Monoidal Topology on Linear Bicategories

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Abstract

Extending endofunctors on the category of sets and functions to the category of sets and relations requires one to introduce a certain amount of laxness. This in turn requires us to consider bicategories rather than ordinary categories. The subject of lax extensions of Set-based functors is one of the fundamental components of monoidal topology, an active area of research in categorical algebra.

The recent theory of linear bicategories, due to Cockett, Koslowski and Seely, is an extension of the usual notion of bicategory to include a second composition in a way analogous to the two connectives of multiplicative linear logic. It turns out that the category of sets and relations has a second composition making it a linear bicategory. The goal of this thesis is first to define the notion of lax extension of Set-based functors to linear bicategories, and then demonstrate crucial properties of our definition.

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Introduction

Category theory is more than a summation of mathematical objects and their morphisms. The routine of reaxiomatization of mathematical objects into categories (groups, rings, vector spaces, and on and on) may give one the mistaken impression of category theory as a second order tool of mathematical abstraction. However, as deep category theoretic investigation into mathematics and logic over prior 5 decades has shown, those objects which arise from fundamental aspects of reality such as distance, order, convergence, and probability are far from arbitrary, and are in fact categorical in nature. Indeed, monoidal topology stems from the thesis that fundamental structures are themselves categories. Early discoveries by monoidal topologists have focused on "triples", relational algebras, and enriched category theory to study these structures. Modern studies in monoidal topology have generalized all of the above as categories of lax algebras with respect to a lax extension of a fixed monad, while incorporating quantales, categories of generalized relations, bicategories and their maps, as well as higher enriched-categories into the interests of the subject.

Recent research in [9] supports the connection between monoidal topology and linear bicategories, with the key result that $\mathcal{Q}$-**Rel** is a linear bicategory when $\mathcal{Q}$ is a quantale possessing linearly distributive categorical structure. This opens many new avenues of inquiry into connections between monoidal topology and linear bicategories. A particularly critical component of the monoidal topological framework, outlined in [25], which has not yet been studied in the context of linear bicategories is the lax extension of **Set** functors and monads to $\mathcal{Q}$-**Rel**. In this thesis, we will develop an analogous linear extension of **Set** functors and monads to $\mathcal{Q}$-**Rel** when $\mathcal{Q}$-**Rel** is a linear bicategory. We will also prove certain conditions under which an appropriate linear bicategorical notion of the Barr extension is a linear extension of monads, and prove that the linear Barr extension of some familiar **Set** monads, such as the ultrafilter monad, are linear extension of monads in **Rel**.

It is well known that **Rel** is a locally posetal bicategory [5, §2.7]. Bicategories, as first defined in [5] are the notion of a category equipped with additional second order morphisms, known as 2-cells or morphisms between morphisms. These 2-cells have an associative unital composition, known as the "vertical" composition. The composition of 1-cells acts as a bifunctor on the second order morphisms, allowing for a second "horizontal" composition. **Rel** is an example of a locally ordered bicategory with a simple notion of equality between morphisms, but more generally one may abstract the notion of equality of a pair of morphisms as an isomorphism of 2-cells between them. This allows for more interesting structures between objects, such as in proof theory for example, where one may prefer that various proofs of a single implication be related up to a rewriting process rather than sheer equality. As an example, viewing a category **Set** as a locally discrete bicategory [5, §2.7], the only relationship we may express between morphisms is identity. However, in a locally posetal bicategory such as **Rel** one may also express inequality between morphisms, and thus making possible the lax algebraic methods of monoidal topology.

Lax extensions of **Set** endofunctors are a core component to growing areas of research in mathematics and theoretical computer science. In monoidal topology, lax extensions play a key role in the generalization of the Eilenberg-Moore category of a **Set** monad $\mathbb{T}$ to a lax algebraic version with respect to the category $\mathcal{Q}$-**Rel** of sets and $\mathcal{Q}$ valued relations where $\mathcal{Q}$ is a quantale. In coalgebraic logic, a theory studied in [36], the extension of F-coalgebras in the category **Set** to lax coalgebras in $\mathcal{Q}$-**Rel** allows for the generalization of bisimulation to a setting of multi-valued logic (or fuzzy logic), as studied in [35] which suggests metric F-coalgebras and a notion of metric bisimulation. We will mostly focus on examples and connections relating linear bicategories and monoidal topology, though it is hoped that research on lax extensions could have consequences of interest outside of this scope.

Monoidal topology is a research area providing a unifying framework for order, metric, and topology, wherein monoids play a recurring central role as a fundamental building block. Its methods are lax algebraic, introducing inequalities in place of usual equalities, and categorical. The theory of monoidal topology has three principal roots. The earliest root comes from Manes' research in 1969 [32], showing the result that the Eilenberg-Moore category $\mathbf{Set}^{\mathbb{\beta}}$ of the ultrafilter monad $\mathbb{\beta}$ is equivalent to the category **CompHaus** of compact Hausdorff spaces and continuous maps. In 1970, Barr extends this result in [2]. In this paper, Barr shows how one may extend a **Set** endofunctor T satisfying the Beck-Chevalley condition to a **Rel** endofunctor $\overline{T}$. Furthermore, by relaxing the Eilenberg-Moore requirements, one may derive a lax Eilenberg-Moore category of relational algebras with respect to the extension $\overline{\mathbb{T}}$ of a **Set** monad $\mathbb{T}$. Using this result Barr presents **Top**, the category of all topological spaces and continuous maps, as a lax Eilenberg-Moore category of the ultrafilter monad. The third root of monoidal topology comes from Lawvere in [29], who expresses metric spaces using enriched category theory in 1973. An enriched category $\mathcal{C}$ over a monoidal category $\mathcal{V}$, as defined in [25, I§4.10], is a collection $\mathsf{ob}(\mathcal{C})$ of objects together with an *enrichment* $\mathcal{C}(-,-)$ assigning to each pair of objects A, B in $\mathsf{ob}(\mathcal{C})$ an object $\mathcal{C}(A,B)$ in $\mathsf{ob}(\mathcal{V})$ such that this assignment satisfies associativity and unitality axioms. A **Set**-category is the usual notion of category, while a **Cat**-category corresponds to the notion of bicategory ([25, I§4.10], referred to as 2-category). One also has a category $\mathcal{V}$-**Cat** comprised of $\mathcal{V}$-categories and $\mathcal{V}$-functors which one may read about in [26] for a symmetric $\mathcal{V}$. Lawvere showed that when $\mathcal{V}$ is equal to the symmetric quantale $P_+ = ([0,\infty]^{\mathrm{op}}, +)$ that P_+-**Cat** corresponds to a notion of generalized metric space, and furthermore that **2-Cat** is equivalent to the category **Ord** of ordered sets and monotone maps.

In subsequent decades, these historical roots have been synthesized into a common framework combining categorical enrichment over a quantale, and the category of lax Eilenberg-Moore algebras with respect to a monad $\mathbb{T}$, making a parameterized lax Eilenberg-Moore category $(\mathbb{T}, \mathcal{Q})$-**Cat** with respect to a quantale $\mathcal{Q}$ and a lax extension $\hat{\mathbb{T}}$ of the **Set** monad $\mathbb{T}$ to $\mathcal{Q}$-**Rel**. This is called the category of lax algebras with respect to $\hat{\mathbb{T}}$. Within this framework, we have the following categorical equivalences, demonstrating the generalization of order, metric, and topology within monoidal topology.

$$\mathbf{Ord} \cong (\mathbb{I}, 2)\text{-}\mathbf{Cat}, \quad \mathbf{Met} \cong (\mathbb{I}, P_{+})\text{-}\mathbf{Cat}, \quad \text{and } \mathbf{Top} \cong (\beta, 2)\text{-}\mathbf{Cat}.$$

Linear bicategories form a generalization of bicategories and linearly distributive categories, and they are exemplified by the dual tensor par composition structure of **Rel**. Originating from Cockett, Koslowski, Seely in [14], linear bicategories find their roots in proof theory and the description of non-commutative linear logic, a non-commutative version of the resource-sensitive logic introduced by Girard in [23]. Earlier research towards non-commutative linear logic such as the cyclic linear sequent calculus in [40, §2.5] replace the structural rule of exchange with a rule of *cyclic exchange*, reordering of formulas by cyclic permutation, however this still does not provide enough noncommutativity for $\otimes, \oplus$ to represent the time sensitive nature of resource consumption. This motivates use of a tensor like composition, since composition is by nature non-commutative. Linear bicategories form a generalization of bicategories, and generalize linearly distributive categories [16] analogous to the way bicategories generalize monoidal categories [14, p. 3]. They have a bicategorical $\otimes$ ("tensor") operation, and an additional bicategorical operation $\oplus$ ("cotensor" or "par"), which together satisfy coherence conditions and *linear distributivity.* This differs from other contexts for linear logic such as $*$-autonomous categories, in which the par is not primitive, but is defined post hoc of linear implication and linear negation $(\text{-})^{\perp}$ via the identity $A^{\perp}$. In a linear bicategory, linear negation is not assumed and may not be present, however there is an elegant categorical notion of linear adjoints which results in a generalized description of negation. An interesting set of examples of linear bicategories comes from [9], most notable for our purpose is the category $\mathcal{Q}$-**Rel** of $\mathcal{Q}$-valued relations, where $\mathcal{Q}$ is a linearly distributive quantale.

The use of monoids as well as the lax algebraic approach of monoidal topology establishes its relationship to monoidal category theory. Examples of monoids within algebra and topology are abundant, some examples (see [31, VII§3]) of monoids in algebra include rings, which are the monoids in the category **AbGrp** of abelian groups and group homomorphisms, and the associative unital K-algebras (defined in [28, p. 121]) where K is a commutative ring, which are the monoids in the category K-**Mod** of K-modules and K-linear maps. Furthermore, monoids are abundant in monoidal topology, for example unital quantales are the monoids in the category **Sup** of complete lattices and join preserving maps; the preorders on a set X are the monoids in the hom-set $\mathbf{Rel}(X,X)$, and $\mathcal{Q}$-categories (X,a) are the monoids in $\mathcal{Q}$-$\mathbf{Rel}(X,X)$. Linear bicategory theory may also be viewed as a relative of monoidal category theory. Many core mathematical objects in monoidal topology, such as lax functors, lax natural transformations, and monoidal categories, have linear bicategorical analogues: linear functors, linear natural transformations, and linearly distributive categories respectively. **Rel** is one of the motivating examples for linear bicategories [14, p. 6]. The $\otimes$ composition is regular relational composition and the cotensor composition is the dual relational composition, known since the time of Peirce and discussed in [38], that is for relations $R \subseteq X \times Y$, $S \subseteq Y \times Z$,

$$R \oplus S = \{(x,z) \in X \times Z \,:\, \forall y \in Y.\ xRy \text{ or } ySz\}.$$

As studied in [9], a quantale $\mathcal{Q}$ with a pair of monoidal operations $\otimes$, $\oplus$ making it into a linearly distributive category allows us to define a linear bicategorical structure on $\mathcal{Q}$-**Rel**. Thus the connection between monoidal topology and linear bicategories is made explicit.

Contents of this thesis

Chapters 1 and 2 provide the appropriate preliminary material on monoidal topology and lax extension of functors. These chapters, as well as this introduction, are heavily influenced by the contents of *Monoidal Topology* [25], by W. Tholen, D. Hoffman, and G. Seal. Unless otherwise made explicit, all statements come from here, and the reader is directed to this book in case of any statement with a missing reference. Chapter 3 provides the preliminary material for linear bicategories, most of it is from the 2 part series on linear bicategories [14, 17] by R. Cockett, J. Koslowski, R. Seely, except for the generalization of the transit map of example 3.1.2.VIII which was suggested by my advisor Richard Blute. Chapters 4 to 6 contain my original research on linear extensions of monads, with citations to others as appropriate. Chapter 4 contains the basic definition and theorems of linear extensions of functors and the linear Barr extension, while chapter 5 proves that certain functors from monoidal topology, such as the ultrafilter functor, have linear extensions to **Rel**, and proves an important general theorem about linear extensions from **Set** to $\mathcal{Q}$-**Rel** when $\mathcal{Q}$ is a Girard quantale. Chapter 6 defines linear extensions of monads and proves a condition under which a **Set** monad has a linear Barr extension. Through the course of my research I have also discovered a linear bicategorical analogue of the theorem [25, III§1.2.1] on lean quantales and maps in $\mathcal{Q}$-**Rel**, which is the topic of chapter 7. Finally, we include a postscript chapter 8 at the end providing potential directions for future work.

Chapter 1

Preliminaries for lax extensions

The definition of linear extension is built upon the definition of lax extension of **Set** functors to $\mathcal{Q}$-**Rel**, the category of sets and $\mathcal{Q}$-valued relations. As the setting with $\mathcal{Q}$-**Rel** is critical to further study, one should first gain some background on $\mathcal{Q}$-**Rel** before proceeding. In this chapter we will attempt to familiarize the reader with the basic prerequisite material to understand lax extensions, covered in chapter 2, as well as material on ultrafilters and the ultrafilter functor, which is a prominent target of lax extension in monoidal topology, and a target of linear extension in this thesis. In order to lay the necessary groundwork for the category $\mathcal{Q}$-**Rel** of sets and $\mathcal{Q}$-valued relations, we will need to study quantales. Quantales in turn require background knowledge on lattices, which shall be roughly outlined by [25, II§1.9, II§1.10] from the Monoidal Topology book. We shall next cover material on $\mathcal{Q}$-**Rel** and the map construction as defined in [25, III§1.1, III1.2]. Finally, we will cover background on ultrafilters and the ultrafilter monad following roughly the material of [25, II§1.12, II§1.13].

1.1 Lattices

The concept of a lattice has existed for over a century. They were first studied in depth by Dedekind in [20], referred to as "Dualgruppen". Later, lattice theory and complete lattices were formally introduced by Birkhoff in [7]. In this section we will provide background on lattice theory for the definition of quantales with the books Birkhoff, Grätzer, Hofmann et al. as references.

A *lattice* is defined simply as follows:

Definition 1.1.1 (See [6, I§4]). A *lattice* is a partially ordered set $(X, \leq)$ such that for each pair of elements $x, y \in X$, the set $\{x, y\}$ has a least upper bound (also supremum or l.u.b) in X denoted $x \vee y$, as well as a greatest lower bound (infimum or g.l.b) in X denoted $x \wedge y$.

Definition 1.1.2 (See [6, I§4]). A lattice X is *complete* if X contains the supremum and infimum of all subsets A of X.

We may call $x \vee y$ the "join" of x and y, and $x \wedge y$ the "meet" of x and y. Furthermore, we may denote the supremum of a subset A of X as $\bigvee A$ and the infimum of A as $\bigwedge A$.

Proposition 1.1.3 (See [24, Lemma 2]). *If $(X, \leq)$ is a lattice, then X is a poset containing the suprema and infima of all finite nonempty subsets of X.*

The converse of the above proposition is obvious since for each $x, y \in X$, $\{x, y\}$ is a finite set, so that X contains $x \wedge y, x \vee y$thus X is a lattice if and only if X is a poset containing the suprema and infima of all finite nonempty subsets.

By the above proposition, we may equivalently define a lattice X as a poset containing infima and suprema of all finite non-empty subsets. Furthermore, a lattice X possesses all finite infima and suprema (including the empty set) provided that $\bigvee \varnothing$ and $\bigwedge \varnothing$ are defined. A poset (or lattice) which has a maximum and minimum element is called *bounded* [24, pp. 5, 10].

Proposition 1.1.4. *Let X be a poset. X has a maximum element $\top$ if and only if $\bigwedge \varnothing$ is defined in X. Dually, X has a minimum element $\bot$ if and only if $\bigvee \varnothing$ is defined in X. Furthermore $\bigwedge \varnothing = \top$, and $\bigvee \varnothing = \bot$.*

Proof: This is shown in [24, p. 5]. The fact that every element of X is an upper bound for $\varnothing$ holds vacuously, and hence if an l.u.b for $\varnothing$ exists then it is the least element of X so that $\bigvee \varnothing = \bot$. Dually, it follows that $\bigwedge \varnothing = \top$. ∎

By the above proposition, a bounded lattice is a poset containing all finite infima and suprema. This notion of bounded lattice is the default in Hofmann et al., in fact it is the definition, see [25, I§1.9].

Lattices may also be defined purely algebraically, and there is an equivalnece between this algebraic characterization of lattices and the posetal definition of lattices. In the following proposition we highlight this characterization for bounded lattices.

Theorem 1.1.5. *Let X be a bounded lattice. The binary operations $\wedge, \vee$, together with nullary operations (elements) $\bot, \top$ satisfy the following:*

1. *$(X, \wedge, \top)$, $(X, \vee, \bot)$ form commutative monoids;*
2. *$\wedge, \vee$ are idempotent, that is $x \wedge x = x = x \vee x$;*
3. *$\wedge, \vee$ satisfy the law of absorption, that is*

$$x \wedge (x \vee y) = x = x \vee (x \wedge y). \tag{1.1.e1}$$

Furthermore, a nonempty set X with binary operations $\wedge, \vee$ and nullary operations $\bot, \top$ is a bounded lattice with the partial ordering $\leq$ such that

$$x \leq y \iff x \vee y = y, \quad \text{or dually} \quad x \leq y \iff x \wedge y = x. \tag{1.1.e2}$$

Proof: The equivalence of the algebraic and posetal characterizations above is stated as fact in [25, I§1.9], however it follows from the analagous theorems for (not necessarily bounded) lattices such as [24, Theorem 3 I§1.10], and [6, Theorem 8, I§5]. We must additionally show that a bounded lattice has $(X, \wedge)$, $(X, \vee)$ which are monoids and not just semigroups and conversely that such a universal algebra yields a lattice which is bounded. Let $x \in X$, clearly $x \vee \bot = x$ since $\bot \leq x$ and $x \leq x$. Dually $x \wedge \top = x$, so that $(X, \wedge, \top)$, $(X, \vee, \bot)$ are monoids. Conversely if X is a universal algebra with operations as above, then $x \vee \bot = x \implies \bot \leq x$ and dually $\top \wedge x = x \implies x \leq \top$ so that $(X, \leq)$ with $\leq$ as above is bounded. ∎

We note the following examples of lattices.

Examples 1.1.6.

(I) The powerset $\mathcal{P}X$ of a set X is partially ordered by $\leq = \subseteq$, and it has meet and join equal to $\wedge = \cap, \vee = \cup$ as well as bounds $\top = X, \bot = \varnothing$. By the above $\mathcal{P}X$ is a bounded lattice, furthermore it is a *frame*: a complete lattice satisfying the distributive law, that is for each $x \in X$, $\{x'_i : i \in I\} \subseteq X$ $x \wedge \bigvee_{i \in I} x'_i = \bigvee_{i \in I} x \wedge x'_i$.

(II) A topology τ on the set X is a *subframe* of $\mathcal{P}X$, that is a subset of $\mathcal{P}X$ closed under finite meets and arbitrary joins as in [25, II§1.9]. The join is also the union of sets, however the meet is the interior of the intersection of sets.

(III) If X is a lattice, then X^{op} is a lattice with $\leq_{X^{\mathrm{op}}} = \geq_X$, $\vee_{X^{\mathrm{op}}} = \wedge_X$, $\wedge_{X^{\mathrm{op}}} = \vee_X$, $\bot_{X^{\mathrm{op}}} = \top_X, \top_{X^{\mathrm{op}}} = \bot_X$. Furthermore, statements with lattices involving $\wedge, \vee, \bot, \top, \leq$ still hold after interchanging with $\vee, \wedge, \top, \bot, \geq$ respectively. This is the duality principle of [24, p. 13], and it also follows from the duality principle on categories [see 1, §3].

1.2 Quantales

One of the most fundamental preliminary notions of monoidal topology is the notion of a quantale, appearing in the category $\mathcal{Q}$-**Rel** of $\mathcal{Q}$-valued relations, the category $\mathcal{Q}$-**Cat** of $\mathcal{Q}$-categories and $\mathcal{Q}$-functors, as well as the categories $(\mathbb{T}, \mathcal{Q})$-**Cat** of lax algebras with respect to a lax extension $\hat{\mathbb{T}}$ of a monad $\mathbb{T}$. Quantales were originally introduced by Mulvey in [33, p. 102], concerned with providing a lattice theoretic setting for C^* algebras, as well as a constructive foundation for quantum mechanics. The following definition is from [34, §2.1.1].

Definition 1.2.1 (See [34, §2.1.1]). A *quantale* is a complete lattice $\mathcal{Q}$ with an associative binary operation $\otimes : \mathcal{Q} \times \mathcal{Q} \to \mathcal{Q}$ which preserves suprema on the left and the right, i.e. for all $q \in \mathcal{Q}$ and subsets $\{p_i : i \in I\} \subseteq \mathcal{Q}$, the following holds:

$$q \otimes \left(\bigvee_{i \in I} p_i \right) = \bigvee_{i \in I} q \otimes p_i, \quad \text{and} \quad \left(\bigvee_{i \in I} p_i \right) \otimes q = \bigvee_{i \in I} p_i \otimes q. \tag{1.2.e1}$$

A quantale $\mathcal{Q}$ is called *unital* when the operation $\otimes$ has an identity element $k \in \mathcal{Q}$, that is when $\langle \mathcal{Q}, \otimes, k \rangle$ forms a monoid. $\mathcal{Q}$ is called *commutative* when the operation $\otimes$ is commutative.

In the above, eq. (1.2.e1) also says that $q \otimes (\text{-})$, and $(\text{-}) \otimes q$ are *sup-maps*. A map $f : X \to Y$ between complete lattices is a sup-map if and only if it is a *left adjoint* [25, Corollary II§1.8.3]. A monotone map $f : X \to Y$ between posets is a left-adjoint provided there exists a unique map $g : Y \to X$ satisfying inequalities $g \,;\, f \leq 1_Y$ and $1_X \leq f \,;\, g$ up to the pointwise ordering [25, II§1.5]. Therefore we denote the right adjoints $q \otimes (\text{-})$ and $(\text{-}) \otimes q$ as $q \multimap (\text{-})$ and $(\text{-}) \multimapinv q$ respectively.

Examples 1.2.2.

(I) The trivial quantale, denoted 1, has underlying lattice as the unique trivial lattice with set $\{*\}$ and the unique binary operation $* \otimes * \mapsto *$.

(II) $2 = \{\bot, \top\}$, the unique lattice with 2 elements, is a complete lattice with meet equal to logical conjunction, and join equal to logical disjunction. 2 is made a quantale by setting the tensor equal to , that is $\otimes = \wedge$.

(III) A *frame* is a complete lattice L such that for each $a \in L$, $a \wedge (\text{-})$ is a sup-map (see examples of frames in examples 1.1.6). A frame may also be called a *locale*, however the morphisms between locales and frames differ, summed up by the statement "**Loc** = (**Frames**)$^{\mathrm{op}}$", see [22].

(IV) $P_+ := ([0,\infty]^{\mathrm{op}}, +)$, where $(\text{-})^{\mathrm{op}}$ denotes reversal of order, is a complete lattice with $\top = 0, \bot = \infty$ and quantale operation $\otimes = +$. This is also known as the Lawvere quantale for its use in Lawvere's generalized metric spaces [29].

(V) Let $(S, \otimes)$ be a semigroup. The powerset $\mathcal{P}S$ is a quantale with the following composition $\dot{\otimes}$. Let A, B be subsets of S:

$$A \dot{\otimes} B := \{a \otimes b \,:\, a \in A, b \in B\} \tag{1.2.e2}$$

(VI) As described in [9, §4.10], one may construct a quantale from a cancellative monoid. Recall that a monoid $(M, \otimes, e)$ is cancellative (on both sides) provided that for all $a, b, c \in M$, $a \otimes b = c \otimes b$ if and only if $a = c$, and $b \otimes a = b \otimes c$ if and only if $a = c$. The underlying set M, viewed as a discrete poset, may be completed to a lattice $\bar{M} = M \sqcup \{\bot, \top\}$ by adjoining top and bottom elements. Furthermore, we may extend the operation $\otimes$ to a join preserving operation $\bar{\otimes}$ on $\bar{M}$ as follows. Let $a \in M \sqcup \{\top\}$, $b \in \bar{M}$:

$$a \bar{\otimes} \top = \top = \top \bar{\otimes} a, \quad \text{and} \quad b \bar{\otimes} \bot = \bot = \bot \bar{\otimes} b. \tag{1.2.e3}$$

1.3 The category $\mathcal{Q}$-Rel

In this section we introduce the category $\mathcal{Q}$-**Rel** which generalizes the notion of a relation over a quantale $\mathcal{Q}$. The material of this section as well as the definition follows from [25, III.§1.1]. Recall that a relation between sets A, B is a subset $R \subseteq A \times B$ where we say $a\ R\ b$, or a and b are related by R, if and only if $(a, b) \in R$. That is for each $(a, b) \in A\times B$, $a\ R\ b$ evaluates to either true or false, 0 or 1, so that R may be viewed as a map from $A\times B$ to the set $2 = \{\bot, \top\}$, see example 1.2.2.II. In the context of monoidal topology we would like to also capture quantitative information such as distance and probability, and therefore we exchange 2 in the above for a unital quantale $(\mathcal{Q}, \otimes, k)$. We call a $\mathcal{Q}$-valued map $R\colon X \times Y \to \mathcal{Q}$ a $\mathcal{Q}$-relation, which may be denoted as $R\colon X \nrightarrow Y$, an arrow with domain X and codomain Y. $\mathcal{Q}$-relations have a composition operation (;) (also written $\otimes$), defined via "matrix multiplication" as follows. Let $R\colon X \nrightarrow Y$ and $S\colon Y \nrightarrow Z$:

$$(R\,;S)(x,z) = \bigvee_{y\in Y} R(x,y) \otimes S(y,z). \tag{1.3.e1}$$

Proposition 1.3.1. *As defined in eq.* (1.3.e1), (;) *is an associative composition operation on composable $\mathcal{Q}$-relations with identities* $\mathrm{E}_X\colon X \to X$, *where*

$$\mathrm{E}_X(x,x') = \begin{cases} k & \text{if } x = x' \\ \bot & \text{otherwise} \end{cases}. \tag{1.3.e2}$$

Proof: Let $R\colon X \nrightarrow Y$, $S\colon Y \nrightarrow Z$, $T\colon Z \nrightarrow W$ be $\mathcal{Q}$-relations. Let $x \in X, w \in W$ be an arbitrary pair of elements so that we have the following:

$$(R\,;(S\,;T))(x,w)$$

$$
\begin{aligned}
&= \bigvee_{y \in Y} R(x,y) \otimes \left(\bigvee_{z \in Z} S(y,z) \otimes T(z,w) \right) \\
&= \bigvee_{y \in Y} \bigvee_{z \in Z} R(x,y) \otimes (S(y,z) \otimes T(z,w)) && (\otimes \text{ preserves suprema}) \\
&= \bigvee_{\substack{y \in Y \\ z \in Z}} (R(x,y) \otimes S(y,z)) \otimes T(z,w) && (\text{Associativity of } \otimes) \\
&= \bigvee_{z \in Z} \left(\bigvee_{y \in Y} R(x,y) \otimes S(y,z) \right) \otimes T(z,w) \\
&= (R\,;(S\,;T))(x,w)
\end{aligned}
$$

If X or W are empty, the above holds vacuously. Hence $R\,;(S\,;T)$ and $(R\,;S)\,;T$ are equal pointwise so that the composition is associative.

Let $x, \in X$, $y \in Y$, and let E_X be the identity $\mathcal{Q}$-relation with respect to X:

$$
\begin{aligned}
&\mathrm{E}_X\,; R(x,y) \\
&= \bigvee_{x' \in X} \mathrm{E}_X(x,x') \otimes R(x',y) \\
&= \bigvee_{x' \in X} \begin{cases} k \otimes R(x,y) = R(x,y) & \text{if } x' = x \\ \bot \otimes R(x',y) = \bot & \text{otherwise .} \end{cases} \\
&= R(x,y)
\end{aligned}
$$

The identity holds on the right similarly. Thus, (;) is associative and unital. ∎

By the above proposition, $\mathcal{Q}$-**Rel** is a category.

We will denote $\leq_{X,Y}$ and $\leq_{\mathcal{Q}}$ both as $\leq$ when the intended order is clear from context. Composition of $\mathcal{Q}$-relations preserves this ordering. In other words, for all

$R, R' \colon X \nrightarrow Y, S \colon Y \nrightarrow Z, T \colon W \nrightarrow X,$

$$R \leq_{X,Y} R' \implies R\,;S \leq R'\,;S \text{ and } T\,;R \leq T\,;R' \tag{1.3.e3}$$

We note that when $\mathcal{Q} = 2$, that $\mathcal{Q}$-**Rel** is isomorphic to the category **Rel**. That is 2-**Rel** $\cong$ **Rel**.

A $\mathcal{Q}$-relation $R \colon X \nrightarrow Y$ (more generally a 1-cell R in a locally posetal bicategory, see section 2.1) is said to be a left adjoint provided there exists $S \colon Y \to Z$ such that $\mathrm{E}_X \leq R \otimes S$ and $S \otimes R \leq \mathrm{E}_Y$. Furthermore, S is called a right adjoint to R and we write $R \dashv S$ to denote the adjunction. Those $\mathcal{Q}$-relations which are left adjoints are called *maps* as they tend to represent a **Set**-like substructure in a bicategory **B**. As defined and described in [10, Definition 1.5], the sub-bicategory of maps is Map(**B**). In $\mathcal{Q}$-**Rel** the maps $R : X \nrightarrow Y$ are precisely the set maps embedded into $\mathcal{Q}$-**Rel** if and only if $\mathcal{Q}$ posesses certain properties.

A unital quantale $\mathcal{Q}$ is said to be *integral* provided that $k = \top$. Furthermore, a quantale $\mathcal{Q}$ is said to be *lean* provided that for all $x, y \in \mathcal{Q}$

$$(x \vee y = \top \text{ and } x \otimes y = \bot) \implies (x = \top \text{ or } y = \top). \tag{1.3.e4}$$

We characterise Map($\mathcal{Q}$-**Rel**) precisely in the following proposition.

Proposition 1.3.2 (See [25, Proposition III§1.2.1]). *Let $\mathcal{Q}$ be a unital quantale which is integral. All left adjoint $\mathcal{Q}$-relations are* **Set***-maps, that is for each left adjoint $R : X \nrightarrow Y$ there exists a function $f : X \to Y$ such that $f_\circ = R$, if and only if $\mathcal{Q}$ is lean.*

1.4 Filters and ultrafilters

Filters serve as a generalized notion of convergence for topological spaces, with *ultrafilters* arising as maximal objects among filters. With respect to topological spaces, the set of all ultrafilters arises in the Stone-Cêch compactification, and in category theory the ultrafilter functor arises as an adjoint in many places. In the following section we shall cover material on filters and ultrafilters from [25, II§1.12, II§1.13] as well as [39, Chapter 4 §12] and [18] where noted.

We use the definition from [25, II§1.13] which defines filters on a meet-semilattice X, before narrowing study to filters on the powerset $\mathcal{P}X$ of a set X. In [18, p. 37] filters are defined on a *boolean algebra* X, a bounded lattice with a notion of boolean negation. In the following we let X be a lattice in order to fit with examples from section 1.1.

Definition 1.4.1. A *filter* $\mathfrak{f}$ on a lattice X is a subset of X such that the following holds:

i) $\mathfrak{f}$ is *nonempty*. When X contains a top element $\top \in \mathfrak{f}$ as in a bounded lattice, we say $\top \in \mathfrak{f}$.

ii) If $a \in \mathfrak{f}$, then $\uparrow_X a \subseteq \mathfrak{f}$. In other words, $\mathfrak{f}$ is *upwards closed*.

iii) If $a, b \in \mathfrak{f}$ then $a \wedge b \in \mathfrak{f}$. In other words, $\mathfrak{f}$ is *closed under finite meets*.

A *proper filter* also has

4. $\mathfrak{f}$ is a proper subset of $\mathcal{P}X$. When the lattice contains a minimum element $\perp$ as in a bounded lattice, we say $\perp \notin \mathfrak{f}$.

In this text, we will make the assumption that a filter $\mathfrak{f}$ is proper unless otherwise noted. When we refer to a filter on the set X, we mean a filter on the (complete) lattice $\mathcal{P}X$ (see example 1.1.6.I). In this case we replace $\bot \in X$ with the empty set $\varnothing$, meets $\wedge$ with set intersection $\cap$, and $\leq$ with set inclusion $\subseteq$.

Examples 1.4.2.

(I) Let X be a topological space with a topology $\tau \subseteq \mathcal{P}X$ of open sets. The set $\mathcal{U}_x = \{V \in \mathcal{P}X : x \in V; \exists U \in \tau, V \subseteq U\}$ of neighbourhoods of the point $x \in X$ is a filter on the set X, [39, Examples §12.2].

(II) The set $\mathcal{O}_x = \{U \in \tau : x \in U\}$ of open neighbourhoods with respect to the point $x \in X$ is a filter on the lattice τ, as described in example 1.1.6.II.

(III) Let X and Y be sets, let $f : X \to Y$ be a function, and let $\mathfrak{F} \subseteq \mathcal{P}X$ be a family of sets. We denote $f[\mathfrak{F}]$ to be the family of sets $\{B \subseteq Y : f^{-1}(B) \in \mathfrak{F}\} = \mathcal{P}f(\mathfrak{F})$. When $\mathfrak{F}$ is a filter, $f[\mathfrak{F}]$ is a filter on the set Y, and we call this the *image filter* of $\mathfrak{F}$ with respect to f. Furthermore, when $\mathfrak{F}$ is an ultrafilter, $f[\mathfrak{F}]$ is also an ultrafilter [25, II§1.12(a)].

(IV) Let X be a lattice. Given a nonempty subset $\mathfrak{f} \subseteq X$ which is closed under finite infima and does not contain $\bot$, we may form a filter $\mathfrak{F} := \uparrow \mathfrak{f}$. $\mathfrak{f}$ is called a *filter base* or *prefilter*, see [25, I§1.12].

For the remainder of the section we shall focus on ultrafilters. Generally, these are maximal filters (w.r.t inclusion) among proper filters on a meet semilattice, X. In many sources such as [18, p. 39] and [4, p. 14], X is also expected to be a boolean algebra. We will focus on ultrafilters which are maximal proper filters on a set X (that is with respect to the lattice $\mathcal{P}X$), consistent with [25, II§1.13].

Definition 1.4.3. An *ultrafilter* on the set X is a proper filter on X that is maximal with respect to inclusion.

Examples 1.4.4.

(I) The *principal ultrafilter* with respect to an element $x \in X$ is equal to $\uparrow_{\mathcal{P}X} \{x\}$, that is the set of all sets containing the singleton $\{x\}$ [25, I§1.13(c)].

(II) Following from Zorn's lemma, every filter on a set X is contained in at least one ultrafilter, [25, Proposition II§1.13.2].

(III) When X is an infinite set, there exists at least one non-principal ultrafilter on X [4, ch. 1. Lemma 3.8]. The set $\{A^{\complement} : A \subseteq X \text{ is finite}\}$ of cofinite sets is a filter base, and is therefore contained in a proper filter by example 1.4.2.IV. Furthermore it follows from example 1.4.4.II that this filter is contained in some ultrafilter. If $\mathscr{F}$ is such an ultrafilter, then $\mathscr{F}$ is not principal. $\mathscr{F}$ is sometimes called the Frechet filter, as in [39, 12.2(c)] where $X = (a, \infty) \subseteq \mathbb{R}$.

(IV) When $\mathfrak{A}$ is an ultrafilter on the set X the image filter $f[\mathfrak{A}]$ is an ultrafilter on the set Y, [25, II§1.13(a)].

Proposition 1.4.5 ([25, Lemma I§1.13.1]). *Let $\mathfrak{F}$ be a filter on the set X. The following are equivalent:*

i. $\mathfrak{F}$ is an ultrafilter on X;

ii. For all $A, B \subseteq X$, if $A \cup B \in \mathfrak{F}$ then $A \in \mathfrak{F}$ or $B \in \mathfrak{F}$;

iii. For each subset $A \subseteq X$, $A \in \mathfrak{F}$ or $A^{\complement} \in \mathfrak{F}$;

For a proof of proposition 1.4.5, see the proof in [25, Lemma I§1.13.1].

Proposition 1.4.6 (As in [25, Examples III§1.10.3(3)]). *Let X be a set and let $\mathfrak{F}$ be an ultrafilter on X. Then for all $A \subseteq X$, $A \in \mathfrak{F}$ if and only if for all $B \in \mathfrak{F}$, $A \cap B \neq \varnothing$.*

Proof: Let $\mathfrak{F}$ be an ultrafilter on the set X, and let $\mathfrak{A}$ be the set of all subsets $A \subseteq X$ such that for each $B \in \mathfrak{F}$, $A \cap B \neq \varnothing$. Clearly, $\mathfrak{F} \subseteq \mathfrak{A}$ as $B \cap B' \neq \varnothing$ for each $B, B' \in \mathfrak{F}$. Furthermore, let $A \in \mathfrak{A}$, then $\mathfrak{a} := \mathfrak{F} \cup \{A \cap B : B \in \mathfrak{F}\}$ forms a filter base for a filter $\uparrow \mathfrak{a}$ on X containing $\mathfrak{F}$. Since $\mathfrak{F}$ is an ultrafilter $\uparrow \mathfrak{a} \subseteq \mathfrak{F}$, and hence $A \in \mathfrak{F}$. Thus $\mathfrak{F} = \mathfrak{A}$. ∎

The converse does not hold in general, for example let $\mathfrak{F} := \{\{1,2,3\},\{1,2\},\{1,3\},\{2,3\}\}$. $\mathfrak{F}$ has the above property, but it is not closed under finite infima, so it fails to be a filter.

1.4A Filters in topology

Filters are one way of generalizing the notion of convergence in metric spaces to all topological spaces. Like sequences, filters can have points of convergence.

Definition 1.4.7. Let X be a topological space with topology τ, and let $\mathfrak{F}$ be a filter on the set X. One says that $\mathfrak{F}$ *converges* to a point $x \in X$ provided that every neighbourhood of x is contained in $\mathfrak{F}$. In other words, $\mathcal{U}_x \subseteq \mathfrak{F}$.

We include the following properties about ultrafilters and convergence in the following theorem.

Theorem 1.4.8. *Let X be a topological space.*

1. *X is a Hausdorff space if and only if every filter on the set X converges to at most one point. [4, ch. 1 Lemma 5.3]*

2. *X is a compact space if and only if every ultrafilter on the set X converges to at least one point [4, ch. 1 Corollary 5.5].*

Item 2 in the above is analogous to the equivalence of compactness with sequential compactness for metric spaces, i.e. that a metric space is compact if and only if every sequence has a convergent subsequence.

1.4B The ultrafilter monad

Let X be a set. We define a topological space βX, the set of ultrafilters on X, as follows. First, we define the underlying set:

$$\beta X := \{\mathfrak{A} \subseteq \mathcal{P}X : \mathfrak{A} \text{ is an ultrafilter on } X\}$$

Furthermore, βX possesses an underlying topology, generated as follows. Let $\widehat{A} := \{\mathfrak{A} \in \beta X : A \in \mathfrak{A}\}$, the set of ultrafilters contained in A. Then βX has the topology $\tau_{\beta X} := \{\widehat{A} : A \subseteq X\}$, which is sometimes referred to as the "hull-kernel" topology, as in [2, p. 1].

Let $f : X \to Y$ be a function. We shall denote $\beta f : \beta X \to \beta Y$ to mean a function taking ultrafilters $\mathfrak{A} \in \beta X$ to $\beta f(\mathfrak{A}) = \{B \subseteq Y : f^{-1}(B) \in \mathfrak{A}\}$, also known as the image filter $f[\mathfrak{A}]$, see example 1.4.4.IV. This is the same mapping done by $\mathcal{P}^2$ to f with the domain and codomain of $\mathcal{P}^2 f$ restricted to βX and βY respectively [25, Example II§3.1.1(5)]. That β is a functor from **Set** to **Set** follows from the functoriality of $\mathcal{P}^2$.

There is an embedding map $e_X : X \to \beta X$ which takes an element $x \in X$ to the principal ultrafilter generated by x. Furthermore, there is an operation $m_X : \beta\beta X \to \beta X$, defined as follows. Let $\mathscr{A}$ be an ultrafilter of ultrafilters on X, that is $\mathscr{A} \in \beta\beta X$, then

$$m_X(\mathscr{A}) := \{A \subseteq X : \{\mathfrak{A} \in \beta X : A \in \mathfrak{A}\} \in \mathscr{A}\} \tag{1.4.e1}$$

$$= \{A \subseteq X : \widehat{A} \in \mathscr{A}\}. \tag{1.4.e2}$$

Which is an ultrafilter. The function m_X is sometimes referred to as the *filtered sum*, or the *Kowalsky sum*.

Theorem 1.4.9. *Let $\mathbb{\beta} = (\beta, m, e)$, with β, m, and e as defined above. Then $\mathbb{\beta}$ is a monad.*

We note theorem 1.4.9 holds following [25, II§3.1.1(5)], with the details sorted out in detail in [32, Proposition 5.5].

Chapter 2

Lax extensions of functors

The theory of linear extensions builds directly on the theory of lax extensions present in the field of monoidal topology. After covering the necessary groundwork in chapter 1, we are ready to begin the task of defining lax extension of **Set** functors to $\mathcal{Q}$-**Rel**. Following roughly the material in the Monoidal Topology [25, II§4.5, II§4.6, II§4.7] we first introduce terminology of *locally posetal bicategories*, what are referred to as *ordered categories* throughout [25], as well as their *lax functors* and *lax natural transformations*. Then we shall cover the theory of lax extensions of **Set** functors and monads to $\mathcal{Q}$-**Rel** from [25, III§1.4, III§1.5]. We will finish off the chapter by introducing a specific class of lax extensions known as the Barr extension [25, III§1.10, III§1.11].

2.1 Bicategories

Bicategories, introduced by Bénabou in [5] are a notion of category in which the associativity and unitality properties of composition may not hold up to equality, but more generally, up to an isomorphism between morphisms in a category where morphisms are the objects. A motivating example is the bicategory **Bim** (as in

[5, §2.5]) whose objects are unital rings and arrows are bimodules, with the tensor product as composition. It is well known that the tensor product of modules is neither associative nor unital, but that it may be considered so up to isomorphism in a category of bimodules and bimodule homomorphisms. As in **Bim**, in a general bicategory we expect that no matter how associativity and unitality isomorphisms are applied they result in the same isomorphism. In other words that all diagrams involving these isomorphisms commute. This is contingent on the commutativity of certain diagrams of higher order morphisms, referred to as coherence conditions or coherence requirements. The specific coherence requirements for a bicategory are analagous to those for a monoid in a monoidal category and includes the infamous pentagon diagram for associativity and the triangle diagrams for left and right unitors. The original paper [5] is an excellent source on bicategory theory with a breadth of examples. [30] is a concise introduction to bicategories with a neat proof of the coherence theorem. One may refer also to bicategories in Maclane's textbook [31, XII§6] and the section [31, VII§2] for discussion on coherence.

We will provide a definition of bicategories based on the original "local definition" found in [5, §1.1], with the following additional differences. We allow the objects to be a collection instead of a set, as in Leinster's definition found in [30, §1], and we make reference to a colleciton of 1-cells as in [31, XII§6] which is in line with the definition of linear bicategory as in [14, Definition 2.1]. Furthermore, the specification of the identity arrows and unitors is in line with [31, XII§6].

Definition 2.1.1. A *bicategory* denoted $\mathbf{B}$ is comprised of the following data:

i) A collection $\mathbf{B}_0$ or $\mathsf{ob}(\mathbf{B})$ of objects, or 0-cells.

ii) For each pair X, Y in B_0, a category $\mathbf{B}(X, Y)$ whose objects are called 1-cells and whose morphisms are called 2-cells. 1-cells f, g in $\mathbf{B}(X, Y)$ are said to have domain X and codomain Y, written $f, g\colon X \to Y$. A morphism α of $\mathbf{B}(X, Y)$ between 1-cells $f, g\colon X \to Y$ is called a 2-cell, denoted $\alpha\colon f \Rightarrow g$. In the locally posetal case $\mathbf{B}(X,Y)$ is a poset with a partial ordering denoted $\leq_{X,Y}$, and we say $f \leq g$ if there exists a 2-cell between them.

iii) A composition operation on composable pairs of arrows denoted by $\otimes$ (or (;)). This is an X,Y,Z-indexed family of functors $\otimes_{X,Y,Z} \colon \mathbf{B}(X,Y) \times \mathbf{B}(Y,Z) \to \mathbf{B}(X,Z)$. In the locally posetal case $\mathbf{B}(X,Y) \times \mathbf{B}(Y,Z)$ is the product of posets and $\otimes_{X,Y,Z}$ is a monotone map. Fixing a 1-cell $f \colon X \to Y$ specifies a functor $f \otimes (\text{-}) \colon \mathbf{B}(Y,Z) \to \mathbf{B}(X,Z)$ mapping pre-composition by f on 1-cells and mapping $\alpha \mapsto 1_f \otimes \alpha$ on 2-cells. $(\text{-}) \otimes f$ is defined similarly using post-composition.

iv) An X-indexed family of 1-cells $\mathrm{E} := \{\mathrm{E}_X \colon X \to X : X \in \mathbf{B}_0\}$ where $\mathrm{E}_X \colon X \to X$ is called the identity arrow on X.

v) The operation $\otimes$ is associative up to a family of natural isomorphisms a indexed by $X,Y,Z,W \in \mathbf{B}_0$, that is $a_{X,Y,Z} \colon (\otimes_{X,Y,Z} \times 1) \,;\, \otimes_{X,Z,W} \overset{\sim}{\Rightarrow} (1 \times \otimes_{Y,Z,W}) \,;\, \otimes_{X,Y,W}$, and for all triples f,g,h of composable 1-cells we write $a_{f,g,h} \colon f \otimes (g \otimes h) \overset{\sim}{\Rightarrow} (f \otimes g) \otimes h$. In the locally posetal case this condition is equivalent to requiring $f \otimes (g \otimes h) = (f \otimes g) \otimes h$.

vi) The operation $\otimes$ is unital with respect to the identity arrows, on the right and on the left, up to families of natural isomorphisms $\ell_{X,Y} \colon \mathrm{E}_X \otimes_{X,X,Y} (\text{-}) \overset{\sim}{\rightarrow} 1$ and $r_{X,Y} \colon (\text{-}) \otimes_{X,Y,Y} \mathrm{E}_Y \overset{\sim}{\rightarrow} 1$. For each $f \colon X \to Y$ we write $\ell_f \colon \mathrm{E}_X \otimes f \overset{\sim}{\Rightarrow} f$ and $r_f \colon f \otimes \mathrm{E}_X \overset{\sim}{\Rightarrow} f$. In the locally posetal case this condition is equivalent to requiring $\mathrm{E}_X \otimes f = f = f \otimes \mathrm{E}_Y$.

vii) The associator and unitors satisfy certain coherence requirements which may be referenced in [5, p. 6], [30, p. 3], or [31, p. 282]. In the locally posetal case, this condition holds trivially.

The composition in $\mathbf{B}(X,Y)$ is referred to as the vertical composition while the composition (;) is referred to as the horizontal composition. The 1-cells and 2-cells form a category under vertical composition. We shall refer to this category as $\mathbf{B}_1$, and we shall also refer to the collection of 1-cells as $\mathbf{B}_1$. In the case where B_1 forms an ordered class, morphisms are unique up to domain and codomain so that all diagrams, including naturality and coherence diagrams, commute trivially. Furthermore, when $\mathbf{B}$ is a locally posetal bicategory all isomorphisms between 1-cells are identities, so that $\otimes$ becomes a strictly associative and unital categorical composition. Bicategories of this kind are referred to as *ordered categories* throughout the Monoidal Topology book (See [25, II§4.5], and which we shall refer to as locally ordered (or locally preordered) bicategories. A *locally posetal bicategory* shall refer to the more specific case where the only isomorphisms between 1-cells are identities.

As discussed above, the horizontal composition must additionally possess associativity and unitality isomorphisms satisfying the coherence conditions, however, in the locally posetal (more generally, the preordered) case the 2-cells represent inequalities, and isomorphisms between 1-cells are unique so that the coherence properties hold trivially. Thus we may refrain from including these conditions for locally posetal bicategories.

The functoriality of (;) holds if and only if (-) ; Y and X ; (-) are functors (see proposition 1 of [31, II§3]). In the locally posetal case, this says that (;) is monotone if and only if

$$\forall f, f' : A \to B,\ g, g' : B \to C$$
$$(f \leq f' \implies f ; g \leq f' ; g) \text{ and } (g \leq g' \implies f ; g \leq f ; g').$$

In other words, condition 2.1.1.iii says that composition in $\mathbf{B}$ is a binary operation which is monotone on both the left and right side.

In the above, one may replace all instances of locally posetal with *ordered* and **Pos** with **Ord**, the category of preordered sets and monotone maps, to define *locally* *ordered bicategory*. The coherence conditions also commute trivially in this case, and furthermore every locally posetal bicategory is also a locally ordered bicategory. In this thesis we will focus only on examples of bicategories with locally posetal structure.

We list the following examples of bicategories and locally posetal bicategories, many of which may be found in [5].

Examples 2.1.2.

(I) Every category $\mathcal{C}$ may be viewed as a bicategory $\bar{\mathcal{C}}$. The 0-cells of $\bar{\mathcal{C}}$ are the objects of $\mathcal{C}$ and the 1-cells are the morphisms of $\mathcal{C}$. The category $\bar{\mathcal{C}}(X, Y)$ is the discrete category whose objects are specified by the morphisms in $\mathcal{C}(X, Y)$. This is known as a *locally discrete* bicategory, that is a bicategory in which the 2-cells form a discrete category.

(II) A 2-*category* is a bicategory in which the associativity and unitality 2-cells are identities, in other words, a bicategory in which the composition is strictly associative and unital. The category of categories **Cat** is a primordial example of a 2-category, with 0-cells the collection of categories, 1-cells the collection of functors, and 2-cells the natural transformations between functors.

(III) **Rel** is a locally posetal bicategory. As previously noted, the relations sharing domain and codomain are ordered under inclusion, and furthermore one may verify that the composition in **Rel** is monotone with respect to this ordering. Let $R \leq R' : X \nrightarrow Y, S \leq S' : Y \nrightarrow Z$ such that $R \subseteq R'$, and $S \subseteq S'$. Suppose that $(x, z) \in R\,;S$. $x(R\,;S)z$ holds if and only if there exists $y \in Y$ such that $x\ R\ y \vee y\ S\ z$, however that implies $x\ R'\ y \wedge y\ S'\ z$ so that $x\ R'\,;S'\ z$ and thus $R\,;S \subseteq R'\,;S'$. Hence $(x, z) \in R'\,;S'$ so that $(;)$ is monotone.

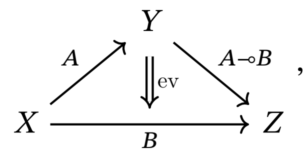
(IV) More generally, for any quantale $\mathcal{Q}$, $\mathcal{Q}$-**Rel** is a locally posetal bicategory. Recall that a $\mathcal{Q}$-relation $R : X \nrightarrow Y$ is equivalently a map $X \times Y \to \mathcal{Q}$ so that hom-sets inherit the pointwise partial ordering. Let $R, R' : X \nrightarrow Y, S, S' : Y \nrightarrow Z$ be $\mathcal{Q}$-relations such that $R \leq R', S \leq S'$ and let $x \in X, z \in Z$. By definition, $R \otimes S(x,z) = \bigvee_{y \in Y} R(x,y) \otimes_{\mathcal{Q}} S(y,z)$, and since $\otimes_{\mathcal{Q}}$ is a monotone map we have $R(x,y) \otimes S(y,z) \leq R'(x,y) \otimes S'(y,z)$. Thus $\bigvee_{y \in Y} R(x,y) \otimes S(y,z) \leq \bigvee_{y \in Y} R'(x,y) \otimes S'(y,z)$, and hence by the pointwise ordering $R \,; S \leq R' \,; S'$.

(V) Further examples of bicategories come from the theory of enriched categories. Bicategories are the same as categories (weakly) enriched over **Cat**. Furthermore, as ordered bicategories are categories enriched over **Ord** [25, II§4.5], locally posetal bicategories are categories enriched over **Pos**. Enriched categories are defined in [25, II§4.10], and a depth of information may be found in Kelly's book on enriched categories [26, §1.2].

(VI) Whereas a category $\mathcal{C}$ has one dual structure $\mathcal{C}^{\mathrm{op}}$, a bicategory $\mathbf{B}$ has two such. $\mathbf{B}^{\mathrm{op}}$ is the bicategory with the same objects as $\mathbf{B}$, and with reversed 1-cells and 1-cell composition. That is $\mathbf{B}^{\mathrm{op}}(A,B) = \mathbf{B}(B,A)$. The other dual $\mathbf{B}^{\mathrm{co}}$ is constructed with the same 0-cells as in $\mathbf{B}$ but with reversed 2-cells, that is with $\mathbf{B}^{\mathrm{co}}(A,B) = \mathbf{B}(A,B)^{\mathrm{op}}$. The compositions in $\mathbf{B}^{\mathrm{op}}$ and $\mathbf{B}^{\mathrm{co}}$ work as expected, see [5, §3] for the dual composition as well as construction of the associators and unitors. These dual constructions may be taken in tandem to form $\mathbf{B}^{\mathrm{coop}}$ or $\mathbf{B}^{\mathrm{opco}}$, which are identical.

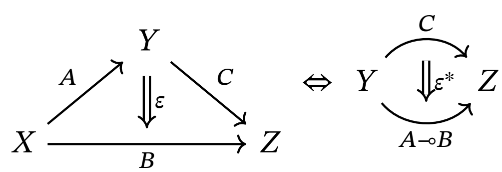
An important notion in the category of categories is that of Kan-extensions, as Maclane writes in his section famously titled "All concepts are Kan extensions" [31, X§7] "The notion of Kan extension subsumes all the other fundamental concepts in category theory.". In a bicategory this concept is generalized to that of a *right extension* as defined by Street in [37, §1] for 2-categories, a definition descended from Kan extensions for enriched categories as in [21, p. 39]. To keep more consistent with the symbology of linear logic, we use the notation from [14].

Definition 2.1.3. A right extension of the 1-cell B along A, if it exists, is a 1-cell denoted $A \multimap B$ as in [14], together with a 2-cell ev as in the following diagram,



, (2.1.e1)

such that for every $C\colon Y \to Z$ there is a bijection, denoted curry or $(\text{-})^*$, between 2-cells of signature $\varepsilon\colon A \otimes C \to B$ and $\mathrm{curry}(\varepsilon)\colon C \to A \multimap B$,



(2.1.e2)

and furthermore that $\mathrm{curry}(\varepsilon)\,;\mathrm{ev} = \varepsilon$.

In the locally posetal case, a right extension is a 1-cell $A \multimap B$ such that for $A\,;C \le B$ if and only if $C \le A \multimap B$. The 1-cell $A \multimap B$ may also be denoted $\hom_{X((,A)}, B)$ as in [37] or $X\mathbf{Mod}(A,B)$ as in [9], and a right extension may also be referred to as a right hom as in [14]. When $\mathbf{B}$ contains all left and right homs, it is called biclosed. A left lifting in $\mathbf{B}$ of $A\colon Z \to X$ is a right extension in $\mathbf{B}^{\mathrm{co}}$, a right lifting is a right extension in $\mathbf{B}^{\mathrm{op}}$, and a left lifting is a right extension in $\mathbf{B}^{\mathrm{coop}}$. A bicategory with all extensions and liftings is called *biclosed*, see [9, §6].

2.2 Lax functors

A lax functor is a notion of functor between bicategories in which we replace the usual equational assumptions of functoriality $Ff \cdot Fg = F(f \cdot g)$ and $F1_X = 1_{FX}$ with the relaxed assumptions that there exist certain families of 2-cells $FR \otimes FS \Rightarrow F(R \otimes S)$ and $F\mathrm{E}_X \Rightarrow \mathrm{E}_{FX}$ satisfying coherence. The 2-cells replacing equality may be isomorphisms, in which case we refer to these as *pseudofunctors* (homomorphisms of bicategories in Bénabou's terms). However, more generally these may not be isomorphisms, and thus the general term is *lax functor* (morphism of bicategories in Bénabou's terms).

The following definition comes from [25, p. II.4.6], which is the definition of morphism of bicategories translated for the locally posetal case. More generally we require that coherence conditions hold, which one may see in [5, Definition 4.1]. As noted previously, diagrams of 2-cells commute trivially in the locally posetal (ordered) case, therefore we may omit the coherence conditions in the following.

Definition 2.2.1. See [25, II§4.6]Let X, Y, Z be objects of $\mathbf{B}$, and let $R, R' : X \to Y$ and $S, S' : Y \to Z$ be 1-cells. A *lax functor* $\hat{F} \colon \mathbf{B} \to \mathbf{C}$ between locally posetal bicategories $\mathbf{B}$ and $\mathbf{C}$ is comprised of a map $\hat{F} \colon \mathsf{ob}(\mathbf{B}) \to \mathsf{ob}(\mathbf{C})$ of objects and an X, Y-indexed family of maps $\hat{F}_{X,Y} \colon \mathbf{B}(X, Y) \to \mathcal{D}(\hat{F}X, \hat{F}Y)$ of hom-sets indexed by $X, Y \in \mathsf{ob}(\mathbf{B})$ which satisfy the following conditions:

i) $R \leq R' \implies \hat{F}_{X,Y}(R) \leq \hat{F}_{X,Y}(R')$ (monotonicity);

ii) $\hat{F}_{X,Y}(R); \hat{F}_{Y,Z}(S) \leq F_{X,Z}(R; S)$ (lax preservation of composition);

iii) $\mathrm{E}_{\hat{F}X} \leq \hat{F}_{X,X}\mathrm{E}_X$, that is $\hat{F}$ (lax preservation of identites).

Examples 2.2.2.

(I) Any functor $F : \mathcal{C} \to \mathcal{D}$ is a lax functor between $\mathcal{C}$ and $\mathcal{D}$ as locally posetal bicategories with hom-sets ordered discretely.

(II) Lax functors are equivalently described as the enriched functors of **Pos**-categories.

(III) Lax extensions provide examples of lax functors. See proposition 2.3.2. Therefore, each entry of examples 2.3.3 provides an example of a lax functor.

2.3 Lax extensions of functors

There is an inclusion functor $(\text{-})_\circ : \mathbf{Set} \hookrightarrow \mathcal{Q}\text{-}\mathbf{Rel}$ which is injective on objects and faithful. Let k be the unit of $\mathcal{Q}$, X, Y be sets and $f : X \to Y$ be a function, then $(\text{-})_\circ$ is defined as follows:

$$X_\circ := X \tag{2.3.e1}$$

$$f_\circ(x,y) := \begin{cases} k & \text{if } f(x) = y \\ \bot & \text{otherwise} \end{cases} \tag{2.3.e2}$$

We may therefore think of **Set** as a subcategory of $\mathcal{Q}$-**Rel** and $\mathcal{Q}$-**Rel** as an extension of **Set**. We may omit $(\text{-})_\circ$ when its use is implicit from context.

Definition 2.3.1. A *lax extension* $\hat{F}$ of a functor $F : \mathbf{Set} \to \mathbf{Set}$ is a collection of maps $\hat{F}_{X,Y} : \mathcal{Q}\text{-}\mathbf{Rel}(X,Y) \to \mathcal{Q}\text{-}\mathbf{Rel}(FX,FY))$ indexed by pairs of sets $X, Y \in \mathsf{ob}(\mathbf{Set})$ satisfying the following conditions. Let X, Y, Z be sets, $r, r' : X \nrightarrow Y, s : Y \nrightarrow Z$ be relations, and let $f : X \to Y$ be a function:

i) $r \leq r' \implies \hat{F}_{X,Y} r \leq \hat{F}_{X,Y} r'$

ii) $\hat{F}_{X,Y} r \,;\, \hat{F}_{Y,Z} s \leq \hat{F}(r \,;\, s)$

iii) $Ff \leq \hat{F}_{X,Y} f$, and $\hat{F}(f^\circ) \leq (Ff)^\circ$

As noted in [25, III§1.4.1], conditions (2.3.1.i), (2.3.1.ii) are the same as in lax functors, and (2.3.1.iii) above implies (2.2.1.iii) of lax functors. In other words, $\hat{F}$ is a lax functor satisfying (2.3.1.iii). Therefore, we may write the following proposition.

Proposition 2.3.2. *A* lax extension *of a functor* $F : \mathbf{Set} \to \mathbf{Set}$ *to* $\mathcal{Q}$-**Rel** *is a lax functor* $\hat{F} : \mathcal{Q}\text{-}\mathbf{Rel} \to \mathcal{Q}\text{-}\mathbf{Rel}$ *satisfying the following extension condition:*

$$\forall f : X \to Y,$$

$$(Ff) \leq \hat{F}f \quad and \quad (Ff)^\circ \leq \hat{F}(f^\circ)$$

Examples 2.3.3.

(I) For each quantale $\mathcal{Q}$, the identity functor on $\mathcal{Q}$-**Rel** is a lax extension of the identity functor on **Set**.

(II) Every functor F has a maximal lax extension $\hat{F}$ which sends each $r : X \nrightarrow Y$ in $\mathcal{Q}$-**Rel** to a $\mathcal{Q}$-relation $\hat{F}r(x, y) = \top$ for each $x \in FX, y \in FY$

(III) The covariant powerset functor P has a lax extension P_L such that for each $r : X \nrightarrow Y, A \subseteq X, B \subseteq Y$,

$$A\ P_L r\ B \iff B \subseteq (A)r \iff \forall x \in A \exists y \in B x\ r\ y \tag{2.3.e3}$$

Where $(A)r$ is the set $\{y \in Y : \exists x \in Ax\ r\ y\}$. We may also define $r(B) = (B)r^\circ = \{x \in X : \exists y \in Bx\ r\ y\}$ and another lax extension P_R

$$A\ P_R r\ B \iff A \subseteq r(B) \iff \forall y \in B \exists x \in Ax\ r\ y \tag{2.3.e4}$$

(IV) The free monoid monad has a functor $\mathscr{L}$ which maps X to $X^* = \bigsqcup_{n \geq 0} X^n$, the finite sequences of its elements, and maps functions $\mathscr{L}f(\langle x_i \rangle_{i=1}^n) = \langle f(x_i) \rangle_{i=1}^n$. This functor has a lax extension $\overline{\mathscr{L}}$ such that for each $r : X \nrightarrow Y, \langle x_i \rangle_{i=1}^n \in \mathscr{L}X, \langle y \rangle_{=1}^{\in} \mathscr{L}Y$

$$\{x\}_{i<n}\ \mathscr{L}r\ \{y\}_{i<m} \iff m = n \text{ and } \forall i \leq n\ x_i\ r\ y_i \tag{2.3.e5}$$

A lax extension $\hat{F}$ of F is called *flat* if $\hat{F}1_X = 1_{FX}$. One may show that equivalently, $\hat{F}$ is a flat lax extension if $\hat{F}(f^\circ) = (Ff)^\circ$.

2.3A Lax extensions of monads

In order to extend a **Set** monad to $\mathcal{Q}$-**Rel**, we extend its underlying functor. The unit and multiplication need not be extended, but they must satisfy some laws.

Definition 2.3.4. A lax extension of a monad $\mathbb{T} = (T, m, e)$ is a lax extension $\hat{T}$ of the functor T such that m, e are oplax with respect to $\hat{T}$. In other words,for all $\mathcal{Q}$-relations $R \colon X \nrightarrow Y$, the following are lax commutative diagrams.

$$\begin{array}{ccc} TTX & \xrightarrow{m_X} & TX \\ {\scriptstyle \hat{T}\hat{T}R}\big\downarrow & \geq & \big\downarrow{\scriptstyle \hat{T}R} \\ TTY & \xrightarrow[m_Y]{} & TY \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{e_X} & TX \\ {\scriptstyle R}\big\downarrow & \geq & \big\downarrow{\scriptstyle \hat{T}R} \\ Y & \xrightarrow[e_Y]{} & TY \end{array} \tag{2.3.e6}$$

Any **Set** monad $\mathbb{T} = (T, m, e)$ such that T satisfies the *Beck-Chevalley condition* (which we shall define shortly in definition 2.4.2) has a lax extension to **Rel** given by the Barr extension, as we shall describe in the following section.

2.4 The Barr extension

When we view the relation $R \colon X \nrightarrow Y$ in **Rel** as a subset $|R|$ of the product $X \times Y$ we can take projection maps $\pi_1 \colon R \to X$, $\pi_2 \colon R \to Y$, such that $R = \pi_1^\circ ; \pi_2$. In [2, p. 42], Barr defines a special type of extension to **Rel** using this information (a lax extension to $\mathcal{Q}$-**Rel** with $\mathcal{Q} = 2$). The following is a slight variation on [25, III§1.10.1], where we make the underlying collection of maps explicit.

Definition 2.4.1. *The Barr extension* of a **Set** functor F denoted $\overline{F}$, is a collection of maps $\overline{F}_{X,Y} \colon \mathbf{Rel}(X, Y) \to \mathbf{Rel}(FX, FY)$ defined at a relation $R \colon X \nrightarrow Y$ in terms of the projection maps $\pi_1 \colon |R| \to X, \pi_2 \colon |R| \to Y$, that is

$$\overline{F}R := (F\pi_1)^\circ ; F\pi_2. \tag{2.4.e1}$$

Pointwise, the Barr extension is given for $x \in FX, y \in FY$ by

$$x\, \overline{F}R\, y \iff \exists w \in F\,|R|\,.(\overline{F}\pi_1(w) = x \wedge \overline{F}\pi_2(w) = y). \tag{2.4.e2}$$

In his paper [2], Barr shows that for a **Set** endofunctor T and composable relations R, S, that $\overline{T}(R\,;S) \leq \overline{T}R\,;\overline{T}S$. However as we will soon see, equality does not necessarily hold in general. The following is an important condition with respect to the Barr extension. We take the definition from [25, III§1.11.2], expanding the definition of BC-squares.

Definition 2.4.2. A **Set** functor F satisfies the Beck-Chevalley condition provided that for each $h_1 \colon W \to X$, $h_2 \colon W \to Y$, $f \colon X \to Z$, $g \colon Y \to Z$ such that $h_1\,;f = h_2\,;g$ we have that

$$h_1^\circ\,;h_2 = f\,;g^\circ \implies F(h_1)^\circ\,;Fh_2 = Ff\,;(Fg)^\circ. \tag{2.4.e3}$$

The Barr extension $\overline{F}$ of a **Set** functor F is a lax extension if and only if F satisfies the Beck-Chevalley condition. This is an explicit statement in [13, §1.3] and follows from a theorem of the same section.

Theorem 2.4.3 (See [13, §1.3]). *Let F be a* **Set** *endofunctor. The following are equivalent:*

1. *There is a unique functor $\overline{F} \colon$* **Rel** $\to$ **Rel** *preserving the involution* $(\text{-})^\circ$ *that extends the functor F.*
2. *The functor F satisfies the Beck-Chevalley condition.*

Examples 2.4.4.

(I) The ultrafilter functor β satisfies the Beck-Chevalley condition, as noted in [25, III§1.12.3(3)], and has Barr extension $\overline{\beta}$ such that

$$\mathfrak{A}\ \overline{\beta}R\ \mathfrak{B} \iff R[\mathfrak{A}] \subseteq \mathfrak{B}$$

where $R[\mathfrak{A}]$ is the filter generated by the filter base $\{(A)R : A \in \mathfrak{A}\}$. It may also be defined as

$$\mathfrak{A}\ R\ \mathfrak{B} \iff \forall A \in \mathfrak{A}, B \in \mathfrak{B}\ \exists x \in A, y \in B. x\ R\ y$$

(II) The double powerset functor does not preserve weak pullbacks. This is proven at the end of the appendix of [36], and with a simpler example in [13, p. 6.6].

Chapter 3

Linear bicategories and linear logic

In the following chapter, we will provide the background on linear logic and linear bicategories necessary for our research on linear extensions in subsequent chapters.

Remark 3.0.1 (Notation). In the literature on lattices, monoidal topology, as well as linear logic, different notations are used for the units. In the case of bounded lattices X, sources such as [6] use $\mathsf{0}$ to denote the minimum element of a lattice and $\mathsf{1}$ is used to denote the maximum element. This conflicts with other usage for these symbols such as in bicategory theory where $\mathsf{1}$ or I denotes the unit in a bicategory (as in [5]), or monoidal topology in which the Lawvere quantale has $\mathsf{0}$ as its maximum element. [25] instead uses $\bot, \top$ to denote bottom and top elements of a lattice and k for the $\otimes$ identity of a quantale. However, this in turn conflicts with the usage of $\bot, \top$ in linearly distributive categories and linear bicategories in [14, 16] where $\bot, \top$ represent the identity elements with respect to $\oplus, \otimes$.

We have chosen to resolve this conflict of notation by maintaining the same notation as [25], and choosing new symbols for the identity arrows for $\otimes, \oplus$ which have become E, D respectively. The letter 'E' is familiar as an identity in many contexts, whereas 'D' often takes the place of a cyclic dualizing element, becoming the identity element of the $\oplus$ in a Girard quantale and a Girard bicategory as in [9].

3.1 Linear bicategories

In short, a linear bicategory **B** as defined in [14, Definition 2.1] is a bicategory possessing a second bicategory structure with respect to a second bicategorical composition operation $\oplus$ such that the two bicategorical compositions satisfy a special condition known as *linear distributivity*, as we shall describe below. The main example of a linear bicategory comes from the category **Rel** of sets and relations, whose bicategory structure is described in example 2.1.2.III. **Rel** furthermore may be imbued with a linear bicategorical structure as given in [14, §2.3]. The operation $\otimes$ is ordinary composition of relations, and $\oplus$ is the dual composition, defined as follows. Let $R\colon X \nrightarrow Y, S\colon Y \nrightarrow Z$ be relations, then:

$$R \otimes S = \{(x,z) \in X \times Z : \exists y \in Y.(x\, R\, y \wedge y\, S\, z)\} \tag{3.1.e1}$$

$$R \oplus S := \{(x,z) \in X \times Z : \forall y \in Y.(x\, R\, y \vee y\, S\, z)\}. \tag{3.1.e2}$$

We are mainly concerned with the linear bicategorical structure of $\mathcal{Q}$-**Rel** when $\mathcal{Q}$ is a linearly distributive quantale, which is locally posetal in general. Thus we shall focus on the simplified case when the linear bicategory is locally posetal. In the following definition we adapt linear bicategories for the locally posetal case, as we have done for bicategories in section 2.1. We base our definition on the original paper [14] as well as the definition from [9] by Blute et al., which runs closer to the definition of bicategory as in [5]. Our definition differs most from [9] in condition 3.1.1.iii, in which we describe and label the bicategories $\mathbf{B}_\otimes, \mathbf{B}_\oplus$ as part of our definition in order to retrieve the associators and unitors as included in [14].

Definition 3.1.1. A *linear bicategory* $\mathbf{B}$ is comprised of the following data:

i) A collection of objects $\mathbf{B_0}$, or 0-cells.

ii) A category $\mathbf{B}(X, Y)$ assigned to each ordered pair (X, Y) of 0-cells. We call the objects f, g of $\mathbf{B}(X, Y)$ 1-cells which are considered to have domain X and codomain Y, written $f, g \colon X \to Y$. We call the morphisms α between 1-cells f, g of $\mathbf{B}(X, Y)$ 2-cells, written using a double arrow as in $\alpha \colon f \Rightarrow g$. In the locally posetal case $\mathbf{B}(X, Y)$ is a poset with partial ordering $\leq_{X,Y}$. The collection of all 1-cells will be denoted $\mathbf{B_1}$.

iii) Two bicategory structures sharing the collection B_0 of 0-cells given above: The first bicategory, denoted $\mathbf{B}_{\otimes}$, has the category assignment $\mathbf{B}_{\otimes}(X, Y) := \mathbf{B}(X, Y)$. The second bicategory, denoted $\mathbf{B}_{\oplus}$, assigns $\mathbf{B}_{\oplus}(X, Y) := \mathbf{B}(X, Y)^{\mathrm{op}}$, the dual category to the above. The composition operation in $\mathbf{B}_{\otimes}$ is called "tensor" and denoted by $\otimes$, while the composition operation in $\mathbf{B}_{\oplus}$ is called "cotensor" (or "par") and denoted $\oplus$. The respective families of identity arrows in $\mathbf{B}_{\otimes}$, and $\mathbf{B}_{\oplus}$ are denoted E, D. Furthemore the respective associators and unitors are $a_{\otimes}, \ell_{\otimes}, r_{\otimes}$, and $a_{\oplus}, \ell_{\oplus}, r_{\oplus}$.

iv) A relationship called "linear distributivity" is satisfied between the tensor and cotensor, which is specified by the following natural transformations:

$$\delta^{\mathrm{L}} \colon (1 \times \oplus) ; \otimes \to (\otimes \times 1) ; \oplus, \quad \text{and} \quad \delta^{\mathrm{R}} \colon (\oplus \times 1) ; \otimes \to (1 \times \otimes) ; \oplus. \tag{3.1.e3}$$

Thus, for all morphisms $A \colon X \to Y$, $B \colon Y \to Z$, $C \colon Z \to W$ there exist the following 2-cells:

$$\delta^{\mathrm{L}} \colon A \otimes (B \oplus C) \Rightarrow (A \otimes B) \oplus C \tag{3.1.e4}$$

$$\delta^{\mathrm{R}} \colon (A \oplus B) \otimes C \Rightarrow A \oplus (B \otimes C) \tag{3.1.e5}$$

These must satisfy the coherence conditions referenced in [14, p. 10], analagous to those conditions for the linear distributivities of linear distributive categories given in [16, Also Definition 2.1](therein referred to as "weakly distributive categories").

Linear bicategories, as with bicategories, require the satisfaction of several commutative diagrams, specified in [14, Definition 2.1]. However, in the locally posetal case these coherence diagrams commute trivially, and thus we may omit them from the above definition.

Examples 3.1.2.

(I) Every bicategory **B** forms a linear bicategory with $\otimes = \oplus$ [14, §2.3]. The linear distributivities are just the associator. Thus linear bicategories form a generalization of bicategories.

(II) If **B** is a linear bicategory, then $\mathbf{B}^{\mathrm{co}}$ and $\mathbf{B}^{\mathrm{op}}$ are linear bicategories [14, §2.1]. $\mathbf{B}^{\mathrm{co}}$ is the linear bicategory produced from **B** by interchanging $\otimes, \oplus$ and reversing the 2-cells (i.e replacing $\leq_{X,Y}$ with $\geq_{X,Y}$), and $\mathbf{B}^{\mathrm{op}}$ is the linear bicategory produced from **B** by reversing the direction of 1-cells and reversing the direction of composition.

(III) As outlined in [14, §2.3] and the beginning of this section, **Rel** is a locally posetal linear bicategory. Furthermore, as we show in example 3.2.2.III, $\mathcal{Q}$-**Rel** forms a locally posetal linear bicategory when $\mathcal{Q}$ is a linearly distributive quantale.

(IV) Linearly distributive categories, as described in [16, §2] (therein referred to as "weakly distributive categories") may be viewed as one object linear bicategories [14, Example 2.3 (4)]. This is essentially analagous to the relationship between monoidal categories and bicategories.

(V) A rich source of linearly distributive categories are $*$-autonomous categories, as described in [3, I§4] which may be conceived as monoidal categories in which every object A has a dual $A^{\perp}$.

(VI) Shift monoids are an example of discrete linearly distributive categories which are not necessarily $*$-autonomous, see [16, §5.1]. A shift monoid is a commutative monoid $(M, \otimes, e)$ possessing a distinguished invertible element $d \in M$ such that we may define a second monoidal operation $\oplus$ with unit d where $x \oplus y = x \otimes y \otimes d^{-1}$ for each $x, y \in M$. The associativity of $\otimes$ implies that $x \otimes (y \oplus z) \leq (x \otimes y) \oplus z$ and $(x \oplus y) \otimes z \leq x \oplus (y \otimes z)$, so that $\otimes, \oplus$ satisfies linear distributivity.

(VII) Another situation in which a linear bicategory arises is as in the transit map example of [Fig. 1. in 14, p. 7]. Imagine we are planning a day trip in downtown Ottawa going from place to place by various means of transportation. We will choose which places we would like to stop at and the route of travel based on the data of a graph such as fig. 3.1.

Nodes in fig. 3.1 represent locations, and arrows represent travel plans between domain and codomain nodes. An important consideration for us is the total time of our daytrip, and so we have marked down a travel time for each of the arrows in fig. 3.1 (separately we could instead consider other factors such as cost, as in [14, p. 7]). We may compose travel plans $A\colon X \to Y, B\colon Y \to Z$, with total times a, b respectively by writing $A \otimes B\colon X \to Z$ which denotes the trip taken by first taking trip A, followed by trip B, with total time $a + b$. An arrow between travel plans sharing domain and codomain represents a comparison in the total times. There is a second composition, denoted $\oplus$ in which $A \oplus B$ represents the trip by first taking A, followed by a stop at Y with recommended time¸, then followed by the trip B, with total time $a+b+$¸. This is just like the shift addition of a commutative monoid example 3.1.2.VI, except

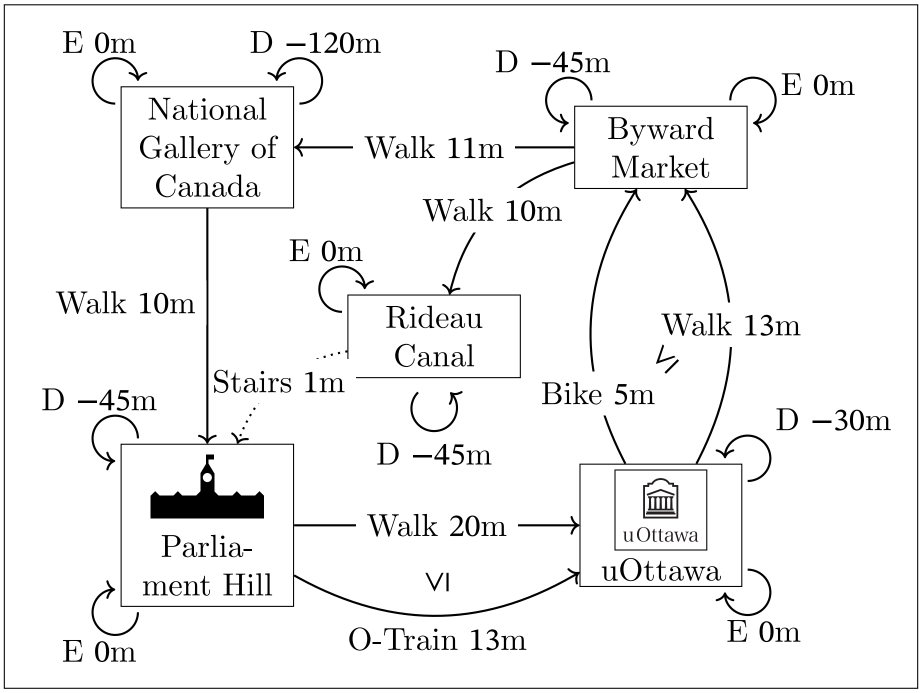


Figure 3.1: Downtown Ottawa and Travel Times

that the "shift" is determined by the object Y. One last piece of information are the arrows E, D, E_Y represents an instantaneous trip at Y which no time passes and no distance is travelled, while D_Y represents a trip undoing a stopover at Y with total time $-_{\varsigma}$. D, E become identity elements with respect to the compositions $\otimes, \oplus$, furthermore $\otimes, \oplus$ are monotone. Thus the graph fig. 3.1 is closed under these operations and forms a linear bicategory which is locally ordered.

(VIII) Example 3.1.2.VII is a generalization of a shift addition on $\mathbb{R}$, wherein the shift element changes depending on the interceding object Y in the composition $\oplus_{X,Y,Z}$. This generalizes to a monoidal preorder $(\mathcal{M}, \otimes, \mathrm{E})$ with shifted tensor in place of $\mathbb{R}$ and shift addition, as follows. Let $\mathbf{B}_0$ be some collection of objects, and let $\bar{\mathbf{B}}_1$ be a collection of arrows (as in fig. 3.1) each assigned a domain and codomain in $\mathbf{B}_0$. Let $\bar{\phi}$ be an assignment $\bar{\mathbf{B}}_1 \to \mathsf{ob}(\mathcal{M})$ of a quantity in $\mathcal{M}$ (such as distance, time as in example 3.1.2.VII) to each arrow and let ψ be an assignment from $\mathbf{B}_0$ to the tensor invertible elements of $\mathcal{M}$ for use in the shift monoid operation, with tensor inverse $(\psi(X)^{-1}, s^L_{\psi(X)}, s^R_{\psi(Y)})$ as in [16, §5.2]. Each tensor invertible element in $\mathcal{M}$ introduces a linearly distributive categorical structure on $\mathcal{M}$ via the shifted tensor operation $\oplus$ of [16, Proposition 5.3], which we shall use to define a linear bicategory.

Let $\mathbf{B}_1$ be the arrows from $\bar{\mathbf{B}}_1$ closed under formal composition via operations $\otimes, \oplus$ union the $\mathbf{B}_0$-indexed families E, D of arrows where $\mathrm{E}_X : X \to X, \mathrm{D}_X : X \to X$ are arrows with domain and codomain equal to X for each $X \in \mathbf{B}_0$. We let $\phi : \mathbf{B}_1 \to \mathcal{M}$ be the map generated inductively from $\bar{\phi}$ such that $\phi(A) = \bar{\phi}(A)$ for $A \in \bar{\mathbf{B}}_1$, $\phi(\mathrm{E}_X) = \mathrm{E}$, $\phi(\mathrm{D}_X) = \psi(X)$, and $\phi(A \otimes B) := \phi(A) \otimes_{\mathcal{M}} \phi(B)$ and $\phi(A \oplus B) = \phi(A) \otimes_{\mathcal{M}} (\psi(Y)^{-1} \otimes_{\mathcal{M}} \phi(B))$ for composable $A : X \to Y, B : Y \to Z \in \mathbf{B}_1$. We may now define a category $\mathbf{B}(X, Y)$ as the collection of arrows in $\mathbf{B}_1$ posessing domain X and codomain Y, with 2-cells between A, B as in $(\mathbf{B}(X, Y))(A, B) = \mathcal{M}(\phi(A), \phi(B))$, and composition as in $\mathcal{M}$. Then the formal operations $\otimes, \oplus$ are associative unital operations forming two bicategorical structures. The associators and unitors may be assigned to the associators and unitors in $\mathcal{M}$ with $a_{\otimes A,B,C} = a_{\phi(A),\phi(B),\phi(C)}$, $\ell_{\otimes A} = \ell_{\phi(A)}$, and $r_{\otimes A} = r_{\phi(A)}$. The associator for the $\oplus$ components are defined by composing the necessary $\otimes$ associators $\bar{a_{\oplus}} = a_{\otimes} ; 1 \otimes a_{\otimes}$ as in [16, §5.2] to make arrows $a_{\oplus A,B,C} : (A \otimes (\psi(Y)^{-1} \otimes B)) \otimes (\psi(Z)^{-1} \otimes C) \to A \otimes (\psi(Y)^{-1} \otimes (B \otimes (\psi(Z)^{-1} \otimes C)))$, and $\bar{\ell_{\oplus}}, \bar{r_{\oplus}}$ are simlarly formed as defined in [16, §5.2]. These associators and unitors form bicategories $\mathbf{B}_{\otimes}$ and $\bar{\mathbf{B}_{\oplus}}$ satisfying coherence and naturality trivially since all diagrams are diagrams in the preordered category $\mathcal{M}$. Construct $\mathbf{B}_{\oplus} := \bar{\mathbf{B}_{\oplus}}^{\mathrm{co}}$ as given by the dual construction as described in section 2.1 to form a bicategory structure on $\mathbf{B}(X, Y)^{\mathrm{op}}$ as required by condition 3.1.1.iii. The pair $\mathbf{B}_{\otimes}, \mathbf{B}_{\oplus}$ additionally satisfies linear distributivity defining the $\delta^{\mathrm{L}}, \delta^{\mathrm{R}}$ as they are in $\mathcal{M}$.

3.2 Linearly distributive quantales

Linearly distributive quantales, or LD-quantales, as introduced in [9, §4.1], are the quantale analogue of *linearly distributive categories* (as described in [16], referred to as *weakly distributive categories*), a model for linear logic in which the $\otimes$ and $\oplus$ are primitive. LD-quantales are used in [9] to introduce linear bicategorical structure on the category $\mathcal{Q}$-**Rel**, which will be the target for linear extensions of **Set** functors.

Definition 3.2.1. An LD-quantale is a complete lattice $\mathcal{Q}$ with associative binary operations $\otimes\colon \mathcal{Q}\times\mathcal{Q}\to\mathcal{Q}$ and $\oplus\colon \mathcal{Q}^{\mathrm{op}}\times\mathcal{Q}^{\mathrm{op}}\to\mathcal{Q}^{\mathrm{op}}$, with respective units $k_\otimes$,$k_\oplus$ such that $(\mathcal{Q},\otimes,k_\otimes)$, $(\mathcal{Q}^{\mathrm{op}},\oplus,k_\oplus)$ are unital quantales and for all $a,b,c\in\mathcal{Q}$ the following linear distributivities hold:

$$\delta^L\colon\ a\otimes(b\oplus c)\le(a\otimes b)\oplus c \tag{3.2.e1}$$

$$\delta^R\colon\ (a\oplus b)\otimes c\le a\oplus(b\otimes c) \tag{3.2.e2}$$

Examples 3.2.2.

(I) As described in [9, §4.10], one may construct an LD-quantale $\bar{M}$ from a cancellative shift monoid $(M,\otimes,e,d)$. The d-shifted tensor $\oplus$ inherits cancellativity from $\otimes$ so that we may construct a pair of quantales $\bar{M}_\otimes=(\bar{M},\bar{\otimes},e)$, and $\bar{M}_\oplus=(\bar{M}^{\mathrm{op}},\bar{\oplus},d)$ as outlined in example 1.2.2.VI. As noted in example 3.1.2.VI, the operations $\otimes,\oplus$ satisfy linear distributivity, and hence $\bar{M}_\otimes,\bar{M}_\oplus$ forms an LD-quantale.

(II) Any LD-quantale (any linearly distributive category for that matter [14, §2.3(4)])$\mathcal{Q}$ may be viewed as a locally posetal linear bicategory with a single 0-cell $*$, 1-cells as the elements of $\mathcal{Q}$, and 2-cells as the ordering in $\mathcal{Q}$.

(III) From [9], when $(\mathcal{Q}, \otimes, \oplus)$ is an LD-quantale, $\mathcal{Q}$-**Rel** forms a linear bicategory with the following compositions. Let $R\colon X \nrightarrow Y$, $S\colon Y \nrightarrow Z$:

$$R \otimes S(x,z) = \bigvee_{y' \in Y} R(x,y') \otimes S(y',z), \quad \text{and} \quad R \oplus S(x,z) = \bigwedge_{y' \in Y} R(x,y') \oplus S(y',z). \tag{3.2.e3}$$

This composition has the following identities:

$$\mathrm{E}(x,x') = \begin{cases} k_{\otimes} & \text{if } x = x' \\ \bot & \text{otherwise} \end{cases}, \quad \text{and} \quad \mathrm{D}(x,x') = \begin{cases} k_{\oplus} & \text{if } x = x' \\ \top & \text{otherwise} \end{cases} \tag{3.2.e4}$$

Furthermore, $\mathcal{Q}$-**Rel** with $\otimes, \oplus$ as defined in eq. (3.2.e3) is a linear bicategory if and only if $(\mathcal{Q}, \otimes, \oplus)$ is an LD-quantale. See [9, §4.6] for details.

One important example of LD-quantales are *Girard quantales*. Girard quantales as defined by Yetter in [40, Definition 1.6] are a type of quantale with a notion of duality, as in a version of Girard's linear logic termed "cyclic linear logic". That is a version of noncommutative linear logic where terms may be rearranged only up to cyclic permutation. We adapt the following definition from the chapter on Girard quantales in [34, Chapter 6]

Definition 3.2.3. A quantale $\mathcal{Q}$ is called *Girard* provided there exists an element $d \in \mathcal{Q}$ such that the following holds for all $a, b \in \mathcal{Q}$:

$$a \multimap d = d \multimapinv a \qquad (d \text{ is cyclic}) \tag{3.2.e5}$$

$$(a \multimap d) \multimap d = a \qquad (d \text{ is dualizing}) \tag{3.2.e6}$$

We denote the linear negation of an element $a \in \mathcal{Q}$ as $a^{\bot} := a \multimap d$. As stated in [34, p. 140], Girard quantales may be regarded as non-commutative posetal examples of Barr's $*$-autonomous categories as described in the book [3, I§4]. Girard quantales are unital with unit k determined by $k = d^{\bot}$.

Examples 3.2.4.

(I) The unique 2 element quantale $2 = \{\bot, \top\}$ is a trivial example of a Girard quantale with binary operation $\wedge$ and cyclic dualizing element $\bot$, such that $a^{\bot} = \neg a$ for each $a \in 2$.

(II) The extended real line $\bar{\mathbb{R}} = \mathbb{R} \sqcup \{-\infty, \infty\}$ equipped with addition $+$ is a Girard quantale with cyclic dualizing element 0 making $a^{\bot} = -a$ for each $a \in \bar{\mathbb{R}}$. The subquantale $\mathbb{Z}_{\infty} = \mathbb{Z} \sqcup \{-\infty, \infty\}$ of extended integers is also a Girard quantale. [9, §1].

A Girard quantale is an LD-quantale with the standard construction $a \oplus b = a^{\bot} \multimap b$ [16, §4.5]. This is handled in full generality with respect to linearly distributive categories in [8], which shows that one may form a linearly distributive category using a proof net construction from a (not necessarily symmetric) $*$-autonomous category $\mathcal{A}$, so that $\mathcal{A}$ must satisfy the linear distributivity axioms $\delta^{\mathrm{R}}, \delta^{\mathrm{L}}$.

3.3 Background on Girard bicategories

Girard bicategories, as introduced in [9] are a bicategorical version of Barr's $*$-autonomous categories [see 3, I§4]. One may consider them as a linear bicategory with a notion of involutive negation. In reference to this thesis, one may consider their relationship with linear bicategories as analagous to the relationship between linearly distributive quantales and Girard quantales. The main motivating example of a Girard bicategory in connection to linear extension is $\mathcal{Q}$-**Rel** which is a Girard bicategory (more specifically, a *Girard quantaloid*) if and only if $\mathcal{Q}$ is a Girard quantale [9, Proposition 2.6]. In this section we shall introduce the definition of Girard bicategory and properties which we will use in the main theorem of section 5.3. We refer the reader to [9, §5] for a comprehensive introduction as well as historical background.

Definition 3.3.1 (See [9, Definition 5.6]). Let **B** be a biclosed bicategory. **B** is a *Girard bicategory* provided there exists an X-indexed family of 1-cells $\mathcal{D}_X : X \to X$ which is *cyclic* and *dualizing*. That is, for each 1-cell $A : X \to Y$, the 2-cells, as determined by extension and lifting (see definition 2.1.3), are invertible ($\mathcal{D}$ is dualizing),

$$\delta_{X,A} : A \Rightarrow (\mathcal{D}_Y \circ\!- A) -\!\circ \mathcal{D}_Y \tag{3.3.e1}$$

furthermore, there exists a family of 2-cells $\theta_A : (\mathcal{D}_Y \circ\!- A) \Rightarrow (A -\!\circ \mathcal{D}_X)$, natural in A, and such that the diagram from [9, Definition 5.5] commutes ($\mathcal{D}$ is cyclic).

We denote the linear negation of a 1-cell A as $A^\perp$ where $A^\perp := A -\!\circ \mathcal{D}_X$. Note that one may construct a 2-cell $\delta_{X,A} ; (\theta_A -\!\circ \mathcal{D}_Y)$ from A to $(A^\perp)^\perp$, which is invertible since $\delta_{X,A}$ and θ_A are invertible. A Girard bicategory posesses linear bicategorical structure where the second composition is defined $A \oplus B := (B^\perp \otimes A^\perp)^\perp$ and $\mathrm{D}_X = \mathcal{D}_X$ [9, Theorem 5.13].

In the main theorem of section 5.3 we shall use the following lemma from [9].

Lemma 3.3.2 (See [9, Lemma §5.12]). *If* **B** *is a Girard bicategory, then for each pair of 1-cells* $A : X \to Y$, $B : Y \to Z$ *there exists an invertible 2-cell* $A^\perp -\!\circ B \to (B^\perp \otimes A^\perp)^\perp$.

In other words, the above lemma states that in a Girard bicategory $A^\perp -\!\circ B \cong A \oplus B$. Note that dually for the right lifting we should have $A \circ\!- B^\perp \cong A \oplus B$. We shall end this section with some examples of a Girard Bicategory.

Examples 3.3.3.

(I) As stated in the beginning of this section, the quantaloid $\mathcal{Q}$-**Rel** is a Girard bicategory if and only if $\mathcal{Q}$ is a Girard quantale.

(II) A Girard quantale $\langle \mathcal{Q}, \otimes, d \rangle$, as defined in definition 3.2.3, is an example of a one object Girard bicategory **B** with $\mathbf{B}_0 = \{*\}$, $\mathbf{B}(*,*) = \mathcal{Q}$, and cyclic dualizing family $\mathcal{D}$ with $\mathcal{D}_* = d$. More generally, a $*$-autonomous category is an example of a one object Girard bicategory [9, Example 5.7]

3.4 Linear functors

In the following definition we introduce the notion of linear functor adapted from [14] for locally posetal linear bicategories.

Definition 3.4.1. A *linear functor* $\vec{F}$ between locally posetal linear bicategories $\mathbf{B}, \mathbf{C}$ is comprised of a triple $(\vec{F}, \vec{F}_{\otimes}, \vec{F}_{\oplus})$ where, when applied to 0-cells of $\mathbf{B}$, $\vec{F}: \mathbf{B}_0 \to \mathbf{C}_0$ is a map, and $\vec{F}_{\otimes}, \vec{F}_{\oplus}$ are collections of functors (monotone maps) indexed by pairs of 0-cells $X, Y \in \mathbf{B}_0$.

$$\vec{F}_{\otimes X,Y}: \mathbf{B}(X, Y) \to \mathbf{C}(\vec{F}X, \vec{F}Y), \text{ and} \tag{3.4.e1}$$

$$\vec{F}_{\oplus X,Y}: \mathbf{B}(X, Y)^{\mathrm{op}} \to \mathbf{C}(\vec{F}X, \vec{F}Y)^{\mathrm{op}}. \tag{3.4.e2}$$

The subscripts X, Y may be omitted when they are clear from context. The triple $(\vec{F}, \vec{F}_{\otimes}, \vec{F}_{\oplus})$ must satisfy the following. Let $R, R' \in \mathbf{B}(X, Y), S \in \mathbf{B}(Y, Z)$:

i) Lax preservation of composition

$$\vec{F}_{\otimes X,Y}R \otimes_{\mathbf{C}} \vec{F}_{\otimes Y,Z}S \leq \vec{F}_{\otimes X,Z}(R \otimes_{\mathbf{B}} S), \tag{3.4.e3}$$

and oplax preservation of composition

$$\vec{F}_{\oplus X,Z}(R \oplus_{\mathbf{B}} S) \leq \vec{F}_{\oplus X,Y}R \oplus_{\mathbf{C}} \vec{F}_{\oplus Y,Z}S; \tag{3.4.e4}$$

ii) lax preservation of identities

$$\mathrm{E}_{\vec{F}X} \leq \vec{F}_{\otimes X,X}\mathrm{E}_X \tag{3.4.e5}$$

and oplax preservation of identities

$$\vec{F}_{\oplus X,X}\mathrm{D}_X \leq \mathrm{D}_{\vec{F}X}; \tag{3.4.e6}$$

iii) finally, $\vec{F}_{\otimes}, \vec{F}_{\oplus}$ satisfy the following 4 inequalities, known as *linear strengths*:

$$\nu_{\otimes}^{\mathrm{R}}: \ \vec{F}_{\otimes X,Z}(R \oplus R') \leq \vec{F}_{\oplus X,Y}(R) \oplus \vec{F}_{\otimes Y,Z}(R'); \tag{3.4.e7}$$

$$\nu_{\otimes}^{\mathrm{L}}: \ \vec{F}_{\otimes X,Z}(R \oplus R') \leq \vec{F}_{\otimes X,Y}(R) \oplus \vec{F}_{\oplus Y,Z}(R'); \tag{3.4.e8}$$

$$\nu_{\oplus}^{\mathrm{R}}: \ \vec{F}_{\otimes X,Y}(R) \otimes \vec{F}_{\oplus Y,Z}(R') \leq \vec{F}_{\oplus X,Z}(R \otimes R'); \tag{3.4.e9}$$

$$\nu_{\oplus}^{\mathrm{L}}: \ \vec{F}_{\oplus X,Z}(R) \otimes \vec{F}_{\otimes X,Y}(R') \leq \vec{F}_{\oplus Y,Z}(R \otimes R'). \tag{3.4.e10}$$

One also says that the pair $(\vec{F}_{\otimes}, \vec{F}_{\oplus})$ has linear strength, provided it satisfies each of $\nu_{\otimes}^{\mathrm{R}}, \nu_{\otimes}^{\mathrm{L}}, \nu_{\oplus}^{\mathrm{R}}, \nu_{\oplus}^{\mathrm{L}}$ (see condition 3.4.1.iii above). A linear functor may be understood as a pair of lax functors, as we define in the following proposition.

Proposition 3.4.2. *A* linear functor $\vec{F} : \mathbf{B} \to \mathbf{C}$ *between locally posetal linear bicategories is comprised of a pair* $(\hat{F}, \check{F})$ *where* $\hat{F} : \mathbf{B}_{\otimes} \to \mathbf{C}_{\otimes}$ *is a lax functor with respect to* $\otimes$ *composition,* $\check{F} : \mathbf{B}_{\oplus} \to \mathbf{C}_{\oplus}$ *is a colax functor with respect to* $\oplus$ *composition, such that the pair share the same underlying map on 0-cells, and satisfies the 4 linear strengths in condition 3.4.1.iii.*

Proof: Let $(\hat{F}, \check{F})$ be a pair as in the above. We will denote F as the underlying map on 0-cells shared by $\hat{F}, \check{F}$. By definition, $\hat{F}$ is a pair $(F, \hat{F})$ in which $\hat{F}$ also denotes the underlying collection of X, Y-indexed monotone maps $\hat{F}_{X,Y} : \mathbf{B}(X, Y) \to \mathbf{C}(FX, FY)$. Furthermore, by definition $\check{F}$ is a pair $(F, \check{F})$ in which $\check{F}$ denotes the underlying collection of X, Y-indexed monotone maps $\check{F}_{X,Y} : \mathbf{B}(X, Y) \to \mathbf{C}(FX, FY)$. Thus $(F, \hat{F}, \check{F})$, denoted $\vec{F}$, is a triple as in definition 3.4.1. By definition the collection $\hat{F}$ satisfies lax preservation of composition and lax preservation of units, and $\check{F}$

satisfies oplax preservation of composition and oplax preservation of units, so that $\vec{F}$ satisfies conditions 3.4.1.i and 3.4.1.ii. Furthermore, it was an initial assumption that $\hat{F}$ and $\check{F}$ satisfy condition 3.4.1.iii, hence $(F, \hat{F}, \check{F})$ forms a linear functor. ∎

One may obtain a linear functor from a linear extension by proposition 4.1.2, therefore one may see examples 4.1.3 for examples of linear functors.

3.5 Linear adjoints

In general, linear bicategories do not possess a negation operator. However, there is an elegant categorical notion which we can use to describe negation. These are *linear adjunctions*. In the following, we will adapt linear adjunctions from [14, §3.1] in the setting of locally posetal linear bicategories.

Definition 3.5.1. Let $\mathbf{B}$ be a locally posetal linear bicategory. A left linear adjunction in $\mathbf{B}$ is a pair of 1-cells $A\colon X \to Y, B\colon Y \to X$ in $\mathbf{B}_1$ such that

$$\mathrm{E}_X \le A \oplus B, \quad \text{and} \quad B \otimes A \le \mathrm{D}_Y. \tag{3.5.e1}$$

In this case we say that A is a left linear adjoint to B, and B is a right linear adjoint to A, and write $A \dashv\!\!| B$.

Any left (resp. right) linear adjoints are unique up to isomorphism [14, Cor. 3.4], and therefore one may say that A is *the* left linear adjoint to B. Furthermore, in a locally posetal linear bicategory, A and B are called *cyclic* linear adjoints if and only if $A \dashv\!\!| B$ and $B \dashv\!\!| A$. We denote this as $A \,|\!\!\vdash\!\!\dashv\!\!|\, B$.

One key fact of linear functors is that they preserve linear adjoints in the following sense:

Proposition 3.5.2. *Let* $\mathbf{B}, \mathbf{C}$ *be locally posetal linear bicategories,* $A\colon X \to Y, B\colon Y \to$

$X \in \mathbf{B}_1$, and $\vec{F}: \mathbf{B} \to \mathbf{C}$. If $A \dashv\!\!\mid B$, then $\vec{F}_{\oplus}A \dashv\!\!\mid \vec{F}_{\otimes}B$, and $\vec{F}_{\otimes}A \dashv\!\!\mid \vec{F}_{\oplus}B$.

This is a fact mentioned in [14, §2.1], for which we include a proof here.

Proof:

$$
\begin{aligned}
& \mathrm{E}_X \leq A \oplus B \\
\Longrightarrow\ & \vec{F}(\mathrm{E}_X) \leq \vec{F}_{\otimes}(A \oplus B) && \text{(by monotonicity of } \vec{F}\text{)} \\
\Longrightarrow\ & \mathrm{E}_{\vec{F}X} \leq \vec{F}_{\otimes}(A \oplus B) && \text{(by lax preservation of identities)} \\
\Longrightarrow\ & \mathrm{E}_{\vec{F}X} \leq \vec{F}_{\oplus}A \oplus \vec{F}_{\otimes}B, \text{ and} && \text{(by } \nu_{\otimes}^{\mathrm{R}}\text{)} \\
& \mathrm{E}_{\vec{F}X} \leq \vec{F}_{\otimes}A \oplus \vec{F}_{\oplus}B. && \text{(by } \nu_{\otimes}^{\mathrm{L}}\text{)}
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
& \mathrm{D}_X \geq A \oplus B \\
\Longrightarrow\ & \vec{F}(\mathrm{D}_X) \geq \vec{F}_{\oplus}(A \oplus B) && \text{(by monotonicity)} \\
\Longrightarrow\ & \mathrm{D}_{\vec{F}X} \geq \vec{F}_{\oplus}(A \oplus B) && \text{(by oplax preservation of identities)} \\
\Longrightarrow\ & \mathrm{D}_{\vec{F}X} \geq \vec{F}_{\otimes}A \oplus \vec{F}_{\oplus}B, \text{ and} && \text{(by } \nu_{\oplus}^{\mathrm{R}}\text{)} \\
& \mathrm{D}_{\vec{F}X} \geq \vec{F}_{\oplus}A \oplus \vec{F}_{\otimes}B. && \text{(by } \nu_{\oplus}^{\mathrm{L}}\text{)}
\end{aligned}
$$

∎

Linear adjoints have the useful property that they are unique (up to isomorphism in the general case), which we phrase as a proposition:

Proposition 3.5.3. *If $A \dashv\!\!\mid B$ and $A \dashv\!\!\mid B'$, then $B = B'$.*

The above fact is pointed out in [14, Corollary 3.4] as a consequence of linear "mates" (see [14, Lemma 3.3]).

Examples 3.5.4.

In **Rel**, each relation $R : X \nrightarrow Y$ is left linear adjoint to $R^{\perp}$ the inverse complement of R. Recall, this is defined as follows:

$$(y,x) \in R^{\perp} \iff (x,y) \notin R \tag{3.5.e2}$$

In **Rel**, $R \oplus R^{\perp}(x,x')$ holds if and only if for all $y \in Y$ $(x,y) \in R$ or $(y,x) \in R^{\perp}$, but by eq. (3.5.e2) this statement is a tautology, so that $\mathrm{D}_X \leq R \oplus R^{\perp}$. $R^{\perp} \otimes R \leq \mathrm{E}_Y$ holds dually.

More generally, the canonical embedding $R_{\circ}$ of a relation $R\colon X \nrightarrow Y$ into $\mathcal{Q}$-**Rel** is a linear adjoint by lemma 6.1.3.

When $\mathcal{Q}$ is a Girard quantale, each one cell $A\colon X \nrightarrow Y$ has a left linear adjoint $\dashv\!\mid A$ in $\mathcal{Q}$-**Rel** where for each $x \in X, y \in Y$, $A^{\perp}(x,y) = A(x,y)^{\perp}$ [9, Lemma 4.23].

3.6 Composition of linear functors

Let $\mathbf{B}, \mathbf{C}, \mathbf{D}$ be locally posetal linear bicategories and let $\mathbf{Lin(B,C)}$ denote the linear functors from $\mathbf{B}$ to $\mathbf{C}$. We shall define composition of linear functors as a map $(;)\colon \mathbf{Lin(B,C)} \times \mathbf{Lin(C,D)} \to \mathbf{Lin(B,D)}$ taking a pair of linear functors $\vec{F}, \vec{G}$ to the obvious composite linear functor $\vec{F};\vec{G}$ which we shall define with the following data.

$$\begin{gathered}(\vec{F}\,;\vec{G})X := \vec{G}(\vec{F}X), \\ (\vec{F}\,;\vec{G})_{\otimes_{X,Y}}(f) = \vec{G}_{\otimes}(\vec{F}_{\otimes}f) \quad \text{and} \quad (\vec{F}\,;\vec{G})_{\oplus_{X,Y}}(f) = \vec{G}_{\oplus}(\vec{F}_{\oplus}f)\end{gathered}. \tag{3.6.e1}$$

Proposition 3.6.1. *Let $\vec{F}\colon \mathbf{B} \to \mathbf{C}, \vec{G}\colon \mathbf{C} \to \mathbf{D}$ be linear functors, then $\vec{F}\,;\vec{G}$ is a linear functor, as defined in eq.* (3.6.e1)*.*

Proof: Let X, Y, Z be 0-cells of $\mathbf{B}$, and $R, R'\colon X \to Y, S\colon Y \to Z$ be 1-cells of $\mathbf{B}$. Since the composite of two monotone maps is a monotone map, $R \leq R'$ implies that $(\vec{F}\,;\vec{G})_{\otimes}(R) = \vec{G}_{\otimes}(\vec{F}_{\otimes}R) \leq \vec{G}_{\otimes}(\vec{F}_{\otimes}R') = (\vec{F}\,;\vec{G})_{\otimes}(R')$ so that $(\vec{F}\,;\vec{G})_{\otimes}$ is monotone. This continues to hold replacing the subscript $\otimes$ with $\oplus$ in the above so that $(\vec{F}\,;\vec{G})_\oplus$ is also monotone.

By the lax preservation of composition and monotonicity of $\vec{F},\vec{G}$ we have

$$\vec{G}_\otimes(\vec{F}_\otimes(R \otimes_{\mathbf{B}} S)) \tag{3.6.e2}$$
$$\leq \vec{G}_\otimes(\vec{F}_\otimes R \otimes_{\mathbf{C}} \vec{F}_\otimes S) \tag{3.6.e3}$$
$$\leq \vec{G}_\otimes(\vec{F}_\otimes R) \otimes_{\mathbf{D}} \vec{G}_\otimes(\vec{F}_\otimes S) \tag{3.6.e4}$$

Which is $(\vec{F}\,;\vec{G})_\otimes(R) \otimes_{\mathbf{D}} (\vec{F}\,;\vec{G})_\otimes(S)$. Similarly, by monotonicity and oplax preservation of composition of $\vec{F},\vec{G}$ we have

$$\vec{G}_\oplus(\vec{F}_\oplus(R \oplus_{\mathbf{B}} S)) \tag{3.6.e5}$$
$$\geq \vec{G}_\oplus(\vec{F}_\oplus R \oplus_{\mathbf{C}} \vec{F}_\oplus S) \tag{3.6.e6}$$
$$\geq \vec{G}_\oplus(\vec{F}_\oplus R) \oplus_{\mathbf{D}} \vec{G}_\oplus(\vec{F}_\oplus S) \tag{3.6.e7}$$

Which is $(\vec{F}\,;\vec{G})_\oplus(R) \oplus_{\mathbf{D}} (\vec{F}\,;\vec{G})_\oplus(S)$. Together the above implies that $\vec{F}\,;\vec{G}$ satisfies lax and oplax preservation of composition. Next we want to show lax and oplax preservation of identities. By monotonicity and lax preservation of identities of $\vec{F},\vec{G}$ we have

$$(\vec{F}\,;\vec{G})_\otimes(\mathrm{E}_X) \geq \vec{G}_\otimes(\vec{F}_\otimes(\mathrm{E}_X)) \tag{3.6.e8}$$
$$\geq \vec{G}_\otimes(\mathrm{E}_{\vec{F}X}) \tag{3.6.e9}$$
$$\geq \mathrm{E}_{\vec{G}(\vec{F}X)} \tag{3.6.e10}$$

Which is $\mathrm{E}_{(\vec{F};\vec{G})X}$. Additionally, by monotonicity and oplax preservation of identities of $\vec{F},\vec{G}$ we have

$$\vec{G}_{\oplus}(\vec{F}_{\oplus}(\mathrm{D}_X)) \leq \vec{G}_{\oplus}(\mathrm{D}_{\vec{F}X}) \tag{3.6.e11}$$

$$\leq \mathrm{D}_{\vec{G}(\vec{F}X)} \tag{3.6.e12}$$

Which is $\mathrm{D}_{(\vec{F};\vec{G})X}$, so that $\vec{F}\,;\vec{G}$ laxly and oplaxly preserves identities.

What remains to be shown is that $\vec{F}\,;\vec{G}$ satisfies the 4 linear strengths of condition 3.4.1.iii. Let $R \in \mathbf{B}(X,Y),\ S \in \mathbf{B}(Y,Z)$.

$$(\vec{F}\,;\vec{G})_{\otimes}(R \oplus S) \leq \vec{G}_{\otimes}(\vec{F}_{\otimes}(R \oplus S)) \tag{3.6.e13}$$

$$\leq \vec{G}_{\otimes}(\vec{F}_{\oplus}R \oplus \vec{F}_{\otimes}S) \qquad \text{By } \nu_{\otimes}^{\mathrm{R}} \text{ of } \vec{F} \tag{3.6.e14}$$

$$\leq \vec{G}_{\oplus}(\vec{F}_{\oplus}R) \oplus \vec{G}_{\otimes}(\vec{F}_{\otimes}S) \qquad \text{By } \ \text{ of } \vec{G} \tag{3.6.e15}$$

Which is $(\vec{F}\,;\vec{G})_{\oplus}(R) \oplus (\vec{F}\,;\vec{G})_{\otimes}(S)$. The linear strength ${\nu_{\otimes}}^{L}$ follows similarly. Next we check ${\nu_{\oplus}}^{R}$

$$(\vec{F}\,;\vec{G})_{\oplus}(R \otimes S) \geq \vec{G}_{\oplus}(\vec{F}_{\oplus}(R \otimes S)) \tag{3.6.e16}$$

$$\geq \vec{G}_{\oplus}(\vec{F}_{\otimes}R \otimes \vec{F}_{\oplus}S) \qquad \text{By } {\nu_{\oplus}}^{R} \text{ of } \vec{F} \tag{3.6.e17}$$

$$\geq \vec{G}_{\otimes}(\vec{F}_{\otimes}R) \otimes \vec{G}_{\oplus}(\vec{F}_{\oplus}S) \qquad \text{By } {\nu_{\oplus}}^{R} \text{ of } \vec{G} \tag{3.6.e18}$$

And ${\nu_{\oplus}}^{L}$ follows similarly as well so that $\vec{F}\,;\vec{G}$ satisfies the 4 linear strengths. Together with the above, we conclude that $\vec{F}\,;\vec{G}$ is a linear functor. ∎

Proposition 3.6.2. *The operation* (;) *as defined in eq.* (3.6.e1) *is associative.*

Proof: Let $\vec{F}: \mathbf{B} \to \mathbf{C},\ \vec{G}: \mathbf{C} \to \mathbf{D},\ \vec{H}: \mathbf{D} \to \mathbf{E}$ be linear functors of locally posetal linear bicategories, $X, Y \in \mathbf{B}_0$, $f \in \mathbf{B}(X, Y)$. Then on 0-cells,

$$(\vec{F}\,;\vec{G})\,;\vec{H}(X) = \vec{H}(\vec{F}\,;\vec{G}(X)) \tag{3.6.e19}$$

$$= \vec{H}(\vec{G}(\vec{F}(X))) \tag{3.6.e20}$$

$$= \vec{G}\,;\vec{H}(\vec{F}(X)) \tag{3.6.e21}$$

$$= \vec{F}\,;(\vec{G}\,;\vec{H})(X) \tag{3.6.e22}$$

So that $(\vec{F};\vec{G});\vec{H}$ and $\vec{F};(\vec{G};\vec{H})$ are equal on 0-cells. Furthermore, as collections of monotone maps they are equal

$$((\vec{F}\,;\vec{G})\,;\vec{H})_{\otimes}(f) = \vec{H}_{\otimes}((\vec{F}\,;\vec{G})_{\otimes}(f)) \tag{3.6.e23}$$

$$= \vec{H}_{\otimes}(\vec{G}_{\otimes}(\vec{F}_{\otimes}f)) \tag{3.6.e24}$$

$$= (\vec{G}\,;\vec{H})_{\otimes}(\vec{F}_{\otimes}f) \tag{3.6.e25}$$

$$= (\vec{F}\,;(\vec{G}\,;\vec{H}))_{\otimes}(f) \tag{3.6.e26}$$

Which holds similarly for the other collection of monotone maps replacing $\otimes$ in the above with $\oplus$. Thus, since they are equal on 0-cells and 1-cells, $(\vec{F}\,;\vec{G})\,;\vec{H}$ and $\vec{F}\,;(\vec{G}\,;\vec{H})$ are equal as linear functors so that (;) is associative. ∎

Proposition 3.6.3. *The operation* (;) *as defined in eq.* (3.6.e1) *has left and right identities.*

Proof: Let $\vec{I}_{\mathbf{B}} : \mathbf{B} \to \mathbf{B}, \vec{I}_{\mathbf{C}} : \mathbf{C} \to \mathbf{C}$ be the identity linear functors for $\mathbf{B}, \mathbf{C}$ respectively. Then for the maps of 0-cells $\vec{F}\,;\vec{I}_{\mathbf{C}}X = \vec{I}_{\mathbf{C}}(\vec{F}X) = \vec{F}X$, and $\vec{I}_{\mathbf{B}}\,;\vec{F}X = \vec{F}(\vec{I}_{\mathbf{B}}X) = \vec{F}X$. Furthermore, for the maps of 1-cells $(\vec{F}\,;\vec{I}_{\mathbf{C}})_{\otimes}f = \vec{I}_{\mathbf{C}\otimes}(\vec{F}_{\otimes}f) = \vec{F}_{\otimes}f$, and $(\vec{I}_{\mathbf{B}}\,;\vec{F})_{\otimes}f = \vec{F}_{\otimes}(\vec{I}_{\mathbf{B}\otimes}f) = \vec{F}_{\otimes}f$, which also holds when replacing $\otimes$ by $\oplus$. Thus $\vec{F}\,;\vec{I}_{\mathbf{C}} = \vec{F}$ and $\vec{I}_{\mathbf{B}}\,;\vec{F} = \vec{F}$ so that the identity linear functors are the identities with respect to (;). ∎

The following corollary is an immediate result of propositions 3.6.1 to 3.6.3.

Corollary 3.6.4. *The collection of locally posetal linear bicategories and linear functors together with composition* (;) *as in eq.* (3.6.e1) *forms a category* $\mathbf{LinBicat_{Pos}}$.

In corollary 3.7.4 we will show that $\mathbf{Lin(B, C)}$, the collection of linear functors from $\mathbf{B}$ to $\mathbf{C}$, forms a category with *linear natural transformations* as morphisms as defined in the subsequent section.

3.7 Linear natural transformations

As noted at the start of this chapter, a definition for linear natural transformation may be found at [17, §5.1], which we will adapt to our setting of locally posetal linear bicategories.

Definition 3.7.1. Let $\mathbf{B}, \mathbf{C}$ be locally posetal linear bicategories, and let $\vec{F}, \vec{G} : \mathbf{B} \to \mathbf{C}$ be a pair of linear functors. A *linear natural transformation* $\vec{\alpha} : \vec{F} \to \vec{G}$ is comprised of a lax natural transformation $\vec{\alpha}_\otimes : \vec{F}_\otimes \to \vec{G}_\otimes$, and an opcolax natural transformation $\vec{\alpha}_\oplus : \vec{G}_\oplus \to \vec{F}_\oplus$ such that for each 1-cell $X \in \mathbf{B}_0$, $\vec{\alpha}_{\otimes X}, \vec{\alpha}_{\oplus X}$ are cyclic linear adjoints (that is $\vec{\alpha}_{\otimes X} \Vdash\!\dashv \vec{\alpha}_{\oplus X}$). By lax and opcolax natural transformations, we mean $\vec{\alpha}_\otimes, \vec{\alpha}_\oplus$ are collections of $\mathbf{B}_0$-indexed 1-cells such that for each $X, Y \in \mathbf{B}_0, R : X \to Y$, we have

$$\vec{\alpha}_{\otimes X} \otimes \vec{G}_\otimes R \leq \vec{F}_\otimes R \otimes \vec{\alpha}_{\otimes Y}, \quad \text{and} \quad \vec{G}_\oplus R \oplus \vec{\alpha}_{\oplus X} \geq \vec{\alpha}_{\oplus Y} \oplus \vec{F}_\oplus R \tag{3.7.e1}$$

[17] presents linear bicategories as the *representable* case of a *polybicategory*, in which the domains and codomains of 2-cells may be typed paths of 1-cells. However, in the locally posetal case these details may be omitted. Indeed condition 3 of §5.1 has the extra data of a B_1 indexed collection of "multi-2-cells" satisfying certain naturality conditions, but these multi-2-cells are already accounted for by representing

2-cells in the transformations $\vec{\alpha}_\otimes, \vec{\alpha}_\oplus$, and the naturality conditions hold trivially due to the uniqueness of 2-cells in a locally posetal linear bicategory.

Example 3.7.2. Let $\mathbf{B}, \mathbf{C}$ be locally posetal linear bicategories and let $\vec{F} : \mathbf{B} \to \mathbf{C}$ be a linear functor. There is an identity natural transformation $\mathbb{1}_{\vec{F}} : \vec{F} \to \vec{F}$ with components ${\mathbb{1}_\otimes}_X = \mathrm{E}_{\vec{F}X}$ and ${\mathbb{1}_\oplus}_X = \mathrm{D}_{\vec{F}X}$.

There will be further examples of linear natural transformations in later sections. For now we would like to show that linear natural transformations have a composition forming a one object bicategory **Lin(B, C)** of linear functors from **B** to **C** and linear natural transformations.

Proposition 3.7.3. *Let $\vec{F}, \vec{G}, \vec{H}$ be linear functors between* **B** *and* **C** *with linear natural transformations $\vec{\alpha} : \vec{F} \to \vec{G}, \vec{\beta} : \vec{G} \to \vec{H}$. The obvious component-wise operation $\vec{\alpha} ; \vec{\beta}$ on linear natural transformations, given as follows, is an associative and unital composition.*

$$(\vec{\alpha} ; \vec{\beta})_\otimes = \vec{\alpha}_\otimes \otimes \vec{\beta}_\otimes, \quad \textit{and} \quad (\vec{\alpha} ; \vec{\beta})_\oplus = \vec{\beta}_\oplus \oplus \vec{\alpha}_\oplus \tag{3.7.e2}$$

The above composition is as stated in [17] subsequent to §5.3. Associativity follows from the composition of lax natural transformations and the composition of linear adjoints. Unitality follows from example 3.7.2. Thus we have the following corollary.

Corollary 3.7.4. *Let* **B**, **C** *be locally posetal linear bicategories. Then* **Lin(B, C)**, *the collection of linear functors from* **B** *to* **C**, *forms a category whose morphisms are linear natural transformations and whose composition* (;) *is defined as in proposition 3.7.3.*

A *colinear* natural transformation ("co" in the bicategorical sense) is a linear natural transformation in which the inequalities in eq. (3.7.e1) are reversed. That is a colinear natural transformation satisfies

$$\vec{\alpha}_{\otimes X} \otimes \vec{G}_{\otimes} R \geq \vec{F}_{\otimes} R \otimes \vec{\alpha}_{\otimes Y}, \quad \text{and} \quad \vec{G}_{\oplus} R \oplus \vec{\alpha}_{\oplus X} \leq \vec{\alpha}_{\oplus Y} \oplus \vec{F}_{\oplus} R. \tag{3.7.e3}$$

Chapter 4

Linear Extensions

After covering the preliminary material, we are finally ready to define a notion of lax extension in the linear bicategorical setting. In the following chapter we will define linear extensions, the linear Barr extensions, and prove basic theorems and propositions on linear extensions.

Recall from the chapter on linear bicategories that $\mathcal{Q}$-**Rel** is a linear bicategory when $\mathcal{Q}$ is a linear quantale. In this case $\mathcal{Q}$ has two quantale structures, $(\mathcal{Q}, \otimes, k_\otimes)$ and $(\mathcal{Q}^{\mathrm{op}}, \oplus, k_\oplus)$. We may refer to these as $\mathcal{Q}_\otimes$ and $\mathcal{Q}_\oplus$ respectively, which is consistent with the notation for linear bicategories. When we pass to the linear bicategory $\mathcal{Q}$-**Rel**, we note that $\mathcal{Q}_\otimes$-**Rel** = $\mathcal{Q}$-**Rel**$_\otimes$ and $\mathcal{Q}_\oplus$-**Rel** = $\mathcal{Q}$-**Rel**$_\oplus$, that is they are isomorphic as categories and their hom-sets are isomorphic as preordered sets. We may also use $\mathcal{Q}$-**Rel** to denote $\mathcal{Q}_\otimes$-**Rel** when it is clear from context. We also note that $\mathcal{Q}$-**Rel**$_\oplus$ has canonical embeddings just like $(\text{-})_\circ, (\text{-})^\circ$ from **Set** to $\mathcal{Q}$-**Rel**$_\otimes$, which we shall denote $(\text{-})_\bullet : \mathbf{Set} \to \mathcal{Q}\text{-}\mathbf{Rel}_\oplus$, and $(\text{-})^\bullet : \mathbf{Set}^{\mathrm{op}} \to \mathcal{Q}\text{-}\mathbf{Rel}_\oplus$. Therefore when $\mathcal{Q}$-**Rel** is a linear bicategory, we may consider extensions on $\mathcal{Q}$-**Rel**$_\oplus$ as well, which must be oplax with respect to the base linear bicategory.

4.1 Linear extensions of functors

When we have a pair of lax extensions $(\hat{F}, \check{F})$ of a **Set** functor F, to $\mathcal{Q}_\otimes$-**Rel** and $\mathcal{Q}_\oplus$-**Rel** respectively, we may attempt to form a linear functor $\vec{F} = (F, \hat{F}, \check{F})$ which is succesful provided that $\hat{F}, \check{F}$ possess linear strength. In this case we call the pair a linear extension.

Definition 4.1.1. Let $\mathcal{Q}$ be a linearly distributive quantale. A *linear extension* of the **Set** functor $F : \mathbf{Set} \to \mathbf{Set}$ to $\mathcal{Q}$-**Rel** is a pair $(\vec{F}_\otimes, \vec{F}_\oplus)$, denoted $\vec{F}$, such that $\vec{F}_\otimes$ is a lax extension with respect to $\mathcal{Q}$-$\mathbf{Rel}_\otimes$ (or $\mathcal{Q}_\otimes$-**Rel**) and $\vec{F}_\oplus$ is a lax extension with respect to $\mathcal{Q}$-$\mathbf{Rel}_\oplus$ (or $\mathcal{Q}_\oplus$-**Rel**). That is, for each $\mathcal{Q}$-relation $f : X \to Y$,

$$(Ff)_\circ \leq \vec{F}_\otimes(f_\circ), \quad \text{and} \quad (Ff)^\circ \leq \vec{F}_\otimes(f^\circ) \tag{4.1.e1}$$

and,

$$\vec{F}_\oplus(f_\bullet) \leq (Ff)_\bullet, \quad \text{and} \quad \vec{F}_\oplus(f^\bullet) \leq (Ff)^\bullet. \tag{4.1.e2}$$

Similar to how a lax extensions of F corresponds to a lax functor satisfying the extension condition (as in proposition 2.3.2), a linear extension corresponds to a linear functor satisfying eqs. (4.1.e1) and (4.1.e2) above, which we shall refer to as the *linear extension condition.*

Proposition 4.1.2. *Let $\mathcal{Q}$ be an LD quantale, let F be a* **Set** *endofunctor, and let $\vec{F} = (\vec{F}, \vec{F}_\otimes, \vec{F}_\oplus)$ be a linear endofunctor (as in definition 3.4.1) on $\mathcal{Q}$-***Rel** *satisfying the* linear extension condition*, that is $\vec{F}_\otimes$ satisfies eq.* (4.1.e1) *and $\vec{F}_\oplus$ satisfies eq.* (4.1.e2)*. Then $(\vec{F}_\otimes, \vec{F}_\oplus)$ forms a linear extension of F to $\mathcal{Q}$-***Rel** *(as in definition 4.1.1). Conversely, a linear extension $\vec{F} = (\vec{F}_\otimes, \vec{F}_\oplus)$ of the* **Set** *endofunctor F to $\mathcal{Q}$-***Rel** *corresponds to a linear functor $(F, \vec{F}_\otimes, \vec{F}_\oplus)$ with the mapping on objects from F.*

Proof: Suppose that $\vec{F} = (\vec{F}, \vec{F}_\otimes, \vec{F}_\oplus)$ forms a linear functor as above, then by proposition 2.3.2 $\vec{F}_{\otimes}$ is a lax extension of F to $\mathcal{Q}$-$\mathbf{Rel}_{\otimes}$ since it satisfies eq. (4.1.e1), which is the lax extension condition. Similarly, $\vec{F}_{\oplus}$ is a lax extension of F to $\mathcal{Q}$-$\mathbf{Rel}_{\oplus}$. Since $\vec{F}$ is a linear functor the pair $(\vec{F}_{\otimes}, \vec{F}_{\oplus})$ has linear strength, hence it is a linear extension.

Conversely suppose $\vec{F}$ is a linear extension of F as above. Clearly $\vec{F}_{\otimes}, \vec{F}_{\oplus}$ becomes a lax functor oplax functor pair based on proposition 2.3.2 and our observation that lax functors in $\mathcal{Q}$-$\mathbf{Rel}_{\oplus}$ are oplax with respect to the base linear bicategory, and furthermore this pair has linear strength so that by proposition 3.4.2 $\vec{F}$ one obtains a linear functor. ∎

We now show some examples of linear extension. By proposition 4.1.2, these also provide examples of linear functors.

Examples 4.1.3.

(I) Every **Set** functor F has the following linear extension to $\mathcal{Q}$-**Rel**. Let $\vec{F}\ R : X \nrightarrow Y$, then

$$\vec{F}_{\otimes}(R) := \top, \quad \text{and} \quad \vec{F}_{\oplus}(R) := \bot. \tag{4.1.e3}$$

Where $\top, \bot$ are respectively the top and bottom element of $\mathcal{Q}$-$\mathbf{Rel}(FX, FY)$. We know that $\vec{F}_{\otimes}$ as defined above is always a lax extension of F so that lax functoriality of $\vec{F}_{\otimes}$ and the lax extension condition hold trivially. It may also be easily checked that oplax functoriality of $\vec{F}_{\oplus}$ holds, and that the linear strengths hold, since $\bot \otimes \top = \bot$ and $\bot \oplus \top = \top$.

(II) The identity linear functor $\vec{I}$ on $\mathcal{Q}$-**Rel** is a linear extension of the identity functor $I_{\mathbf{Set}}$ on **Set**, where $\vec{I}_{\otimes} = I_{\mathcal{Q}\text{-}\mathbf{Rel}}, \vec{I}_{\oplus} = I_{\mathcal{Q}\text{-}\mathbf{Rel}}$.

(III) Let $\vec{\mathscr{L}}_{\otimes} = \overline{\mathscr{L}}$ be the Barr extension of the free monoid monad, and let $\vec{\mathscr{L}}_{\oplus} = \overline{\mathscr{L}}((\text{-})^{\bot})^{\bot}$. Pointwise this yields the following definition. Let $R : X \to$

$Y, \langle x_i \rangle_{i=1}^n \in \mathscr{L}X, \langle y_j \rangle_{j=1}^m \in \mathscr{L}Y$

$$\langle x_i \rangle_{i=1}^n \vec{\mathscr{L}}_{\oplus} R \langle y_j \rangle_{j=1}^m \iff m \neq n \text{ or } \exists i.1 \leq i \leq n \text{ and } \ x_i \, R \, y_i \tag{4.1.e4}$$

This is a linear extension of the free monoid functor, however we will save the details for later.

Although trivial examples such as example 4.1.3.I are easy to check, less trivial linear extensions such as example 4.1.3.III require much more work to prove. We note that in example 4.1.3.I that with respect to the partial ordering on relations, $\vec{F}_{\otimes}$ is "dual" to $\vec{F}_{\oplus}$. More generally, we would like to define the dual $\check{F}$ of a lax functor $\hat{F}$, as it turns out that more generally $\vec{F} = (\hat{F}, \check{F})$ often becomes a linear extension of F.

Definition 4.1.4. Let $\mathcal{Q}$ be a Girard quantale. The *dual* of a lax functor $\hat{F}$: $\mathcal{Q}$-**Rel** $\to$ $\mathcal{Q}$-**Rel** is the functor $\check{F} := \hat{F}((\text{-})^{\perp})^{\perp} = (\text{-})^{\perp} \,;\, \hat{F} \,;\, (\text{-})^{\perp}$, where $(\text{-})^{\perp}$ is the identity on objects and maps morphisms $R : X \nrightarrow Y$ in $\mathcal{Q}$-**Rel** to $\check{F}_{X,Y} R = \hat{F}(R^{\perp})^{\perp}$, where $R^{\perp} = R \multimap \mathrm{D}_X$.

Proposition 4.1.5. *Let $\mathcal{Q}$ be a Girard quantale and F be a* **Set** *functor with a lax extension $\hat{F}$. Then the dual of $\hat{F}$, denoted $\check{F}$, is a lax functor on $\mathcal{Q}$-*$\mathbf{Rel}_{\oplus}$*. That is, $\check{F}$ is oplax with respect to the dual composition $R \oplus S = (S^{\perp} \otimes R^{\perp})^{\perp}$ and the usual ordering on $\mathcal{Q}$-*$\mathbf{Rel}$.

Proof:

Let $\hat{F}$ be a lax functor and let $\check{F}$ be the dual of $\hat{F}$. First we will prove that $\check{F}$ is an oplax functor.

(Monotonicity):

The map $(\text{-}) \multimap \mathrm{D}$ is an order reversing operation and when $\mathcal{Q}$ is Girard $(\text{-})^{\perp} = (\text{-}) \multimap \mathrm{D}$. Therefore, if $R, R' : X \nrightarrow Y$ are such that $R \leq R'$, then

$$R'^{\perp} \leq R^{\perp}, \tag{4.1.e5}$$

$$\Longrightarrow \hat{F}(R'^{\perp}) \leq \hat{F}(R^{\perp}), \tag{4.1.e6}$$

$$\Longrightarrow \hat{F}(R^{\perp})^{\perp} \leq \hat{F}(R'^{\perp})^{\perp}. \tag{4.1.e7}$$

Thus by definition $\check{F}R \leq \check{F}R'$, so that $\check{F}$ is monotone.

(Oplax preservation of compostion w.r.t $\oplus$): Let $S : Y \nrightarrow Z$. Then $\hat{F}(S^{\perp} \otimes R^{\perp}) \leq \hat{F}(S^{\perp}) \otimes \hat{F}(R^{\perp})$ as a lax functor laxly preserves $\otimes$ composition. Thus since $(\text{-})^{\perp}$ is an order reversing operation, we have the following

$$\hat{F}(S^{\perp} \otimes R^{\perp})^{\perp} \leq (\hat{F}(S^{\perp}) \otimes \hat{F}(R^{\perp}))^{\perp}. \tag{4.1.e8}$$

Linear negation is involutive so that $((\text{-})^{\perp})^{\perp}$ is the identity making $\hat{F}((\text{-})^{\perp}) = (\hat{F}((\text{-})^{\perp})^{\perp})^{\perp}$ which is equal to $(\check{F}(\text{-}))^{\perp}$ by definition of $\check{F}$, and hence

$$\Longrightarrow \check{F}((S^{\perp} \otimes R^{\perp})^{\perp}) \leq (\check{F}S^{\perp} \otimes \check{F}R^{\perp})^{\perp}. \tag{4.1.e9}$$

Furthermore, when $\mathcal{Q}$ is Girard the $\otimes, \oplus$ compositions in $\mathcal{Q}$-**Rel** satisfy $A \otimes B = (B^{\perp} \oplus A^{\perp})^{\perp}$, so that

$$\Longrightarrow \check{F}(R \oplus S) \leq \check{F}R \oplus \check{F}S. \tag{4.1.e10}$$

Hence, $\check{F}$ preserves composition oplaxly with respect to $\oplus$.

(Oplax preservation of the unit):

Let $\mathrm{E}_X : X \nrightarrow X$ be the identity on X with respect to $\otimes$. Since $\hat{F}$ is a lax functor, we have $\mathrm{E}_{\hat{F}X} \leq \hat{F}\mathrm{E}_X$. Thus, the following:

$$\mathrm{E}_{\hat{F}X} \leq \hat{F}\mathrm{E}_X \tag{4.1.e11}$$

$$\Longrightarrow \hat{F}\mathrm{E}_X{}^{\perp} \leq \mathrm{E}_{\hat{F}X}{}^{\perp} \tag{4.1.e12}$$

$$\Longrightarrow \hat{F}((\mathrm{E}_X{}^{\perp})^{\perp})^{\perp} \leq \mathrm{E}_{\hat{F}X}{}^{\perp} \tag{4.1.e13}$$

$$\Longrightarrow \check{F}(\mathrm{E}_X{}^{\perp}) \leq \mathrm{E}_{\hat{F}X}{}^{\perp} \tag{4.1.e14}$$

Note that $\mathrm{D}_X : X \nrightarrow X$, the identity with respect to $\oplus$, is the linear adjoint of E_X. That is, $\mathrm{D}_X = (\mathrm{E}_X)^{\perp}$. Furthermore, $\hat{F}X$ and $\check{F}X$ are both equal to FX. Thus,

$$\Longrightarrow \check{F}\mathrm{D}_X \leq \mathrm{D}_{\check{F}X}. \tag{4.1.e15}$$

Hence, $\check{F}$ oplaxly preserves the unit. We also conclude that $\check{F}$ is an oplax functor.

∎

Proposition 4.1.6. *Let $\mathcal{Q}$ be a Girard quantale and let F be a* **Set** *functor with a lax extension $\hat{F}$. Then the dual functor $\check{F} = (\text{-})^{\perp} ; \hat{F} ; (\text{-})^{\perp}$ is a lax extension of F to $\mathcal{Q}$-$\mathbf{Rel}_{\oplus}$ with respect to its reversed ordering.*

Proof: In proposition 4.1.5 we prove that $\check{F}$ is an oplax functor, or a lax functor on $\mathcal{Q}$-$\mathbf{Rel}_{\oplus}$. Thus we need only prove that the lax extension condition holds in $\mathcal{Q}$-$\mathbf{Rel}_{\oplus}$, that is the condition eq. (4.1.e2). Observe that $\hat{F} ; (\text{-})^{\perp} = (\text{-})^{\perp} ; \check{F}$ which follows from the definition of $\check{F}$ and the fact $(\text{-})^{\perp} ; (\text{-})^{\perp}$ is the identity. Recall that $\mathcal{Q}$-$\mathbf{Rel}_{\oplus} = \mathcal{Q}_{\oplus}$-$\mathbf{Rel}$, with the embeddings $(\text{-})_{\bullet}, (\text{-})^{\bullet}$. Also observe that $(\text{-})_{\bullet} = (\text{-})^{\circ} ; (\text{-})^{\perp}$ and $(\text{-})^{\bullet} = (\text{-})_{\circ} ; (\text{-})^{\perp}$, which follows from lemma 6.1.3.

Let $f : X \to Y$ be a function. Starting from the lax extension condition on $\mathcal{Q}$-$\mathbf{Rel}_{\otimes}$ eq. (4.1.e1) and after applying $(\text{-})^{\perp}$ we have

$$(\hat{F}(f_{\circ}))^{\perp} \leq ((Ff)_{\circ})^{\perp}, \text{ and } (\hat{F}(f^{\circ}))^{\perp} \leq ((Ff)^{\circ})^{\perp}$$

Following from the first observation, we have

$$\Longrightarrow \check{F}((f_\circ)^\perp) \leq ((Ff)_\circ)^\perp, \text{ and } \check{F}((f^\circ)^\perp) \leq ((Ff)^\circ)^\perp.$$

Following from the second observation, we have

$$\Longrightarrow \vec{F}_\oplus(f_\bullet) \leq (Ff)_\bullet, \text{ and } \vec{F}_\oplus(f^\bullet) \leq (Ff)^\bullet.$$

Which is the oplax extension condition eq. (4.1.e2). Thus $\check{F}$ is a lax extension from **Set** to $\mathcal{Q}$-$\mathbf{Rel}_\oplus$. ∎

4.2 The linear Barr extension

Recall the Barr extension from monoidal topology which provides a lax extension to **Rel** for a **Set** functor F satisfying the BC condition (see section 2.4). We now introduce an analagous type of linear exension based on the Barr extension, which we shall call the *linear Barr extension.*

Definition 4.2.1. The *linear Barr extension* of a **Set** functor F is defined to be the pair $\vec{F} = (\vec{F}_\otimes, \vec{F}_\oplus)$ where $\vec{F}_\otimes$ is the Barr extension $\overline{F}$ of F to **Rel**, and $\vec{F}_\oplus$ is the dual functor $\overline{F}((\text{-})^\perp)^\perp = (\text{-})^\perp\,; F\,; (\text{-})^\perp$.

It is not obvious that the linear Barr extension of a **Set** functor F is a linear extension of F to **Rel** in general. If F satisfies the Beck-Chevalley condition, then the fact that $\vec{F}_\otimes = \overline{F}$ is a lax extension of F follows from theorem 2.4.3, and thus the dual $\overline{F}((\text{-})^\perp)^\perp$ is an oplax extension following from proposition 4.1.5, however it must be checked that all of the linear strengths hold. We save the details of this for chapter 5. However, the following lemmas greatly simplify our later work.

Lemma 4.2.2. *Let $\mathcal{Q}$ be a Girard quantale, and let $\hat{F}$ be a lax functor on $\mathcal{Q}$-**Rel***

satisfying $\hat{F}((\text{-})^\circ) = (\hat{F}(\text{-}))^\circ$. *Then the pair denoted by* $\vec{F}$ *with* $(\vec{F}_\otimes, \vec{F}_\oplus) = (\hat{F}, \hat{F}((\text{-})^\perp)^\perp)$ *satisfies the linear strength* $\nu_\otimes^{\mathrm{R}}$ *if and only if it satisfies* $\nu_\otimes^{\mathrm{L}}$ *(see condition 3.4.1.iii). Dually,* $\nu_\oplus^{\mathrm{R}}$ *holds if and only if* $\nu_\oplus^{\mathrm{L}}$ *holds.*

Proof: Let $\hat{F}$ be a lax functor commuting with the involution as above. We note that $(\text{-})^\circ$ commutes with $(\text{-})^\perp$, which implies that the dual of $\hat{F}$ also commutes with $(\text{-})^\circ$.

Suppose that $\vec{F}$ satisfies $\nu_\otimes^{\mathrm{R}}$ for all $R\colon X \nrightarrow Y, S\colon Y \nrightarrow Z$, then $\nu_\otimes^{\mathrm{R}}$ is satisfied for S°, R°. Thus we have the following:

$$\nu_\otimes^{\mathrm{R}}:\ \vec{F}_\otimes(R^\circ \oplus S^\circ) \le \vec{F}_\oplus(R^\circ) \oplus \vec{F}_\otimes(S^\circ) \tag{4.2.e1}$$

$$\iff \vec{F}_\otimes(R^\circ \oplus S^\circ)^\circ \le (\vec{F}_\oplus(R^\circ) \oplus \vec{F}_\otimes(S^\circ))^\circ \tag{4.2.e2}$$

$$\iff \vec{F}_\otimes((R^\circ \oplus S^\circ)^\circ) \le (\vec{F}_\oplus R^\circ \oplus \vec{F}_\otimes S^\circ)^\circ \tag{4.2.e3}$$

$$\iff \vec{F}_\otimes(S \oplus R) \le \vec{F}_\otimes S \oplus \vec{F}_\oplus R \tag{4.2.e4}$$

Which is $\nu_\otimes^{\mathrm{L}}$. Thus $\nu_\otimes^{\mathrm{R}}$ holds if and only if $\nu_\otimes^{\mathrm{L}}$ holds. One also shows dually that $\nu_\oplus^{\mathrm{R}}$ holds if and only if $\nu_\otimes^{\mathrm{L}}$ holds. ∎

Lemma 4.2.3. *Let* $\mathcal{Q}$ *be a Girard quantale, and let* $\hat{F}$ *be any lax functor on* $\mathcal{Q}$-**Rel**. *Then the pair* $(\vec{F}_\otimes, \vec{F}_\oplus) = (\hat{F}, \hat{F}((\text{-})^\perp)^\perp)$ *denoted by* $\vec{F}$ *satisfies the linear strength* $\nu_\otimes^{\mathrm{R}}$ *if and only if it satisfies* $\nu_\oplus^{\mathrm{L}}$ *(see condition 3.4.1.iii). Dually,* $\nu_\oplus^{\mathrm{R}}$ *holds if and only if* $\nu_\otimes^{\mathrm{L}}$ *holds.*

Proof: Suppose that $\nu_\otimes^{\mathrm{R}}$ holds as above.

$$\nu_\otimes^{\mathrm{R}}:\ \vec{F}_\otimes(R^\perp \oplus S^\perp) \le \vec{F}_\oplus(R^\perp) \oplus \vec{F}_\otimes(S^\perp) \tag{4.2.e5}$$

$$\iff (\vec{F}_\oplus(R^\perp) \oplus \vec{F}_\otimes(S^\perp))^\perp \le \vec{F}_\otimes(R^\perp \oplus S^\perp)^\perp \tag{4.2.e6}$$

Notice that $\vec{F}_{\otimes}(R^{\perp}) = (\vec{F}_{\otimes}(R^{\perp})^{\perp})^{\perp} = \vec{F}_{\oplus}R^{\perp}$, and similarly $\vec{F}_{\otimes}R^{\perp} = \vec{F}_{\oplus}(R^{\perp})$, thus

$$\iff (\vec{F}_{\otimes}R^{\perp} \oplus \vec{F}_{\oplus}S^{\perp})^{\perp} \leq \vec{F}_{\oplus}((R^{\perp} \oplus S^{\perp})^{\perp}) \tag{4.2.e7}$$

$$\iff \vec{F}_{\oplus}S \otimes \vec{F}_{\otimes}R \leq \vec{F}_{\oplus}(S \otimes R) \tag{4.2.e8}$$

Which is $\nu^{\mathrm{L}}_{\oplus}$. Thus $\nu^{\mathrm{R}}_{\otimes}$ holds if and only if $\nu^{\mathrm{L}}_{\oplus}$. Dually, $\nu^{\mathrm{R}}_{\oplus}$ holds if and only if $\nu^{\mathrm{L}}_{\otimes}$ holds. ∎

We have shown that when $\hat{F}$ is a lax functor commuting with the involution, the linear strengths are related as follows:

$$\begin{array}{ccc} \nu^{\mathrm{L}}_{\otimes} & \Longleftrightarrow & \nu^{\mathrm{R}}_{\otimes} \\ & \nwarrow\!\!\!\!\searrow \ \nearrow\!\!\!\!\swarrow & \\ \nu^{\mathrm{L}}_{\oplus} & \Longleftrightarrow & \nu^{\mathrm{R}}_{\oplus} \end{array} \tag{4.2.e9}$$

Therefore we only need to prove that $\hat{F}$ and its dual satisfy one linear strength. We formulate this statement as the following corollary.

Corollary 4.2.4. *Let $\mathcal{Q}$ be a Girard quantale, and let $\hat{F}$ be a lax functor on $\mathcal{Q}$-**Rel** satisfying $\hat{F}((\text{-})^{\circ}) = (\hat{F}(\text{-}))^{\circ}$. If the pair $(\vec{F}_{\otimes}, \vec{F}_{\oplus}) = (\hat{F}, \hat{F}((\text{-})^{\perp})^{\perp})$ denoted by $\vec{F}$ satisfies one of the linear strengths (see condition 3.4.1.iii), then it satsfies all linear strengths.*

We note that the Barr extension is a particular example of a lax functor satisfying corollary 4.2.4, and thus we have a further corollary when $\vec{F}$ is the linear Barr extension.

Corollary 4.2.5. *Let F be a* **Set** *functor satisfying the BC condition. Then the linear Barr extension $\vec{F}$ is a linear extension of F to* **Rel** *provided the pair $(\vec{F}_{\otimes}, \vec{F}_{\oplus})$ satisfies at least one linear strength (see condition 3.4.1.iii).*

Proof: (-)$^\circ$ commutes with the Barr extension, [25, III.1.10.2(2)]. Let $R : X \nrightarrow Y$. As in equation eq. (2.4.e1), we may represent R as the compositions $\pi_1^\circ ; \pi_2$ and R° as $p_1^\circ ; p_2$ where π_1, π_2, and p_1, p_2 are the projections for R, and R° respectively. The underlying graphs of R, R° are isomorphic up to the bijection $\sigma : |R^\circ| \to |R|$ which swaps coordinates, and clearly $p_1 = \sigma ; \pi_2$, $p_2 = \sigma ; \pi_1$, thus we have the following:

$$\overline{F}(R^\circ) = \overline{F}(p_1^\circ ; p_2) \tag{4.2.e10}$$

$$= (Fp_1)^\circ ; Fp_2 \tag{4.2.e11}$$

$$= (F(\sigma ; \pi_2))^\circ ; F(\sigma ; \pi_1) \tag{4.2.e12}$$

$$= (F\sigma ; F\pi_2)^\circ ; F\sigma ; F\pi_1 \tag{4.2.e13}$$

$$= (F\pi_2)^\circ ; (F\sigma)^\circ ; F\sigma ; F\pi_1 \tag{4.2.e14}$$

Since functors preserve isomorphisms, $F\sigma$ is also an isomorphism. Furthermore, if $f : X \to Y$ is an isomorphism in **Set** then $f^\circ = f^{-1}$ in **Rel**. Thus,

$$= (F\pi_2)^\circ ; \mathrm{E}_{F|R|} ; F\pi_1 \tag{4.2.e15}$$

$$= ((^\circ F\pi_1) ; F\pi_2)^\circ. \tag{4.2.e16}$$

Which is just $(\overline{F}R)^\circ$ by definition, so that $\overline{F}(R^\circ) = (\overline{F}R)^\circ$. Now we apply corollary 4.2.4, showing that $\vec{F}$ is a linear extension, the desired result. ∎

Thus in the following chapter we may apply the above lemma and demonstrate some specific examples of linear extensions.

4.3 General lax and linear extensions

Our notion of linear extension from section 4.1 is defined in line with the lax extensions of functors from [25] taken to the linear bicategorical setting, and hence we only extend functors on **Set** to (lax, linear) functors on $\mathcal{Q}$-**Rel**. However, other examples of lax extensions exist such as lax extensions of $\mathcal{Q}$-**Cat** endofunctors to $\mathcal{Q}$-**Mod** or lax extensions of **Set**/$\mathcal{Q}_0$ functors to $\mathcal{Q}$-**Rel** when $\mathcal{Q}$ is a quantaloid, see [27, §3]. This provides motivation for a more general definition of lax and linear extensions of functors where **Set** and $\mathcal{Q}$-**Rel** are replaced by locally posetal bicategories $\mathbf{B}, \mathbf{C}$, as we shall attempt to define in this section.

First, recall the notion of lax extension of a functor F from **Set** to $\mathcal{Q}$-**Rel** where we extend the maps $F_{X,Y} : \mathbf{Set}(X, Y) \to \mathbf{Set}(FX, FY)$ to maps $\hat{F}_{X,Y} : \mathcal{Q}\text{-}\mathbf{Rel}(X, Y) \to \mathcal{Q}\text{-}\mathbf{Rel}(FX, FY)$ in such a way that it satisfies the axioms of monotonicity, lax preservation of composition, and lax preservation of identities, as well as the lax extension condition with respect to the canonical embeddings $(\text{-})_\circ : \mathbf{Set} \to \mathcal{Q}\text{-}\mathbf{Rel}$ and $(\text{-})^\circ : \mathbf{Set} \to \mathbf{Rel}$. More generally one may define lax extensions between any two locally posetal bicategories $\mathbf{B}, \mathbf{C}$ provided that there exist embeddings $(\text{-})_\circ : \mathbf{B} \to \mathbf{C}$ and $(\text{-})^\circ : \mathbf{B}^{\text{op}} \to \mathbf{C}$ which are bijective on objects and injective on hom-sets. Thus we define a general lax extension as follows.

Definition 4.3.1. Let $\mathbf{B}, \mathbf{C}$ be locally posetal bicategories such that there exist pseudofunctors $(\text{-})_\circ : \mathbf{B} \to \mathbf{C}$ (called the "embedding") and $(\text{-})^\circ : \mathbf{B}^{\text{op}} \to \mathbf{C}$, which are bijective on objects and faithful, and let $F : \mathbf{B} \to \mathbf{B}$ be a lax functor. A lax extension $\hat{F}$ of F to $\mathbf{C}$ is a collection of maps $\hat{F}_{X,Y} : \mathbf{C}(X_\circ, Y_\circ) \to \mathbf{C}((FX)_\circ, (FY)_\circ)$ indexed by pairs $X, Y \in \mathsf{ob}(\mathbf{B})$ such that the following holds. Let $R, R' \in \mathbf{C}(X_\circ, Y_\circ), S \in \mathbf{C}(Y_\circ, Z_\circ), f \in \mathbf{B}(X, Y)$ with $X, Y, Z \in B_0$:

i) If $R \leq R'$, then $\hat{F}_{X_\circ,Y_\circ}(R) \leq \hat{F}_{X_\circ,Y_\circ}(R')$ (monotonicity);

ii) $\hat{F}_{X_\circ,Y_\circ}R \otimes \hat{F}_{Y_\circ,Z_\circ}S \leq \hat{F}_{X_\circ,Z_\circ}(R \otimes S)$ (lax preservation of compostion);

iii) $\mathrm{E}_{FX_\circ} \leq \hat{F}_{X_\circ, X_\circ}(\mathrm{E}_{X_\circ})$ (lax preservation of identities);

iv) $(F_{X,Y}f)_\circ \leq \hat{F}_{X_\circ, Y_\circ}(f_\circ)$ and $(F_{X,Y}f)^\circ \leq \hat{F}_{Y^\circ, X^\circ}(f^\circ)$ (lax extension criterion).

Conditions 4.3.1.i to 4.3.1.iv come from the usual conditions for lax extensions as in section 2.3. $(\text{-})_\circ, (\text{-})^\circ$ are the analogues to the usual embeddings from **Set** to $\mathcal{Q}$-**Rel**, but also to the embeddings $(\text{-})_* : \mathcal{Q}\text{-}\mathbf{Cat} \to \mathcal{Q}\text{-}\mathbf{Mod}$, $(\text{-})^* : \mathcal{Q}\text{-}\mathbf{Cat}^{\mathrm{op}} \to \mathcal{Q}\text{-}\mathbf{Mod}$ as in [27, §3]. An important fact of lax extensions from **Set** to $\mathcal{Q}$-**Rel** is that **Set** and $\mathcal{Q}$-**Rel** have the same objects and $(\text{-})_\circ, (\text{-})^\circ$ have the identity map on objects. This fact allows a lax extension to form a lax functor as in proposition 2.3.2. We generalize this condition by making $(\text{-})_\circ, (\text{-})^\circ$ bijective on objects.

We now use definition 4.3.1 to define a general linear extension.

Definition 4.3.2. Let $\mathbf{B}, \mathbf{C}$ be locally posetal linear bicategories such that there exist pseudofunctors $(\text{-})_\circ : \mathbf{B}_\otimes \to \mathbf{C}_\otimes$ (called the "$\otimes$-embedding") and $(\text{-})^\circ : {\mathbf{B}_\otimes}^{\mathrm{op}} \to \mathbf{C}_\otimes$, and dually $(\text{-})_\bullet : \mathbf{B}_\oplus \to \mathbf{C}_\oplus$ (called the "$\oplus$-embedding") and $(\text{-})^\bullet : {\mathbf{B}_\oplus}^{\mathrm{op}} \to \mathbf{C}_\oplus$ which are bijective on objects and faithful, and let $\vec{F} : \mathbf{B} \to \mathbf{B}$ be a linear functor. A linear extension $\hat{F}$ of F to $\mathbf{C}$ is a collection of maps $\vec{F}_{\otimes X,Y} : \mathbf{C}(X_\circ, Y_\circ) \to \mathbf{C}((FX)_\circ, (FY)_\circ)$ and $\vec{F}_{\oplus X,Y} : \mathbf{C}(X_\circ, Y_\circ)^{\mathrm{op}} \to \mathbf{C}((FX)_\circ, (FY)_\circ)^{\mathrm{op}}$ indexed by pairs $X, Y \in \mathsf{ob}(\mathbf{B})$ such that $\vec{F}_\otimes, (\text{-})_\circ, (\text{-})^\circ$ and $\vec{F}_\oplus, (\text{-})_\bullet, (\text{-})^\bullet$ form lax extensions of F such that $\vec{F}_\otimes, \vec{F}_\oplus$ satisfy 4 linear strengths as in condition 3.4.1.iii.

Chapter 5

Examples of linear extensions

In the following chapter we will prove the existence of linear extensions, working through calculations in sections 5.1 and 5.2 to prove that the free monoid functor and the ultrafilter functor have linear extensions. Then in section 5.3 we will prove a general theorem on the lax and linear functors, and thus lax and linear extensions, between Girard bicategories (see section 3.3).

5.1 Linear extension of the free monoid monad

The free monoid monad, also known as the list monad, $\mathscr{L}\colon \mathbf{Set} \to \mathbf{Set}$ is the monad induced by the adjunction $F \dashv G$ (as described in [25, Examples II.3.1.1(2)]), where G is the forgetful functor from the category **Mon** of monoids and monoid morphisms to **Set**, and F is the free functor from **Set** to **Mon**. $\mathscr{L}$ takes a set X to the set of finite sequences in X, denoted $\mathscr{L}X = \coprod_{i=0}^{\infty} X^i$, while $\mathscr{L}$ takes each function $f\colon X \to Y$ to the pointwise application of f, that is the function $\mathscr{L}f = \langle x_i \rangle_{i=1}^n \mapsto \langle f(x_i) \rangle_{i=1}^n$. The empty sequence in $\mathscr{L}X$, or sequence of length 0, will be denoted $*_X$ and $\mathscr{L}f(*_X) = *_Y$.

We would like to prove the following proposition showing that the free monoid functor has a linear extension from **Set** to **Rel**.

Proposition 5.1.1. *Let $\vec{\mathscr{L}}$ be the linear Barr extension of the free monoid functor $\mathscr{L}$. Then $\vec{\mathscr{L}}$ is a linear extension of $\mathscr{L}$ from* **Set** *to* **Rel***.*

Proof: It is known that $\mathscr{L}$ satisfies the Beck-Chevalley condition and therefore the Barr extension $\overline{\mathscr{L}}$ is a flat lax extension of $\mathscr{L}$ from **Set** to **Rel** (see [25, Following V.1.4.3 Remarks] and [25, Exercise III.1.Q]). It follows from corollary 4.2.5 that $\vec{\mathscr{L}}$ is a linear extension of $\mathscr{L}$ if and only if $\vec{\mathscr{L}}$ satisfies one linear strength, and in this case we shall check $\nu^{\mathrm{L}}_{\otimes}$. We first note the pointwise definitions of $\vec{\mathscr{L}}_{\otimes}, \vec{\mathscr{L}}_{\oplus}$, based on eq. (2.4.e2) which shall be used in our calculations. In the following let $\mathbb{N}_k$ denote the set $\{1, \ldots, k\}$ when k is a positive integer and $\mathbb{N}_0 = \varnothing$, let $R : X \nrightarrow Y$ be a relation, and let $\langle x_i \rangle_{i=1}^{n} \in \mathscr{L}X$, and $\langle y_i \rangle_{i=1}^{m} \in \mathscr{L}Y$ be sequences, with n, m non-negative integers:

$$\langle x_i \rangle_{i=1}^{n} \vec{\mathscr{L}}_{\otimes}\, R\, \langle y_i \rangle_{i=1}^{m} \quad \Longleftrightarrow \quad m = n \wedge \forall i \in \mathbb{N}_n . x_i\, R\, y_i \tag{5.1.e1}$$

and,

$$\langle x_i \rangle_{i=1}^{n} \vec{\mathscr{L}}_{\oplus}\, R\, \langle y_i \rangle_{i=1}^{m} \quad \Longleftrightarrow \quad m \neq n \vee \exists i \in \mathbb{N}_n . x_i\, R\, y_i \tag{5.1.e2}$$

Now we would like to prove that the linear strength $\nu^{\mathrm{L}}_{\otimes}$ holds. Let $S : Y \nrightarrow Z$ be any relation. Written pointwise with respect to $\mathscr{L}$, $\nu^{\mathrm{L}}_{\otimes}$ translates as follows:

$$\langle x_i \rangle_{i=1}^{n}\ \vec{\mathscr{L}}_{\otimes}(R \oplus S)\ \langle z_i \rangle_{i=1}^{m} \implies \langle x_i \rangle_{i=1}^{n}\ (\vec{\mathscr{L}}_{\otimes} R \oplus \vec{\mathscr{L}}_{\oplus} S)\ \langle z_i \rangle_{i=1}^{m} \tag{5.1.e3}$$

For the right hand side of eq. (5.1.e3) we have the following expansion based on eqs. (5.1.e1) and (5.1.e2):

$$\begin{aligned}
&\langle x_i \rangle_{i=1}^{n}\ (\vec{\mathscr{L}}_{\otimes} R \oplus \vec{\mathscr{L}}_{\oplus} S)\ \langle z_i \rangle_{i=1}^{m} \\
\Longleftrightarrow\ & \forall \langle y_i \rangle_{i=1}^{k} \in \mathscr{L}Y . (\langle x_i \rangle_{i=1}^{n}\ \vec{\mathscr{L}}_{\otimes} R\ \langle y_i \rangle_{i=1}^{k} \vee \langle y_i \rangle_{i=1}^{k} \vec{\mathscr{L}}_{\oplus} S \langle z_i \rangle_{i=1}^{m}) \\
\Longleftrightarrow\ & \forall \langle y_i \rangle_{i=1}^{k} \in \mathscr{L}Y . ((n = k \wedge \forall i \in \mathbb{N}_n . x_i\, R\, y_i) \vee (k \neq m \vee \exists i \in \mathbb{N}_m . y_i\, S\, z_i))
\end{aligned}$$

Furthermore, we may expand the left hand side of eq. (5.1.e3) as follows:

$$\begin{aligned} & \langle x_i \rangle_{i=1}^{n} \ \vec{\mathscr{L}}_{\otimes}(R \oplus S) \ \langle z_i \rangle_{i=1}^{m} \\ \iff & m = n \wedge \forall i \in \mathbb{N}_n . x_i \ \vec{\mathscr{L}}_{\otimes}(R \oplus S) \ z_i \\ \iff & m = n \wedge \forall i \in \mathbb{N}_n \forall y \in Y.(x_i \ R \ y \vee y \ S \ z_i) \end{aligned}$$

If the predicate $x_i \ R \ y \vee y \ S \ z_i$ holds for all elements (i, y) in the product $\mathbb{N}_n \times Y$ then $\forall (i, y) \in S.(x_i \ R \ y \vee y \ S \ z_i)$ holds over all subsets of $S \subseteq \mathbb{N}_n \times Y$, including those subsets which may be regarded as sequences of length $k = n$. Conversely, if $\forall (i, y_i) \in \langle y_i \rangle_{i=1}^{k} . (x_i \ R \ y_i \vee y_i \ S \ z_i)$ holds for all sequences $\langle y_i \rangle_{i=1}^{k} \in \mathscr{L}Y$ with length $k = n$ then $x_i \ R \ y \vee y \ S \ z_i$ holds over the product $\mathbb{N}_n \times Y$. Thus we have the following:

$$\iff m = n \wedge \forall \langle y_i \rangle_{i=1}^{k} \in \mathscr{L}Y.(k \neq m \vee \forall i \in \mathbb{N}_n.(x_i \ R \ y_i \vee y_i \ S \ z_i))$$

Now we will show that the left hand side implies the right hand side so that $\nu_{\otimes}^{\mathrm{L}}$ holds. Suppose that $m = n$ and let $\langle y_i \rangle_{i=1}^{k} \in \mathscr{L}Y$, we must show that

$$\begin{aligned} & k \neq m \vee \forall i \in \mathbb{N}_n.(x_i \ R \ y_i \vee y_i \ S \ z_i) \\ & \implies (n = k \wedge \forall i \in \mathbb{N}_n . x_i \ R \ y_i) \vee k \neq m \vee \exists i \in \mathbb{N}_m . y_i \ S \ z_i . \end{aligned} \tag{5.1.e4}$$

Suppose that the left hand side holds. If $k \neq m$, then clearly the right side also holds since $k \neq m$ is a term of the disjunction on the right. Otherwise, if $k = m$ then $\forall i \in \mathbb{N}_n.(x_i \ R \ y_i \vee y_i \ S \ z_i)$ holds. Recall that for predicates p, q on a set X that $\forall x \in X.(p(x) \vee q(x)) \implies \forall x \in X.p(x) \vee \exists x \in X.q(x)$. Thus the above implies $\forall i \in \mathbb{N}_n . x_i \ R \ y_i \vee \exists i . y_i \ S \ z_i$. Since $k = m = n$, if $\exists i \in \mathbb{N}_n . y_i \ S \ z_i$ holds then the right side holds since this is a term in the disjunction on the right. Otherwise if $\forall i \in \mathbb{N}_n . x_i \ R \ y_i$ holds, then since $k = m = n$ we have $n = k \wedge \forall i \in \mathbb{N}_n . x_i \ R \ y_i$, which is also a term of the disjunction on the right. We have shown that eq. (5.1.e4) holds in both cases, when $k \neq m$ and when $k = m$, thus eq. (5.1.e3) holds so that $\vec{\mathscr{L}}$ satisfies $\nu^{\mathrm{L}}_{\otimes}$. By corollary 4.2.5, $\vec{\mathscr{L}}$ constitutes a linear extension of $\mathscr{L}$. ∎

5.2 Linear extension of the ultrafilter functor

The ultrafilter functor $\beta\colon \mathbf{Set} \to \mathbf{Set}$, as previously described in section 1.4B is a functor taking each set X to the set βX of all ultrafilters on X, and each function $f\colon X \to Y$ to the function $\beta f = \mathfrak{a} \mapsto f[\mathfrak{a}]$, where $f[\mathfrak{a}]$ is the image filter with respect to f of $\mathfrak{a}$, as described in example 1.4.2.III. The ultrafilter functor satisfies the Beck Chevalley condition (see [25, Examples III.1.12.3(3)]), and therefore has a linear Barr extension $\vec{\beta} = (\overline{\beta}, (\text{-})^{\perp} ; \overline{\beta} ; (\text{-})^{\perp})$, where $\overline{\beta}$ is the Barr extension of β. Let $R\colon X \nrightarrow Y$ be a relation, and let $\mathfrak{a} \in \beta X$ and $\mathfrak{b} \in \beta Y$ be ultrafilters. $\vec{\beta}_{\otimes} = \overline{\beta}$ may equivalently be written pointwise as follows using eq. (2.4.e2) (see also [25, Examples III.1.10.3(3)]):

$$\mathfrak{a}\ \vec{\beta}_{\otimes} R\ \mathfrak{b} \iff \forall A \in \mathfrak{a}, B \in \mathfrak{b} \exists x \in A, y \in B. x\ R\ y. \tag{5.2.e1}$$

Furthermore, $\vec{\beta}_{\oplus}$ may be written pointwise as follows:

$$\mathfrak{a}\ \vec{\beta}_{\oplus} R\ \mathfrak{b} \iff \exists A \in \mathfrak{a}, B \in \mathfrak{b} \forall x \in A, y \in B. x\ R\ y. \tag{5.2.e2}$$

As in the previous section, we would like to prove that this in fact forms a linear extension, which we shall formulate as a proposition as follows.

Proposition 5.2.1. *Let $\vec{\beta}$ be the linear Barr extension of β, the ultrafilter functor. Then $\vec{\beta}$ is a linear extension of β from* **Set** *to* **Rel**.

Proof: As in the proof of proposition 5.1.1, since β satisfies the Beck-Chevalley condition it follows from corollary 4.2.5 that $\vec{\beta}$ is a linear extension of β from **Set** to **Rel** if it satisfies $\nu^{\mathrm{L}}_{\otimes}$. Let $R\colon X \nrightarrow Y$, $S\colon Y \nrightarrow Z$ be relations, and let $\mathfrak{a} \in \beta X$ and $\mathfrak{c} \in \beta Z$ be ultrafilters. $\nu^{\mathrm{L}}_{\otimes}$ may be expressed pointwise for $\vec{\beta}$ as follows:

$$\mathfrak{a}\ \vec{\beta}_{\otimes}(R \oplus S)\ \mathfrak{c} \implies \mathfrak{a}\ \vec{\beta}_{\otimes}R \oplus \vec{\beta}_{\oplus}S\ \mathfrak{c}. \tag{5.2.e3}$$

The right hand side of eq. (5.2.e3) expands as follows:

$$\begin{aligned} &\mathfrak{a}\ \vec{\beta}_{\otimes}R \oplus \vec{\beta}_{\oplus}S\ \mathfrak{c} \\ \iff &\forall \mathfrak{b} \in \beta Y.(\mathfrak{a}\ \vec{\beta}_{\otimes}R\ \mathfrak{b} \vee \mathfrak{b}\ \vec{\beta}_{\oplus}S\ \mathfrak{c}) \\ \iff &\forall \mathfrak{b} \in \beta Y.(\mathfrak{a}\ \vec{\beta}_{\otimes}R\ \mathfrak{b} \vee \exists B \in \mathfrak{b}, C \in \mathfrak{c} \forall y \in B, z \in C.y\ S\ z). \end{aligned} \tag{5.2.e4}$$

Furthermore, the left hand side is equivalent to the following:

$$\begin{aligned} &\mathfrak{a}\ \vec{\beta}_{\otimes}(R \oplus S)\ \mathfrak{c} \\ \iff &\forall A \in \mathfrak{a}, C \in \mathfrak{c} \exists x \in A, z \in C.x\ R \oplus S\ z \\ \iff &\forall A \in \mathfrak{a}, C \in \mathfrak{c} \exists x \in A, z \in C \forall y \in Y.(x\ R\ y \vee y\ S\ z) \end{aligned}$$

Let p be the predicate $\exists x \in A, z \in C \forall y \in B.(x\ R\ y \vee y\ S\ z)$ defined over triples $A \subseteq X$, $B \subseteq Y$, $C \subseteq Z$ of sets. Then the proposition from the previous line translates to $\forall A \in \mathfrak{a}, C \in \mathfrak{c}.p(A, Y, C)$. Let $\mathfrak{b} \in \beta Y$ be an ultrafilter and let $B \in \mathfrak{b}$ be a subset of Y. Clearly, if $x\ R\ y \vee y\ S\ z$ holds for all $y \in Y$, then it holds for all $y \in B$ so that $p(A, Y, C)$ implies $p(A, B, C)$ using the same choice of x, z, and $\forall \mathfrak{b} \in \beta Y, B \in \mathfrak{b}.p(A, B, C)$ follows via universal generalization. Conversely, if $\forall \mathfrak{b} \in \beta Y, B \in \mathfrak{b}.p(A, B, C)$ holds, then $p(A, Y, C)$ follows since Y is contained in every ultrafilter $\mathfrak{b} \in \beta Y$. Thus we have the following equivalence:

$$\iff \forall A \in \mathfrak{a}, C \in \mathfrak{c} \forall \mathfrak{b} \in \beta Y, B \in \mathfrak{b} \exists x \in A, z \in C \forall y \in B.(x\ R\ y \vee y\ S\ z)$$

This may be rearranged as follows since all choices are independent, except for B which depends on first choosing an ultrafilter $\mathfrak{b}$.

$$\iff \forall \mathfrak{b} \in \beta Y, A \in \mathfrak{a}, B \in \mathfrak{b}, C \in \mathfrak{c} \exists x \in A, z \in C \forall y \in B.(x\ R\ y \vee y\ S\ z). \quad (5.2.e5)$$

Let $\mathfrak{b} \in \beta Y$. By the above equivalences, proving that $\vec{\beta}$ satisfies $\nu_{\otimes}^{\mathrm{L}}$ amounts to proving the following implication:

$$\begin{aligned} &\forall A \in \mathfrak{a}, B \in \mathfrak{b}, C \in \mathfrak{c} \exists x \in A, z \in C \forall y \in B.(x\ R\ y \vee y\ S\ z) \\ &\implies \mathfrak{a}\ \vec{\beta}_{\otimes} R\ \mathfrak{b} \vee \exists B \in \mathfrak{b}, C \in \mathfrak{c} \forall y \in B, z \in C. y\ S\ z \end{aligned} \quad (5.2.e6)$$

Suppose that the left hand side of eq. (5.2.e6) holds. If $\mathfrak{a}\ \vec{\beta}_{\otimes} R\ \mathfrak{b}$ holds then the right hand side of eq. (5.2.e6) must hold, since it is a term in the disjunction. Suppose otherwise that $\neg(\mathfrak{a}\ \vec{\beta}_{\otimes} R\ \mathfrak{b})$ holds, that is the negation of eq. (5.2.e1) written $\exists A \in \mathfrak{a}, B \in \mathfrak{b} \forall x \in A, y \in B.\neg(x\ R\ y)$, then the right side of eq. (5.2.e6) holds if and only if there exist sets $B_0 \in \mathfrak{b}, C_0 \in \mathfrak{c}$ such that $\forall y \in B_0, z \in C_0. y\ S\ z$. By $\neg(\mathfrak{a}\ \vec{\beta}_{\otimes} R\ \mathfrak{b})$ there exist sets $A_0 \in \mathfrak{a}, B_0 \in \mathfrak{b}$ such that $\forall x \in A_0, y \in B_0. x \not R\ y$. Define a subset C_0 of Z as $C_0 := \{z \in Z : \forall y \in \mathfrak{b}. y\ S\ z\}$. Since $\mathfrak{c}$ is an ultrafilter, either C_0 or the complement $C_0^{\complement}$ is an element of $\mathfrak{c}$. According to the left hand side of eq. (5.2.e6), for an arbitrary element C of $\mathfrak{c}$ there exist elements $x \in A_0$, $z \in C$ such that for all y in B_0 $x\ R\ y \vee y\ S\ z$ holds. However, all pairs of elements $x \in A_0, y \in B_0$ have the property $\neg(x\ R\ y)$, so that by disjunctive syllogism $y\ S\ z$ holds. Thus $\exists z \in C \forall y \in B_0 y\ S\ z$. In other words, $C \cap C_0$ is nonempty, and thus since $\mathfrak{b}$ is an ultrafilter, by proposition 1.4.6 C_0 is an element of $\mathfrak{b}$. Hence $B_0 \in \mathfrak{b}, C_0 \in \mathfrak{c}$ are sets such that $\forall y \in B_0, z \in C_0. y\ S\ z$ so that eq. (5.2.e6) holds and $\vec{\beta}$ satisfies $\nu_{\otimes}^{\mathrm{L}}$. By corollary 4.2.5, $\vec{\beta}$ is a linear extension of β from **Set** to **Rel**. ∎

As previously noted, every linear extension corresponds to a linear functor using the same mapping on morphisms as in proposition 4.1.2. In the following we note certain properties of the ultrafilter functor and ultrafilters. The first uncovers a situation involving the ultrafilter functor and logical quantifiers.

In general, the existential and universal quantifier are inflexible with respect to order. For example, what is often an undergraduate's first exposure to quantifiers is the statement $\forall\varepsilon > 0\exists\delta > 0\forall x \in (b,c).\|x-a\| < \delta \rightarrow \|f(x)-L\| < \varepsilon$, which must not be mistaken for $\exists\varepsilon > 0\forall\delta > 0\forall x \in (b,c).\|x-a\| < \delta \rightarrow \|f(x)-L\| < \varepsilon$. However, when ultrafilters are involved, a universal quantifier and an existential quantifier may be swapped in the following setup.

Proposition 5.2.2. *Let p be a predicate on the set X and let $\mathfrak{a} \in \beta X$ be an ultrafilter on the set X. Then*

$$\forall A \in \mathfrak{a}\exists x \in A.p(x) \iff \exists A \in \mathfrak{a}\forall x \in A.p(x). \tag{5.2.e7}$$

Proof:

($\Longleftarrow$): Suppose that the right hand side of eq. (5.2.e7) holds. Then there exists an element A_0 in $\mathfrak{a}$ such that $\forall x \in A_0.p(x)$ holds. Let A be an arbitrary element of $\mathfrak{a}$. Since $\mathfrak{a}$ is a filter, $A \cap A_0$ is nonempty, so let x be an element of $A \cap A_0$. Since $x \in A_0$, $p(x)$ holds so that $\exists x \in A.p(x)$ holds, and by universal generalization we conclude that the left hand side holds.

($\Longrightarrow$): Suppose instead that the left hand side of eq. (5.2.e7) holds. Define $A_0 := \{x \in X : p(x)\}$, the support of p. $X \in \mathfrak{a}$ and X must be nonempty since there are no ultrafilters on the empty set. Thus $p(x)$ occurs for some $x \in X$ so that A_0 is nonempty. Furthermore, since $\mathfrak{a}$ is an ultrafilter it either contains A_0 or $A_0^{\complement}$. However, if $A_0^{\complement} \in \mathfrak{a}$ this contradicts the left hand side, since there does not exist an element in $A_0^{\complement}$ such that $p(x)$ holds. Thus $A_0 \in \mathfrak{a}$. By existential generalization, we may conclude the right hand side. ∎

The next property of $\vec{\beta}$ pertains to the components $\vec{F}_{\otimes}, \vec{F}_{\oplus}$ of a general linear functor $\vec{F}$. It is known that one may construct an arrow $A \otimes B \to A \oplus B$ in a $*$-autonomous category, such that in a Girard bicategory one may expect a similar arrow $\vec{F}_{\otimes}A \Rightarrow \vec{F}_{\oplus}A$ exists with respect to a 1-cell $A : X \to Y$. However $\vec{\beta}$ provides a counterexample in the form of the following proposition.

Proposition 5.2.3. *Let $R : X \nrightarrow Y$ be a relation. Then $\vec{\beta}_{\oplus}R \leq \vec{\beta}_{\otimes}R$ with respect to the pointwise ordering of relations in* **Rel***.*

Proof: Let $\mathfrak{a} \in \beta X, \mathfrak{b} \in \beta Y$ be ultrafilters and suppose that $\mathfrak{a}\ \vec{\beta}_{\oplus}R\ \mathfrak{b}$ holds. Then by the pointwise definition of $\vec{\beta}_{\otimes}$ as in eq. (5.2.e2) there exist sets $A_0 \in \mathfrak{a}, B_0 \in \mathfrak{b}$ such that $\forall x \in A_0, y \in B_0 . x\ R\ y$ holds. Now let $A \in \mathfrak{a}, B \in \mathfrak{b}$ be arbitrary elements. Since $\mathfrak{a}, \mathfrak{b}$ are ultrafilters, the intersections $A \cap A_0, B \cap B_0$ are elements of $\mathfrak{a}, \mathfrak{b}$ respectively and furthermore are nonempty since ultrafilters do not contain the empty set. Take $x_0 \in A \cap A_0, y_0 \in B_0$ so that $x_0\ R\ y_0$. $\exists x \in A, y \in B . x\ R\ y$ follows from existential generalization, and then $\forall A \in \mathfrak{a}, B \in \mathfrak{b} \exists x \in A, y \in B . x\ R\ y$ follows from universal generalization, which is equivalent to $\mathfrak{a}\vec{\beta}_{\otimes}R\mathfrak{b}$ as in eq. (5.2.e1). Thus $\vec{\beta}_{\oplus}R \leq \vec{\beta}_{\otimes}R$ in general.

∎

Following proposition 5.2.3 one may wonder if $\vec{F}_{\oplus} = \vec{F}_{\otimes}$, however this fails due to the following counterexample

Proposition 5.2.4. *There exists a relation $R : X \nrightarrow Y$ such that $\vec{\beta}_{\oplus}R \neq \vec{\beta}_{\otimes}R$.*

Proof: Let X be an infinite set, and let $\mathfrak{a}, \mathfrak{a}'$ be ultrafilters on the set X. Suppose that $\mathfrak{a}\ \vec{\beta}_{\oplus}\mathrm{E}_X\ \mathfrak{a}'$. By the pointwise definition (see eq. (5.2.e2)) there must exist sets $A \in \mathfrak{a}, A' \in \mathfrak{a}'$ such that $x\ \mathrm{E}_X\ x'$, that is $x = x'$ for all $x \in A, x' \in A'$. This occurs only if one of A or A' is empty, or A, A' are singleton sets containing the same element, however ultrafilters do not contain the empty set so that $A = \{x\} = A'$ for some $x \in X$. An ultrafilter containing a singleton set is a principal ultrafilter, in other words it is the ultrafilter generated by that singleton set, thus $\mathfrak{a}$ and $\mathfrak{a}'$ are the same ultrafilter generated by the singleton set $\{x\}$. Therefore in general $\mathfrak{a}\vec{\beta}_{\oplus}\mathrm{E}_X\mathfrak{a}'$ holds if and only if $\mathfrak{a} = \mathfrak{a}'$ is a principal ultrafilter. As pointed out at the beginning of this section, since $\vec{\beta}$ is a flat lax extension, $\vec{\beta}_{\otimes}\mathrm{E}_X = \mathrm{E}_{\beta X}$ so that $\mathfrak{a}\ \vec{\beta}_{\otimes}\mathrm{E}_X\ \mathfrak{a}$ holds for all ultrafilters on X. Since X is an infinite set, there exists an ultrafilter $\mathfrak{a}$ containing the filter base of cofinite sets as described in example 1.4.4.III, which is a non-principal ultrafilter. Thus $\mathfrak{a}\ \vec{\beta}_{\otimes}\mathrm{E}_X\ \mathfrak{a}$ but not $\mathfrak{a}\ \vec{\beta}_{\oplus}\mathrm{E}_X\ \mathfrak{a}$, and one may say that $\vec{\beta}_{\oplus} \neq \vec{\beta}_{\otimes}$ in general. ∎

By the above propositions, one may write $\vec{\beta}_{\oplus} < \vec{\beta}_{\otimes}$, with respect to the obvious pointwise ordering of functors into **Rel** sharing the same object map.

5.3 Correspondence of linear and lax functors

As linear bicategories form a generalization of bicategories, linear functors also form a generalization of lax functors $F\colon \mathbf{B} \to \mathbf{B}'$ when $\mathbf{B}, \mathbf{B}'$ are bicategories with absolute right and left homs (See [14, §2.1]). In the case where $\mathbf{B}, \mathbf{B}'$ are *Girard bicategories*, we may make an even stronger statement, that lax functors are not only generalized by linear functors, but they are in fact determined by the same data. In this chapter we will prove this correspondence holds in the locally posetal case, which is of priority interest for application to lax extension, while further generalization requires a proof of coherence.

In the introduction of [15, pp. 156–157] it is briefly mentioned that functors of $*$-autonomous categories induce a pair of lax functors possessing linear strength. When $\hat{F}$ is a lax functor between locally posetal Girard bicategories $\mathbf{B}, \mathbf{B}'$, it induces an oplax functor, as we prove in proposition 4.1.5. Furthermore, we will prove that $(\hat{F}, \check{F})$ have linear strength.

We shall require the following preliminary results which we will use in conjunction to prove bijective correspondence of lax and linear functors.

The following is a simple proposition in bicategory theory for which we shall provide proof.

Lemma 5.3.1. *Let* $\mathbf{B}, \mathbf{B}'$ *be bicategories and* F *be a lax functor such that there are right extensions* $(A \multimap B, \phi)$ *and* $(FA \multimap FB, \psi)$*. Then there exists a 2-cell* $F(A \multimap B) \Rightarrow FA \multimap FB$*.*

Proof: Suppose we have $\mathbf{B}, \mathbf{B}', F, A : X \to Y, B : X \to Z$ as in the above. Since F is a lax functor, we have a canonical 2-cell $FA \otimes F(A \multimap B) \Rightarrow F(A \otimes (A \multimap B))$ by lax preservation of composition (denoted ϵ), and furthermore we have a 2-cell $F\phi : F(A \otimes (A \multimap B)) \Rightarrow FB$ by functoriality. By the universal property of $FA \multimap FB$ the composition of 2-cells maps uniquely to $\mathrm{curry}_{\psi}(\epsilon ; F\phi) : F(A \multimap B) \Rightarrow FA \multimap FB$. ∎

We note that dually for the right lifting we have a 2-cell $F(A \multimapinv B) \Rightarrow FA \multimapinv FB$. Next is a lemma on correspondence for linear natural transformations and lax natural transformations.

Lemma 5.3.2. *Let* $\mathbf{B}, \mathbf{B}'$ *be locally posetal Girard bicategories such that* $F, G : \mathbf{B} \to \mathbf{B}'$ *are lax functors and let* $\alpha : F \to G$ *be a lax natural transformation between them. Then* $\vec{\alpha} = (\alpha, \alpha^{\perp})$ *is a linear natural transformation between the linear functors* $\vec{F} = \langle F, F, (\text{-})^{\perp} ; F ; (\text{-})^{\perp} \rangle$ *and* $\vec{G} = \langle G, G, (\text{-})^{\perp} ; G ; (\text{-})^{\perp} \rangle$*.*

Proof: This is clear, since $(\text{-})^{\perp}$ reverses the order of domain, codomain, as well as the partial orderso that $\alpha_{\oplus_R} : GR \to FR$ becomes an oplax natural transformation. In a Girard bicategory $f \Vdash\!\!\dashv f^{\perp}$ for all 1-cells f. Thus $\alpha_X \Vdash\!\!\dashv {\alpha_X}^{\perp}$ so that $\vec{\alpha} = (\alpha, \alpha^{\perp})$ forms a linear natural transformation. ∎

Next is the main lemma we will use in our theorem.

Lemma 5.3.3. *Let* $\hat{F}$ *be a lax functor between Girard bicategories* $\mathbf{B}, \mathbf{B}'$*, and let* $\check{F}$ *be the dual of* $\hat{F}$*, that is* $\check{F} = (\text{-})^{\perp} ; \hat{F} ; (\text{-})^{\perp}$*. Then for all pairs* A, B *of composable*

1-cells in **B***, we have a 2-cell*

$$\hat{F}(A \oplus B) \Rightarrow \check{F}A \oplus \hat{F}B \textit{ in } \mathbf{B}'. \tag{5.3.e1}$$

Proof: Let $A\colon X \to Y$, and $B\colon Y \to Z$ be a pair of 1-cells in **B**. Since **B** is a Girard bicategory, by lemma 3.3.2 there is an invertible 2-cell, which we shall denote α, from $A \oplus B$ to $A^{\perp} \multimap B$. By functoriality, $F(\alpha)$ is an invertible 2-cell $\hat{F}(A \oplus B) \Rightarrow \hat{F}(A^{\perp} \multimap B)$. Furthermore by lemma 5.3.1, there is a 2-cell $\beta\colon \hat{F}(A^{\perp} \multimap B) \Rightarrow \hat{F}(A^{\perp}) \multimap \hat{F}B$. We may furthermore apply a 2-cell $\gamma \multimap \hat{F}B\colon \hat{F}(A^{\perp}) \multimap \hat{F}B \Rightarrow ((\hat{F}(A^{\perp}))^{\perp})^{\perp} \multimap \hat{F}B$ where $\gamma\colon \hat{F}(A^{\perp}) \Rightarrow (\hat{F}(A^{\perp})^{\perp})^{\perp}$ is the invertible 2-cell described in the previous section. However, $(\hat{F}((\text{-})^{\perp})^{\perp})^{\perp} = \check{F}(\text{-})^{\perp}$ so that $\gamma \multimap \hat{F}B\colon \hat{F}(A^{\perp}) \multimap \hat{F}B \Rightarrow (\check{F}A)^{\perp} \multimap \hat{F}B$. The codomain of this 2-cell is isomorphic to $\check{F}A \oplus \hat{F}B$ by lemma 3.3.2 through a 2-cell we shall denote α'^{-1}. By composing the above, $F\alpha \,;\, \beta \,;\, (\gamma \multimap \hat{F}B) \,;\, \alpha'^{-1}$ is a 2-cell matching 5.3.e1. ∎

One may also prove using the dual lemmas for right liftings to show that there is a 2-cell $\hat{F}(A \oplus B) \Rightarrow \hat{F}A \oplus \check{F}B$. We shall now prove the main theorem.

Theorem 5.3.4. *Let* $\mathbf{B}, \mathbf{B}'$ *be locally posetal Girard bicategories. There is an isomorphism of categories between* **Lax**$(\mathbf{B}, \mathbf{B}')$ *and* **Lin**$(\mathbf{B}, \mathbf{B}')$*.*

Proof: Let $\hat{F}$ be a lax functor. By proposition 4.1.5 the dual $\check{F}$ of $\hat{F}$ is oplax with respect to $\mathcal{Q}$-**Rel**. By lemma 5.3.3 and its dual lemma there exist 2-cells such that the pair $(\hat{F}, \check{F})$ satisfies the following linear strengths (see condition 3.4.1.ii):

$$\nu_{\otimes}^{\mathrm{R}}\colon \hat{F}(A \oplus B) \leq \check{F}A \oplus \hat{F}B, \text{ and}$$
$$\nu_{\otimes}^{\mathrm{L}}\colon \hat{F}(A \oplus B) \leq \hat{F}A \oplus \check{F}B.$$

Applying lemma 4.2.3 in the more general setting of a locally posetal Girard bicategory shows that $\nu_{\oplus}^{\mathrm{L}}$ and $\nu_{\oplus}^{\mathrm{R}}$ also hold. Thus the pair $(\hat{F}, \check{F})$ constitutes a linear functor by proposition 3.4.2 and so we may map lax functors injectively to linear functors via the assignment $\Phi(\hat{F}) = (\hat{F}, \check{F})$. Let $\alpha\colon F \to G$ be a lax natural transformation between lax functors $F, G\colon \mathbf{B} \to \mathbf{B}'$, then by lemma 5.3.2 $\vec{\alpha} := (\alpha, \alpha^{\perp})$ is a linear natural transformation $\vec{\alpha}\colon \vec{F} \Rightarrow \vec{G}$ and assign $\Phi(\alpha) = \vec{\alpha}$. Φ is a functor since $\Phi(\alpha\,;\beta) = ((\alpha\,;\beta, (\alpha\,;\beta)^{\perp})) = (\alpha\,;\beta, \beta^{\perp}\,;\alpha^{\perp})$ and by composition of linear natural transformations (see eq. (3.7.e2)), this is equal to $(\alpha, \alpha^{\perp})\,;(\beta, \beta^{\perp}) = \Phi(\alpha)\,;\Phi(\beta)$. Furthermore $\Phi(\mathbb{1}) = \vec{\mathbb{1}}$ so that $\Phi\colon \mathbf{Lax}(\mathbf{B}, \mathbf{B}') \to \mathbf{Lin}(\mathbf{B}, \mathbf{B}')$ is a functor.

The inverse mapping Ψ takes a linear functor $\vec{F}$ to a lax functor by forgetting the $\oplus$ component, that is $\Psi(\vec{F}) = \vec{F}_{\otimes}$ and $\Psi(\vec{\alpha}) = \vec{\alpha}_{\otimes}$. The functoriality of $\Psi : \mathbf{Lin}(\mathbf{B}, \mathbf{B}') \to \mathbf{Lax}(\mathbf{B}, \mathbf{B}')$ should be clear, and furthermore, $\Psi(\Phi(\hat{F})) = \hat{F}$, and $\Psi(\Phi(\alpha)) = \alpha$. Let $A\colon X \to Y$ be any 1-cell in $\mathbf{B}$ and define $\vec{F}' := \Phi(\Psi(\vec{F})), \vec{\alpha}' := \Phi(\Psi(\vec{\alpha}))$. Clearly $A \dashv\!\!| \, A^{\perp}$ since $\mathbf{B}$ is a Girard bicategory. By proposition 3.5.2 linear functors preserve linear adjoints so that $\vec{F}_{\otimes}A$ is a left (hence cyclic) linear adjoint to $\vec{F}_{\oplus}(A^{\perp})$ and therefore, since $\mathbf{B}'$ is posetal, $(\vec{F}_{\oplus}(A^{\perp}))^{\perp}$ is equal to $\vec{F}_{\otimes}A$, and dually $\vec{F}_{\oplus}A = (\vec{F}_{\otimes}(A^{\perp}))^{\perp}$. Thus $\vec{F}_{\oplus}$ is determined by $\vec{F}_{\otimes}$, so that $\vec{F}_{\oplus} = \vec{F}'_{\oplus}$. Same argument with linear adjoints applies to the linear natural transformation so that $\vec{\alpha}_{\oplus}$ is determined by $\vec{\alpha}_{\otimes}$. Thus $\Phi(\Psi(\vec{F})) = \vec{F}, \Phi(\Psi(\vec{\alpha})) = \alpha$ and hence Φ, Ψ comprise an isomorphism of categories so that $\mathbf{Lax}(\mathbf{B}, \mathbf{B}') \cong \mathbf{Lin}(\mathbf{B}, \mathbf{B}')$. ∎

Chapter 6

Linear extensions of monads and the linear Barr extension

Thus far we have established linear bicategorical analogues for lax extensions (linear extensions), the Barr extension (the linear Barr extension), and verified some linear Barr extensions for familiar **Set** endofunctors such as the free monoid functor and the ultrafilter functor. In the following chapter, we will add linear extensions of monads to this list, and establish a general condition under which the linear Barr extension is a linear extension of monads.

6.1 Linear extension of monads

Recall from section 2.3A that a lax extension of the **Set** monad $\mathbb{T} = (T, m, e)$ to $\mathcal{Q}$-**Rel** is a triple $\hat{\mathbb{T}} = (\hat{T}, \hat{m}, \hat{e})$ comprised of a lax extension $\hat{T}$ of T making m, e into colax natural transformations $m : \hat{T} ; \hat{T} \to \hat{T}$ and $e : \hat{I}_{\mathcal{Q}\text{-}\mathbf{Rel}} \to \hat{T}$. An analagous notion in a linear bicategory should then be a triple $\vec{\mathbb{T}} = (\vec{T}, \vec{m}, \vec{e})$ comprised of a linear extension $\vec{T}$ of the monad T, as well as *colinear* natural transformations $\vec{m} : \vec{T} ; \vec{T} \to \vec{T}$, $\vec{e} : \vec{I}_{\mathcal{Q}\text{-}\mathbf{Rel}} \to \vec{T}$. We shall call this notion a *linear extension of the* *monad* $\mathbb{T}$, and we note that this terminology should not be confused with the notion of a linear monad internal to a linear bicategory, as defined in [14, Definition 4.13].

We define a linear extension of monads formally as follows, and will expand on what we mean by *induced colinear natural transformations.*

Definition 6.1.1. A *linear extension of monads* $\vec{\mathbb{T}}$ of the **Set** monad $\mathbb{T} = (T, m, e)$ to $\mathcal{Q}$-**Rel** is a linear extension $\vec{T}$ of T to $\mathcal{Q}$-**Rel** such that $\vec{T}$ induces *colinear* natural transformations $\vec{m} : \vec{T}\vec{T} \to \vec{T}, \vec{e} : \vec{I} \to \vec{T}$ from m, e.

For lax extension of monads, what we mean more specifically when we say $\hat{T}$ makes m a colax natural transformation is that the embedding $m_\circ$ of m, where $(m_\circ)_X = (m_X)_\circ$, is a colax natural transformation $m_\circ : \hat{T}\hat{T} \to \hat{T}$, and similarly $e_\circ : I_{\mathcal{Q}\text{-}\mathbf{Rel}} \to \hat{T}$. We mean something similar for linear extensions of monads and induced colinear transformations.

Definition 6.1.2. Let F, G be **Set** endofunctors with linear extensions $\vec{F}, \vec{G}$ to $\mathcal{Q}$-**Rel**, and let $\alpha : F \to G$ be a natural transformation. We denote $\vec{\alpha}$ for the pair of $\mathsf{ob}(\mathbf{Set})$-indexed collections of $\mathcal{Q}$-relations $(\vec{\alpha}_\otimes)_X : \vec{F}X \to \vec{G}X$, $(\vec{\alpha}_\oplus)_X : \vec{G}X \to \vec{F}X$, such that

$$(\vec{\alpha}_\otimes)_X = (\alpha_X)_\circ, \quad \text{and} \quad (\vec{\alpha}_\oplus)_X = (\alpha_X)^\bullet. \tag{6.1.e1}$$

Where $(\text{-})^\bullet = ((\text{-})_\bullet)^\circ$. When $\vec{\alpha}$ is a linear natural transformation from $\vec{F}$ to $\vec{G}$, we call this the *induced linear natural transformation* with respect to the linear extensions $\vec{F}, \vec{G}$. We may also say that the extensions $\vec{F}, \vec{G}$ induce $\vec{\alpha} : \vec{F} \to \vec{G}$.

In order for $\vec{\alpha}$ to be an induced linear natural transformation, we must at least have that $\vec{\alpha}_\otimes = \alpha_\circ, \vec{\alpha}_\oplus = \alpha^\bullet$ are cyclic linear adjoints, therefore we prove the following lemma.

Lemma 6.1.3. *Let $\mathcal{Q}$ be a linearly distributive quantale, and let $R\colon X \nrightarrow Y$ be a morphism in* **Rel**. *Then $R_\circ$ is cyclic linear adjoint to $R^\bullet$ in $\mathcal{Q}$-**Rel** (written $R_\circ \Vdash\!\!\dashv\!\!\Vdash R^\bullet$) where, when applied to relations, the embeddings $(\text{-})_\circ : \mathbf{Rel} \to \mathcal{Q}\text{-}\mathbf{Rel}_\otimes$*

and $(\text{-})^{\bullet} : \mathbf{Rel}^{\mathrm{op}} \to \mathcal{Q}\text{-}\mathbf{Rel}_{\oplus}$ *are defined as follows:*

$$R_{\circ}(x,y) = \begin{cases} k_{\otimes} & \textit{if } x\,R\,y \\ \bot & \textit{otherwise} \end{cases}, \quad \textit{and} \quad R^{\bullet}(y,x) = \begin{cases} k_{\oplus} & \textit{if } x\,R\,y \\ \top & \textit{otherwise} \end{cases}. \tag{6.1.e2}$$

Furthermore, $(\text{-})^{\circ} : \mathbf{Rel}^{\mathrm{op}} \to \mathcal{Q}\text{-}\mathbf{Rel}_{\otimes}, (\text{-})_{\bullet} : \mathbf{Rel} \to \mathcal{Q}\text{-}\mathbf{Rel}_{\oplus}$ *are defined dually.*

Proof: Let $(x,y) \in X \times Y$ be an ordered pair, then:

$$(R_{\circ} \oplus R^{\bullet})(x,x) = \bigwedge_{y' \in Y} R_{\circ}(x,y') \oplus R^{\bullet}(y',x) \tag{6.1.e3}$$

$$= \bigwedge_{y' \in Y} \begin{cases} k_{\otimes} \oplus k_{\oplus} (= k_{\otimes}) & \text{if } x\,R\,y' \\ \bot \oplus \top (= \top) & \text{otherwise} \end{cases}. \tag{6.1.e4}$$

Thus $\mathrm{E}_X(x,x) = k_{\otimes} \leq (R_{\circ} \oplus R^{\bullet})(x,x)$ so that $\mathrm{E}_X \leq R_{\circ} \oplus R^{\bullet}$. Furthermore,

$$(R^{\bullet} \otimes R_{\circ})(y,y) = \bigvee_{x' \in X} R^{\bullet}(y,x') \otimes R_{\circ}(x',y) \tag{6.1.e5}$$

$$= \bigvee_{x' \in X} \begin{cases} k_{\oplus} \otimes k_{\otimes} (= k_{\oplus}) & \text{if } x'\,R\,y \\ \top \otimes \bot (= \bot) & \text{otherwise} \end{cases}, \tag{6.1.e6}$$

so that $(R^{\bullet} \otimes R_{\circ})(y,y) \leq k_{\oplus} = \mathrm{D}_Y(y,y)$. Hence $R_{\circ} \dashv\!\!| \, R^{\bullet}$, and dually $R_{\bullet} \dashv\!\!| \, R^{\circ}$.

We note that since $\otimes, \oplus$ act commutatively on pairs from the set $\{\bot, \top, k_{\otimes}, k_{\oplus}\} \subseteq \mathcal{Q}$, we have $R_{\circ}(x,y) \otimes R^{\bullet}(y,x) = R^{\bullet}(y,x) \otimes R_{\circ}(x,y)$ and $R^{\bullet}(y,x) \oplus R_{\circ}(x,y) = R_{\circ}(x,y) \oplus R^{\bullet}(y,x)$ so these evaluate the same as in the respective case statements of eq. (6.1.e6), and eq. (6.1.e4). Thus, we have $k_{\otimes} = \mathrm{E}_Y(y,y) \leq (R^{\bullet} \oplus R_{\circ})(y,y)$ and $(R_{\circ} \otimes R^{\bullet})(x,x) \leq k_{\oplus} = \mathrm{D}_Y(y,y)$ so that $\mathrm{E}_Y \leq R^{\bullet} \oplus R_{\circ}$ and $R_{\circ} \otimes R^{\bullet} \leq \mathrm{D}_X$. Hence $R^{\bullet} \dashv\!\!| \, R_{\circ}$, and we conclude $R_{\circ} \, |\!\!\vdash\!\!\dashv\!\!| \, R^{\bullet}$. ∎

By lemma 6.1.3, the maps $\alpha_{\otimes X}, \alpha_{\oplus X}$ as in definition 6.1.2 are automatically cyclic linear adjoints. Therefore, to prove $\vec{\alpha}$ is an induced linear natural transformation, we need only prove the lax and oplax conditions in eq. (3.7.e1). This will come into use in the next section on the linear Barr extension.

Examples 6.1.4.

(I) The identity linear monad $\vec{\mathbb{I}}$ is a linear extension of the identity monad $\mathbb{I} = (I, m, e)$ from **Set** to $\mathcal{Q}$-**Rel**. Clearly $\vec{I};\vec{I} = \vec{I}$, and the induced maps are given by $\vec{e}_\otimes = \vec{m}_\otimes =$ E, and $\vec{e}_\oplus = \vec{m}_\oplus =$ D. These satisfy the inequalities of eq. (3.7.e3), holding with equality. It follows from lemma 6.1.3 that these are induced colinear natural transformations and furthermore $\vec{\mathbb{I}}_{\mathcal{Q}\text{-}\mathbf{Rel}}$ is a linear extension of the monad $\mathbb{I}_{\mathbf{Set}}$.

(II) In section 6.2, we will determine that when the linear Barr extension $\vec{T}$ of a **Set** functor T is a linear extension of T to **Rel**, it induces a linear extension $\vec{\mathbb{T}}$ of a monad $\mathbb{T} = \langle T, m, e\rangle$ to **Rel**. Thus it follows from proposition 5.1.1 that the linear Barr extension of the list functor induces a linear extension of monads, and furthermore it follows from proposition 5.2.1 that the linear Barr extension of the ultrafilter functor induces a linear extension of monads.

6.2 The linear Barr extension of monads

We have shown previously that a **Set** endofunctor satisfying the BC condition and preserving linear strengths has a linear Barr extension $\vec{T}$. Furthermore, given in [25, §1.12.1] if one has a **Set** monad $\mathbb{T} = (T, m, e)$ satisfying BC, then the Barr extension yields a lax extension of monads $\overline{\mathbb{T}} = (\overline{T}, m, e)$. In this section we seek to establish an analagous result for the linear Barr extension.

Lemma 6.2.1. *Let S, T be* **Set** *endofunctors that satisfy the BC condition such that S and its dual and T and its dual satisfy the 4 linear strengths, and let $\alpha\colon S \to T$ be a natural transformation. Then the linear Barr extensions $\vec{S}, \vec{T}$ to* **Rel** *induce a* *colinear transformation* $\vec{\alpha}\colon \vec{S} \to \vec{T}$.

Proof: It follows from [25, III§1.12] that $\vec{\alpha}_{\otimes}\colon \vec{S}_{\otimes} \to \vec{T}_{\otimes}$ is a colax natural transformation, where $\vec{S}_{\otimes} = \overline{S}, \vec{T}_{\otimes} = \overline{T}$. That is for each $R\colon X \twoheadrightarrow Y$ we have the inequality $\vec{\alpha}_{\otimes X} \otimes \vec{T}_{\otimes}R \geq \vec{S}_{\otimes}R \otimes \vec{\alpha}_{\otimes Y}$. We obtain the other inequality to make $\vec{\alpha}$ a colinear natural transformation via De Morgan duality as follows.

$$\vec{\alpha}_{\otimes Y} \otimes \vec{T}_{\otimes}(R^{\perp}) \geq \vec{S}_{\otimes}(R^{\perp}) \otimes \vec{\alpha}_{\otimes X} \tag{6.2.e1}$$

$$\Longrightarrow (\vec{T}_{\otimes}(R^{\perp}))^{\perp} \oplus (\vec{\alpha}_{\otimes Y})^{\perp} \leq (\vec{\alpha}_{\otimes X})^{\perp} \oplus (\vec{S}_{\otimes}(R^{\perp}))^{\perp} \tag{6.2.e2}$$

$$\Longrightarrow \vec{T}_{\oplus}R \oplus \vec{\alpha}_{\oplus Y} \leq \vec{\alpha}_{\oplus X} \oplus \vec{S}_{\oplus}R \tag{6.2.e3}$$

Thus $\vec{T}_{\oplus}R \oplus \vec{\alpha}_{\oplus X} \leq \vec{\alpha}_{\oplus Y} \oplus \vec{S}_{\oplus}R$ as desired. In conjunction with the above, $\vec{\alpha}_{\otimes}, \vec{\alpha}_{\oplus}$ are cyclic linear adjoints via lemma 6.1.3 so that we may conclude $\vec{\alpha}\colon \vec{S} \to \vec{T}$ is an induced colinear natural transformation. ∎

Lemma 6.2.2. *Let T be a set endofunctor satisfying the BC condition. If the Barr extension $\overline{T}$ and its dual $(\overline{T}((\text{-})^{\perp}))^{\perp}$ satisfy the 4 linear strengths, then the linear Barr extension $\vec{T}\colon$ **Rel** $\to$ **Rel** satisfies $\overrightarrow{TT} = \vec{T}\,;\vec{T}$, where $\overrightarrow{TT}$ is the linear Barr extension of TT. Furthermore, $\overrightarrow{TT}$ is a linear extension of TT.*

Proof: The components of $\overrightarrow{TT}$ are pairwise equal to the following:

$$\overrightarrow{TT} = (\overline{TT}, (\text{-})^{\perp}\,; \overline{TT}\,; (\text{-})^{\perp}) \tag{6.2.e4}$$

$$= (\overline{T}\,; \overline{T}, ((\text{-})^{\perp}\,; \overline{T}\,; (\text{-})^{\perp})\,; ((\text{-})^{\perp}\,; \overline{T}\,; (\text{-})^{\perp})) \quad \text{By [25, III§1.12] } \overline{TT} = \overline{T}\,\overline{T} \tag{6.2.e5}$$

$$= (\vec{T}_{\otimes}\,; \vec{T}_{\otimes}, \vec{T}_{\oplus}\,; \vec{T}_{\oplus}) \tag{6.2.e6}$$

$$= \vec{T}\,; \vec{T} \tag{6.2.e7}$$

Since $\vec{T}$ is a linear extension, the components of the associated linear functor $\vec{T}\,;\vec{T}$ laxly and colaxly preserves composition and identities, and $\vec{T}_{\otimes}, \vec{T}_{\oplus}$ satisfy the 4 linear strengths. Thus the linear Barr extension $\overrightarrow{TT}$, which is componentwise equal to $\vec{T}\,;\vec{T}$, must be a linear extension of TT to **Rel**. ∎

The following main result of this section follows easily from the above lemmas.

Theorem 6.2.3. *Let* $\mathbb{T} = (T, m, e)$ *be a* **Set** *monad. If* T *satisfies the Beck-Chevalley condition and the Barr extension* $\overline{T}$ *and its dual* $(\overline{T}((\text{-})^{\perp}))^{\perp}$ *satisfy the 4 linear strengths, then the linear Barr extension* $\vec{\mathbb{T}} = (\vec{T}, \vec{m}, \vec{e})$ *is a linear extension of* $\mathbb{T}$ *to* **Rel**.

Proof: It follows from the proof of lemma 6.2.2 that $\overrightarrow{TT} = \vec{T}\vec{T}$ and its dual satisfy the 4 linear strengths. Therefore we may apply lemma 6.2.1 so that we obtain induced colinear natural transformations $\vec{m}\colon\ \overrightarrow{TT} \to \vec{T}$, $\vec{e}\colon\ \vec{I} \to \vec{T}$ from m, e respectively. It also follows from lemma 6.2.2 that $\overrightarrow{TT} = \vec{T}\,;\vec{T}$ so that $\vec{m}$ is an induced colinear natural transformation $\vec{T}\vec{T} \to \vec{T}$. Thus $\vec{\mathbb{T}} = (\vec{T}, \vec{m}, \vec{e})$ is a linear extension of the monad $\mathbb{T}$. ∎

Examples 6.2.4.

(I) Applying theorem 6.2.3 following proposition 5.1.1, the linear free monoid monad $(\mathscr{L}, m, e)$ has a linear Barr extension of monads with triple $(\vec{\mathscr{L}}, \vec{m}, \vec{e})$ where the unit and multiplication have the following $\oplus$-components, determined via linear negation:

$$\vec{m}_{\oplus}(\langle\langle x^i_{j\,j}\rangle_{j=1}^{n_i}\rangle_{i=1}^{n}, \langle x'_i\rangle_{i=1}^{m}) = \begin{cases} \perp & \text{if } m(\langle\langle x^i_{j\,j}\rangle_{j=1}^{n_i}\rangle_{i=1}^{n}) = \langle x'_i\rangle_{i=1}^{l} \\ \top & \text{otherwise} \end{cases} \tag{6.2.e8}$$

Where $\langle {x'}_i \rangle_{i=1}^{l}$ is the concatenated sequence, and

$$\vec{e}_{\oplus}(\langle x_i \rangle_{i=1}^{n}, x') = \begin{cases} \bot & \text{if } \langle x_i \rangle_{i=1}^{n} = \langle x' \rangle \\ \top & \text{otherwise} \end{cases} . \tag{6.2.e9}$$

(II) Similar to the previous example, applying theorem 6.2.3 with respect to β it follows from proposition 5.2.1 that the linear Barr extension of monads is a linear extension of monads.

Chapter 7

Linear adjoints in $\mathcal{Q}$-Rel

When the quantale $\mathcal{Q}$ is *integral* and *lean*, it is a theorem that the maps in $\mathcal{Q}$-**Rel**, that is the $\mathcal{Q}$-relations with left adjoints, are exactly the **Set** maps in $\mathcal{Q}$-**Rel** [25, III§1.2.1]. When $\mathcal{Q}$-**Rel** is a linear bicategory we may in addition define left linear adjoints as in section 3.5, and the question that remains is, what structure do the left linear adjoints possess? It turns out that there is an analagous result for the left linear adjoints, namely, when $\mathcal{Q}$ is *linearly integral* and *linearly lean*, the left linear adjoints are exactly the 2-*relations* in $\mathcal{Q}$-**Rel**.

7.1 Linearly integral and linearly lean LD-quantales

Let $\mathcal{Q}$ be a linearly distributive quantale. We shall call $\mathcal{Q}$ *linearly integral* when both $\mathcal{Q}_\otimes$ and $\mathcal{Q}_\oplus$ are integral, that is $k_\otimes = \top, k_\oplus = \bot$. We shall call $\mathcal{Q}$ *linearly lean* provided that, for all $u, v \in \mathcal{Q}$, the following condition holds:

$$u \oplus v = \top \text{ and } v \otimes u = \bot \implies u = \top \text{ or } v = \top. \qquad (7.1.e1)$$

Note that $\mathcal{Q}$ may be viewed as a linear bicategory with a single object (see example 3.2.2.II), thus $\mathcal{Q}$ has linear adjoints. When $\mathcal{Q}$ is linearly integral, eq. (7.1.e1) essentially tells us that the only linear adjoints in $\mathcal{Q}$ are $\top$ and $\bot$.

Proposition 7.1.1. *Let $\mathcal{Q}$ be a linearly distributive quantale that is linearly integral. If $\mathcal{Q}$ is linearly lean, then $\mathcal{Q}_{\otimes}$ is lean and $\mathcal{Q}_{\oplus}$ is lean.*

Proof: Let $u, v \in \mathcal{Q}$ and suppose that $u \vee v = \top$ and $u \otimes v = \bot$. Since $u = u \oplus \bot$, and $v = \bot \oplus v$, by monotonicity of $\oplus$, $u \vee v \leq u \oplus v$ so that $\top = u \oplus v$. By the linear leanness of $\mathcal{Q}$ we must have $u = \top$ or $v = \top$, and thus $\mathcal{Q}_{\otimes}$ is lean.

To show $\mathcal{Q}_{\oplus}$ is lean, we remark on the following. First, note that linear integrality tells us that $u = \top \implies v = v \otimes u = \bot$ and similarly $v = \top \implies u = v \otimes u = \bot$ so that swapping $\otimes$ with $\oplus$ and $\top$ with $\bot$ in eq. (7.1.e1) results in an equivalent condition. Hence $\mathcal{Q}^{\mathrm{co}}$ is also linearly lean, so that $\mathcal{Q}^{\mathrm{co}}{}_{\otimes}$, which is equal to $\mathcal{Q}_{\oplus}$, is lean by the above proof. ∎

Examples 7.1.2.

(I) The lattice $[0, \infty]^{\mathrm{op}}$, with $a \otimes b = \max(a, b)$ and $a \oplus b = \min(a, b)$ is a linearly distributive quantale which is linearly integral and linearly lean. More generally $(\mathscr{C}, \wedge, \vee)$ is linearly lean and linearly integral for any complete chain $\mathscr{C}$. That it is linearly integral follows since the minimum and maximum elements of a lattice are the identity for $\vee, \wedge$ as described in the algebraic characterization of lattices found in theorem 1.1.5. Since it is totally ordered $x \leq y$ or $y \leq x$ for each $x, y \in \mathscr{C}$, furthermore $x \wedge y = x$ if and only if $x \leq y$ so that $x \vee y = y$ or $x \vee y = x$ as in theorem 1.1.5. Thus $x \oplus y = \top$ implies $x = \top$ or $y = \top$ so that $\mathscr{C}$ is linearly lean.

(II) Let X be a set with $|X| > 2$. The powerset $\mathcal{P}X$ of X is a frame, furthermore $\mathcal{P}X^{\mathrm{op}}$ is also a frame so that $\mathcal{P}X$ forms an LD quantale. This is clearly linearly integral, but it is not linearly lean since for $x \in X$, $\{x\} \otimes \{x\}^{\complement} = \varnothing$ and $\{x\} \oplus \{x\}^{\complement} = X$.

(III) The LD quantale induced by a shift monoid $\langle M, \otimes, e, d\rangle$ as in example 3.1.2.VI with invertible element $d \neq e$ is linearly lean, since $a \otimes b = \top$ if and only if one of a, b is $\top$ as defined in [9], but it is not linearly integral since the $\otimes$ identity is e and the $\oplus$ identity is d.

(IV) Note that being linearly lean is a stronger condition than $\mathcal{Q}_\otimes, \mathcal{Q}_\oplus$ both being lean. Consider the following linearly distributive quantale $\dot{3}$.

The underlying lattice of $\dot{3}$ is the three-chain $\{\bot, e, \top\}$, with $\otimes$ and $\oplus$ determined by $e \otimes e = \bot, e \oplus e = \top$ and respective identities $\top, \bot$. This is the same as the MV-algebra induced on $\{0, \frac{1}{2}, 1\}$ as described by Chang in [11, p. 473]. $\dot{3}$ is linearly integral and $\dot{3}_\otimes, \dot{3}_\oplus$ are lean, however $\dot{3}$ is not linearly lean, since $e \otimes e = \bot, e \oplus e = \top$, but $e \neq \top$.

7.2 2-relations and linear adjoints in $\mathcal{Q}$-Rel

A 2-*relation* in $\mathcal{Q}$-**Rel** is a relation in the sense of **Rel**. More specifically, these are the $\mathcal{Q}$-relations mapped to by the embedding of **Rel** in $\mathcal{Q}$-**Rel**. We define the embedding $(\text{-})_\circ$: **Rel** → $\mathcal{Q}$-**Rel** as the following, which is an extension from **Set** to **Rel** of the canonical embedding $(\text{-})_\circ$: **Set** → $\mathcal{Q}$-**Rel** from eq. (2.3.e1). This is defined as follows:

$$R_\circ(x, y) = \begin{cases} k_\otimes & \text{if } x\ R\ y \\ \bot & \text{otherwise} \end{cases} \tag{7.2.e1}$$

Recall that a quantale is *integral* when $k = \top$, and *lean* when for all $u, v \in \mathcal{Q}$,

$$u \wedge v = \bot \ \text{ and } u \otimes v = \top \implies u = \top \ \text{ or } \ v = \top. \tag{7.2.e2}$$

The following is the main result of this chapter.

Theorem 7.2.1. *Let $\mathcal{Q}$ be a linearly distributive quantale that is linearly integral.*

*Then $\mathcal{Q}$ is linearly lean if and only if the 2-relations in $\mathcal{Q}$-**Rel** are the same as the linear adjoints in $\mathcal{Q}$-**Rel**.*

In our proof of the above theorem we will make use of the following lemma:

Lemma 7.2.2. *When $\mathcal{Q}$ is linearly integral, $R\colon X \nrightarrow Y$ is a 2-relation if and only if for each $(x, y) \in X \times Y$, $R(x, y) \in \{\bot, \top\}$.*

Proof: (**Lemma 7.2.2**)

When $\mathcal{Q}$ is linearly integral, we have $k_\otimes = \top, k_\oplus = \bot$ so that via eq. (6.1.e2) $R\colon X \nrightarrow Y$ in **Rel** is embedded into $\mathcal{Q}$-**Rel** as follows:

$$R_\circ(x, y) = \begin{cases} \top & \text{if } x\,R\,y \\ \bot & \text{otherwise} \end{cases} \tag{7.2.e3}$$

Thus, for each $(x, y) \in X \times Y$, the 2-relation $R_\circ$ has $R_\circ(x, y) \in \{\bot, \top\}$. Conversely, let $R\colon X \nrightarrow Y$ be a $\mathcal{Q}$-relation such that $R(x, y) \in \{\bot, \top\}$. Then R is the image of the relation R' where $xR'y \iff R(x, y) = \top$. Hence, when $\mathcal{Q}$ is linearly integral, a $\mathcal{Q}$-relation $R\colon X \nrightarrow Y$ is a 2-relation if and only if $R(x, y) \in \{\bot, \top\}$. ∎

Proof: (**Theorem 7.2.1**)

Suppose that $\mathcal{Q}$ is linearly lean. Let $R\colon X \nrightarrow Y$ be a left linear adjoint in $\mathcal{Q}$-**Rel**, and let $S\colon Y \nrightarrow X$ be the right linear adjoint to R. Then $\mathrm{E}_X \le R \oplus S$ and $S \otimes R \le \mathrm{D}_Y$. That is, for each $(x, y) \in X \times Y$

$$\top = \mathrm{E}_X(x, x) \tag{7.2.e4}$$

$$\le R \oplus S(x, x) = \bigwedge_{y' \in Y} R(x, y') \oplus S(y', x) \tag{7.2.e5}$$

$$\le R(x, y) \oplus S(y, x), \tag{7.2.e6}$$

and

$$\begin{aligned} \bot &= \mathrm{D}_Y(x,x) && (7.2.e7)\\ &\geq S\otimes R(y,y) = \bigvee_{x'\in X} S(y,x')\otimes R(x',y) && (7.2.e8)\\ &\geq S(y,x)\otimes R(x,y). && (7.2.e9)\end{aligned}$$

Thus for each $(x,y)\in X\times Y$ we have $R(x,y)\oplus S(y,x)=\top$ and $S(y,x)\otimes R(x,y)=\bot$, and so by the linear leanness of $\mathcal{Q}$, we have either $R(x,y)=\top$ or $S(y,x)=\top$, and in the latter case by linear integrality $R(x,y)=R(x,y)\otimes S(x,y)=\bot$. Hence $R(x,y)\in\{\bot,\top\}$ so that by lemma 7.2.2 R is a 2-relation in $\mathcal{Q}$-**Rel**. Furthermore, for any linearly distributive quantale $\mathcal{Q}$, when $R\colon X\nrightarrow Y$ is a **Rel** relation, $R_\circ$ is left linear adjoint to $R^\bullet$ (equal to $(R_\bullet)^\circ$) in $\mathcal{Q}$-**Rel** via lemma 6.1.3. Thus when $\mathcal{Q}$ is linearly lean, the 2-relations in $\mathcal{Q}$-**Rel** are the left linear adjoints.

Conversely, suppose that the 2-relations in $\mathcal{Q}$-**Rel** are the left linear adjoints. That is, via lemma 7.2.2, $R\colon X\nrightarrow Y$ in $\mathcal{Q}$-**Rel** is a left linear adjoint if and only if for each $(x,y)\in X\times Y$, $R(x,y)\in\{\bot,\top\}$. Let $u,v\in\mathcal{Q}$ and suppose $u\oplus v=\top$, and $v\otimes u=\bot$. There are $\mathcal{Q}$-relations $R_u,S_v\colon\{*\}\nrightarrow\{*\}$ where $R_u(*,*)=u$ and $S_v(*,*)=v$. Then $R_u\oplus S_v(*,*)=R_u(*,*)\oplus S_v(*,*)=u\oplus v=\top$, and $S_v\otimes R_u(*,*)=S_v(*,*)\otimes R_u(*,*)=v\otimes u=\bot$. Thus $\mathrm{E}_{\{*\}}\leq R_u\oplus S_v$ and $S_v\otimes R_u\leq \mathrm{D}_{\{*\}}$ so that R_u is a left linear adjoint and hence a 2-relation in $\mathcal{Q}$-**Rel**. Therefore, $u=R_u(*,*)\in\{\bot,\top\}$ so that either $u=\top$ or $u=\bot$. In the latter case $v=u\oplus v=\top$ so that $u=\top$ or $v=\top$. Thus, $\mathcal{Q}$ is linearly lean. ∎

According to lemma 6.1.3, the 2-relations in $\mathcal{Q}$-**Rel** are also cyclic linear adjoints, and all cyclic linear adjoints are left linear adjoints. This bears a strong resemblance to the linear map construction CMap(**B**), a bicategory whose 1-cells are the cyclic linear adjoints of a linear bicategory **B** (defined in [14, Definition

4.2]), is a sort of sub-linear bicategory of **B** as a consequence of [14, p. 4.4]. Therefore our result is analagous to the result on maps in $\mathcal{Q}$-**Rel** from [25, III§1.2.1]. Thus we conclude with the following corollary, which holds via theorem 7.2.1.

Corollary 7.2.3. *Let $\mathcal{Q}$ be a linearly distributive quantale so that $\mathcal{Q}$-**Rel** is a linear bicategory. If $\mathcal{Q}$ is linearly integral, then the cyclic linear maps in $\mathcal{Q}$-**Rel** are the 2-relations in $\mathcal{Q}$-**Rel** if and only if $\mathcal{Q}$ is linearly lean.*

We conclude this chapter with an example illustrating the main result.

Example 7.2.4. An example of a linearly integral quantale which is not linearly lean is $\mathcal{P}X$ with $|X| \geq 2$, where $\otimes = \cap, \oplus = \cup, \top = X, \bot = \varnothing$. There are $\mathcal{Q}$-relations A, B such that $A(*, *) = x$ and $B(*, *) = x^{\complement}$. Then A is left linear adjoint to B. However for each relation R on $\{*\}$, $R_{\circ}(*, *) = X$ or $R_{\circ}(*, *) = \varnothing$, so $R_{\circ} \neq A$. Thus, in the case $\mathcal{Q} = \mathcal{P}X$, linear adjoints in $\mathcal{P}X$-**Rel** are not the same as the 2-relations.

Chapter 8

Postscript

At the outset, the goal of this thesis was to define linear extensions of **Set** functors and monads to $\mathcal{Q}$-**Rel** based on the lax extensions of monoidal topology as presented in [25]. In addition we have defined a linear Barr extension of monads, a linear bicategorical analogue of the map construction, as well as a full characterization of linear extensions to $\mathcal{Q}$-**Rel** when $\mathcal{Q}$ is Girard. However, there is much left available for future exploration which we shall address here.

A category of critical interest to monoidal topology is the category $(\mathbb{T}, \mathcal{Q})$-**Cat** of lax-algebras with respect to a monad $\mathbb{T}$. One hope for future research would be the explication of linear T-algebras with respect to a linear extension $\vec{\mathbb{T}}$ of a monad $\mathbb{T}$, that is a linear bicategorical structure on the category $(\mathbb{T}, \mathcal{Q})$-**Cat** as described in [25, III§1.3]. Since $(\mathbb{T}, \mathcal{Q})$-**Cat** generalizes such fundamental structures as **Ord**, **Met**, **Top** and many others surveyed in [19, Table 1], it would seem quite interesting and valuable to imbue $(\mathbb{T}, \mathcal{Q})$-**Cat** with a nontrivial linear bicategorical structure. A start of exploration into linear bicategorical structures in monoidal topology may be found in [9], for example $\mathcal{Q}$-**Mod**, the locally posetal bicategory of $\mathcal{Q}$-categories and $\mathcal{Q}$-bimodules ordered pointwise. On the other hand, it may be possible (and equally interesting) that $(\mathbb{T}, \mathcal{Q})$-**Cat** admits no linear bicategorical structure except for the trivial one. Since the morphisms of $\mathcal{Q}$-**Cat** behave more like **Set** functions instead of relations, it is possible there are no interesting linear structures here. One last route of investigation on this topic may be to consider the alternate presentation of the lax $\mathbb{T}$-algebras as presented in [25, Part IV], which could shed more light onto this question. If this presentation is more amicable to linear bicategory structures, then one may expect some on $(\mathbb{T}, \mathcal{Q})$-**Cat**.

One other avenue for further research iterating on the results of this thesis would be to explore generalization of theorems proven only in the locally posetal case. For example, theorem 5.3.4 which may hold in the more general case pending proof of the coherence conditions of linear distributivity, which must be satisfied by the 2-cell determined in lemma 5.3.3. Another case is our generalization of the transit map example in example 3.1.2.VII to an ordered monoidal category example 3.1.2.VIII which could be generalized to any monoidal category. While most coherence conditions hold via projection of diagrams into M and the coherence therein as described by [16], the arguments for certain coherence and naturality diagrams in which multiple cotensors are involved need to be checked.

One final area to explore further would be the linear map construction and the cyclic linear map construction. One objective would be to consider a theorem like theorem 7.2.1 in the case where $\mathcal{Q}$ is not linearly integral. We expect a similar characterization as in the proposition of [12, p. 179]. In a bicategory of relations the maps form a discrete substructure as in [10], one wonders if there are comparable theorems to be made for the linear maps. Furthermore, Theorem 7.2.1 in conjunction with [12, p. 179] and in lieu of an undiscovered interesting $\oplus$ on **Set**, indicates that one might expect the structure of the linear adjoints (inherently relational and richly ordered) to be in contrast to the structure of the adjoints (inherently **Set**-like and discreetly ordered). If such a theorem could be nailed down, it can tell us things about lax extension, which as in variations described in [25, 27], involves extending endofucntors on **Set**-like structures (respectively **Set**, $\mathcal{Q}$-**Cat**) to relational structures (respectively $\mathcal{Q}$-**Rel**, $\mathcal{Q}$-**Mod**). It may be that this is a necessary and desired trait characterizing lax extension and its generalization to all bicategories.

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