Matrix Problems and their Relation to the Representation Theory of Quivers and Posets

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Abstract

Techniques from the theory of matrix problems have proven to be helpful for studying problems within representation theory. In particular, matrix problems are well suited to use in problems related to classifying indecomposable representations of quivers and of posets. However, throughout the literature, there are many different types of matrix problems and little clarification of the relationships between them. In this thesis, we choose six types of matrix problems, place them all within a common framework and find correspondences between them. Moreover, we show that their use in the classification of finite-dimensional representations of quivers and posets are, in general, well-founded. Additionally, we investigate a direct relationship between the problem of classifying quiver representations and the problem of classifying poset representations.

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Dedication

For my wife, Elizabeth.

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Chapter 0

Introduction

The genesis of representation theory is found in a letter written to F.G. Frobenius from R. Dedekind in the year 1896 [9]. This letter contained an observation that seeded the representation theory of groups, a method which uses matrices to study groups. Since then, representation theory has grown to include methods of studying objects such as associative algebras, Lie algebras, quivers and posets. The utility of representation theory lies in the fact that it allows one to study abstract algebraic objects with linear algebraic techniques. Thus, it is no surprise that a topic called *matrix problem theory* may be, somehow, related to representation theory.

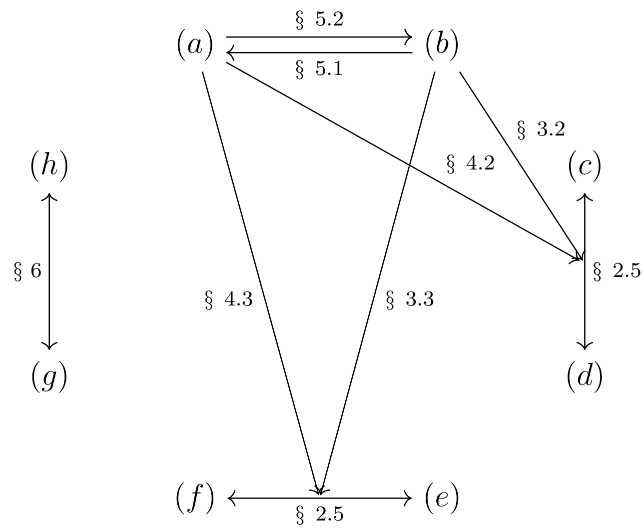
Any non-zero matrix with entries in a field $\Bbbk$ can be reduced to the form $\left(\begin{smallmatrix} \mathbb{I} & 0 \\ 0 & 0 \end{smallmatrix}\right)$, where $\mathbb{I}$ is the identity matrix and each 0 is a block zero matrix, by using elementary column and row transformations. Take a block matrix

$$M = \left(\begin{array}{c|c|c} M_{1,1} & \cdots & M_{1,n} \\ \hline \vdots & \ddots & \vdots \\ \hline M_{m,1} & \cdots & M_{m,n} \end{array}\right)$$

with arbitrarily sized blocks. Consider the question: to what form can M be reduced with elementary column and row transformations that are subject to restrictions as to which blocks they may operate upon? This question forms the basis of what has come to be known as matrix problem theory. Any number of 'matrix problems' may be constructed by considering different block structures and different restrictions as to which transformations are permitted. It turns out that matrix problems are found in many branches of representation theory [26]. In fact, matrix problems have been found to be particularly successful when applied to the representations of quivers and posets [6, 21, 22]. However, providing a general definition for the term 'matrix problem' has proven to be difficult. Hence, there are currently many distinct definitions populating the related literature.

We choose six definitions of matrix problems and classify each as either a *linear matrix problem* or a *categorical matrix problem*. Informally, a linear matrix problem consists of a set of matrices $\mathscr{M}$ upon which a group G acts, and an equivalence relation $\sim_G$ on $\mathscr{M}$ that is described by the orbits of the group action. A categorical matrix problem consists of a functor category $\mathcal{C}$ and an equivalence relation $\sim_{\mathcal{C}}$ on the objects of $\mathcal{C}$ that is described by the isoclasses of $\mathcal{C}$. Beyond these six types of matrix problems, there are also matrix problems presented in terms of vector space categories [30], vectroids [13], spectroids and modules over aggregates [7].

By a *classification* of a matrix problem, we mean a classification of its equivalence relation. An important goal within the theory of matrix problems is to study the classification problem, that is, the question of determining whether it is possible to classify a given matrix problem and then, if so, describing the classification scheme. The classification problem is related to the question of classifying the indecomposable representations of some algebraic object. Therefore, much of the literature on matrix problem theory is focused on the classification problem. However, because of the many definitions of matrix problems, it can be a difficult process to enter this subject and reconcile these differences. Therefore, this text does not focus on the classification problem. Instead, in an attempt to begin such a reconciliation, we will study the six chosen classes of matrix problems and their relationship to the representations of quivers and posets. After covering some background information and introducing the matrix problems, we follow the course depicted by the following diagram.



The nodes in the diagram are:
(a) quiver representations,
(b) poset representations
(c) Gabriel-Roĭter matrix problems,
(d) similarity matrix problems,
(e) linking matrix problems,
(f) basic algebra matrix problems,
(g) differential graded category matrix problems, and
(h) bimodule over a category with co-algebra structure matrix problems.

Each arrow of the diagram corresponds to a topic, which we present in the section number labeling the arrow.

$(c) \xleftrightarrow{\S\ 2.5} (d)$ We discuss an equivalence of similarity matrix problems and Gabriel-Roĭter matrix problems.

$(e) \xleftrightarrow{\S\ 2.5} (f)$ We discuss an equivalence of linking matrix problems and basic algebra matrix problems.

$(b) \xrightarrow{\S\ 3.2} (c) \leftrightarrow (d)$ The problem of classifying representations of a poset can be formulated using a Gabriel-Roĭter matrix problem. It follows from an equivalence between Gabriel-Roĭter matrix problems and similarity matrix problems that the problem of classifying representations of posets can also be formulated using a similarity matrix problem.

$(b) \xrightarrow{\S\ 3.3} (e) \leftrightarrow (f)$ The problem of classifying representations of a poset can be formulated using a linking matrix problem. It follows from an equivalence between similarity matrix problems and basic algebra matrix problems that the problem of classifying representations of posets can also be formulated using a basic algebra matrix problem.

$(a) \xrightarrow{\S\ 4.2} (c) \leftrightarrow (d)$ The problem of classifying representations of a quiver can be formulated using a similarity matrix problem. It follows from an equivalence between similarity matrix problems and Gabriel-Roĭter matrix problems that the problem of classifying representations of quivers can also be formulated as a Gabriel-Roĭter matrix problem.

$(a) \xrightarrow{\S\ 4.3} (e) \leftrightarrow (f)$ The problem of classifying representations of a quiver can be formulated using a linking matrix problem. It follows from an equivalence between linking matrix problems and basic algebra matrix problems that the problem of classifying representations of quivers can also be formulated using a basic algebra matrix problem.

$(a) \xleftrightarrow{\S\ 5.1} (b)$ We show that the problem of classifying representations of a poset can be formulated as the problem of classifying the representations of a quiver.

$(b) \xleftrightarrow{\S\ 5.2} (a)$ There is no general method, known to the author, of formulating the problem of classifying representations of a quiver with the problem of classifying representations of a poset. We attempt several naive methods and show why each method does not work.

$(h) \xleftrightarrow{\S\ 6} (g)$ We introduce the notions of a differential graded category (DGC) matrix problem and a bocs matrix problem. After developing a correspondence between semi-free DGCs and grouplike bocses, we show that the category of representations of a grouplike bocs is equivalent to the category of representations of the corresponding semi-free DGC.

The theory of matrix problems has seen great success. Examples include the proof of the famed ‘tame and wild dichotomy’ by Drozd [4] or the Brauer-Thrall conjecture [23, 25] by Nazarova and Roĭter. Such success ought to motivate further study of matrix problems. Hopefully, this text will help to solidify the foundations of this subject.

Outline of the New Material

Here, we outline which material contained in this text is new or partially new.

Chapter 2 discusses linear matrix problems. Definition 2.0.13 (linear matrix problem) is new, though it is merely a natural generalization of the Gabriel-Roĭter, similarity, linking and basic algebra matrix problems described in that chapter. Although the statement of Proposition 2.5.1 is mentioned in [29, Ex. 1.2] as is the idea of a proof, we provide a full proof in order to formalize this idea.

Chapter 3 shows how the problem of classifying representations of a poset can be studied by using matrix problems. Section 3.1 contains new information. Section 3.2 generalizes examples from [7, Ch. 1]. Section 3.3 generalizes an example [29, Ex. 2.2] and provides details not contained in the initial example.

Chapter 4 shows how the problem of classifying representations of a quiver can be studied by using matrix problems. Everything contained in Sections 4.1 and 4.2 is new. While there is an example in [29, Ex. 1.1] that does relate, without details, a specific quiver to a similarity matrix problem, we give a generalization of this example using a new construction and provide all appropriate details. Similarly, [29, Ex. 2.1] provides an example, without details, that associates a linking matrix problem to a specific quiver. We generalize this example in Section 4.3 by using a new construction and provide all of the necessary details. The main results (Theorems 4.2.1 and 4.3.1) of Chapter 4 are, as far as the author is aware, explicitly stated and proven for the first time.

Chapter 5 directly relates the question of classifying quiver representations to the question of classifying poset representations. The discussion in Section 5.1 largely follows from ideas presented in [30, Sec. 2.2, Sec. 3.1]. However, everything after (and not including) Lemma 5.1.6 is new information. Section 5.2 discusses the problem of classifying representations of a quiver with the representations of a poset. There does not appear to be any information on this subject contained in the literature. Thus, the author believes that everything in this section is new.

Chapter 6 discusses categorical matrix problems. Definition 6.1.6 is a new definition for a bigraph. Proposition 6.1.4 is explicitly stated and proven for the first time, however it seems to be assumed in [28]. The construction in Section 6.1.3 is sketched in [28, Sec. 2]. We provide all missing details. We provide new proofs for Propositions 6.1.14 and 6.1.15, both of which are stated in [28, Sec. 2] without proof. Definition 6.1.16 (differential graded category matrix problem) and Definition 6.2.12 (bocs matrix problem) are new definitions that follow naturally from discussion found in [2, 27, 28]. We provide details, omitted in [27, Prop. 2], in the construction of a correspondence between DGCs and bocses. Also, the proof in Section 6.3 showing the equivalence between the category of representations of a bocs and the category of representations of the corresponding DGC is new.

Chapter 1

Background Material

We first introduce notation that will be used throughout. Denote the set of non-negative integers by $\mathbb{N}$. Let $\Bbbk$ be an algebraically closed field.

Definition 1.0.1 (Composition)**.** A *composition of length* n of a non-negative integer d is an n-tuple of non-negative integers $(d_1, \dots, d_n)$ such that $d_1 + \cdots + d_n = d$.

We will often need to fix a composition for some $d \in \mathbb{N}$, which we will denote by $\underline{d}$. There will be no need to consider two distinct compositions of the same integer, so this notation should not cause any confusion.

We use the following notation:

- $\Bbbk^{n \times m}$, with $n, m \in \mathbb{N}$, denotes the set of all $n \times m$ matrices with entries in $\Bbbk$;
- $\Bbbk^{\underline{d} \times \underline{d}'}$, where $d, d' \in \mathbb{N}$ and $\underline{d}$ and $\underline{d}'$ are compositions of length n and m, respectively, denotes the set of block matrices

$$\left\{(M_{i,j})\colon 1 \le i \le n,\ 1 \le j \le m,\ M_{i,j} \in \Bbbk^{d_i \times d'_j}\right\};$$

- $GL(\underline{n})$, where $n \in \mathbb{N}$ and $\underline{n}$ is a composition of length m, is the group $GL(n, \Bbbk)$ such that each element $G = (G_{i,j})_{i,j=1}^{m} \in GL(\underline{n})$ is considered as a block matrix where block $G_{i,j}$ is size $n_i \times n_j$;

- $\mathbb{I}$ denotes the identity matrix which, if the size is not clear from context, will be written as $\mathbb{I}_n$ to indicate an $n \times n$ matrix; and
- $E_{i,j}$ denotes the matrix, whose size will be clear from context, with 1 in the (i, j)-entry and zeros elsewhere.

This text will use both tuples of matrices and block matrices. Tuples will always be denoted with a single subscript, for instance as $(M_i)_{i=1}^n$, for matrices $M_1, \dots, M_n$. Block matrices will always be denoted with a double subscript, for instance $(M_{i,j})_{i,j=1}^n$ for matrices $M_{i,j}$, with $1 \leq i, j \leq n$.

The analogy between linear maps and matrices is incomplete, as there does not exist a matrix corresponding to the unique mapping from an n-dimensional vector space to the 0-dimensional vector space. Thus we introduce a $0 \times n$ sized *empty matrix*, denoted $\mathscr{I}_{0,n}$. Similarly, we introduce an $m \times 0$ sized empty matrix, denoted $\mathscr{I}_{m,0}$, that corresponds to the linear mapping from the 0-dimensional vector space to an m-dimensional vector space. Let $\mathscr{I}_{0,n}\mathscr{I}_{n,m} = \mathscr{I}_{0,m}$, $\mathscr{I}_{n,m}\mathscr{I}_{m,0} = \mathscr{I}_{n,0}$ and $\mathscr{I}_{n,0}\mathscr{I}_{0,m} = 0$.

1.1 Introduction to Categories

1.1.1 Basics

This section presents the background in algebra and category theory that we will need.

Definition 1.1.1 (Associative Algebra)**.** A *unital associative algebra* A is a $\Bbbk$-bimodule together with $\Bbbk$-bimodule morphisms

- $m\colon A \otimes_{\Bbbk} A \to A$ (multiplication), and
- $i\colon \Bbbk \to A$ (unit)

such that the diagrams

$$\begin{array}{ccc} A \otimes_{\Bbbk} A \otimes_{\Bbbk} A & \xrightarrow{m \otimes 1} & A \otimes_{\Bbbk} A \\ {\scriptstyle 1\otimes m}\downarrow & & \downarrow{\scriptstyle m} \\ A \otimes_{\Bbbk} A & \xrightarrow[m]{} & A \end{array} \quad , \quad \begin{array}{ccccc} \Bbbk \otimes_{\Bbbk} A & \xrightarrow{\cong} & A & \xleftarrow{\cong} & A \otimes_{\Bbbk} \Bbbk \\ {\scriptstyle i\otimes 1}\downarrow & \nearrow{\scriptstyle m} & & \nwarrow{\scriptstyle m} & \downarrow{\scriptstyle 1\otimes i} \\ A \otimes_{\Bbbk} A & & & & A \otimes_{\Bbbk} A \end{array}$$

commute. A *non-unital associative algebra* is an associative algebra without the unit axiom and without the requirement that the second diagram commutes.

Definition 1.1.2 (Category)**.** A *category* $\mathcal{C}$ consists of

- a class of objects $\mathrm{Ob}(\mathcal{C})$,
- a set of morphisms $\mathcal{C}(x, y)$ for all $x, y \in \mathrm{Ob}(\mathcal{C})$,
- a composition $\mathcal{C}(y, z) \times \mathcal{C}(x, y) \to \mathcal{C}(x, z)$, $(f, g) \mapsto fg$ for all $x, y, z \in \mathrm{Ob}(\mathcal{C})$,
- an identity morphism $1_x \in \mathcal{C}(x, x)$ for every $x \in \mathrm{Ob}(\mathcal{C})$,
- *(associativity axiom)* $(hg)f = h(gf)$ for $f \in \mathcal{C}(w, x)$, $g \in \mathcal{C}(x, y)$, $h \in \mathcal{C}(y, z)$, and
- *(unity axiom)*: $1_y f = f = f 1_x$ for all $f \in \mathcal{C}(x, y)$.

We will occasionally write $f \circ g$ to denote the composition of morphisms f and g in circumstances where clarification is added and helpful. Also, we will denote the collection of all morphisms of $\mathcal{C}$ by $\hom(\mathcal{C})$. The terms $\mathcal{C}$-*morphism* or $\mathcal{C}$-*object* refer to morphisms or objects in C.

Definition 1.1.3 (Opposite Category)**.** Given a category $\mathcal{C}$, the *opposite category* $\mathcal{C}^{\mathrm{op}}$ is the category with $\mathrm{Ob}(\mathcal{C}^{\mathrm{op}}) = \mathrm{Ob}(\mathcal{C})$ and $\mathcal{C}^{\mathrm{op}}(x, y) = \mathcal{C}(y, x)$ for all $\mathcal{C}^{\mathrm{op}}$-objects x and y.

That is, the opposite category $\mathcal{C}^{\mathrm{op}}$ is the original category $\mathcal{C}$ with the morphisms reversed. If f is any $\mathcal{C}$-morphism, we refer to the corresponding $\mathcal{C}^{\mathrm{op}}$-morphism f^{op} as the *dual* of f.

Definition 1.1.4 (Initial, Terminal and Zero Objects)**.** Let $\mathcal{C}$ be a category and x be a $\mathcal{C}$-object. We call x an *initial object* if, for every $y \in \mathrm{Ob}(\mathcal{C})$, there exists exactly one morphism $x \to y$. We call x a *terminal object* if, for every $y \in \mathrm{Ob}(\mathcal{C})$, there exists exactly one morphism $y \to x$. If x is both initial and terminal, then we call x a *zero object*.

Initial and terminal objects serve as an example of duality. An initial object in a category $\mathcal{C}$ is terminal in $\mathcal{C}^{\mathrm{op}}$. Similarly, a terminal object in $\mathcal{C}$ is initial in $\mathcal{C}^{\mathrm{op}}$. Not every category contains an initial or terminal object and, moreover, a category may contain one and not the other. It is an easy exercise to show that an initial or terminal object is unique up to unique isomorphism. In the category **Set** of sets and set maps, any singleton is terminal and the empty set is initial. In the category **Grp** of groups and group homomorphisms, any trivial group is a zero object.

Definition 1.1.5 (Subcategory)**.** Let $\mathcal{C}$ be a category. A *subcategory* $\mathcal{D}$ of $\mathcal{C}$, written $\mathcal{D} \subseteq \mathcal{C}$, consists of a subset $\mathrm{Ob}(\mathcal{D})$ of $\mathcal{C}$-objects and a subset $\hom(\mathcal{D})$ of $\mathcal{C}$-morphisms such that

- $1_x \in \mathcal{D}(x, x)$ for every $\mathcal{D}$-object x,
- $x, y \in \mathrm{Ob}(\mathcal{D})$ for every $\mathcal{D}$-morphism $f \colon x \to y$, and
- if f and g are $\mathcal{D}$-morphisms such that fg is a $\mathcal{C}$-morphism, then fg is a $\mathcal{D}$-morphism.

There are three particular classes of subcategories that are important for our considerations.

Definition 1.1.6 (Full, Replete, and Wide Subcategories)**.** Let $\mathcal{C}$ be a category with subcategory $\mathcal{D}$. Then $\mathcal{D}$ is a *full subcategory* if $\mathcal{D}(x, y) = \mathcal{C}(x, y)$ for every pair $x, y \in \mathrm{Ob}(\mathcal{D})$. We call $\mathcal{D}$ a *wide subcategory* if $\mathrm{Ob}(\mathcal{C}) = \mathrm{Ob}(\mathcal{D})$. Finally, $\mathcal{D}$ is a *replete subcategory* if, for any $\mathcal{C}$-isomorphism $f \colon x \to y$ such that $x \in \mathrm{Ob}(\mathcal{D})$, then $y \in \mathrm{Ob}(\mathcal{D})$ and $f \in \mathcal{D}(x, y)$.

Definition 1.1.7 (Functor)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A *functor* $F\colon \mathcal{C} \to \mathcal{D}$ consists of an object map

$$F_{\text{ob}}\colon \text{ Ob}(\mathcal{C}) \to \text{Ob}(\mathcal{D})$$

and a morphism map

$$F_{\text{mor}}\colon \text{ Hom}(\mathcal{C}) \to \text{Hom}(\mathcal{D})$$

such that $F_{\text{mor}}(1_x) = 1_{F_{\text{ob}}(x)}$ for all $x \in \text{Ob}(\mathcal{C})$, and $F_{\text{mor}}(fg) = F_{\text{mor}}(f)F_{\text{mor}}(g)$ for all composable $\mathcal{C}$-morphisms f and g.

Occasionally, we will write Fx instead of $F(x)$ to denote the image of an object x under a functor F. This will be particularly helpful in expressions containing other brackets. Also, when the context is clear, we will omit the subscripts of the object and morphism maps.

The morphism map of a functor F induces a collection of set maps

$$F_{x,y}\colon \mathcal{C}(x, y) \to \mathcal{D}(Fx, Fy)$$

between hom-sets. Several important classes of functors are described by the behavior of their induced set maps.

Definition 1.1.8 (Full, Faithful, Dense Functors)**.** A functor $F\colon \mathcal{C} \to \mathcal{D}$ is said to be *full* if, for every pair $x, y \in \text{Ob}(\mathcal{C})$, the morphism mapping

$$F_{x,y}\colon \mathcal{C}(x, y) \to \mathcal{D}(Fx, Fy)$$

is surjective. If $F_{x,y}$ is injective for every pair $x, y \in \text{Ob}(\mathcal{C})$, then F is said to be *faithful.* If every $\mathcal{D}$-object is isomorphic to a $\mathcal{D}$-object $F(x)$ for some $\mathcal{C}$-object x, then F is *dense.*

The term *essentially surjective* is a common alternative to the term dense. We will call a functor *fully faithful* if it is both full and faithful.

Definition 1.1.9 (Natural Transformation)**.** Consider categories $\mathcal{C}$ and $\mathcal{D}$ and functors $F, G\colon \mathcal{C} \to \mathcal{D}$. A *natural transformation* $\eta\colon F \to G$ consists of a $\mathcal{D}$-morphism $\eta_x\colon F(x) \to G(x)$ for every $\mathcal{C}$-object x such that, for every $\mathcal{C}$-morphism $f\colon x \to y$, the diagram

$$\begin{array}{ccc} F(x) & \xrightarrow{F(f)} & F(y) \\ {\scriptstyle \eta_x}\downarrow & & \downarrow{\scriptstyle \eta_y} \\ G(x) & \xrightarrow{G(f)} & G(y) \end{array}$$

commutes. If η_x is an isomorphism for every $\mathcal{C}$-object x, then η is a *natural isomorphism* and F and G are said to be *naturally isomorphic*.

There exist two distinct, but related notions of composing natural transformations. Figure 1.1 contains diagrams depicting the compositions.

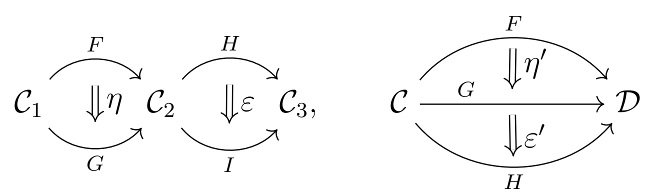


Figure 1.1: Diagram of horizontal composition (right) and vertical composition (left) of natural transformations

Definition 1.1.10 (Composition of Natural Transformations)**.** Let $\eta\colon F \to G$ be a natural transformation between functors $F, G\colon \mathcal{C}_1 \to \mathcal{C}_2$ and $\varepsilon\colon H \to I$ be a natural transformation between functors $H, I\colon \mathcal{C}_2 \to \mathcal{C}_3$. Define the *horizontal composition* of η and ε to be the natural transformation $\varepsilon\eta\colon HF \to IG$ between the functors $HF, IG\colon \mathcal{C}_1 \to \mathcal{C}_3$ with components $(\varepsilon\eta)_x = \varepsilon_{Gx} \circ H(\eta_x)$.

Let $\eta'\colon F \to G$ and $\varepsilon'\colon G \to H$ be natural transformations between functors $F, G, H\colon \mathcal{C} \to \mathcal{D}$. Define the *vertical composition* of η' and ε' to be the natural transformation $\varepsilon'\eta'\colon F \to H$ with components $(\varepsilon'\eta')_x = \varepsilon'_x\eta'_x$. The natural transformation $1_F\colon F \to F$ with components $1_{F(x)}\colon F(x) \to F(x)$ serves as an identity, with regard to vertical composition, for any functor F. A basic computation shows that vertical composition is associative.

We now show two ways in which to obtain a new natural transformation from an old natural transformation. Consider functors

$$F\colon \mathcal{C}_1 \to \mathcal{C}_2, \qquad G\colon \mathcal{C}_1 \to \mathcal{C}_2,$$
$$I\colon \mathcal{C}_0 \to \mathcal{C}_1, \qquad H\colon \mathcal{C}_2 \to \mathcal{C}_3,$$

and a natural transformation $\eta\colon F \to G$. Then, we obtain the natural transformation

$$H\eta\colon HF \to HG$$

with components $(H\eta)_x = H\eta_x$, and the natural transformation

$$\eta I\colon FI \to GI$$

with components $(\eta I)_x = \eta_{I(x)}$.

Natural transformations provide a meaningful way to compare functors.

Definition 1.1.11 (Isomorphic, Equivalent Categories)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Denote by $1_\mathcal{C}$ and $1_\mathcal{D}$ the identity functors on $\mathcal{C}$ and $\mathcal{D}$, respectively. Then $\mathcal{C}$ and $\mathcal{D}$ are said to be *isomorphic* if there exist functors $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{C}$ such that $F \circ G = 1_\mathcal{D}$ and $G \circ F = 1_\mathcal{C}$. If $F \circ G \simeq 1_\mathcal{D}$ and $G \circ F \simeq 1_\mathcal{C}$, where '$\simeq$' denotes a natural isomorphism, then $\mathcal{C}$ and $\mathcal{D}$ are said to be *equivalent*.

Theorem 1.1.12 ([19, Thm. 1, p. 93])**.** *A functor F is an equivalence of categories if and only if F is dense and fully faithful.*

Definition 1.1.13 (Adjunction)**.** An *adjunction* between categories $\mathcal{C}$ and $\mathcal{D}$ consists of a pair of functors

$$F\colon \mathcal{C} \to \mathcal{D} \text{ and } G\colon \mathcal{D} \to \mathcal{C},$$

and a pair of natural transformations

$$\epsilon \colon GF \to 1_{\mathcal{C}} \text{ (counit) and } \eta \colon 1_{\mathcal{D}} \to FG \text{ (unit)},$$

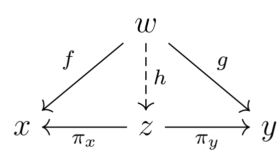
such that $1_F = F\epsilon \circ \eta F$ and $1_G = \epsilon G \circ G\eta$.

We call G the *left adjoint* to F and F the *right adjoint* to G and write $G \dashv F$. This is one of many formulations of an adjunction. For additional perspectives on adjoints, consult [19, Sec. 4].

Definition 1.1.14 (Product, Coproduct, Biproduct)**.** Consider objects x and y in a category $\mathcal{C}$. The *product* of x and y is an object z equipped with morphisms

$$\pi_x \colon z \to x \text{ and } \pi_y \colon z \to y$$

such that, for any $w \in \mathrm{Ob}(\mathcal{C})$ and morphisms $f \in \mathcal{C}(w, x)$ and $g \in \mathcal{C}(w, y)$, there exists a unique morphism $h \colon w \to z$ such that the diagram



commutes. We call π_x and π_y the *projection maps*.

A *coproduct* is the dual of the product. The morphisms dual to the projection maps are called the *injection maps* and are denoted by ι_x and ι_y.

We call z a *biproduct* of x and y if the relations

$$\pi_x \iota_x = 1_x, \quad \pi_y \iota_y = 1_y, \quad \iota_x \pi_x + \iota_y \pi_y = 1_z$$

hold for the diagram

$$x \underset{\pi_x}{\overset{\iota_x}{\rightleftarrows}} z \underset{\pi_y}{\overset{\iota_y}{\leftrightarrows}} y.$$

Products generalize to n-ary products for any non-negative integer n. When $n = 0$, we obtain the *empty product* which is a terminal object. Coproducts also generalize to n-ary coproducts and it follows from duality that the empty coproduct is an initial object. A category is said to *have all finite products* if all n-ary products exist for all non-negative integers n. The dual notion is that a category *has all finite coproducts*. However, in general, the n-ary product or coproduct of any collection of n objects does not necessarily exist.

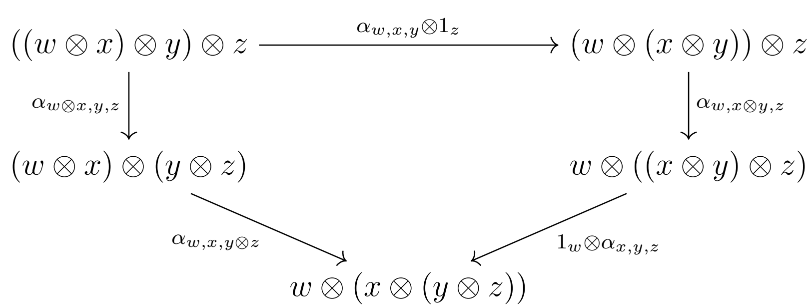
Definition 1.1.15 (Monoidal Category)**.** A *monoidal category* is a category $\mathcal{C}$ equipped with a tensor product in the form of a bifunctor

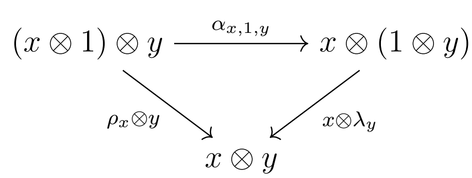
$$\otimes\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C},$$

an identity object $1 \in \mathrm{Ob}(\mathcal{C})$, and three natural isomophisms:

- (associativity) α with components $\alpha_{x,y,z}\colon (x \otimes y) \otimes z \cong x \otimes (y \otimes z)$,
- (left identity) λ with components $\lambda_x\colon 1 \otimes x \cong x$, and
- (right identity) ρ with components $\rho_x\colon x \otimes 1 \cong x$,

such that the diagrams





commute for all objects $w, x, y, z \in \mathcal{C}$. These diagrams are called the *coherence conditions*.

A monoidal category $(\mathcal{C}, \otimes, 1, \alpha, \lambda, \rho)$ can be denoted by $(\mathcal{C}, \otimes, 1)$ if α, λ and ρ are clear or even simply by $\mathcal{C}$.

Example 1.1.16. Examples of monoidal categories include:

- the category **Set** of sets with the Cartesian product and any singleton as the unit;
- the category $\Bbbk$-**mod** of finite-dimensional left $\Bbbk$-modules with the usual tensor product and $\Bbbk$ as the unit; or
- the category of $\Bbbk$-algebras with the usual tensor product and $\Bbbk$ as the unit.

In general, any category having all finite products or all finite coproducts is monoidal by taking the unit to be the empty product. The following example of a monoidal category is important to our discussion of differential graded categories, which is contained in Section 6.1.

Example 1.1.17 (Category of Cochain Complexes in $\Bbbk$-**mod**)**.** A *cochain complex* $M^\bullet$ of $\Bbbk$-modules is a diagram

$$0 \xrightarrow{d_0} M_1 \xrightarrow{d_1} M_2 \xrightarrow{d_2} \cdots$$

in $\Bbbk$-**mod** such that $d_{n+1}d_n = 0$ for every non-negative integer n. A morphism of cochain complexes

$$(f_i)_{i\in\mathbb{N}}\colon M^\bullet \to N^\bullet$$

is a collection of $\Bbbk$-linear maps $f_i\colon M_i \to N_i$ such that the diagram

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & M_{n-1} & \xrightarrow{d_{n-1}} & M_n & \xrightarrow{d_n} & M_{n+1} & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle f_{n-1}} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n+1}} & & \\ \cdots & \longrightarrow & N_{n-1} & \xrightarrow[e_{n-1}]{} & N_n & \xrightarrow[e_n]{} & N_{n+1} & \longrightarrow & \cdots \end{array}$$

commutes. Associativity and composition follow from basic computations. Thus, we have the *category of cochain complexes* in $\Bbbk$-**mod**.

Let $M^\bullet$ and $N^\bullet$ be cochain complexes with differentials d and e, respectively. The tensor product $M^\bullet \otimes N^\bullet$ is the cochain complex

$$\cdots \xrightarrow{c_{n-2}} \left(\bigoplus_{\substack{i+j\\=n-1}} M_i \otimes N_j \right)_{n-1} \xrightarrow{c_{n-1}} \left(\bigoplus_{\substack{i+j\\=n}} M_i \otimes N_j \right)_n \xrightarrow{c_n} \cdots$$

where $c_n = \bigoplus_{i+j=n} d_i \otimes e_j$. The cochain complex

$$0 \longrightarrow \Bbbk \longrightarrow 0 \longrightarrow \cdots$$

serves as the identity object. Thus, the category of cochain complexes in $\Bbbk$-**mod** is monoidal. The monoidal category of chain complexes follow from duality.

Now that we have seen some examples, let us look at a structure preserving map between monoidal categories.

Definition 1.1.18 (Monoidal Functor)**.** Let $(\mathcal{C}, \otimes, 1_{\mathcal{C}})$ and $(\mathcal{D}, \boxtimes, 1_{\mathcal{D}})$ be monoidal categories. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is *monoidal* if there exists a natural transformation η with components

$$\eta_{x,y}\colon Fx \boxtimes Fy \to F(x \otimes y),$$

and a morphism $f\colon 1_{\mathcal{D}} \to F(1_{\mathcal{C}})$ such that, for all objects $x, y, z \in \mathrm{Ob}(\mathcal{C})$, the diagrams

$$\begin{array}{ccc}
(Fx \boxtimes Fy) \boxtimes Fz & \xrightarrow{\alpha_{\mathcal{D}}} & Fx \boxtimes (Fy \boxtimes Fz) \\
{\scriptstyle \eta_{x,y} \boxtimes 1_{Fz}} \downarrow & & \downarrow {\scriptstyle 1_{Fx} \boxtimes \eta_{y,z}} \\
F(x \otimes y) \boxtimes Fz & & Fx \boxtimes F(y \otimes z) \\
{\scriptstyle \eta_{x \otimes y, z}} \downarrow & & \downarrow {\scriptstyle \eta_{x, y \otimes z}} \\
F((x \otimes y) \otimes z) & \xrightarrow{F\alpha_{\mathcal{C}}} & F(x \otimes (y \otimes z))
\end{array}$$

$$\begin{array}{ccc}
Fx \boxtimes 1_{\mathcal{D}} & \xrightarrow{1_{Fa} \boxtimes f} & Fx \boxtimes F1_{\mathcal{C}} \\
{\scriptstyle \rho_{\mathcal{D}}} \downarrow & & \downarrow {\scriptstyle \eta_{x, 1_{\mathcal{C}}}} \\
Fx & \xleftarrow{F\rho_{\mathcal{C}}} & F(x \otimes 1_{\mathcal{C}})
\end{array}
\qquad
\begin{array}{ccc}
1_{\mathcal{D}} \boxtimes Fy & \xrightarrow{f \boxtimes 1_{Fy}} & F1_{\mathcal{C}} \boxtimes Fy \\
{\scriptstyle \lambda_{\mathcal{D}}} \downarrow & & \downarrow {\scriptstyle \eta_{1_{\mathcal{C}}, y}} \\
Fy & \xleftarrow{F\lambda_{\mathcal{C}}} & F(1_{\mathcal{C}} \otimes y)
\end{array}$$

commute in $\mathcal{D}$. A monoidal functor is *strong* when η and f are invertible.

Definition 1.1.19 (Enriched Category)**.** Let $(\mathcal{D}, \otimes, 1, \alpha, \lambda, \rho)$ be a monoidal category. A category $\mathcal{C}$ that is *enriched over* $\mathcal{D}$ is given by the following data:

- a class $\mathrm{Ob}(\mathcal{C})$ of $\mathcal{C}$-objects,

- a $\mathcal{D}$-object $\mathcal{C}(x,y)$ for every pair $x,\ y \in \mathrm{Ob}(\mathcal{C})$,
- a $\mathcal{D}$-morphism $\mathrm{id}_x \colon 1 \to \mathcal{C}(x,x)$ for all $x \in \mathrm{Ob}(\mathcal{C})$, and
- a $\mathcal{D}$-morphism $\circ_{xyz} \colon \mathcal{C}(y,z) \otimes \mathcal{C}(x,y) \to \mathcal{C}(x,z)$ for all $x,\ y,\ z \in \mathrm{Ob}(\mathcal{C})$

with the additional requirement that for all $w,x,y,z \in \mathrm{Ob}(\mathcal{C})$, the diagrams

$$\begin{tikzcd}
(\mathcal{C}(y,z) \otimes \mathcal{C}(x,y)) \otimes \mathcal{C}(w,x) \arrow[rr, "\alpha"] \arrow[d, "\circ_{xyz} \otimes 1_{\mathcal{C}(w,x)}"'] & & \mathcal{C}(y,z) \otimes (\mathcal{C}(x,y) \otimes \mathcal{C}(w,x)) \arrow[d, "1_{\mathcal{C}(y,z)} \otimes \circ_{wxy}"] \\
\mathcal{C}(x,y) \otimes \mathcal{C}(w,x) \arrow[dr, "\circ_{wxy}"'] & & \mathcal{C}(y,z) \otimes \mathcal{C}(w,y) \arrow[dl, "\circ_{wyz}"] \\
& \mathcal{C}(w,z) &
\end{tikzcd}$$

$$\begin{tikzcd}
1 \otimes \mathcal{C}(x,y) \arrow[r, "\lambda"] \arrow[d, "\mathrm{id}_y \otimes 1_{C(x,y)}"'] & \mathcal{C}(x,y) & \mathcal{C}(x,y) \otimes 1 \arrow[l, "\rho"'] \arrow[d, "1_{\mathcal{C}(x,y)} \otimes \mathrm{id}_x"] \\
\mathcal{C}(y,y) \otimes \mathcal{C}(x,y) \arrow[ur, "\circ_{xyy}"'] & & \mathcal{C}(x,y) \otimes \mathcal{C}(x,x) \arrow[ul, "\circ_{xxy}"]
\end{tikzcd}$$

commute in $\mathcal{D}$.

A category enriched over $\mathcal{D}$ is alternatively called a *$\mathcal{D}$-category.* Many examples of enriched categories will be used throughout this text. In fact, the class of enriched categories known as $\Bbbk$-linear categories are central to the theory of matrix problems. They will be discussed in the next section.

Example 1.1.20. We present some examples of enriched categories.

- Every category $\mathcal{C}$ is enriched over the category **Set**. Although, some authors use a definition of a category that only requires the morphism spaces to be a class. In this case, a category where every morphism space is a proper set is called *locally small.* Then, locally small categories are enriched over **Set**.
- The category $\Bbbk$-**mod** of finite-dimensional left $\Bbbk$-modules is enriched over itself.
- A ring can be considered as a category with a single object enriched over the category **Ab** of abelian groups. Here, the ring elements correspond to the mor-

phisms, multiplication corresponds to composition and addition follows from the group structure of the hom-set.

Definition 1.1.21 (Enriched Functor)**.** If $\mathcal{C}$ and $\mathcal{C}'$ are both $\mathcal{D}$-categories, then a $\mathcal{D}$-*enriched functor* $F\colon \mathcal{C} \to \mathcal{C}'$ consists of a set map $F\colon \operatorname{Ob}(\mathcal{C}) \to \operatorname{Ob}(\mathcal{C}')$ and, for every pair x, y of $\mathcal{C}$-objects, a $\mathcal{D}$-morphism

$$F_{x,y}\colon \mathcal{C}(x,y) \to \mathcal{C}'(Fx,Fy),$$

such that the diagrams

$$\begin{array}{ccc} \mathcal{C}(y,z)\otimes\mathcal{C}(x,y) & \xrightarrow{\circ_{xyz}} & \mathcal{C}(x,z) \\ \downarrow{\scriptstyle F_{yz}\otimes F_{xy}} & & \downarrow{\scriptstyle F_{xz}} \\ \mathcal{C}'(Fy,Fz)\otimes\mathcal{C}'(Fx,Fy) & \xrightarrow[\circ_{FxFyFz}]{} & \mathcal{C}'(Fx,Fz), \end{array} \qquad \begin{array}{ccc} 1 & \xrightarrow{\mathrm{id}_x} & \mathcal{C}(x,x) \\ & {\scriptstyle \mathrm{id}_{Fx}}\searrow \quad \swarrow{\scriptstyle F_{xx}} & \\ & \mathcal{C}'(Fx,Fx) & \end{array}$$

commute in $\mathcal{D}$.

Definition 1.1.22 (Pre-addititve, Additive Category)**.** A category is *pre-additive* if it is enriched over **Ab**. A category is *additive* if it is pre-additive, contains a zero object and has a biproduct $a \oplus b$ for all objects a, b.

Equivalently, a category $\mathcal{C}$ is pre-additive if every hom-set is an abelian group and the composition of morphisms is bilinear.

1.1.2 Linear Categories

The focus of this section is to introduce a class of categories that play an important role in matrix problem theory. Such categories will primarily be used in the discussions concerning those matrix problems defined by differential graded categories (DGCs) or bimodules over categories with coalgebra structure (bocses).

Definition 1.1.23 ($\Bbbk$-Linear Category)**.** A $\Bbbk$-*linear category* $\mathcal{C}$ is a category that is enriched over $\Bbbk$-**mod**. A functor that is enriched over $\Bbbk$-**mod** is a $\Bbbk$-*linear functor.*

Remark 1.1.24. There appears to be a discrepancy over the use of the term $\Bbbk$-*linear category*. In both [11] and [14], the term $\Bbbk$-*category* is used instead of $\Bbbk$-linear category.

The concepts of ideals and factor categories follow from the theory of modules.

Definition 1.1.25 (Ideal)**.** An *ideal* I of a $\Bbbk$-linear category $\mathcal{C}$ is a collection of $\mathcal{C}$-morphisms such that, for any $w, x, y, z \in \mathcal{C}$, we have

- $I(x, y) = I \cap \mathcal{C}(x, y)$ is a submodule of $\mathcal{C}(x, y)$,
- if $f \in I(w, x)$, $g \in \mathcal{C}(x, y)$ and $h \in \mathcal{C}(y, z)$, then $gf \in I(w, y)$ and $hg \in I(x, z)$, and
- $I(x, x) \neq \mathcal{C}(x, x)$ for any $x \in \mathrm{Ob}(\mathcal{C})$.

Definition 1.1.26 (Factor Category)**.** Let I be an ideal of a $\Bbbk$-linear category $\mathcal{C}$. Denote by $\mathcal{C}/I$ the category with objects $\mathrm{Ob}(\mathcal{C}/I) = \mathrm{Ob}(\mathcal{C})$ and whose morphism spaces are defined to be

$$\mathcal{C}/I(x, y) = \mathcal{C}(x, y)/I(x, y) = \{[f] : f \in \mathcal{C}(x, y)\},$$

where $[f] = \{f + h : h \in I(x, y)\}$ is the equivalence class of f. Associativity and composition in $\mathcal{C}/I$ follow naturally from $\mathcal{C}$. We call C/I a *factor category*.

The operations, $[f] + [g] = [f + g]$ and $r[f] = [rf]$, are well-defined and so $\mathcal{C}/I(x, y)$ is a $\Bbbk$-module. In particular, $\mathcal{C}/I$ is a $\Bbbk$-linear category.

Definition 1.1.27 (Projection Functor)**.** Let $\mathcal{C}$ be a $\Bbbk$-linear category $\mathcal{C}$ with factor category $\mathcal{C}/I$ for some ideal I of $\mathcal{C}$. Define a functor $\Pi \colon \mathcal{C} \to \mathcal{C}/I$ by letting the object map be $\Pi(x) = x$ for all $x \in \mathrm{Ob}(\mathcal{C})$ and letting the morphism map be $\Pi(f) = [f]$ (see Definition 1.1.26) for every $\mathcal{C}$-morphism f. We call Π a *projection functor*.

1.2 Posets

The concept of a partial order is fairly pervasive throughout mathematics. However, the notion of a representation of a partial order is quite new, having been introduced by Nazarova and Roĭter in the middle of the twentieth century in connection with the second Brauer-Thrall Conjecture (see [22]). Matrix problems associated to the representation theory of posets form an important class of matrix problems. But prior to studying the connection between matrix problems and the representation theory of posets, we must become familiar with the notion of a poset and its representations. The present section is devoted to this end.

Definition 1.2.1 (Poset)**.** A *partially ordered set* or *poset* $(P, \preceq)$ is a binary relation $\preceq$ on a set P that is reflexive, anti-symmetric and transitive.

Usually, we will denote a poset $(P, \preceq)$ by P. Two elements $a, b \in P$ are said to be *comparable* if $a \preceq b$ or $b \preceq a$. If $a \prec b$, meaning $a \preceq b$ and $a \neq b$, and there does not exist any $c \in P$ such that $a \prec c \prec b$, then we say that b *covers* a. A poset in which every pair of elements is comparable is a *total order*.

For the remainder of this section, let P be a poset with n elements.

Definition 1.2.2 (Representation of a Poset)**.** A *representation of* P is a tuple of matrices

$$(M_p)_{p \in P},$$

such that, for some $d_0 \in \mathbb{N}$ and every $p \in P$, we have that $M_p \in \mathbb{k}^{d_0 \times d_p}$ where $d_p \in \mathbb{N}$. By letting $\underline{d} = (d_p)_{p \in P}$, we call the pair $(d_0, \underline{d})$ the *dimension* of M.

There are two notions of a poset representation found in the literature. Definition 1.2.2 follows [7, p. 7]. The notion of a P-space (see Definition 5.1.1) is occasionally taken to be the definition of a poset representation (see [3, Sec. 1] and [21, p. 1]).

Definition 1.2.3 (Category of Finite Dimensional Representations)**.** Let $(M_p)_{p \in P}$ and $(N_p)_{p \in P}$ be representations of P with dimensions $(d_0, \underline{d})$ and $(e_0, \underline{e})$, respectively. A *morphism of representations* $(A, B)\colon (M_p) \to (N_p)$ is a pair of matrices

$$(A, B) \in \mathbb{k}^{e_0 \times d_0} \times \mathbb{k}^{\underline{e} \times \underline{d}}$$

where $B_{q,p} = 0$ whenever $q \npreceq p$ and such that the equality

$$AM_p = \sum_{q \preceq p} N_q B_{q,p}$$

holds for each $p \in P$. The composition of morphisms (C, D) and (A, B) is defined by

$$(C, D) \circ (A, B) = (CA, DB).$$

The identity morphisms are pairs of identity matrices of suitable size. Associativity follows from a basic calculation. Thus, we have a category P-**rep** of representations of P and their morphisms.

Example 1.2.4. Consider the poset $(S, \preceq)$ consisting of a set $S = \{a, b, c\}$ with a single relation $a \prec b$. Then the morphism

$$\left(\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{smallmatrix}\right), \begin{pmatrix} \begin{smallmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \end{smallmatrix} \\ \begin{smallmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{smallmatrix} \\ \begin{smallmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{smallmatrix} \end{pmatrix} \right) : \left(\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right) \right) \to \left(\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \\ 0 & \frac{1}{2} \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \\ -\frac{1}{2} & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 0 \end{smallmatrix}\right) \right)$$

is in the category S-**rep**.

For the remainder of this section, we will present some properties of the category P-**rep**. First, recall that the direct sum of matrices A and B is

$$A \oplus B = \left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right).$$

We define the direct sum of representations $(M_p)_{p \in P}$ and $(N_p)_{p \in P}$ to be

$$(M_p)_{p \in P} \oplus (N_p)_{p \in P} = (M_p \oplus N_p)_{p \in P}. \tag{1.2.1}$$

As with the case for the direct sum of modules, Equation (1.2.1) describes the coproduct of M and N. This can be verified with a basic calculation using the injection maps

$$\iota_M = \left(\begin{pmatrix} \mathbb{I}_{d_0} \\ 0 \end{pmatrix}, \bigoplus_{i=1}^{n} \begin{pmatrix} \mathbb{I}_{d_i} \\ 0 \end{pmatrix} \right) \text{ and } \iota_N = \left(\begin{pmatrix} 0 \\ \mathbb{I}_{e_0} \end{pmatrix}, \bigoplus_{i=1}^{n} \begin{pmatrix} 0 \\ \mathbb{I}_{e_i} \end{pmatrix} \right).$$

To comply with the definition of a morphism of poset representations, we consider the second term of the injection maps to have a block matrix structure in the obvious way.

Evidently, the direct sum $M \oplus N$ is also a product of M and N. This parallels the direct sum of modules in the finite-dimensional case. A basic calculation using the projection maps

$$\pi_M = \left(\begin{pmatrix} \mathbb{I}_{d_0} & 0 \end{pmatrix}, \bigoplus_{i=1}^{n} \begin{pmatrix} \mathbb{I}_{d_i} & 0 \end{pmatrix} \right) \text{ and } \pi_N = \left(\begin{pmatrix} 0 & \mathbb{I}_{e_0} \end{pmatrix}, \bigoplus_{i=1}^{n} \begin{pmatrix} 0 & \mathbb{I}_{e_i} \end{pmatrix} \right)$$

will verify that $M \oplus N$ is, indeed, a product. Because the direct sum coincides with both the product and coproduct, the next lemma is not surprising.

Lemma 1.2.5 (Biproduct of Representations)**.** *All binary biproducts exist in* P-**rep**.

Proof: For all diagrams

$$M \underset{\pi_M}{\overset{\iota_M}{\rightleftarrows}} M \oplus N \underset{\pi_N}{\overset{\iota_M}{\leftrightarrows}} N$$

in P-**rep**, the relations

$$\pi_M \circ \iota_M = 1_M, \quad \pi_N \circ \iota_N = 1_N, \quad \iota_M \circ \pi_M + \iota_N \circ \pi_N = 1_{M \oplus N}$$

of P-**rep**-morphisms follow from a simple calculation on the matrices involved. ∎

Proposition 1.2.6. *The category* P-**rep** *is* $\Bbbk$*-linear and additive.*

Proof: We will first show that P-**rep** is $\Bbbk$-linear. Let $(A, B), (A', B') \colon M \to N$ be morphisms between representations M and N of P. Then, for all $p \in P$, we have that

$$AM_p = \sum_{q \preceq p} N_q B_{q,p} \text{ and } A'M_p = \sum_{q \preceq p} N_q B'_{q,p}$$

which implies that

$$(A + A')M_p = \sum_{q \preceq p} N_q(B_{q,p} + B'_{q,p}).$$

Hence, $(A + A', B + B')\colon M \to N$ is a morphism. Clearly, if $c \in \Bbbk$, then $c(A, B) = (cA, cB)$ is a morphism. Bilinearity follows easily.

We now show P-**rep** is additive. The zero object is the empty matrix $\mathscr{I}_{0,0}$. Because it is $\Bbbk$-linear, P-**rep** is enriched over **Ab**. Lastly, it follows from Lemma 1.2.5 that P-**rep** has all finite biproducts. ∎

1.3 Quivers

Quivers, which can be traced back to Gabriel in [6], were created as a tool to study representations of finite-dimensional algebras. The class of matrix problems associated to the study of quiver representations form an important part of the theory of matrix problems. We introduce quivers and their representations in this section.

Definition 1.3.1 (Category of Quivers)**.** A *quiver* $Q = (Q_N, Q_A, h, t)$ is a quadruple consisting of a set of nodes Q_N, a set of arrows Q_A and two maps $h, t\colon Q_A \to Q_N$ assigning a head and tail, respectively, to each arrow. A *quiver morphism*

$$(f_N, f_A)\colon Q \to Q'$$

is a pair of set maps

$$f_N\colon Q_N \to Q'_N \text{ and } f_A\colon Q_A \to Q'_A$$

where f_A sends any arrow $\eta \xrightarrow{\alpha} \eta'$ of Q to an arrow $f_N(\eta) \xrightarrow{f_A(\alpha)} f_N(\eta')$ of Q'. Composition of morphisms (f_N, f_A) and (g_N, g_A) is defined by

$$(f_N, f_A) \circ (g_N, g_A) = (f_N \circ g_N, f_A \circ g_A).$$

Associativity follows from a basic calculation. Thus, we obtain the category **Quiv** of quivers.

A quiver with only finitely many nodes and arrows is called a *finite quiver*. We will work strictly in the full subcategory **quiv** of finite quivers in the sequel.

Definition 1.3.2 (Path)**.** A *path of length* n in a quiver is a sequence of arrows $\alpha_n\alpha_{n-1}\cdots\alpha_1$ such that $h(\alpha_i) = t(\alpha_{i+1})$ for $i = 1, \ldots, n-1$.

$$t(\rho)\ \bullet \xrightarrow{\alpha_1} \bullet \xrightarrow{\alpha_2} \bullet \quad \cdots \quad \bullet \xrightarrow{\alpha_n} \bullet\, h(\rho)$$

Figure 1.2: A path ρ

Consider a path $\rho = \alpha_n\alpha_{n-1}\cdots\alpha_1$. The head $h(\rho)$ of ρ is the node $h(\alpha_n)$ and the tail $t(\rho)$ of ρ is the node $t(\alpha_1)$. If $h(\rho) = t(\rho)$, then ρ is called a *cycle*. In particular, a cycle of length 1 is a *loop* and of length 0 is an *empty path*. By convention, every node η has an associated empty path ε_η. Denote the length of ρ by $|\rho|$.

Definition 1.3.3 (Path Poset)**.** Let Q be a quiver. The *path poset* $(P_Q, \sqsubseteq)$ consists of the set P_Q of all paths in Q with the relation $\sqsubseteq$ defined by letting

$$\alpha_{j+m}\alpha_{j+m-1}\cdots\alpha_j \sqsubseteq \alpha_n\alpha_{n-1}\cdots\alpha_1$$

if j and m are non-negative integers such that $1 \leq j \leq n$ and $j + m \leq n$.

In the sequel, we will use the relation $\sqsubseteq$ on the set of paths of a quiver without explicitly referring to the path poset of the quiver.

Definition 1.3.4 (Path Algebra)**.** The *path algebra* $\Bbbk Q$ of a quiver Q is the $\Bbbk$-algebra spanned by the set of all paths in Q and with the multiplication of paths

$$\rho_1 = \alpha_n\alpha_{n-1}\cdots\alpha_1, \quad \rho_2 = \beta_m\beta_{m-1}\cdots\beta_1,$$

defined as

$$\rho_2\rho_1 = \begin{cases} \beta_m \cdots \beta_1 \alpha_n \cdots \alpha_1, & \text{if } h(\rho_1) = t(\rho_2), \\ 0, & \text{if } h(\rho_1) \neq t(\rho_2), \end{cases}$$

which is then extended bilinearly.

Observe that the path algebra has a decomposition

$$\Bbbk Q = \bigoplus_{i \in \mathbb{N}} \Bbbk Q_i$$

where $\Bbbk Q_i$ is the subspace spanned by paths of length i. In other words, we say that the path algebra is *graded*, a term that will be formally defined in Section 6.1.

Example 1.3.5. Let R be the quiver

$$\alpha \circlearrowleft \bullet_1 .$$

The paths in R are the finite compositions of α and the empty path ε. Thus, $\Bbbk R$ has basis $\{\varepsilon, \alpha, \alpha^2, \dots\}$ and $\Bbbk R \cong \Bbbk[X]$.

The next items we will introduce are arrow ideals and quiver relations. They will not play a role in the matrix problems that we study. Regardless, these notions are central in the study of quivers. Studying matrix problems that are related to quivers bound by an ideal (see Definition 1.3.6) would be a natural extension of the ideas in this thesis.

Definition 1.3.6 (Ideals)**.** Let Q be a quiver. Denote by R_Q^m the two-sided ideal of the path algebra $\Bbbk Q$ generated by all paths of length m. When $m = 1$, we obtain the *arrow ideal* R_Q. Note that there exists a chain of ideals

$$R_Q \supseteq R_Q^2 \supseteq R_Q^3 \supseteq R_Q^4 \supseteq \dots .$$

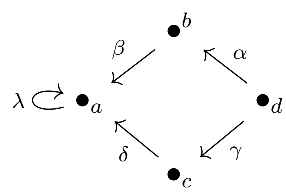
We say that an ideal I is *admissible* if $R_Q^m \subseteq I \subseteq R_Q^2$ for some integer $m \geq 2$. In the case that I is admissible, the pair (Q, I) is called a *bound quiver* and the quotient algebra $\Bbbk Q/I$ is called a *bound quiver algebra*.

Definition 1.3.7 (Quiver Relations)**.** A *relation* on a quiver Q is a $\Bbbk$-linear combination

$$\sum_{i=1}^{n} k_i\rho_i, \text{ with } k_i \in \Bbbk,$$

of paths ρ_i of length at least 2 and such that $h(p_i) = h(\rho_j)$ and $t(p_i) = t(\rho_j)$ for $i, j = 1, \ldots, n$. Given a set of relations $\mathcal{R}$, we say that Q is *bound by* $\mathcal{R}$ if the ideal $\langle\mathcal{R}\rangle$ generated by $\mathcal{R}$ is admissible.

Example 1.3.8. Let S be the quiver



and $\mathcal{R} = \{\beta\alpha - \delta\gamma, \lambda\beta, \lambda^3\}$ a set of relations on S. Then $\langle\mathcal{R}\rangle$ is the ideal generated by $\mathcal{R}$. Let us show that $\langle\mathcal{R}\rangle$ is admissible. It follows by inspection that $\langle\mathcal{R}\rangle \subseteq R_S^2$. We now show that $\langle\mathcal{R}\rangle \supseteq R_S^4$. Suppose that ρ is a path of length at least 4. If the tail of ρ is any of the nodes a, b, or c, then $\lambda^3 \sqsubseteq \rho$ which implies that $\rho \in \langle\mathcal{R}\rangle$. If the tail of ρ is the node d, then either

- $\rho \sqsupseteq \lambda\beta$ which implies that $\rho \in \langle\mathcal{R}\rangle$, or
- $\rho \sqsupseteq \lambda^2\delta\gamma = \lambda^2(\beta\alpha - \delta\gamma) + \lambda^2\beta\alpha$ which implies that $\rho \in \langle\mathcal{R}\rangle$.

Thus $\langle\mathcal{R}\rangle$ is admissible.

For the remainder of this section, we will introduce the necessary concepts to define the category of representations of a quiver Q. This category is where the theory of quivers meets the theory of matrix problems.

Definition 1.3.9 (Quiver Representation)**.** A $\Bbbk$-*linear representation of a quiver* Q is a pair

$$(\{V_\eta\}_{\eta\in Q_N}, \{f_\alpha\}_{\alpha\in Q_A}),$$

consisting of a set $\{V_\eta\}_{\eta\in Q_N}$ of $\Bbbk$-modules indexed by Q_N, and another set $\{f_\alpha\}_{\alpha\in Q_A}$ containing a $\Bbbk$-linear map $f_\alpha\colon V_\eta \to V_{\eta'}$ for every arrow $\eta \xrightarrow{\alpha} \eta'$ in Q.

For the remainder of the thesis, we will only refer to a quiver *representation* instead of a $\Bbbk$-linear representation. Typically, we will write a quiver representation as $(\{V_\eta\},\{f_\alpha\})$ when the indexing sets are clear or are of no importance.

A quiver representation is *finite-dimensional* if each $\Bbbk$-module V_η has a finite basis. In this case, there is a well-founded notion of dimension.

Definition 1.3.10 (Graded Dimension)**.** The *graded dimension* of a representation $(\{V_\eta\},\{f_\alpha\})$ for a quiver Q is the $|Q_N|$-tuple $\underline{d} = (\dim V_\eta)_{\eta\in Q_N}$.

Note that the graded dimension is a composition $\underline{d}$ of

$$d = \sum_{\eta\in Q_N} \dim V_\eta.$$

It is useful, in some instances, to make the graded dimension into an ordered tuple $\underline{d} = (d_i)_{i=1}^n$ by ordering the nodes of Q, for instance, by fixing labels $\eta_1,\ldots,\eta_n$ and letting $d_i = \dim V_{\eta_i}$. We will do this in Chapter 4.

Definition 1.3.11 (Category of Quiver Representations)**.** Let $(\{V_\eta\},\{f_\alpha\})$ and $(\{W_\eta\},\{g_\alpha\})$ be representations of a quiver Q. Then a *morphism of quiver representations*

$$\psi = (\psi_\eta)_{\eta\in Q_N}\colon\ (\{V_\eta\},\{f_\alpha\}) \to (\{W_\eta\},\{g_\alpha\})$$

is a family of linear maps $\psi_\eta\colon V_\eta \to W_\eta$ indexed by Q_N such that, for every arrow $\eta \xrightarrow{\alpha} \eta'$ of Q, the diagram

$$\begin{array}{ccc} V_\eta & \xrightarrow{f_\alpha} & V_{\eta'} \\ {\scriptstyle \psi_\eta}\downarrow & & \downarrow{\scriptstyle \psi_{\eta'}} \\ W_\eta & \xrightarrow[g_\alpha]{} & W_{\eta'} \end{array}$$

commutes. If ψ_η is invertible for each $\eta \in Q_N$, then ψ is an *isomorphism* of representations. The composition of morphisms $(\theta_\eta)_{\eta\in Q_N}$, $(\psi_\eta)_{\eta\in Q_N}$ is defined by

$$(\theta_\eta)_{\eta\in Q_N} \circ (\psi_\eta)_{\eta\in Q_N} = (\theta_\eta \circ \psi_\eta)_{\eta\in Q_N}.$$

Associativity follows from a basic computation. Thus we have the category Q-**Rep** of representations of Q.

For the remainder of the thesis, we will be working strictly in the subcategory Q-**rep** of finite-dimensional representations of Q.

Proposition 1.3.12 ([11, p. 70])**.** *Let Q be a finite quiver. Then Q-**rep** is a $\mathbb{k}$-linear, abelian category.*

Chapter 2

Linear Matrix Problems

In this chapter, we introduce matrix problems. In particular, we discuss four types of matrix problems that are found in the literature: Gabriel-Roĭter, similarity, linking and basic algebra matrix problems. A commonality between these classes is that they can be placed into the following framework.

Definition 2.0.13 (Linear Matrix Problem)**.** Let $\mathscr{M} \subseteq \Bbbk^{m\times n}$ be a set of matrices and $G \subseteq GL(m) \times GL(n)$ be a subgroup with an action $G \times \mathscr{M} \to \mathscr{M}$. A *linear matrix problem* $(\mathscr{M}, \sim)$ is the equivalence relation $\sim$ on $\mathscr{M}$ defined, for all $x, y \in \mathscr{M}$, by letting $x \sim y$ if and only if x and y are in the same G-orbit.

As mentioned in the introduction, when given a linear matrix problem, a typical goal is to classify the equivalence classes if possible. However, we will not study the question of determining classification schemes. The object of this chapter will be to introduce the four types of linear matrix problems listed above and to show that Gabriel-Roĭter matrix problems and similarity matrix problems are, in some sense, equivalent and that linking matrix problems and algebra matrix problems are, in a similar sense, equivalent. In Chapters 3 and 4, we will show how the linear matrix problems relate to representations of posets and quivers, respectively.

Remark 2.0.14. When defining specific matrix problems in the subsequent sections, we will be using the set $\Bbbk^{\underline{m}\times\underline{n}}$ as opposed to $\Bbbk^{m\times n}$ and, similarly, we will use the group $GL(\underline{n})$ instead of $GL(n)$. The advantage of using compositions will become apparent in Chapters 3 and 4 where we will often need to refer to blocks within matrices.

2.1 Gabriel-Roĭter Matrix Problems

Fix compositions $\underline{m}$ and $\underline{n}$ for non-negative integers m and n. Let $\Lambda \subseteq GL(\underline{m})\times GL(\underline{n})$ be a subgroup and $\mathscr{M} \subseteq \Bbbk^{\underline{m}\times\underline{n}}$ be a set of matrices that is closed under the group action

$$\Lambda \times \mathscr{M} \to \mathscr{M}, \quad (A,B)\cdot M = AMB^{-1}. \tag{2.1.1}$$

We will say that $\mathscr{M}$ is *closed under* Λ*-equivalence*.

The equivalence relation $\sim_\Lambda$ defined by the orbits of the action from Equation (2.1.1) will be called the Λ*-equivalence relation*. Stated precisely, we have, for any $M, N \in \mathscr{M}$, that $M \sim_\Lambda N$ if and only if $\Lambda \cdot M = \Lambda \cdot N$. The next definition can be found in [7, Sec. 1].

Definition 2.1.1 (Gabriel-Roĭter Matrix Problems)**.** A *Gabriel-Roĭter matrix problem* is a pair

$$(\mathscr{M}, \sim_\Lambda),$$

consisting of the Λ-equivalence relation $\sim_\Lambda$ on a set $\mathscr{M} \in \Bbbk^{\underline{m}\times\underline{n}}$ that is closed under Λ-equivalence, for some subgroup $\Lambda \subseteq GL(\underline{m}) \times GL(\underline{n})$.

Example 2.1.2. Let $\mathscr{M} = \Bbbk^{\underline{m}\times\underline{n}}$ where $\underline{m} = (m) \in \mathbb{N}$ and $\underline{n} = (n_1, n_2) \in \mathbb{N}^2$. That is, $\mathscr{M}$ contains block matrices of the form $(A_{1,i})_{i=1}^2$ where $A_{1,1} \in \Bbbk^{m\times n_1}$ and $A_{1,2} \in \Bbbk^{m\times n_2}$. Let $\Lambda = GL(m) \times (GL(n_1) \oplus GL(n_2))$. Then the matrix problem $(\mathscr{M}, \sim_\Lambda)$ corresponds to the classification of the equivalence classes on $\mathscr{M}$ arising from arbitrary row transformations and from column transformations that are restricted to each block.

2.2 Similarity Matrix Problems

Let $\Omega \subseteq GL(\underline{n})$ be a subgroup and $\mathscr{N} \subseteq \mathbb{k}^{\underline{n}\times\underline{n}}$ be a set of square matrices that is closed under the group action

$$\Omega \times \mathscr{N} \to \mathscr{N}, \quad G \cdot M = GMG^{-1}. \tag{2.2.1}$$

In this case, we say that $\mathscr{N}$ is *closed under* Ω*-similarity*.

Denote by $\sim_\Omega$ the Ω-*similarity equivalence relation* defined by the orbits of the action from Equation (2.2.1). Stated precisely, we have, for any $M, N \in \mathscr{N}$, that $M \sim_\Omega N$ if and only if $\Omega \cdot M = \Omega \cdot N$.

Definition 2.2.1 (Similarity Matrix Problems)**.** A *similarity matrix problem* is a pair

$$(\mathscr{N}, \sim_\Omega)$$

consisting of the Ω-similarity equivalence relation $\sim_\Omega$ on a set $\mathscr{N} \subseteq \mathbb{k}^{\underline{n}\times\underline{n}}$ that is closed under Ω-similarity, for some subgroup $\Omega \subseteq GL(\underline{n})$.

In this next example, we will refrain from using compositions as it offers no advantages (cf. Remark 2.0.14).

Example 2.2.2. Two pairs of $n \times n$ matrices $(M, N), (M', N')$ are said to be *simultaneously similar* if there exists a non-singular $n \times n$ matrix S such that

$$(M', N') = (SMS^{-1}, SNS^{-1}).$$

The similarity matrix problem $(\mathbb{k}^{n\times n} \times \mathbb{k}^{n\times n}, \sim_{GL(n)})$ corresponds to the classification of the pairs of simultaneously similar matrices.

Example 2.2.3. The similarity matrix problem $(\mathbb{k}^{n\times n}, \sim_{GL(n)})$ corresponds to classifying the usual similarity classes of $n \times n$ matrices. The Jordan normal form provides a classification of this matrix problem.

2.3 Linking Matrix Problems

The definition of a linking matrix problem is quite technical and the background material required is not trivial. Especially when compared to the definitions for Gabriel-Roĭter or similarity matrix problems. The benefit, however, of linking matrix problems is that there is a natural way to associate a linking matrix problem to the category of representations for either a quiver or a poset. This will become evident in Sections 3.3 and 4.3.

The content of this section follows [29, Sec. 2]. However, we introduce several terms not found in that paper. Throughout this section, let $T = \{1, 2, \dots, t\}$ for some non-negative integer t and let $\sim$ be an equivalence relation on T.

Definition 2.3.1 (Linking Matrix)**.** A non-zero matrix $A = (a_{ij})$ in $\Bbbk^{t \times t}$ is called a $T^{\sim}$*-linking matrix* if there are equivalence classes I and J in $T/{\sim}$ such that $a_{ij} \neq 0$ implies that $i \in I$ and $j \in J$.

We will alternatively say that A *links* I to J. Suppose that a matrix A links I to J and a matrix A' links I' to J'. If $J = I'$, then AA' links I to J'. But if $J \neq I'$, then $AA' = 0$. Also, the equivalence classes that are linked by a linking matrix are unique in the sense that, given a matrix A that links I to J and links I' to J', it follows that $I = I'$ and $J = J'$.

Definition 2.3.2 (Step-Sequence)**.** Let $\underline{n} = (n_1, n_2, \dots, n_t)$ be a composition of length t for some non-negative integer n. Then $\underline{n}$ is a $T^{\sim}$*-step sequence* if, for all $i, j \in T$, we have that $n_i = n_j$ whenever $i \sim j$.

Let $\underline{n} = (n_1, \dots, n_t)$ be a $T^{\sim}$-step sequence and $A = (a_{i,j})_{i,j \in T}$ be a matrix linking H to K, where $H, K \in T/{\sim}$. Pick any $h \in H$ and $k \in K$. For each pair

$$(\ell, r) \in \{1, 2, \dots, n_h\} \times \{1, 2, \dots, n_k\},$$

denote by

$$A^{[\ell,r]} = \left(A^{[\ell,r]}_{i,j}\right)_{i,j\in T} \tag{2.3.1}$$

the $\underline{n} \times \underline{n}$ matrix with blocks

$$A^{[\ell,r]}_{ij} = \begin{cases} a_{ij}E_{\ell r}, & \text{if } i \in H \text{ and } j \in K, \\ 0, & \text{otherwise.} \end{cases}$$

Note that because $\underline{n}$ is a $T^{\sim}$-step sequence, n_h and n_k do not depend on the choice of h and k in their respective classes. A matrix of the form (2.3.1) is said to be A-*expanded*.

Let us look at a simple example of an expanded matrix.

Example 2.3.3. Let $S = \{1, 2, 3\}$ and $\sim_S$ be the equivalence relation so that $S/\sim_S = \{\{1\}, \{2, 3\}\}$. Consider the $S^{\sim_S}$-step sequence $\underline{n} = (2, 3, 3)$ and the matrix

$$A = \begin{pmatrix} 0 & a & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad a, b \in \Bbbk.$$

that links $\{1\}$ to $\{2, 3\}$. The collection of all A-expanded matrices is

$$\{A^{[1,1]}, A^{[1,2]}, A^{[1,3]}, A^{[2,1]}, A^{[2,2]}, A^{[2,3]}\}.$$

We will only illustrate the construction of $A^{[1,2]} = \left(A^{[1,2]}_{i,j}\right)_{i,j\in S}$. The others are constructed in an analogous manner. We have that

- $A^{[1,2]}_{ij} = 0$ is an $n_i \times n_j$ matrix for $(i, j) \neq (1, 2), (1, 3)$,
- $A^{[1,2]}_{12} = aE_{12}$ is a 2×3 matrix, and
- $A^{[1,2]}_{13} = bE_{12}$ is a 2×3 matrix.

That is

$$A^{[1,2]} = \left(\begin{array}{cc|ccc|ccc} 0 & 0 & 0 & a & 0 & 0 & b & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

Now that we have provided some of the necessary background, we can introduce the ingredients of a linking matrix problem.

Definition 2.3.4 (Linking Triple)**.** A $T^{\sim}$*-linking triple* $(\underline{n}, \mathcal{P}, \mathcal{V})$ consists of

- a $T^{\sim}$-step sequence $\underline{n}$,
- an empty or finite set $\mathcal{P}$ of nilpotent, upper triangular $T^{\sim}$-linking matrices, and
- a non-empty, finite set $\mathcal{V}$ of $T^{\sim}$-linking matrices.

Denote the multiplicative closure of $\mathcal{P}$ by $\overline{\mathcal{P}}$. Then $\overline{\mathcal{P}}$ is either an empty set or a finite set of nilpotent, upper triangular $T^{\sim}$-linking matrices. Let $\overline{\mathcal{V}}$ denote the closure of $\mathcal{V}$ with respect to right and left multiplication by $\overline{\mathcal{P}}$. Because $\overline{\mathcal{P}}$ is an empty or finite set, $\overline{\mathcal{V}}$ is a finite set of $T^{\sim}$-linking matrices.

The next definitions will present the objects that will provide the set of matrices and the equivalence relation for the definition of a linking matrix problem. For the remainder of the section, denote a $T^{\sim}$-linking triple $(\underline{n}, \mathcal{P}, \mathcal{V})$ by $\mathcal{R}$.

Definition 2.3.5 (Link Module)**.** The $\mathcal{R}$*-link module* $\mathscr{O}$ is the $\Bbbk$-module generated by the V-expanded matrices $V^{[\ell,r]}$ for every non-zero $V \in \overline{\mathcal{V}}$.

Note that the $\mathcal{R}$-link module $\mathscr{O}$ consists of $\underline{n} \times \underline{n}$ matrices.

Definition 2.3.6 (Link Transformation Group)**.** The $\mathcal{R}$*-link transformation group* L is the subgroup of $GL(\underline{n})$ generated by all matrices G such that either

(i) $G = (G_{i,j})_{i,j \in T}$ is a block matrix where $G_{i,i} = G_{j,j}$ whenever $i \sim j$ and $G_{i,j} = 0$ whenever $i \neq j$, or

(ii) $G = \left(\mathbb{I} + cP^{[\ell,r]}\right)$ where $c \in \Bbbk$ and $P^{[\ell,r]}$ is a P-expanded matrix for some $P \in \overline{\mathcal{P}}$.

Lemma 2.3.7. *If $\mathscr{O}$ and L are, respectively, the $\mathcal{R}$-link module and $\mathcal{R}$-link transformation group of a $T^{\sim}$-linking triple $\mathcal{R} = (\underline{n}, \mathcal{P}, \mathcal{V})$, then L acts on $\mathscr{O}$ by*

$$L \times \mathscr{O} \to \mathscr{O}, \quad G \cdot M = GMG^{-1}. \tag{2.3.2}$$

Proof: The statement follows from a basic, though lengthy, calculation. ∎

Denote by $\sim_L$ the $\mathcal{R}$-*link equivalence relation* defined by the orbits of the action from Equation (2.3.2). Stated precisely, we have, for any $M, N \in \mathscr{O}$, that $M \sim_L N$ if and only if $L \cdot M = L \cdot N$.

Prior to defining a linking matrix problem, let us try to understand the intuition of the role played by link modules and link transformation groups, as well as their interaction via the action described above. By drawing an analogy to the definition of a similarity matrix problem, a link module will be playing the role that was filled by the set $\mathscr{N}$ in Definition 2.2.1, and a link transformation group will be playing the role filled by Ω.

Given an $\mathcal{R}$-link transformation group L and an $\mathcal{R}$-link module $\mathscr{O}$, the action of L on $\mathscr{O}$ is not as complicated as it might seem. To describe this action, we first introduce the notion of a *stripe*. Consider a matrix $M \in \mathscr{O}$ and let $r \in T$. Then the r^{th} *horizontal stripe* $M_{r,\bullet}$ consists of the n_r rows of M numbered from $(n_1+\cdots+n_{r-1}+1)$ through $(n_1 + \cdots + n_r)$. The r^{th} *vertical stripe* $M_{\bullet,r}$ consists of the n_r columns of M numbered from $(n_1 + \cdots + n_{r-1} + 1)$ through $(n_1 + \cdots + n_r)$.

Take an element $G \in L$ of the form (i) from Definition 2.3.6. The action (see Equation (2.3.2)) of G on M corresponds to a row transformation occurring simultaneously in all horizontal stripes in such a way that if $i \sim j$, then the same row transformation occurs in the stripes $M_{i,\bullet}$ and $M_{j,\bullet}$. This is followed by the inverse transformation occurring simultaneously in all vertical stripes.

Take an element $G' = \left(\mathbb{I} + cP^{[\ell,r]}\right)$ of L with the form (ii) from Definition 2.3.6. Write $P^{[\ell,r]} = (p_{ij})$. The action of G' corresponds to taking the r^{th} row of the horizontal stripe $M_{\bullet j}$ and adding to it, cp_{ij} times the ℓ^{th} row of the horizontal stripe $M_{\bullet i}$ simultaneously for all $i \in I$ and $j \in J$. This is followed by the inverse column transformation within M.

Essentially, the action is performing elementary row and column transformations on the elements of the link module in such a way that the structure of the link group determines which transformations are permitted. Compare this description of the action to the discussion in the second paragraph of the Introduction (Chapter 0).

Now, let us define a linking matrix problem.

Definition 2.3.8 (Linking Matrix Problems)**.** Let $\mathcal{R} = (\underline{n}, \mathcal{P}, \mathcal{V})$ be a $T^{\sim}$-linking triple with $\mathcal{R}$-link module $\mathscr{O}$ and $\mathcal{R}$-link transformation group L. The *linking matrix problem* of $\mathcal{R}$ is the pair

$$(\mathscr{O}, \sim_L) \tag{2.3.3}$$

where $\sim_L$ is the $\mathcal{R}$-link equivalence relation.

Because both the link module and link transformation group are uniquely determined by a linking triple, every linking matrix problem $(\mathscr{O}, \sim_L)$ is uniquely determined by a linking triple. Therefore, in practice, it will suffice to present a linking matrix problem by a linking triple.

The following example illustrates a linking matrix problem by relating it to the representations of a quiver. A general relationship between classifying quiver representations and linking matrix problems will be discussed in Chapter 4.

Example 2.3.9. Let $S = \{1, 2, 3, 4, 5\}$ and $\sim_S$ be the equivalence relation so that $S/{\sim_S} = \{\{1\}, \{2, 3\}, \{4\}, \{5\}\}$. Also, let $\mathcal{P} = \emptyset$ and

$$\mathcal{V} = \{E_{12}, E_{13}, E_{14}, E_{15}\} \subseteq \Bbbk^{5\times 5}.$$

It is clear that $\mathcal{V}$ contains linking matrices. Hence

$$\mathcal{R} = (\underline{n}, \emptyset, \{E_{12}, E_{13}, E_{14}, E_{15}\})$$

is a linking triple for any $S^{\sim_S}$-step sequence $\underline{n} = (n_1, n_2, n_2, n_4, n_5)$.

We have that $\overline{\mathcal{P}} = \emptyset$ and $\overline{\mathcal{V}} = \{E_{12}, E_{13}, E_{14}, E_{15}\}$. The $\mathcal{R}$-link module $\mathscr{O}$ is the $\Bbbk$-module generated by all V-expanded matrices for the non-zero elements $V \in \overline{\mathcal{V}}$.

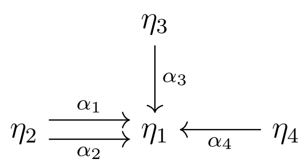
Thus, the elements of $\mathscr{O}$ are the block matrices $M = (M_{i,j})$ in $\Bbbk^{\underline{n} \times \underline{n}}$ such that $M_{i,j} = 0$ when $i \neq 1$ or $j = 1$.

The $\mathcal{R}$-link transformation group L for $\mathcal{R}$ is generated by elements with the form (i) from Definition 2.3.6. Because $\overline{\mathcal{P}}$ is empty, there are no generators with the form (ii). Thus L consists of elements of the form

$$G_1 \oplus G_2 \oplus G_2 \oplus G_3 \oplus G_4,$$

where $G_i \in GL(n_i)$.

The linking matrix problem $(\mathscr{O}, \sim_L)$ corresponds to the problem of classifying the isoclasses of Q-**rep**, where Q is the quiver



Let $(\{V_i\}, \{f_i\})$ and $(\{W_i\}, \{g_i\})$ be representations of Q. Pick a basis for each V_i and W_i. By a slight abuse of notation, let each f_i and g_i also denote the corresponding transformation matrices. Then the representations are isomorphic if and only if there exists a $G \in L$ such that

$$G \begin{pmatrix} 0 & f_1 & f_2 & f_3 & f_4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} G^{-1} = \begin{pmatrix} 0 & g_1 & g_2 & g_3 & g_4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

2.4 Basic Algebra Matrix Problems

Much like the definition of a linking matrix problem, the definition of a basic algebra matrix problem is rather technical. The primary benefit of this definition is that it is well suited for performing computations. In particular, there exists an algorithm called the *Belitskiĭ algorithm* which reduces a pair of square matrices to a canonical form by transformations of simultaneous similarity (cf. Example 2.2.2). Given a basic algebra matrix problem, the machinery of the Belitskiĭ algorithm is able to determine, in a finite number of steps, whether two matrices are in the same equivalence class. However, this is outside the scope of this thesis, and so interested readers should consult [1] or [29, Sec. 1], the latter of which relates the algorithm to linear matrix problems.

The concepts of a basic matrix algebra and a reduced matrix algebra are central to basic algebra matrix problems. Thus, much of this section is devoted to studying these types of algebras. Throughout this section, let $T = \{1, 2, \dots, t\}$ for some non-negative integer t.

We begin by recalling some algebraic facts. A $\Bbbk$-algebra Λ can be considered as a right module over itself. We denote this by Λ_Λ. Hence, if Λ is finite-dimensional, then by the Krull-Schmidt Theorem, $\Lambda_\Lambda = P_1 \oplus \cdots \oplus P_n$ where $P_1, \dots, P_n$ are indecomposable right ideals in Λ. The notion of an idempotent is closely related to such a decomposition.

Definition 2.4.1 (Primitive Orthogonal Idempotents)**.** Let Λ be a $\Bbbk$-algebra. An element $e \in \Lambda$ is *idempotent* if $e^2 = e$. Idempotents $e_1, e_2 \in \Lambda$ are *orthogonal* if $e_1 e_2 = e_2 e_1 = 0$. An idempotent is *primitive* if it cannot be written as a sum of two non-zero idempotents. Any set of non-zero, primitive, pairwise orthogonal idempotents $\{e_1, e_2, \dots, e_n\} \subseteq \Lambda$ such that $e_1 + \cdots + e_n = 1$ is called a *complete set of primitive orthogonal idempotents.*

Observe that an algebra Λ with a complete set of primitive orthogonal idempotents $\{e_1, e_2, \dots e_n\}$ admits the decomposition

$$\Lambda = \left(\sum_{i=1}^{n} e_i\right) \Lambda \left(\sum_{i=1}^{n} e_i\right) = \bigoplus_{i,j=1}^{n} e_i \Lambda e_j.$$

Definition 2.4.2 (Basic Matrix Algebra)**.** An algebra $\Gamma \subseteq \Bbbk^{n \times n}$ of upper triangular matrices is called a *basic matrix algebra* if

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ & \ddots & \vdots \\ 0 & & a_{nn} \end{pmatrix} \in \Gamma \text{ implies that } \begin{pmatrix} a_{11} & & 0 \\ & \ddots & \\ 0 & & a_{nn} \end{pmatrix} \in \Gamma.$$

Any basic matrix algebra Γ admits a decomposition

$$\Gamma = \mathcal{D} \oplus \mathcal{U} \tag{2.4.1}$$

where $\mathcal{D}$ is the subalgebra consisting of diagonal matrices and $\mathcal{U}$ is the subalgebra consisting of strictly upper triangular matrices. Basic matrix algebras are part of a larger class of algebras known as basic algebras.

Definition 2.4.3 (Basic Algebra)**.** An algebra Λ with a complete set of primitive orthogonal idempotents $\{e_1, e_2, \ldots, e_n\}$ is *basic* if $e_i\Lambda \not\cong e_j\Lambda$ as right Λ-modules for all $i \neq j$.

The notion of a Jacobson radical is important to the study of ring theory. We will use this concept to relate basic matrix algebras to basic algebras.

Definition 2.4.4 (Jacobson Radical)**.** Given an $\Bbbk$-algebra Λ, the Jacobson radical $\operatorname{rad}\Lambda$ is the intersection of all maximal ideals of Λ.

Proposition 2.4.5 ([11, Prop. 6.2, page 33])**.** *A finite-dimensional* $\Bbbk$*-algebra* Λ *is basic if and only if* $\Lambda/\operatorname{rad}\Lambda$ *is isomorphic to a product* $\Bbbk \times \cdots \times \Bbbk$ *of copies of* $\Bbbk$*.*

Proposition 2.4.6. *A basic matrix algebra is a basic algebra.*

Proof: For a basic matrix algebra Γ, we have the decomposition $\Gamma = \mathcal{D} \oplus \mathcal{U}$ as in (2.4.1). Then $\operatorname{rad}\Gamma = \mathcal{U}$ and so $\Gamma/\operatorname{rad}\Gamma \cong \mathcal{D}$. Looking ahead at Lemma 2.4.10 (a), we see that $\mathcal{D}$ has a basis $E_1, \ldots, E_r$ consisting of idempotents. Hence, each E_i has entries in $\{0, 1\}$ and so $\mathcal{D} \cong \Bbbk^r$. It then follows from Proposition 2.4.5 that Γ is a basic algebra. ∎

For the remainder of this chapter, we are only interested in basic matrix algebras and not in other types of basic algebras.

The equivalence relation used in the definition of a basic algebra matrix problem (Definition 2.4.13) will arise from an action by a type of algebra called a reduced matrix algebra, whose definition we will now present.

Definition 2.4.7 (Reduced Matrix Algebra)**.** Let $\underline{n}$ be a composition of length t for some non-negative integer n. An algebra Γ of $\underline{n}\times\underline{n}$ matrices is called a *reduced matrix algebra* if

- there exists an equivalence relation $\sim$ on T,
- there exists a family of (possibly trivial) systems of linear equations

$$\left\{ \sum_{\substack{i<j \\ i\in I, j\in J}} c_{ij}^{(\ell)} x_{ij} = 0 \colon 1 \le \ell \le q_{IJ},\ c_{ij}^{(\ell)} \in \Bbbk \right\}_{I,J\in T/\sim} \tag{2.4.2}$$

  indexed by pairs in $T/\sim$, and
- Γ consists of all block upper triangular matrices $G = (G_{ij})_{i,j\in T}$ such that $G_{ii} = G_{jj}$ whenever $i \sim j$ and, for every pair $I, J \in T/\sim$, the equation

$$\sum_{\substack{i<j \\ i\in I, j\in J}} c_{ij}^{(\ell)} G_{ij} = 0$$

  holds for $1 \le \ell \le q_{I,J}$.

Let us provide an example of a reduced algebra.

Example 2.4.8. Let $S = \{1, 2, 3\}$ and $\sim$ be the equivalence relation such that $S/\sim = \{\{1, 2\}, \{3\}\}$. Associate the system of equations $\{x_{13} + 2x_{23} = 0\}$ to the pair $\{1, 2\}, \{3\}$ in $S/\sim$ and the system of equations $\{x_{12} = 0\}$ to the pair $\{1, 2\}, \{1, 2\}$ in $S/\sim$. It follows from this data that the algebra consisting of block matrices of the form

$$S = \begin{pmatrix} S_1 & 0 & S_3 \\ 0 & S_1 & -\frac{1}{2}S_3 \\ 0 & 0 & S_2 \end{pmatrix}$$

where

$$S_1 \in \Bbbk^{2\times 2}, \quad S_2 \in \Bbbk^{3\times 3}, \quad S_3 \in \Bbbk^{2\times 3}$$

is a reduced $\underline{n} \times \underline{n}$ matrix algebra, where $\underline{n} = (2, 2, 3)$.

The next theorem ensures us that using a reduced algebra in the definition of a basic algebra matrix problem is not overly restrictive because, actually, any matrix algebra can be reduced.

Theorem 2.4.9 ([29, Thm. 1.1])**.** *For every algebra Λ of square matrices, there exists a non-singular matrix P such that $P^{-1}\Lambda P$ is a reduced matrix algebra.*

The following lemma will present some properties of basic matrix algebras which make them suitable for use in defining a linear matrix problem. In particular, any basic matrix algebra is decomposable and can be viewed as a reduced algebra. Thus, there exists a family of systems of linear equations which the entries of the matrices must satisfy. This decomposition and family of systems of equations will later be exploited in our definition of a basic algebra matrix problem.

Lemma 2.4.10 ([29, Lem. 2.1])**.** *Let Γ be a basic matrix algebra of $t \times t$, upper triangular matrices. Then, let $\mathcal{D}$ be subalgebra of diagonal matrices in Γ and let $\mathcal{U}$ be the subalgebra of matrices in Γ with zero diagonal.*

(a) *There is a complete set of primitive orthogonal idempotents $\{E_1, \ldots, E_r\}$ of Γ which forms a basis of $\mathcal{D}$. Then, as a $\Bbbk$-module, Γ admits the decomposition*

$$\Gamma = \mathcal{D} \oplus \mathcal{U} = \left(\bigoplus_{i=1}^{r} \Bbbk E_i \right) \oplus \left(\bigoplus_{i,j=1}^{r} E_i \mathcal{U} E_j \right). \tag{2.4.3}$$

(b) *Let $T/{\sim} = \{I_1, \ldots, I_r\}$ where I_i is the set of indices j such that the (j, j)-entry of the basis vector E_i is equal to 1 (see Lemma 2.4.10 (a)). There exists a family $\mathcal{F}$ of systems of equations with the form described in* (2.4.2) *such that, for each pair $I_i, I_j \in T/{\sim}$ of equivalence classes, the solutions to the (I_i, I_j)-system form* *the space* $E_i\mathcal{U}E_j$ *(see Equation* (2.4.3)*). Let* $\underline{t} = (1, 1, \dots, 1)$*. Then* Γ *is the reduced algebra of* $\underline{t} \times \underline{t}$ *matrices given by* $T/\sim$ *and* $\mathcal{F}$*.*

We now provide a construction with which we can obtain new reduced algebras from a basic matrix algebra Γ. In the following definition, we will use the equivalence relation $\sim$ and family $\mathcal{F}$ of systems of linear equations associated to Γ as described in Lemma 2.4.10 (b).

Definition 2.4.11 (Reduction of a basic matrix algebra)**.** Let $\underline{n}$ be a $T^{\sim}$-step sequence. The $\underline{n}$*-reduction of* Γ is the reduced algebra $\Gamma_{\underline{n}}$ consisting of all $\underline{n} \times \underline{n}$ matrices whose blocks satisfy $\mathcal{F}$.

Note the resemblance of the reduction of a basic matrix algebra to an expanded matrix. Now, the following observation is used in the next definition. Given a $\Bbbk$-module $\mathscr{P}$ of $t \times t$ matrices such that $\Gamma\mathscr{P} \subseteq \mathscr{P}$ and $\mathscr{P}\Gamma \subseteq \mathscr{P}$, for a basic matrix algebra Γ, there exists a decomposition

$$\mathscr{P} = \left(\sum_{i=1}^{r} E_i\right) \mathscr{P} \left(\sum_{j=1}^{r} E_j\right) = \bigoplus_{i,j=1}^{r} E_i \mathscr{P} E_j. \tag{2.4.4}$$

It follows from Lemma 2.4.10 (b) that there is a family of system of linear equations

$$\left\{ \sum_{\substack{(i,j)\in\\ I\times J}} d_{ij}^{(\ell)} x_{ij} = 0 \colon 1 \le \ell \le p_{I,J}, d_{ij}^{(\ell)} \in \Bbbk \right\}_{I,J \in T/\sim} \tag{2.4.5}$$

such that $\mathscr{P}$ consists of all matrices $M = (m_{i,j})_{i,j=1}^{t}$ whose entries satisfy (2.4.5).

Definition 2.4.12 (Stepification)**.** Let $\underline{n}$ be a $T^{\sim}$-step sequence and let $\mathscr{P}$ be a $\Bbbk$-module of $t \times t$ matrices satisfying $\Gamma\mathscr{P} \subseteq \mathscr{P}$ and $\mathscr{P}\Gamma \subseteq \mathscr{P}$ for a basic matrix algebra Γ. The $\underline{n}$*-stepification* $\mathscr{P}_{\underline{n}}$ of $\mathscr{P}$ is the space of all $\underline{n} \times \underline{n}$ matrices $(M_{i,j})_{i,j \in T}$ whose blocks satisfy (2.4.5).

We can now define the main concept of the section. For this definition, let $\Gamma \subseteq \Bbbk^{t\times t}$ be a basic matrix algebra and $\mathscr{P} \subseteq \Bbbk^{t\times t}$ be a $\Bbbk$-module such that $\Gamma\mathscr{P} \subseteq \mathscr{P}$ and $\mathscr{P}\Gamma \subseteq \mathscr{P}$. We obtain $T/\!\sim$ from Γ (see Lemma 2.4.10 (b)). Let $\Gamma_{\underline{n}}$ be the $\underline{n}$-reduction of Γ and let $\mathscr{P}_{\underline{n}}$ be the $\underline{n}$-stepification of $\mathscr{P}$ for some $T^{\sim}$-step sequence $\underline{n}$.

Definition 2.4.13 (Basic Algebra Matrix Problem)**.** A *basic algebra matrix problem* is a pair

$$(\mathscr{P}_{\underline{n}}, \sim_{\Gamma_{\underline{n}}})$$

where $\sim_{\Gamma_{\underline{n}}}$ is the equivalence relation on $\mathscr{P}_{\underline{n}}$ defined by letting $M \sim_{\Gamma_{\underline{n}}} N$ if and only if there exists an invertible element $G \in \Gamma_{\underline{n}}$ such that $GMG^{-1} = N$.

2.5 Relationships Between Linear Matrix Problems

In this section, we present two relationships. One between similarity matrix problems and Gabriel-Roĭter matrix problems and the other between linking matrix problems and basic algebra matrix problems. The statement of Proposition 2.5.1 is essentially from [29, Ex. 2.1], though we make the statement explicit and provide a formal proof. Proposition 2.5.2 is found in [29, Thm. 2.1].

Proposition 2.5.1. *The following describes a correspondence between similarity matrix problems and Gabriel-Roĭter matrix problems.*

(i) Given any Gabriel-Roĭter matrix problem $(\mathscr{M}, \sim_\Lambda)$, there exists a similarity matrix problem $(\mathscr{N}, \sim_\Omega)$ and a set bijection $\phi\colon \mathscr{M} \to \mathscr{N}$ such that, for any $M, M' \in \mathscr{M}$, we have that $M \sim_\Lambda M'$ if and only if $\phi(M) \sim_\Omega \phi(M')$.

(ii) Given any similarity matrix problem $(\mathscr{N}, \sim_\Omega)$, there exists a Gabriel-Roĭter matrix problem $(\mathscr{M}, \sim_\Lambda)$ and a set bijection $\theta\colon \mathscr{N} \to \mathscr{M}$ such that, for any $N, N' \in \mathscr{N}$, we have that $N \sim_\Omega N'$ if and only if $\theta(N) \sim_\Lambda \theta(N')$.

Proof:

(i) Let $(\mathscr{M}, \sim_\Lambda)$ be a Gabriel-Roĭter matrix problem for some subgroup $\Lambda \subseteq GL(\underline{m}) \times GL(\underline{n})$ and set $\mathscr{M}$ of $\underline{m} \times \underline{n}$ matrices, where $\underline{m} = (m_1, \ldots, m_r)$ and $\underline{n} = (n_1, \ldots, n_s)$. Let $\underline{t} = (m_1, \ldots, m_r, n_1, \ldots, n_s)$ and define a subgroup

$$\Omega = \{\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right) : (A, B) \in \Lambda\}$$

of $GL(\underline{t})$ and a set $\mathscr{N} = \{\left(\begin{smallmatrix} 0 & M \\ 0 & 0 \end{smallmatrix}\right) : M \in \mathscr{M}\}$ of $\underline{t} \times \underline{t}$ matrices. The set map $\phi\colon \mathscr{M} \to \mathscr{N}$, defined by $\phi(M) = \left(\begin{smallmatrix} 0 & M \\ 0 & 0 \end{smallmatrix}\right)$, is a bijection. Then

$$\phi(M) = \left(\begin{smallmatrix} 0 & M \\ 0 & 0 \end{smallmatrix}\right) \sim_\Omega \left(\begin{smallmatrix} 0 & M' \\ 0 & 0 \end{smallmatrix}\right) = \phi(M')$$

means that there exists an element $\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right) \in \Omega$ such that

$$\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & M \\ 0 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} A^{-1} & 0 \\ 0 & B^{-1} \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & M' \\ 0 & 0 \end{smallmatrix}\right).$$

Equivalently, $AMB^{-1} = M'$ which gives us that $M \sim_\Lambda M'$.

(ii) This follows immediately from letting $\mathscr{N} = \mathscr{M}$, $\Lambda = \{(A, A)\colon A \in \Omega\}$ and defining θ to be the identity mapping.

∎

Proposition 2.5.2 ([29, Thm. 2.1])**.** *The following describes a correspondence between linking matrix problems and basic algebra matrix problems.*

(i) Let $(\mathscr{O}, \sim_L)$ be a linking matrix problem for some $T^{\sim}$-linking triple $(\underline{n}, \mathcal{P}, \mathcal{V})$ and let $(\mathscr{P}_{\underline{n}}, \sim_{\Gamma_{\underline{n}}})$ be the basic algebra matrix problem such that

- *$\mathscr{P}_{\underline{n}}$ is the $\underline{n}$-stepification of the $\Bbbk$-module $\mathscr{P}$ that is generated by the elements of $\overline{\mathcal{V}}$, and*
- *$\Gamma_{\underline{n}}$ is the $\underline{n}$-reduction of the basic algebra Γ generated by the elements of $\mathcal{P}$ and all matrices $E_I = \sum_{i \in I} E_{i,i}$ for every $I \in T/{\sim}$.*

Then $\mathscr{O} = \mathscr{P}_{\underline{n}}$ *and, moreover,* $\sim_L$ *and* $\sim_{\Gamma_{\underline{n}}}$ *describe the same equivalence relation.*

(ii) Let $(\mathscr{P}_{\underline{n}}, \sim_{\Gamma_{\underline{n}}})$ *be a basic algebra matrix problem. Let* $(\mathscr{O}, \sim_L)$ *be the linking matrix problem given by the* $T^{\sim}$*-linking triple* $(\underline{n}, \mathcal{P}, \mathcal{V})$ *where*

- $T/{\sim} = \{I_1, I_2, \ldots, I_r\}$ *(see Lemma 2.4.10 (b)),*
- $\mathcal{P} = \bigcup_{i,j=1}^{r} \mathcal{B}_{i,j}$ *where* $\mathcal{B}_{i,j}$ *is the distinguished basis for the space* $E_i \mathcal{C} E_j$ *(see Lemma 2.4.10 (a)), and*
- $\mathcal{V} = \bigcup_{i,j=1}^{r} \mathcal{B}_{i,j}$ *where* $\mathcal{B}_{i,j}$ *is the distinguished basis for the space* $E_i \mathscr{P} E_j$ *(see equation* (2.4.4)*).*

Then $\mathscr{O} = \mathscr{P}_{\underline{n}}$ *and, moreover,* $\sim_L$ *and* $\sim_{\Gamma_{\underline{n}}}$ *describe the same equivalence relation.*

Chapter 3

Classifying Poset Representations with Linear Matrix Problems

This chapter discusses the problem of classifying representations of a poset P by using linear matrix problems. After a brief discussion about the category P-**rep**, we then show that there exists a Gabriel-Roĭter matrix problem whose classification implies a classification of P-**rep**. We will also show that there exists a linking matrix problem whose classification implies a classification of P-**rep**. Then, it will follow from Propositions 2.5.1 and 2.5.2 that there exists a similarity matrix problem and basic algebra matrix problem, each of whose classification implies a classification of P-**rep**.

For this chapter, let $(P, \preceq)$ be a poset with n elements $p_1, p_2, \ldots, p_n$ labeled so that $i \leq j$ whenever $p_i \preceq p_j$. Then, for any object $(M_p)_{p \in P}$ of P-**rep**, we will write $(M_i)_{i=1}^n$ instead where $M_j = M_{p_j}$ for $j = 1, 2, \ldots, n$.

3.1 Posets and their Representations

We begin with the following observation.

Lemma 3.1.1. *If* $M \cong N$ *in* P-**rep**, *then* $\dim M = \dim N$.

Proof: Suppose $\dim M = (d_0, \underline{d})$ and $\dim N = (e_0, \underline{e})$. Given an isomorphism $(A, B)\colon M \to N$, it follows that $A \in GL(d_0)$, $B \in GL(\underline{e})$, and

$$AM_j = \sum_{i:p_i \preceq p_j} N_i B_{i,j}$$

for $j = 1, 2, \ldots, n$. It follows from properties of matrix multiplication that $d_i = e_i$ for $i = 0, 1, \ldots, n$. ∎

Denote by P-$\mathbf{rep}^{(d_0,\underline{d})}$ the full subcategory of P-**rep** consisting of representations with dimension $(d_0, \underline{d})$. This category will play a crucial role in our discussions. The key property of P-$\mathbf{rep}^{(d_0,\underline{d})}$ for our purposes is described in the next lemma.

Lemma 3.1.2. *The category* P-$\mathbf{rep}^{(d_0,\underline{d})}$ *is replete.*

Proof: The statement follows immediately from Lemma 3.1.1. ∎

It follows from Lemma 3.1.2 that a classification of P-$\mathbf{rep}^{(d_0,\underline{d})}$ for an arbitrary dimension $(d_0, \underline{d})$ implies a classification of P-**rep**. Therefore, we will work strictly in the subcategory P-$\mathbf{rep}^{(d_0,\underline{d})}$ for the remainder of this chapter.

Before we bring matrix problems into the picture, let us see what a classification of a category poset representations looks like with the following simple example.

Example 3.1.3. Given the poset $P = \{p\}$, then $\mathrm{Ob}\left(P\text{-}\mathbf{rep}^{(m,(n))}\right) = \Bbbk^{m\times n}$. Two representations M and N are isomorphic when there exists $(A, B) \in GL(m) \times GL(n)$ such that $AMB^{-1} = N$. Stated another way, $M \cong N$ when M is equivalent to N in the usual sense of matrix equivalence. Therefore, the classification of P-$\mathbf{rep}^{(m,(n))}$ is given by the rank of the objects.

3.2 Poset Representations and Gabriel-Roĭter Matrix Problems

This section is devoted to showing that the problem of classifying representations of a poset P can be formulated with a Gabriel-Roĭter matrix problem.

Our first step is to define an appropriate Gabriel-Roĭter matrix problem considering our poset P. Define $\mathscr{M}$ to be the set whose elements are the block matrices $(M_{1,i})_{i=1}^{n} \in \Bbbk^{d_0 \times \underline{d}}$ such that $(M_{1,1}, M_{1,2}, \ldots, M_{1,n})$ is an object of P-$\mathbf{rep}^{(d_0,\underline{d})}$. Let Λ be the collection of isomorphisms in P-$\mathbf{rep}^{(d_0,\underline{d})}$. Then Λ is a subgroup of $GL(d_0) \times GL(\underline{d})$ and acts on $\mathscr{M}$ by $(A, B) \cdot M = AMB^{-1}$. The pair $(\mathscr{M}, \sim_\Lambda)$ is a Gabriel-Roĭter matrix problem.

To formalize the association between $\mathrm{Ob}\left(P\text{-}\mathbf{rep}^{(d_0,\underline{d})}\right)$ and $\mathscr{M}$, we define a set mapping $\Theta\colon \mathrm{Ob}\left(P\text{-}\mathbf{rep}^{(d_0,\underline{d})}\right) \to \mathscr{M}$ by letting

$$\Theta(M_1, M_2, \ldots, M_n) = (M_1 \ M_2 \ \cdots \ M_n).$$

A basic calculation shows that Θ is a bijection.

Theorem 3.2.1. *Let $M, N \in \mathrm{Ob}(P\text{-}\mathbf{rep}^{(d_0,\underline{d})})$. Then $M \cong N$ in P-$\mathbf{rep}^{(d_0,\underline{d})}$ if and only if $\Theta(M) \sim_\Lambda \Theta(N)$ in $\mathscr{M}$.*

Proof: A morphism of poset representations $(A, B)\colon M \to N$ is an isomorphism if and only if $A \in GL(d_0)$, $B \in GL(\underline{d})$ and the equality

$$AM_j = \sum_{i : p_i \preceq p_j} N_i B_{i,j}$$

holds for $j = 1, \ldots, n$. Equivalently, $A\Theta(M) = \Theta(N)B$, which means that $\Theta(M) \sim_\Lambda \Theta(N)$. ∎

3.3 Poset Representations and Linking Matrix Problems

Having just shown that a Gabriel-Roĭter matrix problem can be used to classify $P\text{-}\mathbf{rep}^{(d_0,\underline{d})}$, we will take a similar approach to show that a linking matrix problem can also be used.

Our first task is to define an appropriate linking triple. This will provide a linking matrix problem $(\mathscr{O}, \sim_L)$ which we can compare to $P\text{-}\mathbf{rep}^{(d_0,\underline{d})}$.

To associate a linking triple to $P\text{-}\mathbf{rep}^{(d_0,\underline{d})}$, consider the set $T = \{1, 2, \ldots, n+1\}$ with the equality equivalence relation $\sim$. That is, $T/{\sim} = \{\{1\}, \ldots, \{n+1\}\}$. Let $\underline{m} = (d_1, \ldots, d_n, d_0)$, which is a $T^{\sim}$-step-sequence. Define subsets

$$\mathcal{P} = \{E_{i,j} \colon i, j \in T \text{ such that } p_i \prec p_j\} \text{ and}$$
$$\mathcal{V} = \{E_{n+1,i} \colon i = 1, 2, \ldots, n\}$$

of $\Bbbk^{(n+1)\times(n+1)}$. Because of how we labeled the poset P, it follows that $\mathcal{P}$ has strictly upper triangular, hence nilpotent, matrices. Also, because $\sim$ is the equality equivalence relation, both $\mathcal{P}$ and $\mathcal{V}$ are linking matrices. Thus $\mathcal{R} = (\underline{m}, \mathcal{P}, \mathcal{V})$ is a $T^{\sim}$-linking triple.

To define a linking matrix problem, we need the $\mathcal{R}$-link module and $\mathcal{R}$-link transformation group. Because $\mathcal{P}$ contains only matrices of the form $E_{i,j}$ where $i \neq j$, it follows that $\overline{\mathcal{P}} = \mathcal{P} \cup \{0\}$. Then we have that $\overline{\mathcal{P}}\mathcal{V} = \{0\}$ and $\mathcal{V}\overline{\mathcal{P}} \subseteq \mathcal{V} \cup \{0\}$. Hence $\overline{\mathcal{V}} = \mathcal{V} \cup \{0\}$. Therefore, the $\mathcal{R}$-link module $\mathscr{O}$ consists of all $\underline{m} \times \underline{m}$ matrices $(M_{i,j})_{i,j\in T}$ such that $M_{i,j} = 0$ when $i \neq n+1$ or $j = n+1$.

The $\mathcal{R}$-link transformation group L is the subgroup of $GL(\underline{m})$ generated by elements $\bigoplus_{i\in T} G_i$, where $G_i \in GL(m_i)$, and by elements $\mathbb{I} + cP^{[\ell,r]}$, where $c \in \Bbbk$ and $P^{[\ell,r]}$ is a P-expanded matrix for some $P \in \overline{\mathcal{P}}$. A more direct description is that L consists of the block matrices $(G_{i,j})_{i,j\in T} \in \Bbbk^{\underline{m}\times\underline{m}}$ such that $G_{i,i} \in GL(m_i)$ and $G_{i,j} = 0$ if $p_i \not\preceq p_j$.

We obtain the linking matrix problem $(\mathscr{O}, \sim_L)$ which we will now compare to the category P-$\mathbf{rep}^{(d_0,\underline{d})}$. Define a set map

$$\Theta\colon \operatorname{Ob}\left(P\text{-}\mathbf{rep}^{(d_0,\underline{d})}\right) \to \mathscr{O},$$

that sends a representation $M = (M_i)_{i=1}^n$ to the block matrix $\Theta(M) = (\Theta(M)_{i,j})_{i,j\in T}$ where $\Theta(M)_{n+1,i} = M_i$ for $i = 1, 2, \ldots, n$ and all other blocks are 0. A basic calculation shows that this is a bijection.

Theorem 3.3.1. *Let $M, N \in \operatorname{Ob}\left(P\text{-}\mathbf{rep}^{(d_0,\underline{d})}\right)$. Then $M \cong N$ in P-$\mathbf{rep}^{(d_0,\underline{d})}$ if and only if $\Theta(M) \sim_L \Theta(N)$ in $\mathscr{O}$.*

Proof: A morphism $(A, B)\colon M \to N$ of representations is an isomorphism if and only if there exists an element $(A, B) \in GL(d_0) \times GL(\underline{d})$ such that the equality

$$AM_j = \sum_{i:p_i \preceq p_j} N_i B_{i,j}$$

holds for $j = 1, 2, \ldots, n$. Consider the element $B \oplus A$ of L. Then

$$(B \oplus A)\Theta(M) = \Theta(N)(B \oplus A),$$

or equivalently, $\Theta(M) \sim_L \Theta(N)$. ∎

Chapter 4

Classifying Quiver Representations with Linear Matrix Problems

In this chapter, we will show that the problem of classifying representations of a quiver can be formulated as the problem of classifying the equivalence classes of some linear matrix problem. First, we will consider a quiver Q and affix to it a labeling system. Then, we will define a subcategory of Q-**rep** whose classification implies a classification of Q-**rep**. Finally, we will relate Q-**rep** to a similarity matrix problem and linking matrix problem.

4.1 Quivers and their Representations

Let Q be a finite quiver with nodes $Q_N = \{\eta_1, \ldots, \eta_p\}$ and arrows $Q_A = \{\alpha_1, \ldots, \alpha_q\}$. Order Q_N and Q_A by their index. Consider the elements of the set

$$Z = Q_A \times \mathbb{Z}_2 \tag{4.1.1}$$

to be labels of Q. That is, the element $(\alpha_i, 0)$ is the label for the tail of the arrow α_i and $(\alpha_i, 1)$ is the label for the head of α_i. Throughout this section, Z will be used in various contexts as an indexing set. Therefore, it is helpful to provide Z with the lexicographical ordering.

Define a map

$$\sigma\colon Z \to Q_N, \quad \sigma(\alpha_i, \delta) = \begin{cases} q_T(\alpha_i), & \text{if } \delta = 0, \\ q_H(\alpha_i), & \text{if } \delta = 1. \end{cases} \tag{4.1.2}$$

This map assigns the corresponding node to the head and tail of each arrow.

Much like how we found a convenient subcategory of P-**rep** to work in, there exists an analogous subcategory of Q-**rep**. Let $Q\text{-}\mathbf{rep}^{(\underline{d})}$ be the full subcategory of Q-**rep** consisting of representations with graded dimension $\underline{d} = (d_1, d_2, \ldots, d_p)$.

Lemma 4.1.1. *The subcategory* $Q\text{-}\mathbf{rep}^{(\underline{d})}$ *is replete.*

Proof: It follows immediately from the fact that isomorphic representations have the same graded dimension. ∎

Lemma 4.1.1 implies that, to classify Q-**rep**, it is sufficient to classify the category $Q\text{-}\mathbf{rep}^{(\underline{d})}$ for an arbitrary $\underline{d}$. To further simplify our work, we consider an equivalent subcategory of $Q\text{-}\mathbf{rep}^{(\underline{d})}$.

Let $Q\text{-}\underline{\mathbf{rep}}^{(\underline{d})}$ be the full subcategory of $Q\text{-}\mathbf{rep}^{(\underline{d})}$ consisting of the representations that associate the module $\Bbbk^{d_i}$ to the node η_i for $i = 1, 2, \ldots, p$.

Proposition 4.1.2. *The categories* $Q\text{-}\mathbf{rep}^{(\underline{d})}$ *and* $Q\text{-}\underline{\mathbf{rep}}^{(\underline{d})}$ *are equivalent.*

Proof: The category $Q\text{-}\underline{\mathbf{rep}}^{(\underline{d})}$ is full by definition and faithful because it is a subcategory of Q-**rep**. It remains to show that it is also a dense subcategory.

Let $(\{V_i\}_{i=1}^p, \{f_i\}_{i=1}^q)$ be a representation in $Q\text{-}\mathbf{rep}^{(\underline{d})}$. There are $\Bbbk$-linear isomorphisms $\iota_i\colon V_i \to \Bbbk^{d_i}$ for $i = 1, 2, \ldots, p$. Consider the representation $\left(\left\{\Bbbk^{d_i}\right\}, \{g_i\}\right)$ in $Q\text{-}\underline{\mathbf{rep}}^{(\underline{d})}$ that associates the $\Bbbk$-linear map $g_i = \iota_k f_i \iota_j^{-1}$ to each arrow $\eta_j \xrightarrow{\alpha_i} \eta_k$ of Q. Then

$$(\iota_i)_{i=1}^p \colon (\{V_i\}, \{f_i\}) \to \left(\left\{\Bbbk^{d_i}\right\}, \{g_i\}\right)$$

is an isomorphism of quiver representations. ∎

It follows from this equivalence of categories that we can work exclusively in Q-$\underline{\mathbf{rep}}^{(d)}$. This offers the advantage of providing a distinguished basis for each $\Bbbk$-module associated to each node of Q.

Before we look at using matrix problems to classify quiver representations, we will look at an example of a well known result in the classification quiver representations.

Example 4.1.3. A quiver is of *finite type* if it has finitely many isoclasses of indecomposable representations. Gabriel's Theorem classifies the indecomposable representations of all quivers of finite type [6, 12]. This theorem states that

- a connected quiver is of finite type if and only if its underlying graph is one of the Dynkin diagrams A_n, D_n, E_6, E_7, or E_8 (see Figure 4.1), and
- the indecomposable representations of a connected quiver are in one-to-one correspondence with the positive roots of the root system of the associated Dynkin diagram.

See [10] for more on Dynkin diagrams and root systems.

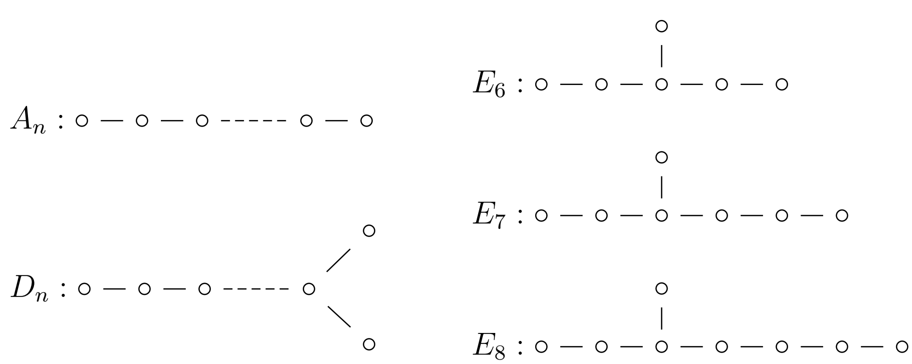


Figure 4.1: Dynkin Diagrams: A_n, D_n, E_6, E_7, E_8, with $n \geq 1$ for A_n and $n \geq 4$ for D_n, where n is the number of nodes.

4.2 Quiver Representations and Similarity Matrix Problems

In this section, we show that the problem of classifying finite-dimensional representations of finite quivers can be formulated as a problem of classifying the equivalence classes for some similarity matrix problem. It will then follow from Proposition 2.5.1 that the problem of classifying finite-dimensional representations of finite quivers can be formulated as a Gabriel-Roĭter matrix problem.

We must first define an appropriate similarity matrix problem. Recall the set $Z = Q_A \times \mathbb{Z}_2$ of labels described in (4.1.1). Consider the $|Z|$-tuple $\underline{m} = (m_z)_{z\in Z}$ where $m_z = d_j$ if $\sigma(z) = \eta_j$. Note that an element $z \in Z$ has the form $z = (z_1, z_2)$. Define a set

$$\mathscr{N} = \{N \in \Bbbk^{\underline{m}\times\underline{m}} \colon N_{z,z'} = 0 \text{ if } z_1 \neq z_1', z_2 \neq 1 \text{ or } z_2' \neq 0\}. \tag{4.2.1}$$

Let Ω be the subgroup of $\bigoplus_{z\in Z} GL(m_z)$ consisting of all block diagonal matrices

$$G = \bigoplus_{z\in Z} G_z$$

such that $G_z = G_{z'}$ if $\sigma(z) = \sigma(z')$. Note that Ω acts on $\mathscr{N}$ by $G \cdot N = GNG^{-1}$. Thus, we have a similarity matrix problem $(\mathscr{N}, \sim_\Omega)$.

Following the strategy used in Sections 3.2 and 3.3, we define a set bijection between $\mathrm{Ob}\left(Q\text{-}\underline{\mathbf{rep}}^{(\underline{d})}\right)$ and $\mathscr{N}$. Observe that every representation in $Q\text{-}\underline{\mathbf{rep}}^{(\underline{d})}$ can be written as $\left(\left\{\Bbbk^{d_i}\right\}_{i=1}^{p}, \{N_i\}_{i=1}^{q}\right)$ where each N_i is the transformation matrix, with regard to the distinguished basis, of the linear mapping associated to the arrow $\alpha_i \in Q_A$. Define a set mapping

$$\Theta\colon\ \mathrm{Ob}\left(Q\text{-}\underline{\mathbf{rep}}^{(\underline{d})}\right) \to \mathscr{N}$$

by sending each representation $\left(\left\{\Bbbk^{d_i}\right\}, \{N_i\}\right)$ to the $\underline{m}\times\underline{m}$ block matrix $\Theta\left(\left\{\Bbbk^{d_i}\right\}, \{N_i\}\right)$

whose (z, z')-block is defined as

$$\Theta\left(\left\{\Bbbk^{d_i}\right\}, \{N_i\}\right)_{z,z'} = \begin{cases} N_j, & \text{if } z = (\alpha_j, 1) \text{ and } z' = (\alpha_j, 0), \\ 0, & \text{otherwise.} \end{cases}$$

A basic calculation shows that Θ is a bijection of sets.

Theorem 4.2.1. *Let* $x = \left(\left\{\Bbbk^{d_i}\right\}, \{M_i\}\right)$ *and* $y = \left(\left\{\Bbbk^{d_i}\right\}, \{N_i\}\right)$ *be representations in* Q-$\underline{\mathbf{rep}}^{(d)}$. *Then* $x \cong y$ *in* Q-$\underline{\mathbf{rep}}^{(d)}$ *if and only if* $\Theta(x) \sim_\Omega \Theta(y)$ *in* $\mathscr{M}$.

Proof: Let $\theta = (\theta_i)_{i=1}^p : x \to y$ be an isomorphism of representations. Then, for every arrow $\eta_j \xrightarrow{\alpha_h} \eta_k$ of Q, we have that

$$M_h = \Theta(x)_{(\alpha_h,1),(\alpha_h,0)} \text{ and } N_h = \Theta(y)_{(\alpha_h,1),(\alpha_h,0)} \tag{4.2.2}$$

by the definition of Θ, and we also have that the diagram

$$\begin{array}{ccc} \Bbbk^{d_j} & \xrightarrow{M_h} & \Bbbk^{d_k} \\ {\scriptstyle\theta_j}\downarrow & & \downarrow{\scriptstyle\theta_k} \\ \Bbbk^{d_j} & \xrightarrow[N_h]{} & \Bbbk^{d_k} \end{array} \tag{4.2.3}$$

commutes. Consider the element $G = \bigoplus_{z \in Z} G_z$ in Ω where, for each $z \in Z$, G_z is the transformation matrix for the map θ_j if $\sigma(z) = \eta_j$. Hence, for each arrow $\eta_j \xrightarrow{\alpha_h} \eta_k$, Diagram (4.2.3) can be rewritten as

$$\begin{array}{ccc} \Bbbk^{d_j} & \xrightarrow{\Theta(x)_{(\alpha_h,1),(\alpha_h,0)}} & \Bbbk^{d_k} \\ {\scriptstyle G_{(\alpha_h,0)}}\downarrow & & \downarrow{\scriptstyle G_{(\alpha_h,1)}} \\ \Bbbk^{d_j} & \xrightarrow[\Theta(y)_{(\alpha_h,1),(\alpha_h,0)}]{} & \Bbbk^{d_k}. \end{array} \tag{4.2.4}$$

But then $G\Theta(x) = \Theta(y)G$ and so $\Theta(x) \sim_\Omega \Theta(y)$.

Conversely, assume there is a matrix $G = \bigoplus_{z \in Z} G_z$ in Ω such that $G\Theta(x) = \Theta(y)G$. Then, Diagram (4.2.4) commutes for $h = 1, \ldots, q$. Define a morphism of representations $\theta = (\theta_i)_{i=1}^p : x \to y$ by letting

$$\theta_i = \begin{cases} G_z, & \text{if } z \in \sigma^{-1}(\eta_i), \\ 1, & \text{if } \sigma^{-1}(\eta_i) = \emptyset, \end{cases}$$

for $i = 1, \ldots, p$. The definition of Ω implies that $G_z = G_{z'}$ if $\sigma(z) = \sigma(z')$. Hence, the choice of $z \in \sigma^{-1}(\eta_i)$ does not effect the definition of θ_i and so θ is well-defined. Now, for every arrow $\eta_j \xrightarrow{\alpha_h} \eta_k$ of Q, it follows from (4.2.2) and the commutativity of Diagram (4.2.4) that Diagram (4.2.3) commutes and so θ is a morphism of representations. Moreover, because its components are invertible, θ is an isomorphism. ∎

4.3 Quiver Representations and Linking Matrix Problems

This section is devoted to showing that the problem of classifying the isoclasses in Q-**rep** can be formulated as the problem of classifying the equivalence classes of some linking matrix problem. The set Z of (4.1.1) and the subcategory Q-$\underline{\textbf{rep}}^{(\underline{d})}$ described in Section 4.1 will be used.

We must first define an appropriate linking matrix problem. In doing so, we will use Z instead of the set $T = \{1, 2, \ldots, t\}$ which we used in Section 2.3. Let $\sim$ be the equivalence relation on Z whose classes are $I_i = \sigma^{-1}(\eta_i)$ for $i = 1, \ldots, p$. Let $\underline{n} = (n_z)_{z \in Z}$ be the $|Z|$-tuple such that $n_z = d_j$ if $\sigma(z) = \eta_j$. It follows, for any $z, z' \in Z$, that $z \sim z'$ implies that $n_z = n_{z'}$ and, therefore, $\underline{n}$ is a step-sequence. Let

$$\mathcal{V} = \{E_{z,z'} \colon z, z' \in Z,\ z_1 = z'_1\ z_2 = 1 \text{ and } z'_2 = 0\}, \text{ and}$$
$$\mathcal{P} = \emptyset.$$

Thus we have a $Z^{\sim}$-linking triple $\mathcal{R} = (\underline{n}, \emptyset, \mathcal{V})$. The next step is to determine the $\mathcal{R}$-link module and $\mathcal{R}$-link transformation group. Recall that $\overline{\mathcal{P}}$ denotes the multiplicative closure of $\mathcal{P}$ and $\overline{\mathcal{V}}$ is the closure of $\mathcal{V}$ with respect to multiplication by elements of $\overline{\mathcal{P}}$. Thus, $\overline{\mathcal{P}} = \emptyset$ and $\overline{\mathcal{V}} = \mathcal{V}$. The $\mathcal{R}$-link module $\mathscr{O}$ consists of all $\underline{n} \times \underline{n}$ block matrices $(M_{z,z'})_{z,z' \in Z}$ such that $M_{z,z'} = 0$ if $E_{z,z'} \notin \mathcal{V}$. The link transformation group L consists of matrices $\bigoplus_{z\in Z} G_z$ such that $G_z \in GL(n_z)$ and $G_z = G_{z'}$ if $z \sim z'$. We now have the linking matrix problem $(\mathscr{O}, \sim_L)$.

From this point until the end of the section, we will basically be repeating the end of Section 4.2 in the context of linking matrix problems. Even the proof of Theorem 4.3.1 will follow exactly the same line of reasoning as Theorem 4.2.1.

Recall that every representation in $Q\text{-}\underline{\mathbf{rep}}^{(\underline{d})}$ has the form $\left(\left\{\Bbbk^{d_i}\right\}, \{M_i\}\right)$ where the linear maps associated to the arrows are given as matrices. Define a set map

$$\Theta\colon \operatorname{Ob}\left(Q\text{-}\underline{\mathbf{rep}}^{(\underline{d})}\right) \to \mathscr{O}, \tag{4.3.1}$$

where each representation $\left(\left\{\Bbbk^{d_i}\right\}, \{M_i\}\right)$ is sent to the block matrix $\Theta\left(\left\{\Bbbk^{d_i}\right\}, \{M_i\}\right)$ whose (z, z')-block is defined as

$$\Theta\left(\left\{\Bbbk^{d_i}\right\}, \{M_i\}\right)_{z,z'} = \begin{cases} M_i, & \text{if } z = (\alpha_i, 1) \text{ and } z' = (\alpha_i, 0), \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to show that Θ is a bijection. We now prove our main result.

Theorem 4.3.1. *Let $x = \left(\left\{\Bbbk^{d_i}\right\}, \{M_i\}\right)$ and $y = \left(\left\{\Bbbk^{d_i}\right\}, \{N_i\}\right)$ be representations in $Q\text{-}\underline{\mathbf{rep}}^{(\underline{d})}$. Then $x \cong y$ in $Q\text{-}\underline{\mathbf{rep}}^{(\underline{d})}$ if and only if $\Theta(x) \sim_L \Theta(y)$ in $\mathscr{O}$.*

Proof: Let $\theta = (\theta_i)_{i=1}^{p} : x \to y$ be an isomorphism of representations. Then, for every arrow $\eta_j \xrightarrow{\alpha_h} \eta_k$ of Q, we have that

$$M_h = \Theta(x)_{(\alpha_h,1),(\alpha_h,0)} \text{ and } N_h = \Theta(y)_{(\alpha_h,1),(\alpha_i,0)} \tag{4.3.2}$$

by the definition of Θ, and we also have that the diagram

$$\begin{array}{ccc} \Bbbk^{d_j} & \xrightarrow{M_h} & \Bbbk^{d_k} \\ {\scriptstyle\theta_j}\downarrow & & \downarrow{\scriptstyle\theta_k} \\ \Bbbk^{d_j} & \xrightarrow[N_h]{} & \Bbbk^{d_k} \end{array} \tag{4.3.3}$$

commutes. Consider the element $G = \bigoplus_{z\in Z} G_z$ of L where, for each $z \in Z$, G_z is the transformation matrix for the map θ_j if $\sigma(z) = \eta_j$. Hence, for each arrow $\eta_j \xrightarrow{\alpha_h} \eta_k$, Diagram (4.3.3) can be rewritten as

$$\begin{array}{ccc} \Bbbk^{d_j} & \xrightarrow{\Theta(x)_{(\alpha_h,1),(\alpha_h,0)}} & \Bbbk^{d_k} \\ {\scriptstyle G_{(\alpha_h,0)}}\downarrow & & \downarrow{\scriptstyle G_{(\alpha_h,1)}} \\ \Bbbk^{d_j} & \xrightarrow[\Theta(y)_{(\alpha_h,1),(\alpha_h,0)}]{} & \Bbbk^{d_k}. \end{array} \tag{4.3.4}$$

But then $G\Theta(x) = \Theta(y)G$ and so $\Theta(x) \sim_L \Theta(y)$.

Conversely, assume there is a matrix $G = \bigoplus_{z \in Z} G_z$ in L such that $G\Theta(x) = \Theta(y)G$. Then, Diagram (4.3.4) commutes for $h = 1, \ldots, q$. Define a morphism of representations $\theta = (\theta_i)_{i=1}^p : x \to y$ by letting

$$\theta_i = \begin{cases} G_z, & \text{if } z \in \sigma^{-1}(\eta_i), \\ 1, & \text{if } \sigma^{-1}(\eta_i) = \emptyset, \end{cases}$$

for $i = 1, \ldots, p$. By the definition of L implies that $G_z = G_{z'}$ if $\sigma(z) = \sigma(z')$. Hence, the choice of $z \in \sigma^{-1}(\eta_i)$ does not effect the definition of θ_i and so θ is well-defined. Now, for every arrow $\eta_j \xrightarrow{\alpha_h} \eta_k$ of Q, it follows from (4.3.2) and the commutativity of Diagram (4.3.4) that Diagram (4.3.3) commutes and so θ is a morphism of representations. Moreover, because its components are invertible, θ is an isomorphism. ∎

Chapter 5

Relating Poset Representations and Quiver Representations

This chapter will not focus on matrix problems, but on directly relating the problem of classifying representations of a poset to the problem of classifying the representations of some quiver. The ideas in the first section will expand the realm of possible techniques to use when attempting to classify poset representations. In the second section, we will see that the problem of classifying quiver representations does not easily admit a method to incorporate the techniques of classifying poset representations. To the author's knowledge, there is no sufficiently general method and so we will investigate several naive attempts which cannot provide the desired equivalence.

5.1 Classifying Poset Representations with Quiver Representations

In this section, we show that the problem of classifying certain representations of a poset can be formulated as the problem of classifying representations of a quiver. Which representations the word 'certain' refers to will be discussed below.

The first half of this section is devoted to introducing an object called a P-space which is closely related to poset representations. In the second half, we will use the category of P-spaces to define a functor P-**rep** $\to Q_P$-**rep** where the quiver Q_P will be introduced as needed. We desire our functor to reflect isomorphisms to ensure that isoclasses are reflected. However, to achieve this, we will need to restrict P-**rep** to a suitable subcategory.

In order to reduce notation throughout this section, we will denote any linear mapping $\Bbbk^r \to \Bbbk^s$ and its transformation matrix, with regard to the distinguished basis, in the same manner. Let P be a poset with elements $p_1, \ldots, p_n$ labeled so that $p_i \preceq p_j$ implies that $i \leq j$.

Definition 5.1.1 (P-Spaces, [22, p. 31])**.** The category P-**sp** of P-spaces consists of

- objects $(V_0; \{V_i\}_{i=1}^n)$ where V_0 is a finite-dimensional $\Bbbk$-module and $\{V_i\}$ is a collection of subspaces of V_0 such that $V_i \subseteq V_j$ if $p_i \preceq p_j$, and
- morphisms $f\colon (V_0; \{V_i\}_{i=1}^n) \to (W_0; \{W_i\}_{i=1}^n)$ where $f\colon V_0 \to W_0$ is a $\Bbbk$-linear map such that $f(V_i) \subseteq W_i$ for $i = 1, 2, \ldots, n$.

Composition is naturally defined and associativity follows from a basic calculation.

We will usually suppress the range on the indices and denote a P-space as $(V_0; \{V_i\})$. The next step is to define a functor P-**rep** $\to P$-**sp**. The following lemma will ensure that this functor is well-defined.

Lemma 5.1.2. *Let M, N be representations of P and $(A, B)\colon M \to N$ be a morphism of poset representations. For each element $p_j \in P$, the diagram*

$$\begin{array}{ccc}
\bigoplus_{i:p_i \preceq p_j} \Bbbk^{d_i} & \xrightarrow{\sum_{i:p_i\preceq p_j} M_i} & \Bbbk^{d_0} \\
\Big\downarrow {\scriptstyle (B_{r,s})_{p_r,p_s \preceq p_j}} & & \Big\downarrow {\scriptstyle A} \\
\bigoplus_{i:p_i \preceq p_j} \Bbbk^{e_i} & \xrightarrow[\sum_{i:p_i\preceq p_j} N_i]{} & \Bbbk^{e_0}.
\end{array}$$

commutes.

Proof: For $\bigoplus_{i:p_i \preceq p_j} x_i \in \bigoplus_{i:p_i \preceq p_j} \Bbbk^{d_i}$, we have that

$$A \circ \sum_{i:p_i \preceq p_j} M_i \left(\bigoplus_{i:p_i \preceq p_j} x_i \right) = \sum_{i:p_i \preceq p_j} AM_i(x_i)$$

and

$$\sum_{i:p_i \preceq p_j} N_i \circ (B_{r,s})_{r,s:p_r,p_s \preceq p_j} \left(\bigoplus_{i:p_i \preceq p_j} x_i \right) = \sum_{i:p_i \preceq p_j} \left(\sum_{k:p_k \preceq p_i} N_k B_{k,i}(x_i) \right).$$

But for every i such that $p_i \preceq p_j$, it follows from the definition of a morphism in P-**rep** that $AM_i(x_i) = \sum_{k:p_k \preceq p_i} N_k B_{k,i}(x_i)$. ∎

We can now introduce a functor which will play an important role throughout this section. After it is defined, we will explore some of its properties. Denote by

$$H\colon P\text{-}\mathbf{rep} \to P\text{-}\mathbf{sp} \tag{5.1.1}$$

the functor that is defined on objects by

$$H(M) = \left(\Bbbk^{d_0};\ \left\{ \sum_{i:p_i \preceq p_j} \operatorname{Im}(M_i) \right\} \right),$$

and on morphisms by $H(A, B) = A$. It follows from Lemma 5.1.2 that

$$A \left(\sum_{i:p_i \preceq p_j} \operatorname{Im}(M_j) \right) \subseteq \sum_{i:p_i \preceq p_j} \operatorname{Im}(N_j) \tag{5.1.2}$$

for $j = 1, \ldots, n$. Hence, $H(A, B)$ is a morphism of P-spaces.

Lemma 5.1.3. *The functor $H\colon P$-**rep** $\to P$-**sp** is full and dense.*

Proof: First, we show fullness. Given poset representations $M = (M_i)$ and $N = (N_i)$ with $\dim M = (d_0, \underline{d})$ and $\dim N = (e_0, \underline{e})$, consider a morphism of P-spaces $f\colon H(M) \to H(N)$. Note that

$$H(M) = \left(\Bbbk^{d_0}; \left\{ \sum_{i:p_i \preceq p_j} \operatorname{Im}(M_i) \right\} \right) \text{ and } H(N) = \left(\Bbbk^{e_0}; \left\{ \sum_{i:p_i \preceq p_j} \operatorname{Im}(N_i) \right\} \right).$$

Hence $f\colon \Bbbk^{d_0} \to \Bbbk^{e_0}$ is a linear map. Write $A = f$. It follows from Lemma 5.1.2 that $A(\operatorname{Im} M_j) \subseteq \sum_{i:p_i \preceq p_j} \operatorname{Im}(N_i)$ for $j = 1, \ldots, n$. Hence, we can choose linear maps $B_{i,j}\colon \Bbbk^{d_j} \to \Bbbk^{e_i}$ for $i, j = 1, 2, \ldots, n$ where $B_{i,j} = 0$ for all i and j where $p_i \not\preceq p_j$ and such that the diagram

$$\begin{array}{ccc} \Bbbk^{d_j} & \xrightarrow{\quad M_j \quad} & \Bbbk^{d_0} \\ {\scriptstyle (B_{i,j})_{i:p_i \preceq p_j}}\downarrow & & \downarrow{\scriptstyle A} \\ \bigoplus_{i:p_i \preceq p_j} \Bbbk^{e_i} & \xrightarrow[\sum_{i:p_i \preceq p_j} N_i]{} & \Bbbk^{e_0} \end{array} \tag{5.1.3}$$

commutes for $j = 1, 2, \ldots, n$.

Let $B = (B_{i,j})_{i,j=1}^{n}$. It follows from the commutativity Diagram (5.1.3) that $(A, B)\colon M \to N$ is a morphism in P-**rep** and, in particular, $H(A, B) = A = f$.

We will now show that H is dense. Let $(V_0; \{V_i\})$ be a P-space. Let $W_0 = V_0$. For every j such that p_j is minimal in P, let $W_j = V_j$. Now, for any j such that W_i is defined when $p_i \prec p_j$, let W_j be a subspace of V_j such that the decomposition

$$V_j = W_j \oplus \sum_{i:p_i \prec p_j} W_i$$

holds. Note that $\bigoplus_{i=1}^{n} W_i \subseteq W$.

If we write $\dim W_i = d_i$ for $i = 0, 1, 2, \ldots, n$, then there exists invertible linear maps $g_i\colon W_i \to \Bbbk^{d_i}$. Let $M = (M_i)_{i=1}^{n}$ be the representation of P where each M_i is the linear map defined so that the diagram

$$\begin{array}{ccc} W_i & \hookrightarrow & W \\ {\scriptstyle g_i^{-1}}\uparrow & & \downarrow{\scriptstyle g_0} \\ \Bbbk^{d_i} & \xrightarrow{M_i} & \Bbbk^{d_0} \end{array}$$

commutes. Then $\dim M = (d_0, \underline{d})$, with $\underline{d} = (d_1, d_2, \ldots, d_n)$.

We claim that $g_0\colon H(M) \to (V_0; \{V_i\})$ is an isomorphism of P-spaces. Note that

$$H(M) = \left(\Bbbk^{d_0}, \left\{ \sum_{i:p_i \preceq p_j} \operatorname{Im}(M_i) \right\} \right).$$

Then $g_0(V_0) = \Bbbk^{d_0}$ (recall $V_0 = W_0$) and, for $j = 1, 2, \ldots, n$, the diagram

$$\begin{array}{ccccc}
V_j & & \xrightarrow{\;\cong\;} & & \bigoplus_{i:p_i \preceq p_j} W_i \\
\big\downarrow & & & & \big\downarrow \cong \; \oplus g_i \\
V_0 & \xrightarrow{\;g_0\;} & \Bbbk^{d_0} & \xleftarrow{\;\sum_{i:p_i \preceq p_j} M_i\;} & \bigoplus_{i:p_i \preceq p_j} \Bbbk^{d_i}
\end{array}$$

commutes. It follows that $g_0(V_j) = \sum_{i:p_i \preceq p_j} \mathrm{Im}(M_i)$. Hence $H(M) \cong (V_0; \{V_i\})$. ∎

Example 5.1.4. The functor H is not faithful. For example, consider a poset $R = \{q, p\}$ with no relations. Then both morphisms

$$\left(\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)\right), \left(\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)\right) : \left(\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)\right) \to \left(\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)\right)$$

are mapped, by H, to the zero morphism.

Example 5.1.5. The functor H does not reflect isomorphisms. Again, consider the poset $R = \{q, p\}$ with no relations. Then the representations

$$\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right) \text{ and } \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right),$$

are clearly not isomorphic because they have different dimensions. Regardless, H maps both morphisms to the same R-space $(\Bbbk^3, \{\Bbbk, \Bbbk\})$.

Because H is does not reflect isomorphisms, additional work is required to obtain a functor that does reflect isomorphisms. We can obtain such a functor by restricting H to a suitable subcategory of P-**rep**. Let P-**rep**$_0$ denote the full subcategory of P-**rep** consisting of all representations $(M_i)_{i=1}^n$ of P such that

(i) M_j is injective for $j = 1, \ldots, n$, and

(ii) $\mathrm{Im}(M_j) \bigcap \sum_{i:p_i \prec p_j} \mathrm{Im}(M_i) = 0$, for $j = 1, \ldots, n$.

Every such representation M in P-$\mathbf{rep}_0$, where $\dim M = (d_0, (d_1, \ldots, d_n))$, is isomorphic to a representation in P-$\mathbf{rep}_0$ of the form

$$\left(\begin{pmatrix} 0 \\ \mathbb{I}_{d_1} \\ 0 \end{pmatrix}, \cdots, \begin{pmatrix} 0 \\ \mathbb{I}_{d_n} \\ 0 \end{pmatrix} \right)$$

where, if $p_i \prec p_j$, then a 1 in the k^{th} row of $\begin{pmatrix} 0 \\ \mathbb{I}_{d_i} \\ 0 \end{pmatrix}$ implies that the k^{th} row of $\begin{pmatrix} 0 \\ \mathbb{I}_{d_j} \\ 0 \end{pmatrix}$ contains only zeros.

Lemma 5.1.6. *The restriction $H_0 \colon P\text{-}\mathbf{rep}_0 \to P\text{-}\mathbf{sp}$ of H reflects isomorphisms.*

Proof: Let M and N be representations in P-$\mathbf{rep}_0$, where $\dim M = (d_0, \underline{d})$ and $\dim N = (e_0, \underline{e})$, chosen such that there is an isomorphism $f \colon H_0(M) \to H_0(N)$ of P-spaces. It follows from the fullness of H_0, which follows from the fullness of H, that there exists a morphism $(A, B) \colon M \to N$ such that $H(A, B) = f = A$. Hence A is an isomorphism. Denote the restriction of A to $\sum_{i: p_i \preceq p_j} \mathrm{Im}(M_i)$ by A_j. It remains to show that $B = (B_{i,j})_{i,j=1}^{n}$ is an isomorphism.

Suppose that p_j is minimal. It follows from Lemma 5.1.2 that the diagram

$$\begin{array}{ccc} \mathbb{k}^{d_j} & \xrightarrow{M_j} & \mathrm{Im}(M_j) \\ {\scriptstyle B_{j,j}} \downarrow & & \downarrow {\scriptstyle A_j} \\ \mathbb{k}^{e_j} & \xrightarrow[N_j]{} & \mathrm{Im}(N_j) \end{array} \tag{5.1.4}$$

commutes. Because M_j and N_j are injective as stipulated by the definition of P-$\mathbf{rep}_0$, they are both isomorphisms in Diagram (5.1.4). Because A is injective, it follows that the restriction A_j is also injective. Moreover, the equality $AM_j = N_j$, which holds by the definition of poset morphisms, implies that A_j is surjective. Therefore, A_j is an isomorphism. It follows that $B_{j,j}$ must be an isomorphism.

We proceed with an induction argument. Assume for j with $1 \leq j \leq n$, that B_{ii} is an isomorphism for all i such that $p_i \prec p_j$. The definition of morphisms in P-$\mathbf{rep}$

implies that the diagram

$$\begin{array}{ccc} \Bbbk^{d_j} & \xrightarrow{M_j} & \sum\limits_{i:p_i \preceq p_j} \operatorname{Im}(M_j) \\ {\scriptstyle (B_{i,j})_{i:p_i \preceq p_j}} \downarrow & & \downarrow {\scriptstyle A_j} \\ \bigoplus\limits_{i:p_i \preceq p_j} \Bbbk^{e_i} & \xrightarrow[(N_i)_{i:p_i \preceq p_j}]{} & \sum\limits_{i:p_i \preceq p_j} \operatorname{Im}(N_j) \end{array} \tag{5.1.5}$$

commutes. From the definition of P-$\mathbf{rep}_0$, the restriction listed in (ii) in particular, we get the decompositions

$$\sum_{i:p_i \preceq p_j} \operatorname{Im}(M_j) = \operatorname{Im}(M_j) \oplus \sum_{i:p_i \prec p_j} \operatorname{Im}(M_i), \text{ and}$$

$$\sum_{i:p_i \preceq p_j} \operatorname{Im}(N_j) = \operatorname{Im}(N_j) \oplus \sum_{i:p_i \prec p_j} \operatorname{Im}(N_i),$$

from which it follows that

$$A_j \colon \operatorname{Im}(M_j) \oplus \sum_{i:p_i \prec p_j} \operatorname{Im}(M_i) \to \operatorname{Im}(N_j) \oplus \sum_{i:p_i \prec p_j} \operatorname{Im}(N_i).$$

Thus, A_j has the form $\begin{pmatrix} A_j^{(1)} & A_j^{(2)} \\ A_j^{(3)} & A_j^{(4)} \end{pmatrix}$. In particular, we have that

$$A_j^{(1)} \colon \operatorname{Im}(M_j) \to \operatorname{Im}(N_j),$$

is an isomorphism because A_j is an isomorphism. Hence, by restricting Diagram (5.1.5) we obtain the commutative diagram

$$\begin{array}{ccc} \Bbbk^{d_j} & \xrightarrow{M_j} & \operatorname{Im}(M_j) \\ {\scriptstyle B_{jj}} \downarrow & & \downarrow {\scriptstyle A_j^{(1)}} \\ \Bbbk^{e_j} & \xrightarrow[N_j]{} & \operatorname{Im}(N_j). \end{array} \tag{5.1.6}$$

Because M_j and N_j are injective, they are isomorphisms in Diagram (5.1.6). Because $A_j^{(1)}$ is also an isomorphism, it follows that B_{jj} is an isomorphism.

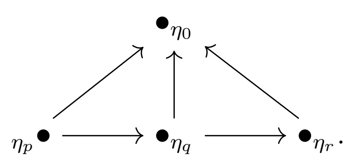
We conclude that B_{ii} is an isomorphism for $i = 1, 2, \ldots, n$. Recall that B is an upper triangular matrix because of how the elements of P were labeled. Thus B is an isomorphism. ∎

The representations contained in the subcategory P-$\mathbf{rep}_0$ are those 'certain representations of a poset' that we mentioned in the opening remarks of this section. Having found a suitable subcategory of poset representations, we can now associate a quiver to P and compare their respective categories of representations.

To the poset P, we associate the quiver Q_P with the node set $\{\eta_i\}_{i=0}^n$ and with arrows

- $\eta_i \xrightarrow{\alpha_{i,0}} \eta_0$ for $i = 1, \ldots, n$, and
- $\eta_j \xrightarrow{\alpha_{j,k}} \eta_k$ for all j, k such that p_k covers p_j.

Example 5.1.7. Associate the quiver



to the poset $\{p, q, r\}$ with the relations $p \prec q \prec r$.

We now present the functor that we will eventually pre-compose with the functor H_0 to obtain our main result. Let

$$R\colon P\text{-}\mathbf{sp} \to Q_P\text{-}\mathbf{rep} \tag{5.1.7}$$

be the functor that maps each object $(V_0; \{V_i\})$ to the quiver representation $R(V_0; \{V_i\})$ that associates

- V_i to node η_i for $i = 0, 1, 2, \ldots, n$, and
- the inclusion morphism to each arrow,

and that maps morphisms $f\colon (V_0, \{V_i\}) \to (W_0, \{W_i\})$ to

$$(R(f)_i)_{i=0}^n\colon R(V_0, \{V_i\}) \to R(W_0; \{W_i\})$$

where $R(f)_0 = f$ and $R(f)_i = f|_{V_i}$ for $i = 1, 2 \ldots, n$.

Proposition 5.1.8. *The functor R is fully faithful.*

Proof: Faithfulness follows directly from the definition of R on morphisms.

It remains to show fullness. Consider P-spaces V and W. It follows from the definition of R, that the representation $R(V)$ in Q_P-**rep** associates an inclusion map $V_i \hookrightarrow V_0$ to each arrow $\eta_i \xrightarrow{\alpha_{i,0}} \eta_0$, and likewise for $R(W)$. Let $(\theta_i)_{i=0}^n \colon V \to W$ be a morphism in Q_P-**rep**. Then the diagram

$$\begin{array}{ccc} V_i & \hookrightarrow & V_0 \\ \downarrow{\scriptstyle \theta_i} & & \downarrow{\scriptstyle \theta_0} \\ W_i & \hookrightarrow & W_0 \end{array}$$

commutes for $i = 1, 2, \ldots, n$. Hence $\theta_0(V_i) \subseteq W_i$ and so $\theta_0 \colon (V_0; \{V_i\}) \to (W_0; \{W_i\})$ is a morphism of P-spaces such that $R(\theta_0) = (\theta_i)_{i=0}^n$. Therefore, H is full. ∎

Example 5.1.9. The functor R is not dense. Consider a representation V in Q_P-**rep** consisting of non-trivial vector spaces $\{V_i\}_{i=0}^n$ and only zero maps. It follows that for any representation $(\{W_i\}_{i=0}^n, \{f_{i,j}\})$ in the image of R such that $V_i \cong W_i$ for $i = 0, 1, 2, \ldots, n$, the diagram

$$\begin{array}{ccc} V_i & \xrightarrow{0} & V_0 \\ {\scriptstyle \cong}\downarrow & & \downarrow{\scriptstyle \cong} \\ W_i & \underset{f_{i,0}}{\hookrightarrow} & W_0 \end{array}$$

cannot commute because $f_{i,0}$ is the inclusion mapping. Hence V is not isomorphic to any representation in the image of R.

For the main result of this section, we consider the composition

$$P\text{-}\mathbf{rep}_0 \xrightarrow{H_0} P\text{-}\mathbf{sp} \xrightarrow{R} Q_P\text{-}\mathbf{rep}$$

of functors H_0 and R.

Theorem 5.1.10. *Poset representations M and N are isomorphic in P-$\mathbf{rep}_0$ if and only if the quiver representations $RH_0(M)$ and $RH_0(N)$ are isomorphic in Q_P-$\mathbf{rep}$.*

Proof: We have $H_0(M) \cong H_0(N)$ because R is full and faithful. Then $M \cong N$ by Lemma 5.1.6. ∎

5.2 Classifying Quiver Representations with Poset Representations

The author is not aware of a general method to classify quiver representations by using a classification of representations for some poset. Of course, this does not imply that it is impossible. In this section, we will discuss three approaches, each for a different naive attempt at constructing such a method. Each approach leads to a functor that cannot reflect isomorphisms. If a functor does not reflect isomorphisms, then a classification of the category in the codomain of the functor will not imply a classification of the category in the domain of the functor.

Our presentation for each approach will follow the same pattern. First, we describe the general idea behind the approach. Then, we explore a simplified example that is chosen so as to be easily generalized.

For the first two approaches, we use the quiver

$$_{\eta}\bullet \xrightarrow{\ \alpha\ } \bullet_{\eta'}$$

which we denote by Q_1. Recall that classifying the isoclasses of Q_1-$\underline{\mathbf{rep}}^{(\underline{d})}$ for an arbitrary dimension $\underline{d}$ implies a classification of Q_1-$\mathbf{rep}$. We will work strictly in the category Q_1-$\underline{\mathbf{rep}}^{(2,1)}$, whose objects are all representations of the form

$$\Bbbk^2 \xrightarrow{\ f\ } \Bbbk$$

for some $\Bbbk$-linear map f. We will use the fact that that the diagram

$$\begin{array}{ccc} \Bbbk^2 & \xrightarrow{(1\ 0)} & \Bbbk \\ \cong\big\downarrow & & \big\downarrow\cong \\ \Bbbk^2 & \xrightarrow[0]{} & \Bbbk, \end{array} \tag{5.2.1}$$

does not commute for any choice of isomorphisms.

5.2.1 A First Approach

This approach considers any quiver Q with at least one non-loop arrow. Denote by P_Q the path poset of Q. The idea behind this approach is to define a mapping

$$\theta\colon \operatorname{Ob}\left(Q\text{-}\underline{\mathbf{rep}}^{(\underline{d})}\right) \to \operatorname{Ob}(P_Q\text{-}\mathbf{rep}),$$

for some fixed composition $\underline{d}$, that sends each representation x of Q to a representation

$$\left(\iota_{h(\rho)} M^{(\rho)}_{n_\rho} M^{(\rho)}_{n_\rho - 1} \cdots M^{(\rho)}_1\right)_{\rho \in P_Q}$$

of P_Q where, to each path $\rho = \alpha^{(\rho)}_{n_\rho} \cdots \alpha^{(\rho)}_1$, we associate a product of matrices

$$\iota_{h(\rho)} M^{(\rho)}_{n_\rho} M^{(\rho)}_{n_\rho - 1} \cdots M^{(\rho)}_1$$

where

- each $M^{(\rho)}_i$ is the transformation matrix, with respect to the distinguished basis, that x associates to the arrow $\alpha^{(\rho)}_i$, and
- $\iota_{h(\rho)}\colon \Bbbk^{d_{h(\rho)}} \hookrightarrow \Bbbk^d$ is an inclusion map where, recall, $h(\rho)$ denotes the head of ρ.

Then any functor whose object map is θ cannot reflect isomorphisms.

Instead of proving this general statement, we will prove it for the case where $Q = Q_1$. This is easily generalized to any quiver with a non-loop arrow.

Let $(P_{Q_1}, \preceq)$ be the path poset of Q_1. That is, $P_{Q_1} = \{\varepsilon_\eta, \varepsilon_{\eta'}, \alpha\}$ with relations $\varepsilon_\eta \prec \alpha$ and $\varepsilon_{\eta'} \prec \alpha$. We will present representations of P_{Q_1} as an ordered triple $(M_\eta, M_{\eta'}, M_\alpha)$ of matrices.

Define a mapping

$$\theta_1\colon \operatorname{Ob}\left(Q_1\text{-}\underline{\mathbf{rep}}^{(2,1)}\right) \to \operatorname{Ob}(P_{Q_1}\text{-}\mathbf{rep}), \tag{5.2.2}$$

that sends a representation $\Bbbk^2 \xrightarrow{(a\ b)} \Bbbk$ in $\operatorname{Ob}\left(Q_1\text{-}\underline{\mathbf{rep}}^{(2,1)}\right)$ to the representation

$$\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ a & b \end{pmatrix}\right)$$

of the poset P_{Q_1}.

Proposition 5.2.1. *Any functor Q_1-$\underline{\mathbf{rep}}^{(2,1)} \to P_{Q_1}$-$\mathbf{rep}$ whose object map is θ_1 (see Equation* (5.2.2)*) cannot reflect isomorphisms.*

Proof: Let x denote the representation $\Bbbk^2 \xrightarrow{(1\ 0)} \Bbbk$ in Q_1-$\underline{\mathbf{rep}}^{(2,1)}$ and let y denote the representation $\Bbbk^2 \xrightarrow{(0\ 0)} \Bbbk$. It follows from the non-commutativity of Diagram (5.2.1) that $x \ncong y$. Then we have that

$$\theta_1(x) = \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}\right) \text{ and } \theta_1(y) = \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}\right).$$

If we let

$$B = \left(\begin{array}{cc|c|cc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

then $(\mathbb{I}, B)\colon \theta_1(x) \to \theta_1(y)$ is an isomorphism in P_Q-$\mathbf{rep}$. ∎

5.2.2 A Second Approach

The next approach we consider holds for any quiver Q with at least one arrow and no cycles. The idea is to let P be a poset whose elements are the nodes of Q and with relations $\eta \preceq \eta'$ for all $\eta, \eta' \in P$ such that there is a path in Q from η to η'. Then define a mapping

$$\theta\colon \operatorname{Ob}\left(Q\text{-}\underline{\mathbf{rep}}^{(\underline{d})}\right) \to \operatorname{Ob}(P\text{-}\mathbf{rep}),$$

for some fixed composition $\underline{d}$, that sends each representation x of Q to the representation

$$\left(\left(M_{1i}^{(\eta)}\right)_{i=1}^{m_\eta}\right)_{\eta\in P}$$

of P, where for each $\eta \in P$, the block matrix $\left(M_{1i}^{(\eta)}\right)_{i=1}^{m_\eta}$ is defined as follows. Denote the paths with head η by $\alpha_{n_i}^{(i)} \cdots \alpha_1^{(i)}$, for $i = 1, 2, \ldots, m_\eta$. Define $M_{1i}^{(\eta)} = \iota_\eta A_{n_i}^{(i)} \cdots A_1^{(i)}$ where

- for $j = 1, \ldots, n_i$, $A_j^{(i)}$ is the transformation matrix, with respect to the distinguished basis, that x associates to the arrow $\alpha_j^{(i)}$, and
- $\iota_\eta \colon \mathbb{k}^{d_\eta} \hookrightarrow \mathbb{k}^d$ is an inclusion map.

Then any functor whose object map is θ cannot reflect isomorphisms.

Similar to the first approach, we will only prove this statement for the quiver Q_1. But this can be generalized to any quiver with at least one arrow and no cycles.

We still work in the category Q_1-$\underline{\mathbf{rep}}^{(2,1)}$ for the same quiver Q_1. Let $(P_1, \preceq)$ be the poset with $P_1 = \{\eta, \eta'\}$ and the relation $\eta \prec \eta'$. Define a mapping

$$\theta_2 \colon \operatorname{Ob}\left(Q_1\text{-}\underline{\mathbf{rep}}^{(2,1)}\right) \to \operatorname{Ob}(P_1\text{-}\mathbf{rep}), \tag{5.2.3}$$

that sends a representation $\mathbb{k}^2 \xrightarrow{(a\ b)} \mathbb{k}$ in $\operatorname{Ob}\left(Q_1\text{-}\underline{\mathbf{rep}}^{(2,1)}\right)$ to the representation

$$\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & a & b \end{pmatrix}\right)$$

of the poset P_1.

Proposition 5.2.2. *Any functor Q_1-$\underline{\mathbf{rep}}^{(2,1)} \to P_1$-$\mathbf{rep}$ whose object map is θ_2 (see Equation* (5.2.3)*) cannot reflect isomorphisms.*

Proof: Let x denote the representation $\mathbb{k}^2 \xrightarrow{(1\ 0)} \mathbb{k}$ in Q_1-$\underline{\mathbf{rep}}^{(2,1)}$ and left y denote the representation $\mathbb{k}^2 \xrightarrow{(0\ 0)} \mathbb{k}$. It follows from the non-commutativity of Diagram (5.2.1) that $x \ncong y$. Then we have that

$$\theta_2(x) = \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}\right) \text{ and } \theta_2(y) = \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}\right).$$

If we let

$$B = \left(\begin{array}{cc:ccc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

then $(\mathbb{I}, B)\colon \theta_2(x) \to \theta_2(y)$ is an isomorphism in P_1-**rep**. ∎

5.2.3 The Third Approach

The final approach we consider holds for any quiver Q with a path of length two and no cycles. The idea is to consider a poset P whose elements are the nodes of Q and with relations $\eta \preceq \eta'$ for all $\eta, \eta' \in P$ such that there is a path in Q from η to η'. First, recall that $t(\rho)$ denotes the tail of the path ρ. Now, define a mapping

$$\theta\colon \operatorname{Ob}\left(Q\text{-}\underline{\mathbf{rep}}^{(\underline{d})}\right) \to \operatorname{Ob}\left(P\text{-}\mathbf{rep}\right),$$

for some fixed composition $\underline{d}$, that sends a representation x of Q to a representation

$$\left(\sum_{\rho: t(\rho)=\eta} \iota_{h(\rho)} M^{(\rho)}_{n_\rho} M^{(\rho)}_{n_\rho - 1} \cdots M^{(\rho)}_1\right)_{\eta \in P}$$

of P such that, for each $\eta \in P$, we define the sum of matrices

$$\sum_{\rho: t(\rho)=\eta} \iota_{h(\rho)} M^{(\rho)}_{n_\rho} M^{(\rho)}_{n_\rho - 1} \cdots M^{(\rho)}_1$$

as follows. Each term of the sum is indexed by a path $\rho = \alpha^{(\rho)}_{n_\rho} \cdots \alpha^{(\rho)}_1$ and is a product of the matrices $M^{(\rho)}_j$, for $j = 1, \ldots, n_\rho$, and $\iota_{h(\rho)}$ where

- $M^{(\rho)}_j$ is the transformation matrix, with respect to the distinguished basis, that x associates to the arrow $\alpha^{(\rho)}_j$, and
- $\iota_{h(\rho)}\colon \mathbb{k}^{d_\eta} \to \mathbb{k}^d$ is an inclusion map.

Then any functor whose object map is θ cannot reflect isomorphisms. We will prove this in the case of the quiver Q_2

$$\eta_1 \bullet \xrightarrow{\alpha_1} \bullet_{\eta_2} \xrightarrow{\alpha_2} \bullet_{\eta_3}.$$

However, this case is easily generalized to any quiver Q with a path of length two and no cycles.

Let $(P_2, \preceq)$ be the poset with set $P_2 = \{\eta_1, \eta_2, \eta_3\}$ and relations $\eta_1 \prec \eta_2 \prec \eta_3$. We will work in the category $Q\text{-}\underline{\mathbf{rep}}^{(3,3,3)}$. Note that the diagram

$$\begin{array}{ccc} \Bbbk^3 & \xrightarrow{\begin{pmatrix} 1&0&0\\0&0&0\\0&0&0 \end{pmatrix}} & \Bbbk^3 \\ \cong\downarrow & & \downarrow\cong \\ \Bbbk^3 & \xrightarrow[\begin{pmatrix} 1&0&0\\0&0&0\\0&0&1 \end{pmatrix}]{} & \Bbbk^3, \end{array} \tag{5.2.4}$$

does not commute for any choice of isomorphisms because the matrices have different ranks. This fact will be used later.

Define a mapping

$$\theta_3\colon \operatorname{Ob}\left(Q_2\text{-}\underline{\mathbf{rep}}^{(3,3,3)}\right) \to \operatorname{Ob}(P_2\text{-}\mathbf{rep}), \tag{5.2.5}$$

that sends a representation $\Bbbk^3 \xrightarrow{M_1} \Bbbk^3 \xrightarrow{M_2} \Bbbk^3$ in $\operatorname{Ob}\left(Q_2\text{-}\underline{\mathbf{rep}}^{(3,3,3)}\right)$ to the representation

$$\left(\begin{pmatrix} \mathbb{I}_3 \\ M_1 \\ M_2M_1 \end{pmatrix}, \begin{pmatrix} 0 \\ \mathbb{I}_3 \\ M_2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \mathbb{I}_3 \end{pmatrix}\right)$$

in $\operatorname{Ob}(P_2\text{-}\mathbf{rep})$.

Proposition 5.2.3. *Any functor $Q_2\text{-}\underline{\mathbf{rep}}^{(3,3,3)} \to P_2\text{-}\mathbf{rep}$ whose object map is θ_3 (see Equation (5.2.5)) cannot reflect isomorphisms.*

Proof: Let x and y be the representations, respectively,

$$\Bbbk^3 \xrightarrow{\begin{pmatrix} 1&0&0\\0&1&0\\0&0&0 \end{pmatrix}} \Bbbk^3 \xrightarrow{\begin{pmatrix} 1&0&0\\0&0&0\\0&0&0 \end{pmatrix}} \Bbbk^3 \quad \text{and} \quad \Bbbk^3 \xrightarrow{\begin{pmatrix} 1&0&0\\0&1&0\\0&0&0 \end{pmatrix}} \Bbbk^3 \xrightarrow{\begin{pmatrix} 1&0&0\\0&0&0\\0&0&1 \end{pmatrix}} \Bbbk^3.$$

It follows from the non-commutativity of Diagram (5.2.4) that $x \ncong y$.

Then we have that

$$\theta_3(x) = \left(\begin{pmatrix} 1&0&0\\0&1&0\\0&0&1\\ \\ 1&0&0\\0&1&0\\0&0&0\\ \\ 1&0&0\\0&0&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0\\0&0&0\\0&0&0\\ \\ 1&0&0\\0&1&0\\0&0&1\\ \\ 1&0&0\\0&0&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0\\0&0&0\\0&0&0\\ \\ 0&0&0\\0&0&0\\0&0&0\\ \\ 1&0&0\\0&1&0\\0&0&1 \end{pmatrix} \right), \text{ and}$$

$$\theta_3(y) = \left(\begin{pmatrix} 1&0&0\\0&1&0\\0&0&1\\ \\ 1&0&0\\0&1&0\\0&0&0\\ \\ 1&0&0\\0&0&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0\\0&0&0\\0&0&0\\ \\ 1&0&0\\0&1&0\\0&0&1\\ \\ 1&0&0\\0&0&0\\0&0&1 \end{pmatrix} \begin{pmatrix} 0&0&0\\0&0&0\\0&0&0\\ \\ 0&0&0\\0&0&0\\0&0&0\\ \\ 1&0&0\\0&1&0\\0&0&1 \end{pmatrix} \right).$$

The only difference between $\theta_3(x)$ and $\theta_3(y)$ is the $(9,3)$-entry of the second term. Thus,

$$(\mathbb{I} + E_{9,6}, \mathbb{I}) : \theta_3(x) \to \theta_3(y)$$

is an isomorphism in P_2-**rep**. ∎

Chapter 6

Categorical Matrix Problems

In this chapter, we introduce two types of matrix problems, both of which use a categorical framework. The first type of matrix problem we discuss is defined via representations of differential graded categories (DGCs) which Kleiner and Roĭter introduced in [28]. These representations formed the first attempt at developing a general theory of matrix problems. The second type of matrix problem we discuss is defined via the related notion of representations of bimodules over a category with co-algebra structure (bocs) which Roĭter introduced in [27]. The benefit of DGC matrix problems and bocs matrix problems is that, in each language, there exists an algorithm which accepts as an input, a category of representations of a DGC or a bocs and outputs an equivalent category whose structure is preferable to work with in the context of classifying its isoclasses. We will not study this algorithm here and point interested readers to [27] and [28]. Instead, we will study both DGC and bocs matrix problems and then show how they are related.

6.1 Differential Graded Category Matrix Problems

While differential graded categories (DGCs) were not introduced specifically for the study of matrix problems, they were used in [28] by Roĭter and Kleiner to this end. Roĭter and Kleiner did, however, introduce the concept of a representation of a DGC. This section explores DGCs and their use in matrix problems. For interested readers, Keller [14] provides an excellent survey of DGCs.

6.1.1 Differential Graded Categories

Definition 6.1.1 (Graded Module)**.** A $\Bbbk$-module M with a decomposition

$$M = \bigoplus_{i \in \mathbb{N}} M_i,$$

where each M_i is a $\Bbbk$-module is called *graded.* A linear map

$$f\colon \bigoplus_{i \in \mathbb{N}} M_i \to \bigoplus_{i \in \mathbb{N}} N_i$$

between graded modules is a *morphism of graded modules* of degree n if $f(M_i) \subseteq N_{i+n}$ for all i. Define the tensor product of graded modules to be $M \otimes_{\Bbbk} N = \bigoplus_{i \in \mathbb{N}} (M \otimes_{\Bbbk} N)_i$ where

$$(M \otimes_{\Bbbk} N)_i = \bigoplus_{j+k=i} M_j \otimes_{\Bbbk} N_k.$$

The natural isomorphims associated to associativity, right and left identity morphisms are graded. The unit object is $\Bbbk \oplus 0 \oplus \cdots$.

The category $\Bbbk$-**gmod** of graded finite-dimensional left $\Bbbk$-modules with graded module morphisms of degree 0 is monoidal. Composition of morphisms in this category is obvious and associativity follows.

There is a natural embedding of categories $\Bbbk$-**mod** $\hookrightarrow$ $\Bbbk$-**gmod** by sending any non-graded $\Bbbk$-module M to the trivially graded $\Bbbk$-module $M = \bigoplus_{i \in \mathbb{N}} M_i$ where $M_0 = M$ and $M_i = \{0\}$ for $i \geq 1$.

Definition 6.1.2 (Graded Category)**.** A category is called *graded* if it is enriched over $\Bbbk$-**gmod**. Similarly, a functor that is enriched over $\Bbbk$-**gmod** is said to be *graded*.

Given a graded category $\mathcal{C}$, denote by I_n the ideal of $\mathcal{C}$ generated by all $\mathcal{C}$-morphisms with degree n. Thus, we obtain the factor category $\mathcal{C}/I_n$, which we will denote by $\mathcal{C}_{(n)}$. This is an important subcategory for our purposes. Particularly when $i = 1$. This will become evident when we introduce the category of representations of a DGC.

Definition 6.1.3 (Semi-Free Graded Category)**.** A graded category $\mathcal{C}$ is *semi-free over* $\mathcal{C}_{(i)}$ if, for any graded category $\mathcal{D}$, every functor $F_{(i)} \colon \mathcal{C}_{(i)} \to \mathcal{D}_{(i)}$ can be uniquely extended to a functor $F \colon \mathcal{C} \to \mathcal{D}$. A category $\mathcal{C}$ that is semi-free over $\mathcal{C}_{(1)}$ is simply called *semi-free*.

Proposition 6.1.4. *Every morphism of a semi-free graded category* $\mathcal{C}$ *is a composite of morphisms of degree* 0 *or* 1*.*

Proof: By contradiction, suppose that there is a $\mathcal{C}$-morphism f whose degree is strictly greater than 1 and that is not a composition of any non-identity morphisms. The identity functor $I_{(1)} \colon \mathcal{C}_{(1)} \to \mathcal{C}_{(1)}$ extends to the identity functor $I \colon \mathcal{C} \to \mathcal{C}$, of course. But is also extends to any functor $I' \colon \mathcal{C} \to \mathcal{C}$ such that $I'(g) = g$ for all morphisms g with $\deg g = 1$ and $I'(f) = 0$. ∎

For the next definition, recall the notion of a cochain complex from Example 1.1.17.

Definition 6.1.5 (Differential Graded Category)**.** A category that is enriched over the category of cochain complexes of left $\Bbbk$-modules is called a *differential graded category* or a *DGC*. A functor that is enriched over the category of cochain complexes of left $\Bbbk$-modules is called a *DG functor*.

There is an another definition of a DGC from [28, p. 321] that is quite useful. Let $\mathcal{C}$ be a graded category. Introduce a mapping d defined so that, for all $\mathcal{C}$-objects x and y, there is a graded $\Bbbk$-module endomorphism $d\colon \mathcal{C}(x,y) \to \mathcal{C}(x,y)$ of degree 1. Then $\mathcal{C}$ is a DGC if, for all composable $\mathcal{C}$-morphisms f, g, the (graded) Leibniz formula

$$d(fg) = d(f)g + (-1)^{\deg f} f d(g) \tag{6.1.1}$$

holds and $d^2 = 0$. We call d the *differential* of $\mathcal{C}$.

Let us make two helpful observations. First, we see that the Leibniz formula generalizes to

$$\begin{aligned} d(f_1 f_2 \cdots f_n) = d(f_1) f_2 \cdots f_n + (-1)^{\deg f_1} f_1 d(f_2) f_3 \cdots f_n + \cdots \\ + (-1)^{\sum_{i=1}^{k-1} \deg f_i} f_1 \cdots f_{k-1} d(f_k) f_{k+1} \cdots f_n + \cdots \\ + (-1)^{\sum_{i=1}^{n-1} \deg f_i} f_1 \cdots f_{n-1} d(f_n). \end{aligned} \tag{6.1.2}$$

The second observation follows from the definition of a graded $\Bbbk$-linear mapping. That is, given a differential d, then for every pair of $\mathcal{C}$-objects x and y and for every $i \in \mathbb{N}$, we obtain a $\Bbbk$-linear mapping $d\colon \mathcal{C}(x,y)_i \to \mathcal{C}(x,y)_{i+1}$.

For the remainder of this section, all functors between graded categories are graded functors and all functors between DGCs are DG functors.

6.1.2 Graphs as Generators

This section introduces a slight generalization of a quiver, called a bigraph.

Definition 6.1.6 (Bigraph)**.** Let $\mathcal{C}_\mathrm{b}$ be the category

$$A_0 \underset{t_0}{\overset{h_0}{\rightrightarrows}} N \underset{t_1}{\overset{h_1}{\leftleftarrows}} A_1.$$

A *bigraph* is a functor $B\colon \mathcal{C}_\mathrm{b} \to \mathbf{Set}$.

We will refer to the image of the $\mathcal{C}_b$-objects in **Set** as the set of nodes B_N, the set of arrows B_{A_0} of type 0 and the set of arrows B_{A_1} of type 1. Intuitively, a bigraph can be thought of as a two-colored quiver where B_{A_0} contains arrows of one color and B_{A_1} contains arrows of another color.

Definition 6.1.7 (Category of Bigraphs)**.** A *bigraph morphism* is a natural transformation between two bigraphs. With vertical composition of natural transformations serving as the composition of bigraph morphisms, we obtain the category **Bgrph** of bigraphs with the full subcategory **bgrph** of bigraphs whose sets of nodes and arrows are finite.

We will be working strictly in the category **bgrph**.

Definition 6.1.8 (Bigraph Path)**.** Consider a bigraph B. A sequence of arrows $\alpha_n\alpha_{n-1}\cdots\alpha_1$ is a *path* of length n if, for $i = 1, 2, \ldots, n-1$ and $j_i, j_{i+1} \in \{0, 1\}$, we have that $B(h_{j_i})(\alpha_i) = B(t_{j_{i+1}})(\alpha_{i+1})$ where $\alpha_i \in B_{A_{j_i}}$.

Informally, a path is a sequence of arrows such that the head of any arrow coincides with the tail of the following arrow. There are no restrictions placed on the color of each arrow in a path. Concepts such as the head or tail of a path, cycles, empty paths and path multiplication carry over from the quiver case.

There are two canonical embeddings of the category **quiv** into **bgrph**. Simply assign to a quiver $Q = (Q_N, Q_A, h, t)$ the bigraph with $B_N = Q_N$, $B_{A_0} = Q_A$ and $B_{A_1} = \emptyset$, or alternatively, $B_N = Q_N$, $B_{A_0} = \emptyset$ and $B_{A_1} = Q_A$. Thus, we can think of **quiv** as a subcategory of **bgrph** in two natural ways. We will utilize both embeddings in the sequel.

Definition 6.1.9 (Graded Bigraph)**.** A bigraph B is called *graded* if there exists a mapping

$$\deg\colon B_{A_0} \bigcup B_{A_1} \to \mathbb{N}$$

such that $\alpha, \alpha' \in B_{A_i}$ implies that $\deg\alpha = \deg\alpha'$ for $i = 0, 1$.

A degree is naturally assigned to a path $\rho = \alpha_n \alpha_{n-1} \cdots \alpha_1$ by letting $\deg \rho = \sum_{i=1}^{n} \deg \alpha_i$.

Definition 6.1.10 (Path Category)**.** Let B be a graded bigraph. Denote by $\mathcal{P}_B$ the category whose objects are the nodes of B and whose hom-sets $\mathcal{P}_B(\eta, \eta')$ consist of all paths from η to η'. The homs-sets are graded by degree, identities are the trivial paths and composition follows from path multiplication. Associativity, then, is obvious. We call $\mathcal{P}_B$ the *path category* of B.

Definition 6.1.11 (Graded Path Category)**.** Let B be a graded bigraph. Denote by $\mathcal{U}_B$ the category whose objects are the nodes of B and whose hom-sets

$$\mathcal{U}_B(\eta, \eta') = \bigoplus_{i \in \mathbb{N}} \mathcal{U}_B(\eta, \eta')_i$$

are the graded $\Bbbk$-modules where each $\mathcal{U}_B(\eta, \eta')_i$ is generated by the set of paths of degree i from η to η'. Identities are the trivial paths and composition follows from path multiplication. Then associativity follows trivially. We call $\mathcal{U}_B$ the *graded path category* of B.

Overloading the symbol, we denote by α the $\mathcal{U}_B$-morphism arising from an arrow α of B. Context should clarify whether we are referring to the $\mathcal{U}_B$-morphism or arrow in B.

Proposition 6.1.12. *Let B be a graded bigraph such that $\deg \alpha \leq n$ for any arrow α. Then the graded path category $\mathcal{U}_B$ is semi-free over $(\mathcal{U}_B)_{(n)}$.*

Proof: This follows immediately from the fact that any functor from $\mathcal{U}_B$ is uniquely determined by its definition on the morphisms α arising from the arrows α of B. ∎

Proposition 6.1.13. *Let B be a graded bigraph. Consider a mapping d introduced on the hom-sets of the graded category $\mathcal{U}_B$ in such a way that, for all $\eta, \eta' \in \mathrm{Ob}(\mathcal{U}_B)$,*

$$d\colon \mathcal{U}_B(\eta,\eta') \to \mathcal{U}_B(\eta,\eta')$$

is a graded $\Bbbk$-linear map of degree 1 and the Leibniz rule holds. Then $\mathcal{U}_B$ is a DGC if $d^2(\alpha) = 0$ on every $\mathcal{U}_B$-morphism α arising from an arrow α of B.

Proof: An arbitrary morphism $f \in \mathcal{U}_B(\eta,\eta')$, for any $\eta,\eta' \in \mathrm{Ob}(\mathcal{U}_B)$, has the form

$$f = \bigoplus_{j\in\mathbb{N}} \left(\sum_{i=1}^{N} k_i \rho_{ij} \right),$$

where $k_i \in \Bbbk$ and $\rho_{ij} = \alpha_{i,j}^{(r_{ij})} \cdots \alpha_{i,j}^{(1)}$ are paths in B from η to η' of degree j. When restricted to $\mathcal{U}_B(\eta,\eta')_j$, d^2 is $\Bbbk$-linear, which implies, for any $j \in \mathbb{N}$, that

$$d^2\left(\sum k_i\rho_{ij}\right) = \sum k_i d^2(\rho_{ij}) = \sum k_i d^2\left(\alpha_{i,j}^{(r_{ij})} \cdots \alpha_{i,j}^{(1)}\right).$$

But our assumption that $d^2\left(\alpha_{i,j}^{(k)}\right) = 0$ for $k = 1, 2, \ldots, r_{ij}$ implies $d^2(\rho_{ij}) = 0$ by the Leibniz formula. Hence $d^2(f) = 0$. ∎

6.1.3 Category of Representations of DGCs

In this section, we will take a semi-free differential graded category $\mathcal{C}$ and a $\Bbbk$-linear category $\mathcal{D}$ and then construct a new $\Bbbk$-linear category $\mathcal{R}(\mathcal{C},\mathcal{D})$ called the category of representations of $\mathcal{C}$ in $\mathcal{D}$. This construction follows [28, Sec. 2], though we add many details that are not included in that paper.

We are interested in the case when $\mathcal{C}$ is the graded path category of a graded bigraph B whose arrows have degree 0 or 1. This is sufficiently general to include the method for using a DGC matrix problem (see Definition 6.1.16) to classify representations of a quiver or a poset. Proposition 6.1.12 ensures that $\mathcal{C}$ is semi-free. We assume that there is a differential on $\mathcal{C}$ so that it is a DGC. Though there is not a canonical way in which to define a differential on a graded path category, it is always possible to introduce one. For instance, the trivial differential will turn any graded category into a DGC. The definition of $\mathcal{R}(\mathcal{C}, \mathcal{D})$ will hold for any choice of differential of $\mathcal{C}$.

We break this construction into several parts:

Step 1 Constructing the DGC $\overline{\mathcal{C}}$ from $\mathcal{C}$;

Step 2 Constructing the DGC $\overline{\mathcal{D}}$ from $\mathcal{D}$; and

Step 3 Defining $\mathcal{R}(\mathcal{C}, \mathcal{D})$ with the use of $\overline{\mathcal{C}}$ and $\overline{\mathcal{D}}$.

After completing the construction, we will show that $\mathcal{R}(\mathcal{C}, \mathcal{D})$ satisfies the category axioms.

Step 1: Construct $\overline{\mathcal{C}}$

Let $\mathcal{C}$ be a semi-free DGC with differential $d_{\mathcal{C}}$. As mentioned above, $\mathcal{C}$ is the graded path category for some graded bigraph B. Let B^* be the bigraph obtained by augmenting B with degree 1 loops e_x for every node x. Let $\overline{\mathcal{C}}$ be the graded path category of B^*. Observe that there is a natural inclusion functor $\mathcal{C} \hookrightarrow \overline{\mathcal{C}}$ which allows us to consider $\mathcal{C}$ as a subcategory of $\overline{\mathcal{C}}$.

Define a differential $d_{\overline{\mathcal{C}}}$ on $\overline{\mathcal{C}}$ by letting

$$d_{\overline{\mathcal{C}}}(e_x) = e_x^2$$

for every $\mathcal{C}$-object x and, for any morphism $f\colon x \to y$, letting

$$d_{\overline{\mathcal{C}}}(f) = d_{\mathcal{C}}(f) + fe_x - (-1)^{\deg f} e_y f$$

if f is a $\mathcal{C}$-morphism. Then extend $d_{\overline{\mathcal{C}}}$ by linearity and to all composite morphisms with the Leibniz formula.

The following proposition is stated without proof in [28, p. 323].

Proposition 6.1.14. *The differential graded category $\overline{\mathcal{C}}$ is semi-free.*

Proof: Let $\mathcal{E}$ be a graded category and $F_{(1)}\colon \overline{\mathcal{C}}_{(1)} \to \mathcal{E}_{(1)}$ be a functor. Given any morphism

$$f = \bigoplus_{j=0}^{m} \left(\sum_{i=0}^{n} k_{ij} f_{ij} \right)$$

in $\overline{\mathcal{C}}$, we have that $f_{ij} = f_{ij}^{(\ell)} f_{ij}^{(\ell-1)} \cdots f_{ij}^{(1)}$ where each term $f_{ij}^{(k)}$ is an arrow in B^* of degree 0 or 1. Thus for any functor $F\colon \overline{\mathcal{C}} \to \mathcal{E}$ extending $F_{(1)}$, the equality

$$F(f) = \bigoplus_{j=0}^{m} \left(\sum_{i=0}^{n} k_{ij} F_{(1)} \left(f_{ij}^{(\ell)} \right) F_{(1)} \left(f_{ij}^{(\ell-1)} \right) \cdots F_{(1)} \left(f_{ij}^{(1)} \right) \right)$$

holds and so $F_{(1)}$ uniquely determines F. ∎

Step 2: Construct $\overline{\mathcal{D}}$

Using $\mathcal{D}$, we will now define a category $\overline{\mathcal{D}}$ which is a slight generalization of the category $\mathcal{D} \times \mathcal{D}$. Define the objects of $\overline{\mathcal{D}}$ to be $\mathrm{Ob}\left(\overline{\mathcal{D}}\right) = \mathrm{Ob}(\mathcal{D} \times \mathcal{D})$. Define the hom-sets to be

$$\overline{\mathcal{D}}(x,y) = \overline{\mathcal{D}}(x,y)_0 \bigoplus \overline{\mathcal{D}}(x,y)_1,$$

for $x = (x_1, x_2)$, $y = (y_1, y_2) \in \mathrm{Ob}\left(\overline{\mathcal{D}}\right)$, where

$$\overline{\mathcal{D}}(x,y)_0 = \left\{ \left(\begin{smallmatrix} f_{11} & 0 \\ 0 & f_{22} \end{smallmatrix}\right) : f_{ii} \in \mathcal{D}(x_i, y_i) \text{ for } i = 1,2 \right\}, \text{ and}$$
$$\overline{\mathcal{D}}(x,y)_1 = \left\{ \left(\begin{smallmatrix} 0 & f_{12} \\ 0 & 0 \end{smallmatrix}\right) : f_{12} \in \mathcal{D}(x_1, y_2) \right\}.$$

Composition is defined for $\overline{\mathcal{D}}$-morphisms

$$f = \left(\begin{smallmatrix} f_{11} & f_{12} \\ 0 & f_{22} \end{smallmatrix}\right) \in \overline{D}(x,y) \text{ and } g = \left(\begin{smallmatrix} g_{11} & g_{12} \\ 0 & g_{22} \end{smallmatrix}\right) \in \overline{D}(y,z)$$

by letting

$$gf = \begin{pmatrix} g_{11}f_{11} & g_{12}f_{11} + g_{22}f_{12} \\ 0 & g_{22}f_{22} \end{pmatrix}. \tag{6.1.3}$$

Associativity follows from a simple calculation.

Finally, we make $\overline{\mathcal{D}}$ a DGC with the trivial differential $d_{\overline{\mathcal{D}}}(f) = 0$ for all morphisms $f \in \overline{\mathcal{D}}$. The reason we take $d_{\overline{\mathcal{D}}}(f) = 0$ is to ensure that the morphisms in the category of representations $\mathcal{R}(\mathcal{C}, \mathcal{D})$, which are DG functors, have a well-defined composition.

Step 3: Define $\mathcal{R}(\mathcal{C}, \mathcal{D})$

The category $\mathcal{R}(\mathcal{C}, \mathcal{D})$ of representations of $\mathcal{C}$ into $\mathcal{D}$ can now be defined. The objects of $\mathcal{R}(\mathcal{C}, \mathcal{D})$ are $\Bbbk$-linear functors $F \colon \mathcal{C}_{(0)} \to \mathcal{D}$. The morphisms will consist of certain $\Bbbk$-linear, DG functors $\Psi \colon \overline{\mathcal{C}} \to \overline{\mathcal{D}}$ which will require some additional work to describe.

Observe that any graded functor $\Psi \colon \overline{\mathcal{C}} \to \overline{\mathcal{D}}$ will send a degree 0 morphism $f \in \mathcal{C}(x, y)$, for any objects x and y, to a $\overline{\mathcal{D}}$-morphism $\Psi(f)$ with degree 0. Hence, we have that

$$\Psi(f) = \begin{pmatrix} \Psi_{11}(f) & 0 \\ 0 & \Psi_{22}(f) \end{pmatrix} \colon (\Psi(x)_1, \Psi(x)_2) \to (\Psi(y)_1, \Psi(y)_2)$$

where $\Psi_{ii}(f) \colon \Psi(x)_i \to \Psi(y)_i$ is a $\mathcal{D}$-morphism for $i = 1, 2$. Thus we can consider Ψ_{ii} as a functor $\Psi_{ii} \colon \mathcal{C}_{(0)} \to \mathcal{D}$ with object map $x \mapsto \Psi_i(x)$ and morphism map $f \mapsto \Psi_{ii}(f)$.

The set of morphisms between objects F and G consist of the DG functors $\Psi \colon \overline{\mathcal{C}} \to \overline{\mathcal{D}}$ that map

- objects $x \mapsto (F(x), G(x))$,
- degree 0 morphisms $f \mapsto \begin{pmatrix} F(f) & 0 \\ 0 & G(f) \end{pmatrix}$,
- degree 1 morphisms $g \mapsto \begin{pmatrix} 0 & \Psi_{12}(g) \\ 0 & 0 \end{pmatrix}$, where $\Psi_{12}(g) \in \mathcal{D}(\Psi_1(x), \Psi_2(y))$ if $g \in \overline{\mathcal{C}}(x, y)$, and
- degree n morphisms $h \mapsto 0$ for all $n \geq 2$.

Intuitively, these morphisms can be thought of as generalized natural transformations. It follows from this definition of morphisms that any morphism $\Psi \colon F \to G$ of $\mathcal{R}(\mathcal{C}, \mathcal{D})$ is completely described by Ψ_{12}. The remaining data is provided by objects F and G.

We have claimed that $\mathcal{R}(\mathcal{C},\mathcal{D})$ is a $\Bbbk$-linear category and will now describe the $\Bbbk$-module structure of the hom-sets. Given morphisms $\Psi,\Theta\colon F\to G$ from $\mathcal{R}(\mathcal{C},\mathcal{D})$, let $(\Psi+\Theta)\colon F\to G$ be the morphism where

$$(\Psi+\Theta)_{12}(g)=\Psi_{12}(g)+\Theta_{12}(g)$$

for any degree 1 morphism $g\in\overline{\mathcal{C}}(x,y)$. This is well-defined because $\Psi_{12}(g)$ and $\Theta_{12}(g)$ are morphisms in $\mathcal{D}(F(x),G(y))$ which is a $\Bbbk$-module. Also, given an element $c\in\Bbbk$, let $c\Psi$ be the morphism where $(c\Psi)_{12}(g)=c\Psi_{12}(g)$.

Composition in $\mathcal{R}(\mathcal{C},\mathcal{D})$ is not a trivial matter. Consider a diagram

$$F\xrightarrow{\Theta}G\xrightarrow{\Psi}H$$

in $\mathcal{R}(\mathcal{C},\mathcal{D})$. To describe the morphism $\Psi\Theta$, we first recall Proposition 6.1.4 which says that, because $\overline{\mathcal{C}}$ is semi-free, any morphism of degree 2 is a composition of degree 1 morphisms. In particular, if g is a $\overline{\mathcal{C}}$-morphism of degree 1, then its differential $d_{\overline{\mathcal{C}}}(g)$ has a degree of 2. Thus,

$$d_{\overline{\mathcal{C}}}(g)=\sum_{i=1}^{m}g_ig_i',$$

for some $\overline{\mathcal{C}}$-morphisms g_i and g_i' of degree 1. Hence, $\Psi_{12}(g_i)$ and $\Theta_{12}(g_i')$ are defined for $i=1,\ldots,m$. Define the composition $\Psi\Theta$ to be the morphism where

$$(\Psi\Theta)_{12}(g)=\sum_{i=1}^{m}\Psi_{12}(g_i)\Theta_{12}(g_i') \tag{6.1.4}$$

for all $\overline{\mathcal{C}}$-morphisms g with degree 1.

We now define the identity morphisms $\Upsilon_F\colon F\to F$ in $\mathcal{R}(\mathcal{C},\mathcal{D})$. For all of the $\overline{\mathcal{C}}$-endomorphisms e_x of degree 1, let

$$(\Upsilon_F)_{12}(e_x)=1_{F(x)}.$$

This extends, by composition of $\overline{\mathcal{D}}$-morphisms, to all $\overline{\mathcal{C}}$-morphisms with degree 1 that factor through any of the endomorphisms e_x. Otherwise, we let

$$(\Upsilon_F)_{12}(g)=0$$

if g is a $\overline{\mathcal{C}}$-morphism of degree 1 that does not factor through any e_x.

The following proposition is stated without proof in [28, Prop. 1].

Proposition 6.1.15. *If $\mathcal{C}$ is a semi-free, differential graded category and $\mathcal{D}$ is a $\Bbbk$-linear category, then $\mathcal{R}(\mathcal{C},\mathcal{D})$ is a category.*

Proof: The following proof will take place in three parts. We show that

(a) composition is well-defined,

(b) Υ_F is an identity morphism for every object F in $\mathcal{R}(\mathcal{C},\mathcal{D})$, and

(c) composition of morphisms is associative.

Part (a)

Given composable morphisms Ψ and Θ of $\mathcal{R}(\mathcal{C},\mathcal{D})$, we must show that the composition $\Psi\Theta\colon \overline{\mathcal{C}} \to \overline{\mathcal{D}}$ is a DG functor. It is clear that $\Psi\Theta$ respects the grading and so it suffices to show that $\Psi\Theta(d_{\overline{\mathcal{C}}}(f)) = d_{\overline{\mathcal{D}}}(\Psi\Theta(f))$ for any $\overline{\mathcal{C}}$-morphism f. But $d_{\overline{\mathcal{D}}}$ is trivial and so we may simply show that $\Psi\Theta(d_{\overline{\mathcal{C}}}(f)) = 0$.

First, suppose that $\deg f \geq 1$. Then $\deg d_{\overline{\mathcal{C}}}(f) \geq 2$ which implies that $\Psi\Theta(d_{\overline{\mathcal{C}}}(f)) = 0$. Now, suppose that $\deg f = 0$. Then

$$\Psi\Theta\left(d_{\overline{\mathcal{C}}}(f)\right) = \begin{pmatrix} 0 & (\Psi\Theta)_{12}\left(d_{\overline{\mathcal{C}}}(f)\right) \\ 0 & 0 \end{pmatrix}.$$

But $d^2_{\overline{\mathcal{C}}}(f) = 0$ and so it follows from the definition of composition in $\mathcal{R}(\mathcal{C},\mathcal{D})$ that $(\Psi\Theta)_{12}\left(d_{\overline{\mathcal{C}}}(f)\right) = \Psi_{12}(0)\Theta_{12}(0) = 0$.

Part (b)

Consider the morphisms $\Psi\colon F \to G$ and $\Upsilon_F\colon F \to F$ of $\mathcal{R}(\mathcal{C},\mathcal{D})$. To show that Υ_F is a right identity, it suffices to show that $(\Psi\Upsilon_F)_{12}(g) = \Psi_{12}(g)$ for any $\overline{\mathcal{C}}$-morphism $g\colon x \to y$ with degree 1. First, suppose that g does not factor through e_x for any $x \in \operatorname{Ob}\left(\overline{\mathcal{C}}\right)$. We have that

$$d_{\overline{\mathcal{C}}}(g) = \sum_{i=1}^{n} g_i g_i' + g e_x + e_y g$$

for some $\overline{\mathcal{C}}$-morphisms g_i and g_i' with degree 1 and such that $g_i \neq e_y$ and $g_i' \neq e_x$. But the equality

$$\begin{aligned}(\Psi\Upsilon_F)_{12}(g) &= \sum_{i=1}^{n} \Psi_{12}(g_i)\Upsilon_{F12}(g_i') + \Psi_{12}(g)\Upsilon_{F12}(e_x) + \Psi_{12}(e_y)\Upsilon_{F12}(g)\\ &= \sum_{i=1}^{n} \Psi_{12}(g_i)(0) + \Psi_{12}(g)1_{F(x)} + \Psi_{12}(e_y)(0)\\ &= \Psi_{12}(g)\end{aligned}$$

follows from Equation (6.1.4). Now, suppose that $x = y$ and that $g = e_x$. Then $d_{\overline{\mathcal{C}}}(e_x) = e_x^2$. Hence, the equality

$$(\Psi\Upsilon_F)_{12}(e_x) = \Psi_{12}(e_x)\Upsilon_{F12}(e_x) = \Psi_{12}(e_x)1_{F(x)} = \Psi_{12}(e_x)$$

holds. This extends, by composition of $\overline{\mathcal{D}}$-morphisms, to any $\overline{\mathcal{C}}$-morphism of degree 1 that factors through e_x. Thus Υ_F is a right identity. A similar argument shows that Υ_F is also a left identity.

Part (c)

It suffices to check that the associativity of morphism composition holds for degree 1 morphisms. Let f be an arbitrary $\overline{\mathcal{C}}$-morphism with $\deg f = 1$. Then

$$d_{\overline{\mathcal{C}}}(f) = \sum_{i=1}^{n} f_i f_i'$$

for some $\overline{\mathcal{C}}$-morphisms f_i and f_i' of degree 1. For $i = 1, \ldots, n$, we also have that

$$d_{\overline{\mathcal{C}}}(f_i) = \sum_{j=1}^{n_0} g_{ij} g_{ij}' \text{ and } d_{\overline{\mathcal{C}}}(f_i') = \sum_{j=1}^{n_1} h_{ij} h_{ij}'$$

for $\overline{\mathcal{C}}$-morphisms g_{ij}, g_{ij}', h_{ij} and h_{ij}', each with degree 1. To simplify the indexing, let $t = \max\{n, n_0, n_1\}$ and write

$$d_{\overline{\mathcal{C}}}(f) = \sum_{i=1}^{t} f_i f_i', \; d_{\overline{\mathcal{C}}}(f_i) = \sum_{j=1}^{t} g_{ij} g_{ij}', \text{ and } d_{\overline{\mathcal{C}}}(f_i') = \sum_{j=1}^{t} h_{ij} h_{ij}',$$

by letting $f_i f_i' = 0$ for $i > n$, $g_{ij} g_{ij}' = 0$ for $j > n_0$ and $h_{ij} h_{ij}' = 0$ for $j > n_1$. Note that

$$\begin{aligned} d_{\overline{\mathcal{C}}}^2(f) &= \sum_{i=1}^{t} d_{\overline{\mathcal{C}}}(f_i f_i') \\ &= \sum_{i=1}^{t} f_i d_{\overline{\mathcal{C}}}(f_i') - d_{\overline{\mathcal{C}}}(f_i) f_i' \\ &= \sum_{i,j=1}^{t} f_i g_{ij} g_{ij}' - h_{ij} h_{ij}' f_i' \\ &= 0. \end{aligned}$$

Therefore,

$$\sum_{i,j=1}^{t} f_i g_{ij} g_{ij}' = \sum_{i,j=1}^{t} h_{ij} h_{ij}' f_i'. \tag{6.1.5}$$

Because $\overline{\mathcal{C}}$ was freely generated, there are no superfluous relations in the collection of its morphisms. Hence, if we let $T = \{1, 2, \ldots, t\}$, there is a set bijection

$$\theta \colon T \times T \to T \times T$$

such that the $(i,j)^{\text{th}}$ term of the left hand side of (6.1.5) equals the $\theta(i,j)^{\text{th}}$ term of the right hand side for all $(i,j) \in T \times T$. For each (i,j) such that $f_i g_{ij} g_{ij}'$ is composed of non-zero morphisms, it follows from the equality $f_i g_{ij} g_{ij}' = h_{k\ell} h_{k\ell}' f_k'$, where $\theta(i,j) = (k,\ell)$, and from the lack of relations on the $\overline{\mathcal{C}}$-morphisms that $f_i = h_{k\ell}$, $g_{i,j} = h_{k\ell}'$ and $g_{ij}' = f_k'$. Hence, for any composable morphisms Ψ, Θ and Φ in $\mathcal{R}(\mathcal{C}, \mathcal{D})$, we have the equality

$$(\Psi\Theta)\Phi(f) = \sum_{i,j=1}^{t} \Psi(g_{ij})\Theta(g_{ij}')\Phi(f_i') = \sum_{i,j=1}^{t} \Psi(f_i)\Theta(h_{ij})\Phi(h_{ij}') = \Psi(\Theta\Phi)(f)$$

and so associativity holds. ∎

Therefore, $\mathcal{R}(\mathcal{C}, \mathcal{D})$ is a category. Using this category, we present a definition of a DGC matrix problem in a similar framework as used for a linear matrix problem.

Definition 6.1.16 (DGC Matrix Problem)**.** Let $\mathcal{C}$ be the graded path category of some bigraph and let $\mathcal{D}$ be a $\Bbbk$-linear category. A *DGC Matrix Problem* is a pair

$$(\mathcal{R}(\mathcal{C}, \mathcal{D}), \sim_{\mathcal{R}})$$

where, for any $F, G \in \mathrm{Ob}(\mathcal{R}(\mathcal{C}, \mathcal{D}))$, we have $F \sim_{\mathcal{R}} G$ if and only $F \cong G$ in $\mathcal{R}(\mathcal{C}, \mathcal{D})$.

Traditionally, $\mathcal{D}$ is taken to be $\Bbbk$-**mod**.

6.2 Bocs Matrix Problems

We now change focus from differential graded categories to the closely related notion of a bocs.

Definition 6.2.1 (Bimodule over a Category)**.** Let $\mathcal{C}_0$ and $\mathcal{C}_1$ be $\Bbbk$-linear categories. A left $\mathcal{C}_0$-*module* M_{L} is a $\Bbbk$-linear functor

$$M_{\mathrm{L}} \colon \mathcal{C}_0 \to \Bbbk\text{-}\mathbf{mod}.$$

A right $\mathcal{C}_0$-*module* M_{R} is a $\Bbbk$-linear functor

$$M_{\mathrm{R}} \colon \mathcal{C}_0^{\mathrm{op}} \to \Bbbk\text{-}\mathbf{mod}.$$

A $(\mathcal{C}_0, \mathcal{C}_1)$-*bimodule* M is a $\Bbbk$-bilinear functor

$$M \colon \mathcal{C}_1^{\mathrm{op}} \times \mathcal{C}_0 \to \Bbbk\text{-}\mathbf{mod}.$$

By the *elements* of a $(\mathcal{C}_0, \mathcal{C}_1)$-bimodule M, we mean every $m \in M(x, y)$ for all pairs $(x, y) \in \mathrm{Ob}(\mathcal{C}_1^{\mathrm{op}} \times \mathcal{C}_0)$. The left action from $\mathcal{C}_0$ and the right action from $\mathcal{C}_1$ is defined, for every $x, x' \in \mathrm{Ob}(\mathcal{C}_1)$ and $y, y' \in \mathrm{Ob}(\mathcal{C}_0)$, to be

$$\mathcal{C}_0(y, y') \times M(x, y) \times \mathcal{C}_1(x', x) \to M, \quad (g, m, f) \mapsto gmf, \tag{6.2.1}$$

where $gmf = M(f^{\mathrm{op}}, g)(m) \in M(x', y')$ for a morphism $(f^{\mathrm{op}}, g) \colon (x, y) \to (x', y')$ of $\mathcal{C}_1^{\mathrm{op}} \times \mathcal{C}_0$.

Example 6.2.2. Let $\mathcal{C}$ be a $\Bbbk$-linear category and x, y be $\mathcal{C}$-objects. The functors

$$\mathcal{C}(x,-)\colon \mathcal{C} \to \Bbbk\text{-}\mathbf{mod},\ \mathcal{C}(-,y)\colon \mathcal{C}^{\mathrm{op}} \to \Bbbk\text{-}\mathbf{mod},\ \text{and } \mathcal{C}(-,-)\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \Bbbk\text{-}\mathbf{mod}$$

are, respectively, a left $\mathcal{C}$-module, a right $\mathcal{C}$-module and a $\mathcal{C}$-bimodule. A module with any of these forms is called a *principal $\mathcal{C}$-(bi)module.*

Definition 6.2.3 (Bimodule Morphism)**.** Let M and N be $(\mathcal{C}_0, \mathcal{C}_1)$-bimodules. A *morphism of $(\mathcal{C}_0, \mathcal{C}_1)$-bimodules* is a natural transformation from M to N.

A $(\mathcal{C}_0, \mathcal{C}_1)$-bimodule morphism resembles a morphism of bimodules over a ring. In particular, if $\theta\colon M \to N$ is a morphism of $(\mathcal{C}_0, \mathcal{C}_1)$-bimodules, then there is a $\Bbbk$-linear mapping $\theta_{x,y}\colon M(x,y) \to N(x,y)$ for all objects (x,y) of $\mathcal{C}_1^{\mathrm{op}} \times \mathcal{C}_0$ such that, for any morphism $(f^{\mathrm{op}}, g)\colon (x,y) \to (x',y')$ in $\mathcal{C}_1^{\mathrm{op}} \times \mathcal{C}_0$, the diagram

$$\begin{array}{ccc} M(x,y) & \xrightarrow{M(f^{\mathrm{op}},g)} & M(x',y') \\ {\scriptstyle \theta_{x,y}}\downarrow & & \downarrow{\scriptstyle \theta_{x',y'}} \\ N(x,y) & \xrightarrow[N(f^{\mathrm{op}},g)]{} & N(x',y'). \end{array}$$

commutes. Thus, for any element $m \in M(x,y)$, we have that

$$\theta_{x',y'} M(f^{\mathrm{op}}, g)(m) = N(f^{\mathrm{op}}, g)\theta_{x,y}(m).$$

But using the notation we set in Equation (6.2.1), we get that $\theta_{x',y'}(gmf) = g\theta_{x,y}(m)f$.

Another similarity to a bimodule over a ring is that we can use $\otimes$ or hom to obtain a new bimodule from two suitable bimodules. Let M be a $(\mathcal{C}_0, \mathcal{C}_1)$-bimodule and N be a $(\mathcal{C}_1, \mathcal{C}_2)$-bimodule. The *tensor product* $M \otimes_{\mathcal{C}_1} N$ is the $(\mathcal{C}_0, \mathcal{C}_2)$-bimodule

$$M \otimes_{\mathcal{C}_1} N\colon \mathcal{C}_2^{\mathrm{op}} \times \mathcal{C}_0 \to \Bbbk\text{-}\mathbf{mod}$$

such that, for each $(x,y) \in \mathcal{C}_2^{\mathrm{op}} \times \mathcal{C}_0$, the $\Bbbk$-module $(M \otimes_{\mathcal{C}_1} N)(x,y)$ is the factor space

$$\bigoplus_{z \in \mathcal{C}_1} M(z,y) \otimes_{\Bbbk} N(x,z) / \langle mc \otimes n - m \otimes cn \rangle,$$

for $m \in M(z,y)$, $n \in N(x,z)$ and $c \in \mathcal{C}_1(z,z)$. The principal $\mathcal{C}_1$-bimodule $\mathcal{C}_1(-,-)$ serves as a unit for the tensor.

Now, let P be a $(\mathcal{C}_2,\mathcal{C}_1)$-bimodule. The $(\mathcal{C}_2,\mathcal{C}_0)$-bimodule

$$\hom_{\mathcal{C}_1}(M,P)\colon \mathcal{C}_0^{\mathrm{op}} \times \mathcal{C}_2 \to \Bbbk\textbf{-mod}$$

has, for every $(x,y) \in \mathcal{C}_0^{\mathrm{op}} \times \mathcal{C}_2$, the value

$$\hom_{\mathcal{C}_1}(M,P)(x,y)$$

which is the space of $\mathcal{C}_1$-linear maps $f\colon M(-,x) \to P(y,-)$. We can, similarly, construct a $(\mathcal{C}_0,\mathcal{C}_2)$-bimodule $\hom_{\mathcal{C}_1}(M,P')$ using M and a $(\mathcal{C}_0,\mathcal{C}_2)$-bimodule P'.

Given $\Bbbk$-linear categories $\mathcal{C}_0$ and $\mathcal{C}_1$, denote by F_{xy} the $(\mathcal{C}_0,\mathcal{C}_1)$-bimodule

$$\mathcal{C}_1(-,x) \otimes_{\Bbbk} \mathcal{C}_0(y,-)\colon \mathcal{C}_1^{\mathrm{op}} \times \mathcal{C}_0 \to \Bbbk\textbf{-mod}. \tag{6.2.2}$$

Note that $1_x \otimes_{\Bbbk} 1_y$ generates F_{xy} in the usual sense of bimodule generators.

Definition 6.2.4 (Free Bimodule)**.** A $(\mathcal{C}_0,\mathcal{C}_1)$-bimodule M is *free* if

$$M \cong \bigoplus_{i=1}^{n} F_{x_i y_i},\ x_i \in \mathcal{C}_1, y_i \in \mathcal{C}_0,$$

where $F_{x_i y_i}$ is described in Equation (6.2.2). Also, M is said to be *freely generated* by the elements $m_1,\dots,m_n$, with $m_i \in M(x_i,y_i)$, if the mapping

$$\bigoplus_{i=1}^{n} F_{x_i y_i} \to M, \quad 1_{x_i} \otimes_{\Bbbk} 1_{y_i} \mapsto m_i$$

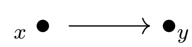
is an isomorphism.

There is a natural way to represent a finite generating system of a $\mathcal{C}$-bimodule with a quiver. Suppose that a $\mathcal{C}$-bimodule M is finitely generated by elements $m_1,\dots,m_n$. Then for $i = 1,\dots,n$, we have that $m_i \in M(x_i,y_i)$ for some $\mathcal{C}$-objects x_i and y_i. Associate to M, the quiver Q_M with nodes $\mathrm{Ob}(\mathcal{C})$ and with arrows $x_i \xrightarrow{\alpha_i} y_i$ for $i = 1,\dots,n$.

We are particularly interested in $\mathcal{C}$-bimodules when $\mathcal{C}$ is a graded path category generated by a quiver $Q_{\mathcal{C}}$ which, for the sake of following Definition 6.1.11, can be considered as a graded bigraph with only degree 0 arrows. Observe that $(Q_{\mathcal{C}})_N = \mathrm{Ob}(\mathcal{C})$. Then by superimposing Q_M and Q_C, we can obtain a bigraph $B_{M,\mathcal{C}}$ that we will associate to the free bimodule M over the graded path category $\mathcal{C}$. More precisely, $B_{M,\mathcal{C}}$ is the bigraph where $(B_{M,\mathcal{C}})_N = \mathrm{Ob}(\mathcal{C})$ is the set of nodes, $(B_{M,\mathcal{C}})_{A_0} = (Q_{\mathcal{C}})_A$ is the set of degree 0 arrows, and $(B_{M,\mathcal{C}})_{A_1} = (Q_M)_A$ is the set of degree 1 arrows.

When depicting such a bigraph, dashed arrows will have degree 0 and solid arrows will have degree 1. The following example shows how to obtain a bigraph from a free bimodule over a graded path category.

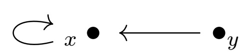
Example 6.2.5. Suppose that $\mathcal{C}$ is the $\Bbbk$-linear category generated by the diagram



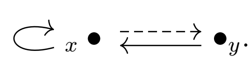
and that M is a free $\mathcal{C}$-bimodule and is isomorphic to the $\mathcal{C}$-bimodule

$$\mathcal{C}(-,x) \otimes_{\Bbbk} \mathcal{C}(x,-) \oplus \mathcal{C}(-,y) \otimes_{\Bbbk} \mathcal{C}(x,-).$$

Then Q_M is the quiver



and $B_{M,\mathcal{C}}$ is the bigraph



The elements of M are the combined elements of the $\Bbbk$-modules $M(x,x)$, $M(x,y)$, $M(y,x)$ and $M(y,y)$, where

$$M(x,x) \cong \mathcal{C}(x,x) \otimes_{\Bbbk} \mathcal{C}(x,x) \oplus \mathcal{C}(x,y) \otimes_{\Bbbk} \mathcal{C}(x,x) \cong \Bbbk \oplus \Bbbk,$$

$$M(x,y) \cong \mathcal{C}(x,x) \otimes_{\Bbbk} \mathcal{C}(x,y) \oplus \mathcal{C}(x,y) \otimes_{\Bbbk} \mathcal{C}(x,y) \cong \Bbbk \oplus \Bbbk,$$

$$M(y,x) \cong \mathcal{C}(y,x) \otimes_{\Bbbk} \mathcal{C}(x,x) \oplus \mathcal{C}(y,y) \otimes_{\Bbbk} \mathcal{C}(x,x) \cong \Bbbk, \text{ and}$$

$$M(y,y) \cong \mathcal{C}(y,x) \otimes_{\Bbbk} \mathcal{C}(x,y) \oplus \mathcal{C}(y,y) \otimes_{\Bbbk} \mathcal{C}(x,y) \cong \Bbbk.$$

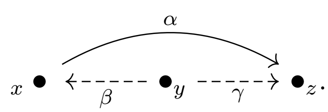
There is also a natural way to associate a free $\mathcal{C}$-bimodule M to a given bigraph $B = (B_N, B_{A_0}, B_{A_1})$. Grade the bigraph B by letting the arrows of B_{A_0} have degree 0 and the arrows of B_{A_1} have degree 1. Let $\mathcal{C}$ be the graded path category (with the trivial grading) generated by the paths of degree 0. Consider all of the degree 1 arrows $x_i \xrightarrow{\alpha_i} y_i$, for $i = 1, \ldots, n$. Then let M be the $\mathcal{C}$-bimodule

$$\bigoplus_{i=1}^{n} \mathcal{C}(-, x_i) \otimes_{\mathbb{k}} \mathcal{C}(y_i, -) \colon \mathcal{C}^{\text{op}} \times \mathcal{C} \to \mathbb{k}\textbf{-mod}.$$

We say that M is *generated* by the bigraph B.

The following example illustrates how to obtain a free bimodule over a graded path category from a bigraph.

Example 6.2.6. Consider the bigraph



Then $\mathcal{C}$ is the graded path category generated from the dashed (degree 0) arrows. Thus $\mathcal{C}$ has three objects and the non-trivial hom-sets are

$$\mathcal{C}(x, x) \cong \mathcal{C}(y, y) \cong \mathcal{C}(z, z) \cong \mathcal{C}(y, x) \cong \mathcal{C}(y, z) \cong \mathbb{k}.$$

Then M, freely generated by the solid (degree 1) arrows, consists of non-trivial $\mathbb{k}$-modules

$$M(x, z) \cong M(y, z) \cong \mathbb{k}.$$

Evidently, these spaces describe the functor

$$\mathcal{C}(-, x) \otimes \mathcal{C}(z, -) \colon \mathcal{C}^{\text{op}} \times \mathcal{C} \to \mathbb{k}\textbf{-mod}.$$

Hence $M \cong \mathcal{C}(-, x) \otimes \mathcal{C}(z, -)$.

Definition 6.2.7 (Co-algebra over a Category)**.** Let $\mathcal{C}$ be a $\mathbb{k}$-linear category. A *co-algebra over* $\mathcal{C}$ consists of a $\mathcal{C}$-bimodule M together with $\mathcal{C}$-linear mappings

- $\Delta\colon M \to M \otimes_{\mathcal{C}} M$ (comultiplication), and
- $\varepsilon\colon M \to \mathcal{C}$ (co-unit)

such that the diagrams

$$\begin{array}{ccc} M & \xrightarrow{\Delta} & M \otimes_{\mathcal{C}} M \\ {\scriptstyle\Delta}\downarrow & & \downarrow{\scriptstyle 1\otimes_{\mathcal{C}}\Delta} \\ M \otimes_{\mathcal{C}} M & \xrightarrow{\Delta\otimes_{\mathcal{C}}1} & M \otimes_{\mathcal{C}} M \otimes_{\mathcal{C}} M \end{array} \qquad \begin{array}{ccccc} M \otimes_{\mathcal{C}} M & \xleftarrow{\Delta} & M & \xrightarrow{\Delta} & M \otimes_{\mathcal{C}} M \\ {\scriptstyle\varepsilon\otimes_{\mathcal{C}}1}\downarrow & & \| & & \downarrow{\scriptstyle 1\otimes_{\mathcal{C}}\varepsilon} \\ \mathcal{C} \otimes_{\mathcal{C}} M & \xleftarrow{\cong} & M & \xrightarrow{\cong} & M \otimes_{\mathcal{C}} \mathcal{C} \end{array}$$

commute. The left diagram depicts the *co-associativity* axiom, and the right diagram depicts the *co-unity* axiom.

Definition 6.2.8 (Bocs)**.** A *bimodule over a category with co-algebra structure* or a *bocs* $\mathscr{B} = (\mathcal{C}, M)$ is a pair consisting of a $\Bbbk$-linear category $\mathcal{C}$ and a $\mathcal{C}$-co-algebra M. The kernel $\ker \mathscr{B}$ of the bocs $\mathscr{B}$ is the kernel $\ker \varepsilon$ of the co-unit.

Let $\mathscr{B} = (\mathcal{C}, M)$ and $\mathscr{C} = (\mathcal{C}', M')$ be two bocses. A morphism

$$(\theta_1, \theta_2)\colon \mathscr{B} \to \mathscr{C}$$

of bocses is a pair consisting of

- a $\Bbbk$-linear functor $\theta_1\colon \mathcal{C} \to \mathcal{C}'$, and
- a $\mathcal{C}$-bimodule morphism $\theta_2\colon M \to M'$ where M' is considered as a $\mathcal{C}$-bimodule via the functor θ_1,

such that the diagrams

$$\begin{array}{ccc} M & \xrightarrow{\varepsilon} & \mathcal{C} \\ {\scriptstyle\theta_2}\downarrow & & \downarrow{\scriptstyle\theta_1} \\ M' & \xrightarrow{\varepsilon'} & \mathcal{C}' \end{array} \qquad \begin{array}{ccccc} M & \xrightarrow{\Delta} & M \otimes_{\mathcal{C}} M & \xrightarrow{\theta_2\otimes\theta_2} & M' \otimes_{\mathcal{C}} M' \\ {\scriptstyle\theta_2}\downarrow & & & & \downarrow{\scriptstyle\pi} \\ M' & & \xrightarrow{\qquad\Delta'\qquad} & & M' \otimes_{\mathcal{C}'} M' \end{array}$$

where π is the natural map, commute. Composition is defined by letting

$$(\theta_1, \theta_2) \circ (\theta_1', \theta_2') = (\theta_1 \circ \theta_1', \theta_2 \circ \theta_2')$$

and associativity follows from the associativity of functors and bimodule morphisms. Thus, we have a category **bocs** containing bocses and their morphisms.

We now introduce an important class of bocses which, as we will show in Section 6.3, naturally corresponds to the class of semi-free DGCs.

Definition 6.2.9 (Grouplike Bocs)**.** Let $\mathscr{B} = (\mathcal{C}, M)$ be a bocs. A *grouplike element* of $\mathscr{B}$ is an element $u_x \in M(x,x)$, for any $x \in \mathrm{Ob}(\mathcal{C})$, such that $\Delta(u_x) = u_x \otimes_{\mathcal{C}} u_x$. If there exists a grouplike element $u_x \in M(x,x)$ for every $x \in \mathrm{Ob}(\mathcal{C})$, then we say that the bocs $\mathscr{B}$ is *grouplike*.

The term *normal bocs* is used by Roĭter in [27, p. 303], but later papers have adopted the term grouplike from the study of co-algebras. We will keep with this convention. For more on grouplike elements and co-algebras, interested readers should consult [31].

We should also point out that the definition of a bocs in [27, p. 301] requires that the co-unit be surjective. However, the author is not aware of any subsequent definitions of a bocs appearing in the literature which requires a surjective co-unit and so we do not include it. However, it is easy to see that ε is surjective if, for every $\mathcal{C}$-object x, the identity morphism 1_x is in the image of ε. In particular, this holds for a grouplike bocs because the co-unit property implies that $\varepsilon(u_x) = 1_x$ for every grouplike element u_x.

We will now show how to generate a grouplike bocs with a graded bigraph whose arrows are of degree 0 and 1. We have already seen how a bigraph can generate both a graded path category $\mathcal{D}$ and a bimodule M over a category $\mathcal{C}$. It turns out that $\mathcal{C} = \mathcal{D}_0$, where $\mathcal{D}_0$ is the subcategory consisting of the degree 0 morphisms of $\mathcal{D}$. Also, for all objects x and y, we have that $M(x,y) = \mathcal{D}(x,y)_1$ where $\mathcal{D}(x,y)_1$ consists of the $\mathcal{D}$-morphisms from x to y with degree 1. Recall that there exists a differential which turns $\mathcal{D}$ into a DGC. Any such differential is defined on the elements of M. Therefore, we can assign a differential to any bimodule over a category that is generated from a bigraph.

Given any bigraph B, let B^* be the bigraph obtained by augmenting B by adding a degree 1 loop to every node. We denote the loop introduced onto the node x by u_x. Then B^* generates a bimodule M over a category $\mathcal{C}$. For the pair $(\mathcal{C}, M)$ to be a bocs, we must place a co-algebra structure onto M. Define the co-unit ε on the generators (that is, arrows of B^*) by letting

$$\varepsilon(u_x) = 1_x$$

for every degree 1 loop u_x, and letting

$$\varepsilon(m) = 0$$

for all other generators. Then extend ε to the rest of M by $\mathcal{C}$-linearity. Thus, we think of the initial bigraph B of generating the kernel of the bocs. Let d be any differential associated to the $\mathcal{C}$-bimodule generated by B. Define a comultiplication Δ on the generators of M by letting

$$\Delta(u_x) = u_x \otimes_{\mathcal{C}} u_x$$

for all degree 1 loops u_x, and letting

$$\Delta(m) = d(m) + m \otimes_{\mathcal{C}} u_x + u_y \otimes_{\mathcal{C}} m$$

for all $m \in \ker \varepsilon$ where $m \in M(x, y)$. Extend Δ to the rest of M by $\mathcal{C}$-linearity. It is clear that M is a co-algebra over $\mathcal{C}$ with grouplike elements u_x, and so $(\mathcal{C}, M)$ is a grouplike bocs. We say that $(\mathcal{C}, M)$ is generated by the pair (B, d).

Definition 6.2.10 (Category of Representations of a Bocs)**.** Let $\mathscr{B} = (\mathcal{C}, M)$ be a bocs. Denote by $\mathscr{B}$-**rep** the category whose objects are the left $\mathcal{C}$-modules

$$F \colon \mathcal{C} \to \Bbbk\text{-}\mathbf{mod}$$

and whose hom-sets $\mathscr{B}$-**rep**(F, G), for $F, G \in \mathrm{Ob}(\mathscr{B}\text{-}\mathbf{rep})$, consist of the $\mathcal{C}$-bimodule morphisms

$$f \colon M \otimes_{\mathcal{C}} F \to G.$$

The composition of morphisms $f\colon F \to G$ and $g\colon G \to H$ is defined to be the $\mathcal{C}$-bimodule morphism

$$M \otimes_{\mathcal{C}} F \xrightarrow{\Delta \otimes 1} M \otimes_{\mathcal{C}} M \otimes_{\mathcal{C}} F \xrightarrow{1 \otimes f} M \otimes_{\mathcal{C}} G \xrightarrow{g} H.$$

The identity morphism on an object F is the mapping

$$M \otimes_{\mathcal{C}} F \xrightarrow{\varepsilon \otimes 1} \mathcal{C} \otimes_{\mathcal{C}} F \xrightarrow{\cong} F.$$

Associativity follows from a simple calculation. Call $\mathscr{B}$-**rep** the *category of representations* of $\mathscr{B} = (\mathcal{C}, M)$.

Example 6.2.11. A *principal bocs* $\mathscr{B}$ is a pair $(\mathcal{C}, \mathcal{C})$ for any $\Bbbk$-linear category $\mathcal{C}$. Then there is a co-algebra structure on the $\mathcal{C}$-bimodule

$$\mathcal{C}(-,-)\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \Bbbk\textbf{-mod}$$

by defining the comultiplication $\Delta\colon \mathcal{C} \to \mathcal{C} \otimes_{\mathcal{C}} \mathcal{C}$ to be an isomorphism and the co-unit $\varepsilon\colon \mathcal{C} \to \mathcal{C}$ to be the identity. Then the category of representations $\mathscr{B}$-**rep** of the bocs $\mathscr{B} = (\mathcal{C}, \mathcal{C})$ is exactly the category $\mathcal{C}$-**mod** of left $\mathcal{C}$-modules.

Definition 6.2.12 (Bocs Matrix Problem)**.** A *bocs matrix problem* is a pair

$$(\mathscr{B}\textbf{-rep}, \sim_{\mathscr{B}})$$

where $\mathscr{B}$ is a bocs and $\sim_{\mathscr{B}}$ is the equivalence relation on $\mathrm{Ob}(\mathscr{B}\textbf{-rep})$ such that, for all $x, y \in \mathrm{Ob}(\mathscr{B}\textbf{-rep})$, we have that $x \sim_{\mathscr{B}} y$ if and only if $x \cong y$ in $\mathscr{B}$-**rep**.

6.3 Correspondence Between DGCs and Bocses

In this section, we describe the relationship between DGCs and bocses. In particular, we are interested in the natural correspondence noted in [27, Prop. 2] between semi-free DGCs and grouplike bocses. In that paper, Roĭter provides the details regarding the association of a semi-free DGC to a given grouplike bocs and, therefore, we will only sketch this here. However, the reverse association is omitted from the paper and so we will provide a detailed account of the process of associating a grouplike bocs to a semi-free DGC.

6.3.1 A DGC Constructed from a Bocs

Let $\mathscr{B} = (\mathcal{C}, M)$ be a grouplike bocs that is generated by a pair (B, d) for some bigraph B and differential d. Let $K = \ker \mathscr{B}$ and denote the grouplike elements by u_x.

Denote by $T(K)$ the category with objects $\mathrm{Ob}(\mathcal{C})$ and morphism spaces

$$T(K)(x,y) = \bigoplus_{i \in \mathbb{N}} T(K)(x,y)_i \text{ for } x, y \in \mathrm{Ob}(T(K)), \tag{6.3.1}$$

where

$$\begin{aligned}
&T(K)(x,y)_0 = \mathcal{C}(x,y),\\
&T(K)(x,y)_1 = K(x,y), \text{ and}\\
&T(K)(x,y)_n = (K \otimes_{\mathcal{C}} \cdots \otimes_{\mathcal{C}} K)(x,y) \quad (n \text{ terms}).
\end{aligned}$$

This category is called the *graded tensor category* of K.

Let us introduce an operator δ so that for every pair of $T(K)$-objects x and y, there is a graded $\mathbb{k}$-linear map

$$\delta \colon T(K)(x,y) \to T(K)(x,y) \tag{6.3.2}$$

of degree 1. For any $f \in T(K)(x,y)_0 = \mathcal{C}(x,y)$, define

$$\delta(f) = f u_x - u_y f. \tag{6.3.3}$$

Now, for any $m \in T(K)(x,y)_1 = K(x,y)$, define

$$\delta(m) = \Delta(m) - m \otimes_{\mathcal{C}} u_x - u_y \otimes_{\mathcal{C}} m, \tag{6.3.4}$$

where Δ is the comultiplication map of the co-algebra M. Extend δ to higher degrees of $T(K)$ by the Leibniz formula. To ensure that δ is well-defined we provide the following lemmas.

Lemma 6.3.1 ([27, Lem. 1])**.** *We have that* $\delta(f) \in K$ *(see Equation* (6.3.3)*) and* $\delta(m) \in K \otimes_{\mathcal{C}} K$ *(see Equation* (6.3.4)*).*

Then next proposition ensures that δ is a differential, and that we have, in fact, associated a semi-free DGC to a grouplike bocs.

Proposition 6.3.2 ([27, Prop. 2])**.** *The operator* δ *described in* (6.3.2) *is a differential and* $(T(K), \delta)$ *is a semi-free DGC.*

6.3.2 A Bocs Constructed from a DGC

In this section, we complete the correspondence by associating a grouplike bocs to a given semi-free DGC. As mentioned in the introduction to this section, we will provide details of this association that were omitted in [27, Prop. 2].

Let $\mathcal{D} = \bigoplus_{i \in \mathbb{N}} \mathcal{D}_i$ be a semi-free DGC with differential δ. Recall the construction described in the first step of Section 6.1.3. By performing this construction using $\mathcal{D}$, we obtain the semi-free DGC $\overline{\mathcal{D}}$ with differential $\overline{\delta}$. Denote by e_x the endomorphisms of degree 1 that were introduced in this construction. Let $\mathcal{C}$ be the category with $\mathrm{Ob}(\mathcal{C}) = \mathrm{Ob}(\mathcal{D})$ and hom-sets $\mathcal{C}(x,y) = \overline{\mathcal{D}}(x,y)_0$ where $\overline{\mathcal{D}}(x,y)_0$ consists of degree 0 morphisms. Now, define the $\mathcal{C}$-bimodule $M\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \Bbbk\text{-}\mathbf{mod}$ by letting $M(x,y) = \overline{\mathcal{D}}(x,y)_1$ for all $x, y \in \mathrm{Ob}(\mathcal{C})$, where $\overline{D}(x,y)_1$ consists of degree 1 morphisms. Define a $\mathcal{C}$-bimodule morphism

$$\varepsilon\colon M \to \mathcal{C}, \tag{6.3.5}$$

by letting $\varepsilon(e_x) = 1_x$ for all $x \in \mathrm{Ob}(\mathcal{C})$ and extending by $\mathcal{C}$-linearity. Then let $\varepsilon(m) = 0$ for all other $m \in M$. Define another $\mathcal{C}$-bimodule morphism

$$\Delta\colon M \to M \otimes M, \tag{6.3.6}$$

by letting

- $\Delta(m) = \delta(m) + m \otimes e_x + e_y \otimes m$ for $m \in M(x, y)$ and $m \in \ker \varepsilon$,
- $\Delta(e_x) = e_x \otimes e_x$ for all $x \in \mathrm{Ob}(\mathcal{C})$, and

extending by $\mathcal{C}$-linearity.

Observe that the element $e_x \in M(x, x)$ is grouplike for every $x \in \mathrm{Ob}(\mathcal{C})$. Therefore, to show that $(\mathcal{C}, M)$ is a grouplike bocs, it is only necessary to show that M is a co-algebra.

Proposition 6.3.3. *The $\mathcal{C}$-bimodule M is a co-algebra with co-unit ε (see Equation* (6.3.5)*) and comultiplication Δ (see Equation* (6.3.6)*).*

Proof: We will first show that the co-algebra axioms hold for elements of M that do not belong to $\ker \varepsilon$. For this, it suffices to show that the axioms hold for any grouplike element e_x. Co-associativity follows from the calculation:

$$(1 \otimes \Delta) \circ \Delta(e_x) = e_x \otimes e_x \otimes e_x = (\Delta \otimes 1) \circ \Delta(e_x).$$

Co-unity follows from the calculation:

$$\begin{aligned}(\varepsilon \otimes 1) \circ \Delta(e_x) &= (\varepsilon \otimes 1)(e_x \otimes e_x) \\ &= e_x \\ &= (1 \otimes \varepsilon)(e_x \otimes e_x) = (1 \otimes \varepsilon) \otimes \circ \Delta(e_x).\end{aligned}$$

We now show that the co-algebra axioms hold for an element $m \in M(x, y)$, with $x, y \in \mathrm{Ob}(\mathcal{C})$, such that $m \in \ker \varepsilon$. Therefore, m is a $\mathcal{D}$-morphism. Write

$$\delta(m) = \sum_{i=1}^{n} m_i \otimes m_i',$$

where, for $i = 1, \ldots, n$, both m_i and m_i' are $\mathcal{D}$-morphisms with degree 1. This implies that $m_i, m_i' \in \ker \varepsilon$. We will use this fact below.

First, we will show that the co-associativity axiom holds. We have that

$$(1 \otimes \Delta) \circ \Delta(m) = \sum_{i=1}^{n} m_i \otimes \Delta(m_i') + m \otimes \Delta(e_x) + e_y \otimes \Delta(m)$$

$$
\begin{aligned}
&= \sum_{i=1}^{n} m_i \otimes \delta(m_i') + \sum_{i=1}^{n} m_i \otimes m_i' \otimes e_x + \sum_{i=1}^{n} m_i \otimes e_{x_i} \otimes m_i' \\
&\quad + m \otimes e_x \otimes e_x + \sum_{i=1}^{n} e_y \otimes m_i \otimes m_i' \\
&\quad + e_y \otimes m \otimes e_x + e_y \otimes e_y \otimes m.
\end{aligned}
$$

We also have

$$
\begin{aligned}
(\Delta \otimes 1) \circ \Delta(m) &= \sum_{i=1}^{n} \Delta(m_i) \otimes m_i' + \Delta(m) \otimes e_x + \Delta(e_y) \otimes m \\
&= \sum_{i=1}^{n} \delta(m_i) \otimes m_i' + \sum_{i=1}^{n} m_i \otimes e_{x_i} \otimes m_i' + \sum_{i=1}^{n} e_y \otimes m_i \otimes m_i' \\
&\quad + \sum_{i=1}^{n} m_i \otimes m_i' \otimes e_x + m \otimes e_x \otimes e_x \\
&\quad + e_y \otimes m \otimes e_x + e_y \otimes e_y \otimes m.
\end{aligned}
$$

The equation

$$
\delta^2(m) = \sum_{i=1}^{n} \delta(m_i) \otimes m_i' - \sum_{i}^{n} m_i \otimes \delta(m_i') = 0
$$

implies that $\sum_{i=1}^{n} \delta(m_i) \otimes m_i' = \sum_{i=1}^{n} m_i \otimes \delta(m_i')$. Then it is clear that

$$
(1 \otimes \Delta) \circ \Delta(m) = (\Delta \otimes 1) \circ \Delta(m).
$$

Finally, we show that the co-unity axiom holds. We have that

$$
\begin{aligned}
(\varepsilon \otimes 1) \circ \Delta(m) &= (\varepsilon \otimes 1) \left(\sum_{i=1}^{n} m_i \otimes m_i' + m \otimes e_x + e_y \otimes m \right) \\
&= \sum_{i=1}^{n} \varepsilon(m_i) \otimes m_i' + \varepsilon(m) \otimes e_x + \varepsilon(e_y) \otimes m \\
&= m
\end{aligned}
$$

and

$$
(1 \otimes \varepsilon) \circ \Delta(m) = (1 \otimes \varepsilon) \left(\sum_{i=1}^{n} m_i \otimes m_i' + m \otimes e_x + e_y \otimes m \right)
$$

$$= \sum_{i=1}^{n} m_i \otimes \varepsilon(m_i') + m \otimes \varepsilon(e_x) + e_y \otimes \varepsilon(m)$$
$$= m.$$

∎

6.3.3 Main Result

In this section, we prove that the categories of representations of any grouplike bocs and its corresponding DGC are equivalent. As described in Section 6.3.1, let $\mathscr{B} = (\mathcal{C}, M)$ be a grouplike bocs with kernel K and let $T(K)$ be the semi-free DGC obtained from $\mathscr{B}$. Their respective categories of representations are $\mathscr{B}$-**rep** and $\mathcal{R}(T(K), \Bbbk\text{-}\mathbf{mod})$, the latter of which we will denote by $\mathcal{R}$.

Recall the category $\overline{T(K)}$ that is used in the construction of $\mathcal{R}$, and that to obtain $\overline{T(K)}$, we introduce an endomorphism e_x of degree 1 for every $x \in \mathrm{Ob}(\mathcal{C})$. We will identify each e_x with the grouplike element u_x. The next lemma justifies this identification.

Lemma 6.3.4. *For all $\mathcal{C}$-objects x and y, we have that $M(x, y) = \overline{T(K)}(x, y)_1$ as $\Bbbk$-modules.*

Proof: Consider an arbitrary element $\sum_{i=1}^{n} c_i m_i \in M(x, y)$, where $c_i \in \Bbbk$. Then, for $i = 1, 2, \ldots, n$, we have that either $m_i \in K$ or $m_i = g u_z f$ for some $\mathcal{C}$-morphisms $f\colon x \to z$, $g\colon z \to y$ and grouplike element $u_z \in M(z, z)$. If $m_i \in K$, then $m_i \in K(x, y) = T(K)(x, y)_1 \subseteq \overline{T(K)}(x, y)_1$ for some $x, y \in \mathrm{Ob}(\mathcal{C})$. If $m_i = g u_z f$, then we identify this with the element $g e_z f \in \overline{T(K)}(x, y)_1$ and so we have that $m_i \in \overline{T(K)}(x, y)_1$. It follows that $\sum_{i=1}^{n} c_i m_i \in \overline{T(K)}(x, y)_1$, and hence $M(x, y) \subseteq \overline{T(K)}(x, y)_1$.

To show that $\overline{T(K)}(x, y)_1 \subseteq M(x, y)$, we interchange the roles of e_z and u_z and use the same argument. ∎

Considering Lemma 6.3.4, we now compare the categories $\mathscr{B}$-**rep** and $\mathcal{R}$. Both categories have the same objects, namely the $\Bbbk$-linear functors $F\colon \mathcal{C} \to \Bbbk$-**mod**. We introduce a functor

$$S\colon \mathscr{B}\text{-}\mathbf{rep} \to \mathcal{R}, \tag{6.3.7}$$

defined as follows. Let the object map be defined by $S(F) = F$ for all representations F of $\mathscr{B}$. Prior to defining the morphism map of S, recall that

$$\mathscr{B}\text{-}\mathbf{rep}(F, G) = \hom_{\mathcal{C}}(M \otimes_{\mathcal{C}} F, G)$$

and, also, that any morphism Ψ in $\mathcal{R}(\mathcal{C}, \mathcal{D})$ is completely described by Ψ_{12}. Then, for any morphism $\theta \in \mathscr{B}$-**rep**(F, G), let $S(\theta) \in \mathcal{R}(S(F), S(G)) = \mathcal{R}(F, G)$ be the morphism where

$$S(\theta)_{12}(m)\colon F(x) \to G(y), \quad v \mapsto \theta(m \otimes_{\mathcal{C}} v)$$

is a $\Bbbk$-linear map. Note that $S(\theta)_{12}$ is well-defined on degree 1 morphisms because of Lemma 6.3.4.

Theorem 6.3.5. *The functor S (see Equation (6.3.7)) is an equivalence of the categories $\mathscr{B}$-**rep** and $\mathcal{R}$.*

Proof: It follows from Theorem 1.1.12 that it suffices to show that S is dense and fully faithful. It follows immediately that S is dense because $S\left(\mathrm{Ob}\left(\mathscr{B}\text{-}\mathbf{rep}\right)\right) = \mathcal{R}$.

We now show that S is full. Let $\psi\colon F \to G$ be an $\mathcal{R}$-morphism. Thus

$$\psi\colon \overline{T(K)} \to \overline{\mathbf{mod}}_{\Bbbk}$$

is a functor which, for each $\overline{T(K)}$-morphism $m\colon x \to y$ with $\deg m = 1$, provides a $\Bbbk$-linear mapping $\psi_{12}(m)\colon F(x) \to G(y)$. Define a mapping $\theta\colon M \otimes_{\mathcal{C}} F \to G$ by letting

$$\theta\left(\sum_i m_i \otimes_{\mathcal{C}} v_i\right) = \sum_i \psi_{12}(m_i)(v_i).$$

To ensure that $\theta \in \mathscr{B}\text{-}\mathbf{rep}(F,G)$, we show that it is $\mathcal{C}$-linear. Consider an element $m \otimes_{\mathcal{C}} v \in (M \otimes_{\mathcal{C}} F)(y)$ where $m \in M(x,y)$ and $v \in F(x)$. Let $h\colon y \to z$ be a $\mathcal{C}$-morphism. Then

$$\theta(hm \otimes_{\mathcal{C}} v) = \psi_{12}(hm)(v)\colon F(x) \to G(z)$$

is a $\Bbbk$-linear map. Note that $hm \in M(x,z)$ is a degree 1 morphism in $\overline{T(K)}$ and so we have that

$$\psi(hm) = \psi(h)\psi(m) = \begin{pmatrix} F(h) & 0 \\ 0 & G(h) \end{pmatrix} \circ \begin{pmatrix} 0 & \psi_{12}(m) \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & G(h)\circ\psi_{12}(m) \\ 0 & 0 \end{pmatrix}$$

(the composition is described in Equation (6.1.3)). In particular, we have that $\psi_{12}(hm) = G(h)\psi_{12}(m)$. Hence

$$\theta(hm \otimes_{\mathcal{C}} v) = \psi_{12}(hm)(v) = G(h)\psi_{12}(m)(v) = G(h)\theta(m \otimes_{\mathcal{C}} v) = h\theta(m \otimes_{\mathcal{C}} v).$$

Recall that the expression $h\theta(m \otimes_{\mathcal{C}} v)$ above follows the notational convention described in Equation (6.2.1). Hence, θ is $\mathcal{C}$-linear and, therefore, S is full.

We now show that S is faithful. Consider morphisms $\theta, \theta' \in \mathscr{B}\text{-}\mathbf{rep}(F,G)$, for some $F, G \in \mathrm{Ob}(\mathscr{B}\text{-}\mathbf{rep})$, such that $\theta \neq \theta'$. That is, $\theta, \theta'\colon M \otimes_{\mathcal{C}} F \to G$ are distinct $\mathcal{C}$-linear maps and so there exists an element $m \otimes_{\mathcal{C}} v \in M \otimes_{\mathcal{C}} F$ such that

$$\theta(m \otimes_{\mathcal{C}} v) \neq \theta'(m \otimes_{\mathcal{C}} v). \tag{6.3.8}$$

Because m is a morphism in $\overline{T(K)}$ with $\deg m = 1$, both $S(\theta)_{12}(m)$ and $S(\theta')_{12}(m)$ are defined. It follows from Equation (6.3.8) that $S(\theta)_{12}(m)(v) \neq S(\theta')_{12}(m)(v)$ and so $S(\theta) \neq S(\theta')$. ∎

Conclusion

We have discussed six different types of matrix problems found throughout the literature. In particular, we have found several relationships between these matrix problems and have shown that applications of linear matrix problems to the classification of representations of both quivers and posets are well-founded.

This work can certainly be expanded upon. One might hope to explore further relationships between the matrix problems discussed here. Additionally, it would be helpful to explore relationships between any matrix problems that were not discussed here. For instance, how are vector space category matrix problems, vectroid matrix problems, spectroid matrix problems and modules over aggregate matrix problems related? How do they relate to the matrix problems discussed in this text?

Much of the literature on matrix problems focuses on their application to problems in representation theory. Perhaps by studying the many different formulations of matrix problems, we can find a sufficiently general definition for a matrix problem that encompasses all of the current definitions.

An essentially equivalent goal was the impetus of Roĭter in writing [27] and [28]. As he wrote in the introduction to [27], he had hoped to reduce the theory of matrix problems to a formal calculus. Since those papers were written, it appears as if the theory of matrix problems has had an influx of many different interpretations. Hopefully, this thesis is able to have a meaningful impact on the foundations of the subject and move us a step closer to realizing Roĭter's vision.

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