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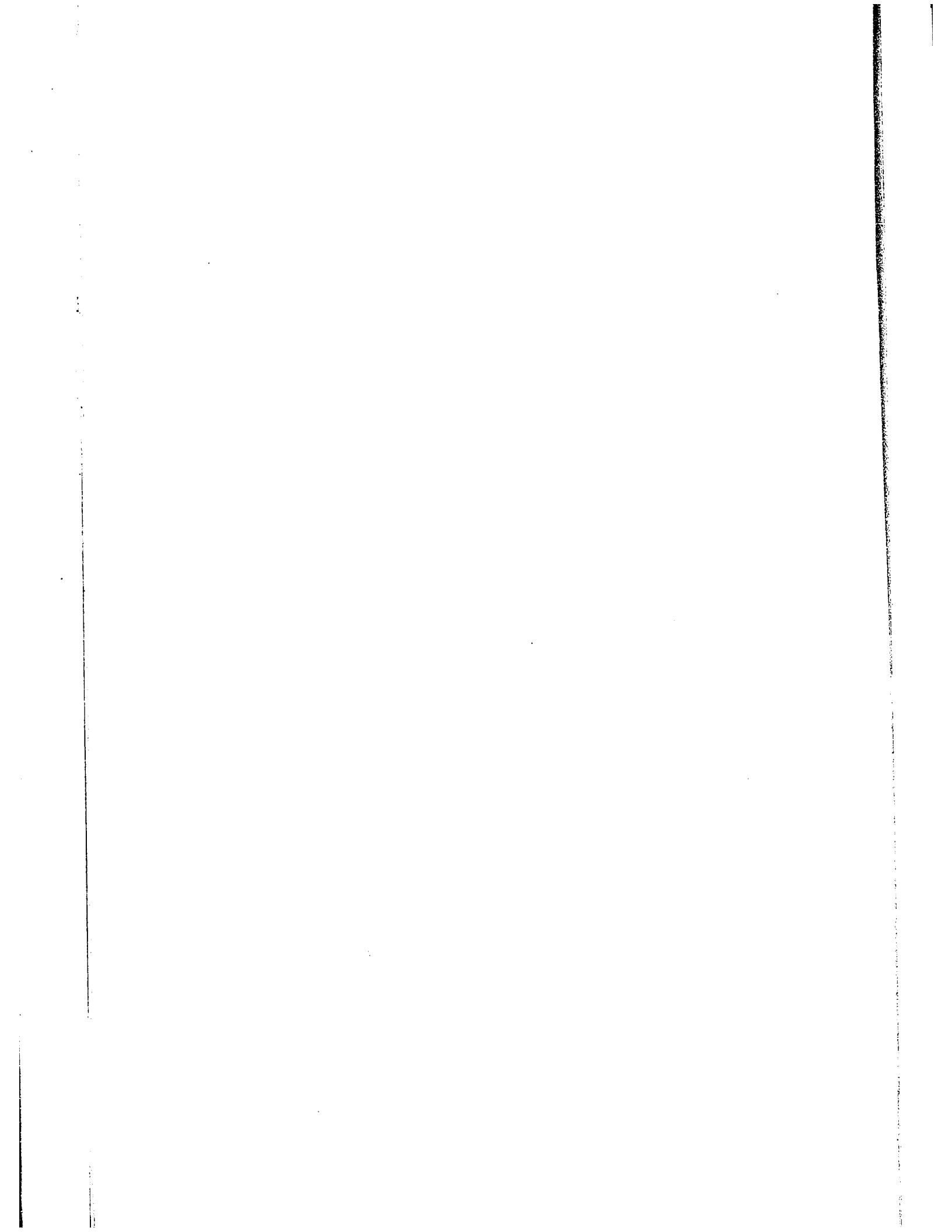
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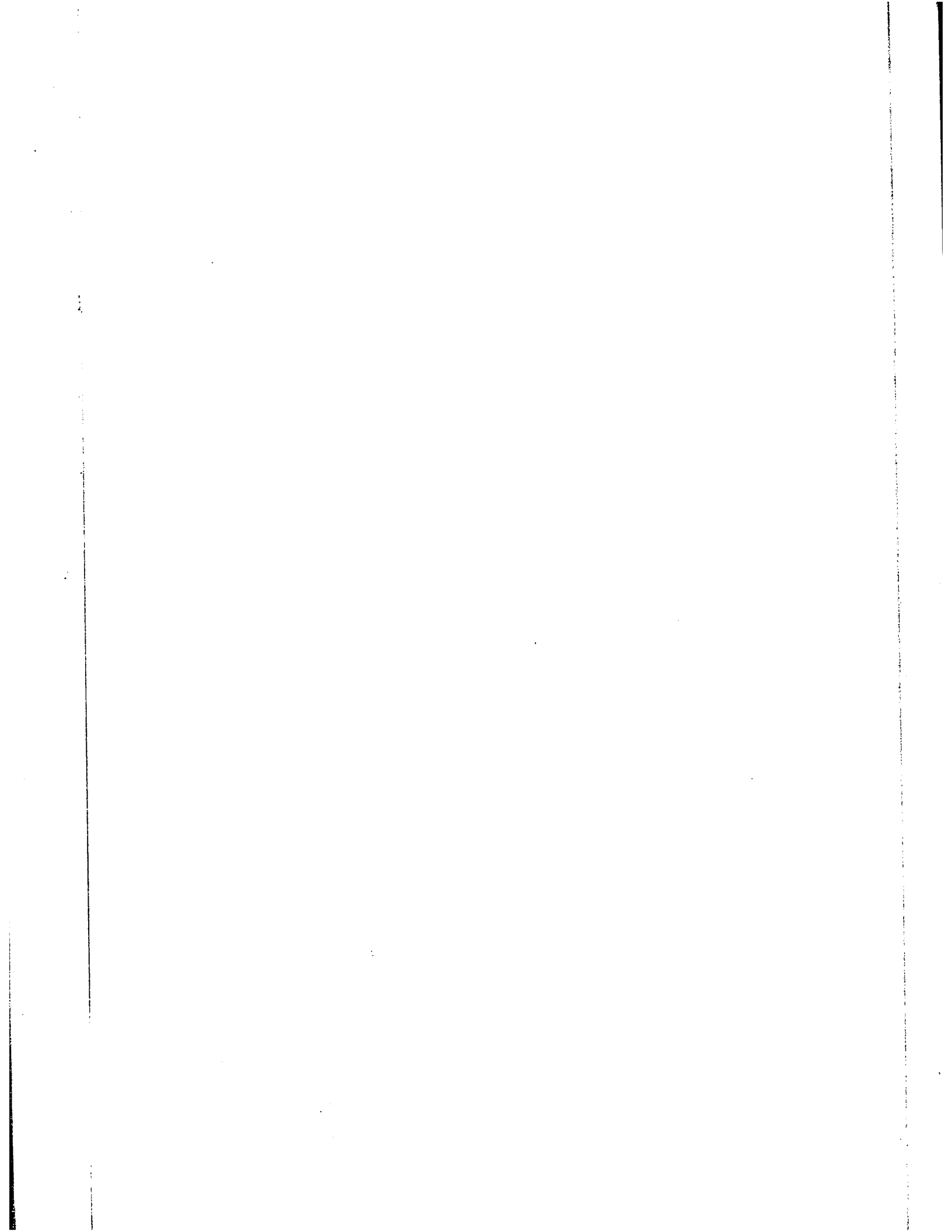
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**OPERATORS IN THE COHOMOLOGY
OF NILPOTENT LIE ALGEBRAS**

by

Alain Claude LeBel, B.Sc.

A M.Sc. Thesis submitted to the School of Graduate Studies and Research
in partial fulfilment of the requirements for the Master's degree in Mathematics†

University of Ottawa
Ottawa, Ontario
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ABSTRACT FOR "OPERATORS IN THE
COHOMOLOGY OF NILPOTENT LIE ALGEBRAS"

ALAIN LEBEL

Operators in the cohomology of Lie algebras are defined, and fundamental results are proven. The central representation is shown to be useful, in particular cases, for proving the Toral Rank Conjecture (TRC), which states the logarithm of the total dimension of any nilpotent Lie algebra's cohomology space is greater than or equal to the dimension of the centre.

Central representation and secondary operators are used to find hypercube-like structures in the cohomology of the free two-step nilpotent Lie algebras with two, three and four generators. Also, a theorem about the new operators acting on the cohomology of the Heisenberg Lie algebras, and how these operators interact with Poincaré duality for this case, is proven.

INTRODUCTION

This paper is a M.Sc. thesis, submitted to the School of Graduate Studies and Research to fulfil the final requirements for the Master's degree in Mathematics at the University of Ottawa. The thesis deals with the cohomology of Lie algebras, with particular emphasis on nilpotent Lie algebras. The purpose of the research was to answer some questions regarding certain operators in cohomology. These operators were invented by Barry Jessup and Grant Cairns with the *Toral Rank Conjecture* in mind. The Toral Rank Conjecture (or *TRC*, for short), claims that the total dimension of the cohomology of a differentiable manifold M is greater than or equal to $2^{\text{rank}(M)}$, where $\text{rank}(M)$ is the dimension of the largest torus acting freely on M .

The breakdown of the thesis is as follows.

Chapter I: Cohomology of Lie algebras.

The necessary background for the main results of the thesis, the first section explains how to compute the cohomology of any Lie algebra. Some examples are shown in section 2, for clarity. In section 3, some classical theorems regarding the cohomology of manifolds are given in order to demonstrate the veracity of the TRC for certain kinds of manifolds. Section 4 focuses on nilpotent Lie algebras, giving known results about their cohomology spaces and motivating the study of a version of the TRC for this vast, unclassified family of Lie algebras.

Chapter II: Operators in the cohomology of Lie algebras.

Here, the central representation is defined, motivated by a link with the TRC. However, as seen in early examples, more operators are needed. Examples of "higher operators", and how they help form hypercubes, are given at the end of section 1 to motivate the more general hypercubes (section 2), higher operators with one parameter (section 3) and secondary operators with two parameters (section 4).

Chapter III: Higher operators in the cohomology of the Heisenberg Lie algebras.

The purpose of this chapter is to prove the following:

Theorem: Let m be a positive integer. The higher operators op_k are such that

$$\text{op}_k |_{H^{m+k+1}(\mathfrak{h}_m)} : H^{m+k+1}(\mathfrak{h}_m) \longrightarrow H^{m-k}(\mathfrak{h}_m)$$

are isomorphisms, and are zero everywhere else they are defined.

Furthermore, suppose $\alpha, \beta \in H^*(\mathfrak{h}_m)$ are homogeneous Poincaré duals, meaning $\alpha \wedge \beta$ is non-trivial in $H^{2m+1}(\mathfrak{h}_m)$, with $|\beta| < |\alpha|$. Then $\text{op}_k(\alpha)$ and $\text{op}_k^{-1}(\beta)$ are also Poincaré duals for some integer k such that $0 \leq k \leq m$.

The statement of the theorem relies heavily on the definitions and results of chapters I and II. Its proof relies on the Hodge star map and the Laplace-Beltrami operator (explained in section 2), a version of Hodge star on cohomology (in section 3) and the Lefschetz decomposition theorem (section 4). In section 5, these ideas are combined with Santharoubane's theorem on the cohomology of the Heisenberg Lie algebras to prove the above theorem.

Chapter IV: Free two-step nilpotent Lie algebras.

The first three sections explain the results from Stefan Sigg's Ph.D. thesis, regarding the cohomology of free two-step nilpotent Lie algebras. In section 1, the central idea from Sigg's paper is explained. We then explain why we are allowed to use his results from homology for our purposes in cohomology. Section 2 is a review of the representation theory needed to prove Sigg's results, and section 3 gives the results.

Section 4 uses Sigg's theorems to investigate higher operators in the cohomology of the free two-steps. Bigradation is used to help prove three theorems about secondary operators that hold for all free two-steps, and some structures that resemble the hypercubes from Chapter II are shown.

Chapter V: Open problems.

This chapter is a brief summary of the unanswered questions the interested reader may wish to investigate. These open problems include observations about higher operators in general and specific questions that stem from Chapter IV.

I owe the proof of Theorem I.12 and most of the proofs of Chapter II to Barry Jessup. Moreover, Dr. Jessup was the first to notice (through his Mathematica® program) different 3-cubes (including the "cheater" ones) in the cohomology of the free two-step nilpotent Lie algebra on three generators.

Section 5 from Chapter III and section 4 from Chapter IV are my own. In particular for Chapter III, I proved the main theorem that is mentioned above (Theorem III.1) and I showed Santharoubane's theorem quickly follows from the Lefschetz decomposition theorem. For Chapter IV, the three theorems that follow from the use of bigradation (called Theorem IV.A, Theorem IV.B and Theorem IV.C) are my own, as is the "6-cube-like structure" in the cohomology of the free two-step on four generators.

I thank all those people who, directly or indirectly, helped me during my research for (and writing of) this thesis. In particular, thanks to Grant Cairns for his guidance and friendship. Indeed, the generosity of the entire staff at the LaTrobe University School of Mathematics will not be forgotten. Thanks to Monica Nevins,

whose fortunate timing allowed me to follow her excellent course on representation theory in the autumn of 2000; even after the course was finished, her enthusiastic teaching was invaluable in the research related to Chapter IV. Thanks, also, to Lisa Langsetmo, for inviting me to speak about the Jessup cohomology operators at Johns Hopkins University.

Special thanks go to my thesis supervisor, Dr. Barry Jessup, for his generosity, wisdom, encouragement and good humour. Without his advice on such diverse topics as research methods and writing style, this paper could not have been completed within the desired amount of time.

Finally, I dedicate this thesis to my father and mother, Reginald and Alice LeBel, for whom I feel deep gratitude. None of my work would be possible without their love and support.

Alain Claude LeBel, B.Sc.

August 2, 2001

CHAPTER I: COHOMOLOGY OF LIE ALGEBRAS

All vector spaces in this chapter are assumed to be finite dimensional over a field \mathbb{F} of characteristic zero unless stated otherwise. If V is a vector space, V^* denotes its dual space.

§1 BASIC DEFINITIONS AND FUNDAMENTAL RESULTS

1.1 Lie algebras. A Lie algebra L is a vector space equipped with a bilinear operator $[\cdot, \cdot] : L \times L \rightarrow L$ called the *bracket*, which is antisymmetric, i.e.

$$[x, y] = -[y, x],$$

and satisfies the *Jacobi identity*

$$[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0,$$

for all x, y, z in L .

If L is a Lie algebra, let $C^0(L) = L$ and inductively define $C^i(L) = [L, C^{i-1}(L)]$. Then, L is said to be *nilpotent* if there exists k in \mathbb{N} such that $C^k(L) = \{0\}$. Similarly, let $C_0(L) = L$ and inductively define $C_i(L) = [C_{i-1}(L), C_{i-1}(L)]$. One says that L is *solvable* if $C_k(L) = \{0\}$ for some k . A *semisimple Lie algebra* is any Lie algebra with no non-trivial solvable ideals. We say that two elements of L *commute* if their bracket is zero and the *centre* of L is the subspace of all elements that commute with everything in L . The centre is an ideal and is denoted $Z(L)$. A Lie algebra is *reductive* if it is the direct sum of its centre and a semisimple subalgebra.

1.2 Exterior algebras. The proofs of results in this subsection can be found in [Gre]. All of the definitions and results are also summarised (without proofs) in the preliminary chapter of [GHV3].

If V is a vector space, let ΛV denote its exterior algebra, which may be taken to be the tensor algebra over V , quotiented by the two-sided ideal generated by $\{x \otimes x \mid x \in V\}$. It is a graded-commutative, associative algebra with identity and the product of two elements α and β is usually denoted by $\alpha \wedge \beta$. Graded-commutativity means that

$$\alpha \wedge \beta = (-1)^{|\alpha| \cdot |\beta|} \beta \wedge \alpha,$$

where $|\alpha|$ is the degree of α and $|\beta|$ is the degree of β . Where the context is clear, we shall also denote the product as $\alpha\beta$ or $\alpha\cdot\beta$. Any basis of V is a system of generators. Indeed, if $\{e_1, \dots, e_n\}$ is such a basis, then $\{e_{\nu_1} \wedge \dots \wedge e_{\nu_k} \mid 1 \leq \nu_1 < \dots < \nu_k \leq n\}$ is a basis for $\Lambda^k V$, the subspace of all elements of degree k . Consequently, $\Lambda^k V$ is of dimension $\binom{n}{k}$ when $0 \leq k \leq n$ and is of dimension 0, otherwise. Hence,

$$\Lambda V = \bigoplus_{k=0}^n \Lambda^k V$$

and $\dim \Lambda V = 2^n$. An element α in ΛV is said to be *homogeneous* if $\alpha \in \Lambda^k V$ for some k and we say the degree of α is $|\alpha| = k$. While the above description assumes the generating set consists of elements of degree 1, one can also define exterior algebras over generating sets of elements of odd degree.

Note that ΛV has the following universal property: if $\phi : V \times V \rightarrow W$ is an antisymmetric, bilinear map into a vector space W , then there exists a unique linear map $\bar{\phi} : \Lambda^2 V \rightarrow W$ such that $\phi(x, y) = \bar{\phi}(x \wedge y)$ for all x, y in V [Gre, §5.2–5.3]. Also note that $\Lambda^1 V = V$ and $\Lambda^0 V = \mathbb{F}$.

We remark that $\Lambda^p(V^*)$ is canonically isomorphic to $(\Lambda^p V)^*$ via the action defined by

$$\langle f_1^* \wedge \dots \wedge f_p^*, x_1 \wedge \dots \wedge x_p \rangle = \det(\langle f_i^*, x_j \rangle),$$

where each $f_i^* \in V^*$ and each $x_j \in V$. Note that $\langle a, b \rangle = ab$ for all $a, b \in \Lambda^0 V = \mathbb{F}$. If we define $\langle \Lambda^p(V^*), \Lambda^q V \rangle = \{0\}$ when $p \neq q$, this makes $\Lambda V^* \cong (\Lambda V)^*$ and we'll use this isomorphism to identify the two algebras. Elements in $\Lambda^k V^*$ are often called *k-forms on V*.

Definition 1. A derivation of degree k is a linear map $\delta : \Lambda V \rightarrow \Lambda V$ such that $\delta(\Lambda^i V) \subseteq \Lambda^{i+k} V$ and such that the graded Leibniz rule is satisfied:

$$\delta(a \wedge b) = (\delta a) \wedge b + (-1)^{k|a|} a \wedge (\delta b),$$

for all b and homogeneous a in ΛV . Derivations on ΛV^* are similarly defined.

Because of the Leibniz rule, derivations on ΛV are uniquely determined by their action on V [Gre, §5.11]. Moreover, given a linear map $f : V \rightarrow \Lambda^{k+1} V$, one can, using the Leibniz formula, define a unique derivation δ of degree k on ΛV such that $\delta|_V = f$.

Definition 2. For any $x \in V$, let $i_x : \Lambda V^* \rightarrow \Lambda V^*$ be the derivation of degree -1 defined by $i_x(y^*) = \langle y^*, x \rangle$ for all $y^* \in V^*$.

One can show that i_x is the dual map of left-multiplication by x on ΛV [Gre, §5.13–5.14]. Similarly, for any $y^* \in \Lambda V^*$, one can define the derivation i_{y^*} on ΛV . As a final remark, recall that

$$\Lambda(U \oplus V) \cong \Lambda U \otimes \Lambda V$$

when U and V are vector spaces [Gre, §5.15].

1.3 Lie algebra cohomology. Since the bracket $[\cdot, \cdot] : L \times L \rightarrow L$ is bilinear and antisymmetric, one can define a unique linear map $\partial : \Lambda^2 L \rightarrow \Lambda^1 L$ satisfying $\partial(x \wedge y) = [x, y]$. Consider its dual map $d : \Lambda^1 L^* \rightarrow \Lambda^2 L^*$, the map that satisfies

$$\langle dz^*, x \wedge y \rangle = \langle z^*, [x, y] \rangle$$

for all x, y in L and for all z^* in L^* . As mentioned after Definition 1, d can be extended to a derivation of degree 1 on all of ΛL^* . Now define $d_i = d|_{\Lambda^i L^*}$ and note that $d_0 = 0$. Obviously, when $\Lambda^i L^*$ or $\Lambda^{i+1} L^*$ is trivial, then d_i is also trivial. Thus, $d_q = 0$ when $q < 0$ or when $q \geq \dim L$.

Proposition 3. $d^2 = 0$

Proof. First note that d^2 is a derivation of degree 2 on ΛV :

$$\begin{aligned} d^2(\alpha \wedge \beta) &= d(d\alpha \wedge \beta + (-1)^{|\alpha|} \alpha \wedge d\beta) \\ &= d^2\alpha \wedge \beta + (-1)^{|\alpha|+1} d\alpha \wedge d\beta + (-1)^{|\alpha|} d\alpha \wedge d\beta + (-1)^{2|\alpha|} \alpha \wedge d^2\beta = d^2\alpha \wedge \beta + \alpha \wedge d^2\beta. \end{aligned}$$

Thus, $d^2 = 0$ is equivalent to $d_2 \circ d_1 = 0$ which, as will be shown, is equivalent to the Jacobi identity for Lie algebras.

Define $\Psi : \Lambda^3 L \rightarrow \Lambda^2 L$ by

$$\Psi(xyz) := \partial(xy)z + \partial(zx)y + \partial(yz)x,$$

for any x, y, z in L . The Jacobi identity is equivalent to $\partial \circ \Psi = 0$, so if one can show that d_2 is the dual of Ψ , then the result will follow. If $a, b \in L^*$ and if $x, y, z \in L$, then

$$\langle \Psi^*(ab), xyz \rangle = \langle ab, \partial(xy)z + \partial(zx)y + \partial(yz)x \rangle.$$

But,

$$\begin{aligned} \langle ab, \partial(xy)z \rangle &= -\langle ab, z\partial(xy) \rangle \\ &= -\langle i_z(ab), \partial(xy) \rangle \\ &= -\langle a, z \rangle \langle b, \partial(xy) \rangle + \langle b, z \rangle \langle a, \partial(xy) \rangle \\ &= -\langle a, z \rangle \langle d_1 b, xy \rangle + \langle b, z \rangle \langle d_1 a, xy \rangle \end{aligned}$$

Thus,

$$\begin{aligned} (1) \quad \langle \Psi^*(ab), xyz \rangle &= -\langle a, z \rangle \langle d_1 b, xy \rangle + \langle b, z \rangle \langle d_1 a, xy \rangle \\ &\quad - \langle a, y \rangle \langle d_1 b, zx \rangle + \langle b, y \rangle \langle d_1 a, zx \rangle \\ &\quad - \langle a, x \rangle \langle d_1 b, yz \rangle + \langle b, x \rangle \langle d_1 a, yz \rangle. \end{aligned}$$

On the other hand,

$$\langle (d_1 a)b, xyz \rangle = \langle d_1 a, i_b(xyz) \rangle = \langle b, x \rangle \langle d_1 a, yz \rangle - \langle b, y \rangle \langle d_1 a, xz \rangle + \langle b, z \rangle \langle d_1 a, xy \rangle,$$

so

$$\begin{aligned} \langle d_2(ab), xyz \rangle &= \langle (d_1 a)b - a(d_1 b), xyz \rangle \\ &= \langle d_1 a, i_b(xyz) \rangle - \langle d_1 b, i_a(xyz) \rangle \\ &= \langle b, x \rangle \langle d_1 a, yz \rangle - \langle b, y \rangle \langle d_1 a, xz \rangle + \langle b, z \rangle \langle d_1 a, xy \rangle \\ &\quad - \langle a, x \rangle \langle d_1 b, yz \rangle + \langle a, y \rangle \langle d_1 b, xz \rangle - \langle a, z \rangle \langle d_1 b, xy \rangle \end{aligned}$$

Comparison of (1) with this last equation gives $\Psi^* = d_2$, hence the desired result. \square

Definition 4. *The derivation d , defined above, is called the differential on ΛL^* . $(\Lambda L^*, d)$ is called the Koszul complex of L and is a graded-commutative differential algebra.*

In the usual language of algebraic topology, one defines the space of cocycles to be $Z^*(L) = \ker d$ (not to be confused with the centre of L) and the space of coboundaries to be $B^*(L) = \text{im } d$. By Proposition 3, all coboundaries are also cocycles. We may now give the following definition:

Definition 5. *The cohomology space of a Lie algebra L (with trivial coefficients) is defined by*

$$H^*(L) = Z^*(L)/B^*(L).$$

Since d is (homogeneous) of degree 1, these spaces are graded. We can define $Z^p(L) = \ker d_p$, the space of p -cocycles, and $B^p(L) = \text{im } d_{p-1}$, the space of p -coboundaries. Define the p^{th} cohomology space of L to be the quotient space $H^p(L) = Z^p(L)/B^p(L)$. Note that $Z^*(L) = \bigoplus_i Z^i(L)$, $B^*(L) = \bigoplus_i B^i(L)$ and $H^*(L) = \bigoplus_i H^i(L)$. Because of the graded Leibniz rule, $B^*(L)$ is an ideal in the subalgebra $Z^*(L)$, so $H^*(L)$ inherits a product from ΛL^* , making $(H^*(L), \wedge)$ a graded-commutative, associative algebra called the cohomology algebra. If α is a cocycle, we denote its cohomology class, $\alpha + B^*(L)$, simply by $[\alpha]$.

The i^{th} Betti number of L , $b_i(L)$, is the dimension of the i^{th} cohomology space of L . When there is no ambiguity, we write the i^{th} Betti number simply as b_i .

§2 EXAMPLES OF LIE ALGEBRA COHOMOLOGY

2.1 The abelian Lie algebras. Let $L = \mathbb{F}^n$ be an abelian Lie algebra, where all brackets are defined to be zero. The differential d is therefore also trivial, so $Z^*(L) = \Lambda L^*$ and $B^*(L) = \{0\}$. Hence, $H^*(L) = \Lambda L^*$.

2.2 A nilpotent example. Let \mathfrak{h}_1 denote the first Heisenberg Lie algebra. This is the first in a class of Lie algebras which will be discussed more thoroughly in the third chapter; for now, it suffices to say that \mathfrak{h}_1 has a basis $\{x, y, z\}$ such that $[x, y] = z$ and z spans the centre.

Consider the dual basis $\{x^*, y^*, z^*\}$ of \mathfrak{h}_1^* . One can see that $dz^* = x^* \wedge y^*$ since $[x, y] = z$, and $dx^* = 0 = dy^*$. By the graded Leibniz rule, we have $d(x^* \wedge z^*) = 0 = d(y^* \wedge z^*)$. That $d(x^* \wedge y^*) = d^2 z^* = 0$ follows from Proposition 3, and that $d(\Lambda^0 \mathfrak{h}_1^*) = \{0\}$ and $d(\Lambda^3 \mathfrak{h}_1^*) = \{0\}$ follows from the general remark that $d_0 = 0$ and $d_q = 0$ when $q \geq 3 = \dim \mathfrak{h}_1$. Therefore, one can describe $H^*(\mathfrak{h}_1)$ as follows:

$$\begin{aligned} H^0(\mathfrak{h}_1) &= \mathbb{F}; \\ H^1(\mathfrak{h}_1) &= \text{span} \{[x^*], [y^*]\}; \\ H^2(\mathfrak{h}_1) &= \text{span} \{[x^* \wedge z^*], [y^* \wedge z^*]\} \\ &= \text{span} \{x^* \wedge z^*, y^* \wedge z^*, x^* \wedge y^*\} / \text{span} \{x^* \wedge y^*\}; \\ H^3(\mathfrak{h}_1) &= \text{span} \{[x^* \wedge y^* \wedge z^*]\}. \end{aligned}$$

2.3 A solvable example. Let $L = \text{span} \{x, y \mid [x, y] = x\}$. Then, $d_1(x^*) = x^* \wedge y^*$ and $d_1(y^*) = 0$, so $H^*(L)$ is just the exterior algebra with a single generator: $[y^*]$.

2.4 A semisimple example. Consider $L = \mathfrak{sl}_2 \mathbb{C}$ with its usual basis $\{x, h, y\}$ such that $[x, y] = h$, $[h, x] = 2x$ and $[h, y] = -2y$. With the dual basis $\{x^*, h^*, y^*\}$ of L^* , one finds $dh^* = x^* \wedge y^*$, $dx^* = 2h^* \wedge x^*$ and $dy^* = -2h^* \wedge y^*$. Hence, d_1 is an isomorphism, so $Z^1(L) = \{0\}$ and $B^2(L) = \Lambda^2 L^*$. Therefore, $H^*(L)$ is

$$\begin{aligned} H^0(L) &= \mathbb{C}; \\ H^1(L) &= \{0\}; \\ H^2(L) &= \{0\}; \\ H^3(L) &= \text{span} \{[x^* \wedge h^* \wedge y^*]\}. \end{aligned}$$

Thus, $H^*(\mathfrak{sl}_2 \mathbb{C})$ is actually an exterior algebra over a single generator of degree 3, namely $[x^* \wedge h^* \wedge y^*]$.

§3 GENERAL RESULTS ABOUT LIE ALGEBRA COHOMOLOGY

If G is a compact connected Lie group, then its de Rham cohomology can be computed directly from the cohomology of the corresponding Lie algebra, since the two are isomorphic [GHV2, Chapter IV, §4]. Moreover, Hopf and Samelson showed that the de Rham cohomology of a compact connected Lie group is a Hopf algebra [Hop, Sam].

Definition 6. A graded connected Hopf algebra is a pair (A, Δ) where A is a graded, connected algebra ("graded, connected" meaning $A = A^0 \oplus \bar{A}$, where $\bar{A} = \bigoplus_{i=1}^{\infty} A^i$ and $A^0 = \mathbb{F}$) and $\Delta : A \rightarrow A \otimes A$ is a homomorphism of algebras satisfying (for all $x \in A$)

$$\Delta(x) = x \otimes 1 + 1 \otimes x + \bar{\Delta}(x), \quad \bar{\Delta}(x) \in \bar{A} \otimes \bar{A}$$

such that Δ is co-associative, so

$$\begin{array}{ccc} A & \xrightarrow{\Delta} & A \otimes A \\ \Delta \downarrow & & \downarrow \Delta \otimes \text{id}_A \\ A \otimes A & \xrightarrow{\text{id}_A \otimes \Delta} & A \otimes A \otimes A \end{array}$$

commutes, and has a co-unit ε , so

$$\begin{array}{ccc} A & \xrightarrow{\Delta} & A \otimes A \\ \Delta \downarrow & & \downarrow \text{id}_A \otimes \varepsilon \\ A \otimes A & \xrightarrow{\varepsilon \otimes \text{id}_A} & A \end{array}$$

commutes and $(\text{id}_A \otimes \varepsilon) \circ \Delta = \text{id}_A = (\varepsilon \otimes \text{id}_A) \circ \Delta$.

An element $x \in \bar{A}$ is primitive if $\bar{\Delta}(x) = 0$. The set of primitives of A , denoted P_A , form a vector subspace of A .

Theorem 7 (Hopf's Theorem). Any finite dimensional graded, commutative Hopf algebra A over a field of characteristic 0 is an exterior algebra generated by the subspace of primitives, i.e. $A = \Lambda P_A$, where P_A has a basis consisting of homogeneous elements of odd degree.

Coming back to our Lie group G , the multiplication

$$\mu : G \times G \longrightarrow G$$

induces a map

$$\mu^* : H^*(G) \longrightarrow H^*(G \times G).$$

That G is connected implies $H^*(G)$ is a graded, connected algebra. That G is compact means the Künneth isomorphism holds [GHV2, §0.14]:

$$\kappa : H^*(G) \otimes H^*(G) \xrightarrow{\cong} H^*(G \times G).$$

This allows one to define

$$\Delta = \kappa^{-1} \circ \mu^* : H^*(G) \longrightarrow H^*(G) \otimes H^*(G),$$

which gives $H^*(G)$ a Hopf algebra structure. We therefore have:

Theorem 8 (Hopf-Samelson Theorem). *Let G be a compact connected Lie group. Then*

$$H^*(G) \cong \Lambda P_{H^*(G)}.$$

Furthermore,

$$\dim P_{H^*(G)} = \dim T,$$

where T is the maximal torus in G .

For a historical look at how the Hopf-Samelson Theorem came about and for an overview of how the original proofs were done, see [Die, Chapter VI, §2]. A proof of Hopf's Theorem can be found in [GHV2, Chapter IV, §4, Theorem IV] or [MM]. The fact that $\dim P_{H^*(G)} = \dim T$ can be found in [FTT, Chapter 1, §4].

Theorem 8 implies that if L is a Lie algebra corresponding to a compact connected Lie group, then $H^*(L)$ is an exterior algebra generated by primitives of odd degree. This can be generalised to hold for *all* reductive Lie algebras, as proven in [GHV3, Chapter V, §5.18 (theorem III) and §10.23]:

Theorem 9. *Let L be a reductive Lie algebra. Then*

$$H^*(L) = \Lambda P_{H^*(L)}.$$

Furthermore, $\dim P_{H^(L)}$ equals the dimension of the Cartan subalgebra of L , hence*

$$\dim P_{H^*(L)} \geq \dim Z(L).$$

The toral rank, $\text{rank}(M)$, of a smooth manifold M is the dimension of the largest torus which acts freely on M . For any compact connected Lie group G , Theorem 8 implies

$$\dim H^*(G) = 2^{\text{rank}(G)}.$$

If H is a closed subspace of G , then G/H is a smooth manifold called a *homogeneous space* and, in fact,

$$\dim H^*(G/H) \geq 2^{\text{rank}(G/H)},$$

as shown in [FTT].

The *Toral Rank Conjecture (TRC)*, first posed by Halperin [Hal1, Hal2] in 1968, states that

$$(TRC) \quad \dim H^*(M) \geq 2^{\text{rank}(M)}.$$

Apart from the evidence already supplied by compact connected Lie groups and by homogeneous spaces, the TRC is known to be true for Kähler manifolds and for all manifolds of dimension at most 6 [Hil].

§4 THE COHOMOLOGY OF NILPOTENT LIE ALGEBRAS

We now discuss a version of the TRC for nilpotent Lie algebras. We start with some well-known facts about this family of Lie algebras.

Theorem 10. *Nilpotent Lie algebras enjoy Poincaré duality: $b_i = b_{n-i}$ when $0 \leq i \leq n$, where n is the dimension of the Lie algebra. In particular, $b_0 = b_n = 1$.*

This was first proven by Jean-Louis Koszul, a student of Élie Cartan. In his Ph.D. thesis [Kos], Koszul wrote a complete overview of Lie algebra homology and cohomology, continuing the work of Chevalley and Eilenberg [CE]. Some theorems of Hopf and Samelson regarding homogeneous spaces are also proven, in this paper, using only the tools of linear algebra. According to Koszul, all previous proofs of the results had been difficult topological proofs [Kos, introduction].

Definition 11. *A Poincaré duality algebra is a finite dimensional positively graded associative algebra $A = \bigoplus_{p=0}^n A^p$ such that*

- (i) $\dim A^n = 1$ and
- (ii) *For some basis vector $e^* \in (A^n)^*$, the bilinear functions*

$$\langle , \rangle' : A^p \times A^{n-p} \longrightarrow \mathbb{F}$$

given by

$$\langle a, b \rangle' = \langle e^*, ab \rangle, \quad a \in A^p, b \in A^{n-p}$$

are nondegenerate.

Note that A satisfies $\dim A^p = \dim A^{n-p}$ for $p = 0, \dots, n$.

A proof that the cohomology algebras of nilpotent Lie algebras are, in fact, Poincaré duality algebras (hence enjoy Poincaré duality) will be given in the third chapter, using the Hodge \star map and the Laplace-Beltrami operator Δ . Another proof of the Poincaré duality of $H^*(L)$, when L is a nilpotent Lie algebra, may be found in [GHV3, Chapter V, §2].

The structure of the cohomology of nilpotent Lie algebras is not as well classified as it is for reductive Lie algebras. However:

Theorem 12. *When L is nilpotent, $b_i(L) \geq 2$ when $0 < i < \dim L$.*

This was first proven by Dixmier [Dix]. For the sake of completeness, we include the following short proof.

Proof. Given a nilpotent Lie algebra L , choose an ideal K of codimension 1, so that in L^* , $K^\perp = \{x \in L^* \mid \langle x, k \rangle = 0, \forall k \in K\} = \text{span}\{u^*\}$, where u^* is the dual of some $u \in L$. Because K is an ideal, $du^* = 0$. Choose a complement $X \subset L^*$ so that $\text{span}\{u^*\} \oplus X = L^*$. Then, $X^\perp = \text{span}\{u\}$ is a one-dimensional complement for K in L .

The Koszul complex of L may then be written as $(\Lambda u^* \otimes \Lambda X, d)$. Moreover, if $\alpha \in \Lambda X$, we may write

$$d\alpha = \bar{d}\alpha + u^*\theta(\alpha),$$

where $\bar{d}\alpha$, and $\theta(\alpha)$ both belong to ΛX . In fact, defining the projection map

$$\rho : \Lambda u^* \otimes \Lambda X \longrightarrow \Lambda X$$

by $\rho(u^*\alpha) = 0$ and $\rho(\alpha) = \alpha$, we explicitly have $\bar{d} = \rho \circ d$ and $\theta = i_u \circ d$.

A short computation then shows that

- (1) \bar{d} is a derivation of ΛX of degree 1,
- (2) $\bar{d}^2 = 0$,
- (3) $\theta(x) = (\text{ad } u)^t(x)$ for all $x \in X$, where $(\text{ad } v)w = [v, w]$ for any $v, w \in L$.
- (4) θ extends $(\text{ad } u)^t$ as a derivation of ΛX of degree 0, and
- (5) $\bar{d}\theta + \theta\bar{d} = 0$.

Thus, $(\Lambda X, \bar{d})$ is the Koszul complex of K and (5) implies that $\theta : \Lambda X \rightarrow \Lambda X$ induces a derivation of degree zero $\theta^* : H^*(\Lambda X) \rightarrow H^*(\Lambda X)$. Because L is nilpotent as a Lie algebra, so is θ and hence so too is θ^* .

Now consider the following short exact sequence of differential complexes:

$$0 \longrightarrow (\Lambda X, \bar{d}) \xrightarrow{p} (\Lambda u^* \otimes \Lambda X, d) \xrightarrow{q} (\Lambda X, \bar{d}) \longrightarrow 0,$$

where $p(\beta) = u^*\beta$ and $q(\alpha + u^*\beta) = \alpha$ for $\alpha, \beta \in \Lambda X$.

The long exact sequence in cohomology induced by this is of the form

$$\begin{aligned} \dots \longrightarrow H^{k-1}(\Lambda X, \bar{d}) \xrightarrow{\partial} H^{k-1}(\Lambda X, \bar{d}) \xrightarrow{p^*} \\ H^k(\Lambda u^* \otimes \Lambda X, d) \xrightarrow{q^*} H^k(\Lambda X, \bar{d}) \xrightarrow{\partial} H^k(\Lambda X, \bar{d}) \longrightarrow \dots, \end{aligned}$$

where the connecting homomorphism ∂ satisfies $\partial(\alpha) = \theta^*(\alpha)$.

In particular, we have the short exact sequence

$$0 \longrightarrow H^{k-1}(\Lambda X, \bar{d})/(\text{im } \theta^*)^{k-1} \xrightarrow{p^*} H^k(\Lambda u^* \otimes \Lambda X, d) \xrightarrow{q^*} (\ker \theta^*)^k \longrightarrow 0,$$

with $(\text{im } \theta^*)^{k-1} = \text{im } \theta^* \cap H^{k-1}(\Lambda X, \bar{d})$ and $(\ker \theta^*)^k = \ker \theta^* \cap H^k(\Lambda X, \bar{d})$, so that

$$\begin{aligned} H^k(\Lambda u^* \otimes \Lambda X, d) &\cong H^{k-1}(\Lambda X, \bar{d})/(\text{im } \theta^*)^{k-1} \oplus (\ker \theta^*)^k \\ &\cong (\ker \theta^*)^{k-1} \oplus (\ker \theta^*)^k \end{aligned}$$

However, $\theta^*|_{H^j(\wedge X)}$ is a nilpotent linear map, so for $1 \leq j \leq n$, we have $(\ker \theta^*)^j \neq 0$. Hence, if $1 < k < n$,

$$\dim H^k(L) = \dim(\ker \theta^*)^{k-1} + \dim(\ker \theta^*)^k \geq 2.$$

□

Some upper bounds are also known:

Theorem 13. *If L is a non-abelian nilpotent Lie algebra of dimension $n \geq 3$, then for $i = 1, \dots, n-1$,*

$$\dim H^i(L) \leq \binom{n}{i} - \binom{n-2}{i-1}.$$

The proof is omitted here but can be found in [CJ1]. The bound is actually equality whenever $L = \mathfrak{h}_1 \oplus \mathbb{F}^{n-3}$.

The Toral Rank Conjecture (TRC) for nilpotent Lie algebras. *If L is a nilpotent Lie algebra, then*

$$(\text{nil-TRC}) \quad \dim H^*(L) \geq 2^{\dim Z(L)}.$$

The connection between both versions of the TRC is as follows. If N is a nilpotent Lie group and D a discrete subgroup with N/D compact, then N/D is called a *nilmanifold* and its toral rank is known to be the dimension of the centre of the (nilpotent) Lie algebra L of N (see [CJ1]). Moreover, it is classical that $H^*(N/D) \cong H^*(L)$ [Nom]. One may thus state the TRC for nilmanifolds in terms of nilpotent Lie algebras, as above.

A more general version of the Lie algebra TRC (stated as above, but not requiring L to be nilpotent) holds whenever L is reductive (by Theorem 9). However, for *nilpotent* Lie algebras, the conjecture is still open. Henceforth, only the TRC for nilpotent Lie algebras will be considered.

In [CJ1], the TRC has been shown to be true for any nilpotent Lie algebra L satisfying at least one of the following three conditions:

- (a) $\dim Z(L) \leq 5$,
- (b) $\dim L/Z(L) \leq 7$,
- (c) $\dim L \leq 14$.

Define an n -step nilpotent Lie algebra to be any Lie algebra L such that $C^n(L) = \{0\}$. When L is 1-step, or abelian, Example 2.1 tells us $\dim H^*(L) = 2^{\dim Z(L)}$. Hence, the TRC is satisfied for all 1-step nilpotent Lie algebras.

Theorem 14. *The TRC holds for all 2-step nilpotent Lie algebras, i.e. for all Lie algebras L such that $C^2(L) = [L, [L, L]] = \{0\}$.*

This was first proven in [DS]; a simpler proof from [CJ1] will complete this chapter.

A *bigraded differential space* (V, d) is a bigraded vector space $V = \bigoplus_{p,q} V_q^p$ whose differential is homogeneous of bidegree $(1, -1)$, i.e. $d : V_q^p \rightarrow V_{q-1}^{p+1}$. For any bigraded space $V = \bigoplus_{p,q} V_q^p$ with $\dim V_q^p < \infty$ for all p, q , the Poincaré series of V is $U_V(t, s) = \sum_{p,q} (\dim V_q^p) t^p s^q$ and the Poincaré-Koszul series U_V of V is

$$U_V(t) := U_V(t, -t) = \sum_{r \geq 0} c_r(V) t^r,$$

where $c_r(V) = \sum_{p+q=r} (-1)^q \dim V_q^p$. Thus,

$$c_r(V) = c_r(H^*(V)),$$

since each subspace $\bigoplus_{p+q=r} V_q^p$ is invariant under d . This is a well-known result. The proof is given here, for completeness.

Lemma 15. *If (W, δ) is a finite dimensional graded differential space with $W = \bigoplus_{q=0}^n W_q$ and $\delta : W_q \rightarrow W_{q-1}$, then*

$$\sum_{q=0}^n (-1)^q \dim W_q = \sum_{q=0}^n (-1)^q \dim H^q(W).$$

Proof. Letting $\delta_q = \delta|_{W_q}$, each W_q can be decomposed as $W_q = A_q \oplus \ker \delta_q = A_q \oplus B_q \oplus \text{im } \delta_{q+1}$, where $\ker \delta_q = B_q \oplus \text{im } \delta_{q+1}$. Note that $\delta|_{A_q}$ is injective, so $\dim A_q = \dim(\text{im } \delta_q)$. Also note that $\dim H^q(W) = \dim B_q = b_q$ is the q^{th} Betti number. Thus, one can inductively show that

$$\sum_{q=0}^m (-1)^q \dim W_q = \dim A_0 + \sum_{q=0}^m (-1)^q b_q + \dim(\text{im } \delta_{m+1})$$

for $0 \leq m \leq n$. Noting that $\dim A_0 = 0$ and $\dim(\text{im } \delta_{n+1}) = 0$ proves the lemma. \square

If $X = X_q^p$ is a bigraded space of dimension 1, the exterior algebra $\Lambda X = \mathbb{F} \oplus X$ is bigraded by putting the field in bidegree $(0, 0)$ and X in bidegree (p, q) . Then $U_{\Lambda X}(t) = 1 + (-1)^q t^{p+q}$.

A straightforward computation using the standard bigradation of the tensor product of two bigraded spaces, namely

$$(V \otimes W)_q^p = \bigoplus_{\substack{(p_1, q_1) + (p_2, q_2) \\ = (p, q)}} (V_{q_1}^{p_1} \otimes W_{q_2}^{p_2}),$$

shows that $U_{V \otimes W}(t, s) = U_V(t, s) \cdot U_W(t, s)$. In particular, $U_{V \otimes W}(t) = U_V(t) \cdot U_W(t)$.

For example, consider $(V, d) = (\Lambda X, d)$, where $X = \sum_i X_i$ with each X_i a bigraded vector space homogeneous of bidegree (p_i, q_i) and d is a derivation of bidegree $(1, -1)$. Then V is a bigraded differential space, and the above remarks establish

$$U_{H^*(V)}(t) = \prod_i (1 + (-1)^{q_i} t^{p_i+q_i})^{\dim X_i}.$$

In particular, the Koszul complex of a *homogeneous graded Lie algebra* L , (i.e. $L = \bigoplus_{i>0} L_i$ with $[L_i, L_j] \subset L_{i+j}$) is a bigraded differential space $(\Lambda X, d)$ where $X = \bigoplus_i X_i$ and $X_i = (L_{i+1})^*$ is homogeneous of bidegree $(1, i)$. Thus,

$$U_{H^*(L)}(t) = \prod_i (1 - (-t)^i)^{\dim L_i}.$$

For a polynomial $h(t) = \sum_r a_r t^r$, the length of h is $\|h\| = \sum_r |a_r|$. Hence, if V is a bigraded differential space with $\dim V_q^p < \infty$, then $\dim H(V) \geq \|U_V(t)\|$, and therefore:

Proposition 16. [CJ1, Proposition 3] *If $L = \bigoplus_{i>0} L_i$ is a finite dimensional homogeneous graded Lie algebra, then*

$$\dim H^*(L) \geq \left\| \prod_i (1 - t^i)^{\dim L_i} \right\|$$

Theorem 14 follows from the above remarks and Proposition 16 as follows:

Proof of Theorem 14. If L is a 2-step nilpotent Lie algebra, then there exists a grading $L = L_1 \oplus L_2$ such that $L_1 \cong L/Z(L)$ and $L_2 = Z(L)$. Letting $m = \dim L_1$ and $z = \dim Z(L)$, we have $U_{H^*(L)}(t) = (1+t)^m (1-t^2)^z$. Multiplying both sides by $(1-t)^m$, we find that $(1-t)^m U_{H^*(L)}(t) = (1-t^2)^{m+z}$. Upon noting that $\|(1-t^i)^j\| = 2^j$, for any $i, j \in \mathbb{N}$, and that $\|h \cdot k\| \leq \|h\| \cdot \|k\|$ for any polynomials $h, k \in \mathbb{F}[t]$, we obtain $\dim H^*(L) \geq 2^z = 2^{\dim Z(L)}$. \square

CHAPTER II: OPERATORS IN THE COHOMOLOGY OF LIE ALGEBRAS

The purpose of this chapter is to define collections of operators on cohomology that depend on elements in the centre $Z(L)$ of a Lie algebra L . It is hoped that these operators will lead to a proof of the nil-TRC (recall the *Toral Rank Conjecture for nilpotent Lie algebras* from Chapter I).

All vector spaces in this chapter are finite dimensional over a field \mathbb{F} of characteristic zero. For any vector space X , denote the set of endomorphisms on X by $\text{End}(X)$. Note that $\text{End}(X)$ is an associative algebra, where multiplication is composition of functions.

§1 CENTRAL REPRESENTATIONS

Recall that when V is a vector space with dual space V^* , there exists a linear map $V \rightarrow \text{End}(\Lambda V^*)$ that sends each x in V to i_x , a derivation of degree -1 on ΛV^* defined by $i_x(v^*) = \langle v^*, x \rangle$ for all v^* in V^* . In fact, this map can be extended to a linear map

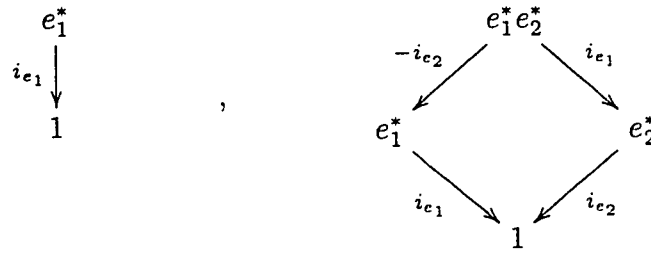
$$\begin{aligned} i : \Lambda V &\longrightarrow \text{End}(\Lambda V^*) \\ x_1 \wedge \cdots \wedge x_p &\longmapsto i_{x_p} \circ \cdots \circ i_{x_1} \end{aligned}$$

by recalling that i_x is the dual of left-multiplication by x [Gre, §5.13]. Since $xy + yx = 0$ for all x and y in V , then $i_x i_y = -i_y i_x$, which shows $i : \Lambda V \rightarrow \text{End}(\Lambda V^*)$ is a homomorphism of associative algebras.

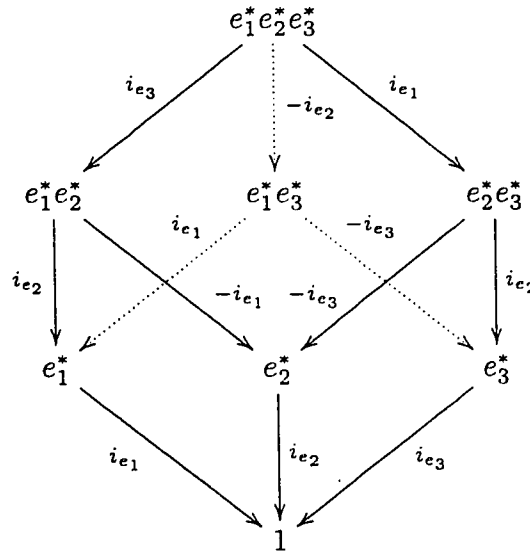
Definition 1.

- (i) A representation of an associative algebra A is a pair (ρ, W) , where W is a vector space and $\rho : A \rightarrow \text{End}(W)$ is a homomorphism of associative algebras. A representation is faithful if ρ is injective. A subrepresentation U of W is a subspace of W that is invariant under $\rho(A)$.
- (ii) Call the pair $(i, \Lambda V^*)$ the standard representation of ΛV .

Example 1. Supposing V has basis $\{e_1, \dots, e_n\}$, and its dual space V^* has dual basis $\{e_1^*, \dots, e_n^*\}$, then the standard representation, that is to say the action of ΛV on ΛV^* via the map i , can be drawn as follows, for $n = 1, 2$ and 3 :



and



We now consider any Lie algebra L and $i|_{Z(L)} : Z(L) \rightarrow \text{End}(\Lambda L^*)$.

Proposition 2. *If d is the differential (Definition I.4), then $di_z + i_zd = 0$ if and only if $z \in Z(L)$.*

Proof. One can verify that $i_zd + di_z$ is a derivation on ΛL^* for any z in L , so it suffices to verify the proposition on L^* . Since $di_z(L^*) = \{0\}$, for all z in L , we only need to check that $i_zda^* = 0$ for all $a^* \in L^*$ iff $z \in Z(L)$. In fact, $\langle i_zda^*, x \rangle = \langle da^*, z \wedge x \rangle = \langle a^*, [z, x] \rangle$ is zero for all a^* in L^* iff $[z, x] = 0$ for all $x \in L$, i.e. z is in $Z(L)$. \square

Hence, whenever $\zeta \in \Lambda Z(L)$, i_ζ (anti-)commutes with the differential. It therefore sends cocycles to cocycles and coboundaries to coboundaries, so

$$i_\zeta^* : H^*(L) \longrightarrow H^*(L)$$

$$[\alpha] \longmapsto [i_\zeta(\alpha)]$$

is well-defined.

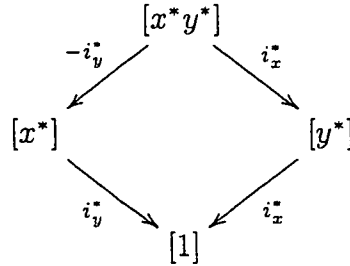
Definition 3. For a Lie algebra L , define the central representation to be the representation $(i^*, H^*(L))$ of $\Lambda Z(L)$, where $Z(L)$ is the centre of L , where

$$\begin{aligned} i^* : \Lambda Z(L) &\longrightarrow \text{End}(H^*(L)) \\ \zeta &\longmapsto i_\zeta^*, \end{aligned}$$

and where $i_\zeta^* : H^*(L) \rightarrow H^*(L)$ sends each $[\alpha]$ to $[i_\zeta(\alpha)]$.

We now consider some examples of central representations, with diagrams similar to those of the standard representation:

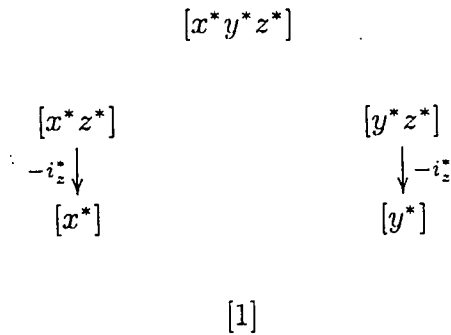
Example 2. Consider the abelian Lie algebra, with basis $\{x, y\}$, all brackets trivial and dual basis $\{x^*, y^*\}$. We draw the picture of the central representation on $H^*(\mathbb{C}^2)$ as follows:



Note that this representation is naturally isomorphic to the standard representation of $\Lambda Z(\mathbb{C}^2)$. In general, $(i^*, H^*(\mathbb{C}^n))$, the central representation corresponding to the n dimensional abelian Lie algebra, is isomorphic to the standard representation $(i, \Lambda Z(\mathbb{C}^n)^*)$ of $\Lambda Z(\mathbb{C}^n)$ (see Example I.2.1).

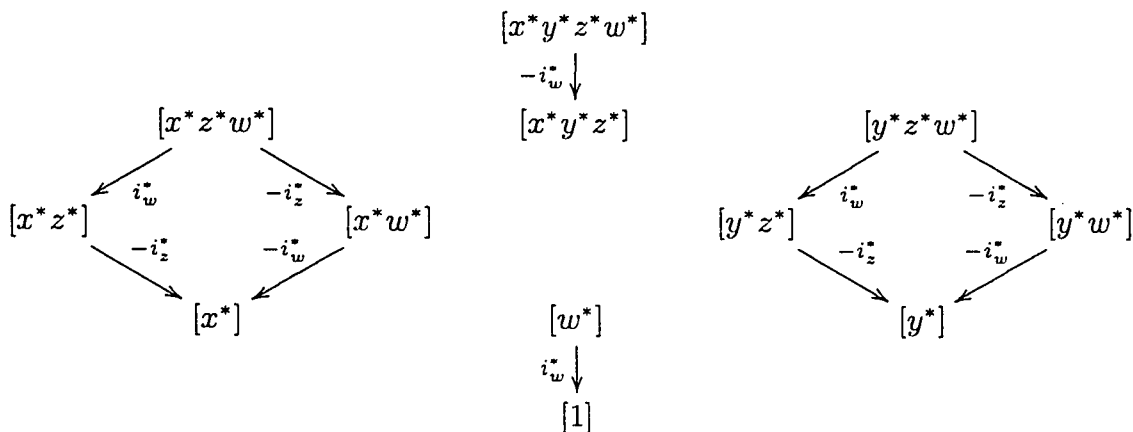
The reader is reminded to be careful not to confuse $Z(L)^*$, the dual space of the centre of L , with $Z^*(L)$, the subspace of cocycles in ΛL^* .

Example 3. The first Heisenberg Lie algebra, $\mathfrak{h}_1 = \text{span}\{x, y, z \mid [x, y] = z\}$ has a one-dimensional centre spanned by z . Let $\{x^*, y^*, z^*\}$ be the dual basis of \mathfrak{h}_1 and $H^*(\mathfrak{h}_1) = \text{span}\{1, x^*, y^*, x^*z^*, y^*z^*, x^*y^*z^*\}$ (as seen in Example I.2.2).



Note the two isomorphic copies of $\Lambda z = \Lambda \text{span}\{z\}$ in the forms of $x^* \otimes \Lambda z$ and $y^* \otimes \Lambda z$, two distinct indecomposable Λz -representations isomorphic to Λz^* , in the picture of $H^*(\mathfrak{h}_1)$.

Example 4. Consider $L = \mathfrak{h}_1 \times \mathbb{F} = \text{span}\{x, y, z, w \mid [x, y] = z\}$ with dual basis $\{x^*, y^*, z^*, w^*\}$. It has a two dimensional centre: $Z(L) = \text{span}\{z, w\}$. Without showing the computations, here is a picture of the cohomology with central representation:



Note the two copies of $\Lambda Z(L)^*$ in the picture. For instance, the map $\phi : \Lambda Z(L)^* \rightarrow H^*(L)$ defined by $\phi(\alpha) = [x^* \wedge \alpha]$ is injective, hence $\Lambda Z(L)^*$ is isomorphic to $\phi(\Lambda Z(L)^*) \subset H^*(L)$.

The central representation is faithful, by definition, if i^* is injective. In fact, as shown in the following lemma from [CJ2], the faithfulness of the central representation for a Lie algebra L implies the existence of a subrepresentation isomorphic to the standard representation of $\Lambda Z(L)$.

Define an orientation class for $Z(L)$ to be a non-zero element $t_Z \in \Lambda^{\dim Z(L)} Z(L)$.

Lemma 4. *Let L be a Lie algebra with centre $Z(L)$. The central representation $(i^*, H^*(L))$ is faithful if and only if there exists $\alpha \in H^*(L)$ such that $i^*(t_Z)\alpha \neq 0$ for any orientation class t_Z . Moreover, if the central representation is faithful, then L satisfies the nil-TRC.*

Proof. The “only if” part of the first statement is clear, so suppose there is such an α and choose an orientation class t_Z . By the Poincaré duality of $\Lambda Z(L)$, for any $0 \neq \beta \in \Lambda Z(L)$ there exists $\delta \in \Lambda Z(L)$ with $\beta \cdot \delta = t_Z$, so that $0 \neq i^*(t_Z)\alpha = i^*(\beta)i^*(\delta)\alpha$. In particular, $i^*(\beta) \neq 0$, so i^* is injective.

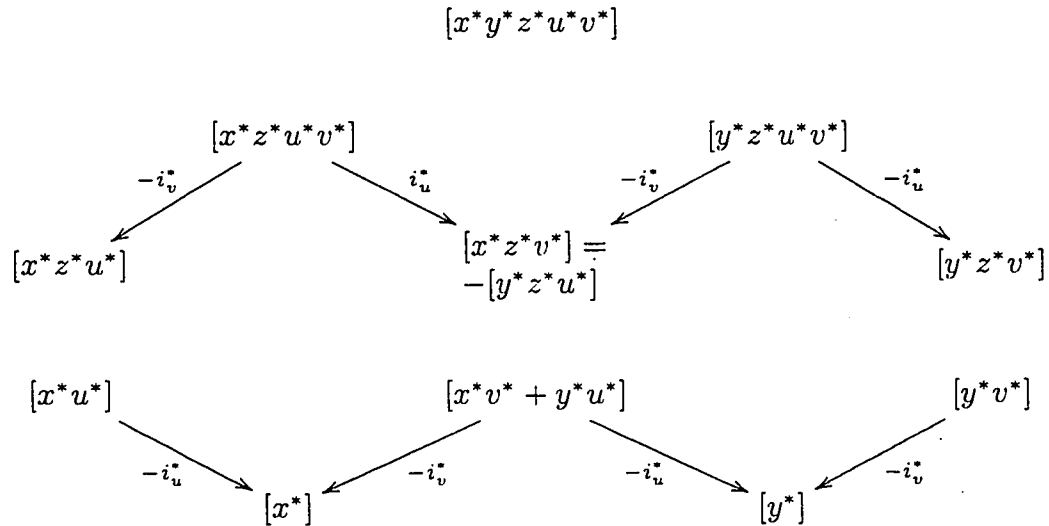
By the first part, choose $\alpha \in H^*(L)$, an orientation class t_Z for $Z(L)$ with $i^*(t_Z)\alpha \neq 0$ and consider the linear map $e_\alpha : \Lambda Z(L) \rightarrow H^*(L)$ defined by evaluation of $i^*(\beta)$ on α :

$$e_\alpha : \beta \longmapsto i^*(\beta)\alpha.$$

Exactly as above, the Poincaré duality of $\Lambda Z(L)$ shows that e_α is injective. Indeed, this shows that $i^*(\Lambda Z)\alpha$ is an indecomposable ΛZ -subrepresentation of $H^*(L)$ which is isomorphic to $\Lambda Z(L)^*$, establishing the result. \square

The faithfulness of the central representation is equivalent to the presence of at least one subrepresentation isomorphic to the standard representation of the (exterior algebra over the) centre, by Lemma 4. Thus the previous three examples have faithful representations. However, not all Lie algebras induce a faithful central representation on their cohomology space:

Example 5. Consider the free 3-step nilpotent Lie algebra on 2 generators. Free 2-step nilpotent Lie algebras are explained in Chapter IV; the free 3-step is a slight generalisation. It is actually isomorphic to $\mathcal{F}(\mathfrak{h}_1) = \text{span}\{x, y, z, u, v\}$, the 2-dimensional "central extension" of \mathfrak{h}_1 with products $[x, y] = z$, $[x, z] = u$ and $[y, z] = v$. The centre of $\mathcal{F}(\mathfrak{h}_1)$ is generated by u and v , and the central representation structure of $H^*(\mathcal{F}(\mathfrak{h}_1))$ is depicted in the diagram below:



[1]

If there was an isomorphic copy of the standard representation, then we would see diamond patterns in the diagram, as in Examples 2 and 4. We resolve this by defining two new operators on the top class $t = x^*y^*z^*u^*v^*$.

Define

$$\text{op}\{u, u\}[t] := [i_u d^{-1} i_u(t)].$$

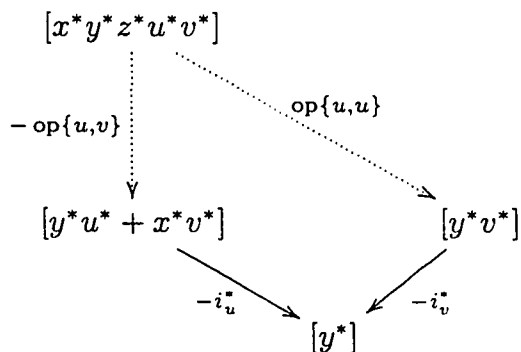
With $i_u(t) = -x^*y^*z^*v^* = d(-y^*u^*v^*)$, one finds $\text{op}\{u, u\}[t] = y^*v^*$. Also, define

$$\text{op}\{u, v\}[t] := [(i_v d^{-1} i_u + i_u d^{-1} i_v)t].$$

With $i_v d^{-1} i_u(t) = i_v(-y^* u^* v^*) = -y^* u^*$ and $i_u d^{-1} i_v(t) = i_u d^{-1}(x^* y^* z^* u^*) = i_u(x^* u^* v^*) = -x^* v^*$, we have $\text{op}\{u, v\}[t] = -[x^* v^* + y^* u^*]$. Note that

$$i_u^* \text{op}\{u, v\}[t] = [y^*] = -i_v^* \text{op}\{u, u\}[t],$$

which “mimics” the identity $i_u^* i_v^* = -i_v^* i_u^*$. In fact,



is a subspace of $H^*(\mathcal{F}(\mathfrak{h}_1))$ isomorphic to $\Lambda Z(\mathcal{F}(\mathfrak{h}_1))$ as vector spaces.

Our goal is to define operators like these ones, defined on certain subspaces of the cohomology of any Lie algebra L with a non-trivial centre. For now, the interest in them is that they seem to “repair” the non-faithfulness of the central representations, as in this last example. In other words, when an n -dimensional centre’s representation does not provide enough edges to form an n -cube, these new operators seem to add more edges to complete the picture.

§2 HYPERCUBES AND FAITHFUL REPRESENTATIONS

Define a *hypercube in n dimensions*, or an *n -cube*, to be a connected graph with 2^n vertices, where each vertex adjoins exactly n edges. For our purposes, we will always consider directed graphs, and we divide n -cubes into $n + 1$ levels (numbered $0, \dots, n$). Level 0 will have one vertex with n edges directed outwards from it. For $i = 1, \dots, n - 1$, level i will have $\binom{n}{i}$ vertices in it, with each vertex having i edges directed to it from level $i - 1$, and $n - i$ edges directed from it to level $i + 1$. Level n will have a single vertex in it, with n edges directed to it from the n vertices in level $n - 1$.

If C denotes an n -cube, let $\text{vertices}(C)$ denote the set of vertices in C and let $\text{edges}(C)$ denote the set of edges in C .

Such hypercubes can be found in graded algebras, where “levels” refer to distinct degrees, “vertices” refer to linearly independent vectors and “directed edges” refer to occurrences of maps connecting two vertices. This is formalised as follows:

Definition 5. Let $A = \bigoplus_i A^i$ be a \mathbb{Z} -graded vector space. Let C denote an n -cube, divided into levels as described above, with the vertices partitioned into subsets C_0, \dots, C_n of vertices(C), with C_i containing the vertices from level i of C .

Define a graded representation of C to be a pair of maps f and g with

$$\text{vertices}(C) \xrightarrow{f} A \text{ and } \text{edges}(C) \xrightarrow{g} \left\{ \begin{array}{l} \text{some operators} \\ \text{defined on subspaces of } A \end{array} \right\} \subset \text{End}(A)$$

such that for some integers $\mu_0 > \dots > \mu_n$, $f(C_i) \subseteq A^{\mu_i}$ and g maps a directed edge from vertex v_1 to vertex v_2 to an operator connecting $f(v_1)$ to $f(v_2)$, i.e. if e is an edge such that $v_1 \xrightarrow{e} v_2$ is a subgraph of C , then $g(e)(f(v_1)) = f(v_2)$.

Say a graded representation of C in A is a faithful representation of C (or A contains a faithful copy of C) if f is injective.

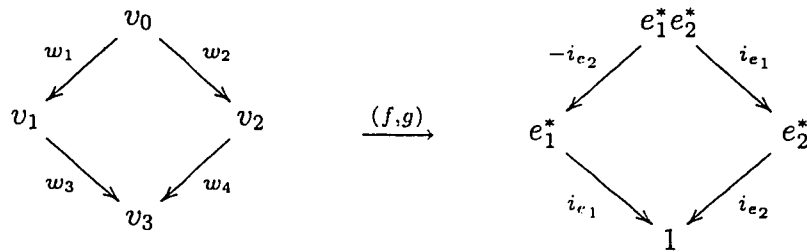
If f and g define a faithful representation of an n -cube in A , then we have (for some integers $\mu_0 > \dots > \mu_n$):

- (0) There exists a vector $x \neq 0$ in A^{μ_0} and there exists n maps of degree $\mu_0 - \mu_1$ that map x to n linearly independent non-trivial vectors in A^{μ_1} .

For $j = 1, \dots, n-1$:

- (j) In A^{μ_j} , the $\binom{n}{j}$ linearly non-trivial vectors mapped to from $A^{\mu_{j-1}}$ are each mapped to a subset of $\binom{n}{j+1}$ linearly independent non-trivial vectors in $A^{\mu_{j+1}}$.
- (n) In A^{μ_n} , there exists a non-trivial vector mapped to from n linearly independent vectors in $A^{\mu_{n-1}}$.

We note that g is not necessarily injective. For example,



where the object on the left is a 2-cube and the object on the right is the diagram of the standard representation of a 2-dimensional centre Z (as in Examples 1 and 2). It is clearly a faithful representation of the 2-cube: f maps the vertices $\{v_0, v_1, v_2, v_3\}$ to the basis $\{e_1^* e_2^*, e_1^*, e_2^*, 1\}$ of ΛZ^* . However, g is not injective. In fact, g maps the edges $\{w_1, w_2, w_3, w_4\}$ to $\text{span}\{i_{e_1}, i_{e_2}\}$.

In analogy with the central representation, where the dimension of the span of the image of g is n , we want our maps g to be as "uninjective" as possible. We

want to keep the number of different operators small (and the basis of $\text{im}(g)$ even smaller, when considering the range of g as a vector space), since there is nothing particularly useful about defining a special map for each of the $n2^{n-1}$ edges in an n -cube.

When there is a faithful representation of an n -cube in A , say A contains an n -cube or there is a hypercube of dimension n in A . Note that, when this happens, $\dim A \geq 2^n$.

For a Lie algebra L with a centre of dimension n , showing $H^*(L)$ contains an n -cube implies the TRC for the Lie algebra. The most natural way to do this is by showing the central representation is faithful. However, while $L = \mathcal{F}(\mathfrak{h}_1)$ clearly satisfies the TRC ($2^{\dim Z(L)} = 4$ and $\dim H^*(L) = 12$), the central representation is not faithful. To find a 2-cube in its central representation's diagram, we had to find more edges by defining more maps $Z(L) \rightarrow \text{End}(H^*(L))$, where L is $\mathcal{F}(\mathfrak{h}_1)$. We generalise these operators in the following two sections.

§3 HIGHER OPERATORS WITH ONE PARAMETER

Recall the operator $\text{op}\{u, u\}$ defined on the top class of $H^*(\mathcal{F}(\mathfrak{h}_1))$ in Example 5 (section 1). The purpose of this section is to generalise this operator to the cohomology of an arbitrary Lie algebra with a non-trivial centre. Throughout this section, let L be a Lie algebra such that $Z(L) \neq \{0\}$, let L^* be its dual and let z be a non-trivial element of $Z(L)$. The following definitions and lemmas assume z is fixed throughout the entire section. We want to define maps from subspaces of $H^*(L)$ to quotient spaces of $H^*(L)$. For this, we'll need a few definitions and lemmas.

Definition 6. Define subspaces $W_j(L) \subseteq \Lambda L^*$ as follows: Let $W_{-1}(L) = \{0\}$ and let $W_0(L) = Z^*(L) = \ker d$. Inductively define

$$W_j(L) := d^{-1}i_z(W_{j-1}(L)) = (d^{-1}i_z)^j W_0(L).$$

When there is no ambiguity, one simply writes W_j for $W_j(L)$. Let $W_j^p = W_j \cap \Lambda^p L^*$. For emphasis on which element of $Z(L)$ is being used, $W_j(L)$ is often written as $W_j(L; z)$.

Lemma 7. Each W_j is a subspace of ΛL^* , $W_j \subseteq W_{j+1}$ and $i_z(W_j) \subseteq W_0$ for all $j \geq 0$.

Proof. The pre-image of a subspace by a linear map is always a subspace. The fact that each W_j is a subspace is therefore trivial.

It is clearly true that $i_z W_0 \subseteq W_0$, since i_z sends cocycles to cocycles. If $\alpha \in W_j$, then $di_z \alpha = -i_z d\alpha = -i_z^2 \alpha' = 0$ for some $\alpha' \in W_{j-1}$. Thus, $i_z(W_j) \subseteq Z^*(L) = W_0$.

To show $W_j \subseteq W_{j+1}$, we proceed by induction, the case $j = 0$ being obvious. Assume $W_{j-1} \subseteq W_j$. If $\gamma \in W_j$, then there exists $\gamma' \in W_{j-1}$ such that $i_z \gamma' = d\gamma$. Since $\gamma' \in W_{j-1} \subseteq W_j$, it follows that $\gamma \in W_{j+1}$. Thus, $W_j \subseteq W_{j+1}$. \square

Definition 8. Define the subspace $K_j(L)$ of $H^*(L)$ to be

$$K_j(L) = \left\{ [\alpha] \in H^*(L) \mid \exists (\alpha_0, \alpha_1, \dots, \alpha_{j+1}) \in \Lambda L^* \times \dots \times \Lambda L^* \text{ such that} \right. \\ \left. \alpha_0 \in [\alpha] \text{ and } i_z \alpha_k = d\alpha_{k+1} \text{ for } k = 0, \dots, j \right\}.$$

Once again, when there is no ambiguity, write K_j for $K_j(L)$. Let $K_j^p = K_j \cap H^p(L)$. By convention, say $K_{-1}(L) = H^*(L)$. For emphasis on the element of $Z(L)$ being considered, $K_j(L)$ is often written as $K_j(L; z)$.

Remark 9. If $(\alpha_0, \dots, \alpha_{j+1})$ is any $(j+2)$ -tuple which shows $[\alpha]$ is in K_j , then each $\alpha_k \in W_k$. The proof of this proceeds inductively, where $\alpha_0 \in [\alpha]$ implies $\alpha_0 \in Z^*(L) = W_0$ and, if $\alpha_{k-1} \in W_{k-1}$ and $d\alpha_k = i_z \alpha_{k-1}$, then $\alpha_k \in W_k = d^{-1}i_z(W_{k-1})$.

Definition 10. For $j \geq 0$, define the one-parameter higher operators

$$\text{op}_j : K_{j-1}^p \longrightarrow H^{p-2j-1}(L) / \left[i_z \left(W_{j-1}^{p-2j} \right) \right],$$

as follows: if $[\alpha] \in K_{j-1}^p$ and $(\alpha_0, \dots, \alpha_j)$ is a representing $(j+1)$ -tuple, then

$$\text{op}_j[\alpha] = [i_z \alpha_j] + \left[i_z \left(W_{j-1}^{p-2j} \right) \right].$$

The operator op_0 is called primary. The operator op_1 is called a one-parameter secondary operator (or just a secondary operator) and is often written as $\text{op}\{z, z\}$ (for reasons that will become clear in the next section). The operator op_2 is sometimes called a tertiary operator. For emphasis on the element of $Z(L)$ being used, op_j may be written as $\text{op}_j\{z\}$.

Now we prove that the one-parameter higher operators are well-defined.

First note that, by the conventions $W_{-1} = \{0\}$ and $K_{-1} = H^*(L)$,

$$\text{op}_0 = i_z^* : H^p(L) \longrightarrow H^{p-1}(L),$$

which is already well-defined.

Suppose $[\alpha] \in K_j^p$ and $(\alpha_0, \dots, \alpha_{j+1})$ is a representing $(j+2)$ -tuple. By Remark 9, each $\alpha_k \in W_k$, so by Lemma 7 each $i_z \alpha_k \in Z^*(L)$. Thus, $[i_z \alpha_k] \in H^{p-2k-1}(L)$.

Now suppose $(\alpha_0, \dots, \alpha_j)$ and $(\alpha'_0, \dots, \alpha'_j)$ are two $(j+1)$ -tuples showing $[\alpha_0] = [\alpha'_0] \in K_{j-1}$. Then, $\alpha_0 - \alpha'_0 = d\beta$, for some $\beta \in \Lambda L^*$, and $i_z(\alpha_k - \alpha'_k) = d(\alpha_{k+1} - \alpha'_{k+1})$ for $k = 0, \dots, j-1$. Hence $d(\alpha_1 - \alpha'_1) = i_z d\beta = -di_z\beta$ so

$$\alpha_1 - \alpha'_1 + i_z\beta \in W_0.$$

Thus, $d(\alpha_2 - \alpha'_2) = i_z(\alpha_1 - \alpha'_1) = i_z(\alpha_1 - \alpha'_1 + i_z\beta_0) \in i_z W_0$, so $(\alpha_2 - \alpha'_2) \in W_1$, showing that $i_z(\alpha_2 - \alpha'_2) \in i_z W_1$.

Continue inductively: supposing $i_z(\alpha_k - \alpha'_k) \in i_z W_{k-1}$,

$$d(\alpha_{k+1} - \alpha'_{k+1}) = i_z(\alpha_k - \alpha'_k) \in i_z(W_{k-1})$$

implies $\alpha_{k+1} - \alpha'_{k+1} \in W_k$ which implies $i_z(\alpha_{k+1} - \alpha'_{k+1}) \in i_z(W_k)$. Then, we have $[i_z\alpha_j] - [i_z\alpha'_j] \in [i_z W_{j-1}]$, so $\text{op}_j([\alpha])$ is well-defined.

Remark 11. We record a fact that will be used in the next section:

$$\text{op}_1 : H^p(L) \cap \ker \text{op}_0 \longrightarrow H^{p-3}(L) / i_z^* H^{p-2}(L),$$

since $K_0 = \{[\alpha] \in H^*(L) \mid i_z\alpha = d\beta \text{ for some } \beta \in \Lambda L^*\} = \ker \text{op}_0$.

Lemma 12. For $j \geq 0$, $K_{j+1} \subseteq K_j$. Furthermore, $K_j = \ker \text{op}_j$.

Proof. That $K_{j+1} \subseteq K_j$ follows from the fact that if $(\alpha_0, \dots, \alpha_{j+2})$ shows $[\alpha] \in K_{j+1}$, then $(\alpha_0, \dots, \alpha_{j+1})$ certainly shows $[\alpha] \in K_j$.

For the second part, if $[\alpha] \in K_j \subset K_{j-1}$ and $(\alpha_0, \dots, \alpha_{j+1})$ is a representing $(j+2)$ -tuple, then

$$\text{op}_j[\alpha] = [i_z\alpha_j] + [i_z W_{j-1}] = [d\alpha_{j+1}] + [i_z W_{j-1}] = [i_z W_{j-1}] = 0.$$

Thus, $[\alpha] \in \ker \text{op}_j$, and so $K_j \subseteq \ker \text{op}_j$.

Conversely, suppose $[\alpha] \in K_{j-1}$, with representing $(j+1)$ -tuple $(\alpha_0, \dots, \alpha_j)$, is such that $[\alpha] \in \ker \text{op}_j$, so $i_z\alpha_j = d\gamma + i_z w_{j-1}$ for some $\gamma \in \Lambda L^*$ and some $w_{j-1} \in W_{j-1}$. Set $\alpha'_{j+1} = \gamma$ and $\alpha'_j = \alpha_j - w_{j-1}$. Then

$$d\alpha'_j = d\alpha_j - dw_{j-1} = i_z\alpha_{j-1} - i_z w_{j-2}$$

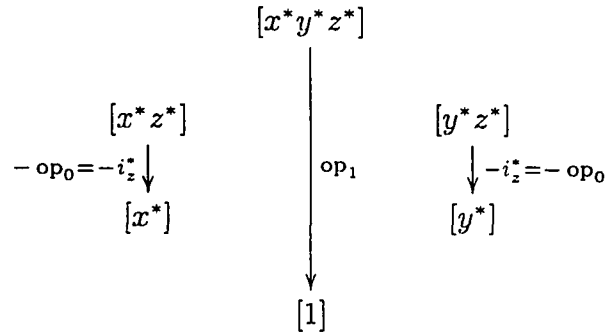
for some $w_{j-2} \in W_{j-2}$. Set $\alpha'_{j-1} = \alpha_{j-1} - w_{j-2}$.

Continue by induction, arriving at $\alpha'_1 = \alpha_1 - w_0$ for some $w_0 \in W_0 = Z^*(L)$, so that $d\alpha'_1 = d\alpha_1 - dw_0 = d\alpha_1 = i_z\alpha_0$. Let $\alpha'_0 = \alpha_0$. Then $(\alpha'_0, \dots, \alpha'_{j+1})$ exhibits $[\alpha]$ as a member of K_j . Thus, $\ker \text{op}_j \subseteq K_j$ and hence $K_j = \ker \text{op}_j$. \square

Example 6. The top class of $H^*(\mathfrak{h}_1)$ is $[x^*y^*z^*]$ (see Example 3). Since

$$\text{op}_0[x^*y^*z^*] = i_z^*[x^*y^*z^*] = [x^*y^*] = [dz^*]$$

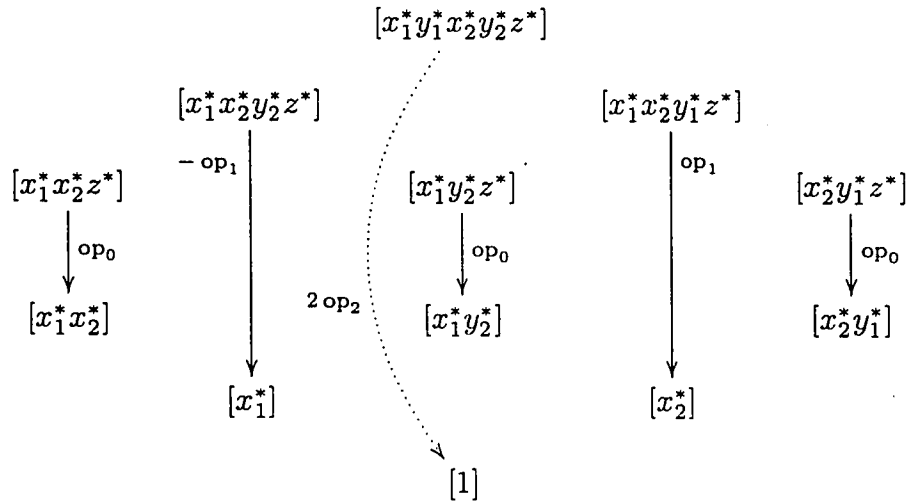
is trivial, we can define $op_1 = op\{z, z\}$ on it. By definition, $op_1[x^*y^*z^*] = [i_z z^*] = [1]$. Since $i_z^* H^1(\mathfrak{h}_1)$ is trivial in $H^0(\mathfrak{h}_1)$, op_1 is actually a well-defined map from $H^3(\mathfrak{h}_1)$ to $H^0(\mathfrak{h}_1)$. Hence, we can now draw the following complete diagram:



Similarly, we can consider the second Heisenberg Lie algebra,

$$\mathfrak{h}_2 = \{x_1, x_2, y_1, y_2, z \mid [x_1, y_1] = [x_2, y_2] = z\},$$

which has the following *partial* diagram for $H^*(\mathfrak{h}_2)$, with higher operators included:



There exist many examples of non-trivial one-parameter higher operators on the cohomology spaces of an infinite family of Lie algebras called the Heisenberg Lie algebras. These are described in Chapter III, where some facts about the operators' interactions with Poincaré duality (for those cases) are also proven.

We end this section with a lemma from [CJ2] that holds for *all* nilpotent Lie algebras, which demonstrates a kind of duality for these higher operators.

A Lie algebra L is called *unimodular* if $x \in L$ implies that the trace of $ad\ x$ is zero (where $ad\ x : L \rightarrow L$ sends each y to $[x, y]$). Note that all nilpotent Lie algebras are unimodular.

Definition 13. We say that a pair $(a, b) \in H^*(L) \times H^*(L)$ are Poincaré duals if $a \cdot b$ is of degree T and is non-trivial in $H^T(L)$, where $T = \dim L$.

Suppose L is unimodular with a non-trivial centre. Fix a non-zero $z \in Z(L)$ and choose $w \in L^*$ with $\langle w, z \rangle = 1$. If we denote $\langle z \rangle^\perp$ by U , there is a decomposition $\Lambda L^* = \Lambda U \otimes \Lambda w$ with $dw \in \Lambda U$.

Suppose $0 \neq [dw]^k \in H^*(L/\langle z \rangle)$ but that $[dw]^{k+1} = 0$ there, so that for some $\mu \in \Lambda^{2k+1}U$, $\sigma := w(dw)^k + \mu$ is a cocycle but not a coboundary in $H^*(L)$. For example, if $z \in Z(L) \cap [L, L]$, we know that $k \geq 1$. We assume this henceforth.

Lemma 14. If t is the top class of $H^*(L)$,

- (i) $\text{op}_j \sigma = 0$ for $j = 0, \dots, k-1$, and $\text{op}_k \sigma = 1$, where σ is defined above.
- (ii) $\text{op}_j t = 0$ for $j = 0, \dots, k-1$,
- (iii) for some cocycle β representing $\text{op}_k t$, we have $[\beta] \cdot [\sigma] = t$, and
- (iv) β is not a representing cocycle for any op_j , $1 \leq j \leq k-1$, so that $\text{op}_k t = [[\beta]] \neq 0$.

Proof. The proof of Statement (i) is straightforward. Let t_U denote the top class in ΛU . Since $0 \neq [dw]^k \in H^*(\Lambda U)$, by the Poincaré duality of $H^*(\Lambda U)$, there is a cocycle $\beta \in \Lambda U$ with $(dw)^k \beta = t_U$. Then, $w(dw)^k \beta = t$, so that $i_z t = (dw)^k \beta = d(w(dw)^{k-1}b)$. In this way, one shows that (ii) holds.

To see (iii), note that in the definition of $\text{op}_{k+1} t$, we may take β as a representing cocycle (before considering ambiguities). Since $w(dw)^k \beta = t$ and $\mu \beta \in \Lambda^n U = 0$, we have $[\beta] \cdot [w(dw)^k + \mu] = t$, and so in particular, $[\beta] \neq 0$.

Suppose β were a representing cocycle for some higher operation op_j , $j \geq 1$. Then, there would be a $(j+1)$ -tuple $(\beta_0, \beta_1, \dots, \beta_j) \in \Lambda L^* \times \dots \times \Lambda L^*$ such that

$$\begin{aligned} 0 &= d\beta_0 \\ i_z \beta_0 &= d\beta_1 \\ i_z \beta_1 &= d\beta_2 \\ &\vdots \\ i_z \beta_{j-1} &= d\beta_j \\ i_z \beta_j &= \beta. \end{aligned}$$

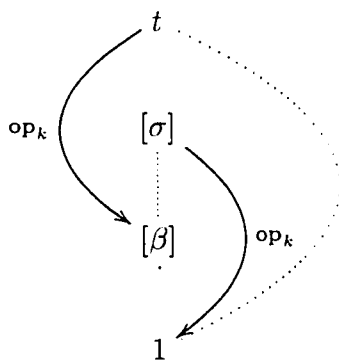
Hence, $\beta_j = w\beta + \varepsilon$ and $\beta_i = wd\beta_{i+1} + \varepsilon_i$ for some $\varepsilon, \varepsilon_i \in \Lambda U$, and $i = 0, \dots, j-1$. Thus, $d\beta_i = (dw)d\beta_{i+1} + d\varepsilon_i$.

Then,

$$\begin{aligned}
 0 &= d\beta_0 = dw \cdot d\beta_1 + d\varepsilon_0 \\
 &= (dw)(dwd\beta_2 + d\varepsilon_1) + d\varepsilon_0 \\
 &= (dw)^2 d\beta_2 + d(\varepsilon_1 dw + \varepsilon_0) \\
 &= (dw)^2 (dwd\beta_3 + d\varepsilon_2) + d(\varepsilon_1 dw + \varepsilon_0) \\
 &= (dw)^3 d\beta_3 + d(\varepsilon_2 (dw)^2 + \varepsilon_1 dw + \varepsilon_0) \\
 &\vdots \\
 &= (dw)^j d\beta_j + d(\varepsilon_{j-1} (dw)^{j-1} + \dots + \varepsilon_2 (dw)^2 + \varepsilon_1 dw + \varepsilon_0) \\
 &= (dw)^j (dw\beta + d\varepsilon) + d(\varepsilon_{j-1} (dw)^{j-1} + \dots + \varepsilon_2 (dw)^2 + \varepsilon_1 dw + \varepsilon_0) \\
 &= (dw)^{j+1} \beta + d(\varepsilon (dw)^j + \varepsilon_{j-1} (dw)^{j-1} + \dots + \varepsilon_2 (dw)^2 + \varepsilon_1 dw + \varepsilon_0)
 \end{aligned}$$

This shows that $[(dw)^{j+1}\beta] = 0$ in $H^*(\Lambda U)$, but the proof of (ii) shows that $[(dw)^i\beta] \neq 0$ in $H^*(\Lambda U)$ for $i = 1, \dots, k$, yielding $j \geq k$. \square

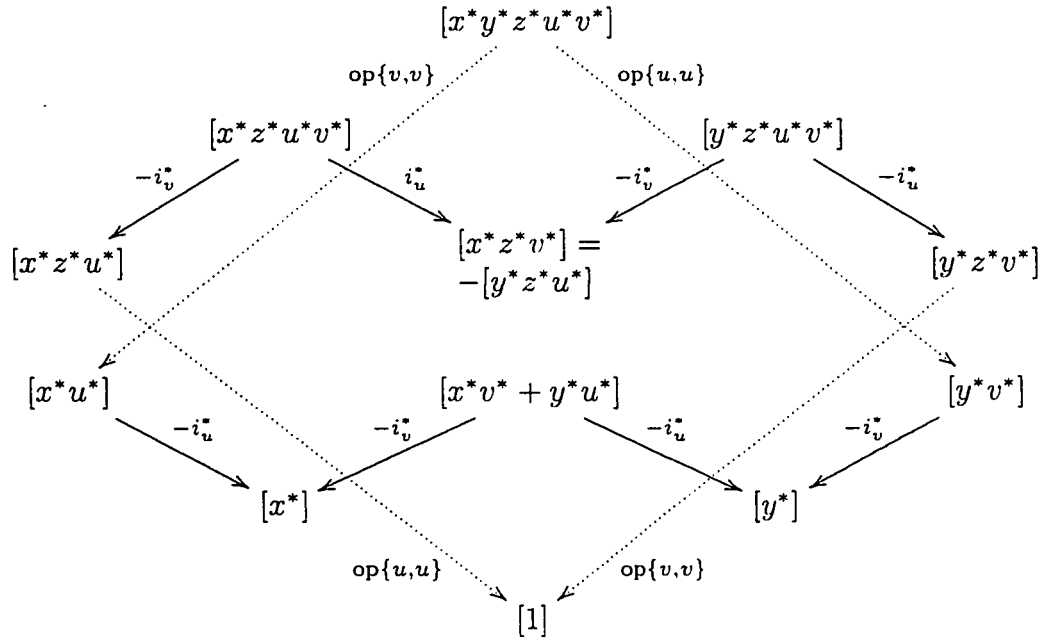
The lemma can be summarised with this diagram of dual operations, where dotted lines denote Poincaré duality:



The pairs $([\sigma], [\beta])$ and $(t, 1)$ are Poincaré duals.

Example 7. Recall Example 5, with $H^*(\mathcal{F}(h_1))$. Now we can add all 1-parameter

secondary operators to its diagram:



Note that we *still* don't have any complete 2-cubes using only our 1-parameter secondary operators. The purpose of the next section is to define a different kind of secondary operator, called 2-parameter secondary operators, which resemble the $op\{u, v\}$ operator that was defined on the top class in Example 5.

§4 SECONDARY OPERATORS WITH TWO PARAMETERS

Recall the operator $op\{u, v\}$ defined on the top class of $H^*(\mathcal{F}(\mathfrak{h}_1))$ in Example 5 (section 1). The purpose of this section is to generalise this secondary operator with two parameters on the cohomology of an arbitrary Lie algebra with a non-trivial centre. Let L be a Lie algebra with a non-trivial centre, let L^* be its dual space and let u and v be two non-trivial elements in $Z(L)$.

Definition 15. Define the subspace $K_0(L; u, v)$ of $H^*(L)$ to be

$$K_0(L; u, v) = \left\{ [\alpha] \in H^*(L) \mid \exists(\alpha_0, \alpha_1, \alpha'_1) \in \Lambda L^* \times \Lambda L^* \times \Lambda L^* \text{ such that} \right. \\ \left. \alpha_0 \in [\alpha], d(\alpha_1) = i_u \alpha_0 \text{ and } d(\alpha'_1) = i_v \alpha_0 \right\}$$

As usual, $K_0^p(L; u, v)$ denotes $K_0(L; u, v) \cap H^p(L)$.

Notice that if $[\alpha] \in H^p(L)$, then α_0 is in $Z^p(L) = W_0(L; u) = W_0(L; v)$, α_1 is in $W_1(L; u)$ and α'_1 is in $W_1(L; v)$.

Lemma 16. For all $u, v \in Z(L)$, $K_0(L; u, v) = K_0(L; u) \cap K_0(L; v)$.

Proof. If $(\alpha_0, \alpha_1, \alpha'_1)$, with $i_u(\alpha_0) = d\alpha_1$ and $i_v(\alpha_0) = d\alpha'_1$, shows $[\alpha] \in K_0(L; u, v)$, then (α_0, α_1) shows $[\alpha] \in K_0(L; u)$ and (α_0, α'_1) shows $[\alpha] \in K_0(L; v)$. Thus, $K_0(L; u, v) \subseteq K_0(L; u) \cap K_0(L; v)$.

Conversely, suppose $[\alpha] \in K_0(L; u) \cap K_0(L; v)$. By Definition 8, there exists (α_0, α_1) in $\Lambda L^* \times \Lambda L^*$ such that $i_u\alpha_0 = d\alpha_1$ with $\alpha_0 \in [\alpha]$, and there exists (β_0, β_1) in $\Lambda L^* \times \Lambda L^*$ such that $i_v\beta_0 = d\beta_1$ with $\beta_0 \in [\alpha]$. Thus, $[\alpha] = [\alpha_0] = [\beta_0]$ and $\beta_0 = \alpha_0 + d\gamma$ for some $\gamma \in \Lambda L^*$. We can therefore write $d\beta_1 = i_v\beta_0 = i_v(\alpha_0 + d\gamma)$, which implies $d(\beta_1 - \gamma) = i_v\alpha_0$. Letting $\alpha'_1 = \beta_1 - \gamma$, we have $(\alpha_0, \alpha_1, \alpha'_1)$ shows $[\alpha] \in K_0(L; u, v)$. Hence, $K_0(L; u, v) = K_0(L; u) \cap K_0(L; v)$. \square

Definition 17. Define the 2-parameter secondary operators, with parameters $u, v \in Z(L)$, to be the maps

$$\text{op}\{u, v\} : K_0^p(L; u, v) \longrightarrow H^p(L) / [i_u W_0(L) + i_v W_0(L)]$$

defined as follows: if $[\alpha] \in K_0^p(L; u, v)$, then for some $\alpha_0 \in [\alpha]$

- (i) there exists $\alpha_1 \in W_1^{p-2}(L; u)$ such that $d\alpha_1 = i_u\alpha_0$.
- (ii) there exists $\alpha'_1 \in W_1^{p-2}(L; v)$ such that $d\alpha'_1 = i_v\alpha_0$.

Let

$$\text{op}\{u, v\}[\alpha] = \frac{1}{2} [i_v\alpha_1 + i_u\alpha'_1] + [i_u W_0(L) + i_v W_0(L)].$$

There are some things to prove for this definition to make sense.

First, if $[\alpha]$ is as above (with α_0, α_1 and α'_1 also as above), then we must show that $i_v\alpha_1 + i_u\alpha'_1$ is a cocycle. Indeed,

$$\begin{aligned} d(i_v\alpha_1 + i_u\alpha'_1) &= di_v\alpha_1 + di_u\alpha'_1 = -i_v d\alpha_1 - i_u d\alpha'_1 \\ &= -i_v i_u \alpha_0 - i_u i_v \alpha_0 = -(i_v i_u + i_u i_v)\alpha_0 = 0 \end{aligned}$$

since $i_v i_u + i_u i_v = 0$.

Second, suppose $(\beta_0, \beta_1, \beta'_1)$ is another triple showing $[\alpha] \in K_0(L; u, v)$, i.e., $\beta_0 \in [\alpha]$, $i_u\beta_0 = d\beta_1$ and $i_v\beta_0 = d\beta'_1$. We want to show that

$$[i_v\alpha_1 + i_u\alpha'_1] - [i_v\beta_1 + i_u\beta'_1] \in [i_u W_0(L) + i_v W_0(L)].$$

Say $\alpha_0 - \beta_0 = d\varepsilon$, since $[\alpha] = [\alpha_0] = [\beta_0]$. We find $d(\alpha_1 - \beta_1) = i_u(\alpha_0 - \beta_0) = i_u d\varepsilon = d(-i_u\varepsilon)$. So $\alpha_1 - \beta_1 + i_u\varepsilon \in Z^*(L) = W_0(L)$. Likewise, $\alpha'_1 - \beta'_1 + i_v\varepsilon \in Z^*(L) = W_0(L)$. Thus,

$$\begin{aligned} &(i_v\alpha_1 + i_u\alpha'_1) - (i_v\beta_1 + i_u\beta'_1) \\ &= i_v(\alpha_1 - \beta_1) + i_u(\alpha'_1 - \beta'_1) \\ &= i_v(\alpha_1 - \beta_1 + i_u\varepsilon) - i_v i_u \varepsilon + i_u(\alpha'_1 - \beta'_1 + i_v\varepsilon) - i_u i_v \varepsilon \\ &= i_v \underbrace{(\alpha_1 - \beta_1 + i_u\varepsilon)}_{\in W_0(L)} + i_u \underbrace{(\alpha'_1 - \beta'_1 + i_v\varepsilon)}_{\in W_0(L)} - \underbrace{(i_v i_u + i_u i_v)\varepsilon}_{=0} \end{aligned}$$

Therefore, $\text{op}\{u, v\}$ is well-defined.

The 2-parameter secondary operators are often simply called secondary operators. Note that no assumptions are made about the linear independence of u and v . In fact, if $u = v$, then $\text{op}\{u, v\} = \text{op}\{u, u\}$, the secondary operator with single parameter u defined in section 3.

Example 7 (cont.). In Example 5, with the cohomology of $L = \mathcal{F}(h_1)$, we defined $\text{op}\{u, v\}[t] = [i_u d^{-1} i_v(t) + i_v d^{-1} i_u(t)]$. This is, up to scalar multiple, our 2-parameter secondary operator from Definition 16:

$$\text{op}\{u, v\}[\alpha] = \frac{1}{2}[i_u d^{-1} i_v(\alpha) + i_v d^{-1} i_u(\alpha)] + i_u^* H^{|\alpha|-2}(L) + i_v^* H^{|\alpha|-2}(L)$$

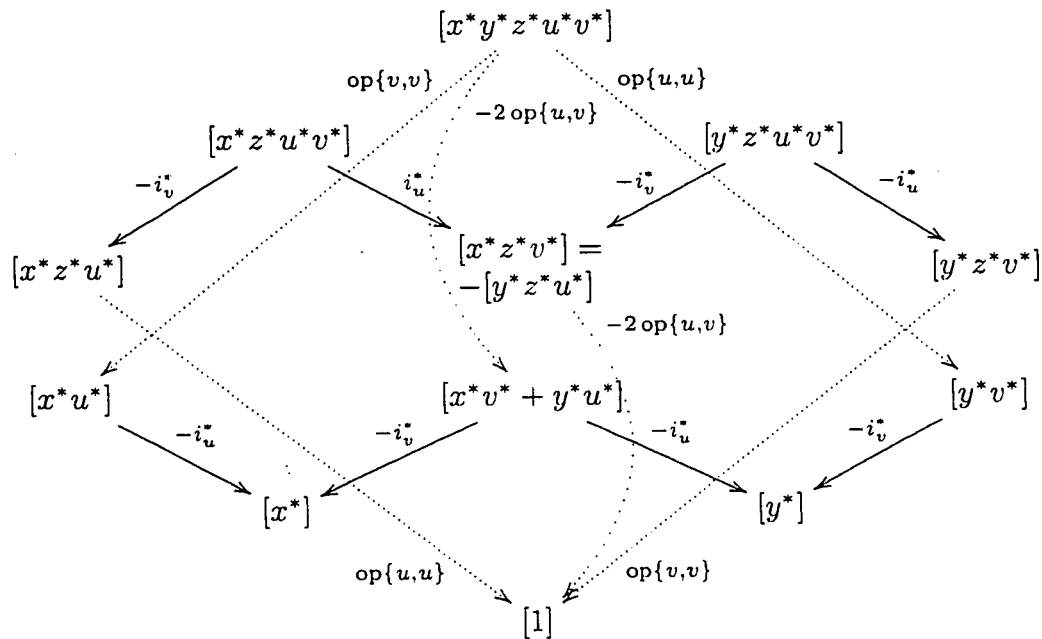
for all $\alpha \in Z^*(L)$. Actually, if $[\alpha] \in H^5(L)$, then $i_u Z^3(L) \subseteq B^2(L)$, $i_v Z^3(L) \subseteq B^2(L)$ and $H^5(L) = \ker i_u^* = \ker i_v^*$. Hence,

$$\text{op}\{u, v\} : H^5(\mathcal{F}(h_1)) \longrightarrow H^2(\mathcal{F}(h_1))$$

is a well-defined map. Likewise, $i_u Z^1(L) = i_v Z^1(L) = \{0\}$ and $H^3(L) = \ker i_u^* = \ker i_v^*$, so

$$\text{op}\{u, v\} : H^3(\mathcal{F}(h_1)) \longrightarrow H^0(\mathcal{F}(h_1))$$

is also well-defined. Hence, we get the following complete diagram for $H^*(\mathcal{F}(h_1))$:



CHAPTER III: HIGHER OPERATORS IN THE COHOMOLOGY OF THE HEISENBERG LIE ALGEBRAS

All vector spaces in this chapter are assumed to be finite dimensional over a field \mathbb{F} of characteristic zero, unless stated otherwise.

§1 INTRODUCTION

If m is a positive integer, the Heisenberg Lie algebra \mathfrak{h}_m is

$$\mathfrak{h}_m = \text{span} \{z, x_1, y_1, \dots, x_m, y_m \mid [x_j, y_j] = z \text{ for } j = 1, \dots, m\}.$$

It is clear that $\dim \mathfrak{h}_m = 2m + 1$ and \mathfrak{h}_m has a centre of dimension 1, spanned by z . In Chapter I (subsection 2.2), the cohomology of the first Heisenberg Lie algebra, \mathfrak{h}_1 , was explicitly computed. In Chapter II (Example 6), the operators $\text{op}_k = \text{op}_k\{z\}$ were computed for the first two Heisenberg Lie algebras, \mathfrak{h}_1 and \mathfrak{h}_2 .

It is the purpose of this chapter to prove the following theorem about the cohomology of the Heisenberg Lie algebras:

Theorem 1. *Let m be a positive integer.*

(i) *The higher operators op_k are such that*

$$\text{op}_k \mid_{H^{m+k+1}(\mathfrak{h}_m)} : H^{m+k+1}(\mathfrak{h}_m) \longrightarrow H^{m-k}(\mathfrak{h}_m)$$

are isomorphisms, and are zero everywhere else they are defined.

(ii) *Furthermore, suppose $\alpha, \beta \in H^*(\mathfrak{h}_m)$ are homogeneous Poincaré duals (as defined in Chapter II) with $|\beta| < |\alpha|$. Then $\text{op}_k(\alpha)$ and $\text{op}_k^{-1}(\beta)$ are also Poincaré duals for some $k \in \{0, 1, \dots, m\}$.*

To prove this theorem, we will need the Laplace-Beltrami operator and the Hodge \star map on ΛL^* , together with a fundamental theorem about them; these definitions and results are given in section 2. In section 3, we will define a variation of Hodge \star on cohomology and hence prove Poincaré duality for the nilpotent Lie algebras (Theorem I.10). In section 4, some useful theorems of Hodge and Lefschetz are stated, the proofs being taken from [Wel]. Finally, in section 5, the main results of Santharoubane's paper on the cohomology of the Heisenberg Lie algebras and Theorem 1 are shown to follow from the main result of Lefschetz. We also give a descriptive spanning set for a subspace of $\Lambda \mathfrak{h}_m^*$ which is canonically isomorphic to the cohomology.

§2 THE LAPLACE-BELTRAMI OPERATOR

Let L be a Lie algebra of dimension n , with ordered basis $\{e_1, \dots, e_n\}$. Let L^* be the dual space with dual ordered basis $\{e_1^*, \dots, e_n^*\}$ such that $\langle e_j^*, e_k \rangle = \delta_{jk}$.

Define the *Hodge \star map* on ΛL^* (with respect to this ordered basis) by

$$\begin{aligned} \star : \Lambda^j L^* &\longrightarrow \Lambda^{n-j} L^* \\ e_{\sigma(1)}^* \cdots e_{\sigma(j)}^* &\longmapsto \text{sign}(\sigma) \cdot e_{\sigma(j+1)}^* \cdots e_{\sigma(n)}^*, \end{aligned}$$

where σ is a permutation on $\{1, \dots, n\}$ such that $\sigma(1) < \sigma(2) < \dots < \sigma(j)$, and uniquely extending the map to all of ΛL^* by linearity.

Note that if τ is also such a permutation, then $\text{sign}(\sigma) \cdot e_{\sigma(j+1)}^* \cdots e_{\sigma(n)}^* = \text{sign}(\tau) \cdot e_{\tau(j+1)}^* \cdots e_{\tau(n)}^*$ (by the properties of multiplication in exterior algebras), so Hodge \star is a well-defined map.

Define the projection map $\Pi_j : \Lambda L^* \rightarrow \Lambda^j L^*$ and let $\omega = \sum_j (-1)^{j(n-j)} \Pi_j$. One can check that $\star\star = \omega$. Moreover, when n is even, then $\omega = \sum_j (-1)^j \Pi_j$ and when n is odd, then $\omega = \text{id}_{\Lambda L^*}$.

As in [ACK] and [Wel], one can define an inner product on $\Lambda^j L^*$ by

$$\langle \alpha, \beta \rangle_j = \star(\alpha \cdot \star(\beta)), \quad \text{for all } \alpha, \beta \in \Lambda^j L^*.$$

This induces a non-degenerate inner product $\langle \cdot, \cdot \rangle_\star$ on all of ΛL^* by defining $\langle \alpha, \beta \rangle_\star$ to be $\langle \alpha, \beta \rangle_k$ when $|\alpha| = |\beta| = k$, and $\langle \alpha, \beta \rangle := 0$ when α and β are homogeneous but $|\alpha| \neq |\beta|$.

We remark that this last definition can also be made by choosing an orientation class λ of ΛL and defining the inner product by $\langle \alpha, \beta \rangle_\lambda = i_\lambda(\alpha \cdot (\star\beta))$. This way, $\langle \alpha, \beta \rangle_\lambda$ is automatically zero whenever α and β are homogeneous with $|\alpha| \neq |\beta|$. Also, the two inner products (old and new) are the same up to a non-zero scalar multiple. If one chooses $\lambda = e_1^* \cdots e_n^*$, the two definitions are equal: $\langle \alpha, \beta \rangle_\lambda = \langle \alpha, \beta \rangle_\star$, for all α and β in ΛL^* .

For any Lie algebra, the *adjoint map* $\text{ad} : L \rightarrow \text{End}(L)$ is defined by $\text{ad}(x)y = [x, y]$. Let $\gamma \in L^*$ be such that $\langle \gamma, x \rangle = \text{Tr}(\text{ad}(x))$ for all $x \in L$.

Define $D(\alpha) = d\alpha - \gamma\alpha$. Note that $d\gamma = 0$, so $D^2 = 0$. While D is not a derivation, one has

$$(1) \quad D(\alpha\beta) = (D\alpha)\beta + (-1)^{|\alpha|}\alpha(d\beta) = (d\alpha)\beta + (-1)^{|\alpha|}\alpha(D\beta).$$

If $\alpha \in \Lambda^{n-1} L^*$, then $d\alpha = \gamma \cdot \alpha$ and so $D(\Lambda^{n-1} L^*) = \{0\}$.

Finally, define the linear map $\partial : \Lambda^j L \rightarrow \Lambda^{j-1} L$ by $\partial = (-1)^{n(j+1)+1} \star D \star$.

Definition 2. The Laplace-Beltrami operator is the map $\Delta = d\partial + \partial d$.

For ease of notation, let $\Delta_j := \Delta|_{\Lambda^j L^*}$.

While Beltrami was the first to generalise Laplacians to Riemannian manifolds, Hodge was the first to generalise Beltrami's Laplacian to all j -forms, for $0 \leq j \leq n$. Hence, the operator Δ is often called the *Hodge Laplacian* [Die, §2.VII.1D]. However, as in [ACK], Δ will be referred to as the Laplace-Beltrami operator to avoid the redundancy of two maps being called "Hodge maps".

The following results will be useful throughout this paper. Their proofs are from [ACK].

Lemma 3. [ACK, Lemma 2.2]

- (a) The map ∂ is the adjoint of d , that is to say $\langle d\alpha, \beta \rangle_* = \langle \alpha, \partial\beta \rangle_*$;
- (b) the Laplace-Beltrami operator Δ is self-adjoint, that is to say $\langle \Delta\alpha, \beta \rangle_* = \langle \alpha, \Delta\beta \rangle_*$;
- (c) $\ker \Delta = \ker d \cap \ker \partial$.

Proof. (a) Let $j \in \{0, \dots, n\}$ and let $\alpha \in \Lambda^j L^*$, $\beta \in \Lambda^{j+1} L^*$. We are required to show that $\star(d\alpha \cdot \star\beta) = \star(\alpha \cdot \star\partial\beta)$ or, equivalently, $(d\alpha) \cdot \star\beta = \alpha \cdot \star\partial\beta$. Since $\alpha \cdot \star\beta \in \Lambda^{n-1} L^*$, we have $D(\alpha \cdot \star\beta) = 0$ and so,

$$\begin{aligned} \alpha \cdot \star\partial\beta &= \alpha \cdot \star(-1)^{n(j+2)+1} \star D \star \beta = (-1)^{n(j+2)+1+j(n-j)} \alpha \cdot D \star \beta \\ &= -(-1)^j \alpha \cdot D \star \beta = d\alpha \cdot \star\beta, \text{ by equation (1).} \end{aligned}$$

(b) By part (a),

$$\begin{aligned} \langle \Delta\alpha, \beta \rangle_* &= \langle d\partial\alpha, \beta \rangle_* + \langle \partial d\alpha, \beta \rangle_* = \langle \partial\alpha, \partial\beta \rangle_* + \langle d\alpha, d\beta \rangle_* \\ &= \langle \alpha, d\partial\beta \rangle_* + \langle \alpha, \partial d\beta \rangle_* = \langle \alpha, \Delta\beta \rangle_*. \end{aligned}$$

(c) That $\ker d \cap \ker \partial \subseteq \ker \Delta$ is clear. Conversely, if $\alpha \in \ker \Delta$, then by part (a),

$$0 = \langle \Delta\alpha, \alpha \rangle_* = \langle d\partial\alpha + \partial d\alpha, \alpha \rangle_* = \langle \partial\alpha, \partial\alpha \rangle_* + \langle d\alpha, d\alpha \rangle_*,$$

which implies $\partial\alpha = 0$ and $d\alpha = 0$, as required. \square

Theorem 4 (The Hodge decomposition theorem for Lie algebras).

[ACK, Theorem 2.3]

- (a) $\Lambda L^* = \text{im } \Delta \oplus \ker \Delta$;
- (b) $\Lambda L^* = \text{im } d \oplus \text{im } \partial \oplus \ker \Delta$;
- (c) $\ker d = \text{im } d \oplus \ker \Delta$.

Proof.

- (a) First, we claim that $\text{im } \Delta \cap \ker \Delta = \{0\}$. To show this, suppose that $\beta \in \text{im } \Delta \cap \ker \Delta$. Then $\beta = \Delta\alpha$ for some α and $0 = \Delta\beta = \Delta^2\alpha$. Hence, by Lemma 3(b),

$$0 = \langle \Delta^2\alpha, \alpha \rangle_* = \langle \Delta\alpha, \Delta\alpha \rangle_*.$$

So $\beta = \Delta\alpha = 0$. Thus, $\text{im } \Delta \cap \ker \Delta = \{0\}$ and so $\Lambda L^* = \text{im } \Delta \oplus \ker \Delta$ for dimension reasons.

(b) Note that $\text{im } \Delta \subset \text{im } d + \text{im } \partial$, by the definition of Δ . So, by (a), it suffices to establish the following 3 conditions: (i) $\text{im } d \cap \text{im } \partial = \{0\}$, (ii) $\text{im } d \cap \ker \Delta = \{0\}$ and (iii) $\text{im } \partial \cap \ker \Delta = \{0\}$.

(i) Suppose that $\alpha = d\beta$ and $\alpha = \partial\gamma$ for some β and γ . Then, by Lemma 3(a), $\langle \alpha, \alpha \rangle_* = \langle d\beta, \partial\gamma \rangle_* = \langle d^2\beta, \gamma \rangle_* = 0$. Hence $\alpha = 0$ as required.

(ii) Suppose that $\alpha = d\beta$ for some β and that $\Delta\alpha = 0$. Then by Lemma 3(c), $\partial\alpha = 0$. So $0 = \langle \partial\alpha, \beta \rangle_* = \langle \partial d\beta, \beta \rangle_* = \langle d\beta, d\beta \rangle_*$, which gives $d\beta = 0$. Hence $\alpha = 0$ as required.

(iii) Suppose that $\alpha = \partial\beta$ for some β and that $\Delta\alpha = 0$. Then by Lemma 3(c), $d\alpha = 0$. So $0 = \langle d\alpha, \beta \rangle_* = \langle d\partial\beta, \beta \rangle_* = \langle \partial\beta, \partial\beta \rangle_*$, which gives $\partial\beta = 0$. Hence $\alpha = 0$ as required.

(c) By part (b), since $\text{im } d \oplus \ker \Delta \subset \ker d$, it suffices to show that $\ker d \cap \text{im } \partial = \{0\}$. Suppose that $\alpha \in \ker d$ and that $\alpha = \partial\beta$ for some $\beta \in \Lambda L^*$. Then

$$0 = \langle d\alpha, \beta \rangle_* = \langle \alpha, \partial\beta \rangle_* = \langle \partial\beta, \partial\beta \rangle_*.$$

Hence $\alpha = \partial\beta = 0$, as required. \square

Remark 5. *In the language of cohomology, Theorem 4(c) can be reworded as*

$$Z^k(L) = B^k(L) \oplus \ker \Delta_k, \text{ for } k = 0, \dots, n.$$

Hence, $\ker \Delta_k$ is canonically isomorphic to $H^k(L)$ for all Lie algebras L .

§3 HODGE \star ON COHOMOLOGY

The purpose of this section is to define a Hodge \star map on the cohomology of unimodular Lie algebras. First, we remark that on cocycles, $\Delta = d\partial$. We start with a couple of lemmas.

Lemma 6 (The Coboundary Lemma). *The mapping*

$$\Delta|_{B^*(L)} : B^*(L) \rightarrow B^*(L)$$

is an isomorphism. In fact, if $0 \leq k \leq n$, then $D\star$ restricted to the k -coboundaries is an isomorphism from $B^k(L)$ to $B^{n-k+1}(L)$.

Furthermore, $B^(L)$ is the perpendicular complement of $\ker \Delta$ in $Z^*(L)$, with respect to the inner product \langle, \rangle_* .*

Proof. By Theorem 4(c), $B^*(L) \cap \ker \Delta = \{0\}$, hence Δ restricted to the coboundaries is injective. Since $\text{im } (d\partial) \subseteq \text{im } d = B^*(L)$, then $\Delta(B^*(L)) \subseteq B^*(L)$. That $\Delta : B^*(L) \rightarrow B^*(L)$ is an isomorphism now follows by a dimension argument.

We show the existence of a left-inverse of $D\star$ restricted to coboundaries as follows, using the shorthand $\star|_{B^k(L)} = \star_k$. Since $d\partial_k = (-1)^{n(k+1)+1}d\star D\star_k$ is an isomorphism on the boundaries, $d\star D\star_k$ has a well-defined inverse; call this inverse φ . It follows that

$$(\varphi \circ d\star_{n-k+1}) \circ D\star_k = \varphi \circ (d\star D\star_k) = \text{id}_{B^k(L)}.$$

Hence,

$$D\star_k : B^k(L) \longrightarrow B^{n-k+1}(L)$$

has a left inverse, so is injective and $\dim B^k(L) \leq \dim B^{n-k+1}(L)$. Likewise, $D\star_{n-k+1} : B^{n-k+1}(L) \rightarrow B^k(L)$ is injective, so $\dim B^k(L) \geq \dim B^{n-k+1}(L)$. Thus, $\dim B^k(L) = \dim B^{n-k+1}(L)$ and $D\star_k$ is an isomorphism.

To prove the last part, suppose $\alpha \in \ker \Delta$ and $\beta \in B^*(L)$. Then, $\beta = \Delta\beta'$ for some $\beta' \in B^*(L)$ (by the first part of the lemma). Since Δ is self-adjoint,

$$\langle \beta, \alpha \rangle_\star = \langle \Delta\beta', \alpha \rangle_\star = \langle \beta', \Delta\alpha \rangle_\star = \langle \beta', 0 \rangle_\star = 0.$$

□

Consider the unimodular Lie algebras, that is to say those Lie algebras such that the trace of the adjoint map is always zero, so $\gamma = 0$. All nilpotent Lie algebras are unimodular, so for the purposes of this thesis (which deals mainly with nilpotent Lie algebras), $\partial = (-1)^{n(k+1)+1}\star d\star$ on $\Lambda^k L^*$.

Lemma 7. *For all unimodular Lie algebras L ,*

$$\ker \Delta = \{ \alpha \in Z^*(L) \mid \star\alpha \in Z^*(L) \} = Z^*(L) \cap \star Z^*(L).$$

Therefore, $\star|_{\ker \Delta} : \ker \Delta \rightarrow \ker \Delta$ is an isomorphism.

Proof. Suppose $\alpha \in \ker \Delta_j$. By Lemma 3(b), $\ker \Delta_j = \ker d_j \cap \ker \partial_j \subseteq Z^j(L)$. This implies $0 = \partial_j \alpha = (-1)^{n(j+1)+1}\star d(\star\alpha)$. Since \star is an isomorphism, $d(\star\alpha) = 0$ and $\star\alpha \in \ker d_{n-j} = Z^{n-j}(\alpha)$. Conversely, if $\alpha \in Z^j(L)$ is such that $\star\alpha \in Z^{n-j}(L)$, then

$$\Delta\alpha = d\partial(\alpha) = (-1)^{n(j+1)+1}d\star(d(\star\alpha)) = (-1)^{n(j+1)+1}d\star(0) = 0.$$

That \star is an automorphism of $\ker \Delta$ follows from its injectivity and a dimension argument. □

Definition 8.

(i) For any Lie algebra L , let

$$\pi : Z^*(L) = B^*(L) \oplus \ker \Delta \longrightarrow \ker \Delta$$

be the projection map onto $\ker \Delta$.

(ii) Now define $\bar{\star} : H^*(L) \rightarrow H^*(L)$ by $\bar{\star}[\alpha] = [\star\pi(\alpha)]$.

To see that $\bar{\star}$ is well-defined, it suffices to notice that π , hence $\star\pi$, sends coboundaries to 0 and that $\star\pi$ sends cocycles to cocycles, by Lemma 7.

Theorem 9. For all unimodular Lie algebras L , $\bar{\star}$ is an isomorphism with the property

$$\bar{\star}\bar{\star}|_{H^j(L)} = (-1)^{j(n-j)} \cdot \text{id}_{H^j(L)},$$

where $n = \dim L$.

Proof. We note that $\text{im}(\star\pi) \subseteq \ker \Delta$, by Lemma 7. It follows that $\pi(\star\pi) = \star\pi$, hence

$$\bar{\star}\bar{\star}[\alpha] = [\star\pi \star\pi(\alpha)] = [\star\star\pi(\alpha)] = [\omega\pi(\alpha)].$$

That $[\alpha] = [\pi(\alpha)]$ follows from Theorem 4(c), which tells us any cocycle α can be uniquely written as $\beta + \pi(\alpha)$, for some $\beta \in B^*(L)$, so $\alpha - \pi(\alpha) \in B^*(L)$. \square

We end this section by proving Theorem I.10 from Chapter I. Recall that for a Lie algebra L , $b_j = \dim H^j(L)$ is the j^{th} Betti number. Since all nilpotent Lie algebras are unimodular, we restate the theorem in more generality, as a corollary to Theorem 9:

Theorem 10. Unimodular Lie algebras enjoy Poincaré duality: $b_j = b_{n-j}$ when $0 \leq j \leq n$, where n is the dimension of the Lie algebra. In particular, $b_0 = b_n = 1$.

Proof. The first part follows from the isomorphism $\bar{\star} : H^j(L) \rightarrow H^{n-j}(L)$ given in Theorem 9. It has already been shown (in Chapter I) that $b_0 = 1$ for all Lie algebras, so the second part follows from the first. \square

In particular, if L is unimodular, then $H^*(L)$ is a Poincaré duality algebra. Recall:

Definition I.11. A Poincaré duality algebra is a finite dimensional positively graded associative algebra $A = \bigoplus_{p=0}^n A^p$ such that

- (i) $\dim A^n = 1$ and
- (ii) For some basis vector $e^* \in (A^n)^*$, the bilinear functions

$$\langle \cdot, \cdot \rangle' : A^p \times A^{n-p} \longrightarrow \mathbb{F}$$

given by

$$\langle a, b \rangle' = \langle e^*, ab \rangle, \quad a \in A^p, b \in A^{n-p}$$

are non-degenerate.

That multiplication $H^j(L) \times H^{n-j}(L) \rightarrow H^n(L)$ is non-degenerate follows from that fact that $[\alpha] = [\pi\alpha] \in H^j(L)$ implies $\bar{\star}[\alpha] = [\star\pi\alpha] \in H^{n-j}(L)$, and $[\alpha] \cdot \bar{\star}[\alpha] = [\pi\alpha \cdot \star\pi\alpha] \in H^n(L)$ is non-zero since $0 \neq \langle \pi\alpha, \pi\alpha \rangle_\star = \star(\pi\alpha \cdot \star\pi\alpha)$ implies $0 \neq \pi\alpha \cdot \star\pi\alpha \in Z^n(L) = \Lambda^n L^\star \cong H^n(L)$, hence $[\pi\alpha \cdot \star\pi\alpha]$ is non-trivial in $H^n(L)$.

§4 THE LEFSCHETZ RESULT

In this section, we concentrate on \mathfrak{h}_m , the m^{th} Heisenberg Lie algebra, with basis $\{z, x_1, y_1, \dots, x_m, y_m\}$ such that $[x_j, y_j] = z$ for $j = 1, \dots, m$.

Suppose that $\{z^*, x_1^*, y_1^*, \dots, x_m^*, y_m^*\}$ is the ordered dual basis. Say \star is the Hodge star map on $\Lambda\mathfrak{h}_m^*$ with respect to this ordered basis.

Let U be the subspace of $\Lambda\mathfrak{h}_m^*$ with ordered basis $\{x_1^*, y_1^*, \dots, x_m^*, y_m^*\}$, so that $dU = \{0\}$. Say \star is the Hodge star map on ΛU with respect to this ordered basis. Thus, $\Lambda\mathfrak{h}_m^* = \Lambda U \oplus z^* \Lambda U$ and, if $\alpha \in \Lambda U$, then

$$\star\alpha = (-1)^{|\alpha|} z^* \star\alpha.$$

Let $\Omega = dz^* = \sum_j x_j^* y_j^*$. Hence,

$$\Omega^m = \underbrace{\Omega \cdots \Omega}_{m \text{ times}} = m! (x_1^* y_1^* x_2^* y_2^* \cdots x_m^* y_m^*).$$

The purpose of this section is to understand how multiplication by Ω acts on ΛU .

The main results of this section are taken from [Wel]. We will be working in vector spaces over the field $\mathbb{F}(i)$ the field extension of \mathbb{F} by i , where i is such that $i^2 = -1$. Define, $U_c := U \otimes_{\mathbb{F}} \mathbb{F}(i)$.

If the field \mathbb{F} already contains an element i that squares to -1 (for example, $\mathbb{F} = \mathbb{C}$), then $\mathbb{F} = \mathbb{F}(i)$ and $U_c = U$. Otherwise, U_c is like the complexification of U . The results we require, once they've been proven over $\mathbb{F}(i)$, can then be generalised to hold for ΛU defined over \mathbb{F} . In [Wel], it is assumed that the field \mathbb{F} is \mathbb{R} , hence $\mathbb{C} = \mathbb{F}(i)$. However, the proofs only require a field of characteristic 0.

Consider $E^{1,0}$ and $E^{0,1}$, the vector subspaces of U_c with dimension m and respective bases $\{z_1, \dots, z_m\}$ and $\{\bar{z}_1, \dots, \bar{z}_m\}$, where each $z_j = x_j^* + iy_j^*$ and $\bar{z}_j = x_j^* - iy_j^*$. One can show that $U_c = E^{1,0} \oplus E^{0,1}$.

The decomposition $U_c = E^{1,0} \oplus E^{0,1}$ gives a bigradation $\Lambda U_c = \bigoplus_{p,q} E^{p,q}$, where $E^{p,q} = \Lambda^p E^{1,0} \otimes \Lambda^q E^{0,1}$. Each $E^{p,q}$ has a basis

$$\{ z_{\mu_1} \cdots z_{\mu_p} \bar{z}_{\nu_1} \cdots \bar{z}_{\nu_q} \mid 1 \leq \mu_1 < \cdots < \mu_p \leq m, 1 \leq \nu_1 < \cdots < \nu_q \leq m \}.$$

Define $\Pi_{p,q}$ to be the projection map $\Lambda U_c \rightarrow E^{p,q}$ and let $C = \sum_{p,q} i^{p-q} \Pi_{p,q}$.

Note that $\{x_1^*, y_1^*, \dots, x_m^*, y_m^*\}$ is another basis for U_c . We can therefore extend the definition of $\underline{\star}$ on ΛU to ΛU_c by defining $\underline{\star}(i\alpha) := i\underline{\star}(\alpha)$ for all $\alpha \in \Lambda U$. Thus, $\underline{\star}$ becomes an $\mathbb{F}(i)$ -linear Hodge star map on ΛU_c (if it wasn't already) defined with respect to the basis $\{x_1^*, y_1^*, \dots, x_m^*, y_m^*\}$, and not with respect to the basis $\{z_1, \bar{z}_1, \dots, z_m, \bar{z}_m\}$.

If we consider $\sum_j z_j \bar{z}_j$, we notice

$$\Omega = \sum_{j=1}^m x_j^* y_j^* = \frac{i}{2} \sum_{j=1}^m z_j \bar{z}_j.$$

Recall the natural projection map, $\Pi_j : \Lambda U_c \rightarrow \Lambda^j U_c$ with $\omega = \sum_j (-1)^{2mj+j} \Pi_j = \sum_j (-1)^j \Pi_j$, an automorphism on ΛU_c such that $\omega^2 = 1$. Define $\Psi : \Lambda U_c \rightarrow \Lambda U_c$ by $\Psi(\alpha) = \Omega \cdot \alpha$ and the adjoint map $\Gamma := \Psi^*$ (adjoint with respect to the Hermitian inner product $\langle \cdot, \cdot \rangle_{\underline{\star}}$). One can show that $\Gamma = \omega \underline{\star} \Psi \underline{\star}$, given $\underline{\star} \underline{\star} = \omega$ and (as it is straightforward to show) $\underline{\star} \omega = \omega \underline{\star}$.

It is necessary, for what follows, to introduce some multi-indexing notation. If

$$\mu = \{\mu_1, \dots, \mu_k\} \subseteq \{1, \dots, m\},$$

then it will be understood that the μ_j 's are in increasing order and x_{μ}^*, y_{μ}^* and z_{μ} will denote the products

$$x_{\mu_1}^* \cdots x_{\mu_k}^*, y_{\mu_1}^* \cdots y_{\mu_k}^* \text{ and } z_{\mu_1} \cdots z_{\mu_k},$$

respectively. Let $|\mu| = k$ denote the length of μ and let $\mu' = \{1, \dots, m\} - \mu$ (set difference).

We let

$$w_{\mu} = \prod_{j \in \mu} z_j \bar{z}_j = (-2i)^{|\mu|} \prod_{j \in \mu} x_j^* y_j^*$$

and note that the ordering of the factors in this last product is irrelevant.

Any element of ΛU_c can be uniquely written in the form

$$\sum_{\mu, \nu, \tau} c_{\mu, \nu, \tau} \cdot z_{\mu} \bar{z}_{\nu} w_{\tau},$$

where $c_{\mu, \nu, \tau} \in \mathbb{F}(i)$ and the sum is over all mutually disjoint multi-indices μ, ν and τ . This is because $\{z_1, \dots, z_m, \bar{z}_1, \dots, \bar{z}_m\}$ is a basis for U_c , so $\{z_{\mu} \bar{z}_{\nu} \mid \mu \text{ and } \nu \text{ multi-indices}\}$ is a basis for ΛU_c . If $\tau = \mu \cap \nu$, then $z_{\mu} \bar{z}_{\nu} = (\pm) z_{\mu-\tau} \bar{z}_{\nu-\tau} w_{\tau}$. Hence an alternative basis for ΛU_c is $\{z_{\mu} \bar{z}_{\nu} w_{\tau} \mid \mu, \nu \text{ and } \tau \text{ are pairwise disjoint}\}$.

Lemma 12. [Wel, Lemma V.1.2] *Suppose μ , ν and τ are mutually disjoint multi-indices. Then*

$$\star(z_\mu \wedge \bar{z}_\nu w_\tau) = \gamma(a, b, c) z_\mu \bar{z}_\nu \wedge w_{(\mu \cup \nu \cup \tau)},$$

for a non-vanishing constant $\gamma(a, b, c) \in \mathbb{F}(i)$, with $a = |\mu|$, $b = |\nu|$ and $c = |\tau|$. Moreover,

$$\gamma(a, b, c) = i^{a-b} (-1)^{\binom{p+1}{2} + c} (-2i)^{p-m}$$

where $p = a + b + 2c$ is the total degree of $z_\mu \bar{z}_\nu w_\tau$.

Proof. The proof can be found in [Wel]. \square

For any two endomorphisms f and g , their commutator $fg - gf$ is denoted $[f, g]$.

Proposition 13. [Wel, Proposition V.1.1] *Let Ω , \star , ω , C , Ψ and Γ be as above. Then,*

- (i) $\star \Pi_{p,q} = \Pi_{m-q, m-p} \star$;
- (ii) $[\Psi, \omega] = [\Gamma, \omega] = [\Psi, C] = [\Gamma, C] = 0$;
- (iii) $[\Gamma, \Psi] = \sum_{j=0}^{2m} (m-j) \Pi_j$.

Proof. Part (i) follows from Lemma 12.

Part (ii) follows from that fact that Ψ and Γ are homogeneous $\mathbb{F}(i)$ -operators.

Observe that

$$\Psi(z_\mu \bar{z}_\nu w_\tau) = \frac{i}{2} \left(\sum_{j=1}^m z_j \bar{z}_j \right) z_\mu \bar{z}_\nu w_\tau = \frac{i}{2} z_\mu \bar{z}_\nu \left(\sum_{j \in (\mu \cup \nu \cup \tau)'} w_{\tau \cup \{j\}} \right),$$

On the other hand, we can show, using Lemma 12 and tedious, straightforward computations, that

$$\Gamma(z_\mu \bar{z}_\nu w_\tau) = \frac{2}{i} z_\mu \bar{z}_\nu \left(\sum_{j \in \tau} w_{\tau - \{j\}} \right).$$

Using these formulas and assuming $z_\mu \bar{z}_\nu w_\tau$ has total degree p , one can show

$$(\Gamma\Psi - \Psi\Gamma) z_\mu \bar{z}_\nu w_\tau = (m-p) z_\mu \bar{z}_\nu w_\tau,$$

and part (iii) of the Proposition follows. \square

Definition 14. *Say $v \in \Lambda^p U_c$ is a primitive p -vector if $\Gamma v = 0$; i.e. if $\Psi \star v = 0$.*

For $k \in \mathbb{Z}$, let $k^+ = \max\{k, 0\}$.

Theorem 15. [Wel, Theorem V.1.4 and Corollary V.1.5]

(i) If v is a primitive p -vector, then $\Psi^q v = 0$ when $q \geq (m - p + 1)^+$. Furthermore, if $0 \leq s \leq r$, then $[\Gamma^s, \Psi^r]v = \alpha_{r,s,p} \Psi^{r-s} v$ for some integer $\alpha_{r,s,p}$. In particular, $[\Gamma^r, \Psi^r]v = c_{r,p} v$, where

$$c_{r,p} = r!(m - p - r + 1)(m - p - r + 2) \cdots (m - p) = \frac{r!(m - p)!}{(m - p - r)!},$$

when $r \geq 1$, and $c_{0,p} = 0$.

(ii) There are no non-trivial primitive vectors of degree $p > m$.

Proof. Again, the proof is in [Wel]. \square

Theorem 16. [Wel, Theorem V.1.6]

Let v be a primitive p -vector and assume $0 \leq r \leq m - p$. Then

$$\star \Psi^r v = \eta(p, r, m) \Psi^{m-p-r} C v,$$

where $\eta(p, r, m)$ is a non-vanishing constant depending only on p , r and n , and $\star \Psi^r v = 0$, if $r > m - p$. Moreover,

$$\eta(p, r, m) = (-1)^{p(p+1)/2} \frac{r!}{(m - p - r)!}.$$

Proof. In [Wel]. \square

Theorem 17 (Lefschetz's primitive decomposition theorem).

[Wel, Theorem V.1.8] Let $v \in \Lambda^p U_c$. Then one can uniquely write v in the form

$$(2) \quad v = \sum_{r \geq (p-m)^+} \Psi^r v_r,$$

where for each $r \geq (p - m)^+$, v_r is a primitive $(p - 2r)$ -vector. Moreover, each v_r can be expressed in the form $v_r = \sum_s a_{p,r,s} \Psi^s \Gamma^{r+s} v$, where $a_{p,r,s} \in \mathbb{Q} \hookrightarrow \mathbb{F}$.

A partial proof can be found in [Wel]. However, the general case for existence is left as an exercise. We therefore include the following completed proof.

Proof. For the uniqueness of representation (2), it suffices to show that if, for any n such that $\frac{p-2n}{2} \geq 0$,

$$w_0 + \Psi^1 w_1 + \cdots + \Psi^n w_n = 0,$$

where the w_r are primitive $(p-2r)$ -forms, then each $w_r = 0$. We can assume $p \leq m$. By applying Γ^n , we obtain

$$(3) \quad \Gamma^n w_0 + \Gamma^n \Psi^1 w_1 + \cdots + \Gamma^n \Psi^n w_n = 0.$$

Since w_r is primitive, we can use Theorem 15 to obtain $\Gamma^r \Psi^r w_r = c_{r,p-2r} w_r$ where $c_{r,p-2r}$ is a non-zero constant equal to $[\Gamma^r, \Psi^r]$ acting on a primitive $(p-2r)$ -vector (see Theorem 15). We can rewrite (3) as

$$\begin{aligned} \Gamma^n w_0 + \Gamma^{n-1}(\Gamma \Psi w_1) + \Gamma^{n-2}(\Gamma^2 \Psi^2 w_2) + \cdots + \Gamma^n(\Psi^n w_n) \\ = \Gamma^n w_0 + c_{1,p-2} \Gamma^{n-1} w_1 + \cdots + c_{n,p-2n} w_n = 0. \end{aligned}$$

Since the w_r are assumed to be primitive, all terms vanish, and we are left with $c_{n,p-2n} w_n = 0$, and hence $w_n = 0$. Similarly, one can show each $w_{n-k} = 0$ by induction on k .

For the proof of existence, write c_r as shorthand for $c_{r,p-2r}$. Let $p \in \{0, \dots, 2m\}$ and let $n = \lfloor \frac{p}{2} \rfloor$. Suppose $v \in \Lambda^p U_c$. Starting with v_n and working down to v_0 , inductively define, for $r = 1, \dots, n$,

$$v_r = c_r^{-1} \Gamma^r \left(v - \sum_{j=r+1}^n \Psi^j v_j \right)$$

and $v_0 = v - \sum_{j=1}^n \Psi^j v_j$. With v of the desired form,

$$v = v_0 + \Psi^1 v_1 + \Psi^2 v_2 + \cdots + \Psi^n v_n,$$

it suffices to show that each v_r is primitive and of the form $\sum_s a_{p,r,s} \Psi^s \Gamma^{r+s} v$.

That $v_n = c_n^{-1} \Gamma^n v$ is primitive is straightforward: it is of degree $p - 2n = p - 2\lfloor \frac{p}{2} \rfloor = 0$ or 1 , and Γ is homogeneous of degree -2 . Also, v_n is trivially of the correct form.

One can show that each v_r is primitive, for $r = 1, \dots, n$, by assuming v_{r+1} is primitive and applying Γ to v_r :

$$\begin{aligned} \Gamma v_r &= c_r^{-1} \left(\Gamma^{r+1} v - \Gamma^{r+1} \sum_{j=r+1}^n \Psi^j v_j \right) \\ &= c_r^{-1} \left(\underbrace{-\Gamma^{r+1} \Psi^{r+1} v_{r+1}}_{=c_{r+1} v_{r+1}} + \underbrace{\Gamma^{r+1} \left(v - \sum_{j=r+2}^n \Psi^j v_j \right)}_{=c_{r+1} v_{r+1}} \right) = 0. \end{aligned}$$

That each v_r is of the desired form can be shown by assuming that for each $j \geq r+1$, v_j is primitive and of the desired form, and $\Gamma^r \Psi^j v_j = [\Gamma^r \Psi^j] v_j = \alpha v_j$ for some integer α , by Theorem 15(i).

That v_0 is primitive and of the desired form must be shown separately, but only because $c_0 = 0$. The proof is similar to the cases c_1, \dots, c_{n-1} . \square

Corollary 18. [Wel, Corollary V.1.9] *Suppose that $v \in \Lambda^p U_c$.*

(i) *If $\Psi^k v = 0$, then the vectors v_r appearing in the primitive decomposition of v from Theorem 17 vanish if $r \geq (p - m + k)^+$; i.e.,*

$$v = \sum_{r=(p-m)^+}^{(p-m+k)^+-1} \Psi^r v_r.$$

(ii) *If $p \leq m$ and $\Psi^{m-p} v = 0$, then $v = 0$. In other words,*

$$\Psi^{m-p} : \Lambda^p U_c \longrightarrow \Lambda^{2m-p} U_c$$

is an isomorphism.

Proof. By Theorem 17,

$$0 = \Psi^k v = \sum_{r \geq (p-m)^+} \Psi^{k+r} v_r.$$

By Theorem 16, since v_r is primitive, $\Psi^q v_r = 0$ if $q \geq (m - (p - 2r) + 1)^+$, which implies that $\Psi^{r+k} v_r = 0$ if $r < (p - m + k)$. So we have

$$0 = \sum_{r \geq (p+k-m)^+} \Psi^{k+r} v_r.$$

The total degree of each term in this expression is $2k + p$, and thus we have a primitive decomposition of the $(p + 2k)$ -vector 0 given by

$$0 = \sum_{q \geq (p+2k-m)^+} \Psi^q v_{q-k},$$

from which it follows from the uniqueness of such a decomposition that each $v_{q-k} = 0$, $q \geq (p + 2k - m)^+$. In other words, $v_r = 0$, $r \geq (p + k - m)^+$, as desired for part (i).

The injectivity of Ψ^{m-p} in part (ii) is a special case of part (i): letting each $k = m - p = -(p - m)$, it follows that $v = \sum_{r=0}^{-1} \Psi^r v_r$ is trivial. The isomorphism follows from a dimension argument. \square

Corollary 19. [Wel, Corollary V.1.10]

Let v be in $\Lambda^p U_c$. Then v is primitive if and only if the following two conditions hold:

- (i) $p \leq m$ and
- (ii) $\Psi^{m-p+1}v = 0$.

Proof. This follows from the primitive decomposition of Corollary 18(a). If v is primitive, then $v = v_0$, and p must be less than or equal to m , and by Theorem 15(i), $\Psi^{m-p+1}v = 0$. Conversely, supposing (i) and (ii) are satisfied, there is only one term in the primitive decomposition given by Corollary 18(a), i.e.

$$v = \sum_{r=(p-n)^+}^{(p-n+n-p+1)^+-1} \Psi^r v_r = \sum_{r=0}^0 \Psi^r v_r = v_0.$$

Hence, $v = v_0$ is primitive. \square

The results of the last two corollaries still hold if we restrict ourselves to the underlying \mathbb{F} -algebra ΛU in ΛU_c . If \mathbb{F} contains no i such that $i^2 = -1$, then $\Lambda U_c = \Lambda U \oplus_{\mathbb{F}} i\Lambda U$ and it is straightforward to verify that $\Psi = \Psi_1 \oplus \Psi_2$ where $\Psi_1 : \Lambda U \rightarrow \Lambda U$ sends α to $\Omega \cdot \alpha$ and where $\Psi_2 : i\Lambda U \rightarrow i\Lambda U$ sends $i\alpha$ to $i(\Omega \cdot \alpha)$, for all $\alpha \in \Lambda U$. We can therefore drop assumptions about the field \mathbb{F} and summarise the important results from Corollaries 18(b) and 19 as follows:

Theorem 20 (The Lefschetz theorem). Let \mathbb{F} be any field of characteristic zero. Let U be the vector space over the field \mathbb{F} with ordered basis $\{x_1^*, y_1^*, \dots, x_m^*, y_m^*\}$. Let $\Psi : \Lambda U \rightarrow \Lambda U$ denote multiplication by $\sum_j x_j^* y_j^*$ and let \star denote the Hodge star map on ΛU defined with respect to the ordered basis. Then

- (i) the map

$$\Psi^k : \Lambda^{m-k} U \longrightarrow \Lambda^{m+k} U$$

is an isomorphism for $k = 0, \dots, m$;

- (ii) as subspaces of $\Lambda^{m-k} U$,

$$\ker \Psi^{k+1} = \ker (\Psi \star)$$

for $k = 0, \dots, m$;

- (iii) $\Psi : \Lambda^p U \rightarrow \Lambda^{p+2} U$ is onto when $p \geq m$ and is injective when $p < m$.

Proof. The first two statements are, respectively, Corollary 18(b) and Corollary 19. The third statement follows from (i). \square

We end this section by giving an explicit structure of the subspace $\ker \Gamma$ of primitive vectors. This is not a result from [Wel], but it is included for the sake of completeness. First we note that Γ is not a derivation. If it was, then $0 = \Gamma(w^2) = (\Gamma w)w + w(\Gamma w) = 2w$, hence $w = 0$, for all $w \in \Lambda^1 U_c$. However, Γ does act like a derivation in the following sense:

Lemma 21. *Suppose σ and τ are multi-indices such that $\sigma \cap \tau = \emptyset$, $\alpha \in \text{span}\{w_\mu \mid \mu \subseteq \tau\}$ and $\beta \in \text{span}\{w_\mu \mid \mu \subseteq \sigma\}$. Then*

$$(4) \quad \Gamma(\alpha\beta) = (\Gamma\alpha)\beta + \alpha(\Gamma\beta).$$

In other words, if α and β don't have common indices, then Γ acts like a derivation on their product.

Proof. By the bilinearity of both sides of equation (4), it suffices to check on basis vectors. Suppose $\alpha = w_\mu$ and $\beta = w_\nu$ where $\mu \cap \nu = \emptyset$. Then

$$\begin{aligned} \Gamma(\alpha\beta) &= \frac{2}{i} \sum_{j \in \mu \cup \nu} w_{(\mu \cup \nu) - j} = \frac{2}{i} \sum_{j \in \mu} w_{(\mu - \{j\}) \cup \nu} + \frac{2}{i} \sum_{j \in \nu} w_{\mu \cup (\nu - \{j\})} \\ &= (\Gamma w_\mu) w_\nu + w_\mu (\Gamma w_\nu) = (\Gamma\alpha)\beta + \alpha(\Gamma\beta) \end{aligned}$$

□

Theorem 22. *The following is a spanning set for all primitive vectors in $\Lambda^p U$, when $p \leq m$: all $z_\mu \bar{z}_\nu \prod_{a=1}^l (w_{j_a} - w_{k_a})$ such that $|\mu \cup \nu \cup \{j_1, k_1, \dots, j_l, k_l\}| = p$, $\mu \cap \nu = \emptyset$, $|\mu \cup \nu| = p - 2l$, and where the pair $j_a < k_a$ precedes $j_b < k_b$ in the list $\{j_1, k_1, \dots, j_l, k_l\}$ iff $j_a < j_b$. When $p > m$, the only primitive in $\Lambda^p U$ is 0.*

Proof. That there are no primitives in $\Lambda^p U$ when $p > m$ is Corollary 19, so suppose $p \leq m$. That the elements of the above form are primitive follows from Lemma 21 and the fact that the indices are all disjoint:

$$\begin{aligned} &\Gamma \left(z_\mu \bar{z}_\nu \prod_{a=1}^l (w_{j_a} - w_{k_a}) \right) \\ &= z_\mu \bar{z}_\nu \sum_{a=1}^l \Gamma(w_{j_a} - w_{k_a}) \left((w_{j_1} - w_{k_1}) \cdots \widehat{(w_{j_a} - w_{k_a})} \cdots (w_{j_l} - w_{k_l}) \right), \end{aligned}$$

where $\Gamma(w_{j_a} - w_{k_a}) = \frac{2}{i} - \frac{2}{i} = 0$.

The proof of the other inclusion proceeds by induction on m , where the case $m = 1$ is straightforward. Let $U_m = \text{span}\{x_1^*, y_1^*, \dots, x_{m-1}^*, y_{m-1}^*\}$ and suppose the theorem holds on ΛU_m . If $\alpha \in \Lambda^p U$, then $\alpha = aw_m + b + \gamma_1 x_m^* + \gamma_2 y_m^*$ for some $a \in \Lambda^{p-2} U_m$, $b \in \Lambda^p U_m$ and $\gamma_1, \gamma_2 \in \Lambda^{p-1} U_m$. Thus $\Gamma\alpha = \Gamma(a)w_m + a + \Gamma(b) + \Gamma(\gamma_1)x_m^* + \Gamma(\gamma_2)y_m^*$. If, furthermore, α is primitive, then $\Gamma(a) = 0$, $a = -\Gamma(b)$ and $\Gamma(\gamma_1) = \Gamma(\gamma_2) = 0$. Hence, $\alpha = b - \Gamma(b)w_m + \gamma_1 x_m^* + \gamma_2 y_m^*$, where $\gamma_1 x_m^* + \gamma_2 y_m^*$ is of the desired form, by induction hypothesis. Therefore, it suffices to show $b - \Gamma(b)w_m$ is of the desired form.

By the induction hypothesis,

$$(5) \quad \Gamma(b) = \sum c_{\mu,\nu,j,k} \cdot z_\mu \bar{z}_\nu (w_{j_1} - w_{k_1}) \cdots (w_{j_l} - w_{k_l})$$

the sum over all disjoint multi-indices μ, ν and $j \cup k$ in $\{1, \dots, m-1\}$, where $j \cup k = \{j_1, k_1, \dots, j_l, k_l\}$ are as in the statement of the theorem, and where $c_{\mu,\nu,j,k}$ are scalars. Note that, for degree reasons, $|\mu| + |\nu| + 2l \leq p-2$, hence $|\mu| + |\nu| + 2l < m-1$. Thus, $\mu \cup \nu \cup j \cup k \neq \{1, \dots, m-1\}$, and therefore, for each summand in equation (5), one can choose an index $b_{\mu,\nu,j,k}$ such that $b_{\mu,\nu,j,k} \notin \mu \cup \nu \cup j \cup k$. Let

$$b' := \sum c_{\mu,\nu,j,k} \cdot z_\mu \bar{z}_\nu (w_{j_1} - w_{k_1}) \cdots (w_{j_l} - w_{k_l}) w_{b_{\mu,\nu,j,k}}.$$

It is straightforward to check, using Lemma 21, that $\Gamma(b) = \Gamma(b')$. Hence, $b = b' + \kappa$, where $\kappa \in \ker \Gamma \cap \Lambda U_m$, hence is also of the desired form. Therefore,

$$\begin{aligned} b - \Gamma(b)w_m &= -\Gamma(b')w_m + b' + \kappa \\ &= -\sum c_{\mu,\nu,j,k} \cdot z_\mu \bar{z}_\nu (w_{j_1} - w_{k_1}) \cdots (w_{j_l} - w_{k_l})w_m + \\ &\quad \sum c_{\mu,\nu,j,k} \cdot z_\mu \bar{z}_\nu (w_{j_1} - w_{k_1}) \cdots (w_{j_l} - w_{k_l})w_{b_{\mu,\nu,j,k}} + \kappa \\ &= \sum c_{\mu,\nu,j,k} \cdot z_\mu \bar{z}_\nu (w_{j_1} - w_{k_1}) \cdots (w_{j_l} - w_{k_l})(w_{b_{\mu,\nu,j,k}} - w_m) + \kappa. \end{aligned}$$

□

§5 PROOF OF THEOREM 1 AND SANTHAROUBANE'S THEOREM

First, we prove the following theorem from [San] on the structure of the cohomology of the Heisenberg Lie algebras. We start by recalling that $\Lambda \mathfrak{h}_m^* = \Lambda U \oplus z^* \Lambda U$ and, if $\alpha \in \Lambda^l U$, then $\star \alpha = (-1)^l z^* \underline{\alpha}$, hence

$$d \star \alpha = (-1)^{|\alpha|} \Psi \underline{\alpha}.$$

Theorem 23 (Santharoubane's Theorem). *Let $m \in \mathbb{N}$. Then*

- (i) *the map $\Psi : \Lambda^{k-2}U \rightarrow B^k(\mathfrak{h}_m)$ is an isomorphism for $k = 0, \dots, m$, and $B^p(\mathfrak{h}_m) = \Lambda^p U$ when $p > m$;*
- (ii) *$Z^k(\mathfrak{h}_m) = \Lambda^k U$ for $k = 0, \dots, m$;*
- (iii) *the first m Betti numbers of \mathfrak{h}_m are $b_0 = 1$, $b_1 = 2m$ and, for $j = 2, \dots, m$, $b_j = \binom{2m}{j} - \binom{2m}{j-2}$.*

Proof. Any $\alpha \in \Lambda \mathfrak{h}_m^*$ can be uniquely written as $\alpha = \alpha' + z^* \alpha''$ for some α' and α'' in ΛU . With $d\alpha' = 0$ and $d(z^* \alpha'') = \Psi(\alpha'')$, we find $\Psi : \Lambda U \rightarrow B^*(\mathfrak{h}_m)$ is an onto map and $\Lambda U \subseteq Z^*(L)$. Theorem 20(iii) therefore implies part (i). To see part (ii), note that for any $\beta \in \Lambda^k U$ with $0 \leq k \leq m-1$, $z^* \beta$ is not a cocycle: $d(z^* \beta) = \Psi(\beta) \neq 0$, by Theorem 20(iii). Finally, part (iii) follows from (i) and (ii), since $b_k = \dim H^k(\mathfrak{h}_m) = \dim Z^k(\mathfrak{h}_m) - \dim B^k(\mathfrak{h}_m) = \dim \Lambda^k U - \dim \Lambda^{k-2} U$ for $k = 0, \dots, m$. □

Note that the remaining Betti numbers of \mathfrak{h}_m can be deduced by Poincaré duality.

Lemma 24. For $k = 0, \dots, m$, $\ker \Delta_{m-k} = \Lambda^{m-k}U \cap \ker \Psi^{k+1}$.

Proof. If $\alpha \in \Lambda^{m-k}U \cap \ker \Psi^{k+1}$, then by Theorem 20(ii), $\alpha \in \ker(\Psi_{\star})$. However, with $d \star \alpha = (-1)^{m-k} \Psi_{\star} \alpha$, we have

$$\Delta_{m-k} \alpha = d \partial_{m-k} \alpha = (-1)^{(2m+1)(m-k+1)+1} d \star d \star \alpha = d \star \Psi_{\star} \alpha = 0.$$

Thus, $\Lambda^{m-k}U \cap \ker \Psi^{k+1} \subseteq \ker \Delta_{m-k}$.

Conversely, if $\alpha \in \ker \Delta_{m-k}$, then $\alpha \in \Lambda^{m-k}U$, by Theorem 23(ii). Furthermore, $\star \alpha \in Z^{m+k+1}(\mathfrak{h}_m)$, by Lemma 7, hence $0 = d \star \alpha = (-1)^{m-k} \Psi_{\star} \alpha$. Therefore, by Theorem 20(ii), $\alpha \in \ker \Psi^{k+1}$. \square

To prove Theorem 1, the reader is reminded of the definitions of $\text{op}_k\{x\}$, $W_j^k(L; x)$ and $K_j^k(L; x)$, for an arbitrary Lie algebra L with $x \in Z(L)$, as defined in Chapter II. Below, we assume that op_k is defined on $H^*(\mathfrak{h}_m)$ with respect to the z in the basis of \mathfrak{h}_m .

Lemma 25. When $0 \leq j \leq k \leq m$, $W_j^{m-k} = Z^{m-k}(\mathfrak{h}_m)$. Moreover, $i_z(W_j^{m-k}) = \{0\}$.

Proof. The second statement follows immediately from the first, since when $m-k \leq m$, we then have $Z^{m-k}(\mathfrak{h}_m) = \Lambda^{m-k}U$, by Theorem 23(ii). Since each $W_j \subseteq W_{j+1}$, we have

$$Z^{m-k}(\mathfrak{h}_m) = W_0^{m-k} \subseteq W_1^{m-k} \subseteq \dots \subseteq W_k^{m-k},$$

so it suffices to show that $W_k^{m-k} \subseteq Z^{m-k}(\mathfrak{h}_m)$. By definition, $\alpha_k \in W_k^{m-k}$ iff there exists a $(k+1)$ -tuple

$$(\alpha_0, \dots, \alpha_k) \in W_0^{m+k} \times W_1^{m+k-2} \times \dots \times W_k^{m-k}$$

such that $i_z \alpha_j = d \alpha_{j+1}$ for $j = 0, \dots, k-1$. Note that $d \alpha_0 = 0$. Since $d \alpha = dz^* \cdot i_z \alpha$, we have $\Psi^l(i_z \alpha_k) = i_z \alpha_{k-l}$ for $l = 0, \dots, k$ and $\Psi^k(i_z \alpha_k) = d \alpha_0 = 0$. However, $\Psi^k : \Lambda^{m-k}U \rightarrow \Lambda^{m+k}U$ is injective, so $i_z \alpha_k = 0$ and, consequently, $\alpha_k \in \Lambda^{m-k}U = Z^{m-k}(\mathfrak{h}_m)$. \square

Recall the projection map $\pi : Z^*(\mathfrak{h}_m) \rightarrow \ker \Delta$ from Definition 8. Say $\pi_l := \pi|_{Z^l(\mathfrak{h}_m)}$. Note that $[\alpha] = [\pi(\alpha)]$ whenever $\alpha \in Z^*(\mathfrak{h}_m)$, and $\ker \pi_l = B^l(\mathfrak{h}_m)$. Also, by Lemma 24,

$$\Psi^{k+1} \pi_{m-k}(\alpha) = 0$$

for any $\alpha \in Z^{m-k}(\mathfrak{h}_m)$, for $k = 0, \dots, m$. Therefore, $z^* \Psi^k \pi_{m-k}(\alpha)$ is a cocycle and we may define:

Definition 26. For $k \geq 0$, let

$$\begin{aligned}\Phi_k : H^{m-k}(\mathfrak{h}_m) &\longrightarrow H^{m+k+1}(\mathfrak{h}_m) \\ [\alpha] &\longmapsto [z^* \Psi^k \pi(\alpha)].\end{aligned}$$

One last result before proving the main theorem:

Lemma 27. Suppose $0 \leq k \leq m$. Then, when restricted to $H^{m-k}(\mathfrak{h}_m)$, we find

$$\bar{\star} \Phi_k \bar{\star} \Phi_k = (k!)^2 (-1)^{m-k} \text{id}_{H^{m-k}(\mathfrak{h}_m)}.$$

Proof. Suppose $[\alpha] \in H^{m-k}(\mathfrak{h}_m)$. Then $[\alpha] = [v]$ for a unique primitive $v \in \Lambda^{m-k}U$ (in fact, by Lemma 24, $v = \pi(v) = \pi(\alpha)$). Thus, $\bar{\star} \Phi_k(\alpha) = [\star z^* \Psi^k v] = [\star \Psi^k v]$. From Theorem 16, $\star \Psi^k v = (-1)^{(m-k)(m-k+1)/2} k! C v$. Hence,

$$\bar{\star} \Phi_k \bar{\star} \Phi_k[\alpha] = (k!)^2 [C^2 \pi(\alpha)].$$

If we can show the following two results, the lemma will follow:

- (i) If $i \notin \mathbb{F}$, we can write $\Lambda U_c = \Lambda U \oplus_{\mathbb{F}} i \Lambda U$. Then if α is in $\Lambda^k U$, $C\alpha$ is also in $\Lambda^k U$.
- (ii) Also, $C^2|_{\Lambda^k U} = (-1)^k \text{id}_{\Lambda^k U}$.

These results both hold when $k = 0$ (clearly) and when $k = 1$: any $\alpha \in \Lambda^1 U$ can be written as a linear combination of x_j^* 's and y_j^* 's. But $x_j^* = \frac{z_j + \bar{z}_j}{2}$ and $y_j^* = \frac{z_j - \bar{z}_j}{2i}$, and one can show that $Cx_j^* = -y_j^*$ and $Cy_j^* = x_j^*$.

Suppose both results hold on $\Lambda^{k-1}U$. By linearity, we can subdivide into the following three subcases.

Case 1: With k even, $\alpha = aw_\tau$, where $a = (-2i)^{-|\tau|} = (-2i)^{(-k)}$. Then $C\alpha = a \cdot Cw_\tau = ai^{|\tau|-|\tau|} w_\tau = aw_\tau = \alpha$. Thus, (i) holds and, with k even, part (ii) trivially holds.

Case 2: $\alpha = x_j x_\mu y_\nu$, where $j \notin \mu$ and $j \notin \nu$ and where $x_\mu y_\nu \in \Lambda^{k-1}U$.

Case 3: $\alpha = y_j x_\mu y_\nu$, where $j \notin \mu$ and $j \notin \nu$ and where $x_\mu y_\nu \in \Lambda^{k-1}U$.

For these last two cases, the proofs that (i) and (ii) hold follow from the following fact: if $\alpha' = z_\mu \bar{z}_\nu \in \Lambda^{k-1}U_c$ is such that $j \notin \mu$ and $j \notin \nu$, then $C(x_j^* \alpha') = -y_j^* C\alpha'$ and $C(y_j^* \alpha') = x_j^* C\alpha'$. Indeed,

$$\begin{aligned}C(x_j^* \alpha') &= \frac{1}{2} C((z_j + \bar{z}_j) z_\mu \bar{z}_\nu) = \frac{1}{2} C(z_j z_\mu \bar{z}_\nu) + \frac{1}{2} C(\bar{z}_j z_\mu \bar{z}_\nu) \\ &= \frac{1}{2} \left(i^{|\mu|+1-|\nu|} z_j z_\mu \bar{z}_\nu + i^{|\mu|-|\nu|-1} \bar{z}_j z_\mu \bar{z}_\nu \right) = \frac{1}{2} (iz_j - i\bar{z}_j) C(z_\mu \bar{z}_\nu) = -y_j^* C\alpha'\end{aligned}$$

and, likewise, $C(y_j^* \alpha') = x_j^* C\alpha'$. So (i) holds for Cases 2 and 3 by induction hypothesis. Furthermore, $C^2(x_j^* \alpha') = -x_j^* C^2(\alpha')$, so (ii) holds for Case 2 (and similarly for Case 3) by induction hypothesis. \square

We restate a stronger version of Theorem 1 as follows:

Theorem 1. Let m be a positive integer and suppose $k \in \{0, \dots, m\}$.

(i) The higher operators op_k on $H^*(\mathfrak{h}_m)$ are such that

$$\text{op}_k |_{H^{m+k+1}(\mathfrak{h}_m)} : H^{m+k+1}(\mathfrak{h}_m) \longrightarrow H^{m-k}(\mathfrak{h}_m)$$

is a well-defined isomorphism, and is zero everywhere else it is defined. Moreover, its inverse is

$$\Phi_k : H^{m-k}(\mathfrak{h}_m) \longrightarrow H^{m+k+1}(\mathfrak{h}_m).$$

(ii) If $[\alpha] \in H^{m+k+1}(\mathfrak{h}_m)$ and $[\beta] \in H^{m-k}(\mathfrak{h}_m)$ are Poincaré duals, i.e. $[\alpha] \cdot [\beta] = t$ for some non-trivial top class $t \in H^{2m+1}(\mathfrak{h}_m)$, then $\text{op}_k[\alpha]$ and $\Phi_k[\beta]$ are also Poincaré duals. Indeed,

$$(\text{op}_k[\alpha]) \cdot (\Phi_k[\beta]) = (-1)^{m-k} \cdot t.$$

Moreover, on $\Lambda^{m-k}U$,

$$(-1)^{m-k}(k!)^2 \text{op}_k \bar{\star} = \bar{\star} \Phi_k.$$

Proof. Lemma 25 establishes $i_z(W_{k-1}^{m-k+1}) = \{0\}$, so $\text{op}_k : K_{k-1}^{m+k+1} \rightarrow H^{m-k}(\mathfrak{h}_m)$.

Suppose $c \in Z^{m-k}(\mathfrak{h}_m)$. Then, if $\beta_p = z^* \cdot \Omega^{k-p} \cdot \pi(c)$ for $p = 0, 1, \dots, k$, we have $[\beta_0] = \Phi_k[c]$ and, with a short computation, $(\beta_0, \dots, \beta_k)$ shows that $\Phi_k[c] \in K_{k-1}^{m+k+1}$. Moreover, $i_z \beta_k = \pi(c)$, so that $\text{op}_k \circ \Phi_k = \text{id}_{H^{m-k}(\mathfrak{h}_m)}$. This, together with Poincaré duality, proves that $K_{k-1}^{m+k+1} = H^{m+k+1}(\mathfrak{h}_m)$ and op_k and Φ_k are isomorphisms which are inverse to each other.

Now we want to show that

$$\text{op}_l : K_{l-1}^j \rightarrow H^{j-2l-1}(\mathfrak{h}_m) / [i_z W_{l-1}^{j-2l}]$$

is the zero map when $j \neq m + l + 1$.

First note that, by definition, we always have $\text{op}_l(K_{l-1}^j) \subseteq i_z W_{l-1}^{j-l}$. We now consider the three possible cases:

Case 1: $j < m + l + 1$. Then $j - 2l \leq m - l$, so suppose $j - 2l = m - k$ for some $k \geq l$. Hence, $i_z(W_{l-1}^{j-2l}) = i_z(W_{l-1}^{m-k}) = \{0\}$, by Lemma 25, and so $\text{op}_l = 0$.

Case 2: $j \geq m + 2l + 2$. Then $j - 2l - 1 \geq m + 1$, so $i_z(W_{l-1}^{j-2l}) \subseteq \Lambda^{j-2l-1}U = B^{j-2l-1}(\mathfrak{h}_m)$, by Theorem 23(i), and so op_l is trivial (in the sense that it maps to the boundaries).

Case 3: $m + l + 2 \leq j \leq m + 2l + 1$. Here, $j - 2l - 1 \leq m$, so by the fact that $\text{op}_{m+1+2l-j}$ is an isomorphism onto $H^{j-2l-1}(\mathfrak{h}_m)$,

$$H^{j-2l-1}(\mathfrak{h}_m) = \text{im } \text{op}_{m+1+2l-j} \subseteq [i_z W_{l-1}^{j-2l}] \subseteq H^{j-2l-1}(\mathfrak{h}_m),$$

which shows that $[i_z W_{l-1}^{j-2l}] = H^{j-2l-1}(\mathfrak{h}_m)$, establishing that op_l is trivial (in the sense that it maps to its ambiguity).

This establishes the first part of the theorem.

Let $[\alpha]$ and $[\beta]$ be as in the statement of the theorem. From part (i), we can find $\alpha' \in Z^{m-k}(\mathfrak{h}_m)$ such that $[z^*(dz^*)^k \alpha'] = [\alpha]$. Also, $z(dz)^k \pi(\beta) \in Z^{m+k+1}(\mathfrak{h}_m)$. Note that $op_k[\alpha] = [\alpha']$ and $\Phi_k[\beta] = [z^*(dz^*)^k \pi(\beta)]$. Hence, since multiplication is well-defined on cohomology,

$$t = [\alpha] \cdot [\beta] = [z^*(dz^*)^k \alpha' \cdot \pi(\beta)] = (-1)^{m-k} [\alpha' \cdot z^*(dz^*)^k \pi(\beta)].$$

Hence, $(-1)^{m-k} \cdot t = (op_k[\alpha]) \cdot (\Phi_k[\beta])$.

Finally, by Lemma 27,

$$\bar{\star} \Phi_k \bar{\star} \Phi_k = (k!)^2 (-1)^{m-k} \text{id}_{H^{m-k}(\mathfrak{h}_m)}.$$

Applying op_k to the left of both sides of this equation, part (i) implies

$$(k!)^2 (-1)^{m-k} op_k = \bar{\star} \Phi_k \bar{\star} \Phi_k op_k = \bar{\star} \Phi_k \bar{\star}.$$

Finally, since $\bar{\star} \bar{\star} = \omega = \text{id}_{H^*(\mathfrak{h}_m)}$, we have $(k!)^2 (-1)^{m-k} op_k \bar{\star} = \bar{\star} \Phi_k$. \square

CHAPTER IV: FREE TWO-STEP NILPOTENT LIE ALGEBRAS

Though the TRC is known to be true for two-step nilpotent Lie algebras (Theorem I.14), many proofs exist. The subject of this chapter is an approach that might lead to an alternate proof that the TRC holds for the free two-step nilpotent Lie algebras, and consequently for *all* two-step nilpotent Lie algebras. It is hoped that the new method, if successful, may lead to similar results for more general nilpotent Lie algebras by giving insight into the structure of their cohomology.

This chapter closely follows Stefan Sigg's *Laplacian and Homology of Free Two-Step Nilpotent Lie algebras* [Sig]. This chapter describes Sigg's paper (with the relevant representation theory included) and explains how we can use his theorems for our purposes, even though Sigg's paper deals with homology on the exterior algebra, while we are interested in the cohomology on the dual of the exterior algebra. One bulky proof from [Sig], that of Theorem 16, is omitted, but the proof relies strictly on Casimir operators and two formulas from homology that turn out to be the same for cohomology. All of section 4, namely the use of bigradation and Sigg's decomposition to find hypercubes in cohomology, is original work.

Throughout this chapter, our field \mathbb{F} will be \mathbb{C} , the field of the complex numbers.

§1 \mathbf{GL}_r ACTIONS ON THE FREE TWO-STEPS

Let \mathfrak{f}_r denote the free Lie algebra of rank $r \geq 2$ over the complex numbers and define the *free two-step nilpotent Lie algebra of rank r* to be

$$\mathfrak{m}_r = \mathfrak{f}_r / C^2(\mathfrak{f}_r) = \mathfrak{f}_r / [\mathfrak{f}_r, [\mathfrak{f}_r, \mathfrak{f}_r]].$$

We let $\{e_1, \dots, e_r\}$ be a basis for the generating set and denote its span by E_r . Let $\{f_{ij} = [e_i, e_j] \mid 1 \leq i < j \leq r\}$ be a basis for $F_r := [\mathfrak{m}_r, \mathfrak{m}_r] = Z(\mathfrak{m}_r)$. Hence, $\mathfrak{m}_r = E_r \oplus F_r$, $\Lambda \mathfrak{m}_r^* = \Lambda E_r^* \otimes \Lambda F_r^*$ and

$$\mathfrak{m}_r = \text{span} \left\{ \{e_i\}_{i=1}^r \cup \{f_{ij} \mid 1 \leq i < j \leq r \mid [e_i, e_j] = f_{ij}\} \right\}.$$

We also have the usual dual basis $\{e_i^*\}_{i=1}^r$ for E_r^* and $\{f_{ij}^*\}_{1 \leq i < j \leq r}$ for F_r^* . Note that $dE_r^* = \{0\}$, and $d : F_r^* \xrightarrow{\cong} \Lambda^2 E_r^*$ is an isomorphism sending each f_{ij}^* to $e_i^* \wedge e_j^*$.

We define an action of $\mathbf{GL}_r := \mathbf{GL}(r, \mathbb{C})$, the general linear group of all invertible $r \times r$ matrices, on E_r^* as follows. Using the isomorphism $E_r^* = \bigoplus_{i=1}^r \mathbb{C}e_i^* \cong \mathbb{C}^r$, define \mathbf{GL}_r to act on E_r^* as the *standard representation*. Note that this is *not* the usual

adjoint representation on the dual space. This induces an action on $F_r^* \cong \Lambda^2 E_r^*$ as follows: if $A \in \mathbf{GL}_r$, then

$$A \cdot f_{ij}^* = A \cdot d^{-1}(e_i^* \wedge e_j^*) := d^{-1}\left((Ae_i^*) \wedge (Ae_j^*)\right).$$

In turn, this induces a natural action on all of Λm_r^* (as described by Proposition 5, below). Note that the action of \mathbf{GL}_r commutes with the differential.

Homology and Sigg's results.

Instead of defining the action of \mathbf{GL}_r on E_r^* , Sigg defines it (more naturally) to be the standard representation on E_r . The theorems from §3, below, were therefore originally intended for the homology on Λm_r . Their proofs use the boundary operator $\bar{\partial} : \Lambda L \rightarrow \Lambda L$ given by

$$(1) \quad \bar{\partial}_p(x_1 \wedge \cdots \wedge x_p) = \sum_{i < j} (-1)^{i+j+1} [x_i, x_j] \wedge x_1 \wedge \cdots \widehat{x}_i \cdots \widehat{x}_j \cdots \wedge x_p$$

with $\bar{\partial}_p = 0$ when $p \leq 1$. That $\bar{\partial}_p \bar{\partial}_{p+1} = 0$ follows from the Jacobi identity (a classical result [Kos]). The p th homology space is defined to be

$$H_p(L) = \ker \bar{\partial}_p / \text{im } \bar{\partial}_{p+1}.$$

Given any Lie algebra L with a positive Hermitian form $\langle \cdot, \cdot \rangle_L$, Sigg defines a positive Hermitian form $\langle \cdot, \cdot \rangle$ on ΛL , by setting $\langle \Lambda^p L, \Lambda^q L \rangle = \{0\}$ when $p \neq q$ and

$$\langle x_1 \wedge \cdots \wedge x_p, y_1 \wedge \cdots \wedge y_p \rangle = \det(\langle x_i, y_j \rangle_L).$$

He then defines \bar{d}_p to be the adjoint of $\bar{\partial}_{p+1}$ with respect to this inner product, and proves:

Lemma 1. [Sig, Lemma 8]. *Let $\{e_1, \dots, e_n\}$ be an orthonormal basis of L with $[e_i, e_j] = \sum_{k=1}^n c_{ij}^k e_k$ for $1 \leq i < j \leq n$. Then*

$$(2) \quad \bar{d}_p(x_1 \wedge \cdots \wedge x_p) = \sum_{i=1}^p (-1)^{i+1} \bar{d}_1(x_i) \wedge x_1 \wedge \cdots \widehat{x}_i \cdots \wedge x_p,$$

where $\bar{d}_1(e_k) = \sum_{i < j} c_{ij}^k e_i \wedge e_j$ ($k = 1, \dots, n$).

In the proofs of his theorems, Sigg uses results from representation theory, the Laplacian operator defined by $\bar{\Delta}_p = \bar{d}_{p-1} \bar{\partial}_p + \bar{\partial}_{p+1} \bar{d}_p$ and formulas (1) and (2). In the two sections that follow, the only difference from Sigg's paper is that we use the Laplacian Δ as defined in Chapter III. The following lemma tells us that our Laplacian Δ has the same properties as Sigg's Laplacian $\bar{\Delta}$.

Lemma 2. *The differential d (from Definition I.4) and its adjoint ∂ (as defined in chapter 3) satisfy similar equations on ΛL^* :*

$$(3) \quad d_p(x_1^* \wedge \cdots \wedge x_p^*) = \sum_{i=1}^p (-1)^{i+1} d_1(x_i^*) \wedge x_1^* \wedge \cdots \widehat{x_i^*} \cdots \wedge x_p^*,$$

and

$$(4) \quad \partial_p(x_1^* \wedge \cdots \wedge x_p^*) = \sum_{i < j} (-1)^{i+j+1} \partial_2(x_i^* \wedge x_j^*) \wedge x_1^* \wedge \cdots \widehat{x_i^*} \cdots \widehat{x_j^*} \cdots \wedge x_p^*,$$

where $\partial_2(e_i^* \wedge e_j^*) = \sum_{k=1}^n c_{ij}^k e_k^*$ and $\partial_i = 0$ when $i \leq 1$.

Proof. That d_p satisfies (3) is a direct consequence of the Leibniz rule. That an adjoint of d_{p-1} satisfies formula (4) is given by Lemma 1. By the uniqueness of adjoint maps, ∂_p satisfies (4). \square

Note that our action of \mathbf{GL}_r commutes with ∂ , hence with Δ . Therefore, in all the proofs from [Sig] that follow (in the next sections), the only modifications made from the original is that everything takes place in the dual space of $\Lambda \mathfrak{m}_r$ (with our own Laplacian Δ which has the same desirable properties as $\overline{\Delta}$), with the action on the basis of \mathfrak{m}_r^* defined exactly as Sigg's action on the basis of \mathfrak{m}_r (i.e. we don't use the adjoint representation, as Sigg does for the cohomology in the last section of his paper).

Henceforth, we will drop the stars (*) from the basis vectors of \mathfrak{m}_r^* , for ease of notation.

§2 GENERAL REPRESENTATIONS OF \mathbf{GL}_r

In this section, we recall some general representation theory, as well as some results about representations of \mathbf{GL}_r . As with Sigg's paper, this section closely follows [FuHa].

2.1 The switch from the Lie group to the Lie algebra.

Definition 3.

- (a) Given a Lie group G , a representation of the Lie group on a vector space V is a homomorphism of Lie groups $\rho : G \rightarrow \mathbf{GL}(V)$, denoted (ρ, V) .
- (b) Given a Lie algebra \mathfrak{g} , a representation of the Lie algebra on a vector space V is a homomorphism of Lie algebras $\varrho \mathfrak{g} \rightarrow \mathfrak{gl}(V)$, denoted (ϱ, V) .

Theorem 4. *Given a Lie group representation $\rho : G \rightarrow \mathbf{GL}(V)$, where G is a connected and simply connected Lie group, we can instead consider the derivative representation $D\rho : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ of the corresponding Lie algebra, with*

$$D\rho(A) = \left. \frac{d}{ds} \right|_{s=0} \rho(\exp(sA)), \quad A \in \mathfrak{g},$$

where \mathfrak{g} is the Lie algebra of G , without any loss of information. That is, representations of a connected and simply connected Lie group are in one-to-one correspondence with representations of its Lie algebra.

A proof can be found in [FuHa] or in [GHV2, Chapter I, §3].

In particular, a decomposition of a G -representation into irreducible subrepresentations is also valid for the corresponding \mathfrak{g} -representation. As an example, the following proposition shows how the usual group action on tensor products induces Lie group and Lie algebra actions on the exterior algebra:

Proposition 5. *Let ρ be a representation of a Lie group G on V .*

(a) *By setting*

$$x \cdot (v_1 \wedge \cdots \wedge v_p) = \rho(x)v_1 \wedge \rho(x)v_2 \wedge \cdots \wedge \rho(x)v_p$$

for $x \in G$ and $v_i \in V$, we obtain a representation of G on the exterior algebra ΛV .

(b) *For the derivative representation, we get*

$$A \cdot (v_1 \wedge \cdots \wedge v_p) = \sum_{i=1}^p v_1 \wedge \cdots \wedge D\rho(A)(v_i) \wedge \cdots \wedge v_p,$$

with $A \in \mathfrak{g}$ and $v_i \in V$.

We will employ several tools from representation theory, such as the *Casimir operator* and *weight space decomposition*, as will be seen throughout this chapter.

To better understand representations of \mathbf{GL}_r , it helps to understand the representations of \mathbf{SL}_r .

2.2 Representations of \mathbf{SL}_r . This subsection summarises the needed results from [FuHa, Chapters 14 and 15].

Let $\mathbf{SL}_r := \mathbf{SL}(r, \mathbb{C})$ denote the *special linear group*, the subgroup of \mathbf{GL}_r of all matrices of determinant 1. Its Lie group representation can be identified with the Lie algebra representation of $\mathfrak{sl}_r := \mathfrak{sl}(r, \mathbb{C})$ (by Theorem 4), a semisimple Lie algebra.

As with all semisimple Lie algebras, \mathfrak{sl}_r can be decomposed as

$$\mathfrak{sl}_r \mathbb{C} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha \right),$$

where \mathfrak{h} denotes the Cartan subalgebra (a maximal abelian subalgebra of diagonalisable elements), the set of roots Φ is a certain finite set of nonzero elements in \mathfrak{h}^* , and each root space,

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{sl}_r \mid [H, x] = \alpha(H)x, \text{ for all } H \in \mathfrak{h}\},$$

is one-dimensional.

If V is any representation of \mathfrak{sl}_r , then \mathfrak{h} acts semisimply on V , meaning $V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda$ where each $V_\lambda = \{v \in V \mid H \cdot v = \lambda(H)v, \forall H \in \mathfrak{h}\}$. If V_λ is non-trivial, then it is called a weight space of \mathfrak{sl}_r , its vectors are weight vectors and λ is called a weight.

Remark 6. Note that if $x \in \mathfrak{g}_\alpha$, then $x : V_\lambda \longrightarrow V_{\lambda+\alpha}$.

Remark 7. Given an irreducible representation V and a weight λ of \mathfrak{sl}_r on V , one can completely determine the weight space decomposition of V by repeated applications of the action of the \mathfrak{g}_α 's on V_λ (since $\mathfrak{sl}_r \cdot V_\lambda$ is a subrepresentation of V , hence is all of V).

We note that

$$\mathfrak{h} = \left\{ \left(\begin{array}{cccc} a_1 & 0 & \dots & 0 \\ 0 & a_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_r \end{array} \right) \mid a_1 + \dots + a_r = 0 \right\}$$

and that, if we define the functional L_i by

$$L_i \left(\begin{array}{cccc} a_1 & 0 & \dots & 0 \\ 0 & a_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_r \end{array} \right) = a_i,$$

then

$$\mathfrak{h}^* = \text{span}\{L_1, \dots, L_r\} / \text{span}\{L_1 + \dots + L_r\}.$$

Let E_{ij} denote the $r \times r$ matrix with a 1 as the (i, j) entry and zeroes elsewhere. One can show that $\Phi = \{L_i - L_j \mid i \neq j\}$ is the complete set of roots, and that each $\mathfrak{g}_{L_i - L_j} = \text{span}\{E_{ij}\}$. In an arbitrary representation V , each weight λ can be written as

$$\lambda = \lambda_1 L_1 + \dots + \lambda_r L_r = (\lambda_1 - \lambda_r) L_1 + \dots + (\lambda_{r-1} - \lambda_r) L_{r-1}.$$

Each $\lambda_i - \lambda_r$ must be an integer [FuHa, Chapter 15]. Thus, each weight sits on a *weight lattice*

$$\Lambda_W = \mathbb{Z}[L_1, \dots, L_r] / \{L_1 + \dots + L_r\} \subset \mathfrak{h}^*.$$

Remarks 6 and 7 tell us that any two weights in an irreducible representation will differ by an integer linear combination of roots. Hence, for irreducible representations, weights are congruent modulo the *root lattice*, $\Lambda_R \subset \mathfrak{h}^*$, generated by integer linear combinations of the roots.

Supposing V to be irreducible and finite dimensional, there must be some λ such that $E_{ij}(V_\lambda) = \{0\}$ whenever $i < j$. This happens when $\lambda = \lambda_1 L_1 + \dots + \lambda_r L_r$ is such that $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{r-1} \geq \lambda_r = 0$ with $\sum_{i=1}^r \lambda_i$ maximal. Call this λ the *highest weight* and call vectors in V_λ the *highest weight vectors*. By remark 7, V is completely determined by repeated actions of E_{ji} , with $i < j$, on V_λ . In fact, the highest weight vector is unique up to scalar multiple (i.e. V_λ is one-dimensional when λ is a highest weight) [FuHa, Chapter 15]. We write $V = V(\lambda) = V(\lambda_1, \dots, \lambda_r)$. It is a classical result that all finite-dimensional irreducible representations of $\mathfrak{sl}_r \mathbb{C}$ are in one-to-one correspondence with all $V(\lambda_1, \dots, \lambda_r)$ such that $\lambda_r = 0$ [FuHa, Proposition 15.15].

We will henceforth write $V(\lambda) = V(\lambda_1, \dots, \lambda_i)$, where $i = \max\{j \mid \lambda_j \neq 0\}$, to avoid denoting an indefinite number of zeroes. Note that $V(0) \cong \mathbb{C}$, the trivial representation.

2.3 Representations of \mathbf{GL}_r . There is a precise relationship between representations of \mathbf{GL}_r and those of its subgroup \mathbf{SL}_r . In fact, if D_k denotes the one-dimensional representation of \mathbf{GL}_r given by the k^{th} power of the determinant, then

Proposition 8. [FuHa, Proposition 15.47] *Every irreducible complex representation of $\mathbf{GL}_r \mathbb{C}$ is isomorphic to*

$$V(\lambda_1, \dots, \lambda_{r-1}, \lambda_r) = V(\lambda_1 - \lambda_r, \dots, \lambda_{r-1} - \lambda_r, 0) \otimes D_{\lambda_r},$$

for some unique $\lambda = \lambda_1 L_1 + \dots + \lambda_r L_r \in \mathbb{Z}[L_1, \dots, L_r]$ with $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r$.

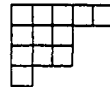
Note that it is not necessarily true that $\lambda_r \geq 0$. The action of the group \mathbf{GL}_r on $V(\lambda_1, \dots, \lambda_{r-1}, 0)$ is exactly the same as the action of \mathbf{SL}_r on $V(\lambda_1, \dots, \lambda_{r-1}, 0)$ when $\lambda_i \geq 0$. Each irreducible representation of \mathbf{GL}_r with $\lambda_r \geq 0$ can be represented by a unique *Young diagram*.

2.4 Young diagrams and Frobenius notation. Let \mathcal{P}_r denote the set of all partitions $\lambda = (\lambda_1, \dots, \lambda_r)$ such that $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r \geq 0$. According to [Mac, §I.A.8], there exists a bijection

$$\begin{aligned} \mathcal{P}_r &\xrightarrow{1:1} \mathcal{M}_r^{\text{pol}} \\ \lambda &\longmapsto [V(\lambda)], \end{aligned}$$

where $\mathcal{M}_r^{\text{pol}}$ is the set of equivalence classes of irreducible polynomial representations of \mathbf{GL}_r (meaning the matrix entries of the corresponding representation maps are given by polynomials in the entries of the matrices of \mathbf{GL}_r), with equivalence classes determined by isomorphisms of representations.

We will use two additional notations to describe partitions. The first is the *Young diagram* $Y(\lambda)$ of a partition λ , which is the graphical arrangement of λ_i boxes in the i th row. For example, the Young diagram of $\lambda = (5, 3, 3, 1)$ is



We identify a box in a Young diagram with the pair (i, j) that numbers the row i and the column j of the box. For each box (i, j) in $Y(\lambda)$, we associate the *hook number* $h(i, j)$, which is the number of boxes directly below or directly to the right, including the box itself once. In $Y(5, 3, 3, 1)$, for example, we have $h(1, 2) = 6$ and $h(2, 3) = 2$. The *conjugate* λ' of a partition λ is defined by $\lambda'_i = \#\{j \mid \lambda_j \geq i\}$, i.e. one gets the Young diagram of λ' by transposing that of λ .

The other notation that we will use in this chapter is due to Frobenius. Let $d = d_\lambda$ be the number of boxes on the diagonal of $Y(\lambda)$. Set $\alpha_i = \lambda_i - i$ and $\beta_i = \lambda'_i - i$ for $i = 1, \dots, d$, i.e., α_i equals the number of boxes in the i th row to the right of the diagonal and β_i equals the number of boxes in the i th column below the diagonal. We have $\alpha_1 > \dots > \alpha_d \geq 0$ and $\beta_1 > \dots > \beta_d \geq 0$ and write $\lambda = (\alpha \mid \beta) = (\alpha_1, \dots, \alpha_d \mid \beta_1, \dots, \beta_d)$. In our example, we have $\lambda = (5, 3, 3, 1) = (4, 1, 0 \mid 3, 1, 0)$ and $\lambda' = (4, 3, 3, 1, 1) = (3, 1, 0 \mid 4, 1, 0)$. The conjugate of $(\alpha \mid \beta)$ is $(\beta \mid \alpha)$ and a partition λ is called *self-conjugate* if $\lambda = \lambda'$, i.e., $Y(\lambda)$ is symmetric, or $\alpha = \beta$ in the Frobenius notation. Define the *length* of λ to be $l(\lambda) = \max\{i \mid \lambda_i > 0\}$. Finally, one calls $|\lambda| = \sum_{i=1}^r \lambda_i$ the *absolute value* of λ .[†] Henceforth, a weight $\lambda = \lambda_1 L_1 + \dots + \lambda_r L_r$ will be denoted as a partition $\lambda = (\lambda_1, \dots, \lambda_r) \in \mathcal{P}_r$.

2.5 Casimir operator. Let $\{E_{ij} \mid 1 \leq i, j \leq r\}$ be the standard basis of \mathfrak{gl}_r and let $b : \mathfrak{gl}_r \times \mathfrak{gl}_r \rightarrow \mathbb{C}$ be the bilinear form defined by $b(A, B) = \text{Trace}(AB)$. The form b is nondegenerate and it is invariant in the sense that $b([A, B], C) = b(A, [B, C])$ holds for all A, B and $C \in \mathfrak{gl}_r$. The dual of each E_{ij} with respect to b is E_{ji} and for each \mathfrak{gl}_r representation $\varrho : \mathfrak{gl}_r \rightarrow \mathfrak{gl}(V)$, the *Casimir operator* $\mathfrak{C}_\varrho \in \mathfrak{gl}(V)$ is defined to be

$$\mathfrak{C}_\varrho = \sum_{i,j=1}^r \varrho(E_{ij})\varrho(E_{ji}).$$

[†]In fact, what I call "absolute value", Macdonald calls "weight", as [Mac] is not really a book on representation theory. I mention this only to ease the pain of anyone who actually takes the trouble of looking at my references.

It is a well-known fact that the Casimir operator commutes with the corresponding representation and thus acts as multiplication by a scalar on irreducible representations (by Schur's lemma) [FuHa, §C.2].

Proposition 9. [Sig, Proposition 2] *Let (V, ρ) be a \mathbf{GL}_r representation, $W \subset V$ an irreducible subspace with highest weight $\lambda = (\lambda_1, \dots, \lambda_r) \in \mathcal{P}_r$, and $\varrho = D\rho$ be the corresponding representation of \mathfrak{gl}_r . Then $\mathfrak{C}_\varrho|_W = c(\lambda)\text{id}_W$, where*

$$c(\lambda) = \sum_{i=1}^r \lambda_i^2 - 2i\lambda_i + (r+1)\lambda_i.$$

Proof. Let $w_\lambda \in W$ be a vector of highest weight, i.e. $\varrho(E_{ji})w_\lambda = 0$ for all $i > j$ and $\varrho(E_{ii})w_\lambda = \lambda_i w_\lambda$ for $i = 1, \dots, r$. Thus we have

$$\begin{aligned} \mathfrak{C}_\varrho w_\lambda &= \sum_{i \neq j} \varrho(E_{ij})\varrho(E_{ji})w_\lambda + \left(\sum_{i=1}^r \lambda_i^2 \right) w_\lambda \\ &= \sum_{i < j} \varrho([E_{ij}, E_{ji}])w_\lambda + \left(\sum_{i=1}^r \lambda_i^2 \right) w_\lambda \\ &= \sum_{i < j} \varrho(E_{ii} - E_{jj})w_\lambda + \left(\sum_{i=1}^r \lambda_i^2 \right) w_\lambda \\ &= \left(\sum_{i=1}^r (r-i)\lambda_i - \sum_{i=1}^r (i-1)\lambda_i + \sum_{i=1}^r \lambda_i^2 \right) w_\lambda \\ &= \sum_{i=1}^r (\lambda_i^2 - 2i\lambda_i + (r+1)\lambda_i) w_\lambda. \end{aligned}$$

□

§3 DECOMPOSITION OF COHOMOLOGY

3.1 Sigg's Theorem. We now return to the free two-step nilpotent Lie algebras \mathfrak{m}_r of the first section. Recall that $E_r^* = \bigoplus_{i=1}^r \mathbb{C}e_i \cong \mathbb{C}^r$ is treated as the standard representation of \mathbf{GL}_r , hence of \mathfrak{gl}_r , and that the action of \mathbf{GL}_r on F_r^* is induced

by the isomorphism $d : F_r^* \xrightarrow{\cong} \Lambda^2 E_r^*$. Thus, if $A \in \mathfrak{gl}_r$, then

$$\begin{aligned} A \cdot (e_{\alpha_1} \cdots e_{\alpha_k} \cdot f_{\beta_1, \gamma_1} \cdots f_{\beta_l, \gamma_l}) = \\ \sum_{i=1}^k e_{\alpha_1} \cdots \underbrace{Ae_{\alpha_i}}_{\text{instead of } e_{\alpha_i}} \cdots e_{\alpha_k} \cdot f_{\beta_1, \gamma_1} \cdots f_{\beta_l, \gamma_l} + \\ \sum_{j=1}^l e_{\alpha_1} \cdots e_{\alpha_k} \cdot f_{\beta_1, \gamma_1} \cdots \underbrace{d^{-1}(Ae_{\beta_j} \wedge e_{\gamma_j} + e_{\beta_j} \wedge Ae_{\gamma_j})}_{\text{instead of } f_{\beta_j, \gamma_j}} \cdots f_{\beta_l, \gamma_l}. \end{aligned}$$

As mentioned, d commutes with the action of \mathbf{GL}_r . With $E_{ii}(e_j) = \delta_{ij}$, the weight $\mu = (\mu_1, \dots, \mu_r)$ of a typical basis vector $e_{\alpha_1} \wedge \cdots \wedge e_{\alpha_k} \wedge f_{\beta_1, \gamma_1} \wedge \cdots \wedge f_{\beta_m, \gamma_m}$ of $\Lambda^{k+m} \mathfrak{m}_r^*$ is given by

$$\mu_i = \#\{j \mid \alpha_j = i\} + \#\{s \mid \beta_s = i\} + \#\{t \mid \gamma_t = i\} \geq 0$$

for $i = 1, \dots, r$. Note that since $\Lambda \mathfrak{m}_r^*$ is clearly a polynomial representation, its irreducible subrepresentations can (and will) be represented by Young diagrams.

Let us look at the cochain complex first and see what we can say about the multiplicities m_λ in the decomposition $\Lambda \mathfrak{m}_r^* \cong \bigoplus_{\lambda \in \mathcal{P}_r} (m_\lambda V(\lambda))$. First, $\Lambda^k E_r^* \cong \Lambda^k(\mathbb{C}^r)$ is irreducible and equivalent to $V(1^k) = V(\underbrace{1, 1, \dots, 1}_{k \text{ times}})$ [FuHa, Chapter 15].

On the other hand, there is the following result.

Lemma 10. [Sig, Lemma 3] *For $0 \leq m \leq \binom{r}{2}$ we let \mathcal{B}_m denote the set of all partitions $\mu \in \mathcal{P}_r$ with $|\mu| = 2m$ whose Frobenius notation is $(\alpha_1, \dots, \alpha_d \mid \alpha_1 + 1, \dots, \alpha_d + 1)$. Then*

$$\Lambda^m F_r^* \cong \Lambda^m(\Lambda^2 E_r^*) \cong \bigoplus_{\mu \in \mathcal{B}_m} V(\mu).$$

Proof. Sigg gives the following direct proof, but he does refer the reader to a certain character identity found in [Mac, I.5, Ex.10]. The proof is based on an induction over m , the cases $m = 0$ and $m = 1$ being clear. We thus assume that the assertion holds for $m - 1$. If $\mu = (\alpha_1, \dots, \alpha_d \mid \alpha_1 + 1, \dots, \alpha_d + 1) \in \mathcal{B}_m$, then we set

$$\begin{aligned} w_\mu = f_{12} \wedge \cdots \wedge f_{1, \alpha_1 + 2} \wedge f_{23} \wedge \cdots \wedge f_{2, \alpha_2 + 3} \wedge f_{34} \wedge \cdots \\ \cdots \wedge f_{d, d+1} \wedge \cdots \wedge f_{d, \alpha_d + d+1}. \end{aligned}$$

This is a highest weight for all $\mu \in \mathcal{B}_m$ and we denote the corresponding irreducible subrepresentation in $\Lambda^m F_r^*$ by $\mathbf{GL}_r \cdot w_\mu$. It follows that $W = \bigoplus_{\mu \in \mathcal{B}_m} \mathbf{GL}_r \cdot w_\mu$

is a subrepresentation of $\Lambda^m F_r^*$. In order to show equality we consider the map $\phi_m : \Lambda^{m-1} F_r^* \otimes F_r^* \rightarrow \Lambda^m F_r^*$ with $\phi_m(v \otimes f_{ij}) = v \wedge f_{ij}$. Obviously, ϕ_m is a surjective \mathbf{GL}_r morphism (meaning it commutes with the action of \mathbf{GL}_r). We are done if we can show that the image of ϕ_m lies in W . The induction hypothesis gives $\Lambda^{m-1} F_r^* \cong \bigoplus_{\mu \in \mathcal{B}_{m-1}} \mathbf{GL}_r \cdot w_\mu$. Furthermore we define the map $\sim : \mathcal{B}_m \rightarrow \mathcal{B}_{m-1}$ by setting

$$\begin{aligned} \tilde{\mu} &= (\alpha_1, \dots, \alpha_d \mid \alpha_1 + 1, \dots, \alpha_d + 1)^\sim \\ &:= \begin{cases} (\alpha_1, \dots, \alpha_{d-1} \mid \alpha_1 + 1, \dots, \alpha_{d-1} + 1), & \text{if } \alpha_d = 0, \\ (\alpha_1, \dots, \alpha_{d-1}, \alpha_d - 1 \mid \alpha_1 + 1, \dots, \alpha_{d-1} + 1, \alpha_d), & \text{if } \alpha_d \neq 0. \end{cases} \end{aligned}$$

and noting that $|\mu| = 2d + 2 \sum_{i=1}^d \alpha_i$, but $|\tilde{\mu}| = |\mu| - 2$.

Then $\phi_m(w_{\tilde{\mu}} \otimes f_{d, \alpha_d + d + 1}) = w_\mu$ and since ϕ_m is invariant under the \mathbf{GL}_r action, the proof is complete. \square

The initial data that follows from Lemma 10 can be pictured as

$$\begin{aligned} F_r^* &\cong V(1, 1) \hat{=} \begin{array}{|c|} \hline \square \\ \hline \end{array} \quad (r \geq 2), \\ \Lambda^2 F_r^* &\cong V(2, 1, 1) \hat{=} \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array} \quad (r \geq 3), \\ \Lambda^3 F_r^* &\cong V(3, 1, 1, 1) \oplus V(2, 2, 2) \hat{=} \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & & \\ \hline \square & & \\ \hline \end{array} \oplus \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \quad (r \geq 4). \end{aligned}$$

The next step is the decomposition of the tensor products $\Lambda^k E_r^* \otimes \Lambda^m F_r^*$. This is done in the following lemma, which is also known as Pieri's formula.

Lemma 11. [Sig, Lemma 4] *For $\mu \in \mathcal{P}_r$ and $0 \leq k \leq r$, we let \mathcal{T}_μ^k denote the set of all partitions $\lambda \in \mathcal{P}_r$ whose Young diagram $Y(\lambda)$ can be obtained by adding k boxes to that of μ but at most one box in each row. This may be written formally as $\mathcal{T}_\mu^k = \{\lambda \in \mathcal{P}_r \mid |\lambda| = |\mu| + k, 0 \leq \lambda_i - \mu_i \leq 1\}$. Then*

$$V(1^k) \otimes V(\mu) \cong \bigoplus_{\lambda \in \mathcal{T}_\mu^k} V(\lambda).$$

Proof. See [FuHa, Sect. A.1, Ex. A.32(v)] or [Mac I.5.17]. \square

Putting together the information of Lemmas 10 and 11, we obtain a pretty good understanding of the \mathbf{GL}_r structure of the exterior algebra, which Sigg summarises in the following:

Theorem 12. [Sig, Theorem 5] For $0 \leq p \leq \binom{r+1}{2}$, we set $\mathcal{A}_p = \{(k, m) \in \mathbb{Z}^2 \mid 0 \leq k \leq r, 0 \leq m \leq \binom{r}{2}, k + m = p\}$. Then

$$\Lambda^p \mathfrak{m}_r^* \cong \bigoplus_{p=0}^{r(r+1)/2} \bigoplus_{(k,m) \in \mathcal{A}_p} \bigoplus_{\mu \in \mathcal{B}_m} \bigoplus_{\lambda \in \mathcal{T}_\mu^k} V(\lambda),$$

where the sets \mathcal{B}_m and \mathcal{T}_μ^k are defined in Lemmas 10 and 11, respectively. Note that we can specify the decomposition of the p -forms as

$$\Lambda^p \mathfrak{m}_r^* \cong \bigoplus_{(k,m) \in \mathcal{A}_p} \bigoplus_{\mu \in \mathcal{B}_m} \bigoplus_{\lambda \in \mathcal{T}_\mu^k} V(\lambda).$$

Here the initial data for $r \geq 4$ is

$$\begin{aligned} \mathfrak{m}_r^* &\cong \square \oplus \begin{array}{|c|} \hline \square \\ \hline \end{array}, \\ \Lambda^2 \mathfrak{m}_r^* &\cong \begin{array}{|c|} \hline \square \\ \hline \end{array} \oplus \begin{array}{|c|} \hline \square \\ \square \\ \hline \end{array} \oplus \begin{array}{|c|} \hline \square & \square \\ \hline \end{array} \oplus \begin{array}{|c|} \hline \square & \square & \square \\ \hline \end{array}, \\ \Lambda^3 \mathfrak{m}_r^* &\cong \begin{array}{|c|} \hline \square \\ \square \\ \hline \end{array} \oplus \begin{array}{|c|} \hline \square & \square \\ \square \\ \hline \end{array} \oplus \begin{array}{|c|} \hline \square & \square \\ \square & \square \\ \hline \end{array} \oplus \begin{array}{|c|} \hline \square & \square & \square \\ \square & \square & \square \\ \hline \end{array} \oplus \begin{array}{|c|} \hline \square & \square & \square & \square \\ \square & \square & \square & \square \\ \hline \end{array} \oplus \begin{array}{|c|} \hline \square & \square & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \square & \square \\ \hline \end{array} \oplus \begin{array}{|c|} \hline \square & \square & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \square & \square \\ \hline \end{array}. \end{aligned}$$

If one continues to produce the decomposition of the exterior powers in the above manner, an interesting phenomenon appears:

Proposition 13. [Sig, Proposition 6] Let m_λ be the multiplicity of $V(\lambda)$ in the decomposition of $\Lambda^p \mathfrak{m}_r^*$. If $\lambda \in \mathcal{P}_r$ is self-conjugate (i.e., $\lambda = \lambda'$ or $Y(\lambda)$ is symmetric), then $m_\lambda = 1$ and $V(\lambda)$ appears in the decomposition of $\Lambda^{d_\lambda} E_r^* \otimes \Lambda^{(|\lambda| - d_\lambda)/2} F_r^* \subset \Lambda^{(|\lambda| + d_\lambda)/2} \mathfrak{m}_r^*$.

Proof. According to Theorem 12, the multiplicity m_λ of a Young diagram $Y(\lambda)$ equals the number of distinct ways of getting from $Y(\lambda)$ to a Young diagram of a partition of the form $(\alpha_1, \dots, \alpha_{d_\lambda} \mid \alpha_1 + 1, \dots, \alpha_{d_\lambda} + 1)$

by removing a certain number of boxes, but at most one in each row. If λ is self-conjugate, i.e. $Y(\lambda)$ is symmetric, then there is exactly one such possibility: the removal of one box in each of the first d_λ rows. Recall that d_λ denotes the number of diagonal boxes of $Y(\lambda)$. In the notation of Theorem 12, we then have $k = d_\lambda \leq r$ and $2m = |\lambda| - d_\lambda \leq \binom{r}{2}$, i.e. $0 \leq p = k + m = (|\lambda| + d_\lambda)/2 \leq \binom{r+1}{2} = \dim \mathfrak{m}_r^*$. \square

Since the boundary map ∂ and the differential d are homomorphisms of representations (i.e. they commute with the action of \mathbf{GL}_r), we get the “ \supseteq part” of the following theorem, which we refer to as “Sigg’s theorem”, even though Sigg’s version really deals with homology but this version deals with cohomology.

Theorem 14 (Sigg’s Theorem). [Sig, Theorem 7] *Let S_p denote the set of all self-conjugate partitions λ with $|\lambda| = 2p - d_\lambda$ and $l(\lambda) \leq r$. Then the decomposition into irreducibles of the p^{th} cohomology space of the free two-step nilpotent Lie algebras of rank r as a representation of \mathbf{GL}_r is given by*

$$H^p(\mathfrak{m}_r) \cong \bigoplus_{\lambda \in S_p} V(\lambda).$$

The proof of this theorem will be in the next section. For now, a useful corollary is mentioned, given the following dimension formula from [FuHa, §6.1]:

$$\dim V(\lambda) = \prod_{(i,j) \in Y(\lambda)} \frac{r - i + j}{h(i,j)},$$

where $h(i,j)$ denotes the hook number of the box (i,j) in the Young diagram $Y(\lambda)$.

Corollary 15. [Sig, Corollary 14] *The Betti numbers of the free two-step nilpotent Lie algebras of rank r are given by*

$$b_p(\mathfrak{m}_r) = \dim H^p(\mathfrak{m}_r) = \sum_{\lambda \in S_p} \prod_{(i,j) \in Y(\lambda)} \frac{r - i + j}{h(i,j)},$$

where S_p is described in Theorem 14.

3.2 Proof of Sigg’s Theorem. As mentioned, d , ∂ and Δ commute with the \mathbf{GL}_r action. Also, from Theorem III.4, $\ker d = \text{im } d \oplus \ker \Delta$, so $\ker \Delta_p$ is canonically isomorphic to $H^p(\mathfrak{m}_r)$. Sigg shows that Δ acts by multiplication by a scalar on irreducible subrepresentations.

Theorem 16. [Sig, Theorem 10] *Let $\varrho = D\rho : \mathfrak{gl}_r \rightarrow \mathfrak{gl}(\Lambda \mathfrak{m}_r^*)$ denote the Lie algebra homomorphism of our \mathbf{GL}_r action. Set $\mathcal{D} = \sum_{i=1}^r \varrho(E_{ii})$ and consider the Casimir operator with respect to ϱ , i.e. $\mathcal{C} = \sum_{1 \leq i,j \leq r} \varrho(E_{ij})\varrho(E_{ji})$. Then*

$$\Delta = \frac{1}{2}(r\mathcal{D} - \mathcal{C}).$$

The proof can be found in [Sig].

Theorem 17. [Sig, Theorem 11] *Identify $V(\lambda)$ with an irreducible subrepresentation of $\Lambda\mathfrak{m}_r^*$ with highest weight $\lambda = (\alpha_1, \dots, \alpha_{d_\lambda} \mid \beta_1, \dots, \beta_{d_\lambda})$. Then*

$$\Delta|_{V(\lambda)} = Q(\lambda) \cdot \text{id}_{V(\lambda)},$$

where

$$Q(\lambda) = \frac{1}{2} \sum_{i=1}^{d_\lambda} Q(\alpha_i \mid \beta_i) = \frac{1}{2} \sum_{i=1}^{d_\lambda} (\alpha_i + \beta_i + 1)(\beta_i - \alpha_i) \in \mathbb{Z}.$$

Proof. The operator \mathfrak{D} commutes with ϱ and reduces to a multiplication with $|\lambda| = \sum_{i=1}^r \lambda_i$ on each irreducible component with highest weight λ . According to Proposition 9, the eigenvalue $c(\lambda)$ of the Casimir operator on $V(\lambda)$ is given by

$$c(\lambda) = \sum_{i=1}^r \lambda_i^2 - 2i\lambda_i + (r+1)\lambda_i.$$

Since $\Delta = \frac{1}{2}(r\mathfrak{D} - \mathfrak{C})$ we arrive at the following formula for the Laplace eigenvalue:

$$(5) \quad Q(\lambda) = -\frac{1}{2} \sum_{i=1}^r (\lambda_i^2 - 2i\lambda_i + \lambda_i).$$

In order to prove that the above expression for $Q(\lambda)$ is correct, we proceed with an induction on $d = d_\lambda$. Assume $d = 1$, i.e., $\lambda = (\alpha \mid \beta)$ with certain $\alpha, \beta \in \mathbb{N} \cup \{0\}$. Then $\lambda = (\alpha + 1, 1, \dots, 1)$ and $l(\lambda) = \beta + 1$. According to formula (5) we then get $Q(\alpha \mid \beta) = \frac{1}{2}(\alpha + \beta + 1)(\beta - \alpha)$. Now we suppose that the theorem holds for $d - 1$. If we define $\tilde{\lambda} = (\alpha_1, \dots, \alpha_{d-1} \mid \beta_1, \dots, \beta_{d-1})$ it remains to be shown that

$$2Q(\lambda) - 2Q(\tilde{\lambda}) = (\alpha_d + \beta_d + 1)(\beta_d - \alpha_d).$$

It follows immediately from the definition of the Frobenius notation that

$$\tilde{\lambda} = (\lambda_1, \dots, \lambda_{d-1}, \lambda_d - \alpha_d - 1, \lambda_{d+1} - 1, \dots, \lambda_{d+\beta_d} - 1, \lambda_{d+\beta_d+1}, \dots, \lambda_r),$$

and with formula (5) we compute

$$\begin{aligned}
-2Q(\bar{\lambda}) &= -2Q(\lambda_1, \dots, \lambda_{d-1}) \\
&\quad + (\lambda_d - \alpha_d - 1)^2 - 2d(\lambda_d - \alpha_d - 1) + \lambda_d - \alpha_d - 1 \\
&\quad - 2Q(\lambda_{d+1} - 1, \dots, \lambda_{d+\beta_d} - 1) - 2Q(\lambda_{d+\beta_d+1}, \dots, \lambda_r) \\
&= -2Q(\lambda) + (\alpha_d + 1)(2d + \alpha_d - 2\lambda_d) \\
&\quad + 2 \sum_{i=d+1}^{d+\beta_d} i - 2 \sum_{i=d+1}^{d+\beta_d} \lambda_i \\
&= -2Q(\lambda) - (\alpha_d + 1)\alpha_d + (d + \beta_d)(d + \beta_d + 1) \\
&\quad - d(d + 1) - 2 \sum_{i=d+1}^{d+\beta_d} d \\
&= -2Q(\lambda) - \alpha_d^2 - \alpha_d + \beta_d^2 + \beta_d \\
&= -2Q(\lambda) + (\alpha_d + \beta_d + 1)(\beta_d - \alpha_d),
\end{aligned}$$

which completes the proof. \square

Proof of Theorem 14. It follows from Sigg's results on the decomposition of the cochain complex into irreducibles (see Theorem 12) that each partition λ with Frobenius notation $(\alpha_1, \dots, \alpha_d \mid \beta_1, \dots, \beta_d)$ that occurs with a positive multiplicity satisfies the relationship $\alpha_i \leq \beta_i$ for $i = 1, \dots, d$. If we are given such a partition, the formula for the Laplace eigenvalues in Theorem 17 says that $Q(\lambda)$ is zero if and only if $\alpha_i = \beta_i$ for all $i = 1, \dots, d$. This is exactly the definition for self-conjugate partitions. According to Proposition 13, the multiplicity of a self-conjugate partition is 1, and since the kernel of the Laplacian is a complete set of cohomology class representatives (Remark III.5), the theorem is proved. \square

Sigg ends his paper by remarking that in the case of free three-step nilpotent Lie algebras, there is a partition that appears with multiplicity 2 in the second exterior power, but only one copy survives in the homology. Hence, Sigg's theorem does not generalise to k -step nilpotent Lie algebras for $k \geq 3$.

3.3 Cohomology of the first three free two-steps.

Here is a breakdown of $H^*(\mathfrak{m}_2)$ in terms of Young diagrams.

$$\begin{aligned}
H^0(\mathfrak{m}_2) &= \mathbb{C}, \text{ as usual;} \\
H^1(\mathfrak{m}_2) &\cong V(1) \hat{=} \square, \text{ with a highest weight vector of } e_1; \\
H^2(\mathfrak{m}_2) &\cong V(2, 1) \hat{=} \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array}, \text{ with a highest weight vector of } e_1 f_{12}; \\
H^3(\mathfrak{m}_2) &\cong V(2, 2) \hat{=} \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}, \text{ with a highest weight vector of } e_1 e_2 f_{12}.
\end{aligned}$$

Of course, $H^0(\mathfrak{m}_r) = \mathbb{C}$, for all $r \in \mathbb{N}$.

In fact, $H^1(\mathfrak{m}_r) \hat{=} \square$ and $H^2(\mathfrak{m}_r) \hat{=} \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array}$ for all r such that $r \geq 2$. This is because if $\lambda \in \mathcal{P}_r$ is such that $l(\lambda) = r$, then each $\lambda_i \geq 1$. If, furthermore, λ is symmetric, then $\lambda_1 = r$. Thus, $|\lambda| \geq 2r - 1$, hence $|\lambda| + d_\lambda \geq 2r$ and therefore, if $V(\lambda) \hookrightarrow H^p(\mathfrak{m}_r)$, then $2p = |\lambda| + d_\lambda \geq 2r$, by Sigg's theorem, and $p \geq r$. We have therefore proven the following:

Lemma 18. *If $\lambda \in \mathcal{P}_r$ is such that $\lambda \notin \mathcal{P}_{r-1}$, then $|\lambda| \geq 2r - 1$. Furthermore, if $V(\lambda) \hookrightarrow H^p(\mathfrak{m}_r)$, then $p \geq l(\lambda) = r$.*

We now do a similar breakdown for $H^*(\mathfrak{m}_3)$. Write "h.w.v." as shorthand for "highest weight vector". That the given highest weight vectors are correct is easily checked by counting the different indices.

$$H^0(\mathfrak{m}_3) = \mathbb{C};$$

$$H^1(\mathfrak{m}_3) \cong V(1) \hat{=} \square, \text{ with h.w.v. } e_1;$$

$$H^2(\mathfrak{m}_3) \cong V(2,1) \hat{=} \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array}, \text{ with h.w.v. } e_1 f_{12};$$

$$H^3(\mathfrak{m}_3) \cong V(2,2) \oplus V(3,1,1) \hat{=} \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} \oplus \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array},$$

with respective h.w.v.'s of $e_1 e_2 f_{12}$ and $e_1 f_{12} f_{13}$;

$$H^4(\mathfrak{m}_3) \cong V(3,2,1) \hat{=} \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}, \text{ with h.w.v. } e_1 e_2 f_{12} f_{13};$$

$$H^5(\mathfrak{m}_3) \cong V(3,3,2) \hat{=} \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}, \text{ with h.w.v. } e_1 e_2 f_{12} f_{13} f_{23};$$

$$H^6(\mathfrak{m}_3) \cong V(3,3,3) \hat{=} \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}, \text{ with h.w.v. } e_1 e_2 e_3 f_{12} f_{13} f_{23}.$$

Finally, because we'll need the results later, here is a breakdown for $H^*(\mathfrak{m}_4)$. Note that highest weight vectors don't change from $H^*(\mathfrak{m}_3)$ when $l(\lambda) < 4$. Also note

that $H^i(\mathfrak{m}_3)$ and $H^i(\mathfrak{m}_4)$ have the same Young diagrams when $i = 0, 1, 2$ and 3 .

$$H^0(\mathfrak{m}_4) = \mathbb{C};$$

$$H^1(\mathfrak{m}_4) \cong V(1) \hat{=} \square, \text{ with h.w.v. } e_1;$$

$$H^2(\mathfrak{m}_4) \cong V(2, 1) \hat{=} \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array}, \text{ with h.w.v. } e_1 f_{12};$$

$$H^3(\mathfrak{m}_4) \cong V(2, 2) \oplus V(3, 1, 1) \hat{=} \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \oplus \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & & \\ \hline \end{array},$$

with respective h.w.v.'s of $e_1 e_2 f_{12}$ and $e_1 f_{12} f_{13}$;

$$H^4(\mathfrak{m}_4) \cong V(3, 2, 1) \oplus V(4, 1, 1, 1) \hat{=} \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \\ \hline \square & & \\ \hline \end{array} \oplus \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & & & \\ \hline \square & & & \\ \hline \end{array},$$

with respective h.w.v.'s of $e_1 e_2 f_{12} f_{13}$ and $e_1 f_{12} f_{13} f_{14}$;

$$H^5(\mathfrak{m}_4) \cong V(3, 3, 2) \oplus V(4, 2, 1, 1) \hat{=} \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \square & & \\ \hline \end{array} \oplus \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & \square & & \\ \hline \square & & & \\ \hline \end{array},$$

with respective h.w.v.'s of $e_1 e_2 f_{12} f_{13} f_{23}$ and $e_1 e_2 f_{12} f_{13} f_{14}$;

$$H^6(\mathfrak{m}_4) \cong V(3, 3, 3) \oplus V(4, 3, 2, 1) \hat{=} \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \square & & \\ \hline \end{array} \oplus \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \\ \hline \square & \square & & \\ \hline \square & & & \\ \hline \end{array},$$

with respective h.w.v.'s of $e_1 e_2 e_3 f_{12} f_{13} f_{23}$ and $e_1 e_2 f_{12} f_{13} f_{14} f_{23}$;

$$H^7(\mathfrak{m}_4) \cong V(4, 3, 3, 1) \oplus V(4, 4, 2, 2) \hat{=} \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \\ \hline \square & \square & & \\ \hline \square & & & \\ \hline \end{array} \oplus \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \\ \hline \square & \square & & \\ \hline \square & & & \\ \hline \end{array},$$

with respective h.w.v.'s of $e_1 e_2 e_3 f_{12} f_{13} f_{14} f_{23}$
and $e_1 e_2 f_{12} f_{13} f_{14} f_{23} f_{24}$;

$$H^8(\mathfrak{m}_4) \cong V(4, 4, 3, 2) \hat{=} \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \\ \hline \square & \square & & \\ \hline \square & & & \\ \hline \end{array}, \text{ with h.w.v. } e_1 e_2 e_3 f_{12} f_{13} f_{14} f_{23} f_{24}.$$

$$H^9(\mathfrak{m}_4) \cong V(4, 4, 4, 3) \hat{=} \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \\ \hline \square & \square & \square & \\ \hline \square & & & \\ \hline \end{array}, \text{ with h.w.v. } e_1 e_2 e_3 f_{12} f_{13} f_{14} f_{23} f_{24} f_{34}.$$

$$H^{10}(\mathfrak{m}_4) \cong V(4, 4, 4, 4) \hat{=} \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \\ \hline \square & \square & \square & \\ \hline \square & & & \\ \hline \end{array}, \text{ with h.w.v. } e_1 e_2 e_3 e_4 f_{12} f_{13} f_{14} f_{23} f_{24} f_{34}.$$

§4 HIGHER OPERATORS ON $H^*(\mathfrak{m}_r)$

4.1 Searching for hypercubes. Recall the definitions of hypercubes and of the central representation, as described in Chapter II. In short, if $z = \dim Z(\mathfrak{m}_r) = \binom{r}{2}$, then we are looking for a z -cube divided into $z + 1$ levels as follows (with $0 \leq \nu_z < \dots < \nu_1 < \nu_0 \leq z + r$).

Level 0 of cube: There is a single vector in $H^{\nu_0}(\mathfrak{m}_r)$, with z operators (all primary, all secondary or all higher, as defined in Chapter II) mapping it to z linearly

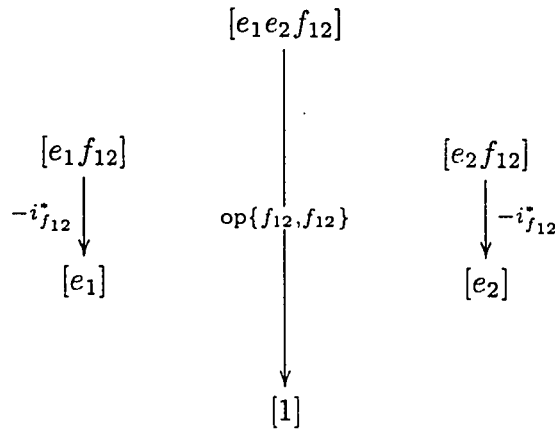
independent vectors in $H^{\nu_1}(\mathfrak{m}_r)$.

Level i of cube: There are $\binom{z}{i}$ linearly independent vectors in $H^{\nu_i}(\mathfrak{m}_r)$, each mapped to by exactly i operators coming from level $i-1$, and each mapping to $i+1$ linearly independent vectors in level $i+1$.

Level z of cube: There is a single vector in $H^{\nu_z}(\mathfrak{m}_r)$, mapped to by z operators from the z linearly independent vectors in level $z-1$.

4.2 Example: $r = 2$.

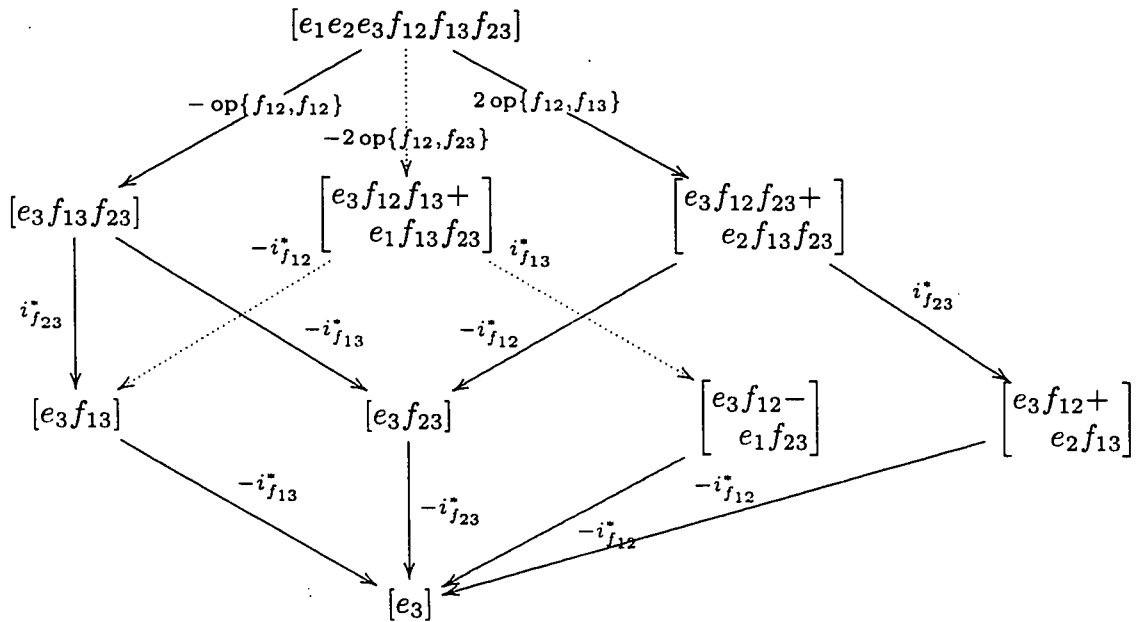
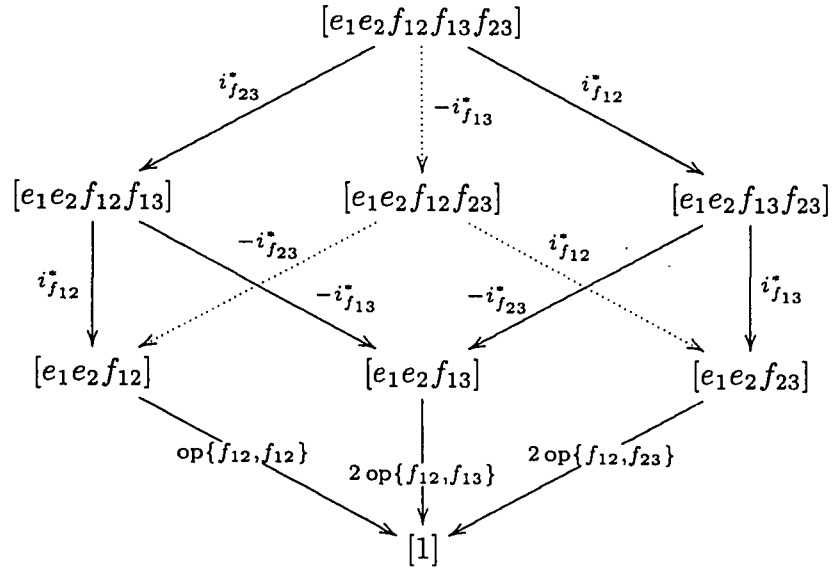
When $r = 2$, then $z = 1$ and we want two linearly independent vectors connected by a primary or secondary operator. Since \mathfrak{m}_2 is isomorphic to \mathfrak{h}_1 , we have already seen (in Chapter II and in Chapter III) that 1-cubes exist in $H^*(\mathfrak{m}_2)$. In fact, primary operators form 1-cubes bridging $H^2(\mathfrak{m}_2)$ and $H^1(\mathfrak{m}_2)$, and a single secondary operator, $\text{op}\{f_{12}, f_{12}\}$, maps any top class to a "bottom class" (a non-trivial element of the field $H^0(\mathfrak{m}_r)$).



4.3 Example: $r = 3$.

When $r = 3$, then $z = 3$ and we want levels 0 and 3 of our 3-cube to each contain one vertex, and levels 1 and 2 to each contain 3 vertices. Here are two examples of desired 3-cubes. They are dual to each other, in the sense that multiplication of the span of level i in the first diagram by the span of level $3-i$ in the second

diagram is non-degenerate.



The first cube really is an honest 3-cube. However, the second one only satisfies our requirements if we identify $[e_3f_{12} + e_2f_{13}]$ with $[e_3f_{12} - e_1f_{23}]$, which we can because $-i_{f_{12}}^*$ sends both to $[e_3]$. This sort of "cheating" identification will be

used to find what looks like a 6-cube in $H^*(\mathfrak{m}_4)$ (see subsection 4.7). In fact, while searching for $\binom{\tau}{2}$ -cubes in $H^*(\mathfrak{m}_\tau)$, when $\tau \geq 4$, some extra tools are needed. We are now ready to use Sigg's results to analyse the actions of the central representation on the cohomology of \mathfrak{m}_τ .

4.4 Bigradation on $\Lambda\mathfrak{m}_\tau^*$.

If $V = \bigoplus_{p,q} V_q^p$ is a bigraded vector space, a subspace U of V is called a *bigraded subspace* if $U = \bigoplus_{p,q} U \cap V_q^p$. For example, the images and kernels of bihomogeneous endomorphisms on V are bigraded subspaces.

Define $\tilde{X}_q^p := \Lambda^{p-q} E_\tau^* \otimes \Lambda^q F_\tau^* \subseteq \Lambda^p \mathfrak{m}_\tau^*$, so $\Lambda\mathfrak{m}_\tau^* = \bigoplus_{p,q} \tilde{X}_q^p$ is a bigraded space. First note that the differential d is a homomorphism of bidegree $(1, -1)$:

$$d: \tilde{X}_q^p \longrightarrow \tilde{X}_{q-1}^{p+1}.$$

Hence, $\ker d$ and $\operatorname{im} d$ are bigraded subspaces, and we can define a bigradation on the cohomology:

$$H_q^p(\mathfrak{m}_\tau) = \tilde{X}_q^p \cap \ker d / \tilde{X}_q^p \cap \operatorname{im} d.$$

Letting $T = \binom{\tau+1}{2} = \dim \mathfrak{m}_\tau^*$ and $z = \dim Z(\mathfrak{m}_\tau) = \dim F_\tau^* = \binom{\tau}{2}$, we have $\star(\tilde{X}_q^p) = \tilde{X}_{z-q}^{T-p}$, where \star is the Hodge star map defined in chapter III. Thus, ∂ is bihomogeneous of degree $(-1, 1)$, and $\Delta = d\partial + \partial d$ is bihomogeneous of degree $(0, 0)$. Therefore, $\ker \Delta$ is also bigraded and the decomposition $\ker d = \ker \Delta \oplus \operatorname{im} d$ is one of bigraded subspaces, meaning

$$\tilde{X}_q^p \cap \ker d = \left(\tilde{X}_q^p \cap \ker \Delta \right) \oplus \left(\tilde{X}_q^p \cap \operatorname{im} d \right).$$

Let $X_q^p = \tilde{X}_q^p \cap \ker \Delta_p$ and note that X_q^p is canonically isomorphic to $H_q^p(\mathfrak{m}_\tau)$.

Lemma 19. For $p \in \{0, \dots, T\}$ and $q \in \{0, \dots, z\}$,

$$\tilde{\star}(H_q^p(\mathfrak{m}_\tau)) = H_{z-q}^{T-p}(\mathfrak{m}_\tau),$$

where $\tilde{\star}$ is given in Definition III.8.

Proof. Recall from Chapter III the projection map $\pi: \ker d = \ker \Delta \oplus \operatorname{im} d \rightarrow \ker \Delta$. We have $\pi: \tilde{X}_q^p \cap \ker d \rightarrow \tilde{X}_q^p \cap \ker \Delta$. Therefore, if $\alpha \in \tilde{X}_q^p \cap \ker \Delta_p$, then

$$\tilde{\star}[\alpha] = [\star\pi(\alpha)] \in \left[\tilde{X}_{z-q}^{T-p} \cap \ker \Delta_{T-p} \right] = \left[X_{z-q}^{T-p} \right] = H_{z-q}^{T-p}(\mathfrak{m}_\tau).$$

Since $\tilde{\star}\tilde{\star} = (-1)^{p(T-p)} \operatorname{id}_{H^*(\mathfrak{m}_\tau)}$ on $H^*(\mathfrak{m}_\tau)$, by Theorem III.9, and since

$$\tilde{\star}: H_{z-q}^{T-p}(\mathfrak{m}_\tau) \longrightarrow H_q^p(\mathfrak{m}_\tau),$$

we have $\tilde{\star} : H_q^p(\mathfrak{m}_r) \rightarrow H_{z-q}^{T-p}(\mathfrak{m}_r)$ is an isomorphism with inverse

$$(-1)^{p(T-p)} \tilde{\star} |_{H_{z-q}^{T-p}(\mathfrak{m}_r)}.$$

□.

We note that \tilde{X}_q^p is invariant under the \mathbf{GL}_r action, and so decomposes as a sum

$$\tilde{X}_q^p = \bigoplus_{\lambda \in \mathcal{P}_r^{p,q}} V(\lambda)$$

for some subset $\mathcal{P}_r^{p,q}$ of \mathcal{P}_r , where $V(\lambda)$ is an irreducible subspace of $\Lambda^p \mathfrak{m}_r^*$ with highest weight λ . Namely, $\mathcal{P}_r^{p,q}$ is the set of all $\lambda \in \mathcal{P}_r$ such that $|\lambda| = p + q$ (since $|\lambda|$ is precisely the number of indices in the highest weight vector) and such that $V(\lambda) \subseteq \Lambda^p \mathfrak{m}_r^*$ is non-trivial.

As seen in subsection 3.2, $\ker \Delta_p = \bigoplus_{\lambda \in \mathcal{S}_p} V(\lambda)$ is canonically isomorphic to $H^p(\mathfrak{m}_r)$, where \mathcal{S}_p is a subset of \mathcal{P}_r defined in Theorem 14 and where $V(\lambda)$ is an irreducible subrepresentation of $\Lambda^p \mathfrak{m}_r^*$ with highest weight λ . Since

$$\Lambda^p \mathfrak{m}_r^* = \bigoplus_q \tilde{X}_q^p = \bigoplus_q \bigoplus_{\lambda \in \mathcal{P}_r^{p,q}} V(\lambda),$$

we can decompose $\ker \Delta_p = \bigoplus_q \bigoplus_{\lambda \in \mathcal{P}_r^{p,q} \cap \mathcal{S}_p} V(\lambda)$. Furthermore, $\tilde{X}_q^p \cap \ker \Delta_p = \bigoplus_{\lambda \in \mathcal{P}_r^{p,q} \cap \mathcal{S}_p} V(\lambda)$.

Note that $i_{f_{jk}}$ is of bidegree $(-1, -1)$, hence all primary operators on $H^*(\mathfrak{m}_r)$ are also of bidegree $(-1, -1)$. As an immediate consequence, we show that $H^*(\mathfrak{m}_3)$ contains no 3-cubes whose edges are all primary operators. We first translate our decomposition of $H^*(\mathfrak{m}_3)$ into the language of bigradation, where H_q^p is shorthand for $H_q^p(\mathfrak{m}_r)$:

$$\begin{aligned} H^6(\mathfrak{m}_3) &= H_3^6 &= V(3, 3, 3); \\ H^5(\mathfrak{m}_3) &= H_3^5 &= V(3, 3, 2); \\ H^4(\mathfrak{m}_3) &= H_2^4 &= V(3, 2, 1); \\ H^3(\mathfrak{m}_3) &= H_2^3 \oplus H_1^3 &= V(3, 1, 1) \oplus V(2, 2); \\ H^2(\mathfrak{m}_3) &= H_1^2 &= V(2, 1); \\ H^1(\mathfrak{m}_3) &= H_0^1 &= V(1); \\ H^0(\mathfrak{m}_3) &= H_0^0 &= V(0). \end{aligned}$$

Thus, each i_z^* being of bidegree $(-1, -1)$, the two longest unbroken chains of primary operators are $H_3^5 \rightarrow H_2^4 \rightarrow H_1^3$ and $H_2^3 \rightarrow H_1^2 \rightarrow H_0^1$. They are both of

"length" 3, but a 3-cube has 4 levels. Hence, there are no desired 3-cubes in $H^*(\mathfrak{m}_3)$ with all edges primary operators.

4.5 Example: $r = 4$.

When $r = 4$, then $z = 6$ and we want the seven levels of our 6-cube to contain 1, 6, 15, 20, 15, 6 and 1 vertices (i.e. linearly independent vectors), respectively. To find these, we first translate into bigradations our decomposition of $H^*(\mathfrak{m}_4)$ from subsection 3.3, where H_q^p is shorthand for $H_q^p(\mathfrak{m}_4)$:

$$\begin{aligned}
H^{10}(\mathfrak{m}_4) &= H_6^{10} = V(4, 4, 4, 4); \\
H^9(\mathfrak{m}_4) &= H_6^9 = V(4, 4, 4, 3); \\
H^8(\mathfrak{m}_4) &= H_5^8 = V(4, 4, 3, 2); \\
H^7(\mathfrak{m}_4) &= H_4^7 \oplus H_5^7 = V(4, 3, 3, 1) \oplus V(4, 4, 2, 2); \\
H^6(\mathfrak{m}_4) &= H_3^6 \oplus H_4^6 = V(3, 3, 3) \oplus V(4, 3, 2, 1); \\
H^5(\mathfrak{m}_4) &= H_3^5 = V(3, 3, 2) \oplus V(4, 2, 1, 1); \\
H^4(\mathfrak{m}_4) &= H_3^4 \oplus H_2^4 = V(4, 1, 1, 1) \oplus V(3, 2, 1); \\
H^3(\mathfrak{m}_4) &= H_2^3 \oplus H_1^3 = V(3, 1, 1) \oplus V(2, 2); \\
H^2(\mathfrak{m}_4) &= H_1^2 = V(2, 1); \\
H^1(\mathfrak{m}_4) &= H_0^1 = V(1); \\
H^0(\mathfrak{m}_4) &= H_0^0 = V(0).
\end{aligned}$$

With the exception of $p = 5$, where $H^5(\mathfrak{m}_4) = H_3^5 \cong V(3, 3, 2, 0) \oplus V(4, 2, 1, 1)$, each H_q^p in this example corresponds to a single irreducible subrepresentation of $H^p(\mathfrak{m}_4)$. We find the following three distinct unbroken chains of maximal length, where right arrows are collections of possible primary operators:

$$\begin{aligned}
H_6^9 &\rightarrow H_5^8 \rightarrow H_4^7 \rightarrow H_3^6, \\
H_5^7 &\rightarrow H_4^6 \rightarrow H_3^5 \rightarrow H_2^4 \rightarrow H_1^3 \quad \text{and} \\
H_3^4 &\rightarrow H_2^3 \rightarrow H_1^2 \rightarrow H_0^1.
\end{aligned}$$

With each of these three chains, the highest weight vector of the left-most irreducible subrepresentation is mapped to the highest weight vector of the right-most irreducible subrepresentation by a composition of primary operators:

$$\begin{aligned}
-i_{f_{34}}i_{f_{24}}i_{f_{14}} : e_1e_2e_3f_{12}f_{13}f_{14}f_{23}f_{24}f_{34} &\in H_6^9 \mapsto e_1e_2e_3f_{12}f_{13}f_{23} \in H_3^6 \\
i_{f_{24}}i_{f_{23}}i_{f_{14}}i_{f_{13}} : e_1e_2f_{12}f_{13}f_{14}f_{23}f_{24} &\in H_5^7 \mapsto e_1e_2f_{12} \in H_1^3 \\
-i_{f_{14}}i_{f_{13}}i_{f_{12}} : e_1f_{12}f_{13}f_{14} &\in H_3^4 \mapsto e_1 \in H_0^1
\end{aligned}$$

However, there are no six-cubes of the desired form in $H^*(\mathfrak{m}_4)$ with all edges primary operators; there are not even enough levels! In other words, the central representation $i^* : \Lambda Z(\mathfrak{m}_4) \rightarrow \text{End}(H^*(\mathfrak{m}_4))$ (Definition II.3) is not faithful.

Ignoring ambiguity for a moment, recall that $\text{op}\{f_{kl}, f_{mn}\}[\alpha] = [i_{f_{kl}}d^{-1}i_{f_{mn}}(\alpha) + i_{f_{mn}}d^{-1}i_{f_{kl}}(\alpha)]$. We note that secondary operators are of bidegree $(-3, -1)$. It can now easily be checked that any non-trivial secondary operator, if it exists, would have one the following four forms:

$$H_6^{10} \xrightarrow{\text{sec}} H_5^7, \quad H_4^7 \xrightarrow{\text{sec}} H_3^4, \quad H_3^6 \xrightarrow{\text{sec}} H_2^3 \quad \text{or} \quad H_1^3 \xrightarrow{\text{sec}} H_0^0.$$

Thus, if there exists a hypercube in $H^*(\mathfrak{m}_4)$ of the desired form, it would have one of the following three forms, where short unlabelled right arrows represent collections of primary operators and long "sec" arrows represent collections of secondary operators.

$$H_6^{10} \xrightarrow{\text{sec}} H_5^7 \rightarrow H_4^6 \rightarrow H_3^5 \rightarrow H_2^4 \rightarrow H_1^3 \xrightarrow{\text{sec}} H_0^0,$$

$$H_6^9 \rightarrow H_5^8 \rightarrow H_4^7 \xrightarrow{\text{sec}} H_3^4 \rightarrow H_2^3 \rightarrow H_1^2 \rightarrow H_0^1, \quad \text{or}$$

$$H_6^9 \rightarrow H_5^8 \rightarrow H_4^7 \rightarrow H_3^6 \xrightarrow{\text{sec}} H_2^3 \rightarrow H_1^2 \rightarrow H_0^1.$$

For each H_q^p implicated in a potential hypercube, either the $p - q$'s are all even, or they all odd. This holds for $r \geq 2$, because primary operators send H_q^p to H_{q-1}^{p-1} (so $p - q$ is sent to $(p - 1) - (q - 1) = p - q$) and secondary operators send H_q^p to H_{q-1}^{p-3} (so $p - q$ is sent to $p - q - 2$).

A *Mathematica*® program, written for the purposes of finding a 6-cube in $H^*(\mathfrak{m}_4)$, found that there are no 6-cubes of the second or third form (i.e. the vertex in level 0 is a single vector in $H^9(\mathfrak{m}_4)$). There are not enough linearly independent vectors in H_3^4 in the images of the secondary operators, and there are not enough linearly independent vectors in H_3^6 in the images of any three composed primary operators. Twenty are needed in each, but:

Mathematica® Observations.

- (a) *There are, at most, only 14 linearly independent elements in $H^6(\mathfrak{m}_4)$ that are in the image of some composition of three primary operators;*
- (b) *there are only 15 non-trivial, linearly independent elements in $H^4(\mathfrak{m}_4)$ that are in the image of some secondary operator.*

Efforts were therefore concentrated on finding a 6-cube of the first form:

$$H_6^{10} \xrightarrow{\text{sec}} H_5^7 \rightarrow H_4^6 \rightarrow H_3^5 \rightarrow H_2^4 \rightarrow H_1^3 \xrightarrow{\text{sec}} H_0^0.$$

We note two observations regarding secondary operators on $H^*(\mathfrak{m}_4)$, the first about secondary operators from $H^{10}(\mathfrak{m}_4)$ to $H^7(\mathfrak{m}_4)$ and the second about secondary operators from $H^3(\mathfrak{m}_4)$ to $H^0(\mathfrak{m}_4)$. Their generalisations for all $r \geq 3$ are also given.

First, $i_w^*(H^1(\mathfrak{m}_4)) = \{0\}$, for all $w \in F_r^*$, so secondary operators defined from $H^3(\mathfrak{m}_4)$ to $H^0(\mathfrak{m}_4)$ (hence from H_1^3 to H_0^0), if they exist, have no ambiguity. This does hold in general, given that $H^1(\mathfrak{m}_r) = \text{span} \{[e_i] \mid i = 1, \dots, r\}$. What also holds in general is:

Theorem A. *For every non-trivial vector $w \in V(2, 2) \subset H^3(\mathfrak{m}_r)$, there exists a secondary operator $\text{op} \{f_{kl}, f_{mn}\}$ such that $\text{op} \{f_{kl}, f_{mn}\}(w) \neq 0$. In fact, each of these secondary operators sends some scalar multiple of some weight vector to 1.*

Proof. Note that $v = [e_1 e_2 f_{12}]$ is a highest weight vector of $V(2, 2)$.

For the first part, note that $w \in V(2, 2) = \mathbf{GL}_r \cdot v$, so

$$w = \left[\sum_{\mu_1 < \mu_2} \sum_{\nu_1 < \nu_2} a_{\nu_1, \nu_2}^{\mu_1, \mu_2} \cdot e_{\nu_1} e_{\nu_2} f_{\mu_1, \mu_2} \right],$$

for some set $\{a_{\nu_1, \nu_2}^{\mu_1, \mu_2} \in \mathbb{C} \mid 1 \leq \nu_1 < \nu_2 \leq r, 1 \leq \mu_1 < \mu_2 \leq r\}$. This sum representing w will contain some $a_{m, n}^{k, l} \cdot e_k e_l f_{mn}$ such that $0 \neq a_{m, n}^{k, l} \neq -a_{k, l}^{m, n}$ (note that $e_k e_l f_{mn} - e_m e_n f_{kl} = df_{kl} f_{mn}$ is a coboundary, so if w is a linear combination of these, it is trivial). Thus,

$$\text{op} \{f_{kl}, f_{mn}\} [w] = \frac{a_{m, n}^{k, l} + a_{k, l}^{m, n}}{2} \neq 0.$$

For the second part, we divide the proof into three subcases, with $k \neq l$, $m \neq n$ and no assumptions are made about ordering. The first case is $\text{op} \{f_{kl}, f_{kl}\}$, which sends $E_{k, 1}^2 E_{l, 2}^2(v) = [4e_k e_l f_{kl}]$ to 4. The second case is $\text{op} \{f_{kn}, f_{mn}\}$ where k, m and n are distinct; it sends $E_{k, 1} E_{m, 1} E_{n, 2}^2(v) = [2(e_k e_n f_{mn} + e_m e_n f_{kn})]$ to 2. Finally, $\text{op} \{f_{kl}, f_{mn}\}$, where k, l, m and n are distinct, sends the following vector to 1:

$$E_{k, 1} E_{l, 2} E_{m, 1} E_{n, 2}(v) = [e_k e_l f_{mn} + e_m e_n f_{kl} + e_k e_n f_{kl} + e_m e_l f_{kn}].$$

□

The second observation from $H^*(\mathfrak{m}_4)$ is that, for any $w_1, w_2 \in F_4^*$, $\text{op} \{w_1, w_2\}$ maps $H^{10}(\mathfrak{m}_4)$ into H_5^7 (if we ignore the ambiguity for a moment). Furthermore, for all $w \in F_4^*$, $i_w^* : H^8(\mathfrak{m}_4) \rightarrow H_4^7$. Thus, the onto map

$$p_{w_1, w_2} : H_5^7 \oplus H_4^7 / [i_{w_1}^* H^8(\mathfrak{m}_4) + i_{w_2}^* H^8(\mathfrak{m}_4)] \longrightarrow H_5^7,$$

defined by $p_{w_1, w_2}(\alpha + \beta + [i_{w_1}^* H^8(\mathfrak{m}_4) + i_{w_2}^* H^8(\mathfrak{m}_4)]) = \alpha$, whenever $\alpha \in H_5^7$ and $\beta \in H_4^7$, is well-defined. We can now define the operator

$$\widetilde{\text{op}} \{w_1, w_2\} : H^{10}(\mathfrak{m}_4) \rightarrow H^7(\mathfrak{m}_4)$$

by $\widetilde{\text{op}} \{w_1, w_2\} := p_{w_1, w_2} \cdot \text{op} \{w_1, w_2\}$.

With $H_5^7 = V(4, 4, 2, 2)$, this last result generalises as follows: Let V_r denote $V(r, \dots, r, r-2, r-2)$, the unique irreducible subrepresentation of $H^*(\mathfrak{m}_r)$ whose Young diagram consists of an $r \times r$ square of boxes with the bottom right 2×2 square missing. By composing the secondary operators $\text{op} \{w_1, w_2\}$ with maps p_{w_1, w_2} (defined as with $H^*(\mathfrak{m}_4)$, but in more generality below), we obtain well-defined maps $\widetilde{\text{op}} \{w_1, w_2\} : H^T(\mathfrak{m}_r) \rightarrow H^{T-3}(\mathfrak{m}_r)$ for any $w_1, w_2 \in F_r^*$. In fact, if t denotes the top class $e_1 \cdots e_r f_{12} f_{13} \cdots f_{r-1, r} \in \Lambda^T \mathfrak{m}_r$, we find

Theorem B. *Every $\widetilde{\text{op}} \{f_{kl}, f_{mn}\} [t]$ is non-trivial in $V_r \subseteq H^{T-3}(\mathfrak{m}_r)$, for $r \geq 2$. Moreover, $V_r = \text{span} \{ \widetilde{\text{op}} \{f_{kl}, f_{mn}\} [t] \mid 1 \leq k < l \leq r, 1 \leq m < n \leq r \}$.*

That V_r is in $H^{T-3}(\mathfrak{m}_r)$ is easily verified by Sigg's theorem:

$$2(T-3) - d_\lambda = 2 \left(\binom{r+1}{2} - 3 \right) - (r-2) = r^2 - 4 = |\lambda|,$$

$$\text{when } \lambda = (\underbrace{r, \dots, r}_{r-2 \text{ times}}, r-2, r-2).$$

The proof of Theorem B will require a few lemmas and a formal definition of $\widetilde{\text{op}}$. These are relegated to subsection 4.6. We end this subsection by observing that secondary operators from H_4^7 to H_3^4 have no ambiguity, since $H_4^5 = \{0\}$ in this case. This does not hold in general, as

$$H^5(\mathfrak{m}_r) = H_4^5(\mathfrak{m}_r) \oplus H_3^5(\mathfrak{m}_r) = V(5, 1, 1, 1, 1) \oplus V(4, 2, 1, 1) \oplus V(3, 3, 2)$$

when $r \geq 5$. We prove this as follows.

The calculations for breaking down $H^*(\mathfrak{m}_5)$ into irreducible subrepresentations are not shown, but

$$H^5(\mathfrak{m}_5) = V(5, 1, 1, 1, 1) \oplus V(4, 2, 1, 1) \oplus V(3, 3, 2).$$

It suffices to show that $H_2^5 = \{0\}$ when $r \geq 2$ and, when $r \geq 6$, no new partitions $\lambda \in \mathcal{P}_r$ with $l(\lambda) = r$ are such that $V(\lambda) \hookrightarrow \ker \Delta_5 \cong H^5(\mathfrak{m}_r)$.

That $H_2^5 = \{0\}$ certainly holds when $r \leq 4$. Thus, if $r \geq 5$ and $\lambda \in \mathcal{P}_r$ is such that $\lambda \notin \mathcal{P}_4$, then Lemma 18 implies $|\lambda| \geq 9$. Hence, if $V(\lambda) \hookrightarrow H^5(\mathfrak{m}_r)$, then $V(\lambda) \hookrightarrow H_q^5$ where $q \geq 4$.

To show that, for $r \geq 6$, $H^5(\mathfrak{m}_r)$ has no more irreducible subrepresentations than $H^5(\mathfrak{m}_5)$, note that if $l(\lambda) \geq 6$, then $|\lambda| \geq 11$ and $V(\lambda) \hookrightarrow H_q^5$ implies $q \geq 6$, so $V(\lambda) \hookrightarrow \widetilde{X}_q^5 = \Lambda^5 E_r^* \otimes \Lambda^{5-q} F_r^* = \{0\}$, since $5 - q < 0$.

This also proves that all primary operators on $H_3^6(\mathfrak{m}_r) \subset H^6(\mathfrak{m}_r)$ are trivial, and hence

Theorem C. For all $r \geq 4$, all secondary operators are defined on $H_3^6(\mathfrak{m}_r)$.

In other words, $H_3^6(\mathfrak{m}_r) \subseteq K_0^6(\mathfrak{m}_r)$, where K_j^k is from Definition II.8.

4.6 Proof of Theorem B.

Recall that $T = \binom{r+1}{2} = \dim \mathfrak{m}_r$ and that $z = \dim Z(\mathfrak{m}_r) = T - r$.

Lemma 20. The mapping $d: \tilde{X}_z^p \rightarrow \tilde{X}_{z-1}^{p+1}$ is injective when $p \in \{0\} \cup \mathbb{N}$.

Proof.

$$\begin{aligned} d: \tilde{X}_z^p &\rightarrow \tilde{X}_{z-1}^{p+1} \text{ is injective} && \text{iff} \\ \partial: \tilde{X}_{z-1}^{p+1} &\rightarrow \tilde{X}_z^p \text{ is onto} && \text{iff} \\ d: \tilde{X}_1^{T-p-1} &\rightarrow \tilde{X}_0^{T-p} \text{ is onto} && \text{iff} \\ d: (\Lambda^{T-p-2} E_r^*) \otimes F_r^* &\rightarrow \Lambda^{T-p} E_r^* \text{ is onto} && \text{(which it clearly is).} \end{aligned}$$

□

Lemma 21. $H^{T-3}(\mathfrak{m}_r) = V_r \oplus H_{z-2}^{T-3}(\mathfrak{m}_r)$.

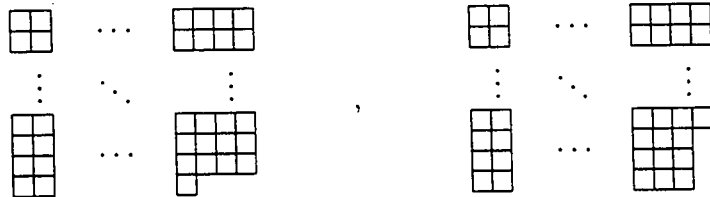
Proof. Since $\tilde{X}_{z-j}^{T-3} = \Lambda^{r+j-3} E_r^* \otimes \Lambda^{z-j} F_r^*$ is trivial when $j > 3$ or when $j < 0$, we have

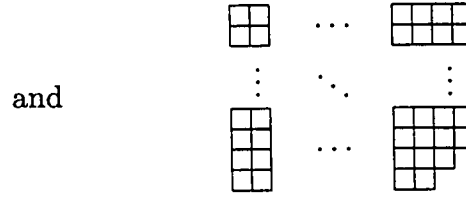
$$\Lambda^{T-3} \mathfrak{m}_r^* = \tilde{X}_z^{T-3} \oplus \tilde{X}_{z-1}^{T-3} \oplus \tilde{X}_{z-2}^{T-3} \oplus \tilde{X}_{z-3}^{T-3}.$$

That $\tilde{X}_{z-3}^{T-3} \subseteq B^{T-3}(\mathfrak{m}_r)$ follows from the observation that

$$\begin{aligned} e_1 \cdots e_r \cdot f_{12} \cdots \widehat{f_{ij}} \cdots \widehat{f_{kl}} \cdots \widehat{f_{mn}} \cdots f_{r-1,r} \\ = d \left((\pm) e_1 \cdots \widehat{e_i} \cdots \widehat{e_j} \cdots e_r \cdot f_{12} \cdots \widehat{f_{kl}} \cdots \widehat{f_{mn}} \cdots f_{r-1,r} \right). \end{aligned}$$

Lemma 20 implies $\tilde{X}_z^{T-3} \cap Z^{T-3}(\mathfrak{m}_r) = \{0\}$. Thus, $H^{T-3}(\mathfrak{m}_r) = H_{z-1}^{T-3} \oplus H_{z-2}^{T-3}$, where H_{z-1}^{T-3} is the direct sum of all $V(\lambda)$ with $\lambda \in S_r$ such that $|\lambda| = (T-3) + (z-1) = \binom{r+1}{2} + \binom{r}{2} - 4 = r^2 - 4$. However, the only three partitions $\mu \in \mathcal{P}_r$ satisfying $|\mu| = r^2 - 3$ have Young diagrams which are $r \times r$ grids of squares with three squares missing from the bottom-right corner, as follows:





All partitions in \mathcal{P}_r satisfying $|\lambda| = r^2 - 4$ can be obtained by removing one box from the bottom-right corner of any of the above Young diagrams. Thus, the only *symmetric* partition such that $|\lambda| = r^2 - 4$ is $\lambda = (r, \dots, r, r-2, r-2)$, which is obtained by removing the diagonal box from the third of the above diagrams. Hence $H_{z-1}^{T-3} = V_r$. \square

We are now ready to define secondary operators on the top class. By subsection 3.3, $H^2(\mathfrak{m}_r) = V(2, 1) = H_1^2$ when $r \geq 2$. Hence,

$$H^{T-2}(\mathfrak{m}_r) = \bar{\kappa}H^2(\mathfrak{m}_r) = \bar{\kappa}H_1^2 = H_{z-1}^{T-2}$$

and $i_w^*H^{T-2}(\mathfrak{m}_r) \subseteq H_{z-2}^{T-3}$. Thus, $\widetilde{\text{op}} \{w_1, w_2\} : H^T(\mathfrak{m}_r) \longrightarrow H_{z-2}^{T-3}$ can be defined without ambiguity, for any $w_1, w_2 \in F_r^*$:

Definition 22. Let $\widetilde{\text{op}} \{w_1, w_2\} := p_{w_1, w_2} \cdot \text{op} \{w_1, w_2\}$, where the onto map

$$p_{w_1, w_2} : (H_{z-1}^{T-3} \oplus H_{z-2}^{T-3}) / [i_{w_1}^*H^{T-2}(\mathfrak{m}_r) + i_{w_2}^*H^{T-2}(\mathfrak{m}_r)] \longrightarrow H_{z-1}^{T-3}$$

is given by $p_{w_1, w_2}(\alpha + \beta + [i_{w_1}^*H^{T-2}(\mathfrak{m}_r) + i_{w_2}^*H^{T-2}(\mathfrak{m}_r)]) = \alpha$, whenever $\alpha \in H_{z-1}^{T-2}$ and $\beta \in H_{z-2}^{T-2}$.

Henceforth, let θ_{ij} denote the position of f_{ij} in the ordered list

$$\{f_{12}, f_{13}, \dots, f_{r-1, r}\}.$$

For example, $\theta_{1,2} = 1$ and $\theta_{r-1, r} = z$. Thus, $f := f_{12} \cdots f_{r-1, r} = (-1)^{\theta_{ij}+1} \widehat{f_{ij}} \cdot \widehat{f_{ij}}$, where $\widehat{f_{ij}}$ is shorthand for $f_{12} \cdots \widehat{f_{ij}} \cdots f_{r-1, r}$. Therefore, $t = (-1)^{\theta_{ij}+r+1} f_{ij} \cdot e_1 \cdots e_r \cdot \widehat{f_{ij}}$ and one can show

$$(6) \quad \widetilde{\text{op}} \{f_{kl}, f_{mn}\} [t] = \frac{1}{2} \left[(-1)^{\theta_{mn}+k+l+r} \widehat{e_k e_l f_{mn}} + (-1)^{\theta_{kl}+m+n+r} \widehat{e_m e_n f_{kl}} \right],$$

where $\widehat{e_i e_j}$ is shorthand for $e_1 \cdots \widehat{e_i} \cdots \widehat{e_j} \cdots e_r$.

Lemma 23. *No $\widetilde{\text{op}} \{f_{kl}, f_{mn}\} [t]$ is a coboundary.*

Proof. By Lemma 20, $d : \widetilde{X}_z^{T-4} \rightarrow H_{z-1}^{T-3}$ is injective. Hence, all elements of the form $d(e_1 \cdots \widehat{e}_k \cdots \widehat{e}_l \cdots \widehat{e}_m \cdots \widehat{e}_n \cdots e_r f)$, with $k < l < m < n$, are linearly independent, where $f = f_{12} \cdots f_{r-1,r}$ and

$$\{e_1 \cdots \widehat{e}_k \cdots \widehat{e}_l \cdots \widehat{e}_m \cdots \widehat{e}_n \cdots e_r f \mid 1 \leq k < l < m < n \leq r\}$$

is a basis for \widetilde{X}_z^{T-4} . Note that

$$\begin{aligned} d(e_1 \cdots \widehat{e}_k \cdots \widehat{e}_l \cdots \widehat{e}_m \cdots \widehat{e}_n \cdots e_r f) &= (-1)^{r-4} [\\ &(-1)^{k+l+\theta_{kl}+1} \widehat{e}_m \widehat{e}_n \widehat{f}_{kl} + (-1)^{m+n+\theta_{mn}+1} \widehat{e}_k \widehat{e}_l \widehat{f}_{mn} + \\ &(-1)^{k+m+\theta_{km}} \widehat{e}_l \widehat{e}_n \widehat{f}_{km} + (-1)^{l+n+\theta_{ln}} \widehat{e}_k \widehat{e}_m \widehat{f}_{ln} + \\ &(-1)^{k+n+\theta_{kn}+1} \widehat{e}_l \widehat{e}_m \widehat{f}_{kn} + (-1)^{l+m+\theta_{lm}+1} \widehat{e}_k \widehat{e}_n \widehat{f}_{lm}]. \end{aligned}$$

Also, $\{d(e_1 \cdots \widehat{e}_k \cdots \widehat{e}_l \cdots \widehat{e}_m \cdots \widehat{e}_n \cdots e_r f) \mid 1 \leq k < l < m < n \leq r\}$ is a basis for $B^{T-3}(\mathfrak{m}_r) \cap H_{z-1}^{T-3}$.

We now show that, for fixed k, l, m and n ,

$$\widetilde{\text{op}} \{f_{kl}, f_{mn}\} [t] = \frac{1}{2} \left[(-1)^{\theta_{kl}+m+n+r} \widehat{e}_m \widehat{e}_n \widehat{f}_{kl} + (-1)^{\theta_{mn}+k+l+r} \widehat{e}_k \widehat{e}_l \widehat{f}_{mn} \right]$$

is linearly independent from all coboundaries. Suppose

$$0 = a \cdot \widetilde{\text{op}} \{f_{kl}, f_{mn}\} [t] + \sum_{1 \leq w < x < y < z \leq r} \alpha_{wxyz} d(e_1 \cdots \widehat{e}_w \cdots \widehat{e}_x \cdots \widehat{e}_y \cdots \widehat{e}_z \cdots e_r f)$$

for some scalars a and $\{\alpha_{wxyz} \mid 1 \leq w < x < y < z \leq r\}$. We want to show $a = 0$ and each $\alpha_{wxyz} = 0$. For ease of notation, say $\alpha_{wxyz} = \alpha_{\sigma(w)\sigma(x)\sigma(y)\sigma(z)}$ for every permutation σ on $\{w, x, y, z\}$. So we can say

$$\begin{aligned} 0 &= ((-1)^{\theta_{kl}+m+n+r} a + (-1)^{k+l+\theta_{kl}+1} \alpha_{klmn}) \widehat{e}_m \widehat{e}_n \widehat{f}_{kl} + \\ &((-1)^{\theta_{mn}+k+l+r} a + (-1)^{m+n+\theta_{mn}+1} \alpha_{klmn}) \widehat{e}_k \widehat{e}_l \widehat{f}_{mn} + \\ &(-1)^{k+m+\theta_{km}} \alpha_{klmn} \widehat{e}_l \widehat{e}_n \widehat{f}_{km} + (-1)^{l+n+\theta_{ln}} \alpha_{klmn} \widehat{e}_k \widehat{e}_m \widehat{f}_{ln} + \\ &(-1)^{k+n+\theta_{kn}+1} \alpha_{klmn} \widehat{e}_l \widehat{e}_m \widehat{f}_{kn} + (-1)^{l+m+\theta_{lm}+1} \alpha_{klmn} \widehat{e}_k \widehat{e}_n \widehat{f}_{lm} \\ &+ \sum_{\substack{w < x, y < z \\ \{w, x, y, z\} \neq \{k, l, m, n\}}} \alpha_{wxyz} (-1)^{y+z+\theta_{yz}+\tau_{wxyz}} \widehat{e}_w \widehat{e}_x \widehat{f}_{yz}. \end{aligned}$$

where $\tau_{wxyz} = 0$ if $w < y < x < z$ or $y < w < z < x$, and $\tau_{wxyz} = 1$ otherwise. Thus,

$$\begin{aligned} 0 = & ((-1)^{\theta_{kl}+m+n+r} a + (-1)^{k+l+\theta_{kl}+1} \alpha_{klmn}) \widehat{e_m e_n f_{kl}} + \\ & ((-1)^{\theta_{mn}+k+l+r} a + (-1)^{m+n+\theta_{mn}+1} \alpha_{klmn}) \widehat{e_k e_l f_{mn}} + \\ & + \sum (-1)^{y+z+\theta_{yz}+\tau_{wxyz}} \alpha_{wxyz} \widehat{e_w e_x f_{yz}}. \end{aligned}$$

where the last summation is over all distinct $\{w, x, y, z\}$ such that $w < x$, $y < z$ and, if $\{w, x\} = \{k, l\}$ (respectively, $\{m, n\}$), then $\{y, z\} \neq \{m, n\}$ (respectively, $\{k, l\}$).

The linear independence of the $\{\widehat{e_w e_x f_{yz}} \mid w < x, y < z\}$ therefore implies each $\alpha_{wxyz} = 0$ and $a = (-1)^{k+l+m+n+1} \alpha_{klmn}$, hence $a = 0$. \square

Lemma 24. On Λm_r^* , $i_{f_{mn}} \cdot E_{kl} - E_{kl} \cdot i_{f_{mn}} = i_{E_{lk}(f_{mn})}$ when $k \neq l$ and $m \neq n$.

Proof. It is a fundamental result that the commutator $d_1 d_2 - d_2 d_1$ of two derivations d_1 and d_2 is again a derivation. Hence, $i_{f_{mn}} \cdot E_{kl} - E_{kl} \cdot i_{f_{mn}}$ and $i_{E_{lk}(f_{mn})}$ are two derivations of degree -1 and it suffices to show that they are equal on m_r^* . In fact, both are trivial on $E_r^* = \text{span}\{e_a \mid a = 1, \dots, r\}$ and it suffices to check on F_r^* . Note that $E_{kj} \cdot i_{f_{mn}}(F_r^*) = \{0\}$, so it suffices to check that $i_{f_{mn}} \cdot E_{kj} = i_{E_{lk}(f_{mn})}$ on F_r^* . We divide the proof into three subcases: $\{k, l\} = \{m, n\}$, $k \notin \{m, n\}$ and $k \in \{m, n\}$ but $l \notin \{m, n\}$.

Case 1: If $\{k, l\} = \{m, n\}$, then $i_{E_{lk}(f_{kl})} = i_{f_{ll}} = i_0 = 0$ and, since $E_{kl}(f_{ab}) \neq \pm f_{kl}$, $i_{f_{kl}} \cdot E_{kl}(f_{ab}) = 0$, for all $f_{ab} \in F_r^*$.

Case 2: If $k \notin \{m, n\}$ then $i_{E_{lk}(f_{mn})} = i_0 = 0$ and $i_{f_{mn}} \cdot E_{kl}(f_{ab}) = 0$, since $E_{kl}(f_{ab}) = f_{kb}, f_{ak}$ or 0 , when $a < b$.

Case 3: If $k \in \{m, n\}$ and $l \notin \{m, n\}$, suppose without loss of generality that $k = m$ (the case $k = n$ being analogous). Then $i_{E_{lk}(f_{mn})} = i_{f_{ln}}$. It suffices to see, for $1 \leq a < b \leq r$,

$$i_{f_{mn}} \cdot E_{kl}(f_{ab}) = \left. \begin{array}{l} 0, \quad \text{if } l \neq a \text{ or } b; \\ i_{f_{mn}} f_{ak} = \begin{cases} -1, & \text{if } a = n; \\ 0, & \text{otherwise;} \end{cases}, \quad \text{if } l = b; \\ i_{f_{mn}} f_{mb} = \begin{cases} 1, & \text{if } b = m; \\ 0, & \text{otherwise;} \end{cases}, \quad \text{if } l = a; \end{array} \right\} = i_{f_{ln}}(f_{ab}).$$

\square

Lemma 25. For all $i \neq j$, $k \neq l$ and $m \neq n$,

$$\text{op} \{f_{kl}, f_{mn}\} \cdot E_{ij} |_{H^3(\mathfrak{m}_r)} = \left(\text{op} \{E_{ji}(f_{kl}), f_{mn}\} + \text{op} \{f_{kl}, E_{ji}(f_{mn})\} \right) \Big|_{H^3(\mathfrak{m}_r)}$$

and

$$E_{ij} \cdot \widetilde{\text{op}} \{f_{kl}, f_{mn}\} |_{H^T(\mathfrak{m}_r)} = \left(-\widetilde{\text{op}} \{E_{ji}(f_{kl}), f_{mn}\} - \widetilde{\text{op}} \{f_{kl}, E_{ji}(f_{mn})\} \right) \Big|_{H^T(\mathfrak{m}_r)}$$

Proof. We start by proving the more general result on $H^p(\mathfrak{m}_r)$,

$$(7) \quad \text{op} \{f_{kl}, f_{mn}\} \cdot E_{ij} - E_{ij} \cdot \text{op} \{f_{kl}, f_{mn}\} = \text{op} \{E_{ji}(f_{kl}), f_{mn}\} + \text{op} \{f_{kl}, E_{ji}(f_{mn})\},$$

where we assume the ambiguity of every secondary operator on $H^p(\mathfrak{m}_r)$ to be trivial, so equation (7) is well-defined. This is certainly the case if $p = 3$, or if we restrict ourselves to the top class ($p = T$) and replace our operators by the $\widetilde{\text{op}} \{f_{kl}, f_{mn}\}$'s by applying $p_{f_{kl}, f_{mn}}$ to both sides of equation (8), below:

Defining the quotient map $q : Z^{p-3}(\mathfrak{m}_r) \rightarrow Z^{p-3}(\mathfrak{m}_r)/B^{p-3}(\mathfrak{m}_r) = H^{p-3}(\mathfrak{m}_r)$, we compute:

$$\begin{aligned} E_{ij} \cdot \text{op} \{f_{kl}, f_{mn}\} [\alpha] &= E_{ij} \cdot q \left(i_{f_{kl}} d^{-1} i_{f_{mn}} + i_{f_{mn}} d^{-1} i_{f_{kl}} \right) \left(\frac{\alpha}{2} \right) \\ &= q \left(E_{ij} \cdot i_{f_{kl}} d^{-1} i_{f_{mn}} + E_{ij} \cdot i_{f_{mn}} d^{-1} i_{f_{kl}} \right) \left(\frac{\alpha}{2} \right) \\ &= q \left((-i_{E_{ji}(f_{kl})} + i_{f_{kl}} E_{ij}) d^{-1} i_{f_{mn}} + \right. \\ &\quad \left. (-i_{E_{ji}(f_{mn})} + i_{f_{mn}} E_{ij}) d^{-1} i_{f_{kl}} \right) \left(\frac{\alpha}{2} \right) \\ &= q \left(-i_{E_{ji}(f_{kl})} d^{-1} i_{f_{mn}} + i_{f_{kl}} d^{-1} E_{ij} i_{f_{mn}} \right. \\ &\quad \left. - i_{E_{ji}(f_{mn})} d^{-1} i_{f_{kl}} i_{f_{mn}} d^{-1} E_{ij} i_{f_{kl}} \right) \left(\frac{\alpha}{2} \right) \\ &= q \left(-i_{E_{ji}(f_{kl})} d^{-1} i_{f_{mn}} + i_{E_{ji}(f_{kl})} d^{-1} i_{f_{mn}} + i_{f_{kl}} d^{-1} i_{f_{mn}} E_{ij} \right. \\ &\quad \left. - i_{E_{ji}(f_{mn})} d^{-1} i_{f_{kl}} + i_{E_{ji}(f_{mn})} d^{-1} i_{f_{kl}} \right. \\ &\quad \left. + i_{f_{mn}} d^{-1} i_{f_{kl}} E_{ij} \right) \left(\frac{\alpha}{2} \right) \end{aligned}$$

so

$$(8) \quad E_{ij} \cdot \text{op} \{f_{kl}, f_{mn}\} [\alpha] = \left(-\text{op} \{E_{ji}(f_{kl}), f_{mn}\} - \text{op} \{f_{kl}, E_{ji}(f_{mn})\} \right. \\ \left. + \text{op} \{f_{kl}, f_{mn}\} \cdot E_{ij} \right) [\alpha].$$

The first part of the lemma now follows since $E_{ij}(\mathbb{C}) = \{0\}$. The second part also follows since $E_{ij}[t] = \{0\}$ for any $t \in H^T(\mathfrak{m}_r)$. \square

We are now ready to prove Theorem B.

Proof of Theorem B. Lemmas 21 and 23 imply the first part of the theorem (i.e. that every $\widetilde{\text{op}} \{f_{kl}, f_{mn}\}$ is non-trivial and in V_r).

First we verify that $\widehat{e_{r-1}e_r f_{r-1,r}} := e_1 \cdots e_{r-2} f_{12} \cdots f_{r-2,r}$ is a highest weight vector of V_r . It suffices to check that $E_{ij}(\widehat{e_{r-1}e_r f_{r-1,r}}) = 0$ when $i < j$ (which is straightforward) and to count indices to verify

$$E_{ii}(\widehat{e_{r-1}e_r f_{r-1,r}}) = \begin{cases} r \cdot \widehat{e_{r-1}e_r f_{r-1,r}}, & \text{if } 1 \leq i \leq r-2, \\ (r-2) \cdot \widehat{e_{r-1}e_r f_{r-1,r}}, & \text{if } i = r-1 \text{ or } r. \end{cases}$$

By equation (6), $\widetilde{\text{op}} \{f_{r-1,r}, f_{r-1,r}\}[t] = (-1)^{T-1} \widehat{e_{r-1}e_r f_{r-1,r}}$, so it is a highest weight vector of V_r . Representation theory tells us that repeated applications of E_{ji} , with $i < j$, to $\widetilde{\text{op}} \{f_{r-1,r}, f_{r-1,r}\}[t]$ gives us a complete basis for $V_r \subset H^{T-3}(\mathfrak{m}_r)$. Indeed, Lemma 25 tells us that repeated applications of E_{ji} to $\widetilde{\text{op}} \{f_{r-1,r}, f_{r-1,r}\}[t]$ will give a basis consisting of linear combinations of $\widetilde{\text{op}} \{f_{kl}, f_{mn}\}[t]$'s. \square

4.7 A 6-cube. No actual representations of 6-cubes have yet been found in $H^*(\mathfrak{m}_4)$. However, if certain identifications on $H^*(\mathfrak{m}_4)$ are made, something that looks like a representation of the 6-cube can be constructed. Henceforth, "primary operators" will refer specifically to those primary operators of the form $i_{f_{jk}}^*$ unless otherwise specified. We start with level 0 of our 6-cube being the top class, $t = e_1 e_2 e_3 e_4 f_{12} f_{13} f_{14} f_{23} f_{24} f_{34}$, and level 1 having the vertices

$$\begin{aligned} v_{12} &:= \widetilde{\text{op}} \{f_{12}, f_{12}\}[t] = [e_3 e_4 \widehat{f_{12}}], \\ v_{13} &:= -2\widetilde{\text{op}} \{f_{12}, f_{13}\}[t] = [e_3 e_4 \widehat{f_{13}} + e_2 e_4 \widehat{f_{12}}], \\ v_{14} &:= 2\widetilde{\text{op}} \{f_{12}, f_{14}\}[t] = [e_3 e_4 \widehat{f_{14}} + e_2 e_3 \widehat{f_{12}}], \\ v_{23} &:= -2\widetilde{\text{op}} \{f_{12}, f_{23}\}[t] = [e_3 e_4 \widehat{f_{23}} + e_1 e_4 \widehat{f_{12}}], \\ v_{24} &:= 2\widetilde{\text{op}} \{f_{12}, f_{24}\}[t] = [e_3 e_4 \widehat{f_{24}} - e_1 e_3 \widehat{f_{12}}] \text{ and} \\ v_{34} &:= -2\widetilde{\text{op}} \{f_{12}, f_{34}\}[t] = [e_3 e_4 \widehat{f_{34}} - e_1 e_2 \widehat{f_{12}}]. \end{aligned}$$

So the first two levels of the 6-cube are as we want them to be. For level 2, we expect to find 15 linearly independent vectors, each mapped from level 1 by two

primary operators (up to sign). However, what we find are

$$\begin{aligned} [e_3 e_4 f_{14} f_{23} f_{24} f_{34}] &= i_{f_{13}}^*(v_{12}) = i_{f_{12}}^*(v_{13}), \\ [e_3 e_4 f_{13} f_{23} f_{24} f_{34}] &= -i_{f_{14}}^*(v_{12}) = i_{f_{12}}^*(v_{14}), \\ [e_3 e_4 f_{13} f_{14} f_{24} f_{34}] &= i_{f_{23}}^*(v_{12}) = i_{f_{12}}^*(v_{23}), \\ [e_3 e_4 f_{13} f_{14} f_{23} f_{34}] &= -i_{f_{24}}^*(v_{12}) = i_{f_{12}}^*(v_{24}) \text{ and} \\ [e_3 e_4 f_{13} f_{14} f_{23} f_{24}] &= i_{f_{34}}^*(v_{12}) = i_{f_{12}}^*(v_{34}), \end{aligned}$$

which account for five vectors, and the following ten pairs of vectors, where each of the twenty vectors only has one primary operator mapping to it from level 1:

$$\begin{aligned} -i_{f_{14}}^*(v_{13}) &= [e_3 e_4 f_{12} f_{23} f_{24} f_{34} + e_2 e_4 f_{13} f_{23} f_{24} f_{34}] \\ \text{and } -i_{f_{13}}^*(v_{14}) &= [e_3 e_4 f_{12} f_{23} f_{24} f_{34} - e_2 e_3 f_{14} f_{23} f_{24} f_{34}], \\ i_{f_{23}}^*(v_{13}) &= [e_3 e_4 f_{12} f_{14} f_{24} f_{34} + e_2 e_4 f_{13} f_{14} f_{24} f_{34}] \\ \text{and } -i_{f_{13}}^*(v_{23}) &= [e_3 e_4 f_{12} f_{14} f_{24} f_{34} - e_1 e_4 f_{14} f_{23} f_{24} f_{34}], \\ -i_{f_{24}}^*(v_{13}) &= [e_3 e_4 f_{12} f_{14} f_{23} f_{34} + e_2 e_4 f_{13} f_{14} f_{23} f_{34}] \\ \text{and } -i_{f_{13}}^*(v_{24}) &= [e_3 e_4 f_{12} f_{14} f_{23} f_{34} + e_1 e_3 f_{14} f_{23} f_{24} f_{34}], \\ i_{f_{34}}^*(v_{13}) &= [e_3 e_4 f_{12} f_{14} f_{23} f_{24} + e_2 e_4 f_{13} f_{14} f_{23} f_{24}] \\ \text{and } -i_{f_{13}}^*(v_{34}) &= [e_3 e_4 f_{12} f_{14} f_{23} f_{24} + e_1 e_2 f_{14} f_{23} f_{24} f_{34}], \\ i_{f_{23}}^*(v_{14}) &= [e_3 e_4 f_{12} f_{13} f_{24} f_{34} + e_2 e_3 f_{13} f_{14} f_{24} f_{34}] \\ \text{and } i_{f_{14}}^*(v_{23}) &= [e_3 e_4 f_{12} f_{13} f_{24} f_{34} - e_1 e_4 f_{13} f_{23} f_{24} f_{34}], \\ -i_{f_{24}}^*(v_{14}) &= [e_3 e_4 f_{12} f_{13} f_{23} f_{34} + e_2 e_3 f_{13} f_{14} f_{23} f_{34}] \\ \text{and } i_{f_{14}}^*(v_{24}) &= [e_3 e_4 f_{12} f_{13} f_{23} f_{34} + e_1 e_3 f_{13} f_{23} f_{24} f_{34}], \\ i_{f_{34}}^*(v_{14}) &= [e_3 e_4 f_{12} f_{13} f_{23} f_{24} + e_2 e_3 f_{13} f_{23} f_{24} f_{34}] \\ \text{and } i_{f_{14}}^*(v_{34}) &= [e_3 e_4 f_{12} f_{13} f_{23} f_{24} - e_1 e_2 f_{13} f_{23} f_{24} f_{34}], \\ -i_{f_{24}}^*(v_{23}) &= [e_3 e_4 f_{12} f_{13} f_{14} f_{34} + e_1 e_4 f_{13} f_{14} f_{23} f_{24}] \\ \text{and } -i_{f_{23}}^*(v_{24}) &= [e_3 e_4 f_{12} f_{13} f_{14} f_{34} + e_1 e_3 f_{13} f_{14} f_{24} f_{34}], \\ i_{f_{34}}^*(v_{23}) &= [e_3 e_4 f_{12} f_{13} f_{14} f_{24} + e_1 e_4 f_{13} f_{14} f_{23} f_{24}] \\ \text{and } -i_{f_{23}}^*(v_{34}) &= [e_3 e_4 f_{12} f_{13} f_{14} f_{24} - e_1 e_2 f_{13} f_{14} f_{24} f_{34}], \\ i_{f_{34}}^*(v_{24}) &= [e_3 e_4 f_{12} f_{13} f_{14} f_{23} - e_1 e_3 f_{13} f_{14} f_{23} f_{24}] \\ \text{and } i_{f_{24}}^*(v_{34}) &= [e_3 e_4 f_{12} f_{13} f_{14} f_{23} + e_1 e_2 f_{13} f_{14} f_{23} f_{34}]. \end{aligned}$$

We define one of the vertices in level 2 to be the set which is the first pair:

$$\begin{aligned} &\overline{e_3 e_4 f_{12} f_{23} f_{24} f_{34}} := \\ &\{ [e_3 e_4 f_{12} f_{23} f_{24} f_{34} - e_2 e_4 f_{13} f_{23} f_{24} f_{34}], [e_3 e_4 f_{12} f_{14} f_{24} f_{34} + e_2 e_4 f_{13} f_{14} f_{24} f_{34}] \}. \end{aligned}$$

The set $\overline{e_3e_4f_{12}f_{23}f_{24}f_{34}}$ now has two primary operators mapping to it from level 1. Likewise, we identify the other nine pairs in the list to obtain sets that are mapped to from level 1 by two primary operators. Hence, level 2 of our hypercube consists of the following vertices:

$$\begin{aligned} & [\overline{e_3e_4f_{14}f_{23}f_{24}f_{34}}], [\overline{e_3e_4f_{13}f_{23}f_{24}f_{34}}], [\overline{e_3e_4f_{13}f_{14}f_{24}f_{34}}], [\overline{e_3e_4f_{13}f_{14}f_{23}f_{34}}], \\ & [\overline{e_3e_4f_{13}f_{14}f_{23}f_{24}}], [\overline{e_3e_4f_{12}f_{23}f_{24}f_{34}}], [\overline{e_3e_4f_{12}f_{14}f_{24}f_{34}}], [\overline{e_3e_4f_{12}f_{14}f_{23}f_{34}}], \\ & [\overline{e_3e_4f_{12}f_{14}f_{23}f_{24}}], [\overline{e_3e_4f_{12}f_{13}f_{24}f_{34}}], [\overline{e_3e_4f_{12}f_{13}f_{23}f_{34}}], [\overline{e_3e_4f_{12}f_{13}f_{23}f_{24}}], \\ & [\overline{e_3e_4f_{12}f_{13}f_{14}f_{34}}], [\overline{e_3e_4f_{12}f_{13}f_{14}f_{24}}], [\overline{e_3e_4f_{12}f_{13}f_{14}f_{23}}], \end{aligned}$$

where each $\overline{e_3e_4f_{12}f_{\alpha_1\beta_1}f_{\alpha_2\beta_2}f_{\alpha_3\beta_3}}$ is a set containing two distinct elements of the form

$$[e_3e_4f_{12}f_{\alpha_1\beta_1}f_{\alpha_2\beta_2}f_{\alpha_3\beta_3} + \text{something}],$$

where the "somethings" may be read from the list of pairs.

We also note that $i_{f_{12}}^*(v_{12}) = 0$, and we obtain the following vectors which we ignore since none are of the form $[e_3e_4f_{12}f_{\alpha_1\beta_1}f_{\alpha_2\beta_2}f_{\alpha_3\beta_3} + \text{something}]$:

$$\begin{aligned} i_{f_{13}}^*(v_{13}) &= [e_2e_4f_{14}f_{23}f_{24}f_{34}], & -i_{f_{14}}^*(v_{14}) &= [e_2e_3f_{13}f_{23}f_{24}f_{34}], \\ i_{f_{23}}^*(v_{23}) &= [e_1e_4f_{13}f_{14}f_{24}f_{34}], & -i_{f_{24}}^*(v_{24}) &= [e_1e_3f_{13}f_{14}f_{23}f_{24}], \\ & & \text{and } i_{f_{34}}^*(v_{34}) &= [e_1e_2f_{13}f_{14}f_{23}f_{24}]. \end{aligned}$$

The operator $i_{f_{12}}^*$ acting on $\overline{e_3e_4f_{12}f_{23}f_{24}f_{34}}$ is well-defined, since it sends every element in the set to $[e_3e_4f_{23}f_{24}f_{34}]$. Indeed, it is straightforward to check that $i_{f_{12}}^*$ is well-defined on any of the identified pairs of vectors in level 2 of the 6-cube. However, if we apply $-i_{f_{23}}^*$ to each element in $\overline{e_3e_4f_{12}f_{23}f_{24}f_{34}}$, we obtain $[e_3e_4f_{12}f_{24}f_{34} + e_2e_4f_{13}f_{24}f_{34}]$ and $[e_3e_4f_{12}f_{24}f_{34} - e_2e_3f_{14}f_{24}f_{34}]$. Thus, $i_{f_{23}}^*$ is not well-defined on this vertex unless there is a vertex set $\overline{e_3e_4f_{12}f_{24}f_{34}}$ in level 3 which contains these two vectors. In fact, let level 3 of our 6-cube be

$$\begin{aligned} & [\overline{e_3e_4f_{23}f_{24}f_{34}}], [\overline{e_3e_4f_{14}f_{24}f_{34}}], [\overline{e_3e_4f_{14}f_{23}f_{34}}], [\overline{e_3e_4f_{14}f_{23}f_{24}}], [\overline{e_3e_4f_{13}f_{24}f_{34}}], \\ & [\overline{e_3e_4f_{13}f_{23}f_{34}}], [\overline{e_3e_4f_{13}f_{23}f_{24}}], [\overline{e_3e_4f_{13}f_{14}f_{34}}], [\overline{e_3e_4f_{13}f_{14}f_{24}}], [\overline{e_3e_4f_{13}f_{14}f_{23}}], \\ & [\overline{e_3e_4f_{12}f_{24}f_{34}}], [\overline{e_3e_4f_{12}f_{23}f_{34}}], [\overline{e_3e_4f_{12}f_{23}f_{24}}], [\overline{e_3e_4f_{12}f_{14}f_{34}}], [\overline{e_3e_4f_{12}f_{14}f_{24}}], \\ & [\overline{e_3e_4f_{12}f_{14}f_{23}}], [\overline{e_3e_4f_{12}f_{13}f_{34}}], [\overline{e_3e_4f_{12}f_{13}f_{24}}], [\overline{e_3e_4f_{12}f_{13}f_{23}}], [\overline{e_3e_4f_{12}f_{13}f_{14}}], \end{aligned}$$

where

$$\begin{aligned} \overline{e_3e_4f_{12}f_{24}f_{34}} &= \left\{ \begin{aligned} [e_3e_4f_{12}f_{24}f_{34} + e_2e_4f_{13}f_{24}f_{34}] &= i_{f_{23}}i_{f_{14}}(v_{13}) = -i_{f_{14}}i_{f_{23}}(v_{13}), \\ [e_3e_4f_{12}f_{24}f_{34} - e_2e_3f_{14}f_{24}f_{34}] &= i_{f_{23}}i_{f_{13}}(v_{14}) = -i_{f_{13}}i_{f_{23}}(v_{14}), \\ [e_3e_4f_{12}f_{24}f_{34} + e_1e_4f_{23}f_{24}f_{34}] &= i_{f_{14}}i_{f_{13}}(v_{23}) = -i_{f_{13}}i_{f_{14}}(v_{23}) \end{aligned} \right\}. \end{aligned}$$

Likewise, each $\overline{e_3 e_4 f_{12} f_{\alpha_1 \beta_1} f_{\alpha_2 \beta_2}}$ is defined to be a similar triple of vectors, with three primary operators mapping to the set from three vertex sets in level 2. Primary operators between our level 2 and our level 3 are well-defined:

$$i_{f_{\alpha_j \beta_j}}^* : \overline{e_3 e_4 f_{12} f_{\alpha_1 \beta_1} \cdots f_{\alpha_3 \beta_3}} \longrightarrow \overline{e_3 e_4 f_{12} f_{\alpha_1 \beta_1} \cdots \widehat{f_{\alpha_j \beta_j}} \cdots f_{\alpha_3 \beta_3}},$$

for $j = 1, 2, 3$. To demonstrate these last statements, it suffices to look at one of the vertex sets and say that the proofs for the other vertex sets are similar.

While there are six primary operators going to $\overline{e_3 e_4 f_{12} f_{24} f_{34}}$ from $H^6(\mathfrak{h}_m)$, there are in fact only three edges going to it from the vertices of the form $\overline{e_3 e_4 f_{12} f_{23} f_{24} f_{34}}$, $\overline{e_3 e_4 f_{12} f_{14} f_{24} f_{34}}$ and $\overline{e_3 e_4 f_{12} f_{13} f_{24} f_{34}}$.

Similarly, we define level 4 to be the following 15 vertices:

$$\begin{aligned} & [\overline{e_3 e_4 f_{24} f_{34}}], [\overline{e_3 e_4 f_{23} f_{34}}], [\overline{e_3 e_4 f_{23} f_{24}}], [\overline{e_3 e_4 f_{14} f_{34}}], [\overline{e_3 e_4 f_{14} f_{24}}], \\ & [\overline{e_3 e_4 f_{14} f_{23}}], [\overline{e_3 e_4 f_{13} f_{34}}], [\overline{e_3 e_4 f_{13} f_{24}}], [\overline{e_3 e_4 f_{13} f_{23}}], [\overline{e_3 e_4 f_{13} f_{14}}], \\ & \overline{e_3 e_4 f_{12} f_{34}}, \overline{e_3 e_4 f_{12} f_{24}}, \overline{e_3 e_4 f_{12} f_{23}}, \overline{e_3 e_4 f_{12} f_{14}}, \overline{e_3 e_4 f_{12} f_{14}}, \end{aligned}$$

where the five vertex sets are

$$\begin{aligned} \overline{e_3 e_4 f_{12} f_{34}} = \{ & [e_3 e_4 f_{12} f_{34} + e_2 e_4 f_{13} f_{34}], [e_3 e_4 f_{12} f_{34} - e_2 e_3 f_{14} f_{34}], \\ & [e_3 e_4 f_{12} f_{34} - e_1 e_4 f_{23} f_{34}], [e_3 e_4 f_{12} f_{34} + e_1 e_3 f_{24} f_{34}]\}, \end{aligned}$$

$$\begin{aligned} \overline{e_3 e_4 f_{12} f_{24}} = \{ & [e_3 e_4 f_{12} f_{24} + e_2 e_4 f_{13} f_{24}], [e_3 e_4 f_{12} f_{24} - e_2 e_3 f_{14} f_{24}], \\ & [e_3 e_4 f_{12} f_{24} - e_1 e_4 f_{23} f_{24}], [e_3 e_4 f_{12} f_{24} + e_1 e_2 f_{24} f_{34}]\}, \end{aligned}$$

$$\begin{aligned} \overline{e_3 e_4 f_{12} f_{23}} = \{ & [e_3 e_4 f_{12} f_{23} + e_2 e_4 f_{13} f_{23}], [e_3 e_4 f_{12} f_{23} - e_2 e_3 f_{14} f_{23}], \\ & [e_3 e_4 f_{12} f_{23} - e_1 e_3 f_{23} f_{24}], [e_3 e_4 f_{12} f_{23} + e_1 e_2 f_{23} f_{34}]\}, \end{aligned}$$

$$\begin{aligned} \overline{e_3 e_4 f_{12} f_{14}} = \{ & [e_3 e_4 f_{12} f_{14} + e_2 e_4 f_{13} f_{14}], [e_3 e_4 f_{12} f_{14} + e_1 e_4 f_{14} f_{23}], \\ & [e_3 e_4 f_{12} f_{14} - e_1 e_3 f_{14} f_{24}], [e_3 e_4 f_{12} f_{14} + e_1 e_2 f_{14} f_{34}]\}, \end{aligned}$$

$$\begin{aligned} \overline{e_3 e_4 f_{12} f_{13}} = \{ & [e_3 e_4 f_{12} f_{13} + e_2 e_3 f_{13} f_{14}], [e_3 e_4 f_{12} f_{13} + e_1 e_4 f_{13} f_{23}], \\ & [e_3 e_4 f_{12} f_{13} - e_1 e_3 f_{13} f_{24}], [e_3 e_4 f_{12} f_{13} + e_1 e_2 f_{13} f_{34}]\}. \end{aligned}$$

Again, we observe the well-definedness of primary operators from level 3 to level 4. One can verify that, for each vertex set, there are four primary operators going from the vertex sets in level 3 to it.

One may also verify that, so far, each vector in our 6-cube, whether a vertex itself or an element of a vertex set, is non-trivial. In fact, since the primary and secondary operators are well-defined, it suffices to check this fact on the 30 vectors that appear in level 4.

Now we consider level 5 in a similar fashion. First, of the six vertices we expect to find, we have the following five linearly independent vectors:

$$[e_3e_4f_{13}], [e_3e_4f_{14}], [e_3e_4f_{23}], [e_3e_4f_{24}], [e_3e_4f_{34}].$$

Each of these is mapped to from level 4 by five primary operators in a well-defined manner. For example,

$$[e_3e_4f_{34}] = i_{f_{13}}^*[e_3e_4f_{13}f_{34}] = i_{f_{14}}^*[e_3e_4f_{14}f_{34}] = i_{f_{23}}^*[e_3e_4f_{23}f_{34}] = i_{f_{24}}^*[e_3e_4f_{24}f_{34}]$$

and $i_{f_{12}}^*(\overline{e_3e_4f_{12}f_{34}}) = \{[e_3e_4f_{34}]\}$. Moreover, as we would expect, each of these is mapped to level 6 by secondary operators. Namely, if we define the single vertex in level 6 to be 1, then the above vertices are mapped to 1 by, respectively,

$$2 \text{ op } \{f_{13}, f_{34}\}, 2 \text{ op } \{f_{14}, f_{34}\}, 2 \text{ op } \{f_{23}, f_{34}\}, 2 \text{ op } \{f_{24}, f_{34}\}, \text{ op } \{f_{34}, f_{34}\}.$$

As for the sixth vertex in level 5, define

$$\overline{e_3e_4f_{12}} = \{[e_3e_4f_{12} - e_1e_2f_{34}], [e_3e_4f_{12} + e_1e_3f_{24}], \\ [e_3e_4f_{12} - e_1e_4f_{23}], [e_3e_4f_{12} + e_2e_3f_{14}], [e_3e_4f_{12} - e_2e_4f_{13}]\}$$

Four of the vectors in this set are linearly independent. However, $e_3e_4f_{12} - e_1e_2f_{34} = d(f_{12}f_{34})$, so the first element in the list, $[e_3e_4f_{12} - e_1e_2f_{34}]$, is trivial in $H^*(\mathfrak{h}_m)$. Thus, the secondary operator $2 \text{ op } \{f_{12}, f_{34}\}$ sends four of the vectors in $\overline{e_3e_4f_{12}}$ to 1, but sends one vector to 0. The secondary operator is therefore not well-defined on this vertex. Furthermore, if one wanted to create an equivalence relation based on these $\overline{e_3e_4f_{\alpha_1\beta_1} \cdots f_{\alpha_k\beta_k}}$ vertex sets, and hence find a hypercube in a quotient space of $H^*(\mathfrak{m}_4)$, one would find the equivalence classes induced by $\overline{e_3e_4f_{12}}$ imply one of the vertices in level 5 is trivial in the quotient space.

CHAPTER V: OPEN PROBLEMS

In every example seen, if α and β are Poincaré duals in $H^*(L)$, where L is a nilpotent Lie algebra, and there exist non-trivial elements $\text{op}_k \alpha$ and $\text{op}_k^{-1} \beta$, then these two elements are also Poincaré duals. The same holds for all examples of 2-parameter secondary operators. Therefore, a problem to explore is that of proving this property holds for *all* nilpotent Lie algebras, or of finding a counter-example.

Of course, the proof of the Toral Rank Conjecture is still open.

One might wish to find a general proof of the TRC on free two-step nilpotent Lie algebras by using the results of Chapter IV. By looking at the free two-steps on 5, 6 or 7 generators, one might see whether or not the “cheater” $\binom{r}{2}$ -cube from subsection 4.7 generalises in a satisfying manner. If not, one might consider *other* subgraphs in the diagrams representing $H^*(\mathfrak{m}_r)$, with “edges” being operators and the number of “vertices” (i.e. linearly independent vectors) being greater than or equal to 2^r , in general. It is the author’s opinion that the bigradation from subsection 4.4 can be a powerful tool for this end, and it is with a touch of regret that this project must be postponed indefinitely due to lack of time.

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