AN APPROACH TO NON-LINEAR SEPARATION

-THE VERTEX WEIGHT METHOD-

by

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ABSTRACT

In this dissertation we will study the problem of non-linear separation. As usual, a Boolean function F of n binary variables, $x_1, \ldots, x_n$, x_i 1, 0 , i = 1, . . . , n, will be represented by a set of vertices C in the n-dimensional Euclidean space, E^n, where each vertex has n binary-valued components, 1, 0 .

Any two disjoint sets of vertices, C_1 and C_2, can be separated by a hypersurface, which can be represented by an algebraic equation, called a discriminant function of degree r, 0 r n.

A ∅ transformation of a vertex is a mapping which transfers a vertex into real value domain. The transferred vertex is a product of its components and its associated weight, called vertex weight. A discriminant function, called ∅-function, is defined as the summation of the ∅ transformations of the vertices in C_1 minus the summation of the ∅ transformations of the vertices in C_2, plus the summation of the ∅ transformations of the don't care vertices, if any. The weights of the vertices belonging to C_1 and C_2 are restricted to be greater than zero, while the weights of the don't care vertices are unrestricted, and may be any real value.

In this dissertation a method is given to find the minimum degree ∅-function for two disjoint sets C_1 and C_2 in E^n and to reduce the number of terms in this ∅-function by finding a suitable set of vertex weights for all vertices. As a special case a ∅-function of degree 1 can be obtained by using this method for a first order threshold function.

It is shown that for symmetrical sets (representing symmetrical Boolean functions), the method to find the minimum degree ∅-function can be simplified.

Furthermore, the concept of imperfect ∅-function is introduced. It is shown that two disjoint sets C_1 and C_2 can be discriminated by a set of simpler imperfect ∅-functions instesd of a single more complicated ∅-function.

CONTENTS

CHAPTER I

INTRODUCTION

In recent years logical elements on the threshold principle have received considerable attention. In fact, the switching function produced by a single threshold device is relatively complex in comparison to that produced by the usual AND and OR gates. Furthermore, threshold logic is not limited in scope to logical design only. Its mathematical model is applicable to the decision rules in pattern recognition and adaptive control systems.

Many papers on threshold logic deal with the properties and the testing and realization of threshold functions [43 up to 60]. The fundamental properties of threshold functions are predicated by a set of inequalities given as relation (2-1), where the unknowns are the weights and the threshold value. The methods to realize a threshold function basically consist in solving a set of linear inequalities. This has been first studied by McNaugton [28], Muroga et al [30], and Winder [41], and later by many others. Threshold logic in its original studies is restricted in application, because single threshold elements can realize only a small fraction of the total number of Boolean functions. Recently, non-linear threshold functions, i.e. Boolean functions which cannot be realized by a single threshold element, have been intensively studied. Most of the works can be classified into the following three areas:

(1) Compound synthesis,
(2) Multithreshold threshold function realization,
(3) High-order threshold function realization.

Compound systhesis of threshold logic elements is the realization of a general Boolean function with a number of threshold logic elements. General compound systhesis has been studied by Akers [61], Minnick [29], Winder [42], Miller and Winder [79], Sheng [86], and many others (see references III), and that for symmetric Boolean functions by Kaut z [75], Sheng [85] , and others. The maximum number of threshold logic elements in compound synthesis is 2^n , where n is the number of binary variables.

The realization of multithreshold threshold functions has been first studied by Atada [3] in 1964 , and later by Haring [67], Haring and Okori [68], Haring and Dieghnis [69], and Mow and Fu [82]. The maximum number of thresholds needed to realize a Boolean function is 2^n-1, n being the number of variables.

There have not appeared many papers dealing with the high-order threshold function realization. Golomb [20]has used the Rademach-Walsh functions to find a set of invariants for classification of Boolean functions, which is precisely the coefficients of nth degree discriminant functions. The concept of high-order threshold functions has been developed by Nilsson [35], and Krishnan [26]. It was shown that the order of the threshold function is at most n. Kaszerman (88) has used AND gates for sets of variables to realize high-order threshold functions.

From the geometrical point of view, compound synthesis uses more than one hyperplane to separate two disjoint sets of vertices C_1 and C_2 in the n-dimensional Euclidean space into different regions. Each region contains only the vertices belonging to either C_1 or C_2. Multithreshold threshold function realization uses several parallel hyperplanes to separate the vertices of C_1 and C_2 into different regions. Vertices in one region belong to the same category and vertices in two neighbouring regions do not belong to the same category. High-order threshold function realization uses a single hypersurface to separate C_1 from C_2.

For instance, in an example given by Moore (see [42]), a Boolean function of 12 variables is symmetrical in a set of 3 variables $\{x_{1_1}, x_{1_2}, x_{1_3}\}$, in a set of 4 variables , $\{x_{2_1}, x_{2_2}, x_{2_3}, x_{2_4}\}$, and in a set of 5 variables, $\{x_{3_1}, x_{3_2}, x_{3_3}, x_{3_4}, x_{3_5}\}$. Using Sheng's notation for symmetrical function [55], Moore's function is

$$F = x_1^3(x_2^4+x_2^3x_3^2+x_2^2x_3^3+x_2x_3^4+x_3^5)+x_1^2(x_2^4x_3^2+x_2^3x_3^4+x_2^2x_3^5)$$

$$+x_1(x_2^4x_3^4+x_2^3x_3^5)+x_2^4x_3^5 \quad , \qquad (1\text{-}1)$$

where x_i^j means $\sum_{i_1<\cdots<i_j} x_{i_{i_1}} \cdots x_{i_{i_j}}$. For instance $x_1^2 = x_{1_1}x_{1_2}+x_{1_1}x_{1_3}+x_{1_2}x_{1_3}$.

Then we can obtain the hypersurface with our method, as follows:

$$\phi(X) = 8\sum_i x'_{1_i} + 10\sum_j x_{2_j} + 8\sum_k x_{3_k} + 4\sum_{i,j} x_{1_i}x_{1_j} - 75. \qquad (1\text{-}2)$$

where

$$X = (x_{1_1}, x_{1_2}, x_{1_3}, x_{2_1}, \ldots, x_{2_4}, x_{3_1}, \ldots, x_{3_5}).$$

It is known that Moore's function is not a threshold function of first order. But it can be realized by a 2nd degree ϕ-function with terms of $\sum_{i,j} x_{1_i}x_{1_j}$. Equation (1-2) is found to be the minimum degree ϕ-function of Moore's function, and also the realization with a minimum number of terms. Since the realization of high-order threshold functions uses only one hypersurface to separate set C_1 from set C_2, where $C_1 \cap C_2 = \theta$, it is more direct than the compound synthesis and the multithreshold threshold function realization. The method presented in this dissertation is intended to find the minimum degree ϕ-function and to reduce the number of terms in this ϕ-function.

In Chapter II, the general principles of forming a high-degree ϕ-function from vertex weights are described. Sets of vertices instead of Boolean functions are used throughout this dissertation. The sets in the n-dimensional Euclidean space, E^n, can be mapped into the expanded E^{n_r} space, where

$$n_r = \sum_{i=1}^{r} \binom{n}{i} .$$

Then an r-degree ϕ-function, which discriminates the disjoint sets C_1 and C_2, can be constructed as a first degree function in the E^m space, where $n \leq m \leq n_r$.

In Chapter III, some theorems on the minimization of the degree of ϕ-functions and on the elimination of certain terms of ϕ-functions are presented. The method to minimize the order and to eliminate certain terms of the ϕ-function is reduced to the solution of a homogeneous system with restricted and/or unrestricted variables, where the weights of vertices belonging to C_1 and C_2 are represented by restricted variables, and weights of the don't care vertices by unrestricted variables. A procedure to solve this system is given. The procedure to construct the ϕ-function is based on the method to solve the homogeneous system. In this procedure, we first find the realizability of ϕ-function with degree n. Next we try to find the realizability of the ϕ-function with degree n-1, and so on. Finally the minimum degree ϕ-function is found. After that we eliminate the terms of the ϕ-function one by one. The whole process is carried out systematically.

In Chapter IV, we apply the vertex weight method to symmetric sets of vertices (symmetric Boolean functions). Because the vertex weights of vertices with the same number of 1's in the n-tuple are the same, the number of homogeneous equations for symmetric sets is condensed from 2^n to n. The coefficients of the homogeneous equations are in Pascal triangle relation.

In Chapter V, we apply the vertex weight method to form imperfect ϕ-functions, which are the discriminant functions with errors of discrimination being either in C_1, or C_2, or in both. Here we form a set of imperfect ϕ-functions with errors either all in C_1 or all in C_2. This set of imperfect ϕ-functions can be of any degree from 1 to n-1. Using this set of imperfect ϕ-functions, C_1 and C_2 can be discriminated.

In Chapter VI, some concluding remarks are given.

CHAPTER II

ϕ TRANSFORMATION

1. Introduction:

A Boolean function $F(x_1, x_2, \ldots, x_n)$ of n binary variables $(x_1, x_2, \ldots, x_n)$ is called a threshold function if and only if:

$$F(x_1, x_2, \ldots, x_n) = b_1 \quad \text{if} \quad \sum_{i=1}^{n} a_i x_i \geq T \qquad (2\text{-}1\text{-a})$$

and

$$F(x_1, x_2, \ldots, x_n) = b_2 \quad \text{if} \quad \sum_{i=1}^{n} a_i x_i < T \qquad (2\text{-}1\text{-b})$$

where

$F(x_1, x_2, \ldots, x_n)$ is a Boolean function of $(x_1, x_2, \ldots, x_n)$,

$\sum_{i=1}^{n} a_i x_i$ is an algebraic function of $(x_1, x_2, \ldots, x_n)$,

$(x_1, x_2, \ldots, x_n)$ is an ordered n-tuple or vector of n binary variables,

x_i is a Boolean variable assuming a value of either b_1 or b_2 for $i = 1, 2, \ldots, n$,

a_i is a coefficient called the weight of x_i for $i = 1, 2, \ldots, n$,

T is a constant of real value called the threshold value and

b_1, b_2 are the two possible real values for the binary variables or Boolean functions.

In the present study we choose $b_1 = 1$ and $b_2 = 0$.

We now set

$$(x_1, x_2, \ldots, x_n) = X$$

and

$$(a_1, a_2, \ldots, a_n) = A \quad .$$

Then

$$F(X) = 1 \quad \text{if} \quad A \cdot X \geq T \quad \text{or} \quad A \cdot X - T \geq 0 \tag{2-2-a}$$

and

$$F(X) = 0 \quad \text{if} \quad A \cdot X < T \quad \text{or} \quad A \cdot X - T < 0 \quad . \tag{2-2-b}$$

Here $A \cdot X - T = 0$ is called a discriminant function, and Boolean function $F(X)$ is said to be realized by this discriminant function [31, 33, 37, 49].

There are several methods [references II] available for finding the discriminant function corresponding to a given threshold function. Since threshold functions constitute only a small fraction of the set of all Boolean functions for a given number of variables, n, especially when n is very large [42], it is of interest to extend these methods to finding the discriminant function for an arbitrary Boolean function.

In the problem of statistical recognition, the discriminant function is derived by Bayes' decision rule. Chow [11] has shown that a Boolean function $F(X)$ is a threshold function, if and only if there exist a binomial distribution $p = (p_1, p_2)$ and two independent probability distributions

$$p(X/C_1) = \prod_{i=1}^{n} p(x_i/C_1) \quad , \qquad C_1 = \{X/F(X) = 1\} \tag{2-3-a}$$

and

$$p(X/C_2) = \prod_{i=1}^{n} p(x_i/C_2) \quad , \qquad C_2 = \{X/F(X) = 0\} \tag{2-3-b}$$

such that their associated recognition function realizes F(X). The proof of the existence of such distributions has been given by Chow. But it is difficult in practice to find those probability distributions to form a discriminant function with no errors for realizing F(X).

A Boolean function which can be realized by a discriminant function of degree r is called an rth order threshold function, or a threshold function of order r. It has been proved that any Boolean function of n variables can be realized as a threshold function of order r, $r \leq n$ [35, 26].

Krishnan [26]has extended Chow's work to include rth order threshold functions, but the problem how to find the discriminant function still remains.

Golomb [20] has used the Rademacher-Walsh functions, a set of orthogonal functions, to construct a set of invariants for classifying Boolean functions. Coleman [13] hasdrives the same orthogonal set from Fourier transformation, and employs it in his representation of switching functions. This is precisely an example of using ϕ-functions to realize high-order threshold functions. These ϕ-functions have also been discussed by Bahadur [4], Dertouzos [17], Chow [90], and Abend et al [89], and others.

A ϕ-function of degree r can be written as:

$$\phi(X) = a_o + \sum_{i_1} a_{i_1} x_{i_1} + \sum_{i_1<i_2} a_{i_1 i_2} x_{i_1} x_{i_2} + \cdots + \sum_{i_1< \cdots <i_r} a_{i_1 \cdots i_r} x_{i_1} \cdots x_{i_r}$$

$$= a_o + \sum_{j=1}^{r} \sum_{i_1< \cdots <i_j} a_{i_1 \cdots i_j} x_{i_1} \cdots x_{i_j} , \quad i_1, \ldots, i_j = 1, \ldots, n. \tag{2-4}$$

If the products of variables are called <u>secondary variables,</u> and if the highest degree of these products of variables in $\phi(X)$ be r, then the E^n space can be expanded into a new E^{n_r} space where $n_r = \sum_{i=1}^{r} \binom{n}{i}$ and thus $\phi(X)$ becomes the discriminant function of a first order threshold function in the E^{n_r} space. In other words, any Boolean function can be considered as a first order threshold function in a certain E^{n_r} space. Here all the variables except the secondary variables will be called primary variables. As a matter of fact, the secondary variables, which are the products of primary variables, are all dependent.

Sometimes, however, all of the secondary variables in the E^{n_r} space will not be necessary and some of them can thus be eliminated. In such a case the ϕ-function can also be taken as a discriminant function of a first order threshold function in an Euclidean space with a dimension less than n_r.

In the present dissertation we shall suggest a method for constructing a ϕ-function for the realization of Boolean function by assigning a weight to each vertex which will then be transferred into the domain of ϕ-function. The value of this weight will be restricted and have to be greater than zero if the vertex belongs to C_1 or C_2 as defined in Equations (2-3), and will be unrestricted, i.e. the weight can have any real value, if the vertex is a don't care condition. As a special case, a first degree ϕ-function can be obtained by using this method for a first order threshold function.

2. Set of Vertices:

In the E^n space there will be a total of 2^n vertices and each component of the vertices has a value of either 1 or 0. The finite unit set in the E^n space is the set of all the 2^n vertices and is denoted by U; the empty set in E^n is denoted by θ.

A Boolean function F(X) of n variables, X, can be denoted by a set of vertices in E^n such that

$$C_1 = \{X/F(X) = 1\} \tag{2-5}$$

and its complement

$$C_2 = \{X/F(X) = 0\} = U \setminus C_1 \quad . \tag{2-6}$$

In subsequent discussions we shall use these sets of vertices in the E^n space to represent Boolean functions. For instance, a set of vertices C_1 in the E^n space which is linearly separable from its complement represents the first order threshold function F(X) of n variables, where

$$C_1 = \{X/F(X) = 1\} \quad .$$

3. Expansion of the E^n Spaces:

Since any secondary variable is the product of a subset of the primary variables. For instance, consider a set of vertices

$$C_1 = \{X_1, X_2, X_3\}$$

in E^3 with components x_1, x_2, and x_3, as shown in Table 2-1-(a).

Table 2-1

	x_1	x_2	x_3
X_1	1	1	0
X_2	1	0	1
X_3	0	0	1

(a)

	x_1	x_2	x_3	x_1x_2
X_1	1	1	0	1
X_2	1	0	1	0
X_3	0	0	1	0

(b)

A secondary variable obtained from E^3 is x_1x_2, which is the product x_1 and x_2 in E^3 space. Thus an E^4 space is formed as shown in Table 2-1-(b). Here the secondary variable depends on the primary variables x_1 and x_2.

The maximum number of secondary variables in E^{n_r} is

$$n_r - n = \sum_{i=2}^{r} \binom{n}{i} . \qquad (2\text{-}7)$$

It is clear that the number of vertices in a set of vertices in E^n is the same as the number of vertices in its expanded space. Also the components of the primary variables for each vertex in E^{n_r} are the same as those of the primary variables in the corresponding vertex in the E^n space. Because of this, the sets and the vertices in E^n are denoted by the same symbols as their corresponding sets and vertices in the expanded spaces without any ambiguity.

We next derive some important theorems.

Theorem 1: A necessary and sufficient condition for any two sets of vertices C_1 and C_2 in the E^m space to be linearly separable is that there exists a weight vector A in E^m, such that

$$A \cdot X_i > A \cdot X_j \ , \quad \text{for all } X_i \in C_1 \ , \quad \text{and } X_j \in C_2 \ . \qquad (2\text{-}8)$$

The proof of this theorem is obvious.

Lemma 1: If any two sets of vertices C_1 and C_2 in the E^m space are linearly separable, then they are also linearly separable in any E^{m+1} space, where $E^n \subset E^m \subset E^{m+1}$, n is the number of primary variables.

Lemma 2: If any two sets of vertices C_1 and C_2 in the E^m space are linearly separable, then they are also linearly separable in any E^s space, where $E^m \subset E^s$, and $n \leq m \leq s \leq n_n$, where

$$n_n = \sum_{i=1}^{n} \binom{n}{i} .$$

Theorem 2: If any two sets of vertices C_1 and C_2 in the E^s space are not linearly separable, then they are also not linearly separable in any E^m space, where $E^n \subset E^m \subset E^s$, and $n \leq m \leq s \leq n_n$.

The proof of this theorem can be obtained directly from Lemmas 1 and 2.

4. <u>$\emptyset$ Transformation:</u>

<u>Definition 1</u>: Let $X_1 = (x_1, \ldots, x_n)$ be a vertex in E^n. Then

$$\emptyset_{X_1}(X) = w_{X_1}\, x_1 x_2 \ldots x_n \tag{2-9}$$

where w_{X_1} is a real value greater than zero, called <u>vertex weight</u>, and $x_1 x_2 \ldots x_n$ is the product of the primary variables in the same form as in X_1. This $\emptyset_{X_1}(X)$ will be called <u>$\emptyset$ transformation</u> of vertex X_1, and simply written as $\emptyset_{X_1}$.

<u>Definition 2</u>: A discriminant function discriminating two disjoint sets of vertices C_1 and C_2 in E^n, which is formed from $\emptyset$ transformations of the vertices in E^n is called a <u>$\emptyset$-function.</u>

<u>Definition 3</u>: The transformation on real values, between the negated and the unnegated forms of a variable in the domain of $\emptyset$-function will be defined as :

$$x_i = (1 - \overline{x}_i) \tag{2-10-a}$$

and

$$\overline{x}_i = (1 - x_i) \tag{2-10-b}$$

Definition 4: In a $\emptyset$-function each primary variable can be arbitrarily chosen either in negated or in unnegated form but not in both. This set of primary variables is called the positivization of the $\emptyset$-function, denoted by P which is represented in an ordered n-tuple form. A component of a vertex, having the same form as of the corresponding primary variable in the positivization, will be transferred to 1, otherwise zero in the $\emptyset$-function.

From the above definitions we can see that for any vertex other than $(x_1, x_2, \ldots, x_n)$, Equation (2-9) must be equal to zero. Assume that there is another vertex, $X_2 = (\bar{x}_1, x_2, \ldots, x_n)$. Then we have

$$\emptyset_{X_2}(X) = w_{X_2} \bar{x}_1 x_2 \ldots x_n \tag{2-11}$$

Thus,

$$\emptyset_{X_2}(X) > 0 \quad , \quad \text{if } X = X_2$$

and

$$\emptyset_{X_2}(X) = 0 \quad , \quad \text{if } X \neq X_2 \; .$$

However, in the domain of $\emptyset$-function the positivization of the primary variables can be arbitrarily assigned without affecting the results of the $\emptyset$-function. For instance, let the positivization be $(x_1, x_2, \ldots, x_n)$. By

Equations (2-10), Equation (2-11) can be rewritten as:

$$\phi_{C_2}(X) = w_{X_2}\bar{x}_1x_2\cdots x_n = w_{X_2}(1 - x_1)x_2\cdots x_n$$

$$= w_{X_2}(x_2\cdots x_n - x_1\cdots x_n) \ .$$

In general, if the positivization in the domain of ϕ-function is $(x_1, x_2, \cdots, x_n)$, then for the vertex $X_i = (x_{i_1}, \cdots, x_{i_r}, \bar{x}_{i_{r+1}}, \cdots, \bar{x}_{i_n})$ we have:

$$\phi_{X_i}(X) = w_{X_i}x_{i_1}\cdots x_{i_r}\bar{x}_{i_{r+1}}\cdots\bar{x}_{i_n}$$

$$= w_{X_i}x_{i_1}\cdots x_{i_r}(1 - x_{i_{r+1}})\cdots(1 - x_{i_n})$$

$$= w_{X_i}x_{i_1}\cdots x_{i_r}(1 - \sum_{j=r+1}^{n} x_{i_j} + \sum_{r<i_j<i_k} x_{i_j}x_{i_k} + \cdots$$

$$+ (-1)^{n-r}x_{r+1}\cdots x_n) \qquad (2\text{-}12)$$

Equation (2-12) shows that the number of terms with positive signs is the same as the number of terms with negative signs except in the case when $r=n$, where there is only a single term which is positive.

<u>Theorem 3:</u> A ϕ-function that discriminates the set C_1 from its complement $U\backslash C_1$ can be obtained directly by summing the ϕ_{X_i}'s for all $X_i \in C_1$.

<u>Proof:</u> Set

$$\phi_{C_1}(X) = \sum_{X_i \in C_1} \phi_{X_i} \qquad (2\text{-}13)$$

For sake of simplicity, we omit P_i in the ϕ-function if P_i is not specially defined.

where

$$w_{X_i} > 0 \ .$$

Then

$$\phi_{C_1}(X_i) = w_{X_i} > 0 \ , \quad \text{if } X_i \in C_1$$

and

$$\phi_{C_1}(X_i) = 0 \qquad , \quad \text{if } X_i \notin C_1 \ .$$

Q.E.D.

Theorem 4: In the ϕ transformation the following relations exist:

$$\begin{aligned} &(1) \quad \phi_{\bigcup_i C_i} = \sum_i \phi_{C_i} \\ &(2) \quad \phi_{\bigcap_i C_i} = \prod_i \phi_{C_i} \end{aligned} \qquad (2\text{-}14)$$

where all $C_i \in U$.

Proof: For (1), assume

$$C_1 = \bigcup_{i=1}^{r} C_{1i} \ .$$

Set

$$\phi_{C_1}(X) = \phi_{C_{11}}(X) + \cdots + \phi_{C_{1r}}(X) = \sum_{i=1}^{r} \phi_{C_{1i}}(X)$$

If $X_j \in C_1$, then there must be some subsets C_{1k}, $k \in \{1, \cdots, r\}$, such that $X_j \in C_{1k}$, and we have:

$$\phi_{C_{1k}}(X_j) > 0 \ ,$$

That is,

$$\phi_{C_1}(X_j) > 0 \ .$$

If $X_j \notin C_1$, i.e. $X_j \notin C_{1i}$ for all $i = 1, \cdots, r$, then

$$\phi_{C_{1i}}(X_j) = 0 \quad \text{for all } i = 1, \cdots, r.$$

That is,

$$\phi_{C_1}(X_j) = 0 \ .$$

Then (1) is proved.

For (2), we assume

$$C_1 = \bigcap_{i=1}^{r} C_{1i} \ .$$

Set

$$\phi_{C_1}(X) = \phi_{C_{11}}(X) \cdots \phi_{C_{1r}}(X) = \prod_{i=1}^{r} \phi_{C_{1i}}(X)$$

If $X_j \in C_1$, i.e. $X_j \in C_{1i}$ for all $i \in \{1, \cdots, r\}$, then we have:

$$\phi_{C_{1i}}(X_j) > 0 \ ,$$

i.e.

$$\phi_{C_1}(X_j) > 0 \ .$$

If $X_j \notin C_1$, then there is at least one i, $i = 1, \cdots, r$, such that $X_j \notin C_{1i}$,

i.e.

$$\phi_{C_{1i}}(X_j) = 0$$

i.e.

$$\phi_{C_1}(X_j) = 0 \quad .$$

Then (2) is proved.

Q.E.D.

The aforementioned theorems only consider the case of a ϕ-function discriminating a set from its complement. In general, we shall consider a ϕ-function discriminating any set of vertices C_1 from any other set C_2, where $C_1 \cap C_2 = \theta$.

<u>Theorem 5:</u> Let C_1 and C_2 be two sets of vertices and let $C_1 \cap C_2 = \theta$ and $C_1, C_2 \in U$. Then there is a $\phi_{C_1|C_2}$, such that

$$\phi_{C_1|C_2} = \phi_{C_1} - \phi_{C_2} \tag{2-15}$$

which discriminates C_1 from C_2.

<u>Proof:</u> If $X_i \in C_1$, then by Theorem 3, $\phi_{C_1}(X_i) > 0$ and $\phi_{C_2}(X_i) = 0$, i.e.

$$\phi_{C_1|C_2}(X_i) > 0 \quad .$$

If $X_i \in C_2$, then by Theorem 3, $\phi_{C_1}(X_i) = 0$ and $\phi_{C_2}(X_i) > 0$, i.e.

$$\phi_{C_1|C_2}(X_i) < 0 \quad .$$

If $X_i \notin C_1 \cup C_2$, then by Thm 3, $\phi_{C_1}(X_i) = 0$, and $\phi_{C_2}(X_i) = 0$, i.e.

$$\phi_{C_1|C_2}(X_i) = 0 \quad .$$

Q.E.D.

Equation (2-15) is a discriminant function which separates the sets C_1 and C_2. If there exists a don't care set $U \backslash (C_1 \cup C_2)$, then Theorem 5 still holds, because $U \backslash (C_1 \cup C_2)$ can be suitably separated into two subsets C'_1 and C'_2, where $C'_1 \cap C'_2 = \theta$. Then $\phi_{(C_1 \cup C'_1) | (C_2 \cup C'_2)}$ is the discriminant function separating $C_1 \cup C'_1$ from $C_2 \cup C'_2$, which obviously also separates C_1 from C_2. In Chapter III we shall develop a method to find C'_1 and C'_2 for obtaining the minimum degree ϕ-function and reducing the number of terms of the ϕ-function.

<u>Definition 5:</u> All the ϕ-functions will be called equivalent ϕ-functions if and only if these ϕ-functions discriminate the set C_1 from set C_2 where $C_1 \cap C_2 = \theta$, $C_1, C_2 \in U$.

CHAPTER III

MINIMIZATION OF DEGREE AND NUMBER OF TERMS IN $\emptyset$-FUNCTIONS

1. Theory of Minimization of Degree of $\emptyset$-function:

Recall Equation (2-12)

$$\emptyset_{X_i(P_1)}(X) = w_{X_i}(x_{i_1}\dots x_{i_r} - x_{i_1}\dots x_{i_r}\sum_{j>r}^{n} x_{i_j} + \dots + (-1)^{n-r} x_1\dots x_n) \tag{2-12}$$

where $P_1 = (x_{i_1}, \dots, x_{i_n})$, r is the number of primary variables with same form in both P_1 and X_i, $r \geq 0$. It is evident that the number of terms in $\emptyset_{X_i(P_1)}$ is equat to $2^{(n-r)}$. Since w_{X_i} is greater than zero, the sign of a term depends on the number of primary variables in the term. That is the terms with degree r, r+2, ... are positive and the terms with degree r+1, r+3, ... are negative. The number of positive terms is equal to the number of negative terms except only when n=r.

Let $d^m_{X_i(P_1)}$ denote the sign of a term in a $\emptyset$-function, i.e., $d^m_{X_i(P_1)} \in \{-1, +1\}$, where X_i is the vertex, P_1 is the positivization and m is the number of primary variables in the term. It is obvious that

$$d^r_{X_i(P_1)} = d^{r+2}_{X_i(P_1)} = \dots = +1$$

and

$$d^{r+1}_{X_i(P_1)} = d^{r+3}_{X_i(P_1)} = \dots = -1$$

Then Equation (2-12) becomes

$$\emptyset_{X_i(P_1)} = w_{X_i}(d^r_{X_i(P_1)}\, x_{i_1}\dots x_{i_r} + d^{r+1}_{X_i(P_1)} x_{i_1}\dots x_{i_r}\sum_{j>r} x_{i_j} + \dots + d^n_{X_i(P_1)}\, x_{i_1}\dots x_{i_n}. \tag{3-3}$$

For simplicity, (P_1) in d's will be ometted.

Definition 6: A $\emptyset$-function having a minimum degree amongst all equivalent $\emptyset$-functions will be called a minimum degree $\emptyset$-function.

Theorem 6: The degree of the minimum degree $\emptyset$-functions that discriminate the set of vertices C_1 from set of vertices C_2 in the set U.

where $C_1 \cap C_2 = \theta$, is independent of the positivization of the primary variables in the domain of ϕ-function.

Proof: Let

$$C_1 = \{X_1^1, X_2^1, \ldots, X_k^1\}$$

and

$$C_2 = \{X_1^2, X_2^2, \ldots, X_q^2\} \quad .$$

Then by Theorems 3 and 5, we obtain:

$$\phi_{C_1 | C_2}(P_i) = \phi_{X_1^1}(P_i) + \cdots + \phi_{X_k^1}(P_i) - \phi_{X_1^2}(P_i) - \cdots - \phi_{X_q^2}(P_i) \quad .$$

Without loss of generality, we can put

$$P_i = (x_1, x_2, \ldots, x_n) \quad .$$

Then

$$\begin{aligned}
\phi_{C_1 | C_2}(P_i) = {} & (d^n_{X_1^1}\, w_{X_1^1} + \cdots + d^n_{X_k^1}\, w_{X_k^1} - d^n_{X_1^2}\, w_{X_1^2} - \cdots - d^n_{X_q^2}\, w_{X_q^2}) x_1 \cdots x_n \\
& + \sum_{i_1 < \cdots < i_{n-1}} \Big(\sum_{X_j^1 \in (i_1, \ldots, i_{n-1})} d^{n-1}_{X_j^1}\, w_{X_j^1} \\
& - \sum_{X_j^2 \in (i_1, \ldots, i_{n-1})} d^{n-1}_{X_j^2}\, w_{X_j^2} \Big) x_{i_1} \cdots x_{i_{n-1}} \\
& + \sum_{i_1 < \cdots < i_{n-2}} \Big(\sum_{X_j^1 \in (i_1, \ldots, i_{n-2})} d^{n-2}_{X_j^1}\, w_{X_j^1} \\
& - \sum_{X_j^2 \in (i_1, \ldots, i_{n-2})} d^{n-2}_{X_j^2}\, w_{X_j^2} \Big) x_{i_1} \cdots x_{i_{n-2}} + \cdots \\
& + \sum_{i_1} \Big(\sum_{X_j^1 \in (i_1)} d^1_{X_j^1}\, w_{X_j^1} - \sum_{X_j^2 \in (i_1)} d^1_{X_j^2}\, w_{X_j^2} \Big) x_{i_1} \\
& + \Big(\sum_{X_j^1 \in (i_o)} d^o_{X_j^1}\, w_{X_j^1} - \sum_{X_j^2 \in (i_o)} d^o_{X_j^2}\, w_{X_j^2} \Big)
\end{aligned} \qquad (3\text{-}4)$$

where $X_j^k \in (i_1, i_2, \cdots, i_r)$, $k = 1, 2$, means all $X_j^k \in C_k$ such that $\phi_{X_j^k}(P_i)$ has the term $x_{i_1} x_{i_2} \cdots x_{i_r}$ with non-zero coefficients, and i_o is the constant term. The sign of w's depends on $d^m_{X_j^k}$, where $k = 1$ or 2, $0 \leq m \leq n$, and the signs in Equation (3-4) will be determined for any fixed P_i. Now we assume that a minimum degree ϕ-function $\phi_{C_1 | C_2(P_i)}$ has a degree r. Then there must be a set $W = \{ w_{X_1^1}, \cdots, w_{X_k^1}, w_{X_1^2}, \cdots w_{X_q^2} \}$ which makes the coefficients of the terms with powers higher than r in $\phi_{C_1 | C_2(P_i)}$ zero, i.e.

$$
\begin{aligned}
&\sum_{X_j^1 \in (i_1, \cdots, i_n)} d^n_{X_j^1} w_{X_j^1} - \sum_{X_j^2 \in (i_1, \cdots, i_n)} d^n_{X_j^2} w_{X_j^2} = 0 \\
&\sum_{X_j^1 \in (i_1, \cdots, i_{n-1})} d^{n-1}_{X_j^1} w_{X_j^1} - \sum_{X_j^2 \in (i_1, \cdots, i_{n-1})} d^{n-1}_{X_j^2} w_{X_j^2} = 0 \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&\sum_{X_j^1 \in (i_1, \cdots, i_{n-r+1})} d^{n-r+1}_{X_j^1} w_{X_j^1} - \sum_{X_j^2 \in (i_1, \cdots, i_{n-r+1})} d^{n-r+1}_{X_j^2} w_{X_j^2} = 0 \, .
\end{aligned}
\qquad (3\text{-}5)
$$

Let P_j be another positivization such that $P_j \neq P_i$. The transformation of $\phi_{C_1 | C_2(P_i)}$ to $\phi_{C_1 | C_2(P_j)}$ can be obtained by using Equation (2-10). It is sufficient if we can prove now that Equation (3-5) still holds when P_i is transferred to P_j in Equation (3-4). Since the transformation of Equation (2-10) does not produce any term with a degree higher than that of the original one in the ϕ-function, and since the sign in every equation of (3-5) depends on the associated secondary variables, the relation (3-5) will always exist. If $\phi_{C_1 | C_2(P_j)}$ has a degree less than r, then it will contradict our assumption.

Q.E.D.

Theorem 7: Let C_1 be a set of vertices in the set U. If C_1 can be decomposed into several $(n-i)$-cubes where $i \leq r$, then the minimum degree ϕ_{C_1} will have a degree not greater than r.

Proof: Let all the vertices of an $(n-i)$-cube in C_1 be

$$C_{1i} = \bigcup_{\text{all } (x^*_{i+1}, \cdots, x^*_n)} (x_1, \cdots, x_i, x^*_{i+1}, \cdots, x^*_n) \ .$$

Set all the vertex weights equal to 1 and $P_i = (x_1, \cdots, x_n)$. Then by Theorem 3, we have:

$$\phi_{C_{1i}}(P_i) = x_1 \cdots x_i \Big(\sum_{\text{all } (x^*_{i+1}, \cdots, x^*_n)} x^*_{i+1} \cdots x^*_n \Big)$$

$$= x_1 \cdots x_i [x_{i+1} \cdots x_n + (1-x_{i+1}) x_{i+2} \cdots x_n + \cdots + (1-x_{i+1}) \cdots (1-x_n)]$$

The function in the square brackets is symmetrical, and equals 1. Thus

$$\phi_{C_{1i}}(P_i) = x_1 \cdots x_i \ .$$

Hence by Theorems 4 and 6, this theorem is proved.

Q.E.D.

Corollary 7a: Let F be a Boolean function in INDF (irredundant normal disjunctive form) in which the maximum number of variables in a term is t. Then F can be realized by a minimum degree ϕ-function of degree r, where $r \leq t$.

Theorem 8: The minimum number of terms in a ϕ-function which discriminates the set C_1 from C_2, where $C_1 \cap C_2 = \theta$ in the set U, will be determined by the positivization of the primary variables and the vertex weights in the domain of ϕ-function.

Proof: From Equation (3-4), we see that the coefficient of a secondary variable $(x_{i_1} \cdots x_{i_r})$ in the ϕ-function is

$$\sum_{X_j^1 \in (i_1, \cdots, i_r)} d^r_{X_j^1} w_{X_j^1} - \sum_{X_j^2 \in (i_1, \cdots, i_r)} d^r_{X_j^2} w_{X_j^2} .$$

The minimum number of terms in the ϕ-function is given by the maximum number of terms whose coefficients equal zero, i.e.

$$\sum_{X_j^1 \in (i_1, \cdots, i_r)} d^r_{X_j^1} w_{X_j^1} - \sum_{X_j^2 \in (i_1, \cdots, i_r)} d^r_{X_j^2} w_{X_j^2} = 0 . \tag{3-6}$$

These equations depend on two factors: One is the set W itself, and the other is the d's, which is determined by the positivization.

Q.E.D.

Consequently, we can conclude that if the vertex weights are found, then the weights of the variables are also determined. The vertex weights of the vertices which are specified to belong to either C_1 or C_2 will be called restricted weights, for they must be greater than zero. The vertex weights of don't care vertices will be called unrestricted weights, for they can be of any real values.

Let C_1 and C_2 be two sets of vertices in U, where $C_1 \cap C_2 = \theta$, and let positivization $P_i = (x_1, \cdots, x_n)$, where

$$C_1 = \{X_1^1, X_2^1, \cdots, X_k^1\} ,$$

$$C_2 = \{X_1^2, X_2^2, \cdots, X_q^2\} .$$

By Theorem 5, we have:

$$\phi_{C_1 | C_2}(P_i) = \sum_{j=1}^{k} \phi_{X_j^1}(P_i) - \sum_{j=1}^{q} \phi_{X_j^2}(P_i)$$

$$= (d^n_{X^1_1} w_{X^1_1} + \cdots + d^n_{X^1_k} w_{X^1_k} - d^n_{X^2_1} w_{X^2_1} - \cdots - d^n_{X^2_q} w_{X^2_q}) x_1 \cdots x_n$$

$$+ \sum_{i_1 < \cdots < i_{n-1}} \Big(\sum_{X^1_j \in (i_1, \ldots, i_{n-1})} d^{n-1}_{X^1_j} w_{X^1_j} - \sum_{X^2_j \in (i_1, \ldots, i_{n-1})} d^{n-1}_{X^2_j} w_{X^2_j} \Big) x_{i_1} \cdots x_{i_{n-1}}$$

$$+ \cdots\cdots\cdots\cdots\cdots$$

$$+ \Big(\sum_{X^1_j \in (i_0)} d^0_{X^1_j} w_{X^1_j} - \sum_{X^2_j \in (i_0)} d^0_{X^2_j} w_{X^2_j} \Big) \qquad (3\text{-}7)$$

In order to show the relation between the vertex weights and the variable weights in Equation (3-7) more clearly, a matrix Q is constructed as shown in Table 3-1.

Table 3-1 Q matrix

	X^1_1	......	......	......	X^1_k	X^2_1	......	......	......	X^2_q
1										
x_1										
$\vdots$										
x_n										
$\vdots$										
$x_1 x_2$										
$\vdots$										
$x_{i_1} \cdots x_{i_r}$										
$\vdots$										
$x_1 \cdots x_n$										

In the Q matrix, the column designators give the expansion of $\phi_{X(P_i)}$, $X \in C_1$, $(-\phi_{X(P_i)}, X \in C_2)$ with vertex weight equal to 1, and entries of the $\phi_{X(P_i)}$ column show its coefficients of the constant, of the primary variables, and of the secondary variables with respect to P_i, which are shown on the left side of the matrix as distinct row designators. The entry values can be 1, 0, or -1. Then the coefficient of the secondary variable $x_{i_1} \cdots x_{i_r}$ in the ϕ-functionis equal to

$$\vec{b}_{i_1 \cdots i_r} \cdot W$$

where $\vec{b}_{i_1 \cdots i_r}$ is the row vector of Q with the associated variable $x_{i_1} \cdots x_{i_r}$, and

$$W = (w_{X_1^1}, \cdots, w_{X_k^1}, w_{X_1^2}, \cdots, w_{X_q^2}) .$$

It is obvious that by Equation (3-7),

$$\phi_{C_1 | C_2(P_i)} = \widetilde{X} \, Q \, W^T = \widetilde{X} \, A^T \tag{3-8}$$

where $\widetilde{X} = (1, x_1, \ldots, x_n, x_1 x_2, \ldots, x_{i_1} x_{i_2} \cdots x_{i_r}, \ldots, x_1 \cdots x_n)$

$$A = (a_o, a_1, \ldots, a_n, a_{n-1}, \ldots, a_{2^n-1})$$

and W^T and A^T are the transposes of W and A, respectively.

2. Solution of Homogeneous Systems with Restricted Variables and/or Unrestricted Variables:

If a set of vertex weights satisfying Equations (3-5) can be found, then the minimum degree ϕ-function can also be obtained. The problem of finding this set of vertex weights is equivalent to that of solving a homogeneous system with restricted variables and/or unrestricted variables, representing the restricted and the unrestricted vertex weights, respectively.

Assume that there is a homogeneous system expressed as follows:

$$
\begin{aligned}
&b_{11}z_1 + b_{12}z_2 + \cdots + b_{1n}z_n = 0 \\
&b_{21}z_1 + b_{22}z_2 + \cdots + b_{2n}z_n = 0 \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
&b_{m1}z_1 + b_{m2}z_2 + \cdots + b_{mn}z_n = 0 \ .
\end{aligned}
\tag{3-9}
$$

The system in (3-9) has a matrix of coefficients $B = (b_{ij})$, $i = 1, \cdots, m$, $j = 1, \cdots, n$, where $b_{ij} = 1, 0$, or -1. Then we can rewrite (3-9) as:

$$B\,Z^T = 0 \tag{3-10}$$

where $Z = (z_1, z_2, \cdots, z_n)$.

Suppose we can find k independent equations in the system (3-9), where $k \le m$. If $k \ge n$, we will only have a trivial solution. If $k < n$, then in these k independent equations we can arbitrarily assign k dependent variables $z_{i_1}, \cdots, z_{i_k}$ and $(n-k)$ independent variables $z_{j_1}, \cdots, z_{j_{n-k}}$ such that (See [18] part I)

$$
\begin{aligned}
&z_{i_1} = f_1(z_{j_1}, z_{j_2}, \cdots, z_{j_{n-k}}) \\
&z_{i_2} = f_2(z_{j_1}, z_{j_2}, \cdots, z_{j_{n-k}}) \\
&\cdots\cdots\cdots\cdots\cdots \\
&z_{i_k} = f_k(z_{j_1}, z_{j_2}, \cdots, z_{j_{n-k}}) \ .
\end{aligned}
\tag{3-11}
$$

Equations (3-11) can be obtained as follows. We first choose an arbitrary dependent variable in any equation of (3-9) with a coefficient other than zero, then operate on B with respect to the row of the corresponding equation in (3-9) to obtain a row reduced echolon matrix, R, in which the entries are all zero in this particular column except in the chosen row. For instance, if we choose z_1 as the dependent variable in row 1 of (3-9), where $b_{11} \neq 0$, then in B, we substitute $(\frac{-b_{i1}}{b_{11}} b_{1j} + b_{ij})$, $j = 1, \cdots, n$ for the ith row, $i = 2, \cdots, m$, and obtain a reduced matrix R_1 from B. It is obvious that the solutions of Equations (3-9) are the same as the solutions of

$$R_1 Z^T = 0 \; .$$

Continuing the same procedure from R_1 to form the reduced matrix R_2, finally we can find k dependent variables in k independent equations of (3-9), and the remaining equations all become of the form $0 = 0$. The reduced matrix in the final stage is called the reduced echolon matrix, R. And the solutions of (3-9) are the same as the solutions of

$$R Z^T = 0. \tag{3-12}$$

It is to be noted that the rows of this R reduced echolon matrix are not accurately in the same order as that of a standard echolon form.

There are four forms of the equations in (3-11), viz.

$$z_{i_s} = -\sum_{j_k} c_{j_k} z_{j_k} \tag{3-13}$$

$$z_{i_s} = \sum_{j_r} c_{j_r} z_{j_r} - \sum_{j_k} c_{j_k} z_{j_k} \tag{3-14}$$

$$z_{i_s} = \sum_{j_r} c_{j_r} z_{j_r} \tag{3-15}$$

$$z_{i_s} = 0 \tag{3-16}$$

where $c_{j_k}, c_{j_r} > 0$. Assume that all the variables are restricted, i.e. $z_i > 0$, $i = 1, \cdots, n$. Then there can be no solution for the system if there is any equation of the form of (3-13) or (3-16). An equation of the form of (3-13) or (3-16) is called a contradiction equation. In the B matrix the associated row of the contradiction equation has the character that the entries other than zero are either all positive, or all negative. If all the equation of (3-11) are of the form of (3-15), then we can obtain the solution by assigning independent variables any values greater than zero. Here, only equations of the form of (3-14) do not give a direct answer. Since $z_{i_s} > 0$, we can change the equality (3-14) into an inequality

$$\sum_{j_r} c_{j_r} z_{j_r} - \sum_{j_k} c_{j_k} z_{j_k} > 0 \,, \tag{3-17}$$

or

$$\sum_{j_r} c_{j_r} z_{j_r} > \sum_{j_k} c_{j_k} z_{j_k} \,. \tag{3-18}$$

To solve the inequalities of the form of (3-18) is a problem of linear programming [21,39]. There are several established methods which can be used to solve these inequalities with integer solutions. Here we will use the successive higher ordering method of Sheng and Hwa [56] to solve these inequalities.

Now, if some of the variables in the system are unrestricted, i.e. if these variables can have any real value, the possibility of the system to have a solution increases. In this case, the method stated above has to be modified.

As in Equation (3-11), all f's are linear equations of independent variables. The unrestricted dependent variables always have a solution, if the independent variables are real. Thus, we can take the equations in (3-11) with unrestricted dependent variables as <u>don't care equations</u>. And the contradiction equations of the form of (3-13) and (3-16) containing some unrestricted variables are actually not contradictory. The system has solutions if the equations in (3-11) excepting the don't care equations have solutions. In order to have more don't care equations, we prefer to choose the unrestricted variables as the dependent variables.

In this dissertation we are interested in the integral solution of the system. In such a case the integer coefficients of the functions f's in relation (3-11) aremostly desired. The change of a fractional coefficient to integer coefficient can be done by transferring the associated variable. For instance, assume that there is a term in f_i as $(r_1/r_2)z_i$, where r_1, r_2 are integers without a common factor, and r_2 is greater than 1. If we set

$$z_i = r_2 z_i' \quad , \qquad \text{for all } f\text{'s},$$

then this term becomes $r_1 z_i'$. This process is to be continued until all the coefficients in f's become integers.

Theoretically we can choose any entry in a row, with a value other than zero, to be the coefficient of an associate dependent variable in the row. For instance in matrix B of Equation (3-10) we choose such an entry in a certain row, and after operation we have the reduced matrix R'. If we choose a different entry in same or a different row, we will have a different reduced matrix R''. Then

$$B\ Z = 0 \tag{3-10}$$

$$R'\ Z = 0 \tag{3-19}$$

$$R''Z = 0 \tag{3-20}$$

Here we can see that the solutions of (3-20) are also the solutions of (3-10) and (3-19). If there is no solution of (3-20), then there is also no solution of (3-10) and (3-19).

In order to avoid the occurence of equations of the form of (3-14) as for as practicable, we will establish a decision rule to choose an entry for coefficient of an associated dependent variable in a row according to a certain preference order. Here M denotes the sum of the entries in a row all with the same sign, and M_p denotes the sum of the entries in the same row with the opposite sign.

The preference order for choosing an entry of an associated restricted dependent variable is as follows:

I. Choosing a row
   (1) with the smallest M,
   (2) with the smallest value of an entry or entries of M,
   (3) with smallest ratio of M/M_p ,
   (4) with M of positive sign,
   (5) with the smallest row index.

II. Choosing an entry of M with the smallest column index.

For the preference order for choosing an entry of an associated unrestricted dependent variable, we establish the decision rule as follows:

(1) Between $b_{i,j}$ and $b_{k,m}$, $b_{i,j}$ is chosen if $i<k$.

(2) Between $b_{i,j}$ and $b_{i,k}$, $b_{i,j}$ is chosen if $j<k$.

The procedure to solve a homogeneous system with restricted and/ or unrestricted variables is as follows:

Step 1) Form a matrix of coefficients $B = (b_{ij})$ for the homogeneous system as in Equation (3-9). Check the contradiction relations in B.

A. If any, stop the process.

B. If none, proceed to Step 2).

Step 2) Find $b_{i,j}$ according to the preference order for choosing an entry of an associated unrestricted dependent variable.

A. If there is any, proceed to Step 3).

B. If there is none, proceed to Step 5).

Step 3) Operate on B with row operation with respect to row $\vec{b}_i$ to obtain a row reduced matrix.

A. If there is any contradiction row, stop the process.

B. If there is no contradiction row, proceed to Step 4).

Step 4) Check the entries of the associated dependent variables in each row except the rows with all zero entries.

A. If every row has one, proceed to Step 8).

B. If any row has none, proceed to Step 2).

Step 5) Find $b_{i,j}$ according to the preference order for choosing an entry of an associated restricted dependent variable, M starts from 1.

A. If there is any, proceed to Step 6) and set $M = 1$.

B. If there is none, repeat Step 5) and set $M \rightarrow M+1$.

Step 6) Operate on B with row operation with respect to row $\vec{b}_i$ to obtain a row reduced matrix.

A. If there is any contradiction row, stop the process.

B. If there is no contradiction row, proceed to Step 7).

Step 7) Check the entries of the associated dependent variables in each row except the rows with all zero entries.

A. If every row has one, proceed to Step 8).

B. If any row has none, proceed to Step 5).

Step 8) Form equations as in (3-11). Find the relations as in (3-14), and form the inequalities as in (3-18), except the don't care equations.

A. If there is any, proceed to Step 9).

B. If there is none, proceed to Step 10).

Step 9) Use the successive higher ordering method to find the solution of the inequalities.

A. If there is no solution, stop.

B. If there is a solution, proceed to Step 10).

Step 10) Find all the values of w's. Here we assign the possible minimum value for each component of W.

We choose here a simple example for illustration.

<u>Example 3-1:</u> The B matrix of a homogeneous system is as follows:

Table 3-2 B matrix

	1	2	3	4	5	6	7	8
1	1			-1			-1	-1
2				-1	1		1	-1
3		1				-1		-1
4			1	-1	-1			-1
5			-1	1	1	-1		
6		1				1		1

In the above z_2 and z_6 are unrestricted variables, and the rest are restricted variables. The blank entries in Table 3-2 are the entries with value zero.

Step 1) The B matrix is already shown above.

Step 2) We next find b_{32}. Since condition A is satisfied, we now proceed to Step 3).

Step 3) Operating B matrix with respect to $\vec{b}_3$, we obtain the R_1 matrix for B, as shown below.

Table 3-3 R_1 matrix

	1	2	3	4	5	6	7	8
1	1			-1			-1	-1
2				-1	1		1	-1
3		1				-1		-1
4			1	-1	-1			-1
5			-1	1	1	-1		
6						1		1

$1/2\,(\vec{b}_6 - \vec{b}_3) \longrightarrow \vec{b}_6$

1: entry of the associated dependent variable.

Since condition B is now satisfied, we proceed to Step 4).

Step 4) Since condition B is satisfied, we proceed to Step 2).

Step 2) Then we find b_{56}. Since condition A is now satisfied, we proceed to Step 3).

Step 3) Operating B matrix with respect to $\vec{b}_5$, we obtain the R_2 matrix for B, shown in Table 3-4.

Table 3-4 R_2 matrix

	1	2	3	4	5	6	7	8	
1	1			-1			-1	-1	
2				-1	1		1	-1	
3		1	1	-1	-1			-1	$\vec{b}_3 - \vec{b}_5 \longrightarrow \vec{b}_3$
4			1	-1	-1			-1	
5			-1	1	1	-1			
6			-1	1	1			1	$\vec{b}_6 + \vec{b}_5 \longrightarrow \vec{b}_6$

Since condition B is satisfied, we now proceed to Step 4).

Step 4) Since condition B is satisfied, we next proceed to Step 2).

Step 2) Since condition B is satisfied, we proceed to Step 5).

Step 5) We now find b_{11}. Since condition A is satisfied, we proceed to Step 6).

Step 6) There is no operation needed, so R_3 is the same as R_2. Since condition B is satisfied, we proceed to Step 7).

Step 7) Since condition B is satisfied, we proceed to Step 5).

Step 5) b_{43} is found. Since condition B is satisfied, proceed to Step 6).

Step 6) Operating R_3 with respect to $\vec{b}_4$, we obtain the R_4 matrix for B, shown in Table 3-5.

Table 3-5 R_4 matrix

	1	2	3	4	5	6	7	8	
1	1			-1			-1	-1	
2				-1	1		1	-1	
3		1							$\vec{b}_3 - \vec{b}_4 \longrightarrow \vec{b}_3$
4			1	-1	-1			-1	
5						-1		-1	$\vec{b}_5 + \vec{b}_4 \longrightarrow \vec{b}_5$
6									$\vec{b}_6 + \vec{b}_4 \longrightarrow \vec{b}_6$

Since condition B is satisfied, we proceed to Step 7).

Step 7) Since condition B is satisfied, we proceed to Step 5).

Step 5) Since condition B is satisfied, we set M = 2 and repeat Step 5).

Step 5) We find b_{24}. Since condition B is satisfied, we proceed to Step 6).

Step 6) We obtain R_5 matrix for B as shown in Table 3-6.

Table 3-6 R_5 matrix

	1	2	3	4	5	6	7	8	
1	1				-1		-2		$\vec{b}_1 - \vec{b}_2 \longrightarrow \vec{b}_1$
2				-1	1		1	-1	
3		1							
4			1		-2		-1		$\vec{b}_4 - \vec{b}_2 \longrightarrow \vec{b}_4$
5						-1		-1	
6									

Since condition B is satisfied, we proceed to Step 7).

Step 7) Since condition A is satisfied, we proceed to Step 8).

Step 8) We obtain:

$z_1 = z_5 + 2z_7 = f_1$,

$z_2 = 0 \qquad = f_2$,

$z_3 = 2z_5 + z_7 = f_3$,

$z_4 = z_5 + z_7 - z_8 = f_4$,

$z_6 = -z_8 \qquad = f_6$.

The inequalities of the form (3-18), except the don't care equations, are formed from f_3 and f_4, i.e.

$2z_5 > z_7$,

$z_5 + z_7 > z_8$.

Since condition A is satisfied, we now proceed to Step 9).

Step 9) Using the successive higher ordering method, we obtain:

$z_5 = z_7 = z_8 = 1$.

Since condition B is satisfied, we proceed to Step 10).

Step 10) Substituting values of z_5, z_7 and z_8 into f_1, f_2, f_3, f_4 and f_6, we obtain

$z_1 = 3$, $z_2 = 0$, $z_3 = 1$, $z_4 = 1$, $z_6 = -1$.

3. <u>Relations between Vertex Weights and Variable Weights:</u>

Recall Equation (3-8), i.e.

$$\phi_{C_1 | C_2}(P_i) = \widetilde{X}\,Q\,W^T = \widetilde{X}\,A^T \tag{3-8}$$

where Q is determined by P_i, and by the vertices in C_1 and C_2.

Assume

$$U = C_1 \cup C_2 \quad ,$$

then Q is a $2^n x 2^n$ matrix. No matter what P_i is, no two sets of distinct primary variables between P_i and a vertex X_j are the same. Here by distant primary variables we mean primary variables which appear in different forms in the two vertices. Then after suitable interchange of the columns of X's in the Q matrix, we can eventually have a Q matrix of such a form that the entries above a diagonal are all zeros and the entries on the diagonal are all non-zeros. This means that all the rows in the Q matrix are independent and all the columns are independent too. That is, the Q matrix is non-singular.

From Equation (3-8), we have

$$Q\,W^T = A^T \; . \tag{3-19}$$

Multiplying both sides of Equation (3-19) by the inverse of Q on the left, we obtain:

$$W^T = Q^{-1}\,A^T \; . \tag{3-20}$$

Let W be a set of vertex weights, such that

$$\widetilde{X}\, Q\, W^T > 0 \quad , \quad \text{if } X \in C_1 \tag{3-21-a}$$

and

$$\widetilde{X}\, Q\, W^T < 0 \quad , \quad \text{if } X \in C_2 \tag{3-21-b}$$

where $X = (1, X)$.

By Equation (3-19), there must be a vector A such that

$$\widetilde{X}\, A^T > 0 \quad , \quad \text{if } X \in C_1 \tag{3-22-a}$$

and

$$\widetilde{X}\, A^T < 0 \quad , \quad \text{if } X \in C_2 \; . \tag{3-22-b}$$

By Equation (3-20), the reverse is also true. That is, if there is an A vector which satisfied Equations (3-22), then there must be a W vector which satisfies Equations (3-21).

Assume that $A = (a_0, a_1, \cdots, a_{2^n-1})$ is variable weight vector of the first degree ϕ -function, that is, the weights of the primary variables $a_1, a_2, \cdots, a_n$ are not all equal to zero, and the weights of the secondary variables are

$$a_{n+1} = a_{n+2} = \cdots = a_{2^n-1} = 0 \; .$$

Set

$$Q = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{12^n} \\ b_{21} & b_{22} & \cdots & b_{22^n} \\ \cdots & \cdots & \cdots & \cdots \\ b_{2^n1} & b_{2^n2} & \cdots & b_{2^n2^n} \end{pmatrix}$$

$$Q^{-1} = \frac{1}{N}\begin{bmatrix} B_{11} & B_{12} & \cdots & B_{12^n} \\ & \cdots\cdots\cdots\cdots & & \\ B_{2^n1} & B_{2^n2} & \cdots & B_{2^n2^n} \end{bmatrix}$$

where b_{ij}, B_{ij}, N are all integers. Then

$$W^T = Q^{-1} A^T = \frac{1}{N}\begin{bmatrix} B_{11}\, a_o + \cdots + B_{1(n+1)}\, a_n \\ B_{21}\, a_o + \cdots + B_{2(n+1)}\, a_n \\ \cdots\cdots\cdots\cdots \\ B_{2^n1}\, a_o + \cdots + B_{2^n(n+1)}\, a_n \end{bmatrix} \tag{3-23}$$

Since $W > 0$, we have:

$$\begin{aligned} &B_{11}\, a_o + \ldots + B_{1(n+1)}\, a_n > 0 \\ &B_{21}\, a_o + \ldots + B_{2(n+1)}\, a_n > 0 \\ &\qquad\cdots\cdots\cdots\cdots \\ &B_{2^n1}\, a_o + \ldots + B_{2^n(n+1)}\, a_n > 0 \;. \end{aligned} \tag{3-24}$$

Solving the system of (3-24), the variable weights of $a_o, a_1, \ldots, a_n$ are obtained.

On the other hand we have:

$$Q\,W^T = \begin{bmatrix} b_{11} & \cdots & b_{12^n} \\ b_{21} & \cdots & b_{22^n} \\ & \cdots\cdots & \\ b_{2^n1} & \cdots & b_{2^n2^n} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_{2^n} \end{bmatrix} = \begin{bmatrix} a_o \\ a_1 \\ \vdots \\ a_{2^n-1} \end{bmatrix} \tag{3-25}$$

That is,

$$
\begin{aligned}
& b_{(n+2)1}\, w_1 + \cdots + b_{(n+2)2^n}\, w_{2^n} = 0 \\
& b_{(n+3)1}\, w_1 + \cdots + b_{(n+3)2^n}\, w_{2^n} = 0 \\
& \qquad \cdots\cdots\cdots\cdots\cdots\cdots \\
& b_{2^n 1}\, w_1 + \cdots + b_{2^n 2^n}\, w_{2^n} = 0 \;.
\end{aligned}
\qquad (3\text{-}26)
$$

Solving the system of (3-26), the vertex weights of $w_1, \cdots, w_{2^n}$ are obtained. The method to obtain the vertex weights will be called the <u>vertex weight method.</u>

The systems in (3-24) and (3-26) can be extended to the case of rth degree ϕ-function without difficulty. These extended systems are shown as follows:

$$
\begin{aligned}
& B_{11}\, a_o + \cdots + B_{1(n_r+1)}\, a_{n_r} > 0 \\
& B_{21}\, a_o + \cdots + B_{2(n_r+1)}\, a_{n_r} > 0 \\
& \qquad \cdots\cdots\cdots\cdots\cdots \\
& B_{2^n 1}\, a_o + \cdots + B_{2^n (n_r+1)}\, a_{n_r} > 0 \;.
\end{aligned}
\qquad (3\text{-}27)
$$

and

$$
\begin{aligned}
& b_{(n_r+2)1}\, w_1 + \cdots + b_{(n_r+2)2^n}\, w_{2^n} = 0 \\
& b_{(n_r+3)1}\, w_1 + \cdots + b_{(n_r+3)2^n}\, w_{2^n} = 0 \\
& \qquad \cdots\cdots\cdots\cdots\cdots\cdots \\
& b_{2^n 1}\, w_1 + \cdots + b_{2^n 2^n}\, w_{2^n} = 0
\end{aligned}
\qquad (3\text{-}28)
$$

where $n_r = \sum_{i=1}^{r} \binom{n}{i}$.

On the basis of the above discussion, Theorem 9 is now derived.

Theorem 9: Two sets of vertices C_1 and C_2 in U, where $C_1 \cap C_2 = \theta$, can be discriminated by a minimum degree ϕ-function of degree r if and only is the system of (3-28) exists.

4. Relations between the Vertex Weight Method and the Rademacher-Walsh Orthogonal Functions Method:

Define a set of 2^n polynormals in E^n as follows:

$$
\begin{aligned}
\psi_o(X) &= 1 \\
\psi_1(X) &= 2x_1 - 1 \\
\psi_2(X) &= 2x_2 - 1 \\
&\cdots\cdots\cdots\cdots \\
\psi_n(X) &= 2x_n - 1 \\
\psi_{n+1}(X) &= (2x_1 - 1)(2x_2 - 1) \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
\psi_{2^n-1}(X) &= \prod_{i=1}^{n} (2x_i - 1)
\end{aligned}
\qquad (3\text{-}29)
$$

where $X \in U$, and $\psi_j(X)$, for $j > n$, is a finite product of $\psi_1, \psi_2, \cdots, \psi_n$. This set of polynormals is precisely the Rademacher-Walsh functions.

For any real valued functions f and g in vector space of E^{2^n}, the inner product of f and g is defined as:

$$< f.g > = 2^{-n} (f(X_1).g(X_1) + f(X_2).g(X_2) + \cdots + f(X_{2^n}).g(X_{2^n}))$$

$$= 2^{-n} \sum_{X\in U} f(X).g(X) \tag{3-30}$$

Consequently, any real-valued function $f(X)$ on U can be expanded as a unique linear combination of ψ's as:

$$f(X) = \sum_{i=0}^{2^n-1} c_i\psi_i(X) \tag{3-31}$$

and the expansion coefficients are:

$$c_i = < f.\psi_i > = 2^{-n} \sum_{X\in U} f(X).\psi_i(X) \tag{3-32}$$

Golomb [20] has used this orthogonal set to find a set of invariants

$$T_o, T_1, \cdots, T_{2^n-1}$$

for each equivalent family of Boolean functions, where

$$T_i = \max (c_i, 1-c_i) .$$

A set of Boolean functions is said to belong to an equivalent family when any two functions in the set differ only by permution and/or complementation of their variables.

Coleman [13] derives the same orthogonal set from Fourier transformation, and employs it in his representation of switching functions.

Bahadur [4] uses the Rademacher-Walsh orthogonal functions for finding the ratio of given probability distribution $p(X)$, $p(X) \geq 0$, and $\sum_{X\in U} p(X) = 1$, to the joint probability distribution of the x_i's when the x_i's are independently distributed. Chow [10] revises Bahadur's expansion for directly finding the conditional probability distribution as:

$$p(X/C_k) = \sum_{i=0}^{2^n-1} c_i(k)\, \psi_i(X) \tag{3-33}$$

with

$$c_i(k) = 2^{-n} \sum_{X \in U} p(X/C_k)\, \psi_i(X) \tag{3-34}$$

where k is the index of the disjoint sets of vertices in U.

In order to make Equations (3-31) and (3-32) more systematic, we construct an expansion matrix as shown in Table 3-7, where the entry of $(\psi_i, X_j) = f(X_j)\, \psi_i(X)$. Then

$$c_i = \langle f.\psi_i \rangle = 2^{-n} \sum_{j=1}^{2^n} (\psi_i, X_j) \tag{3-35}$$

Table 3-7 Orthogonal Expansion Matrix

	X_1	X_2	....	X_{2^n}
ψ_o	$f(X_1)\psi_o(X_1)$	$f(X_2)\psi_0(X_2)$	....	$f(X_{2^n})\psi_o(X_{2^n})$
ψ_1	$f(X_1)\psi_1(X_1)$	$f(X_2)\psi_1(X_2)$	....	$f(X_{2^n})\psi_1(X_{2^n})$
$\vdots$	...	....	....	....
ψ_n	$f(X_1)\psi_n(X_1)$	$f(X_2)\psi_n(X_2)$	....	$f(X_{2^n})\psi_n(X_{2^n})$
ψ_{n+1}	$f(X_1)\psi_{n+1}(X_1)$	$f(X_2)\psi_{n+1}(X_2)$	....	$f(X_{2^n})\psi_{n+1}(X_{2^n})$
$\vdots$	...	....	....	....
ψ_{2^n-1}	$f(X_1)\psi_{2^n-1}(X_1)$	$f(X_2)\psi_{2^n-1}(X_2)$	....	$f(X_{2^n})\psi_{2^n-1}(X_{2^n})$

Compare this matrix with the Q matrix in Table 3-1. If $k + q = 2^n$, and $P_i = (x_1, \cdots, x_n)$ and the vertices in the Q matrix, $X_1^1, X_2^1, \ldots, X_k^1, X_1^2, \ldots, X_q^2$, are relabeled as the corresponding vertices in Table 3-7, then we obtain a relation between the entries of these two matrices as :

$$(x_{i_1} \cdots x_{i_j}, X_i) = \frac{1}{2^{n-j}} \left[\sum_{\psi_k \in (i_1, \cdots, i_j)} (-1)^{k-j} (\psi_k, X_i) \right] \qquad (3\text{-}36)$$

where (ψ_k, X_i) is the value of an entry in orthogonal expansion, and $\psi_k \in (i_1, \cdots, i_j)$ means all the ψ_k orthogonal functions which contain the primary variables of $x_{i_1}, \ldots, x_{i_j}$, $(x_{i_1} \cdots x_{i_j}, X_i)$ being an entry in Q matrix with row designator $x_{i_1} \cdots x_{i_j}$ and column designator X_i. In the present method the ϕ-function can transfer any two disjoint sets, C_1 and C_2 in E^n, into whole domain of real value. And in the orthogonal expansion employed by Golomb [20] and Colemen [13], the disjoint sets C_1 and C_2 must be $C_1 \cap C_2 = \theta$, and f(X) function transfers C_1 and C_2 into two different point in the domain of real values. Actually the f(X) functions with orthogonal bases are the nth degree discriminant functions without decreasing the degree and terms by proper adjustment of c_i's.

5. Some Discussions about Minimum Weights:

Let I be a row vector with all 2^n entries equal to 1. Then by Equation (3-19), we have:

$$I\,Q\,W^T = I\,A^T$$

i.e.

$$\sum_{j=1}^{2^n} \sum_{i=1}^{2^n} W_{X_j}\, b_{ij} = \sum_{i=0}^{2^n-1} a_i$$

Since

$$\sum_{i=1}^{2^n} b_{ij} = 0 \ , \quad \text{when } X_j \neq P_i$$
$$= 1 \ , \quad \text{when } X_j = P_i \in C_1$$
$$= -1 \ , \quad \text{when } X_j = P_i \in C_2$$

we have:

$$w_{X_j} b_{2^n j} = \sum_{i=0}^{2^n-1} a_i \ , \quad \text{when } X_j = P_i \tag{3-37}$$

where

$$b_{2^n j} = 1 \quad \text{if } X_j \in C_1$$

$$b_{2^n j} = -1 \quad \text{if } X_j \in C_2 \ .$$

That is,

$$w_{P_i} = \left| \sum_{i=0}^{2^n-1} a_i \right| . \tag{3-38}$$

By Theorem 3, we obtain:

$$w_{X_i} = \left| \phi_{C_1 | C_2 (P_i)}(X_i) \right| . \tag{3-39}$$

Then

$$w_{\overline{P}_i} = |a_o| \tag{3-40}$$

where $\overline{P}$ is a vertex with all primary variables negated to the correspon-ding variables in P_i.

For a first order threshold function, it has been proved that there is always a set of positive weights, $-a_o, a_1, \cdots, a_n$, with respect to a certain P_i (here we assume $P_i = P_1 = (x_1, x_2, \cdots, x_n)$ without loss of generality such that

$$\phi_{C_1 | C_2}(P_1) = a_o + a_1 x_1 + \cdots + a_n x_n \ .$$

Then

$$\sum_{i=0}^{2^n-1} a_i = \sum_{i=1}^{n} a_i - |a_o| \ .$$

Now by Equations (3-38) and (3-40), we obtain:

$$w_{P_1} + w_{\overline{P}_1} = \left| \sum_{i=1}^{n} a_i - |a_o| \right| + |a_o| = \left| \sum_{i=1}^{n} a_i \right|$$

i.e.

$$w_{P_1} + 2w_{\overline{P}_1} = \sum_{i=1}^{n} a_i + |a_o| = \sum_{i=1}^{n} a_i + w_{\overline{P}_1} \ . \qquad (3\text{-}41)$$

Assume W to be the set of minimum positive weights for the first degree ϕ-function, that is, minimum $\sum_{i=1}^{2^n} w$. Thus, since w_{P_1} and $w_{\overline{P}_1}$ are all minimum positive values, their sum is also a minimum. Therefore in the case of first degree ϕ-function, we have:

$$\min(w_{P_i} + 2w_{\overline{P}_i}) = \min_{P_i} \sum_{i=0}^{n} |a_i| \ .$$

Now, we transfer x_1 to $\overline{x}_1$ in the domain of ϕ-function. Setting $P_2 = (\overline{x}_1, x_2, \cdots, x_n)$, we have:

$$\phi_{C_1 | C_2(P_2)} = a_o + a_1(1 - \bar{x}_1) + \cdots + a_n x_n$$
$$= (a_o + a_1) - a_1 \bar{x}_1 + \cdots + a_n x_n$$

i.e.

$$w_{P_2} + 2w_{\bar{P}_2} = \sum_{i=1}^{n} a_i + \left| a_1 - |a_o| \right| = \sum_{i=1}^{n} a_i + w_{\bar{P}_2} \quad . \tag{3-42}$$

Comparing (3-41) and (3-42), we have:

$$\min_{P_k} \sum_{i=0}^{n} a_i \lesseqgtr \min_{P_j} \sum_{i=0}^{n} a_i \quad \text{if} \quad w_{\bar{P}_k} \lesseqgtr w_{\bar{P}_j} \tag{3-43}$$

Equation (3-43) is true for any P_j and P_k. Equation (3-43) shows that the minimum set of a's depends on the minimum value of $w_{\bar{P}_i}$. For a fixed P_i, the minimum W set will give the minimum set A.

For the rth degree ϕ-function, $r > 1$, the relation of inequalities (3-43) does not depend on minimum $w_{\bar{P}_i}$ only, since the absolute values of a's of the rth degree terms are not changed for any transformation of P_i, but the absolute values of a's of the terms with degree less than r are changed for the transformation.

Now, if we define the minimum set A for a minimum degree ϕ-function as

$$\min \left| \sum_{i=1}^{2^n - 1} a_i \right| \tag{3-44}$$

then for a fixed P_i, the minimum W set will imply the minimum set A.

6. Procedure of Vertex Weight Method:

We put

$$\phi_{C_1 | C_2(P_i)}(X) = \phi_{C_1(P_i)}(X) - \phi_{C_2(P_i)}(X) + \phi_{U\setminus(C_1 \cup C_2)(P_i)}(X) \qquad (3\text{-}45)$$

where

$$\begin{aligned}\phi_{C_1 | C_2(P_i)}(X) &\geq 1, \quad \text{if } X \in C_1 \\ &\leq -1, \quad \text{if } X \in C_2 \\ &= \text{any integer}, \quad \text{if } X \in U\setminus(C_1 \cup C_2).\end{aligned}$$

The weights of the vertices belonging to C_1 or C_2 are restricted to be greater than zero, and the weights of vertices belonging to $U\setminus(C_1 \cup C_2)$ are unrestricted and can be any integer. Here the restricted weights will be the restricted variables and unrestricted weights will be the unrestricted variables in the homogeneous system.

According to the homogeneous system of (3-28), we first find the minimum degree ϕ-function. We start to construct the ϕ-function with degree n. If there is no contradiction in system (3-28) with $r = n$, then we decrease r and put $r = n-1$. If there is no contradiction in system with $r = n-1$, then we decrease r to $r = n-2$, and so on, until a minimum r for the system of (3-28) without contradiction is obtained.

After the minimum degree ϕ-function is obtained, we still can eliminate the number of terms in the minimum degree ϕ-function by adjusting the vertex weights. Our method for the elimination of the number of terms in the minimum degree ϕ-function is by adding $\vec{b}_{i_1 \ldots i_k} \cdot W = 0$ of an associated secondary variable $x_{i_1} \cdots x_{i_k}$ with k less than r to the system of (3-28). If there is no contradiction in the extended system, we absorb it in the system. If there is a contradiction, we delete it. Then the next homogeneous equation is jointed to the system, and so on, until all the equations of the coefficients of the secondary variables with degree less than r are tested.

Finally the solution for W is obtained. If the solution of an unrestricted weight is greater than zero, we can classify the associated vertex into set C_1. If it is less than zero, we can classify the vertex into set C_2. If it is equal to zero, we can classify it into either C_1 or C_2, or into some other set depending on how we define the gap between C_1 and C_2 in the domain of ϕ-function.

After W has been obtained, the variable weights vector A can be easily calculated by Equation(3-19).

Here we do not consider eliminating the primary variables. If it is desired, then Step 2) of the following procedure will be changed, that is, "N>1" in Step 2) will be replaced by "N > 0".

The method of solution of the homogeneous system as discussed above has to be modified. The first step in the method is to be deleted, and Step 10) and the stop process in the method has to be led back to the main steps of the procedure.

The procedure is as follows:

Step 1) Form an expansion matrix as in Table 3-1 with respect to $P = (x_1, \cdots x_n)$.

Step 2) Check index N to see if $N > 1$. N starts from $N = n$.

A. If $N > 1$, proceed to Step 3).

B. If $N \not> 1$, proceed to Step 10).

Step 3) Check Q_N to see if there is any contradiction row. Q_N is the matrix of rows in Q with associated secondary variables of degree N.

A. If any, proceed to Step 6).

B. If none, proceed to Step 4).

Step 4) Form the B matrix which is the joint of Q'_N and the former reduced matrix R.

$$B = \begin{pmatrix} Q'_N \\ R \end{pmatrix}$$

where Q'_N is the reduced Q_N with respect to depent variables in R. When $N = n$, $R = 0$.

Step 5) Solve this homogeneous system by means of the procedure stated in Section 2 of this chapter.

A. If there is any solution, rename the reduced echelon matrix of B as R, set $N \longrightarrow N - 1$, and proceed to Step 2).

B. If there is no solution, proceed to Step 6).

Step 6) Check index I to see if $I \leq n+1$. I starts at

$$I = 2^n - \sum_{j=1}^{N} \binom{n}{j}$$

which denotes the index number of the lowest row in Q_N.

A. If $I \leq n+1$, proceed to Step 10).

B. If $I \not\leq n+1$, proceed to Step 7).

Step 7) Check the row $\vec{b}_I$ to see if there is any contradiction.

A. If there is any contradiction, delete this row, proceed to Step 6), and set $I \longrightarrow I - 1$.

B. If no contradiction, proceed to Step 8).

Step 8) Form B which is the joint matrix of the reduced matrix of $\vec{b}_I$ and R, and proceed to Step 9).

Step 9) Solve the system by means of the method stated in Section 6.

A. If there is solution, proceed to Step 6), rename the above reduced B matrix as R, and set $I \longrightarrow I-1$.

B. If there is no solution, proceed to Step 6), delete $\vec{b}_I$, and set $I \longrightarrow I-1$.

Step 10) From the solution of R, find W, and form the ϕ-function.

7. <u>Example 3-2:</u>

In E^4,

$C_1 = \{ 1111, 1110, 1011, 1001, 0100, 0000 \}$

$C_2 = \{ 1101, 1000, 0111, 0001 \}$.

Step 1) The expansion matrix Q is formed as shown below.

Table 3-8

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	
	1111	1110	1011	1001	0100	0000	1101	1000	0111	0001	1100	1010	0110	0101	0011	0010	
1						1											Q_o
2						-1		-1									Q_1
3					1	-1											
4						-1										1	
5						-1				-1							
6					-1	1		1			1						Q_2
7						1		1				1				-1	
8				1		1		1		1							
9					-1	1							1			-1	
10					-1	1				1				1			
11						1				1					1	-1	
12		1			1	-1		-1			-1	-1	-1			1	Q_3
13				-1	1	-1	-1	-1		-1	-1			-1			
14			1	-1		-1		-1		-1		-1			-1	1	
15					1	-1			-1	-1			-1	-1	-1	1	
16	1	-1	-1	1	-1	1	1	1	1	1	1	1	1	1	1	-1	Q_4

where the row designators are listed below:

row	designator	row	designator	row	designator
1	1	7	x_1x_3	13	$x_1x_2x_4$
2	x_1	8	x_1x_4	14	$x_1x_3x_4$
3	x_2	9	x_2x_3	15	$x_2x_3x_4$
4	x_3	10	x_2x_4	16	$x_1x_2x_3x_4$
5	x_4	11	x_3x_4		
6	x_1x_2	12	$x_1x_2x_3$		

In this matrix all zero entries are left blank.

Step 2) $N = 4$. The condition A is satisfied. So we proceed to Step 3).

Step 3) The condition B is satisfied. So we proceed to Step 4).

Step 4) Here $R = 0$, $N = 4$.

Therefore, $B = Q_4$.

Step 5) We next proceed to the method of solving the homogeneous system.

5-2) We find $b_{16\ 11}$. Since the condition A is satisfied, we proceed to Step 5-3).

5-3) Since the condition B is satisfied, we proceed to Step 5-4).

5-4) Since the condition A is satisfied, we proceed to Step 5-8).

5-8) Since the condition B is satisfied, we proceed to the main step again.

The condition A of Step 5) is satisfied. So we proceed to Step 2), and set $N = 3$, and reduced B as R.

Step 2) $N = 3$. The condition A is satisfied, so we proceed to Step 3).

Step 3) We check Q_3. Since the condition B is satisfied, we proceed to Step 4).

Step 4) We obtain B matrix, as shown below.

Table 3-9

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
12	1		-1	1			1		1	1				1	1	
13	1	-1	-1						1			1	1		1	-1
14			1	-1		-1		-1		-1		-1			-1	1
15					1	-1			-1	-1			-1	-1	-1	1
16	1	-1	-1	1	-1	1	1	1	1	1	1	1	1	1	1	-1

-1: the coefficient of dependent variable.

Step 5) We proceed to the method of solving the homogeneous system.

5-2) We find $b_{12\ 14}$. Since condition A is satisfied, we proceed to 5-3).

5-3) Operating the Table 3-9 of B with respsct to row $\vec{b}_{12}$, we obtain

Table 3-10

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
12	1		-1	1			1		1	1				1	1	
13	1	-1	-1						1			1	1		1	-1
14			1	-1		-1		-1		-1		-1			-1	1
15	1		-1	1	1	-1	1						-1			1
16		-1			-1	1		1			1	1	1			-1

Since the condition B is satisfied, we proceed to Step 5-4).

5-4) Since condition B is satisfied, we proceed to Step 5-2).

5-2) We find $b_{13\ 12}$. Since the condition A is satisfied, we proceed to Step 5-3).

5-3) Operating Table 3-10 of B with respect to row $\vec{b}_{13}$, we obtain:

Table 3-11

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
12	1		-1	1			1		1	1				1	1	
13	1	-1	-1						1			1	1		1	-1
14	1	-1		-1		-1		-1	1	-1			1			
15	1		-1	1	1	-1	1						-1			1
16	-1		1		-1	1		1	-1		1				-1	

Since the condition B is satisfied, we proceed to Step 5-4).

5-4) Since the condition B is satisfied, we proceed to Step 5-2).

5-2) We find $b_{14\ 13}$. Since the condition A is satisfied, we proceed to Step 5-3).

5-3) Operating Table 3-11 of B with respect to row $\vec{b}_{14}$ we obtain Table 3-12.

Table 3-12

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
12	1		-1	1			1		1	1				1	1	
13			-1	1		1		1		1		1			1	-1
14	1	-1		-1		-1		-1	1	-1			1			
15	2	-1	-1		1	-2	1	-1	1	-1						1
16	-1		1		-1	1		1	-1		1				-1	

Since the condition B is satisfied, we proceed to Step 5-4).

5-4) Since the condition B is satisfied, we proceed to Step 5-2).

5-2) We next find $b_{15\ 16}$. Since the condition A is satisfied, we proceed to Step 5-3).

5-3) Operating the Table 3-12 of B, we obtain

Table 3-13 R matrix

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
12	1		-1	1			1		1	1				1	1	
13	2	-1	-2	1	1	-1	1		1			1			1	
14	1	-1		-1		-1		-1	1	-1			1			
15	2	-1	-1		1	-2	1	-1	1	-1						1
16	-1		1		-1	1		1	-1		1				-1	

Since the condition B is satisfied, we proceed to Step 5-4).

5-4) Since the condition A is satisfied, we proceed to Step 5-8).

5-8) Since the condition B is satisfied, we proceed to the main step.

The condition A of Step 5) is satisfied. So we proceed to Step 2). and set N = 2 and rename R as shown in Table 3-13.

Step 2) N = 2. The condition A is satisfied, we proceed to Step 3).

Step 3) We now check Q_2. There is a contradiction in row 8. Since the condition A is satisfied, we proceed to Step 6).

Step 6) Here I = 11. Since the condition B is satisfied, we proceed to Step 7).

Step 7) Since the condition B is satisfied, we proceed to Step 8).

Step 8) The B matrix is found as shown below.

Table 3-14

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
11	2	-1	-1		1	-1	1	-1	1						1	
12	1		-1	1			1		1	1				1	1	
13	2	-1	-2	1	1	-1	1		1			1			1	
14	1	-1		-1		-1		-1	1	-1			1			
15	2	-1	-1		1	-2	1	-1	1	-1						1
16	-1		1		-1	1		1	-1		1				-1	

We now proceed to Step 9).

Step 9) We proceed with the method of solving the homogeneous system.

9-2) We find $b_{11\ 14}$. Since the condition A is satisfied, we proceed to Step 9-3).

9-3) Operating the Table 3-14, we obtain Table 3-15.

Table 3-15 R matrix

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
11	2	-1	-1		1	-1	1	-1	1						1	
12	-1	1		1	-1	1		1		1				1		
13			-1	1				1				1				
14	1	-1		-1		-1		-1	1	-1			1			
15	2	-1	-1		1	-2	1	-1	1	-1						1
16	1	-1					1				1					

Since the condition B is satisfied, we proceed to Step 9-4).

9-4) Since the condition A is satisfied, we proceed to Step 9-8).

9-8) Since the condition B is satisfied, we proceed to the main step again.

Since the condition A is satisfied, we proceed to Step 6), and rename B as R as shown in Table 3-15, and set I = 10.

Step 7) We now check $\vec{b}_{10}$. Since the condition B is satisfied, we proceed to Step 8).

Step 8) Form matrix

$$B = \begin{pmatrix} R \\ \vec{b}_{10} \end{pmatrix}$$ as shown in Table 3-16.

Table 3-16

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
10	1	-1		-1				-1								
11	2	-1	-1		1	-1	1	-1	1						1	
12	-1	1		1	-1	1		1		1				1		
13			-1	1				1				1				
14	1	-1		-1		-1		-1	1	-1			1			
15	2	-1	-1		1	-2	1	-1	1	-1						1
16	1	-1					1				1					

We next proceed to Step 9).

Step 9) We proceed with the method of solving the homogeneous system.

9-2) The condition B is satisfied, so we proceed to Step 9-5).

9-5) As M = 1 so we find $b_{10\,1}$ and proceed to 9-6).

9-6) Operating Table 3-16 of B, we obtain

Table 3-17 R matrix

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
10	1	-1		-1				-1								
11		1	-1	2	1	-1	1	1	1						1	
12					-1	1				1				1		
13			-1	1				1				1				
14						-1			1	-1			1			
15		1	-1	2	1	-2	1	1	1	-1						1
16				1			1	1			1					

Since the condition B is satisfied, we proceed to Step 9-7).

9-7) Since the condition A is satisfied, we proceed to Step 9-8).

9-8) From $\vec{b}_{10}$, we obtain:

$$w_1 = w_2 + w_4 + w_8 .$$

Since the condition B is satisfied, we proceed to the main step again.

Since the condition A is satisfied, we proceed to Step 6), and rename B as R in Table 3-17, and set I = 9.

Step 6) Since the condition B is satisfied, we proceed to Step 7).

Step 7) We mext check $\vec{b}_9$. Since the condition B is satisfied, we proceed to Step 8).

Step 8) Form matrix

$$B = \begin{pmatrix} R \\ \vec{b}_9 \end{pmatrix}$$ as shown in Table 3-18.

Table 3-18

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
9		1	-1	2			1	1								
10	1	-1		-1				-1								
11		1	-1	2	1	-1	1	1	1						1	
12					-1	1				1				1		
13			-1	1				1				1				
14						-1			1	-1			1			
15		1	-1	2	1	-2	1	1	1	-1						1
16				1			1	1			1					

Then we proceed to Step 9)

Step 9) We proceed with the method of solving the homogeneous system.

9-2) Since the condition B is satisfied, we proceed to Step 9-5).

9-5) As M = 1, we find $b_{9\,3}$ and proceed to 9-6).

9-6) We obtain R matrix, as shown below.

Table 3-19 R matrix

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
9		1	-1	2			1	1								
10	1	-1		-1				-1								
11					1	-1			1						1	
12					-1	1				1				1		
13		-1		-1			-1					1				
14						-1			1	-1			1			
15					1	-2			1	-1						1
16				1			1	1			1					

Since the condition B is satisfied, we proceed to Step 9-7).

9-7) Since the condition A is satisfied, we proceed to Step 9-8).

9-8) From $\vec{b}_9$ and $\vec{b}_{10}$, we obtain:

$$w_3 = w_2 + 2w_4 + w_7 + w_8 ,$$

$$w_1 = w_2 + w_4 + w_8 .$$

Since the condition B is satisfied, we proceed to the main Step again.

Since the condition A of Step 9) is satisfied, we proceed to Step 6), and rename B as R in Table 3-19, and set I = 8.

Step 6) The condition B is satisfied, so we proceed to Step 7).

Step 7) We check $\vec{b}_8$. Since the condition A is satisfied, we delete $\vec{b}_8$ and set I = 7 and proceed to Step 6).

Step 6) Since the condition B is satisfied, we proceed to Step 7).

Step 7) We now check $\vec{b}_7$. Since the condition B is satisfied, we proceed to Step 8).

Step 8) Form matrix

$$B = \begin{bmatrix} R \\ \vec{b}_7 \end{bmatrix}$$ as shown in Table 3-20.

Table 3-20

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
7																
9		1	-1	2			1	1								
10	1	-1		-1				-1								
11					1	-1			1						1	
12					-1	1				1				1		
13		-1		-1			-1					1				
14						-1			1	-1			1			
15					1	-2			1	-1						1
16				1			1	1			1					

We proceed to Step 9).

Step 9) We proceed with the method to solve the homogeneous system.

9-2) Since the condition B is satisfied, we proceed to Step 9-5).

9-5) As M = 2 we find $b_{7\,6}$. Since the condition A is satisfied, we proceed to Step 9-6).

9-6) The R matrix is obtained as given below.

Table 3-21 R matrix

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
7		1		1	1	-1	1	1	1	-1						
9		1	-1	2			1	1								
10	1	-1		-1				-1								
11		-1		-1			-1	-1		1					1	
12		1		1			1	1	1					1		
13		-1		-1			-1					1				
14		-1		-1	-1		-1	-1					1			
15		-2		-2	-1		-2	-2	-1	1						1
16				1			1	1			1					

Since the condition B is satisfied, we proceed to Step 9-7).

9-7) Since the condition A is satisfied, we proceed to Step 9-8).

9-8) From $\vec{b}_7$, $\vec{b}_9$, and $\vec{b}_{10}$, we obtain:

$$w_6 = w_2 + w_4 + w_5 + w_7 + w_8 + w_9 \quad w_{10} ,$$

$$w_3 = w_2 + 2w_4 + w_7 + w_8 ,$$

$$w_1 = w_2 + w_4 + w_8 .$$

From f_6, we have:

$w_2 + w_4 + w_5 + w_7 + w_8 + w_9 > w_{10}$.

Since the condition A is satisfied, we proceed to Step 9-9).

9-9) Since the condition B is satisfied, we proceed to the main step again.

Since the condition A is satisfied, we proceed to Step 6), and set $I = 6$.

Step 6) Since the condition B is satisfied, we proceed to Step 7).

Step 7) We check $\vec{b}_6$. Since the condition B is satisfied, we proceed to Step 8).

Step 8) Form matrix

$$B = \begin{bmatrix} R \\ \vec{b}_6 \end{bmatrix}$$ as shown in Table 3-22.

Table 3-22

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
6		1						1	1	-1						
7		1		1	1	-1	1	1	1	-1						
9		1	-1	2			1	1								
10	1	-1		-1				-1								
11		-1		-1			-1	-1		1					1	
12		1		1			1	1	1					1		
13		-1		-1			-1					1				
14		-1		-1	-1		-1	-1					1			
15		-2		-2	-1		-2	-2	-1	1						1
16				1			1	1			1					

We now proceed to Step 9).

Step 9) We proceed with the method to solve the homogeneous system

9-2) Since the condition B is satisfied, we proceed to Step 9-5).

9-5) $M = 1$, so we find $b_{6\,10}$. Since the condition A is satisfied, we proceed to 9-6).

9-6) The R matrix is obtained as given below.

Table 3-23 R matrix

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
6		1						1	1	-1						
7				1	1	-1	1									
9		1	-1	2			1	1								
10	1	-1		-1				-1								
11				-1			-1		1						1	
12		1		1			1	1	1					1		
13		-1		-1			-1					1				
14		-1		-1	-1		-1	-1					1			
15		-1		-2	-1		-2	-1								1
16				1			1	1			1					

Since the condition B is satisfied, we proceed to Step 9-7).

9-7) Since the condition A is satisfied, we proceed to Step 9-8).

9-8) From $\vec{b}_6$, $\vec{b}_7$, $\vec{b}_9$, and $\vec{b}_{10}$, we have:

$$w_{10} = w_2 + w_8 + w_9 ,$$

$$w_6 = w_4 + w_5 + w_7 , \qquad (3\text{-}46)$$

$$w_3 = w_2 + 2w_4 + w_7 + w_8 ,$$

$$w_1 = w_2 + w_4 + w_8 .$$

Since the condition B is satisfied, we proceed to the main step.

The condition A of Step 9) is satisfied, so we proceed to Step 6), and rename B as R, in Table 3-23, and set I = 5.

Step 6) Since the condition A is satisfied, we proceed to Step 10).

Step 10) From Equations (3-46), we have:

$$w_2 = w_4 = w_5 = w_7 = w_8 = w_9 = 1$$

and

$$w_{10} = 3 \quad w_6 = 3 \quad w_3 = 5 \quad w_1 = 3$$

and by the last R in Table 3-23, we find

$$w_{14} = -w_2 - w_4 - w_7 - w_8 - w_9 = -5 \ ,$$

$$w_{15} = w_4 + w_7 - w_9 = 1 \ ,$$

$$w_{11} = -w_4 - w_7 - w_8 = -3 \ ,$$

$$w_{13} = w_2 + w_4 + w_5 + w_7 + w_8 = 5 \ ,$$

$$w_{12} = w_2 + w_4 + w_7 = 3 \ ,$$

$$w_{16} = w_2 + 2w_4 + w_5 + 2w_7 + w_8 = 7 \ ,$$

We finally obtain

$$\phi(X) = 3 - 4x_1 - 2x_2 + 4x_3 - 6x_4 + 8x_1x_4$$

where X_{12}, X_{13}, $X_{16} \in C_1$ and X_{11}, $X_{14} \in C_2$.

CHAPTER IV

ϕ-FUNCTIONS OF SYMMETRICAL SETS

1. Q Matrix for Symmetrical Sets:

To find the minimum degree ϕ-function for a symmetrical Boolean function is simpler than the genereal case. It has been proved that the variable weights of the symmetrical variables in a first order threshold function can be equal [9]. Assume that all the vertices in E^n space are separated into disjoint subsets as:

$$s_n \;,\; s_{n-1} \;,\; \cdots \;,\; s_i \;,\; \cdots \;,\; s_o$$

where s_i denotes a subset of vertices, each with i 1's in the positivization of $P_1 = (x_1, x_2, \cdots, x_n)$. Then a symmetrical Boolean function with respect to P_1 can be denoted by a set C_1, called a symmetrical set, such that

$$C_1 = \bigcup_i s_i \;.$$

Arrange the ordering of the vertices in the Q matrix in the decreasing order of subscripts of s_i. Then we can obtain a matrix with entries on the upper side of the digonal being all zero.

For instance, in E^3, $C_1 = s_1 \cup s_o$, and the Q-matrix will be as shown in Table 4-1.

It is obvious that if the function is a first order threshold function, the variable weights of the primary variables can be equal, and in turn, the vertex weights in a s_i will also be equal. Thus for a symmetrical

Table 4-1 Q-matrix

	s_3	s_2			s_1			s_o
	111	011	101	110	001	010	100	000
1								1
x_1							1	-1
x_2						1		-1
x_3					1			-1
x_1x_2				-1		-1	-1	1
x_1x_3			-1		-1		-1	1
x_2x_3		-1			-1	-1		1
$x_1x_2x_3$	-1	1	1	1	1	1	1	-1

set, no matter of what order the ϕ-function may be, the vertex weights of the vertices in the same s_i, $i = 0, \cdots, n$, can be taken to be equal. Hence the relation of (3-19)

$$A^T = Q\, W^T$$

can be simplified.

For the above example, we compare the variable weights:

$$a_{12} = -w_{110} - w_{010} - w_{100} + w_{000} \,,$$

$$a_{13} = -w_{101} - w_{001} - w_{100} + w_{000} \,,$$

$$a_{23} = -w_{011} - w_{001} - w_{010} + w_{000} \,.$$

Set

$$w_2 = w_{110} = w_{101} = w_{011} \,,$$

$$w_1 = w_{100} = w_{010} = w_{001} \,,$$

$w_0 = w_{000}$.

Then

$a_{12} = a_{13} = a_{23} = -w_2 - 2w_1 + w_0$,

where w_i gives the vertex weight in subset s_i. Then the Q matrix is reduced to that shown in Table 4-2.

Table 4-2

	s_3	s_2	s_1	s_0
1				1
x_{i_1}			1	-1
$x_{i_1} x_{i_2}$		-1	-2	1
$x_1 x_2 x_3$	-1	3	3	1

It is interesting to note that the values of entries in the table show the relations of a Pascal triangle. The values of the entries in each row depend on the degree of the associated variable, and the sign of each entry depends on whether s_i belongs to C_1 or C_2.

In genereal, the Q matrix of a symmetrical set will be as shown in Table 4-3.

In the table, $d_{j,i}$ determines the sign of entry (j, i), and the first column on the left side of the matrix shows the number of primary variables in the corresponding secondary variable of each row. Here

$$d_{j,j} = +1 \quad , \quad \text{if } s_j \in C_1 \quad , \tag{4-1}$$

$$d_{j,j} = -1 \quad , \quad \text{if } s_j \in C_2 \quad ,$$

Table 4-3 Q matrix of symmetrical sets

		s_n	$\cdots$	s_j	$\cdots$	s_i	$\cdots$	s_2	s_1	s_0
0	1									$d_{0,0}$
1	x_{i_1}								$d_{1,1}$	$d_{1,0}$
2	$x_{i_1}x_{i_2}$							$d_{2,2}$	$2d_{2,1}$	$d_{2,0}$
3	$x_{i_1}x_{i_2}x_{i_3}$							$\cdots$	$\cdots$	$\cdots$
$\vdots$						$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
j	$x_{i_1}\cdots x_{i_j}$			$d_{j,j}$	$\cdots$	$\binom{j}{i}d_{j,i}$	$\cdots$	$\cdots$	$\cdots$	$d_{j,0}$
$\vdots$			$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	
n	$x_1\cdots x_n$	$d_{n,n}$	$\cdots$	$\cdots$	$\cdots$	$\binom{n}{i}d_{n,i}$	$\cdots$	$\cdots$	$\cdots$	$d_{n,0}$

and

$$d_{j,i} = -d_{(j-1),i} \,, \tag{4-2}$$

where $j = 0, 1, \cdots, n$ and $i = 0, 1, \cdots, j$. Equations (4-1) and (4-2) show the important relations between the subsets of s_i's and the signs in corresponding columns.

The above Q matrix is utilized to find the ϕ-functions for symmetrical sets.

Definition 7: A symmetrical set is called an rth order symmetrical set if it can be realized by a minimum ϕ-function with degree r.

Definition 8: Set the subsets $\{s_i\}$ in E^n in sequence of decreasing i, $i = n, \cdots, 1, 0$, i.e.

$$s_n\ ,\quad s_{n-1}\ ,\quad \ldots\ ,\quad s_2\ ,\quad s_1\ ,\quad s_0\ .$$

A run of s_i's belonging to the same set C_1 or C_2 is called a partition block.

Later in this chapter we will show the relation between the number of partition blocks of a symmetrical set and its order.

2. Homogeneous Systems for Symmetrical sets:

By Theorem 9, the homogeneous system for symmetrical sets of any order can be obtained from Table 4-3. For instance, the homogenous system for $(n-2)$ order symmetrical sets is:

$$d_{n-1,n-1}w_{n-1} + \binom{n-1}{n-2}d_{n-1,n-2}w_{n-2} + \cdots + \binom{n-1}{i}d_{n-1,i}w_i + \cdots + d_{n-1,0}w_0 = 0\ , \tag{4-3}$$

$$d_{n,n}w_n + \binom{n}{n-1}d_{n,n-1}w_{n-1} + \cdots + \binom{n}{i}d_{n,i}w_i + \cdots + d_{n,0}w_0 = 0\ .$$

In order to show the relation of the vertex weights more clearly, here we choose the weights of $d_{i,i}$, $i = n, n-1, \ldots$ terms as dependent variables of the equations. Then by Equ. (4-2) the reduced homogeneous system of (4-3) is:

$$\begin{aligned} &d_{n-1,n-1}w_{n-1} + \cdots + \binom{n-1}{i}d_{n-1,i}w_i + \cdots + d_{n-1,0}w_0 = 0\ , \\ &d_{n,n}w_n + \binom{n}{n-1}d_{n-1,n-2}w_{n-2} + \cdots + \binom{n}{i}\binom{n-i-1}{i}d_{n-1,i}w_i \\ &\qquad + \cdots + \binom{n-1}{i}d_{n-1,0}w_{n-1,0} = 0\ . \end{aligned} \tag{4-4}$$

In general, for rth order set we have the system as follows:

$$\begin{array}{ll}
r+1 & d_{r+1,r+1}w_{r+1} + \cdots + \binom{r+1}{i} d_{r+1,i}w_i + \cdots + d_{r+1,0}w_0 = 0 , \\
\vdots & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
k & d_{k,k}w_k + \cdots + \binom{k}{i} d_{k,i}w_i + \cdots + d_{k,0}w_0 = 0 , \\
\vdots & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
n & d_{n,n}w_n + \cdots + \binom{n}{i} d_{n,i}w_i + \cdots + d_{n,0}w_0 = 0 ,
\end{array} \tag{4-5}$$

where $n \geq k \geq r+1$. The numbers on the left side of the system indicate the corresponding numbers in the first column of Table 4-3. The reduced form is:

$$\begin{array}{ll}
r+1 & d_{r+1,r+1}w_{r+1} + \cdots + \binom{r+1}{i} d_{r+1,i}w_i + \cdots + d_{r+1,0}w_0 = 0 , \\
\vdots & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
k & d_{k,k}w_k + \cdots + \binom{k}{i}\binom{k-i-1}{k-r-1} d_{r+1,i}w_i + \cdots + \binom{k-1}{k-r-1} d_{r+1,0}w_0 = 0 , \\
\vdots & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
n & d_{n,n}w_n + \cdots + \binom{n}{i}\binom{n-i-1}{n-r-1} d_{r+1,i}w_i + \cdots + \binom{n-1}{n-r-1} d_{r+1,0}w_0 = 0 ,
\end{array} \tag{4-6}$$

where $0 \leq i \leq r$.

3. Examples to Show Relation between Number of Partition Blocks and Order of Symmetrical Sets:

Using the Q matrix of symmetrical function as given in Table 4-3, we have found all ϕ-functions of the symmetrical sets of variables up to 5. The order of the symmetrical sets for each n is listed below. Here we use the notions

$s_{ijk\cdots} = s_i \cup s_j \cup s_k \cup \cdots \in C_1$,

$s_\theta = \theta$.

Table 4-4 The Order of the Symmetrical Sets for n up to 5

n = 1

zero order sets	s_{01}	s_θ				
1st order sets	s_0	s_1				

n = 2

zero order sets	s_{012}	s_θ				
1st order sets	s_0	s_2	s_{01}	s_{12}		
2nd order sets	s_1	s_{02}				

n = 3

zero order sets	s_{0123}	s_θ				
1st order sets	s_0	s_3	s_{01}	s_{23}	s_{012}	s_{123}
2nd order sets	s_1	s_2	s_{03}	s_{12}	s_{013}	s_{023}
3rd prder sets	s_{02}	s_{13}				

n = 4

zero order sets	s_{01234}	s_θ				
1st order sets	s_0	s_4	s_{01}	s_{34}	s_{012}	s_{234}
	s_{0123}	s_{1234}				
2nd order sets	s_1	s_2	s_3	s_{04}	s_{12}	s_{23}

	s_{014}	s_{034}	s_{123}	s_{0124}	s_{0234}	s_{0134}
3rd order sets	s_{02}	s_{03}	s_{14}	s_{24}	s_{013}	s_{023}
	s_{124}	s_{134}				
4th order sets	s_{024}	s_{13}				

$n = 5$

zero order sets	s_{012345}	s_{θ}				
1st order sets	s_{0}	s_{5}	s_{01}	s_{45}	s_{012}	s_{345}
	s_{0123}	s_{2345}	s_{01234}	s_{12345}		
2nd order sets	s_{1}	s_{2}	s_{3}	s_{4}	s_{05}	s_{12}
	s_{23}	s_{34}	s_{015}	s_{045}	s_{123}	s_{234}
	s_{0125}	s_{0145}	s_{0345}	s_{1234}	s_{01235}	s_{01245}
	s_{01345}	s_{02345}				
3rd order sets	s_{02}	s_{03}	s_{04}	s_{15}	s_{25}	s_{35}
	s_{013}	s_{014}	s_{023}	s_{034}	s_{125}	s_{145}
	s_{235}	s_{245}	s_{0124}	s_{0134}	s_{0234}	s_{1235}
	s_{1245}	s_{1345}				
4th order sets	s_{13}	s_{14}	s_{24}	s_{025}	s_{035}	s_{124}
	s_{134}	s_{0135}	s_{0235}	s_{0245}		

5th order sets s_{024} s_{135}

From the above results we find the following interesting relation:

$$r = m - 1 \tag{4-7}$$

where r is the order of the symmetrical set and m is the number of partition blocks.

For instance, when $n = 5$, the sequence of a subset is

$\underline{s_5}$, s_4 , $\underline{s_3 \ , \ s_2 \ , \ s_1}$, s_0 .

where $m = 4$. Then s_{1235}, and s_{04} are 3rd order symmetrical sets. The under lined s_i's here belong to C_1, and the rest to C_2, or vise versa.

4. <u>Some Discussions about Partition Blocks and Order of Symmetrical Sets:</u>

From Section 3 we find that when $n = 5$, the relation (4-7) exists. It has not been proved that relation (4-7) exists in general for any positive integer n. Here we show that when $r \leq 1$ and $r \geq n-1$, the relation (4-7) exists for any positive integer n.

(1) $r = 0$:

According to Eqn. (4-6), the reduced system is :

$$\begin{aligned}
d_{1,1}w_1 + d_{1,0}w_0 &= 0 \ , \\
&\cdots\cdots\cdots\cdots \\
d_{k,k}w_k \qquad\qquad + d_{1,0}w_0 &= 0 \ , \\
&\cdots\cdots\cdots\cdots \\
d_{n,n}w_n \qquad\qquad\qquad + d_{1,0}w_0 &= 0.
\end{aligned} \tag{4-8}$$

If $d_{1,0} = -1$, then all $d_{i,i} = +1$, $i = 1, \cdots, n$. Otherwise, (4-8) does not have a solution. Hence by Eqn. (4-1), we have

$$\underline{s_n, \quad \ldots, \quad s_1, \quad s_0},$$

in one block. Similarly, if $d_{1,0} = +1$, the symmetric subsets are in one block.

(2) $r = 1$:

According Eqn. (4-6) we have the reduction system for a symmetrical set with order 1 as:

$$\begin{aligned}
d_{2,2}w_2 + 2d_{2,1}w_1 + d_{2,0}w_0 &= 0 , \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\
d_{k,k}w_k \qquad + \binom{k}{1} d_{2,1}w_1 + (k-1)d_{2,0}w_0 &= 0 , \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\
d_{n,n}w_n \qquad + nd_{2,1}w_1 + (n-1)d_{2,0}w_0 &= 0 .
\end{aligned} \tag{4-9}$$

Then the inequalities of these homogeneous systems are:

$$\begin{aligned}
\frac{1}{d_{2,2}} (2d_{2,1}w_1 + d_{2,0}w_0) &< 0 , \\
\cdots\cdots\cdots\cdots\cdots\cdots & \\
\frac{1}{d_{k,k}} \left(\binom{k}{1} d_{2,1}w_1 + (k-1)d_{2,0}w_0 \right) &< 0 , \\
\cdots\cdots\cdots\cdots\cdots\cdots & \\
\frac{1}{d_{n,n}} \left(\binom{n}{1} d_{2,1}w_1 + (n-1)d_{2,0}w_0 \right) &< 0 .
\end{aligned} \tag{4-10}$$

In (4-10), there are four possibilities for $d_{2,1}$ and $d_{2,0}$.

Table 4-5

	$d_{2,1}$	$d_{2,0}$
1	+1	+1
2	+1	-1
3	-1	+1
4	-1	-1

We will show that the relation (4-7) exists in each case of Table 4-5.

1) In this case, from (4-10) we have:

$$d_{2,2} = \cdots = d_{i,i} = \cdots = d_{n,n} = -1 \ .$$

By Eqns. (4-1) and (4-2), we have:

$$s_n \ , \quad s_{n-1} \ , \quad \ldots \ , \quad s_1 \ , \quad \underline{s_0} \ .$$

The relation (4-7) therefore exists.

2) Assume

$$d_{k,k} = +1$$

$$d_{j,j} = -1$$

From (4-10), we have:

$$k\,w_1 < (k-1)\,w_0 \ , \tag{4-11}$$

$$j\,w_1 > (j-1)\,w_0 \ . \tag{4-12}$$

By successive higher ordering method* [56], from (4-12), we obtain:

$$w_1 > \frac{j-1}{j} w_0 ,$$

or

$$w_1 = \Delta w_1 + \frac{j-1}{j} w_0 . \qquad (4\text{-}13)$$

Substituting (4-13) into (4-11), we obtain:

$$(k-1)\, w_0 > k\, \Delta w_1 + \frac{k(j-1)}{j} w_0 . \qquad (4\text{-}14)$$

Comparing the coefficients of w_0 on both sides of inequalities (4-14), we obtain:

$$\frac{k-1}{\frac{k(j-1)}{j}} = \frac{(k-1)}{k} \; \frac{j}{(j-1)} \quad \begin{aligned} &> 1 , \quad \text{if } k > j , \\ &< 1 , \quad \text{if } k < j . \end{aligned} \qquad (4\text{-}15)$$

If there is no contradiction in (4-14), (4-15) must be greater than 1. Then we must have

$$k > j . \qquad (4\text{-}16)$$

That is, if

$$d_{2,2} = \cdots = d_{i,i} = -1 ,$$

then we have

---

* Here the coefficients can be fractions. Integralization of them is not necessary.

$d_{i+1,i+1} = \cdots = d_{n,n} = +1$.

By (4-1) and (4-2) we have:

$\underline{s_n \;, \quad \cdots \;, \quad s_{i+1}} \;, \quad s_i \;, \quad \cdots \;, \quad s_0 \;,$

and so the relation of (4-7) exists.

3) Similarly as in 2), we assume

$d_{k,k} = +1$,

$d_{j,j} = -1$.

Then we have:

$$k\, w_1 > (k-1)\, w_0 \;, \tag{4-17}$$

$$j\, w_1 < (j-1)\, w_0 \;. \tag{4-18}$$

From (4-18), we obtain:

$$w_0 = \Delta w_0 + \frac{j}{j-1}\, w_1 \;. \tag{4-19}$$

Substituting (4-19) into (4-17), we obtain:

$$k\, w_1 > (k-1)\, \Delta w_0 + \frac{j(k-1)}{(j-1)}\, w_1 \;. \tag{4-20}$$

Comparing the coefficients of w_1 on both sides of (4-20), we obtain:

$$\frac{k}{\frac{j(k-1)}{j-1}} = \frac{(j-1)}{j}\,\frac{k}{(k-1)} \begin{cases} > 1 \;, & \text{if } k < j \;, \\ < 1 \;, & \text{if } k > j \;. \end{cases} \tag{4-21}$$

If there is no contradiction then we have

$k > j$.

By Eqns. (4-1) and (4-2), that is, if

$d_{2,2} = \cdots = d_{i,i} = +1$,

then we have:

$d_{i+1,i+1} = \cdots = d_{n,n} = -1$.

By Eqns. (4-1) and (4-2), we have

$\underline{s_n \ , \ \ldots \ , \ s_{i+1}} \ , \ s_i \ , \ \ldots \ , s_0$,

and the relation (4-7) exists.

4) It is clear that if there is no contradiction in (4-9), we will have

$d_{i,i} = +1$, $i = 2, \cdots, n.$

That is, we have:

$\underline{s_n \ , \ \ldots \ , \ s_1} \ , \ s_0$,

and the relation (4-7) exists.

For n degree ϕ-function, the row n of (4-5) cannot be zero, that is,

$$d_{n,n}w_n + \binom{n}{n-1} d_{n,n-1}w_{n-1} + \ldots + \binom{n}{i} d_{n,i}w_i + \ldots + d_{n,0}w_0 \neq 0 \ .$$

In that case there must be

$$d_{n,n} = d_{n,n-1} = \ldots = d_{n,i} = \ldots = d_{n,0} = -1 \text{ or } +1 \ .$$

Then by (4-1) and (4-2) , we can easily find that the number of blocks is

$m = n + 1$.

For (n-1) degree ϕ-function, we have:

$$d_{n,n}w_n + \binom{n}{n-1} d_{n,n-1}w_{n-1} + \cdots + \binom{n}{i} d_{n,i}w_i + \cdots + d_{n,0}w_0 = 0 \ .$$

It is clear that if there is any $d_{n,j}$ differing from others, then the system has a solution. Then by (4-1) and (4-2), we find that the number of blocks in this case is:

$$m = n \ .$$

We have tried some examples with upto n=10, and found that the relation (4-7) still holds. Therefore it is our strong conjecture that (4-7) is a general relation for any value of n, although so far we have not obtained a proof.

CHAPTER V

IMPERFECT ϕ-FUNCTIONS

1. Introduction:

In Chapter III we presented a method to form a ϕ-function discriminating a set of vertices C_1 from another set C_2 in E^n space, where $C_1 \cap C_2 = \theta$. A ϕ-function which cannot discriminate the disjoint sets C_1 and C_2 will be called an imperfect ϕ-function.

Similar to Ito and Fukunaga [74], we define the error rate of an imperfect ϕ-function as:

$$e = \frac{n_e}{N_1 + N_2} \tag{5-1}$$

where

n_e: number of mis-discriminated vertices in C_1 and C_2 by the imperfect ϕ-function,

N_1: number of vertices in set C_1,

N_2: number of vertices in set C_2.

The mis-discriminated vertices can be either all in C_1, or all in C_2, or in both. Geometrically, for first degree ϕ-functions, a plane is defined as a single-sided small error plane [19] if the mis-discriminated vertices are either all in C_1, or all in C_2, which situation has been discussed by Mattson [78], and a plane is defined as a double-sided small error plane if the mis-discriminated vertices are in both C_1 and C_2, which situation has been discussed by Minnick [29] and others. Here we will generalize the hyperplanes to hypersurfaces and therefore will not restrict our imperfect ϕ-functions to the first degree.

Now we will use a superscript for the error rate, e, to denote the number of primary and secondary variables of the ϕ-function, and use a subscript to denote the set to which the mis-discriminated vertices belong. Then in the extreme case, the number of primary and secondary variables is zero, i.e. the ϕ-function is a constant function, which obviously discriminates the vertices either all to C_1 or all to C_2. Thus we have:

$$e_1^0 = \frac{N_1}{N_1 + N_2} \quad ,$$

and

$$e_2^0 = \frac{N_2}{N_1 + N_2} \quad .$$

If we define

$$e_{min}^0 = \min(e_1^0, e_2^0) \tag{5-2}$$

then we have:

$$e_{min}^0 \leq 1/2 \quad .$$

2. Realization of Imperfect ϕ-functions:

The imperfect ϕ-functions can be formed by means of the vertex weight method. We concluded that in this method the unrestricted variables of weights can be any real values, and the don't care equations in the homogeneous system are all not contradictory. Hence in the process of solving the homogeneous system, we can eliminate any contradictory relation in the system by assigning it a don't care equation, and the dependent variable in such an equation is an unrestricted variable. In this case the associated vertices of these unrestricted variables of weights might be misdiscriminated into the wrong set. This kind of contradictory relations may occur explicitly in the rows of the Q-matrices or reduced matrices, or implicitly in the process of solving for the weights with successive higher ordering method.

We shall give an example to illustrate the process of solution of this homogeneous system for imperfect ϕ-functions. However, in the example the number of mis-discriminated vertices is not exactly minimum.

<u>Example 5-1:</u> We are to find a first degree imperfect ϕ-function for Example 3-1. In Table 3-23, row 8 is missing. Now we add row 8 back to this table, and reduce its entries as shown in Table 5-1.

Table 5-1

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
6		1						1	1	-1						
7				1	1	-1	1									
8		1		2	1		1	2	1							
9		1	-1	2			1	1								
10	1	-1		-1				-1								
11		-1		-1			-1	-1	-1					-1		
12				-1			-1		1						1	
13				-1			-1	-1			-1					
14		-1		-1	-1		-1	-1					1			
15		-1		-1			-1					1				
16		1		2	1		2	1								-1

In Table 5-1, $C_1 = \{X_1, X_2, X_3, X_4, X_5, X_6\} = \{1111, 1110, 1011, 1001, 0100, 0000\}$, $C_2 = \{X_7, X_8, X_9, X_{10}\} = \{1101, 1000, 0111, 0001\}$, and $\bigcirc$ is the entry of associated unrestricted dependent variables.

Since row 8 is contradictory, we choose $b_{8\ 9}$. After reducing the matrix of Table 5-1, we obtain Table 5-2.

Table 5-2

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
6				-2	-1		-1	-1		-1						
7				1	1	-1	1									
8		1		2	1		1	2	1							
9		1	-1	2			1	1								
10	1	1		-1				-1								
11				1	1			1						-1		
12		-1		-3	-1		-2	-2							1	
13				-1			-1	-1			-1					
14		-1		-1	-1		-1	-1					1			
15		-1		-1			-1					1				
16		1		2	1		2	1								-1

In Table 5-2, row 6 is a contradiction which had an associated dependent variable already. We can cancel the previous assignment to find a new dependent variable, or just change the previous one to be unrestricted. Here we choose the latter case, i.e. we choose $b_{6\ 10}$ again.

Solving the system in Table 5-2, we obtain the vertex weights as follows:

w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8	w_9	w_{10}	w_{11}	w_{12}	w_{13}	w_{14}	w_{15}	w_{16}
3	1	5	1	1	3	1	1	-7	-5	-3	3	5	3	9	7

and the discriminant function is.

$$\phi(X) = 3 - 4x_1 - 2x_1 + 4x_3 + 2x_4 \tag{5-3}$$

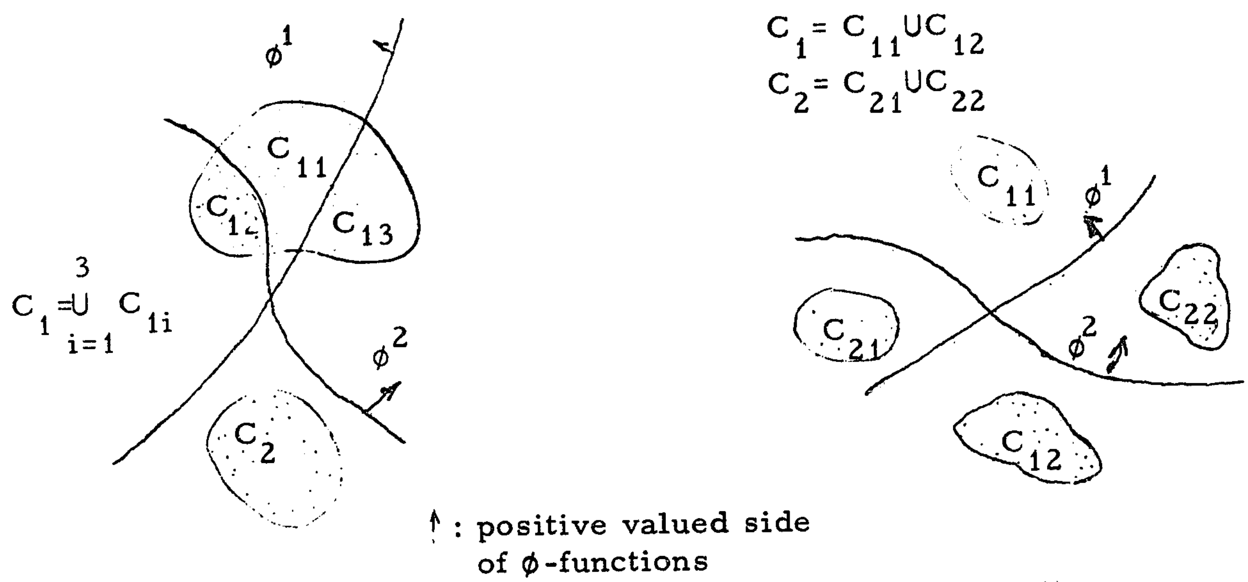
where X_9 and X_{10} are mis-discriminated. Then

$$e = 2/10 = 0.2 \quad .$$

Comparing the ϕ-function of (5-3) with that in Example 3-2, we can see that the simplicity of the ϕ-function is gained at the expense of error.

3. Imperfect ϕ-function for Compound Synthesis:

Compound synthisis has been studied by Minnick [29], Winder [42], Sheng [86] and others. In this chapter we will use the imperfect ϕ-function method for compound synthesis. In our method hypersurfaces are used instead of hyperplanes, to separate two disjoint sets C_1 and C_2 in E^n. Furthermore, these hypersurfaces satisfy the cases of both with or without don't care vertices.



(a) (b)

Fig. 5-1

One situation is shown in Fig. 5-1-(a), where ϕ^1 and ϕ^2 are two single-sided error hypersurfaces. Using the same symbols as in Chapter II, we have the Boolean functions

$$F_1 = F_{11} + F_{13} + F_{12} = (F_{11} + F_{12}) + (F_{11} + F_{13})$$

$$= F_{\phi^1} + F_{\phi^2} \ , \tag{5-4}$$

where F_{ϕ^1} is realized by ϕ^1, and F_{ϕ^2} is realized by ϕ^2. On the other hand, we have:

$$F_2 = \overline{(F_{11} + F_{12})}\,\overline{(F_{11} + F_{13})} = \overline{F}_{\phi^1}\,\overline{F}_{\phi^2} \ . \tag{5-5}$$

In other words, ϕ^1 and ϕ^2 can realize

$$C_1 = \{X/F(X) = 1\}$$

and

$$C_2 = \{X/F(X) = 0\} \ ,$$

that is,

if $X \in C_1$, then $\phi^i(X) > 0$, for at least one i,

if $X \in C_2$, then $\phi^i(X) < 0$, for all i's.

Another situation is shown in Fig. 5-1-(b), where

$$F_1 = F_{11} + F_{12} = F_{\phi^1} F_{\phi^2} + \overline{F}_{\phi^1}\,\overline{F}_{\phi^2} \ , \tag{5-6}$$

$$F_2 = F_{\phi^1}\,\overline{F}_{\phi^2} + \overline{F}_{\phi^1} F_{\phi^2} \ . \tag{5-7}$$

In this case,

if $X \in C_1$, then $\phi^1(X) < 0$ and $\phi^2(X) < 0$ or

$\phi^1(X) > 0$ and $\phi^2(X) > 0$,

if $X \in C_2$, then $\phi^1(X) > 0$ and $\phi^2(X) < 0$ or

$\phi^1(X) < 0$ and $\phi^2(X) > 0$.

According to our method, the imperfect ϕ-functions for both cases of Fig. 5-1-(a) and 5-1-(b), can be constructed in the same way. The only difference is that for single-sided error hypersurfaces, the dependent vertex weights in contradiction equations are chosen either all in C_1, or all in C_2, and for double-sided error hypersurfaces, the dependent vertex weights in contradiction equations are chosen arbitrarly in C_1 or in C_2.

For instance, let a Boolean function F_1 be the combination of three first order threshold functions F_{11}, F_{12}, and F_{13}, such that

$$F_1 = (F_{11} + F_{12})\, F_{13} \quad . \tag{5-8}$$

Geometrically, it is shown in Fig. 5-2.

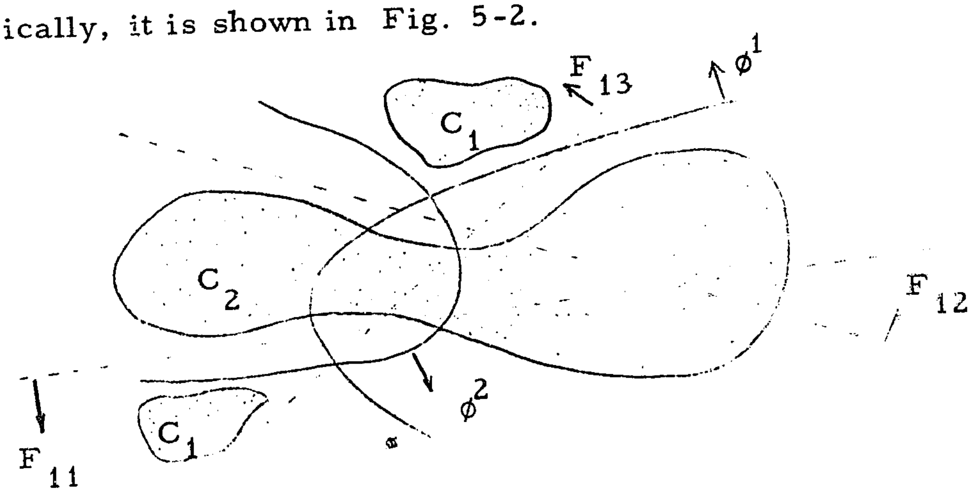


Fig 5-2.

From Fig. 5-2, we have

$$F_1 = F_{\phi}1 \, F_{\phi}2 = (F_{11} + F_{12}) \, F_{13} \quad .$$

Here it is not necessary that

$$F_{11} + F_{12} = F_{\phi}1 \quad ,$$

and

$$F_{13} = F_{\phi}2 \quad .$$

Now, the remaining problem is how to find those imperfect ϕ-functions.

4. Theory:

Definition 9: For disjoint sets of vertices C_1 and C_2 in E^m space with fixed primary and secondary variables, we always can find imperfect ϕ-functions with different error rates. Those with minimum error rate are called minimum error rate imperfect ϕ-functions in E^m space. This minimum error rate will be denoted by e^m_{min}.

Theorem 10: $e^m_{min} \leq e^s_{min}$,

where $E^m \supset E^s$.

Proof: By Theorem 9 of Chapter III, and Table 3-1, we know that the ϕ-function in the E^s space must satisfy the homogeneous system

$$\vec{b}_{i_{s+1}} . W = 0$$

$$\vec{b}_{i_{s+2}} . W = 0$$

$$\begin{aligned}
&\cdots\cdots\cdots \\
&\vec{b}_{i_m} \cdot W = 0 \\
&\vec{b}_{i_{m+1}} \cdot W = 0 \\
&\cdots\cdots\cdots\cdot \\
&\vec{b}_{2^n-1} \cdot W = 0
\end{aligned} \tag{5-9}$$

where $\vec{b}$'s are the row vectors in Q matrix of C_1 and C_2, and the designators of $\vec{b}$'s in the Q matrix are not the components of E^s space, and W is the vertex weight vector.

Similarly, we can obtain the homogeneous system for the ϕ-function in E^m space. Since $E^s \subset E^m$, we have:

$$\begin{aligned}
&\vec{b}_{i_{m+1}} \cdot W = 0 \ , \\
&\vec{b}_{i_{m+2}} \cdot W = 0 \ , \\
&\cdots\cdots\cdots\cdots \\
&\vec{b}_{2^n-1} \cdot W = 0 \ .
\end{aligned} \tag{5-10}$$

Since system (5-10) is a part of system (5-9), the solutions of (5-9) must be the solutions of (5-10), but the reverse is not true. Thus the contradiction relations produced in (5-10) are the contradiction relations in (5-9). That is , the number of mis-discriminated vertices in E^m divided by a minimum error rate imperfect ϕ-function is not larger than that in E^s space produced by a minimum error rate imperfect ϕ-function.

In other words, the solution of (5-9) forms a hyperplane in E^s , and the solution of (5-10) forms a hypersurface in E^s.

Theorem 11: For disjoint sets C_1 and C_2 in E^n space, where n is the number of primary variables, we always can find an imperfect ϕ-function with error rate less than $e^0_{min} = \min(e^0_1, e^0_2)$.

Proof: It suffices to prove that if all the vertices except one, X_i, in C_1 be assigned as don't care vertices, then there must be a first degree ϕ-function separating X_i from the vertices of C_2. It is well known that if C_1 contains one vertex only, then there must be a first degree ϕ-function separating C_1 from its complement. Then the error rate of this imperfect ϕ-function is

$$e^n_1 = \frac{N_1-1}{N_1+N_2} < \frac{N_1}{N_1+N_2} = e^0_1 .$$

Similarly, we can obtain

$$e^n_2 = \frac{N_2-1}{N_1+N_2} < \frac{N_2}{N_1+N_2} = e^0_2 .$$

Q.E.D.

Corollary 11a: For two disjoint sets of vertices C_1 and C_2, there exists an imperfect ϕ-function with a degree r, $0 < r < n$, such that all mis-discriminated vertices either all belong to C_1, or all belong to C_2, and the error rate of the imperfect ϕ-function is less than e^0_1 or e^0_2, respectively.

Theorem 12: Let e_ϕ be the error rate of an imperfect ϕ-function, and $e_{-\phi}$ that of an imperfect ϕ-function with a sign different from the former one. Then

$$\min(e_\phi, e_{-\phi}) \leq 1/2$$

for any degree of ϕ-functions.

Proof: Let

$$e_\phi = \frac{n_{e_1} + n_{e_2}}{N_1 + N_2} = b \leq 1, \quad \text{for any degree of } \phi\text{-function}$$

where n_{e_1} is the number of mis-discriminated vertices in C_1, and n_{e_2} that in C_2. Then

$$e_{-\phi} = \frac{(N_1 - n_{e_1}) + (N_2 - n_{e_2})}{N_1 + N_2} = 1 - \frac{n_{e_1} + n_{e_2}}{N_1 + N_2} = 1 - b .$$

Thus we have

$$\min(e_\phi, e_{-\phi}) \leq 1/2 . \tag{5-11}$$

The equality holds for $b = 1/2$.

Q.E.D.

5. Realization of Non-linear Separable Disjoint Sets with Imperfect ϕ-functions:

As stated before, there are two kinds of imperfect ϕ-functions, one single-sided and the other double-sided. Here we only study the single-sided imperfect ϕ-functions.

Our objective is to find the single-sided imperfect ϕ-functions with errors as small as possible.

How to find an imperfect ϕ-function was given in Section 2 of this chapter. In the solution, the set of vertices in C_1 with weights greater than zero forms the associated subset of vertices of the imperfect ϕ-function.

For instance, assume that all the mis-discriminated vertices are in C_1. We can find a set of imperfect ϕ-functions $\{\phi_{11}, \phi_{12}, \cdots, \phi_{1r}\}$ in the following sequence:

1) Form ϕ_{11} which discriminate C_{11} from C_2, where $C_{11} \subset C_1$, and C_{11} contains at least one vertex of C_1.

2) Consider all vertices in C_{11} as don't care vertices. Form ϕ_{12} which discriminate C_{12} from C_2, where $C_{12} \not\subset C_{11}$, and $C_{12} \subset C_1$.

. . . .

i) Consider all vertices in $\bigcup_{j=1}^{i-1} C_{1j}$ as don't care vertices. Find ϕ_{1i} which discriminates C_{1i} from C_2, $C_{1i} \subset C_1$, where

$$C_{1i} \not\subset \bigcup_{j=1}^{i-1} C_{1j} .$$

If $C_1 = \bigcup_{j=1}^{i} C_{1j}$, stop the process. If not, go to the next step i+1).

Theorem 13: There exists a set of imperfect ϕ-functions $\{\phi_{11}, \ldots, \phi_{1r}\}$, $0 < r \leq N_1$, of any degree greater than zero but less than n, where n is the number of primary variables with associated subsets of vertices $\{C_{11}, \ldots, C_{1r}\}$ such that

$$C_1 = \bigcup_{j=1}^{r} C_{1j}.$$

Proof: By Theorem 11 and Corollary 11a, we can always find a set of imperfect ϕ-functions $\{\phi_{11}, \cdots, \phi_{1r}\}$ with associated subsets $\{C_{11}, \ldots, C_{1r}\}$ such that

$$C_{1i} \not\subset \bigcup_{j \neq i} C_{1j} \quad \text{for all } i \text{ and } j, \text{ and } C_{1i}, C_{1j} \subset C_1.$$

Since the number of vertices in C_1 is finite, for certain value of r, $r \leq N_1$, there must be

$$C_1 = \bigcup_{j=1}^{r} C_{1j} \quad . \tag{5-12}$$

Q.E.D.

Corollary 13a: For two disjoint sets C_1 and C_2, there must be a set of imperfect ϕ-functions $\{\phi_{21}, \phi_{22}, \ldots, \phi_{2k}\}$ with associated subsets $\{C_{21}, C_{22}, \ldots, C_{2k}\}$ such that

$$C_2 = \bigcup_{j=1}^{k} C_{2j} \quad .$$

If the set of imperfect ϕ-functions is found, then the following relations are satisfied:

$$\begin{aligned} &\text{If} \quad X \in C_1 , \quad \text{then } \phi_{1i}(X) > 0 , \quad \text{for at lease one } i, \\ &\text{If} \quad X \in C_2 , \quad \text{then } \phi_{1i}(X) < 0 , \quad \text{for all } i. \end{aligned} \tag{5-13}$$

The degree of each imperfect ϕ-function of $\{\phi_{11}, \cdots, \phi_{1r}\}$ can be any number greater than zero and less than n. Moreover, it is clear that the larger the dimension of the imperfect ϕ-functions, the smaller the number of r. If we do not impose any limitation on the degree of $\{\phi_{11}, \ldots, \phi_{1r}\}$, then any two disjoint sets C_1 and C_2 can be discriminated by a single ϕ-function. Assume that the degree of $\{\phi_{11}, \ldots, \phi_{1r}\}$ is limited to d, $0 < d < n$. The procedure to find these imperfect ϕ-functions is as follows:

Step 1) In Q matrix of C_1 and C_2, form the homogeneous system for the dth degree imperfect ϕ-functions.

Step 2) Solve the homogeneous system. If there are contradiction relations in the process, then set one of the variables of the vertex weights in C_1 as unrestricted variable for every contradiction equation, until the system has a solution. By Theorem 9 and Corollary 11a, the solution always exists, with which the associated subset of vertices is not empty.

Step 3) Form ϕ_{1i} and C_{1i}.

Step 4) Form $\bigcup_{j=1}^{i} C_{1i}$.

A) If $\bigcup_{j=1}^{i} C_{1j} = C_1$, proceed to Step 5).

B) If not, set all the vertices in $\bigcup_{j=1}^{i} C_{1j}$ as don't care vertices.

Then set $i \longrightarrow i+1$, and proceed to Step 2).

Step 5) Form $\{\phi_{11}, \cdots, \phi_{1r}\}$.

In the above procedure, we set all the associated subsets $\{C_{11}, \cdots, C_{1r}\}$ to be contained in C_1. It is also possible to find a set of associated subsets $\{C_{21}, \cdots, C_{2k}\}$ all to be contained in C_2. In that case the relations of (5-13) become:

$$\begin{array}{llll} \text{If} \quad X \in C_1 , & \text{then} \quad \phi_{2i}(X) > 0 , & \text{for all } i, & \\ & & & (5\text{-}14) \\ \text{If} \quad X \in C_2 , & \text{then} \quad \phi_{2i}(X) < 0 , & \text{for at least one } i. & \end{array}$$

The preference of choosing the imperfect ϕ-function set of the form of $\{\phi_{11}, \cdots, \phi_{1r}\}$ or of $\{\phi_{21}, \cdots, \phi_{2k}\}$ depends on the number, degree and dimension of these functions. We have not found any definite criterior to determine which set of the imperfect ϕ-functions is preferred. Roughly we can choose one of the two according to the relative sizes of C_1 and C_2, i.e. if

$N_1 < N_2$, $\{\phi_{11}, \cdots, \phi_{1r}\}$ is chosen,

$N_1 > N_2$, $\{\phi_{21}, \cdots, \phi_{2k}\}$ is chosen,

$N_1 = N_2$, any one may be chosen.

6. <u>Example 5-2:</u>

$C_1 = \{1111, 1110, 1101, 1010, 0110, 0101, 0011\}$

$= \{X_1, X_2, X_3, X_4, X_5, X_6, X_7\}$,

$C_2 = \{1000, 0111, 0100, 0010, 0001, 0000\}$

$= \{X_8, X_9, X_{10}, X_{11}, X_{12}, X_{13}\}$.

I. Find $\{\phi_{11}, \ldots, \phi_{1r}\}$ set.

Step 1) Q matrix of C_1 and C_2 is shown in Table 5-3.

Table 5-3

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1													-1			
2								-1					1			
3										-1			1			
4											-1		1			
5												-1	1			
6								1		1			-1	1		
7				1				1			1		-1			
8								1				1	-1			1
9					1					1	1		-1			
10						1				1		1	-1			
11							1				1	1	-1			
12		1		-1	-1			-1		-1	-1		1	-1		
13			1			-1		-1		-1		-1	1	-1		-1
14				-1			-1	-1			-1	-1	1		1	-1
15					-1	-1	-1		-1	-1	-1	-1	1			
16	1	-1	-1	1	1	1	1	1	1	1	1	1	-1	1	-1	1

where the row designators are listed below:

row	designator	row	designator	row	designator
1	1	7	x_1x_3	13	$x_1x_2x_4$
2	x_1	8	x_1x_4	14	$x_1x_3x_4$
3	x_2	9	x_2x_3	15	$x_2x_3x_4$
4	x_3	10	x_2x_4	16	$x_1x_2x_3x_4$
5	x_4	11	x_3x_4		
6	x_1x_2	12	$x_1x_2x_3$		

Step 2) The reduced matrix of Table 5-3 is shown in Table 5-4.

Table 5-4

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
6					-1			1			-1			1		
7				1				1			1		-1			
8					1			1	1							1
9				-1	1			-1		1						
10				-1	-1			-1	-1		-1	1				
11					1		1		1		1					
12		1		-1	-1						-1					
13			1					1	1		-1					
14				-1	1				1						1	
15				-1		-1		-1	-1							
16	1			-1					1		-1					

From Table 5-4, we obtain

w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8	w_9	w_{10}	w_{11}	w_{12}	w_{13}	w_{14}	w_{15}	w_{16}
3	5	1	1	1	-3	-5	1	1	1	3	7	5	3	-1	-3

Step 3)

$\phi_{11}(X) = 4x_1 + 4x_2 + 2x_3 - 2x_4 - 5$,

$C_{11} = \{X_1, X_2, X_3, X_4, X_5\}$.

Step 4) Condition B is satisfied, proceed to Step 2), and set C_{11} to be don't care vertices.

Step 2) Form the reduced matrix as shown in Table 5-5.

Table 5-5

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
6							1	1	1					1		
7				1				1			1		-1			
8							-1	1			-1					1
9				-1			-1	-1	-1	1	-1					
10				-1			1	-1				1				
11					1		1		1		1					
12		1		-1			1		1							
13			1					1	1		-1					
14				-1			-1				-1				1	
15				-1		-1		-1	-1							
16	1			-1					1		-1					

From Table 5-5, we obtain

w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8	w_9	w_{10}	w_{11}	w_{12}	w_{13}	w_{14}	w_{15}	w_{16}
5	1	1	3	-5	-5	1	1	1	9	3	3	7	-3	7	3

Step 3)

$\phi_{12}(X) = 6x_1 - 2x_2 + 4x_3 + 4x_4 - 7$,

$C_{12} = \{X_1, X_2, X_3, X_4, X_7\}$.

Step 4) Condition B is satisfied, proceed to Step 2), and set $C_{11} \cup C_{12}$ to be don't care vertices.

Step 2) Form the reduced matrix as shown in Table 5-6.

Table 5-6

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
6						-1		1				-1		1		
7						1				1		1	-1			
8						-1		1		-1						1
9				-1		-1		-1	-1							
10						2			1	1	-1	1				
11						1	1		1			1				
12		1						1	1			-1				
13			1			-2		1		-1		-1				
14								1	1	-1					1	
15					-1	-1			-1	-1						
16	1					-1		1	1	-1		-1				

From Table 5-6, we obtain

w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8	w_9	w_{10}	w_{11}	w_{12}	w_{13}	w_{14}	w_{15}	w_{16}
1	-1	3	-3	-3	1	-3	1	1	1	5	1	3	1	-1	1

Step 3)

$\phi_{13}(X) = 2x_1 + 2x_2 - 2x_3 + 2x_4 - 3$,

$C_{13} = \{X_1, X_3, X_6\}$.

Step 4) Condition A is satisfied, proceed to Step 5) .

Step 5)

$\{\phi_{11}, \phi_{12}, \phi_{13}\}$,

$\{C_{11}, C_{12}, C_{13}\}$

II. Find $\{\phi_{21}, \ldots, \phi_{2k}\}$ set.

Similiarly as in I, we obtain

$$\phi_{21}(X) = 2x_1 + 2x_2 + 2x_3 + 2x_4 - 3 ,$$

$$\phi_{22}(X) = 2x_1 - 2x_2 - 2x_3 - 2x_4 + 5 ,$$

and

$$C_{21} = \{X_8, X_{10}, X_{11}, X_{12}, X_{13}\} ,$$

$$C_{22} = \{X_9\} ,$$

$$C_2 = C_{21} \cup C_{22} .$$

CHAPTER VI

CONCLUSIONS

In this dissertation the approach to the non-linear separation of disjoint sets in E^n space is different from other established methods. Here we first try to find the vertex weights of the vertices, and then we form the discriminant function for the disjoint sets C_1 and C_2 in E^n. However, if we do not attempt to obtain the minimum degree of the ϕ-function, we can freely assign the vertex weights any real values greater than zero if the weights are restricted, and any real values if the weight are not restricted. If there are no unrestricted vertex weights, and each vertex weight is assigned to be 1, by this method, we can also get the invariants similar to Golomb's, but with different values. Nevertheless the bases employed here are not orthogonal.

In this method the positivization does not affect the minimum degree of the ϕ-function. For the first degree ϕ-function, all the weights of primary variables can be positive if a certain positivization is cho sen. But for the higher degree ϕ-functions, it is not true that a positive set of weights of primary and secondary variables always exists.

For instance, for $C_1 = \{11, 00\}$, and $C_2 = \{10, 01\}$, the discriminate function is of second degree and the weights of the primary variables are always different in sign from the weights of secondary variables.

The linear programming method for the realization of first order threshold functions always has maximum set and minimum set. These sets are employed for reducing the inequalities of the weights of the primary variables. Because of fixed positivization, the method suggested in this dissertation does not count those of maximum and minimum sets.

But the inequalities to be used for the solution of the homogeneous system are reduced only to the case of Eqn. (3-14). It is seen in Example 3-2 that in last reduced matrix in Table 3-23, the homogeneous system has no inequalities. The other parts of the procedure are all straight for - ward.

It is well known that the first order threshold function is k-asummable [38], where $k \geq 1$. Here in the vertex weight method, sometimes the contradiction relations of (3-13) and (3-16) appear in some reduced matrices. These contradiction relations are also due to the asummability property to the high-order threshold functions in the expanded space. Because of these contradiction relations, cut and try process in the successive higher ordering method is reduced.

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