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Kesten's Yaglom Limit Counterexample

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# Kesten's Yaglom limit counterexample

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# Abstract

We consider a positive matrix $Q$, with entries $\{1, 2, \ldots\}$. Kesten showed that if there exists a constant $1 \leq L < \infty$ and sequences $u_1 < u_2 < \ldots$ and $d_1 < d_2 < \ldots$ such that $Q(i, j) = 0$ whenever $i < u_r < u_r + L < j$ or $i > d_r + L > d_r > j$ for some $r$, then if $Q$ is also substochastic, it has the strong ratio limit property, that is,

$$\lim_{n \to \infty} \frac{Q^{n+m}(i, j)}{Q^n(k, l)} = R^{-m} \frac{f(i)\mu(j)}{f(k)\mu(l)}$$

for a suitable $R$ and some $R^{-1}$-harmonic function $f$ and $R^{-1}$-invariant measure $\mu$. In this thesis, we delve into a counterexample of Kesten that shows that for a positive irreducible matrix $Q$, that is the transition probability matrix of a Markov chain $\{X_n\}$, the Yaglom limit theorem is not valid if we drop one of the restriction imposed above, even when $Q$ has a minimal normalized quasi-stationary distribution. In the last section, we show the ratio limit property must also then fail.

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# Dedication

I dedicate this work to my Mom, Dad and Aunt.

# Contents

# Chapter 1

# Introduction

Let $Q = \{Q(i,j)\}_{i,j\in S}$ be a positive matrix on a countable state space $S$. (In this paper $Q$ being positive means that $Q(i,j) \geq 0$ for all $i,j \in S$.) We say $Q$ has the *strong ratio limit property* (SRLP) if there exist strictly positive constants $R$, $\mu(i)$, $f(i)$, $i \in S$, such that

(1.1) $$\lim_{n\to\infty} \frac{Q^{n+m}(i,j)}{Q^n(k,l)} = R^{-m}\frac{f(i)\mu(j)}{f(k)\mu(l)} \quad i,j,k,l \in S,\ m \in \mathbb{Z}.$$

We say $Q$ is *irreducible*, if for each $i,j \in S$ there exists an $n = n(i,j)$ with $Q^n(i,j) > 0$. If $Q$ is irreducible, it is called *aperiodic*, if $\gcd\{n : Q^n(i,j) > 0\} = 1$. We also say $Q$ is *R-recurrent*, when

(1.2) $$\sum_n Q^n(i,j)R^{-n} = \infty,$$

where

(1.3) $$R^{-1} = \lim_{n\to\infty}[Q^n(i,j)]^{1/n}.$$

It was observed by Vere-Jones [25], that for an *irreducible aperiodic* matrix $Q$, the limit $R^{-1}$ exists and is independent of $i,j$; that is, if (1.2) holds for some $i,j$, then it holds for all $i,j$.

The SRLP is also closely related to the *age distribution* of an absorbing Markov chain. Assume the chain $\{X_n; n = 0,1,\ldots\}$ has state space $\varphi = \{0,1,\ldots\}$, that $\{0\}$ is the single absorbing state and that the time to absorption has *finite* expectation. Introduce the instantaneous return process $\{\tilde{X}_n; n = 0,1,\ldots\}$ obtained by allowing $\{X_n\}$ to hit $\{0\}$ and then instantly returning it to $\{1\}$ and repeating this procedure so that each time $\{\tilde{X}_n\}$ hits $\{0\}$ it moves immediately to $\{1\}$ and then evolves with the same transition probabilities as $\{X_n\}$ until next hitting $\{0\}$. Let $T_n$ be the time measured backwards from $n$ to the last time $\{\tilde{X}_n\}$ hit $\{0\}$. Assuming $\varphi$ is irreducible and aperiodic for $\{X_n\}$, Levikson [13] shows that for each $j \in \varphi$, $\lim_{n\to\infty} P_i\{T_n = k| \tilde{X}_n = j\}$ exists and is called the limiting age distribution. Pakes [18] (Theorem 1 and corollary Section 2) showed that if the return chain $\{\tilde{X}_n\}$ is *R-recurrent* and satisfies the SRLP, then the limiting age distribution exists.

A measure $\mu$ on $S_0$ is *normalized quasi-stationary distribution* if it is a distribution probability and

$$P_\mu\{X_n = i|\tau > n\} = \mu(i), \quad \text{for all} \quad i \in S_0,\ n \geq 1,$$

where $S_0 = \{1, 2, \ldots\}$ and $\tau$ is a stopping time(Kesten [10]). Closely related with the notation of quasi-stationary distribution is so-called *Yaglom limit*; this is,

$$\lim_{n\to\infty} \frac{P\{X_n = j|X_0 = i\}}{P\{\tau > n|X_0 = i\}} = \lim_{n\to\infty} \frac{Q^n(i,j)}{\sum_{l\in S_0} Q^n(i,l)},$$

where $Q := (P(i,j))_{i,j\in S_0} =$ restriction of $P$ to $S_0 \times S_0$.

The existence of the Yaglom limit was established for *branching processes* in the pioneering work by Yaglom [27], for *symmetric random walks* by Seneta [21], for *left continuous random walks* by Pakes [17], and for *birth and death processes* by Good [9], Kijima and Seneta [11] and Van Doorn [24]. In the case of *finite* Markov chains, there is only one quasi-stationary distribution (Darroch and Seneta [3]).

There is a close connection between the existence of a normalized quasi-stationary distribution and the existence of the Yaglom limit; that is, if the Yaglom limit exists for all $j$, then it is a normalized quasi-stationary distribution (Vere-Jones [26], Theorem 1 and 2), Ferrari [6].

By the Perron-Frobenuis theorem, Gantmacher [7] chapter XIII, any *finite irreducible aperiodic* matrix $Q$ has the SRLP.

As was observed by Kingman and Orey [12] for an *irreducible recurrent aperiodic* Markov chain with *countable state space* indexed by *discrete time*, if the probability of returning to $i$ by time $N$ is greater than $\epsilon$, where $N$ and $\epsilon$ are positive numbers independent of $i$, then the SRLP holds. This condition can be taken as a very weak form of spatial homogeneity.

Pruitt [19] showed that, if ${}_kp^n_{ij}$ is the probability, starting from $i$, of being at $j$ at step $n$ without visiting $k$ at steps 1,2,...,$n-1$, then for a *R-recurrent* chain, SRLP will hold if

$$(\star) \qquad \limsup_{n\to\infty} \frac{P^{(n+1)}_{00}}{P^{(n)}_{00}} \leq R^{-1}.$$

He also showed that $(\star)$ is always satisfied for a *reversible chain*; that is reversibility implies $(\star)$ (and therefore the SRLP if the chain is also $R$-recurrent).

The papers listed above all assume that $Q$ is $R$-recurrent or that $Q$ is the transition probability matrix of reversible Markov chain. Kesten [10] has proven the SRLP for a more general class of sub-Markov chains on $S_0 = \{1, 2, \ldots\}$, which are in general neither $R$-recurrent nor reversible. For a *positive irreducible* matrix $Q$ on $S_0$, consider following restrictions:

- (1.4u) There exists a constant $1 \leq L < \infty$ and an infinite sequence $1 \leq u_1 < u_2 < \ldots$ such that $Q(i,j) = 0$ whenever $i < u_r < u_r + L < j$ for some $r$. Moreover, there exist constants $\delta_0$ and $M < \infty$, and for each $i \in \bigcup_r [u_r, u_r + L)$, there exist integers $1 \leq n' = n'(i), n'' = n''(i) \leq M$ such that $Q^{n'}(i, i+1) \geq \delta_0$, $Q^{n''}(i+1, i) \geq \delta_0$ ;

- (1.4d) There exists a constant $1 \leq L < \infty$ and an infinite sequence $1 \leq d_1 < d_2 < \ldots$ such that $Q(i,j) = 0$, whenever $i > d_r > d_r + L > j$ for some $r$. Moreover, there exist constants $\delta_0$ and $M < \infty$, and for each $i \in \bigcup_r [d_r, d_r + L)$, there exist integers $1 \leq n' = n'(i), n'' = n''(i) \leq M$ such that $Q^{n'}(i, i+1) \geq \delta_0$, $Q^{n''}(i+1, i) \geq \delta_0$ ;

- (1.5) There exist constants $\delta_1$ and $N < \infty$ and for each $i \in S_0$, there exist integers $1 \leq k_1, \ldots,\ k_r \leq N$ (with $k_j = k_j(i)$ and $r = r(i)$) such that $Q^{k_s} \geq \delta_1$ for $1 \leq s \leq r$, and $\gcd(k_1, \ldots, k_r) = 1$.

For a positive irreducible matrix $Q$ that is the transition probability matrix of a Markov chain $\{X_n\}$ and satisfies conditions (1.4)–(1.5) Kesten [10] proved Theorems 1.0.1–2.

**Theorem 1.0.1.** *Let* $0 < s < \infty$ *and assume that* $Q$ *is irreducible. If* $Q$ *satisfies (1.4u) and (1.5), then up to a multiplicative constant,* $Q$ *has at most one positive* $s$*-harmonic function. If* $Q$ *satisfies (1.4d) and (1.5), then, up to a multiplicative constant,* $Q$ *has at most one positive* $s$*-invariant measure.*

**Theorem 1.0.2.** *Let* $Q$ *be irreducible and substochastic ( that is,* $\sum_j Q(i,j) \leq 1$ *for all* $i \in S_0$*). If* $Q$ *satisfies (1.4 u), (1.4d), and (1.5), then* $R$ *defined by (1.3) is independent of the choice of* $i, j$ *in (1.3), and* $1 \leq R < \infty$*. Moreover, for all* $i \in S_0$ *there exist* $f(i) > 0$ *and* $\mu(i) > 0$ *such that:*

$$f(i) = R(Qf)(i), \quad i \in S_0;$$

$$\mu(i) = R(\mu Q)(i), \quad i \in S_0,$$

*and such that (1.1) holds for* $i, j, k, l \in S_0$*. In addition, if we assume that for some* $L_1 < \infty$*,*

$$\sum_{j \in S_0} Q(i,j) = 1 \quad \text{for } i \geq L_1,$$

*and that for some* $i_0$*,*

$$\sum_{j \in S_0} Q^n(i_0, j) \to 0 \quad (n \to \infty) \quad \text{and} \quad R > 1,$$

*then we can even take the measure* $\mu$ *to be a probability measure on* $S_0$*; that is,*

$$\sum_{j \in S_0} \mu(j) = 1,$$

*and with this normalization,*

$$\lim_{n \to \infty} \frac{Q^{n+m}(i,j)}{\sum_{i \in S_0} Q^n(k,l)} = R^{-m} \frac{f(i)}{f(k)} \mu(j), \quad i, j, k \in S_0.$$

*Kesten [10]*

In this thesis, we delve into a counterexample of Kesten that shows that the limit in (1.1) does not exist if one of the conditions (1.4)—(1.5) does not hold, even when $Q$ has a minimal normalized quasi-stationary distribution. Failure of the ratio limit theorem in this example results from the failure of the Yaglom limit.

The failure of the Yaglom limit in this example is due to the fact that (1.4) does not hold; this means that we cannot drop (1.4) entirely.

We begin by constructing the Markov chain $\{X_n\}$ in Section 3.1. Then in Section 3.2, we show that the absorbing state, $\tau$, has an exponentially bounded tail; by Theorem 2.2.1, this is necessary and sufficient for $\{X_n\}$ to have a normalized quasi-stationary distribution.

In Section 3.3, we try to reduce the proof of failure of the Yaglom limit, to an estimate for return probabilities of nearest-neighbour on $\mathbb{Z}$. In this section, we temporarily work with a new Markov chain $\{\tilde{X}_n\}$, which does not agree with $\{X_n\}$ on certain intervals.

In Section 3.4, we show the existence of a universal constant, which provides the required estimate on these return probabilities. In Section 3.5, we show the failure of the Yaglom limit by applying the results in Section 3.3 and 3.4 recursively. Finally in Section 3.6, we show that the failure of Yaglom limit implies the failure of the ratio limit property.

# Chapter 2

# Background

## 2.1 Definitions

Most definitions are from Billingsley [2], unless otherwise indicated.

**Definition 1.** ***(field)*** *Let $\Omega$ be a completely arbitrary nonempty space. A class of $\Omega$ is called a field, if it contains $\Omega$ itself and is closed under the formation of complements and finite unions.*

**Definition 2.** ***($\sigma$-field)*** *A class of $\Omega$ is a $\sigma$-field if it is a field and if it is also closed under the formation of countable unions.*

**Definition 3.** ***(probability measures)*** *A set function is a real-valued function defined on some class of subsets of $\Omega$. A set function $P$ on a field is a probability measure, if it satisfies following conditions:*

1. $0 \leq P(A) \leq 1$ *for* $A$ *in* $\mathcal{F}$;
2. $P(\emptyset) = 0, \;\; P(\Omega) = 1$;
3. *if* $A_1, \; A_2, \ldots$ *is a disjoint sequence of* $\mathcal{F}$*-sets and if* $\bigcup_{k=1}^{\infty} A_k \in \mathcal{F}$*, then*

$$P(\bigcup_{k=1}^{\infty} A_k) = \sum_{k=1}^{\infty} P(A_k).$$

**Definition 4.** ***(probability measure space)*** *If $\mathcal{F}$ is a $\sigma$-field in $\Omega$ and $P$ is a probability measure on $\mathcal{F}$, the triple $(\Omega,\mathcal{F},P)$ is called a probability measure space.*

**Definition 5.** ***(Markov chain)*** *Let $S$ be a finite or countable set. Suppose that to each pair $i$ and $j$ in $S$, a nonnegative number $P_{ij}$ is assigned such that for all $i$ in $S$ $\sum_{j\in S} P_{ij} = 1$. Let $X_0, X_1, X_2, \ldots$ be a sequence of random variables whose ranges are contained in $S$. The sequence is a Markov chain or Markov process if*

$$P\{X_{n+1} = j|X_0 = i_0, \ldots, X_n = i_n\} = P\{X_{n+1} = j|X_n = i_n\},$$

*for every $n$ and every sequence $i_0, \ldots, i_n$ in $S$ for which $P\{X_0 = i_0, \ldots, X_n = i_n\} > 0$. The set $S$ is the state space of the process.*

**Definition 6.** ***(nearest neighbour random walks)*** *A one-dimensional nearest neighbour random walk is a Markov chain whose state space is a finite or infinite subset of integers such that if it is in state $i$ after a single transition, either stays in $i$ or moves to $i-1$ or $i+1$.*

**Definition 7.** ***(simple random variable)*** *Let $(\Omega, \mathcal{F}, P)$ be an arbitrary probability space, and let $X$ be a real-valued function on $\Omega$; $X$ is a simple random variable if it has finite range and if*

$$\{\omega : X(\omega) = x\} \in \mathcal{F}$$

*for each real $x$.*

**Definition 8.** ***($\sigma$-field generated by a random variable)*** *The $\sigma$-field $\sigma(X)$ generated by $X$ is the smallest $\sigma$-field with respect to which $X$ is measurable; that is $\sigma(X)$ is the intersection of all $\sigma$-fields with respect to which $X$ is measurable.*

**Definition 9.** ***(Taylor series)*** *A Taylor series is a series expansion of a function about a point. A one-dimensional Taylor series is an expansion of a real function $f(x)$ about a point $x = a$ given by*

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^n(a)}{n!}(x-a)^n + \ldots$$

**Definition 10.** ***(power series)*** *An infinite series of the form*

$$\sum_{k=0}^{\infty} a_k(z-z_0)^k = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \ldots$$

*where the coefficients $a_k$ are complex numbers, is called a power series in $z - z_0$.*

**Definition 11.** ***(radius of convergence)*** *A power series will converge only for certain values of $x$. In general, there is always an interval $(-R, R)$ in which a power series converges, and $R$ is called the radius of convergence. The quantity $R$ is called the radius of convergence, because in the case of a power series with complex coefficients, the values of $x$ with $|x| < R$ form an open disk with radius $R$.*

**Definition 12.** ***(analyticity at a point)*** *A complex function $w = f(z)$ is said to be analytic at a point $z_0$ if $f$ is differentiable at $z_0$ and at every point in some neighbourhood of $z_0$.*

**Definition 13.** ***(singular point)*** *A point $z$ at which a complex function $w = f(z)$ fails to be analytic is called a singular point of $f$.*

**Definition 14.** ***(Stirling's formula)***

$$n! \sim \sqrt{2\pi} n^{n+\frac{1}{2}}\, e^{-n},$$

*where the sign $\sim$ is used to indicate that a ratio of two sides tends to one as $n \longrightarrow \infty$. Feller [4]*

**Definition 15.** ***(indicator function)*** *The indicator function of a set $A$ is the function on $\Omega$ that assumes the value 1 on $A$ and 0 on $A^c$; it is denoted $I_A$.*

**Definition 16.** ***(expected value)*** *A simple random variable in the form*

$$X = \sum_i x_i I_{A_i},$$

*is assigned expected value or mean value*

$$E[X] = E\left[\sum_i x_i I_{A_i}\right] = \sum_i x_i P(A_i).$$

**Definition 17.** ***(variance)*** *If $E[X] = m$, the variance of $X$ is*

$$Var[X] = E[(X-m)^2] = E[X^2] - m^2.$$

**Definition 18.** ***(conditional probability)*** *If $P(A) > 0$ , the conditional probability of $B$ given $A$ is defined via*

$$P(B|A) = \frac{P(A \cap B)}{P(A)}.$$

**Definition 19.** ***(conditional expectation)*** *If $P(A) > 0$ , and $Y = \sum_j y_j I_{B_j}$ is a simple random variable, the conditional expected value of $Y$ given $A$ is defined by*

$$E[Y|A] = \sum_j y_j P(B_j|A).$$

**Definition 20.** ***(independent events)*** *The events $E_1, E_2, \ldots, E_n$ are said to be independent if for every subset $E_{1'}, E_{2'}, \ldots, E_{r'}$ and $0 < r' \leq n$, of these events*

$$P(E_{1'} \cap E_{2'} \ldots \cap E_{r'}) = P(E_{1'})P(E_{2'}) \ldots P(E_{r'}).$$

*Ross [20]*

**Definition 21.** ***(distribution function)*** *The distribution function of a random variable $X$ is defined by*

$$F(X) = \mu(-\infty, x] = P[X \leq x]$$

*for real $x$. $F$ is right-continuous, nondecreasing and*

$$\lim_{x \to -\infty} F(x) = 0, \quad \lim_{x \to \infty} F(x) = 1.$$

**Definition 22.** ***(density function)*** *The probability mass function of a discrete random variable $X$ is given by*

$$f_X(x) = P(X = x) \ \ \textit{for all } x.$$

*The probability density function of a continuous random variable $X$ is the function $f_X(x)$ defined by*

$$F_X(x) = \int_{-\infty}^{x} f_X(t)dt \ \ \textit{for all } x.$$

**Definition 23.** ***(normal distribution)*** *Let $X$ be a random variable such that $E(X) = \mu$, $Var(X) = \sigma^2$; then we say $X$ is a normal random variable or simply $X$ is normally distributed with parameters $\mu$ and $Var(X) = \sigma^2$, if the density of $X$ is given by*

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(x-\mu)^2}{2\sigma^2}}.$$

*The density function is a bell-shaped curve that is symmetric around $\mu$. Ross [20]*

**Definition 24.** ***(geometric random variable)*** *Suppose that independent trials, each having a probability $p$ of being a success, are performed until a success occurs. If we let $X$ be the number of trails required until the first success , then $X$ is said to be a geometric random variable with parameter $p$. Its probability mass function is given by*

$$p(n) = P\{X = n\} = (1-p)^{n-1}p. \quad n = 1, 2, \ldots$$

**Definition 25.** ***(convergence in probability)*** *If for random variables $X_n$*

$$\lim_{n\to\infty} P[|X_n - X| \geq \epsilon] = 0$$

*holds for each positive $\epsilon$, then $X_n$ is said to converge to $X$ in probability.*

**Definition 26.** ***(convergence with probability 1)*** *If for random variables $X_n$*

$$P[\limsup_{n\to\infty}(|X_n - X| \geq \epsilon)] = 0$$

*holds for each $\epsilon$ (rational or not), then $X_n$ is said to converge to $X$ with probability 1.*

**Definition 27.** ***(expected value as integral)*** *The expected value of a random variable $X$ on $(\Omega,\mathcal{F},P)$ is the integral of $X$ with respect to the measure $P$,*

$$E[X] = \int X dP.$$

**Definition 28.** ***(conditional expected value)*** *Suppose that $X$ is an integrable random variable on $(\Omega,\mathcal{F},P)$ and that $\mathcal{G}$ is a $\sigma$-field in $\mathcal{F}$. There exists a random variable $E[X|\mathcal{G}]$, called the conditional expected value of $X$ given $\mathcal{G}$, having these two properties:*

1. $E[X|\mathcal{G}]$ *is measurable* $\mathcal{G}$ *and integrable.*

2. $E[X|\mathcal{G}]$ *satisfies the functional equation*

$$\int_G E[X|\mathcal{G}]dP = \int_G XdP, \quad G \in \mathcal{G}.$$

**Definition 29.** ***(martingale)*** *Let* $X_1,\ X_2,\ldots$ *be a sequence of random variables on a probability space* $(\Omega,\mathcal{F},P)$*, and let* $\mathcal{F}_1,\ \mathcal{F}_2,\ldots$ *be a sequence of* $\sigma$*-fields in* $\mathcal{F}$*. The sequence* $\{(X_n,\mathcal{F}_n) : n = 1,2,\ldots\}$ *is a martingale if the following conditions hold*

1. $\mathcal{F}_n \subset \mathcal{F}_{n+1}$;

2. $X_n$ *is measurable with respect to* $\mathcal{F}_n$;

3. $E[|X_n|] < \infty$;

4. *with probability 1,* $E[X_{n+1}|\ \mathcal{F}_n] = X_n$.

**Definition 30.** ***(supermartingale)*** *The sequence of random variables* $X_n$ *is a supermartingale relative to* $\sigma$*-fields* $\mathcal{F}_n$ *if 1,2, and 3 of the definition of martingale hold and in place of 4,*

$$E[X_{n+1}|\ \mathcal{F}_n] \leq X_n \text{ with probability } 1.$$

**Definition 31.** ***(Markov's inequality)*** *Let* $(\Omega,\mathcal{F},P)$ *be an arbitrary probability space, and let* $X$ *be a real-valued function on* $\Omega$*. If* $X$ *is nonnegative, then for positive* $\alpha$

$$P\{X \geq \alpha\} \leq \frac{1}{\alpha}E[X].$$

**Definition 32.** ***(stochastic process)*** *A stochastic process* $\{X(t), t \in T\}$ *is a collection of random variables; that is, for each* $t$ *in* $T$*,* $X(t)$ *is a random variable, where* $T$ *is the index set of the process. Ross [20]*

**Definition 33.** ***(tightness)*** *A sequence of probability measures* $\mu_n$ *on* $(R^1,\mathcal{R}^1)$ *is said to be tight, if for each* $\epsilon$ *there exists a finite interval* $(a,b]$ *such that*

$$\mu_n(a,b] > 1-\epsilon \text{ for all } n\ .$$

**Definition 34.** ***(absorbing state)*** *A state is absorbing if on once being entered, it is never left.*

**Definition 35.** ***(strong ratio limit property(SRLP))****Let* $Q$=$\{Q(i,j)\}_{i,j\in S}$ *be a positive matrix on the countable state space* $S$ *(here* $Q$ *is positive means that* $Q(i,j) \geq 0$ *for all* $i,j \in S$*). We say that* $Q$ *has the strong ratio limit property (SRLP) if there exist strictly positive constants* $R$, $\mu(i)$ *and* $f(i)$ $(i \in S)$*, such that*

$$\lim_{n\to\infty} \frac{Q^{n+m}(i,j)}{Q^n(k,l)} = R^{-m}\frac{f(i)\mu(j)}{f(k)\mu(l)} \qquad i,j,k,l \in S,\ m \in \mathbb{Z}.$$

*Kesten [10]*

**Definition 36.** ***(periodicity)*** *A state* $i$ *has period* $k$*, if any return to state* $i$ *must occur in some multiple of* $k$ *time steps and* $k$ *is the largest number with this property. Formally the period of state is defined as*

$$k = \gcd\{n : P(X_n = i|\ X_0 = i) > 0\},$$

*where* gcd *is the greatest common divisor.*

**Definition 37.** ***(irreducible)*** *Let* $Q$ *be as above then* $Q$ *is irreducible if for each* $i,j \in S$ *there exists an* $n = n(i,j)$ *with* $Q^n(i,j) > 0$*. Kesten [10]*

**Definition 38.** ***(aperiodic)*** *If* $Q$ *is irreducible it is called aperiodic, if* $\gcd\{n : Q^n(i,j) > 0\} = 1$*; this condition does not depend on* $i$ *in the irreducible case. Kesten [10]*

**Definition 39.** ***(recurrence)*** *A state* $i$ *is said to be transient, if, starting at state* $i$*, there is a nonzero probability that we will never return to* $i$*. Formally, let the random variable* $T_i$ *be the next return time to state* $i$*, and set*

$$T_i = \min\{n : X_n = i|\ X_0 = i\};$$

*the state* $i$ *is transient if*

$$P(T_i < \infty) < 1.$$

*If the state* $i$ *is not transient, then it is said to be recurrent.*

**Definition 40.** *(**R-recurrent**) $Q$ is $R$-recurrent if*

$$\sum Q^n(i,j)R^{-n} = \infty,$$

*where*

$$R^{-1} = \lim_{n\to\infty}[Q^n(i,j)]^{1/n}.$$

*Kesten [10]*

**Definition 41.** *(**ergodic**) An irreducible aperiodic positive recurrent chain is called ergodic. Asmussen [1]*

**Definition 42.** *(**harmonic function**) $f$ is called s-harmonic for $Q$ if*

$$Qf(i) = \sum_{j\in S_0} Q(i,j)f(j) = sf(i), \qquad \text{for all} \quad i \in S_0$$

*where*

$$S_0 = \{1, 2, \ldots\}.$$

*Kesten [10]*

**Definition 43.** *(**invariant measure**) A measure $\mu$ on $S_0$ is called an s-invariant measure for $Q$ if*

$$\mu Q(i) := \sum_{j\in S_0} \mu(j)Q(j,i) = s\mu(i), \qquad \text{for all} \quad i \in S_0.$$

*Kesten [10]*

**Definition 44.** *(**stationary distribution**) Let $S$ be a finite or countable set. Suppose that the chain has initial probabilities $\pi_i$ satisfying*

$$\sum_{i\in S} \pi_i p_{ij} = \pi_j, \qquad j \in S.$$

*It then follows by induction that*

$$\sum_{i\in S} \pi_i p_{ij}^{(n)} = \pi_j, \qquad j \in S, \quad n = 0, 1, 2, \ldots.$$

*If $\pi_i$ is the probability that $X_0 = i$, then $\pi_j$ is the probability that $X_n = j$; thus the probability of $[X_n = j]$ is the same for all $n$. A set of probabilities satisfying this condition is called a stationary distribution.*

**Definition 45.** ***(normalized quasi-stationary)*** *A measure $\mu$ on $S_0$ is called a normalized quasi-stationary distribution if it is a distribution probability and if*

$$P_\mu\{X_n = i|\tau > n\} = \mu(i), \qquad \text{for all} \quad i \in S_0,\ n \geq 1.$$

*Kesten [10]*

**Definition 46.** ***(reversible process)*** *The transition matrix is said to be reversible if and only if there is a sequence $\{\rho_i\}$ of positive constants such that $\rho_i p_{ij} = \rho_j p_{ji}$ for every $i$ and $j$. Pruitt [19]*

**Definition 47.** ***(branching process)*** *In probability theory, a branching process is a Markov process that models a population in which each individual in generation $n$ produces some random number of individuals in generation $n+1$, according to a fixed probability distribution that does not vary from individual to individual.*

**Definition 48.** ***(birth and death process)*** *The birth-death process is a special case of a continuous-time Markov process where the states represent the current size of a population and where the transitions are limited to births and deaths.*

## 2.2 Theorems

**Theorem 2.2.1.** *Let $X_n$ be a Markov chain on $S_0 \cup \{0\}$ with 0 as absorbing state and $\tau$ is a stopping time. Let $Q$ be defined by*

$$Q := (P(i,j))_{i,j \in S_0} = \textit{restriction of } P \textit{ to } \; S_0 \times S_0.$$

*Assume that $Q$ is irreducible, that absorption of $\{X_n\}$ at 0 is certain, and that*

$$\lim_{i \to \infty} P\{\tau \leq n | X_0 = i\} = 0 \quad \textit{for all } n \geq 1.$$

*Then*

$$P\{\tau > n | X_0 = i\} \to 0 \textit{ exponentially in } n \textit{ for } \; \textit{some fixed } i \in S_0,$$

*is necessary and sufficient for $X_n$ to have a normalized quasi-stationary distribution. Kesten [10]*

**Theorem 2.2.2** (central limit theorem). *Let $X_1, \; X_2, \ldots$ be a sequence of independent, identically distributed random variables each with mean $\mu$ and variance $\sigma^2$ . then the distribution of*

$$\frac{X_1 + X_2 + \ldots + X_n - n\mu}{\sigma\sqrt{n}}$$

*tends to the standard normal as $n \to \infty$. Ross [20]*

**Theorem 2.2.3** (Cauchy-Hadamard Theorem). *The reciprocal of the radius of convergence of the Taylor series*

$$a_0 + a_1 z + a_2 z^2 + \ldots$$

*is given by*

$$\limsup_{n \to \infty} (|a_n|)^{\frac{1}{n}} = R^{-1}.$$

# Chapter 3

# A counterexample

## 3.1 Constructing the counterexample

In this section, we want to construct a Markov chain $\{X_n\}$ on $\mathbb{Z}_+$ for which the Yaglom limit does not exist, despite the existence of a normalized quasi-distribution. We will show in Section 3.6 that the failure of the existence of the Yaglom limit implies failure of the ratio limit property. Moreover $Q$ has at least two distinct $R^{-1}$-invariant probability measures. We construct our Markov chain $\{X_n\}$ on $\mathbb{Z}_+$ with an absorbing state $\delta$ rather than on $S_0 \cup \{0\}$ . We can transform it into a Markov chain on $S_0 \cup \{0\}$ by identifying $\delta$ with 0, defined by $\tau = \inf\{n \geq 0 : X_n = \delta\}$; we will show that the following limit does not exist for some fixed $j$:

$$\lim_{n \to \infty} P\{X_n = j | X_0 = 0, \tau > n\}.$$

In our example, $|X_{n+1} - X_n| \leq 1$, as long as $\tau > n + 1$.

The transition probability matrix $P$ of our chain is:

$$P(\delta,\delta) = 1; \tag{3.1}$$

$$P(0,\delta) = \alpha \in (0,1),\ P(0,0) = r_0 \in [\tfrac{1}{4},\tfrac{1}{2}], \tag{3.2}$$

$$P(0,1) = P(0,-1) = \frac{1}{2}(1-\alpha-r_0);$$

$$\text{if } j>0,\quad P(j,j) = r_j \in [\tfrac{1}{4},\tfrac{1}{2}],\quad P(j,j+1) = (1-r_j)p, \tag{3.3}$$

$$P(j,j-1) = (1-r_j)q.$$

$$\text{if } j<0,\quad P(j,j) = r_j \in [\tfrac{1}{4},\tfrac{1}{2}],\quad P(j,j-1) = (1-r_j)p, \tag{3.4}$$

$$P(j,j+1) = (1-r_j)q.$$

where

$$1-\alpha-r_j > 0,\quad p+q=1,\quad 0<p<\tfrac{1}{2}<q. \tag{3.5}$$

From these equations, we see that $\delta$ is an absorbing state. Since $\gcd\{n : Q^n(i,j) > 0\} = 1$, $Q$ is aperiodic. It is also possible to reach any point from any other point; thus for all $i,j$, there exists $n$ such that $Q^n(i,j) > 0$ (so $Q$ is *irreducible*). The Markov chain also satisfies (1.5) (there exist constants $\delta_1$ and $N < \infty$ and for each $i \in S_0$, there exist integers $1 \le k_1, \dots, k_r \le N$ with $k_j = k_j(i)$ and $r = r(i)$ such that $Q^{k_s}(i,i) \ge \delta_1$ for $1 \le s \le r$, and $\gcd(k_1,\dots,k_r) = 1$). When $X$ is in state $j$, it will either stay there with probability $r_j$ or will move to state $j+1$ or $j-1$ (like a nearest neighbour walk) with drift $q-p>0$ towards 0; since $0<p<\frac{1}{2}<q$. From Theorem 2.2.1, the normalized quasi-stationary distribution will also exist. The main difficulty is to prove $\lim_{n\to\infty} P\{X_n = j | X_0 > 0, \tau > n\}$ does not exist for some fixed $j$. The idea to prove this is to choose the $r_j$ such that for certain times $(n_k^-)$, $X_n$ will be more likely to enter $(-\infty,-1]$ after visiting 0, for the last time, and for certain times $(n_k^+)$, $X_n$ will be more likely to enter to $[1,\infty)$ after visiting 0 for the last time. Since between successive visits to 0 our chain has to stay either in $(-\infty,-1]$ or $[1,\infty)$, the value of

$$P\{X_n > 0 | X_0 = 0, \tau > n\} = \sum_{j=1}^{\infty} P\{X_n = j | X_0 = 0, \tau > n\}$$

will be larger for $n = n_k^+$ than for $n = n_k^-$. So $\lim_{n\to\infty} P\{X_n = j | X_0 > 0, \tau > n\}$ does not exist for some fixed $j$.

## 3.2 An exponentially bounded tail for $\tau$

In this section, we want to show that for a Markov chain $\{X_n\}_{n\geq 0}$ on $(\mathbb{Z}\bigcup\{\delta\})$ with transition probability $P$ of the form (3.1)–(3.5) $\tau$, defined by $\tau = \inf\{n \geq 0; X_n = \delta\}$, has an exponentially bounded tail. By Theorem 2.2.1, this is necessary and sufficient for $\{X_n\}$ to have a normalized quasi-stationary distribution. Note that, in our example $|X_{n+1} - X_n| \leq 1$, as long as $\tau > n+1$; that is, $\lim_{i\to\infty} P\{\tau \leq n | X_0 = i\} = 0 \quad$ for all $n \geq 1$.

Let $Q$ be the restriction of $P$ to $\mathbb{Z}\times\mathbb{Z}$. If $r_j \in [\frac{1}{4}, \frac{1}{2}]$ for all $j$ and $0 < \alpha < 1$, then we have to show that after identification of $\mathbb{Z}\cup\{\delta\}$ with $S_0 \cup \{\delta\}$, then

$$P\{\tau > n|\ X_0 = i\} \to 0 \text{ exponentially in } n \text{ for some fixed } i \text{ in } S_0.$$

We start by defining $\sigma_0 = 0$ and $\sigma_1 < \sigma_2 < \sigma_3 < \ldots$ as the successive times (greater than zero) at which $X_n = 0$.

Now take $X_0 = 0$, then for $X_1$ we have one of these possibilities:

either

$$X_1 = \delta \text{ and } \tau = 1,$$

or

$$X_1 = 0 \text{ and } \sigma_1 = 1,$$

or

$$X_1 = \pm 1 \text{ and } \sigma_1 > 1.$$

For the sake of argument, let $X_1 = 1$. So we have $X_n > 0$ for $n < \sigma_1$, $X_{\sigma_1} = 0$, and for each $n < \sigma_1$,

$$\begin{aligned}
E\{X_{n+1} - X_n|X_n\}& \\
&= 0 + 1 \times P\{X_{n+1} - X_n = 1|\ X_n\} + (-1)\times P\{X_{n+1} - X_n = -1|\ X_n\}\\
&= (1 - r_{X_n})p + (1 - r_{X_n})q\\
&= (1 - r_{X_n})(p - q) = -(1 - r_{X_n})|q - p|\\
&\leq -\frac{1}{2}|q - p|
\end{aligned}$$

(as long as $r_j \in [\frac{1}{4}, \frac{1}{2}]$ ). We will use this to show the following lemma.

**Lemma 3.2.1.** *The sequence* $\left(X_{n\wedge\sigma_1} + \frac{1}{2}(n\wedge\sigma_1)|q-p|\right)_n$ *is a positive supermartingale with increments bounded by 2.*

*Proof.* It is obvious that $X_{n\wedge\sigma_1}+\frac{1}{2}(n\wedge\sigma_1)|q-p|$ is positive, as both terms are positive. The sequence is a supermartingale because

$E\{X_{n+1\wedge\sigma_1} + \frac{1}{2}(n+1\wedge\sigma_1)|q-p| - (X_{n\wedge\sigma_1} + \frac{1}{2}(n\wedge\sigma_1)|q-p|)|\ X_n\}$

$$\begin{aligned} &= E\{X_{n+1} + \frac{1}{2}|q-p| - X_n|\ X_n\} \\ &= \frac{1}{2}|q-p| + E\{X_{n+1} - X_n|\ X_n\} \\ &\le \frac{1}{2}|q-p| - \frac{1}{2}|q-p| \\ &= 0. \end{aligned}$$

To show that increments are bounded by 2, let $Y_{n+1} = X_{n+1\wedge\sigma_1} + \frac{1}{2}(n+1\wedge\sigma_1)|q-p|$ and $Y_n = X_{n\wedge\sigma_1} + \frac{1}{2}(n\wedge\sigma_1)|q-p|$, then

$$\begin{aligned} |Y_{n+1} - Y_n| &= |(X_{n+1\wedge\sigma_1} - X_{n\wedge\sigma_1}) + \frac{1}{2}|q-p|((n+1\wedge\sigma_1) - (n\wedge\sigma_1))\ | \\ &\le |X_{n+1\wedge\sigma_1} - X_{n\wedge\sigma_1}| + \frac{1}{2}|q-p| \\ &\le 1+1=2. \end{aligned}$$

□

**Lemma 3.2.2.** *For some constants* $C_1, C_2 > 0$

$$P\{\sigma_1 \ge k|\ X_1 = 1\} \le C_1 \exp\{-C_2 k\}.$$

*Proof.* We have

$$\begin{aligned} P\{\sigma_1 \ge k|\ X_1 = 1\} &= P\{X_1 > 0, \dots, X_{k-1} > 0, X_{k\wedge\sigma_1} \ge 0|\ X_1 = 1\} \\ &\le P\{X_{k\wedge\sigma_1} \ge 0|\ X_1 = 1\} \\ &= P\{X_{k\wedge\sigma_1} + \frac{1}{2}(k\wedge\sigma_1)|q-p| \ge \frac{1}{2}k|q-p|\ |\ X_1 = 1\} \quad (1) \end{aligned}$$

(as $k\wedge\sigma_1 = k$ for $\sigma_1 \ge k$). For $n=1$, the value of $X_{n\wedge\sigma_1} + \frac{1}{2}(n\wedge\sigma_1)|q-p|$ is $X_1 + \frac{1}{2}(1\wedge\sigma_1)|q-p| = 1 + \frac{1}{2}|q-p|$. Subtract this from both sides of the inequality inside (1), we obtain

$$P\{X_{k\wedge\sigma_1}+\tfrac{1}{2}(k\wedge\sigma_1)|q-p|-(X_1+\tfrac{1}{2}(1\wedge\sigma_1))\geq \tfrac{1}{2}k|q-p|-(1+\tfrac{1}{2}|q-p|)|\ X_1=1\}$$
$$=P\{X_{k\wedge\sigma_1}+\frac{1}{2}(k\wedge\sigma_1)|q-p|-(X_1+\frac{1}{2}(1\wedge\sigma_1))\geq \frac{1}{2}(k-1)|q-p|-1|\ X_1=1\}.$$

From Lemma 3.2.1, $X_{n\wedge\sigma_1}+\frac{1}{2}(n\wedge\sigma_1)|q-p|$ is a positive supermartingale with increments bounded by 2, so we can use standard exponential bounds [15] to show for some constants $C_1, C_2>0$

$$P\{X_{k\wedge\sigma_1}+\tfrac{1}{2}(k\wedge\sigma_1)|q-p|-(X_1+\tfrac{1}{2}(1\wedge\sigma_1))\geq \tfrac{1}{2}(k-1)|q-p|-1\mid X_1=1\}$$
$$\leq C_1\exp\{-C_2k\},$$

so we have

$$P\{\sigma_1\geq k|\ X_1=1\}\leq C_1\exp\{-C_2k\}.$$

□

The same estimate holds when $X_1=-1$. Therefore for $k\geq 1$

(3.6) $$P_0\{\sigma_1>k|\ X_1=\pm 1\}=\ P_0\{\sigma_1>k|\ X_1\neq\delta\}\leq C_1\exp\{-C_2k\}.$$

**Lemma 3.2.3.** *If for $k\geq 1$ and for some constants $C_1, C_2>0$,*

$$P_0\{\sigma_1>k|\ X_1\neq\delta\}\leq C_1\exp\{-C_2k\},$$

*then for $s<C_2$,*

$$E_0\{\exp(s\sigma_1)|\ X_1\neq\delta\}<\infty.$$

*Proof.* We have from (3.6)

$$\begin{aligned}E_0\{\exp(s\sigma_1)|\ X_1\neq\delta\} &= \sum_{k=1}^{\infty}P_0\{e^{s\sigma_1}\geq k|\ X_1\neq\delta\}\\ &= \sum_{k=1}^{\infty}P_0\left\{\sigma_1\geq\frac{\ln k}{s}|\ X_1\neq\delta\right\}\\ &\leq \sum_{k=1}^{\infty}C_1e^{-C_2\frac{\ln k}{s}}.\end{aligned}$$

The last inequality follows as replacing $k$ by $\frac{\ln k}{s}$ in our hypothesis. Now

$$\sum_{k=1}^{\infty} C_1 e^{-C_2 \frac{\ln k}{s}} = \sum_{k=1}^{\infty} C_1 k^{\frac{-C_2}{s}},$$

and this will converge if $\frac{C_2}{s} > 1$ . So if $C_2 > s$, we have

$$E_0\{\exp(s\sigma_1)|\ X_1 \neq \delta\} < \infty.$$

□

Now define $v = \inf\{i : \sigma_i = \infty\}$; then on $\{X_0 = 0\}$, for $n > \sigma_{v-1}$, $X_n \neq 0$. So when $\{\sigma_j < \infty\}$, $\sigma_{j+1} - \sigma_j$ has the same distribution as $\sigma_1$. Thus

$$P_0\{\sigma_{j+1} - \sigma_j > k|\ \sigma_j < \infty; X_{\sigma_{j+1}} \neq \delta\} = P_0\{\sigma_1 > k|\ X_1 \neq \delta\}.$$

Hence, if $\{\sigma_j < \infty, X_{\sigma_j+1} \neq \delta\}$, the chain returns to zero almost surely. It follows that almost surely on $\{X_0 = 0\}$,

$$X_{\sigma_{v-1}+1} = \delta \quad \text{or equivalently} \quad \tau = \sigma_{v-1} + 1,$$

since for $n > \sigma_{v-1}$, $X_n \neq 0$.
In the following lemma, we will show that $E_0\{e^{s\tau}\} < \infty$ and this will permit us to show that the distribution of $\tau$ has an exponentially bounded tail.

**Lemma 3.2.4.** *There exists $s_0 > 0$, such that*

$$E_0\{e^{s_0\tau}\} < \infty.$$

*Proof.* We have $\tau = \sigma_{v-1} + 1$. Therefore, we have

$$\begin{aligned} E_0\{e^{s\tau}\} &= E_0\{e^{s(\sigma_{v-1}+1)}\} \\ &= \sum_{k=0}^{\infty} E_0\{e^{s(\sigma_k+1)}; v = k+1\} \\ &= \sum_{k=0}^{\infty} E_0\{e^{s(\sigma_k+1)}; \sigma_k < \infty, X_{\sigma_k+1} = \delta\}. \end{aligned} \tag{2}$$

Now let $\mathcal{F}_k$ be the $\sigma$-field generated by $\{X_0, \ldots, X_k\}$. Then
$E_0\{e^{s(\sigma_k+1)}; \sigma_k < \infty, X_{\sigma_k+1} = \delta\}$

$$\begin{aligned} &= E_0\{E_0\{e^{s(\sigma_k+1)} I[\sigma_k < \infty] I[X_{\sigma_k+1} = \delta] | \mathcal{F}_{\sigma_k}\}\} \\ &= E_0\{e^{s(\sigma_k+1)} I[\sigma_k < \infty] P_0\{X_{\sigma_k+1} = \delta | \mathcal{F}_{\sigma_k}\}\} \\ &= \alpha E_0\{e^{s(\sigma_k+1)} I[\sigma_k < \infty]\}. \end{aligned} \tag{3}$$

The last equality, (3), is because $X_{\sigma_k} = 0$ almost everywhere on $\{\sigma_k < \infty\}$ and from before, we have $P(0, \delta) = \alpha$. Moreover,

$$\{\sigma_k < \infty\} = \{\sigma_{k-1} < \infty, X_{\sigma_{k-1}+1} \neq \delta\}.$$

So
$E_0\{e^{s(\sigma_k+1)} I[\sigma_k < \infty]\}$

$$\begin{aligned} &= E_0\{e^{s(\sigma_k+1)} I[\sigma_{k-1} < \infty, X_{\sigma_{k-1}+1} \neq \delta]\} \\ &= E_0\{e^{s(\sigma_k - \sigma_{k-1} + \sigma_{k-1}+1)} I[\sigma_{k-1} < \infty] I[X_{\sigma_{k-1}+1} \neq \delta]\} \\ &= E_0\{e^{s(\sigma_k - \sigma_{k-1})} I[X_{\sigma_{k-1}+1} \neq \delta] e^{s(\sigma_{k-1}+1)} I[\sigma_{k-1} < \infty]\} \\ &= E_0\{E_0\{e^{s(\sigma_k - \sigma_{k-1})} I[X_{\sigma_{k-1}+1} \neq \delta] e^{s(\sigma_{k-1}+1)} I[\sigma_{k-1} < \infty]\} | \mathcal{F}_{\sigma_{k-1}}\} \\ &= E_0\{E_0\{e^{s(\sigma_1)} I[X_{\sigma_{k-1}+1} \neq \delta] e^{s(\sigma_{k-1}+1)} I[\sigma_{k-1} < \infty]\} | \mathcal{F}_{\sigma_{k-1}}\} \\ &= E_0\{E_0\{e^{s(\sigma_1)} I[X_1 \neq \delta] e^{s(\sigma_{k-1}+1)} I[\sigma_{k-1} < \infty]\} | \mathcal{F}_{\sigma_{k-1}}\} \\ &= E_0\{e^{s(\sigma_1)} I[X_1 \neq \delta]\} E_0\{e^{s(\sigma_{k-1}+1)} I[\sigma_{k-1} < \infty]\}. \end{aligned}$$

Now from Lemmas 3.2.2 and 3.2.3,

$$E_0\{\exp(s\sigma_1) | X_1 \neq \delta\} < \infty \text{ if } s < C_2.$$

Thus we can find $s_0 > 0$ for which

$$\begin{aligned} E_0\{e^{s_0(\sigma_1)} I[X_1 \neq \delta]\} &= E_0\{e^{s_0(\sigma_1)} | I[X_1 \neq \delta]\} P_0\{X_1 \neq \delta\} \\ &\leq \frac{1 - \frac{\alpha}{2}}{1 - \alpha} P_0\{X_1 \neq \delta\} \\ &= \frac{1 - \frac{\alpha}{2}}{1 - \alpha}(1 - \alpha) \\ &= 1 - \frac{\alpha}{2}. \end{aligned}$$

So, we have

$$\begin{aligned} E_0\{e^{s_0(\sigma_k+1)}I[\sigma_k<\infty]\} &\le (1-\frac{\alpha}{2})E_0\{e^{s_0(\sigma_{k-1}+1)}I[\sigma_{k-1}<\infty]\} \\ &\le (1-\frac{\alpha}{2})^2E_0\{e^{s_0(\sigma_{k-2}+1)}I[\sigma_{k-2}<\infty]\} \\ &\le \dots \\ &\le (1-\frac{\alpha}{2})^k\exp(s_0). \end{aligned} \tag{4}$$

From (2),(3) and (4) we have,

$$\begin{aligned} E_0\{e^{s_0\tau}\} &= \sum_{k=0}^{\infty}E_0\{e^{s_0(\sigma_k+1)};\sigma_k<\infty,X_{\sigma_k+1}=\delta\} \\ &= \sum_{k=0}^{\infty}\alpha E_0\{e^{s_0(\sigma_k+1)}I[\sigma_k<\infty]\} \\ &\le \sum_{k=0}^{\infty}\alpha(1-\frac{\alpha}{2})^k\exp(s_0) \\ &\le \alpha\exp(s_0)\sum_{k=0}^{\infty}(1-\frac{\alpha}{2})^k \\ &\le \alpha\exp(s_0)\frac{2}{\alpha}<\infty \end{aligned}$$

as desired. □

**Corollary 3.2.1.** *For some fixed $i\in S_0$,*

$$P\{\tau>n|X_0=i\}\to 0 \textit{ exponentially in } n.$$

*Proof.* From Lemma 3.2.4 we know, there exists $M<\infty$ such that $M=E_0\{e^{s_0\tau}\}$. By Markov's inequality, we have

$$\begin{aligned} P\{\tau\ge n|X_0=i\} &= P\{e^{s_0\tau}\ge e^{s_0n}|X_0=i\} \\ &\le \frac{E_0\{e^{s_0\tau}\}}{e^{s_0n}} \\ &= Me^{-s_0n}\to 0 \text{ exponentially in } n \text{ for some fixed } i\in S_0. \end{aligned}$$

□

**Lemma 3.2.5.** *If $E_0\{e^{s_0\tau}\} < \infty$ for $s_0 > 0$, then $\tau$ has an exponentially bounded tail.*

*Proof.* Since $E_0\{e^{s_0\tau}\} < \infty$, there exist $C$ such that

$$\begin{aligned} C &= E_0\{e^{s_0\tau}\} \\ &= \sum_{n=0}^{\infty} e^{s_0 n} P_0\{\tau = n\} \\ &\geq \sum_{n=k}^{\infty} e^{s_0 n} P_0\{\tau = n\} \\ &\geq e^{s_0 k} \sum_{n=k}^{\infty} P_0\{\tau = n\} \\ &= e^{s_0 k} P_0\{\tau \geq k\}, \end{aligned}$$

so we have

$$P_0\{\tau \geq k\} \leq C e^{-s_0 k}.$$

□

## 3.3 An estimate for return probabilities of nearest-neighbour walks on $\mathbb{Z}$

In this section, we want to reduce the proof of failure of the existence of the Yaglom limit to an estimate for return probabilities of nearest-nighbour walks on $\mathbb{Z}$.
We start with the following general result on positive matrices of Vere-Jones [25] which we will use in this section.

**Lemma 3.3.1.** *If $Q$ is a positive, irreducible and aperiodic matrix, then*

$$\lim_{n\to\infty}[Q^n(i,j)]^{\frac{1}{n}} =: R^{-1} \in [0,\infty]$$

*exists for all $i,j$ and is independent of $i,j$.*

**Lemma 3.3.2.** *If $Q$ is a substochastic irreducible matrix which satisfies* (1.5), *then for fixed $i,j$*

$$\lim_{n\to\infty}\frac{Q^{n+1}(i,j)}{Q^n(i,j)} = \frac{1}{R}$$

*for the same $R$ as in previous lemma; Moreover, $1 \le R < \infty$. Kesten [10]*

Let $X_n$ be as in Section 3.2; then $X_j > 0$ means $X_j \in [1,\infty)$, and we exclude $X_j = \delta$. Now define

$$g_n^+ = P_0\{n < \sigma_1 < \infty, X_j > 0 \text{ for } 0 < j \le n\}$$

and

$$g_n^- = P_0\{n < \sigma_1 < \infty, X_j < 0 \text{ for } 0 < j \le n\}.$$

Now let

$$g_n = g_n^+ + g_n^- = P_0\{n < \sigma_1 < \infty\}.$$

This is the probability of starting from zero and not reaching zero for the first time by time $n$. Let $f_n$ be the probability of starting from 0 and reaching 0 for the first time at time $n$.

$$f_n = P_0\{\sigma_1 = n\},$$

and $u_n$ be the probability of starting from zero and reaching zero again after $n$ steps (not necessarily for the first time);

$$u_n = P_0\{X_n = 0\} = P^n(0,0).$$

Obviously, all these functions depend on $r_j$. We choose $r_j$ recursively, together with sequences of times $n_j^{\pm}$.
Choose integers

$$a_1 = 1 < a_2 < a_3 < \dots \tag{3.7}$$

$$b_1 = 1 < b_2 < b_3 < \dots \tag{3.8}$$

and select numbers

$$c_k, d_k \in [\tfrac{1}{4}, \tfrac{1}{2}], \tag{3.9}$$

such that

$$r_j = c_k \quad \text{for } a_k \leq j < a_{k+1}; \tag{3.10}$$

$$r_j = d_k \ \text{ for } -b_{k+1} \leq j < -b_k, \ k \geq 1. \tag{3.11}$$

That is, the $r_j$ is taken constant on certain intervals. We can also choose $c_k$ and $d_k$ such that

$$\tfrac{1}{4} \leq r_0 < d_1 < c_1 < d_2 < c_2 < \dots < \tfrac{1}{2}, \tag{3.12}$$

with

$$\lim_{k\to\infty} c_k = \lim_{k\to\infty} d_k = \tfrac{1}{2}. \tag{3.13}$$

This means that as $k$ increases, $r_j$ (which is either equal to $c_k$ or $d_k$) is also increasing. Consequently $(1-r_j)$ will be smaller for larger $k$, hence so will be $(1-r_j)q$. This implies that we will have less chance to return to origin for larger value of $k$.
Now, we want to construct a new Markov chain, $\{\tilde{X}_n\}$, from $\{X_n\}$ which satisfies (3.1)–(3.5), but with $r_j$ replaced by $\tilde{r_j}$. For any $1 \leq l < k$, let:

$$\tilde{r_j} = r_j = c_l \text{ for } a_l \leq j < a_{l+1};$$

$$\tilde{r_j} = r_j = d_l \text{ for } -b_{l+1} \leq j < -b_l.$$

Moreover, let

$$\tilde{r_0} = r_0;$$

$$\tilde{r_j} = c_k \text{ for } j \geq a_k;$$

$$\tilde{r_j} = d_k \text{ for } j \leq -b_k.$$

So $\tilde{r_j}$ agree with $r_j$ for $-b_k < j < a_k$. For any other entity $A$, we just use notation $\tilde{A}$, for the choice $\{\tilde{X}_n\}$.
Now, we construct the process $\{\tilde{X}_n\}$ out of two ingredients:

- The sequence of steps which behaves like a nearest neighbour walk; that is, $\{\tilde{X}_{n+1}\} \neq \{\tilde{X}_n\}$.
- The sequence of steps which stands still; that is, $\{\tilde{X}_{n+1}\} = \{\tilde{X}_n\}$.

Define $\{\rho_i\}_i$, $0 \leq \rho_0 < \rho_1 < \ldots$ to be the successive values of $n \geq 0$ for which $\{\tilde{X}_{n+1}\} \neq \{\tilde{X}_n\}$, so $\tilde{X}_{\rho_i} - \tilde{X}_{\rho_{i-1}} = \pm 1$. Hence $\{\tilde{X}_{\rho_i}\}_{i=0}$ considers only the steps on which $\tilde{X}_n$ does not stay put.
Define

$$Y_i = \tilde{X}_{\rho_i} - \tilde{X}_{\rho_{i-1}} \quad i \geq 1;$$

and

$$T_l = \sum_{i=1}^{l} Y_i + \tilde{X}_0 = (\tilde{X}_{\rho_1} - \tilde{X}_0) + (\tilde{X}_{\rho_2} - \tilde{X}_{\rho_1}) + \ldots + (\tilde{X}_{\rho_l} - \tilde{X}_{\rho_{l-1}}) + \tilde{X}_0 = \tilde{X}_{\rho_l},$$

and let

$$\lambda_i = \rho_i - \rho_{i-1} \quad i \geq 1 \quad \text{and} \quad \lambda_0 = \rho_0.$$

In fact $\lambda_i$ is the time required to jump from one state to the next one. So $\lambda_i = t$ means for $(t-1)$ units of time we have stayed at the same state and at the time $t$ we have jumped that is,

$$\tilde{X}_{\rho_{i-1}+1} = \tilde{X}_{\rho_{i-1}+2} = \ldots = \tilde{X}_{\rho_{i-1}+t-2} = \tilde{X}_{\rho_{i-1}+t-1}$$

and

$$\tilde{X}_{\rho_{i-1}+t} = \tilde{X}_{\rho_i}.$$

So, we have for $l \geq 1$ on $\{T_l = j\} = \{\tilde{X}_{\rho_{l-1}+1} = j\}$,

$$\tilde{P}\{\lambda_l = t |\ T_0, \ldots, T_l, \lambda_0, \ldots, \lambda_{l-1}\} = \tilde{r}_j^{t-1}(1 - \tilde{r}_j), \quad t = 1, 2, \ldots$$

while on $\{T_0 = 0\} = \{\tilde{X}_0 = 0\}$,

$$\tilde{P}\{\lambda_0 = t |\ T_0\} = \tilde{r}_0^t(1 - \tilde{r}_0), \quad t \geq 0.$$

Thus, we see

$$\tilde{X}_n = T_0 \quad \text{for} \quad n \leq \lambda_0 = \rho_0;$$

and for $l \geq 0$

$$\tilde{X}_n = T_{l+1} \quad \text{for} \quad \lambda_0 + \ldots + \lambda_l < n \leq \lambda_0 + \ldots + \lambda_{l+1}.$$

So $\{T_l > 0\}_{l \geq 0}$ is a Markov chain and even stronger, on $\{T_l > 0\}$,

$$\tilde{P}\{T_{l+1} = T_l + 1 |\ T_0, \ldots, T_l, \lambda_0, \ldots, \lambda_{l-1}\} = \frac{p(1 - \tilde{r}_j)}{(1 - \tilde{r}_j)} = p.$$

$$\tilde{P}\{T_{l+1} = T_l - 1 |\ T_0, \ldots, T_l, \lambda_0, \ldots, \lambda_{l-1}\} = \frac{q(1 - \tilde{r}_j)}{(1 - \tilde{r}_j)} = q,$$

and we have on $\{T_l < 0\}$:

$$\tilde{P}\{T_{l+1} = T_l + 1 |\ T_0, \ldots, T_l, \lambda_0, \ldots, \lambda_{l-1}\} = \frac{q(1 - \tilde{r}_j)}{(1 - \tilde{r}_j)} = q.$$

$$\tilde{P}\{T_{l+1} = T_l - 1 |\ T_0, \ldots, T_l, \lambda_0, \ldots, \lambda_{l-1}\} = \frac{p(1-\tilde{r_j})}{(1-\tilde{r_j})} = p.$$

Finally on $\{T_l = 0\}$,

$$\tilde{P}\{T_{l+1} = \delta |\ T_0, \ldots, T_l, \lambda_0, \ldots, \lambda_{l-1}\} = \frac{\alpha}{1-r_0}.$$

(as $P\{X_{l+1} = \delta\} = \alpha$ and $P\{X_{n+1} \neq X_n\} = (1 - r_0)$ when $X_n = 0$)

Moreover

$$\tilde{P}\{T_{l+1} = \pm 1 |\ T_0, \ldots, T_l, \lambda_0, \ldots, \lambda_{l-1}\} = \frac{1}{2}(1 - \frac{\alpha}{1-r_0}),$$

(as we know the process does not stay put and for $\{T_l = 0\}$, the probability of jumping to the left is the same as jumping to the right).

For our new Markov chain, $\{\tilde{X}_n\}$, we will prove the following lemmas.

The next lemma shows that for certain times $n_k^+$, it is more likely that $\tilde{X}_n$ entered $[1, \infty)$ at the last time that it left 0.

**Lemma 3.3.3.** *Let $k \geq 2$ and assume that $a_0, \ldots, a_k$, $b_0, \ldots b_k$, $r_0$, $c_1, \ldots, c_{k-1}$, $d_1, \ldots, d_k$ have already been chosen in accordance with* (3.7) – (3.12). *Then there exist $m_k^+ \leq n_k^+ < \infty$ such that uniformly for*

$$d_k < \frac{1}{4} + \frac{d_k}{2} \leq c_k \leq \frac{1}{2} \quad \textit{and} \quad a_{k+1}, b_{k+1} > n,$$

*we have*

$$\frac{\tilde{g_n}^+}{\tilde{g_n}^-} \geq k \textit{ for } n \geq m_k^+.$$

*Proof.* We begin by calculating $\tilde{g_n}^-$:

$$\begin{aligned}
\tilde{g}_n^- &= \tilde{P}_0\{n < \tilde{\sigma}_1 < \infty,\ \tilde{X}_j < 0 \ \text{ for } \ 0 < j \leq n\} \\
&= \tilde{P}_0\{\tilde{\sigma}_1 > n,\ \tilde{X}_j < 0 \ \text{ for } \ 0 < j \leq n\} \\
&= \tilde{P}_0\{\tilde{X}_j < 0 \ \text{ for } \ 0 < j \leq \tilde{\sigma}_1,\ \tilde{\sigma}_1 > n\} \\
&= \tilde{P}_0\{\tilde{X}_{\rho_i} < 0 \ \text{ for } \ 0 < \rho_i < \tilde{\sigma}_1,\ \tilde{\sigma}_1 > n\}.
\end{aligned}$$

The last inequality follows because we can remove the steps at which $\tilde{X}_j = \tilde{X}_{j+1}$. Now due to the fact that starting at zero and returning to zero for the first time requires an even number of steps, we have,

$$\begin{aligned}\tilde{g}_n^- &= \sum_{l=1}^{\infty} \tilde{P}_0\{\tilde{X}_{\rho_i} < 0 \ \text{ for } \ 0 < \rho_i < 2l, \tilde{X}_{2l} = 0, \lambda_0 = 0, \tilde{\sigma}_1 = \lambda_1 + \ldots + \lambda_{2l-1} + 1 > n\} \\ &= \sum_{l=1}^{\infty} \tilde{P}_0\{T_t < 0 \ \text{ for } \ 0 < t < 2l, T_{2l} = 0, \lambda_0 = 0, \lambda_1 + \ldots + \lambda_{2l-1} \geq n\}. \qquad (5)\end{aligned}$$

The $l$th summand in the right-hand side of (5) is the probability of those paths which return to 0 for the first time after $2l$ nonzero steps through strictly negative values of $T$, and the return occurs at time greater than $n$.

We have required that $\lambda_0 = 0$; otherwise $\tilde{\sigma}_1 = 1$. Now conditioned on the values $T_0, \ldots, T_{2l}$ with $T_i \leq 0$ for $1 \leq i < 2l$, the random variables $\lambda_1, \ldots, \lambda_{2l-1}$ are independent geometric variables with distribution

$$\tilde{P}\{\lambda_l = t | \ T_0, \ldots, T_l, \lambda_0, \ldots, \lambda_{l-1}\} = \tilde{r}_j^{t-1}(1 - \tilde{r}_j); \ \text{ hence}$$

$$\begin{aligned}\tilde{P}\{\lambda_l \geq t | \ T_0, \ldots, T_{2l}\} &= \sum_{k=t}^{\infty} \tilde{r}_{T_i}^{k-1}(1 - \tilde{r}_{T_i}) \\ &= \frac{\tilde{r}_{T_i}^{t-1}(1 - \tilde{r}_{T_i})}{(1 - \tilde{r}_{T_i})} \\ &= \tilde{r}_{T_i}^{t-1} \quad \text{ for } \ 1 \leq i \leq 2l - 1.\end{aligned}$$

Moreover on $T_i \leq 0$, there exist $j \leq k$ such that $\tilde{r}_{T_i} = d_j$. Since

$$\frac{1}{4} \leq r_0 < d_1 < c_1 < d_2 < \ldots < d_k < c_k < \frac{1}{2},$$

we have that for all $j$, $d_j \leq d_k$ that is $\tilde{r}_j^{t-1} \leq d_k^{t-1}$. So if we replace $\tilde{r}_j$ by $d_k$ for all $j \geq 1$, then all the holding times $\lambda_i$ will be stochastically increased. For $\tilde{g}_n^-$ we have

$$\begin{aligned}\tilde{g}_n{}^- &= \sum_{l=1}^{\infty} \tilde{P}_0\{T_t < 0 \ \text{ for } \ 0 < t < 2l, T_{2l} = 0, \lambda_0 = 0, \lambda_1 + \ldots + \lambda_{2l-1} \geq n\} \\ &= \sum_{l=1}^{\infty} \tilde{P}_0\{T_1 < 0, \ldots, T_{2l-1} < 0, T_{2l} = 0, \lambda_0 = 0, \ \lambda_1 + \ldots + \lambda_{2l-1} \geq n\} \\ &= \sum_{l=1}^{\infty} \tilde{P}\{\lambda_1 + \ldots + \lambda_{2l-1} \geq n | \ T_0, T_1, \ldots, T_{2l}\} \tilde{P}\{T_0, \ldots, T_{2l-1} < 0\}. \qquad (6)\end{aligned}$$

From (6) we can conclude that, if we replace $\tilde{r}_j$ by $d_k$ for all $j \geq 1$, $\tilde{g}_n^-$ will also increase.

Now after this replacement, $\tilde{X}$ behaves on the half line $(-\infty, -1]$ as nearest neighbour walk, $\{U_n\}_{n\geq 0}$ with iid steps. Then $U_n$ stands still with probability $d_k$ and moves one unit to the right with probability $(1-d_k)q$ and one unit to the left with probability $(1-d_k)p$. For such a walk, we can explicitly calculate the probability of staying in $(-\infty, -1]$ during the time interval $[1, n]$; from this we obtain for $n \geq 1$,

$$\begin{aligned} \tilde{g}_n^- &\leq \tilde{P}_0\{\tilde{X}_1 = -1\}P_{-1}\{U_j < 0, 1 \leq j \leq n-1\} \\ &= \frac{1}{2}(1-\alpha-r_0) \times A. \end{aligned}$$

Now to calculate $A$:

Let $U_{j,n}$ be the probability that the process reaches zero, starting from state $j$, after the $n$th step and let $\tau^* = \min\{n; U_n = 0\}$. Now define the generating function

$$(*) \qquad U_j(s) = \textstyle\sum_{n=0}^{\infty} U_{j,n}s^n = \sum_{n=0}^{\infty} P_j(\tau^* = n)s^n$$

then

$$U_{j,n+1} = p(1-d_k)U_{j+1,n} + q(1-d_k)U_{j-1,n} + d_kU_{j,n};$$

multiplying both side by $s^{n+1}$ and adding for $n = 0, 1, 2, \ldots$, we deduce

$$\begin{aligned} U_j(s) &= sp(1-d_k)U_{j+1}(s) + sq(1-d_k)U_{j-1}(s) + sd_kU_j(s) \\ (1-sd_k)U_j(s) &= sp(1-d_k)U_{j+1}(s) + sq(1-d_k)U_{j-1}(s) \end{aligned}$$

Since the transition probabilities are the same starting from any point, we have

$$\frac{U_{j+1}(s)}{U_j(s)} = \frac{U_{j+2}(s)}{U_{j+1}(s)} = \ldots = G(s),$$

where $G(s)$ is independent of $j$. Therefore,

$$(1-sd_k)U_{j-1}(s)G(s) = sp(1-d_k)U_{j-1}(s)G^2(s) + sq(1-d_k)U_{j-1}(s).$$

Dividing both sides by $U_{j-1}(s)$,

$$\begin{aligned} (1-sd_k)G(s) &= sp(1-d_k)G^2(s) + sq(1-d_k) \\ G(s) &= \frac{(1-d_ks) - [(1-d_ks)^2 - 4(1-d_k)^2pqs^2]^{\frac{1}{2}}}{2(1-d_k)ps}. \end{aligned}$$

Now from (*), we can calculate

$$A = \sum_{l \geq n} (\text{coefficient of } \ s^l \text{ in } G(s))$$

Now from Cauchy-Hadamard theorem, we have

$$r = \limsup_{n \to \infty} [\text{coefficient of } s^l \text{ in } \ G(s)]^{\frac{1}{l}},$$

where $r$ is the radius of convergence of $G(s)$; therefore $r$ must be smaller than size of smallest singularity of $G(s)$. So we have

$$\begin{aligned}
\limsup_{l \to \infty} [\text{coefficient of } s^l \text{ in } G(s)]^{\frac{1}{l}} &\leq [\text{size of smallest singularity of G(s)}]^{-1} \\
&= d_k + (1 - d_k)\sqrt{4pq} \\
&< d_k + (1 - d_k) \\
&< 1.
\end{aligned}$$

It follows that

$$\limsup_{n \to \infty} [\tilde{g}_n^-]^{\frac{1}{n}} \leq d_k + (1 - d_k)\sqrt{4pq}.$$

In the other direction, we have

$$\begin{aligned}
\tilde{g}_n^+ &= \tilde{P}_0\{\tilde{X}_j > 0 \text{ for } 0 < j \leq n\} \\
&\geq \tilde{P}_0\{\tilde{X}. \text{ hits } a_k \ \text{ at } \ \text{the } \ \text{time } \zeta \leq a_k^2 \text{ and } \tilde{X}_j > 0 \text{ for } 0 < j < \zeta \\
&\qquad \text{and } \tilde{X}_j \geq a_k \text{ for } \zeta \leq j \leq \zeta + n\} \\
&= \tilde{P}_0\{\tilde{X}. \text{ hits } a_k \text{ at the time } \zeta \leq a_k^2 \text{ and } \tilde{X}_j > 0 \text{ for } 0 < j < \zeta\} \\
&\quad \times \tilde{P}_{a_k}\{\tilde{X}_j \geq a_k \text{ for } 0 \leq j \leq n\}.
\end{aligned}$$

For $j \geq a_k$, $\tilde{r}_j = c_k$; therefore, on $[a_k, \infty)$, $X$ behaves like a nearest neighbour walk $\{S_n\}$ with iid steps, with probability

$$\begin{aligned}
P\{S_{n+1} = S_n + 1\} &= (1 - c_k)p, \\
P\{S_{n+1} = S_n\} &= c_k, \\
P\{S_{n+1} = S_n - 1\} &= (1 - c_k)q.
\end{aligned}$$

Thus

$$\tilde{g}_n^+ \geq C_3 P_0\{S_t \geq 0 \text{ for } 0 \leq t \leq n\},$$

where $C_3 = \tilde{P}_0\{\tilde{X}_. \text{ hits } a_k \text{ at the time } \zeta \leq a_k^2 \text{ and } \tilde{X}_j > 0 \text{ for } 0 < j < \zeta\}$ which depends on $\alpha, r_0, a_1, \ldots, a_k, c_0, \ldots, c_{k-1}$ but not on $c_k, b_0, \ldots, b_k, d_0, \ldots, d_k$ or $n$. Now define $s_0$, such that $\exp(s_0) = \sqrt{\frac{q}{p}}$ and define

$$\psi = \psi(\exp(s_0)) = (1-c_k)p\exp(s_0) + c_k + (1-c_k)q\exp(-s_0) = c_k + (1-c_k)\sqrt{4pq}.$$

Introduce the transformed random walk $\{S_n^*\}_{n\geq 0}$,

$$P\{S_{n+1}^* = S_n^*\} = \frac{c_k}{\psi};$$

$$P\{S_{n+1}^* = S_n^* + 1\} = \frac{e^{s_0 \times 1}}{\psi}(1-c_k)p = \frac{1}{2}(1 - \frac{c_k}{\psi});$$

$$P\{S_{n+1}^* = S_n^* - 1\} = \frac{e^{s_0 \times -1}}{\psi}(1-c_k)q = \frac{1}{2}(1 - \frac{c_k}{\psi}).$$

This definition implies that

$$P_x\{S_n = y\} = \psi\ \exp\left(-(y-x)\right)P_x\{S_n^* = y\}.$$

We know for $A_1, A_2, \ldots, A_n$ we have,

$$P(A_1 \cap A_2 \cap \ldots A_n) = P(A_1)P(A_2|\ A_1)P(A_3|\ A_1 \cap A_2)\ldots P(A_n|\ A_1 \cap A_2 \cap \ldots A_{n-1}).$$

So for any set $\Xi = \Xi_n \subset \mathbb{Z}^{l+1}$, using the fact that $\{S_l\}$ is a Markov chain we have

$$
\begin{aligned}
P_{x_0}\{(S_0,\dots,S_l) \in \Xi, S_l = y\} \\
&= P\{(S_0,\dots,S_l) = (x_0,x_1,\dots,x_{l-1},y), \text{ for } x_0,x_1,\dots,x_{l-1} \in \Xi\} \\
&= P\{S_l = y|\ S_0 = x_0,\dots,S_{l-1} = x_{l-1}\}P\{S_{l-1} = x_{l-1}|\ S_0 = x_0,\dots,S_{l-2} = x_{l-2}\} \\
&\quad \dots P\{S_2 = x_2|\ S_0 = x_0, S_1 = x_1\}P\{S_1 = x_1|\ S_0 = x_0\} \underbrace{P\{S_0 = x_0\}}_{1} \\
&= P_{x_{l-1}}\{S_l = y\} \times P_{x_{l-2}}\{S_{l-1} = x_{l-1}\}\dots P_{x_1}\{S_2 = x_2\} \times P_{x_0}\{S_1 = x_1\} \times 1 \\
&= \frac{\psi}{e^{s_0(y-x_{l-1})}} P\{S_l^* = y|\ S_{l-1}^* = x_{l-1}\} \\
&\quad \times \frac{\psi}{e^{s_0(x_{l-1}-x_{l-2})}} P\{S_{l-1}^* = x_{l-1}|\ S_{l-2}^* = x_{l-2}\} \\
&\quad \ddots \\
&\quad \times \frac{\psi}{e^{s_0(x_2-x_1)}} P\{S_2^* = x_2|\ S_1^* = x_1\} \\
&\quad \times \frac{\psi}{e^{s_0(x_1-x_0)}} P\{S_1^* = x_0\} \\
&= \frac{\psi^l}{e^{s_0(y-x_0)}} P_{x_0}\{(S_0^*,\dots,S_l^*) = (x_0,\dots,y)\} \\
&= \psi^l \times e^{s_0(-(y-x))} P_{x_0}\{(S_0^*,\dots,S_l^*) \in \Xi, S_y^* = y\}
\end{aligned}
$$

Now we want to obtain a lower bound for $P_0\{S_t \geq 0, \text{for } 0 \leq t \leq n\}$. From the simple combinatorial fact that on the event $\{S_0 = S_l = 0\}$, there exists at least one $\{0 \leq j \leq l\}$ such that

$$S_k - S_j \geq 0 \ \text{ for } \ j \leq k \leq j + l$$

(with $S_k$ interpreted as $S_{k-1}$ if $k > l$). Therefore,

$$P_0\{S_l = 0, S_t \geq 0, 0 \leq t \leq l\} = P_0\{S_t \geq 0, 0 < t < l|\ S_l = 0\}P_0\{S_l = 0\}.$$

As observed by Spitzer [22], Theorem 2.1 and the beginning of Section 3,

$$
\begin{aligned}
P_0\{S_l = 0, S_t \geq 0, 0 \leq t \leq l\} &\geq \frac{1}{l+1} P_0\{S_l = 0\} \\
&= \frac{1}{l+1} \psi^l P_0\{S_l^* = 0\} \\
&\geq C_4 \psi^l (l+1)^{-\frac{3}{2}},
\end{aligned}
$$

for some constant $C_k > 0$ and all $l \geq 0$. The last inequality follows from the local limit theorem, as observed by McDonald [14], Section 2 Lemma 0. The standard proof of this limit theorem Gnedenko and Kolmogorov [8], Section 49 shows that this estimate is uniform in $c_k \in [\frac{1}{4}, \frac{1}{2}]$, so we can take $C_4$ independent of $c_k$, provided $\frac{1}{4} \le c_k \le \frac{1}{2}$. Then we also have uniformly for $c_k$ satisfying $d_k < \frac{1}{4} + \frac{d_k}{2} \le c_k \le \frac{1}{2}$,

$$\begin{aligned} \tilde{g_n}^+ &\ge C_3 P_0\{S_t \ge 0 \text{ for } 0 \le t \le n\} \\ &\ge C_3 P_0\{S_n = 0, S_t \ge 0 \text{ for } 0 \le t \le n\} \\ &\ge C_3 C_4 \psi^n (n+1)^{-\frac{3}{2}}. \end{aligned}$$

Now we have

$$\psi = c_k + (1 - c_k)\sqrt{4pq} \ge d_k + (1 - d_k)\sqrt{4pq},$$

since

$$d_k < \frac{1}{4} + \frac{d_k}{2} \le c_k \le \frac{1}{2}.$$

Now we have both

$$\tilde{g_n^+} \ge C_3 C_4 \psi^n (n+1)^{-\frac{3}{2}},$$

and

$$\limsup_{n\to\infty} [\tilde{g_n^-}]^{\frac{1}{n}} \le d_k + (1 - d_k)\sqrt{4pq} < \psi; \text{ since}$$

$$\limsup_{n\to\infty} [\tilde{g_n^-}] < \psi^n.$$

Hence, we have

$$\frac{\tilde{g_n}^+}{\tilde{g_n}^-} \ge \frac{C_3 C_4 \psi^n (n+1)^{-\frac{3}{2}}}{\psi^n}.$$

This means we can find an $m_k^+$ such that

$$\frac{\tilde{g_n}^+}{\tilde{g_n}^-} \ge k \text{ for } n \ge m_k^+.$$

□

**Corollary 3.3.1.** *Under the conditions* $d_k < \frac{1}{4} + \frac{d_k}{2} \le c_k \le \frac{1}{2}$ *and* $a_{k+1}, b_{k+1} > n$,

$$\frac{\tilde{g_n}^+}{\tilde{g_n}^-} \ge k \quad \textit{for} \quad n \ge m_k^+;$$

*implies*

$$\frac{g_n^+}{g_n^-} \ge k \quad \textit{for} \quad n \ge m_k^+.$$

*Proof.* Clearly $|X_j| \leq n$ and $|\tilde{X}_j| \leq n$ for $j \leq n$, If $X_0 = 0$ , $\tilde{X}_0 = 0$. Therefore, $g_n^{\pm}$ does not depend on the choice of $r_j$ for $|j| \geq n$. Hence, $\tilde{g}_l^{\pm}$ and $g_l^{\pm}$ for $l \leq m_k^+$ are independent of the further choice of $a_j$, $b_j$, $c_j$ and $d_j$ with $j \geq k+1$ as long as we take $a_{k+1}$, $b_{k+1} > m_k^+$. □

**Lemma 3.3.4.** *Uniformly for $c_k$ satisfying*

$$d_k \leq \frac{1}{4} + \frac{1}{2}d_k \leq c_k \leq \frac{1}{2},$$

*we have*

$$\lim_{n\to\infty} \frac{\tilde{u}_{n+1}(i,j)}{\tilde{u}_n(i,j)} = \frac{1}{R_k^+},$$

*where* $R_k^+ = [c_k + (1-c_k)\sqrt{4pq}]^{-1}$.

*Proof.* Note that

$$\tilde{u}_n = \tilde{P}_0\{\tilde{X}_n = 0\} = \tilde{Q}^n(0,0).$$

We already know from Lemma 3.3.1 and 3.3.2 that $\lim_{n\to\infty}(\tilde{u}_n)^{\frac{1}{n}}$ and $\lim_{n\to\infty}\frac{\tilde{u}_{n+1}(i,j)}{\tilde{u}_n(i,j)}$ exist; therefore we just need to show uniformity and determine the explicit value of the limits. As observed by Kesten [10](proof of Lemma 4), uniformity will follow if we can show uniformly for $c_k$ satisfying $d_k \leq \frac{1}{4} + \frac{1}{2}d_k \leq c_k \leq \frac{1}{2}$;

$$\left[\left(\frac{\tilde{Q}-\delta I}{1-\delta}\right)^n (0,0)\right]^{\frac{1}{n}} \text{ converges to its limit,}$$

for some $0 < \delta < \frac{1}{2}\min_j \tilde{r}_j = \frac{1}{2}r_0$. To simplify instead of $\left[\left(\frac{\tilde{Q}-\delta I}{1-\delta}\right)^n (0,0)\right]^{\frac{1}{n}}$, we prove

$$(**) \qquad \lim_{n\to\infty}[\tilde{Q}^n(0,0)]^{\frac{1}{n}} = \lim_{n\to\infty}[\tilde{u}_n]^{\frac{1}{n}} \to \frac{1}{R_k^+}$$

uniformly for $c_k$, satisfying $d_k \leq \frac{1}{4} + \frac{1}{2}d_k \leq c_k \leq \frac{1}{2}$.
Call

$$T_l = \tilde{X}_{\rho_l},$$

$$\lambda_i = \rho_i - \rho_{i-1} \quad i \geq 1 \quad \text{and} \quad \lambda_0 = \rho_0.$$

Now define

$$\nu(m) = \text{number of indices } i \in [0, m-1] \text{ for which } T_i \in [a_k, \infty)$$

and let $\xi(m)$, $\kappa(m)$ be the total holding times during the first $m-1$ steps of $T$, in points outside $[a_k, \infty)$ and inside $[a_k, \infty)$, respectively. Thus

$$\xi(m) = \sum_{i=0}^{m-1} \lambda_i I\left[T_i \in (-\infty, ak)\right];$$

$$\kappa(m) = \sum_{i=0}^{m-1} \lambda_i I\left[T_i \in [a_k, \infty)\right].$$

So $\kappa(m)$ is the sum of $\nu(m)$ geometric variables with parameter $c_k$. Now we decompose $\tilde{u}_n$ in terms of the values of $m$ for which $\rho_{m-1} < n \leq \rho_m$ and the value of $T_m$; we have

$$\tilde{X}_n = T_{l+1} \quad \lambda_0 + \ldots + \lambda_l \leq n \leq \lambda_0 + \ldots + \lambda_{l+1}.$$

Hence,

$$\begin{aligned}
\tilde{u}_n &= \tilde{P}_0\{\tilde{X}_n = 0\} \\
&= \sum_{l=0}^{\infty}\sum_{r=0}^{n}\sum_{s=r}^{n}\sum_{t=0}^{n} P_0\{T_{2l} = 0, \xi(2l) = r, \xi(2l+1) = s, \nu(2l) = t, n-s \leq \kappa(2l) \leq n-r\}.
\end{aligned}$$

The $l$th summand is the probability of a path returning to 0 for the first time after $2l$ steps. We have started from zero and have reached zero again after $n$ steps; so the total holding time during $2l-1$ steps of $T$ outside $[a_k, \infty)$ can vary between zero and $n$,

$$\xi(2l) = r, \quad 0 \leq r \leq n.$$

Hence, the total holding times during the $2l$ steps of $T$ outside $[a_k, \infty)$ can vary between $r$ and $n$,

$$\xi(2l+1) = s \quad r \leq s \leq n.$$

Therefore, the total holding times during the first $2l-1$ steps inside $[a_k, \infty)$ can vary between $n-s$ and $n-r$,

$$n - s \leq \kappa(2l) < n - r.$$

So, we have

$$\begin{aligned}
\tilde{u}_n &= \tilde{P}_0\{\tilde{X}_n = 0\} \\
&= \sum_{l=0}^{\infty}\sum_{r=0}^{n}\sum_{s=r}^{n}\sum_{t=0}^{n} P_0\{T_{2l}=0, \xi(2l)=r, \xi(2l+1)=s, \nu(2l)=t, n-s \le \kappa(2l) \le n-r\} \\
&= \sum_{l=0}^{\infty}\sum_{r=0}^{n}\sum_{s=r}^{n}\sum_{t=0}^{n} P_0\{T_{2l}=0, \xi(2l)=r, \xi(2l+1)=s, \nu(2l)=t\} \times \\
&\qquad P\left\{n-s \le \sum_{1}^{l} L_j < n-r\right\}. \qquad (7)
\end{aligned}$$

Here the $L_j$ are iid random variables, each with the distribution

$$P\{L_j = t\} = c_k^{t-1}(1-c_k), \quad t \ge 1.$$

From (7), the event

$$\{T_{2l}=0, \xi(2l)=r, \xi(2l+1)=s, \nu(2l)=t\},$$

only depends on the T-process. The holding times of $\tilde{u}_n$ on $c_k$ emanates from the last factor, which equals

$$P\{\sum_{1}^{t} L_j \in [n-s, n-r)\} = \sum_{m=n-s}^{n-r-1} \binom{m-1}{t-1} c_k^{m-t}(1-c_k)^t.$$

If we take $\eta$ to be a small positive number and $l \ge 0$, such that

$$\frac{1}{4}(1+\eta)^l \le c_k \le \frac{1}{4}(1+\eta)^{l+1},$$

and define

$$P^*\{L_j = t\} = (\frac{1}{4}(1+\eta)^l)^{t-1}(1-\frac{1}{4}(1+\eta)^l), \quad t \ge 1,$$

then we have

$$P^*\{L_j = t\} = CP\{L_j = t\},$$

where

$$\begin{aligned}
C &= \frac{c_k^{t-1}(1-c_k)}{(\frac{1}{4}(1+\eta)^l)^{t-1}(1-\frac{1}{4}(1+\eta)^l)} \\
&\le \frac{(\frac{1}{4}(1+\eta)^l)^{t-1}(1-\frac{1}{4}(1+\eta)^{l+1})}{(\frac{1}{4}(1+\eta)^l)^{t-1}(1-\frac{1}{4}(1+\eta)^l)} \\
&\le \frac{(1-\frac{1}{4}(1+\eta)^{l+1})}{(1-\frac{1}{4}(1+\eta)^l)}.
\end{aligned}$$

Similarly

$$C \geq \frac{(1-\frac{1}{4}(1+\eta)^l)}{(1-\frac{1}{4}(1+\eta)^{l+1})}.$$

If we replace $c_k$ by $\frac{1}{4}(1+\eta)^l$ and $\frac{1}{4}(1+\eta)^{l+1}$, we will have

$$\tfrac{1}{(1+\eta)}[\tfrac{(1-\frac{1}{4}(1+\eta)^l)}{(1-\frac{1}{4}(1+\eta)^{l+1})}] \leq \tfrac{(1-\frac{1}{4}(1+\eta)^l)}{(1-\frac{1}{4}(1+\eta)^{l+1})} \leq C \leq \tfrac{(1-\frac{1}{4}(1+\eta)^{l+1})}{(1-\frac{1}{4}(1+\eta)^l)} \leq (1+\eta)[\tfrac{(1-\frac{1}{4}(1+\eta)^{l+1})}{(1-\frac{1}{4}(1+\eta)^l)}],$$

for each fixed $\eta \geq 0$ and $l \geq 0$ with $(1+\eta) \leq 2$. We have

$$P\{L_j = t\} = \frac{P^*\{L_j = t\}}{C};$$

Hence,

$$P^*\{L_j = t\}(1+\eta)[\frac{(1-\frac{1}{4}(1+\eta)^{l+1})}{(1-\frac{1}{4}(1+\eta)^l)}] \leq P\{L_j = t\} \leq P^*\{L_j = t\}(1+\eta)^{-1}[\frac{(1-\frac{1}{4}(1+\eta)^l)}{(1-\frac{1}{4}(1+\eta)^{l+1})}].$$

Now we can find a lower bound for $\tilde{u}_n$, on replacing $c_k$ by $\frac{1}{4}(1+\eta)^{l+1}$ and an upper bound on replacing $c_k$ by $\frac{1}{4}(1+\eta)^l$.

$$\frac{1}{(1+\eta)^n}\left[\frac{1-\frac{1}{4}(1+\eta)^{l+1}}{1-\frac{1}{4}(1+\eta)^l}\right][\tilde{u}_n \text{ calculated with } c_k \text{ replaced by } \frac{1}{4}(1+\eta)^l]$$

$$\leq \tilde{u}_n \leq$$

$$(1+\eta)^n\left[\frac{1-\frac{1}{4}(1+\eta)^l}{1-\frac{1}{4}(1+\eta)^{l+1}}\right][\tilde{u}_n \text{ calculated with } c_k \text{ replaced by } \frac{1}{4}(1+\eta)^{l+1}].$$

Since we are taking $c_k$ in $[\frac{1}{4}, \frac{1}{2}]$, we see $[\tilde{Q}^n(0,0)]^{\frac{1}{n}} = [\tilde{u}_n]^{\frac{1}{n}} \to \frac{1}{R_k^+}$, for some choice of $R_k^+$, is implied by convergence of

$$[\tilde{u}_n \text{ calculated with } c_k \text{ replaced by } \frac{1}{4}(1+\eta)^l]^{\frac{1}{n}}$$

for each fixed $\eta \geq 0$ and $l \geq 0$ with $(1+\eta) \leq 2$ (this convergence is known from Lemma 3.3.1). Thus uniformity in (**) holds; hence,

$$\lim_{n\to\infty} \frac{\tilde{u}_{n+1}(i,j)}{\tilde{u}_n(i,j)} = \frac{1}{R_k^+}$$

holds uniformly for $c_k$ satisfying $d_k \leq \frac{1}{4} + \frac{1}{2}d_k \leq c_k \leq \frac{1}{2}$. Now we need to show that $R_k^+$ is the same given in $R_k^+ = [c_k + (1-c_k)\sqrt{4pq}]^{-1}$. By Lemma 3.3.1 is follows that

$\lim_{n\to\infty}(\tilde{u}_n)^{\frac{1}{n}}$ exists, and we have shown uniformity. From standard renewal theory we know that

$$\sum_{n=0}^{\infty}\tilde{u}_n s^n = [1-\sum_{n=1}^{\infty}\tilde{u}_n s^n]^{-1}.$$

Thus

$$\begin{aligned}
\sum_{n=0}^{\infty}\tilde{u}_n s^n &= [1-\sum_{n=1}^{\infty}\tilde{u}_n s^n]^{-1} \\
&= [1-\sum_{n=1}^{\infty}\tilde{f}_n s^n]^{-1} \\
&= [1-\sum_{n=1}^{\infty}P_0\{\sigma_1 = n\}s^n]^{-1} \\
&= [1-\sum_{n=1}^{\infty}(P_0\{n-1<\sigma_1<\infty\} - P_0\{n<\sigma_1<\infty\})\, s^n]^{-1} \\
&= [1-\sum_{n=1}^{\infty}(\tilde{g}_{n-1}-\tilde{g}_n)s^n]^{-1} \\
&= [1-\sum_{n=1}^{\infty}\tilde{g}_{n-1}s^n + \sum_{n=1}^{\infty}\tilde{g}_n s^n]^{-1} \\
&= [1-s\sum_{n=1}^{\infty}\tilde{g}_{n-1}s^{n-1} + \sum_{n=0}^{\infty}\tilde{g}_n s^n - \tilde{g}_0]^{-1} \\
&= [1-s\sum_{n=0}^{\infty}\tilde{g}_n s^n + \sum_{n=0}^{\infty}\tilde{g}_n s^n - (1-\alpha)]^{-1} \\
&= [\alpha + (1-s)\sum_{n=0}^{\infty}\tilde{g}_n s^n]^{-1}, \qquad (8)
\end{aligned}$$

(recall that $\tilde{g}_0 = \tilde{P}_0\{\sigma_1 < \infty\} = 1-\alpha$). We have shown in Lemma 3.3.3,

$$\tilde{g}_n \leq \frac{1}{2}(1-\alpha-r_0)\sum_{l\geq n} \text{coefficient of } s^l \text{ in } G(s).$$

Hence

$$
\begin{aligned}
\sum_{n=1}^{\infty} \tilde{g}_n^- s^n &\leq \sum_{n=1}^{\infty} \frac{1}{2}(1-\alpha-r_0) \sum_{l \geq n} \left(\text{coefficient of } s^l \text{ in } G(s)\right) s^n \\
&= \frac{1}{2}(1-\alpha-r_0) \sum_{n=1}^{\infty} \sum_{l \geq n} \left(\text{coefficient of } s^l \text{ in } G(s)\right) s^n \\
&= \frac{1}{2}(1-\alpha-r_0)[\text{coefficient of } s \text{ in } G(s) \times s + \\
&\quad \ldots + \text{ coefficient of } s^n \text{ in } G(s) \times s^n + \ldots] \\
&= \frac{1}{2}(1-\alpha-r_0)\,[G(s) - G(1)] \\
&\leq \frac{1}{2}(1-\alpha-r_0)\frac{s}{s-1}\,[G(s) - G(1)]\,.
\end{aligned}
$$

Let *H(s)* be defined by replacing $d_k$ by $c_k$ in $G(s)$, that is

$$
H(s) = \frac{(1-c_k s) - [(1-c_k s)^2 - 4(1-c_k)^2 pqs^2]^{\frac{1}{2}}}{2(1-c_k)ps}.
$$

Then for

$$
0 \leq s \leq s_1 := \text{ smallest singularity of } \frac{s}{s-1}[H(s) - H(1)],
$$

we have

$$
\sum_{n=1}^{\infty} \tilde{g}_n^- s^n \leq \frac{1}{2}(1-\alpha-r_0)\frac{s}{s-1}[H(s) - H(1)].
$$

Similarly,

$$
\sum_{n=1}^{\infty} \tilde{g}_n^+ s^n \leq \frac{1}{2}(1-\alpha-r_0)\frac{s}{s-1}[H(s) - H(1)].
$$

Thus,

$$
\begin{aligned}
\sum_{n=0}^{\infty} \tilde{g}_n s^n &= \tilde{g}_0 + \sum_{n=1}^{\infty} \tilde{g}_n s^n \\
&= \tilde{g}_0 + \sum_{n=1}^{\infty} \tilde{g}_n^+ s^n + \sum_{n=1}^{\infty} \tilde{g}_n^- s^n \\
&\leq \tilde{g}_0 + (1-\alpha-r_0)\frac{s}{s-1}[H(s) - H(1)].
\end{aligned}
$$

Finally, from (8), for $1 \le s < s_1$

$$\begin{aligned}
\sum_{n=0}^{\infty} \tilde{u}_n s^n &= [\alpha + (1-s)\sum_{n=0}^{\infty} \tilde{g}_n s^n]^{-1} \\
&\le [\alpha + \left(\tilde{g}_0 + (1-\alpha-r_0)\frac{s}{s-1}[H(s)-H(1)]\right)(1-s)]^{-1} \\
&= [\alpha + (1-\alpha)(1-s) - (1-\alpha-r_0)s[H(s)-H(1)]]^{-1}.
\end{aligned}$$

The right side cannot be analytic beyond $|s| = \lim_{n\to\infty} [\tilde{u}_n]^{-\frac{1}{n}}$. This is because $\sum_{n=0}^{\infty} \tilde{u}_n s^n$ is a power series and it converges when $(\tilde{u}_n s^n)^{\frac{1}{n}} \le 1$; so $s \le (\frac{1}{\tilde{u}_n})^{\frac{1}{n}}$. Moreover, if $\alpha > 1 - r_0 - \frac{1}{2}p$ we have $(1-\alpha-r_0) < 2p$; thus

$$\alpha + (1-s)(1-\alpha) - (1-\alpha-r_0)s(H(s) - \underbrace{H(1)}_{1})$$

$$= \alpha + 1 - \alpha - s + s\alpha - (1-\alpha-r_0)s(H(s)-1)$$

$$= 1 - r_0 s - (1-\alpha-r_0)sH(s).$$

The right side is positive for $0 \le s < [c_k + (1-c_k)\sqrt{4pq}\,]^{-1}$. It suffices to check the inequality for $s = [c_k + (1-c_k)\sqrt{4pq}\,]^{-1}$; since $H$ is a power series with positive coefficient. The smallest singularity of $[\alpha+(1-s)(1-\alpha)-(1-\alpha-r_0)s(H(s)-H(1))]^{-1}$ is at $[c_k + (1-c_k)\sqrt{4pq}\,]^{-1}$; hence

$$\lim_{n\to\infty} [\tilde{u}_n]^{\frac{1}{n}} \le c_k + (1-c_k)\sqrt{4pq}.$$

In order to show $\lim_{n\to\infty}[\tilde{u}_n]^{\frac{1}{n}} = c_k + (1-c_k)\sqrt{4pq}$ , it would suffice to show also

$$\lim_{n\to\infty} [\tilde{u}_n]^{\frac{1}{n}} \ge c_k + (1-c_k)\sqrt{4pq}.$$

We have

$$\begin{aligned}
\tilde{P}_0\{\tilde{\tau} > n\} &= \sum_{l=n+1}^{\infty} \tilde{P}_0\{\tilde{\tau} = l\} \\
&= \sum_{l=n+1}^{\infty} \tilde{P}_0\{\tilde{X}_{l-1} = 0\}\tilde{P}\{\tilde{X}_l = \delta\} \\
&= \sum_{l=n+1}^{\infty} \tilde{P}_0\{\tilde{X}_{l-1} = 0\}\alpha \\
&= \alpha \sum_{m=n}^{\infty} \tilde{U}_m. \qquad (9)
\end{aligned}$$

Using the fact that $\lim_{n\to\infty} \frac{\tilde{U}_{n+1}}{\tilde{U}_n} = \frac{1}{R_k^+}$, where $R_k^+ = [c_k + (1-c_k)\sqrt{4pq}]^{-1}$, we have

$$\begin{aligned}
\lim_{n\to\infty} \sum_{m=n}^{\infty} \tilde{u}_m &= \lim_{n\to\infty} (\tilde{u}_n + \tilde{u}_{n+1} + \ldots) \\
&\sim \tilde{u}_n + \frac{\tilde{u}_n}{R_k^+} + \frac{\tilde{u}_n}{R_k^{+2}} \\
&= \tilde{u}_n \sum_{k=0}^{\infty} \left(\frac{1}{R_k{}^+}\right)^k \\
&= \tilde{u}_n \left(\frac{R_k^+}{R_k^+ - 1}\right). \qquad (10)
\end{aligned}$$

From (9) and (10),

$$\begin{aligned}
\lim_{n\to\infty} \tilde{P}_0\{\tilde{\tau} > n\} &= \alpha \lim_{n\to\infty} \sum_{m=n}^{\infty} \tilde{u}_m \\
&\sim \alpha\tilde{u}_n \left(\frac{R_k^+}{R_k^+ - 1}\right); \text{ and so} \\
\lim_{n\to\infty} \frac{\tilde{P}_0\{\tilde{\tau} > n\}}{\tilde{u}_n} &= \alpha \frac{R_k^+}{R_k^+ - 1};
\end{aligned}$$

as $\lim_{n\to\infty} \frac{\tilde{U}_{n+1}}{\tilde{U}_n} = \frac{1}{R_k^+}$ is uniform in $c_k$, this convergence is uniform in $c_k$ satisfying $d_k < \frac{1}{4} + \frac{d_k}{2} \le c_k \le \frac{1}{2}$. Now we have

$$\begin{aligned}
\lim_{n\to\infty} [\tilde{u}_n]^{\frac{1}{n}} &\ge \lim_{n\to\infty} [\tilde{P}_0\{\tilde{\tau} > n\}]^{\frac{1}{n}} \\
&\ge \limsup_{n\to\infty} [\tilde{g}_n^+]^{\frac{1}{n}}. \qquad (11)
\end{aligned}$$

We have shown before in the beginning of this section that

$$\tilde{g}_n^+ \geq c_3 c_4 \psi^n (n+1)^{\frac{-3}{2}},$$

where $c_3, c_4 > 0$; so

$$\limsup_{n\to\infty} [\tilde{g}_n^+]^{\frac{1}{n}} \geq \psi = c_k + (1-c_k)\sqrt{4pq}.$$

From (11),

$$\lim_{n\to\infty} [\tilde{u}_n]^{\frac{1}{n}} \geq \psi$$

and we have shown before that

$$\lim_{n\to\infty} [\tilde{u}_n]^{\frac{1}{n}} \leq \psi.$$

These two together imply

$$\lim_{n\to\infty} [\tilde{u}_n]^{\frac{1}{n}} = \psi.$$

Moreover, we know from Lemma 3.3.1

$$\lim_{n\to\infty} [\tilde{u}_n]^{\frac{1}{n}} = \frac{1}{R_k^+}, \quad R_k^+ \in [0,\infty);$$

thus

$$R_k^+ = [\psi]^{-1} = [c_k + (1-c_k)\sqrt{4pq}]^{-1}.$$

□

**Lemma 3.3.5.** *Uniformly for $d_k < \frac{1}{4} + \frac{d_k}{2} \leq c_k \leq \frac{1}{2}$, $a_{k+1}$, $b_{k+1} > n$, and $m_k^+ \leq n_k^+ < \infty$, we have*

$$\tilde{P}_0\{\tilde{X}_n > 0 | \tilde{\tau} > n\} \geq \frac{R_k^+ - 1}{\alpha R_k^+}\left[\sum_{l=1}^{m_k^+} \tilde{g}_l^+ (R_k^+)^l + \frac{\tilde{g}_n}{4\tilde{u}_n}\right] - \frac{1}{k}$$

*for $n \geq n_k^+$, where*

$$R_k^+ = [c_k + (1-c_k)\sqrt{4pq}]^{-1}.$$

*Proof.* We have

$$\tilde{P}_0\{\tilde{X}_n > 0 | \tilde{\tau} > n\} = \frac{\tilde{P}_0\{\tilde{X}_n > 0 \cap \tilde{\tau} > n\}}{\tilde{P}_0\{\tilde{\tau} > n\}} = \frac{1}{\tilde{P}_0\{\tilde{\tau} > n\}} \tilde{P}_0\{\tilde{X}_n \neq \delta, \tilde{X}_n > 0\}.$$

A standard last-exit decomposition shows that

$$\tilde{P}_0\{\tilde{X}_n \neq \delta, \tilde{X}_n > 0\} = \sum_{l=0}^{n-1} \tilde{u}_l \tilde{g}^+_{n-l} = \sum\nolimits_1 + \sum\nolimits_2,$$

where $\sum_1$ and $\sum_2$ are the sums over $0 \leq l < n - m_k^+$ and over $n - m_k^+ \leq l < n$, respectively. So

$$\tilde{P}_0\{\tilde{X}_n > 0 | \tilde{\tau} > n\} = \frac{1}{\tilde{P}_0\{\tilde{\tau} > n\}} \left(\sum\nolimits_1 + \sum\nolimits_2\right).$$

As shown previously(Lemma 3.3.4),

$$\lim_{n\to\infty} \frac{\tilde{P}_0\{\tilde{\tau} > n\}}{\tilde{u}_n} = \alpha \frac{R_k^+}{R_k^+ - 1}.$$

So

$$\begin{aligned}
\lim_{n\to\infty} \frac{\sum_2}{\tilde{P}_0\{\tilde{\tau} > n\}} &= \left(\sum_{l=n-m_k^+}^{n} \tilde{g}^+_{n-l} \tilde{u}_l\right) \frac{R_k^+ - 1}{\tilde{u}_n(\alpha R_k^+)} \\
&= \sum_{l=n-m_k^+}^{n} \tilde{g}^+_{n-l} \left(\frac{\tilde{u}_n}{\tilde{u}_l}\right)^{-1} \frac{R_k^+ - 1}{\tilde{u}_n(\alpha R_k^+)} \\
&= \sum_{l=1}^{m_k^+} \tilde{g}^+_l (R_k^+)^l \frac{R_k^+ - 1}{\alpha R_k^+};
\end{aligned}$$

The last equality follows from $\lim_{n\to\infty} \frac{\tilde{u}_{n+1}}{\tilde{u}_n} = \frac{1}{R_k^+}$. Again this convergence is uniform for $c_k$ satisfying, $d_k < \frac{1}{4} + \frac{d_k}{2} \leq c_k \leq \frac{1}{2}$.

On the other hand, we have for $n \geq m_k^+$

$$
\begin{aligned}
\sum\nolimits_1 &= \sum_{l=0}^{n-m_k^+} \tilde{u}_l \tilde{g}_{n-l}^+ \\
&= \underbrace{\tilde{u}_0}_{1} \tilde{g}_n^+ + \sum_{l=1}^{n-m_k^+} \tilde{u}_l \tilde{g}_{n-l}^+ \\
&\geq \tilde{g}_n^+ \\
&= \frac{k\tilde{g}_n^+ + \tilde{g}_n^+}{(k+1)\tilde{g}_n^+} \\
&\geq \frac{k\tilde{g}_n^+ + k\tilde{g}_n^-}{(k+1)} \\
&= (1+\frac{1}{k})^{-1}(\tilde{g}_n^+ + \tilde{g}_n^-) \\
&\geq \frac{1}{2}\tilde{g}_n
\end{aligned}
$$

(by using Lemma 3.3.3, $\frac{\tilde{g}_n^+}{\tilde{g}_n^-} \geq k$ for $n \geq m_k^+$). Thus for $n \geq m_k^+$ sufficiently large but independent of $c_k$,

$$
\frac{\sum_1}{\tilde{P}_0\{\tilde{\tau} > n\}} \geq \frac{1}{4}\tilde{g}_n \times \frac{R_k^+ - 1}{\alpha R_k^+ \tilde{u}_n}.
$$

So these observation show that there exists an $n \geq m_k^+$ (independent of $c_k$) such that if $d_k < \frac{1}{4} + \frac{d_k}{2} \leq c_k \leq \frac{1}{2}$ for $n \geq n_k^+$, then

$$
\begin{aligned}
\tilde{P}_0\{\tilde{X}_n > 0 | \tilde{\tau} > n\} &= \frac{\sum_1 + \sum_2}{\tilde{P}_0\{\tilde{\tau} > n\}} \\
&\geq \sum_{l=1}^{m_k^+} \tilde{g}_l^+ (R_k^+)^l \frac{R_k^+ - 1}{\alpha R_k^+} + \frac{1}{4} \times \frac{R_k^+ - 1}{\alpha R_k^+} \frac{\tilde{g}_n}{\tilde{u}_n} \\
&\geq \frac{R_k^+ - 1}{\alpha R_k^+} \left[ \sum_{l=1}^{m_k^+} \tilde{g}_l^+ (R_k^+)^l + \frac{\tilde{g}_n}{4\tilde{u}_n} \right] - \frac{1}{k}.
\end{aligned}
$$

□

**Corollary 3.3.2.** *Under the condition $d_k < \frac{1}{4} + \frac{d_k}{2} \leq c_k \leq \frac{1}{2}$ and $a_{k+1}, b_{k+1} > n$, we can drop the tildes in Lemma 3.3.4 and 3.3.5.*

*Proof.* We have $|X_i| \leq n$, $|\tilde{X}_i| \leq n$ for $i \leq n$ if $X_0 = \tilde{X}_0 = 0$. As long as we take $a_{k+1}$, $b_{k+1} > n \geq m_k^+$, we have that $g_l^{\pm}$ and $g_l$ for $l \leq m_k^+$ are independent of further choices of $a_j$, $b_j$ , $c_j$ and $d_j$ with $j \geq k+1$. □

We can also conclude from $[\tilde{u}_n]^{\frac{1}{n}} \longrightarrow \frac{1}{R_{k+}}$ uniformly for $c_k$ satisfying $d_k < \frac{1}{4} + \frac{d_k}{2} \leq c_k \leq \frac{1}{2}$, so that we may choose $n_k^+ \leq n$ large enough that

$$\left| [\tilde{u}_n]^{\frac{1}{n}} - \frac{1}{R_{k+}} \right| < \frac{1}{k}.$$

**Lemma 3.3.6.** *If* $a_0, \ldots, a_k, b_0, \ldots, b_k, b_{k+1}, c_0, \ldots, c_k, d_0, \ldots, d_k$ *have already been chosen as before then there* $m_k^- \leq n_k^-$ *such that uniformly for*

$$c_k < \frac{1}{4} + \frac{c_k}{2} \leq d_{k+1} \leq \frac{1}{2},$$

*and*

$$a_{k+1}, b_{k+2} > n$$

*we have*

$$\frac{g_n^-}{g_n^+} \geq k \quad for\ n \geq m_k^-$$

*and*

$$P_0\{X_n > 0 | \tau > n\} \leq \frac{R_k^- - 1}{\alpha R_k^-} \sum_{l=1}^{m_k^-} g_l^+ (R_k^-)^l + \frac{2}{k} \quad for\ n \geq n_k^-,$$

*where*

$$R_k^- = [d_{k+1} + (1 - d_{k+1})\sqrt{4pq}]^{-1}.$$

*Moreover, the probabilities* $g_l^+$ *for* $l \leq m_k^-$ *do not depend on further choice of* $a_{k+1}$, $c_{k+1}$ *or* $c_j$, $d_j$, $a_j$, $b_j$ *with* $j \geq k+2$, *provided we take* $a_{k+1}$, $b_{k+2} > n_k^-$.

*Proof.* This also is proven in the same way as Lemmas 3.3.3–5 by interchanging the positive and negative half lines. Only for calculating

$$P_0\{X_n > 0 | \tau > n\} \leq \frac{R_k^- - 1}{\alpha R_k^-} \sum_{l=1}^{m_k^-} g_l^+ (R_k^-)^l + \frac{2}{k} \quad \text{for } n \geq n_k^-;$$

we have here again

$$\tilde{P}_0\{\tilde{X}_n \neq \delta, \tilde{X}_n > 0\} = \sum_{l=0}^{n-1} \tilde{u}_l \tilde{g}^+_{n+1} = \sum\nolimits_1 + \sum\nolimits_2,$$

where $\sum_1$ and $\sum_2$ are the sums over $0 \leq l < n - m_k^+$ and over $n - m_k^+ \leq l < n$ respectively, and we have

$$\lim_{n\to\infty} \frac{\sum_2}{\tilde{P}_0\{\tilde{\tau} \geq n\}} = \sum_{l=1}^{m_k^+} \tilde{g}_l^+ (R_k^+)^l \frac{R_k^+ - 1}{\alpha R_k^+}.$$

This time, we estimate $\sum_1$ from above. We have $n \geq m_k^-$

$$\frac{\tilde{g}_n^-}{\tilde{g}_n^+} \geq k \quad \text{for } n \geq m_k^-,$$

since

$$\begin{aligned}
\sum\nolimits_1 &= \sum_{l=0}^{n-m_k^-} \tilde{u}_l \tilde{g}^+_{n-1} \\
&\leq \frac{1}{k} \sum_{l=0}^{n-m_k^-} \tilde{u}_l \tilde{g}^-_{n-1} \\
&\leq \frac{1}{k} \tilde{P}_0\{\tilde{X}_n \neq \delta, \tilde{X}_n < 0\} \\
&\leq \frac{1}{k} \tilde{P}_0\{\tilde{\tau} > n\}, \text{ therefore we have,}
\end{aligned}$$

$$\tilde{P}_0\{\tilde{X}_n > 0 | \tilde{\tau} > n\} = \frac{\sum_1 + \sum_2}{\tilde{P}_0\{\tilde{\tau} > n\}} \leq \frac{R_k^- - 1}{\alpha R_k^-} \sum_{l=1}^{m_k^-} \tilde{g}_l^+ (R_k^-)^l + \frac{2}{k} \quad \text{for } n \geq n_k^-.$$

By Corollary 3.3.1 we can again drop the tildes; therefore,

$$P_0\{X_n > 0 | \tau > n\} \leq \frac{R_k^- - 1}{\alpha R_k^-} \sum_{l=1}^{m_k^-} g_l^+ (R_k^-)^l + \frac{2}{k} \quad \text{for } n \geq n_k^-.$$

□

## 3.4 Existence of a universal lower bound

In this section, we want to find a universal constant $C_5 > 0$, which is independent of $\alpha$, $r_0$, $a_i$, $b_i$, $c_i$ and $d_i$, with following property. If for some $k \geq 2$, $a_1, \ldots,$ $b_1, \ldots b_k$ and $\alpha$, $r_0$, $c_1, \ldots, c_{k-1}$, $d_1, \ldots d_k$ have already been chosen as before, then there exist constants $N_k$, $q_k$ such that for all choices of $c_k$ which satisfy $c_k < \frac{1}{4} + \frac{c_k}{2} \leq d_{k+1} \leq \frac{1}{2}$, and for all $n \geq N_k$,

$$\max_{0 \leq p \leq 2q_k+1} \frac{\tilde{g}_{n-p}}{\tilde{U}_{n-p}} \geq C_5.$$

In this section we again use the Markov chain $\{\tilde{X}\}$ and the random walks $\{S\}$ and $\{S^*\}$, where $\{\tilde{X}\}$ has the following transition probabilities

$$P(\delta, \delta) = 1;$$

$$p(0, \delta) = \alpha \in (0,1), \quad P(0,0) = r_0 \in [\frac{1}{4}, \frac{1}{2}],$$

$$P(0,1) = P(0,-1) = \frac{1}{2}(1 - \alpha - r_0);$$

$$\text{if } j > 0, \quad P(j,j) = r_j \in [\frac{1}{4}, \frac{1}{2}], \quad P(j, j+1) = (1 - r_j)p,$$

$$P(j, j-1) = (1 - r_j)q.$$

$$\text{if } j < 0, \quad P(j,j) = r_j \in [\frac{1}{4}, \frac{1}{2}], \quad P(j, j-1) = (1 - r_j)p,$$

$$P(j, j+1) = (1 - r_j)q.$$

Here

$$1 - \alpha - r_j > 0, \quad p + q = 1, \quad 0 < p < \frac{1}{2} < q.$$

But with $r_j$ replaced by $c_k$ or $d_k$, if $j \geq a_k$ or $j \leq -b_k$. Further

$$\begin{aligned} P\{S_{n+1} = S_n\} &= c_k, \\ P\{S_{n+1} = S_n + 1\} &= (1 - c_k)p, \\ P\{S_{n+1} = S_n - 1\} &= (1 - c_k)q, \quad \text{while} \\ P\{S^*_{n+1} = S^*_n\} &= c_k R^+_k, \\ P\{S^*_{n+1} = S^*_n \pm 1\} &= \frac{1}{2}(1 - c_k R^+_k). \end{aligned}$$

Note that $\psi = (R_k^+)^{-1}$ and all the processes depend on $k$. Let $X^*$ be the Markov chain with transition probability measure as above, except the $r_j$ is replaced by $c_k$ for all $j \neq 0$ and $P^*$ is the probability measure governing this chain.

**Lemma 3.4.1.** *Let $T$ and $\lambda$ as in Section 3.3, then we have*

$$\tilde{u}_n \leq \sum_{l \geq n} P_0^*\{X_l^* = 0\}.$$

*Proof.* We see that $\tilde{u}_n$ is bounded by

$$\begin{aligned}
\sum_{l\geq n} \tilde{u}_l &= \sum_{l\geq n} \tilde{P}_0\{\tilde{X}_l = 0\} \\
&= \sum_{l\geq n} \tilde{E}_0\{I(\tilde{X}_l = 0)\} \\
&= \tilde{E}_0\{\text{number of } l \geq n \ \text{ with } \tilde{X}_l = 0\} \\
&= \tilde{E}_0\{\text{number of } l \geq n \ \text{ with } \tilde{X}_l = 0 \text{ and has not left zero yet }\} \\
&\quad + \tilde{E}_0\{\text{number of } l \geq n \ \text{ with } \tilde{X}_l = 0 \text{ and has left zero}\} \\
&= \tilde{E}_0\{(\lambda_0 - n + 1)^+\} \\
&\quad + \sum_{j=1}^{\infty} \tilde{E}_0 \left\{ I[T_{2j} = 0] \left[ \sum_0^{2j} (\lambda_i - n + 1)^+ - \sum_0^{2j-1} (\lambda_i - n + 1)^+ \right] \right\}. \quad (12)
\end{aligned}$$

The last equality, (12), results from $\lambda$ being the holding time of the $\tilde{X}_n$ at zero. Since $l \geq n$, we take the max of the $(\lambda_0 - n + 1)^+$ and zero. The $j$th summand for the second part is for the paths which return to zero for the first time after $2j$ nonzero steps. Now consider

$$\sum_0^{2j} (\lambda_i - n + 1)^+ - \sum_0^{2j-1} (\lambda_i - n + 1)^+;$$

this is an increasing function of all the $\lambda_i$. The right side of (12) increases if we replace $\lambda_j$ by $L_j$ for $j \geq 1$, where conditionally on $T_i$, they are independent and $L_i$ has the distribution $P\{L_j = t\} = c_k^{t-1}(1 - c_k)$, if $X_{T_i} \neq 0$ and $L_i = \lambda_i$, if $X_{T_i} = 0$. This means we are just replacing $r_j$ by $c_k$ for all $j \neq 0$; therefore,

$$\begin{aligned}\sum_{l \geq n} \tilde{u}_l &\leq \tilde{E}_0\{(\lambda_0 - n + 1)^+\} + \sum_{j=1}^{\infty} \tilde{E}_0 \left\{ I[T_{2j} = 0] \left[ \sum_0^{2j} (L_i - n + 1)^+ - \sum_0^{2j-1} (L_i - n + 1)^+ \right] \right\} \\ &= \sum_{l \geq n} P_0^*\{X_l^* = 0\}.\end{aligned}$$

□

**Lemma 3.4.2.** *Let*

$$f_l^* = P_0^*\{X_l^* = 0, X_t^* \neq 0 \text{ for } 0 < t < l\}$$

*and*

$$h_l = P_0\{S_l = 0, S_t > 0 \text{ for } 0 < t < l\},$$

*then we have*

$$f_l^* \leq h_l, \quad l \geq 1.$$

*Proof.* For $l = 1$ we have

$$f_l^* = r_0 \leq c_k = P_0\{S_1 = 0\}.$$

When $l > 1$, this means that either $X_1^* = 1$ or $X_1^* = -1$, so we have

$$\begin{aligned} f_l^* &= P_0^*\{X_l^* = 1, X_\cdot^* \text{ first returns to 0 at time } l\} \\ &\quad + P_0^*\{X_l^* = -1, X_\cdot^* \text{ first returns to 0 at time } l\} \\ &= \frac{1}{2}(1 - \alpha - r_0) P_1^*\{X_\cdot^* \text{ first visits to 0 at time } \ l - 1\} \\ &\quad + \frac{1}{2}(1 - \alpha - r_0) P_{-1}^*\{X_\cdot^* \text{ first visits to 0 at time } l - 1\} \\ &= (1 - \alpha - r_0) P_1^*\{ \text{ first visits to 0 at time } \ l - 1\}, \end{aligned}$$

because for $X_\cdot^* > 0$ ($X_\cdot^* < 0$), $X_\cdot^*$ (respectively $-X_\cdot^*$) behaves like $S_\cdot$. Finally,

$$\begin{aligned} 1 - \alpha - r_0 \leq \frac{1}{2}p &\leq (1 - c_k) \\ &= P_0\{S_1 = 1\}. \end{aligned}$$

So for $l > 1$,

$$\begin{aligned} f_l^* &\leq P_0\{S_1 = 1\}P_1\{S_\cdot \text{ first visits 0 at time } l-1\} \\ &= h_l. \end{aligned}$$

□

**Corollary 3.4.1.** *For $l \geq 1$ we have*

$$P_0^*\{X_l^* = 0\} \leq (R_k^+)^{-1}P_0\{S_l^* = 0, S_t^* > 0 \textit{ for } 0 < t < l\}.$$

*Proof.* If our first return to zero is at the $l$th step, then

$$P_0^*\{X_l^* = 0\} = f_l^*.$$

If the second return to zero is at the $l$th step, then

$$P_0^*\{X_l^* = 0\} = \sum_{n_1+n_2=l} f_{n_1}^* f_{n_2}^*.$$

If the $r$th return to zero is at the $l$th step, then

$$P_0^*\{X_l^* = 0\} = \sum_{r\geq l}\sum_{n_1+n_2+\ldots+n_r=l} f_{n_1}^* f_{n_2}^* \cdots f_{n_r}^*.$$

By Lemma 3.4.2,

$$\begin{aligned} P_0^*\{X_l^* = 0\} &\leq \sum_{r\geq 1}\sum_{n_1+n_2+\ldots+n_r=l} h_{n_1}h_{n_2}\ldots h_{n_r} \\ &\leq \sum_{r\geq l} P_0\{r\text{th return of } S_\cdot \text{ is at time } l \text{ and } S_t \geq 0, 0 < t < l\} \\ &= P_0\{S_l = 0,\ S_t \geq 0,\ 0 < t < l\} \\ &= (R_k^+)^{-l}P_0\{S_l^* = 0, S_t^* \geq 0, 0 < t < l\}. \end{aligned}$$

□

**Corollary 3.4.2.** *For some universal constant $C_6 < \infty$,*

$$P_0^*\{X_l^* = 0\} \leq C_6(l+1)^{-\frac{3}{2}}(R_k^+)^{-l}, \quad l \geq 0.$$

*Proof.* We begin by showing that for some universal $C_7 < \infty$,

$$P_0\{S_l^* = 0, S_t^* > 0,\ 0 < t < l\} \le C_7(l+1)^{-\frac{3}{2}},\quad l \ge 1.$$

We know that $S^*_\cdot$ stands still with probability $\frac{c_k}{\psi} = c_k R_k^+$ at each step; otherwise, moves like a simple symmetric random walk. Ignoring the times that $S^*_\cdot$ stands still, the probability of starting from zero and returning to zero after $2p$ jumps can be calculated by using Catalan numbers. We have

$$C_p = \frac{1}{p+1}\binom{2p}{p} = \frac{2}{p+1}\binom{2p-1}{p}.$$

Since $S^*$ moves to right and left with equal probability and there are $2p$ steps, the probability is $2^{-2p}$. The chain can stand still at each point that it visits as well; let $\zeta_i$ be the iid geometric random variable representing this behaviour at the $i$th point it visits. Then we have

$$P\{\zeta_i = r\} = (c_k R_k^+)^r(1 - c_k R_k^+),\quad r \ge 0.$$

All together we have

$$P_0\{S_l^* = 0, S_t^* > 0,\ 0 < t < l\} = \sum_{p=0}^{\infty} P\left\{\sum_{i=0}^{2p} \zeta_i = l - 2p\right\} \frac{2}{p+1}\binom{2p-1}{p} 2^{-2p}.$$

From Sterling's formula, it follows that there is a universal constant $C_8$ such that,

$$\begin{aligned}\sum_{p=0}^{\infty} P\left\{\sum_{i=0}^{2p} \zeta_i = l - 2p\right\} \frac{2}{p+1}\binom{2p-1}{p} 2^{-2p} &\le \sum_{p=0}^{\infty} P\left\{\sum_{i=0}^{2p} \zeta_i = l - 2p\right\} C_8(p+1)^{-\frac{3}{2}} \\ &= \sum_{p=0}^{[\frac{l}{2}]} P\left\{\sum_{i=0}^{2p} \zeta_i = l - 2p\right\} C_8(p+1)^{-\frac{3}{2}}.\end{aligned}$$

We have

$$\begin{aligned}\left\{\sum_{i=0}^{2p} \zeta_i = l - 2p\right\} &= \left\{\sum_{i=0}^{2p} \zeta_i = l - (2p+1) + 1\right\} \\ &= \left\{\sum_{i=0}^{2p} \zeta_i + 1 = l + 1\right\},\end{aligned}$$

and for a constant $0 \le t \le 1$, $\{\zeta_i + 1\} \sim$ Geometric($t$). Now by using the central limit theorem for some constants $k_0, k_1$, we have

$$\left\{\sum_{i=0}^{2p} \zeta_i + 1\right\} \sim \text{Normal}(k_0(2p+1), k_1(2p+1)).$$

Thus

$$\begin{aligned} P\left\{\sum_{i=0}^{2p} \zeta_i = l - 2p\right\} &\sim \frac{1}{\sqrt{2\pi}(2p+1)^{\frac{1}{2}}(k_1)^{\frac{1}{2}}} \exp\left(-\frac{(l+1-k_0(2p+1))^2}{2(2p+1)k_1}\right) \\ &\le \frac{1}{k_2(2p+1)^{\frac{1}{2}}}\left(\frac{2p+1}{(l+1-k_0(2p+1))^2}\right)^{\frac{5}{2}}. \end{aligned}$$

The last inequality follows from $x^{\frac{5}{2}} < e^x$, hence $x^{-\frac{5}{2}} > e^{-x}$. So

$$\begin{aligned} \sum_{p=0}^{[\frac{l}{2}]} P\left\{\sum_{i=0}^{2p} \zeta_i = l - 2p\right\} \frac{C_8}{(p+1)^{\frac{3}{2}}} &\le \sum_{p=0}^{[\frac{l}{2}]} \frac{KC_8}{(2p+1)^{\frac{1}{2}}} \frac{(2p+1)^{\frac{5}{2}}}{(l+1-k_0(2p+1))^5} \frac{1}{(p+1)^{\frac{3}{2}}} \\ &\le C_8 K\left([\frac{l}{2}]+1\right) \frac{(l+1)^{\frac{5}{2}}}{(k_0+1)^2(l+1)^5} \\ &\le C_7(l+1)\frac{(l+1)^{\frac{5}{2}}}{(l+1)^5} \\ &= C_7(l+1)^{-\frac{3}{2}}, \end{aligned}$$

for some universal $C_7 < \infty$. By Corollary 3.4.1,

$$\begin{aligned} P_0^*\{X_l^* = 0\} &\le (R_k^+)^{-1} P_0\{S_l^* = 0, S_t^* > 0 \text{ for } 0 < t < l\} \\ &\le (R_k^+)^{-1} C_7 (l+1)^{-\frac{3}{2}} \\ &\le C_6(l+1)^{-\frac{3}{2}}(R_k^+)^{-l} \quad l \ge 0. \end{aligned}$$

□

**Corollary 3.4.3.** *We have*

$$\sum_{p=0}^{\infty} \tilde{u}_p (R_k^+)^p \le \sum_{p=0}^{\infty}\sum_{l \ge p} C_6(l+1)^{-\frac{3}{2}}(R_k^+)^{p-l} \le C_{12} < \infty,$$

*where $C_{12}$ is universal.*

*Proof.* From Corollary 3.4.2 and Lemma 3.4.1, we have

$$\begin{aligned}
\sum_{p=0}^{\infty} \tilde{u}_p (R_k^+)^p &\le \sum_{p=0}^{\infty} \sum_{l \ge p} \tilde{u}_l (R_k^+)^p \\
&\le \sum_{p=0}^{\infty} \sum_{l \ge p} P_0^*\{X_l^* = 0\} (R_k^+)^p \\
&\le \sum_{p=0}^{\infty} \sum_{l \ge p} C_6 (l+1)^{-\frac{3}{2}} (R_k^+)^{-l} (R_k^+)^p \\
&= \sum_{p=0}^{\infty} \sum_{l \ge p} C_6 (l+1)^{-\frac{3}{2}} (R_k^+)^{p-l} \le C_{12} < \infty.
\end{aligned}$$

□

**Corollary 3.4.4.** *For some universal $C_9$, we have*

$$\tilde{P}_0\{\tilde{X}_n = 0 \textit{ and } \tilde{X}_s = 0 \textit{ for some } r \le s \le n-1\} \le C_9 (R_k^+)^{-n} (n+1)^{-\frac{3}{2}} (r+1)^{-\frac{1}{2}}.$$

*Proof.* From Corollary 3.4.2 and Lemma 3.4.1 we have

$$\begin{aligned}
\tilde{P}_0\{\tilde{X}_n &= 0 \text{ and } \tilde{X}_s = 0 \text{ for some } r \le s \le n-1\} \\
&\le \sum_{s=r}^{n-r} \tilde{u}_s \tilde{u}_{n-s} \\
&\le \sum_{s=r}^{n-r} \sum_{l \ge s} P_0^*\{X_l^* = 0\} \sum_{t \ge n-s} P_0^*\{X_t^* = 0\} \\
&\le C_6^2 \sum_{s=r}^{n-r} \sum_{l \ge s} (l+1)^{-\frac{3}{2}} (R_k^+)^{-l} \sum_{t \ge n-s} (t+1)^{-\frac{3}{2}} (R_k^+)^{-t} \\
&\le C_6^2 (R_k^+)^{-s} (R_k^+)^{-(n-s)} \sum_{s=r}^{n-r} \sum_{l \ge s} (l+1)^{-\frac{3}{2}} \sum_{t \ge n-s} (t+1)^{-\frac{3}{2}} \\
&\le C_9 (R_k^+)^{-n} (n+1)^{-\frac{3}{2}} (r+1)^{-\frac{1}{2}}.
\end{aligned}$$

□

Since $C_9$ is a universal constant, for any $\epsilon > 0$ there exists $q_0 = q_0(\epsilon) < \infty$, which depends only on $\epsilon$, such that if $r \ge q_0$, then

$$C_9 (R_k^+)^{-n} (n+1)^{-\frac{3}{2}} (r+1)^{-\frac{1}{2}} \le \epsilon (n+1)^{-\frac{3}{2}} (R_k^+)^{-n}.$$

**Lemma 3.4.3.** *If $r \geq q_0$ and $n \geq N_k$, then*

$$\tilde{u}_n \leq 2\tilde{P}_0\{\tilde{X}_n = 0 \text{ and } \tilde{X}_s \neq 0 \text{ for } r \leq s \leq n-r\}$$

*Proof.* We have
$\tilde{P}_0\{\tilde{X}_n = 0 \text{ and } \tilde{X}_s = 0 \text{ for } r \leq s \leq n-r\}$

$$\begin{aligned} &= \left|\tilde{u}_n - P\{\tilde{X}_n = 0 \text{ and } \tilde{X}_s \neq 0 \text{ for } r \leq s \leq n-r\}\right| \\ &\leq \epsilon(n+1)^{-\frac{3}{2}}(R_k^+)^{-n}. \end{aligned} \tag{13}$$

As was shown previously, Lemma 3.3.4, there exists $N_k$ such that for all $n \geq N_k$,

$$\lim_{n\to\infty} \tilde{u}_n = \left(\frac{R_k^+ - 1}{\alpha R_k^+}\right) \tilde{P}_0\{\tilde{\tau} > n\} \text{ and } \tilde{g}_n \geq C_3C_4(R_k^+)^{-n}(n+1)^{-\frac{3}{2}}.$$

Hence

$$\begin{aligned} \tilde{u}_n &\geq C_{10}\tilde{P}_0\{\tilde{\tau} > n\} \\ &\geq C_{10}\tilde{g}_n \\ &\geq C_{10}C_3C_4(R_k^+)^{-n}(n+1)^{-\frac{3}{2}}. \end{aligned} \tag{14}$$

Moreover, $C_{10}$ and $N_k$ depend on $\alpha, r_0, a_i, b_i, d_i, i \leq k$ and $c_i,\ i \leq k-1$, but not on $c_k$. From Lemma 3.4.1 and Corollary 3.4.2, we have

$$\begin{aligned} \tilde{u}_n &\leq \sum_{l\geq n} \tilde{u}_l \\ &\leq \sum_{l\geq n} P_0^*\{X_l^* = 0\} \\ &\leq \sum_{l\geq n} C_6(l+1)^{-\frac{3}{2}}(R_k^+)^{-l} \\ &\leq C_{11}(R_k^+)^{-n}(n+1)^{-\frac{3}{2}}. \end{aligned}$$

Choosing $\epsilon = C_{10}C_3C_4/2$, we find that for $q_k := q_0(C_{10}C_3C_4/2)$; this is independent of $c_k$. From (13) and (14), we have

$$\begin{aligned}
\tilde{u}_n &- P\{\tilde{X}_n = 0 \text{ and } \tilde{X}_s \neq 0 \text{ for } r \leq s \leq n-r\} \\
&\leq |\tilde{u}_n - P\{\tilde{X}_n = 0 \text{ and } \tilde{X}_s \neq 0 \text{ for } r \leq s \leq n-r\}| \\
&\leq \epsilon(n+1)^{-\frac{3}{2}}(R_k^+)^{-n} \\
&\leq C_{10}C_3C_4/2(n+1)^{-\frac{3}{2}}(R_k^+)^{-n} \\
&\leq \frac{1}{2}\tilde{u}_n.
\end{aligned}$$

The last inequality shows that

$$\tilde{u}_n \leq 2\tilde{P}_0\{\tilde{X}_n = 0 \text{ and } \tilde{X}_s \neq 0 \text{ for } r \leq s \leq n-r\}$$

for $r \geq q_0$ and $n \geq N_k$. □

**Lemma 3.4.4.** *There exists a universal constant $C_5 > 0$, which is independent of $\alpha, r_0$, $a_i$, $b_i$, $c_i$ and $d_i$, $i \geq 1$, with the following property. If, for some $k \geq 2$, $a_1, \dots$, $b_1, \dots$, $b_k$ and $\alpha$, $r_0$, $c_1, \dots$, $c_{k-1}$, $d_1, \dots$, $d_k$ have already been chosen as before, then there exist constants $N_k$, $q_k$ such that for all choices of $c_k$ which satisfy $c_k < \frac{1}{4} + \frac{c_k}{2} \leq d_{k+1} \leq \frac{1}{2}$, and for all $n \geq N_k$,*

$$\max_{0 \leq p \leq 2q_k+1} \frac{\tilde{g}_{n-p}}{\tilde{U}_{n-p}} \geq C_5.$$

*Proof.* We have from Lemma 3.4.3, that

$$\tilde{u}_n \leq 2\tilde{P}_0\{\tilde{X}_n = 0 \text{ and } \tilde{X}_s \neq 0 \text{ for } r \leq s \leq n-r\},$$

for $r \geq q_0$ and $n \geq N_k$. The probability on the right side for $r = 1$ is very similar to $\tilde{g}_n$. Standard decomposition with respect to the last time in $[0, r-1]$ and the first time in $[n-r+1, n]$ at which $\tilde{X}_. = 0$, shows that

$$\begin{aligned}
\tilde{P}_0\{\tilde{X}_n = 0 \text{ and } \tilde{X}_s \neq 0 \text{ for } r \leq s \leq n-r\} &= \sum_{p=0}^{r-1}\sum_{q=0}^{r-1} \tilde{u}_p \tilde{f}_{n-p-q} \tilde{u}_q \\
&\leq \sum_{p=0}^{r-1}\sum_{q=0}^{r-1} \tilde{u}_p \tilde{g}_{n-p-q-1} \tilde{u}_q \\
&\leq \max_{p,q \leq q_k} \tilde{g}_{n-p-q-1} \sum_{p=0}^{r-1} \tilde{u}_p \sum_{q=0}^{r-1} \tilde{u}_q. \quad (15)
\end{aligned}$$

We have from Lemma 3.4.1 and Corollary 3.4.2

$$\begin{aligned}\tilde{u}_n &\leq \sum_{l\geq n} \tilde{u}_l \\ &\leq \sum_{l\geq n} P_0^*\{X_l^* = 0\} \\ &\leq \sum_{l\geq n} C_6(l+1)^{-\frac{3}{2}}(R_k^+)^{-l} \quad l \geq 0. \end{aligned} \tag{16}$$

Moreover, from Corollary 3.4.3

$$\begin{aligned}\sum_{p=0}^{\infty} \tilde{u}_p (R_k^+)^p &\leq \sum_{p=0}^{\infty}\sum_{l\geq n} C_6(l+1)^{-\frac{3}{2}}(R_k^+)^{p-l} \\ &\leq C_{12} < \infty. \end{aligned} \tag{17}$$

So from (15), (16), (17) and Lemma 3.4.3, we have

$$\begin{aligned}\tilde{u}_n &\leq 2\tilde{P}_0\{\tilde{X}_n = 0 \text{ and } \tilde{X}_s \neq 0 \text{ for } r \leq s \leq n-r\} \\ &\leq 2\max_{p,q\leq q_k} \tilde{g}_{n-p-q-1} \sum_{p=0}^{r-1}\tilde{u}_p \sum_{q=0}^{r-1}\tilde{u}_q \\ &= 2\max_{p,q\leq q_k} \tilde{g}_{n-p-q-1} \sum_{p=0}^{r-1}\tilde{u}_p \frac{(R_k^+)^p}{(R_k^+)^p} \sum_{q=0}^{r-1}\tilde{u}_q \frac{(R_k^+)^q}{(R_k^+)^q} \\ &\leq 2\max_{p,q\leq q_k} \tilde{g}_{n-p-q-1}(R_k^+)^{-p}(R_k^+)^{-q} \sum_{p=0}^{r-1}\tilde{u}_p(R_k^+)^p \sum_{q=0}^{r-1}\tilde{u}_q(R_k^+)^q \\ &\leq 2C_{12}^2 \max_{p,q\leq q_k} \tilde{g}_{n-p-q-1}(R_k^+)^{-p-q} \quad \text{for } n \geq N_k,\ r = q_k. \end{aligned}$$

So

$$\max_{p,q\leq q_k} \frac{\tilde{u}_n}{\tilde{u}_{n-p-q-1}} \leq 2C_{12}^2 \max_{p,q\leq q_k} \frac{\tilde{g}_{n-p-q-1}}{\tilde{u}_{n-p-q-1}}(R_k^+)^{-p-q}.$$

Since we know that for fixed $p, q$

$$\frac{\tilde{u}_n}{\tilde{u}_{n-p-q}} \to (R_k^+)^{-p-q} \quad \text{as} \quad n \to \infty$$

uniformly for $c_k$. We can increase $N_k$, if necessary, so that

$$\begin{aligned}\max_{0\leq p\leq 2q_k+1} \frac{\tilde{g}_{n-p}}{\tilde{u}_{n-p}} &= \max_{p,q\leq q_k} \frac{\tilde{g}_{n-p-q-1}}{\tilde{u}_{n-p-q-1}} \\ &\geq \frac{1}{4}(C_{12}^2 R_k^+)^{-1} \quad \text{for } n \geq N_k. \end{aligned}$$

□

## 3.5 Failure of the Yaglom limit

In this section, we want to show the failure of the Yaglom limit, by applying the results of Sections 3 and 4 recursively. First we need to show that, the conditional distribution of $X_n$, given $\tau \geq n$, is tight. Let $\mu_n(j) = P_0\{X_n = j | \tau > n\}$. We have to show for all $n$ and for given $\epsilon > 0$, there exists a finite interval $(a, b]$ such that

$$\mu_n(a, b] \geq 1 - \epsilon$$

or equivalently

$$1 - \mu_n(a, b] \leq \epsilon.$$

For all $n$, we have

$$\begin{aligned} 1 - \mu_n(-A, A] &= P_0\{|X_n| \geq A \mid \tau > n\} \\ &\leq P_0\{\tau \geq A + n | \tau > n\} \\ &\leq \sum_{l=n+A}^{\infty} P_0\{X_l = 0| \tau > n\} \\ &= \frac{\sum_{l=n+A}^{\infty} P_0\{X_l = 0, \ \tau > n\}}{\sum_{l=n}^{\infty} P_0\{X_n = 0\}\alpha} \\ &= \frac{\sum_{l=n+A}^{\infty} P_0\{X_l = 0\}}{\sum_{l=n}^{\infty} P_0\{X_n = 0\}\alpha} \longrightarrow \frac{1}{\alpha}\frac{R^{-n-A}}{R^{-n}} \longrightarrow \frac{1}{\alpha}R^{-A}; \end{aligned}$$

by taking $A$ large, we have for given $\epsilon$, $\frac{1}{\alpha}R^{-A} < \epsilon$.

Now assume that $\alpha$, $r_0, a_1, \ldots, a_k$, $b_1, \ldots, b_k$, $c_1, \ldots, c_{k-1}$ and $d_1, \ldots, d_k$ have already been chosen as before. By Lemma 3.3.3 and Corollary 3.3.1, we can choose $m_k^+$, $n_k^+$ so that

$$\frac{g_n^+}{g_n^-} \geq k \text{ for } n \geq m_k^+,$$

and

$$P_0\{X_n > 0|\tau > n\} \geq \frac{R_k^+ - 1}{\alpha R_k^+}\left[\sum_{l=1}^{m_k^+} g_l^+(R_k^+)^l + \frac{g_n}{4u_n}\right] - \frac{1}{k}$$

for $n \geq n_k^+$, where

$$R_k^+ = [c_k + (1 - c_k)\sqrt{4pq}]^{-1}.$$

We can choose $c_k$ so close to $\frac{1}{2}$ such that,

$$d_k < \frac{1}{4} + \frac{d_k}{2} \le c_k \le \frac{1}{2}$$

also holds. We have shown that

$$\lim_{k\to\infty} [R_k^+]^{-1} = R^{-1}.$$

Since $|g_k^+| \le 1$, as $k \to \infty$, we have

$$\sum_{l=1}^{m_k^+} g_k^+(R_k^+)^l \to \sum_{l=1}^{m_k^+} g_k^+(R)^l.$$

Hence, $m_k^+$ does not depend on the choice of $c_k^+$. We have

$$\left| \frac{R_k^+ - 1}{\alpha R_k^+} \sum_{l=1}^{m_k^+} g_k^+(R_k^+)^l - \frac{R-1}{\alpha R} \sum_{l=1}^{m_k^+} g_k^+(R)^l \right| \le \frac{1}{k}.$$

Finally, we pick $N_k \ge n_k^+$ and $b_{n+1} = N_k + 1$ so that

$$\frac{\tilde{g}_{N_k}}{\tilde{u}_{N_k}} \ge C_5.$$

Next we choose $m_k^-$, $n_k^-$ such that by Lemma 3.3.6,

$$\frac{g_n^-}{g_n^+} \ge k \quad \text{for } n \ge m_k^-$$

and

$$P_0\{X_n > 0 | \tau > n\} \le \frac{R_k^- - 1}{\alpha R_k^-} \sum_{l=1}^{m_k^-} g_l^+(R_k^-)^l + \frac{2}{k} \quad for\ n \ge n_k^-,$$

where $R_k^- = [d_{k+1} + (1 - d_{k+1})\sqrt{4pq}\,]^{-1}$ holds. We choose $d_{k+1}$ sufficiently close to $\frac{1}{2}$ so that $c_k < \frac{1}{4} + \frac{c_k}{2} \le d_{k+1} \le \frac{1}{2}$ also holds. Again we have

$$\left| \frac{R_k^- - 1}{\alpha R_k^-} \sum_{l=1}^{m_k^-} g_k^+(R_k^-)^l - \frac{R-1}{\alpha R} \sum_{l=1}^{m_k^-} g_k^+(R)^l \right| \le \frac{1}{k}.$$

Finally we pick $a_{k+1} \geq (N_k \vee n_k^-) + 1$. We repeat the procedure with $k$ replaced by $k+1$, etc. Now for $n = N_k$ and $n = n_k^-$; we have

$$\sum_{l=1}^{\infty} g_l^+ R^l < \infty.$$

That is because,

$$1 \geq \tilde{P}_0\{\tilde{X}_n > 0 | \tilde{\tau} > n\} \geq \frac{R_k^+ - 1}{\alpha R_k^+} \left[ \sum_{l=1}^{m_k^+} \tilde{g}_l^+ (R_k^+)^l + \frac{\tilde{g}_n}{4\tilde{u}_n} \right] - \frac{1}{k}.$$

As $k \to \infty$ we have

$$1 \geq \frac{R-1}{\alpha R} \sum_{l=1}^{\infty} g_l^+ R^l + \frac{g_n}{4u_n};$$

that is,

$$\sum_{l=1}^{\infty} g_l^+ R^l \leq (1 - \frac{g_n}{4u_n})(\frac{\alpha R}{R-1}) < \infty.$$

Now from Lemmas 3.3.5 and 3.3.6, we have

$$\begin{aligned}
&\limsup_{n\to\infty} P_0\{X_n > 0 | \tau > n\} - \liminf_{n\to\infty} P_0\{X_n > 0|\tau > n\} \\
\geq\ & \limsup_{n\to\infty} \left( \frac{R_k^+ - 1}{\alpha R_k^+} [\sum_{l=1}^{m_k^+} \tilde{g}_l^+ (R_k^+)^l + \frac{\tilde{g}_n}{4\tilde{u}_n}] - \frac{1}{k} \right) - \liminf_{n\to\infty} \left( \frac{R_k^- - 1}{\alpha R_k^-} \sum_{l=1}^{m_k^-} g_l^+ (R_k^-)^l + \frac{2}{k} \right) \\
\geq\ & \limsup_{k\to\infty} \frac{R-1}{\alpha R} \frac{g_{N_k}}{4u_{N_k}} \qquad (k \to \infty \text{ as } n \to \infty) \\
=\ & \frac{R-1}{4\alpha R} \limsup_{k\to\infty} \frac{\tilde{g}_{N_k}}{\tilde{u}_{N_k}} \geq \frac{R-1}{4\alpha R} C_5.
\end{aligned}$$

This shows that $\lim_{n\to\infty} P_0\{X_n > 0 | \tau > n\}$ does not exist. This implies the failure of the Yaglom limit; since we have the conditional distribution of $X_n$ given $\tau > n$, is tight. By using Theorem 25.10 in Billingsley [2] to interchange sum and limit, we have

$$\begin{aligned}
\lim_{n\to\infty} P_0\{X_n > 0 | \tau > n\} &= \lim_{n\to\infty} \sum_{j>0} P\{X_n = j |\ X_0 = 0,\ \tau > n\} \\
&= \sum_{j>0} \lim_{n\to\infty} P\{X_n = j |\ X_0 = 0,\ \tau > n\}.
\end{aligned}$$

Since we have shown $\lim_{n\to\infty} P_0\{X_n > 0|\tau > n\}$ does not exist, $\lim_{n\to\infty} P\{X_n = j|\ X_0 = 0,\ \tau > n\}$ does not exist for some fixed $j$; This is the failure of the Yaglom Limit. This failure of the existence of the Yaglom limit is due to the fact that (1.4) does not hold for our chain. The example satisfies all the restrictions in Theorem 1.0.1 and 1.0.2 except (1.4). To see that (1.4) fails, note that after the identification $\mathbb{Z}_+ \cup \{\delta\}$ with $S_0 \cup \{0\}$, we will have $Q^n(2j, 2j-1) = P\{X_n = -j|X_0 = j\}$ and this equation is equal to zero whenever $n < 2j$, since our chain behaves as if it were like a nearest neighbour walk. Similarly $Q^n(2j, 2j+1) = Q^n(2j-1, 2j) = Q^n(2j-1, 2j-2) = 0$ for all large $j$, and this violates the second part of $(1.4u)$ and of $(1.4d)$; Nor can any other identification of states lead to a walk satisfying.

For example suppose $s_i$ were identified with $i$, so that $Q^n(i,j) = P\{X_n = s_j|X_n = s_i\}$. Now let $u > 0$ ; without lost of generality, we may assume that $s_u > 0$ .

Let $s = \min\{s_i : 1 \leq i < u\}$ and assume $s < 0$ and define $w$ by $s - 1 = s_w$. We must have $w \geq u$, from the definition $s = \min\{s_i : 1 \leq i < u\}$ and where $w < u$, then $s_w \leq u$, but $s_w = s-1$. Now if (1.4) were satisfied, there should exist $1 \leq L < \infty$ such that $Q(i,v) = 0$ whenever $i < u < u+L < v$. So we have to take $u + L \geq w$ (because for some $i < u$ $s_i = s$, and $Q(i,w) = P\{X_1 = s-1|X_0 = s\} > 0$, so $w$ cannot be greater than $u+L$). But now $[u, u+L)$ contains some $k$ with $s_u > 0$ *and* $s_w = s-1 < 0$ sicne we have $s_u > 0$ and $s_w \leq 0$ and $k$ must becomes large when $u$ become large. Then $Q^n(k, k+1) = P\{X_n = s_{k+1}|X_0 = s_k\}$ and this will be zero whenever $n < |s_k - s_{k+1}|$. For fixed $n$ and $L$, this will actually be the case for large enough $n$, so (1.4) cannot hold.

## 3.6 Failure of the ratio limit property

In this section we want to show that, in our example the ratio limit property also fails. The equations (3.1)–(3.4) shows that $\delta$ is an absorbing state and the absorption is possible only from state zero. So for some $L_1 < \infty$,

$$Q(i,0) = 0 \text{ for } i \geq L_1;$$

where $S_0 = \{1, 2, \ldots\}$. So

$$\begin{aligned} \sum_{j \in S_0} Q(i,j) &= 1 - Q(i,0) \\ &= 1 - 0 = 1. \end{aligned} \tag{18}$$

Moreover, when $X$ is in state $j \neq 0$, it will either stay there with probability $r_j$ or moves to state $j+1$ or $j-1$ (like a nearest neighbour walk) with drift $q - p > 0$ towards 0 (since $0 < p < \frac{1}{2} < q$). So for some $i_0$ we have $Q^n(i_0, 0) = 1$; hence

$$\begin{aligned} \sum_{j \in S_0} Q^n(i_0,j) &= 1 - Q^n(i_0,0) \\ &= 1 - 1 = 0. \end{aligned} \tag{19}$$

By Kesten [10] Section 2, the ratio limit property ,(18) and (19) would imply the existence of

$$\lim_{n \to \infty} P\{X_n = j | \, X_0 = i, \tau > n\}.$$

Since we have shown in Section 3.4 this limit does not exist for our example, the ratio limit property must also fail.

# Bibliography

[1] Asmussen, S. (2000) *Applied probability and queues.* 2th edn, Springer-Verlag, Berlin.

[2] Billingsley, P. (1986) *Probability and Measure.* 3rd edn, Wiley, NewYork.

[3] Darroch, J. N. and Seneta, E. (1965) *On quasi-stationary distributions in absorbing discrete-time finite Markov chains.* J. Appl. Probability 2 88–100. (Reviewer: R. Getoor) 60.65

[4] Feller, W. (1968) *An introduction to probability theory and its applications. Vol. I.* Third edition John Wiley and Sons, Inc., New York-London-Sydney.

[5] Feller, W. (1971) *An introduction to probability theory and its applications. Vol. II.* Second edition John Wiley and Sons, Inc., New York-London-Sydney.

[6] Ferrari, P. A., Kesten, H., Martnez, S., Picco, P. (1995) *Existence of quasi stationary distributions.* A renewal dynamical approach, Ann. Probab. Vol 23, No.2. 501-521.

[7] Gantmacher, F. R. (1959) *The theory of matrices.* Chelsea, Newyork.

[8] Gnedenko, B. V. and Kolmogorov, A.N.(1986) *Limit Distributions for Sums of Independent Random Variables.*, 2nd edn

[9] Good, P (1968) *The limiting behavior of transient birth and death processes conditioned on survival.* J. Austral. Math. Soc. 8 716–722. (Reviewer: H. Kesten) 60.67

[10] Kesten, H. (1995) *A ratio limit theorem for (sub) Markov chains on 1,2,... with bounded jumps.*, Adv. Appl. Prob. 27 ,652 691.

[11] Kijima, M. and Seneta, E.(1992) *Some results for quasi-stationary distributions of birth-death processes.* J. Appl. Probab. 28 , No. 3, 503–511. (Reviewer: Erik A. van Doorn) 60J80

[12] Kingman, J. F. C. and Orey, S. (1964)*Ratio Limit Theorems for Markov Chains.* Proceedings of the American Mathematical Society, Vol. 15, No. 6, 907-910.

[13] Levikson, B. (1977) *The age distribution of Markov processes.* J.Appl. Prob 14, 492-506.

[14] McDonald, D. (1994) *On local limit theorems for integer valued random variables.* Teor. vernoiatnostei i ee Primen, Vol.24, No.3, 607-614.

[15] Neveu, J. (1972) *Martingales temps discret.* 154–155.

[16] Orey, S. (1961) *Strong ratio limit property.* Bull. Amer. Math. Soc.67, 571-574

[17] Pakes, A. G. (1973) *Conditional limit theorems for a left-continuous random walk.* J. Appl. Probability 10, 39–53. (Reviewer: D. J. Daley) 60J15

[18] Pakes, A. G. (1978) *on the age distribution of a Markov chain.* Stoch. Proc. Appl. Prob 15, 65-77.

[19] Pruitt, W. E. (1965) *Strong ratio limit property for R-recurrent Markov chain.* Proceedings of the American Mathematical Society, Vol. 16, No. 2, 196-200.

[20] Ross, Sh. M. (2003) *introduction to probability models.* 8th edn.

[21] Seneta, E and Vere-Jones, D. (1966) *On quasi-stationary distributions in discrete-time Markov chains with a denumerable infinity of states.* J. Appl. Probab, 3 403-434.

[22] Spitzer, F. (1956) *A combinatorial lemma and its application to probability theory.* Trans. Amer. Math. Soc. 82, 323-339.

[23] Stroock, D. W. (2005) *An introduction to Markov processes, Graduate Texts in Mathematics.*, Springer-Verlag, Berlin.

[24] Van Doorn, E. A. (1991) *Quasi-stationary distributions and convergence to quasi-stationarity of birth-death processes.* Adv. in Appl. Probab. 23 , No. 4, 683–700. (Reviewer: E. Seneta) 60J80 (60J27)

[25] Vere-Jones, D. (1967) *Ergodic properties of nonnegative matrices I.*, Pacific J. Math. 361–386.

[26] Vere-Jones, D. (1969) *Some limit theorem for evanescent process.*, Austral. J. Statist 11, 67-78.

[27] Yaglom, A. M. (1947) *Certain limit theorems of the theory of branching random processes.* (Russian) Doklady Akad. Nauk SSSR (N.S.) 56, 795–798. (Reviewer: M. Love) 60.0X