell, n) +\!\!+ \text{tail}(\ell, n)$ by proposition 4.6.3. The inequality then follows directly by applying proposition B.4.6. The second part follows from proposition B.4.8. ∎

B.5 First results - cZ

Proposition B.5.1. $cZ(\ell) \leq \text{len}(\ell)$

Proof. We show that $cZ(\ell) \dot{-} \text{len}(\ell) = 0$ by induction on ℓ. The base case is clear; for the inductive step, we compute

$$cZ(0 :: \ell) \dot{-} \text{len}(0 :: \ell) = (1 + cZ(\ell)) \dot{-} (1 + \text{len}(\ell)) = cZ(\ell) \dot{-} \text{len}(\ell),$$

$$cZ(1 :: \ell) \dot{-} \mathrm{len}(1 :: \ell) = cZ(\ell) \dot{-} (1 + \mathrm{len}(\ell)) = (cZ(\ell) \dot{-} \mathrm{len}(\ell)) \dot{-} 1.$$

Thus this term satisfies the recursion $f(b :: \ell) = f(\ell) \dot{-} b$. ∎

Proposition B.5.2. $cZ(\ell_1 +\!\!+ \ell_2) = cZ(\ell_1) + cZ(\ell_2)$.

Proof. We go by induction on ℓ_1. The base case $\ell_1 = \varnothing$ is easy; for the induction step, we compute

$$cZ(b :: \ell +\!\!+ \ell_2) = cZ(b :: (\ell +\!\!+ \ell_2)) = \overline{b} + cZ(\ell +\!\!+ \ell_2)$$

and

$$cZ(b :: \ell) + cZ(\ell_2) = \overline{b} + cZ(\ell) + cZ(\ell_2),$$

so both satisfy the same recurrence. ∎

Proposition B.5.3. Let $\ell \in L(\{0, 1\})$ and $n < \mathrm{len}(\ell)$. If $\mathrm{nth}(\ell, n) = 0$, then

$$chZ(\ell, sn) = s(chZ(\ell, n))$$

Proof. Since $n < \mathrm{len}(\ell)$, by definition, we have

$$\begin{aligned}\mathrm{head}(\ell, sn) &= \mathrm{head}(\ell, n) +\!\!+ \mathrm{nthSingleton}(\ell, n)\\ &= \mathrm{head}(\ell, n) +\!\!+ [\mathrm{nth}(\ell, n)] = \mathrm{head}(\ell, n) +\!\!+ [0].\end{aligned}$$

Then, by proposition B.5.2, we have

$$\begin{aligned}chZ(\ell, sn) = cZ(\mathrm{head}(\ell, sn)) = cZ(\mathrm{head}(\ell, n) +\!\!+ [0]) = cZ(\mathrm{head}(\ell, n)) + cZ([0])\\ = chZ(\ell, n) + 1,\end{aligned}$$

as desired. ∎

B.6 Intermediate results - technical facts

Proposition B.6.1. Let $\ell \in L(\{0,1\})$. Then

1. $cZ(tZ(\ell)) = cZ(\ell)$
2. $cZ(tpZ(\ell)) = cZ(\ell) \dot{-} 1$

Proof. We start with the first statement, going by induction on ℓ. For the base case, $cZ(tZ(\varnothing)) = cZ(\varnothing)$. For the inductive step,

$$cZ(tZ(0 :: \ell)) = cZ(0 :: \ell) = 1 + cZ(\ell)$$

and

$$cZ(tZ(1 :: \ell)) = cZ(tZ(\ell)),$$

while $cZ(1 :: \ell) = cZ(\ell)$. Thus both functions satisfy

$$f(b :: \ell) = \begin{cases} 1 + cZ(\ell) & \text{if } b = 0 \\ f(\ell) & \text{if } b = 1 \end{cases}.$$

For the second statement, we again go by induction on ℓ. For the base case, we have $cZ(tpZ(\varnothing)) = cZ(\varnothing)$. For the inductive step,

$$cZ(tpZ(0 :: \ell)) = cZ(\ell) \qquad \text{and} \qquad cZ(tpZ(1 :: \ell)) = cZ(tpZ(\ell)).$$

Thus both functions satisfy

$$f(b :: \ell) = \begin{cases} cZ(\ell) & \text{if } b = 0 \\ f(\ell) & \text{if } b = 1 \end{cases}.$$

This completes the proof. ∎

Proposition B.6.2. Let $\ell \in L(\{0,1\})$ and $k \in N$. Then

1. $cZ(tpkZ(\ell, k)) = cZ(\ell) \dot{-} sk$
2. $cZ(tkZ(\ell, k)) = cZ(\ell) \dot{-} k$

Proof. For the first part, we go by induction on k. In the base case, we have

$$cZ(tpkZ(\ell, 0)) = cZ(tpZ(\ell)) = cZ(\ell) \dot{-} 1$$

by proposition B.6.1.

For the inductive step, we claim that both functions satisfy $f(\ell, sk) = f(\ell, k) \dot{-} 1$. This is clear for $cZ(\ell) \dot{-} sk$; for the other term, we compute

$$cZ(tpkZ(\ell, sk)) = cZ(tpZ(tpkZ(\ell, k))) = cZ(tpkZ(\ell, k)) \dot{-} 1$$

(using propositions B.6.1 and B.3.4). This finishes the proof.

For the second part, we go by cases on whether k is zero or a successor. For the zero case, we use proposition B.6.1 to compute

$$cZ(tkZ(\ell, 0)) = cz(tZ(\ell)) = cZ(\ell).$$

For the successor case, we use propositions B.3.5 and B.6.1, as well as the first part, to compute

$$cZ(tkZ(\ell, sk)) = cZ(tZ(tpkZ(\ell, k))) = cZ(tpkZ(\ell, k)) = cZ(\ell) \dot{-} sk.$$

That's all we needed to show. ∎

Proposition B.6.3. Let $\ell \in L(\{0,1\})$. Then:

1. $kZ(\ell, cZ(\ell)) = \text{len}(\ell)$;
2. If $k < cZ(\ell)$, then $kZ(\ell, k) < \text{len}(\ell)$.

Proof. The first part is proved by induction on ℓ: we show that $kZ(\ell, cZ(\ell))$ satisfies the defining equations of len. For the base case, we compute

$$kZ(\varnothing, cZ(\varnothing)) = kZ(\varnothing, 0) = zZ(\varnothing) = 0.$$

For the inductive step, we compute

$$\begin{aligned} kZ(0 :: \ell, cZ(0 :: \ell)) &= kZ(0 :: \ell, 1 + cZ(\ell)) \\ &= pzZ(0 :: \ell) + kZ(tpZ(0 :: \ell), cZ(\ell)) = 1 + kZ(\ell, cZ(\ell)) \end{aligned}$$

and

$$kZ(1 :: \ell, cZ(1 :: \ell)) = kZ(1 :: \ell, cZ(\ell)) = 1 + kZ(\ell, cZ(\ell)).$$

(Note: the last equality is from proposition B.3.3.) Thus $kZ(b :: \ell, cZ(b :: \ell)) = 1 + kZ(\ell, cZ(\ell))$, as desired.

For the second part, we first use proposition B.3.1 to compute

$$\text{len}(tkZ(\ell, k)) = \text{len}(\text{tail}(\ell, kZ(\ell, k))) = \text{len}(\ell) \dot{-} kZ(\ell, k).$$

So, to get the result, it suffices to show $\mathrm{len}(tkZ(\ell,k)) > 0$. Using proposition B.5.1, proposition B.6.2, and the assumption $cZ(\ell) \geq sk$, we compute

$$\mathrm{len}(tkZ(\ell,k)) \geq cZ(tkZ(\ell,k)) = cZ(\ell) \dot{-} k \geq sk \dot{-} k = 1 > 0.$$

This is what we needed to show. ∎

Proposition B.6.4. We have

$$pzz(\ell) = \begin{cases} 1 + zZ(\ell) & \text{if } 0 < cZ(\ell) \\ zZ(\ell) & \text{else} \end{cases}$$

and

$$pkz(\ell,k) = \begin{cases} 1 + kZ(\ell,k) & \text{if } k < cZ(\ell) \\ kZ(\ell,k) & \text{else} \end{cases}.$$

Proof. We start with the first equality, which we prove by induction on ℓ. For $\ell = \varnothing$, note that $pzZ(\varnothing) = zZ(\varnothing) = 0$, and the if-then reduces to the else case because $0 < cZ(\varnothing) = 0$ is impossible. So, we get equality.

In the inductive case $\ell = x :: \ell'$, we split into cases depending on x. If $x = 0$, we have $pzZ(0 :: \ell') = 1$ and

$$\begin{cases} 1 + zZ(0 :: \ell') & \text{if } 0 < cZ(0 :: \ell') \\ zZ(0 :: \ell') & \text{else} \end{cases} = \begin{cases} 1 & \text{if } 0 < 1 + cZ(\ell') \\ 0 & \text{else} \end{cases} = 1.$$

(Note that we reduce to the "if" case because $0 < 1 + cZ(\ell')$ is true.) If $x = 1$, then we have $pzZ(1 :: \ell') = 1 + pzZ(\ell')$ and

$$\begin{aligned} \begin{cases} 1 + zZ(1 :: \ell') & \text{if } 0 < cZ(1 :: \ell') \\ zZ(1 :: \ell') & \text{else} \end{cases} &= \begin{cases} 1 + 1 + zZ(\ell') & \text{if } 0 < cZ(\ell') \\ 1 + zZ(\ell') & \text{else} \end{cases} \\ &= 1 + \begin{cases} 1 + zZ(\ell') & \text{if } 0 < cZ(\ell') \\ zZ(\ell') & \text{else} \end{cases}. \end{aligned}$$

So, the same recurrence is satisfied.

We now move on to the second equality, which is proved by induction on k, which is to say we must show the right hand side satisfies the same inductive definition as pkZ. The base case $k = 0$ reduces to the first equality, so that's done. For the inductive step, we compute

$$\begin{aligned} \begin{cases} 1 + kZ(\ell, sk) & \text{if } sk < cZ(\ell) \\ kZ(\ell, sk) & \text{else} \end{cases} &= \begin{cases} 1 + pzZ(\ell) + kZ(tpZ(\ell), k) & \text{if } sk < cZ(\ell) \\ pzZ(\ell) + kZ(tpZ(\ell), k) & \text{else} \end{cases} \\ &= pzZ(\ell) + \begin{cases} 1 + kZ(tpZ(\ell), k) & \text{if } sk < cZ(\ell) \\ kZ(tpZ(\ell), k) & \text{else} \end{cases}. \end{aligned}$$

To ensure that this is the same recurrence as pkZ, it just remains to check that $sk < cZ(\ell)$ is the same condition as $k < cZ(tpZ(\ell))$. By proposition B.6.1, the latter condition is $k < P(cZ(\ell))$, so we need to check that $sk < m$ is the same as $k < Pm$ (where $m = cZ(\ell)$).

However, $ssk \dot{-} m \neq sk \dot{-} Pm$ in general, so we will instead appeal to proposition 3.3.4 (noting that $a < b$ means $sa \dot{-} b = 0$, and $sa \dot{-} b \geq 1$ is equivalent to $b \leq a$ by proposition 3.2.4). If $sk < m$, then $m > 0$, so $m = sPm$. Then $0 = ssk \dot{-} m = ssk \dot{-} sPm = sk \dot{-} Pm$, so $k < Pm$. On the other hand, if $sk \geq m$, then $Pm \leq Psk = k$ (by proposition 3.2.2). ∎

B.7 Intermediate results - key facts

Lemma B.7.1. Let $\ell \in L(\{0,1\})$. Then:

- $tkZ(\ell, cZ(\ell)) = \varnothing$;
- If $cZ(\ell) > 0$, then $\mathrm{len}(tkZ(\ell, P(cZ(\ell)))) > 0$.

Proof. For the first part, we simply use propositions B.3.1, B.6.3, and 4.3.3 to compute

$$\mathrm{len}\Big(tkZ(\ell, cZ(\ell))\Big) = \mathrm{len}\Big(\mathrm{tail}(\ell, kZ(\ell, cZ(\ell)))\Big) = \mathrm{len}\Big(\mathrm{tail}(\ell, \mathrm{len}(\ell))\Big) = \mathrm{len}(\varnothing) = 0.$$

The conclusion follows by proposition 4.3.2.

For the second part, we similarly use proposition B.3.1 to compute

$$\mathrm{len}\Big(tkZ(\ell,\ P(cZ(\ell)))\Big) = \mathrm{len}\Big(\mathrm{tail}(\ell,\ kZ(\ell, P(cZ(\ell))))\Big) = \mathrm{len}(\ell) \dot{-} kZ(\ell,\ P(cZ(\ell))).$$

Since $cZ(\ell) > 0$, we have $P(cZ(\ell)) < cZ(\ell)$, so $kZ(\ell, P(cZ(\ell))) < \mathrm{len}(\ell)$ by proposition B.6.3. Thus the above term must be greater than zero. ∎

Lemma B.7.2. Let $\ell \in L(\{0,1\})$.

- If $n \leq \mathrm{len}(\ell)$, then
$$n \leq kZ(\ell, chZ(\ell, n)).$$
- If $n \leq \mathrm{len}(\ell)$ and $chZ(\ell, n) > 0$, then
$$kZ(\ell, P(chZ(\ell, n))) < n.$$

Proof. We start with the first part. Using the first part of lemma B.7.1, along with proposition B.3.1, we compute

$$\begin{aligned}\varnothing &= tkZ\Big(\text{head}(\ell, n),\ cZ(\text{head}(\ell, n))\Big)\\ &= \text{tail}\Big(\text{head}(\ell, n),\ kZ\Big(\text{head}(\ell, n), cZ(\text{head}(\ell, n))\Big)\Big).\end{aligned}$$

If we calculate the length of both sides, we get

$$0 = \text{len}(\text{head}(\ell, n)) \dot{-} kZ\Big(\text{head}(\ell, n), chZ(\ell, n)\Big).$$

Since $n \leq \text{len}(\ell)$, corollary 4.6.4 tells us $\text{len}(\text{head}(\ell, n)) = n$. Thus we conclude that

$$n \leq kZ\Big(\text{head}(\ell, n),\ chZ(\ell, n)\Big),$$

and by applying corollary B.4.9, we conclude that $n \leq kZ(\ell, chZ(\ell, n))$.

Now, we prove the second part. Since $chZ(\ell, n) > 0$, we apply the second part of lemma B.7.1 to $\text{head}(\ell, n)$. This tells us that

$$\begin{aligned}0 &< \text{len}\Big(tkZ\Big(\text{head}(\ell, n),\ P(chZ(\ell, n))\Big)\Big)\\ &= \text{len}\Big(\text{tail}\Big(\text{head}(\ell, n),\ kZ(\text{head}(\ell, n), P(chZ(\ell, n)))\Big)\Big)\\ &= \text{len}(\text{head}(\ell, n)) \dot{-} kZ\Big(\text{head}(\ell, n),\ P(chZ(\ell, n))\Big)\\ &= n \dot{-} kZ\Big(\text{head}(\ell, n),\ P(chZ(\ell, n))\Big).\end{aligned}$$

Thus, $kZ(\text{head}(\ell, n),\ P(chZ(\ell, n))) < n$. (Note: in this calculation, we also used proposition B.3.1, corollary 4.6.4, and the assumption $n \leq \text{len}(\ell)$.)

To conclude the proof, we'd like to show that

$$kZ(\text{head}(\ell, n),\ P(chZ(\ell, n))) = kZ(\ell,\ P(chZ(\ell, n)))$$

by applying the second part of corollary B.4.9. To do so, we just need to check that $P(chZ(\ell, n)) < n$ (we already know that $n \leq \text{len}(\ell)$). For this, we use the fact that $chZ(\ell, n) > 0$ to compute

$$P(chZ(\ell, n)) < chZ(\ell, n) \leq \text{len}(\text{head}(\ell, n)) = n.$$

Note that, in this calculation, we also used proposition B.5.1, corollary 4.6.4, and the assumption $n \leq \text{len}(\ell)$. ∎

Lemma B.7.3. If $m < cZ(\ell)$, then $kZ(\ell, m) < kZ(\ell, sm)$.

Proof. Since $m < cZ(\ell)$, by propositions B.3.6 and B.6.4, we have

$$kZ(\ell, sm) = pkZ(\ell, m) + zZ(tpkZ(\ell, m)) \geq pkZ(\ell, m) = 1 + kZ(\ell, m) > kZ(\ell, m),$$

as desired. ∎

B.8 Final results

Proposition B.8.1. If $k < cZ(\ell)$, then $\mathrm{nth}(\ell, kZ(\ell, k)) = 0$.

Proof. First, note that $\mathrm{nth}(\ell, kZ(\ell, k))$ is well-defined because $k < cZ(\ell)$ implies that $kZ(\ell, k) < \mathrm{len}(\ell)$ (by proposition B.6.3). This also lets us say (by proposition 4.4.3) that

$$\mathrm{nth}(\ell, kZ(\ell, k)) = \mathrm{nthDef}(\ell, kZ(\ell, k), \mathrm{def}) = \mathrm{nthDef}(\ell, kZ(\ell, k), 0).$$

So, it suffices to show that $\mathrm{nthDef}(\ell, kZ(\ell, k), 0) = 0$ for all ℓ, k.

We prove this by induction on k. The base case $k = 0$ reduces to showing that $\mathrm{nthDef}(\ell, zZ(\ell), 0) = 0$, which we can do by induction on ℓ. For $\ell = \varnothing$, we check

$$\mathrm{nthDef}(\varnothing, zZ(\varnothing), 0) = \mathrm{nthDef}(\varnothing, 0, 0) = 0.$$

For the induction step, we can split into the cases $b = 0$, $b = 1$ and check that

$$\mathrm{nthDef}(b :: \ell, zZ(b :: \ell), 0) = \begin{cases} 0 & \text{if } b = 0 \\ \mathrm{nthDef}(\ell, zZ(\ell), 0) & \text{if } b = 1 \end{cases}.$$

This inductive characterization is shared by the zero map, so the two must be equal.

Thus, we have established the base case for the induction on k. For the inductive step, we compute:

$$\begin{aligned} \mathrm{nthDef}(\ell, kZ(\ell, sk), 0) &= \mathrm{nthDef}(\ell,\ pzZ(\ell) + kZ(tpZ(\ell), k),\ 0) \\ &= \mathrm{nthDef}(\mathrm{tail}(\ell, pzZ(\ell)),\ kZ(tpZ(\ell), k),\ 0) \\ &= \mathrm{nthDef}(tpZ(\ell),\ kZ(tpZ(\ell), k),\ 0). \end{aligned}$$

We used proposition B.3.1 for this last equality. (We also use proposition 4.4.5.)

So, we've shown that the term $f(\ell, k) = \mathrm{nthDef}(\ell, kZ(\ell, k), 0)$ satisfies $f(\ell, 0) = 0$ and $f(\ell, sk) = f(tpZ(\ell), k)$. This inductive characterization (using the inductive scheme of theorem 3.5.3) is also satisfied by $f(\ell, k) = 0$, so we have the desired equality. ∎

Proposition B.8.2. Let $\ell \in L(\{0,1\})$ and $n < \text{len}(\ell)$.

If $\text{nth}(\ell, n) = 0$, then $chZ(\ell, n) < cZ(\ell)$ and $n = kZ(\ell, chZ(\ell, n))$.

Proof. By the first part of lemma B.7.2, we have $n \leq kZ(\ell, chZ(\ell, n))$.

On the other hand, we can also apply proposition B.5.3 to note that

$$chZ(\ell, sn) = s(chZ(\ell, n)) > 0.$$

This (combined with $sn \leq \text{len}(\ell)$, because $n < \text{len}(\ell)$) means we can apply the second part of lemma B.7.2 to sn. This gives

$$kZ(\ell, P(chZ(\ell, sn)) < sn.$$

Since $chZ(\ell, sn) = s(chZ(\ell, n))$, this reduces to

$$kZ(\ell, chZ(\ell, n)) < sn.$$

Finally, after applying the definition of $<$ and cancelling the s on each side, we get

$$kZ(\ell, chZ(\ell, n)) \leq n.$$

Since we've established both inequalities, we conclude that $n = kZ(\ell, chZ(\ell, n))$. ∎

Bibliography

[Borc 94a] F. Borceux. *Handbook of categorical algebra. 1*. Vol. 50 of *Encyclopedia of Mathematics and its Applications*, Cambridge University Press, Cambridge, 1994. Basic category theory.

[Borc 94b] F. Borceux. *Handbook of categorical algebra. 2*. Vol. 51 of *Encyclopedia of Mathematics and its Applications*, Cambridge University Press, Cambridge, 1994. Categories and structures.

[Carb 93] A. Carboni, S. Lack, and R. F. C. Walters. "Introduction to extensive and distributive categories". *J. Pure Appl. Algebra*, Vol. 84, No. 2, pp. 145–158, 1993.

[Cock 90] J. R. B. Cockett. "List-arithmetic distributive categories: locoi". *J. Pure Appl. Algebra*, Vol. 66, No. 1, pp. 1–29, 1990.

[Desr 25] S. Desrochers. "The List Object Endofunctor is Polynomial". 2025.

[Dijk 20] J. van Dijk and A. G. Oldenziel. "Gödel incompleteness through Arithmetic Universes after A. Joyal". 2020.

[favi] N. C. F. (https://math.stackexchange.com/users/501886/nafavier). "If a complemented relation is symmetric, is its complement symmetric?". Mathematics Stack Exchange. URL:https://math.stackexchange.com/q/5069744 (version: 2025-05-23).

[Gamb 13] N. Gambino and J. Kock. "Polynomial functors and polynomial monads". *Math. Proc. Cambridge Philos. Soc.*, Vol. 154, No. 1, pp. 153–192, 2013.

[Good 57] R. L. Goodstein. *Recursive number theory: A development of recursive arithmetic in a logic-free equation calculus*. North-Holland Publishing Co., Amsterdam, 1957.

[Hazr 20] S. Hazratpour and S. Vickers. "Fibrations of AU-contexts beget fibrations of toposes". *Theory Appl. Categ.*, Vol. 35, pp. Paper No. 16, 562–593, 2020.

[Jay 93a] C. B. Jay. "Matrices, Monads and the Fast Fourier Transform". Tech. Rep. UTS-SOCS-93.13, 1993.

[Jay 93b] C. B. Jay. "Tail recursion through universal invariants". pp. 151–189, 1993. 4th Summer Conference on Category Theory and Computer Science (Paris, 1991).

[Jay 94] C. B. Jay and J. R. B. Cockett. "Shapely types and shape polymorphism". In: *Programming languages and systems—ESOP '94 (Edinburgh, 1994)*, pp. 302–316, Springer, Berlin, 1994.

[John 02] P. T. Johnstone. *Sketches of an elephant: a topos theory compendium. Vol. 1*. Vol. 43 of *Oxford Logic Guides*, The Clarendon Press, Oxford University Press, New York, 2002.

[Joya 95] A. Joyal and I. Moerdijk. *Algebraic set theory*. Vol. 220 of *London Mathematical Society Lecture Note Series*, Cambridge University Press, Cambridge, 1995.

[Maie 05] M. E. Maietti. "Modular correspondence between dependent type theories and categories including pretopoi and topoi". *Math. Structures Comput. Sci.*, Vol. 15, No. 6, pp. 1089–1149, 2005.

[Maie 10] M. E. Maietti. "Joyal's arithmetic universe as list-arithmetic pretopos". *Theory Appl. Categ.*, Vol. 24, pp. No. 3, 39–83, 2010.

[Maie 12] M. E. Maietti and S. Vickers. "An induction principle for consequence in arithmetic universes". *J. Pure Appl. Algebra*, Vol. 216, No. 8-9, pp. 2049–2067, 2012.

[Maie 24] M. E. Maietti and D. Trotta. "Quotients, pure existential completions and arithmetic universes". 2024.

[Nief 82] S. B. Niefield. "Cartesianness: topological spaces, uniform spaces, and affine schemes". *J. Pure Appl. Algebra*, Vol. 23, No. 2, pp. 147–167, 1982.

[Roma 89] L. Román. "Cartesian categories with natural numbers object". *J. Pure Appl. Algebra*, Vol. 58, No. 3, pp. 267–278, 1989.

[Uemu 22] T. Uemura. "The universal exponentiable arrow". *J. Pure Appl. Algebra*, Vol. 226, No. 7, pp. Paper No. 106991, 32, 2022.

[Vick 19] S. Vickers. "Sketches for arithmetic universes". *J. Log. Anal.*, Vol. 11, pp. Paper No. FT4, 56, 2019.

[Vick 99] S. Vickers. "Topical categories of domains". *Math. Structures Comput. Sci.*, Vol. 9, No. 5, pp. 569–616, 1999.

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