

The Shapley Value for games with a finite number of effort levels

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Abstract:

*The Shapley value is a compensation scheme that distributes the surplus of a joint work among its different contributors. It implicitly assumes that players can exert two levels of effort such as high or low, and that all players choose the high level. We extend this concept to an environment in which each player might choose a different level of effort from a finite list of two or more alternatives. We provide four axioms that uniquely characterize the new concept.*

## **1. Introduction:**

In his seminal paper called *A Value for n-person Games*, Shapley (1953) defines four axioms- symmetry, efficiency, additivity and null player- which he uses to characterize a compensation scheme that distributes the surplus of a joint work among its different contributors. This scheme is popularly known as the *Shapley value*. Shapley proved that this value is the unique function satisfying the four aforementioned axioms.

Shapley doesn't make any assumptions on the effort that an individual exerts when he contributes to the surplus. However, there implicitly two stages of effort, "not working" or "working". Shapley also assumes that all players choose to work. These two assumptions are critical in the derivation of the Shapley value. In reality, people generally have the option to choose from a richer set of effort levels. Furthermore, some workers might not work for some exogenous reason.

In this paper, we extend the Shapley value to a more general environment where each individual can choose a different level of effort from a finite ordered set of options. We provide four axioms which uniquely characterize our new concept.

The plan of the paper is as follows. The next section relates our paper to the literature. Section 3 introduces the mathematical and economic concepts that will be used throughout this paper. Section 4 defines our new value. Section 5 presents a numerical example. In Section 6, we provide four axioms and show that they uniquely characterize our new value. Section 7 concludes.

## **2. Literature review:**

The Shapley value has been studied and extended to many other environments (see, for example, Serrano (2013) for a review of the literature). Young (1985) provided a different characterization of the Shapley value. He introduced a new axiom named monotonicity, which he defined as “a general principle of fair division which states that as the underlying of a problem change, the solution should change in parallel fashion.” (Young, 1985, page 65). He also replaced the additivity and null player axioms by a new axiom, which allowed him to provide a more economically meaningful axiomatization of the Shapley value. Neyman (1989) proved two different theorems on the uniqueness of the Shapley value using Young’s axioms and the strongly positive axiom.

In another paper, de Clippel and Serrano (2008) extend the Shapley value to environments involving externalities. They use two different methodologies in order to reach their conclusions. They first provide four axioms: efficiency, anonymity, monotonicity and marginality that included externalities. These axioms are applied to the extended Shapley Value in order to find the payoff of all players in the game around an externality free value which lead to two results. The first one gives bounds on players’ payoffs. The second one gives a solution

when one individual finds a way to avoid all externalities. Combining both results, they obtain a solution that is compared to a Pigouvian transfer. With the second methodology, they find conditions to the coalition structure that state that when all axioms are applied, the value is efficient. This paper is interesting for our study, because it shows us how to use axioms that have been modified in another situation and how to apply them to a new value.

The Shapley value has also been studied by Feltkamp (1995) where he defines a new axiomatic characterization of this value and the Banzhaf value for simple games, control games, and transferable games. He argues that weakening the additivity axiom does not lead to a value different from the Shapley value.

Our environment is similar to the one proposed by Hsiao and Raghavan (1993) and Freixas and Zwicker (2003). They assume that players can choose from a finite set of options their level of efforts or support, and a function maps the effort profile into a collective output. Pongou, Tchantcho and Tedjeugang (2012, 2013, 2014) and Guemmegne and Pongou (2014) provide different interpretations of this environment. Hsiao and Raghavan (1993) and Freixas (2005b) generalize the Shapley value to multi-choice cooperative games, but their values differ from ours in significant respects.

The Shapley value has been compared to other concepts in numerous other studies. For example, studies have been interested in identifying conditions under which the Shapley value ordinarily coincides with the Banzhaf value and the influence relation introduced by Isbell (1958) and their generalizations (see, e.g., Freixas (2005), Tchantcho, Lambo, Pongou and Engoulou (2008), Freixas, Marciniak and Pons (2012) and Pongou, Tchantcho and Tedjeugang (2014)), and the references therein). In future research, it will be interesting to compare our extended Shapley value to these other concepts as done in the aforementioned studies.

### 3. Preliminaries:

In this section we introduce preliminary definitions. The set  $N = \{1, \dots, n\}$  denotes a non-empty set of individuals or players in the game.

A subset  $S$  of  $N$ , is called a *coalition* and a *multi-choice game* is defined by a triplet  $(N, T, v)$ , where  $T = \{0, 1, 2, \dots, t\}$  is a finite set of effort levels, and  $v$  a production function which maps each effort profile into a real output. The highest level of effort is  $t$ , followed by  $t-1$ , and so on. A player who has effort level 0 is assumed to be unemployed, and a player who chooses effort level 1 is assumed to be giving the lowest possible effort.

For each set  $S$ ,  $T^S$  is the set of all possible vectors of effort for players in  $S$ . An element  $z$  of  $T^S$  can be written as  $(z_1, \dots, z_s) = z$  where every  $z_i \in T$  is the stage of effort of the  $i^{\text{th}}$  player in  $S$ .

**Definition 1:** A *cooperative game* with players  $N = \{1, 2, \dots, n\}$  is a positive real valued function  $v(z)$  defined on an effort set,  $T^N$  such that  $v((0, \dots, 0)) = 0$ . So we have that  $v : T^N \longrightarrow \mathbb{R}_+$

In the course of this paper, we use the symbol,  $\preceq$  in the summations of our generalized Shapley value. Let  $\bar{z}$  be the given effort set corresponding to all players in  $N$ , for a vector  $z$  in  $T^S$ , we say that  $z \preceq \bar{z}$  if  $z$  is a projection of  $\bar{z}$  such that for the  $i^{\text{th}}$  player if  $z_i \neq \bar{z}_i$ , then  $z_i = 0$ . In other words, if the player enters the game he has to bring the stage of effort given in the vector  $\bar{z}$ . If it's different, we consider that the player hasn't entered the game yet so his effort level is 0. We will also be using the vector  $e_i$  which is defined as a vector where all entries are 0 except for the  $i^{\text{th}}$  spot that is 1.

**Definition 2:** A *unanimity game* is a cooperative game  $v$  such that:

$$v_y(z) = \begin{cases} 1 & \text{if } y \preceq z \\ 0 & \text{if } y \not\preceq z \end{cases}$$

**Definition 3:** An *allocation procedure* for the  $i^{\text{th}}$  player is a function  $\varphi_i$  that associates a cooperative game  $v$  to an *allocation*  $\omega_i$  which is normally the part of the surplus going to the  $i^{\text{th}}$  player.

### 4. Generalization:

We would now like to extend the Shapley value. Let us recall the traditional value for games with 2 levels of effort:

$$\varphi_i(v) = \sum_{S: i \in S} \frac{(s)(n-s-1)!}{n!} [v(S_{+i}) - v(S)] \quad (1)$$

Where  $S_{+i} = S \cup \{i\}$

To generalize this value, we first have to write the marginal contribution ( $v^i(S)$ ) of the  $i^{\text{th}}$  player in terms of vectors since  $v$  is defined on the effort set  $T^N$ . It is done below:

$$\varphi_i(v) = \sum_{\substack{z \leq \bar{z} \\ z_i = 0}} \frac{|z|!(n - |z| - 1)!}{n!} [v(z + \bar{z}_i e_i) - v(z)] \quad (2)$$

where  $|z|$  is the number of players  $i$  such that  $z_i > 0$ .

We can also show that equation (2) can be written in another way if we apply the summation on all  $z_i \geq 1$ :

In this case, the player has entered the game since  $z_i \geq 1$ , so we have that  $v(z) - v(z - \bar{z}_i e_i)$  is the marginal contribution of player  $i$ , and the permutation fraction in front of the marginal contribution would now be  $\frac{(|z|-1)!(n-|z|)!}{n!}$ . So it is also equivalent to write equation (2) as follows:

$$\varphi_i(v) = \sum_{\substack{z \leq \bar{z} \\ z_i \geq 1}} \frac{(|z| - 1)!(n - |z|)!}{n!} [v(z) - v(z - \bar{z}_i e_i)] \quad (3)$$

Here we use the small letters  $n, s$  to indicate the cardinality of the sets  $N, S$  and we define  $|z|$  as the number of players in  $N$  that have entered the game so we have that  $|z| = |\{i \in N; z_i \geq 1\}|$ .

To simplify all calculations in this paper, we use equation (2) since (2) and (3) are equivalent.

#### 4.1 – The generalization of the Shapley Value:

Our goal in this section is to prove that with the equations (2), we can derive the Shapley value given above by equation (1). Let us start by giving a brief comparison between our concept and Shapley's. When Shapley or Young wrote their papers, no precision on the effort of each player was given but we can see that there are two stages of effort: “working” and “not working”. Applying this to our environment, we can define all sets that we have to use. First we have that  $N = \{1, 2, \dots, n\}$  with a set of effort levels that can be written as  $T = \{0, 1\}$  where 0 is not working and 1 is working and by the definition of our  $\bar{z}$ , we have that  $\bar{z} = \underbrace{(1, 1, 1, \dots, 1)}_{n \text{ times}}$  since they suppose that when a player enters, he's working. Second, we also have the coalition  $S \subseteq N$ .

First let us look at the marginal contribution with these new notations.

For any player  $i \in N$  since  $z_i = 0$  or  $1$  and  $\bar{z}_i = 1$  the marginal contribution is now given by:

$$mc_i(z) = v(z + e_i) - v(z)$$

$z \in T^N$  so all  $z$  might have some  $z_i = 0$  which means that the  $i^{\text{th}}$  player has not yet entered the game. For a coalition  $S$ , we can write  $z_i = 0$  if  $i \notin S$  and so the first equation can be written as:

$$mc_i(z(S)) = v(z(S) + e_i) - v(z(S))$$

Here we write  $z$  as  $z(S)$  in order to show that our vector  $z$  varies according to the set  $S$ .

For our next step, we need to eliminate all vectors and transform them in sets. According to Young and Shapley's definition, players that have entered the game, will always have an effort level of 1. So we conclude that in our vector  $z$ , having  $z_i(S) = 1$  can be written as  $i \in S$ . In the same way, if  $z_i(S) = 0$  we have that  $i \notin S$ . We can also substitute  $z(S) + e_i$  by  $S_{+i}$  because when  $z_i(S) = 0$  in the vector  $z$  and that we add  $z(S)$  by  $e_i$ , we get a vector  $z$  in which  $z_i(S) = 1 \Rightarrow i \in S$  as we saw. We now have to subtract the gain where player  $i$  has entered the game by the initial game that doesn't include player  $i$ . Using the sets, our marginal contribution can now be written as:

$$mc_i(S) = v(S_{+i}) - v(S) \text{ if } i \in S$$

$$\Rightarrow mc_i(S) = v^i(S) \text{ where } v^i(S) = v(S_{+i}) - v(S) \text{ in equation (1)}$$

We have shown that the marginal contributions of all players  $i$  are the same but now we have to see if the permutation fraction in front of the marginal contributions can also be written in another way. For our coalition  $S$ , let  $S_0 = \{i \in S; z_i = 0\}$  and  $S_1 = \{i \in S; z_i = 1\}$ . We have that  $S = S_0 \cup S_1$  because we know that  $z_i(S) = 0$  or  $1$ . We see that  $\sum_{i \in S} z_i(S) =$  number of players that have entered the game  $= |S_1|$ . Now if we have that  $z_i = 0 \Rightarrow i \notin S$  and so we find that the permutation of all players will be  $\frac{s!(n-s-1)!}{n!}$ , since before player  $i$  enters the game, we first permute all elements in  $S$  and after player  $i$  has entered the game we are left with a permutation of all other players in  $N$  which has  $n - s - 1$  elements. This permutation fraction is exactly the value that Young and Shapley found in their papers. We have shown that our equation:  $\varphi_i(v) = \sum_{\substack{z \leq \bar{z} \\ z_i = 0}} \frac{|z|!(n-|z|-1)!}{n!} [v(z + \bar{z}_i e_i) - v(z)]$ , can lead to the Shapley value even if we are in a different environment.

■

## 5. Numerical example:

Let  $N = \{1,2,3\}$ ,  $T = \{0,1,2,3\}$

Let  $v$  be defined as follows:

$z$	$v(z)$	$Z$	$v(z)$
(0,0,0)	0	(2,2,2)	120
(1,0,0)	70	(2,2,3)	130
(2,0,0)	80	(2,3,1)	120
(3,0,0)	90	(2,3,2)	130
(0,1,0)	70	(2,3,3)	140
(0,2,0)	80	(3,1,1)	110
(0,3,0)	90	(3,1,2)	120
(0,0,1)	70	(3,1,3)	130
(0,0,2)	80	(3,2,1)	120
(0,0,3)	90	(3,2,2)	130
(1,1,0)	80	(3,2,3)	140
(1,2,0)	90	(3,3,1)	130
(1,3,0)	100	(3,3,2)	140
(2,1,0)	90	(3,3,3)	150
(2,2,0)	100	(0,1,1)	80
(2,3,0)	110	(0,1,2)	90
(3,1,0)	100	(0,1,3)	100
(3,2,0)	110	(0,2,1)	90
(3,3,0)	120	(0,2,2)	100
(1,1,1)	90	(0,2,3)	110
(1,1,2)	100	(0,3,1)	100
(1,1,3)	110	(0,3,2)	110
(1,2,1)	100	(0,3,3)	120
(1,2,2)	110	(1,0,1)	80
(1,2,3)	120	(2,0,1)	90
(1,3,1)	110	(3,0,1)	100
(1,3,2)	120	(1,0,2)	90
(1,3,3)	130	(1,0,3)	100
(2,1,1)	100	(2,0,2)	100
(2,1,2)	110	(2,0,3)	110
(2,1,3)	120	(3,0,2)	110
(2,2,1)	110	(3,0,3)	120

Our first example will be with the given effort vector  $\bar{z} = (2,0,3)$  :

Let us find  $\varphi_1(v)$ ,  $\varphi_2(v)$  and  $\varphi_3(v)$ :

$$\varphi_1(v) = \sum_{\substack{z \preceq \bar{z} \\ z_1=0}} \frac{|z|!(n-|z|-1)!}{n!} [v(z + \bar{z}_1 e_1) - v(z)]$$

Here, the only vectors  $z$  satisfying  $z \preceq \bar{z}$  and  $z_1 = 0$  are  $(0,0,3)$  and  $(0,0,0)$ . So our summation becomes:

$$\begin{aligned} \varphi_1(v) &= \frac{1!(3-1-1)!}{3!} [v((0,0,3) + (2,0,0)) - v(0,0,3)] \\ &\quad + \frac{0!(3-0-1)!}{3!} [v((0,0,0) + (2,0,0)) - v(0,0,0)] \\ &= \frac{1}{6} [v(2,0,3) - v(0,0,3)] + \frac{2}{6} [v(2,0,0) - v(0,0,0)] \\ &= \frac{1}{6} (110 - 90) + \frac{1}{3} (80 - 0) = 30 \end{aligned}$$

Now we find  $\varphi_2(v)$ :

$$\varphi_2(v) = \sum_{\substack{z \preceq \bar{z} \\ z_2=0}} \frac{|z|!(n-|z|-1)!}{n!} [v(z + \bar{z}_2 e_2) - v(z)]$$

Here, the only vectors  $z$  satisfying  $z \preceq \bar{z}$  and  $z_2 = 0$  are  $(0,0,3)$ ,  $(0,0,0)$ ,  $(2,0,0)$  and  $(2,0,3)$ . So our summation becomes:

$$\begin{aligned} \varphi_2(v) &= \frac{1!(3-1-1)!}{3!} [v((0,0,3) + (0,0,0)) - v(0,0,3)] \\ &\quad + \frac{0!(3-0-1)!}{3!} [v((0,0,0) + (0,0,0)) - v(0,0,0)] \\ &\quad + \frac{1!(3-1-1)!}{3!} [v((2,0,0) + (0,0,0)) - v(2,0,0)] \\ &\quad + \frac{2!(3-2-1)!}{3!} [v((2,0,3) + (0,0,0)) - v(2,0,3)] = 0 \end{aligned}$$

Now for  $\varphi_3(v)$ :

$$\varphi_3(v) = \sum_{\substack{z \preceq \bar{z} \\ z_3=0}} \frac{|z|!(n-|z|-1)!}{n!} [v(z + \bar{z}_3 e_3) - v(z)]$$

Here, the only vectors  $z$  satisfying  $z \preceq \bar{z}$  and  $z_3 = 0$  are  $(2,0,0)$  and  $(0,0,0)$ . So our summation becomes:

$$\begin{aligned}
\varphi_3(v) &= \frac{1!(3-1-1)!}{3!} [v((2,0,0) + (0,0,3)) - v(2,0,0)] \\
&\quad + \frac{0!(3-0-1)!}{3!} [v((0,0,0) + (0,0,3)) - v(0,0,0)] \\
&= \frac{1}{6} [v(2,0,3) - v(2,0,0)] + \frac{2}{6} [v(0,0,3) - v(0,0,0)] \\
&= \frac{1}{6} (110 - 80) + \frac{1}{3} (90 - 0) = 35
\end{aligned}$$

With this numerical example, we conclude that our expression works even if all individuals have different stages of effort. The second player has no gain since he has not entered and player 3 has more gain than 1 since his effort and marginal contribution are higher than player 1.

Let us study another example where all players have the same given effort,  $\bar{z} = (3,3,3)$ , to see if all gains will be equal also:

For player 1, the only vectors  $z$  satisfying  $z \preceq \bar{z}$  and  $z_1 = 0$  are  $(0,0,3)$ ,  $(0,0,0)$ ,  $(0,3,3)$  and  $(0,3,0)$ . So, player 1's payoff is:

$$\begin{aligned}
\varphi_1(v) &= \frac{1!(3-1-1)!}{3!} [v((0,0,3) + (3,0,0)) - v(0,0,3)] \\
&\quad + \frac{0!(3-0-1)!}{3!} [v((0,0,0) + (3,0,0)) - v(0,0,0)] \\
&\quad + \frac{2!(3-2-1)!}{3!} [v((0,3,3) + (3,0,0)) - v(0,3,3)] \\
&\quad + \frac{1!(3-1-1)!}{3!} [v((0,3,0) + (3,0,0)) - v(0,3,0)] \\
&= \frac{1}{6} [v(3,0,3) - v(0,0,3)] + \frac{2}{6} [v(3,0,0) - v(0,0,0)] + \frac{2}{6} [v(3,3,3) - v(0,3,3)] \\
&\quad + \frac{1}{6} [v(3,3,0) - v(0,3,0)] \\
&= \frac{1}{6} (120 - 90) + \frac{1}{3} (90 - 0) + \frac{1}{3} (150 - 120) + \frac{1}{6} (120 - 90) = 50
\end{aligned}$$

For player 2, the only vectors  $z$  satisfying  $z \preceq \bar{z}$  and  $z_2 = 0$  are  $(0,0,3)$ ,  $(0,0,0)$ ,  $(3,0,3)$  and  $(3,0,0)$ . So our summation becomes:

$$\begin{aligned}
\varphi_2(v) &= \frac{1!(3-1-1)!}{3!} [v((0,0,3) + (0,3,0)) - v(0,0,3)] \\
&\quad + \frac{0!(3-0-1)!}{3!} [v((0,0,0) + (0,3,0)) - v(0,0,0)] \\
&\quad + \frac{2!(3-2-1)!}{3!} [v((3,0,3) + (0,3,0)) - v(3,0,3)] \\
&\quad + \frac{1!(3-1-1)!}{3!} [v((3,0,0) + (0,3,0)) - v(3,0,0)]
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{6} [v(0,3,3) - v(0,0,3)] + \frac{2}{6} [v(0,3,0) - v(0,0,0)] + \frac{2}{6} [v(3,3,3) - v(3,0,3)] \\
&\quad + \frac{1}{6} [v(3,3,0) - v(3,0,0)] \\
&= \frac{1}{6} (120 - 90) + \frac{1}{3} (90 - 0) + \frac{1}{3} (150 - 120) + \frac{1}{6} (120 - 90) = 50
\end{aligned}$$

For player 3, the only vectors  $z$  satisfying the fact that  $z \preceq \bar{z}$  and that  $z_3 = 0$  are  $(0,3,0)$ ,  $(0,0,0)$ ,  $(3,3,0)$  and  $(3,0,0)$ . So our summation becomes:

$$\begin{aligned}
\varphi_3(v) &= \frac{1!(3-1-1)!}{3!} [v((0,3,0) + (0,0,3)) - v(0,3,0)] \\
&\quad + \frac{0!(3-0-1)!}{3!} [v((0,0,0) + (0,0,3)) - v(0,0,0)] \\
&\quad + \frac{2!(3-2-1)!}{3!} [v((3,3,0) + (0,0,3)) - v(3,3,0)] \\
&\quad + \frac{1!(3-1-1)!}{3!} [v((3,0,0) + (0,0,3)) - v(3,0,0)] \\
&= \frac{1}{6} [v(0,3,3) - v(0,3,0)] + \frac{2}{6} [v(0,0,3) - v(0,0,0)] + \frac{2}{6} [v(3,3,3) - v(3,3,0)] \\
&\quad + \frac{1}{6} [v(3,0,3) - v(3,0,0)] \\
&= \frac{1}{6} (120 - 90) + \frac{1}{3} (90 - 0) + \frac{1}{3} (150 - 120) + \frac{1}{6} (120 - 90) = 50
\end{aligned}$$

We have shown in this example that when all level of effort are identical, all players will have the same fraction of the surplus which is  $v(3,3,3) \frac{1}{3} = 50$ . We can also see that unlike the other example, the gain is higher for each individual since his effort is higher. This tells us that the gain of each player is not only related to his effort given but also to the effort given by the other person entering the game.

## 6. Axiomatization:

In this section, we'll be defining four axioms that uniquely characterize the Shapley value:

Axiom 1:

We say that an allocation procedure  $\varphi_i$  is *symmetric* if for all permutation  $\pi$  of  $N$ , and for any cooperative game  $v$ ,  $\varphi_{\pi i}(\pi v) = \varphi_i(v)$  where  $\pi v(z) = v(\pi z)$ .

Axiom 2:

We say that an allocation procedure  $\varphi_i$  satisfies the *additivity* axiom if for any two cooperative games  $v$  and  $w$ ,  $\varphi_i(v + w) = \varphi_i(v) + \varphi_i(w)$ .

Axiom 3:

For any cooperative game  $v$  the null player axiom is stated as follows: if a player  $i \in N$  upon entering a game does not change the collective output, then  $\varphi_i(v) = 0$ , which means that his payoff is zero.

Axiom 4:

We say that an allocation procedure  $\varphi_i$  is *efficient* if for any cooperative game  $v$ ,  $\sum_{i \in N} \varphi_i(v) = v(\bar{z})$  where  $\bar{z}$  is the predetermined effort profile.

*Theorem:* Let  $v$  a game,  $N$  the set of all players and  $T^N$  a set of effort profiles.. The value  $\varphi(v)$  given by (2) is the unique function satisfying efficiency, additivity, symmetry and the null player axiom.

We first need to show that our equation given by (2) satisfies to our four axioms.

**Proof.**

Axiom 1:

Let  $mc_i$  be the marginal contribution of player  $i$ . For two identical players  $i, j \in N$  such that  $z_i = 0$  and  $z_j = 0$  and  $\bar{z}_i = \bar{z}_j$  we have that  $mc_i = v(z + \bar{z}_i e_i) - v(z)$  and  $mc_j = v(z + \bar{z}_j e_j) - v(z)$  by our formula.

Let us now substitute  $i$  and  $j$ , since  $z_i = 0 = z_j$ . The two players are identical, so we have that  $mc_i = mc_j \Rightarrow v(z + \bar{z}_i e_i) - v(z) = v(z + \bar{z}_j e_j) - v(z) \Rightarrow v(z + \bar{z}_i e_i) = v(z + \bar{z}_j e_j) \Rightarrow \varphi_i = \varphi_j$

Axiom 2:

We want to show that our value is additive. Let  $v$  and  $w$  be cooperative games.

$$\begin{aligned} \varphi_i(v + w) &= \sum_{\substack{z \leq \bar{z} \\ z_i = 0}} \frac{|z|! (n - |z| - 1)!}{n!} [(v + w)(z + \bar{z}_i e_i) - (v + w)(z)] \\ &= \sum_{\substack{z \leq \bar{z} \\ z_i = 0}} \frac{|z|! (n - |z| - 1)!}{n!} [v(z + \bar{z}_i e_i) + w(z + \bar{z}_i e_i) - [v(z) + w(z)]] \end{aligned}$$

$$\begin{aligned}
&= \sum_{\substack{z \ll \bar{z} \\ z_i=0}} \frac{|z|!(n-|z|-1)!}{n!} [[v(z + \bar{z}_i e_i) - v(z)] + [w(z + \bar{z}_i e_i) - w(z)]] \\
&= \sum_{\substack{z \ll \bar{z} \\ z_i=0}} \frac{|z|!(n-|z|-1)!}{n!} [v(z + \bar{z}_i e_i) - v(z)] + \sum_{\substack{z \ll \bar{z} \\ z_i=0}} \frac{|z|!(n-|z|-1)!}{n!} [w(z + \bar{z}_i e_i) - w(z)] \\
&= \varphi_i(v) + \varphi_i(w)
\end{aligned}$$

Axiom 3:

If player  $i$  does not change the output after entering a game  $v$ , then it means that  $v(z + \bar{z}_i e_i) - v(z) = 0$  for any  $z$ . It follows that:

$$\begin{aligned}
\varphi_i(v) &= \sum_{\substack{z \ll \bar{z} \\ z_i=0}} \frac{|z|!(n-|z|-1)!}{n!} [v(z + \bar{z}_i e_i) - v(z)] \\
&= 0.
\end{aligned}$$

Axiom 4:

For this axiom, let us suppose that every player enters the game as follows; player 1 enters first, then player 2 then 3, continuing like this, the last player entering is the  $n^{\text{th}}$  player. We also have a given vector,  $\bar{z} = (\bar{z}_1, \dots, \bar{z}_n)$ , of effort associated to every player in  $N$ . Let  $z_0 = (0, \dots, 0)$  be the initial state in which players have not entered yet.

For the first step, we have that player 1 enters the game, so his marginal contribution will be given by:

$$\varphi_1 = v(\underbrace{z_0 + \bar{z}_1 e_1}_{y_1}) - v(z_0)$$

For the second step, player 2 enters so his marginal contribution is now:

$$\varphi_2 = v(\underbrace{y_1 + \bar{z}_2 e_2}_{y_2}) - v(y_1)$$

For the third step, player 3 enters so his marginal contribution is now:

$$\varphi_3 = v(\underbrace{y_2 + \bar{z}_3 e_3}_{y_3}) - v(y_2)$$

⋮

In our last step, the  $n^{\text{th}}$  player enters the game so his marginal contribution is now:

$$\varphi_n = v(\underbrace{y_{n-1} + \bar{z}_n e_n}_{y_n}) - v(y_{n-1}) = v(\bar{z}) - v(y_{n-1})$$

And so if we apply the summation, we have that :

$$\sum_{i=1}^n \varphi_i = v(\bar{z}) - v(z_0) \text{ since all other terms } v(y_i) \text{ are cancelled out}$$

$$\Rightarrow \sum_{i=1}^n \varphi_i = v(\bar{z}) \text{ since } v(z_0) = 0 \text{ by assumption.}$$

We can repeat the process for any order of entry of players in the game, and we will reach the same conclusion.

- Now let  $\xi$  be an *allocation procedure* satisfying efficiency, additivity, symmetry and that obeys to the null player axiom. We need to show that for all game  $v$ ,  $\xi(v) = \varphi(v)$ .

We denote by  $I(v) = \{z \preceq \bar{z}; \exists y \preceq z \text{ with } v(y) \neq 0\}$  the set of all vectors  $z$  that have a non-zero image. The theorem is proved by induction on the cardinality of  $I(v)$ , referred to  $|I(v)|$ , because given a cooperative game  $v$ , showing that  $\xi(v) = \varphi(v)$  for any number of elements in  $I(v)$  is equivalent to showing that  $\xi(v) = \varphi(v)$  for any given expression for  $v$ . It's important to enumerate three important properties of  $I(v)$  that will be used throughout this proof:

i)  $I(w \pm v) \subseteq I(w) \cup I(v)$

Proof: Let  $z \in I(w + v)$ ,  
 $\Rightarrow z \preceq \bar{z}$  and  $\exists y \preceq z$  with  $(v + w)[y] \neq 0$   
 $\Rightarrow z \preceq \bar{z}$  and  $\exists y \preceq z$  with  $v(y) + w(y) \neq 0$  by the additivity of  $v$

We have three cases :

1. If  $v(y) = 0$  and  $w(y) \neq 0 \Rightarrow z \in I(w)$  since  $w(y) \neq 0 \Rightarrow z \in I(w) \cup I(v)$
2. If  $w(y) = 0$  and  $v(y) \neq 0 \Rightarrow z \in I(v)$  since  $v(y) \neq 0 \Rightarrow z \in I(w) \cup I(v)$
3. If  $v(y) \neq 0$  and  $w(y) \neq 0 \Rightarrow z \in I(w)$  and  $z \in I(v) \Rightarrow z \in I(w) \cup I(v)$

So in all cases, we have that  $z \in I(w) \cup I(v)$   
 $\Rightarrow I(w \pm v) \subseteq I(w) \cup I(v)$

■

ii)  $I(v_x) \subset I(v)$

Proof: Let  $z \in I(v_x)$ ,  
 $\Rightarrow z \preceq \bar{z}$  and  $\exists y \preceq z$  with  $v_x(y) \neq 0$   
 $\Rightarrow$  since  $v_x(y) \neq 0$ , by the definition of a unanimity game defined in the third section, we have that  $v_x(y) = 1$  and that  $y \preceq x$   
 $\Rightarrow$  if for one subgame  $v_x$  we have that  $v_x(y) \neq 0$ , it's clear that for the game  $v$ , we can use the same  $z$  and find a  $y$  such that  $v(y) \neq 0$   
 $\Rightarrow z \in I(v)$   
 $\Rightarrow I(v_x) \subset I(v)$

■

iii) if  $I(v_x) \subset I(v) \Rightarrow |I(v_x)| < |I(v)|$

Proof: Let  $\alpha = |I(v)|$  This proof is done by induction on  $\alpha$ ,

If  $\alpha = 1 \Rightarrow$  there is only one element in  $I(v)$  so the only way that our strict inclusion,  $I(v_x) \subset I(v)$ , can be satisfied, is if  $I(v_x) = \emptyset$   
 $\Rightarrow |I(v_x)| = 0 \Rightarrow |I(v_x)| < \alpha$

Now we suppose that  $|I(v_x)| < a$  for all  $a = 1, \dots, n$  and we need to show that the inequality is satisfied for  $a = n + 1$ .

We have that  $I(v_x) \subset I(v) \Rightarrow \exists z \in I(v)$  such that  $z \notin I(v_x)$

$$\Rightarrow |I(v) - \{z\}| = a - 1 = n$$

By our induction hypothesis, we have that  $|I(v_x)| < |I(v) - \{z\}| = n$

$$\Rightarrow |I(v_x)| < n < n + 1$$

So our implication is satisfied for any  $a$  ■

Now we have all the elements to prove the theorem.

Let  $|I(v)| = 0$ . It implies that for all  $z$ , there exists no  $y \preceq z$  such that  $v(y) \neq 0$ . So we conclude that  $v$  is the zero game or that the only  $y$  satisfying this equation is the zero vector. By the null player axiom, we conclude that  $\xi_i(v) = 0$  and as shown above,  $\varphi_i(v) = 0$ . So for  $|I(v)| = 0$  we conclude that  $\xi(v) = \varphi(v)$ .

Assume now that  $\xi(v) = \varphi(v)$  for  $|I(v)| \leq k$ , where  $k = 0, 1, \dots$ , let us show that  $\xi(v) = \varphi(v)$  for  $|I(v)| = k + 1$

Let  $w$  be the minimal element of  $I(v)$ , let us first show that  $\xi(v_w) = \varphi(v_w)$ .

First, if  $w_i = 0$  then  $v_w(w + w_i e_i) - v_w(w) = 0$ , implying by the null player axiom that  $\xi_i(v_w) = 0 = \varphi_i(v_w)$ .

Second, if  $w_i \geq 1$ , we now use the fact that  $\xi_i(v_w)$  is efficient, we have that:

$$\sum_{j \in N} \xi_j(v_w) = v(\bar{z})$$

Since we have symmetry, we know that for all identical players  $i, j \in N$ ,  $\xi_i(v_w) = \xi_j(v_w)$ , using this fact, we can write  $\xi_i(v_w)$  as:

$$\xi_i(v_w) = \sum_{\substack{z \preceq \bar{z} \\ z_i = 0}} \frac{|z|! (n - |z| - 1)!}{n!} [v_w(z + \bar{z}_i e_i) - v_w(z)] = \varphi_i(v_w)$$

So we conclude that for  $w_i \geq 1$ , we also have that  $\xi(v_w) = \varphi(v_w)$ .

Now that we have shown that it is satisfied for  $v_w$  we now need to show that it is true for any game  $v$ . Let us use the set  $I(v - v_w) = \{z \preceq \bar{z} ; \exists y \preceq z \text{ with } (v - v_w)[y] \neq 0\}$ . This set also satisfies properties i) and ii). So we have that :

$I(v - v_w) \subset [I(v) \cup I(v_w)]$  by property i) and that  $[I(v) \cup I(v_w)] \subset I(v)$  by property ii) since  $I(v_w)$  is included in  $I(v)$

$$\Rightarrow I(v - v_w) \subset I(v)$$

$$\Rightarrow |I(v - v_w)| < |I(v)| \text{ by property iii)}$$

$$\Rightarrow |I(v - v_w)| < k + 1 \text{ since } |I(v)| = k + 1 \text{ by the induction assumption}$$

We also have that:

$$I(v_w) \subset I(v) \text{ by ii)}$$

$$\Rightarrow |I(v_w)| < |I(v)| \text{ by iii)}$$

$$\Rightarrow |I(v_w)| < k + 1 \text{ since } |I(v)| = k + 1 \text{ by the induction assumption}$$

Thus by the induction hypothesis, we have that  $\xi(v - v_w) = \varphi(v - v_w)$  (\*) since  $|I(v - v_w)| \leq k$  and that  $\xi(v_w) = \varphi(v_w)$  (\*\*) since  $|I(v_w)| \leq k$

We now apply the additivity of  $\varphi$  and  $\xi$  to write  $\varphi$  and  $\xi$  in another way:

$$\begin{aligned} \xi(v) &= \xi(v_w) + \xi(v - v_w) \text{ and} \\ \varphi(v) &= \varphi(v_w) + \varphi(v - v_w) \\ \Rightarrow \xi(v) &= \varphi(v) \text{ by (*) and (**)} \end{aligned}$$

■

## 7. Conclusion:

We generalized the Shapley value to a game in which players might choose a level of effort from a finite set of alternatives. The classical value only applies to environments where players either work or not, and it is derived under the assumption that all players work. In our generalization of this value, we consider environments with more than two levels of effort, and we assume that each player might have a different level of effort, therefore covering the possibility of certain players not working. We therefore address the limitation of the traditional Shapley value as it does not consider that players may have outside options and choose either not to work for an organization, or to only devote a limited amount of their time to it.

We also extend the four axioms (symmetric, additivity, efficiency, null player) characterizing the classical Shapley value to our environment, and show that these four axioms uniquely characterize our extended value.

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