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The Unitary Group as an Invariant of a Simple Unital C*-Algebra

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THE UNITARY GROUP AS AN INVARIANT OF A SIMPLE UNITAL C^*−ALGEBRA

By

Ahmed Al-Rawashdeh, B.Sc., M.S.

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Abstract

In 1954, H. Dye proved that the unitary groups of von Neumann factors not of type I_{2n} determine the algebraic type of factors. Using Dye's result, M. Broise showed that any isomorphism between the unitary groups of two von Neumann factors not of type I_n is implemented by a linear or a conjugate linear $*-$isomorphism between the factors. Using Dye's approach, A. Booth proved that two simple unital AF-algebras are isomorphic if and only if their unitary groups are (algebraically) isomorphic. In the first part of this thesis, we extend Booth's result to a larger class of amenable unital C^*-algebras. If φ is an isomorphism between the unitary groups of two unital C^*-algebras, it induces a bijective map θ_φ between the sets of projections of the algebras. For some UHF-algebras, we construct an automorphism φ of their unitary group, such that θ_φ does not preserve the orthogonality of projections. For a large class of unital C^*-algebras, we show that θ_φ is always an orthoisomorphism. This class includes in particular the Cuntz algebras $\mathcal{O}_n$, $2 \leq n \leq \infty$, and the simple unital AF-algebras having 2-divisible K_0-group. If φ is a continuous automorphism of the unitary group of a UHF-algebra A, we show that φ is implemented by a linear or a conjugate linear $*-$automorphism of A.

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Dedication

I dedicate this work to my parents.

Contents

Chapter 1

Introduction

For two unital C^*-algebras $\mathfrak{A}_1$ and $\mathfrak{A}_2$, A. Paterson proves in [Pt] the following result:

Theorem 1.0.0.1. *[Pt, Th.4.6] The unitary groups of $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are isometrically isomorphic with respect to the metrics induced by the norms of $\mathfrak{A}_1$ and $\mathfrak{A}_2$ if and only if $\mathfrak{A}_i$ is the direct sum of closed ideals I_i, J_i $(i = 1, 2)$, where I_1 and I_2 are $*-$isomorphic and J_1 and J_2 are $*-$antiisomorphic.*

He asks also the following question:
What can one say about unital C^*-algebras $\mathfrak{A}_1$ and $\mathfrak{A}_2$ which are such that there is a (topological and algebraic) isomorphism from $\mathcal{U}(\mathfrak{A}_1)$ onto $\mathcal{U}(\mathfrak{A}_2)$?

Let us elaborate Paterson's question as follows:
[Q.1] *If $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are two simple unital C^*-algebras whose unitary groups are isomorphic (as abstract groups), are $\mathfrak{A}_1$ and $\mathfrak{A}_2$ either $*-$isomorphic or $*-$antiisomorphic ?*
[Q.2] *If $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are simple, unital C^*-algebras and φ is an isomorphism between their unitary groups, does φ extend to a $*-$isomorphism or $*-$antiisomorphism between $\mathfrak{A}_1$ and $\mathfrak{A}_2$?*

In this thesis, we study both questions. Before stating our results, let us review what is known about these questions. For continuous von Neumann factors, [Q.1] and [Q.2] were answered by H. Dye in [Dy] combined with M. Broise's result in [Br]. Indeed:

Theorem 1.0.0.2. *[Dy, Th.2] Let N and M be two von Neumann factors not of type I_{2n}, and φ be a group isomorphism between their unitary groups $\mathcal{U}(N)$ and $\mathcal{U}(M)$. Then there exists a linear or a conjugate linear $*-$isomorphism Ψ of N on M which implements φ in the following sense: for every $u \in \mathcal{U}(N)$, $\varphi(u) = \lambda(u)\Psi(u)$ for some (possibly discontinuous) character λ of $\mathcal{U}(N)$.*

In [Br, Th.1], M. Broise shows that the unitary group of a factor not of type I_n has no non-trivial characters. Therefore using Dye's result, he obtains

Theorem 1.0.0.3. *If N and M are two von Neumann factors not of type $I_n (n < \infty)$, then any isomorphism between their unitary groups is implemented by a linear or a conjugate linear $*-$isomorphism between the factors.*

Generalizing Dye's approach, A. Booth proves in [Bo], that any isomorphism between the unitary groups of two simple unital approximately finite C^*-algebras (AF-algebras) A and B induces an order isomorphism between the $K-$groups $K_0(A)$ and $K_0(B)$ sending $[1_A]$ to $[1_B]$. Using Elliott's classification theorem for AF-algebras, A. Booth answers [Q.1] by showing:

Theorem 1.0.0.4. *[Bo, p.88] If A and B are simple, unital AF-algebras with isomorphic unitary groups, then A and B are isomorphic as C^*-algebras.*

In Chapter 4, we answer [Q.1] affirmatively for a large class of unital amenable C^*-algebras, which includes the Cuntz algebras and the irrational rotation algebras. In Chapter 6, we solve [Q.2] for UHF-algebras, if the isomorphism φ is continuous.

Let $\mathfrak{A}$ and $\mathfrak{B}$ be two unital C^*-algebras, and φ be an isomorphism between their unitary groups. Let us recall Dye's strategy. As φ sends self-adjoint unitaries to self-adjoint unitaries, then a natural mapping θ_φ between the sets of projections $\mathcal{P}(\mathfrak{A})$ and $\mathcal{P}(\mathfrak{B})$ can be defined by setting

$$I - 2\theta_\varphi(p) = \varphi(I - 2p), \;\; p \in \mathcal{P}(A).$$

Such a map θ_φ is a **projection orthoisomorphism** between the C^*-algebras $\mathfrak{A}$ and $\mathfrak{B}$ if it is a bijective map which preserves orthogonality of projections, i.e., if $p, q \in \mathcal{P}(\mathfrak{A})$, then

$$pq = 0 \text{ if and only if } \theta(p)\theta(q) = 0.$$

Now, let us ask the following question:
[Q.3] *When is the induced map θ_φ an orthoisomorphism ?*

For von Neumann factors, Dye answered it by proving:

Theorem 1.0.0.5. *[Dy, Le.13] Let M and N be two factors. If φ is an isomorphism between $\mathcal{U}(M)$, $\mathcal{U}(N)$ and M is not of type I_{2n} $(n \geq 1)$, then θ_φ is an orthoisomorphism.*

Moreover, Dye outlined the construction of an automorphism φ of $\mathcal{U}(\mathbb{M}_{2n}(\mathbb{C}))$ whose induced map $\theta_\varphi : \mathcal{P}(\mathbb{M}_{2n}(\mathbb{C})) \to \mathcal{P}(\mathbb{M}_{2n}(\mathbb{C}))$ is not an orthoisomorphism.

We now give a detailed presentation of the chapters of this thesis. In Chapter 2, we present the background material we shall need. We have used the following references: [B], [Da], [Ef], [Go], [M], [RS] and [WO].

In Chapter 3, we introduce the results of Dye ([Dy]) and Booth ([Bo]). One of the main tools used by Dye in his study and that we shall also need in Chapter 6, is a family of projections of $\mathbb{M}_n(A)$, for a C^*-algebra A and $n \geq 3$. If $V_n(A)$ denotes the direct sum of n copies of A, considered as a left Hilbert A-module, Dye considers the following projections of $\mathbb{M}_n(A)$: for each a in A and $1 \leq i \neq j \leq n$, $P_{i,j}(a)$ denotes the projection whose range consists of all left multiples of the vector with 1 in the i^{th} place, a in the j^{th} place, and zeros elsewhere. If $\{E_{i,j}\}_{i,j}^n$ denotes the standard system of matrix units of $\mathbb{M}_n(\mathbb{C})$, then

$$P_{i,j}(a) = (1+aa^*)^{-1}\otimes E_{i,i}+(1+aa^*)^{-1}a\otimes E_{i,j}+a^*(1+aa^*)^{-1}\otimes E_{j,i}+a^*(1+aa^*)^{-1}a\otimes E_{j,j}.$$

In the second part of this chapter, we present results ([Bo]) that A. Booth obtained in his master thesis (1998). In [Bo], he proves that if $\mathfrak{A}$ and $\mathfrak{B}$ are simple, unital AF-algebras with isomorphic unitary groups, then $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic as

C^*-algebras. Let φ be an isomorphism between $\mathcal{U}(\mathfrak{A})$ and $\mathcal{U}(\mathfrak{B})$. To θ_φ, Booth associates an orthoisomorphism $\tilde{\theta}_\varphi$ from $\mathcal{P}(\mathfrak{A})$ to $\mathcal{P}(\mathfrak{B})$, preserving unitary equivalence of projections and he proves that $\tilde{\theta}_\varphi$ induces an order isomorphism of the scaled dimension groups $K_0(\mathfrak{A})$ and $K_0(\mathfrak{B})$. By Elliott's theorem, $\mathfrak{A}$ and $\mathfrak{B}$ are then $*-$isomorphic.

In Chapter 4, we solve [Q.1] by extending Booth's method to a larger class of finite C^*-algebras and purely infinite C^*-algebras. Let $\mathcal{F}$ denote the collection of all simple unital stably finite C^*-algebras of real rank zero which have cancellation and whose K_0-group is a weakly unperforated interpolation group. If A and B belong to $\mathcal{F}$, or if they are simple purely infinite C^*-algebras and if φ is an isomorphism from $\mathcal{U}(A)$ and $\mathcal{U}(B)$, then we show that φ induces an order isomorphism between $K_0(A)$ and $K_0(B)$, sending $[1_A]$ to $[1_B]$. In particular, if A and B are both irrational rotation algebras or both Cuntz algebras with isomorphic unitary groups, then A and B are isomorphic as C^*-algebras.

More generally, using Elliott's classification theorem, we show that simple, unital $A\mathbb{T}-$algebras of real rank zero, where unitary and K_1 groups are isomorphic, are also isomorphic as C^*-algebras. If the isomorphism between the unitary groups is topological isomorphism, the assumption on the K_1-groups is not necessary. In particular, the C^*-crossed product associated to two Cantor minimal systems are isomorphic if and only if their unitary groups are isomorphic.

For simple, unital purely infinite C^*-algebras, we study the case of Kirchberg algebras, and we prove that two Kirchberg algebras are $*-$isomorphic if and only if their unitary groups are isomorphic.

In Chapter 5, we study [Q.3]. Let A be a unital C^*-algebra and φ be an automorphism of $\mathcal{U}(A)$; we prove that if A belongs to one of the following classes of C^*-algebras:

- *Cuntz algebras* $\mathcal{O}_n$*,* $2 \leq n \leq \infty$*,*
- C^*-*algebras of class* $\mathcal{F}$ *and whose* K_0-*groups are 2-divisible,*
- *UHF-algebras whose generalized integers are* $(3^{n_3}, 5^{n_5}, \ldots)$*,*

- *Simple purely infinite C^*-algebras whose K_0-groups are 2-divisible,*
- *C^*-algebras which have a normalized faithful trace τ, such that*

$$\tau(u) = \tau(\varphi(u)),$$

for every self-adjoint unitary u in the C^*-algebra,

then the induced map θ_φ is an orthoisomorphism of $\mathcal{P}(A)$.
On the other hand, if A is a UHF-algebra whose generalized integer is such that

$$\overline{n} = (2^{n_2}, 3^{n_3}, 5^{n_5}, \dots), \text{ where } 1 \leq n_2 < \infty \text{ and } 0 \leq n_p \leq \infty$$

for every prime $p > 2$, we construct an automorphism φ of $\mathcal{U}(A)$, whose induced map θ_φ is not an orthoisomorphism. This construction generalizes Booth's precise description of Dye's example of such an automorphism of $\mathcal{U}(\mathbb{M}_{2n}(\mathbb{C}))$.

In this chapter, we also show that if A is a unital C^*-algebra, $\varphi \in \text{Aut}(\mathcal{U}(A))$ and A has a $\varphi-$invariant faithful trace, then φ is implemented by a $*-$isomorphism of the algebra A.

In Chapter 6, we discuss the extension problem of question [Q.2]. To a unital C^*-algebra A and a system of matrix units $\{e_{i,j}\}_{i,j=1}^n$, $n \geq 3$ of A, we associate as in Dye, the family of projections $P_{i,j}(a)$, for $a \in e_{1,1}Ae_{1,1}$, $1 \leq i \neq j \leq n$. If Γ denotes the subgroup of $\mathcal{U}(A)$ generated by the self-adjoint unitaries of the form $1 - 2P_{i,j}(a)$, for $a \in e_{1,1}Ae_{1,1}$, $1 \leq i \neq j \leq n$, we generalize [Dy], Lemma 6 as follows:

Theorem 1.0.0.6. *Let A and B be two unital C^*-algebras and for some $n \geq 3$, let $\{e_1, \cdots, e_n\}$ be a partition of the unity in A, consisting of orthogonal equivalent projections. If $\varphi : \mathcal{U}(A) \longrightarrow \mathcal{U}(B)$ is an isomorphism such that θ_φ is a projection orthoisomorphism, then there exists a unique $\psi : A \to B$, which is a direct sum of linear and conjugate linear $*$-isomorphisms, such that $\psi(u) = \varphi(u)$ for all $u \in \Gamma$.*

In particular, we study the UHF-algebra case and we show that, if A is a UHF-algebra and φ is an automorphism of $\mathcal{U}(A)$, such that θ_φ is an orthoisomorphism of $\mathcal{P}(A)$, then there exists a linear or a conjugate linear $*-$automorphism ψ which coincides with φ on the subgroup of $\mathcal{U}(A)$ generated by $\{\Gamma, K_\infty\}$, where K_∞ denotes the subgroup generated by the self-adjoint unitaries in A_∞ $(= \cup_{n\geq 1} A_n)$. Moreover, if A is a UHF-algebra and θ_φ is not necessarily an orthoisomorphism, we show that there exists a linear or a conjugate linear $*-$automorphism ψ which coincides on K_∞ with φ up to a character of $\mathcal{U}(A)$.

Finally, we discuss the case where φ is a continuous automorphism of the unitary group of the UHF-algebras, and we show

Theorem 1.0.0.7. *Let A be a UHF-algebra. If φ is a continuous automorphism of $\mathcal{U}(A)$, then φ is implemented by a linear or a conjugate linear $*-$automorphism ψ of A.*

Chapter 2

Background Material and Some Basic Results

In this chapter, we recall some background material, basic definitions, and some main theorems. Also, we prove some results which will be used in the next chapters.

2.1 Dimension Groups

2.1.1 Ordered Groups

A set $\mathcal{I}$ with a partial order relation $\leq$ is said to be a **partially ordered set**, and will often be denoted by $(\mathcal{I}, \leq)$. Two elements x and y of $\mathcal{I}$ are **comparable** if either $x \leq y$ or $x \geq y$. A partial order on a set $\mathcal{I}$ in which any two elements are comparable is called a **total order**, and a set with a total order is called a **totally ordered set**.

Definition 2.1.1.1. *A partially ordered set $(\mathcal{I}, \leq)$ is said to be upward directed if every finite subset of $\mathcal{I}$ has an upper bound in $\mathcal{I}$, and is downward directed if every finite subset of $\mathcal{I}$ has a lower bound in $\mathcal{I}$.*

Definition 2.1.1.2. *A* **partially ordered abelian group** *is an abelian group G with a specified partial order $\leq$ which is translation-invariant, i.e. for any $x, y, z \in G$ we have*

$$x \leq y \Rightarrow x + z \leq y + z.$$

A **totally ordered abelian group** *is a partially ordered abelian group in which $\leq$ is a total order.*

Example 2.1.1.3. *Consider the following examples:*

1. *The additive group $\mathbb{Z}$ with the partial order $x \leq y$ if and only if $0 \leq y - x$ is a partially ordered abelian group (in fact, a totally ordered abelian group). This will be referred to as the* **usual ordering** *on $\mathbb{Z}$. Similarly, the additive groups $\mathbb{Q}$ and $\mathbb{R}$ with their usual order $\leq$ are totally ordered abelian groups.*

2. *The additive group $\mathbb{C}$ is a partially ordered abelian group with $z \leq w$ if and only if $w - z \in \mathbb{R}$ and $w - z \geq 0$ in $\mathbb{R}$.*

3. *We can define another ordering on $\mathbb{Z}$ by setting $x \leq y$ if and only if $y - x$ is non-negative and even. Then $\mathbb{Z}$ is a partially ordered abelian group, (but not a totally ordered abelian group).*

4. *Let $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in \mathbb{Z}^n$. Say that $x \leq y$ if and only if $x_i \leq y_i$ for every $i = 1, \ldots, n$, where x_i and y_i are compared using the usual ordering on $\mathbb{Z}$. Then $\mathbb{Z}^n$ is a partially ordered abelian group. This ordering on $\mathbb{Z}^n$ is called the* ***simplicial ordering***.

Definition 2.1.1.4. *A* ***cone*** *in an abelian group G is a subset C of G containing $0 \in G$ and C is closed under addition. A cone C is called a* ***strict cone*** *if 0 is the only element $x \in C$ for which $-x \in C$.*

Definition 2.1.1.5. *Let G be a partially ordered abelian group. Let $x \in G$. We say that x is a* ***positive element*** *of G if $0 \leq x$, and we say that x is a* ***strictly positive element*** *of G if $0 < x$.*

We denote by G^+ the set of all positive elements of G. Then G^+ is a strict cone, called the positive cone of G. Conversely, given a strict cone C in G we can define a relation $\leq_C$ on G by

$$x \leq_C y \Longleftrightarrow y - x \in C.$$

Then the relation $\leq_C$ is a translation invariant partial order, therefore $(G, \leq_C)$ is a partially ordered abelian group.

Example 2.1.1.6. *For each of the examples in (2.1.1.3), the positive cone is as follows:*

1. $\mathbb{Z}^+$ *is the set of non-negative integers,* $\mathbb{Z}^+ = \{0, 1, 2, \ldots\}$. *Similarly for* $\mathbb{R}^+$ *and* $\mathbb{Q}^+$.
2. $\mathbb{C}^+ = \mathbb{R}^+$.
3. $\mathbb{Z}^+ = 2\mathbb{Z}^+$.
4. $(\mathbb{Z}^n)^+ = \{(x_i);\ x_i \in \mathbb{Z}^+\}$.

Proposition 2.1.1.7. *[Go, 1.3] Let H be a subgroup of a partially ordered abelian group G. Then the following are equivalent:*

1. *H is an upward directed partially ordered set,*
2. *H is a downward directed partially ordered set,*
3. *H is generated (as a subset of G) by a subset of G^+,*
4. *All elements of H have the form $x - y$ for some $x, y \in H^+$.*

Definition 2.1.1.8. *A* ***directed subgroup*** *of a partially ordered abelian group G is any subgroup H of G which satisfies the equivalent conditions of Proposition (2.1.1.7). If G is a directed subgroup of itself then we say that G is* ***directed,*** *or that G is a* ***directed abelian group.***

Definition 2.1.1.9. *Let G be a partially ordered abelian group. We say that G is* ***unperforated*** *if for any $x \in G$ and any $n \in \mathbb{N}$, then nx is positive only if x is positive.*

Example 2.1.1.10. *If $\mathbb{Z}$ is made into a partially ordered abelian group with positive cone $\{0, 2, 4, \ldots\}$, then $\mathbb{Z}$ is perforated.*

Let G be an abelian group together with a subset P of G, called the **positive cone**, such that

1. $P + P \subset P$,

2. $P - P = G$,

3. $P \cap (-P) = \{0\}$,

4. If $a \in G$ and $na \in P$ for some $n \in \mathbb{N}$, then $a \in P$.

Then G is a directed, unperforated partially ordered abelian group. Conversely, any directed, unperforated, partially ordered abelian group satisfies (1-4) with $P = G^+$.

Let G be a directed, unperforated, partially ordered abelian group. If $a, b \in G$ and $na \leq nb$ for some $n \in \mathbb{N}$, then $a \leq b$. Indeed,

$$na \leq nb \Rightarrow 0 \leq nb - na = n(b-a) \Rightarrow n(b-a) \in G^+ \Rightarrow b - a \in G^+ \Rightarrow a \leq b.$$

So, in particular, $na = 0$ implies that $a = 0$; therefore a directed, unperforated, partially ordered abelian group is torsion free.

Definition 2.1.1.11. *Let G and H be two partially ordered abelian groups. A* ***positive homomorphism*** *from G to H is a group homomorphism $f : G \longrightarrow H$ such that*

$$f(G^+) \subseteq H^+.$$

Note that a homomorphism is positive if and only if it is order preserving (that is, $x \leq y$ implies $f(x) \leq f(y)$, for any x and $y \in G$).

Definition 2.1.1.12. *Let G be a partially ordered abelian group, and let $u \in G^+$. We say that u is an* ***order unit*** *in G if for any $x \in G$, there is an $n \in \mathbb{N}$ such that $x < nu$. We say that a positive homomorphism $f : G \longrightarrow H$ is a* ***normalized*** *if $f(u) = v$, where u and v are order units of G and H respectively.*

Example 2.1.1.13. *Any strictly positive element of $\mathbb{R}$ is an order unit.*

2.1.2 Grothendieck Groups

For a given commutative semigroup $\mathcal{H}$, the Grothendieck group $\mathcal{G}(\mathcal{H})$ is the smallest abelian group containing $\mathcal{H}$. Recall that a commutative semigroup $\mathcal{H}$ is said to satisfy **cancellation** if

$$x + y \; = \; z + y \implies x = z \text{ for any } x, y, z \; \in \; \mathcal{H}.$$

Definition 2.1.2.1. *Let $\mathcal{H}$ be an abelian (additive) semigroup. Denote by $\equiv$ the equivalence relation given on $\mathcal{H} \times \mathcal{H}$ by*

$$(h_1, k_1) \equiv (h_2, k_2) \iff \exists\, x \; \in \; \mathcal{H} \text{ such that } h_1 + k_2 + x = h_2 + k_1 + x.$$

The ***Grothendieck group*** *of $\mathcal{H}$ is*

$$\mathcal{G}(\mathcal{H}) = \mathcal{H} \times \mathcal{H} / \equiv, \text{ where}$$

addition is defined by

$$[(h_1, k_1)] + [(h_2, k_2)] \; = \; [(h_1 + h_2, k_1 + k_2)].$$

The identity I is the class of (x, x), for any $x \; \in \; \mathcal{H}$.

Remark 2.1.2.2. *Thinking of an element $[(h, k)] \; \in \; \mathcal{G}(\mathcal{H})$ as the formal difference $h - k$, it is natural to represent $h \; \in \; \mathcal{H}$ in $\mathcal{G}(\mathcal{H})$ as $[(h, 0)]$.*

Example 2.1.2.3. $\mathcal{G}(\mathbb{Z}^+) \; = \; \mathbb{Z}$, $\mathcal{G}(\mathbb{R}^+) \; = \; \mathbb{R}$, $\mathcal{G}(\mathbb{Q}^+) \; = \; \mathbb{Q}$.

2.1.3 Dimension Groups

Proposition 2.1.3.1. *[Go, 2.1] For a partially ordered abelian group G, the following are equivalent:*

1. *Given $x_1, x_2, y_1, y_2 \; \in \; G$ such that $x_i \leq y_j$ for all i, j, there exists an element $z \; \in \; G$ such that $x_i \leq z \leq y_j$ for all i, j.*

2. *Given $x, y_1, y_2 \; \in \; G^+$ such that $x \leq y_1 + y_2$, there exist $x_1, x_2 \; \in \; G^+$ such that $x = x_1 + x_2$ and $x_i \leq y_i$ for all i.*

3. *Given* $x_1, x_2, y_1, y_2 \in G^+$ *such that* $x_1 + x_2 = y_1 + y_2$, *there exist elements* $z_{11}, z_{12}, z_{21}, z_{22} \in G^+$ *such that* $x_i = z_{i1} + z_{i2}$ *for each* i *and* $y_j = z_{1j} + z_{2j}$ *for each* j.

Definition 2.1.3.2. *A partially ordered abelian group G is said to satisfy the* ***Riesz interpolation property****, provided G satisfies condition (1) of Proposition (2.1.3.1), and is said to satisfy the* ***Riesz decomposition properties*** *if conditions (2) and (3) of Proposition (2.1.3.1) hold in G.*

Proposition 2.1.3.3. *[Go, 2.2] Let G be an interpolation group. Then we have the following:*

1. *Given* $x_i, y_j \in G$ *such that* $x_i \leq y_j$ *for all* $i = 1, \dots n$, $j = 1, \dots k$, *there exists* $z \in G$ *such that* $x_i \leq z \leq y_j$.

2. *Given* $x, y_1, \dots y_k \in G^+$ *such that* $x \leq \sum_{j=1}^{k} y_j$ *there exists* $x_1, \dots, x_k \in G^+$ *such that* $x = \sum_{i=1}^{k} x_i$ *and* $x_j \leq y_j$ *for all* j.

3. *Given* $x_1, \dots, x_n, y_1, \dots, y_k \in G^+$ *such that* $\sum_i x_i = \sum_j y_j$, *there exist* $z_{ij} \in G^+$ *such that*
$$x_i = \sum_j z_{ij} \text{ and } y_j = \sum_i z_{ij}$$
for all i, j.

Definition 2.1.3.4. *A* ***dimension group*** *is a directed, unperforated, partially ordered abelian group which satisfies the Riesz interpolation property. A* ***scale*** *on a dimension group G is a subset* Σ *of* G^+ *which satisfies:*

1. *For each* $a \in G^+$ *there are* $a_1, \dots, a_n \in \Sigma$ *such that* $a = a_1 + \dots + a_n$.

2. *If* $0 \leq a \leq b$ *with* $b \in \Sigma$, *then* $a \in \Sigma$.

3. *Given* $a, b \in \Sigma$, *there is* $c \in \Sigma$ *with* $a \leq c$ *and* $b \leq c$.

We call $(G, \Sigma(G))$ *a* ***scaled dimension group****.*

Let $(G, \Sigma(G))$ and $(H, \Sigma(H))$ be scaled dimension groups. Let $f : G \to H$ be a positive homomorphism. We call f a **contraction** if $f(\Sigma(G)) \subseteq \Sigma(H)$. If f is invertible and f^{-1} is also a contraction, then we call f an isomorphism of scaled dimension groups.

Consider the examples in (2.1.1.3):

1. $\mathbb{Z}$ with the usual ordering is a dimension group, for any $n \in \mathbb{Z}^+ \backslash \{0\}$ the set $[0, n] = \{0, 1, 2, \ldots, n\}$ is a scale on $\mathbb{Z}$.

2. $\mathbb{C}$ with that ordering is not directed, and hence not a dimension group.

3. $\mathbb{Z}$ in this case is neither unperforated nor directed, hence is not a dimension group.

4. $\mathbb{Z}^n$ with the simplicial ordering is a dimension group, and the scale are the sets $[0, n_1] \times \ldots \times [0, n_k]$, with $n_1, \ldots, n_k \in \mathbb{Z}^+ \backslash \{0\}$.

Let G be a dimension group with an order unit u. Then the set

$$[0, u] = \{x \in G \mid 0 \leq x \leq u\}$$

is a scale on G.

Definition 2.1.3.5. *Let $(G, \Sigma(G))$ and $(H, \Sigma(H))$ be scaled dimension groups. A map $f : \Sigma(G) \to \Sigma(H)$ is a **scale homomorphism** if whenever $a = b + c$, for some $a, b, c \in \Sigma(G)$, it follows that $f(a) = f(b) + f(c)$. If f is invertible and f^{-1} is also a scale homomorphism then f is called a **scale isomorphism**.*

The following is the extension theorem of scale homomorphisms (scale isomorphisms) to contractions (isomorphism of scaled dimension groups), which was proved by E. Effros in [EF, L.7.3]

Proposition 2.1.3.6. *Any scale homomorphism $f : \Sigma(G) \to \Sigma(H)$ extends to a unique contraction $\tilde{f} : G \to H$. If f is a scale isomorphism, then $\tilde{f}$ is an isomorphism of scaled dimension groups.*

Proof: Extend f to an additive map $f_1 : G^+ \to H^+$ by

$$f_1(a_1 + \cdots + a_n) = f(a_1) + \cdots + f(a_n).$$

To see that this is well-defined suppose that $a_1 + \cdots a_n = b_1 + \cdots b_m$. From the Riesz interpolation property, there exist $c_{ij} > 0$ with $a_i = \sum_j c_{ij}$, $b_j = \sum_i c_{ij}$, and we have

$$\sum_i f(a_i) = \sum_{ij} f(c_{ij}) = \sum_j f(b_j).$$

Extend f_1 to $G = G^+ - G^+$ by setting $f_2(a-b) = f_1(a) - f_1(b)$, where $a, b \in G^+$. f_2 is well defined since if $a - b = a' - b'$, then $a + b' = a' + b$ which implies $f_2(a) + f_2(b') = f_2(a') + f_2(b)$. Hence f_2 is the desired extension. □

Remark 2.1.3.7. *The extension of f in the previous proposition is based on the following:*

1. *the scale Σ generates G as an abelian group.*

2. *G satisfies the Riesz interpolation property.*

Definition 2.1.3.8. *A **convex subset** of a partially ordered set G is a set $H \subseteq G$ such that*

$$(x, z \in H, y \in G, x \leq y \leq z) \Longrightarrow y \in H.$$

*A convex subgroup of a partially ordered abelian group G is a subgroup H of G which is a convex subset of G. An **ideal** of a partially ordered abelian group G is any directed convex subgroup of G. A partially ordered abelian group G is **simple** if G is nonzero and directed (so G is a non-zero ideal of itself) and the only ideals of G are $\{0\}$ and G.*

2.2 C^*–Algebras

2.2.1 Basic Definitions

Definition 2.2.1.1. *An involutive algebra A is a complex algebra given together with an involution* $\begin{cases} A \longrightarrow A \\ a \longmapsto a^* \end{cases}$ *such that*

$$(a+b)^* = a^* + b^*$$

$$(\lambda a)^* = \bar{\lambda} a^*$$

$$(ab)^* = b^* a^*$$

$$(a^*)^* = a$$

for all $a, b \in A$ and $\lambda \in \mathbb{C}$.

Recall that a Banach algebra A is an algebra over $\mathbb{C}$ that has a norm $\|.\|$ relative to which A is Banach apace and $\|ab\| \leq \|a\| \, \|b\|$ for all $a, b \in A$.

Definition 2.2.1.2. *A C^*-algebra is a Banach involutive algebra A such that $\|a^*a\| = \|a\|^2$ for all $a \in A$.*

Definition 2.2.1.3 (Inductive limit of C^*-algebras). *Let A_i, $i \in \mathcal{I}$ be C^*- algebras and let $\Phi_{ij} : A_i \to A_j$ be C^*-algebra morphisms. It follows that $\{A_i, \Phi_{i,j}\}$ is a normed inductive system of normed algebras, hence has a direct limit A_∞ in the category of normed algebras. Moreover, A_∞ has a $*-$algebra structure and satisfies $\|a^*a\| = \|a\|^2$. Hence we construct the (unique) C^*-**algebra direct limit** A of the directed system $\{A_i, \Phi_{i,j}\}$ to be $\overline{A_\infty}$.*

A $*-$homomorphism $\Phi : A \to B$ between two C^*-algebras is a linear map such that

$$\Phi(ab) = \Phi(a)\Phi(b)$$

$$\Phi(a^*) = \Phi(a)^*$$

for all $a, b \in A$.

Definition 2.2.1.4. *Let A be a unital C^*-algebra.*

1. *$a \in A$ is **normal** if $a^*a = aa^*$,*

2. *$a \in A$ is **self adjoint** if $a^* = a$, and $B \subseteq A$ is self adjoint if B is closed under involution,*

3. *$a \in A$ is **positive** if there exists $b \in A$ such that $a = b^*b$,*

4. *$p \in A$ is a **projection** if $p^* = p = p^2$,*

5. $w \in A$ *is an* ***isometry*** *if* $w^*w = 1$,

6. $w \in A$ *is a* ***partial isometry*** *if* w^*w *is a projection,*

7. $u \in A$ *is a* ***unitary*** *if* $uu^* = u^*u = 1$.

We will denote by $\mathcal{U}(A)$ *the unitary group of* A.

Example 2.2.1.5. *The following are* C^*-*algebras, with the indicated involutions and norms:*

1. $\mathbb{C}$, *with the complex conjugation as involution.*

2. *The complex matrices* $\mathbb{M}_n(\mathbb{C})$, *with the involution* $(a_{ij})^* = (\overline{a_{ij}})^t$, *and*

$$||A|| = sup\{||A\zeta||;\ \zeta \in \mathbb{C}^n \text{ such that } ||\zeta|| = 1\}.$$

3. *For a given Hilbert space* $\mathcal{H}$, *the space of all bounded linear operators* $\mathcal{B}(\mathcal{H})$ *is a* C^*-*algebra with the usual adjoint operator, and for* $T \in \mathcal{B}(\mathcal{H})$, *the norm of* T *is*

$$||T|| = sup\{||T(x)||;\ ||x|| \leq 1\}.$$

4. *Let* X *be a compact space. Then the set* $C(X)$ *of all continuous complex valued functions on* X, *with* $f^*(x) = \overline{f(x)}$ *is a* C^*-*algebra and the norm defined by*

$$||f|| = sup_{x \in X}||f(x)||.$$

An **ideal** in a C^*-algebra A, is defined to be a norm-closed two sided ideal. A is said to be **simple** if there is no proper ideal in A ([Da, p.12]).

Proposition 2.2.1.6. *If* A *is a simple unital* C^*-*algebra, then the centre of* A *is*

$$\mathcal{Z}(A) = \{\alpha I |\ \alpha \in \mathbb{C}\}.$$

Definition 2.2.1.7. *The* C^*-*matrix algebra* $\mathbb{M}_n(A)$, *of a* C^*-*algebra* A *is defined to be the algebra of the* $n \times n$ *matrices* $(a_{i,j})$ *over* A *with operations:*

$$\alpha(a_{i,j}) = (\alpha a_{i,j})(\alpha \text{ scalar}),\ \ (a_{i,j}) + (b_{i,j}) = (a_{i,j} + b_{i,j})$$

$$(a_{i,j})^* = (a^*_{j,i}) \text{ and } (a_{i,j})(b_{i,j}) = (\sum_{k=1}^{n} a_{i,k}b_{k,j}).$$

2.2.2 Spectrum of C^*−Algebras

The spectral theory for C^*−algebras holds for more general algebras. Consider A to be a unital Banach algebra.

Definition 2.2.2.1. *For each $a \in A$, the **spectrum** of a is the subset*

$$\sigma(a) = \{\lambda \in \mathbb{C} |\ \lambda - a \text{ is not invertible in } A\}$$

of the complex plane.

Lemma 2.2.2.2. *[HJ, 4.11] For each $a \in A$, the sequence $(\|a^n\|^{\frac{1}{n}})_{n\geq 1}$ is convergent and*

$$lim_{n\to\infty}\|a^n\|^{\frac{1}{n}} = inf_{n\geq 1}\|a^n\|^{\frac{1}{n}} \leq \|a\|.$$

Definition 2.2.2.3. *[HJ, 4.12] The **spectral radius** of $a \in A$ is the real number*

$$\rho(a) = lim_{n\to\infty}\|a^n\|^{\frac{1}{n}}.$$

Equivalently, $\rho(a)^{-1}$ is the radius of convergence of the series $\sum_{n=0}^{\infty}\lambda^n a^n$. Observe that $\rho(a) \leq \|a\|$. If A is a C^*−algebra and a is normal element, then $\rho(a) = \|a\|$.

Proposition 2.2.2.4. *[HJ, 4.14] For each $a \in A$, the spectrum $\sigma(a)$ is a non empty compact subset of $\mathbb{C}$ which is contained in the closed disc of radius $\rho(a)$, centered at the origin.*

Proposition 2.2.2.5. *[HJ, 4.20] Let A be a unital C^*−algebra, then we have:*
(i) if $a \in A$ is self adjoint, then its spectrum is in $\mathbb{R}$.
(ii) if $u \in A$ is unitary, then its spectrum is in the unit circle of the complex plane.
(iii) for each $a \in A$, the spectrum of a^* is $\{\lambda \in \mathbb{C};\ \bar{\lambda} \in \sigma(a)\}$.

Theorem 2.2.2.6. *(Continuous functional calculus)[HJ, 4.24] Let A be a unital C^*−algebra, and $a \in A$ be a self-adjoint element. Let $C(\sigma(a))$ be the C^*−algebra of continuous functions on the spectrum of a, then a map*

$$\begin{cases} C(\sigma(a)) \longrightarrow A \\ \quad f \longmapsto f(a) \end{cases}$$

defines a morphism of C^*−algebras, moreover $\sigma(f(a)) = f(\sigma(a))$, for all $f \in C(\sigma(a))$.

Now we prove the following result:

Theorem 2.2.2.7. *If A is a unital C^*-algebra, then any element of A can be written as a linear combinations of four unitaries.*

Proof: If $a \in A_{sa}$ with $\|a\| \leq 1$, then $\rho(a) \subseteq [-1, 1]$. Let $f(t) = \sqrt{1-t^2}$, so $f \in C(\sigma(a))$ therefore by using the continuous functional calculus, we have $\sqrt{1-a^2} \in A$. Define u_1, u_2 to be as follows:

$$u_1 = a + i\sqrt{1-a^2}$$

$$u_2 = a - i\sqrt{1-a^2}.$$

Notice that u_1 and u_2 belong to $\mathcal{U}(A)$, moreover $u_1^* = u_2$, and $a = (u_1 + u_2)/2$. Now, if $a \in A_{sa}$ and $\|a\| > 1$, define $a' = a/2\|a\|$, then $a' \in A_{sa}$, and we get that $a = \|a\|u_1 + \|a\|u_2$, hence the theorem follows for the self adjoint unitaries. Finally if $a \in A$ then, $a = a_1 + a_2 i$, where a_1 and a_2 are in A_{sa}, therefore the theorem follows.□

2.2.3 Projections in C^*-Algebras

Let us denote by $\mathcal{P}(A)$ the set of all projections in a C^*-algebra A. This set is a partially ordered by the following partial ordering: For $p_1, p_2 \in \mathcal{P}(A)$, $p_1 \leq p_2$ whenever $p_1p_2 = p_2p_1 = p_1$. Also two projections p_1, p_2 are **orthogonal** if $p_1p_2 = 0$. Note that $0 \leq p \leq 1$ for any projection p. If $p \leq q$, then $q-p$ is a projection orthogonal to p, the projection $1-p$ is called the **orthogonal complement** of p. The sum of two projections p and q is a projection precisely when p and q are orthogonal.

Not all C^*-algebras have plenty of projections, the following example shows that some algebras have only trivial projections.

Example 2.2.3.1. *If X is a compact subset, then p is a projection in $C(X)$ if and only if there exists a clopen subset B of X such that $p = \chi_B$, where χ is the characteristic function on X; therefore if X is connected, then either $p = 0$ or $p = 1$.*

For two commuting projections p and q, we define their symmetric difference

$$p\Delta q = p + q - 2pq.$$

If p and q are two orthogonal projections, then $p\Delta q = p + q$. The following properties of the symmetric difference of commuting projections can be verified by directly:

$$\begin{aligned} p\Delta(q\Delta r) &= (p\Delta q)\Delta r \\ (1-p)\Delta q &= 1 - p\Delta q \\ (p_1\Delta p_2 \ldots \Delta p_n)r &= p_1 r\Delta p_2 r\Delta \ldots \Delta p_n r. \end{aligned}$$

Definition 2.2.3.2. *[WO, 5.2.1] Projections p and q in a unital C^*-algebra A are said to be:*

- ***equivalent**, denoted $p \sim q$, when there is a partial isometry $v \in A$ such that $p = v^*v$ and $q = vv^*$;*
- ***unitarily equivalent**, denoted $p \sim_u q$, when there is a unitary $u \in A$ such that $p = u^*qu$;*
- ***homotopically equivalent**, denoted $p \sim_h q$, when p and q are connected by a norm-continuous path of projections in A.*

Let p and q be two projections in a C^*-algebra A. If p is equivalent to a subprojection of q, then we write $p \preceq q$, and if p is equivalent to a proper subprojection of q, then we write $p \prec q$.

Lemma 2.2.3.3. *[WO, 5.2.10] If p and q are projections in A, then*

$$p \sim_h q \Longrightarrow p \sim_u q \Longrightarrow p \sim q.$$

Lemma 2.2.3.4. *[WO, 5.2.5] Let p and q be projections in a unital C^*-algebra A. Then $p \sim_u q$ if and only if $p \sim q$ and $1 - p \sim 1 - q$.*

Proposition 2.2.3.5. *[WO, 5.2.6] If p and q are projections in A such that $\|p-q\| < 1$, then they are homotopically equivalent.*

Proposition 2.2.3.6. *[HJ, 4.28] Let A be a unital C^*-algebra. If there exists a tower $A_1 \subset A_2 \subset \ldots$ of C^*-subalgebras of A such that $A_\infty = \bigcup_{n\geq 1} A_n$ is dense in A, then for each projection $e \in A$, there exists a unitary $u \in A$ such that $ueu^* \in A_\infty$.*

Definition 2.2.3.7. *Let A be a C^*-algebra, and B be a C^*-subalgebra of A. We say that B is a* ***hereditary*** *$*-$subalgebra of A if*

$$[a \in A,\, b \in B,\, 0 \leq a \leq b] \Longrightarrow a \in B.$$

Example 2.2.3.8. *Obviously, $\{0\}$ and A are hereditary C^*-subalgebras of A, and any intersection of hereditary C^*-subalgebras is also hereditary.*

Example 2.2.3.9. *Let A be a C^*-algebra, and $p \in \mathcal{P}(A)$. Then pAp is also a C^*-algebra with unit p. Clearly pAp is a $*-$algebra with identity p. To show that pAp is closed, suppose that x is an accumulation point of pAp. Let (x_n) be a sequence in pAp converging to some x in A. Since $\|p\| \leq 1$ we see that the sequence (px_np) converges to pxp. However $px_np = x_n$ since $x_n \in pAp$ for all n. Then $x = lim_{n\to\infty} px_np = pxp$, and hence $x \in pAp$. Hence pAp is a C^*-algebra; it is thus a hereditary C^*-subalgebra of A.*

Remark 2.2.3.10. *A C^*-subalgebra of a simple C^*-algebra need not be simple; as a trivial example, $\mathbb{M}_2 \oplus \mathbb{M}_2 \subset \mathbb{M}_4$, shows.*

Theorem 2.2.3.11. *[Mu, 3.2.8] A hereditary C^*-subalgebra of simple C^*-algebra is simple.*

Definition 2.2.3.12. *[Da, p.101] A C^*-algebra A is said to be* ***finite*** *if for any two equivalent projections p and q such that $p \leq q$, then $p = q$. Moreover, A is said to be* ***stably finite*** *if $\mathbb{M}_n(A)$ is finite for all $n \geq 1$.*

Example 2.2.3.13. *Any finite dimensional C^*-algebra is stably finite.*

2.2.4 AF-Algebras

We first mention the following results about the finite dimensional C^*-algebras.

Lemma 2.2.4.1. *[Da, p.74] A finite dimensional C^*-algebra has a unit.*

Lemma 2.2.4.2. *[Da, Th.III.1.1] Every finite dimensional C^*-algebra A is isomorphic (as C^*-algebras) to a direct sum of full matrix algebras*

$$A \simeq \mathbb{M}_{n_1} \oplus \ldots \oplus \mathbb{M}_{n_k} \textit{ for some } n_i, k \in \mathbb{N}.$$

Now we define a set of matrix units of C^*-algebras, and we prove some basic results, which will be used in the next sections.

Definition 2.2.4.3. *If B is a unital C^*-algebra, then a set $\{e^r_{i,j}\} \subseteq B, 1 \leq i, j \leq n$ and $1 \leq r \leq m$, is said to be a* ***system of matrix units*** *in B if it satisfies:*

$$e^r_{i,j}e^r_{j,k} = e^r_{i,k}, e^r_{i,j}e^s_{k,l} = 0 \text{ if } r \neq s \text{ or } j \neq k, \ (e^r_{i,j})^* = e^r_{j,i}, \sum_{i,r}^{n,m} e^r_{i,i} = 1$$

and $e_{i,i}$ are projections for every i.

Proposition 2.2.4.4. *Let A be any unital C^*-algebra. Then A has a set of matrix units $\{e_{i,j}\}$, $1 \leq i, j \leq n$ if and only if there exists a set of n pairwise orthogonal equivalent projections which constitutes a partition of unity.*

Proof: Let $\{e_{i,j}\}$, $1 \leq i, j \leq n$ be a system of matrix units of A. For every i, define $r_i = e_{i,i}$, so it is clear that these r_i's are orthogonal projections such that $\sum_{i=1}^n r_i = 1$. Now we prove that $r_1 \sim_u r_i$, and the general case is similar. Let $u = e_{1,i} + e_{i,1} + \sum_{l\neq 1, l\neq i}^n e_{l,l}$ so, u is a self-adjoint unitary in A, and $ue_{i,i}u = e_{1,1}$ therefore the set $\{r_1, r_2, \dots r_n\}$ forms a partition of unity. Conversely, as r_i is equivalent to r_1 for every i, let s_i be a partial isometry such that $s_i^*s_i = r_1$ and $s_is_i^* = r_i$. Define $e_{i,j} = s_is_j^*$, and we claim that the set $\{e_{i,j}, 1 \leq i, j \leq n\}$ forms a set of matrix units of A; we show that as follows:

(i) For every i, $e_{i,i} = r_i$ is a projection and $\sum_{i=1}^n e_{i,i} = 1$,

(ii) $(e_{i,j})^* = s_js_i^* = e_{j,i}$,

(iii) $e_{i,j}e_{j,k} = s_is_j^*s_js_k^* = s_ir_1s_k^* = e_{i,k}$,

(iv) If $j \neq k$, then $e_{i,j}e_{k,l} = s_is_j^*s_ks_l^* = s_i(s_j^*r_j)(r_ks_k)s_l^* = 0$. Therefore, we have proved our claim. □

Proposition 2.2.4.5. *Let A be any unital C^*-algebra. Then A has a system of matrix units if and only if there exists $n \in \mathbb{N}$ such that $\mathbb{M}_n(\mathbb{C})$ can be embedded in A.*

Proof: If $\{e_{i,j}, 1 \leq i, j \leq n\}$ forms a set of matrix units of A, then define the map

$$\Psi : \mathbb{M}_n(\mathbb{C}) \longrightarrow A \quad \text{by}$$

$$(a_{i,j})_{n\times n} \longmapsto \sum_{i,j=1}^{n} a_{i,j}e_{i,j}.$$

It is clear that Ψ is an embedding of $\mathbb{M}_n(\mathbb{C})$ into A. Conversely assume that, for some $n \in \mathbb{N}$, there exists an embedding

$$\Psi : \mathbb{M}_n(\mathbb{C}) \longrightarrow A$$

As $\{E_{i,j},\ 1 \leq i, j \leq n\}$, where $E_{i,j}$ denotes the matrix with 1 placed in $i \times j$ and zeros elsewhere, forms a system of matrix units of $\mathbb{M}_n(\mathbb{C})$, then $\{\Psi(E_{i,j}),\ 1 \leq i, j \leq n\}$ forms a system of matrix units of A, hence the result follows. □

Proposition 2.2.4.6. *If A is a unital C^*-algebra, and for some n, $\{e_{i,j}\}_{i,j=1}^{n}$, $\{f_{i,j}\}_{i,j=1}^{n}$ are two systems of matrix units of A such that $e_{1,1} \sim f_{1,1}$, then there exists a unitary u in A such that, $u(e_{i,j})u^* = f_{i,j}$, for every $1 \leq i, j \leq n$.*

Proof: Let v be a partial isometry such that $v^*v = e_{1,1}$ and $vv^* = f_{1,1}$. Define $u = \sum_{j=1}^{n} f_{j,1}ve_{1,j}$, then

$$uu^* = \sum_{j,k}^{n} f_{j,1}ve_{1,j}e_{k,1}v^*f_{1,k} = \sum_{j}^{n} f_{j,1}ve_{1,1}v^*f_{1,j} = 1.$$

Similarly, $u^*u = 1$, therefore u is a unitary in A and

$$u(e_{i,j})u^* = f_{i,1}ve_{1,1}v^*f_{1,j} = f_{i,j}.$$

□

Definition 2.2.4.7. *[HJ, 5.6] A C^*-algebra A is an **approximately finite** algebra (AF-algebra) if $A = \overline{\cup A_n}$ such that for every $n \geq 1$, A_n is a finite dimensional C^*-subalgebra of A, and $A_n \subseteq A_{n+1}$. In other words A is an AF-algebra if it is the C^*-algebraic direct limit of a directed system of finite dimensional C^*-algebras and C^*-algebra homomorphisms.*

Consider the following examples:

Example 2.2.4.8. *For each $n \in \mathbb{N}$, let $A_n = \mathbb{M}_n(\mathbb{C})$, and let $\Phi_{n,n+1}$ be the embedding of A_n into A_{n+1}, given by*

$$\Phi_{n,n+1}(A) = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \in \mathbb{M}_{n+1}, \text{ for any } A \text{ in } \mathbb{M}_n.$$

Then the corresponding AF-algebra is isomorphic to $\mathbb{K}$, where $\mathbb{K}$ denotes the compact operators on the separable, infinite dimensional Hilbert space $\mathcal{H} \simeq \ell_2$.

Example 2.2.4.9. *[HJ, 5.7] The algebra of continuous functions on the Cantor ternary set X is a commutative AF-algebra. For each $n \in \mathbb{N}$, define an algebra A_n of continuous functions on X as follows:*
A_0 is the algebra of constant functions,
A_1 is the algebra of functions constant on $X \cap [0, \frac{1}{3}]$ and $X \cap [\frac{2}{3}, 1]$, and so on
A_n is the algebra of functions constant on $X \cap [\frac{j}{3^n}, \frac{j+1}{3^n}]$ for each $j \in \{0, 1, \ldots, 3^n - 1\}$. Then

$$A_0 \subset A_1 \subset \ldots \subset A_n \subset A_{n+1} \subset \ldots$$

is a nested sequence of finite dimensional abelian C^*–algebras, and the corresponding AF-algebra is the C^*–algebra of continuous functions on the Cantor set X.

Now we introduce another example of AF-algebras, the **UHF-algebras**.

Example 2.2.4.10. *Let $(k_j)_{j\geq 1}$ be a sequence of integers, with $k_j \geq 2$. For each $n \geq 1$, the algebra*

$$A_n = \bigotimes_{j=1}^{n} \mathbb{M}_{k_j}(\mathbb{C}) \approx \mathbb{M}_{k_1 k_2 \ldots k_n}(\mathbb{C})$$

is a full matrix algebra, and the embedding from A_n to A_{n+1} is given by

$$x \longmapsto x \otimes 1_{k_{n+1}}.$$

The resulting inductive limit C^*–algebra

$$\bigotimes_{j=1}^{\infty} \mathbb{M}_{k_j}(\mathbb{C})$$

*is an AF-algebra called a **UHF-algebra**.*

Let A be a UHF-algebra. If for every j, $k_j = 2$, then A is called a 2-infinity UHF-algebra, and if for every j, $k_j = 3$, then A is called a 3-infinity UHF-algebra.

Theorem 2.2.4.11. *[Da, Th.III.5.2] Any UHF-algebra is simple.*

Theorem 2.2.4.12. *[Da, IV.2.3] AF-algebras are stably finite.*

Proposition 2.2.4.13. *[Da, IV.1.3] If $A = \overline{\cup_{n\geq 1} A_n}$ is an AF-algebra, then every projection in A is equivalent to a projection in $\cup_{n\geq 1} A_n$. Moreover, if p and q are in $\cup_{n\geq 1} A_n$ and equivalent in A, then they are equivalent in $\cup_{n\geq 1} A_n$.*

2.3 K-Theory

2.3.1 K_0-Groups

Let $\tilde{\mathcal{P}}(A)$ denote the collection of all projections in $\bigcup_{n\geq 1} \mathbb{M}_n(A)$. This is a semigroup under the operation of direct sum, in the following sense: If p and q are in $\tilde{\mathcal{P}}(A)$, then

$$p \oplus q = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}.$$

Definition 2.3.1.1. *[Da, p.99] Two projections in $\tilde{\mathcal{P}}(A)$, say $p \in \mathbb{M}_m(A)$ and $q \in \mathbb{M}_n(A)$ with $m \leq n$, are equivalent ($p \sim q$) if $p \oplus 0_{n-m}$ is equivalent to q in $\mathbb{M}_n(A)$. Say that p and q are* ***stably*** *equivalent (denoted $p \approx q$) if there is a projection $r \in \tilde{\mathcal{P}}(A)$ such that $p \oplus r \sim q \oplus r$.*

Stable equivalence is an equivalence relation. Let $K_0^+(A)$ denote the collection of all stable equivalence classes, denoted by $[p]$ for $p \in \tilde{\mathcal{P}}(A)$.

Lemma 2.3.1.2. *[Da, IV.1.4] Let p, p', q, q' and r be projections in $\tilde{\mathcal{P}}(A)$. Then*
(i) If $p \approx p'$ and $q \approx q'$, then $p \oplus q \approx p' \oplus q'$.
(ii) $p \oplus q \sim q \oplus p$.
(iii) If $pq = 0$ in $\mathbb{M}_n(A)$, then $p + q \sim p \oplus q$.
(iv) If $p \oplus r \approx q \oplus r$, then $p \approx q$.

Corollary 2.3.1.3. *$K_0^+(A)$ is an abelian cancellation semigroup with the operation*

$$[p] + [q] := [p \oplus q]$$

and with zero element $[0]$.

Proof: By the previous lemma (i), the sum operation on $K_0^+(A)$ is well defined. By part (ii), it is commutative. By (iv), $[p] + [r] = [q] + [r]$ implies that $[p] = [q]$, so $K_0^+(A)$ has the cancellation property. Associativity follows because direct sum is an associative operation. Finally it is clear that $[0]$ is a zero element. □

Note that an equivalence of two projections implies a stable equivalence, simply by choosing $r = 0$. A C^*-algebra A satisfies the **cancellation** property if stable equivalence implies equivalence.

Theorem 2.3.1.4. *[Da, IV.1.6] If A is an AF-algebra, then A satisfies cancellation, and $K_0^+(A)$ is generated by projections in A.*

Remark 2.3.1.5. *Two projections in $\mathbb{M}_n(\mathbb{C})$ are equivalent if and only if they have the same rank.*

Definition 2.3.1.6. *[Da, p.100] If A is a C^*-algebra, then $K_0(A)$ denotes the Grothendieck group of the semigroup $K_0^+(A)$, recall* §2.1.2.

Consider the following elementary examples:

Example 2.3.1.7. $K_0(\mathbb{M}_n(\mathbb{C})) = \mathbb{Z}$.

Example 2.3.1.8. *Let A be the finite dimensional C^*-algebra $\mathbb{M}_{n_1} \oplus \cdots \oplus \mathbb{M}_{n_k}$. Then two projections $p = p_1 \oplus \cdots \oplus p_k$ and $q = q_1 \oplus \cdots \oplus q_k$ in some matrix algebra over A are equivalent if and only if $rank(p_i) = rank(q_i)$ for all $1 \le i \le k$. Therefore $K_0(A) = \mathbb{Z}^k$.*

Define an order relation on $K_0(A)$ by setting $x \le y$ if $y - x \in K_0^+(A)$. Then we have the following proposition:

Proposition 2.3.1.9. *[Da, IV.2.2] Let A be a C^*-algebra. Then*
(i) $\leq$ is a reflexive and transitive relation.
(ii) If $x \leq y$, then $x + z \leq y + z$ for all $z \in K_0(A)$.
(iii) If A is unital, then $[1]$ is an order unit for $K_0(A)$.

Recall that a strict cone of an abelian group G is a subset C of G such that $0 \in C$, C is closed under addition and 0 is the only element $x \in C$ for which $-x \in C$.

Theorem 2.3.1.10. *[Da, IV.2.4] If A is a stably finite C^*-algebra, then $K_0^+(A)$ is a strict cone.*

Corollary 2.3.1.11. *If A is stably finite, then $K_0(A)$ is a partially ordered group.*

Proof: By the previous theorem $K_0^+(A)$ is a strict cone. Therefore the relation $\leq$ is a partial order. □

The following remark is a characterization of a dimension group. Recall the Effros-Handelman-Shen Theorem ([EHS]).

Remark 2.3.1.12. *Every dimension group is of the form $K_0(A)$ for some AF-algebra A.*

If ϕ is a $*-$homomorphism between two C^*-algebras A and B, then define the $*-$homomorphisms $\phi^{(n)}$ of $\mathbb{M}_n(A)$ into $\mathbb{M}_n(B)$ obtained by applying ϕ to each matrix entry. So we can define a natural map

$$\phi_* = K_0(\phi) : \ K_0(A) \longrightarrow K_0(B) \ \text{by}$$

$$\phi_*([p]) \ = \ [\phi^{(n)}(p)], \ \text{for all}\, p \in \tilde{\mathcal{P}}(A).$$

Moreover, if ϕ is unital, then so is ϕ_*.

Theorem 2.3.1.13. *[Da, IV.3.3] If $A = \overline{\cup A_n}$ is an AF-algebra, then*

$$K_0(A) = \varinjlim K_0(A_n)$$

as a scaled dimension group.

Note that for an AF-algebra A, $K_0^+(A) = \varinjlim K_0^+(A_n)$ and $\Sigma(A) = \varinjlim \Sigma(A_n)$.

Example 2.3.1.14. *Consider the C^*-algebra of compact operators $\mathcal{K}$. Think of $\mathcal{K}$ as a direct limit of $\mathbb{M}_n(\mathbb{C})$, with the embedding $\phi_{mn} : \mathbb{M}_m \to \mathbb{M}_n$ given by $\phi_{mn}(A) = A \oplus 0_{n-m}$. The corresponding imbedding ϕ_{mn*} of $K_0(\mathbb{M}_m) = (\mathbb{Z}, \mathbb{Z}^+, [0, m])$ into $K_0(\mathbb{M}_n) = (\mathbb{Z}, \mathbb{Z}^+, [0, n])$ is the identity map. Thus $K_0(\mathcal{K}) = (\mathbb{Z}, \mathbb{Z}^+, \mathbb{Z}^+)$.*

Now let us recall Elliott's classification result for AF-algebras:

Theorem 2.3.1.15. *[Da, IV.4.3] Two AF-algebras A and B are $*-$isomorphic if and only if their scaled dimension groups are isomorphic. Moreover, given a scaled dimension group isomorphism $\rho : K_0(A) \to K_0(B)$, there is a $*-$ isomorphism ϕ of A onto B such that $\phi_* = \rho$.*

2.3.2 K_1-Groups

Let A be a unital C^*-algebra. By $GL_n(A)$ (resp. $\mathcal{U}_n(A)$) we denote the invertible (resp. unitary) group of $\mathbb{M}_n(A)$. The connected component of the identity will be denoted by $GL_n(A)_0$ and $\mathcal{U}_n(A)_0$. A **homotopy** between elements x and y is a continuous path of elements, which starts at x and ends at y.

Definition 2.3.2.1. *[WO, 4.1.3] If A is a unital C^*-algebra, then define the following: $\mathbb{M}_\infty(A) = \cup_{n=1}^\infty \mathbb{M}_n(A)$, $GL_\infty(A) = \cup_{n=1}^\infty GL_n(A)$, and $\mathcal{U}_\infty = \cup_{n=1}^\infty \mathcal{U}_n(A)$.*

Lemma 2.3.2.2. *[WO, 4.2.1] If x, y are elements of $\mathbb{M}_n(A)$ such that x is invertible and $||x - y|| < \frac{1}{||x^{-1}||}$, then the elements $x + t(y - x)$, $t \in [0, 1]$, are all invertible in $\mathbb{M}_n(A)$.*

Proposition 2.3.2.3. *[WO, 4.2.4] If u, v are unitaries of a unital C^*-algebra A with $||u - v|| < 2$, then u is homotopic to v.*

Remark 2.3.2.4. *The connected components coincide with the path components.*

Proposition 2.3.2.5. *[WO, 4.2.6] For each $n \in \mathbb{N}$, or $n = \infty$, we have that*

$$GL_n(A)/GL_n(A)_0 = \mathcal{U}_n(A)/\mathcal{U}_n(A)_0.$$

Definition 2.3.2.6. *[WO, 7.1.1] Let A be a unital C^*−algebra. Then*

$$K_1(A) = GL_\infty(A)/GL_\infty(A)_0 = \mathcal{U}_\infty(A)/\mathcal{U}_\infty(A)_0.$$

When u is a unitary or invertible $n \times n$ matrix, $[u] \in K_1(A)$ denotes the connected component containing $diag(u, 1_\infty)$.

Proposition 2.3.2.7. *[WO, 7.1.2] $K_1(A)$ is a commutative group with a multiplication defined by*

$$[u][v] = [uv] = [diag(u,v)].$$

Proposition 2.3.2.8 (Homotopy Invariance of K_1). *[WO, 7.1.6] K_1 is a covariant, homotopy invariant functor from the category of C^*−algebras to the category of abelian groups.*

Example 2.3.2.9. *[WO, 7.1.11]*

1. $K_1(\mathbb{C}) = 0$.

2. $K_1(A) = 0$ *for any AF-algebra A.*

3. $K_1(A \oplus B) = K_1(A) \oplus K_1(B)$.

4. *For any abelian group G there is a C^*−algebra A with $K_1(A) \simeq G$.*

2.4 Infinite C^*−Algebras

A projection p in a C^*−algebra is said to be **finite** if $p \sim q \leq p$, implies $p = q$. Hence in a finite C^*−algebra, all projections are finite. Now we study another type of C^*−algebras in which not all the projections are finite. A projection p in a C^*−algebra A is said to be **infinite** projection if p is equivalent to one of its proper subprojections, and A is called an infinite C^*−algebra if it contains an infinite projection.

Definition 2.4.0.1. *[B, 6.11.1] A unital C^*−algebra is* ***properly infinite*** *if it contains two orthogonal projections equivalent to the identity.*

Theorem 2.4.0.2. *[B, 6.11.8] If A is a properly infinite (unital) C^*-algebra, then*

$$K_0^+(A) = K_0(A) = \Sigma(A).$$

Proposition 2.4.0.3. *Let A be a simple unital C^*-algebra. The following are equivalent:*
(i) A is infinite.
(ii) A is properly infinite.
(iii) A contains a sequence of mutually orthogonal equivalent nonzero projections.

Proof: (ii) implies (i) is trivial.
(i) implies (iii) : Let $(p_n)_{n\geq 1}$ be a sequence of mutually orthogonal nonzero equivalent projections. Let $p = p_1$, and u_n a partial isometry with $u_n^* u_n = p$ and $u_n u_n^* = p_n$ for each n, so $p = u_n^* p_n u_n$. Then there are elements x_i with $1 = \sum_{i=1}^n x_i^* p x_i = \sum_{i=1}^n p_i$ and $v = \sum_{i=1}^n u_i x_i$, then $1 = v^* q v$, so vq is an isometry with range projection $qvv^* \leq q$. Similarly, there is a projection equivalent to 1 under $\sum_{i=kn+1}^{k(n+1)} p_i$ for every k. □

The following is an even stronger notion of infiniteness:

Definition 2.4.0.4. *[B, 6.11.4] A unital C^*-algebra A is* ***purely infinite*** *if $A \neq \mathbb{C}$ and, for every nonzero $a \in A$, there are $x, y \in A$ with $xay = 1$.*

If A is purely infinite, then any matrix algebra over A is also purely infinite ([B, p.46]). The Cuntz algebra is an example of a purely infinite C^*-algebra; it will be discussed in §2.6.3.

Proposition 2.4.0.5. *[B, 6.11.5] Let A be a simple unital C^*-algebra. Then A is purely infinite if and only if every nonzero hereditary C^*-subalgebra of A contains an infinite projection. In particular, every nonzero projection in a purely infinite C^*-algebra is infinite.*

Proposition 2.4.0.6. *[Cu1, 1.5] In any C^*-algebra A, the following hold:*
(i) If p, q are infinite projections and $pq = 0$, then $p + q$ is an infinite projection.
(ii) If p is an infinite projection, and $p' \sim p$, then p' is an infinite projection,

(iii) If p and q are infinite projections, there exists an infinite projection p' such that $p \sim p'$ and $p' < q$, moreover $q - p'$ is an infinite projection,
(iv) If q is a projection in A, which majorizes an infinite projection in A, then q is an infinite projection.

Recall that in a C^*-algebra A which has cancellation, two projections are equivalent if and only if they have the same K_0-class. For the set of infinite projections, we have the following results.

Theorem 2.4.0.7. *[B, 6.11.9] Two infinite projections in a simple unital C^*-algebra are equivalent if and only if they have the same K_0-class. Two infinite projections with infinite complements which have the same K_0-class are unitarily equivalent. Two nontrivial projections with the same K_0-class in a purely infinite C^*-algebra are unitarily equivalent.*

By the preceding results about infinite projections, we give condition so that an infinite C^*-algebra may have a set of matrix units of some size.

Theorem 2.4.0.8. *Let A be an infinite C^*-algebra, and let p be a non-trivial projection in A, such that p and $1-p$ are infinite projections. If $[1] = n[p]$ in $K_0(A)$, for some $n \geq 3$, then A has a system of matrix units of dimension n.*

Proof: As p and $1-p$ are infinite projections, there exists a projection $p_1 \sim p$ and $p_1 < 1-p$, moreover $1-(p+p_1)$ is also an infinite projection. Again there exists a projection $p_2 \sim p$ and $p_2 < 1-(p+p_1)$ with $1-(p+p_1+p_2)$ is an infinite projection, then we continue applying the same argument to find a projection $p_{n-1} \sim p$ and $p_{n-1} < 1-(p+p_1+\cdots+p_{n-2})$. Therefore, $[1] = [p+p_1+\cdots+p_{n-1}]$, which implies that $1 \sim (p+p_1+\cdots+p_{n-1})$.

Let s be a partial isometry such that $ss^* = p+p_1+\cdots+p_{n-1}$ and $s^*s = 1$. For every $1 \leq i \leq n-1$, put $q_i = s^*p_is$, also let $q = s^*ps$. If $i \neq j$, then

$$q_iq_j = s^*p_iss^*p_js = s^*p_ip_js = 0.$$

Note that $q_i \sim p_i$, $q \sim p$, therefore $\{q, q_i\,;\ 1 \leq i \leq n-1\}$ is a set of orthogonal equivalent projections, moreover

$$\begin{aligned} q + q_1 + \cdots + q_{n-1} &= s^*ps + s^*p_1s + \cdots + s^*p_{n-1}s \\ &= s^*(ss^*)s \\ &= 1. \end{aligned}$$

Therefore, the set $\{q, q_i\,;\ 1 \le i \le n-1\}$ forms a partition of the unity consisting of orthogonal equivalent projections, and hence A has a set of matrix units of dimension n. □

Recall that for any unital C^*-algebra A, $\mathcal{U}(A)_0$ is the identity component of the unitary group of A.

Theorem 2.4.0.9. *[Cu1, p.188] If A is a purely infinite simple C^*-algebra, then*
(i) $K_0(A) = \{[p] \mid p$ is a non-zero projection in $A\}$,
(ii) $K_1(A) \simeq \mathcal{U}(A)/\mathcal{U}(A)_0$.

Theorem 2.4.0.10. *[Le, 3.8] If A is a simple, unital purely infinite C^*-algebra, then the set of self adjoint unitaries of A forms a set of generators for $\mathcal{U}_0(A)$.*

Let A be a simple, unital purely infinite C^*-algebra. Recall from Theorem (2.4.0.9)(ii) that $K_1(A) \simeq \mathcal{U}(A)/\mathcal{U}(A)_0$. Also in this cace, M. Leen ([Le, Th.3.8]) proved that $\mathcal{U}(A)_0$ is generated by commutators of elements of $\mathcal{U}(A)_0$. Hence if λ is a character of $\mathcal{U}(A)$, its restrict to $\mathcal{U}(A)_0$ is trivial.

We show that λ can be extended to a character of $K_1(A)$ as follows: If $[u] = [v]$, then

$$1 = \lambda(uv^*) = \lambda(u)\overline{\lambda(v)}$$

hence $\lambda(u) = \lambda(v)$. Now define $\tilde{\lambda} : K_1(A) \longrightarrow \mathbb{S}^1$ by $[u] \mapsto \lambda(u)$. Moreover,

$$\tilde{\lambda}([u][v]) = \tilde{\lambda}([uv]) = \lambda(u)\lambda(v) = \tilde{\lambda}([u])\tilde{\lambda}([v]),$$

so $\tilde{\lambda}$ is a character of $K_1(A)$.

We denote by $\hat{\mathcal{U}}(A)$ and $\hat{K_1}(A)$ the groups of characters of $\mathcal{U}(A)$ and $K_1(A)$ respectively. Let π denote the quotient map from $\mathcal{U}(A)$ to $\mathcal{U}(A)/\mathcal{U}(A)_0$.

Proposition 2.4.0.11. *If A is a simple, unital purely infinite C^*-algebra, then $\hat{K_1}(A)$ and $\hat{\mathcal{U}}(A)$ are isomorphic as groups.*

Proof: We show that the map $\Lambda : \hat{K_1}(A) \longrightarrow \hat{\mathcal{U}}(A)$ defined by $\Lambda(\lambda) = \lambda \circ \pi$ is an isomorphism.

(i) Λ is a homomorphism: If λ_1, λ_2 belong to $\hat{K_1}(A)$ and $u \in \mathcal{U}(A)$, then

$$\begin{aligned} ((\lambda_1\lambda_2) \circ \pi)(u) &= (\lambda_1\lambda_2)(\pi(u)) \\ &= \lambda_1(\pi(u))\lambda_2(\pi(u)) \\ &= ((\lambda_1 \circ \pi)(\lambda_2 \circ \pi))(u) \end{aligned}$$

hence $\Lambda(\lambda_1\lambda_2) = \Lambda(\lambda_1)\Lambda(\lambda_2)$.

(ii) Λ is bijective: If $\Lambda(\lambda) = 1$ as the trivial character of $\mathcal{U}(A)$, then for every $u \in \mathcal{U}(A)$ we have

$$1 = \Lambda(\lambda)(u)(\lambda(\pi(u))$$

therefore, $\lambda = 1$ as the trivial character of $K_1(A)$, hence Λ is injective.

If μ belongs to $\hat{\mathcal{U}}(A)$, then as we showed above, μ can be extended to a character λ of $K_1(A)$, such that $\lambda(\pi(u)) = \mu(u)$, therefore $\Lambda(\lambda) = \mu$, hence Λ is surjective. □

2.5 Properties of State Space

2.5.1 Stable Rank and Real Rank Zero

Let A be a unital C^*-algebra. Set

$$lg_n(A) = \left\{ (x_1, \ldots, x_n) \in A^n | \sum_{i=1}^{n} x_i^* x_i \text{ is invertible} \right\}.$$

The **stable rank** of A, denoted $sr(A)$, is the smallest n such that $lg_n(A)$ is dense in A^n. If $lg_n(A)$ is never dense, set $sr(A) = \infty$. Note that $sr(A) = 1$ if and only if the invertible elements of A are dense in A.

Theorem 2.5.1.1. *[B, p.36] Let A be a C^*-algebra. If A has stable rank one, then any matrix algebra over A has also stable rank one.*

Definition 2.5.1.2. *[B, p.37] A C^*-algebra A has* ***real rank zero*** *if the invertible self-adjoint elements of A are dense in A_{sa}.*

Proposition 2.5.1.3. *[Da, Pro. V.7.2] Every AF-algebra has real rank zero.*

Theorem 2.5.1.4. *[B, 6.5.6] Let A be a C^*-algebra. Then A has real rank zero if and only if the set of self-adjoint elements of A of finite spectrum is dense in A_{sa}.*

Proposition 2.5.1.5. *[B, 6.5.7] Let A be a C^*-algebra with real rank zero. Then A has cancellation if and only if $sr(A) = 1$.*

Proposition 2.5.1.6. *[ET, p.27] Let A be a C^*-algebra. Then A has cancellation if and only if for every n, equivalent projections in $\mathbb{M}_n(A)$ are unitarily equivalent in $\mathbb{M}_n(A)$.*

2.5.2 Traces and FCQ

Definition 2.5.2.1. *Let A be a C^*-algebra. A* ***quasitrace*** *on A is a map*
$\tau : A \longrightarrow \mathbb{C}$ such that
(i) $\tau(a^*a) = \tau(aa^*) \geq 0$,
(ii) $\tau(\lambda a) = \lambda\tau(a)$, $\lambda \in \mathbb{C}$,
(iii) τ is linear on any commutative C^*-subalgebra B of A,
(iv) τ extends to $\tau' : \mathbb{M}_n(A) \longrightarrow \mathbb{C}$ that satisfies (i),(ii) and (iii).

Let us denote by $QT(A)$ the space of all quasi-traces on A.

Definition 2.5.2.2. *A trace on a C^*-algebra A is a linear map $\tau : A \longrightarrow \mathbb{C}$ such that $\tau(ab) = \tau(ba)$, and $\tau(a^*a) \geq 0$ (i.e. τ is positive). In case A is unital, τ is called a* ***normalized trace*** *if $\tau(1) = 1$. Then τ is said to be a* ***faithful*** *trace if $a \geq 0$ and $\tau(a) = 0$ implies that $a = 0$.*

It is clear that every trace is a quasi-trace. However, the converse is true for many C^*-algebras, e.g. exact C^*-algebras. these will be discussed in section §2.8.

Theorem 2.5.2.3. *[B, p.41] In exact C^*-algebras, every quasitrace is a trace.*

Definition 2.5.2.4. *[B, 6.8.1] A state on a scaled ordered group (G, G^+, u) is an order preserving homomorphism $s : G \longrightarrow \mathbb{R}$ such that $s(u) = 1$.*

Now given a quasi-trace τ of a C^*−algebra A, extend τ to a quasitrace on $\mathbb{M}_n(A)$. If A has cancellation, then we can extend τ to a state on $K_0(A)$. First define $s : K_0^+(A) \longrightarrow \mathbb{C}$ by $s([p]) = \tau(p)$. It is clear that s is a well-defined order preserving normalized homomorphism. Extend s to $K_0(A)$ linearly. We define a map $\chi : QT(A) \longrightarrow S(K_0(A))$, sending τ to the extension s.

Theorem 2.5.2.5. *[B, 6.9.1] If A is a stably finite, unital C^*−algebra with real rank zero then the map $\chi : QT(A) \longrightarrow S(K_0(A))$ is a bijection.*

Evans and Takesaki studied the fundamental comparability questions in [ET]. In the following, we state the FCQ (V.I and V.II):

- **Fundamental Comparability Question, Version 1(FCQ1)**[ET, p.22]: Let A be a simple C^*−algebra. If p, q are nonzero projections in A with $\tau(p) \leq \tau(q)$ for every trace τ on A, is it true that $p \preceq q$?

- **Fundamental Comparability Question, Version 2(FCQ2)**[ET, p.22]: Let A be a simple C^*−algebra. If p, q are nonzero projections in A with $\tau(p) < \tau(q)$ for every trace τ on A, is it true that $p \prec q$?

Theorem 2.5.2.6. *[ET, p.22] Any AF-algebra satisfies FCQ2.*

2.5.3 Finite C^*−Algebras and Systems of Matrix Units

In this section we show that some finite C^*−algebras have a systems of matrix units. In particular, this true for simple, unital AF-algebras, whose K_0−group is 2-divisible.

Theorem 2.5.3.1. *Let A be a stably finite, simple, unital C^*−algebra with real rank zero such that $K_0(A)$ is unperforated and has cancellation. If there exists $p \in \mathcal{P}(A)\backslash\{0\}$ such that $n[p] = [1]$ in $K_0(A)$ for some $n > 1$, then A has a system of matrix units of dimension n.*

Proof: As $K_0(A)$ is unperforated, it has no torsion elements, hence $p \neq 1$. Since $n[p] = [1]$, we have $(n-1)[p] = [1-p]$, so $[p] \leq [1-p]$. If $[p] = [1-p]$, then $n = 2$ and $1 = p + (1-p)$ is a sum of two orthogonal equivalent projections, which means that A has a system of matrix units of dimension 2. If $[p] < [1-p]$, then by [B, 6.9.2], there exists a projection $r_1 \leq 1-p$ such that $p \sim r_1$. If $r_1 = 1-p$, then $n = 2$, so we may assume that $r_1 < 1-p$. As

$$[1 - (p + r_1)] = (n-2)[p],$$

$[p] \leq [1-(p+r_1)]$, note that the equality gives the case $n = 3$, i.e. the unity is a sum of three orthogonal equivalent projections, so the theorem holds. If $[p] < [1-(p+r_1)]$, then there exists a projection $r_2 < 1-(p+r_1)$ such that $p \sim r_2$. Continuing to apply the same argument, we get a projection $r_{n-2} < 1 - (p + r_1 + \cdots + r_{n-3})$ such that $p \sim r_{n-2}$. Now as

$$[1 - (p + r_1 + \cdots + r_{n-2})] = [1] - (n-1)[p] = [p],$$

we have that

$$1 = (p + r_1 + \cdots + r_{n-2}) + (1 - (p + r_1 + \cdots + r_{n-2})),$$

which forms a partition of unity consisting of $n-$orthogonal equivalent projections, hence A has a system of matrix units of dimension n. □

In particular, we have the following result for AF-algebras:

Corollary 2.5.3.2. *If A is a simple, unital AF-algebra such that $n[p] = [1]$ in $K_0(A)$ for some $n > 1$, and some $p \in \mathcal{P}(A) \backslash \{0\}$, then A has a system of matrix units of dimension n.*

Proof: Follows directly from the preceding theorem. □

Now we prove the following result for AF-algebras such that its K_0-group is 2-divisible, which in particular will be used in the last chapter of the thesis.

Proposition 2.5.3.3. *If A is a simple, unital AF-algebra, such that $K_0(A)$ is 2-divisible, then for any $k > 1$, A has a system of matrix units of dimension 2^k.*

Proof: As $K_0(A)$ is a 2-divisible group, then $[1] = 2[p_1]$, for some $p_1 \in \mathcal{P}(A) \backslash \{0, 1\}$. Then continue applying the same argument and for any $k > 1$, we get $[1] = 2^k[p]$, for some non-trivial projection p of A. Hence Corollary (2.5.3.2) proves that A has a system of matrix units of dimension 2^k. □

2.6 C^*-Algebra Examples

In this section, we introduce different examples of simple unital C^*-algebras. We discuss the irrational rotation algebras, and we give some main properties. Also we introduce the approximately divisible C^*-algebras, which generalize both simple unital AF-algebras, and the irrational rotation algebras. And we introduce the Cuntz algebras, which are one of the main examples of purely infinite C^*-algebras.

2.6.1 Approximately Divisible C^*-Algebras

B. Blackadar and A. Kumjian studied the case of simple approximately divisible C^*-algebras ([BK]), and they proved main results of these C^*-algebras and their $K-$theory.

Definition 2.6.1.1. *[BK, 1.1] A finite dimensional C^*-algebra is completely non-commutative (cncfd) if it has no commutative central summands (i.e., no abelian central projections).*

Definition 2.6.1.2. *[BK, 1.2] A separable unital C^*-algebra is approximately divisible if, for every $x_1, \ldots x_n \in A$ and $\epsilon > 0$, there is a completely noncommutative finite dimensional C^*-subalgebra B of A, containing the unit of A, such that $\| [x_i, y] \| < \epsilon \, \forall i = 1, 2, \ldots, n$ and for all $y \in B_1$, where B_1 is the unit ball of B.*

If B is a C^*-subalgebra of A, then B' denotes the commutant of B in A.

Theorem 2.6.1.3. *[BK, 1.3] Let A be an approximately divisible (separable unital) C^*-algebra. Then*
(a) A can be written as $\overline{\cup A_n}$, where $A'_n \cap A_{n+1}$ contains a completely noncommutative unital finite dimensional C^*-algebra (in particular, the A_n are strictly increasing).

Conversely, any (separable unital) C^*-algebra which can be written in such a way is approximately divisible.

(b) There is a decreasing sequence (D_n) of unital subalgebras of A such that D_n is an AF-algebra with no abelian projections and $A = \overline{[\cup(A \cap D'_n)]}$

Theorem 2.6.1.4. *[BK, 1.4] Let A be an approximately divisible (separable unital) C^*-algebra. Then*

(a) If A is finite, it is stably finite. If A is infinite, it is purely infinite.

(b) A has stable rank 1 if and only if it is finite.

(c) A satisfies all the Fundamental Comparability Questions.

(d) A has real rank zero if and only if the projections of A distinguish quasitraces (i.e. if τ_1 and τ_2 are quasitraces on A with $\tau_1(p) = \tau_2(p)$ for all projections $p \in A$, then $\tau_1 = \tau_2$), in particular, if A has a unique quasitrace, then A has real rank zero.

(e) If A is exact, then A has real rank zero if and only if the projections of A distinguish the tracial states of A. In particular, if A has a unique tracial state, then A has real rank zero.

Definition 2.6.1.5. *[BK, 2.6] A standard completely noncommutative finite dimensional C^*-algebra is a C^*-algebra isomorphic to $\mathbb{M}_2, \mathbb{M}_3$ or $\mathbb{M}_2 \oplus \mathbb{M}_3$.*

Every cncfd C^*-algebra contains a (unital) standard cncfd C^*-subalgebra. In fact, if $n \geq 7$, then $\mathbb{M}_n$ contains a unital copy of $M_2 \oplus M_3$, (see [BK, p.270]).

Proposition 2.6.1.6. *[BK, 2.7] A C^*-algebra is approximately divisible if and only if for every $x_1, x_2, \ldots, x_n \in A$ and $\epsilon > 0$, there is a standard cncfd C^*-subalgebra B of A with*

$$\|[x_k, e^r_{i,j}]\| < \epsilon, \textit{ for all } k, i, j \textit{ and } r, \textit{ for a set of matrix units } \{e^r_{i,j}\} \textit{ of } B.$$

Proposition 2.6.1.7. *[BK, 3.11] Let A be a simple approximately divisible algebra. Then $K_1(A) \simeq \mathcal{U}(A)/\mathcal{U}(A)_0$, where $\mathcal{U}(A)$ is the unitary group of A and $\mathcal{U}(A)_0$ is the connected component of the identity.*

Proposition 2.6.1.8. *[BK, 4.1] Every infinite dimensional simple, unital AF-algebra is approximately divisible.*

Theorem 2.6.1.9. *[El1, Th.2] Let A be a simple unital C^*-algebra, not of finite dimension. If A is an inductive limit of a sequence of finite direct sums of matrix algebras over the continuous functions on the unit circle $C(\mathbb{T})$, then A is approximately divisible.*

Now we prove the following lemma, which shows that in approximately divisible C^*-algebras, projections can be divided into smaller subprojections.

Lemma 2.6.1.10. *Let A be a unital approximately divisible C^*-algebra. If $p \in \mathcal{P}(A)$, then either*
(i) $p = p_1 + p_2$, such that $p_1 p_2 = 0$, and $p_1 \sim p_2$, or
(ii) $p = p_1 + p_2 + p_3$, such that p_i's are pairwise orthogonal equivalent projection, or
(iii) $p = p_1 + p_2 + q_1 + q_2 + q_3$ pairwise orthogonal projections such that $p_1 \sim p_2$ and all q_i's are equivalent projections.

Proof: Recall from Theorem (2.6.1.3)(a) that A can be written as $\overline{\cup_{n\geq 1} A_n}$, where A_n are strictly increasing, and $B \subseteq A_n' \cap A_{n+1}$ such that B is either isomorphic to $\mathbb{M}_2$, $\mathbb{M}_3$ or $\mathbb{M}_2 \oplus \mathbb{M}_3$. We may assume that $p \in A_\infty$. Then we consider the possibilities of B:
(i) If $B \cong \mathbb{M}_2$ and $p \in A_n$ for some n, then for some system of matrix units $\{e_{i,j};\ 1 \leq i, j \leq 2\}$ of B, we have

$$e_{i,j} p = p e_{i,j};\ \ 1 \leq i, j \leq 2\,,$$

$$p = pe_{1,1} + pe_{2,2},\ \ (pe_{i,i})(pe_{i,i}) = pe_{i,i} = (pe_{i,i})^*$$

$$pe_{1,1} = (pe_{1,2})(pe_{1,2})^* \text{ and } pe_{2,2} = (pe_{1,2}^*)(pe_{1,2})$$

Hence, $pe_{1,1} \sim pe_{2,2}$.
(ii) If $B \cong \mathbb{M}_3$, then the proof is similar to (i).
(iii) If $B \cong \mathbb{M}_2 \oplus \mathbb{M}_3$, then let $\{e_{i,j}^2,\ e_{k,l}^3\}$ $1 \leq i, j \leq 2$ and $1 \leq k, l \leq 3$ be a system of matrix units of B. As $\sum_{i,r}^{3,2} e_{i,i}^r = 1$, then

$$p = pe_{1,1}^1 + pe_{2,2}^1 + pe_{1,1}^2 + pe_{2,2}^2 + pe_{3,3}^3,$$

$$pe_{1,1}^1 = (pe_{1,2}^1)(pe_{1,2}^1)^*,\ pe_{2,2}^1 = (pe_{1,2}^1)^*(pe_{1,2}^1)$$

$$pe_{i,i}^2 = (pe_{i,j}^2)(pe_{i,j})^* \text{ and } pe_{i,j}^2 = (pe_{i,j})^*(pe_{i,j}^2) \text{ for } 1 \leq i,j \leq 3.$$

Now if $p \in A$, then by Proposition (2.2.3.6), p is unitarily equivalent to a projection in A_∞, apply the first part of the proof on this projection, therefore the lemma is checked. □

2.6.2 Irrational Rotation Algebras

In this section, we study a class of C^*−algebras that has been studied intensively in recent years. The canonical model acts on the circle T which we will think of as $\mathbb{R}/\mathbb{Z}$ via the map $t \mapsto z(t) = \exp(2\pi i t)$. Fix an irrational number θ. Let $\mathcal{H} = L^2(\mathbb{R}/\mathbb{Z})$ and consider two unitary operators on $\mathcal{H}$, the operator $U = M_{z(t)}$ of multiplication by the unimodular function $z(t)$, and V, the operator of rotation by θ. That is

$$Uf(t) = z(t)f(t) \text{ and } Vf(t) = f(t-\theta)$$

A simple calculation yields

$$VUf(t) = (Uf)(t-\theta) = z(t-\theta)f(t-\theta)$$

$$= \exp(-2\pi i\theta)z(t)(Vf)(t) = \exp(-2\pi i\theta)UVf(t)$$

Hence

$$UV = \exp(2\pi i\theta)VU \tag{1}$$

Definition 2.6.2.1. *[Da, p.166] A C^*−algebra A_θ is universal for the relation (1) provided that it is generated by two unitaries u_1, v_1 satisfying (1) and if $B = C^*(u,v)$ is any other C^*−algebra, where u and v satisfying (1), then there is a homomorphism of B onto A_θ which carries u_1 to u and v_1 to v.*

Definition 2.6.2.2. *[Da, p.167] For θ an irrational number, the irrational rotation algebra A_θ is a universal C^*−algebra generated by two unitaries u, v that satisfy (1).*

Theorem 2.6.2.3. *[Da, VI.1.2] The irrational rotation algebra A_θ has a faithful unital scalar valued trace τ, which is unique.*

Theorem 2.6.2.4. *[Da, VI.1.4] A_θ is simple.*

Now we have the following result which was proved by M. Rieffel.

Theorem 2.6.2.5. *[Da, VI.2.1]* $\tau(\mathcal{P}(A_\theta)) = (\mathbb{Z}+\theta\mathbb{Z}) \cap [0,1]$.

Theorem 2.6.2.6. *[RS, p.52] For any irrational rotation algebra A_θ, we have:*

1. $K_0(A_\theta) = (\mathbb{Z}+\theta\mathbb{Z}) \subseteq \mathbb{R}$.

2. $K_1(A_\theta) = \mathbb{Z}^2$.

In particular A_θ is not an AF-algebra, while $K_0(A_\theta)$ is a dimension group. Indeed A_θ can be embedded in an AF-algebra (this is due to Pimsner and Voiculescu) with the same K_0-group.

$$(G, G^+, \Sigma(G)) = (\mathbb{Z}+\theta\mathbb{Z}, (\mathbb{Z}+\theta\mathbb{Z}) \cap \mathbb{R}^+, (\mathbb{Z}+\theta\mathbb{Z}) \cap [0,1]).$$

Theorem 2.6.2.7. *[Da, VI.5.3] Two irrational rotation algebras A_θ and A_η are isomorphic if and only if $\eta = \pm\,\theta \bmod \mathbb{Z}$.*

Theorem 2.6.2.8. *[Ri] The irrational rotation algebra A_θ has the FCQ.*

Theorem 2.6.2.9. *[Ri] The irrational rotation algebra A_θ has cancellation.*

Theorem 2.6.2.10. *[EE, §5] The irrational rotation algebra A_θ is isomorphic to the inductive limit of a sequence of direct sums of two matrix algebras over $C(\mathbb{T})$.*

Corollary 2.6.2.11. *The irrational rotation algebra A_θ is approximately divisible.*

Proof: Directly follows from the previous theorem and Theorem (2.6.1.9). □

2.6.3 The Cuntz Algebras

The first interesting examples of purely infinite, simple C^*−algebras were given by Cuntz in [Cu2] where he introduced the class of C^*−algebras $\mathcal{O}_n$, for $2 \leq n \leq \infty$, now called the Cuntz algebras.

Definition 2.6.3.1. *[Cu2] The* ***Cuntz algebra*** $\mathcal{O}_n$*, where* $2 \leq n < \infty$*, is the universal unital* C^*-*algebra which is generated by isometries* $s_1, s_2, \ldots, s_n$*, such that*

$$\sum_{i=1}^{n} s_i s_i^* = 1,$$

with $s_i^* s_j = 0$ *when* $i \neq j$*. The Cuntz algebra* $\mathcal{O}_\infty$ *is the universal unital* C^*-*algebra generated by an infinite sequence of isometries* $s_1, s_2, s_3, \ldots$ *with mutually orthogonal range of projections* $s_j s_j^*$*.*

For every $i = 1, 2, \ldots, n$, and $2 \leq n \leq \infty$ the projection $s_i s_i^*$ will be denoted by p_i, and we call such projections the **standard projections** of $\mathcal{O}_n$, therefore, $p_i \sim 1$ for every $i = 1, 2, \ldots, n$ since $s_i^* s_i = 1$, hence $[p_i] = [1]$. In case $n < \infty$, the standard projections form an orthogonal partition of unity in $\mathcal{O}_n$.

Theorem 2.6.3.2. *[Cu2] For every* $2 \leq n \leq \infty$*, the Cuntz algebras* $\mathcal{O}_n$ *are simple.*

Remark 2.6.3.3. *By using the universal property we have that, if* $t_1, \ldots, t_n$ *are any isometries, such that* $\sum_{i=1}^{n} t_i t_i^* = 1$*, then* $C^*(t_1, \ldots, t_n)$ *is isomorphic to the Cuntz algebra* $\mathcal{O}_n$*.*

Theorem 2.6.3.4. *[Cu2] For all* $2 \leq n \leq \infty$*, the Cuntz algebras* $\mathcal{O}_n$ *are purely infinite.*

Theorem 2.6.3.5. *[Da, V.6.7] If* $n \neq m$*, then* $\mathcal{O}_n$ *and* $\mathcal{O}_m$ *are not isomorphic.*

Proposition 2.6.3.6. *[Da, V.7.5] If* $n \geq 2$*, then* $\mathcal{O}_n$ *has real rank zero.*

The following theorems summarize the K-theory results of the Cuntz algebras.

Theorem 2.6.3.7. *[Cu1, 3.7, 3.8] (i)* $K_0(\mathcal{O}_n) \cong \mathbb{Z}_{n-1}$*,*
(ii) $K_1(\mathcal{O}_n) \cong 0$.

Being a torsion group, $K_0(\mathcal{O}_n)$ is not a dimension group. And for $\mathcal{O}_\infty$, we have the following results.

Theorem 2.6.3.8. *[Cu1, 3.11] (i)* $K_0(\mathcal{O}_\infty) \cong \mathbb{Z}$*,*
(ii) $K_1(\mathcal{O}_\infty) \cong 0$.

Recall that, for a purely infinite simple, unital C^*-algebra A,

$$K_0(A) = \{\, [p] \mid p \in \mathcal{P}(A)\backslash\{0,1\}\, \}.$$

Indeed for the Cuntz algebras $\mathcal{O}_n$, where $n < \infty$, we have the following:

Proposition 2.6.3.9. *If $x \in K_0(\mathcal{O}_n)$, $n < \infty$, then $x = [p_1+p_2+\cdots+p_s]$, for some $1 \leq s \leq n-1$.*

Proof: Let ψ be the isomorphism from $K_0(\mathcal{O}_n)$ onto $\mathbb{Z}_{n-1}$ such that $\psi([1]) = 1$. If $x \in K_0(\mathcal{O}_n)$, then $\psi(x) \in \mathbb{Z}_{n-1}$. Therefore,

$$\begin{aligned}
1 &= \psi([p_1]) \\
2 &= \psi([p_1+p_2]) \\
&\vdots \\
n-2 &= \psi([p_1+p_2+\cdots+p_{n-2}]), \text{ and} \\
0 &= \psi([p_1+p_2\cdots+p_{n-1}]).
\end{aligned}$$

□

Now we prove the following results for Cuntz algebras:

Lemma 2.6.3.10. *In $\mathcal{O}_n$, $p_k \sim_u p_l$, for $1 \leq k, l \leq n$.*

Proof: As $p_k = s_k s_k^*$ and $p_l = s_l s_l^*$, we know that $p_k \sim p_l \sim 1$. Moreover

$$1 - p_k = \sum_{i=1, i\neq k}^{n} p_i \text{ and } 1 - p_l = \sum_{i=1, i\neq l}^{n} p_i$$

Let $t = s_k s_l^*$ and $a = t + \sum_{i\neq k, i\neq l}^{n} p_i$. Then $t^*t = p_l, tt^* = p_k, aa^* = 1 - p_l$ and $a^*a = 1 - p_k$, therefore $1 - p_k \sim 1 - p_l$, hence $p_k \sim_u p_l$. □

Lemma 2.6.3.11. *If A is a unital C^*-algebra and $p \in \mathcal{P}(A)$ such that $p \sim 1$, then there exists a $*-$isomorphism $A \longrightarrow pAp$ which maps 1 to p.*

Proof: Let v be a partial isometry such that $v^*v = p$ and $vv^* = 1$. Define $\Delta_v : A \longrightarrow pAp$ by $x \longmapsto v^*xv$. Δ_v is well defined since, $v^*xv = v^*vv^*xvv^*v = p(v^*xv)p \in pAp$. Linearity is clear, Δ_v preserves the adjoints and $\Delta_v(xy) = v^*xyv = v^*xvv^*yv = \Delta_v(x)\Delta_v(y)$. Also Δ_v^{-1} is defined by $y \longmapsto vyv^*$, that ends the proof. □

Lemma 2.6.3.12. *Let p be a projection of the Cuntz algebra $\mathcal{O}_n$. If p is equivalent to 1, then p can be written as a sum of n unitarily equivalent orthogonal projections and each of them is equivalent to p.*

Proof: In $\mathcal{O}_n$, we have $1 = p_1 + p_2 + \cdots + p_n$. Let v be a partial isometry such that $v^*v = p$ and $vv^* = 1$, therefore we have,

$$p = \Delta_v(p_1) + \Delta_v(p_2) + \cdots + \Delta_v(p_n).$$

Let $\Delta_v(p_i) = q_i, i = 1, 2, \ldots, n$. Then $q_iq_j = \Delta_v(p_ip_j) = 0$, moreover, $q_i \sim_u q_j$ since Δ_v preserves unitarily equivalence of projections. As $p_i \sim 1$, therefore $p_i = u_i^*u_i$ and $u_iu_i^* = 1$, so $q_i = \Delta_v(u_i^*)\Delta_v(u_i)$ and $p = \Delta_v(u_i)\Delta_v(u_i^*)$, hence we have $q_i \sim p \ \forall \ i = 1, 2, \ldots, n$. □

2.7 von Neumann Algebras

Definition 2.7.0.1. *Let $\mathcal{H}$ be a Hilbert space and $\mathcal{M}$ be a subset of $\mathcal{B}(\mathcal{H})$. The **commutant** of $\mathcal{M}$, denoted $\mathcal{M}'$, is the set of all elements of $\mathcal{B}(\mathcal{H})$ which commute with all elements of $\mathcal{M}$. The bicommutant $(\mathcal{M}')'$ of $\mathcal{M}$ will be denoted by $\mathcal{M}''$.*

Example 2.7.0.2. $\mathcal{B}(\mathcal{H})' = \mathbb{C}.1_{\mathcal{B}(\mathcal{H})}$.

Remarks 2.7.0.3. *(i) $\mathcal{M} \subset \mathcal{M}''$,*
(ii) If $\mathcal{M}$ and $\mathcal{N}$ are subsets of $\mathcal{B}(\mathcal{H})$ and $\mathcal{M} \subset \mathcal{N}$, then $\mathcal{N}' \subset \mathcal{M}'$ and $\mathcal{M}'' \subset \mathcal{N}''$, and so on we have:

$$\mathcal{M}' = \mathcal{M}''' = \mathcal{M}^5 = \ldots \quad \text{and} \quad \mathcal{M} \subset \mathcal{M}'' = \mathcal{M}^4 = \ldots$$

Definition 2.7.0.4. *A $*-$subalgebra $\mathcal{M}$ of $\mathcal{B}(\mathcal{H})$ is called a **von Neumann** if $\mathcal{M} = \mathcal{M}''$.*

Example 2.7.0.5. *The algebra $\mathcal{B}(\mathcal{H})$ is a von Neumann algebra.*

Definition 2.7.0.6. *A **factor** is a von Neumann algebra $\mathcal{M}$ whose center $\mathcal{Z}(\mathcal{M}) = \mathcal{M} \cap \mathcal{M}'$ only contains the scalar operators.*

Example 2.7.0.7. *$\mathcal{B}(\mathcal{H})$ is a factor.*

The following proposition is von Neumann's density theorem in the finite dimensional case.

Proposition 2.7.0.8. *If A is an involutive subalgebra of $\mathbb{M}_n(\mathbb{C})$ which contains the unit matrix, then A is a von Neumann algebra on $\mathbb{C}^n$.*

Proof: See [HJ, Pr.2.13]. □

Example 2.7.0.9. *Let $(X, \mathcal{F}, \mu)$ be a separable $\sigma-$finite measure space. Let $\mathcal{M} = \{m_f : f \in L^\infty(X,\mu)\}$, where m_f denotes the multiplication operator on $L^2(X,\mu)$: $(m_f\zeta)(x) = f(x)\zeta(x)$, $\zeta \in L^2(X,\mu)$. Then $\mathcal{M}$ is an abelian von Neumann algebra acting on a separable Hilbert space. Moreover, $\mathcal{M}$ is $*-$isomorphic to $L^\infty(X,\mu)$.*

Let $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra and $x \in \mathcal{M}$ be a self adjoint element. Then the spectral projections of x belong to $\mathcal{M}$. Indeed the spectral theorem ([Ru, p.324]) states that there exists a unique resolution of the identity $E : \mathcal{B}(\sigma(x)) \to \mathcal{M} \subset \mathcal{B}(\mathcal{H})$ defined on the Borel subsets of the spectrum of x which satisfies

$$x = \int_{\sigma(x)} \lambda dE(\lambda).$$

In addition, every projection $E(w)$ belongs to $\mathcal{M}$.

Let $\mathcal{M}$ be a factor. A **dimension function** on $\mathcal{M}$ is a function

$$D : \mathcal{P}(\mathcal{M}) \longrightarrow [0,\infty] \text{ such that}$$

(i) $p \sim q \iff D(p) = D(q)$,
(ii) $D(p \oplus q) = D(p) + D(q)$,
(iii) p is a finite iff $D(p) < \infty$.

Definition 2.7.0.10. *Let $\mathcal{M}$ be a factor. Then the **type** of $\mathcal{M}$ is defined to be as follows:*
*(i) I_n, if $D(\mathcal{P}(\mathcal{M})) = \{1, 2, \dots n\}$, and $\mathcal{M}$ is called a **finite discrete** factor. If $n = \infty$, then it is called an **infinite discrete** factor,*
*(ii) II_1, if $D(\mathcal{P}(\mathcal{M})) = [0,1]$, and $\mathcal{M}$ is called a **finite continuous** factor,*
*(iii) II_∞, if $D(\mathcal{P}(\mathcal{M})) = [0,\infty]$, and $\mathcal{M}$ is called an **infinite continuous** factor,*
*(iv) III, if $D(\mathcal{P}(\mathcal{M})) = \{0,\infty\}$, and $\mathcal{M}$ is called a **purely infinite** factor.*

Example 2.7.0.11. $\mathbb{M}_n(\mathbb{C})$ *is a finite factor.*

Example 2.7.0.12. *Let G be a discrete group, and $\ell^2(G) = \{f : G \to \mathbb{C}; \sum_G |f(g)|^2 < \infty\}$. Define the map*

$$u : G \longrightarrow \mathcal{B}(\ell^2(G))$$

$$\text{by} \quad (u_g f)(\zeta) = f(g^{-1}\zeta).$$

Then $\{u_g | \, g \in G\}''$ is a von Neumann algebra denoted by $vN(G)$. If all the conjugate classes in G are infinite (G is icc), then $vN(G)$ is a II_1 factor.

2.8 Nuclear and Exact C^*-Algebras

For each pair of C^*-algebras A and B, let $A \otimes_{min} B$ ($A \otimes_{max} B$) denote the minimal (maximal) tensor product of A and B, respectively.

Definition 2.8.0.1. *[RS, 2.1.1] A C^*-algebra A is said to be* ***nuclear*** *if the canonical surjection $A \otimes_{max} B \to A \otimes_{min} B$ is injective (and hence an isomorphism) for every C^*-algebra B.*

A C^*-algebra A is nuclear if and only if its bidual A^{**} is an injective von Neumann algebra ([RS, p.25]).

Proposition 2.8.0.2. *[RS, 2.1.2] (i) All abelian C^*-algebras are nuclear.*
(ii) An inductive limit of nuclear C^*-algebras is nuclear.
(iii) If A and B are nuclear, then so is $A \otimes_{min} B$.

Example 2.8.0.3. *Let A be an inductive limit C^*-algebra of a sequence of finite direct sums of matrix algebras over the continuous functions on the unit circle $\mathbb{T}$. As any finite dimensional C^*-algebra F, and $C(\mathbb{T})$ is nuclear, then $F \otimes_{min} C(\mathbb{T})$ is so. Therefore by Proposition (2.8.0.2)(ii), we have that A is nuclear. In particular any irrational rotation algebra is nuclear.*

Definition 2.8.0.4. *[RS, 6.1.9] A C^*-algebra A is said to be* ***exact*** *if every short exact sequence of C^*-algebras*

$$0 \longrightarrow I \xrightarrow{i} B \xrightarrow{\pi} B/I \longrightarrow 0$$

induces an exact sequence

$$0 \longrightarrow A \otimes_{min} I \xrightarrow{id_A \otimes i} A \otimes_{min} B \xrightarrow{id_A \otimes \pi} A \otimes_{min} B/I \longrightarrow 0.$$

Theorem 2.8.0.5. *[RS, p.94] Every nuclear C^*-algebra is exact.*

Proposition 2.8.0.6. *[RS, 6.1.10] (i) Every sub C^*-algebra of an exact C^*-algebra is exact.*
(ii) An inductive limit of exact C^*-algebras is exact.

Theorem 2.8.0.7 (Kirchberg's Exact Embedding Theorem). *[RS, 6.3.11] A separable C^*- algebra A is exact if and only if there is an injective $*-$homomorphism $\iota: A \to \mathcal{O}_2$.*

Chapter 3

Dye's and Booth's Results

In this chapter, we recall known results proved by H. Dye in 1954: The unitary groups of von Neumann factors not of type I_{2n} determine the algebraic type of the factors. In the first section, we give a summary of Dye's result for factors. In the second section, we recall A. Booth's result from his M.Sc. thesis in 1998 concerning simple, unital AF-algebras.

3.1 Dye's Results

H. Dye proved that an isomorphism between the unitary groups of two von Neumann factors not of type I_{2n} is implemented by an isomorphism or an antiisomorphism of the factors themselves.

Let A and B be simple unital C^*−algebras, and let φ be an isomorphism between their unitary groups $\mathcal{U}(A)$ and $\mathcal{U}(B)$. Then φ maps the self-adjoint unitaries of A onto the self-adjoint unitaries of B, and therefore defines a natural mapping θ_φ between the sets of projections $\mathcal{P}(A)$ and $\mathcal{P}(B)$ via

$$1 - 2\theta_\varphi(p) = \varphi(1 - 2p).$$

Notation 3.1.0.1. *The quadruple $(A, B, \varphi, \theta_\varphi)$ consists of simple unital C^*−algebras A and B, $\varphi : \mathcal{U}(A) \to \mathcal{U}(B)$ is a group isomorphism and $\theta_\varphi : \mathcal{P}(A) \to \mathcal{P}(B)$ is the* *induced map between their sets of projections.*

The map θ_φ has the following properties:

Lemma 3.1.0.2. *[Dy, L.9] Let $(A, B, \varphi, \theta_\varphi)$ be as in (3.1.0.1). Then*
(i) $\theta_\varphi(upu^*) = \varphi(u)\theta_\varphi(p)\varphi(u)^*$,
(ii) $\varphi(-1) = -1$,
(iii) $\theta_\varphi(0) = 0$, $\theta_\varphi(1) = 1$, and $\theta_\varphi(1-p) = 1 - \theta_\varphi(p)$,
(iv) θ_φ preserves commutativity and the symmetric difference of commuting projections.

The following concept will be crucial in the next chapters.

Definition 3.1.0.3. *[Dy, p.75] A* ***projection orthoisomorphism*** *between two C^*-algebras A and B is a one-to-one mapping θ from $\mathcal{P}(A)$, the set of projections in A, onto $\mathcal{P}(B)$, the set of projections in B, which preserves orthogonality, i.e., if $p, q \in \mathcal{P}(A)$, then*

$$pq = 0 \textit{ if and only if } \theta(p)\theta(q) = 0.$$

We collect certain properties of these mappings:

Lemma 3.1.0.4. *[Dy, L.1] Any projection orthoisomorphism θ between C^*-algebras A and B preserves the following: 0, 1, the orthocomplement $1-p$ of p, order, and commutativity.*

Theorem 3.1.0.5. *[Dy, L.13] If A is a factor not of type $I_{2n} (n \geq 1)$, then θ_φ is an orthoisomorphism.*

Dye associates to any element a of a C^*-algebra A, a family of projections in the C^*-matrix algebra $\mathbb{M}_n(A)$ $(n \geq 2)$, denoted $P_{i,j}(a)$. He used them to prove his main results. For $n \geq 2$ and $a \in A$, we shall also use the set of projections $\{P_{i.j}(a),\ 1 \leq i, j \leq n\}$ in the next few chapters. Now we remind the reader of their properties.

Definition 3.1.0.6. *[Dy, p.76] Let A be a unital C^*-algebra, and let $\{E_{i,j}\}_{i,j}^n$ be the standard system of matrix units of $\mathbb{M}_n(\mathbb{C})$. For each $a \in A$ and $1 \leq i \neq j \leq n$, define the following projection of $\mathbb{M}_n(A)$ by*

$$P_{i,j}(a) := (1+aa^*)^{-1} \otimes E_{i,i} + (1+aa^*)^{-1} a \otimes E_{i,j} + a^*(1+aa^*)^{-1} \otimes E_{j,i} + a^*(1+aa^*)^{-1} a \otimes E_{j,j}.$$

If $a = 0$, then $P_{i,j}(a)$ is the ith diagonal matrix unit E_i of $\mathbb{M}_n(A)$.

In particular, if $n = 3$, then

$$P_{1,2}(a) = \begin{pmatrix} (1+aa^*)^{-1} & (1+aa^*)^{-1}a & 0 \\ a^*(1+aa^*)^{-1} & a^*(1+aa^*)^{-1}a & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

In the Hilbert A-module $A \oplus \cdots \oplus A$ (n times), $P_{i,j}(a)$ is the projection in $\mathbb{M}_n(A)$ whose range consists of all left multiples of the vector with 1 in the *ith* place, a in the *jth* place, and zeroes elsewhere.

Lemma 3.1.0.7. *[Dy, L.3] (i) $P_{i,j}(-a^*) = e_i + e_j - P_{i,j}(a)$,*
(ii) $P_{j,i}(b) = P_{i,j}(a)$ if and only if $b = a^{-1}$.

Dye used these projections to prove:

Theorem 3.1.0.8. *[Dy, L.6] Let θ be a projection orthoisomorphism between the C^*-matrix algebras $\mathbb{M}_n(A)$ and $\mathbb{M}_n(B)$ $(n \geq 3)$, under which the diagonal matrix units e_i, and more generally, projections of the form $P_{i,j}(a)$ correspond. Then there exists a linear mapping Ψ of $\mathbb{M}_n(A)$ onto $\mathbb{M}_n(B)$ being the direct sum of a $*$-isomorphism and $*-$antiisomorphism, with the property that $\Psi(P_{i,j}(a)) = \theta(P_{i,j}(a))$.*

Following [Dy, Lemma 2], let $\mathfrak{P}$ be the collection of all projections of the C^*-matrix algebra $\mathbb{M}_n(A)$ having as its rang the submodule $[x(e, a_2, \ldots, a_n) | x \in A]$ of $A \oplus \cdots \oplus A$ (more generally, e is in any place), where e is a projection of A and $a_2, \ldots, a_n$ are elements of A satisfying $ea_i = a_i$. Clearly, $\mathfrak{P}$ properly contains $\{P_{i,j}(a), a \in A \text{ and } 1 \leq i \neq j \leq n\}$.

Lemma 3.1.0.9. *Let A be a C^*- algebra. If θ is an orthoisomorphism of $\mathcal{P}(\mathbb{M}_n(A))$ such that θ leaves all the projections $P_{i,j}(a)$ fixed, then all projections in $\mathfrak{P}$ are also fixed.*

Proof: See first part in the proof of Lemma 7 of [Dy]. □

Theorem 3.1.0.10. *[Dy, Th.2] Let* N *and* M *be two von Neumann factors not of type* I_{2n}*, and* φ *be a group isomorphism between their unitary groups* $\mathcal{U}(N)$ *and* $\mathcal{U}(M)$*. Then there exists a linear or a conjugate linear* $*-$*isomorphism* Ψ *of* N *on* M *which implements* φ *in the following sense: for every* $u \in \mathcal{U}(N)$, $\varphi(u) = \lambda(u)\Psi(u)$ *for some (possibly discontinuous) character* λ *of* $\mathcal{U}(N)$.

Kadison [Ka] proved that the unitary group of a factor not of type I_n has no non-trivial continuous characters. In [Br, Th.1], Broise extended this result by proving that the unitary group of a factor not of type I_n has no non-trivial characters. Therefore Dye's result ([Dy, Th.2]) becomes:

Theorem 3.1.0.11. *If* N *and* M *are two von Neumann factors not of type* I_{2n}*, then any isomorphism between their unitary groups is implemented by either linear or conjugate linear* $*-$*isomorphism between the factors themselves.*

3.2 Booth's Results

Let A and B be two simple, unital C^*-algebras, and let $\varphi : \mathcal{U}(A) \to \mathcal{U}(B)$ be an isomorphism of their unitary groups. The associated map θ_φ from $\mathcal{P}(A)$ to $\mathcal{P}(B)$ is not necessarily an orthoisomorphism (see section §5.3). In [Bo], A. Booth associates to θ_φ a projection orthoisomorphism $\tilde{\theta}_\varphi$ which preserves the unitary equivalence of projections, and then shows that $\tilde{\theta}_\varphi$ induces an isomorphism of the scaled dimension groups from $K_0(A)$ to $K_0(B)$. Using Elliott's theorem, Booth obtains:

Theorem 3.2.0.1. *[Bo, p.88] If* A *and* B *are simple, unital AF-algebras with isomorphic unitary groups, then* A *and* B *are isomorphic as* C^*-*algebras.*

Let us now review in more details the definition of $\tilde{\theta}_\varphi$.

Let $\hat{\mathbb{S}}^1$ denote the group of characters of the unit circle $\mathbb{S}^1$. Recall that the inverse of the character χ is given by

$$\chi^{-1}(\mu) = \overline{\chi}(\mu) = \overline{\chi(\mu)}, \text{ for all } \mu \in \mathbb{S}^1.$$

Theorem 3.2.0.2. *[Bo, 3.1.0.5] If $(A, B, \varphi, \theta_\varphi)$ is as in (3.1.0.1), and p is a non-trivial projection of A, then there are elements a_p and b_p in $\hat{\mathbb{S}}^1$ such that*

$$\varphi(\mu p + 1 - p) = a_p(\mu)\theta_\varphi(p) + b_p(\mu)(1 - \theta_\varphi(p)), \text{ for all } \mu \in \mathbb{S}^1.$$

Definition 3.2.0.3 (The map c). *[Bo, 3.2.1.2] If $(A, B, \varphi, \theta_\varphi)$ is as in (3.1.0.1), let $c : \mathcal{P}(A)\backslash\{0,1\} \longrightarrow \hat{\mathbb{S}}^1$ be the map defined by*

$$c(p) = c_p = a_p\overline{b}_p \in \hat{\mathbb{S}}^1.$$

We collect main properties of c in the following propositions.

Proposition 3.2.0.4. *[Bo, 3.2.1.3] Let $(A, B, \varphi, \theta_\varphi)$ be as is in (3.1.0.1). The map $c : \mathcal{P}(A)\backslash\{0,1\} \longrightarrow \hat{\mathbb{S}}^1$ has the following properties:*
(i) If p is unitarily equivalent to q, then $c_p = c_q$.
(ii) For any $p \in \mathcal{P}(A)\backslash\{0,1\}$, the map $c_p : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ is a group automorphism of $\mathbb{S}^1$. So in fact

$$c : \mathcal{P}(A)\backslash\{0,1\} \longrightarrow Aut(\mathbb{S}^1).$$

(iii) For any $p \in \mathcal{P}(A)\backslash\{0,1\}$, we have $c_p = c_{1-p}$.

Proposition 3.2.0.5. *[Bo, 3.2.2.3] Let $(A, B, \varphi, \theta_\varphi)$ be as in (3.1.0.1). If $p, q \in \mathcal{P}(A)\backslash\{0,1\}$ are such that $pq = 0$ and $p + q = r < 1$, then exactly one of the following holds:*

$$\theta_\varphi(p)\theta_\varphi(q) = 0 \iff c_p = c_q = c_r \tag{2}$$

$$\theta_\varphi(1-p)\theta_\varphi(1-q) = 0 \iff c_p = c_q = \overline{c_r} \tag{3}$$

$$\theta_\varphi(1-p)\theta_\varphi(q) = 0 \iff c_p = \overline{c_q} = c_r \tag{4}$$

$$\theta_\varphi(p)\theta_\varphi(1-q) = 0 \iff \overline{c_p} = c_q = c_r \tag{5}$$

Let $(A, B, \varphi, \theta_\varphi)$ be as in (3.1.0.1). If the map c is constant, then from the previous proposition we have that θ_φ is an orthoisomorphism. If the image of c contains only two elements c_p and $\bar{c}_p$, for some $p \in \mathcal{P}(A)\backslash\{0,1\}$, then define

$$\mathcal{P}_{c_p} = \{q \in \mathcal{P}(A)\backslash\{0,1\};\ c_q = c_p\}$$

and

$$\mathcal{P}_{\bar{c}_p} = \{q \in \mathcal{P}(A)\backslash\{0,1\};\ c_q = \bar{c}_p\}.$$

Clearly, $\mathcal{P}_{c_p}$ and $\mathcal{P}_{\bar{c}_p}$ form a partition of $\mathcal{P}(A)\backslash\{0,1\}$.

Now for a given $(A, B, \varphi, \theta_\varphi)$, suppose there is a projection $p \in \mathcal{P}(A)\backslash\{0,1\}$ such that $c(\mathcal{P}(A)\backslash\{0,1\}) = \{c_p, \bar{c}_p\}$. Define $\tilde{\theta}_\varphi : \mathcal{P}(A) \to \mathcal{P}(B)$ by

$$\tilde{\theta}_\varphi(q) = \begin{cases} \theta_\varphi(q) & \text{if } q \in \mathcal{P}_{c_p} \\ 1 - \theta_\varphi(q) & \text{if } q \in \mathcal{P}_{\bar{c}_p} \\ 1 & \text{if } q = 1 \\ 0 & \text{if } q = 0 \end{cases}$$

Then we have:

Proposition 3.2.0.6. *[Bo, 3.2.2.9] Let $(A, B, \varphi, \theta_\varphi)$ be as in (3.1.0.1). If there is a projection $p \in \mathcal{P}(A)\backslash\{0,1\}$ such that $c(\mathcal{P}(A)\backslash\{0,1\}) = \{c_p, \bar{c}_p\}$, then the map $\tilde{\theta}_\varphi$ is an orthoisomorphism and preserves unitary equivalence.*

In [Bo, 3.2.3.8], Booth introduced the following sufficient condition on simple unital C^*-algebras A guaranteeing the image of the map c consists of at most two elements.

Definition 3.2.0.7. *[Bo, 3.2.3.8] Let A be a C^*-algebra. A is said to be* ***oddly decomposable*** *if for every pair $p, q \in \mathcal{P}(A)\backslash\{0,1\}$ there is an odd integer $n \geq 3$ and a decomposition of q as a sum $q = \sum_{i=1}^{n} r_i$ of pairwise orthogonal projections $r_i \in \mathcal{P}(A)\backslash\{0,1\}$, such that each r_i is unitarily equivalent to some projection $\tilde{r}_i < p$.*

Proposition 3.2.0.8. *[Bo, 3.2.3.10] Let $(A, B, \varphi, \theta_\varphi)$ be as in (3.1.0.1). If A is oddly decomposable, then for any projection $p \in \mathcal{P}(A)\backslash\{0,1\}$ we have $c(\mathcal{P}(A)\backslash\{0,1\}) \subseteq \{c_p, \bar{c}_p\}$.*

As a consequence of the preceding, we have:

Theorem 3.2.0.9. *Let $(A, B, \varphi, \theta_\varphi)$ be as in (3.1.0.1). If A is oddly decomposable, then φ induces an orthoisomorphism between $\mathcal{P}(A)$ and $\mathcal{P}(B)$ that preserves unitary equivalence of projections.*

Booth proved the following results, which he used in order to prove Proposition (3.2.0.6):

Lemma 3.2.0.10. *[Bo, Le.3.2.3.13] Let $(A, B, \varphi, \theta_\varphi)$ be as in (3.1.0.1), and suppose also that A is oddly decomposable. Let $p \in \mathcal{P}(A)\backslash\{0,1\}$. Suppose that p_1 and p_2 are two orthogonal projections of $\mathcal{P}_{c_p}$ such that $p_1 + p_2 \neq 1$ and $\theta(p_1)\theta(p_2) = 0$. Then θ preserves orthogonality in $\mathcal{P}_{c_p}$.*

Proposition 3.2.0.11. *[Bo, Pro.3.2.3.14] Let $(A, B, \varphi, \theta_\varphi)$ be as in (3.1.0.1), and suppose also that A is oddly decomposable. Let $p \in \mathcal{P}(A)\backslash\{0,1\}$. If $\mathcal{P}_{c_p} \neq \mathcal{P}(A)\backslash\{0,1\}$, then θ preserves orthogonality in $\mathcal{P}_{c_p}$ (respectively in $\mathcal{P}_{\bar{c}_p}$) and flips orthogonality in $\mathcal{P}_{\bar{c}_p}$ (respectively $\mathcal{P}_{c_p}$).*

Lemma 3.2.0.12. *[Bo, Cor.3.2.3.15] Let $(A, B, \varphi, \theta_\varphi)$ be as in (3.1.0.1), and suppose also that A is oddly decomposable. Let $p \in \mathcal{P}(A)\backslash\{0,1\}$ be such that θ preserves orthogonality on $\mathcal{P}_{c_p}$ and flips orthogonality on $\mathcal{P}_{\bar{c}_p}$. Suppose that p_1 and p_2 are two orthogonal projections in $\mathcal{P}(A)\backslash\{0,1\}$ such that $p_1 + p_2 = r \neq 1$. Then*

1. *$p_1, p_2 \in \mathcal{P}_{c_p} \Rightarrow r \in \mathcal{P}_{c_p}$,*
2. *$p_1 \in \mathcal{P}_{\bar{c}_p}$ and $p_2 \in \mathcal{P}_{\bar{c}_p} \Rightarrow r \in \mathcal{P}_{c_p}$,*
3. *$p_1 \in \mathcal{P}_{c_p}$ and $p_2 \in \mathcal{P}_{\bar{c}_p} \Rightarrow r \in \mathcal{P}_{\bar{c}_p}$.*

Finally, in [Bo, Pro.3.4.0.1], Booth then shows the following

Proposition 3.2.0.13. *[Bo, 3.4.0.1] Every simple unital infinite dimensional AF-algebra is oddly decomposable.*

Chapter 4

Extensions of Booth's Result

In this chapter, we extend Booth's result to a larger class of finite unital C^*-algebras as well as to purely infinite unital C^*-algebras. We prove that an isomorphism between unitary groups induces an isomorphism between the corresponding scaled ordered groups (K_0-groups) for many simple unital C^*-algebras.

Recall that for simple unital C^*-algebra, a group isomorphism φ between the unitary groups induces a map θ_φ between the sets of projections of the C^*-algebras. When the C^*-algebras are oddly decomposable we construct from θ_φ a projection orthoisomorphism that preserves unitary equivalence of projections. If the C^*-algebras have cancellation, then this map induces an isomorphism between the scaled ordered groups (K_0-groups).

We also study when φ induces an isomorphism between K_1-groups, or even $*-$isomorphisms between the C^*-algebras themselves.

4.1 Finite C^*-Algebras of Real Rank Zero

4.1.1 An Isomorphism Between the K_0-Groups

Let $\mathcal{F}$ denote the collection of all C^*-algebras A such that
(i) A is a simple, unital, stably finite C^*-algebra with real rank zero,
(ii) $K_0(A)$ is a weakly unperforated interpolation group, with cancellation.

The set $\mathcal{F}$ is non-empty as the irrational rotation algebras A_θ belong to $\mathcal{F}$.

In this section, we prove that if A and B are two C^*-algebras which belong to $\mathcal{F}$ such that their unitary groups are isomorphic, then their K_0-groups are isomorphic, in particular, we prove that two irrational rotation algebras having isomorphic unitary groups are isomorphic.

For any unital C^*-algebra A, B. Blackadar in [B, p.32] defined the scale of A as

$$\Sigma(A) = \{[p] |\, p \text{ is a projection in } A\}.$$

If A has cancellation, then $\Sigma(A)$ is a hereditary subset of $K_0^+(A)$ (see [B, p.38]), so if $0 \leq [p] \leq [1]$, then $[p] \in \Sigma(A)$, on the other hand if $[p] \in \Sigma(A)$, then $[p] \leq [1]$ hence

$$\Sigma(A) = [0, [1]].$$

Definition 4.1.1.1. *Let A be any C^*-algebra. An element $a \in A$ is full if the two sided ideal of A generated by a is all of A, i.e. if there exist elements $x_i, y_i (i = 1, \ldots, r)$ with $1 = \sum_{i=1}^r x_i a y_i$. If $a \geq 0$ is full, then there exist $x_1, x_2, \ldots, x_r$ in A with*

$$1 = \sum_{i=1}^{r} x_i a x_i^*.$$

Proposition 4.1.1.2. *[B, 6.3.5] If A is stably finite and every non-zero idempotent in $\mathbb{M}_\infty(A)$ is full, then $K_0(A)$ is a simple ordered group.*

As a consequence result, we have

Corollary 4.1.1.3. *If A is a stably finite simple C^*-algebra, then $K_0(A)$ is a simple ordered group.*

Proof: Follows directly from the previous proposition. □

Theorem 4.1.1.4. *Every C^*-algebra in $\mathcal{F}$ is oddly decomposable.*

Proof: Let p, q be non-trivial projections of A. Then $[p]$ and $[q]$ are in $\Sigma(A)$; as $K_0(A)$ is a non-cyclic simple interpolation group, by [Go, Le.14.5], there exists $y > 0$ such that $y < [q]$. As y is an order unit, there exists $k \in \mathbb{N}$ such that

$$[p] \leq ky.$$

By the Riesz interpolation property, there exist $a_1, \dots, a_k \in K_0(A)^+$ such that

$$[p] = a_1 + \cdots a_k, \quad 0 < a_i \leq y < [q], \text{ and } 1 \leq i \leq k.$$

Hence for all $1 \leq i \leq k$, $a_i = [p_i]$, and $p_i \in \mathcal{P}(A)$. Moreover by [B, 6.9.2], the projections $p_i's$ can be chosen to be orthogonal. Then we have $[p] = [p_1 + \cdots + p_k]$ and for every $1 \leq i \leq k$, $[p_i] < [q]$, and again using [B, 6.9.2], we have that $p_i \sim r_i \leq q$, moreover as $K_0(A)$ has cancellation, $p_i \sim_u r_i \leq q$ and also there exists a unitary u of A such that

$$p = \sum_{i=1}^{k} u p_i u^*.$$

If k is an odd number exceeding three, then we are done. Otherwise we can increase the number k as follows: As $a_1 > 0$, by [Go, Le.14.5], there exists $0 < a_{1,1} < a_1$, then $[p] = a_{1,1} + (a_1 - a_{1,1}) + a_2 + \cdots + a_k$, and we complete the proof as in the first part. In particular, if $k = 1$, i.e. $[p] < [q]$, using the same argument of [Go, 14.5], there exists $a > 0$ such that $2a < [p]$. Then we have

$$[p] = a + a + ([p] - 2a),$$

notice that $a < [q]$ and $[p] - 2a < [q]$, therefore as we did above, they can be realized as orthogonal projections of A, hence the theorem follows. □

Theorem 4.1.1.5. *Let A and B be two C^*-algebras which belong to $\mathcal{F}$. If φ is an isomorphism between $\mathcal{U}(A)$ and $\mathcal{U}(B)$, then φ induces a projection orthoisomorphism from $\mathcal{P}(A)$ to $\mathcal{P}(B)$, that preserves unitary equivalence of projections.*

Proof: Follows directly from Theorem (4.1.1.4) and Theorem (3.2.0.9). □

Proposition 4.1.1.6. *[Go, 1.9] If G is an ordered group and $S \subseteq G^+$, then the convex subgroup $< S >$ generated by S is an ideal of G.*

Proposition 4.1.1.7. *If A is a simple, unital, stably finite C^*-algebra, then $\Sigma(A)$ generates $K_0(A)$.*

Proof: As A is stably finite and simple, $K_0(A)$ is a simple ordered group by Proposition (4.1.1.2). The convex subgroup generated by the scale of A is an ideal of

$K_0(A)$ by Proposition (4.1.1.6), hence is equal to $K_0(A)$. $\square$

Recall from Remark (2.1.3.7) that the previous proposition together with A having Riesz interpolation property will enable us to use the extension theorem (Proposition (2.1.3.6)) of scale homomorphisms. We now prove the main theorem of this section.

Theorem 4.1.1.8. *Let A and B be C^*-algebras in $\mathcal{F}$. If $\mathcal{U}(A)$ and $\mathcal{U}(B)$ are isomorphic, then $K_0(A)$ and $K_0(B)$ are isomorphic as scaled ordered groups.*

Proof: From Theorem (4.1.1.5) we obtain a map $\tilde{\theta} : \mathcal{P}(A) \to \mathcal{P}(B)$; this is an orthoisomorphism that preserves unitary equivalence of projections. If $[p] = [q]$, since A has cancellation then $p \sim_u q$, therefore $\tilde{\theta}(p) \sim \tilde{\theta}(q)$ and hence $[\tilde{\theta}(p)] = [\tilde{\theta}(q)]$. Then we can define

$$\tilde{\theta}_* : \Sigma(A) \longrightarrow \Sigma(B) \text{ by}$$
$$[p] \longmapsto [\tilde{\theta}(p)].$$

Let us show that $\tilde{\theta}_*$ is a scale homomorphism: Suppose x, y and $z \in \Sigma(A)$ and $z = x + y$.

So $x = [p]$ and $y = [q]$, for some projections p and q in A. Then $[p] < [1 - q]$, as $K_0(A)$ is weakly unperforated, therefore by [B, 6.9.2], it follows that $p \sim q_1 \leq 1 - q$, therefore $x = [q_1]$, and $y = [q]$, q_1 and q are orthogonal projections in A, indeed

$$\begin{aligned}
\tilde{\theta}_*(z) &= \tilde{\theta}_*([q_1] + [q]) \\
&= \tilde{\theta}_*([q_1 + q]) \\
&= [\tilde{\theta}(q_1) + \tilde{\theta}(q)] \\
&= [\tilde{\theta}(q_1)] + [\tilde{\theta}(q)] \\
&= \tilde{\theta}_*(x) + \tilde{\theta}_*(y).
\end{aligned}$$

So $\tilde{\theta}_*$ is a scale homomorphism. Since the inverse of $\tilde{\theta}_*$ under composition is given by $\tilde{\theta}_*^{-1}[q] = [\tilde{\theta}^{-1}(q)]$ for any $q \in \mathcal{P}(B)$, we see that $\tilde{\theta}_*^{-1}$ is also a scale homomorphism and hence $\tilde{\theta}_*$ is a scale isomorphism. Then by Proposition (2.1.3.6), we get that $\tilde{\theta}_*$ extends to an isomorphism between $K_0(A)$ and $K_0(B)$. $\square$

As a consequence of the previous theorem, we have the following result for the irrational rotation algebras.

Corollary 4.1.1.9. *If A_θ and A_η are two irrational rotation algebras with isomorphic unitary groups, then $A_\theta \simeq A_\eta$ as C^*-algebras.*

Proof: By the previous theorem we get

$$K_0(A_\theta) \cong K_0(A_\eta)$$

therefore, $\mathbb{Z} + \theta\mathbb{Z} = \mathbb{Z} + \eta\mathbb{Z}$, which implies that $\theta = \pm\eta \mod \mathbb{Z}$, hence by Theorem (2.6.2.7) we get $A_\theta \simeq A_\eta$ as C^*-algebras. □

4.1.2 $A\mathbb{T}-$Algebras

We proved in the last section that the unitary groups of the irrational rotation algebras determine the algebraic type. Here we are going to extend this result to a larger class of C^*-algebras. By an $A\mathbb{T}-$algebra, we mean the C^*-algebra inductive limit of a sequence of finite direct sums of matrix algebras over the continuous functions on the unite circle $\mathbb{T}$. Recall from Theorem (2.6.2.10) that the irrational rotation algebra is an $A\mathbb{T}-$algebra. Every simple, unital $A\mathbb{T}-$algebra is approximately divisible (see [El1, Th.2]).
Now let us show the following consequence of this result:

Proposition 4.1.2.1. *If A is a simple, unital $A\mathbb{T}-$algebra of real rank zero, then A belongs to $\mathcal{F}$.*

Proof: By [RS, Pro.3.2.4], A has stable rank one. As A is a simple, unital $A\mathbb{T}-$algebra, it is approximately divisible, then it is finite by Theorem (2.6.1.4)(b) and hence stable finite by Theorem (2.6.1.4)(a). As A has real rank zero, $K_0(A)$ has cancellation by Proposition (2.5.1.5). From [RS, p.48], $K_0(A)$ is weakly unperforated. As A has real rank zero, we get that $K_0(A)$ has the Riesz interpolation property [BK, 3.15]. Hence A belongs to $\mathcal{F}$. □

Let A and B be simple unital $A\mathbb{T}-$algebras of real rank zero, and let φ be an isomorphism between their unitary groups. So far we have proved that such C^*-algebras have the same scaled ordered groups. In this section, we extend the result to obtain a $*-$isomorphism between the C^*-algebras themselves. To do this, we shall use Elliott's theorem ([El2]). In the case of simple, unital $A\mathbb{T}-$algebras of real rank zero (as in [RS, p.51]), Elliott's theorem ([El2]) reduces to the following:

Theorem 4.1.2.2. *[El2] If A and B are two simple, unital $A\mathbb{T}-$algebras of real rank zero, then A is isomorphic to B if and only if there are group isomorphisms $\alpha_0 : K_0(A) \to K_0(B)$ and $\alpha_1 : K_1(A) \to K_1(B)$ such that $\alpha_0(K_0(A)^+) = K_0(B)^+$ and $\alpha_0([1_A]_0) = [1_B]_0$. For each pair of isomorphisms (α_0, α_1), there is an isomorphism $\alpha : A \to B$ such that $K_0(\alpha) = \alpha_0$ and $K_1(\alpha) = \alpha_1$.*

The following theorem shows that we can construct an isomorphism between the K_1-groups in the case where the isomorphism between the unitary groups is assumed to be continuous.

Theorem 4.1.2.3. *If A and B are simple, unital $A\mathbb{T}-$algebras and φ is a topological isomorphism between $\mathcal{U}(A)$ and $\mathcal{U}(B)$, then*

$$K_1(A) \simeq K_1(B).$$

Proof: By Proposition (2.6.1.7), it is enough to show that $\mathcal{U}(A)/\mathcal{U}(A)_0 \simeq \mathcal{U}(B)/\mathcal{U}(B)_0$. Let Ψ denote the homomorphism from $\mathcal{U}(A)$ to $\mathcal{U}(B)/\mathcal{U}(B)_0$, given by $\Psi = \pi \circ \varphi$ where π is the quotient map from $\mathcal{U}(B)$ to $\mathcal{U}(B)/\mathcal{U}(B)_0$. Therefore Ψ is surjective. Let us check that ker $\Psi = \mathcal{U}(A)_0$. If $u \in \mathcal{U}(A)_0$, there is a continuous map $u_t : [0,1] \to \mathcal{U}(A)$ such that $u_0 = u$ and $u_1 = 1_A$, whence the map

$$\varphi \circ u_t : [0,1] \longrightarrow \mathcal{U}(B)$$

is continuous. Indeed, $\varphi \circ u_0 = \varphi(u)$ and $\varphi \circ u_1 = 1_B$ then $\Psi(u) = \mathcal{U}(B)_0$ and hence $u \in$ ker Ψ.

Conversely, If $u \in$ ker Ψ, then $\varphi(u) \in \mathcal{U}(B)_0$, and thus there exists a continuous map $v_t : [0,1] \to \mathcal{U}(B)$ such that $v_0 = \varphi(u)$ and $v_1 = 1_B$. Therefore the map

$$\varphi^{-1} \circ v_t : [0,1] \longrightarrow \mathcal{U}(A)$$

is continuous. $\varphi^{-1} \circ v_0 = u$ and $\varphi^{-1} \circ v_1 = 1_A$ hence, $u \in \mathcal{U}(A)_0$. Therefore the theorem is verified. □

Now we have the following:

Theorem 4.1.2.4. *Let A and B be two simple, unital $A\mathbb{T}-$algebras of real rank zero. If φ is an isomorphism between $\mathcal{U}(A)$ and $\mathcal{U}(B)$ such that $K_1(A) \simeq K_1(B)$, then A and B are $*-$isomorphic.*

Proof: As A and B belong to $\mathcal{F}$ (by Proposition (4.1.2.1)), and applying Theorem (4.1.1.8), it follows that φ induces an isomorphism α_0 between the $K_0(A)$ and $K_0(B)$ such that $\alpha_0(K_0(A)^+) = K_0(B)^+$ and $\alpha_0([1_A]_0) = [1_B]_0$. By Theorem (4.1.2.2), A and B are isomorphic as C^*-algebras. □

The following result is similar

Theorem 4.1.2.5. *Let A and B be two simple, unital $A\mathbb{T}-$algebras of real rank zero. If φ is an isomorphism between the topological groups $\mathcal{U}(A)$ and $\mathcal{U}(B)$, then A and B are $*-$isomorphic.*

Proof: As φ is a topological isomorphism, it follows from Theorem (4.1.2.3) that

$$K_1(A) \simeq K_1(B).$$

Hence the result follows directly From Theorem (4.1.2.2). □

4.1.3 C^*-Algebra Associated to Cantor Minimal Systems

In this section, we define the associated C^*-algebra $C(X) \times_T \mathbb{Z}$ of a Cantor minimal system (X, T), and we give some properties. As a good application, we obtain consequently from the previous section a result for $C(X) \times_T \mathbb{Z}$, that the unitary groups of these algebras determine their algebraic types. We introduce the notion of topological full groups of a minimal system studied in [GPS2], and we make the link with the unitary groups of $C(X) \times_T \mathbb{Z}$.

Definition 4.1.3.1. *(**Minimal System**). Let X be a compact metric space and let $T : X \to X$ be a homeomorphism. Then (X,T) is called a (topological) dynamical system. We say that (X,T) is minimal if $TA = A$, where A is a closed subset of X, implies that $A = \emptyset$ or $A = X$. This is equivalent to all $T-$orbits being dense, i.e. $Orb_T(x)^- = \{T^n(x) | \, n \in \mathbb{Z}\}^- = X$, for all $x \in X$.*

Let us recall the connection between dynamical systems and C^*-algebras (see [Pu1] and [To] for a complete treatment).

The homeomorphism T gives rise to a $*-$automorphism of $C(X)$, also denoted T, by setting $T(f) = f \circ T^{-1}$, $f \in C(X)$. The associated C^*-crossed product, which we denote by $C(X) \times_T \mathbb{Z}$, is the universal C^*-algebra generated by $C(X)$ and a unitary element u satisfying $ufu^* = T(f)$, for all f in $C(X)$. Assume that (X,T) is minimal. Then we may realize the crossed product as follows: Let μ be any $T-$invariant probability measure on X. Then $C(X) \times_T \mathbb{Z}$ is isomorphic to the C^*-algebra acting on the Hilbert space $\mathcal{H} = L^2(X,\mu)$ and is generated by the multiplication operators $m_f \in \mathcal{B}(\mathcal{H})$ such that $g \longmapsto fg$ and the unitary operator

$$u = u_T : \; g \longmapsto g \circ T^{-1},$$

for every $f \in C(X)$ and $g \in \mathcal{H}$, $C(X)$ is naturally embedded in $C(X) \times_T \mathbb{Z}$ as a maximal abelian C^*-subalgebra. The minimality of T implies that $C(X) \times_T \mathbb{Z}$ is a simple C^*-algebra. There is a one to one correspondence between the set of normalized traces on $C(X) \times_T \mathbb{Z}$ and the set of $T-$invariant probability measures on X.

We now assume that (X,T) is a minimal Cantor system, i.e. X is the Cantor set and summarize some of the properties of $C(X) \times_T \mathbb{Z}$ (see for example [Pu1], [Pu2], [GPS1] and [El2]).

Theorem 4.1.3.2. *If (X,T) is a Cantor minimal system, then*
(i) $K_0(C(X) \times_T \mathbb{Z})$ is a simple dimension group and $K_1(C(X) \times_T \mathbb{Z}) = \mathbb{Z}$,
(ii) $C(X) \times_T \mathbb{Z}$ has real rank zero,
(iii) $C(X) \times_T \mathbb{Z}$ is an $A\mathbb{T}-$algebra.

Recall that if p is a projection of $C(X) \times_T \mathbb{Z}$, then $p \sim_u \chi_V$, for some clopen subset V of X, (see [Pu2, Th.2.1]).

Theorem 4.1.3.3. *Let (X_1, T_1) and (X_2, T_2) be two Cantor minimal systems. If φ is an isomorphism between the unitary groups of $C(X_1) \times_{T_1} \mathbb{Z}$ and $C(X_2) \times_{T_2} \mathbb{Z}$, then*

$$C(X_1) \times_{T_1} \mathbb{Z} \simeq C(X_2) \times_{T_2} \mathbb{Z}$$

as C^*-algebras.

Proof: As $C(X) \times_T \mathbb{Z}$ is a simple, unital $A\mathbb{T}-$algebra of real rank zero, and $K_1(C(X) \times_T \mathbb{Z}) = \mathbb{Z}$, the result follows directly from Theorem (4.1.2.4). □

To a Cantor minimal system (X, T) can be associated several types of full groups, which were studied in [GPS2]. Let us recall these definitions. Let Homeo(X) denote the group of all homeomorphisms of X.

Definition 4.1.3.4. *[GPS2, 1.1] Let (X, T) be a dynamical system. The full group $[T]$ of (X, T) is the subgroup of all homeomorphisms γ of X such that*

$$\gamma(x) \in Orb_T(x), \quad \textit{for all}\, x \in X.$$

Remark 4.1.3.5. *[GPS2, 1.2] To any $\gamma \in [T]$ is associated a map $n: X \to \mathbb{Z}$, defined by*

$$\gamma(x) = T^{n(x)}(x), \quad \textit{for}\, x \in X.$$

Definition 4.1.3.6. *[GPS2, 2.1] If (X, T) is a Cantor minimal system, then the topological full group $\tau[T]$ of $[T]$ is subgroup of all homeomorphisms $\gamma \in [T]$, whose associated map $n: X \to \mathbb{Z}$ is continuous.*

Let $\gamma \in \tau[T]$ and for each $k \in \mathbb{Z}$,

$$X_k = \{x \in X;\, \gamma(x) = T^k(x)\} = n^{-1}(\{k\}).$$

Then $(X_k)_{k\in\mathbb{Z}}$ is a finite partition of X into clopen sets such that

$$X = \coprod_{k\in\mathbb{Z}} X_k = \coprod_{k\in\mathbb{Z}} T^k(X_k),$$

where $\coprod$ denotes the disjoint union. Therefore, $\tau[T]$ is a countable group.

Definition 4.1.3.7. *[GPS2, 4.5] If (X_1, T_1) and (X_2, T_2) are two dynamical systems, they are (topologically)* ***orbit equivalent*** *if there exists a homeomorphism $F : X_1 \to X_2$ such that*

$$F(Orb_{T_1}(x)) = Orb_{T_2}(F(x)), \ \ for\ all\ \ x \in X_1.$$

Such a map is called an ***orbit map****.*

Definition 4.1.3.8. *[GPS2, 4.9] Let (X_1, T_1) and (X_2, T_2) be two minimal systems that are (topologically) orbit equivalent. We say that (X_1, T_1) and (X_2, T_2) are* ***strong orbit equivalent*** *if there exists an orbit map $F : X_1 \to X_2$ so that the associated orbit cocycles each have at most one point of discontinuity.*

Let (X, T) be a Cantor minimal system. For all $x \in X$, let

$$Orb_T^+(x) = \{T^k(x);\ k \geq 1\}$$

denote the forward orbit of x.

Definition 4.1.3.9. *[GPS2, 4.10] If (X, T) is a minimal system and $y \in X$, we denote by $\tau[T]_y$ the subgroup of $\tau[T]$ consisting of those γ such that $\gamma(Orb_T^+(y)) = Orb_T^+(y)$.*

Combining Theorem (4.1.3.3) with [GPS2], Corollary 4.11, we obtain:

Theorem 4.1.3.10. *For $i = 1, 2$ let (X_i, T_i) be two Cantor minimal systems. Then the following are equivalent:*
(i) The unitary groups $\mathcal{U}(C(X_1) \times_{T_1} \mathbb{Z})$ and $\mathcal{U}(C(X_2) \times_{T_2} \mathbb{Z})$ are isomorphic as abstract groups,
(ii) The dimension groups $K_0(C(X_i) \times_{T_i} \mathbb{Z})$, $i = 1, 2$, are isomorphic by a map preserving the distinguished order units,
(iii) $C^*(X_1, T_1)$ and $C^*(X_2, T_2)$ are isomorphic,
(iv) For any $y_i \in X_i$, $i = 1, 2$, $\tau[T_1]_{y_1}$ and $\tau[T_2]_{y_2}$ are isomorphic as abstract groups,
(v) (X_1, T_1) and (X_2, T_2) are strong orbit equivalent.

4.2 The Kirchberg Algebra Case

In this section, we study the classification of a special class of simple purely infinite C^*−algebras.

Definition 4.2.0.1. *A **Kirchberg algebra** is a separable, simple, unital, purely infinite C^*−algebra which is nuclear.*

Let A and B be two Kirchberg algebras. We prove in this section that if their unitary groups are isomorphic, then the algebras are also isomorphic. We shall do this by showing that such algebras have the same K−theory.

Theorem 4.2.0.2. *[RS, Th.8.4.1] Two Kirchberg algebras A and B are $*$-isomorphic if and only if there are isomorphisms $\alpha_0 : K_0(A) \to K_0(B)$ and $\alpha_1 : K_1(A) \to K_1(B)$ with $\alpha_0([1_A]_0) = [1_B]_0$. For each such pair of isomorphisms, there is an isomorphism $\varphi : A \to B$ with $K_0(\varphi) = \alpha_0$ and $K_1(\varphi) = \alpha_1$.*

Let φ be an isomorphism between $\mathcal{U}(A)$ and $\mathcal{U}(B)$, where A and B be as given above. We show that from φ we construct an orthoisomorphism between the sets of projections which preserves the unitarily equivalence of projections. To do this we prove the following theorem.

Theorem 4.2.0.3. *Every simple, unital purely infinite C^*−algebra A is oddly decomposable.*

Proof: Let $p, q \in \mathcal{P}(A)\backslash\{0, 1\}$. As q is an infinite projection, there exists $q' < q$ such that $q' \sim q$. As p and q' are infinite projections, using Proposition (2.4.0.6), there exists $p_1 \sim q'$, $p_1 < p$ and $p - p_1$ is an infinite projection. Similarly there exist a projection $p_2 < p - p_1$ such that $p_2 \sim q'$, and a projection $r < q'$ such that $r \sim p - p_1 - p_2$. Then by Theorem (2.4.0.7), we get that

$$p_1 \sim_u q' < q, \ \ p_2 \sim_u q' < q \text{ and } p - p_1 - p_2 \sim_u r < q.$$

As $p = p_1 + p_2 + (p - p_1 - p_2)$, is written as a sum of three orthogonal projections, such that each of them is unitarily equivalent to a proper subprojection of q, A is oddly decomposable. □

Theorem 4.2.0.4. *If A and B are simple, unital purely infinite C^*-algebras, then an isomorphism between their unitary groups induces an orthoisomorphism between $\mathcal{P}(A)$ and $\mathcal{P}(B)$ preserving the unitary equivalence of projections.*

Proof: By the previous theorem, A is oddly decomposable and by Theorem (3.2.0.9), the proof is completed. □

Let A be an infinite C^*-algebra. As 1 is an infinite projection of A, we obtain from Theorem (2.4.0.9)(i).

Proposition 4.2.0.5. *If A is a simple, unital, purely infinite C^*-algebra, then*

$$K_0(A) = \{[p] \mid p \in \mathcal{P}(A)\backslash\{0,1\}\}.$$

Recall that if A is a simple, unital, purely infinite C^*-algebra, then by Theorem (2.4.0.2), we have

$$K_0^+(A) = K_0(A) = \Sigma(A).$$

Theorem 4.2.0.6. *If A and B are two simple, unital, purely infinite C^*-algebras and φ is an isomorphism between $\mathcal{U}(A)$ and $\mathcal{U}(B)$, then $K_0(A)$ and $K_0(B)$ are isomorphic as ordered groups.*

Proof: Let $\tilde{\theta} : \mathcal{P}(A) \longrightarrow \mathcal{P}(B)$ be the orthoisomorphism constructed from φ, in Theorem (4.2.0.4). It preserves the unitary equivalence of projections. If p and q are two non-trivial projections of A, with the same K_0-classes, then by Theorem (2.4.0.7), $p \sim_u q$, which implies $\tilde{\theta}(p) \sim_u \tilde{\theta}(q)$, and hence $[\tilde{\theta}(p)] = [\tilde{\theta}(q)]$ in $K_0(B)$. Then we can define $\psi : K_0(A) \longrightarrow K_0(B)$ by $\psi([p]) = [\tilde{\theta}(p)]$, for every $p \in \mathcal{P}(A)\backslash\{0,1\}$. Now let us check that ψ is an isomorphism:
(i) ψ is a homomorphism: Let $[p]$ and $[q] \in K_0(A)$. Pick p_1, p_2 two orthogonal infinite projections in A, such that $p_1 + p_2 < 1$. There exists $r_1 \sim p$ and $r_1 < p_1$. Also there exists $r_2 \sim q$ and $r_2 < p_2$. Then r_1 and r_2 are two orthogonal infinite projections of A

with infinite complement, such that $[r_1] = [p]$ and $[r_2] = [q]$, therefore

$$\begin{aligned}
\psi([p]+[q]) &= \psi([r_1]+[r_2]) \\
&= \psi([r_1+r_2]) \\
&= [\tilde{\theta}(r_1)+\tilde{\theta}(r_2)] \\
&= [\tilde{\theta}(r_1)]+[\tilde{\theta}(r_2)] \\
&= \psi([p])+\psi([q]).
\end{aligned}$$

(ii) ψ is bijection: If $\psi([p]) = \psi([q])$, where p and q are non-trivial projections, then $\tilde{\theta}(p)$ and $\tilde{\theta}(q)$ are also non-trivial projections with the same K_0-class, hence $\tilde{\theta}(p) \sim_u \tilde{\theta}(q)$, which implies that $[p] = [q]$. It's clear that ψ is surjective. Hence we have that ψ is an isomorphism i.e. $K_0(A) \simeq K_0(B)$ as a scaled ordered groups. □

In particular, we have the following result for the Cuntz algebras $(\mathcal{O}_n)$, with $n < \infty$.

Corollary 4.2.0.7. *Two Cuntz algebras $\mathcal{O}_n$ and $\mathcal{O}_m$ are isomorphic if and only if their unitary groups are isomorphic.*

Proof: As $K_0(\mathcal{O}_n) = \mathbb{Z}_{n-1}$, the result follows directly from the previous theorem.□

Theorem 4.2.0.8. *Let A and B be two simple, unital, purely infinite C^*-algebras. If φ is an isomorphism between $\mathcal{U}(A)$ and $\mathcal{U}(B)$, then the groups $K_1(A)$ and $K_1(B)$ are isomorphic.*

Proof: By Theorem (2.4.0.9)(ii), it is enough to show that $\mathcal{U}(A)/\mathcal{U}(A)_0$ and $\mathcal{U}(B)/\mathcal{U}(B)_0$ are isomorphic. By Leen's result stated in Theorem (2.4.0.10), the connected component of the unitary group of a simple, unital, purely infinite C^*-algebra A is generated by the self-adjoint unitaries of A. Let $\pi : \mathcal{U}(B) \longrightarrow \mathcal{U}(B)/\mathcal{U}(B)_0$ be the quotient homomorphism, and let

$$\Psi : \mathcal{U}(A) \longrightarrow \mathcal{U}(B)/\mathcal{U}(B)_0$$

be the surjective homomorphism defined by $\Psi(u) = \pi(\varphi(u))$, for every $u \in \mathcal{U}(A)$. If u is a self-adjoint unitary of A, then so is $\varphi(u)$ in B and therefore $\mathcal{U}(A)_0 \subseteq \ker(\Psi)$.

Conversely if $u \in \ker(\Psi)$, then $\varphi(u) \in \mathcal{U}(B)_0$, and by Leen's theorem again, $u \in \mathcal{U}(A)_0$. Therefore $\mathcal{U}(A)/\mathcal{U}(A)_0$ and $\mathcal{U}(B)/\mathcal{U}(B)_0$ are isomorphic. □

Theorem 4.2.0.9. *Two Kirchberg algebras A and B are $*-$isomorphic if and only if their unitary groups are isomorphic.*

Proof: Follows directly from Theorem (4.2.0.2), Theorem (4.2.0.6) and the previous result. □

4.3 A Counter Example in a Non-simple Case

In this section we give an example of two C^*-algebras whose unitary groups are isomorphic however the algebras themselves are not $*-$isomorphic. The example is given in the non-simple C^*-algebra $C(X)$, where X is a compact set. We use Milutin's theorem which is stated as follows:

Theorem 4.3.0.1 (Milutin). *[Va, p.494] If X and Y are two compact, metrizable spaces which are non-countable, then $C(X,\mathbb{R}) \simeq C(Y,\mathbb{R})$ as Banach spaces.*

Let us recall the following results of V. Pestov [Pe]. Let ζ denote the group homomorphism from $C(X,\mathbb{T})$ to the cohomotopy group $\pi^1(X)$ assigning to every mapping its homotopy class. Denote by $C^0(X,\mathbb{T})$ the kernel of ζ. Let X be a topological space and θ be the map of the linear space $C(X,\mathbb{R})$ to the group $C(X,\mathbb{T})$, given by $\theta(f) = \exp(2\pi i f)$. The image of $C(X,\mathbb{R})$ under θ is contained in $C^0(X,\mathbb{T})$ and θ is an additive group homomorphism.

If $x_0 \in X$, then let $C(X,x_0,\mathbb{R}) = \{f \in C(X,\mathbb{R});\ f(x_0) = 0\}$, $C(X,x_0,\mathbb{T}) = \{f \in C(X,\mathbb{T});\ f(x_0) = 1\}$, $C^0(X,x_0,\mathbb{T}) = \{f \in C^0(X,\mathbb{T});\ f(x_0) = 1\}$. Obviously, θ maps $C(X,x_0,\mathbb{R})$ to $C^0(X,x_0,\mathbb{T})$. Denote by θ_0 the restriction of θ to $C(X,x_0,\mathbb{R})$.

Proposition 4.3.0.2. *[Pe, Pro.13] Let X be a path-connected space and let $x_0 \in X$. Then the map $\theta_0 : C(X,x_0,\mathbb{R}) \to C^0(X,x_0,\mathbb{T})$ is an algebraic isomorphism.*

For every element $x_0 \in X$, the groups $C^0(X,\mathbb{T})$ and $C^0(X,x_0,\mathbb{T}) \oplus \mathbb{T}$ are isomorphic under the mapping $f \mapsto (f.f(x_0)^{-1}, f(x_0))$. Similarly, the groups $C(X,x_0,\mathbb{R}) \oplus \mathbb{R}$ and $C(X,\mathbb{R})$ under the mapping $f \mapsto (f - f(x_0), f(x_0))$, (see [Pe, Lemma 7]).

now consider the following short exact sequence:

$$0 \longrightarrow C^0(X,\mathbb{T}) \xrightarrow{\iota} C(X,\mathbb{T}) \xrightarrow{\zeta} \pi^1(X) \longrightarrow 0.$$

If X is compact, then $C(X,\mathbb{T})$ splits, i.e. $C(X,\mathbb{T}) = C^0(X,\mathbb{T}) \oplus \pi^1(X)$. Now let us prove the following lemma:

Lemma 4.3.0.3. *Let X and Y be two compact spaces. If $C(Y,\mathbb{R})$ and $C(X,\mathbb{R})$ are isomorphic as Banach spaces, then there is an isomorphism between $C(Y,\mathbb{R})$ and $C(X,\mathbb{R})$ which sends 1 (as a constant function) to itself and hence sends all constant functions to constants.*

Proof: Let ψ denote the isomorphism from $C(Y,\mathbb{R})$ onto $C(X,\mathbb{R})$. If $x_0 \in X$, and $k \in \mathbb{R}\backslash\{-1\}$, then we define $\varphi_k : C(X,\mathbb{R}) \to C(X,\mathbb{R})$ by $g \mapsto g + kg(x_0)$. It is clear that φ_k is a linear map and $\varphi_k(1) = 1 + k$.
The map φ_k is surjective: If $h \in C(X,\mathbb{R})$, then $h - \frac{k}{k+1}h(x_0) \in C(X,\mathbb{R})$ and

$$\varphi_k(h - \frac{k}{k+1}h(x_0)) = h + kh(x_0) - \frac{k}{k+1}h(x_0)\varphi_k(1) = h.$$

Now to show that φ_k is injective, let $g \in \ker(\varphi_k)$. Then for every $x \in X$, $g(x) + kg(x_0) = 0$ and in particular, $(k+1)g(x_0) = 0$, therefore $g = 0$, hence φ_k is a bijective.

Let $\psi(1) = f$. As f is a non-zero function which belongs to $C(X,\mathbb{R})$, there exists $x_0 \in X$ such that $|f(x_0)| = \|f\|_\infty$. Let $k = 2\text{sign}(f(x_0))$. Then for all $x \in X$,

$$\begin{aligned}
\varphi_k(f)(x) &= f(x) + kf(x_0) \\
&= f(x) + 2\text{sign}(f(x_0)).f(x_0) \\
&= f(x) + 2|f(x_0)| > 0.
\end{aligned}$$

The map $\psi_1 = \varphi_k \circ \psi$ is an isomorphism from $C(Y,\mathbb{R})$ onto $C(X,\mathbb{R})$ with $\psi_1(1) > 0$. Then define $\Phi : C(Y,\mathbb{R}) \to C(X,\mathbb{R})$ by $g \mapsto \frac{1}{\psi_1(1)}\psi_1(g)$ and hence the lemma is checked. □

Now we introduce the following main example:

Example 4.3.0.4. Consider $X = [0,1]$ and $Y = [0,1] \times [0,1]$ as subspaces of the usual topology of $\mathbb{R}$. As X and Y are not homeomorphic topological spaces, the

C^*−algebras $C(X)$ and $C(Y)$ are not $*$−isomorphic.
Claim: $C(X,\mathbb{T}) \simeq C(Y,\mathbb{T})$ as abstract groups.
As X and Y are both contractible subsets of $\mathbb{R}$, their cohomology groups $H^q(X) = H^q(Y) = 0$ for all $q > 0$. Then by [Hus, Corollary 10.2], the cohomotopy groups $\pi^1(X)$ and $\pi^1(Y)$ are trivial. As X and Y are both compact metrizable non-countable spaces, there exists a Banach space-isomorphism Φ from $C(X,\mathbb{R})$ to $C(Y,\mathbb{R})$, by Milutin's theorem. By Lemma (4.3.0.3), we may assume that Φ maps constant functions onto themselves. Now define $\psi : C(X,x_0,\mathbb{R}) \to C(Y,y_0,\mathbb{R})$ by $f \mapsto \Phi(f) - \Phi(f)(y_0)$. It is clear that ψ is a linear. If $g \in C(Y,y_0,\mathbb{R})$, then $h = \Phi^{-1}(g) - \Phi^{-1}(g)(x_0) \in C(X,x_0,\mathbb{R})$ and $\psi(h) = g$, hence ψ is a surjective. If $\psi(f) = 0$, then for all $y \in Y$, $\Phi(f)(y) = \Phi(f)(y_0)$, therefore $\Phi(f)$ is a constant function of Y and then $f = 0$. Hence ψ is an isomorphism. By Proposition (4.3.0.2), we have that $C^0(X,\mathbb{T}) \simeq C^0(Y,\mathbb{T})$, hence $C(X,\mathbb{T}) \simeq C(Y,\mathbb{T})$. □

Chapter 5

About the Induced Map θ_φ

Recall that any isomorphism φ between the unitary groups of two unital C^*- algebras induces a map θ_φ between their sets of projections.

In this chapter, we prove that this map θ_φ is an orthoisomorphism for some classes of C^*-algebras including the Cuntz algebras $\mathcal{O}_n$, $2 \leq n \leq \infty$, simple unital purely infinite C^*-algebras whose K_0-group is 2-divisible, and a large class of finite C^*-algebras of real rank zero (including AF-algebras) whose K_0-group is 2-divisible. In the last section, we also give examples which show that the induced map θ_φ need not be an orthoisomorphism.

Notation 5.0.0.1. *In this chapter, the quadruple (A, B, φ, θ) will represent the following: A and B are simple, unital C^*-algebras, φ is an isomorphism between their unitary groups, and $\theta = \theta_\varphi$ is the induced map between the sets of projections.*

5.1 C^*-Algebras With $\varphi-$Compatible Faithful Traces

Let (A, B, φ, θ) be as in (5.0.0.1). Suppose that τ_A and τ_B are traces on A and B respectively. They are $\varphi-$compatible if for all $u \in \mathcal{U}(A)$,

$$\tau_A(u) = \tau_B(\varphi(u)).$$

In this section, we show that if A and B have $\varphi-$compatible faithful traces, then θ_φ is a projection orthoisomorphism, moreover φ is implemented by a $*-$isomorphism between the algebras themselves.

Lemma 5.1.0.1. *Let A be a unital C^*-algebra, and φ be an automorphism of $\mathcal{U}(A)$. If τ is a $\varphi-$compatible trace on A, then*

$$\tau(\theta_\varphi(p)) = \tau(p) \text{ for all } p \in \mathcal{P}(A).$$

Proof: If $p \in \mathcal{P}(A)$, then $p = \frac{1-u}{2}$ for some self adjoint unitary u. Therefore

$$\theta_\varphi(p) = \frac{1-\varphi(u)}{2} \Longrightarrow \tau(\theta_\varphi(p)) = \frac{1-\tau(\varphi(u))}{2} = \tau(p).$$

□

Let A be a unital C^*-algebra and let φ be an automorphism of $\mathcal{U}(A)$. Recall from the definition of θ_φ that if φ is extendible to a $*-$isomorphism ψ of A, then so is θ_φ, hence θ_φ is an orthoisomorphism.

Theorem 5.1.0.2. *Let (A, B, φ, θ) be as in (5.0.0.1). If there exist faithful, normalized traces τ_A and τ_B on A and B respectively, such that*

$$\tau_A(u) = \tau_B(\varphi(u)) \text{ for every self-adjoint } u \in \mathcal{U}(A),$$

then θ is a projection orthoisomorphism.

Proof: Assume that p and q are non-trivial orthogonal projections in A. Let u, v be the unitaries defined by $u = 1 - 2p$ and $v = 1 - 2q$. If $x = \theta(p)\theta(q)$, then

$$\begin{aligned} xx^* &= \frac{1}{8}[(1-\varphi(u))(1-\varphi(v))(1-\varphi(u))] \\ &= \frac{1}{8}[2 - 2\varphi(u) - \varphi(v) + \varphi(uv) + \varphi(vu) - \varphi(uvu)]. \end{aligned}$$

Applying τ_B, we get:

$$\begin{aligned} \tau_B(xx^*) &= \frac{1}{8}[2 - 2\tau_A(u) - \tau_A(v) + \tau_A(uv) + \tau_A(vu) - \tau_A(uvu)] \\ &= \tau_A(pqp) \\ &= 0. \end{aligned}$$

As τ_B is faithful, $x = 0$. □

Theorem 5.1.0.3. *Let (A, B, φ, θ) be as in (5.0.0.1). If A and B have $\varphi-$compatible faithful traces τ_A and τ_B, then φ is implemented by a $*-$isomorphism between the algebras themselves.*

Proof: If $\sum_{i=1}^{n} \alpha_i u_i = 0$, then $\sum_{i,j}^{n} \alpha_i \overline{\alpha_j} u_i u_j^* = 0$, therefore

$$\begin{aligned} 0 &= \tau_A(\sum_{i,j}^{n} \alpha_i \overline{\alpha_j} u_i u_j^*) \\ &= \sum_{i,j}^{n} \alpha_i \overline{\alpha_j} \tau_A(u_i u_j^*) \\ &= \tau_B(\sum_{i,j}^{n} \alpha_i \overline{\alpha_j} \varphi(u_i)\varphi(u_j)^*) \\ &= \tau_B(\sum_{i}^{n} \alpha_i \varphi(u_i) \sum_{j}^{n} \overline{\alpha_j} \varphi(u_j)^*). \end{aligned}$$

As τ_B is a faithful trace, $\sum_{i}^{n} \alpha_i \varphi(u_i) = 0$. As every element of A can be written as a finite linear combinations of $\mathcal{U}(A)$, then let $\varphi' : A \longrightarrow A$ denote the map defined by

$$\varphi'(\sum_{i=1}^{n} \alpha_i u_i) = \sum_{i=1}^{n} \alpha_i \varphi(u_i).$$

Now we show that φ' is a $*-$isomorphism. (i) It is clear that $\varphi'(x^*) = \varphi'(x)^*$, and φ' is a linear map. Also if $x = \sum_{i=1}^{n} \alpha_i u_i$ and $y = \sum_{j=1}^{m} \gamma_j v_j$, where for every i and j u_i, v_j are unitaries in A, then

$$\begin{aligned} \varphi'(xy) &= \varphi'(\sum_{i,j=1}^{n,m} \alpha_i \gamma_j u_i v_j) \\ &= \sum_{i,j=1}^{n,m} \alpha_i \gamma_j \varphi(u_i)\varphi(v_j) \\ &= \varphi'(x)\varphi'(y) \end{aligned}$$

hence the map φ' is a $*-$homomorphism.

(ii) φ' is bijective: If $y = \sum_{i=1}^{n} \alpha_i u_i \in A$, then there exist v_i, $i = 1, 2 \ldots, n$, such that $\varphi(v_i) = u_i$. Let $x = \sum_{i=1}^{n} \alpha_i v_i$ therefore $\varphi'(x) = y$, hence it is surjective.

Now let $x = \sum_{i=1}^{n} \alpha_i u_i$. If $\varphi'(x) = 0$, then

$$
\begin{aligned}
0 &= \tau_B(\sum_{i,j}^{n} \alpha_i\overline{\alpha_j}\varphi(u_i)\varphi(u_j)^*) \\
&= \sum_{i,j}^{n} \alpha_i\overline{\alpha_j}\tau_A(u_iu_j) \\
&= \tau_A(\sum_{i}^{n} \alpha_iu_i \sum_{j}^{n} \overline{\alpha_j}u_j^*) \\
&= \tau_A(xx^*).
\end{aligned}
$$

As τ_A is a faithful trace, $x = 0$. Therefore φ' is injective, and hence the theorem is proved. □

Remark 5.1.0.4. *Let A be a simple, unital C^*-algebra with a non-trivial character λ and an isomorphism φ on $\mathcal{U}(A)$ such that*

$$\tau(u) = \lambda(u)\tau(\varphi(u)), \textit{ for all } u \in \mathcal{U}(A),$$

where τ is a normalized faithful trace on A. Then a proof similar to Theorem (5.1.0.3) shows that φ is implemented by a $*-$isomorphism on A up to the character λ. However, it is not clear that such a C^*-algebra and isomorphism exist.

5.2 On the Cuntz Algebras $\mathcal{O}_n$

In §4.2 we proved that, if φ is an isomorphism between the unitary groups of two Cuntz algebras $\mathcal{O}_n$ and $\mathcal{O}_m$, then $n = m$.

Let φ be an automorphism of $\mathcal{U}(\mathcal{O}_n)$, where $2 \leq n \leq \infty$. In this section we prove that the induced map θ_φ is an orthoisomorphism of $\mathcal{P}(\mathcal{O}_n)$.

Recall that the Cuntz algebra $\mathcal{O}_n$ is the universal C^*-algebra generated by isometries $s_1, s_2, \ldots, s_n$, such that $\sum_{i=1}^{n} s_is_i^* = 1$, and the Cuntz algebra $\mathcal{O}_\infty$ is the universal C^*-algebra generated by an infinite sequence of isometries with mutually orthogonal range projections $s_js_j^*$. We will denote by p_j the standard projections $s_js_j^*$.

Proposition 5.2.0.1. *In $K_0(\mathcal{O}_n)$, we have $[1] = n[\theta_\varphi(p_1)]$.*

Proof: From Lemma (2.6.3.10) we have $p_i \sim_u p_j$, therefore $\theta_\varphi(p_i) \sim_u \theta_\varphi(p_j)$, hence they have the same K_0-class, on the other hand $\{\theta_\varphi(p_i)\,;\, i = 1, 2, \ldots, n\}$ forms a partition of the unity, so the proposition is proved. □

5.2.1 General Results on the $c-$Map

Let (A, B, φ, θ) be as in (5.0.0.1). Recall that to any non-trivial projection $p \in \mathcal{P}(A)$, Booth associated ([Bo] or see §3.2, 3.2.0.3) an automorphism c_p of $\mathbb{S}^1$.

Lemma 5.2.1.1. *Let (A, B, φ, θ) be as in (5.0.0.1). If for any non-trivial projections p and q in A, $c_p = c_q$, then θ is an orthoisomorphism.*

Proof: Let p and q be two orthogonal projections of A. Then $\theta(p)\theta(q) = 0$ follows from the definition of θ if $p + q = 1$, and from Proposition (3.2.0.5)(2) if $p + q < 1$. □

The converse holds if A satisfies the following condition:

$$\{\forall\, p, q \in \mathcal{P}(A)\backslash\{0,1\}\,, \exists\, p_1 \leq p \text{ and } q_1 \leq q\,;\, p_1, q_1 \neq 0, \text{ such that } p_1 \sim_u q_1\,\} \qquad (\ddagger)$$

Lemma 5.2.1.2. *Let (A, B, φ, θ) be as in (5.0.0.1). If A satisfies the condition $(\ddagger)$ and if θ is an orthoisomorphism, then for any two non-trivial projections p and q in A, $c_p = c_q$.*

Proof: If $p, q \in \mathcal{P}(A)\backslash\{0,1\}$, then by $(\ddagger)$ there exist nonzero projections p_1 and q_1 such that $p_1 \leq p$, $q_1 \leq q$ and $p_1 \sim_u q_1$, and hence $c_{p_1} = c_{q_1}$. As $p = p_1 + (p - p_1)$ and $q = q_1 + (q - q_1)$, by Proposition (3.2.0.5)(2)

$$\theta(p_1)\theta(p - p_1) = 0 \Longleftrightarrow c_p = c_{p_1} = c_{p-p_1}$$
$$\theta(q_1)\theta(q - q_1) = 0 \Longleftrightarrow c_q = c_{q_1} = c_{q-q_1}$$

and therefore $c_p = c_q$. □

Proposition 5.2.1.3. *Any purely infinite unital C^*-algebra satisfies the condition $(\ddagger)$.*

Proof: Let A be a purely infinite unital C^*-algebra. If $p, q \in \mathcal{P}(A)\backslash\{0,1\}$, then there exists $p_1 < p$ such that $p_1 \sim q$, and there exists $q_1 < q$ such that $q_1 \sim p_1$. Now p_1 and q_1 are two infinite with infinite complement projections of A, indeed they have the same K_0-classes, therefore $q_1 \sim_u p_1$. □

Proposition 5.2.1.4. *[Bo, 3.2.3.10] Any simple unital oddly decomposable C^*-algebra satisfies the condition (‡).*

Remark 5.2.1.5. *Many simple, unital C^*-algebras satisfy condition (‡). In particular, this applies if A is a simple, unital, stably finite C^*-algebra of real rank zero and $K_0(A)$ is a weakly unperforated interpolation group with cancellation.*

For simple, unital C^*-algebras, we prove the following two results:

Lemma 5.2.1.6. *Let (A, B, φ, θ) be as in (5.0.0.1). If p and q are two non-trivial orthogonal projections of A such that $p \sim_u q$ and $p + q < 1$, then either $\theta(p)\theta(q) = 0$ or $\theta(p)\theta(q) = \theta(1-p)\Delta\theta(q)$.*

Proof: Assume that $p + q = r < 1$. Then p and q are unitarily equivalent projections, whence $c_p = c_q$. By Proposition (3.2.0.5),

$$\begin{aligned} \text{either } c_p = c_q = c_r; &\quad \text{this occurs iff } \theta(p)\theta(q) = 0 \\ \text{or } c_p = c_q = \overline{c_r}; &\quad \text{this occurs iff } \theta(1-p)\theta(1-q) = 0. \end{aligned}$$

It is easy to check that

$$\theta(1-p)\theta(1-q) = 0 \Longleftrightarrow \theta(p)\theta(q) = \theta(1-p)\Delta\theta(q).$$

□

Now let us prove the following lemma (see [Dy, Lemma 12]):

Lemma 5.2.1.7. *Let (A, B, φ, θ) be as in (5.0.0.1). If $p_1, p_2, \ldots, p_{2n+1}$ are unitarily equivalent orthogonal projections of A that form a partition of unity, then there exist $i \neq j$ such that $\theta(p_i)\theta(p_j) = 0$. Moreover, $\theta(p_r)\theta(p_s) = 0$ for all $r \neq s$.*

Proof: By Lemma (5.2.1.6), we may assume that

$$\theta(p_i)\theta(p_j) = \theta(1-p_i)\Delta\theta(p_j), \text{ for all } i \neq j.$$

Now, $1 - p_1 = p_2\Delta p_3\Delta\dots\Delta p_n$ therefore, $1 - \theta(p_1) = \theta(p_2)\Delta\theta(p_3)\Delta\dots\Delta\theta(p_n)$, multiply both sides by $\theta(p_1)$ to get,

$$\begin{aligned} 0 &= \theta(p_1)\theta(p_2)\Delta\theta(p_1)\theta(p_3)\Delta\dots\Delta\theta(p_1)\theta(p_n) \\ &= [\theta(1-p_1)\Delta\theta(p_2)]\Delta[\theta(1-p_1)\Delta\theta(p_3)]\Delta\dots\Delta[\theta(1-p_1)\Delta\theta(p_n)]. \end{aligned}$$

Note that,

$$[\theta(1-p_1)\Delta\theta(p_i)]\Delta[\theta(1-p_1)\Delta\theta(p_j)] = \theta(p_i)\Delta\theta(p_j);\ \forall\, i \neq j.$$

Since n is odd, we have that $0 = \theta(p_2)\Delta\theta(p_3)\Delta\dots\Delta\theta(p_n)$, therefore $\theta(p_1) = 1$, which is a contradiction since $p_1 \neq 1$ and θ preserves 1. Therefore for some $i \neq j$, $\theta(p_i)\theta(p_j) = 0$. By Proposition (3.2.0.5)(2) and Lemma (3.2.0.12), $c_{p_i} = c_{p_j} = c_{p_i+p_j}$. As all the projections p_r's are equivalent and for all $r \neq s$, $p_r + p_s$ is also equivalent to $p_i + p_j$, it follows $\theta(p_r)\theta(p_s) = 0$. □

Let A be a unital C^*-algebra which has a system of matrix units $\{e_{i,j}\}_{i,j=1}^n$. By Proposition (2.2.4.4), $\{e_{i,i},\ i = 1,\dots,n\}$ forms a partition of unity consisting of orthogonal unitarily equivalent projections. Then by Proposition (3.2.0.4)(i), all the $c_{e_{i,i}}$ $(i = 1, 2, \dots, n)$ are equal, and we call this automorphism c_1 $(= c_{e_{1,1}} = c_{e_{2,2}} = \dots = c_{e_{n,n}})$. Also note that $[e_{i,i} + e_{j,j}] = [e_{s,s} + e_{r,r}]$ for all $i \neq j$ and $s \neq r$. Then $e_{i,i} + e_{j,j} \sim_u e_{s,s} + e_{r,r}$, by c_2 we mean $c_{e_{i,i}+e_{j,j}}$ for all $i \neq j$, and we continue to define

$$c_k = c_{e_{1,1}+e_{2,2}+\dots+e_{k,k}} \text{ for every } k \leq n-1.$$

5.2.2 Results for $\mathcal{O}_n$, $n < \infty$

Let φ be an automorphism of the unitary group of the Cuntz algebra $\mathcal{O}_n$, and let $\theta : \mathcal{P}(\mathcal{O}_n) \to \mathcal{P}(\mathcal{O}_n)$ be the associated bijection. In this subsection, we will denote this data $(\mathcal{O}_n, \varphi, \theta)$. The standard projection $s_j s_j^* = p_j$ is denoted by p_j.

Lemma 5.2.2.1. *Let $(\mathcal{O}_n, \varphi, \theta)$ be as above. Then for all $1 \leq r \neq s \leq n$, we have that $\theta(p_r)\theta(p_s) = 0$.*

Proof: As the standard projections of $\mathcal{O}_n$ are unitarily equivalent and form a partition of unity, i.e., $1 = p_1 + p_2 + \cdots + p_n$, and $\{p_i\}$ are pairwise orthogonal unitarily equivalent. If n is odd, then the result follows directly from (5.2.1.7).
If n is even, then by Lemma (2.6.3.12), p_1 can be written as a sum $p_1 = p_{11} + p_{12} + \cdots + p_{1n}$ of n non-trivial projections equivalent to p_1. For all $i = 1, 2, \ldots, n$, as $p_{1i} \sim_u p_1$, we have that

$$1 = p_{11} + p_{12} + \cdots + p_{1n} + p_2 + p_3 + \cdots + p_n,$$

is a sum of an odd number of unitarily equivalent orthogonal projections. By Lemma (5.2.1.7), for all $r \neq s$, it follows $\theta(p_r)\theta(p_s) = 0$. □

Theorem 5.2.2.2. *Let $(\mathcal{O}_n, \varphi, \theta)$ be as above. Then the map θ is an orthoisomorphism.*

Proof: Recall that any non-trivial projection of $\mathcal{O}_n$ is unitarily equivalent to a sum $p_1 + \cdots + p_k$ of standard projections for some $1 \leq k \leq n-1$. By (5.2.1.1), the proof of the theorem will be completed if we show that $c_1 = c_k$, for all $1 \leq k \leq n-1$.

From the previous theorem, we have that $c_1 = c_2$. As for every $k \leq n-1$,

$$\begin{aligned} \theta(p_1 + p_2 + \cdots p_{k-1})\theta(p_k) &= [\theta(p_1)\Delta\theta(p_2)\Delta \ldots \Delta\theta(p_{k-1})]\theta(p_k) \\ &= \theta(p_1)\theta(p_k)\Delta\theta(p_2)\theta(p_k)\Delta \ldots \Delta\theta(p_{k-1})\theta(p_k) \\ &= 0, \end{aligned}$$

by Proposition (3.2.0.5) we have $c_1 = c_{k-1} = c_k$, hence the theorem follows. □

5.2.3 The $\mathcal{O}_\infty$ Case

In this subsection, we show that if φ is an automorphism of $\mathcal{U}(\mathcal{O}_\infty)$, then θ is an orthoisomorphism of $\mathcal{P}(\mathcal{O}_n)$. We recall the following:

Theorem 5.2.3.1. *[Cu1, 3.12] In $\mathcal{O}_\infty$, every projection is equivalent to a projection either of the form $\sum_{i=1}^{k} s_i s_i^*$ $(1 \leq k < \infty)$ or $1 - \sum_{i=1}^{k} s_i s_i^*$ $(1 \leq k < \infty)$.*

Proposition 5.2.3.2. *[RS, 4.2.3] Let A be a C^*-algebra and let p be a projection in A.*
(i) There is a unital embedding $\iota : \mathcal{O}_\infty \to pAp$ if and only if p is properly infinite.
(ii) If p is a full, properly infinite projection and $[p] = 0$ in $K_0(A)$, then there is a unital embedding $\iota : \mathcal{O}_2 \to pAp$.

As $\mathcal{O}_\infty$ is a purely infinite and simple C^*-algebra, it is oddly decomposable. If φ is an automorphism of $\mathcal{U}(\mathcal{O}_\infty)$, then by Booth's result, the image of the map c contains at most two elements. By Proposition (3.2.0.11), there exists a partition $\{\mathcal{P}_e, \mathcal{P}_o\}$ be the partition of $\mathcal{P}(\mathcal{O}_\infty)\backslash\{0,1\}$, such that the induced map θ_φ preserves orthogonality on $\mathcal{P}_e$, and flips orthogonality on $\mathcal{P}_o$.

Lemma 5.2.3.3. *Let $(\mathcal{O}_\infty, \varphi, \theta)$ be given and $\mathcal{P}_e$, $\mathcal{P}_o$ be as above. Then for all $1 \le i < \infty$, the standard projection p_i belongs to $\mathcal{P}_e$, and hence, so does every finite sum of p_i's.*

Proof: If $q = 1 - p_1$. Then q is a full, properly infinite projection, with $[q] = 0$ in $K_0(\mathcal{O}_\infty)$, therefore by Proposition (5.2.3.2)(ii) there exists a unital embedding $\iota : \mathcal{O}_2 \to q(\mathcal{O}_\infty)q$. And therefore, q can be written as a sum of two orthogonal unitarily equivalent projections, q_1 and q_2, so $c_{q_1} = c_{q_2}$, which means that either both of q_1 and q_2 are in $\mathcal{P}_e$ or in $\mathcal{P}_o$, and in both cases (Lemma (3.2.0.12)), q belongs to $\mathcal{P}_e$. As $c_{1-p_1} = c_{p_1}$, p_1 belongs to $\mathcal{P}_e$, hence so does p_i for every i, and again by Lemma (3.2.0.12), any finite sum of such projections is in $\mathcal{P}_e$. □

Theorem 5.2.3.4. *For $(\mathcal{O}_\infty, \varphi, \theta)$, the map θ is an orthoisomorphism.*

Proof: By Proposition (5.2.1.1), it is enough to show that $q \in \mathcal{P}_e$, for every non-trivial projection q in $\mathcal{O}_\infty$. Let q be a non-trivial projection of $\mathcal{O}_\infty$. As by Theorem (5.2.3.1), either

$$q \sim_u \sum_{i=1}^{k} p_i\,, \quad \text{or} \quad q \sim_u 1 - \sum_{i=1}^{k} p_i\,,$$

for every $1 \le k < \infty$, we then get in both cases that $c_q = c_{(\sum_{i=1}^{k} p_i)}$, and by Lemma (5.2.3.3), $q \in \mathcal{P}_e$, and the theorem follows. □

5.3 Purely Infinite C^*-Algebras with 2-Divisible K_0-Group

Let A be a simple, unital purely infinite C^*-algebra. Then by Theorem (4.2.0.3), A is oddly decomposable. If φ is an automorphism of $\mathcal{U}(A)$, then let $\{\mathcal{P}_e, \mathcal{P}_o\}$ be the partition of $\mathcal{P}(A)\backslash\{0,1\}$, such that the induced map θ_φ preserves orthogonality on $\mathcal{P}_e$, and flips orthogonality on $\mathcal{P}_o$. In this section, we study the case of A where $K_0(A)$ is a 2-divisible group, and then we prove the following:

Theorem 5.3.0.1. *Let (A, B, φ, θ) be as in (5.0.0.1). If A is a purely infinite C^*-algebra whose K_0-group is 2-divisible, then θ is an orthoisomorphism.*

Proof: Let p be a non-trivial projection in A. By Lemma (5.2.1.1), it is enough to show that $p \in \mathcal{P}_e$. As $K_0(A)$ is 2-divisible, $[p] = 2[q]$, where q is a non-trivial projection in A. As q and $1 - q$ are infinite projections, there exists an infinite projection q' in A such that $q \sim q'$ and $q' < 1 - q$. Therefore $[p] = [q + q']$. The projections p and $q + q'$ are infinite and having infinite complement projections, then by Theorem (2.4.0.7), $p \sim_u q + q'$ and hence $c_p = c_{q+q'}$. As $c_q = c_{q'}$, either q and q' belong to $\mathcal{P}_e$ or to $\mathcal{P}_o$, and in both cases we have that their sum $q + q'$ belongs to $\mathcal{P}_e$, hence $p \in \mathcal{P}_e$, and then the theorem follows. □

5.4 Finite C^*-Algebras with 2-Divisible K_0-Group

As in chapter 4, let $\mathcal{F}$ denote the class of all simple, unital, stably finite C^*-algebras A of real rank zero which have cancellation, and whose $K_0(A)$ is a weakly unperforated interpolation group.

A C^*-algebra belonging to $\mathcal{F}$ is oddly decomposable by Theorem (4.1.1.4). As in §5.3, If φ is an automorphism of $\mathcal{U}(A)$, let $\{\mathcal{P}_e, \mathcal{P}_o\}$ denote the partition of $\mathcal{P}(A)\backslash\{0,1\}$, such that the induced map θ_φ preserves orthogonality on $\mathcal{P}_e$, and flips orthogonality on $\mathcal{P}_o$.

Theorem 5.4.0.1. *Let (A, B, φ, θ) be as in (5.0.0.1). If A belongs to $\mathcal{F}$, and $K_0(A)$ is a 2-divisible group, then θ is an orthoisomorphism.*

Proof: Let p be a non-trivial projection of A. As $K_0(A)$ is 2-divisible, there exists an element a in the scale of $K_0(A)$, with $[p] = 2a$, and therefore a non-trivial projection q of A, with $[p] = 2[q]$. As $[q] < [1 - q]$, and by [B, 6.9.2], there exists a projection $q' \leq 1 - q$ and $q \sim q'$. Then as $[p] = [q + q']$, $p \sim_u q + q'$ and $c_p = c_{q+q'}$. As $c_q = c_{q'}$, either q and q' belong to $\mathcal{P}_e$ or to $\mathcal{P}_o$, and in both cases we have that their sum $q + q'$ belongs to $\mathcal{P}_e$, hence $p \in \mathcal{P}_e$, and then the theorem follows. □

5.5 The UHF-Algebras

Let $(p_j)_{j \geq 1}$ be the sequence of prime numbers. If A is a UHF-algebra, we denote by $\bar{n} = (p_j^{n_j})_{j \geq 1}$ its generalized integer. We have the following result:

Theorem 5.5.0.1. *Let A be a UHF-algebra with generalized integer $\bar{n}$. If the exponent n_1 of $p_1 = 2$ is either zero or is infinite in $\bar{n}$ and if φ is an automorphism of $\mathcal{U}(A)$, then the induced map θ_φ is an orthoisomorphism of $\mathcal{P}(A)$.*

Proof: By Theorem (5.4.0.1), we can assume that $n_1 = 0$. For any sequence $(m_j)_{j \geq 1}$ of integers with $0 \leq m_j \leq n_j$ and $m_j = 0$ for all but finitely many j, let $n = \prod_{j \geq 1} p_j^{m_j}$. By Lemma (5.2.1.7), for any partition $\{q_i,\ i = 1, \ldots, n\}$ of the unity consisting of orthogonal equivalent projections, we have $\theta_\varphi(q_i)\theta_\varphi(q_j) = 0,\ i \neq j$. If $\{\mathcal{P}_e, \mathcal{P}_o\}$ denotes the partition of $\mathcal{P}(A) \backslash \{0, 1\}$ induced by θ_φ, (i.e. θ_φ preserves orthogonality on $\mathcal{P}_e$, and flips orthogonality on $\mathcal{P}_o$), then for all $1 \leq i \leq n$, $q_i \in \mathcal{P}_e$, by Lemma (3.2.0.12).

For any $q \in \mathcal{P}(A) \backslash \{0, 1\}$, there exists a number n as above, such that $\tau(q) = \frac{k}{n}$, $0 < k < n$. As q is unitarily equivalent to a sum of $k-$orthogonal equivalent projections $(q_i)_{i=1}^k$, with $\tau(q_i) = \frac{1}{n}$, then $q \in \mathcal{P}_e$, hence the theorem is checked. □

5.6 A θ_φ which is not an Orthoisomorphism

In this section, we construct examples of automorphisms φ of the unitary groups of some UHF-algebras A, whose induced map θ_φ are not orthoisomorphisms of the projections of A.

5.6.1 Some Useful Tools

In [HS1, p.245] de la Harpe and Skandalis defined a determinant (H-S determinant), for C^*-algebras. Let E be a Banach space, and A be a C^*-algebra which has a continuous trace $\tau : A \to E$. They defined Δ_τ as follows

$$\Delta_\tau : GL^0_\infty(A) \longrightarrow E/\tau(K_0(A)),$$

if $x = \exp(y)$, then $\Delta_\tau(x) = \frac{1}{2\pi i}\tau(y)$.

Proposition 5.6.1.1. *[HS1, Pro.2] (i) Δ_τ is a homomorphism,*
(ii) If $A = \mathbb{M}_n(\mathbb{C})$, and τ is the usual trace, then the restriction to $GL^0_1(A) = GL(n, \mathbb{C})$ of $\exp(2\pi i \Delta_\tau)$ is the usual determinant.

Let A be a UHF-algebra and τ be its unique normalized trace. In [HS3, p.194], de la Harpe and Skandalis showed that

$$\mathcal{U}(A)/D\mathcal{U}(A) \simeq \mathbb{R}/\tau(K_0(A)),$$

i.e. $D\mathcal{U}(A)$ is equal to the $\ker(\Delta_\tau)$.

Let K denote the normal subgroup of $\mathcal{U}(A)$ generated by the self-adjoint unitaries of A. Define $\pi : \mathbb{R}/\mathbb{Z} \times K \to \mathcal{U}(A)$ by $\pi(\theta, k) = \exp(2\pi i\theta)k$. If $u \in \mathcal{U}(A)$ and $\theta \in \Delta_\tau(u)$, then $\Delta_\tau(\exp(-2\pi i\theta)u) = 0$ and using de la Harpe and Skandalis results (see [HS3, p.194] and [HS2, Th.1]), we obtain $\exp(-2\pi i\theta)u \in K$, moreover $\pi(\theta, \exp(-2\pi i\theta)u) = u$, so π is a surjective homomorphism, therefore we have that

$$\mathcal{U}(A) \simeq (\mathbb{R}/\mathbb{Z} \times K)/\ker(\pi).$$

For all $n \in \mathbb{Z}^+$, let K_n be the normal subgroup of $\mathcal{U}_n = \mathcal{U}(\mathbb{M}_n(\mathbb{C}))$ generated by the self-adjoint unitaries. Let us denote by Δ_n (resp. $\tilde{\Delta}_n$) the H-S determinant associated to the usual (resp. normalized) trace on $\mathbb{M}_n(\mathbb{C})$.

By Proposition (5.6.1.1), the usual determinant on $\mathcal{U}_n$ is given by

$$\det(u) = \exp(2\pi i \Delta_n(u)) = \exp(2\pi i n \tilde{\Delta}_n(u)).$$

In [Dy, p.87] Dye showed that

$$\begin{aligned} K_n &= \{u \in \mathcal{U}_n | \ \det(u) = \pm 1\} \\ &= \left\{u \in \mathcal{U}_n | \ \Delta_n(u) \in \frac{1}{2}\mathbb{Z}\right\} \\ &= \left\{u \in \mathcal{U}_n | \ \tilde{\Delta}_n(u) \in \frac{1}{2n}\mathbb{Z}\right\}. \end{aligned}$$

Now for every $n \in \mathbb{Z}^+$, let $\pi_n : \mathbb{R}/\mathbb{Z} \times K_n \longrightarrow \mathcal{U}_n$ denote the map $\pi(\theta, k) = \exp(2\pi i \theta)k$. Then

$$\ker(\pi_n) = \left\{(\theta, \exp(-2\pi i \theta)1_n) | \ \theta \in (\frac{1}{2n})\mathbb{Z}\right\},$$

and hence

$$\mathcal{U}_n \simeq (\mathbb{R}/\mathbb{Z} \times K_n)/\ker(\pi_n).$$

Following Dye, we define the automorphism $V_n : K_n \to K_n$ by $V_n(k) = \det(k)k$.

5.6.2 The Construction

For all $(n \geq 2)$, Dye constructed in [Dy, p.87] an automorphism φ of $\mathcal{U}_{2n}$ such that for every projection $p \in \mathbb{M}_{2n}(\mathbb{C})$, of odd rank, $\operatorname{rank}(\theta_\varphi(p)) = 2n - \operatorname{rank}(p)$. A detailed discussion of Dye's example can be found [Bo, App. A].

If A is a UHF-algebra whose generalized integer $\bar{n}$ is of the form $(2^{n_2}, 3^{n_3}, 5^{n_5}, \ldots)$, with $1 \leq n_2 < \infty$ and $0 \leq n_p \leq \infty$, for every prime $p > 2$. We generalize Booth's description to construct an automorphism φ of $\mathcal{U}(A)$ for which θ_φ is not an orthoisomorphism of $\mathcal{P}(A)$.

Before constructing φ, let us first recall that if we consider $\mathbb{R}$ as a $\mathbb{Q}-$vector space and if H is a basis, containing 1, of $\mathbb{R}$, then

$$\mathbb{R} \simeq \mathbb{Q} \oplus \left(\bigoplus_{h \in H, h \neq 1} \mathbb{Q} \right),$$

and

$$\mathbb{R}/\mathbb{Z} \simeq \mathbb{Q}/\mathbb{Z} \oplus \left(\bigoplus_{h \in H,\, h \neq 1} \mathbb{Q} \right).$$

Denoting by $\mathbb{Z}(p^\infty) = \mathbb{Z}[\frac{1}{p}]/\mathbb{Z} = \left\{ [\frac{a}{p^n}];\ a \in \mathbb{Z},\ n \in \mathbb{Z}^+ \right\}$, we have

$$\mathbb{Q}/\mathbb{Z} \simeq \bigoplus_{p \text{ prime}} \mathbb{Z}(p^\infty),$$

hence

$$\mathbb{R}/\mathbb{Z} \simeq \mathbb{Z}(2^\infty) \oplus \left(\bigoplus_{p \text{ prime} \neq 2} \mathbb{Z}(p^\infty) \right) \oplus \left(\bigoplus_{h \in H,\, h \neq 1} \mathbb{Q} \right). \tag{6}$$

Then we have

Lemma 5.6.2.1. *Let m be an integer ≥ 1. Then there exists an automorphism χ of $\mathbb{R}/\mathbb{Z}$ such that for each odd integer q, we have*

$$\chi\left(\left[\frac{1}{2^{m+1}q} \right] \right) = (2^m q + 1) \left[\frac{1}{2^{m+1}q} \right].$$

Proof: Let ξ be the map of $\mathbb{Z}(2^\infty)$ defined by $\xi([\frac{1}{2^n}]) = \left[\frac{2^m+1}{2^n}\right]$, for all $n \in \mathbb{Z}^+$. It is easy to check that ξ is an automorphism. Then using (6), we define an automorphism χ of $\mathbb{R}/\mathbb{Z}$ by $\chi = \xi \oplus (\bigoplus \mathrm{id}) \oplus (\bigoplus \mathrm{id})$.

As $\gcd(2^{m+1}, q) = 1$, there exist $k,\ l \in \mathbb{Z}$ (with l odd) such that $1 = 2^{m+1}k + ql$, hence

$$\left[\frac{1}{2^{m+1}q} \right] = \left[\frac{l}{2^{m+1}} \right] + \left[\frac{k}{q} \right],$$

with $\left[\frac{l}{2^{m+1}}\right] \in \mathbb{Z}(2^\infty)$ and $\left[\frac{k}{q}\right] \in \bigoplus_{p \neq 2} \mathbb{Z}(p^\infty)$. As

$$\begin{aligned} \chi\left(\left[\frac{1}{2^{m+1}q} \right] \right) = \left[\frac{k}{q} \right] + \left[\frac{(2^m+1)l}{2^{m+1}} \right] &= \left[\frac{2^{m+1}k + (2^m+1)lq}{2^{m+1}q} \right] \\ &= \left[\frac{1 + 2^m lq}{2^{m+1}q} \right] = \left[\frac{1 + 2^m q}{2^{m+1}q} \right], \end{aligned}$$

the lemma is checked. □

Let A be a UHF-algebra whose generalized integer $\bar{n}$ is of the form $(2^{n_2}, 3^{n_3}, 5^{n_5}, \dots)$ with $1 \leq n_2 < \infty$ and $0 \leq n_p \leq \infty$, for every prime $p > 2$. There exists an increasing sequence $(A_n)_{n \geq 1}$ of finite dimensional full matrix algebras with $A = \overline{\cup_{n \geq 1} A_n}$ and

$$A_n = \mathbb{M}_{2^{n_2}} \otimes \mathbb{M}_{p_1} \otimes \cdots \otimes \mathbb{M}_{p_n}$$

$$\simeq \mathbb{M}_{2^{n_2} p_1 \dots p_n},$$

where the p_i's are odd prime numbers. Let $s_0 = 2^{n_2}$ and for every $n \in \mathbb{Z}^+$, denote $2^{n_2} p_1 \dots p_n$ by s_n. Then

$$\mathbb{M}_{s_0} \xrightarrow{i_1} \mathbb{M}_{s_1} \xrightarrow{i_2} \mathbb{M}_{s_2} \longrightarrow \cdots \longrightarrow \mathbb{M}_{s_n} \longrightarrow \cdots$$

and

$$K_0(A) \simeq \{\frac{p}{s_n}, \text{ where } p \in \mathbb{Z} \text{ and } n \in \mathbb{Z}^+\}.$$

Lemma 5.6.2.2. *Let A be a UHF-algebra as above. Then*

$$K = \left\{ u \in \mathcal{U}(A) |\ \Delta_\tau(u) \in \frac{1}{2}\tau(K_0(A)) \right\}.$$

Proof: Let $v = 1 - 2p$ be an involution of A. Therefore $v = 1 - 2p$, for some projection p, moreover

$$\exp(\pi i p) = \exp(\pi i)p + (1 - p) = v.$$

Now apply Δ_τ to v, we get $\Delta_\tau(v) = \Delta_\tau(\exp(\pi i p)) = (1/2)\tau(p)$, and hence $\Delta_\tau(v) \in \frac{1}{2}\tau(K_0(A))$.
Conversely, we can choose an involution v_0 in A such that $\Delta_\tau(v_0) \in \frac{1}{2}\tau(K_0(A))$, fix such v_0. If $u \in \mathcal{U}(A)$ such that $\Delta_\tau(u) \in \frac{1}{2}\tau(K_0(A))$, then

$$\Delta_\tau(uv_0) = \Delta_\tau(u) + \Delta_\tau(v_0) = 0,$$

mod $\tau(K_0(A))$, using de la Harpe and Skandalis results (see [HS3, p.194] and [HS2, Th.1]), we get that $uv_0 \in K$ and hence $u = (uv_0)v_0 \in K$. □

Consequently, we have:

$$K = \left\{ u \in \mathcal{U}(A) |\ \Delta_\tau(u) = \frac{p}{2s_n}, \text{ for some } p \in \mathbb{Z}, \text{ and some } n \in \mathbb{Z}^+ \right\}.$$

Also recall the surjective homomorphism $\pi : \mathbb{R}/\mathbb{Z} \times K \to \mathcal{U}(A)$. If $(\theta, k) \in \mathbb{R}/\mathbb{Z} \times K$ such that $\exp(2\pi i\theta)k = 1$, then $k = \exp(-2\pi i\theta).1$, and

$$\Delta_\tau(\exp(-2\pi i\theta).1) = \frac{1}{2\pi i}\tau(-2\pi i\theta.1) = -\theta,$$

therefore $\theta = \frac{p}{2s_n}$, for some $p \in \mathbb{Z}$ and $n \geq 0$, and hence

$$\ker(\pi) = \left\{(\theta, \exp(-2\pi i\theta).1) |\ \theta = \frac{p}{2s_n} \text{ for some } p \in \mathbb{Z} \text{ and some } n \in \mathbb{Z}^+\right\}.$$

Now we define an automorphism V on K as follows: if $k \in K$, then for some $p \in \mathbb{Z}$ and for some positive integer n we have that $\Delta_\tau(k) = \frac{p}{2s_n}$, then we define the image of k to be

$$\begin{aligned} V(k) &:= (\exp(2\pi i s_n \Delta_\tau(k))).k \\ &= \exp(\pi i p).k \\ &= \begin{cases} k; & \text{if p is even} \\ -k; & \text{if p is odd} \end{cases} \end{aligned}$$

As s_n are odd numbers, the map V is well-defined.

Now let φ_{s_n} and φ be the automorphisms on $\mathbb{R}/\mathbb{Z} \times K_{s_n}$ and $\mathbb{R}/\mathbb{Z} \times K$ defined by $\varphi_{s_n} = (\chi, V_{s_n})$ and $\varphi = (\chi, V)$. If $\theta = \left[\frac{1}{2s_n}\right] = \frac{1}{2s_n} + \mathbb{Z}$, then

$$\begin{aligned} \varphi_{s_n}(\theta, \exp(-2\pi i\theta)1_{s_n}) &= (\chi(\theta), V_{s_n}(\exp(-2\pi i\theta)1_{s_n})) \\ &= (\chi(\theta), \det(\exp(-2\pi i\theta)1_{s_n}).\exp(-2\pi i\theta)1_{s_n}) \\ &= (\chi(\theta), \exp(-2\pi i\theta s_n).\exp(-2\pi i\theta)1_{s_n}) \\ &= (\chi(\theta), \exp(-2\pi i\theta(s_n+1))1_{s_n}). \end{aligned}$$

As $\chi(\theta) = \left[\frac{s_n+1}{2s_n}\right]$, we proved that $\varphi_{s_n}(\ker(\pi_{s_n})) \subseteq \ker(\pi_{s_n})$. Therefore, we define the automorphisms

$$\varphi_{s_n} : (\mathbb{R}/\mathbb{Z} \times K_{s_n})/\ker(\pi_{s_n}) \longrightarrow (\mathbb{R}/\mathbb{Z} \times K_{s_n})/\ker(\pi_{s_n})$$

as follows, if $u = [(\theta, k)]$, then $\varphi_{s_n}(u) := [(\chi(\theta), V_{s_n}(k))]$.

Also, if $\theta = \left[\frac{p}{2s_n}\right]$ for some $p \in \mathbb{Z}$ and $n \in \mathbb{Z}^+$, then $\Delta_\tau(\exp(-2\pi i\theta).1) = \frac{-p}{2s_n}$. As $\theta + \frac{p}{2} = (1+s_n)\theta$,

$$V(\exp(-2\pi i\theta).1) = \exp(-2\pi i(\theta + \frac{p}{2})).1 = \exp(-2\pi i(1+s_n)\theta).1$$

and hence we proved that $\varphi(\ker(\pi)) \subseteq \ker(\pi)$. Then we define the automorphism φ of $\mathcal{U}(A)$ as follows, if $u = [(\theta, k)]$, then $\varphi(u) := [(\chi(\theta), V(k))]$.

If $k \in K$, then $k = [(0, k)]$ moreover $\varphi(k) = V(k)$, similarly $\varphi_{s_n}|_{K_{s_n}} = V_{s_n}$, for every $n \in \mathbb{Z}^+$.

Now let us prove the following lemmas:

Lemma 5.6.2.3. *The following diagram is commutative:*

$$\begin{array}{ccc} \mathcal{U}_{s_n} & \xrightarrow{i} & \mathcal{U}_{s_{n+1}} \\ \varphi_{s_n}\downarrow & & \downarrow\varphi_{s_{n+1}} \\ \mathcal{U}_{s_n} & \xrightarrow[i]{} & \mathcal{U}_{s_{n+1}} \end{array}$$

Proof: It is enough to check for $k \in K_{s_n}$,

$$i(V_{s_n}(k)) = i(\exp(2\pi i\Delta_{s_n}(k)).k) = \exp(2\pi i\Delta_{s_n}(k)).k \otimes 1_{P_{n+1}}.$$

On the other hand,

$$\begin{aligned} V_{s_{n+1}}(i(k)) &= V_{s_{n+1}}(k \otimes 1_{p_{n+1}}) \\ &= \exp(2\pi i\Delta_{s_{n+1}}(k \otimes 1_{p_{n+1}})).k \otimes 1_{p_{n+1}} \\ &= \exp(2\pi i p_{n+1}\Delta_{s_n}(k)).k \otimes 1_{p_{n+1}} \\ &= \exp(2\pi i\Delta_{s_n}(k)).k \otimes 1_{p_{n+1}}, \end{aligned}$$

and hence the diagram is commutative. □

Lemma 5.6.2.4. *Let $n \in \mathbb{Z}^+$. Then $\varphi|_{\mathcal{U}_{s_n}} = \varphi_{s_n}$.*

Proof: We show that $V|_{K_{s_n}} = V_{s_n}$. If $u \in K_{s_n}$, then

$$\Delta_\tau(u) = \tilde{\Delta}_{s_n}(u) = \frac{\sum_{j=1}^{s_n}\theta_j}{s_n},$$

therefore

$$V(u) = \exp(2\pi i\Delta_{s_n}(u)).u = \det(u).u = V_{s_n}(u),$$

hence the lemma is established. □

The Conclusion: The map θ_φ between the sets of projections of A, induced by φ, is not an orthoisomorphism. Hence φ is not extendable to a $*-$isomorphism on the algebra A.

Proof: Let p and q be orthogonal projections of odd ranks in $\mathbb{M}_{s_n}(\mathbb{C})$ for some $n \in \mathbb{Z}^+$. Let u and w be the self-adjoint unitaries which correspond to p and q respectively. It is easy to check that $\varphi(u) = -u$ and $\varphi(w) = -w$, and hence

$$\theta_\varphi(p)\theta_\varphi(q) = (1-p)(1-q) \neq 0,$$

which proves that θ_φ is not an orthoisomorphism. □

Moreover, we show the following property of the constructed automorphism φ.

Proposition 5.6.2.5. *Let A be as before. Then there is a character η of $\mathcal{U}(A)$ such that $\varphi(u) = \eta(u)u$, for every $u \in \mathcal{U}(A)$.*

Proof: Recall that for all $k \in K$, $V(k) = \text{sign}(k)k$, where $\text{sign}(k) \in \{\pm 1\}$. If $u \in \mathcal{U}(A)$, then $u = \mu k$, for some $\mu \in \mathbb{S}^1$ and $k \in K$, moreover

$$\begin{aligned} \varphi(u) &= \chi(\mu)V(k) \\ &= \chi(\mu)\text{sign}(k)k \\ &= [\chi(\mu)\text{sign}(k)\bar{\mu}]\mu k \\ &= [\chi(\mu)\text{sign}(k)\bar{\mu}]u \end{aligned}$$

therefore, $\varphi(u)u^* \in \mathbb{S}^1$, and the map $\eta : \mathcal{U}(A) \to \mathbb{S}^1$, defined by $\eta(u) = \varphi(u)u^*$ is a character of $\mathcal{U}(A)$. □

5.6.3 More About θ_φ for UHF-Algebras

Let A be a unital UHF-algebra as in the previous section, i.e. A has the generalized integer:

$$\bar{n} = (2^{n_2}, 3^{n_3}, 5^{n_5}, \ldots), \text{ where } 1 \leq n_2 < \infty \text{ and } 0 \leq n_p \leq \infty, \text{ for every prime } p > 2.$$

For every $n \geq 0$, and let s_n be also defined as before. Denote by $\{E^{s_n}_{i,j}\}^{s_n}_{i,j=1}$ the standard set of matrix units of $\mathbb{M}_{s_n}(\mathbb{C})$. Let τ denote the normalized faithful trace of A . For such C^*-algebra we define the following:

Definition 5.6.3.1. *Let A be a C^*-algebra as above, and $p \in \mathcal{P}(A)$ therefore*

$$[p] = \frac{m}{s_n} \in \tau(\Sigma(A)),$$

for some $n \geq 0$ and some $0 \leq m \leq s_n$. Then p is a projection of even rank if m is even number, and of odd rank if m is odd number.

Lemma 5.6.3.2. *Let A be a unital UHF-algebra, such that its generalized integer has $1 \leq n_2 < \infty$, and φ be an automorphism of $\mathcal{U}(A)$. If θ_φ is not an orthoisomorphism, then even rank projections are trace-preserved and odd rank projections are trace-flipped by θ_φ.*

Proof: As A is oddly decomposable, by Booth's result the image of the map c contains at most two elements. By Proposition (3.2.0.11), let $\{\mathcal{P}_e, \mathcal{P}_o\}$ be the partition of $\mathcal{P}(A)\backslash\{0,1\}$, such that the induced map θ_φ preserves orthogonality on $\mathcal{P}_e$, and flips orthogonality on $\mathcal{P}_o$.

The image (under the map c) of a projection in $\mathcal{P}(\mathbb{M}_{s_n}(\mathbb{C}))$ of rank i will be denoted by $c^{s_n}_i$. If $E^{s_n}_{11} \in \mathcal{P}_e$, for some $n \geq 0$, then any finite orthogonal sum of $E^{s_n}_{ii}$, $1 \leq i \leq s_n$, will also be in $\mathcal{P}_e$. As $c^{s_n}_1 = c^{s_{n+1}}_{p_{n+1}}$, we have $E^{s_{n+1}}_{11} \in \mathcal{P}_e$. Therefore the map c is constant, which is a contradiction. Then for all $n \geq 0$, we have $E^{s_n}_{11} \in \mathcal{P}_o$, i.e. for any $i \neq j$, $\theta_\varphi(1 - E^{s_n}_{ii})\theta_\varphi(1 - E^{s_n}_{jj}) = 0$. Therefore

$$1 = \theta_\varphi(\sum_{i=1}^{s_n} E^{s_n}_{ii}) = \Delta^{s_n}_{i=1}\theta_\varphi(E^{s_n}_{ii}) = \sum_{i=1}^{s_n} \theta_\varphi(1 - E^{s_n}_{ii}),$$

and then

$$1 = \tau(\sum_{i=1}^{s_n} \theta_\varphi(1 - E^{s_n}_{ii})) = s_n\tau(\theta_\varphi(1 - E^{s_n}_{11})),$$

hence we have that $\tau(\theta_\varphi(1 - E^{s_n}_{11})) = \frac{1}{s_n} = \tau(E^{s_n}_{11})$.

If $p \in \mathcal{P}(A)$, then $p \sim_u \sum_{i=1}^{m} E^{s_n}_{ii}$ for some $n \geq 0$. If p is an even rank projection, then m is an even number, moreover

$$\theta_\varphi(\sum_{i=1}^{m} E^{s_n}_{ii}) = \Delta^{m}_{i=1}(1 - \theta_\varphi(E^{s_n}_{ii})) = \sum_{i=1}^{m} \theta_\varphi(1 - E^{s_n}_{ii}),$$

and then

$$\tau(\theta_\varphi(p)) = m(\tau(\theta_\varphi(1 - E_{ii}^{s_n}))) = \frac{m}{s_n} = \tau(p),$$

therefore the trace of p is preserved by θ_φ.

If p is an odd rank projection, then in this case m is an odd number and

$$\theta_\varphi(\sum_{i=1}^{m} E_{ii}^{s_n}) = 1 - \Delta_{i=1}^{m}(1 - \theta_\varphi(E_{ii}^{s_n})) = 1 - \sum_{i=1}^{m} \theta_\varphi(1 - E_{ii}^{s_n}),$$

therefore we have

$$\begin{aligned}
\tau(\theta_\varphi(p)) &= 1 - m(\tau(\theta_\varphi(1 - E_{11}^{s_n}))) \\
&= 1 - m\tau(E_{11}^{s_n}) \\
&= 1 - \tau(p),
\end{aligned}$$

therefore the trace of p is flipped by θ_φ. □

Consequently, we prove the following:

Theorem 5.6.3.3. *Let A be a UHF-algebra, such that its generalized integer has $1 \leq n_2 < \infty$. If φ_1 and φ_2 are two automorphisms of $\mathcal{U}(A)$ whose induced maps θ_{φ_1}, θ_{φ_2} (between the sets of projections) are not orthoisomorphisms, then $\theta_{\varphi_1} \circ \theta_{\varphi_2}$ is an orthoisomorphism.*

Proof: Let $\varphi = \varphi_1 \circ \varphi_2$, then $\theta_\varphi = \theta_{\varphi_1} \circ \theta_{\varphi_2}$. By the previous lemma, we have that all the projections of A are trace-preserved by θ_φ, therefore the theorem follows directly from Theorem (5.1.0.2). □

Chapter 6

Automorphisms of the Unitary Groups and Their Extension

Let A and B be two unital C^*−algebras, and $\varphi : \mathcal{U}(A) \to \mathcal{U}(B)$ be a group isomorphism, whose induced map $\theta_\varphi : \mathcal{P}(A) \to \mathcal{P}(B)$ is an orthoisomorphism. Following Dye's strategy in [Dy], we show in this chapter that there exist Jordan $*$−isomorphisms between A and B, which coincide with φ on special subgroups of $\mathcal{U}(A)$.

Also we study the case of UHF-algebras, and we show that if A is a UHF-algebra and φ is a continuous automorphism of $\mathcal{U}(A)$, then φ is implemented by a linear or a conjugate linear $*$−automorphism of A.

6.1 Technical Results

Recall that a C^*−algebra A can be written as a $(n \times n) - C^*$−matrix algebra if (and only if) it contains an $(n \times n)$ system of matrix units $\{e_{i,j}\}_{i,j=1}^n$. Let $N = \{a \in A \mid ae_{1,1} = e_{1,1}a = a\} = e_{1,1}Ae_{1,1}$ denote the corner algebra.

Let A be a unital C^*−algebra, and $\{e_{i,j}\}_{i,j=1}^n$ be a system of matrix units in A. Then any element $x \in A$ is equal to $\sum_{i,j=1}^n e_{i,1}(e_{1,i}xe_{j,1})e_{1,j}$ and $(e_{1,i}xe_{j,1})_{i,j=1}^n \in \mathbb{M}_n(e_{1,1}Ae_{1,1})$. Conversely if $(a_{i,j}) \in \mathbb{M}_n(e_{1,1}Ae_{1,1})$, then $\sum_{i,j}^n e_{i,1}a_{i,j}e_{1,j} \in A$.

Let A be a unital C^*-algebra, and for some $n \in \mathbb{Z}^+$, $\{e_1, \ldots, e_n\}$ be a partition of the unity consisting of orthogonal equivalent projections. For all i, j, let Part.Isom(e_j, e_i) denote the set of all partial isometries $u \in A$ such that $u^*u = e_j$ and $uu^* = e_i$.

Definition 6.1.0.1. *Let A be a unital C^*-algebra, and for some $n \in \mathbb{Z}^+$, $\{e_1, \ldots, e_n\}$ be a partition of the unity consisting of orthogonal equivalent projections. For $1 \leq i \neq j \leq n$:*
(a) If $u \in$ Part.Isom(e_j, e_i), then $p_{i,j}(u)$ will denote the projection of A given by

$$p_{i,j}(u) := \frac{1}{2}(e_i + e_j + e_i u e_j + e_j u^* e_i).$$

(b) $\Gamma_{i,j}$ will denote the set of self-adjoint unitaries $1 - 2p_{i,j}(u)$, $u \in$ Part.Isom(e_j, e_i).

Lemma 6.1.0.2. *If A is a unital C^*-algebra, and $\{e_1, \ldots, e_n\}$ $(n \in \mathbb{Z}^+)$ is a partition of the unity, then for $1 \leq i \neq j \leq n$,*

$$\Gamma_{i,j} = \left\{x \in \mathcal{U}(A) \mid x^2 = 1,\ xe_k = e_k,\ \forall\, k \notin \{i, j\},\ \text{and}\ xe_ix = e_j\right\}.$$

Proof: Let $u \in$ Part.Isom(e_j, e_i), and $x = 1 - 2p_{i,j}(u)$. Then $x = 1 - (e_i + e_j + e_iue_j + e_ju^*e_i)$. If $k \notin \{i, j\}$, then

$$xe_k = e_k - (e_i + e_j + e_iue_j + e_ju^*e_i)e_k = e_k.$$

Also, we have that

$$\begin{aligned} xe_ix &= x(e_i - (e_i + e_iue_j)) \\ &= -x(e_iue_j) \\ &= -(e_iue_j - (e_iue_j + e_j)) \\ &= e_j. \end{aligned}$$

Conversely, if x is a self-adjoint unitary in A such that $xe_k = e_k$ for all $k \notin \{i, j\}$, $xe_ix = e_j$, then define $u = xe_j$, so $u^*u = e_j$ and $uu^* = e_i$, hence $u \in$ Part.Isom(e_j, e_i), moreover

$$\begin{aligned} 1-2p_{i,j}(-u) &= 1-(e_i+e_j-e_ixe_j-e_jxe_i) \\ &= 1-(e_i+e_j-xe_j-xe_i) \\ &= \sum_{k\notin\{i,j\}}^{n} e_k + xe_j + xe_i \\ &= x(\sum_{k\notin\{i,j\}}^{n} e_k + e_j + e_i) \\ &= x. \end{aligned}$$

□

Remark 6.1.0.3. *If A is a unital C^*-algebra, and $\{e_{i,j}\}_{i,j}^{n}$ is a $(n \times n)-$system of matrix units in A, then the map which associates to any $v \in$ Part.Isom$(e_{j,j}, e_{i,i})$ the unitary $e_{1,i}ve_{j,1}$ of $e_{1,1}Ae_{1,1}$ is a bijection.*

Now let A be a unital C^*-algebra, and $\{e_{i,j}\}_{i,j=1}^{n}$ be a system of matrix units in A. Recall (§3.1) that to any $a \in e_{1,1}Ae_{1,1}$ and $1 \leq i \neq j \leq n$, Dye associates a projection $P_{i,j}(a)$ of A. If $v \in \mathcal{U}(e_{1,1}Ae_{1,1})$, then $e_{i,1}ve_{1,j} \in$ Part.Isom(e_j, e_i) and

$$p_{i,j}(e_{i,1}ve_{1,j}) = \frac{1}{2}(e_i + e_j + e_{i,1}ve_{1,j} + e_{j,1}v^*e_{1,i}) = P_{i,j}(v).$$

Lemma 6.1.0.4. *Let A be a unital C^*-algebra, and $\{e_{i,j}\}_{i,j=1}^{n}$, $\{f_{i,j}\}_{i,j=1}^{n}$ be two systems of matrix units in A, such that $e_{i,i} = f_{i,i}$, for all $1 \leq i \leq n$. If $v \in \mathcal{U}(e_{1,1}Ae_{1,1})$, then there exists $u \in \mathcal{U}(e_{1,1}Ae_{1,1})$ such that*

$$P_{i,j}^{f}(v) = P_{i,j}^{e}(u),$$

where $P_{i,j}^{e}(.)$ (resp.$P_{i,j}^{f}(.)$) denotes the projection $P_{i,j}(.)$ corresponds to $\{e_{i,j}\}_{i,j=1}^{n}$ (resp.$\{f_{i,j}\}_{i,j=1}^{n}$).

Proof: For every $1 \leq i, j \leq n$, recall that

$$P_{i,j}^{f}(v) = \frac{1}{2}(f_{i,i} + f_{j,j} + f_{i,1}vf_{1,j} + f_{j,1}v^*f_{1,i}) \in \mathcal{P}(A).$$

Note that $e_{1,j}f_{j,1}$ is a unitary in $e_{1,1}Ae_{1,1}$ hence so is $u = e_{1,i}f_{i,1}vf_{1,j}e_{j,1}$, and it's easy to check that $P_{i,j}^{f}(v) = P_{i,j}^{e}(u)$. □

6.2 An Extension Result for Unital C^*-Algebras

Let A and B be two unital C^*-algebras, φ be an isomorphism between $\mathcal{U}(A)$ and $\mathcal{U}(B)$, and θ_φ be the induced map between the projections $\mathcal{P}(A)$ and $\mathcal{P}(B)$.

Let $\{e_i,\ 1 \leq i \leq n\}$ be a partition of the unity in A, consisting of orthogonal equivalent projections. If

$$\theta : \mathcal{P}(A) \longrightarrow \mathcal{P}(B),$$

is a projection orthoisomorphism, preserving equivalence classes of projections, then $\{f_i = \theta(e_i),\ 1 \leq i \leq n\}$ forms a partition of the unity in B consisting of pairwise orthogonal equivalent projections, that we will call the induced partition of the unity in B from θ and $\{e_{i,i} = e_i,\ 1 \leq i \leq n\}$.

Lemma 6.2.0.1. *Let A, B and φ be as above such that θ_φ is an orthoisomorphism. If $\{e_1, \cdots, e_n\}$ is a partition of the unity, consisting of orthogonal equivalent projections in A, and $\{f_1, \cdots, f_n\}$ is the induced partition of the unity in B, then for every $u \in$ Part.Isom(e_j, e_i), $(1 \leq i \neq j \leq n)$, we have that $\theta_\varphi(p_{i,j}(u)) = p_{i,j}(v)$, for some $v \in$ Part.Isom(f_j, f_i).*

Proof: If $u \in$ Part.Isom(e_j, e_i), and $x = 1 - 2p_{i,j}(u)$, then by Lemma (6.1.0.2), $xe_ix = e_j$, and

$$f_j = \theta_\varphi(xe_ix) = \varphi(x)f_i\varphi(x).$$

If $k \notin \{i, j\}$, then $p_{i,j}(u)e_k = 0$, and as θ_φ is an orthoisomorphism, $\theta_\varphi(p_{i,j}(u))f_k = 0$, therefore $\varphi(x)f_k = f_k$. By Lemma (6.1.0.2), there exists $w \in$ Part.Isom(f_j, f_i) such that $\varphi(x) = 1 - 2p_{i,j}(w)$. □

Recall the following result proved by Dye (in the first part of [Dy, Le.8]).

Proposition 6.2.0.2. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be two unital C^*-algebras. If $\theta : \mathcal{P}(\mathrm{M}_n(\mathfrak{A})) \to \mathcal{P}(\mathrm{M}_n(\mathfrak{B}))$ is an orthoisomorphism, such that for every $1 \leq i, j \leq n$ and $u \in \mathcal{U}(\mathfrak{A})$, $\theta(P_{i,j}(u)) = P_{i,j}(w)$, for some $w \in \mathcal{U}(\mathfrak{B})$, then $\theta(P_{i,j}(a)) = P_{i,j}(b)$, where a, b belong to $\mathfrak{A}$, $\mathfrak{B}$ respectively.*

Lemma 6.2.0.3. *Let $\{e_{i,j}\}_{i,j=1}^n$, and $\{f_{i,j}\}_{i,j=1}^n$ be two systems of matrix units in A and B respectively such that $\theta_\varphi(e_{i,i}) = f_{i,i}$. If θ_φ is a projection orthoisomorphism, then for every $a \in e_{1,1}Ae_{1,1}$ and $1 \leq i \neq j \leq n$, $\theta_\varphi(P_{i,j}(a)) = P_{i,j}(b)$, for some $b \in f_{1,1}Bf_{1,1}$.*

Proof: If $v \in \mathcal{U}(e_{1,1}Ae_{1,1})$, then

$$\theta_\varphi(P_{i,j}(v)) = \theta_\varphi(p_{i,j}(e_{i,1}ve_{1,j})) = p_{i,j}(w),$$

for some $w \in$ Part.Isom(f_j, f_i), by Lemma (6.2.0.1). Then $u = f_{1,i}wf_{j,1}$ is a unitary of $f_{1,1}Bf_{1,1}$, and

$$\begin{aligned} \theta_\varphi(P_{i,j}(v)) &= p_{i,j}(w) \\ &= p_{i,j}(f_{i,1}(f_{1,i}wf_{j,1})f_{1,j}) \\ &= P_{i,j}(u). \end{aligned}$$

The result now follows from Proposition (6.2.0.2). □

Definition 6.2.0.4. *Let A be a unital C^*-algebra. If for $n \in \mathbb{Z}^+$, $\{e_{i,j}\}_{i,j=1}^n$ is a system of matrix units in A, then we denote by Γ the subgroup of $\mathcal{U}(A)$ generated by*

$$\{u \in \mathcal{U}_{sa}(A) |\, u = 1 - 2P_{i,j}(a),\ \textit{for some}\ a \in e_{1,1}Ae_{1,1},\ 1 \leq i \neq j \leq n\}.$$

Remark 6.2.0.5. *The subgroup Γ depends only on the partition of the unity $\{e_{i,i}\}_{i=1}^n$. Indeed if $\{e_{i,j}\}_{i,j=1}^n$, and $\{f_{i,j}\}_{i,j=1}^n$ are two systems of matrix units in A, such that $e_{i,i} = f_{i,i}$ and if $a \in e_{1,1}Ae_{1,1}$, then it is easy to check that $P_{i,j}^e(a) = P_{i,j}^f(f_{1,i}e_{i,1}ae_{1,j}f_{j,1})$.*

Remark 6.2.0.6. *In [Dy, p.79], Dye proves that any Jordan $*-$isomorphism of a C^*-matrix algebra $\mathbb{M}_n(\mathfrak{A})$ $(n \geq 3)$ which leaves all the $P_{i,j}(a)$ fixed is the identity. Therefore if ψ and η are two $*-$isomorphisms or $*-$antiisomorphisms between two C^*-matrix algebras $\mathbb{M}_n(\mathfrak{A})$ and $\mathbb{M}_n(\mathfrak{B})$ such that $\psi(P_{i,j}(a)) = \eta(P_{i,j}(a))$, then $\psi = \eta$.*

Remark 6.2.0.7. *Recall that if η is a $*-$antihomomorphism between two C^*-algebras $\mathfrak{A}$ and $\mathfrak{B}$, then the map $\eta_1 : \mathfrak{A} \to \mathfrak{B}$, defined by $\eta_1(x) = (\eta(x))^*$, for all $x \in \mathfrak{A}$, is a conjugate linear $*-$homomorphism.*

Finally, we prove the main theorem of this section:

Theorem 6.2.0.8. *Let A and B be two unital C^*-algebras and for some $n \geq 3$, let $\{e_1, \cdots, e_n\}$ be a partition of the unity in A, consisting of orthogonal equivalent projections. If $\varphi : \mathcal{U}(A) \longrightarrow \mathcal{U}(B)$ is an isomorphism such that θ_φ is a projection orthoisomorphism, then there exists a unique $\psi : A \to B$, which is a direct sum of linear and conjugate linear $*$-isomorphisms, such that $\psi(u) = \varphi(u)$ for all $u \in \Gamma$, where Γ is defined in (6.2.0.4).*

Proof: As θ_φ is an orthoisomorphism which preserves unitary equivalence of projections, let $\{f_1, \cdots, f_n\}$ be the partition of unity in B, induced by $\{e_1, \cdots, e_n\}$ and θ_φ. By Lemma (6.2.0.3), for every $a \in e_{1,1}Ae_{1,1}$ and $1 \leq i \neq j \leq n$, $\theta_\varphi(P_{i,j}(a)) = P_{i,j}(b)$, for some $b \in f_{1,1}Bf_{1,1}$. Then as θ_φ satisfies the hypotheses of Lemma (3.1.0.8) ([Dye, Le.6]), there exists a linear map

$$\psi : A \longrightarrow B,$$

which is a direct sum of a $*$-isomorphism and a $*$-antiisomorphism, such that

$$\psi(P_{i,j}(a)) = \theta_\varphi(P_{i,j}(a)),$$

for every $a \in e_{1,1}Ae_{1,1}$ and $1 \leq i \neq j \leq n$. By Remark (6.2.0.6), such ψ is unique, and by Remark (6.2.0.7), we can assume that ψ is a direct sum of linear and conjugate linear $*$-isomorphisms.

Now if $u = 1 - 2P_{i,j}(a)$, for some $a \in e_{1,1}Ae_{1,1}$ and $1 \leq i, j \leq n$, then u is a self-adjoint unitary of A and

$$\begin{aligned}\psi(u) &= \psi(1 - 2P_{i,j}(a)) \\ &= (1 - 2\psi(P_{i,j}(a)) \\ &= (1 - 2\theta_\varphi(P_{i,j}(a)) \\ &= \varphi(u).\end{aligned}$$

Hence $\psi(u) = \varphi(u)$, for all $u \in \Gamma$. $\square$

6.3 The Case of UHF-Algebras

Let $\mathcal{A}$ be a UHF-algebra. We fix an increasing sequence $(A_m)_{m\geq 1}$ of subalgebras of $\mathcal{A}$, such that for all $m \geq 1$, A_m is isomorphic to a $(k_m \times k_m)-$full matrix algebra, and we denote $\cup_{m\geq 1} A_m$ by $\mathcal{A}_\infty$. We can assume that $n = k_1 \geq 3$ and let $\{e_1, \cdots, e_n\}$ be a partition of the unity in A_1. As in (3.1.0.6), We will consider the subset

$$\{P_{i,j}(a);\ a \in\ e_1\mathcal{A}e_1,\ 1 \leq i \neq j \leq n\}$$

of projections of $\mathcal{A}$.

If φ is an automorphism of $\mathcal{U}(\mathcal{A})$ such that the induced map θ_φ is an orthoisomorphism of $\mathcal{P}(\mathcal{A})$, then extending Theorem (6.2.0.8), we show that there exists a linear or conjugate linear $*-$automorphism ψ of $\mathcal{A}$ such that $\varphi(u) = \psi(u)$, for every self-adjoint u unitary in $\mathcal{A}_\infty$. Denote by K_∞ the subgroup of $\mathcal{U}(\mathcal{A})$ generated by all self-adjoint unitaries of $\mathcal{A}_\infty$.

Lemma 6.3.0.1. *Let φ be an automorphism of $\mathcal{U}(\mathcal{A})$, such that θ_φ is an orthoisomorphism of $\mathcal{P}(\mathcal{A})$. If $\theta_\varphi(P_{i,j}(a)) = P_{i,j}(a)$, for all $a \in\ e_1\mathcal{A}e_1$, and $1 \leq i \neq j \leq n$, then $\theta_\varphi(p) = p$, for all $p \in\ \mathcal{P}(\mathcal{A}_\infty)$.*

Proof: If $p \in\ \mathcal{P}(\mathcal{A}_\infty)$, then for some $k \geq 1$, $p \in\ \mathcal{P}(A_k)$, i.e. p is a projection in the matrix algebra over the von Neumann algebra $e_1A_ke_1$. Then the restriction of θ_φ on $\mathcal{P}(A_k)$ is an orthoisomorphism onto its image. Moreover for any $a \in\ e_1A_ke_1$, and $\leq i \neq j \leq n$, the projection $P_{i,j}(a)$ is fixed under θ_φ. Therefore by the same argument in the proof of [Dy, Le.7], there exists a subprojection q of p, where q is a projection of $\mathbb{M}_n(e_1A_ke_1)$ having as its range the submodule $[x(e, a_2, \ldots, a_n)|\ x \in\ e_1A_ke_1]$ of $e_1A_ke_1 \oplus \cdots \oplus e_1A_ke_1$. By Lemma (3.1.0.9), $\theta_\varphi(q) = q$. Let $\mathfrak{I}$ denote the set

$$\{q \in\ \mathcal{P}(\mathbb{M}_n(e_1A_ke_1))\mid q \leq p \text{ and } \theta_\varphi(q) = q\},$$

then $\mathfrak{I}$ is a non-empty set. Let $q_1 = sup\{q;\ q \in\ \mathfrak{I}\}$, therefore $\theta_\varphi(q_1) = q_1$. If $q_1 < p$, then there exists a projection q of $\mathbb{M}_n(e_1A_ke_1)$ such that $q \leq p - q_1$ and $\theta_\varphi(q) = q$, therefore $q_1 + q$ is a $\theta_\varphi-$invariant which is a contradiction, hence $q_1 = p$. □

Therefore in the case of the UHF-algebras, Theorem (6.2.0.8), can be rewritten as follows:

Theorem 6.3.0.2. *Let $\mathcal{A}$ be a UHF-algebra as above. If φ is an automorphism of $\mathcal{U}(\mathcal{A})$, such that θ_φ is an orthoisomorphism of $\mathcal{P}(\mathcal{A})$, then there exists a (unique) linear or conjugate linear $*-$automorphism ψ of $\mathcal{A}$ such that ψ coincides with φ on the subgroup generated by $\{\Gamma,\ K_\infty\}$, where Γ is defined in (6.2.0.4).*

Proof: As $\mathcal{A}$ is simple, by Theorem (6.2.0.8), there exists a unique linear or conjugate linear $*-$automorphism ψ of $\mathcal{A}$ which agree with φ on Γ; hence $\theta_\varphi(P_{i,j}(a)) = \psi(P_{i,j}(a))$, for every $a \in e_1\mathcal{A}e_1$ and $1 \leq i, j \leq n$. By Lemma (6.3.0.1), we have $\psi(p) = \theta_\varphi(p)$, for all $p \in \mathcal{P}(\mathcal{A}_\infty)$, and therefore $\psi(u) = \varphi(u)$, for every self-adjoint unitary u of $\mathcal{A}_\infty$, hence the theorem is checked. □

In the following result, we discuss the extension of φ as in Theorem (6.3.0.2) without assuming that θ_φ is an orthoisomorphism, and we show

Theorem 6.3.0.3. *If φ is an automorphism of the unitary group of a UHF-algebra $\mathcal{A}$, then there exists a linear or a conjugate linear $*-$automorphism ψ of $\mathcal{A}$, such that ψ coincides with φ up to a character of $\mathcal{U}(\mathcal{A})$, on K_∞.*

Proof: We have the following two cases:

Case I: If θ_φ is an orthoisomorphism of $\mathcal{P}(\mathcal{A})$, then the result follows directly from Theorem (6.3.0.2).

Case II: If θ_φ is not an orthoisomorphism of $\mathcal{P}(\mathcal{A})$, then by Theorem (5.4.0.1) and Theorem (5.5.0.1), $1 \leq n_2 < \infty$. Recall from §5.6.2 the constructed automorphism φ_K of $\mathcal{U}(\mathcal{A})$, such that θ_{φ_K} is not an orthoisomorphism. Then define $\varphi_1 = \varphi_K^{-1} \circ \varphi$, as $\theta_\varphi = \theta_{\varphi_K} \circ \theta_{\varphi_1}$, by Theorem (5.6.3.3), the map θ_{φ_1} is an orthoisomorphism. Then by Theorem (6.3.0.2,) there exists a linear or conjugate linear $*-$automorphism ψ of $\mathcal{A}$, such that $\psi = \varphi_1$ on K_∞, which implies that $\varphi(u) = \varphi_K(\psi(u))$, for all $u \in K_\infty$.

Recall from §5.6.2 that for some character η of $\mathcal{U}(\mathcal{A})$, $\varphi_K(u) = \eta(u)u$, for all $u \in \mathcal{U}(\mathcal{A})$. Therefore for all $u \in K_\infty$, we have

$$\varphi(u) = \eta(\psi(u))\psi(u).$$

As $\eta \circ \psi$ is a character of $\mathcal{U}(\mathcal{A})$, the theorem follows. □

For a UHF-algebra A, de la Harpe and Skandalis ([HS3, p.194]) proved that

$$\mathcal{U}(A)/D\mathcal{U}(A) \simeq \mathbb{R}/\tau(K_0(A)),$$

where $D\mathcal{U}(A)$ denotes the subgroup of commutators of $\mathcal{U}(A)$. Therefore we have

Proposition 6.3.0.4. *If A is a unital UHF-algebra, then* $\widehat{\mathcal{U}}(A) \simeq \{\lambda \in \hat{\mathbb{R}}_d;\ \lambda|_{\tau(K_0(A))} = 1\}$, *where $\mathbb{R}_d$ denotes $\mathbb{R}$ endowed with the discrete topology.*

In particular, the only continuous character of $\mathcal{U}(A)$ is the trivial character (as $\tau(K_0(A))$ is a dense subgroup of $\mathbb{R}$, [Ef, Cor.4.7]).

Let $\mathcal{A}$, K and K_∞ be as above. As K_∞ is dense in K, we have

Proposition 6.3.0.5. *If φ is a continuous automorphism of $\mathcal{U}(\mathcal{A})$, such that $\varphi(u) = u$, for all $u \in K_\infty$, then $\varphi(u) = u$ is the identity on K.*

Proposition 6.3.0.6. *If φ is a continuous automorphism of the unitary group of a UHF-algebra $\mathcal{A}$, then θ_φ is an orthoisomorphism of $\mathcal{P}(\mathcal{A})$.*

Proof: If θ_φ is not an orthoisomorphism of $\mathcal{P}(\mathcal{A})$, then by Theorem (5.4.0.1) and Theorem (5.5.0.1), $1 \leq n_2 < \infty$. Recall from §5.6.2 the constructed automorphism φ_K of $\mathcal{U}(\mathcal{A})$, such that θ_{φ_K} is not an orthoisomorphism, moreover the restriction of φ_K on K is the continuous automorphism V of K. Therefore $\varphi \circ \varphi_K$ is an automorphism of $\mathcal{U}(\mathcal{A})$, by Theorem (5.6.3.3), we have that $\theta_{\varphi \circ \varphi_K}$ is an orthoisomorphism of $\mathcal{P}(\mathcal{A})$. Then by Theorem (6.3.0.2), there exists a linear or a conjugate linear $*-$automorphism ψ of $\mathcal{A}$ such that for all $u \in K_\infty$, $\psi(u) = \varphi \circ \varphi_K(u)$, and therefore for all $u \in K$, we have that

$$V(u) = \varphi^{-1} \circ \psi(u).$$

As φ is a continuous automorphism of $\mathcal{U}(A)$, so is $\varphi^{-1} \circ \psi$, which is a contradiction and hence the proposition is now checked. □

Lemma 6.3.0.7. *Let A be a real rank zero C^*-algebra. If an element $x \in A$ commutes with all the projections of A, then x belongs to the center $\mathcal{Z}(A)$ of A.*

Proof: Let $y \in A$ be a self-adjoint element. We want to show that $xy = yx$. As A has real rank zero, the self-adjoint elements with finite spectrum of A are dense in the set of self-adjoint elements. If $\epsilon > 0$, then there exists a self-adjoint element z with finite spectrum such that $\|y - z\| < \epsilon/2$. But $z = \sum_{i=1}^{k} \lambda_i r_i$ where for each i, r_i is a projection in A. Therefore $xr_i = r_i x$, for every $i = 1, 2, \ldots, k$ and hence $xz = zx$. And then

$$\|xy - yx\| \leq \|x\| \, \|y - z\| + \|z - y\| \, \|x\| \; < \; \epsilon\|x\|.$$

As ϵ is arbitrary, $xy = yx$ and hence $x \in \mathcal{Z}(A)$. □

Finally, we have the following result:

Theorem 6.3.0.8. *Let $\mathcal{A}$ be a UHF-algebra as above. If φ is a continuous automorphism of $\mathcal{U}(\mathcal{A})$, then φ is implemented by a linear or a conjugate linear $*$-automorphism ψ of $\mathcal{A}$.*

Proof: By Proposition (6.3.0.6), the induced map θ_φ is an orthoisomorphism of $\mathcal{P}(\mathcal{A})$. By Theorem (6.3.0.2) and Proposition (6.3.0.5), there is an automorphism ψ of $\mathcal{A}$, such that $\psi = \varphi$ on K.

Define φ_1 on $\mathcal{U}(\mathcal{A})$ by $\varphi_1 = \psi^{-1} \circ \varphi$. If $u \in \mathcal{U}(\mathcal{A})$ and $v \in K$, then

$$\varphi_1(u)v = \varphi_1(uv) = (uvu^{-1})\varphi_1(u),$$

therefore, $u^{-1}\varphi_1(u) \in \mathcal{Z}(K)$ and this implies that $u^{-1}\varphi_1(u)$ commutes with all projections in $\mathcal{A}$. Then by Lemma (6.3.0.7), $u^{-1}\varphi_1(u) \in \mathcal{Z}(\mathcal{A})$, therefore $u^{-1}\varphi_1(u) = \lambda(u)$ and $\lambda(u)$ is in the unite circle, and that's true for every unitary u, so we get $\lambda : \mathcal{U}(\mathcal{A}) \to \mathbb{S}^1$ is a character, moreover $\varphi(u) = \lambda(u)\psi(u)$. As φ is continuous, λ is continuous character of $\mathcal{U}(\mathcal{A})$, hence $\lambda = 1$.

□

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