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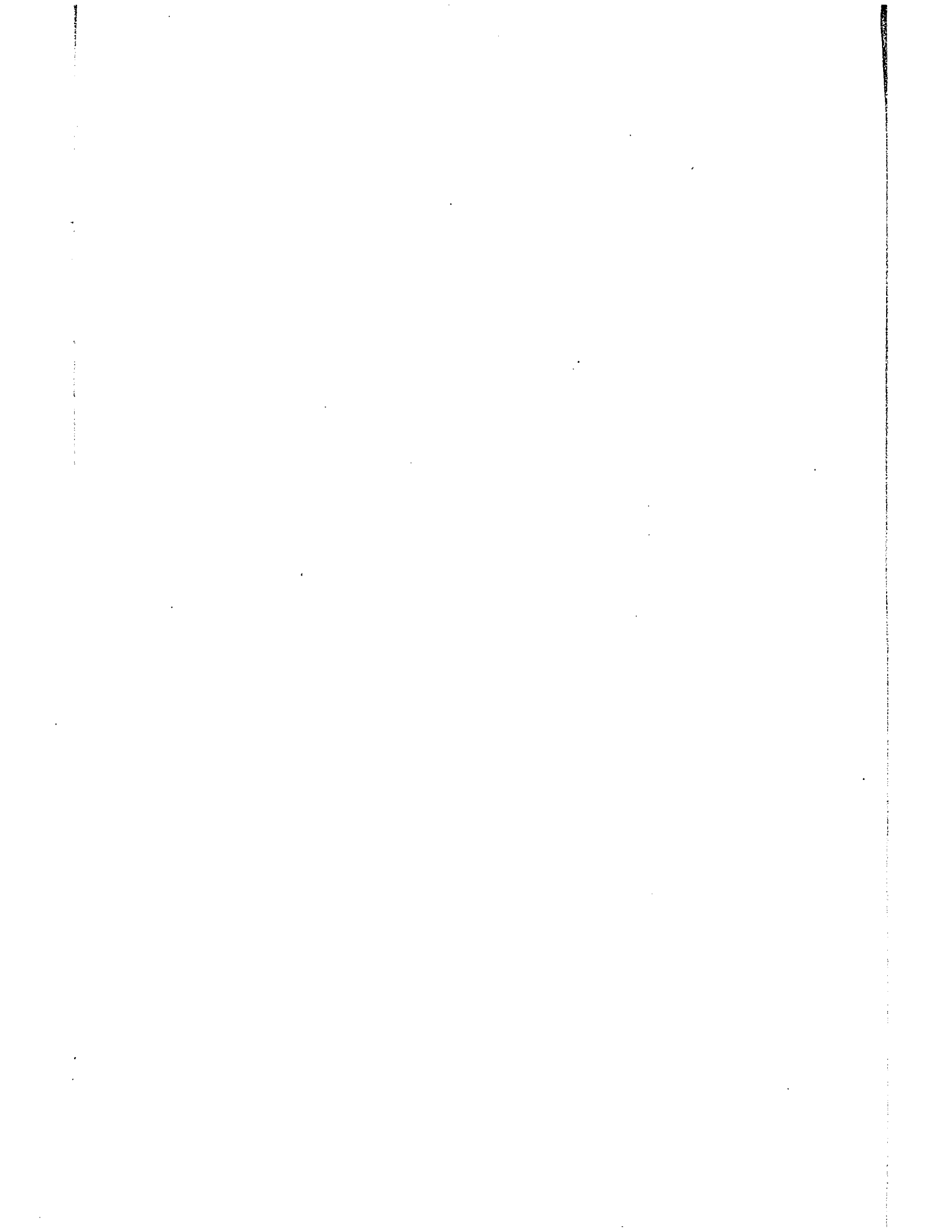
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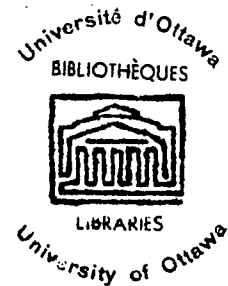
Ext AND THE SQUARE LEMMA

by

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Submitted in Partial Fulfillment
of the Requirements for the Degree of
Master of Science

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ABSTRACT

In his book "Lectures on Rings and Modules" Professor J. Lambek introduced Ext^1 based on the "Square Lemma". In this thesis his approach has been used to define Ext^n for any positive integer n . This definition will be called symmetric definition of Ext^n , since it is symmetric in both projective and injective modules. The equivalence of this new definition of Ext^n and the ordinary one has been shown. This symmetric definition together with the Square Lemma gives us an elegant method of diagram chasing, by squares instead of elements. Some applications are given.

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CHAPTER I
PRELIMINARIES

1.1 Introduction

Homology theory is a powerful technique in studying the structures of rings. Often we have to deal with a lot of arrows (homomorphisms) and many complicated diagrams. In order to show some relationship between two objects (modules) in a given diagram a hide and seek game of chasing diagram by elements from one corner to another, then to another etc., is necessary, and in most cases this is not an easy job and is uninteresting. For example, see the proof of Exact sequence of homology theorem in (2). By introducing the Square Lemma of (4), and the symmetric definition of Ext^n , we are able to chase the diagrams by squares instead of by elements, which is a more elegant treatment (see the proofs of Exact sequence of homology theorem in Appendix, and the theorem on Exact sequence of Ext^n in first variable in Chapter III).

Based on the Square Lemma, Ext^1 was first introduced by Professor J. Lambek. In this thesis this approach has been extended to any positive integer n , by showing there does exist an isomorphism of two particular R -modules, and the common value of this isomorphism is taken to be Ext^n . This definition is symmetric in projective and injective modules, and is different from the ordinary one given in (2). Their equivalence is shown in Chapter II. If a statement concerning projective module is proved based on ordinary definition of Ext^n , it is not always the case that a corresponding statement concerning injective module follows easily, quite the contrary, one might go a long way before

the truth of the corresponding statement for injectives is established. (Compare the proofs of theorems on Exact sequence of Ext^n in first and second variables in (2)). However, in using the symmetric definition of Ext^n , if a property concerning a projective module is true, it is also true when the word projective is replaced by injective, and vice versa. In whatever way we may define Ext^n , the most important point of all is that Ext^n should be independent of resolutions used to define it. The proof of this fact is quite long when the ordinary definition is used. However, this can be seen directly from our symmetric definition of Ext^n . Consider the statement: " $f: A \rightarrow B$ induces $f^*: \text{Ext}^n(A, C) \rightarrow \text{Ext}^n(B, C)$ for each n and is independent of the resolution is used to define Ext^n ". The truth of this statement cannot be seen easily if the ordinary definition of Ext^n is used, yet it is almost a direct consequence of the symmetric definition.

1.2 Notations

All modules which appear in the following are left R -modules over a ring R with unity. We shall simply call a left R -module an R -module without any further explanation.

Given any two R -modules A and B , we shall follow the notation of the book by Lambek (4), denote the additive abelian group of all R -homomorphisms from A into B by $B \circ A$, instead of the usual notation $\text{Hom}(A, B)$. If $f: B \rightarrow C$ is an R -homomorphism, we denote the induced homomorphisms by:

$$B \oplus A \xrightarrow{f \oplus 1} C \oplus A$$

$$A \oplus C \xrightarrow{1 \oplus f} A \oplus B$$

where 1 denotes the identity homomorphism on A.

1.3 Basic definitions

1. Exact sequence

Suppose $\{M_i\}$ is a non-empty collection of R-modules with a corresponding collection of mappings $f_i: M_i \rightarrow M_{i-1}$ such that $\text{Ker} f_i = \text{Im} f_{i-1}$. Then the sequence

$$\dots \rightarrow M_{i-1} \xrightarrow{f_{i-1}} M_i \xrightarrow{f_i} M_{i-1} \rightarrow \dots$$

is called an exact sequence of R-modules.

2. Projective module

An R-module P is called a projective R-module if the diagram:

$$\begin{array}{ccc} & P & \\ & \downarrow \mu & \\ A & \xrightarrow{\pi} B & \longrightarrow 0 \end{array} \quad \text{exact}$$

can be embedded in the diagram

$$\begin{array}{ccccc}
 & & P & & \\
 & \bar{\mu} \swarrow & \downarrow \mu & & \\
 A & \xrightarrow{\pi} & B & \longrightarrow & 0
 \end{array}$$

in such a way that the latter diagram is commutative i.e. $\pi\bar{\mu} = \mu$

3. Injective module

An R-module I is called injective if the diagram

$$\begin{array}{ccccc}
 0 & \longrightarrow & A & \xrightarrow{\pi} & B & \text{exact} \\
 & & \mu \downarrow & & & \\
 & & I & & &
 \end{array}$$

can be embedded in the diagram

$$\begin{array}{ccccc}
 0 & \longrightarrow & A & \xrightarrow{\pi} & B & \text{exact} \\
 & & \mu \downarrow & \swarrow \bar{\mu} & & \\
 & & I & & &
 \end{array}$$

in such a way that the latter diagram is commutative, i.e. $\mu = \bar{\mu}\pi$.

4. Splitting of exact sequence

An exact sequence $M \xrightarrow{\mu} S \longrightarrow 0$ is said to be split if there is a map $S \xrightarrow{p} M$ such that μp is identity on S. An exact sequence $0 \longrightarrow S \xrightarrow{j} M$ splits if there is a map $M \xrightarrow{k} S$ such that kj is identity on S.

5. Projective and injective resolutions

Given any module A, an exact sequence

$$\dots \longrightarrow P_n \xrightarrow{d_n} \dots \longrightarrow P_1 \xrightarrow{d_1} P \longrightarrow A \longrightarrow 0$$

with all P_i projective, is called a projective resolution of A . An exact sequence:

$$0 \longrightarrow A \longrightarrow I \xrightarrow{c_1} I_1 \xrightarrow{c_2} \dots \longrightarrow I_n \xrightarrow{c_{n-1}} \dots$$

with each I_i injective, is called an injective resolution of A .

Note that, given any module A , there exists a projective resolution of A , and an injective resolution of A , since every R -module is a homomorphic image of a projective module and can be embedded as a submodule of an injective module. (p.83, p.90 (4)).

6. Exact functor

(For definitions of category and functor refer to (5) and (6)).

Let Ω be the category of all left R -modules, a covariant functor T is said to be exact, if whenever $0 \longrightarrow E_1 \xrightarrow{\phi} E \xrightarrow{\psi} E_2 \longrightarrow 0$ is an exact sequence of R -modules, the sequence

$$0 \longrightarrow T(E_1) \xrightarrow{T(\psi)} T(E) \xrightarrow{T(\phi)} T(E_2) \longrightarrow 0$$

is also exact.

Remark: There is a corresponding definition for contravariant functor. (Refer to (6)).

1.4 Basic Lemmas

Lemma 1 For any module D .

- (i) If the sequence $0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} L$ is exact at A and B , the induced sequence

$$0 \longrightarrow A\emptyset D \xrightarrow{\alpha\emptyset 1} B\emptyset D \xrightarrow{\beta\emptyset 1} L\emptyset D$$

of abelian groups is exact,

- (ii) If the sequence $M \xrightarrow{\sigma} B \xrightarrow{\gamma} C \longrightarrow 0$ is exact at B and C, the induced sequence

$$0 \longrightarrow D\emptyset C \xrightarrow{1\emptyset\gamma} D\emptyset B \xrightarrow{1\emptyset\sigma} D\emptyset M$$

of abelian groups is exact.

(Refer to p. 23, (5)).

Lemma 2

The following properties of module D are equivalent

- (i) D is projective
 (ii) For each epimorphism $B \xrightarrow{\sigma} C \longrightarrow 0$
 $B\emptyset D \xrightarrow{\sigma\emptyset 1} C\emptyset D$ is an epimorphism
 (iii) If $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is a short exact sequence, so is

$$0 \longrightarrow A\emptyset D \longrightarrow B\emptyset D \longrightarrow C\emptyset D \longrightarrow 0$$

(Refer to p. 23 (5)).

Lemma 3

The following properties of module D are equivalent.

- (i) D is injective
 (ii) For each $0 \longrightarrow A \xrightarrow{\gamma} B$ monomorphism
 $D\emptyset B \longrightarrow D\emptyset A$ is an epimorphism
 (iii) If $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is a short exact sequence, so is

$$0 \longrightarrow D\emptyset C \longrightarrow D\emptyset B \longrightarrow D\emptyset A \longrightarrow 0$$

(Refer to p. 24 (5)).

Lemma 4

Given the following homomorphism of R-modules

$$A \xrightarrow{\alpha} A' \xrightarrow{\alpha'} A'' \quad B \xrightarrow{\beta} B' \xrightarrow{\beta'} B''$$

then $(\beta' \circ \alpha) \cdot (\beta \circ \alpha') = (\beta' \cdot \beta) \circ (\alpha' \cdot \alpha)$. In particular,
 if I denotes the identity mapping of A, B, A' and B'
 then $(1 \circ \alpha) \cdot (\beta \circ 1) = \beta \circ \alpha = (\beta \circ 1) \cdot (1 \circ \alpha)$.

(Refer to p. 122 (4)).

Lemma 5

Let T be an exact covariant functor and let $E_1 \longrightarrow E \longrightarrow E_2$
 be an exact sequence of R -modules, then the sequence
 $T(E_1) \longrightarrow T(E) \longrightarrow T(E_2)$ is also exact.

(Refer to p. 147 (6)).

Corollary: An exact covariant functor preserves long
 exact sequence.

Proof: A direct consequence of Lemma 5.

Remarks

1. The corresponding statement of Lemma 5. and its
 corollary for contravariant functor is also true
 (Refer to (6)).
2. In view of definition 5. in 1.3 and the above Lemmas
 $\circ P$ is an exact covariant functor if P is projective;
 and $I \circ$ is an exact contravariant functor if I is
 injective

1.5 Square Lemma

Here we introduce the Lemma which plays a central role
 throughout the remaining chapters.

Square Lemma: In the diagram

$$\begin{array}{ccccc}
 & \lambda & & \mu & \\
 A & \longrightarrow & B & \longrightarrow & C \\
 \alpha \downarrow & & \beta \downarrow & & \downarrow \gamma \\
 & 1 & & 2 & \\
 D & \longrightarrow & E & \longrightarrow & F \\
 & \lambda' & & \mu' &
 \end{array}$$

if both squares are commutative and both top and bottom rows are exact, then

$$\frac{\text{Im}(B \rightarrow E) \cap \text{Im}(D \rightarrow E)}{\text{Im}(A \rightarrow E)} = \frac{\text{Ker}(B \rightarrow F)}{\text{Ker}(B \rightarrow C) + \text{Ker}(B \rightarrow E)}$$

Proof: Consider

$$\begin{aligned} \text{Im } \beta \cap \text{Im } \lambda' &= \text{Im } \beta \cap \text{Ker } \mu' \\ &= \{\beta(b) \mid \mu' \beta(b) = 0\} && \text{since the bottom row is exact} \\ &= \{\beta(b) \mid \gamma \mu(b) = 0\} && \text{since square 2 is commutative} \\ &= \beta(\text{Ker}(\gamma \cdot \mu)) \end{aligned}$$

$$\begin{aligned} \text{Im}(\beta \cdot \lambda) &= \beta(\text{Im } \lambda) \\ &= \beta(\text{Ker } \mu) && \text{since the top row is exact} \\ &= \beta(\text{Ker } \mu + \text{Ker } \beta) && \text{since } \beta(\text{Ker } \beta) = 0 \end{aligned}$$

on the other hand we have the following isomorphisms

$$\beta(\text{Ker}(\gamma \cdot \mu)) \cong \text{Ker}(\gamma \cdot \mu) / \text{Ker } \beta$$

and
$$\beta(\text{Ker } \mu + \text{Ker } \beta) \cong (\text{Ker } \mu + \text{Ker } \beta) / \text{Ker } \beta$$

By the 2nd isomorphism theorem in group theory (Refer to (1)), we have

$$\frac{\beta(\text{Ker}(\gamma \cdot \mu))}{\beta(\text{Ker } \mu + \text{Ker } \beta)} \cong \frac{\text{Ker}(\gamma \cdot \mu) / \text{Ker } \beta}{(\text{Ker } \mu + \text{Ker } \beta) / \text{Ker } \beta} \Rightarrow \frac{\text{Ker}(\gamma \cdot \mu) / \text{Ker } \beta}{(\text{Ker } \mu + \text{Ker } \beta) / \text{Ker } \beta} \cong \frac{\text{Ker}(\gamma \cdot \mu)}{\text{Ker } \mu + \text{Ker } \beta}$$

Finally we get

$$\frac{\text{Im } \beta \cap \text{Im } \lambda'}{\text{Im}(\beta \cdot \lambda)} \cong \frac{\text{Ker}(\gamma \cdot \mu)}{\text{Ker } \mu + \text{Ker } \beta}$$

For simplicity, we sometimes denote the left hand side as Im 1, this means the image of square 1; and the right hand side as Ker 2, the kernel of square 2. The conclusion of the Lemma then becomes

$$\text{Im } 1 \cong \text{Ker } 2.$$

CHAPTER II

Ext^n

Ext^1 will be defined after proposition 1. In order to present Ext^n clearly and be more familiar with its development, we introduce proposition 2 and define Ext^2 at the end of its proof. Then Ext^n will be defined in general for each positive integer n , by showing there does exist an isomorphism from a particular module onto another particular module.

2.1 Ext^1 and Ext^2

Proposition 1 (Refer to (4)).

$$\text{Let } 0 \longrightarrow P'_1 \xrightarrow{\beta} P \xrightarrow{\alpha} A \longrightarrow 0$$

$$0 \longrightarrow B \xrightarrow{\beta'} I \xrightarrow{\alpha'} I'_1 \longrightarrow 0$$

be two exact sequences with P projective and I injective, then

$$\frac{B\emptyset P'_1}{\text{Im}(B\emptyset P \rightarrow B\emptyset P'_1)} \cong \frac{I'_1\emptyset A}{\text{Im}(I\emptyset A \rightarrow I'_1\emptyset A)}$$

i.e.
$$\frac{B\phi P_1}{\text{Im}(B\phi P \rightarrow B\phi P'_1)} \cong \frac{I_1'\phi A}{\text{Im}(I_1\phi A \rightarrow I_1'\phi A)}$$

Define the common value of this isomorphism to be the Extension product of degree 1 of A and B, and denoted by

$$\text{Ext}^1(A, B).$$

Note that, as usual, $\text{Ext}^1(A, B)$ is only defined up to isomorphism.

Proposition 2.

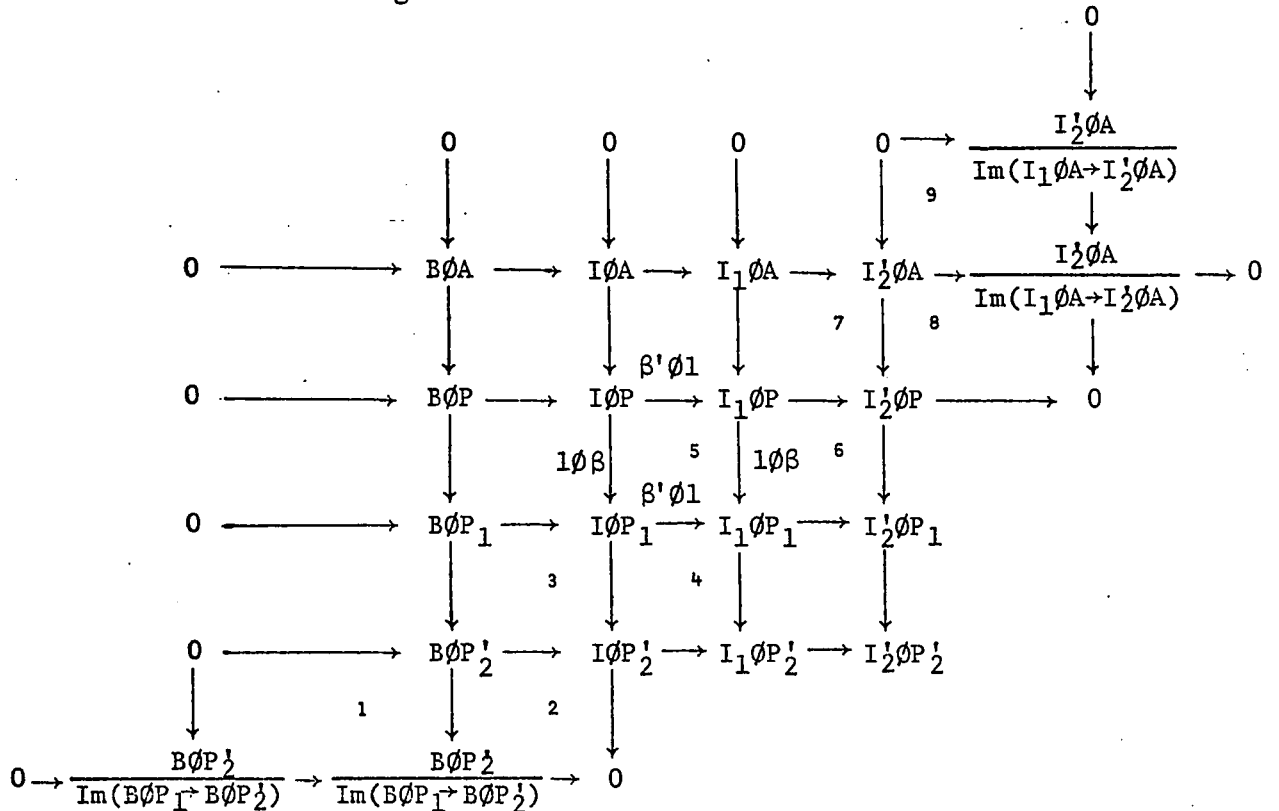
Let
$$0 \rightarrow P_2 \xrightarrow{\gamma} P_1 \xrightarrow{\beta} P \xrightarrow{\alpha} A \rightarrow 0$$

$$0 \rightarrow B \xrightarrow{\gamma'} I \xrightarrow{\beta'} I_1 \xrightarrow{\alpha'} I_2 \rightarrow 0$$

be two exact sequences of R-modules with P, P₁ projective and I, I₁ injective, then

$$\frac{B\phi P_2}{\text{Im}(B\phi P_1 \rightarrow B\phi P_2)} \cong \frac{I_2'\phi A}{\text{Im}(I_1\phi A \rightarrow I_2'\phi A)}$$

Proof: Consider the diagram



We observe that the third and the fourth rows in the diagram are exact, as well as the third and the fourth columns. This is because P and P_1 are projective and I and I_1 are injective. Then apply Lemma 5 of Chapter I.

It remains to show the exactness of some portions of sequences in this diagram which are necessary for our diagram chase. The fact that the following two sequences

$$0 \longrightarrow B\emptyset P'_2 \longrightarrow I\emptyset P'_2$$

$$0 \longrightarrow I'_2 \emptyset A \longrightarrow I'_2 \emptyset P$$

are exact follows from Lemma 1 of Chapter I. The exactness of the last row and the last column are obvious.

The commutativity of each square in the diagram is also obvious. For example, consider square 5. The compounded mappings from $I\emptyset P$ to $I_1 \emptyset P_1$ are

$$(1\emptyset\beta) \cdot (\beta' \emptyset 1) = \beta' \emptyset \beta$$

$$(\beta' \emptyset 1) \cdot (1\emptyset\beta) = \beta' \emptyset \beta$$

by Lemma 4, i.e.

$$(1\emptyset\beta) \cdot (\beta' \emptyset 1) = (\beta' \emptyset 1) \cdot (1\emptyset\beta).$$

The commutativity of all other squares can be argued in a similar way.

Thus we may use the square lemma in the following diagram chase:

$$\text{Im } 1 \cong \text{Ker } 2 \cong \text{Im } 3 \cong \text{Ker } 4 \cong \text{Im } 5 \cong \text{Ker } 6 \cong \text{Im } 7 \cong \text{Ker } 8 \cong \text{Im } 9$$

i.e.
$$\frac{B\phi P'_2}{\text{Im}(B\phi P'_1 \rightarrow B\phi P'_2)} \cong \frac{I_2\phi A}{\text{Im}(I_1\phi A \rightarrow I_2\phi A)}$$

Define the common value of the isomorphism to be the extension product of degree 2 of A and B, denoted by

$$\text{Ext}^2(A, B).$$

Remark: From the definitions of $\text{Ext}^1(A, B)$ and $\text{Ext}^2(A, B)$, one can see how $\text{Ext}^n(A, B)$ should be defined.

2.2 Ext^n

In the following we shall define Ext^n for each positive integer n. Given any two R-modules A and B, let

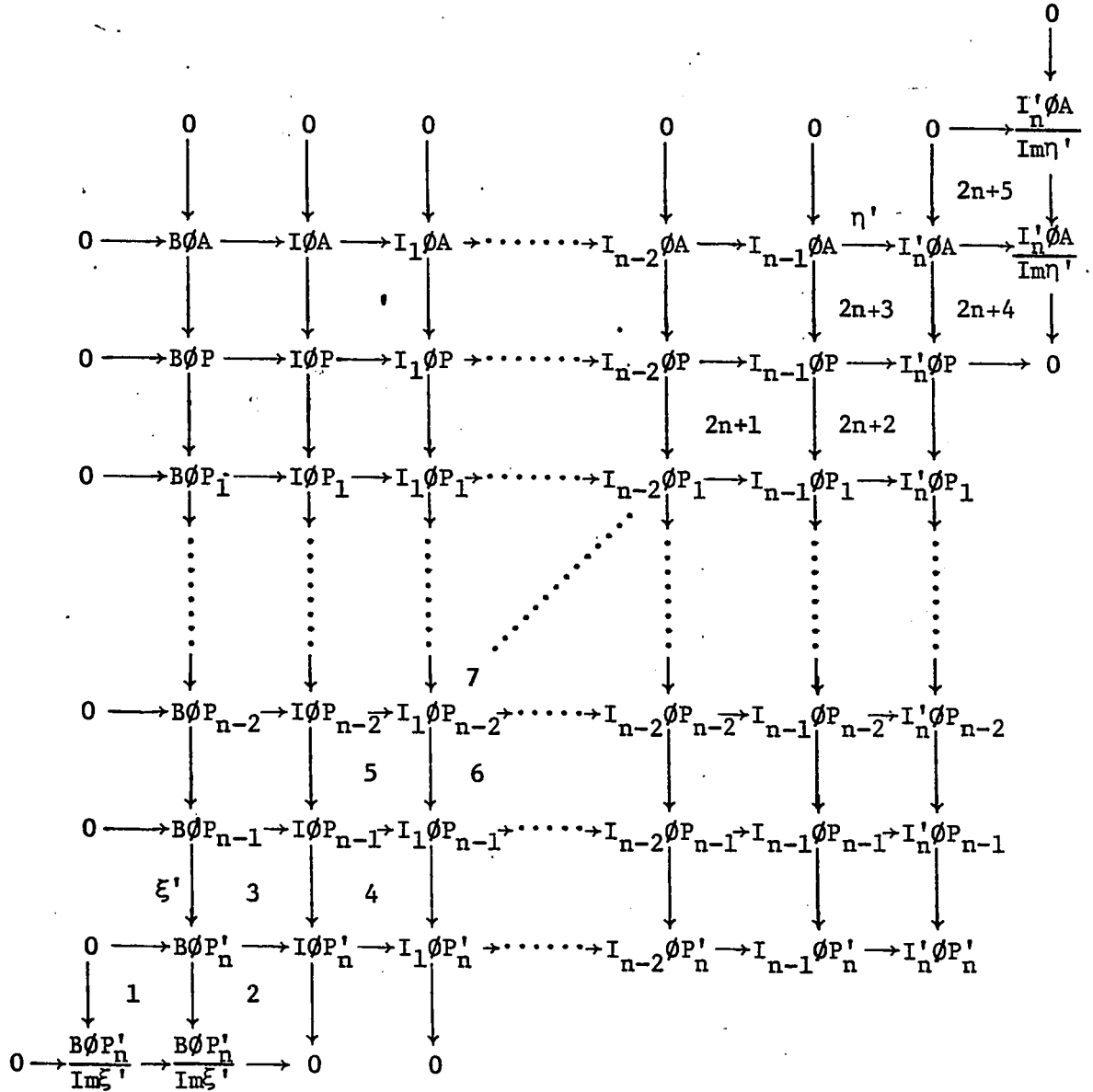
$$\begin{aligned} 0 \rightarrow P'_n \xrightarrow{\xi} P_{n-1} \xrightarrow{d_{n-1}} P_{n-2} \rightarrow \dots \rightarrow P_1 \xrightarrow{d_1} P \rightarrow A \rightarrow 0 \\ 0 \rightarrow B \rightarrow I \xrightarrow{c_1} I_1 \xrightarrow{c_2} \dots \rightarrow I_{n-2} \xrightarrow{c_{n-1}} I_{n-1} \xrightarrow{\eta} I'_n \rightarrow 0 \end{aligned}$$

be two exact sequences of R-modules with P, P_1, \dots, P_{n-1} projective and I, I_1, \dots, I_{n-1} injective, where

$$P'_n = \text{Ker } d_{n-1}$$

$$I'_n = I_{n-1} / \text{Im } c_{n-1}.$$

Consider the following diagram.



It is clear that all squares commute. In fact, a typical square is

$$\begin{array}{ccc}
 I_i \emptyset P_j & \xrightarrow{c_i \emptyset l_{P_j}} & I_{i+1} \emptyset P_j \\
 \downarrow l \emptyset d_{j+1} & & \downarrow l \emptyset d_{j+1} \\
 I_i \emptyset P_{j+1} & \xrightarrow{c_i \emptyset l_{P_{j+1}}} & I_{i+1} \emptyset P_{j+1}
 \end{array}$$

The compounded mappings from $I_i \emptyset P_j$ to $I_{i+1} \emptyset P_{j+1}$ are

$$(l_{I_{i+1}} \emptyset d_{j+1}) \cdot (c_i \emptyset l_{P_j}) = c_i \emptyset d_{j+1}$$

$$(c_i \emptyset l_{P_{j+1}}) \cdot (l_{I_i} \emptyset d_{j+1}) = c_i \emptyset d_{j+1}$$

The fact that equalities hold follows from Lemma 4, Chapter I. This shows the commutativity of the above square. Note that, as we mentioned in Propositions 1 and 2, square 1 and square $2n+5$ are trivially commutative.

Since each P, P_1, \dots, P_{n-1} is projective and each I, I_1, \dots, I_{n-1} is injective, all columns except the first and last columns are exact, and all rows except the top and bottom rows are exact. In fact, a typical row is

$$0 \rightarrow B \emptyset P_i \rightarrow I \emptyset P_i \rightarrow \dots \rightarrow I_n' \emptyset P_i \rightarrow 0 \quad i \neq n.$$

This is exact because $\emptyset P_i$ with P_i projective is an exact functor, hence preserves the long exact sequence by corollary to Lemma 5, Chapter I. Similar argument holds for the exactness of columns.

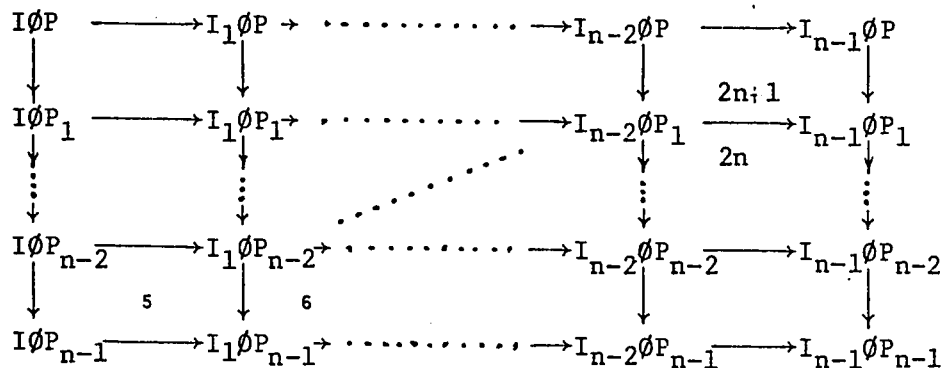
In order to reach the conclusion

$$\frac{I_n' \emptyset A}{\text{Im} \eta'} \cong \frac{B \emptyset P_n'}{\text{Im} \xi'}$$

we must find a suitable route for square chasing. Let us call each square in the big diagram a unit square. For each n , we define Ext^n starting from exact sequences of $n+1$ terms*, and construct a basic diagram consisting of $(n+1) \times (n+1)$ unit squares, then expand the diagram to $(n+3) \times (n+3)$ unit squares by putting two particular modules at the corner of this square, one on the bottom left corner and the other on the upper right corner. The route for square chasing is via those squares along the diagonal. To be precise: the route consists of all $n+3$ squares along the main diagonal and those $n+2$ squares situated immediately below the diagonal. Hence there are $2n+5$ squares involved. We arrange them in increasing order starting from left corner sq 1, sq 2,, sq $2n+4$, sq $2n+5$ in such a manner that all those odd number squares are on the main diagonal, and all even number squares are immediately below the diagonal. Our main purpose is to show

$$\text{Im } 1 \cong \text{Ker } 2 \cong \text{Im } 3 \cong \text{Ker } 4 \cong \dots \cong \text{Im } 2n+3 \cong \text{Ker } 2n+4 \cong \text{Im } 2n+5,$$

However, we are not entitled to claim this unless we make some more explanation. As mentioned before, the following diagram is the largest possible subdiagram of the big diagram that is both exact and commutative.



* An exact sequence $0 \rightarrow N_1 \rightarrow N_2 \rightarrow \dots \rightarrow N_n \rightarrow 0$ is called an exact sequence of n terms.

Therefore we don't have to worry whenever our chasing route is inside this diagram. We have

$$\text{Im } 5 \cong \dots \cong \text{Ker } 2n \cong \text{Im } 2n+1$$

The routine proof is thus omitted.

Lemmas 1, 2, 3 enable us to have the following isomorphisms

$$\text{Im } 1 \cong \text{Ker } 2 \cong \text{Im } 3 \cong \text{Ker } 4 \cong \text{Im } 5$$

$$\text{Im } 2n+1 \cong \text{Ker } 2n+2 \cong \text{Im } 2n+3 \cong \text{Ker } 2n+4 \cong \text{Im } 2n+5$$

Finally we have

$$\text{Im } 1 \cong \text{Im } 2n+5$$

i.e.

$$\frac{B\phi P'_n}{\text{Im}(B\phi P_{n-1} \rightarrow B\phi P'_n)} \cong \frac{I'_n \phi A}{\text{Im}(I_{n-1} \phi A \rightarrow I'_n \phi A)}$$

Define the common value of this isomorphism to be

$$\text{Ext}^n(A, B).$$

2.3 Equivalence of the two definitions of Ext^n

In this section, we shall prove that our symmetric definition is equivalent to the ordinary definition of Ext^n , which is described as follows.

Let A be a given R-module and let

$$\dots \rightarrow P_n \xrightarrow{d_n} \dots \rightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P \rightarrow A \rightarrow 0$$

be a projective resolution of A. Apply $B\phi$, and we get

$$0 \rightarrow B\phi P \xrightarrow{d'_1} B\phi P_1 \xrightarrow{d'_2} B\phi P_2 \rightarrow \dots \xrightarrow{d'_n} B\phi P_n \xrightarrow{d'_{n+1}} \dots$$

then define $\text{Ext}^n(A, B) = \text{Ker } d'_{n+1} / \text{Im } d'_n$ for $n \geq 1$.

Before proving the equivalence, we introduce another Lemma.

Lemma 7. Given a commutative square

$$\begin{array}{ccc}
 & \lambda & \\
 A & \longrightarrow & B \\
 \alpha \downarrow & & \downarrow \beta \\
 A' & \longrightarrow & B' \\
 & \lambda' &
 \end{array}$$

there exists a canonical homomorphism

$$\phi: \text{Ker}\alpha \rightarrow \text{Ker}\beta, \quad \phi': \text{Cok}\alpha \rightarrow \text{Cok}\beta$$

so that the following two additional squares commute

$$\begin{array}{ccccccc}
 0 & \rightarrow & \text{Ker}\alpha & \rightarrow & A & \rightarrow & A' & \rightarrow & \text{Cok}\alpha & \rightarrow & 0 \\
 & & \downarrow \phi & & \downarrow \lambda & & \downarrow \lambda' & & \downarrow \phi' & & \\
 0 & \rightarrow & \text{Ker}\beta & \rightarrow & B & \rightarrow & B' & \rightarrow & \text{Cok}\beta & \rightarrow & 0
 \end{array}$$

where $\text{Cok}\alpha = A' / \text{Im}\alpha$, $\text{Cok}\beta = B' / \text{Im}\beta$.

In particular if λ, λ' are zero (identity) homomorphisms, the induced homomorphisms ϕ, ϕ' are zero (identity) homomorphisms respectively. If moreover,

$$\begin{array}{ccc}
 & \mu & \\
 B & \longrightarrow & C \\
 \beta \downarrow & & \downarrow \gamma \\
 B' & \longrightarrow & C' \\
 & \mu' &
 \end{array}$$

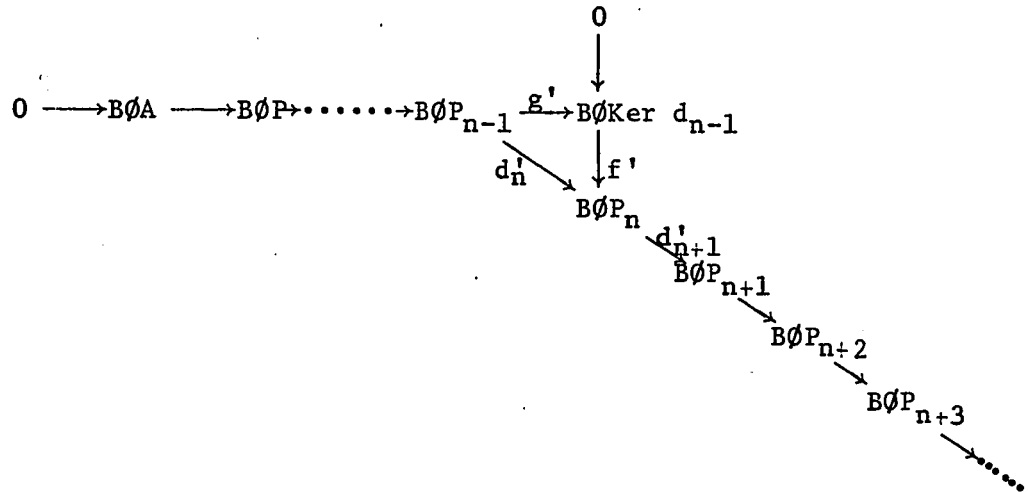
is another commutative square such that

$$\begin{array}{ccccccc}
 0 & \longrightarrow & A & \xrightarrow{\lambda} & B & \xrightarrow{\mu} & C & \longrightarrow & 0 \\
 0 & \longrightarrow & A' & \xrightarrow{\lambda'} & B' & \xrightarrow{\mu'} & C' & \longrightarrow & 0
 \end{array}$$

where g is the inclusion homomorphism and f is an epimorphism from P_n onto $\text{Ker } d_{n-1}$. Note that the triangle is commutative. We shall prove

$$\frac{B\phi \text{Ker } d_{n-1}}{\text{Im}(B\phi P_{n-1} \rightarrow B\phi \text{Ker } d_{n-1})} \cong \frac{\text{Ker}(B\phi P_n \rightarrow B\phi P_{n+1})}{\text{Im}(B\phi P_{n-1} \rightarrow B\phi P_n)}$$

where $\text{Ker } d_{n-1} = P'_n$ in the symmetric definition of Ext^n , and P'_n is not necessarily projective. By applying $B\phi$ to the above diagram we get



Observe that the triangle is commutative. In fact

$$g' = 1\phi g$$

$$f' = 1\phi f$$

$$f' \cdot g' = (1\phi g) \cdot (1\phi f) = 1\phi g \cdot f = 1\phi d_n = d'_n$$

by Lemma 4.

After applying $B\phi$, we have the sequence

$$0 \longrightarrow B\phi \text{Ker } d_{n-1} \xrightarrow{f'} B\phi P_n \xrightarrow{d'_{n+1}} B\phi P_{n+1}$$

exact.

Hence f is a monomorphism and $\text{Im } f' = \text{Ker } d'_{n+1}$. In other words we have

$$f': \quad B\emptyset\text{Ker } d_{n-1} \longrightarrow \text{Ker } d'_{n+1}$$

an isomorphism.

Now $\text{Im } d'_n = N$ is a submodule of $\text{Ker } d'_{n+1}$. Let $M = f'^{-1}(N) = f'^{-1}(\text{Im } d'_n)$. This is a submodule of $B\emptyset\text{Ker } d_{n-1}$. Therefore f' induces an isomorphism

$$\frac{B\emptyset\text{Ker } d_{n-1}}{M} \cong \frac{\text{Ker } d'_{n+1}}{N}$$

This is a consequence of the following remark:

Let R be a ring and A, B R -modules, and $f: A \rightarrow B$ an R -epimorphism. If $N \subseteq B$ is a submodule of B , let $M = f^{-1}(N)$, then f induces an R -isomorphism $A/M \cong B/N$. The proof of this fact is simple. (Refer to (6)).

Since f' is a monomorphism and the triangle in the above diagram is commutative, we have

$$\frac{B\emptyset\text{Ker } d_{n-1}}{\text{Im } g'} \cong \frac{\text{Ker } d'_{n+1}}{\text{Im } d'_n}$$

which is the same as

$$\frac{B\emptyset\text{Ker } d_{n-1}}{\text{Im}(B\emptyset P_{n-1} \rightarrow B\emptyset\text{Ker } d_{n-1})} \cong \frac{\text{Ker } d'_{n+1}}{\text{Im}(B\emptyset P_{n-1} \rightarrow B\emptyset P_n)}$$

and recall the Symmetric definition of Ext^n , we have

$$\text{Ker } d_{n-1} = P'_n$$

$$\text{Ext}^n(A, B) = \frac{\text{Ker } d'_{n+1}}{\text{Im } d'_n} \cong \frac{B \cap \text{Ker } d_{n-1}}{\text{Im } g'} = \frac{B \cap P'_n}{\text{Im}(B \cap P_{n-1} \rightarrow B \cap P'_n)}$$

Hence we reach the agreement of the two definitions.

2.4 Some remarks

Remark 1

Ext^n was defined, so far only for $n \geq 1$. However Ext^n , when $n=0$, still has its meaning. In general, we define Ext^n by starting from two exact sequences of $n+2$ terms*. Naturally, when $n=0$, we define Ext^0 by starting from two exact sequences of 2 terms. A precise description is as follows: Let A and B be any two given R -modules, and let

$$\begin{array}{ccccccc} 0 & \longrightarrow & P_0 & \longrightarrow & A & \longrightarrow & 0 \\ & & & & & & \\ 0 & \longrightarrow & B & \longrightarrow & I_0 & \longrightarrow & 0 \end{array}$$

be two exact sequences. But then $P_0 \cong A$, $B \cong I_0$. Consider the following diagram.

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & B \cap A & \longrightarrow & I_0 \cap A & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & B \cap P_0 & \longrightarrow & I_0 \cap P_0 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array}$$

* Refer to the foot note on P. 16.

Now define $\text{Ext}^0(A, B)$ to be the common value of the isomorphism

$$B \otimes P_0 \cong I_0 \otimes A$$

Actually we have $B \otimes A \cong B \otimes P_0 \cong I_0 \otimes A \cong B \otimes A$. Since $P_0 \cong A$, $B \cong I_0$, $\text{Ext}^0(A, B)$ can be denoted simply by $B \otimes A$.

Remark 2

It is worth noting that the ordinary definition of Ext^n can also be produced by using injective resolution. (Refer to P.55 (2)).

That is, if

$$0 \longrightarrow B \longrightarrow I \xrightarrow{c_1} I_1 \xrightarrow{c_2} I_2 \longrightarrow \dots \longrightarrow I_n \longrightarrow \dots$$

is an injective resolution of B , then Hom it into

$$0 \longrightarrow I \otimes A \xrightarrow{c'_1} I_1 \otimes A \xrightarrow{c'_2} \dots \longrightarrow I_n \otimes A \longrightarrow \dots$$

and define $\text{Ext}^n(A, B) = \frac{\text{Ker } c'_{n+1}}{\text{Im } c'_n}$

Argue similarly by considering injective modules in 2.3, we can reach the conclusion

$$\frac{I'_n \otimes A}{\text{Im}(I_{n-1} \otimes A \rightarrow I'_n \otimes A)} \cong \frac{\text{Ker}(I_n \otimes A \rightarrow I_{n+1} \otimes A)}{\text{Im}(I_{n-1} \otimes A \rightarrow I_n \otimes A)}$$

where $I'_n = I_{n-1} / \text{Im } c_{n-1}$ as mentioned in 2.2. This result is useful to us when we are trying to simplify some proofs of theorems in Chapter III of (2).

Remark 3

We have defined the common value of the following isomorphism to be $\text{Ext}^n(A,B)$,

$$\frac{B\phi P'_n}{\text{Im}(B\phi P_{n-1} \rightarrow B\phi P'_n)} \cong \frac{I'_n\phi A}{\text{Im}(I_{n-1}\phi A \rightarrow I'_n\phi A)}$$

Notice that whatever the resolution of B we take as in

$$0 \longrightarrow B \longrightarrow I_1 \longrightarrow I_1 \longrightarrow \dots \longrightarrow I'_n \longrightarrow 0$$

the left hand side expression remains the same. On the other hand, the different choice of resolution of A

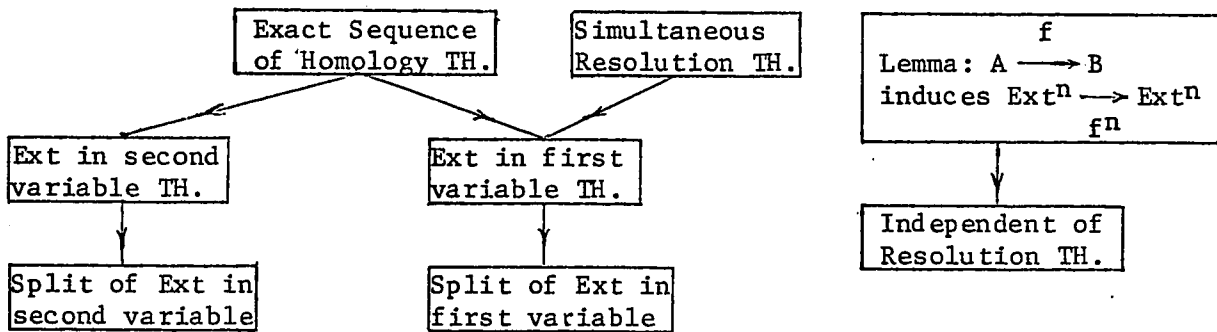
$$0 \longrightarrow P'_n \longrightarrow \dots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P \longrightarrow A \longrightarrow 0$$

does not affect the right hand side expression. Hence both sides can only depend on A and B, and not on the particular choice of resolution of A or B. This means $\text{Ext}^n(A,B)$ is independent of the resolutions used to define it.

CHAPTER III

SOME APPLICATIONS OF Ext^n .

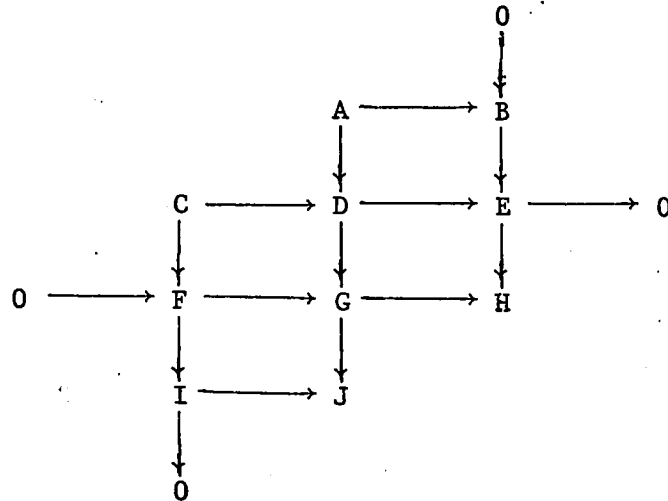
As an application of the Square Lemma and the Symmetric definition of Ext^n , we shall rewrite Chapter 3 in the book 'Rings and Homology' by Jans. Except for those basic properties of $\text{Hom}(,)$ and various terminologies we have mentioned before, the chapter can indeed be simplified to a considerable extent. The following diagram gives a general description of the material presented in chapter 3 of (3).



The main object of that chapter is to establish theorems on the Exact sequences of first (second) variable of Ext^n , which are extremely important to the rest of the book. In the following, we shall establish these two theorems directly and simultaneously by the Square Lemma and the Symmetric definition of Ext^n without using the Exact sequence of homology theorem and the Simultaneous resolution theorem. The proofs of these two theorems are quite long. We shall also give various different approaches to some other lemmas in this chapter by using the Symmetric definition of Ext^n .

First of all we need two lemmas which are also applicable to the proof of the Exact Sequence of Homology Theorem, which will be given in the appendix.

Lemma 8. Given a diagram

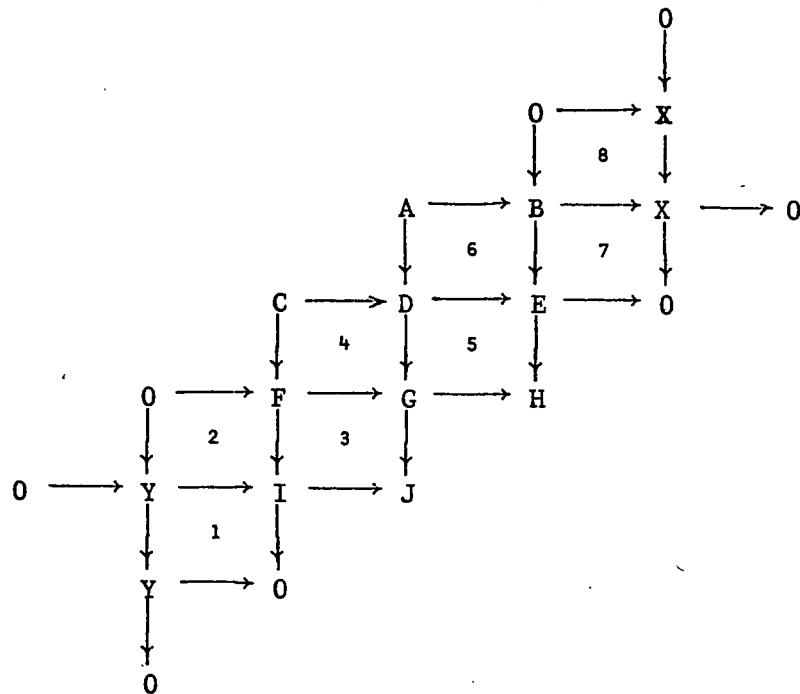


in which all rows and columns are exact and all squares commute. Then there exists a 'connecting homomorphism' $\theta: A \rightarrow B \rightarrow I \rightarrow J$ so that $A \rightarrow B \rightarrow I \rightarrow J$ is exact.

Proof: Let $X = \text{Cok}(A \rightarrow B)$, $Y = \text{Ker}(I \rightarrow J)$, we obtain exact sequences

$$A \rightarrow B \rightarrow X \rightarrow 0 \qquad 0 \rightarrow Y \rightarrow I \rightarrow J$$

Our purpose is to show $X \cong Y$. We enlarge the diagram to

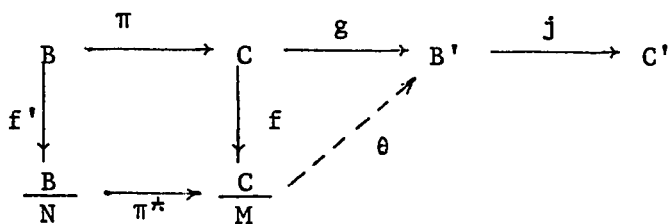


Note that this diagram is exact and commutative. By the Square Lemma

$$Y = \text{Ker} 1 \cong \text{Im } 2 \cong \text{Ker } 3 \cong \text{Im } 4 \cong \text{Ker } 5 \cong \\ \cong \text{Im } 6 \cong \text{Ker } 7 \cong \text{Im } 8 = X$$

This completes the proof of the lemma.

Lemma 9. Given a diagram



in which $B \rightarrow C \rightarrow B' \rightarrow C'$ is an exact sequence, π^* is the induced map by π so that $\pi^*f' = f\pi$, and N, M are any submodules of B, C respectively such that $M \supseteq \pi(N)$ and f, f' are the natural epimorphisms. Then there exists an induced homomorphism $\theta: C/M \rightarrow B'$ such that $B/N \rightarrow C/M \xrightarrow{\theta} B' \rightarrow C'$ is exact provided $M \subseteq \text{Im } \pi$

Proof: Since $\text{Ker } f = M \subseteq \text{Im } \pi = \text{Ker } g$, we have a well defined homomorphism $\theta: C/M \rightarrow B'$ which sent $\theta(\bar{c}) = g(c)$ where $\bar{c} = c + M$, $c \in C$ so that $\theta f(c) = \theta(\bar{c}) = g(c)$.

To show the exactness at B' , consider $\theta f(C) = g(C) = \text{Ker } j \Rightarrow \theta(C/M) = \text{Ker } j$, i.e. $\text{Im } \theta = \text{Ker } j$.

Next, we claim $\text{Ker } \theta = \text{Im } \pi^*$. Suppose we have $\text{Im } \pi^* \subsetneq \text{Ker } \theta$, and there exists $\bar{c} \in \text{Ker } \theta$, $\bar{c} \notin \text{Im } \pi^*$. Since f is epimorphism, $\theta f = g$, then $f^{-1}(\bar{c}) \subseteq \text{Ker } g = \text{Im } \pi \Rightarrow \exists \pi^{-1}(f^{-1}(\bar{c})) \subseteq B$. But notice that our square is commutative, hence $\pi^*f'(\pi^{-1}(f^{-1}(\bar{c}))) = f\pi(\pi^{-1}(f^{-1}(\bar{c}))) = f(f^{-1}(\bar{c})) = \bar{c} \in \text{Im } \pi^*$. This is a contradiction to our assumption. On the other hand it is clear that we have $\text{Im } \pi^* \subseteq \text{Ker } \theta$ since $\theta\pi^*(b+N) = \theta(\pi(b) + M) = g\pi(b) = 0$. This proves the exactness at C/M . Hence the lemma is proved.

Theorem 1. (The exact sequence of the first variable of Ext.)

If $0 \rightarrow A \xrightarrow{j} B \xrightarrow{\pi} D \rightarrow 0$ is exact, it induces the long exact sequence

$$\begin{aligned} \dots &\rightarrow \text{Ext}^{n-1}(A, C) \xrightarrow{\theta} \text{Ext}^n(D, C) \rightarrow \text{Ext}^n(B, C) \rightarrow \\ &\rightarrow \text{Ext}^n(A, C) \rightarrow \text{Ext}^{n+1}(D, C) \rightarrow \dots \end{aligned}$$

Proof: Let $0 \rightarrow C \rightarrow I \rightarrow I_1 \rightarrow \dots \rightarrow I_{n-1} \rightarrow I'_n \rightarrow 0$ be an exact sequence with I, I_1, \dots, I_{n-1} injective and $I'_n = I_{n-1}/\text{Im}(I_{n-2} \rightarrow I_{n-1})$ as in the definition of Ext^n . We claim that in the following diagrams all rows and columns are exact and all squares commute by considering Lemma 7 Chapter II, and the properties of ϕ as stated in Lemma 3, and Lemma 4 Chapter I.

$$\begin{array}{ccccc}
 & & 0 & & 0 \\
 & & \downarrow & & \downarrow \\
 & & \text{Ker}(I_{n-1}\phi_B \rightarrow I'_n\phi_B) & \xrightarrow{\pi^0} & \text{Ker}(I_{n-1}\phi_A \rightarrow I'_n\phi_A) \\
 & & \downarrow & & \downarrow \\
 0 & \longrightarrow & I_{n-1}\phi_D & \longrightarrow & I_{n-1}\phi_B & \xrightarrow{\pi} & I_{n-1}\phi_A \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & I'_n\phi_D & \longrightarrow & I'_n\phi_B & \longrightarrow & I'_n\phi_A \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & \frac{I'_n\phi_D}{\text{Im}(I_{n-1}\phi_D \rightarrow I'_n\phi_D)} & \longrightarrow & \frac{I'_n\phi_B}{\text{Im}(I_{n-1}\phi_B \rightarrow I'_n\phi_B)} & \longrightarrow & \frac{I'_n\phi_A}{\text{Im}(I_{n-1}\phi_A \rightarrow I'_n\phi_A)} \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

Hence by Lemma 8 and 9, we have following diagram exact and commutative

$$\begin{array}{ccccccc}
 \text{Ker}(I_{n-1}\phi_B \rightarrow I'_n\phi_B) & \xrightarrow{\pi^0} & \text{Ker}(I_{n-1}\phi_A \rightarrow I'_n\phi_A) & \xrightarrow{g} & \frac{I'_n\phi_D}{\text{Im}(I_{n-1}\phi_D \rightarrow I'_n\phi_D)} & \longrightarrow & \frac{I'_n\phi_B}{\text{Im}(I_{n-1}\phi_B \rightarrow I'_n\phi_B)} \\
 \downarrow f' & & \downarrow f & & \nearrow \theta & & \\
 \frac{\text{Ker}(I_{n-1}\phi_B \rightarrow I'_n\phi_B)}{\text{Im}(I_{n-2}\phi_B \rightarrow I_{n-1}\phi_B)} & \xrightarrow{\pi^*} & \frac{\text{Ker}(I_{n-1}\phi_A \rightarrow I'_n\phi_A)}{\text{Im}(I_{n-2}\phi_A \rightarrow I_{n-1}\phi_A)} & & & &
 \end{array}$$

where g is the connecting homomorphism of Lemma 8 and θ is induced by g . π^* is induced by π^0 , which in turn, is induced by π . All we need to verify is that the above diagram really satisfies the hypothesis of Lemma 9. Indeed it is true that

$$\pi(\text{Im } I_{n-2}\phi_B \rightarrow I_{n-1}\phi_B) = \text{Im}(I_{n-2}\phi_A \rightarrow I_{n-1}\phi_A).$$

Since we have the commutative squares,

$$\begin{array}{ccccc} \downarrow & & \downarrow & & \\ I_{n-2}\phi_B & \longrightarrow & I_{n-2}\phi_A & \longrightarrow & 0 \quad I_{n-2} \text{ is injective} \\ \downarrow & & \downarrow & & \\ I_{n-1}\phi_B & \longrightarrow & I_{n-1}\phi_A & & \\ \downarrow & & \downarrow & & \\ I'_n\phi_B & \longrightarrow & I'_n\phi_A & & \end{array}$$

of course then

$$\pi^0(\text{Im } I_{n-2}\phi_B \rightarrow I_{n-1}\phi_B) = \text{Im}(I_{n-2}\phi_A \rightarrow I_{n-1}\phi_A).$$

The condition $\text{Ker } f = \text{Im}(I_{n-2}\phi_A \rightarrow I_{n-1}\phi_A) \subseteq \text{Im } \pi^0$ also follows from this fact.

In order to reach our final goal, we claim once more, that

$$\text{Ker}(I_{n-1}\phi_A \longrightarrow I'_n\phi_A) = \text{Ker}(I_{n-1}\phi_A \rightarrow I_n\phi_A)$$

where I_n is injective. But this is clear if we consider the situation in the following diagrams.

$$\begin{array}{ccccccc} & & & 0 & & & \\ & & & \downarrow & & & \\ \dots & \rightarrow & I_{n-2} & \rightarrow & I_{n-1} & \rightarrow & I_{n-1}/\text{Im}(I_{n-2} \rightarrow I_{n-1}) \rightarrow 0 \\ & & & & \searrow & & \downarrow \\ & & & & & & I_n \rightarrow \dots \end{array}$$

$$\begin{array}{ccccccc} & & & 0 & & & \\ & & & \downarrow & & & \\ \dots & \rightarrow & I_{n-2}\phi_A & \rightarrow & I_{n-1}\phi_A & \rightarrow & I'_n\phi_A \\ & & & & \searrow & & \downarrow \\ & & & & & & I_n\phi_A \rightarrow \dots \end{array}$$

where $I_{n-1}/\text{Im}(I_{n-2} \rightarrow I_{n-1}) = I'_n$. Hence we have

$$\frac{\text{Ker}(I_{n-1}\phi_A \rightarrow I'_n\phi_A)}{\text{Im}(I_{n-2}\phi_A \rightarrow I_{n-1}\phi_A)} \cong \frac{\text{Ker}(I_{n-1}\phi_A \rightarrow I_n\phi_A)}{\text{Im}(I_{n-2}\phi_A \rightarrow I_{n-1}\phi_A)} \cong \frac{I'_{n-1}\phi_A}{\text{Im}(I_{n-2}\phi_A \rightarrow I'_{n-1}\phi_A)}$$

The isomorphism comes from 2.3 and 2.4 Remark 2, chapter II. Similarly,

$$\frac{\text{Ker}(I_{n-1}\phi_B \rightarrow I'_n\phi_B)}{\text{Im}(I_{n-2}\phi_B \rightarrow I_{n-1}\phi_B)} \cong \frac{\text{Ker}(I_{n-1}\phi_B \rightarrow I_n\phi_B)}{\text{Im}(I_{n-2}\phi_B \rightarrow I_{n-1}\phi_B)} \cong \frac{I'_{n-1}\phi_B}{\text{Im}(I_{n-2}\phi_B \rightarrow I'_{n-1}\phi_B)}$$

Therefore we have following sequence exact,

$$\begin{array}{ccccc} \frac{I'_{n-1}\phi_B}{\text{Im}(I_{n-2}\phi_B \rightarrow I'_{n-1}\phi_B)} & \xrightarrow{\pi^*} & \frac{I'_{n-1}\phi_A}{\text{Im}(I_{n-2}\phi_A \rightarrow I'_{n-1}\phi_A)} & \xrightarrow{\theta} & \frac{I'_n\phi_D}{\text{Im}(I_{n-1}\phi_D \rightarrow I'_n\phi_D)} \\ & & & & \downarrow j^* \\ & & & & \frac{I'_n\phi_B}{\text{Im}(I_{n-1}\phi_B \rightarrow I'_n\phi_B)} \end{array}$$

Now recall our Symmetric definition, we immediately recognize that the sequence

$$\text{Ext}^{n-1}(B, C) \xrightarrow{\pi^*} \text{Ext}^{n-1}(A, C) \xrightarrow{\theta} \text{Ext}^n(D, C) \xrightarrow{j^*} \text{Ext}^n(B, C)$$

is exact, for all $n \geq 1$. In other words, we get the long exact sequence

$$\begin{array}{ccccccc} \longrightarrow & \text{Ext}^{n-1}(A, C) & \xrightarrow{\theta} & \text{Ext}^n(D, C) & \longrightarrow & \text{Ext}^n(B, C) & \longrightarrow \\ & & & & & & \downarrow \theta \\ & & & & & & \text{Ext}^{n+1}(D, C) \end{array}$$

Remark : As a special case, consider $n = 0$. We remark that the long exact sequence of Ext^n starts from

$$0 \rightarrow C\phi D \rightarrow C\phi B \rightarrow C\phi A \xrightarrow{\theta} \text{Ext}^1(D, C) \rightarrow \dots$$

First of all recall the discussion of Ext^0 given in Remark 1, 2.4 chapter II that

$$\text{Ext}^0(D, C) = C\phi D, \quad \text{Ext}^0(B, C) = C\phi B, \quad \text{Ext}^0(A, C) = C\phi A$$

Consider the following diagram,

$$\begin{array}{ccccccc}
 & & & & 0 & & 0 \\
 & & & & \downarrow & & \downarrow \\
 & & & & C\phi B & \longrightarrow & C\phi A \\
 & & & & \downarrow & & \downarrow \\
 & & I\phi D & \longrightarrow & I\phi B & \longrightarrow & I\phi A \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & I_1^1\phi D & \longrightarrow & I_1^1\phi B & \longrightarrow & I_1^1\phi A \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & \frac{I_1^1\phi D}{\text{Im}(I\phi D \rightarrow I_1^1\phi D)} & \longrightarrow & \frac{I_1^1\phi B}{\text{Im}(I\phi B \rightarrow I_1^1\phi B)} & &
 \end{array}$$

where $\frac{I_1^1\phi D}{\text{Im}(I\phi D \rightarrow I_1^1\phi D)} = \text{Ext}^1(D, C), \quad \frac{I_1^1\phi B}{\text{Im}(I\phi B \rightarrow I_1^1\phi B)} = \text{Ext}^1(B, C).$

The commutativity and exactness of the diagram is obvious (Look at the diagram of Ext^1). By Lemma 8, there exists a connecting homomorphism θ such that the following sequence exact,

$$0 \rightarrow C\phi D \rightarrow C\phi B \rightarrow C\phi A \xrightarrow{\theta} \text{Ext}^1(D, C) \rightarrow \text{Ext}^1(B, C) \rightarrow \dots$$

Note that, we get the result without using Lemma 9 at all !

Theorem 2 (Exact sequence of the second variable of Ext)

If $0 \rightarrow C \rightarrow E \rightarrow F \rightarrow 0$ is exact, it induces the following

long exact sequence,

$$\begin{aligned} \dots \rightarrow \text{Ext}^{n-1}(A, F) \xrightarrow{\theta} \text{Ext}^n(A, C) \rightarrow \text{Ext}^n(A, E) \rightarrow \\ \rightarrow \text{Ext}^n(A, F) \xrightarrow{\theta} \text{Ext}^{n+1}(A, C) \rightarrow \dots \end{aligned}$$

Proof: Since our definition of Ext^n is symmetric in injective and projective modules, the proof of this theorem is done in exactly the same way as the proof of the exact sequence in the first variable of Ext by starting from the projective resolution of A

$$0 \rightarrow P'_n \rightarrow P_{n-1} \rightarrow \dots \rightarrow P_1 \rightarrow P \rightarrow A \rightarrow 0$$

where $P'_n = \text{Ker}(P_{n-1} \rightarrow P_{n-2})$, and is not necessarily a projective module.

Before we proceed to another important theorem of this chapter, Splitting of Ext , we shall prove another lemma.

Lemma 10. If $f: A \rightarrow B$ is an R -homomorphism, then there are induced maps $f^n: \text{Ext}^n(B, C) \rightarrow \text{Ext}^n(A, C)$ for every C and $n \geq 0$.

Proof: Consider the following commutative diagram,

$$\begin{array}{ccc} & f & \\ A & \longrightarrow & B \\ \downarrow & & \downarrow \\ & f & \\ A & \longrightarrow & B \end{array}$$

By applying θ we have, in the following diagram square 1 is commutative, where $0 \rightarrow C \rightarrow I \rightarrow I_1 \rightarrow \dots \rightarrow I_{n-1} \rightarrow I'_n$ is as in the definition of Ext^n . The commutativity of square 2 follows from Lemma 7.

$$\begin{array}{ccc}
 I_{n-1} \phi_B & \longrightarrow & I_{n-1} \phi_A \\
 \downarrow & & \downarrow \\
 & 1 & \\
 I'_n \phi_B & \longrightarrow & I'_n \phi_A \\
 \downarrow & & \downarrow \\
 & 2 & \\
 \frac{I'_n \phi_B}{\text{Im}(I_{n-1} \phi_B \rightarrow I'_n \phi_B)} & \xrightarrow{f^n} & \frac{I'_n \phi_A}{\text{Im}(I_{n-1} \phi_A \rightarrow I'_n \phi_A)}
 \end{array}$$

where $\frac{I'_n \phi_B}{\text{Im}(I_{n-1} \phi_B \rightarrow I'_n \phi_B)} = \text{Ext}^n(B, C)$, $\frac{I'_n \phi_A}{\text{Im}(I_{n-1} \phi_A \rightarrow I'_n \phi_A)} = \text{Ext}^n(A, C)$.

Hence, for each n , f , and any R -module C , there is a mapping $f^n: \text{Ext}^n(B, C) \rightarrow \text{Ext}^n(A, C)$. Note that if f is a zero map, the induced f^n is a zero map, and if f is an identity map on A , f^n is an automorphism on $\text{Ext}^n(A, C)$ by Lemma 7. Also the induced map f^n is independent of resolutions used to define Ext^n , since by Remark 3,2.4, chapter II, the Symmetric definition is independent of resolutions.

We state the corresponding situation in form of another lemma.

Lemma 11. If $g: C \rightarrow D$ is an R -homomorphism it induces $g^n: \text{Ext}^n(A, C) \rightarrow \text{Ext}^n(A, D)$ for each $n \geq 0$ and any R -module A .

Proof: Similar to Lemma 10, only the modules I_{n-1} , I'_n should be replaced by P_{n-1} , P'_n in the proof.

Theorem 3. (Splitting of Ext)

If $0 \rightarrow A \xrightarrow{j} B \xrightarrow{\pi} D \rightarrow 0$ is exact and splits, then for each n and C

$$0 \rightarrow \text{Ext}^n(D, C) \xrightarrow{\pi^n} \text{Ext}^n(B, C) \xrightarrow{j^n} \text{Ext}^n(A, C) \rightarrow 0$$

is exact and splits.

Proof:

Suppose that ρ gives the splitting, that is $\pi\rho$ is the identity on D . Then we can construct k so that kj is the identity on A . But then $j^n k^n$ is identity on $\text{Ext}^n(A, C)$ and $\rho^n \pi^n$ is identity on $\text{Ext}^n(D, C)$. These facts, together with the exact sequence of Ext in the first variable, give the desired exactness and splitting in the conclusion of the theorem.

We state the corresponding theorem in the second variable.

Theorem 4 If $0 \rightarrow C \xrightarrow{j} E \xrightarrow{\pi} F \rightarrow 0$ is exact and split, then

$$0 \rightarrow \text{Ext}^n(A, C) \xrightarrow{j^n} \text{Ext}^n(A, E) \xrightarrow{\pi^n} \text{Ext}^n(A, F) \rightarrow 0$$

is also split exact.

Proof:

The proof is similar to the above.

Next we prove a Lemma which enables us to compute Ext^n for certain special modules.

Lemma 12

$\text{Ext}^n(A, C) = 0$ for all C and $n \geq 1$ if and only if A is projective.

Proof:

If A is projective, consider the diagram of Ext^n :

$$\begin{array}{ccccccc}
 & & & & & & 0 \\
 & & & & & & \downarrow \\
 & & & & & & I'_n \phi A \\
 & & & & & \xrightarrow{\text{Im}(I_{n-1} \phi A \rightarrow I'_n \phi A)} & \\
 0 & \xrightarrow{B \phi A} & I \phi A & \xrightarrow{\dots} & I_{n-1} \phi A & \xrightarrow{I'_n \phi A} & I'_n \phi A \xrightarrow{\text{Im}(I_{n-1} \phi A \rightarrow I'_n \phi A)} 0 \\
 & \downarrow & \downarrow & & \downarrow & \downarrow & \downarrow \\
 0 & \xrightarrow{B \phi P} & I \phi P & \xrightarrow{\dots} & I_{n-1} \phi P & \xrightarrow{I'_n \phi P} & I'_n \phi P \xrightarrow{\dots} 0 \\
 & \vdots & \vdots & & \vdots & \vdots & \vdots
 \end{array}$$

By Lemma 2, $\text{Im}(I_{n-1} \phi A \rightarrow I'_n \phi A) = I'_n \phi A$. Hence

$$\text{Ext}^n(A, C) = \frac{I'_n \phi A}{\text{Im}(I_{n-1} \phi A \rightarrow I'_n \phi A)} = 0, \text{ for all } C \text{ and } n \geq 1.$$

Conversely, if $\text{Ext}^1(A, C) = 0$ for all C and if

$$0 \longleftarrow C \xrightarrow{\pi} E \longrightarrow F \longrightarrow 0$$

is exact, then by the exact sequence of Ext^n in the second variable we have that

$$0 \longrightarrow \text{Hom}(A, C) \longrightarrow \text{Hom}(A, E) \xrightarrow{\pi''} \text{Hom}(A, F) \xrightarrow{\theta} \text{Ext}^1(A, C)$$

is exact. But this says that π'' is an epimorphism, which, since it is true for all $E \xrightarrow{\pi} F \longrightarrow 0$, is just another way of saying that A is projective.

We state the corresponding property for the injective module without proof.

Lemma 13

$$\text{Ext}^n(A, C) = 0 \text{ for all } A \text{ and } n \geq 1 \text{ if and only if } C \text{ is injective.}$$

APPENDIX

Exact Sequence of Homology Theorem

If $0 \rightarrow A \xrightarrow{j} B \xrightarrow{\pi} C \rightarrow 0$ is an exact sequence of complexes, it induces the following long exact sequence of homology,

$$\dots \rightarrow H_{n+1}(C) \xrightarrow{\theta} H_n(A) \xrightarrow{j^*} H_n(B) \xrightarrow{\pi^*} H_n(C) \rightarrow \dots$$

Proof:

For the definition of a complex and maps between complex refer to Chapter 3 in "Rings and Homology".

By definition of complex mappings j, π , we have following commutative diagram

$$\begin{array}{ccccccc}
 & 0 & & 0 & & 0 & & 0 & & \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 \dots & \rightarrow A_{n+2} & \longrightarrow & A_{n+1} & \xrightarrow{d_{n+1}^A} & A_n & \xrightarrow{d_n^A} & A_{n-1} & \rightarrow & \dots \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 \dots & \rightarrow B_{n+2} & \longrightarrow & B_{n+1} & \xrightarrow{d_{n+1}^B} & B_n & \xrightarrow{d_n^B} & B_{n-1} & \rightarrow & \dots \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 \dots & \rightarrow C_{n+2} & \longrightarrow & C_{n+1} & \xrightarrow{d_{n+1}^C} & C_n & \xrightarrow{d_n^C} & C_{n-1} & \rightarrow & \dots \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & 0 & & 0 & & 0 & & 0 & &
 \end{array}$$

Now we claim in the following diagram all rows and columns are exact and all squares commute.

$$\begin{array}{ccccccc}
 & & & 0 & & 0 & \\
 & & & \downarrow & & \downarrow & \\
 & & & \text{Ker } d_n^B & \xrightarrow{\pi^0} & \text{Ker } d_n^C & \\
 & & & \downarrow & & \downarrow & \\
 (*) & 0 & \longrightarrow & A_n & \longrightarrow & B_n & \longrightarrow C_n \longrightarrow 0 \\
 & & & \downarrow & & \downarrow & \\
 & 0 & \longrightarrow & \text{Ker } d_{n-1}^A & \longrightarrow & \text{Ker } d_{n-1}^B & \longrightarrow \text{Ker } d_{n-1}^C \\
 & & & \downarrow & & \downarrow & \\
 & & & \text{Ker } d_{n-1}^A & & \text{Ker } d_{n-1}^B & & \text{Ker } d_{n-1}^C \\
 & & & \downarrow & & \downarrow & & \downarrow \\
 & & & \text{Im } d_n^A & \longrightarrow & \text{Im } d_n^B & \longrightarrow & \text{Im } d_n^C \\
 & & & \downarrow & & \downarrow & & \downarrow \\
 & & & 0 & & 0 & & 0
 \end{array}$$

Since we have following diagram commutative and exact

$$\begin{array}{ccccccc}
 & & & \downarrow & & \downarrow & & \downarrow & \\
 0 & \longrightarrow & A_{n-1} & \longrightarrow & B_{n-1} & \longrightarrow & C_{n-1} & \longrightarrow 0 \\
 & & \downarrow d_{n-1}^A & & \downarrow d_{n-1}^B & & \downarrow d_{n-1}^C & \\
 0 & \longrightarrow & A_{n-2} & \longrightarrow & B_{n-2} & \longrightarrow & C_{n-2} & \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow &
 \end{array}$$

and $\text{Im } d_n^A \subseteq \text{Ker } d_{n-1}^A$, $\text{Im } d_n^B \subseteq \text{Ker } d_{n-1}^B$, $\text{Im } d_n^C \subseteq \text{Ker } d_{n-1}^C$, it follows from Lemma 7, Section 2.3 (Chapter II), the third row in the diagram (*) is exact, and every thing is exact and commutes. Whence the diagram (*) satisfies the condition stated in the Lemma 8, Chapter 3, so there exists a connecting homomorphism $g: \text{Ker } d_n^C \rightarrow \text{Ker } d_{n-1}^A / \text{Im } d_n^A$ such that the horizontal sequence in the following diagram is exact.

$$\begin{array}{ccccccc}
 \text{Ker } d_n^B & \xrightarrow{\pi^0} & \text{Ker } d_n^C & \xrightarrow{g} & \text{Ker } d_{n-1}^A & \longrightarrow & \text{Ker } d_{n-1}^B \\
 f' \downarrow & & f \downarrow & & \nearrow \theta & & \downarrow \\
 \text{Ker } d_n^B & \xrightarrow{\pi^*} & \text{Ker } d_n^C & & \text{Im } d_n^A & & \text{Im } d_n^B \\
 \downarrow & & \downarrow & & & & \downarrow \\
 \text{Im } d_{n+1}^B & & \text{Im } d_{n+1}^C & & & & 0 \\
 \downarrow & & \downarrow & & & & \\
 0 & & 0 & & & &
 \end{array}$$

The commutativity of the square in the above diagram is clear if we notice the fact $\pi(\text{Im } d_{n+1}^B) = \text{Im } d_{n+1}^C$ ($\Rightarrow \pi^0(\text{Im } d_{n+1}^B) = \text{Im } d_{n+1}^C$) in the big diagram at the beginning, where π induces π^0 which in turn induces π^* .

f, f' are the natural epimorphisms. Now the commutativity of the big diagram also enables us to claim that $\text{Ker } f = \text{Im } d_{n+1}^C \subseteq \text{Im } \pi^0 = \text{Ker } g$. Hence the above diagram satisfies all the hypotheses of the Lemma 9, Chapter III. Therefore, there exists a homomorphism θ , such that the following sequence is exact for all $n \geq 2$.

$$\begin{array}{ccccccc} \text{Ker } d_n^B & \xrightarrow{\pi^*} & \text{Ker } d_n^C & \xrightarrow{\theta} & \text{Ker } d_{n-1}^A & \xrightarrow{j^*} & \text{Ker } d_{n-1}^B \\ \text{Im } d_{n-1}^B & & \text{Im } d_{n+1}^C & & \text{Im } d_n^A & & \text{Im } d_n^B \end{array}$$

i.e., we get a long exact sequence of homology.

$$\dots \xrightarrow{\theta} H_n(A) \xrightarrow{j^*} H_n(B) \xrightarrow{\pi^*} H_n(C) \xrightarrow{\theta} H_{n-1}(A) \xrightarrow{\theta} H_{n-1}(B) \rightarrow \dots$$

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