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UNIVERSITÉ D'OTTAWA
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QUEUES WITH STATE DEPENDENT SERVICES RATE AND HYSTERESIS

BY

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ABSTRACT

A queue with service rate varying with the queue length is analysed.

The variation contains some hysteresis: The service rate is accelerated when the queue reaches a specified length and the fast rate is maintained until the queue reaches a shorter specified length.

Explicit expressions are derived to evaluate:

- the average waiting time and average queue length
- the average length of the fast rate periods
- the average number of fast rate periods.

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1. THE PROBLEM

1.1 Queueing systems are often controlled by reducing the service time when the queue is too long. A common example would be a cashier in a super-market: when the queue is long an additional employee is used to put the groceries in bags. When the queue is shorter the additional employee can be given other duties, when the cashier is assisted her mean service time per customer is reduced.

The service time is a function of the queue length. However, to avoid having to call the additional employee frequently he is asked to stay in until the queue is somewhat shorter than when he was first called. We will call "v" the length of the queue when the employee is called and "u" (u < v) the length of the queue when he is released.

The service time is a function of the queue length with "hysteresis". This phenomena is very common: another example would be in telephony the operators who answer the call when one dials "0" have, on their positions, a lamp that comes on when the queue of waiting calls is long (v), then the operators hurry up. The lamp comes off when the queue is shorter (u).

In general, when it is undesirable to change the service rate too frequently, hysteresis is often used. In this report the frequency of change of service rate, the duration of the accelerated service rate and the mean waiting time of a request are evaluated.

1.2 Surprisingly, it does not seem that this type of queue has been analysed in the literature:

R.W. Conway and W.L. Maxwell have considered the case where the service rate μ is a function of the queue length k following a law of the form $\mu(k) = k^c \mu_0$ (for any c). In [1] they have analysed the single server case and then they generalized to the case of many servers [2]. (M/M/1 and M/M/m)

C.M. Harris has studied in [3] the case where the service rate is defined by a stochastic process indexed by the queue length (M/G/1).

In [4] P.J. Courtor's and J. Georges have generalized the work of R.W. Conway and W.L. Maxwell to the case where the service rate is not exponential (M/G/1) and both the arrival and the service rates vary with the queue length.

1.3 The method we have used to solve this problem is rather straightforward. The only originality is in the definition of 2-dimensional states: one dimension defines the queue length and the other the service time. This makes it possible to apply classical queueing theory techniques. We are also proposing an original and very simple way to calculate the mean time during which the service time is continuously shorter. This method can be applied to the classical problem of mean busy period. The method used in Section 4 to calculate the roots of two non linear equations with two unknowns is interesting, it is a generalization of the "secant" method. It is surprisingly stable and can be extended to more than two dimensions in a very simple manner.

The notation we have used is adopted from Kleinrock [5].

2. THE MODEL

2.1 General

Consider a queueing system with "m" servers, Poisson arrivals and exponential service times (M/M/m). The mean arrival rate is λ requests per second. The mean service rate is a function of the number of requests in the system with some hysteresis: As long as the number of requests in the system has not yet reached "v" the mean service rate per server is "μ" per second. Once the number of requests reaches "v" the mean service rate per server becomes "βμ" and remains at this new value until the number of requests drop to a lower value "u". At "u" the mean service rate per server becomes "μ" again. When the number of requests reaches "v" again, the mean service rate becomes "βμ". The variation of the mean service rate per server is described in Figure 1, note that m < u < v.

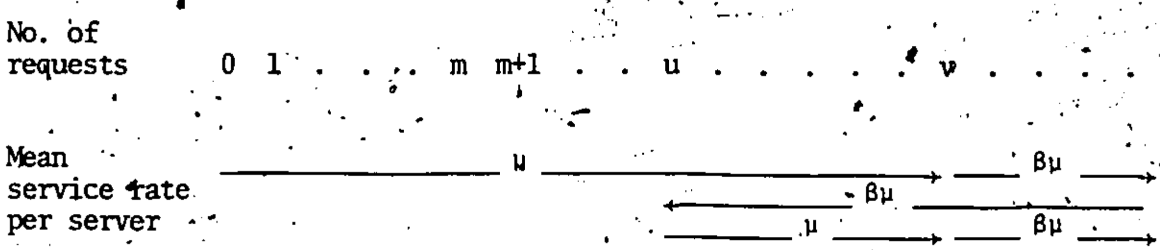


FIGURE 1.

MEAN SERVICE RATE PER SERVER

2.2 Definition of States

To analyse this system, it is necessary to define its states. A state is characterized by the number of requests in the system "k" and by the appropriate mean service rate "μ" or "$\beta\mu$". (k,0) will characterise the states with mean service rate "μ"; (k,1) will characterise the states with mean service rate "$\beta\mu$". Only the following states are possible:

(k,0) for $0 \leq k \leq v - 1$

(k,1) for $k \geq u - 1$

This is illustrated in Figure 2.

Number of requests "k"	0	...	m	...	u	u + 1	.....	v - 1	v	
states	(0,0)		(m,0)		(u,0)	(u+1,0)	...	(v-1,0)		
						(u+1,1)		(v-1,1)	(v,1)	....

FIGURE 2

DEFINTION OF STATES

Let P(k,0) be the probability of being in state (k,0)

$$\text{then} \quad P(k,0) = 0 \quad \text{for } k < 0 \text{ or } k \geq v$$

$$P(k,1) = 0 \quad \text{for } k \leq u$$

2.3 State Equations

To calculate the values of P(k,0) and P(k,1) we will establish some relationships between them. If we consider all the possible states:

$$\sum_{k=0}^{v-1} P(k,0) + \sum_{k=u+1}^{\infty} P(k,1) = 1 \qquad (2.1)$$

Figure 3 indicates the probabilities of moving from one state to another (Transition probabilities). Since the arrivals follow a Poisson process with mean arrival rate "λ" the probability of a new arrival in dt is λdt.

Similarily the probability of a termination of service is μdt (or βμdt) multiplied by the number of requests being served. This number is always ≤m (number of servers). In Figure 3 the "dt" are not included for clarity.

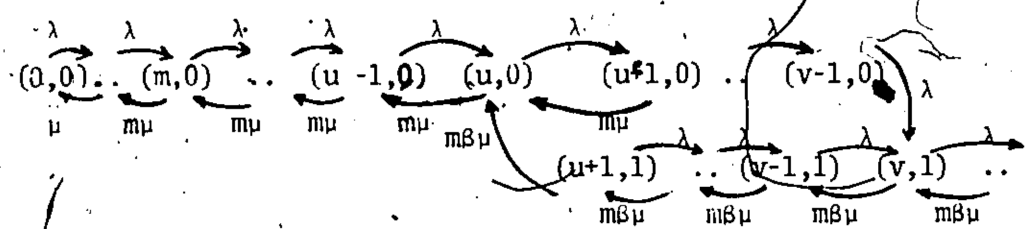
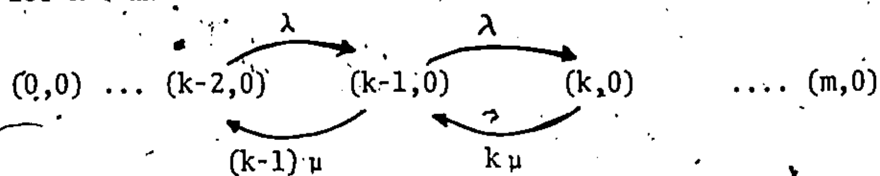


FIGURE 3

Transition probabilities

When the system is in equilibrium the probability of entering a state should be equal to the probability of leaving it.

2.3.1 Consider the states (k-2,0) , (k-1,0) and (k,0) for $k \leq m$:



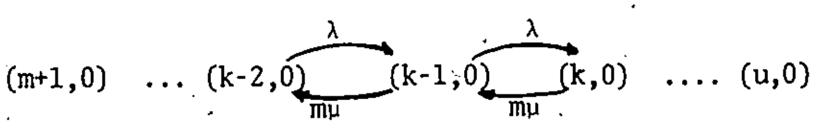
$$\lambda P(k-2,0) + k\mu P(k,0) = \lambda P(k-1,0) + (k-1)\mu P(k-1,0)$$

entering (k-1,0) = leaving (k-1,0)

Let $\rho = \lambda/m\mu$

$$k P(k,0) = (\rho m + k-1) P(k-1,0) - \rho m P(k-2,0) \quad \text{for } k \leq m \qquad (2.2)$$

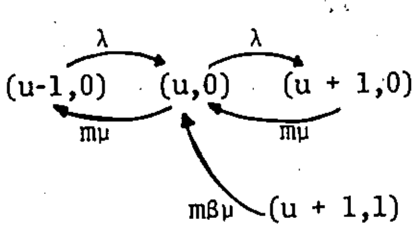
2.3 Consider these 3 states for $m + 1 \leq k \leq u$



$$\lambda P(k-2,0) + m\mu P(k,o) = \lambda P(k-1,0) + m\mu P(k-1,0)$$

$$P(k,0) = (\rho + 1) P(k-1,0) - \rho P(k-2,0) \quad m + 1 \leq k \leq u \qquad (2.3)$$

2.3.3 Consider now all the states surrounding (u,0) in Figure 3



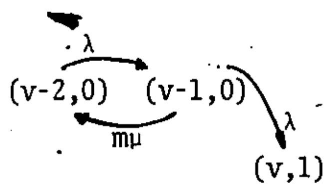
$$\lambda P(u-1,0) + m\mu P(u+1,0) + m\beta\mu P(u+1,1) = \lambda P(u,0) + m\mu P(u,0)$$

$$P(u+1,0) + \beta P(u+1,1) = (\rho+1)P(u,0) - \rho P(u-1,0) \qquad (2.4)$$

2.3.4 Consider the states (k-2,0), (k-1,0) and (k,0) for $u+2 \leq k \leq v-1$, the derivation is identical to 2.3.2 hence

$$P(k,0) = (\rho+1) \; P(k-1,0) - \rho P(k-2,0) \tag{2.5}$$

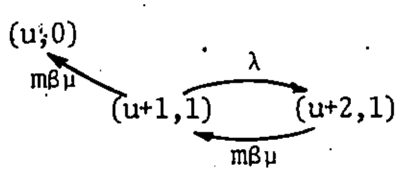
2.3.5 Consider the states surrounding (v-1,0)



$$\lambda P(v-2,0) = \lambda P(v-1,0) + m\mu \; P(v-1,0)$$

$$P(v-1,0) = \frac{\rho}{1+\rho} \; P(v-2,0) \tag{2.6}$$

2.3.6 Consider the states surrounding (u+1,1)

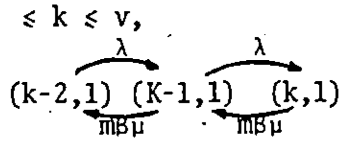


$$m\beta\mu P(u+2,1) = \lambda \; P(u+1,1) + m\beta\mu P(u+1,1)$$

$$P(u+2,1) = (\frac{\rho}{\beta}+1) \; P(u+1,1) \tag{2.7}$$

2.3.7 Consider the states (k-2,1), (k-1,1) and (k,1)

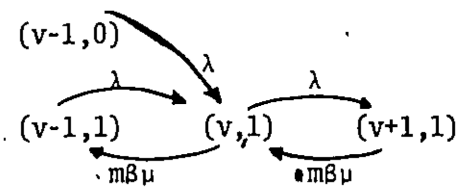
for $u+3 \leq k \leq v$,



$$\lambda P(k-2,1) + m\beta\mu P(k,1) = \lambda P(k-1,1) + mP\mu P(k-1,1)$$

$$P(k,1) = \left(\frac{\rho}{\beta} + 1\right) P(k-1,1) - \frac{\rho}{\beta} P(k-2,1) \qquad (2.8)$$

2.3.8 Consider the states surrounding (v,1)



$$\lambda P(v-1,0) + \lambda P(v-1,1) + m\beta\mu P(v+1,1) =$$

$$\lambda P(v,1) + m\beta\mu P(v,1)$$

$$P(v+1,1) = \left(\frac{\rho}{\beta} + 1\right) P(v,1) - \frac{\rho}{\beta}[P(v-1,0) + P(v-1,1)] \qquad (2.9)$$

2.3.9 And finally consider the states

(k-2,1), (k-1,1) and (k,1) for $k \geq v+2$

the relationship is identical to 2.3.7

$$P(k,1) = \left(\frac{\rho}{\beta} + 1\right) P(k-1,1) - \frac{\rho}{\beta} P(k-2,1) \qquad (2.10)$$

2.3.10 In summary the state equations are:

$$\sum_{k=0}^{v-1} P(k,0) + \sum_{k=u+1}^{\infty} P(k,1) = 1 \qquad (2.1)$$

$$k\,P(k,0) = (\rho m + k - 1)\,P(k-1,o) - \rho m\,P(k-2,0) \qquad 0 \leq k \leq m \quad (2.2)$$

$$P(k,0) = (\rho+1)\,P(k-1,0) - \rho\,P(k-2,0) \qquad m+1 \leq k \leq u \quad (2.3)$$

$$P(u+1,0) + \beta P(u+1,1) = (\rho+1)P(u,0) - \rho P(u-1,0) \qquad (2.4)$$

$$P(k,0) = (\rho+1)\,P(k-1,0) - \rho\,P(k-2,0) \qquad u+2 \leq k \leq v-1 \quad (2.5)$$

$$P(v-1,0) = (\rho\ (\rho+1))\ P(v-2,0) \tag{2.6}$$

$$P(u+2,1) = ((\rho/\beta) + 1)\ \ P(u+1,1) \tag{2.7}$$

$$P(k,1) = ((\rho/\beta) + 1)\ P(k-1,1) - (\rho/\beta)P(k-2,1) \qquad u+3\leq k\leq v \tag{2.8}$$

$$P(v+1,1) = ((\rho/\beta)+1)P(v,1) - (\rho/\beta)[P(v-1,1)+P(v-1,0)] \tag{2.9}$$

$$P(k,1) = ((\rho/\beta)+1)\ P(k-1,1) - (\rho/\beta)\ P(k-2,1) \qquad k\geq v+2 \tag{2.10}$$

3. THE SOLUTION

3.1 Solution of the state equations:

Using (2.2) and (2.3) we will calculate all the P(k,0) for $k \leq u$ as a function of P(0,0). Knowing P(u,0) and solving (2.5) we get P(k,0) for $u + 2 \leq k \leq v - 1$ in function of P(0,0) and of P(u + 1, 0). Applying the equation (2.6) we get P(u + 1, 0) as a function of P(0,0) and (2.4) gives P(u + 1,1) as a function of P(0,0). (2.7), (2.8), (2.9) and (2.10) give P(k,1) for $k \geq u + 2$ as a function of P(0,0). At this point all the probabilities are calculated as functions of P(0,0). Solving (2.1) gives P(0,0).

3.1.1 Equation (2.2) gives

$$k\,P(k,0) = (\rho m + k-1)\,P(k-1,0) - \rho m\,P(k-2,0) \qquad 0 \le k \le m$$

$$P(k,0) = 0 \quad \text{for } k < 0$$

Therefore:

$$P(1,0) = \rho m\,P(0,0)$$

$$2\,P(2,0) = (\rho m + 1)\,P(1,0) - \rho m\,P(0,0)$$

$$= ((\rho m + 1)\,\rho m - \rho m)\,P(0,0)$$

$$= (\rho m)^2\,P(0,0)$$

$$P(2,0) = \frac{(\rho m)^2}{2 \times 1}\,P(0,0)$$

Let us assume that $P(q,0) = \frac{(\rho m)^q}{q!}\,P(0,0)$

for $q = k-1$ and $q = k-2$

$$\text{Then}\quad k\,P(k,0) = \left[(\rho m + k-1)\frac{(\rho m)^{k-1}}{(k-1)!} - \rho m\frac{(\rho m)^{k-2}}{(k-2)!}\right] P(0,0)$$

$$= \frac{(\rho m)^k + (k-1)(\rho m)^{k-1} - (k-1)(\rho m)^{k-1}}{(k-1)!}\,P(0,0)$$

$$P(k,0) = \frac{(\rho m)^k}{k!}\,P(0,0)$$

Therefore if $P(q,0) = \frac{(\rho m)^q}{q!}\,P(0,0)$ is true for $q = k-1$ and $q = k-2$ it is also true for $q = k$. Since it is true for $q = 0$ and $q = 1$ it is true for $q = 3$ and $q = 4$ etc... up to $q = m$.

Therefore $P(k,0) = \frac{(\rho m)^k}{k!} P(0,0)$ $\qquad 0 \leq k \leq m \quad (3.1)$

3.1.2 Let us solve equation (2.3)

$$P(k,0) = (\rho+1) \; P(k-1,0) - \rho \; P(k-2,0) \qquad m+1 \leq k \leq u$$

From (3.1) we know that $P(m-1,0) = \frac{(\rho m)^{m-1}}{(m-1)!} P(0,0)$

and $P(m,0) = \frac{(\rho m)^m}{m!} P(0,0)$

Applying (2.3) $P(m+1,0) = \left[(\rho+1) \frac{(\rho m)^m}{m!} - \rho \frac{(\rho m)^{m-1}}{(m-1)!}\right] P(0,0)$

$$= \frac{\rho^{m+1} \; m^m + (\rho m)^m - (\rho m)^m}{m!} P(0,0)$$

$$= \rho^{m+1} \frac{m^m}{m!} P(0,0)$$

Similarly (2.4) $P(m+2,0) = \left[(\rho+1)\rho^{m+1} \frac{m^m}{m!} - \rho \frac{(\rho m)^m}{m!}\right] P(0,0)$

$$= \frac{\rho^{m+2} \; m^m + \rho^{m+1} \; m^m - \rho^{m+1} \; m^m}{m!} P(0,0)$$

$$= \rho^{m+2} \frac{m^m}{m!} P(0,0)$$

Let us assume that $P(q,0) = \rho^q \frac{m^m}{m!} P(0,0)$

for $q = k-1$ and $q = k-2$ then applying (2.3)

$$P(k,0) = \left[(\rho+1) \quad \rho^{k-1} \frac{m^m}{m!} - \rho x \rho^{k-2} \frac{m^m}{m!} \right] \quad P(0,0)$$

$$= \frac{\rho^k m^m + \rho^{k-1} m^m - \rho^{k-1} m^m}{m!} \qquad P(0,0)$$

$$= \rho^k \frac{m^m}{m!} \quad P(0,0)$$

If $P(q,0) = \rho^q \frac{m^m}{m!} P(0,0)$ is true for $q = k-1$ and $q = k-2$ it is also true for $q = k$ and since it is true for $q = m+1$ and $q = m+2$ it is true for all q satisfying $m+1 \le q \le u$.

Therefore $P(k,0) = \rho^k \frac{m^m}{m!} \quad P(0,0)$ $\qquad m+1 \le k \le u \qquad$ (3.2)

3.1.3 Let us skip (2.4) and try to solve (2.5)

$$P(k,0) = (\rho+1) \cdot P(k-1,0) - \rho \, P(k-2,0) \qquad u+2 \le k \le v-1$$

We can get $P(u,0)$ from (3.2) but we do not have $P(u+1,0)$. We will calculate these $P(k,0)$ as functions of $P(u+1,0)$ and $P(0,0)$:

$$P(u+2,0) = (\rho+1) \, P(u+1,0) - \rho \, P(u,0)$$

$$P(u+3,0) = (\rho+1) \, P(u+2,0) - \rho \, P(u+1,0)$$

$$= (\rho+1) \, (\rho+1) \, P(u+1,0) - (\rho+1) \, \rho \, P(u,0) - \rho \, P(u+1,0)$$

$$= (\rho^2 + \rho + 1) \, P(u+1,0) - (\rho^2+\rho) \, P(u,0)$$

A pattern seems to emerge, let us try

$$P(q,0) = (\rho^{q-u-1} + \rho^{q-u-2} + \ldots + 1) \, P(u+1,0) - (\rho^{q-u-1} + \rho^{q-u-2} + \ldots + \rho) P(u,0)$$

From Appendix 1 the geometrical sums can be calculated:

$$P(q,0) = \frac{1-\rho^{q-u}}{1-\rho} P(u+1,0) - \frac{\rho-\rho^{q-u}}{1-\rho} P(u,0)$$

Let us assume that it is true for q = k-1 and q = k-2

$$P(k,0) = (\rho+1) P(k-1,0) - \rho P(k-2,0)$$

$$= \frac{1}{1-\rho}\Big[(\rho+1)(1-\rho^{k-1-u}) P(u+1,0) - (\rho+1)(\rho-\rho^{k-1-u}) P(u,0)$$

$$- \rho(1-\rho^{k-2-u}) P(u+1,0) + \rho(\rho-\rho^{k-2-u}) P(u,o)\Big]$$

$$= \frac{1}{1-\rho}\Big[(1-\rho^{k-u}) P(u+1,0) - (\rho-\rho^{k-u}) P(u,0)\Big]$$

$$P(k,0) = \frac{1-\rho^{k-u}}{1-\rho} P(u+1,0) - \frac{\rho-\rho^{k-u}}{1-\rho} P(u,0)$$

And since the relationship was true for q = u + 2 and u +3, it is true for all q in $u+2 \leq q \leq v-1$

$$P(k,0) = \frac{1-\rho^{k-u}}{1-\rho} P(u+1,0) - \frac{\rho-\rho^{k-u}}{1-p} P(u,0) \qquad u+2 \leq k \leq v-1$$

From 2.12 $P(u,0) = \rho^{u} \frac{m^{m}}{m!} P(0,0)$

Therefore

$$P(k,0) = \frac{1-\rho^{k-u}}{1-\rho} P(u+1,0) - \frac{\rho^{u+1}-\rho^{k}}{1-\rho} \frac{m^{m}}{m!} P(0,0) \qquad u+2 \leq k \leq v-1 \qquad (3.3)$$

3.1.4 Relationship (2.6) was $P(v-1,0) = \frac{\rho}{1+\rho} P(v-2,0)$

Since we can calculate $P(v-1,0)$ and $P(v-2,0)$ from (3.3), (2.6) will give a relationship between $P(u+1,0)$ and $P(0,0)$:

$$P(v-1,0) = \frac{1-\rho^{v-u-1}}{1-\rho} P(U+1,0) - \frac{\rho^{u+1} - \rho^{v-1}}{1-\rho} \frac{m^m}{m!} P(0,0)$$

$$P(v-2,0) = \frac{1-\rho^{v-u-2}}{1-\rho} P(u+1,0) - \frac{\rho^{u+1} - \rho^{v-2}}{1-\rho} \frac{m^m}{m!} P(0,0)$$

$$\frac{\rho}{1+\rho} P(v-2,0) = \frac{\rho-\rho^{v-u-1}}{1-\rho^2} P(u+1,0) - \frac{\rho^{u+2} - \rho^{v-1}}{1-\rho^2} \frac{m^m}{m!} P(0,0)$$

and applying (2.6)

$$\frac{1-\rho^{v-u-1}}{1-\rho} P(u+1,0) - \frac{\rho^{u+1}\rho^{v-1}}{1-\rho} \frac{m^m}{m!} P(0,0) = \frac{\rho-\rho^{v-u-1}}{1-\rho^2} P(u+1,0) - \frac{\rho^{u+1}\rho^{v-1}}{1-\rho} \frac{m^m}{m!} P(0,0)$$

$$\left[(1+\rho)(1-\rho^{v-u+1})\right] - (\rho-\rho^{v-u-1}) \; P(u+1,0) = \left[(1+\rho)(\rho^{u+1}\rho^{v-1}) - (\rho^{u+2}-\rho^{v-1})\right] \frac{m^m}{m!} P(0,0)$$

$$P(u+1,0) = \frac{\rho^{u+1} - \rho^{v}}{1-\rho^{v-u}} \frac{m^m}{m!} P(0,0) \qquad (3.4)$$

3.1.5 Now we can go back to (2.4)

$$P(u+1,0) + \beta P(u+1,1) = (\rho+1) P(u,0) - \rho P(u-1,0)$$

From (3.2) $P(u,0) = \rho^u \frac{m^m}{m!} P(0,0)$

$$P(u-1,0) = \rho^{u-1} \frac{m^m}{m!} P(0,0)$$

From (3.4) $P(u+1,0) = \frac{\rho^{u+1} - \rho^{v}}{1-\rho^{v-u}} \quad \frac{m^m}{m!} \; P(0,0)$

Therefore $\beta P(u+1,1) = \left[(\rho+1)\rho^{u} - \rho x \rho^{u-1} - \frac{\rho^{u+1}-\rho^{v}}{1-\rho^{v-u}}\right] \frac{m^m}{m!} \quad P(0,0)$

$$= \frac{\rho^{u+1}-\rho^{v+1}-\rho^{u+1} + \rho^{v}}{1 - \rho^{v-u}} \quad \frac{m^m}{m!} \quad P(0,0)$$

$$P(u+1,1) = \frac{\rho^{v}\,(1-\rho)}{\beta(1-\rho^{v-u})} \quad \frac{m^m}{m!} P(0,0) \qquad (3.5)$$

3.1.6 We can also solve (3.3)

$$P(k,0) = \frac{1-\rho^{k-u}}{1-\rho} \; P(u+1,0) - \frac{\rho^{u+1} - \rho^{k}}{1-\rho} \quad \frac{m^m}{m!} \; P(0,0) \qquad u+2 \leq k \leq v-1$$

From (3.4) $P(u+1,0) = \frac{\rho^{u+1} - \rho^{v}}{1 - \rho^{v-u}} \times \frac{m^m}{m!} \quad P(0,0)$

$$P(k,0) = \left[\frac{1-\rho^{k-u}}{1-\rho} \times \frac{\rho^{u+1}-\rho^{v}}{1-\rho^{v-u}} - \frac{\rho^{u+1}-\rho^{k}}{1-p}\right] \frac{m^m}{m!} \; P(0,0)$$

$$= \frac{\rho^{u+1} - \rho^{k+1} - \rho^{v} + \rho^{k+v-u} - (\rho^{u+1} - \rho^{v+1} - \rho^{k} + \rho^{k+v-u})}{(1-\rho)\;(1-\rho^{v-u})} \frac{m^m}{m!} \quad P(0,0)$$

$$= \frac{\rho^{k} - \rho^{k+1} - (\rho^{v} - \rho^{v+1})}{(1-\rho)\;(1-\rho^{v-u})} \quad \frac{m^m}{m!} \quad P(0,0)$$

$$= \frac{\rho^{k}\;(1-\rho) - \rho^{v}\;(1-\rho)}{(1-\rho)\;(1-\rho^{v-u})} \quad \frac{m^m}{m!} \quad P(0,0)$$

$$P(k,0) = \frac{\rho^k - \rho^v}{1-\rho^{v-u}} \frac{m^m}{m!} P(0,0) \qquad u+2 \leq k \leq v-1 \quad (3.6)$$

One can note that (3.4) is a special case of (3.6). So that (3.6) is valid for $u + 1 \leq k \leq v-1$.

3.1.7 Relation (2.7) gives $P(u+2,1)$:

$$P(u+2,1) = ((\rho/\beta) + 1) \quad P(u+1,1)$$

Applying (3.5) → $P(u+2,1) = (1 + (\rho/\beta)) \dfrac{\rho^v (1-\rho) m^m}{\beta(1-\rho^{v-u})m!} P(0,0)$

By multiplying numerator and denominator by $1 - (\rho/\beta)$

$$P(u+2,1) = \frac{1-\rho}{\beta-\rho} \frac{\rho^v}{1-\rho^{v-u}} (1-(\rho/\beta)^2) \frac{m^m}{m!} P(0,0) \qquad (3.7)$$

3.1.8 And now we solve (2.8)

$$P(k,1) = (1 + (\rho/\beta)) P(k-1,1) - (\rho/\beta) P(k-2,1) \qquad u+3 \leq k \leq v$$

$$P(u+3,1) = (1+(\rho/\beta)) P(u+2,1) - (\rho/\beta) P(u+1,1)$$

$$= \left[(1+(\rho/\beta)) \frac{1-\rho}{\beta-\rho} \times \frac{\rho^v}{1-\rho^{v-u}} (1-(\rho/\beta)^2) - (\rho/\beta) \frac{1-\rho}{\beta} \frac{\rho}{1-\rho^{v-u}}\right] \frac{m^m}{m!} P(0,0)$$

$$= \frac{(1-\rho)\rho^v}{1-\rho^{v-u}} \left[(1+(\rho/\beta)) \frac{1-(\rho/\beta)^2}{\beta-\rho} - (\rho/\beta)^2\right] \frac{m^m}{m!} P(0,0)$$

$$= \frac{(1-\rho)\rho^v}{1-\rho^{v-u}} \left[\frac{1 + (\rho/\beta) - (\rho/\beta)^2 - (\rho/\beta)^3 - (\rho/\beta) + (\rho/\beta)^2}{\beta-\rho}\right] \frac{m^m}{m!} P(0,0)$$

$$= \frac{(1-\rho)\rho^{v}}{1-\rho^{v-u}} \quad \frac{1 - (\rho/\beta)^{3}}{\beta-\rho} \quad \frac{m^{m}}{m!} \quad P(0)$$

$$= \frac{1-\rho}{\beta-\rho} \quad \frac{\rho^{v}}{1-\rho^{v-u}} \quad (1 - (\rho/\beta)^{3}) \quad \frac{m^{m}}{m!} \quad P(0,0)$$

From $P(u+2,1)$ and $P(u+3,1)$ a pattern seems to emerge, let us try:

$$P(q,1) = \frac{1-\rho}{\beta-\rho} \quad \frac{\rho^{v}}{1-\rho^{v-u}} \quad (1 - (\rho/\beta)^{q-u}) \quad \frac{m^{m}}{m!} \quad P(0,0)$$

Assume it is true for k-1, k-2 and calculate it for k:

$$P(k,1) = (1+(\rho/\beta)) \quad P(k-1,1) - (\rho/\beta) \quad P(k-2,2)$$

$$= \frac{1-\rho}{\beta-\rho} \quad \frac{\rho^{v}}{1-\rho^{v-u}} \left[(1 + (\rho/\beta))(1-(\rho/\beta)^{k-1-u}) - (\rho/\beta)(1 - (\rho/\beta)^{k-2-u}) \right] \frac{m^{m}}{m!} \quad P(0,0)$$

$$= \frac{1-\rho}{\beta-\rho} \quad \frac{\rho^{v}}{1-\rho^{v-u}} \left[1 - (\rho/\beta)^{k-u} \right] \frac{m^{m}}{m!} \quad P(0,0)$$

Since it is also true for k then it is true for all q such that $u + 3 \leq q \leq v$.

And noticing that $P(u+2,1)$ and $P(u+1,1)$ satisfy the same relationship we conclude that:

$$P(k,1) = \frac{1-\rho}{\beta-\rho} \quad \frac{\rho^{v}}{1-\rho^{v-u}} \quad (1-(\rho/\beta)^{k-u}) \quad \frac{m^{m}}{m!} \quad P(0,0) \qquad u+1 \leq k \leq v \qquad (3.8)$$

3.1.9 Now we can solve (2.9)

$$P(v+1,1) = (1+(\rho/\beta)) \quad P(v,1) - (\rho/\beta) \quad [P(v-1,1) + P(v-1,0)]$$

from (3.6) $P(v-1,0) = \dfrac{\rho^{v-1}-\rho^{v}}{1-\rho^{v-u}} \quad \dfrac{m^{m}}{m!} \quad P(0,0)$

from (3.8) $P(v-1,1) = \frac{1-\rho}{\beta-\rho} \frac{\rho^v}{1-\rho^{v-u}} (1-(\rho/\beta)^{v-u-1}) \frac{m^m}{m!} P(0,0)$

and $P(v,1) = \frac{1-\rho}{\beta-\rho} \frac{\rho^v}{1-\rho^{v-u}} (1 - (\rho/\beta)^{v-u}) \frac{m^m}{m!} P(0,0)$

$$P(v+1,1) = \frac{1}{1-\rho^{v-u}} \left[(1+(\rho/\beta)) \frac{(1-\rho)}{\beta-\rho} \times \rho^v (1-(\rho/\beta)^{v-u}) - (\rho/\beta) \left(\frac{1-\rho}{\beta-\rho} \rho^v (1-(\rho/\beta)^{v-u-1}) + \rho^{v-1} \rho^v\right)\right] \frac{m^m}{m!} P(0,0)$$

$$= \frac{1 - \rho}{(1-\rho^{v-u})(\beta-\rho)} \left[(1 + (\rho/\beta)) \rho^v (1-(\rho/\beta)^{v-u}) - (\rho/\beta)(\rho^v-\rho^v(\rho/\beta)^{v-u-1} + \rho^{v-1}(\beta-\rho))\right] \frac{m^m}{m!} P(0,0)$$

$$= \frac{(1-\rho)\rho^v}{(1-\rho^{v-u})(\beta-\rho)} \left[1 + (\rho/\beta)-(\rho/\beta)^{v-u}-(\rho/\beta)^{v-u+1}-(\rho/\beta)+(\rho/\beta)^{v-u}-1+(\rho/\beta)\right] \frac{m^m}{m!} P(0,0)$$

$$= \frac{(1-\rho)\rho^v}{(1-\rho^{v-u})(\beta-\rho)} \left[(\rho/\beta) - (\rho/\beta)^{v-u+1}\right] \frac{m^m}{m!} P(0,0)$$

$$P(v+1,1) = \frac{1-\rho}{\beta-\rho} \frac{\rho^v}{1-\rho^{v-u}} \left((\rho/\beta) - (\rho/\beta)^{v+1-u}\right) \frac{m^m}{m!} P(0,0)$$

$$= \frac{1-\rho}{\beta-\rho} \frac{1-(\rho/\beta)^{v-u}}{1-\rho^{v-u}} \rho^v \frac{\rho}{\beta} \frac{m^m}{m!} P(0,0) \qquad (3.9)$$

3.1.10 Finally we can solve 2.10

$$P(k,1) = (1+\rho/\beta)P(k-1,1) - (\rho/\beta)P(k-2,1) \quad k \geq v+2 \qquad k\geq v+2$$

$$P(v+2,1) = (1+\rho/\beta)P(v+1,1) - (\rho/\beta)P(v,1)$$

Applying (3.8) and (3.9)

$$P(v+2,1) = \frac{1-\rho}{\beta-\rho} \frac{1-(\rho/\beta)^{v-u}}{1-\rho^{v-u}} \rho^v \left[(1+\rho/\beta)\,\rho/\beta - (\rho/\beta)\right] \frac{m^m}{m!} P(0,0)$$

$$= \frac{1-\rho}{\beta-\rho} \frac{1-(\rho/\beta)^{v-u}}{1-\rho^{v-u}} \rho^v (\rho/\beta)^2 \frac{m^m}{m!} P(0,0)$$

A pattern can be tried:

$$P(q,1) = \frac{1-\rho}{\beta-\rho} \frac{1-(\rho/\beta)^{v-u}}{1-\rho^{v-u}} \rho^v (\rho/\beta)^{q-v} \frac{m^m}{m!} P(0,0)$$

$$= \frac{1-\rho}{\beta-\rho} \frac{1-(\rho/\beta)^{v-u}}{1-\rho^{v-u}} \beta^v (\rho/\beta)^q \frac{m^m}{m!} P(0,0)$$

Assume it is true for q = k-1 and q = k-2, calculate q =k applying (2.10)

$$P(k,1) = \frac{1-\rho}{\beta-\rho} \frac{1-(\rho/\beta)^{v-u}}{1-\rho^{v-u}} \beta^v \left[(1+\rho/\beta)(\rho/\beta)^{k-1} - (\rho/\beta)(\rho/\beta)^{k-2}\right] \frac{m^m}{m!} P(0,0)$$

$$= \frac{1-\rho}{\beta-\rho} \frac{1-(\rho/\beta)^{v-u}}{1-\rho^{v-u}} \beta^v (\rho/\beta)^k \frac{m^m}{m!} P(0,0)$$

Since the relationship was true for q = v+1 and v+2 it is true for all q

Therefore

$$P(k,1) = \frac{1-\rho}{\beta-\rho} \frac{1-(\rho/\beta)^{v-u}}{1-\rho^{v-u}} \beta^v (\rho/\beta)^k \frac{m^m}{m!} P(0,0) \qquad k\geq v+2 \quad (3.10)$$

Note that P(v+1,1) and P(v,1) satisfy this relationship.

3.1.11 All the probabilities have been calculated as functions of P(0,0)

Let $G = \frac{m^m}{m!} P(0,0)$. Let us regroup the probabilities for clarity:

$$0 \le k \le m-1 \qquad P(k,0) = \frac{(\rho m)^k}{k!} \frac{m!}{m^m} G \tag{3.1}$$

$$m \le k \le u \qquad P(k,0) = \rho^k G \tag{3.2}$$

$$u+1 \le k \le v-1 \qquad P(k,0) = \frac{\rho^k - \rho^v}{1-\rho^{v-u}} G \tag{3.6}$$

$$u+1 \le k \le v-1 \qquad P(k,1) = \frac{1-\rho}{\beta-\rho} \rho^v \frac{1-(\rho/\beta)^{k-u}}{1-\rho^{v-u}} G \tag{3.8}$$

$$v \le k \qquad P(k,1) = \frac{1-\rho}{\beta-\rho} \beta^v \frac{1-(\rho/\beta)^{v-u}}{1-\rho^{v-u}} (\rho/\beta)^k G \tag{3.10}$$

Note: G will be given in (3.19)

3.1.12 Now we can apply (2.1)

$$\sum_{k=0}^{v-1} P(k,0) + \sum_{k=u+1}^{\infty} P(k,1) = 1$$

$$\sum_{k=o}^{m-1} P(k,0) = G \frac{m!}{m^m} \sum_{k=0}^{m-1} \frac{(\rho m)^k}{k!} \tag{3.11}$$

$$\sum_{k=m}^{u} P(k,0) = G \sum_{k=m}^{u} \rho^k = G \frac{\rho^m - \rho^{u+1}}{1-\rho} \tag{3.12}$$

$$\sum_{k=u+1}^{v-1} P(k,0) = \frac{G}{1 - \rho^{v-u}} \left[\left(\sum_{k=u+1}^{v-1} \rho^k \right) - (v-u-1)\rho^v \right]$$

$$= \frac{G}{1 - \rho^{v-u}} \left[\frac{\rho^{u+1} - \rho^v}{1 - \rho} - (v-u-1)\, \rho^v \right]$$

$$= G \cdot \frac{\rho^{u+1} - (v-u)\rho^v + (v-u-1)\rho^{v+1}}{(1-\rho)\quad(1-\rho^{v-u})} \qquad (3.13)$$

$$\sum_{k=u+1}^{v-1} P(k,1) = \frac{G(1-\rho)\rho^v}{(\beta-\rho)(1-\rho^{v-u})} \quad (v-u-1) - \frac{(\rho/\beta) - (\rho/\beta)^{v-u}}{1 - \rho/\beta}$$

$$= \frac{G(1-\rho)\rho^v \left[(v-u-1) - (v-u)\,(\rho/\beta) + (\rho/\beta)^{v-u} \right]}{(\beta-\rho)\ (1 - \rho^{v-u})\ (1-\rho/\beta)} \qquad (3.14)$$

$$\sum_{k=v}^{\infty} P(k,1) = \frac{G(1-\rho)(1-(\rho/\beta)^{v-u})}{(\beta-\rho)(1-\rho^{v-u})} \quad \beta^v \Big/ \frac{(\rho/\beta)^v}{1 - \rho/\beta} \qquad \text{note that } \rho/\beta < 1$$

$$= \frac{G(1-\rho)\rho^v \ (1 - (\rho/\beta)^{v-u})}{(\beta-\rho)\ (1-\rho^{v-u})\ (1-\rho/\beta)} \qquad (3.15)$$

Combining (3.14) and (3.15)

$$\sum_{k=u+1}^{\infty} P(k,1) = \frac{G\,(1-\rho)\rho^{v}}{(\beta-\rho)(1-\rho^{v-u}(1-(\rho/\beta))} \quad [(v-u)\ (1-\rho/\beta)]$$

$$= \frac{G(1-\rho)\ (v-u)\ \rho^{v}}{(\beta-\rho)(1-\rho^{v-u})} \qquad (3.16)$$

Note that this is the probability of being in any state with a service rate "$\beta\mu$". We will reuse this fact later.

Combining (3.12) and (3.13)

$$\sum_{k=m}^{v-1} P(k,0) = G\ \frac{(\rho^{m} + \rho^{u+1})(1-\rho^{v-u}) + \rho^{u+1} - (v-u)\ \rho^{v} + (v-u-1)\rho^{v+1}}{(1-\rho)\ (1-\rho^{v-u})}$$

$$= G\left[\frac{\rho^{m}}{1-\rho} + \frac{(v-u)\ \rho^{v+1} - (v-U)\ \rho^{v}}{(1-\rho)\ (1-\rho^{v-u})}\right]$$

$$= G\left[\frac{\rho^{m}}{1-\rho} - \frac{(v-u)\ \rho^{v}}{1-\rho^{v-u}}\right] \qquad (3.17)$$

And combining (3.16) and (3.17)

$$\sum_{k=m}^{v-1} P(k,0) + \sum_{k=u+1}^{\infty} P(k,1) =$$

$$= G\left[\frac{\rho^{m}}{1-\rho} + \frac{(v-u)\ \rho^{v}\ \ (1-\rho-\ (\beta-\rho))}{(\beta-\rho)\ (1-\rho^{v-u})}\right]$$

$$= G\left[\frac{\rho^m}{1-\rho} - \frac{(\beta-1)\ (v-u)\rho^v}{(\beta-\rho)(1-\rho^{v-u})}\right] \qquad (3.18)$$

And applying (2.1)

$$\frac{1}{G} = \frac{\rho^m}{1-\rho} - \frac{m!}{m^m} \sum_{k=o}^{m-1} \frac{(\rho m)^k}{k!} - \frac{(\beta-1)(v-u)\rho^v}{(\beta-\rho)(1-\rho^{v-u})} \qquad (3.19)$$

The equations of section 3.1.11 and (3.19) give all the probability of states

Since $G = \frac{m^m}{m!} P(0,0)$

$$P(0,0) = 1/\left[\frac{(\rho m)^m}{m!(1-\rho)} + \sum_{k=o}^{m-1} \frac{(\rho m)^k}{k!} - \frac{m^m}{m!} \frac{(\beta-1)(v-u)\rho^v}{(\beta-\rho)(1-\rho^{v-u})}\right] \qquad (3.20)$$

3.2 Some interesting probabilities:

In section 3.1 the probabilities of being in the various states have been calculated. Some combinations of these probabilities are particularly interesting.

3.2.1 $B = \sum_{k=m}^{v-1} P(k,0) + \sum_{k=u+1}^{\infty} P(k,1)$

Applying (3.18) and (3.19)

$$B = \frac{\frac{\rho^m}{1-\rho} - \frac{(\beta-1)(v-u)\rho^v}{(\beta-\rho)(1-\rho^{v-u})}}{\frac{\rho^m}{1-\rho} \times \frac{m!}{m^m} \sum_{k=0}^{m-1} \frac{(\rho m)^k}{k!} - \frac{(\beta-1)(v-u)\rho^v}{(\beta-\rho)(1-\rho^{v-u})}} \qquad (3.21)$$

It is interesting to note that when $\beta=1$ (No variation of service rate)

$$B_{.} = \frac{1}{1+(1-\rho)\frac{m!}{(\rho m)^m}\sum_{k=0}^{m-1}\frac{(\rho m)^k}{k!}}$$

which is the classical form for the Erlang delay formula.

3.2.2 Probability that the system is using the service rate $\beta\mu$: B_β

$$B_\beta = \sum_{k=u+1}^{\infty} P(k,1) = \frac{(1-\rho)(v-u)\rho^v}{(\beta-\rho)(1-\rho^{v-u})} G \qquad (3.16)$$

Applying (3.19)

$$B_\beta = \frac{\dfrac{(1-\rho)(v-u)\rho^v}{(\beta-\rho)(1-\rho^{v-u})}}{\dfrac{\rho^m}{1-\rho} + \dfrac{m!}{m^m}\sum_{k=0}^{m-1}\dfrac{(\rho m)^k}{k!} - \dfrac{(\beta-1)(v-u)\rho^v}{(\beta-\rho)(1-\rho^{v-u})}} \qquad (3.22)$$

If we consider a time interval T, during a fraction of time $B_\beta \times T$ the service rate is $\beta\mu$. Therefore the number of requests served with a service rate $\beta\mu$ during T is: $m\beta\mu \times B_\beta \times T$ (3.23)

3.2.3 Frequency of change of service rate: F_c

The system is ready for a change of service rate when it is in the state (v-1,0).

$$P(v-1,0) = \frac{\rho^{v-1} - \rho^v}{1-\rho^{v-u}} G \qquad (3.6)$$

Let us consider a time interval T; the system is ready for a change during

P(v-1,0) x T. Since the rate of arrival is λ, the number of changes of service rate during P(v-1,0) x T is:

$$F_c \times T = \frac{\rho^{v-1} - \rho^{v}}{1 - \rho^{v-u}} \quad G \times \lambda \times T \tag{3.24}$$

3.3 Mean control period and mean busy period:

3.3.1 The mean control period is the mean period during which the system is using continuously a service rate $\beta\mu = t_\beta$. This mean period is the ratio of the total time spent with the service rate $\beta\mu$ over the number of times the service rate $\beta\mu$ was started. Applying the results of [3.18] and [3.24].

$$t_\beta = \frac{B_\beta \times T}{F_c}$$

$$t_\beta = \frac{\dfrac{(1-\rho)(v-u)\rho^{v}}{(\beta-\rho)(1-\rho^{v-u})}}{\dfrac{(\rho^{v-1} - \rho^{v})\lambda}{(1-\rho^{v-u})}}$$

$$t_\beta = \frac{(v-u)\rho}{\lambda(\beta-\rho)} \tag{3.25}$$

3.3.2 The mean busy period:

The classical mean busy period is the mean period during which the servers were continuously all busy, t_m:

The total of the busy periods during a time T is given by

$$B \times T = \left[\frac{\rho^m}{1-\rho} - \frac{(\beta-1)(v-u)\rho^v}{(\beta-\rho)(1-\rho^{v-u})} \right] G \times T$$

In T the total periods during which the system is ready to start a busy period is given by:

$$P(m-1,0) \times T = \frac{(\rho m)^{m-1}}{(m-1)!} \quad \frac{m!}{m^m} \quad \times G \times T$$

$$= \rho^{m-1} \quad G \quad T$$

The number of arrivals during these periods is:

$$\rho^{m-1} \times G \times T \times \lambda$$

Therefore the mean busy period is given by:

$$t_m = \frac{B x T}{\rho^{m-1} x G x T x \lambda} = \frac{1}{\lambda \rho^{m-1}} \left[\frac{\rho^m}{1-\rho} - \frac{(\beta-1)(v-u)\rho^v}{(\beta-\rho)(1-\rho^{v-u})} \right] \qquad (3.26)$$

Again we notice that when $\beta=1$

$$t_m = \frac{\rho}{\lambda(1-\rho)}$$

which is the classical mean busy period.

3.4 Mean queue length

The service rate has been changed in order to decrease the queue length. To quantify this effect let us calculate the mean queue length:

$$\bar{q} = \sum_{k=m}^{v-1} (k-m)\ P(k,0) + \sum_{k=u+1}^{\infty} (k-m)\ P(k,1)$$

$$\bar{q} = \sum_{k=m}^{v-1} k\ P(k,0) + \sum_{k=u+1}^{\infty} kP(k,1) - mB \qquad (3.27)$$

3.4.1 Letus calculate first $k_1 = \sum_{k=m}^{u} kP(k,0)$

From (3.2) $k_1 = G \sum_{k=m}^{u} k\ \rho^k$ and applying A.3

$$k_1 = \frac{G}{(1-\rho)^2} \left[m\rho^m - (m-1)\rho^{m+1} - (u+1)\rho^{u+1} + u\rho^{u+2} \right]$$

3.4.2 $k_2 = \sum_{k=u+1}^{v-1} kP(k,0) = G \sum_{k=u+1}^{v-1} k\ \frac{\rho^k - \rho^v}{1-\rho^{v-u}}$ (3.6)

$$k_2 = \frac{-G\rho^v}{1-\rho^{v-u}} \times \frac{(v+u)(v-u-1)}{2} + \frac{G}{1-\rho^{v-u}} \sum_{k=u+1}^{v-1} k\rho^k$$

$$k_2 = \frac{-G(v+u)(v-u-1)\rho^v}{2(1-\rho^{v-u})} +$$

$$\frac{G}{(1-\rho^{v-u})(1-\rho)^2} \left[(u+1)\rho^{u+1} - u\rho^{u+2} - v\rho^v + (v-1)\rho^{v+1} \right]$$

3.4.3 $$k_3 = \sum_{k=u+1}^{v-1} kP(k,1) = \frac{(1-\rho)\rho^v G}{(\beta-\rho)(1-\rho^{v-u})} \sum_{k=u+1}^{v-1} (k-k\ (\rho/\beta)^{k-u})$$

$$k_3 = \frac{G(1-\rho)(v+u)(v-u-1)\rho^v}{2(\beta-\rho)(1-\rho^{v-u})} - \frac{G(1-\rho)\rho^{v-u}}{(\beta-\rho)(1-\rho^{v-u})\beta^{-u}} \sum_{k=u+1}^{v-1} k\ (\rho/\beta)^k$$

$$k_3 = \frac{G(1-\rho)(v+u)(v-u-1)\rho^v}{2(\beta-\rho)(1-\rho^{v-u})} - \frac{G(1-\rho)\rho^{v-u}}{(\beta-\rho)(1-\rho^{v-u})\beta^{-u}(1-\rho/\beta)^2} \left[(u+1)(\rho/\beta)^{u+1} - u(\rho/\beta)^{u+2} - v(\rho/\beta)^v + (v-1)(\rho/\beta)^{v+1}\right]$$

3.4.4 $$k_4 = \sum_{v}^{\infty} kP(k,1) = \frac{(1-\rho)\beta^v(1-(\rho/\beta)^{v-u})G}{(\beta-\rho)(1-\rho^{v-u})} \sum_{v}^{\infty} k\ (\rho/\beta)^k$$

$$k_4 = \frac{G(1-\rho)(1-(\rho/\beta)^{v-u})\beta^v}{(\beta-\rho)(1-\rho^{v-u})(1-\rho/\beta)^2} \left[v\ (\rho/\beta)^v - (v-1)\ (\rho/\beta)^{v+1}\right]$$

$$= \frac{G(1-\rho)(\beta^{v-u} - \rho^{v-u})}{(\beta-\rho)(1-\rho^{v-u})\beta^{-u}(1-\rho/\beta)^2} \left[v\ (\rho/\beta)^v - (v-1)\ (\rho/\beta)^{v+1}\right]$$

3.4.5 Letus combine $k_3 + k_4$

$$k_3 + k_4 = \frac{G(1-\rho)}{(\beta-\rho)(1-\rho^{v-u})\beta^{-u}(1-\rho/\beta)^2} \left[v(\rho/\beta)^v \beta^{v-u} - (v-1)\ (\rho/\beta)^{v+1}\beta^{v-u} - (u+1)(\rho/\beta)^{u+1}\rho^{v-u} + u(\rho/\beta)^{u+2}\rho^{v-u}\right]$$

$$+ \frac{G(1-\rho)(v+u)(v-u-1)\rho^v}{2(\beta-\rho)(1-\rho^{v-u})}$$

$$k_3 + k_4 = \frac{G(1-\rho)}{(\beta-\rho)(1-\rho^{v-u})}\left[\frac{v\rho^v - (v-1)\frac{\rho^{v+1}}{\beta} - (u+1)\frac{\rho^{v+1}}{\beta} + u\frac{\rho^{v+2}}{\beta^2}}{(1-\rho/\beta)^2} + \frac{(v+u)(v-u-1)\rho^v}{2}\right]$$

$$k_3 + k_4 = \frac{G(1-\rho)\rho^v}{(\beta-\rho)(1-\rho^{v-u})}\left[\frac{v\beta^2 - (v+u)\beta\rho + u\rho^2}{(\beta-\rho)^2} + \frac{(v+u)(v-u-1)}{2}\right]$$

$$= \frac{G(1-\rho)\rho^v}{(\beta-\rho)(1-\rho^{v-u})}\left[\frac{v\beta-u\rho}{\beta-\rho} + \frac{(v+u)(v-u-1)}{2}\right]$$

3.4.6 Combining $k_1 + k_2$

$$k_1 + k_2 = \frac{G}{(1-\rho)^2}\left[m\rho^m - (m-1)\rho^{m+1}\right] - \frac{G\rho^v(v+u)(v-u-1)}{2(1-\rho^{v-u})}$$

$$+ \frac{G}{(1-\rho^{v-u})(1-\rho)^2}\left[(-(u+1)\rho^{u+1} + u\rho^{u+2})(1-\rho^{v-u}) + (u+1)\rho^{u+1} - u\rho^{u+2} - v\rho^v + (v-1)\rho^{v+1}\right]$$

$$= \frac{G}{(1-\rho)^2}\left[m\rho^m - (m-1)\rho^{m+1}\right] - \frac{G\rho^v(v+u)(v-u-1)}{2(1-\rho^{v-u})}$$

$$+ \frac{G}{(1-\rho^{v-u})(1-\rho)^2}\left[+(u+1)\rho^{v+1} - u\rho^{v+2} - v\rho^v + (v-1)\rho^{v+1}\right]$$

$$= \frac{G}{(1-\rho)^2}\left[m\rho^m - (m-1)\rho^{m+1}\right] - \frac{G\rho^v(v+u)(v-u-1)}{2(1-\rho^{v-u})}$$

$$- \frac{G\rho^{v}}{(1-\rho^{v-u})(1-\rho)^2} \quad (u\rho^2 - (v+u)\rho + v)$$

3.4.6 And finally

$$k_1+k_2+k_3+k_4 = \frac{G\rho^m}{(1-\rho)^2} \quad (m - (m-1)\rho)$$

$$+ \frac{G\rho^v(v+u)(v-u-1)}{2(1-\rho^{v-u})} \left(\frac{1-\rho}{\beta-\rho} - 1\right)$$

$$+ \frac{G\rho^v}{(1-\rho^{v-u})} \left[\frac{(1-\rho)(v\beta-u\rho)}{(\beta-\rho)^2} + \frac{(u\rho-v)(1-\rho)}{(1-\rho)^2}\right]$$

Let k_5 be the last term:

$$k_5 = \frac{G\rho^v}{1-\rho^{v-u}} \quad \frac{(1-\rho)^2\ (v\beta - u\rho) + (\beta-\rho)^2\ (u\rho-v)}{(\beta-\rho)^2\ (1-\rho)}$$

$$= \frac{G\rho^v}{(1-\rho^{v-u})(\beta-\rho)^2(1-\rho)} \left[(1-2\rho+\rho^2)(v\beta-u\rho) + (\beta^2-2\beta\rho+\rho^2)(u\rho-v)\right]$$

$$= \frac{G\rho^v}{(1-\rho^{v-u})(\beta-\rho)^2(1-\rho)} \left[v\beta -2v\beta\rho+v\beta\rho^2-u\rho+2u\rho^2-u\rho^3 \right.$$
$$\left. +u\beta^2\rho-2u\beta\rho^2+u\rho^3-v\beta^2+2v\beta\rho-v\rho^2\right]$$

$$= \frac{G\rho^v}{(1-\rho^{v-u})(\beta-\rho)^2(1-\rho)} \left[v\beta(1-\beta)-u(1-\beta^2)\rho+2u(1-\beta)\rho^2 -v(1-\beta)\rho^2\right]$$

$$= \frac{G\rho^v(1-\beta)}{(1-\rho^{v-u})(\beta-\rho)^2(1-\rho)} \quad (v\beta-u(1+\beta)\rho + (2u-v)\rho^2)$$

And going back to $k_1+k_2+k_3+k_4$ =

$$\frac{G\rho^m}{(1-\rho)^2}\,(m - (m-1)\rho)$$

$$- \frac{G\rho^v(\beta-1)}{(1-\rho^{v-u})(\beta-\rho)}\left[\frac{(v+u)(v-u-1)}{2} + \frac{v\beta - u(1+\beta)\rho + (2u-v)\rho^2}{(\beta-\rho)(1-\rho)}\right]$$

If the last square parenthesis is k_6:

$$2(\beta-\rho)(1-\rho)\;k_6 = (v+u)(v-u-1)(\beta-\rho)(1-\rho)$$

$$+ 2\,v\beta - 2u(1+\beta)\rho + (4u-2v)\rho^2$$

$$= (v^2-u^2)(\beta-\rho)(1-\rho) - (v-u)(\beta-(\beta+1)\rho + \rho^2)$$

$$+ 2v - 2u(1+\beta)\rho + (4u-2v)\rho^2$$

$$= (v^2-u^2)(\beta-\rho)(1-\rho) + v\beta - u\beta + v(\beta+1)\rho - u(\beta+1)\rho + (3u-3v)\rho^2$$

$$= (v^2-u^2)(\beta-\rho)(1-\rho) + (v-u)\;(\beta+(\beta+1)\rho-3\rho^2)$$

$$= (v^2-u^2)(\beta-\rho)(1-\rho) + (v-u)\left[(\beta+\rho)(1+\rho) - 4\rho^2\right]$$

And finally: $k_1+k_2+k_3+k_4$ =

$$\frac{G\,^m(m-(m-v\;)}{(1-\rho)^2} - \frac{G(\beta-1)\;(v-u)\rho^v}{2(1-\rho^{v-u})(\beta-\rho)^2(1-\rho)}\left[\begin{array}{l}(v+u)(\beta-\rho)(1-\rho)\\ \quad + (\beta+\rho)(1+\rho) - 4\rho^2\end{array}\right]$$

3.4.7 And now using (3.27)

$$\bar{q} = \frac{G\rho^m(m - m\rho - \rho)}{(1-\rho)^2} - mB$$

$$- \frac{G(\beta-1)(v-u)\rho^v}{2(\beta-\rho)^2(1-\rho)(1-\rho^{v-u})}\left[(v+u)(\beta-\rho)(1-\rho)+(\beta+\rho)(1+\rho) - 4\rho^2\right]$$

However $B = \left[\frac{\rho^m}{1-\rho} - \frac{(\beta-1)(v-u)\rho^v}{(\beta-\rho)(1-\rho^{v-u})}\right] G$ (3.21)

$$\frac{\bar{q}}{G} = \frac{\rho^m(m-m\rho+\rho)}{(1-\rho)^2} - \frac{m\rho^m}{1-\rho}$$

$$- \frac{(\beta-1)(v-u)\rho^v}{(\beta-\rho)(1-\rho^{v-u})}\left[\frac{(v+u)(\beta-\rho)(1-\rho) + (\beta+\rho)(1+\rho)-4\rho^2}{2(\beta-\rho)(1-\rho)} - m\right]$$

$$\frac{\bar{q}}{G} = \frac{\rho^{m+1}}{(1-\rho)^2} - \frac{(\beta-1)(v-u)\rho^v}{(\beta-\rho)(1-\rho^{v-u})}\left[\frac{v+u}{2} + \frac{(\beta+\rho)(1+\rho) - 4\rho^2}{2(\beta-\rho)(1-\rho)} - m\right] \quad (3.28)$$

G is given in (3.19)

3.5 Mean Waiting Time

And now we can apply Little's formula [6]: The mean waiting time $\bar{w} = \frac{\bar{q}}{\lambda}$

Therefore:

$$\bar{w} = \frac{G}{\lambda}\left\{\frac{\rho^{m+1}}{(1-\rho)^2} - \frac{(\beta-1)(v-u)\rho^v}{(\beta-\rho)(1-\rho^{v-u})}\left[\frac{v+u}{2} - \frac{(\beta+\rho)(1+\rho) - 4\rho^2}{2(\beta-\rho)(1-\rho)} - m\right]\right\} \quad (3.29)$$

Furthermore, if we consider the average delay of the requests that encounter some delay:

$$\bar{w}_d = \frac{\bar{w}}{B} \leftarrow (\bar{w} = B\bar{w}_d + (1-B)0)$$

And $B = \left[\frac{\rho^m}{1-\rho} - \frac{(\beta-1)(v-u)\rho^v}{(\beta-\rho)(1-\rho^{v-u})}\right] G$

Let $R = \frac{(\beta-1)(v-u)\rho^v}{(\beta-\rho)(1-\rho^{v-u})}$

then

$$\bar{w}_d = \frac{1}{\lambda} \times \frac{\frac{\rho^{m+1}}{(1-\rho)^2} - R\left(\frac{v+u}{2} - \frac{(\beta+\rho)(1+\rho)-4\rho^2}{2(\beta-\rho)(1-\rho)} - m\right)}{\frac{\rho^m}{1-\rho} - R} \quad (3.30)$$

And for $\beta = 1$, $R = 0$ and $\bar{w}_d = \frac{\rho}{\lambda(1-\rho)}$ again this is the classical value.

4. THE APPLICATION

4.1 In a particular application, it will be necessary to determine at which queue lengths the control (shorter service time) should be put on and turned off.

It will be interesting to evaluate the effect of this control, on the capacity of the system. It will be desirable to make sure that the control is not activated when the traffic is low and is in full use when the traffic is high.

To verify all this properties a general program has been developed. It is then applied to many situations. The operators of a telephone service, the cashiers of a super market, the tellers of a bank and the data link of a packet transmission system.

4.2 Output: Before describing the program it might be advantageous to have a look at the output (Fig. 4).

HT	is the service time
BETAG	is the given value of β (reduction of service time)
FCIG	is the desired time between activation of the control (Inverse of the Frequency (F_c)
TBETAG	is the desired duration of the control (t_β)
WBG	is the desired mean waiting time ($\bar{w}$)
M	is the number of servers (m)
OF	is the queue length at which the control is turned off
ON	is the queue length at which the control is put on
AN	is the traffic capacity (number of requests served during a service time HT) to meet the mean waiting time (WBG). It will represent the normal load.
DN	is the improvement in normal traffic capacity due to the control (expressed in percent).

AV — is the traffic capacity to achieve a mean waiting time (5 * WBG). It will represent the overload capacity.

DV — is the percent improvement in overload capacity due to the control.

BBETN — is the probability that the system is using the control at any given time during normal load (B_β).

BBETV — same as BBETN but for overload.

TBETN — is the mean duration of the control (t_β) for normal load.

TBETV — same as TBET but for overload.

FCIV — the time between activation of the controls for normal load.

KLGT — is an indicator of the number of iterations required to calculate the line. The units and tens digit is the number of iterations on a single function. The hundreds is the number of iterations on a double function.

Now let us consider some numbers. Figure 4 analyses the case of a bank. We can consider the case of a single queue and the customers go to the first free teller. (When there is one queue per teller, and the customers move to an idle teller even if they were queueing for a different teller, the situation is essentially the same. The order of service will be different but this will not affect the average waiting time.)

Let us assume that the average service time is 2 minutes (HT), the tellers are asked not to check whether the saving books are up to date when the control is on, and this reduces their service time by 20% (BETAG = 1/2). The desired mean waiting time is 2 minutes for normal load. And during this normal load the tellers should not be asked to hurry up more than once during 2 hours (FCIG) and the control period should last about 10 minutes (TBETAG).

For 7 tellers the control should be put on when there are about 21 customers waiting and it should be turned off when there are only 12. The normal capacity (mean waiting time 2 minutes) will be 6.6 customers during 2 minutes. This is 5% more than if there were no control. The overload capacity is 8.1 customers during 2 minutes (mean waiting time 10 minutes) and this is 19% more than if there was no control . In normal load the control is applied during 8.3% of the time and in overload it is applied 81.3% of the time. It will last 9.9 minutes in normal load and 65.5 minutes in overload. In overload the control will be applied every 80.6 minutes. (It took 36 single iterations and 7 double iterations to calculate this line.)

On this example it is interesting to note that it is often possible to obtain a substantial increase in capacity without applying the control all the time. One would expect that since the acceleration of service rate is 20% (BETAG-1) the maximum capacity increase will be in the order of 20%. In the case of 7 tellers a 19% increase can be achieved even though the control is applied only 81% of the time.

It is also useful to note that when the load is normal the control is introduced seldomly (once every 2 hours) and lasts only 10 minutes. In an overload however it will be introduced every 80 minutes and last 65 minutes so that at all times a frequent change of procedures is avoided to prevent irritating the tellers. This is the whole idea behind the hysteresis.

Figure 5 applies this technique to a super market, where the service time is 5 or 10 minutes and can be accelerated by 60% when another employee put the groceries in bags.

Figure 6 describes the case of telephone operators with holding times varying between 30 and 60 seconds.

Figure 7 applies to a date link transmitting packets with mean duration varying between 10 and 20 ms.

4.3 The program: Now let us consider the program used to produce these tables.

Figure 8: Program TDCT: It is the control program. After reading the input data and setting up the output headings it will calculate the normal load without control ANNC. This is done by calling first DEFPAR to define the parameters and then INVER. It is an inversion routine that calculates the traffic capacity for a given mean waiting time.

The normal load with controls depends on the length of the queue at which the controls are introduced. An iteration will be done to calculate simultaneously AN, V and D. INVERD is called, it will be described later.

Once V and D are determined, the overload traffics are evaluated. (AVNC and AV).

Figure 9. INVER and INVERD

INVER is an inversion routine that applies to monotonic functions. For a function C(A) it calculates A for a given C. The secant method is used but the straight line is replaced by a curve fit of the form $A=XC^{Y}$.

INVERD is a double inversion it will solve: C1(A1,A2)=0 and C2(A1,A2)=0 First it will call INVER to get A1(A2) by inversing C1(A1,A2)=0 and then by substitution it derives C2(A1(A2),A2) that is C2E(A2) which is inversed using exactly the same logic as INVER. This principle can be extended to many dimensions.

Figure 10: TDCS

This subroutine calculates the functions derived in section 3. It also ensures the proper linking with INVER to calculate any parameter (A,V or D) to satisfy any criteria (BBETA, FCI, TBETA or WB).

Figure 11: TDCC

This program allows a more direct call to TDCS to calculate any of the parameters without producing the table of Figure 4.

5. THE CONCLUSION

A program is available to evaluate the effect of a control with hysteresis. It has been applied to a variety of cases to illustrate its generality.

Whenever there is some penalty in applying too often a control, it is useful to be able to choose the queue lengths at which the control will be put on and turned off, so that it is automatically not implemented during normal load but it is in full use during overload. It is also important to limit the frequency of introduction of the control to avoid irritating an operator.

When the control is costly or represents a degradation of service (BETAG-1) it is interesting to note that the improvement of overload capacity (DV) is larger than the relative degradation of service (BBETV x (BETAG-1)).

References:

1. R.W. Conway and M.L. Maxwell: "A Queueing Model with State Dependent Service Rate", Journal of Industrial Engineering, March-April 1961. pp. 132-136.

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5. L. Kleinrock: "Queueing Systems" John Wiley & Sons.

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APPENDIX 1. USEFUL RELATIONSHIPS

The following relations are often used in the text:

- $$\sum_{k=i}^{j} k = \frac{1}{2}(j+i)(j-i+1) \qquad (A.1)$$

- $$\sum_{k=i}^{j} a^k = \frac{a^i - a^{j+1}}{1-a} \qquad (A.2)$$

- $\sum_{k=i}^{j} ka^k$: This sum is not classical, let us calculate it:

$$\sum_{k=i}^{j} ka^k = a\frac{d}{da}\int_a \sum_{k=i}^{j} ka^{k-1}\,da$$

$$= a\frac{d}{da} \sum_{k=i}^{j} \int_a ka^{k-1}\,da$$

$$= a\frac{d}{da} \sum_{k=i}^{j} a^k = a\frac{d}{da} \frac{a^i - a^{j+1}}{1-a}$$

$$= \frac{a}{(1-a)^2}\left[(1-a)(ia^{i-1} - (j+1)a^j) + (a^i - a^{j+1})\right]$$

$$= \frac{a}{(1-a)^2}\left[ia^{i-1} - (i-1)a^i - (j+1)a^j + ja^{j+1}\right]$$

$$= \left[ia^i - (i-1)a^{i+1} - (j+1)a^{j+1} + ja^{j+2}\right]/(1-a)^2 \qquad (A.3)$$

- $$(1+x)^i = \sum_{k=o}^{i} \binom{i}{k} x^i \qquad (A.4)$$

APPENDIX 2 SUMMARY OF RESULTS

Definitions: $R = \frac{(\beta-1)(v-u)\rho^{v}}{(\beta-\rho)(1-\rho^{v-u})}$ $\beta>\rho$

$$G^{-1} = \frac{\rho^{m}}{1-\rho} + \frac{m!}{m^{m}} \sum_{k=o}^{m-1} \frac{(\rho m)^{k}}{k!} - R$$

$$P(k,0) = \frac{(\rho m)^{k}}{k!} \frac{m!}{m^{m}} G \qquad 0\leq k\leq m-1 \qquad (3.1)$$

$$P(k,0) = \rho^{k} G \qquad m\leq k\leq u \qquad (3.2)$$

$$P(k,0) = \frac{\rho^{k}-\rho^{v}}{1-\rho^{v-u}} G \qquad u+1\leq k\leq v-1 \qquad (3.6)$$

$$P(k,1) = \frac{(1-\rho)\rho^{v}}{(\beta-\rho)} \frac{1-(\rho/\beta)^{k-u}}{1-\rho^{v-u}} G \qquad u+1\leq k\leq v-1 \qquad (3.8)$$

$$P(k,1) = \frac{(1-\rho)\beta^{v}}{(\beta-\rho)} \frac{1-(\rho/\beta)^{v-u}}{1-\rho^{v-u}} (\rho/\beta)^{k} G \qquad v\leq k \qquad (3.10)$$

Blocking: $B = \left(\frac{\rho^{m}}{1-\rho} - R\right) G$ (3.21)

Use of $\beta\mu$: $B_{\beta} = \frac{1-\rho}{\beta-1} RG$ (3.22)

Rate of change: $F_{c} = \lambda \frac{\rho^{v-1}-\rho^{v}}{1-\rho^{v-u}} G$ (3.24)

Control period: $t_{\beta} = \frac{(v-u)\rho}{\lambda(\beta-\rho)}$ (3.25)

Busy period: $t_{m} = \frac{1}{\lambda\rho^{m-1}}\left(\frac{\rho^{m}}{1-\rho} - R\right)$ (3.26)

Queue length: $$\bar{q} = \left(\frac{\rho^{m+1}}{(1-\rho)^2} - R\left[\frac{v+u}{2} + \frac{(\beta+\rho)(1+\rho) - 4\rho^2}{2(\beta-\rho)(1-\rho)} - m\right]\right) G \quad (3.28)$$

Delay of delayed calls: $$\bar{w}_d = \frac{\bar{q}}{\lambda B} \quad (3.30)$$

NOTATION:

- m Number of servers
- β Service rate with control/normal service rate
- λ arrival rate
- μ normal service rate
- ρ λ /(μm)
- v Queue length at which the control is put on
- u Queue length at which the control is turned off

HT = 1   BETAG = 1.20   FCIG = 120.   TBETAG = 5.   WBG = 2.0

M	OF	ON	AN	DN	AV	DV	BBETN	BBETV	TBETN	TBETV	FCIV	KLGT
1	7	9	0.8	5	1.1	17	0.041	0.520	5.0	17.1	32.9	540
2	10	14	1.7	4	2.3	19	0.047	0.701	5.7	29.2	41.7	493
3	17	21	2.7	3	3.4	18	0.036	0.746	4.4	25.2	33.8	844
4	21	26	3.7	3	4.6	19	0.038	0.799	4.6	30.6	38.3	843
5	25	31	4.7	3	5.8	19	0.039	0.849	4.7	39.4	46.5	846
6	30	37	5.7	2	7.0	19	0.039	0.865	4.7	43.1	49.8	849
7	34	42	6.7	2	8.2	20	0.039	0.883	4.8	48.8	55.3	846
8	38	47	7.8	2	9.4	19	0.040	0.885	4.9	48.8	55.2	738

HT = 2   BETAG = 1.20   FCIG = 120.   TBETAG = 10.   WBG = 2.0

M	OF	ON	AN	DN	AV	DV	BBETN	BBETV	TBETN	TBETV	FCIV	KLGT
1	2	4	0.7	10	1.0	19	0.068	0.486	8.2	25.0	51.5	951
2	3	7	1.5	7	2.2	19	0.074	0.646	9.0	36.5	56.5	938
3	5	10	2.5	6	3.4	19	0.076	0.698	9.2	42.7	61.1	943
4	7	13	3.5	6	4.6	19	0.078	0.734	9.4	48.1	65.5	941
5	9	16	4.5	5	5.7	19	0.080	0.763	9.5	53.6	70.2	636
6	11	19	5.6	5	6.9	19	0.080	0.785	9.8	58.2	74.2	737
7	12	21	6.6	5	8.1	19	0.083	0.813	9.9	65.5	80.6	736
8	15	25	7.6	5	9.3	19	0.082	0.830	10.1	71.8	86.4	736

FIGURE 4: APPLICATION TO A BANK

4.1

HT = 3 BETAG = 1.20 FCIG = 120. TBETAG = 15. WBG = 2.0

M	OF	ON	AN	DN	AV	DV	BBETN	BBETV	TBETN	TBETV	FCIV	KLGT
1	0	3	0.7	19	1.0	21	0.118	0.496	14.2	50.6	102.0	2560
2	0	5	1.4	11	2.1	21	0.122	0.644	14.2	52.1	80.8	428
3	1	7	2.4	9	3.3	20	0.128	0.685	15.0	58.9	85.9	734
4	2	9	3.4	9	4.5	19	0.125	0.712	15.1	64.5	90.6	736
5	4	11	4.4	7	5.7	19	0.110	0.728	13.1	61.3	84.2	738
6	5	13	5.4	7	6.8	19	0.117	0.750	13.6	67.4	89.9	837
7	6	15	6.5	7	8.0	19	0.120	0.769	14.0	73.3	95.3	836
8	6	16	7.5	7	9.2	19	0.124	0.794	14.3	81.3	102.4	1038

HT = 4 BETAG = 1.20 FCIG = 120. TBETAG = 20. WBG = 2.0

M	OF	ON	AN	DN	AV	DV	BBETN	BBETV	TBETN	TBETV	FCIV	KLGT
1	0	2	0.6	20	1.0	22	0.106	0.419	12.4	33.9	81.0	532
2	0	4	1.3	12	2.0	21	0.112	0.592	13.2	44.2	74.6	429
3	0	5	2.3	11	3.2	21	0.135	0.676	16.1	54.1	80.0	326
4	0	7	3.3	10	4.4	20	0.148	0.701	18.6	71.0	101.3	325
5	0	8	4.3	10	5.6	20	0.166	0.735	20.1	78.6	106.9	327
6	1	10	5.3	9	6.8	20	0.164	0.741	19.5	84.5	114.0	301542
7	1	11	6.4	9	8.0	20	0.166	0.770	19.9	94.1	122.3	735
8	2	13	7.4	9	9.2	20	0.170	0.773	20.3	98.0	126.9	531

4.2

HT = 1 BETAG = 1.10 FCIG = 120. TBETAG = 5. WBG = 2.0

M	OF	ON	AN	DN	AV	DV	BBETN	BBETV	TBETN	TBETV	FCIV	KLGT
1	6	8	0.8	3	1.0	9	0.053	0.580	6.4	27.4	47.3	2461
2	11	14	1.7	2	2.1	9	0.048	0.602	5.6	24.7	41.0	433
3	18	21	2.7	2	3.2	9	0.040	0.635	4.7	22.0	34.6	430
4	22	26	3.7	2	4.3	9	0.046	0.700	5.5	29.0	41.5	430
5	29	33	4.7	2	5.4	9	0.040	0.726	4.8	27.1	37.3	844
6	32	37	5.7	2	6.4	9	0.046	0.760	5.4	32.9	43.4	533
7	39	44	6.7	1	7.5	10	0.040	0.777	4.9	31.1	40.0	841
8	41	47	7.7	2	8.6	10	0.047	0.794	5.4	35.6	44.9	631

HT = 2 BETAG = 1.10 FCIG = 120. TBETAG = 10. WBG = 2.0

M	OF	ON	AN	DN	AV	DV	BBETN	BBETV	TBETN	TBETV	FCIV	KLGT
1	2	4	0.7	6	1.0	10	0.084	0.440	9.6	27.5	62.4	536
2	3	7	1.5	4	2.0	10	0.092	0.665	11.0	51.0	76.7	2352
3	6	10	2.4	3	3.1	9	0.078	0.614	9.4	37.6	61.1	737
4	8	13	3.4	3	4.2	9	0.090	0.642	10.5	44.2	68.9	428
5	10	15	4.4	3	5.3	10	0.079	0.687	9.3	44.8	65.2	634
6	12	18	5.4	3	6.4	9	0.087	0.708	10.3	52.2	73.7	634
7	15	21	6.4	3	7.5	10	0.080	0.725	9.5	50.9	70.2	632
8	16	23	7.5	3	8.6	9	0.088	0.738	10.4	56.3	76.2	634

4.3

HT = 3 BETAG = 1.10 FCIG = 120. TBETAG = 15. WBG = 2.0

M	OF	ON	AN	DN	AV	DV	BBETN	BBETV	TBETN	TBETV	FCIV	KLGT
1	0	2	0.6	10	0.9	11	0.117	0.460	14.1	32.9	71.6	431
2	0	4	1.3	6	1.9	10	0.128	0.632	15.4	45.1	71.4	325
3	1	6	2.3	5	3.0	10	0.129	0.719	15.0	68.5	95.3	2655
4	2	8	3.3	5	4.0	8	0.136	0.666	16.3	61.4	92.3	322
5	4	10	4.3	4	5.2	10	0.122	0.670	14.7	59.2	88.3	632
6	6	12	5.3	4	6.3	10	0.117	0.679	13.6	57.6	84.7	733
7	7	14	6.3	4	7.4	10	0.121	0.696	14.9	65.7	94.4	628
8	9	16	7.3	4	8.5	10	0.116	0.705	13.8	63.9	90.7	530

HT = 4 BETAG = 1.10 FCIG = 120. TBETAG = 20. WBG = 2.0

M	OF	ON	AN	DN	AV	DV	BBETN	BBETV	TBETN	TBETV	FCIV	KLGT
1	0	2	0.6	10	0.9	11	0.105	0.391	12.2	36.0	92.0	428
2	0	3	1.2	7	1.9	11	0.115	0.592	14.3	36.0	60.8	324
3	0	5	2.2	6	2.9	10	0.144	0.619	17.6	56.3	90.9	323
4	0	6	3.2	6	4.0	10	0.167	0.665	20.1	66.0	99.2	325
5	0	7	4.2	6	5.0	8	0.187	0.695	22.1	74.9	107.8	322
6	2	9	5.1	5	6.2	10	0.161	0.678	19.2	72.3	106.6	531
7	4	11	6.1	5	7.3	10	0.148	0.670	18.0	69.8	104.0	631
8	3	11	7.2	5	8.4	10	0.171	0.716	19.7	80.9	113.1	730

4.4

HT = 5 BETAG = 1.60 FCIG = 120. TBETAG = 30. WBG = 5.0

M	OF	ON	AN	DN	AV	DV	BBETN	BBETV	TBETN	TBETV	FCIV	KLGT
1	0	3	1.0	54	1.4	64	0.207	0.672	25.8	88.3	131.4	746
2	0	6	1.9	34	3.0	63	0.195	0.846	23.8	138.2	163.5	539
3	0	9	3.1	29	4.6	62	0.213	0.887	26.1	201.9	227.7	534
4	0	12	4.2	27	6.2	62	0.229	0.907	26.8	254.3	280.3	531
5	0	15	5.4	26	7.8	61	0.237	0.927	28.3	327.3	353.2	429
6	0	17	6.6	25	9.4	61	0.243	0.938	28.2	368.2	392.5	429
7	0	20	7.8	24	11.0	61	0.247	0.946	29.3	427.6	452.2	428
8	0	23	9.0	23	12.6	61	0.250	0.951	29.4	474.3	499.0	428

HT = 10 BETAG = 1.60 FCIG = 120. TBETAG = 60. WBG = 5.0

M	OF	ON	AN	DN	AV	DV	BBETN	BBETV	TBETN	TBETV	FCIV	KLGT
1	0	2	0.9	71	1.3	67	0.164	0.517	20.4	72.2	139.7	536
2	0	3	1.6	41	2.8	66	0.171	0.750	20.3	75.9	101.2	433
3	0	5	2.8	35	4.4	63	0.191	0.799	23.9	117.5	147.0	433
4	0	6	3.9	32	6.0	63	0.208	0.840	25.1	138.7	165.1	435
5	0	8	5.2	31	7.6	62	0.231	0.863	26.9	179.8	208.2	434
6	0	9	6.3	29	9.2	62	0.231	0.884	28.0	201.9	228.3	636
7	0	10	7.5	28	10.8	62	0.237	0.899	28.2	223.7	248.8	301646
8	0	12	8.7	27	12.3	61	0.234	0.906	29.3	255.6	282.0	834

FIGURE 5: APPLICATION TO A SUPER MARKET

HT = 30   BETAG = 1.10   FCIG = 3600.   TBETAG = 60.   WBG = 2.0

M	OF	ON	AN	DN	AV	DV	BBETN	BBETV	TBETN	TBETV	FCIV	KLGT
2	0	2	0.5	4	1.1	8	0.010	0.121	35.7	53.5	444.0	317
4	0	4	1.9	2	2.9	7	0.014	0.163	48.9	82.2	503.9	214
6	0	6	3.5	2	4.9	6	0.016	0.181	56.9	105.6	583.3	317
8	0	7	5.2	2	6.9	6	0.018	0.224	62.7	113.0	505.4	316
10	1	9	7.0	1	8.9	6	0.017	0.211	60.0	116.9	553.8	519
12	1	10	8.8	1	11.0	6	0.018	0.237	61.4	123.6	521.9	317
14	2	12	10.6	1	13.0	6	0.017	0.224	63.0	126.9	566.1	319
16	7	15	13.6	1	14.8	3	0.017	0.151	59.6	87.0	575.0	319
18	7	16	15.5	1	16.9	3	0.018	0.167	63.0	92.1	551.7	319
20	9	18	17.4	1	18.8	3	0.017	0.152	59.3	85.5	561.6	319
22	9	19	19.4	1	20.9	3	0.018	0.165	62.5	90.3	548.0	319
24	11	21	21.3	1	22.9	3	0.017	0.151	59.4	84.5	560.3	318
26	11	22	23.3	1	24.9	3	0.018	0.161	62.4	88.9	551.9	318
28	12	23	25.3	1	26.9	3	0.017	0.163	59.6	84.8	519.2	318
30	13	24	27.2	0	28.9	3	0.017	0.167	56.6	81.4	487.2	422
32	16	28	29.3	1	30.8	2	0.016	0.133	60.9	82.5	622.2	217
36	18	31	33.2	1	34.9	2	0.017	0.132	61.2	82.4	625.7	218
40	20	34	37.2	1	38.9	2	0.017	0.129	61.6	82.2	636.2	218
44	21	36	41.1	1	43.0	2	0.017	0.136	61.9	82.7	608.3	218
48	23	39	45.1	1	47.0	2	0.017	0.132	62.3	82.5	623.3	218
52	24	41	49.1	1	51.1	2	0.017	0.138	62.7	83.0	603.7	218
56	22	39	52.8	0	55.5	3	0.016	0.186	58.1	83.0	446.1	426
60	23	41	56.8	0	59.5	3	0.016	0.181	58.6	82.5	455.2	427
64	24	43	60.7	0	63.3	2	0.016	0.166	59.1	80.5	485.3	424

FIGURE 6: APPLICATION TO TELEPHONE OPERATORS

6.1

HT = 45 BETAG = 1.10 FCIG = 3600. TBETAG = 90. WBG = 2.0

M	OF	ON	AN	DN	AV	DV	BBETN	BBETV	TBETN	TBETV	FCIV	KLGT
2	0	1	0.4	5	0.9	9	0.010	0.119	35.6	35.5	298.8	317
4	0	3	1.7	3	2.7	6	0.014	0.131	50.1	78.7	599.6	316
6	0	4	3.3	2	4.6	6	0.017	0.168	58.6	90.1	537.1	317
8	0	6	4.9	2	6.5	6	0.019	0.162	64.7	119.2	737.2	317
10	0	7	6.6	2	8.5	6	0.020	0.182	69.3	127.9	702.4	317
12	0	8	8.4	2	10.6	6	0.021	0.198	73.0	136.3	688.3	317
14	0	9	10.2	2	12.6	6	0.021	0.212	77.9	144.7	682.1	319
16	3	11	13.4	1	14.6	3	0.024	0.167	85.4	120.0	719.8	523
18	3	12	15.4	1	16.6	3	0.025	0.177	91.1	126.6	716.6	319
20	4	13	17.3	1	18.6	3	0.024	0.175	85.6	118.8	679.9	524
22	4	14	19.2	1	20.6	3	0.025	0.182	90.7	125.0	685.4	420
24	5	15	21.2	1	22.6	3	0.024	0.179	86.0	118.2	659.6	625
26	5	16	23.1	1	24.6	3	0.025	0.185	90.7	123.9	669.6	419
28	6	17	25.1	1	26.6	3	0.024	0.181	86.5	117.9	650.5	624
30	6	18	27.1	1	28.6	3	0.025	0.186	91.0	123.2	662.7	419
32	7	19	29.0	1	30.6	3	0.024	0.182	87.2	117.7	648.2	624
36	8	21	32.9	1	34.6	3	0.025	0.183	87.8	118.0	644.8	524
40	9	23	36.9	1	38.7	2	0.025	0.182	88.5	118.0	648.9	524
44	9	24	40.8	1	42.7	3	0.025	0.197	89.2	119.3	606.3	525
48	10	26	44.8	1	46.8	2	0.025	0.194	89.9	119.3	616.5	525
52	11	28	48.7	1	50.8	2	0.025	0.190	90.5	119.3	627.7	525
56	11	29	52.7	1	54.9	2	0.026	0.201	91.1	120.5	598.9	526
60	13	32	56.7	1	58.8	2	0.026	0.183	92.2	119.3	652.4	321
64	14	34	60.7	1	62.9	2	0.026	0.179	92.8	119.3	665.7	321

HT = 60 BETAG = 1.10 FCIG = 3600. TBETAG = 120. WBG = 2[illegible]0

M	OP	ON	AN	DN	AV	DV	BBETN	BBETV	TBETN	TBETV	FCIV	KLGT
2	0	1	0.4	7	0.8	9	0.010	0.085	35.0	43.5	511.3	316
4	0	2	1.6	3	2.5	7	0.014	0.130	50.4	63.9	491.6	316
6	0	3	3.1	3	4.4	7	0.017	0.154	59.4	81.2	526.8	317
8	0	5	4.7	2	6.3	6	0.019	0.139	65.9	118.3	851.2	318
10	0	6	6.4	2	8.2	6	0.020	0.152	71.8	129.5	849.5	319
12	0	6	8.1	2	10.3	6	0.021	0.189	76.0	122.3	648.0	319
14	0	7	9.9	2	12.3	6	0.022	0.196	79.5	134.0	682.9	320
16	0	9	13.3	2	14.4	3	0.033	0.183	120.9	171.3	938.4	320
18	0	10	15.3	2	16.4	3	0.034	0.188	125.5	178.2	949.5	319
20	0	10	17.2	2	18.5	3	0.035	0.211	129.5	169.4	802.7	420
22	1	11	19.1	1	20.4	3	0.032	0.201	118.2	159.4	791.0	624
24	1	12	21.1	2	22.4	3	0.036	0.204	124.6	166.3	815.3	522
26	2	13	23.0	1	24.4	3	0.033	0.195	118.5	157.6	807.5	625
28	2	14	25.0	1	26.4	3	0.033	0.197	124.1	164.1	832.7	623
30	2	14	26.9	1	28.4	3	0.033	0.213	118.9	158.2	741.1	624
32	3	15	28.9	1	30.4	3	0.032	0.204	114.2	151.0	740.5	624
36	3	16	32.8	1	34.5	3	0.032	0.221	115.2	152.5	691.3	623
40	4	18	36.8	1	38.5	3	0.033	0.211	116.3	152.1	721.3	725
44	4	19	40.7	1	42.5	3	0.033	0.223	117.3	153.3	688.1	724
48	4	20	44.7	1	46.6	3	0.033	0.233	118.3	154.5	661.8	726
52	5	22	48.6	1	50.6	2	0.033	0.222	119.2	154.0	694.0	828
56	5	23	52.6	1	54.6	2	0.034	0.230	120.1	155.1	673.1	828
60	5	24	56.6	1	58.7	2	0.034	0.238	120.9	156.1	656.1	625
64	6	25	60.5	1	62.7	2	0.032	0.237	115.4	148.7	628.4	727

6.3

HT = 30 BETAG = 1.10 FCIG = 7200. TBETAG = 60. WBG = 2.0

M	OF	ON	AN	DN	AV	DV	BBETN	BBETV	TBETN	TBETV	FCIV	KLGT
2	0	2	0.5	3	1.1	8	0.006	0.121	43.8	53.5	444.0	215
4	0	5	1.9	1	2.9	5	0.008	0.126	58.9	100.6	799.1	214
6	1	7	3.5	1	4.8	5	0.008	0.135	57.9	102.8	760.7	520
8	2	9	5.2	1	6.8	5	0.009	0.140	58.3	106.8	761.7	315
10	3	11	7.0	1	8.8	5	0.008	0.142	59.3	110.7	781.1	317
12	3	12	8.8	1	10.9	5	0.009	0.169	60.7	117.2	692.5	317
14	4	14	10.6	0	12.9	5	0.009	0.165	61.9	120.5	728.8	217
16	11	19	13.6	1	14.7	2	0.008	0.087	59.6	83.0	955.6	218
18	12	21	15.5	1	16.7	2	0.009	0.088	62.9	86.9	990.3	218
20	14	23	17.4	1	18.7	2	0.008	0.084	59.2	81.1	968.3	218
22	15	25	19.4	1	20.7	2	0.009	0.084	62.4	84.8	1010.2	218
24	17	27	21.3	1	22.6	2	0.008	0.080	59.2	79.8	997.0	217
26	17	28	23.3	1	24.7	2	0.009	0.088	62.2	83.8	952.1	217
28	19	30	25.3	1	26.6	2	0.008	0.084	59.7	79.5	949.6	217
30	19	31	27.2	0	28.7	2	0.009	0.091	62.1	83.2	918.6	217
32	21	33	29.2	1	30.7	2	0.008	0.086	59.7	79.2	922.2	218
36	23	36	33.1	1	34.7	2	0.008	0.087	60.1	79.1	907.3	219
40	25	39	37.0	1	38.7	2	0.008	0.088	60.4	79.0	900.7	220
44	28	43	41.0	0	42.7	1	0.008	0.082	60.8	78.5	958.0	219
48	30	46	44.9	0	46.7	1	0.008	0.083	61.1	78.6	949.8	219
52	32	49	48.9	0	50.7	1	0.008	0.083	61.4	78.6	952.0	218
56	33	51	52.9	0	54.8	2	0.008	0.087	61.7	79.0	906.9	218
60	35	54	56.8	0	58.8	1	0.008	0.086	62.1	79.0	916.0	218
64	37	57	60.8	0	62.8	1	0.008	0.085	62.4	79.0	926.5	217

6.4

HT = 45 BETAG = 1.10 FCIG = 7200. TBETAG = 90. WBG = 2.0

M	OF	ON	AN	DN	AV	DV	BBETN	BBETV	TBETN	TBETV	FCIV	KLGT
2	0	2	0.4	4	0.9	7	0.006	0.069	45.5	69.7	1009.0	316
4	0	4	1.7	2	2.7	5	0.009	0.096	62.0	103.0	1076.3	316
6	0	5	3.2	2	4.6	6	0.010	0.134	71.7	110.5	822.1	317
8	0	7	4.9	1	6.5	5	0.011	0.135	78.6	136.5	1014.5	317
10	0	8	6.6	1	8.5	5	0.012	0.156	83.9	143.4	917.7	317
12	0	9	8.3	1	10.5	5	0.013	0.173	87.9	150.4	866.8	317
14	0	11	10.1	1	12.5	5	0.013	0.167	92.3	170.4	1017.8	317
16	6	14	13.3	1	14.5	2	0.012	0.102	84.0	115.5	1127.6	525
18	6	15	15.3	1	16.5	2	0.013	0.112	89.5	121.9	1085.3	319
20	7	17	17.2	1	18.4	2	0.013	0.105	94.0	126.5	1199.8	319
22	8	18	19.1	1	20.4	2	0.013	0.107	89.1	119.4	1113.6	319
24	8	19	21.1	1	22.4	2	0.013	0.114	93.4	124.8	1093.5	319
26	10	21	23.0	1	24.4	2	0.013	0.102	89.1	117.6	1152.9	318
28	10	22	25.0	1	26.4	2	0.013	0.108	93.2	122.7	1139.7	318
30	13	25	27.1	1	28.3	2	0.012	0.087	90.9	115.5	1322.2	216
32	11	24	28.9	1	30.4	2	0.013	0.113	93.3	121.9	1083.5	318
36	13	26	32.8	0	34.4	2	0.012	0.111	85.7	112.4	1008.3	423
40	14	28	36.7	0	38.4	2	0.012	0.114	86.4	112.6	986.9	423
44	15	30	40.6	0	42.4	2	0.012	0.117	87.1	113.1	966.2	424
48	16	32	44.6	0	46.4	2	0.012	0.118	87.8	113.3	957.2	424
52	17	34	48.5	0	50.5	2	0.012	0.119	88.4	113.6	952.6	424
56	18	36	52.5	0	54.5	2	0.012	0.120	89.1	113.8	951.4	424
60	24	43	56.7	1	58.2	1	0.013	0.081	91.9	110.0	1363.0	217
64	26	46	60.7	1	62.2	1	0.013	0.077	92.5	109.8	1434.0	217

6.5

HT = 60   BETAG = 1.10   FCIG = 7200.   TBETAG = 120.   WBG = 2.0

N	OF	ON	AN	DN	AV	DV	BBETN	BBETV	TBETN	TBETV	FCIV	KLGT
2	0	1	0.4	5	0.8	9	0.006	0.085	46.1	43.5	511.3	317
4	0	3	1.6	2	2.5	6	0.009	0.092	63.9	94.1	1023.8	316
6	0	4	3.1	2	4.3	6	0.011	0.120	74.2	106.3	884.9	317
8	0	6	4.7	2	6.2	5	0.012	0.112	81.4	139.5	1239.9	318
10	0	7	6.3	1	8.2	5	0.013	0.128	86.9	148.5	1163.2	318
12	0	8	8.1	1	10.1	5	0.013	0.139	91.4	157.3	1128.5	318
14	0	9	9.8	1	12.1	5	0.013	0.150	96.7	166.1	1109.9	320
16	2	11	13.2	1	14.4	3	0.017	0.128	123.3	166.7	1304.9	626
18	3	12	15.1	1	16.3	3	0.016	0.126	116.0	155.6	1233.7	524
20	4	14	17.1	1	18.3	2	0.017	0.113	122.3	161.2	1422.2	319
22	5	15	19.0	1	20.3	2	0.016	0.112	115.8	152.0	1356.4	524
24	5	16	21.0	1	22.2	2	0.017	0.117	121.9	158.9	1358.4	319
26	6	17	22.9	1	24.2	2	0.016	0.115	116.1	150.9	1312.5	524
28	6	18	24.9	1	26.2	2	0.017	0.119	121.8	157.3	1324.0	318
30	7	19	26.8	1	28.2	2	0.016	0.116	116.6	150.2	1290.8	523
32	7	20	28.8	1	30.2	2	0.017	0.119	122.1	156.1	1307.9	318
36	8	22	32.7	1	34.2	2	0.017	0.119	122.5	155.3	1303.4	317
40	9	24	36.7	1	38.2	2	0.017	0.118	123.0	154.6	1307.2	318
44	9	25	40.6	1	42.2	2	0.017	0.128	123.6	155.1	1208.2	318
48	10	27	44.6	1	46.2	2	0.018	0.126	124.1	154.6	1227.5	321
52	11	29	48.5	1	50.2	2	0.018	0.125	124.7	154.4	1239.0	321
56	11	30	52.5	1	54.2	2	0.018	0.132	125.3	155.0	1170.7	321
60	12	32	56.5	1	58.2	2	0.018	0.129	125.8	154.6	1197.2	321
64	13	33	60.4	1	62.3	2	0.017	0.132	119.8	147.5	1120.9	525

6.6

HT = 10 BETAG = 1.20 FCIG =600000. TBETAG = 300. WBG = 30.0

M	OF	ON	AN	DN	AV	DV	BBETN	BBETV	TBETN	TBETV	FCIV	KLGT
1	24	35	0.8	0	1.0	4	0.000	0.088	287.0	504.1	5738.3	636
2	32	52	1.7	0	2.1	7	0.000	0.220	299.6	611.0	2778.8	325

HT = 20 BETAG = 1.20 FCIG =600000. TBETAG = 600. WBG = 30.0

M	OF	ON	AN	DN	AV	DV	BBETN	BBETV	TBETN	TBETV	FCIV	KLGT
1	6	21	0.7	0	1.0	8	0.001	0.165	622.4	1344.1	8163.2	325
2	4	30	1.6	0	2.1	12	0.001	0.366	611.8	1731.7	4736.3	325

HT = 10 BETAG = 1.20 FCIG =600000. TBETAG =1000. WBG =100.0

M	OF	ON	AN	DN	AV	DV	BBETN	BBETV	TBETN	TBETV	FCIV	KLGT
1	45	73	0.9	0	1.1	9	0.002	0.354	1020.5	2155.7	6089.7	432
2	70	119	1.9	0	2.3	17	0.002	0.766	999.5	5732.6	7290.5	431

HT = 20 BETAG = 1.20 FCIG =600000. TBETAG =2000. WBG =100.0

M	OF	ON	AN	DN	AV	DV	BBETN	BBETV	TBETN	TBETV	FCIV	KLGT
1	10	43	0.9	0	1.1	13	0.003	0.475	2023.4	6024.7	12688.3	325
2	10	67	1.8	0	2.3	18	0.003	0.775	2000.9	12439.7	16055.6	329

FIGURE 7: APPLICATION TO A DATA LINK

```
      COMMON /BINV/AINV,CINV,NTEST
      COMMON /BINV2/AINV2,CINV2,NTEST2
      COMMON /BPAR/AH,AL,AP,CH,CL,CM,Y,KLM,KI,KLT
      COMMON /BPAR2/A2H,A2L,A2P,C2H,C2L,C2M,Y2,KL2M,KL2,KL2T
      COMMON /BTDC/ A,M,HT,BETA,V,D,MAXD,BBETA,FCI,TBETA,WB,INVA,INVC
      COMMON /BTDC2/ INVA2,INVC2
      INTEGER HTL,HTH,HTE,NHT,OF,CN,DN,DV
C
      EXTERNAL TDCS,TDCS1,TDCS2
C
C
    5 WRITE(6,20)
   20 FORMAT(/' TYPE ML,MH,ME,HTL,HTH,HTE,BETAG,FCIG,RTBG,WBG,NTEST,'
     1,'KLM')
      READ(7,*)ML,MH,MEG,HTL,HTH,HTE,BETAG,FCIG,RTBG,WBG,NTEST,KLM
      IF(ML.EQ.0)STOP
      IF(KLM.EQ.0)GO TO 5
      AP=.01
      A2P=AP
      NTEST2=NTEST
      KL2M=KLM
      FCIH=1.05*FCIG
      FCIL=.95*FCIG
      WBL=.99*WBG
      WBGV=5.*WBG
      WBLV=.99*WBGV
C
      DO 1000  NHT=HTL,HTH,HTE
      ME=MEG
      HT=NHT
      TBETAG=RTBG*HT
      TBETAH=1.05*TBETAG
      LINE=0
      WRITE(8,30)NHT,BETAG,FCIG,TBETAG,WBG
   30 FORMAT('1HT =',I3,5X,'BETAG =',F5.2,5X,'FCIG =',F7.0,5X
     1,'TBETAG =',F5.0,5X,'WBG =',F5.1///'   M    OF   ON      AN  DN     AV'
     2,' DV   BBETN  BBETV   TBETN  TBETV    FCIV  KLGT'/)
      M=ML
      GO TO 101
C
C
  100 M=M+ME
      IF(M.EQ.32)ME=2*ME
      IF(M.GT.MH)GO TO 1000
  101 KLGT=0
      YA=1./M
      AHG=.99*M
      AHB=AHG*BETAG
      ALG=.1*M
C
C GET NORMAL LOAD
      CALL DEFPAR(1,4,.9*M,AHG,ALG,WBG,WBL,1E70,YA,1.,1.,M+2.,1.)
      KLT=0
      CALL INVER(TDCS)
      IF(NTEST.GE.1)WRITE(6,102)BBETA,FCI,TBETA,WB
```

FIGURE 8: PROGRAM TDCT

```
  102 FORMAT(4E13.4)
      KLGT=KLGT+KL
      ANNC=AINV
C
      BETA=BETAG
      RC=ANNC/M
      ANDA=ANNC/HT
      D=TBETAG*ANDA*(BETA-RC)/RC
      D=INT(D+.5)
      V=M+D
      CALL TDCS
      IF(NTEST.GE.1) WRITE(6,102) BBETA,FCI,TBETA,WB
C
C IF(FCI.GT.FCIG) TBETAG WILL NOT BE SATISFIED,SET MAXD=1
C                 TDCS WILL USE D=V-M.
      IF(FCI.GT.FCIG) MAXD=1
C
C START ITERATION FOR AN,V AND D
C
      V1=V
      D1=D
      AN=ANNC
  110 CALL DEFPAR(2,2,V1,100.*M,M+1.,FCIH,FCIL,1E70,.15,AN,BETAG,V1,D1)
      CALL DPAB2(1,4,AN,AHE,ALG,WBG,WBL,1E70,YA)
      KLT=0
      KL2T=0
      CALL INVERD(TDCS1,TDCS2)
      AN=AINV2
      V=INT(V+.5)
      D=INT(D+.5)
      KLGT=KLGT+KLT+100*KL2T
C
  140 IF(TBETA.LE.TBETAH)GO TO 150
      MAXD=0
      RC=AN/M
      ANDA=AN/HT
      D3=TBETAG*ANDA*(BETA-RC)/RC
      D3=INT(D3+.5)
      IF(D.EQ.D3)GO TO 150
      D1=D3
      V1=M+D1
      IF(KL2T.LT.2*KLM)GO TO 110
C
  150 BBETN=BBETA
      TBETN=TBETA
      DN=INT(100.*(AN-ANNC)/ANNC+.5)
C
C GET OVERLOAD
      CALL DEFPAR(1,4,ANNC,AHG,.9*ANNC,WBGV,WBLV,1E70,YA,AN,1.,V,D)
      KLT=0
      CALL INVER(TDCS)
      IF(NTEST.GE.1) WRITE(6,102) BBETA,FCI,TBETA,WB
      KLGT=KLGT+KL
      AVNC=AINV
C
```

```
      AH=AHB
      BETA=BETAG
      KLT=0
      CALL INVER(TDCS)
      IF(NTEST.GE.1) WRITE(6,102) BBETA,FCI,TBETA,WB
      KLGT=KLGT+KL
      AV=AINV
      BBETV=BBETA
      TBETV=TBETA
      FCIV=FCI
      DV=INT(100.*(AV-AVNC)/AVNC+.5)
C
C
      ON=V-M
      OF=ON-D
      WRITE(8,200) M,OF,ON,AN,DN,AV,DV,BBETN,BBETV,TBETN,TBETV
     1,FCIV,KLGT
      WRITE(6,200) M,OF,ON,AN,DN,AV,DV,BBETN,BBETV,TBETN,TBETV
     1,FCIV,KLGT
  200 FORMAT(I3,I5,I4,2(F7.1,I3),1X,2F7.3,1X,2F7.1,F8.1,I6)
      LINE=LINE+1
      IF(LINE.EQ.4*(LINE/4)) WRITE(8,210)
  210 FORMAT(1X)
      GO TO 100
 1000 CONTINUE
      GO TO 5
      END
C
C
      SUBROUTINE DEFPAR(NP1,NP2,P3,P4,P5,P6,P7,P8,P9,P10,P11,P12,P13)
      COMMON /BINV/AINV,CINV,NTEST
      COMMON /BPAR/AH,AL,AM,CH,CL,CM,Y,KLM,KI,KLT
      COMMON /BTDC/ A,M,HT,BETA,V,D,MAXD,BBETA,FC,TBETA,WB,INVA,INVC
      INVA=NP1
      INVC=NP2
      AINV=P3
      AH=P4
      AL=P5
      CH=P6
      CL=P7
      CM=P8
      Y=P9
      A=P10
      BETA=P11
      V=P12
      D=P13
      IF(NTEST.GE.1) WRITE(6,10) NP1,NP2,P3,P4,P5,P6,P7,P8,P9,P10,P11,P12
     1,P13
   10 FORMAT(/2I3,3E12.4/6X,3E12.4/F8.4,F10.4,F6.2,2F8.1)
      RETURN
      END
C
      SUBROUTINE DPAR2(NP1,NP2,P3,P4,P5,P6,P7,P8,P9)
      COMMON /BINV2/AINV2,CINV2,NTEST2
```

```
      COMMON /BPAR2/A2H,A2L,A2P,C2H,C2L,C2M,Y2,KL2M,KL2,KL2T
      COMMON /BTDC/ A,M,HT,BETA,V,D,MAXD,BBETA,FC,TBETA,WB,INVA,INVC
      COMMON /BTDC2/ INVA2,INVC2
      INVA2=NP1
      INVC2=NP2
      AINV2=P3
      A2H=P4
      A2L=P5
      C2H=P6
      C2L=P7
      C2M=P8
      Y2=P9
      IF(NTEST2.GE.1)WRITE(6,10)NP1,NP2,P3,P4,P5,P6,P7,P8,P9,A,BETA,V,D
   10 FORMAT(/2I3,3E12.4/6X,3E12.4/F8.4,F10.4,F6.2,2F8.1)
      RETURN
      END
```

```
      SUBROUTINE INVER(SN)
      COMMON /BINV/A,C,NTEST
      COMMON /BPAR/AH,AL,AP,CH,CL,CM,Y,KLM,KL,KLT
C
C
C THIS SUBROUTINE CALCULATES THE INVERSE OF A MONOTONIC CONTINUOUS
C FUNCTION C(A) WHERE C IS DEFINED BY SN (EXTERNAL IN CALLING PROGRAM)
C
C C CAN CHANGE SIGN IN THE ITERATION, BUT NOT A.
C IF REQUIRED A SHIFT OF A SHOULD BE DONE EXTERNALLY.
C
C AH,AL BOUNDS ON ALLOWABLE A
C AP RELATIVE ERROR ON A
C
C CH,CL BOUNDS ON DESIRED C
C CM MAX OF C - ERROR KL=1000+KL
C
C Y INITIAL POWER OF CURVE FIT A=X*C**Y
C KL NUMBER OF LOOPS
C KLM MAX OF KL - ERROR KL=2000+KL
C KLT SUM OF ALL KL
C NTEST IF GE 1 WRITE ON 6 PROGRESS OF ITERATION
C
C IF CONSECUTIVE C'S ARE EQUAL - ERROR KL=3000+KL
C IF A,AH,AL PRODUCE C=0 - ERROR KL=4000+KL
C
C NP IN OUTPUT SHOWS THE PATH FOLLOWED BY THE ITERATION
C     =1 A2 AND A USED FOR NEXT ITERATION
C     =2 A1 AND A USED FOR NEXT ITERATION
C     =3 A AND OLD Y USED FOR NEXT ITERATION
C
C CD IN OUTPUT IS THE SHIFT ON C (INTERNAL ONLY)
C
C
C
      IF(AH.LT.1E-20)AH=1E20
      IF(ABS(Y).LT.1E-20)Y=.1
      IF(CM.LT.1E-20)CM=1E20
      CG=(CH+CL)/2.
      IREAD=1
      NP=0
      CD=0.
C
C GET FIRST POINT
   10 CALL SN
      IF(C*CG.LE.0.)CD=-2.*C
      IF(ABS(C*CG).LE.1E-20)CD=CG
      KL=1
      IF(NTEST.GE.1)WRITE(6,15)
   15 FORMAT(' KL  Y      NP A',11X,'C',11X,'CD')
      IF(NTEST.GE.1)WRITE(6,25)KL,Y,NP,A,C,CD
   25 FORMAT(I3,F7.3,I3,3E12.4)
      IF(C.LE.CH.AND.C.GE.CL)GO TO 2000
      IF(KL.GE.KLM)GO TO 1200
C
```

FIGURE 9: SUBROUTINE INVER AND INVERD

```
  CHECK BAD INPUT
      IF(ABS(A).GT.1E-70.AND.ABS(C).GT.1E-70)GO TO 90
      GO TO (30,40,1400),INBAD
   30 A=AH
      INBAD=2
      GO TO 10
   40 A=AL
      INBAD=3
      GO TO 10
C
   90 A1=0.
      C1=0.
      NP=3
C
C START OF ITERATION
C
C CALCULATE NEW A AND C
  100 A2=A
      C2=C
  110 IF(ABS(Y).GT.10.)Y=SIGN(10.,Y)
      IF(ABS(Y).LT.1E-6) Y=SIGN(1E-6,Y)
      A=A2*((CG+CD)/(C2+CD))**Y
      IF(A.GT.AH) A=AH
      IF(A.LT.AL) A=AL
      IF(ABS(A-A2).GT.1E-20)GO TO 120
      IF(ABS(A-AH).GT.1E-20)GO TO 1300
      GO TO 2000
  120 CALL SN
      IF((C+CD)*(CG+CD).LE.0.)CD=-2.*C
      IF(ABS((C+CD)*(CG+CD)).LE.1E-20)CD=-CG
      KL=KL+1
      IF(NTEST.GE.1)WRITE(6,25)KL,Y,NP,A,C,CD
      IF(C.GT.CM)GO TO 1100
C
C COMPARE WITH BOUNDS
      IF(C.LE.CH.AND.C.GE.CL)GO TO 2000
      IF((C-C2)*(A-A2))160,1300,150
C INCREASING FUNCTION
  150 IF(C.LT.CL.AND.A.GE.AH)GO TO 2000
      IF(C.GT.CH.AND.A.LE.AL)GO TO 2000
      GO TO 200
C DECREASING FUNCTION
  160 IF(C.LT.CL.AND.A.LE.AL)GO TO 2000
      IF(C.GT.CH.AND.A.GE.AH)GO TO 2000
  200 IF(KL.GE.KIM)GO TO 1200
C
C THE FOLLOWING SECTION IS THE MAIN CONTROL OF THE ITERATION.
C IT IS IMPORTANT TO SELECT A1 CAREFULLY. A PREFERENCE IS GIVEN TO
C THE OLD A2 OR A1 IN AN ATTEMPT TO SURROUND CG.
C HOWEVER IF C,C2 OR C1 ARE FAR FROM CG THEY WILL BE IGNORED.
C
  IF C IS WORSE THAN C2 EXCHANGE A,A2 (BAD Y)
      IF(ABS(CG-C).LT.ABS(CG-C2))GO TO 250
      A1=A
      C1=C
```

```
      Y=ALOG(A2/A1)/ALOG((C2+CD)/(C1+CD))
      GO TO 110
C
  250 IF((CG-C)*(CG-C2).LE.0.)GO TO 300
      IF((CG-C)*(CG-C1).LE.0.)GO TO 400
C
C C,C1,C2 SAME SIDE OF CG
      IF(ABS(A-A2).LE.AP*.5*A)GO TO 2000
      GO TO 500
C
C C,C2 EITHER SIDE OF CG
  300 IF(ABS(A-A2).LE.AP*A)GO TO 2000
      GO TO 500
C
C C,C1 EITHER SIDE OF CG
  400 IF(ABS(A-A1).LE.AP*A)GO TO 2000
      IF(ABS(A1*C1).GT.1E-20)GO TO 600
      IF(A.GT.AL.AND.A.LT.AH)GO TO 500
C
C DROP A1,A2 KEEP A AND OLD Y
  450 NP=3
      GO TO 100
C
C DROP A1 CONSIDERE A,A2
  500 IF(ABS(CG-C).LE..2*ABS(CG-C2))GO TO 100
      A1=A2
      C1=C2
      NP=1
      GO TO 700
C
C CONSIDERE A,A1
  600 IF(ABS(CG-C2).LE..2*ABS(CG-C1))GO TO 500
      NP=2
C
  700 Y=ALOG(A/A1)/ALOG((C+CD)/(C1+CD))
      GO TO 100
C
C END OF ITERATION
 1100 KL=1000+KL
      GO TO 2000
 1200 KL=2000+KL
      GO TO 2000
 1300 KL=3000+KL
      GO TO 2000
 1400 KL=KL+4000
 2000 KLT=KLT+KL
      RETURN
      END
```

```
      SUBROUTINE INVERD(SN1,SN2)
      COMMON /BINV2/A,C,NTEST
      COMMON /BPAR2/AH,AL,AP,CH,CL,CM,Y,KLM,KL,KLT
C
C
C THIS SUBROUTINE CALCULATES THE INVERSE OF TWO MONOTONIC CONTINUOUS
C FUNCTIONS C1(A1,A2) WHERE C1 IS DEFINED BY SN1 (EXTERNAL) AND
C            C2(A1,A2) WHERE C2 IS DEFINED BY SN2 (EXTERNAL).
C
C METHOD: INVER IS CALLED TO INVERSE C1 AND OBTAIN A1(A2)
C          THEN C2( A1(A2),A2) IS INVERSED AS A SINGLE FUNCTION
C          OF A SINGLE VARIABLE C2E(A2).
C
C Y1 (Y IN BPAR) IS THE CURVE FIT FOR C1(A1,CONSTANTE)
C Y2 (Y IN BPAR2) IS THE CURVE FIT FOR C2E(A2)
C
C THE LOGIC IN INVERD IS IDENTICAL TO INVER EXCEPT FOR
C THE CALL TO INVER TO GET A1(A2).
C
C C CAN CHANGE SIGN IN THE ITERATION, BUT NOT A.
C IF REQUIRED A SHIFT OF A SHOULD BE DONE EXTERNALLY.
C
C AH,AL BOUNDS ON ALLOWABLE A
C AP RELATIVE ERROR ON A
C
C CH,CL BOUNDS ON DESIRED C
C CM MAX OF C - ERROR KL=1000+KL
C
C Y INITIAL POWER OF CURVE FIT A=X*C**Y
C KL NUMBER OF LOOPS
C KLM MAX OF KL - ERROR KL=2000+KL
C KLT SUM OF ALL KL
C NTEST IF GE 1 WRITE ON 6 PROGRESS OF ITERATION
C
C IF CONSECUTIVE C'S ARE EQUAL - ERROR KL=3000+KL
C IF A,AH,AL PRODUCE C=0 - ERROR KL=4000+KL
C
C NP IN OUTPUT SHOWS THE PATH FOLLOWED BY THE ITERATION
C     =1 A2 AND A USED FOR NEXT ITERATION
C     =2 A1 AND A USED FOR NEXT ITERATION
C     =3 A AND OLD Y USED FOR NEXT ITERATION
C
C CD IN OUTPUT IS THE SHIFT ON C (INTERNAL ONLY)
C
C
C
      IF(AH.LT.1E-20) AH=1E20
      IF(ABS(Y).LT.1E-20) Y=.1
      IF(CM.LT.1E-20) CM=1E20
      CG=(CH+CL)/2.
      INBAD=1
      NP=0
      CD=0.
C
C GET FIRST POINT
```

```
   10 CALL INVER(SN1)
      CALL SN2
      IF(C*CG.LE.0.)CD=-2.*C
      IF(ABS(C*CG).LE.1E-20)CD=CG
      KL=1
      IF(NTEST.GE.1)WRITE(6,15)
   15 FORMAT(/'      ITER 2.    KL   Y        NP A',11X,'C',11X,'CD')
      IF(NTEST.GE.1)WRITE(6,25)KL,Y,NP,A,C,CD
   25 FORMAT(/'-     ITER 2.   ',I3,F7.3,I3,3E12.4/)
      IF(C.LE.CH.AND.C.GE.CL)GO TO 2000
      IF(KL.GE.KLM)GO TO 1200
C
C CHECK BAD INPUT
      IF(ABS(A).GT.1E-70.AND.ABS(C).GT.1E-70)GO TO 90
      GO TO (30,40,1400),INBAD
   30 A=AH
      INBAD=2
      GO TO 10
   40 A=AL
      INBAD=3
      GO TO 10
C
   90 A1=0.
      C1=0.
      NP=3
C
C START OF ITERATION
C
  100 A2=A
      C2=C
  110 IF(ABS(Y).GT.10.)Y=SIGN(10.,Y)
      IF(ABS(Y).LT.1E-6)Y=SIGN(1E-6,Y)
      A=A2*((CG+CD)/(C2+CD))**Y
      IF(A.GT.AH)A=AH
      IF(A.LT.AL)A=AL
      IF(ABS(A-A2).GT.1E-20)GO TO 120
      IF(ABS(A-AH).GT.1E-20)GO TO 1300
      GO TO 2000
C
  120 CALL INVER(SN1)
      CALL SN2
      IF((C+CD)*(CG+CD).LE.0.)CD=-2.*C
      IF(ABS((C+CD)*(CG+CD)).LE.1E-20)CD=-CG
      KL=KL+1
      IF(NTEST.GE.1)WRITE(6,25)KL,Y,NP,A,C,CD
      IF(C.GT.CM)GO TO 1100
C
C COMPARE WITH BOUNDS
      IF(C.LE.CH.AND.C.GE.CL)GO TO 2000
      IF((C-C2)*(A-A2))160,1300,150
C INCREASING FUNCTION
  150 IF(C.LT.CL.AND.A.GE.AH)GO TO 2000
      IF(C.GT.CH.AND.A.LE.AL)GO TO 2000
      GO TO 200
C DECREASING FUNCTION
```

```
  160 IF(C.LT.CL.AND.A.LE.AL)GO TO 2000
      IF(C.GT.CH.AND.A.GE.AH)GO TO 2000
  200 IF(KL.GE.KLM)GO TO 1200
C
C THE FOLLOWING SECTION IS THE MAIN CONTROL OF THE ITERATION.
C IT IS IMPORTANT TO SELECT A1 CAREFULLY. A PREFERENCE IS GIVEN TO
C THE OLD A2 OR A1 IN AN ATTEMPT TO SURROUND CG.
C HOWEVER IF C,C2 OR C1 ARE FAR FROM CG THEY WILL BE IGNORED.
C
C IF C IS WORSE THAN C2 EXCHANGE A,A2 (BAD Y)
      IF(ABS(CG-C).LT.ABS(CG-C2))GO TO 250
      A1=A
      C1=C
      Y=ALOG(A2/A1)/ALOG((C2+CD)/(C1+CD))
      GO TO 110
C
  250 IF((CG-C)*(CG-C2).LE.0.)GO TO 300
      IF((CG-C)*(CG-C1).LE.0.)GO TO 400
C
C C,C1,C2 SAME SIDE OF CG
      IF(ABS(A-A2).LE.AP*.5*A)GO TO 2000
      GO TO 500
C
C C,C2 EITHER SIDE OF CG
  300 IF(ABS(A-A2).LE.AP*A)GO TO 2000
      GO TO 500
C
C C,C1 EITHER SIDE OF CG
  400 IF(ABS(A-A1).LE.AP*A)GO TO 2000
      IF(ABS(A1*C1).GT.1E-20)GO TO 600
      IF(A.GT.AL.AND.A.LT.AH)GO TO 500
C
C DROP A1,A2 KEEP A AND OLD Y
  450 NP=3
      GO TO 100
C
C DROP A1 CONSIDERE A,A2
  500 IF(ABS(CG-C).LE..2*ABS(CG-C2))GO TO 100
      A1=A2
      C1=C2
      NP=1
      GO TO 700
C
C CONSIDERE A,A1
  600 IF(ABS(CG-C2).LE..2*ABS(CG-C1))GO TO 500
      NP=2
C
  700 Y=ALOG(A/A1)/ALOG((C+CD)/(C1+CD))
      GO TO 100
C
C END OF ITERATION
 1100 KL=1000+KL
      GO TO 2000
 1200 KL=2000+KL
      GO TO 2000
```

```
 1300 KL=3000+KL
      GO TO 2000
 1400 KL=KL+4000
 2000 KLT=KLT+KL
      RETURN
      END
```

```
      SUBROUTINE TDCS
      COMMON /BINV/AINV,CINV,NTEST
      COMMON /BINV2/AINV2,CINV2,NTEST2
      COMMON /BPAR/AH,AL,AP,CH,CL,CM,Y,KLM,KL,KLT
      COMMON /BPAR2/A2H,A2L,A2P,C2H,C2L,C2M,Y2,KL2M,KL2T
      COMMON /BTDC/A,M,HT,BETA,V,D,MAXD,BBETA,PCI,TBETA,WB,INVA,INVC
      COMMON /BTDC2/INVA2,INVC2
      DIMENSION AAR(2),CAR(2)
C
C
      NEN=1
      INVAE=INVA
      INVCE=INVC
      GO TO 10
C
C ENTRY 1 FOR DOUBLE INVERSION
      ENTRY TDCS1
      NEN=1
      INVAE=INVA
      INVCE=INVC
C
C UPDATE ON NEW VALUES CHANGED IN INVEED
      IF(INVA2.EQ.1) A=AINV2
      IF(INVA2.EQ.2) V=AINV2
      IF(INVA2.EQ.3) D=AINV2
      GO TO 10
C
C ENTRY 2 FOR DOUBLE INVERSION
      ENTRY TDCS2
      NEN=2
      INVAE=INVA2
      INVCE=INVC2
C
C SET VARIABLES FOR INVER
C
   10 AAR(1)=AINV
      AAR(2)=AINV2
      IF(INVAE.EQ.1) A=AAR(NEN)
      IF(INVAE.EQ.2) V=AAR(NEN)
      IF(INVAE.EQ.3) D=AAR(NEN)
C
C PASSING MAXD=1 TO THIS SUBROUTINE IS A CODE REQUESTING THAT
C D BE MAXIMUM. D=V-M ,OR V=M+D.
      IF(MAXD.NE.1) GO TO 20
      IF(INVAE.EQ.2) D=V-M
      IF(INVAE.EQ.3) V=M+D
   20 IF(D.LT.1.) D=1.
C
      RC=A/M
      ANDA=A/HT
      ALRC=ALOG(RC)
      IF(RC.LE.0.991) GO TO 30
      IF(RC.GT.1.009) GO TO 40
      A=A*1.02
      RC=A/M
```

FIGURE 10: SUBROUTINE TDCS

```
      ANDA=A/HT
      ALRC=ALOG(RC)
      GO TO 42
   30 RCM=0.
      RCV=0.
      RCD=0.
      IF(M*ALRC.GT.-165.)RCM=EXP(M*ALRC)
      IF(V*ALRC.GT.-165.)RCV=EXP(V*ALRC)
      IF(D*ALRC.GT.-165.)RCD=EXP(D*ALRC)
      GO TO 45
   40 RCM=1E10
      RCV=1E10
      RCD=1E10
   42 IF(M*ALRC.LT.165)RCM=EXP(M*ALRC)
      IF(V*ALRC.LT.165)RCV=EXP(V*ALRC)
      IF(D*ALRC.LT.165)RCD=EXP(D*ALRC)
   45 IF(M.GT.15)GO TO 100
C
      X1=1./M
      X2=1.
      POIS=X2
      M1=M-1
      DO 50 K=1,M1
      X1=X1*(K+1)/M
      X2=X2*RC*M/K
      POIS=POIS+X2
   50 CONTINUE
      GO TO 200
C
  100 X1=EXP((A+ALGAMA(M+1.))/(M*ALOG(M*1.)))
      X3=1./(9.*M)
      POIS=.5*ERFC((RC**(1./3.)-(1.-X3))/SQRT(2.*X3))
C
  200 GZ=RCM/(1.-RC)+X1*POIS
      R=(BETA-1.)*D*RCV/((BETA-RC)*(1.-RCD))
      G=1./(GZ-R)
      WB=(G/ANDA)*(RC*RCM/(1.-RC)**2-R*(V-D/2.+((BETA+RC)*(1.+RC)
     C  -4.*RC**2)/(2.*(BETA-RC)*(1.-RC))-M))
      BBETA=0.
      IF((BETA-1.).GT.1E-10)BBETA=(1.-RC)/(BETA-1.)*R*G
      FCI=1./(ANDA*G*(RCV/RC-RCV)/(1.-RCD))
      TBETA=D*RC/(ANDA*(BETA-RC))
      IF(NTEST.GE.2)WRITE(6,210)A,M,HT,BETA,RC,ANDA,RCV,RCD,RCM,X1,X2
     1  ,POIS,GZ,R,G,BBETA,FCI,TBETA,WB
  210 FORMAT(F10.2,I5,2F6.1/5F10.5/6F12.5/4F10.5)
      IF(INVCE.EQ.1)CAR(NEN)=BBETA
      IF(INVCE.EQ.2)CAR(NEN)=FCI
      IF(INVCE.EQ.3)CAR(NEN)=TBETA
      IF(INVCE.EQ.4)CAR(NEN)=WB
      CINV=CAR(1)
      CINV2=CAR(2)
      RETURN
      END
```

FILE: TDCC FORTRAN A BELL-NORTHERN RESEARCH TIME= 11:48:23 DATE: 08/22/75

```
      COMMON /BINV/AINV,CINV,NTEST
      COMMON /BPAR/AH,AL,AP,CH,CL,CM,Y,KLM,KI,KLT
      COMMON /BTDC/ A,M,HT,BETA,V,D,MAXD,BBETA,PCI,TBETA,WB,INVA,INVC
C
      EXTERNAL TDCSD
C
C
    5 WRITE(6,20)
   20 FORMAT(' TYPE M HT A BETA V D MAXD INVA INVC CH Y NTEST KLM')
      READ (5,*) M,HT,A,BETA,V,D,MAXD,INVA,INVC,CH,Y,NTEST,KLM
      IF(M.EQ.0) STOP
      IF(KLM.EQ.0)GO TO 5
C
      AP=.01
      CL=.95*CH
      CM=1000.*CH
C
      GO TO (110,120,130),INVA
  110 AINV=A
      AH=.99*M*BETA
      AL=.1*M
      GO TO 200
  120 AINV=V
      AH=10*M
      AL=M+D
      GO TO 200
  130 AINV=D
      AH=V-M
      AL=1.
  200 CALL INVER1(TDCSD)
      WRITE(6,1000) M,HT,A,BETA,V,D,INVA,INVC,CH,Y,KL,BBETA,PCI,TBETA,WB
 1000 FORMAT(I5,F4.0,F8.2,F6.2,2F9.2,2I4,E12.3,F7.3,I5/4E12.3/)
      GO TO 5
      END
```

FIGURE 11: PROGRAM TDCC