VIBRATION OF TRIANGULAR PLATES

by

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A thesis
presented to the School of Graduate Studies
of the University of Ottawa
in partial fulfillment of the
requirements for the degree of
Master Of Applied Science
in
Department of Mechanical Engineering
University of Ottawa, Ottawa

OTTAWA, Ontario, 1982

ACKNOWLEDGEMENTS

It is with deep gratitude and appreciation that I acknowledge the continuous guidance and support of my supervisor Prof. S. Mirza. Research work under him has been a stimulating and rewarding experience.

Thanks are also due to Prof. R. Natarajan for his helpful suggestions.

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NOMENCLATURE

SYMBOL		DESCRIPTION
a	-	Root Chord
A	-	Element Area in XY-Plane
A*	-	Element Area in -Plane
b	-	Trailing Edge Dimension
D	-	Plate Flexural Rigidity
D0	-	$D\|_{(0,0)}$ ($= E t_0^3/(12(1-\nu^2))$)
D*	-	D/D0
[D]	-	Flexural Rigidity Matrix
d.o.f.	-	Degrees of Freedom
E	-	Modulus of Elasticity
G	-	Shear Modulus
i,j,k	-	Dummy subscripts or Superscripts
$[K^*_{ss}]^{(e)}$	-	Element Stiffness Matrix
$[K^*_{ss}]$	-	Global Stiffness Matrix
L1,L2,L3	-	Area Coordinates
$[M]^{*(e)}$	-	Element Mass MAtrix
$[M^*]$	-	Global Mass Matrix
M_x, M_y, M_{xy}	-	Moment Components
$[N_i]$	-	Shape Functions
q	-	Transversely Distributed Load Per Unit Area

Q_x, Q_y	-	Shear Force / Unit Length
h(x,y)	-	Plate Thickness
T0	-	h(0,0)
T1	-	h(a,0)
T2	-	h(-b*Sin θ ,bCos θ)
U	-	Strain Energy
u,v,w	-	Displacement Components
w*	-	w/a
x,y,z	-	Cartesian Coordinates

DESCRIPTION OF GREEK SYMBOLS

α	Non-dimensional frequency Parameter $\left(\frac{\omega}{2\pi} a^2 \sqrt{\frac{\rho h_0}{D_0}} \right)$
θ	Trailing Edge Sweepback Angle
θ_x, θ_y	Rotation About The X-axis and Y-axis Respectively
ϕ	Aspect Ratio (= b/a)
ν	Poissons Ratio
ξ, η	Non-Dimensional Coordinates
ω	Natural Vibration Frequency of Plate
$\epsilon_x, \epsilon_y, \epsilon_z$	Normal Strains
π	3.1416...
$\{w\}^{ne}$	Vector of Nodal d.o.f. for Element
$\{\chi\}$	Vector of Plate Curvatures
	Normal Strains
∇^{*2}, ∇^{w2}	Non-Dimensional Form of
∇^2	Harmonic Operator ($\partial^2/\partial x^2 + \partial^2/\partial y^2$)
$\Diamond^4(,)$	'Die Operator'

SUPERSCRIPTS

*	Equivalent non-dimensional Quantities
e	Refers to an Element
ne	Refers to the Element Nodes

Chapter 1

INTRODUCTION

1.1 SCOPE OF THE PROBLEM

During recent years, the analysis of the response of structures to dynamic loadings has become increasingly important. More and more emphasis is being given in design problems on the aspects related to the dynamic behaviour as it is found that merely the static analysis is inadequate.

Stiffened or unstiffened flat plates of various geometries are extensively used in many structures like aircrafts, ships, missiles, containers, sea-platforms etc.. These are subjected to different types of static and time-varying forces in their working environment.

For the study of the dynamic response of plates, whether to periodic loading or random loading or to transient shock, it is essential to know the modes and periods of natural oscillation. This is termed as the free-vibration analysis. The present study is aimed at the free-vibration analysis of homogeneous triangular plate of arbitrary planform and varying thickness.

1.2 HISTORICAL BACKGROUND

The transverse vibration of homogeneous plates has been a subject of study since the early nineteenth century . However, the mathematical complexity involved in its analysis has limited the exact solutions to plates of certain specific geometries and boundary conditions. The analysis gets even more involved and complicated for plates with varying thickness. Thus for the solution of plates of complex geometries and varying thickness, one has to make use of the approximate solution methods.

A review of the existing literature reveals that the vibration analysis of rectangular, circular, trapezoidal and rhombic plates has been investigated by many researchers. However, for triangular and other polygonal plates, because of the complications posed by their geometry, the published literature is rather limited. This is particularly true for triangular plates of arbitrary planform and with complicating effects like non-uniform thickness. Most of the existing analytical and experimental work in the field of plate vibration has been reviewed by Leissa [1,2,3].

Focussing our attention on the the available literature on triangular plate vibration, it is convenient to classify it into : (1) works dealing directly with the analytical and experimental methods for triangular plates, and (2) works in which the triangular plate problem is addressed as one of the trial problems for testing a numerical technique.

Most of the earlier works have been restricted to specific triangular geometries. The isoceles right triangular cantilever plate problem has been investigated by Christenson [46] using the gridwork method. Hanson and Touvila [47] studied similar problems using experimental techniques. The general isoceles triangle has been investigated by Kumaraswamy and Cadambe [48] and by Ota,Hamada,Tarumoto [49].

Williams, Yeow and Brinson [7] used the trilinear coordinates to develop solution of a simply supported equilateral plate. Whereas Negm, Chander and Donaldson [10] have applied the extended field method of approximate analysis for forced vibration of clamped right triangle and clamped-simple-clamped isoceles triangular plates.

Gustafson, Stokey and Zorowski [9] made an experimental study of cantilevered triangular plates while Hewitt,Mazumdar and Jagannathan [50] have obtained the frequencies and mode shapes of clamped equilateral triangular viscoelestic plate using the concept of contour lines of equal deflection.

In some of the papers dealing with the formulation of new triangular plate bending elements [17,23], the triangular isoceles cantilevered plate vibration problem has been dealt with as a trial problem.

1.3 OUTLINE OF THE PRESENT STUDY

In structural analysis the two popular approximate methods, both based on mathematical approximations, are the Finite Difference Method and the Energy or Variational Approach. Finte Element Method, which is a variant of the latter, is a very powerful and attractive technique by virtue of its versatality in application and the relative ease with which complex plate geometries can be accurately modelled by an assemblage of finite elements and the solutions determined using modern digital computers. The unified approach in discretization of the Finite Element Method makes it eminently suitable for completely arbitrary and complex geometries and boundary conditions.

In the present study of the natural modes and frequencies of general arbitrary tria n les, the Displacement Finite Element formulation is carried out in terms of suitable non-dimensional parameters using the nine degree-of-freedom triangular plate bending element developed by Bazeley et al.. The effect of sweepback and thickness variation is studied for plates of different aspect ratios. The computer program developed incorporating this element has provisions for automatic mesh generation and also for graphical representation of the deflected configuration of plate for each mode together with the 'nodal line' or points of zero deflection. Consistent stiffness and mass matrices are generated using numerical integration. As a trial several test-problems are run for rectangular, rhombic, delta (right-angle) triangle and equilateral triangular plates. A remarkably good agreement is observed between the results obtained using the present approach and exact, experimental and other approximate results available in literature for such problems. For the arbitrary triangular plate with linear variation of thickness in the two coordinate directions, a large body of data has been generated for various values of the four principal non-dimensional geometric parameters, viz., the aspect ratio, sweepback angle, and the two thickness taper ratios.

Chapter II

GENERAL INTRODUCTION TO PLATE THEORY

A plate can be defined as a structural component for which one dimension, referred to as the thickness 'h', is much smaller compared to the other two linear dimensions. Like all structures, plates are essentially three dimensional, however, by introducing certain simplifying approximations an adequate analysis of stresses and moments in a plate, characterized by the fact that its thickness is small, can be performed by treating it as a two-dimensional problem. These approximations consist of the introduction of particular simplifications into the governing equations of the mathematical theory of elasticity. The motion of the plate is then described as the deflection of its central plane.

2.1 ASSUMPTIONS OF PLATE THEORY

It is well known that for the majority of technical applications the Kirchhoff-Love's classical theory of plates (thin plate small- deflection theory of bending) yields sufficiently accurate results. The fundamental simplifying assumptions adopted in the classical theory are:

1. The middle surface of the plate remains unstrained during bending; thus, it is a neutral surface.

2. Normals to the middle surface before deformation remain normal to the same surface after deformation. (this assumption does not imply that transverse shear strain is necessarily zero. The assumption implies that the transverse shear is so small that any distortion of transverse sections caused by the existence of transverse shear strain makes a negligible contribution to displacements.)

3. Normal stresses in the direction transverse to the plate are small compared to other stresses; thus, they are neglected.

4. The deflection of the middle plane is small. The slope of the deflected plate in any direction is small so that its square may be neglected in comparison with unity.

These assumptions are applicable and justified for the cases of bending of thin-plates when the deflections are small. For the analysis of thick plates and when the deflections are not small (exception; when the plate bends into a developable surface) a more general theory is required owing to the fact that the assumptions are no longer appropriate.

2.2 BASIC PLATE RELATIONS AND EQUATIONS

The equation defining small lateral deflection of the middle surface of a thin plate subjected to lateral loads may be formulated by eliminating less significant terms from the equation of three- dimensional elasticity. Consequently, the governing plate equation can be derived in a concise and straightforward manner.

Using rectangular coordinates coinciding with the principal material axes, the stress-strain relations for an orthotropic elastic body are given by

$$\begin{aligned}
\epsilon_x &= \frac{1}{E_x}\left(\sigma_x - \nu_{xy}\sigma_y - \nu_{xz}\sigma_z\right) \\
\epsilon_y &= \frac{1}{E_y}\left(\sigma_y - \nu_{yx}\sigma_x - \nu_{yz}\sigma_z\right) \qquad (2.1) \\
\epsilon_z &= \frac{1}{E_z}\left(\sigma_z - \nu_{zx}\sigma_x - \nu_{zy}\sigma_y\right)
\end{aligned}$$

$$\gamma_{xy} = \frac{1}{G_{xy}}\tau_{xy} \; ; \quad \gamma_{xz} = \frac{1}{G_{xz}}\tau_{xz} \; ; \quad \gamma_{yz} = \frac{1}{G_{yz}}\tau_{yz}$$

The constants E , ν , and G represent the modulus of elasticity , Poisson's ratio , and shear modulus of elasticity respectively.

The simplifying assumptions in the classical theory of plates can be mathematically formulated as

$$\sigma_z = 0 \ , \ \epsilon_z = 0 \ , \ \gamma_{xz} = \gamma_{yz} = 0$$

$$\text{or} \qquad (2.2)$$

$$\frac{1}{E_z} = 0 \ , \ \frac{1}{G_{xz}} = 0 \ , \ \frac{1}{G_{yz}} = 0$$

For a geometric linear theory (thin-plate small deflection theory) in which displacement components are small compared to the plate thickness 'h', the strain displacement relations are:

$$\epsilon_x = \frac{\partial u}{\partial x} \ , \ \epsilon_y = \frac{\partial v}{\partial y} \ , \ \epsilon_z = \frac{\partial w}{\partial z} \qquad (2.3)$$

$$\gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \ , \ \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \ , \ \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}$$

Because of the assumptions , $\epsilon_z=0$, the deflection function depends, in rectilinear coordinates, only on variables x and y , thus for static problems

$$w=w(x,y)$$

Inroducing the simplifications , the stress-strain relations become

$$\epsilon_x = \frac{1}{E_x}(\sigma_x - \nu_{xy}\sigma_y)$$
$$\epsilon_y = \frac{1}{E_y}(\sigma_y - \nu_{yx}\sigma_x) \qquad (2.4)$$
$$\gamma_{xy} = \frac{1}{G_{xy}}\tau_{xy} \;;\; \epsilon_z = 0 = \gamma_{xz} = \gamma_{yz}$$

For an isotropic plate

$$E_x = E_y = E \;,\; G_{xy} = G = \frac{E}{2(1+\nu)}$$
$$\nu_{xy} = \nu_{yx} = \nu \qquad (2.5)$$

Consequently

$$\epsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y)$$
$$\epsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x)$$
$$\gamma_{xy} = \frac{1}{G}\tau_{xy} \qquad (2.6)$$
$$\epsilon_z = \gamma_{xz} = 0$$

2.3 STRESS AND MOMENTS RELATIONS

Considering now the state of stress in a plate with an arbitrary small deflection w(x,y) by taking a differntial element cut out of the plate by two pairs of planes parallel to xz and yz plates,[4]

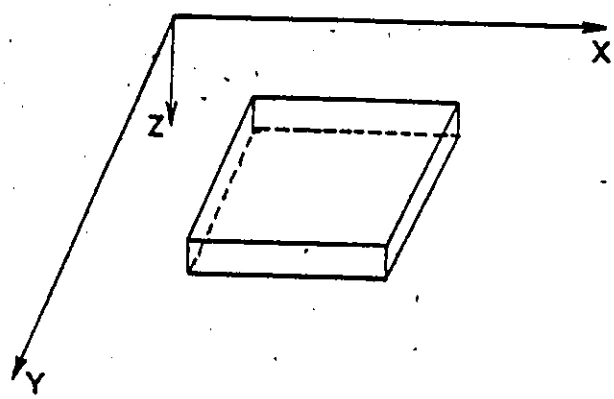


Figure 1: : DIFFERNTIAL PLATE ELEMENT

The middle plane is a neutral plane and accordingly the displacements u and v in the xy-plane at a distance z from the mid plane are given by:

$$u = -z\,\partial w/\partial x$$
$$v = -z\,\partial w/\partial y \qquad (2.7)$$

The strains in this xy-plane are therefore

$$
\begin{aligned}
\epsilon_x &= \partial u/\partial x = -z\,\partial^2 w/\partial x^2 \\
\epsilon_y &= \partial v/\partial y = -z\,\partial^2 w/\partial y^2 \\
\gamma_{xy} &= \partial u/\partial y + \partial v/\partial x = -2z\,\partial^2 w/\partial x\,\partial y
\end{aligned}
\qquad (2.8)
$$

Substituting equations (2.8) into equations (2.6) yields

$$
\begin{aligned}
\sigma_x &= -\frac{Ez}{1-\nu^2}\left(\partial^2 w/\partial x^2 + \nu\,\partial^2 w/\partial y^2\right) \\
\sigma_y &= -\frac{Ez}{1-\nu^2}\left(\partial^2 w/\partial y^2 + \nu\,\partial^2 w/\partial x^2\right) \\
\tau_{xy} &= -\frac{Ez}{1+\nu}\,\partial^2 w/\partial x\,\partial y
\end{aligned}
\qquad (2.9)
$$

These stresses vary linearly through the thickness of the plate and are equivalent to moments per unit length acting on an element of the plate

$$
\begin{aligned}
M_x &= \int_{-h/2}^{h/2} \sigma_x z\,dz = -D\left(\partial^2 w/\partial x^2 + \nu\,\partial^2 w/\partial y^2\right) \\
M_y &= \int_{-h/2}^{h/2} \sigma_y z\,dz = -D\left(\partial^2 w/\partial x^2 + \nu\,\partial^2 w/\partial x^2\right) \qquad (2.10) \\
M_{xy} &= -\int_{-h/2}^{h/2} \tau_{xy} z\,dz = D(1-\nu)\,\partial^2 w/\partial x\,\partial y
\end{aligned}
$$

where D , the flexural rigidity of the plate, $= Eh^3/(12(1-\nu^2))$, [4]

2.4 EQUATIONS FOR EQUILIBRIUM

Consider an element dxdy of the plate subjected to a uniformly distributed load per unit area q.

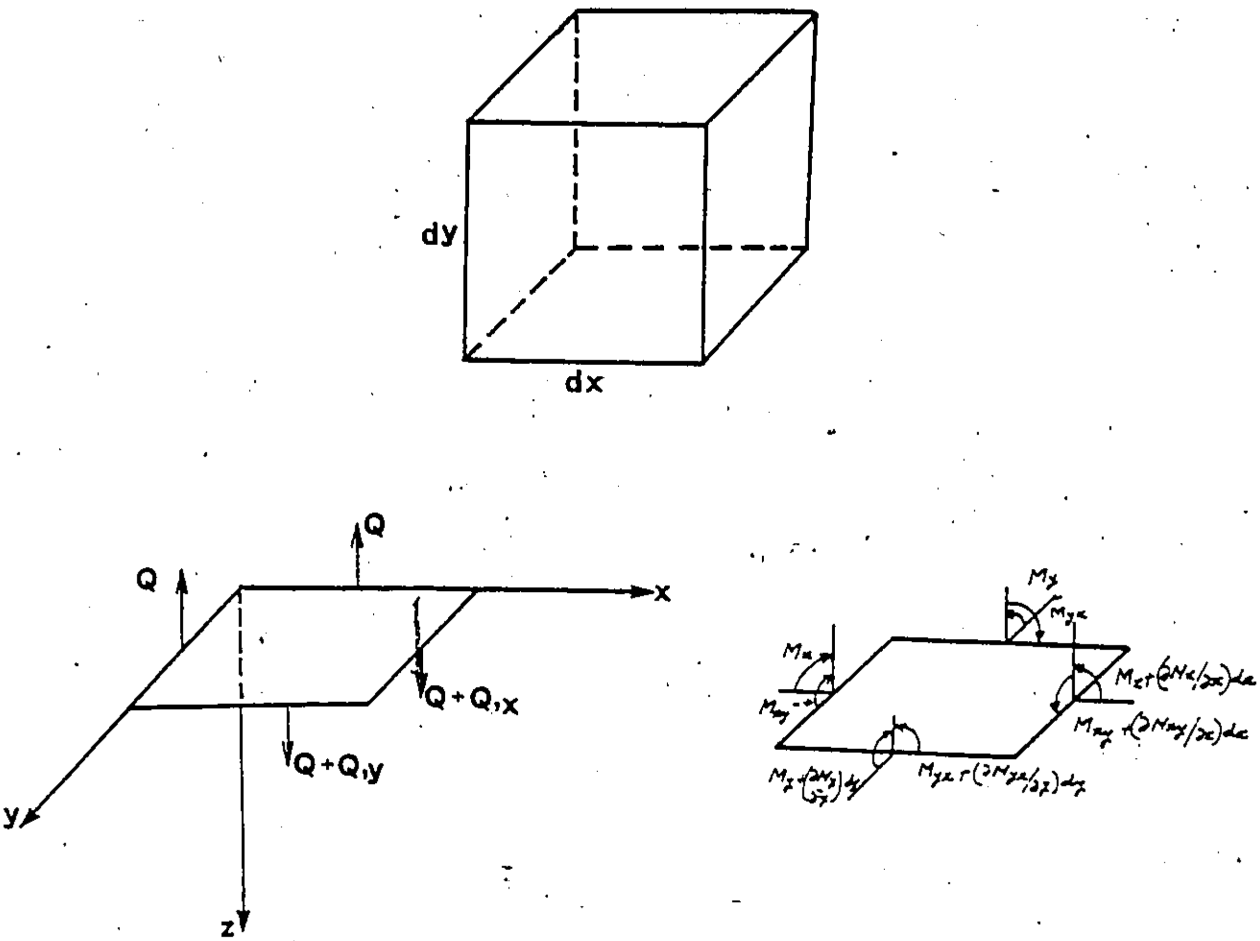


Figure 2: FORCES AND MOMENTS ON DIFFERENTIAL PLATE ELEMENT

2.4.1 FORCE EQUILIBRIUM

Projecting all the forces acting on the element onto the z-axis

$$\frac{\partial Q_x}{\partial x} dx\,dy + \frac{\partial Q_y}{\partial y} dy\,dx + q\,dx\,dy = 0 \qquad (2.11)$$

which gives

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + q = 0 \qquad (2.12)$$

2.4.2 MOMENT EQUILIBRIUM

Taking moments of all forces with respect to the x-axis

$$\frac{\partial M_{xy}}{\partial x} dx\,dy - \frac{\partial M_y}{\partial y} dx\,dy + Q_y\,dx\,dy = 0 \qquad (2.13)$$

(Higher order quanties arising from the moment of load q and the moment due to change in force Qx are neglected)

The equation of equilibrium expressed in terms of the derivatives of the moments and the applied loading

$$\frac{\partial^2 M_x}{\partial x^2} - 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + q = 0 \qquad (2.14)$$

The governing differntial equation for the lateral deflection of the plate can be obtained by substituting the moment-curvature equations in the equilibrium equation

$$\frac{\partial^2}{\partial x^2}\left[D\left(\frac{\partial^2 w}{\partial x^2}+\nu\frac{\partial^2 w}{\partial y^2}\right)\right]+2(1-\nu)\frac{\partial^2}{\partial x\partial y}\left[D\frac{\partial^2 w}{\partial x\partial y}\right]+\frac{\partial^2}{\partial y^2}\left[D\left(\frac{\partial^2 w}{\partial y^2}+\nu\frac{\partial^2 w}{\partial x^2}\right)\right]=q \qquad (2.15)$$

where the plate flexural rigidity D=D(x,y)

Writing this in the invariant form [43] ,

$$\nabla^2(D\nabla^2 w)-(1-\nu)\Diamond^4(D,w)=q \qquad (2.16)$$

where

$$\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2} \qquad (2.17)$$

And the 'die operator' is defined by

$$\Diamond^4(D,w)=\frac{1}{2}\left[(\nabla^2 D)(\nabla^2 w)+\nabla^2(D\nabla^2 w+w\nabla^2 D)\right]-\frac{1}{4}\left[\nabla^4(D,w)+D\nabla^4 w+w\nabla^4 D\right]$$

$$\equiv\frac{\partial^2 D}{\partial x^2}\frac{\partial^2 w}{\partial y^2}-2\frac{\partial^2 D}{\partial x\partial y}\cdot\frac{\partial^2 w}{\partial x\partial y}+\frac{\partial^2 D}{\partial y^2}\frac{\partial^2 w}{\partial x^2} \qquad (2.18)$$

2.5 NON-DIMENSIONALIZING SCHEME

At this point we introduce the following non-dimensional parameters

$$\phi = b/a \quad ; \quad \xi = x/a \quad ; \quad \eta = y/b$$
$$w^* = w/a \quad ; \quad D^* = D/D_0 \qquad (2.19)$$

With these definitions the differential operators can also be written in the non-dimensional form as,

$$\begin{aligned}\nabla^2 &= \frac{\partial^2}{a^2 \partial \xi^2} + \frac{\partial^2}{b^2 \partial \eta^2} \\ &= \frac{1}{a^2}\left(\frac{\partial^2}{\partial \xi^2} + \frac{1}{\phi^2}\frac{\partial^2}{\partial \eta^2}\right) = \frac{1}{a^2}\nabla^{*2} \\ &= \frac{1}{b^2}\left(\phi^2 \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2}\right) = \frac{1}{b^2}\nabla'^{*2}\end{aligned} \qquad (2.20)$$

Similarly the 'die operator can be simplified as

$$\begin{aligned}\Diamond^4(D, w) &= \frac{\partial^2 D}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - 2\frac{\partial^2 D}{\partial x \partial y}\frac{\partial^2 w}{\partial x \partial y} + \frac{\partial^2 D}{\partial y^2}\frac{\partial^2 w}{\partial x^2} \qquad (2.21) \\ &= \frac{D_0}{ab^2}\left[\frac{\partial^2 D^*}{\partial \xi^2}\frac{\partial^2 w^*}{\partial \eta^2} - 2\frac{\partial^2 D^*}{\partial \xi \partial \eta}\frac{\partial^2 w^*}{\partial \xi \partial \eta} + \frac{\partial^2 D^*}{\partial \eta^2}\frac{\partial^2 w^*}{\partial \xi^2}\right]\end{aligned}$$

$$\Diamond^4(D, w) = \frac{D_0}{ab^2}\Diamond^{*4}(D^*, w^*) \qquad (2.22)$$

Expressing in terms of the non-dimensional parameters

$$D\nabla^2 w = \frac{a D_0}{b^2}\left(D^* \nabla'^{*2} w^*\right)$$

and

$$\nabla^2\left(D\nabla^2 w\right) = \frac{D_0}{ab^2}\nabla^{*2}\left(D^* \nabla'^{*2} w^*\right)$$

Substituting in eqn. (2.16)

$$\frac{D_0}{ab^2}\nabla^{*2}\left(D^*\nabla'^{*2}w^*\right) - (1-\nu)\frac{D_0}{ab^2}\Diamond^{*4}\left(D^*, w^*\right) = q$$

or

$$\nabla^{*2}\left(D^*\nabla'^{*2}w^*\right) - (1-\nu)\Diamond^{*4}\left(D^*, w^*\right)$$
$$= q\frac{ab^2}{D_0} = \frac{q a^3 \phi^2}{D_0}$$

And as $\phi^2\nabla^{*2} = \nabla'^{*2}$

This can be rewritten as

$$\phi^2\nabla^{*2}\left(D^*\nabla^{*2}w^*\right) - (1-\nu)\Diamond^{*4}\left(D^*, w^*\right)$$
$$= q\frac{a^3\phi^2}{D_0}$$

2.6 DIFFERENTIAL EQUATION FOR PLATE VIBRATION

The fourth order partial differential equation is written in the invariant form , in terms of the non-dimensional parameters , as

$$\nabla^{*2}\left(D^{*}\nabla'^{*2}w^{*}\right)-(1-\nu)\,\Diamond^{*4}\left(D^{*}, w^{*}\right)=\frac{q a^{3}\phi^{2}}{D_{0}} \qquad (2.23)$$

Where the operators ∇^{*2} and ∇'^{*2} and the other parameters have been defined earlier (section 2.5).

This equation governs the bending of plates subjected to static lateral loading , q, where the flexural rigidity parameter D^{*}, the lateral deflection parameter w^{*} , and the static lateral loading q are all functions of the spatial coordinates. In other words

$$D^{*}=D^{*}(\xi,\eta)\ ,\ w^{*}=w^{*}(\xi,\eta)\ ,\ q=q(\xi,\eta)$$

In the case of the free-vibration of a plate, the surface loading $q(\xi,\eta)$, will be replaced by inertial body force acting. It can be easily seen that the magnitude of the inertial force acting on the differential element , $d\xi d\eta$, is ($\rho h\, d\xi d\eta (d^{2}w/dt^{2})$

$= \rho h\, d\xi d\eta\, a\, d^{2}w^{*}/dt^{9}) - \rho$ is the mass density of the plate, h is the thickness and $d\xi d\eta$ is the area of the differntial element. The direction in which this force acts is opposite to the direction of positive lateral direction) or in other words the negative z-direction. Replacing the static lateral force per unit area, $q(\xi,\eta)$, by the inertial body force per unit area---the governing equation for the free vibration of plates can be obtained as

$$\nabla^{*2}(D^*\nabla'^{*2}w^*)-(1-\nu)\Diamond^{*4}(D^*,w^*) = -\left(\rho h\frac{d^2w}{dt^2}a^3\phi^2\right)\frac{1}{D_0}$$

or

$$\nabla^{*2}(D^*\nabla'^{*2}w^*)-(1-\nu)\Diamond^{*4}(D^*,w^*) + \frac{\rho h a^4\phi^2}{D_0}\frac{d^2w^*}{dt^2} = 0 \qquad (2.24)$$

Where now the non-dimensional displacement w^* is a funtion of the two spatial coordinates and time.

$$w^* = w^*(\xi,\eta,t)$$

Using separation of variables , $w(\xi,\eta,t)$ can be written as

$$w^*(\xi,\eta,t) = w^*(\xi,\eta)\,.\,T(t) \qquad (2.25)$$

Substituting in equation (2.24)

$$T(t)\left[\nabla^{*2}D^*\nabla'^{*2}w^*(\xi,\eta)-(1-\nu)\Diamond^{*4}(D^*,w^*(\xi,\eta))\right] = -\frac{\rho h a^4\phi^2}{D_0}w^*(\xi,\eta)\frac{d^2T(t)}{dt^2}$$

$$\frac{D_0}{\rho h a^4\phi^2 w^*(\xi,\eta)}\left[\nabla^{*2}D^*\nabla'^{*2}w^*(\xi,\eta)-(1-\nu)\Diamond^{*4}(D^*,w^*(\xi,\eta))\right] = -\frac{d^2T(t)}{dt^2}\Big/T(t) \qquad (2.26)$$

As the LHS is a function of spatial coordinates only and RHS a funtion of time only, for this relation to hold good

LHS = RHS= A constant

Denoting this constant by ω^2, a real positive number

$$\frac{d^2 T(t)}{dt^2} + \omega^2 T(t) = 0 \qquad (2.27)$$

which has a solution of the form

$$T(t) = A \sin(\omega t + \gamma) \qquad (2.28)$$

where A and γ are constants.

The constant (ω^2) is limited to be a positive real number because otherwise the solution of the equation would give an unstable diverging solution. The LHS of the equation gives

$$\left[\nabla^{*2} D^* \nabla^{*2} w^*(\xi,\eta) - (1-\nu)\Diamond^{*4}\left(D^*, w^*(\xi,\eta)\right)\right] - \frac{\omega^2 \rho h a^4 w^*(\xi,\eta)}{D_0} = 0 \qquad (2.29)$$

a homogeneous fourth order partial differntial equation governing the free vibration of a plate.

2.7 BOUNDARY CONDITIONS

The governing differntial equation together with the appropriate conditions of the plate at the boundaries in terms of the lateral deflection of the middle surface $w(x,y)$ constitute the complete system of equations to be solved.

Three types of boundary conditions are considered here

2.7.1 SIMPLY SUPPORTED EDGE CONDITIONS:

The conditions on a simply supported edge parallel to the y-axis at x=a, are

$$w\Big|_{x=a} = 0$$

$$M_x\Big|_{x=a} = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right)_{x=a} = 0 \qquad (2.30)$$

Since the change of w with respect to the y-coordinate vanishes along this edge , these conditions become

$$w\Big|_{x=a} = 0 \qquad \frac{\partial^2 w}{\partial x^2}\Big|_{x=a} = 0 \qquad (2.31)$$

2.7.2 CLAMPED EDGE CONDITIONS

If a plate boundary is clamped, the deflection and the normal slope of the middle surface must vanish at the boundary. This can be expressed as

$$w=0 \qquad (2.32)$$

$\partial w/\partial n=0$, n being normal to the edge.

Thus on a clamped edge parallel to the y-axis at x=a, the boundary conditions are

$$w\Big|_{x=a} = 0 \qquad \frac{\partial w}{\partial x}\Big|_{x=a} = 0 \qquad (2.33)$$

The boundary conditions on a clamped edge parallel to the x-axis at $y = b$ are

$$w\Big|_{y=b} = 0$$

$$\frac{\partial w}{\partial y}\Big|_{y=b} = 0 \qquad (2.34)$$

2.7.3 FREE EDGE CONDITION

Kirchoff first showed t hat for the thin plate theory two boundary conditions at the free edge could be formulated involving the bending moment, twisting moment and the shear force. One condition corresponds to equating the bending moment stress resultant to zero; and the other corresponds to setting equal to zero an expression involving both the twisting moment shear stress resultant and the transverse shear stress resultant.

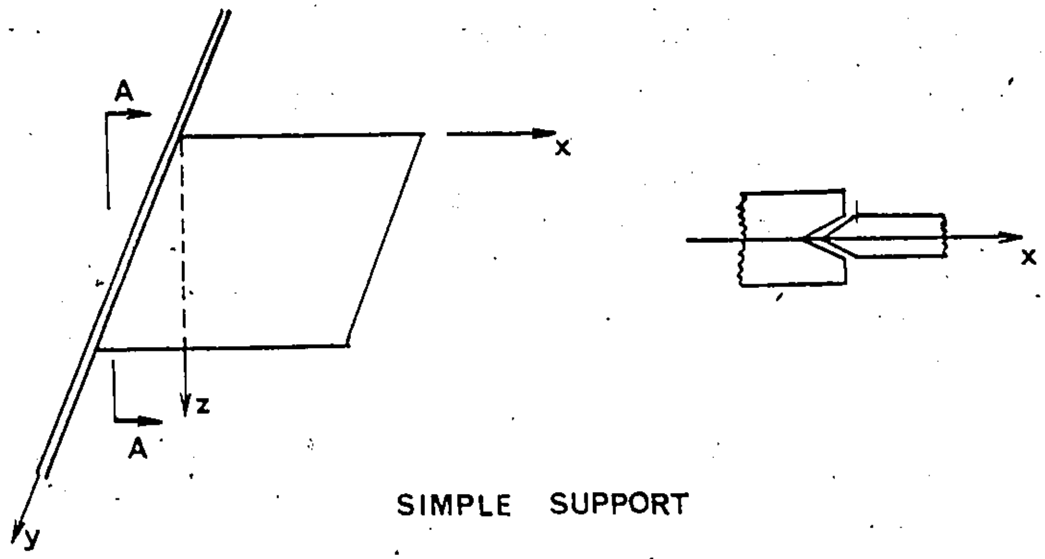
The boundary conditions on a free edge parallel to the y-axis at $x = a$ are,

$$\left(\partial^3 w/\partial x^3 + (2-\nu)\, \partial^3 w/\partial y^2 \right)_{x=a} = 0$$

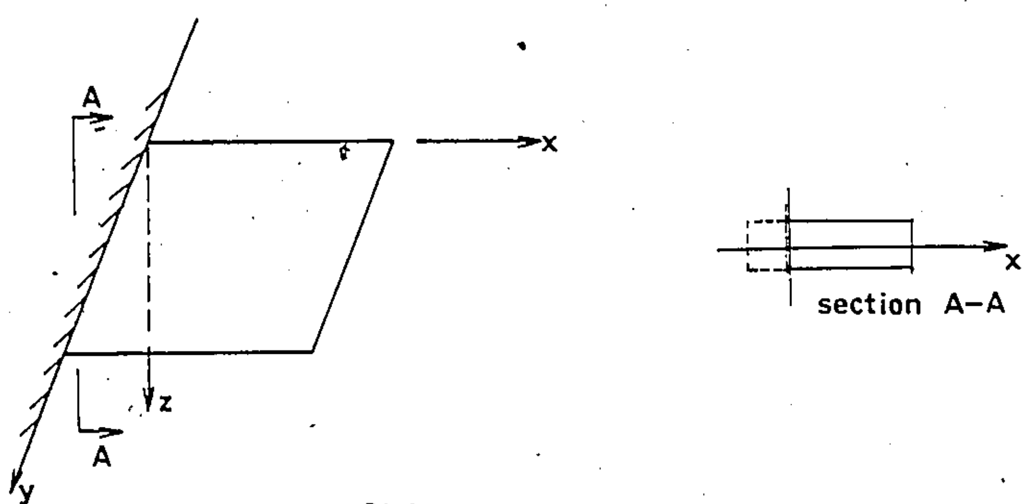
$$\left(\partial^2 w/\partial x^2 + \nu\, \partial^2 w/\partial y^2 \right)_{x=a} = 0$$

2.7.4 CONTINUITY CONDITIONS

In some cases the plate is assumed to be resting along an intermediat line support parallel to the fixed edge. Along this edge we impose the continuity conditions.



SIMPLE SUPPORT



CLAMPED SUPPORT

Figure 3: BOUNDARY CONDITIONS

2.8 METHODS FOR SOLUTION

The thin plate deflection is governed by a fourth-order partial differential equation. All solutions for the lateral deflections must satisfy the governing equation and the appropriate plate boundary conditions. Integration of this equation to yield exact analytical solutions of the deflection w(X,Y) poses considerable problems. Except for simple configurations of loading and shape, such as axisymmetrically loaded circular plates or rectangular plates with certain edge conditions , the solution of the governing plate equation is extremely difficult. The most powerful method is the Fourier Series Method, but, unfortunately of all the combinations of boundary conditions possible only very few can be tackled using this method.

The following are the different types of solution methods for an exact analytical solution:

1. Navier Solution: applies only to the limited category of simply supported rectangular plates. In this method the deflection can always be represented in the form of a double trigonometric series.

$$w(x,y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \qquad (2.36)$$

2. Levy Solution: for solution of rectangular plates with two opposite edges simply supported and arbi-

trary boundary conditions on the remaining two sides. M. Levy suggested the solution in the form

$$w(x,y) = \lim_{k \to \infty} \sum_{m=1}^{k} Y_m(y) \sin \frac{m x}{a} \qquad (2.37)$$

3. Levy Solution in Combination With the Principle of Superposition : For rectangular plates with arbitrary boundary conditions imposed on all four sides. This solution has been successfully used for a large number of problems [6].
4. Conformal Mapping Method: Used for the analysis of vibrating polygonal plates yielding very satisfactory convergence. However, it is difficult to obtain mapping funtions for a plate with a complicated shape. Only fundamental frequency can be determined-- higher modes are not possible.

The order of mathematical complexity increases manyfolds when the flexural rigidity, D, is not constant(as in the present problem of plate with varing thickness). Thus recourse has to be made to the approximate techniques for the solution of the problem. The different approximate methods are briefly discussed below.

Energy Methods: These are based on the fact that the governing equation of the deformed elastic body can be derived by minimizing the energy associated with deformation and loading. Applications of the energy methods are effective in situations involving irregular shapes,non-uniform loads,variable cross-section and anisotropic materials.

1. Principle of Virtual Work: It is essentially the statement of equilibrium. According to this, the first-order variation of the total potential energy is zero for any virtual displacement--or, the total potential energy is stationary.

2. Rayleigh-Ritz: involves the setting up of a power series to represent the plate deflection w satisfying at least the geometric boundary conditions and substituting this in the energy integrals and minimizing them with respect to the various parameters of the power series. This results in the solution of a number of simultaneous equations to obtain the eigen-values.

3. Modified Rayleigh-Ritz Method Incorporating Lagrange Multipliers: In this approach use of Lagrange Multipliers is made to enforce boundary conditions or constraints not satisfied by the assumed series of the Ritz method. In this way it can handle problems with unusual constraints.

Weighted Residual Methods: Substituting the series of assumed functions in the governing differential equation a "residue" (error) will be obtained. This method involves minimizing this error by taking the integral of the weighted error over the domain to be zero. Their are two methods associated with this. These are:

(1) Galerkin's Method

(2) Collocation Method

Finite Difference Method: Direct approximation to the governing equation is made in terms of a finite number of values of the unknown function, selected at strategic mesh points. The governing differntial equation and the expressions defining the boundary conditions are replaced with equivalent difference equations. The solution of the problem is thus reduced to the simultaneous solution of a set of algebraic equations written for every nodal point within the domain.With the Finite Difference Method some fairly difficult problems can be treated;but, it becomes difficult to use when we encounter irregular geometry or unusual specifications of boundary conditions.

Finite Strip Method: The domain is assumed to be divided into two systems of strips at right angles to one another, each strip regarded as functioning as a beam. This method permits qualitative analysis of the plate behavior with ease but is less adequate, in general, in obtaining accurate quantitative results. Its application is limited to rectangular plate geometry with arbitrary boundary conditions.

Finite Element Method: The basic concept of the method, when applied to problems of structural analysis, is that a continuum(the total structure) can be modelled analytically by its subdivisions into regions(the finite element) in each of which the behavior is described by a seperate set of assumed functions representing stresses or displacements in that region. In common with the alternative procedures for the accomplishment of numerical solutions for practical problems in structural mechanics, the Finite Element Method requires the formation and solution of a system of algebraic equations. The special advantages of the method reside in its suitability for automation of the equation formation process and in the ability to represent highly irregular and complex structures and loading situations. The method involves extensive calculations but because of the repetitive nature of these computations it is ideally suited for programming for a solution using a digital computer.

Of all the approximate methods described above, by far the most powerful and versatile tool is the Finite Element Method. Within the scope of plate analysis, particularly for arbitrary plate geometry and boundary conditions the Finite Element Method is found to be most convinient to use and has proved highly successful. The technique and its application to the present problem will be discussed in details in Chapter III.

Chapter III

FORMULATION OF THE PROBLEM

The need for the approximate solution techniques to describe the free-vibration of plates with arbitrary planform and thickness variation is apparent when one considers the mathematical complexity involved in such investigations. Although the governing equation describing the dynamic behaviour of plates is relatively simple to write, exact solutions of real problems are seldom possible. This is particularly true for non-simple plate geometry (e.g., arbitrary triangular and quadilateral planforms) and boundary conditions. The solution gets even more involved for the case of plates with varying rigidity (e.g., due to thickness variation in $D=Eh^3/12(1-\nu^2)$) as can be seen from the governing differential equation discussed in article 2.6. A rigorous solution for such problems is impracticable.

The real-life problems do not always involve simple rectangular or regular triangular shapes. The plates with complicated non-simple geometries being not amenable by the available rigorous solution techniques, one has to consider the approximate techniques for such problems. A satisfactory numerical solution would first require a suitable scheme for the reduction of the infinite degrees of freedom of the continuum to finite number. To accomplish this 'discretization' two classical approaches both based on mathematical approximations are possible. The first is one in which a direct approximation to the governing equation is made in terms of a finite number of values of the unknown function, selected at strategic mesh points. This is the well-known method of FINITE DIFFERENCES.

The variational, or energy, formulation of the problem is the second alternative. Here solution of the differential equation is replaced by an equivalent problem of minimizing a certain functional which is defined by a suitable integral of the unknown function and its derivates. In structural problems, such an integral can in fact be the total potential energy of the system. Taking a discrete number of unknown parameters to define the unknown function, the approximate solution will be obtained by minimizing with respect to these parameters. This is the principle of the well-known Ritz Method.

A particular variant of this approach often used for 'discretization' from the engineering viewpoint is the FINITE ELEMENT METHOD. The structure can be considered as subdivided, a priori, into sections or elements and the attention is then focussed on the unknown values of displacements (or force) at nodal points where such elements are joined. In each of such elements, the behaviour is described by a seperate set of assumed functions representing the displacements or stress in that region. The approximating functions (sometimes called interpolation functions) are defined in terms of the values of the field variables at specified points called nodes or nodal points. For two-dimensional or three-dimensional continuum the structure is similarly subdivided, but not by points as in the one- dimensional case, but by lines and surfaces. One of the great advantages of the Finite Element Method over other numerical methods is the ease with which arbitrarily shaped regions can be modelled by an assemblage of finite elements and analysed accurately. The method involves extensive computations but, because of the repetitive nature of these computations, it is ideally suited for programming for a solution using a high-speed digital computer. The paper by Turner, Clough, Martins, and Topp [44] in which a continuous structure, which was part of an aircraft, was analysed as an assemblage of two-dimensional elements. This may be regarded as a starting point of the Finite Element Method.

In common with the alternative procedures of numerical solutions for practical problems in structural mechanics, the FEM requires the formation and solution of systems of algebraic equations from the governing differential equation describing the behaviour of the structure.

There are four principal methods for determining the element properties for the construction of the algebraic equations for the element.

1. DIRECT APPROACH: Originating from the direct stiffness method of structural analysis, this is used only for relatively simple problems.

2. VARIATIONAL APPROACH: A more advanced approach for the determination of the element properties is the Variational Approach. Using calculus of variations it involves extremizing a functional. For solid mechanics the functional could be the potential energy, the complementary potential energy, or some derivatives of these, as in Reissner\s variational principle. There are two methods associated with the variational approach,namely,

   (a) Displacement Finite Element Method

   (b) Force Finite Element Method

   The Displacement Finite Element Method (Stiffness Method) : Here the displacements of the nodes are chosen as the unknowns. In this approach the compatibility conditions in and among elements are initially satisfied. Then the governing equations in terms of the nodal variable are written for each nodal point using equilibrium conditions.

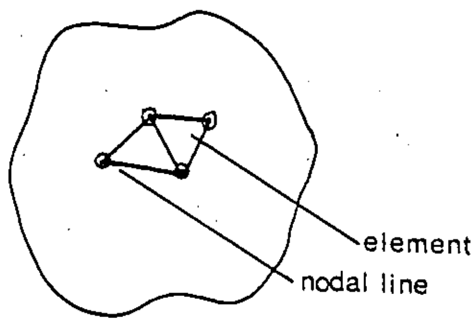
The Force Finite Element Method(Flexibility Method): Here the internal forces are chosen as the unknowns. In this approach the equilibrium conditions are used first to generate the governing equations. The compatibility conditions are then introduced to develop additional equations that might be necessary to obtain a solution.

3. WEIGHTED RESIDUAL APPROACH:An even more versatile approach with its basis entirely in mathematics is Weighted Residual Approach. It starts from the governing equation and can be used for problems for which we do not have a funtional.

4. ENERGY BALANCE:The determination of the element properties with this approach relies on the balance of thermal and/or mechanical energy of a system and requires no variational statement.

Regardless of the approach used, the application of the Finite Element Method for the solution of the continuum problem, follows a certain step-by-step procedure.

1. SUBDIVISION OF THE CONTINUUM domain into elements. The two important parameters, viz., the shape of the elements used and the number of such elements are decided by the nature of the given problem, engineering judgement and experience.



2. SELECT INTERPOLATION FUNCTIONS to represent the variation of the field variable over the element. The order of the element depends on the number of nodes assigned to the element, the number and nature of unknowns at each node, (these could be the magnitude of the field variable and its derivatives) and certain continuity requirements imposed at the nodes and the element boundaries.

3. DERIVE THE ELEMENT PROPERTIES: Once the elements and their interpolation functions have been selected, the matrix equations describing properties of individual

elements are derived using one of the four approaches mentioned earlier.

4. ASSEMBLE THE ELEMENT PROPERTIES TO OBTAIN SYSTEM(GLOBAL) EQUATIONS AND APPLY BOUNDARY CONDITIONS taking care of the continuity of the nodal variables at the nodes. The contribution of each finite element to the overall system properties leads to a final system of equations involving the nodal variables. Depending on the nature of the problem this could be linear or non-linear system of simultaneous algebraic equations or an eigen-value problem (as in the case of dynamic analysis).

5. SOLVE THE SYSTEM OF EQUATIONS for the determination of the values of the field variables at the nodes. Many standard solution techniques are now available to accomplish this.

6. INTERPRETATIONS: from the values of the nodal variables the distribution of the field variable over an entire element domain can be known using the shape functions. Also other important parameters like stresses and strains and velocity distribution etc. can be determined using the appropriate set of relations (stress-displacement and stress-strain relatins, or pressure-velocity relations etc.).

3.1 DISPLACEMENT FORMULATION OF THE PLATE VIBRATION PROBLEM

The aim of the present thesis problem is to present accurate data regarding the natural frequencies and mode-shapes of the triangular plates which would be of considerable use and help to the designer. As for the accurate finite element modelling of the arbitrary plate geometry is concerned, an obvious choice of the shape of the finite elements to be used is the triangular plate element.The problem then, at this stage, is to select a triangular element most suitable (considering factors such as the convenience and ease in formulation, availability in explicit form the element properties, accuracy, computational effort and reliability) for the specific problem of linear-free-vibration analysis of triangular plates.

The plate can be imagined as subdivided into elements interconnected at specific nodes and along the nodal lines. If the displacements at the nodes associated with a typical element define uniquely the displacement within it, then the process of minimization of potential energy will result in stiffness and mass matrices.

Neglecting the effect of shear, the bending strain energy of an isotropic plate of flexural rigidity D can be written as

$$V = \frac{1}{2}\iint_A D\left[\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + \left(\frac{\partial^2 w}{\partial y^2}\right)^2 + 2\nu\left(\frac{\partial^2 w}{\partial x^2}\right)\left(\frac{\partial^2 w}{\partial y^2}\right) + 2(1-\nu)\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\right] dx\,dy \qquad (3.1)$$

This can be written in the equivalent matrix form as

$$V = \frac{1}{2}\iint_A [\chi]^T [D] \{\chi\}\, dx\,dy \qquad (3.2)$$

where,

$$\{\chi\} = \left\{-\frac{\partial^2 w}{\partial x^2},\ -\frac{\partial^2 w}{\partial y^2},\ 2\frac{\partial^2 w}{\partial x \partial y}\right\}^T \qquad (3.3)$$

and rigidity matrix [D] for isotropic materials

$$[D] = \frac{E h^3}{12(1-\nu^2)}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \qquad (3.4)$$

The RHS of equation (3.2) can now be expressed in terms of the non-dimensional quantities defined in chapter II, namely,

$$\xi = x/a \quad , \quad \eta = y/b$$
$$\phi = b/a \quad , \quad w^* = w/a \tag{3.5}$$

where 'a' and 'b' are the characteristic dimensions of the plate as shown in Fig. 4.

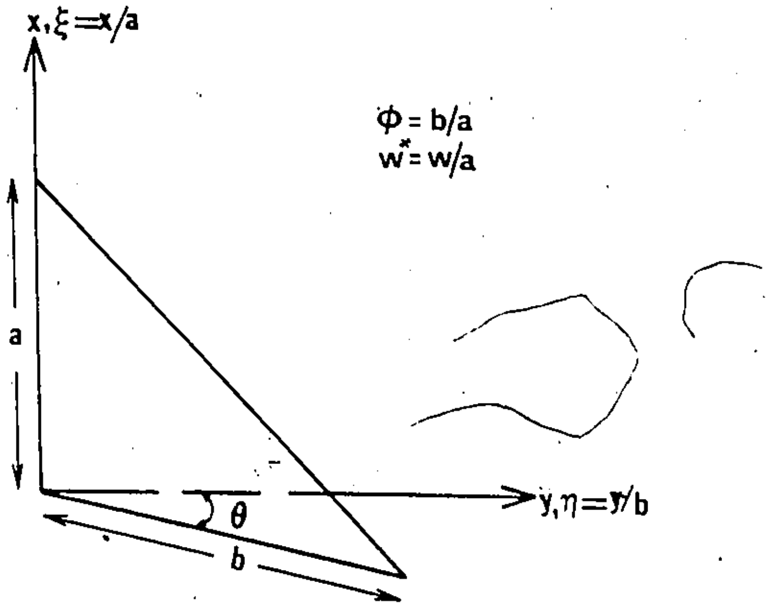


Figure 4: PLATE GEOMETRY

The quantity {K} is written in the form

$$\left[\chi\right] = \frac{1}{a}\begin{bmatrix} -\partial^2 w^*/\partial\xi^2 \\ -\frac{1}{\phi^2}\,\partial^2 w^*/\partial\eta^2 \\ \frac{2}{\phi}\,\partial^2 w^*/\partial\xi\partial\eta \end{bmatrix}$$

In the symbolic form we write

$$\left[\chi \right] = \frac{1}{a} \left[\chi^* \right] \tag{3.7}$$

where '*' denotes the equivalent non-dimensional form. Substituting eqn.(3.7) in eqn.(3.2) the strain energy

$$U = \frac{\phi}{2} \iint_{A^*} \left[\chi^* \right] \left[D \right] \left[\chi^* \right] d\xi d\eta \tag{3.8}$$

Finally, strain energy U can be written as,

$$U = \frac{1}{2} \iint_{A} \frac{1}{a^2} \left[\chi^* \right] \left[D \right] \left[\chi^* \right] dx\, dy \tag{3.9}$$

In the displacement finite-element formulation the displacement w(x,y) is expressed in terms of the nodal variables.

$$w(x,y) = \lfloor N(x,y) \rfloor \{ w \}^{ne}$$

where $N(x,y) = \lfloor N_1, N_2, \text{----} \rfloor$ are the 'shape functions' or the set of assumed displacement pattern in each element and $\{w\}^{ne}$ are the element nodal variables.

3.1.1 DISPLACEMENT FUNCTIONS REQUIREMENT

In the formulation of the total potential energy of the element the transverse displacement w(x,y) is the primary variable. The strain energy expression (eqn. 3.1) involves the second derivatives of the displacement w(x,y). To ensure that the approximate solution converges to the correct solution as we increase the number of elements, and to make it meaningful to assemble the global system of equations from the element equations, the set of assumed displacement patterns in each element should belong to the admissible class of functions satisfying the following requirements:

1. Compatibility:

   (a) The assumed w(x,y) must be continuous and have continuous first derivatives inside each element.

   (b) w(x,y) and its normal derivate $\partial w/\partial n$ (the normal slope) must be uniquely specified along any element interface S (where n is normal to S) by nodal displacement selected on S.

2. Completeness: All rigid body displacement states and uniform strain(constant curvature) states must be included in the expansion. In other words, the six terms $1, x, y, x^2, xy, y^2$ (or their equivalent in other coordinate system) must be included in the set of element displacement modes.

Enforcing this C1 continuity requirments presents a problem. It is found that it is impossible to specify simple polynomial expressions for the shape functions ensuring full compatibility when only w and its slopes are prescribed at the nodes. Considerable difficulties have been encountered in the developement of suitable plate elements. Reiterating: these are caused by the need for both displacement and slope compatibility along the lines connecting the elements, which in turn, is caused by the dependence of the internal energy on the second derivatives of the normal displacement.

A way of getting around this difficulty would be to define additional nodal degrees of freedom, leading to the generation of slope and deflection compatible elements with nodal degrees of freedom being

$$w;\ \partial w/\partial x,\ \partial w/\partial y,\ \partial^2 w/\partial x^2,\ \partial^2 w/\partial y^2,\ \partial^2 w/\partial x\partial y \tag{3.10}$$

$$w,\ \partial w/\partial x,\ \partial w/\partial y,\ \partial^2 w/\partial x\partial y$$

Also, by introducing the mid-side nodes with the normal slope $\partial w/\partial n$ as the nodal variable leads to the 21-DOF element described by Argyris, Bell, Bosshard, Irons and Visser [18].

The reduced eighteen-degree-of-freedom version is developed by Argyris, Bell, Cowper et al. [18]. An essentially similar but more complicated formulation has been developed by Butlin and Ford [18].

3.2 NON-CONFORMING ELEMENTS

It is found that the compatibility requirement is not actually necessary for strain energy convergence. 'Non-Conforming' elements which violate slope continuity between corners have been used extensively and often with good results.

If any group of Non-Conforming elements can represent rigid body motion and constant curvature states exactly, the Finite Element solutions will converge to the true value of strain energy, though not necessarily in monotonic form [22,18]

Numerical results for many non-conforming elements have been presented [18,21] and for many problems shown to give better results than with conforming elements. Although their derivation is simpler, the use of non-conforming elements in general purpose programs has certain disadvantages: (a) energy convergence is sensitive to the mesh division pattern, (b) curvatures and bending moments may not converge even if the strain energy does, (c) no error control is available (d) lack of assuredness of an upper bound or lower bound solution. Admittedly these are not very versatile and reliable for programs dealing with a wide variety of problems, the so called "General Purpose" programs, i.e. they might work beautifully for one type of problem but fail when used in a different type of analysis. However, for programs where just one type of problem, say, the free- vibration analysis of thin plates, is concerned a simple conforming element can prove to be very effective and efficient. In fact in many problems reported in [21,18], the results of the non-conforming element are as good as or even better than the results obtained by using conforming elements. To make a judicious and effective use of the non-conforming element it is imperative to study thoroughly the behaviour of the element in a similar problem for which results are available, and run a few check-problems to recognize the peculiarities, if any, of the particular element.

Several displacement models with various degrees of refinement have been proposed as well as elements for analysis based on equilibrium,mixed and hybrid variational principles. A review is presented in Appendix A. Thus, to select a suitable element from a long list of alternatives, it is important to consider which alternative provides the best balance in usage, taking into consideration factors such as simplicity of formulation, versatality of application, reliability, computational effort and accuracy.

From the study of the various triangular elements available (Appendix A) it can be concluded that for the problem requiring highly accurate results and for the development of general-purpose computer programs where reliability, versatality and accuracy are most important the use of high-precision elements (HSM, A-9, etc.)is recommended.

However, in the formulation of programs for some specific problems, the use of the relatively simple elements could prove highly efficient and lead to results of acceptable accuracy with substantial savings of effort, time and money. From the designer's viewpoint, the comparative study of the solution considered in [26] the 9 DOF element of Bazeley et al. is the simplest available element of acceptable accuracy.If more accurate results are required, the A-9 or the HSM is recommended.

The simple non-conforming nine-degree-of-freedom triangular element developed by Bazeley et al.[21] violates the continuity of slope conditions. However because of its conceptual simplicity in formulation, convenience and economy it has been used in a number of structural problems [16,18,21] dealing with the free vibration of plates and shown to give results with acceptable accuracy. The complete derivation of the shape funtions for this element is presented in Reference 21. For this element it has been proved [18] that the strain energy would always converge when the mesh is generated by three sets of parrallel straight lines.

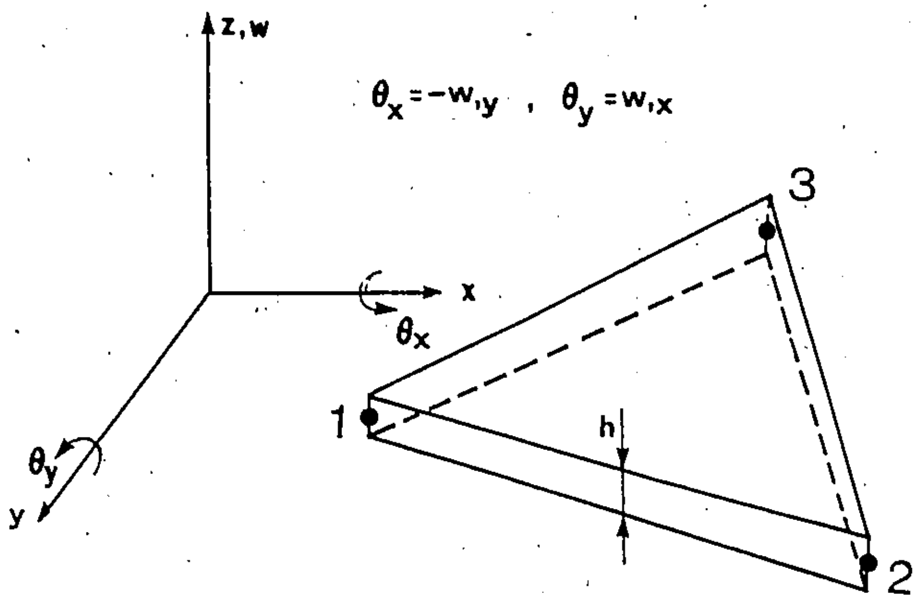


Figure 5: ELEMENT WITH NODAL DOF

The three degrees of freedom at each node are:

The displacement, 'w'

Rotation about X-axis, $\theta_x \left(= -\partial w/\partial y \right)$ (3.11)

Rotation about Y-axis, $\theta_y \left(= \partial w/\partial x \right)$

Thus the displacement w(x,y) can be expressed in terms of the shape functions N(x,y) and the nodal variables $\{w\}^{ne}$ as:

$$w(x,y) = \lfloor N(x,y) \rfloor \{w\}^{ne}$$

This expression can be written in terms of the area coordinates Li and the non-dimensional parameters defined in section 3.1, as

$$a\left(\frac{w}{a}\right) = \sum_{i=1}^{3} \begin{bmatrix} N_i^1 & N_i^2 & N_i^3 \end{bmatrix} \begin{bmatrix} a(w/a) \\ -\frac{a}{b} \left.\frac{\partial(w/a)}{\partial(y/b)}\right|_i \\ \frac{a}{a} \left.\frac{\partial(w/a)}{\partial(x/a)}\right|_i \end{bmatrix} \quad (3.12)$$

With a little algebraic manipulation it can be seen that,

$$w^* = \sum_{i=1}^{3} \begin{bmatrix} N_i^{1*} & N_i^{2*} & N_i^{3*} \end{bmatrix} \begin{bmatrix} W_i^* \\ -\left.\frac{\partial w^*}{\partial \eta}\right|_i \\ \left.\frac{\partial w^*}{\partial \xi}\right|_i \end{bmatrix} \quad 3.13)$$

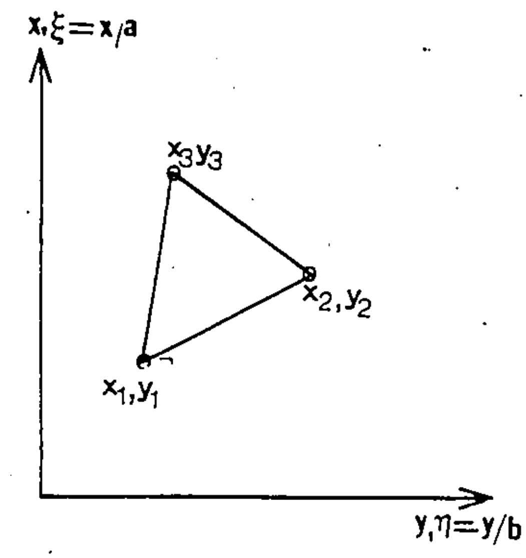


Figure 6: SYSTEM OF ELEMENT COORDINATES

The shape functions N_i^j expressed explicitly in terms of the area coordinates,

$$N_1^1 = L_1 + L_1^2 L_2 + L_2^2 L_3 - L_1 L_2^2 - L_1 L_3^2$$

$$N_1^2 = b_3\left(L_1^2 L_2 + \tfrac{1}{2} L_1 L_2 L_3\right) - b_2\left(L_3 L_1^2 + \tfrac{1}{2} L_1 L_2 L_3\right)$$

$$N_1^3 = c_3\left(L_1^2 L_2 + \tfrac{1}{2} L_1 L_2 L_3\right) - c_2\left(L_3 L_1^2 + \tfrac{1}{2} L_1 L_2 L_3\right)$$

$$N_2^1 = L_2 + L_2^2 L_3 + L_3^2 L_1 - L_2 L_3^2 - L_2 L_1^2$$

$$N_2^2 = b_1\left(L_2^2 L_3 + \tfrac{1}{2} L_1 L_2 L_3\right) - b_3\left(L_1 L_2^2 + \tfrac{1}{2} L_1 L_2 L_3\right)$$

$$N_2^3 = c_1\left(L_2^2 L_3 + \tfrac{1}{2} L_1 L_2 L_3\right) - c_3\left(L_1 L_2^2 + \tfrac{1}{2} L_1 L_2 L_3\right)$$

$$N_3^1 = L_3 + L_3^2 L_1 + L_1^2 L_2 - L_3 L_1^2 - L_3 L_2^2$$

$$N_3^2 = b_2\left(L_3^2 L_1 + \tfrac{1}{2} L_1 L_2 L_3\right) - b_1\left(L_2 L_3^2 + \tfrac{1}{2} L_1 L_2 L_3\right)$$

$$N_3^3 = c_2\left(L_3^2 L_1 + \tfrac{1}{2} L_1 L_2 L_3\right) - c_1\left(L_2 L_3^2 + \tfrac{1}{2} L_1 L_2 L_3\right)$$

where $b_1 = y_2 - y_3 \qquad c_1 = x_3 - x_2$

$b_2 = y_3 - y_1 \qquad c_2 = x_1 - x_3$

$b_3 = y_1 - y_2 \qquad c_3 = x_2 - x_1$

(x_i, y_i) being the coordinates of the vertex i

Bending strain energy for the element as given earlier in equation (3.9)

$$U^{(e)} = \frac{\phi}{2} \iint_{\Delta^{*(e)}} \left[K^*\right] \left[D\right] \left[K^*\right] d\xi\, d\eta$$

At this stage we note that

$$\left[K^*\right] = \begin{bmatrix} -\partial^2 w^*/\partial\xi^2 \\ -1/\phi^2\left(\partial^2 w^*/\partial\eta^2\right) \\ 2/\phi\left(\partial^2 w^*/\partial\xi\partial\eta\right) \end{bmatrix} \doteq \begin{bmatrix} -\partial^2 N^*/\partial\xi^2 \\ -1/\phi^2\left(\partial^2 N^*/\partial\eta^2\right) \\ 2/\phi\left(\partial^2 N^*/\partial\xi\partial\eta\right) \end{bmatrix} \left\{w^*\right\}^{ne} \qquad (3.14)$$

and consequently the strain energy is given by

$$U^{(e)} = \frac{1}{2} \left\{w^*\right\}^{ne} \left[K_{ss}^*\right] \left\{w^*\right\}^{ne} \qquad (3.15)$$

where,

$$\left[K_{ss}^*\right] = \phi \iint_{\Delta^*} \left[K_N^*\right]^T \left[D\right] \left[K_N^*\right] d\xi\, d\eta \qquad (3.16)$$

In the expanded form the stiffness matrix $[K^*_{ss}]$ of the element is,

$$K^*_{ij_{ss}} = \phi \iint_{\Delta^*} D\, A^*_{ij}\, d\xi\, d\eta \qquad (3.17)$$

$$A^*_{ij} = \Bigg[\left(\frac{\partial^2 N_i^*}{\partial \xi^2} + \frac{\nu}{\phi^2}\frac{\partial^2 N_i^*}{\partial \eta^2}\right)\frac{\partial^2 N_j^*}{\partial \xi^2} + \left(\frac{1}{\phi^2}\frac{\partial^2 N_i^*}{\partial \eta^2} + \nu\frac{\partial^2 N_i^*}{\partial \xi^2}\right)\frac{\partial^2 N_j^*}{\partial \xi^2}\cdot\frac{1}{\phi^2}$$

$$+ \frac{2(1-\nu)}{\phi}\frac{\partial^2 N_i^*}{\partial \xi\,\partial \eta}\,\frac{\partial^2 N_j^*}{\partial \xi\,\partial \eta}\Bigg] \qquad (3.18)$$

The flexural rigidity D can be expressed as

$$D = D_0\,(h/h_0)^3 \qquad (3.19)$$

where $D_0 = E h_0^3/[12(1-\nu^2)]$ and h_0 is some characteristic thickness.

Introducing this in equation (3.17) we get

$$K^*_{ij_{ss}} = \phi\, D_0 \iint_{\Delta^*} (h/h_0)^3\, A^*_{ij}\, d\xi\, d\eta \qquad (3.20)$$

This integral can be evaluated using the Numerical Integration technique (section 4.4) after substituting the appropriate function for the thickness 'h'.

The final global stiffness matrix is assembled by appropriately combining the contributions of the constituent elements of the plate.

3.3 ELEMENT MASS MATRIX

Neglecting rotary inertia effect, the expression for the Kinetic Energy is,

$$T = \frac{1}{2} \iint_{\Delta} \omega^2 \rho h \, w^T . w \, dx \, dy \tag{3.21}$$

$$T = \frac{1}{2} a^2 \omega^2 \rho h_0 \iint_{\Delta} (h/h_0) \, w^{*T} . w^* \, dx \, dy$$

Substituting $w^* = \lfloor N^* \rfloor \{w^*\}^{ne}$, and non-dimensionalizing various quantities we get,

$$T = \frac{1}{2} \rho \omega^2 h_0 a^3 b \{w^*\}^{ne^T} \iint_{\Delta^*} (h/h_0) \, N^{*T} . N^* \, d\xi \, d\eta \, \{w^*\}^{ne} \tag{3.22}$$

Thus,

$$\left[M^*\right]^{(e)} = \iint_{\Delta^*} (h/h_0) \, N^{*T} . N^* \, d\xi \, d\eta \tag{3.23}$$

The system of equations for the free vibration problem would now look like

$$\left\{ D_0 \left[K^*_{ss}\right] - \omega^2 \rho a^4 \phi h_0 \left[M^*\right] \right\} \{w^*\}^{ne} = 0 \tag{3.24}$$

Rewriting this as

$$\left[\left[K^*_{ss}\right] - \frac{\omega^2 \rho a^4 \phi h_0}{D_0} \left[M^*\right] \right] \{w^*\}^{ne} = 0 \tag{3.25}$$

where [K_{ss}^*] and [M^*] are the final global stiffness and mass matrices respectively, assembled from the element matries [K_{ss}^*]$^{(e)}$ and [M^*]$^{(e)}$.

We thus have an eigenproblem of the form

$$\left[K - \alpha^2 M\right]\{w\}^{ne} = 0 \qquad (3{\cdot}26)$$

with the non-dimensional frequency parameter α^2 as its eigenvalues

$$\alpha^2 = \frac{\rho \omega^2 a^4 \phi h_o}{D_o}$$

3.4 THICKNESS VARIATION

The spatial variation of the thickness 'h' needs to be specified in order to determine the coefficients of the stiffness matrix and the mass matrix. As can be seen from eqn.(3.20) and eqn.(3.23), the integrand of Kij* involves a cubic power of 'h' and that of Mij* involves first power. Theoretically any order of thickness variation can be handled when the integrals are evaluated numerically using the quadrature formulas. However, with higher orders of thickness variation, because of the presence of 'h**3' in Kij* , a very high order of integration scheme is needed for exact integration, thereby paying the penalty by way of computing time.

In the present problem, a linear variation of thickness, 'h', in the two coordinate directions has been assumed.

$$H(x, y) = H_0 + m_1 x + m_2 y \tag{3.27}$$

This can be written in terms of the non-dimensional parameters as

$$\frac{H}{H_0} = 1 + m_1^* \xi + m_2^* \eta \tag{3.28}$$

where

$$m_1^* = \frac{m_1 a}{H_0} \quad ; \quad m_2^* = \frac{m_2 b}{H_0} \tag{3.29}$$

Defining the two ratios

$$H_x^* = \frac{H/_{\substack{\xi=1\\ \eta=0}}}{H_0} = \frac{H/_{\xi=1,\,\eta=0}}{H_0} \tag{3.30}$$

$$H_y^* = H_B/H_0$$

From eqn.(3.28),

$$m_1^* = H_x^* - 1.$$

(3.31)

and

$$m_2^* = \left(H_y^* - 1\right) \sec\theta + \phi \tan\theta \left(H_x^* - 1\right)$$

Rewriting eqn.(3.28),

$$H/H_0 = 1 + (H_x^* - 1)\xi + \left[(H_y^* - 1)\sec\theta + \phi\tan\theta\,(H_x^* - 1)\right]\eta \tag{3.32}$$

This can be substituted for the thickness taper ratio in the expressions for Kij* and Mij*.

In the above procedure the so-called consistent mass concept is adopted. In this case, the mass coefficients are derived by integration of the same displacement functions which are employed in the formulation of the stiffness matrix. However it is pointed out [19] that the consistent mass matrix provides a better approximation to the inertia forces acting in the structure than is obtained for the elastic forces from the corresponding stiffness matrix. This is because the mass matrix is derived from the assumed displacement functions whereas the stiffness matrix involves derivatives of the displacement functions, and the derivatives are represented with less accuracy than are the displacements. Thus it would be reasonable to employ lower order interpolation functions in formulating the mass matrix. However, in practice the introduction of additional displacement functions for this purpose complicates rather than simplifies the overall analysis.

In general, the only simplified mass matrix of practical significance is the lumped mass matrix. The diagonal form of the lumped mass matrix can be of significant advantage both in the solution of the eigenvalue problem. However, in many cases , particularly when higher order elements are used, it is difficult to derive satisfactory lumping procedures. Moreover the consistent mass displacement type formulation offers a major advantage in that vibration frequencies computed from such models represents upper bounds to the true vibration frequencies if continuity is maintained in the finite element formulation.

3.5 SOLUTION OF THE EIGENPROBLEM

The generalized Eigenproblem is of the form

$$\underline{K}\ \underline{\phi} = \omega^2 \underline{M}\ \underline{\phi} \tag{3.33}$$

in which the stiffness and mass matrices are symmetric.

The following different efficient algorithms have been developed for the solution of the Eigenproblem in structural mechanics [18].

1. Householder - QR -Inverse Iteration (HQRI)
2. Generalized Jacobi Iteration
3. Determinant Search
4. Subspace Iteration

The optimum choice depends upon the characteristics of the structural problem under consideration particularly on the order and the bandwidths of the matrices, and the number of eigenvalues and eigenvectors to be computed. Each of these methods is discussed in brief:-

3.5.1 HOUSE HOLDER - QR - INVERSE ITERATION:

To use this method, it is necessary to transform the eigen-problem to the standard form

$$\underline{A}\,\underline{X} = \underline{\lambda}\,\underline{X}$$

and then the HQRI solution is carried out in three steps.

(a) Matrix A is reduced to the tridiagonal form using the householder's procedure.

(b) all N eigenvalues of the tri-diagonal matrix are obtained by QR iteration.

(c) inverse iteration is used To calculate the required number of eigenvectors of the tri-diagonal matrix, which are then transformed to the eigenvectors of the original matrix.

HQRI is very efficient if many eigenvalues & eigenvectors are required and if the matrix is full or has large band-width--arising from coordinate transformation or static con-densation. It is inefficient for small bandwidths or for the problem of large system but few eigenvalues are needed.

3.5.2 GENERALIZED JACOBI ITERATION (GJI)

This is an attractive alternative to HQRI as the preliminary transformation to standard form is not required.

$\underline{K}\underline{\phi} = \omega^2 \underline{M}\,\underline{\phi}$ is solved directly using generalized Jacobi rotation matrices to transform $\underline{K}$ and $\underline{M}$ simultaneously to diagonal form.

GJI is most efficient if off-diagonal terms in both matrices are small or if only a few off-diagonal terms exist. Like the HQRI, GJI is effective in obtaining all the eigenvalues and eigenvectors of the system.

3.5.3 DETERMINANT SEARCH

For matrices with small bandwidths a DS algorithm provides a very efficient solution. The algorithm uses triangular factorization and vector inverse iteration directly on the general problem $K\phi = \omega^2 M\phi$ and solves for the required eigenvalues and vectors in succession from the least dominant pair upwards. In the eigenproblem M can be diagonal, with zero diagonal elements, or maybe banded positive definite.

3.5.4 SUBSPACE ITERATION

It is a very effective procedure for finding a relatively small number of eigenvalues and eigenvectors of large structural systems. It involves the repeated application of Ritz method, in which computed eigenvectors from one step are used as the trial basis vectors for the next iteration.

The objective of the Subspace Iteration analysis is the calculation of p eigenvectors and eigenvalues satisfying

$$\underline{K}\,\underline{\phi} = \underline{M}\,\underline{\phi}\,\underline{\Omega}^2$$

in which the eigenvectors $\underline{\phi}$ from an M-orthonormal basis of the p-dimensional least dominant subspace of the operators K and M. The solution is carried out by simultaneous iteration with q linearly independent vectors, where q > p. In the k-th iteration step, the vectors span the q-dimensional subspace Ek and 'best' eigenvalue and eigenvector approximations are calculated i.e.,when vectors span the p-dimensional least dominant subspace the required eigenvalue and eigenvectors are obtained.

V* ◄--- starting set of q starting vectors

For the k-th step, the procedure is to solve for V_k from

$$\underline{K}\,\underline{\bar{V}}_k = \underline{M}\,\underline{V}_{k-1} \qquad (3.34)$$

Then the projections of the operations K & M onto the subspace Ek is given by

$$\underline{K}_k = \underline{\bar{V}}_k^T\,\underline{K}\,\bar{V}_k \qquad (3.35)$$

$$\underline{M}_k = \underline{\bar{V}}_k^T\,\underline{K}\,\bar{V}_k$$

solving the eigenproblem of the projected operators leads finally to the k-th improved approximation to eigen-vectors.

$$V_k = V_k Q_k \qquad (3.36)$$

Provided that the starting subspace is not orthogonal to any of the required eigenvectors, the process converges to the desired results, i.e.

$$\underline{\Omega}_k^2 \longrightarrow \underline{\Omega}^2 \text{ and } \underline{V}_k \longrightarrow \phi \text{ as } k \longrightarrow \infty \qquad (3.37)$$

Chapter IV

DEVELOPMENT OF THE COMPUTER PROGRAM

4.1 INTRODUCTION

Having established the underlying mathematics of the Finite Element Method and the formulation of the element equations for the plate vibration problem, a computer program has to be developed to evaluate, assemble and solve the resulting eigenvalue problem.

In this chapter the details of computer codes for the free vibration analysis of arbitrary triangular plate have been discussed. The discussion also includes comments on debugging, verification and documentation phases. Special care has been taken to construct the FORTRAN program in its simplest form. Further, coding can be easily understood with the aid of the many comment statements appearing throughout the program.

The program consists of essentially three units as shown in the figure below.

<u>UNIT:1</u> SET UP ARRAYS READ INPUT DATA GENERATE MESH TOPOLOGY

<u>UNIT:2</u> SET UP AND SOLVE THE EIGENVALUE PROBLEM

<u>UNIT:3</u> PRINT AND/OR PLOT SOLUTIONS CALCULATE, IF REQUIRED, QUANTITIES DERIVED FROM SOLUTION

<u>UNIT 1</u>:

The reading and generation of the arrays of data that define the finite element model of the problem are performed by UNIT 1 of the program.

UNIT 2:

The actual finite element calculations as described in Chapter II are carried out by this main body of the program. In this part of the program element matrices and vectors are calculated and assembled, boundary conditions are imposed, and the global set of equations is solved to determine the frequencies and nodal -point values of the displacement giving the modal shape.

UNIT 3:

This is concerned with the presentation of the results. The frequencies are tabulated and the mode shapes plotted showing the deflected plate with the 'nodal lines'.

The overall logic of the Finite Element Program developed for the present problem is shown in Table 1. The aim of the program is to keep it simple yet efficient and versatile. Although in the present problem the non-conforming 9 DOF triangular element is incorporated, with minimal modifications the same program can handle different types of elements.

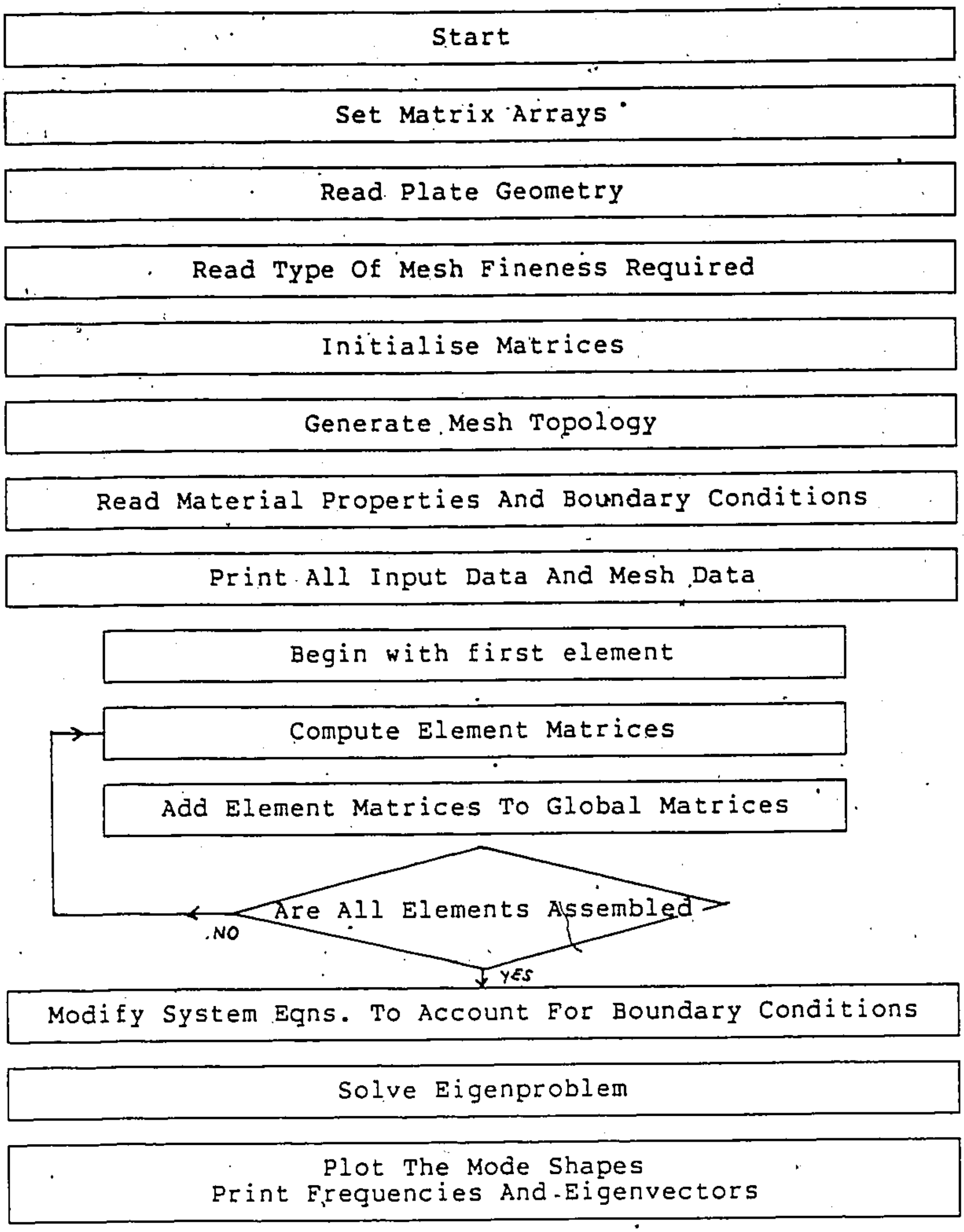


TABLE 1

OVERALL LOGIC OF THE COMPUTER PROGRAM

In the first half of this chapter the programming aspects of the different building blocks used in the program will be discussed one by one.
These are:

1. Mesh Generation

2. Stiffness Matrix and Mass Matrix Calculations

3. The Eigenvalue Solver

4. Mode-shape plotting Routine

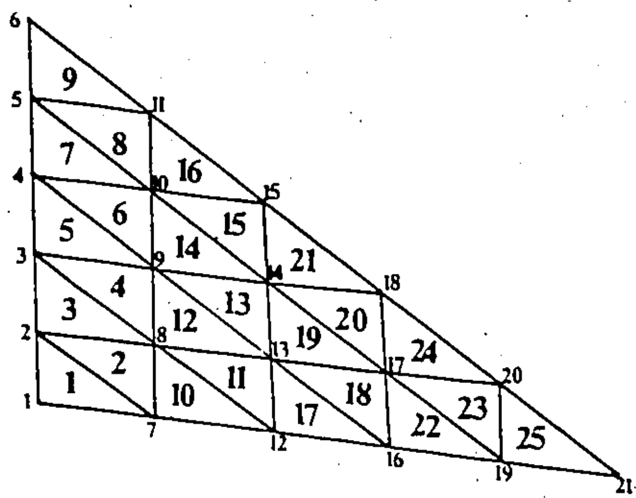
Later the total structure of the program incorporating all these units would be considered with detailed descripition of the entire listing.

4.2 MESH GENERATION

Even for the case of fairly small-size Finite Element problems, data prepration (nodal numbers and their coordinates; element numbers and their definition) becomes quite an elaborate and tedious task and a potential source of error. The presence of such errors will bring about incorrect results and if detected at that stage it would mean more runs on the computer after correction. However, if the errors remain undetected and such incorrect results are used for making decisions and judgements, there maybe serious repercussions. It is therefore important to eliminate such errors in data and this can be achieved to a large extent by automatic mesh generation. Using as input the minimal ammount of necessary information to describe the geometry of the domain and the desired fineness of the mesh divisions, the required data is generated by the computer.

In the present program the elementary routine developed by Y.K.Cheung and M.F.Yoe [20] has been suitably modified for more effective use in the current problem under study. The objective is to generate a mesh pattern that would divide the arbitrary triangular plate under consideration into triangular elements using three sets of parallel straight lines (As stated earlier in secton (3.2), for this non-conforming triangular element the strain energy would always converge if the elements are generated by three sets of parallel lines).

The obvious choice of the mesh pattern for the triangular plate will then be:



The input data needed to generate this type of mesh is:

1. Total number of generating line segments ($\overline{16}$;$\overline{7,11}$,...), NY ('NY'=5 in the above fig.)

2. Weighting factors, CONX and CONY, along the spanwise and chordwise directions respectively.

3. Aspect Ratio, PHI (=b/a)

4. Sweepback Angle, THETA(=θ)

The output consists of firstly the ξ and η coordinates of each point with its corresponding number and secondly the element number with its element definition. Both nodal numbers and element numbers are arranged in sequential order.

The plate in the X,Y-plane is first transformed to the ξ,η-plane using the transformation

$$\xi = x/a$$
$$\eta = y/b$$
$$\phi = b/a$$

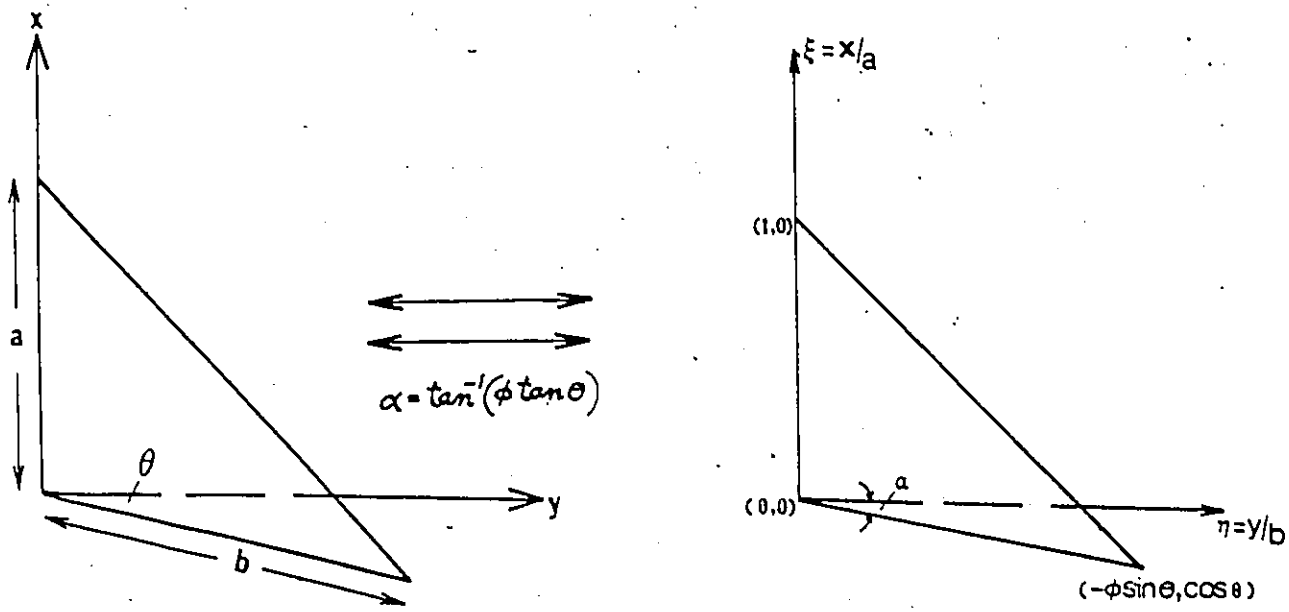
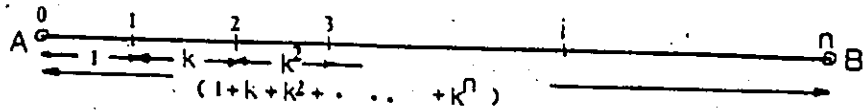


Figure 7: COORDINATE TRANSFORMATION

Depending on the value of the weighting factor CONX the size of the intervals along the generating lines are fixed.

CONX < 1	intervals along the generating lines will become progressively shorter.
CONX = 1	intervals stay equal
CONX > 1	intervals become progressively longer.

Coordinates of point i along the generating line AB are given by:



$$x_i = (x_B - x_A)\,\frac{\sum_{j=1}^{i} k^{j-1}}{\sum_{j=1}^{n} k^{j-1}} + x_A \qquad (4.1)$$

Summing the geometric progression,

$$x_i = (x_B - x_A)\,\frac{k^{i}-1}{k^{n}-1} + x_A \qquad (4.2)$$

Similarly

$$y_i = (y_B - y_A)\,\frac{k^{i}-1}{k^{n}-1} + y_a$$

Corresponding to CONX, a weighting factor CONY has been introduced along the spanwise direction which controls the size of the elements along that direction.

This is useful when, for example, referring to the present problem a finer mesh is desired close to the constrained edge of the plate so that when the displacement at the nodes are plotted and points of zero displacement determined a better representation is obtained of the nature of the 'nodal-line' close to the edge.

However it should be noted that the condition that the triangular elements be generated by three sets of parallel lines would require

CONX=CONY

The node numbers are labelled correctly as we go from one generating line to the next, by increasing the number by one everytime we come across a point.

Example: For the different parameters defined as,

$$PHI=1.5$$

$$\theta=15^{\circ}$$

$$CONX=1.00$$

$$CONY=1.00$$

$$NY=5$$

the mesh with node numbers, node coordinates and element numbers as shown in Fig. 8 is generated. The complete mesh data is transferred to the main program to carry out further calculations.

With merely increasing or decreasing the value of NY, the mesh can be made finer or coarser as desired.

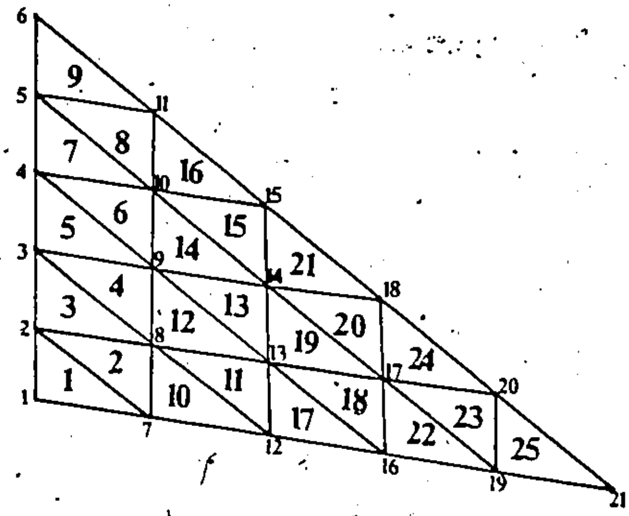


Figure 8: TYPICAL FINITE ELEMENT MESH FOR NY=5

4.3 GENERATION OF ELEMENT MATRICES

The structural stiffness matrix is generated by appropriately assembling all the element stiffness matrices. Thus the primary step is to obtain the stiffness matrix for each element.

From Eqn.(3.20) the coefficient $(K_{ij}^{*(e)})$ is given by

$$K_{ij}^{*(e)} = \phi D_0 \iint_{\Delta^*} (h/h_0)^3 A_{ij}^* \, d\xi \, d\eta = \iint_{\Delta^*} F_s^*(L_1, L_2, L_3) \, d\xi \, d\eta \qquad (4.3)$$

A_{ij}^* (eqn. 3.18) involves the second partial derivatives of the shape functions [N] with respect to the spatial coordinates ξ and η. The function [Ni] being a cubic function in area coordinates Li, the quantity $A_{ij}^{*(e)}$ would then be a linear function in Li. If the thickness 'h' is constant over the entire element, Equation(4.3) can be integrated explicitly using the following formula:

$$\iint_{A^{(e)}} L_1^{\alpha} L_2^{\beta} L_3^{\gamma} \, dA^{(e)} = \frac{\alpha!\beta!\gamma!}{(\alpha+\beta+\gamma+2)!} 2A^{(e)} \text{ ; } A^{(e)} = \text{AREA OF ELEMENT} \quad (4.4)$$

However for the case of variable thickness plate,

$$h=h(L1,L2,L3)$$

and as the expression for $K_{ij}^{*(e)}$ involves the cubic power of 'h', even a linear variation of 'h' makes $F_s^*(L1,L2,L3)$ quite involved for explicit integration. Thus numerical integration formulas are used to evaluate the integrals.

4.3.1 ELEMENT MASS MATRIX:

From the eqn.(3.23) in chapter III, the expression for the mass matrix coefficient is:

$$m_{ij}^{*(e)} = \iint_{A^*} (h/h_o) \, N_i^* \cdot N_j^* \, d\xi \, d\eta = \iint_{A^*} F_m^* \, d\xi \, d\eta \quad (4.5)$$

In this case, N being a cubic function and h a linear function, F_m^* would therefore be a 7-th order function of the area coordinates and will have to be integrated numerically.

4.4 NUMERICAL INTEGRATION

An important part of Finite Element Analysis is the evaluation of integrals in the element equations. In many cases these integrals cannot be evaluated in closed form and in others the term-by-term multiplication and integration could be highly cumbersome and time consuming. In such cases the approximate numerical methods are used.

Quadrature formulas approximate the definite integral of a function by a properly weighted sum of particular values of the function at suitably distributed points within the domain of integration. In the present problem the integrands in the stiffness and mass matrix calculations are polynomials in area coordinates and the integration has to be performed over the triangular region. Suitable quadrature rules of the Newton-Cotes type and the Gauss & Radau type have been derived by Silvester, Irons and Hammer [27].

The highly efficient Gaussian type formulas which are fully symmetric with the three vertices of the triangle have been derived by Hammer and co-workers [27]

For integration over a triangle of area A the formulas are of the form:

$$\iint_A f\, dA = A \sum_{i=1}^{N} W_i\, f(\xi_i, \eta_i, \zeta_i) \qquad (4.6)$$

where,

ξ_i, η_i, ζ_i =area coordinates of the i-th sampling point

W_i =weight associated with the i-th sampling point

N = number of sampling points

The values of the constants W_i, ξ_i, η_i and ζ_i for various formulas are listed in table [2]. Because of triangular symmetry of the formulas the sampling points occur in groups of six, three, or one. If a sampling point has area coordinates (α, β, γ) , $\alpha \neq \beta \neq \gamma$, then there are also five symmetrically disposed points with coordinates (α, γ, β), (β, α, γ), (β, γ, α), (γ, α, β), and (γ, β, α) all of which have the same weight. A sampling point which has two equal coordinates, for example, (α, β, β) occurs as a member of a trio of sampling points with equal weights the coordinates of the other two being (β, α, β) and (β, β, α). occur at the centroid.

TABLE 2

WEIGHTS AND AREA COORDINATES FOR NUMERICAL INTEGRATION

	ξ_i	η_i	ζ_i	w_i
3 POINT FORMULA	1/6	1/6	2/3	1/3
	1/6	2/3	1/6	1/3
(DEGREE OF PRECISION: 2)	2/3	1/6	1/6	1/3
3 POINT FORMULA	1/2	1/2	0	1/3
	1/2	0	1/2	1/3
(DEGREE OF PRECISION: 2)	0	1/2	1/2	1/3
	1/3	1/3	1/3	0.225
7 POINT FORMULA	0.7974269853	0.1012865073	0.1012865073	0.1259391805
	0.1012865073	0.1012865073	0.7974269853	"
	0.1012865073	0.7974269853	0.1012865073	"
	0.4701420641	0.4701420641	0.0597158717	0.1323941527
(DEGREE OF PRECISION: 5)	0.4701420641	0.0597158717	0.4701420641	"
	0.0597158717	0.4701420641	0.4701420641	"

The essential steps in the element matrix calculations for K^*_{ij} and M^*_{ij} are listed below:

1. Specify the (ξ, η) coordinates $(\xi 1, \eta 1)$, $(\xi 2, \eta 2)$ and $(\xi 3, \eta 3)$ of nodal points of each element.
2. Calculate the area A and the coefficients BI, CI, BJ, CJ, BM AND CM.
3. Specify a set of l integration points $(L_1, L_2, L_3)^l$, $l=1,2,3 \ldots I_l$, and quadrature weights W_l for the triangular element domain.
4. Calculate values of the shape functions Ni, and its second partial derivatives $\frac{\partial^2 N_i^*}{\partial \xi^2}$, $\frac{\partial^2 N_i^*}{\partial \eta^2}$, $\frac{\partial^2 N_i^*}{\partial \xi \partial \eta}$, at the integration points $(L_1, L_2, L_3)^l$.
5. Calculate the thickness 'h' at each point of integration.
6. Using the results of step 2 through 5 , determine the value of the integrands in eqn.(3.20) and eqn.(3.23) at the integration point $(L1, L2, L3)^l$ and multiply each by $(W_l A)$
7. Sum the number computed in step 6 in accordance with eqn.(4.6) to obtain K^*_{ij} and M^*_{ij}.

TABLE 3

ELEMENT STIFFNESS MATRIX FLOWCHART

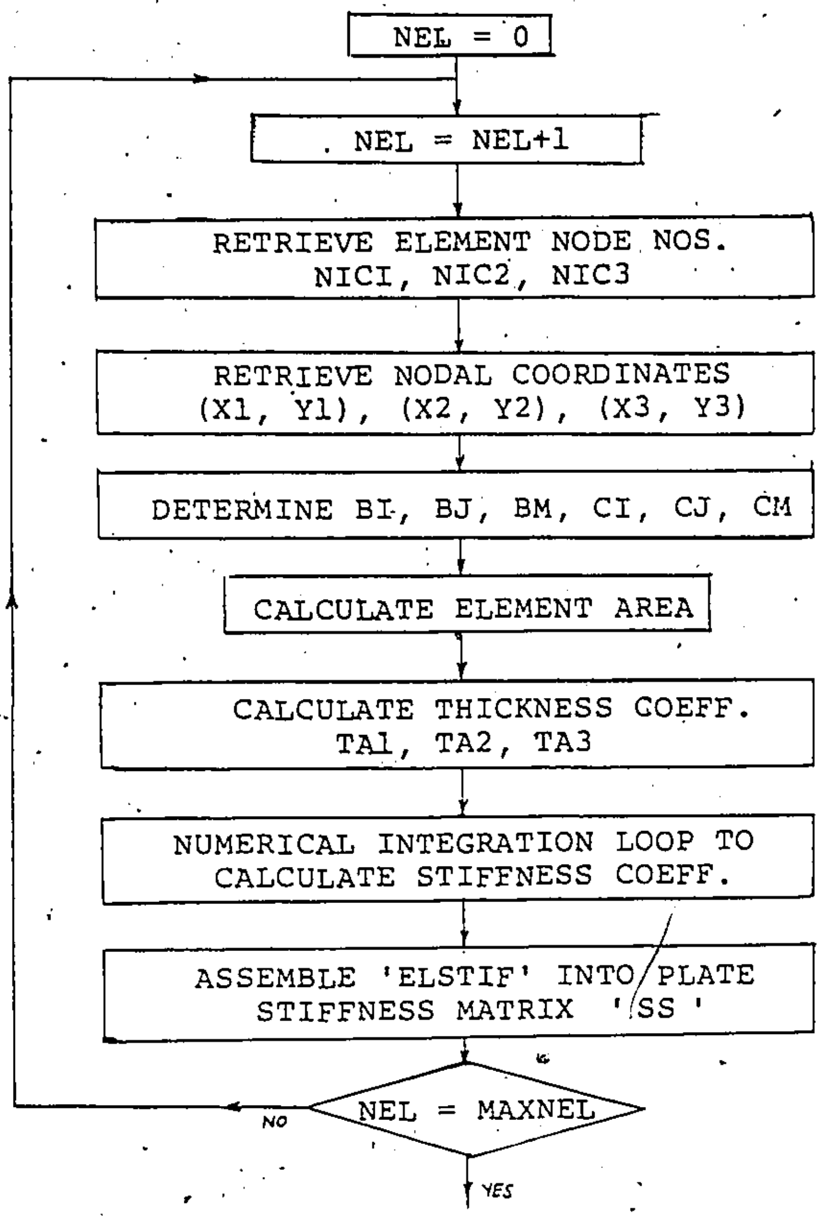
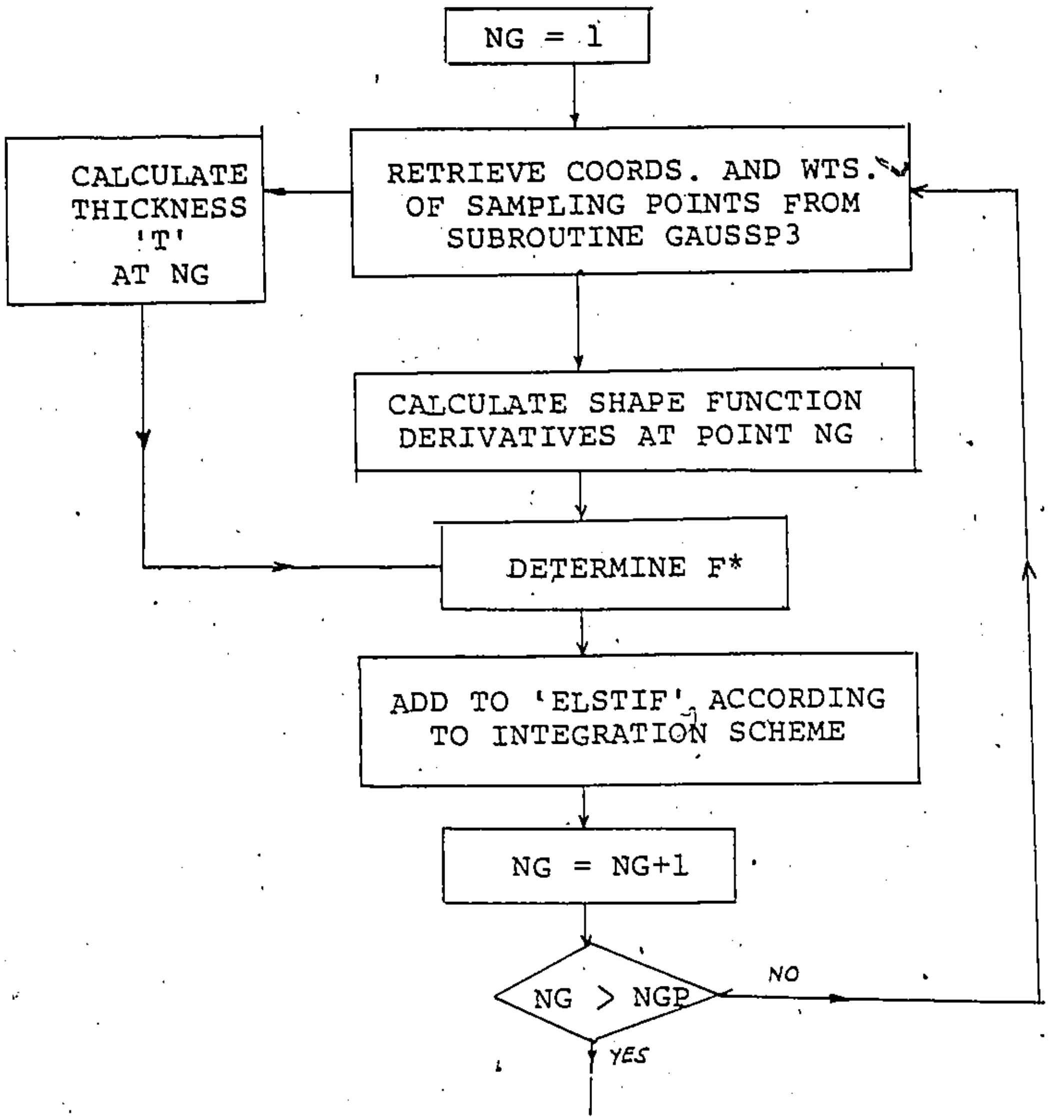


TABLE 4

NUMERICAL INTEGRATION LOOP



4.5 THE EIGENVALUE SOLVER

After having assembled the structural stiffness and mass matrices and after having introduced the boundary conditions, the next step invloves the solution of the resulting eigenproblem. In the present problem, the Generalized Jacobi Method of solution section (3.51) has been used. In this method the solution of the generalized eigenproblem $K\phi = \lambda M\phi$, $M \neq I$, is sought employing the basic properties of the eigenvectors in the matrix ,

$$\bar{\Phi} K \bar{\Phi} = \Lambda$$
$$\bar{\Phi}^T M \bar{\Phi} = I \qquad (4.7)$$

The nxn matrix is unique and is constructed by iteration. The idea is to diagonalize K and M using successive pre- and postmultiplication by matrices P_k^T and P_k , respectively, where k=1,2...

Defining k_1=K and M_1 =M

$$K_{i+1} = P_i^T K_i P_i \quad , \quad i = 1, k$$

Similarly

$$M_{i+1} = P_i^T M_k P_i \quad , \quad i = 1, k \qquad (4.8)$$

P_i's are selected to bring K_k and M_k closer to diagonal form, such that

$$K_{k+1} \longrightarrow \Lambda \quad \text{AND} \quad M_{k+1} \longrightarrow I \quad \text{as } k \to \infty \qquad (4.9)$$

Thus with l being the last iteration

$$\Phi = P_1 P_2 \,\ldots\, P_l \qquad (4.10)$$

In practice it is not necessary that M_{k+1} converges to I and K_{k+1} to Λ, but they only need to converge to diagonal form. Namely, if

$$K_{k+1} \longrightarrow \operatorname{diag}(K_l) \quad \text{and} \quad M_{k+1} \longrightarrow \operatorname{diag}(M_l) \quad \text{as } k \to \infty$$

$$\Lambda = \operatorname{diag}\left(\frac{K_l^{i+1}}{M_l^{i+1}}\right) \qquad (4.11)$$

$$\Phi = P_1 P_2 \ldots P_l \operatorname{diag}\left(\frac{1}{\sqrt{M_l^{l+1}}}\right)$$

P_k is an orthogonal matrix which in the Jacobi Method is a rotation matrix which is selected in such a way that the off-diagonal element in K_k and M_k is zeroed simultaneously. If element (i,j) is to be zeroed, the corresponding orthogonal matrix P_k is

$$P_k = \begin{array}{cc} \begin{array}{ccccc} & & i\text{th} & j\text{th} & \end{array} & \\ \begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & \alpha & \\ & & \gamma & 1 & \\ & & & & \ddots \end{bmatrix} & \begin{array}{c} \\ \\ i\text{th} \\ j\text{th} \\ \\ \end{array} \end{array} \qquad (4.12)$$

Performing the multiplications $P_k^T K_k P_k$ and $P_k^T M_k P_k$ and using the condition that $k_{ij}^{(k+1)}$ and $m_{ij}^{(k+1)}$ shall be zero gives,

$$\gamma = -\bar{k}_{ii}^{(k)}/x \quad ; \quad \alpha = \bar{k}_{jj}^{(k)}/x \qquad (4.13)$$

where

$$\gamma = -k_{ii}^{(k)}/x \quad ; \quad \alpha = k_{jj}^{(k)}/x$$

$$k_{ii}^{(k)} = k_{ii}^{(k)} m_{ij}^{(k)} - m_{ii}^{(k)} k_{ij}^{(k)}$$

$$k_{jj}^{(k)} \quad k_{jj}^{(k)} m_{ij}^{(k)} - m_{jj}^{(k)} k_{ij}^{(k)}$$

$$k^{(k)} \quad k_{ii}^{(k)} m_{jj}^{(k)} - k_{jj}^{(k)} m_{ii}^{(k)}$$

$$x = \frac{k^{(k)}}{2} + \mathrm{Sign}\left(k^{(k)}\right) \left(\frac{k^{(k)}}{2}\right)^2 + k_{ii}^{(k)} k_{jj}^{(k)}$$

The above relations are used when M is a positive definite full or banded mass matrix. In that case

$$\left(\frac{k^{(k)}}{2}\right)^2 + k_{ii}^{(k)} k_{jj}^{(k)} > 0 \qquad (4.14)$$

In addition, det $P_k \neq 0$ is a necessary condition for the algorithm to work. The actual soluton procedure may be summarized in the following steps

1. Initialize the threshold for the sweep. Typically the threshold used for sweep m may be 10^{-2m}.

2. For all (i,j) with i<j calculate coupling factors

$[(k_{ij}^{(k)})^2/k_{ii}^{(k)} k_{jj}^{(k)}]^{1/2}$ and $[(m_{ij}^{*})^2/m_{ii}^{(k)} m_{jj}^{(k)}]^{1/2}$ and applying a transformation if the factor is larger than the current threshold.

3. Check the convergence by comparing successive eigenvalue approximated by testing if all off-diagonal elements are small enough.

The Subroutine JACOBI given in Ref. 19 is used in this computer program. The Jacobi Method solves simultaneously for all eigenvalues and corresponding eigenvectors. However in large structural problems when the order of K and M is large, this method would prove inefficient when we require only some eigenpairs. For such large systems more effective solution methods are available which solve for only a few specific eigenpairs (e.g., Subspace Iteration Method).

When the order of the matrices is relatively small, as in the present problem, the solution of the eigenproblem is not very expensive and the simplicity and elegance of the Jacobi solution makes it attractive.

4.6 PLOTTING OF MODE SHAPES

Having determined the nodal-point values of displacement together with frequency for each mode by the JACOBI routine, it is desired to present these results in a suitable format.

We are first interested in determining the coordinates of the points on the plate where the displacement w is zero. This is achieved by considering the chordwise stations, namely, $\overline{1,6}$; $\overline{7,11}$; $\overline{12,15}$; $\overline{16,18}$; $\overline{19,20}$ and the spanwise stations $\overline{1,21}$; $\overline{2,20}$; $\overline{3,18}$; $\overline{4,15}$; $\overline{5,11}$. Along each of these stations the displacement at each node is scanned as we move from one node to the next. If a change of sign is observed between two consecutive nodes, the point of zero displacement is interpolated from the coordinates of the nodes and the values of the displacement at each node. If i and i+1 denote the two consecutive nodes with coordinates (Xi,Yi) and (Xi+1,Yi+1) respectively and the displacements Wi and Wi+1, the coordinates of zero displacement are given by (using linear interpolation):

$$x_n = x_i + (x_{i+1} - x_i) * |w_i| / (|w_i| + |w_{i+1}|)$$

$$y_n = y_i + (y_{i+1} - y_i) * |w_i| / (|w_i| + |w_{i+1}|)$$

The MODSHP plotting routine executes the following steps for each mode:

1. Determines the points of zero displacement along all the chordwise and spanwise stations.

2. Plots the geometry of the plate in dotted lines.

3. Designates the points of zero displacement with a symbol.

4. Using the displacements at the nodes on the boundary of the plate, it plots the deflected geometry in solid lines.

5. Prints the values of the different parameters and also the frequency.

All operations, including the generation of the sequence of node numbers along the stations for any value of NY, are automatic.

The following variables have to be transfered to the MODSHP subroutine from the main program:

NY: No. of generating lines

NV: No. of modes to be plotted

XX: Matrix of eigenvectors

COORDS : (X,Y) coordinates of the nodes

MAXNOD : MAX. No. of nodes in the mesh

THETA : Sweepback angle θ

EIG : Vector of eigenvalues

NCASE : Problem identification

PHI : Aspect Ratio

HX,HY : Thickness Taper Ratio

R (SPAN/ROOTCORD) =1.5

T.E. SWEEPBACK ANGLE=0.0°

T1/T0=0.0 T2/T0=0.5

N.D. FREQ FOR MODE 1=0.384

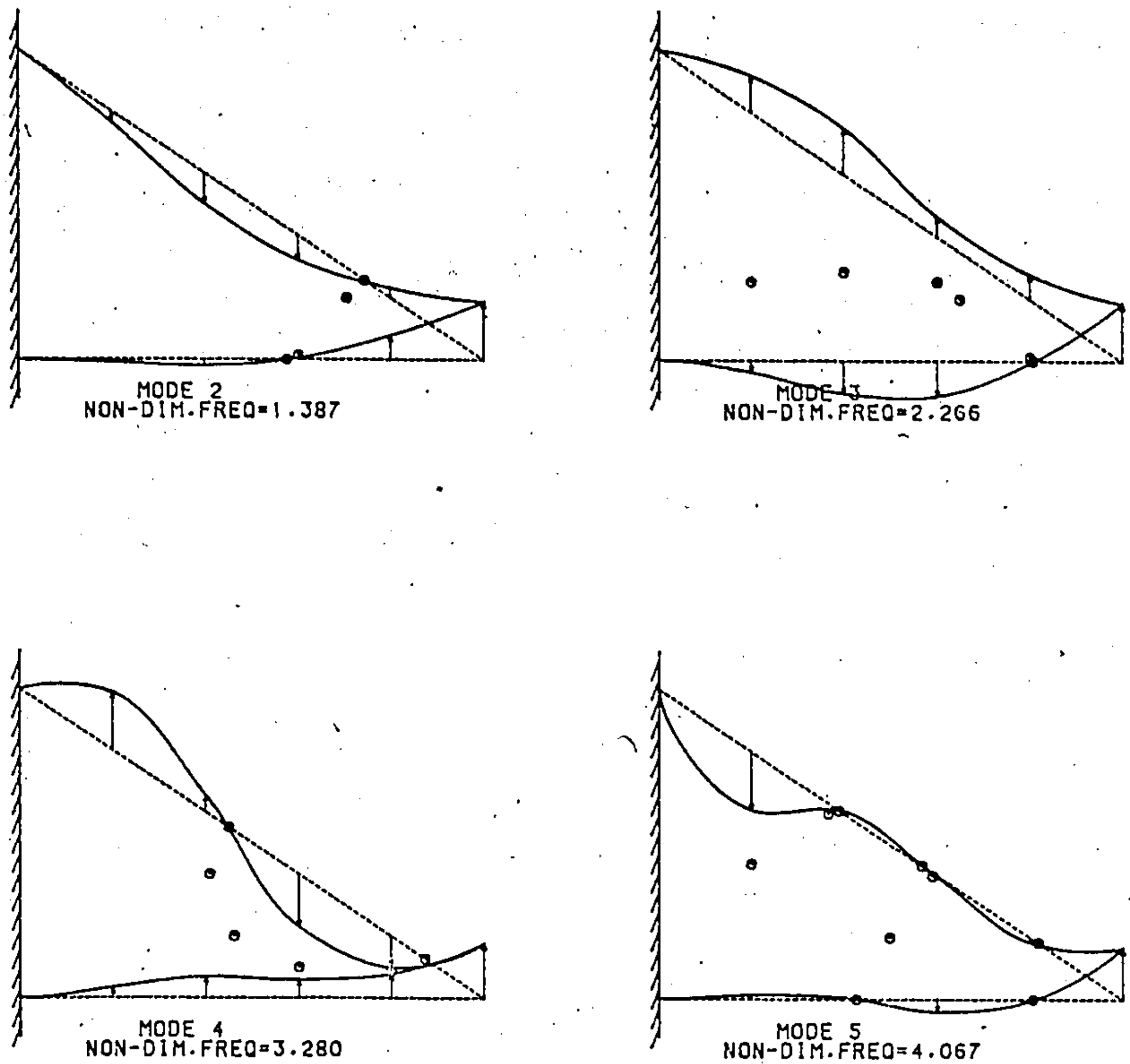


Figure 9: SAMPLE PLOT FROM "MODSHP" ROUTINE

4.7 VERIFICATION

The relatively simple problem of the natural frequencies of a cantilevered rectangular plate is a convenient example to study the verification and performance of the program as some results are available in literature [18].

The Finite Element solutions were so designed that the effects of the mesh pattern, fineness of subdivision, different orders numerical integration scheme, single precision and double precision could be studied. Table 5 shows the eight differently oriented meshes and/or different fineness ratios considered.

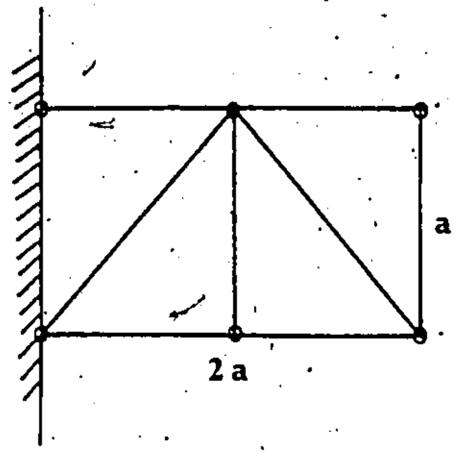
A very coarse division into only four elements, mesh (a), for which results are available using the same element, Ref[18], confirms the proper functioning of the program.

It can be concluded from the table of frequencies for the various mesh patterns that the numerical results do show a converging trend though as expected not monotonic, for the mesh (f), (g) and (h). Although for each individual cases the results may be good but the same cannot be said about the convergence for these meshes.

Also, from Table(6) it is observed that Double Precision has negligible improvement on the accuracy of the results. The same table shows the results obtained from different integration schemes. A comparasion would show that a 3-point integration for the stiffness matrix. and 3-point integration for the mass matrix gives adequate results.

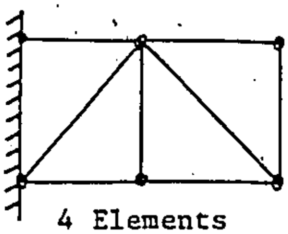
TABLE 5

VERIFICATION-PROBLEM

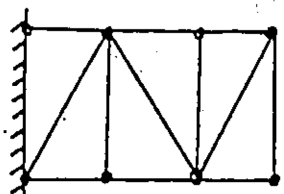


$E = 30\times10^6$ lb/in^2
$\nu = 0.3$
$a = 1$ in
$\rho = 0.283$ lb/in^3

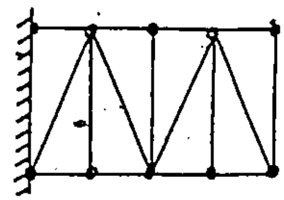
	EXACT	ZIENKIEWICZ
$\omega_1 =$	846	826.0
$\omega_2 =$	3638	3728
$\omega_3 =$	5266	5157
$\omega_4 =$	11870	12055



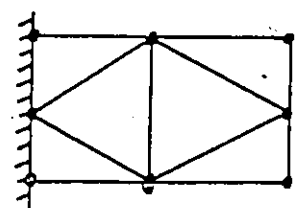
4 Elements
(a)



6 Elements
(b)

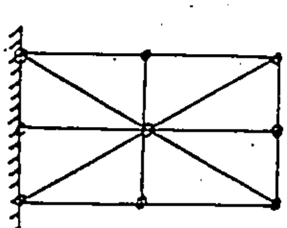


8 Elements
(c)

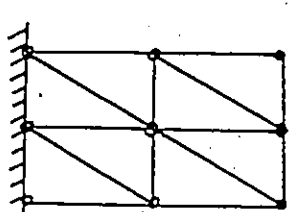


6 Elements
(d)

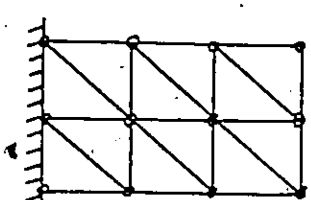
(a)	(b)	(c)	(d)
836.98		819.3	837.09
3938.75		4212.48	3564.59
4912.38		5139.81	6951.00
11895.82		11988.82	15400.00



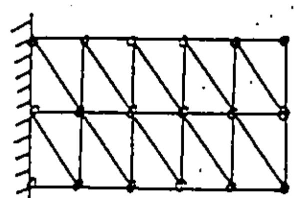
8 Elements
(e)



8 Elements
(f)



12 Elements
(g)

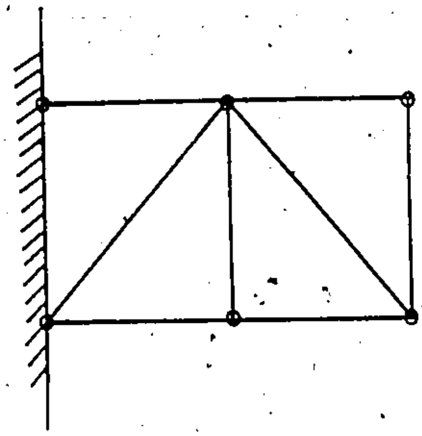


20 Elements
(h)

(e)	(f)	(g)	(h)
863.38	869.98	782.32	836.93
4385.15	3835.50	3578.60	3623.90
7614.00	7553.00	5580.00	5192.00
13678.00	13523.00	12043.00	11805.00

TABLE 6

STUDY OF DIFFERENT INTEGRATION SCHEMES



$E = 30x10^6$ lb/in^2
$\nu = 0.3$
a = 0.1 in
$\rho = 0.283$ lb/in^3

EXACT	ZIENKIEWICZ
$\omega_1 = 846$	826.0
$\omega_2 = 3638$	3728
$\omega_3 = 5266$	5157
$\omega_4 = 11870$	12055

Int. Scheme For Stiffness Matrix \ Integration Scheme For Mass Matrix	1 POINT	3 POINTS		7 POINTS
1 POINT	447.79 1101.48 3908.86 7711.24			
		SINGLE PRECISION	DOUBLE PRECISION	
3 POINTS ($^2/3$, $^2/3$, $^1/6$, 0.333...)		836.98 3938.75 4912.38 11895.82	836.97 3938.70 4912.37 11895.08	837.29 3977.39 4998.35 12487.24
3 POINTS (0.5,0.5,0.0 0.333...)		838.25 4083.19 50026.76 12398.44		

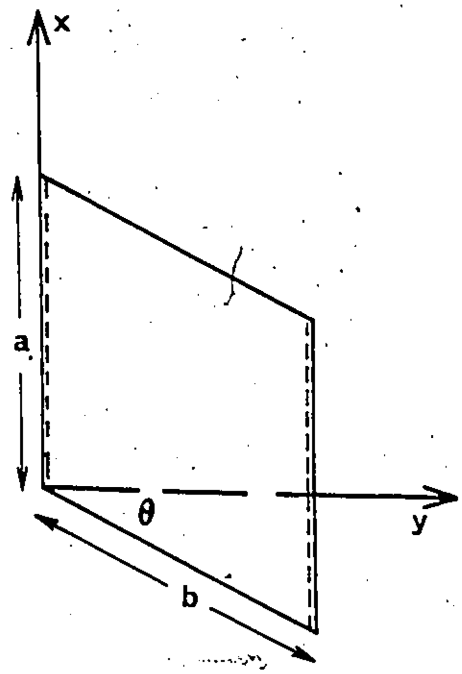
4.8 TRIAL PROBLEM

To check the suitability of the 9 deegres-of freedom element in the analysis of free vibration of skew plates, the problem of the skew-plate simply supported at the two parallel edges and free at the other two is considered. This problem has been investigated by T.Mizusawa et al. [12] using the modified Rayleigh-Ritz method with Lagarange multipliers. For a rectangular plate with similar boundary conditions the results are compared with D.Gorman's [6].

Table (7) gives a comparison between the results obtained by Mizusawa [12] and results from the present FEM study. First four modes for skew angles from 0 to 45 deg. have been investigated. For 0 deg. (square plate) the results for =0.33... have been compared with D.Gorman's in Table (8). The results for both these cases evidentally show that for this problem of skew plate the degree of accuracy obtained using the simple 9 DOF non-conforming element gives very satisfactory results for all values of the skew angle in the range of 0 to 45 degrees. The percentage dicrepancy (with results obtained by Mizusawa et al.) in the non-dimensional frequency parameter α is in the range of -1.55 to +3.1

TABLE 7

NON-DIMENSIONAL FREQUENCIES FOR SKEW-PLATE



Non Dimensional Frequency Parameter

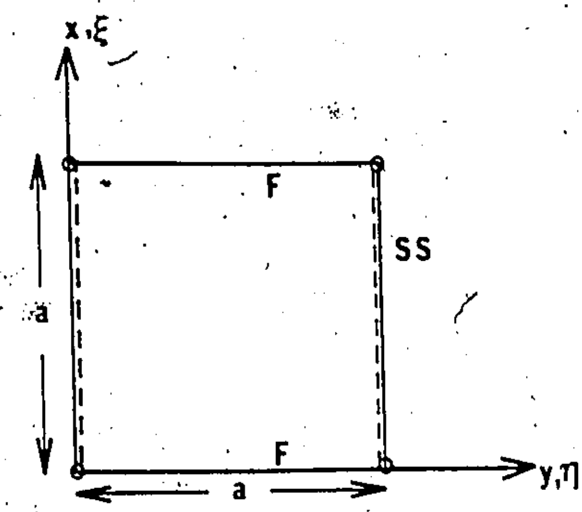
$$\alpha = \frac{\omega a^2}{\Pi^2}\sqrt{\frac{\zeta h}{D}}$$

$\phi = 1.0 \quad \upsilon = 0.3$

		MIZUSAWA [12] 1	BABU & REDDY [12] 2	PRESENT STUDY (9 D.O.F. Element) 3	% DISCREPANCY between 3 & 1
Mode 1	θ = 0°	0.9759	0.9766	1.00058	(2.52%)
	θ = 15°	1.033	1.039	1.0406	(0.73%)
	θ = 30°	1.229	1.263	1.2327	(0.3%)
	θ = 45°	1.655	1.819	1.7076	(3.1%)
Mode 2	θ = 0°	1.635	1.636	1.6663	(1.9%)
	θ = 15°	1.670	1.819	1.644	(-1.55%)
	θ = 30°	1.791	2.469	1.7648	(-1.46%)
	θ = 45°	2.056	4.084	2.0964	(1.96%)

TABLE 8

SQUARE PLATE



$$\alpha = \omega a^2 \sqrt{\frac{\rho h}{D}}$$

$\nu = 1/3$

	D.GORMAN	PRESENT STUDY (9 DOF ELEMENT)
α_1	9.568	9.803469
α_2	15.88	16.22731

Chapter V

PRESENTATION AND DISCUSSION OF RESULTS

It will be noted that there are four geometric parameters associated with the triangular plate of arbitrary planform with linear thickness variation along the two coordinate directions. These are:

1. Aspect Ratio (PHI=b/a)
2. Sweepback Angle (THETA)
3. Thickness Taper Ratio Along the X-Axis (=T1/T0)
4. Thickness Taper Ratio Along the Y-Axis (=T2/T0)

In the current study, three types of boundary conditions have been considered. These are:

1. Clamped-Free-Free Plate
2. Plate Clamped Along One Edge and With Simple Supprot at Opposite Tip.
3. Support Along Chord at an Arbitrary Location.

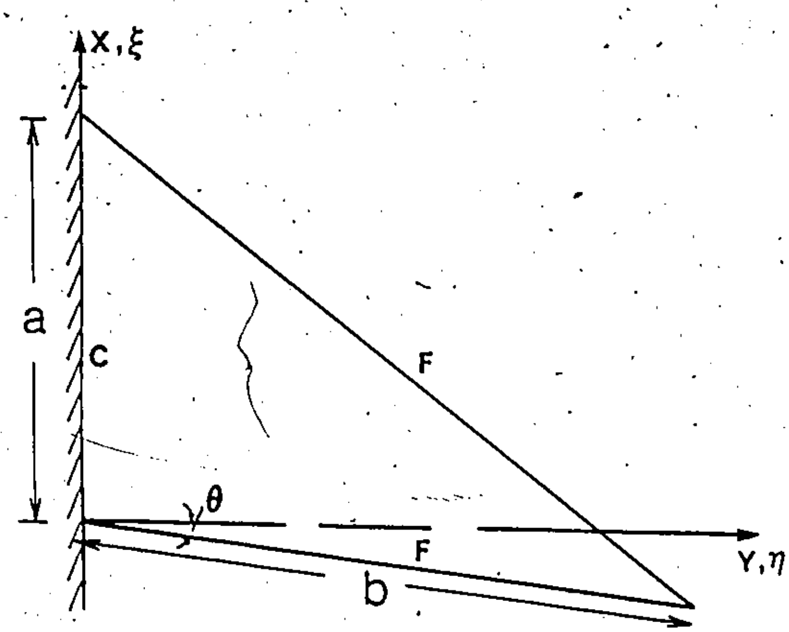


Figure 10: CLAMPED-FREE -FREE PLATE

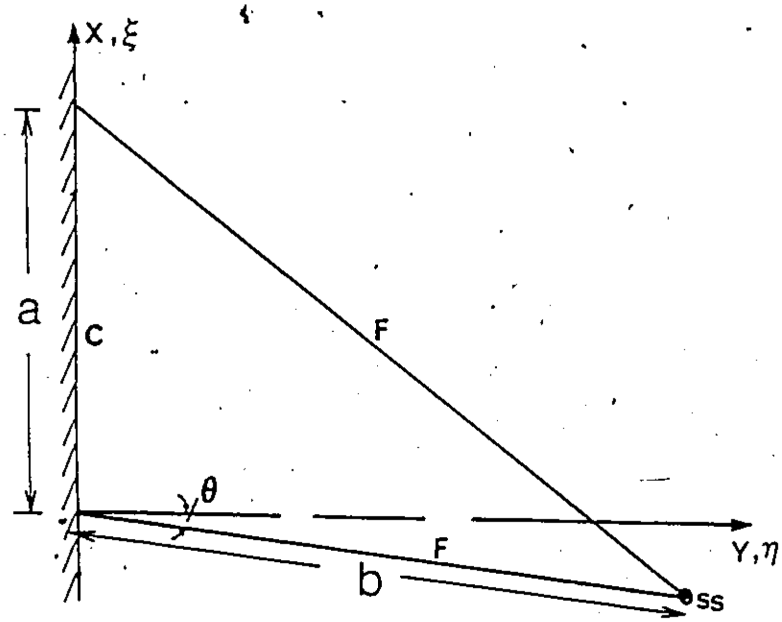


Figure 11: PLATE WITH SIMPLE SUPPORT AT TIP

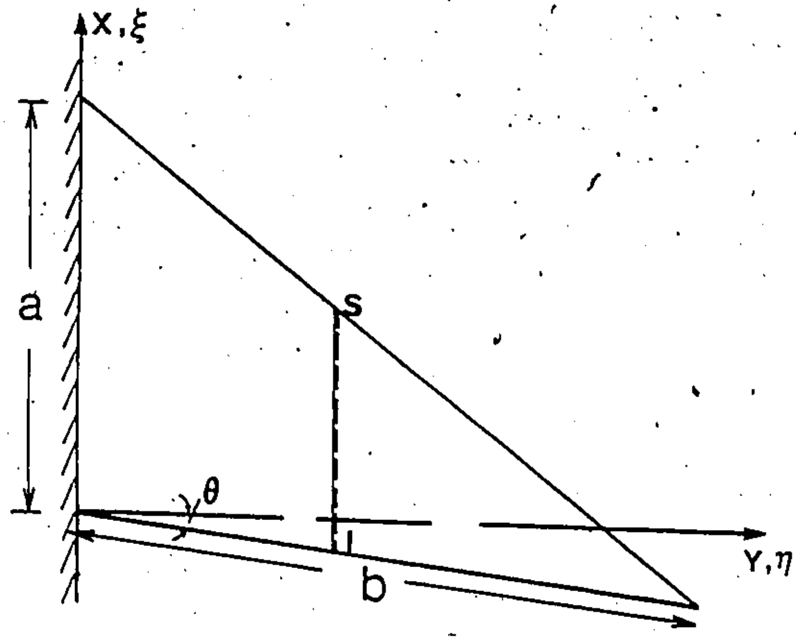


Figure 12: PLATE WITH INTERMEDIATE SUPPORT ALONG $\overline{IS}$

In the present study two locations of the intermediate support have been considered,

(1) (2/3)*b from the root

(2) (1/3)*b from root

In the Finite Element Analysis, before proceeding with the computations for a general case, it is important to examine the convergence of the solution as the number of the elements is increased. This is particularly essential for non-conforming elements because as mentioned earlier the convergence may not be monotonic for such cases. This study is important in order to determine the required degree of mesh-fineness for accurate results. As expected, it is found that convergence rates could differ depending upon the boundary conditions. It will be noted that although a mesh of 25 elements will provide good convergence for cantilever and simple-support-at tip cases, tables (9,10,11), the same mesh-fineness is found to be inadequate for the pin-pin-pin case, table (17). For this particular boundary condition 49 elements are required to give very good results. This study is presented in Tables (9,13).

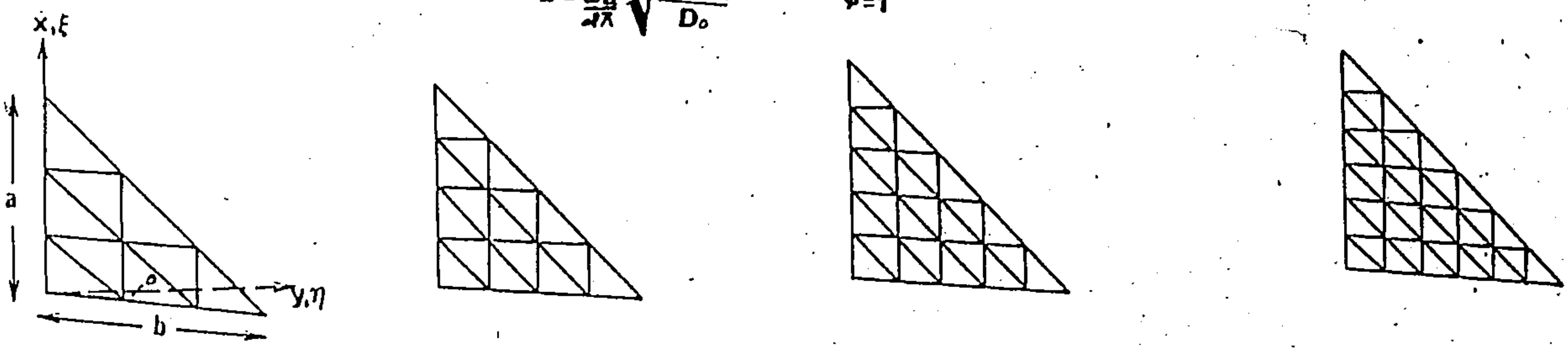
It can be appreciated that a definite converging trend is observed in all cases considered and that convergence is faster for larger aspect ratio table (10). Also, for lower sweepback angles and lower modes a monotonic convergence is observed whereas for higher angles the convergence is not always monotonic. This type of non-monotonic convergence would be, in fact, expected when using non-conforming elements.

A study of the convergence for different taper-ratios along the span is also presented. Again a mesh of 25 elements is found to provide adequate accuracy tables (12,13).

TABLE 9

CONVERGENCE STUDY FOR THE FIRST THREE MODES

$$\omega^* = \frac{\omega a^2}{4\pi}\sqrt{\frac{\rho h_0 \phi}{D_0}} \qquad \phi = 1$$



CASE I: CLAMPED - FREE - FREE

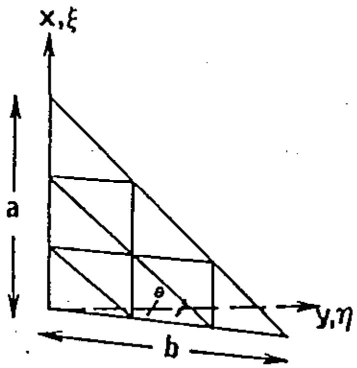
$\theta = 0^\circ$	$\omega_1^* =$	0.9780	0.9789	0.9803	0.9806
	$\omega_2^* =$	3.6140	3.6490	3.6700	3.6850
	$\omega_3^* =$	5.4200	5.3500	5.2980	5.2690
$\theta = 15^\circ$		0.9262	0.9225	0.9214	0.9208
		3.2996	3.3370	3.3610	3.3780
		5.4450	5.4150	5.4040	5.3960
$\theta = 30^\circ$		0.9285	0.9210	0.9175	0.9155
		3.2860	3.3360	3.3580	3.3730
		5.7570	5.6860	5.7210	5.7270

CASE II: SIMPLE SUPPORT AT TIP

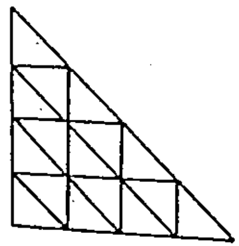
$\theta = 0^\circ$	2.1903	2.1909	2.1930	2.1946
$\theta = 15^\circ$	2.0257	2.0262	2.0278	2.0293
$\theta = 30^\circ$	2.0284	2.0182	2.0158	2.0143

CONVERGENCE STUDY FOR THE FIRST THREE MODES

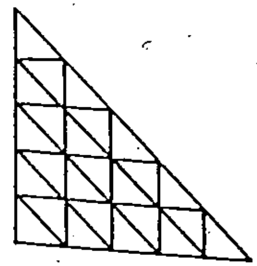
$$\omega^* = \frac{\omega a^2}{\alpha \pi}\sqrt{\frac{\rho h_o \phi}{D_o}}$$



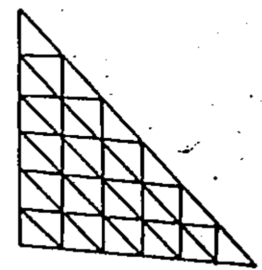
9 Elements



16 Elements



25 Elements



36 Elements

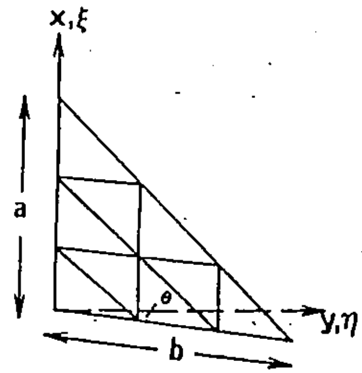
CLAMPED - FREE - FREE
PHI=1.5,T1/T0=1.0,T2/T0=1.0

		9 Elements	16 Elements	25 Elements	36 Elements
θ = 0°	ω_1^*=	0.5706556	0.5667946	0.5644583	0.5636487
	ω_2^*=	2.421236	2.401223	2.387995	2.378997
	ω_3^*=	3.821351	3.665667	3.5984496	3.562855
θ = 15°		0.5448197	0.5351644	0.5315418	0.5305679
		2.225107	2.199078	2.183338	2.174659
		3.771692	3.681818	3.663935	3.648115
θ = 30°		0.5438868	0.5316031	0.5277157	0.5245181
		2.229136	2.163352	2.137704	2.127153
		3.891870	3.863667	3.883361	3.872328

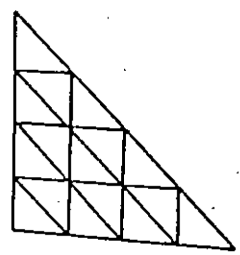
TABLE 11

CONVERGENCE STUDY FOR THE FIRST THREE MODES

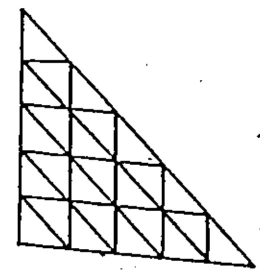
$$\omega^* = \frac{\omega a^2}{2\pi}\sqrt{\frac{\rho h_o \phi}{D_o}}$$



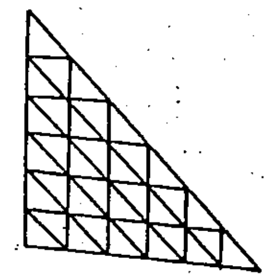
9 Elements



16 Elements



25 Elements



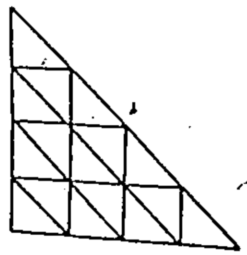
36 Elements

CLAMPED - FREE - FREE
PHI=1.0,T1/T0=0.5,T2/T0=0.5

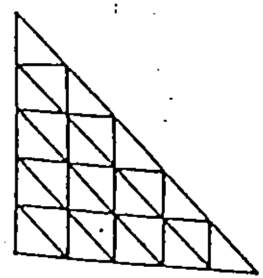
		9 Elements	16 Elements	25 Elements	36 Elements
θ = 0°	ω_1^*=	0.8249868	0.8310289	0.834	0.8352062
	ω_2^*=	2.584136	2.612841	2.626	2.633901
	ω_3^*=	3.973449	3.993781	3.994	3.994514
θ = 15°		0.7999657	0.8016699	0.802	0.802377
		2.458675	2.495823	2.513	2.522657
		4.031327	4.055263	4.076	4.084534
θ = 30°		0.8078614	0.8055599	0.804	0.8022137
		2.542387	2.57981	2.594	2.602136
		4.391871	4.305427	4.348	4.360456

TABLE 12

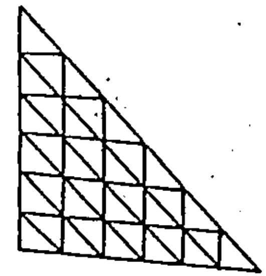
CONVERGENCE STUDY FOR THE FIRST THREE MODES



16 Elements



25 Elements



36 Elements

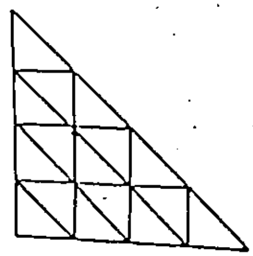
CLAMPED-FREE-FREE
PHI=1.0, θ=0°, T1/TO=1.0

$$\alpha = \frac{\omega a^2}{\alpha\pi}\sqrt{\frac{\rho h_o \phi}{D_o}}$$

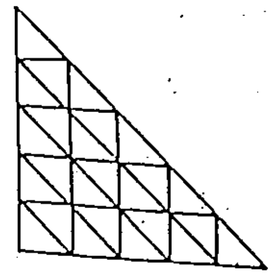
	16 Elements	25 Elements	36 Elements
	0.9787197	0.9802521	0.9785239
T2/TO = 1.0	3.648932	3.670290	3.7067192
	5.349922	5.298110	5.1813211
	1.006261	1.012816	1.0123432
T2/TO = 0.5	3.206299	3.256839	3.2591835
	4.645007	4.539062	4.5443323
	1.152066	1.176900	1.172034
T2/TO = 0.0	2.680064	2.807508	2.7867704
	3.953564	3.894597	3.8948721

CONVERGENCE STUDY FOR THE FIRST THREE MODES

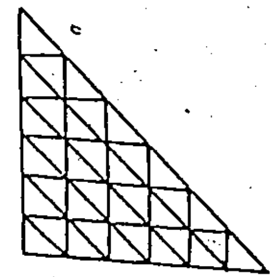
TABLE 13



16 Elements



25 Elements



36 Elements

SIMPLE SUPPORT AT TIP
PHI=1.0, $\theta=0^{\circ}$, T1/T0=1.0

$$\Omega = \frac{\omega a^2}{\alpha \pi}\sqrt{\frac{\rho h_0 \phi}{D_0}}$$

	16 Elements	25 Elements	36 Elements
	2.190894	2.209877	2.2079495
T2/T0 = 1.0	4.567907	4.494894	4.5085256
	7.199363	7.12387	7.0858833
	1.797460	1.816950	1.8084246
T2/T0 = 0.5	4.047188	4.011166	4.0183651
	5.760814	5.660016	5.62056
	1.353626	1.335699	1.3055747
T2/T0 = 0.0	3.176282	3.153546	3.0933346
	4.223971	4.073532	4.04012

It can be concluded from the convergence study that primarily the type of boundary conditions dictates the mesh fineness required for accurate results. In all further computations for cantilever and the simple support at tip boundary conditions a mesh of 25 elements has been used. Thirty-six elements have been used for the two cases with intermediate supports. Whereas for the pin-pin-pin case , 49 elements have been used. Any further increase in the number of elements used does not provide any substantial improvement in the accuracy of the results. However, if still better accuracy is justified for certain problems, a finer mesh can be used at the expense of computation time.

5.1 TEST-PROBLEMS FOR TRIANGULAR PLATE

Problem 1: Table (14) gives a comparison between the first six natural frequencies for a delta plate (PHI=1, THETA=0), clamped at the root for which some FEM results are available using other sophisticated triangular plate bending elements. Vibration experiments on such plates are reported by Gustafson et al. [9]. The agreement between the calculated and experimental frequencies appeares to be consistent and very good. The nodal line patterns associated with the first five modes are shown in Table (15). The experimental lines are shown in dotted lines. The agreement between predicted and experimental results is very good. An excellent agreement is observed between the results obtained using "high-precision" element [23] and the frequencies obtained in the present study using a relatively simple element.

<u>Problem</u> <u>2</u>: Using the trilinear coordinates, R.Williams et al. [7] have developed the solution for the natural frequencies of a simple- supported equilateral triangular plate.

Table (16) shows a comparision between the results in [7] and the results obtained in this thesis using the present FE program for the same problem. As can be seen from table (16) there is excellent agreement on the frequencies.

<u>Problem</u> <u>3</u>: Another problem studied is the delta (right angle) trianglular plate simply supported along the three edges. Table (17) shows that for this case more elements are required for good convergence. The results obtained using 49 elements are found to be in very close agreement with the results obtained by D.Gorman [private communication].

TABLE 14

NATURAL FREQUENCIES OF CANTILEVERED DELTA TRIANGULAR PLATE

Non-dimensional Frequency $= \frac{\omega a^2}{2\pi}\sqrt{\frac{\rho h}{D}}$

MODE	EXPT. VALUE (REF.9)	PRESENT ANALYSIS	HIGH PRECISION ELEMENT (REF.23)	SHELL ELEMENT REF. ( )	DKT ELEMENT (REF.17)	HSM ELEMENT (REF.17)
		N = 5	N = 3	N = 4	N = 4	N = 4
1	0.9238393	0.980252 (6.1%)*	0.980072 (6.08%)**	0.9827507	0.9238393	0.9318726
2	3.6418013	3.670290 (0.78%)	3.7301685 (2.42%)	3.5748564	3.149087	3.2481654
3	5.0878106	5.298110 (4.13%)	5.1976002 (2.17%)	4.7611196	4.1666491	4.3942617
4	8.702834	8.899158 (2.25%)	8.9384799 (2.7%)	8.6492781	7.2595024	7.7281166
5	11.809076	11.85497 (0.39%)	12.194678 (3.26%)	10.978959	8.8688573	9.7793076
6	15.477655	15.830580	15.88468 (2.63%)	13.8442	10.810258	12.368735

* Percent discrepancy between 1 & 2

** Percent discrepancy between 1 & 3

TABLE 15

MODE-SHAPES OF CANTILEVERED DELTA PLATE

$\phi = 1.0$

$\theta = 0°$

$T_2/T_0 = 1.0$

$T_1/T_0 = 1.0$

--- EXPT.

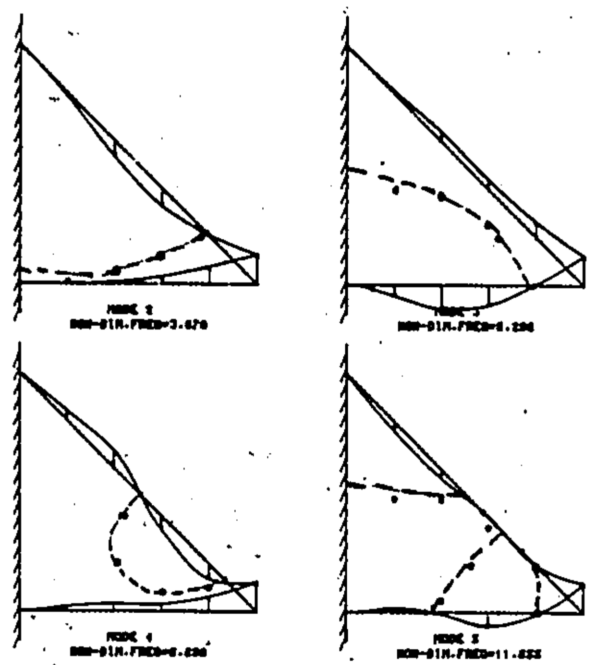
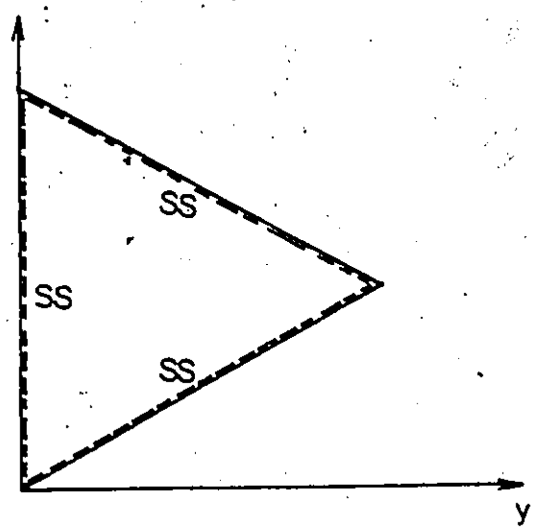


TABLE 16

NATURAL FREQUENCIES OF SIMPLE-SUPPORTED EQUILATERAL PLATE

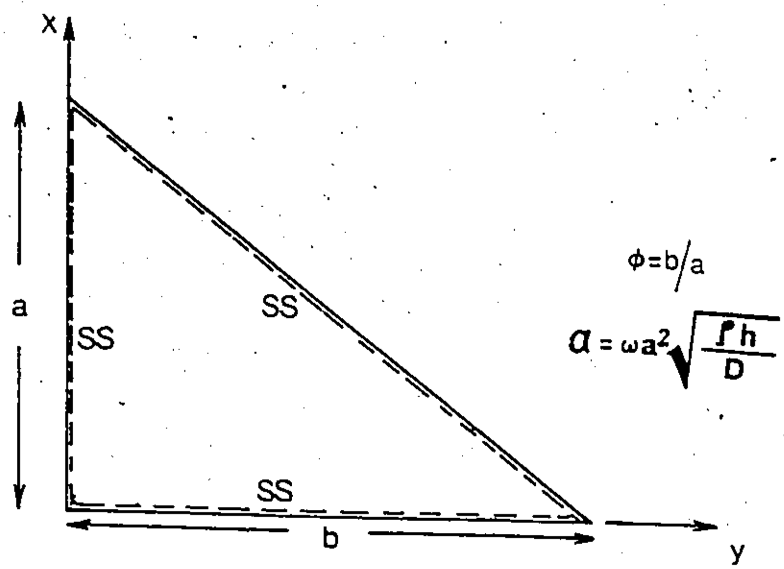


$$\alpha_{mn} = \frac{\omega_{mn}}{2\pi} a^2 \sqrt{\frac{\rho h}{D}}$$

MODE	R. WILLIAMS ET AL [Ref. 7]	PRESENT STUDY	% DISC.
m=n=1	8.380	8.295991	
n=1 m=2	19.5308	19.98189	
n=2 m=1	19.5308	20.34328	
m=2 n=2	33.32	32.32584	

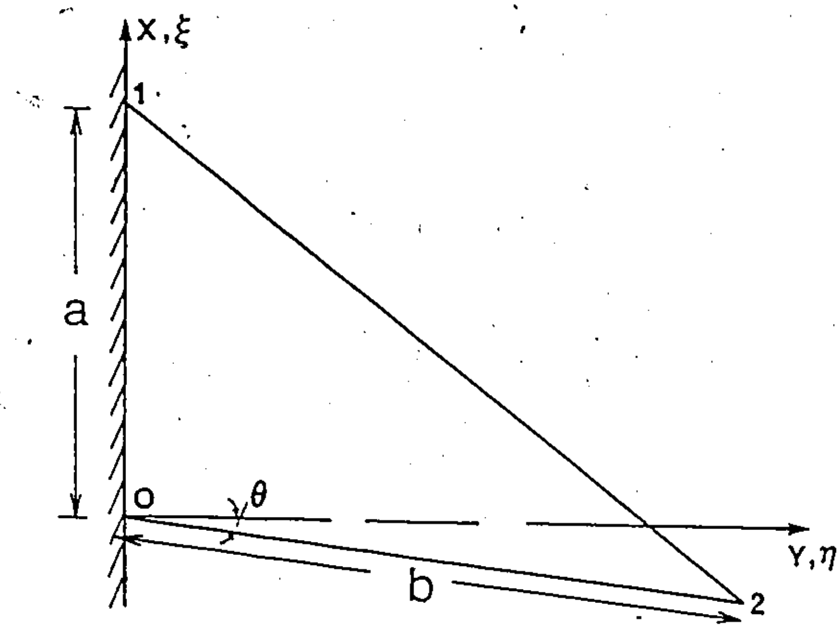
TABLE 17

FREQUENCIES FOR SIMPLE-SUPPORTED RIGHT TRIANGULAR PLATE



	D. GORMAN 51	PRESENT METHOD 25 ELEMENTS	PRESENT METHOD 36 ELEMENTS	PRESENT METHOD 49 ELEMENTS
$\phi = 1.0$	$\alpha_1 =$ 49.35	46.7967	48.24670	48.51588
	$\alpha_2 =$ 98.70	91.6803	94.52025	95.70825
	$\alpha_3 =$ 128.30	113.4275	120.6834	122.3324
	$\alpha_4 =$ 167.80	134.1078	153.9639	160.5054
$\phi = 1.5$	34.28		33.5009	33.6928
	65.59		63.1047	63.9128
	91.85		85.4672	87.3232
	106.40		103.1617	104.3955
$\phi = 2.0$	27.76		27.0901	27.2546
	49.89		48.2962	48.8727
	74.67		69.4294	72.0442
	81.32		76.9656	78.2961

A large body of data has been generated for each case of the boundary conditions. Three values of Aspect Ratio (1.0,1.5 and 2.0), four values of the Sweepback angle (0,15,30,45) and three values (1.0,0.5 and 0.0) for each Thickness Taper Ratio have been considered. For all possible combination of these four parameters, and for various boundary conditions, the first six natural frequencies and the corresponding mode shapes are computed. The computed non-dimensional parameter α, defined as $\frac{\omega a^2}{2\pi}\sqrt{\frac{\rho h_0 g}{D_0}}$ is suitably tabulated to give a comparative study of the effect of sweepback, change in T1/T0 and change in aspect ratio for a fixed T2/T0. These frequencies are presented in Tables (18-29). Computer plots of the mode shapes for each case is plotted giving the configuration of the deflected plate and the 'nodal points'. All computations are done with value of Poisson's ratio of 0.3. A typical mesh for each case of the boundary conditions is shown in Figs. (10,11,12,13).



PHI = a/b

THETA = θ

$T_2/T_0 = h|_2/h|_0$

$T_1/T_0 = h|_1/h|_0$

$$\alpha = \frac{\omega a^2}{2\pi}\sqrt{\frac{\rho h_o \phi}{D_o}}$$

TABLE 18

CLAMPED - FREE - FREE

T2/TO = 1.0

PHI	T1/TO = 1.0				T /TO = 0.5				T1/TO = 0.0			
	0°	15°	30°	45°	0°	15°	30°	45°	0°	15°	30°	45°
1.0	0.980	0.921	0.917	0.950	0.806	0.784	0.799	0.838	0.678	0.672	0.700	0.739
	3.670	3.361	3.358	3.656	2.971	2.824	2.937	3.318	2.334	2.355	2.584	3.036
	5.298	5.404	5.722	6.504	4.723	4.823	5.090	5.849	4.135	4.175	4.473	5.308
	8.899	8.260	8.617	10.018	7.131	6.879	7.476	9.119	5.045	5.266	6.043	7.896
	11.855	10.959	10.730	11.612	9.930	9.268	9.246	10.008	7.191	6.967	7.418	8.724
	15.831	15.898	16.373	18.170	13.182		12.988	15.058	9.789	9.092	9.112	10.560
1.5	0.564	0.532	0.528	0.545	0.451	0.439	0.447	0.466	0.365	0.368	0.381	0.397
	2.388	2.183	2.138	2.231	1.963	1.846	1.868	2.002	1.587	1.563	1.649	1.810
	3.598	3.664	3.883	4.411	3.064	3.147	3.346	3.875	2.643	2.729	2.925	3.484
	6.126	5.731	5.840	6.390	5.179	4.990	5.253	5.761	4.082	4.154	4.668	5.196
	8.547	7.954	7.990	9.110	7.031	6.563	6.650	7.860	5.430	5.14	5.296	6.631
	12.514	11.897	11.646	12.156	10.407	9.931	9.877	10.641	7.824	7.419	7.636	8.818
2.0	0.377	0.358	0.355	0.368	0.297	0.290	0.296	0.309	0.238	0.239	0.247	0.260
	1.660	1.545	1.516	1.565	1.384	1.315	1.324	1.394	1.151	1.128	1.171	1.253
	2.870	2.892	3.067	3.453	2.375	2.426	2.597	3.009	2.013	2.084	2.252	2.683
	4.376	4.195	4.309	4.762	3.792	3.712	3.875	4.202	3.227	3.263	3.493	3.734
	6.908	6.439	6.583	7.719	5.691	5.329	5.501	6.677	4.500	4.299	4.554	5.785
	9.441	9.289	9.255	9.529	8.369	8.312	8.308	8.585	6.981	6.767	7.052	7.658

TABLE 19

CLAMPED - FREE - FREE

T2/T0 = 0.5

PHI	T1/T0 = 1.0				T1/T0 = 0.5				T1/T0 = 0.0			
	0°	15°	30°	45°	0°	15°	30°	45°	0°	15°	30°	45°
1.0	1.008	0.940	0.922	0.934	0.834	0.802	0.804	0.822	0.703	0.697	0.711	0.731
	3.222	2.955	2.934	3.113	2.626	2.513	2.594	2.827	2.144	2.181	2.349	2.606
	4.611	4.679	4.940	5.639	3.994	4.076	4.348	5.104	3.404	3.513	3.888	4.778
	7.247	6.815	7.141	8.065	5.839	5.769	6.355	7.360	4.307	4.646	5.482	6.768
	9.518	8.785	8.726	9.943	7.805	7.265	7.279	8.492	5.578	5.605	6.090	7.338
	13.959	13.245	13.293	14.366	10.723	10.446	10.825	12.349	7.300	6.917	7.485	9.813
1.5	0.582	0.543	0.531	0.536	0.466	0.450	0.450	0.456	0.384	0.382	0.388	0.395
	2.022	1.856	1.808	1.857	1.662	1.573	1.579	1.648	1.387	1.369	1.413	1.482
	3.238	3.253	3.398	3.796	2.678	2.735	2.911	3.369	2.266	2.370	2.606	3.129
	4.805	4.560	4.709	5.235	4.036	3.970	4.195	4.629	3.280	3.447	3.764	4.136
	6.925	6.447	6.494	7.195	5.525	5.224	5.422	6.341	4.067	4.023	4.487	5.725
	9.341	8.829	8.670	9.252	8.157	7.584	7.474	8.047	5.865	5.645	5.978	6.962
2.0	0.390	0.366	0.358	0.362	0.307	0.297	0.297	0.301	0.250	0.249	0.253	0.258
	1.389	1.294	1.263	1.292	1.146	1.093	1.093	1.132	0.967	0.950	0.970	1.006
	2.606	2.574	2.651	2.862	2.106	2.127	2.247	2.518	1.783	1.849	2.008	2.312
	3.400	3.321	3.512	4.060	2.895	2.895	3.079	3.498	2.476	2.554	2.738	3.088
	5.555	5.186	5.268	5.627	4.419	4.216	4.444	5.045	3.365	3.382	3.801	4.572
	7.015	6.758	6.651	7.395	6.127	5.940	5.803	6.330	5.145	4.960	5.031	5.628

TABLE 20

CLAMPED - FREE - FREE

T2/TO = 0.0

PHI	T1/TO = 1.0				T1/TO = 0.5				T1/TO = 0.0			
	0°	15°	30°	45°	0°	15°	30°	45°	0°	15°	30°	45°
1.0	1.155	1.072	1.043	1.043	0.995	0.954	0.947	0.958	0.931	0.912	0.915	0.928
	2.690	2.481	2.441	2.519	2.230	2.152	2.192	2.308	2.085	2.075	2.141	2.259
	3.912	3.857	4.004	4.470	3.233	3.280	3.558	4.125	2.958	3.186	3.606	4.174
	5.077	4.983	5.321	6.137	4.242	4.402	4.815	5.678	4.033	4.261	4.767	5.923
	6.725	6.319	6.550	7.641	5.209	5.110	5.657	7.020	4.213	4.653	5.668	7.118
	9.114	8.608	8.505	9.518	7.270	6.986	7.149	8.252	5.569	5.862	6.507	8.191
1.5	0.669	0.621	0.601	0.599	0.557	0.535	0.529	0.531	0.512	0.503	0.502	0.505
	1.640	1.510	1.463	1.478	1.339	1.277	1.270	1.298	1.226	1.200	1.207	1.226
	2.775	2.606	2.565	2.648	2.206	2.159	2.216	2.347	2.026	2.062	2.154	2.258
	3.214	3.245	3.466	3.933	2.688	2.795	3.026	3.530	2.552	2.731	3.055	3.550
	4.678	4.360	4.447	4.993	3.605	3.561	3.857	4.400	3.122	3.387	3.770	4.326
	5.881	5.774	5.916	6.748	4.790	4.703	4.862	5.734	3.859	4.018	4.553	5.668
2.0	0.450	0.420	0.407	0.407	0.367	0.354	0.350	0.353	0.334	0.329	0.329	0.331
	1.126	1.047	1.018	1.032	0.908	0.870	0.866	0.887	0.822	0.806	0.809	0.818
	2.063	1.912	1.864	1.896	1.627	1.567	1.583	1.652	1.468	1.461	1.495	1.541
	2.465	2.530	2.703	2.959	2.026	2.128	2.306	2.587	1.965	2.097	2.294	2.499
	3.404	3.223	3.315	3.813	2.666	2.654	2.852	3.305	2.393	2.533	2.777	3.312
	4.570	4.519	4.782	5.667	3.602	3.612	3.879	4.656	3.018	3.218	3.665	4.248

TABLE 21

SIMPLE SUPPORT AT TIP

T2/To - 1.0

PHI	T1/TO = 1.0				T1/TO = 0.5				T1/TO = 0.0			
	0°	15°	30°	45°	0°	15°	30°	45°	0°	15°	30°	45°
1.0	2.193	2.028	2.016	2.126	1.893	1.799	1.837	1.973	1.607	1.593	1.683	1.840
	4.556	4.441	4.593	5.230	3.816	3.786	4.010	4.731	3.129	3.195	3.521	4.345
	7.106	6.953	7.364	8.258	6.291	6.341	6.819	7.576	5.043	5.211	5.871	6.910
	10.081	9.224	9.304	10.869	8.007	7.453	7.696	9.359	5.904	5.867	6.346	7.892
	15.009	14.025	13.941	14.801	12.188	11.453	11.680	13.128	8.349	7.704	8.060	10.079
	16.038	16.016	17.258	20.405	13.870	13.847	14.831	16.708	10.975	11.713	11.265	12.638
1.5	1.309	1.219	1.201	1.244	1.131	1.077	1.084	1.136	0.975	0.959	0.988	1.042
	3.309	3.145	3.171	3.446	2.692	2.629	2.738	3.106	2.202	2.223	2.407	2.855
	4.427	4.411	4.665	5.263	3.963	3.999	4.209	4.671	3.535	3.644	3.820	4.188
	7.634	6.978	7.028	7.723	6.129	5.730	5.998	6.954	4.485	4.368	4.889	6.216
	10.126	9.508	9.336	10.216	8.846	8.208	8.037	8.758	6.917	6.470	6.579	7.323
	12.968	12.496	13.314	15.458	10.434	10.396	10.925	12.533	8.869	8.478	8.634	9.887
2.0	0.885	0.834	0.824	0.853	0.764	0.734	0.739	0.772	0.665	0.654	0.670	0.703
	2.590	2.418	2.406	2.540	2.112	2.032	2.091	2.292	1.745	1.737	1.850	2.107
	3.176	3.264	3.534	4.130	2.806	2.894	3.101	3.566	2.509	2.606	2.766	3.124
	5.951	5.511	5.557	5.862	4.947	4.660	4.877	5.351	3.913	3.811	4.200	4.897
	7.740	7.368	7.315	8.391	6.780	6.433	6.306	7.133	5.735	5.483	5.453	6.138
	10.932	10.591	11.593	13.712	9.167	9.035	9.810	11.477	7.270	7.287	7.801	9.429

TABLE 22

SIMPLE SUPPORT AT TIP

T2/T0 = 0.5

PHI	T1/T0 = 1.0				T1/T0 = 0.5				T1/T0 = 0.0			
	0°	15°	30°	45°	0°	15°	30°	45°	0°	15°	30°	45°
1.0	1.791	1.663	1.646	1.706	1.549	1.481	1.500	1.573	1.354	1.341	1.388	1.455
	4.031	3.856	3.949	4.416	3.300	3.264	3.478	4.044	2.690	2.813	3.165	3.812
	5.649	5.629	5.977	6.765	4.99	5.097	5.417	6.145	4.302	4.559	4.866	5.680
	8.293	7.639	7.856	9.216	6.448	6.118	6.588	8.158	4.543	4.680	5.478	7.227
	11.867	11.003	10.672	11.247	9.781	9.180	9.197	9.912	6.286	6.192	7.024	8.710
	13.105	12.968	13.977	16.427	10.948	11.069	11.937	13.808	8.537	8.462	8.678	10.430
1.5	1.052	0.980	0.962	0.985	0.900	0.861	0.862	0.887	0.784	0.774	0.787	0.8096
	2.858	2.639	2.604	2.744	2.309	2.220	2.281	2.482	1.912	1.938	2.080	2.310
	3.566	3.648	3.905	4.444	3.113	3.207	3.418	3.917	2.756	2.865	3.065	3.604
	6.062	5.564	5.609	6.138	4.817	4.588	4.859	5.470	3.530	3.660	4.226	4.938
	7.882	7.487	7.491	8.443	6.671	6.274	6.293	7.220	5.061	4.862	5.137	6.294
	10.430	10.155	10.971	12.187	8.236	8.275	8.906	10.374	6.380	6.414	6.774	8.323
2.0	0.709	0.667	0.656	0.673	0.602	0.580	0.580	0.598	0.526	0.518	0.526	0.540
	2.095	1.937	1.898	1.965	1.729	1.646	1.663	1.763	1.460	1.452	1.513	1.620
	2.731	2.818	3.059	3.503	2.297	2.403	2.613	3.041	2.009	2.127	2.326	2.773
	4.538	4.229	4.241	4.556	3.740	3.586	3.729	4.012	2.970	3.034	3.330	3.592
	6.117	5.863	6.012	7.114	5.089	4.860	4.980	5.999	4.100	3.973	4.198	5.293
	8.453	8.323	9.214	10.676	6.841	6.918	7.762	8.932	5.380	5.573	6.177	7.390

TABLE 23

SIMPLE SUPPORT AT TIP

T2/T0 = 0.0

PHI	T1/T0 = 1.0				T1/T0 = 0.5				T1/T0 = 0.0			
	0°	15°	30°	45°	0°	15°	30°	45°	0°	15°	30°	45°
1.0	1.314	1.216	1.187	1.212	1.129	1.081	1.076	1.106	1.043	1.026	1.031	1.057
	3.104	2.849	2.815	2.996	2.538	2.454	2.513	2.713	2.354	2.368	2.445	2.604
	4.130	4.156	4.380	5.025	3.515	3.513	3.864	4.595	3.215	3.433	3.910	4.680
	5.709	5.414	5.724	6.575	4.552	4.650	5.167	6.061	4.125	4.635	5.144	6.277
	7.331	6.891	7.078	8.310	5.756	5.484	5.873	7.380	4.741	4.771	5.709	7.625
	9.410	9.287	9.919	11.040	7.419	7.541	8.007	9.349	5.925	6.244	7.105	8.993
1.5	0.767	0.712	0.693	0.699	0.636	0.612	0.608	0.617	0.578	0.571	0.572	0.579
	1.987	1.812	1.766	1.822	1.603	1.529	1.528	1.591	1.451	1.432	1.440	1.473
	2.949	2.968	3.098	3.501	2.417	2.479	2.643	3.042	2.260	2.388	2.576	2.811
	3.996	3.779	3.964	4.580	3.180	3.188	3.442	4.015	2.931	3.088	3.362	4.107
	5.459	5.154	5.228	5.957	4.245	4.060	4.291	5.089	3.462	3.633	4.216	4.889
	6.581	6.389	6.616	7.463	5.262	5.344	5.542	6.346	4.476	4.611	5.025	6.286
2.0	0.516	0.483	0.472	0.478	0.420	0.407	0.406	0.414	0.379	0.376	0.378	0.383
	1.880	1.283	1.252	1.285	1.106	1.064	1.063	1.100	0.991	0.981	0.987	1.003
	2.387	2.366	2.433	2.670	1.911	1.931	2.028	2.270	1.790	1.827	1.902	2.010
	2.936	2.871	3.102	3.711	2.350	2.408	2.658	3.192	2.204	2.352	2.613	3.185
	4.438	4.270	4.407	5.109	3.420	3.444	3.530	4.105	2.870	3.021	3.343	3.722
	5.409	5.205	5.358	6.208	4.264	4.288	4.436	5.190	3.840	3.847	4.119	5.041

TABLE 24

SIMPLE SUPPORT AT STN.
(2/3)b FROM ROOT
T2/TO = 1.0

PHI	T1/TO = 1.0				T1/TO = 0.5				T1/TO = 0.0			
	0°	15°	30°	45°	0°	15°	30°	45°	0°	15°	30°	45°
1.0	4.510	4.384	4.541	5.033	3.934	3.955	4.183	4.650	3.230	3.457	3.852	4.297
	6.467	6.037	6.384	7.841	5.238	4.966	5.386	6.909	4.091	3.992	4.439	6.073
	8.855	8.518	8.743	9.798	7.856	7.557	7.829	8.871	6.051	5.936	6.537	7.907
	13.651	13.358	13.851	15.882	10.706	10.514	10.996	12.830	7.669	7.527	8.044	9.824
	18.116	17.836	19.292	21.653	15.072	15.408	16.940	19.330	10.081	9.711	10.247	12.565
	20.649	21.175	22.444	28.080	16.470	16.205	17.513	21.695	11.651	12.571	14.570	17.143
1.5	2.572	2.508	2.604	2.884	2.223	2.234	2.359	2.612	1.902	1.990	2.142	2.364
	4.235	3.901	4.008	4.688	3.552	3.321	3.505	4.232	2.877	2.783	3.069	3.851
	5.844	5.663	5.750	6.349	5.067	4.916	5.011	5.592	4.268	4.162	4.354	5.020
	10.625	10.908	11.016	12.415	8.385	8.244	8.703	10.298	5.846	5.797	6.288	7.997
	12.159	11.363	12.185	14.749	10.177	9.997	11.024	13.123	7.944	8.194	9.213	10.865
	15.804	15.973	16.028	16.972	13.440	13.272	13.272	15.208	9.967	9.454	9.998	12.158
2.0	1.708	1.674	1.751	1.965	1.466	1.477	1.569	1.759	1.261	1.311	1.411	1.570
	3.003	2.788	2.846	3.321	2.584	2.422	2.519	2.988	2.194	2.105	2.254	2.709
	4.540	4.405	4.498	4.950	3.847	3.756	3.865	4.336	3.255	3.216	3.377	3.923
	7.759	7.947	8.719	9.867	6.567	6.859	7.474	8.535	5.122	5.193	5.684	7.059
	10.144	9.237	9.648	12.307	8.152	7.495	8.079	10.409	6.299	6.373	7.278	8.884
	13.723	13.129	12.779	13.169	11.482	11.437	11.611	12.351	8.870	8.587	9.008	11.143

TABLE 25

SIMPLE SUPPORT AT STN.
(2/3)b FROM ROOT
T2/TO = 0.5

PHI	T1/TO = 1.0				T1/TO = 0.5				T1/TO = 0.0			
	0°	15°	30°	45°	0°	15°	30°	45°	0°	15°	30°	45°
	3.708	3.540	3.578	3.869	3.214	3.164	3.244	3.522	2.710	2.828	2.953	3.217
	5.313	5.076	5.495	6.747	4.209	4.135	4.664	6.025	3.245	3.327	3.993	5.453
	7.091	6.874	7.114	8.135	6.105	5.964	6.253	7.203	4.769	4.859	5.403	6.516
1.0	11.764	11.480	11.906	13.614	8.933	8.767	9.284	11.146	5.985	6.117	6.826	8.765
	14.684	14.064	14.566	15.978	12.314	12.462	13.073	14.160	7.719	7.770	8.836	11.732
	16.565	16.538	18.150	22.057	13.484	13.815	15.004	18.367	9.250	10.113	11.525	13.147
	2.123	2.040	2.076	2.259	1.806	1.787	1.845	2.010	1.541	1.578	1.649	1.796
	3.311	3.109	3.247	3.773	2.726	2.620	2.825	3.349	2.230	2.240	2.515	3.001
	4.854	4.697	4.812	5.460	4.041	3.949	4.095	4.770	3.291	3.322	3.591	4.401
1.5	8.700	8.845	8.946	9.353	6.791	6.817	7.276	8.288	4.599	4.759	5.447	7.031
	9.994	9.109	9.812	11.446	8.113	7.865	8.493	9.679	6.041	6.377	7.471	8.457
	12.182	11.297	11.224	12.946	10.403	9.830	9.934	11.433	7.568	7.254	7.604	10.088
	1.412	1.367	1.406	1.558	1.188	1.183	1.233	1.367	1.013	1.035	1.030	1.204
	2.303	2.171	2.250	2.634	1.930	1.851	1.960	2.313	1.639	1.617	1.750	2.045
	3.826	3.703	3.792	4.227	3.128	3.066	3.191	3.669	2.590	2.625	2.835	3.413
2.0	6.148	6.283	6.733	7.120	5.106	5.422	5.980	6.459	3.942	4.189	4.830	5.760
	8.193	7.579	8.885	9.190	6.422	5.950	6.451	7.965	4.728	4.839	5.517	6.933
	9.637	8.781	8.523	10.477	8.458	7.799	7.614	8.758	6.680	6.389	6.658	7.624

TABLE 26

SIMPLE SUPPORT AT STN.
(2/3)b FROM ROOT.
T2/T0 = 0.0

PHI	T1/T0 = 1.0				T1/T0 = 0.5				T1/T0 = 0.0			
	0°	15°	30°	45°	0°	15°	30°	45°	0°	15°	30°	45°
1.0	2.844	2.666	2.647	2.780	2.425	2.350	2.383	2.532	2.235	2.238	2.311	2.472
	4.058	4.011	4.404	5.199	3.170	3.285	3.786	4.622	2.725	2.948	3.427	4.137
	5.434	5.260	5.532	6.649	4.389	4.373	4.739	5.910	3.766	4.004	4.647	6.008
	7.534	7.142	7.198	8.025	6.273	6.140	6.389	7.229	4.963	5.300	6.067	7.157
	9.650	9.567	10.203	12.080	7.046	7.153	7.895	9.917	5.821	6.081	6.823	9.089
	11.320	11.037	11.950	13.933	9.209	9.538	10.494	12.154	6.666	7.274	8.623	11.229
1.5	1.675	1.580	1.575	1.673	1.386	1.350	1.369	1.462	1.257	1.257	1.289	1.362
	2.377	2.302	2.448	2.808	1.882	1.891	2.074	2.411	1.665	1.713	1.873	2.128
	3.787	3.601	3.639	3.915	2.977	2.930	3.089	3.461	2.638	2.765	3.051	3.421
	4.634	4.391	4.375	4.885	3.814	3.727	3.787	4.275	3.409	3.481	3.681	4.249
	6.875	7.006	7.613	8.540	5.024	5.220	5.834	6.963	3.879	4.296	5.126	6.523
	8.282	7.594	8.014	10.132	6.366	6.167	6.733	8.447	4.698	5.093	5.965	7.506
2.0	1.130	1.080	1.094	1.194	0.920	0.905	0.932	1.024	0.827	0.831	0.857	0.916
	1.616	1.559	1.642	1.895	1.281	1.273	1.372	1.591	1.138	1.153	1.237	1.391
	2.940	2.750	2.712	2.832	2.294	2.231	2.296	2.477	2.074	2.108	2.227	2.405
	3.346	3.261	3.335	3.755	2.718	2.710	2.821	3.250	2.454	2.552	2.749	3.161
	4.812	4.976	5.713	6.914	3.668	3.950	4.763	3.599	3.077	3.502	4.226	4.971
	6.658	6.244	6.498	8.115	4.940	4.715	4.930	6.490	3.695	3.862	4.334	5.563

TABLE 27

SIMPLE SUPPORT AT STN.
(1/3)b FROM ROOT
$T2/T_O = 1.0$

PHI	T1/TO = 1.0				T1/TO = 0.5				T1/TO = 0.0			
	0°	15°	30°	45°	0°	15°	30°	45°	0°	15°	30°	45°
1.0	2.194	2.072	2.071	2.147	1.924	1.858	1.888	1.970	1.685	1.669	1.724	1.806
	8.175	7.478	7.469	8.176	7.128	6.663	6.816	7.653	6.110	5.886	6.208	7.174
	12.022	12.162	12.703	14.227	11.098	11.270	11.762	13.204	7.705	7.705	8.877	12.145
	20.051	18.612	19.232	22.106	17.489	16.699	17.752	20.812	10.206	10.373	10.855	12.280
	25.862	25.151	24.448	27.722			18.056	24.002	14.188	13.806	15.647	19.526
	27.102	25.916	27.668	35.473	22.942	21.310	21.565	24.806	14.568	14.431	15.915	21.506
1.5	1.271	1.204	1.195	1.226	1.094	1.059	1.069	1.104	0.942	0.934	0.958	0.992
	5.395	4.943	4.862	5.064	4.743	4.416	4.434	4.703	4.111	3.919	4.041	4.375
	8.244	8.274	8.651	10.002	7.398	7.471	7.811	9.094	6.652	6.750	7.063	8.306
	14.293	13.465	13.854	15.041	12.834	12.294	12.899	14.079	7.151	7.030	7.929	10.672
	19.251	17.851	17.864	20.735	13.962	13.186	14.037	18.212	11.277	11.048	11.922	13.133
	21.037	19.656	20.345	24.877	17.059	15.844	15.867	18.647	11.818	11.546	13.761	16.611
2.0	0.850	0.812	0.808	0.826	0.725	0.706	0.713	0.737	0.618	0.615	0.632	0.656
	3.799	3.551	3.522	3.644	3.370	3.185	3.209	3.368	2.965	2.847	2.924	3.116
	6.592	6.521	6.809	7.975	5.817	5.809	6.081	7.162	5.149	5.197	5.459	6.478
	10.551	10.297	10.874	11.901	9.587	9.439	10.072	11.103	8.645	7.338	8.902	10.270
	16.181	14.949	14.821	16.969	14.454	13.414	13.452	15.549	8.658	8.615	9.287	11.138
	20.996	18.496	19.023	22.765	14.478	13.475	13.479	16.741	12.832	12.064	12.218	14.277

TABLE 28

SIMPLE SUPPORT AT STN.
(1/3)b FROM ROOT
$T2/T_O = 0.5$

PHI	T1/TO = 1.0				T1/TO = 0.5				T1/TO = 0.0			
	0°	15°	30°	45°	0°	15°	30°	45°	0°	15°	30°	45°
	1.862	1.746	1.726	1.760	1.593	1.534	1.545	1.584	1.371	1.358	1.390	1.429
	6.172	5.659	5.627	6.039	5.239	4.947	5.079	5.589	4.387	4.330	4.617	5.205
1.0	8.960	9.029	9.420	10.697	7.994	8.111	8.500	9.780	7.045	6.974	7.695	9.067
	14.224	13.315	13.870	15.847	12.133	11.783	12.669	14.673		7.221	8.541	13.561
	18.403	17.333	17.674	19.989	15.613	14.728	15.214	17.806	9.661	9.985	11.387	13.624
	25.379	24.617	25.048	28.260	15.722	15.466	17.150	23.929	12.463	11.940	12.565	15.421
	1.080	1.014	0.996	1.009	0.902	0.870	0.869	0.884	0.760	0.753	0.765	0,778
	3.962	3.639	3.562	3.665	3.384	3.179	3.187	3.331	2.872	2.789	2.877	3.045
1.5	6.278	6.264	6.547	7.520	5.406	5.462	5.761	6.762	4.668	4.788	5.134	6.192
	9.842	9.348	9.531	10.129	8.605	8.392	8.727	9.244	5.643	5.899	7.852	8.438
	13.474	12.532	12.778	15.093	11.370	10.644	11.008	13.439	7.297	7.430	7.985	10.221
	18.861	17.612	17.498	19.290	12.152	11.712	12.659	16.781	9.283	8.801	9.297	11.908
	0.724	0.684	0.673	0.680	0.596	0.578	0.578	0.587	0.497	0.494	0.502	0.510
	2.755	2.578	2.541	2.598	2.363	2.250	2.258	2.337	2.030	1.977	2.024	2.114
2.0	5.043	4.954	5.178	6.002	4.265	4.257	4.501	5.331	3.644	3.712	3.991	4.825
	7.197	7.077	7.353	7.808	6.328	6.332	6.679	7.090	5.524	5.648	6.043	6.426
	10.988	10.113	10.241	12.187	9.447	8.796	9.003	10.820	5.631	5.844	7.162	9.674
	14.988	14.391	14.521	16.240	11.757	11.048	11.943	14.036	7.920	7.554	7.926	9.964

TABLE 29

SIMPLE SUPPORT AT STN.
(1/3)b FROM ROOT
$T2/T_O = 0.0$

PHI	T1/TO = 1.0				T1/TO = 0.5				T1/TO = 0.0			
	0°	15°	30°	45°	0°	15°	30°	45°	0°	15°	30°	45°
1.0	1.720	1.604	1.570	1.576	1.477	1.424	1.423	1.442	1.378	1.359	1.373	1.393
	4.007	3.707	3.673	3.832	3.313	3.206	3.299	3.520	3.091	3.092	3.235	3.462
	5.913	5.816	6.072	6.965	4.855	4.943	5.393	6.440	4.418	4.783	5.475	6.485
	7.788	7.691	8.192	9.367	6.506	6.796	7.426	8.574	6.083	6.538	7.263	8.813
	10.063	9.455	9.725	11.322	7.926	7.808	8.515	10.460	6.508	7.170	8.612	10.779
	14.690	13.749	13.878	15.543	11.068	10.836	11.381	13.281	6.841	8.596	9.977	12.604
1.5	1.004	0.934	0.906	0.905	0.834	0.802	0.795	0.799	0.765	0.753	0.754	0.756
	2.475	2.283	2.220	2.250	2.012	1.923	1.923	1.977	1.839	1.806	1.824	1.857
	4.155	3.907	3.865	4.001	3.298	3.241	3.351	3.571	3.021	3.105	3.286	3.463
	4.933	4.976	5.302	6.188	4.107	4.280	4.623	5.496	3.832	4.139	4.620	5.624
	7.324	6.921	7.217	8.261	5.638	5.635	6.202	7.187	4.809	5.271	5.930	6.607
	9.904	9.713	9.949	11.627	7.945	7.609	7.920	9.648	5.043	6.258	7.147	9.543
2.0	0.676	0.633	0.614	0.614	0.551	0.532	0.527	0.531	0.500	0.494	0.495	0.496
	1.713	1.600	1.564	1.586	1.376	1.324	1.326	1.366	1.243	1.223	1.232	1.249
	3.087	2.880	2.823	2.846	2.438	2.369	2.410	2.494	2.200	2.222	2.302	2.376
	3.778	3.847	4.146	4.944	3.079	3.230	3.536	4.274	2.943	3.176	3.597	4.138
	5.646	5.473	5.900	7.078	4.329	4.393	4.911	5.803	3.800	4.065	4.425	5.030
	7.449	7.578	8.079	9.575	5.975	6.043	6.362	7.772	4.901	5.134	5.862	7.623

Figs. 13-15 show the variation of fundamental frequency vs. thickness taper ratio along the trailing edge, T2/T0, for different skew angles and taper ratios along the root (T1/T0). These are presented for fixed Aspect Ratio. It is interesting to note that:

(1) With decreasing T2/T0, the frequency increases for all skew angles and T1/T0 ratios.

(2) A similar trend is observed with decreasing T1/T0. However it is noted that the spread in the fundamental frequency with change in T2/T0 reduces with decreasing T1/T0.

(3) With increasing sweepback, the frequency first decreases, attains a minimum and then rises again. This trend is observed for all T1/T0 ratios. It is also noted that with decreasing T1/T0, the minimum occurs at lower sweepback angles.

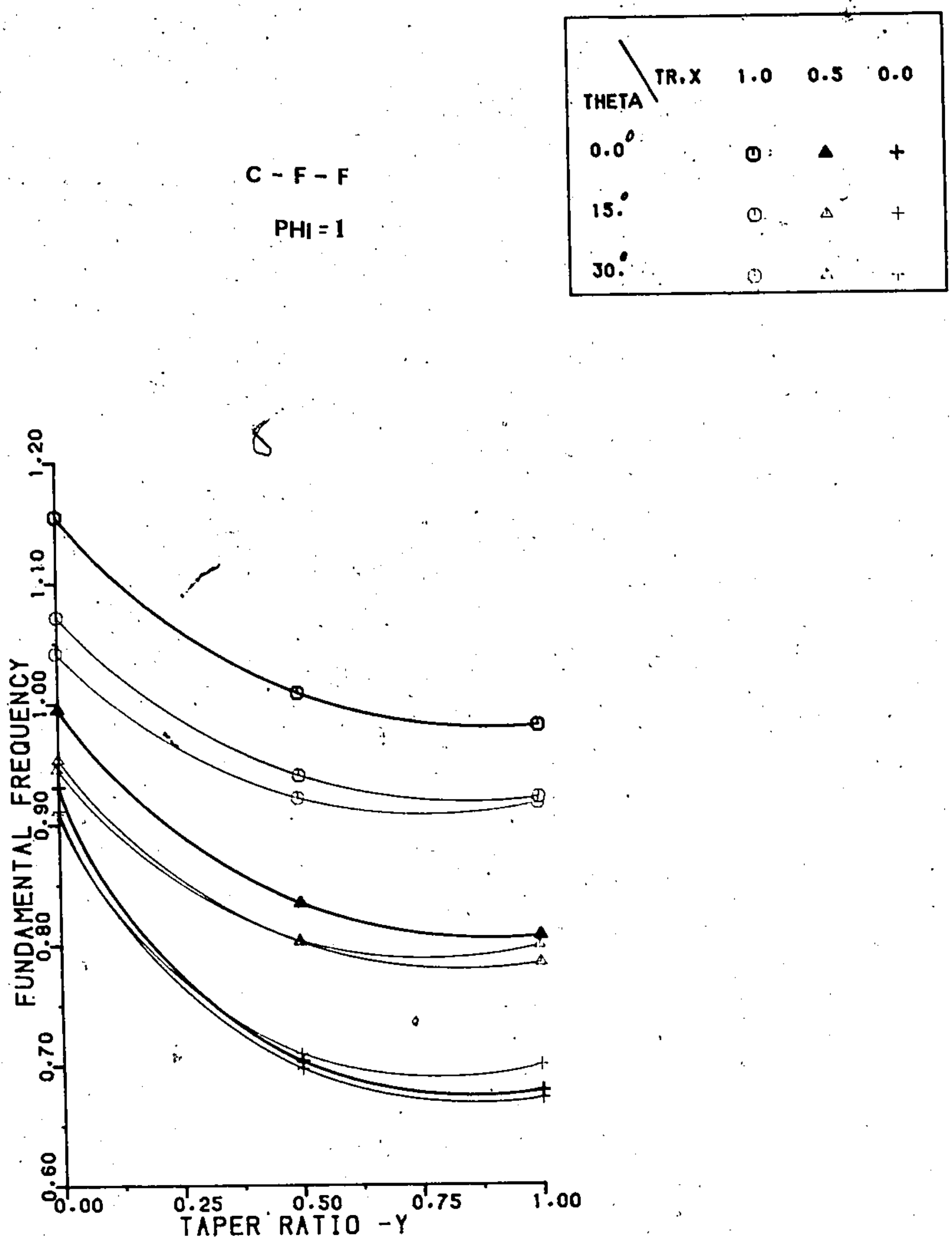
(4) The frequency shows lesser dependence on the sweepback for larger aspect ratios and decreasing T2/T0 and T1/T0.

5.2 CONCLUSIONS

A solution procedure based on the Finite Element Method has been developed for studying the free vibration of triangular plates of arbitrary planform and with linearly varying thickness. Many sample problems have been solved to evaluate the effectiveness of the method for the free vibration of plates. The method is shown to yield results with good accuracy. The results for the skew plate studied, Table (7), showed that the solution procedure was stable over a broad range of the sweepback angle (skew angle) --- 0 to 45 degrees, for the first four modes

However, no attempt was made to study the stability of the method beyond 45 degrees. Results for all trial problems (rectangular,square, equilateral and right triangular plates) show a very good agreement with the results available for these cases.

It can . thus be concluded that although a relatively simple element was used, it produces excellent results from the point of view of accuracy and computational costs. The results obtained compare remarkably well with those available for delta and equilateral triangular plates. A maximum descrepancy of six percent is observed. The performance and effectiveness of the element was thoroughly investigated and with a very good degree of confidence the extensive solutions for the current problem were developed. The non-dimensional frequency parameter and the mode shapes obtained for all the cases have been presented. For lower modes and for boundary conditions considered, the element performs satisfactorily. The solution was found to be stable over a broad range (0-45) of the sweepback angle. For plates with boundary conditions other than those considered, it would be recommended to check the convergence to select the mesh fineness required for acceptable degree of accuracy. For problems demanding still better accuracy, the use of more sophisticated elements (A-9,HSM etc.) is suggested.



THETA
TR.X
1.0
0.5
0.0
0.0°
15.°
30.°
C - F - F
PHI = 1
FUNDAMENTAL FREQUENCY
TAPER RATIO -Y
0.60
0.70
0.80
0.90
1.00
1.10
1.20
0.00
0.25
0.50
0.75
1.00

FIGURE 13

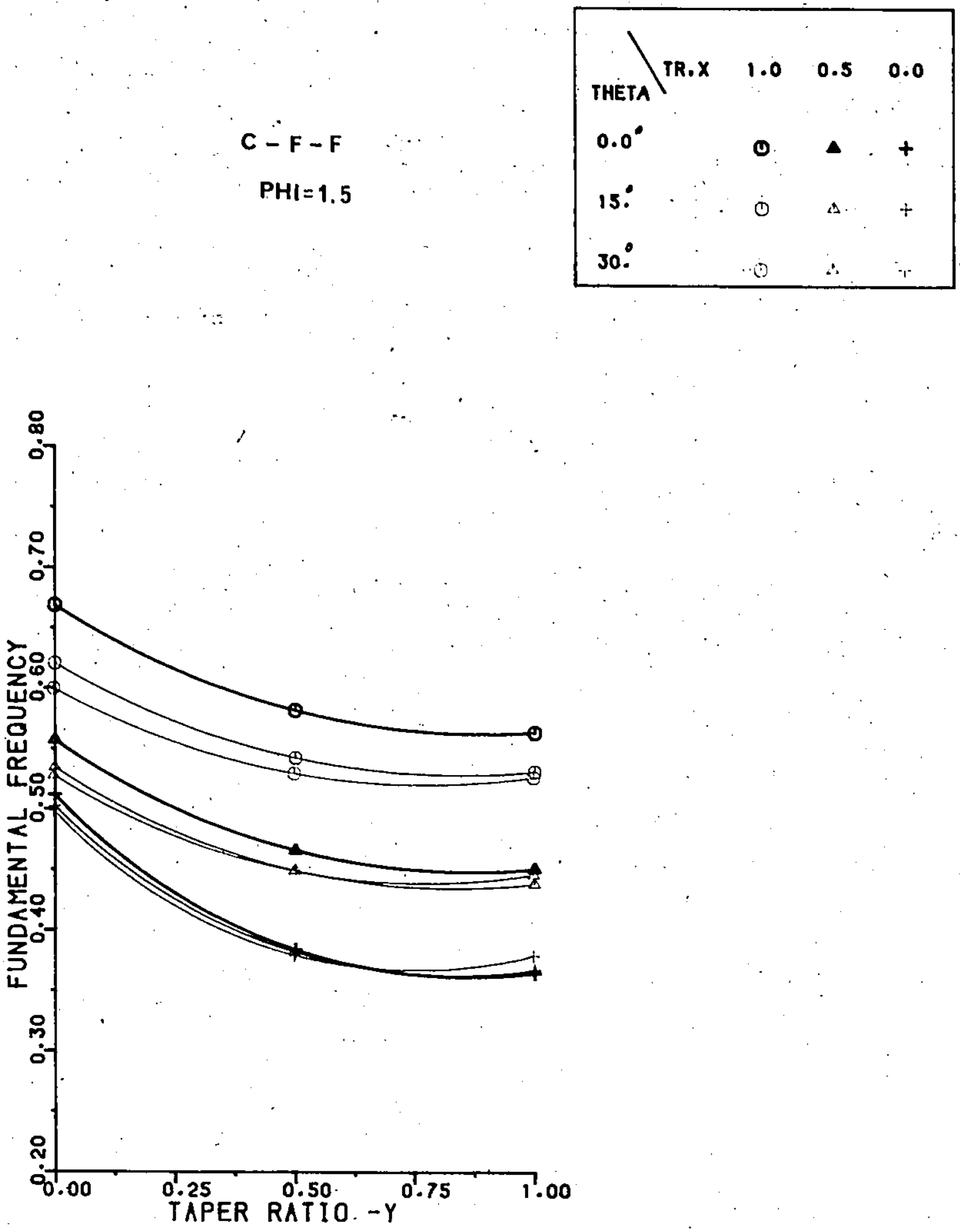
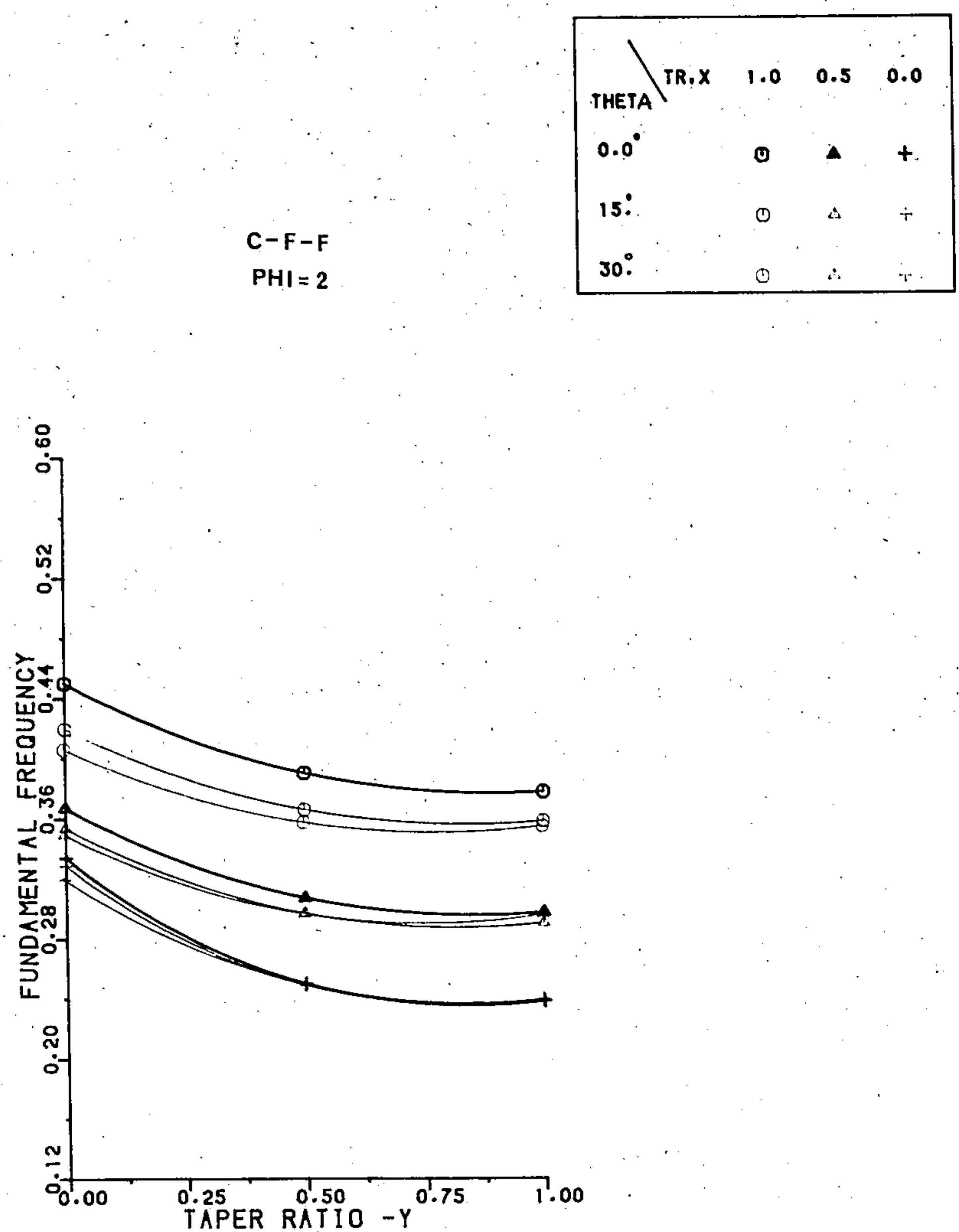

C - F - F
PHI=1.5
TR.X
THETA
1.0
0.5
0.0
0.0°
15.°
30.°
FUNDAMENTAL FREQUENCY
0.20
0.30
0.40
0.50
0.60
0.70
0.80
0.00
0.25
0.50
0.75
1.00
TAPER RATIO -Y

FIGURE 14



TR, X
THETA
1.0
0.5
0.0
0.0°
15°
30°
C-F-F
PHI = 2
FUNDAMENTAL FREQUENCY
0.60
0.52
0.44
0.36
0.28
0.20
0.12
0.00
0.25
0.50
0.75
1.00
TAPER RATIO -Y

FIG. 15

Appendix A

TRIANGULAR PLATE BENDING FINITE ELEMENT

Formulation of plate and shell elements has been an area of active research in the Finite Element Method. This is due to the many difficulties arising in the satisfaction of the completeness and compatability requirements for such elements. Various theories have been used to get around these difficulties and in the process, a large number of elements are available-simple as well as highly sophisticated ones. Unfortunately, most of these are not always reliable, in the sense of consistency of stability of solutions. Further, their theoretical formulation is highly involved and intricate. Triangular plate bending elements having displacement and rotations at the corner modes as degrees of freedom(DOF) are particularly appealing for many practical reasons; e.g., arbitrary plate geometry, general supports and cutouts can be modelled accurately. Several theoretical and numerical studies on plate bending finite elements have appeared in the past two decades. A detailed recent study of triangular plate bending elements with three DOF at the corner nodes (displacement w and rotations $\theta x, \theta y$) has been done by J.L.Batoz et. al.[17]. It is apparent that accurate and efficient elements can be developed for the analysis of specific problems. Thus, at this stage, to select an element suitable for the particular problem of the vibration of plates, it is important to consider which alternative provides the best balance in usage. The alternatives and the factors governing selections are: Simplicity in formulation, computational effort, reliability and accuracy.

A.1 REVIEW OF 9-DOF TRIANGULAR PLATE BENDING ELEMENTS

The available triangular plate bending elements with 9-DOF: the displacement w and rotations θx and θy at the three corner nodes, have been classified [17] as:

1. Displacement models based on Kirchoff thin-plate theory
2. Hybrid stress models based also on Kirchhoff Plate Theory
3. Displacement models derived from the theory of plates with transverse shear deformation.

Clough & Tocher[25] in 1965, reported the element formulations based on the principle of minimum potential energy, where displacements w, rot. $\theta x = -\partial w/\partial y$ and $\theta y = \partial w/\partial x$, are required to be continuous. Elements labelled A,T and T-10 in this Ref. are based on cubic polynomials in local coordinates x,y. Their ineffectiveness is due to; incompleteness [A], incompatability(A,T,T-10), sensitivity to mesh orientation (A,T) and singularity (T). It was then realised that it is impossible to formulate a compatible triagular element with 9 DOF. With a simple polynomial expansion for W. One of the first compatible triangular elements is the HCT element [17]. Its formulation is based on subdivision of the complete element into three subtriangles. An incomplete cubic(9-term) polynomial is used in each subregion for the displacement and the normal slope along the exterior edge of each region varies linearly. The formulation of this element has also been discussed using area- coordinates [22] where it is called LCCT-9. The HCT element has frequently been regarded as a reference element for bending analysis of plates, mainly because of the extensive numerical results presented with its formulation. However, the formulation involves cumbersome algebraic manipulation and element is rather stiff. Several studies(e.g. order of convergence, error estimates) have been conducted for this element[37].

A non-conforming element satisfying the rigid body mode and the constant strain states is presented in [21]. The shape functions expressed in area coordinates are given explicitly in reference [18]. In the same reference, numerical results using this element are presented for both static and dynamic analysis. In many cases, it is shown that the results obtained using this simple non-conforming element are superior to those obtained using sophisticated conforming elements. The use of this element is recommended, specially when convenience and economy is important, as the study of this element reports the convergence to the correct answer, when the mesh is generated by three sets of equally spaced parallel lines.

Reported in the same Ref.[21] is a conforming element obtained by appropriate superposition of polynomials and rational shape functions in area coordinates. Due to the presence of these rational functions, a very high-order numerical integration scheme is needed to evaluate the stiffness matrix. In the results for free vibrations of the plates using these elements, better results are reported using the non-conforming element for the problems considered [21].

A.Razzaque [24] has presented the A-9 element described from the conforming element by replacing the true second-derivatives of the shape functions with smoothed derivatives. These pseudo-derivatives are the least square linerar polynomial of the true second derivatives and the result is that only three numerical points are needed to evaluate the stiffness matrix. The pseudo-derivatives are not given explicitly, but good numerical results are reported for several static problems. The CPT element [30] is a 9-DOF triangular element used in the computer program ICES-STRUDL II. Two subtriangles are used in the formulation. The normal slope continuity is enforced along the three sides but the transverse displacement is not inter-element continuous along one side. only a few numerical results using this elements have been published.

A.1.1 HYBRID STRESS ELEMENTS

The hybrid stress (or hybrid mixed) methods [31] is one approach to get around the difficulties posed by the compatibility requirements for the formulation of pure displacement modes. This element is derived from Kirchhoff plate theory and has a linear distribution of the bending moments in the interior normal slope ($w_{,n}$) variation along the edges of the element. This element has been studied by various authors [32-35]. The derivation of the stiffness matrix of hybrid stress elements appears to be rather cumbersome and the evaluation of the element matrices appears to involve more algebraic manipulations and computer storage than comparative displacement models. Numerical results available with this element show that it is a very effective 9-DOF triangular element[17].

Another way of deriving this element has been the hybrid displacement approach instead of the hybrid stress model [35,36]. Another class of 9-DOF triangular elements for the analysis of thin plates can be obtained with the so-called simplified hybrid displacement method [37] by using a 10-term cubic polynomial for w and correcting the stiffness matrix to restore continuity of w,n along the sides. The resulting 10-DOF are reduced to 9 using static condensation.

A drawback of this element is that the stiffness matrix in some cases can be singular (after the introduction of boundary conditions) [38] and hence not reliable.

A.1.2 DISCRETE KIRCHHOFF THEORY ELEMENTS

Based on the discrete Kirchhoff theory for bending of thin plates these elements are formulated including transverse shear deformation . In this case the independent quantities are the deflection w, and the rotations x and y and only Co-continuity requirements need to be satisfied. The transverse shear energy is neglected and the Kirchhoff hypothesis is introduced in a discrete way along the edges of the element to relate the rotations to the transverse displacement. This approach has been used to formulate 9 DOF triangular bending elements [39,40]. However they are not well known for several reasons----The formulation is neither simple nor attractive [17] and is 'mathematically cloudy' [41]. Practical applications are reported to be difficult and complicated [42,45].

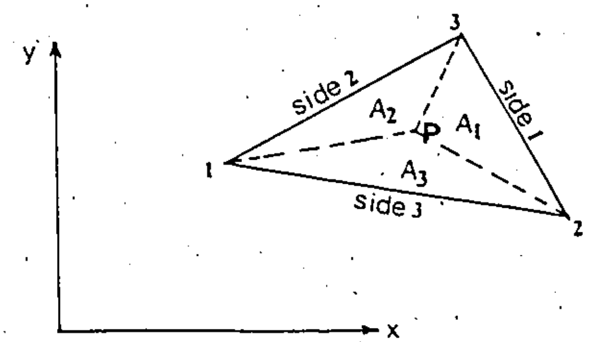
The aim of the present thesis problem is to present accurate data regarding the natural frequencies and mode-shapes of the triangular plates which would be of considerable use and help to the designer. As for the accurate finite element modelling of the arbitrary plate geometry is concerned, an obvious choice of the shape of the finite elements to be used is the triangular plate element.The problem then, at this stage, is to select a triangular element most suitable (considering factors such as the convenience and ease in formulation, availability in explicit form the element properties, accuracy, conputational effort and reliability) for the specific problem of linear-free-vibration analysis of triangular plates.

Appendix B

AREA COORDINATES

(TRIANGULAR COORDINATES OR NATURAL COORDINATES)

In the case of the triangle it is found that the area coordinates are very useful and convinient in defining the interpolation procedures. Illustrated below is a scheme of designation of points used in area coordinates.



Identifying the sides of the elements by the opposite vertex; e.g., side 1 is opposite of the vertex point 1. A general point P within the triangle is seen to uniquely subdivide the triangle into three triangles of area A_1, A_2, and A_3, where the subscripts are identified with the adjacent side. The triangular coordinates L_i (i=1,2,3) are, by definition, the ratios of the areas A_i to the total area A; i.e.,

$$L_i = \frac{A_i}{A} \quad , i = 1, 2, 3 \tag{3.1}$$

Evidently the sum of these areas Ai's must equal A :

$$A_1 + A_2 + A_3 = A \qquad (B.2)$$

and, dividing both sides by A,

$$L_1 + L_2 + L_3 = 1 \qquad (B.3)$$

The rectangular coordinates, x and y, can be written in terms of the area coordinates as,

$$\begin{aligned} x &= L_1 x_1 + L_2 x_2 + L_3 x_3 \\ y &= L_1 y_1 + L_2 y_2 + L_3 y_3 \end{aligned} \qquad (B.4)$$

To solve for L_1, L_2, and L_3 in terms of the known x and y coordinates of the vertices, eqns.(B3) and (B4) can be written as,

$$\begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix} \qquad (B.5)$$

and, by inversion

$$L_i = \frac{1}{2A}\left(b_{0i} + b_{1i}x + b_{2i}y\right), \quad i = 1, 2, 3 \qquad (B.6)$$

where

$$bo_i = x_{i+1} y_{i+2} - x_{i+2} y_{i+1} \; ; \; b1_i = y_{i+1} - y_{i+2} \; ; \; b2_i = x_{i+2} - x_{i+1}$$

$$i = 1, 2, 3 \text{ taken cyclicly}$$

and

$$A = \frac{1}{2} \det \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} = \text{Area } 123 \qquad (B.7)$$

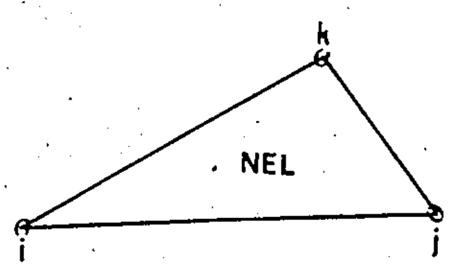
therefore, to any point P(x,y) within the triangle there corresponds a unique set of area coordinates $L1, L2, L3$ (which are not independent but are related by eqn. (B3).).

Appendix C

PROGRAMMING DETAILS

C.1 CODING FOR ASSEMBLING THE STRUCTURAL MATRICES

The address of the element matrix coefficients in the global matrix can be easily determined from the element definitions LNODS(NEL,I), I=1,(no. of nodes per element) for the element number NEL(NEL=1,MAXNEL)



LNODS(NEL,1)=i

LNODS(NEL,2)=j

LNODS(NEL,3)=k

The numbers i,j,k are the numbers corresponding to the three nodes of the element NEL.

The element matrix corresponding to this element is a 9x9 matrix if there are 3 DOF per node.

$$
\begin{array}{cc}
 & \begin{array}{ccccc} i_1 & i_2 & i_3 & \cdots & k_3 \end{array} \\
\begin{array}{c} i_1 \\ i_2 \\ \vdots \\ k_3 \end{array} & \left[\begin{array}{ccccc} a_{11} & a_{12} & \cdots & & a_{19} \\ a_{21} & a_{22} & & & \\ \vdots & & & & \\ a_{91} & & & & a_{99} \end{array} \right]
\end{array}
$$

The three rows corresponding to the i-th node are destined for the three rows following [(i-1)*3]-th. row of the global matrix. Again for each row, the column numbers are determined in exactly the same way.

---------For Element No. "NEL" ---------------

```
DO 50 I=1,NNODZ
DO 49 II=1,NVABZ
ISTRST=(LNODS(NEL,I)-1)*NVABZ+II
IELEMT=(I-1)*NVABZ+II
DO 48 J=1,NNODZ
DO 47 JJ=1,NVABZ
JSTRST=(LNODS(NEL,J)-1)*NVABZ+JJ
JELEMT=(J-1)*NVABZ+JJ
SS(ISTRST,JSTRST)=SS(ISTRST,JSTRST)+ELSTIF(IELEMT,JELEMT)
CMM(ISTRST,JSTRST)=CMM(ISTRST,JSTRST)+ELMM(IELEMT,JELEMT)
47      CONTINUE
48      CONTINUE
49      CONTINUE
50      CONTINUE
```

ISTRST,JSTRST: destination in global matrix of of the coefficient (IELEMT,JELEMT) of the element.

C.2 BOUNDARY CONDITIONS

After assembling the structural stiffness matrix and mass matrix the eigenproblem to be solved is of the form:

$$-(K-\omega^2 M)w = 0$$

subject the prescribed boundary conditions.

The variable "MAXFIX" indicates the number of nodes with constraints. Such constraints are necessary to eliminate the rigid body modes associated with the structural matrices.

The nodal fixity is indicated to the program by a three digit fixity code, corresponding to the displacement w ; rotation about X-axis θ_x ; rotation about Y-axis θ_y . A digit set to zero indicates associated DOF is free to take any value while for the digit set to unity the associated DOF is constrained.

If the fixity code corresponding to the n-th DOF is unity then the diagonal coefficients corresponding to this n-th DOF in both the stiffness matrix and mass matrix are made unity and the rest of the coefficients in the corresponding row and column are made zero.

C.3 PREPARATION OF DATA

After selecting the element type and mesh divisions, the input data is extracted directly from the sketch of the plate. Normally this would involve complete information concerning element and node numbering, nodal coordinates, material properties and boundary condition. The use of the automatic mesh generation routine, however, greatly simplifies this task of data preparation by eliminating the need for element and node numbering. The scheme for non-dimensionalization reduces further the number of quantities to be prescribed. The following are the four groups of data required by the program:

1. Problem Identification
2. Data for triangular element mesh generation.
3. Material Properties.
4. Specification of Boundary Condition.

Classification 1 : Problem Identification

FORMAT : I5

Card columns used : 1-5

Number of cards in this group : 2

Explanation : These two cards identify the type of problem being analysed.

Card 1 (4b̸)1 if the same element type is used for the entire plate.

(4b̸)2 if more than one type of elements are being used.

Note: This card is included with the view of possible extension of the present program to include more than one element types.

Card 2 (4b̸)0 For cantilevered plate

(4b̸)1 for plate clamped at root and with simple support at tip.

(4b̸)2 Clamped plate with internal simple support at a distance of b/3 from the tip.

(4b̸)3 Simple support at 2b/3 from tip.

(4b̸)n Indicates a case other than
n > 3 those mentioned above.

<u>Classification 2</u> : Data for plate geometry

To generate the mesh topology using the automatic mesh generation routine, the non-dimensional parameters defining the geometry of the plate--PHI,THETA,HX,HY; and the mesh characteristics--NY,CONX,CONY have to be prescribed.

FORMAT:I5

Card columns used:1-5

Number of cards : 1

Explanation : This specifies the desired mesh fineness. It denotes the number of intervals the root-chord and trailing edge (AO and OB respectively) are each divided into.

FORMAT : F5.2,5X,F5.2

Card columns used : 1-15

Number of cards in this group: 3

Explanation:

Card 1	Value of aspect ratio,PHI, and sweepback angle THETA.
Card 2	Weighting factors CONX and CONY.
Card 3	Thickness taper ratios HX (along X) and HY (along Y).

Classification 3: Material Properties

Before the element stiffness coefficients can be evaluated element material properties must be available. In this program it is assumed that all elements have the same isotropic properties. As this program is formulated in terms of non-dimensional parameters, only the Poisson's ratio need be specified.

FORMAT: F5.2

Card columns used: 1-5

Number of cards: 1

Explanation: Value of Poisson's ratio "PR".

Classification 4 : Boundary conditions

NOTE: For NCASE=0 (Cantilever plate) and NCASE=1 (Clamped at root and simple support at tip)--the boundary conditions are taken care of automatically in the program.

NCASE=2 and NCASE=3 (Intermediate supports at b/3 and 2b/3 from the tip respectively). Input of boundary conditions is not required for these cases if NY=6 (36 elements ; 28 nodes). The program has provision to supply the B.C.

For cases other than those mentioned above, the nodal fixities have to be prescribed using a three digit fixity code. The first digit refers to the displacement 'w'; the second and third to the rotations θx and θy respectively.

A digit set to zero indicates that the associated DOF is free to move while for the case of the digit set to unity the associated DOF is constrained.

Example: If node number 'i' is clamped
'j' is simply supported
and 'k' is free

The nodal fixities would then be:

Node #	'w'	'θx'	'θy'
i	1	1	1
j	1	0	0
k	0	0	0

Initially (0,0,0) is set for all nodes; thus only the fixities of the constrained nodes need be specified.

Format: I5 6
Card columns used: 1-5
Number of cards in this group: 1
Explanation: Specifies MAXFIX, the number of nodes with constraints.

FORMAT; 4I5

Card columns used: 1-20

Number of cards in this group: MAXFIX

Explanation: The node number followed by its fixity codes for 'w','θx' & 'θy' each with I5 Format.

C.4 PRINT OUT OF DATA AND MESH CHARACTERISTICS

The input data and the element and node numbering as generated by the mesh generation routine are printed out in a routine manner. The inclusion of such a data image allows the checking of input data easily and quickly. Also, the mesh details giving the element definition, the nodal coordinates and the nodal fixities are suitably presented for easy reference.

TABLE 13

SAMPLE DATA PRINTOUT

TOTAL NUMBER OF ELEMENTS= 25
NODES = 21
LOADS= 0
FIXITIES= 6

ELEMENT DEFINITIONS

ELEMENT	NODE1	NODE2	NODE3
1	1	8	2
2	1	7	8
3	2	9	3
4	2	8	9
5	3	10	4
6	3	9	10
7	4	11	5
8	4	10	11
9	5	11	6
10	7	13	8
11	7	12	13
12	8	14	9
13	8	13	14
14	9	15	10
15	9	14	15
16	10	15	11
17	12	17	13
18	12	16	17
19	13	18	14
20	13	17	18
21	14	18	15
22	16	20	17
23	16	19	20
24	17	20	18
25	19	21	20

NODAL CO-ORDINATES

NODE	X	Y
1	0.0	0.0
2	0.200	0.0
3	0.400	0.0
4	0.600	0.0
5	0.800	0.0
6	1.000	0.0
7	-0.052	0.173
8	0.148	0.173
9	0.348	0.173
10	0.548	0.173
11	0.748	0.173
12	-0.104	0.346
13	0.096	0.346
14	0.296	0.346
15	0.496	0.346
16	-0.155	0.530
17	0.045	0.530
18	0.245	0.530
19	-0.207	0.773
20	-0.007	0.773
21	-0.259	0.906

APPLIED LOADS AND FIXITIES

NODE	APPLIED LOADS			FIXITIES		
1	0.0	0.0	0.0	1	1	1
2	0.0	0.0	0.0	1	1	1
3	0.0	0.0	0.0	1	1	1
4	0.0	0.0	0.0	1	1	1
5	0.0	0.0	0.0	1	1	1
6	0.0	0.0	0.0	1	1	1
7	0.0	0.0	0.0	0	0	0
8	0.0	0.0	0.0	0	0	0
9	0.0	0.0	0.0	0	0	0
10	0.0	0.0	0.0	0	0	0
11	0.0	0.0	0.0	0	0	0
12	0.0	0.0	0.0	0	0	0
13	0.0	0.0	0.0	0	0	0
14	0.0	0.0	0.0	0	0	0
15	0.0	0.0	0.0	0	0	0
16	0.0	0.0	0.0	0	0	0
17	0.0	0.0	0.0	0	0	0
18	0.0	0.0	0.0	0	0	0
19	0.0	0.0	0.0	0	0	0
20	0.0	0.0	0.0	0	0	0
21	0.0	0.0	0.0	0	0	0

C.5 DEFINITION OF SYMBOLS AND PROGRAM LISTING

DEFINITION OF SYMBOLS

VARIABLE NAME	DESCRIPTION
A	Coefficient of Shape Function Derivatives
ALFA1, ALFA2, ALFA3	Area Coordinates of Integration Points
AREA	Element Area
BI	$y_2 - y_3$
BJ	$y_3 - y_1$
BM	$y_1 - y_2$
CI	$x_3 - x_2$
CJ	$x_1 - x_3$
CM	$x_2 - x_1$
CONX	Weighting Factor Along X-Axis
CONY	Weighting Factor Along Y-Axis
Coords(I,J)	Nodal Coordinates
CMM(,)	Global Mass Matrix
EIG(J)	Eigenvalue, J-th.
ELMM(,)	Element Mass Matrix
ELSTIF(,)	Element Stiffness Matrix
FN(J)	Shape Function, J-th.

HX	Taper Ratio Along X
HY	Taper Ratio Along Y
LNODS(I,J)	Element Definition
MAXFIX	Maximum Number Of Nodal Fixities
MAXNOD	Maximum Number Of Nodes
MAXNEL	Maximum Number of Elements
MAXVAR	Maximum Number Of Variables
NCASE	Problem Identification Code
NY	Mesh Fineness Factor
NNODZ	Number of Nodes Per Element
NVABZ	Number of Variables Per Node
N2X(I)	$\partial^2 N/\partial x^2$
N2Y(I)	$\partial^2 N/\partial y^2$
N2XY(I)	$\partial^2 N/\partial x \partial y$
NGP	Number of Gaussian Integration Points
NV	Number of Eigenvalues Required
NI	Maximum Number of Iterations for JACOBI

NN	Eigenvector is Normalized w.r.t. NN-th. DOF
NFIX	Maximum Number of constrained Nodes
PHI	Aspect Ratio
PT(,)	Area Coordinates of Sampling Points for Numerical Integration
PR	Poisson's Ratio
SS(,)	Structural Stiffness Matrix
T	Plate Thickness
THETA	Trailing Edge Sweepback Angle
W	Weight of The Gaussian Samling Point
XX(,)	Eigen Vector

SUBROUTINES

AGEN	Automatic Triangular Element Mesh Generation
GAUSS3 GAUSS7	Area Coordinates and Weights for 3-Point and 7-Point GAussian Quadrature Schemes
ITER	Jacobi Iteration for Eigen-Pairs Solution
MAX	Determining Maximum Value

MATINV	Matrix Inversion
MODSHP	Modeshape Plotting Routine

C.6 LISTING OF THE COMPUTER PROGRAM

```
C     -------------------------------------------------------------------
C
C     FINITE ELEMENT PROGRAM FOR DETERMINING THE MODE SHAPES AND
C
C     FREQUENCIES OF VARIABLE THICKNESS ARBITRARY TRIANGULAR PLATES
C
C     USING A 9 DEGREE OF FREEDOM GENERAL TRIANGULAR PLATE BENDING
C
C     ELEMENT ******** MODIFIED BY**************
C
C
C-------------------------------------------------------------------
C
C      DATA INPUT CODING
C
C-------------------------------------------------------------------
      REAL N2X,N2Y,N2XY
      DIMENSION LNODS(100,3),COORDS(100,2),ALOAD(50,3),NFIX(50,3)
      DIMENSION ESTIF(9,9),SS(90,90),ELMM(9,9),CMM(90,90)
      DIMENSION SL(90),D(3,3)
      DIMENSION SC(90,90),EIG(90),XX(90,25),XB(90),XA(90)
     1,SYM(25)
      DIMENSION A(9,6)
      DIMENSION N2X(9),N2Y(9),N2XY(9),FN(9)
      DIMENSION REACT(3),W(13),PT(13,3)
C
C     NVABZ IS THE NUMBER OF VARIABLES PER NODE
C     NNODZ IS THE NUMBER OF NODES PER ELEMENT
C
      NVABZ=3
      NNODZ=3
C     -------------------------
C
C      READ IN DATA
C
C     -------------------------
      READ(5,2000) NELM
      READ(5,2000)NCASE
      READ(5,2000)NY
      READ(5,2001)PHI,THETA
      READ(5,2001)CONX,CONY
      READ(5,2001)HX,HY
      READ(5,2002)PR
      XI=1.0
      YA=1.0
C
C
      WRITE(6,3000)PHI,NCASE,HX,HY,THETA
C
C
      MAXFIX=NY+1
      PI=3.1415926
      XI=XI*COS(THETA*PI/180.0)
C
C
      YA=1.0
      XI=SQRT((PHI*SIN(THETA*PI/180.))**2+(COS(THETA*PI/180.))**2)
      ALFA=ATAN(PHI*TAN(THETA*PI/180.))*180./PI
      XI=XI*COS(ALFA.PI/180.)
C     -------------------------------------------------------
```



```
C
C
C      CALL SUBROUTINE FOR AUTOMATIC MESH GENERATION
C
C
C     -----------------------------------------------------------
      CALL AGEN(NY,XI,YA,ALFA,CONX,CONY,COORDS,LNODS,MAXNOD,
     1MAXNEL)
      DO 21 I=1,MAXNOD
      Z=COORDS(I,1)
      COORDS(I,1)=COORDS(I,2)
      COORDS(I,2)=Z
   21 CONTINUE
      WRITE(6,1002)MAXNEL,MAXNOD,MAXLOD,MAXFIX
      NDOF=NVABZ*NNODZ
      NDT=NDOF*(MAXNEL+1)
C     ---------------------------------
C
C     INITIALIZE VECTORS AND MATRICES
C
C     ---------------------------------
      MAXVAR=MAXNOD*NVABZ
      DO 2 I=1,MAXNOD
      DO 1 J=1,3
      ALOAD(I,J)=0.0
      NFIX(I,J)=0
    1 CONTINUE
    2 CONTINUE
      DO 3 I=1,MAXVAR
      SL(I)=0.0
      DO 3 J=1,MAXVAR
      SS(I,J)=0.0
      CMM(I,J)=0.0
    3 CONTINUE
      DO 6 I=1,9
      DO 5 J=1,9
      ESTIF(I,J)=0.0
      ELMM(I,J)=0.0
    5 CONTINUE
    6 CONTINUE
C     -------------------------------
C
C      READ/WRITE ELEMENT DEFINITIONS
C
C     -------------------------------
      WRITE(6,1003)
      DO 11 NEL=1,MAXNEL
      WRITE(6,1005) [illegible](LNODS(NEL,I),I=1,NNODZ)
   11 CONTINUE
C
C     WRITE NODAL COORDINATES
C
      WRITE(6,1004)
      DO 12 NOD=1,MAXNOD
      WRITE(6,1006)[illegible](COORDS(NOD,I),I=1,2)
   12 CONTINUE
C     -------------------------------------
C
C      READ IN ELEMENT FIXITIES (NFIX)
C          IF NCASE > 3
C
C     -------------------------------------
```

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```
      C
            IF(NCASE.LT.3) GO TO 36
            READ(5,2000)MAXFIX
            DO 34 I=1,MAXFIX
   34       READ(5,2003)NIC,(NFIX(NIC,J),J=1,NVABZ)
            GO TO 37
   36       CONTINUE
            DO 33 I=1,MAXFIX
            DO 33 J=1,NVABZ
            NFIX(I,J)=1
   33       CONTINUE
            NFIX(MAXNOD,1)=NCASE
            WRITE(6,1007)
   37       DO 16 I=1,MAXNOD
            WRITE(6,1012)I,(ALOAD(I,J),J=1,NVABZ),(NFIX(I,J),J=1,NVABZ)
   16       CONTINUE
      C
      C     READ IN ELASTIC PROPERTIES AND DENSITY
      C
            READ(5,2002)PR
      C
      C
            TM1=HA-1.0
            TM2=(HA-1.)*PHI*TAN(THETA*PI/180.)+(HY-1.)*(1./COS(THETA*PI/180.))
      C
            WRITE(6,1014)PR
      C
      C     -----------------------------------------------------------
      C
      C     SET UP ELEMENTAL STRUCTURAL STIFFNESS MATRIX
      C
      C-----------------------------------------------------------------
            D1=YM/(12.*(1.-PR**2))
            D(1,1)=D1
            D(1,2)=PR*D1
            D(1,3)=0.0
            D(2,1)=D(1,2)
            D(2,2)=D(1,1)
            D(2,3)=0.0
            D(3,1)=0.0
            D(3,2)=0.0
            D(3,3)=D1*(1.0-PR)/2.0
            DO 100 NEL=1,MAXNEL
      C
      C     SET UP ELEMENT GEOMETRY
      C
            NIC1=LNODS(NEL,1)
            NIC2=LNODS(NEL,2)
            NIC3=LNODS(NEL,3)
      C
      C
            X1=COORDS(NIC1,1)
            Y1=COORDS(NIC1,2)
            X2=COORDS(NIC2,1)
            Y2=COORDS(NIC2,2)
            X3=COORDS(NIC3,1)
            Y3=COORDS(NIC3,2)
      C
      C
            AREA=(X2*Y3-X3*Y2-X1*(Y3-Y2)+Y1*(X3-X2))/2.0
            AREA=ABS(AREA)
            BI=Y2-Y3
```



```
            CI=X3-X2
            BJ=Y3-Y1
            CJ=X1-X3
            BM=Y1-Y2
            CM=X2-X1
            TA1=TM1*X1+TM2*Y1
            TA2=TM1*X2+TM2*Y2
            TA3=TM1*X3+TM2*Y3
            DO 50 I=1,9
            DO 50 J=1,9
            ELSTIF(I,J)=0.0
   50       ELMA(I,J)=0.0
      C
      C     DETERMINE BTDB FOR EACH GAUSS POINT
      C     RECALL THE AREA COORDINATES AND WEIGHTS OF EACH G. POINT
      C
            IF(TM1.EQ.0.0)GOTO500
            CALL GAUSS7(NGP,PT,W)
            GO TO 501
  500       IF(TM2.EQ.0.0)GO TO 502
            CALL GAUSS7(NGP,PT,W)
            GO TO 501
  502       CALL GAUSS3(NGP,PT,W)
  501       DO 101 JJ=1,NGP
            ALFA1=PT(JJ,1)
            ALFA2=PT(JJ,2)
            ALFA3=PT(JJ,3)
            WT=W(JJ)
      C
      C     DETERMINE THE THICKNESS AT EACH GAUSS POINT
      C
            T=1.+TA1*ALFA1+TA2*ALFA2+TA3*ALFA3
      C-----------------------------------
      C
      C
      C DETERMINE THE COEFF. INVOLVED IN THE SHAPE FUNCTION DERIVATIVES
      C
      C
      C----------------------------------------
            A(1,1)=2.0*(ALFA2+ALFA3)
            A(1,2)=-2.0*ALFA1
            A(1,3)=-2.0*ALFA1
            A(1,4)=2.0*ALFA1-2.0*ALFA2
            A(1,5)=0.0
            A(1,6)=2.0*(ALFA1-ALFA3)
            A(2,1)=2.0*(BJ*ALFA2-BJ*ALFA3)
            A(2,2)=0.0
            A(2,3)=0.0
            A(2,4)=2.0*ALFA1*BM+2.0*(BM*ALFA3-BJ*ALFA3)
            A(2,5)=2.0*(BJ*ALFA1+BM*ALFA1)
            A(2,6)=2.0*(BM*ALFA2-BJ*ALFA2)-2.0*BJ*ALFA1
            A(3,1)=2.0*(CM*ALFA2-CJ*ALFA3)
            A(3,2)=0.
            A(3,3)=0.
            A(3,4)=2.0*ALFA1*CM+2.0*(CM*ALFA3-CJ*ALFA3)
            A(3,5)=2.0*(CM*ALFA1-CJ*ALFA1)
            A(3,6)=2.0*ALFA1*CJ+2.0*(CM*ALFA2-CJ*ALFA2)
            A(4,1)= 2.0*ALFA1
            A(4,2)= 2.0*ALFA1
            A(4,3)
            A(4,4)            ALFA1)
```

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```
0154          A(4,6)=0.0
0155          A(5,1)=0.0
0156          A(5,2)=2.*(BI*ALFA3-BM*ALFA1)
0157          A(5,3)=0.0
0158          A(5,4)=.5*(BI*ALFA3-BM*ALFA3)-2.*BM*ALFA2
0159          A(5,5)=2.*BI*ALFA2+.5*(BI*ALFA1-BM*ALFA1)
0160          A(5,6)=.5*(BI*ALFA2-BM*ALFA2)
0161          A(6,1)=0.0
0162          A(6,2)=2.*(CI*ALFA3-CM*ALFA1)
0163          A(6,3)=0.0
0164          A(6,4)=.5*(CI*ALFA3-CM*ALFA3)-2.0*CM*ALFA2
0165          A(6,5)=2.*CI*ALFA2+.5*(CI*ALFA1-CM*ALFA1)
0166          A(6,6)=.5*(CI*ALFA2-CM*ALFA2)
0167          A(7,1)=-2.*ALFA3
0168          A(7,2)=-2.*ALFA3
0169          A(7,3)=2.*(ALFA1+ALFA2)
0170          A(7,4)=0.0
0171          A(7,5)=2.*(ALFA3-ALFA2)
0172          A(7,6)=2.*(ALFA3-ALFA1)
0173          A(8,1)=0.0
0174          A(8,2)=0.0
0175          A(8,3)=2.*(BJ*ALFA1-BI*ALFA2)
0176          A(8,4)=.5*(BJ*ALFA3-BI*ALFA3)
0177          A(8,5)=-2.*BI*ALFA3+.5*(BJ*ALFA1-BI*ALFA1)
0178          A(8,6)=2.*BJ*ALFA3+.5*(BJ*ALFA2-BI*ALFA2)
0179          A(9,1)=0.0
0180          A(9,2)=0.0
0181          A(9,3)=2.0*(CJ*ALFA1-CI*ALFA2)
0182          A(9,4)=.5*(CJ*ALFA3-CI*ALFA3)
0183          A(9,5)=-2.*CI*ALFA3+.5*(CJ*ALFA1-CI*ALFA1)
0184          A(9,6)=2.*CJ*ALFA3+.5*(CJ*ALFA2-CI*ALFA2)
        C     ------------------------------------------------------------
        C
        C     DETERMINATION OF SHAPE FUNCTION SECOND DERIVATIVES
        C
        C     ------------------------------------------------------------
0185          DO 102 I=1,9
0186          N2X(I)=((BI**2)*A(I,1)+(BJ**2)*A(I,2)+(BM**2)*A(I,3)
             1+2.*BI*BJ*A(I,4)+2.*BM*BJ*A(I,5)
             1+2.*BM*BI*A(I,6))/(4.0*(AREA**2))
0187          N2Y(I)=((CI**2)*A(I,1)+(CJ**2)*A(I,2)+(CM**2)*A(I,3)
             1+2.*CI*CJ*A(I,4)+2.*CM*CJ*A(I,5)
             1+2.*CM*CI*A(I,6))/(4.*(AREA**2))
0188          N2XY(I)=(BI*CI*A(I,1)+BJ*CJ*A(I,2)+BM*CM*A(I,3)
             1+(BI*CJ+BJ*CI)*A(I,4)+(BJ*CM+BM*CJ)*A(I,5)
             1+(BM*CI+BI*CM)*A(I,6))/(4.*(AREA**2))
0189      102 CONTINUE
        C
        C
        C     CONTRIBUTION TO THE ELEMENT STIFFNESS MATRIX
        C
        C
0190          DO 103 I=1,9
0191          DO 103 J=1,9
0192          Q1=(N2X(I)+PR*N2Y(I)/PHI/PHI)*N2X(J)
0193          Q2=(N2Y(I)/PHI/PHI+PR*N2X(I))*N2Y(J)/PHI/PHI
0194          Q3=2.0*(1.0-PR)*N2XY(I)*N2XY(J)/PHI/PHI
0195          Q4=WT*AREA*(T**3)
0196          ELSTIF(I,J)=ELSTIF(I,J)+(Q1+Q2+Q3)*Q4*PHI
0197      103 CONTINUE
0198      101 CONTINUE
        C
```



```
        C
        C     DETERMINING THE ELEMENT MASS MATRIX ELMM
        C
        C
0199          CALL GAUSS3(NGM,PT,W)
0200          DO 302 JM=1,NGM
0201          ALFA1=PT(JM,1)
0202          ALFA2=PT(JM,2)
0203          ALFA3=PT(JM,3)
0204          WT=W(JM)
0205          T=1.+TA1*ALFA1+TA2*ALFA2+TA3*ALFA3
        C
        C     ------------------------------------
        C
        C     DETERMINING THE SHAPE FUNCTIONS
        C
        C     ------------------------------------
        C
0206          FN(1)=ALFA1*(1.0-ALFA2**2-ALFA3**2)+(ALFA1**2)*(ALFA2
             1+ALFA3)
        C
0207          FN(2)=BM*(ALFA1*ALFA1*ALFA2+.5*ALFA1*ALFA2*ALFA3)
             1-BJ*(ALFA3*ALFA1*ALFA1+.5*ALFA1*ALFA2*ALFA3)
        C
0208          FN(3)=(ALFA1**2)*(CM*ALFA2-CJ*ALFA3)+.5*ALFA1*ALFA2*ALFA3
             1*(CM-CJ)
        C
0209          FN(4)=ALFA2*(1.+ALFA2*ALFA3+ALFA2*ALFA1-ALFA3**2-ALFA1**2)
        C
0210          FN(5)=(ALFA2**2)*(BI*ALFA3-BM*ALFA1)+.5*ALFA1*ALFA2*ALFA3
             1*(BI-BM)
        C
0211          FN(6)=(ALFA2**2)*(CI*ALFA3-CM*ALFA1)+.5*ALFA1*ALFA2*ALFA3
             1*(CI-CM)
        C
0212          FN(7)=ALFA3*(1.0+ALFA3*ALFA1+ALFA3*ALFA2-ALFA1**2-ALFA2**2)
        C
0213          FN(8)=(ALFA3**2)*(BJ*ALFA1-BI*ALFA2)+.5*ALFA1*ALFA2*ALFA3
             1*(BJ-BI)
        C
0214          FN(9)=(ALFA3**2)*(CJ*ALFA1-CI*ALFA2)+.5*ALFA1*ALFA2*ALFA3
             1*(CJ-CI)
        C
        C
        C     CONTRIBUTION TO THE MASS MATRIX ELMM
        C
0215          DO 301 I=1,9
0216          DO 301 J=1,9
0217          ELMM(I,J)=ELMM(I,J)+AREA*WT*T*FN(I)*FN(J)
0218      301 CONTINUE
0219      302 CONTINUE
        C
        C
        C
        C
        C     ASSEMBLING THE GLOBAL STIFFNESS AND MASS MATRICES
        C
0220          DO 43 I=1,NCODE
0221          DO 43 II=1,NCODE
0222          [illegible]
0223          [illegible]
```

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```
            DO 47 J=1,NNODZ
            DO 46 JJ=1,NVARZ
            ISTRST=(NOD(NEL,J)-1)*NVARZ+JJ
            JELEMT=(J-1)*NVARZ+JJ
            SS(ISTRST,JSTRST)=SS(ISTRST,JSTRST)+ELSTIF(IELEMT,JELEMT)
            CMM(ISTRST,JSTRST)=CMM(ISTRST,JSTRST)+ELMM(IELEMT,JELEMT)
   46       CONTINUE
   47       CONTINUE
   48       CONTINUE
   49       CONTINUE
  100       CONTINUE
C            WRITE(6,99999)((SS(I,J),J=1,9),I=1,9)
C            WRITE(6,99999)((CMM(I,J),J=1,9),I=1,9)
99999       FORMAT(//,2X,9E12.5)
C
C
C           APPLYING BOUNDARY CONDITIONS
C
C
            DO 511 I=1,MAXNOD
            DO 511 J=1,NVARZ
            IF(NFIX(I,J).EQ.0)GO TO 511
            NS=(I-1)*NVARZ+J
            DO 51 N=1,MAXVAR
            SS(NS,N)=0.0
            SS(N,NS)=0.0
            SS(NS,NS)=1.0
            CMM(NS,N)=0.0
            CMM(N,NS)=0.0
            CMM(NS,NS)=1.0
   51       CONTINUE
  511       CONTINUE
C            WRITE(6,99999)((SS(I,J),J=1,MAXVAR),I=1,MAXVAR)
C
C         CALL SUBROUTINE JACOBI
C
C
C            CALL JACOBI(SS,CMM,XX,EIG,DX,MAXVAR,1.E-05,15,1,6)
C
C
C           DETERMINING INVERSE OF THE STIFFNESS MATRIX
C
C
            CALL MATINV(SS,MAXVAR)
            DO 52 I=1,MAXVAR
            DO 123 J=1,MAXVAR
            SC(I,J)=0.0
            DO 512 K=1,MAXVAR
            SC(I,J)=SC(I,J)+SS(I,K)*CMM(K,J)
  512       CONTINUE
  123       CONTINUE
   52       CONTINUE
C
C
C           NV=NUMBER OF MODES REQUIRED
C
C           NI=MAX. NUMBER OF ITERATIONS.
C
C           NN=EIGEN VECTOR IS NORMALIZED WITH RESPECT TO
C           THE NN-TH DEGREE OF FREEDOM
C
C
```



```
            NV=6
            NI=40
            IF(NFIX(MAXNOD,1).EQ.0)GO TO 333
            NN=MAXVAR-17
            GO TO 335
  333       NN=MAXVAR-2
  335       CALL ITER(NN,NV,NI,MAXVAR,SC,CMM,XX,XB,EIG,SMM)
            DO 53 I=1,NV
            EIG(I)=1.0/EIG(I)
C
C
C            AX=8.0
C            BY=12.0
C            EIG(I)=EIG(I)*D1*TO*TO/RO/PHI/(AX**4)
            EIG(I)=SQRT(EIG(I))/(3.1415926*2.)
            WRITE(6,5000)I,EIG(I)
            WRITE(6,7000)
            DO 54 J=1,MAXNOD
            M=1+(J-1)*3
            WRITE(6,6001)J,XX(M,I),XX(M+1,I),XX(M+2,I)
   54       CONTINUE
   53       CONTINUE
            TR=TC/(TC+TM1*COORDS(MAXNOD,1)+TM2*COORDS(MAXNOD,2))
C
C
            DO 31 I=1,MAXNOD
            Z=COORDS(I,1)
            COORDS(I,1)=COORDS(I,2)*8.0+PHI
            COORDS(I,2)=Z*8.0
   31       CONTINUE
            CALL MODSHP(NY,NV,XX,COORDS,MAXNOD,THETA,EIG,TR,NCASE,PHI,HX,HY,
           1NGRPH)
 1001       FORMAT(4I5)
 1002       FORMAT(1H1,5X,'TOTAL NUMBER OF ELEMENTS=',I5,/,25X,
           1'NODES =',I5,//,25X,'LOADS=',I5,//,22X,'FIXITIES=',I5)
 1003       FORMAT(///,5X,'ELEMENT DEFINITIONS',//,5X,'ELEMENT',2X,
           1'NODE1',2X,'NODE2',2X,'NODE3',/)
 1004       FORMAT(3I5)
 1005       FORMAT(5X,I5,3(2X,I5))
 1006       FORMAT(///,5X,'NODAL CO-ORDINATES',//,5X,'NODE',6X,'X',9X,
           1'Y',/)
 1007       FORMAT(I5,2(5X,F10.6))
 1008       FORMAT(5X,I4,2 10.3)
 1010       FORMAT(I5,2(5X,F10.3))
 1011       FORMAT(4I5)
 1009       FORMAT(///,5X,'APPLIED LOADS AND FIXITIES',///,5X,'NODE',
           15X,'APPLIED LOADS',15X,'FIXITIES',/)
 1012       FORMAT(5X,I4,5X,3E10.3,3I5)
 1013       FORMAT(E10.3)
 1014       FORMAT(///,5X,'MATERIAL PROPERTIES',//,5X,'POISSON S RATIO=',F5.2)
 5000       FORMAT(1H1,5X,'MODE NUMBER =',I2,5X,'NON-DIM. FREQ. =',E15.7)
 6000       FORMAT(/,E13.5,/)
 1015       FORMAT(I5,[illegible].7,2X,[illegible].7,2X,[illegible].7,2X,F9.7)
 1016       FORMAT(I5)
 7000       FORMAT(//,[illegible],'NODE NUMBER',[illegible],5X,'DISPLACEMENTS',5X,
           1[illegible]'ROTATIONS',[illegible]'ROTATION',/,22X,'W',[illegible],'-ANY',5X,'-AX',//)
 6001       FORMAT(//,I10,[illegible])
 1100       FORMAT([illegible])
           1[illegible]
 [illegible] FORMAT(///,[illegible]'PHI=',[illegible],//,[illegible]'HX=',
           1[illegible]'THETA',[illegible],//)
 [illegible] FORMAT([illegible])
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FORTRAN IV G LEVEL  21                MAIN          DATE = 82252        13/28/57          PAGE 0002
      2001   FORMAT(F5.2,5X,F5.2)
      2002   FORMAT(F5.2)
      2003   FORMAT(4I5)
      1200   CONTINUE
             STOP
             END
```

```
FORTRAN IV G LEVEL  21                MAIN          DATE = 82252        13/28/57          PAGE 0001
      C
      C
      C          SUBROUTINE MATINV
      C
      C
             SUBROUTINE MATINV(A,N)
             DIMENSION A(90,90)
             DO 19 I=1,N
             Z=A(I,I)
             A(I,I)=1.0
             DO 69 J=1,N
        69   A(I,J)=A(I,J)/Z
             DO 19 K=1,N
             IF(K-I)9,19,9
        9    Z=A(K,I)
             A(K,I)=0.0
             DO 49 J=1,N
        49   A(K,J)=A(K,J)-Z*A(I,J)
        19   CONTINUE
             RETURN
             END
```

```
FORTRAN IV G LEVEL  21                MAIN          DATE = 82252        13/28/57          PAGE 0001
      C ******************************************************************
      C
      C
             SUBROUTINE ITER(NN,NV,NI,NB,SC,AM,XA,XB,EIG,SMN)
      C
      C     ITERATION TECHNIQUE TO CALCUTE THE NATURAL
      C
      C     FREQUENCY AND MODE SHAPES
      C
      C ******************************************************************
      C
      C          MN=MODE NUMBER
      C
      C          NV=NUMBER OF MODES REQUIRED
      C
      C      EIG(J)=J-TH EIGEN VALUE
      C
      C      XX(I,J)=J-TH EIGEN VECTOR
      C
      C      NN=DISPLACEMENT VECTOR IS NORMALIZED WITH
      C         RESPECT TO THE NN-TH DEGREE OF FREEDOM
      C
      C
             REAL XB(90),XA(90,25),SC(90,90),AM(90,90),EIG(90),SMN(20)
             MN=1
        1    CONTINUE
             DO 10 I=1,NB
        10   XA(I,MN)=0.0
             XA(NN,MN)=1.0
             IT=0
        2    CONTINUE
             IT=IT+1
             DO 20 K=1,NB
             XB(K)=0.0
             DO 20 L=1,NB
        20   XB(K)=XB(K)+SC(K,L)*XA(L,MN)
             EIG(MN)=XB(NN)
             DO 30 M=1,NB
        30   XA(M,MN)=XB(M)/EIG(MN)
             IF(IT-NI)2,2,3
        3    CONTINUE
             SUM=0.0
             DO 50 I=1,NB
             XB(I)=0.0
             DO 40 J=1,NB
        40   XB(I)=XB(I)+AM(I,J)*XA(J,MN)
        50   SUM=SUM+XA(I,MN)*XB(I)
             SMN(MN)=SUM
             DO 60 I=1,NB
             EXI=EIG(MN)*XA(I,MN)/SUM
             DO 60 J=1,NB
        60   SC(I,J)=SC(I,J)-EXI*XB(J)
             MN=MN+1
             IF(MN-NV)1,1,4
        4    CONTINUE
             RETURN
             END
```

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```
      C
      C
      C
            SUBROUTINE AGEN(NY,XB,YA,THETA,CONX,CONY,COORDS,LNODS,
           1MAXNOD,MAXNEL)
      C
      C*****************************************************************
      C
      C
      C     SUBROUTINE FOR AUTOMATIC MESH GENERATION
      C
      C
      C*****************************************************************
            DIMENSION NX(20),YL(20),XF(20),YF(20),NOD(4),SUM1(15),XL(20)
            DIMENSION COORDS(100,2),LNODS(100,3)
      C
      C
      C     READ/WRITE NO. OF LINES AND WT. FACTOR
      C
      C
            WRITE(6,1001)NY,XB,YA,THETA
            WRITE(6,1002)CONX,CONY
            NX(1)=NY
            CON=CONY
            PI=3.1415926
            DO 100 I=1,NY
            IF(CONX.EQ.1.0)GO TO 13
            XF(I)=XB*((CONX**(I-1))-1.)/(CONX**NY-1.)
            GO TO 14
   13       XF(I)=(I-1)*XB/NY
   14       NX(I+1)=NX(I)-1
            YF(I)=-XF(I)*TAN(THETA*PI/180.)
            XL(I)=XF(I)
            YL(I)=(1.0-(XL(I)/XB))*YA+YF(I)
            WRITE(6,1000)NX(I),XF(I),YF(I),XL(I),YL(I)
  100       CONTINUE
            N=0
      C
      C     CALCUTE NODE NO. AND THEIR CORRESPONDING X,Y COORD.
      C
            DO 350 I=1,NY
            NXI=NX(I)+1
            SUM1(1)=0.0
            SUM1(2)=1.0
            SUM=1.0
            IF(NXI-2)190,291,190
  190       DO 250 K=3,NXI
            SUM1(K)=SUM1(K-1)*CON
            SUM=SUM+SUM1(K)
  250       CONTINUE
  291       CONTINUE
            X=XF(I)
            Y=YF(I)
            DO 300 J=1,NXI
            N=N+1
            X=(XL(I)-XF(I))*SUM1(J)/SUM+X
            Y=(YL(I)-YF(I))*SUM1(J)/SUM+Y
      C     WRITE(6,1430)X,Y,N
            COORDS(N,1)=X
            COORDS(N,2)=Y
  300       CONTINUE
  350       CONTINUE
```



```
            COORDS(N+1,1)=XB
            COORDS(N+1,2)=-XB*TAN(THETA*PI/180.)
      C     WRITE(6,1430)COORDS(N+1,1),COORDS(N+1,2),MAXNOD
            MAXNOD=N+1
            N=0
            NSUM=0
            NYI=NY-1
            DO600 I=1,NYI
            NXI=NX(I)
            DO 500 J=1,NXI
            IF(J-NXI)379,371,379
  371       IF(NX(I+1)-NX(I))380,379,401
      C
      C     MESH VARIATION
      C
  379       NOD(1)=J+NSUM
            NOD(2)=NOD(1)+1
            NOD(3)=NOD(2)+NXI+1
            NOD(4)=NOD(3)-1
            GO TO 412
  380       NOD(1)=NOD(2)
            NOD(2)=NOD(1)+1
            NOD(4)=0
            GO TO 412
      C
  401       NOD(1)=J+NSUM
            NOD(2)=NOD(1)+1
            NOD(3)=NOD(2)+NXI+1
            NOD(4)=NOD(3)-1
            N=N+1
            LNODS(N,1)=NOD(1)
            LNODS(N,2)=NOD(3)
            LNODS(N,3)=NOD(2)
      C     WRITE(6,1470)N,NOD(1),NOD(2),NOD(3)
            N=N+1
            LNODS(N,1)=NOD(1)
            LNODS(N,2)=NOD(3)
            LNODS(N,3)=NOD(4)
            NOD(1)=NOD(2)
            NOD(2)=NOD(3)+1
            NOD(4)=0
  412       N=N+1
            LNODS(N,1)=NOD(1)
            LNODS(N,2)=NOD(3)
            LNODS(N,3)=NOD(2)
      C     WRITE(6,1470)N,NOD(1),NOD(2),NOD(3)
      C
      C
      C     SKIP A TRIANGLE IF FOURTH NODE OF QUAD IS ZERO
      C
            IF(NOD(4))432,434,433
  433       N=N+1
            LNODS(N,1)=NOD(1)
            LNODS(N,2)=NOD(4)
            LNODS(N,3)=NOD(3)
      C     WRITE(6,1470)N,NOD(1),NOD(3),NOD(4)
  434       CONTINUE
  500       CONTINUE
            NSUM=NSUM+NXI+1
  600       CONTINUE
            MAXNEL=N+1
            [illegible]
            [illegible]
```

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FORTRAN IV G LEVEL 21 AGUN DATE = 82252 13/28/57 PAGE 0003

```
      LNOD(NUMAXNEL,3)=MAXNOD-1
 1100 FORMAT(I5,5X,F5.2,5X,F5.2,5X,F5.2,5X,F5.2,5X,F5.2)
 1000 FORMAT(I5,5X,4I10.5)
 1001 FORMAT('1 CONTROL DATA',///,'NO. OF INTERVALS=',I5,
     1///,I5X,'U=',F10.5,///,15X,'A=',F10.5,///,11X,'THETA=',F10.5)
 1002 FORMAT(//,'CON-X=',F7.4,//,'CON-Y=',F7.4,/)
 1431 FORMAT('1 X COORDINATE    Y COORDINATE    NODAL POINT')
 1430 FORMAT(2F10.5,I4)
 1432 FORMAT('1  ELEMENT NO.    NODAL POINT OF THE ELEMENT')
 1470 FORMAT(I8,5X,3I8)
 1471 CONTINUE
      RETURN
      END
```

FORTRAN IV G LEVEL 21 MAIN DATE = 82252 13/28/57 PAGE 0001

```
C ***********************************
C
C        SUBROUTINE GAUSSPT(NGP,PT,W)
C
C ***********************************
      SUBROUTINE GAUSS7(NGP,PT,W)
      DIMENSION PT(13,3),W(13)
      NGP=7
      PT(1,1)=1.0/3.0
      PT(1,2)=1./3.
      PT(1,3)=1./3.
      W(1)=0.225
      PT(2,1)=0.797426985353087
      PT(2,2)=0.101286507323456
      PT(2,3)=PT(2,2)
      W(2)=0.125939180544827
      PT(3,1)=PT(2,2)
      PT(3,2)=PT(2,1)
      PT(3,3)=PT(2,2)
      W(3)=W(2)
      PT(4,1)=PT(3,1)
      PT(4,2)=PT(4,1)
      PT(4,3)=PT(3,2)
      W(4)=W(2)
      PT(5,1)=0.470142064105115
      PT(5,2)=PT(5,1)
      PT(5,3)=0.059715871789770
      W(5)=0.132394152788506
      PT(6,1)=PT(5,1)
      PT(6,2)=PT(5,3)
      PT(6,3)=PT(5,1)
      W(6)=W(5)
      PT(7,1)=PT(6,2)
      PT(7,2)=PT(6,1)
      PT(7,3)=PT(6,3)
      W(7)=W(6)
      RETURN
      END
```

FORTRAN IV G LEVEL 21 MAIN DATE = 82252 13/28/57 PAGE 0001

```
C *******************************************
C
C        SUBROUTINE GAUSSP3(NGP,PT,W)
C
C *******************************************
      SUBROUTINE GAUSS3(NGP,PT,W)
      DIMENSION PT(13,3),W(13)
      NGP=3
      PT(1,1)=2.0/3.0
      PT(1,2)=1.0/6.0
      PT(1,3)=PT(1,2)
      W(1)=1.0/3.0
      PT(2,1)=1.0/6.0
      PT(2,2)=PT(1,1)
      PT(2,3)=PT(2,1)
      W(2)=W(1)
      PT(3,1)=PT(2,1)
      PT(3,2)=PT(3,1)
      PT(3,3)=PT(1,1)
      W(3)=W(1)
      RETURN
      END
```

FORTRAN IV G LEVEL 21 MAIN DATE = 82252 13/28/57 PAGE 0001

```
C
C
      SUBROUTINE MAX(A,Z,M,N,MODE)
      DIMENSION A(90,25)
    1 Z=ABS(A(1,MODE))
      M=1
      DO 2 I=2,N
      II=(I-1)*3+1
      Y=ABS(A(II,MODE))
      IF(Y-Z)2,2,3
    3 Z=Y
      M=I
    2 CONTINUE
    4 RETURN
      END
```

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```
      C------------------------------------------------------------
      C
      C
      C     SUBROUTINE FOR PLOTTING THE MODE SHAPES
      C
      C
      C------------------------------------------------------------
            SUBROUTINE MODSHP(NY,NV,XX,COORDS,MAXNOD,THETA,EIG,TH,NCASE,PHI,
           1HX,HY,NGRPH)
            DIMENSION NOD(25,50),XX(90,25),COORDS(100,2),X(90),Y(90),EIG(90)
            PI=3.1415926
            SUP=PHI
            TF=HX
      C
            IF(NGRPH.EQ.1)CALL PLOTS(105.0,27.5)
            IF(NGRPH.EQ.2)CALL PLOT(35.0,0.0,-3)
            IF(NGRPH.EQ.3)CALL PLOT(60.0,0.0,-3)
      C      CALL NEWPEN(2)
            F=0.8
            CALL FACTOR(F)
            CALL SYMBOL(4.5,26.5,0.3,21HR(SPAN/ROOTCORD)      =,0.0,21)
            CALL NUMBER(999.0,999.0,0.3,SUP,0.,1)
            CALL SYMBOL(4.5,25.5,0.3,21HT.E. SWEEPBACK ANGLE=,0.0,21)
            CALL NUMBER(999.0,999.0,0.3,THETA,0.,1)
            CALL SYMBOL(999.0,999.0,0.3,80,0.0,-1)
            CALL SYMBOL(4.5,24.5,0.3,6HT1/T0=,0.0,6)
            CALL NUMBER(999.0,999.0,0.3,HX,0.0,1)
            CALL SYMBOL(999.,999.,0.3,12H        T2/T0=,0.0,12)
            CALL NUMBER(999.,999.,0.3,HY,0.,1)
            CALL SYMBOL(4.5,23.5,0.3,21HN.D. FREQ FOR MODE 1=,0.0,21)
            CALL NUMBER(999.0,999.0,0.3,EIG(1),0.0,3)
            IF(NCASE.EQ.0)GO TO 93
            CALL SYMBOL(4.5,22.5,0.3,21HSIMPLE SUPPORT AT TIP,0.0,21)
   93       F=0.5/PHI
            CALL FACTOR(F)
    9       NV1=NV-1
            DO 7000 MODE=2,NV1
            CALL MAX(XX,Z,MAXM,MAXNOD,MODE)
            DO 11 M=1,MAXNOD
            MM=(M-1)*3+1
            XX(MM,MODE)=XX(MM,MODE)/Z
   11       CONTINUE
            WRITE(6,9000)Z
 9000       FORMAT(//,5X,'MAX. DISPL=',E15.7,//)
            DO 150 I=1,MAXNOD
            II=(I-1)*3+1
  150       XX(II,MODE)=XX(II,MODE)*PHI
            MO=MODE
            NCOUNT=0
            NOD(1,1)=1
            NY1=NY+1
            DO 66 NST=2,NY1
            IF(NST.EQ.NY1)GO TO 50
            GO TO 49
   50       N1=NY1
            NOD(N1,1)=N1
            DO 51 NN=2,N1
   51       NOD(N1,NN)=NOD(N1,NN-1)+NY-NN+2
            NN=N1-1
            GO TO 55
   49       CONTINUE
            N1=NST-1
```



```
            NOD(NST,1)=NOD(N1,1)+(NY-NST+3)
            N=NY-NST+2
            DO 20 I=2,N
   20       NOD(NST,I)=NOD(NST,I-1)+1
   95       NN=N-1
   55       DO 65 I=1,NN
            M=1+(NOD(NST,I)-1)*3
            W1=XX(M,MODE)
            M=1+(NOD(NST,I+1)-1)*3
            W2=XX(M,MODE)
            IF(W2.EQ.0.0)GO TO 65
            IF((W1/W2).LT.0.0)GO TO 60
            GO TO 65
   60       NCOUNT=NCOUNT+1
            AX=COORDS(NOD(NST,I+1),1)-COORDS(NOD(NST,I),1)
            X(NCOUNT)=COORDS(NOD(NST,I),1)+AX*ABS(W1)/(ABS(W1)+ABS(W2))
            AY=COORDS(NOD(NST,I+1),2)-COORDS(NOD(NST,I),2)
            Y(NCOUNT)=COORDS(NOD(NST,I),2)+AY*ABS(W1)/(ABS(W1)+ABS(W2))
   65       CONTINUE
   66       CONTINUE
   25       DO 96 NST=1,NY
            NOD(NST,1)=NST
            N=NY-NST+2
            DO 90 I=2,N
   90       NOD(NST,I)=NOD(NST,I-1)+NY-I+3
            N1=N-1
            DO 15 I=1,N1
            M=1+(NOD(NST,I)-1)*3
            W1=XX(M,MODE)
            M=1+(NOD(NST,I+1)-1)*3
            W2=XX(M,MODE)
            IF(W2.EQ.0.0)GO TO 15
            IF((W1/W2).LT.0.0)GO TO 10
            GO TO 15
   10       NCOUNT=NCOUNT+1
            AX=COORDS(NOD(NST,I+1),1)-COORDS(NOD(NST,I),1)
            X(NCOUNT)=COORDS(NOD(NST,I),1)+AX*ABS(W1)/(ABS(W1)+ABS(W2))
            AY=COORDS(NOD(NST,I+1),2)-COORDS(NOD(NST,I),2)
            Y(NCOUNT)=COORDS(NOD(NST,I),2)+AY*ABS(W1)/(ABS(W1)+ABS(W2))
   15       CONTINUE
   96       CONTINUE
 1000       FORMAT(//,5X,(5X,I5))
 7000       CONTINUE
            WRITE(6,2000)MODE,NCOUNT,((X(I),Y(I)),I=1,NCOUNT)
 2000       FORMAT(//,3X,I5,3X,I5,//,3X,(E15.7,3X,E15.7))
            IF(MODE.NE.3 .AND. MODE.NE.5)GO TO 40
            AMODE=[illegible]*(COORDS(MAXNOD,1)/PHI/COS(THETA*PI/180.))
            WRITE(6,3000)AMODE
 3000       FORMAT(//,5X,F6.3)
            YM[illegible]=[illegible].0
            IF(THETA.GT.10.0)YMODE=7.0
            IF([illegible].0)YMODE=YMODE+COORDS(NY+1,2)+3.0
            GO TO [illegible]
   40       XM[illegible].0
            Y[illegible].0
            IF(THETA.GT.15.0)YMODE=7.0
            IF([illegible].EQ.2)YMODE=YMODE+COORDS(NY+1,2)+3.0
   42       CONTINUE
      C     [illegible](2)
            XM[illegible]*PHI
            [illegible]*PHI
            [illegible]
```

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```
0114            CALL PLOT(0.0,-1.0,2)
0115            X1=COORDS(NY+1,1)
0116            X2=COORDS(NY+1,2)+1.0
0117            CALL PLOT(X1,X2,2)
0118            CALL PLOT(X1,COORDS(NY+1,2),3)
0119            CALL DASHP(COORDS(MAXNOD,1),COORDS(MAXNOD,2),.1)
0120            CALL PLOT(COORDS(MAXNOD,1),COORDS(MAXNOD,2),3)
0121            CALL DASHP(0.0,0.0,.1)
0122            CALL PLOT(0.0,0.0,3)
0123            DELTA=0.25
0124            DO 59 IK=1,25
0125            C1=-1.0+DELTA
0126            IF(C1.GT.X2)GO TO 59
0127            CALL PLOT(0.0,C1,3)
0128            CX=C1-0.5
0129            CALL PLOT(-0.2,CX,2)
0130            DELTA=DELTA+0.5
0131      59    CONTINUE
0132            SC=0.0
0133            IF(THETA.EQ.15.0)SC=1.0
0134            IF(THETA.EQ.30.0)SC=3.25
0135            S1X=2.75
0136            S1Y=-1.0-SC
0137            S2X=1.75
0138            S2Y=-1.5-SC
0139            SIZESB=0.25*PHI
0140            SIZENB=0.25*PHI
0141            CALL SYMBOL(S1X,S1Y,SIZESB,6H MODE ,0.0,6)
0142            CALL NUMBER(999.0,999.0,SIZENB,FLOAT(MD),0.,-1)
0143            CALL SYMBOL(S2X,S2Y,SIZESB,13HNON-DIM.FREQ= ,0.0,13)
0144            CALL NUMBER(999.0,999.0,SIZENB,EIG(MODE),0.0,3)
      C          CALL SYMBOL(999.0,999.0,SIZESB,4H CPS,0.0,4)
      C         IF(MODE.LT.5)GO TO 501
0145      502   DO 500 I=1,NCOUNT
0146            SYB=0.15*PHI
0147            CALL SYMBOL(X(I),Y(I),SYB,1,0.,-1)
0148      500   CONTINUE
0149            GO TO 503
0150      501   NC=NCOUNT+1
0151            X(NC)=0.0
0152            X(NC+1)=1.0
0153            Y(NC)=0.0
0154            Y(NC+1)=1.0
0155            CALL FLINE(X,Y,-NCOUNT,1,1,1)
      C
      C         FOR ISOMETRIC VIEW
      C
      C
0156      503   CONTINUE
      C503       CALL NEWPEN(2)
0157            ARS=0.1*PHI
      C         DO 83 NST=1,NY
0158            NST=1
0159            N=NY-NST+2
0160            IF(NST.EQ.1)N=N+1
0161            DO 82 NM=2,N
0162            NA=NM-1
0163            M1=1+(NOD(NST,NA)-1)*3
0164            CALL PLOT(COORDS(NOD(NST,NA),1),COORDS(NOD(NST,NA),2),3)
0165            X(NM)=COORDS(NOD(NST,NA),1)
0166            Y(NM)=COORDS(NOD(NST,NA),2)+XX(M1,MODE)
0167            IF(XX(M1,MODE).GT.0.0)GO TO 81
```



```
0168            CALL DASHP(X(NM),Y(NM),0.075)
0169            IF(ABS(XX(M1,MODE)).LT.0.25)GO TO 82
0170            CALL ARCHD(X(NM),COORDS(NOD(NST,NA),2),X(NM),Y(NM),ARS,0.0,04)
0171            GO TO 82
0172      81    CALL PLOT(X(NM),Y(NM),2)
0173            IF(ABS(XX(M1,MODE)).LT.0.25)GO TO 82
0174            CALL ARCHD(X(NM),COORDS(NOD(NST,NA),2),X(NM),Y(NM),ARS,0.0,04)
0175      82    CONTINUE
0176            X(1)=0.0
0177            Y(1)=0.0
0178            X(2)=.2
0179            Y(2)=0.0
0180            X(N+1)=0.0
0181            X(N+2)=1.0
0182            Y(N+1)=0.0
0183            Y(N+2)=1.0
0184            CALL FLINE(X,Y,-N,1,0,1)
0185      83    CONTINUE
0186            N1=NY+1
0187            NOD(N1,1)=N1
0188            DO 84 NA=2,N1
0189      84    NOD(N1,NA)=NOD(N1,NA-1)+NY-NA+2
0190            DO 89 NA=1,N1
0191            M2=1+(NOD(N1,NA)-1)*3
0192            X(NA)=COORDS(NOD(N1,NA),1)
0193            CALL PLOT(COORDS(NOD(N1,NA),1),COORDS(NOD(N1,NA),2),3)
0194            Y(NA)=COORDS(NOD(N1,NA),2)+XX(M2,MODE)
0195            IF(XX(M2,MODE).GT.0.0)GO TO 88
0196            IF(XX(M2,MODE).EQ.0.0)GO TO89
0197            CALL DASHP(X(NA),Y(NA),0.075)
0198            IF(ABS(XX(M2,MODE)).LT.0.25)GO TO 89
0199            CALL ARCHD(X(NA),COORDS(NOD(N1,NA),2),X(NA),Y(NA),ARS,0.0,04)
0200            GO TO 89
0201      88    CALL PLOT(X(NA),Y(NA),2)
0202            IF(ABS(XX(M2,MODE)).LT.0.25)GO TO 89
0203            CALL ARCHD(X(NA),COORDS(NOD(N1,NA),2),X(NA),Y(NA),ARS,0.0,04)
0204      89    CONTINUE
0205            X(N1+1)=0.
0206            X(N1+2)=1.0
0207            Y(N1+1)=0.0
0208            Y(N1+2)=1.0
0209            CALL FLINE(X,Y,-N1,1,0,1)
0210            XORG=XMODE
0211            YORG=YMODE
0212            CALL PLOT(XORG,YORG,-3)
0213      5000  CONTINUE
0214            IF(IORPH.EQ.2)CALL PLOT(0.0,0.0,999)
0215            RETURN
0216            END
```

```
0001          SUBROUTINE JACOBI(A,B,X,EIGV,D,N,RTOL,NSMAX,IFPR,IOUT)
        C.......................................................................
        C.
        C.  PROGRAM
        C.     TO SOLVE THE GENERALIZED EIGENPROBLEM USING THE
        C.     GENERALIZED JACOBI ITERATION
        C.
        C. - - INPUT VARIABLES - -
        C.       A(N,N)     =STIFFNESS MATRIX(ASSUMED POSITIVE DEFINITE)
        C.       B(N,N)     =MASS MATRIX(ASSUMED POSITIVE DEFINITE)
        C.       X(N,N)     =MATRIX STORING EIGEN VECTORS ON SOLN EXIT
        C.       EIGV(N)    =VECTOR STORING EIGEN VALUES
        C.       D(N)       =WORKING VECTOR
        C.       N          =ORDER OF MATRICES A AND B
        C.       RTOL       =CONVERGENCE TOLERANCE(USUALLY SET TO 10.**-12)
        C.       NSMAX      =MAXIMUM NUMBER OF SWEEPS ALLOWED(USU. SET TO 15)
        C.       IFPR       =FLAG FOR PRINTING DURING ITERATION
        C.            EQ.0   NO PRINTING
        C.            EQ.1   INTERMIDIATE RESULTS ARE PRINTED
        C.       IOUT       =OUTPUT DEVICE NUMBER
        C.
        C. - - OUTPUT - -
        C.       A(N,N)     =DIAG. STIFFNESS MATRIX
        C.       B(N,N)     =DIAG. MASS MATRIX
        C.       X(N,N)     =EIGENVECTORS STORED COLUMNWISE
        C.       EIGV(N)    =EIGENVALUES
        C.
        C.......................................................................
0002          DIMENSION A(90,90),B(90,90),X(90,90),EIGV(90),D(90)
        C
        C     INITIALIZE EIGENVALUE AND EIGENVECTORS
        C
0003          DO 10 I=1,N
0004          IF(A(I,I).GT.0. .AND. B(I,I).GT.0.) GO TO 4
0005          WRITE(6,2020)
0006          STOP
0007     4    D(I)=A(I,I)/B(I,I)
0008    10    EIGV(I)=D(I)
0009          DO 30 I=1,N
0010          DO 21 J=1,N
0011    21    X(I,J)=0.
0012    30    X(I,I)=1.
0013          IF(N.EQ.1) RETURN
        C
        C     INITIALIZE SWEEP COUNTER AND EIGEN ITERATION
        C
0014          NSWEEP=0
0015          NR=N-1
0016    40    NSWEEP=NSWEEP+1
0017          IF(IFPR.EQ.1) WRITE(6,2000)NSWEEP
        C
        C     CHECK IF PRESENT OFF-DIAG. ELEMENT IS LARGE ENOUGH TO REQUIRE ZEROING
        C
0018          EPS=(.01**NSWEEP)**2
0019          DO 210 J=1,NR
0020          JJ=J+1
0021          DO 210 K=JJ,N
0022          EPTOLA=(A(J,K)*A(J,K))/(A(J,J)*A(K,K))
0023          EPTOLB=(B(J,K)*B(J,K))/(B(J,J)*B(K,K))
0024          IF(EPTOLA.LT.EPS .AND. EPTOLB.LT.EPS) GO TO 210
        C
        C     IF ZEROING IS REQD CAL....
```



```
        C
0025          AKK=A(K,K)*B(J,K)-B(K,K)*A(J,K)
0026          AJJ=A(J,J)*B(J,K)-B(J,J)*A(J,K)
0027          AB=A(J,J)*B(K,K)-A(K,K)*B(J,J)
0028          CHECK=(AB*AB+4.*AKK*AJJ)/4.
0029          IF(CHECK)50,60,60
0030    50    WRITE(6,2020)
0031          STOP
0032    60    SQCH=SQRT(CHECK)
0033          D1=AB/2.+SQCH
0034          D2=AB/2.-SQCH
0035          DEN=D1
0036          IF(ABS(D2).GT.ABS(D1))DEN=D2
0037          IF(DEN) 80,70,80
0038    70    CA=0.
0039          CG=-A(J,K)/A(K,K)
0040          GO TO 90
0041    80    CA=AKK/DEN
0042          CG=-AJJ/DEN
        C
        C     PERFORM .......
        C
0043    90    IF(N-2) 100,190,100
0044   100    JP1=J+1
0045          JM1=J-1
0046          KP1=K+1
0047          KM1=K-1
0048          IF(JM1-1)130,110,110
0049   110    DO 120 I=1,JM1
0050          AJ=A(I,J)
0051          BJ=B(I,J)
0052          AK=A(I,K)
0053          BK=B(I,K)
0054          A(I,J)=AJ+CG*AK
0055          B(I,J)=BJ+CG*BK
0056          A(I,K)=AK+CA*AJ
0057   120    B(I,K)=BK+CA*BJ
0058   130    IF(KP1-N)140,140,160
0059   140    DO 150 I=KP1,N
0060          AJ=A(J,I)
0061          BJ=B(J,I)
0062          AK=A(K,I)
0063          BK=B(K,I)
0064          A(J,I)=AJ+CG*AK
0065          B(J,I)=BJ+CG*BK
0066          A(K,I)=AK+CA*AJ
0067   150    B(K,I)=BK+CA*BJ
0068   160    IF(JP1-KM1)170,170,190
0069   170    DO 180 I=JP1,KM1
0070          AJ=A(J,I)
0071          BJ=B(J,I)
0072          AK=A(I,K)
0073          BK=B(I,K)
0074          A(J,I)=AJ+CG*AK
0075          B(J,I)=BJ+CG*BK
0076          A(I,K)=AK+CA*AJ
0077   180    B(I,K)=BK+CA*BJ
0078   190    AK=A(K,K)
0079          BK=B(K,K)
0080          A(K,K)=AK+2.*CA*A(J,K)+CA*CA*A(J,J)
0081          B(K,K)=[illegible]
0082          A(J,J)=[illegible]
```

```
            B(J,J)=B(J,J)+2.*CG*B(J,K)+CG*CG*BK
            A(J,K)=0.
            B(J,K)=0.
      C
      C     UPDATE......
      C
            DO 200 I=1,N
            XJ=X(I,J)
            XK=X(I,K)
            X(I,J)=XJ+CG*XK
      200   X(I,K)=XK+CA*XJ
      210   CONTINUE
      C
      C     UPDATE....
      C
            DO 220 I=1,N
            IF(A(I,I).GT.0. .AND. B(I,I).GT.0.)GO TO 220
            WRITE(6,2020)
            STOP
      220   EIGV(I)=A(I,I)/B(I,I)
            IF(IFPR.EQ.0)GO TO 230
            WRITE(6,2030)
            WRITE(6,2010)(EIGV(I),I=1,N)
      C
      C     CHECK.....
      C
      230   DO 240 I=1,N
            TOL=RTOL*D(I)
            DIF=ABS(EIGV(I)-D(I))
            IF(DIF.GT.TOL)GO TO 280
      240   CONTINUE
      C
      C     CHECK......
      C
            EPS=RTOL**2
            DO 250 J=1,NR
            JJ=J+1
            DO 250 K=JJ,N
            EPSA=(A(J,K)*A(J,K))/(A(J,J)*A(K,K))
            EPSB=(B(J,K)*B(J,K))/(B(J,J)*B(K,K))
            IF((EPSA.LT.EPS).AND.(EPSB.LT.EPS))GO TO 250
            GO TO 280
      250   CONTINUE
      C
      C     FILL......
      C
      255   DO 260 I=1,N
            DO 260 J=1,N
            A(J,I)=A(I,J)
      260   B(J,I)=B(I,J)
            DO 270 J=1,N
            BB=SQRT(B(J,J))
            DO 270 K=1,N
      270   X(K,J)=X(K,J)/BB
            RETURN
      C
      C
      C
      280   DO 290 I=1,N
      290   D(I)=EIGV(I)
            IF(NSWEEP.LT.NSMAX)GO TO 40
            GO TO 255
```



```
      2000  FORMAT(//,'SWEEP NUMBER IN *JACOBI*=',I4)
      2010  FORMAT(1H0,6E20.12)
      2020  FORMAT(//,'ERROR**********',//)
      2030  FORMAT(//,'CURRENT EIGENVALUES IN *JACOBI* ARE',/)
            END
```

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