Frege on Ideal Language, Multiple Analyses, and Identity

OLYA HASHEMI SHAHROUDI

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Abstract

My thesis is partly historical and partly critical. First, I will lay down a brief overview of Frege's semantics and ontology based on his major works such as *Begriffsschrift*, "On Sense and Reference", "On Concept and Object", "Thought" and so on. The goal of this part of my research is to get a clear picture of Frege's system and see what features of it are arbitrary and in those cases what other options exist. Next, I will use the secondary literature to find out how other scholars have interpreted Frege. I intend to use some modern logical tools such as lambda calculus in order to analyze Frege's views more perspicuously. Then I will draw my conclusions about his ontology and semantics.

Frege seems to have believed that we approach the world via thought, but have no access to thought except via language. Hence, his enquiry into the nature of the world was conducted via enquiries into the nature of language. However, he knew that natural language sometimes muddles thought, so he tried to create an artificial formula language that would be able to capture the logical structure of the world itself as it is reflected in thought, better than any natural language. I will discuss the importance and the role of an ideal language in science, and will try to determine from Frege's scattered remarks what characteristics he thinks such a language must have. I will then consider Frege's own formalized language, as first presented in the 1879 *Begriffsschrift* and as further developed in his later writings. I will then discuss Frege's semantics, focussing mainly on his theory of multiple analyses and his notion of thought and conceptual content. Finally, I will provide a detailed study of Frege's theories of identity, in *Begriffsschrift* and "On Sense and Reference", that were critical in the development of his semantics and ontology.

Content

Introduction

Although language has been a matter of concern throughout the history of philosophy, Gottlob Frege was among the first philosophers who placed language at the centre of their philosophical investigations. Frege believed that if the language used in mathematics and science were replaced by an ideal language, both mathematics and science would gain the precision they require, and devising an ideal language became a key part of his philosophical investigations. What he came up with was a formula-language modelled on that of arithmetic and algebra. He called his ideal language *Begriffsschrift* or Conceptual Notation. He believed that his ideal language was able to represent the conceptual content of an expression precisely. However, in the course of constructing and investigating his ideal language, he developed further theories which give rise to at least apparent inconsistency in his logical system. My goal in writing this thesis is to explain Frege's ideal language and to provide a critical explanation of some of his theories such as the theory of multiple analyses, the theory of sense, and his theories of identity.

Frege's ideal language has only two main syntactical categories: proper names which designate objects and function names which designate functions. Frege claims that every expression expresses a sense which is different from what it designates (its reference). He calls the sense of a declarative sentence the *thought* it conveys. He further argues that there are different ways of dividing a sentence into function/argument names. Such variations in the analysis of a sentence appear to reveal

ambiguities in the formulas of the *Begriffsschrift*, making it look like Frege's ideal language is not as ideal as he thinks. Yet Frege clearly believed that these different analyses do not change the conceptual content or the thought of a sentence. One of the questions I will seek to answer is whether this view is defensible.

In the first chapter, I explain why Frege was keen to develop an ideal language, and then I lay out the main features of his ideal language. In the second chapter, I will interpret his theory of multiple analysis and his notion of conceptual content and thought, arguing that his views on these matters can be defended against claims of inconsistency. The third chapter concerns Frege's theories of identity in *Begriffsschrift* and "On Sense and Reference" and their complications. In the same chapter, I will examine Frege's mature theory of identity in "On Sense and Reference" in light of his views on multiple analyses, using lambda calculus notation in order to distinguish a number of different possible readings of one and the same identity statement. Also, I will argue, based only on the theory of multiple analyses, that identity statements of the forms 'a=a' and 'a=b' have different cognitive values, and hence that Frege's theory of sense may not have been needed to solve the puzzle of the cognitive values of identity statements.

Chapter One: An Overview of Frege's System

1-1) Ideal Language and Ordinary Language

In the nineteenth-century Gottlob Frege tried to replace incomplete or fallacious proofs and definitions that he claimed had been accepted in mathematics with correct and complete ones. He believed that there were gaps in the chains of proofs in mathematics. He saw that the rules of inference used in mathematical proofs were almost never explicitly stated. Euclid, for instance, had axioms and theorems but there was no mention of the rules of inference used in his proofs, and little had changed since. These incomplete proofs, according to Frege, depend essentially on the intuition of mathematicians and he insisted that this might lead to fallacious conclusions. As well, by the time of Frege, the notion of number had had been given a variety of different definitions; for Frege, this ambiguity is not acceptable for mathematics. His intention, then, was to provide a firm foundation for mathematics, relying not on the intuition of mathematicians, but on a logical system of the highest precision.

Logic, he thought, requires exact definitions and inference rules, and the structure of a formal system wherein mathematical statements were supposed to be represented must be purely logical. Frege also believed that psychology and subjectivity should be banned from mathematics. For him, the references of mathematical expressions are abstract objects, such as numbers, that are independent of human minds, and accordingly he rejects any subjective or psychological interpretation of mathematics. The ultimate idea of Frege was to reformulate proofs of mathematics

within a purely logical and formal system of arguments, one that would exclude any error, subjectivity, or ambiguity. For Frege, a reformulation of mathematics based on logic includes developing a formula language adequate for the expression of all mathematical statements, together with rules of inference adequate for the formalization of any mathematical reasoning.

He discovered that ordinary language prevented scientific statements from being as clear and precise as they should be. Consider, for example, the sentence 'Horses do not fly'. Grammar suggests, and traditional logic agrees, that 'horses' here refers to something. However, in the equally true statement 'Unicorns do not exist', which has the same grammatical form as the former sentence, 'unicorn', according to Frege, should not play the same role. Ordinary language here is misleading, Frege thinks, because in two sentences with the same grammatical form, the subjects play radically different roles; 'unicorns' is a non-referring term while 'horses' refers. Furthermore, unlike flying, which is a property that can be attributed to horses, Frege does not think that existence is a property.[1] He prevents the mistake of taking existence as a property in his ideal language along the following lines: the former sentence is rendered as '$\forall_x(H_x \rightarrow \neg F_x)$' and the latter as '$\forall_x(\neg U_x)$'.[2] These two sentences clearly differ in grammatical structure; the former expresses a relation between two concepts, while the latter is about the objects falling under a concept. The difference in the syntactical structure of these two formulas reflects the logical difference. Many such examples convinced him that he had to deal with ordinary language and its problems in order to construct a clear and exact logical

[1] See for example "Dialogue with Puenjer on Existence," *Posthumous Writings*, pp. 52-67.
[2] *Begriffsschrift* (Halle, 1879), §12. I use the translation of T.W.Bynum in G. Frege, *Conceptual Notation and Related Article*s (Oxford University Press, 1972). Cf. "Dialogue with Pünjer on Existence," *Posthumous Writings*, trans. Long and White, ed. H. Hermes, et al. (Oxford: Blackwell, 1979), pp. 53-67.

language that would serve the purposes of mathematics. To do so, he tried to discover the roots of the inadequacies of ordinary language.[3] In a manuscript entitled "Logic", for instance, he writes: "It is the business of the logician to conduct an unceasing struggle against psychology and those parts of language and grammar which fail to give untrammeled expression to what is logical."[4] He realized that one of the sources of ambiguity in a language resides in the imprecision of the relation between its linguistic expressions and the world —i.e. its semantics.

There is considerable debate about the relationship between natural language and the philosophical study of what exists, or, ontology. There are two prominent views with respect to this relation: the strong relativity thesis holds that our ontology depends heavily on language, but the weaker view holds that ontology is independent of language. Based on the strong relativity thesis, the furniture of the world depends on the concepts and the language we possess. Thought is essentially dependent upon language and since we grasp reality in thought it is language that determines our ontology.

On the weaker view, by contrast, the aforementioned considerations do not by themselves support the idea that the existence and the nature of the things in the world are dependent on what we think of them. According to the weaker view, no one can deny that language constrains what we say about what there is, but it goes no farther than this. Language only shapes the way we conceive the world, not the world we conceive; language is descriptive rather than creative. According to the weaker view, although ontology is independent of language, we still need to discuss ontology. Thus, a

[3] *Posthumous Writings*, p. 253.
[4] *Ibid.*, p. 6-7.

philosophical investigation of language matters even for a philosopher such as Frege who holds the weaker view.

For Frege, thought is only accessible to us through language and thus we have no choice in approaching thoughts but to start with ordinary language, and to struggle against its imperfections.[5] In "Negation" (1918), Frege claims that it is "not the least of the logician's tasks to indicate the pitfalls laid by language in the way of the thinker"[6]. In his 1918 "Thought", he describes his motives for investigating language as follows:

> I cannot put a thought[7] in the hands of my readers with the request that they should examine it from all sides. Something in itself not perceptible by sense, the thought is presented to the reader —and I must be content with that— wrapped up in a perceptible linguistic form. .... So one fights against language, and I am compelled to occupy myself with language although it is not my proper concern here.[8]

Here, Frege presents us with a problem that lies at the heart of his attitude toward ordinary language. On one hand, he has to rely on it in order to get access to what he calls thoughts; ordinary language seems to mirror thought.[9] On the other hand, language often muddles thought and in that sense it is not a perfect mirror. For this reason, there is a need for a more logically perfect language, one that would better display the structure of thought. Accordingly, Frege had no choice other than considering the study of language to be an important part of his investigations. He felt

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[5] *Begriffsschrift*, §3.
[6] *The Frege Reader*, ed. Michael Beaney (Oxford: Blackwell, 1997), p. 354.
[7] By 'thought', Frege here means the sense of an assertoric sentence.
[8] *The Frege Reader*, p. 333, footnote D.
[9] See, for example, "Compound Thoughts," *Collected Papers*, p.390: "It is astonishing what language can do. With a few syllables it can express an incalculable number of thoughts, so that even if a thought has been grasped by an inhabitant of the Earth for the very first time, a form of words can be found in which it will be understood by someone else to whom it is entirely new. This would not be possible if we could not distinguish parts in the thought corresponding to the parts of a sentence, so that the structure of a sentence can serve as a picture of the structure of the thought."

that he had to develop a logical symbolism in order to illustrate accurately the conceptual content of thought. The grammar of this logical language would guarantee the correctness of inferences.[10] In his first major work, *Begriffsschrift* (1879),[11] he developed a system of logical notation that he believed adequately displays the "conceptual content" that is not always apparent in ordinary language expressions. Later, in his 1882 "On the Scientific Justification of a Conceptual Notation," he explains the need for such a logical system as follows:

> [Ordinary][12] language can be compared to the hand, which despite its adaptability to the most diverse tasks is still inadequate. We build for ourselves artificial hands, tools for particular purposes, which work with more accuracy than the hand can provide.... We need a system of symbols from which every ambiguity is banned, which has a strict logical form from which the content cannot escape.[13]

Frege used mathematical symbols to improve logical notation, and applied this new logical system of symbols to better express thought. He realized that the imperfections of ordinary language stood in the way of developing a logically ideal language. He explains this development, which led him from mathematics to the study of language, in his 1919 "Notes for Ludwig Darmstaedter":

> I started out from mathematics. The most pressing need, it seemed to me, was to provide this science with a better foundation. I soon realized that number is not a heap, a series of things, nor a property of a heap either, but that in stating a number that we have arrived at as the result of counting we are making a statement about a concept. The logical imperfections of language stood in the

[10] " On the Scientific Justification of a Conceptual Notation," *Conceptual Notation and Related Articles,* p. 84-85.
[11] "*Begriffsschrift*" is usually translated as "Concept Script" or "Conceptual Notation".
[12] 'ordinary' is supplied by the translator. In Frege's assay a contrast is drawn between language (*die Sprache*) and a formula language (*Formelsprache*)
[13] *Conceptual Notation and Related Articles*, p. 86.

> way of such investigations. I tried to overcome these obstacles with my *Concept-Script*. In this way I was led from mathematics to logic.[14]

Frege mentions that Leibniz also noticed the need for an appropriate system of symbols and, hence, tried to create a universal system (universal characteristic) for all the sciences, which was exactly what Frege was looking for.[15] Although Frege couldn't achieve this goal, he believed that his *Begriffsschrift* was a big step towards it. According to him, his new logical system of symbols captured the logical structure of thought better than any natural language.

Frege believed that one of the logical impurities of ordinary language that stood in the way of his logical investigation into the foundations of arithmetic was its grammar. Long before Frege, in traditional syllogistic logic, a proposition or a sentence was analyzed into three grammatical constituents, namely, the subject term, the copula, and the predicate term.[16] Grammatically speaking, the subject of a sentence is the expression that indicates what the sentence is about, or as Frege says, the subject of the sentence is "the concept with which the judgment is chiefly concerned."[17] The predicate, by contrast, is the expression that indicates what is being said about the subject. So, for example, in the sentence 'Fido is a dog', 'Fido' is the subject and 'dog' is the predicate. Similarly, in the sentence 'Dogs are animals', 'dog' is the subject and 'animal' the predicate.

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[14] *Posthumous Writings*, p. 253.
[15] *The Frege Reader*, p. 50.
[16] Often, 'subject' and 'predicate' were used ambiguously: sometimes to refer to linguistic expressions and sometimes to what they name. This ambiguity can be avoided by saying that the grammatical subject is the name whereas the logical subject would be what the name stands for (an individual, species, or universal); the grammatical predicate, similarly would be an adjective or common noun while the logical predicate would be a universal or a property.
[17] *Conceptual Notation and Related Articles*, p. 113.

Nonetheless, Frege believed that for centuries, logicians were misled by the superficial features of natural language. That is, when logicians based the logical distinction between subject and predicate on the grammatical distinction between subject and predicate expressions, they followed the grammar of natural language too closely, and this prevented them from realizing that natural language is inadequate to capture the logical structure of thought, or of the world itself as it is reflected in thought. Frege confesses that even he took this wrong approach for a while; he says: "In my first draft of a formula language, I was misled by the example of [ordinary] language into forming judgments by combining subject and predicate. I soon became convinced, however, that this was an obstacle to my special goal and led only to useless prolixity."[18] He argues that since the grammatical distinction between subject and predicate did not correspond to any logically significant distinction between components of the thought underlying a sentence, it had no place in an ideal language:

> A distinction of *subject* and *predicate* does not *occur* in my way of representing a judgement. To justify this, I note that the contents[19] of two judgements can differ in two ways: first, it may be the case that [all] the consequences which can be derived from the first judgement combined with certain others can always be derived also from the second judgment combined with the same others. The two propositions, "At Plataea the Greeks defeated the Persians" and "At Plataea the Persians were defeated by the Greeks", differ in the first way. Even if one can perceive a slight difference in sense, the agreement [of sense] still predominates. Now I call the part of the content which is the same in both

[18] *Begriffsschrift*, § 3.
[19] In *Begriffsschrift*, Frege first distinguishes between the content of a sentence and the psychological attitudes (such as judgment, assertion, denial and so on) that someone might take towards that content. So, for instance, the content of the sentence 'Snow is white' is the proposition that snow is white, apart from any attitude taken towards it. He further asserts that not every content can be judged; "the tree", for example, cannot become a judgement. As a consequence, he distinguishes between two different kinds of contents: those that can, and those that cannot, be the content of a judgement.

> the *conceptual content*. … *only this* is meaningful for [our] "conceptual notation"[20]

It is obvious that the two aforementioned examples have different arrangements of words and different subjects and predicates; in the former 'the Greeks' appears to be the grammatical subject, whereas in the latter it is 'the Persians'. Yet, according to Frege, they both have the same logical consequences. Frege concludes that it is irrelevant to the conceptual content of a sentence which term of the sentence plays the grammatical role of the subject or the predicate. Therefore, according to Frege, sentences of natural language that have different grammatical subjects or predicates need not have different expressions in a logically ideal language. In other words, sentences of natural language with different subjects and predicates might be translated to the same sentence in an ideal language. For Frege, only that which influences the logical consequences of a judgment– i.e., the conceptual content of a sentence –is to be expressed in a logically perfect language.

Frege pointed out another problem with the grammatical distinction of subject-predicate: such a distinction cannot do justice to the conceptual content of general propositions (general facts) such as (1) 'The dog is an animal.' According to the traditional view, the expression 'The dog' (which designates an Aristotelian universal) is the subject and 'animal' is the predicate. However, on Frege's understanding the words 'The dog' and 'animal' both express concepts, and hence the actual content of the proposition is that if anything is a dog then it is an animal or, in symbols, $\forall x\,(Dx \rightarrow Mx)$.

[20] *Begriffsschrift*, §3. Frege further claims that one can imagine a language in which there is only one predicate, 'is a fact', which is applied to all the sentences as its subjects; in such a language, the subject would display everything of logical significance. So, for example, the proposition 'At Plataea the Greeks defeated the Persians' could be expressed as: The defeat of the Persians by the Greeks at Plataea is a fact. (*Ibid.*). As we can see, the subject in this language contains the entire thought, i.e., everything that matters for logical analysis.

In fact, this proposition is about a relation between two concepts ([Dx] and [Mx]); a general claim that whatever falls under the first concept also falls under the second. It means that the logical structure of general propositions is radically different from other propositions such as that expressed by the sentence (2) 'The Prime Minister is a Conservative,' in which the subject, 'The Prime Minister', is a proper name referring to one object. The logical notation of sentence (2) would be <λx.Cx >(p)[21] or Cp, in which 'C' stands for the property of being conservative and 'p' stands for the Prime Minister. Hence, natural language fails to capture the logical difference between two markedly different propositions. According to standard grammar, the terms 'The dog' and 'The Prime Minister' are considered to be the same type of name, being located at the same position in the proposition serving as the subjects of the propositions. Also the properties of being an animal and being a Conservative both serve as the predicates of the propositions.

On one hand, ordinary language presents the same grammatical structure for what Frege sees as totally different thought-contents (conceptual content) and, on the other hand, it lacks an exact correspondence between syntactical categories and their ontological counterparts. Due to these problems,[22] Frege felt the need for a logically ideal language, one that, unlike traditional logic, does not follow the grammar of ordinary language, and does allow us to properly display the distinction between these different kinds of propositions and differentiate distinct ontological categories. In the

[21]This means that the object designated by 'p' has the property designated by 'C'.
[22] Another problem of ordinary language would be the problem of non-referring names. Frege realized that there are singular terms, not only in fictional but also in scientific contexts, which do not refer to anything in the world. In "On Sense and Reference" (1892), Frege says: "The expression 'the least rapidly convergent series'… has no reference, since for every given convergent series, another convergent, but less rapidly convergent, series can be found." (*The Frege Reader*, p. 153.)

preface of *Begriffsschrift,* he describes the relation between ordinary and ideal language as follows:

> I can make the relation of my "conceptual notation" to ordinary language clearest if I compare it to the relation of the microscope to the eye. The latter, because of the range of its applicability and because of the ease with which it can adapt itself to the most varied circumstances, has a great superiority over the microscope. Of course, viewed as an optical instrument it reveals many imperfections, which usually remain unnoticed only because of its intimate connection with mental life. But as soon as scientific purposes place strong requirements upon sharpness of resolution, the eye proves to be inadequate. On the other hand, the microscope is perfectly suited for just such purposes; but, for this very reason, it is useless for all others.[23]

In *Begriffsschrift,* Frege developed a new symbol system which, he believed, captures the logical structure of thought, or even of the world as reflected in thought. While ordinary language has nouns, verbs, adjectives, adverbs, prepositions and so on, and its propositions are analysed based on the grammatical distinction between subject and predicate, Frege's ideal language in *Begriffsschrift* simply requires two sorts of linguistic terms, function names and proper names, and he analyzes propositions based on the mathematical distinction between function and argument, to which I now turn.

1-2) Frege's Syntactical Categories and Their Ontological Counterparts

As we saw, Frege found the traditional analysis of sentences into subject-copula-predicate inadequate for the purposes of logic. In its place, he proposes an analysis in terms of *functions* and *arguments*, inspired by the analysis of arithmetical statements. An arithmetical sentence such as (3.2)+1, for example, could be parsed as the linear function f(x)=(3.x) +1, which maps integers to integers, and the number 2 as the

[23] *Conceptual Notation and Related Articles*, p. 104-5.

argument. In this example, the function (3.x) +1 maps 2 to 7 [(3.2)+1=7]. Inserting other numbers such as 4 or 5 as arguments would yield 13 or 16, respectively.

Frege then noticed that the same kind of analysis that he applied in mathematics could be applied to other linguistic expressions as well. For instance, 'The capital of Germany' can be rendered in terms of 'The capital of x'—a function which maps every country to its capital city— and 'Germany' the argument. This function maps 'Germany', 'France', and 'Canada' to 'Berlin', 'Paris', and 'Ottawa', respectively. Judging that he could apply this technique of analysis across the board, Frege made the function/argument distinction the basis of his ideal language (Conceptual Notation).[24]

In *Begriffsschrift*, he considers function and argument as linguistic objects. That is, he applies these two terms to the expressions themselves, rather than to what they stand for:

> *If in an expression … a simple or a compound sign has one or more occurrences and if we regard that sign as replaceable in all or some of these occurrences by something else (but everywhere by the same thing), then we call the part that remains invariant in the expression a* ***function****, and the replaceable part the* ***argument*** *of the function.*[25]

For instance, in the sentence 'Carbon dioxide is heavier than hydrogen' the symbol 'carbon dioxide' can be considered as replaceable by other linguistic symbols (such as 'nitrogen') and, hence, as the variable component of the sentence, and what remains, namely '--- is heavier than hydrogen', as the constant component of the sentence. Frege calls the constant component a function, and the variable one its argument.[26] In his later

[24] *Begriffsschrift, 1879, §2,* p. 113.
[25] *Ibid*., §9, p. 22. Bolding is mine.
[26] *Ibid*., §2, p. 126.

works, Frege adds precision to this account, distinguishing signs from what they signify. What in *Begriffsschrift* is called a function, he later calls a *function name*, and what he then called an argument, later a *proper name*. As a result, in Frege's ideal language there are only two basic syntactical categories: singular terms that Frege calls *proper names*, and function names. Consequently, he lists all expressions of his ideal language under these two main categories: proper names and function names.

Although Frege never seems to explicitly hold that ontology depends on language, there seems to be an almost perfect resemblance between his ontological and syntactical categories. From this resemblance, some commentators—such as Mendelsohn—go so far as to claim that Frege's syntactic categories were simply projected onto the world.[27] This interpretation is based on Frege's claim that just as in his ideal language there are proper names and function names, so too are there distinct kinds of entities (objects and functions) that are the referents of each type of expression.[28] In Frege's ontology there are only these two kinds of entities, namely, objects and functions. A proper name designates an object, real or abstract, and a function name designates a function:

1.) Some signs are *in need of completion* (e.g. =). If supplementation yields a sentence, then the sign **refers** to a concept (in the case of two arguments, a *relation*). More generally: ***function***.
2.) Those parts of a sentence that are not in need of supplementation are called *proper names*; they **refer** to ***objects***.[29]

For Frege, objects and functions are fundamentally distinct: "*Objects* stand opposed to functions. Accordingly, I count as an *object* everything that is not a function."[30]

[27] Richard L. Mendelsohn. *The Philosophy of Gottlob Frege* (Cambridge University Press, 2005), p. 70.
[28] Frege, 1971, p. 32-3.
[29] *Frege's Lectures on Logic: Carnap's Student Notes, 1910-1914*, Edited and translated by E. Reck and S. Awodey (Chicago: Open Court, 2004), p. 87. Boldface is added.

Likewise, for him, *proper name* and *function name* are two different linguistic categories. Furthermore, for Frege, objects, like their syntactical counterparts (proper names), are saturated entities and complete by themselves, whereas functions, like their syntactical counterparts (function names), are unsaturated entities and incomplete by themselves. However, when he compares functions to their signs as being incomplete and unable to subsist on their own, he makes it hard to imagine the possibility of having entities as functions in the same way that there are objects in the world.[31]

In a language, there are different kinds of proper names: to begin with, there are genuine proper names such as 'Aristotle', and definite descriptions such as 'the morning star'. Every proper name, he says, is saturated and complete by itself, and also designates a saturated object. Somewhat surprisingly, Frege claims that, like genuine names and definite descriptions, those *sentences* that express complete thoughts are also proper names, and refer to their objects, namely their truth values. He says "the names '$2^2=4$' and '$3>2$' *refer to* the same truth-value, which I call for short *the True*. Likewise, for me, '$3^2=4$' and '$1>2$' *refer to* the same truth-value, which I call for short *the False*, exactly as the name '2^2' *refers to* the number Four."[32] Thus, he considers a complete sentence to be a proper name, and the True or the False its referent. For instance, '2+3=5' designates the True and '2+3=4' refers to the False.

Function names, on the other hand, are unsaturated, in need of completion by argument(s)/proper names and each designates a function, a category that includes

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[30] *Frege: Basic Laws of Arithmetic,* Edited and translated by P. Ebert, M. Rossberg, C. Wright (Oxford University Press, 2013) I, §2. p. 7.
[31] "A concept is unsaturated in that it requires something to fall under it; hence it cannot exist on its own." *Philosophical and Mathematical Correspondence (University of Chicago Press,* 1980). p. 101. And also, "Comments on Sense and Meaning", 1892-1895, p. 119. Also, *Posthumous Writings*, 1979, p. 17.
[32] *Basic Laws* I, §2. p. 7.

concepts and relations among others. According to Frege, there are fundamentally different kinds of functions based on the number and levels of their argument places. Corresponding distinctions then carry over to function names. To begin with, functions are divided into two general categories: functions of one argument such as ξ+2, and functions of two or more arguments, for instance, ξ+ζ. The latter, he writes, stands

> *…in need of double completion insofar as a function with one argument is obtained after their completion by one argument has been effected. Only after yet another completion do we arrive at an object, and this object is then called the value of the function for the two arguments.*[33]

Frege also divides functions (and thus function names) in another manner, classifying them as first level, second level, and third level and so on. First level functions are functions whose arguments are objects, such as ξ+ζ in which ξ and ζ each mark a place that is to be completed by an object, for instance, 2 and 3. Second level functions, by contrast, are those whose arguments are first level functions. For instance, Frege considers universal quantifiers to be second-level functions, i.e., $\forall_x \psi_x$ in which ψ_x marks a place for a first level function of one argument such as ξ=ξ (ξ is identical to itself), and also $\forall_x\forall_y \psi_{(x,y)}$ in which $\psi_{(x,y)}$ marks a place for a first level function of two arguments such as (ξ=ζ)→ (ζ= ξ); as, for instance, in $\forall_x\forall_y$ (x=y)→(y=x) which states that identity is a symmetrical relation. Third level functions are functions whose arguments are second level functions ; for example, $\forall_\psi (F_x(\psi_x))$ in which 'F_x' indicates an argument place for a second level function which in turn takes first-level functions of one argument as its

[33]*Frege: Basic Laws of Arithmetic I*, 2013, §4, p. 8.

arguments.[34] The hierarchy of levels appears to be infinite; quantifiers over n^{th}-level functions, for example, deliver $(n+1)^{st}$-level functions.

Within levels, Frege again distinguishes functions according to the number of their argument-places. This function ξ+ζ, for instance, is a first level function of two arguments while ξ= ξ is a first level function of one argument. Frege acknowledges all possible combinations of levels and numbers of arguments. He also makes it clear that just as the distinction of function and object is vital, so too is the distinction of different levels of functions, and also the distinction of functions of different numbers of arguments.[35]

Frege also introduces mixed or unequal-levelled functions.[36] He gives the following example: "The differential quotient is … to be regarded as a function with two arguments, of which the one has to be a first-level function with one argument, and the other an object. We can therefore call it an unequal-levelled function with two arguments."[37] That is, $\left(\frac{df(x)}{dx}\right)_{(x=t)}$ represents a function of two arguments, f(x) and x , the former marking a place for a first-level function and the latter an object. Similarly, the integral:

$$\int_0^t f(x)dx,$$

[34] *Frege: Basic Laws of Arithmetic I*, §24, p. 41.
[35] "On Concept and Object", *Translations from the Philosophical Writings of Gottlob Frege*, Ed. P. Geach and M. Black (Oxford: Blackwell, 1970), p. 51.
[36] *Frege: Basic Laws of Arithmetic I*, §22, p. 39. See also "Function and Concept" in Geach and Black, p. 40.
[37] *Ibid.*, §22, p. 39.

would be a mixed-levelled function, whereas $\forall_x(F_{(x)} \rightarrow G_{(x)})$ is a second-level function with two first–level functions, $F_{(x)}$ and $G_{(x)}$, as its arguments. Hence, according to Frege, the former equation is an unequal-levelled function and the latter equal-levelled.

On the basis of the distinction between different kinds of functions, Frege proposes a new, mathematically inspired, understanding of concepts and relations:

> '$\xi^2=4$' and '$\xi>2$' were not accepted as names of functions, as I accepted them to be. But with this it is acknowledged at the same time that the range of function-values cannot remain restricted to numbers; for if I take the numbers 0, 1, 2, 3, one after the other, as the argument of the function, $\xi^2=4$, then I do not obtain numbers.
>
> '$0^2=4$', '$1^2=4$', $2^2=4$', '$3^2=4$'
>
> Are expressions of thoughts, some true, some false. I express it like this: the value of the function $\xi^2=4$ is either the *truth-value* of the true, or that of the false….it seems appropriate simply to call a *concept* any function whose value is always a truth value.[38]

First-level functions, for instance, can be divided into functions that map every argument/object to a truth value and functions that don't. So too for second-level functions: some, such as quantifiers, map all first-level functions to truth-values; others, such as φ(2), do not. This function would have a truth value as its value for some arguments such as $\xi+\xi=\xi.\xi$, while it would have other objects as its value for other arguments such as $\xi+1$.[39] However, only functions of one argument (e.x. $\xi+\xi=5$) which map every argument to a truth value are called *concepts*. Functions of two or more arguments, which map every n-tuple of arguments to a truth value, are called *relations*

[38] *Ibid.*, §2-§3, p. 6-8.
[39] *Ibid.*, §22, p. 38.

(e.x. ξ+ζ=5); and for Frege, relations, which are functions of more than one argument, are fundamentally different from concepts, which have only one argument.[40]

1-3) Expression, Sense, and Reference

There is a clear distinction between linguistic expressions and what they stand for; that is, when we have a name which designates an entity there are, at least, two levels, the level of language where the name belongs, and the level of reference where the entity belongs. Frege, at the beginning, only had these two levels in his semantics. However, he came to think that they could not by themselves provide an adequate account of meaning. He discovered various puzzles regarding the cognitive value of identity statements,[41] opaque contexts, and non-referring names, which, he thought, could not be solved with a two-level semantics. To solve these problems, he postulates a third level of *sense*.[42]

To better understand Frege's motive, it is helpful to review one of the puzzles that Frege encountered: the problem of names that are used meaningfully in everyday language, but do not refer to anything. There is a longstanding debate about the existence of the entities apparently referred to by names such as 'Pegasus', and names of the things that may not exist, for example 'Homer', and names of the things that could not possibly exist, for instance, 'the round square'. All of these names seem to be meaningful and are used in ordinary discourse, but they may not have a reference. Some believe that since these names make sense, they must refer to something. Meinong

[40] *Ibid*., §4, p. 8.
[41] I will explain this issue in detail in the third chapter.
[42] See Frege's letter to *Husserl in Philosophical and Mathematical Correspondence*, p. 63.

claimed that with names such as 'Pegasus', we have an object but a non-existing one.[43] By contrast, the later Russell argues that ordinary proper names such as 'Scott' are, in fact, disguised definite descriptions; they are not logically proper names,[44] but rather abbreviations for definite descriptions. In addition, he calls definite descriptions, such as 'the author of *Waverley'*, incomplete symbols because, he claims, they have no meaning on their own; when a definite description occurs in a sentence, the whole sentence has a meaning, but the definite description does not designate a constituent of the proposition expressed by the sentence.[45] For instance, when we say 'Pegasus has a pure white colour', it means that 'There is an x such that x is a winged divine stallion, and for all y, if y is a winged divine stallion then y=x, and x has a pure white colour.' In this manner, Russell claims, the sentence does not express a proposition that has Pegasus as a constituent; instead, it expresses a proposition whose constituents are the property of being a winged divine stallion, the property of having pure white colour, and so on. The proposition is meaningful even though there is no entity corresponding to the name 'Pegasus' (existing or non-existing).[46] Frege's theory of sense, by contrast, allows for a proper name to express a sense that fails to refer to an object:

> It may perhaps be granted that every grammatically well-formed expression representing a proper name always has a sense. But this is not to say that to the sense there also corresponds a reference. The words 'the celestial body most

[43] A. Meinong, "The Theory of Objects," in R. Chisholm, ed., *Realism and the background of Phenomenology* (NY: Free Press, 1960), pp. 83-6. Cf. B. Russell "On Denoting," *Mind*, New Series, Vol. 14, No. 56 (Oct, 1905) 479-493, pp. 482-4.
[44] According to him the only logically proper names are 'This' and 'That' in cases where we are acquainted with the designated object; for instance, in the sentence 'This is white, 'This' is the logically proper name. See *The Philosophy of Logical Atomism* (Chicago: Open Court, 1985), Section II.
[45] *The Philosophy of Logical Atomism*, 1985, p. 122.
[46] *The Philosophy of Logical Atomism*, pp. 118-120.

> distant from the Earth' have a sense, but it is very doubtful if they also have a reference.... In grasping a sense, one is not certainly assured of a reference.[47]

For instance, the name 'Odysseus' expresses a sense that might also be expressed by the description 'the Greek king of Ithaca and a hero of Homer's epic poem'. This sense does not present an object, and so the name fails to refer. That is, as long as a non-referring proper name has a sense it can function in the language without positing any entity as its reference in our ontology.

Moreover, for Frege, the sense and reference of a sign are both objective; so they should not be confused with what Frege calls ideas, which are subjective, belonging to an individual mind, and cannot be communicated to another person. In short, Frege states that "[a] proper name (word, sign, sign combination, expression) *expresses* its sense, *stands for* or *designates* its reference."[48]

Frege further expands his theory of sense and reference to cover declarative sentences. The sentence, he claims, "contains a thought."[49] This thought is not subjective and does not depend on the mind of the speaker; on the contrary, it is objective and thus can be grasped by anyone. He asks whether the thought is the sense or the reference of a sentence and, if the former, whether it also has a reference. Frege adopts the following substitution principle in order to find an answer to his question: "[I]f we now replace one word of the sentence by another having the same reference, but a different sense, this can have no bearing upon the reference of the sentence."[50] For instance, 'the morning star' and 'the evening star' have the same reference, but when we replace one with the other in a sentence, they produce different thoughts, because

[47] "On Sense and Reference," 1892, p. 28.
[48] *Ibid*., p. 31.
[49] *Ibid*., p. 32.
[50] *Ibid*., p. 31.

someone may not know that they are both names for the same object; the thought of the original sentence differs from the thought of the new sentence, but the reference, according to the substitution principle, remains the same. Hence, Frege concludes that the thought contained in a sentence cannot be its reference, but rather is its sense.

Following that, Frege claims that the reference of a declarative sentence, if any, must be its truth value.[51] He argues that once we start replacing the parts of a sentence with other expressions which are co-referential, we can transform the whole sentence so completely that nothing remains constant through the whole series of substitutions except the reference, whatever it is, and, since the only thing that remains the same is the truth value, this must be the reference of the sentence. Frege further argues:

> We have seen that the reference of a sentence may always be sought, whenever the reference of its components is involved; and that this is the case when and only when we are inquiring after the truth-value. We are therefore driven into accepting the truth-value of a sentence as constituting its reference.[52]

Correspondingly, Frege claims that if a proper name which has no reference is used in a sentence, the sentence has no reference; to find out the truth value of a sentence, we need the reference of the proper name of which the predicate is affirmed or denied. Thus, the lack of a reference of a part of a sentence leads to the lack of reference of the whole sentence.[53] It is obvious from the works of other philosophers[54] that there is no compulsion to draw this conclusion that sentences designate truth-values, however, this

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[51] *Ibid*., p. 32-3. For an enlightening discussion of Frege's argument about why we have a truth value as the reference of a sentence see Mendelsohn's *The Philosophy of Gottlob Frege*, chapter 8. See also A. Church, *Introduction to Mathematical Logic*, Vol. I, Princeton University Press, 1956. pp. 24-25.
[52] *Ibid*., pp. 33-4.
[53] *Ibid*., p. 33.
[54] Russell in *The Philosophy of Logical Atomism*, for instance, argues that every sentence refers to a *fact*, which is an object in the world. Similarly, Wittgenstein, in *Tractatus Logico-Philosophicus,* believes that the reference of any sentence is a *state of affairs*. It is to say that there are as many objects as there are different sentences.

view is in accordance with Frege's use of sentences in his ideal language.[55] Frege's theory compared to others, also, has the benefit of simplicity at the level of reference, positing only two additional objects, the True and the False. "If now the truth value of a sentence is its reference, then on the one hand all true sentences have the same reference [the True] and so, on the other hand, do all the false sentences [the False]".[56]

Frege later expands the sense/reference distinction to cover all linguistic expressions. In particular, a function name must also have a sense and a reference.[57] According to Frege, functions are individuated extensionally, by their value ranges; if two function names have the same value range, they refer to the same function: for example, the two function names 'ξ+2' and '3+(ξ-1)' always give us the same value for the same argument-value, so they designate the same function at the level of reference; it means that 'ξ+2' and '3+(ξ-1)' are two coextensive function names; they pick out the same reference while expressing different senses. In his "Comments on Sense and Meaning", Frege introduces a notation for the identity of co-extensive concepts which is a second level function, unlike the notation of identity for proper names (=) which is a first level function.[58]

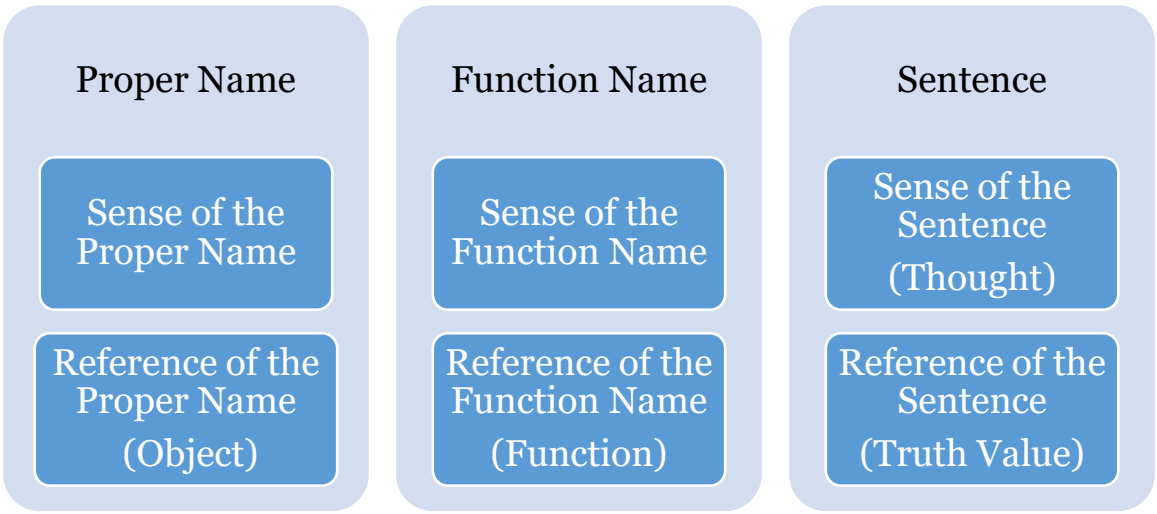
Frege's theory of sense addresses not only the puzzles about the cognitive value of identity statements and the substitution of proper names, but also the alteration of cognitive value as a general phenomenon that happens under all kinds of substitutions of co-referential terms, and in any kind of statement or context, opaque (so and so believes that, so and so is afraid of something, or so and so said that…etc.), and non-

[55] For more information, see Joan Weiner, *Frege Explained*, 2004. pp. 94-101.
[56] "On Sense and Reference," 1892, p. 35.
[57] *The Frege Reader*: "Comments on Sinn and Bedeutung", 1997, pp. 172-3.
[58] *Ibid*., p. 177.

opaque. Substitution of coextensive function names or co-referential proper names does not always preserve the cognitive value of the whole expression. Thus, similar to co-referential proper names, we can have functions that are co-extensive and when we substitute one for the other it will change the cognitive value. For instance, in a true sentence (which refers to the True) such as 'John knows that x+2= (x+3)-1' if we replace the function (ξ+3)-1 with its coextensive function $\xi + \log_{10}(100)$, John may not know that this new equation $x+2= x + \log_{10}(100)$ is true as well. He may not have the required knowledge about logarithms. Hence, although the references of the parts of the sentence remain the same, the whole sentence 'John knows that $x+2= x + \log_{10}(100)$' now refers to the False.

In this chapter, I explained Frege's motivations for constructing an ideal language. I also laid down a brief explanation of Frege's system only to provide the basis for my discussions in the next two chapters. In brief, there are three levels in Frege's semantics: the level of language, the level of sense and the level of reference. According to him, a linguistic sign expresses its sense, and by virtue of its sense designates its reference. Here is an overall diagram of Frege's semantics, similar to what he gives in a letter to Husserl.[59]

[59]*Philosophical and Mathematical Correspondence*, 1980, "Letter to Husserl, 1891," p. 63.


Proper Name
Sense of the Proper Name
Reference of the Proper Name (Object)
Function Name
Sense of the Function Name
Reference of the Function Name (Function)
Sentence
Sense of the Sentence (Thought)
Reference of the Sentence (Truth Value)

Chapter Two: The Alternative Analyses of a Sentence

According to Frege, a sentence can be analysed in number of different ways into function names and proper names, and similarly, thoughts into functions and arguments, while the conceptual content of the sentence remains the same under all of these different analyses. I believe that Frege's theory of multiple analyses creates challenges for both his theory of identity and his idea of formulating an ideal language. In this chapter, I first explain what an argument and a function in a sentence are for Frege, and discuss his view regarding the analysis of a given sentence (and its thought) into argument(s) and function(s) in a number of different ways. I then introduce the lambda operator in order to facilitate my discussion of Frege's theory of multiple analyses. In the last section, I explain how Frege's theory of multiple analyses affects his notion of the conceptual content of a sentence and how it eventually might obscure the meaning of the formulas in his ideal language, *Begriffsschrift*. I explain why it seems that Frege's theory of multiple analyses ascribes different conceptual contents to the same *Begriffsschrift* formula, apparently giving rise to a one-to-many relation between a sentence and its conceptual content even in his ideal language. I clarify what Frege means by conceptual content, and discuss whether or not some commentators are right in assuming that the theory of multiple analyses is inconsistent with Frege's notion of a thought. I conclude by suggesting how Frege's theory of multiple analyses might be integrated into his ideal language based on a specific interpretation of his notion of

conceptual content. The discussion of the apparent inconsistency between Frege's theory of multiple analyses and his theory of identity will be left to the next chapter.

2-1) On the Extraction of Function Names from Sentences

According to Frege's mature semantics, as set out in the *Basic Laws*, a function name is an expression that has one or more empty places and is in need of completion. We obtain a function name from a proper name (according to Frege sentences are also proper names) by removing a part of it;[60] for instance, in the sentence 'Aristotle is wise', when we remove 'Aristotle', as the variable component, what remains is the function name '—is wise', which is regarded as the constant component of the whole sentence. The symbol which is regarded as variable (in our case 'Aristotle'), is considered to be replaceable by other symbols such as 'Plato'. Frege's examples in *Begriffsschrift, §9* suggest that he then considered functions and arguments to be linguistic objects; he applies these two terms to expressions rather than to what they stand for:

> *If in an expression, whose content need not be capable of becoming a judgment, a simple or a compound sign has one or more occurrences and if we regard that sign as replaceable in all or some of these occurrences by something else (but everywhere by the same thing), then we call the part that remains invariant in the expression a function, and the replaceable part the argument of the function.*[61]

[60] *Basic Laws* I, §26. Frege here presents the hierarchy of functions by his method of decomposing a complex proper name: by removing two proper names from a proper name, we can get a first-level function name of two arguments and so on. Also by removing a first-level function name from a proper name, we can get a second-level function name and so on.
[61] *Ibid*., §9, p. 22.

For instance, he explains, in the sentence 'Carbon dioxide is heavier than hydrogen' the symbol 'carbon dioxide' can be considered as replaceable by other symbols (such as 'nitrogen'), hence as the variable component of the sentence, and what remains, '—— is heavier than hydrogen', as the constant component of the sentence. Of the two components, variable and constant, he calls the latter a function and the former its argument.[62] However, Frege later adds precision to his account, distinguishing signs from what they signify: according to him, there are function *names* and proper *names* at the level of language, whereas *functions* and *arguments* are at the level of reference. Also, there is a third level in his semantics: every function name and proper name has a *sense*.

Frege introduces the function/argument distinction in a way that makes it seem that there is no unique answer to the question of which expressions in a sentence stand for functions, and which for arguments. In *Begriffsschrift*, he asks us to consider a sentence such as 'Hydrogen is lighter than carbon dioxide', and he examines different ways of analyzing this sentence into functions and arguments. He says:

> Let us suppose that the circumstance that hydrogen is lighter than carbon dioxide is expressed in our formula language. Then, in place of the symbol for hydrogen, we can insert the symbol for oxygen or for nitrogen. By this means, the sense is altered in such a way that "oxygen" or "nitrogen" enters into the relations in which "hydrogen" stood before. If we think of an expression as variable in this way, it divides into [1] a constant component which represents the totality of the relations and [2] the symbol which is regarded as replaceable by others and which denotes the object which stands in these relations. I call the first component a function, the second its argument. *This distinction has*

[62] *Ibid.*, §2, p. 126.

> *nothing to do with the conceptual content, but only with our way of viewing it.*[63]

As a result, Frege maintains that it depends on us which part of a sentence we regard as constant (or function name) and which part as variable (or argument). For example, "Hydrogen is lighter than carbon dioxide" can be analyzed into a kind of function name (the relation name) "—is lighter than ---" and its arguments "hydrogen" and "carbon dioxide." Or, it can be analyzed into another kind of function name (the concept name) "— is lighter than carbon dioxide" and its argument "hydrogen." Frege later expands these multiple divisions to the level of sense and claims that a thought corresponding to a sentence can also be analyzed into parts in different ways.[64]

In short, for Frege a sentence and its thought can be analyzed in a number of ways; we decide what part of a sentence (or its thought) is variable and what part is constant. As such, the number of argument-places in the sentence is also a matter of decision. For instance, we can divide the sentence '5≠3' into a saturated part '5' as the argument and an unsaturated part of not being equal to 3 or ('x≠3') as the function name. Alternatively, we can consider 5 and 3 as arguments and (x≠y) as the unsaturated relation of not being equal to. The sentence also can be analyzed into '3' as the argument and the unsaturated part the concept of not being equal to 5 or ('5≠x').

However, Frege claims that when either a function or argument is "indeterminate",[65] we cannot arbitrarily divide a sentence (or its thought) into different parts, since the sentence is already intrinsically divided into variable and constant parts

[63] *Begriffsschrift*, 1879, § 9, emphasis added.
[64] *The Frege Reader*, (1997), "On Concept and Object", p. 188. Also *Posthumous Writings*, p. 255. And many other places.
[65] *Begriffsschrift*, §9. Frege does not give an explanation for the term 'indeterminate', however, his examples suggest that indeterminacy is indicated by using variables rather than constants.

(function and argument). For instance, the sentence '$\forall_x$ (x+0=x)' has an indeterminate part which is marked by 'x', while in the sentence '2+0=2', everything is determinate. So, in the first case we have no alternative to considering x+0=x as the function, whereas in the second case we can recognize more than one function, for example, x+y=2, 2+x=2, and so on. In other words, in a universally quantified sentence, the argument and the function name cannot be distinguished totally arbitrarily. To explain this point, in *Begriffsschrift* Frege gives the following sentence: "Whatever arbitrary positive integer we take as argument for 'being representable as the sum of four squares', the [resulting] proposition is always true." In this sentence we have a claim of the form $\forall_x(P_x \rightarrow Q_x)$, where the introduction of the universal quantifier forces us to explicitly indicate a function, $P_x \rightarrow Q_x$. It means that "the whole splits up into function and argument according to its own content, and not just according to our way of looking at it."[66]

So far, I have been discussing examples of arbitrarily choosing arguments and functions in sentences. However, according to Frege, the function/argument analysis can be applied also to those linguistic components that do not by themselves constitute complete sentences,[67] that is, the distinction can be applied to any proper name; expressions such as 'the capital city of Canada', for instance, can be analyzed into 'the capital city of—' and 'Canada'.

[66] *Begriffsschrift*, 1879, § 9.
[67] *Ibid*., § 9.

2-2) Lambda Calculus: Function and Predicate Abstracts

The lambda operator is a device introduced by Alonzo Church for forming names of functions.[68] I shall discuss this operator here because by it I will be able to illustrate more clearly the difference of the structures of Frege's alternative analyses of one and the same sentence. Similar to quantifiers, lambda [λ] is a variable binding operator. Consider, for example, the term '2+3'. If we choose to view this term as expressing the application of the addition function to the pair <2, 3>, the term can be rendered as <λx,y. x+y>(2,3). But we can also view the term as the application of the function of adding three to the argument 2, which can be rendered as <λx. x+3>(2), or as <λx. 2+y>(3). Finally, the term can be viewed as involving the application of a second level function, which can be rendered as <λ*, 2*3>(+), etc.

As we saw in section 1-2, Frege treats concepts as a special kind of function that if saturated by an argument always give us either the True or the False as its value. The lambda operator can be used for forming names of concepts too. For instance, if 'Px' stands for 'x is a philosopher', then '<λx. Px>' stands for the concept or property of being a philosopher, and '<λx. Px> (a)' should be read as follows: the object designated by 'a' has the property <λx. Px>.[69]

The use of the lambda operator provides a clearer view of the different analyses of a simple statement into functions and arguments; for instance, 'Brutus killed Brutus' (K_{bb}), could be analyzed in any of the following ways (and more besides):

1) The predicate 'killing Brutus' applies to Brutus: $<\lambda_x. K_{xb}>$ (b)

[68] http://plato.stanford.edu/entries/church-turing/
[69] Fitting and Mendelsohn, 1998. p. 197.

2) The predicate 'being killed by Brutus' applies to Brutus: $<\lambda_x. K_{bx}>$ (b)

3) The relation of killing applies to the ordered pair of <Brutus, Brutus> : $<\lambda_{x,y}. K_{xy}>$ (b,b)

4) The predicate 'killing oneself' applies to Brutus: $<\lambda_x. K_{xx}>$ (b)

5) The (second level) predicate 'is a relation that Brutus bears to Brutus' applies to killing: $<\lambda_{\phi}. \phi_{bb}>$ (K)

6) $<\lambda_{\phi,x}. \phi_{xx}>$ (K,b)

7) $<\lambda_{\phi,x,y}. \phi_{xy}>$ (K,b,b)

Nonetheless, Frege believes that all of these different readings have the same conceptual content. Therefore, he does not see any need for an operator to express these different readings in his ideal language, for, according to him, a single formula can clearly and unambiguously express the conceptual content of all these sentences. I shall come back to this point in the following section and will consider whether Frege may be mistaken in this regard.

2-3) The Problem of Different Analyses of *Begriffsschrift* Formulas

A number of commentators believe that the account of different analyses raises difficulties and perhaps even contradictions in the explanation of Frege's semantics in relation to the notion of conceptual content and the theory of sense. Once Frege introduces the notion of function in *Begriffsschrift,* he complicates the relation of formulas to their conceptual contents in his ideal language, giving rise to some

difficulties.[70] One might think that, in order to avoid ambiguity, for any given conceptual content there must be one and only one reading for each formula in the ideal language. However, Frege claims that, in any given sentence, generally there is no unique answer to the question of which expressions stand for functions, and which for arguments. Therefore, based on his own claim, his ideal language inevitably permits various analyses for one and the same formula (and its associated conceptual content) and this, according to Frege, is an important source of ambiguity for a language. On the other hand, these various readings, contrary to what Frege says, may well not convey the same conceptual content. Thus, based on Frege's own analysis, it might turn out that no ideal language can really be all that "ideal". To examine this point, however, we must first clarify what Frege means by conceptual content.

2-3-1) The Notion of Conceptual Content

Frege's early account of semantics, given in *Begriffsschrift* and *Foundations of Arithmetic,* is based on the notion of *conceptual content*,[71] that is, on that part of meaning that is relevant to logical inferences. According to this account, the same conceptual content can be represented by more than one sentence; for instance, the two

[70] Frege, in *Begriffsschrift* and later on, sometimes talks about content as being the same in two expressions, and other times about conceptual content (as part of the content). However, a number of passages suggest that the notion of content is a more general notion within which the conceptual content resides; so it seems reasonable to assume that Frege sometimes uses the word 'content' when he means conceptual content. It also seems that he later drops the notion of conceptual content of a sentence in favour of sense and reference; the conceptual content of a sentence then becomes its thought plus its truth value.

[71] Frege does not use this term in his later works, and of course, in my thesis, I have no problem replacing most of the occurrences of this term by either 'reference', 'thought', 'thought and reference', or 'content', However, since I wanted to talk about that core notion which Frege tried to capture in his ideal language i.e. , the *part* of the content which is meaningful for the conceptual notation and is significant for logical inferences, and since Frege throughout his works does not refer to the same notion when he talks about that *part* (he sometimes takes the reference, other times the thought, and in other places the sense and reference together as the key *part*) I decided to use the term 'conceptual content' when I'm referring to that significant *part* of the content of an expression.

propositions "At Plataea the Greeks defeated the Persians" and "At Plataea the Persians were defeated by the Greeks" have the same *conceptual content,* which, for Frege, is the part of the content of these two sentences that is the same in both. In §2 of *Begriffsschrift*, Frege distinguishes between two different kinds of contents: those that can be judged and those that cannot. He claims that declarative sentences (or what he also calls judgments) have judgeable contents and other expressions do not; "the tree" for example cannot be judged,[72] that is, it is not true or false and, thus, it has non-judgeable content; whereas, the two sentences just mentioned above have judgeable contents.

In §3 of the same work, Frege states that "… the contents of two judgments can differ in two ways: first, it may be the case that [all] the consequences which can be derived from the first judgment combined with certain others can always be derived also from the second judgment combined with the same others; secondly, this may not be the case,"[73] Frege here provides a contextual definition[74] for the notion of judgeable content and, thus, illustrates what it means for two sentences to have the *same* content. I can formulate what he explains about "having the same content" as follows: two sentences Q and R, have the same content if and only if, for all sentences X and sets of sentences P:

$$P, Q \vdash X \text{ iff } P, R \vdash X$$ [75]

[72] *Begriffsschrift*, 1879, §2, p. 112.
[73] *Ibid*., p. 112.
[74] Since Frege cannot define this primitive logical notion in terms of more fundamental notions, he gives hints in his discussion to clarify as much as possible what this contextual definition means.
[75] In this formula, the single turnstile '⊢' is taken in the modern sense, not Frege's combination of judgment stroke and content stroke. This sign in its modern sense denotes syntactic consequence, or provability.

Thus we have an equivalence relation,[76] *sameness of* (judgeable) content, but no explicit definition of the content as such.

In §8 of *Begriffsschrift,*[77] Frege introduces the notation '≡' for expressing the relation of having the same content between two singular terms, such as the geometrical names 'A' and 'B', which have non-judgeable contents; the sentence 'A≡B' means that "*the symbol A and the symbol B have the same conceptual content, so that we can always replace A by B and vice versa*."[78] In *Begriffsschrift,* Frege has not yet developed his theory of sense and reference, but his explanation of conceptual content makes it clear that neither the different modes of determining an object—or what Frege later calls sense—nor the different expressions that correspond to these different modes of determination can be considered as the shared part [the conceptual content] between these singular terms. Thus, from the context, it appears that what Frege calls the conceptual content of two names 'A' and 'B', is what they stand for, which he would later call their reference.

Based on what we saw above, Frege provides two different clarifications for the notion of conceptual content. On one hand, Frege's explanation for the equality of non-judgeable contents of symbols 'A' and 'B' in the above example points towards the reference of the symbols as their conceptual content. On the other hand, in regard to the conceptual content of sentences, Frege's explanation above makes it clear which

[76] An equivalence relation has three characteristics: it is reflexive, transitive, and symmetric: A has the same content as A. If A has the same content as B, and B has the same content as C, then A has the same content as C. Also, if A has the same content as B then B has the same content as A.
[77] *Begriffsschrift*, p. 124. "Identity of content differs from conditionality and negation by relating to names, not to contents. Although symbols are usually only representative of their contents…they at once appear *in propria persona* as soon as they are combined by the symbol for identity of content, for this signifies the circumstance that the two names have the same content."
[78] *Ibid*., p. 126.

sentences have the same conceptual content, but it does not determine what the conceptual content of a sentence is. That is, although we can say that the aforementioned sentences Q and R have the same conceptual content, we are in no position to tell whether or not the conceptual content of sentence Q is or is not, for example, Julius Caesar.[79]

To explain, Frege argues that the conceptual content of two sentences is the same if and only if the logical consequences that can be derived from these two sentences are the same. This definition allows us to determine when two sentences A and B have the same conceptual content, but given the sentence A and Caesar we cannot say whether the statement 'the conceptual content of A= Caesar' is true or false, because we do not have the conceptual content of A yet.

Similarly, when I say that I was born in the same city as my brother, as long as I do not say which city it is, no one would know my birth-place. You know that 'my birth-place=my brother's birth-place' is a true identity statement, but based on this statement, you cannot say whether 'my birth-place=Tehran' is true or false. Although, it seems as if in the statement 'my birth-place=my brother's birth-place' we are dealing with the object 'my birth-place', we do not have the object yet.

Frege also gives an example of a contextual definition,[80] when he says that the direction of line a is identical to the direction of line b if and only if a is parallel to b;[81] this is a contextual definition for the direction of a line. He says that we have a way to decide whether the direction of line a is identical to the direction of line b, but we cannot

[79] *The Foundation of Arithmetic*, Page 62. § 56.
[80] See Beaney's "Logic and Metaphysics in Early Analytic Philosophy," in Haaparanta and Koskinen, eds., *Categories of Being*, by (Oxford University Press, 2012), pp. 258-266.
[81] *Ibid.*, §§ 64-67.

say if the direction of line a=c, when c is not given as the direction of a line. Although we have a singular term (a name) 'the direction of line a' which should identify an object, we see from this example that it does not do so. From the definition we just get the sameness of the directions, but not the direction itself. We have the same situation for Frege's definition of conceptual content of a sentence. According to his definition, we are given the sameness of the conceptual content, but not the conceptual content itself.

For singular terms, however, this is not the case. To be able to replace one name with another one we have to know exactly what the two names refer to, for instance, if two names 'A' and 'B' both refer to Julius Caesar then we can replace one with the other in a formula. So, the conceptual content or the reference of the names needs to be defined before any substitution happens. Thus, Frege appears to give an explicit definition of the conceptual content of names or expressions with non-judgeable contents, whereas he only gives a contextual definition for judgeable contents.[82]

Frege's account of conceptual content, however, shifts from one context to another. In a letter to Husserl of May 14, 1891 and in "On Sense and Reference", Frege presents a distinction between sense and reference within the conceptual content of a sentence. He says that "[W]hat I used to call judgeable content is now divided into thought and truth value."[83] In some places he says that conceptual content of a sentence is its thought (sense) plus its reference, but in other places he talks about only the sense of a sentence as its conceptual content:

[82] *Ibid*., pp. 113, 124-5.
[83] Letter to Husserl of 24 May, 1891, *Philosophical and Mathematical Correspondence*, p. 63; cf. *Basic Laws I*, p. x.

> In logic, one must decide to regard equipollent propositions as differing only according to form. After the assertoric force with which they may have been uttered is subtracted, equipollent propositions have something in common in their content, and this is what I call the thought they express.[84]

Also, he says that "[i]t is only with this part of the content that logic is concerned. I call anything else that goes to make up the content of a sentence the coloring of the thought."[85] Since, here, Frege takes only the thought as the conceptual content of a sentence, then for him, sentences like 'not (not A)' and 'A' have the same sense and, as a result, have the same conceptual content.[86] This seems to be in tension with what he says in *Begriffsschrift*, where he writes that "[s]ince only this [the conceptual content] is meaningful for [our] "conceptual notation", we need not distinguish between propositions which have the same conceptual content."[87]

It is commonly believed that Frege changed his account of conceptual content, not only for sentences, but for singular terms too. I believe that this interpretation may be mistaken, and Frege's account of conceptual content of a singular term (non-judgeable content) might have remained the same throughout his works. Even in his later works, Frege sometimes states that having the same conceptual content for singular terms means that they have the same reference, even though they have different senses. For instance, in "Function and Concept", he claims that the sign '2.2^3+2' has the same conceptual content as the signs '18' or '3.6'; here, having the same

[84] *Ibid.*, p. 67.
[85] *Posthumous Writings,* p. 198.
[86] "Compound Thoughts," in *Collected Papers on Mathematics, Logic, and Philosophy*, trans. Max Black *et al.*, ed. B. McGuinness (Oxford: Blackwell, 1984), p. 399.
[87] *Begriffsschrift*, §3.

reference seems to be sufficient for having the same conceptual content. In fact, Frege uses the terms '*Inhalt'* and '*Bedeutung'* interchangeably in this passage.[88]

As far as I am aware, in none of his later works does Frege give a new definition for the conceptual content of a singular term. He explicitly divides only a *judgeable* content (or the conceptual content of a sentence) into thought and truth value and says nothing new about non-judgeable contents. After all, for Frege, there is no truth value or thought involved with the non-judgeable contents. Thus, the possibility remains open that a non-judgeable content remains the same as it was before, namely, the reference. In this case, in Frege's syntactical categories, proper names could also be either expressions which have their sense (thought) plus reference as their conceptual content (i.e., declarative sentences), or expressions which have their reference as their conceptual content (i.e., singular terms). In fact, Frege himself was aware of this distinction, treating sentences differently from other proper names. In his letter to Husserl (1891), for instance, he separates sentences from other proper names in his diagram (see p. 29 above), even though he still regards sentences as a kind of proper name. This means that Frege's theory of multiple analyses may well interact with two distinct notions of conceptual content. For this reason, I will examine the effects of multiple analyses on non-judgeable and judgeable contents separately. And I will show that Frege theory of multiple analyses is only applicable to the conceptual content of sentences, and not to the conceptual content of singular terms.

[88] *Collected Papers,* p. 138. *Kleine Schriften*, ed. I. Angelelli (Hildesheim: Olms, 1967), p. 126.

2-3.2) Multiple Analyses and Non-Judgeable Contents

Frege says that an expression and its sense (but not its reference) can be analyzed in different ways.[89] Also, I argued that, for Frege, the conceptual content of a singular term might still be its reference. If so, the conceptual content of a singular term cannot be analyzed differently and will remain the same under the multiple analyses of the singular term.

To clear up this discussion, let's revisit Frege's "On Sense and Reference" where he wonders why two identity statements such as P: 'the morning star= the morning star' and Q: 'the morning star= the evening star' have different cognitive values.[90] He argues that the difference between the cognitive values of P and Q cannot be based on the difference between the reference of 'the morning star' and 'the evening star', because they both have the same reference. Also, the difference cannot be about the different signs themselves, otherwise the identity statement would be about the linguistic entities, and not about the facts about Venus.[91] Thus, Frege concludes that the difference between the cognitive values of sentences P and Q is the outcome of the difference between the senses of 'the morning star' and 'the evening star'. Furthermore, in a sentence, only those expressions are interchangeable *salva veritate* that have the same conceptual content. Given this fact and the fact that we can replace 'the morning star' by 'the evening star' in a true identity statement and get another identity statement that is equally true, we conclude that these two expressions have the same conceptual content. On the other hand, the only thing that the expressions 'the morning star' and 'the

[89] "On Concept and Object" p. 199 (*Collected Papers*, p. 188). Also, *Posthumous Writings*, p. 255.
[90] The cognitive value of a sentence is the contribution that grasping or understanding the sentence makes to a thinker's epistemic perspective.
[91] I will explain this matter in detail in my chapter on identity.

evening star' have in common is their reference. Therefore, we can conclude that the references of these expressions are their conceptual contents. Since Frege applies multiple analyses only to the expressions and their senses (thoughts), not to their references, the theory of multiple analyses is not applicable to the conceptual content of a singular term, as some commentators believe.

However, one might say that if the conceptual content of a name is its reference, then what about examples such as 'I think that the morning star is very bright'? In this sentence, if we replace 'morning star' by 'evening star,' the resulting sentence is not necessarily a true sentence anymore, whereas based on Frege's explanation whenever we replace a name by another one with the same reference, the resulting sentence must retain the same truth value. Do we not then have a counterexample to the claim that the conceptual content of a name is its reference? Doesn't this example show that although the names are co-referential, they do not have the same conceptual content? Frege himself, in "On Sense and Reference", sees this problem and replies to the objection by claiming that in such sentences a name does not refer to its ordinary reference, but has an indirect reference. He says that "one might just as well claim that the reference of 'morning star' is not Venus, since one may not always say 'Venus' in place of 'morning star'. One has the right to conclude… that 'morning star' does not always stand for the planet Venus, *viz.* when the word has its indirect reference."[92] He suggests that we have to apply quotation marks to words that do not have their customary reference. In our example, 'morning star' does not refer to Venus, but to the string of words or the sign 'morning star', and does not have the same reference as 'evening star' and,

[92] "On Sense and Reference," p. 45.

consequently, cannot be replaced by that *salva veritate*.[93] In other words, in our example, 'morning star' and 'evening star' do not have the same conceptual content. Therefore, the claim that the conceptual content of a name is its reference, is not refuted by such examples.

Pavel Tichy appears to disagree with the view that the theory of multiple analyses has nothing to do with the reference of a name, and claims that the different analyses could be applied not only to the sense of a name but also to its reference. As such, one may conclude that, even though the conceptual content of a name may only be its reference, yet it can be affected by the multiple analyses. Tichy argues that Frege tried to solve the problem of logical adhesion by introducing functions as unsaturated entities. He argues that

> The Function combines with the argument into self-contained *whole*. '*W*hat holds the two together?' Frege asked. Why is it that the whole does not disintegrate to its two ingredients? Frege saw a key to answering these questions in the assumption that a Function by itself is not self-contained. It contains a logical gap which needs filling. This is why the Function latches on to its argument, sticking to it as if though a suction effect.[94]

Tichy reads Frege as claiming that the whole (the object), which results from saturating a function with an argument, is the value of the function for that argument. Thus, according to Tichy, Frege's line of thought presupposes that objects such as number two and the True, consist of functions and arguments. For instance, the number two is the result of "the saturation of the first-level Function log ξ by the Object 100,"[95] or "the

[93] *Ibid*., pp. 44-5.
[94] *The Foundation of Frege's Logic*, 2012. p. 27.
[95] *Ibid*., p. 41.

saturation of the second-level Function ϕ(100) by the first-level Function log ξ."[96] So, it is the integrity of the number two that, according to Frege, would be protected by the unsaturatedness of functions. In other words, it is the unsaturated nature of the functions which allows the number two to be an integral whole, rather than a mere set of parts, preventing its disintegration into function and argument. Tichy concludes that, according to Frege, all these different analyses into functions and arguments must coexist in the object (reference of the name) itself.[97] For example, since 2=5-3, the number 5, 3 and the subtraction function would be part of the number 2. It is to say that, according to Frege, the theory of multiple analyses could be applied to the objects at the level of reference and, hence, it affects the conceptual contents of the singular terms.

However, for me, it is strange to believe that we can find, for instance, any trace of Aristotle in Aristotle's hat (Aristotle could not be a part of Aristotle's hat). Even Frege himself denies such a conclusion saying that "[t]hings are different in the domain of meaning [reference]. We cannot say that Sweden is a part of the capital of Sweden. The same object can be the meaning [reference] of different expressions, and any one of them can have a sense different from any other."[98] Besides, as far as I am aware, there is no textual evidence in Frege's works that supports Tichy's interpretation. Quite to the contrary, whenever Frege talks about multiple analyses, it seems as if what is analyzable is at the level of expression and sense, not at the level of reference; which suggests that if we can analyze or carve up the conceptual content of a *sentence* in various ways, it is

[96] *Ibid*., p. 41.
[97] *Ibid*., pp. 40-42.
[98] *Posthumous Writings*. p. 255.

because we are dealing with its *thought*.[99] In no way do Frege's explanations suggest that we can carve up the number two (the reference or object) in different ways. Further, every time that Frege talks about different analyses of an expression other than sentences, he considers only either the expression itself or its sense to be analyzed, but never its conceptual content (or reference). Accordingly, when he talks about carving up or analyzing the conceptual content of an expression, he is taking only sentences into consideration. As a final point, there is no textual evidence to prove that Frege considered the reference of the parts of a name as the parts of its reference.[100] Hence, Frege never would have agreed that the parts that we discover in expressions and their corresponding senses could be discerned in the object itself. It means that Frege's theory of multiple analyses does not apply to the conceptual content of a singular term, and that Tichy's interpretation, as I understand it, seems to be inconsistent with Frege's view.

2-3.3) Multiple Analyses and Judgeable Contents

Based on what I explained, I believe that Frege applies his theory of multiple analyses only to the conceptual content of sentences, and not to the conceptual content of singular terms. However, even the application of this theory to the conceptual content of a sentence (or what Frege later calls the thought of a sentence) creates difficulties. In this section, I will examine, first, if Frege's theory of multiple analyses is inconsistent with his accounts of the thought of a sentence and, second, since conceptual content is what Frege wants to be expressed without ambiguity in his *Begriffsschrift* formulas, I

[99] *The Foundation of Arithmetic*. p. 67.
[100] Frege in "On Sense and Reference" pages 35 and 36 does say that a judgment is a distinction of parts within a truth value, but he only was talking about judgeable contents. He later retracts that statement as well. See, for example, *Frege's Lectures on Logic,* p. 87.

will investigate whether Frege's multiple analyses lead to imprecision in his ideal language.

As to the first question, based on Frege's theory of multiple analyses,

(1) a thought can be analyzed differently into parts.[101]

Frege also argues that he does not start from concepts to form a thought, but rather gets to the parts by splitting up the thought; thus

(2) the realization of a complete thought precedes concept formation.[102]

It is only possible to get to different concepts from one and the same sentence if we accept the first claim. For instance, in our previous example, 'Brutus killed Brutus' (K_{bb}), we can have different parts for the same thought; in the third analysis in section 2-2, $<\lambda_{x,y}.K_{xy}>(b,b)$, we have a relation (function with two arguments), whereas in the fourth analysis, $<\lambda_x. K_{xx}>$ (b), we have a concept (function with one argument). For Frege, functions with two arguments and functions of one argument are intrinsically different,[103] yet the same thought can present either of these intrinsically distinct functions as a part.

However, Frege in other places claims that

(3) "…thoughts have parts out of which they are built up. And these parts, these building blocks, correspond to groups of sounds, out of which the sentence expressing the thought is built up, so that the construction of the sentence out of

[101] *The Frege Reader*, (1997), "On Concept and Object," p. 188.
[102] *Posthumous Writings*, 1979. p. 253. Also p. 17.
[103] *Basic Laws* I, p. 37.

parts of a sentence corresponds to the construction of a thought out of parts of a thought."[104]

The received interpretation of claim (3) suggests that, according to Frege, a thought has fundamental parts like its atoms, and the thought of a sentence must have only one fixed set of constituent parts. This claim also suggests that a thought can be assembled from concepts. Based on (1) and (2), for Frege, a thought can be analyzed into different sets of constituent parts, whereas, based on the received interpretation of (3), there is only one such set of parts for each thought. Thus, there seems to be inconsistency in Frege's account of thought.

A number of philosophers tried to justify these apparently contradictory claims. Dummett, for instance, argues that in Frege's semantics we are dealing with two different notions of analysis, and even though Frege does not explicitly (or even intentionally) distinguish these two notions, he uses them in practice.[105] Dummett calls these two different kinds of analyzing a thought, *decomposition* and *analysis*. He claims that whenever Frege talks about the process of dividing a thought into its parts in different ways (as in claims (1) and (2)), he is talking about the decomposition of the thought, and whenever Frege talks about the fundamental constituents of a thought from which the thought is built up (as in claim (3)), he is talking about the analysis of the thought. For Dummett, analysis corresponds to the process of dividing a molecule into its constituent atoms; there is only one way of doing this, and we will eventually reach the fundamental constituents. Decomposition, by contrast, corresponds to the division of a country according to its regions, spoken languages, or other features; there

[104] *Posthumous Writings,* 1979, p. 225.
[105] *The Interpretation of Frege's Philosophy*, 1981. Section: Alternative Analyses, pp. 261-291.

are many different ways to do this and every time we get different parts that in fact could not ultimately be considered to be the fundamental constituents of the country. If we accept Dummett's distinction between two different uses of the term "analysis" in Frege's philosophy, then the inconsistency between claims (1), (2), and (3) arguably vanishes. Because, when Frege is talking about analyzing a thought in different ways, every time getting different constituents, the notion of analysis is different from analyzing a thought in a way that gives us a fixed set of constituents which are the fundamental constituents of the thought. Frege is using two different processes in analyzing a thought; in the first process (claims number 1,2) we get different sets of functions and arguments based on our way of parsing the thought, whereas in the second process we are dealing with only one set of functions and arguments that are the main constituents of the thought. So, the difference between Frege's claims is due to the difference between two different notions of analysis: the process of analysis of a thought in the claims (1) and (2) is different from the claim (3). However, I will later approach this question from a different perspective.

2-4) The Challenge of Multiple Analyses for Frege's Ideal Language

Even though we can resolve the above inconsistency among Frege's claims based on Dummett's interpretation, I believe that Frege's theory of multiple analyses creates yet another challenge for his notion of thought and, eventually, for his ideal language. Contrary to what Frege holds, it seems that different analyses of a sentence result in different thoughts for the same sentence and, thus, cause ambiguities in the conceptual notations of his ideal language. In an unpublished essay of 1881, Frege says that in the conceptual content of the sentence '$2^4=16$', the number 2 can be replaced by something

else (e.g., -2 or 3).[106] Thus, we can replace 2 by x, and come up with the reading '$x^4=16$' and '2'. This reading means that the sentence '$2^4=16$' can be split up into two parts; the proper name (2) and the function name ($x^4=16$: x is a fourth root of 16). The same sentence can also be split up differently if we take '4' as variable and come up with the reading '$2^x=16$, 4'. This second reading means that the sentence can be split up into the proper name '4' and the function name 'x is the log of 16 to the base 2'. Consequently, the same sentence can be rendered as (1) '2 is a fourth root of 16' [$<\lambda_x.\ x^4=16>(2)$ or $\sqrt[4]{16} = 2$], and (2) '4 is a logarithm of 16 to the base 2' [$<\lambda_x.\ 2^x=16>(4)$ or $\log_2 16 = 4$].

However, it would not be guaranteed that these different notational readings express the same thought or conceptual content. To explain, it is quite possible that one understands the first reading and knows that 2 is a fourth root of 16, whereas s/he does not understand the second reading and wonders if it is true that 4 is the log of 16 to the base 2. So, we do not have sufficient reason to take these two readings to convey the same thought. In fact, Frege himself says that "[a]nybody who did not know that the evening star is the morning star might hold the one to be true, the other false" and from here he concludes that the sentences 'the morning star is a body illuminated by the Sun' and 'the evening star is a body illuminated by the Sun' express different thoughts.[107] Frege also claims that according to the laws of logic and in relation to inference, concept names, like proper names, can replace each other *salva veritate,* if the extension of the concepts is the same.[108] He adds that the thought of the sentence under such a replacement would be altered, but its reference will not. It means that it is possible for

[106] "Boole's Logical Calculus and the Concept-script," *Posthumous Writings*, p. 16.
[107] "On Sense and Reference", p. 41.
[108] "Comments on Sense and Meaning," *Posthumous Writings*, p. 118.

someone, who does not know that two concepts have the same extension, to take a sentence with the first concept as true and the same sentence with the second concept occupying the place of the first as false. Similarly, we can conclude that, according to Frege, if one cannot see that two different readings of the same sentence convey the same thought, then the thoughts of the two readings are different. It seems that based on the theory of multiple analyses, in Frege's ideal language there would be a one-to-many relation between a sentence and different thoughts or conceptual contents. This result appears to violate Frege's notion of thought and also seems to undermine his attempt to build an ideal language.

Even if we accept Frege's claim that multiple analyses do not change the thought of a formula and, hence, does not give rise to a one-to-many relation between a formula and different thoughts, the theory of multiple analyses would still create another problem for his ideal language. According to the theory of multiple analyses, we could have different *Begriffsschrift* formulas that all express the same conceptual content, for example, (A→(B→C)) and (B→(A→C)). However, Frege's way of introducing the conceptual content in *Begriffsschrift* suggests that for all the sentences with the same conceptual content there should be only one conceptual notation in the ideal language. To explain, according to him, the only thing that a conceptual notation expresses is the conceptual content of the sentence, or the part of a sentence that is necessary for a logical inference. Frege says that "[s]ince *only this* [conceptual content] is meaningful for "conceptual notation", we do not distinguish between propositions [sentences] which have the same conceptual content."[109] Hence, to have different conceptual notations

[109] *Begriffsschrift*, §3.

such as $\sqrt[4]{16} = 2$ and $\log_2 16 = 4$ that according to Frege, express the same conceptual content or thought seems to go against his stated intention. However, we see that he allows these notations in his ideal language. This suggests that there is also a many-to-one relation between *Begriffsschrift* formulas and their conceptual content. This is in tension with Frege's desire that in the ideal language there should be a one-to-one relation between formulas and conceptual contents.

2-5) Resolving the Challenge of Multiple Analyses for Ideal Language

The theory of multiple analyses seems to be incompatible with Frege's program to formulate an ideal language. That is, since *Begriffsschrift*, Frege had been trying to develop a language within which the conceptual content (thought) of a sentence could be expressed precisely and unambiguously. But, based on my discussion in section 2-4, the possibility of various ways of analyzing a sentence into functions and arguments seems to result in different conceptual concepts for any given formula in Frege's ideal language, and eventually to the problem of one-to-many relation between a sentence and its conceptual content or thought. Also, in Frege's ideal language, different formulas may have the same conceptual content, which creates the problem of many-to-one relation between a conceptual content and its corresponding sentences. These both seem incompatible with the notion of an ideal language which is supposed to match conceptual contents and formulas one-to-one. If we hold that there must be a one-to-one relation between formulas and conceptual contents in an ideal language, then, based on what I discussed in section 2.4, Frege's attempt to develop an ideal language would seem to be in vain.

However, I believe that we might interpret both Frege's notion of conceptual content and his definition of ideal language in a way that meets the above challenge. I think that we can find hints for such an interpretation in Frege's own works. Although Frege may have at first intended to replace all the different sentences with the same conceptual content by a single formula in his ideal language, such that each formula expresses exactly one conceptual content, and conversely, it is clear that he allowed that there could be more than one formula for each conceptual content.[110] Furthermore, Frege does not consider a formula in isolation, but rather as embedded in a logical system and based on the role that it plays in logic. That may be why, in *Begriffsschrift* §3, he defines conceptual content of judgeable content contextually. If I am right in this interpretation, then Frege's system might well be able to stand up to the challenge explained above. I will now investigate this possibility.

2-5.1) The Problem of One-to-Many Relation

In section 2-4, I interpreted Frege's example about a sentence containing the phrase 'morning star' or 'evening star' in a way that would not guarantee that the different readings of a sentence convey the same thought, since these different readings may have different cognitive values; someone might understand one and not understand the other. At another place, likewise, Frege suggests a cognitive test for identity of thought:

> …Two sentences A and B can stand in such a relation that anyone who recognizes the content of A as true must thereby also recognize the content of B as true and, conversely, that anyone who recognizes the content of B must straightaway also recognize the content of A as true. (*Equipollence*) It is here being assumed that there is no difficulty in grasping the content of A and B.

[110] See "Compound Thoughts", p. 4, and *The Foundations of Arithmetic*, §64, and also *Basic laws* I, II, §10.

> …one has to separate off from the content of a sentence the part that alone can be accepted as true or rejected as false. I call this part the thought expressed by the sentence. It is the same in equipollent sentences of the kind given above.[111]

This passage could also be interpreted to suggest that if two sentences have different cognitive values then their thoughts may be different too. This, in turn, might be taken to mean that the difference in cognitive values, i.e., in how people take things to be true, is the basis for the difference in the thoughts. Such a subjective, psychological view, however, would seem to go against the core of Frege's philosophy, in particular his claim that "[t]he psychological is to be sharply separated from the logical, the subjective from the objective."[112] Thus, this interpretation would seem to be untenable.

To explain, it is true that, in his example of 'the morning star' and 'the evening star', Frege says that if we replace one word of a sentence with another one that has the same reference but a different sense, then the thought of the sentence would change, because someone may take the first thought as being true, while the second one as false. However, we might consider Frege as not basing his argument based on the claim that 'if a person does not understand the two sentences in the same way, then the sentences express different thoughts.' On the contrary, we might say that, for Frege, since two words that are replaced by each other have different senses, then the two resulting sentences express different thoughts. That a person understands two sentences differently is *evidence* for such a claim, but not the *reason* why it is true; that is, since the senses of the sentences are different, the cognitive value of the sentences are different, but not necessarily *vice versa*.[113] Hence, we might interpret Frege as merely claiming that if a word in a sentence is replaced by another one with a different sense,

[111] *Posthumous Writings*, pp. 197-8.
[112] *The Foundations of Arithmetic*, p. 17.
[113] "On Sense and Reference," p. 41.

then the thought of the resulting sentence will be different from the thought of the original sentence. Although this interpretation of Frege's claim is not entirely in accordance with the way Frege puts things (namely, in terms of what people take to be true), it is in accordance with his total philosophy, because in the claim so understood no psychology is involved. This new interpretation resolves the problem caused by the previous interpretation in the section 2-4; namely, the problem that different analyses of a sentence appear to result in a one-to-many relation between a sentence and its conceptual content in an ideal language.

Further to the above point, let's review Frege's own example;[114] the formula '$2^4=16$'could be divided in many ways resulting in different sets of function(s) and argument(s) as its parts. Based on what I concluded in section 2-4, it would seem that we have a different thought or conceptual content every time that we analyze the formula differently. We could come up, for instance, with these two readings; (1) '2 is a fourth root of 16' [$<\lambda_x.\ x^4=16>(2)$ or $\sqrt[4]{16} = 2$], and (2) '4 is a logarithm of 16 to the base 2' [$<\lambda_x.\ 2^x=16>(4)$ or $\log_2 16 = 4$], or etc. However, it seems quite possible that one understands the first reading and knows that 2 is a fourth root of 16, whereas s/he wonders if it is true that 4 is the log of 16 to the base 2. So, we do not have sufficient reason to take these two readings to convey the same thought. Yet, Frege would disagree with this result, arguing that, if we understand these two sentences perfectly, then as soon as we accept the first one as true we immediately accept the second one as true. If we do not do this it is because we do not understand them perfectly, because there may be difficulty in grasping the content of these two readings, not because they have

[114] "Boole's Logical Calculus and the Concept-script," *Posthumous Writings*, p. 16.

different thoughts.[115] Frege constantly says that "[w]hether the [truth] or falsity of a thought can be apprehended with greater or less difficulty is of no matter from a logical point of view, for the difference is a psychological one."[116] He would further argue that since all of these readings have the same logical consequences, all of them have the same conceptual content. In all of his works, Frege clearly confirms that the logical consequences derived from two sentences *determine* whether they have the same conceptual content: if the same consequences can be derived from sentence A joined with an arbitrary set of other sentences, and from sentence B joined with the same set of sentences, then sentences A and B have the same conceptual content.

Further to this point, Frege uses equivalence classes to define what he meant by numbers in *The Foundations of Arithmetic*, so the same option would seem to be open to him here. When we have an equivalence relation we can form an equivalence class for each sentence. When, for example, we have Fa^Ga, we may deduce any of the following: ∃x((Fx^Gx) & x=a) or ∃x((Fa^Gx) & x=a) or ∃x((Fx^Ga) & x=a). (Frege's system does not have an operator to represent the existential quantifier, but it is equivalent to '¬∀x¬', and his substitution rule for the universal quantifier yields basically the same result.)[117] According to his system, for example, we can instantiate from the universal formula ∀x∀y(x+y=y+x) and deduce 3+2=2+3, but we also can get the same sentence from ∀x(x+3=3+x) or ∀*(3*2=2*3). Similarly, we can obtain K_{bb} from any of the universally quantified formulas $\forall_x K_{xb}$, $\forall_x K_{bx}$, or $\forall_x K_{xx}$. For Frege conceptual content is what matters for logical inference, so if all the different analyses of a formula yield the

[115] *Posthumous Writings*, p. 197.
[116] "Compound Thoughts," p. 12.
[117] *Basic Laws I*, §20. We would also need the law for the substitution of identicals to obtain the above results.

same consequences in logic, then all of them have the same conceptual content and thus convey the same thought. This provides further support for the claim that the theory of multiple analyses does not result in a one-to-many relation between a sentence and its thought in an ideal language.

Finally, note that replacing a part of a sentence with another one with the same reference and different sense is fundamentally different from what happens in multiple analyses of a sentence. In the first case, we replace one name with the other one while the rest of the sentence remains the same, whereas in the different readings of a sentence we are not replacing a part of a sentence with another part that has the same reference and different sense; rather, we are dividing the same sentence over and over into parts that each time the parts have totally different references and senses. Thus, the examples of replacing one name by another in a sentence is irrelevant to the discussion of multiple analyses, and cannot be used to support the idea that there are as many thoughts as there are readings. The multiple analysis of a sentence is similar to dividing a rectangle once into two triangles, and another time into three smaller rectangles or any other shapes that we like. Although, the sets of shapes (parts) are very different in each division, the original rectangle remains the same. The same analogy can be used for a thought that can be analyzed in different ways. Based on my discussion in this section, Frege's system is sound as long as we keep psychological interpretations away from the logical ones, the cognitive from the thought, and also keep in mind the analogy that I've just explained. Consequently, the theory of multiple analyses would not result in a one-to-many relation of a sentence to conceptual contents in Frege's ideal language.

2-5.2) The Problem of Many-to-One Relation

However, I believe that the problem of many-to-one relation in Frege's ideal language is yet to be solved. Frege says that

> The judgment 'Line a is parallel to line b', in symbols: a || b, can be taken as an identity. If we do this, we obtain the concept of direction, and say: 'The direction of line a is equal to the direction of line b'. Thus we replace the symbol || by the more generic symbol =, through removing what is specific in the content of the former and dividing it between a and b. We carve up the content in a way different from the original way, and this yields us a new concept.[118]

In this passage, Frege argues that we can carve up the same content in different ways. He also says that two sentences in the passage convey the same sense but in different ways.[119] Famously, Frege later stated a logical law which allows us to derive the two aforementioned sentences from each other.[120] Recalling the definition of the sameness of conceptual content in term of logical consequences we see that these formulas have the same conceptual content. This indicates that the same conceptual content could be expressed by different *Begriffsschrift* formulas.

Frege repeatedly claims that two very different formulas have the same conceptual content, provided that they are inter-derivable. He explains that the statement that 'two functions have the same value for the same arguments' is the same as saying that 'the value-range of the first function is equal to the value-range of the second one', because these two statements are logically interchangeable. It means that according to Frege, for example, '$x^2-4x=x(x-4)$' and '$\grave{\varepsilon}(\varepsilon^2-4\varepsilon)=\grave{\alpha}(\alpha.(\alpha-4))$' express the same thought and refer to the same truth value, because they have the same logical

[118] *The Foundations of Arithmetic*. § 64, p. 67.
[119] *Ibid*., p. 11.
[120] *Basic laws* I, §20 (V).

consequences.[121] Based on these considerations, Frege seems to allow different *Begriffsschrift* formulas for the same conceptual content. Thus, although in the beginning Frege believed that there must be a one-to-one relation between a sentence and its conceptual content in an ideal language, he later realizes that it is not an attainable requirement. Accordingly, the theory of multiple analyses of a sentence is in accordance with Frege's ideal language, because a many-to-one correspondence between formulas and conceptual contents is allowed.

Incidentally, it occurred to me that it might have been better if Frege presented conceptual content as a third notion distinct from sense and reference. Frege claims that '¬¬B' and 'B' convey the same thought[122]. However, it seems to me that '¬¬B' and 'B', express two different modes of presentation of one and the same truth value, so it seems reasonable to suppose that they must have two different senses. But, these two sentences have the same logical consequences, though they have the same conceptual content. Thus, contrary to what Frege suggests, sameness of sense must somehow be different from sameness of conceptual content. [123]

2-5.3) Concluding Point

Finally, it should be noticed that an important aspect of Frege's ideal language is that there are no function names in isolation, and even a first level function name unaccompanied by proper names as arguments must be attached to a second level function name. Frege claims that he gets to the parts of a thought, concepts and

[121] "Function and Concept", 1891. pp. 10, 11. It should be mentioned that in *Basic laws* I, II §10, he says that these two sentences are "gleichbedeutend", synonymous. However, some translated that as co-referential.
[122] "Compound Thoughts," p. 10.
[123] What Frege says in *Begriffsschrift* and *The Foundations of Arithmetic* about conceptual content, he later says about thought.

arguments, by analyzing the thought;[124] he continues that we should not start from previously formed concepts, but the opposite, we should arrive at the concepts by splitting up the content of a possible judgment.[125] It means that in Frege's ideal language there are only complete expressions and no unsaturated entities in isolation. Yet Frege in other places talks about parts of a thought that have something corresponding to them as their reference.[126] He calls these parts the building blocks of a thought.[127] So, it seems that Frege's theory of thought varies from time to time.

To resolve the apparent inconsistency of Frege's notion of thought, we need to consider the way he introduces the notion of a concept (or of a function in general). For him, the claim that a judgeable content is analyzable does not presuppose that the content must already be itself articulated (i.e., the different concepts resulting from the analysis are already formed). For, Frege claims, the ideas of these concepts could not be formed apart from the objects that fall under them. The behavior of concepts could be compared to the behavior of the atom: "we suppose an atom never to be found on its own, but only combined with others, moving out of one combination only in order to enter immediately into another".[128] Thus, for Frege, a sign for a function never appears without an argument.

In addition, Frege's theory of multiple analyses allows us to distinguish all of these functions and arguments within a complete formula by the fact that all of these readings are mutually deducible according to the laws of logic. So, it is that logical

[124] *Posthumous Writings*. p. 253: "I do not begin with concepts and put them together to form a thought or judgment; I come by the parts of a thought by analysing a thought."
[125] *Ibid*., p. 17.
[126] *Ibid*., p. 255.
[127] *Ibid*., p. 225.
[128] *Ibid*., p. 17.

machinery that is going to give us all the different readings of a sentence, by showing us the way to get from the first reading to the second one; for instance, the function K_{xx} is embedded in the formula K_{bb} and so on. The theory of multiple analyses allows us to get a different set of building blocks from each analysis of a same thought.

In Summary, if we interpret Frege's line of thought in this way that, firstly, conceptual content is defined contextually and, secondly, conceptual content is what matters for logical inference and, thirdly, all the multiple analyses of a sentence will somehow yield the same logical consequences, and, lastly, no psychological view should be allowed in, then Frege's system will meet the challenge of multiple analyses. I have to add that it would have profited Frege if he had developed a notation such as lambda calculus in his *Begriffsschrift,* which could distinguish these different readings and display the logical rules that allow one reading to be deduced from the other.[129]

Chapter Three: Analyzing Identity Statements

The use of identity statements in both ideal and natural languages is unavoidable, yet analyzing these statements appears to be problematic. In examining Frege's point of view on identity, first, I will critically examine his theory of identity in *Begriffsschrift* and will explain the difficulties in this theory which compelled Frege to develop his second theory of identity in his later work "On Sense and Reference." However, I will show that his latter theory of identity also faces complications that may have been overlooked by Frege.

[129] This is an interesting extension that is suggested by some interpreters such as Mendelsohn, in his book *The Philosophy of Gottlob Frege*, Chapter 6. Also see references in Tichy page 42.

In order to shed more light on the question of the nature of identity, I will further evaluate Frege's position regarding identity statements with reference to his theory of multiple analyses. At the beginning of "On Sense and Reference," Frege asks if equality [identity] is a relation. However, it seems that he assumes that equality *is* a relation without further ado, since he is only occupied with the terms of an identity statement rather than whether it is a relation or not. Using Frege's notion of multiple analyses, I will argue that this is an important oversight. I will apply predicate abstraction notation in analyzing identity statements to distinguish a number of different possible readings as to the nature of identity statements. I will then argue that the puzzle of cognitive significance of identity statements such as 'a=a' or 'a=b', raised by Frege at the beginning of "On Sense and Reference", might well be solved without appealing to the notion of sense.

3-1) Frege's Theory of Identity in *Begriffsschrift*

In *Begriffsschrift*, Frege claims that identity statements are different from the rest in that in an identity statement the symbols stand for themselves, whereas elsewhere they stand for their designata. For example, in a statement such as (1) 'Aristotle wrote the *Organon'*, the name 'Aristotle' refers to a person who is the subject of the sentence, but once the name 'Aristotle' enters into an identity statement, it stands for itself. Hence, according to the *Begriffsschrift* account, in identity statements such as (2) 'Aristotle = The author of *Organon'*, the name 'Aristotle' as a linguistic expression enters into the identity relation, but not the person who is designated by this name. It appears that, according to Frege, the sentence (1) talks about the objective world outside language and in it the term 'Aristotle' refers to a person, whereas the sentence (2) is only about

linguistic signs and the term ‘Aristotle’ there merely stands for itself. It seems that Frege radically changes his semantics in interpreting identities. But why does he do this? Why, according to him, does the word ‘Frege’ refer to a person except when it is used in an identity statement?

The answer to this question rests on the reason behind the formation of his formula language, the Conceptual Notation. In *Begriffsschrift,* Frege introduces his conceptual notation in order to express the conceptual content of a sign or a sentence, and to avoid the ambiguities of ordinary language. Conceptual content is the part of the content of a name or a sentence which is important for logical inferences. Contrary to the case of ordinary language, Frege’s conceptual notation is supposed to introduce only one name for every content. But, Frege realizes that if in his ideal language for every content there is only one name, then in such a language a true and informative identity statement of the form ‘a=b’ cannot be given a satisfactory interpretation, because such a statement can be interpreted as being true only if ‘a’ and ‘b’ have the same content. To explain this point, consider the following example: if ‘a’ designates α and ‘a=b’ is a true statement, then ‘b’ must designate α too, because according to Frege the identity statement means that the two names have the same conceptual content. Therefore, all identity statements of the forms ‘a=a’ and ‘a=b’ should have the same conceptual notation of the form α=α, which is not informative, because it is merely a rehearsal of the Identity Law that everything is identical to itself. Furthermore, it would transform all identity statements into logical truths. Thus, if the ideal language introduced only one name for every content, then the identity sign would have no significance in the

ideal language of *Begriffsschrift*.[130] However, the usefulness of the identity sign in arithmetic and science and its importance for the conceptual content of mathematical formulae is too obvious to be overlooked. Hence, Frege had to provide an explanation of the content of identity statements in a way that makes the statements informative.

Frege sees only two options in *Begriffsschrift*: in an identity statement 'a=b', the conceptual content of the two signs 'a' and 'b', the relata of the identity relation, could be either entities designated by the signs, or the signs themselves. Having excluded the first option, he concludes that once a sign enters into an identity relation, it no longer represents its corresponding content, but rather represents itself. Accordingly, in the content of an identity statement, unlike other statements, we must include the signs. And, as a result, the content designated by a name would be the name itself. It suggests that the identity relation is not a relation between contents, but between symbols and, hence, Frege claims: "[i]dentity of content differs from conditionality and negation in that it applies to names and not to contents."[131]

However, Frege realizes that his claim may give the impression that an identity statement concerns only linguistic facts, and not facts about the world. So, it may appear that an identity statement would pertain only to linguistic expressions, and there would be no point in having different symbols in *Begriffsschrift* that present the same content. Consequently, there would be no reason to include the identity sign in a conceptual language such as *Begriffsschrift*. To prevent this misunderstanding, Frege says that an

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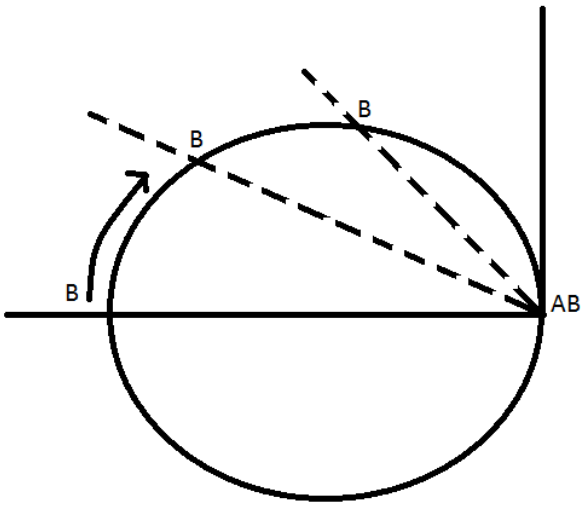
[130] More to this point, Wittgenstein believes that identity-sign has no significance for an ideal language. He says that "Identity of object I express by identity of sign and not by using a sign for identity. Difference of objects I express by difference of signs. … Roughly speaking, to say of *two* things that they are identical is nonsense, and to say of *one* thing that it is identical with itself is to say nothing at all…. The identity-sign, therefore, is not an essential constituent of conceptual notation." *Tractatus*, §5.53-§5.535.

[131] *Begriffsschrift*, §8.

identity statement is not just about the expression, because with each name there is associated what he calls a *mode of determination*.[132] Frege does not explain this new term; he only gives an example to illustrate what he means by it. He remarks that in some cases we need two names for the same content since the content is given by different modes of determination, hence, having different names for the same content is not an insignificant matter. In his example, Frege fixes a point A on the perimeter of a circle. Now imagine a straight line passing through A that intersects with this circle at the point B (the point of intersection). Next imagine that this line rotates about the point A.



When the line is tangent to the circle, the point of intersection of the line and circle (B) overlaps with point A. It is understandable, Frege says, that at this time 'A' and 'B' name the same point, but differently, the former determining the point as "directly given through intuition" while the latter determines it as the intersection of a circle and a line. It is also clear, he says, that we could not have the same name from the beginning,

[132] It seems that 'mode of determination' here means what Frege would later call the *sense* of the linguistic expression.

because the content of B varies with the angle of the line AB. Accordingly, Frege claims that "[t]he need for a symbol for identity of content thus rests on the following: the same content can be fully determined in different ways; but *that*, in a particular case, *the same content* is actually given by *two modes of determination* is the content of a *judgment*."[133] To form this judgment, one needs two names, each with its own mode of determination, and an identity symbol to relate those two names. Frege concludes that if 'A ≡ B'[134] is true, it follows that the symbol 'A' and the symbol 'B' have the same conceptual content although their modes of determination differ. Giving this argument, Frege believes that his identity theory is satisfactory.

3-2) Some Difficulties with the *Begriffsschrift* Theory

Frege's proposed solution, however, creates a number of complications in his logical system. One problem arises from the fact that most of the time there aren't as many names as there are objects. For instance, when we talk about real numbers, do we want to say that ∀x∀y (x+y=y+x) holds for all real numbers, or only for those that have names? When we quantify over a domain, we want to quantify over all the possible objects in that domain, not only the ones that have names. Therefore, if in an identity statement such as 'a=b' the relation holds between names, then, once we have formulas with variables such as 'x=y', they would appear to hold only for objects that have names.

In a reply to Thau and Caplan, Richard Heck[135] brings up another complication regarding Frege's theory of identity. Heck argues that when we quantify, for instance, in

[133] *Begriffsschrift*, §8.
[134] In *Begriffsschrift*, Frege uses this symbol '≡' as the sign for equality, however he later returns to the original sign of equality '='.
[135] "Frege on Identity and Identity-Statements: A Reply to Thau and Caplan," p. 4.

the formula ∀x∀y F(x,y), the values of the variables are objects, not names. Yet, in the *Begriffsschrift* theory of identity, Frege claims that the identity relation is about names, so, in the formula, 'x=y' the values of variables 'x' and 'y' are names. This creates a difficulty, because formulas such as 'F(x, y)' and 'x=y' can both occur in the scope of the same quantifier. For example, Frege claims that, for all x and y, if x=y then f(x)→f(y): while in the first statement, based on *Begriffsschrift* theory, 'x' and 'y' are to be replaced by names, in the latter, according to Frege, they must be replaced by objects. Hence, the claim '∀x∀y (x=y →f(x)→f(y))' threatens to become incoherent. As we can see, Frege's semantic account of identity forces him to take 'a' as standing for a name in the identity statement while 'a' stands for an object in the function.

Another problem is that Frege thinks that it is only in identity statements that we are confronted with this problem of whether names stand for themselves or for their contents. However, names in other statements may also face a similar challenge. For instance, the names 'The author of *Nicomachean Ethics*' and 'Aristotle' both stand for the same person and therefore have the same content. Hence, if *Begriffsschrift* formulas are only supposed to represent the content of the names, then the conceptual notation for these two names should be the same. As a result, there must be only one *Begriffsschrift* notation that corresponds to all the different names and descriptions that have the same conceptual content. This would have been particularly inconvenient in arithmetic, in which we need to have many different names for the same number to formulate mathematical theorems. For instance, Frege could only have '2' as the conceptual notation of '$\log_{10}100$', or '6/3', or any other expression that refers to the number 2, and thus no theorem of mathematics could be formulated in his ideal

language. Frege knew full well that identity is needed in mathematics, thus it seems to me that his argument should have addressed this issue as well.

The next objection would be that Frege's argument in *Begriffsschrift* about the terms that enter into an identity relation is incomplete. He introduces the new concept of 'the mode of determination', which apparently plays an important role for identity, but it is not clear how this new concept is supposed to fit into his logical system. On one hand, Frege's justification for the need for an identity sign is based on the fact that the same content could be given by different modes of determination, and these different modes of determination, in turn, must be expressed by different signs. On the other hand, according to the arguments in the *Begriffsschrift*, the mode of determination cannot be part of the content of the names since two names with the same content could have different modes of determination, as is the case for 'A' and 'B'.

It looks like that Frege adds a new level to his semantics by saying that 'A' and 'B' correspond to different ways of determining the same content: there is the sign at the level of language, and there is the object designated by the sign at the level of reference, and besides that, there is the mode of determination that the sign corresponds to, which apparently is neither at the level of language nor at the level of reference. Frege does not explain what exactly the mode of determination is; however, it is clear that he includes the mode of determination in the content of a judgment about identity. He says "that in a particular case *two ways of determining it [an object]* really yields the *same result* is the content of a *judgment*."[136] However, in *Begriffsschrift,* Frege gives '—A≡ B' as the content of the judgment '⊢A=B', which does not represent the modes of determination.

[136] *Begriffsschrift*, §8.

So, contrary to what Frege says above, the mode of determination is not presented in his formulation of the content of a judgment about identity. Neither is the mode of determination signified in the content of an identity statement itself: in §8, Frege concludes that a true statement such as 'A ≡ B' *means* that "*The sign A and the sign B have the same conceptual content, so that we can everywhere put B for A and vice versa.*"[137] According to this, no trace of mode of determination can be found either in the content of the identity statement or the content of the judgment about identity, even though, according to Frege, the mode of determination is of significance for identity statements. Whatever the mode of determination in *Begriffsschrift* means, it looks like it is what Frege later calls the sense of a name.

Finally, I noticed that although Frege in *Begriffsschrift* claims that in identity statements symbols stand for themselves, he does not reject the idea that they concern their contents as well. When Frege claims that in the statement 'a=b', 'a' and 'b' stand for themselves, clearly the identity statement does not mean that the sign '*a*' and the sign '*b*' are identical, because they obviously are not. In fact, in *Begriffsschrift* §8, Frege himself states that it may appear that the identity relation pertains only to the linguistic expressions and not to the content, but this appearance is deceptive. From what Frege says, one might conclude that since a name designates an object, if an identity statement is about names that designate the same content, then an identity statement is equally about *the names and the thing* that is designated by the names.[138]

[137] *Begriffsschrift*, §8.
[138] Cf. Mendelsohn. *The Philosophy of Gottlob Frege*. Section 4.3.

It seems obvious that the *Begriffsschrift* theory of identity creates some problems for Frege's system. He recognizes some of them and later, in an attempt to solve these problems, he presents his theory of sense and a new theory of identity. To this I now turn.

3-3) Frege's Theory of Identity in "On Sense and Reference"

In a later article entitled "On Sense and Reference", Frege modifies his theory of identity. He asks if identity is a relation and, if so, whether it is a relation between signs (names) or between objects. He then tries to give an answer to this identity puzzle: How do we account for the difference in cognitive significance between two true statements of the forms 'a=a' and 'a=b'? 'a=a', he claims, is *a priori* and analytic[139] while 'a=b' usually contains new knowledge that cannot be known *a priori*, hence, it is not analytic.

Frege explains why in *Begriffsschrift* he wrongly claimed that identity must be a relation between names. He argues that if we regard identity as a relation between the contents of the signs then 'a=a' and 'a=b' could not differ in cognitive value and both sentences would express the same trivial fact that an object is identical to itself. For instance, the two identity statements 'The Morning star is the Morning star' and 'The Morning star is The Evening star' will both convey the trivial fact that Venus is Venus, whether named "The Morning star" or "The Evening star". Hence, he concluded, in *Begriffsschrift*, that identity should be a relation between signs.

[139] According to *The Foundations of Arithmetic* §3, a statement is analytic if it can be derived by logical laws and (logically justifiable) definitions alone: otherwise, it is synthetic. Apparently, Frege thought that synthetic and analytic statements have different cognitive values.

Later, however, Frege claims that if identity was a relation between two names which are arbitrarily chosen to name the same object, then identity statements would not express facts about the world but rather about language. He argues that since the connection between a sign and its referent is arbitrary, if identity statements were just about the signs, they would only express our linguistic choices rather than what he calls proper knowledge.[140] For example, if for the sake of brevity, we decide to call Barack Obama Barry, then 'Barack Obama=Barry' would be a true statement just by virtue of our arbitrarily deciding to give Obama the latter name. If the proposed theory were correct then all identity statements would be like this one. However, there are other identity statements whose truth depends on more than our arbitrary choices, for instance, $\log_{10}10{,}000=2^2$; we acknowledge this statement as true not by stipulation of names, but by virtue of a relation among numbers which are objects. And we need mathematical knowledge to be able to calculate both sides of the identity sign.

Thus, to solve the puzzle of identity, Frege introduces the mode of presentation which he calls the sense, and argues that "A difference [of cognitive values] can arise only if the difference between the signs corresponds to a difference in the mode of presentation of that which is designated."[141] In other words, the difference between the cognitive values of two sentences 'a=a' and 'a=b' is not due to the difference between the shapes of the signs 'a' and 'b' or to the fact that these two signs are arbitrarily chosen to name the same thing, but rather because these two signs have different senses, while sharing reference. Henceforth, Frege says that a name not only refers to something, but also it has a sense that contains its mode of presentation. Suppose, for example, we have

[140] "On Sense and Reference," 1892, p. 26.
[141] *Ibid*., p. 26.

four lines a, b, c, d that all intersect in the same point. Now we can call this point 'the point of intersection of a and b', 'the point of intersection of c and d', or etc. These different names express different modes of presenting the same object.[142]

According to Frege's theory of sense, "the difference between the signs corresponds to a difference in the mode of presentation of that which is designated";[143] for instance, 'the point of intersection of a and b' and 'the point of intersection of c and d' have the same reference but different senses. Accordingly, he concludes that in an informative identity statement such as "a=b" the claim is that two different signs with two different senses designate the same object. For Frege, the sense or the mode of designation of a sign is objective and not arbitrary (senses being the third realm entities),[144] hence it enables us to represent extra-linguistic knowledge about the world; this is why we can express proper knowledge in an identity statement. Frege holds the difference in the thoughts of the two identity statements accountable for the difference in their cognitive values: the identity statements 'a=a' and 'a=b' have different cognitive values because they express different thoughts. And, the two sentences express different thoughts because 'a' and 'b' have different senses.

3-4) Some Difficulties with the "On Sense and Reference" Discussions

Some criticisms have been raised regarding Frege's discussions in "On Sense and Reference". In this section, I will consider some of them and will explain why Frege did not see them as objections to his theory of sense.

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[142] Frege does not, however, further clarify if the mode of presentation is the sense itself or if it is a part of the sense.

[143] "On Sense and Reference," p. 26.

[144] "Thoughts," p. 69 (*The Frege Reader*, p. 337).

A major problem arises with Frege's theory of sense when it comes to determining the sense of a proper name. Contrary to the case of definite descriptions such as 'the morning star', when we apply Frege's theory of sense to a proper name like 'Aristotle' or 'John', it seems like there is no reasonable way to ascribe a sense to it. Frege, in a footnote in "On Sense and Reference," proposes different possible senses for a proper name and concludes that since one can ascribe a variety of senses to a proper name, it is better not use such names in an ideal language:

> In the case of actual proper name such as 'Aristotle' opinions as to the sense may differ. It might, for instance, be taken to be the following: the pupil of Plato and teacher of Alexander the Great. Anybody who does this will attach another sense to the sentence 'Aristotle was born in Stagira' than will a man who takes as the sense of the name: the teacher of Alexander the Great who was born in Stagira. So long as the reference remains the same, such variations of sense may be tolerated, although they are to be avoided in the theoretical structure of a demonstrative science and ought not to occur in a perfect language.[145]

But none of the possible senses for 'Aristotle' that Frege proposes here appears to be consistent with his arguments on identity in "On Sense and Reference". Being the pupil of Plato and the teacher of Alexander the Great, for example, cannot be a sense for 'Aristotle', because we learn something when we find out that Aristotle was the student of Plato, unless the name were defined in that way, which does not seem to be the case for ordinary proper names. The cognitive value of 'Aristotle=Aristotle' differs from that of 'Aristotle=The pupil of Plato and the teacher of Alexander'. In the latter statement, accordingly, the two sides of the identity sign must have different senses. The same goes for any other similar suggestion that one may make as to the sense of 'Aristotle'.

[145] "On Sense and Reference," p. 27.

Moreover, as Searle pointed out, if the sense of 'Aristotle' was the same as the sense of 'The pupil of Plato and the teacher of Alexander', then the claim that Aristotle is the pupil of Plato and the teacher of Alexander would be a necessary truth, which seems plainly wrong.[146] It would also be an analytic truth which seems even more implausible. Frege thus seems unable to provide senses for ordinary proper names. Although this can be considered a major problem for the analysis of ordinary language, Frege does not see it as a problem for his ideal language, because for scientific purposes we simply "stipulate that for every proper name there shall be just one associated manner of presentation of the object so designated."[147] Thus for example, we might stipulate that the sense of 'Aristotle' is to be [The pupil of Plato and the teacher of Alexander], and it is this stipulation that would make the claim 'Aristotle is etc.' analytic and hence necessary in Frege's ideal language.

However, Frege does not suggest any criteria for choosing a definite sense from different possibilities, hence, it seems that the sense of a name can be chosen arbitrarily from the various equivalent definite descriptions. Accordingly, the cognitive value of so-called trivial identity statements such as 'Aristotle = Aristotle' may not be *a priori* after all, because these two sign-occurrences may have different senses even if they both refer to the same person. That is, we might choose as the sense of the term 'Aristotle' on the left side 'the author of the *Organon* and as the sense of the term 'Aristotle' on the right side 'the author of the *Nicomachean Ethics*'. In this case, the equation would no longer be analytic, for one might not know that the Aristotle who wrote the *Organon* is the same Aristotle who wrote the *Nicomachean Ethics*. Such examples seem to show that an

[146] J. Searle, "Proper Names," Mind 67 (1958), 166-73, p. 172; cf. S. Kripke, *Naming and Necessity* (Cambridge; Harvard University Press, 1980), pp. 61f.
[147] "Thoughts," p. 66 (*Collected Papers*, p. 359).

identity statement of the form a=a can either be analytic or synthetic. The identity statement 'a=a' is thus potentially just as informative as 'a=b', supposing both to be true. Furthermore, if a proper name such as 'Aristotle' may carry different senses for different people, then could it not be said that the sense of a simple name is somehow subjective and indefinite?[148] As above, this problem could be removed by arbitrarily specifying a single sense in an ideal language, but it remains a problem for ordinary language.

Another challenge regarding Frege's discussion of identity is that he assumes that all identity statements of the form 'a=a' are true.[149] However, in natural language,[150] an identity statement of the form 'a=a' might in some circumstances be false, when, i.e., two individuals have the same name. For example, if I have a neighbor called 'John O'Connor' and then I hear in the news that a fellow called John O'Connor won the lottery, then I can ask myself "is this lucky man John O'Connor the same person as my neighbor John O'Connor?" In this case, the question "Is John O'Connor John O'Connor" is anything but trivial. Accordingly, the cognitive value of the statement 'a=a' could be the same as the cognitive value of the statement 'a=b', hence, it may not always be true. It suggests that 'John=John' can be just as informative as 'John=my neighbor' and it can be false. This shows that it is not the case that every identity statement of the kind 'a=a' is a truth, still less that all are analytic. Frege, of course, rules out such cases in his ideal language. He says that in an ideal language every proper name should have one and only one definite sense via which a name designates its reference.

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[148] See Dummett, *Frege: Philosophy of Language*, pp. 102f. See Also, J. McDowell, "On the Sense and Reference of a Proper Name," *Mind*, Vol. 86, No. 342 (pp. 159-185), 1977, pp. 165, 182f.
[149] "On Sense and Reference," p. 25 (*Collected Papers*, p.157).
[150] Frege's examples in "On Sense and Reference" are all from ordinary language.

The last issue regarding Frege's arguments that I would like to consider is raised by Mendelsohn:[151] he says that it is not only in identity statements that we are confronted with the problem of cognitive value. For instance, while the two names 'The author of *Waverley'* and 'Scott' have the same content according to the *Begriffsschrift*, the statements "The author of *Waverley* authored *Waverley*" and "Scott wrote *Waverley*" have different cognitive values; the former statement is analytic (supposing that there is an author of *Waverley*), whereas the latter is informative and synthetic. However, an attempt to represent the content of these two sentences in conceptual notation would be confronted by the same problem raised by Frege; both statements will turn into an analytic and uninformative *Begriffsschrift* formula, and both will have the same cognitive value. This suggests that the problem of cognitive value will be raised for all the statements in which there are two different proper names that refer to the same object, and Frege's theory of sense should include them as well as identity statements.

3-5) Identity from a Broader Perspective

Frege's conceptual system gives rise to a number of different possibilities for the semantics of identity statements. In this section, I will illustrate and explain some of these possibilities and then, in the next section, explain why, given his aims, Frege justifiably rejects some of them.

[151] *The Philosophy of Gottlob Frege*, p. 30. In the previous section, I raised the same problem, but from a different perspective.

As we saw before, according to Frege's semantics, there are three levels: the level of language, the level of sense, and the level of reference. In the case of singular terms, we have:

Name

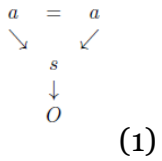
↓

Sense

↓

Object

In addition, there are two general forms for identity statements: every identity statement is either of the form 'a=a' or 'a=b'. In view of that, we could have different identity statements in the following manner:[152]



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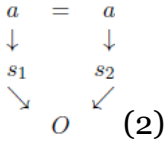

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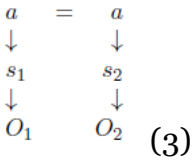

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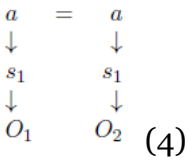

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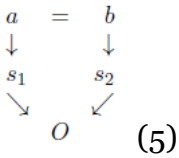

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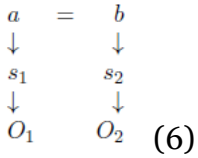

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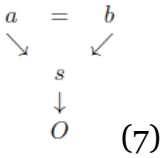

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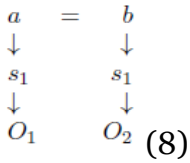


(8)

[152] 'S' stands for the sense at the level of sense, and 'O' stands for the object at the level of reference.

According to Frege, an identity statement of the form 'a=a' is an analytic truth. This statement does not provide us with new information; it only expresses the fact that an object is identical to itself. (1), illustrates what Frege believes of such a statement in his ideal language: a proper name which has one single definite sense and always refers to the one and the same object. 'The Morning star=The Morning star' is an example of this kind of identity statement. However, there are other semantically different identity statements which can be presented by the form of 'a=a'. In (2), for instance, 'Aristotle=Aristotle', we have a case in which two occurrences of the same name-type refer to the same object while having different senses (for the explanation see the previous section). (3), on the other hand, shows a case in which the two occurrences have different senses and different references; for instance, 'John O'Connor = John O'Connor'. Additionally, (3), could be an instance of homophonies, in which an expression has unrelated senses and references in different settings. For instance, in the two sentences 'I went to the bank to deposit some money' and 'I went to the bank to watch the sunset', the expression 'the bank' designates two unrelated objects and expresses two distinct senses.

Similarly, in (4), we have expressions that have the same sense but different references in different contexts. Words such as 'here', 'now', 'then', 'I', and etc. would be examples according to some theories of indexicals.[153] According to these theories, the word 'now' conveys the same sense every time it is said or heard by an English speaking person, but it designates different times when uttered at different times. The word 'now' in the sentence 'I am at home now' said at 11:30 am designates a different time than the

[153] While, according to Frege, the sense of an expression determines its reference, indexicals are an apparent counterexample.

word ‘now’ in the sentence ‘I am at home now’ said at 11:30 pm. Another example would be the pronouns such as ‘She’; ‘She is a philosophy student’ can be uttered by any philosophy student and every time the word ‘She’ would refer to the person who articulates that statement. Given this, the statements ‘She=She’ and ‘now=now’ can be false even though the two tokens have the same sense.

Furthermore, according to Frege, an identity statement of the form ‘a=b’ is an informative statement which shows a relation between two senses expressed by two different names that designate the same object: (5) is the case in which two names that have different senses designate the same object. ‘The Morning star=The Evening star’ is the example given by Frege himself. There is also (6), which is an illustration of a false identity statement, for instance, ‘Aristotle=Plato’. Nonetheless, there are other situations in which the identity statement of the form ‘a=b’ may embody a semantically different identity statement. For instance, (7) which is an identity statement of the form ‘a=b’, could be an instance of synonyms. This statement only expresses linguistic knowledge; a relation between two different names that have the same sense and reference; ‘A=¬¬A’ would be an instance of this case, according to Frege.[154] There may also be situations, depicted in (8), in which two words are synonyms but in different contexts refer to different things. This case belongs especially to indexical synonyms, which can replace each other. For instance, in the sentences ‘John is in the classroom now’ said at 10:30 am and ‘John is in the classroom at this time’ said at 11 am, the two words ‘now’ and ‘this time’ might well refer to two different time segments while having the same sense.

[154] “Compound Thoughts,” p. 44. (Collected Papers, p. 399).

As these examples show, Frege's semantics appears to allow these eight different cases for identity statements. However, as we shall see, he only accepts four of them (1, 5, 6, and 7) as legitimate in his ideal language.

3-6) Frege's Perspective

Frege excludes some of the instances of identity statements above from his ideal language because he believes that they involve ambiguity either at the level of sense or the level of reference, or they violate the principle that sense determines reference. For Frege, 'Aristotle=Aristotle' is never informative, because he sets up his ideal language in a way that we cannot have two senses expressed by one and the same expression,[155] which would create ambiguity on the level of sense.[156] As a result, cases (2) and (3) are not allowed in Frege's ideal language. Also, according to Frege, we cannot have two references for the same expression.[157] Hence, case (4) would not arise either. Finally, case (8) is ruled out because Frege will not permit the same sense to present different referents.

Although, according to Frege, there should be no reference or sense ambiguity in an ideal language, it seems to me that it would have been reasonable to study all of these possibilities before constructing an ideal language, as he did in some similar cases.[158] For instance, in his article "The Thought", Frege talks about how natural language is ambiguous, and does not fit the thought precisely. He says that thought-wise, the contents of ordinary language statements, at times, have extra parts that are not

[155] "On Sense and Reference," pp. 27-28.
[156] "Thought," 1918, pp. 65-6.
[157] Since, according to Frege, sense determines the reference, if every expression could only have one definite sense, then consequently, we cannot have more than one reference for every expression.
[158] Most of Frege's examples are ordinary language examples.

important, and at times, they have less content than the thought. For instance, the statement 'I haven't yet finished my thesis' means that 'I have not finished my thesis' while at the same time, it hints that 'I was supposed to finish it already'. According to Frege, the hinting part is extra information that, logically speaking, is of no importance for the thought. In contrast, the statement 'I will submit my thesis today' would convey different thoughts on different days. In order to apprehend the thought correctly, Frege argues, a time indication is needed to show the time that the statement was uttered. Interestingly, Frege says "… the time of utterance is part of the expression of the thought."[159] For instance, when I say 'I'm going to school now' at different times, we are in fact dealing with two different *expressions*. The word 'now' uttered today is a different expression from the word 'now' uttered yesterday. So, 'now=now' will not yield an instance of case (4), because it will be false only if we are dealing with different expressions. In general, Frege appears to believe that these and related problems involving indexicals would be resolved if we showed the time or the place at which the statement is uttered, and indicate exactly the person who uttered the statement.

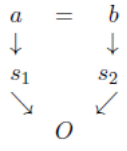
Frege tries to ban all the identity statements that have sense or reference ambiguity from his ideal language. However, as a result of using ordinary language, the statements actually made in science, logic, and arithmetic may not be compatible with his view. If we are to remove the imperfections, we must know exactly what they are. And it is clear that Frege was trying to do this. Many of his most important articles, including "On Sense and Reference", "Object and Concept" and "The Thought" are largely devoted to the analysis of natural language. By analysing natural language,

[159] "Thought," 1918, p. 64.

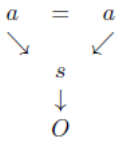
Frege explains the functions of the language and reveals the roots of the ambiguities in order to avoid them in his ideal language. In fact, the shortcomings of natural language were a main motivation of Frege's attempt to build an ideal language. Thus, it seems to me that it would have been better if Frege had studied all the possible cases of identity by the same token.

3-7) Identity in the Light of Multiple Analyses: Is Identity a Relation?

Besides those identity cases that I have already discussed in the previous section, there is another one, which I call (9), that cannot on Frege's principles be overlooked in his ideal language. At the beginning of "On Sense and Reference", Frege asks if identity (equality) is a relation, but later he simply supposes without argument that it is. It seems that he assumes that since identity statements of the form a=b can be analyzed as <λx,y. x=y>(a,b), where the relation of 'identity' is applied to the pair <a, b>, identity *must be* a relation in which two terms should enter. In fact, Frege only seems to consider the two statements 'a=b' and 'a=a' in the following manner:

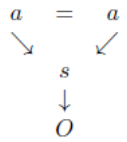


(5) <λxy . x=y>(a,b)



(1) <λx,y. x=y>(a,a).

Frege thus takes identity to be a relation in both cases. Yet statement (1) could be analyzed differently, making identity a concept:



(9) <λx. x=x>(a).

For, based on Frege's theory of multiple analyses, a simple identity statement of the form a=a yields two readings: first, <λx. x=x>(a), where the concept of 'being self-identical' is applied to the object a.[160] According to this reading, we have a function of one argument. In the second reading, <λx,y. x=y>(a,a), we get a function of two arguments. Given these two readings of the same identity statement 'a=a', it seems that an identity sign can be considered both as the name of a function of one argument (concept) and as the name of a function of two arguments (relation). For instance, we can analyze the sentence 'Aristotle = Aristotle' in two different ways: one analysis makes identity a predicate applied to the man Aristotle, and the other one makes it a relation applied to the pair of arguments <Aristotle, Aristotle>.

Furthermore, Frege could have argued that if, in our analysis, we take the equality sign '=' as the constant part (hence, the function), and everything else (the object names that are present in the identity statements 'a=a' and 'a=b') as variables and replaceable (hence, the arguments), then the two statements 'a=a' and 'a=b' would have different cognitive values, because the identity sign in the sentence 'a=a' could refer to either a concept or a relation, while if the identity sign in the sentence 'a=b' is taken to represent the function, it can only refer to a relation. And since concepts and relations are two fundamentally distinct kinds of functions, it might be argued that the two sentences have different cognitive values. Hence, Frege may not have needed the notion

[160] *Basic Laws* I, § 26. Also, § 30.

of sense to justify the difference between the cognitive values of the two identity statements ‘a=a’ and ‘a=b’. For, could it not be the difference between the two identity signs, one referring only to the relation <λx,y. x=y> another one referring to the concept <λx. x=x> and the relation <λx,y. x=y>, that gives rise to the difference in the cognitive values of the two sentences ‘a=b’ and ‘a=a’?

In summary, based on Frege’s theory of multiple analyses, we are not obliged to take identity as a relation in all cases. So, the answer to Frege’s question ‘Is equality a relation?’ would be ‘yes’ and ‘no’, based on the different ways of analyzing identity statements. Nonetheless, Frege’s answer is ‘yes’, because, I believe, he does not consider the possibility of different analyses when he was arguing on identity. Frege fails to see that based on the different readings of the same statement of the form a=a, we could get two distinct functions for identity, one time a function of two arguments<λx,y. x=y>(a,a) and another time a function of one argument<λx. x=x>(a). The former reading is taken by Frege as the suggestion for identity being a relation, while the later one is more compatible with his discussion in *Begriffsschrift* §9. This also means that the puzzle of different cognitive values of the identity statements ‘a=a’ and ‘a=b’, which was raised by Frege at the beginning of “On Sense and Reference”, might have been solved by simply differentiating between the two functions designated by the identity signs in the two sentences. That is, the identity sign in the statement ‘a=a’ designates a concept (x=x) while the identity sign in the statement ‘a=b’ designate a relation (x=y) and, according to Frege, concept and relation are fundamentally different functions. Thus, He could have held the difference between the two functions designated by the identity signs in those two statements accountable for the difference between the cognitive values of them,

with no need to introduce the theory of sense in order to solve the identity puzzle. However, the sense/reference distinction still is needed for the problem of opaque contexts or non-referring names.[161]

[161] See my discussion in the first chapter, section 1-3: Expression, Sense, and Reference.

Conclusion

Frege began by investigating the foundations of mathematics and came to the conclusion that, in general, mathematical arguments were not as precise as they needed to be. He believed that the root of the problem lay in the imperfections of the language used to formulate the arguments of mathematics and science. In order to deal with this problem, he developed an ideal language —Conceptual Notation or *Begriffsschrift*— that is supposed to enable us to precisely represent the conceptual content of expressions, or that part of their content that is significant for logical inferences. A recognition of the shortcomings of natural language was thus a main motivation of Frege's attempt to build an ideal language. His further remarks on the workings of this ideal language have, however, given rise to at least the appearance of inconsistency.

In this work, my primary aim has been to explain Frege's ideal language and to scrutinize some of the theories he developed —namely, the theory of multiple analyses, the theory of sense, and the theory of identity— with regard to the problems that they may raise for the semantics of his ideal language. The theory of multiple analyses, for instance, seems to generate ambiguity in Frege's ideal language. In the Conceptual Notation, every expression is either a function name or a proper name, and every expression has both a sense and a reference. The sense of a declarative sentence is called the thought of the sentence, which is supposed to be displayed by the conceptual notation. In *Begriffsschrift*, Frege uses the notion of conceptual content as the important part of a content that should be displayed in the conceptual notation, but he later divides the conceptual content of a sentence into thought (sense) plus truth-value (reference). According to Frege's theory of multiple analysis, the thought of a sentence

can be parsed into functions and arguments in various ways, yet the thought-content or the conceptual content of the sentence is said to remain the same. At first glance, it seems that these multiple analyses create ambiguity in the formulas of Frege's ideal language, because they may represent different thoughts. There would thus appear to be a one-to-many relation between a sentence and its conceptual content in an ideal language. Yet Frege does not appear to give an adequate explanation of the relation between a sentence and its different analyses.

As I explained in my second chapter, it also seems as if these different readings correspond to different thoughts (conceptual contents), because someone may take one reading as true while deeming the other one false. This suggests that these readings have different cognitive values, and hence correspond to different thoughts (conceptual contents). Could Frege allow such multiple decompositions of a sentence in his ideal language, which is supposed to be unambiguous and clear? Should not such variations in the analysis of a sentence be reflected in differences in conceptual notation? Finally, do Frege's theories create inconsistency and is his ideal language, like ordinary language, inevitably ambiguous?

I have argued that there is no inconsistency in Frege's system regarding the abovementioned theory. First, I recalled that Frege defines conceptual content contextually. This contextual definition narrows down the class of all the sentences that have the same conceptual content but, unlike the conceptual content of a singular term, it does not give us the conceptual content itself. Also, since this conceptual content is what matters for logical inferences, all the different analyses of a formula have the same logical consequences and thus have the same conceptual content, even if someone may

not be able to recognize this. So, according to Frege, all the different analyses of one and the same formula have the same conceptual content and are allowed to be substituted for each other in an ideal language. Hence, I believe that Frege's system meets the challenge of multiple analyses and the integrity of his logical system remains intact.

Frege's discussion of identity, by contrast, seems to me to be unsatisfactory not only for the case of ordinary language but also for his ideal language. The application of the theory of multiple analyses to identity statements provides us a number of different readings, including one that Frege seems to have overlooked in his analysis of such statements. He appears to consider identity only as a relation. However, identity can be a relation or a concept based on the different ways of analyzing an identity statement: A simple identity statement of the form 'a=a', for example, yields the following two readings: 1) $<\lambda_x.\ x=x>(a)$, where the concept of 'being self-identical' is applied to the object *a*. According to this reading, we have a function of one argument. In the second reading 2) $<\lambda_{x,y}.\ x=y>(a,a)$, the relation of 'identity' is applied to the pair <a, a>. According to the second reading, we get a function of two arguments. Considering these two readings of the identity statement 'a=a', the main challenge would be whether we should consider the identity sign as the name of a function of one argument (concept), or as the name of a function of two arguments (relation)? To this, my answer is *both*, but Frege does not seem to have considered this ontological question (the difference between two distinct kinds of functions reached at by different analysis of the same content). Moreover, the difference between the cognitive values of identity statements of the forms 'a=a' and 'a=b' can, I have argued, be explained without postulating the notion of sense as Frege does. The identity function x=y is a relation, while the identity function

x=x is a concept, hence, the difference between these two functions can be held responsible for the difference between the cognitive values of the two identity statements ‘a=a’ and ‘a=b’.

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