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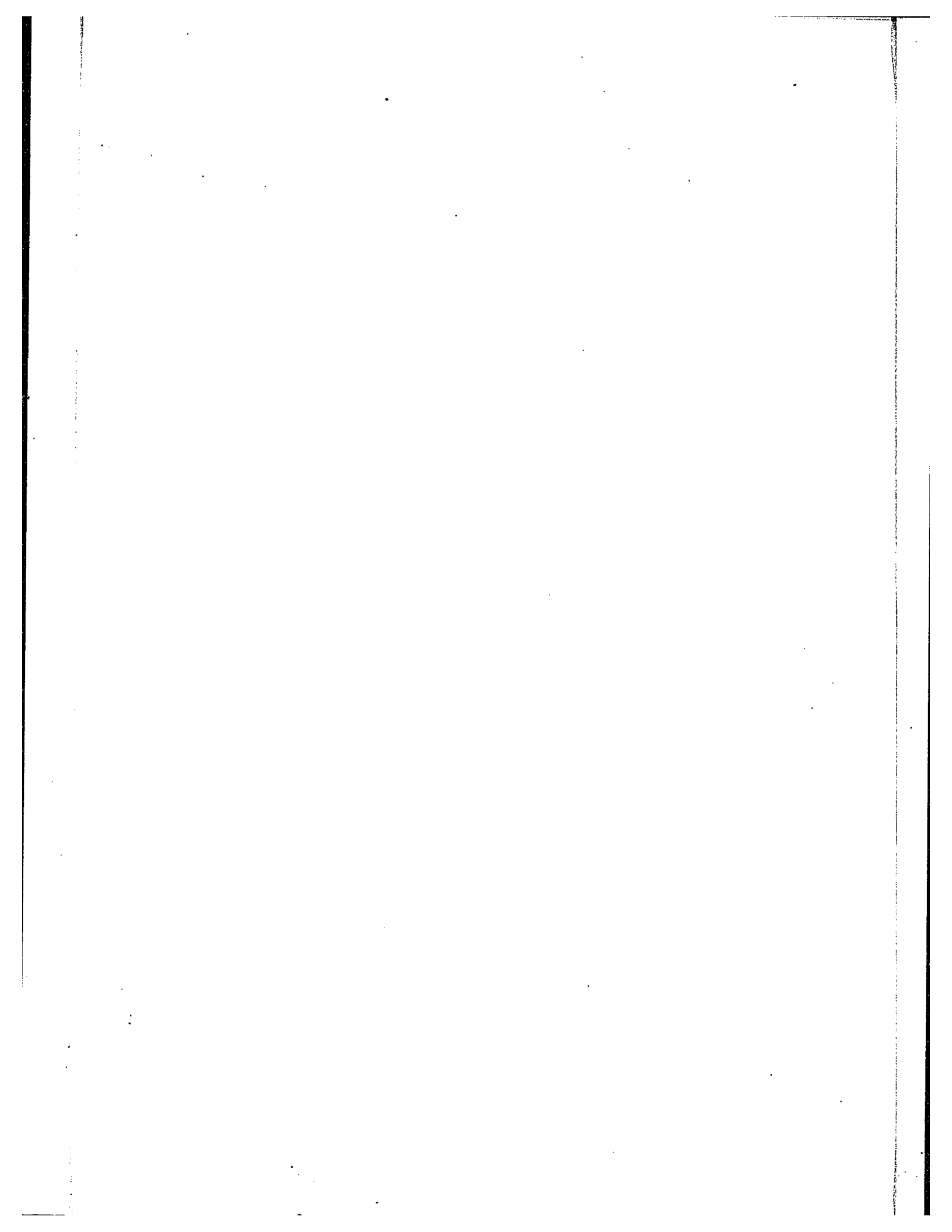
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FLOW OF FLUIDS THROUGH POROUS MEDIA

BY

CARLOS TIU

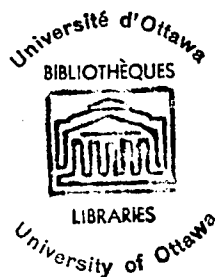
**A thesis submitted in partial fulfillment of the requirement for the
degree of Master of Science**

In the

Department of Chemical Engineering

University of Ottawa

April, 1965



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ACKNOWLEDGMENT

The author is deeply indebted to his director, Professor William Kosicki, who through his advice, guidance, encouragement, and personal understanding, made this thesis possible.

He is also grateful to both faculty and fellow students, particularly to his colleagues, Mr. C. J. Hsu and Mr. C. H. Chou, for their valuable suggestions and criticisms throughout the course of this study.

He wishes to thank the National Research Council for the financial assistance under the grant No. A-1602.

Errors inevitably creep into a work like this, suggestions for improvement and criticisms will be greatly appreciated.

C. Tiu

TABLE OF CONTENTS

Chapter	Page
I. Abstract.....	1
II. Introduction.....	2
III. Literature Survey.....	6
IV. Development of Equations.....	14
V. Approximate Solution to One-Dimensional Constant Pressure Filtration Problem.....	32
VI. Discussion of Results.....	43
VII. Summary and Conclusion.....	49
VIII. Appendices	
a. Appendix-A: Tables.....	57
b. Appendix-B: Derivation of the General Transient Equation for Flow of Incompressible Fluids Through Porous Media.....	67
c. Appendix-C: Supplement to Integral Method	73
IX. Nomenclature.....	75
X. References.....	78

LIST OF TABLES

Table	Page
1. Summary of the Equations for Flow of Fluid Through Compressible Porous Media	51
2. Equations for Constant Pressure Filtration of Newtonian Fluids.....	56
A-1. Summary of Differential Operations Involving the ∇ - and $\nabla\nabla$ - Operator in General Curvilinear Coordinates.....	57
A-2. Unit Vectors and Scale Factors of Various Coordinate Systems.....	59
A-3. Differential Operations Involving the ∇ - and $\nabla\nabla$ - Operator in the General Equation for Flow of Fluid Through Compressible Porous Media.....	62
A-4. The Equation of Continuity in Various Coordinate Systems for Flow of Fluid Through Compressible Porous Media.....	66

I- ABSTRACT

A three-dimensional transient equation for the flow of any time-independent incompressible fluid, Newtonian or non-Newtonian, through compressible porous media has been derived. Equations applicable to Ostwald-de Waele fluids, Bingham fluids, and Newtonian fluids are deduced directly from the general transient equation. It is shown that the one-dimensional filtration equation derived by Tiller (1) is only a special case of the general flow problem.

An equation describing the flow of a compressible fluid (ideal gas) through compressible porous media is also developed. It reduces to the ordinary unsteady flow equation through incompressible porous media when the porosity and specific cake resistance are constant.

The usefulness of the integral method in arriving at a reasonable solution to the one-dimensional constant pressure filtration problem is tested utilizing a linear pressure profile through the cake.

II - INTRODUCTION

The flow of fluids through porous material is a subject of importance in many fields of engineering. Such diversified fields as soil mechanics, ground water hydrology, petroleum engineering, water purification, industrial filtration, ceramic engineering and powder metallurgy all rely heavily upon it as fundamental to their individual problems.

There exist several mechanisms of fluid flow through porous materials. The primary mechanism is, of course, of a purely "mechanical" nature, namely, flow as a result of an applied force in the form of a pressure differential. Ordinarily, this is the flow that is encountered in most situations concerning porous media. However, flow may also occur as a result of applied electrical or thermal gradients, or under a centrifugal field.

The mathematical theory of flow through porous media is generally formulated with Darcy's law being taken as the fundamental law of flow. The law states that "the volumetric rate of flow of a homogeneous fluid through a porous medium is proportional to the driving force per unit volume which acts on the fluid and to the cross-sectional area normal to the direction of flow and inversely proportional

to the viscosity of the fluid". Mathematically, it is expressed as

$$\dot{V} = - \frac{k}{\eta} (\nabla P + \sigma \nabla h) \quad (1)$$

where the medium is assumed to be isotropic, that is, having the same permeability to flow in all directions. Despite its universal application, this law is subject to several limitations:

- a) The fluid is homogeneous, or single phase fluid.
- b) There is no interaction of the fluid and the porous media.
- c) The flow rates are kept relatively small. The region in which Darcy's law is valid is called the "viscous" or "laminaar" range, and that of high rates when Darcy's law breaks down is known as "turbulent".
- d) In the case of gases, there are sometimes deviations from Darcy's law at moderate and low flow rates which are believed due to slip flow. This phenomenon is observed if the pore diameters, become comparable with, or less than, the molecular mean free paths of the flowing gas.

Darcy's law in itself is not sufficient to determine the flow pattern in a porous medium for a given set of boundary conditions as it contains three unknowns (\dot{V} , P , σ). Two further equations are therefore required for the complete specification of the problem. These are the equation of state of the fluid and the equation of continuity.

After a differential form of Darcy's law has been derived for incompressible porous media, attempts have been made to generalize this law for flow of fluids through compressible porous materials. A most important application is in filtration where the cake is compressible. Tiller (1) has treated this subject extensively, and has derived a one-dimensional filtration equation for a Newtonian fluid taking into account that the superficial velocity varies throughout the cake. To date, a general equation for flow of fluids through porous media applicable to all fluids and systems of flow has not been presented.

Physical laws must be independent of any particular coordinate systems used in describing them mathematically, if they are to be valid. Konicki (2) has proposed a general discharge equation which includes the fluid model properties and parameters of the flow geometry that can be applied to any time-independent fluids and any flow geometry. This equation is expressed as

$$\frac{2Ux}{r_H} = \frac{1}{a} T_0^{-b/a} \int_{T_y}^{T_0} T^{\frac{b}{a} - 1} f(T) dT \quad (2)$$

where T_0 is defined as

$$T_o = r_H \left(- \frac{dP'}{dx} \right) \quad (3)$$

It is the purpose of this work to derive a basic differential equation for flow of generalized fluids through compressible porous media based on this new discharge equation (Eq. 2). It will be shown that the equation derived in this paper reduces to the one-dimensional equation for Newtonian fluids as given by Tiller.

In the later section of this work, the usefulness of the integral method will be tested in solving the one-dimensional constant pressure filtration equation using a simple linear pressure profile through the cake.

III - LITERATURE SURVEY

A vast amount of literature on the subject of fluid flow through porous materials has been published in the journals. There are various books which also treat this subject extensively. Muskat (3) treats the flow of homogeneous Newtonian fluids with principal emphasis on incompressible fluids and the solution of Laplace's equation. Carman (4) gives a thorough treatment on the gas permeability and different types of flow of gases into or through porous media. Scheidegger (5) presents a general literature survey on this subject designed as a reference for research workers. Collins (6) emphasizes the formulation of the problem which is designed as an expository textbook for general use.

The theory of laminar flow of fluids through homogeneous porous media is based on a classical experiment originally performed by Darcy (7). Using the idea of hydraulic radius, Blake (8) derived a relationship analogous to Poiseuille's law for flow of fluids through packed column. The permeability is given by

$$k = \frac{d_m^2}{k_o (36)} \frac{E^3}{(1 - E)^2} \quad (4)$$

where $k_o = 2$, if the packed column is regarded as a bundle of parallel circular capillary tubes. However, in laminar flow the assumption of

mean hydraulic radius frequently gives too large a throughput for a given pressure gradient. One would expect that the permeability should be somewhat smaller than given by Eq. 4. A second assumption implicitly made in the development is that the path of the fluid going through the bed is of length which is equal to the length of the packed column. This assumption is not true since the liquid traverses a very tortuous path. Experimental measurement indicates that k_o is somewhere between 4 and 5. Using the value $k_o = 25/6$, the permeability in the well-known Blake-Kozeny equation becomes (9)

$$k = \frac{d_m^2}{150} \frac{E^3}{(1 - E)^2} \quad (5)$$

Unaware of Blake's paper, Kozeny (10) also applied the mean hydraulic radius principle and tortuosity factor to modify Hagen-Poiseuille's equation which relates the permeability to the porosity. Later, Carman (11) modified the Kozeny theory by introducing the concept of specific surface into the permeability. Mathematically, the permeability is

$$k = \frac{E^3}{5 S_o^2 (1 - E)^2} \quad (6)$$

This equation is consistent with Blake's equation (Eq. 5) where the specific surface, S_o (the total particle surface/the volume of particles), is related to mean particle diameter d_m by

$$S_o = 6/d_m \quad (6)$$

The famous Kozeny-Carman equation which takes the form

$$\dot{V}_x = \frac{E^3}{3 S_o^2 \eta (1 - E)^2} \left(\frac{dP}{dx} \right) \quad (7)$$

is derived for viscous flow in granular beds by the assumption of perfectly random packing of discrete particles and through the use of a mean hydraulic pore diameter. Recently, Kosicki and Hsu (12) extended the Kozeny-Carman relationship to flows involving non-Newtonian fluids described by the Ostwald-de Waele model.

Numerous authors (1, 13, 14) have applied the Kozeny equation to the filtration problem. In the filtration of solids from suspensions or in any process in which a liquid flows through the interstices of a solid bed, the direction of flow is in the direction of decreasing pressure. The solids are supported by a screen, cloth, or other solid bed known as the septum or filter medium. In ordinary filtration practice, the solids closest to the septum are packed more densely than the rest of the cake. The mechanical compacting (or compression) pressure, P_c , varies throughout the bed. At any given point in the bed, it is equal to the frictional pressure drop past the particle at that point plus the cumulative frictional pressure drop resulting from all the particles behind it. Thus, the compressive

effect increases with the depth of the cake and reaches a maximum at the septum surface. The porosity is a minimum at the point of contact between the cake and the septum and is maximum at the surface where the liquid enters. Since the filtration resistance depends upon the porosity, the resistance steadily increases as the liquid passes through the cake if the solids are compressible. In order to treat quantitatively the filtration problem, it is necessary to have the relationship between the rate of flow, pressure on the liquid, and the porosity in addition to the various parameters.

In the conventional analysis of constant pressure filtration (14), three approximations are made which may be invalid under certain circumstances. These are a) filtration resistance is constant, b) the ratio of mass of wet to mass of dry cake is constant, and c) at any instant the rate of flow is constant throughout the cake. The first two postulates are primarily in error in the initial period of filtration, while the third assumption involving the constancy of flow rate may be strikingly in error for thick slurries.

It was first shown theoretically in 1953 by Tiller (15) that filtrate volume versus time curves for constant pressure filtration were not perfect parabolas, and later, the same author (1) derived a new partial differential equation for flow through compressible media in which the flow rate varies with the cake thickness. For practical

considerations, variation in cake resistance and the ratio of the mass of wet to mass of dry cake can be neglected when the filtration lasts for more than a few minutes. The equation developed by Tiller for one-dimensional constant pressure filtration is

$$S_c \frac{\partial^2 P_s}{\partial x^2} = S_c \left(\frac{\partial P_s}{\partial x} \right)^2 \frac{d}{dP_s} \ln \alpha_x (1-E) - \mu \rho_s \alpha_x (1-E) \left(\frac{dE}{dP_s} \right) \left(\frac{\partial P_s}{\partial t} \right) \quad (8)$$

or, in terms of weight of cake, it is

$$S_c \frac{\partial^2 P_s}{\partial W_x^2} = S_c \left(\frac{\partial P_s}{\partial W_x} \right)^2 \frac{d}{dP_s} \ln \alpha_x + \frac{\mu \alpha_x}{\rho_s} \frac{d}{dP_s} \ln (1-E) \frac{\partial P_s}{\partial t} \quad (9)$$

It will be shown in this work that the above equation is a special case which can be deduced from the general equation derived hereinafter. Recently (16), a new definition of filtration resistance was developed in view of the new theory (1) of the variation of flow with respect to distance through a filter cake.

Shchelkachev (17) also proposed a theory describing the flow through compressible porous media based on the assumption that

- a) both fluid and medium are elastic bodies following Hooke's law;
- b) the permeability of the medium does not change during compression.

In contrast to the investigations of Tiller, the theory of Shchelkachev

is incomplete inasmuch as no dependence of the permeability on the prevailing fluid pressure has been assumed.

The unsteady flow of compressible liquids through incompressible porous media has been treated extensively in a number of textbooks (3, 6). Solutions of flow problems for this system are facilitated by assuming a linear relation between the density and pressure of the liquid. By contrast, the study of the unsteady flow of gases has been less fruitful than the study of liquid systems, because of the nature of the density relationship which must be used to express the thermodynamic properties of a gas. The primary difficulty is that the basic differential equation which describes such gas flow, is non linear, and it has not been possible as yet to express an exact analytical solution to this equation for general boundary and initial conditions. Numerous approximate methods such as the numerical method, linear approximation, electrical analog simulation, perturbation techniques, etc. have been tried to solve this non linear problem (18, 19, 20, 21). However, to the author's knowledge, a solution of the partial differential equation describing the flow of fluids through compressible porous media, even for the simple one-dimensional filtration problem relating to incompressible Newtonian fluids, developed by Tiller (Eqs. 8 and 9), has not been obtained. The usefulness of the

integral method in arriving at a reasonable solution to the problem was tested by utilizing a linear pressure profile through the cake.

The integral method was first introduced by Pohlhausen (22) in order to solve non linear boundary layer problems in fluid mechanics. A modern account of the von Karman-Pohlhausen method may be found in Schlichting (23). Goodman (24) used this method to solve a variety of heat transfer problems including the phase change problems that involve moving boundaries. The method, however, is general and equally appropriate for solving any problem governed by a diffusion-type equation. As a matter of fact, work has been done elsewhere using this method to solve the unsteady flow of gases through incompressible porous media. The solutions thus obtained, although not exact, are often sufficiently accurate for engineering purposes. Since filtration is a moving boundary problem, it is believed that the integral method may be applied equally well to this type of problem. It is also possible in principle to extend this method to solve the flow of compressible gases through compressible porous materials.

A review of the literature appearing in a multitude of journals and books concerning the flow of fluids through porous media indicates

that a general flow equation for generalized fluids, particularly, non-Newtonian fluids, through compressible porous material has not yet been proposed. Hence, it is worthwhile and desirable that more effort be exerted in this particular field of research.

IV - DEVELOPMENT OF EQUATIONS

The flow of fluids through porous media in laminar flow is governed by Darcy's law (Eq. 1). However, the applicability of the original Darcy's equation is restricted to simple Newtonian fluids only. It is the object here to develop a general differential equation describing the flow of any time-independent fluid through compressible porous materials.

Kozicki (2) has proposed a general discharge equation for any time-independent fluid and any flow geometry under the assumptions that a) the system is isothermal, b) fluid is incompressible with constant fluid properties, and c) flow is steady, laminar and unidirectional. Mathematically, this equation is

$$\frac{2U_x}{r_H} = \frac{1}{a} T_0^{-\frac{b}{a}} \int_{T_y}^{T_0} T^{\frac{b}{a}-1} f(T) dT \quad (2)$$

where $T_0 = r_H (-dP'/dx)$

$T_y =$ yield stress

$a, b =$ functions of flow geometry only

$f(T) = dU_i/dx_j$ which is a function of the fluid model obtained from the shear stress and shear rate relationship;

T can be T_{rz} for flow in pipe, T_{xy} for slit flow, etc.

Analogous to Darcy's law, it is postulated that Eq. (2) can be extended to three-dimensional flow. Thus, Eq. (2) becomes

$$\bar{U} = \frac{r_H}{2k} |\bar{T}_0|^{-\frac{b}{a}-1} \int_{T_y}^{|\bar{T}_0|} T^{\frac{b}{a}-1} f(T) dT (\bar{T}_0) \quad (10)$$

in the most general case, where \bar{T}_0 is defined as

$$\bar{T}_0 = -r_H (\nabla P) \quad (11)$$

It should be noted here that Eq. (10) is in vector form, and $\bar{T}_0^{-\frac{b}{a}}$ is a vector quantity having a magnitude $|\bar{T}_0^{-\frac{b}{a}}|$ and in the direction of pressure gradient, i. e.,

$$\bar{T}_0^{-\frac{b}{a}} = |\bar{T}_0|^{-\frac{b}{a}-1} (\bar{T}_0) \quad (12)$$

For the case in which a fluid is flowing through a bed of solids, the true velocity \bar{U} in the interstices of the solids is given by the superficial velocity \dot{V} divided by the porosity, or

$$\bar{U} = \frac{\dot{V}}{E} \quad (13)$$

The hydraulic radius r_H is defined as

$$\begin{aligned} r_H &= \frac{\text{cross section available for flow}}{\text{wetted perimeter}} \\ &= \frac{\text{void volume}}{\text{surface of solids}} \\ &= \frac{E}{S_0 (1-E)} \end{aligned} \quad (14)$$

where $E/(1 - E)$ represents the ratio of the void volume to the volume of the solids, and the specific surface, S_o , is the surface area of solids per unit volume of solids.

The substitution of Eqs. (13) and (14) into Eq. (10) gives the superficial velocity of any time-independent fluid flows through a solid bed in terms of pressure gradient and porosity of the bed, i. e.

$$\dot{V} = \frac{E^2}{2\mu S_o (1 - E)} \left| T_o \right|^{-\frac{b}{a} - 1} \int_{T_y}^{|T_o|} T^{\frac{b}{a} - 1} f(T) dT (\bar{T}_o) \quad (15)$$

The above equation (Eq. 15), together with the equation of continuity, serves as the fundamental equation in the derivation of the basic differential equation for flow of fluids through porous material.

(F) Modified Equation of Continuity

The equation of continuity is derived in usual fashion by applying the law of conservation of material with no sources or sinks, i. e.,

$$\text{Input} - \text{Output} = \text{Accumulation} \quad (16)$$

For this particular case in which a fluid is flowing in a porous medium, the porosity should be taken into account.

In an arbitrary surface, there is enclosed a volume of fluid which has a specific weight and moves with a superficial velocity \dot{V} . At any time, the material within the volume is

$$M = \iiint_v E \sigma dv \quad (17)$$

The time rate of change of this material is

$$\begin{aligned} \frac{\partial M}{\partial t} &= \frac{\partial}{\partial t} \iiint_v E \sigma dv \\ &= \iiint_v \frac{\partial}{\partial t} (E \sigma) dv \end{aligned} \quad (18)$$

The material leaving a surface s per unit time is

$$\dot{M} = \iint_s \sigma \dot{V} \cdot \bar{n} ds \quad (19)$$

The application of divergence theorem shows that

$$\iint_s \sigma \dot{V} \cdot \bar{n} ds = \iiint_v \nabla \cdot (\sigma \dot{V}) dv \quad (20)$$

The substitution of Eqs. (18), (19) and (20) into Eq. (16) yields

$$\iiint_v \left[\nabla \cdot (\sigma \dot{V}) + \frac{\partial}{\partial t} (E \sigma) \right] dv = 0 \quad (21)$$

Since the volume element v is chosen arbitrarily, the integrand in Eq. (21), assumed continuous, must be identically zero. Hence,

$$\nabla \cdot (\sigma \dot{V}) = - \frac{\partial}{\partial t} (E \sigma) \quad (22)$$

Equation (22) is the equation of continuity for flow of fluids through porous medium. This equation reduces to the ordinary equation of continuity if $E = 1$.

(II) General Flow Equation for Incompressible Fluid Flow Through Compressible Porous Media

For the general case in which an incompressible fluid is flowing through a compressible porous medium, the specific weight, γ , of the fluid is constant. Thus, the equation of continuity (Eq. 22) reduces to

$$\nabla \cdot \dot{V} = - \frac{\partial E}{\partial t} \quad (23)$$

When Eq. (15) is substituted into Eq. (23), one obtains (see Appendix B for the complete derivation),

$$A_1 (\nabla P' \cdot \nabla E) + A_2 (\nabla P' \cdot \nabla P' \cdot \nabla \nabla P') + A_3 (\nabla^2 P') = 2a S_o \frac{\partial E}{\partial t} \quad (24)$$

where

$$A_1 = S_o \frac{b}{a} \left[\frac{E^{1-\frac{b}{a}} (2-E-\frac{b}{a})}{(1-E)^2 - \frac{b}{a}} \right] |\nabla P'|^{-\frac{b}{a}-1} \int_{T_y}^{|T_o|} g(T) dT + \frac{E}{(1-E)^2} |\nabla P'|^{-1} f(|T_o|) \quad (25)$$

$$A_2 = -(\frac{b}{a} + 1) S_o \frac{b}{a} \left[\frac{E^{2-\frac{b}{a}}}{(1-E)^{1-\frac{b}{a}}} \right] |\nabla P'|^{-3\frac{b}{a}-1} \int_{T_y}^{|T_o|} g(T) dT + \frac{E^2}{(1-E)} |\nabla P'|^{-3} f(|T_o|) \quad (26)$$

$$A_3 = S_o \frac{b}{a} \left[\frac{E^{2-\frac{b}{a}}}{(1-E)^{1-\frac{b}{a}}} \right] |\nabla P'|^{-\frac{b}{a}-1} \int_{T_y}^{|T_o|} g(T) dT \quad (27)$$

and

$$g(T) = T^{\frac{b}{a}-1} f(T) \quad (28)$$

Equation (24) is the general transient flow equation for any time-independent incompressible fluid flowing through a compressible porous medium. Since there is no specification of the coordinate system used in the derivation, and the vector operator ∇ and dyadic operator $\nabla\nabla$ are invariant quantities, this flow equation is therefore applicable to any coordinate systems (Table A-1 in Appendix-A gives the summary of the differential operation involving the ∇ and $\nabla\nabla$ -operator in general curvilinear coordinates).

In most engineering applications, one directional flow situation is usually encountered. Thus, it is desirable to express Eq. (24) in two commonly used one dimensional forms.

Case II-a. One-Dimensional Flow in x-Direction in Cartesian

Coordinates

$$A_1 \left(\frac{\partial P'}{\partial x} \right) \left(\frac{\partial E}{\partial x} \right) + A_2 \left(\frac{\partial P'}{\partial x} \right)^2 \left(\frac{\partial^2 P'}{\partial x^2} \right) + A_3 \frac{\partial^2 P'}{\partial x^2} = 2a S_o \frac{\partial E}{\partial t} \quad (29)$$

The coefficients A_1 , A_2 and A_3 are defined the same way as Eq. (25) to (27) except that the absolute value of the pressure gradient becomes

$$\frac{\partial P'}{\partial x}$$

Case II-b. One-Dimensional Flow in Radial Direction in Cylindrical

Coordinates

$$A_1 \left(\frac{\partial P'}{\partial r} \right) \left(\frac{\partial E}{\partial r} \right) + A_2 \left(\frac{\partial P'}{\partial r} \right)^2 \left(\frac{\partial^2 P'}{\partial r^2} \right) + A_3 \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial P'}{\partial r} \right) = 2a S_0 \frac{\partial E}{\partial t} \quad (30)$$

In this case, the absolute value of the pressure gradient in the coefficients A_1 , A_2 and A_3 reduces to $\frac{\partial P'}{\partial r}$.

(III) Application of the General Flow Equation to Specific Fluid

Models

In this section, the general flow equation (Eq. 24) is applied to two useful non-Newtonian models, namely, Ostwald-de Waele model and Bingham model, and to the simple Newtonian fluid. The steady-state rheological behavior of most fluids in a simple flow geometry can be expressed by the generalized form (9):

$$T_{yx} = -\eta' \frac{dU_x}{dy} \quad (31)$$

where the apparent viscosity may be expressed as a function of either dU_x/dy or T_{yx} . If η' is independent of the rate of shear, the behavior is Newtonian, with $\eta' = \eta$ (viscosity). The Newtonian fluids are consequently represented by

$$T_{yx} = -\eta \frac{dU_x}{dy} \quad (32)$$

In the region of shear in which η' decreases with increasing rate of shear ($-dU_x/dy$), the behavior is termed as pseudoplastic.

This region can be well represented by the Ostwald-de Waele model or power law model:

$$T_{yx} = -K \left| \frac{dU_x}{dy} \right|^{n-1} \frac{dU_x}{dy} \quad (33)$$

A fluid which has the property to remain rigid when the shear stress is of smaller magnitude than the yield stress T_y but flows somewhat like a Newtonian fluid when the shear stress exceeds T_y is described reasonably accurately by the Bingham model.

Mathematically, it is represented by

$$T_{yx} = -\eta \frac{dU_x}{dy} + T_y \quad \text{if } |T_{yx}| > T_y \quad (34a)$$

$$\frac{dU_x}{dy} = 0 \quad \text{if } |T_{yx}| < T_y \quad (34b)$$

Case III-a. General Flow Equation for Ostwald-de Waele Fluid

For this particular fluid model, $f(T)$ from Eq. (33) is

$$f(T) = -dU_x/dy = \left(\frac{T}{K}\right)^{\frac{1}{n}} \quad (35)$$

When Eq. (35) is substituted into Eq. (24), one gets

$$\begin{aligned} & A_{1p} (\nabla P' \cdot \nabla E) + A_{2p} (\nabla P' \cdot \nabla P' \cdot \nabla \nabla P') + A_{3p} (\nabla^2 P') \\ & = K_1 \frac{\partial E}{\partial t} \end{aligned} \quad (36)$$

where

$$A_{1P} = \left| \nabla P' \right| \frac{1-n}{n} \left[\frac{E^{1+\frac{1}{n}} (2-E+\frac{1}{n})}{(1-E)^{2+\frac{1}{n}}} \right] \quad (37)$$

$$A_{2P} = \frac{1-n}{n} \left| \nabla P' \right| \frac{1-3n}{n} \left[\frac{E^{2+\frac{1}{n}}}{(1-E)^{1+\frac{1}{n}}} \right] \quad (38)$$

$$A_{3P} = \left| \nabla P' \right| \frac{1-n}{n} \left[\frac{E^{2+\frac{1}{n}}}{(1-E)^{1+\frac{1}{n}}} \right] \quad (39)$$

and

$$K_1 = \frac{2(a+bn) K^{\frac{1}{n}} S_o^{1+\frac{1}{n}}}{n} \quad (40)$$

Equation (36) is the general flow equation describing the flow of an Ostwald-de Waele fluid through a compressible porous medium. This equation is also obtained by first evaluating the superficial velocity for the Ostwald-de Waele fluid, that is, by substituting Eq. (35) into Eq. (15), or

$$\dot{V} = -E \left(\frac{r_H}{K} \right)^{\frac{1}{n}} \frac{r_H^n}{2(a+bn)} \left| \nabla P' \right|^{-1+\frac{1}{n}} (\nabla P') \quad (41)$$

and then combining this velocity with the equation of continuity (Eq. 23).

Case III-b. General Flow Equation for Bingham Fluid

The function $f(T)$ for the Bingham model is obtained from Eq. (34a), i.e.

$$f(T) = \frac{T - T_y}{\eta} \quad \text{for} \quad |T| > T_y \quad (42a)$$

Thus, using this expression to evaluate the integral in Eq. (24) gives the general flow equation for a Bingham plastic fluid through compressible porous material. It is

$$A_{1b} (\nabla P' \cdot \nabla E) + A_{2b} (\nabla P' \cdot \nabla P' \cdot \nabla \nabla P') + A_{3b} (\nabla^2 P')$$

$$= 2a S_o \frac{\partial E}{\partial t} \quad (43)$$

where

$$A_{1b} = \frac{S_o^{b/a}}{\eta} \left[\frac{E^{1-b/a} (2-E-\frac{b}{a})}{(1-E)^{2-b/a}} \right] |\nabla P'|^{-\frac{b}{a}-1} h(|T_o|)$$

$$+ \frac{E}{(1-E)^2} |\nabla P'|^{-1} f(|T_o|) \quad (44)$$

$$A_{2b} = -(1+\frac{b}{a}) \frac{S_o^{b/a}}{\eta} \left[\frac{E^{2-b/a}}{(1-E)^{1-b/a}} \right] |\nabla P'|^{3-b/a} h(|T_o|)$$

$$+ \frac{E^2}{(1-E)} |\nabla P'|^{-3} f(|T_o|) \quad (45)$$

$$A_{3b} = \frac{S_o^{b/a}}{\eta} \left[\frac{E^{2-b/a}}{(1-E)^{1-b/a}} \right] |\nabla P'|^{-b/a-1} h(|T_o|) \quad (46)$$

and

$$h(|T_o|) = \frac{|T_o|^{\frac{b}{a}+1} - T_y^{\frac{b}{a}+1}}{1+b/a} - \frac{T_y}{b/a} (|T_o|^{b/a} - T_y^{b/a}) \quad (47)$$

and

$$f(|T_o|) = \frac{|T_o| - T_y}{\eta} \quad (42b)$$

Case III-c. General Flow Equation for Newtonian Fluids

Newtonian fluid is a special type of fluid that follows Newton's law of viscosity. The function $f(T)$ defined by Eq. (32) is

$$f(T) = \frac{T}{\eta} \quad (48)$$

When Eq. (48) is substituted into Eq. (15), one obtains the superficial velocity

$$\dot{V} = - \frac{E^3}{2 \eta S_o^2 (a+b) (1-E)^2} (\nabla P') \quad (49)$$

When Eq. (49) is combined with the equation of continuity (Eq. 23), it yields

$$\nabla^2 P' + \frac{3-E}{E(1-E)} \left[\nabla P' \cdot \nabla E \right] = \frac{2 \eta S_o^2 (a+b) (1-E)^2}{E^3} \left(\frac{\partial E}{\partial t} \right) \quad (50)$$

Equation (50) is the equation describing the transient flow of any incompressible Newtonian fluid through compressible porous medium. It is interesting to note that this equation can also be obtained directly from the equation describing the flow of pseudoplastic fluids (Eq. 36) by setting $n = 1$ and $K = \eta$. Therefore, this is compatible with the theory describing the rheological behavior of the fluids.

One of the most important applications of flow of fluids through porous media is in filtration. Tiller (1) has derived a one-dimensional

partial differential equation (Eq. 8) describing the filtration of a Newtonian slurry under the constant pressure process. It will be shown here that his equation is only a special case which can be reduced directly from the general flow equation derived in this paper.

When the flow of a Newtonian fluid is in the x-direction only, the general flow equation for Newtonian fluids (Eq. 50) reduces to

$$\frac{\partial^2 P'}{\partial x^2} + \frac{3-E}{E(1-E)} \left(\frac{\partial P'}{\partial x} \right) \left(\frac{\partial E}{\partial x} \right) = \frac{2\eta S_o^2 (a+b)(1-E)^2}{E^3} \left(\frac{\partial E}{\partial t} \right) \quad (51)$$

In the conventional filtration analysis, a specific cake resistance is usually defined as

$$\alpha_x = \frac{2 S_o^2 (a+b)(1-E)}{E^3 \rho_s} \quad (52)$$

If this definition is used in Eq. (51), it simplifies to

$$\frac{\partial^2 P'}{\partial x^2} - \left(\frac{\partial P'}{\partial x} \right) \frac{\partial}{\partial x} \left[\ln \alpha_x (1-E) \right] = \eta \rho_s \alpha_x (1-E) \left(\frac{\partial E}{\partial t} \right) \quad (53)$$

Equation (53) is the general one-dimensional filtration equation for Newtonian fluids in terms of a potential function P' . For the case of constant pressure filtration and negligible gravitational force (i. e., $P' = P$),

$$P_a = P + P_s = \text{constant} \quad (54)$$

whence

$$dP = -dP_s \quad (55)$$

If the variable P' in Eq. (53) is replaced in terms of P_s by Eq. (55), it becomes

$$\frac{\partial^2 P_s}{\partial x^2} - \left(\frac{\partial P_s}{\partial x} \right) \frac{\partial}{\partial x} \left[\ln a_x (1-E) \right] = -\eta \rho_s a_x (1-E) \left(\frac{\partial E}{\partial t} \right) \quad (56)$$

Tiller (15, 25, 26) has shown that for moderately compressible materials, it is possible to represent both porosity E and local specific cake resistance a_x by power functions of compressive pressure.

For example:

$$a_x = a_o P_s^\gamma \quad \text{if} \quad P_s > P_i \quad (57a)$$

$$a_x = a_i = a_o P_i^\gamma \quad \text{if} \quad P_s < P_i \quad (57b)$$

$$E = E_o P_s^{-\lambda} \quad \text{if} \quad P_s > P_i \quad (57c)$$

$$E = E_i = E_o P_i^{-\lambda} \quad \text{if} \quad P_s < P_i \quad (57d)$$

where P_i is a low pressure in the range of 0.1 to 1.0 psia, and a_o ,

E_o , γ , λ are constants obtained experimentally. With the above

relationships, Eq. (56) becomes

$$\frac{\partial^2 P_s}{\partial x^2} - \left(\frac{\partial P_s}{\partial x} \right) \frac{d}{dP_s} \left[\ln a_x (1-E) \right] = -\eta \rho_s a_x (1-E) \left(\frac{dE}{dP_s} \right) \left(\frac{\partial P_s}{\partial t} \right) \quad (58)$$

In Eq. (58), P_s is the only dependent variable since the other variables are expressed in terms of the compressive pressure. In principle,

there exists a solution to this differential equation expressing the solid compressive pressure, P_s , as a unique function of position in the cake and time.

In filtration, it is sometimes convenient to express the independent variable in terms of the weight of dry solid in the cake, W_x , rather than the distance x . The weight of solid per square foot in an element of thickness dx is given by

$$\begin{aligned} dW_x &= (\text{density}) (\text{volume of solids/square foot cross-section}) \\ &= \rho_s (1 - E) dx \end{aligned} \quad (59)$$

Thus, if the variable x in Eq. (58) is replaced by W_x using Eq. (59), one obtains

$$\frac{\partial^2 P_s}{\partial W_x^2} = \left(\frac{\partial P_s}{\partial W_x} \right)^2 \frac{d}{dP_s} \ln a_x + \frac{\eta_x}{\rho_s} \frac{d}{dP_s} \ln (1 - E) \left(\frac{\partial P_s}{\partial t} \right) \quad (60)$$

(IV) General Flow Equation for Compressible Fluid (Ideal Gas) Flow Through Compressible Porous Media

The discharge equation proposed by Kozicki (Eq. 2) is not directly applicable to gases since it is derived under the assumption of incompressible fluid. However, since gases behave as Newtonian fluids, the flow equation can be derived directly using Darcy's law.

Darcy's law states that

$$\dot{V} = - \frac{k}{\eta} (\nabla P + \sigma \nabla h) \quad (1)$$

If the specific weight is a function of the pressure only, or constant, use can be made of a potential function $\bar{\Phi}$ defined by the relationship

$$\bar{\Phi} = h + \int_{P_0}^P \frac{dP}{\sigma} \quad (61)$$

which allows writing Eq. (1) in the form

$$\dot{V} = - \frac{k\sigma}{\eta} \nabla \bar{\Phi} \quad (62)$$

If the pressure gradient is large compared to the term $\sigma \nabla h$ such as is often the case with gases, then Eq. (1) or Eq. (62) is simply

$$\dot{V} = - \frac{k}{\eta} \nabla P \quad (63)$$

For the case of compressible fluids, the modified equation of continuity is

$$\nabla \cdot (\sigma \dot{V}) + \frac{\partial}{\partial t} (E\sigma) = 0 \quad (22)$$

When Eq. (63) is substituted in Eq. (20), one gets

$$\nabla \cdot \left[\frac{\sigma k}{\eta} \nabla P \right] = E \frac{\partial \sigma}{\partial t} + \sigma \frac{\partial E}{\partial t} \quad (64)$$

For this particular case, gases are assumed to behave ideally under a moderate pressure range, i. e.,

$$\sigma = \frac{P}{RT} M_0 \quad (65)$$

Viscosity is taken as constant under an isothermal system. This is true only for gases at low density in which viscosity has no dependence on pressure. Thus, Eq. (64) becomes

$$\nabla \cdot [Pk \nabla P] = \eta \left[P \frac{\partial E}{\partial t} + E \frac{\partial P}{\partial t} \right] \quad (66)$$

As the flow through porous media can also be expressed by the Kozeny equation, a comparison of the coefficients shows that the permeability k is related to the porosity and specific cake resistance by

$$k = \frac{1}{c_x \rho_s (1-E)} \quad (67)$$

When this definition of permeability is used in Eq. (66), it yields

$$\nabla^2 P^2 - \nabla P^2 \nabla \left[\ln c_x (1-E) \right] = 2\eta \rho_s c_x (1-E) \left[P \frac{\partial E}{\partial t} + E \frac{\partial P}{\partial t} \right] \quad (68)$$

Equation (68) is the basic differential equation for the flow of an ideal gas through a porous medium. For one-dimensional axial flow, it reduces to

$$\frac{\partial^2 P^2}{\partial x^2} - \left(\frac{\partial P^2}{\partial x} \right) \frac{\partial}{\partial x} \left[\ln c_x (1-E) \right] = 2\eta \rho_s c_x (1-E) \left[P \frac{\partial E}{\partial t} + E \frac{\partial P}{\partial t} \right] \quad (69)$$

and for radial flow, it becomes

$$\frac{1}{r} \left[\frac{\partial}{\partial r} \left(r \frac{\partial P^2}{\partial r} \right) \right] - \left(\frac{\partial P^2}{\partial r} \right) \frac{\partial}{\partial r} \left[\ln c_x (1-E) \right] = 2\eta \rho_s a_x (1-E) \left[F \frac{\partial E}{\partial t} + E \frac{\partial P}{\partial t} \right] \quad (70)$$

In the case of constant pressure filtration where the solid compressive pressure is related to hydraulic pressure by Eq. (54), and if porosity and specific cake resistance can be expressed as functions of compressive pressure alone, then Eq. (68) becomes

$$\nabla^2 P_s - \left[\frac{1}{P_s - P_s} + \frac{d}{dP_s} \ln c_x (1-E) \right] (\nabla P_s)^2 = -\eta \rho_s a_x (1-E) \left[\frac{dE}{dP_s} - \frac{E}{P_s - P_s} \right] \left(\frac{\partial P_s}{\partial t} \right) \quad (71)$$

Similarly, Eq. (69) simplifies to

$$\frac{\partial^2 P_s}{\partial x^2} - \left[\frac{1}{P_s - P_s} + \frac{d}{dP_s} \ln c_x (1-E) \right] \left(\frac{\partial P_s}{\partial x} \right)^2 = -\eta \rho_s a_x (1-E) \left[\frac{dE}{dP_s} - \frac{E}{P_s - P_s} \right] \left(\frac{\partial P_s}{\partial t} \right) \quad (72)$$

It is interesting to note that if the medium is incompressible, that is, both the porosity and specific cake resistance are constants, the general flow equation (Eq. 68) reduces to

$$\nabla^2 P^2 = \frac{2E\eta}{k} \frac{\partial P}{\partial t} \quad (73)$$

which is the well-known unsteady isothermal flow equation applicable to an ideal gas in an incompressible porous medium. For one-dimensional axial flow, Eq. (69) simplifies to

$$\frac{\partial}{\partial x} \left(P \frac{\partial P}{\partial x} \right) = \frac{E\eta}{k} \frac{\partial P}{\partial t} \quad (74)$$

and similarly, Eq. (70) reduces to

$$\frac{\partial^2 P^2}{\partial r^2} + \frac{1}{r} \frac{\partial P^2}{\partial r} = \frac{2E\eta}{k} \frac{\partial P}{\partial t} \quad (75)$$

for one-dimensional radial flow.

V APPROXIMATE SOLUTION TO ONE-DIMENSIONAL
CONSTANT PRESSURE FILTRATION PROBLEM

The non-linear partial differential equations are frequently solved by numerical methods. However, due to the laborious computation involved in the numerical technique, mathematicians are apt to use other simpler approximations to solve non-linear problems, if possible. One of the most useful approximations is the integral method. It has been shown by numerous authors (22, 24) that this method is appropriate to any diffusive-type problem, linear or non-linear. Hence, it is possible that the integral method could be employed to solve a moving boundary filtration problem.

A constant pressure filtration process as shown in Fig. (1) is considered here. In the general case, the superficial velocity varies throughout the cake thickness due to the compressibility of the cake. The solid pressure is zero at the cake surface, and maximum at the supported medium.

The governing partial differential equation is

$$\frac{\partial^2 P_s}{\partial x^2} = \left(\frac{\partial P_s}{\partial x} \right)^2 \frac{d}{dP_s} \left[\ln e_x (1-E) \right] - \eta \rho_s a_x (1-E) \left(\frac{dE}{dP_s} \right) \left(\frac{\partial P_s}{\partial t} \right) \quad (58)$$

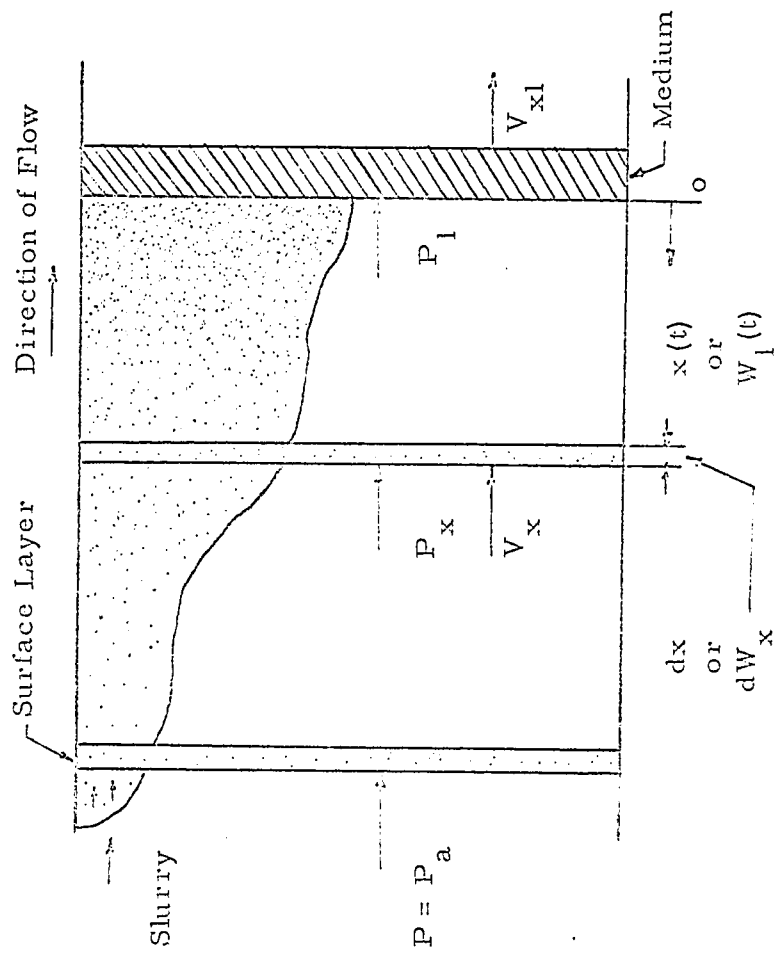


FIGURE 1. SCHEMATIC DIAGRAM OF FILTER BED

or, in terms of the weight of solid, it is

$$\frac{\partial^2 P_s}{\partial W_x^2} = \left(\frac{\partial P_s}{\partial W_x} \right)^2 \frac{d}{dP_s} \ln a_x + \frac{\mu a_x}{g_c \rho_s} \frac{d}{dP_s} \ln(1-E) \left(\frac{\partial P_s}{\partial t} \right) \quad (60)$$

Tiller (26) has shown that the porosity E and local specific cake resistance a_x can be approximately represented by power functions of compressive pressure, viz.,

$$a_x = a_o P_s^\gamma \quad (57a)$$

$$E = E_o P_s^{-\lambda} \quad (57c)$$

The substitution of Eq. (57a) and (57c) in Eq. (60) gives

$$\frac{\partial^2 P_s}{\partial W_x^2} = \left(\frac{\gamma}{P_s} \right) \left(\frac{\partial P_s}{\partial W_x} \right)^2 + \frac{\mu a_o E_o}{(144) g_c \rho_s} \left(\frac{P_s^{\gamma-\lambda-1}}{1 - E_o P_s^{-\lambda}} \right) \left(\frac{\partial P_s}{\partial t} \right) \quad (76)$$

The integral method concept involves first assuming a profile. For instance, in the present case, a pressure profile is assumed and substituted in the original partial differential equation. The resulting differential equation is then integrated with respect to one of the independent variables. Thus, the differential equation will, thereby, be satisfied only on the average. The number of independent variables in the original equation is reduced by one after the integration. In this particular case, Eq. (76) is integrated with respect to weight of solid W_x which reduces the partial differential equation to ordinary differential equation in terms of time variable, or

$$\int_0^{W_1(t)} \left(\frac{1 - E_0 P_s^{-\lambda}}{P_s^{\gamma - \lambda - 1}} \right) \left(\frac{\partial^2 P_s}{\partial W_x^2} - \frac{\gamma}{P_s} \left(\frac{\partial P_s}{\partial W_x} \right)^2 \right) dW_x$$

$$= K_2 \frac{d}{dt} \int_0^{W_1(t)} P_s dW_x \quad (77)$$

where

$$K_2 = \frac{\mu \pi_0 E_0 \lambda}{(144) g_c \rho_s} \quad (78)$$

For simplicity, solid compressive pressure is assumed to be a linear function of the weight of solid deposited, i. e.,

$$P_s = a_0 + a_1 W_x \quad (79)$$

where the coefficients a_0 and a_1 may depend on time t . In order to evaluate these coefficients, the pressure profile must satisfy the following boundary conditions:

B. C. (I):

$$P_s (W_x = W_1(t), t) = 0 \quad (80)$$

This condition states that the solid compressive pressure at the surface layer is zero, since the hydraulic pressure at this point is equal to the applied pressure.

B. C. (II):

$$P_s (W_x = 0, t) = P_a - P_1 \quad (81)$$

where P_1 is the pressure at the septum which is related to the exit velocity V_{xl} by

$$P_1 = \frac{\mu R_m}{S_c} V_{xl} \quad (82)$$

It should be noted here that B. C. (II) is not a known boundary condition.

In the general case, the exit filtrate velocity is related to the rate of change in solid in the cake by

$$V_{xl} = \frac{dq}{dt} = \left(\frac{1-mX}{\rho X} \right) \frac{dW_x}{dt} - \frac{W_x}{\rho} \frac{dm}{dt} \quad (83)$$

When the rate of change of the mass of wet to mass of dry solid becomes negligible, which is true only after a short filtration period has elapsed, then the second term in the right-hand side of Eq. (83) approaches zero, and the rate of filtration is proportional to the rate of deposition of solid, or

$$V_{xl} = \left(\frac{1-mX}{\rho X} \right) \frac{dW_x}{dt} \quad (84)$$

B. C. (II) is conveniently represented by

$$P_s (W_x = 0, t) = f_1(V_{xl}) \quad (85)$$

When these two boundary conditions (Eqs. 80 and 85) are used in Eq. (79), the pressure profile is found to be

$$P_s = f_1(V_{xl}) \left[1 - \frac{W_x}{W_1(t)} \right] \quad (86)$$

The integrals in Eq. (77) are now evaluated using the assumed pressure profile (Eq. 86). It simplifies to an ordinary differential equation, or

$$\frac{f_2(V_{x1})}{W_1} = \frac{K_2}{2} \frac{d}{dt} [f_1(V_{x1}) W_1] \quad (87)$$

where

$$f_2(V_{x1}) = \gamma \left[\frac{E_0 \{f_1(V_{x1})\}^{2-\gamma}}{1-\gamma} - \frac{\{f_1(V_{x1})\}^{\lambda-\gamma+2}}{\lambda-\gamma+1} \right] \quad (88)$$

Equation (87) is the differential equation describing the rate of growth of solid in the filtration operation. This equation can not be integrated analytically since $f_1(V_{x1})$ from B. C. (II) is itself a function of rate of growth of solid. The non-linearity in the equation makes the solution extremely difficult. However, two simple cases will be treated here:

Case 1: Filtrate and time relationship is parabolic.

The experiment often shows that after a short period of filtration, the lines of the reciprocal of the rate of change of filtrate volume vs. the filtrate volume are virtually straight which may be represented by

$$dt/dq = (2/s_1)q + b_1 \quad (89)$$

or upon integration

$$s_1 t = q^2 + s_1 b_1 q \quad (90)$$

Equation (90) is the well-known parabolic equation for constant pressure filtration. In terms of the superficial velocity, it is

$$v_{x1} = dq/dt = (b_1^2 + \frac{4t}{s_1})^{-\frac{1}{2}} \quad (91)$$

With Eq. (91), B. C. II (Eq. 81) can now be approximately represented by

$$P_s (W_x = 0, t) = F_a - \frac{\mu R_m}{g_c} (b_1^2 + \frac{4t}{s_1})^{-\frac{1}{2}} = g_1(t) \quad (92)$$

When Eqs. (89) and (92) are substituted in Eq. (79), the pressure profile becomes

$$P_s = g_1(t) \left[1 - \frac{W_x}{W_1} \right] \quad (93)$$

Using this new pressure profile, Eq. (77) is integrated into

$$\frac{g_2(t)}{W_1} = \frac{K_2}{2} \frac{d}{dt} [g_1(t) W_1] \quad (94)$$

where

$$g_2(t) = \gamma \left[\frac{E_o \{g_1(t)\}^{2-\gamma}}{1-\gamma} - \frac{\{g_1(t)\}^{\lambda-\gamma+2}}{\lambda-\gamma+1} \right] \quad (95)$$

The integration of Eq. (94) with respect to time and making use of the initial condition (i. e., $W_1 = 0$ when $t = 0$) yields

$$W_1 = \frac{1}{g_1(t)} \left[\frac{4}{K_2} \int_0^t g_1(t) g_2(t) dt \right]^{\frac{1}{2}} \quad (96)$$

Equation (96) is the solution to this problem when the filtrate and time relationship is represented approximately by the usual parabolic function. This solution can be used as a trial function which may lead to a more rigorous solution to the problem by substituting the solution into B. C. (II) and obtaining an explicit boundary condition at $W_x = 0$. Then, the differential equation is resolved to obtain a new relationship between the solid deposited and time.

Case 2: Medium Resistance is Negligible

In the case of negligible medium resistance, the solution is greatly simplified since B. C. (II) becomes constant which is equal to the applied pressure P_a . With this new boundary condition, the pressure profile (Eq. 79) now takes the form

$$P_s = P_a \left(1 - \frac{W_x}{W_1}\right) \quad (97)$$

The partial differential equation (Eq. 77) is then integrated into

$$\frac{K'_2}{W_1} = \frac{K_2 P_a}{2} \frac{dW_1}{dt} \quad (98)$$

where

$$K'_2 = \gamma \left[\frac{E_o P_a^{2-\gamma}}{1-\gamma} - \frac{P_a^{\lambda-\gamma+2}}{\lambda-\gamma+1} \right] \quad (99)$$

Finally, the integration of Eq. (98) using the initial condition when $t = 0$, $W_1 = 0$ gives

$$W_1 = \left[\frac{4K'_2}{K_2 P_a} t \right]^{\frac{1}{2}} \quad (100)$$

In the filtration problem, it is usually desired to express the solution in terms of filtrate volume rather than weight of solid.

Thus, substituting Eq. (99) into Eq. (84) gives

$$V_{x1} = \left(\frac{1-m X}{\rho X} \right) \left(\frac{K'_2}{K_2 P_a} t \right)^{\frac{1}{2}} \quad (101)$$

or, in terms of filtrate volume, it is

$$q = 2 \left(\frac{1-m X}{\rho X} \right) \left(\frac{K'_2 t}{K_2 P_a} \right)^{\frac{1}{2}} \quad (102)$$

Equation (102) resembles the well-known parabolic function describing the filtrate volume and time relation.

Example:

Talc is to be filtered at a constant pressure of 15 lb. force/sq. in. under the following conditions:

- $\mu = 0.001$ lb. mass/(ft.) (sec.)
- $\rho = 62.4$ lb. mass/cu. ft.
- $\rho_s = 167.0$ lb. mass/cu. ft.
- $X = 0.20$, mass fraction solids in slurry

$$\alpha = 1.85 (10^{11}) \text{ ft./lb. mass (numerical integration)}$$

$$E_1 = 0.900, \text{ extrapolation from } 1.1 \text{ lb./sq. in.}$$

$$E_{av} = 0.826 \text{ (numerical integration)}$$

$$m = 2.75$$

$$m_1 = 4.36$$

Neglect medium resistance and calculate the time necessary to produce a cake 2 in. in thickness.

Solution:

The problem will be solved by two methods: namely, conventional method and integral method.

A. Conventional Method* - Velocity is assumed to be constant throughout the cake which is equal to the exit velocity V_{x1} .

The mass of dry solids per square foot for a 2 in. cake is given by

$$W = \rho_s (1 - E_{av}) L = 4.84 \text{ lb. mass/sq. ft.} \quad (\text{E-1})$$

and the filtrate volume becomes

$$q = \frac{1 - mX}{X} \quad W = 0.171 \text{ cu. ft./sq. ft.} \quad (\text{E-2})$$

* This solution is reproduced from Tiller (1) for the purpose of comparison. Readers are referred to the original paper (reference 1) for the details of the solution.

Calculating the conventional time one gets

$$t = \frac{\mu \rho X a g^2}{2 g_c P_a (1 - mX)} = 1,100 \text{ sec.} \quad (\text{E-3})$$

B. Integral Method

From the foregoing section, the time and weight of solid relationship is given by

$$t = \frac{K_2 P_a}{4 K'_2} W^2 \quad (\text{E-4})$$

where

$$K_2 = \frac{\mu a_o E_o \lambda}{(144) g_c \rho_s} \quad (\text{E-5})$$

$$K'_2 = \gamma \left[\frac{E_o P_a^{2-\gamma}}{1-\gamma} - \frac{P_s^{\lambda-\gamma+2}}{\lambda-\gamma+1} \right] \quad (\text{E-6})$$

The experimental constants for Talc C are taken from the data given by Tiller (26) which are:

$E_o = 0.86$	$\lambda = 0.054$
$b = 6.26$	$a = 0.075$
$d = 3.11$	$\gamma = 0.506$
$j = 39.6 (10^{12})$	

and a_o is computed from

$$a_o = \frac{aj}{\rho_s} E_o^{-(b+d)} = 0.738 \times 10^{11}$$

Using this data,

$$K_2 = \frac{0.001 (0.738 \times 10^{11}) (0.86) (0.054)}{32.17 (167.0) (144)} = 4.4$$

$$K'_2 = 0.506 \left[\frac{0.86 (15)^{1.494}}{0.494} - \frac{(15)^{1.548}}{0.548} \right] = -10.85$$

$$t = \frac{4.4 (15)}{4 (-10.85)} (4.84)^2 = -35.6 \text{ sec.}$$

VI - DISCUSSION OF RESULTS

The general discharge equation for any flow geometry (Eq. 2) has been proposed under the assumptions that the system is isothermal, the fluid is incompressible with constant flow characteristics, and the flow is steady, laminar and unidirectional. It has been shown elsewhere (28) that the proposed equation agrees reasonably well with the existing data for flow in parallel plates, pipes, annuli, and rectangular channels. It is postulated in this work that the equation is equally appropriate for flow through packed beds or porous media. In fact, Hsu (27) has obtained recently some experimental data to support this postulation.

Heuristically, Eq. (2) is extended to the three-dimensional case as shown in Eq. (10). It should be noted that \bar{T}_0 is by definition a vector quantity in the direction of pressure gradient as defined by Eq. (11), which must not be confused with shear stress where the latter is a tensor.

In the case of flow of fluids through porous medium, the superficial velocity in the solid bed is represented by Eq. (15) using the concepts of porosity and hydraulic radius. Equation (22) is the modified equation of continuity for flow of fluids through porous medium. It can be seen that when $E = 1$, the equation

reduces to the ordinary equation of continuity where the medium has no solid particle; on the other hand, when $E = 0$, it means that there is no free space available to flow, and the medium is entirely solid. The substitution of the superficial velocity into the equation of continuity gives the general differential equation for flow of any time-independent fluid through compressible porous media (Eq. 24).

The general flow equation is derived without the specification of any coordinate system. Furthermore, the vector differential operator ∇ and dyadic differential operator $\nabla\nabla$ are invariant quantities. Thus, the equation derived is applicable to any curvilinear orthogonal coordinate system.

The Ostwald-de Waele model and Bingham model are two useful non-Newtonian fluid models. The equations describing the flow of these two fluid models (Eqs. 36 and 43 respectively) are reduced directly by introducing the corresponding shear rate relationship, $f(T)$, into the general transient flow equation. It is expected that these equations can also be obtained by combining the respective discharge equation for each fluid with the equation of continuity.

An important application of flow of fluids through porous media is in filtration. Tiller has derived a partial differential equation (Eq. 8) describing a constant pressure filtration process by considering the variation of fluid velocity throughout the cake. The equation is

based upon the assumption that the specific cake resistance α_x and porosity E are functions of compressive pressure P_g alone, which is equivalent to assuming that equilibrium porosities are reached instantaneously with changes in pressure. However, the porosity changes do not occur instantaneously in an actual case, hence, Eq. (8) can be viewed only as an approximation in the search for the best differential equation describing the flow through compressible porous media.

When the concepts of permeability, porosity and specific cake resistance are introduced to the general flow equation, it is shown that the one-dimensional flow equation for incompressible Newtonian fluids simplifies to the equation developed by Tiller.

Equation (68) is the general transient flow equation for compressible gases through compressible porous media. If both porosity and specific cake resistance are constant, then the equation reduces to the ordinary unsteady flow equation for gases through incompressible porous media. Rigorously, the assumption of constant gas permeability (or cake resistance) is invalid even in an incompressible porous medium, since in an actual case, gas does not stick to the walls of the pores as required by Darcy's law and a phenomenon termed "slip" occurs. The slipping of the fluid along

the pore walls gives rise to an apparent dependence of permeability on pressure which is commonly known as Klinkenberg effect (6).

The general flow equation is a three-dimensional second order non-linear partial differential equation. Integration of this equation is extremely difficult if not impossible. In order to make use of the equation, it is advisable to carefully examine the physical conditions and to make use of all the known facts to simplify the equation as much as possible before attempting the solution. It has been pointed out before that the one-dimensional constant pressure filtration is just a simplified case of the general flow problem. However, even this simple case does not have an exact analytical solution. The integral method was used here to arrive at an approximate solution to the moving boundary problem.

Clearly, the solution obtained by the application of a linear pressure profile in the integral method is inadmissible. The use of a two-coefficient linear profile permitted only two boundary conditions (i.e., Eqs. (80) and (81)) to be imposed. There is a third boundary condition,* associated with the movement of the boundary, which must also be satisfied. Thus, the solution obtained with the linear profile is under-constrained.

The third boundary condition states that the weight of solid deposited by the slurry on the surface of the cake per unit time is equal

* see Appendix-C for the third boundary condition

to the rate of increase of the weight of filter cake. This is a necessary condition and relates the growth of the cake thickness to the solid concentration in the slurry.

As stated, a unique solution to the problem must satisfy three boundary conditions. Furthermore, the linear profile assumed does not contribute to the second derivative term which appears in the original partial differential equation. Hence, a three-coefficient profile must also be assumed to retain the second derivative in the differential equation. It is difficult to establish a general criterion regarding the admissibility of the approximate solution, but judging from the success encountered in the application of the integral method to diffusive-type nonlinear problems, it is believed that a second order pressure profile should lead to a better result. Unfortunately, a quadratic profile does not lead to a simple analytical solution.

Additional factors which should be taken into consideration in the evaluation of the final results are the limitations of the porosity and specific cake resistance functions. Obviously, the assumed functions (i. e., Eqs. (57a) and (57c)) are not applicable at the surface of the cake, since at the surface the solid pressure is zero, which implies a zero cake resistance and an infinite porosity. Hence, other functions for the porosity and specific cake resistance must be used in the region of the surface layer. Disagreement may also be attributed

to the discrepancies between the values of the experimental constants used in the example which were presented by Tiller (26).

The integral method is generally accepted (24) as a useful approximate method of solving nonlinear partial differential equations. However, an approximation may contain some irrevocable errors in the final numerical result. Thus, the question arises as to how to eliminate, or at least reduce, such errors and thereby improve the accuracy. Many improvements have been suggested in the literature (24). Some of the proposed improvements involve: a) using a higher order polynomial or other profile such as the logarithmic or trigonometric function; in this case, there is no a priori guarantee of an improvement; b) resort to the method of weighted residuals; and c) the method of Yang (29) - an iterative procedure utilizing the first approximation to achieve an improved profile. The details of these techniques are rather involved and beyond the scope of the present investigation. A detailed account is given by Goodman (24).

VII - SUMMARY AND CONCLUSION

In the first part of this work, the general transient equations describing the flow of fluids through compressible porous media are developed. Table 1 gives a summary of the equations derived in this paper. These include equations for flow of Ostwald-de Waele fluids, Bingham fluids, compressible and incompressible Newtonian fluids through compressible porous media. Table 2 shows the constant pressure filtration equations for Newtonian fluids. It is seen that Tiller's one-dimensional filtration equation is just a simple case which can be reduced directly from the general three-dimensional equation derived here.

The foregoing derivations have been made without the specification of coordinate system using the vector and tensor notation. Hence, the equations obtained are applicable to any orthogonal curvilinear coordinate system. The details of vector and tensor manipulations for different coordinate systems are tabulated in Appendix-A.

The second part is devoted to the solution of a one-dimensional constant pressure filtration problem utilizing the well-known integral method. In the most general case, the boundary condition at the supported medium or septum is an implicit function of the rate of change of solid cake thickness which makes the solution to the problem

extremely difficult. Nevertheless, explicit solutions are obtained for the cases: a) assuming time and filtrate volume are related parabolically, and b) resistance at the supported medium is negligible.

In summary, it appears that a two-coefficient linear pressure profile cannot be used to obtain a solution by the integral method. The solution thus obtained is under-constrained as a result of the fact that one boundary condition (associated with the movement of the cake surface) cannot be imposed on the solution. This condition relates the growth of cake thickness to the solid concentration of the slurry. Hence, a pressure profile, with three or more coefficients, that satisfies all three boundary conditions must be used. It is difficult to give a general criterion regarding the admissibility of the results yielded by this approximate method. However, judging from the success of the applications of the integral method to diffusive-type nonlinear problems, it is believed that a second or higher order profile should yield a reasonable solution. Unfortunately, a quadratic profile does not lead to a simple analytical solution.

TABLE I
SUMMARY OF THE EQUATIONS FOR FLOW OF
FLUIDS THROUGH COMPRESSIBLE POROUS MEDIA

I. Incompressible Fluids

1) Generalized Fluids

a) Three-Dimensional Flow

$$A_1 (\nabla P' \cdot \nabla E) + A_2 (\nabla P' \cdot \nabla P' \cdot \nabla \nabla P') + A_3 (\nabla^2 P') = 2\alpha S_0 \frac{\partial E}{\partial t} \quad (1A)$$

where

$$A_1 = S_0^{b/a} \left[\frac{E^{1-b/a} (2-E-b/a)}{(1-E)^{2-b/a}} |\nabla P'|^{-b/a-1} \right] \int_{T_y}^{|T_0|} g(T) dT + \frac{E}{(1-E)^2} |\nabla P'|^{-1} f(|T_0|) \quad (1B)$$

$$A_2 = -(1 + \frac{b}{a}) S_0^{b/a} \left[\frac{E^{2-b/a}}{(1-E)^{1-b/a}} |\nabla P'|^{-3-b/a} \right] \int_{T_y}^{|T_0|} g(T) dT + \frac{E^2}{(1-E)} |\nabla P'|^{-3} f(|T_0|) \quad (1C)$$

$$A_3 = S_0^{b/a} \left[\frac{E^{2-b/a}}{(1-E)^{1-b/a}} |\nabla P'|^{-b/a-1} \right] \int_{T_y}^{|T_0|} g(T) dT \quad (1D)$$

Table 1 - Continued

and

$$g(T) = T^{b/a-1} f(T) \quad (1E)$$

$$T_0 = r_H (\nabla P') \quad (1F)$$

b) One-Dimensional Flow in Cartesian Coordinate

$$A_1 \left[\left(\frac{\partial P'}{\partial x} \right) \left(\frac{\partial E}{\partial x} \right) \right] + A_2 \left[\left(\frac{\partial P'}{\partial x} \right)^2 \left(\frac{\partial^2 P'}{\partial x^2} \right) \right] + A_3 \left[\frac{\partial^2 P'}{\partial x^2} \right] =$$

$$2aS_0 \frac{\partial E}{\partial t} \quad (1G)$$

where $|\nabla P'|$ in the coefficients A_1 , A_2 and A_3 is

$$|\nabla P'| = \frac{\partial P'}{\partial x}$$

c) One-Dimensional Radial Flow in Cylindrical Coordinate

$$A_1 \left[\left(\frac{\partial P'}{\partial r} \right) \left(\frac{\partial E}{\partial r} \right) \right] + A_2 \left[\left(\frac{\partial P'}{\partial r} \right)^2 \left(\frac{\partial^2 P'}{\partial r^2} \right) \right] + A_3 \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial P'}{\partial r} \right) \right] =$$

$$2aS_0 \frac{\partial E}{\partial t} \quad (1H)$$

where $|\nabla P'|$ in the coefficients A_1 , A_2 and A_3 is

$$|\nabla P'| = \frac{\partial P'}{\partial r}$$

d) Ostwald-de Waele Fluids (Pseudoplastic Fluids)

$$A_{1p} (\nabla P' \cdot \nabla E) + A_{2p} (\nabla P' \cdot \nabla P' \cdot \nabla \nabla P') + A_{3p} (\nabla^2 P') = K_1 \frac{\partial E}{\partial t} \quad (1I)$$

Table 1 - Continued

where

$$A_{1p} = |\nabla P'| \frac{1-n}{n} \left[\frac{E^{1+\frac{1}{n}} (2-E+\frac{1}{n})}{(1-E)^{2+\frac{1}{n}}} \right] \quad (1J)$$

$$A_{2p} = \frac{1-n}{n} |\nabla P'| \frac{1-3n}{n} \left[\frac{E^{2+\frac{1}{n}}}{(1-E)^{1+\frac{1}{n}}} \right] \quad (1K)$$

$$A_{3p} = |\nabla P'| \frac{1-n}{n} \left[\frac{E^{2+\frac{1}{n}}}{(1-E)^{1+\frac{1}{n}}} \right] \quad (1L)$$

and

$$K_1 = \frac{2(a+bn) K_0^{\frac{1}{n}} S_0^{1+\frac{1}{n}}}{n} \quad (1M)$$

3) Bingham Plastic Fluids

$$A_{1b} (\nabla P' \cdot \nabla E) + A_{2b} (\nabla P' \cdot \nabla P' \cdot \nabla \nabla P') + A_{3b} (\nabla^2 P') = 2a\delta_0 \frac{\partial E}{\partial t} \quad (1N)$$

where

$$A_{1b} = \frac{S_0^{b/a}}{\eta} \left[\frac{E^{1-b/a} (2-E-b/a)}{(1-E)^{2-b/a}} |\nabla P'|^{-\frac{b}{a}-1} \right] h(|T_0|) + \frac{E}{(1-E)^2}$$

$$|\nabla P'|^{-1} \frac{|T_0| - T_y}{\eta} \quad (1O)$$

$$A_{2b} = -\left(1 + \frac{b}{a}\right) \frac{S_0^{b/a}}{\eta} \left[\frac{E^{2-b/a}}{(1-E)^{1-b/a}} |\nabla P'|^{-3-b/a} \right] h(|T_0|) +$$

$$\frac{E^2}{(1-E)} |\nabla P'|^{-3} \frac{|T_0| - T_y}{\eta} \quad (1P)$$

Table 1 - continued

$$A_{3b} = \frac{S_o^{b/a}}{\eta} \left[\frac{E^{2-b/a}}{(1-E)^{1-b/a}} |\nabla P'|^{-b/a-1} \right] h(|T_o|) \quad (1Q)$$

and

$$h(|T_o|) = \frac{|T_o|^{b/a+1} - T_y^{b/a+1}}{1 + b/a} - \frac{a T_y}{b} (|T_o|^{b/a} - T_y^{b/a}) \quad (1R)$$

4) Newtonian Fluids

a) Three-Dimensional Flow

$$\nabla^2 P' + \frac{3-E}{E(1-E)} [\nabla P' \cdot \nabla E] = \frac{2\eta(a+b)S_o^2(1-E)^2}{E^3} \frac{\partial E}{\partial t} \quad (1S)$$

b) One-Dimensional Flow in Cartesian Coordinate

$$\frac{\partial^2 P'}{\partial x^2} - \left(\frac{\partial P'}{\partial x}\right) \frac{\partial}{\partial x} [\ln a_x (1-E)] = \eta \rho_s a_x (1-E) \frac{\partial E}{\partial t} \quad (1T)$$

c) One-Dimensional Radial Flow in Cylindrical Coordinate

$$\frac{1}{r} \left[\frac{\partial}{\partial r} \left(r \frac{\partial P'}{\partial r} \right) \right] - \left(\frac{\partial P'}{\partial r}\right) \frac{\partial}{\partial r} [\ln a_x (1-E)] = \eta \rho_s a_x (1-E) \frac{\partial E}{\partial t} \quad (1U)$$

where the specific cake resistance is defined as

$$a_x = \frac{2(a+b)S_o^2(1-E)}{E^3 \rho_s} \quad (1V)$$

Table 1 - continued

II. Compressible Fluids (Ideal Gas)

a) Three-Dimensional Flow

$$\nabla^2 P^2 - \nabla P^2 \cdot \nabla [\ln a_x (1 - E)] = 2\eta \rho_s a_x (1 - E) \left[P \frac{\partial E}{\partial t} + E \frac{\partial P}{\partial t} \right] \quad (1W)$$

b) One-Dimensional Flow in Cartesian Coordinate

$$\frac{\partial^2 P^2}{\partial x^2} - \left(\frac{\partial P^2}{\partial x} \right) \frac{\partial}{\partial x} [\ln a_x (1 - E)] = 2\eta \rho_s a_x (1 - E) \left[P \frac{\partial E}{\partial t} + E \frac{\partial P}{\partial t} \right] \quad (1X)$$

c) One-Dimensional Radial Flow in Cylindrical Coordinate

$$\frac{1}{r} \left[\frac{\partial}{\partial r} \left(r \frac{\partial P^2}{\partial r} \right) \right] - \left(\frac{\partial P^2}{\partial r} \right) \frac{\partial}{\partial r} [\ln a_x (1 - E)] = 2\eta \rho_s a_x (1 - E) \left[P \frac{\partial E}{\partial t} + E \frac{\partial P}{\partial t} \right] \quad (1Y)$$

TABLE 2
EQUATIONS FOR CONSTANT PRESSURE FILTRATION
OF NEWTONIAN FLUIDS

I. Incompressible Fluids

a) Three-Dimensional Flow

$$\nabla^2 P_s - \frac{1}{\alpha_x (1-E)} (\nabla P_s)^2 \frac{d}{dP_s} \alpha_x (1-E) = -\eta \rho_s \alpha_x (1-E) \frac{dE}{dP_s} \left(\frac{\partial P_s}{\partial t} \right) \quad (2A)$$

b) One-Dimensional Flow

$$\frac{\partial^2 P_s}{\partial x^2} - \left(\frac{\partial P_s}{\partial x} \right)^2 \frac{d}{dP_s} \ln \alpha_x (1-E) = -\eta \rho_s \alpha_x (1-E) \frac{dE}{dP_s} \left(\frac{\partial P_s}{\partial t} \right) \quad (2B)$$

II. Compressible Fluids (Ideal Gas)

a) Three-Dimensional Flow

$$\begin{aligned} \nabla^2 P_s - \frac{1}{P_a - P_s} + \frac{d}{dP_s} \ln \alpha_x (1-E) (\nabla P_s)^2 \\ = -\eta \rho_s \alpha_x (1-E) \frac{dE}{dP_s} - \frac{E}{P_a - P_s} \left(\frac{\partial P_s}{\partial t} \right) \end{aligned} \quad (2C)$$

b) One-Dimensional Flow

$$\begin{aligned} \frac{\partial^2 P_s}{\partial x^2} - \frac{1}{P_a - P_s} + \frac{d}{dP_s} \ln \alpha_x (1-E) \left(\frac{\partial P_s}{\partial x} \right)^2 \\ = -\eta \rho_s \alpha_x (1-E) \frac{dE}{dP_s} - \frac{E}{P_a - P_s} \left(\frac{\partial P_s}{\partial t} \right) \end{aligned} \quad (2D)$$

VIII - APPENDIX

APPENDIX-A

TABLE A-1

SUMMARY OF DIFFERENTIAL OPERATIONS INVOLVING THE
 ∇ -AND $\nabla\nabla$ - OPERATOR IN GENERAL CURVILINEAR
COORDINATES

$$\nabla\psi = \frac{e_1}{h_1} \frac{\partial\psi}{\partial u_1} + \frac{e_2}{h_2} \frac{\partial\psi}{\partial u_2} + \frac{e_3}{h_3} \frac{\partial\psi}{\partial u_3} \quad (A1-1)$$

$$\nabla \cdot \bar{A} = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial u_1} (A_1 h_2 h_3) + \frac{\partial}{\partial u_2} (A_2 h_1 h_3) + \frac{\partial}{\partial u_3} (A_3 h_1 h_2) \right] \quad (A1-2)$$

$$\nabla \times \bar{A} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 e_1 & h_2 e_2 & h_3 e_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix} \quad (A1-3)$$

$$\nabla^2 \psi = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial u_1} \left(\frac{h_2 h_3}{h_1} \frac{\partial\psi}{\partial u_1} \right) + \frac{\partial}{\partial u_2} \left(\frac{h_3 h_1}{h_2} \frac{\partial\psi}{\partial u_2} \right) + \frac{\partial}{\partial u_3} \left(\frac{h_1 h_2}{h_3} \frac{\partial\psi}{\partial u_3} \right) \right] \quad (A1-4)$$

Table A-1 - continued

$$\begin{aligned} \nabla \nabla \psi = & \frac{e_1 e_1}{h_1^2} \frac{\partial^2 \psi}{\partial U_1^2} + \frac{e_1 e_2}{h_1 h_2} \frac{\partial}{\partial U_1} \frac{\partial \psi}{\partial U_2} + \frac{e_1 e_3}{h_1 h_3} \frac{\partial}{\partial U_1} \frac{\partial \psi}{\partial U_3} \\ & + \frac{e_2 e_1}{h_2 h_1} \frac{\partial}{\partial U_2} \frac{\partial \psi}{\partial U_1} + \frac{e_2 e_2}{h_2^2} \frac{\partial^2 \psi}{\partial U_2^2} + \frac{e_2 e_3}{h_2 h_3} \frac{\partial}{\partial U_2} \frac{\partial \psi}{\partial U_3} \\ & + \frac{e_3 e_1}{h_3 h_1} \frac{\partial}{\partial U_3} \frac{\partial \psi}{\partial U_1} + \frac{e_3 e_2}{h_3 h_2} \frac{\partial}{\partial U_3} \frac{\partial \psi}{\partial U_2} + \frac{e_3 e_3}{h_3^2} \frac{\partial^2 \psi}{\partial U_3^2} \end{aligned} \quad (A1-5)$$

TABLE A-2

UNIT VECTORS AND SCALE FACTORS OF VARIOUS
COORDINATE SYSTEMS

1. Cartesian Coordinates (x, y, z):

$$\underline{e}_1 = \underline{i}, \underline{e}_2 = \underline{j}, \underline{e}_3 = \underline{k} \quad (A2-1)$$

$$h_1 = h_2 = h_3 = 1 \quad (A2-2)$$

2. Cylindrical Coordinates (r, θ , z):

$$x = r \cos \theta, y = r \sin \theta, z = z$$

$$\underline{e}_1 = \underline{e}_r = \cos \theta \underline{i} + \sin \theta \underline{j}$$
$$\underline{e}_2 = \underline{e}_\theta = -\sin \theta \underline{i} + \cos \theta \underline{j} \quad (A2-3)$$

$$\underline{e}_3 = \underline{e}_z = \underline{k}$$

$$h_1 = h_r = 1, h_2 = h_\theta = r, h_3 = h_z = 1 \quad (A2-4)$$

3. Spherical Coordinates (r, θ , ϕ):

$$x = r \sin \theta \cos \phi, y = r \sin \theta \sin \phi, z = r \cos \theta$$

$$\underline{e}_1 = \underline{e}_r = \sin \theta \cos \phi \underline{i} + \sin \theta \sin \phi \underline{j} + \cos \theta \underline{k}$$
$$\underline{e}_2 = \underline{e}_\theta = \cos \theta \cos \phi \underline{i} + \cos \theta \sin \phi \underline{j} - \sin \theta \underline{k} \quad (A2-5)$$

$$\underline{e}_3 = \underline{e}_\phi = -\sin \phi \underline{i} + \cos \phi \underline{j}$$

$$h_1 = h_r = 1, h_2 = h_\theta = r, h_3 = h_\phi = r \sin \theta \quad (A2-6)$$

Table A-2 - continued

4. Other Orthogonal Coordinate Systems:

a) Ellipsoidal Coordinates (λ, μ, ν):

$$\frac{x^2}{a^2 - \lambda} + \frac{y^2}{b^2 - \lambda} + \frac{z^2}{c^2 - \lambda} = 1 \quad \lambda < c^2 < b^2 < a^2$$

$$\frac{x^2}{a^2 - \mu} + \frac{y^2}{b^2 - \mu} + \frac{z^2}{c^2 - \mu} = 1 \quad c^2 < \mu < b^2 < a^2 \quad (A2-7)$$

$$\frac{x^2}{a^2 - \nu} + \frac{y^2}{b^2 - \nu} + \frac{z^2}{c^2 - \nu} = 1 \quad c^2 < b^2 < \nu < a^2$$

$$\underline{e}_1 = \underline{e}_\lambda, \quad \underline{e}_2 = \underline{e}_\mu, \quad \underline{e}_3 = \underline{e}_\nu \quad (A2-8)$$

$$h_\lambda = 1/2 \sqrt{\frac{(\mu - \lambda)(\nu - \lambda)}{(a^2 - \lambda)(b^2 - \lambda)(c^2 - \lambda)}}$$

$$h_\mu = 1/2 \sqrt{\frac{(\nu - \mu)(\lambda - \mu)}{(a^2 - \mu)(b^2 - \mu)(c^2 - \mu)}} \quad (A2-9)$$

$$h_\nu = 1/2 \sqrt{\frac{(\lambda - \nu)(\mu - \nu)}{(a^2 - \nu)(b^2 - \nu)(c^2 - \nu)}}$$

b) Oblate Spheroidal Coordinate (ξ, η, ϕ)

$$\begin{aligned} x &= a \cosh \xi \cos \eta \cos \phi & \text{where } \xi &\geq 0 \\ y &= a \cosh \xi \cos \eta \sin \phi & -\frac{\pi}{2} &\leq \eta \leq \frac{\pi}{2} \\ z &= a \sinh \xi \sin \eta & 0 &\leq \phi < 2\pi \end{aligned} \quad (A2-10)$$

Table A-2 - continued

$$\underline{e}_1 = \underline{e}_\xi \cdot \underline{e}_2 = \underline{e}_\eta \cdot \underline{e}_3 = \underline{e}_\phi \quad (\text{A2-11})$$

$$h_\xi = h_\eta = a \sqrt{\sinh^2 \xi + \sin^2 \eta} \quad (\text{A2-12})$$

$$h_\phi = a \cosh \xi \cos \eta$$

TABLE A-3

DIFFERENTIAL OPERATIONS INVOLVING THE ∇ - AND $\nabla\nabla$ -
OPERATOR IN THE GENERAL EQUATIONS FOR FLOW OF
FLUIDS THROUGH COMPRESSIBLE POROUS MEDIA

1. Cartesian Coordinates (x, y, z):

$$\nabla P' \cdot \nabla E = \left(\frac{\partial P'}{\partial x} \right) \left(\frac{\partial E}{\partial x} \right) + \left(\frac{\partial P'}{\partial y} \right) \left(\frac{\partial E}{\partial y} \right) + \left(\frac{\partial P'}{\partial z} \right) \left(\frac{\partial E}{\partial z} \right) \quad (A3-1)$$

$$\begin{aligned} \nabla\nabla P' &= \underline{ii} \frac{\partial^2 P'}{\partial x^2} + \underline{ij} \frac{\partial}{\partial x} \frac{\partial P'}{\partial y} + \underline{ik} \frac{\partial}{\partial x} \frac{\partial P'}{\partial z} \\ &+ \underline{ji} \frac{\partial}{\partial y} \frac{\partial P'}{\partial x} + \underline{jj} \frac{\partial^2 P'}{\partial y^2} + \underline{jk} \frac{\partial}{\partial y} \frac{\partial P'}{\partial z} \\ &+ \underline{ki} \frac{\partial}{\partial z} \frac{\partial P'}{\partial x} + \underline{kj} \frac{\partial}{\partial z} \frac{\partial P'}{\partial y} + \underline{kk} \frac{\partial^2 P'}{\partial z^2} \end{aligned} \quad (A3-2)$$

$$\nabla P' \cdot \nabla\nabla P' = \frac{\partial P'}{\partial x} \frac{\partial}{\partial x} (\nabla P') + \frac{\partial P'}{\partial y} \frac{\partial}{\partial y} (\nabla P') + \frac{\partial P'}{\partial z} \frac{\partial}{\partial z} (\nabla P') \quad (A3-3)$$

$$\begin{aligned} \nabla P' \cdot \nabla P' \cdot \nabla\nabla P' &= \frac{\partial P'}{\partial x} \left(\frac{\partial P'}{\partial x} \frac{\partial^2 P'}{\partial x^2} + \frac{\partial P'}{\partial y} \frac{\partial}{\partial y} \frac{\partial P'}{\partial x} + \frac{\partial P'}{\partial z} \frac{\partial}{\partial z} \frac{\partial P'}{\partial x} \right) \\ &+ \frac{\partial P'}{\partial y} \left(\frac{\partial P'}{\partial x} \frac{\partial}{\partial x} \frac{\partial P'}{\partial y} + \frac{\partial P'}{\partial y} \frac{\partial^2 P'}{\partial y^2} + \frac{\partial P'}{\partial z} \frac{\partial}{\partial z} \frac{\partial P'}{\partial y} \right) \\ &+ \frac{\partial P'}{\partial z} \left(\frac{\partial P'}{\partial x} \frac{\partial}{\partial x} \frac{\partial P'}{\partial z} + \frac{\partial P'}{\partial y} \frac{\partial}{\partial y} \frac{\partial P'}{\partial z} + \frac{\partial P'}{\partial z} \frac{\partial^2 P'}{\partial z^2} \right) \end{aligned}$$

(A3-4)

Table A-3 - continued

$$\nabla^2 P' = \frac{\partial^2 P'}{\partial x^2} + \frac{\partial^2 P'}{\partial y^2} + \frac{\partial^2 P'}{\partial z^2} \quad (A3-5)$$

$$\nabla P' = \left[\left(\frac{\partial P'}{\partial x} \right)^2 + \left(\frac{\partial P'}{\partial y} \right)^2 + \left(\frac{\partial P'}{\partial z} \right)^2 \right]^{\frac{1}{2}} \quad (A3-6)$$

2. Cylindrical Coordinates (r, θ, z):

$$\nabla P' \cdot \nabla \Phi = \left(\frac{\partial P'}{\partial r} \right) \left(\frac{\partial \Phi}{\partial r} \right) + \frac{1}{r^2} \left(\frac{\partial P'}{\partial \theta} \right) \left(\frac{\partial \Phi}{\partial \theta} \right) + \left(\frac{\partial P'}{\partial z} \right) \left(\frac{\partial \Phi}{\partial z} \right) \quad (A3-7)$$

$$\begin{aligned} \nabla \nabla P' = & \frac{e_r e_r}{r^2} \frac{\partial^2 P'}{\partial r^2} + \frac{e_r e_\theta}{r} \frac{\partial}{\partial r} \frac{\partial P'}{\partial \theta} + \frac{e_r e_z}{r} \frac{\partial}{\partial r} \frac{\partial P'}{\partial z} \\ & + \frac{e_\theta e_r}{r} \frac{\partial}{\partial \theta} \frac{\partial P'}{\partial r} + \frac{e_\theta e_\theta}{r^2} \frac{\partial^2 P'}{\partial \theta^2} + \frac{e_\theta e_z}{r} \frac{\partial}{\partial \theta} \frac{\partial P'}{\partial z} \\ & + \frac{e_z e_r}{r} \frac{\partial}{\partial z} \frac{\partial P'}{\partial r} + \frac{e_z e_\theta}{r} \frac{\partial}{\partial z} \frac{\partial P'}{\partial \theta} + \frac{e_z e_z}{r^2} \frac{\partial^2 P'}{\partial z^2} \end{aligned} \quad (A3-8)$$

$$\nabla P' \cdot \nabla \nabla P' = \frac{\partial P'}{\partial r} \frac{\partial}{\partial r} (\nabla P') + \frac{1}{r^2} \frac{\partial P'}{\partial \theta} \frac{\partial}{\partial \theta} (\nabla P') + \frac{\partial P'}{\partial z} \frac{\partial}{\partial z} (\nabla P') \quad (A3-9)$$

$$\begin{aligned} \nabla P' \cdot \nabla P' \cdot \nabla \nabla P' = & \frac{\partial P'}{\partial r} \left(\frac{\partial P'}{\partial r} \frac{\partial^2 P'}{\partial r^2} + \frac{1}{r^2} \frac{\partial P'}{\partial \theta} \frac{\partial}{\partial \theta} \frac{\partial P'}{\partial r} \right. \\ & \left. + \frac{\partial P'}{\partial z} \frac{\partial}{\partial z} \frac{\partial P'}{\partial r} \right) + \frac{1}{r^2} \frac{\partial P'}{\partial \theta} \left(\frac{\partial P'}{\partial r} \frac{\partial}{\partial r} \frac{\partial P'}{\partial \theta} + \frac{1}{r} \frac{\partial P'}{\partial \theta} \right. \\ & \left. \frac{\partial^2 P'}{\partial \theta^2} + \frac{\partial P'}{\partial z} \frac{\partial}{\partial z} \frac{\partial P'}{\partial \theta} \right) + \frac{\partial P'}{\partial z} \left(\frac{\partial P'}{\partial r} \frac{\partial}{\partial r} \frac{\partial P'}{\partial z} + \frac{1}{r^2} \frac{\partial P'}{\partial \theta} \right. \\ & \left. \frac{\partial}{\partial \theta} \frac{\partial P'}{\partial z} + \frac{\partial P'}{\partial z} \frac{\partial^2 P'}{\partial z^2} \right) \end{aligned} \quad (A3-10)$$

Table A-3 - continued

$$\nabla^2 P' = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial P'}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 P'}{\partial \theta^2} + \frac{\partial^2 P'}{\partial \phi^2} \quad (\text{A3-11})$$

$$|\nabla P'| = \left[\left(\frac{\partial P'}{\partial r} \right)^2 + \left(\frac{1}{r} \frac{\partial P'}{\partial \theta} \right)^2 + \left(\frac{\partial P'}{\partial \phi} \right)^2 \right]^{\frac{1}{2}} \quad (\text{A3-12})$$

3. Spherical Coordinates (r, θ , ϕ):

$$\nabla P' \cdot \nabla E = \left(\frac{\partial P'}{\partial r} \right) \left(\frac{\partial E}{\partial r} \right) + \frac{1}{r^2} \left(\frac{\partial P'}{\partial \theta} \right) \left(\frac{\partial E}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \left(\frac{\partial P'}{\partial \phi} \right) \left(\frac{\partial E}{\partial \phi} \right) \quad (\text{A3-13})$$

$$\begin{aligned} \nabla \nabla P' = & \frac{e_r e_r}{r} \frac{\partial^2 P'}{\partial r^2} + \frac{e_r e_\theta}{r} \frac{\partial}{\partial r} \frac{\partial P'}{\partial \theta} + \frac{e_r e_\phi}{r \sin \theta} \frac{\partial}{\partial r} \frac{\partial P'}{\partial \phi} \\ & + \frac{e_\theta e_r}{r} \frac{\partial}{\partial \theta} \frac{\partial P'}{\partial r} + \frac{e_\theta e_\theta}{r^2} \frac{\partial^2 P'}{\partial \theta^2} + \frac{e_\theta e_\phi}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \frac{\partial P'}{\partial \phi} \end{aligned} \quad (\text{A3-14})$$

$$+ \frac{e_\phi e_r}{r \sin \theta} \frac{\partial}{\partial \phi} \frac{\partial P'}{\partial r} + \frac{e_\phi e_\theta}{r^2 \sin \theta} \frac{\partial}{\partial \phi} \frac{\partial P'}{\partial \theta} + \frac{e_\phi e_\phi}{r^2 \sin^2 \theta} \frac{\partial^2 P'}{\partial \phi^2}$$

$$\nabla P' \cdot \nabla \nabla P' = \frac{\partial P'}{\partial r} \frac{\partial}{\partial r} (\nabla P') + \frac{1}{r^2} \frac{\partial P'}{\partial \theta} \frac{\partial}{\partial \theta} (\nabla P') + \frac{1}{r^2 \sin^2 \theta}$$

$$\frac{\partial P'}{\partial \phi} \frac{\partial}{\partial \phi} (\nabla P') \quad (\text{A3-15})$$

$$\nabla P' \cdot \nabla P' \cdot \nabla \nabla P' = \frac{\partial P'}{\partial r} \left(\frac{\partial P'}{\partial r} \frac{\partial^2 P'}{\partial r^2} + \frac{1}{r^2} \frac{\partial P'}{\partial \theta} \frac{\partial \partial P'}{\partial \theta \partial r} + \frac{1}{r^2 \sin^2 \theta}$$

$$\frac{\partial P'}{\partial \phi} \frac{\partial \partial P'}{\partial \phi \partial r} \right) + \frac{1}{r^2} \frac{\partial P'}{\partial \theta} \left(\frac{\partial P'}{\partial r} \frac{\partial}{\partial r} \frac{\partial P'}{\partial \theta} + \frac{1}{r^2} \frac{\partial P'}{\partial \theta} \frac{\partial^2 P'}{\partial \theta^2} \right) \quad (\text{A3-16})$$

$$+ \frac{1}{r^2 \sin^2 \theta} \frac{\partial P'}{\partial \phi} \frac{\partial}{\partial \phi} \left(\frac{\partial P'}{\partial r} \frac{\partial}{\partial r} \frac{\partial P'}{\partial \phi} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial P'}{\partial \phi} \left(\frac{\partial P'}{\partial r} \frac{\partial}{\partial r} \frac{\partial P'}{\partial \theta} \right)$$

Table A3 - continued

$$\begin{aligned}
 & + \frac{1}{r^2} \frac{\partial P'}{\partial \theta} \frac{\partial}{\partial \theta} \frac{\partial P'}{\partial \beta} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial P'}{\partial \beta} \frac{\partial^2 P'}{\partial \beta^2} \\
 \nabla^2 P' = & \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial P'}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial P'}{\partial \theta} \right) \\
 & + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 P'}{\partial \beta^2}
 \end{aligned} \tag{A3-17}$$

$$|\nabla P'| = \left[\left(\frac{\partial P'}{\partial r} \right)^2 + \left(\frac{1}{r} \frac{\partial P'}{\partial \theta} \right)^2 + \left(\frac{1}{r \sin \theta} \frac{\partial P'}{\partial \beta} \right)^2 \right]^{\frac{1}{2}} \tag{A3-18}$$

Vector and dyadic operations of the ellipsoidal and oblate spheroidal coordinates are not included here because of the complexity involved. However, these operations can be obtained from general equation given by Table A-1 directly.

TABLE A-4

THE EQUATION OF CONTINUITY IN VARIOUS COORDINATE
SYSTEMS FOR FLOW OF FLUIDS THROUGH
COMPRESSIBLE POROUS MEDIA

1. General Coordinates:

$$\nabla \cdot (\sigma \mathbf{E} \bar{\mathbf{U}}) = - \frac{\partial}{\partial t} (E\sigma) \quad (\text{A4-1})$$

or
$$\nabla \cdot (\sigma \dot{\mathbf{V}}) = - \frac{\partial}{\partial t} (E\sigma) \quad (\text{A4-2})$$

a) Cartesian Coordinates (x, y, z):

$$\frac{\partial}{\partial t} (E\sigma) + \frac{\partial}{\partial x} (\sigma E U_x) + \frac{\partial}{\partial y} (\sigma E U_y) + \frac{\partial}{\partial z} (\sigma E U_z) = 0 \quad (\text{A4-3})$$

b) Cylindrical Coordinates (r, θ , z):

$$\frac{\partial}{\partial t} (E\sigma) + \frac{1}{r} \frac{\partial}{\partial r} (r\sigma E U_r) + \frac{1}{r} \frac{\partial}{\partial \theta} (\sigma E U_\theta) + \frac{\partial}{\partial z} (\sigma E U_z) = 0 \quad (\text{A4-4})$$

c) Spherical Coordinates (r, θ , ϕ):

$$\frac{\partial}{\partial t} (E\sigma) + \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \sigma E U_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sigma E U_\theta \sin \theta) \\ + \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} (\sigma E U_\phi) = 0 \quad (\text{A4-5})$$

APPENDIX B

DERIVATION OF THE GENERAL TRANSIENT EQUATION

FOR FLOW OF INCOMPRESSIBLE FLUIDS

THROUGH POROUS MEDIA

The general discharge equation is

$$\bar{U} = \frac{r_H}{2a} |T_o|^{-b/a-1} \int_{T_y}^{|T_o|} T^{b/a-1} f(T) dT (\bar{T}_o) \quad (B-1)$$

The hydraulic radius is defined as

$$r_H = \frac{E}{S_o(1-E)} \quad (B-2)$$

and the superficial velocity in the porous medium is

$$\dot{V} = E\bar{U} \quad (B-3)$$

When Eqs. (B-2) and (B-3) are substituted into Eq. (B-1), it yields

$$\dot{V} = \frac{E^2}{2aS_o(1-E)} |T_o|^{-b/a-1} \int_{T_y}^{|T_o|} T^{b/a-1} f(T) dT (\bar{T}_o) \quad (B-4)$$

The equation of continuity for flow of incompressible fluids through porous media is

$$\nabla \cdot \dot{V} = -\frac{\partial E}{\partial t} \quad (B-5)$$

When Eq. (B-4) is introduced into Eq. (B-5), it becomes

$$\nabla \cdot \left[\frac{E^2}{2aS_0(1-E)} |T_0|^{-b/a-1} \int_{T_y}^{|T_0|} T^{b/a-1} f(T) dT (\bar{T}_0) \right] = -\frac{\partial E}{\partial t} \quad (B-6)$$

The operation of the left-hand side of Eq. (B-6) is as follows:

$$\begin{aligned} & \nabla \cdot \left[\frac{E^2}{2aS_0(1-E)} |T_0|^{-b/a-1} \int_{T_y}^{|T_0|} g(T) dT (\bar{T}_0) \right] = \frac{1}{2aS_0} \\ & \left[\frac{E^2}{(1-E)} |T_0|^{-b/a-1} \int_{T_y}^{|T_0|} g(T) dT (\nabla \cdot \bar{T}_0) + \bar{T}_0 \cdot \int_{T_y}^{|T_0|} g(T) dT \right. \\ & \left. \nabla \left\{ \frac{E^2}{(1-E)} |T_0|^{-b/a-1} \right\} + \bar{T}_0 \cdot \frac{E^2}{(1-E)} |T_0|^{-b/a-1} \nabla \int_{T_y}^{|T_0|} g(T) dT \right] \quad (B-7) \end{aligned}$$

where

$$g(T) = T^{b/a-1} f(T) \quad (B-8)$$

The task remained is to evaluate Eq. (B-7) term by term. First,

the divergence of \bar{T}_0 is

$$\begin{aligned} \nabla \cdot \bar{T}_0 &= \nabla \cdot \frac{E}{S_0(1-E)} \quad (-\nabla P) \\ &= \frac{E}{S_0(1-E)} (-\nabla^2 P) + \frac{1}{S_0(1-E)^2} (-\nabla P) \cdot (\nabla E) \quad (B-9) \end{aligned}$$

Then, the evaluation of

$$\nabla \cdot \left[\frac{E^2}{(1-E)} |T_0|^{-b/a-1} \right]$$

$$\begin{aligned}
 &= \left[\frac{E}{S_0 (1-E)} \left| -\nabla P' \right| \right]^{-b/a-1} \nabla \left[\frac{E^2}{(1-E)} \right] + \frac{E^2}{(1-E)} \nabla \left[\frac{E}{S_0 (1-E)} \left| -\nabla P' \right| \right]^{-b/a-1} \\
 &= \left[\frac{E}{S_0 (1-E)} \left| -\nabla P' \right| \right]^{-b/a-1} \left[\frac{2E-E^2}{(1-E)^2} \right] \nabla E \\
 &+ \left(-\frac{b}{a} - 1 \right) \frac{E^2}{(1-E)} \left[\frac{E \left| -\nabla P' \right|}{S_0 (1-E)} \right]^{-b/a-2} \left[\frac{E \left| -\nabla P' \right|}{S_0 (1-E)} \right]^{-1} (-\nabla P') \cdot (-\nabla P') \\
 &+ \left[\frac{\left| -\nabla P' \right|}{S_0 (1-E)^2} \nabla E \right] \\
 &= S_0^{1+b/a} \left| -\nabla P' \right|^{-1-b/a} \left[\frac{E^{-b/a} (1-E-b/a)}{(1-E)^{1-b/a}} \right] \nabla E \\
 &- \left(\frac{b+a}{a} \right) \left| -\nabla P' \right|^{-b/a-3} \left[\frac{E^{1-b/a}}{(1-E)^{-b/a}} \right] S_0^{1+b/a} (-\nabla P') \cdot (-\nabla P')
 \end{aligned}
 \tag{B-10}$$

The gradient of the integral is

$$\nabla \int_{T_y}^{|T_0|} g(T) dT = \int_{T_y}^{|T_0|} \nabla g(T) dT + g(|T_0|) \nabla |T_0| - g(T_y) \nabla T_y
 \tag{B-11}$$

Since $g(T)$ is not a function of position, and T_y is constant, the first and third terms in the right-hand side of Eq. (B-11) are equal to zero, thus

$$\nabla \int_{T_y}^{|T_0|} g(T) dT = g(T_0) \nabla \left[\frac{E}{S_0 (1-E)} \left| -\nabla P' \right| \right]$$

$$= g(|T_o|) \left[\frac{E |\nabla P'|^{-1}}{S_o (1-E)} (-\nabla P') \cdot (-\nabla \nabla P') + \frac{|\nabla P'|}{S_o (1-E)^2} \nabla E \right] \quad (B-12)$$

The substitution of Eqs. (B-9), (B-10), and (B-12) into Eq. (B-7) gives

$$2 a S_o \nabla \cdot \dot{V} = \left(\frac{|\nabla P'|}{S_o} \right)^{-b/a-1} \frac{E^{1-b/a}}{(1-E)^{-b/a}} \int_{T_y}^{|T_o|} g(T) dT$$

$$\left[\frac{E}{S_o (1-E)} (-\nabla^2 P') + \frac{1}{S_o (1-E)^2} (-\nabla P') \cdot (\nabla E) \right] + \frac{E}{S_o (1-E)} (-\nabla P')$$

$$\int_{T_y}^{|T_o|} g(T) dT \left[\frac{|\nabla P'|}{S_o} \right]^{-b/a-1} \left[\frac{E^{-b/a} (1-E-b/a)}{(1-E)^{1-b/a}} \right] \nabla E$$

$$+ \frac{E}{S_o (1-E)} (-\nabla P') \cdot \int_{T_y}^{|T_o|} g(T) dT \left(-\frac{b}{a} - 1 \right) S_o^{1+b/a} |\nabla P'|^{-b/a-3}$$

$$\left[\frac{E^{1-b/a}}{(1-E)^{-b/a}} \right] (-\nabla P') \cdot (-\nabla \nabla P') + \frac{E}{S_o (1-E)} (-\nabla P') \cdot \frac{E^2}{(1-E)}$$

$$\left[\frac{E |\nabla P'|}{S_o (1-E)} \right]^{-b/a-1} g(|T_o|) \left[\frac{E |\nabla P'|^{-1}}{S_o (1-E)} (-\nabla P') \cdot (-\nabla \nabla P') + \frac{|\nabla P'|}{S_o (1-E)^2} \nabla E \right]$$

$$= (-\nabla P') \cdot (\nabla E) \left[S_o^{b/a} |\nabla P'|^{-b/a-1} \frac{E^{1-b/a}}{(1-E)^{2-b/a}} \int_{T_y}^{|T_o|} g(T) dT \right]$$

$$\begin{aligned}
 & + S_0^{b/a} |\nabla P'|^{-b/a-1} \frac{E^{1-b/a} (1-E-b/a)}{(1-E)^{2-b/a}} \int_{T_y}^{T_0} g(T) dT \\
 & + \frac{E |\nabla P'|^{-1}}{(1-E)^2} f(|T_0|) \\
 & + (-\nabla P') \cdot (-\nabla P') \cdot (-\nabla \nabla P') \left[\left(-\frac{b}{a}-1\right) S_0^{b/a} |\nabla P'|^{-b/a-3} \frac{E^{2-b/a}}{(1-E)^{1-b/a}} \right. \\
 & \left. \int_{T_y}^{T_0} g(T) dT + \frac{E^2}{(1-E)} |\nabla P'|^{-3} f(|T_0|) \right] \\
 & + (-\nabla^2 P') \left[S_0^{b/a} |\nabla P'|^{-b/a-1} \frac{E^{2-b/a}}{(1-E)^{1-b/a}} \int_{T_y}^{T_0} g(T) dT \right] \tag{B-13}
 \end{aligned}$$

Finally, introducing Eq. (B-13) into Eq. (B-5), one gets the general three-dimensional transient equation for flow of incompressible fluids through compressible porous media, or

$$A_1 [\nabla P' \cdot \nabla E] + A_2 [\nabla P' \cdot \nabla P' \cdot \nabla \nabla P'] + A_3 [\nabla^2 P'] = 2a S_0 \frac{\partial E}{\partial t} \tag{B-14}$$

where

$$A_1 = \left[S_0^{b/a} |\nabla P'|^{-b/a-1} \frac{E^{1-b/a} (2-E-b/a)}{(1-E)^{2-b/a}} \int_{T_y}^{T_0} g(T) dT \right]$$

$$+ \frac{E}{(1-E)^2} |\nabla P'|^{-1} f(|T_0|) \quad (B-15)$$

$$A_2 = \left[- \left(\frac{a+b}{a} \right) S_0^{b/a} |\nabla P'|^{-b/a-3} \frac{E^{2-b/a}}{(1-E)^{1-b/a}} \int_{T_y}^{|T_0|} g(T) dT \right. \\ \left. + \frac{E^2}{(1-E)} |\nabla P'|^{-3} f(|T_0|) \right] \quad (B-16)$$

$$A_3 = \left[S_0^{b/a} |\nabla P'|^{-b/a-1} \frac{E^{2-b/a}}{(1-E)^{1-b/a}} \int_{T_y}^{|T_0|} g(T) dT \right] \quad (B-17)$$

The equations for flow of Ostwald-de Waele fluid, Bingham fluid, and Newtonian fluid are reduced directly from the general transient equation (Eq. B-14) by substituting their respective functions of $f(T)$ in the coefficients A_1 , A_2 and A_3 .

Appendix-C

Supplement to Integral Method

The Third Boundary Condition:

The condition states that the solid deposited on the surface of the cake per unit time is equal to the rate of increase in the weight of the filter cake. It follows, therefore, that

$$\frac{dW_1}{dt} = \frac{\rho X}{1 - m_1 X} V_{xi} \quad \text{at } W_x = W_1 \quad (C-1)$$

where

- V_{xi} = velocity of the liquid at the surface
- X = fraction of solid in the slurry
- m_1 = ratio of mass of wet to dry solid at the surface layer
- ρ = density of the liquid

From the Kozeny equation, the liquid velocity is related to the pressure gradient by

$$g_c \frac{dP_s}{dW_x} = \frac{\alpha_x \mu V_{xi}}{X} \quad \text{at } W_x = W_1 \quad (C-2)$$

Substitution of Eq. (C-2) into (C-1) gives

$$\frac{g_c dP_s}{\alpha_x \mu dW_x} = \frac{1 - m_1 X}{X} \frac{dW_1}{dt} \quad \text{at } W_x = W_1 \quad (C-3)$$

Equation (C-3) is the third boundary condition which relates the rate of growth of cake to the pressure gradient and solid concentration in the slurry. It is obvious that the profile will involve the term dW_1/dt

which when substituted in the integral equation yields a second order differential equation in terms of $W_1(t)$, and there is only one initial condition (i. e., $W_1 = 0$ at $t = 0$). To circumvent this difficulty, the condition

$$P_s(W_x = W_1, t) = 0$$

is differentiated with respect to time which gives

$$\frac{\partial P_s}{\partial W_x} \frac{dW_1}{dt} + \frac{\partial P_s}{\partial t} = 0 \quad (C-4)$$

Then, substitution of $\partial P_s / \partial t$ from the original differential equation gives a condition in terms of pressure derivative with respect to weight of solid only. This is

$$\frac{1}{\rho_s} \frac{d}{dP_s} \ln(1-E) \left(\frac{\partial P_s}{\partial W_x} \right)^2 = - \frac{1-m_1 X}{\rho X} \left[\frac{\partial^2 P_s}{\partial W_x^2} - \left(\frac{\partial P_s}{\partial W_x} \right)^2 \frac{d}{dP_s} \ln a_x \right]$$

at $W_x = W_1$ (C-5)

Equation (C-5) is the third boundary condition which must be satisfied by the assumed pressure profile. Hence, a second or higher order profile with three or more coefficients must be assumed.

NOMENCLATURE

\bar{A}	=	arbitrary vector
a, b	=	functions of flow geometry defined in Eq. (2)
b_1	=	intercept defined in Eq. (89)
d_m	=	mean particle diameter, ft.
E	=	porosity, dimensionless
g_c	=	conversion factor, $32.17 \text{ lb}_m\text{-ft}/\text{lb}_f\text{-sec}^2$
h	=	elevation, ft.
h_i	=	scalar factors in coordinate system
k	=	permeability, sq. ft.
k_o	=	Kozeny constant defined in Eq. (4)
K	=	fluid consistency index, $\text{lb}_f\text{-sec}^n/\text{sq. ft.}$
m	=	ratio of the mass of wet to dry solid, dimensionless
m_i	=	value of m in infinitesimal surface layer of cake, dimensionless
M	=	material in an arbitrary volume element, lb_f
M_o	=	molecular weight
n	=	flow behavior index, dimensionless
\bar{n}	=	outward unit normal
P	=	hydraulic pressure, $\text{lb}_f/\text{sq. ft.}$
P_1	=	hydraulic pressure at the septum, $\text{lb}_f/\text{sq. ft.}$
P_a	=	applied pressure, $\text{lb}_f/\text{sq. ft.}$
P_s	=	solid compressive pressure, $\text{lb}_f/\text{sq. ft.}$

- P' = $(P + \sigma h)$, $lb_f/sq. ft.$
 q = filtrate volume, $cu. ft./sq. ft.$
 R = gas constant
 r_H = hydraulic radius, $ft.$
 R_m = resistance of the septum, $1/ft.$
 s = arbitrary surface, $sq. ft.$
 s_1 = slope defined in Eq. (89)
 S_o = specific surface of solid, or surface area per unit volume of solid, $sq. ft./cu. ft.$
 t = time, $sec.$
 T' = temperature
 T = shear stress which can be T_{rz} for pipe flow, or T_{yx} for slit flow
 \bar{T}_o = $r_H (-\nabla P')$, $lb_f/sq. ft.$
 T_y = yield stress, $lb_f/sq. ft.$
 T_{yx} = shear stress acts on a plane perpendicular to y-axis and in the x-direction, $lb_f/sq. ft.$
 \bar{U} = true velocity of the fluid, with subscript, scalar component, ft/sec
 U_i = coordinate axes, $i = 1, 2, 3$
 v = arbitrary volume element, $cu. ft.$
 $\cdot V$ = vector volumetric rate of flow per unit area or superficial velocity, with subscript, scalar component, $ft./sec.$
 V_{x1} = exit velocity in the filter cake x-direction, $ft./sec.$
 W = total mass of dry solid per unit area, $lb_m/sq. ft.$
 W_x = mass of dry solid per unit area at distance x from the supported medium, $lb_m/sq. ft.$

- $W_1(t)$ = mass of dry solid measured to the surface of the cake, $lb_m/sq. ft.$
 X = fraction solids in slurry, dimensionless
 $f(T)$ = dU_1/dx_p , function of fluid model obtained from shear stress and shear rate relationship, 1/sec.
 $g(T)$ = $T^{b/a-1} f(T)$ defined in Eq. (28)
 α_o, E_o = experimental constants defined in Eq. (57)
 γ, λ = experimental constants defined in Eq. (57)
 α_x = specific cake resistance, $ft./lb_m$
 ρ = density of liquid, $lb_m/cu. ft.$
 ρ_s = true density of solid, $lb_m/cu. ft.$
 μ = viscosity, $lb_m/sec-ft.$
 η = viscosity, $lb_f-sec./sq. ft.$
 η' = apparent viscosity, $lb_f-sec./sq. ft.$
 σ = specific weight, $lb_f/cu. ft.$
 Φ = potential function defined in Eq. (61), ft.
 Ψ = scalar function
 ∇ = vector operator del
 $\nabla \cdot$ = divergence of a vector
 $\nabla \times$ = curl of a vector
 ∇^2 = Laplacian operator
 $\nabla \nabla$ = dyadic operator
 $|\nabla P'|$ = magnitude of the vector $\nabla P'$ and \bar{T}_o respectively
 $|\bar{T}_o|$

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