Premonoidal $*$-Categories and Algebraic Quantum Field Theory

By

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Abstract

Algebraic Quantum Field Theory (AQFT) is a mathematically rigorous framework that was developed to model the interaction of quantum mechanics and relativity. In AQFT, quantum mechanics is modelled by C^*-algebras of observables and relativity is usually modelled in Minkowski space. In this thesis we will consider a generalization of AQFT which was inspired by the work of Abramsky and Coecke on *abstract quantum mechanics* [1, 2]. In their work, Abramsky and Coecke develop a categorical framework that captures many of the essential features of finite-dimensional quantum mechanics.

In our setting we develop a categorified version of AQFT, which we call *premonoidal C^*-quantum field theory*, and in the process we establish many analogues of classical results from AQFT. Along the way we also exhibit a number of new concepts, such as a *von Neumann category*, and prove several properties they possess.

We also establish some results that could lead to proving a premonoidal version of the classical Doplicher-Roberts theorem, and conjecture a possible solution to constructing a fibre-functor. Lastly we look at two variations on AQFT in which a causal order on double cones in Minkowski space is considered.

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Dedication

This work is dedicated to my loving partner Kim and my bossy cat Gilby.

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Chapter 1

Introduction

1.1 General Overview

Algebraic Quantum Field theory (AQFT) is a mathematically rigorous framework for modelling the interaction of quantum mechanics in its C^*-algebra formulation and relativity, usually modelled in Minkowski space. Moreover as its name suggests this theory is an algebraic approach to standard quantum field theory [4, 16]. As such many of its aspects can be motivated by seeing how they have a counterpart in the usual QFT approach, see [4, 16] for explicit connections. From a mathematical point of view an AQFT is essentially a well-behaved functor. Out of the numerous references we could give for this subject we suggest the ones we found least intimidating to the beginner, they are [17, 16, 32].

Let us briefly sketch here some of the main ideas involved in AQFT. We start by considering Minkowski space, or more generally a Lorentz manifold, as a set equipped with an order $\ll$ on its elements, see Section 2.3 for a precise definition. The order $\ll$ gives a notion of a causality where one interprets $x \ll y$ as x causally precedes y. One uses this order to define a *double cone* which is simply an interval with respect to the order $\ll$, i.e. a set of the form

$$D = \{x \mid a \ll x \ll b\}$$

is the double cone determined by a and b. The set of double cones forms a partially ordered set with respect to subset inclusion, hence it forms a category. An AQFT is then an assignment of a C^*-algebra to each double cone U. Thus we have a map

$$U \longmapsto \mathbb{A}(U).$$

The algebra $\mathbb{A}(U)$ is referred to as the *algebra of local observables* and the interpretation of this algebra is as follows. $\mathbb{A}(U)$ is the C^*-algebra generated by observables measurable in the region U, where an observable measurable in U consists of any observable measured in a region of space O during a period of time T such that $T \times O \subseteq U$ [4].

There are numerous properties that one can demand be satisfied by this assignment. Of these we single out two of particular interest. The first property is what is referred to in the literature as *isotony* . An AQFT satisfies isotony if $\mathbb{A}(U) \subseteq \mathbb{A}(V)$ whenever $U \subseteq V$. Thus an AQFT satisfying isotony is what a category theorist would call a functor $\mathbb{A} : K \longrightarrow$ **C*-AlgI** where the codomain is the category of C^*-algebras and inclusions and K is the poset of double cones in Minkowski space. Physically isotony is saying that any observable which can be measured in U can also be measured in any larger region V.

The second property of interest is a statement concerning the causal structure of spacetime and what implications this has for the local algebras. To state this condition we need first to define the *quasi-local algebra*. Given an AQFT which satisfies isotony then the set $\{\mathbb{A}(U) \mid U \in K\}$ is a directed set with respect to subset inclusion, since the set of double cones is a directed poset. Hence one can form the directed colimit which is simply the norm-closure of the algebra

$$\bigcup_{U \in K} \mathbb{A}(U).$$

The resulting C^*-algebra is denoted $\hat{\mathbb{A}}$ and is called the quasi-local algebra. Moreover one can view each local algebra $\mathbb{A}(U)$ as a subalgebra of $\hat{\mathbb{A}}$ in a canonical way. We can now state the second property known as *microcausality or Einstein causality.* An AQFT, satisfying isotony, satisfies microcausality if whenever U and V are *spacelike separated* regions then the local algebras $\mathbb{A}(U)$ and $\mathbb{A}(V)$ commute with each other elementwise in $\hat{\mathbb{A}}$. The microcausality assumption is based on a fundamental principle of relativity that states events occurring in U and events occurring V, where U and V are spacelike separated, cannot influence each other. Thus a measurement in U followed by a measurement in V is equivalent to the opposite order of measurements. Thus in any larger region O containing U and V the observables corresponding to the measurements in U and V respectively will commute with each other. There are typically other axioms, for example involving the Poincaré group, but for us these will mostly be peripheral, with the main focus on the isotony and microcausality axioms.

The second influence on this work is the recent *abstract quantum mechanics* of Abramsky and Coecke [1, 2]. See also Selinger [36] for a categorical axiomatization of the notion of completely positive map, which is fundamental in the interpretation of quantum mechanics. There the authors abstract away from the category of finite-dimensional Hilbert spaces and develop the notion of a *dagger compact closed category.* This abstract framework captures many of the essential features necessary to express quantum mechanical concepts. They go on to show that much of quantum mechanics can still be carried out in this more abstract setting, and that it also provides insight into underlying structures.

The authors show for example that dagger compact closed categories provide sufficient structure to model protocols such as quantum teleportation or entanglement swapping. The correctness of the interpretation basically just amounts to the coherence equations of the theory.

One of the features of this encoding of the teleportation protocol is that it does not take into account the fact that teleportation takes place in spacetime. We believe that an appropriate modification of AQFT would allow for a modelling of such protocols in a way which takes spacetime explicitly into account. More specifically following the philosophy of AQFT one should associate to each region in spacetime some sort of category. But exactly what sort of category one should associate is somewhat elusive. A reasonable first guess would be to assign a dagger compact closed category to each double cone.

Continuing down this road, it is then evident what an appropriate notion of isotony could be; what is not evident is how to express the microcausality axiom. For argument's sake suppose that for each double cone U we associate some type of category

$\mathbb{A}(U)$. Whatever choice we make we should do so in a way that a traditional AQFT is a degenerate example of such a thing. i.e. every one-object such category should be a $*$-algebra. On the other hand a crucial element in Abramsky and Coecke's abstract quantum mechanics is the existence of a tensor product. Thus bearing this in mind our categories, $\mathbb{A}(U)$, should also be equipped with some kind of tensor product. Now it is well known that a one-object monoidal category is the same thing as a commutative monoid. The commutativity is a consequence of bifunctoriality of the tensor product. Hence imposing a monoidal structure on our categories is too strong since our one-object categories are supposed to be $*$-algebras which may not be commutative.

If we drop the requirement that tensor is a bifunctor from the definition of monoidal category and simply ask that $A \otimes -$ and $- \otimes A$ are endofunctors for all objects A, we obtain the concept of a *premonoidal category*, as introduced by Power and Robinson [31]. One then has that one-object premonoidal categories are the same thing as monoids. Thus we propose that our categories be obtained by modifying the notion of dagger compact closed category, replacing the monoidal structure with premonoidal structure.

We claim that the usual bifunctoriality equation can then be used to capture microcausality. Indeed given two premonoidal subcategories $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{C}$ we say that $\mathcal{A}$ and $\mathcal{B}$ *commute* with each other in $\mathcal{C}$ if for all arrows $f \in \mathcal{A}(A, A')$ and $g \in \mathcal{B}(B, B')$ the equations

$$A' \otimes g \circ f \otimes B = f \otimes B' \circ A \otimes g$$
$$g \otimes A' \circ B \otimes f = B' \otimes f \circ g \otimes A$$

hold. Note that in the case that $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are monoids this amounts to saying that the submonoids commute with each other. Thus we will express microcausality by saying that if U and V are spacelike separated double cones then the premonoidal categories $\mathbb{A}(U)$ and $\mathbb{A}(V)$ must commute with each other in $\hat{\mathbb{A}}$. (We also mention the recent work of Coecke and Lal, [8], in which they interpret microcausality using a partially defined tensor product.

Coincidentally, the above analysis leads to a natural point of departure for the true goal of this thesis, which is to categorify the traditional notion of AQFT. That is to say, the ultimate goal of this thesis is to develop an abstract approach to AQFT where the C^*-algebras are replaced by certain types of categories, which we define, and then to redevelop AQFT in this more abstract setting. Indeed we will show that much of the so called *DHR analysis* can be redone in this new theory and that certain notions such as *Haag duality* have interesting counterparts here which provide surprising mathematical insight into these concepts.

The initial reason we became interested in AQFT is rooted in the astonishing *Doplicher-Roberts theorem.* The theorem shows that the category of *physically relevant representations* of the quasi-local algebra $\hat{\mathbb{A}}$ is equivalent to the category of representations of an essentially unique compact (super)group. This theorem is presented in [12] and its physical significance is explained in [13]. An alternate proof, by Müger, is given in the appendix of [17] and its importance is discussed by Halvorson in the main body of this article. Müger states this result in a very abstract manner where the statement makes no reference to the category of physically meaningful representations of $\hat{\mathbb{A}}$. Instead the statement is that any STC^* is equivalent to the category of representations of an essentially unique compact (super)group, where an STC^*(symmetric tensor $*$-category plus extra structure) is an abstraction of the category of representations of the quasi-local algebra. Note that this result is explicitly about monoidal rather than premonoidal categories. It therefore makes sense to look at the Doplicher-Roberts theorem in the premonoidal setting.

Indeed we will examine this problem of proving the Doplicher-Roberts theorem in the premonoidal setting. In the process we develop a premonoidal theory of STC^*'s which we call SPC^*'s. We also define many premonoidal analogues of standard notions from the theory of tensor $*$-categories including also the notions of conjugate objects, compact closure, dimension theory etc. While we don't have a complete solution to this question, we indicate possible forms the solution could take.

In addition to these two major themes this thesis also proposes some other variations of AQFT. Namely we will consider a modification of the category of *localized transportable endomorphisms of* $\hat{\mathbb{A}}$ where for each double cone $U \subset M$ in Minkowski space we will view U as a spacetime in its own right. Then given any AQFT $\mathbb{A}$, we consider the analogous notion of localized transportable endomorphisms of $\mathbb{A}(U)$ instead of $\hat{\mathbb{A}}$. These considerations give a net of categories denoted Δ_U. Next we introduce a second partial order, $\sqsubseteq$, on the set of double cones which we interpret as a causal order. Using this partial order we examine under what conditions it is possible to obtain a functor $\Delta_U \longrightarrow \Delta_V$ whenever $U \sqsubseteq V$. It turns out that the theory of *Hilbert C^*-modules* and *Rieffel induction* provide some possible solutions to these questions. This second ordering should be thought of as a causal order on subsets and we propose a framework which interacts with this new order on double cones. The idea here is that to each double cone U we will associate a dagger compact closed category $\mathbb{A}(U)$ and require that this assignment be functorial with respect to the causal ordering. Now it is not necessarily the case that the poset of double cones will be directed with respect to $\sqsubseteq$ and so we cannot construct the directed colimit of the categories $\mathbb{A}(U)$. Instead we will consider the *Grothendieck category* associated to the functor $\mathbb{A}$, denoted $\mathbf{G}(\mathbb{A})$. We then go on to show that one can model the teleportation protocol in this category.

1.2 Chapter Descriptions

We begin with several expository chapters, giving most of the basic material we need on manifolds and spacetime, functional analysis and category theory. Chapter 2 provides the necessary background in smooth manifolds and linear algebra to tackle the basics of semi-Riemannian geometry and several concepts from relativity. In particular we explain the two orderings on points in any spacetime manifold, and then we specialize to Minkowski space.

Next we give a brief summary of key elements in the theory of operator algebras in Chapter 3. Chapter 4 gives a quick introduction to the theory of Hilbert C^*-modules and Rieffel induction. These concepts are needed when we deal with one of our proposed variations of an AQFT.

Then in Chapter 5 we give a review of monoidal categories and related notions of tensor $*$-categories. Next we present premonoidal categories following Power and Robinson in [31]. We also prove several of our own results concerning premonoidal categories which we use later on. Chapter 7 gives an abbreviated presentation of the classical Doplicher-Roberts theorem, in the style of Müger and Chapter 8 provides a moderately detailed description of the so called DHR analysis as well as a brief introduction to algebraic quantum field theory.

Chapter 9 develops the premonoidal analogues of notions from the theory of tensor $*$-categories and serves as background for Chapters 10 and 11. At last we arrive at Chapter 10 where we develop an abstract approach to AQFT, which we refer to as *Premonoidal* C^**-Quantum Field Theory* or PC^* QFT for short. We are able to establish many of the results presented in Chapter 8 here in this new setting. Finally in Chapter 11 we conjecture a premonoidal version of the Doplicher-Roberts theorem.

In Chapter 12, we present one of our proposed variations of AQFT in which a second order on double cones, modelling causality, is considered. We then build a category in which the teleportation protocol can be encoded. Chapter 13 deals with our second proposed variation on AQFT and following this, we conclude in Chapter 14 with a look towards future work.

1.3 New Results

As a courtesy to the reader we provide here an indication of what results contained in the thesis are new.

- Chapters 2, 3, and 4 are background material and contain no new results.
- Chapter 5 also consists mainly of background material with the exception of Remark 5.6.10 which is new.
- In Chapter 6 all of the material in section 6.1 is known. In section 6.2 Theorem 6.2.2, Lemma 6.2.3, and Example 6.2.4 are new as well as all results in section 6.3.
- Chapters 7, and 8 are survey chapters and do not contain any new results.
- Chapters 9, 10, and 11 represent the main contributions of this thesis, and all the results found therein are new.

- All material found in Chapter12 is new with the exception of the beginning of section 12.4 including up to Theorem 12.4.2.

- Finally, all of the results in Chapter 13 are new as well.

Chapter 2

Spacetime and Causality

We assume the reader is familiar with the notion of a smooth manifold. A suitable reference is [29].

2.1 Linear Algebra

The underlying mathematical structure of general relativity is that of a *Lorentz manifold*. In this theory spacetime is modelled by a Lorentz manifold. In order to define the concept of a Lorentz manifold we must first introduce the concept of covariant tensors on a real vector space. For this section we follow the treatment found in [27].

Fix a finite-dimensional real vector space $\mathcal{V}$ with $dim_{\mathbb{R}}\mathcal{V} = n$.

Definition 2.1.1. A **bilinear form** on $\mathcal{V}$ is a map $\beta : \mathcal{V} \times \mathcal{V} \longrightarrow \mathbb{R}$ such that the induced maps $\beta(-, v) : \mathcal{V} \longrightarrow \mathbb{R}$ and $\beta(v, -) : \mathcal{V} \longrightarrow \mathbb{R}$ are linear. The space of all bilinear forms on $\mathcal{V}$ is denoted $\mathcal{T}^2(\mathcal{V})$ and its elements are also referred to as **covariant tensors of rank two** on $\mathcal{V}$. A bilinear form β is **symmetric** if $\beta(v, w) = \beta(w, v)$ for all v, $w \in V$. A symmetric bilinear form β is **nondegenerate** if $\beta(v, w) = 0$ for all $w \in \mathcal{V}$ implies $v = 0$. i.e. the assignment $v \mapsto \beta(v, -)$ is injective.

The following is standard terminology.

Definition 2.1.2. A **scalar product g** on $\mathcal{V}$ is a nondegenerate symmetric bilinear form on $\mathcal{V}$.

Notice that a scalar product as defined above is not required to be positive definite, only nondegenerate. Consider the following example.

Example 2.1.3. Define $\mathbf{g} : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ by $\mathbf{g}(v,w) = v^1w^1 + v^2w^2 + \cdots + v^{n-1}w^{n-1} - v^nw^n$. Then $\mathbf{g}$ is a scalar product which is not positive definite.

From now on assume that $\mathbf{g}$ is a scalar product on $\mathcal{V}$. Then we say that $v,\, w \in \mathcal{V}$ are *orthogonal* if $\mathbf{g}(v,w) = 0$. We say that $u \in \mathcal{V}$ is a *unit vector* if $\mathbf{g}(u,u) = \pm 1$. Finally if $\{e_1, \ldots, e_n\}$ is a basis for $\mathcal{V}$ whose elements are pairwise orthogonal unit vectors then we call such a basis *orthonormal*. We now state a result which establishes the existence of orthonormal bases for spaces with scalar products.

Theorem 2.1.4. Suppose $\mathbf{g} : \mathcal{V} \times \mathcal{V} \longrightarrow \mathbb{R}$ is a scalar product on an n-dimensional real vector space $\mathcal{V}$. Then there exists a basis $\{e_1, \ldots, e_n\}$ for $\mathcal{V}$ such that $\mathbf{g}(e_i, e_j) = \pm\delta_{i,j}$ for $i,\, j = 1, \ldots, n$. Moreover the number of basis vectors e_i for which $\mathbf{g}(e_i, e_i) = -1$ is the same for any such basis.

The number $r = |\{e_i \mid \mathbf{g}(e_i, e_i) = -1\}|$ is called the index of $\mathbf{g}$. For simplicity we will assume that all orthonormal bases are indexed in such a way that all of these e_i appear at the beginning of the list so that if $\{e_1, \ldots e_r, e_{r+1}, \ldots, e_n\}$ is an orthonormal basis then $\mathbf{g}(e_i, e_i) = -1$ for $i = 1, \ldots r$ and $\mathbf{g}(e_i, e_i) = 1$ for $i = r+1, \ldots n$. Thus if $v = \sum_{i=1}^n v^i e_i$ and $w = \sum_{i=1}^n w^i e_i$ then we have

$$\mathbf{g}(v,w) = -v^1w^1 - \cdots - v^rw^r + v^{r+1}w^{r+1} + \cdots + v^nw^n.$$

2.2 Semi-Riemannian Manifolds

We suppose in this section that M is a smooth n-dimensional manifold. References for the material in this section are [28] and [29].

Definition 2.2.1. A **covariant tensor field of rank two** on M is a map A assigning to each $p \in M$ an a bilinear form A_p on the tangent space $T_p(M)$.

If (U, φ) is a coordinate chart with coordinate functions $x^1, \ldots x^n$ then

$$A_p = \sum_{i,j} A_p(\frac{\partial}{\partial x^i}|_p, \frac{\partial}{\partial x^j}|_p) dx^i_p \otimes dx^j_p. \tag{1}$$

The functions $A_{ij} : U \longrightarrow \mathbb{R}$ defined by $A_{ij}(p) = A_p(\frac{\partial}{\partial x^i}|_p, \frac{\partial}{\partial x^j}|_p)$ are called the *components* of A relative to (U, φ).

Definition 2.2.2. A covariant tensor field of rank two on M is **smooth** if its components relative to (U, φ) are smooth real valued functions for all charts (U, φ) in some atlas for M.

Definition 2.2.3. A **metric tensor** $\mathbf{g}$ on a smooth manifold M is a smooth covariant tensor field of rank two on M such that each $\mathbf{g}_p$ is a scalar product and the index of $\mathbf{g}_p$ is independent of $p \in M$. A **semi-Riemannian manifold** is a smooth manifold M equipped with a metric tensor $\mathbf{g}$.

Suppose we are given a semi-Riemannian manifold M with metric tensor $\mathbf{g}$, then the *index* of M is defined to be the index of $\mathbf{g}_p$ where $p \in M$. So the index of M is an integer r with $0 \leq r \leq n = dim\ M$. If $r = 0$, then M is called a *Riemannian manifold* and we see that each symmetric nondegenerate bilinear form $\mathbf{g}_p$ on T_pM is in fact positive definite. Thus each tangent space is equipped with an inner product. If $r = 1$ and $n = dim\ M \geq 2$ then M is called a *Lorentz manifold*.

Example 2.2.4. Consider the smooth manifold $\mathbb{R}^n$. Then for each $p \in \mathbb{R}^n$ we have a canonical isomorphism of vector spaces $\mathbb{R}^n \cong T_p\mathbb{R}^n$ which we denote by $v \mapsto v_p$. This allows us to define a scalar product on $T_p\mathbb{R}^n$ as follows:

$$\langle v_p, w_p \rangle = v \cdot w \tag{2}$$

where the right hand side is the inner product of v and w in $\mathbb{R}^n$. Thus we obtain a metric tensor $\mathbf{g}$ on $\mathbb{R}^n$ defined by $\mathbf{g}_p(v_p, w_p) = \langle v_p, w_p \rangle$. This metric tensor makes $\mathbb{R}^n$ a Riemannian manifold. We can modify the above construction to obtain a semi-Riemannian manifold as follows. For $0 < r \leq n$ we get a new metric tensor

$$\mathbf{g}(v_p, w_p) = -\sum_{i=1}^{r} v^i w^i + \sum_{i=r+1}^{n} v^i w^i \tag{3}$$

of index r where $v = (v^1, \dots, v^n)$ and $w = (w^1, \dots, w^n)$. This makes $\mathbb{R}^n$ into a semi-Riemannian manifold which we denote by $\mathbb{R}^n_r$. When $n \geq 2$ and $r = 1$ $\mathbb{R}^n_1$ is called *Minkowski n-space.*

The theory of special relativity is concerned with the Lorentz manifold $\mathbb{R}^4_1$. The space $\mathbb{R}^4_1$ is usually referred to simply as *Minkowski space* rather than Minkowski 4-space.

2.3 Causality

For this section we follow the presentation given in [29]. Let $(M, \mathbf{g})$ be a fixed semi-Riemannian manifold.

Remark 2.3.1. As a convenient notation we will write $\langle v, w \rangle$ in place of $\mathbf{g}_p(v, w)$ to denote the scalar product of tangent vectors v and $w \in T_pM$.

Definition 2.3.2. Let $v \in T_pM$ be a tangent vector to M. Then v is

spacelike	if $\langle v, v \rangle > 0$ or $v = 0$,
null (lightlike)	if $\langle v, v \rangle = 0$ and $v \neq 0$,
timelike	if $\langle v, v \rangle < 0$.

The set of null vectors in T_pM is called the **nullcone** at $p \in M$.

Thus each tangent vector to M is one of these three types, and is referred to as its *causal character* i.e. the causal character of $v \in T_pM$ is either spacelike, null, or timelike. We may also extend these notions to smooth curves in M as follows.

Definition 2.3.3. A smooth curve α in M is **spacelike** if all tangent vectors $\alpha'(t)$ are spacelike. One defines **timelike** and **null/lightlike** curves similarly.

Now suppose that $(M, \mathbf{g})$ is a Lorentz manifold and let $\mathcal{T}_p$ denote the set of timelike vectors in T_pM.

Definition 2.3.4. Let $u \in \mathcal{T}_p$ be a timelike tangent vector to M. The **timecone** of T_pM containing u is the set

$$C(u) = \{v \in \mathcal{T}_p \mid \langle u, v\rangle < 0\}. \tag{4}$$

The **opposite timecone** is

$$C(-u) = -C(u) = \{v \in \mathcal{T}_p \mid \langle u, v\rangle > 0\}. \tag{5}$$

In fact one can show that $\mathcal{T}_p$ is the disjoint union of the timecones $C(u)$ and $C(-u)$. Moreover it is also the case that $u \in C(v) \Leftrightarrow v \in C(u) \Leftrightarrow C(u) = C(v)$. One further useful property of timecones is that they are *convex* in the sense that if v, and $w \in C(u)$ and $a, b \geq 0$, both not zero, then $au+bv \in C(u)$. Thus in the tangent space T_pM of a Lorentz manifold M there are precisely two timecones however there is no canoncial choice of timecone in each of these tangent spaces. Selecting a timecone in T_pM is said to *time-orient* T_pM. Thus a natural question in Lorentz geometry, and an important one in general relativity, is: Is there a way of time-orienting each tangent space of M in a continuous manner? Suppose we have a function τ which associates to each $p \in M$ a timecone τ_p in T_pM. Then τ is *smooth* if for each $p \in M$ there exists a smooth vector field V on some neighbourhood U of p such that $V_q \in \tau_q$ for all $q \in U$. In this case τ is called a *time-orientation* of M. M is said to be *time-orientable* in case there exists a time-orientation of M. Finally, to *time-orient* M amounts to choosing a specific time-orientation on M.

Example 2.3.5. Minkowski n-space, $\mathbb{R}^n_1$ is time-orientable. If $u^0, \dots, u^{n-1}$ denote the natural coordinates on $\mathbb{R}^n_1$ then the tangent vector $\frac{\partial}{\partial u^0}|_p$ is a timelike vector in $T_p\mathbb{R}^n_1$ and thus determines a timecone τ_p. Since $\frac{\partial}{\partial u^0}$ is a smooth vector field τ is indeed a time-orientation on $\mathbb{R}^n_1$.

This example suggests the following equivalent characterization of time-orientability.

Lemma 2.3.6. A Lorentz manifold M is time-orientable if and only if there exists a timelike vector field X on M.

Another key notion in a Lorentz manifold is that of a *causal* vector. A vector $v \in T_pM$ is *causal* if it is either null or timelike. Then for any timelike vector $v \in T_pM$ we define the *causal cone* containing v to be the set $\overline{C}(v) = \{w \mid w \text{ is causal and } \langle v, w\rangle < 0\}$. Lastly we say that a curve in M is a *causal curve* if all its tangent vectors are causal.

Now suppose that $(M, \mathbf{g})$ is a *spacetime*, that is to say $(M, \mathbf{g})$ is a time-orientable Lorentz manifold with a fixed time orientation τ. Then a tangent vector $v \in T_pM$ is *future-pointing* if $v \in \overline{\tau_p}$. Equivalently if $X \in \mathcal{X}(M)$ is a timelike vector field on M which determines the time orientation on M then $v \in T_pM$ is future-pointing if and only if $\langle X_p, v\rangle \leq 0$. Similarly the tangent vector $v \in T_pM$ is *past-pointing* if $v \in -\overline{\tau_p}$, i.e. $\langle X_p, v\rangle \geq 0$. A timelike curve is *future-directed* if each of its velocity vectors is future-pointing and similarly a causal curve is *future-directed* if each of its velocity vectors is future-pointing. Dually one can define *past-directed* timelike (respectively causal) curves by saying that all the velocity vectors must be past-pointing. These concepts lead to extremely important binary relations on the set M, which are referred to by O'Neil as *causality relations*.

Definition 2.3.7. If p and $q \in M$ we define two relations on M as follows:

1. $p \ll q$ if there is a future-directed timelike curve in M from p to q.

2. $p < q$ if there is a future-directed causal curve in M from p to q.

3. $p \leq q$ if either $p = q$ or $p < q$.

4. If $A \subseteq M$ then the **chronological future** of A is the set

$$I^+(A) = \{q' \in M \mid \exists p' \in A \text{ with } p' \ll q'\}. \tag{6}$$

5. Similarly if $A \subseteq M$ then the **causal future** of A is the set

$$J^+(A) = \{q' \in M \mid \exists p' \in A \text{ with } p' \leq q'\} \tag{7}$$

Note that for any $p \in M$ define $I^+(p) = I^+(\{p\}) = \{q \in M \mid p \ll q\}$ and similarly for $J^+(p)$. Replacing "future" with "past" in the above definitions one obtains the sets $I^-(A)$ and $J^-(A)$, the chronological (respectively causal) past of A.

Lemma 2.3.8. The relations $\ll$ and $<$ are transitive. If M does not contain any closed causal curves then $\leq$ is antisymmetric.

Lemma 2.3.9. The sets $I^+(p)$ and $I^-(p)$ are open in M for all $p \in M$. More generally the sets $I^+(A)$ and $I^-(A)$ are open in M for any subset A of M.

Thus for p and $q \in M$ we can define the set $I(p,q) = \{r \in M \mid p \ll r \ll q\}$ which is open since it is the intersection of the two open sets $I^+(p)$ and $I^-(q)$.

Remark 2.3.10. If $x \ll z$ in M then it is a non-trivial fact that there are infinitely many $y \in M$ such that $x \ll y \ll z$, c.f. [29] p.402. Hence it follows that $I(x,z) \neq \emptyset$ if and only if $x \ll z$.

Definition 2.3.11. Let p and $q \in M$ be such that $p \ll q$ then the set $I(p,q)$ is called the **open double cone** with vertices p and q.

Theorem 2.3.12 (c.f. [30] Prop.4.21, Def.4.22 p.33)**.** The collection $\{I(p,q) \mid p,\, q \in M\}$ forms a basis for a topology on M called the **Alexandrov topology**.

The following theorem of Kronheimer and Penrose gives a characterization of when the Alexandrov topology agrees with the manifold topology.

Theorem 2.3.13. Given a spacetime M the following conditions are equivalent.

1. M is strongly causal.

2. The Alexandrov topology is in agreement with the manifold topology.

3. The Alexandrov topology is Hausdorff.

Remark 2.3.14 (c.f. [30] Def.4.4 p.27)**.** Note a spacetime M is *strongly causal* at $p \in M$ if for every open neighbourhood U of p there exists an open set $Q \subseteq U$, with $p \in Q$, which is *order convex* with respect to the order $\ll$, i.e. if x, and $y \in Q$ and $x \ll z \ll y$ then $z \in Q$. Then we say that M is *strongly causal* if it is strongly causal at each of its points.

2.4 Causality in Minkowski Space

Since the majority of AQFT is concerned with Minkowski space we will give focus on this specific case of a spacetime. Thus throughout this section M will denote 4-dimensional Minkowski space $M = \{(t, x, y, z) \in \mathbb{R}^4\}$ and given vectors $v = (t_1, x_1, y_1, z_1)$ and $w = (t_2, x_2, y_2, z_2) \in M$ then as usual we define their inner product by

$$\langle v, w\rangle = -t_1t_2 + x_1x_2 + y_1y_2 + z_1z_2.$$

Then translating previous notions to our setting we see that a vector $v = (t, x, y, z)$ in Minkowski space is:

1. *timelike* if $\langle v, v\rangle < 0$, i.e. $t^2 > x^2 + y^2 + z^2$
2. *null/lightlike* if $\langle v, v\rangle = 0$, i.e. $t^2 = x^2 + y^2 + z^2$
3. *spacelike* if $\langle v, v\rangle > 0$, i.e. $t^2 < x^2 + y^2 + z^2$

Thus for example the vector $v_0 = (1, 0, 0, 0)$ is certainly timelike and thus it defines a timelike vector field X on M by $X_p = v_0$ for all $p \in M$. Hence a vector $v = (t, x, y, z)$ is future-pointing in case $\langle v_0, v\rangle \leq 0$, i.e $t \geq 0$. Similarly v is past-pointing if $t \leq 0$. Moreover one can show that for p and $q \in M$ that $p \ll q$ if and only if the vector $\overrightarrow{pq} = q - p$ is timelike and future-pointing and similarly $p < q$ if and only if the vector $\overrightarrow{pq} = q - p$ is causal and future-pointing.

Now for each $p \in M$ the chronological future of p is called the *future timecone of* p and is given by the set $I^+(p) = \{q \in M \mid \overrightarrow{pq}$ is timelike future-pointing$\}$ and similarly $I^-(p)$ is called the *past timecone of* p. We also have $J^+(p) = \{q \in M \mid \overrightarrow{pq}$ is causal future-pointing$\}$ and similarly for $J^-(p)$. Moreover it is also the case that $J^+(p)$ and $J^-(p)$ are closed subsets of Minkowski space and are equal to the closures of the open sets $I^+(p)$ and $I^-(p)$ respectively. Lastly if p and $q \in M$ and $q - p$ is timelike future-pointing then the open double cone with vertices p and q is given by the set

$$\begin{aligned} I(p,q) &= \{z \in M \mid z - p \text{ and } q - z \text{ are timelike future-pointing } \} \\ &= \{z \in M \mid p_0 < z_0 < q_0 \text{ and } \sum_{i=1}^{3}(z_i - p_i)^2 < (z_0 - p_0)^2, \\ &\qquad \text{and } \sum_{i=1}^{3}(q_i - z_i)^2 < (q_0 - z_0)^2\}. \end{aligned}$$

Before closing this section we note that the equivalent conditions of Theorem 2.3.13 are satisfied for Minkowski space and thus the double cones form a basis for the manifold topology.

Chapter 3

Hilbert Spaces and C^*-algebras

Remark 3.0.1. We will assume that all vector spaces given are complex vector spaces unless we explicitly state otherwise. This assumption stands for the whole of Chapter 3. Our main references for this material are [9] and [20].

3.1 Banach Space Preliminaries

We start by recalling some important concepts from functional analysis that are directly relevant to understanding the basic theory of C^*-algebras.

Definition 3.1.1. A **normed space** consists of a vector space V and a mapping $\| \, \| : V \longrightarrow \mathbb{R}$ such that for all $c \in \mathbb{C}$, v, $w \in V$

- $\|v\| \geq 0$ and $\|v\| = 0$ implies $v = 0$ (8)
- $\|cv\| = |c|\|v\|$ (9)
- $\|v + w\| \leq \|v\| + \|w\|$ (triangle inequality) (10)

Thus one sees that if $(V, \| \, \|)$ is a normed space then it can be made into a metric space with metric $d : V \times V \longrightarrow \mathbb{R}$ given by $d(u, v) = \|u - v\|$ for all u, $v \in V$.

It is clear that if $(V, \| \, \|)$ is a normed space then the following maps are continuous with respect to the topology induced by the metric $d(x, y) = \|x - y\| : (x, y) \mapsto x + y$, $(c, x) \mapsto cx$, and $x \mapsto \|x\|$ for all x, $y \in V$ and $c \in \mathbb{C}$.

Definition 3.1.2. A normed space $(V, \| \, \|)$ is called a **Banach space** if the corresponding metric space (V, d) is complete.

It is clear that the complex field $\mathbb{C}$ with the usual absolute value as norm is a Banach space. Hence for any $n \in \mathbb{N}$ it follows that $\mathbb{C}^n$ is a Banach space with norm $\|(c_1, \ldots, c_n)\| = |c_1| + \cdots + |c_n|$. Note that all norms on finite dimensional vector spaces induce the same metric topology and so this norm is not the only norm one could use on this space. Morphisms between Banach spaces will be continuous linear maps, and turns out they have a nice characterization in this setting.

Lemma 3.1.3. [c.f. [20] Theorem 1.5.5 p.40] If $T : V \longrightarrow W$ is a linear map between normed spaces, then the following are equivalent.

1. T is continuous.
2. There exists a real number $C \geq 0$ such that $\|Tx\| \leq C\|x\|$ for all $x \in V$.
3. $\sup\{\|Tx\|/\|x\| \mid x \in V, \, x \neq 0\} < \infty$.
4. $\sup\{\|Tx\| \mid x \in V, \, \|x\| = 1\} < \infty$.

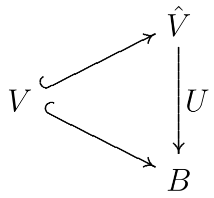
If one and hence all of these conditions are satisfied then the suprema in 3 and 4 are equal to the smallest C satisfying 2.

For any linear operator $T : V \longrightarrow W$ between normed spaces we define $\|T\| \in \mathbb{R}_{\geq 0} \cup \{\infty\}$ to be the suprema in Lemma 3.1.3. Hence T is continuous if and only if $\|T\| < \infty$, and in this case one also has $\|Tx\| \leq \|T\|\|x\|$ for all $x \in V$. Moreover $\|T\|$ is the smallest such real number and is called the *bound* of T. In view of Lemma 3.1.3 one often refers to a continuous linear map between normed spaces as a *bounded* linear operator. Now as one might expect the set of bounded linear maps between normed spaces V and W is again a normed space, denoted $\mathfrak{B}(V, W)$, with the norm of a bounded linear operator T given by its bound $\|T\|$. The following theorem extends this result to the level of Banach spaces.

Theorem 3.1.4 (c.f. [20] Theorem 1.5.6 p.41)**.** If V is a normed space and B is a Banach space then the set of bounded linear operators $\mathfrak{B}(V, B)$ is a Banach space with operator bound as its norm.

Note that every normed space V can be viewed as an everywhere-dense subset of a Banach space $\hat{V}$ which is essentially unique:

Theorem 3.1.5 (c.f. [20] Theorem 1.5.1 p.36)**.** If $(V, \| \|_V)$ is a normed space then there exists a Banach space $(\hat{V}, \| \|_{\hat{V}})$ such that V is a subspace of $\hat{V}$ which is everywhere-dense and $\|v\|_{\hat{V}} = \|v\|_V$ for all $v \in V$. Moreover if $(B, \| \|_B)$ is any other Banach space with these same properties as $\hat{V}$ then there exists a norm preserving epimorphism $U : \hat{V} \longrightarrow B$ such that



commutes. The Banach space $\hat{V}$ is called the **completion** of V.

Note that in the literature norm preserving epimorphisms are sometimes called *isometric isomorphisms.*

3.2 Hilbert Spaces

A particularly important example of a Banach space is a *Hilbert space.* These are Banach spaces whose norm is induced by an inner product.

Definition 3.2.1. An **inner product** on a complex vector space H consists of a function $\langle \, , \rangle : H \times H \longrightarrow \mathbb{C}$ satisfying for all x, y, $z \in H$ and a, $b \in \mathbb{C}$

1. $\langle ax + by, z\rangle = a\langle x, z\rangle + b\langle y, z\rangle$,
2. $\langle x, y\rangle = \overline{\langle y, x\rangle}$,
3. $\langle x, x\rangle \geq 0$.

If one also has that $\langle x, x\rangle = 0$ implies $x = 0$ then the inner product is called a **definite inner product**. An **inner product space** consists of a vector space H equipped with an inner product $\langle \, , \rangle$.

The next lemma summarizes some useful properties of inner products.

Lemma 3.2.2 (c.f. [20] Props.2.1.1 and 2.1.2 pp.78-79)**.** Suppose that $(H, \langle\,,\rangle)$ is an inner product space. Then

1. $|\langle x, y\rangle|^2 \leq \langle x, x\rangle\langle y, y\rangle$ for all x, $y \in H$
2. $\mathfrak{L} = \{z \in H \mid \langle z, z\rangle = 0\}$ is a linear subspace of H and the equation $\langle x+\mathfrak{L}, y+\mathfrak{L}\rangle_1 = \langle x, y\rangle$ defines a definite inner product $\langle\,,\rangle_1$ on the quotient vector space $H/\mathfrak{L}$.
3. $\|x\| = \langle x, x\rangle^{1/2}$ defines a semi-norm on H. If the inner product is definite then one obtains a norm on H in this way.

Thus given any inner product space one can apply the above result to obtain an inner product space with a definite inner product and hence a normed space.

Definition 3.2.3. A **pre-Hilbert space** consists of a normed space $(H, \|\,\|)$ such that $\|x\| = \langle x, x\rangle^{1/2}$ for all $x \in H$ for some definite inner product $\langle\,,\rangle$ on H. If in addition $(H, \|\,\|)$ is a pre-Hilbert space which happens to be a Banach space then we call $(H, \langle\,,\rangle)$ a **Hilbert space**.

Remark 3.2.4. In other words a Hilbert space is an inner product space H with a definite inner product $\langle\,,\rangle$ such that the associated normed space $(H, \|\,\|)$ is complete.

As in the case of Banach spaces one also has

Theorem 3.2.5 (c.f. [20] Prop. 2.1.6 p.80)**.** If H is a pre-Hilbert space then its completion $\hat{H}$ is a Hilbert space.

For morphisms of Hilbert spaces we take bounded linear maps. Thus for Hilbert spaces H and K the set of morphisms from H to K is $\mathfrak{B}(H, K)$, which, as previously mentioned, is a Banach space.

Proposition 3.2.6 (c.f. [20] Theorem 2.4.2 p.101)**.** If H, K, and L are Hilbert spaces and $T \in \mathfrak{B}(H, K)$, then there exists a unique bounded linear map $T^* \in \mathfrak{B}(K, H)$ such that

1. $\langle T^*x, y\rangle_H = \langle x, Ty\rangle_K$ for all $x \in K$ and $y \in H$.

   In addition, if $S \in \mathfrak{B}(H, K)$ and $R \in \mathfrak{B}(K, L)$, then

2. $(aT + bS)^* = \bar{a}T^* + \bar{b}S^*$ for all a, $b \in \mathbb{C}$
3. $(RT)^* = T^*R^*$
4. $(T^*)^* = T$
5. $\|T^*T\| = \|T\|^2$
6. $\|T^*\| = \|T\|$.

The unique bounded linear map T^* is called the **adjoint** of T.

As a convenient short hand we will simply write $\mathfrak{B}(H)$ instead of $\mathfrak{B}(H, H)$ to denote the Banach space of bounded linear maps from a Hilbert/Banach space H to itself. One then quickly sees that for a Hilbert space H, $\mathfrak{B}(H)$ is in fact a $\mathbb{C}$-algebra equipped with an anti-linear map $*$ satisfying equations 3, 4, and 5 of Proposition 3.2.6 for all R, and $T \in \mathfrak{B}(H)$. Moreover the ring multiplication given by composition is in fact continuous. This algebra $\mathfrak{B}(H)$ is an example of a *C^*-algebra*, a concept to be introduced shortly.

Before closing this section we mention some interesting types of operators on Hilbert spaces.

Definition 3.2.7. If H and K are Hilbert spaces then we say that an operator $U : H \longrightarrow K$ is **unitary** if $U^*U = id_H$ and $UU^* = id_K$. We say that an operator $T : H \longrightarrow H$ is: **self-adjoint** if $T^* = T$, **normal** if $T^*T = TT^*$, and **positive** if $\langle Tx, x\rangle \geq 0$ for all $x \in H$.

Remark 3.2.8. Note that the term unitary operator is usually reserved for those operators U whose domain and codomain are equal and also satisfy $U^*U = id = UU^*$.

3.3 C^*-algebras

Definition 3.3.1 (c.f. [20] Def. 3.1.1 p.174)**.** Suppose that $\mathcal{A}$ is an algebra over $\mathbb{C}$ with unit I. Then $\mathcal{A}$ is called a **normed algebra** if $\mathcal{A}$ is a normed space and $\|ab\| \leq \|a\|\|b\|$ for all a, $b \in \mathcal{A}$ and $\|I\| = 1$. If in addition $\mathcal{A}$ is a Banach space then $\mathcal{A}$ is called a **Banach algebra**.

We can now finally state the main definition of this chapter.

Definition 3.3.2. An **involution** on a Banach algebra $\mathcal{A}$ is a function $(-)^* : \mathcal{A} \longrightarrow \mathcal{A}$ such that for all a, $b \in \mathcal{A}$ and λ, and $\mu \in \mathbb{C}$,

1. $(\lambda a + \mu b)^* = \bar{\lambda} a^* + \bar{\mu} b^*$,
2. $(ab)^* = b^* a^*$, and
3. $(a^*)^* = a$.

A C^***-algebra** is a Banach algebra, $\mathcal{A}$, together with an involution $(-)^* : \mathcal{A} \longrightarrow \mathcal{A}$ and satisying

$$\|a^* a\| = \|a\|^2 \qquad (C^*\text{-identity}) \tag{11}$$

A key example of a C^*-algebra arises from the Banach algebra $\mathfrak{B}(H)$, where H is Hilbert space, described in the previous section. The involution is given by taking the adjoint of a bounded linear map. In fact an amazing result know as the GNS construction implies that every C^*-algebra can be viewed as a subalgebra of $\mathfrak{B}(H)$ for a suitable choice of Hilbert space H. Before discussing this result, we present a nontrivial example of a C^*-algebra.

Example 3.3.3. Suppose H is a Hilbert space and let $(H)_1 := \{x \in H \mid \|x\| \leq 1\}$ be the closed unit ball in H. A *compact operator* is a bounded linear operator $T : H \longrightarrow H$ such that the set $T(H)_1 = \{Tx \mid x \in (H)_1\}$ is relatively compact, i.e., has compact closure with respect to the norm topology. If $\mathcal{K}(H)$ denotes the set of compact operators on H, then $\mathcal{K}(H)$ is a C^*-algebra ([33] Cor.1.2 p.2). Moreover, $\mathcal{K}(H)$ contains the identity if and only if H is finite-dimensional.

In order to sketch the GNS construction we need a few definitions. The first notion is a generalization of the notion of eigenvalues of a linear map on a finite dimensional vector space.

Definition 3.3.4. If $\mathcal{A}$ is a Banach algebra and $a \in \mathcal{A}$ then we define the **spectrum of** a to be the subset $sp_{\mathcal{A}}(a)$ of complex numbers given by

$$sp_{\mathcal{A}}(a) = \{\lambda \in \mathbb{C} \mid a - \lambda \mathrm{I} \text{ does not have a two-sided inverse}\} \tag{12}$$

If $\lambda \in sp_{\mathcal{A}}(a)$ we say that λ is a **spectral value** for a.

We will simply write $sp(a)$ to denote the spectrum of a when the algebra $\mathcal{A}$ is clear from the context.

Definition 3.3.5. If $\mathcal{A}$ is a C^*-algebra and $a \in \mathcal{A}$ then we say that a is:

1. **hermitian/self-adjoint** in case $a^* = a$,
2. **unitary** in case $a^*a = I = aa^*$,
3. **normal** in case $a^*a = aa^*$, and
4. **positive** in case a is hermitian and $sp(a) \subset [0, \infty)$.

If a is positive we write $a \geq 0$ and we denote the set of positive elements in $\mathcal{A}$ by $\mathcal{A}^+$.

The positive elements in a C^*-algebra play an important role in the theory. The following proposition is one of the many results concerning this set $\mathcal{A}^+$.

Proposition 3.3.6 (c.f. [9] Theorem 3.6 p.241)**.** If $\mathcal{A}$ is a C^*-algebra, and $a \in \mathcal{A}$, then the following are equivalent.

1. $a \geq 0$.
2. $a = b^2$ for some $b \in \mathcal{A}$
3. $a = x^*x$ for some $x \in \mathcal{A}$.

Definition 3.3.7 (c.f. [9] Def. 5.10 p.250)**.** A linear functional $\rho : \mathcal{A} \longrightarrow \mathbb{C}$ on a C^*-algebra $\mathcal{A}$ is **positive** if $\rho(a) \geq 0$ for all $a \in \mathcal{A}^+$. A positive linear functional ρ is called a **state** if $\|\rho\| = 1$.

Notice that a positive linear functional isn't assumed to be bounded, but it turns out that it always is!

Lemma 3.3.8 (c.f. [9] Prop.5.11 and Cor.5.12 p.250)**.** If ρ is a positive linear functional on a C^*-algebra $\mathcal{A}$ then,

$$|\rho(y^*x)|^2 \leq \rho(y^*y)\rho(x^*x). \tag{13}$$

If ρ is also nonzero then ρ is bounded and $\|\rho\| = \rho(1)$.

We need two more definitions in order to state the main result.

Definition 3.3.9. A **$*$-homomorphism** from a C^*-algebra $\mathcal{A}$ to another C^*-algebra $\mathcal{C}$ consists of a $\mathbb{C}$-algebra homomorphism $\phi : \mathcal{A} \longrightarrow \mathcal{C}$ such that $\phi(a^*) = \phi(a)^*$ for all $a \in \mathcal{A}$. We say that $\mathcal{B} \subseteq \mathcal{A}$ is a C^*-subalgebra of $\mathcal{A}$ if $\mathcal{B}$ is a subalgebra which is norm-closed and for each $b \in \mathcal{B}$ one has $b^* \in \mathcal{B}$.

Note that one does not need to assume that a $*$-homomorphism is continuous, as this is a consequence of the definition (see [20] Theorem 4.1.8). In fact, $\|\phi(a)\| \leq \|a\|$ for all $a \in \mathcal{A}$. The GNS theorem is concerned with the existence of representations of C^*-algebras, so we need the following:

Definition 3.3.10 (c.f. [9] Def.5.1 p.248 and Def.5.6 p.249)**.** A **representation** of a C^*-algebra $\mathcal{A}$ consists of a Hilbert space H and a $*$-homomorphism $\pi : \mathcal{A} \longrightarrow \mathfrak{B}(H)$. A representation is called **cyclic** if there exists a vector $e \in H$ such that the set $\{\phi(a)e \mid a \in \mathcal{A}\}$ is everywhere-dense in H. In this case e is called a **cyclic vector**. Two representations (π_1, H_1) and (π_2, H_2) are **equivalent** if there exists a unitary map $U : H_1 \longrightarrow H_2$ such that $U\pi_1(a)U^{-1} = \pi_2(a)$ for all $a \in \mathcal{A}$.

Theorem 3.3.11 (Gelfand-Naimark-Segal Construction, c.f. [9] Theorem 5.14, p.250)**.** Let $\mathcal{A}$ be a C^*-algebra. Then,

1. if ρ is positive linear functional on $\mathcal{A}$, there is a cyclic representation (π_ρ, H_ρ) of $\mathcal{A}$ such that $\rho(a) = \langle \pi_\rho(a)e, e\rangle$ for all $a \in \mathcal{A}$ where $e \in H$ is the cyclic vector, and

2. if (π, H) is a cyclic representation with cyclic vector e, the equation $\rho(a) = \langle \pi(a)e, e\rangle$ defines a positive linear functional on $\mathcal{A}$. Moreover the representation (π_ρ, H_ρ) one obtains using the process in (1) is equivalent to the representation (π, H).

Note that if ρ is a positive linear functional on a C^*-algebra $\mathcal{A}$ then for any $c > 0$ we have that $c\rho$ is positive as well and in addition the representations π_ρ and $\pi_{c\rho}$ are equivalent. Thus up to equivalence it is sufficient consider states on $\mathcal{A}$ rather than arbitrary positive linear functionals.

Theorem 3.3.12 (c.f. [9] Theorem 5.17 p.253)**.** It $\mathcal{A}$ is a C^*-algebra then there exists a representation (π, H) of $\mathcal{A}$ for which $\pi : \mathcal{A} \longrightarrow \mathfrak{B}(H)$ is an isometry.

3.4 Von Neumann Algebras

For this section we fix a Hilbert space H.

Definition 3.4.1. Let $S \subseteq \mathfrak{B}(H)$ be any subset of bounded linear operators on the Hilbert space H. The **commutant** of S is the set $S' \subseteq \mathfrak{B}(H)$ defined by

$$S' = \{T \in \mathfrak{B}(H) \mid TK = KT \quad \forall K \in S\}. \tag{14}$$

We call the set $S'' = (S')'$ the **double commutant** of S.

It is clear from the definition that if $X \subseteq Y \subseteq \mathfrak{B}(H)$ then $Y' \subseteq X'$ and that $X \subseteq X''$ for all subsets X and Y. Consequently it follows that $S''' = S'$ for all subsets S of $\mathfrak{B}(H)$.

Definition 3.4.2 (c.f. [9] Def.7.1 p.281)**.** If $\mathcal{A}$ is a $*$-subalgebra of $\mathfrak{B}(H)$ such that $\mathcal{A}'' = \mathcal{A}$ then we say that $\mathcal{A}$ is a **von Neumann algebra**.

This definition is not the traditional one found in many texts on the subject but is indeed equivalent. This equivalence is usually referred to as von Neumann's double commutant theorem. In order state this theorem we first need to define some important topologies on $\mathfrak{B}(H)$.

Definition 3.4.3. The **strong-operator topology** on $\mathfrak{B}(H)$ is the topology which has a base of neighborhoods at $T_0 \in \mathfrak{B}(H)$ given by sets of the form

$$V(T_0 : x_1, \ldots, x_m; \epsilon) = \{T \in \mathfrak{B}(H) \mid \|(T - T_0)x_j\| < \epsilon (j = 1, \ldots, m)\}$$

where $x_1, \ldots, x_m \in H$ and $\epsilon > 0$.

Definition 3.4.4. The **weak-operator topology** on $\mathfrak{B}(H)$ is the topology with a base of neighborhoods at each $T_0 \in \mathfrak{B}(H)$ given by sets of the form

$$V(T_0 : (x_1, y_1), \ldots, (x_m, y_m); \epsilon) = \{T \in \mathfrak{B}(H) \mid |\langle (T - T_0)x_j, y_j \rangle| < \epsilon (j = 1, \ldots m)\}$$

where $x_1, \ldots, x_m, y_1, \ldots, y_m \in H$ and $\epsilon > 0$.

Remark 3.4.5. Every set which is open (resp. closed) in the weak-operator topology is open (resp. closed) in the strong-operator topology which is in turn open (resp. closed) in the norm topology on $\mathfrak{B}(H)$.

Theorem 3.4.6 (Double commutant c.f. [20] Theorem 5.3.1 p.326)**.** *If $\mathcal{A}$ is a self-adjoint algebra of operators on a Hilbert space H that contains the identity, then the closure of $\mathcal{A}$ in the weak-operator topology is the same as the closure in the strong-operator topology which in turn is equal to $\mathcal{A}''$.*

We now state some elementary facts concerning von Neumann algebras which can be found in [20]. If $S \subseteq \mathfrak{B}(H)$ then the set $(S \cup S^*)''$, where $S^* = \{a^* \mid a \in S\}$, is the von Neumann algebra generated by S. The set $(S \cup S^*)'$ is always a von Neumann algebra for any subset S of $\mathfrak{B}(H)$.

Specific examples of von Neumann algebras acting on H include $\mathbb{C} \cdot id_H$ and $\mathfrak{B}(H)$. If G is a group and $u : G \longrightarrow \mathfrak{B}(H)$ is a unitary representation of G then the commutant of $u(G) = \{u_g : H \longrightarrow H \mid g \in G\}$ is a von Neumann algebra. Moreover $u(G)'$ is equal to the set of maps in $\mathfrak{B}(H)$ which commute with the group action.

Remark 3.4.7. We note that von Neumann algebras do admit an abstract description which does not make reference to a Hilbert space. It was shown by Sakai [35] that a von Neumann algebra can be defined as a C^*-algebra M which as a Banach space is the dual of some other Banach space M_* called the *predual* which is unique up to isomorphism, i.e., $M = (M_*)^*$ as Banach spaces.

Chapter 4

Hilbert C^*-modules and Induced Representations

In this chapter we take a minimalist approach and present just enough of the theory of Hilbert C^*-modules to be able to state theorems concerning the problem of inducing a representation of a C^*-algebra $\mathcal{B}$ to a representation of another C^*-algebra $\mathcal{A}$. This process is sometimes referred to as *Rieffel induction* after Marc A. Rieffel. Two good references for this material are [24, 33], the latter being the one we use for this chapter.

4.1 Inner Product Modules

Let $\mathcal{A}$ be a fixed C^*-algebra which may not necessarily have an identity.

Definition 4.1.1 (c.f. [33] p.8)**.** A **right-$\mathcal{A}$-module** consists of a complex vector space X and a bilinear mapping $X \times \mathcal{A} \longrightarrow X$, denoted $(x, a) \mapsto x \cdot a$, satisfying the usual equations stating X is a module over the ring $\mathcal{A}$.

Remark 4.1.2. If $\mathcal{A}$ has an identity then we also require that $x \cdot 1_{\mathcal{A}} = x$ for all $x \in X$. In this case requiring X to be a vector space in advance is redundant. When X is a right $\mathcal{A}$-module we will sometimes write $X_{\mathcal{A}}$ to emphasize this.

Definition 4.1.3 (c.f. [33] Def.2.1 p.8)**.** Suppose X is a right $\mathcal{A}$-module equipped with a pairing $\langle \, , \rangle_{\mathcal{A}} : X \times X \longrightarrow \mathcal{A}$ such that conditions

1. $\langle x, \lambda y + \mu z\rangle_{\mathcal{A}} = \lambda\langle x, y\rangle_{\mathcal{A}} + \mu\langle x, z\rangle_{\mathcal{A}}$ for all λ and $\mu \in \mathbb{C}$,

2. $\langle x, y \cdot a\rangle_{\mathcal{A}} = \langle x, y\rangle_{\mathcal{A}} a$,

3. $\langle x, y\rangle_{\mathcal{A}}^* = \langle y, x\rangle_{\mathcal{A}}$,

4. $\langle x, x\rangle_{\mathcal{A}}$ is a positive element of the C^*-algebra $\mathcal{A}$, i.e., $\langle x, x\rangle_{\mathcal{A}} \geq 0$,

5. $\langle x, x\rangle_{\mathcal{A}} = 0$ implies that $x = 0$,

are satisfied. Then, X is called a **right inner product $\mathcal{A}$-module**.

Remark 4.1.4. It is immediate that $\langle\, ,\rangle_{\mathcal{A}}$ is conjugate linear in the first variable.

We pause now to give some basic examples.

Example 4.1.5. In the case $\mathcal{A} = \mathbb{C}$, inner product $\mathbb{C}$-modules are the same thing as complex inner products spaces in which the inner product is conjugate linear in the first variable and linear in the second variable.

Example 4.1.6. $\mathcal{A}$ is an inner product $\mathcal{A}$-module with obvious $\mathcal{A}$-module structure and pairing given by $\langle x, y\rangle_{\mathcal{A}} = x^* y$.

A standard result on complex inner product spaces is the Cauchy-Schwarz inequality, and Example 4.1.5 suggests that this might generalize to arbitrary inner product $\mathcal{A}$-modules. Indeed one has the following result.

Lemma 4.1.7 (c.f. [33] Lem.2.5 p.9)**.** Suppose that X is an inner product $\mathcal{A}$-module and x and $y \in \mathcal{A}$, then

$$\langle x, y\rangle_{\mathcal{A}}^* \langle x, y\rangle_{\mathcal{A}} \leq \|\langle x, x\rangle_{\mathcal{A}}\| \langle y, y\rangle_{\mathcal{A}} \tag{15}$$

as positive elements of the C^*-algebra $\mathcal{A}$.

The Cauchy-Schwarz inequality has the following nice consequence.

Corollary 4.1.8 (c.f. [33] Cor.2.7 p.10)**.** If X is an inner product $\mathcal{A}$-module then the formula

$$\|x\|_{\mathcal{A}} = \|\langle x, x\rangle_{\mathcal{A}}\|^{1/2} \tag{16}$$

defines a norm on X such that $\|x \cdot a\|_{\mathcal{A}} \leq \|x\|_{\mathcal{A}}\|a\|$. Moreover the normed module $(X_{\mathcal{A}}, \|\ \|_{\mathcal{A}})$ is **nondegenerate** in the sense that the elements $x{\cdot}a$ span a dense subspace of X. Explicitly,

$$X \cdot \langle X, X\rangle_{\mathcal{A}} \equiv \operatorname{span}\{x \cdot \langle y, z\rangle_{\mathcal{A}} \mid x, y, z \in X\}$$

is $\|\ \|_{\mathcal{A}}$-dense in X_A.

Definition 4.1.9 (c.f. [33] Def.2.8 p.11)**.** An inner product $\mathcal{A}$-module X is called a **Hilbert C^*-module** or **Hilbert $\mathcal{A}$-module** if X is complete with respect to the norm $\|\ \|_{\mathcal{A}}$. It is a **full** Hilbert $\mathcal{A}$-module if the ideal $I = \operatorname{span}\{\langle x, y\rangle_{\mathcal{A}} \mid x, y \in X\}$ is dense in $\mathcal{A}$.

Example 4.1.10 (c.f. [33] Ex.2.9 p.11)**.** Hilbert $\mathbb{C}$-modules are Hilbert spaces.

Example 4.1.11 (c.f. [33] Ex.2.10 p.11)**.** If $\mathcal{A}$ is a C^*-algebra then $\mathcal{A}_{\mathcal{A}}$ is a Hilbert $\mathcal{A}$-module with $a \cdot b = ab$ and $\langle a, b\rangle_{\mathcal{A}} = a^*b$.

Example 4.1.12 (c.f. [33] Ex.2.12 p.11)**.** Suppose that $p \in \mathcal{A}$ is a projection in the C^*-algebra $\mathcal{A}$. Then the set $\mathcal{A}p \equiv \{ap \mid a \in \mathcal{A}\}$ is a Hilbert $p\mathcal{A}p$-module with $\langle ap, bp\rangle_{p\mathcal{A}p} = pa^*bp$. Then we have that $\|ap\|_{p\mathcal{A}p} = \|ap\|$ and since $\mathcal{A}p$ is a closed linear subspace of $\mathcal{A}$ it follows that it is complete with respect to $\|\ \|_{p\mathcal{A}p} = \|\ \|$. Hence it is a Hilbert $p\mathcal{A}p$-module.

4.2 Adjointable Operators

One might naively expect that given a map between Hilbert modules that it has an *adjoint* as is the case for Hilbert spaces. This is however not automatic.

Definition 4.2.1 (c.f. [33] Def.2.17 p.16)**.** A function $T : X \longrightarrow Y$ between Hilbert $\mathcal{A}$-modules is called **adjointable** if there exists a function $T^* : Y \longrightarrow X$ such that

$$\langle Tx, y\rangle_{\mathcal{A}} = \langle x, T^*y\rangle_{\mathcal{A}} \qquad \text{for all } x \in X, y \in Y. \tag{17}$$

Lemma 4.2.2 (c.f. [33] Lem.2.18 p.16)**.** If $T : X \longrightarrow Y$ is an adjointable map between Hilbert $\mathcal{A}$-modules X and Y then T is a bounded linear map between the underlying Banach spaces X and Y, and T preserves the $\mathcal{A}$-module structures on X, and Y.

If X and Y are Hilbert $\mathcal{A}$-modules then the set of adjointable operators from X to Y is denoted $\mathcal{L}(X, Y)$ and one simply writes $\mathcal{L}(X)$ or $\mathcal{L}(X_{\mathcal{A}})$ when $Y = X$.

Lemma 4.2.3. If $T \in \mathcal{L}(X, Y)$ then T^* is unique and $T^{**} = T$. Moreover $\mathcal{L}(X)$ is a Banach subalgebra of $\mathfrak{B}(X) = \{\text{bounded linear maps on } X\}$ and it is equipped with an involution $T \mapsto T^*$.

The previous lemma foreshadows the following proposition.

Proposition 4.2.4 (c.f. [33] Prop.2.21 p.17)**.** If X is a Hilbert $\mathcal{A}$-module then $\mathcal{L}(X)$ is a C^*-algebra with respect to the operator norm.

Example 4.2.5. Adjointable maps between Hilbert $\mathbb{C}$-modules are the same thing as bounded linear maps between Hilbert spaces.

As mentioned earlier there are examples of bounded $\mathcal{A}$-linear maps between Hilbert $\mathcal{A}$-modules which aren't adjointable. For example see [33] Example 2.19 on page 17 or also see [24] page 8.

Example 4.2.6. If $\mathcal{A}$ is a C^*-algebra, and $a \in \mathcal{A}$, then the map $L_a : \mathcal{A} \longrightarrow \mathcal{A}$ given by $b \mapsto L_a(b) = ab$ defines an adjointable operator on $\mathcal{A}$ with adjoint L_{a^*}.

4.3 Induced Representations: Rieffel Induction

We conclude our brief presentation on Hilbert modules with a discussion on induced representations. For this discussion fix C^*-algebras $\mathcal{A}$ and $\mathcal{B}$, a Hilbert $\mathcal{B}$-module X and a $*$-homomorphism $\rho : \mathcal{A} \longrightarrow \mathcal{L}(X)$. Then $X_{\mathcal{B}}$ becomes a left $\mathcal{A}$-module with $a \cdot x = \rho(a)(x)$. In this case we say $\mathcal{A}$ acts as adjointable operators on $X_{\mathcal{B}}$. Now if $\pi : \mathcal{B} \longrightarrow \mathfrak{B}(H_\pi)$ is a nondegenerate representation of $\mathcal{B}$ then we would like to build a representation of $\mathcal{A}$. i.e., we want to *induce* a representation of $\mathcal{A}$ given a representation of $\mathcal{B}$.

Proposition 4.3.1 (c.f. [33] Prop.2.64 p.23)**.** Suppose $\mathcal{A}$ and $\mathcal{B}$ are C^*-algebras and $X_{\mathcal{B}}$ is a right Hilbert $\mathcal{B}$-module on which $\mathcal{A}$ acts as adjointable operators. If $\pi : \mathcal{B} \longrightarrow \mathfrak{B}(H_\pi)$ is a nondegenerate representation then there is a unique positive semi-definite inner product on the algebraic tensor product $X \odot H_\pi$ satisfying

$$\langle x \otimes h, y \otimes k \rangle = \langle k, \pi(\langle y, x \rangle_{\mathcal{B}})h \rangle. \tag{18}$$

A quick calculation reveals that any vector of the form $v = x \cdot b \otimes h - x \otimes \pi(b)h \in X \odot H_\pi$ has the property that its inner product with any other vector $y \otimes k \in X \odot H_\pi$ is zero. Now applying statement 2 of Lemma 3.2.2 the quotient vector space $X \odot H_\pi / \mathcal{S}$ where $\mathcal{S} = \{z \in X \odot H_\pi \mid \langle z, z \rangle = 0\}$ becomes a pre-Hilbert space with a definite inner product which we also denote $\langle\, , \rangle$. After completing this space we get a Hilbert space which is denoted $X \otimes_{\mathcal{B}} H_\pi$. Moreover vectors $x \otimes h \in X \otimes_{\mathcal{B}} H_\pi$ now have the property that $(x \cdot b) \otimes h = x \otimes \pi(b)h$ for all $b \in \mathcal{B}$. The subscript $\mathcal{B}$ on the tensor symbol is there to emphasize that this tensor product is $\mathcal{B}$-balanced in this sense. As a convenience we will sometimes write $x \otimes_{\mathcal{B}} h$ for the image in $X \otimes_{\mathcal{B}} H_\pi$ of the element $x \otimes h$ in $X \odot H_\pi$.

Theorem 4.3.2 (c.f. [33] Prop.2.66 p.35)**.** Suppose $\mathcal{A}$ and $\mathcal{B}$ are C^*-algebras, $\pi : \mathcal{B} \longrightarrow \mathfrak{B}(H_\pi)$ is representation of $\mathcal{B}$ and that $\mathcal{A}$ acts as adjointable operators on the a Hilbert $\mathcal{B}$-module X. Then

$$\mathrm{Ind}\pi(a)(x \otimes_{\mathcal{B}} h) = (a \cdot x) \otimes_{\mathcal{B}} h \tag{19}$$

extends to a representation $\mathrm{Ind}\pi : \mathcal{A} \longrightarrow X \otimes_{\mathcal{B}} H_\pi$ where $X \otimes_{\mathcal{B}} H_\pi$ is as described in the above discussion. If in addition, $\mathcal{A} \cdot X$ is dense in X then $\mathrm{Ind}\pi$ is a nondegenerate representation of $\mathcal{A}$.

Remark 4.3.3. In general the induced representation $\mathrm{Ind}\pi$ will depend on the Hilbert $\mathcal{B}$-module X and also the homomorphism $\mathcal{A} \longrightarrow \mathcal{L}(X)$. For these reasons we will use the following notations to refer to the induced representation $X - \mathrm{Ind}_{\mathcal{B}}^{\mathcal{A}}\pi$, $\mathrm{Ind}_{\mathcal{B}}^{\mathcal{A}}\pi$, and X-$\mathrm{Ind}\pi$.

Lastly we state one more result which shows that Rieffel induction is functorial.

Theorem 4.3.4 (c.f. [33] Prop.2.69 p.37)**.** Suppose that $\mathcal{A}$, $\mathcal{B}$ and X are as in Theorem 4.3.2. If $\pi_i : \mathcal{B} \longrightarrow \mathfrak{B}(H_i)$ are nondegenerate representations of $\mathcal{B}$ and $T : H_1 \longrightarrow H_2$ is a bounded intertwining operator then the map $1 \otimes T$ given by $x \otimes h \mapsto x \otimes (Th)$ extends to a bounded linear operator $1 \otimes_{\mathcal{B}} T : X \otimes_{\mathcal{B}} H_1 \longrightarrow X \otimes_{\mathcal{B}} H_2$ which intertwines X-Indπ_1 and X-Indπ_2. Moreover the map $T \mapsto 1 \otimes_{\mathcal{B}} T$ is $*$-linear, and if $S : H_2 \longrightarrow H_3$ intertwines π_2 and π_3 then $1 \otimes_{\mathcal{B}} ST = 1 \otimes_{\mathcal{B}} S \circ 1 \otimes_{\mathcal{B}} T$. Consequently the functor X-Ind : $\mathbf{Rep}(\mathcal{B}) \longrightarrow \mathbf{Rep}(\mathcal{A})$ preserves unitary equivalence and direct sums.

Chapter 5

Category Theory

5.1 Monoidal Categories

We start by collecting some important notions from category theory.

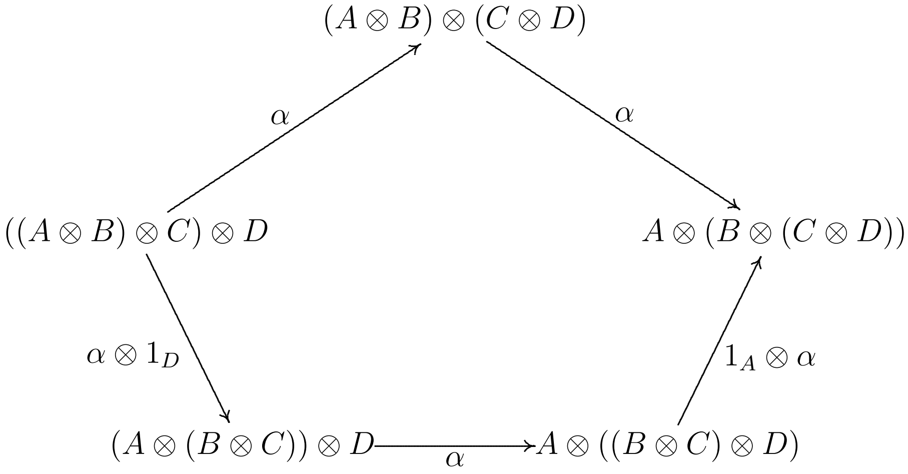
Definition 5.1.1 (c.f. [25]). A **monoidal category** $(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$ consists of a category $\mathcal{C}$, a functor $\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$, an object $I \in |\mathcal{C}|$, and natural isomorphisms $\forall\, A, B, C \in |\mathcal{C}|$:

$$\lambda_A : I \otimes A \longrightarrow A$$

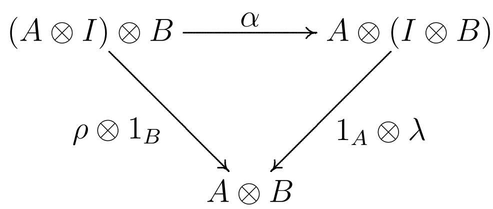
$$\rho_A : A \otimes I \longrightarrow A$$

$$\alpha_{A,B,C} : (A \otimes B) \otimes C \longrightarrow A \otimes (B \otimes C)$$

such that the following diagrams commute



(M1)



(M2)

$$\lambda = \rho \, : \, I \otimes I \longrightarrow I \tag{M3}$$

We say that a monoidal category is **strict** if all components of α, λ, and ρ are identity maps.

Example 5.1.2. Let M be a monoid in the usual sense of the word. Then we can view M as a category whose objects are elements of M and the only arrows are identity arrows. Let $\otimes$ denote the multiplication in M and let $I \in M$ be the unit for this multiplication. Then $(M, \otimes, I)$ is a strict monoidal category, as follows directly from the monoid axioms.

Example 5.1.3. Any category with all finite products (resp. coproducts) is a monoidal category where $\otimes = \times$ (resp. $+$), $I = 1$ (resp. 0). The isomorphisms α, λ, and ρ are determined by the universal property of product (resp. coproduct). In particular the category **Set** is a monoidal category. Unless stated otherwise we will take the monoidal structure on **Set** to be the one given by cartesian product.

Example 5.1.4. Let $\Bbbk$ be a fixed field and consider the category $\mathbf{Vect}_{\Bbbk}$ whose objects are vector spaces over $\Bbbk$ and arrows are $\Bbbk$-linear maps. Then $\mathbf{Vect}_{\Bbbk}$ is a monoidal category where:

$$V \otimes W = V \otimes_{\Bbbk} W$$
$$I = \Bbbk$$

and $\alpha : (U \otimes V) \otimes W \cong U \otimes (V \otimes W)$, $\lambda : \Bbbk \otimes U \cong U$, and $\rho : U \otimes \Bbbk \cong U$ are the usual vector space isomorphisms.

Example 5.1.5. Let $\mathcal{C}$ be a category and consider the category $\mathbf{Func}(\mathcal{C})$ whose objects are endofunctors on $\mathcal{C}$ and arrows are natural transformations between such functors. Then $\mathbf{Func}(\mathcal{C})$ is a strict monoidal category where $F \otimes G = F \circ G$ for functors F and G and $I = id_{\mathcal{C}}$.

We now state a coherence theorem for monoidal categories which can be found in [25]. We give the version from Kock's book [23].

Theorem 5.1.6 (Mac Lane's Coherence Theorem)**.** Let $(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$ be a monoidal category. Every diagram that can be built out of the components of α, λ, and ρ, and identity maps, using composition and monoidal operations, commutes.

We will use this theorem extensively to establish the commutativity of diagrams.

5.2 Monoid Objects

Definition 5.2.1 (c.f. [25])**.** Let $(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$ be a monoidal category. A **monoid** (M, μ, η) in $\mathcal{C}$ is an object $M \in |\mathcal{C}|$ together with two arrows $\mu : M \otimes M \longrightarrow M$, $\eta : I \longrightarrow M$ such that

$$\begin{array}{ccccc} (M \otimes M) \otimes M & \xrightarrow{\alpha} & M \otimes (M \otimes M) & \xrightarrow{1_M \otimes \mu} & M \otimes M \\ \downarrow{\scriptstyle \mu \otimes 1_M} & & & & \downarrow{\scriptstyle \mu} \\ M \otimes M & & \xrightarrow[\mu]{\hphantom{xxxxxxxxxxxx}} & & M \end{array} \qquad \text{(Mon1)}$$

and

$$\begin{array}{ccccc} I \otimes M & \xrightarrow{\eta \otimes 1_M} & M \otimes M & \xleftarrow{1_M \otimes \eta} & M \otimes I \\ & \searrow^{\lambda} & \downarrow{\scriptstyle \mu} & \swarrow_{\rho} & \\ & & M & & \end{array} \tag{Mon2}$$

commute. The map μ is called multiplication and η is called the unit. Equation (Mon1) is called the associativity axiom and (Mon2) is called the unit axiom .

Example 5.2.2. As one might expect, monoids in the monoidal category **Set** are monoids in the usual sense.

Example 5.2.3. Monoids in the monoidal category $\mathbf{Vect}_{\Bbbk}$ are associative $\Bbbk$-algebras.

Example 5.2.4 (c.f. [25])**.** Monoids in the monoidal category $\mathbf{Func}(\mathcal{C})$ are monads .

Definition 5.2.5. A **morphism of monoids** $f : (\,M,\,\mu,\,\eta\,) \longrightarrow (\,M',\,\mu',\,\eta'\,)$ is an arrow $f : M \longrightarrow M'$ such that following diagrams commute:

$$\begin{array}{ccc} M \otimes M & \xrightarrow{f \otimes f} & M' \otimes M' \\ {\scriptstyle \mu}\downarrow & & \downarrow{\scriptstyle \mu'} \\ M & \xrightarrow[f]{} & M' \end{array} \qquad\qquad \begin{array}{ccc} M & \xrightarrow{f} & M' \\ {\scriptstyle \eta}\nwarrow & & \nearrow{\scriptstyle \eta'} \\ & I & \end{array}$$

The following lemma is immediate and can be found in [25].

Lemma 5.2.6. Given a monoidal category $(\,\mathcal{C},\,\otimes, I,\,\alpha,\,\lambda,\,\rho\,)$ the monoids in $\mathcal{C}$ form a category $Mon_{\mathcal{C}}$.

5.3 Monoidal Functors and Natural Transformations

Definition 5.3.1 (c.f. [25])**.** A **monoidal functor** $(\,F,\,d^F,\,e^F\,) : \mathcal{C} \longrightarrow \mathcal{D}$ between monoidal categories $\mathcal{C}$ and $\mathcal{D}$ consists of the following three items

- a functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ between categories;
- morphisms $d^F_{A,B} : F(A) \otimes F(B) \longrightarrow F(A \otimes B)$ in $\mathcal{D}$ for all $A, B \in |\mathcal{C}|$ which are natural in A and B and,
- for tensor units I and I', a morphism $e^F : I' \longrightarrow F(I)$ in $\mathcal{D}$.

These must make the following three diagrams, involving the structural maps α, λ, and ρ, commute in $\mathcal{D}$:

$$\begin{array}{ccc} (F(A) \otimes F(B)) \otimes F(C) & \xrightarrow{\alpha'} & F(A) \otimes (F(B) \otimes F(C)) \\ \downarrow{\scriptstyle d^F_{A,B} \otimes 1} & & \downarrow{\scriptstyle 1 \otimes d^F_{B,C}} \\ F(A \otimes B) \otimes F(C) & & F(A) \otimes F(B \otimes C) \\ \downarrow{\scriptstyle d^F_{A \otimes B,C}} & & \downarrow{\scriptstyle d^F_{A,B \otimes C}} \\ F((A \otimes B) \otimes C) & \xrightarrow[F(\alpha)]{} & F(A \otimes (B \otimes C)) \end{array} \tag{MF1}$$

$$\begin{array}{ccc} F(B) \otimes I' & \xrightarrow{\rho'} & F(B) \\ \downarrow{\scriptstyle 1 \otimes e^F} & & \uparrow{\scriptstyle F(\rho)} \\ F(B) \otimes F(I) & \xrightarrow[d^F_{B,I}]{} & F(B \otimes I) \end{array} \tag{MF2}$$

$$\begin{array}{ccc} I' \otimes F(B) & \xrightarrow{\lambda'} & F(B) \\ \downarrow{\scriptstyle e^F \otimes 1} & & \uparrow{\scriptstyle F(\lambda)} \\ F(I) \otimes F(B) & \xrightarrow[d^F_{I,B}]{} & F(I \otimes B) \end{array} \tag{MF3}$$

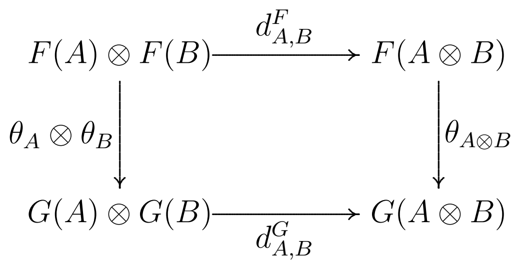
A monoidal functor is said to be **strong** when e^F and all the $d^F_{A,B}$ are isomorphisms, and **strict** when e^F and all the $d^F_{A,B}$ are identities.

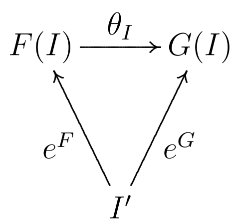
Remark 5.3.2. If (F, d^F, e^F) is a strict monoidal functor then Definition 5.3.1 reduces to the following:

- a functor $F : M \longrightarrow M'$;
- $F(\alpha) = \alpha'$, $F(\lambda) = \lambda'$, $F(\rho) = \rho'$;
- $F(f \otimes g) = F(f) \otimes F(g)$ for all arrows f and g in M.

Next we consider what morphisms of monoidal functors should be.

Definition 5.3.3 (c.f. [25])**.** A **monoidal natural transformation** $\theta : (F, d^F, e^F) \longrightarrow (G, d^G, e^G)$ between monoidal functors (F, d^F, e^F), $(G, d^G, e^G) : M \longrightarrow M'$ is a natural transformation $\theta : F \longrightarrow G$ between the functors F and G such that the following two diagrams



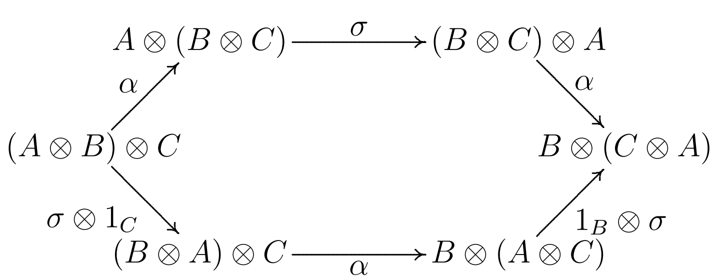


commute.

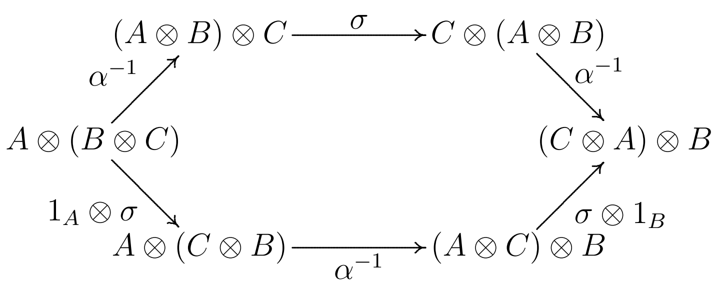
5.4 Braided Monoidal Categories

Braided monoidal categories form an interesting class of monoidal categories.

Definition 5.4.1 (c.f. [18])**.** A natural isomorphism $\sigma_{A,B} : A \otimes B \longrightarrow B \otimes A$ in a monoidal category, $(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$, is a **braiding** if the following diagrams commute:



(B1)



(B2)

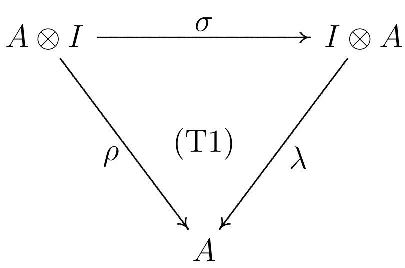
A monoidal category equipped with a braiding is called a **braided monoidal category**.

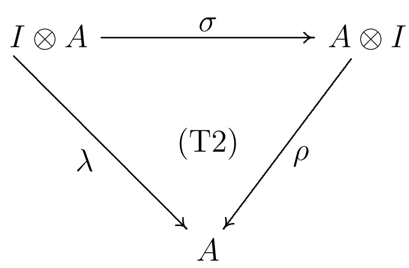
Remark 5.4.2. If $(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho, \sigma)$ is a braided monoidal category then it may happen that $\sigma_{B,A} \circ \sigma_{A,B} = 1_{A \otimes B}$ for all objects A and B in $\mathcal{C}$. In this case we say that $(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho, \sigma)$ is a **symmetric monoidal category**.

Lemma 5.4.3. If $\mathcal{C}$ is a symmetric monoidal category then $Mon_{\mathcal{C}}$ is monoidal.

The following is due to Joyal and Street [18].

Proposition 5.4.4. The following diagrams are commutative in any braided monoidal category $(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho, \sigma)$:





We now state a result of Kelly and Laplaza found in [22].

Proposition 5.4.5. If $\mathcal{C}$ is any monoidal category with tensor unit I, then the monoid $Hom(I, I)$ is commutative. Furthermore the value of the composite $I \cong I \otimes I \xrightarrow{f \otimes g} I \otimes I \cong I$ shows that $f \circ g = g \circ f$.

Remark 5.4.6. We note that $Hom(I, I)$ is sometimes called **the set of scalars**.

5.5 Tensor $*$-Categories

In this section we give a brief overview of *tensor $*$-categories* and related categorical notions which are relevant to AQFT. The reader is referred to the work of Selinger [36], where the detailed definition and properties are given. In [36], he also gives a graphical calculus for representing morphisms in such categories.

Definition 5.5.1. A **dagger structure** on a category $\mathcal{C}$ consists a functor $(-)^* : \mathcal{C}^{op} \longrightarrow \mathcal{C}$ such that $A^* = A$ for all objects A and $(f^*)^* = f$ for all arrows f. A **dagger category** consist of a category $\mathcal{C}$ equipped with a dagger structure $(-)^*$. If $\mathcal{C}$ and $\mathcal{D}$ are dagger categories then a **dagger functor** from $\mathcal{C}$ to $\mathcal{D}$ consists of a functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ such that $F(f^*) = F(f)^*$ for all arrows f in $\mathcal{C}$.

Example 5.5.2. The category **Hilb** of Hilbert spaces with T^* defined to be the adjoint of the bounded linear map T yields a dagger structure on **Hilb**. Similarly the category **Rel** of sets and relations is also a dagger category with $R^* = \bar{R}$ where $\bar{R}$ is the converse relation of R.

Definition 5.5.3. In a dagger category $\mathcal{C}$ we say that a map $f : X \longrightarrow Y$ is an **isometry** if $f^* \circ f = id_X$, we say f is **unitary** if f and f^* are both isometries. If $p : X \longrightarrow X$ then p is called a **projection** if $p = p \circ p = p^*$.

Definition 5.5.4. A **$\mathbb{C}$-linear category** is category $\mathcal{C}$ such that for each pair of objects A and B the homset $\mathcal{C}(A, B)$ is a complex vector space and such that the composition map $\circ : \mathcal{C}(B, C) \times \mathcal{C}(A, B) \longrightarrow \mathcal{C}(A, C)$ is bilinear. A functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ between $\mathbb{C}$-linear categories is called **$\mathbb{C}$-linear** if for all objects A and B in $\mathcal{C}$ the function $F : \mathcal{C}(A, B) \longrightarrow \mathcal{D}(FA, FB)$ is $\mathbb{C}$-linear. An object X in a $\mathbb{C}$-linear category is **irreducible** if $\mathcal{C}(X, X) = \mathbb{C} \cdot id_X$.

Definition 5.5.5. If X and Y are objects in $\mathbb{C}$-linear category $\mathcal{C}$ then a **direct sum** (also called a biproduct) of X and Y consists of an object Z in $\mathcal{C}$ and maps $X \xrightarrow{u_1} Z \xleftarrow{u_2} Y$ and maps $X \xleftarrow{v_1} Z \xrightarrow{v_2} Y$ such that $v_1 \circ u_1 = id_X$, $v_2 \circ u_2 = id_Y$ and $u_1 \circ v_1 + u_2 \circ v_2 = id_Z$. A $\mathbb{C}$-linear category is called **semisimple** if every object is a finite direct sum of irreducible objects.

Definition 5.5.6. If $\mathcal{C}$ is a $\mathbb{C}$-linear category, then a **$*$-operation on $\mathcal{C}$** consists of a dagger structure $(-)^*$ on $\mathcal{C}$ such that the function $(-)^* : \mathcal{C}(A, B) \longrightarrow \mathcal{C}(B, A)$ is antilinear for all objects A and B. A $*$-operation on $\mathbb{C}$-linear category is **positive** if $f^* \circ f = 0$ implies $f = 0$. A **$*$-category** consist of a $\mathbb{C}$-linear category equipped with a positive $*$-operation. A functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ between $*$-categories is called a **$*$-functor** if F is at the same time a $\mathbb{C}$-linear functor and a dagger functor.

Remark 5.5.7. In any $\mathbb{C}$-linear category $\mathcal{C}$ we have that $\mathcal{C}(A, A)$ is a $\mathbb{C}$-algebra. If $\mathcal{C}$ is $*$-category then $\mathcal{C}(A, A)$ is a $*$-algebra.

Definition 5.5.8. A **C^*-category** is a $*$-category $\mathcal{C}$ such that for each pair of objects A, and B the space $\mathcal{C}(A, B)$ comes equipped with a norm, denoted $\| \ \|_{A,B}$, with respect to which it is a Banach space. Moreover these norms must satisfy

1. $\|g \circ f\|_{A,C} \leq \|g\|_{B,C} \|f\|_{A,B}$ for all arrows $f : A \longrightarrow B$, and $g : B \longrightarrow C$, and
2. $\|f^* \circ f\|_{A,A} = \|f\|^2_{A,B}$ for all arrows $f : A \longrightarrow B$.

Remark 5.5.9. In a C^*-category $\mathcal{C}$, the space $\mathcal{C}(A, A)$ is a C^*-algebra for all objects A. Moreover if $\mathcal{A}$ is a C^*-algebra then it can be viewed as a one-object C^*-category in an evident manner.

Another example of a C^*-category is the category **Hilb**, which is of course the motivating example for the definition. Yet another example of a C^*-category (described in Example 5.5.10) arises from considering the set of self-adjoint idempotent elements of a C^*-algebra.

Example 5.5.10 (c.f. [15])**.** Suppose $\mathcal{A}$ is a C^*-algebra, then we define a category $\mathcal{P}(\mathcal{A})$ as follows. An object consists of an element $a \in \mathcal{A}$ such that $a = a^2 = a^*$ and an arrow $r : a \longrightarrow a'$ consists of an element $r \in \mathcal{A}$ such that $ra = r = a'r$. Then composition is given by multiplication in $\mathcal{A}$ and the identity arrow on a is equal to a, i.e. $id_a = a$.

Definition 5.5.11. If $\mathcal{C}$ is a $*$-category then we say that an object X is a **subobject** of Y if there exists and isometry $u : X \longrightarrow Y$ in $\mathcal{C}$, i.e. we have a map $u : X \longrightarrow Y$

such that $u^* \circ u = id_X$. We say that $\mathcal{C}$ has **subobjects** if for every object Y and projection $p : Y \longrightarrow Y$ there exists an object X and an isometry $u : X \longrightarrow Y$ such that $p = u \circ u^*$. Given objects X and Y of $\mathcal{C}$ a **direct sum** of X and Y consists of an object Z and isometries $u : X \longrightarrow Z$ and $v : Y \longrightarrow Z$ such that $u \circ u^* + v \circ v^* = id_Z$. We say that $\mathcal{C}$ **has direct sums** if each pair of objects has a direct sum.

Definition 5.5.12. A **$\mathbb{C}$-linear tensor category** consists of a $\mathbb{C}$-linear category $\mathcal{C}$ equipped with a monoidal structure $(\mathcal{C}, \otimes, \alpha, \lambda, \rho, I)$ such that $\otimes : \mathcal{C}(A, B) \times \mathcal{C}(X, Y) \longrightarrow \mathcal{C}(A \otimes X, B \otimes Y)$ is bilinear for all objects A, B, X, and Y in $\mathcal{C}$. We say that $\mathcal{C}$ is a **symmetric $\mathbb{C}$-linear tensor category** in case the underlying monoidal category is symmetric.

Definition 5.5.13. A **tensor $*$-category** is a $\mathbb{C}$-linear tensor category $\mathcal{C}$ which is also a $*$-category and satisfies $(f \otimes g)^* = f^* \otimes g^*$ for all arrows f and g. Moreover all the components of the structural isomorphisms α, λ, and ρ are required to be unitary. We say that $\mathcal{C}$ is a **symmetric tensor $*$-category** if it also comes equipped with a symmetry for the monoidal structure on $\mathcal{C}$ which is unitary. A functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ between tensor $*$-categories is called a **tensor $*$-functor** (respectively symmetric tensor $*$-functor) if it is a $*$-functor, which is also a monoidal functor (respectively symmetric monoidal functor) and all the accompanying structural maps are unitary.

We present a construction which can be thought of as a categorical extension of the GNS Theorem discussed previously (see Section 3.2 Theorems 3.3.11 and 3.3.12). The following is taken from [15] where the elementary theory of C^* and W^*-categories is developed. If $\mathcal{C}$ is a C^* category then a *state* on $\mathcal{C}$ is a pair (A, ϕ) where A is an object of $\mathcal{C}$ and $\phi : \mathcal{C}(A, A) \longrightarrow \mathbb{C}$ is a positive linear form on the C^*-algebra $\mathcal{C}(A, A)$. A *representation* of a C^* category is a $*$- functor $F : \mathcal{C} \longrightarrow$ **Hilb**. The following proposition is taken from [15].

Proposition 5.5.14 (c.f. Proposition 1.9 in [15])**.** If $\mathcal{C}$ is a C^* category with state (A, ϕ) then there exists a representation $F_\phi : \mathcal{C} \longrightarrow$ **Hilb** of $\mathcal{C}$ and a cyclic vector $v_\phi \in F_\phi(A)$ such that

$$\phi(a) = \langle v_\phi, F_\phi(a) v_\phi \rangle \qquad \forall a \in \mathcal{C}(A, A).$$

Moreover if F is another representation with cyclic vector $v \in F(A)$ such that $\phi(a) = \langle v, F(a)v \rangle$ for all $a \in \mathcal{C}(A, A)$ then there is a unique unitary natural transformation $u : F_\phi \Longrightarrow F$ such that $u_A v_\phi = v$.

Proof. We give just a brief sketch of the proof. For $B \in |\mathcal{C}|$, we build a Hilbert space $F_\phi(B)$ as follows. For a, b in the Banach space, $\mathcal{C}(A, B)$ define a semi-definite inner-product by

$$\langle a, b \rangle = \phi(a^* \circ b).$$

Now define $F_\phi(B)$ as the Hilbert space obtained as the completion of the quotient of $\mathcal{C}(A, B)$ by the vectors of length zero. Thus we have a canonical map $\mathcal{C}(A, B) \longrightarrow F_\phi(B)$ which we denote by $a \mapsto \hat{a}$. Now given $b : B \longrightarrow C$ in $\mathcal{C}$ we define $F_\phi(b) : F_\phi B \longrightarrow F_\phi C$ by $F_\phi(b)(\hat{a}) = \widehat{b \circ a}$. The details that this gives a bounded linear map can be found in [15]. □

The last theorem we state from [15] concerns the existence of faithful representations.

Theorem 5.5.15 (c.f. Proposition 1.14 in [15])**.** If $\mathcal{C}$ is a C^* category then there exists a faithful embedding functor $F : \mathcal{C} \longrightarrow \mathbf{Hilb}$.

Proof. Again we simply sketch the idea of the proof here. Let $F = \bigoplus_\phi F_\phi$ where ϕ ranges over all positive linear forms on the C^*-algebras $\mathcal{C}(A, A)$ and A ranges of over all objects in $\mathcal{C}$. Then clearly F is a representation and it is faithful since if $F(a) = 0$ for some $a \in \mathcal{C}(A, B)$ it follows that $\phi(a^* a) = 0$ for all positive linear functionals on $\mathcal{C}(A, A)$ and so $a = 0$. □

5.6 Conjugates and Traces

In this section we examine the notions of *conjugate object* in a tensor $*$-category and the related notion of a *trace*. If one makes certain assumptions then these notions considered yield the familiar categorical concepts of *compact closed categories* and *traced monoidal categories*.

Definition 5.6.1. If X is an object in tensor $*$-category $\mathcal{C}$ then a **conjugate** of X consists of an object $\overline{X}$ in $\mathcal{C}$ and arrows $r : I \longrightarrow \overline{X} \otimes X$ and $\overline{r} : I \longrightarrow X \otimes \overline{X}$ such that the diagrams

$$\begin{array}{ccccccc} X & \xrightarrow{\lambda_X^{-1}} & I \otimes X & \xrightarrow{\overline{r} \otimes X} & (X \otimes \overline{X}) \otimes X & \xrightarrow{\alpha_{X,\overline{X},X}} & X \otimes (\overline{X} \otimes X) \\ & & & & & & \Big\downarrow X \otimes r^* \\ & & & id_X \searrow & & & X \otimes I \\ & & & & & & \Big\downarrow \rho_X \\ & & & & & & X \end{array}$$

$$\begin{array}{ccccccc} \overline{X} & \xrightarrow{\lambda_{\overline{X}}^{-1}} & I \otimes \overline{X} & \xrightarrow{r \otimes \overline{X}} & (\overline{X} \otimes X) \otimes \overline{X} & \xrightarrow{\alpha_{\overline{X},X,\overline{X}}} & \overline{X} \otimes (X \otimes \overline{X}) \\ & & & & & & \Big\downarrow \overline{X} \otimes \overline{r}^* \\ & & & id_{\overline{X}} \searrow & & & \overline{X} \otimes I \\ & & & & & & \Big\downarrow \rho_{\overline{X}} \\ & & & & & & \overline{X} \end{array}$$

both commute. In this case we say that the triple $(\overline{X}, r, \overline{r})$ is a conjugate of X, and as an abbreviation we will also call just the object part of the triple, $\overline{X}$, a conjugate of X. We say that $\mathcal{C}$ **has conjugates** if every nonzero object has a conjugate.

Remark 5.6.2. Note that if we have a symmetric tensor $*$-category with conjugates and if we choose a conjugate for each object then this will induce a compact closed structure on the category. Indeed if $(\overline{X}, r, \overline{r})$ is a conjugate of X then we can define maps $\eta_X : I \longrightarrow \overline{X} \otimes X$ and $\epsilon_X : X \otimes \overline{X} \longrightarrow I$ by r and $\overline{r}^*$. Then for example one of the required equations for compact closure is

$$\lambda_X \circ (\overline{r}^* \otimes X) \circ \alpha^{-1}_{X,\overline{X},X} \circ (X \otimes r) \circ \rho^{-1}_X = id_X$$

which one can show is simply the image of the first conjugate equation under the functor $(-)^*$. The other equation for compact closure is exactly the same equation as the second conjugate equation.

Definition 5.6.3. If X is an object in a tensor $*$-category then a conjugate of X, $(\overline{X}, r, \overline{r})$, is called a **standard conjugate** in case

$$\begin{array}{ccccc} I & \xrightarrow{\;r\;} & \overline{X} \otimes X & \xrightarrow{\;\overline{X} \otimes s\;} & \overline{X} \otimes X \\ \downarrow{\scriptstyle \overline{r}} & & & & \downarrow{\scriptstyle r^*} \\ X \otimes \overline{X} & \xrightarrow[\;s \otimes \overline{X}\;]{} & X \otimes \overline{X} & \xrightarrow[\;\overline{r}^*\;]{} & I \end{array}$$

commutes for all arrows $s : X \longrightarrow X$.

Definition 5.6.4. A **TC*** consists of tensor $*$-category $\mathcal{C}$ such that the following conditions hold:

1. $\dim \mathcal{C}(X, Y) < \infty$ for all objects X and Y,
2. $\mathcal{C}$ has conjugates,
3. $\mathcal{C}$ has direct sums,
4. $\mathcal{C}$ has subobjects, and
5. the tensor unit I is irreducible.

An **STC*** consists of TC^* such that the underlying tensor $*$-category is symmetric.

The prototypical example of an STC^* is the category $\mathbf{Hilb}_{fd}$ of finite dimensional Hilbert spaces and linear maps. For each Hilbert space H the Hilbert space $\overline{H}$ is given by the dual space and if e_i is basis for H and f_i the corresponding dual basis for $\overline{H}$ then $r(1_{\mathbb{C}}) = \sum f_i \otimes e_i$ and $\overline{r}(1_{\mathbb{C}}) = \sum e_i \otimes f_i$. An important offshoot of this example is the category of finite dimensional unitary representations of a compact group G, $\mathbf{Rep}(G)_{fd}$.

Lemma 5.6.5. Every TC^* is semisimple.

Lemma 5.6.6. If $\mathcal{C}$ is a TC^* then every object X in $\mathcal{C}$ admits a standard conjugate.

Proposition 5.6.7. If $\mathcal{C}$ is a TC^* and X is an object of $\mathcal{C}$ and $(\overline{X}, r, \overline{r})$ is a standard conjugate then the map

$$Tr_X : \mathcal{C}(X,X) \longrightarrow \mathbb{C}$$
$$s \longmapsto r^* \circ (\overline{X} \otimes s) \circ r$$

is independent of the choice of standard conjugate. Moreover we also have that

1. $Tr_X(s \circ t) = Tr_Y(t \circ s)$ for all $s : Y \longrightarrow X$, and $t : X \longrightarrow Y$, and
2. $Tr_{X\otimes Y}(s \otimes t) = Tr_X(s)Tr_y(t)$ for all $s : X \longrightarrow X$ and $t : Y \longrightarrow Y$.

The map Tr_X is called the **trace**.

Thus the proposition shows that in a TC^* there is a notion of trace of an endomorphism the result of which is a scalar. Moreover this trace function satisfies similar properties of the familiar trace in linear algebra. Now as the trace of the identity morphism in linear algebra yields the dimension of the vector space it is sensible to define the dimension an object in general TC^* as such.

Definition 5.6.8. If X is an object in a TC^* then we define the **dimension** of X, denoted $d(X)$, by $d(X) = Tr_X(id_X)$. In other words $d(X) = r^* \circ r$ for any standard conjugate $(\overline{X}, r, \overline{r})$.

This innocent concept of dimension is in fact a powerful tool that plays a key role in the reconstruction theorem. Here are some of its properties.

Theorem 5.6.9. If X and Y are objects in a TC^* then

1. $d(X \oplus Y) = d(X) + d(Y)$,
2. $d(X \otimes Y) = d(X)d(Y)$,
3. $d(\overline{X}) = d(X) \geq 1$, and

4. if $d(X) = 1$ then $X \otimes \overline{X} \cong I$, in which case we say X is **invertible**.

Remark 5.6.10. In any compact closed symmetric monoidal category we can define the trace of an endomorphism $f : X \longrightarrow X$ by $tr_X(f) = \epsilon_X \circ c_{\overline{X},X} \circ \eta_X$ where $\eta_X : I \longrightarrow \overline{X} \otimes X$ and $\epsilon_X : X \otimes \overline{X} \longrightarrow I$ are the usual unit and counit respectively. Notice that this is in general not same as the trace defined in terms of standard conjugates in a TC^*. If X is an object in an STC^* with standard conjugate $(\overline{X}, r, \overline{r})$ then for any $f : X \longrightarrow X$ we will have $Tr_X(f) = tr_X(f)$ provided the diagram

$$\begin{array}{ccc} I & \xrightarrow{\quad r \quad} & \overline{X} \otimes X \\ & {\scriptstyle \overline{r}} \searrow & \big\downarrow {\scriptstyle c_{\overline{X},X}} \\ & & X \otimes \overline{X} \end{array} \tag{20}$$

commutes. This equation first surfaced in a landmark paper by Abramsky and Coecke, [1], where they constructed a categorical framework for doing abstract quantum mechanics. Abramsky and Coecke defined the notion of a *strongly compact closed category* which is a compact closed category equipped with a dagger structure which is a strict symmetric monoidal functor $(-)^* : \mathcal{C}^{op} \longrightarrow \mathcal{C}$ and such that the above diagram commutes. In [36], Selinger proposed the term *dagger compact closed category* for such categories, and they have subsequently become known by this name.

Lemma 5.6.11. If X is an object in STC^* and $(\overline{X}, r, \overline{r})$ is a standard conjugate then the expression

$$\Theta(X) = (r^* \otimes X) \circ (\overline{X} \otimes c_{X,X}) \circ (r \otimes X)$$

defined by:

$$X \xrightarrow{\lambda_X^{-1}} I \otimes X \xrightarrow{r \otimes X} (\overline{X} \otimes X) \otimes X \xrightarrow{\alpha} \overline{X} \otimes (X \otimes X) \xrightarrow{\overline{X} \otimes c_{X,X}} \overline{X} \otimes (X \otimes X) \xrightarrow{\alpha^{-1}} (\overline{X} \otimes X) \otimes X \xrightarrow{r^* \otimes X} I \otimes X \xrightarrow{\lambda} X$$

$$\Theta(X) = \lambda \circ (r^* \otimes X) \circ \alpha^{-1} \circ (\overline{X} \otimes c_{X,X}) \circ \alpha \circ (r \otimes X) \circ \lambda_X^{-1} : X \longrightarrow X$$

is independent of the choice of standard conjugate.

The map $\Theta(X) : X \longrightarrow X$ defined in Lemma 5.6.11 is called the *twist*. We now summarize a few important properties of this map.

Proposition 5.6.12. If $\mathcal{C}$ is an STC^* then $\Theta(X) : X \longrightarrow X$ is a unitary monoidal natural transformation of the identity functor on $\mathcal{C}$. Explicitly we have

1. $\Theta(Y) \circ s = s \circ \Theta(X)$ for all $s : X \longrightarrow Y$,

2. $\Theta(X)^* = \Theta(X)^{-1}$,

3. $\Theta(X \otimes Y) = \Theta(X) \otimes \Theta(Y)$, moreover we also have that

4. $\Theta(X)^2 = id_X$

5. If X and Y are irreducible then $\Theta(X) = \pm id_X$, and if Z is an irreducible direct summand of $X \otimes Y$ then $\Theta(Z) = \omega_X \omega_Y id_Z$ where $\Theta(X) = \omega_X id_X$ and similarly for ω_Y.

We close this section with a final definition and an interesting result related to the theory of TC^*'s.

Definition 5.6.13. If $\mathcal{C}$ is a C^*-category which is also a tensor $*$-category, with respect to the same dagger structure, then we say $\mathcal{C}$ is **C^*-tensor category** in case $\|s \otimes t\| \leq \|s\| \cdot \|t\|$ for all arrows s and t.

Lemma 5.6.14. If $\mathcal{C}$ is a C^*-tensor category with direct sums, and an irreducible tensor unit, then $\dim \mathcal{C}(X, Y) < \infty$ whenever X and Y have conjugates.

Proposition 5.6.15. If $\mathcal{C}$ is a C^*-tensor category which has direct sums, subobjects, conjugates, and an irreducible tensor unit then $\mathcal{C}$ is a TC^*. Conversely if $\mathcal{C}$ is a TC^* then there exist unique norms on $\mathcal{C}(X, Y)$ making $\mathcal{C}$ a C^*-tensor category.

Chapter 6

Premonoidal Categories

6.1 Introduction to Premonoidal Categories

As we saw previously, a monoidal category can be seen as a natural generalization of the notion of a monoid. There is however a simpler generalization of a monoid. Any monoid M can be viewed as a one-object category denoted $M[1]$ and $M[1](1,1) = M$ with $m \circ n = mn$ where the right hand side is multiplication in M. Thus we can think of a small category as a "multi-object monoid" or that categories come from monoids. So a natural question is to ask: what is the analog for the case of monoidal categories? It is a well known fact that the category $M[1]$ is monoidal if and only if the monoid M is commutative (this is due to the fact that $\otimes$ is a bifunctor). Thus monoidal categories are "multi-object commutative monoids". Now as there are plenty of noncommutative monoids one should reverse this process to see what categorical gadget corresponds to the noncommutative monoids. This leads us to the concept of a premonoidal category, a notion defined by Power and Robinson in [31]. We follow their presentation given in [31].

Definition 6.1.1. A **binoidal** category consists of a category $\mathcal{C}$ and functors $H_B : \mathcal{C} \longrightarrow \mathcal{C}$ and $K_B : \mathcal{C} \longrightarrow \mathcal{C}$ for all objects B in $\mathcal{C}$ satisfying $H_B(C) = K_C(B)$ for all pairs of objects B and C in $\mathcal{C}$.

In a binoidal category the object $H_B(C) = K_C(B)$ is denoted $B \otimes C$ and for any arrow $f : X \longrightarrow Y$ we write $B \otimes f$ for $H_B(f)$ and $f \otimes B$ for $K_B(f)$. Thus in this new notation $H_B = B \otimes -$ and $K_B = - \otimes B$. Notice that $- \otimes -$ is only a functor when one of the arguments is fixed, i.e. it is not a bifunctor.

Definition 6.1.2. If $\mathcal{C}$ is a binoidal category and $f : A \longrightarrow C$ is an arrow in $\mathcal{C}$ then f is **central** if for all arrows $g : B \longrightarrow D$ both diagrams

$$
\begin{array}{ccc}
A \otimes B & \xrightarrow{f \otimes B} & C \otimes B \\
{\scriptstyle A \otimes g}\downarrow & & \downarrow{\scriptstyle C \otimes g} \\
A \otimes D & \xrightarrow[f \otimes D]{} & C \otimes D
\end{array}
\qquad\qquad
\begin{array}{ccc}
B \otimes A & \xrightarrow{B \otimes f} & C \otimes B \\
{\scriptstyle g \otimes A}\downarrow & & \downarrow{\scriptstyle g \otimes C} \\
D \otimes A & \xrightarrow[D \otimes f]{} & C \otimes D
\end{array}
$$

commute. The composite $C \otimes g \circ f \otimes B$ will be denoted $f \ltimes g$, and read f left-times g, and the composite $f \otimes D \circ A \otimes g$ will be denoted $f \rtimes g$ and read f right-times g. Moreover if f is central then we will write $f \otimes g$ for $f \ltimes g = f \rtimes g$ and $g \otimes f$ for the other composite.

Now in order to define a premonoidal category we require one more definition. Namely

Definition 6.1.3. If $\mathcal{C}$ is a binoidal category and $G, H : \mathcal{B} \longrightarrow \mathcal{C}$ are functors then a natural transformation $\alpha : G \Longrightarrow H$ is **central** if its components $\alpha_B : G(B) \longrightarrow H(B)$ are central maps in $\mathcal{C}$.

Definition 6.1.4. A **premonoidal category** consists of a binoidal category $\mathcal{C}$ together with a distinguished object $I \in |\mathcal{C}|$ and central natural isomorphisms α, λ and ρ with components $\alpha_{A,B,C} : (A \otimes B) \otimes C \longrightarrow A \otimes (B \otimes C)$, $\lambda_A : I \otimes A \longrightarrow A$, and $\rho_A : A \otimes I \longrightarrow A$. These structural isomorphisms must satisfy the same coherence equations as in the definition of a monoidal category. A premonoidal category is **strict** if the structural maps are identities.

We now pause to present some examples.

Example 6.1.5. If M is a monoid, then $M[1]$ is a one object strict premonoidal category.

Example 6.1.6. If $\mathcal{D}$ is any category then define a new category $\mathcal{C} = [\mathcal{D}, \mathcal{D}]_u$ whose objects are functors $F : \mathcal{D} \longrightarrow \mathcal{D}$ and an arrow $h : F \longrightarrow G$ is a **transformation** i.e. consists of arrows $h_D : FD \longrightarrow GD$ for each $D \in |\mathcal{D}|$. Then $F \otimes G = F \circ G$ for $F, G \in |\mathcal{C}|$ and for any transformation $h : F \longrightarrow G$ define $(H \otimes h)_D = H(h_D)$ and $(h \otimes H)_D = h_{HD}$. Then $\mathcal{C}$ is a strict premonoidal category. If one restricts to transformations which are natural then one obtains a subcategory of $\mathcal{C}$ which is monoidal.

Example 6.1.7. Every monoidal category is a premonoidal category, and every strict monoidal category is a strict premonoidal category.

Definition 6.1.8. If $\mathcal{C}$ is a premonoidal category then the **centre** of $\mathcal{C}$ is the category $\mathcal{Z}(\mathcal{C})$ with objects the same as those of $\mathcal{C}$ and its arrows are the central maps in $\mathcal{C}$.

The following example justifies the choice of the term centre.

Example 6.1.9. If G is a group then it can viewed as a one object premonoidal category $G[1]$. It is easy to show that $\mathcal{Z}(G[1])$ is just the centre of G viewed as one object monoidal category.

This example suggests the following proposition.

Proposition 6.1.10. The centre $\mathcal{Z}(\mathcal{C})$ of a premonoidal category $\mathcal{C}$ is a monoidal category.

This clever observation by Power and Robinson allows them to easily prove the following coherence theorem for premonoidal categories.

Theorem 6.1.11. Every diagram built from the components of the structural natural isomorphisms in definition 6.1.4 of a premonoidal category commutes.

Proof. Given such a diagram in a premonoidal category $\mathcal{C}$ we have by definition that it consists entirely of central maps, therefore it is a diagram in the monoidal category $\mathcal{Z}(\mathcal{C})$. By the coherence theorem for monoidal categories this diagram commutes in $\mathcal{Z}(\mathcal{C})$ and hence in $\mathcal{C}$. □

At this point we conclude this section with a brief discussion on the notion of morphism between premonoidal categories.

Definition 6.1.12. Let $(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$ and $(\mathcal{D}, \otimes, J, \alpha', \lambda', \rho')$ be premonoidal categories. A **premonoidal functor** from $\mathcal{C}$ to $\mathcal{D}$ consists of a functor $F : \mathcal{C} \longrightarrow \mathcal{D}$, which maps central arrows in $\mathcal{C}$ to central arrows in $\mathcal{D}$, and a central natural transformation d^F with components $d^F_{A,B} : (FA) \otimes (FB) \longrightarrow F(A \otimes B)$ and a central map $e^F : J \longrightarrow F(I)$ satisfying diagrams analogous to those for monoidal functors. We say that a premonoidal functor is **strong** if the maps $d_{A,B}$ and e are isomorphisms. A premonoidal functor is said to be **strict** if these maps are identities.

6.2 Examples of Premonoidal Categories

We now outline two more interesting examples.

Example 6.2.1. Let $\mathcal{C}$ be a symmetric monoidal category with symmetry $\tau_{A,B} : A \otimes B \longrightarrow B \otimes A$ and let $S \in |\mathcal{C}|$ be a fixed object. Define a new category $\mathcal{C}_S$ as follows, the objects are the same as those of $\mathcal{C}$ and $\mathcal{C}_S(X, Y) = \mathcal{C}(X \otimes S, Y \otimes S)$. For $Z \in |\mathcal{C}_S|$ and $f \in \mathcal{C}_S(X, Y)$ define $Z \otimes f \in \mathcal{C}_S(Z \otimes X, Z \otimes Y)$ as the composition;

$$\begin{array}{ccccc}
(Z \otimes X) \otimes S & \xrightarrow{\alpha_{Z,X,S}} & Z \otimes (X \otimes S) & \xrightarrow{id_Z \otimes f} & Z \otimes (Y \otimes S) \\
& \searrow^{Z \otimes f} & & & \downarrow{\scriptstyle \alpha^{-1}_{Z,Y,S}} \\
& & & & (Z \otimes Y) \otimes S
\end{array}$$

and then $f \otimes Z \in \mathcal{C}_S(X \otimes Z, Y \otimes Z)$ by

$$\begin{array}{ccccc}
(X \otimes Z) \otimes S & \xrightarrow{\tau_{X,Z} \otimes id_S} & (Z \otimes X) \otimes S & \xrightarrow{Z \otimes f} & (Z \otimes Y) \otimes S \\
& \searrow^{f \otimes Z} & & & \downarrow{\scriptstyle \tau_{Z,Y} \otimes id_S} \\
& & & & (Y \otimes Z) \otimes S.
\end{array}$$

The structural isomorphisms for associativity and units come from the corresponding maps in $\mathcal{C}$.

This example gives a nice construction for transforming a symmetric monoidal category into a premonoidal category. One symmetric monoidal category that will be of particular interest is **Hilb** the category of complex Hilbert spaces and bounded linear maps. If we fix a Hilbert space S with $dim(S) \geq 1$ and we consider the centre of $\mathbf{Hilb}_S$ then the following result holds.

The following is an example of the calculation of the centre of a premonoidal category, followed by an observation regarding the centre of functor categories. These results are new.

Theorem 6.2.2. If S is a Hilbert space with $dimS \geq 1$ then $\mathcal{Z}(\mathbf{Hilb}_S) \simeq \mathbf{Hilb}$.

Proof. If $dim(S) = 1$ then $S \cong \mathbb{C}$ and clearly $\mathbf{Hilb}_S \simeq \mathbf{Hilb}$. So now suppose that $dim(S) > 1$ and that $f \in \mathbf{Hilb}_S(X, Y)$ is central. Then we will show that $f = \hat{f} \otimes id_S$ for some bounded linear map $\hat{f} : X \longrightarrow Y$. Let $B_S = \{h_j \mid j \in J\}$ be an orthonormal basis for S. Then for $a \neq b \in J$ define $T_{a,b} : S \otimes S \longrightarrow S \otimes S$ by $T_{a,b}(h_i \otimes h_j) = (\delta_{i,a}\delta_{j,b} + \delta_{i,b}\delta_{j,a})h_j \otimes h_i$ where $\delta_{p,q} = 1$ if $p = q$ and $\delta_{p,q} = 0$ otherwise. Now notice the that the vector subspace $(S \otimes S)_{a,b} = \{\lambda h_a \otimes h_b + \mu h_b \otimes h_a \mid \lambda, \mu \in \mathbb{C}\}$ is a finite-dimensional linear subspace of $S \otimes S$ and hence is closed. Moreover the map $T_{a,b}$ is then just the projection onto the closed subspace $(S \otimes S)_{a,b}$ followed by a twist and is therefore continuous. Now suppose that $B_X = \{e_i \mid i \in I\}$ and $B_Y = \{g_k \mid k \in K\}$ are orthonormal bases for X and Y respectively. We now compute $f \rtimes T_{a,b} : (X \otimes S) \otimes S \longrightarrow (Y \otimes S) \otimes S$ on basis elements.

$$\begin{aligned} f \rtimes T_{a,b}(e_i \otimes h_j \otimes h_k) &= f \otimes S(X \otimes T_{a,b}(e_i \otimes h_j \otimes h_k)) \\ &= f \otimes S[(\delta_{j,a}\delta_{k,b} + \delta_{j,b}\delta_{k,a})e_i \otimes h_k \otimes h_j] \end{aligned}$$

Now taking $j = k = a \neq b$ we get that

$$\begin{aligned} f \rtimes T_{a,b}(e_i \otimes h_j \otimes h_k) &= f \rtimes T_{a,b}(e_i \otimes h_a \otimes h_a) \\ &= 0. \end{aligned}$$

On the other hand we now calculate $f \ltimes T_{a,b}$ applied to $(e_i \otimes h_j \otimes h_k)$. First observe that

$$\begin{aligned} f \otimes S(e_i \otimes h_j \otimes h_k) &= (\tau_{S,Y} \otimes id_S)(id_S \otimes f)(\tau_{X,S} \otimes id_S)(e_i \otimes h_j \otimes h_k) \\ &= (\tau_{S,Y} \otimes id_S)(h_j \otimes f(e_i \otimes h_k)) \\ &= (\tau_{S,Y} \otimes id_S)h_j \otimes (\sum_{r\in K,\, p\in J} c_{i,k}^{r,p} g_r \otimes h_p) \\ &= (\tau_{S,Y} \otimes id_S)(\sum_{r\in K,\, p\in J} c_{i,k}^{r,p} h_j \otimes g_r \otimes h_p) \\ &= \sum_{r\in K,\, p\in J} c_{i,k}^{r,p} g_r \otimes h_j \otimes h_p. \end{aligned}$$

Thus we have

$$\begin{aligned} f \ltimes T_{a,b}(e_i \otimes h_j \otimes h_k) &= (Y \otimes T_{a,b})(f \otimes S)(e_i \otimes h_j \otimes h_k) \\ &= Y \otimes T_{a,b}(\sum_{r\in K,\, p\in J} c_{i,k}^{r,p} g_r \otimes h_j \otimes h_p) \\ &= \sum_{r\in K,\, p\in J} c_{i,k}^{r,p} g_r \otimes T_{a,b}(h_j \otimes h_p) \\ &= \sum_{r\in K,\, p\in J} c_{i,k}^{r,p} (\delta_{j,a}\delta_{p,b} + \delta_{j,b}\delta_{p,a}) g_r \otimes h_p \otimes h_j. \end{aligned}$$

Now we have assumed that $j = k = a \neq b$ thus the above becomes

$$\sum_{r\in K,\, p\in J} c_{i,a}^{r,p} \delta_{p,b} g_r \otimes h_p \otimes h_a = \sum_{r\in K} c_{i,a}^{r,b} g_r \otimes h_b \otimes h_a.$$

Now as f is central it follows that

$$\sum_{r\in K} c_{i,a}^{r,b} g_r \otimes h_b \otimes h_a = 0.$$

Thus it follows from properties of orthonormal bases that $c_{i,a}^{r,b} = 0$ for all $r \in K$ as long as $a \neq b$. Thus in general we have shown that

$$c_{i,a}^{r,b} = \delta_{a,b} c_{i,a}^{r,b} \tag{21}$$

for all $r \in K$. We now show that $c_{i,a}^{r,a} = c_{i,b}^{r,b}$ for all a, $b \in J$. For this we consider the case when $j = a$ and $k = b$ and $a \neq b \in J$. In this case we get

$$\begin{aligned} f \rtimes T_{a,b}(e_i \otimes h_j \otimes h_k) &= (\delta_{j,a}\delta_{k,b} + \delta_{j,b}\delta_{k,a})(f \otimes S)(e_i \otimes h_j \otimes h_k) \\ &= \sum_{r\in K,\, p\in J} c_{i,j}^{r,p}(\delta_{j,a}\delta_{k,b} + \delta_{j,b}\delta_{k,a}) g_r \otimes h_k \otimes h_p \\ &= \sum_{r\in K,\, p\in J} c_{i,a}^{r,p} g_r \otimes h_b \otimes h_p \\ &= \sum_{r\in K} c_{i,a}^{r,a} g_r \otimes h_b \otimes h_a. \end{aligned}$$

On the other hand

$$\begin{aligned} f \ltimes T_{a,b}(e_i \otimes h_j \otimes h_k) &= \sum_{r\in K,\, p\in J} c_{i,k}^{r,p}(\delta_{j,a}\delta_{p,b} + \delta_{j,b}\delta_{p,a}) g_r \otimes h_p \otimes h_j. \\ &= \sum_{r\in K,\, p\in J} c_{i,b}^{r,p}\delta_{p,b} g_r \otimes h_p \otimes h_a \\ &= \sum_{r\in K} c_{i,b}^{r,b} g_r \otimes h_b \otimes h_a \end{aligned}$$

Now by centrality of f and properties of orthonormal bases it follows that $c_{i,a}^{r,a} = c_{i,b}^{r,b}$ for all $a, b \in J$ and $r \in K$. Thus fix $a \in J$ and define $d_i^r = c_{i,a}^{r,a}$ for all $r \in K$. We now have

$$\begin{aligned} f(e_i \otimes h_a) &= \sum_{r\in K,\, b\in J} c_{i,a}^{r,b} g_r \otimes h_b \\ &= \sum_{r\in K,\, b\in J} \delta_{a,b} c_{i,a}^{r,b} g_r \otimes h_b \\ &= \sum_{r\in K} c_{i,a}^{r,a} g_r \otimes h_a \\ &= \sum_{r\in K} d_i^r g_r \otimes h_a \\ &= (\sum_{r\in K} d_i^r g_r) \otimes h_a \\ &= \widehat{f}(e_i) \otimes h_a. \end{aligned}$$

The map $\widehat{f}$ is defined by the equation $\widehat{f}(e_i) = \sum_{r\in K} d_i^r g_r$. Moreover it is now clear that $f = \widehat{f} \otimes id_S$ as was claimed. Thus the natural inclusion functor $\mathbf{Hilb} \longrightarrow \mathbf{Hilb}_S$

is full and faithful and essentially surjective on objects when restricting the codomain to $\mathcal{Z}(\mathbf{Hilb}_S)$. Hence $\mathcal{Z}(\mathbf{Hilb}_S)$ is equivalent to $\mathbf{Hilb}$. $\square$

Another interesting example to consider the finding the centre of is the premonoidal category in Example 6.1.6. One might be tempted to guess that the centre of this premonoidal category would be the monoidal category whose objects are endofunctors and arrows natural transformations. However this turns out not to be the case.

Lemma 6.2.3. Let $\mathcal{C}$ be a category and let $[\mathcal{C},\mathcal{C}]_u$ be the premonoidal category defined in Example 6.1.6. Then $\mathcal{Z}([\mathcal{C},\mathcal{C}]_u) \subseteq \mathcal{C}^{\mathcal{C}}$.

Proof. We must show that every central map in $[\mathcal{C},\mathcal{C}]_u$ is natural. Let $t : F \longrightarrow G$ be a central map in $[\mathcal{C},\mathcal{C}]_u$ and let $f : A \longrightarrow B$ be any arrow in $\mathcal{C}$. Now let Δ_A and Δ_B denote the constant endofunctors on $\mathcal{C}$, then the family s_X defined by $s_X = f$ for all objects $X \in |\mathcal{C}|$ defines an arrow from Δ_A to Δ_B in $[\mathcal{C},\mathcal{C}]_u$. Now we invoke centrality of t to get that the diagram

$$\begin{array}{ccc} F \otimes \Delta_A = F \circ \Delta_A & \xrightarrow{F \otimes s} & F \otimes \Delta_B = F \circ \Delta_B \\ {\scriptstyle t \otimes \Delta_A}\big\downarrow & & \big\downarrow{\scriptstyle t \otimes \Delta_B} \\ G \otimes \Delta_A = G \circ \Delta_A & \xrightarrow[G \otimes s]{} & G \otimes \Delta_B = G \circ \Delta_B \end{array}$$

commutes. Thus for any object X we have the diagram

$$\begin{array}{ccc} F \otimes \Delta_A X = FA & \xrightarrow{(F \otimes s)_X = F(f)} & F \otimes \Delta_B X = FB \\ {\scriptstyle (t \otimes \Delta_A)_X = t_A}\big\downarrow & & \big\downarrow{\scriptstyle (t \otimes \Delta_B)_X = t_B} \\ G \otimes \Delta_A X = GA & \xrightarrow[(G \otimes s)_X = G(f)]{} & G \otimes \Delta_B X = GB \end{array}$$

which commutes for any arrow $f : A \longrightarrow B$ in $\mathcal{C}$. Thus t is a natural transformation from F to G. In general if $\alpha : F \longrightarrow G$ in $[\mathcal{C},\mathcal{C}]_u$ is natural it is not the case that it will be central. The naturality of α will guarantee that the diagram

$$\begin{array}{ccc} F \otimes H & \xrightarrow{\alpha \otimes H} & G \otimes H \\ {\scriptstyle F \otimes \beta}\downarrow & & \downarrow{\scriptstyle G \otimes \beta} \\ F \otimes K & \xrightarrow[\alpha \otimes K]{} & G \otimes K \end{array}$$

commutes for all arrows $\beta : H \longrightarrow K$ in $[\mathcal{C}, \mathcal{C}]_u$. However the other diagram required for α to be central has no reason to commute. At a typical component this diagram becomes

$$\begin{array}{ccc} HFA & \xrightarrow{H(\alpha_A)} & HGA \\ {\scriptstyle \beta_{FA}}\downarrow & & \downarrow{\scriptstyle \beta_{GA}} \\ KFA & \xrightarrow[K(\alpha_A)]{} & KGA \end{array}$$

whose commutativity is a priori independent of the naturality of α. $\square$

The following example illustrates that in general the above containment of Lemma 6.2.3 is strict.

Example 6.2.4. Consider the group $Q = \{1, -1, i, j, k \mid ij = k,\ jk = i,\ i^2 = j^2 = k^2 = -1\}$ of quaternions. Then we can view Q as a one object category denoted $Q[1]$. Now consider the category $[Q[1], Q[1]]_u$ of Example 6.1.6. Then objects of this premonoidal category are simply group homomorphisms of Q and arrows are merely elements of Q. Now notice that an arrow $g : F \longrightarrow G$ is central if for all $x \in Q$ and group homomorphisms H and $K : Q \longrightarrow Q$ we have

$$G(x)g = gF(x) \tag{22}$$

$$xH(g) = K(g)x. \tag{23}$$

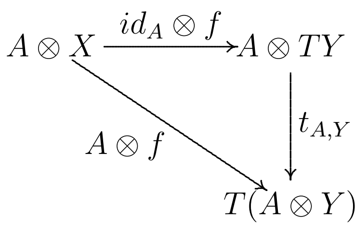
Thus taking $H = K = id_Q$ we see that g must be a central element of the group Q and thus $g = \pm 1$. Now define a group homomorphism $F : Q \longrightarrow Q$ by

$$F(x) = \begin{cases} 1, & \text{if } x = \pm 1 \text{ or } i \\ -1, & \text{if } x = j \text{ or } k \end{cases} \tag{24}$$

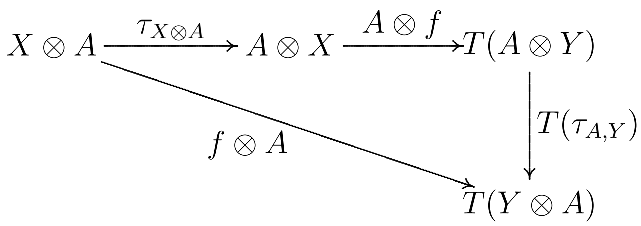
To see that F is a well-defined group homomorphism we must verify that it respects the relations which define Q. Then for example $i \in Q$ is a natural transformation from F to itself since $iF(x) = F(x)i$ for all $x \in Q$ but i is clearly not central in Q since $ij = k = -ji$. Thus the natural transformation i isn't central.

Another example of how premonoidal categories arise is as follows.

Example 6.2.5 (c.f. [31]). Suppose $(\mathcal{C}, \otimes, I, a, l, r, \tau)$ is a symmetric monoidal category and (T, μ, η) is a monad on $\mathcal{C}$ with strength $t_{A,B} : A \otimes TB \longrightarrow T(A \otimes B)$. Then the Kleisli category $\mathcal{C}_T$ has a premonoidal structure. Given objects A, B in $\mathcal{C}$ we define $A \otimes_{\mathcal{C}_T} B = A \otimes B$ and given any arrow $f \in \mathcal{C}_T(X, Y)$ define $A \otimes f$ by the following composite in $\mathcal{C}$



and similarly one defines $f \otimes A$ as the following composite in $\mathcal{C}$



The structural maps are also defined in an evident manner.

6.3 Commutants in Premonoidal Categories

The results of this section are new, and inspired by the theory of *von Neumann algebras*. It will be the basis of our definition of *von Neumann category* below.

Definition 6.3.1. Let $\mathcal{A}$, and $\mathcal{B}$ be premonoidal subcategories of the premonoidal category $\mathcal{C}$. Then we say that $\mathcal{A}$ and $\mathcal{B}$ **commute** if for all arrows $f : A \longrightarrow A'$ in $\mathcal{A}$ and $g : B \longrightarrow B'$ in $\mathcal{B}$ we have that $f \ltimes g = f \rtimes g$ and $g \ltimes f = g \rtimes f$ in $\mathcal{C}$.

An easy calculation shows that if $\mathcal{A}$, and $\mathcal{B}$ are submonoids of a monoid $\mathcal{C}$ then they commute with each other as monoids if and only if they commute as premonoidal categories. In a related note we can also define the *commutant* of a premonoidal subcategory $\mathcal{A}$ of a premonoidal category $\mathcal{C}$. Our invention of the notion of commutant for the setting of premonoidal categories will prove to be an important notion in later chapters. In addition this notion also has links to von Neumann algebras and more general structures that are being explored by the author and his thesis advisor R. Blute.

Theorem 6.3.2. Let $\mathcal{A}$ be a set of objects and arrows in a premonoidal category $\mathcal{C}$. Then the **commutant** of $\mathcal{A}$ with respect to $\mathcal{C}$ will be the category with objects the same as those of $\mathcal{C}$ and its arrows will be arrows $f : A \longrightarrow B$ in $\mathcal{C}$ such that $f \ltimes g = f \rtimes g$ and $g \ltimes f = g \rtimes f$ for all arrows g in $\mathcal{A}$. This category, which will be denoted $\mathcal{A}'$, is premonoidal.

Remark 6.3.3. Before we prove Theorem 6.3.2, we make an important observation. By definition of $\mathcal{A}'$ it has the same objects as $\mathcal{C}$ and if $z : A \longrightarrow B$ is an arrow in $\mathcal{Z}(\mathcal{C})$ then in particular we have that $z \ltimes g = z \rtimes g$ and $g \ltimes z = g \rtimes z$ for all arrows g in $\mathcal{A}$. Thus $\mathcal{A}'$ contains all the central maps in $\mathcal{C}$.

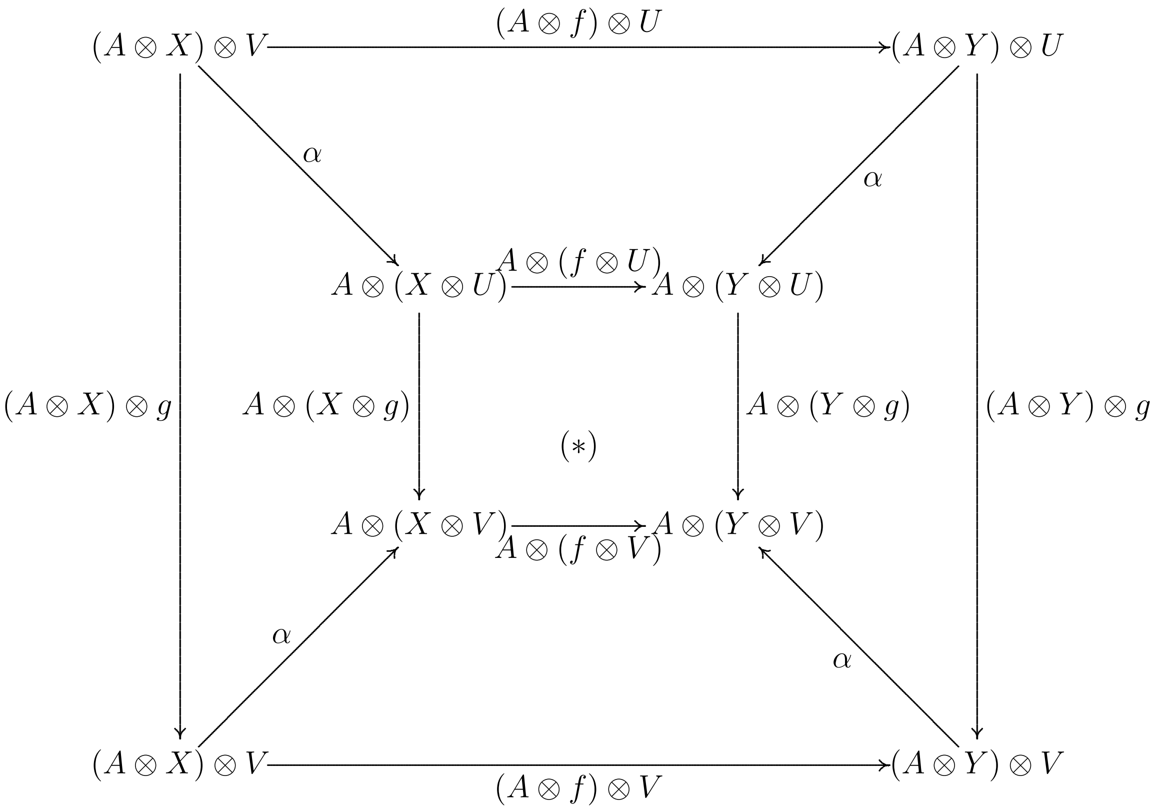
Proof. We start by showing that $\mathcal{A}'$ is a category. For each object A the identity map id_A is a central map in $\mathcal{C}$ and thus a map in $\mathcal{A}'$, so $\mathcal{A}'$ contains identities. Next we must show that given $f : A \longrightarrow B$, and $e : B \longrightarrow C$ in $\mathcal{A}'$ that $e \circ f : A \longrightarrow C$ is an arrow in $\mathcal{A}'$. Indeed let $g : X \longrightarrow Y$ be an arrow in $\mathcal{A}$.

$$\begin{aligned}(e \circ f) \ltimes g &= [C \otimes g][(e \circ f) \otimes X] \\ &= ([C \otimes g][e \otimes X])[f \otimes X] \\ &= [e \otimes Y]([B \otimes g][f \otimes X]) \\ &= ([e \otimes Y][f \otimes Y])[A \otimes g] \\ &= [e \circ f \otimes Y][A \otimes g] \\ &= (e \circ f) \rtimes g\end{aligned}$$

Similarly we can show that $g \ltimes (e \circ f) = g \rtimes (e \circ f)$. Hence $e \circ f$ is an arrow in $\mathcal{A}'$. Clearly this composition is associative and unital thus $\mathcal{A}'$ is a category. Thus by Remark 6.3.3 we have that $\mathcal{Z}(\mathcal{C})$ is a subcategory of $\mathcal{A}'$.

We now establish the premonoidal structure on the commutant category. Given objects A and B of $\mathcal{A}'$ we define $A \otimes_{\mathcal{A}'} B = A \otimes_{\mathcal{C}} B = A \otimes B$. If also $f : X \longrightarrow Y$ in $\mathcal{A}'$ then we define $A \otimes_{\mathcal{A}'} f = A \otimes f$ and $f \otimes_{\mathcal{A}'} A = f \otimes A$. We must verify that both of these arrows are still arrows in $\mathcal{A}'$. For example consider the arrow $A \otimes f : A \otimes X \longrightarrow A \otimes Y$

and let $g : U \longrightarrow V$ be any arrow in $\mathcal{A}$ then



All the inner diagrams except $(*)$ commute on account of naturality of α and diagram $(*)$ commutes since it is just the result of applying the functor $A \otimes (-)$ to the both sides of the equation $f \ltimes g = f \rtimes g$. Hence we have that $(A \otimes f) \ltimes g = (A \otimes f) \rtimes g$ and similarly one can check that $g \ltimes (A \otimes f) = g \rtimes (A \otimes f)$. Hence $(A \otimes f)$ is an arrow in $\mathcal{A}'$ and likewise so is $(f \otimes A)$. Now since $\mathcal{Z}(\mathcal{C}) \subseteq \mathcal{A}'$ it follows that the remaining requirements for $\mathcal{A}'$ to be a premonoidal category are all satisfied since all the relevant diagrams that must commute are diagrams which live in the centre and commute there. □

6.3.1 Maximally Monoidal Categories

In general, computing the commutant of a collection of arrows in a premonoidal category is a priori a nontrivial endeavor. However it is evident that for the case when $\mathcal{A} = \mathcal{Z}(\mathcal{C})$ that $\mathcal{A}' = \mathcal{C}$ and $\mathcal{C}' = \mathcal{A}$. The following lemma provides a nice example of a commutant inside a premonoidal category.

Lemma 6.3.4. Let $\mathcal{C}$ be a category and let $[\mathcal{C}, \mathcal{C}]_u$ be the premonoidal category defined in Example 6.1.6, then $(\mathcal{C}^{\mathcal{C}})' = \mathcal{C}^{\mathcal{C}}$, where $\mathcal{C}^{\mathcal{C}}$ denotes the subcategory of $[\mathcal{C}, \mathcal{C}]_u$ whose arrows are natural transformations.

Proof. Suppose that $\alpha : F \longrightarrow G$ and $\beta : H \longrightarrow K$ are natural transformations. Then the diagram

$$\begin{array}{ccc} F \otimes H & \xrightarrow{\alpha \otimes H} & G \otimes H \\ {\scriptstyle F \otimes \beta}\downarrow & & \downarrow{\scriptstyle G \otimes \beta} \\ F \otimes K & \xrightarrow[\alpha \otimes K]{} & G \otimes K \end{array}$$

commutes since for any object X the diagram

$$\begin{array}{ccc} FHX & \xrightarrow{\alpha_{HX}} & GHX \\ {\scriptstyle F(\beta_X)}\downarrow & & \downarrow{\scriptstyle G(\beta_X)} \\ FKX & \xrightarrow[\alpha_{KX}]{} & GKX \end{array}$$

commutes by naturality of α. Similarly the other required diagram

$$\begin{array}{ccc} H \otimes F & \xrightarrow{H \otimes \alpha} & H \otimes G \\ {\scriptstyle \beta \otimes F}\downarrow & & \downarrow{\scriptstyle \beta \otimes G} \\ K \otimes F & \xrightarrow[K \otimes \alpha]{} & K \otimes G \end{array}$$

will commute on account of the naturality of β. Hence we have established that $\mathcal{C}^{\mathcal{C}} \subseteq (\mathcal{C}^{\mathcal{C}})'$. Thus to complete the proof we must show that any arrow $\beta : H \longrightarrow K$ in $(\mathcal{C}^{\mathcal{C}})'$ is a natural transformation. Indeed consider the previous diagram in the case that $F = \Delta_X$ and $G = \Delta_Y$ are the constant functors and where X and Y are arbitrary objects in $\mathcal{C}$. Then a natural transformation from F to G is simply any arrow $f : X \longrightarrow Y$ in $\mathcal{C}$. So letting $\alpha_A = f$ for all objects A the diagram becomes

$$\begin{array}{ccc} H\Delta_X(A) = HX & \xrightarrow{H(\alpha_A) = H(f)} & H\Delta_Y(A) = HY \\ \Big\downarrow {\scriptstyle \beta_{\Delta_X A} = \beta_X} & & \Big\downarrow {\scriptstyle \beta_{\Delta_Y A} = \beta_Y} \\ K\Delta_X(A) = KX & \xrightarrow[K(\alpha_A) = K(f)]{} & K\Delta_Y(A) = KY \end{array}$$

which commutes by assumption on β. Thus β is a natural transformation and hence we have completed the proof. □

Remark 6.3.5. The above lemma provides an example of a category that is equal to its own commutant, a property that is quite unusual and deserves the prestige of a definition. Suppose that $\mathcal{A}$ is a premonoidal subcategory of a premonoidal category $\mathcal{C}$ then in case $\mathcal{A}' = \mathcal{A}$ we will say $\mathcal{A}$ is **maximally-monoidal** or a **maximal-monoidal subcategory of** $\mathcal{C}$. We will justify the name shortly.

The following proposition summarizes some key properties of maximal-monoidal categories.

Proposition 6.3.6. Let $\mathcal{A}$ be a maximal-monoidal subcategory of the premonoidal category $\mathcal{C}$ then following statements hold

1. $\mathcal{Z}(\mathcal{C}) \subseteq \mathcal{A}$;

2. $\mathcal{A}$ is a monoidal category;

3. $\mathcal{Z}(\mathcal{C}) = \mathcal{A}$ if and only if $\mathcal{C}$ is monoidal;

4. if $\mathcal{B} \subseteq \mathcal{C}$ where $\mathcal{B}$ is monoidal, then $\mathcal{B} \subseteq \mathcal{B}'$;

5. if $\mathcal{B}$ is maximally-monoidal and $\mathcal{A} \subseteq \mathcal{B}$ then $\mathcal{A} = \mathcal{B}$;

6. if $\mathcal{B} \subseteq \mathcal{C}$ is a monoidal category which is maximal with respect to monoidal subcategories of $\mathcal{C}$, and $\mathcal{B}'$ is also monoidal, then $\mathcal{B}$ is maximally-monoidal;

7. if $\mathcal{C}$ contains a monoidal subcategory whose commutant is also monoidal, it contains a maximally-monoidal subcategory.

Proof. 1. Let $\mathcal{A}$ and $\mathcal{C}$ be as above then the first statement is a consequence of Remark 6.3.3 which shows that every central map is a map in the commutant category $\mathcal{A}'$ and using that $\mathcal{A}' = \mathcal{A}$ the result follows.

2. To see that $\mathcal{A}$ is monoidal simply notice that given any arrows f and g they are also both arrows in $\mathcal{A}'$ and hence $f \ltimes g = f \rtimes g = f \otimes g$ and $g \ltimes f = g \rtimes f = g \otimes f$. Thus $\otimes$ is a bifunctor when restricted to the category $\mathcal{A}$ and so this category is monoidal.

3. $\mathcal{Z}(\mathcal{C}) = \mathcal{A}$ if and only if $\mathcal{Z}(\mathcal{C})' = \mathcal{A}' = \mathcal{A}$ but $\mathcal{Z}(\mathcal{C})' = \mathcal{C}$. Thus $\mathcal{Z}(\mathcal{C}) = \mathcal{A}$ if and only if $\mathcal{Z}(\mathcal{C}) = \mathcal{C}$ which occurs if and only if $\mathcal{C}$ is monoidal.

4. if $\mathcal{B} \subseteq \mathcal{C}$ is monoidal then we have that $f \ltimes g = f \rtimes g$, and $g \ltimes f = g \rtimes f$ for all arrows f and g in $\mathcal{B}$. Hence $\mathcal{B} \subseteq \mathcal{B}'$.

5. Suppose $\mathcal{B}' = \mathcal{B}$. Then if $\mathcal{A} \subseteq \mathcal{B}$, taking commutants we get $\mathcal{B}' \subseteq \mathcal{A}'$ so $\mathcal{B} \subseteq \mathcal{A}$, and hence $\mathcal{A} = \mathcal{B}$.

6. Suppose that $\mathcal{B} \subseteq \mathcal{C}$ is monoidal and is maximal in the sense that if $\mathcal{D} \subseteq \mathcal{C}$ is any other monoidal category with $\mathcal{B} \subseteq \mathcal{D}$ then $\mathcal{B} = \mathcal{D}$. We have by 4 that $\mathcal{B} \subseteq \mathcal{B}'$ and thus by maximality $\mathcal{B} = \mathcal{B}'$, since $\mathcal{B}'$ is also monoidal.

7. For simplicity we will assume that $\mathcal{C}$ is a small premonoidal category. Then we will define a set $\mathbb{M}$ as follows

$$\mathbb{M} = \{\mathcal{B} \subseteq \mathcal{C} \mid \mathcal{B} \text{ and } \mathcal{B}' \text{ are monoidal w.r.t. the monoidal structure on } \mathcal{Z}(\mathcal{C})\}.$$

Then $\mathbb{M}$ becomes a partially ordered set with respect to inclusion of subcategories and is clearly nonempty since we have assumed there exists at least one monoidal subcategory whose commutant is also monoidal. We will use Zorn's lemma to show that this set possesses at least one maximal element. Indeed suppose that $T \subseteq \mathbb{M}$ is a totally ordered subset then we must show that it has an upper bound. Define a category $\mathcal{T}$ as follows. Let $|\mathcal{T}| = \bigcup_{\mathcal{B} \in T} |\mathcal{B}|$ and $arr(\mathcal{T}) = \bigcup_{\mathcal{B} \in T} arr(\mathcal{B})$. The fact this defines a category follows from a general lemma that we prove later (Lemma 10.1.2). Moreover clearly $\mathcal{T} \subseteq \mathcal{C}$ since every object of $\mathcal{T}$ belongs to a subcategory of $\mathcal{C}$ as does every morphism.

Furthermore, given objects A and B in $\mathcal{T}$ then there exists monoidal subcategories $\mathcal{B}_1$ and $\mathcal{B}_2$ belonging to T and containing A and B respectively. In addition since T is totally ordered we may assume without loss of generality that $\mathcal{B}_1 \subseteq \mathcal{B}_2$ and thus $A \otimes B$ can seen as an object in this larger category. Hence $A \otimes B$ is thus an object of $\mathcal{T}$. By similar arguments one can show that given a pair of arrows f and g in $\mathcal{T}$ that $f \otimes g$ exists and is an arrow in $\mathcal{T}$. The structural maps that make $\mathcal{T}$ monoidal are given by the structural maps that make $\mathcal{Z}(\mathcal{C})$ monoidal.

The last fact we need to establish before applying Zorn's lemma is to show that $\mathcal{T}'$ is also monoidal. Indeed suppose that f and g are arrows in $\mathcal{T}'$ then for all arrows h in $\mathcal{T}$ we have that $f \ltimes h = f \rtimes h$ and $h \ltimes f = h \rtimes f$ and similarly for g. Thus since $arr(\mathcal{T}) = \bigcup_{\mathcal{B} \in T} arr(\mathcal{B})$ we have that $f, g \in \bigcap_{\mathcal{B} \in T} arr(\mathcal{B}')$. Moreover as $\mathcal{B}'$ is monoidal for each category $\mathcal{B}$ it follows that $f \ltimes g = f \rtimes g$ and $g \ltimes f = g \rtimes f$. Hence $\mathcal{T}'$ is also monoidal. Thus the totally ordered set T has an upper bound.

Therefore by Zorn's lemma the poset $\mathbb{M}$ contains at least one maximal element, call it $\mathcal{M}$. Then by definition as $\mathcal{M} \in \mathbb{M}$ we have that $\mathcal{M}'$ is also monoidal. On the other hand by 4 $\mathcal{M} \subseteq \mathcal{M}'$ and taking commutants and using that $\mathcal{E} \subseteq \mathcal{E}''$ for any premonoidal category, we get that

$$\mathcal{M} \subseteq \mathcal{M}'' \subseteq \mathcal{M}'.$$

Now since both $\mathcal{M}$ and $\mathcal{M}'$ are monoidal then so is $\mathcal{M}''$. Using maximality of

$\mathcal{M}$ we must have that $\mathcal{M} = \mathcal{M}'$.

□

Remark 6.3.7. Part 7 of the proposition actually admits an alternate proof, which is much simpler and does not make use of Zorn's lemma. Further the proof that we now present does not require that $\mathcal{C}$ be small. By assumption there exists a monoidal category $\mathcal{B} \subseteq \mathcal{C}$ such that $\mathcal{B}'$ is also monoidal. Letting $\mathcal{E} = \mathcal{B}'$ we will show that $\mathcal{E}' = \mathcal{E}$. Indeed since by part 4 $\mathcal{B} \subseteq \mathcal{E}$ taking commutants we get that $\mathcal{E}' \subseteq \mathcal{B}' = \mathcal{E}$. Now by assumption $\mathcal{E}$ is monoidal thus applying part 4 again we get that $\mathcal{E} \subseteq \mathcal{E}'$ and hence $\mathcal{E}' = \mathcal{E}$. Thus $\mathcal{E} = \mathcal{B}'$ is a maximal-monoidal subcategory of $\mathcal{C}$.

Remark 6.3.8. Notice that if $\mathcal{A}$ is a maximal-monoidal subcategory of a premonoidal category $\mathcal{C}$ then it is clearly maximal in the sense made precise in Proposition 6.3.6. Furthermore it is also clear that $\mathcal{A}'' = \mathcal{A}$. Now the reader who is familiar with the notion of a von Neumann algebra (see Section 3.4) will have noticed the parallels with our notion of commutant defined here and the corresponding notion for von Neumann algebras. Furthermore a von Neumann algebra that satisfies $M = M' \subseteq \mathfrak{B}(H)$ is called a *maximal abelian von Neumann algebra* since it will be abelian and maximal with respect to the abelian von Neumann algebras on $\mathfrak{B}(H)$ (see [9] pp. 281 for a brief discussion). Thus drawing on this concrete setting as our inspiration we coin the term *maximal-monoidal subcategory*. The use of the adjective monoidal is justified since these categories will be monoidal. As well monoidal subcategories of $\mathcal{C}$ are categories that commute with themselves in the sense of our Definition 6.3.1. Note that we avoid the adjective abelian as this already has a precise meaning in the world of categories that is widely used.

Chapter 7

Reconstruction Theorem for STC^*'s

In this chapter we give an overview of the Doplicher-Roberts reconstruction theorem [12, 13] highlighting the key aspects. For the most part we will present the proof as given by Müger in the appendix of [17] indicating any instance where significant differences occur. We will begin the chapter by making some preliminary definitions and then give a precise statement of the reconstruction theorem. Next we will delve into the heart of the proof and show the existence of an *absorbing monoid*. We then indicate how this key ingredient provides the tool required to prove the main Doplicher-Roberts theorem.

The class of theorems we are considering are generalizations of Pontryagin duality. In Pontryagin duality, one associates to a locally compact abelian group, its class of characters, called its *dual group*. The theorem then states that the original group is isomorphic to its double dual [26].

To generalize to the case of nonabelian groups, one has to use the category of representations, rather than the group of characters. This is because a nonabelian group will have irreducible representations that are not one-dimensional. The Tannaka-Krein theorem [19] is about reconstructing the group from its category of representations.

The basis of the Tannaka-Krein theorem is a fibre functor, thought of as a forgetful functor, to the category of vector spaces. One then recovers the group as the group of structure-preserving endomorphisms of this functor. The remarkable aspect of the Doplicher-Roberts theorem is that no such functor is required. In the proof given by Müger in [17], one takes the data of the theorem, and uses it to construct a fibre functor. The result then follows from the classical Tannaka-Krein theorem.

7.1 Preliminary Definitions and Statement of the Theorem

In order to state the main theorem we will first need to make several definitions.

Definition 7.1.1. If $\mathcal{C}$ is an STC^* (Definition 5.6.4) then it is said to be **even** in case the twist map (Lemma 5.6.11) satisfies $\Theta(X) = id_X$ for all objects $X \in |\mathcal{C}|$.

Examples of such categories include $\mathbf{Hilb}_{fd}$ and $\mathbf{Rep}(G)_{fd}$ where the latter category consists of the finite-dimensional representations of a compact group G.

Definition 7.1.2. A **supergroup** consists of a group G and $k \in G$ which is central and has order 2. A **compact supergroup** consists of a supergroup (G, k) such that G is a compact Hausdorff group. An **isomorphism of of supergroups** between (G, k) and (H, j) consists of a group isomorphism $\phi : G \longrightarrow H$ such that $\phi(k) = j$. In the case that the supergroups are compact we also require the map ϕ to be continuous.

Now given a supergroup (G, k) there is also a notion of a representation.

Definition 7.1.3. Let (G, k) be a compact supergroup. A **representation of** (G, k) consist of a unitary representation (H, π) of G. A **morphism of representations** (H, π) and (K, ψ) of (G, k) consists of a bounded linear map $T : H \longrightarrow K$ such that $T \circ \pi(g) = \psi(g) \circ T$ for all $g \in G$.

Now it is worth noting that since $k \in G$ has order 2 that for any representation (H, π) the maps $P^{\pi}_{\pm} \equiv (id_H \pm \pi(k))/2$ are orthogonal projections. Thus H decomposes as a direct sum $H = H_+ \oplus H_-$ where $H_\pm = P^\pi_\pm(H)$. Furthermore, if $T : (H, \pi) \longrightarrow (K, \psi)$ is a map of representations then

$$\begin{aligned} T(P^\pi_\pm(h)) &= T\frac{(id_H \pm \pi(k))}{2}(h) \\ &= \frac{(T \pm T \circ \pi(k))}{2}(h) \\ &= \frac{(T \pm \psi(k) \circ T)}{2}(h) \\ &= \frac{(id_K \pm \psi(k))}{2}T(h) \end{aligned}$$

which shows that $T(H_\pm) \subseteq K_\pm$. As well since $k \in G$ is central it follows that $\pi(g) \circ \pi(k) = \pi(k) \circ \pi(g)$ for all $g \in G$ and hence $\pi(g)(H_\pm) = H_\pm$. Let $\mathbf{Rep}(G, k)$ denote the category of unitary representations of the compact supergroup (G, k). Then from our observations, one can show that there is an equivalence of C^*-tensor categories $\mathbf{Rep}(G, k) \simeq \mathbf{Rep}(G)$. An interesting feature about the category $\mathbf{Rep}(G, k)$ is the symmetric structure it possesses. Namely if (H, π) and (K, ψ) are two representations then we define $(H, \pi) \otimes (K, \psi) = (H \otimes K, \pi \otimes \psi)$ where $(\pi \otimes \psi)(g) = \pi(g) \otimes \psi(g)$. Now define $\sigma_{H,K} : (H, \pi) \otimes (K, \psi) \longrightarrow (K, \psi) \otimes (H, \pi)$ by

$$\sigma_{H,K} = \tau_{H,K} \circ (id_{H\otimes K} - 2P^\pi_- \otimes P^\psi_-) \tag{25}$$

where $\tau_{H,K} : H \otimes K \longrightarrow K \otimes H$ is the symmetry given on decomposable tensors by $x \otimes y \mapsto y \otimes x$. Note in the case that $G = \mathbb{Z}_2 = \{e, k \mid k^2 = e\}$ that $\mathbf{Rep}(\{e, k\}, k)$ is denoted **SHilb** and is called the category of *super Hilbert spaces*. Clearly there is a forgetful functor $V : \mathbf{Rep}(G, k) \longrightarrow \mathbf{SHilb}$ for any compact supergroup (G, k).

Theorem 7.1.4 (Doplicher-Roberts Reconstruction Theorem)**.** Let $\mathcal{C}$ be an STC^*. There exists a compact supergroup (G, k), which is unique up to isomorphism together with an equivalence of symmetric tensor $*$-categories $F : \mathcal{C} \longrightarrow \mathbf{Rep}(G, k)_{fd}$. Consequently the functor $E = V \circ F : \mathcal{C} \longrightarrow \mathbf{SHilb}$ is a faithful symmetric $*$-preserving tensor functor.

7.2 Reconstruction Toolkit

The existence of a compact supergroup guaranteed from Theorem 7.1.4 stems from the existence of a so called *fibre functor*. In this section we provide the major tools that will be used in the proof of the reconstruction theorem.

Definition 7.2.1. Suppose $\mathcal{C}$ is an STC^*. Then a **fibre functor** for $\mathcal{C}$ is a faithful $\mathbb{C}$-linear tensor functor $E : \mathcal{C} \longrightarrow \mathbf{Vect}_{\mathbb{C}}$. A functor $E : \mathcal{C} \longrightarrow \mathbf{Hilb}$ is called a **$*$-preserving fibre functor** if it is a faithful tensor $*$-functor. E is said to be **symmetric** in case it maps the symmetry of $\mathcal{C}$ to the symmetry in $\mathbf{Vect}_{\mathbb{C}}$ or **Hilb**.

The next important step in getting towards a proof of Theorem 7.1.4 is to show that fibre functors exist. We will now provide the required foundation following the approach of Müger in [17].

Definition 7.2.2 (c.f. B.37 [17])**.** Suppose that $\mathcal{C}$ is an additive tensor category. It is said to be **finitely generated** if there exists an object Z in $\mathcal{C}$ such that every object X is the direct summand of some tensor power of Z.

Thus if $\mathcal{C}$ is finitely generated then there exists an object Z such that any object X_1 is a direct summand of $Z^{\otimes n} = Z \otimes \cdots \otimes Z$ for some $n \in \mathbb{N}$. Thus there exists an object X_2 and maps $u_i : X_i \longrightarrow Z^{\otimes n}$, and $p_i : Z^{\otimes n} \longrightarrow X_i$ for $i = 1, 2$ satisfying the following equations:

$$p_i \circ u_i = id_{X_i} \qquad\qquad u_1 \circ p_1 + u_2 \circ p_2 = id_{Z^{\otimes n}}.$$

Now in order to proceed with the construction of a fibre functor we need to ensure that we can form a certain directed/filtered colimit. In general $\mathcal{C}$ need not have all filtered colimits, however it is possible form its *free filtered cocompletion* which we describe now. Our description is taken from Borceux [7].

Definition 7.2.3 (c.f. [7] Def.1.6.4)**.** If $F : \mathcal{C} \longrightarrow \mathbf{Set}$ is a functor then $\mathbf{Elts}(F)$ is the category called the **elements of** F defined as follows:

1. Objects consist of pairs (A, a) where $A \in |\mathcal{C}|$ and $a \in F(A)$,

2. an arrow $x : (A, a) \longrightarrow (B, b)$ consists of an arrow $x : A \longrightarrow B$ such that $F(x)(a) = b$, and
3. composition is given by composition in $\mathcal{C}$.

Definition 7.2.4 (c.f. [7] Def. 2.13.1)**.** A category $\mathcal{C}$ is **filtered** if

1. it contains at least one object;
2. for all objects A and B there exists an object C and arrows $f : A \longrightarrow C$ and $g : B \longrightarrow C$;
3. for all pairs of parallel arrows $f, g : A \longrightarrow B$ there exists an arrow $h : B \longrightarrow C$ such that $h \circ f = h \circ g$.

Remark 7.2.5. A category $\mathcal{C}$ is called **cofiltered** when $\mathcal{C}^{op}$ is filtered.

Definition 7.2.6 (c.f. [7] Def.6.3.1)**.** A functor $F : \mathcal{C} \longrightarrow \mathbf{Set}$ is **flat** just when the category $\mathbf{Elts}(F)$ is cofiltered.

Proposition 7.2.7 (c.f. [7] Ex.6.7.3)**.** The category of flat functors and natural transformations, denoted $\mathbf{Flat}(\mathcal{C}^{op}, \mathbf{Set})$, from $\mathcal{C}^{op}$ to $\mathbf{Set}$ is the free filtered cocompletion of $\mathcal{C}$.

There is an alternate description of the free filtered cocompletion of a category $\mathcal{C}$, which is usually referred to as $\operatorname{Ind}\mathcal{C}$. This description is slightly more complicated on the surface, however it turns out to be useful. We will give some of the details of this construction which can be found in A.7 of [17].

Indeed let $\mathcal{C}$ be a category. We define a new category $\operatorname{Ind}\mathcal{C}$ as follows. An object consists of a pair (I, F) where I is a small filtered category and $F : I \longrightarrow \mathcal{C}$ is a functor. Now given a pair of objects (I, F) and (J, H) we define the hom-set $\operatorname{Ind}\mathcal{C}((I, F), (J, H))$ by

$$\operatorname{Ind}\mathcal{C}((I, F), (J, H)) = \lim_{i \in |I|} \operatorname*{colim}_{j \in |J|} \mathcal{C}(Fi, Hj).$$

Unpacking this definition one can work out the composition in this category. The category $\mathcal{C}$ can be embedded in $\operatorname{Ind}\mathcal{C}$ by sending each object X to the pair $(\mathbf{1}, F)$

where $\mathbf{1} = \{*\}$ is the discrete category on the one element set and $F : \mathbf{1} \longrightarrow \mathcal{C}$ is given by $F(*) = X$. This embedding turns out to be full and faithful.

Now the real reason that we chose to give a description of $\operatorname{Ind}\mathcal{C}$ was that it is easy to describe the monoidal structure on this category. If $\mathcal{C}$ is a monoidal category, then given objects (I, F) and (J, H) in $\operatorname{Ind}\mathcal{C}$, define

$$(I, F) \otimes (J, H) \equiv (I \times J, F \otimes H)$$

where $F \otimes H : I \times J \longrightarrow \mathcal{C}$ is given by $(F \otimes H)(i, j) = Fi \otimes Hj$.

We finish our discussion on filtered colimits by stating two results that we will require later on.

Theorem 7.2.8 (c.f. [17] A.71)**.** If $\mathcal{C}$ is a category, then $\operatorname{Ind}\mathcal{C}$ has all small filtered colimits. $\operatorname{Ind}\mathcal{C}$ is an abelian category whenever $\mathcal{C}$ is abelian.

Lemma 7.2.9 (c.f.[17] A.72)**.** If X is an object in a TC^* $\mathcal{C}$, then it is projective when viewed as an object of $\operatorname{Ind}\mathcal{C}$.

Now we will turn our attention to how the symmetric groups act in an STC^*. For each $n \in \mathbb{N}$ let $\mathcal{S}_n$ denote the symmetric group on n letters. Then $\mathcal{S}_n$ has the following presentation:

$$\mathcal{S}_n = \langle \sigma_1, \ldots, \sigma_n \mid R1, R2, R3 \rangle$$

$$\sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{when } |i - j| \geq 2 \tag{R1}$$

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \tag{R2}$$

$$\sigma_i^2 = 1 \tag{R3}$$

Definition 7.2.10. Suppose that $\mathcal{C}$ is an STC^* with symmetry $c_{X,Y} : X \otimes Y \longrightarrow Y \otimes X$. Let $X \in |\mathcal{C}|$ and $n \in \mathbb{N}$ then we define a map

$$\Pi_n^X : \mathcal{S}_n \longrightarrow \mathcal{C}(X^{\otimes n}, X^{\otimes n}) \text{ by } \Pi_n^X(\sigma_i) = id_{X^{\otimes i-1}} \otimes c_{X,X} \otimes id_{X^{\otimes n-i-1}}.$$

Lemma 7.2.11 (c.f. [17] B.45)**.** The maps Π_n^X respect the defining relations of $\mathcal{S}_n$ and thus extend to a group homomorphism from the group $\mathcal{S}_n$ to group of unitary of automorphisms of $X^{\otimes n}$.

We will use these maps to produce projections.

Lemma 7.2.12 (c.f. [17] B.47)**.** If $\mathcal{C}$ is an even STC^*, for each object X, we will define orthogonal projections $S_n^X : X^{\otimes n} \longrightarrow X^{\otimes n}$ and $A_n^X : X^{\otimes n} \longrightarrow X^{\otimes n}$ by $S_0^X = A_0^X = id_{X^{\otimes 0}} = id_I$ and for $n \geq 1$

$$S_n^X = \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} \Pi_n^X(\sigma)$$
$$A_n^X = \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} \operatorname{sgn}(\sigma) \Pi_n^X(\sigma).$$

Then S_n^X and A_n^X are orthogonal projections which satisfy

$$\Pi_n^X(\sigma) \circ S_n^X = S_n^X \circ \Pi_n^X(\sigma) = S_n^X$$
$$\Pi_n^X(\sigma) \circ A_n^X = A_n^X \circ \Pi_n^X(\sigma) = \operatorname{sgn}(\sigma) A_n^X.$$

Definition 7.2.13 (c.f. [17] B.48)**.** $S_n(X)$ and $A_n(X)$ denote the subobjects of $X^{\otimes n}$ corresponding to the idempotents S_n^X and A_n^X respectively.

Theorem 7.2.14 (c.f. [17] B.49 and B.50)**.** If X is any non-zero object in an STC^*, then the dimension of X (Definition 5.6.8) satisfies $d(X) \in \mathbb{N}$.

Definition 7.2.15 (c.f. [17] B.51)**.** For any object X, we define the **determinant of** X as the isomorphism class of the object $A_{d(X)}(X)$, and denote it by $\det(X)$.

This concludes all of the relevant definitions that we need. We will now proceed to state several results that combine to yield a sketch of the Doplicher-Roberts reconstruction theorem.

7.3 Fibre Functors and Absorbing Monoids

Lemma 7.3.1 (c.f. [17] B.44)**.** Suppose $\mathcal{C}$ is a TC^* with generator Z and that $\hat{\mathcal{C}}$ is a $\mathbb{C}$-linear strict tensor category containing $\mathcal{C}$ as a full tensor subcategory. If (B, μ, η) is a monoid in $\hat{\mathcal{C}}$ such that

1. $\dim \hat{\mathcal{C}}(I, B) = 1$,

2. there exists $d \in \mathbb{N}$ and an isomorphism $\alpha_Z : (B \otimes Z, \mu \otimes id_Z) \longrightarrow \oplus_d(B, \mu)$ of B-modules.

Define $E : \mathcal{C} \longrightarrow \mathbf{Vect}$ by:

$EX = \hat{\mathcal{C}}(I, B \otimes X)$ and,

$E(s) = (id_B \otimes s) \circ \phi$ for all $s : X \longrightarrow Y$ and $\phi : I \longrightarrow B \otimes X$.

Then, E is a fibre functor.

Thus in order to build a fibre functor all we need to do is find a category $\hat{C}$ that contains a monoid B satisfying the above conditions.

Let $\hat{\mathcal{C}} = \mathrm{Ind}\,\mathcal{C}$ and from now on we assume that $\mathcal{C}$ is an STC^*. Now for any object X in $\mathcal{C}$ there exists an object

$$S(X) = \operatorname*{colim}_{n \longrightarrow \infty} \bigoplus_{i=0}^{n} S_n(X) \in |\hat{\mathcal{C}}|$$

together with monomorphisms $v_n : S_n(X) \longrightarrow S(X)$. This is the usual *symmetric algebra*, familiar from algebraic geometry.

Proposition 7.3.2 (c.f. [17] B.56)**.** If $\mathcal{C}$ is an STC^* then there exists a map $m_{S(X)} : S(X) \otimes S(X) \longrightarrow S(X)$ such that

$$m_{S(X)} \circ v_i \otimes v_j = v_{i+j} \circ m_{S(X)} : S_i(X) \otimes S_j(X) \longrightarrow S(X).$$

Moreover $(S(X), m_{S(X)}, \eta_{S(X)} \equiv v_0)$ is a commutative monoid in $\hat{\mathcal{C}}$.

Lemma 7.3.3. Suppose that $\mathcal{C}$ is an STC^* with generator Z satisfying $\det(Z) \cong I$. Then there exists arrows:

$$s : I \longrightarrow Z^{\otimes d} \qquad s' : Z^{\otimes d} \longrightarrow I$$

such that $s' \circ s = id_I$ and $s \circ s' = A_d^Z$, where $d = d(Z)$.

For a proof of this see the remark in the proof of B.50.

Set Q to be $Q = S(Z)^{\otimes d}$, where $d = d(Z)$ and Z is a generator of $\mathcal{C}$ with $\det(Z) \cong I$. Then $(Q, m_Q, \eta_Q) = (S(Z), m_{S(Z)}, \eta_{S(Z)})^{\otimes d}$ has the structure of a commutative monoid. Moreover the map:

$$m_0 = m_Q \circ (id_Q \otimes (f - \eta_Q)) : Q \longrightarrow Q$$

is a map of Q-modules. Here $f : I \longrightarrow Q$ is the map given by $f = \underbrace{v_1 \otimes \cdots \otimes v_1}_{d} \circ s$. Thus the image of m_0, $j = \operatorname{im} m_0 : (J, \mu_J) \longrightarrow (Q, m_Q)$defines an ideal in $j : J \longrightarrow Q$. Finally we let B be the quotient monoid of Q determined by this ideal. The calculations that B satisfies the hypothesis Lemma 7.3.1 are contained in B.59 and B.61.

Both B.59 and B.61 have as their hypothesis that $\mathcal{C}$ is *even*. Thus we have produced a fibre functor in the finitely generated even case.

Theorem 7.3.4 (c.f. [17] B.40)**.** Every finitely generated even STC^* admits a symmetric fibre functor $E : \mathcal{C} \longrightarrow \mathbf{Vect}$.

Now note that:

Lemma 7.3.5 (c.f. [17] B.38)**.** Let $\mathcal{C}$ be a TC^*. The finitely generated tensor subcategories $\mathcal{C}_i$ of $\mathcal{C}$ form a directed system and $\mathcal{C} = \operatorname{colim}_{i \in I} \mathcal{C}_i$.

This can be used to show that the finitely generated assumption can be eliminated. The next order of business is to produce a $*$-preserving symmetric fibre functor given a symmetric fibre functor $E : \mathcal{C} \longrightarrow \mathbf{Vect}$. Indeed for each object X in $\mathcal{C}$, choose an arbitrary inner product structure on $E(X)$. Since these spaces are finite-dimensional they will automatically be Hilbert spaces. Now define a new functor $\tilde{E}$ as follows:

$$\tilde{E}(X) = E(X) \qquad \text{for all objects } X$$
$$\tilde{E}(s) = E(s^*)^\dagger \qquad \text{for all arrows } s.$$

Then one can prove the following result.

Lemma 7.3.6. Suppose $\mathcal{C}$ is an STC^* and $E : \mathcal{C} \longrightarrow \mathbf{Vect}$ is a symmetric fibre functor then $\tilde{E} : \mathcal{C} \longrightarrow \mathbf{Hilb}$ is a symmetric $*$-preserving fibre functor.

7.4 Tannaka-Krein

In order to produce the compact group we need the classical Tannaka-Krein theorem.

Theorem 7.4.1 (c.f. [17] B.6)**.** Suppose $\mathcal{C}$ is an STC^* and $E : \mathcal{C} \longrightarrow \mathbf{Hilb}$ is a $*$-preserving symmetric fibre functor. Let G_E be the group of unitary monoidal natural transformations of E with topology inherited from $\prod_{X\in|\mathcal{C}|}\mathcal{U}(E(X))$. Then G_E is a compact group and the functor $F : \mathcal{C} \longrightarrow \mathbf{Rep}(G_E)_{fd}$ defined by:

$$FX = (EX, \pi_X) \text{ where } \pi_X(g) = g_X \forall X \in |\mathcal{C}|,$$

is an equivalence of STC^*'s. If E_1, $E_2 : \mathcal{C} \longrightarrow \mathbf{Hilb}$ are $*$-preserving fibre functors then $E_1 \cong E_2$ and $G_{E_1} \cong G_{E_2}$.

One key point to keep in mind is that if a an STC^* admits a fibre functor then necessarily it is even, see [17] B.10. Combining this with previous results it follows that:

Theorem 7.4.2 (c.f. [17] B.12)**.** If $\mathcal{C}$ is an even STC^* then there is a compact group G, unique up to isomorphism, such that there is an equivalence $F : \mathcal{C} \longrightarrow \mathbf{Rep}(G_E)_{fd}$.

Finally we show how one can eliminate the evenness criterion.

Definition 7.4.3. Let $\mathcal{C}$ be an STC^*. Define a new category $\tilde{\mathcal{C}}$ called the **bosonization** of $\mathcal{C}$ as follows. As monoidal $*$-categories $\tilde{\mathcal{C}}$ and $\mathcal{C}$ coincide, but we define a new symmetry by:

$$\tilde{(c)}_{X,Y} = (-1)^{\frac{(1-\Theta(X))(1-\Theta(Y))}{4}} c_{X,Y}$$

for all irreducible objects X and Y and then extend to all objects by naturality.

Lemma 7.4.4. $\tilde{\mathcal{C}}$ is an even STC^*.

Now to see why one gets a supergroup instead of just a group we first need the following result.

Lemma 7.4.5. If G is a compact group, then the unitary monoidal natural transformations of the identity functor on $\mathbf{Ref}(G)_{fd}$ form an abelian group that is isomorphic to the centre $Z(G)$ of G.

Now suppose that $\mathcal{C}$ is an STC^* and consider its bosonization $\tilde{\mathcal{C}}$. Then as $\tilde{\mathcal{C}}$ is an even STC^* we have by Theorem 7.4.2 there is a compact group G such that $\tilde{\mathcal{C}} \simeq \mathbf{Rep}(G)_{fd}$ as STC^*'s. Now as the twist Θ is a unitary monoidal natural transformation of the identity functor (see Proposition 5.6.12) it follows that under this equivalence it corresponds to a unitary monoidal transformation of the identity functor on $\mathbf{Rep}(G)_{fd}$. Hence by Lemma 7.4.5 there exists an element $k \in G$ which is central and satisfies $k^2 = e$. We refer the reader to the proof of B.18 in [17] for the remaining details which show that $\mathcal{C} \simeq \mathbf{Rep}(G,k)_{fd}$ as STC^*'s.

Chapter 8

Algebraic Quantum Field Theory

8.1 The Basic Setup

Algebraic quantum field theory (AQFT) is an algebraic approach to quantum field theory. This is done by extracting the key ideas from quantum field theory and organizing this data into a rigorous mathematical framework. The fundamental notion in AQFT is the concept of a net of local observable algebras indexed by spacetime regions. The spacetime manifold that is usually considered is Minkowski spacetime $M = \mathbb{R}^4_1$, the spacetime of special relativity. This is the spacetime that we will consider as well. In this case the regions of spacetime considered are the double cones.

Definition 8.1.1. If x and $y \in M$ and y is in the causal future of x then the **double cone** determined by x and y is the set (x, y) given by the intersection of the causal future of x with the causal past of y. Denote by K the set of double cones on Minkowski space.

Then K becomes a partially ordered set with respect to the subset ordering, moreover this partially ordered set is directed.

Definition 8.1.2. A **net of observable algebras** over Minkowski space consists of a functor $\mathbb{A} : K \longrightarrow$ **C*-Alg** such that if $O_1 \subseteq O_2$ then the induced map $i_{O_1,O_2} : \mathbb{A}(O_1) \longrightarrow \mathbb{A}(O_2)$ is an isometric $*$-homomorphism.

Given a net of observable algebras $\mathbb{A} : K \longrightarrow$ **C*-Alg** the interpretation of the C^*-algebra $\mathbb{A}(O)$ for a double cone $O \in K$ is that it contains all observables which can be measured in the spacetime region O. For instance Araki, see [4] p.78, gives the example that measuring an observable in a finite spatial region A during a finite time period T can be considered as an observable being measured in the spacetime region $T \times A$. Armed with this intuition we can now proceed by introducing some plausible conditions that should be satisfied by the net of algebras.

Definition 8.1.3. Let $\mathbb{A} : K \longrightarrow$ **C*-Alg** be net of observable algebras then $\mathbb{A}$ satisfies **isotony** if $\mathbb{A}(O_1) \subseteq \mathbb{A}(O_2)$ whenever $O_1 \subseteq O_2 \in K$.

The intuition for isotony is that any observable which can be measured in O_1 can also be measured in a "bigger" region O_2. The next condition that is usually considered is called *microcausality* or *Einstein causality*. In order to state this condition we first need a definition. Note that since K is directed we have that $\{\mathbb{A}(O) \mid O \in K\}$ is a directed system of C^*-algebras thus we can form the directed colimit of this family to get a C^*-algebra $\hat{\mathbb{A}}$, called the quasi-local algebra. Note that $\hat{\mathbb{A}}$ is the completion of $\cup_{O \in K} \mathbb{A}(O)$. Then one can view each $\mathbb{A}(O)$ as a subalgebra of $\hat{\mathbb{A}}$.

Definition 8.1.4. A net $\mathbb{A}$ satisfies **microcausality** if whenever O_1, and $O_2 \in K$ are spacelike separated we have that $[\mathbb{A}(O_1), \mathbb{A}(O_2)] = 0$ in $\hat{\mathbb{A}}$.

The microcausality condition is intended to incorporate the causality features of relativity into the theory. One of the fundamental elements in relativity theory is that events occurring at spacelike separated spacetime locations have no causal relationship. Thus, if O_1 and O_2 are spacelike separated then events in O_1 do not affect events in O_2. Hence if T is an observable measurable in O_1 and S is an observable measurable in O_2 then we can measure T and S simultaneously, since measurement of one does not affect the measurement of the other. Moreover one of the tenets of quantum theory is that simultaneously measurably observables should commute with each other. Thus the microcausality axiom is physically reasonable.

Remark 8.1.5. Note that there many more physically relevant conditions that could be considered, however the two mentioned here are always part of the basic assumptions in any AQFT. We will present a few more of these possibilities in the next section and use these assumptions to build a symmetric monoidal $*$-category known as the **DHR** category. Here DHR stands for Doplicher, Haag, and Roberts.

8.2 Localized Transportable Endomorphisms

For this entire section suppose that we have given a net $\mathbb{A} : K \longrightarrow \mathbf{C^*\text{-}Alg}$ which satisfies isotony and microcausality. Furthermore suppose that (π_0, H_0) is a fixed representation of $\hat{\mathbb{A}}$ which we call the *vacuum representation*.

In physics as well as mathematics one often uses representation theory as a tool for understanding any underlying structure, and AQFT is no exception. In this realm, representations of the C^*-algebra $\hat{\mathbb{A}}$ satisfying a certain criterion, which is sometimes referred to as the selection criterion, are what is considered. The study of these representations is what led to the so called Doplicher-Roberts reconstruction theorem, which shows that from this category of representations of $\hat{\mathbb{A}}$ one can produce a compact group such that its category of finite dimensional unitary representations is equivalent to the original category of representations of $\hat{\mathbb{A}}$.

The category of representations of $\hat{\mathbb{A}}$ is however somewhat complicated to work with directly. To circumvent this issue one considers a special class of $*$-endomorphisms of $\hat{\mathbb{A}}$ and builds a category whose objects consist of these endomorphisms. This category is called $\Delta(\hat{\mathbb{A}})$ and the objects of this category will be chosen in such a way that if $f : \hat{\mathbb{A}} \longrightarrow \hat{\mathbb{A}}$ is an object and a representation $\pi : \hat{\mathbb{A}} \longrightarrow \mathfrak{B}(H)$ satisfies the selection criterion then so does the representation $\pi \circ f : \hat{\mathbb{A}} \longrightarrow \mathfrak{B}(H)$. So without further delay we now define the category $\Delta = \Delta(\hat{\mathbb{A}})$.

Definition 8.2.1. If $O \in K$ is a double cone then we define $\mathbb{A}(O^c)$ to be the C^*-subalgebra of $\hat{\mathbb{A}}$ generated by the set $\cup_{O_1 \perp O \in K} \mathbb{A}(O_1)$ where $O_1 \perp O$ means that O_1 and O are spacelike separated.

Definition 8.2.2. A $*$-homomorphism $\rho : \hat{\mathbb{A}} \longrightarrow \hat{\mathbb{A}}$ is **localized in** $O \in K$ if

$$\rho(a) = a \qquad \forall a \in \mathbb{A}(O^c). \tag{26}$$

We say that ρ is **localizable** if it is localized in some $O \in K$. If ρ is localized in O then we say it is **transportable** if for any double cone O_1 there exists a unitary element $U \in \hat{\mathbb{A}}$ and a $*$-homomorphism $\rho_1 : \hat{\mathbb{A}} \longrightarrow \hat{\mathbb{A}}$ localized in O_1 and satisfying

$$U\rho(a) = \rho_1(a)U \qquad \forall a \in \hat{\mathbb{A}}. \tag{27}$$

If $O \in K$ then we denote by $\Delta(O)$ the set of transportable endomorphisms which are localized in O.

Now we will define a category $\Delta(\hat{\mathbb{A}}) = \Delta$ as follows. The objects of Δ is the set $ob(\Delta) = \cup_{O \in K}\Delta(O)$. Given two objects ρ and δ a morphism from ρ to δ is an element $r \in \hat{\mathbb{A}}$ such that $r\rho(a) = \delta(a)r$ for all $a \in \hat{\mathbb{A}}$. Then composition of arrows is given by multiplication in $\hat{\mathbb{A}}$ and the identity arrow on ρ is given by the identity element $1_{\hat{\mathbb{A}}} \in \hat{\mathbb{A}}$. It is straightforward to verify that this is a category. Moreover it is a routine calculation to show that:

Lemma 8.2.3. The category Δ is a $*$-category.

Next we will show that Δ has direct sums provided the net satisfies three basic assumptions.

Definition 8.2.4 (Property B). A net of von Neumann algebras $\mathbb{V} : K \longrightarrow$ **C***-**Alg** on a Hilbert space H satisfies **property B** if whenever O_1 and O_2 are double cones such that the closure of O_1 is contained in O_2, i.e. $\overline{O_1} \subseteq O_2$ the following implication holds. If $E \in \mathbb{V}(O_1)$ is any nonzero projection then $E = VV^*$ for some isometry $V \in \mathbb{V}(O_2)$.

Remark 8.2.5. Notice that using the vacuum representation of $\hat{\mathbb{A}}$ we obtain for each double cone O a $*$-subalgebra of $\mathfrak{B}(H_0)$ given by $\pi_0(\mathbb{A}(O))$. Thus taking the double commutant of this set, $\pi_0(\mathbb{A}(O))''$, we obtain the smallest von Neumann algebra containing $\pi_0(\mathbb{A}(O))$. We denote this net of von Neumann algebras by $\mathfrak{R}_0$, i.e. $\mathfrak{R}_0(O) = \pi_0(\mathbb{A}(O))''$ for all double cones O.

Definition 8.2.6. We say the net $\mathbb{A} : K \longrightarrow$ **C***-**Alg** satisfies **Haag duality** in case

$$\pi_0(\mathbb{A}(O^c))' = \pi_0(\mathbb{A}(O)) \qquad \text{for all double cones } O \in K. \tag{28}$$

Remark 8.2.7. Note that if the net $\mathbb{A}$ satisfies Haag duality then for each double cone O we have that $\pi_0(\mathbb{A}(O^c))' = \pi_0(\mathbb{A}(O))$ thus taking the double commutant of both sides we get that $\pi_0(\mathbb{A}(O^c))''' = \pi_0(\mathbb{A}(O))''$ but for any subset X of bounded linear operators on a Hilbert space one has $X''' = X'$ and hence $\pi_0(\mathbb{A}(O)) = \pi_0(\mathbb{A}(O^c))' = \pi_0(\mathbb{A}(O^c))''' = \pi_0(\mathbb{A}(O))''$. Thus for each double cone O one has that $\pi_0(\mathbb{A}(O))$ is a von Neumann algebra.

Theorem 8.2.8 (c.f. [17] Prop.8.16 and Prop.8.18)**.** If the net $\mathbb{A}$ satisfies Haag duality, the net $\mathfrak{R}_0$ satisfies property B, the vacuum representation π_0 is faithful, and each $\pi_0(\mathbb{A}(O)) \neq \mathbb{C} \cdot id_{H_0}$ then the $*$-category Δ has direct sums and subobjects.

Remark 8.2.9. Note that if M is a von Neumann algebra on a Hilbert space H then it is generated by its projections [21]. Thus if $M \neq \mathbb{C} \cdot id_H$ then it follows that M contains a nonzero projection $p \in M$ such that $p \neq id_H$. In this case notice that $e = id_H - p = 1_M - p$ is also a nonzero projection in M. Hence there exists projections p and e in M such that $p + e = 1_M$.

We can now proceed to give a sketch of the proof of Theorem 8.2.8, adding some details which do not appear in [17].

Proof. First notice that by Remark 8.2.7 we have that the net of von Neumann algebras $\mathfrak{R}_0$ satisfies $\mathfrak{R}_0(O) = \pi_0(\mathbb{A}(O))$ for all double cones $O \in K$. Now by Remark 8.2.9 we also have that there are projections E and F in $\pi_0(\mathbb{A}(O))$ such that $E+F = id_{H_0}$. Now let $\rho_1 \in \Delta(O_1)$ and $\rho_2 \in \Delta(O_2)$ and let $O \in K$ such that $\overline{O_1 \cup O_2} \subseteq O$. Then it follows that there are projections E and $F \in \pi_0(\mathbb{A}(O))$ such that $E+F = id_{H_0}$ and moreover by property B it follows that we have isometries V_1 and $V_2 \in \pi_0(\mathbb{A}(O))$ such that $E = V_1 V_1^*$ and $F = V_2 V_2^*$. Now as π_0 is faithful it follows that there exists unique elements v_1 and $v_2 \in \mathbb{A}(O)$, with $V_i = \pi_0(v_i)$, satisfying $v_i^* v_i = id$ and $v_1 v_1^* + v_2 v_2^* = id$. Moreover it now also follows that $v_i^* v_j = 0$ whenever $i \neq j$. Thus we define $\rho : \hat{\mathbb{A}} \longrightarrow \hat{\mathbb{A}}$ by

$$\rho(a) = v_1 \rho_1(a) v_1^* + v_2 \rho_2(a) v_2^* \qquad \text{for all } a \in \hat{\mathbb{A}}. \tag{29}$$

Then ρ is a $*$-homomorphism and if $a \in \mathbb{A}(O^c)$ then it follows that $\rho_i(a) = a$ moreover since $v_i \in \mathbb{A}(O)$ we have that $v_i a = a v_i$ for each $a \in \mathbb{A}(O^c)$ hence

$$\begin{aligned}\rho(a) &= v_1\rho_1(a)v_1^* + v_2\rho_2(a)v_2^* \\ &= v_1 a v_1^* + v_2 a v_2^* \\ &= a[v_1v_1^* + v_2v_2^*] \\ &= a \cdot id = a.\end{aligned}$$

Thus ρ is localized in O.

Next we would like to show that ρ is transportable. If $\hat{O}$ is any double cone in K then there exists double cones $\hat{O}_1$ and $\hat{O}_2$ such that $\overline{\hat{O}_1 \cup \hat{O}_2} \subseteq \hat{O}$. Now since both ρ_1 and ρ_2 are transportable it follows that there exists $\hat{\rho}_1$ and $\hat{\rho}_2$ localized at $\hat{O}_1$ and $\hat{O}_2$ respectively. In addition we have unitary elements U_1 and $U_2 \in \hat{\mathbb{A}}$ such that $U_1\rho_1(a) = \hat{\rho}_1(a)U_1$ and $U_2\rho_2(a) = \hat{\rho}_2(a)U_2$ for all $a \in \hat{\mathbb{A}}$. Now arguing as above we find that there exists elements $\hat{v}_1$ and $\hat{v}_2 \in \mathbb{A}(\hat{O})$ such that $\hat{v}_i^*\hat{v}_j = \delta_{i,j} \cdot id$ and $\hat{v}_1\hat{v}_1^* + \hat{v}_2\hat{v}_2^* = id$. Thus $\hat{\rho}(a) = \hat{v}_1\hat{\rho}_1(a)\hat{v}_1^* + \hat{v}_2\hat{\rho}_2(a)\hat{v}_2^*$ defines a $*$-endomorphism of $\hat{\mathbb{A}}$ which is localized in $\hat{O}$. Now define $w \in \hat{\mathbb{A}}$ by $w = \hat{v}_1 U_1 v_1^* + \hat{v}_2 U_2 v_2^*$ one then checks that $w\rho(a) = \hat{\rho}(a)w$ for all $a \in \hat{\mathbb{A}}$ and $w^*w = id = ww^*$. Hence ρ is an object of Δ, and moreover it is easy to see that $v_i : \rho_i \longrightarrow \rho$ are maps in Δ which satisfy the conditions of a direct sum, i.e. $\rho = \rho_1 \oplus \rho_2$.

Next to see that Δ has subobjects suppose that $e : \rho \longrightarrow \rho$ is a projection in Δ for $\rho \in \Delta(O)$. Then one has that for each $a \in \mathbb{A}(O^c)$ that

$$\begin{aligned}\pi_0(e)\pi_0(a) &= \pi_0(ea) \\ &= \pi_0(e\rho(a)) \\ &= \pi_0(\rho(a)e) \\ &= \pi_0(ae) \\ &= \pi_0(a)\pi_0(e).\end{aligned}$$

Thus $\pi_0(e) \in \pi_0(\mathbb{A}(O^c)'$ and so by Haag duality $\pi_0(e) \in \pi_0(\mathbb{A}(O))$. Hence as π_0 is faithful it follows that $e \in \mathbb{A}(O)$. Now pick O_1 such that $\overline{O_1} \subseteq O$ and use property B to get an isometry $v \in \mathbb{A}(O)$ such that $v^*v = E$. Then we define a $*$-homomorphism

$\rho' : \hat{\mathbb{A}} \longrightarrow \hat{\mathbb{A}}$ as follows:

$$\rho'(a) = v^*\rho(a)v \qquad \text{for all } a \in \hat{\mathbb{A}}. \tag{30}$$

Then ρ' is localized in O and also $v\rho'(a) = vv^*\rho(a)v = e\rho(a)v = \rho(a)ev = \rho(a)vv^*v = \rho(a)v$, in other words $v\rho'(a) = \rho(a)v$ for all $a \in \hat{\mathbb{A}}$.

In order to wrap things up we must show that ρ' is transportable. Let O_2 be any double cone and then pick another double cone O_3 in such a way that $\overline{O_3} \subseteq O_2$. Next invoke the transportability of ρ to obtain a map μ localized in O_3 and a unitary arrow $U : \rho \longrightarrow \mu$ in Δ. Thus by composition in Δ we see that $e' = UeU^*$ is a map from μ to μ in Δ, indeed e' is in fact a projection. Hence by a similar argument to the one above for the projection ρ we see that there exists an isometry $v' \in \mathbb{A}(O_2)$ such that $e' = v'v'^*$. So letting $\mu' = v'^*\mu v'$ we get that μ' is localized in O_2 and the element $w = v'^*Uv$ satisfies $w\rho'(a) = \mu'(a)w$ for all $a \in \hat{\mathbb{A}}$ and w is unitary. Thus this shows that ρ' is transportable.

To summarize we have shown that if $e : \rho \longrightarrow \rho$ is a projection in Δ then there exists an object ρ' in Δ and an arrow $v : \rho' \longrightarrow \rho$ such $e = vv^*$ and $v^*v = id$. □

Lemma 8.2.10. If the vacuum representation is irreducible then $\pi_0(\hat{\mathbb{A}})' = \mathbb{C} \cdot id_{H_0}$.

Proof. Note that irreducibility of π_0 means that the only closed subspaces left invariant by the algebra $\pi_0(\hat{\mathbb{A}})$ are (0) and H_0. Now simply apply the following theorem which can be found in [20] p.330.

> *Suppose $F \subset \mathfrak{B}(H)$ is a self-adjoint set of bounded linear operators on a Hilbert space H. If (0) and H are the only closed subspaces of H left invariant by the set F then the commutant F' satisfies $F' = \mathbb{C} \cdot id_H$. The converse also holds.*

□

Corollary 8.2.11. If the vacuum representation is faithful and irreducible then the object $id_{\hat{\mathbb{A}}}$ of Δ is irreducible. In other words $\Delta(id_{\hat{\mathbb{A}}}, id_{\hat{\mathbb{A}}}) = \mathbb{C} \cdot id_{id_{\hat{\mathbb{A}}}}$.

We state one last theorem to close this section.

Theorem 8.2.12. Δ is a C^*-category.

8.3 The Monoidal Structure of Δ

In this section we explore the tensor structure on the category Δ and indicate how it has a symmetry if the spacetime dimension is three or larger.

Remark 8.3.1. For this section we will assume that the net $\hat{\mathbb{A}}$ satisfies the hypotheses of Theorem 8.2.8.

From an abstract nonsense point of view one can easily see how the category Δ is monoidal. Indeed as mentioned earlier we may view any C^*-algebra, $\mathcal{A}$, as a one object C^*-category which we denote by $\mathcal{A}[1]$. Then an endofunctor on $\mathcal{A}[1]$ is the same thing as a monoid endomorphism on the underlying monoid of $\mathcal{A}$. Thus we consider the functor category $Func(\mathcal{A}[1])$ with its usual monoidal structure. Note that a morphism $f : F \longrightarrow G$ in this category is a natural transformation which in this degenerate case corresponds to an element f of $\mathcal{A}$ such that $fF(a) = G(a)f$ for all $a \in \mathcal{A}$. Thus taking $\mathcal{A} = \hat{\mathbb{A}}$ the category Δ can be seen as the full subcategory of $Func(\hat{\mathbb{A}}[1])$ whose objects consist of the $*$-functors which are localizable and transportable. Then Δ will be monoidal as long as the composition of two such functors is again localizable and transportable.

Lemma 8.3.2. If $\rho \in \Delta(O_1)$ and $\sigma \in \Delta(O_2)$ are objects in Δ then $\rho \circ \sigma$ is also an object in Δ and is localized in any O with $O_1 \cup O_2 \subseteq O$.

So one has that Δ is a strict monoidal category where given two objects ρ and σ we define $\rho \otimes \sigma = \rho \circ \sigma$ and if $r : \rho \longrightarrow \rho'$ and $s : \sigma \longrightarrow \sigma'$ then we define $r \otimes s$ by $r \otimes s = r\rho(s)$ which is the same as $\rho'(s)r$. The only obstacle that remains is to build a symmetry map $\varepsilon_{\rho,\sigma} : \rho \otimes \sigma \longrightarrow \sigma \otimes \rho$. We will present following the proof given by Halvorson in [17].

For convenience we state a useful lemma.

Lemma 8.3.3. If $r \in \hat{\mathbb{A}}$ and $ra = ar$ for all $a \in \mathbb{A}(O^c)$ then $r \in \mathbb{A}(O)$. If $T : \rho_1 \longrightarrow \rho_2$ in Δ where $\rho_1 \in \Delta(O_1)$ and $\rho_2 \in \Delta(O_2)$ then $T \in \mathbb{A}(O)$ for any $O \supseteq O_1 \cup O_2$.

Proof. Let $r \in \hat{\mathbb{A}}$ and suppose that $ra = ar$ for all $a \in \mathbb{A}(O^c)$ then we have that $\pi_0(r)\pi_0(a) = \pi_0(a)\pi_0(r)$ for all $a \in \mathbb{A}(O^c)$. Thus $\pi_0(r) \in \pi_0(\mathbb{A}(O^c)'$ and hence by

Haag duality $\pi_0(r) \in \pi_0(\mathbb{A}(O))$. Moreover since π_0 is faithful we have that $r \in \mathbb{A}(O)$. By a similar argument one shows the other statement also holds. □

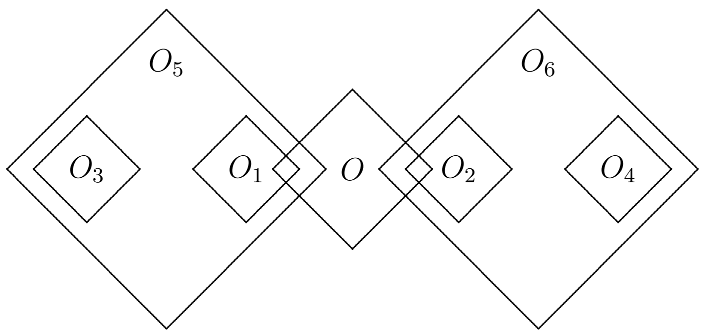
Now given $\rho \in \Delta(O_1)$ and $\sigma \in \Delta(O_2)$ it is instructive to consider $\rho \otimes \sigma$ and $\sigma \otimes \rho$ in the case that O_1 and O_2 are spacelike separated.

Lemma 8.3.4. If $\rho \in \Delta(O_1)$ and $\sigma \in \Delta(O_2)$ and O_1 and O_2 are spacelike separated then $\rho\sigma = \sigma\rho$, i.e., $\rho \otimes \sigma = \sigma \otimes \rho$.

Proof. For this proof we combine elements of the proof given by Halvorson in [17] and by Haag in [16]. First note that since $\cup_{O \in K} \mathbb{A}(O)$ is dense in $\hat{\mathbb{A}}$ is suffices to show that $\rho\sigma(a) = \sigma\rho(a)$ for all $a \in \mathbb{A}(O)$ for each $O \in K$. Thus let $O \in K$ be arbitrary. Then there exists double cones O_i, $i = 2, \ldots, 6$ such that

- O_1 and O_3 are spacelike separated,
- O_3 and O are spacelike separated,
- O_2 and O_4 are spacelike separated,
- O_4 and O are spacelike separated,
- $O_1 \cup O_3 \subseteq O_5$,
- $O_2 \cup O_4 \subseteq O_6$,
- O_5 and O_6 are spacelike separated.

Following Haag we illustrate this situation in the diagram below.



Now since ρ and σ are transportable there exists unitaries U_1 and $U_2 \in \hat{\mathbb{A}}$ and $\rho' \in \Delta(O_3)$, $\sigma' \in \Delta(O_4)$ such that $U_1 : \rho \longrightarrow \rho'$ and $U_2 : \sigma \longrightarrow \sigma'$ are arrows in Δ. Applying Lemma 8.3.3 we get that $U_1 \in \mathbb{A}(O_5)$ and $U_2 \in \mathbb{A}(O_6)$. Since O_5 and O_6 are spacelike separated we therefore have $U_1U_2 = U_2U_1$. Moreover since $\rho \in \Delta(O_1)$ and O_1 and O_6 are spacelike separated we have that $\rho(U_2) = U_2$ and similarly one shows that $\sigma(U_1) = U_1$.

Now let $a \in \mathbb{A}(O)$. Since O is spacelike separated from O_4 and O_3 we have that $\sigma'(a) = a = \rho'(a)$ thus

$$\begin{aligned}\rho\sigma(a) &= \rho(U_2\sigma'(a)U_2^*)\\ &= \rho(U_2aU_2^*)\\ &= \rho(U_2)\rho(a)\rho(U_2)^*\\ &= U_2U_1\rho'(a)U_1^*U_2^*\\ &= U_1U_2aU_2^*U_1^*\\ &= U_1\sigma(a)U_1^*\\ &= \sigma(U_1)\sigma(a)\sigma(U_1^*)\\ &= \sigma(U_1aU_1^*)\\ &= \sigma\rho(a).\end{aligned}$$

Hence $\rho \otimes \sigma = \sigma \otimes \rho$. □

Now suppose that $\rho_i \in \Delta(O_i)$ for $i = 1, 2$. Then pick double cones $\widetilde{O_i}$ which are spacelike separated from each other. Using transportability of ρ_i there exists $\widetilde{\rho_i} \in \Delta(\widetilde{O_i})$ and unitary maps $U_i \in \Delta(\rho_i, \widetilde{\rho_i})$. Furthermore we also have that $U_1 \otimes U_2 \in \Delta(\rho_1 \otimes \rho_2, \widetilde{\rho_1} \otimes \widetilde{\rho_2})$ and since $\widetilde{O_1}$ is spacelike separated from $\widetilde{O_2}$ we have that $\widetilde{\rho_1} \otimes \widetilde{\rho_2} = \widetilde{\rho_2} \otimes \widetilde{\rho_1}$. Hence the composition $(U_2 \otimes U_1)^* \circ (U_1 \otimes U_2)$ is defined and yields a map $\varepsilon_{\rho_1,\rho_2}(U_1, U_2) : \rho_1 \otimes \rho_2 \longrightarrow \rho_2 \otimes \rho_1$. Notice that this map is just given by the formula

$$\varepsilon_{\rho_1,\rho_2}(U_1, U_2) = \rho_2(U_1^*)U_2^*U_1\rho_1(U_2). \tag{31}$$

Remark 8.3.5. The endomorphisms $\widetilde{\rho_i}$ used in the above discussion are sometimes referred to as *spectator morphisms* [17]. The reason for this is that they do not explicitly appear in the definition of $\varepsilon_{\rho_1,\rho_2}(U_1,U_2)$ but rather only implicity through the unitary elements U_i.

Remark 8.3.6. It turns out that this map leads to a braiding if the dimension of spacetime is two or less and otherwise if the spacetime dimension is three or greater one gets a symmetry. We will assume that our Minkowski space is at least three-dimensional for simplicity.

Lemma 8.3.7. Let $\rho_i \in \Delta(O_i)$, $i = 1,2$, and suppose $\widetilde{O_i}$ are spacelike separated double cones and that $\widetilde{\rho_i} \in \Delta(\widetilde{O_i})$. If $W_1 \in \mathbb{A}(\widetilde{O_2}^c)$ and $W_2 \in \mathbb{A}(\widetilde{O_1}^c)$ are unitary elements such that $W_1W_2 = W_2W_1$ then $\varepsilon_{\rho_1,\rho_2}(U_1,U_2) = \varepsilon_{\rho_1,\rho_2}(W_1U_1,W_2U_2)$.

Proof. Since $W_1 \in \mathbb{A}(\widetilde{O_2}^c)$ it follows that $\rho_2(W_1) = W_1$ and similarly $\rho_1(W_2) = W_2$. Now the proof proceeds by expanding the expression $\varepsilon_{\rho_1,\rho_2}(W_1U_1,W_2U_2)$. □

Lemma 8.3.8. Let $\rho_i \in \Delta(O_i)$, $i = 1,2$ and suppose that $\widetilde{O_i}$ are spacelike separated double cones and that $U_i \in \Delta(\rho_i, \widetilde{\rho_i})$, and $U_i' \in \Delta(\rho_i, \widehat{\rho_i})$ are are unitary maps with $\widetilde{\rho_i}$, and $\widehat{\rho_i} \in \Delta(\widetilde{O_i})$. Then we have that $\varepsilon_{\rho_1,\rho_2}(U_1,U_2) = \varepsilon_{\rho_1,\rho_2}(U_1',U_2')$.

Corollary 8.3.9. If $\widetilde{O_i}$, $i = 1,2$, are spacelike separated double cones and $\widehat{O_i}$ is any other pair of spacelike separated double cones such that $\widetilde{O_i} \subseteq \widehat{O_i}$ then $\varepsilon_{\rho_1,\rho_2}(U_1,U_2) = \varepsilon_{\rho_1,\rho_2}(U_1',U_2')$ where $U_i \in \Delta(\rho_i,\widetilde{\rho_i})$, $U_i' \in \Delta(\rho_i,\widehat{\rho_i})$ are are unitary maps with $\widetilde{\rho_i} \in \Delta(\widetilde{O_i})$ and $\widehat{\rho_i} \in \Delta(\widehat{O_i})$.

Proof. Since $\widetilde{O_i} \subseteq \widehat{O_i}$ it follows that $\widetilde{\rho_i} \in \Delta(\widehat{O_i})$ thus applying the previous lemma we get the desired result. □

Remark 8.3.10. Notice that the above corollary implies that if instead $\widehat{O_i} \subseteq \widetilde{O_i}$ that $\varepsilon_{\rho_1,\rho_2}$ remains unchanged. Thus we can either expand or shrink the regions in which the spectator morphisms are localized without affecting the value of $\varepsilon_{\rho_1,\rho_2}(U_1,U_2)$.

Remark 8.3.11. The next step in this argument is to show that given any pair of spacelike separated double cones that $\varepsilon_{\rho_1,\rho_2}(U_1,U_2)$ is independent of their choice. First note that if $(\widetilde{O_1},\widetilde{O_2})$ is a pair of spacelike separated double cones then so are $(\widetilde{O_1}+x,\widetilde{O_2}+x)$ for all $x \in M$. Now we claim that spectator morphisms localized in $\widetilde{O_i}$ give the same definition for ε as spectator morphisms localized in $\widetilde{O_i}+x$.

Indeed by shrinking $\widetilde{O_1}$ we can assume that $\widetilde{O_1}+x$ is spacelike separated from $\widetilde{O_2}$ and moreover that there is another double cone $\widehat{O}$ which is spacelike separated from $\widetilde{O_2}$ and such that $\widetilde{O_1}$, and $\widetilde{O_1}+x \subseteq \widehat{O}$. Thus applying Corollary 8.3.9 we get that $(\widetilde{O_1},\widetilde{O_2})$, $(\widehat{O_1},\widetilde{O_2})$, and $(\widetilde{O_1}+x,\widetilde{O_2})$ all give the same definition for $\varepsilon_{\rho_1,\rho_2}$. Repeating the same process for $\widehat{O_2}$ we get that $(\widetilde{O_1}+x,\widetilde{O_2}+x)$ and $(\widetilde{O_1},\widetilde{O_2})$ give the same definition for $\varepsilon_{\rho_1,\rho_2}$.

Now the next fact is only true if the dimension of spacetime is three or larger. Suppose that $(\widetilde{O_1},\widetilde{O_2})$ is a pair of spacelike separated double cones. Then for any other pair of spacelike separated double cones $(O_1^\flat,O_2^\flat)$ there is a sequence of pairs of spacelike separated double cones $(\widehat{O_i},\overbrace{O_i})$, $i=1,\ldots,n$. Moreover the sequence satisfies that each $(\widehat{O_{i+1}},\overbrace{O_{i+1}})$ is obtained from $(\widehat{O_i},\overbrace{O_i})$ by a translation or by a shrinking or expanding of double cones as in Corollary 8.3.9. Note we also must have that $(\widehat{O_1},\overbrace{O_1})=(\widetilde{O_1},\widetilde{O_1})$ and that $(\widehat{O_n},\overbrace{O_n})=(O_1^\flat,O_2^\flat)$. Thus we have in this case that the value of ε is the same for each pair $(\widehat{O_{i+1}},\overbrace{O_{i+1}})$ and hence is independent of the choice of spacelike separated double cones used to define it.

Lemma 8.3.12. If the dimension of spacetime is three or larger then $\varepsilon_{\rho_2,\rho_1}=\varepsilon^{-1}_{\rho_1,\rho_2}$.

Proof. Suppose that $\rho_i \in \Delta(O_i)$ then choose spacelike separated double cones $\widetilde{O_i}$ in such a way that $\widetilde{O_1}=O_1$. Also choose spectator morphisms $\widetilde{\rho_i}$ with $\widetilde{\rho_1}=\rho_1$, and $U_1=id\in\Delta(\rho_1,\rho_1)$ and $U_2\in\Delta(\rho_2,\widetilde{\rho_2})$ unitaries. Then we get

$$\begin{aligned}\varepsilon_{\rho_1,\rho_2}&=\rho_2(U_1^*)U_2^*U_1\rho_1(U_2)\\&=U_2^*\rho_1(U_2)\end{aligned}$$

and on the other hand

$$\begin{aligned}\varepsilon_{\rho_2,\rho_1}&=\rho_1(U_2^*)U_1^*U_2\rho_2(U_1)\\&=\rho_1(U_2)^*U_2\\&=\varepsilon^*_{\rho_1,\rho_2}.\end{aligned}\tag{32}$$

But $\varepsilon_{\rho_1,\rho_2}$ is unitary and hence we see that $\varepsilon_{\rho_2,\rho_1}=\varepsilon^*_{\rho_1,\rho_2}=\varepsilon^{-1}_{\rho_1,\rho_2}$ □

Finally we can state the last two results of this section which shows that ε is a symmetry for the monoidal category Δ.

Theorem 8.3.13. The map $\varepsilon_{\rho_1,\rho_2}$ is a symmetry on the monoidal category Δ. Furthermore it is the unique symmetry on Δ satisfying $\varepsilon_{\rho_1,\rho_2} = id_{\rho_1\otimes\rho_2}$ whenever $\rho_i \in \Delta(O_i)$ and O_1 and O_2 are spacelike separated.

Lemma 8.3.14. Δ is a symmetric C^*-tensor category.

Remark 8.3.15. The category Δ is an endofunctor category, with monoidal structure given by composition which is a highly non-symmetric operation. It is thus astonishing that this category is symmetric monoidal. In general one would not expect any relationship between $\rho_1 \circ \rho_2$ and $\rho_2 \circ \rho_1$.

8.4 DHR-Representations

For completeness we will present in this section the concept of a *DHR-representation* of $\hat{\mathbb{A}}$ and exhibit the connection they have with localized transportable endomorphisms [16, 17]. Again we assume that the net $\hat{\mathbb{A}}$ is equipped with a vacuum representation (π_0, H_0).

Definition 8.4.1. A representation $\pi : \hat{\mathbb{A}} \longrightarrow \mathfrak{B}(H)$ is called a **DHR-representation** if for each double cone $O \in K$ there exists a unitary map $V_O : H \longrightarrow H_0$ such that

$$V_O \circ \pi(a) = \pi_0(a)V_O \qquad \forall a \in \mathbb{A}(O^c). \tag{33}$$

Remark 8.4.2. The representations in the above definition were proposed by Doplicher, Haag, and Roberts as physically relevant and thus bear the name DHR-representation.

Notice that one can immediately form a category $\mathbf{DHR}_{\mathbb{A}} = \mathbf{DHR}$ whose objects are DHR-representations and arrows are intertwining maps.

Theorem 8.4.3. The assignment $\rho \in \Delta(O) \mapsto F(\rho) = \pi_0 \circ \rho$ extends to a functor $F : \Delta \longrightarrow \mathbf{DHR}$ with $F(s) = \pi_0(s)$ for all arrows s in Δ. If π_0 is faithful and satisfies Haag duality then F is an equivalence of categories.

Remark 8.4.4. We will prove a more general version of this result in Chapter 10 in Theorem 10.2.23.

Remark 8.4.5. The equivalence allows one to endow the category **DHR** with a symmetric C^*-tensor structure and shows that studying Δ is a useful way to study **DHR**.

Chapter 9

Premonoidal $*$-categories & von Neumann categories

In this chapter, we present several new definitions which will be fundamental in our notion of functorial quantum field theory. We also prove some properties of our structures and give examples.

9.1 Premonoidal $*$ -categories

In this section we extend some well known concepts, used in the Doplicher Roberts theorem, to the setting of premonoidal categories. We will use these definitions in later sections to prove an analogous reconstruction theorem.

Definition 9.1.1. An **Ab-premonoidal category** is a premonoidal category $\mathcal{C}$ such that for all objects X, Y in $\mathcal{C}$ the set $\mathcal{C}(X, Y)$ is equipped with an abelian group structure. Moveover if $f, g \in \mathcal{C}(X, Y)$ and $h \in \mathcal{C}(A, X)$ and $k \in \mathcal{C}(Y, B)$ then $(f + g) \circ h = f \circ h + g \circ h$ and $k \circ (f + g) = k \circ f + k \circ g$. In addition we also require that for all objects A the functions $A \otimes - : \mathcal{C}(X, Y) \longrightarrow \mathcal{C}(A \otimes X, A \otimes Y)$ and $- \otimes A : \mathcal{C}(X, Y) \longrightarrow \mathcal{C}(X \otimes A, Y \otimes A)$ are group homomorphisms for all objects X, Y in $\mathcal{C}$.

In many cases the hom-sets in our categories will be more than just abelian groups, they will turn out to have the structure of a complex vector space. The following definition precisely captures this phenomenon.

Definition 9.1.2. A **$\mathbb{C}$-linear premonoidal category** is a a premonoidal category $\mathcal{C}$ in which every hom-set $\mathcal{C}(X,Y)$ is a complex vector space and the composition map $(f,g) \mapsto g \circ f$ is bilinear and the functions $A \otimes - : \mathcal{C}(X,Y) \longrightarrow \mathcal{C}(A \otimes X, A \otimes Y)$ and $- \otimes A : \mathcal{C}(X,Y) \longrightarrow \mathcal{C}(X \otimes A, Y \otimes A)$ are $\mathbb{C}$-linear for all objects A, X, and Y in $\mathcal{C}$.

Definition 9.1.3. A **positive $*$-operation** on a $\mathbb{C}$-linear premonoidal category $\mathcal{C}$ is a family of functions assigning to each arrow $s \in \mathcal{C}(X,Y)$ an arrow $s^* \in \mathcal{C}(Y,X)$ with $(g \circ f)^* = f^* \circ g^*$ for composable arrows f and g and $id_A^* = id_A$. The map $s \mapsto s^*$ must be anti-linear, and satisfy $(s^*)^* = s$ and if $s^* \circ s = 0$ then $s = 0$. We also require that if f is a central map in $\mathcal{C}$ then so is f^* and that for all arrows g in $\mathcal{C}$ and objects A, that $(A \otimes f)^* = A \otimes f^*$ and $(f \otimes A)^* = f^* \otimes A$. Finally if $\mathcal{C}$ is not strict then we will insist that $\alpha^* = \alpha^{-1}$, $\lambda^* = \lambda^{-1}$, and $\rho^* = \rho^{-1}$ and in the case of symmetry that $\tau^* = \tau^{-1}$. A $\mathbb{C}$-linear premonoidal category equipped with a positive $*$-operation is called a **premonoidal $*$-category**.

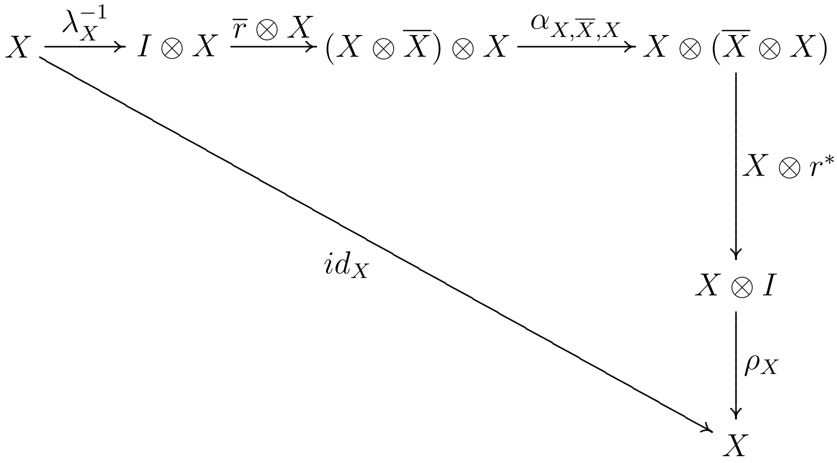
Definition 9.1.4. A **premonoidal $*$-functor** from a premonoidal $*$-category $\mathcal{C}$ to another premonoidal $*$-category $\mathcal{D}$ consists of a premonoidal functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ such that the function $\mathcal{C}(A,B) \longrightarrow \mathcal{D}(FA,FB)$ is $\mathbb{C}$-linear for all objects A and $B \in |\mathcal{C}|$ and $F(s^*) = F(s)^*$ for all arrows s in $\mathcal{C}$.

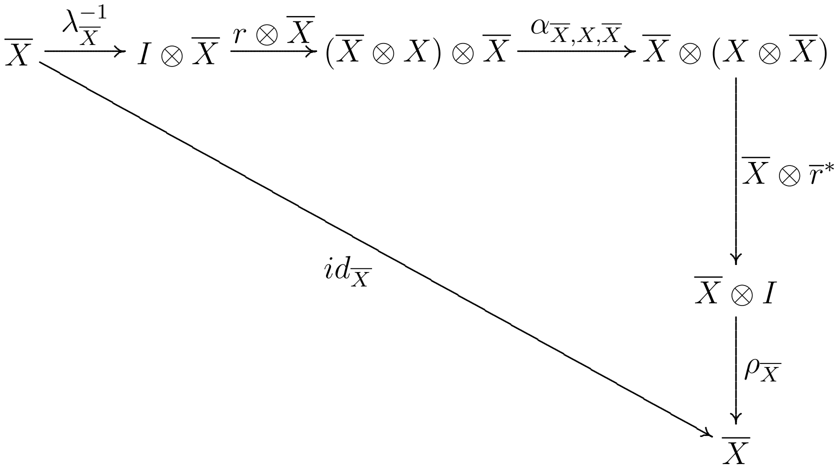
Now in a premonoidal $*$-category, $\mathcal{C}$, we say an arrow $v \in \mathcal{C}(X,Y)$ is an *isometry* if $v^* \circ v = id_X$ and *unitary* if we also have $v \circ v^* = id_Y$. A map $p \in \mathcal{C}(X,X)$ is a *projection* if $p = p \circ p = p^*$. Lastly $\mathcal{C}$ *has subobjects* if for every projection $p \in \mathcal{C}(X,X)$ there exists an isometry $v \in \mathcal{C}(X,Y)$ such that $p = v \circ v^*$. In the case that the map v is central then we say that $\mathcal{C}$ has *central subobjects.*

Definition 9.1.5. Let $\mathcal{C}$ be a premonoidal $*$-category and X, Y objects. A **direct sum** of X and Y consists of an object Z together with central maps $u : X \longrightarrow Z$ and $v : Y \longrightarrow Z$ such that $u^* \circ u = id_X$, $v^* \circ v = id_Y$, and $u \circ u^* + v \circ v^* = id_Z$.

Notice that if $D \cong Z$ then D will also be a direct sum of X and Y. Thus direct sums are only defined up to isomorphism and for this reason we will write $Z \cong X \bigoplus Y$. We now present a notion which is a key ingredient for the Doplicher Roberts result.

Definition 9.1.6. A **conjugate** of an object X in a premonoidal $*$-category $\mathcal{C}$ consists of an object $\overline{X}$ and a pair of arrows $r : I \longrightarrow \overline{X} \otimes X$ and $\overline{r} : I \longrightarrow X \otimes \overline{X}$ such that $r, \overline{r} \in \mathcal{Z}(\mathcal{C})$ and satisfy the conjugate equations.





In this case we say that $(\overline{X}, r, \overline{r})$ is a conjugate of X. We say that $\mathcal{C}$ **has conjugates** every non-zero object has a conjugate.

The following lemma shows that any two conjugates of X are unitarily isomorphic. The proof of this result is the same as the one given in [17] for the case of tensor $*$-categories.

Lemma 9.1.7. Let X be an object in a premonoidal $*$-category $\mathcal{C}$ and $(\overline{X}, r, \overline{r})$, and $(\overline{X}', r', \overline{r}')$ conjugates of X. Then the map $\rho_{\overline{X}'} \circ (\overline{X}' \otimes \overline{r}^*) \circ \alpha_{\overline{X}',X,\overline{X}} \circ (r' \otimes \overline{X}) \circ \lambda_{\overline{X}}^{-1} :$ $\overline{X} \longrightarrow \overline{X}'$ is unitary.

Definition 9.1.8. Let $\mathcal{C}$ be a $\mathbb{C}$-linear premonoidal category. Then an object X is called **irreducible** if $\mathcal{Z}(\mathcal{C})(X,X) = \mathbb{C} \cdot id_X$.

We now give the main definition of this section.

Definition 9.1.9. A **PC*** is a premonoidal $*$-category $\mathcal{C}$ with $\dim \mathcal{Z}(\mathcal{C})(X,Y) < \infty$, for all objects X and Y, and moreover we require that $\mathcal{C}$ has conjugates, direct sums, central subobjects and an irreducible tensor unit I. A **BPC*** is a PC^* with a unitary braiding, and an **SPC*** is a PC^* with a unitary symmetry.

Remark 9.1.10. Note that a monoidal PC^* is referred to as a TC^* in [17]. Similarly a BPC^* is referred to as a BTC^* and an SPC^* as an STC^*.

The following result is now immediate.

Lemma 9.1.11. If $\mathcal{C}$ is a PC^* then the centre $\mathcal{Z}(\mathcal{C})$ is a TC^*. Similarly if $\mathcal{C}$ is BPC^* or an SPC^* then $\mathcal{Z}(\mathcal{C})$ is a BTC^* or an STC^* respectively.

We now proceed by establishing several results concerning PC^* analogous to those for TC^*.

Lemma 9.1.12. If $\mathcal{C}$ is a PC^* then it is semisimple. That is to say every object is a finite direct sum of irreducible objects.

Proof. If $\mathcal{C}$ is a PC^* then $\mathcal{Z}(\mathcal{C})$ is a TC^* and any object in $\mathcal{C}$ is an object in the centre. Hence applying Lemma A.35 in [17] to the centre the result follows. □

Now there is a more general notion which is related to premonoidal $*$-categories. This notion is the premonoidal analogue of C^*-tensor categories considered by Doplicher and Roberts.

Definition 9.1.13. A **premonoidal** C^***-category** is a premonoidal $*$-category $\mathcal{C}$ such that $\mathcal{C}(X,Y)$ is a Banach space with norm denoted by $\|\cdot\|_{X,Y}$ such that $\|s \circ t\|_{X,Z} \leq \|s\|_{X,Y}\|t\|_{Y,Z}$ and $\|s^* \circ s\| = \|s\|^2$ and $\|A \otimes s\|_{A\otimes X, A\otimes Y} = \|s\|_{X,Y} = \|s \otimes A\|_{X\otimes A, Y\otimes A}$ for all $s : X \longrightarrow Y$ and $t : Y \longrightarrow Z$ and objects A.

Definition 9.1.14. If $\mathcal{C}$ and $\mathcal{D}$ are premonoidal C^*-categories then a **premonoidal C^*-functor** from $\mathcal{C}$ to $\mathcal{D}$ consists of a premonoidal functor (F, d^F, e^F) such that F is a premonoidal $*$-functor.

Again by virtue of the definition of premonoidal C^*-categories we have the following result.

Lemma 9.1.15. If $\mathcal{C}$ is a premonoidal C^*-category then $\mathcal{Z}(\mathcal{C})$ is a C^*-tensor category.

Proposition 9.1.16. Let $\mathcal{C}$ be a premonoidal C^*-category with direct sums, and an irreducible tensor unit I. If X and Y have conjugates then $\dim \mathcal{Z}(\mathcal{C})(X,Y) < \infty$. Hence a premonoidal C^*-category with direct sums, conjugates, central subobjects, and an irreducible unit is a PC^*.

Theorem 9.1.17. Suppose that $(\overline{X}, r, \overline{r})$ is a conjugate of X then for all objects Y and Z in a premonoidal $*$-category $\mathcal{C}$ there is a bijection $\mathcal{C}(X \otimes Z, Y) \cong \mathcal{C}(Z, \overline{X} \otimes Y)$.

Proof. Given $f : X \otimes Z \longrightarrow Y$ define $\varphi(f) : Z \longrightarrow \overline{X} \otimes Y$ by

$$\varphi(f) \equiv (\overline{X} \otimes f) \circ (r \otimes Z) \circ \lambda_Z^{-1}.$$

Similarly given $g : Z \longrightarrow \overline{X} \otimes Y$ define $\psi(g) : X \otimes Z \longrightarrow Y$ by

$$\psi(g) \equiv \lambda_Y \circ (\overline{r}^* \otimes Y) \circ (X \otimes g).$$

Now suppose that $f : X \otimes Z \longrightarrow Y$ is given then we will show that $\psi\varphi(f) = f$. We will omit associativity isomorphisms for convenience. Unpacking $\psi\varphi(f)$ we get

$$\begin{aligned}\psi\varphi(f) &= \lambda_Y \circ (\overline{r}^* \otimes Y) \circ (X \otimes \overline{X} \otimes f) \circ (X \otimes r \otimes Z) \circ (X \otimes \lambda_Z^{-1}) \\ &= \lambda_Y \circ (I \otimes f) \circ (\overline{r}^* \otimes X \otimes Z)(X \otimes r \otimes Z) \circ (X \otimes \lambda_Z^{-1}) \quad \text{by centrality of } \overline{r}^* \\ &= f \circ (\lambda_X \otimes Z) \circ (\overline{r}^* \otimes X \otimes Z)(X \otimes r \otimes Z) \circ (X \otimes \lambda_Z^{-1}).\end{aligned}$$

Note since we are omitting associativity isomorphisms we have that $X \otimes \lambda_Z^{-1} = \rho_X^{-1} \otimes Z$ by the coherence axioms for premonoidal categories. Thus we get

$$\psi\varphi(f) = f \circ (\lambda_X \otimes Z) \circ (\overline{r}^* \otimes X \otimes Z)(X \otimes r \otimes Z) \circ (\rho_X^{-1} \otimes Z)$$

and by using the first conjugate equation it follows that

$$\begin{aligned}(\lambda_X \otimes Z) \circ (\overline{r}^* \otimes X \otimes Z)(X \otimes r \otimes Z) \circ (\rho_X^{-1} \otimes Z) &= [\lambda_X \circ (\overline{r}^* \otimes X)(X \otimes r) \circ \rho_X^{-1}] \otimes Z \\ &= id_X \otimes Z \\ &= id_{X \otimes Z}.\end{aligned}$$

Hence $\psi\varphi(f) = f$. By similar arguments we can show $\varphi\psi(g) = g$ for all arrows $g : Z \longrightarrow \overline{X} \otimes Y$. □

9.2 von Neumann Categories

The following definition is our attempt at generalizing the notion of a von Neumann algebra, much like our definition of premonoidal C^*-category is an attempt at generalizing the notion of a C^*-algebra.

Definition 9.2.1. Let $\mathcal{A} \subseteq \mathcal{C}$ be a premonoidal C^*-subcategory of a premonoidal C^*-category $\mathcal{C}$. Then $\mathcal{A}$ is called a **$\mathcal{C}$-von Neumann category** when $\mathcal{A}''(X, Y) = \mathcal{A}(X, Y)$ for all objects X and Y in $\mathcal{A}$. We also say $\mathcal{A}$ is called a **von Neumann category on** $\mathcal{C}$. When $\mathcal{C} = \mathbf{Hilb}_H$ then $\mathcal{A}$ is simply called a **von Neumann category**.

Now a natural question to ask is whether a one-object von Neumann category is a von Neumann algebra or not. Before we provide an answer to this question we need an intermediate result.

Lemma 9.2.2. If $\mathcal{A}$ is a collection of objects and arrows in a premonoidal $*$-category $\mathcal{C}$ closed under $*$, then the commutant $\mathcal{A}'$ is a premonoidal $*$-subcategory of $\mathcal{C}$.

Proof. By Theorem 6.3.2 the category $\mathcal{A}'$ is a premonoidal category, and thus it remains to show that given any arrow $S : B \longrightarrow A$ in $\mathcal{A}$ that S^* is again an arrow in this category. Indeed if $g : X \longrightarrow Y$ is any arrow in $\mathcal{A}$ then

$$\begin{aligned}(g \otimes B) \circ (X \otimes S^*) = (Y \otimes S^*) \circ (g \otimes A) \qquad &\text{iff} \\ (g \otimes B)^{**} \circ (X \otimes S^*) = (Y \otimes S^*) \circ (g \otimes A)^{**}. & \end{aligned} \tag{34}$$

Now using that S is an arrow in the commutant category we get that

$$\begin{aligned}(X \otimes S) \circ (g^* \otimes B) &= (g^* \otimes A) \circ (Y \otimes S) \qquad \text{using properties of } * \\ (X \otimes S) \circ (g \otimes B)^* &= (g \otimes A)^* \circ (Y \otimes S).\end{aligned}$$

Now applying $*$ to both sides of the above equation one obtains the required equation, namely equation 34. Similarly one can show that the other required equation for S^* to be an arrow in $\mathcal{A}'$ also holds. In addition since composition is bilinear we also have that $\mathcal{A}'$ will be $\mathbb{C}$-linear and the positivity of the $*$-operation is also immediate. Thus $\mathcal{A}'$ is a premonoidal $*$-category. □

Theorem 9.2.3. If $\mathcal{A} \subseteq \mathbf{Hilb}_H$ is a von Neumann category then $\mathcal{A}(\mathbb{C}, \mathbb{C})$ has the structure of a von Neumann algebra.

Proof. Let $\mathcal{M} = \mathcal{A}(\mathbb{C}, \mathbb{C})$. First notice that since $\mathcal{A}$ is a category closed under $*$ we have that $T \circ S^* \in \mathcal{M}$ for all $T, S \in \mathcal{M}$. Thus $\mathcal{M}$ is a $*$-subalgebra of $\mathfrak{B}(\mathbb{C} \otimes H)$. We will show that if $S \in \mathcal{A}'(\mathbb{C}, \mathbb{C})$ then $S \circ T = T \circ S$ for all $T \in \mathcal{M}$. Indeed if $S \ltimes T = S \rtimes T$ then the following diagram commutes

$$\begin{array}{ccccccc}
\mathbb{C}\otimes\mathbb{C}\otimes H & & \xrightarrow{\;id_{\mathbb{C}}\otimes S\;} & \mathbb{C}\otimes\mathbb{C}\otimes H & \xrightarrow{\;\tau_{\mathbb{C},\mathbb{C}}\otimes id_H\;} & & \mathbb{C}\otimes\mathbb{C}\otimes H \\
{\scriptstyle \tau_{\mathbb{C},\mathbb{C}}\otimes id_H}\downarrow & & & & & & \downarrow{\scriptstyle id_{\mathbb{C}}\otimes T} \\
\mathbb{C}\otimes\mathbb{C}\otimes H & \xrightarrow[id_{\mathbb{C}}\otimes T]{} & \mathbb{C}\otimes\mathbb{C}\otimes H & \xrightarrow[\tau_{\mathbb{C},\mathbb{C}}\otimes id_H]{} & \mathbb{C}\otimes\mathbb{C}\otimes H & \xrightarrow[id_{\mathbb{C}}\otimes S]{} & \mathbb{C}\otimes\mathbb{C}\otimes H
\end{array}$$

Now the map $\tau_{\mathbb{C},\mathbb{C}} : \mathbb{C} \otimes \mathbb{C} \longrightarrow \mathbb{C} \otimes \mathbb{C}$ is given by $\tau(a \otimes b) = b \otimes a$ but by definition of tensor and linearity of τ it follows that $\tau(a \otimes b) = ab\tau(1 \otimes 1) = ab(1 \otimes 1) = a \otimes b$ and hence $\tau_{\mathbb{C},\mathbb{C}} = id$. Using this fact in the above diagram one obtains

$$(id_{\mathbb{C}} \otimes T) \circ (id_{\mathbb{C}} \otimes S) = (id_{\mathbb{C}} \otimes S) \circ (id_{\mathbb{C}} \otimes T)$$

and thus $id_{\mathbb{C}} \otimes (T \circ S) = id_{\mathbb{C}} \otimes (S \circ T)$ and this occurs if and only if $T \circ S = S \circ T$. Hence we have shown that in fact that $S \in \mathcal{A}'(\mathbb{C}, \mathbb{C})$ if and only if $S \circ T = T \circ S$ for all $T \in \mathcal{M}$. We denote the commutant of the algebra $\mathcal{M}$ in $\mathfrak{B}(\mathbb{C} \otimes H)$ by $\mathcal{M}'$.

Thus we have shown that $\mathcal{M}' = \mathcal{A}'(\mathbb{C},\mathbb{C})$ and all that we used about $\mathcal{A}$ is that it was a premonoidal C^*-category. Hence as $\mathcal{A}'$ is a premonoidal C^*-category we also have that $\mathcal{N}' = \mathcal{A}''(\mathbb{C},\mathbb{C})$, where $\mathcal{N} = \mathcal{A}'(\mathbb{C},\mathbb{C})$. On the other hand $\mathcal{A}$ is a von Neumann category and hence $\mathcal{A}(\mathbb{C},\mathbb{C}) = \mathcal{A}''(\mathbb{C},\mathbb{C})$. Thus $\mathcal{M} = \mathcal{N}'$, and clearly as $\mathcal{M}' = \mathcal{N}$ it follows $\mathcal{M}'' = \mathcal{N}' = \mathcal{M}$ showing that $\mathcal{M}$ is a von Neumann algebra. □

Scholium 9.2.4. If $\mathcal{A} \subseteq \mathbf{Hilb}_H$ is a premonoidal C^*-subcategory then $S \in \mathcal{A}'(\mathbb{C},\mathbb{C})$ if and only if $S \circ T = T \circ S$ for all $T \in \mathcal{A}(\mathbb{C},\mathbb{C})$.

Corollary 9.2.5. Every one-object von Neumann category is a von Neumann algebra.

Thus the above corollary justifies our claim that a von Neumann category is an appropriate generalization of the notion of a von Neumann algebra. Before providing some concrete examples of von Neumann categories we will first establish some analogues of classical results found in the theory of von Neumann algebras.

Proposition 9.2.6. If $\mathcal{A}$ is a set of objects and arrows in a premonoidal C^*-category $\mathcal{C}$ closed under $*$, then $\mathcal{A}'$ is a premonoidal C^*-category. In particular it is a $\mathcal{C}$-von Neumann category.

Proof. By Lemma 9.2.2 we have that $\mathcal{A}$ is a premonoidal $*$-category. Furthermore each hom-set $\mathcal{A}'(X,Y)$ is a normed linear subspace of $\mathcal{C}(X,Y)$ with norm $|| \, ||_{X,Y}$ coming from the C^*-structure on $\mathcal{C}$. These norms already satisfy the required conditions of Definition 9.1.13. Thus it remains to show that each space $\mathcal{A}'(X,Y)$ is complete with respect to its norm, or equivalently that it is a closed subspace of $\mathcal{C}(X,Y)$. We will show that the former is true.

Notice that for any arrow $f : A \longrightarrow C$ the linear map $\zeta_f : \mathcal{C}(B,D) \longrightarrow \mathcal{C}(A \otimes C, B \otimes D)$ given by $\zeta_f(g) = f \ltimes g - f \rtimes g = (C \otimes g) \circ (f \otimes B) - (f \otimes D) \circ (A \otimes g)$ is bounded. Similarly the linear map $\eta_f : \mathcal{C}(B,D) \longrightarrow \mathcal{C}(C \otimes A, D \otimes B)$ given by $\eta_f(g) = g \ltimes f - g \rtimes f = (D \otimes f) \circ (g \otimes A) - (g \otimes C) \circ (B \otimes f)$ is bounded. So let (g_j) be a Cauchy sequence in $\mathcal{A}'(B,D)$. By completeness of $\mathcal{C}(B,D)$ it converges to a map $g = \lim g_j$ in $\mathcal{C}(B, D)$. Now for any arrow $f : A \longrightarrow C$ in $\mathcal{A}$ we have that

$$\begin{aligned}\zeta_f(g) &= \zeta_f(\lim g_j) \\ &= \lim \zeta_f(g_j), \qquad \text{by continuity of } \zeta_f \\ &= \lim 0, \qquad \text{since } \zeta_f(g_j) = 0 \quad \forall\, j \\ &= 0.\end{aligned}$$

Similarly we also have that $\eta_f(g) = 0$ for any arrow f in $\mathcal{A}$ and thus $g \in \mathcal{A}'(B, D)$. Hence we have shown that $\mathcal{A}'(B, D)$ is complete, establishing that $\mathcal{A}'$ is a premonoidal C^*-category.

To see that $\mathcal{A}'$ is a $\mathcal{C}$-von Neumann category we observe that $\mathcal{A} \subseteq \mathcal{A}''$ and taking commutants we get $\mathcal{A}''' \subseteq \mathcal{A}'$. On the other hand we also have that $\mathcal{A}' \subseteq \mathcal{A}'''$ and thus the result follows that $\mathcal{A}''' = \mathcal{A}'$. □

In fact even more is true about the category $\mathcal{A}'$.

Lemma 9.2.7. Let $\mathcal{A}$ and $\mathcal{C}$ be as in Proposition 9.2.6, then $\mathcal{A}'(X, Y)$ is closed in the weak topology for all objects X and Y.

Proof. To show that $\mathcal{A}'(X, Y)$ is closed we will show that it is equal to its closure in the weak topology. Clearly we have that $\mathcal{A}'(X, Y) \subseteq wk - cl\, \mathcal{A}'(X, Y)$. If an element $x \in wk - cl\, \mathcal{A}'(X, Y)$, then we will show that $x \in \mathcal{A}'(X, Y)$. Since x is in the weak closure it follows that there exists a net (x_j) in $\mathcal{A}'(X, Y)$ that converges to x, i.e. $\lim x_j = x$. Now to show that x is an arrow in the commutant category we will show that $\zeta_f(x) = 0$ and $\eta_f(x) = 0$ for all arrows f in $\mathcal{A}$, where ζ_f and η_f are the continuous maps defined in the proof of Proposition 9.2.6.

Our argument relies on the following fundamental fact which can be found in [20] Proposition 1.3.3. p.30.

> *Suppose that $T : V \longrightarrow W$ is a continuous linear map of locally convex spaces. Then T will also be continuous with respect to the weak topologies on V and W.*

Thus this result shows that the maps ζ_f and η_f will also be continuous with respect to the weak topologies. Thus for any arrow f in $\mathcal{A}$ we have that

$$\begin{aligned}\zeta_f(x) &= \zeta_f(\lim x_j)\\ &= \lim \zeta_f(x_j), &&\text{since } \zeta_f \text{ is continuous w.r.t the weak topologies}\\ &= \lim 0, &&\text{since } x_j \in \mathcal{A}'(X,Y)\\ &= 0\end{aligned}$$

Thus $\zeta_f(x) = 0$ and similarly $\eta_f(x) = 0$ for all arrows f in $\mathcal{A}$. Hence $\mathcal{A}'(X,Y)$ is closed in the weak topology for all pairs of objects X and Y. □

Remark 9.2.8. Notice that in the case of a locally convex space with a given initial topology, one can always define a new topology called the weak topology which in general is coarser than the initial one. Thus any weakly closed subset will automatically be closed in the initial topology, but the converse may not be true. Notice also that in the case that $\mathcal{F} \subseteq \mathfrak{B}(H)$, where H is a Hilbert space, that the commutant $\mathcal{F}' \subseteq \mathfrak{B}(H)$ is closed in the weak topology. Thus our Lemma 9.2.7 establishes a nice analog of this classical result. In light of this result it is natural to ask whether von Neumann's Double Commutant theorem (see Theorem 3.4.6) has an analog in our setting. At this point we don't have enough machinery at our disposal to establish such a result.

Now we consider some consequences of our above results.

Definition 9.2.9. Let $\mathcal{A}$ be a collection of objects and arrows in a premonoidal C^*-category $\mathcal{C}$ and let $\mathcal{A}^* = \{x^* \mid x \in \mathcal{A}\}$. Then $\{\mathcal{A} \cup \mathcal{A}^*\}''$ is the **$\mathcal{C}$-von Neumann category generated by $\mathcal{A}$.**

Corollary 9.2.10. If $\mathcal{M}$ is a von Neumann category on $\mathbf{Hilb}_H$ then $\mathcal{M}(K,K)$ is a von Neumann algebra on $K \otimes H$ for all objects K.

Proof. By Lemma 9.2.7 we have that $\mathcal{M}(K,K)$ is weakly closed, and thus applying von Neumann's double commutant theorem we have that the weak closure of a self-adjoint set of operators on a Hilbert space is equal to the double commutant of that set. Since $\mathcal{M}(K, K)$ is already closed we have that it is equal to its double commutant and thus is a von Neumann algebra. □

Remark 9.2.11. The above result, Corollary 9.2.10, is somewhat surprising considering the abstract definition of a von Neumann category, c.f. Definition 9.2.1. A priori, there is no reason that one would expect such a strong connection between our concept of von Neumann category and the classical notion. Our notion of von Neumann category turns out to also be closely related the Ghez-Lima-Roberts notion of W^*-category [15] in the sense that every von Neumann category is also what they call a *concrete W^*-category*. Our notion however has a more algebraic flavour whereas theirs is more topological. In addition another main difference is that our categories have a premonoidal structure and W^*-categories need not have this additional structure.

9.3 Examples of von Neumann Categories

At this point we feel that some examples of von Neumann categories are in order. We will also use this opportunity to draw further parallels between our theory and the classical one.

Example 9.3.1. By Corollary 9.2.5 every von Neumann algebra $\mathcal{M}$ can be viewed as a one-object von Neumann category.

Example 9.3.2. If $\mathcal{C}$ is a premonoidal C^*-category then $\mathcal{C}$ and $\mathcal{Z}(\mathcal{C})$ are $\mathcal{C}$-von Neumann categories. This is clear since $\mathcal{C} = \mathcal{Z}(\mathcal{C})'$. In the case $\mathcal{C} = \mathbf{Hilb}_H$ we see that $\mathcal{Z}(\mathbf{Hilb}_H) \simeq \mathbf{Hilb}$ is a von Neumann category. In the case that $\mathcal{C}$ is a von Neumann algebra viewed as a one-object von Neumann category we get that centre of a von Neumann algebra is again a von Neumann algebra.

The above example motivates the following comparison that we now explore. If H is a Hilbert space then $\mathfrak{B}(H)$ is a von Neumann algebra and the centre of $\mathfrak{B}(H)$ is $\mathbb{C}$. Now by the above example $\mathfrak{B}(H)$ can be viewed as a one-object von Neumann category on $\mathbf{Hilb}_H$ and its centre will be the subcategory with object $\mathbb{C}$ and will have as arrows the central maps on this object. Thus we think of $\mathbf{Hilb}_H$ as a multi-object version of the classical $\mathfrak{B}(H)$ and likewise since $\mathcal{Z}(\mathbf{Hilb}_H) \simeq \mathbf{Hilb}$ we think of $\mathbf{Hilb}$ as a playing the role of the complex numbers $\mathbb{C}$. We now illustrate the analogy with a diagram.

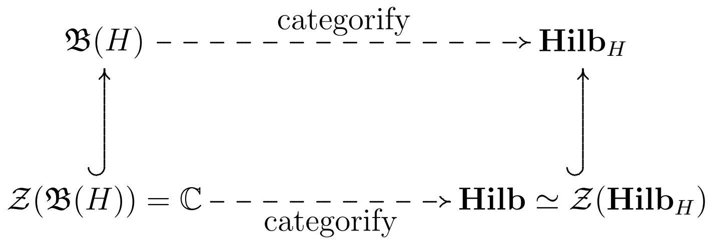


Figure 1: $\mathfrak{B}(H)$ Analogy

Now if $\mathcal{M} \subseteq \mathfrak{B}(H)$ is von Neumann algebra then we can fill in the above diagram as follows.

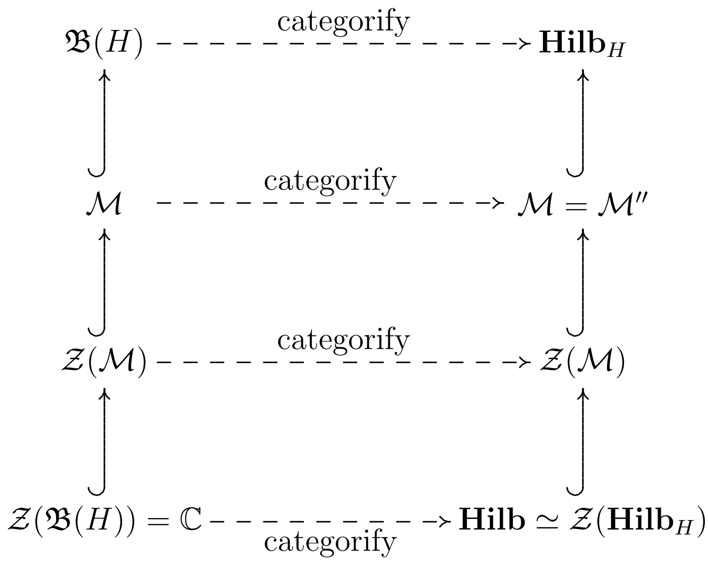


Figure 2: Expanded $\mathfrak{B}(H)$ Analogy

Here in Figure 2 the left column corresponds to the classical setting and the second column corresponds to our categorified setting of von Neumann categories. Notice also that for the sake of our analogy we have simplified things by assuming that

$\mathcal{M}'' = \mathcal{M}$ which entails that it is a *wide subcategory* of $\mathbf{Hilb}_H$ that contains the centre of $\mathbf{Hilb}_H$. Thus using these two analogies for intuition, one can see how our theory matches up with some of the standard features of von Neumann algebras.

Continuing with more examples of von Neumann categories, we will consider premonoidal C^*-categories that arise as a functor category.

Example 9.3.3. Suppose that $\mathcal{D}$ is a C^*-category then let $[\mathcal{D}, \mathcal{D}]_*$ be the premonoidal category whose objects are $*$-functors. An arrow $t : F \longrightarrow G$ consists of a family of maps $t_A : FA \longrightarrow GA$ in $\mathcal{D}$ such that the set $\{\|t_A\|\}$ is bounded. We call these arrows *bounded transformations*. We should note that this example is a premonoidal variation on an example of Ghez-Lima-Robers, [15], of a C^*-category. Now given a map $t : F \longrightarrow G$ then one defines $\|t\| \equiv \sup_A \|t_A\|$, which yields a norm on the linear space $[\mathcal{D}, \mathcal{D}]_*(F, G)$, where addition and scalar multiplication are defined point-wise.

The premonoidal structure on this category is the same as the one described in Example 6.1.6. Namely given two $*$-functors F and G we define $F \otimes G \equiv F \circ G$ which is clearly again a $*$-functor. Further given a transformation $t : F \longrightarrow G$ and a $*$-functor H then we define $(H \otimes t)_A \equiv H(t_A)$ and $(t \otimes H)_A \equiv t_{HA}$. Now it's clear that $\{\|(t \otimes H)_A\|\}$ is bounded and since $\|H(f)\| \leq \|f\|$ for all arrows f it follows that $\{\|(H \otimes t)_A\|\}$ is also bounded. Now let $\mathcal{D}^{\mathcal{D}}$ denote the wide subcategory of $[\mathcal{D}, \mathcal{D}]_*$ whose arrows are the bounded natural transformations. Observing that a constant functor is a $*$-functor one can reuse the proof of Lemma 6.3.4 to show that $(\mathcal{D}^{\mathcal{D}})' = \mathcal{D}^{\mathcal{D}}$. Hence $\mathcal{D}^{\mathcal{D}}$ is a $[\mathcal{D}, \mathcal{D}]_*$-von Neumann category.

As we mentioned in our section on maximally-monoidal categories any monoidal category $\mathcal{A}$ in a premonoidal category $\mathcal{C}$ that satisfies $\mathcal{A}' = \mathcal{A}$ is called a *maximally-monoidal category* and it shares similarities with maximal abelian von Neumann algebras. In the above example, Example 9.3.3, the category $\mathcal{D}^{\mathcal{D}}$ is a maximally-monoidal category which is also a $[\mathcal{D}, \mathcal{D}]_*$-von Neumann category. Thus we propose the term **maximally-monoidal $\mathcal{C}$-von Neumann category** for any maximally-monoidal category $\mathcal{A}$ in a premonoidal C^*-category $\mathcal{C}$. This leads to a more general version of Example 9.3.3.

Example 9.3.4. Every maximally-monoidal $\mathcal{C}$-von Neumann category is a $\mathcal{C}$-von

Neumann category.

Now in the case of $\mathbf{Hilb}_H$ it would be interesting to understand the relationship between von Neumann algebras on $\mathfrak{B}(H)$ and von Neumann categories on $\mathbf{Hilb}_H$. For example, is it the case that every von Neumann category $\mathcal{A}$ is of the form $\mathcal{A}(X,Y) = wk - cl(\mathfrak{B}(X,Y) \otimes_{alg} \mathcal{M})$ where $\mathcal{M} \subseteq \mathfrak{B}(H)$ is a von Neumann algebra? Notice that we could also use the strong closure instead of the weak closure here since they coincide on convex subsets (see [20] p.305 Theorem 5.1.2 for the precise statement of this fact).

Chapter 10

Premonoidal C^*-Quantum Field Theory

This chapter represents our primary new material including our definition of premonoidal C^*-quantum field theory.

10.1 Local Systems of Premonoidal C^*-Categories

In this section we investigate some properties of certain families of premonoidal C^*-categories called *local systems of premonoidal C^*-categories.*

Definition 10.1.1. Let $(K, \leq)$ be a directed poset. A **local system of premonoidal C^*-categories** is a functor $\mathbb{A} : K \longrightarrow$ **PDAG** where **PDAG** denotes the category of (small) premonoidal C^*-categories and premonoidal C^*-functors between them. (We also require that $\mathbb{A}$ satisfy that $\mathbb{A}(U) \subseteq \mathbb{A}(V)$ whenever $U \leq V$ in K.) If $U \leq V$ then we denote the corresponding functor from $\mathbb{A}(U)$ to $\mathbb{A}(V)$ by $i_{U,V}$. The poset K will be called the **index poset**.

Now let $\mathbb{A}$ be a local system of premonoidal C^*-categories with index poset $(K, \leq)$. Then we can associate to such a functor a category which will be denoted $\hat{\mathbb{A}}$ and which will be called the *quasi-local premonoidal C^*-category.* The set of objects is given by a quotient of the disjoint union $|\hat{\mathbb{A}}| = (\coprod_{U \in K} |\mathbb{A}(U)|)/\sim$ and we represent an $x \in |\hat{\mathbb{A}}|$

by the equivalence class of the element (U,a) where $U \in K$ and $a \in |\mathbb{A}(U)|$. The equivalence relation $\sim$ is defined as follows $(U,a) \sim (V,b)$ if there exists $W \in K$ with $U, V \leq W$ and such that $i_{U,W}(a) = i_{V,W}(b)$ in $\mathbb{A}(W)$. Clearly this relation is reflexive and symmetric, and transitivity follows from functoriality of $\mathbb{A}$ and directedness of K. Indeed if $(U,a) \sim (V,b)$ and $(V,b) \sim (W,c)$ then there exists $X, Y \in K$ such that $U, V \leq X$ and $V, W \leq Y$ such that $i_{U,X}(a) = i_{V,X}(b)$ and $i_{V,Y}(b) = i_{W,Y}(c)$. So as K is directed there exist $Z \in K$ with X, $Y \leq Z$. Thus

$$\begin{aligned} i_{U,Z}(a) &= i_{X,Z} i_{U,X}(a) \\ &= i_{X,Z} i_{V,X}(b) \\ &= i_{V,Z}(b) \\ &= i_{Y,Z} i_{V,Y}(b) \\ &= i_{Y,Z} i_{W,Y}(c) \\ &= i_{W,Z}(c) \end{aligned}$$

as required. Similarly the arrows of $\hat{\mathbb{A}}$ are defined as follows $(\hat{\mathbb{A}})_1 = (\coprod_{U \in K} (\mathbb{A}(U))_1)/\sim$. Again arrows of $\hat{\mathbb{A}}$ will be denoted by equivalence classes of elements of the form $(U, f : a \longrightarrow a')$ where $U \in K$ and $f : a \longrightarrow a'$ is an arrow in $\mathbb{A}(U)$. We will also write just (U,f) when no confusion will result. The equivalence class $[(U,f)]$ is an arrow from $[(U,a)]$ to $[(U,a')]$ in $\hat{\mathbb{A}}$. We define $(U, f : a \longrightarrow a') \sim (V, g : b \longrightarrow b')$ if there exists $W \in K$ with $U, V \leq W$ and such that $i_{U,W}(f) = i_{V,W}(g)$ in the category $\mathbb{A}(W)$. Notice it follows that $(U,a) \sim (V,b)$ and $(U,a') \sim (V,b')$. By exactly the same arguments one shows that $\sim$ is an equivalence relation on $\coprod_{U \in K} (\mathbb{A}(U))_1$.

Lemma 10.1.2. If $\mathbb{A} : K \longrightarrow$ **CAT** is functor from a directed poset into the category of categories then $\hat{\mathbb{A}}$ is a category. In particular if $\mathbb{A}$ is a local system of premonoidal C^*- categories then $\hat{\mathbb{A}}$ is a category.

Proof. To show that $\hat{\mathbb{A}}$ is a category we must show $\sim$ is a congruence relation with respect to composition. To this end suppose that $[(U,f)] = [(V,g)] : [(U,a)] = [(V,b)] \longrightarrow [(U,a')] = [(V,b')]$ and $[(X,h)] = [(Y,k)] : [(X,r)] = [(Y,s)] \longrightarrow [(X,r')] = [(Y,s')]$ and $[(U,a')] = [(X,r)]$. Then we must show that $[(X,h)] \circ [(U,f)] =$

$[(Y,k)]\circ[(V,g)]$ where composition is defined as follows, $[(X,h)]\circ[(U,f)] = [(Z,c_{f,h})]$ where X and $U \leq Z$ and $c_{f,h} = i_{X,Z}h \circ i_{U,Z}f$ in $\mathbb{A}(Z)$. Hence let $[(X,h)]\circ[(U,f)] = [(Z, i_{X,Z}h \circ i_{U,Z}f)]$ as above and let $[(Y,k)]\circ[(V,g)] = [(N, i_{Y,N}k \circ i_{V,N}g)]$. Then we must show that $(Z, i_{X,Z}h\circ i_{U,Z}f) \sim (N, i_{Y,N}k\circ i_{V,N}g)$. Since $(U,f) \sim (V,g)$ there exists $W_1 \in K$ such that U, $V \leq W_1$ and $i_{U,W_1}f = i_{V,W_1}g$ and similarly as $(X,h) \sim (Y,k)$ there exists $W_2 \in K$ with X, $Y \leq W_2$ and $i_{X,W_2}h = i_{Y,W_2}k$. Now as K is directed there exists $W \in K$ with Z, N, W_1, $W_2 \leq W$. Furthermore

$$\begin{aligned} i_{Z,W}(i_{X,Z}h \circ i_{U,Z}f) &= i_{X,W}h \circ i_{U,W}f \\ &= i_{W_2,W}i_{X,W_2}h \circ i_{W_1,W}i_{U,W_1}f \\ &= i_{W_2,W}i_{Y,W_2}k \circ i_{W_1,W}i_{V,W_1}g \\ &= i_{Y,W}k \circ i_{V,W}g \\ &= i_{N,W}i_{Y,N}k \circ i_{N,W}i_{V,N}g \\ &= i_{N,W}(i_{Y,N}k \circ i_{V,N}g) \end{aligned}$$

and hence $(Z, i_{X,Z}h \circ i_{U,Z}f) \sim (N, i_{Y,N}k \circ i_{V,N}g)$. One now easily verifies that this composition is associative and unital with identity given by the class of $(U, id_a : a \longrightarrow a)$. □

The results proved thus far within this section are essentially minor variations of some standard results concerning functors from a directed poset into the category **CAT**. However the remaining results that we now present are original and play a significant role in our theory of premonoidal C^*-quantum field theory that we are developing.

Lemma 10.1.3. If $\mathbb{A}$ is a local system of premonoidal C^*- categories then $\hat{\mathbb{A}}$ has a premonoidal structure.

Proof. We now show that $\hat{\mathbb{A}}$ is a premonoidal category. Given objects $[(U,a)]$, and $[(V,b)]$ in $\hat{\mathbb{A}}$ define $[(U,a)] \otimes [(V,b)] = [(W, i_{U,W}a \otimes i_{V,W}b)]$ where $W \in K$ and U, $V \leq W$. Note that if $W' \in K$ with U, $V \leq W'$ then $(W, i_{U,W}a \otimes i_{V,W}b) \sim (W', i_{U,W'}a \otimes i_{V,W'}b)$ holds since by directedness of K there exists $W'' \in K$ and W, $W' \leq W''$.

Thence

$$\begin{aligned} i_{W,W''}(i_{U,W}a \otimes i_{V,W}b) &= i_{W,W''}i_{U,W}a \otimes i_{W,W''}i_{V,W}b \\ &= i_{U,W''}a \otimes i_{V,W''}b \\ &= i_{W',W''}i_{U,W'}a \otimes i_{W',W''}i_{V,W'}b \\ &= i_{W',W''}(i_{U,W'}a \otimes i_{V,W'}b) \end{aligned}$$

as required. Thus the tensor product is independent of the choice of upper bound W. We must also show that it is independent of the choice of representatives for $[(U,a)]$ and $[(V,b)]$. Thus suppose that $(U,a) \sim (U',a')$ and $(V,b) \sim (V',b')$ so that there exists X, $Y \in K$ such that U, $U' \leq X$ and V, $V' \leq Y$ and such that $i_{U,X}a = i_{U',X}a'$ and $i_{V,Y}b = i_{V',Y}b'$. Now pick $Z \in K$ such that X, and $Y \leq Z$. Then by functoriality of $\mathbb{A}$ we have $i_{U,Z}a = i_{U',Z}a'$ and $i_{V,Z}b = i_{V',Z}b'$. Hence by the previous argument, as Z is an upper bound for U, and V we have that $(W, i_{U,W}a \otimes i_{V,W}b) \sim (Z, i_{U,Z}a \otimes i_{V,Z}b)$. But $(Z, i_{U,Z}a \otimes i_{V,Z}b) = (Z, i_{U',Z}a' \otimes i_{V',Z}b')$ and so $(W, i_{U,W}a \otimes i_{V,W}b) \sim (Z, i_{U',Z}a' \otimes i_{V',Z}b')$ as required.

Next suppose that $[(U,a)]$ is an object and $[(V,g)] : [(V,b)] \longrightarrow [(V,b')]$ an arrow in $\hat{\mathbb{A}}$. Define $[(U,a)] \otimes [(V,g)] : [(U,a)] \otimes [(V,b)] \longrightarrow [(U,a)] \otimes [(V,b')]$ by $[(U,a)] \otimes [(V,g)] = [(W, i_{U,W}a \otimes i_{V,W}g)]$ where U, $V \leq W \in K$. By similar arguments to those used above one shows that this operation is well-defined. Functoriality of $[(U,a)] \otimes - : \hat{\mathbb{A}} \longrightarrow \hat{\mathbb{A}}$ follows from the observation that the $i_{U,W}a \otimes - : \mathbb{A}(W) \longrightarrow \mathbb{A}(W)$ are functors for each $W \geq U \in K$. Similarly one defines the functors $- \otimes [(U,a)] : \hat{\mathbb{A}} \longrightarrow \hat{\mathbb{A}}$. The tensor unit in $\hat{\mathbb{A}}$ is the equivalence class of the element (U, I_U) where I_U is the tensor unit in $\mathbb{A}(U)$. Note that $[(U,I_U)] = [(V,I_V)]$ for any U, $V \in K$, so let $I = [(U,I_U)]$.

Now given objects $[(U,a)]$, $[(V,b)]$, and $[(W,c)]$ then $([(U,a)] \otimes [(V,b)]) \otimes [(W,c)] = [(X, (i_{U,X}a \otimes i_{V,X}b) \otimes i_{W,X}c)]$ for any $X \in K$ with U, V, and $W \leq X$. Similarly $[(U,a)] \otimes ([(V,b)] \otimes [(W,c)]) = [(X, i_{U,X}a \otimes (i_{V,X}b \otimes i_{W,X}c))]$ for any $X \in K$ with U, V, and $W \leq X$. Then we define the associativity isomorphism as follows

$$\alpha_{[(U,a)],[(V,b)],[(W,c)]} : [(X, (i_{U,X}a \otimes i_{V,X}b) \otimes i_{W,X}c)] \longrightarrow [(X, i_{U,X}a \otimes (i_{V,X}b \otimes i_{W,X}c))]$$

by

$$\alpha_{[(U,a)],[(V,b)],[(W,c)]} = [(X, \mathfrak{a}_{(i_{U,X}a),(i_{V,X}b),(i_{W,X}c)})]$$

where

$$\mathfrak{a}_{(i_{U,X}a),(i_{V,X}b),(i_{W,X}c)} : (i_{U,X}a \otimes i_{V,X}b) \otimes i_{W,X}c \longrightarrow i_{U,X}a \otimes (i_{V,X}b \otimes i_{W,X}c)$$

is the natural associativity in the premonoidal category $\mathbb{A}(X)$. Since α is defined in terms of the maps $\mathfrak{a}$ it follows that α is natural and satisfies the *Mac Lane pentagon* since the $\mathfrak{a}$ maps do. Now notice that $[(U, I_U)] = [(V, I_V)]$ thus $[(U, I_U)] \otimes [(V, b)] = [(V, I_V)] \otimes [(V, b)] = [(V, I_V \otimes b)]$, similarly $[(V, b)] \otimes [(U, I_U)] = [(V, b \otimes I_V)]$. Thus we define the left and right units as $\lambda_{[(V,b)]} = [(V, \mathfrak{l}_b)]$ and $\rho_{[(V,b)]} = [(V, \mathfrak{r}_b)]$ respectively where $\mathfrak{l}_b : I_V \otimes b \longrightarrow b$ and $\mathfrak{r}_b : b \otimes I_V \longrightarrow b$ are the premonoidal units in the category $\mathbb{A}(V)$. Again the naturality of λ and ρ follow from the naturality of the maps $\mathfrak{l}$ and $\mathfrak{r}$ respectively moreover the structural equations also follow from the fact that they hold for the maps $\mathfrak{l}$ and $\mathfrak{r}$. Hence $(\hat{\mathbb{A}}, \otimes, I, \alpha, \lambda, \rho)$ is a premonoidal category. □

Proposition 10.1.4. If $\mathbb{A}$ is a local system of premonoidal C^*- categories then an arrow $[(U, f)]$ in $\hat{\mathbb{A}}$ is in $\mathcal{Z}(\hat{\mathbb{A}})$ if and only if f is an arrow in $\mathcal{Z}(\mathbb{A}(U))$.

Proof. Suppose that $[(U, f)]$ is central in $\hat{\mathbb{A}}$ with $f : a \longrightarrow a'$ an arrow in $\mathbb{A}(U)$. Let $g : b \longrightarrow b'$ be any arrow in $\mathbb{A}(U)$ so that

$$\begin{aligned}([(U, a')] \otimes [(U, g)]) \circ ([(U, f)] \otimes [(U, b)]) &= [(U, a' \otimes g)] \circ [(U, f \otimes b)] \\ &= [(U, (a' \otimes g) \circ (f \otimes b))]\end{aligned}$$

and on the other hand

$$\begin{aligned}([(U, f)] \otimes [(U, b')]) \circ ([(U, a)] \otimes [(U, g)]) &= [(U, f \otimes b')] \circ [(U, a \otimes g)] \\ &= [(U, (f \otimes b') \circ (a \otimes g))]\end{aligned}$$

and by centrality we have $[(U, (a' \otimes g) \circ (f \otimes b))] = [(U, (f \otimes b') \circ (a \otimes g))]$. So there exists $W \geq U \in K$ such that $i_{U,W}(a \otimes g \circ f \otimes b) = i_{U,W}(f \otimes b' \circ a \otimes g)$. So in the case that $i_{U,W}$ is a faithful functor it follows that $a \otimes g \circ f \otimes b = f \otimes b' \circ a \otimes g$ in $\mathbb{A}(U)$. Similarly one can show that $g \otimes a \circ b \otimes f = b' \otimes f \circ g \otimes a$.

To see that the converse is true suppose that $f : a \longrightarrow a'$ is a central map in $\mathbb{A}(U)$ and let $[(V, g)] : [(V, b)] \longrightarrow [(V, b')]$ be any arrow in $\hat{\mathbb{A}}$ and let $W \in K$ with $U,\ V \leq W$. Then $[(U, f)] \otimes [(V, b)] = [(W, i_{U,W} f \otimes i_{V,W} b)]$ and $[(U, a')] \otimes [(V, g)] = [(W, i_{U,W} a' \otimes i_{V,W} g)]$. Thus

$$\begin{aligned}([(U,a')] \otimes [(V,g)]) \circ ([(U,f)] \otimes [(V,b)]) &= [(W, i_{U,W}a' \otimes i_{V,W}g)] \circ [(W, i_{U,W}f \otimes i_{V,W}b)] \\ &= [(W, (i_{U,W}a' \otimes i_{V,W}g) \circ (i_{U,W}f \otimes i_{V,W}b))] \\ &= [(W, (i_{U,W}f \otimes i_{V,W}b') \circ (i_{U,W}a \otimes i_{V,W}g))].\end{aligned}$$

We have used the fact that the functors $i_{U,W}$ take central maps to central maps in the passage from the second last line to the last line above. We can now rearrange the last line above as follows.

$$\begin{aligned}[(W, (i_{U,W}f \otimes i_{V,W}b') \circ (i_{U,W}a \otimes i_{V,W}g))] &= [(W, i_{U,W}f \otimes i_{V,W}b')] \circ [(W, i_{U,W}a \otimes i_{V,W}g)] \\ &= ([(U,f)] \otimes [(V,b')]) \circ ([(U,a)] \otimes [(V,g)]).\end{aligned}$$

Similarly one shows that the other relevant diagram commutes. Hence $[(U, f)]$ is central in $\hat{\mathbb{A}}$.

□

Our next result concerns the existence of a premonoidal C^*-structure on the category $\hat{\mathbb{A}}$.

Theorem 10.1.5. If $\mathbb{A}$ is a local system of premonoidal C^*- categories then $\hat{\mathbb{A}}$ can be faithfully embedded into a premonoidal C^*- category $\mathfrak{U}(\mathbb{A}) = \mathfrak{U}$. Moreover this embedding is a strict premonoidal C^*- functor which is surjective on objects.

Proof. We have already shown that $\hat{\mathbb{A}}$ has a premonoidal structure. We now endow it with a positive $*$-operation. Indeed given $[(U, f)] : [(U, a)] \longrightarrow [(U, a')]$ define $[(U, f)]^* : [(U, a')] \longrightarrow [(U, a)]$ by $[(U, f)]^* = [(U, f^*)]$. This is well-defined since the functors $i_{U,W} : \mathbb{A}(U) \longrightarrow \mathbb{A}(W)$ commute with the $*$-operation. Moreover $[(U, f)]^{**} = [(U, f^{**}] = [(U, f)]$. To show positivity and anti-linearity of $*$ we must first show that the hom-sets of $\hat{\mathbb{A}}$ are complex vector spaces which behave well with respect to composition, i.e. we must show $\hat{\mathbb{A}}$ is $\mathbb{C}$-linear premonoidal. Indeed given parallel arrows $[(U,f)] : [(U,a)] \longrightarrow [(U,a')]$ and $[(V,g)] : [(V,b)] \longrightarrow [(V,b')]$, then there exists $X \geq U$, and V with $[(U,a)] = [(X, i_{U,X}a)] = [(X, i_{V,X}b)] = [(V,b)]$ and $[(U,a')] = [(X, i_{U,X}a')] = [(X, i_{V,X}b')] = [(V,b')]$. Thus $[(U,f)] = [(X, i_{U,X}f)]$ and $[(V,g)] = [(X, i_{V,X}g)]$. Hence define $[(U,f)] + [(V,g)] = [(X, i_{U,X}f + i_{V,X}g)]$ and $z[(U,f)] = [(U, zf)]$.

A tedious but routine calculation shows that this makes $\hat{\mathbb{A}}$ a $\mathbb{C}$-linear premonoidal category. The zero maps are the equivalence classes $0 = 0_{A,A'} = [(U, 0_{a,a'})] : A = [(U,a)] \longrightarrow A' = [(U,a')]$ where $0_{a,a'} : a \longrightarrow a'$ is the zero map in $\mathbb{A}(U)$.

Now suppose that $[(U,f)]^* \circ [(U,f)] = 0$. Then $[(U, f^* \circ f)] = 0 = [(U,0)]$ and hence $f^* \circ f = 0$ and therefore $f = 0_{a,a'}$ since $\mathbb{A}(U)$ is a premonoidal $*$-category, so $[(U,f)] = 0$ as required. The anti-linearity of $*$ is also easily verified. If $[(U,f)]$ is a central map then it follows that f in $\mathbb{A}(U)$ is central and hence so is f^* by assumption. Thus $[(U,f^*)]$ is central. One can also easily verify with these definitions that $([(U,a)]\otimes[(V,g)])^* = [(U,a)]\otimes[(V,g)]^*$ and $([(V,g)]\otimes[(U,a)])^* = [(V,g)]^*\otimes[(U,a)]$ and that the structural maps α, λ, and ρ are unitary. Hence $\hat{\mathbb{A}}$ is a premonoidal $*$-category.

Lastly given any arrow $[(U,f)] : A \longrightarrow A'$ in $\hat{\mathbb{A}}$ we define its norm by $\|[(U,f)]\|_{A,A'} = \|f\|_{a,a'}$ where $f : a \longrightarrow a'$ in $\mathbb{A}(U)$. If $[(U,f)] = [(V,g)]$, then there exists $W \geq U$, and V in K such that $i_{U,W}f = i_{V,W}g : c \longrightarrow c'$ in $\mathbb{A}(W)$. Now as the map $\mathbb{A}(U)(a,a') \longrightarrow \mathbb{A}(W)(i_{U,W}a, i_{U,W}a')$ is a $*$-monomorphism it follows from elementary results in C^*-algebra theory that this map is norm-preserving. Hence

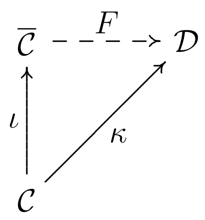
$$\begin{aligned}\|f\|_{a,a'} &= \|i_{U,W}f\|_{i_{U,W}a, i_{U,W}a'} \\ &= \|i_{V,W}g\|_{i_{V,W}b, i_{V,W}b'} \\ &= \|g\|_{b,b'}.\end{aligned}$$

It is now routine to see that with this definition the norm satisfies the triangle inequality, and respects the scalar multiplication and all the other properties which define Banach and C^*- norms. The only issue is whether or not the normed space $\hat{\mathbb{A}}(A,B)$ is complete with respect to the given norm. In general there is no reason for this space to be complete, so we must complete it. We denote by $\mathfrak{U}(A, B)$ the completion of $\hat{\mathbb{A}}(A, B)$ and the resulting space will be a Banach space with a positive $*$-operation. Thus we have built a C^*- category $\mathfrak{U}$ with objects the same as those of $\hat{\mathbb{A}}$ and hom-sets given by the sets $\mathfrak{U}(A, B)$ as defined above.

Furthermore since all the structural maps such as composition, the $*$-operation, etc. are continuous it follows that they lift to the appropriate maps in $\mathfrak{U}$ making it a premonoidal C^*- category. Moreover it is immediate that $\hat{\mathbb{A}}$ embeds faithfully into $\mathfrak{U}$ and that this embedding is a strict premonoidal C^*- functor which is surjective on objects. □

We wish to show that the above construction is a colimit. In order to do that we have developed the following convenient result.

Lemma 10.1.6. Let $\mathcal{C}$ be a premonoidal $*$-category such that for each pair of objects A and B the vector space $\mathcal{C}(A, B)$ has a norm $\| \|_{A,B}$ satisfying the equations in Definition 9.1.13. Then there exists a premonoidal C^*-category $\overline{\mathcal{C}}$ and an isometric embedding $\iota : \mathcal{C} \longrightarrow \overline{\mathcal{C}}$. Moreover if $\mathcal{D}$ is any other premonoidal C^*-category and $\kappa : \mathcal{C} \longrightarrow \mathcal{D}$ an isometric embedding then there exists a unique premonoidal C^*-functor $F : \mathcal{D} \longrightarrow \overline{\mathcal{C}}$ such that the diagram



commutes.

Proof. We begin by defining the category $\overline{\mathcal{C}}$. The objects of $\overline{\mathcal{C}}$ will be the same as those of $\mathcal{C}$. Now for each pair of objects A and B in $\mathcal{C}$ we know that $\mathcal{C}(A, B)$ is a normed vector space. Thus we define $\overline{\mathcal{C}}(A, B)$ to be the Banach space obtained by completing this normed vector space. One particular realization of $\overline{\mathcal{C}}(A, B)$ has, as elements equivalence classes of Cauchy sequences in $\mathcal{C}(A, B)$ where two sequences $(f)_{n\in\mathbb{N}}$ and $(g)_{n\in\mathbb{N}}$ are equivalent if the sequence $(\|f_n - g_n\|_{A,B})_{n\in\mathbb{N}}$ of real numbers converges to 0. Moreover there is a canonical map $\varphi = \varphi_{A,B} : \mathcal{C}(A,B) \longrightarrow \overline{\mathcal{C}}(A,B)$ given by $\varphi(f) = [(f_n = f)_{n\in\mathbb{N}}]$ i.e. f maps to the equivalence class of the constant sequence with value f. One clearly has that $\varphi(f) = \varphi(g)$ if and only if $f = g$ and so φ is injective.

Now suppose that $[(f_n)_{n\in\mathbb{N}}] \in \overline{\mathcal{C}}(A,B)$ and $[(g_n)_{n\in\mathbb{N}}] \in \overline{\mathcal{C}}(B,C)$. Then we define $[(g_n)_{n\in\mathbb{N}}] \circ [(f_n)_{n\in\mathbb{N}}] = [(g_n \circ f_n)_{n\in\mathbb{N}}]$. First we show that the sequence $(g_n \circ f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. First notice that if $(h_n)_{n\in\mathbb{N}}$ is a Cauchy sequence then the sequence of real numbers $(\|h_n\|_{A,B})_{n\in\mathbb{N}}$ is a Cauchy sequence since for any $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $\|h_m - h_n\|_{A,B} < \epsilon$ for all m, $n \geq N$ but $\big|\|h_m\|_{A,B} - \|h_n\|_{A,B}\big| \leq \|h_m - h_n\|_{A,B}$ so $\big|\|h_m\|_{A,B} - \|h_n\|_{A,B}\big| < \epsilon$ for all m, and $n \geq N$. So by completeness of $\mathbb{R}$ the sequence $(\|h_n\|_{A,B})_{n\in\mathbb{N}}$ converges and is therefore bounded. So there exists $H > 0$ such that $H = 1 + \sup\{\|h_n\|_{A,B} \mid n \in \mathbb{N}\}$.

Now let $\varepsilon > 0$. Then since $(f_n)_{n\in\mathbb{N}}$ and $(g_n)_{n\in\mathbb{N}}$ are Cauchy sequences, there exists N_1 and $N_2 \in \mathbb{N}$ such that $\|f_m - f_n\|_{A,B} \leq \varepsilon/2G$ and $\|g_m - g_n\|_{B,C} \leq \varepsilon/2F$ for all m ,$n \geq max(N_1, N_2)$ where $F = 1 + \sup\{\|f_n\|_{A,B} \mid n \in \mathbb{N}\}$ and similarly $G = 1 + \sup\{\|g_n\|_{B,C} \mid n \in \mathbb{N}\}$. Hence for all m ,$n \geq max(N_1, N_2)$

$$\begin{aligned}
\|g_m \circ f_m - g_n \circ f_n\|_{A,C} &= \|g_m \circ f_m - g_m \circ f_n + g_m \circ f_n - g_n \circ f_n\|_{A,C} \\
&\leq \|g_m \circ f_m - g_m \circ f_n\|_{A,C} + \|g_m \circ f_n - g_n \circ f_n\|_{A,B} \\
&= \|g_m \circ (f_m - f_n)\|_{A,C} + \|(g_m - g_n) \circ f_n\|_{A,C} \\
&\leq \|g_m\|_{B,C}\|f_m - f_n\|_{A,B} + \|g_m - g_n\|_{B,C}\|f_n\|_{A,B} \\
&< G\frac{\varepsilon}{2G} + \frac{\varepsilon}{2F}F \\
&= \varepsilon.
\end{aligned}$$

Thus $(g_n \circ f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Next we show that this composition is well-defined. Suppose that $[(f_n)] = [(f'_n)]$ and $[(g_n)] = [(g'_n)]$ then we show that

$(\|g_n \circ f_n - g'_n \circ f'_n\|_{A,C}) \longrightarrow 0$. Indeed

$$\begin{aligned}\|g_n \circ f_n - g'_n \circ f'_n\|_{A,C} &= \|g_n \circ f_n - g_n \circ f'_n + g_n f'_n - g'_n \circ f'_n\|_{A,C} \\ &\leq \|g_n \circ f_n - g_n \circ f'_n\|_{A,C} + \|g_n f'_n - g'_n \circ f'_n\|_{A,C} \\ &\leq \|g_n\|_{B,C}\|f_n - f'_n\|_{A,B} + \|g_n - g'_n\|_{B,C}\|f'_n\|_{A,B} \\ &\leq G\|f_n - f'_n\|_{A,B} + \|g_n - g'_n\|_{B,C}F'\end{aligned}$$

where $G = 1 + \sup\{\|g_n\|_{B,C} \mid n \in \mathbb{N}\}$ and $F' = 1 + \sup\{\|f'_n\|_{A,B} \mid n \in \mathbb{N}\}$. Now for any $\varepsilon > 0$ there exists N_1 and $N_2 \in \mathbb{N}$ such that for all $n \geq N = max(N_1, N_2)$ $\|f_n - f'_n\|_{A,B} < \varepsilon/2G$ and $\|g_n - g'_n\|_{A,B} < \varepsilon/2F'$. Hence for all $n \geq N$ the above chain of inequalities becomes

$$\begin{aligned}\|g_n \circ f_n - g'_n \circ f'_n\|_{A,C} &\leq G\|f_n - f'_n\|_{A,B} + \|g_n - g'_n\|_{B,C}F' \\ &< G\frac{\varepsilon}{2G} + \frac{\varepsilon}{2F'}F' \\ &= \varepsilon.\end{aligned}$$

Hence the sequence $(\|g_n \circ f_n - g'_n \circ f'_n\|_{A,C})$ converges to 0 and so composition is well-defined. The identity map in $\overline{\mathcal{C}}(A, A)$ is given by $id_A = \varphi(1_A)$ where $1_A \in \mathcal{C}(A, A)$ is the identity on A in the category $\mathcal{C}$. The associativity of this composition follows from the associativity of the composition in $\mathcal{C}$. Hence $\overline{\mathcal{C}}$ is a category.

Next we show that if $f : A \longrightarrow B$ and $g : B \longrightarrow C$ in $\mathcal{C}$, then $\varphi(g \circ f) = \varphi(g) \circ \varphi(f)$. Indeed $\varphi(g \circ f) = [(h_n)]$ where $h_n = g \circ f$ for all $n \in \mathbb{N}$; on the other hand $\varphi(g) \circ \varphi(f) = [(g_n = g)] \circ [(f_n = f)] = [(g_n \circ f_n = g \circ f)] = [(h_n)] = \varphi(g \circ f)$. Notice that φ preserves identities by definition. Hence we have a functor $\iota : \mathcal{C} \longrightarrow \overline{\mathcal{C}}$ given by $\iota(A) = A$ for all objects A so given $f : A \longrightarrow B$ in $\mathcal{C}$ define $\iota(f) = \varphi(f)$.

Now we proceed by equipping $\overline{\mathcal{C}}$ with a premonoidal structure as follows. Given objects A and B we define $A \otimes_{\overline{\mathcal{C}}} B = A \otimes_{\mathcal{C}} B$ and in the future we simply abbreviate this by writing $A \otimes B$. Next for each object A in $\overline{\mathcal{C}}$ we must define functors $A \otimes (-)$ and $(-) \otimes A$. Indeed, since for each object A in $\mathcal{C}$ the functions $A \otimes (-) : \mathcal{C}(X, Y) \longrightarrow \mathcal{C}(A \otimes X, A \otimes Y)$ and $(-) \otimes A : \mathcal{C}(X, Y) \longrightarrow \mathcal{C}(X \otimes A, Y \otimes A)$ are continuous, it follows that the induced maps $A \otimes (-) : \overline{\mathcal{C}}(X, Y) \longrightarrow \overline{\mathcal{C}}(A \otimes X, A \otimes Y)$ and $(-) \otimes A : \overline{\mathcal{C}}(X, Y) \longrightarrow \overline{\mathcal{C}}(X \otimes A, Y \otimes A)$ are linear and continuous. Explicitly one has $A \otimes [(f_n)] = [(A \otimes f_n)]$ and $[(f_n)] \otimes A = [(f_n \otimes A)]$. It is now straightforward to verify that $A \otimes (-) : \overline{\mathcal{C}} \longrightarrow \overline{\mathcal{C}}$ and $(-) \otimes A : \overline{\mathcal{C}} \longrightarrow \overline{\mathcal{C}}$ are functors. The structural isomorphisms $\alpha : (A \otimes B) \otimes C \longrightarrow A \otimes (B \otimes C)$, $\lambda : I \otimes A \longrightarrow A$ and $\rho : A \otimes I \longrightarrow A$ are given by $\alpha = \iota(\mathfrak{a}) = \varphi(\mathfrak{a})$, $\lambda = \iota(\mathfrak{l}) = \varphi(\mathfrak{l})$, and $\rho = \iota(\mathfrak{r}) = \varphi(\mathfrak{r})$ respectively where $\mathfrak{a}$, $\mathfrak{l}$, and $\mathfrak{r}$ are the corresponding structural maps in $\mathcal{C}$. To see that these maps are central it is enough to show that ι sends central maps to central maps. Certainly if $f : A \longrightarrow B$ in $\mathcal{C}$ is central and $[(g_n)] : X \longrightarrow Y$ is an arrow in $\overline{\mathcal{C}}$ then

$$\begin{aligned}
(B \otimes [(g_n)]) \circ (\iota(f) \otimes X) &= [(B \otimes g_n)] \circ [(f \otimes X)] \\
&= [(B \otimes g_n \circ f \otimes X)] \\
&= [(f \otimes Y \circ A \otimes g_n)] \\
&= [(f \otimes Y)] \circ [(A \otimes g_n)] \\
&= (\iota(f) \otimes Y) \circ (A \otimes [(g_n)])
\end{aligned}$$

as required. Similarly the other diagram for centrality also commutes in $\overline{\mathcal{C}}$. Analogous calculations also show that α, λ, and ρ satisfy the coherence equations and are natural transformations.

The only piece of structure missing the is $*$- operation on $\overline{\mathcal{C}}$. Given an arrow $[(s_n)] : A \longrightarrow B$ define $[(s_n)]^* = [(s_n^*)]$. To see that $*$ is well-defined it suffices to observe that $(-)^* : \mathcal{C}(A, B) \longrightarrow \mathcal{C}(B, A)$ is continuous. Notice that $\|s\|^2 = \|s^* \circ s\| \leq \|s^*\|\|s\|$ and so $\|s\| \leq \|s^*\|$ similarly $\|s^*\|^2 = \|s \circ s^*\| \leq \|s\|\|s^*\|$ so $\|s^*\| \leq \|s\|$. Thus $\|s^*\| = \|s\|$ and so $(-)^*$ is isometric and hence continuous and thus extends to a continuous map $(-)^* : \overline{\mathcal{C}}(A, B) \longrightarrow \overline{\mathcal{C}}(B, A)$. To see that $(-)^*$ is positive consider $[(s_n)] : A \longrightarrow B$ and suppose that $[(s_n)]^* \circ [(s_n)] = 0$. Then $[(s_n^* \circ s_n)] = [(0)]$ and so we have that $(\|s_n^* \circ s_n\|)_{n \in \mathbb{N}} \longrightarrow 0$ i.e. $(\|s_n\|^2)_{n \in \mathbb{N}} \longrightarrow 0$ and hence $(\|s_n\|)_{n \in \mathbb{N}} \longrightarrow 0$. Thus $[(s_n)] = [(0)]$ as required.

The remaining properties that $(-)^*$ must satisfy follow easily from its definition. Lastly, given an arrow $[(t_n)] : A \longrightarrow B$ define $\|[(t_n)]\|_{A,B} = \lim_{n \to \infty} \|t_n\|_{A,B}$. That this norm makes $\overline{\mathcal{C}}(A, B)$ a Banach space is a standard result in analysis. Moreover the remaining properties that these norms should satisfy, listed in Definition 9.1.13, all follow from properties of limits and the primitive norms coming from $\mathcal{C}$. In addition, the maps $\mathcal{C}(A,B) \longrightarrow \overline{\mathcal{C}}(A,B)$ given by $f \mapsto \varphi(f)$ are isometric embeddings and the image of φ is dense in $\overline{\mathcal{C}}(A,B)$ for all objects A and B. Therefore $\overline{\mathcal{C}}$ is a premonoidal C^*-category and the functor $\iota : \mathcal{C} \longrightarrow \overline{\mathcal{C}}$ is an isometric strict premonoidal C^*-embedding.

Now suppose that $\mathcal{D}$ is a premonoidal C^*-category and $\kappa : \mathcal{C} \longrightarrow \mathcal{D}$ is an isometric embedding. We will define a functor $F : \overline{\mathcal{C}} \longrightarrow \mathcal{D}$ as follows. For each object A in $\overline{\mathcal{C}}$ we have that A is actually an object of $\mathcal{C}$. Thus, we define $FA = \kappa A$. Now if $[(f_n)] : A \longrightarrow B$, then define $F([(f_n)]) = \lim_{n\to\infty} \kappa(f_n)$. To see that this is well-defined we must check that the sequence $(\kappa(f_n))$ is convergent and the limit is independent of the choice of representative. First, as (f_n) is a Cauchy sequence we have for any $\varepsilon > 0$ that there exists $N \in \mathbb{N}$ such that $\|f_m - f_n\|_{A,B} < \varepsilon$ for all m, $n \geq N$ and hence $\|\kappa(f_m) - \kappa(f_n)\|_{\kappa A,\kappa B} = \|\kappa(f_m - f_n)\|_{\kappa A,\kappa B} = \|f_m - f_n\|_{A,B} < \varepsilon$. So $(\kappa(f_n))$ is a Cauchy sequence and thus convergent. Now suppose that $[(f_n)] = [(g_n)]$. Then we have that $(\|f_n - g_n\|_{A,B}) \longrightarrow 0$. Hence we have that

$$\begin{aligned}\|\lim_{n\to\infty} \kappa(f_n - g_n)\|_{\kappa A,\kappa B} &= \lim_{n\to\infty} \|\kappa(f_n - g_n)\|_{\kappa A,\kappa B}\\ &= \lim_{n\to\infty} \|f_n - g_n\|_{A,B}\\ &= 0.\end{aligned}$$

Thus $0 = \lim_{n\to\infty} \kappa(f_n - g_n) = \lim_{n\to\infty} \kappa(f_n) - \lim_{n\to\infty} \kappa(g_n)$ and hence $\lim_{n\to\infty} \kappa(f_n) = \lim_{n\to\infty} \kappa(g_n)$ as required. So F is well-defined. If $[(f_n)] : A \longrightarrow B$ and $[(g_n)] : B \longrightarrow C$ in $\overline{\mathcal{C}}$, then we want to show that $F([(g_n)] \circ [(f_n)]) = F([(g_n)]) \circ F([(f_n)])$. This amounts to showing that

$$\lim_{n\to\infty} (\kappa(g_n \circ f_n)) = \lim_{n\to\infty} \kappa(g_n) \circ \lim_{n\to\infty} \kappa(f_n) \tag{35}$$

If $g = \lim_{n\to\infty} \kappa(g_n)$, then by continuity of composition we have the following

$$\begin{aligned}
\lim_{n\to\infty}(\kappa(g_n \circ f_n)) - \lim_{n\to\infty}\kappa(g_n) \circ \lim_{n\to\infty}\kappa(f_n) &= \lim_{n\to\infty}(\kappa(g_n \circ f_n)) - g \circ \lim_{n\to\infty}\kappa(f_n) \\
&= \lim_{n\to\infty}(\kappa(g_n \circ f_n)) - \lim_{n\to\infty} g \circ \kappa(f_n) \\
&= \lim_{n\to\infty}(\kappa(g_n \circ f_n)) - g \circ \kappa(f_n) \\
&= \lim_{n\to\infty}(\kappa(g_n) - g) \circ \kappa(f_n)
\end{aligned}$$

Now taking norms on both sides we obtain

$$\begin{aligned}
\| \lim_{n\to\infty}(\kappa(g_n \circ f_n)) - \lim_{n\to\infty}\kappa(g_n) \circ \lim_{n\to\infty}\kappa(f_n)\| &= \| \lim_{n\to\infty}(\kappa(g_n) - g) \circ \kappa(f_n)\| \\
&= \lim_{n\to\infty} \|(\kappa(g_n) - g) \circ \kappa(f_n)\| \\
&\leq \lim_{n\to\infty} \|(\kappa(g_n) - g)\|\|\kappa(f_n)\| \\
&= \left(\lim_{n\to\infty} \|\kappa(g_n) - g\|\right)\left(\lim_{n\to\infty} \|\kappa(f_n)\|\right) \\
&= 0 \cdot \lim_{n\to\infty} \|\kappa(f_n)\| \\
&= 0.
\end{aligned}$$

Hence equation 35 holds, and so F preserves composition. Furthermore it is clear that F also preserves identities and is thus a functor. Now as taking limits of convergent sequences is a $\mathbb{C}$-linear operation it follows that F is $\mathbb{C}$-linear. Moreover since $(-)^* : \mathcal{D}(X, Y) \longrightarrow \mathcal{D}(Y, X)$ is continuous κ is a $*$-functor we have that

$$\begin{aligned}
F([(f_n)]^*) &= \lim_{n\to\infty} \kappa(f_n^*) \\
&= \lim_{n\to\infty} \kappa(f_n)^* \\
&= (\lim_{n\to\infty} \kappa(f_n))^* \\
&= (F[(f_n)])^*
\end{aligned}$$

and so F is a $*$-functor.

Now we wish to demonstrate that F is a premonoidal $*$-functor. Since κ is a such a functor we have for all objects A, and B in $\mathcal{C}$ a central natural transformation with components $d^{\kappa}_{A,B} : \kappa A \otimes \kappa B \longrightarrow \kappa(A \otimes B)$ and a central morphism $e^{\kappa} : I_{\mathcal{D}} \longrightarrow \kappa(I_{\mathcal{C}})$. Thus for all objects A and B in $\overline{\mathcal{C}}$ we define $d^F_{A,B} = d^{\kappa}_{A,B}$ and $e^F = e^{\kappa}$. Using arguments similar to those given above one can verify that these morphisms satisfy the relevant requirements and hence F equipped with these morphisms becomes a premonoidal C^*-functor.

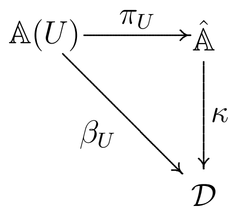
Next we show that $F \circ \iota = \kappa$, indeed for each object $A \in |\mathcal{C}|$ we have that $F\iota(A) = F(A) = \kappa(A)$. Moreover, if $f \in \mathcal{C}(A, B)$ then $F\iota(f) = F[(f_n = f)] = \lim_{n\to\infty} \kappa(f_n = f) = \kappa(f)$. Now suppose that $G : \overline{\mathcal{C}} \longrightarrow \mathcal{D}$ is another premonoidal C^*-functor such that $G \circ \iota = \kappa$. As the maps $\overline{\mathcal{C}}(A, B) \longrightarrow \mathcal{D}(FA, FB)$ and $\overline{\mathcal{C}}(A, B) \longrightarrow \mathcal{D}(GA, GB)$ are $*$-homomorphisms it follows that they are continuous. Hence as $F|_{\iota(\mathcal{C}(A,B))} = G|_{\iota(\mathcal{C}(A,B))}$ and $\iota(\mathcal{C}(A, B))$ is dense in $\overline{\mathcal{C}}(A, B)$ it follows that $F = G$ for all arrows in $\overline{\mathcal{C}}$. Clearly $FA = GA$ for all objects A and hence $G = F$ and therefore F is unique. □

Proposition 10.1.7. If $\mathbb{A} : K \longrightarrow \mathbf{PDAG}$ is a local system of premonoidal C^*-categories then $\mathfrak{U}(\mathbb{A})$ is the object part of the colimit of the functor $\mathbb{A}$.

Proof. For each $U \in K$ we need an arrow $\tau_U : \mathbb{A}(U) \longrightarrow \mathfrak{U}(\mathbb{A})$ in **PDAG**. We define τ in two steps. First we have a functor $\pi_U : \mathbb{A}(U) \longrightarrow \hat{\mathbb{A}}$ given by $\pi_U(a) = [(U, a)]$ for all objects $a \in \mathbb{A}(U)$ and, given an arrow $f : a \longrightarrow b$ in $\mathbb{A}(U)$, we define $\pi_U(f) = [(U, f)] : [(U, a)] \longrightarrow [(U, b)]$. It is immediate that π_U is a functor. Thus we will define $\tau_U = \iota \circ \pi_U : \mathbb{A}(U) \longrightarrow \overline{\hat{\mathbb{A}}}$. Notice that by definition $\overline{\hat{\mathbb{A}}} = \mathfrak{U}(\mathbb{A})$.

Now to see that τ_U is a premonoidal C^*-functor it is enough to show that π_U is, since composition of premonoidal C^*-functors is again such a functor. Indeed for $f : a \longrightarrow b$ in $\mathbb{A}(U)$ we have $\pi_U(f^*) = [(U, f^*)] = [(U, f)]^* = \pi_U(f)^*$. Now given objects a, b of $\mathbb{A}(U)$, we have that $[(U, a)]\otimes[(U, b)] = [(U, a\otimes b)]$ hence $\pi_U(a)\otimes\pi_U(b) = \pi_U(a\otimes b)$. Moreover $I_{\hat{\mathbb{A}}} = [(U, I_U)]$, and so $\pi_U(I_U) = I_{\hat{\mathbb{A}}}$. Thus we have shown that π_U is a strict premonoidal C^*-functor. In addition if $U \leq V \in K$ then we have a strict premonoidal C^*-functor $i_{U,V} : \mathbb{A}(U) \longrightarrow \mathbb{A}(V)$ which satisfies $\pi_U = \pi_V \circ i_{U,V}$ hence $\tau_U = \tau_V \circ i_{U,V}$ whenever $U \leq V \in K$. Hence $(\tau, \mathfrak{U}(\mathbb{A}))$ is a cocone on $\mathbb{A}$ with vertex $\mathfrak{U}(\mathbb{A})$, i.e. $\tau : \mathbb{A} \Longrightarrow \Delta_{\mathfrak{U}(\mathbb{A})}$ is a natural transformation where $\Delta_{\mathfrak{U}(\mathbb{A})} : K \longrightarrow \mathbf{PDAG}$ is the constant functor with value $\mathfrak{U}(\mathbb{A})$.

Now suppose that $(\beta, \mathcal{D})$ is any other cocone on $\mathbb{A}$, i.e., for each $U \in K$ we have premonoidal C^*-functor $\beta_U : \mathbb{A}(U) \longrightarrow \mathcal{D}$ such that if $V \in K$ and $U \leq V$, then $\beta_U = \beta_V \circ i_{U,V}$. Now what we must do is define a premonoidal C^*-functor $G : \mathfrak{U}(\mathbb{A}) \longrightarrow \mathcal{D}$. By Lemma 10.1.6 it suffices to define a premonoidal $*$-functor $\kappa : \hat{\mathbb{A}} \longrightarrow \mathcal{D}$ such that the diagram



commutes for all $U \in K$. Indeed for $[(U, a)] \in |\hat{\mathbb{A}}|$ define $\kappa([(U, a)]) = \beta_U(a)$. To see that this is well-defined, suppose that $(U, a) \sim (V, b)$. Then there exists $W \in K$ with U, $V \leq W$ and $i_{U,W}(a) = i_{V,W}(b)$. Hence

$$\begin{aligned} \kappa([(U, a)]) &= \beta_U(a) \\ &= \beta_W i_{U,W}(a) \\ &= \beta_W i_{V,W}(b) \\ &= \beta_V(b) \\ &= \kappa([(V, b)]). \end{aligned}$$

Similarly if $[(U, f)]$ is an arrow in $\hat{\mathbb{A}}$ then define $\kappa([(U, f)]) = \beta_U(f)$. The verification that this is well-defined is similar to the previous argument. It follows that κ is premonoidal $*$-functor because each β_U is. For example if $[(U, a)] \otimes [(V, b)] = [(W, i_{U,W}a \otimes i_{V,W}b)]$ then we need a map $d^{\kappa}_{[(U,a)],[(V,b)]} : \kappa([(U, a)]) \otimes \kappa([(V, b)]) \longrightarrow \kappa([(W, i_{U,W}a \otimes i_{V,W}b)])$. Thus we need a map $d^{\kappa}_{[(U,a)],[(V,b)]} : \beta_U(a) \otimes \beta_V(b) \longrightarrow \beta_W(i_{U,W}a \otimes i_{V,W}b)$. Notice that $\beta_U(a) = \beta_W i_{U,W}(a)$ and $\beta_V(b) = \beta_W i_{V,W}(b)$ and thus we have a map $d^{\beta_W}_{i_{U,W}a, i_{V,W}b} : \beta_W i_{U,W}(a) \otimes \beta_W i_{V,W}(b) \longrightarrow \beta_W(i_{U,W}a \otimes i_{V,W}b)$ as required.

We must show that these maps are independent of the choice of representatives chosen. Notice that the choice of map d depends only on the upper bound W of U and V. Suppose that $W' \in K$ is an upper bound for U and V then we must show that

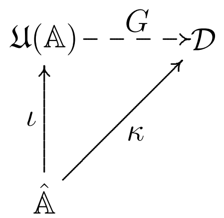
$d^{\beta_{W'}}_{i_{U,W'}a, i_{V,W'}b} = d^{\beta_W}_{i_{U,W}a, i_{V,W}b}$. Since K is directed there exists $Z \in K$ with W, $W' \leq Z$. Moreover it is easy to see that since $\beta_W = \beta_Z i_{W,Z}$ and $\beta_{W'} = \beta_Z i_{W',Z}$ that

$$\begin{aligned}\beta_W i_{U,W}(a) \otimes \beta_W i_{V,W}(b) &= \beta_{W'} i_{U,W'}(a) \otimes \beta_{W'} i_{V,W'}(b)\\ &= \beta_Z i_{U,Z}(a) \otimes \beta_Z i_{V,Z}(b)\end{aligned}$$

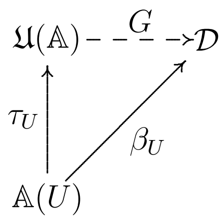
and similarly that

$$\begin{aligned}\beta_W(i_{U,W}a \otimes i_{V,W}b) &= \beta_{W'}(i_{U,W'}a \otimes i_{V,W'}b)\\ &= \beta_Z(i_{U,Z}a \otimes i_{V,Z}b).\end{aligned}$$

Thus $d^{\beta_{W'}}_{i_{U,W'}a, i_{V,W'}b} = d^{\beta_W}_{i_{U,W}a, i_{V,W}b}$ when $d^{\beta_W}_{i_{U,W}a, i_{V,W}b} = d^{\beta_Z}_{i_{U,Z}a, i_{V,Z}b} = d^{\beta_{W'}}_{i_{U,W'}a, i_{V,W'}b}$ for all $Z \in K$ with W, $W' \leq Z$. This last equality follows from the assumption that whenever $O \leq O' \in K$ that $d^{\beta_O}_{x,y} = d^{\beta_{O'}}_{i_{O,O'}x, i_{O,O'}y}$ which follows from the fact that $\beta_O = \beta_{O'} \circ i_{O,O'}$ as premonoidal functors. Now one also clearly has that $\beta_U = \kappa \circ \pi_U$ for all $U \in K$. Now applying Lemma 10.1.6 to the functor κ we get unique a premonoidal C^*-functor $G : \mathfrak{U}(\mathbb{A}) \longrightarrow \mathcal{D}$ such that the diagram



commutes. Thus we obtain the following diagram



which commutes for all $U \in K$. Moreover G is unique, since it is uniquely determined by κ which is uniquely determined by β. Hence $\mathfrak{U}(\mathbb{A})$ has the relevant universal property, and is thus a colimit. □

The following example considers the familiar case where one has a local system of premonoidal C^*- categories given by a directed family of C^*-algebras where each algebra is viewed as a one object premonoidal C^* category. This example illustrates why one needs to pass to the category $\mathfrak{U}$.

Example 10.1.8. Suppose that $(K, \leq)$ is a directed poset, that for each $U \in K$, $\mathbb{A}(U)$ is a C^*-algebra, and if $U \leq V$ then $\mathbb{A}(U) \subseteq \mathbb{A}(V)$. Then since a C^*-algebra is the same thing as a one-object premonoidal C^*- category it follows that $\mathbb{A}$ is a local system of premonoidal C^*- categories. Thus we can construct the category $\hat{\mathbb{A}}$ which amounts to taking a union $\hat{\mathbb{A}} = \bigcup_{U \in K} \mathbb{A}(U)$. Here $\hat{\mathbb{A}}$ is an algebra where given $a, b \in \hat{\mathbb{A}}$ then there exists $U, V \leq W \in K$ with $a \in \mathbb{A}(U) \subseteq \mathbb{A}(W)$ and $b \in \mathbb{A}(V) \subseteq \mathbb{A}(W)$ and thus $ab \in \mathbb{A}(W)$. Since there is a common identity in all the algebras $\mathbb{A}(U)$ it follows that this is the identity of $\hat{\mathbb{A}}$. In general however the algebra $\hat{\mathbb{A}}$ is not complete thus we must complete it to obtain a C^*-algebra.

Now in light of the fact that C^*-algebras are one object premonoidal C^*- categories one could ask what is the natural generalization of a representation of a C^*-algebra to the premonoidal setting. We propose the following definition.

Definition 10.1.9. If $\mathcal{C}$ is a premonoidal C^*- category then a **representation** of $\mathcal{C}$ consists of a Hilbert space H and a premonoidal C^*- functor $\pi : \mathcal{C} \longrightarrow \mathbf{Hilb}_H$. We denote this representation by the pair (H, π).

Remark 10.1.10. Notice that in particular if I is the tensor unit in $\mathcal{C}$ then $\mathcal{C}(I, I)$ is a C^*-algebra and that a representation of $\mathcal{C}$ gives a representation (π_I, H) of this C^*-algebra as follows. If $f : I \longrightarrow I$ then define $\pi_I(f) = H \cong \mathbb{C} \otimes H \xrightarrow{\pi(f)} \mathbb{C} \otimes H \cong H$. Given any premonoidal C^*- category $\mathcal{C}$ we will denote by $\mathcal{I}$ the distinguished C^*-algebra $\mathcal{C}(I, I)$.

We also define the notion of morphism between representations of premonoidal C^*-categories.

Definition 10.1.11. Let (H, π), and (K, φ) be two representations of a premonoidal C^*-category $\mathcal{C}$. A morphism from (H, π) to (K, φ) consists of a family of maps

$\theta_A : \pi(A) \otimes H \longrightarrow \varphi(A) \otimes K$, where $A \in |\mathcal{C}|$, such that for all arrows $f : A \longrightarrow B$ in $\mathcal{C}$ the diagram

$$\begin{array}{ccc} \pi(A) \otimes H & \xrightarrow{\pi(f)} & \pi(B) \otimes H \\ \Big\downarrow{\scriptstyle \theta_A} & & \Big\downarrow{\scriptstyle \theta_B} \\ \varphi(A) \otimes K & \xrightarrow[\varphi(f)]{} & \varphi(B) \otimes K \end{array}$$

Remark 10.1.12. Notice that if (H, π) is a representation of a premonoidal C^*-category $\mathcal{C}$ then there is an associated functor $\hat{\pi} : \mathcal{C} \longrightarrow$ **Hilb** given by $\hat{\pi}(A) = \pi(A) \otimes H$ for all objects A and given $f : A \longrightarrow B$ $\hat{\pi}(f) = \pi(f) : \pi(A) \otimes H \longrightarrow \pi(B) \otimes H$. So given a pair of representations (H, π) and (K, φ) of $\mathcal{C}$ one sees that a morphism from (H, π) to (K, φ) is the same as a natural transformation $\theta : \hat{\pi} \Longrightarrow \hat{\varphi}$.

10.2 Premonoidal DHR Representations

The goal of this section is to generalize the notion of an AQFT $\mathbb{A}$ by replacing each C^*-algebra $\mathbb{A}(U)$ by a premonoidal C^*-category and then also to appropriately modify the notions of **DHR**-representation and also the category of localized transportable endomorphisms of $\hat{\mathbb{A}}$.

We start first by presenting the main definition of this section. Let $(K, \leq)$ denote the poset of open double cones in Minkowski space ordered by subset inclusion.

Definition 10.2.1. A **premonoidal C^*-quantum field theory** (PC^* QFT) consists of a local system of premonoidal C^*-categories $\mathbb{A}$, indexed by $(K, \leq)$, such that

$$U \leq V \text{ then the functor } i_{U,V} : \mathbb{A}(U) \longrightarrow \mathbb{A}(V) \text{ is faithful and,} \tag{36}$$

$$\text{if } U \perp V \text{ then } \mathbb{A}(U) \text{ and } \mathbb{A}(V) \text{ commute in } \mathfrak{U}(\mathbb{A}). \tag{37}$$

Notice that in the case of a PC^* QFT where each $\mathbb{A}(U)$ is simply a C^*-algebra one recovers the usual notion of an AQFT. Now let (H_0, π_0) be a fixed representation of $\mathfrak{U}(\mathbb{A})$, where $\mathbb{A}$ is a PC^* QFT, which we will call the *vacuum representation*. The following definition generalizes the notion of a **DHR**-representation to our setting.

Definition 10.2.2. A **premonoidal DHR-representation** of $\mathbb{A}$ is a representation (H, π) of $\mathfrak{U}(\mathbb{A})$ such that for each $U \in K$ there exists a family $\beta(U)_A : \pi_0(A) \otimes H_0 \longrightarrow \pi(A) \otimes H$, where $A \in |\mathfrak{U}(\mathbb{A})|$ which is unitary and satisfies the following equation, if $f : A \longrightarrow B$ in $\mathbb{A}(V)$, and $V \perp U$, then the diagram

$$\begin{array}{ccc} \pi_0(A) \otimes H_0 & \xrightarrow{\pi_0(f)} & \pi_0(B) \otimes H_0 \\ \Big\downarrow{\scriptstyle \beta(U)_A} & & \Big\downarrow{\scriptstyle \beta(U)_B} \\ \pi(A) \otimes H & \xrightarrow[\pi(f)]{} & \pi(B) \otimes H \end{array}$$

commutes in **Hilb**.

Now we wish to build a category whose objects will be premonoidal **DHR**-representations. The arrows of this category will simply be maps between representations. Call this category **DHR-Rep**. We now leave our discussion of **DHR**-representations to discuss a related category, namely the category of localized transportable endomorphisms of $\mathfrak{U}(\mathbb{A})$. Let $\mathbb{A}$ be a PC^* QFT then we will denote this category by $\mathfrak{D}$. The objects of this category will be certain types of premonoidal C^*-endofunctors $F : \mathfrak{U}(\mathbb{A}) \longrightarrow \mathfrak{U}(\mathbb{A})$. We define these in two steps.

Definition 10.2.3. A premonoidal C^*-functor $F : \mathfrak{U}(\mathbb{A}) \longrightarrow \mathfrak{U}(\mathbb{A})$ is **localized at** $U \in K$ in case that for each $A \in |\mathbb{A}(V)|$, where $V \perp U$, there exists a unitary map $\upsilon_A : FA \longrightarrow A$ such that for all arrows $f : A \longrightarrow B$ in $\mathbb{A}(V)$ the diagram

$$\begin{array}{ccc} FA & \xrightarrow{Ff} & FB \\ \Big\downarrow{\scriptstyle \upsilon_A} & & \Big\downarrow{\scriptstyle \upsilon_B} \\ A & \xrightarrow[f]{} & B \end{array}$$

commutes in $\mathfrak{U}(\mathbb{A})$. We say that a premonoidal C^*-functor is localized if there exists a double cone U at which it is localized.

Definition 10.2.4. Let $F : \mathfrak{U}(\mathbb{A}) \longrightarrow \mathfrak{U}(\mathbb{A})$ be localized at $U \in K$. Then F is **transportable** if for any double cone $V \in K$ there exists a premonoidal C^*-functor $G : \mathfrak{U}(\mathbb{A}) \longrightarrow \mathfrak{U}(\mathbb{A})$ localized at V and a unitary premonoidal natural transformation $\vartheta : F \Longrightarrow G$.

Definition 10.2.5. If U is a double cone then we let $\mathfrak{D}(U)$ be the set of premonoidal C^*-endofunctors on $\mathfrak{U}(\mathbb{A})$ which are transportable and localized at U.

The set of objects for the category $\mathfrak{D}$ is given by the set $|\mathfrak{D}| = \bigcup_{U \in K} \mathfrak{D}(U)$. Next we define the morphisms between two localized transportable functors.

Definition 10.2.6. If F, and $G \in |\mathfrak{D}|$ then a morphism from F to G consists a map $t_A : FA \longrightarrow GA$ in for each object A in $\mathfrak{U}(\mathbb{A})$.

Note that there is no assumption that the morphisms be natural transformations.

Lemma 10.2.7. $\mathfrak{D}$ is a category.

Proof. Suppose that $t : F \longrightarrow G$ and $s : G \longrightarrow K$ are morphisms in $\mathfrak{D}$ then define $s \circ t : F \longrightarrow K$ by $(s \circ t)_A = s_A \circ t_A$. This composition is clearly associative and the arrow id_F defined as $(id_F)_A = id_{FA}$ acts as an identity with respect to this composition. Thus the result follows.

□

The category $\mathfrak{D}$ has much more structure, in particular it has a premonoidal structure. For objects F, and G in $\mathfrak{D}$ we will define $F \otimes G = F \circ G$. To see that this is well-defined we must show that the composite $F \circ G$ is again localized and transportable.

Lemma 10.2.8. If $F \in \mathfrak{D}(U)$ and $G \in \mathfrak{D}(V)$ then $F \circ G \in \mathfrak{D}(W)$ for any double cone $W \in K$ with $U, V \leq W$.

Proof. Let $O \perp W$ where $U, V \leq W \in K$. Then it follows that $U \perp O$ and $V \perp O$ and thus for any $A \in |\mathbb{A}(O)|$ there exists unitary maps $\upsilon_A : FA \longrightarrow A$ and $\omega_A : GA \longrightarrow A$ such that for any arrow $f : A \longrightarrow B$ in $\mathbb{A}(O)$

$$\upsilon_B \circ Ff \circ \upsilon_A^* = f$$
$$\omega_B \circ Gf \circ \omega_A^* = f$$

then we define $\beta_A : FGA \longrightarrow A$ by $\omega A \circ \upsilon_{GA} = \upsilon_A \circ F(\omega_A)$. Then clearly β_A is unitary and for any arrow $f : A \longrightarrow B$ in $\mathbb{A}(O)$ we have

$$\begin{aligned}\beta_B \circ FG(f) \circ \beta_A &= \omega_B \circ (\upsilon_{GB} \circ FG(f) \circ \upsilon^*_{GA}) \circ \omega^*_A \\ &= \omega_B \circ Gf \circ \omega_A \\ &= f\end{aligned}$$

as required. It remains to show that $F \circ G$ is transportable. To this end suppose that $D \in K$ is any double cone then by transportability of F and G there exists premonoidal C^*-functors T_F and $T_G : \mathfrak{U}(\mathbb{A}) \longrightarrow \mathfrak{U}(\mathbb{A})$ which are localized at D and unitary premonoidal natural transformations $\vartheta : F \Longrightarrow T_F$ and $\delta : G \Longrightarrow T_G$. Now since both T_F and T_G are localized at D it follows that $T_F \circ T_G$ is also localized at D and moreover we have a unitary natural transformation $\kappa : FG \Longrightarrow T_F T_G$ given by $\kappa_A = T_F \circ \vartheta_{GA} = \vartheta_{T_G(A)} \circ F\delta_A$ for all $A \in |\mathfrak{U}(\mathbb{A})|$. □

With the aid of this lemma we can now prove the following.

Theorem 10.2.9. The category $\mathfrak{D}$ has the structure of a strict premonoidal category.

Proof. Given objects F and G in $\mathfrak{D}$ we define $F \otimes G$ by $F \otimes G = F \circ G$. This is well-defined according to the previous lemma. Now given $t : F \longrightarrow F'$ in $\mathfrak{D}$ and G any object we define the map $G \otimes t : GF \longrightarrow GF'$ by $(G \otimes t)_A = G(t_A) : GFA \longrightarrow GF'A$ similarly define $t \otimes G : FG \longrightarrow F'G$ by $(t \otimes G)_A = t_{GA} : FGA \longrightarrow F'GA$. Note the tensor unit is the identity functor. This premonoidal structure is identical to the one in Example 6.1.6 in Section 6.1. □

The reason for considering the category $\mathfrak{D}$ is that it is very rich in structure. We now indicate the relationship between this category and **DHR-Rep**.

Lemma 10.2.10. Suppose that $F \in |\mathfrak{D}|$ and let (H_0, π_0) be the vacuum representation. Then $(H_0, \pi_0 \circ F)$ is a **DHR**-representation.

Proof. Let $U \in K$. Since F is transportable there exists $G \in \mathfrak{D}(U)$ and a unitary premonoidal natural transformation $\vartheta : F \Longrightarrow G$. Thus for each $A \in |\mathfrak{U}(\mathbb{A})|$ we have a unitary $\vartheta_A : FA \longrightarrow GA$. Now as G is localized at U it follows that for each $A \in |\mathbb{A}(V)|$, where $U \perp V$, there exists unitary maps $\theta_A : GA \longrightarrow A$ satisfying the diagram in Definition 10.2.3. Thus we for each $A \in |\mathbb{A}(V)|$ we define $\beta(U)_A$: $\pi_0(A) \otimes H_0 \longrightarrow \pi_0 \circ F(A) \otimes H_0$ by

$$\beta(U)_A = \pi_0(\vartheta_A^*) \circ \pi_0(\theta_A^*).$$

Now let $f : A \longrightarrow B$ be an arrow in $\mathbb{A}(V)$ then as π_0 is a functor the following diagram commutes since functors preserve commutative diagrams.

$$\begin{array}{ccc} \pi_0 A \otimes H_0 & \xrightarrow{\pi_0(f)} & \pi_0 B \otimes H_0 \\ {\scriptstyle \pi_0\theta_A^*}\downarrow & & \downarrow{\scriptstyle \pi_0\theta_B^*} \\ \pi_0 GA \otimes H_0 & \xrightarrow[\pi_0(Gf)]{} & \pi_0 GB \otimes H_0 \end{array}$$

Now by the naturality of ϑ and functoriality of π_0 we have that the diagram

$$\begin{array}{ccc} \pi_0 GA \otimes H_0 & \xrightarrow{\pi_0(Gf)} & \pi_0 GB \otimes H_0 \\ {\scriptstyle \pi_0\vartheta_A^*}\downarrow & & \downarrow{\scriptstyle \pi_0\vartheta_B^*} \\ \pi_0 FA \otimes H_0 & \xrightarrow[\pi_0(Ff)]{} & \pi_0 FB \otimes H_0 \end{array}$$

commutes. Hence stacking these two diagrams together we obtain

$$\begin{array}{ccccc} & \pi_0 A \otimes H_0 & \xrightarrow{\pi_0(f)} & \pi_0 B \otimes H_0 & \\ & {\scriptstyle \pi_0\theta_A^*}\downarrow & & \downarrow{\scriptstyle \pi_0\theta_B^*} & \\ \beta(U)_A & \pi_0 GA \otimes H_0 & \xrightarrow[\pi_0(Gf)]{} & \pi_0 GB \otimes H_0 & \beta(U)_B \\ & {\scriptstyle \pi_0\vartheta_A^*}\downarrow & & \downarrow{\scriptstyle \pi_0\vartheta_B^*} & \\ & \pi_0 FA \otimes H_0 & \xrightarrow[\pi_0(Ff)]{} & \pi_0 FB \otimes H_0 & \end{array}$$

which commutes as required. Now it remains to show that if X is an object in $\mathfrak{U}(\mathbb{A})$ that is not an object of $\mathbb{A}(V)$ for any $V \perp U$ that we can construct a unitary map

$\beta(U)_X : \pi_0(X) \otimes H_0 \longrightarrow \pi_0 FX \otimes H_0$. Indeed suppose $X \in |\mathfrak{U}(\mathbb{A})|$ and that $X \notin \mathbb{A}(V)$ for any $V \perp U$. Then, there exists $O_X \in K$ such that $X \in |\mathbb{A}(O_X)|$. Moreover, since for any double cone one can always find another double cone which is spacelike separated from it, we can therefore choose a double cone $\hat{O}_X \in K$ with $\hat{O}_X \perp O_X$. Now, as F is transportable there exists $G^{\hat{O}_X} \in \mathfrak{D}(\hat{O}_X)$ and a unitary premonoidal natural transformation $\gamma_Z^{\hat{O}_X} : FZ \longrightarrow G^{\hat{O}_X}(Z)$. Furthermore, as $G^{\hat{O}_X}$ is localized at $\hat{O}_X$ we also have that for all $W \perp \hat{O}_X$ there are unitary maps $\epsilon_Y^{\hat{O}_X} : G^{\hat{O}_X}(Y) \longrightarrow Y$ for all objects $Y \in |\mathbb{A}(W)|$. Thus as $O_X \perp \hat{O}_X$ it follows that $\epsilon_X^{\hat{O}_X} \circ \gamma_X^{\hat{O}_X} : FX \longrightarrow X$ is a unitary map. Thus we define $\beta(U)_X = \pi_0(\gamma_X^{\hat{O}_X})^* \circ \pi_0(\epsilon_X^{\hat{O}_X})^*$. Hence we have shown that $(H_0, \pi_0 \circ F)$ is a premonoidal **DHR**-representation. □

Now in order for this assignment to be functorial we will require some additional assumptions on the morphisms in $\mathfrak{D}$.

Lemma 10.2.11. If $t : F \longrightarrow G$ is an arrow in $\mathfrak{D}$ which is also a natural transformation from the functor F to G then the there is an induced arrow $(H_0, \pi_0 \circ F) \longrightarrow (H_0, \pi_0 \circ G)$ given by $\pi_0(t_A) : \pi_0 F(A) \longrightarrow \pi_0 G(A)$ for all $A \in |\mathfrak{U}(\mathbb{A})|$.

Proof. Let $f : A \longrightarrow B$ be an arrow in $\mathfrak{U}(\mathbb{A})$ then we must show that the following equation holds

$$\pi_0 G(f) \circ \pi_0(t_A) = \pi_0(t_B) \circ \pi_0 F(f)$$

which is clearly the case by functoriality of π_0 and naturality t. □

Remark 10.2.12. Notice that if π_0 is faithful then $\pi_0(t)$ will be a map of **DHR**-representations if and only if $t : F \longrightarrow G$ is a natural transformation.

Now in the traditional AQFT setting one uses the category whose objects are localized transportable endomorphisms and arrows are natural transformations to study the category of DHR representations. Similarly for our more general setting we will consider a subcategory Δ of $\mathfrak{D}$ whose objects are the same as those of $\mathfrak{D}$ but arrows are those arrows in $\mathfrak{D}$ which are also natural transformations. It is clear that Δ is a monoidal category.

Remark 10.2.13. We will write $F \in \Delta(O)$ to indicate that F is an object of Δ which is localized at O.

We now introduce some technical conditions on the vacuum representation and the net $\mathbb{A}$ which will ensure the functor $\Delta \longrightarrow$ **DHR-Rep** defined above is an equivalence of categories. The first condition is what is known as *Haag duality* in the context of AQFT. Here we present the corresponding analogue for PC^* QFT. In order to state the Haag duality condition we need to first define the notion of a *premonoidal C^*-category generated by a family premonoidal C^*-categories.* More precisely suppose that $\mathcal{C}_{i \in I} \subseteq \mathcal{C}$ are premonoidal C^*-subcategories of the premonoidal C^*-category $\mathcal{C}$ then we will construct a premonoidal C^*-category, denoted $\bigvee_i \mathcal{C}_i \subseteq \mathcal{C}$, such that $\mathcal{C}_j \subseteq \bigvee_i \mathcal{C}_i$ for all $j \in I$ and if $\mathcal{D}$ is any other premonoidal C^*-category with these properties then $\bigvee_i \mathcal{C}_i \subseteq \mathcal{D}$.

We give an inductive definition of $\bigvee_i \mathcal{C}_i$, starting first with the objects of this category. Indeed we define the set $\mathcal{O}$ inductively by $X \in \mathcal{O}$ for all $X \in \bigcup_i |\mathcal{C}_i|$ and if A, $B \in \mathcal{O}$ then $A \otimes B \in \mathcal{O}$ and $B \otimes A \in \mathcal{O}$, and for any triple A, B, and $C \in \mathcal{O}$ then $(A \otimes B) \otimes C \in \mathcal{O}$ and $A \otimes (B \otimes C) \in \mathcal{O}$. Notice that $\mathcal{O} \subseteq |\mathcal{C}|$ and thus two elements A and $B \in \mathcal{O}$ are defined to be equal if and only if they are equal as objects in $\mathcal{C}$.

Now we define another set $\mathcal{A}$ inductively by $f \in \mathcal{A}$ for all $f \in \bigcup_i Arr(\mathcal{C}_i)$ and if f, $g \in \mathcal{A}$ and $A \in \mathcal{O}$ then $f \otimes A \in \mathcal{A}$, and $A \otimes f \in \mathcal{A}$ and if $dom(g) = cod(f)$ then $g \circ f \in \mathcal{A}$. If f and $g \in \mathcal{A}$ are a pair of parallel arrows then we also insist that $cf + bg \in \mathcal{A}$ for all a and $b \in \mathbb{C}$. Moreover for all A, B and $C \in \mathcal{O}$ we require that $\alpha_{A,B,C}$, λ_A, and ρ_A belong to $\mathcal{A}$ as well as their inverses. Again as $\mathcal{A} \subseteq Arr(\mathcal{C})$ we define f and $g \in \mathcal{A}$ to be equal when they are equal as arrows in $\mathcal{C}$.

Now we define $\bigvee_i \mathcal{C}_i$ to be the category with object set equal to the set $\mathcal{O}$ and whose arrows consist of the set $\mathcal{A}$. To prove that this is a category we use induction. First we show that for all $A \in \mathcal{O}$ that $Id_A \in \mathcal{A}$. Suppose $A \in |\mathcal{C}_i|$ for some i then Id_A is an arrow in $\mathcal{C}_i$ and hence $Id_A \in \mathcal{A}$. Now for our induction hypothesis suppose that $Id_A \in \mathcal{A}$ for some $A \in \mathcal{O}$. We will now show that $Id_{A\otimes B}$ and $Id_{B\otimes A} \in \mathcal{A}$ for any $B \in \mathcal{O}$. Indeed $Id_{A\otimes B} = Id_A \otimes B$ and $Id_A \otimes B \in \mathcal{A}$ since by the induction hypothesis $Id_A \in \mathcal{A}$. Similarly we have $Id_{B\otimes A} \in \mathcal{A}$, and hence by induction we have $Id_X \in \mathcal{A}$ for all $X \in \mathcal{O}$. Now clearly $\mathcal{A}$ is closed under composition and hence $\bigvee_i \mathcal{C}_i$

is a category and furthermore for all objects A, and B we have that $\bigvee_i \mathcal{C}_i(A, B)$ is a complex vector space and composition is bilinear with respect to these vector space structures. Next we show that $\mathcal{A}$ is closed under the $*$-operation. Again we do this by induction, thus suppose that $f \in Arr(\mathcal{C}_i)$ for some i then $f^* \in Arr(\mathcal{C}_i)$ and hence $f^* \in \mathcal{A}$. Now suppose that $f \in \mathcal{A}$ such that $f^* \in \mathcal{A}$ then for any $g \in \mathcal{A}$ with $dom(g) = cod(f)$ we have that $g \circ f^* \in \mathcal{A}$ since both f^* and g are in $\mathcal{A}$. To see that $(f \otimes X)^*$ is in $\mathcal{A}$ notice that $(f \otimes X)^* = f^* \otimes X$ and by the induction hypothesis $f^* \in \mathcal{A}$, so $f^* \otimes X \in \mathcal{A}$. Similarly we see that $(X \otimes f)^* \in \mathcal{A}$. In addition since the structural maps are unitary it follows that $\alpha^*_{A,B,C}$, λ^*_A and ρ^*_A belong to $\mathcal{A}$. Finally if $h \in \mathcal{A}$ is of the form $h = af + bg$ for a, $b \in \mathbb{C}$ and f, $g \in \mathcal{A}$ with f^* and $g^* \in \mathcal{A}$ then $h^* = \overline{a}f^* + \overline{b}g^*$, this is a linear combination of elements of $\mathcal{A}$ and is thus an element of $\mathcal{A}$. Therefore by induction we have that $\mathcal{A}$ is closed under the $*$-operation.

To see that $\bigvee_i \mathcal{C}_i$ is a premonoidal C^*-category we must verify that the functors $A \otimes (-) : \bigvee_i \mathcal{C}_i \longrightarrow \bigvee_i \mathcal{C}_i$ are $\mathbb{C}$-linear. But this is immediate since this functor is simply the restriction of $A \otimes (-)$ to the subcategory $\bigvee_i \mathcal{C}_i$ of $\mathcal{C}$. Similarly the functors $(-) \otimes A : \bigvee_i \mathcal{C}_i \longrightarrow \bigvee_i \mathcal{C}_i$ are also $\mathbb{C}$-linear. Finally the structural maps satisfy the required diagrams in $\mathcal{C}$ and hence they also satisfy these diagrams in the subcategory $\bigvee_i \mathcal{C}_i$. Thus $\bigvee_i \mathcal{C}_i$ is a premonoidal $*$-category. For all objects X, and $Y \in |\bigvee_i \mathcal{C}_i|$ we now take the closure of $(\bigvee_i \mathcal{C}_i)(X, Y)$ in $\mathcal{C}(X, Y)$ to guarantee completeness. We will also denote this category $\bigvee_i \mathcal{C}_i$. It now follows that this category is a premonoidal C^*-category. Moreover if $\mathcal{D} \subseteq \mathcal{C}$ and $\mathcal{C}_i \subseteq \mathcal{D}$ for all $i \in I$ then it is clear that $\bigvee_i \mathcal{C}_i \subseteq \mathcal{D}$. Thus $\bigvee_i \mathcal{C}_i$ is the smallest premonoidal C^*-subcategory of $\mathcal{C}$ containing the premonoidal C^*-categories $\mathcal{C}_i$.

Using the above construction we may now define the notion of Haag duality. In order to do this we must make some further assumptions concerning the category Δ.

Remark 10.2.14. From this point forward the only objects of Δ which we will consider are those objects F such that if F is localized at $O \in K$ then for all $O' \in K$ with $O' \perp O$ we have that $F(f) = f$ for all arrows $f : A \longrightarrow B$ in $\mathbb{A}(O')$. In other words, we only consider those functors which are the identity on all categories associated to a spacelike-separated region of O. We also assume that these functors are strict, i.e. preserve the premonoidal structure on the nose. The subcategory consisting of these objects will also be denoted by Δ. We also at this point make the simplifying assumption that all DHR representations (H, π) are *strict* in the sense that the functor π is a strict premonoidal functor. We will also denote this category by **DHR-Rep**.

Definition 10.2.15. A representation (H, π) of $\mathfrak{U}(\mathbb{A})$ is said to satisfy **Haag duality** if the following two conditions hold. The first is a condition on each double cone $O \in K$. Suppose that we have a family of maps

$$\eta_X : \pi(FX) \longrightarrow \pi(GX) \tag{38}$$

in $\mathbf{Hilb}_H$ where $X \in |\mathfrak{U}(\mathbb{A})|$ and F and G are objects of Δ which are localized at $O \in K$. Then $\eta_B \circ \pi(f) = \pi(f) \circ \eta_A$ for all arrows $f : A \longrightarrow B$ in $\bigvee_{O' \perp O} \mathbb{A}(O')$ implies $\eta_A \in \pi(\mathbb{A}(O)(FA, GA))$ for all $A \in |\mathbb{A}(O)|$ and moreover there exists a family of arrows $t_X^O : FX \longrightarrow GX$, one arrow for each object $X \in |\mathfrak{U}(\mathbb{A})|$, such that $\eta_X = \pi(t_X^O)$ for all X. The second condition requires that for all double cones $O \in K$,

$$\pi\left(\bigvee_{O' \perp O} \mathbb{A}(O')\right)' (\pi X, \pi Y) = \pi\left[\mathbb{A}(O)(X, Y)\right] \tag{39}$$

for all objects X, and $Y \in |\mathbb{A}(O)|$.

The above definition is a subtle generalization of Haag duality one normally encounters in AQFT. In particular conditions 38 and 39 will coincide in the special case that one is dealing with a net of C^*-algebras rather than the more general case of premonoidal C^*-categories. In this special case one recovers the usual notion of Haag duality found in Definition 8.2.6. The correctness of our generalization is evidenced by our Theorem 10.2.23, a major result that shows that a certain subcategory of Δ is equivalent to a certain subcategory of **DHR-Rep**. What is surprising about this generalization is that Haag duality in this new setting is not simply the restatement of one condition in more general terms but rather the statement of *two* separate conditions that turn out to coincide in the AQFT setting.

Remark 10.2.16. We now pause to make some additional noteworthy observations about representations (H, π) satisfying Haag duality:

1. for every double cone $O \in K$ the category $\pi\left(\bigvee_{O' \perp O} \mathbb{A}(O')\right)'$ is a premonoidal C^*-subcategory of $\mathbf{Hilb}_H$ which is a von Neumann category;

2. $\pi(\mathbb{A}(O))$ is a $\mathcal{C}$-von Neumann subcategory of the full subcategory $\mathcal{C}$ of $\mathbf{Hilb}_H$ with objects $|\mathcal{C}| = \pi|\mathbb{A}(O)|$;

3. condition 38 is a technical condition that will guarantee that the functor in Theorem 10.2.23 is full.

Before stating the theorem which provides sufficient conditions for the functor $\Delta \longrightarrow$ **DHR-Rep** we must make some further assumptions about the objects in each of these categories.

Definition 10.2.17. Let (H, π) be a premonoidal DHR representation. Then we say that (H, π) is **coherent** if for each double cone U there exists a family of unitary maps $\beta(U)_A : \pi_0(A) \otimes H_0 \longrightarrow \pi(A) \otimes H$ satisfying the conditions of Definition 10.2.2 and the following two additional requirements. For all objects A, B, and $X \in |\mathfrak{U}(\mathbb{A})|$ and arrows $q : A \longrightarrow B$ in $\mathfrak{U}(\mathbb{A})$ the diagrams

$$\begin{array}{ccc} \pi_0(A) \otimes \pi_0(X) \otimes H_0 & \xrightarrow{\beta(U)_{A\otimes X}} & \pi(A) \otimes \pi(X) \otimes H \\ {\scriptstyle \tau_{\pi_0(A),\pi_0(X)} \otimes id_{H_0}} \Big\downarrow & & \Big\downarrow {\scriptstyle \tau_{\pi(A),\pi(X)} \otimes id_H} \\ \pi_0(X) \otimes \pi_0(A) \otimes H_0 & \xrightarrow[\beta(U)_{X\otimes A}]{} & \pi(X) \otimes \pi(A) \otimes H \end{array} \tag{40}$$

$$\begin{array}{ccc} \pi_0(X) \otimes \pi_0(A) \otimes H_0 & \xrightarrow{\beta(U)_{X\otimes A}} & \pi(X) \otimes \pi(A) \otimes H \\ {\scriptstyle id_{\pi_0(X)} \otimes \beta(U)_A} \Big\downarrow & & \Big\downarrow {\scriptstyle id_{\pi(X)} \otimes \pi(q)} \\ \pi_0(X) \otimes \pi(A) \otimes H & & \pi(X) \otimes \pi(B) \otimes H \\ {\scriptstyle id_{\pi_0(X)} \otimes \pi(q)} \Big\downarrow & & \Big\downarrow {\scriptstyle \beta(U)^*_{X\otimes B}} \\ \pi_0(X) \otimes \pi(B) \otimes H & \xrightarrow[id_{\pi_0(X)} \otimes \beta(U)^*_B]{} & \pi_0(X) \otimes \pi_0(B) \otimes H_0 \end{array} \tag{41}$$

must commute in **Hilb**.

The reason we use the term *coherent* is simply because the above conditions are natural equations to impose from a categorical point of view and in the categorical world these types of conditions are usually referred to as coherence conditions. Equation 40 is just a formal way of saying that the family $\beta(U)_Z$ should be compatible with the twist map in **Hilb**. On the other hand an interpretation of equation 41 is less straightforward. One way to think about it is as follows. Given any map $q : A \longrightarrow B$ in $\mathfrak{U}(\mathbb{A})$ we get a bounded linear map $\pi(q) : \pi(A) \otimes H \longrightarrow \pi(B) \otimes H$ in **Hilb**, i.e. a map from $\pi(A)$ to $\pi(B)$ in $\mathbf{Hilb}_H$. Now using the maps $\beta(U)_A$ we can produce a map $\beta(U)^*_B \pi(q)\beta(U)_A : \pi_0(A) \otimes H_0 \longrightarrow \pi_0(B) \otimes H_0$, i.e. a map in $\mathbf{Hilb}_{H_0}$ from $\pi_0(A)$ to $\pi_0(B)$. Now equation 41 simply states the maps from $\pi_0(X) \otimes \pi_0(A) \longrightarrow \pi_0(X) \otimes \pi_0(B)$ in $\mathbf{Hilb}_{H_0}$ given by $\pi_0(X) \otimes \beta(U)^*_B \pi(q)\beta(U)_A$ and $\beta(U)^*_{X\otimes B}[\pi(X) \otimes \pi(q)]\beta(U)_{X\otimes A}$ are equal. We will see in Theorem 10.2.23 another reason for insisting on this equation.

Remark 10.2.18. The conditions of Definition 10.2.17 that define a coherent premonoidal DHR representation can be given a nice categorical interpretation. Suppose that (H, π) is such a representation then we can define a new representation $(H_0, \psi_{U,\pi} = \psi)$ for each double cone U as follows: $\psi(X) = \pi_0 X$ for all objects $X \in |\mathfrak{U}(\mathbb{A})|$ and for any arrow $f : X \longrightarrow Y$ we define $\psi(f) = \beta(U)^*_Y \pi(f)\beta(U)_X$. Now the conditions of Definition 10.2.17 will guarantee that $\psi : \mathfrak{U}(\mathbb{A}) \longrightarrow \mathbf{Hilb}_{H_0}$ will be a strict premonoidal functor. This observation is a key component in our Theorem 10.2.23, and it will be given a proof in the body of the proof of the theorem.

Next we must define the corresponding notion of coherent DHR representation for localized transportable endomorphisms in the category Δ.

Definition 10.2.19. Suppose that F is an object in Δ. Then we say that F is **coherently transportable** if for each double cone $U \in K$ there exists maps $r^U_X = r_X : X \longrightarrow FX$ in $\mathfrak{U}(\mathbb{A})$ which are unitary and for all arrows $f : A \longrightarrow B$ the diagrams

$$\begin{array}{ccc} A\otimes X & \xrightarrow{r_{A\otimes X}} & FA\otimes FX \\ \downarrow{\scriptstyle \tau_{A,X}} & & \downarrow{\scriptstyle \tau_{FA,FX}} \\ X\otimes A & \xrightarrow[r_{X\otimes A}]{} & FX\otimes FA \end{array} \tag{42}$$

$$\begin{array}{ccc} X\otimes A & \xrightarrow{r_{X\otimes A}} & FX\otimes FA \\ \downarrow{\scriptstyle X\otimes r_A} & & \downarrow{\scriptstyle FX\otimes Ff} \\ X\otimes FA & & FX\otimes FB \\ \downarrow{\scriptstyle X\otimes Ff} & & \downarrow{\scriptstyle r^*_{X\otimes B}} \\ X\otimes FB & \xrightarrow[X\otimes r^*_B]{} & X\otimes B \end{array} \tag{43}$$

commute for all objects X. We also require that $F(f)r_A = r_B f$ for all arrows $f : A \longrightarrow B$ in $\bigvee_{V\perp U}\mathbb{A}(V)$.

Lemma 10.2.20. Suppose that F is an object in Δ such that for each double cone $U \in K$ there exists maps $r^U_X = r_X : X \longrightarrow FX$ in $\mathfrak{U}(\mathbb{A})$ which are unitary and central. Suppose further that for all arrows $f : A \longrightarrow B$ in $\bigvee_{V\perp U}\mathbb{A}(V)$ we have $F(f)r_A = r_B f$ and that $r_{A\otimes B} = r_A \otimes r_B$ for all objects A and B. Then F is coherently transportable.

Lemma 10.2.21. Suppose that $F \in |\Delta|$ is coherently transportable, then $(H_0, \pi_0 \circ F)$ is coherent.

Proof. For each $U \in K$ define $\beta(U)_X : \pi_0(X)\otimes H_0 \longrightarrow \pi_0 F(X)\otimes H_0$ by $\pi_0(r^U_X)$ where the maps r^U_X are as in Definition 10.2.19. To see that equation 40 holds simply note it is the image of equation 42 under the functor π_0. Similarly equation 41 holds since it is the image of equation 43 by π_0.

Lastly we must show that the maps $\beta(U)_A$ satisfy the conditions in Definition 10.2.2. Indeed suppose that $f : A \longrightarrow B$ in $\mathbb{A}(V)$ for some $V \perp U$. Then, by assumption, we have $F(f)r^U_B = r^U_A f$ and hence by functoriality of π_0 we have $\pi_0 F(f)\pi_0(r^U_B) = \pi_0(r^U_A)\pi_0(f)$ and hence $\pi_0 F(f)\beta(U)_B = \beta(U)_A \pi_0(f)$.

□

Corollary 10.2.22. Let Δ_c denote the full subcategory of Δ whose objects are those $F \in |\Delta|$ which are coherently transportable. Similarly let **cDHR** denote the full subcategory of **DHR**-**Rep** with objects consisting of the coherent premonoidal DHR representations. Then the functor $\Delta \longrightarrow$ **DHR**-**Rep** restricts to a functor $\Delta_c \longrightarrow$ **cDHR**.

We now state and prove the result which establishes the equivalence between the category Δ_c and **cDHR**. The corresponding result for AQFTs is Proposition 8.57 in [17]. This corresponding result from algebraic quantum field theory is of the utmost importance as far as the so-called *DHR-analysis* is concerned. It is this equivalence of categories that allows one to endow the category of physically relevant representations with a monoidal structure and ultimately show that this category of representations is equivalent to a category of representations of a compact super group via the Doplicher-Roberts reconstruction theorem. In our setting we have developed our own version of the classical result, namely Theorem 10.2.23, and it is our feeling that this equivalence of categories will play a similar role in our theory of premonoidal C^*-quantum field theory.

Theorem 10.2.23. If the vacuum representation (H_0, π_0) of a PC^* QFT $\mathbb{A}$ satisfies Haag duality and the functor $\pi_0 : \mathfrak{U}(\mathbb{A}) \longrightarrow \mathbf{Hilb}_{H_0}$ is faithful and injective on objects then the functor $\Delta_c \longrightarrow$ **cDHR** is an equivalence of categories.

Proof. We start by first establishing some notation. Let us denote the functor which maps an object of $F \in \Delta$ to the premonoidal DHR-representation $(H_0, \pi_0 \circ F)$ by $\mathbb{P}(F)$. Thus given a morphism $t : F \longrightarrow G$ in Δ, we write $\mathbb{P}(t)$ for the map $\pi_0(t) : (H_0, \pi_0 \circ F) \longrightarrow (H_0, \pi_0 \circ G)$ of premonoidal DHR-representations. First we show that $\mathbb{P}$ is faithful. Suppose that $\mathbb{P}(t) = \mathbb{P}(s)$, so that $\mathbb{P}(t)_A = \pi_0(t_A) = \mathbb{P}(s)_A = \pi_0(s_A)$ for all $A \in |\mathfrak{U}(\mathbb{A})|$. Since π_0 is faithful we have that $t_A = s_A$ for all $A \in |\mathfrak{U}(\mathbb{A})|$ and so $t = s$. Therefore $\mathbb{P}$ is faithful.

Now let's see why $\mathbb{P}$ is full. Let $\theta : \mathbb{P}(F) \longrightarrow \mathbb{P}(G)$ and suppose that F is localized at $O_1 \in K$ and G is localized at $O_2 \in K$. Then, as K is directed we can find a double cone $O \in K$ such that O_1, and $O_2 \leq O$. Thus F and G are also localized at $O \in K$. Now suppose that $f : A \longrightarrow B$ is an arrow in $\bigvee_{O' \perp O} \mathbb{A}(O')$ then

$$\begin{aligned}
\theta_B \pi_0(f) = \theta_B \pi_0(F(f)) \quad & \text{since } F \text{ is localized at } O \\
= \pi_0(G(f))\theta_A \quad & \text{since } \theta \text{ is a morphism from } \mathbb{P}(F) \text{ to } \mathbb{P}(G) \\
= \pi_0(f)\theta_A \quad & \text{since } G \text{ is localized at } O.
\end{aligned}$$

Hence by Haag duality it follows that $\theta_A = \pi_0(t^O_A)$ for all objects A. Notice that since $\theta_A : \pi_0 FA \longrightarrow \pi_0 GA$ in $\mathbf{Hilb}_H$ it follows that the maps t^O_A are of the form $FA \longrightarrow GA$. Now to see that this family is natural suppose that $f : X \longrightarrow Y$ in $\mathfrak{U}(\mathbb{A})$ then

$$\begin{aligned}
\pi_0(t^O_Y F(f)) &= \pi_0(t^O_Y)\pi_0(F(f)) \\
&= \theta_Y \pi_0(F(f)) \\
&= \pi_0(G(f))\theta_X \\
&= \pi_0(G(f))\pi_0(t^O_X) \\
&= \pi_0(G(f)t^O_X)
\end{aligned}$$

and so as π_0 is faithful it follows that $t^O_Y F(f) = G(f)t^O_X$. Therefore t^O_X is natural in X. Thus $t : F \longrightarrow G$ is a map in Δ.

Lastly we must show that $\mathbb{P}$ is essentially surjective on objects. Suppose that (H, π) is a coherent DHR representation and let $O \in K$ be any double cone. Then there exists unitary maps $\beta(O)_A : \pi_0(A) \otimes H_0 \longrightarrow \pi(A) \otimes H$ for each $A \in |\mathfrak{U}(\mathbb{A})|$ such that for any $O' \perp O$, we have $\beta(O)_B \pi_0(f) = \pi(f)\beta(O)_A$ for all $f : A \longrightarrow B$ in $\mathbb{A}(O')$. In addition the family $\beta(O)_X$ also satisfies equations 40 and 41 of Definition 10.2.17. Thus we define a new representation (ψ, H_0) as follows. For all objects $X \in \mathfrak{U}(\mathbb{A})$ define $\psi(X) = \pi_0(X)$ and if $f : X \longrightarrow Y$ then we define $\psi(f) : \pi_0(X) \otimes H_0 \longrightarrow \pi_0(Y) \otimes H_0$ by $\psi = \beta(O)^*_Y \pi(f)\beta(O)_X$. Then we claim that $\psi(X) \otimes \psi(f) = \psi(X \otimes f)$

and $\psi(f) \otimes \psi(X) = \psi(f \otimes X)$ for all arrows $f : A \longrightarrow B$ and objects X. Indeed

$$\begin{aligned}\psi(X \otimes f) &= \beta(O)^*_{X\otimes B}[\pi(X \otimes f)]\beta(O)_{X\otimes A} \\ &= \beta(O)^*_{X\otimes B}[\pi(X) \otimes \pi(f)]\beta(O)_{X\otimes A} \\ &= \pi_0(X) \otimes [\beta(O)^*_B \pi(f)\beta(O)_A] \qquad \text{by equation 41} \\ &= \psi(X) \otimes \psi(f)\end{aligned}$$

Similarly using equation 40 and equation 41 one shows that $\psi(f)\otimes\psi(X) = \psi(f\otimes X)$.

Now we want to show that for all arrows $f : A \longrightarrow B$ in $\mathfrak{U}(\mathbb{A})$ that $\psi(f) = \pi_0(h)$ for some arrow h in $\mathfrak{U}(\mathbb{A})$. First we observe that if $f : A \longrightarrow B$ is an arrow in $\mathbb{A}(O')$, where $O' \perp O$ then $\psi(f) = \beta(O)^*_B \pi(f)\beta(O)_A = \pi_0(f)$ by definition. Thus it remains to consider the case of an arbitrary double cone U which is not spacelike separated from O. Since K is directed there exists $W \in K$ with O, $U \leq W$. Hence we also have that $\mathbb{A}(U) \subseteq \mathbb{A}(W)$ and so $f : A \longrightarrow B$ is an arrow in $\mathbb{A}(W)$ as well. Now for any $g : X \longrightarrow Y$ in $\bigvee_{W'\perp W} \mathbb{A}(W')$ we have

$$\begin{aligned}\psi(f) \ltimes \pi_0(g) &= \psi(f) \ltimes \psi(g) \\ &= \psi(B) \otimes \psi(g) \circ \psi(f) \otimes \psi(X) \\ &= \psi(B \otimes g) \circ \psi(f \otimes X) \\ &= \psi(B \otimes g \circ f \otimes X) \\ &= \psi(f \otimes Y \circ A \otimes g) \\ &= \psi(f) \otimes \psi(Y) \circ \psi(A) \otimes \psi(g) \\ &= \psi(f) \rtimes \psi(g) \\ &= \psi(f) \rtimes \pi_0(g).\end{aligned}$$

Similarly one shows that $\pi_0(g) \ltimes \psi(f) = \pi_0(g) \rtimes \psi(f)$ and hence by Haag duality, i.e., equation 39, we have that $\psi(f) \in \pi_0 \left(\bigvee_{W'\perp W} \mathbb{A}(W')\right)' (\pi_0 X, \pi_0 Y) = \pi_0 [\mathbb{A}(W)(X, Y)]$. Hence there exists $f_{\pi_0} : X \longrightarrow Y$ in $\mathbb{A}(W)$ such that $\psi(f) = \pi_0(f_{\pi_0})$. As π_0 is faithful it follows that f_{π_0} is unique. Furthermore we can now view ψ as a functor $\psi : \mathfrak{U}(\mathbb{A}) \longrightarrow \pi_0(\mathfrak{U}(\mathbb{A}))$. Thus we define a functor $\Psi : \mathfrak{U}(\mathbb{A}) \longrightarrow \mathfrak{U}(\mathbb{A})$ by $\Psi = \pi_0^{-1} \circ \psi$. Then $\Psi(X) = X$ for all objects X and clearly if $f \in \mathbb{A}(O')$ where $O' \perp O$ then $\Psi(f) = \pi_0^{-1} \circ \psi(f) = \pi_0^{-1}\pi_0(f) = f$. Thus Ψ is localized at $O \in K$.

Now to see that Ψ is transportable choose any double cone D. Then, as above we can find a family $\beta(D)_X : \pi_0(X) \otimes H_0 \longrightarrow \pi(X) \otimes H$ satisfying the conditions of Definition 10.2.17. Thus we can define a new representation as before (H_0, ϕ) where $\phi(X) = \pi_0(X)$ and $\phi(f) = \beta(D)^*_B \pi(f)\beta(D)_A$ for all objects X and arrows $f : A \longrightarrow B$ in $\mathfrak{U}(\mathbb{A})$. By the above argument we obtain a functor $\Phi : \mathfrak{U}(\mathbb{A}) \longrightarrow \mathfrak{U}(\mathbb{A})$ given by $\Phi = \pi_0^{-1} \circ \phi$. Now we know that Φ is localized at $D \in K$ and thus we simply need to build a unitary natural transformation $\epsilon_X : \Psi(X) \longrightarrow \Phi(X)$, i.e., $\epsilon_X : X \longrightarrow X$. So define $\theta_X : \pi_0(X) \otimes H_0 \longrightarrow \pi_0(X) \otimes H_0$ by $\theta_X = \beta(D)^*_X \circ \beta(O)_X$. Then we claim that θ is a map from the representation $(H_0, \psi = \pi_0 \circ \Psi)$ to $(H_0, \phi = \pi_0 \circ \Phi)$. Indeed let $f : A \longrightarrow B$ be any arrow in $\mathfrak{U}(\mathbb{A})$ then

$$\begin{aligned}
\theta_B \circ \psi(f) &= \beta(D)^*_B \circ \beta(O)_B \circ \beta(O)^*_B \pi(f)\beta(O)_A \\
&= \beta(D)^*_B \pi(f)\beta(O)_A \\
&= \beta(D)^*_B \pi(f)\beta(D)_A \beta(D)^*_A \beta(O)_A \\
&= \phi(f) \circ \theta_A.
\end{aligned}$$

Hence by fullness of $\mathbb{P}$ there exists a natural transformation $\epsilon_X : \Psi(X) \longrightarrow \Phi(X)$ such that $\theta_X = \pi_0(\epsilon_X)$. In addition since the θ_X are unitary and π_0 is faithful, it follows that ϵ_X is also unitary. Thus Ψ is transportable. We now show that the maps $r^D_X := \epsilon^*_X$ satisfy the all the properties listed in Definition 10.2.19.

We start by checking equation 42. Indeed

$$\begin{aligned}
\pi_0(\tau_{\Psi A, \Psi X} \circ r_{A\otimes X}) &= \pi_0(\tau_{X,A}) \circ \pi_0(r_{A\otimes X}) \\
&= [\tau_{\pi_0 X, \pi_0 A} \otimes id_{H_0}] \circ [\beta(O)^*_{A\otimes X}] \circ [\beta(D)_{A\otimes X}] \\
&= [\beta(O)^*_{X\otimes A}][\tau_{\pi A, \pi X} \otimes id_H] \circ [\beta(D)_{A\otimes X}] \\
&= [\beta(O)^*_{X\otimes A}] \circ [\beta(D)_{X\otimes A}] \circ [\tau_{\pi_0 A, \pi_0 X} \otimes id_{H_0}] \\
&= \pi_0(r_{X\otimes A}) \circ \pi_0(\tau_{X,A}) \\
&= \pi_0(r_{X\otimes A} \circ \tau_{X,A})
\end{aligned}$$

Hence as π_0 is faithful we have that equation 42 holds. Next to see that equation 43 is satisfied we will use the same approach. Let $f : A \longrightarrow B$ be any arrow in $\mathfrak{U}(\mathbb{A})$.

Then

$$\begin{aligned}
&\pi_0\left(r^*_{X\otimes B}\circ\Psi(X)\otimes\Psi(f)\circ r_{X\otimes A}\right)\\
&=\pi_0(\epsilon_{X\otimes B})\circ\pi_0\Psi(X)\otimes\pi_0\Psi(f)\circ\pi_0(\epsilon^*_{X\otimes A})\\
&=[\beta(D)^*_{X\otimes B}\beta(O)_{X\otimes B}]\circ[\pi_0X\otimes\psi(f)]\circ[\beta(O)^*_{X\otimes A}\beta(D)_{X\otimes A}] \qquad \text{so by equation 41}\\
&=[\beta(D)^*_{X\otimes B}\beta(O)_{X\otimes B}]\circ[\beta(O)^*_{X\otimes B}[\pi(X)\otimes\pi(f)]\beta(O)_{X\otimes A}]\circ[\beta(O)^*_{X\otimes A}\beta(D)_{X\otimes A}]\\
&=\beta(D)^*_{X\otimes B}\circ[\pi(X)\otimes\pi(f)]\circ\beta(D)_{X\otimes A}\\
&=\pi_0(X)\otimes\phi(f) \qquad \text{by equation 41}\\
&=\pi_0(X)\otimes\beta(D)^*_B\pi(f)\beta(D)_A\\
&=\pi_0(X)\otimes\beta(D)^*_B\beta(O)_B\beta(O)^*_B\pi(f)\beta(O)_A\beta(O)^*\beta(D)_A\\
&=\pi_0(X)\otimes\pi_0(r^*_B)\psi(f)\pi_0(r_A)\\
&=\pi_0(X)\otimes\pi_0(r^*_B)\pi_0\Psi(f)\pi_0(r_A)\\
&=\pi_0(X\otimes r^*_B\Psi(f)r_A).
\end{aligned}$$

Since π_0 is faithful it follows that $r^*_{X\otimes B}\circ[\Psi(X)\otimes\Psi(f)]\circ r_{X\otimes A}=X\otimes r^*_B\Psi(f)r_A$ as required. Lastly given $f:A\longrightarrow B$ in $\bigvee_{D'\perp D}\mathbb{A}(D')$, then $\Phi(f)=f$ and as ϵ_X is natural in X and unitary we have that

$$\begin{aligned}
\Psi(f)r_A&=\Psi(f)\epsilon^*_A\\
&=\epsilon^*_B\Phi(f)\\
&=r_Bf,
\end{aligned}$$

as required. Hence we have shown that Ψ is coherently transportable and so by Lemma 10.2.21 we have that $\mathbb{P}(\Psi)=(H_0,\psi)$ is coherent. In addition we also have that $\mathbb{P}(\Psi)$ is isomorphic to the representation (H,π). Hence the the functor $\mathbb{P}$ is an equivalence of categories. □

10.3 Symmetry Structure on Δ

For this section we will keep the same assumptions that we stated in Remark 10.2.14. We would like to show that if the dimension of spacetime is three or larger, the vacuum representation satisfies Haag duality and is faithful, then Δ can be equipped with a symmetric monoidal structure. Hence to our list of assumptions for this section we also add that the vacuum representation is faithful and satisfies Haag duality. However it turns out these assumptions are insufficient to yield such a result. The obstacle that one encounters is that there is a lack of "uniformity" with regards to the objects of the categories $|\mathbb{A}(O)|$. What we mean by this is that in the traditional AQFT setting we can view each C^*-algebra $\mathbb{A}(O)$ as one object premonoidal C^*-category and then every category $\mathbb{A}(O)$ has the same object set as every other category $\mathbb{A}(O')$. In our more general setting we have not imposed such a restriction on the functor $\mathbb{A}$ and consequently we are unable to exploit the theorems used in the classical DHR analysis to show the category Δ is symmetric.

Instead of giving a complete proof that the category Δ is symmetric, under suitable assumptions, we will simply give an indication of how one might approach such a result.

Lemma 10.3.1 (c.f. Lemma 8.3.3)**.** If $F_1 \in \Delta(O_1)$ and $F_2 \in \Delta(O_2)$ and $t : F_1 \longrightarrow F_2$ is an arrow in Δ then $t_A : F_1A \longrightarrow F_2A$ is an arrow in $\mathbb{A}(O)$ for all $A \in |\mathbb{A}(O)|$ where $O_1 \cup O_2 \subseteq O$.

Proof. By Lemma 10.2.8 it follows that F_i are also localized at O and so for any $f : A \longrightarrow B$ in $\bigvee_{O' \perp O} \mathbb{A}(O')$ we have that $F_i(f) = f$. Furthermore since t is a natural transformation we also have that $t_B F_1(f) = F_2(f) t_A$ and thus $t_B f = f t_A$. Hence we have that

$$\pi_0(t_B)\pi_0(f) = \pi_0(f)\pi_0(t_A)$$

and so by Haag duality we have that $\pi_0(t_A) \in \pi_0[\mathbb{A}(O)(F_1A, F_2A)]$. But π_0 is faithful and hence $t_A : F_1A \longrightarrow F_2A$ is an arrow in $\mathbb{A}(O)$ for all objects A in $\mathbb{A}(O)$. $\square$

The next order of business is to show that $F \circ G = G \circ F$ whenever F and G are localized at spacelike separated double cones. First we need the following definition.

Definition 10.3.2. Let $\mathbb{A}$ be a PC^* QFT. Then we say that $\mathbb{A}$ is **uniform** if $|\mathbb{A}(O)| = |\mathbb{A}(O')|$ for all double cones O, $O' \in K$.

Lemma 10.3.3. If $F \in \Delta(O_1)$ and $G \in \Delta(O_2)$ then $F \circ G = G \circ F$ whenever O_1 and O_2 are spacelike separated and $\mathbb{A}$ is uniform.

Proof. Using the universal property of $\hat{\mathbb{A}}$ it suffices to show that $(F \circ G)|_{\mathbb{A}(O)} = (G \circ F)|_{\mathbb{A}(O)}$ for each $O \in K$. Thus let $O \in K$ be arbitrary. Then there exists double cones $O_2, \ldots, O_6$ such that $O_1 \perp O_3$, $O_3 \perp O$, $O_2 \perp O_4$, $O_4 \perp O$, $O_1 \cup O_3 \subseteq O_5$, $O_2 \cup O_4 \subseteq O_6$, and $O_5 \perp O_6$. (See Lemma 8.3.4 for an explanatory diagram.) Now as F and G are transportable there exists $F' \in \Delta(O_3)$ and $G' \in \Delta(O_4)$ and unitary maps $\vartheta : F \longrightarrow F'$, and $\theta : G \longrightarrow G'$. Applying Lemma 10.3.1 we get that ϑ_A is an arrow in $\mathbb{A}(O_5)$ for all objects A in $\mathbb{A}(O_5)$ and similarly θ_B is an arrow in $\mathbb{A}(O_6)$ for all objects B in $\mathbb{A}(O_6)$. By uniformity it follows that θ_X is an arrow of $\mathbb{A}(_6)$ and ϑ_X is an arrow of $\mathbb{A}(O_5)$ for all objects $X \in |\mathfrak{U}(\mathbb{A})|$. Furthermore uniformity also implies that $FX = X = F'X = GX = G'X$ for all objects X, which means all the objects of Δ will be identity on objects functors.

Appealing to the naturality of ϑ we have that

$$\vartheta_{G'X} \circ F(\theta_X) = F'(\theta_X) \circ \vartheta_{GX}.$$

But F and F' are localized at $O_1, O_3 \subseteq O_5$, and $\theta_X \in \mathbb{A}(O_6)$, so $F(\theta_X) = \theta_X = F'(\theta_X)$ since $O_1, O_3 \perp O_6$. Recalling that all the functors involved are the identity on objects, we get from the naturality of ϑ that

$$\vartheta_X \circ \theta_X = \theta_X \circ \vartheta_X$$

for all objects X. Furthermore since $\vartheta_X \in \mathbb{A}(O_5)$ and $O_2, O_4 \perp O_5$ we also have that $G(\vartheta_X) = \vartheta_X = G'(\vartheta_X)$ for all objects X. We can now show that $F \circ G = G \circ F$. Indeed it is clear that these two functors agree on objects since they are both the identity on objects. So now let $f : X \longrightarrow Y$ be any arrow in $\mathbb{A}(O)$ then since O_3,

$O_4 \perp O$ it follows that $F'(f) = f = G'(f)$. Thus:

$$\begin{aligned} FG(f) &= F(\theta_Y^* G'(f)\theta_X) \\ &= F(\theta_Y^* f \theta_X) \\ &= F(\theta_Y)^* F(f) F(\theta_X) \\ &= \theta_Y^* (\vartheta_Y^* F'(f) \vartheta_X) \theta_X \\ &= \theta_Y^* \vartheta_Y^* f \vartheta_X \theta_X \\ &= \vartheta_Y^* (\theta_Y^* f \theta_X) \vartheta_X \\ &= \vartheta_Y^* G(f) \vartheta_X \\ &= G(\vartheta_Y^*) G(f) G(\vartheta_X) \\ &= G(\vartheta_Y^* f \vartheta_X) \\ &= GF(f). \end{aligned}$$

□

Now supposing that $\mathbb{A}$ is a uniform premonoidal C^*-quantum field theory then for any objects $F_i \in \Delta(O_i)$ we can pick double cones $\widetilde{O_i}$ spacelike to O_i and so by transportability there exists $\tilde{F}_i \in \Delta(\widetilde{O_i})$ and unitary natural transformations $U_i \in \Delta(F_i, \tilde{F}_i)$. Putting all this data together one obtains maps:

$$\varepsilon_{F_1,F_2}(U_1, U_2)_X \equiv F_2(U_{1\,X}^*) U_{2\,X}^* U_{1\,X} F_1(U_{2\,X}) : F_1 F_2 X \longrightarrow F_2 F_1 X.$$

Now by following the presentation that we gave in Section 8.3 it is our suspicion that the category Δ in our more general setting will also be symmetric with symmetric structure given by the map ε that we defined above. Notice that we need to make the assumption that $\mathbb{A}$ is uniform to be able to reuse the arguments given in Section 8.3.

Chapter 11

Towards a Premonoidal Doplicher-Roberts Theorem

In this rather speculative chapter, we begin to put together the ideas necessary to obtain a premonoidal version of the Doplicher-Roberts theorem. We hope to have a definitive result in future work.

Throughout this chapter we will assume that $\mathcal{C}$ is a symmetric premonoidal C^*-category with direct sums, conjugates, central subobjects, and an irreducible tensor unit. Thus in particular $\mathcal{C}$ is an SPC^*. Let's agree to call such a category a *normed* SPC^*.

A premonoidal Doplicher-Roberts theorem would be something like:

> If $\mathcal{C}$ is a normed SPC^*, then there exists a group G and a representation H of G and an equivalence of premonoidal C^*-categories $F : \mathcal{C} \longrightarrow \mathbf{Rep}_{fd}(G)_H$.

One approach to establishing such a result would be to try to mimic the proof presented by Müger in [17]. The first stage of the proof would be to produce a $*$-preserving fibre functor $E : \mathcal{C} \longrightarrow \mathbf{Hilb}_H$ and the second stage would be to imitate the classical Tannaka-Krein construction.

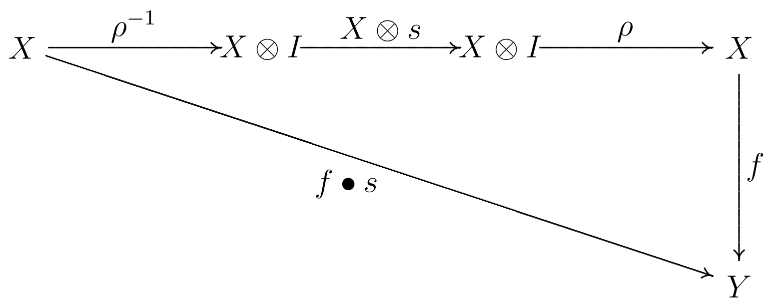
Stage one in the proof of the Doplicher-Roberts theorem is much more difficult than stage two. In other words proving the existence of/constructing a fibre functor is much more involved than producing the compact supergroup from a given fibre functor. Furthermore without a fibre functor we cannot appeal to Tannaka-Krein type arguments, thus for this reason we will focus on how one might construct a fibre functor in the premonoidal setting.

We will start by assuming that we have a fibre functor and see what implications this has. Let H be a fixed Hilbert space and suppose that $E : \mathcal{C} \longrightarrow \mathbf{Hilb}_H$ is a strong premonoidal functor which is faithful and $*$-preserving.

- if E preserves central maps then it restricts to a $*$-preserving fibre functor $E : \mathcal{Z}(\mathcal{C}) \longrightarrow \mathbf{Hilb}$;
- there is a faithful representation $\pi_E^I : \mathcal{C}(I, I) \longrightarrow \mathfrak{B}(H)$ given by $a : I \longrightarrow I \mapsto \pi_E(a) \equiv E(a)$;
- for each object $X \in |\mathcal{C}|$ there is a faithful representation of the C^*-algebra $\mathcal{C}(X, X)$ on the Hilbert space $EX \otimes H$. If $f : X \longrightarrow X$ then define $\pi_E^X(f) = E(f) : EX \otimes H \longrightarrow EX \otimes H$.

Now we see that whatever we construct as a candidate for a fibre functor we should at least have that it satisfies the above three properties. In particular when restricting to the monoidal category $\mathcal{Z}(\mathcal{C})$ we should obtain the classical Doplicher-Roberts theorem. Before we proceed to outline a possible course of action for constructing a fibre functor we make a few observations about the category $\mathcal{C}$.

Let $\mathcal{I}$ denote the C^*-algebra $\mathcal{C}(I, I)$. Then for all objects X and Y we can define an action $\mathcal{C}(X, Y) \times \mathcal{I} \longrightarrow \mathcal{C}(X, Y)$ as follows:



for all $s \in I$ and $f : X \longrightarrow Y$.

Remark 11.0.4. In the case that $\mathcal{C}$ is also a monoidal category then the above action $f \bullet s$ coincides with definition of $\bullet$ given by Abramsky and Coecke in [1](see equation 46 of section 12.4 for our description their definition).

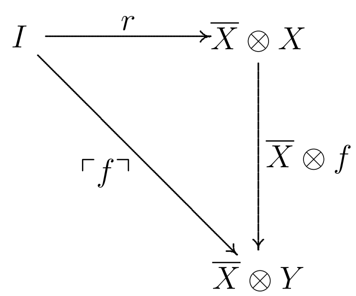
Lemma 11.0.5. If $\mathcal{C}$ is a normed SPC^*, then the Banach space $\mathcal{C}(X, Y)$ is a right $\mathcal{I}$-module.

Proof. Follows from the bilinearity of composition. □

In fact even more is true.

Proposition 11.0.6. Under the above assumptions on the category $\mathcal{C}$ the right $\mathcal{I}$-module $\mathcal{C}(X, Y)$ can be equipped with the structure a Hilbert $\mathcal{I}$-module.

Proof. We need to define an $\mathcal{I}$-valued inner product on $\mathcal{C}(X, Y)$. To this end suppose that $(\overline{X}, r : I \longrightarrow \overline{X} \otimes X, \overline{r} : I \longrightarrow X \otimes \overline{X})$ is a conjugate of X. Then given any $f : X \longrightarrow Y$ we define $\ulcorner f \urcorner : I \longrightarrow \overline{X} \otimes Y$ as follows:



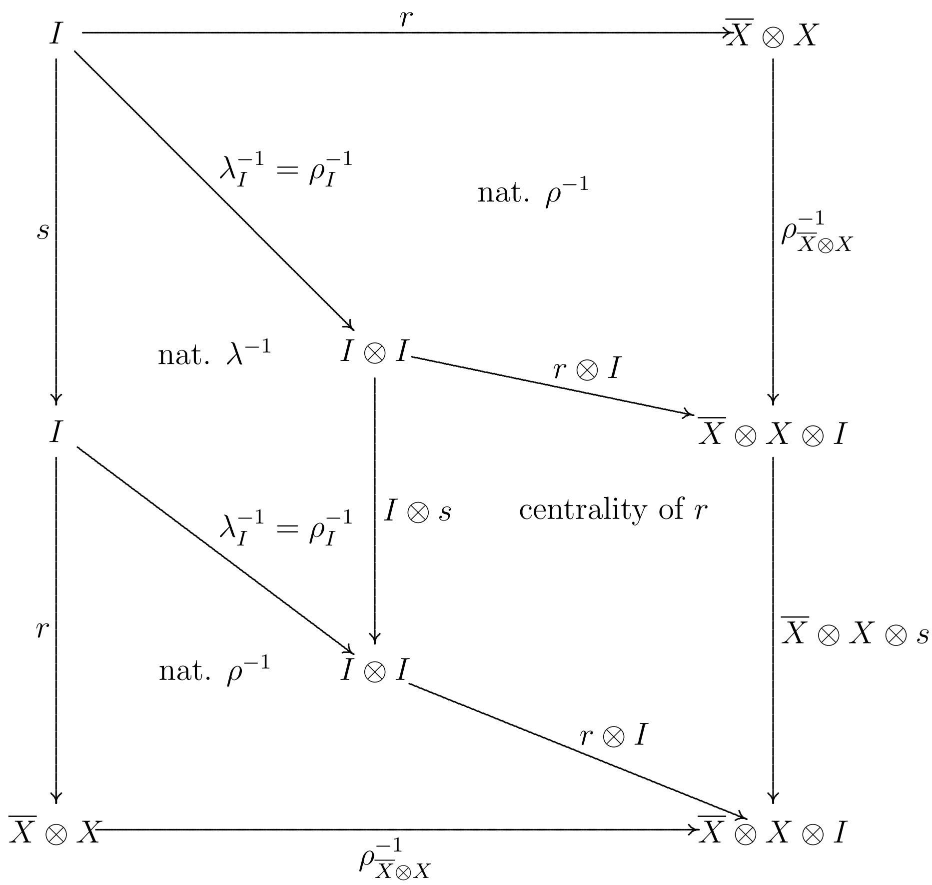
Then given f and $g : X \longrightarrow Y$ we define their inner-product by

$$\langle f, g \rangle_{\mathcal{I}} \equiv \ulcorner f \urcorner^* \circ \ulcorner g \urcorner. \tag{44}$$

We must show that the conditions of Definition 4.1.3 are satisfied. The only condition which might cause a problem is the second condition, all others are straightforward. Indeed given $s : I \longrightarrow I$ and f, $g : X \longrightarrow Y$ then we need to show that

$$\langle f, g \bullet s \rangle_{\mathcal{I}} = \langle f, g \rangle_{\mathcal{I}} \circ s$$
$$\ulcorner f \urcorner^* \circ \ulcorner g \bullet s \urcorner = \ulcorner f \urcorner^* \circ \ulcorner g \urcorner \circ s.$$

This equation follows from the following calculation.



Using the commutativity of the above diagram one can now easily verify Equation 44 holds. To see that $\mathcal{C}(X, Y)$ is a Hilbert $\mathcal{I}$-module it remains to show that it is complete with respect to the norm $\|f\|_{\mathcal{I}} \equiv \|\langle f, f\rangle_{\mathcal{I}}\|^{\frac{1}{2}}_{I,I}$. Notice that:

$$\begin{aligned}\|f\|^2_{\mathcal{I}} &= \|\ulcorner f \urcorner^* \circ \ulcorner f \urcorner\|_{I,I} \\ &= \|\ulcorner f \urcorner\|^2_{I,\overline{X}\otimes Y},\end{aligned}$$

and hence $\|f\|_{\mathcal{I}} = \|\ulcorner f \urcorner\|_{I,\overline{X}\otimes Y}$. Thus (f_n) is a Cauchy sequence with respect to $\| \; \|_{\mathcal{I}}$ if and only if it is a Cauchy sequence with respect to $\| \; \|_{I,\overline{X}\otimes Y}$. Thus as $\mathcal{C}(I, \overline{X} \otimes Y)$ is complete with respect to the norm $\| \; \|_{I,\overline{X}\otimes Y}$ it follows that if (f_n) is a Cauchy sequence with respect to $\| \; \|_{\mathcal{I}}$ then there exists a unique map $g : I \longrightarrow \overline{X} \otimes Y$ such that $\lim \ulcorner f_n \urcorner = g$. Now by Theorem 9.1.17 it follows that there exists a unique map $f : X \longrightarrow Y$ such that $\ulcorner f \urcorner = g$. Moreover it is now clear that (f_n) converges to f in the norm $\| \|_{\mathcal{I}}$ as required. Thus $\mathcal{C}(X, Y)$ is complete with respect the norm $\| \|_{\mathcal{I}}$. $\square$

As an interesting side note it is almost immediate that there are functors $R : \mathcal{C} \longrightarrow \mathcal{I}$-**hMod** from the category $\mathcal{C}$ to the category of right Hilbert $\mathcal{I}$-modules. Indeed for any fixed object X in $\mathcal{C}$ define $RY \equiv \mathcal{C}(X, Y)$ and given any $f : Y \longrightarrow Z$ then $Rf : \mathcal{C}(X, Y) \longrightarrow \mathcal{C}(X, Z)$ is defined by post-composition. It is clear that this preserves composition and identities since R is a hom-functor. The only thing one needs to check is that Rf has and adjoint, but this is also easy to see and its adjoint is given by $(Rf)^* = R(f^*)$.

Now the reason we were considering this situation is that we wanted to indicate the connection between the C^*-algebra $\mathcal{I}$ and the spaces $\mathcal{C}(X, Y)$. Up to this point, we haven't really exploited the fact that we are dealing with a premonoidal category. We will now explore one possible condition that we could impose on $\mathcal{C}$ that gives us a way of relating central maps to arbitrary maps.

Definition 11.0.7. If $\mathcal{C}$ is a normed SPC^*, we say that it is **centrally dense**, if for all pairs of objects X and Y the set

$$\mathcal{Z}(\mathcal{C})(X, Y) \bullet \mathcal{I} \equiv \text{span}\{f \bullet s \mid f \in \mathcal{Z}(\mathcal{C})(X, Y),\ s \in \mathcal{I}\}$$

is $\| \|_{\mathcal{I}}$-dense in $\mathcal{C}(X, Y)$.

Remark 11.0.8. Note that the space $\mathcal{C}(X, Y)$ has two norms on it. Namely the norm $\| \|_{X,Y}$ that comes from the fact that $\mathcal{C}$ is a C^*-tensor category and the other norm $\|f\|_{\mathcal{I}} \equiv \|\langle f, f\rangle_{\mathcal{I}}\|_{I,I}^{\frac{1}{2}}$, coming from the inner-product $\mathcal{I}$-module structure. Comparing these norms we see

$$\begin{aligned}
\|f\|_{\mathcal{I}}^2 &= \|\ulcorner f \urcorner^* \circ \ulcorner f \urcorner\|_{I,I} \\
&= \|\ulcorner f \urcorner\|^2 \\
&= \|(\overline{X} \otimes f) \circ r\|^2 \\
&\leq \|f\|_{X,Y}^2 \|r\|_{I,\overline{X}\otimes Y}^2,
\end{aligned}$$

and thus $\|f\|_{\mathcal{I}} \leq \|f\|_{X,Y}\|r\|_{I,\overline{X}\otimes Y}$. Now it follows from a standard result of point set topology that the $\| \|_{\mathcal{I}}$-topology will be coarser than the $\| \|_{X,Y}$-topology on $\mathcal{C}(X,Y)$. Thus sets which are $\| \|_{\mathcal{I}}$-closed in $\mathcal{C}(X,Y)$ are necessarily also $\| \|_{X,Y}$-closed, however the converse need not hold. Thus a centrally dense category is a priori a weaker notion than requiring the sets $\mathcal{Z}(\mathcal{C})(X,Y) \bullet \mathcal{I}$ to be dense in the$\| \|_{X,Y}$-topology.

We will now give an outline of a possible solution to constructing a fibre functor $E : \mathcal{C} \longrightarrow \mathbf{Hilb}_H$. Let $\mathcal{C}$ be a centrally dense normed SPC^*, then

- by the Doplicher-Roberts theorem there exists a fibre functor $F : \mathcal{Z}(\mathcal{C}) \longrightarrow$ **Hilb**;
- by the GNS construction there exists a faithful representation of the C^*-algebra $\mathcal{I}$, which denote $\pi : \mathcal{I} \longrightarrow \mathfrak{B}(H)$.

Hence we define a functor $E : \mathcal{C} \longrightarrow \mathbf{Hilb}_H$ by

- $EX \equiv FX$ and;
- if $f = \lim_j f_j \bullet s_j$ then define $Ef = \lim_j F(f_j) \otimes \pi(s_j)$.

We do not yet have a proof that E as we have just defined it is indeed well-defined let alone a fibre functor. Thus for the moment we can only conjecture that this yields a fibre functor. Assuming that this procedure yields a fibre functor then our next step would be to try to imitate the classical Tannaka-Krein construction to obtain a premonoidal version of the Doplicher-Roberts reconstruction theorem.

Chapter 12

Extension of AQFT to Causal Orderings

The following chapter is a departure from the previous work, and was inspired by discussions between the author and several colleagues at the Oxford University Computing Laboratory. Discussions with Samson Abramsky, Richard Blute, Bob Coecke, Tim Porter and Jamie Vicary led us to consider the notion of a causal dagger net.

12.1 Causal Dagger Nets

We introduce the notion of *causal dagger net*. Inspired in part by ideas from algebraic quantum field theory, a causal dagger net is a functor from some poset of regions of spacetime to the category of monoidal dagger categories.

One crucial difference with AQFT is that rather than order spacetime regions under subset inclusion, we extend the causal ordering on points to regions. We argue here that, for the purposes of encoding protocols such as quantum teleportation, this is more appropriate.

This brings our notion of functorial QFT more in line with the *causal set* theory of Sorkin [6]. We explore the extent to which the monoidal and dagger structures of the individual categories in the codomain of the functor extend to the dagger net. Such a question makes sense when considering the *Grothendieck category* associated to the net. We show that there are local versions of the monoidal and dagger structures, and argue that such notions have good physical intuition.

12.2 Causal Orderings of Subsets

One of the main goals of this chapter is to develop a framework in which causal evolution can be represented in AQFT by having a second ordering on the poset K of double cones in Minkowski space. In the standard definition of AQFT, one orders K by subset inclusion. However, there is an ordering on individual points in Minkowski space, namely the causal ordering as described in Section 2.3. A second ordering on double cones taking into account the causal ordering on points would then be the appropriate mechanism to model evolution.

This leads to the classic question of the appropriate method of lifting an ordering on elements of a poset to a class of subsets of the poset. This is a standard topic in domain theory and typically goes under the heading *powerdomain theory* [3]. We collect here 5 possible definitions for such an ordering, and discuss their relative merits. It is sensible to think of the poset as arising from the causal ordering on Minkowski space or a Lorentz manifold. But in fact, one can apply these constructions to any partially ordered set.

The first approach we consider is due to Crane and Christensen [10]. If $U, V \in \mathsf{K}$, they define $U \sqsubseteq_{cc} V$ by

$$\forall x \in U, \forall y \in V, x \leq y$$

The result is a strict partial order; the only element of K comparable to itself is the empty set. The real problem from the point of view of modelling causal evolution is exhibited in the following situation. Suppose that one has a sequence of double cones $D_0, \ldots, D_n$ where we imagine that each D_i is the same region in space during some period of time T_i. Also suppose that T_{i+1} is a period of time which occurs after T_i for each i, i.e. we could think of D_0 as a laboratory during some initial time period T_0 and then D_1 would be the same laboratory during some later time period T_2 and so on and so forth. Then the sequence of double cones should be thought of as an evolving sequence. But under the Crane-Christensen ordering D_0 is not less than any D_i until the intersection of the two is empty, i.e. the lowest point of D_i is greater than the highest point of D_0. But to model evolution, one would want this family of double cones to form a chain.

Borrowing some techniques from powerdomain theory we define three more possible orderings, namely the *lower order*, the *upper order*, and the *Egli-Milner order*.

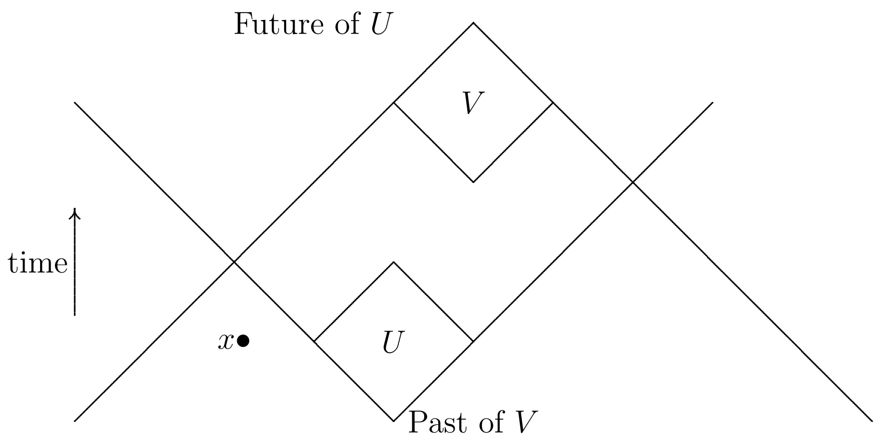
- The *lower order* is defined by saying $U \sqsubseteq_l V$ if

$$\forall x \in U, \exists y \in V, x \leq y.$$

- The *upper ordering* is defined by saying $U \sqsubseteq_u V$ if

$$\forall y \in V, \exists x \in U, x \leq y.$$

- The *Egli-Milner order* is defined by saying $U \sqsubseteq_{EM} V$ if and both $U \sqsubseteq_l V$ and $U \sqsubseteq_u V$ hold. Each of these relations provide a preorder structure on K and in the case of the Egli-Milner order we get that $\sqsubseteq_{EM}$ is antisymmetric. This result follows from the fact that if the collection of subsets are order-convex then the Egli-Milner order is antisymmetric. Thus since any double cone in Minkowski space is order-convex the result follows. Note that a subset C of a poset is *order-convex* if $x \leq y \leq z$ implies $y \in C$ whenever x and $z \in C$. Thus while these orderings are reasonably well-behaved there is a clear sense in which they fail to capture causality. Consider the following diagram



> We see that the point x is in the past of V but has no causal relationship with any point of U. Moreover one can easily verify that $U \sqsubseteq_{EM} V$ and thus one would only be able to obtain partial information about V, even when given complete information about U.

The next order is an order which seems to be original to this setting, we call it the *Bellairs order*. We say that $U \sqsubseteq_B V$ if for every $v \in V$, every maximal descending chain with v as its top element intersects U. This, we believe properly captures causal precedence in the sense that all of the events which have an influence on V must pass through or have passed through U. Thus in theory, one could completely calculate the state of V, based on complete information about U. There is however one strange feature enjoyed by this order, namely if $V \subseteq U$ then $U \sqsubseteq_B V$.

12.3 Causal Dagger Net Structure

Now we define a category DAG as follows. Objects are compact closed dagger categories, and morphisms are strong monoidal dagger functors.

Definition 12.3.1. A **dagger net** is a functor $\mathsf{A} : \mathsf{K} \longrightarrow \mathsf{DAG}$. Evidently, we are taking K with one of the orderings considered above. We further require that if $a \subseteq b$, then $\mathsf{A}(a) \subseteq \mathsf{A}(b)$. Thus A is a functor with respect to both structures.

We note that the second requirement has clear intuitive meaning. If $\mathsf{A}(a)$ is a category encoding all of the processes taking place locally to the region a, then evidently we can view each of these processes as a taking place in any larger region, b, containing a.

Given a dagger net, of immediate concern is the *Grothendieck category,* which can be formed whenever one has a functor from a category into a category of categories. So let $F : \mathcal{C} \longrightarrow \mathsf{CAT}$ be such a functor. The objects of $\mathbf{G}(F)$ are pairs (a, B) with a an object of $\mathcal{C}$ and B an object of $F(a)$. An arrow $(a, B) \longrightarrow (a', B')$ is a pair (f, g) with $f : a \longrightarrow a'$ in $\mathcal{C}$ and $g : F(f)(B) \longrightarrow B'$ in $F(a')$.

The category $\mathbf{G}(\mathsf{A})$ will be the category in which quantum protocols will be encoded.

12.3.1 The case of binary sups

We wish to make K a monoidal category. We also assume throughout that the causal ordering has binary sups. (We will deal with the case this fails next.) We will define $U \otimes V$ as the sup of U and V *with respect to the EM-ordering.* So if $U = [a, a']$ and $V = [b, b']$, then $U \otimes V = [a \vee b, a' \vee b']$. The empty set is the tensor unit. (This is purely formal.)

Lemma 12.3.2. This makes K a symmetric monoidal poset (with respect to EM ordering).

We now note that when K is monoidal, so is $\mathbf{G}(\mathsf{A})$. This result seems to be new.

Lemma 12.3.3. $\mathbf{G}(\mathbf{A})$ is monoidal. The tensor is given by $(a, B) \otimes (a', B') = (a \otimes a', \mathbf{A}(i_a)(B) \otimes \mathbf{A}(i_{a'})(B'))$. Note that the strong monoidality of the functors in **DAG** is needed here. Here, i_a denotes the inequality $a \leq a \otimes a'$. The tensor unit is $(\emptyset, I_{\mathbf{A}(\emptyset)})$ where $I_{\mathbf{A}(\emptyset)} \in |\mathbf{A}(\emptyset)|$ is the tensor unit in the compact closed dagger category $\mathbf{A}(\emptyset)$.

Proof. Let (a, B), (a', B'), and (a'', B'') be objects in $\mathbf{G}(\mathbf{A})$. Then we need to construct a natural isomorphism $\alpha = \alpha_{(a,B),(a',B'),(a'',B'')} : ((a, B) \otimes (a', B')) \otimes (a'', B'') \longrightarrow (a, B) \otimes ((a', B') \otimes (a'', B''))$. We have:

$$((a, B) \otimes (a', B')) \otimes (a'', B'') = (a \otimes a', \mathbf{A}(i_a)(B) \otimes \mathbf{A}(i_{a'})(B')) \otimes (a'', B'')$$

$$= (a \otimes a' \otimes a'', \mathbf{A}(i_{a \otimes a'})(\mathbf{A}(i_a)(B) \otimes \mathbf{A}(i_{a'})(B')) \otimes \mathbf{A}(j_{a''})(B''))$$

$$= (a \otimes a' \otimes a'', (\mathbf{A}(i_{a \otimes a'} i_a)(B) \otimes \mathbf{A}(i_{a \otimes a'} i_{a'})(B')) \otimes \mathbf{A}(j_{a''})(B''))$$

.

Thus

$$((a, B) \otimes (a', B')) \otimes (a'', B'') = (a \otimes a' \otimes a'', (\mathbf{A}(j_a)(B) \otimes \mathbf{A}(j_{a'})(B')) \otimes \mathbf{A}(j_{a''})(B'')).$$

On the other hand

$$(a,B)\otimes((a',B')\otimes(a'',B''))=(a,B)\otimes(a'\otimes a'',\mathbf{A}(k_{a'})(B')\otimes\mathbf{A}(k_{a''})(B''))$$

$$=(a\otimes a'\otimes a'',\mathbf{A}(j_a)(B)\otimes\mathbf{A}(i_{a'\otimes a''})(\mathbf{A}(k_{a'})(B')\otimes\mathbf{A}(k_{a''})(B'')))$$

$$=(a\otimes a'\otimes a'',\mathbf{A}(j_a)(B)\otimes(\mathbf{A}(i_{a'\otimes a''}k_{a'})(B')\otimes\mathbf{A}(i_{a'\otimes a''}k_{a''})(B''))).$$

The last equality follows from the fact that $\mathbf{A}(i_{a'\otimes a''})$ is a strong monoidal functor. We have used $i_{a'\otimes a''}$ to denote the inequality $a'\otimes a''\leq a\otimes a'\otimes a''$, and $k_{a'}$ denotes $a'\leq a'\otimes a''$ and $k_{a''}$ denotes the inequality $a''\leq a'\otimes a''$. With these definitions one sees that $i_{a'\otimes a''}k_{a'}=j_{a'}$ and $i_{a'\otimes a''}k_{a''}=j_{a''}$. Therefore we have

$$(a,B)\otimes((a',B')\otimes(a'',B''))=(a\otimes a'\otimes a'',\mathbf{A}(j_a)(B)\otimes(\mathbf{A}(k_{a'})(B')\otimes\mathbf{A}(k_{a''})(B'')))$$

We see that the first components of both objects are the same. Thus an arrow from one to the other in $\mathbf{G}(\mathbf{A})$ will simply be an arrow in the category $\mathbf{A}(a\otimes a'\otimes a'')$. Now as $\mathbf{A}(a\otimes a'\otimes a'')$ is a monoidal category there is a natural isomorphism $\theta^{a,a',a''}_{X,Y,Z}$: $(X\otimes Y)\otimes Z\longrightarrow X\otimes(Y\otimes Z)$ for all objects X, Y, and Z satisfying the usual coherence conditions. Thus we define $\alpha_{(a,B),(a',B'),(a'',B'')}$: $((a,B)\otimes(a',B'))\otimes(a'',B'')\longrightarrow(a,B)\otimes((a',B')\otimes(a'',B''))$ by

$$\alpha_{(a,B),(a',B'),(a'',B'')}=\left(id,\theta_{\mathbf{A}(k_a)(B),\mathbf{A}(k_{a'})(B'),\mathbf{A}(k_{a''})(B'')}\right).\tag{45}$$

We must also construct natural isomorphisms $\lambda=\lambda_{(a,B)}:(\emptyset,I_{\mathbf{A}(\emptyset)})\otimes(a,B)\longrightarrow(a,B)$ and $\rho=\rho_{(a,B)}:(a,B)\otimes(\emptyset,I_{\mathbf{A}(\emptyset)})\longrightarrow(a,B)$ for all objects (a,B) in $\mathbf{G}(\mathbf{A})$. Let s_a denote the inequality $\emptyset\leq a$, then a quick calculation shows that $(\emptyset,I_{\mathbf{A}(\emptyset)})\otimes(a,B)=(a,\mathbf{A}(s_a)(I_{\mathbf{A}(\emptyset)})\otimes B)$. Similarly $(a,B)\otimes(\emptyset,I_{\mathbf{A}(\emptyset)})=(a,B\otimes\mathbf{A}(s_a)(I_{\mathbf{A}(\emptyset)}))$. In addition since $\mathbf{A}(s_a)$ is a strong monoidal functor the object $\mathbf{A}(s_a)(I_{\mathbf{A}(\emptyset)})$ is the tensor unit in $\mathbf{A}(a)$.

Similarly there exist natural isomorphisms $L^a_X:I_{\mathbf{A}(a)}\otimes X\longrightarrow X$ and R^a_X : $X\otimes I_{\mathbf{A}(a)}\longrightarrow X$, where $I_{\mathbf{A}(a)}$ is the tensor unit in $\mathbf{A}(a)$, satisfying the usual coherence conditions. So define $\lambda_{(a,B)}=(id,L^a_B)$ and $\rho_{(a,B)}=(id,R^a_B)$. The coherence conditions for α, λ, and ρ follow from the fact that each diagram amounts to one in which every object is of the form $(x, -)$ where the first coordinate is the same. Thus commutativity reduces to an instance of the monoidal coherence axioms in the category $\mathbf{A}(x)$.

It should also be verified that this operation $\otimes$ extends to a bifunctor. Indeed let $s : (b, B) \longrightarrow (b', B')$ and $t : (c, C) \longrightarrow (c', C')$ be arrows in $\mathbf{G}(\mathbf{A})$. Let's agree that if $x \leq y$ in K then we denote this arrow by i_x^y. Then it follows that s is of the form $s = (i_b^{b'}, f)$ where $f : (\mathsf{A}i_b^{b'})(B) \longrightarrow B'$ is an arrow in $\mathsf{A}(b')$ and similarly we have that $t = (i_c^{c'}, g)$ where $g : \mathsf{A}(i_c^{c'})(C) \longrightarrow C'$ is an arrow in $\mathsf{A}(c')$. Then we will define

$$\left(b \otimes c, (\mathsf{A}i_b^{b\otimes c})(B) \otimes (\mathsf{A}i_c^{b\otimes c})(C)\right) \xrightarrow{s\otimes t} \left(b' \otimes c', (\mathsf{A}i_{b'}^{b'\otimes c'})(B') \otimes (\mathsf{A}i_{c'}^{b'\otimes c'})(C')\right)$$

as follows. First we must have that $s \otimes t = (i_{b\otimes c}^{b'\otimes c'}, h)$ where

$$h : (\mathsf{A}i_{b\otimes c}^{b'\otimes c'})[(\mathsf{A}i_b^{b\otimes c})(B) \otimes (\mathsf{A}i_c^{b\otimes c})(C)] \longrightarrow (\mathsf{A}i_{b'}^{b'\otimes c'})(B') \otimes (\mathsf{A}i_{c'}^{b'\otimes c'})(C')$$

is an arrow in $\mathsf{A}(b' \otimes c')$. By strictness of the monoidal functors it follows that the domain of h in the above expression reduces to

$$(\mathsf{A}i_b^{b'\otimes c'})(B) \otimes (\mathsf{A}i_c^{b'\otimes c'})(C).$$

Now if we apply the functor $\mathsf{A}(i_{b'}^{b'\otimes c'}) : \mathsf{A}(b') \longrightarrow \mathsf{A}(b' \otimes c')$ to the arrow $f : (\mathsf{A}i_b^{b'})(B) \longrightarrow B'$ we get the arrow

$$\mathsf{A}(i_{b'}^{b'\otimes c'})(f) : (\mathsf{A}i_b^{b'\otimes c'})(B) \longrightarrow \mathsf{A}(i_{b'}^{b'\otimes c'})(B').$$

Similarly using the functor $\mathsf{A}(i_{c'}^{b'\otimes c'}) : \mathsf{A}(c') \longrightarrow \mathsf{A}(b' \otimes c')$ applied to the arrow $g : \mathsf{A}(i_c^{c'})(C) \longrightarrow C'$ yields

$$\mathsf{A}(i_{c'}^{b'\otimes c'})(g) : \mathsf{A}(i_c^{b'\otimes c'})(C) \longrightarrow \mathsf{A}(i_{c'}^{b'\otimes c'})(C').$$

Thus we define h by

$$h = \mathsf{A}(i_{b'}^{b'\otimes c'})(f) \otimes \mathsf{A}(i_{c'}^{b'\otimes c'})(g).$$

Now routine calculations show that the maps α, λ, and ρ are natural transformations and that $\otimes$ is a bifunctor. Hence $\mathbf{G}(\mathbf{A})$ is a monoidal category. $\square$

12.3.2 The general case

The existence of a monoidal structure on $\mathbf{G}(\mathsf{A})$ puts a severe restriction on the types of spacetimes being considered. But in fact to interpret protocols, for example, the quantum teleportation protocol, one need only take tensors "locally" as we now describe. Define a ternary relation T by

$$T = \{(a, b, c) | a, b \leq c \in \mathsf{K}\}$$

Then if $(a, b, c) \in T$, we define:

$$(a, B) \otimes_c (b, B') = (c, \mathsf{A}(i_a^c)(B) \otimes \mathsf{A}(i_b^c)(B'))$$

Now the map i_a^c refers to $a \leq c$ similarly for i_b^c.

Lemma 12.3.4.

- If $(a, b, c) \in T$ and $(a, b, c') \in T$ with $c \leq c'$, there is a canonical map $(a, B) \otimes_c (b, B') \longrightarrow (a, B) \otimes_{c'} (b, B')$.
- If $(a, b, c) \in T$ and if $(a, b', c) \in T$ with $b \leq b'$, then there is a canonical map $(a, B) \otimes_c (b, B') \longrightarrow (a, B) \otimes_c (b', B')$.
- If $(a, b, c) \in T$ and $(a', b, c) \in T$ with $a \leq a'$, then there is a canonical map $(a, B) \otimes_c (b, B') \longrightarrow (a', B) \otimes_c (b, B')$.

12.4 Encoding Protocols in a Causal Set

In [1] the authors establish a categorical framework that captures many of the main ingredients present in finite-dimensional quantum mechanics. In addition the authors go on to show that many quantum protocols can be modelled in their setting of "semi-additive dagger compact closed categories". What we aim to do in this section is to add to their framework by explicitly involving spacetime through the use of causal dagger nets.

We now give a brief description of abstract quantum mechanics. A more detailed account of the *abstract quantum mechanics* of Abramsky and Coecke can be found in §8 of [1]. Let $(\mathcal{C}, \otimes, \alpha, \lambda, \rho, \sigma, I, *, \dagger)$ be a dagger compact closed category with biproducts.

Remark 12.4.1. In the setting of a compact closed dagger category the dagger structure is denoted by $\dagger$ rather than by $*$. Moreover the symbol $*$ is used in this context to denote the conjugate structure, i.e. each object has a conjugate object which is denoted A^* instead of $\overline{A}$.

- State spaces are represented by objects in $\mathcal{C}$
- A *basis* for A consists of a unitary isomorphism $\text{base}_A : n \cdot I \longrightarrow A$ where $n \cdot I = \underbrace{I \oplus \cdots \oplus I}_{n \text{ copies}}$
- The *qubit state space* consists of an object Q and a unitary isomorphism $\text{base}_Q : I \oplus I \longrightarrow Q$
- Given state spaces A and B then $A \otimes B$ is the state space describing the compound system and $\text{base}_{A\otimes B} = (\text{base}_A \otimes \text{base}_B) \circ d^{-1}_{n,m}$ where $d_{n,m} : (n \cdot I) \otimes (m \cdot I) \simeq (nm) \cdot I$ comes from the distributivity isomorphisms of $\otimes$ over $\oplus$
- A *teleportation base* consists of a map $s : I \longrightarrow I$ together with a map $\text{prebase}_T : 4 \cdot I \longrightarrow Q^* \otimes Q$ such that:
  1. $\text{base}_T \equiv s \bullet \text{prebase}_T$ is unitary where $\bullet$ is defined as follows. Suppose that we have maps $r : I \longrightarrow I$ and $f : A \longrightarrow B$ then there is a map $r \bullet f : A \longrightarrow B$ given by
  $$A \xrightarrow{\lambda^{-1}} I \otimes A \xrightarrow{r \otimes f} I \otimes B \xrightarrow{\lambda} B \qquad (46)$$
  2. The four maps $\beta_j : Q \longrightarrow Q$ defined by $\ulcorner \beta_j \urcorner \equiv \text{prebase}_T \circ q_j$ are unitary. Here $q_j : I \longrightarrow 4 \cdot I$ is the canonical coproduct injection onto the jth factor and we are using the fact that in any compact closed category there is a bijection $\mathcal{C}(A, B) \simeq \mathcal{C}(I, A^* \otimes B)$ given by sending a map $f : A \longrightarrow B$ to its *name* $\ulcorner f \urcorner = (id_{A^*} \otimes f) \circ \eta_A$.

3. $2s^\dagger s = id_I$.

- Suppose that we have a teleportation base as above then it defines a *teleportation observation*

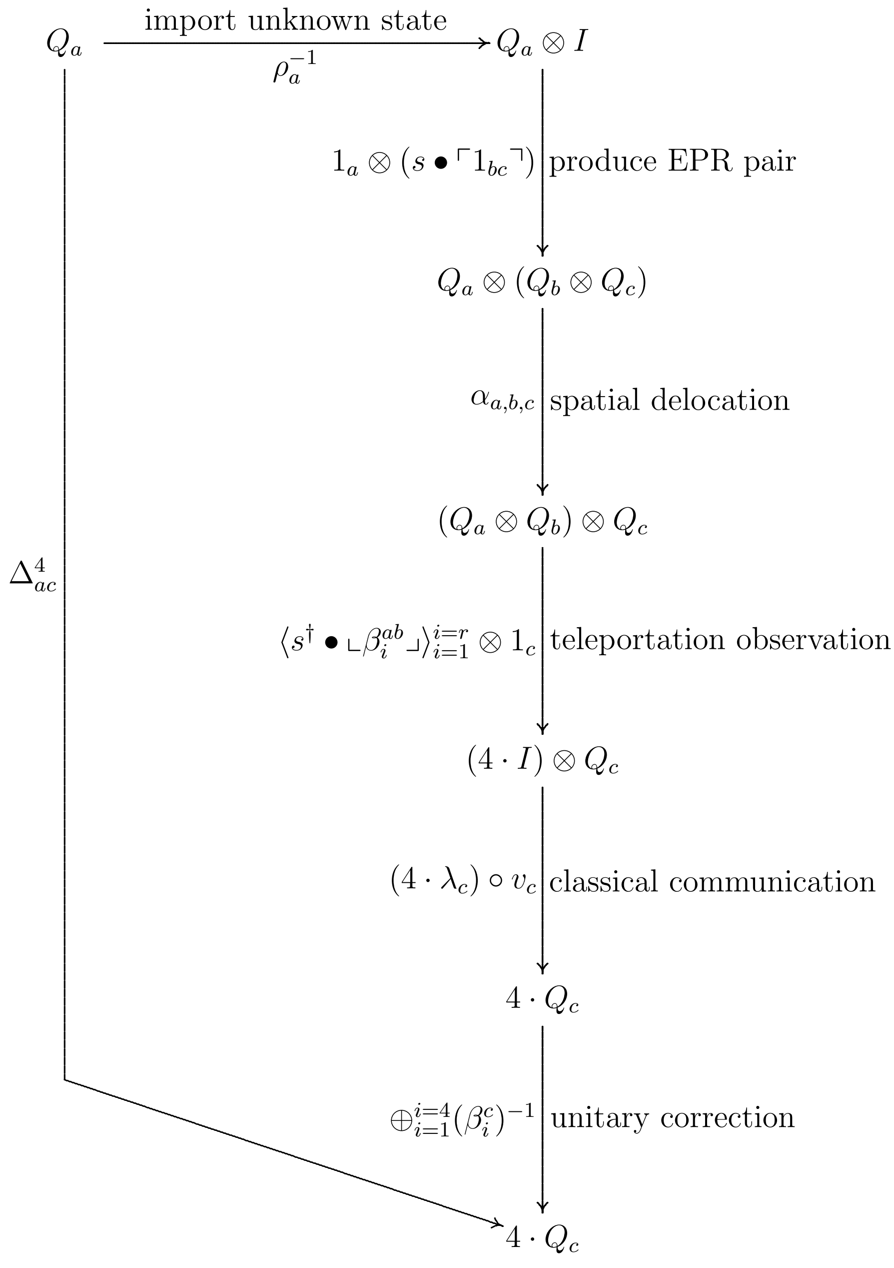
$$\langle s^\dagger \bullet \llcorner \beta_i \lrcorner \rangle_{i=1}^{i=r} : Q \otimes Q^* \longrightarrow 4 \cdot I.$$

Now we wish illustrate the teleportation protocol, as described by Abramsky and Coecke in [1] §9, so that we can refer to it later. Let $\mathcal{C}$ be a dagger compact closed category with biproducts and suppose that it admits a teleportation base $(s : I \longrightarrow I, \text{prebase}_T : 4 \cdot I \longrightarrow Q^* \otimes Q)$. Now we will be considering the teleportation protocol which involves 3 qubits whose state spaces are all equal to Q but we will label them by Q_a, Q_b, and Q_c respectively to distinguish which qubit we are referring to. Now as before we have maps $\beta_j : Q \longrightarrow Q$ which we will label as $\beta_j^{ab} : Q_a \longrightarrow Q_b$ to indicate which qubits are involved and similarly for identities $1_{ab} : Q_a \longrightarrow Q_b$. The final piece of the teleportation protocol is the *labelled, weighted diagonal* which is the map:

$$\Delta_{ac}^4 \equiv \langle s^\dagger s \bullet 1_{ac} \rangle_{i=1}^{i=4} : Q_a \longrightarrow 4 \cdot Q_c.$$

According to [1] this map expresses the appropriate behaviour of the teleportation protocol which is that "...the input qubit is propagated to the output along each branch of the protocol, with 'weight' $s^\dagger s$, corresponding to the probability amplitude for that branch."

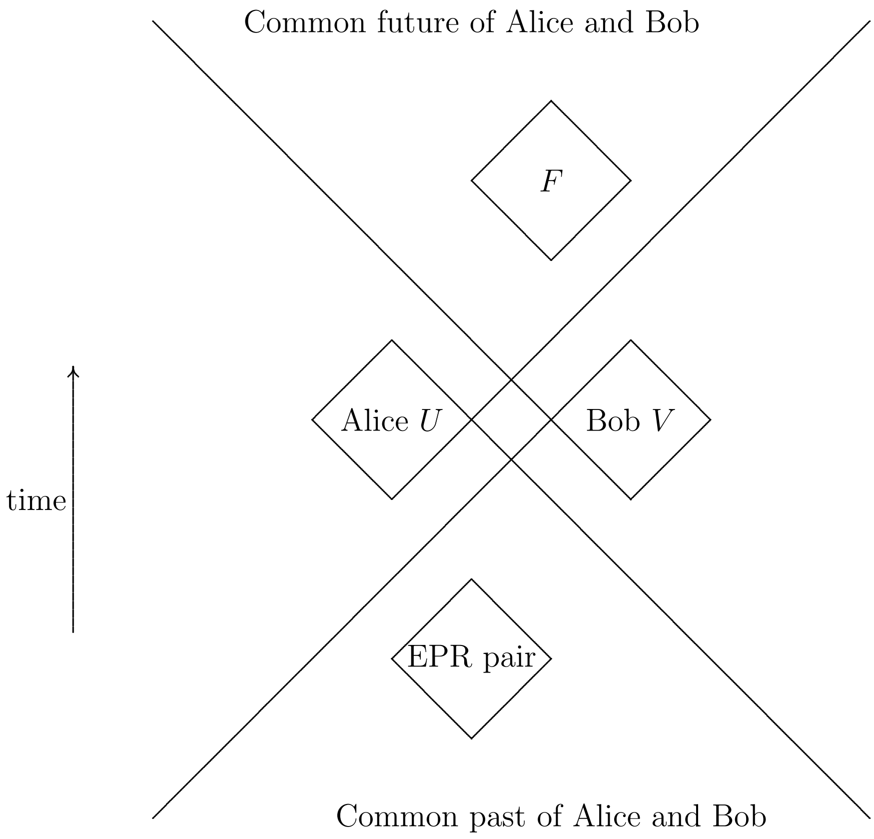
Theorem 12.4.2. The following diagram commutes.



The right-hand side of the above diagram is the formal description of the teleportation protocol, while the commutativity of the diagram shows that the protocol behaves as it should.

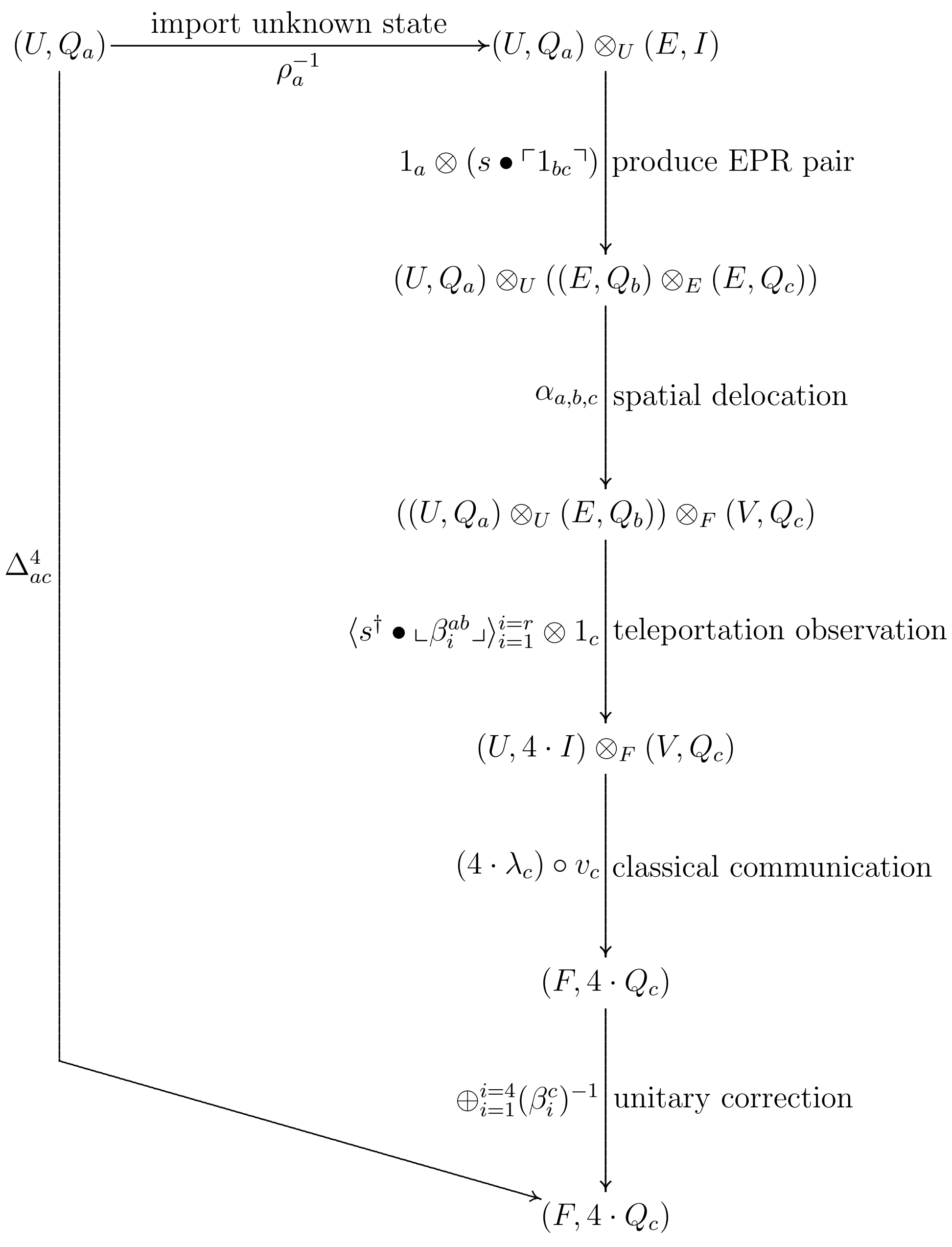
Suppose that $\mathbf{A} : K \longrightarrow \mathbf{DAG}$ is a causal dagger net and that each category

$\mathbf{A}(U)$ is a dagger compact closed category with biproducts. Here we use the symbol $\dagger$ to denote the dagger functor and we use $*$ to denote the compact structure on the category $\mathbf{A}(U)$. We denote the unit and counit by $\eta_X : I \longrightarrow X^* \otimes X$ and $\epsilon_X : X \otimes X^* \longrightarrow I$ respectively. Furthermore we will also assume that these categories have strict compact structure in the sense that $A^{**} = A$, $(A \otimes B)^* = A^* \otimes B^*$, and $I^* = I$. If $\mathbf{G}(\mathbf{A})$ is the associated Grothendieck category, then we have a functor $\pi : \mathbf{G}(\mathbf{A}) \longrightarrow K$ given by $\pi(U, A) = U$ and, given an arrow (f, g), then $\pi(f, g) = f$. Furthermore given any $U \in K$ then the fibre over U, $\pi^{-1}(U)$, is equivalent to $\mathbf{A}(U)$. We will also assume that if $U \leq V \in K$ then $\mathbf{A}(U) \subseteq \mathbf{A}(V)$ and that each category $\mathbf{A}(U)$ admits a teleportation base. Suppose that Alice is in possession of a qubit that she wishes to transmit to Bob. We will illustrate the sequence of events with a diagram of double cones.



Alice is in possession of a qubit, then sometime in Alice and Bob's past an EPR pair was created. After they share this entangled pair of qubits, Alice gets one of these entangled qubits while Bob gets the other one. Then Alice performs a *Bell base measurement* on her system of two qubits and then sends Bob the result of this experiment by means of classical communication. Thus Bob receives this message in the common future of Alice and Bob. It is then that Bob can use this measurement result to perform a unitary correction on his qubit thus obtaining a qubit whose state is that of the original qubit possessed by Alice. We give a formal description of the teleportation protocol using the Grothendieck category $\mathbf{G}(\mathbf{A})$.

Let E denote the double cone in which the EPR pair was created, U and V be the double cones that Alice and Bob are located in during some given time interval. Lastly let F be any double cone which is in the common future of Alice and Bob in which Bob receives a classical communication from Alice. Then the above diagram suggests the following must hold $E \leq U, V \leq W$. The right-hand side of the diagram below is our formal description of the teleportation protocol.



Note that we have omitted writing morphisms in the above diagrams as pairs (f, g) since there is only at most one map $U \longrightarrow V$ in K. Now we claim that the above diagram commutes. This follows immediately from the above theorem, Theorem 12.4.2.

Chapter 13

Local DHR Analysis

In the previous section, we considered the partially ordered set $(M, \leq)$ of spacetime and looked at some of the different possibilities for lifting this order to subsets. In this section we will look some different possibilities on how to incorporate this causal ordering on subsets of M into AQFT. In addition to this, the other main thread of this chapter is to look at AQFT on a local level in the sense that each double cone is a spacetime in its own right. Formulating local versions of notions such as *transportable endomorphism* we look at how causality can be used to relate different double cones and the associated local theories.

13.1 Localized Transportable Endomorphisms

To fix our notation let M denote Minkowski space, and let $(K, \subseteq)$ denote the set of double cones on M ordered under inclusion. In addition let $\mathbb{A} : (K, \subseteq) \longrightarrow \mathbf{C^*} - \mathbf{Alg}$ be an AQFT satisfying isotony and microcausality, and also as usual let $\hat{\mathbb{A}}$ denote the quasi-local algebra of observables.

Now for each $U \in K$ let $K_U = \{O \in K \mid O \subsetneq U\}$. Also for each $O \in K_U$ let $\mathbb{A}_U(O^\perp)$ be the C^*-algebra generated by the set

$$\bigcup_{\substack{O' \in K_U \\ O' \perp O}} \mathbb{A}(O').$$

Definition 13.1.1. A $*$-homomorphism $\rho : \mathbb{A}(U) \longrightarrow \mathbb{A}(U)$, $U \in K$, is **localized at** $O \in K_U$ if $\rho(a) = a$ for all $a \in \mathbb{A}_U(O^\perp)$. We say that an endomorphism $\rho : \mathbb{A}(U) \longrightarrow \mathbb{A}(U)$ is **localized in** U if there is an $O \in K_U$ at which ρ is localized.

Within the class of endomorphisms localized in U are the ones which are also *transportable within* U.

Definition 13.1.2. Let $U \in K$, and suppose $\rho : \mathbb{A}(U) \longrightarrow \mathbb{A}(U)$ is localized in U. Then ρ is **transportable in** U if for each $O \in K_U$ $\exists\, \tau : \mathbb{A}(U) \longrightarrow \mathbb{A}(U)$ localized at O and a unitary element $x \in \mathbb{A}(U)$ such that

$$\rho(a) = x^* \tau(a) x \quad \forall a \in \mathbb{A}(U). \tag{47}$$

For our convenience we let $\Delta_U(O)$ denote the collection of endomorphisms $\mathbb{A}(U) \longrightarrow \mathbb{A}(U)$ which are localized at $O \in K_U$ and transportable in U and we let $\Delta_U = \cup_{O \in K_U} \Delta_U(O)$.

These definitions lead one to consider an important category whose set of objects is Δ_U for U in K. The construction of this category is an instance of the following general result. Let R be a ring with unity and $S \subseteq End(R)$ then there is a category $End_S(R)$ whose objects are elements of S and an arrow $r : f \longrightarrow g$ consists of an element $r \in R$ such that $rf(r') = g(r')r$ for each $r' \in R$.

Lemma 13.1.3. *$End_S(R)$ is a category. If S is a monoid with respect to composition then $End_S(R)$ is a monoidal category.*

Thus by Lemma 13.1.3 it follows that $End_{\Delta_U}(\mathbb{A}(U))$ is a category. In fact more is true.

Lemma 13.1.4. *Let $U \in K$ then Δ_U is a monoid with respect to composition.*

Proof. If $\rho \in \Delta_U$ and $\sigma \in \Delta_U$ are localized at O_1 and O_2 respectively then their composite $\rho \circ \sigma$ will be localized at any $O_3 \supseteq O_1 \cup O_2$. Now as $O_1 \cup O_2 \subsetneq U$ it follows by properties of Minkowski space that there exists a double cone $O_3 \subsetneq U$ with $O_1 \cup O_2 \subseteq O_3$. Transportability within U of the composite is also easily verified. If

$O \subsetneq U$ then there exists x, $y \in \mathbb{A}(U)$ and $*$-homomorphisms ρ', $\sigma' : \mathbb{A}(U) \longrightarrow \mathbb{A}(U)$ localized at O satisfying Equation 47. Then the following calculation

$$\begin{aligned}\rho(\sigma(a)) &= x^*\rho'(\sigma(a))x \\ &= x^*\rho'(y^*\sigma'(a)y)x \\ &= x^*\rho'(y^*)\rho'\sigma'(a)\rho'(y)x \\ &= (\rho'(y)x)^*\rho'\sigma'(a)\rho'(y)x \ \forall a \in \mathbb{A}(U)\end{aligned}$$

shows that $\rho\sigma$ is transportable in U. □

Hence $End_{\Delta_U}(\mathbb{A}(U))$ is a monoidal category for each $U \in K$ called the category of localized and transportable endomorphisms in U.

Now assume that K is also equipped with a second partial order $\sqsubseteq$ which we interpret as a causal order. Then we wish to answer the following question. If $V \sqsubseteq U$ then what additional data are needed to construct a functor between the categories $End_{\Delta_V}(\mathbb{A}(V))$ and $End_{\Delta_U}(\mathbb{A}(U))$? We now give one possible solution to this question. First we make a couple of definitions.

Definition 13.1.5. If U, $V \in K$, and $V \sqsubseteq U$ then a **causal connection** from V to U consists of a pair of functions $\Gamma : K_V \longrightarrow K_U$ and $\Lambda : K_U \longrightarrow K_V$ which are monotone with respect to $\subseteq$ and also satisfy

$$O \subseteq \Lambda\Gamma O \quad \forall O \in K_V \tag{48}$$

$$\Gamma\Lambda O' \subseteq O' \quad \forall O' \in K_U. \tag{49}$$

In other words Λ and Γ are functors with Λ left adjoint to Γ. We will abbreviate this by just writing $(\Lambda \dashv \Gamma)$.

The second definition we need is one that relates the algebras $\mathbb{A}(V)$ and $\mathbb{A}(U)$.

Definition 13.1.6. Suppose $V \sqsubseteq U \in K$ and $(\Lambda \dashv \Gamma)$ is a causal connection. An **embedding projection pair** with respect to $(\Lambda \dashv \Gamma)$ consists of a pair of $*$-homomorphisms (f,g) with $f : \mathbb{A}(U) \longrightarrow \mathbb{A}(V)$ and $g : \mathbb{A}(V) \longrightarrow \mathbb{A}(U)$ satisfying

$$\textit{if } a \in \mathbb{A}_U((\Gamma O)^\perp) \textit{ then } f(a) \in \mathbb{A}_V((\Lambda\Gamma O)^\perp) \tag{50}$$

$$gf(a) = a \quad \forall a \in \mathbb{A}_U((\Gamma O)^\perp). \tag{51}$$

Theorem 13.1.7. Suppose that V, $U \in K$, $(\Lambda \dashv \Gamma)$, and (f,g) are as in Definition 13.1.6. Then given a $*$-homomorphism $\rho : \mathbb{A}(V) \longrightarrow \mathbb{A}(V)$ define $F(\rho) : \mathbb{A}(U) \longrightarrow \mathbb{A}(U)$ by

$$F(\rho)(a) = g\rho f(a) \quad \forall a \in \mathbb{A}(U).$$

If $\rho \in \Delta_V$ then $F(\rho) \in \Delta_U$. In particular $F(\rho) \in \Delta_U(\Gamma O)$ for $\rho \in \Delta_V(O)$.

Proof. Let $a \in \mathbb{A}_U((\Gamma O)^\perp)$ then by assumption $f(a) \in \mathbb{A}_V((\Lambda\Gamma O)^\perp)$. In addition, since $O \subseteq \Lambda\Gamma O$ we have that $\mathbb{A}_V((\Lambda\Gamma O)^\perp) \subseteq \mathbb{A}_V(O^\perp)$. Hence $f(a) \in \mathbb{A}_V(O^\perp)$ and so for $\rho \in \Delta_V(O)$ we have that $\rho(f(a)) = f(a)$ $\forall a \in \mathbb{A}_U((\Gamma O)^\perp)$. Therefore for $a \in \mathbb{A}_U((\Gamma O)^\perp)$

$$\begin{aligned} F(\rho)(a) &= g\rho(f(a)) \\ &= gf(a) \\ &= a \qquad \text{since } a \in \mathbb{A}_U((\Gamma O)^\perp) \end{aligned}$$

Hence $F(\rho)$ is localized at ΓO. It remains to show that $F(\rho)$ is transportable in U. Let $O' \in K_U$, then as ρ is transportable in V there exists $\tau \in \Delta_V(\Lambda O')$ and a unitary element $x \in \mathbb{A}(V)$ such that

$$\rho(a) = x^*\tau(a)x \qquad \forall a \in \mathbb{A}(V).$$

So

$$\begin{aligned} F(\rho)(a) &= g(\rho(f(a))) \\ &= g(x^*\tau(f(a))x) \\ &= g(x)^* g\tau f(a) g(x) \\ &= g(x)^* F(\tau)(a) g(x) \qquad \text{as required.} \end{aligned}$$

Lastly we need check that the endomorphism $F(\tau)$ is localized at O'. Indeed, since $\Gamma\Lambda O' \subseteq O'$ it follows that $\mathbb{A}_U((O')^\perp) \subseteq \mathbb{A}_U((\Gamma\Lambda O')^\perp)$. Thus if $a \in \mathbb{A}_U((O')^\perp)$ then $a \in \mathbb{A}_U((\Gamma\Lambda O')^\perp)$. So $f(a) \in \mathbb{A}_V((\Lambda\Gamma\Lambda O')^\perp)$, but $\mathbb{A}_V((\Lambda\Gamma\Lambda O')^\perp) \subseteq \mathbb{A}_V((\Lambda O')^\perp)$ as $\Lambda O' \subseteq \Lambda\Gamma\Lambda O'$. Therefore $f(a) \in \mathbb{A}_V((\Lambda O')^\perp)$, but τ is localized at $\Lambda O'$ so

$$\begin{aligned} F(\tau)(a) &= g\tau f(a) \\ &= gf(a) \\ &= a \qquad \text{since } a \in \mathbb{A}_U((\Gamma\Lambda O')^\perp) \end{aligned}$$

Hence $F(\tau)$is localized at O'. □

Using Theorem 13.1.7 we can define a functor $\mathcal{F} : End_{\Delta_V}(\mathbb{A}(V)) \longrightarrow End_{\Delta_U}(\mathbb{A}(U))$ which on an object $\rho \in \Delta_V$ is $\mathcal{F}(\rho) = F(\rho)$ and if $r : \rho \longrightarrow \gamma$ is an arrow in $End_{\Delta_V}(\mathbb{A}(V))$ then $\mathcal{F}(r) = g(r)$ is and arrow form $F(\rho)$ to $F(\gamma)$ in $End_{\Delta_U}(\mathbb{A}(U))$.

Lemma 13.1.8. If $fg : \mathbb{A}(V) \longrightarrow \mathbb{A}(V)$ is localized in V and transportable in V to the identity. Then for ρ and $\gamma \in \Delta_V$ there is an isomorphism $m_{\rho,\gamma} : F(\rho\gamma) \longrightarrow F(\rho)F(\gamma)$.

Proof. Since fg is transportable in V to the identity, we have that there exists a unitary $c \in \mathbb{A}(V)$ such that $fg(a) = c^*ac$ for all $a \in \mathbb{A}(V)$. Thus

$$\begin{aligned} F(\rho)F(\gamma)(a) &= g\rho(fg(\gamma f(a))) \\ &= g\rho(c^*\gamma(f(a))c) \\ &= g\rho(c)^* g\rho\gamma(f(a))g\rho(c) \\ &= g\rho(c)^* F(\rho\gamma)(a)g\rho(c) \end{aligned}$$

So $m_{\rho,\gamma}F(\rho\gamma)(a) = F(\rho)F(\gamma)(a)m_{\rho,\gamma}$ for $m_{\rho,\gamma} = g\rho(c)^*$ which is unitary. □

13.2 DHR Representations

In the previous section we were considering localized transportable endomorphisms which were localized and transportable within some "region" $U \in K$. We then constructed the category $End_{\Delta_U}(\mathbb{A}(U))$ and established a functor between two of these categories under certain assumptions. We now wish to provide a more general construction for doing this. In order to proceed we will need to consider a different category $\mathbf{DHR}(U)$ for $U \in K$, which we will show is related to $End_{\Delta_U}(\mathbb{A}(U))$. We start by describing the objects of $\mathbf{DHR}(U)$. Let (π_0, H_0) be a fixed $*$-representation of $\hat{\mathbb{A}}$ which will we refer to as the *vacuum representation*.

Definition 13.2.1. Let $U \in K$ and (π, H_π) be a $*$-representation of $\mathbb{A}(U)$. (π, H_π) is a **DHR-representation in** U if for every $O \in K_U$ there exists a unitary map $T_O : H_\pi \longrightarrow H_0$ such that

$$T_O \pi(a)(h) = \pi_0(a) T_O(h) \quad \forall a \in \mathbb{A}_U(O^\perp),\ h \in H_\pi \tag{52}$$

Thus define $\mathbf{DHR}(U)$ to be the category whose objects are **DHR**-representations in U and arrows are bounded linear intertwining maps.

Theorem 13.2.2. For each $U \in K$ there is a functor $End_{\Delta_U}(\mathbb{A}(U)) \longrightarrow \mathbf{DHR}(U)$.

Proof. Given an endomorphism $\rho : \mathbb{A}(U) \longrightarrow \mathbb{A}(U)$ localized and transportable in U then we define $\mathcal{E}(\rho) : \mathbb{A}(U) \longrightarrow \mathbf{B}(H_0)$ by

$$\mathcal{E}(\rho)(a) = \pi_0(\rho(a)) \quad \forall a \in \mathbb{A}(U). \tag{53}$$

Then it is straightforward to check that $(\mathcal{E}(\rho), H_0)$ is a $*$-representation of $\mathbb{A}(U)$. It is a **DHR**-representation in U because ρ is localized and transportable in U. Now suppose that $r : \rho \longrightarrow \tau$ is an arrow in $End_{\Delta_U}(\mathbb{A}(U))$ then define $\mathcal{E}(r) : (\mathcal{E}(\rho), H_0) \longrightarrow (\mathcal{E}(\tau), H_0)$ by

$$\mathcal{E}(r) = \pi_0(r). \tag{54}$$

Then we have for each $a \in \mathbb{A}(U)$

$$\begin{aligned}\mathcal{E}(r)\mathcal{E}(\rho)(a) &= \pi_0(r)\pi_0(\rho(a))\\ &= \pi_0(r\rho(a))\\ &= \pi_0(\tau(a)r)\\ &= \pi_0(\tau(a))\pi_0(r)\\ &= \mathcal{E}(\tau)(a)\mathcal{E}(r)\end{aligned}$$

and hence $\mathcal{E}(r)$ is an intertwining map as required. Thus we have defined a functor $\mathcal{E} : End_{\Delta_U}(\mathbb{A}(U)) \longrightarrow \mathbf{DHR}(U)$. The functor equations follow from the fact that π_0 is $*$-homomorphism. □

Remark 13.2.3. We suspect that under the appropriate assumptions the functor that we have defined would turn out to be an equivalence of categories. Indeed what one needs is an analogous notion of Haag duality for this setting. Further, by assuming that the vacuum representation is faithful this would guarantee our functor $\mathcal{E}$ to be faithful. Fullness and essential surjectivity should follow from Haag duality.

Now given $V \sqsubseteq U \in K$ we want to build a functor $G : \mathbf{DHR}(V) \longrightarrow \mathbf{DHR}(U)$.

Definition 13.2.4. Suppose that $X = X_{\mathbb{A}(V)}$ is a right Hilbert $\mathbb{A}(V)$-module and $f : \mathbb{A}(U) \longrightarrow \mathcal{L}(X_{\mathbb{A}(V)})$ is a $*$-homomorphism. We say that the $\mathbb{A}(U)$-$\mathbb{A}(V)$ bimodule X is of **DHR-type** if for each $O' \in K_U$ there is a unitary $S_{O'} : X \otimes_{\mathbb{A}(V)} H_0 \longrightarrow H_0$ such that

$$\pi_0(a)S_{O'} = S_{O'}Ind\pi_0(a) \quad \forall a \in \mathbb{A}_U(O'^{\perp}) \tag{55}$$

i.e. $(Ind\pi_0,\ X \otimes_{\mathbb{A}(V)} H_0)$ is a **DHR**-representation in U.

Theorem 13.2.5. Suppose that $V \sqsubseteq U \in K$ and let $X_{\mathbb{A}(V)}$, and f be as in Definition 13.2.4 and let $\Lambda : K_U \longrightarrow K_V$ be a function. Then for any (π, H_π), a **DHR**-representation in V, $(Ind\pi, X \otimes_{\mathbb{A}(V)} H_\pi)$ is a **DHR**-representation in U.

Proof. Suppose that $(\pi,\ H_\pi)$ is a **DHR**-representation in V. We want to show that $(Ind\pi, X \otimes_{\mathbb{A}(V)} H_\pi)$ is a **DHR**-representation in U. Let $O' \in U$ and consider $\Lambda O'$ which is an element of K_V. So as (π, H_π) is a **DHR**-representation in V there exists a unitary map $T_{\Lambda O'} : H_\pi \longrightarrow H_0$ satisfying an equation analogous to that in Definition 13.2.1. Moreover since X is an $\mathbb{A}(U)$-$\mathbb{A}(V)$ bimodule of **DHR**-type we have another unitary map $S_{O'} : X \otimes_{\mathbb{A}(V)} H_0 \longrightarrow H_0$ satisfying the equation in Definition 13.2.4. To show that $(Ind\pi, X \otimes_{\mathbb{A}(V)} H_\pi)$ is a **DHR**-representation in U we will show that $S_{O'}(1 \otimes_{\mathbb{A}(V)} T_{\Lambda O'})Ind\pi(a) = \pi_0(a)S_{O'}(1 \otimes_{\mathbb{A}(V)} T_{\Lambda O'})$ for all $a \in \mathbb{A}(O'^{\perp})$. Indeed

$$
\begin{aligned}
S_{O'}(1 \otimes_{\mathbb{A}(V)} T_{\Lambda O'})Ind\pi(a) &= S_{O'}Ind\pi_0(a)(1 \otimes_{\mathbb{A}(V)} T_{\Lambda O'}) \\
&= \pi_0(a)S_{O'}(1 \otimes_{\mathbb{A}(V)} T_{\Lambda O'})
\end{aligned}
$$

as required. □

Now as $X \otimes_{\mathbb{A}(V)} (-)$ is a functor from the category of $*$-representations of $\mathbb{A}(V)$, $\mathbf{Rep}(\mathbb{A}(V))$, to the category of $*$-representations of $\mathbb{A}(U)$, $\mathbf{Rep}(\mathbb{A}(U))$, it follows that $X \otimes_{\mathbb{A}(V)} (-)$ restricts to a functor $\mathbf{DHR}(V) \longrightarrow \mathbf{DHR}(U)$ since these categories are both full subcategories $\mathbf{Rep}(\mathbb{A}(V))$ and $\mathbf{Rep}(\mathbb{A}(U))$ respectively.

Thus Theorem 13.2.5 provides some sufficient conditions for obtaining a functor $G : \mathbf{DHR}(V) \longrightarrow \mathbf{DHR}(U)$ whenever $V \sqsubseteq U$. The key ingredients are the existence of an $\mathbb{A}(U)$-$\mathbb{A}(V)$ bimodule X of DHR-type and of a function $\Lambda : K_U \longrightarrow K_V$. At this point it is not clear when these things might exist in general. Therefore the next step along this path would be to examine their existence when using one of the specific causal orderings that we mentioned in Chapter 12.2.

Chapter 14

Future Work

Initially the goal of this thesis was to try to establish a categorical version of algebraic quantum field theory. Inspired by the work of Abramsky and Coecke [1], we wanted our theory to incorporate not only elements from AQFT but also elements from their work such as a dagger structure and a tensor product structure. What has resulted from this effort is the development of our premonoidal C^*-quantum field theory. Along the way we explored many exciting research avenues that warrant further investigation. Of these we mention the ones that we find the most intriguing.

1. **von Neumann Categories:**

   (a) Our theory of von Neumann categories seems to be a very promising subject to pursue. As we alluded at the end of Section 9.3, von Neumann categories on $\mathbf{Hilb}_H$ may possibly be classified by the von Neumann algebras on H. This result would depend on an extension of the commutation theorem for tensor products of von Neumann algebras. An elementary proof of this classical result was given by Rieffel and van Daele [34], and it is our hope that by modifying their approach we can establish the required result.

   (b) With the help of such a result we would also be able to classify all the monoidal subcategories of $\mathbf{Hilb}_H$ in terms of the abelian von Neumann algebras on H.

(c) Develop a theory of factors for von Neumann categories.

(d) Mimic classical constructions, such as the crossed product construction, to produce more examples of von Neumann categories.

(e) Establish a double commutant theorem for VNC's.

2. **Premonoidal C^*-Quantum Field Theory:**

   (a) Show that the category Δ in our premonoidal setting can be equipped with a symmetry.

   (b) Look at categorical versions of other common axioms from AQFT such as Property B for example.

   (c) Look for ways of incorporating the theory of von Neumann categories in our theory, in the same way that von Neumann algebras fit into AQFT.

   (d) Investigate connections with other categorical theories of physics, in particular topos theory and its use in physical theories as studied by Doering, Isham, Butterfield, Landsman, ...etc. (See [11])

3. **Premonoidal Category Theory**

   (a) We already considered premonoidal categories with duals/conjugates. It would be interesting to develop this further and look at how this relates to Benton and Hyland's *traced premonoidal categories* [5].

   (b) Of course proving our conjecture about the existence of fibre functors in the premonoidal setting is one of our more pressing research goals. In conjunction with this, another goal is to prove the full Doplicher-Roberts reconstruction theorem in the premonoidal setting.

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