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**MULTIFRACTAL MODELLING AND SPATIAL ANALYSIS WITH GIS:
GOLD POTENTIAL ESTIMATION IN THE MITCHELL-SULPHURETS
AREA, NORTHWESTERN BRITISH COLUMBIA**

by

Qiuming Cheng

A thesis submitted to the School of Graduate Studies in partial fulfilment
of the requirements for the degree of Ph.D. in Earth Sciences

OTTAWA-CARLETON GEOSCIENCE CENTRE

UNIVERSITY OF OTTAWA

OTTAWA, CANADA, 1994



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ISBN 0-612-04867-5

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**MULTIFRACTAL MODELLING AND SPATIAL ANALYSIS WITH GIS: GOLD
POTENTIAL ESTIMATION IN THE MITCHELL-SULPHURETS AREA,
NORTHWESTERN BRITISH COLUMBIA**

submitted by Qiuming Cheng

in partial fulfilment of the requirements for the degree of Doctor of Philosophy

Thesis Supervisor

Chairman, Examining Committee

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ABSTRACT

In combination with statistical modelling and GIS-based (Geographic Information Systems) spatial analysis, fractal and multifractal theory can be used for various purposes in the geosciences. In general, fractal models are applicable to binary patterns whereas measures such as element concentration values and number of points, lengths of line segments or areas of polygons per unit cell are better described by means of multifractal models. Two multifractal models have previously been developed. One of these is based on the fractal dimension spectrum $f(\alpha)$ and the other one on the codimension function $C(\gamma)$. It can be shown that these two multifractal models are identical if $\gamma > D - \alpha(0)$. For $\gamma < D - \alpha(0)$, however, $f(\alpha)$ is a continuous function whereas $C(\gamma)$ becomes a constant. In this thesis, the $f(\alpha)$ model is used.

In general, the multifractal model provides more information about a measure than a fractal model. Several interrelated fractal models with different dimensions can be derived from the multifractal model and these can be used for different purposes. An element concentration-area fractal model is derived for separating geochemical anomalies from background on the basis of element concentration values with multifractal properties. The model's use is demonstrated for two data sets: (1) Au and Au-associated trace elements and oxides in bedrock samples from the Mitchell-Sulphurets mineral district; and (2) Au in stream sediment samples from the Iskut River map sheet, northwestern British Columbia. Perimeters and areas in two-dimensional space may be related by power-law

relationships. The power-law exponents obtained in perimeter-area analysis are associated with, but do not necessarily provide, unbiased estimates of the fractal dimensions of the perimeters and areas. The exponents can be used as unbiased estimates of the dimension of the perimeter (D_L) only if the dimension of the measured area is $D_A = 2$. If $D_A < 2$, the ratio exponent $D_{AL} = 2D_L/D_A > D_L$. Similar relations hold true for surface areas and volumes of fractal geometries in three-dimensional space.

The multifractal model provides new types of functions for modelling the covariance and semivariogram in spatial statistics as well as a new type of second-order intensity function for two-dimensional point processes. These functions are used to describe the underlying spatial structure of Au mineral occurrences in the Iskut River map sheet and for three relatively well known data sets: (1) Cochran's example of the number of tree seedlings per foot along a 200 ft long bed, (2) De Wijs's example of zinc values from a sphalerite-quartz vein near Pulacayo in Bolivia, and (3) Gerrard's example of trees in Lansing Woods.

A new version of the weights of evidence technique based on the box-counting method is proposed for integrating various nonfractal and fractal patterns in mineral potential mapping, and is demonstrated with a case study of Au potential estimation in the Iskut River map sheet. The ordinary weights of evidence method is used for geochemical anomaly separation in the Mitchell-Sulphurets mineral district and gives results similar to those obtained with the fractal element concentration-area method. As another

nonfractal method of geochemical anomaly separation, a new spatial statistical approach based on U-statistics is proposed which uses a moving average with optimal window radius. This method is tested by computer simulation and for geochemical anomaly separation of Au and Au-associated elements in stream sediment samples from the Iskut River map sheet.

The calculations required for the applications of the fractal and multifractal models developed in this thesis are performed with the aid of GIS. The raster model, with functions for converting vector and raster formats into one another, and various types of point and map modelling functions available in most GIS's are advantageous for the implementation of fractal and multifractal analysis.

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ACKNOWLEDGEMENTS

This study was partially funded by an Natural Sciences and Engineering Research Council (NSERC) grant to Dr. Frederik P. Agterberg at the University of Ottawa. Most data were provided by the Geological Survey of Canada (GSC) and initial data processing and GIS analysis work was carried out using computers of the GSC Geomathematics Section. I gratefully acknowledge all support received at GSC. In particular, I wish to acknowledge the help of Dr. S.B. Ballantyne, geochemistry scientist, who provided the geological and geochemical data for this study and gave guidance during a field trip to the Mitchell-Sulphurets area in the summer of 1992. Significant contributions to this project came from members of the GSC Geomathematics Section through encouragement, technical instruction and discussions. Mr. Danny Wright and Dr. Andy Rencz provided technical assistance and allowed usage of their computers in the GSC Geomathematics Section. Dr. Graeme F. Bonham-Carter reviewed most parts of this thesis and suggested many improvements associated with GIS applications. I wish to express my sincerest gratitude to the advice and support of my supervisor Dr. Frederik P. Agterberg, who was always available for helpful discussion and assistance throughout the project. His and Mrs. Agterberg's enthusiasm and hospitality provided me and my family more comfort during the stay in Ottawa. I also like to thank my friends, colleagues and the staff in the Department of Earth Sciences at the University of Ottawa for their help on numerous occasions. Finally, I would like to thank and acknowledge the patience of my wife, Ming Wu, and my son, Tony Hao Cheng, who put up with my many long evenings on the computer.

INTRODUCTION

Spatial characteristics of geological objects must be considered in the classification, recognition and interpretation of patterns that result from underlying geological processes. The geological objects are characterized by distinct physical or chemical properties. However, observed and measured geoscience data commonly reflect the superposition of patterns associated with different geological processes. For example, Au concentration values measured on rock samples usually are the end product of several regional geological processes and Au mineralization of different types. Each of these processes not only contributed to the individual Au values but also produced characteristic spatial patterns.

With the development of new technologies, more geoscience data are becoming available, and patterns of a more complex nature can be studied. Objective, quantitative methods, especially statistical approaches, play an important role in this endeavor. Computer-based geographical information systems (GIS) provide a powerful tool for the handling the spatial data. Newly developed quantitative techniques for spatial data integration, including statistical methods, artificial intelligence and fuzzy logic, can be implemented with GIS. These methods are, therefore, increasingly being used for spatial data analysis in many fields of geoscience.

Recently developed fractal theory provides powerful tools for revealing the complex

nature of spatial objects (Mandelbrot, 1983; Feder, 1988). Fractal models have been applied in various fields of geoscience (Korvin, 1992; Turcotte, 1992; Fowler and Roach, 1992; Agterberg et al., 1993a; Agterberg, 1994a; Cheng and Agterberg, 1994; Cheng et al., 1994a, 1994b, 1994d). This thesis primarily deals with the development and application of fractal/ multifractal methods and their implementation with GIS.

Fractals are irregular geometric sets with Hausdorff dimensions different from their topological dimensions. A fractal generally has non-integer dimension. The ordinary concepts of number, length and area may not be valid for fractals. For example, the length of a fractal line measured in one-dimensional space may be infinite. Mathematical methods developed based on Lebesgue measure may not be valid and need modification. Various fractal models have been developed and used in different fields of science. There are two major types of models: fractal and multifractal models. Fractal models are commonly employed for describing the geometrical nature of fractals and for relationships between fractals. Multifractal models deal with spatially intertwined fractal measures with self-similarity defined on geometrical support, and have been intensively used in physics and chemistry for describing complex quantities with self-similarity (Feder, 1988; Schertzer and Lovejoy, 1991a, 1992; Evertsz and Mandelbrot, 1992; Agterberg, 1994a). It will be shown in this thesis that multifractal models can be used for characterizing spatial objects in the geosciences; for example, Au-associated faults and igneous rocks, patterns of Au concentration values determined on rock samples, and the distribution of Au mineral occurrences within a given area.

In general, various fractal models with different fractal dimensions can be derived from a multifractal model and can be used for different purposes. In this thesis, a concentration-area fractal model will be derived from a multifractal model for geochemical anomaly separation (also see Cheng, Agterberg and Ballantyne, 1994a); statistical models primarily characterized by a single fractal dimension will be derived for analysis of spatial patterns and spatial point processes by means of random variables with fractal properties (also see Cheng and Agterberg, 1994); and a fractal method will be developed for integrating fractal and nonfractal patterns in mineral potential mapping and demonstrated with a case study of Au potential mapping in the Iskut River area (also see Cheng, Agterberg and Bonham-Carter, 1994b). These models will be developed in such a way that they can be readily implemented with GIS. The scope of the study can be summarized as follows:

Chapter 1 introduces the general geology of the study areas selected to create the database for GIS and statistical applications. Two areas were chosen: the Iskut River map sheet, northwestern British Columbia, and the Mitchell-Sulphurets mineral district located within the Iskut River map sheet.

Chapter 2 discusses a nonfractal spatial statistical approach for geochemical anomaly separation. The procedure is demonstrated using concentration values for Au and Au-associated trace elements in stream sediment samples from the Iskut River map sheet.

Chapter 3 introduces the basic concepts of fractal geometry, fractal modelling and their implementation with GIS. The chapter covers the box-counting method, number-size model and contains a detailed discussion of the perimeter-area model (also see Cheng, 1994).

Chapter 4 discusses the concepts of multifractal modelling and introduces two previously developed multifractal models, one based on the fractal spectrum $f(\alpha)$, and the other on the codimension function $C(\gamma)$. The first model is applied to several examples including Au concentration values in bedrock samples and Au-associated faults and igneous rocks in the Mitchell-Sulphurets mineral district.

Chapter 5 introduces a new power-law type fractal model (concentration-area model) derived from the multifractal model (also see Cheng, Agterberg and Ballantyne, 1994a) for geochemical anomaly separation. The model will be illustrated with applications to two datasets: Au-associated trace element and oxides in rock samples from the Mitchell-Sulphurets mineral district, and Au concentration values in stream sediment samples from the Iskut River map sheet.

Chapter 6 deals with the multifractal model and spatial statistics. New results derived from the multifractal model (also see Cheng and Agterberg, 1994) are presented for modelling autocorrelation, covariance and semivariogram functions in spatial statistics. The usefulness of this approach will be demonstrated by means of applications to two

datasets: (a) Cochran's example of number of tree seedlings per foot along a 200 ft long bed; (b) De Wijs's example of zinc values from a sphalerite-quartz vein in Bolivia.

Chapter 7 discusses the use of multifractal models for analysis of spatial point processes. Newly derived relationships between the multifractal model and second-order intensity and $K(r)$ functions (also see Cheng and Agterberg, 1994) are presented. These results will be applied to two datasets: (a) hickory and oak trees in the Lansing Woods, Clinton, Michigan, U.S.A.; (b) Au mineral occurrences in the Iskut River map sheet.

Chapter 8 introduces a new approach as proposed by Cheng, Agterberg and Bonham-Carter (1994b) for fractal and nonfractal pattern integration in mineral potential mapping. This method will be demonstrated with a case study of Au potential estimation in the Iskut River map sheet.

Chapter 1 STUDY AREAS AND DATABASE

1.1 STUDY AREAS AND GENERAL GEOLOGY

Two areas were chosen for study: the Iskut River map sheet (N.T.S. 104B), northwestern British Columbia (Fig. 1.1) for spatial data integration in Au potential mapping at 1:250,000 scale, and the Mitchell-Sulphurets mineral district (located in the northwestern part of the Iskut River map sheet) as a localized area for fractal and multifractal modelling of spatial patterns associated with Au mineralization at 1:20,000 scale.

1.1.1 Iskut River Map Sheet

The Iskut River area is located in northwestern British Columbia (Fig. 1.2). In this relatively isolated area, 6 Au deposits and 177 Au mineral occurrences of predominantly hydrothermal types have been documented (B.C. Minfile Map 104B, 1989; Anderson, 1993). Geological characterization of the Iskut River area includes the following dominant features. The area is underlain by Paleozoic and Mesozoic sedimentary, volcanic and plutonic rocks and has been subjected to low-grade regional metamorphism, heterogeneous penetrative deformation and complex fault history. Paleozoic sedimentary and volcanic rocks are mainly exposed in the central and western parts of the NTS 104B map sheet. Major rock types include greenstones, limestones, shales and clastic sedimentary rocks. Mesozoic assemblages are divided into three major groups (Anderson, 1993): 1 Upper



Fig. 1.1 Location map of the Iskut River map sheet, northwestern British Columbia.

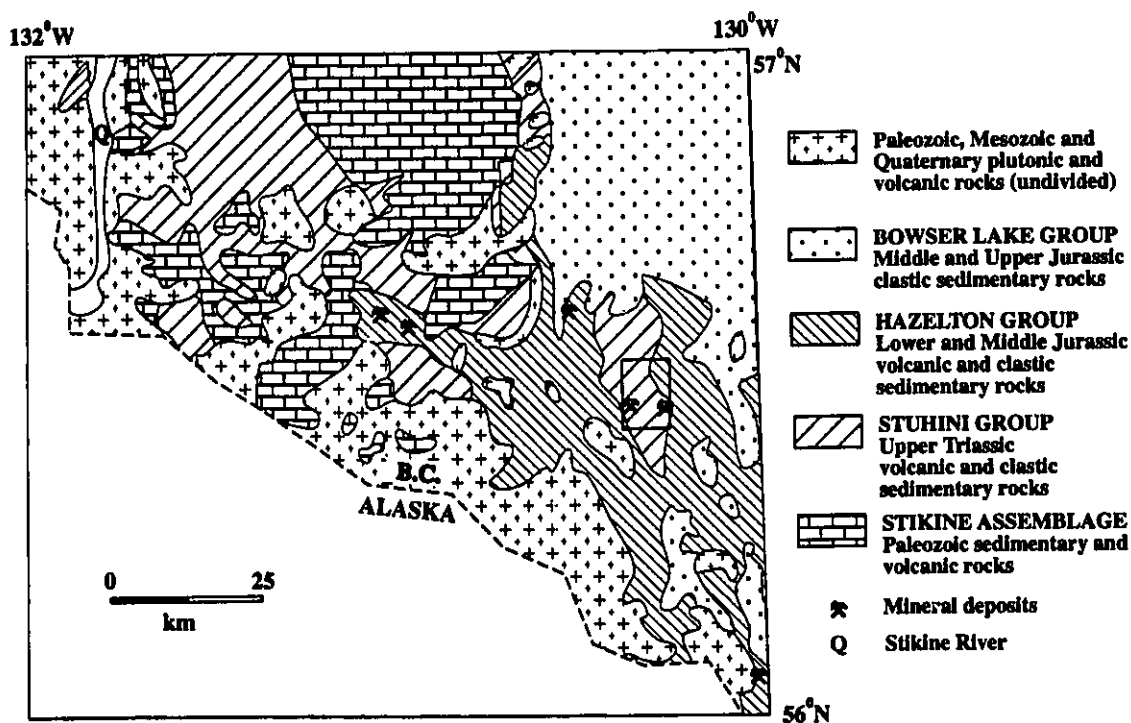


Fig. 1.2 Regional geology map of the Iskut River area (after Anderson, 1993).
Box indicates location of the Mitchell-Sulphurets mineral district.

Triassic Stuhini group (volcanic and clastic sedimentary sequences); 2 Lower and Middle Jurassic Hazelton group (volcanic and clastic sedimentary sequences); 3 Middle and Upper Jurassic Bowser Lake group (clastic sedimentary sequences) which outcrop mainly in the northwestern parts of the map sheet. Paleozoic and early Mesozoic rocks (up to Middle Jurassic) were intruded during two episodes of magmatism. Late Triassic plutonic rocks consist of I-type hornblende-biotite metadiorite, quartz monzonite and monzodiorite. Early Jurassic plutonic activity is characterized in the southwestern parts of NTS 104B by biotite-hornblende granodiorite and quartz monzodiorite intrusions. In the northeastern area alkali-feldspar-rich, biotite- or hornblende-rich syenite, quartz monzonite and alkali-feldspar porphyry intrusions predominate. Stockwork vein-type epithermal precious-metal mineralization and mesothermal base- and precious-metal mineralization are commonly spatially related and may be genetically associated with alkali-feldspar porphyry intrusions (Anderson, 1993; Kirkham et al., 1990). Sedimentary and volcanic clastic rocks of both the Stuhini and Hazelton groups are favourable for Au mineralization in the area (Alldrick et al., 1987; Anderson, 1989, 1993).

Examination and interpretation of gravity and aeromagnetic geophysical data provide useful information for understanding the distribution of the various geological units in the Iskut River (NTS 104B) area. The gravity data show that Paleozoic basement rocks, such as greenstones, have a higher density than the younger volcanic and clastic sedimentary Mesozoic rocks. High-resolution aeromagnetic data show that the Mesozoic rocks have a relatively low magnetic susceptibility signature when compared to the intermediate to

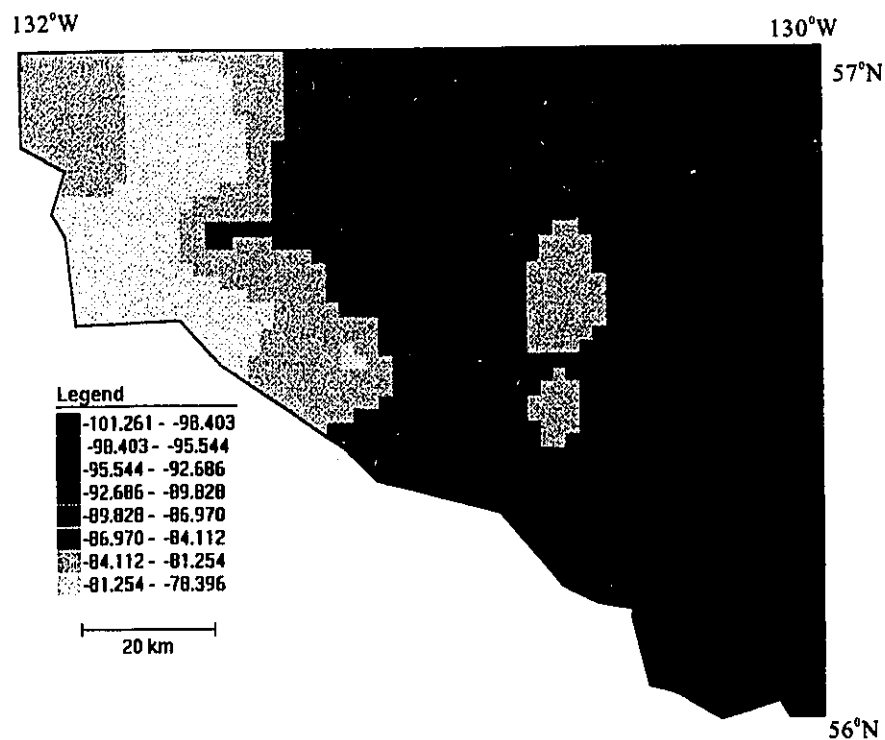


Fig. 1.3 Gravity map of the Iskut River map sheet
(Geological Survey of Canada, 1978)

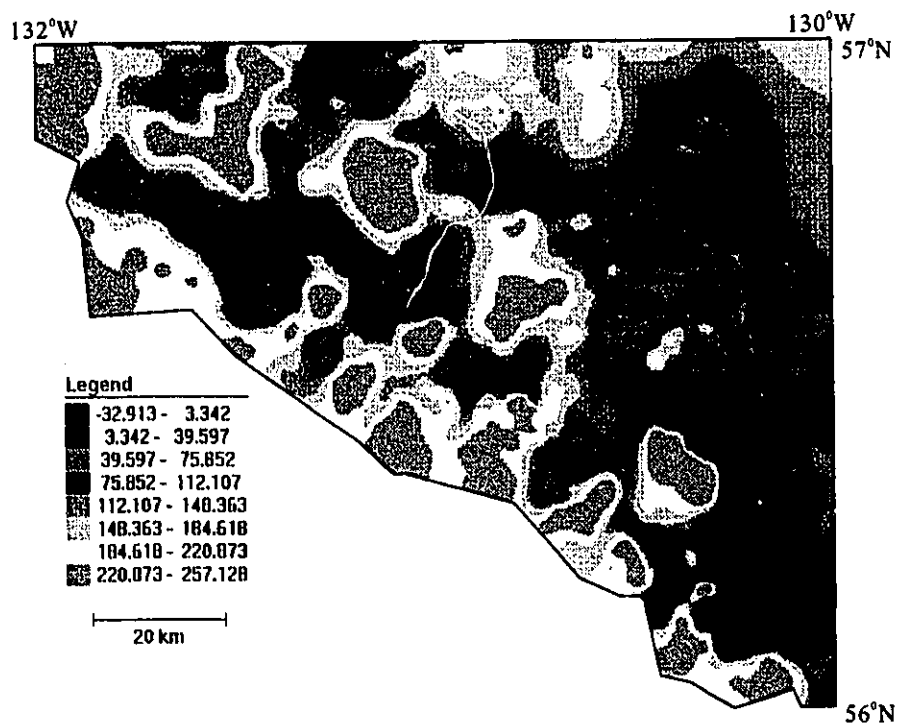


Fig. 1.4 Aeromagnetic map of the Iskut River map sheet
(Geological Survey of Canada, 1978)

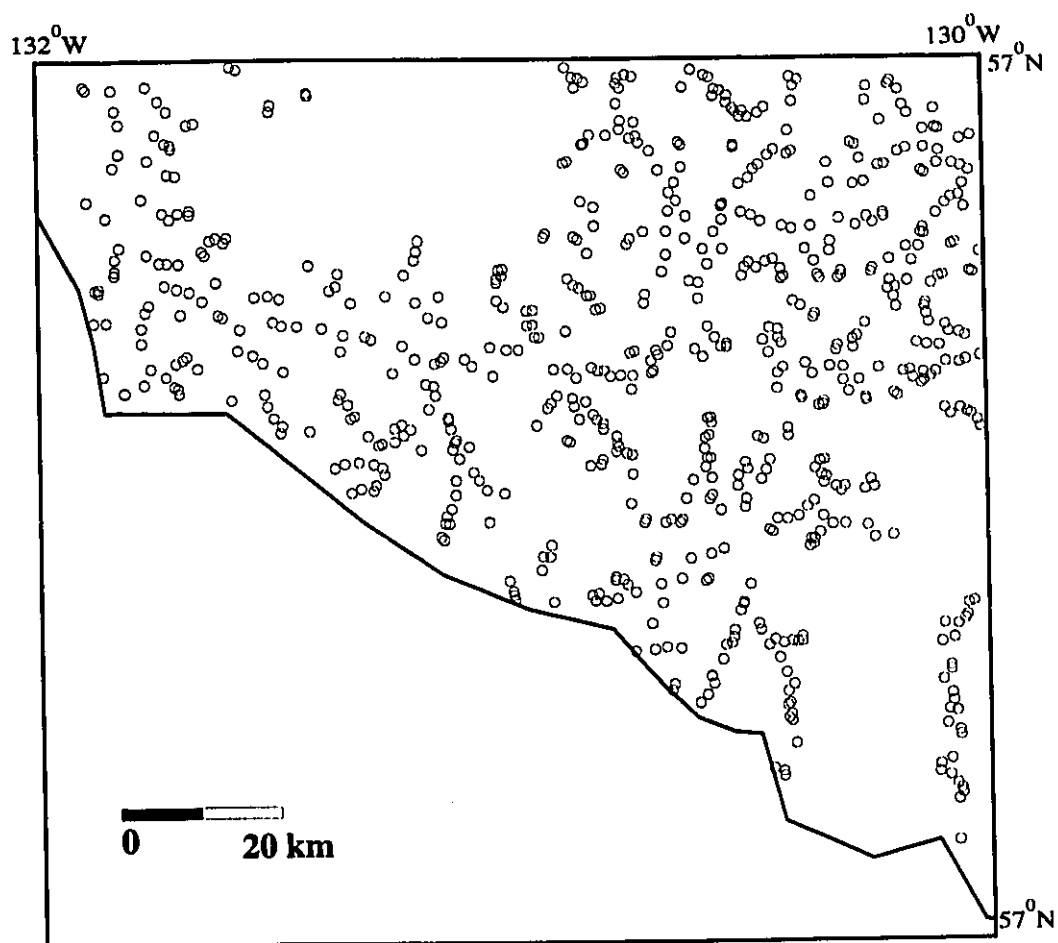
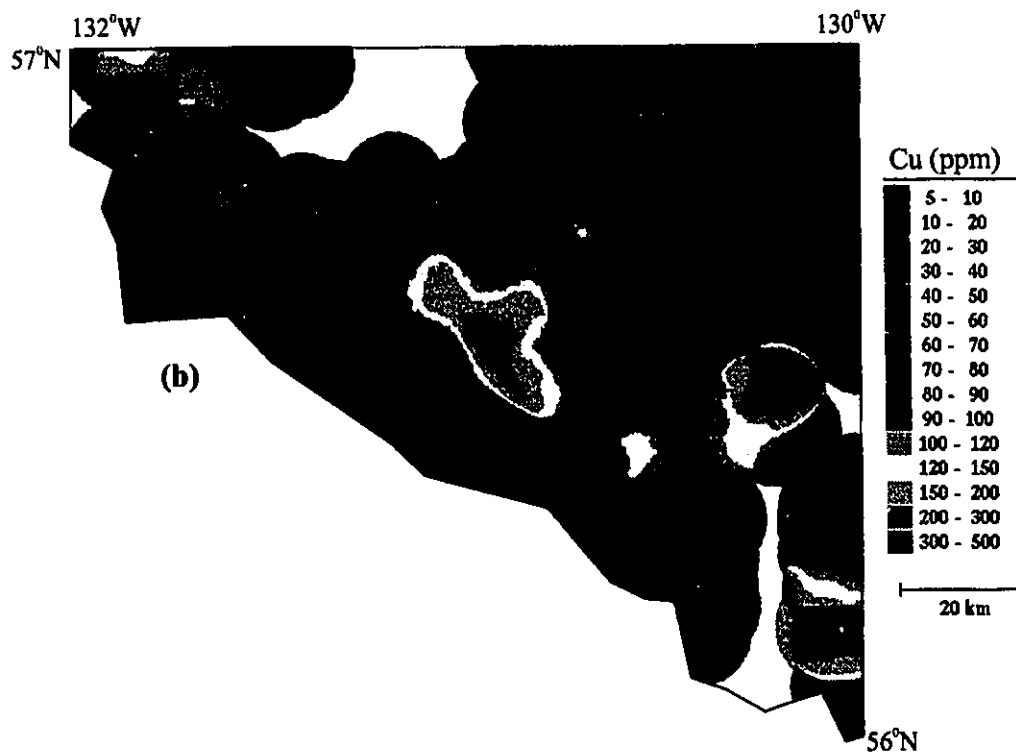
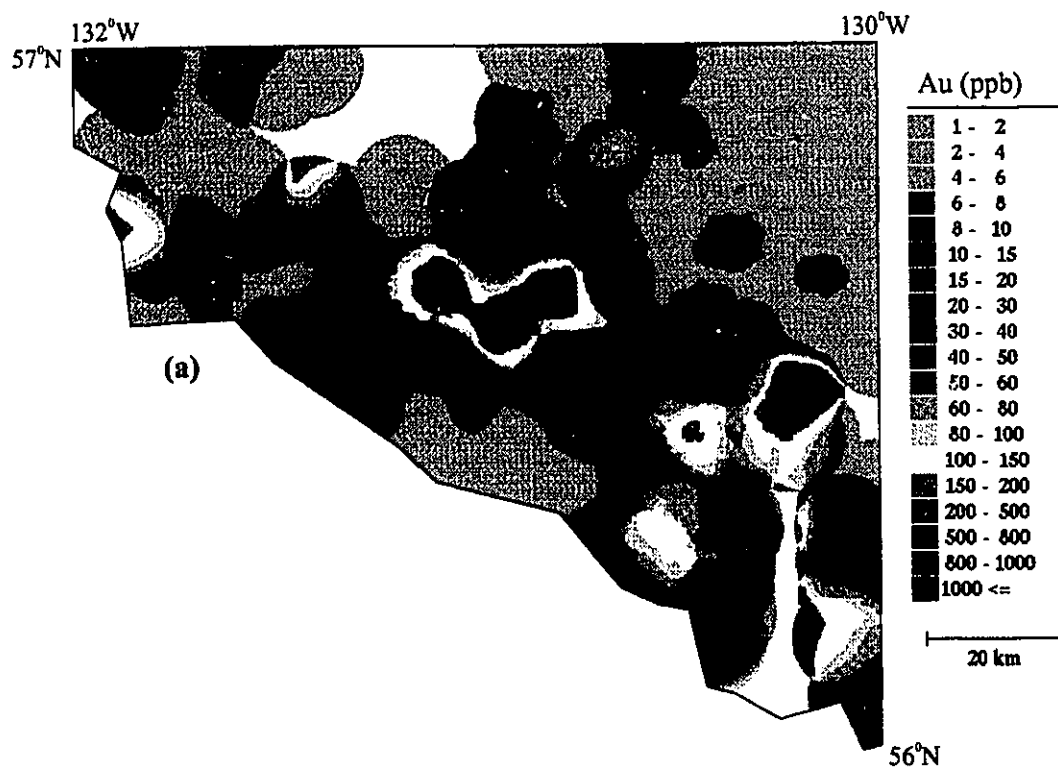


Fig. 1.5 698 stream sediment sample location map, Iskut River area, northwestern British Columbia (GSC Open file 1645, 1988)



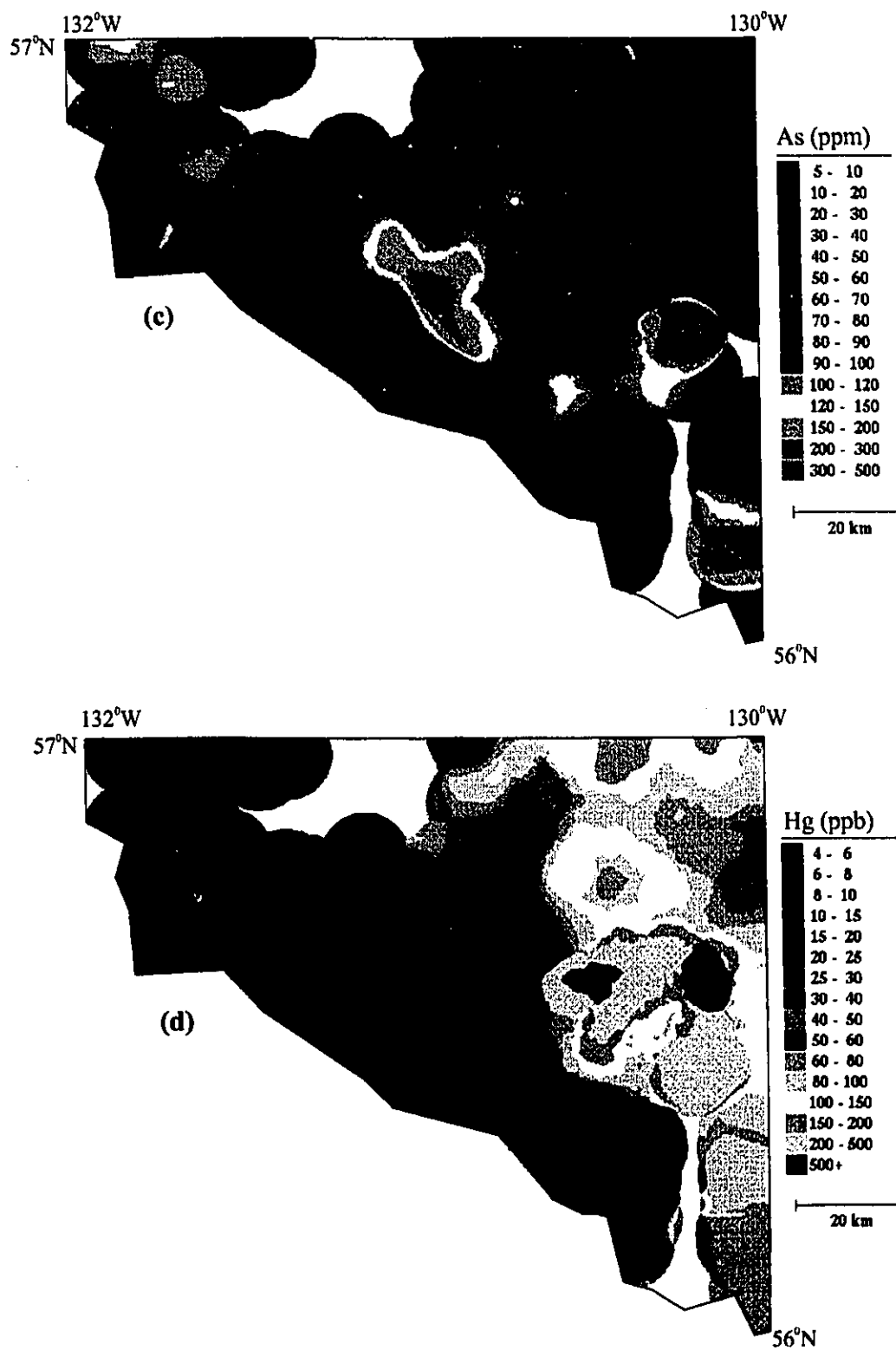


Fig. 1.6 Geochemical contour maps of stream sediment data, Iskut River area (GSC Open file 1645, 1988). (a) Au (ppb), (b) Cu (ppm), (c) As (ppm), (d) Hg (ppb)

mafic intrusive high magnetic anomalies. Therefore, gravity and magnetic maps can be constructed to depict geophysical distinct rock units (Figs. 1.3 and 1.4). Examples of these distinctive features are the two major zones containing abundant intrusions with NW-SE orientation shown on the map.

Interpretation of geochemical data provided by the sites of 698 stream sediment samples taken in the Iskut River NTS 104B map area can also be used to characterize distinct rock units (GSC Open File 1645, 1988). An illustrative example is the clastic sedimentary Bowser Lake group rocks being coincident with high Hg concentrations in stream sediments compared to other geological units (see Fig. 1.6d). These data also provide direct interpretation of anomalies associated with mineralization (B.C. Minfile Map NTS 104B). Au in stream sediment geochemical anomalies can be used as an indicator pattern for Au prediction (see Chapter 8). The regional distribution patterns for Au and the Au-associated elements Cu and As are shown in Figs. 1.6a, b, c, respectively.

1.1.2 Geology and Au-Associated Alteration of the Mitchell-Sulphurets Area

The Mitchell-Sulphurets mineral district, in the northwestern part of the Iskut River map sheet (Fig. 1.2), is noted for extensive alteration zones associated with copper and molybdenum porphyry systems as well gold and silver mineralization (Alldrick and Britton, 1988; Alldrick et al., 1990; Anderson, 1993; Ballantyne, 1990; Britton and Alldrick, 1988; Harris, 1990; Harris and Ballantyne, 1992a, 1992b; Kirkham, 1990;

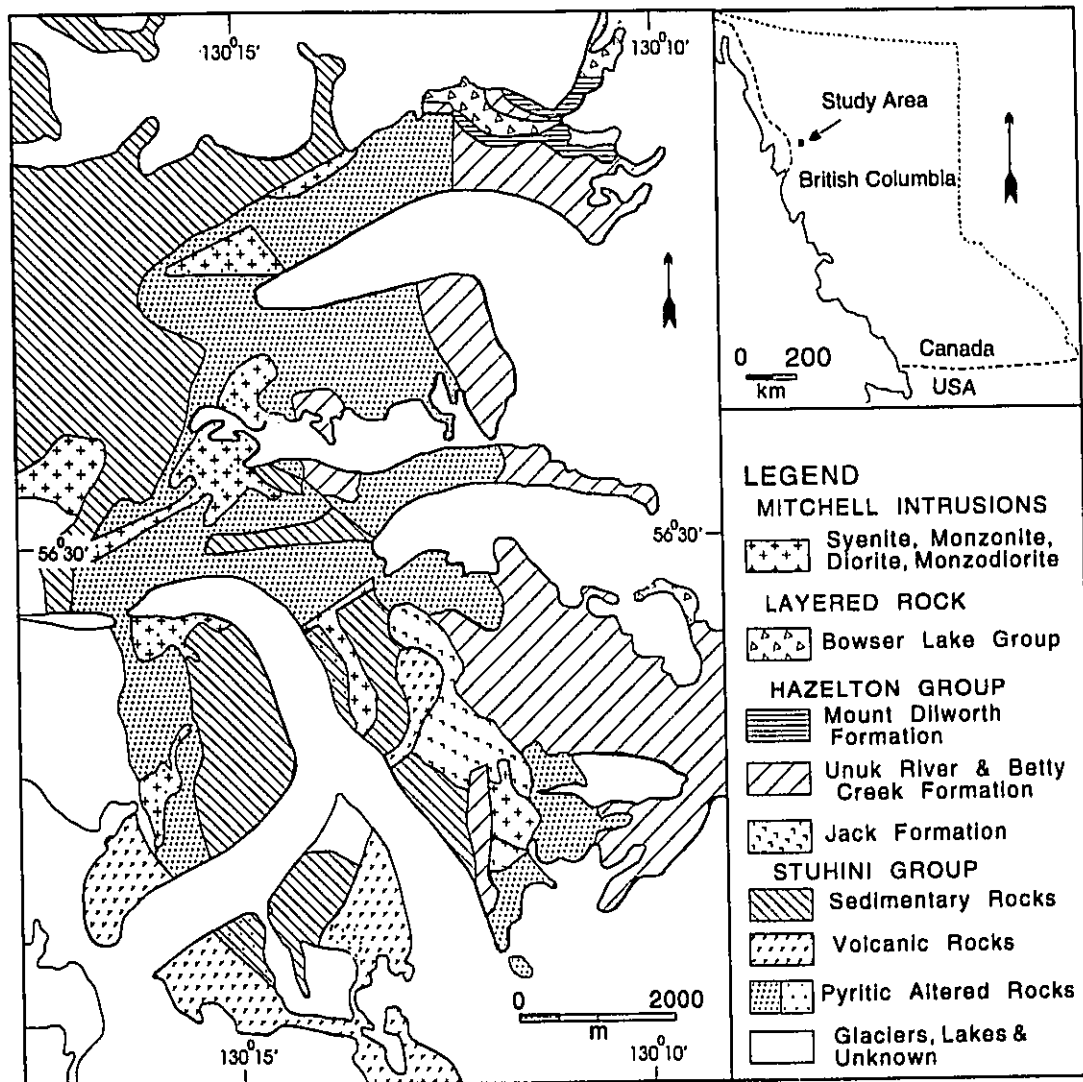


Fig. 1.7 Simplified geology of the Mitchell-Sulphurets mineral district (Digitized from the map compiled by R.V. Kirkham, GSC, personal communication, 1993). Structural information is not included.

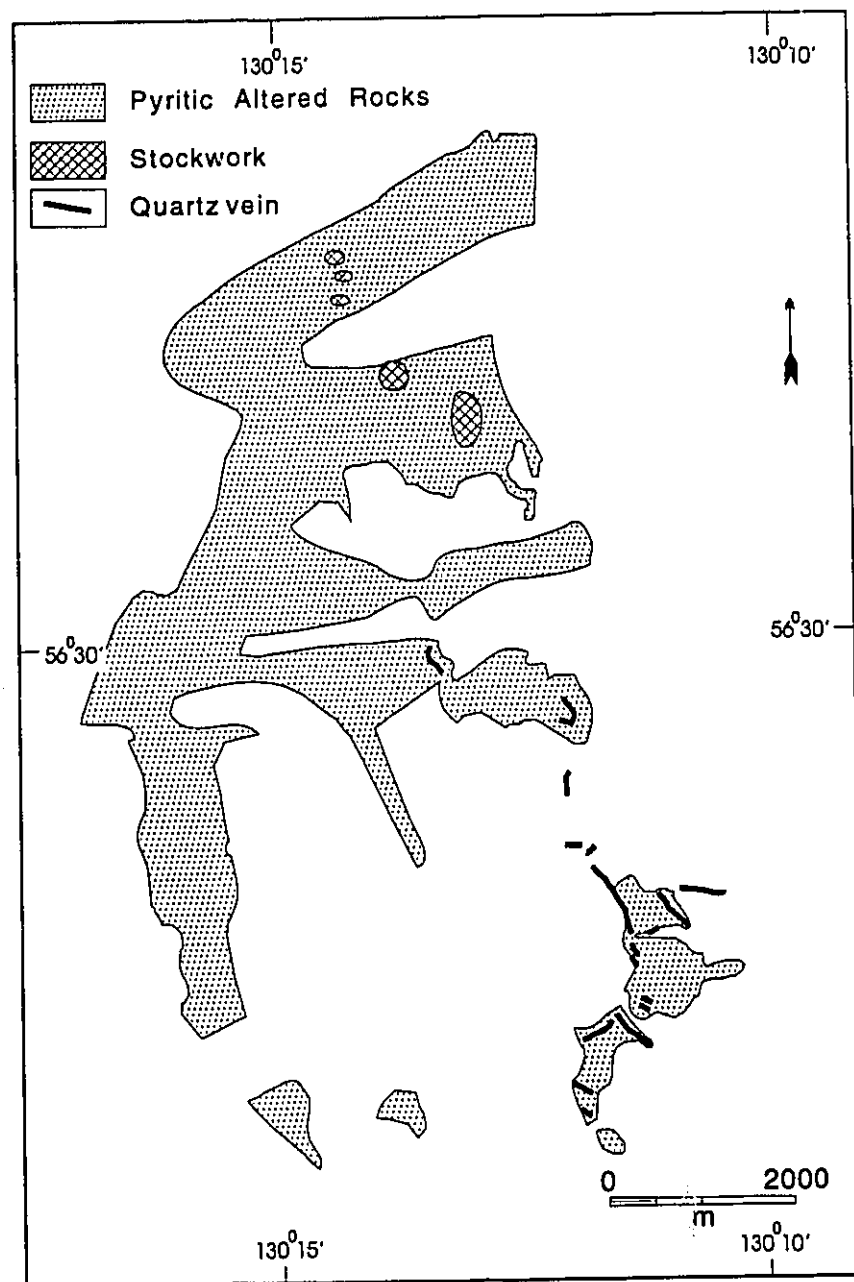


Fig. 1.8 Alteration zones in the Mitchell-Sulphurets mineral district reclassified using SPANS-GIS. Simplified stockwork and vein system compiled by R.V. Kirkham (pers. comm., 1993)

Kirkham et al., 1989, 1990, 1992). The area is underlain by Upper Triassic to Jurassic volcanic and sedimentary rocks, primarily from the Hazelton and Stuhini groups (Fig. 1.7). The central part of the area is underlain by so-called pyritic altered rocks (Fig. 1.8) that host most of the potassic, sulphitic and silicic alteration zones. The contacts of the pyritic altered rocks with the Stuhini group to the west and the Hazelton group to the east consist of faults striking S-N to NE-SW and S-N to NNW-SSE, respectively. Most of the geochemical anomalies in bedrock samples in the area are spatially related to the alteration zones (Cheng, Agterberg and Ballantyne, 1992, 1994a; Grunsky, Cheng and Agterberg, 1994). Vein-stockwork systems occurring in the pyritic altered rocks (Fig. 1.8) show both brittle and ductile deformation which probably took place at relatively great depths (Roach and Macdonald, 1992). Most volcanic and sedimentary rocks in the area are cut by subvolcanic porphyritic intrusions of dioritic, monzonitic, syenitic and low silica granitic composition. Relatively large granitic intrusions are located in the western part of the Mitchell-Sulphurets area and smaller dioritic intrusions in the southeastern part. Regional geological and aeromagnetic maps (Figs. 1.2 and 1.4) also show that the majority of intrusive rocks occur in the western parts the Mitchell-Sulphurets area. Thus, the Mitchell-Sulphurets area is located at the boundary between a western area containing abundant intrusions and an eastern area containing fewer intrusions. This unique feature is interpreted to have strongly influenced the lithogeochemical element distributions associated with the occurrence of different types of Au, Au-Ag and Cu mineralization found in the Mitchell-Sulphurets district. The sericitic-pyritic alteration, extensive quartz vein and stockworks development and several types of gold-bearing zones (Cu, Cu-Mo,

Mo, Ag, Ag-Sb-As) have been interpreted as being genetically related to the emplacement of the intrusions (Alldrick and Britton, 1988; Alldrick et al., 1990; Anderson, 1993; Ballantyne, 1990; Britton and Alldrick, 1988; Harris, 1990; Harris and Ballantyne, 1992a, 1992b; Kirkham, 1990; Kirkham et al., 1989, 1990, 1992; Roach and Macdonald, 1992). In addition, some Au mineralization may be related to younger faults (Alldrick et al., 1987; Kirkham, 1990; Kirkham et al., 1989, 1990, 1992).

1.2 PREVIOUS WORK AND DATABASE

The area is relatively isolated but exploration and geological survey activity had been performed in the past. Before 1960, exploration focus was on copper and on massive and disseminated base metal deposits (see Anderson (1989) for a detailed summary). During the period from 1960 to 1975, a series of regional geological, geophysical and geochemical surveys resulted in the discovery of numerous copper and molybdenum, lead, zinc, gold and silver showings (B.C.Minfile Map 104B, 1989). More recent geoscience data have been compiled to provide a general framework for further exploration; namely, a 1:250,000 scale NTS 104B geological map (Anderson, 1993) and geophysical and geochemical databases (Figs. 1.3 to 1.6) (Geological Survey of Canada, 1978, 1988). Since the 1980's, interest in precious metals has brought exploration activity to unprecedented levels resulting in the discovery of disseminated and high-grade vein-type gold and silver deposits (Alldrick, 1983, 1984, 1985, 1987; Alldrick and Britton, 1988; Anderson, 1989; Britton, 1989; Britton and Alldrick, 1988; Grove, 1986; Harris and

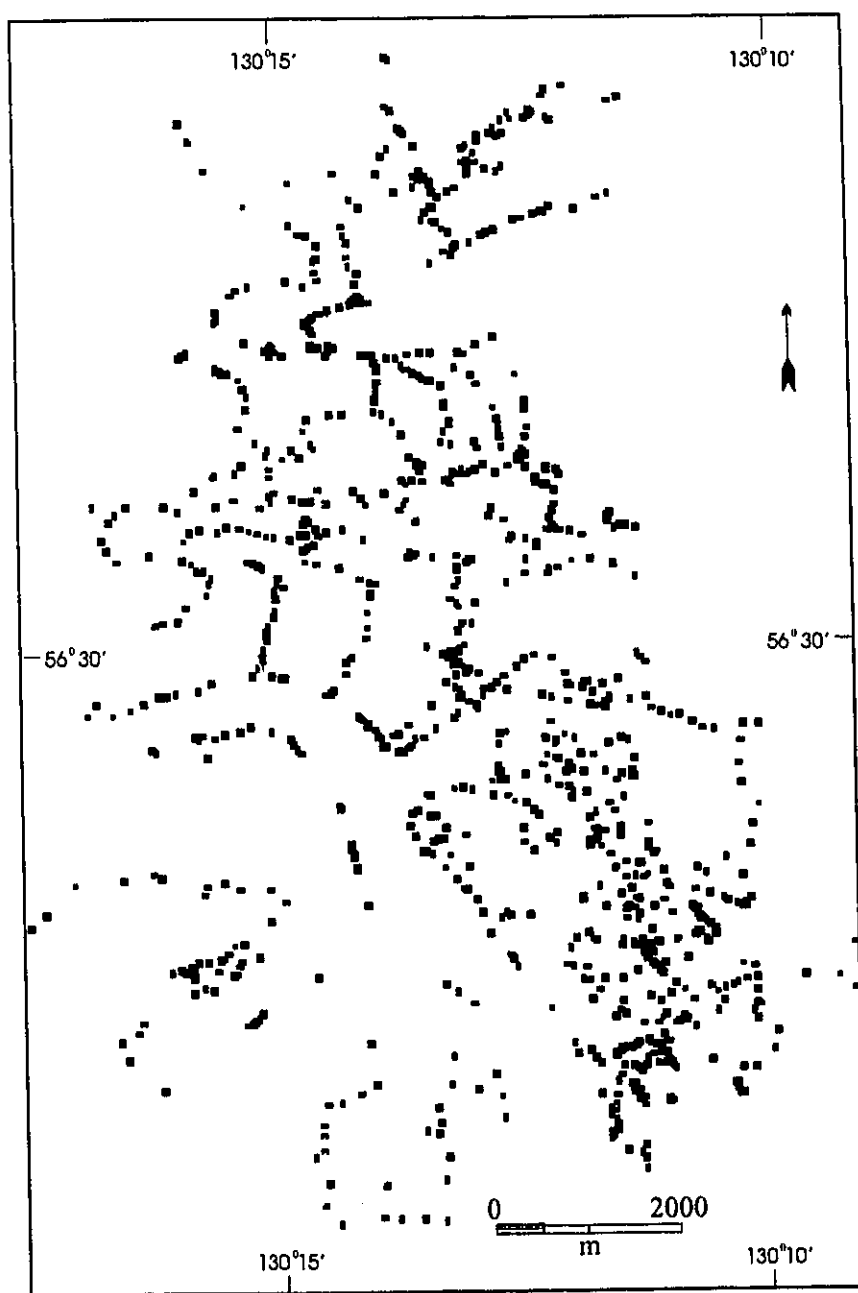
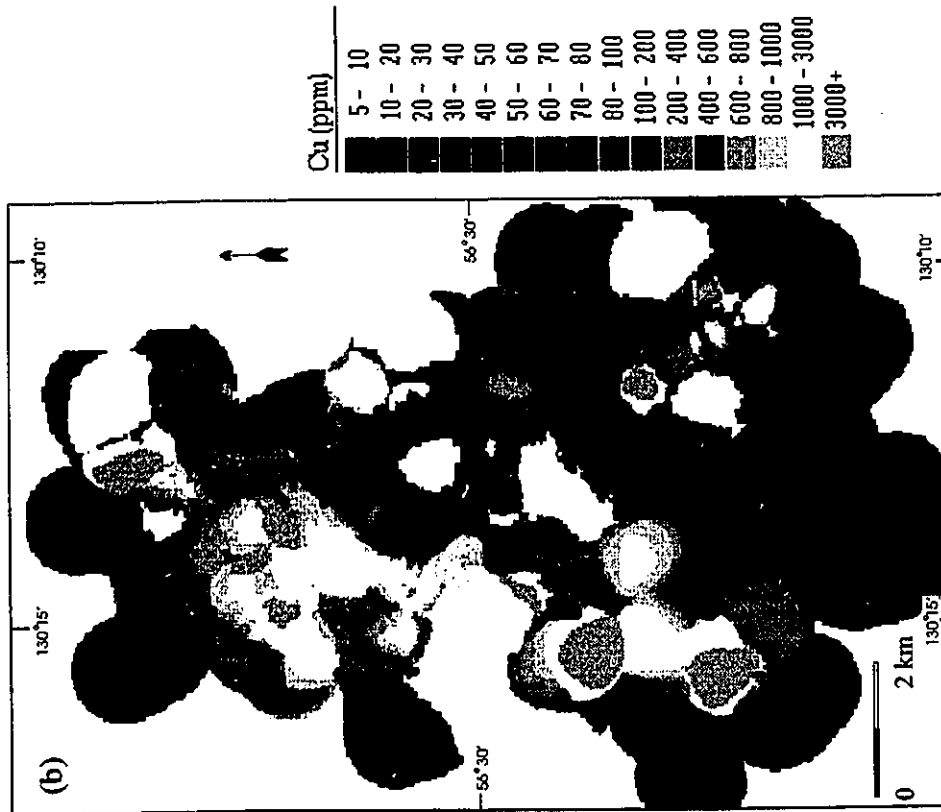
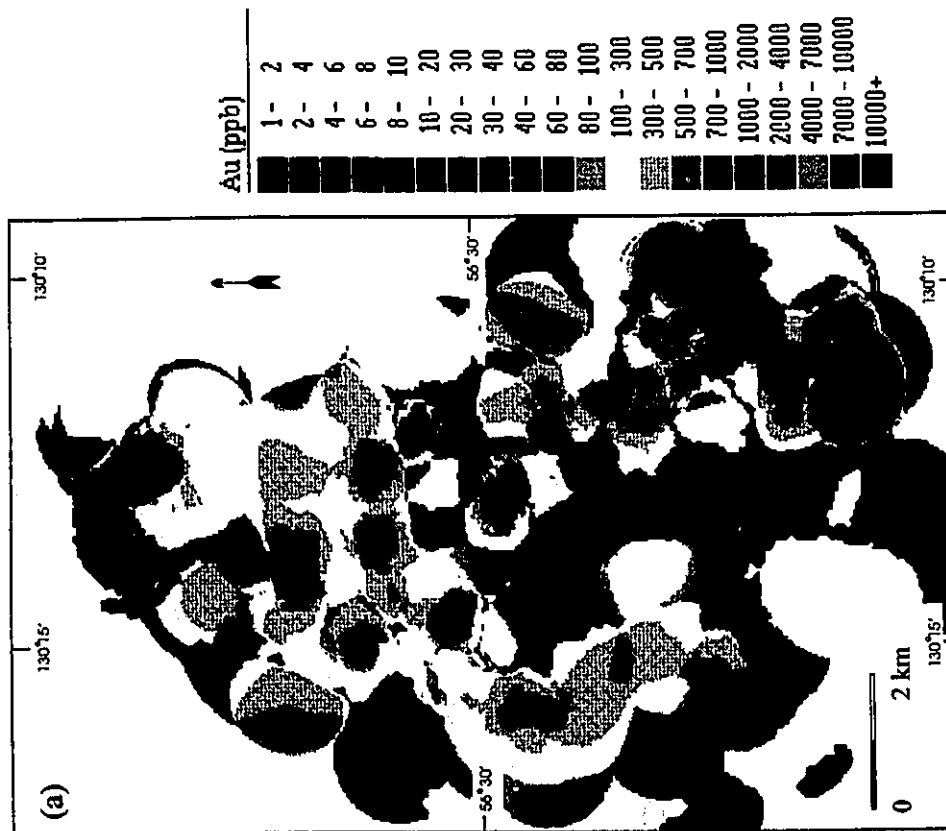
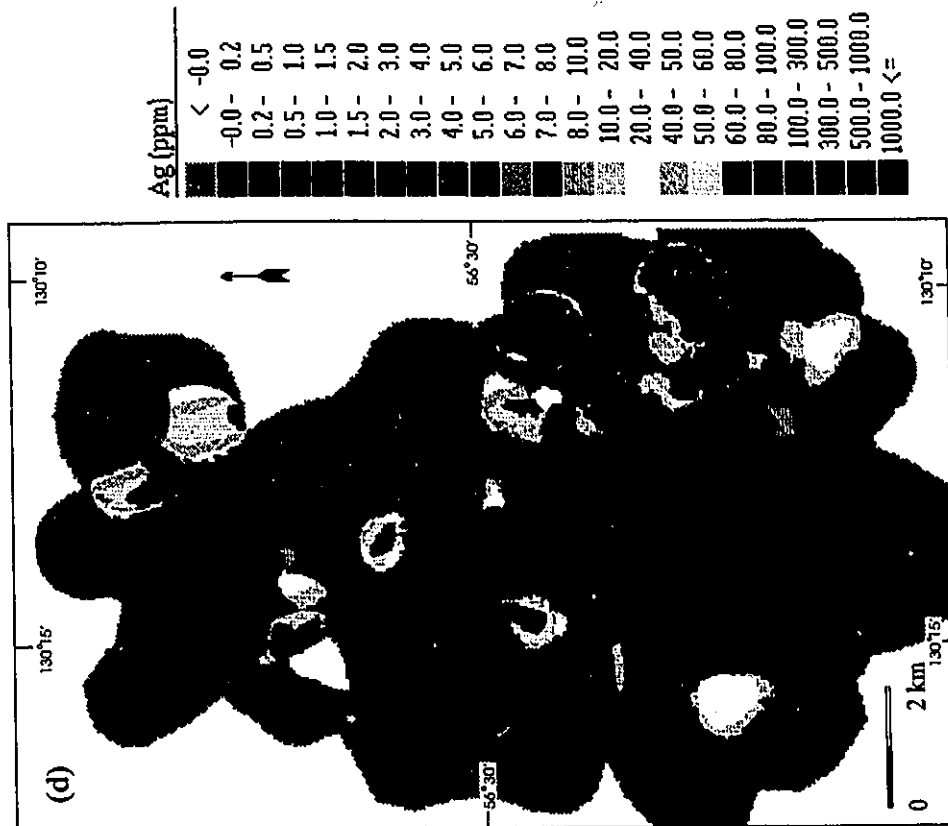
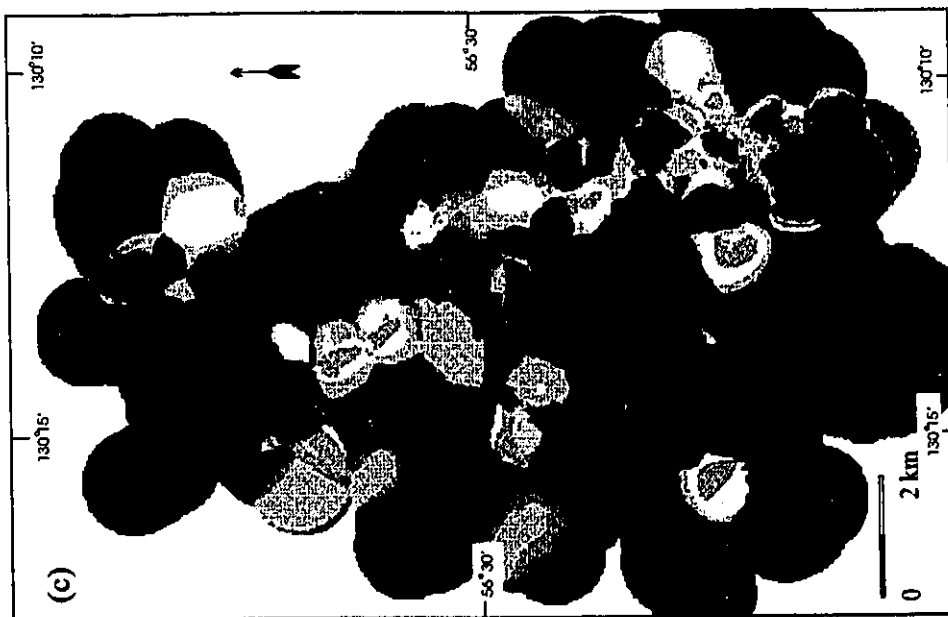


Fig. 1.9 1033 surface bedrock sample location map of the Mitchell-Sulphurets mineral district (data from Ballantyne et al., 1991)





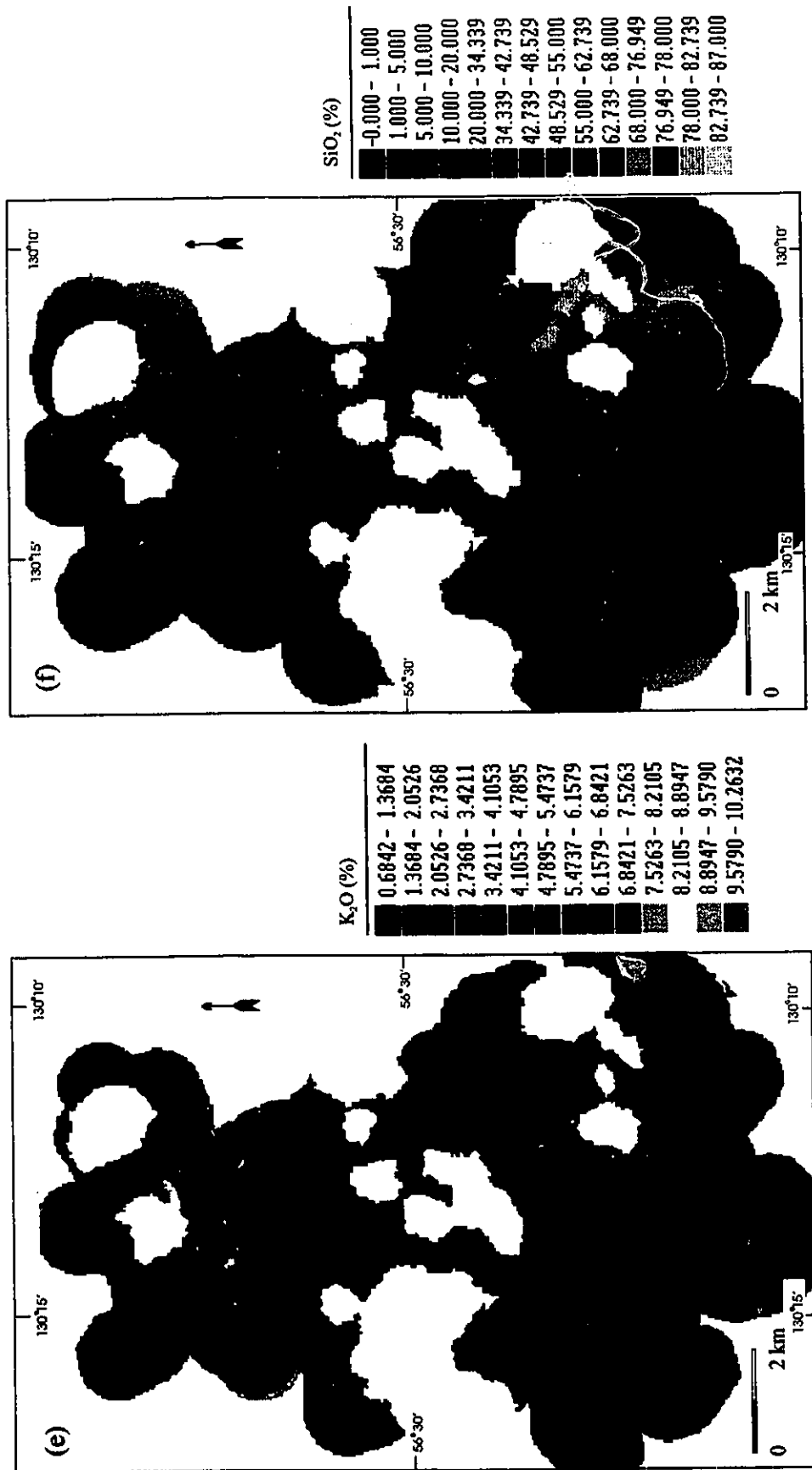


Fig. 1.10 Geochemical contour maps of selected trace elements and oxides in surface bedrock samples from the Mitchell-Sulphurets area. (a) Au(ppb); (b) Cu (ppm); (c) As (ppm); (d) Ag (ppm); (e) K₂O (%); (f) SiO₂ (%).

Ballantyne, 1992a,1992b; Roach and Macdonald, 1992).

Within the 120 km² area covering the Mitchell-Sulphurets district a suite of approximately 1033 lithogeochemical surface bedrock samples were collected during summer fieldwork geological mapping and mineral deposit investigations (1986-1990) by Geological Survey of Canada researchers, R.V. Kirkham, S.B. Ballantyne and D.C. Harris. This sampling covered accessible areas of both altered and unaltered bedrock (Fig. 1.9). S.B. Ballantyne supervised the sample preparation and lithogeochemical analysis which included neutron activation determination of Au plus 33 elements, total acid digestion and ICP analysis and XRF whole rock determinations (Ballantyne, 1990). This material was provided to the author by S.B. Ballantyne and used as the basis for the lithogeochemical statistical analysis presented in this thesis.

The distribution of concentration values for some selected elements in the lithogeochemical data set were used to construct contour maps (Fig. 1.10a, b, c, d, e, f). This new lithogeochemical database and geological mapping conducted by the Geological Survey of Canada were used to compile a new version of the Mitchell-Sulphurets district 1:20,000 scale geological map (Fig. 1.6; R.V. Kirkham, GSC, personal communication, 1993).

Many mineral deposit research reports and papers have been published during the last few years concerned with: study of alteration zones (Ballantyne et al., 1992); geochemistry

(Kirkham, Ballantyne and Harris, 1989; Ballantyne, 1990); Cu-Au deposits (Ballantyne et al., 1992; Kirkham et al., 1992; Harris and Ballantyne, 1992a, 1992b); isotopic study (Alldrick et al., 1990; Godwin et al., 1991); mineralogy (Harris, 1990; Harris and Ballantyne, 1992a, 1992b) and geology (Britton, 1989; Britton et al., 1990; Kirkham, 1990; Kirkham et al., 1989, 1990, 1992).

Statistical characteristics of the lithogeochemical data were examined by using various multivariate analysis procedures (Cheng, Agterberg and Ballantyne, 1992) including cluster and factor analysis by means of SYSTAT/SYGRAPH (SYSTAT Inc., 1990). Several groups of oxides and trace elements emerged using cluster analysis including {CaO, CO₂, MnO}, {Na₂O}, {Al₂O₃, K₂O} and {SiO₂} for oxides, and {Cu, La, Tb, Mo}, {F, V, Co, Cr, Ni}, {Ta, Th, Cl, Sr, Ba}, {Au, Ag, Sb, Eu, Se, W, As} for trace elements. Similar results were obtained by using the factor analysis method; e.g., factor 1 has relatively high scores for {TiO₂, FeO, MgO and P₂O₅}, factor 2 for {Al₂O₃}, factor 3 for {Fe₂O₃} and factor 4 {Al₂O₃, Na₂O}. For trace elements, the factors show the groupings {Zn, Ag, Cd, Sb}, {As, Se, Eu, Tb, Ta, Au, Cu, Pb}, {Cr, Fe, Co, Ni, Tb, Sr} and {Rb, Mo, La, Tb, Cu, V, F}. It was also shown that As, Sb, Ag, Pb, Zn, Cd, Se, Eu Cu Mo F, Nb, S, K₂O and SiO₂ are positively correlated with Au in the alteration zones. The elements Cr, V, Ce, Co, Tb, Te, T, Sr, Sm, W, La, and oxides MnO and CaO are negatively correlated with Au.

Spatial factor analysis was also applied to these multivariate data. The spatial factors were

calculated for the 10 trace elements Zn, As, Mo, Ag, Cd, Sb, Ba, W, Au and Cu and 11 oxides Si_2O , TiO_2 , Al_2O_3 , MnO , total iron, MgO , CaO , Na_2O , K_2O , CO_2 and P_2O_5 with four different orientations (N-S, NE-SW, EW and NW-SE) and various lags (100 to 900 meters). The results obtained show that two major alteration zones can be distinguished from the spatial factors: S-N to NE-SW oriented zones on the west side of Fig. 1.8 have associations of K_2O , Au, Cu, Mo, As, Ba, Cd, W, CaO (negative), SiO_2 (minor); and S-N to NW-SE oriented zones on the east side of Fig. 1.8 have associations of SiO_2 , Al_2O_3 , CaO , As, Ag, Sb, Au and Cd. In Figs. 1.11a and 1.11b two spatial factors are illustrated: the contour map (Fig. 1.11a) of the third spatial factor for 11 oxides which delineates the silicic alteration; Fig. 1.11b of the third spatial factor for 10 trace elements delineates the Au mineralization (Grunsky, Cheng and Agterberg, 1994). Examples of other quantitative techniques will be introduced in later chapters using these same lithogeochemical data.

Data sets for spatial data integration in mineral potential mapping for Au in the Iskut River map sheet are summarized in Table 1.1. Datasets for fractal and multifractal modelling of Au-associated spatial objects in the Mitchell-Sulphurets area are summarized in Table 1.2. In addition, some well-known examples from literature were also used for testing the techniques developed in this thesis, as summarized in Table 1.3.

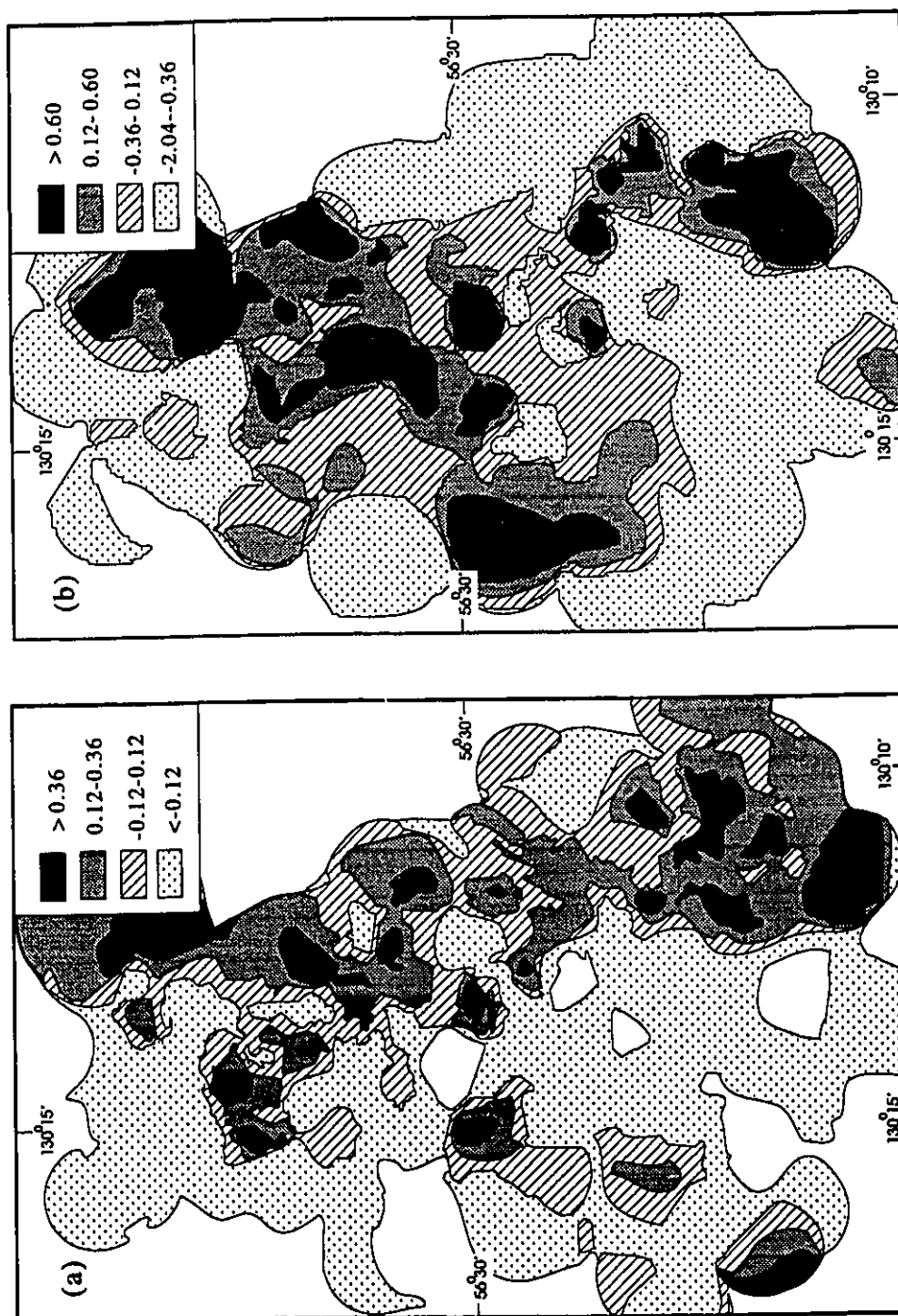


Fig. 1.11 Contour maps of third spatial factors (Grunsky, et al., 1994). (a) 11 oxides, NW-SE direction, 400 m lag, delineates silicic alteration; (b) 10 trace elements, NS direction, 400 m lag, delineates gold mineralization.

Table 1.1 Data sets for Spatial Data Integration for Au mineral potential mapping

DATA TYPE	SCALE	FORMAT	SOURCES
Geological map (N.T.S. 104B)	1:250,000	Vector	Generalized geology base map for NGR stream sediment, GSC Open File 1645, 1988
Geophysical aeromagnetic map	1:250,000	Raster	GSC, 1978
Geophysical gravity map	1:250,000	Raster	GSC, 1978
Geochemical data in stream sediments	1:250,000	Points	NGR stream sediment, GSC Open File 1645, 1988
Au mineral occurrences	1:250,000	Points	B.C. Minfile Map 104B, 1989

Table 1.2 Data sets for fractal and multifractal modelling in the Mitchell-Sulphurets area

DATA TYPE	SCALE	FORMAT	SOURCES
Geological map	1:20,000	Vector	R.V. Kirkham, personal communication. 1993
Geochem. data in bedrock samples	1:20,000	Points	Ballantyne, 1990
Igneous rocks	1:20,000	Vector	Reclassified by SPANS
Faults and thrusts	1:20,000	Vector	Reclassified by SPANS
Au-associated alteration zones	1:20,000	Vector	Reclassified by SPANS
Au mineral occurrences	1:250,000	Points	B.C. Minfile Map 104B, 1989
Glaciers	1:250,000	Vector	Data from DEM NTS 104B

Table 1.3 Data sets of examples from literature

DATA TYPE	FORMAT	SOURCES
Tree seedlings per foot along 200 ft long bed	Continuous value	Cochran, 1963; Rose, 1992
Hickories and Oaks in Lansing Woods, Clinton, Michigan, U.S.A.	Discrete value (spatial point process)	Gerrard, 1969; Diggle, 1983
Zinc values from sphalerite-quartz vein in Pulacayo, Bolivia	Continuous value	De Wijs, 1951; Agterberg, 1994

(Sources are fully documented in References section)

Chapter 2 A SPATIAL STATISTICAL APPROACH TO GEOCHEMICAL ANOMALY SEPARATION

2.1 INTRODUCTION

Geochemical anomaly separation is a basic problem of exploration geochemistry. Statistical theory plays an important role in this field. Methods used include statistical approaches such as moving averages, Kriging and probability graphs (Garrett, 1989; Govett et al., 1975; Miesch, 1981; Sinclair, 1974, 1976, 1991; Stanley, 1988; Stanley and Sinclair, 1989), and fractal methods such as the concentration-area fractal model (Cheng, Agterberg and Ballantyne, 1994a). Various characteristics of samples are considered in these methods, such as (1) element concentration values of samples only (in most statistical methods); (2) the concentration values and spatial characteristics (in Kriging and other various moving average methods); and (3) the concentration values, spatial correlation and the geometry of the anomalous areas (in fractal methods). The common purpose of these methods is illustrated in Fig. 2.1.1 where the concentration value shown might be a moving average or obtained by Kriging; A (anomaly) and B (background) represent two populations to be separated with different spatial characteristics. One purpose of using statistical methods is to set up a threshold (t) so that samples can be classified according to their values: samples with values greater than t are classified as anomalous and those less than t as background. There are different methods of defining the threshold t . Suppose that it is possible to set a lower limit for anomaly (t_1 in Fig.

2.1.1) and an upper limit for background (t_2). A sample with value less than t_1 may be grouped into the background category and a sample with value greater than t_2 can be classified as anomalous. If the two populations A and B are overlapping to some extent, then t_1 and t_2 should be different with $t_1 < t_2$, because one is unable to classify the samples with values between t_1 and t_2 , which could belong to either anomaly or background. Two errors are associated with any classification: Type I (rejection error) occurs when a sample with background value is rejected as a background sample, and Type II (acceptance error) when a sample with anomalous value is accepted as a background sample. These two types of errors are negatively associated. If one type of error is made small by changing the threshold, the other becomes large. In some cases, minimizing one type of error will be optimal and in other applications it is better to minimize the total error (Type I and Type II) (Stanley, 1988). In this chapter, a new approach will be proposed for decreasing the total error (Type I and Type II) by incorporating spatial characteristics of the sample locations. This approach is first demonstrated with the aid of an artificial example and then the method is used for separating geochemical anomalies from background for Au and Au-associated element concentration values in stream sediment samples from the Iskut River map sheet, northwestern British Columbia.

2.2 SPATIAL U*-STATISTICS

Statistical inference can be used for testing a null hypothesis H_0 about the parameters of the frequency distribution of a population such as the mean and variance. Parametric

statistical hypothesis testing usually consists of the following four steps: (1) assuming a group of distributions with unknown parameters θ and distribution functions $F(x, \theta)$ depending on the values of θ ; for example, θ_A and θ_B may correspond to anomalous and background populations in geochemical exploration; (2) creating a null hypothesis about the parameter θ , denoted $H_0: \theta \in \Theta_0$; the total domain of θ , written as Θ , is called the domain of admission for H_0 and $H_1: \theta \in \Theta - \Theta_0$ being the alternative hypothesis; (3) deriving samples $X_1, \dots, X_n$ from the population to construct the statistics $U = U(x_1, \dots, x_n)$; (4) defining two subsets (W and W^c) for U , called rejection and acceptance regions, respectively. If a value of U belongs to W , then the hypothesis (H_0) is rejected; otherwise, if the value of U belongs to W^c , the hypothesis is accepted. The two errors (for rejection and acceptance) can be expressed in the following probability form:

$$e_I = P[U_{H_0} \in W] \quad ; \quad e_{II} = P[U_{H_1} \in W^c] \quad (2.2.1)$$

where U_{H_0} indicates the value of U when H_0 is true, and U_{H_1} the value of U when H_0 is false or H_1 is true. Generally, e_I and e_{II} are negatively associated and depend on several factors such as the definitions of W and W^c , the frequency distribution function, U -statistics and sample size n .

Normally, statistics can be grouped into two categories: for one type, the two errors in Eq.(2.2.1) approach zero as the sample size (n) increases; for the other, however, the two errors will not be close to zero no matter how large the sample size is. For the first type of statistics, increasing the sample size (n) may greatly improve the hypothesis testing

results. In this chapter, several common statistics, such as U, T, χ^2 and F, will be examined for this property (see section 2.4). It will be shown that U-, χ^2 - and F-statistics belong to the first type, and T-statistics belong to the second. Both errors (e_I and e_{II}) in the use of the U-, χ^2 - and F-statistics approach zero as the sample size increases infinitely, whereas the two errors for T-statistics approach a finite value not equal to zero. These properties of the statistics are to be considered in geochemical data analysis as shown in detail for the U-statistics.

It can be shown (see section 2.4) that sample size is the only factor which controls the two errors (e_I and e_{II}). Therefore, in applications of a statistic, one usually gets better results by increasing the sample size. In geochemical exploration, the fact that different sampling density may statistically effect the recognition of anomaly from background can be explained in terms of the classification errors (e_I and e_{II}) as follows. Suppose that two populations (A and B) have normal distributions with means of μ_A and μ_B , and variances σ_A^2 and σ_B^2 , respectively. Samples ($X_1, \dots, X_n$) are collected in order to distinguish between the populations A and B. A new random variable $U = (1/n) \sum X_i$ can be constructed (U - statistic) with the following properties: $EU = \mu_A$ and $\sigma^2(U) = (1/n)\sigma_A^2$ if all samples are from population A, and $EU = \mu_B$ and $\sigma^2(U) = (1/n)\sigma_B^2$ if all samples are from B. The distribution function of U is illustrated in Fig. 2.2.1. The means for A and B are the same in Figs. 2.1.1 and 2.2.1, but the variances for A and B are different by the factor $(1/n)$ which depends on sample size n.

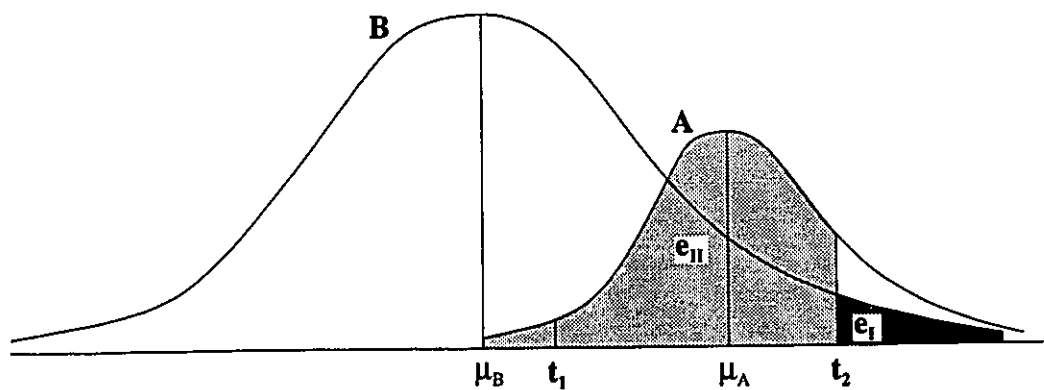


Fig. 2.1.1 Two populations: A: anomaly with mean μ_A . B: background with mean μ_B . t_1 : lower limit of anomaly. t_2 : upper limit of background. e_I and e_{II} : errors type I and II with t_2 as the threshold.

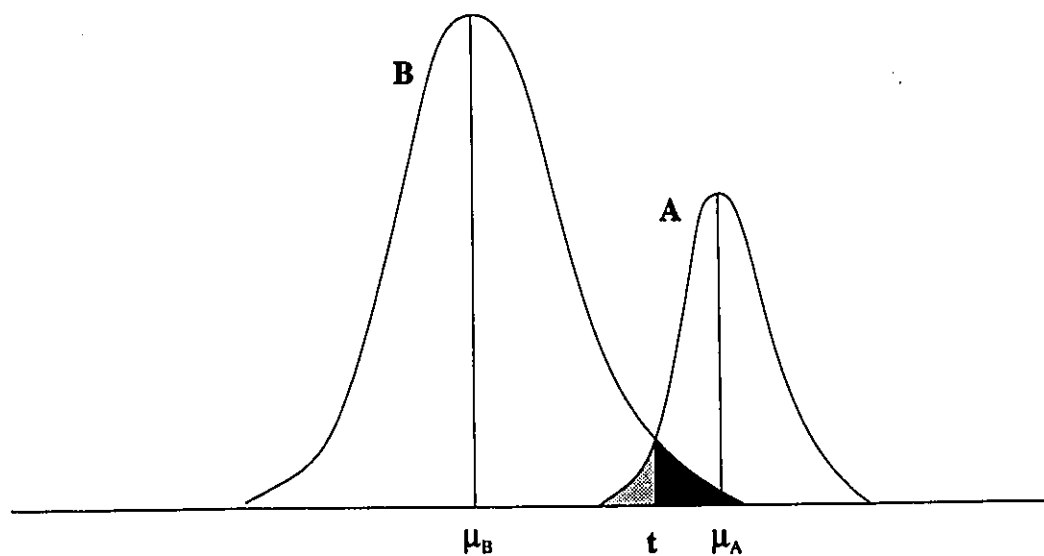


Fig. 2.2.1 Density function of random variable (statistics) U (see text for definition).

More generally, instead of an arithmetic average, a weighted average U can be used with:

$$U = \sum_{i=1}^m w_i X_i \quad (2.2.2)$$

where $0 \leq w_i \leq 1$, $\sum w_i = 1$. Then, $EU = \mu_A$ and $\sigma^2(U) = \sum w_i^2 \sigma_A^2$ if all samples are collected from population A, and $EU = \mu_B$ with $\sigma^2(U) = \sum w_i^2 \sigma_B^2$ if all samples come from B. In general, $1/n \leq \sum w_i^2 \leq 1$; $\sum w_i^2 = 1$, if and only if $i = k$ with $w_k = 1$ (and $w_i = 0$, when $i \neq k$) which implies that $U = X_k$; $\sum w_i^2 = 1/n$, if and only if $w_i = 1/n$, $i = 1, \dots, n$, for the arithmetic average. Different types of weighting in Eq.(2.2.2) would result in a different variance $\sigma^2(U)$ with another factor $\sum w_i^2$ but the smallest variance is reached for the arithmetic average. In the preceding discussion, all values were supposed to originate from a single population (A or B). In general, we do not know which population the values come from. However, by inspecting the geometrical characteristics of the geochemical anomalies, we may notice that anomalous samples often occur in groups of samples which correspond to geographically distinct areas. Values of samples and the geometry of anomalous areas have been considered in anomaly separation (Sinclair, 1991; and Cheng, Agterberg and Ballantyne, 1994a). A experimental study using average values of samples was presented by Govett et al. (1975). These authors reported that for systematical sampling (grid sampling), averaging values of samples is useful for separation between populations. This approach can be generalized as follows.

Suppose that a given area consists of two subareas A and B (Fig.2.2.2) representing anomaly and background, respectively. The area A is circular or elliptical in shape. The

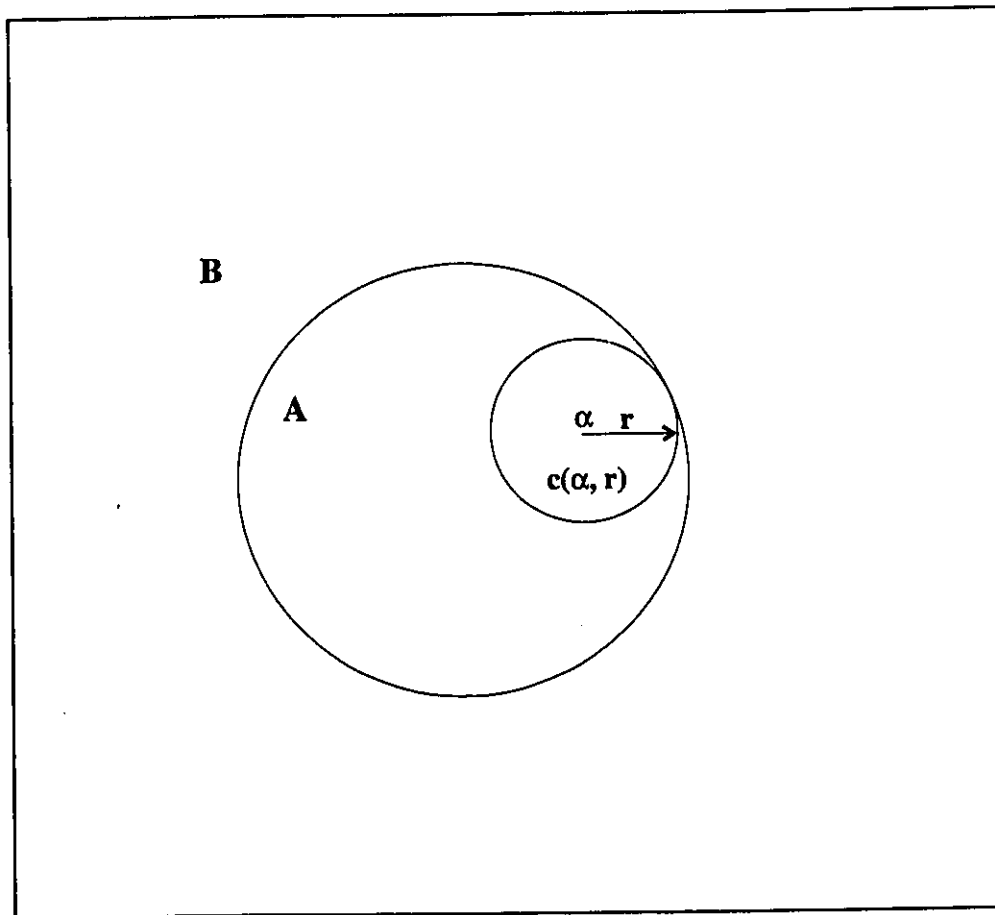


Fig. 2.2.2 Moving windows with varying radius. B: background. A: anomalous area. α : sample location. $c(\alpha, r)$: vicinity around α with a radius of r . Samples in $c(\alpha, r)$ are weighted and averaged.

mixture distribution for samples randomly collected from the area of Fig. 2.2.2 is illustrated in Fig. 2.1.1. Samples with values between the thresholds t_1 and t_2 in Fig. 2.1.1 can not be classified individually. For this reason, a new statistic will be constructed: let α_i indicate the location of the i -th sample in the study area and X_i the value of this sample; choose any sample, say α_i , from the set of samples and create a circular neighbourhood around it denoted as $c(\alpha_i, r)$, where the radius r ($0 \leq r \leq r_{\max}$). The value of r can be assigned in such way that $c(\alpha_i, r) \subset A$ if $\alpha_i \in A$ or $c(\alpha_i, r) \subset B$ if $\alpha_i \in B$. A function for $c(\alpha_i, r)$ can be defined as follows:

$$\mu_{c(\alpha_i, r)}(\alpha_j) = \frac{r - d(\alpha_i, \alpha_j)}{r} \quad (2.2.3)$$

where $d(\alpha_i, \alpha_j) \leq r$ represents distance from sample location α_j to α_i . This function has the properties: $0 \leq \mu_{c(\alpha_i, r)}(\alpha_j) \leq 1$; $\mu_{c(\alpha_i, r)}(\alpha_j) = 1$ if $\alpha_i = \alpha_j$; and $\mu_{c(\alpha_i, r)}(\alpha_j) = 0$ if $\alpha_j \notin c(\alpha_i, r)$, where $\notin$ indicates "does not belong to". A new value $U_i(r)$ for sample α_i can be defined as a linear combination of X_i :

$$U_i(r) = \frac{\bar{X}_i(r) - \mu}{S_i(r)} \quad (2.2.4)$$

with

$$\bar{X}_i(r) = \sum_{j=1}^n w_j(r) X_j \quad ; \quad S_i(r) = \sigma \sqrt{\sum_{j=1}^n w_j^2(r)} \quad ; \quad w_j(r) = \frac{\mu_{c(\alpha_i, r)}(\alpha_j)}{\sum_{j=1}^n \mu_{c(\alpha_i, r)}(\alpha_j)}$$

where μ and σ are the mean and standard deviation of X_i in the whole area. A new statistic $U_i^* = U_i(r_0)$ can be defined where r_0 is the value yielding

$$|U_i(r_0)| = \max_{0 \leq r \leq r_{\max}} |U_i(r)| \quad (2.2.5)$$

The value of U_i^* for sample α_i has the following distribution:

If $\alpha_i \in A$:

$$U_i^* \sim N\left(\frac{\mu_A - \mu}{S_i(r_0)}, \frac{\sigma_A}{\sigma}\right) \quad (2.2.6)$$

and, if $\alpha_i \in B$:

$$U_i^* \sim N\left(\frac{\mu_B - \mu}{S_i(r_0)}, \frac{\sigma_B}{\sigma}\right) \quad (2.2.7)$$

where $\mu_B \leq \mu \leq \mu_A$, so that $U_i^*(r_0)$ has mean $(\mu_A - \mu)/S_i(r_0) > (\mu_A - \mu)/\sigma > 0$ if α_i belongs to A, and $U_i^*(r_0)$ has mean $(\mu_B - \mu)/S_i(r_0) < (\mu_B - \mu)/\sigma < 0$ ($\mu_B < \mu$) if α_i belongs to B. If α_i is located on the boundary between A and B then r_0 should be close to zero and therefore $U_i^*(0)$ should be $(X_i - \mu)/\sigma$ which is equal to the original standardized value of X_i . The value $U_i^*(r_0)$ can be easily classified. The value of $U_i^*(r_0)$ depends on the total number of samples within $c(\alpha_i, r)$ and is also associated with the location of α_i ; e.g., if α_i is located near the boundary between A and B, there can only be relatively few samples in $c(\alpha_i, r)$ and the value of $U_i^*(r_0)$ will be close to the original standardized value of X_i . Of course, the value of $U_i^*(r_0)$ also depends on density of samples collected from the area and, statistically, the sampling density may influence the result of distinguishing between geochemical populations.

2.3 EXAMPLES OF APPLICATION

In the next section, an artificial example is given to demonstrate the method proposed in the previous section, before application to separating Au and Au-associated geochemical anomalies from background for stream sediment data in the Iskut River map sheet, northwestern British Columbia.

2.3.1 An Artificial Example

Suppose 3600 samples with sampling density of 1 sample per 1 km^2 are regularly collected from a mineral district as in Fig. 2.2.2 with total area of 3600 km^2 , consisting of background B (3283 km^2) and anomalous area A (317 km^2). In total, 317 samples are collected from A and 3283 samples from B with the anomalous samples accounting for 8.8% of all samples. The values for anomalous samples belong to a normal distribution with mean of 40 (μ_A) and standard deviation of 10 (s_A). Values for background samples belong to a normal distribution with mean of 20 (μ_B) and standard deviation of 15 (s_B). Frequency histograms are shown in Figs. 2.3.1a and 2.3.1b, respectively. The distribution of mixed anomalous and background samples is shown in Fig. 2.3.1c. In order to separate anomalous samples from background using their frequency distribution, several thresholds are used; e.g., mean + c standard deviations ($c = 0.0, 0.5, 1.0, 1.5, 2.0$ and 2.5). The errors (e_I and e_{II}) of misclassification associated with these thresholds are estimated by means of the following formulae

$$e_I = \{ \text{number of samples from B classified into A} \} / \{ \text{total number of samples in B} \}$$

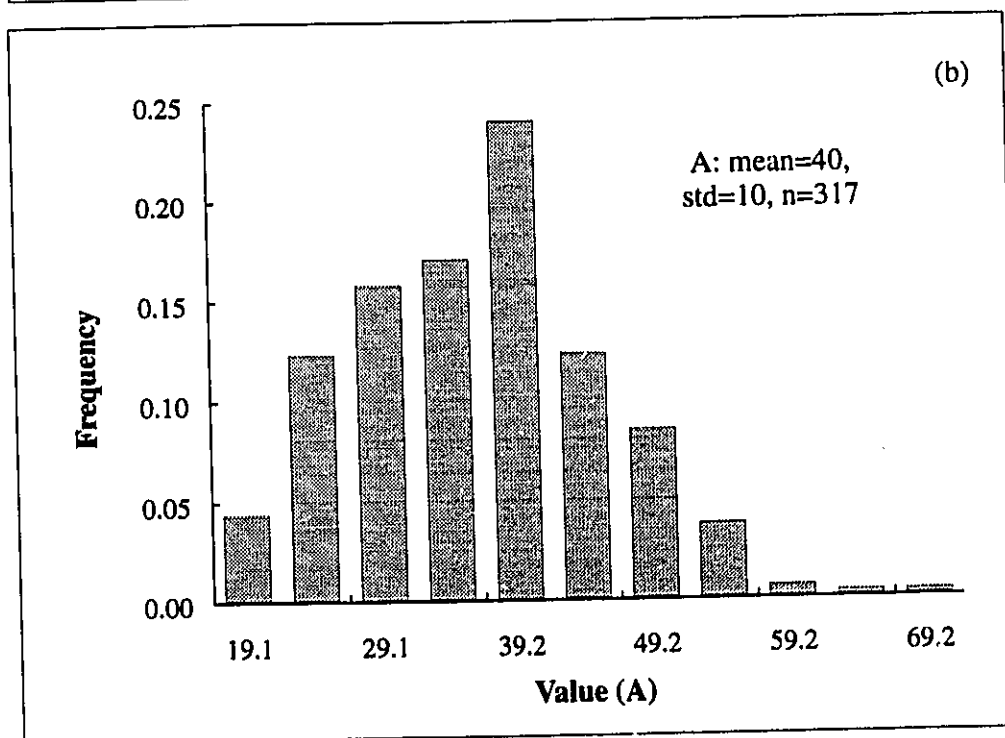
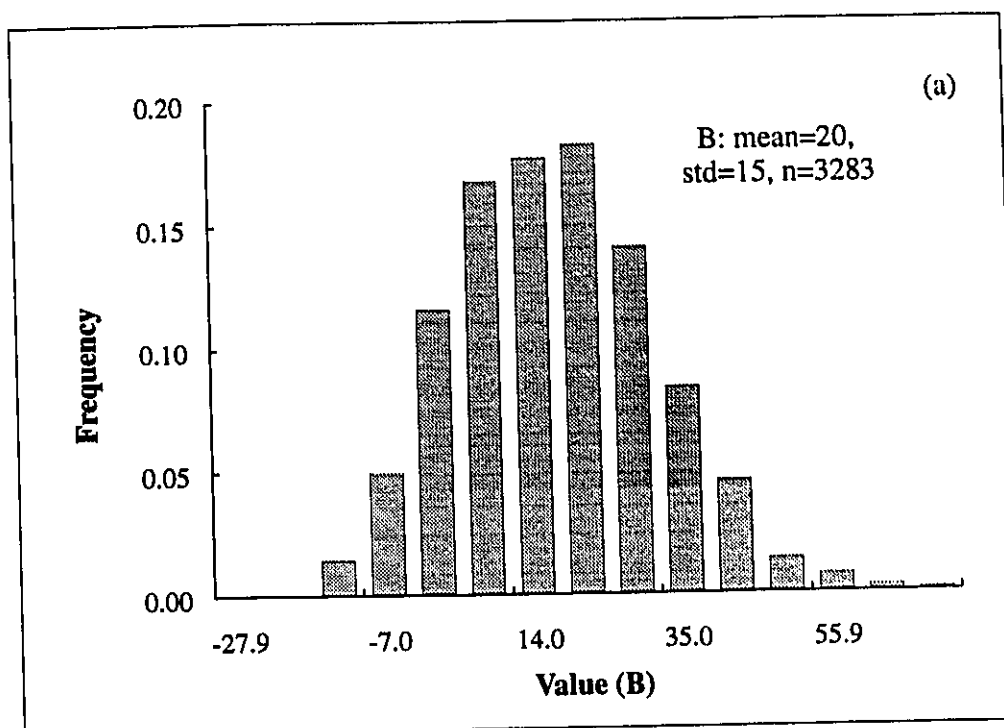
$$e_{II} = \{ \text{number of samples from A classified into B} \} / \{ \text{total number of samples in A} \}$$

The calculated results of e_I and e_{II} for various thresholds are shown in Table 2.3.1. It is shown that results obtained by these thresholds are not satisfactory because errors range from 0.436 to 0.937. The results for $c=0.5$ and 1.5 are shown in Fig. 2.3.2a and 2.3.2b, with total errors ($e_I + e_{II}$) 0.436 and 0.807, respectively.

Table 2.3.1 Errors of classification associated with regional Au values

threshold	mean	mean + 0.5 std	mean + std	mean + 1.5 std	mean + 2 std
e_I	0.458	0.263	0.126	0.047	0.017
e_{II}	0.019	0.173	0.42	0.76	0.92

U^* - statistics were obtained for the same artificial example. Calculations carried out for various distance parameters (r_{max}), including 15 km, 10 km and 6 km, essentially gave the same results. The results for $r_{max}=6$ km are illustrated in Figs. 2.3.1d, 2.3.2c and 2.3.2d. The histogram in Fig. 2.3.1d shows two populations: the one for background B has values less than or close to zero, and the other for anomaly A has values greater than 2. All anomalous samples are located in the right tail of the histogram with values greater than 2; only 4.6 % background samples are misclassified in this area. The errors created by different thresholds ($m+cs$ with $c=0.0, 0.5, 1, 1.5, 2$ and 2.5) are shown in Table 2.3.2. It can be seen that threshold $U^*=2$ provides good results with total error of 0.046, much smaller than those shown in Table 2.3.1. Figs. 2.3.2c and 2.3.2d illustrate the results for classifications with thresholds $m + 0.5$ standard deviation and $U^*=2$. All anomalous samples (317) are correctly classified and only 150 out of 3283 background samples are



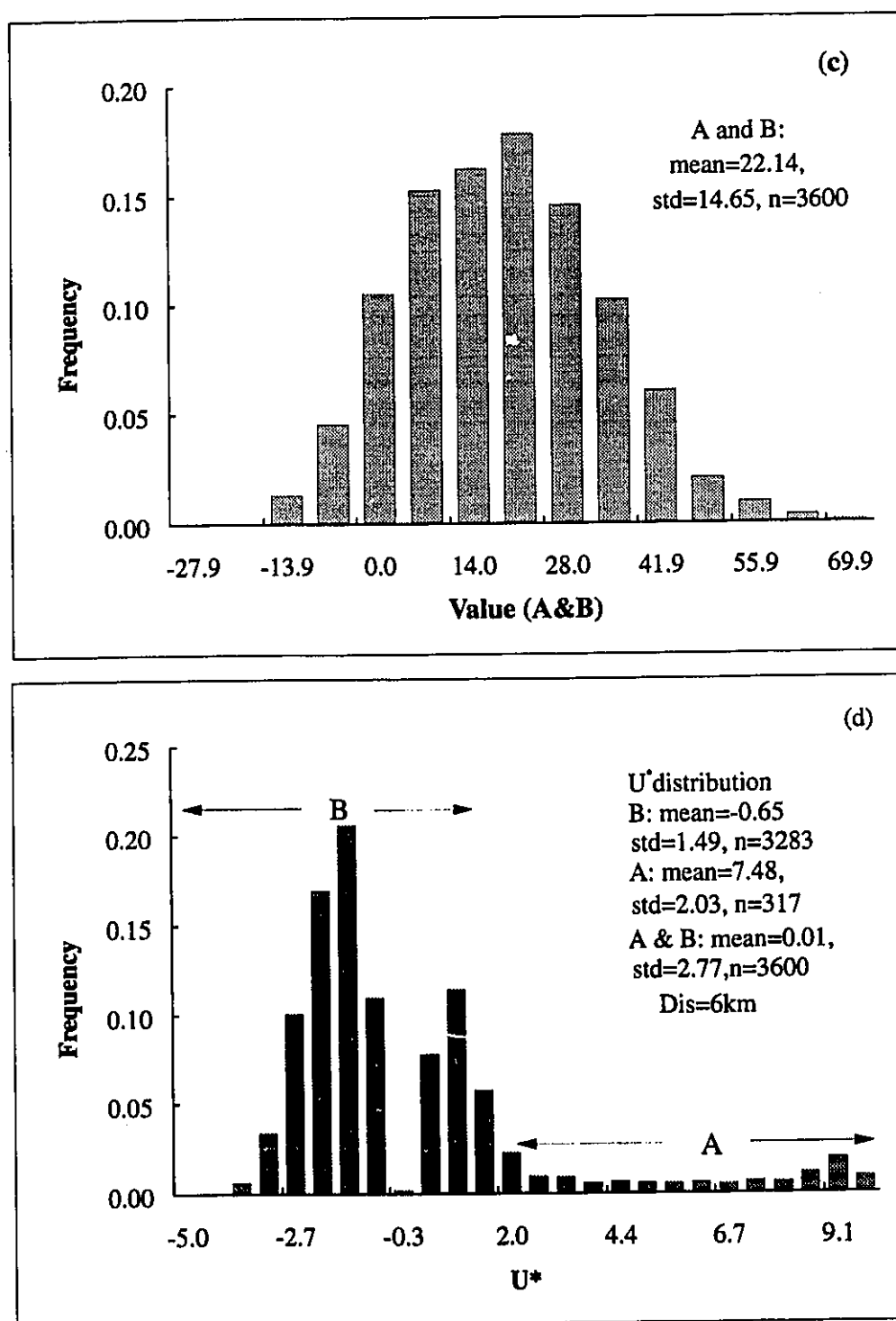
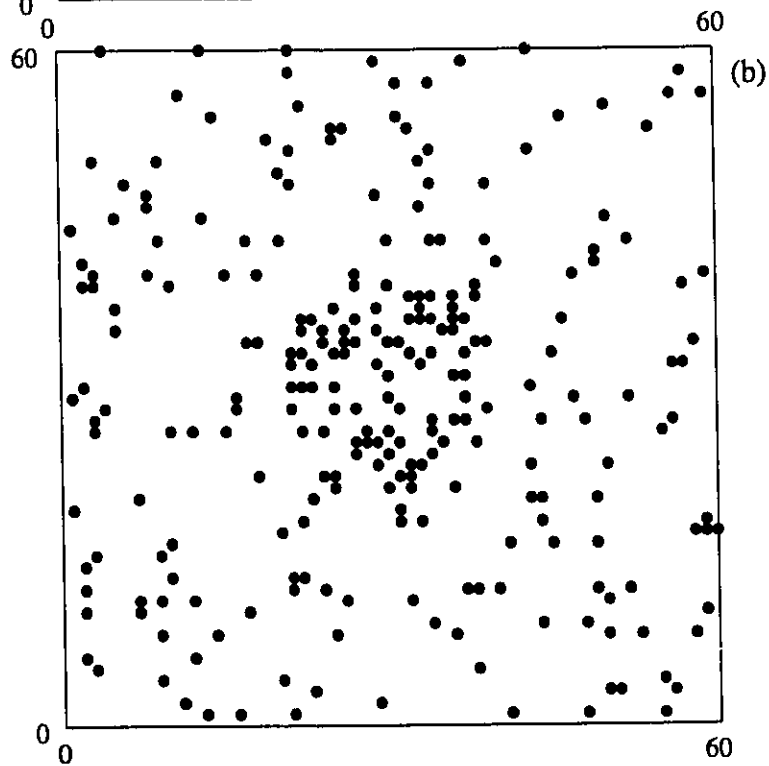
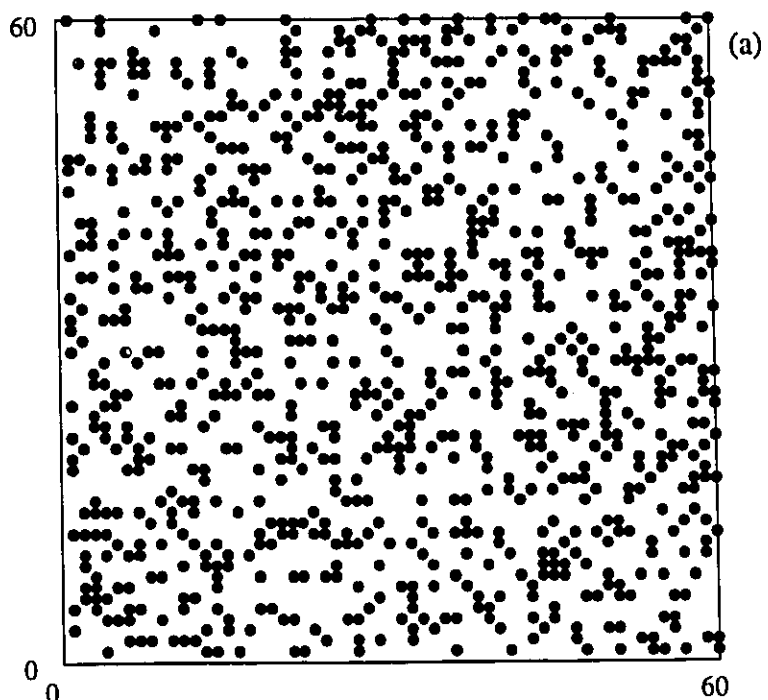


Fig. 2.3.1 Histograms for artificial example. (a) values for 3283 samples (background) derived from normal distribution with $m = 20$, $s = 15$. (b) values for 317 anomalous samples from normal distribution with $m = 40$, $s = 10$. (c) mixed samples in Figs. 2.3.1a and 2.3.1b. (d) U^* -values (see text for definition) for 3600 samples. B: background and A: anomaly.



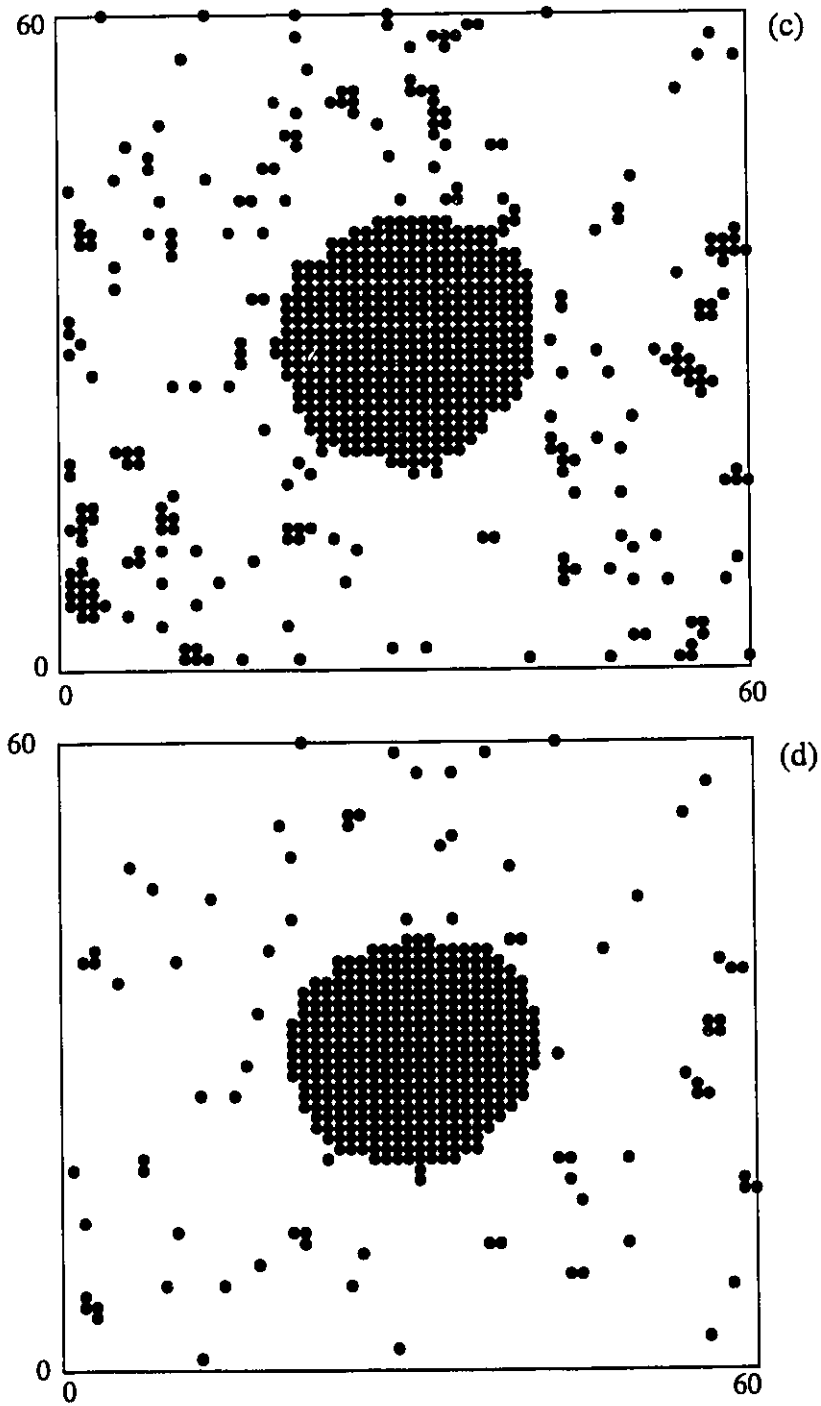


Fig. 2.3.2 Samples classified using different thresholds. (a) threshold of $m+0.5s$ based on histogram of Fig. 2.3.1c with $e_1 + e_{II} = 0.436$; (b) threshold of $m+1.5s$ based on histogram of Fig. 2.3.1c with $e_1 + e_{II} = 0.81$; (c) threshold of $m+0.5s$ based on histogram of Fig. 2.3.1d with $e_1 + e_{II} = 0.10$; (d) threshold of $U' = 2$ based on histogram of Fig. 2.3.1d with $e_1 + e_{II} = 0.042$.

misclassified.

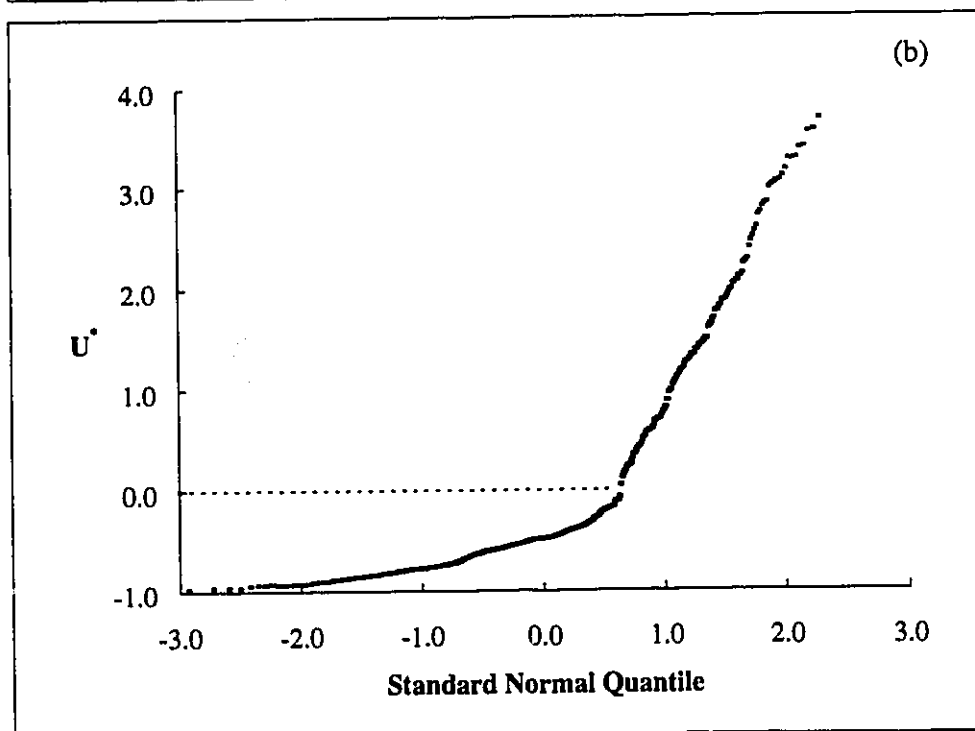
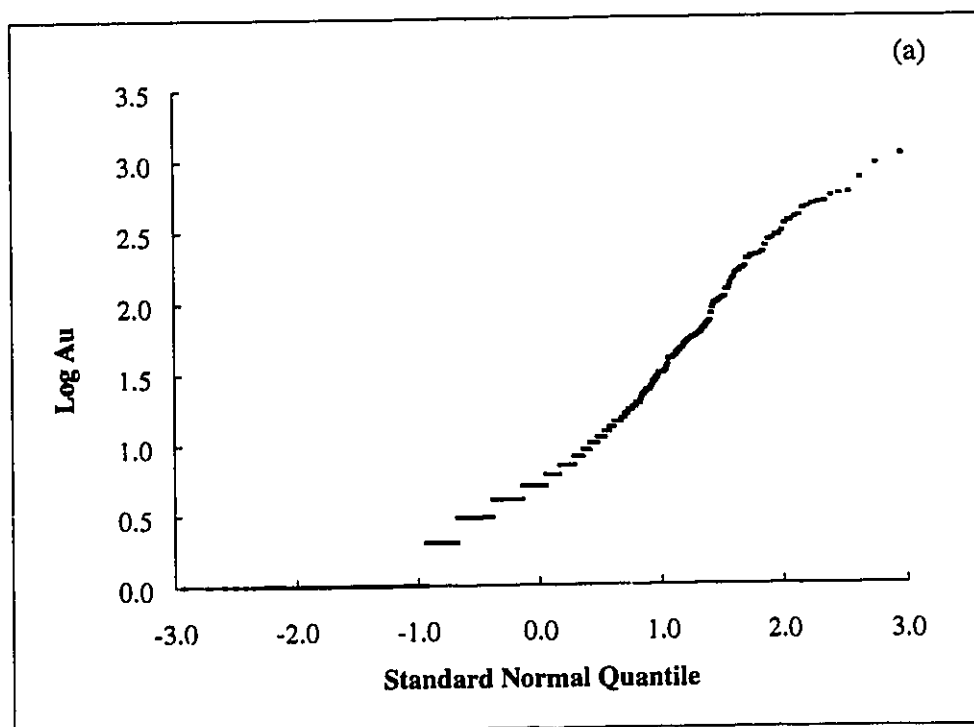
Table 2.3.2 Errors of classification associated with the U^* -statistics

threshold	mean	2	mean + std	mean + 1.5 std	mean + 2 std
e_I	0.314	0.046	0.128	0.0006	0.0
e_{II}	0.0	0.0	0.0003	0.085	0.23

2.3.2 Stream Sediment Data from Iskut River Map Sheet, B.C.

In total, 698 stream sediment samples were collected from the Iskut River area, northwestern British Columbia (Geological Survey of Canada, 1988, GSC Open file 1645). The samples are irregularly distributed in the area (Fig. 1.5), and were analyzed for many trace elements including Au, Ag, Cu, Mo, As, Sb and Pb. The element concentration values of Au, Ag, Cu, As and Sb are spatially associated with Au mineralization. Gold concentration values of the 698 samples are shown in a Q-Q plot (Fig. 2.3.3a) and histogram (Fig. 2.3.3c), respectively. The purpose of this chapter is to delineate Au-associated anomalous areas based on these geochemical values. The anomalous areas can also be delineated by means of fractal methods (Chapter 5; also see Cheng et al., 1994d).

The U^* -statistics method was applied to the stream sediment data with maximum radius $r_{\max}=10$ km. The results are illustrated in Figs. 2.3.4b and 2.3.4d. Fig. 2.3.4b shows two distinct distributions for values of U^* of which one (with $U^*>0$) is close to normal. The



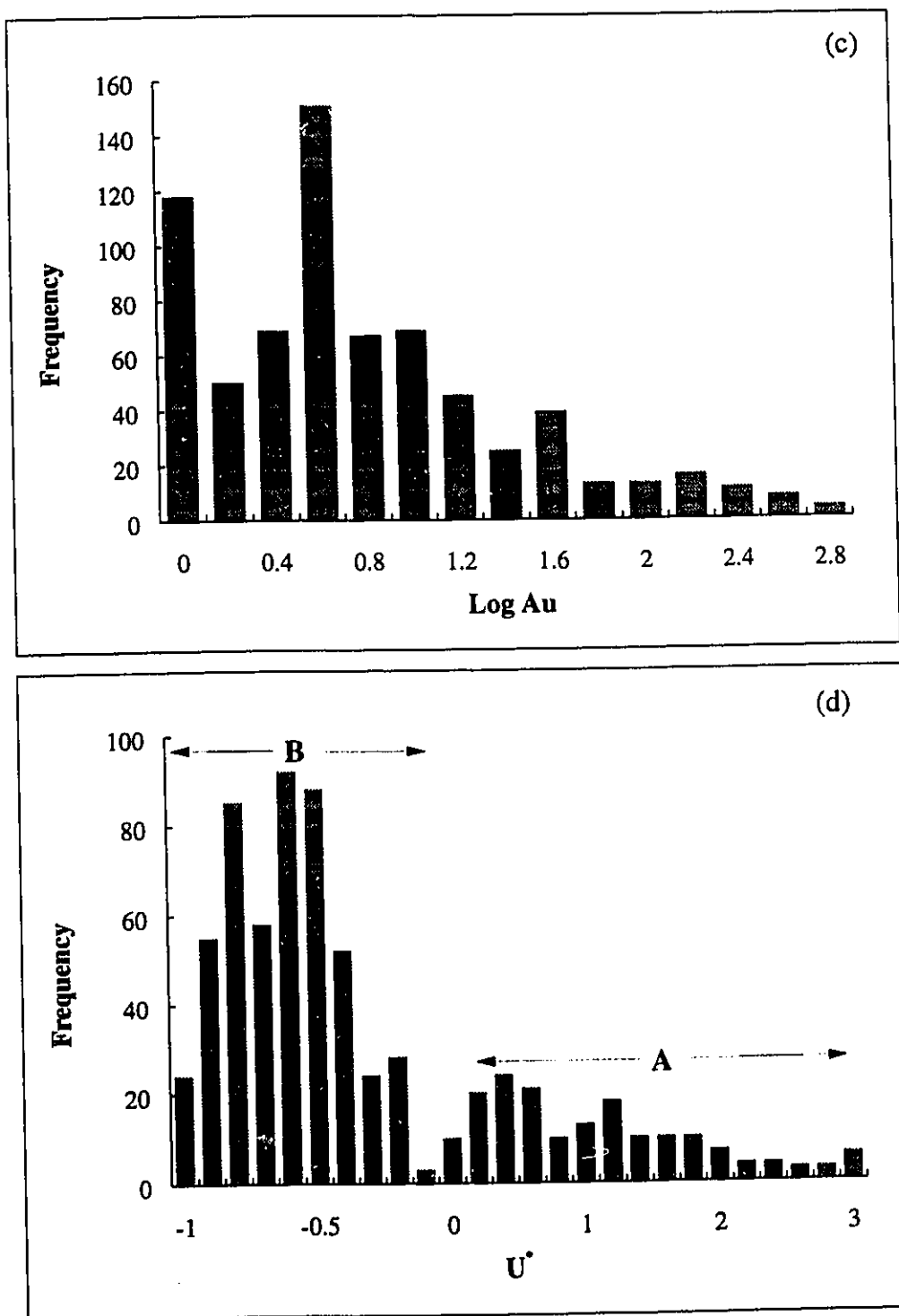


Fig. 2.3.3 Q-Q plots and histograms of Au values for 698 samples. (a) Q-Q plot for base 10 log-transformed data; (b) Q-Q plot for U^* -values; (c) histogram for base 10 log-transformed data; (d) histogram for U^* -values. B: background and A: anomaly

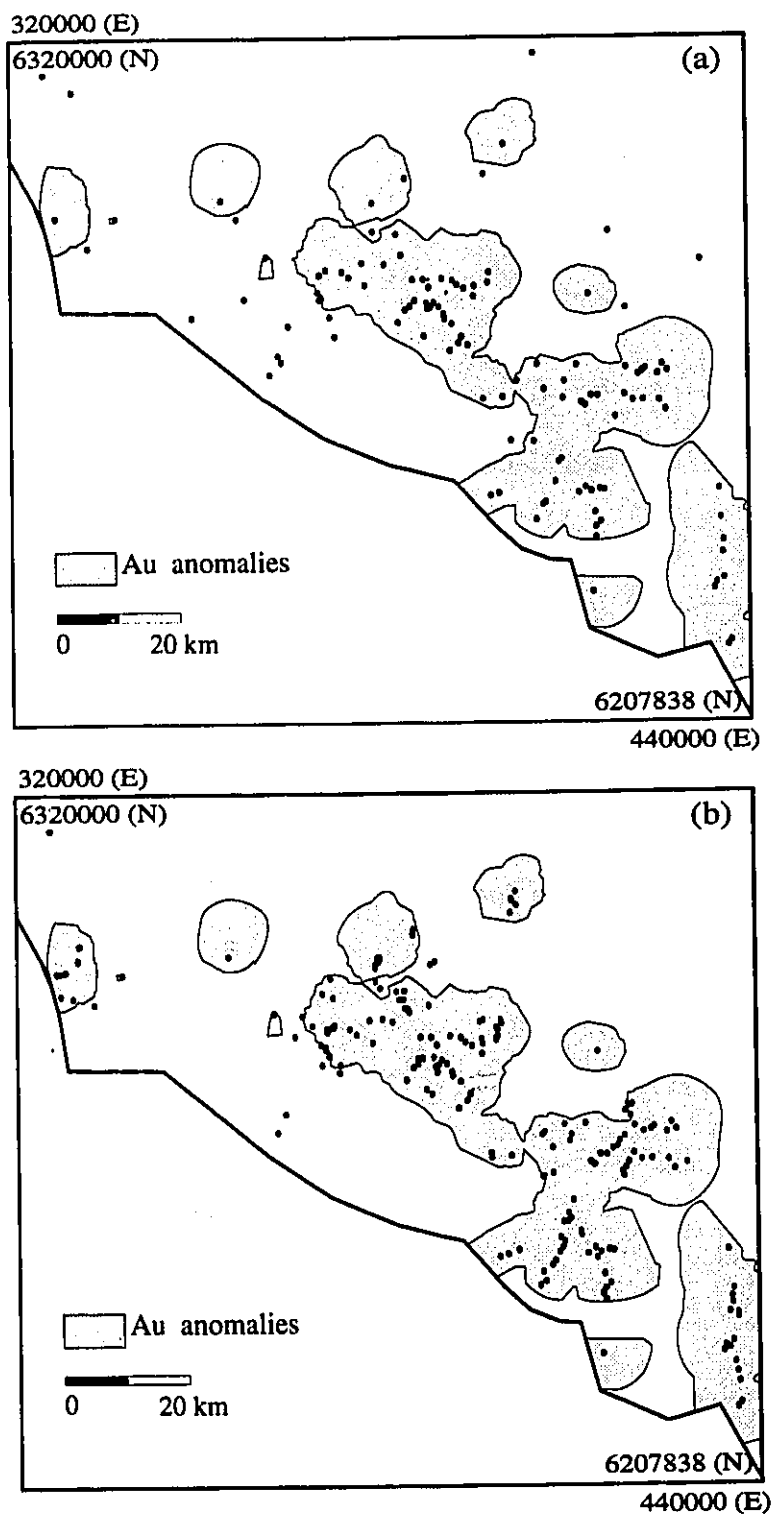
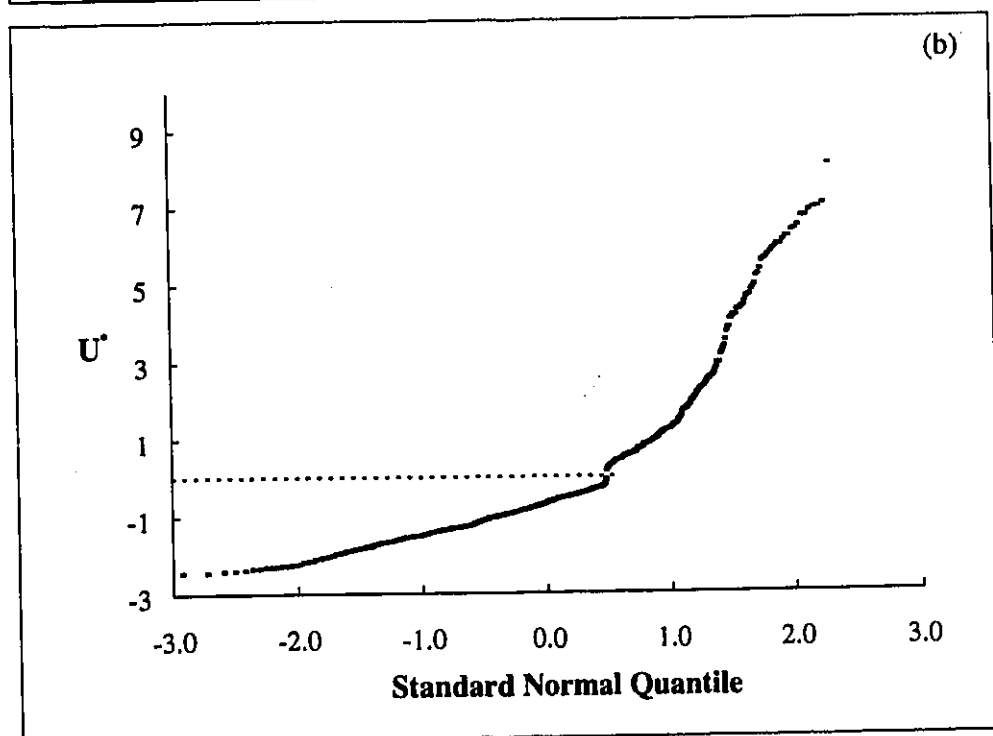
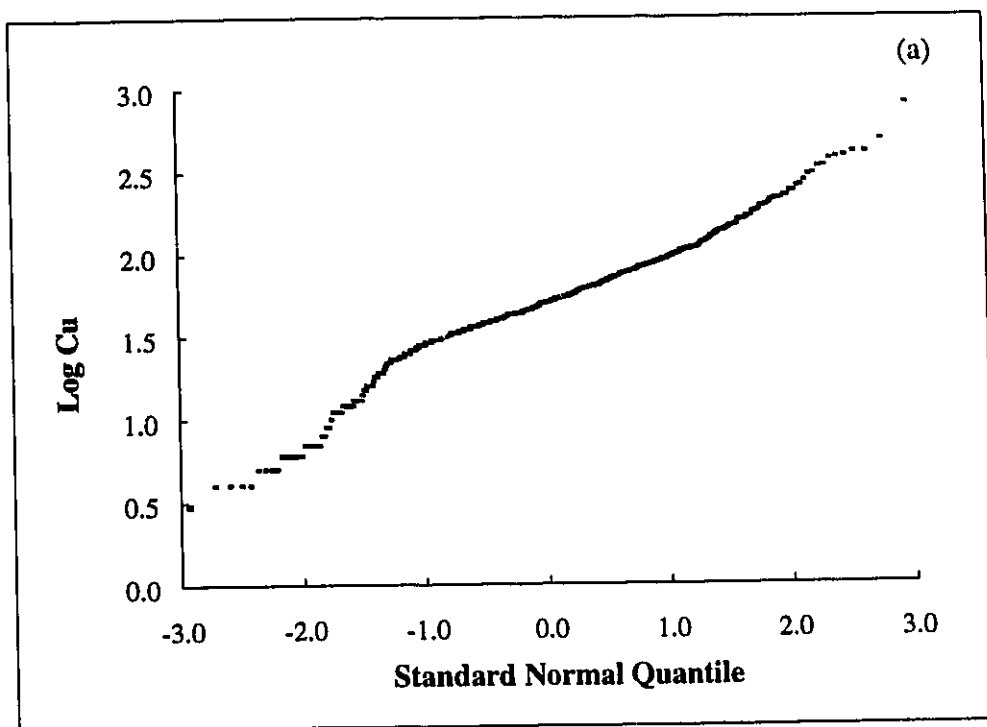


Fig. 2.3.4 Comparison of results obtained by U* - statistics and by fractal method. Shaded areas represent anomalous areas obtained by fractal method. (a) anomalous samples (dots) with Au > 30 ppb; (b) anomalous samples (dots) with U* > 0



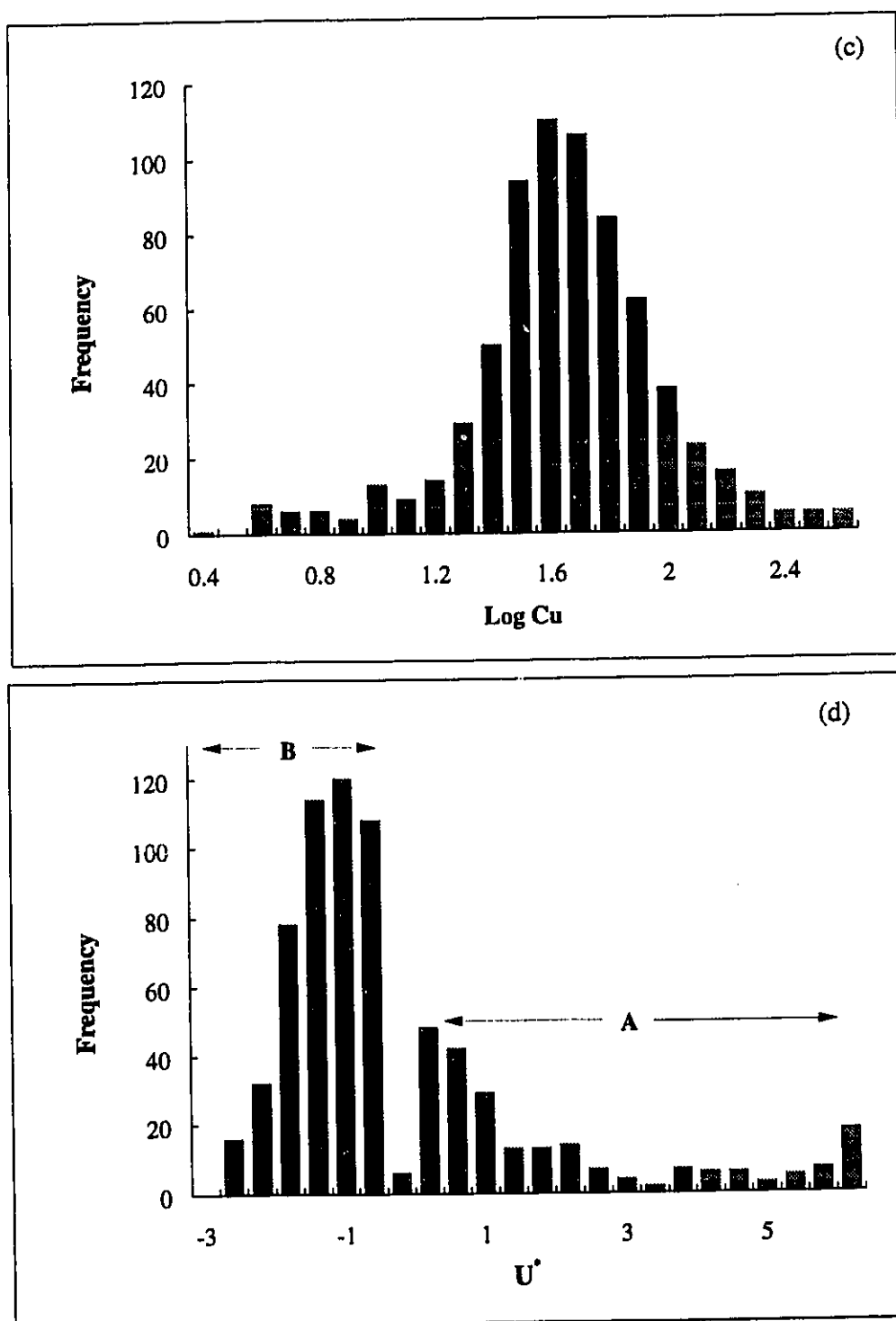
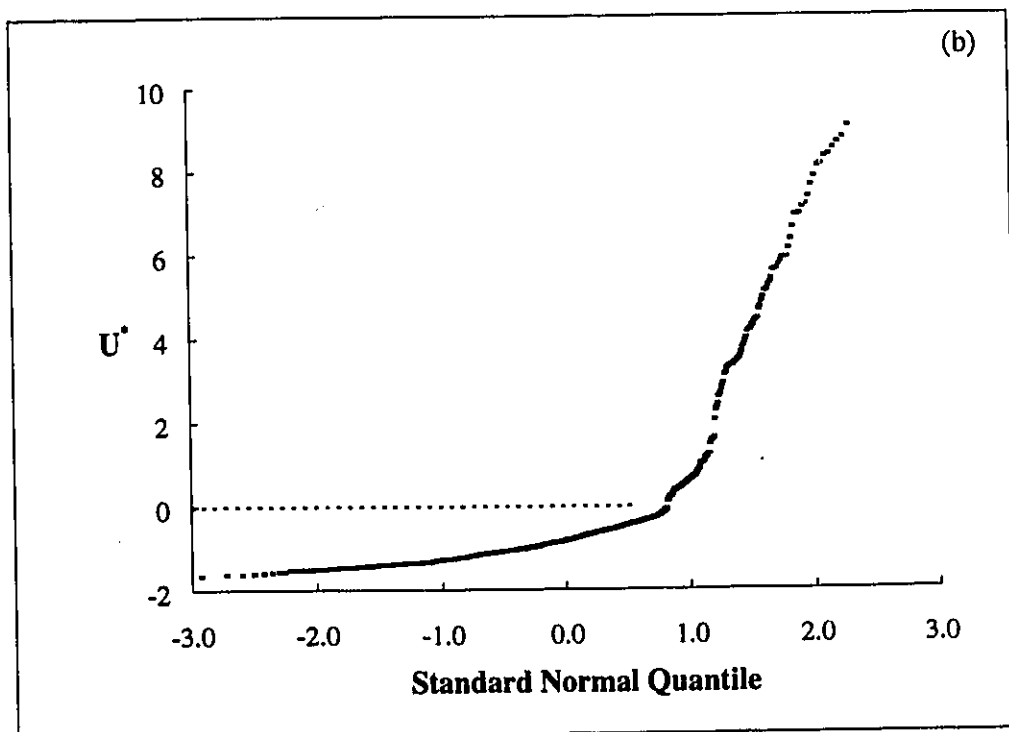
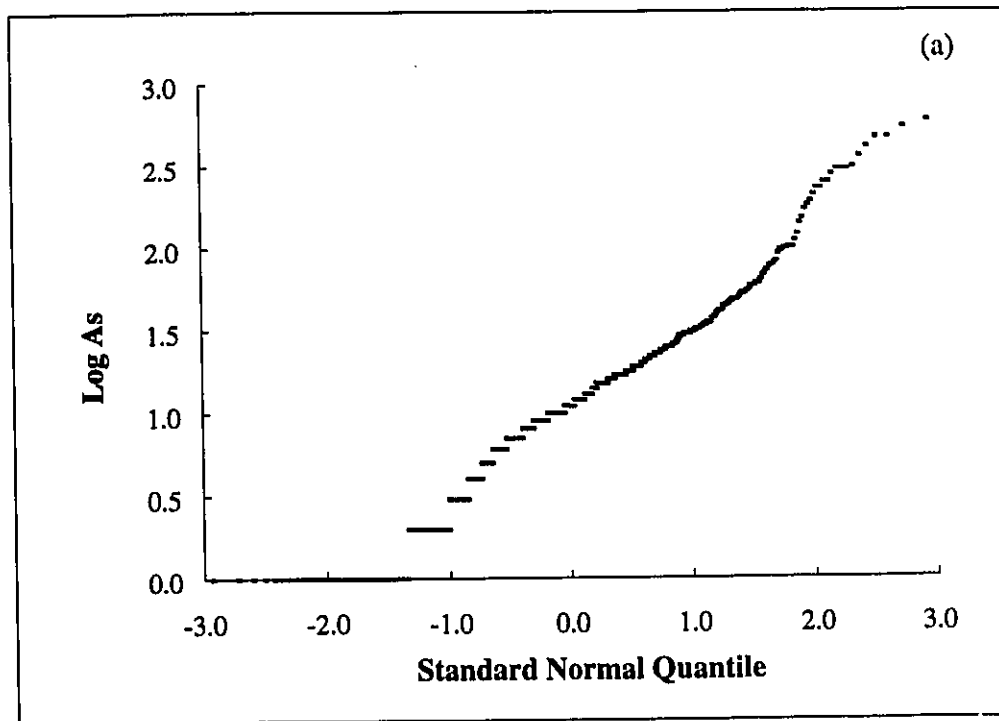


Fig. 2.3.5 Q-Q plots and histograms of Cu values for 698 samples. (a) Q-Q plot for base 10 log-transformed data; (b) Q-Q plot for U' -values; (c) histogram for base 10 log-transformed data; (d) histogram for U' -values. B: background and A: anomaly



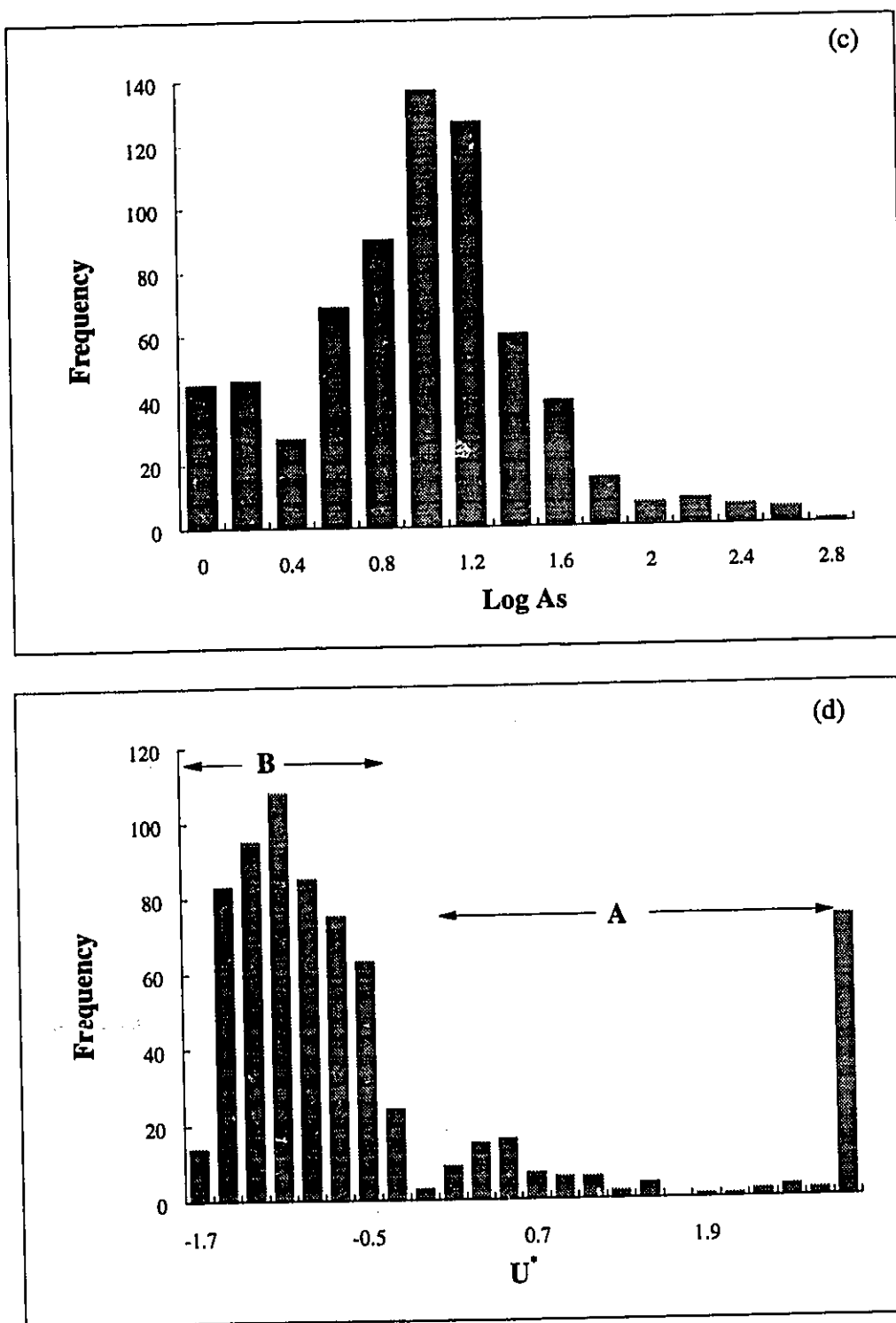


Fig. 2.3.6 Q-Q plots and histograms of A_s values for 698 samples. (a) Q-Q plot for base 10 log-transformed data; (b) Q-Q plot for U^* -values; (c) histogram for base 10 log-transformed data; (d) histogram for U^* -values. B: background and A: anomaly

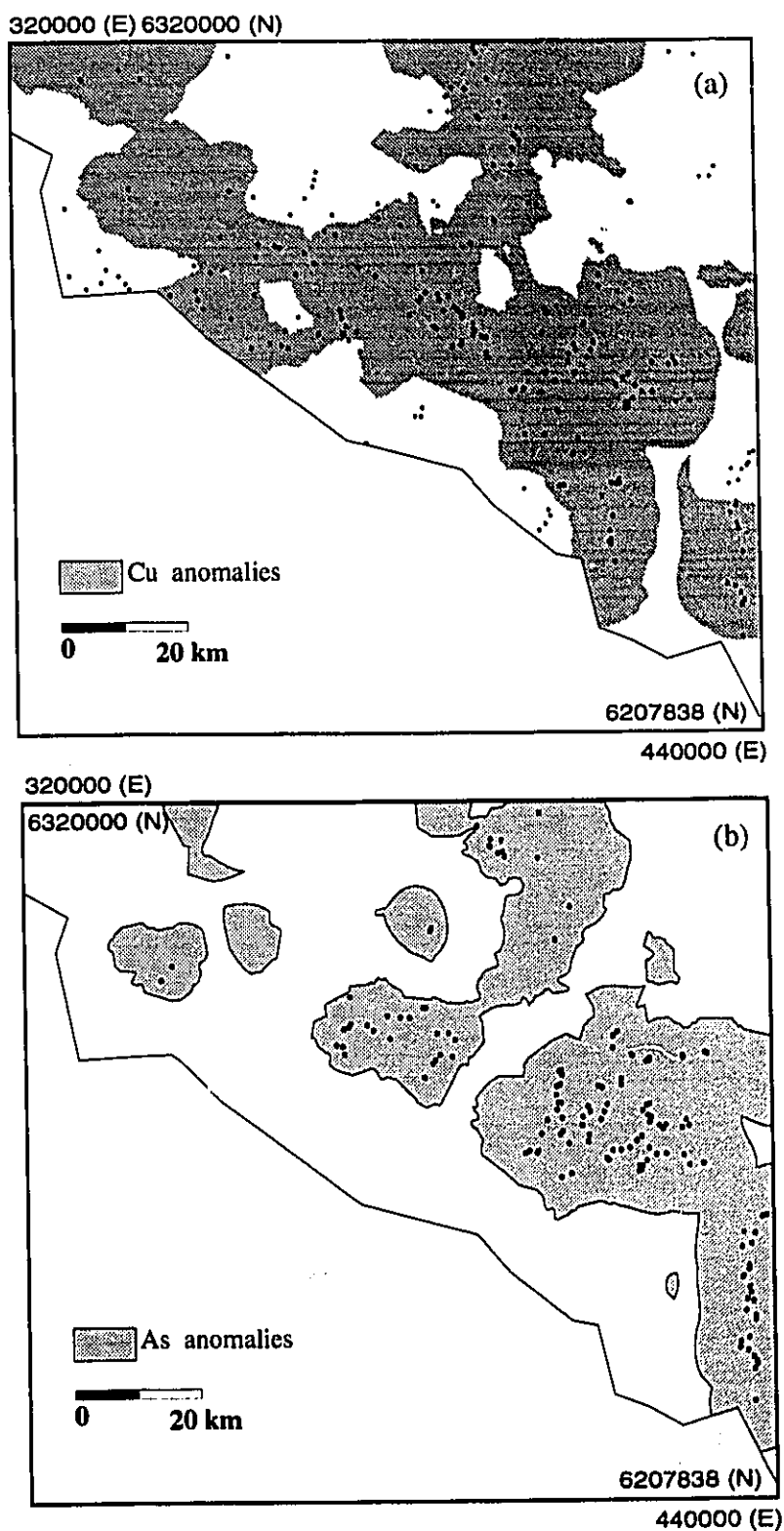


Fig. 2.3.7 Anomalies of stream sediment samples. (a) anomalies with $U > 0$ for Cu; (b) anomalies with $U > 0$ for As.

same results can be seen in Fig. 2.3.3d where background samples (506 samples) are located on the left side ($U^* < 0$) and anomalous samples (192) on the right ($U^* > 0$). The anomalous samples classified by threshold $U^* = 0$ are plotted in Fig. 2.3.4b, which shows that most anomalous samples classified by threshold $U^* = 0$ are located within the anomalous areas delineated by the fractal method (Fig. 2.3.4a). Fig. 2.3.4a shows the locations of samples with Au concentration values greater than 30ppb. There are more anomalous samples within the anomalous areas in Fig. 2.3.4b than in Fig. 2.3.4a, and fewer anomalous samples outside the anomalous areas in Fig. 2.3.4b than in Fig. 2.3.4a. This result implies that the total error of classification for threshold $U^* = 0$ is smaller than for a threshold of $Au = 30ppb$. Similar results were obtained for Cu and As as shown in Figs. 2.3.5 to 2.3.7.

2.4 STATISTICAL PROPERTIES OF U- AND T-STATISTICS

The statistical properties of the total error (e_I and e_{II}) in Eq. (2.2.1) will be examined for several statistics (U , T , χ^2 and F). It will be shown that for U -, χ^2 - and F -statistics the total error approaches zero as the sample size (n) increases; for T -statistics, however, the total error will not be close to zero no matter how large the sample size is.

2.4.1 U- Test

Assume that $X_1, X_2, \dots, X_n$ are values derived from the normal population $N(\mu, \sigma_0^2)$ where

the variance σ_0^2 is known and the mean μ is an unknown parameter. In order to test the null hypothesis $H_0: \mu = \mu_0$, U-statistics can be used with

$$U = \frac{\bar{X} - \mu_0}{\sigma_0} \sqrt{n} \quad (2.4.1)$$

where $\bar{X}$ indicates the sample mean.

For a small value α with $0 < \alpha < 1$, the rejection region W can be determined by

$$W = \{U; |U| \geq U_{\alpha/2}\} \quad (2.4.2)$$

where $U_{\alpha/2}$ is an undetermined threshold. If H_0 is true, then U satisfies the normal distribution of which the density function can be expressed as $\phi(x, 0, 1)$. If H_0 is false, U may satisfy a nonstandard normal distribution with mean

$$\delta = \frac{\mu - \mu_0}{\sigma_0} \sqrt{n}$$

where $\mu \neq \mu_0$ is the population mean. It follows that the errors of the first and second kinds can be expressed as

$$e_I = \int_N \Phi(x, 0, 1) dx$$

$$e_{II} = \int_N \Phi(x, \delta, 1) dx = \int_{-U_{\alpha/2}-\delta}^{U_{\alpha/2}-\delta} \Phi(x, 0, 1) dx$$

Choosing $U_{\alpha/2} = (1/2)\delta$, the sum of e_I and e_{II} becomes

$$e_I + e_{II} = 1 - \int_{-\frac{1}{2}\delta}^{\frac{1}{2}\delta} \Phi(x, 0, 1) dx + \int_{-\frac{3}{2}\delta}^{-\frac{1}{2}\delta} \Phi(x, 0, 1) dx \quad (2.4.3)$$

It is noted that

$$\lim_{n \rightarrow \infty} \delta = \infty$$

and

$$\lim_{n \rightarrow \infty} (e_I + e_{II}) = 0 \quad (2.4.4)$$

2.5.2 T-test

Assume that $X_1, X_2, \dots, X_n$ are derived from the normal distribution $N(\mu, \sigma^2)$ where μ and σ^2 are unknown parameters. In order to test the hypothesis $H_0: \mu = \mu_0$, the following T-statistic is used:

$$T = \frac{\bar{X} - \mu_0}{S_n} \sqrt{n-1}$$

where S_n is the standard deviation of the sample.

If H_0 is true, then T satisfies the central $t(n-1)$ distribution; otherwise, T satisfies the noncentral $t(\delta, n-1)$ distribution where $\delta = \mu - \mu_0$. A rejection area W can be determined as

$$W = \{ |T_{H_0}| \geq t_{\frac{\alpha}{2}} \}$$

where $t_{\alpha/2}$ is an undetermined value, so that

$$\begin{aligned} e_I &= P(|T_{H_0}| \geq t_{\frac{\alpha}{2}}) \\ e_{II} &= P(|T_{H_1}| < t_{\frac{\alpha}{2}}) \end{aligned}$$

In order to discuss the relation between these errors and sample size n , the following theorem is introduced first:

$$\lim_{n \rightarrow \infty} T = X$$

Here, X is a normal random variable and the limit is in probability. According to the definition of Student's t -distribution, T can be expressed as:

$$T = \frac{X}{\sqrt{Y/n}} \quad (2.4.5)$$

where X satisfies the normal distribution, and Y the χ^2 -distribution. Consequently, Y can be expressed as the sum of squares of n normal random variables, and

$$Y = \sum_{i=1}^n X_i^2$$

where $X_i, i=1,2,\dots,n$ are assumed to be independent with identical normal distributions.

Since

$$\frac{1}{n} \sum_{i=1}^n X_i^2 \rightarrow EX_i^2 = \sigma^2, n \rightarrow \infty, (P)$$

where the limit is in probability equal to 1. For uniform convergence, the following two expressions hold true

$$\frac{1}{n} \sum_{i=1}^n X_i^2 \rightarrow \sigma^2$$

and

$$\sqrt{\frac{1}{n} \sum_{i=1}^n X_i^2} \rightarrow \sigma$$

Using the theorem, it follows that

$$\sqrt{\frac{1}{n} \sum_{i=1}^n X_i^2} \rightarrow \sigma, (P)$$

Therefore,

$$T = \frac{X}{\sqrt{Y/n}} \rightarrow \frac{X}{\sigma}, (P)$$

where X/σ satisfies the normal distribution $N(\mu, 1)$. It follows that, if $\delta > 0$,

$$\begin{aligned} \lim_{n \rightarrow \infty} (e_I + e_{II}) &= 1 - \int_{-t_{\alpha/2}}^{t_{\alpha/2}} \Phi(x, 0, 1) dx + \int_{-t_{\alpha/2}}^{t_{\alpha/2}} \Phi(x, \delta, 1) dx \\ &= 1 - \int_{-t_{\alpha/2}}^{t_{\alpha/2}} \Phi(x, 0, 1) dx + \int_{-t_{\alpha/2}-\delta}^{t_{\alpha/2}-\delta} \Phi(x, 0, 1) dx \\ &\geq 1 - \int_{t_{\alpha/2}-\delta}^{t_{\alpha/2}} \Phi(x, 0, 1) dx \geq 1 - \int_{-\delta}^{\delta} \Phi(x, 0, 1) dx > 0 \end{aligned}$$

where, as before, Φ represents the density function of the normal distribution. Similarly,

if $\delta < 0$,

$$\lim_{n \rightarrow \infty} (e_I + e_{II}) \geq 1 - \int_{-t_{\alpha/2}}^{-t_{\alpha/2}-\delta} \Phi(x, 0, 1) dx \geq 1 - \int_{\delta}^{-\delta} \Phi(x, 0, 1) dx > 0$$

For these results use was made of the following properties of the mean and variance of

T. When H_0 is true, the mean and variance of T are

$$ET=0, \sigma^2(T) = \frac{n-1}{n-3}$$

For $n \rightarrow \infty$, T approaches a standard normal variable. When H_0 is false, the mean and variance of T are

$$ET_{H_1} = \delta \frac{\Gamma[\frac{n-2}{2}]}{\Gamma[\frac{n-1}{2}]} \sqrt{\frac{n-1}{2}}$$

$$\sigma^2(T_{H_1}) = \frac{(n-1)(1+\delta^2)}{n-3} - \frac{\delta^2(n-1)}{2} \left\{ \frac{\Gamma[\frac{n-2}{2}]}{\Gamma[\frac{n-1}{2}]} \right\}^2, n > 3$$

where $\Gamma(x)$ is a gamma function. It can be shown that

$$\lim_{n \rightarrow \infty} \left\{ \frac{\Gamma[\frac{n-2}{2}]}{\Gamma[\frac{n-1}{2}]} \sqrt{\frac{n-1}{2}} \right\} = 1$$

From the asymptotic formula

$$\Gamma[x] = \sqrt{2\pi x} x^{x-\frac{1}{2}} e^{-x} [1+r(x)]$$

where

$$|r(x)| \leq e^{\frac{1}{12x} - 1}$$

leading to Napier's definition

$$\lim_{n \rightarrow \infty} \left(1 - \frac{1}{n}\right)^{n-1} = e^{-1}$$

Finally, because

$$\begin{aligned} \frac{\Gamma[\frac{n-2}{2}]}{\Gamma[\frac{n-1}{2}]} \sqrt{\frac{n-1}{2}} &= \frac{(\frac{n-2}{2})^{\frac{n-3}{2}} e^{-\frac{n-2}{2}} [1+r(\frac{n-2}{2})]}{(\frac{n-1}{2})^{\frac{n-2}{2}} e^{-\frac{n-1}{2}} [1+r(\frac{n-1}{2})]} (\frac{n-1}{2})^{\frac{1}{2}} \\ &= (1 - \frac{1}{n-1})^{\frac{n-3}{2}} e^{\frac{1}{2}} \frac{1+r(\frac{n-2}{2})}{1+r(\frac{n-1}{2})} \end{aligned}$$

leading to a version of Stirling's formula for $n!$ with

$$\lim_{n \rightarrow \infty} \frac{\Gamma[\frac{n-2}{2}]}{\Gamma[\frac{n-1}{2}]} \sqrt{\frac{n-1}{2}} = 1$$

it follows that

$$\begin{aligned} \lim_{n \rightarrow \infty} ET_{H_1} &= \delta = \mu - \mu_0 \\ \lim_{n \rightarrow \infty} DT_{H_1} &= 1 + \delta^2 - \delta^2 = 1 \end{aligned}$$

Consequently, the limits of the mean and variance of T are consistent with those of a normal distribution.

2.4.3 χ^2 -test

Assume that $X_1, X_2, \dots, X_n$ are derived from the normal distribution $N(\mu, \sigma^2)$. In order to test the hypothesis $H_0: \sigma^2 = \sigma_0^2$, the following χ^2 -statistic is used:

$$\chi^2 = \frac{nS_n^2}{\sigma_0^2}$$

When H_0 is true, χ^2 is the $\chi^2(n-1)$ distribution. When H_0 is false, $\chi^2 \sigma_0^2 / \sigma^2$ satisfies the

$\chi^2(n-1)$ distribution. The rejection area can be determined as

$$W = \{\chi^2; \chi^2 \geq C_{1\alpha}, \chi^2 \leq C_{2\alpha}, C_{2\alpha} < C_{1\alpha}\}$$

where $C_{1\alpha}$ and $C_{2\alpha}$ are two undetermined constants. Therefore, the two errors can be expressed as

$$\begin{aligned} e_I &= 1 - P(C_{2\alpha} \leq \chi_{H_0}^2 \leq C_{1\alpha}) \\ e_{II} &= P(C_{2\alpha} < \chi_{H_1}^2 < C_{1\alpha}) \end{aligned}$$

It can be proved that $e_I + e_{II} \rightarrow 0$, for $n \rightarrow \infty$. Because, under H_0 , χ^2 and, under H_1 , $\chi^2 \sigma_0^2 / \sigma^2$, are $\chi^2(n-1)$ distributions. The two errors can be calculated as

$$\begin{aligned} e_I &= 1 - P(C_{2\alpha} \leq \sum_{i=1}^{n-1} X_i^2 \leq C_{1\alpha}) \\ e_{II} &= P(C_{2\alpha} < \frac{\sigma_0^2}{\sigma^2} \sum_{i=1}^{n-1} X_i^2 < C_{1\alpha}) \end{aligned}$$

where $X_1, X_2, \dots, X_n$ are independent standard normal random variables.

Assume $\sigma^2 > \sigma_0^2$ without losing generality. The two values of $C_{1\alpha}$ and $C_{2\alpha}$ can be chosen as

$$C_{1\alpha} = (1+\epsilon)(n-1), C_{2\alpha} = (1-\epsilon)(n-1)$$

where $\epsilon > 0$ is a small positive value; e.g.,

$$\epsilon = \frac{1}{2} \frac{\sigma^2 - \sigma_0^2}{\sigma^2 + \sigma_0^2}$$

Therefore,

$$\begin{aligned}
e_I + e_{II} &= 1 - P\left\{\frac{C_{2\alpha}}{n-1} \leq \frac{1}{n-1} \sum_{i=1}^{n-1} X_i^2 \leq \frac{C_{1\alpha}}{n-1}\right\} + P\left\{\frac{C_{2\alpha}}{n-1} \frac{\sigma^2}{\sigma_0^2} < \sum_{i=1}^{n-1} X_i^2 < \frac{C_{1\alpha}}{n-1} \frac{\sigma^2}{\sigma_0^2}\right\} \\
&= 1 - P\left\{\frac{C_{2\alpha}}{n-1} - 1 \leq \frac{1}{n-1} \sum_{i=1}^{n-1} X_i^2 - 1 \leq \frac{C_{1\alpha}}{n-1} - 1\right\} + P\left\{\frac{C_{2\alpha}}{n-1} \frac{\sigma^2}{\sigma_0^2} - 1 < \sum_{i=1}^{n-1} X_i^2 - 1 < \frac{C_{1\alpha}}{n-1} \frac{\sigma^2}{\sigma_0^2} - 1\right\} \\
&\leq 1 + P\left\{\frac{1}{n-1} \sum_{i=1}^{n-1} X_i^2 - 1 < -\epsilon\right\} - P\left\{-\epsilon \leq \frac{1}{n-1} \sum_{i=1}^{n-1} X_i^2 - 1 \leq \epsilon\right\} \\
&\leq 1 + P\left\{\left|\frac{1}{n-1} \sum_{i=1}^{n-1} X_i^2 - 1\right| > \epsilon\right\} - P\left\{\left|\frac{1}{n-1} \sum_{i=1}^{n-1} X_i^2 - 1\right| \leq \epsilon\right\}
\end{aligned}$$

Because

$$\lim_{n \rightarrow \infty} \frac{1}{n-1} \sum_{i=1}^{n-1} X_i^2 - 1 = 0$$

it follows that

$$\lim_{n \rightarrow \infty} (e_I + e_{II}) = 1 + \lim_{n \rightarrow \infty} P\left\{\left|\frac{1}{n-1} \sum_{i=1}^{n-1} X_i^2 - 1\right| > \epsilon\right\} - \lim_{n \rightarrow \infty} P\left\{\left|\frac{1}{n-1} \sum_{i=1}^{n-1} X_i^2 - 1\right| \leq \epsilon\right\} = 0$$

2.4.4 F-test

Assume that $X_1, X_2, \dots, X_n$ are normal with $N(\mu_1, \sigma_1^2)$ and $Y_1, Y_2, \dots, Y_n$ are normal with $N(\mu_2, \sigma_2^2)$ where μ_1, μ_2, σ_1^2 and σ_2^2 are unknown parameters; X_i and Y_j are independent.

In order to test the hypothesis $H_0: \sigma_1^2 = \sigma_2^2$, F-statistics can be used with

$$F = \frac{mS_{1m}^2}{nS_{2n}^2} \frac{n-1}{m-1}$$

When H_0 is true, F satisfies the central $F(x, m-1, n-1)$ distribution; otherwise, $F\sigma_2^2/\sigma_1^2$ satisfies the central $F(x, m-1, n-1)$ distribution. The rejection area can be determined as

$$W = \{F; F \geq f_{1\alpha} \cup F \leq f_{2\alpha}, f_{2\alpha} < f_{1\alpha}\}$$

Therefore,

$$e_I = 1 - P(f_{2\alpha} \leq F \leq f_{1\alpha}) ; e_{II} = P(f_{2\alpha} < \frac{\sigma_2^2}{\sigma_1^2} F < f_{1\alpha})$$

It can be proved that

$$\lim_{n, m \rightarrow \infty} F(m-1, n-1) = 1$$

Consequently,

$$e_I + e_{II} = 1 - P(f_{2\alpha} - 1 \leq F - 1 \leq f_{1\alpha} - 1) + P(\frac{\sigma_1^2}{\sigma_2^2} f_{2\alpha} - 1 < F - 1 < \frac{\sigma_1^2}{\sigma_2^2} f_{1\alpha} - 1)$$

Assume $\sigma_1^2 > \sigma_2^2$ without losing generality, and with ϵ being a small positive value; e.g., $0 < \epsilon < (\sigma_1^2 - \sigma_2^2)/(\sigma_1^2 + \sigma_2^2) < 1$ if $\sigma_1^2 > \sigma_2^2$, or $0 < \epsilon < (\sigma_2^2 - \sigma_1^2)/(\sigma_1^2 + \sigma_2^2) < 1$ if $\sigma_2^2 > \sigma_1^2$. Then, setting $f_{1\alpha} = 1 + \epsilon$ and $f_{2\alpha} = 1 - \epsilon$,

$$\begin{aligned} e_I + e_{II} &\leq 1 - P(-\epsilon \leq F - 1 \leq \epsilon) + P(F - 1 < -\epsilon) \\ &\leq 1 - P(|F - 1| \leq \epsilon) + P(|F - 1| > \epsilon) \end{aligned}$$

and

$$\lim_{n, m \rightarrow \infty} e_I + e_{II} = 0$$

This completes the proof that the U-, χ^2 - and F-statistics belong to the first type of statistic, and T to the second type for which the total error is not close to zero no matter how large the sample size is. In the applications of this chapter (section 2.3), U*-statistics based on U were used, which are of the first type. Similar statistics of the second type

could be constructed based on T by replacing the population parameters used in Eq. (2.2.4) by estimates for the samples.

2.5 CONCLUSION

The parametric statistics can be grouped into two types: for the first type, including U-, χ^2 - and F-statistics, the total misclassification error (rejection and acceptance) approaches zero as the sample size increases infinitely; for the other type, including T-statistics, the total error may not approach zero no matter how many samples are used. A spatial statistic U'-based on U-statistics is proposed using the moving average technique with variable window radius for geochemical anomaly separation. This method is used for Au and Au-associated geochemical anomaly separation in stream sediment samples from the Iskut River map sheet, northwestern British Columbia.

Chapter 3 SPATIAL OBJECTS AND FRACTAL MODELS

3.1 INTRODUCTION

Spatial objects are usually grouped into points, lines, areas, surfaces and volumes, with different spatial dimensions. As more powerful and higher resolution computers become available, more complex objects can be dealt with in GIS applications in the geosciences. Recently developed fractal/multifractal theories provide powerful tools for revealing the complex nature of fractal objects (Mandelbrot, 1983). A fractal is a set with Hausdorff dimension different from its topological dimension. Fractal geometry is characterized by non-integer dimensions. It has been shown that many types of spatial objects in the geosciences can be treated by means of fractal modelling (Korvin, 1992; Turcotte, 1992; Schroeder, 1990; Falconer, 1990; Robert, 1988; Fowler et al., 1989; Fowler and Roach, 1992; Roach and Fowler, 1993; Unwin, 1989; Agterberg et al., 1993a; Cheng and Agterberg, 1994; Cheng, Agterberg and Ballantyne, 1994a).

GIS is also a new subject with rapid development during the past decade, which provides computer-based methods for the information management of spatial data, especially for defining, extracting and analyzing spatial objects from digitized maps. In addition, the raster data model included in most GIS software is advantageous for performing the measurements required for fractal analysis with varying yardsticks for quantification. The ability to make vector and raster conversions in GIS is also useful for implementing the

fractal modelling of spatial objects. In this chapter, concepts of fractal geometry and their implementation with GIS will be discussed.

3.2 FRACTAL OBJECTS AND MEASUREMENTS

In general, mathematical measures are defined on the basis of elementary concepts such as number, length and area. Most mathematical theory was developed from these measures leading to classical mathematics including calculus and probability theory based on Lebesgue measure. In fractal geometry, the elementary concepts of number, length and area are not valid and should be modified. For discussing the concept of fractals and fractal measures, simple examples will be used. A classic example is the coastline. When a coastline is measured at progressively smaller scales, its length generally increases infinitely (Mandelbrot, 1983). In order to measure the length of a coastline, measuring sticks of varying sizes can be used. The coastline as it is represented on a map can be divided into successive curved line segments. One way to estimate its length then is to replace these curved line segments by straight lines and to add the lengths of the resulting straight line segments. The length of coastline obtained in this way depends on the size of the measuring stick used to define the lengths of the segments. For ordinary lines with dimension 1 (e.g., any coastline after drawing it on a map), the estimated length approaches a finite value as the stick size decreases to zero. For a fractal coastline, however, the estimated length will continue to increase indefinitely as stick size decreases to zero. It has been found that, in general, the estimated length of coastlines follows a

power-law relation with the length of the measuring stick δ (Mandelbrot, 1983):

$$L(\delta) = N(\delta) \delta \propto \delta^{1-D} \quad (3.2.1)$$

where $\propto$ indicates proportionality. If the length L and stick size δ are plotted on log-log paper, the result is a straight line with:

$$\text{Log } L(\delta) \propto (1-D) \text{Log } \delta \quad (3.2.2)$$

The slope of this line is $(1-D)$, where D is the fractal dimension. Values of D are typically about 1.25 (Mandelbrot, 1983), although the deeply incised coastline of Norway has $D=1.52$ (Feder, 1988). Similarly, the fractal dimension of a natural surface is generally found to be between 2 and 3.

Figure 3.2.1a illustrates an artificial spatial object (black area) in a square study region. The fractal dimension determined by the box-counting method is as follows. A grid with linear cell size ϵ ($1/2$ to $1/32$) is superimposed on Fig. 3.2.1a. The number of cells containing the object is counted and denoted as $N(\epsilon)$. The quantity $N(\epsilon)$ depends on the cell size ϵ , generally increasing with decrease in ϵ . Fig. 3.2.1b shows estimated results of $N(\epsilon)$ represented in a log-log plot. A straight line fitted to these values has an estimated slope of -1.76. In general, if the value $N_A(\epsilon)\epsilon^2$ approaches a finite value for small ϵ , it can be taken as the estimated area of nonfractal objects. The subscript A refers to area objects, and $N_L(\epsilon)\epsilon$ and $N_E(\epsilon)$ are comparable measures of the length and number of nonfractal line and point objects, respectively. For fractal objects, however, $N(\epsilon)$ has the following power-law relation (Mandelbrot, 1983):

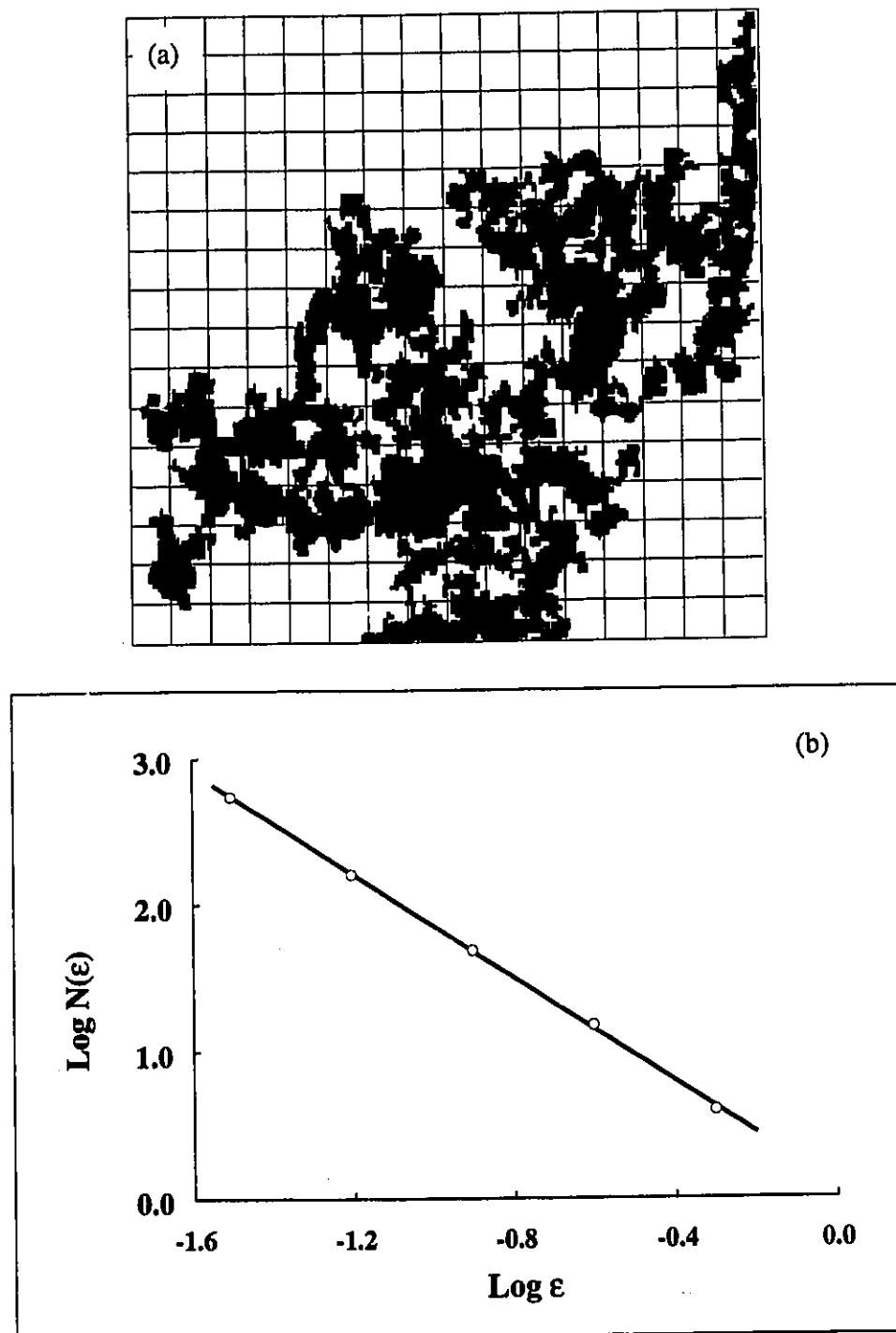


Fig. 3.2.1 Artificial example showing fractal pattern and fractal measurements.
(a) fractal pattern is represented by black areas. Grid size is 1/16;
(b) log-log plot showing power-law relationship between $N(\epsilon)$ and ϵ .

$$N(\epsilon) = c\epsilon^{-D} \quad (3.2.3)$$

$$D = \lim_{\epsilon \rightarrow 0} \frac{\log N(\epsilon)}{\log 1/\epsilon} \quad (3.2.4)$$

where $0 \leq D \leq 2$ is the box-counting fractal dimension of the pattern which can be estimated by least squares together with its standard deviation ($D = 1.76 \pm 0.02$ for the objects in Fig.3.2.1a). This relation can be represented as a linear function on log-log paper:

$$\text{Log } N(\epsilon) = \text{Log } c - D \text{Log } \epsilon \quad (3.2.5)$$

The dimension (D) is a useful measure for fractal patterns. Nonfractal geometries with integer dimension are special cases of fractal geometries. The constant c in Eq. (3.2.3) is also a measure of the geometry. In ordinary geometry it represents area with $D=2$, length of line with $D=1$, or number of points with $D=0$. For fractals in D -dimensional space it is a measure with

$$c = \lim_{\epsilon \rightarrow 0} N(\epsilon) \epsilon^D \quad (3.2.6)$$

Example 3.2.1. Figure 3.2.2 shows the construction of a set by means of an initiator that is a triangle. First, nine subtriangles with one third-length sides are formed. The generator eliminates the triangles except two, those in the lower left and lower right corners. The first generation consists of two lower triangles. Each new generation is obtained by applying the generator to all remaining triangles. This prefractal results in a geometry with zero perimeter and zero area. The length of yardstick of the n -th generation is

$$\epsilon_n = \left(\frac{1}{3}\right)^n \quad (3.2.7)$$

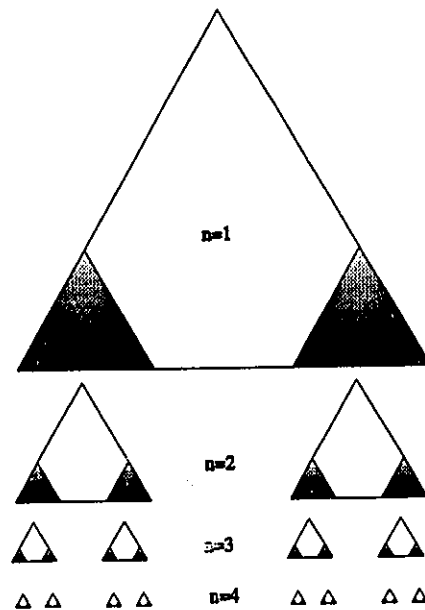


Fig. 3.2.2 Construction of set with fractal dimension $D_L = D_A = 0.6309$. The initiator is a triangle. In each generation, the triangles are divided into nine same-shaped triangles; the generator eliminates seven and leaves two small triangles in the lower corners. Four generations of the prefractal sets are shown in the Figure.

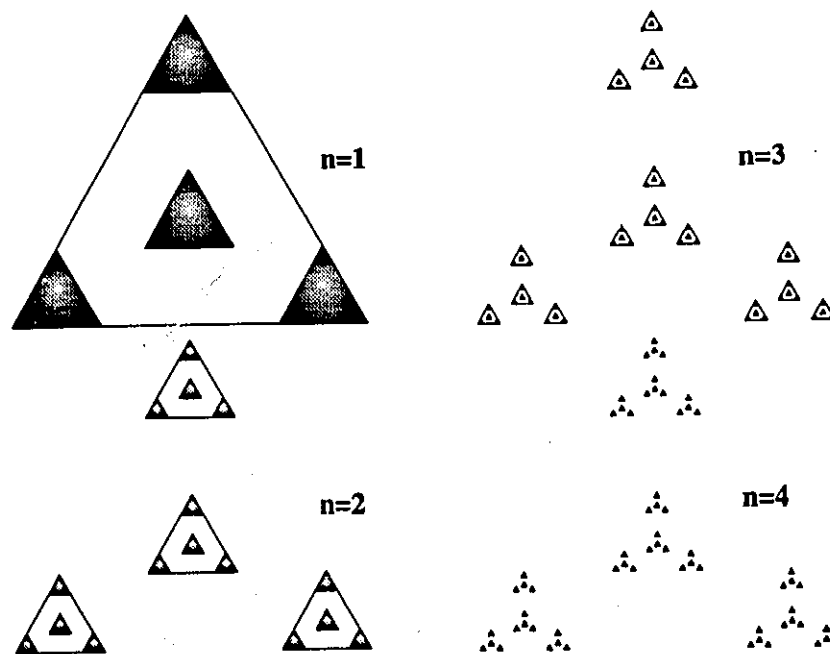


Fig. 3.2.3 Construction of set with fractal dimension $D_L = D_A = 1$. The initiator is a triangle. In each generation, the triangles are divided into sixteen same-shaped triangles; the generator eliminates twelve and leaves four triangles connected with the one at the centre. Four generations of the prefractal sets are shown.

The length and area of the n -th generation prefractal are given by

$$L(e_n) = 3(2)^n e_n \quad (3.2.8)$$

$$A(e_n) = A_0 2^n e_n^2 \quad (3.2.9)$$

where $A_0 = 3^{1/2}/4$. It can be estimated that the dimensions of the perimeter and the area are same with $D_L = D_A = 0.6309$ which is also the dimension of the triadic Cantor set.

Example 3.2.2. Figure 3.2.3 shows the construction of a set by means of an initiator that is a triangle. First, sixteen subtriangles with one fourth-length sides are formed. The generator eliminates all triangles except the four that are shaded. The first generation consists of four triangles linked to the one at the centre. Each new generation is obtained by applying the generator to all remaining triangles. This prefractal results in a geometry consisting of separate dots. The length of yardstick of the n -th generation is

$$e_n = \left(\frac{1}{4}\right)^n \quad (3.2.10)$$

The length and area of the n -th generation prefractal are given by

$$A(e_n) = A_0 4^n e_n^2 \quad (3.2.11)$$

$$L(e_n) = 3(4)^n e_n \quad (3.2.12)$$

where $A_0 = 3^{1/2}/4$. It can be estimated that the dimensions of the perimeter and the area are the same with $D_L = D_A = 1$. This set consists of separate points and has dimension 1, the same as an ordinary line. It illustrates that a fractal set may also have integer

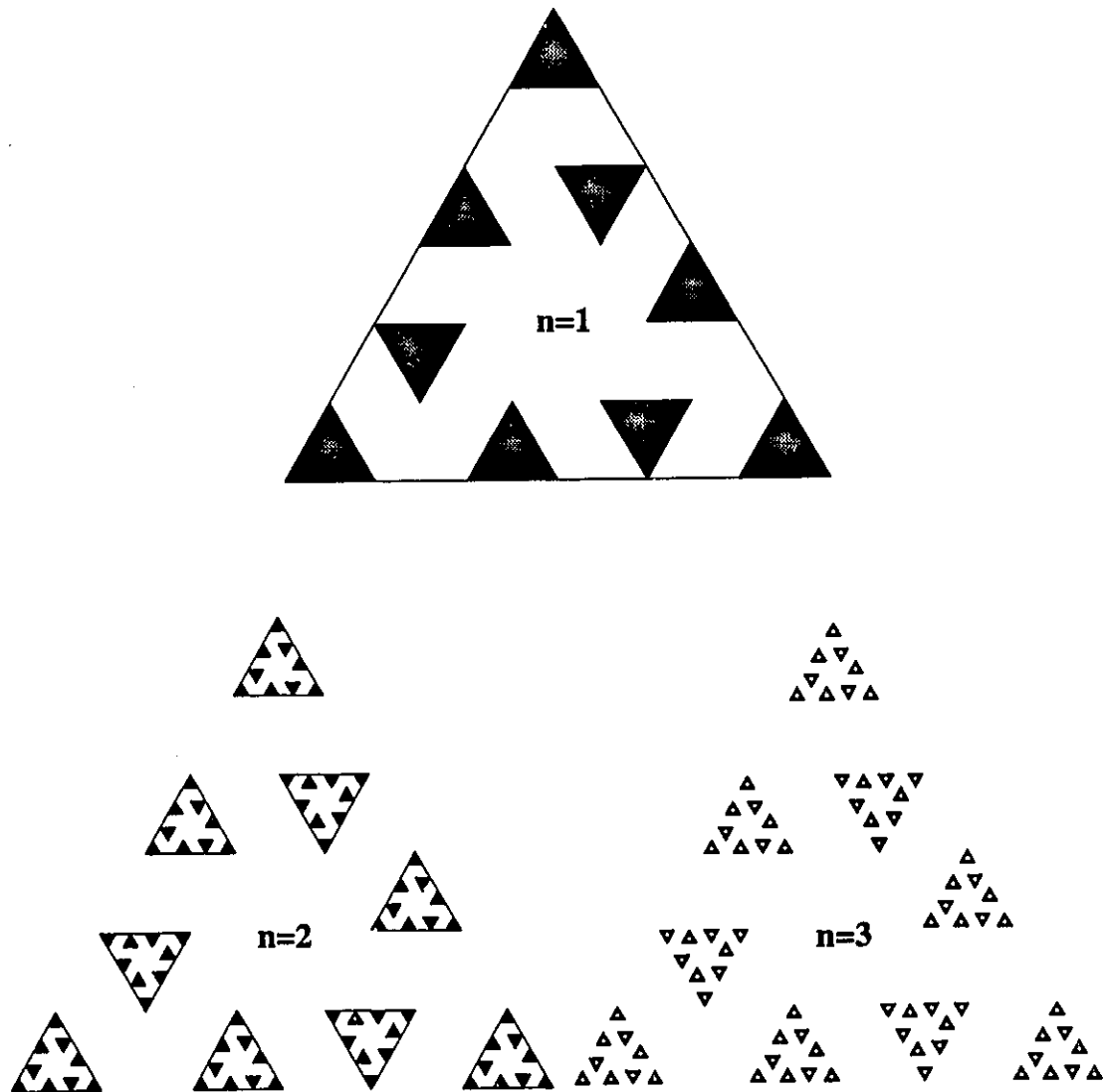


Fig. 3.2.4 Construction of set with fractal dimension $D_1 = D_\lambda = 1.2226$. The initiator is a triangle. In each generation, the triangles are divided into thirty six same shaped triangles; the generator eliminates twenty five and leaves nine triangles disconnected from each other. First to third generations of the prefractal sets are shown

dimension.

Example 3.2.3. Figure 3.2.4 shows the construction of a set by means of an initiator that is a triangle. First, thirty six subtriangles with one sixth-length sides are formed. The generator eliminates twenty seven triangles so that the remaining nine triangles are not connected each other. The first generation consists of nine disconnected triangles. Each new generation is obtained by applying the generator to all remaining triangles. This prefractal results in a geometry consisting of separate dots. The length of yardstick of the n -th generation is

$$e_n = \left(\frac{1}{6}\right)^n \quad (3.2.13)$$

The length and area of the n -t generation prefractal are given by

$$A(e_n) = A_0 9^n e_n^2 \quad (3.2.14)$$

$$L(e_n) = 3(9)^n e_n \quad (3.2.15)$$

where $A_0 = 3^{1/2}/4$. It can be estimated that the dimensions of the perimeter and the area are the same, and $D_L = D_A = 1.2263$ with constants $C_L = 3$ and $C_A = \sqrt{3}/4$.

The preceding examples show that different sets with dimension D ($0 \leq D \leq 2$) can be constructed from a triangle set by a generator which eliminates a portion of the remaining subtriangles in each generation. For example, the initial triangle can be divided into n^2 small same shaped triangles; in each generation, the generator eliminates $n^2 - n^D$ small

triangles and the remaining n^D triangles comprise the prefractal set. The fractal set constructed in this way has fractal dimension of D . These examples show that fractal sets consisting of separate points with dimensions $D=0.6309$, $D=1$ and $D=1.2263$ respectively. Note that one fractal set (example 2) has integer dimension. Moreover, it is shown that it is possible to construct a set which consists of separate points with any dimension D between 0 and 2. In the same way it is possible to construct a set consisting line segments with dimension between 1 and 2. For a D -dimension geometry, the measure C with

$$C = \lim_{\epsilon \rightarrow 0} N(\epsilon) \epsilon^D \quad (3.2.16)$$

is a constant, but the measure in D_1 -space ($D_1 < D$) is infinite, or

$$C_1 = \lim_{\epsilon \rightarrow 0} N(\epsilon) \epsilon^{D_1} = \infty \quad (3.2.17)$$

On the other hand, in D_2 -space ($D_2 > D$) it is equal to zero, or

$$C_2 = \lim_{\epsilon \rightarrow 0} N(\epsilon) \epsilon^{D_2} = 0 \quad (3.2.18)$$

These results hold true for ordinary geometries. For example, a line segment may has finite length in one-dimensional space, zero area in two-dimensional space, and an infinitely large number of points in zero-dimensional space. Therefore, both D and the constant C in Eq. (3.2.3) are measures of geometry.

3.3 BOX-COUNTING METHOD AND IMPLEMENTATION WITH GIS

Two common data structure models: vector and raster models are usually applied for representing spatial objects in GIS (Bonham-Carter, 1994). The vector model represents

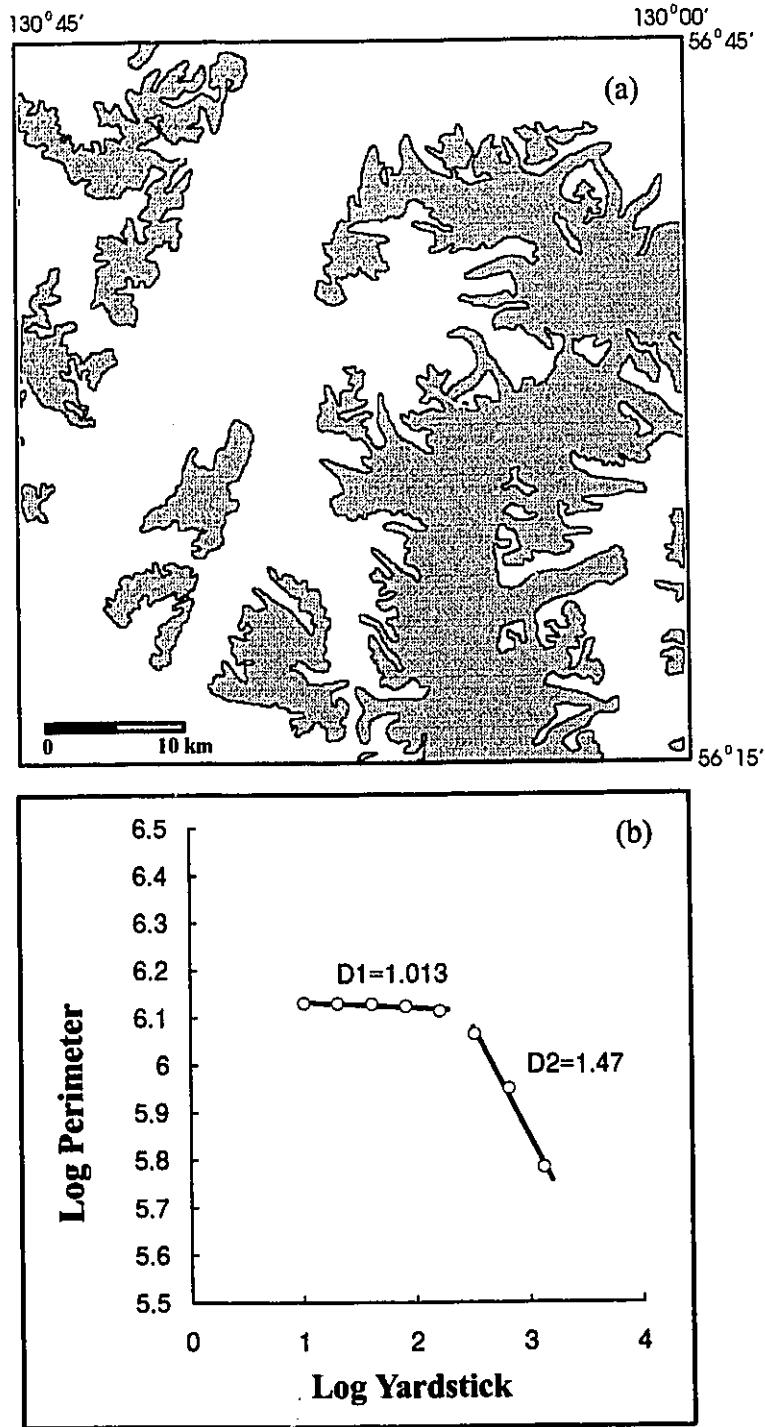


Fig. 3.3.1 Glaciers in the Unuk River area, British Columbia. (a) Outlines of glaciers (shaded areas). Data are in vector format. (b) Power-law relationship between estimated length of glacier outlines and yardstick (represented as quadlevel in SPANS-GIS). Straight lines are least squares fits.

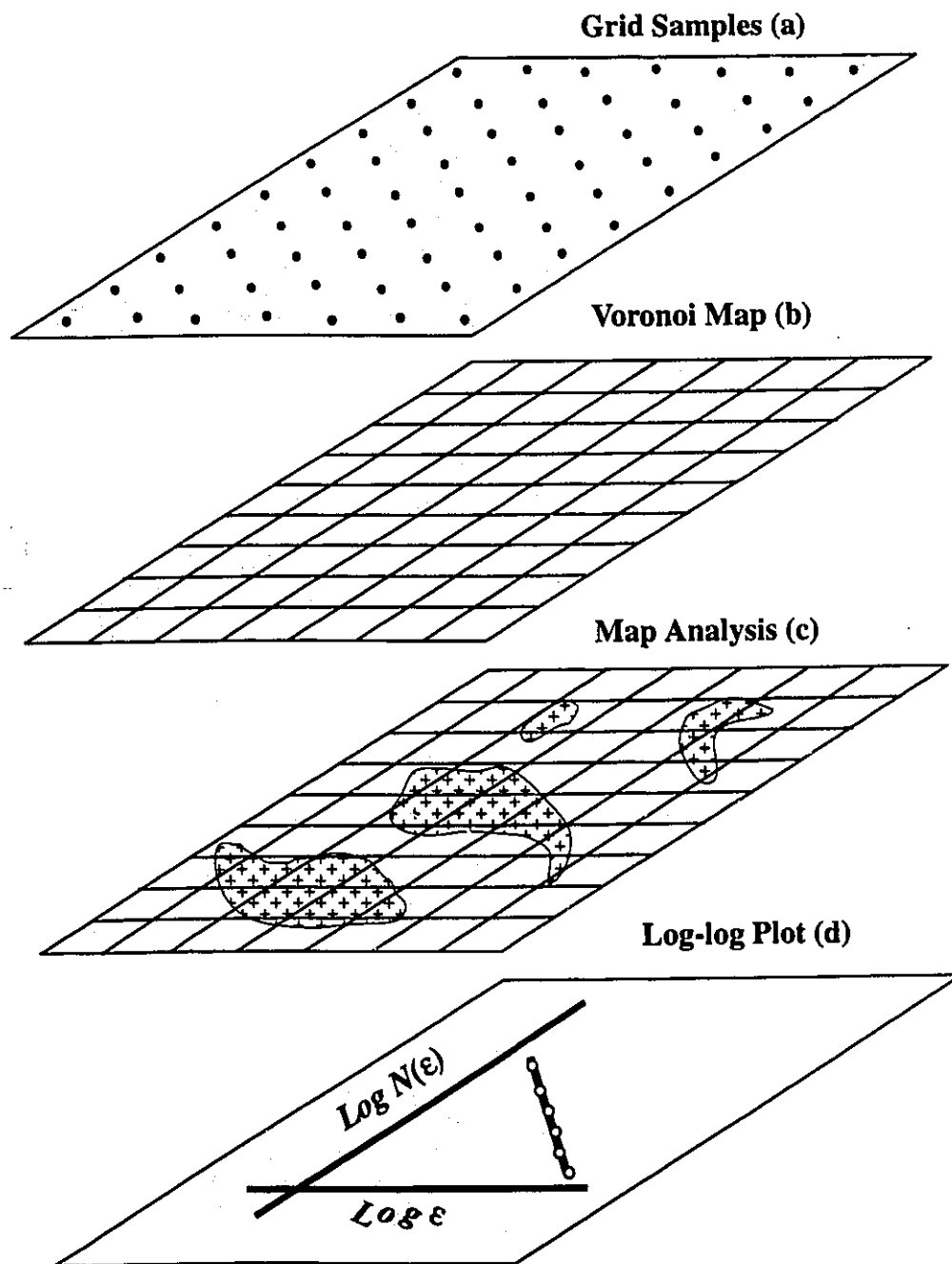


Fig. 3.3.2 Diagrams show the implementation of box-counting method with GIS. (a) grid sampling with point modelling of GIS; (b) cells for different sizes obtained by SPANS Voronoi tessellation; (c) superimposing fractal objects on cells in (b) by SPANS map modelling; (d) fractal modelling.

points as coordinate pairs, lines as strings of coordinate pairs and areas as lines that form closed polygons. Spatial data digitized from maps are generally in vector format with mineral occurrences as points, faults as lines, and rock units as polygons. The raster model, however, represents spatial objects by using cells or pixels with units of constant size. For example, points are represented with one pixel, and lines and areas are represented with groups of pixels. The pixel values can be represented in matrix form. For most commercial GIS software, both raster and vector models are used and can be converted from one structure to the other. For example, a polygon in vector format describing an object can be converted into raster format by representing the polygon as a group of pixels containing the object. On the other hand, areas represented in raster format can also be converted into vector format by connecting edge pixels as lines that form a polygon which contains the polygonal object. Any spatial object can be handled in GIS by means of these two data models. Since spatial objects can be represented by using pixels with varying sizes, the measurements required for fractal models can be made in a GIS. In addition, GIS are commonly applied for defining and extracting spatial objects from a complex map or image, and for spatial modelling. Fig. 3.3.1a shows the glaciers in the Unuk River area, northwestern British Columbia. The vector data for the outlines were obtained from digital elevation data and imported into SPANS-GIS. The estimated lengths of the glacier outlines were obtained using vector measurements with various quadlevels ranging from 8 to 15. The results are shown in log-log plot form (Fig. 3.3.1b). The data are fitted by two straight lines with different slopes corresponding to "textural" ($D_1 = 1.01 \pm 0.003$) and "structural" ($D_2 = 1.47 \pm 0.05$) fractal dimensions (Kaye,

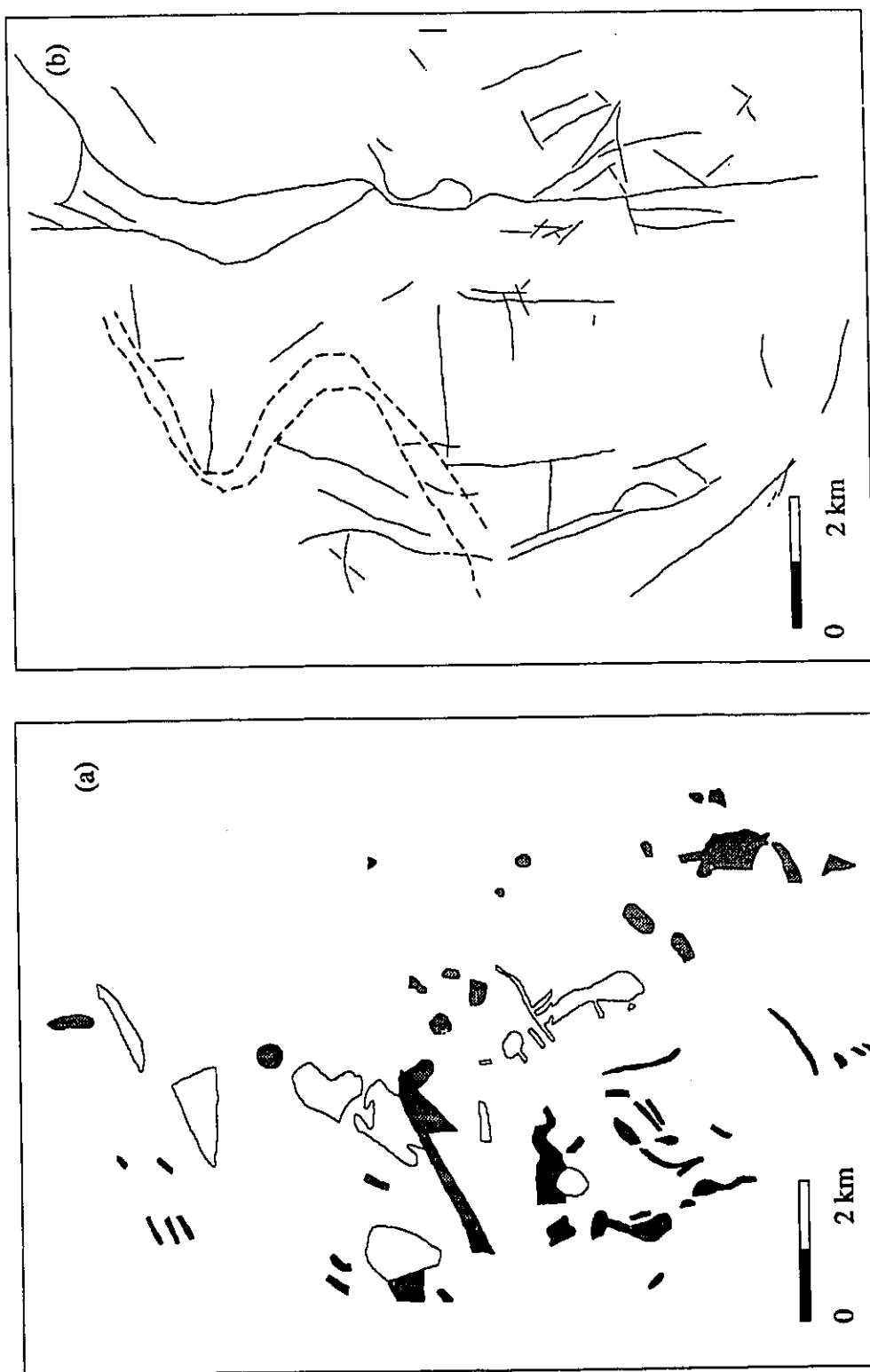


Fig. 3.3.3 Intrusive rocks and faults reclassified from geological map of the Mitchell-Sulphurets mineral district, northwestern British Columbia. (a) shaded areas represent monzonite and granodiorite, and blank polygons represent granite; (b) dashed lines represent thrusts and solid lines are for faults.

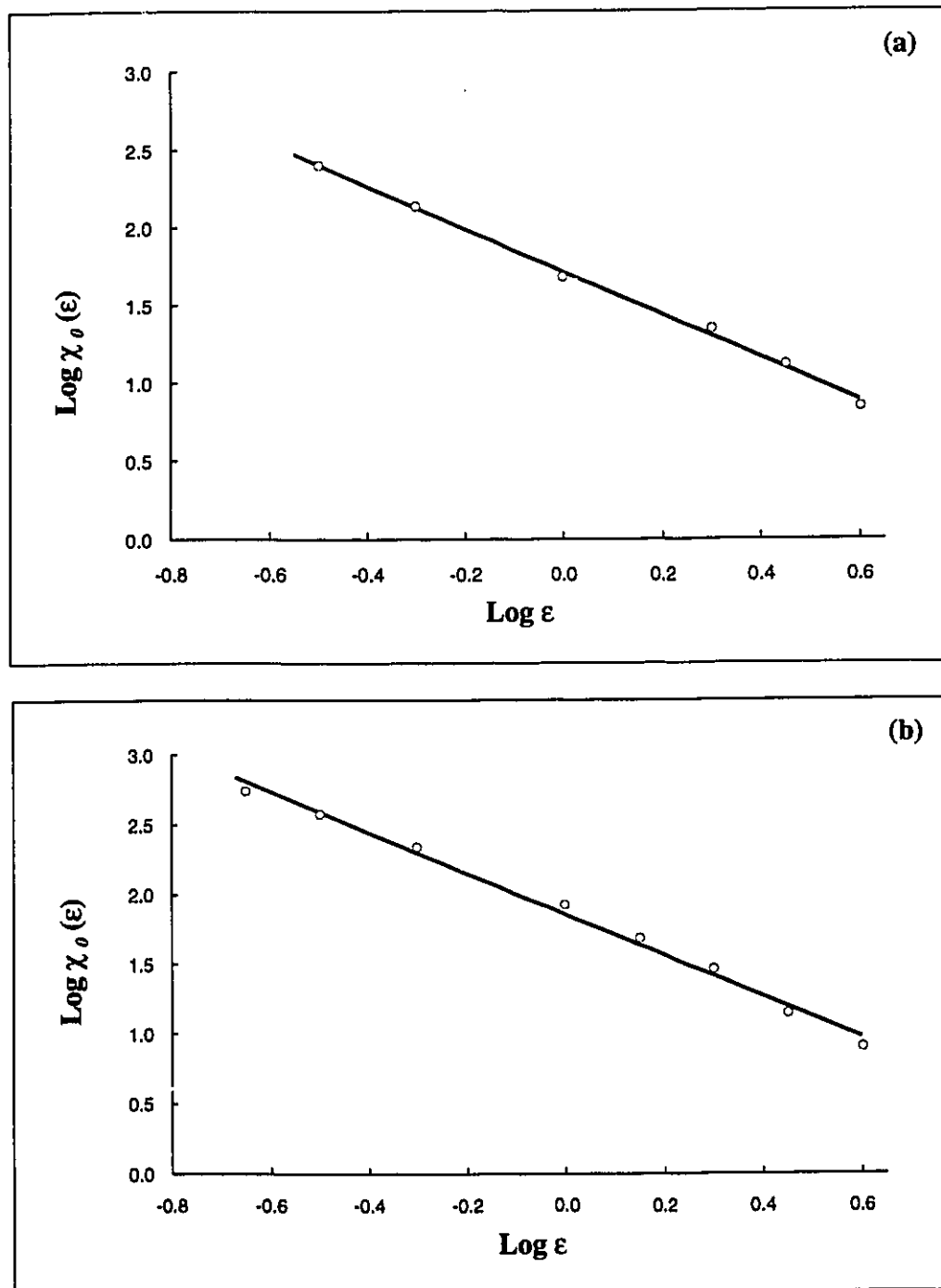


Fig. 3.3.4 Box-counting dimension. (a) Igneous rocks in Fig. 3.3.3a; (b) Faults in Fig. 3.3.3b. Straight lines are the least squares fits.

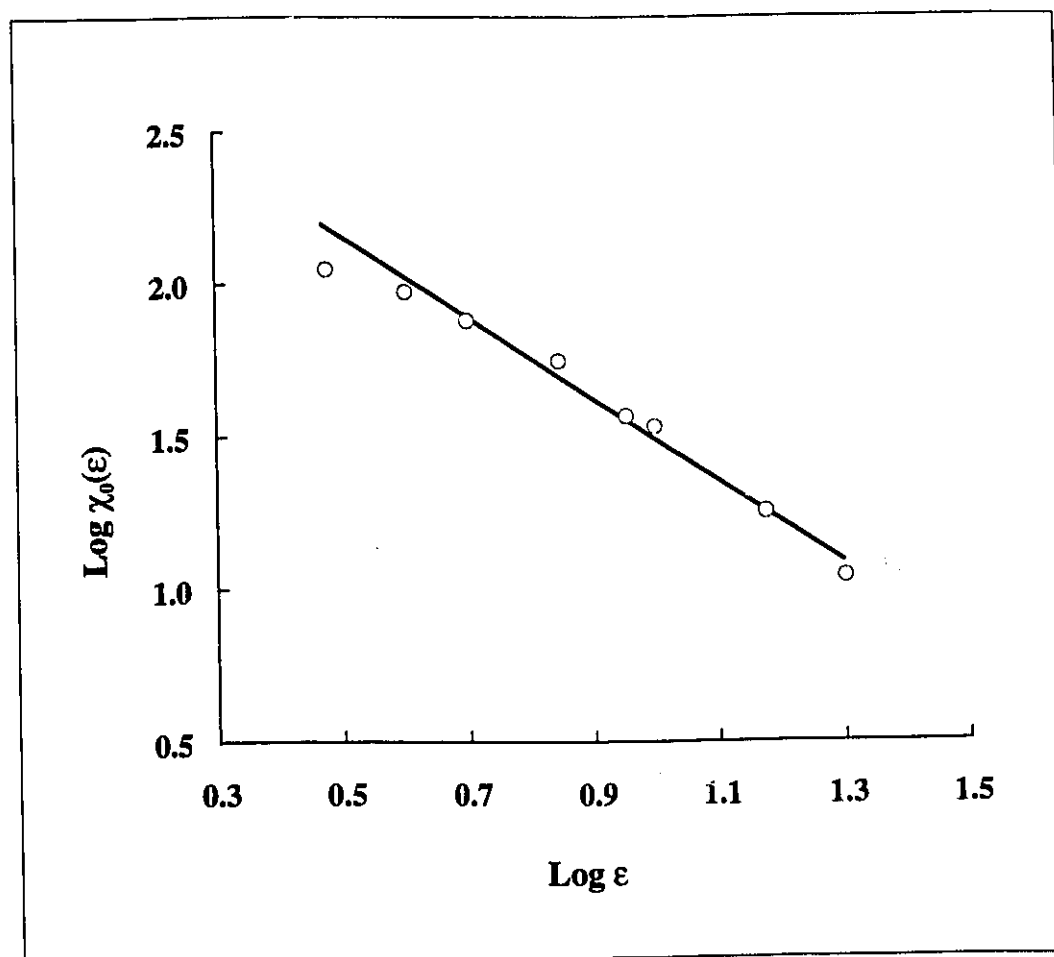


Fig. 3.3.5 Box-counting dimension of gold mineral occurrences in the Iskut River map sheet. Straight line is the least squares fit.

1989), respectively.

The box-counting method is commonly used in fractal analysis. This method can be relatively easily implemented with GIS. Using the following steps (Fig.3.3.2): (1) collect grid samples with different densities by point modelling of GIS; (2) create Voronoi maps based on these samples and obtain grid maps for different grid sizes; (3) superimpose the binary map containing fractal objects with the grid map by map analysis using GIS to obtain the area of each cell intersection with the binary objects; (4) plot the values in a log-log plot to estimate the fractal dimension and measures. The method is illustrated as follows: Fig. 3.3.2a shows a rectangular base map of the Mitchell-Sulphurets mineral district in northwestern British Columbia with area of 128.51 km². Use was made of the point modelling procedure of SPANS-GIS with modelling equation

$$RESULT \{A/\epsilon^2 \text{ IF } ('BASE') = 1; 0\}$$

This means that the sample density is A/ϵ^2 per point within the base map which is a binary map with value 1 for presence and 0 for absence. Various numbers of points are collected from the base map ($\epsilon = 2, 4, 6, \dots, 14$). The results for $\epsilon = 8$ are illustrated in Fig. 3.3.2b. Using the points in Fig. 3.3.2b, a Voronoi map can be created by means of SPANS-GIS to form a grid map. Fig. 3.3.2c shows the overlaps of objects and the grid map from which the area of each grid intersection with the objects can be computed using the map modelling procedure of SPANS-GIS. These values can be used for fractal and multifractal modelling as illustrated in Fig.3.3.2d. Fig. 3.3.3 shows the distribution of igneous rocks and faults in the Mitchell-Sulphurets area. For fractal modelling the number

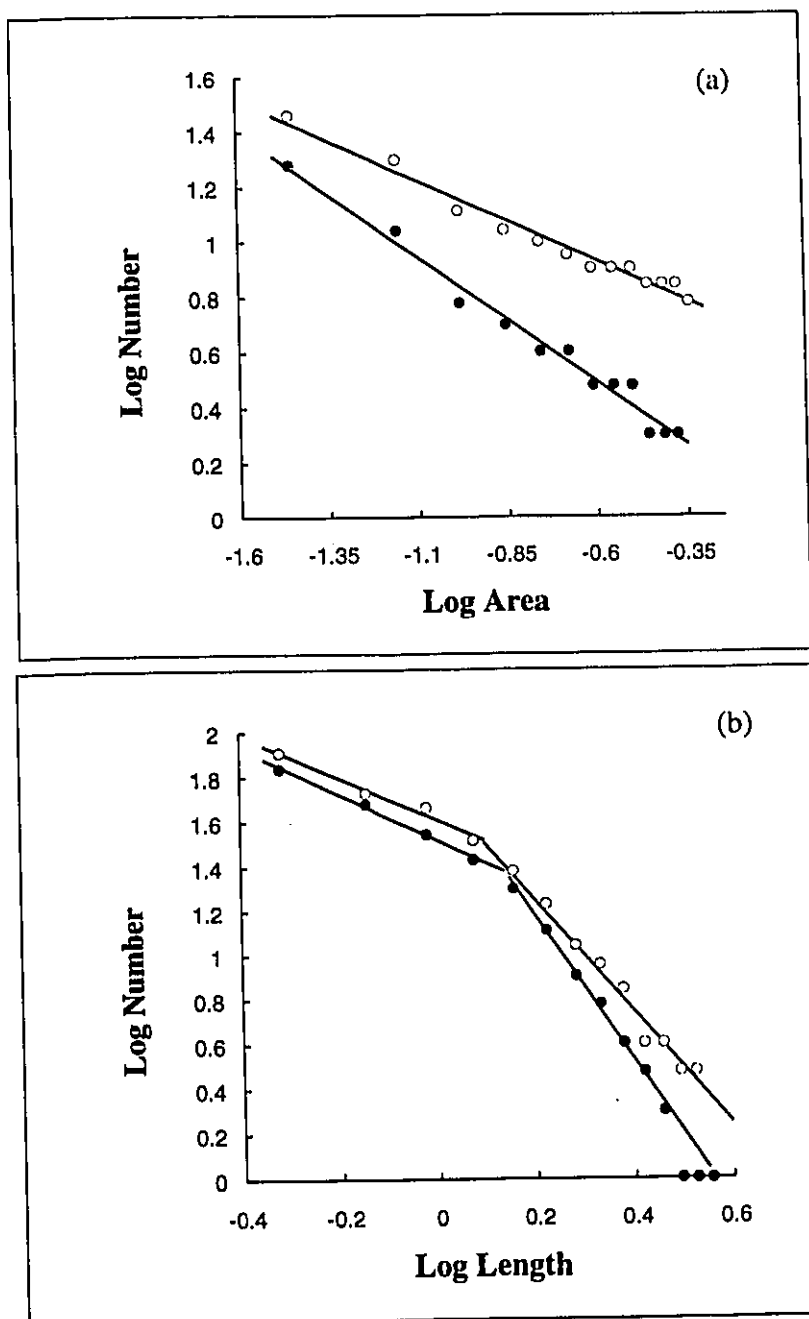


Fig. 3.4.1 Log-log plots show the relationships between number of igneous rocks and faults in Figs. 3.3.3a and 3.3.3b. (a) Empty circles represent the result for all igneous rocks and solid circles are for monzonite and granodiorite only; (b) Empty circles represent result for thrusts and faults and solid circles are for faults only.

of grid cell intersections with igneous rocks are plotted on log-log paper against grid size ϵ (Fig. 3.3.4a). Later, for multifractal modelling, these values will be used to define a measure (see Chapter 4). In Fig. 3.3.4a, a straight line is fitted to the values using the least squares method and the box-counting dimension is estimated to be about 1.404 ± 0.036 . A similar method can be used for vector data such as the faults in Fig. 3.3.3b. In order to treat them as polygon data, the vectors can be dilated to form buffer zones around the vectors. A small dilation distance keeps the buffer zone areas approximately proportional to the lengths of the vectors. Treating each buffer zone as a binary map, the box-counting method can be implemented. The results are shown in Fig. 3.3.4b with estimated box-counting dimension of about 1.712 ± 0.056 . A similar method can be used for point data such as the Au mineral occurrences in the Iskut River map sheet. The number of points in each cell on the grid map can be computed by point modelling in SPANS-GIS. The results for these Au occurrences are shown in Fig. 3.3.5 with estimated box-counting dimension of about 1.335 ± 0.077 .

3.4 FRACTAL MODELLING OF SPATIAL OBJECTS

A fractal model provides a tool for describing the nature of a fractal set, such as estimating its fractal dimension, and modelling the relationships between fractal sets, or between fractal and non-fractal sets. Fractal models can be grouped into two major classes: fractal and multifractal models. In this section two types of fractal models will be introduced and illustrated by means of examples: (1) a number-size model for

describing the relationship between number and size of objects, such as Au-associated intrusive rocks and faults in the Mitchell-Sulphurets area; (2) a perimeter-area model for Au-associated geochemically anomalous areas, and for glaciers. Other fractal and multifractal models will be discussed in later chapters.

3.4.1 Number-size model

Suppose a fractal set consists of similar parts with different sizes. The number of parts (N) and the size (S) then may follow the power-law relation (Korčák, 1940; Mandelbrot, 1967):

$$N(>S) \propto S^{-D} \quad (3.4.1)$$

where $N(>S)$ represents number of parts with size greater than S, and D is constant. This fractal model has been widely used in geoscience for describing the distribution characteristics for various types of spatial objects. For example, Korčák (1940) claimed that $D=1/2$ for islands, Mandelbrot (1975) obtained $D=0.65$; and Fujiwara (1977) studied the fragments resulting from the high velocity impact of a projectile on basalt and reported $D=2.5$. More examples are summarized by Turcotte (1989). It can be seen from the following examples that the model of Eq. (3.4.1) is readily implemented with GIS, because it is easy to use GIS to extract objects and compute their sizes. The areas for intrusive rocks and lengths of thrusts and faults were obtained using vector measurements in SPANS-GIS. The results are shown in Figs. 3.4.1a & b, respectively. Fig. 3.4.1a shows that the number of the intrusive bodies with area greater than A, $N(>A)$, follows a power-

law relation with the size (A), because the values fall on a straight line on the log-log plot. The estimated exponents are $D=0.99\pm0.048$ for monzonite and granodiorite, and 0.59 ± 0.028 for all intrusive bodies. Similarly, Fig. 3.4.1b shows that the number of segments of thrusts or faults with length greater than L, $N(>L)$, follows a power-law relation with L. The length was computed from two nodes representing either end points of faults or their intersection points. Empty circles are for all faults and thrusts, and solid circles are for faults only. Two segments of straight lines can be fitted to these values which implies that the number-size relations for faults only and for faults and thrusts combined have bifractal properties. The two estimated fractal dimensions are 1.0 ± 0.05 and 3.2 ± 0.097 for faults only, and 0.93 ± 0.098 and 2.52 ± 0.11 for thrusts and faults combined.

3.4.2 Perimeter-area model

For a group of similarly shaped sets in two-dimensional space, the ratio of the perimeter (L) and area (A) has the form $p=L/\sqrt{A}$. For example, if the sets consist of circles, squares or equilateral triangles, the values of p equal $2\sqrt{\pi}$, 4 and $6/3^{1/4}$, respectively. As a generalization, the following relationship between perimeters and areas for similarly shaped fractal sets was originally introduced by Mandelbrot (1983)

$$L(\delta) = C\delta^{(1-D)} A(\delta)^{\frac{D}{2}} \quad (3.4.2)$$

where C is a constant and D was used as the fractal dimension of the perimeter L. This relation depends on the value of the yardstick δ which is used for measurement.

Eq.(3.4.2) has been used extensively in physics, earth sciences and other fields; e.g., for cloud and rain perimeter fractal dimension by Lovejoy (1982) and Hentschel and Procaccia (1983); streams and drainage areas (Hack, 1957; Mandelbrot, 1983); fractal dimension of hail clouds (Rys and Walóvogel, 1986); fracture surfaces of titanium specimens (Pande et al., 1987); perimeters and areas of geochemical landscapes (Bölviken et al., 1992). In these applications, the exponent D in Eq.(3.4.2) was used for estimating fractal dimension of perimeter or fractal dimension of fractional Brownian surface (D_s) by means of the relation $D_s = D + 1$ (Mandelbrot, 1983; Mandelbrot, 1985; and Voss, 1985). The technique of slit island analysis (Mandelbrot et al., 1984) was also based on Eq. (3.4.2).

The following modified expression was derived by Lovejoy and Schertzer (1991) on the basis of multifractal theory:

$$L(\delta) = C\delta^{(1-\xi_T)} A(\delta)^{\frac{\xi_T}{2}} \quad (3.4.3)$$

where $\xi_T = 2D(p_T)/D(s_{T2})$; $D(p_T)$ and $D(s_{T2}) \leq 2$ are fractal dimensions for the perimeter and the area of subsets with concentration values above threshold T . This relation holds for a yardstick δ which is small enough to measure the smallest area accurately. For each value of δ , $L(\delta)$ and $A(\delta)$ satisfy the power-law relation:

$$L(\delta) = C_0 A(\delta)^{\frac{\xi_T}{2}} \quad (3.4.4)$$

Korvin (1992) discussed the relationships between length (L), width (W) and area (A) of

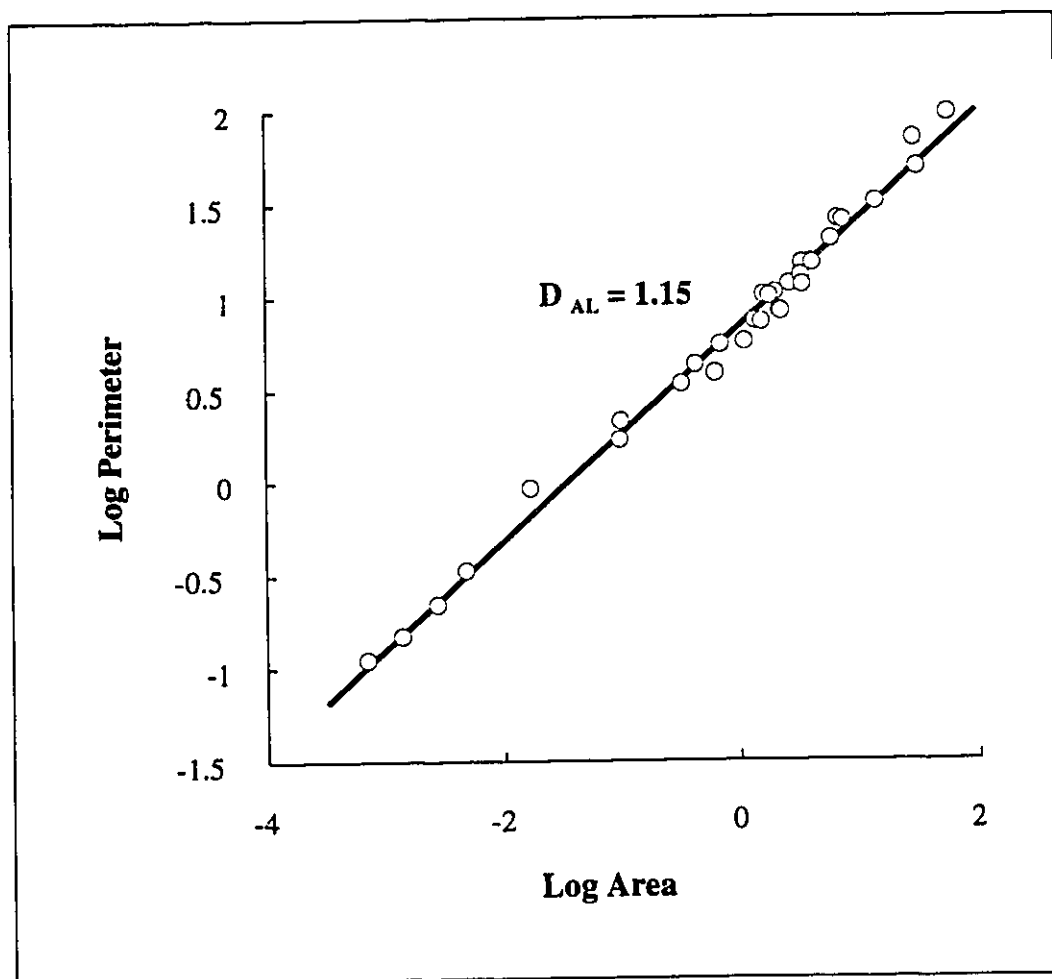


Fig. 3.4.2 Log-log plot showing the relationship between areas and perimeters of the glacier outlines. The values for perimeters and areas were computed by means of vector modelling in SPANS-GIS. Straight line is the least squares fit

river and drainage systems and described Hack's (1957) law by means of the following equations:

$$L \propto W^{P_L} ; \quad A \propto W^{P_X + P_Y} ; \quad L \propto A^{\frac{P_L}{P_X + P_Y}} \quad (3.4.5)$$

where $\propto$ denotes proportionality, and P_L , P_X and P_Y represent the exponents of the power-law relations between L , A and W , respectively.

Eqs. (3.4.4) and (3.4.5) show that length, width, perimeter and area of river and drainage systems may follow power-law relations. In general, relations between any two different measures for fractal geometries with similar shapes follow power-laws. It will be shown that the exponent of a power-law relation of this type is not necessary equal to the fractal dimensions of the two measures although it is associated with them.

Relationships between perimeters, areas and volumes for similarly shaped geometries can be expressed as (also see Cheng, 1994):

$$L(\delta) \propto \delta^{1-D_{AL}} \sqrt{A(\delta)}^{D_{AL}} \quad (3.4.6)$$

$$\frac{L_i(\delta)}{L_j(\delta)} \propto \left[\frac{A_i(\delta)}{A_j(\delta)} \right]^{D_{AL}} \quad (3.4.7)$$

where $D_{AL} = 2 D_L / D_A$; D_L and D_A denote the fractal dimensions of perimeter and area, respectively. For an ordinary surface area with $D_A = 2$, we have $D_{AL} = D_L$. This simplified model has often been used for estimating the fractal dimension D_L . It was demonstrated

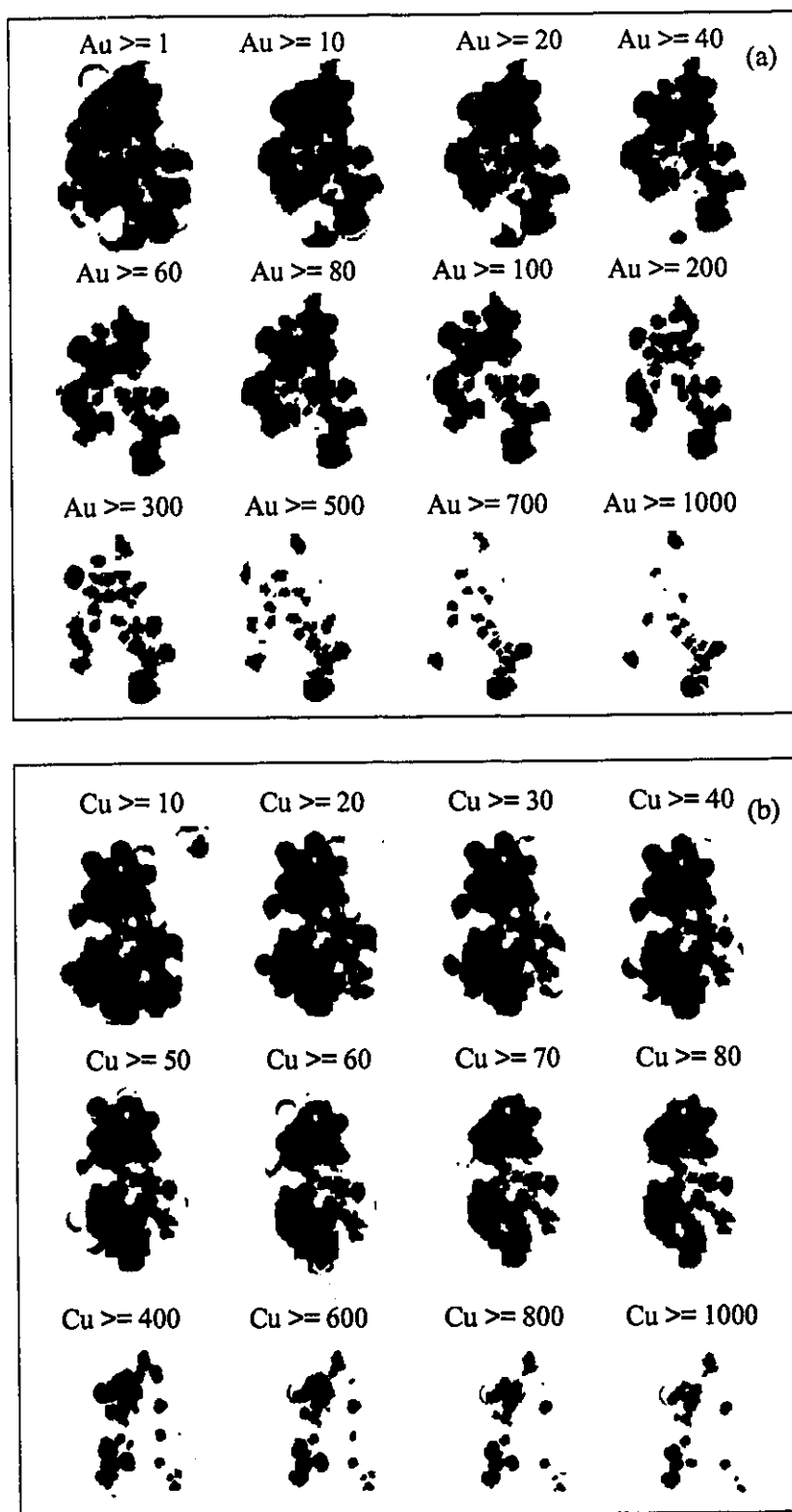
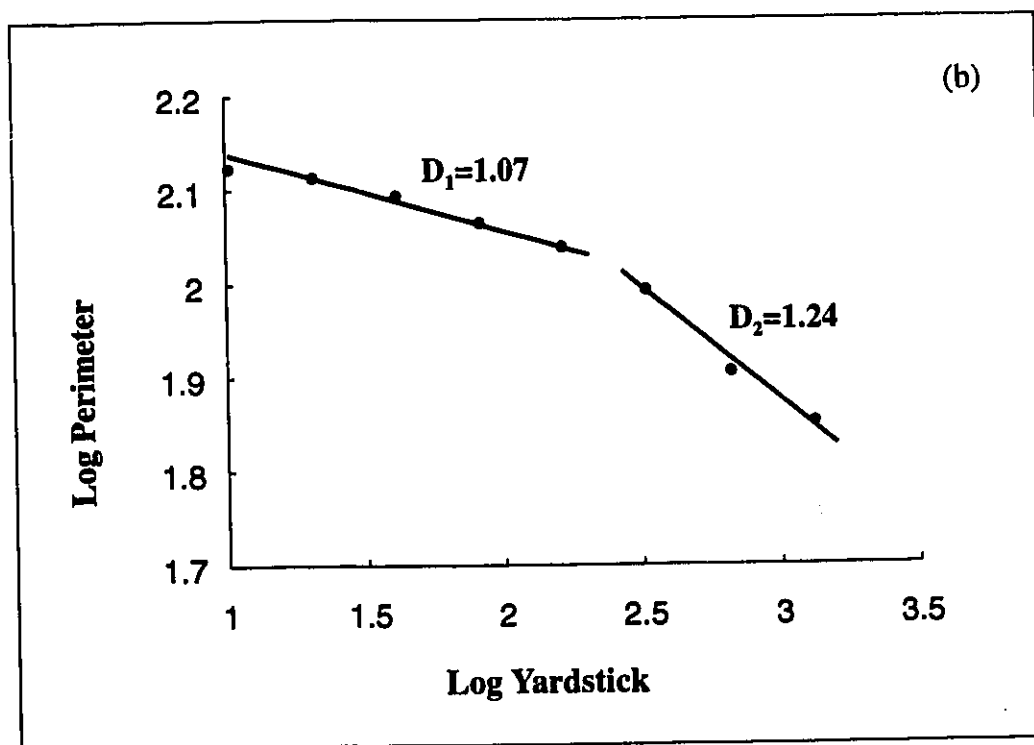
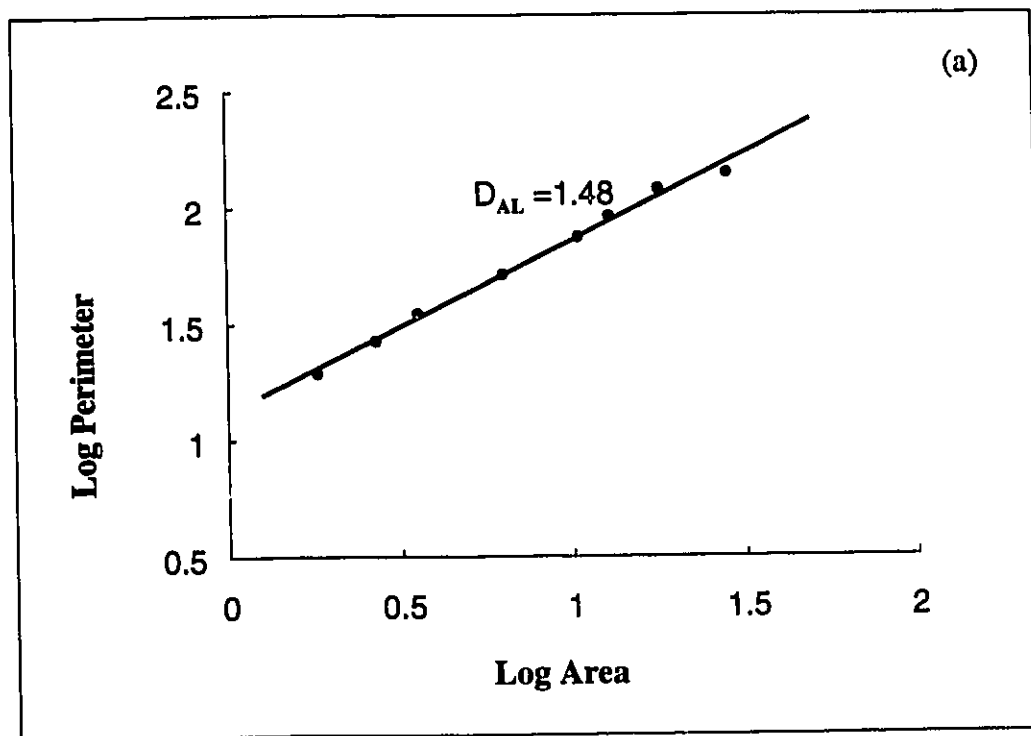
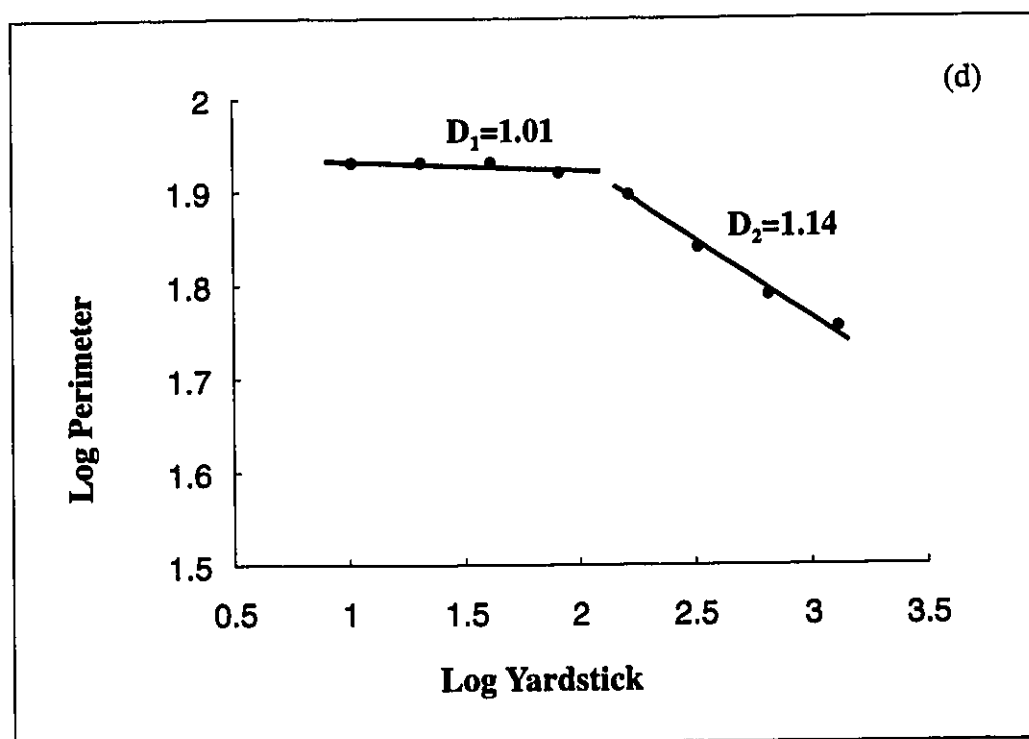
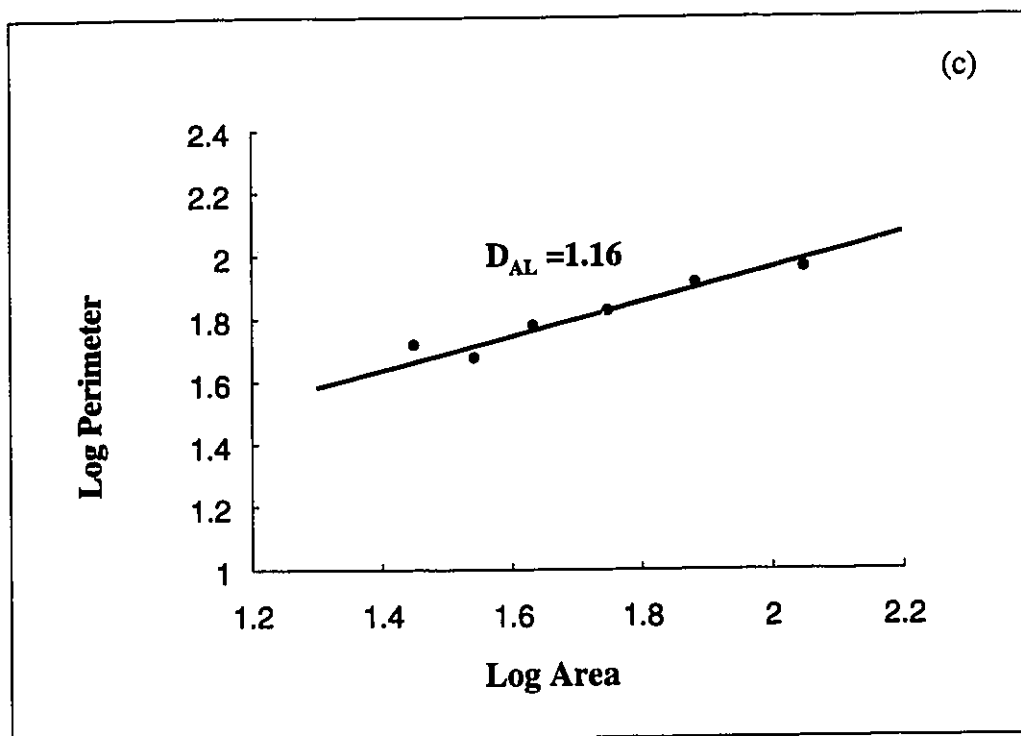


Fig. 3.4.3 Separate contours of Au (ppb) and Cu (ppm) analyzed for 1033 bedrock samples. (a) Au; (b) Cu.





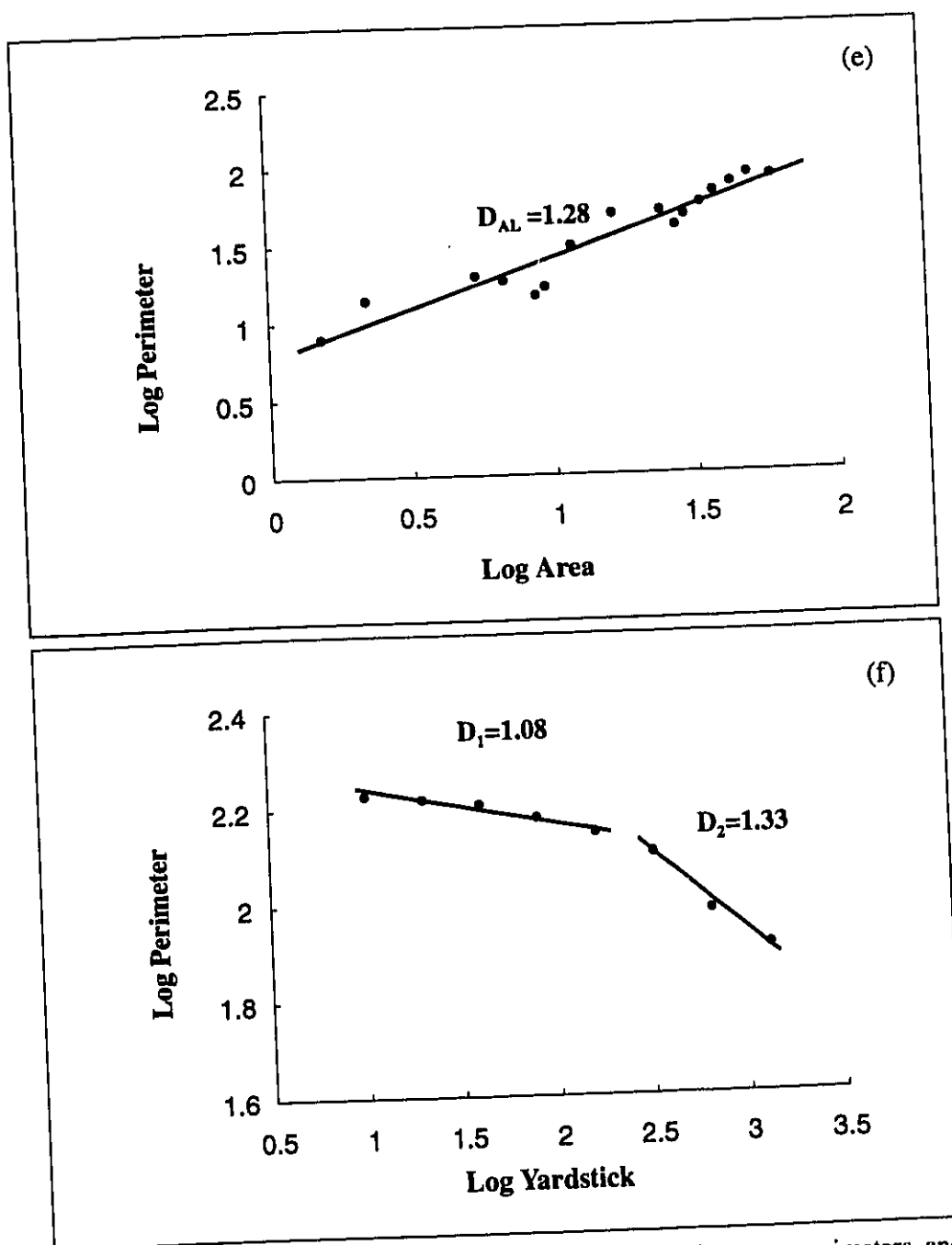


Fig. 3.4.4 Diagrams (a), (c) and (e) show the relationships between perimeters and anomalous areas using contours with different values greater than thresholds for (a) Au, (c) Cu and (e) As. Solid lines obtained by LS fitting. Diagrams (b), (d) and (f) are log-log plots showing the relationships of estimated lengths of the perimeters of anomalous areas for Au, Cu and As with variable yardstick δ . Separate straight lines are for textural (D_1) and structural (D_2) fractal dimensions.

theoretically and by means of simple examples that the case $D_A < 2$ and $D_{AL} \neq D_L$ can exist (section 3.5). The relations of Eqs. (3.4.6) and (3.4.7) are demonstrated by the following two examples.

Example 3.4.1 The perimeters and areas of the glaciers, shown in Fig. 3.4.2, were computed in vector format (Fig. 3.3.1a). A straight line was fitted on the log-log plot, indicating a power-law relation with exponent $D_{AL} = 1.15 \pm 0.01$.

Example 3.4.2 For concentration values of Au and the Au-associated trace elements Cu, Ag and As, several interpolation procedures, including Kriging and so-called potential mapping (available in SPANS) can be used for creating contour maps which represent the spatial distribution of an element. We have used potential mapping. This simple weighted moving average method uses a moving circular window with adjustable parameters to control the weighting of values at neighbouring points. The parameters include radius of the circular window, decay ratio of the weighting function, and maximum number of samples to be included within each window. For this study the parameters selected were: radius of 0.8 km, decay ratio of 0.5 (corresponding to a linear weighting function with weight 0 for samples located at the boundary or outside the moving window, and 1 for samples located at the centre) and maximum of 10 samples per window. (If more than 10 samples occur within the window for a given location, only the 10 nearest points are used to evaluate the surface at that location.)

Figure 3.4.3 shows Au and Cu maps employing single contours to divide the data set into two parts, above and below the contour's value. It can be seen that the shapes of the areas enclosed by successive contours are changing gradually; total enclosed area decreases as the value of the contour increases. From these contours an optimum threshold for separating anomalies from background areas can be selected by means of a log-log plot for the element concentration-area relation. This threshold coincides with a sudden change in the rate of decrease of the area enclosed by high value contours on the log-log plot. Figure 3.4.3 suggests that the contours on the maps for Au and Cu have similar shapes and this would be in agreement with Eqs. (3.4.6) and (3.4.7). Experimental results for Au, Cu and As are shown on log-log plots in Figures 3.4.4a, c and e using perimeters and areas for different contours such as those in Figure 3.4.4 with concentration values above the thresholds previously defined using the element concentration-area method. Computations used the same yardstick of 10.3 meters corresponding to level 15 of the quadtree data structure, as used in SPANS, for the study area delimited in Figure 3.3.3. A quadtree is a hierarchical raster data structure with a variable grid size. A square region (the universe) enclosing the study area is recursively subdivided into quadrants, and can be described as a tree structure. At each level in the tree, the maximum number of square blocks is equal to $2^n \times 2^n$, where n is the level. Thus at level 0, there is $2^0 \times 2^0 = 1$ block; at level 10, there are $2^{10} \times 2^{10} = 1024 \times 1024$ blocks. In the present case with a 337.5km universe, the length of the side of the blocks, or yardstick, at level 15 is equal to 10.3m. At level 14, the yardstick is 20.6m, at level 13 it is 41.2m, and so on. Perimeter length was measured by converting the quadtree raster to a vector format and computation of

line lengths. The boxes for the yardstick ($\delta=10.3$ m) corresponding to level 15 are so small that normally they contain a single sample with a single Au concentration value only (as before, average concentration values were used for boxes with more than one sample). The straight lines on the log-log plots of Figures 3.4.4a, c, and e represent power-laws for the perimeter-area relationships for Au, Cu and As, respectively. The estimated exponents D_{AL} are 1.48 ± 0.024 for Au, 1.16 ± 0.05 for Cu and 1.28 ± 0.06 for As.

Figures 3.4.4 b, d and f show the relationships between estimated lengths of perimeters of the contours for concentration values equal to the optimum threshold values for Au, Cu and As using yardsticks corresponding to different quadlevels in SPANS. These contours can be regarded as the outlines of the anomalous areas, and appear to have two different fractal dimensions. The first one (D_1 in Fig. 3.4.4) for yardsticks less than 300m is equivalent to the so-called "textural" fractal dimension (cf. Kaye, 1989, p. 27). D_1 is close to 1 for the elements considered and can be explained as a result of smoothing during interpolation. The other estimate (D_2 in Fig. 3.4.4), obtained by using yardsticks greater than 300 metres, is equivalent to the "structural" fractal dimension. D_2 ranges from 1.14 ± 0.016 to 1.33 ± 0.042 and is probably more representative of the true geometry of the anomalous areas for the elements considered. It is noted that the least squares estimates of D_1 and D_2 are subject to considerable uncertainty because they are based on relatively few points.

From any two of the three fractal dimensions (D_L for perimeter; D_{AL} for perimeter-area

relation; and D_A for area), the third can be obtained by means of the relation $D_A=2D_L/D_{AL}$. Theoretically, D_A can not be greater than 2. In most applications of the perimeter-area method, it is set equal to 2 (also see Appendix B) so that $D_L=D_{AL}$. The latter relation holds approximately true for Cu and As ($D_{AL}=1.16\pm0.05$ and $D_A=1.96\pm0.11$ for Cu and $D_{AL}=1.28\pm0.06$ and $D_A=2.07\pm0.17$ for As). For Au, the estimated value of D_{AL} (1.48 ± 0.024) is significantly greater than the estimated structural fractal dimension (1.24 ± 0.032). From the relation $D_A=2D_L/D_{AL}$ it follows that $D_A=1.68\pm0.07$ for gold which is less than 2. These results indicate that the distribution of Au in the alteration zones is more irregular than that of Cu or As.

3.5 MATHEMATICAL PROPERTIES OF PERIMETER-AREA FRACTAL MODEL

Suppose two sets of objects have similar shapes but different sizes as shown in the example of Fig. 3.5.1. Size is measured by counting boxes for presence of a feature. The boxes (with widths a and b) for the two objects which are not of the same size in Fig. 3.5.1 can be subdivided into smaller boxes with the same size (δ). The numbers of boxes containing perimeter (N_L) (boxes containing both black and blank areas) and area (N_A) of these two objects are proportional to the size of the subareas (δ) according to the relations:

$$N_{L_2} \propto \left(\frac{\delta}{b}\right)^{D_L} ; \quad N_{A_2} \propto \left(\frac{\delta}{b}\right)^{D_A} \quad (3.5.1)$$

where D_L and D_A represent the dimensions of the perimeter and area. Therefore, the

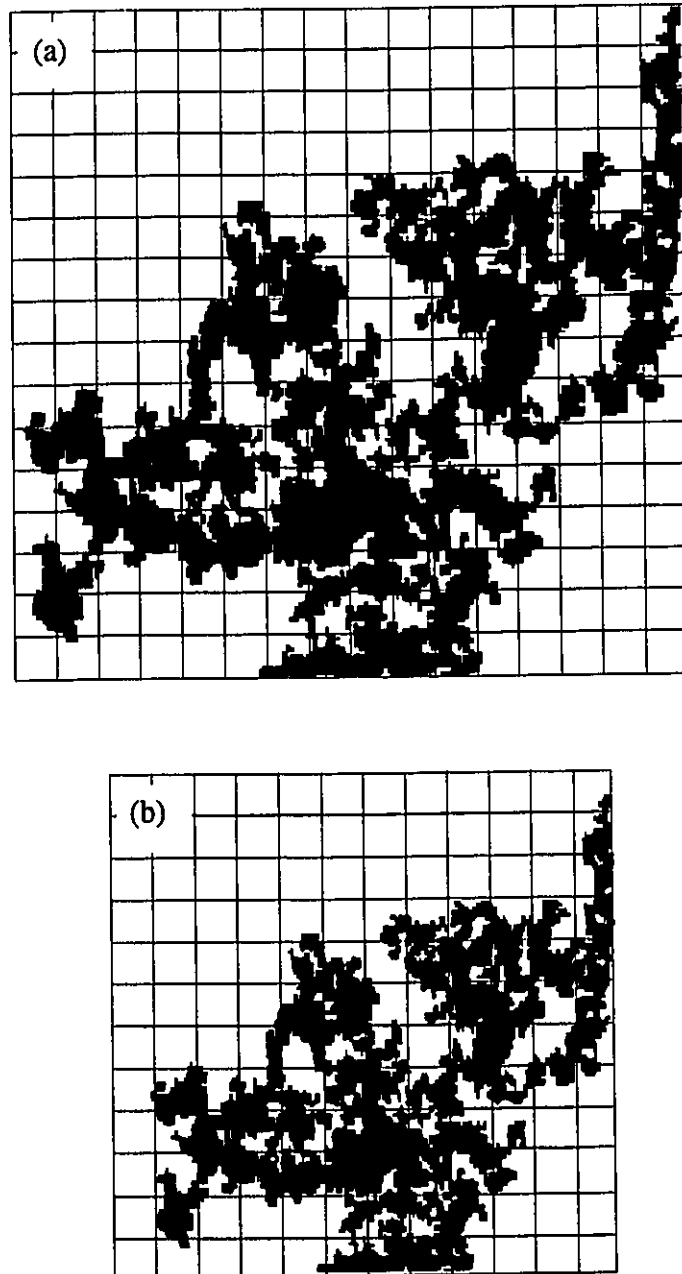


Fig. 3.5.1 Two similar sets measured with same yardstick

estimated lengths of perimeters and areas of the two objects can be written as:

$$\begin{aligned} L_1 &\propto \left(\frac{\delta}{a}\right)^{D_L} \delta ; & A_1 &\propto \left(\frac{\delta}{a}\right)^{D_A} \delta^2 \\ L_2 &\propto \left(\frac{\delta}{b}\right)^{D_L} \delta ; & A_2 &\propto \left(\frac{\delta}{b}\right)^{D_A} \delta^2 \end{aligned} \quad (3.5.2)$$

From Eq. (3.5.2), it follows that:

$$\frac{L_1}{L_2} \propto \left(\frac{b}{a}\right)^{D_L} ; \quad \frac{A_1}{A_2} \propto \left(\frac{b}{a}\right)^{D_A} \quad (3.5.3)$$

Therefore,

$$\frac{L_1}{L_2} \propto \left(\frac{A_1}{A_2}\right)^{\frac{1}{2} D_{AL}} \quad (3.5.4)$$

where the ratio of perimeters (L_1/L_2) and the ratio of areas (A_1/A_2) satisfy a power-law relation with exponent $D_{AL} = 2 D_L/D_A$. This power-law relation is independent of yardstick δ . From Eq.(3.5.2), we also obtain:

$$L(\delta) = C \delta^{(1-D_{AL})} A(\delta)^{\frac{D_{AL}}{2}} \quad (3.5.5)$$

This equation is the same as Eq. (3.4.3) with $\xi_T = D_{AL}$. It follows that the power-law relationship between $L(\delta)$ and $A(\delta)$ depends on the value of δ .

Similarly, the relations between length and volume and between area and volume can be expressed as:

$$\begin{aligned} \frac{L_1}{L_2} &\propto \left(\frac{V_1}{V_2}\right)^{\frac{1}{3} D_{VL}} ; & L(\delta) &= C \delta^{1-D_{VL}} V(\delta)^{\frac{D_{VL}}{2}} \\ \frac{A_1}{A_2} &\propto \left(\frac{V_1}{V_2}\right)^{\frac{2}{3} D_{VA}} ; & A(\delta) &= C \delta^{1-D_{VA}} V(\delta)^{\frac{2}{3} D_{VA}} \end{aligned} \quad (3.5.6)$$

where $D_{VL}=3D_L/D_V$ and $D_{VA}=3D_A/2D_V$. For ordinary geometries with $D_V=3$, $D_A=2$ and $D_L=1$, we have $D_{VA}=D_{VL}=D_{AL}=1$; and for fractal geometries, D_{VA} , D_{VL} and $D_{AL}>1$. If $D_A=2$ in Eqs. (3.5.4) and (3.5.5) corresponds to an ordinary two-dimensional set, then $D_{AL}=D_L$. Only in this case can D_{AL} be used as an unbiased estimate of the fractal dimension of the perimeter L . This special case has been widely used in applications. However, a difference between D_{AL} and D_L has been found in some applications of perimeter-area analysis. For example, by means of slit island analysis, Pande et al. (1987) obtained dimensions greater than those estimated from resolution-length analysis of vertical sections through the fracture surface used in their study. Goodchild (1988) generated random fractal surfaces by means of a stochastic process known as fractional Brownian motion, flooding the surfaces with water to obtain lakes. He checked the relationships between perimeters and areas of these lakes and concluded that the exponent D_{AL} estimated from his data was significantly less than the theoretical value D_S-1 where D_S is the fractal dimension of the surface. Indeed, if $D_A<2$, corresponding to fractal sets as will be shown in examples that follow, the exponent D_{AL} is not an unbiased estimate of D_L . Using D_{AL} rather than D_L we are in error by the ratio $2/D_{AL}>1$ (also see Lovejoy and Schertzer, 1991; and Korvin, 1992). The following examples will be used to demonstrate that Eqs. (3.4.6) and (3.4.2) are identical only for the case of $D_A=2$, but different for $D_A<2$.

Example 3.5.1. Figure 3.5.2 shows the construction of a rectangular Euclidian Sierpinski gasket of which the initiator is a rectangle with unit side. The generator eliminates an upper left subrectangle as shown. The striped areas in Fig. 3.5.2 show the 4-th and the

5-th generations of this prefractal (cf. Feder, 1988, p.17).

For the n -th generation, the yardstick satisfies $\delta_n = (1/2)^n$, and the estimated length of the perimeter, which includes all edges enclosing the striped area, is

$$L(\delta_n) = 6L(\delta_{n-1}) - 4\delta_n \quad (3.5.6)$$

$$\begin{aligned} L(\delta_n) &= 4 \left[3^n - \sum_{k=0}^{n-1} (3^{n-1-k}) \right] \delta_n = 3^n \left[4 - \frac{4}{3} \sum_{k=0}^{n-1} \left(\frac{1}{3} \right)^k \right] \delta_n = \\ &= 3^n \left[4 - \frac{4}{3} \frac{1}{1 - \frac{1}{3}} + o(n) \right] \delta_n = 3^n [2 + o(n)] \delta_n \end{aligned} \quad (3.5.7)$$

where $\lim_{n \rightarrow \infty} o(n) = 0$. By definition, we obtain:

$$D_L = 1 - \lim_{n \rightarrow \infty} \frac{\log(3^n [2 + o(n)] \delta_n)}{\log \delta_n} = \frac{\log(3)}{\log(2)} = 1.58 \quad (3.5.8)$$

The corresponding estimated area is

$$A(\delta_n) = 3^n \delta_n^2 \quad (3.5.9)$$

and the fractal dimension D_A can be estimated as:

$$\frac{L_1(\delta)}{L_2(\delta)} \propto \frac{A_1(\delta)}{A_2(\delta)} \quad ; \quad L(\delta) \propto \delta^{-1} A(\delta) \quad (3.5.10)$$

These expressions are identical to Eqs.(3.5.4) and (3.5.5) with $D_{AL} = 2$.

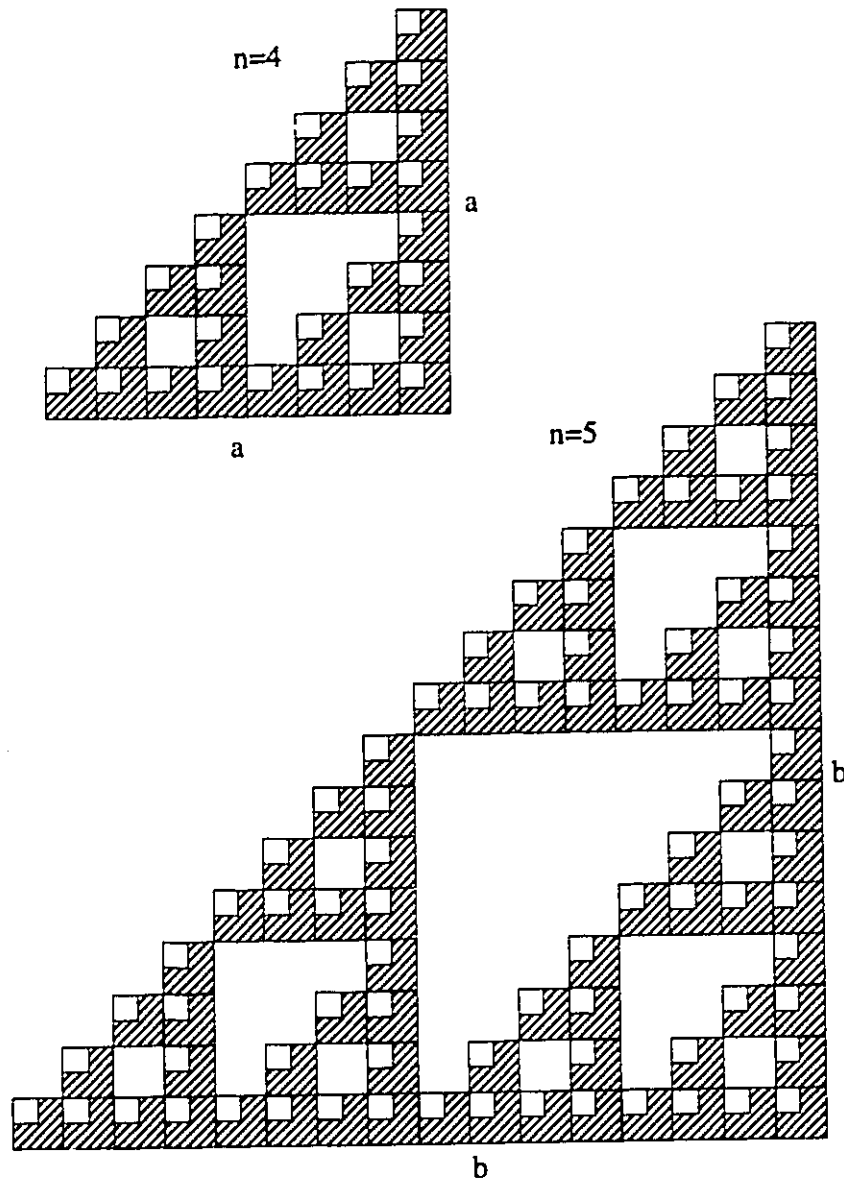


Fig. 3.5.2 Construction of rectangular Euclidian Sierpinski gasket. Initiator is a rectangle; generator eliminates the upper-left sub-rectangle with half length of side. Result shown is for 5-th generation with $n=5$ for yardstick $\delta_n=(1/2)^n$. Dimensions of the perimeter and area are equal with $D_L=D_A=1.58$, whereas the exponent obtained from the perimeter-area relation is $D_{AL}=2$.

Example 3.5.2. Figure 3.5.3 illustrates the construction of a Koch curve with as initiator an equilateral triangle with unit length of sides. Each next generation Koch curve is a curve with sides consisting of 4 line-segments that are $1/3$ times as long as those of the previous generation. The 3-th and 4-th generations of this prefractal are shown in Fig. 3.5.3.

The perimeter and enclosed area of the n -th generation prefractal can be estimated from the relations:

$$L(\delta_n) = 3(4)^n \delta_n \quad (3.5.11)$$

$$N_A(\delta_n) = N_A(\delta_{n-1}) 9 + N_L(\delta_{n-1}) \quad (3.5.12)$$

where N_A is the number of areas and N_L ($N_L(\delta_n) = 3 \times 4^n$) is the number of segments, both of which are measured by using the yardstick δ . The length of each yardstick δ_n for the n -th generation is

$$\delta_n = \left(\frac{1}{3}\right)^n$$

From Eq. (3.5.10) it follows that

$$D_L = 1 - \lim_{n \rightarrow \infty} \frac{\log [3(4)^n \delta_n]}{\log \delta_n} = \frac{\log(4)}{\log(3)} = 1.2618$$

From Eq. (3.5.11) we have

$$\begin{aligned}
A(\delta_n) &= [N_A(\delta_0) 9^n + \sum_{k=0}^{n-1} N_L(\delta_k) 9^{n-1-k}] \delta_n^2 = \\
&= 9^n [1 + \frac{1}{3} \sum_{k=0}^{n-1} (\frac{4}{9})^k] \delta_n^2 = \\
&= 9^n [1 + \frac{1}{3} \times \frac{9}{5} + o(n)] \delta_n^2 = [\frac{8}{5} + o(n)] 9^n \delta_n^2
\end{aligned} \tag{3.5.13}$$

Consequently, the dimension of area is

$$D_A = 2 - \lim_{n \rightarrow \infty} \frac{\log\{[\frac{8}{5} + o(n)] 9^n \delta_n^2\}}{\log \delta_n} = \frac{\log(9)}{\log(3)} = 2$$

so that $D_{AL} = D_L = 1.2618$. For this example,

$$\frac{L_1(\delta_n)}{L_2(\delta_n)} \propto [\frac{A_1(\delta_n)}{A_2(\delta_n)}]^{0.6308} \tag{3.5.14}$$

and both Eqs.(3.4.2) and (3.5.4) are satisfied.

3.6 CONCLUSION

A new model for the relationships between perimeters, areas and volumes of similarly shaped fractal geometries is proposed. It has been shown theoretically and by means of examples that the fractal dimension estimated from the perimeter-area relation is not necessarily the same as the fractal dimensions of the perimeter and the area. Only if $D_A=2$, can the exponent (D_{AL}) estimated from the perimeter-area relation be used as unbiased estimate of the fractal dimension of the perimeter. For $D_A < 2$, this exponent is

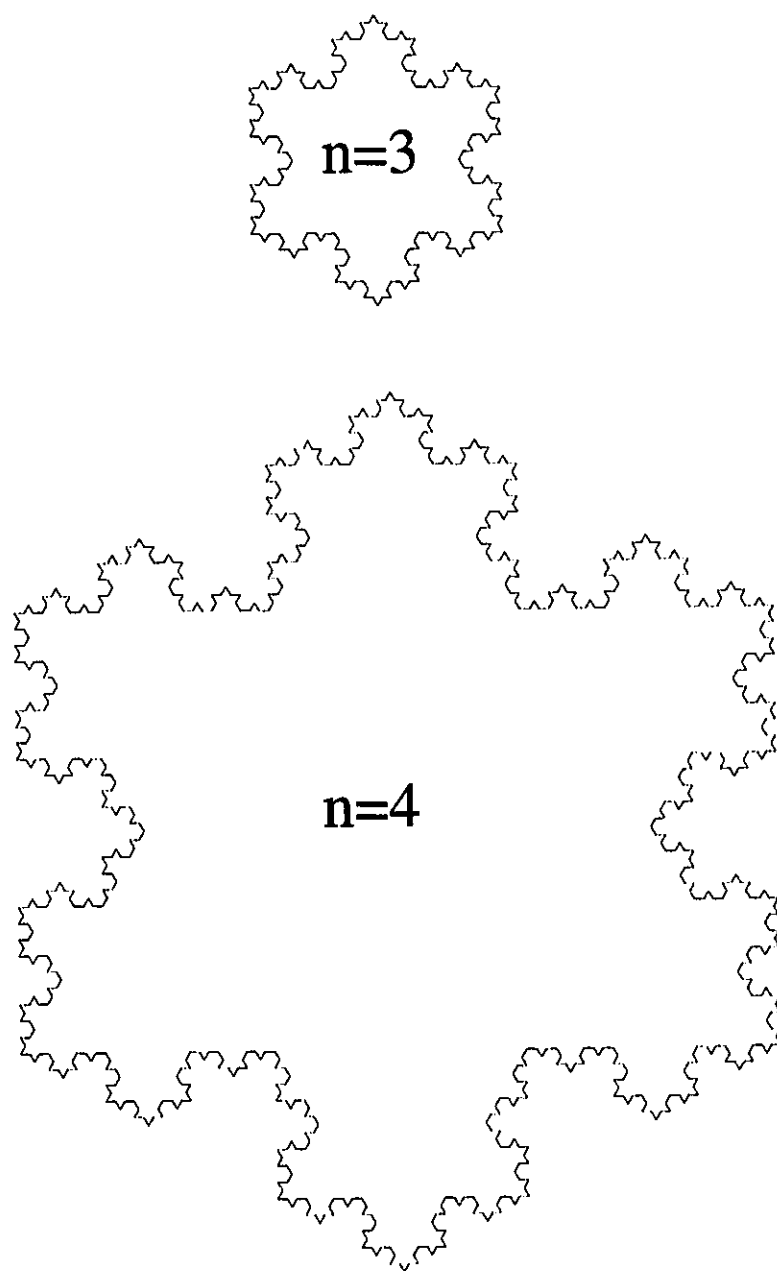


Fig. 3.5.3 Construction of Koch curve. The initiator is a triangle. Each segment of the initiator is replaced by the generator. The first generation of this prefractal is a curve consisting of 4 line segments each of length equal to $(1/3)$ of each of laterals. Results shown are for 3-th and 4-th generation with $n=3$ and $n=4$ for yardstick $(1/3)^n$. For the dimensions: $D_L=1.2618$ and $D_A = 2$; therefore, $D_{AL}=D_L$.

greater than the fractal dimension of the perimeter by the factor $2/D_A > 1$. It was shown that gold is more irregularly distributed than Cu and As in the Mitchell-Sulphurets area. The estimated fractal dimensions D_A of anomalous areas for Au, Cu and As were 1.68 ± 0.07 , 1.96 ± 0.11 and 2.07 ± 0.17 , respectively. With the aid of GIS, it is relatively easy to implement the various fractal models for spatial analysis.

Chapter 4 MULTIFRACTAL MODELLING FOR SPATIAL OBJECTS

4.1 INTRODUCTION

The concepts of multifractals have been developed and applied recently in physics and chemistry where this approach was shown to be useful for the study of the spatial distribution of physical and chemical quantities with geometrical support (Stanley and Meakin, 1988; Feder, 1988; Schertzer and Lovejoy eds., 1991; Evertsz and Mandelbrot, 1992; Agterberg, 1994a). The multifractal model has also been used for describing spatial objects of points, vectors, areas and surfaces dealt with in applications of GIS. For example, Agterberg et al. (1993a) studied Au concentration values in bedrock samples collected from the Mitchell-Sulphurets mineral district; Cheng and Agterberg (1994) applied the multifractal model to characterize the spatial points of tree seedlings in Lansing Woods, Clinton, Michigan, U.S.A.; Agterberg et al. (1994) used the multifractal model to analyze the spatial distributions of fractures in the Lac Du Bonnet batholith, Manitoba. Some of these examples will be summarized in this chapter and later. In addition, examples of faults and polygons of igneous rocks associated with Au mineralization will be discussed.

Fractal geometry and fractal models provide the simplest example of scale invariance, and are useful for characterizing fractal sets. In addition, one may be interested in the measure or field on geometrical sets. For example, in the use of the box-counting method, measure

on objects intersected with each cell can be taken into account in addition to the number of cells. If this measure has self-similar or scale invariance properties, it can be described by means of the multifractal model. Two interrelated multifractal models have been developed based on the fractal dimension function (cf. Evertsz and Mandelbrot, 1992) and codimension function (cf. Schertzer and Lovejoy, 1991a, 1991b), respectively. The general theory of the multifractal models is introduced in section 4.2. The model based on the fractal dimension function will be introduced in section 4.3. The model based on the codimension function and interrelations between the two models are discussed in section 4.4. Applications of the first model to Au concentration values in bedrock samples collected from a mineral district, vectors for faults and polygons for igneous rocks are described in section 4.5. Additional examples will be given in later chapters.

4.2 THE MULTIFRACTAL MODEL

Suppose $\mu(S)$ represents the measure of a set S in $\mathbb{R}^D$, $D = 1, 2$ or 3 . The space is subdivided into cells of the same linear size ϵ (equal intervals in $\mathbb{R}^1$, squares of side ϵ in $\mathbb{R}^2$); $\mu_i(\epsilon)$ denotes the measure on the intersection of S with the i -th cell of size ϵ . For a self-similar measure μ , the logarithm of the measure of the cell divided by the logarithm of the size of the cell is restricted to an interval $[\alpha_{\min}, \alpha_{\max}]$, where $0 \leq \alpha_{\min} \leq \alpha_{\max} \leq \infty$,

$$\alpha_i = \frac{\log \mu_i(\epsilon)}{\log \epsilon} \quad (4.2.1)$$

where α_i is called the coarse Hölder exponent. For each value of α , $N_\epsilon(\alpha)$ represents the

number of cells of size ε having a coarse Hölder exponent α . Now suppose that a cell of size ε has been selected at random from among the total number of cells which is proportional to ε^{-D} . The probability of hitting the value α is $P_\varepsilon(\alpha) = CN_\varepsilon(\alpha)\varepsilon^D$, where C is a constant. In the case of a multifractal measure, this probability no longer tends to a limit as $\varepsilon \rightarrow 0$. Two functions will be considered:

$$f_\varepsilon(\alpha) = -\frac{\log N_\varepsilon(\alpha)}{\log \varepsilon} \quad (4.2.2)$$

$$c_\varepsilon(\alpha) = \frac{\log P_\varepsilon(\alpha)}{\log \varepsilon} \quad (4.2.3)$$

The definition Eq. (4.2.3) differs from $c_\varepsilon(\alpha)$ defined by Evertzs and Mandelbrot (1992) with a negative sign. As $\varepsilon \rightarrow 0$, both $f_\varepsilon(\alpha)$ and $c_\varepsilon(\alpha)$ tend to well-defined limits $f(\alpha)$ and $c(\alpha)$. The relationship between them is

$$c(\alpha) = D - f(\alpha) \quad (4.2.4)$$

The definition of $f(\alpha)$ means that for each α , the number of cells increases for decreasing ε as

$$N_\varepsilon(\alpha) \propto \varepsilon^{-f(\alpha)} \quad (4.2.5)$$

The exponent $f(\alpha)$ is a continuous function of α . The values of $f(\alpha)$ could be interpreted loosely as fractal dimensions of subsets of cells of size ε having coarse Hölder exponent α in the limit $\varepsilon \rightarrow 0$ (Evertsz and Mandelbrot, 1992). Similarly, the values of the function $c(\alpha)$ can be seen as representing the codimensions of the subsets. The function $c(\alpha)$ is not

the same as the codimension function $C(\gamma)$ of Schertzer and Lovejoy (1991a, 1991b) although these two functions are associated with each other as shown in section 4.4. Multifractals are spatially intertwined fractals with a continuous spectrum of fractal dimensions (cf. Stanley and Meakin, 1988; Evertsz and Mandelbrot, 1992; Agterberg, 1994a).

4.3 FRACTAL DIMENSION SPECTRUM $f(\alpha)$ and ITS ESTIMATION

Two different methods have been used to obtain an empirical estimate of $f(\alpha)$ for an arbitrary measure (Evertsz and Mandelbrot, 1992): the histogram method and the method of moments. Generally, the second method provides better results and will be discussed in this section and also used in later chapters.

The method of moments is based on a quantity called the partition function defined as:

$$\chi_q(\varepsilon) = \sum_{i=1}^{N(\varepsilon)} \mu_i^q; \quad -\infty \leq q \leq +\infty. \quad (4.3.1)$$

where $N(\varepsilon)$ is the total number of cells of size ε covering the set S . Taking subsets of cells with α between α and $\alpha + d\alpha$, and replacing the measure $\mu_i(\varepsilon)$ by ε^α ,

$$\chi_q(\varepsilon) = \int_{\alpha_{\min}}^{\alpha_{\max}} N_\varepsilon(\alpha) \varepsilon^{\alpha q} d\alpha \quad (4.3.2)$$

Replacing $N_\varepsilon(\alpha)$ using equation Eq. (4.2.2) gives

$$\chi_q(\epsilon) \propto \int_{\alpha_{\min}}^{\alpha_{\max}} \epsilon^{\alpha q - f(\alpha)} d\alpha \quad (4.3.3)$$

In the limit $\epsilon \rightarrow 0$, the dominant contribution to the integral comes from α 's close to the value that minimizes the exponent $\alpha q - f(\alpha)$. If $f(\alpha)$ is differentiable, the necessary condition is

$$\frac{\partial}{\partial \alpha} \{ \alpha q - f(\alpha) \} = 0$$

or

$$\frac{\partial f(\alpha)}{\partial \alpha} = q \quad (4.3.4)$$

Suppose that the minimum value of $\alpha q - f(\alpha)$ as

$$\tau(q) = \min_{\alpha} [\alpha q - f(\alpha)] = \alpha(q) q - f[\alpha(q)] \quad (4.3.5)$$

Consequently,

$$\chi_q(\epsilon) \propto \epsilon^{\tau(q)} \quad (4.3.6)$$

From Eqs. (4.3.4) and (4.3.5), we obtain

$$\frac{\partial \tau(q)}{\partial q} = \alpha(q) + q \alpha'(q) - f'(\alpha(q)) \alpha'(q) = \alpha(q) \quad (4.3.7)$$

and

$$f(\alpha) = \alpha q - \tau(q) \quad (4.3.8)$$

The function $f(\alpha)$ can be estimated by Eq.(4.3.8). It can be interpreted as the negative of

the Legendre transform of $\tau(q)$.

If the values of $\chi_q(\epsilon)$ are plotted on log-log paper versus ϵ , they follow straight lines for each q with slope equal to the value of $\tau(q)$. From the values of $\tau(q)$, the values of $\alpha(q)$ can be estimated by Eq. (4.3.7), and values of $f(\alpha)$ can be computed by Eq. (4.3.8). In order to get good estimates of $f(\alpha)$, it is necessary to be aware of the following possible characteristics of $\tau(q)$, $\alpha(q)$ and $f(\alpha)$:

1. Symmetry or asymmetry of $f(\alpha)$

Eqs.(4.2.3) and (4.2.4) shows that symmetry of $f(\alpha)$ depends on the distribution of the measure μ . In the simplest case, being the binomial measure created in a multiplicative cascade model (see Chapter 6.4.3 for more discussion), the $f(\alpha)$ curve is shaped like the mathematical symbol $\cap$, symmetrical around $f_{\max}$. Often, the $f(\alpha)$ curve is leaning to one side due to nonbinomial nature of the frequency distribution of the measure μ . Left-sided multifractals were discussed by Mandelbrot and Evertsz (1991).

2. The domain of $f(\alpha)$ is bounded by:

$$\alpha_{\max} = \lim_{q \rightarrow -\infty} \sum_{i=1}^{N(\epsilon)} \frac{\mu_i^q(\epsilon)}{\sum \mu_i^q(\epsilon)} \log_e \mu_i(\epsilon) = N(\mu_{\min}) \log_e \mu_{\min}$$

$$\alpha_{\min} = \lim_{q \rightarrow \infty} \sum_{i=1}^{N(\epsilon)} \frac{\mu_i^q(\epsilon)}{\sum \mu_i^q(\epsilon)} \log_e \mu_i(\epsilon) = N(\mu_{\max}) \log_e \mu_{\max} \quad (4.3.9)$$

where $N(\mu_{\min})$ and $N(\mu_{\max})$ represent the numbers of cells having the smallest measure $\mu_{\min}$

and the largest measures $\mu_{\max}$, respectively.

Therefore,

$$\alpha_{\max} - \alpha_{\min} = \log_{\epsilon} \left[\frac{\frac{N(\mu_{\min})}{\mu_{\min}}}{\frac{N(\mu_{\max})}{\mu_{\max}}} \right] \quad (4.3.10)$$

For given cell size ϵ , the domain of $f(\alpha)$ depends on the ratio of the largest and smallest values of the measure. For example, increasing the cutoff value for Au concentration values in bedrock samples leads to a decreasing domain of $f(\alpha)$ (for further explanation, see section 4.5).

3. When $q \gg 1$, then $\tau(q) = \alpha_{\min} q - f(\alpha_{\min})$, corresponding to a straight line with slope $\alpha_{\min}$ and intercept $f(\alpha_{\min})$. If this line goes through the origin, $f(\alpha_{\min}) = 0$; otherwise, $f(\alpha_{\min}) \neq 0$. Similarly, we have $\tau(q) = \alpha_{\max} q - f(\alpha_{\max})$ when $q \ll 1$.

4. Suppose $q \geq q_{\max}$, $\tau(q) = \alpha_{\max} q - f(\alpha_{\max})$, then $\alpha(q) = \alpha(q_{\max})$ and $f'(\alpha(q)) = \alpha'(q) q + \alpha(q) - \tau'(q) = \alpha_{\max} - \alpha_{\max} = 0$; when $q \leq -q_{\min}$, $\tau(q) = \alpha_{\min} q - f(\alpha_{\min})$, then $\alpha(q) = \alpha(q_{\min})$ and $f'(\alpha(q)) = \alpha'(q) q + \alpha(q) - \tau'(q) = \alpha_{\min} - \alpha_{\min} = 0$; and when $q_{\min} \leq q \leq q_{\max}$, then $f'(\alpha(q)) = q$.

5. If $q=0$, then $f(\alpha(0)) = -\tau(0) = -\log N(\epsilon)/\log \epsilon$, representing the largest dimension or box-counting dimension of the support.

6. Setting $q=1$ in Eq. (4.3.8) gives

$$\alpha(1) = f(\alpha(1)) = \lim_{\epsilon \rightarrow 0} \frac{\sum_{i=1}^{N(\epsilon)} \mu_i \log \mu_i}{\log \epsilon} \quad (4.3.11)$$

The value $f(\alpha(1)) = \alpha(1)$ is called the entropy (or information) dimension of the measure.

At this point the straight line $f(\alpha) = \alpha$ is tangent to the curve of $f(\alpha)$.

7. If one assumes that $\sum \mu_i = 1$, then $\tau(1) = 0$; otherwise, $\tau(1) \neq 0$. In this case, a new partition function can be constructed as

$$\chi_q(\epsilon) = \sum_{i=1}^{N(\epsilon)} \left[\frac{\mu_i}{\sum \mu_i} \right]^q \propto \epsilon^{\tau(q) - q\tau(1)} \quad (4.3.12)$$

The new $\tau^*(q)$ satisfy the relations $\tau^*(q) = \tau(q) - q\tau(1)$ and $\tau^*(1) = 0$. From $\tau^*(q)$, it can be derived that $\alpha^* = \alpha - \tau(1)$ and $f(\alpha^*) = \alpha^* q - \tau^*(q) = f(\alpha)$.

4.4 CODIMENSION FUNCTION $C(\gamma)$ AND RELATIONSHIPS BETWEEN THE TWO MULTIFRACTAL MODELS

A multifractal model based on the codimension $C(\gamma)$ has been developed and applied to describe multifractal fields by Schertzer and Lovejoy (1987, 1991a, 1991b, 1992) and Lavallée et al. (1991). In this model the scale ratio λ ($\lambda = L/\epsilon$, where L is the largest scale which can be set to 1 without loss of generality) is used instead of ϵ itself. The field ρ_λ (ϵ_λ in Schertzer and Lovejoy, 1991a, 1991b) represents the field on each cell with scale ratio λ ($\lambda > 1$), and can be characterized by its probability distribution or by the corresponding law for the statistical moments

$$Pr(\rho_\lambda \geq \lambda^\gamma) \propto \lambda^{-C(\gamma)} \quad (4.4.1)$$

$$\langle \rho_\lambda^q \rangle \propto \lambda^{K(q)} \quad (4.4.2)$$

where $\langle \rangle$ represents mathematical expectation of statistical moment and $C(\gamma)$ is called codimension of subsets with field order above γ . The relations between $K(q)$, $C(\gamma)$ and γ were derived as (Schertzer and Lovejoy, 1991):

$$K(q) = \max_{\gamma} (q\gamma - C(\gamma)) \quad (4.4.3)$$

$$C(\gamma) = \max_q (q\gamma - K(q)) \quad (4.4.4)$$

The characteristics of the functions $C(\gamma)$ and $K(q)$ have been discussed in detail by Schertzer and Lovejoy (1991a) who also proposed a univariate model for $C(\gamma)$ based on three factors. In addition, the relationships between their model and the multifractal model based on $f(\alpha)$ were already studied (Schertzer and Lovejoy, 1991a).

New relations will be derived in this section to associate the fractal dimension function $f(\alpha)$, codimension function $c(\alpha)$ defined in Eq (4.2.3) and codimension function $C(\gamma)$ discussed by Schertzer and Lovejoy (1991a). The field ρ_λ is associated with the multifractal measure μ_i in the relation $\rho_\lambda = \mu_i \epsilon^{-D}$. The relations $\rho_\lambda \propto \lambda^\gamma$ and $\mu_i(\epsilon) \propto \epsilon^\alpha$ yield $\gamma = D - \alpha$. From Eq. (4.2.2) we obtain

$$\begin{aligned} Pr(\rho_\lambda \geq \lambda^\gamma) &= Pr(\mu_i(\epsilon) \geq \epsilon^{-\gamma+D}) = Pr(\mu_i(\epsilon) \geq \epsilon^\alpha) \\ &= \int_{\alpha_{min}}^{\alpha} P_\epsilon(\alpha) d\alpha \propto \int_{\alpha_{min}}^{\alpha} \epsilon^{D-f(\alpha)} d\alpha \end{aligned} \quad (4.4.5)$$

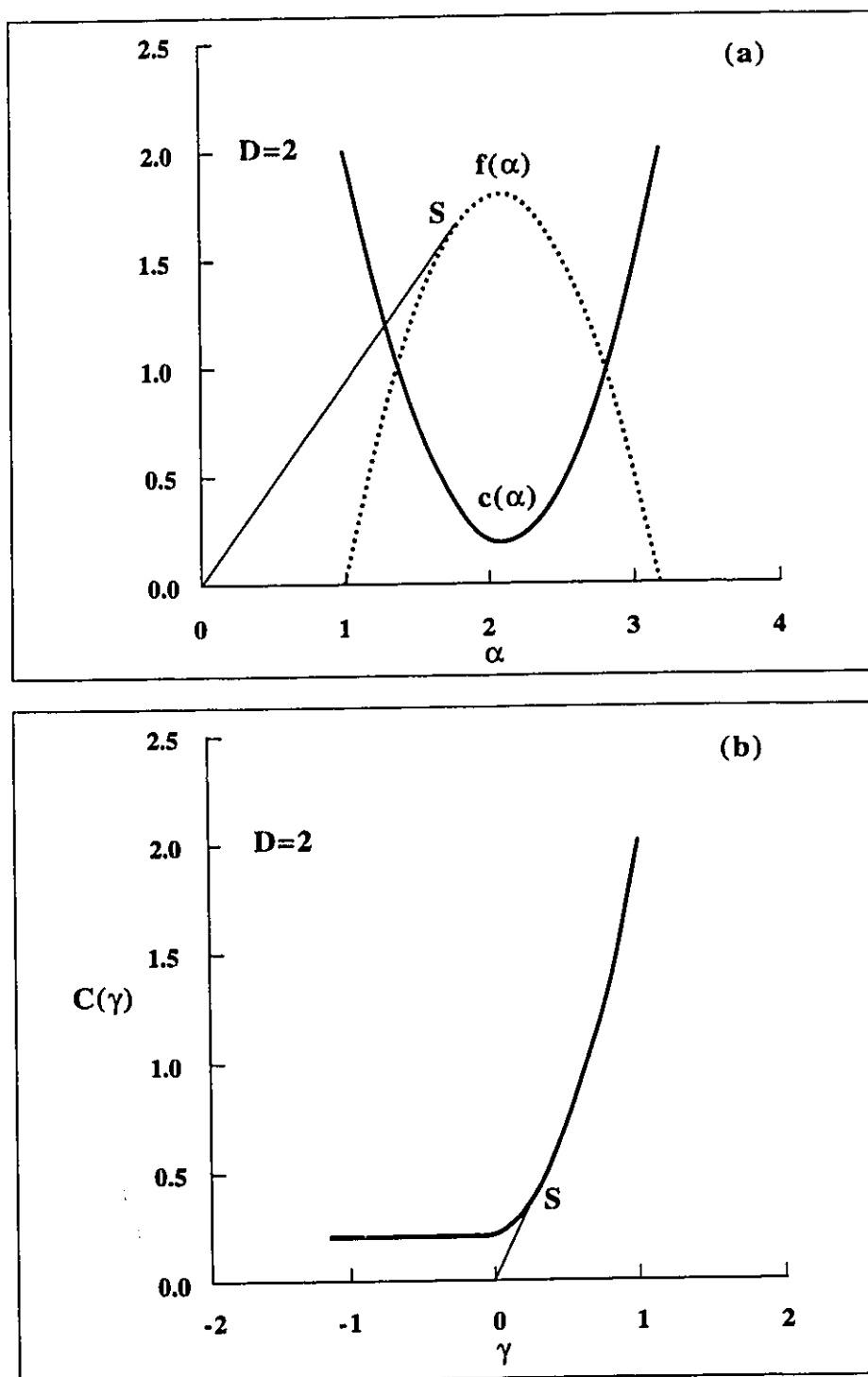


Fig. 4.4.1 Relationships between $f(\alpha)$, $c(\alpha)$ and $C(\gamma)$. (a) $f(\alpha)$ -curve (dotted line) and $c(\alpha)$ -curve (solid line). The values of $c(\alpha)$ were computed using Eq. (4.2.4); (b) $C(\gamma)$ -curve represented by solid line. The values of $C(\gamma)$ were computed using Eq. (4.4.7)

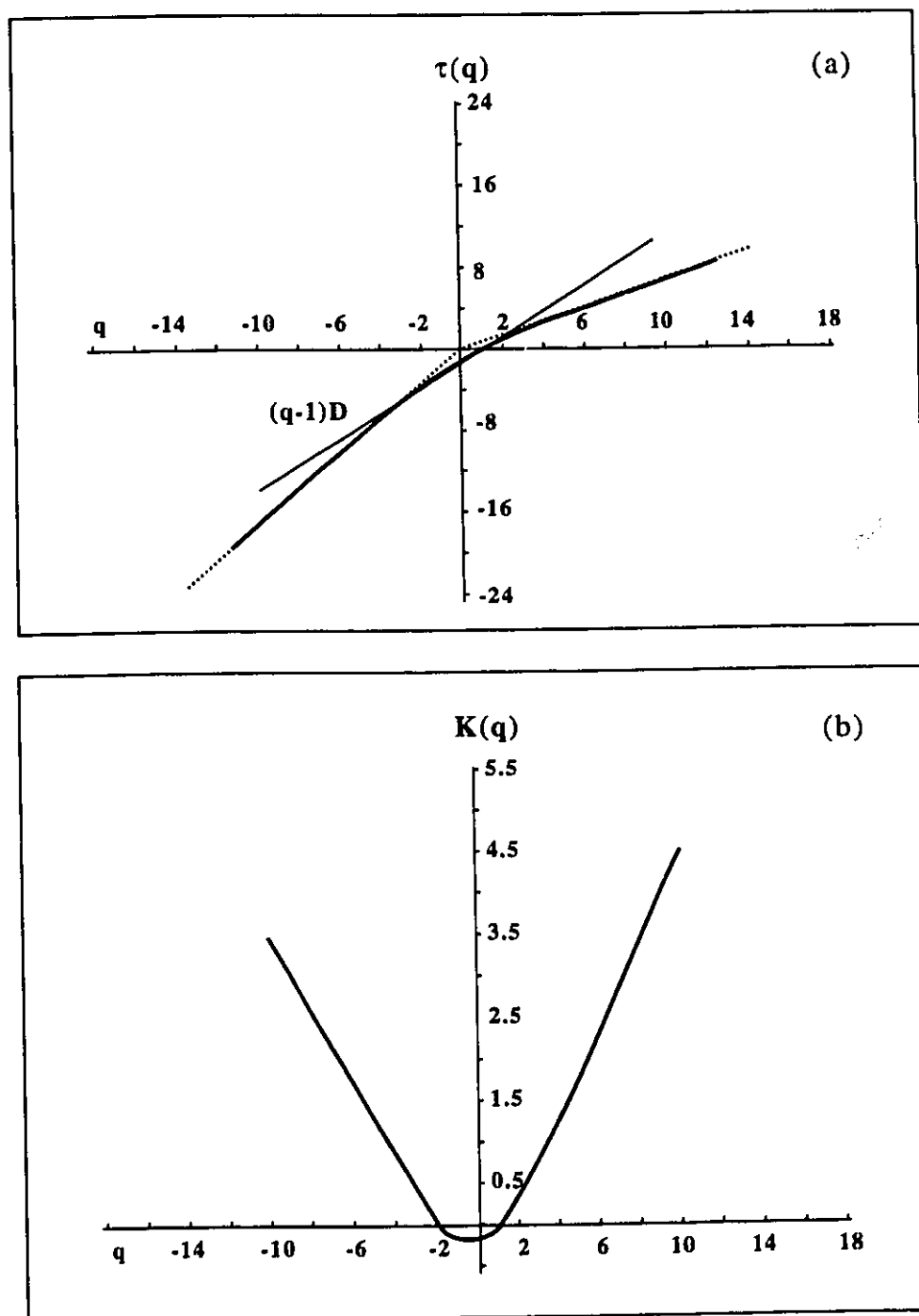


Fig. 4.4.2 Characteristics of the functions $\tau(q)$ and $K(q)$. (a) $\tau(q)$ -curve represented by solid curved line, solid straight line is for $\tau = (q-1) D$ intersecting horizontal axis at $q = 1$. Dotted lines represent $\tau = \alpha_{\max} q - f(\alpha_{\max})$ and $\tau = \alpha_{\min} q - f(\alpha_{\min})$, respectively; (b) $K(q)$ -curve.

Let $C(\gamma)$ denote the minimum value of the exponent and

$$\begin{aligned}
 C(\gamma) &= \min_{[\alpha_{\min}, \alpha]} [D - f(\alpha)] \\
 &= D - \max_{[\alpha_{\min}, \alpha]} f(\alpha) \\
 &= D - \max_{[D - \alpha_{\max}, D - \gamma]} f(D - \gamma)
 \end{aligned} \tag{4.4.6}$$

or, from Eq. (4.2.4),

$$C(\gamma) = \min_{[\alpha_{\min}, \alpha]} C(\alpha) \tag{4.4.7}$$

From Eqs. (4.4.2) and (4.3.6), we get

$$\begin{aligned}
 X_q(z) &= e^{-D} \langle \mu^q(z) \rangle \\
 &= e^{-D} z^{qD} \langle \rho_\lambda^q \rangle \\
 &= e^{-D + qD - K(q)}
 \end{aligned} \tag{4.4.8}$$

Therefore,

$$\tau(q) = (q-1)D - K(q); \quad K(q) = (q-1)D - \tau(q) \tag{4.4.9}$$

Combining the relations of Eqs. (4.4.6) and (4.4.9) gives, if $D - \alpha(0) \leq \gamma \leq D - \alpha_{\min}$:

$$C(\gamma) = D - f(\alpha) = D - \alpha q + \tau(q) = \gamma q - K(q) \tag{4.4.10}$$

This is identical to Eq. (4.4.4) only for values of γ with $D - \alpha(0) \leq \gamma \leq D - \alpha_{\min}$.

The following properties of function $C(\gamma)$ and $K(q)$ are also illustrated in Figs. 4.4.1 and 4.4.2.

1. $C_{\min}(\gamma_{\min}) = D - f_{\max}(\alpha(0))$, corresponding to the codimension of the support since $f_{\max}$

is the box-counting dimension of the support. $C_{\max}(\gamma_{\max}) = D - f(\alpha_{\min})$, corresponding to the codimension of subsets with the smallest coarse Hölder exponent $\alpha_{\min}$ associated with the largest values of μ . If $f_{\max}(\alpha(0)) = D$, as in the binomial multiplicative cascade model, then $C_{\min} = 0$, but in most cases $f_{\max}(\alpha(0)) < D$ and $C_{\min} > 0$. If $f(\alpha_{\min}(+\infty)) = 0$, then $C_{\max}(\gamma_{\max}) = D$ which is the box-counting dimension of the support; and if $f(\alpha_{\min}(+\infty)) < 0$, corresponding to negative dimension (Mandelbrot, 1992), then $C_{\max}(\gamma_{\max}) > D$. Both of these cases can be found in Schertzer and Lovejoy (1991a, 1992).

2. If $f(\alpha)$ is differentiable, then $C'(\gamma) \geq 0$ and it is a continuous function. From Eq. (4.4.6), if $D - \alpha_{\max} \leq \gamma \leq D - \alpha(0)$, then $C(\gamma) = D - f_{\max}(\alpha(0))$ and $C'(\gamma) = 0$, and if $D - \alpha(0) \leq \gamma \leq D - \alpha_{\min}$, then $C(\gamma) = D - f(D - \gamma)$ and $C'(\gamma) = f'(D - \gamma) = f'(\alpha) = q > 0$. This means that function $C(\gamma)$ is a continuous non-decreasing function and $C'(\gamma)$ is a continuous non-negative function.

3. The curve $C(\gamma)$ is tangent with straight line $C = \gamma$ at $\gamma = D - \alpha(1)$, because $C(D - \alpha(1)) = D - f(\alpha(1)) = D - \alpha(1)$ and $C'(D - \alpha(1)) = f'(\alpha(1)) = 1$.

4. $\gamma_{\min} = D - \alpha_{\max}$ and $\gamma_{\max} = D - \alpha_{\min}$. If $\alpha_{\min} < D < \alpha_{\max}$, which is common but not required, then $\gamma_{\min} < 0$ and $\gamma_{\max} > 0$.

5. Since $K(q) = (q-1)D - \tau(q)$, $K'(q) = D - \alpha(q) = \gamma(q)$ which is an increasing function; at $\alpha(q_D) = D$, $K'(q_D) = 0$; if $q > q_D$, $K'(q) > 0$, and if $q < q_D$, then $K'(q) < 0$, implying that

for $q < q_D$, $K(q)$ is a decreasing function; and for $q > q_D$, $K(q)$ is an increasing function with $K(q_D)$ as its minimum value. The slopes along the curve correspond to the values of $\gamma(q)$.

For the so-called fractal β -model, $\tau(q)$ is a straight line with slope α ($\alpha_{\min} = \alpha_{\max} = \alpha$), and the curve $f(\alpha)$ becomes a single spike. In this case, $K(q)$ is a straight line with $K(q) = (D - \alpha)q - D_\beta$. The slope of this line is $D - \alpha$ and the intercept is D_β , where D_β is the dimension of the β -model. Further, $\gamma_{\min} = \gamma_{\max} = D - \alpha$, and the curve $C(\gamma)$ reduces to a single spike.

In conclusion, the two models are essentially the same although they are based on fractal dimension $f(\alpha)$ and codimension $C(\gamma)$, respectively. Comparison shows that the fractal dimension spectrum $f(\alpha)$ and the codimension function $c(\alpha)$ are identical, characterizing not only the higher values of measure or field but also the lower values of the measure or field. Although the codimension function $C(\gamma)$ is associated with $f(\alpha)$ and $c(\alpha)$ in Eqs. (4.4.6) and (4.4.7), information derived from $C(\gamma)$ is only valid for $D - \alpha(0) \leq \gamma$ because $C(\gamma)$ is the codimension of subsets with fields of order greater than γ (or subsets with coarse Hölder exponent less than α). In this respect, $f(\alpha)$ is more useful than $C(\gamma)$. In later Chapters, only the multifractal model based on $f(\alpha)$ will be used.

4.5 MULTIFRACTAL SPECTRA FOR SPATIAL OBJECTS

4.5.1 Lithogeochemical data of Au from surface bedrock samples

Gold in the Earth's crust is irregularly distributed. From a mineral exploration point of

view, most types of gold deposits are complex. The irregularity of gold distribution in rocks is associated with the following factors: 1. Gold is a precious metal with low concentration values (average Au concentration is 2 ppb; see Jensen and Bateman, 1981); 2. Gold ore bodies often occupy relatively small volumes, e.g, gold bearing quartz veins may only be a few meters or even a few centimeters wide; 3. For most gold deposits, mineralization may be attributed to several processes (including hydrothermal activity), which reinforced one another and created spatial associations extending over long distances. As a result, the distribution of gold in a map area is concentrated in relatively small areas. By systematically sampling a given area, the estimated frequency distribution of the gold concentration values tends to be positively skewed. Usually this distribution can be described as lognormal or as a mixture of several lognormal distributions. In general, the probability of getting a sample with a relatively high concentration value is small and that of getting a low value is relatively high. Another factor to be kept in mind is that gold concentration values are determined with respect to an analytical detection limit and this affects the shape of the frequency distribution to some extent. If a series of similar processes are superimposed onto the same area, one result can be a distribution of concentration values with self-similarity in scaling, which can be described by using multifractal theory.

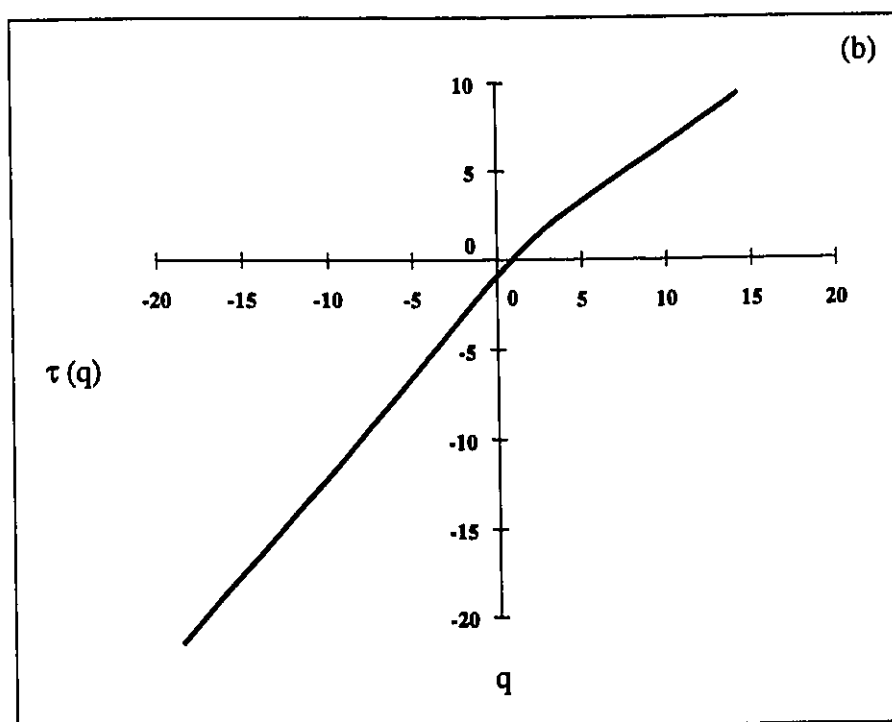
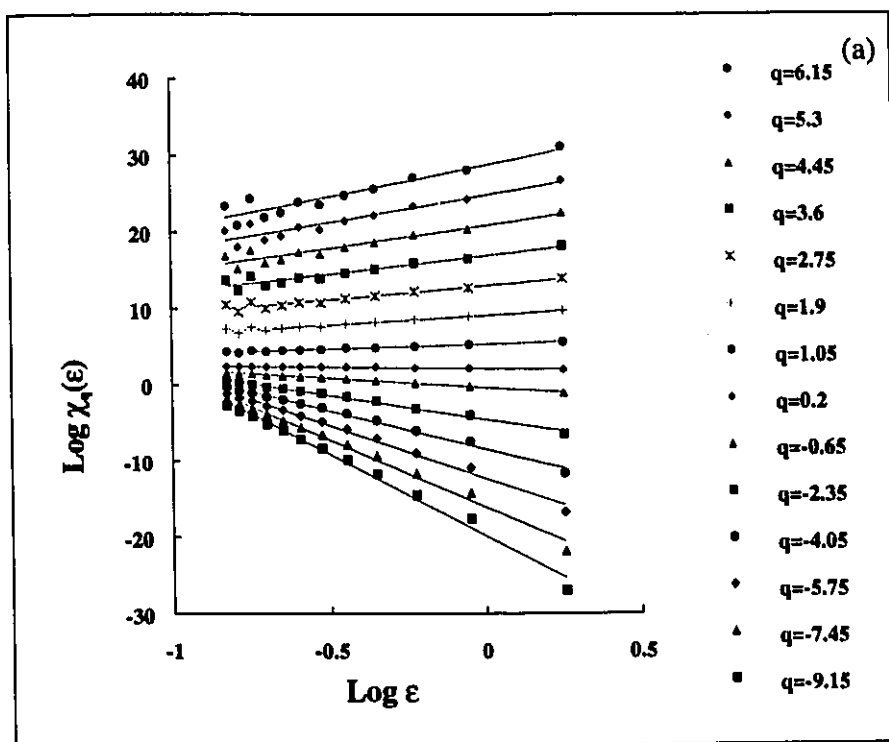
Lithogeochemical data of Au from 1033 bedrock samples were analyzed using the multifractal model. Some results have been published in Agterberg et al. (1993a) and Cheng, Agterberg and Ballantyne (1994a). For multifractal analysis, the total area of about

120 km² can be divided into $N(\epsilon)$ subareas of the same size $\epsilon \times \epsilon$ (ϵ ranging from 1.2 km to 0.1 km). The Au-values of samples within the i -th cell can be denoted as X_{ij} , $j=1, \dots, m(i)$, where $m(i)$ is the total number of samples in the i -th cell. The multifractal measure $\mu_i(\epsilon)$ can be defined as

$$\mu_i(\epsilon) = \frac{\epsilon^2}{m(i)} \sum_{j=1}^{m(i)} X_{ij} \quad (4.5.1)$$

When ϵ is set equal to a small value so that $\mu_i(\epsilon)$ is less than 1, we have $0 \leq \mu_i(\epsilon) \leq 1$, but $\sum \mu_i(\epsilon) \neq 1$. This situation is discussed in more detail in section 4.3.

For comparison, various cutoff values may be used for selecting samples, such as 100 ppb, 200 ppb, 500 ppb and 1000 ppb. Results for cutoff at 100 ppb are shown in Figs. 4.5.1 and 4.5.2. The results for other cases are summarized in Fig. 4.5.3. The values of $\chi_q(\epsilon)$ for the 100 ppb cutoff are plotted against ϵ on log-log paper (Fig. 4.5.1a) with q ranging from -10 to 7. Straight lines were fitted by the least squares method which shows that the measure defined in Eq.(4.5.1) satisfies the multifractal model. The slopes of the straight lines in Fig. 4.5.1a provide estimates of $\tau(q)$ according to Eq. (4.3.6). The values of $\tau(q)$ (Fig. 4.5.1b) show that for $q \ll 1$ the curve of $\tau(q)$ tends to be a straight line with slope $\alpha_{\max}=1.1$ and intercept $f(\alpha_{\max}) > 0$. Similarly, for $q \gg 1$, $\alpha_{\min}=0.65$ and $f(\alpha_{\min})=0$. The values of α (Fig. 4.5.1c) can be estimated from slopes of the curve $\tau(q)$ using the central difference method. The values of the fractal dimension spectrum $f(\alpha)$ (Fig. 4.5.1d) are estimated by Eq. (4.3.8). It can be seen in Fig. 4.5.1d that the box-counting dimension is about 0.9. The entropy dimension denoted as S in Fig. 4.5.1d is about 0.88. At this



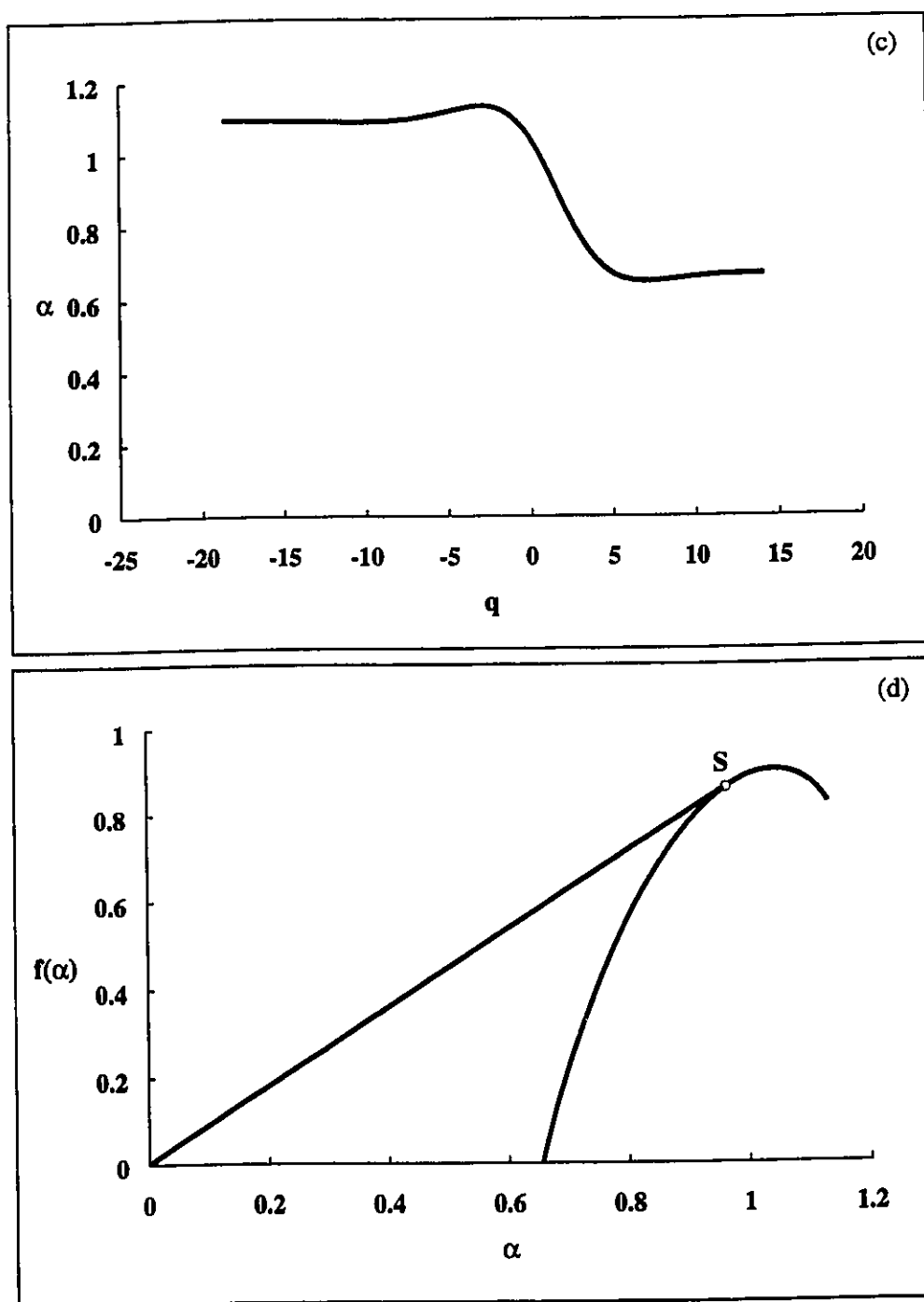


Fig. 4.5.1. Derivation of multifractal spectrum for gold (cutoff value = 100 ppb) in Mitchell-Sulphurets Mineral District. (a) ϵ in km, log base 10, lowest sets of solid triangles and squares are for q equal to -7.45 and -9.15, respectively; (b) $\tau(q)$ values; (c) $\alpha(q)$ values estimated from the slope of $\tau(q)$ -curve by central difference method; (d) Multifractal spectrum for Au > 100 ppb, S represents entropy dimension. See text for further explanation.

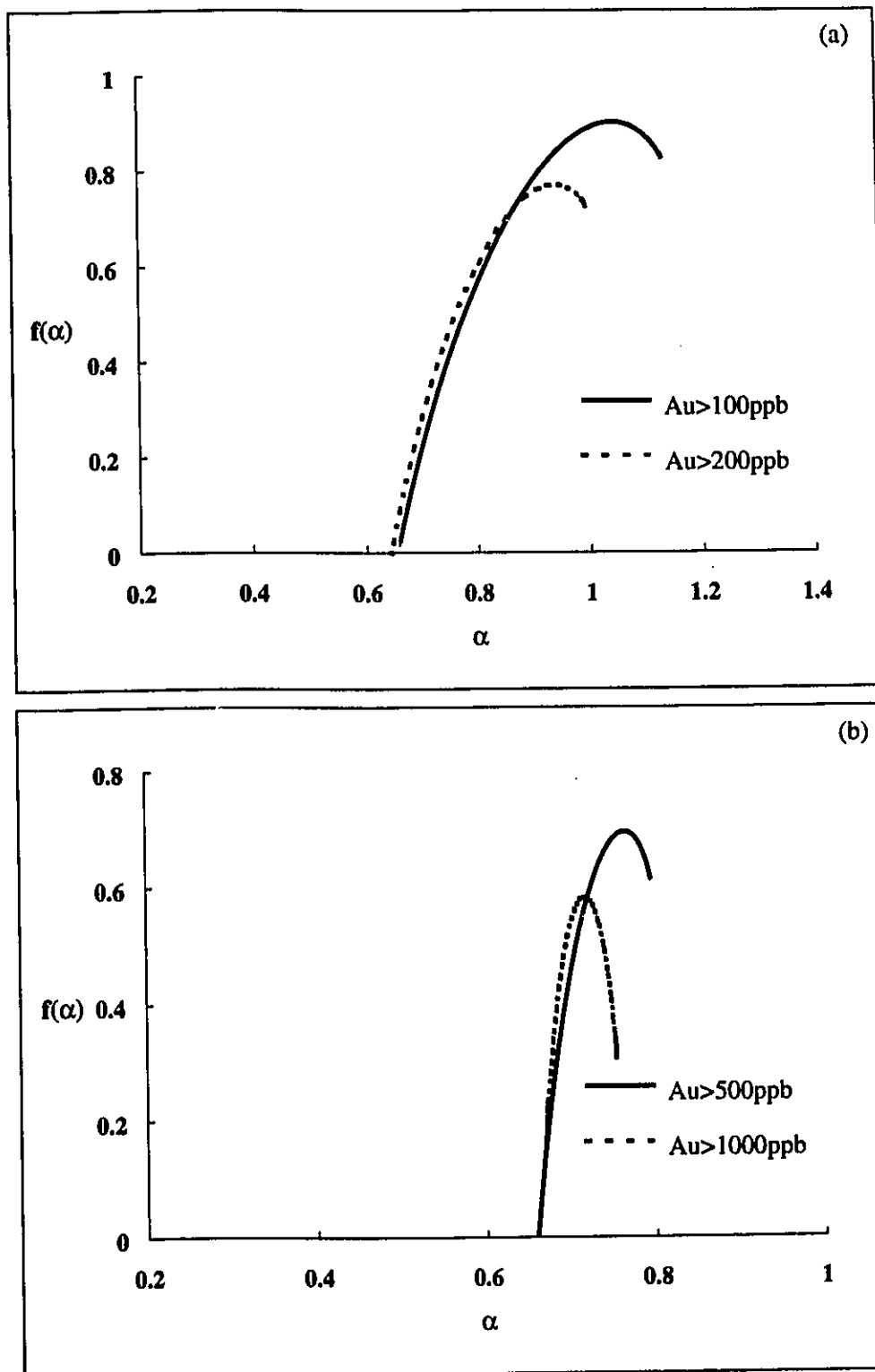


Fig. 4.5.2 Multifractal spectra for four different cutoff values. (a) $\text{Au} > 100 \text{ ppb}$ and 200 ppb ; (b) $\text{Au} > 500 \text{ ppb}$ and 1000 ppb .

point the curve of $f(\alpha)$ is tangent with the line $f(\alpha) = \alpha$. The multifractal spectrum $f(\alpha)$ is a truncated symmetrical curve which is associated with the skew frequency distribution of Au and the cutoff value selected.

The results for other cutoffs are shown in Fig. 4.5.2; for increasing cutoff value of Au the domains of fractal dimension spectra decrease and the fractal dimension values decrease at the same time.

4.5.2 Faults and Igneous Rocks in the Mitchell-Sulphurets Mineral District

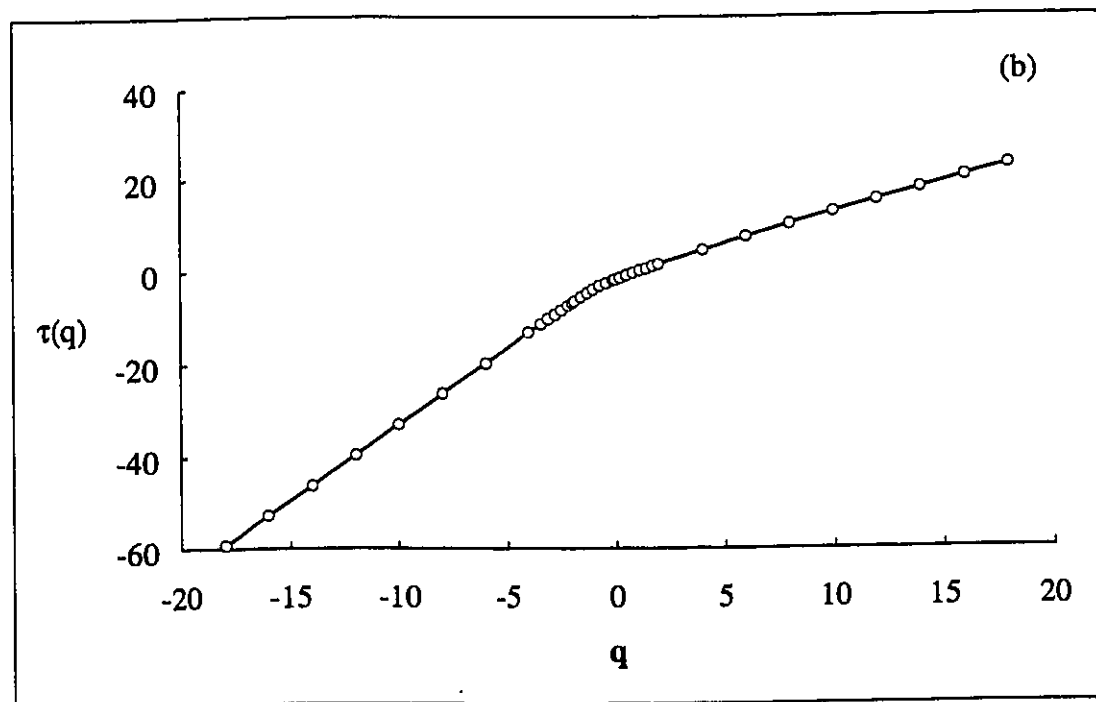
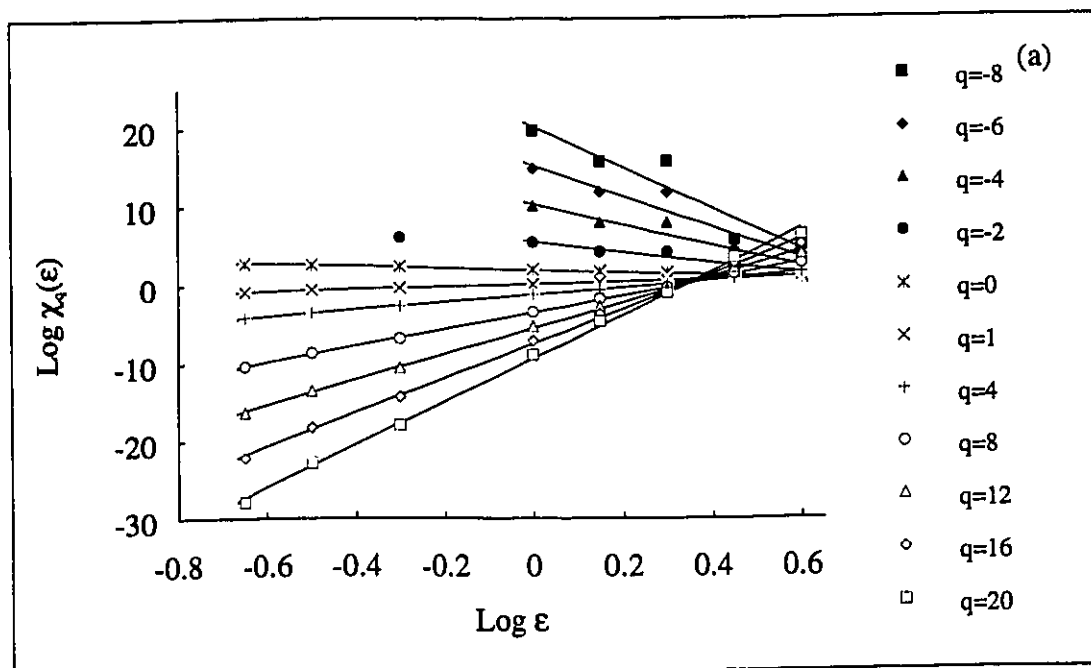
Faults (primarily normal faults and thrusts) and igneous rocks in the Mitchell-Sulphurets mineral district were studied by means of fractal models (number-size model) in Chapter 3. The densities of faults and igneous rocks per unit area can also be described using multifractal models. As discussed in section 4.2, skewly distributed measures and clustered entities may have multifractal properties. The multifractal measures for faults and igneous rocks can be defined as total length of the faults and total area of igneous rocks per cell of size ϵ , respectively. In order to use SPANS-GIS to obtain the required measurements as discussed in Chapter 3, a area map can be created by buffering the faults with a narrow width so that the areas of the buffer zones are approximately proportional to the lengths of the faults. For example, 1-meter buffers were created around all faults to form a binary pattern. The total study area of about 120 km² was divided into subareas of the same linear size ϵ ranging from 0.2 km to 4 km, accomplished by means of GIS

(see Chapter 3). The measure of a cell can be denoted as μ_i (total area of buffers around faults or igneous rocks in the i -th cell). The values of $\chi_q(\epsilon)$ with q ranging from -20 to 20 are illustrated in Figs. (4.5.3) for faults and (4.5.4) for igneous rocks. For $q < -2$, the straight lines show a poor fit on Figs. (4.5.3a) and (4.5.4a). The estimation of the values of $\tau(q)$, α and $f(\alpha)$ is illustrated in Figs. (4.5.3b-d) and (4.5.4b-d), respectively. The box-counting dimensions are estimated to be 1.712 ± 0.056 for faults, and 1.404 ± 0.036 for igneous rocks. The entropy dimensions are 1.64 ± 0.003 for faults, and 1.395 ± 0.0052 for igneous rocks. The spatial distribution of igneous rocks is more irregular than that of faults. Other examples will be given in Chapters 6 and 7, including use of the multifractal model for Au mineral occurrences in the Iskut River map sheet, northwestern British Columbia.

4.6 CONCLUSION

Newly derived relations between $f(\alpha)$, $c(\alpha)$ and $C(\gamma)$ (Eqs. (4.4.6) and (4.4.7)) show that two previously derived, different multifractal models are nearly identical. The fractal spectrum $f(\alpha)$ provides more information about the multifractal measure than the codimension function $C(\gamma)$.

The multifractal model was used for describing the spatial distribution of Au concentration values in bedrock samples, as well as faults and igneous rocks in the Mitchell-Sulphurets mineral district, northwestern British Columbia.



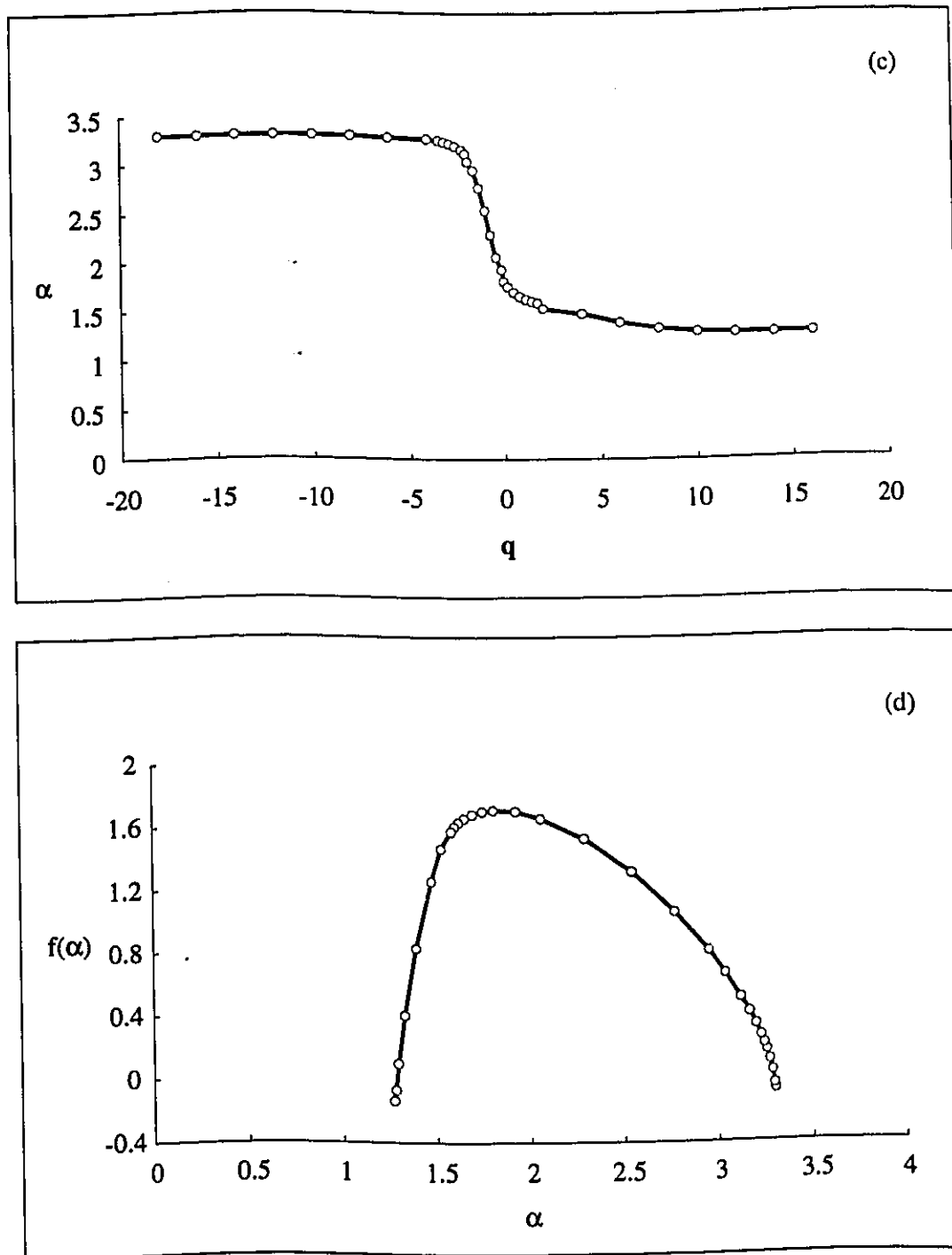
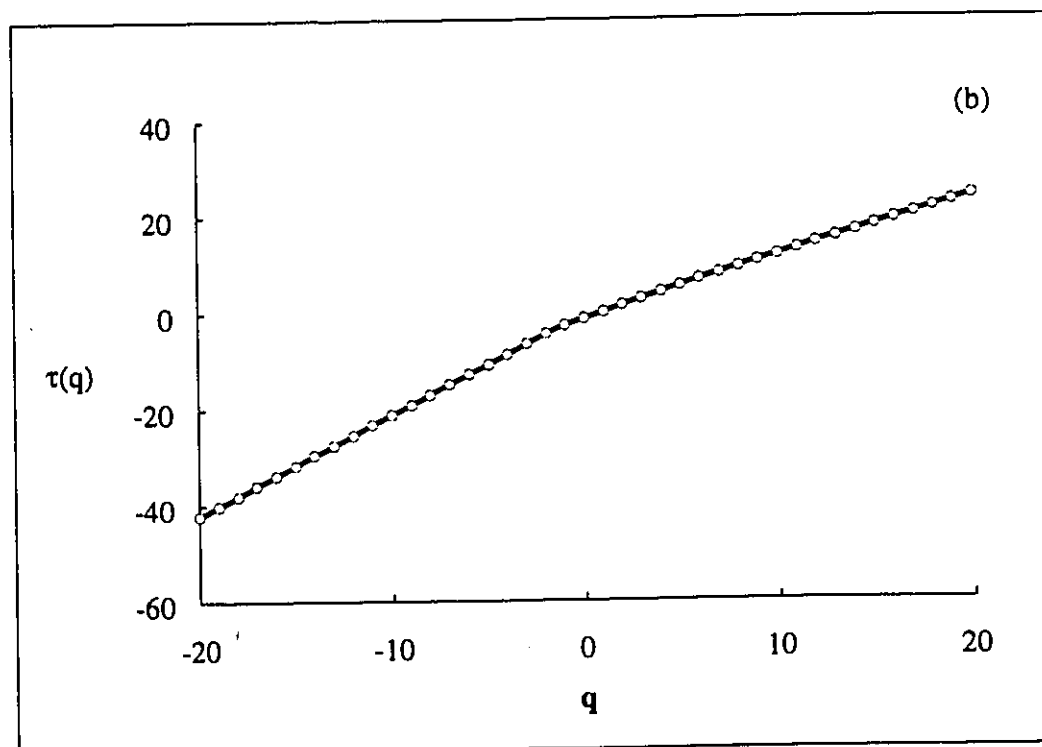
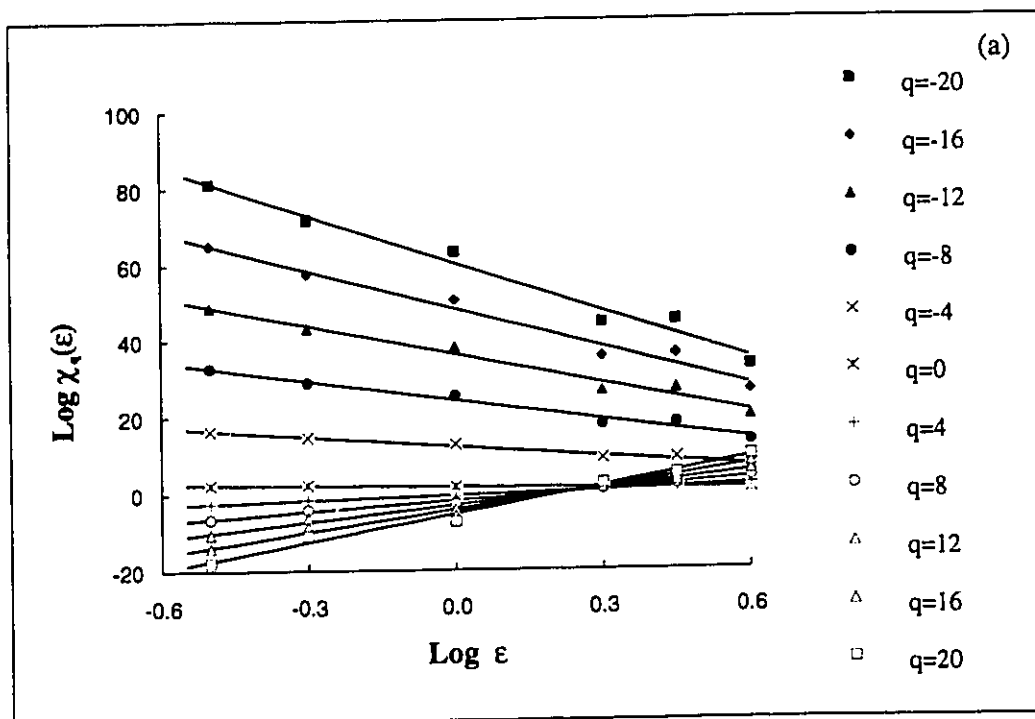


Fig. 4.5.3. Derivation of multifractal spectrum for faults (normal faults and thrusts) in the Mitchell-Sulphurets Mineral District. (a) ϵ in km, log base 10, q ranging from -20 to 20; (b) $\tau(q)$ values; (c) $\alpha(q)$ values estimated from the slope of $\tau(q)$ -curve by central difference method; (d) Multifractal spectrum.



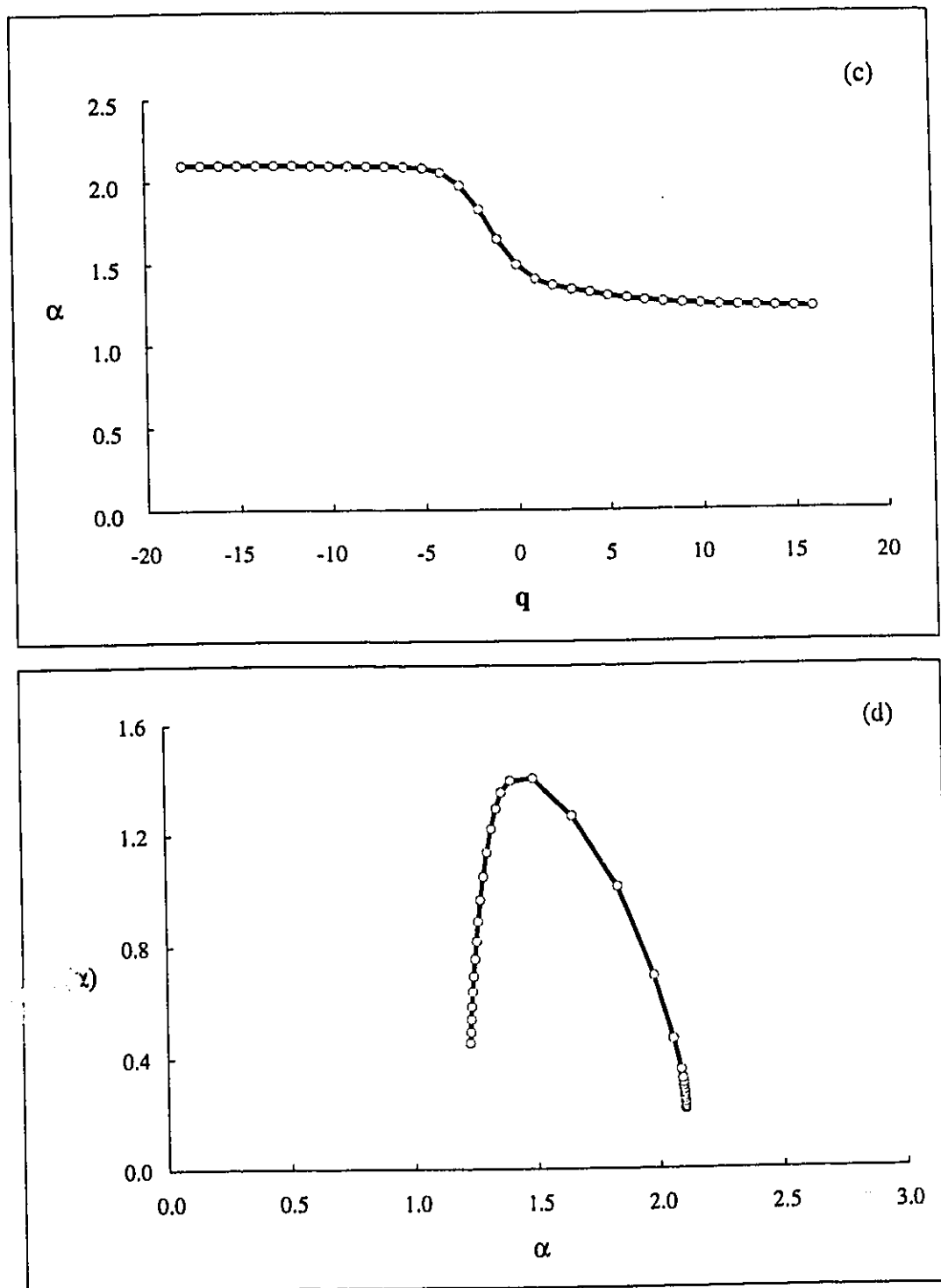


Fig. 4.5.4. Derivation of multifractal spectrum for igneous rocks in the Mitchell-Sulphurets mineral district. (a) ϵ in km, log base 10, q ranging from -20 to 20; (b) $\tau(q)$ values; (c) $\alpha(q)$ values estimated from the slope of $\tau(q)$ -curve by central difference method; (d) Multifractal spectrum.

Chapter 5 USE OF THE MULTIFRACTAL MODEL FOR SEPARATION OF GEOCHEMICAL ANOMALIES FROM BACKGROUND

5.1 INTRODUCTION

As mentioned in Chapter 2, recognition of a geochemical anomaly from among background is a basic problem in geochemical exploration. Spatial distributions of most elements for a given geological-geochemical environment are the end products of a sequence of geological processes, such as volcanic or intrusive activities, sedimentary processes, tectonism, metamorphism and mineralization. During these events or processes some elements may be enriched, possibly forming ore-grade material, and others may be dispersed. The spatial characteristics exhibited by some elements in relation to ore deposits provide guidelines for mineral exploration. One of the major objectives of studies currently being conducted in the Mitchell-Sulphurets mineral district, northwestern British Columbia, is to distinguish lithogeochemical anomalies, assumed to be caused by mineralization, from background element concentrations related to regional geological processes.

Several statistical procedures can be used for helping to delineate geochemical anomalies (see Chapter 2). Some methods employ element concentration values only and require the samples to be collected uniformly over the region of interest; e.g., the identification of some fixed percentage of the highest values works under the assumption that all samples

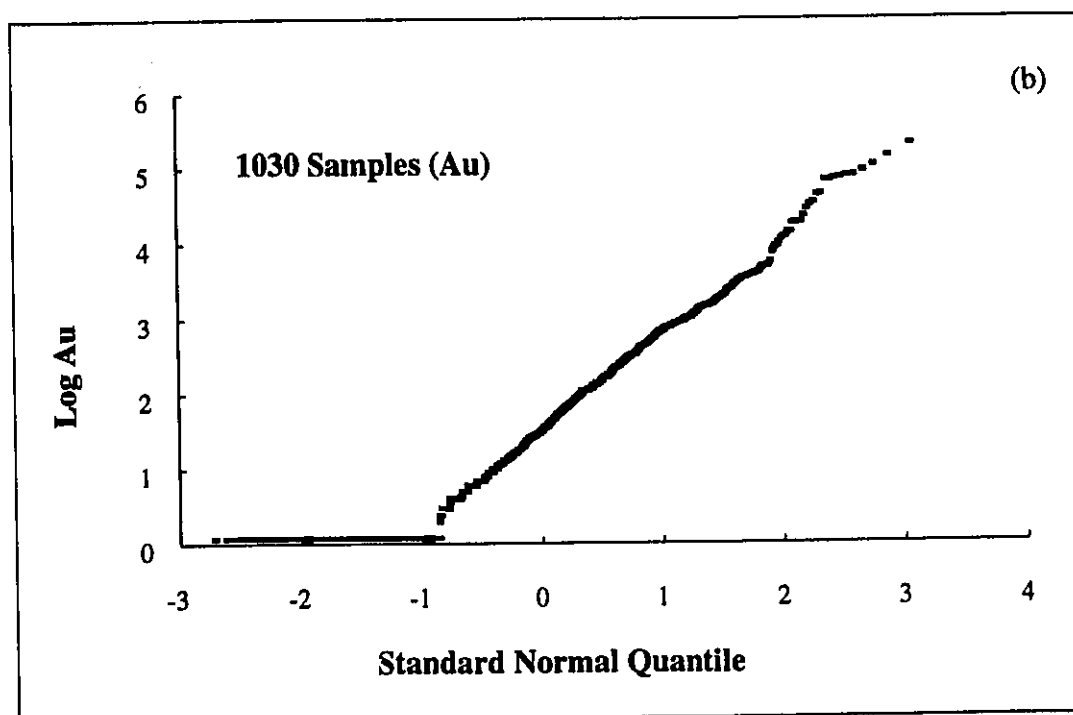
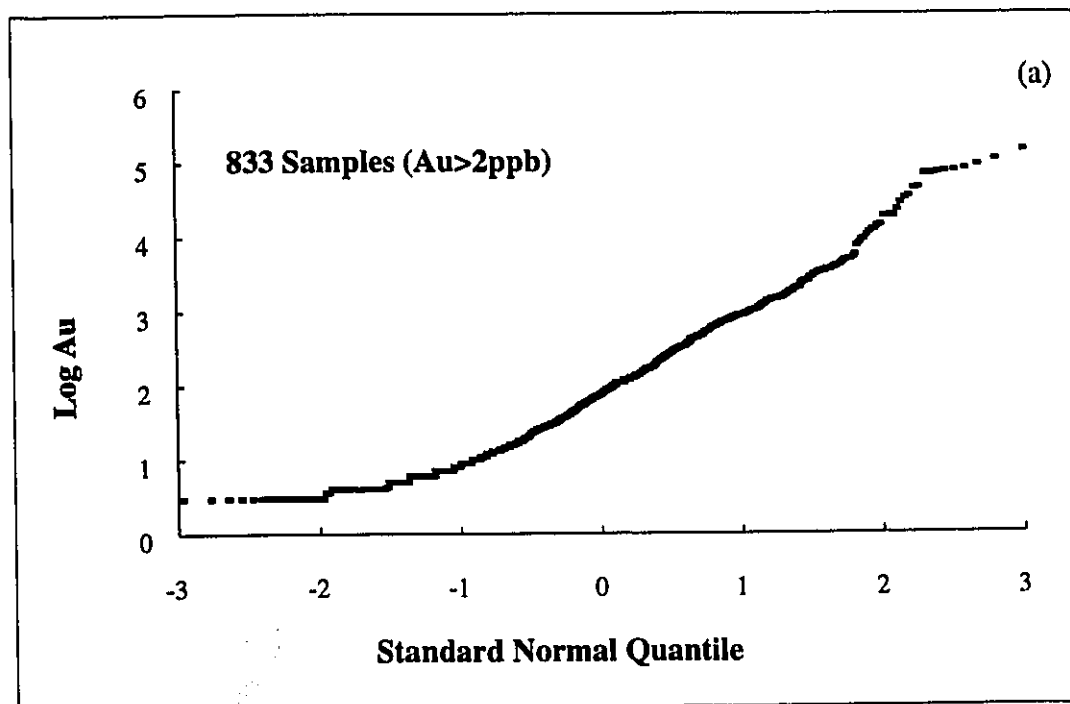
were randomly collected. This procedure may give poor results due to the complexity of the element distributions. It is possible to improve the results by considering both the frequency distribution of element concentration values and their spatial distribution. Values and shapes of anomalies were considered in anomaly recognition methods discussed by Sinclair (1991). Various techniques, based on average values, also have been proposed (e.g., Govett et al., 1975). These authors have shown that combining geometrical shapes of anomalies with frequency distributions of element concentration values can provide more information for distinguishing between populations (also see Chapter 2). In the Mitchell-Sulphurets study, the fractal and multifractal nature of lithogeochemical anomalies is explored (see Chapter 4). The results are applied to successfully separate anomalous from background concentrations in the data set (Cheng, Agterberg and Ballantyne, 1994a).

Applications of fractals to geochemical exploration were suggested by Bölviken et al. (1992). Several models are applied in their paper for describing the fractal nature of stream sediment data. This chapter deals with stream sediment data and lithogeochemical data which generally are more irregularly distributed than stream sediment data. It was shown in Chapter 4 that Au in bedrock samples from the Mitchell-Sulphurets area requires a multifractal instead of a fractal approach. A concentration-area model will be derived based on the multifractal model and used to separate the geochemical anomalies from background for lithogeochemical data and stream sediment data.

5.2 FREQUENCY DISTRIBUTIONS AND SPATIAL INTERPOLATION OF GOLD AND ASSOCIATED TRACE ELEMENTS

Two well-known procedures for rapid characterization of the distribution of a chemical element in a study area are use of Q-Q plots and interpolation maps with geochemical isopleths. Q-Q plots show how closely the elements follow normal, lognormal or other known frequency distributions. For Au and the Au-associated elements such as Cu and As, the lognormal model was tested by means of Q-Q plots. Figure 5.2.1 shows log-transformed data for these elements plotted against normal quantiles, illustrating that most of the trace elements in the study area could be modelled as if they originate from one or more lognormal populations. However, the Q-Q plots do not provide a method for separating anomalies from background in the Mitchell-Sulphurets area. A similar result was shown in Chapter 2 for stream sediment data, but distinct thresholds cannot be derived.

Several interpolation procedures, including Kriging and so-called potential mapping (available in SPANS) can be used for creating contour maps which represent the spatial distribution of an element. We have used potential mapping. This simple weighted moving average method uses a moving circular window with adjustable parameters to control the weighting of values at neighbouring points. The parameters include radius of the circular window, decay ratio of the weighting function, and maximum number of samples to be included within each window. For this study the parameters selected were: radius of 0.8



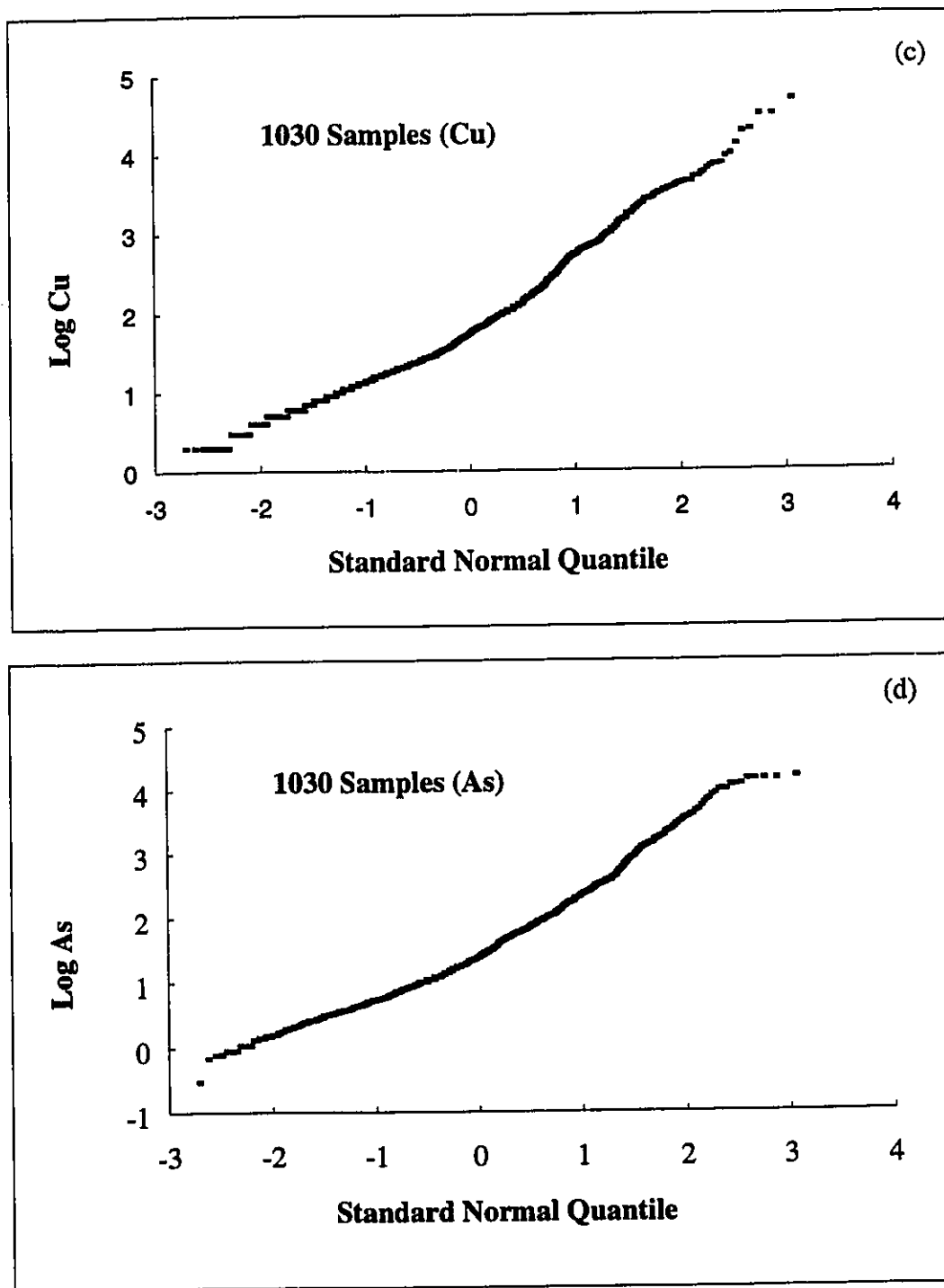


Fig. 5.2.1 Q-Q plots, data are base 10 log-transformed. (a) 833 samples with Au > 2 ppb; (b) Au, all samples; values below detection limit (2 ppb) were set equal to 1.2 ppb; (c) Cu (ppm), all samples; (d) As (ppm), all samples.

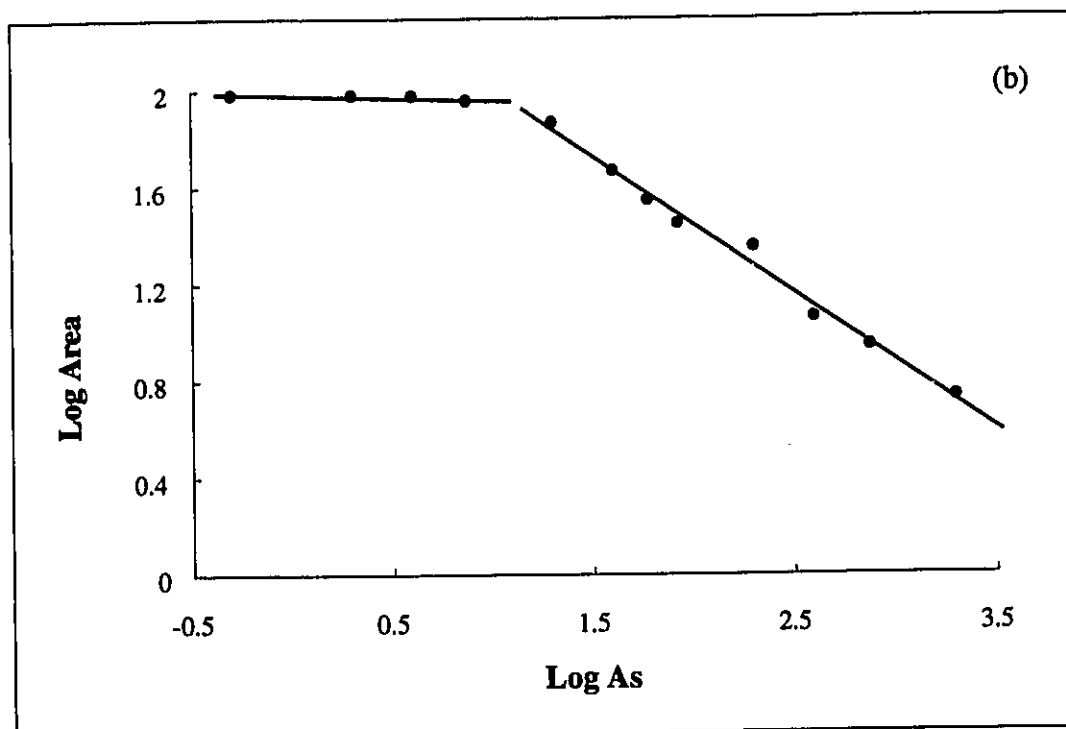
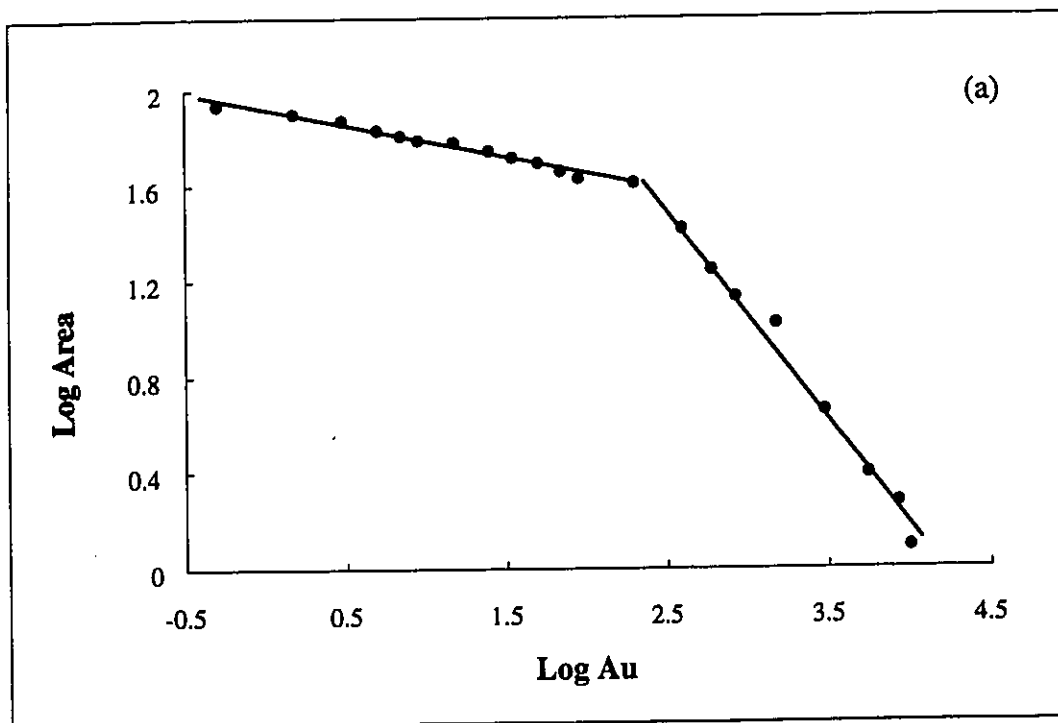
km for lithogeochemical data, and 8 km for stream sediment data, decay ratio of 0.5 (corresponding to a linear weighting function with weight 0 for samples located at the boundary or outside the moving window, and 1 for samples located at the centre) and maximum of 10 samples per window. (If more than 10 samples occur within the window for a given location, only the 10 nearest points are used to evaluate the surface at that location.)

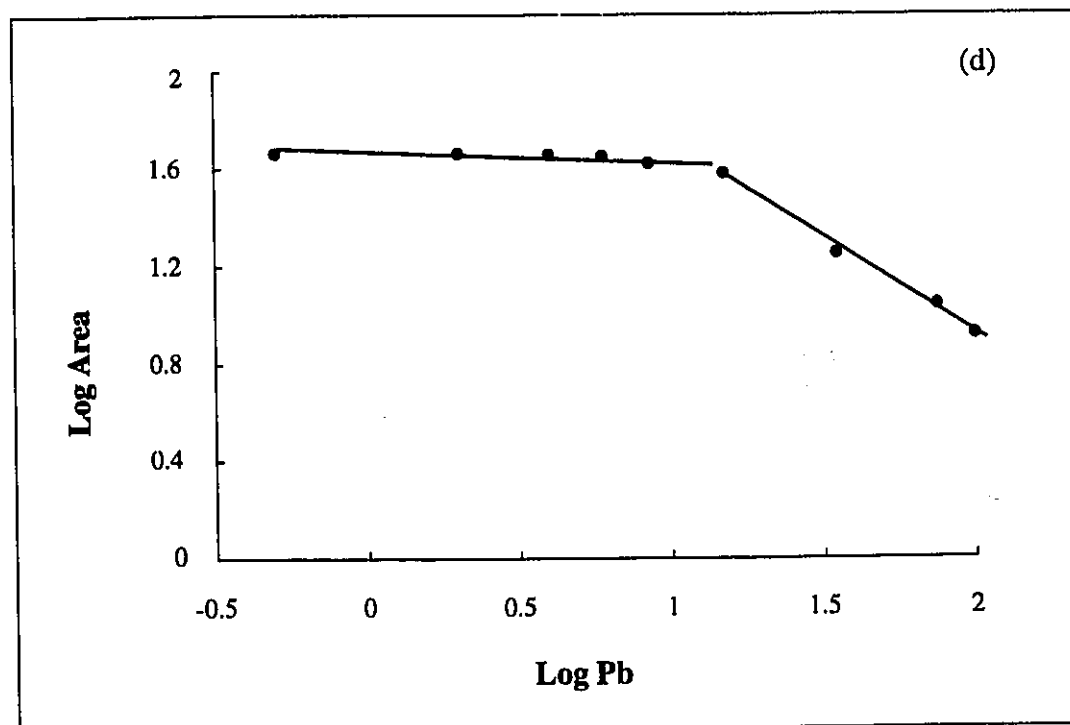
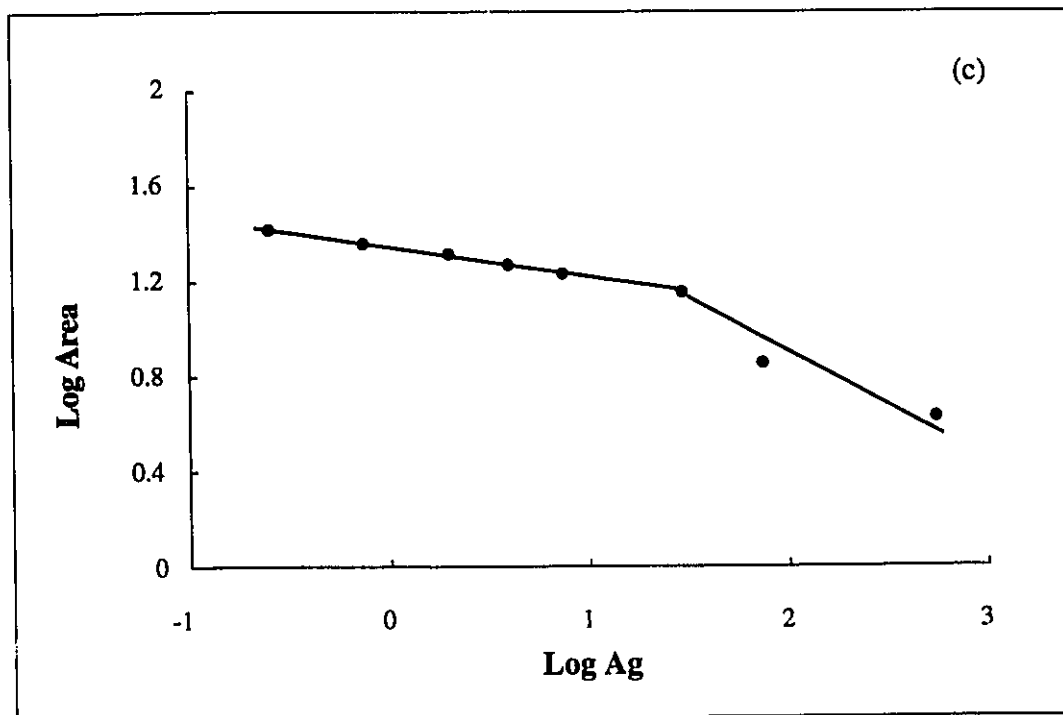
Figure 3.4.3 shows Au and Cu maps employing single contours to divide the data set into two parts, above and below the contour value. It can be seen that the shapes of the areas enclosed by successive contours are changing gradually; total enclosed area decreases as the value of the contour increases. From these contours an optimum threshold for separating anomalies from background areas can be selected by means of a log-log plot for the element concentration-area relation. This threshold coincides with a sudden change in the rate of decrease of the area enclosed by high value contours on the log-log plot.

Let $A(p)$ denote the area with concentration values greater than the contour value p . This implies that $A(p)$ is a decreasing function of p . If v represents the threshold, the following empirical model generally provides a good fit to the data for different elements in the study area:

$$A(p \leq v) \propto p^{-\alpha_1} \quad ; \quad A(p > v) \propto p^{-\alpha_2} \quad (5.2.1)$$

where $\propto$ denotes proportionality.





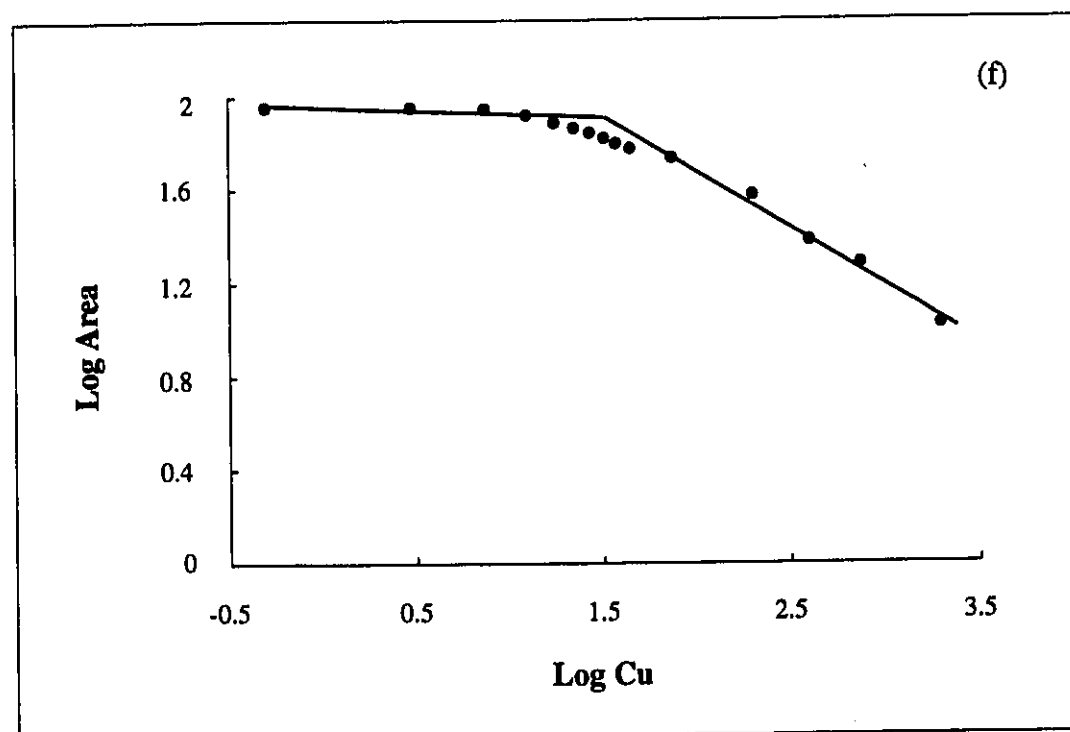
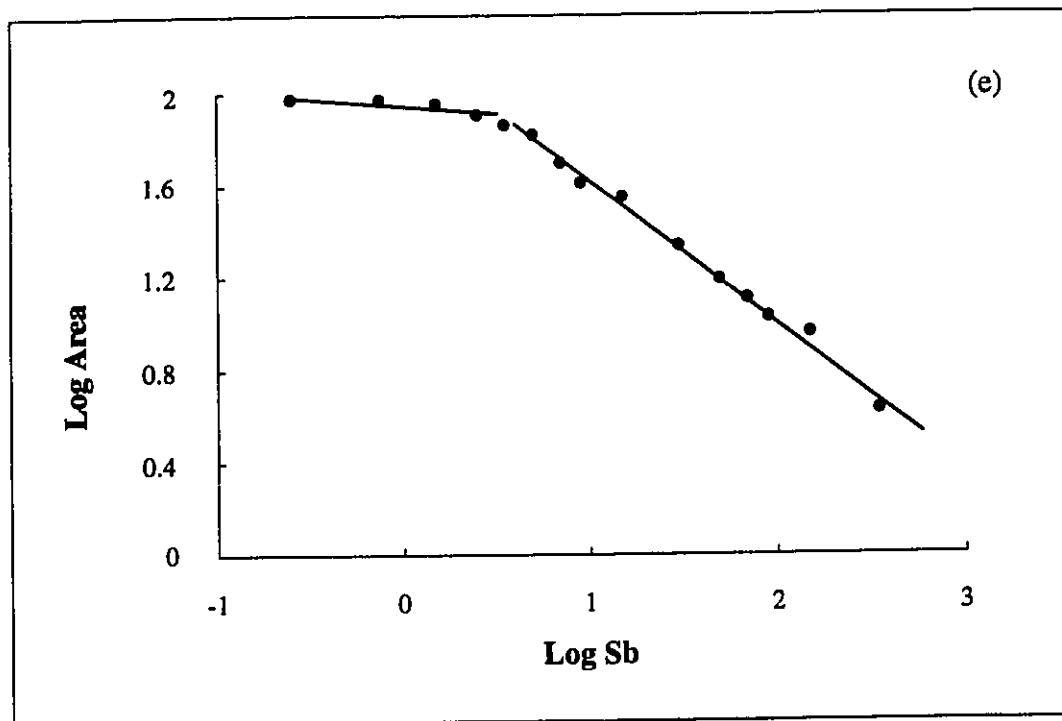


Fig. 5.2.2 Log-log plots representing the relationships between areas $A(\rho)$, bounded by contours, and contour value for various elements of lithogeochemical data. Solid lines obtained by least squares (LS) fitting.

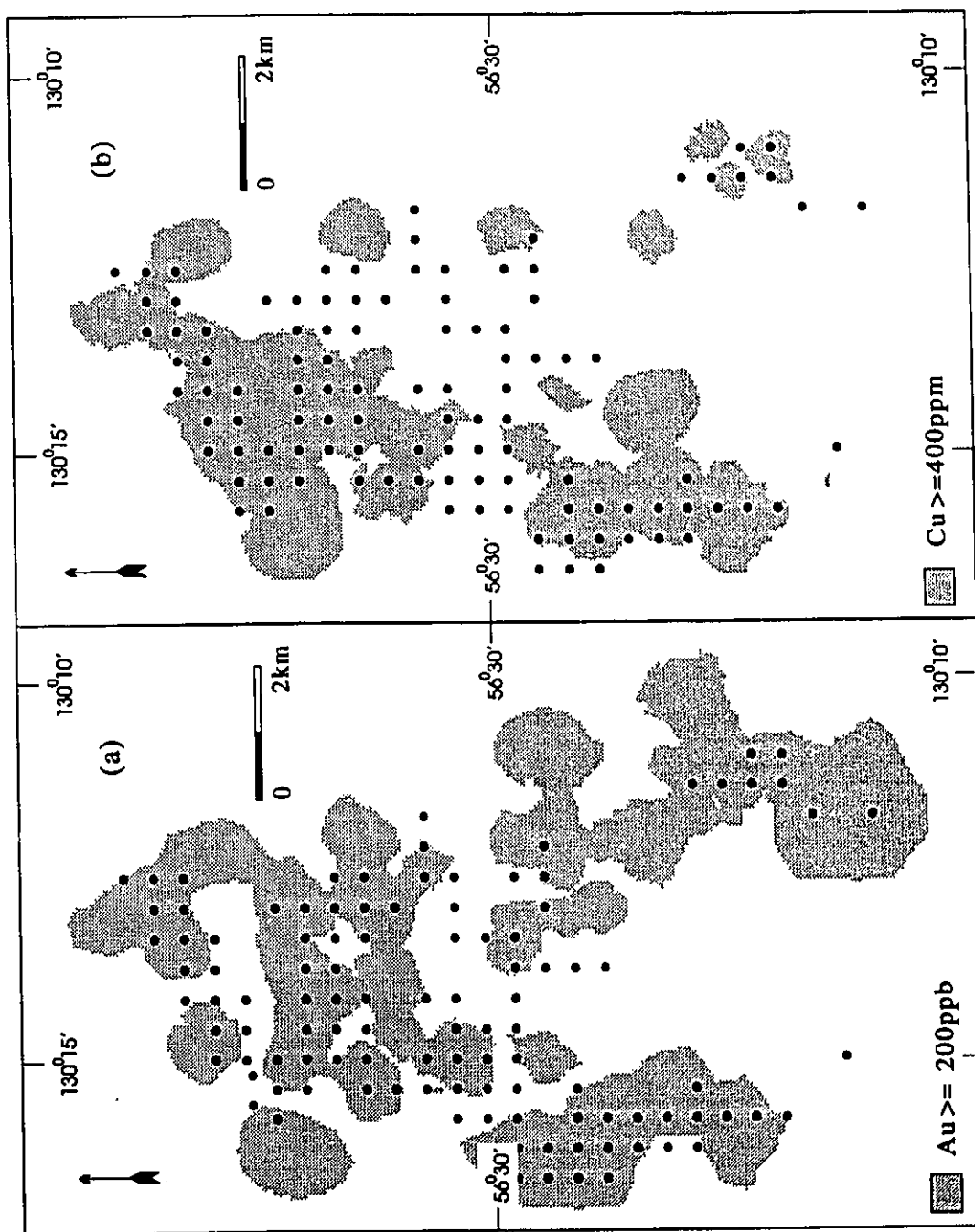


Fig. 5.2.3 Maps showing spatial relationship between anomalies obtained using the fractal method (grey pattern) and alteration zones approximated by 110 black dots. (a) Au; (b) Cu.

Table 5.2.1 Results Obtained by Using the Concentration-Area Method and Weights of Evidence Procedure; α_1 and α_2 are the exponents of the power-law relation for concentration values less and greater than the threshold value (v), respectively.

Elements & Oxides	Power Law			W. of E. v
	v	α_1	α_2	
Au (ppb)	200	0.137	0.879	200
Ag (ppm)	75	0.127	0.820	75
As (ppm)	40	0.064	0.700	40
Sb (ppm)	7	0.061	0.606	7
Zn (ppm)	200	0.059	0.910	200
Pb (ppm)	15	0.026	0.775	15
Cu (ppm)	400	0.092	0.725	400
Mo (ppm)	90	0.478	2.234	90
Fe (ppm)	3.5	0.068	2.898	6.5
Th (ppm)	2.5	0.043	2.030	3.5
Sr (ppm)	225	0.018	2.325	225
Sm (ppm)	7	0.048	3.158	7
W (ppm)	3.5	0.317	3.069	3.5
F (ppm)	150	0.018	1.324	150
Cr (ppm)	15	0.071	0.950	2.5
V (ppm)	125	0.044	4.446	7.5
Ce (ppm)	17.5	0.049	1.935	17.5
Co (ppm)	13.5	0.117	3.545	13.5
Nb (ppm)	12.5	0.314	3.911	6.5
Tb (ppm)	0.225	0.097	1.194	0.325
La (ppm)	15	0.024	2.931	15
SiO ₂ (%)	55	0.011	17.179	62.74
TiO ₂ (%)	0.475	0.052	4.262	0.55
Al ₂ O ₃ (%)	13.5	0.067	17.862	13.5
MnO (%)	2.5	0.135	10.771	0.5
Fe ₂ O ₃ (%)	4.5	0.011	2.386	3.5
CaO (%)	5	0.394	3.985	2
S (%)	1.5	0.232	1.397	1.5

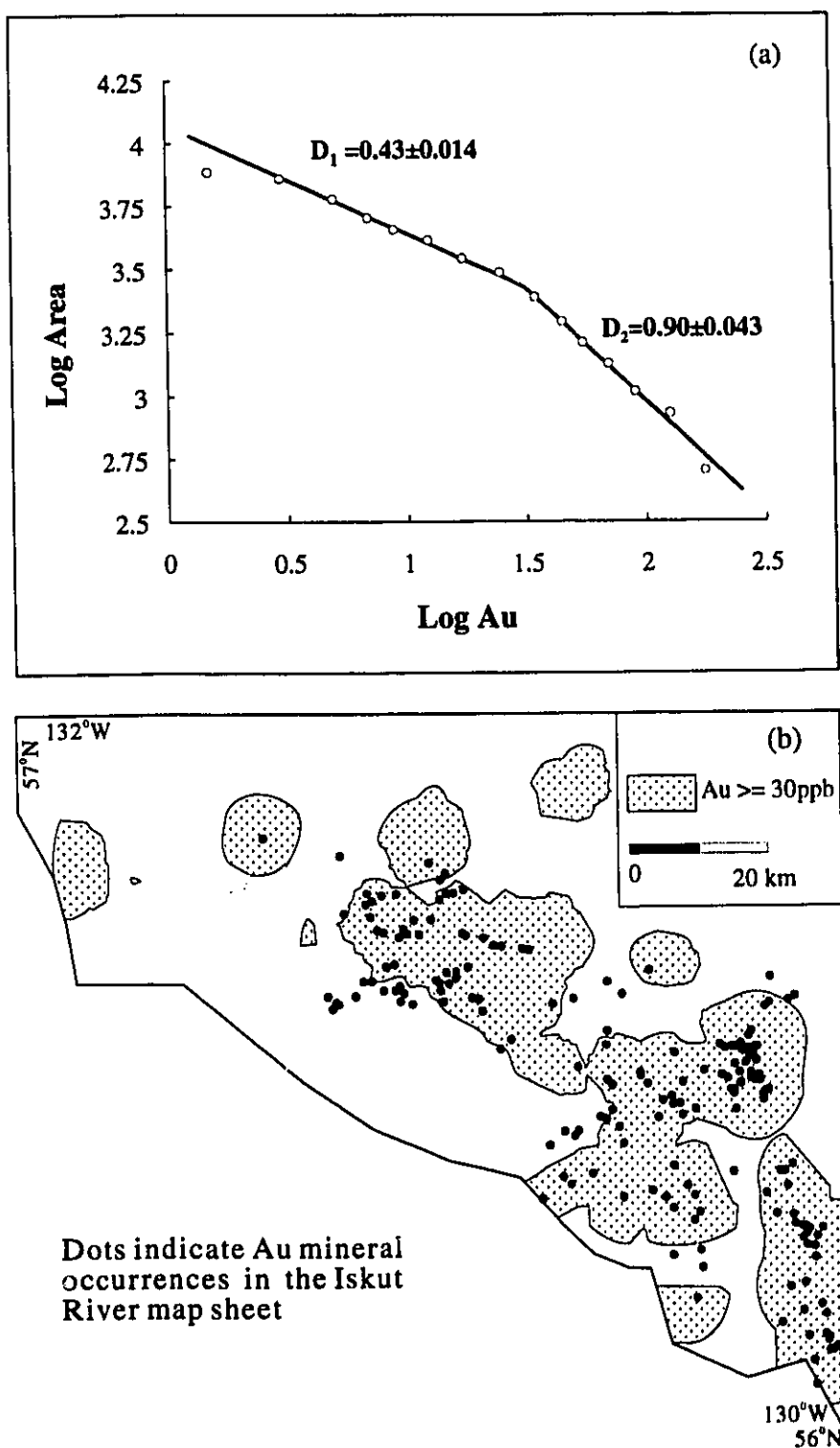


Fig. 5.2.4 Stream sediment data. (a) Log-log plots showing power-law relationship between areas $A(p)$, bounded by contours, and contour value for Au in stream sediment samples. Solid lines obtained by LS fitting; (b) Anomalies of Au with values above the threshold (30 ppb) determined from (a).

Figure 5.2.2 shows log-log plots satisfying the power-law relations of Eq. (5.2.1) for Au, Cu, As, Ag, Sb and Pb. All areas were computed using SPANS from separate contour maps such as those shown in Figure 3.4.3 for Au and Cu. Pairs of estimated exponents and corresponding optimum thresholds for these 6 elements and 22 other elements or oxides are presented in Table 5.2.1. These thresholds delineate anomalous areas. Comparison of the areas above and below the threshold of 200 ppb Au on the contour map (Figs. 3.4.3 and 5.2.3a) with the geological map shows significant spatial correlation between the areas with Au concentration above 200 ppb and Au-associated alteration zones. The same type of correlation with alteration zones applies to Cu (400 ppm threshold; see Fig. 5.2.3a). Note, however, that the Au anomalies are more prominent than the Cu anomalies in the southeastern part of the area.

Similar results were obtained for stream sediment data. Fig. (5.2.4a) shows concentration-area relations for Au values in 698 stream sediment samples. Two straight lines were fitted to the values from which a threshold of Au equal to 30ppb was estimated and used as regional threshold to separate anomalies from background (Fig.5.2.4b). Fig.3.2.4 shows that the result in Fig.5.2.4b is consistent with the result obtained using the spatial statistical method of Chapter 2.

It may be concluded that the log-log plots for the element concentration-area relation provide an excellent method for separating anomalies from background in the study area. This empirical result will be explained by fractal modelling in the next section.

5.3 FRACTAL MODEL FOR GEOCHEMICAL ANOMALY SEPARATION

Rose et al. (1979, p. 34-35) discussed the concept of geochemical anomaly in the following terms. A threshold is the concentration of an indicator element above which a measurement is considered anomalous. In the simplest situation, it is equal to the upper limit of normal background fluctuations: any higher values are anomalies, and lower values constitute background. However, in more complex situations, two or more types of threshold values may be recognized. For example, the anomalies related to ore may be set in a background of higher than normal values, representing a geochemical relief consisting of (1) a low-lying plain of regional background, separated by a regional threshold from (2) a plateau of higher values related to extensive feeble mineralization or dispersion, from which rise (3) the anomalies most closely related to ore, defined by a local threshold (Rose et al., 1979, Fig. 2.8). These authors made a clear distinction between regional and local thresholds.

The thresholds for Au (200 ppb), Cu (400 ppm) and many other elements (Table 5.2.1) are local thresholds and much larger than the regional thresholds for these elements. For example, the regional threshold for Au is about 30 ppb instead of 200 ppb. In terms of regional thresholds nearly the entire Mitchell-Sulphurets area is anomalous. The local thresholds are helpful in the delineation of potential ore bodies.

The background itself can be considered to be the result of extensive feeble

mineralization. Fractals and multifractals share the property of self-similarity in that, on the average, the spatial variability of the feature under study is independent of scale. Fractal models commonly result in power-law relations between the variables of interest. Such relations plot as straight lines on log-log paper. The concentration-area relations of most elements of interest in the Mitchell-Sulphurets area do not satisfy the simple fractal model because different power-laws apply to background and anomalies. It can, however, be assumed that the spatial variability of the elements of interest in the Mitchell-Sulphurets area is multifractal instead of fractal.

A bifractal model implying independent fractal models for background and anomalies does not provide a suitable explanation, because such a model would mean different types of self-similarity within background and anomalies, respectively. It will be shown in this section that the multifractal model provides an excellent explanation for all empirical results described in the previous section. The concentration-area plots resulting from a multifractal model are dominated by two separate power-law relations, thus creating a bifractal appearance. In reality, background and anomalies share the same type of self-similarity.

Concentration-area model

A contour map provides a smoothed version of the spatial distribution of an element. In the previous section contour maps were used to obtain approximate relations between

areas $A(\rho)$ and concentration values ρ , with $A(\rho)$ decreasing for increasing ρ . Conversely, the area with concentration values less than ρ (to be written as $A(T)-A(\rho)$ where $A(T)$ is total area) is an increasing function of ρ . In Appendix A it is shown that if the element concentration per unit area satisfies a fractal or multifractal model, then the area $A(\rho)$ has indeed a power-law type relation with ρ . When the concentration per unit area follows a fractal model, this power-law relation has only one exponent. On the other hand, when the concentration per unit area satisfies a multifractal model with a spectrum of fractal dimensions, then several separate power-law relations between area $A(\rho)$ and ρ can be established. For a range of ρ close to its minimum value $\rho_{\min}$, the predicted multifractal power-law relations are:

$$A(\rho) = C_1 \rho^{-\alpha_1} ; \quad A(T) - A(\rho) = C \rho^\beta \quad (5.3.1)$$

where C_1 and C are constants and α_1 and β are exponents associated with the maximum singularity exponent. For a range of ρ close to its maximum value $\rho_{\max}$, the predicted power-law relation is:

$$A(\rho) = C_2 \rho^{-\alpha_2} \quad (5.3.2)$$

where C_2 is another constant and α_2 is the exponent associated with the minimum singularity exponent (see section 5.4).

The results in Figures 5.2.3, 5.2.4 and 5.2.6b and Table 5.2.1 show that for most elements of interest, the area $A(\rho)$ as determined from contour maps generally has approximately two separate power-law relations with ρ , over restricted ranges of ρ . Figure 5.4.1 shows

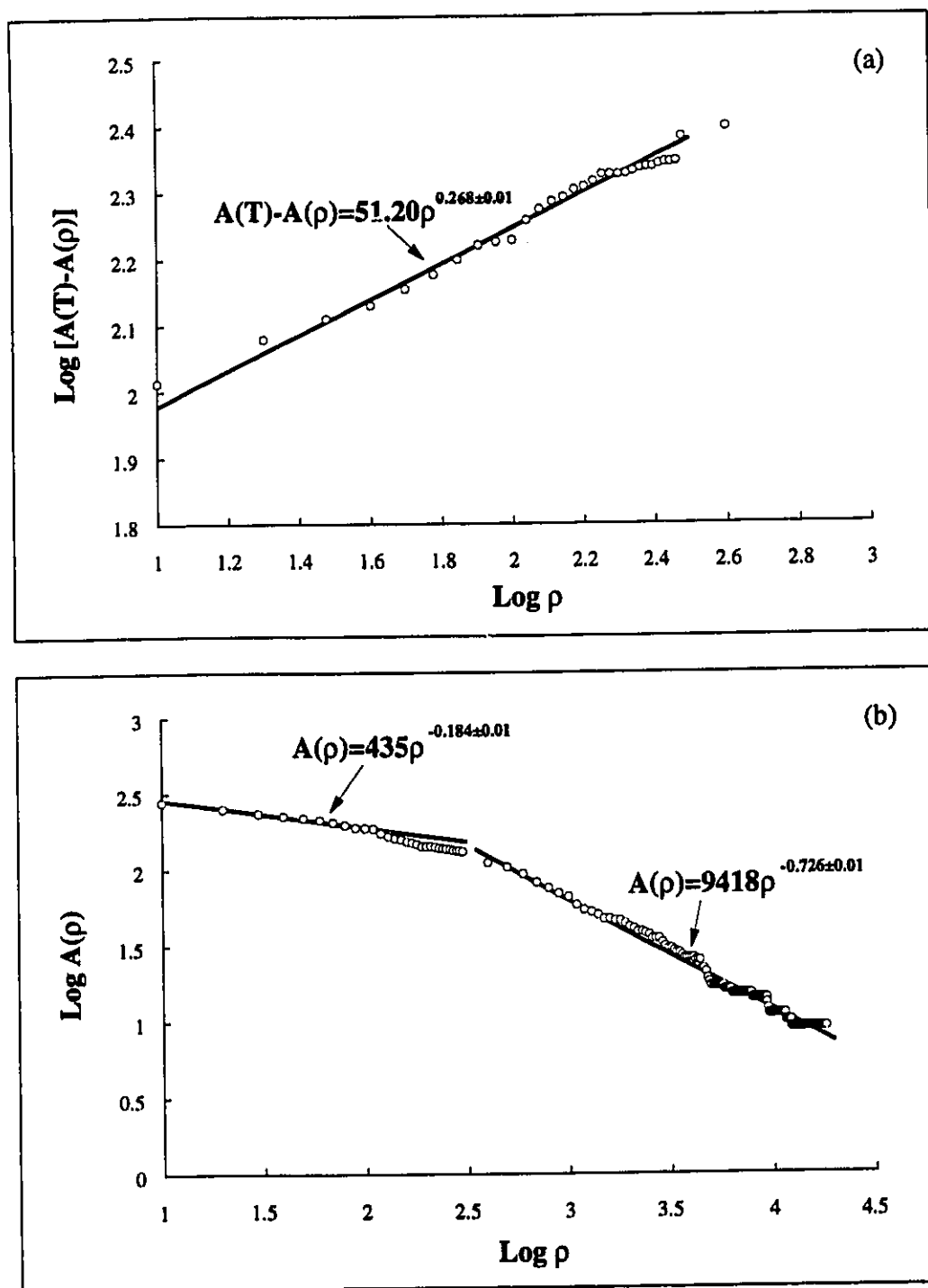


Fig. 5.4.1 Log-log plots for Au representing relationships between areas and concentration values expressed by means of Eqs. (5.3.1) and (5.3.2). The areas were determined by means of box-counting method with a yardstick of 90 meters. Solid lines obtained by LS fitting. (a) Relationship between $A(T)-A(\rho)$ and ρ , representing second relation of Eq. (5.3.1); (b) relationship between $A(\rho)$ and ρ , representing the first relation in Eqs. (5.3.1) and (5.3.2), respectively.

the relationships between $A(\rho)$ and ρ on log-log plots for Au which are of a different type. The areas used for Figure 5.4.1 were obtained by the box-counting of original element concentration values instead of using contours based on interpolation. Box-counting consists of superimposing a grid with cells on the study region. Square cells measuring 90m on a side were used. Each area $A(\rho)$ was set equal to the number of cells (multiplied by cell area) with concentration values greater than ρ . (average concentration values were used for boxes containing more than a single sample). Figure 5.4.1a shows the relation between $A(T)-A(\rho)$ and ρ with Au concentration values ranging from 2 to 200 ppb represented by the second power-law relation of Eq.(5.3.1) with exponent $\beta=0.268\pm0.01$ and constant $C=51.2\pm2.4$. Figure 5.4.1b shows the relation between $A(\rho)$ and ρ with concentration values both greater and less than 200 ppb. The results are equivalent to the power-law relations of Eq. (5.3.1) with $\alpha_1=0.184\pm0.01$, $C_1=435\pm16$ and $\alpha_2=0.726\pm0.01$, $C_2=9418\pm567.8$, respectively. These estimates of the exponents ($\alpha_1=0.184$; $\alpha_2=0.726$) for Au differ from those obtained from the contours of Figures 5.2.2 and 5.2.3 ($\alpha_1=0.137$; $\alpha_2=0.879$; cf. Fig. 5.2.4a, Table 5.2.1). The discrepancies are small and can be ascribed to the fact that entirely different techniques for interpolation and extrapolation of the gold concentration values were used.

5.4 MULTIFRACTAL MODEL AND CONCENTRATION-AREA MODEL

Gold in the Mitchell-Sulphurets mineral district has high concentration values in very small areas, relatively low values in a much larger area, and very low concentration

values in all other rocks. Situations of this type can be analyzed by means of multifractal models (Chapter 4). Relationships between area and element concentration value (amount of metal per unit area) can be derived from the multifractal model as follows.

Eqs. (4.3.5) to (4.3.8) show that the multifractal spectrum $f(\alpha)$ is usually a continuous function with domain from $\alpha_{\min}$ to $\alpha_{\max}$. However, if $\tau(q)$ is a linear function of q , or if α is constant ($\alpha = \alpha_{\min} = \alpha_{\max}$), $f(\alpha)$ is reduced to a constant value as well. Then the multifractal becomes a fractal with single dimension $f(\alpha)$ and singularity exponent α . This situation arises naturally for distributions which are not very irregular, and may also arise artificially when the measures are formed by smoothing data during contouring. Two cases will be considered separately in the following discussion: (1) the measure satisfies the fractal model, and (2) the measure satisfies the multifractal model.

(1). Suppose that the measure for a single set satisfies the fractal model with dimension $f(\alpha)$ and singularity α , both of which are constant. For small cell size ϵ , the estimated area $A(\epsilon)$ and concentration value $\rho(\epsilon)$ then can be expressed as:

$$A(\epsilon) \propto \epsilon^{-f(\alpha)+2} \quad (5.4.1)$$

$$\rho(\epsilon) \propto \epsilon^{\alpha-D} \quad (5.4.2)$$

Elimination of ϵ gives:

$$A(\rho) \propto \rho^{\frac{D-f(\alpha)}{\alpha-D}} \quad (5.4.3)$$

where it is assumed that $0 \leq f(\alpha) \leq D$ and $\alpha \neq D$. For $\alpha=D$, ρ becomes a constant which is independent of $A(\rho)$. Eq. (5.4.3) represents a power-law relation between area $A(\rho)$ and ρ with a negative exponent that is associated with the fractal dimension $f(\alpha)$ and α .

(2) Suppose the measure satisfies a multifractal model with continuous dimension spectrum $f(\alpha)$. Two special kinds (a and b) of situations will be considered: (a) α relatively close to the minimum value $\alpha_{\min}$, and (b) α relatively close to the maximum value $\alpha_{\max}$.

The area for subsets with singularity $\alpha_{\min}^+$ close to $\alpha_{\min}$ can be expressed as:

$$A(\alpha < \alpha_{\min}^+) = \int_{\alpha_{\min}}^{\alpha_{\min}^+} C e^{-f(\alpha) + D} d\alpha \quad (5.4.4)$$

where C is a constant. From Chapter 4 we know that in most cases $f(\alpha)$ is a finite function with $q_{\min} \leq f(\alpha) \leq q_{\max}$. It was also shown in Chapter 4 that $q_{\min}$ and $q_{\max}$ were not large for the Au concentration values (≈ 4). Therefore, for $\alpha_{\min}^+$ close to $\alpha_{\min}$, we have approximately:

$$A(\alpha < \alpha_{\min}^+) \approx C e^{-f(\alpha_{\min}) + D} \int_{\alpha_{\min}}^{\alpha_{\min}^+} e^{-f'(\alpha_{\min})(\alpha - \alpha_{\min})} d\alpha$$

$$\begin{aligned}
& \approx \frac{C e^2}{f'(\alpha_{\min}) \log e} [e^{-f(\alpha_{\min}^+)} - e^{-f(\alpha_{\min})}] \\
& \approx C e^{-f(\alpha_{\min})+2} (\alpha_{\min}^+ - \alpha_{\min})
\end{aligned} \tag{5.4.5}$$

Log-transformation of both sides, gives:

$$\begin{aligned}
& \log A(\alpha < \alpha_{\min}^+) \\
& \approx \log [C e^{-f(\alpha_{\min})+2} (D - \alpha_{\min})] + \log [1 - \frac{D - \alpha_{\min}^+}{D - \alpha_{\min}}]
\end{aligned} \tag{5.4.6}$$

By approximation $\log(1+x) \approx x$ if $|x| \ll 1$,

$$\begin{aligned}
\log A(\alpha < \alpha_{\min}^+) & \approx \log [C e^{D-f(\alpha_{\min})} (D - \alpha_{\min})] - \frac{D - \alpha_{\min}^+}{D - \alpha_{\min}} \\
& \approx \log [C e^{D-f(\alpha_{\min})} (D - \alpha_{\min})] - \frac{\log \frac{\rho}{C_1}}{\log \frac{\rho_{\max}}{C_1}}
\end{aligned} \tag{5.4.7}$$

where C_1 is a constant satisfying the relation $\rho_{\max} = C_1 e^{\alpha_{\min}-D}$. If we set $f(\alpha_{\min})=0$, then Eq.(5.4.7) becomes

$$\log A(\alpha < \alpha_{\min}^+) \approx \log [C e^D (D - \alpha_{\min})] - \frac{\log \frac{\rho}{C_1}}{\log \frac{\rho_{\max}}{C_1}} \tag{5.4.8}$$

which is the same as Eq. (5.3.2) with $\alpha_2 = 1/\log(\rho_{\max}/D)$.

Similarly, an approximate relationship between area of subsets with α close to $\alpha_{\max}$

(written as $\alpha_{\max}^-$) and concentration ρ is:

$$A(\rho) = \int_{\alpha_{\max}^-}^{\alpha_{\max}} C e^{-f(\alpha)+D} d\alpha \quad (5.4.9)$$

$$A(\rho) \approx A(T) - C e^{-f(\alpha_{\max})+D} [\alpha_{\max} - \alpha_{\max}^-] \quad (5.4.10)$$

From Eq (5.4.10) it follows that

$$\begin{aligned} \log A(\rho) &\approx \\ &\approx \log A(T) - \frac{C}{A(T)} e^{-f(\alpha_{\max})+D} [\alpha_{\max} - D + \frac{\log \frac{\rho}{C_1}}{\log \frac{1}{e}}] \end{aligned} \quad (5.4.11)$$

and

$$\begin{aligned} \log [A(T) - A(\rho)] & \\ &\approx \log (C e^{-f(\alpha_{\max})+D} \alpha_{\max}) - \frac{D}{\alpha_{\max}} + \frac{\log \frac{\rho}{C_1}}{\alpha_{\max} \log \frac{1}{e}} \end{aligned} \quad (4.4.12)$$

Eqs. (5.4.11) and (5.4.12) correspond to the two relations of Eq.(5.3.1) with the exponent $\alpha_1 = [C e^{-f(\alpha_{\max})+D}] / [A(T) \log(1/e)]$ and $\beta_1 = 1 / [\alpha_{\max} \log(1/e)]$, respectively.

5.5 COMPARISON WITH WEIGHTS OF EVIDENCE PROCEDURE

The relationship of alteration zones and distribution of trace and major elements can be

quantitatively studied by using the weights of evidence method previously used for mineral potential mapping and in deriving weights of evidence from geoscience contour maps for the prediction of discrete events (Agterberg, 1989a, 1989b, 1992; Agterberg and Bonham-Carter, 1990; Bonham-Carter et al., 1988). In this method, the contrast $C = W^+ - W^-$ is a measure of correlation between points and patterns, and the corresponding t-value ($C/s(C)$) provides a measure of its statistical significance. The weights W^+ and W^- are calculated by means of the following formulae:

$$W^+ = \log \frac{P[A(v) | D]}{P[A(v) | \bar{D}]} ; \quad W^- = \log \frac{P[\bar{A}(v) | D]}{P[\bar{A}(v) | \bar{D}]} \quad (5.5.1)$$

where $A(v)$ is the area enclosed by the contour with concentration value v , and $\bar{A} = A(T) - A(v)$; D and $\bar{D}$ indicate presence and absence of points, respectively. The latter usually represent separate mineral deposits. For combining two binary maps, similar relations for the contrast C and t-value can be expressed as follows:

$$C = \log \left(\frac{T_{11}T_{22}}{T_{21}T_{12}} \right) ; \quad t = \frac{C}{\sqrt{\left(\frac{1}{T_{11}} + \frac{1}{T_{22}} + \frac{1}{T_{12}} + \frac{1}{T_{21}} \right)}} \quad (5.5.2)$$

where T_{11} , T_{12} , T_{21} and T_{22} indicate areas of overlap between the two patterns.

The largest t-values in Figure 5.5.1 indicate that 200 ppb Au and 400ppm Cu are optimum thresholds for separating the contour map into two parts forming the binary patterns which are most strongly correlated with the alteration pattern (Fig.5.2.5). These thresholds are the same as obtained previously by using the element concentration-area

method. In Table 5.3.1 it is shown that nearly all thresholds for elements and oxides resulting from application of weights of evidence duplicate those extracted from the log-log plots for the element concentration-area relations.

5.6 CONCLUSION

An element concentration-area method is proposed based on a multifractal model for separating geochemical anomaly from background by means of a threshold. The method is applicable to elements for which the spatial pattern of concentration values satisfies a multifractal model (also see Section 5.4). The method has been successfully applied to Au, Cu and other Au-associated elements in bedrock samples from the Mitchell-Sulphurets district as well as to Au in stream sediment samples from the Iskut River map sheet.

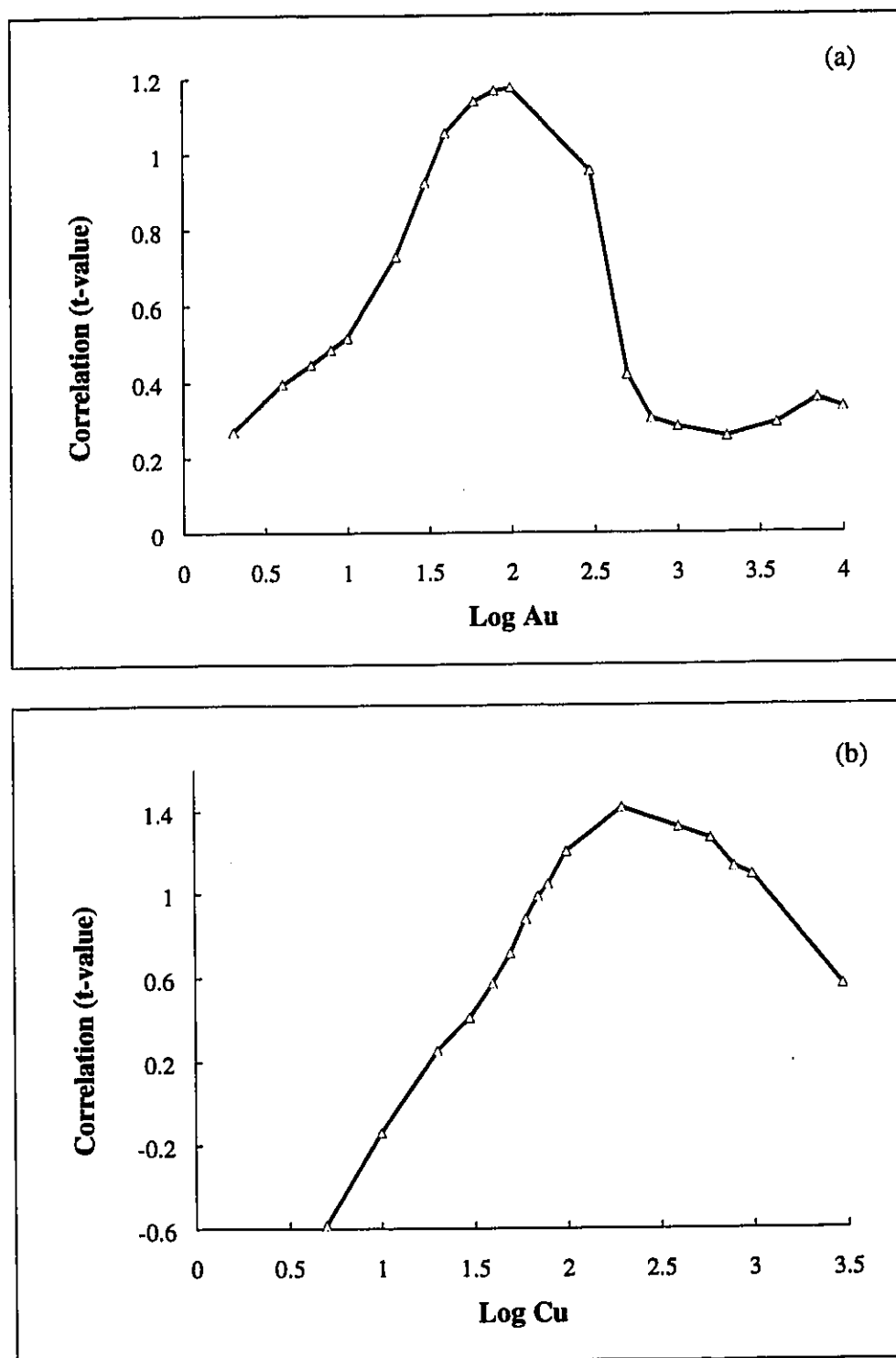


Fig. 5.5.1 Plots showing t-values obtained by weights of evidence method. (a) Au and (b) Cu.

Chapter 6 MULTIFRACTAL MODELLING AND SPATIAL STATISTICS

6.1 INTRODUCTION

Spatial autocorrelation and variograms are useful tools in spatial analysis and geostatistics. Links between fractal/multifractal and geostatistics (correlation and variogram) have been discussed by several investigators: Mandelbrot and Van Ness (1968) showed that the expected value $E\{X_i - X_{i+h}\}^2$ at elevation X_i at point i along a sampling line across the fractal landscape is proportional to h^{2H} , where H is called the Hurst exponent and h represents distance (or "lag"). This model has been used by many others; e.g., Feder (1988, p.171). Meneveau and Sreenivasan (1987) proposed a model $\langle \epsilon_r^2 \rangle \propto \epsilon_0^2 (r/L)^\mu$ in terms of multifractals; Rose (1992) derived a model with power-law type variance and autocorrelation; Agterberg (1994a) reviewed some of the previous work from the two different fields of geostatistics and fractal geometry and suggested that fractal and multifractal models are useful for analysis of self-similar sets and random variables in space and that such models can result in exponential and power-law type spatial covariance models; Cheng and Agterberg (1994) derived an approximate power-law type autocorrelation and covariance, and a logarithmic type of semivariogram from the multifractal model. These results will be introduced and applied to case studies in this Chapter.

It was shown in Chapters 4 and 5 that the multifractal model can be used for

characterizing the Au distribution in a mineral district where the Au values are of high variability and singularity. In this Chapter, theoretical expressions will be derived for autocorrelation, covariance and semivariogram of a random variable with fractal or multifractal properties. These newly derived results will be applied to two datasets: (1) Cochran's example of number of tree seedlings per foot along a 200 ft long bed (data from Cochran, 1963, also see Rose, 1992), and (2) De Wijs's example of zinc values from a sphalerite-quartz vein near Pulacayo in Bolivia (data from De Wijs, 1951; also see Agterberg, 1994a). An advantage of using these relatively well known examples in the two case-history studies is that the results of multifractal modelling can be compared to the published results obtained by means of other statistical techniques.

6.2 AUTOCORRELATION AND SEMIVARIOGRAMS OF RANDOM VARIABLES IN $\mathbf{R}^1$

Suppose that $\xi_i(\epsilon) = \epsilon^{-1} \mu_i(\epsilon)$ with $i=1, \dots, N(\epsilon)$, is a random variable on the interval $(0, L]$. For example, it may represent number of tree seedlings in a one-foot segment from the 200 ft bed used by Cochran. In order to study the distribution and spatial characteristics of $\xi_i(\epsilon)$, a series of samples can be collected. For simplification we assume that these samples are regularly located within $(0, L]$ in that adjoining samples of equal size are taken at a regular interval. Consequently, the interval $(0, L]$ is subdivided into $N(\epsilon)=L/\epsilon$ cells, each of size ϵ , and the average value $\xi_i(k\epsilon)$ of a cell of size $k\epsilon$ can be expressed as

$$\xi_i(k\epsilon) = \frac{1}{k} \sum_{j=(i-1)k+1}^{ik} \xi_j(\epsilon) \quad (6.2.1)$$

with $i=1, \dots, N_k$, where $N(\epsilon)/k-1 < N_k \leq N(\epsilon)/k$ is the total number of cells of size ϵ^k . Variance (σ^2), covariance (cov), autocorrelation (ρ) and semivariogram (γ) of the random variable $\xi_j(\epsilon)$ can be expressed as

$$\sigma^2(\epsilon) = \frac{1}{N(\epsilon)} \sum_{j=1}^{N(\epsilon)} [\xi_j(\epsilon) - \bar{\xi}(\epsilon)]^2 \quad (6.2.2)$$

$$\begin{aligned} \text{cov}(h) &= \sigma_h(\epsilon) \\ &= \frac{1}{N(\epsilon)-h} \sum_{j=1}^{N(\epsilon)-h} [\xi_j(\epsilon) - \bar{\xi}(\epsilon)] [\xi_{j+h}(\epsilon) - \bar{\xi}(\epsilon)] \end{aligned} \quad (6.2.3)$$

$$\rho_h(\epsilon) = \frac{\text{cov}(h)}{\sigma^2(\epsilon)} \quad (6.2.4)$$

$$\gamma_h(\epsilon) = \frac{1}{2[N(\epsilon)-h]} \sum_{j=1}^{N(\epsilon)-h} [\xi_j(\epsilon) - \xi_{j+h}(\epsilon)]^2 \quad (6.2.5)$$

where $\bar{\xi}(\epsilon)$ represents the average value of $\xi_j(\epsilon)$ in $(0, L]$ with $j=1, \dots, N(\epsilon)$, and h is lag. Suppose, as before, that the measure $\mu_j(\epsilon)$ either has the multifractal property and satisfies Eqs. (4.3.6) and (4.3.8), or that its second-order moment has fractal properties so that Eq. (4.3.6) holds true for $q=2$ if modelling is restricted the spatial statistics of Eqs.(6.2.1) to (6.2.5). Setting $q=1$ in Eq. (4.3.6) then gives

$$E[\xi_j(\epsilon)] = \frac{1}{N(\epsilon)} \sum_{j=1}^{N(\epsilon)} \xi_j(\epsilon) = \frac{1}{L} \sum_{j=1}^{N(\epsilon)} \mu_j(\epsilon) = c_1 \epsilon^{\tau(1)} \quad (6.2.6)$$

were c_1 is a constant. In general, $\tau(1)=0$. Assuming that $\xi_j = \xi_j(\epsilon) = E\{\xi_j\}$, $j=1, \dots, N(\epsilon)$, is the expected value of $\xi_j(\epsilon)$ in $(0, L]$ gives $c_1 = \xi$. Setting $q=2$ in Eq. (4.3.6),

$$\begin{aligned} E[\xi_j(\epsilon)]^2 &= \frac{1}{N(\epsilon)} \sum_{j=1}^{N(\epsilon)} [\xi_j(\epsilon)]^2 \\ &= \frac{1}{\epsilon L} \sum_{j=1}^{N(\epsilon)} [\mu_j(\epsilon)]^2 = c_2 \epsilon^{\tau(2)-1} \end{aligned} \quad (6.2.7)$$

where c_2 is another constant. From Eqs. (6.2.6) and (6.2.7),

$$\sigma^2[\xi_i(\epsilon)] = c_2 \epsilon^{\tau(2)-1} - \xi^2 \quad (6.2.8)$$

Consequently,

$$\sigma^2[\xi_i(k\epsilon)] = \sigma^2\left[\frac{1}{k} \sum_{j=(i-1)k}^{ik} \xi_j(\epsilon)\right] = c_2 (k\epsilon)^{\tau(2)-1} - \xi^2 \quad (6.2.9)$$

Combining Eq. (6.2.9) with the following equality (e.g., see Bartlett, 1966, p. 284)

$$\sigma^2\left[\frac{1}{k} \sum_{i=1}^k \xi_i(\epsilon)\right] = \frac{\sigma^2(\epsilon)}{k} \left[1 + 2 \sum_{s=1}^{k-1} \left(1 - \frac{s}{k}\right) \rho_s(\epsilon)\right] \quad (6.2.10)$$

it follows that

$$k^2 [c_2 (k\epsilon)^{\tau(2)-1} - \xi^2] = \sigma^2(\epsilon) \left[k + 2 \sum_{s=1}^{k-1} (k-s) \rho_s(\epsilon)\right] \quad (6.2.11)$$

and, by elementary methods,

$$\begin{aligned} \rho_k(\epsilon) &= \frac{c_2 \epsilon^{\tau(2)-1}}{2\sigma^2(\epsilon)} \\ &= \frac{[(k+1)^{\tau(2)+1} - 2k^{\tau(2)+1} + (k-1)^{\tau(2)+1}] - \xi^2}{2\sigma^2(\epsilon)} \end{aligned} \quad (6.2.12)$$

For small ϵ or if $\xi \ll 1$, the last term in Eq. (6.2.12) can be neglected, and

$$\rho_k(\epsilon) = \frac{1}{2} [(k+1)^{\tau(2)+1} - 2k^{\tau(2)+1} + (k-1)^{\tau(2)+1}] \quad (6.2.13)$$

Similarly,

$$\begin{aligned} \text{cov}(k) &= \\ \frac{1}{2} C_2 \epsilon^{\tau(2)-1} [(k+1)^{1+\tau(2)} - 2k^{1+\tau(2)} + (k-1)^{1+\tau(2)}] - \xi^2 \end{aligned} \quad (6.2.14)$$

and

$$\begin{aligned} \gamma_k(\epsilon) &= \\ C_2 \epsilon^{\tau(2)-1} [1 - \frac{1}{2} \{(k+1)^{1+\tau(2)} - 2k^{1+\tau(2)} + (k-1)^{1+\tau(2)}\}] \end{aligned} \quad (6.2.15)$$

The semivariogram has sill $\gamma_\infty(\epsilon) = C_2 \epsilon^{\tau(2)-1}$ which depends on the value of cell size ϵ . As cell size decreases to zero, the sill increases infinitely. The covariance (cov) also depends on cell size but the autocorrelation function is relatively independent of cell size. Replacement of the second-order difference in Eq. (6.2.13) by the second derivative yields

$$\rho_k(\epsilon) = \frac{1}{2} \{\tau(2) + 1\} \tau(2) k^{\tau(2)-1} \quad (6.2.16)$$

Likewise, Eqs. (6.2.14) and (6.2.15) can be approximated by

$$\text{cov}(k) = \frac{1}{2} C_2 \epsilon^{\tau(2)-1} \{\tau(2) + 1\} \tau(2) k^{\tau(2)-1} - \xi^2 \quad (6.2.17)$$

and

$$\gamma_k(\epsilon) = C_2 \epsilon^{\tau(2)-1} [1 - \frac{1}{2} \{\tau(2) + 1\} \tau(2) k^{\tau(2)-1}] \quad (6.2.18)$$

Hence, provided that $\tau(2)$ is only slightly less than 1,

$$\gamma_k(\epsilon) \sim -C_2 \epsilon^{\tau(2)-1} \log \left[\frac{1}{2} \{ (k+1)^{1+\tau(2)} - 2k^{1+\tau(2)} + (k-1)^{1+\tau(2)} \} \right] \quad (6.2.19)$$

or

$$\gamma_k(\epsilon) \sim -C_2 \epsilon^{\tau(2)-1} \log \left[\frac{1}{2} \{ \tau(2) + 1 \} \tau(2) k^{\tau(2)-1} \right] \quad (6.2.20)$$

This last expression shows that the semivariogram may be approximately linear in $\log k$.

6.3. HURST EXPONENT H AND FAIRFIELD-SMITH'S b

Suppose that η_i represents a random variable with zero mean for length of step (labelled i) of a particle engaged in fractional Brownian motion, moving forwards or backwards along an axis (Mandelbrot, 1983; Feder, 1988, p. 166). After j successive steps η_i , the position ζ_j of this particle is

$$\zeta_j = \sum_{i=1}^j \eta_i \quad (6.3.1)$$

with semivariogram

$$E[\zeta_j - \zeta_{j+k}]^2 \propto k^{2H} \quad (6.3.2)$$

where H is the Hurst exponent with $0 \leq H \leq 1$; $H=1/2$ corresponds to the special case of Brownian motion with independent increments. As shown originally by Mandelbrot and Van Ness (1968), ζ_j can be interpreted as the elevation at a point labelled j along a

sampling line across a fractal landscape with fractal dimension $3-H$. The fractal dimension D for any profile across the landscape equals $D=2-H$. The autocorrelation ρ_k between two elevations ζ_j and ζ_{j+k} satisfies an expression similar Eq. (6.2.13) with $\tau(2)+1=2H$. Consequently, Eq. (6.3.2) is equivalent to

$$E[\zeta_j - \zeta_{j+k}]^2 = E\left[\sum_{i=j+1}^{j+k} \eta_i\right]^2 = c_2 \epsilon^{\tau(2)-1} k^{\tau(2)+1} \quad (6.3.3)$$

as follows from Eq. (6.2.7), and $H=\frac{1}{2}\{\tau(2)+1\}$. Fractional Brownian motion is called persistent if $\frac{1}{2}<H\leq 1$ or $0<\tau(2)\leq 1$, and antipersistent if $0\leq H<\frac{1}{2}$ or $-1\leq\tau(2)<0$. The limiting cases $\tau(2)=1$ and $\tau(2)=-1$ result in $\rho_k(\epsilon) = \text{constant}$, $k=1,2,\dots$

Before the development of multifractal theory, Yaglom (1966) had proposed a model with power-law type second-moment mass exponent similar to Eq. (6.2.7). A power-law type of variance

$$\sigma^2(X_k) \propto k^{2H} \quad (6.3.4)$$

for self-similar processes X_k with stationary increments satisfying Eq. (6.2.13) is discussed by Taqqu (1988). From Eq. (6.2.9), it follows that

$$\sigma^2[\xi(k\epsilon)] = \sigma^2\left[\sum_{i=1}^k \xi_i(\epsilon)\right] = c_2 k^2 (k\epsilon)^{\tau(2)-1} = k^2 \xi^2 \quad (6.3.5)$$

If $\xi=0$, this is similar to Eq. (6.3.4). In practical applications, the condition $\xi=0$ could be forced to be satisfied approximately by subtracting an estimate of the mean from the observed values. However, Eqs. (6.2.7) and (6.2.8) then would become identical, implying

that the multifractal model would not be valid for $q=2$.

In general, the autocorrelation of average values of adjoining cells of size $k\epsilon$ satisfies

$$\rho_1(k\epsilon) = \frac{2\sigma^2[\xi(2k\epsilon)]}{\sigma^2[\xi(k\epsilon)]} - 1 \quad (6.3.6)$$

In combination with Eq. (6.2.9) this yields

$$\log \{ [1 + \rho_1(k\epsilon)] / 2 \} = [1 - 2^{\tau(2)-1}] \frac{C_2(k\epsilon)^{\tau(2)-1}}{\xi^2} \quad (6.3.7)$$

An empirical variance-size relation

$$\sigma^2[\xi(k\epsilon)] = k^{-b} \sigma^2[\xi(\epsilon)] \quad (6.3.8)$$

where k is a constant was originally proposed by Fairfield-Smith (1938). Setting $b=1-\tau(2)$, Eq. (6.3.8) follows from Eq. (6.2.13) if $\xi=0$. However, for $\xi>0$, Rose (1985, 1992) derived Eq. (6.3.6) as well as Eq. (6.3.13) from a postulate on similarity of autocorrelation which can be expressed in the form

$$\rho_1(\epsilon_1) \equiv \rho_1(\epsilon_2) \quad (6.3.9)$$

where ϵ_1 and ϵ_2 are any two different small segments; e.g., $\epsilon_1=\epsilon$ and $\epsilon_2=k\epsilon$. On the other hand, according to Eq. (6.3.7) $\rho(k\epsilon)$ depends on k . It also follows from Eq. (6.3.9) that

$$\rho_1(k\epsilon) = 2^{\tau(2)-1} \quad (6.3.10)$$

This also would imply that autocorrelation between adjoining cells is independent of cell size. If $\xi>0$, Eq. (6.3.10) is approximately equal to Eq. (6.3.7) only if $\tau(2)$ is close to 1.

6.4. APPLICATION TO COCHRAN'S TREE SEEDLINGS EXAMPLE

6.4.1 Tree seedling data

The tree seedling data of Cochran (1963) are numbers of tree seedlings counted in each foot of bed in a 200 ft long bed. Figure 6.4.1 shows the original data and their frequency histogram. Figure 6.4.2 shows the square root transformed data of Fig. 6.4.1 and their frequency histogram. These two data sets were used by Rose (1992) to estimate autocorrelation function and semivariogram.

6.4.2. The multifractal spectrum of the tree seedlings

A measure $\mu_i(\epsilon) = \epsilon \xi_i(\epsilon)$, $i=1, \dots, 200$, was defined as in Eq. (6.2.1) with $\xi_i(\epsilon)$ representing the number of tree seedlings per unit of bed. Cell size ϵ was allowed to vary from 1 to 30 ft (total bed size is 200 ft). The partition function was estimated using Eq. (4.3.1) with the moment q ranging from -35 to 35. The results for original data and square root transformed data were plotted on log-log paper, respectively, (Figs. 6.4.3a and 6.4.4a). For each value of q a straight line was fitted by means of linear regression of $\log \chi(\epsilon)$ on $\log \epsilon$ (ordinary least squares method). Figure 6.4.5 shows the same type of results for $q=2$. In this case, the least squares method gave estimated slope $\hat{\tau}(2)=0.9554 \pm 0.002$ for original data and 0.9908 ± 0.002 for transformed data, results that will be used extensively in the next section during the fitting of autocorrelation and semivariogram functions to the data.

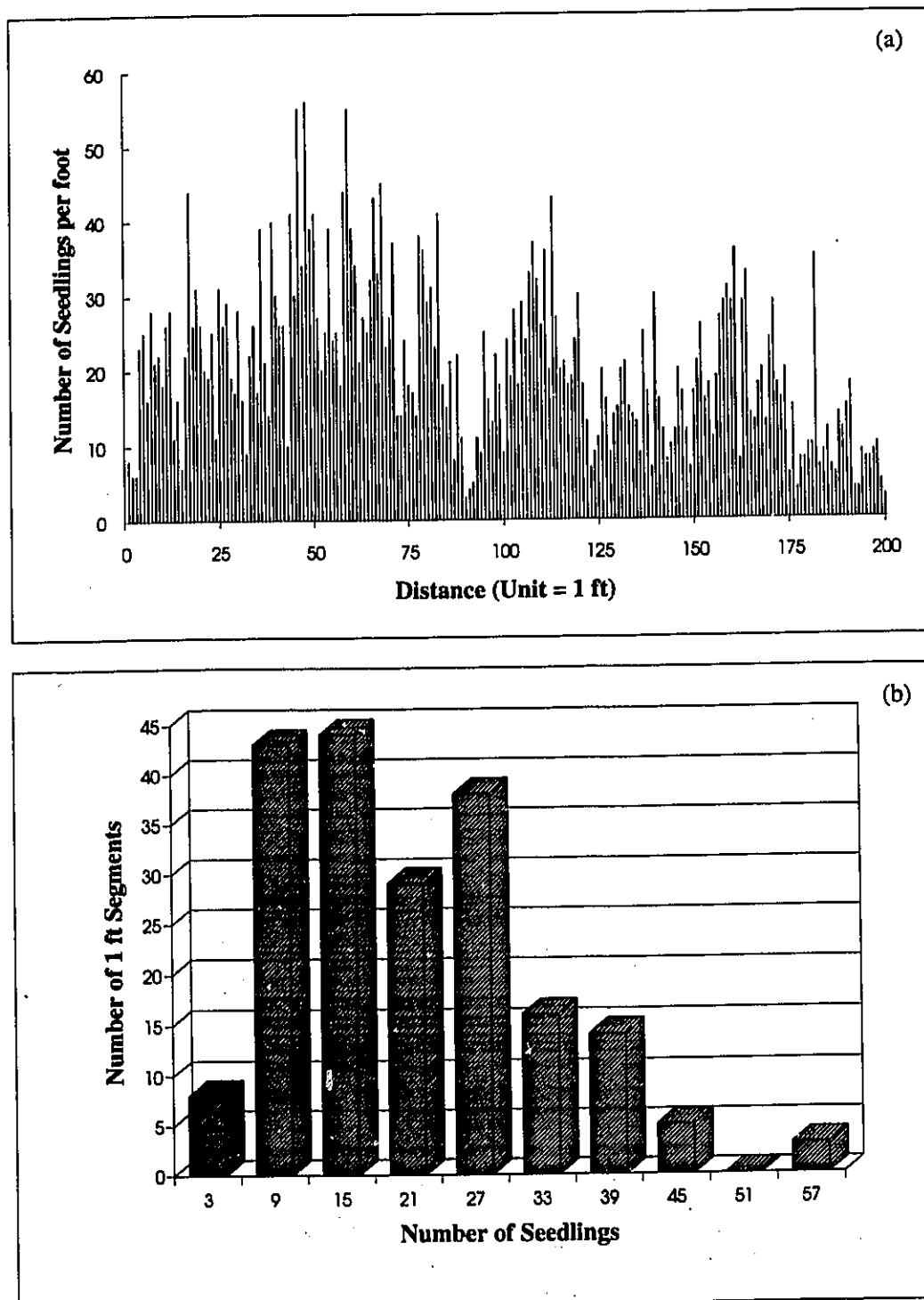


Fig. 6.4.1. Number of tree seedlings (Cochran, 1963; Rose, 1992). (a) Number of tree seedlings in one foot segments along a 200-ft bed; (b) Histogram of the number of tree seedlings.

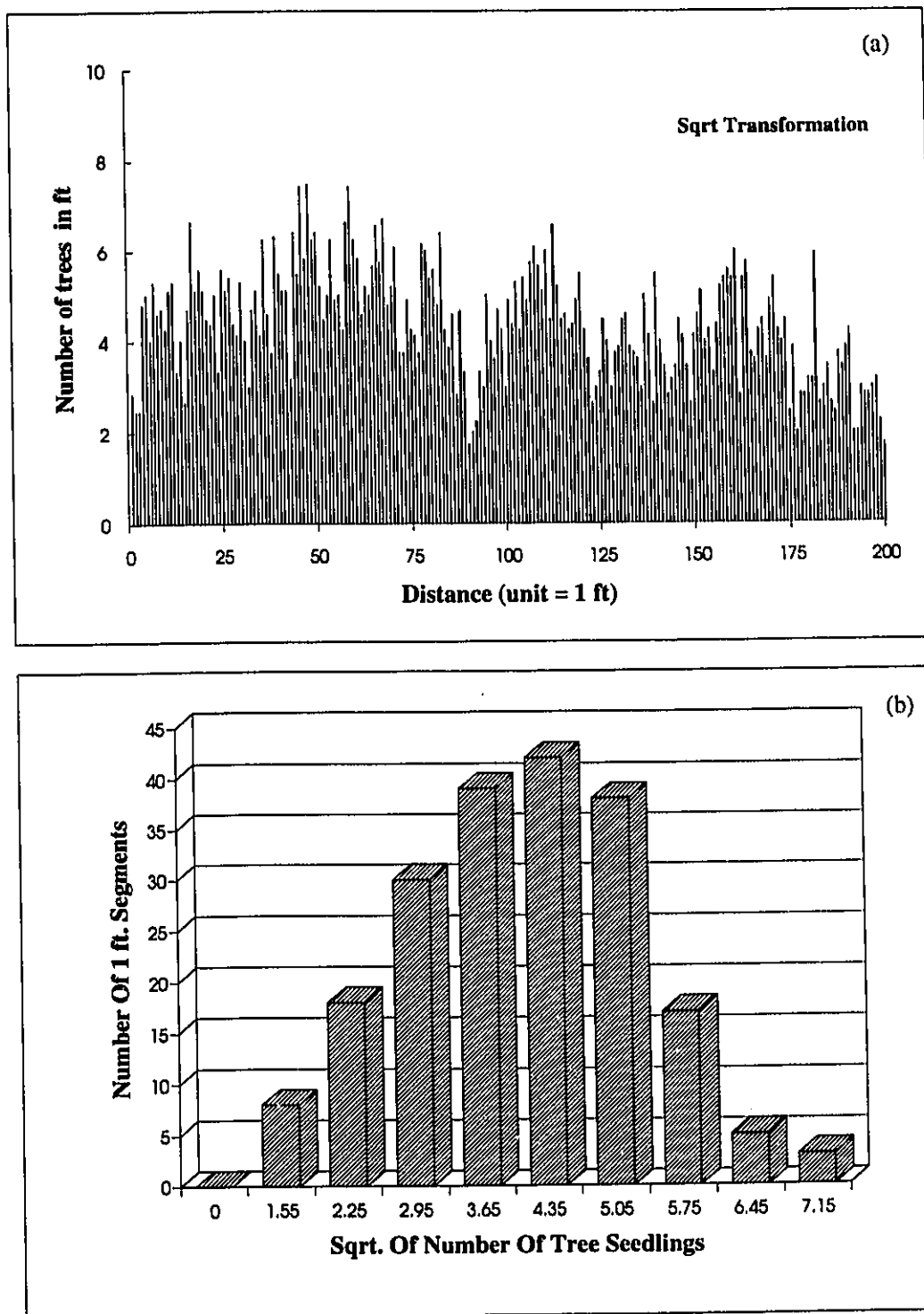
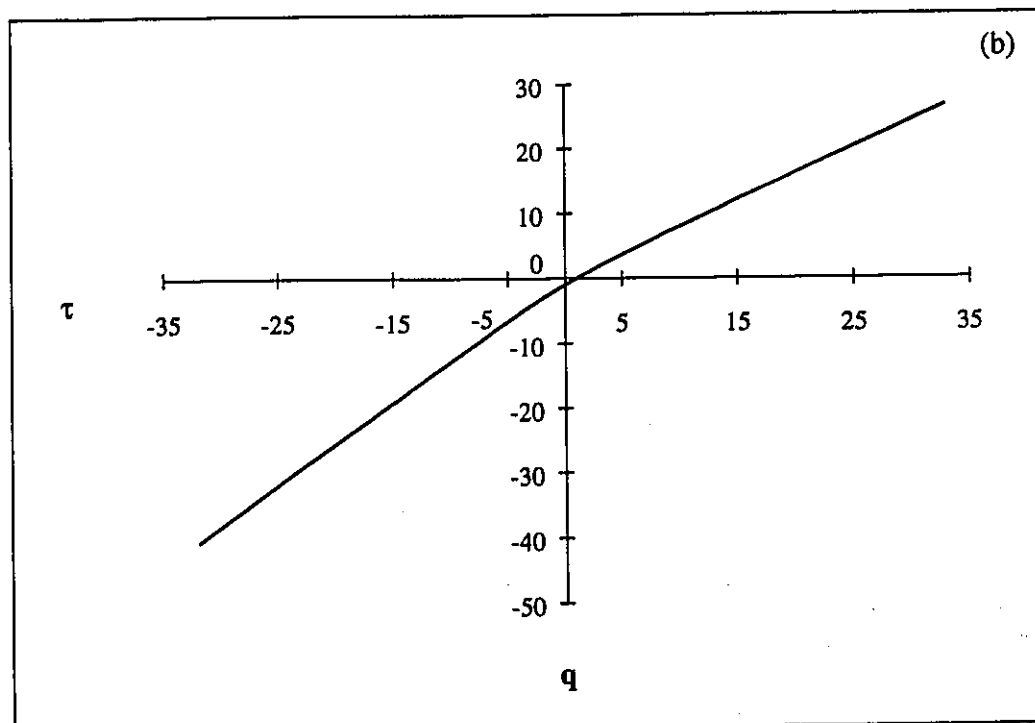
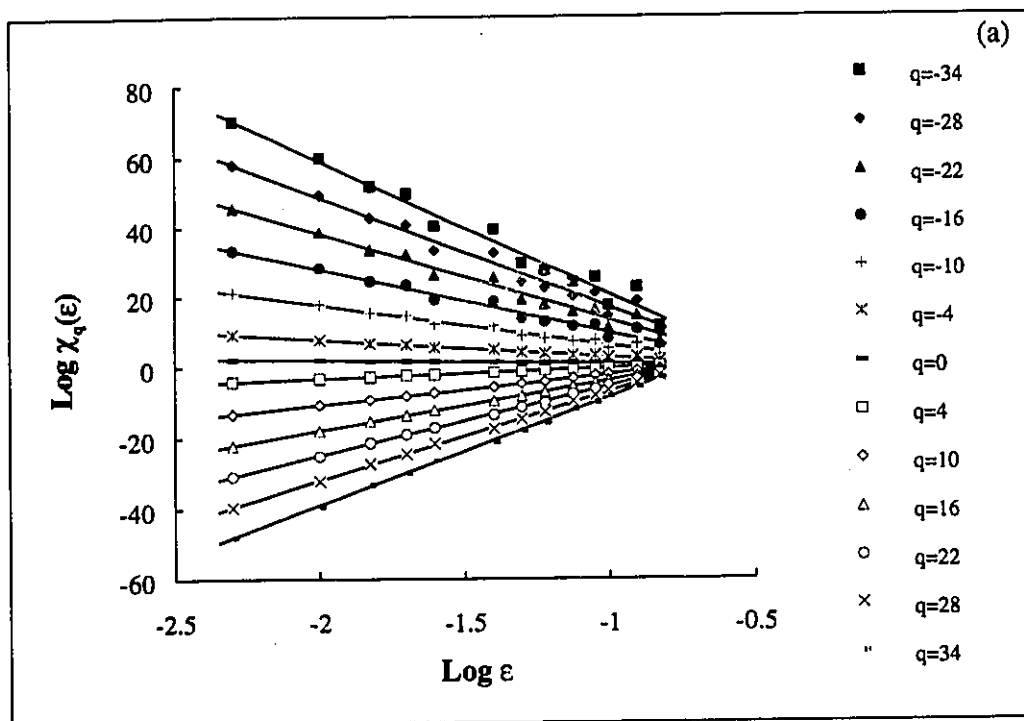


Fig. 6.4.2. Square root transformation for the number of tree seedlings in Fig. 6.4.1; (a) Square root transformed values ; (b) Histogram of the transformed values.



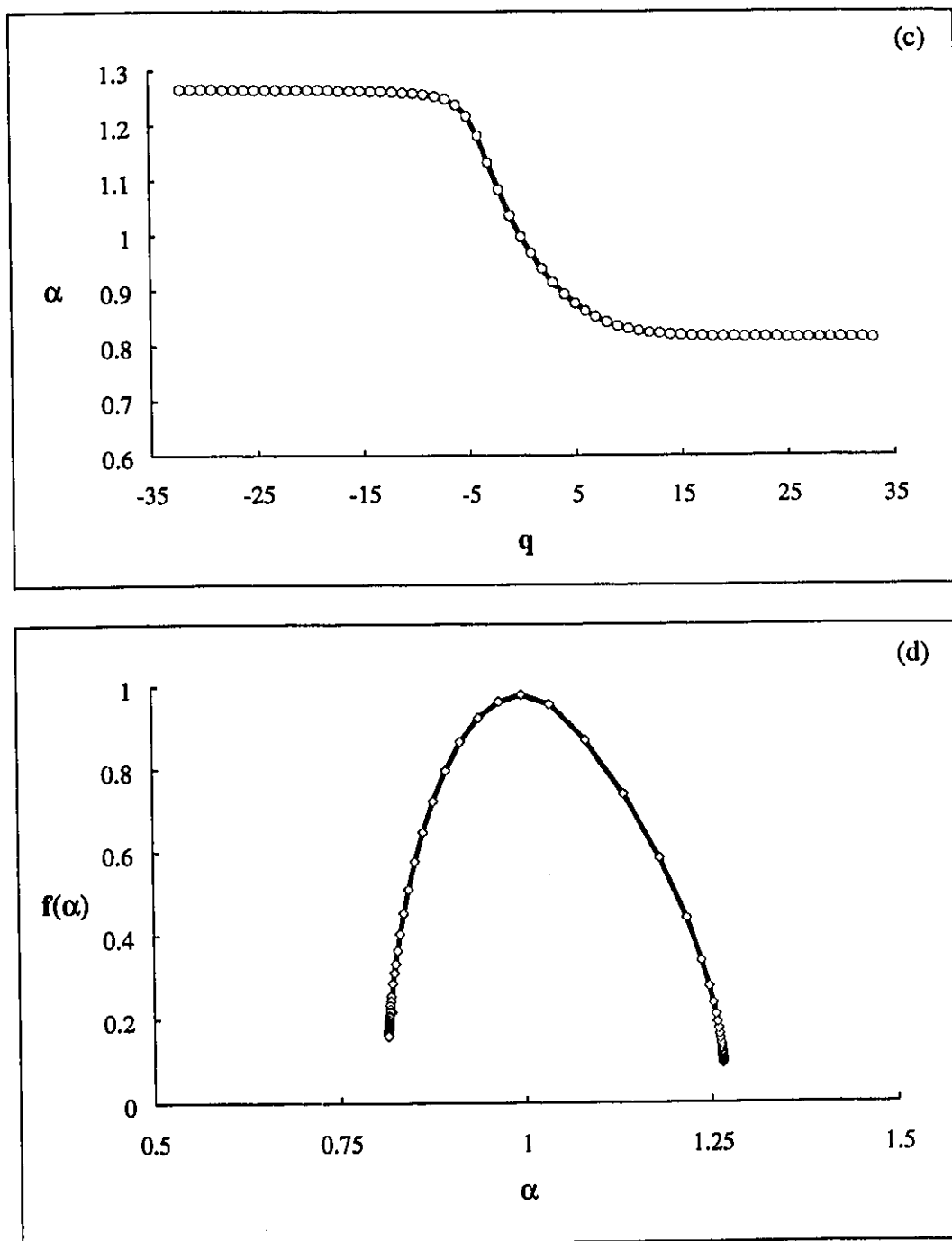
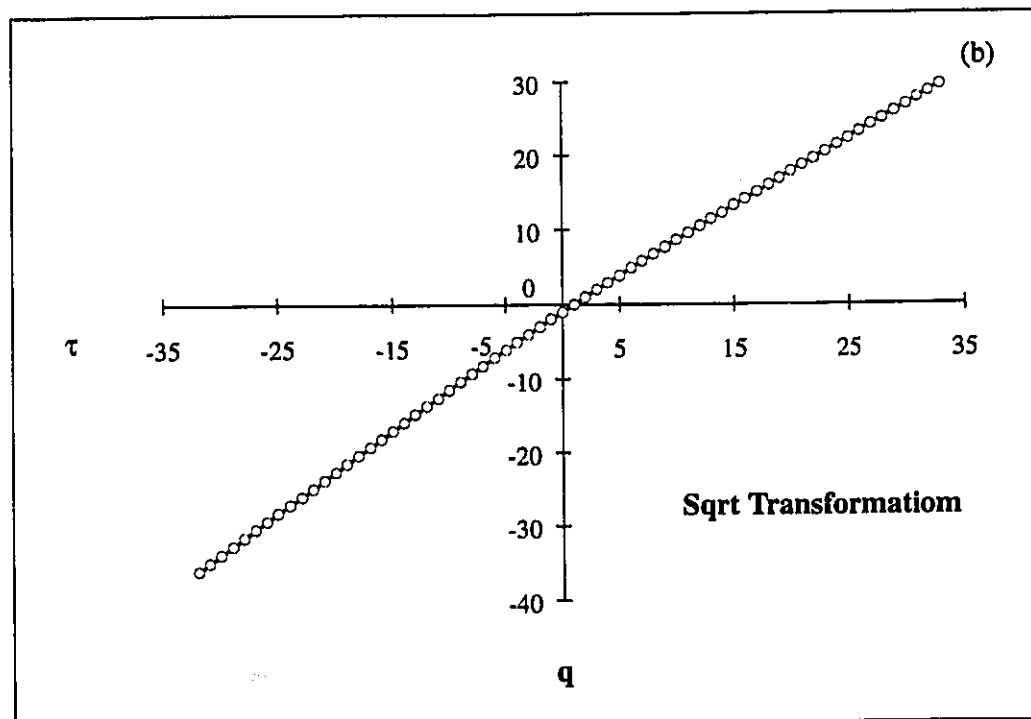
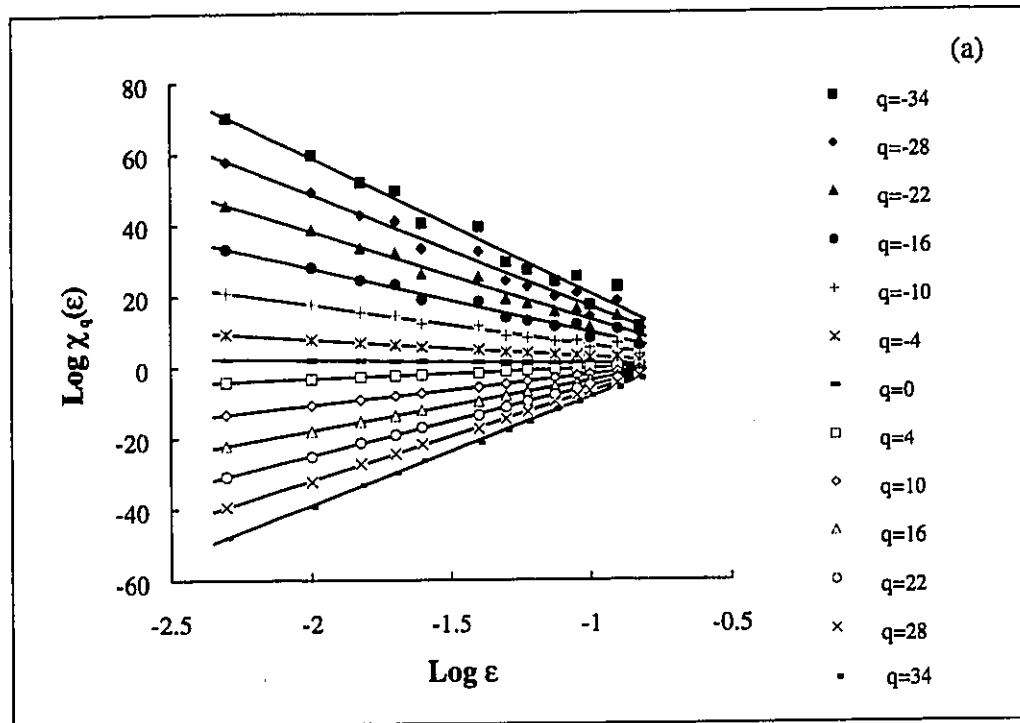


Fig. 6.4.3. Results of multifractal analysis applied to the tree seedlings in Fig. 6.4.1. (a) Log-log plot for relationship between $\chi_q(\epsilon)$ and ϵ ; straight lines obtained by means of method of least squares (LS); (b) Estimates of $\tau(q)$ are the slopes of the straight lines in (a); (c) Singularity α estimated from (b) by means of the central difference method, and (d) multifractal spectrum $f(\alpha)$. Smallest cell size $\epsilon=1/200$ corresponds to 1 ft; logarithmic scales have base 10.



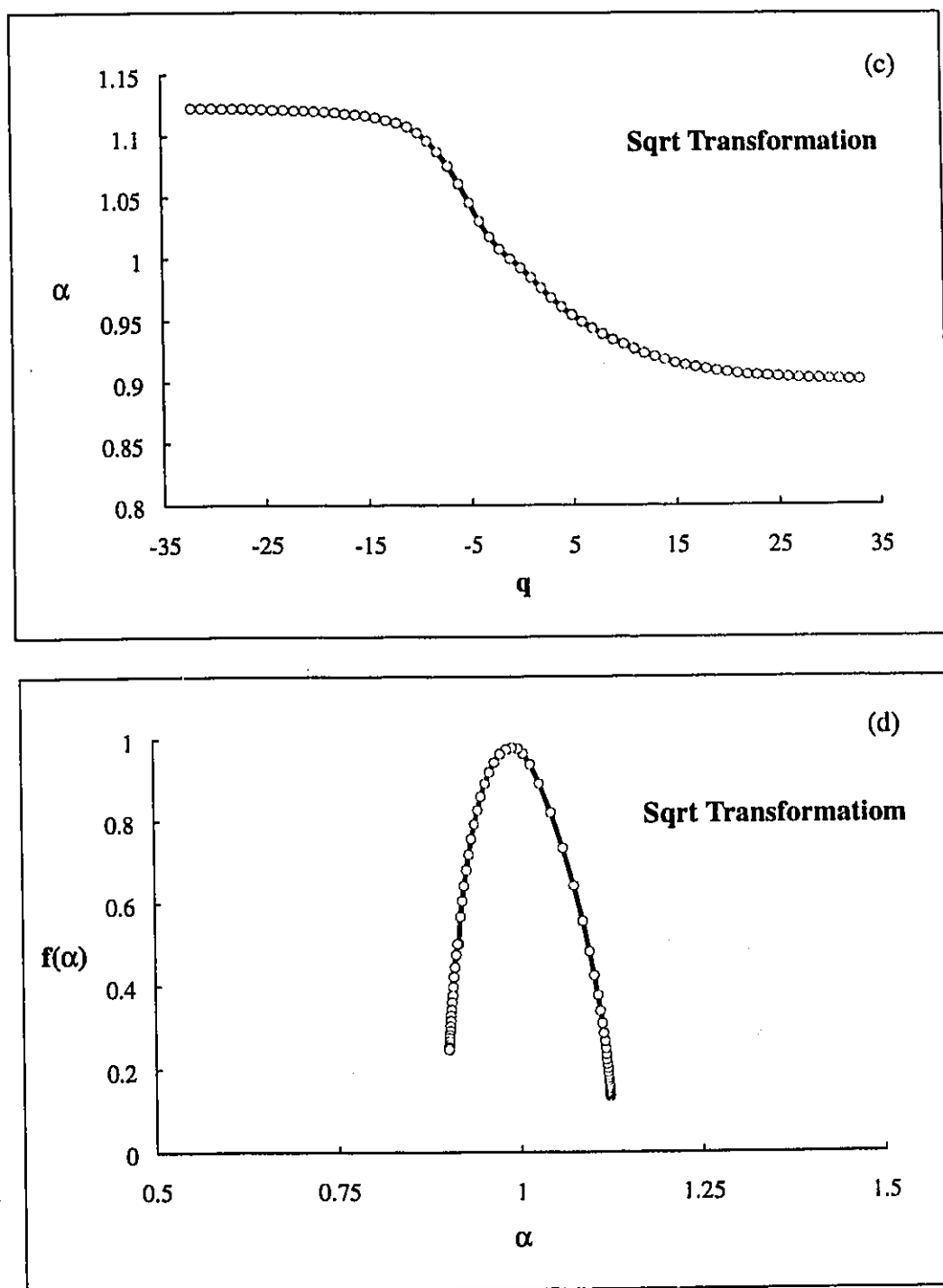


Fig. 6.4.4. Results of multifractal analysis applied to the transformed values for tree seedlings in Fig. 6.4.2. (a) Log-log plot for relationship between $\chi_q(\epsilon)$ and ϵ ; straight lines obtained by means of method of least squares (LS); (b) Estimates of $\tau(q)$ are the slopes of the straight lines in (a); (c) Singularity α estimated from (b) by means of the central difference method, and (d) multifractal spectrum $f(\alpha)$. Smallest cell size $\epsilon=1/200$ corresponds to 1 ft; logarithmic scales have base 10.

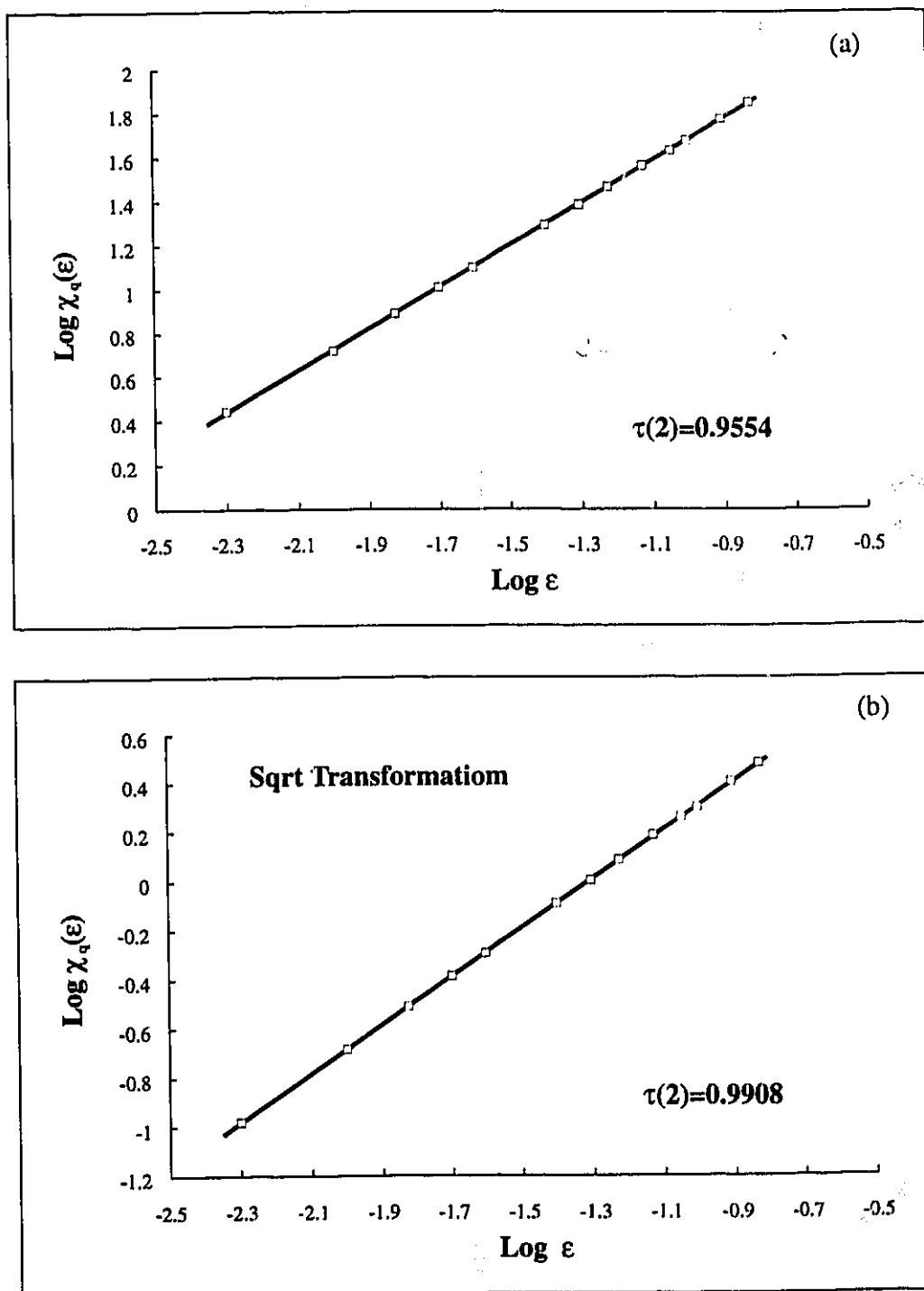


Fig. 6.4.5. Log-log plot for relationship between $\chi_2(\epsilon)$ and ϵ . (a) original data; (b) transformed data.

Values of $\hat{\tau}(q)$ estimated as slopes of all best-fitting straight lines in Figs. 6.4.3a and 6.4.4a were connected by straight-line segments in Figs. 6.4.3b and 6.4.4b. Together these estimates appear to form a continuous curve. Values of the singularity α were estimated by applying the central difference technique to successive pairs of values of $\hat{\tau}(q)$ (Figs. 6.4.3c and 6.4.4c). Finally, the multifractal spectrum of Figs. 6.4.3d and 6.4.4d were obtained using Eq. (4.3.8). The spectrum is approximately symmetrical with maximum value close to 1 and minimum values less than 0.2. The only difference between these two spectra in Figs. 6.4.3d and 6.4.4d is that, due to the different variances of these two cases, the spectral domain for the original data is larger than that for the transformed data (cf. Chapter 4).

6.4.3. Relation to binomial multiplicative cascade model

The binomial multiplicative cascade model in $\mathbb{R}^1$ is perhaps the simplest multifractal model in existence, and has been discussed by many authors including Feder (1988) and Evertsz and Mandelbrot (1992). A realisation of the stochastic version of this model can be obtained as follows (Meneveau and Sreenivasan, 1987; Agterberg, 1994a; Cheng and Agterberg, 1994). At the first stage ($k=1$) in a process consisting of k stages, the interval $(0, L]$ with measure ξL is subdivided into two equal intervals: $(0, L/2]$ with measure $(1+X)\xi L$, and $(L/2, L]$ with $(1-X)\xi L$, where X is a random variable with $\text{pr}(X=d)=\text{pr}(X=-d)=\frac{1}{2}$ ($d>0$). At stage 2 these two intervals are halved again with new measures for the halves defined in the same way as at stage 1. This process is repeated at stages $k = 3, 4, \dots$

At stage k the j -th subinterval has size $\epsilon_j = L/2^k$, and $E\{\xi_j(\epsilon)\} = \xi$. The frequency distributions of $\xi(\epsilon)$ is logbinomial.

Defining a new constant $m = (1+d)/2$, it can be shown (Evertsz and Mandelbrot, 1992, p. 941) that, for $k \rightarrow \infty$,

$$\tau(q) = -\log_2 [m^q + (1-m)^q] \quad (6.4.1)$$

with corresponding expressions of $\alpha(q)$ and $f(\alpha)$ following from (4.3.8). The resulting multifractal spectrum has its maximum at the point where $f(\alpha) = 1$ for $\alpha = \alpha(0)$. It resembles a parabola which is truncated at the bottom because $f(\alpha) \geq 0$. The minimum on the left is reached for $\alpha_{\min} = -\log_2 m$.

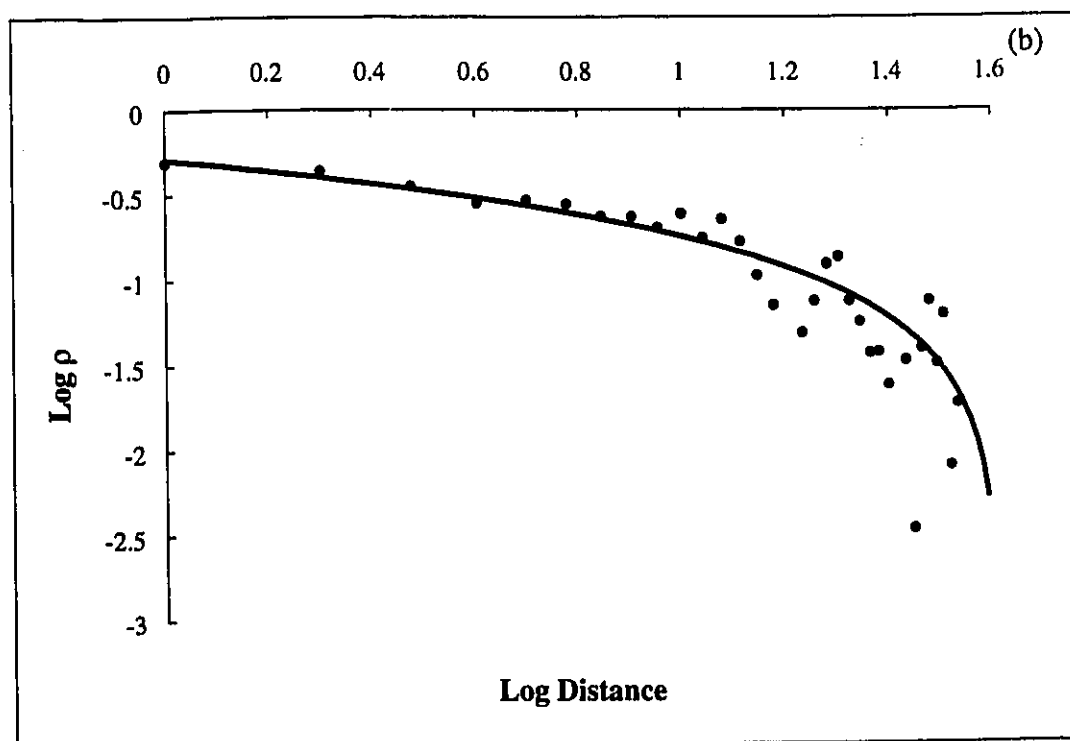
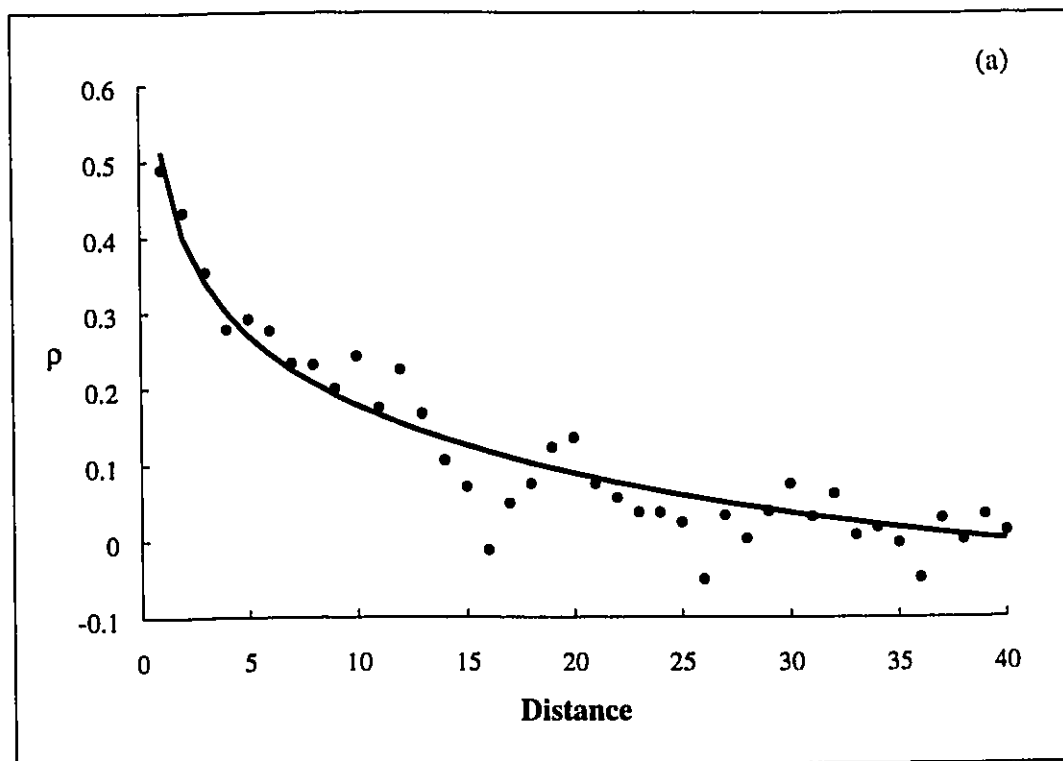
As discussed in the section 6.4.2, the estimate $\hat{\tau}(2) = 0.9554 \pm 0.002$ for original data and 0.9908 ± 0.002 for transformed data are probably very good. Substitution into (6.4.1) gives $-\log_2 \{\hat{m}^2 + (1-\hat{m})^2\} = 0.9554$ for original data (0.9908 for transformed data) and it follows that $\hat{m} = 0.589$ and $\hat{d} = 0.177$ for original data ($\hat{m} = 0.540$ and $\hat{d} = 0.08$ for transformed data). If the binomial model is valid this result would mean that halving any sufficiently long interval within the bed of seedlings gives subintervals with the property that the subinterval with the greater density has a density of seedlings that is enlarged by a factor of approximately 1.18 for original data (1.08 for transformed data). Equivalently, for two adjoining samples which are not too small, the sample with the greater number of seedlings contains approximately 1.43 times as many seedlings as the other sample. As illustrated in Figs. 6.4.3a, the straight lines of which the slopes are used to estimate $\tau(q)$

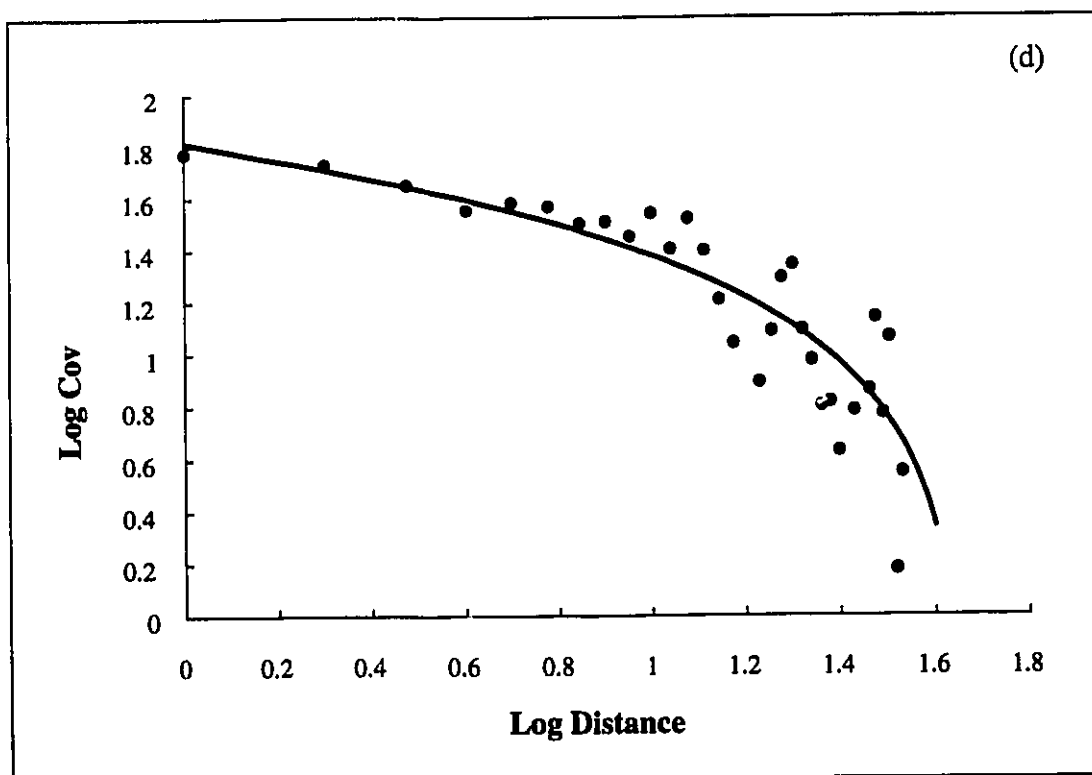
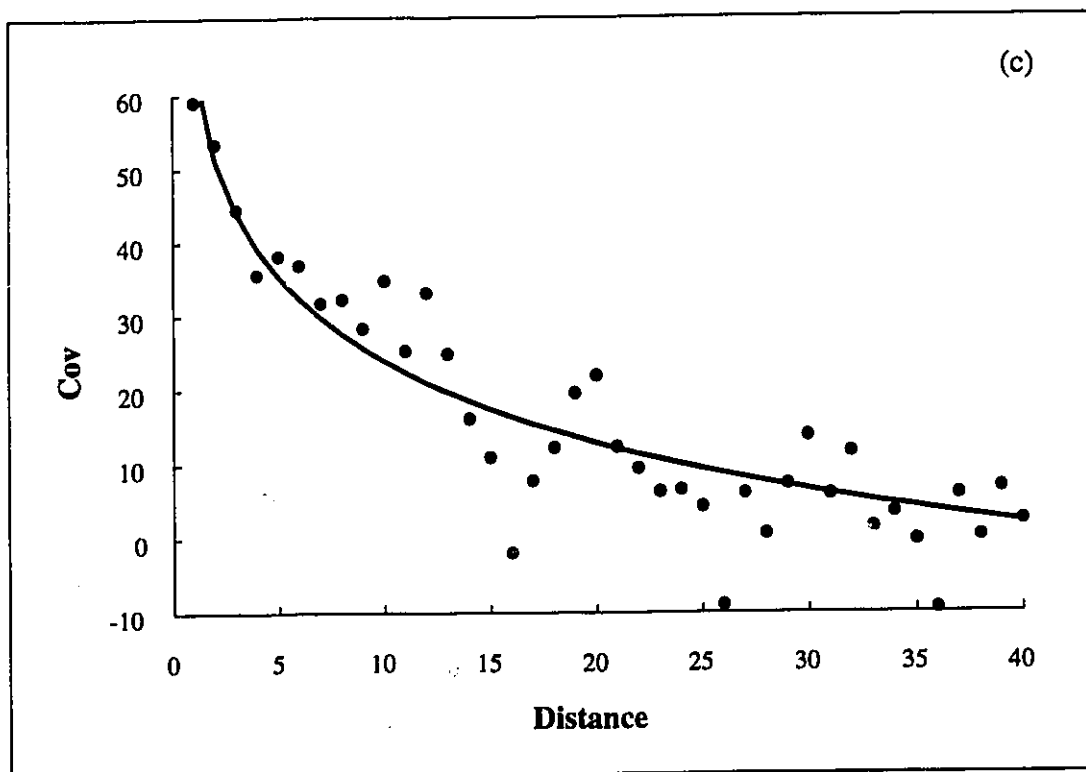
show a better fit for positive values of q than for $q < 0$. Because of Eq. (4.3.8), this result indicates that the left tail of the curve in Fig. 6.4.3d provides a better estimate of its corresponding part in the true multifractal spectrum than the right tail. The left tail is determined by estimates of $\tau(q)$ for the largest values of q which, in turn, are based on the relatively few, largest observed values. Therefore, it is likely that the uncertainty increases toward the left in the left tail of Fig. 6.4.3d. Nevertheless, suppose that $\alpha_{\min} = 0.82$ for original data (0.90 for transformed data) as suggested by the curves in Figs. 6.4.3c and 6.4.3d. This would give $\hat{m} = 0.566$ and $\hat{d} = 0.133$ which differ from the results in the preceding paragraph which were based on $\hat{\tau}(2) = 0.9554 \pm 0.002$.

It may be concluded from the preceding results either that the binomial multiplicative cascade model is valid or that is approximately valid. Assuming that this model is correct, and using $\hat{d} = 0.177$ (as derived from $\hat{\tau}(2) = 0.9554 \pm 0.002$) results in $\alpha_{\min} = 0.76$ for original data (0.889 for transformed data) instead of 0.82 for original data (0.90) for transformed data) as suggested by the curves of Figs. 6.4.3c and 6.4.3d. This small discrepancy could mean that the binomial multiplicative cascade model is not valid. However, it can be assumed that the patterns of Fig. 6.4.3 are the realization of a multifractal model which is either a binomial multiplicative cascade model or only slightly different from it.

6.4.4. Autocorrelation and semivariogram functions

The covariance, autocorrelation and semivariogram values for successive lags in Figs.





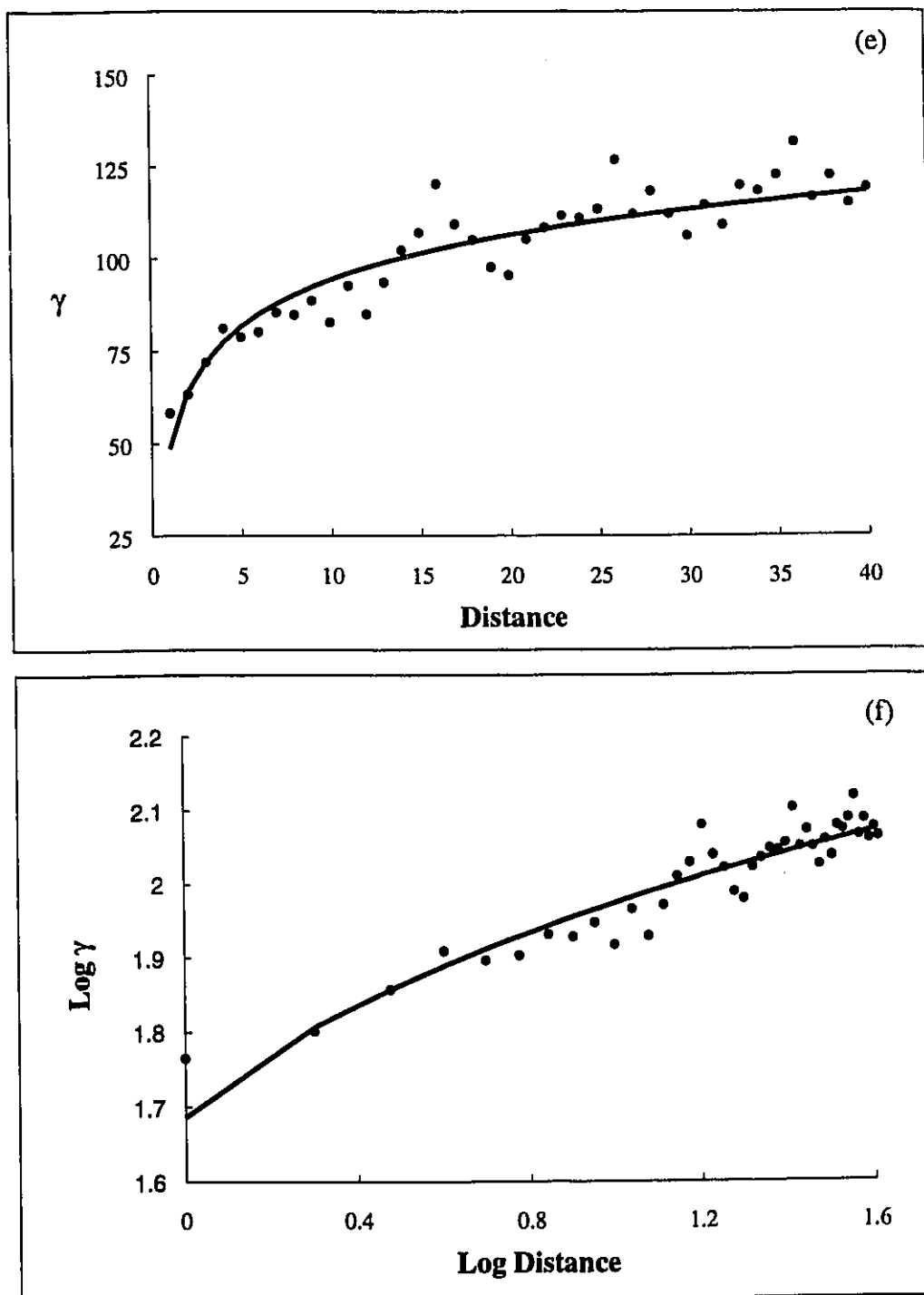
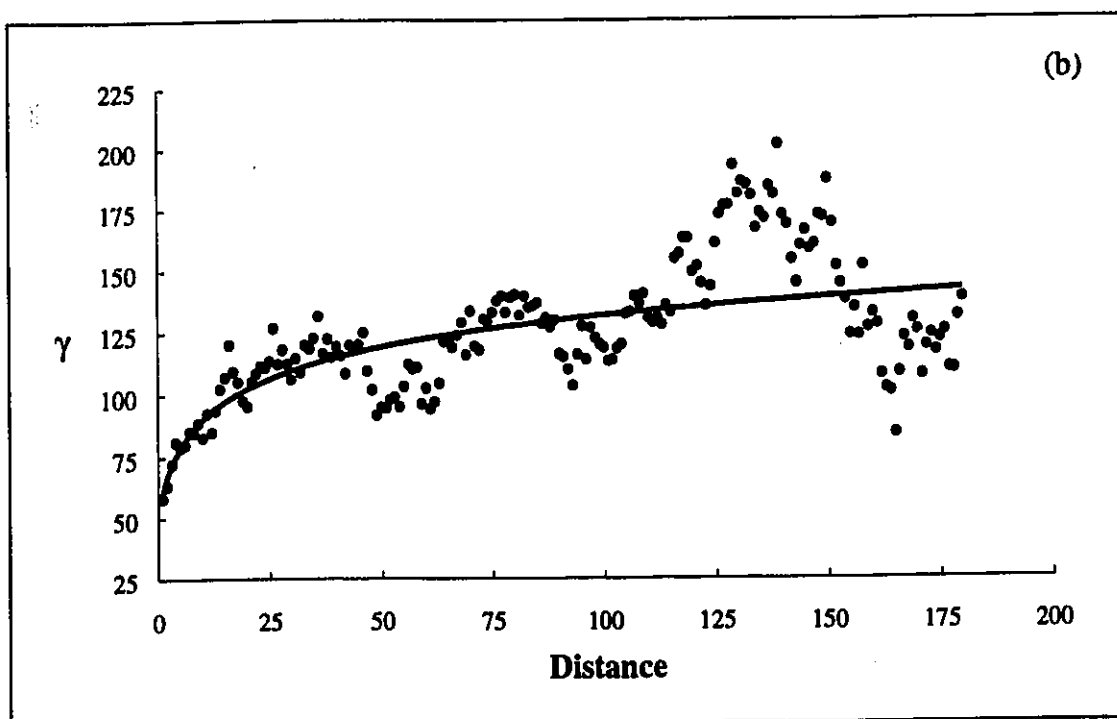
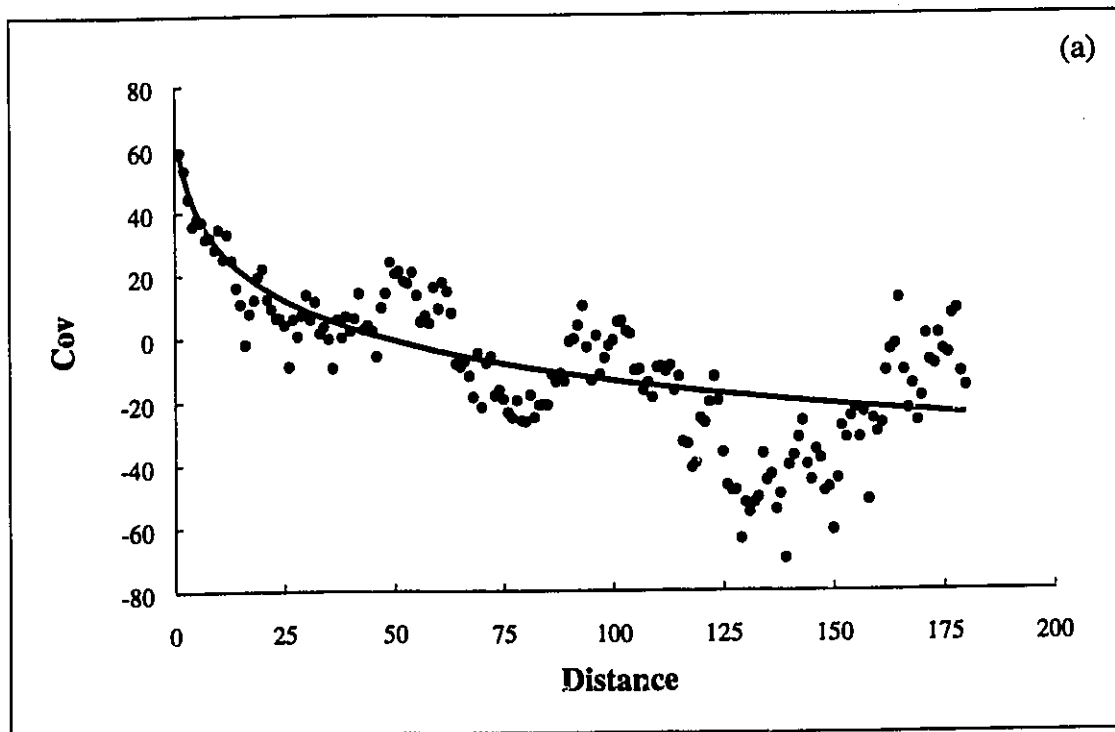


Fig. 6.4.6. Estimation of autocorrelation, covariance and semivariogram functions for tree seedling data in Fig. 6.4.1. (a) Estimates of autocorrelation coefficients (distance ≤ 40 ft); (b) Log-log plot of (a); (c) Estimated covariances; (d) Log-log plot of (c); (e) Estimated semivariogram γ_i ; (f) Log-log plot of (e). Solid lines obtained by linear regression using Eqs. (6.2.13), (6.2.14) and (6.2.15), respectively, after setting $\hat{t}(2)=0.9554$.



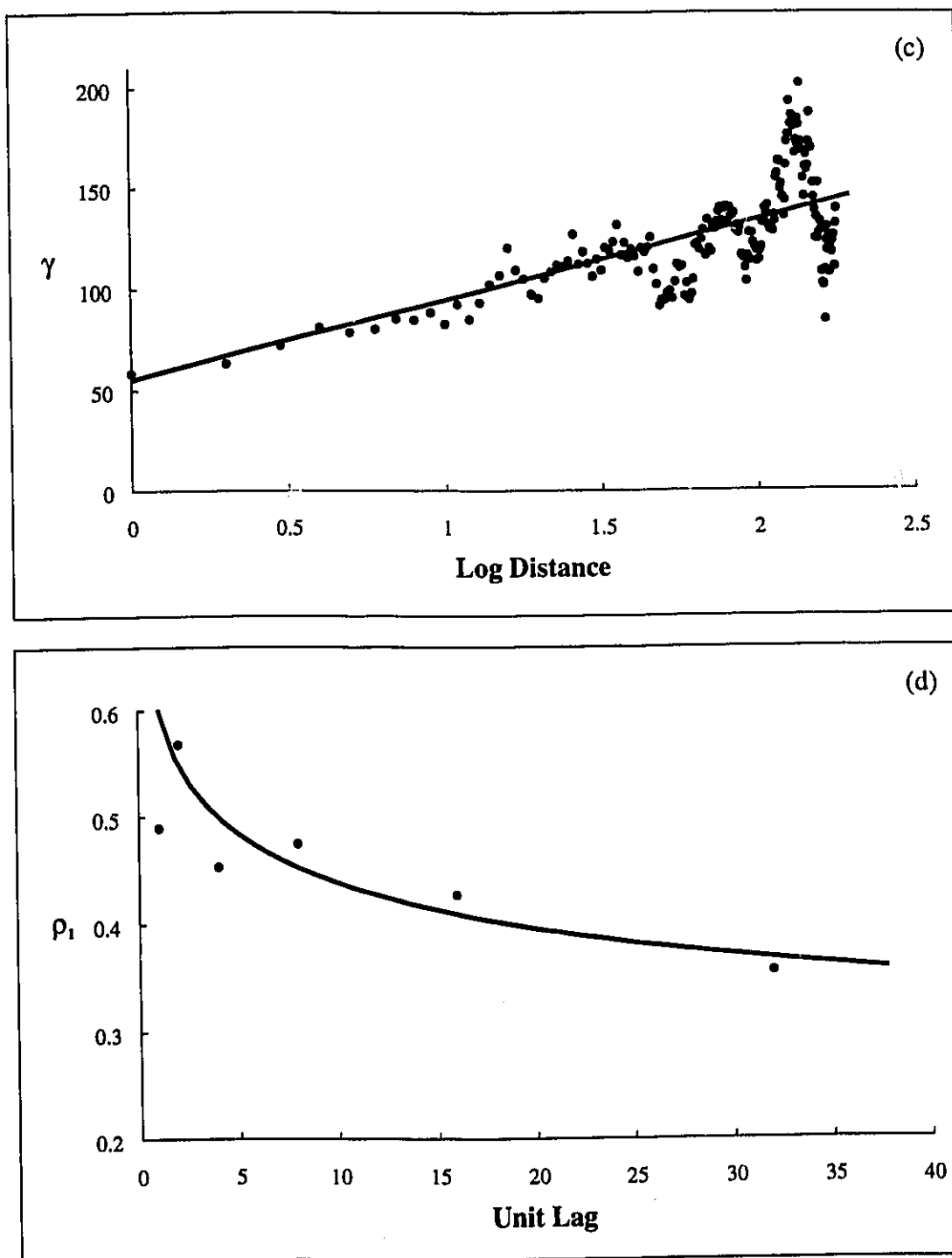
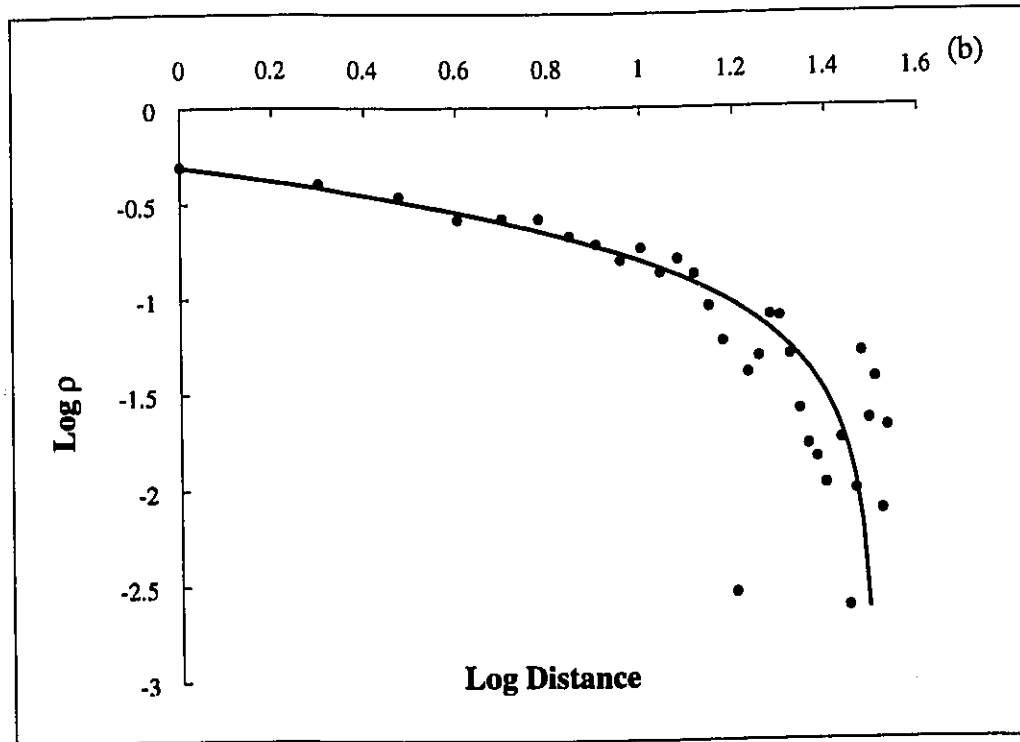
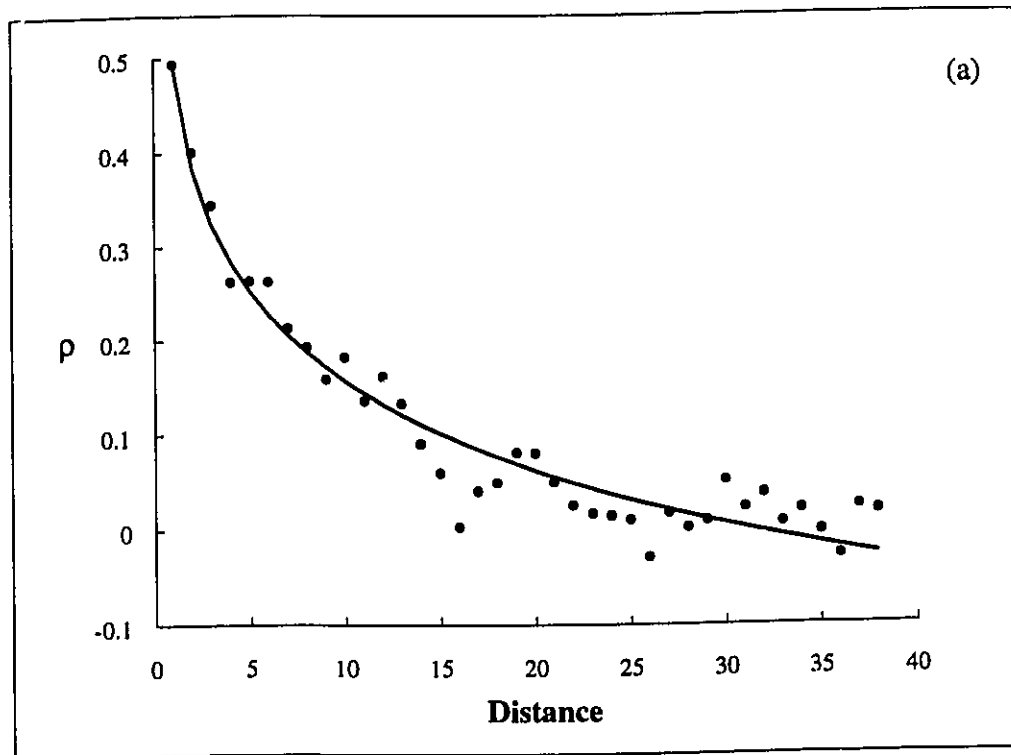
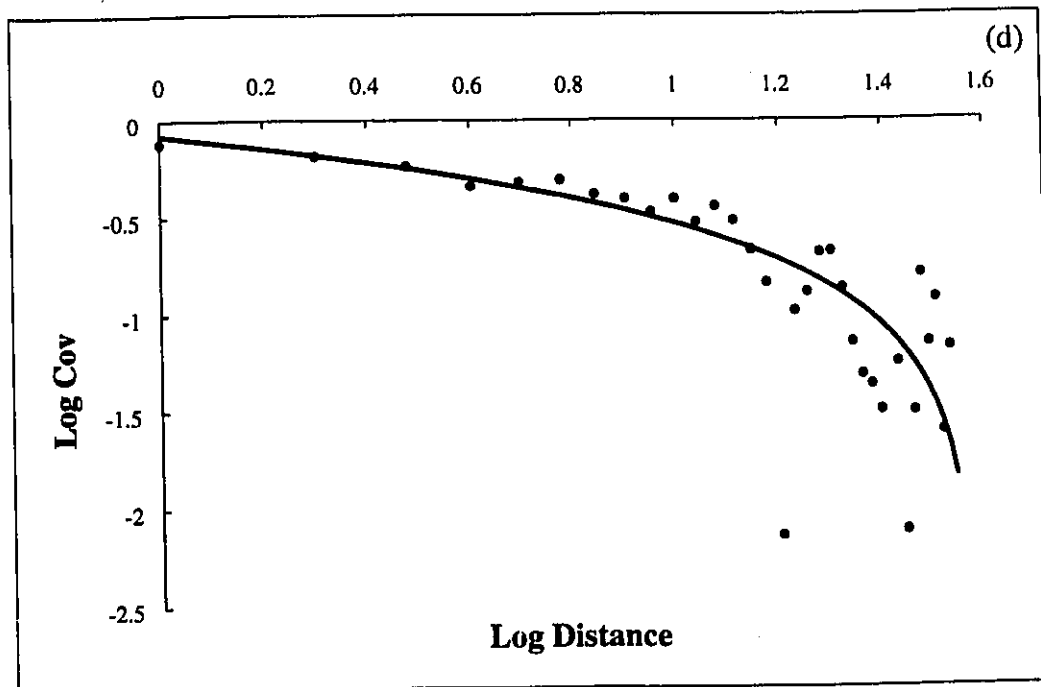
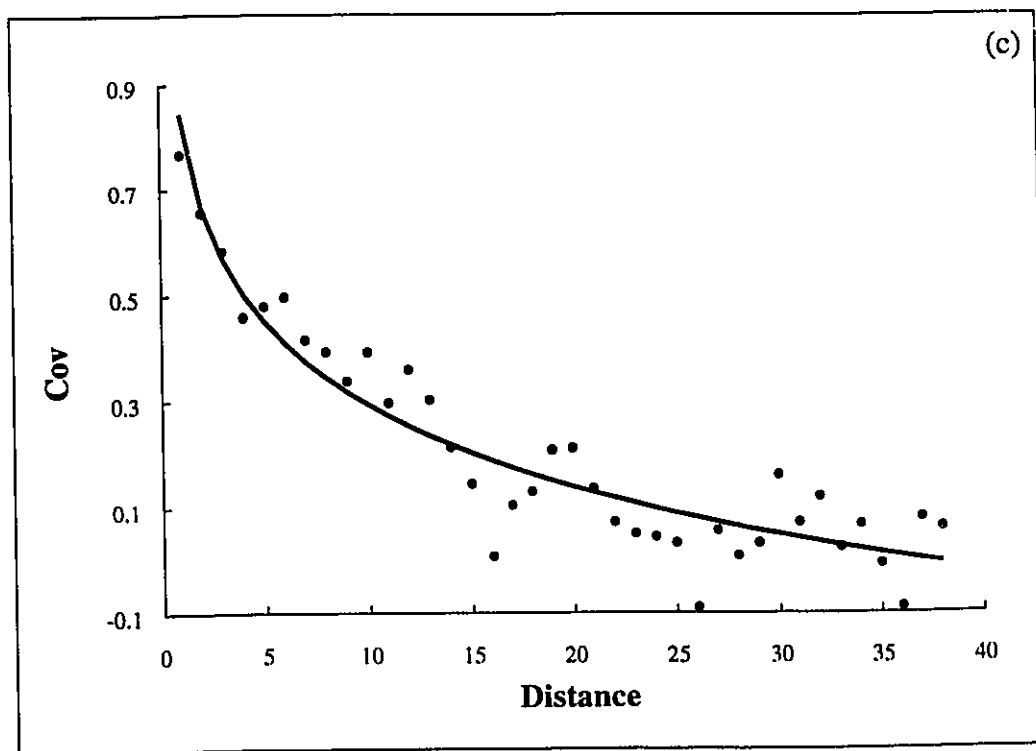


Fig. 6.4.7. First 180 estimated values of (a) covariance and (b) semivariogram. Curves satisfy Eqs. (6.2.14) and (6.2.15) with $\hat{\tau}(2)=0.9554$ and c_2 calculated from sample mean and standard deviation using the 200 original values. (c) Semivariogram with best-fitting straight line showing logarithmic relation. (d) Autocorrelation between adjoining cells with different sizes; curve satisfies Eq. (6.3.7) with $\hat{\tau}(2)=0.9554$.





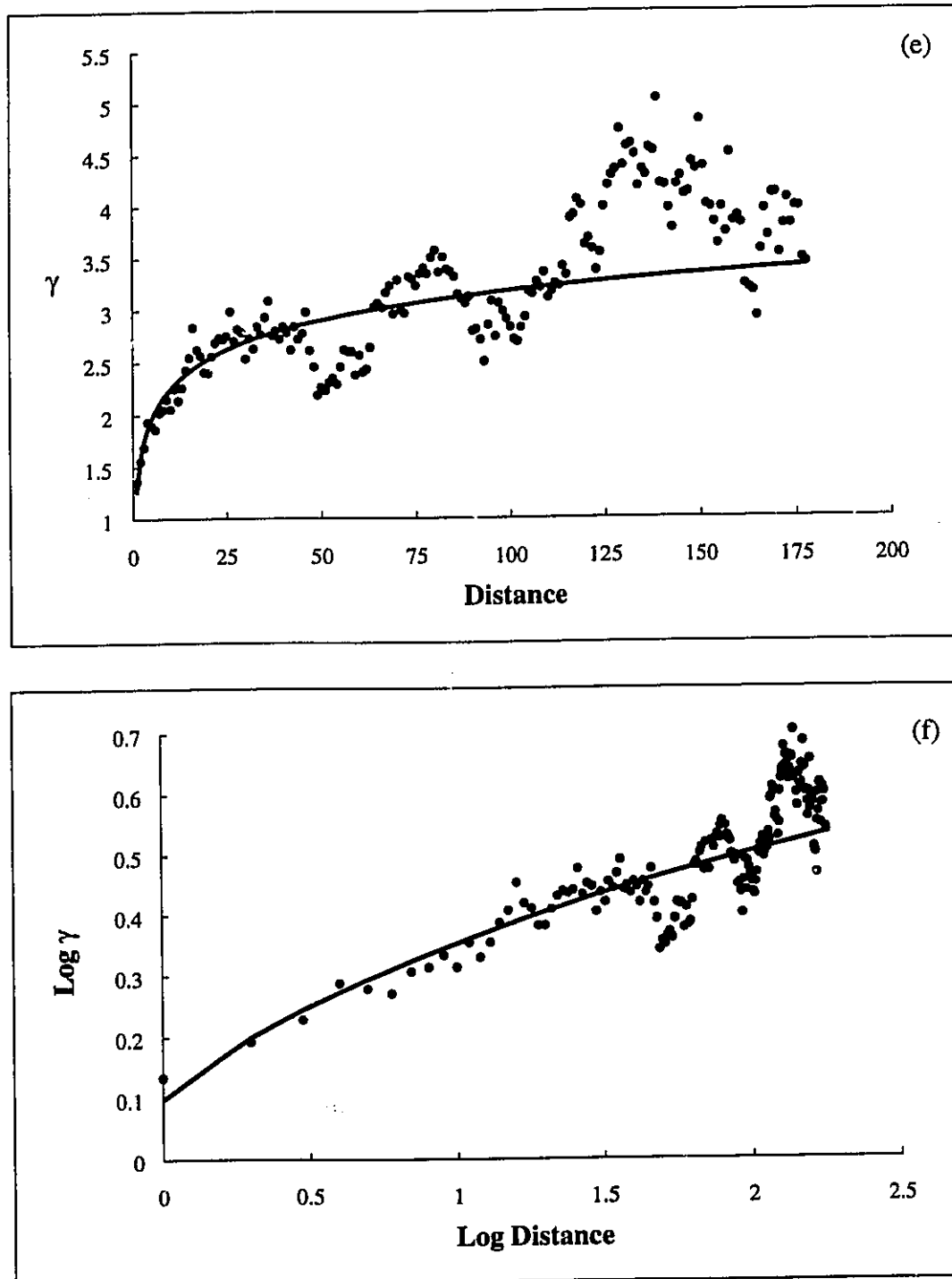


Fig. 6.4.8. Estimation of autocorrelation, covariance and semivariogram functions for transformed data in Fig. 6.4.2. (a) Estimates of autocorrelation coefficients (distance ≤ 40 ft); (b) Log-log plot of (a); (c) Estimated covariances; (d) Log-log plot of (c); (e) Estimated semivariogram γ_k (distance ≤ 180 ft); (f) Log-log plot of (e). Solid lines obtained by linear regression using Eqs. (6.2.13), (6.2.14) and (6.2.15), respectively, after setting $\hat{t}(2)=0.9908$.

6.4.6a to 6.4.8c (Figs. 6.4.6 and 6.4.7 for original data and Fig. 6.4.8 for transformed data) were estimated by means of Eqs. (6.2.3) to (6.2.5). One foot was used as unit of lag. Estimated continuous functions for autocorrelation, covariance and semivariogram satisfying Eqs. (6.2.12), (6.2.14) and (6.2.15) also are shown in Figs. 6.4.6a to 6.4.8c. In all applications the estimate $\hat{\tau}(2)=0.9554$ resulting from Fig. 6.4.5a for original data (0.9908 for transformed data) was used. As a result, the remaining coefficients could be fitted by ordinary least squares. For the first 40 values (lag ≤ 40 ft) of the original data, this gave

$$\begin{aligned}\hat{\rho}(k) &= 1.742 [(k+1)^{1.9554} - 2k^{1.9554} + (k-1)^{1.9554}] - 2.758 \\ \hat{\sigma}(k) &= 215.058 [(k+1)^{1.9554} - 2k^{1.9554} + (k-1)^{1.9554}] - 338.647 \\ \hat{\gamma}(k) &= 494.857 - 237.596 [(k+1)^{1.9554} - 2k^{1.9554} + (k-1)^{1.9554}]\end{aligned}$$

and for the transformed data

$$\begin{aligned}\hat{\rho}(k) &= 7.70 [(k+1)^{1.9908} - 2k^{1.9908} + (k-1)^{1.9908}] - 14.78 - \\ \hat{\sigma}(k) &= 12.65 [(k+1)^{1.9908} - 2k^{1.9908} + (k-1)^{1.9908}] - 24.14 \\ \hat{\gamma}(k) &= 46.83 - 23.08 [(k+1)^{1.9908} - 2k^{1.9908} + (k-1)^{1.9908}]\end{aligned}$$

These results are graphically shown in Figs. 6.4.6 and 6.4.8 using arithmetic as well as logarithmic scales. Figs. 6.4.7a to 6.4.7c show these same functions for original data extended to longer distances. A slightly different procedure was used for estimating the functions shown in Fig. 6.4.7. Instead of using linear regression, the sample mean and standard deviation of the 200 original values of Fig. 6.4.1 ($\bar{x}=20.774$ and $\hat{\sigma}=10.889$) were substituted into (6.2.8) yielding $c_2=434.349$. Figure 6.4.7c shows the logarithmic semivariogram of Eq. (6.2.20) with

$$\hat{\gamma}(k) = 44.096 + 44.494 \log(k)$$

in which the coefficients were obtained by ordinary least squares. Estimated autocorrelation coefficients for values from adjoining cells are shown in Fig. 6.4.9 for different cell sizes. The curves in Fig. 6.4.7d satisfy Eq. (6.3.7) with $\hat{t}(2)=0.9554$, $c_2=434.349$ and $\xi=20.774$ for original data. The fact that these curves are not approximately horizontal indicates that the autocorrelation is not approximately constant for different cell sizes. In this application, Eqs. (6.2.14) and (6.2.15) which result from the multifractal model give better results than those based on Eqs. (6.2.17) and (6.2.18).

6.5. APPLICATION TO De WIJS'S ZINC VALUES EXAMPLE

6.5.1 Zinc value data

De Wijs (1951) studied assay values from the Pulacayo sphalerite-quartz vein in Bolivia. Along a drift 118 channel samples had been obtained at 2.00 meter intervals (see Fig. 6.5.1a). The massive sulphide vein was on average about 0.50 meter wide but all samples were cut over the anticipated stoping width of 1.30 meters. These channel samples provide estimates of the zinc concentration in 2m-long blocks measured along the vein in the direction of the vein. As shown by De Wijs, the zinc concentration values have an approximate log-normal frequency distribution (see Fig. 6.5.1b).

6.5.2 Multifractal spectrum of zinc concentration values

The measure μ is defined by means of Eq. (6.2.1) where ξ indicates the concentration of zinc per unit length. Cell sizes ranging from 2 m to 30 m (234 m as total length) are

used. The estimated values of partition function for moment q ranging from -35 to 35 are plotted on log-log paper, (see Fig. 6.5.2a). The slopes, $\tau(q)$, of the fitted straight lines are shown in Fig. 6.5.2b. From Fig. 6.5.2b, it can be seen that $\tau(0)=-0.976\pm0.011$ (i.e., the box-counting dimension is close to 1), $\tau(1)=0.03\pm0.013$ (close to 0) and $\tau(2)=0.979\pm0.019$. From $\tau(q)$, the exponents α and multifractal dimension spectrum $f(\alpha)$ can be estimated (see Figs. 6.5.2c and 6.5.2d).

6.5.3. Autocorrelation and semivariogram functions

A lag unit of 2m was used for estimating the spatial correlation and semivariogram for the zinc values. The computed values of autocorrelation coefficient, covariance and semivariogram are shown in Fig. 6.5.3 with the plots on the right side being log-log

$$\begin{aligned}\rho(k) &= 4.37 [(k+1)^{1.979} - 2k^{1.979} + (k-1)^{1.979}] - 8.00 \\ \text{Cov}(k) &= 277.65 [(k+1)^{1.979} - 2k^{1.979} + (k-1)^{1.979}] - 508.87 \\ \gamma(k) &= 391.49 - 180.38 [(k+1)^{1.979} - 2k^{1.979} + (k-1)^{1.979}]\end{aligned}$$

plots. The solid lines in Fig. 6.5.3 were fitted by means of Eqs. (6.2.13), (6.2.14) and (6.2.15), respectively. The following estimated relations were obtained for the fitted lines:

$$\begin{aligned}\rho(k) &= 4.37 [(k+1)^{1.979} - 2k^{1.979} + (k-1)^{1.979}] - 8.00 \\ \text{Cov}(k) &= 277.65 [(k+1)^{1.979} - 2k^{1.979} + (k-1)^{1.979}] - 508.87 \\ \gamma(k) &= 391.49 - 180.38 [(k+1)^{1.979} - 2k^{1.979} + (k-1)^{1.979}]\end{aligned}$$

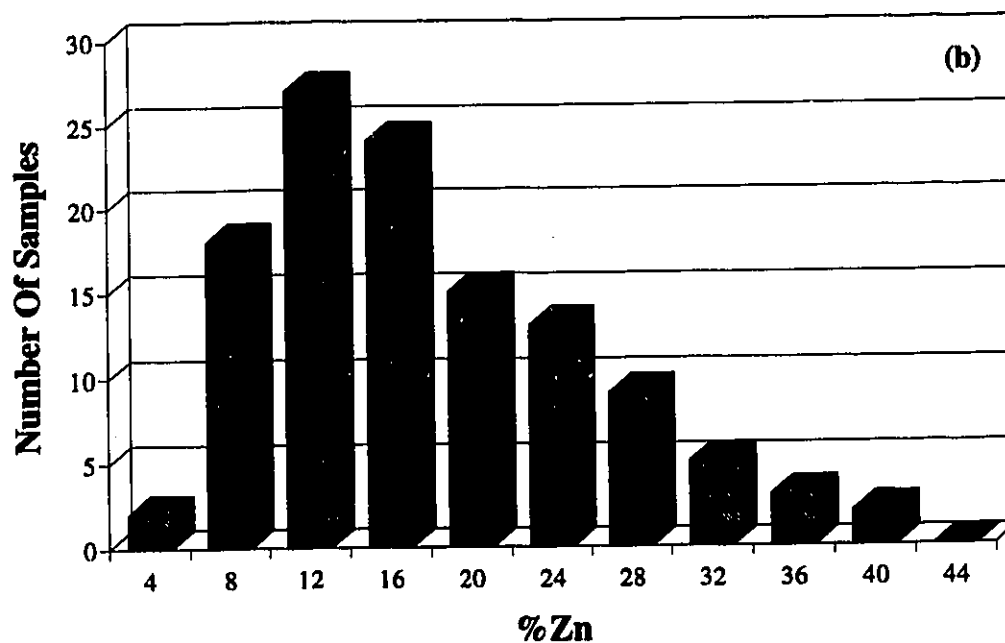
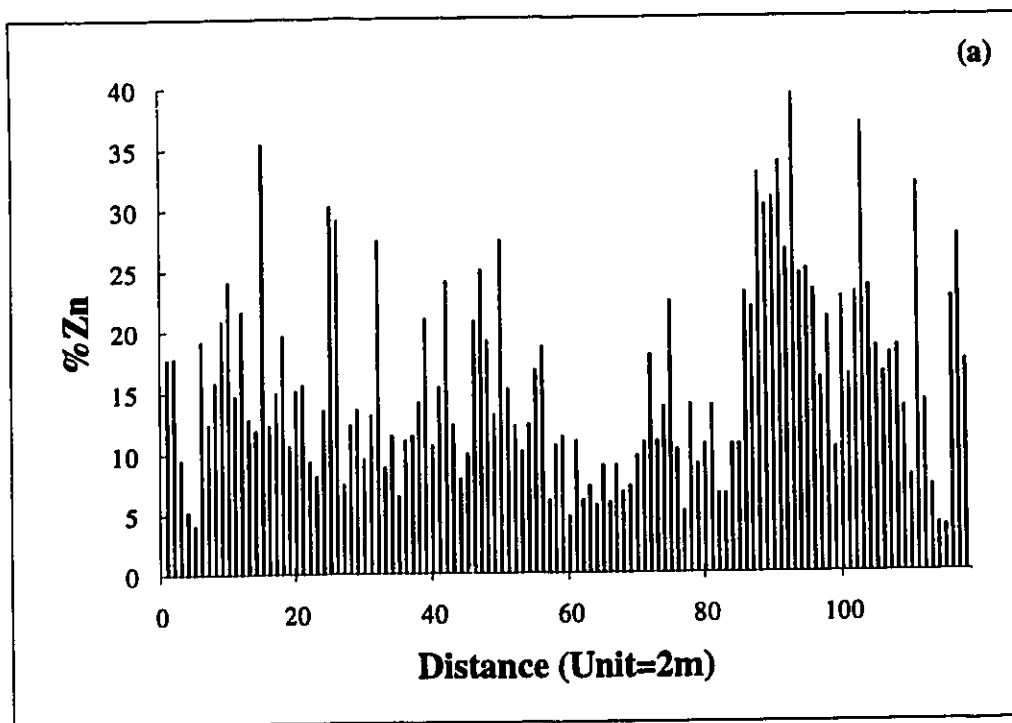
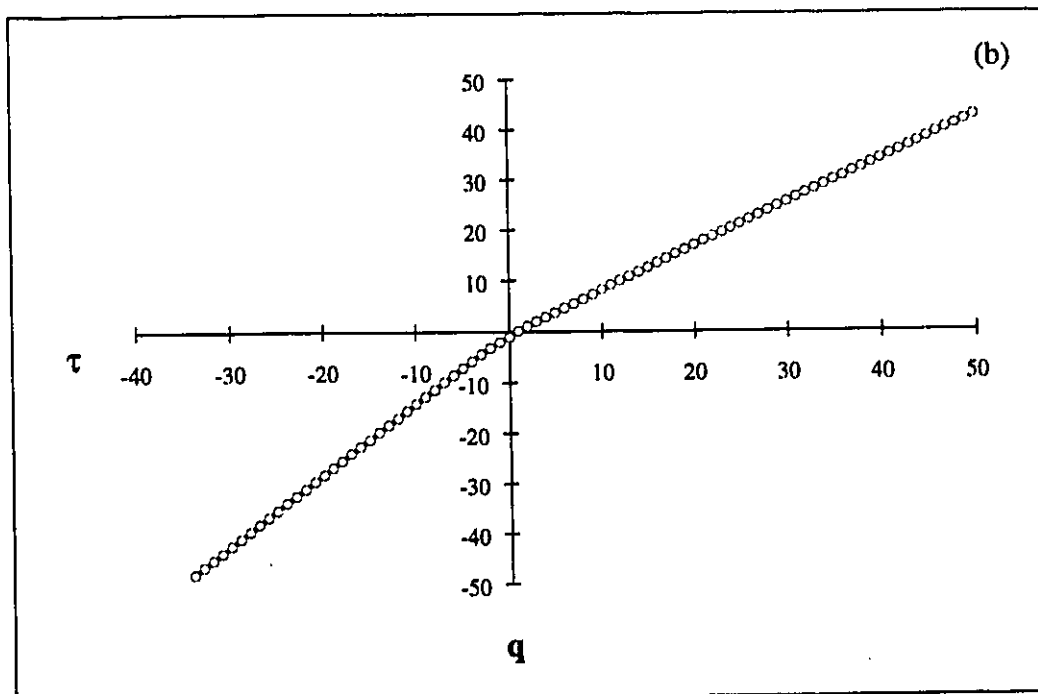
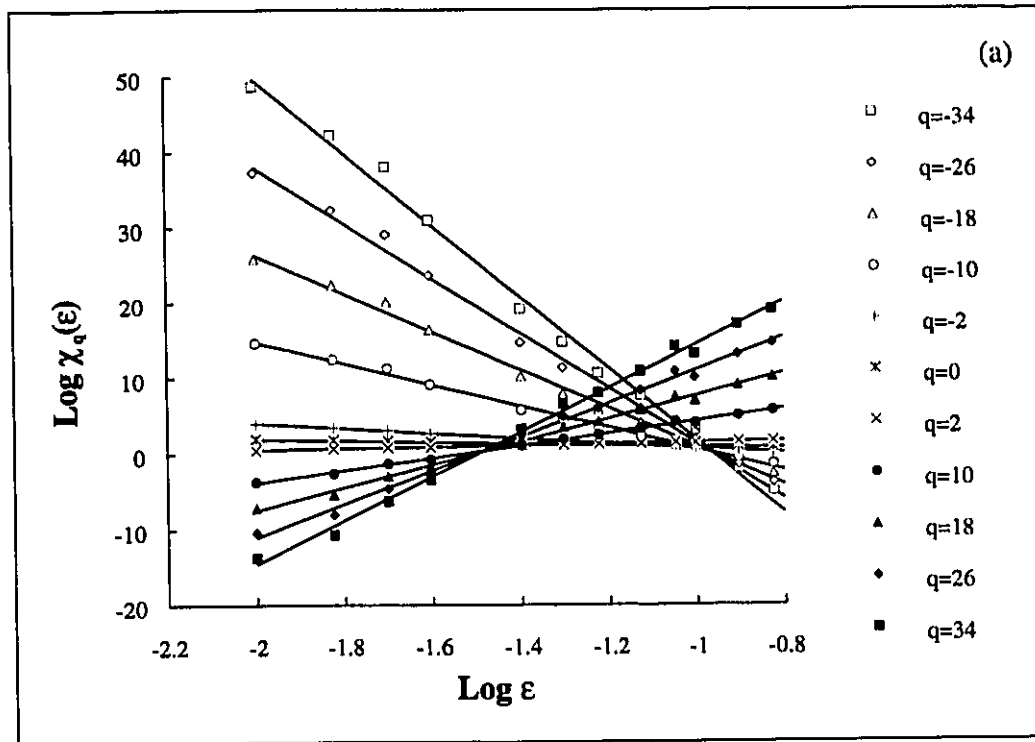


Fig. 6.5.1 (a) Zinc concentration values in 2 meter segments. (b) histogram of the zinc values. The data are from De Wijs (1951).



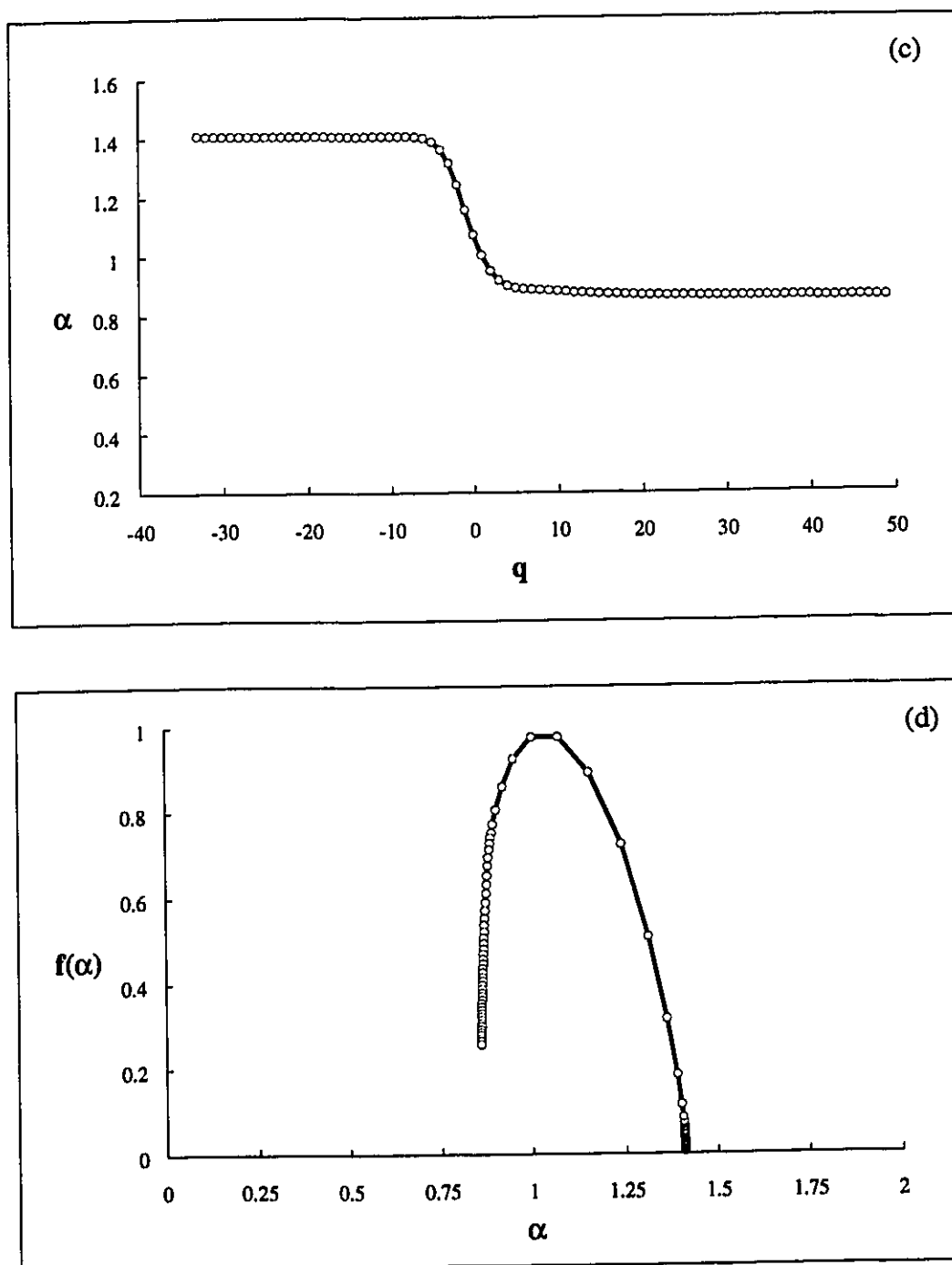


Fig. 6.5.2 Results of multifractal analysis applied to the zinc values in Fig. 6.5.1. (a) Log-log plot for relationship between $\chi_q(\epsilon)$ and ϵ ; straight lines obtained by means of method of least squares (LS); (b) Estimates of $\tau(q)$ are the slopes of the straight lines in (a); (c) Singularity α estimated from (b) by means of the central difference method, and (d) multifractal spectrum $f(\alpha)$. Smallest cell size $\epsilon=1/200$ corresponds to 1 ft; logarithmic scales have base 10.

Matheron (1962, p75) introduced the De Wijsian semivariogram for log-transformed zinc values with:

$$\gamma(k) = \beta \log_e k \quad (6.5.1)$$

where β is a constant, $\beta=0.23$.

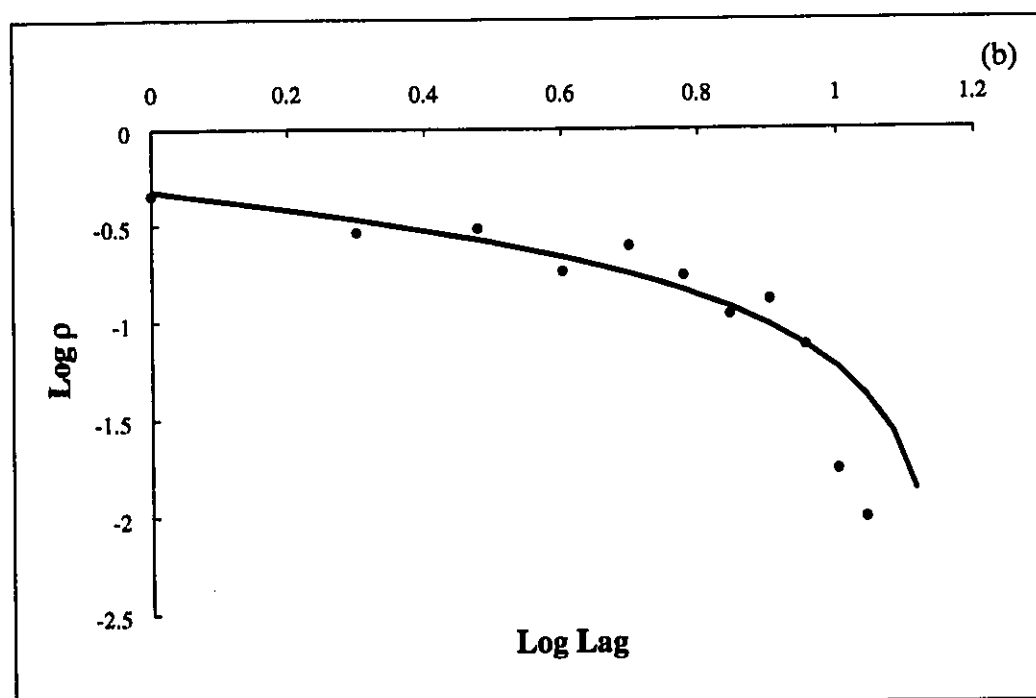
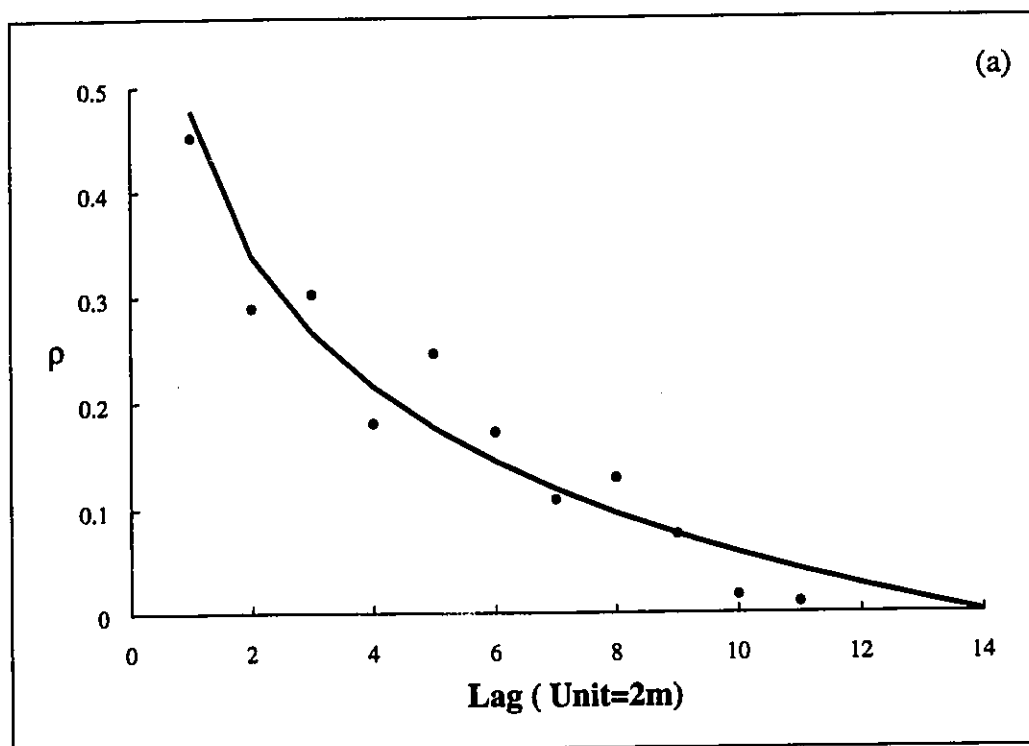
The following power-law type covariance was derived by Agterberg (1994a) from Eq.(6.5.1) based on the relation:

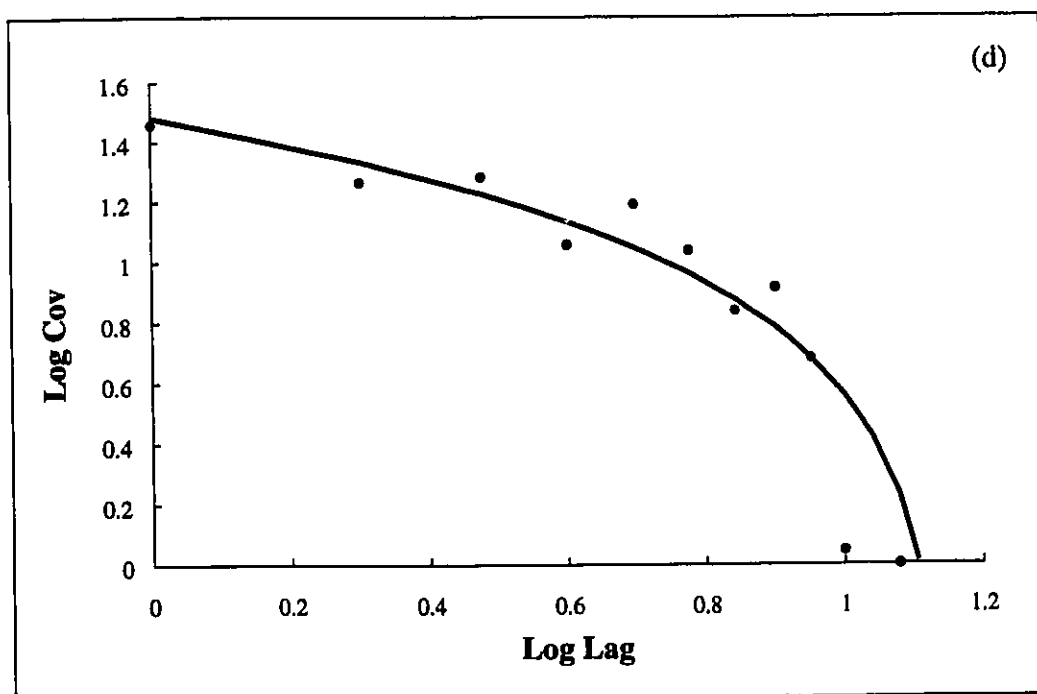
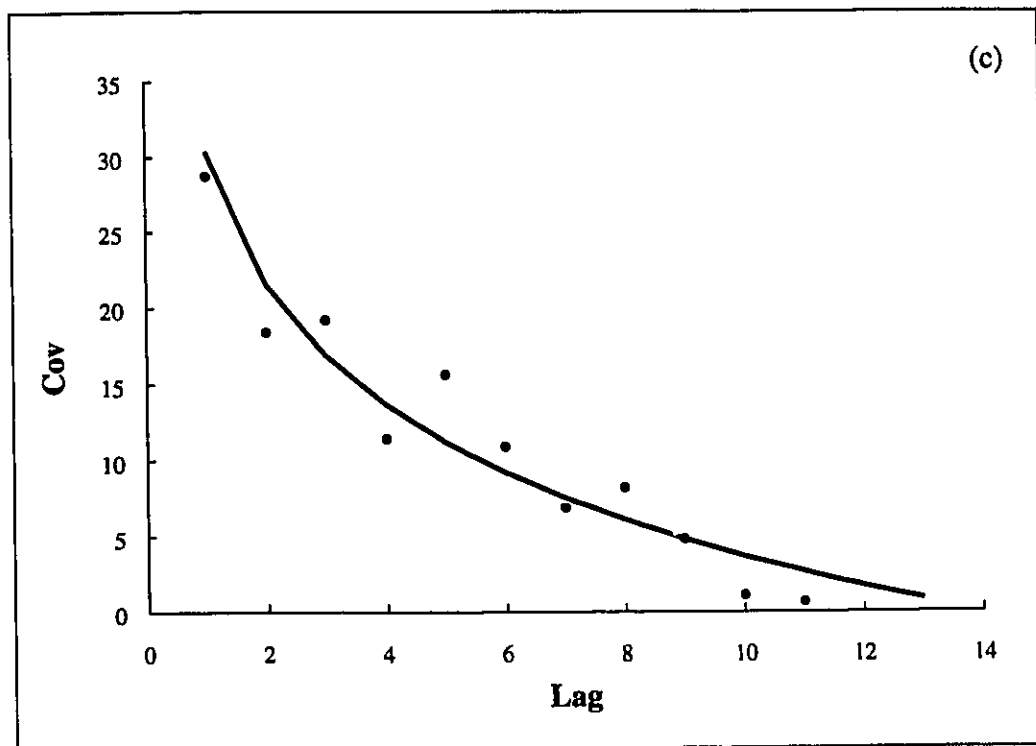
$$Cov(k) = var \ e^{-\gamma(k)} \quad (6.5.2)$$

where $Cov(k)$ is the covariance for regional zinc values and $\gamma(k)$ is the semivariogram for log-transformed values. Relation Eq.(6.5.1) also hold true for the original zinc values. The constant value β depends on the cell size z . Figs. 6.5.5a and b show exponential relations of Eq. (6.2.18) with $\gamma = 785.41 - 763.38 k^{-0.021}$, and logarithmic relation of Eq. (6.2.20) with $\gamma = 22.31 \log(k) + 36.9$.

6.6 CONCLUSION

In general, the multifractal model provides more information about measurements on spatial objects than ordinary fractal models. The mass exponent function $\tau(q)$ is useful for characterizing the underlying structure of the measurements with $-\tau(0)$ corresponding to the largest fractal dimension (= box-counting dimension of the support), and $\tau(2)$ determining the fractal correlation dimension of a random variable. The Hurst exponent





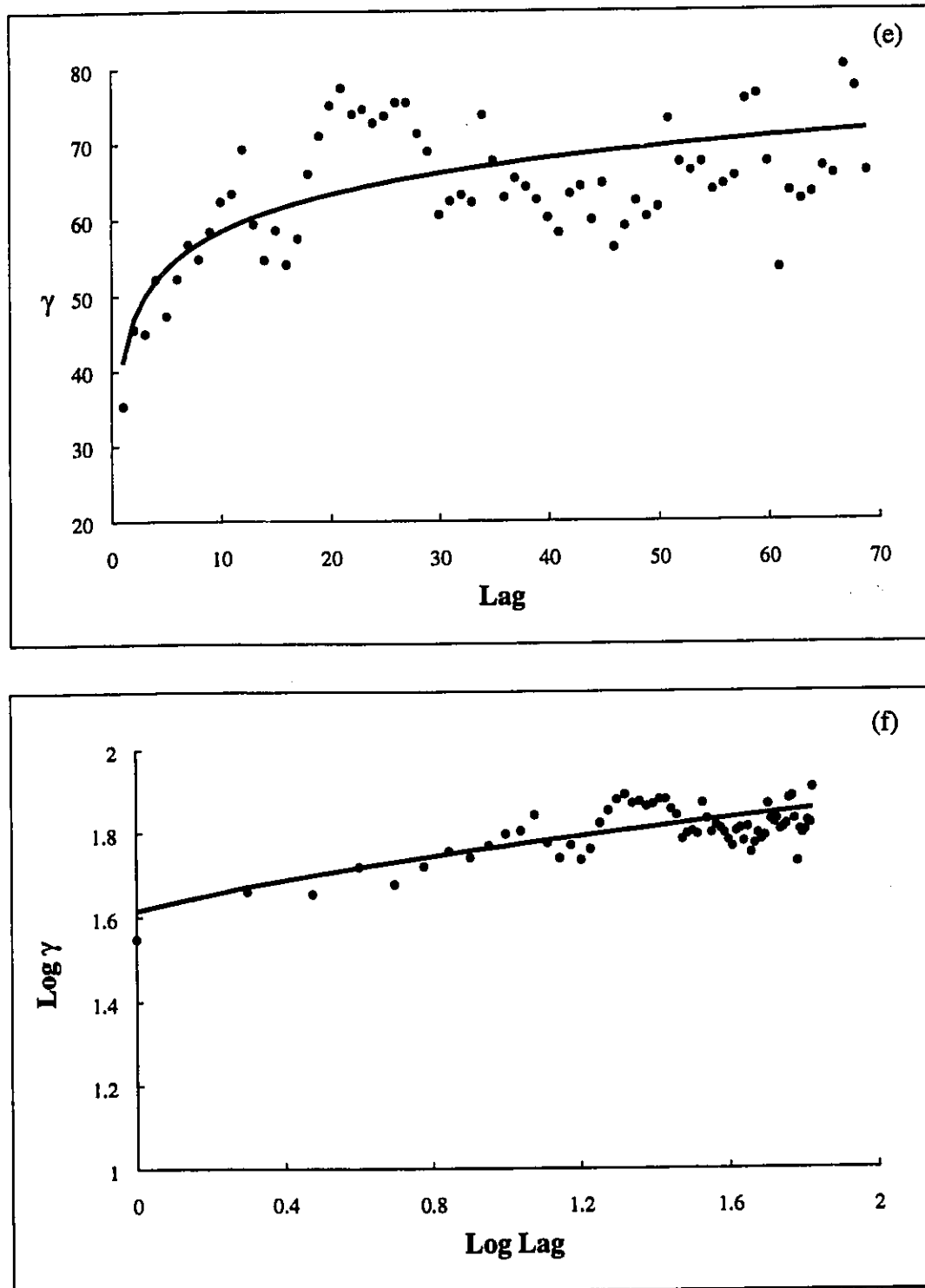


Fig. 6.5.3. Estimation of autocorrelation, covariance and semivariogram functions for zinc values in Fig. 6.5.1. (a) Estimates of autocorrelation coefficients; (b) Log-log plot of (a); (c) Estimated covariances; (d) Log-log plot of (c); (e) Estimated semivariogram γ_k ; (f) Log-log plot of (e). Solid lines obtained by linear regression using Eqs. (6.2.12), (6.2.14) and (6.2.15), respectively, after setting $\hat{\tau}(2)=0.979$.

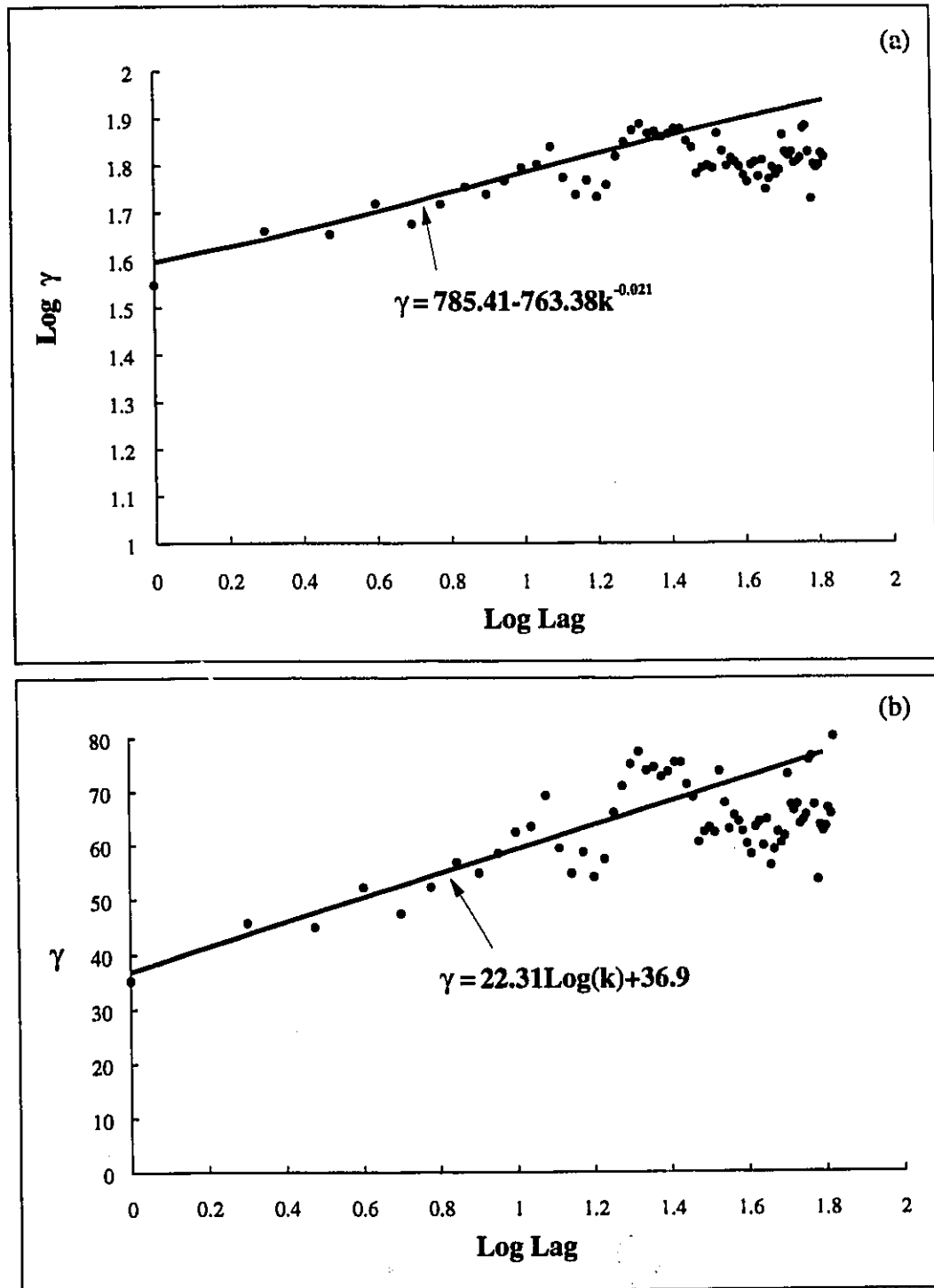


Fig. 6.5.4 Comparison of exponential model and logarithmic model. (a) exponential model Eq. (6.2.18) with exponent $\tau(2)-1=-0.021$; (b) logarithmic model Eq. (6.2.20).

H of a Brownian surface satisfies $H=(1/2)[\tau(2)+1]$.

The multifractal model results in covariance, autocorrelation and semivariogram functions for use in spatial statistics. These functions are primarily characterized by the value of $\tau(2)$.

Chapter 7 MULTIFRACTALS AND SPATIAL POINT PROCESSES

7.1 INTRODUCTION

A spatial point process is any stochastic mechanism which generates a countable set of events X_i in the plane (cf. Diggle, 1983, p.46). The statistical theory of spatial point processes is covered in several books including Diggle (1983); Ripley (1981, 1988) and Stoyan et al., (1987), and has been intensively used in biology and geology.

Spatial point clusters can be modelled as fractals and characterized by their cluster dimension (Mandelbrot, 1983; Feder, 1988; Coleman and Pignonero, 1992; Agterberg, 1993; Agterberg, Cheng and Wright, 1993a). The underlying spatial structure of the spatial point processes may be multifractal instead of fractal (Cheng and Agterberg, 1994). Spatial statistics resulting from the multifractal model will be discussed in this chapter. Spatial statistics for point processes will be introduced in section 7.2; the multifractal model and relations between spatial statistics and multifractal model are discussed in section 7.3; box-counting and cluster determination methods for fractal analysis for point processes are illustrated in section 7.4; spatial dependence of the spatial point processes is discussed in section 7.5; and finally two case studies will be discussed in section 7.6 to demonstrate the methods developed in sections 7.2 to 7.5. These are (a) hickory and oak trees in Lansing Woods, Clinton, Michigan, U.S.A. (Gerrard, 1969; also see Diggle, 1983, Appendix A.4); and (b) Au mineral occurrences in the Iskut River map sheet,

northwestern British Columbia (B.C. Minfile 104B, 1989).

7.2 SPATIAL STATISTICS FOR POINT PROCESSES

Point processes have been intensively studied in spatial statistics (Ripley, 1981, 1988; Diggle, 1983; Cressie, 1991). The following notation can be used, x and y denote two points in the plane (two-dimensional space $\mathbb{R}^2$); $r = \|x - y\|$ is the distance between x and y ; $N(A)$ represents number of events (at points) within the block A with area $|A|$; $E(X)$ is the expectation of a random variable X ; dx and dy are infinitesimal regions which contain the points x and y , respectively; $b_x(r)$ represents the disc with centre x and radius r . The first-order intensity function (λ) and the second-order intensity function (λ_2) are commonly used for describing spatial point patterns in $\mathbb{R}^2$. They are defined as:

$$\lambda(x) = \lim_{|dx| \rightarrow 0} \frac{E[N(dx)]}{|dx|} \quad (7.2.1)$$

$$\lambda_2(x, y) = \lim_{|dx||dy| \rightarrow 0} \frac{E[N(dx)N(dy)]}{|dx||dy|} \quad (7.2.2)$$

For a stationary and isotropic process, $\lambda_2 = \lambda_2(\|x - y\|)$. It will be discussed in later sections that these two functions provide a basis for deriving the statistical moments of number of points per unit area. An alternative characterization of the second-order properties of a stationary and isotropic process is provided by the function $K(r)$. One definition of $K(r)$ (cf. Diggle, 1983, p.47) is:

$$K(r) = \lambda^{-1} E(\text{number of further events within distance } r \text{ of an arbitrary event})$$

The notion of an arbitrary event of the process involves the conceptual limit of random

sampling from a finite population. For a mathematical rigorous discussion see, for example, Daley and Vere-Jones (1972), Stoyan et al. (1987) and Ogata and Katsura (1991). A simplified presentation is as follows (cf. Agterberg, 1993):

$$\lim_{|dx| \rightarrow 0} \frac{E[N(dx)]}{P[N(dx)=1]} = 1 \quad (7.2.3)$$

$$\lim_{|dx||dy| \rightarrow 0} \frac{E[N(dx)N(dy)]}{P[N(dx)=N(dy)=1]} = 1 \quad (7.2.4)$$

which is equivalent to assuming that multiple coincident events cannot occur. Under the conditions Eqs. (7.2.3) and (7.2.4), the conditional intensity of an event at x , given an event at 0, is $\lambda_2(0,x)/\lambda$. It follows that the expected number of further events within distance r of an arbitrary event is

$$\lambda K(r) = \int \int_{b_r(0)} \{\lambda_2(0,x)/\lambda\} dx = 2\pi\lambda^{-1} \int_0^r \lambda_2(r) r dr \quad (7.2.5)$$

Conversely,

$$\lambda_2(r) = \lambda^2 (2\pi r)^{-1} \frac{dK(r)}{dr} \quad (7.2.6)$$

The covariance density can be defined as

$$C(r) = \lambda_2(r) - \lambda^2 \quad (7.2.7)$$

The practical advantage of $K(r)$ is that it can be estimated objectively from a set of data.

The mean and variance of the number of events ($N(A)$) in area A can be determined as follows. In general, the first two moments of $N(A)$ are:

$$E\{N(A)\} = \int_A \lambda(x) dx ; \quad E\{N^2(A)\} = E\left\{\int_A N(dx)\right\}^2 \quad (7.2.8)$$

For a stationary process, (7.2.8) becomes :

$$E\{N(A)\} = \lambda |A| ; \quad E\{N^2(A)\} = \lambda |A| + \int_A \int_A \lambda_2(|x-y|) dx dy \quad (7.2.9)$$

Consequently, the variance is :

$$\sigma^2\{N(A)\} = \int_A \int_A \lambda_2(|x-y|) dx dy + \lambda |A| (1 - \lambda |A|) \quad (7.2.10)$$

7.3 MULTIFRACTAL MODEL FOR SPATIAL POINT PROCESSES

From the multifractal point of view, $N(A)$ can be considered as a measure on area A . Suppose that A is rectangular in shape with size $n\epsilon_1 \times m\epsilon_2$ where $\epsilon_1 = k\epsilon$ and $\epsilon_2 = \ell\epsilon$ with ϵ being the unit of linear size. Consequently, A can be divided into nm subareas $A_{ij}(\epsilon_1, \epsilon_2)$, $i=1, 2, \dots, n$, $j=1, 2, \dots, m$. The number of cells with $N[A_{ij}(\epsilon_1, \epsilon_2)] > 0$ is denoted as $N(\epsilon)$. The partition function of the measure $N[A_{ij}(\epsilon_1, \epsilon_2)]$ is

$$\chi_q(\epsilon_1, \epsilon_2) = \sum_{ij}^{N(\epsilon)} [N[A_{ij}(\epsilon_1, \epsilon_2)]]^q \quad (7.3.1)$$

where $\sum$ is the sum for cells with $N[A_{ij}(\epsilon_1, \epsilon_2)] > 0$. If the measure $N(A)$ satisfies a multifractal model, then the partition function will have following power-law relation with the cell sizes ϵ_1 and ϵ_2 for any moment q or

$$\chi_q(\epsilon_1, \epsilon_2) \propto \epsilon_1^{\tau_1(q)} \epsilon_2^{\tau_2(q)} \quad (7.3.2)$$

For a square with size $\epsilon = \epsilon_1 = \epsilon_2$, it follows that

$$\chi_q(\mathbf{e}, \mathbf{e}) \propto e^{\tau(q)} \quad (7.3.3)$$

where $\tau(q) = \tau_1(q) + \tau_2(q)$. Setting $q=0$ in Eqs.(7.3.2) and (7.3.3) yields $N(\mathbf{e}) \propto e^{\tau(0)}$ and setting $q=1$ in Eq. (7.3.2) yields

$$\begin{aligned} E[MA(\mathbf{e}, \mathbf{e})] &= \frac{1}{nm} \sum_{i=1}^n \sum_{j=1}^m MA_{ij}(\mathbf{e}, \mathbf{e}) \propto \\ &\propto \frac{N(T)}{A(T)} e^{\tau(1)+2} \end{aligned} \quad (7.3.4)$$

By definition

$$\lambda = \lim_{e \rightarrow 0} \frac{E[MA(\mathbf{e}, \mathbf{e})]}{e^2} = \lim_{e \rightarrow 0} \frac{N(T)}{A(T)} e^{\tau(1)} \quad (7.3.5)$$

λ has a finite, positive value only if $\tau(1)=0$. The relationship between the second-order intensity $\lambda_2(r)$ can be derived as follows. Setting $q=2$ in Eq. (7.3.2),

$$E[N^2\{A(k\mathbf{e}, \ell\mathbf{e})\}] = c(k\mathbf{e})^{\tau_1(2)+1}(\ell\mathbf{e})^{\tau_2(2)+1} \quad (7.3.6)$$

where c is a constant. For an stationary, isotropic process, $\tau_1(2) = \tau_2(2) = 1/2\tau(2)$.

Suppose that the area $A(k\mathbf{e}, \ell\mathbf{e})$ consists of $k\ell$ unit cells $A(i, j)$ with $i=1, \dots, k$; $j=1, \dots, \ell$. According to this notation, the area $A[(k-1)\mathbf{e}, \ell\mathbf{e}]$ consists of $(k-1)\ell$ unit cells $A(i, j)$ with $i=1, \dots, k-1$; $j=1, \dots, \ell$. If $\mathbf{e}$ is small, the integral on the right side of Eq. (7.2.9) can be replaced by a sum as (7.3.7):

$$\begin{aligned} E[N^2(A)] &= \lambda |A| \\ &+ e^4 \sum_{i=1}^k \sum_{s=1}^k \sum_{j=1}^{\ell} \sum_{t=1}^{\ell} \lambda_2 [e\{(i-s)^2 + (j-t)^2\}^{\frac{1}{2}}] \end{aligned} \quad (7.3.7)$$

The first-order difference in expected value of $N^2(A)$ in the k -direction satisfies

$$\begin{aligned}
 \nabla_k E[N^2(A(k\mathbf{e}, \ell\mathbf{e}))] &= \\
 &= E[N^2(A(k\mathbf{e}, \ell\mathbf{e}))] - E[N^2(A((k-1)\mathbf{e}, \ell\mathbf{e}))] = \\
 &= \lambda \ell e^2 + \sum_{i=1}^k \sum_{s=1}^k \sum_{j=1}^{\ell} \sum_{t=1}^{\ell} \lambda_2 [e\{(i-s)^2 + (j-t)^2\}^{\frac{1}{2}}] e^4 - \\
 &\quad - \sum_{i=1}^{k-1} \sum_{s=1}^{k-1} \sum_{j=1}^{\ell} \sum_{t=1}^{\ell} \lambda_2 [e\{(i-s)^2 + (j-t)^2\}^{\frac{1}{2}}] e^4 = \quad (7.3.8) \\
 &= \lambda \ell e^2 + \sum_{i=1}^{k-1} \sum_{j=1}^{\ell} \sum_{t=1}^{\ell} \lambda_2 [e\{(i-s)^2 + (j-t)^2\}^{\frac{1}{2}}] e^4 + \\
 &\quad + \sum_{s=1}^k \sum_{j=1}^{\ell} \sum_{t=1}^{\ell} \lambda_2 [e\{(i-s)^2 + (j-t)^2\}^{\frac{1}{2}}] e^4
 \end{aligned}$$

Consequently, in the ℓ direction,

$$\begin{aligned}
 \nabla_{\ell} \nabla_k E[N^2(A(k\mathbf{e}, \ell\mathbf{e}))] &= \\
 &= \nabla_k E[N^2(A(k\mathbf{e}, \ell\mathbf{e}))] - \nabla_k E[N^2(A(k\mathbf{e}, (\ell-1)\mathbf{e}))] = \\
 &= \lambda e^2 + 4 \sum_{i=1}^{k-1} \sum_{j=1}^{\ell-1} \lambda_2 (e\{i^2 + j^2\}^{\frac{1}{2}}) e^4 + 2 \sum_{i=1}^{k-1} \lambda_2 (ie) e^4 + \quad (7.3.9) \\
 &\quad + 2 \sum_{j=1}^{\ell-1} \lambda_2 (je) e^4 + \lambda_2 (0) e^4
 \end{aligned}$$

Application of the same method to the right-hand side of Eq. (7.3.6) shows that Eq.(7.3.9)

is approximately equal to

$$[\tau_1(2) + 1] [\tau_2(2) + 1] [(k-1)\epsilon]^{\tau_1(2)} [(\ell-1)\epsilon]^{\tau_2(2)} \epsilon^2$$

Introducing simplified notation and setting $k=\ell$,

$$\begin{aligned}
\nabla_{kt}E(k) &= \nabla_i \nabla_k E[N^2\{A(k\epsilon, k\epsilon)\}] = \lambda\epsilon^2 + \\
&+ 4 \sum_{i=1}^{k-1} \sum_{j=1}^{k-1} \lambda_2 \{\epsilon(i^2+j^2)^{\frac{1}{2}}\} \epsilon^2 + 4 \sum_{i=1}^{k-1} \lambda_2(i\epsilon) \epsilon^2 + \lambda_2(0) \epsilon^2 \quad (7.3.10) \\
&\propto [\tau_1(2)+1] [\tau_2(2)+1] [(k-1)\epsilon]^{\tau(2)} \epsilon^2
\end{aligned}$$

The corresponding second-order difference is

$$\begin{aligned}
&\nabla_{kt}E(k+1) - 2\nabla_{kt}E(k) + \nabla_{kt}E(k-1) = \\
&= 8 \sum_{i=1}^k [\lambda_2\{\epsilon(i^2+k^2)^{\frac{1}{2}}\} - \lambda_2\{\epsilon(i^2+(k-1)^2)^{\frac{1}{2}}\}] \epsilon^2 + \\
&\quad + 4\lambda_2\{(k+1)\epsilon\sqrt{2}\} \epsilon^2 + 4\lambda_2(k\epsilon\sqrt{2}) \epsilon^2 + \\
&\quad + 4[\lambda_2\{(k+1)\epsilon\} - \lambda_2(k\epsilon)] \epsilon^2 \quad (7.3.11)
\end{aligned}$$

For small ϵ , the right side of (7.3.11) becomes approximately $8\lambda_2(k\epsilon\sqrt{2})\epsilon^2$. Consequently,

$$\begin{aligned}
\lambda_2(k\epsilon\sqrt{2}) &\sim \frac{1}{8} \{\tau_1(2)+1\} \{\tau_2(2)+1\} \epsilon^{\tau(2)} \\
&\quad \{(k+1)^{\tau(2)} - 2k^{\tau(2)} + (k-1)^{\tau(2)}\} \quad (7.3.12)
\end{aligned}$$

Setting $r=k\epsilon\sqrt{2}$ and replacement of the second-order difference for λ_2 by its second derivative results in

$$\lambda_2(r) \sim \frac{c_2}{8} \left\{ \frac{1}{2} \tau(2) + 1 \right\}^2 \tau(2) \{\tau(2)-1\} 2^{1-\frac{1}{2}\tau(2)} r^{\tau(2)-2} \quad (7.3.13)$$

where c_2 is another constant. This shows that the second-order intensity λ_2 has approximately a power-law relation with distance r . The exponent is $\tau(2)-2$. From Eqs. (7.2.2) and (7.3.13) it follows that, for small ϵ ,

$$\begin{aligned}\lambda^2 K(k\epsilon) &= 2\pi \int_0^{k\epsilon} \lambda_2(r) r dr = 2\pi \epsilon^2 \int_0^k \lambda_2(r) r dr \sim \\ &\sim D_1 2^{1-\frac{1}{2}\tau(2)} \epsilon^{\tau(2)} 2\pi k^{\tau(2)}\end{aligned}\quad (7.3.14)$$

where $D_1 = (c/8)[1/2 \tau(2)]^2$ is a constant. Or, using Eqs. (7.2.2) and (7.3.12), and replacing the integral by a sum, yields

$$\begin{aligned}\lambda^2 K(k\epsilon) &\sim 2\pi \epsilon^2 \sum_{s=1}^k \lambda_2(s\epsilon) s \sim \\ &\sim D_1 \epsilon^{\tau(2)} 2^{1-\frac{1}{2}\tau(2)} 2\pi k(k+1) [(k+1)^{\tau(2)-1} - k^{\tau(2)-1}]\end{aligned}\quad (7.3.15)$$

Consequently,

$$K(r) \sim D_2 r^{\tau(2)} \quad (7.3.16)$$

where D_2 is another constant.

$$D_2 = \frac{c\pi}{4} [\tau(2)+1]^2 [\tau(2)-1] 2^{1-\frac{1}{2}\tau(2)} \left\{ \frac{A(T)}{N(T)} \right\}^2 \quad (7.3.17)$$

which represents a power-law relation between $K(r)$ and distance r with exponent $\tau(2)$. In general, however, Eq. (7.3.16) gives better results than Eq. (7.3.13) and the step from Eqs.(7.3.12) to (7.3.13) should be avoided.

7.4 BOX-COUNTING DIMENSION AND CLUSTER DIMENSION

The two methods most commonly used are box-counting and cluster density determination (Mandelbrot, 1983; Agterberg 1993). The box-counting method results in the following

power-law relation

$$N(\epsilon) \propto \epsilon^{-D_b} \quad (7.4.1)$$

where $N(\epsilon)$ is the number of cells containing one or more points and D_b is the so-called box-counting dimension. The cluster density determination method can be performed by centering circles with different radii on all points in the region, counting how many other points occur within these circles and averaging the results. The relation between cluster density and radius is according to a power-law with exponent $D_c - 2$, where D_c is the so-called cluster dimension. Various estimators can be used for obtaining the cluster dimension, e.g., covariance function $C(r)$ (Ogata et al., 1991); second-order intensity function and $K(r)$ function (Agterberg, 1993; Agterberg, Cheng and Wright, 1993a). Carlson (1991) applied both box-counting and cluster density determination methods to characterize ore deposits. His work shows that for distances less than 15km, the cluster dimension $D_c = 0.83$ versus box-counting dimension $D_b = 0.50$. For distances between 15km and 1000km, the two dimensions are $D_c = 1.17$ versus $D_b = 1.51$, respectively. These two dimensions are not necessarily the same. Actually from a multifractal point of view these two dimensions are expected to be different with $D_c \neq D_b$. It is shown in section 7.3 that the second-order intensity function and $K(r)$ function can be used as estimators for computing the cluster dimension $D_c(\approx \tau(2))$.

7.5 SPATIAL INDEPENDENCE OF POINT PROCESSES

Testing for spatial independence of point distributions is a topic of special interest in the

study of spatial point processes. There are three major types of point processes with different point patterns: complete spatial randomness (CSR), regularity, and aggregation. Methods to test for these properties include use of inter-event distances, dispersion or Pearson statistics and Monte Carlo tests (Diggle, 1983). A complete spatial random point process has the following properties:

1. For some $\lambda > 0$, and any finite region A, $N(A)$ has a Poisson distribution with mean $\lambda |A|$ and

$$P[N(A) = k] = \frac{1}{k!} [\lambda |A|]^k e^{-\lambda |A|} \quad (7.5.1)$$

2. For any two disjoint regions A and B, $N(A)$ and $N(B)$ are independent with

$$P[N(A) = x, N(B) = y] = P[N(A) = x] P[N(B) = y] \quad (7.5.2)$$

3. The mean and second moment of $N(A)$ are

$$E[N(A)] = \lambda |A| ; E[N^2(A)] = \lambda^2 |A|^2 + \lambda |A| \quad (7.5.3)$$

4. From Eqs. (7.5.3), (7.2.1) and (7.2.2), we have

$$\lambda(x) = \lambda ; \lambda_2(x, y) = \lambda^2 \quad (7.5.4)$$

5. Combining Eqs. (7.5.4) and (7.2.5) yields

$$K(r) = \pi r^2 \quad (7.5.5)$$

The properties Eqs. (7.5.4) and (7.5.5) can be used for testing CSR of a point process. For example, the estimated values of $\bar{K}(r) - \pi r^2$ can be plotted against r and the departure of the

values from zero means that the point process is not CSR (Diggle, 1983). One can also plot the estimated values $[\bar{K}(r)/\pi]^{1/2}$ against r . Significant departure of the values from r indicates that the point process is not CSR (Diggle, 1983; Agterberg, 1993). Eq.(7.3.5) shows that if $\tau(1)=0$, then λ becomes constant, and Eq.(7.3.13) indicates that if $\tau(2)=2$, λ_2 is constant. Therefore, if $\tau(2)=2$ and $\tau(1)=0$ in Eq.(7.3.16), then $K(r)=2\pi D_1 r^2$ corresponding to CSR. In this special case, the multifractal measure will become a nonfractal measure with $f(\alpha)\equiv\alpha=2$. Usually, $0 < \tau(2) < 2$, indicates a cluster point process and, therefore, $\tau(1)$ is the index which characterizes the first-order intensity λ and $\tau(2)-\tau(1)$ determines the second-order intensity function λ_2 . These indexes can be used to describe the aggregation or CSR of point processes.

According to Pearson's criterion

$$\chi^2 = \frac{\sum_{ij}^{N(\epsilon)} (N(A_{ij}) - \bar{N})^2}{N} \quad (7.5.6)$$

is distributed as χ^2 with $N(\epsilon)-1$ degrees of freedom under CSR, $\bar{N}$ being the average value of $N(A_{ij})$. Significantly small values of χ^2 are of interest because they indicate a tendency towards regularity; very large values indicate aggregation (Diggle, 1983). From Eq. (7.3.1) and (7.3.3), it follows that

$$\chi^2 \sim N(T) \{A(T) e^{\tau(w)-2} - 1\} \quad (7.5.7)$$

which shows that χ^2 would remain constant if $\tau(2) \approx 2$. On the other hand, if $\tau(2) < 2$, then χ^2 would increase infinitely as ϵ tends to zero. Thus, for the purpose of statistical

inference, the case $\tau(2) < 2$ indicating a form of aggregation can be tested against the null hypothesis of CSR.

7.6 APPLICATIONS TO TWO CASE STUDIES

7.6.1 Hickories and Oaks in Lansing Woods

Gerrard (1969) describes an investigation of a 19.6 acre square plot in Lansing Woods, Clinton, Michigan, U.S.A. He provided the locations of 2251 trees (mainly hickories, maples and oaks) in this plot. Maps for 703 hickories and 929 oaks are shown in Figs. 7.6.1a and 7.6.1b, respectively. Diggle (1983) tested the hypothesis of spatial independence of the trees by several methods, reaching the conclusion that, although the underlying spatial point processes are isotropic, the hickories and maples show significant aggregation whereas the CSR model can be accepted for the oaks.

In order to analyze the data by means of the multifractal model, several linear cell sizes ranging from 0.071 to 0.5 were used for computing the partition function. The results for hickories are shown in Figs. 7.6.2a-d, and those for oaks in Figs. 7.6.3a-d. The values of $\tau(1)$ are close to 0 for both hickories and oaks indicating constant first-order intensity. The second-moment mass exponents $\tau(2)$ are estimated to be 1.821 ± 0.025 for hickories (also see Fig. 7.6.4a) and 1.913 ± 0.04 for oaks (Fig. 7.6.4b). Suppose that r_{ij} is the distance between points i and j in A . Values of $K(r)$ were estimated by using Ripley's edge effect

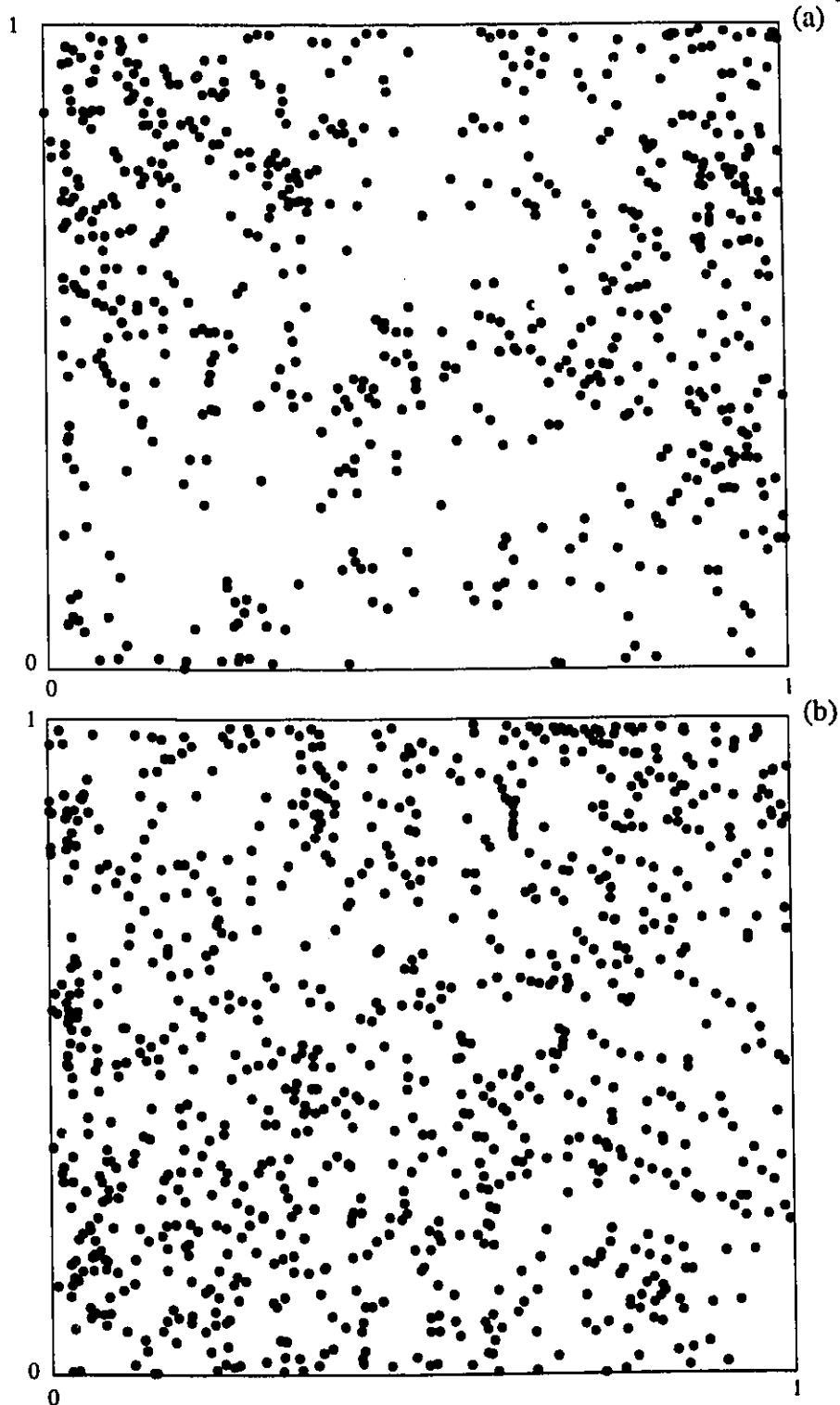
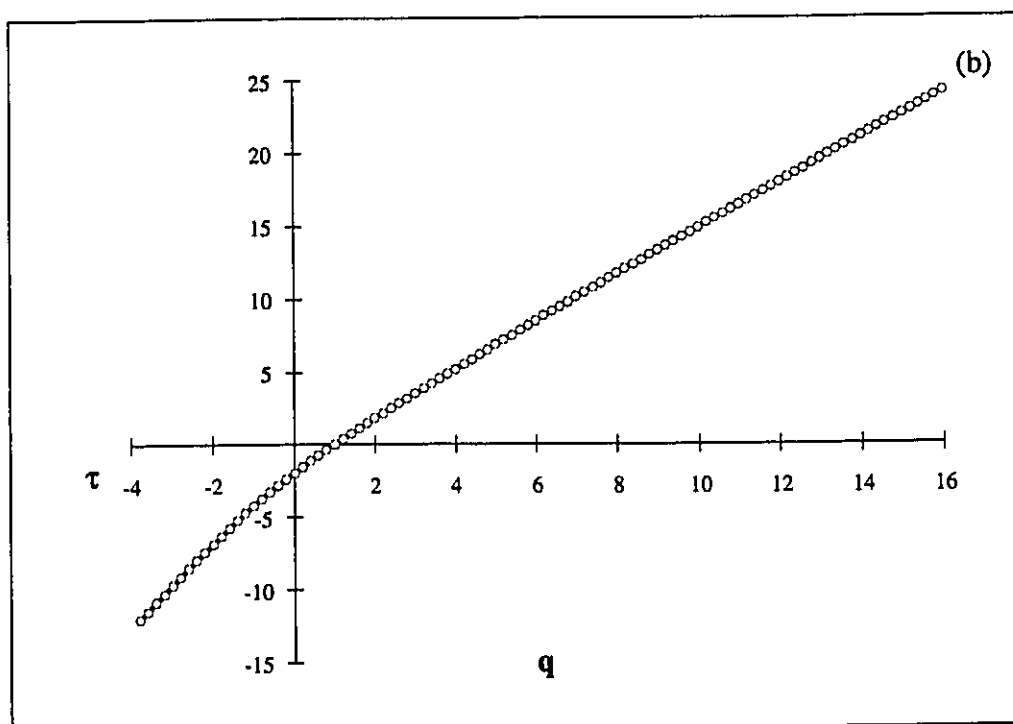
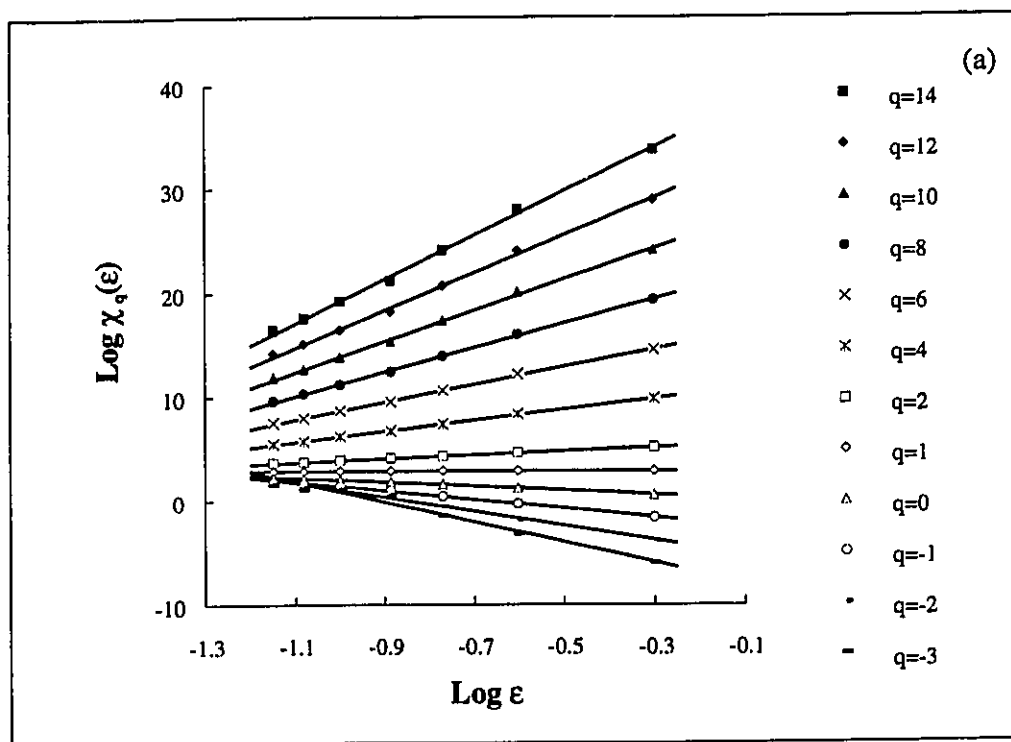


Fig. 7.6.1 Locations of trees in Lansing Woods, Clinton, Michigan, U.S.A. (Gerrard, 1969). (a) hickories; (b) oaks.



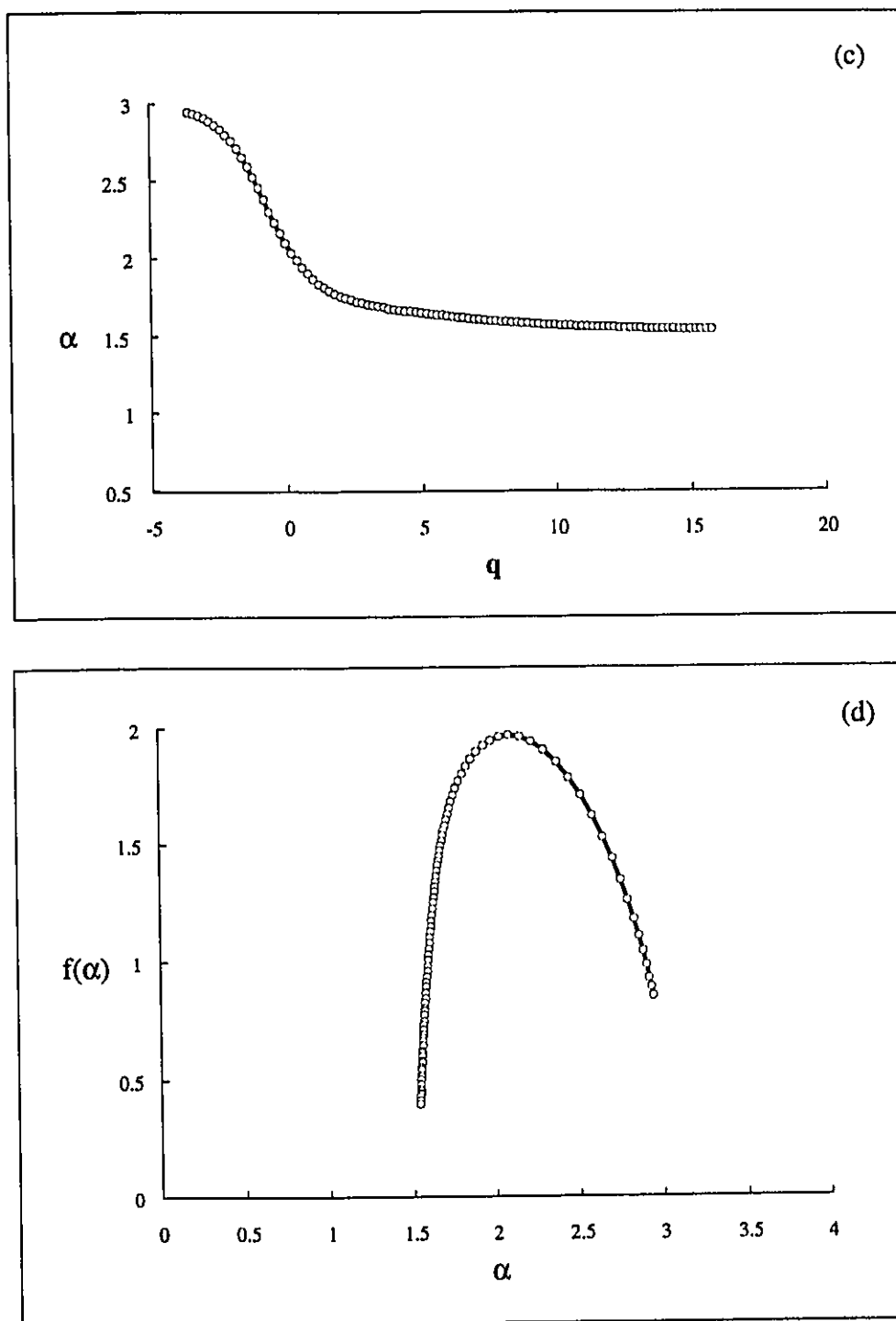
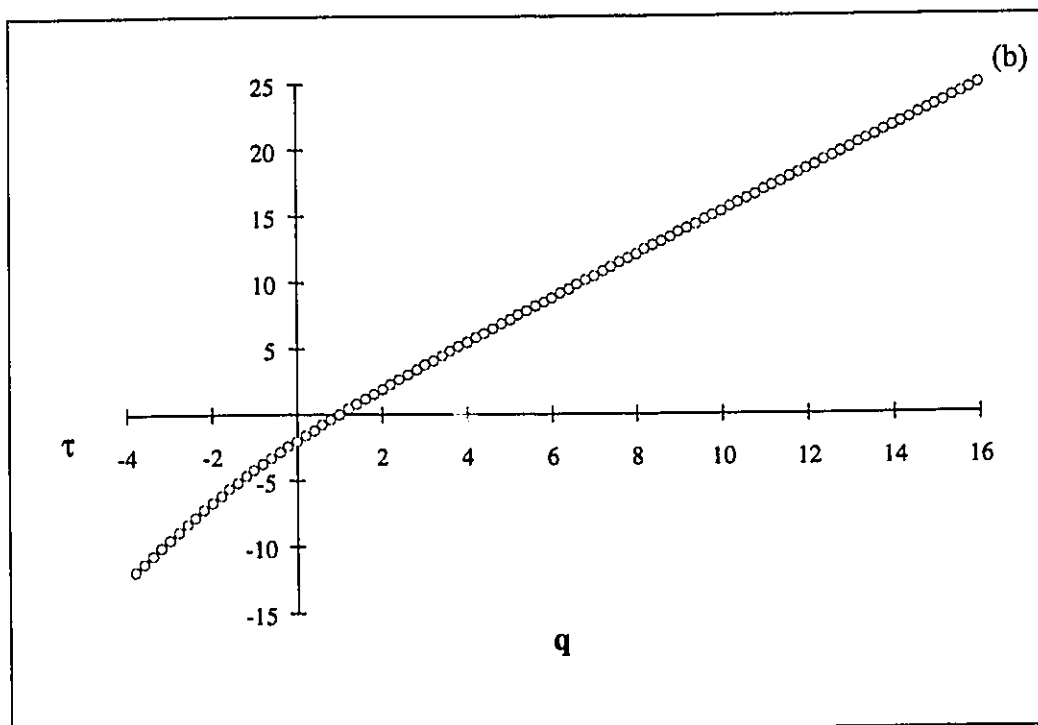
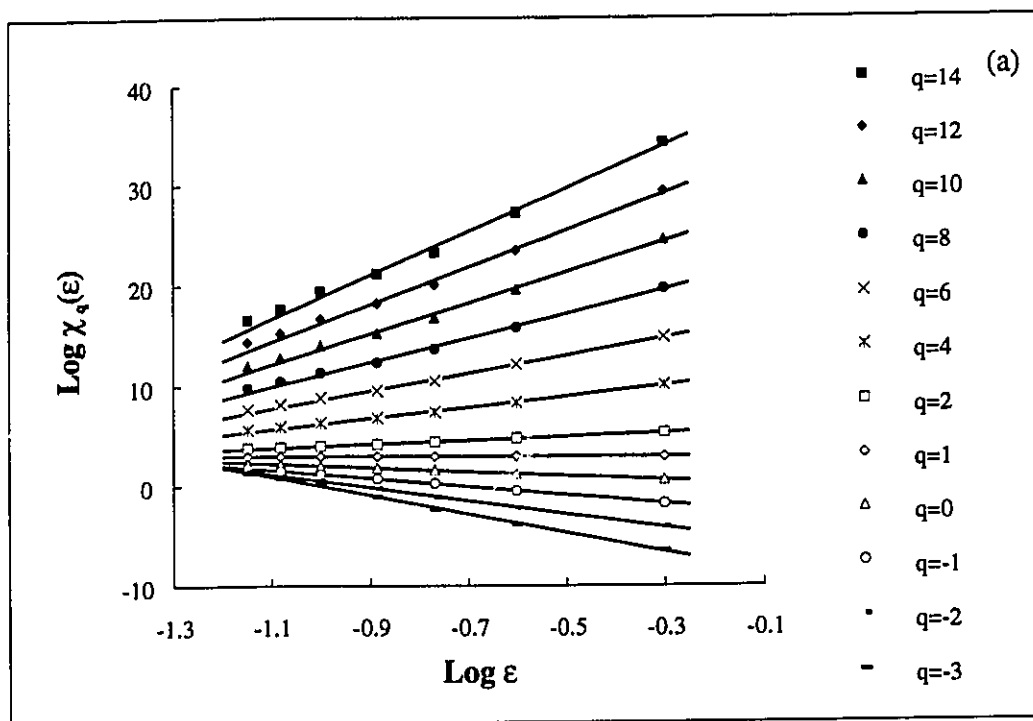


Fig. 7.6.2 Multifractal analysis of Gerrard's example of hickories. (a) Log-log plot of $\chi_q(\epsilon)$ versus ϵ for selected values of q ; (b) $\tau(q)$ versus q . (c) $\alpha(q)$ versus q ; (d) Multifractal spectrum showing $f(\alpha)$ versus α .



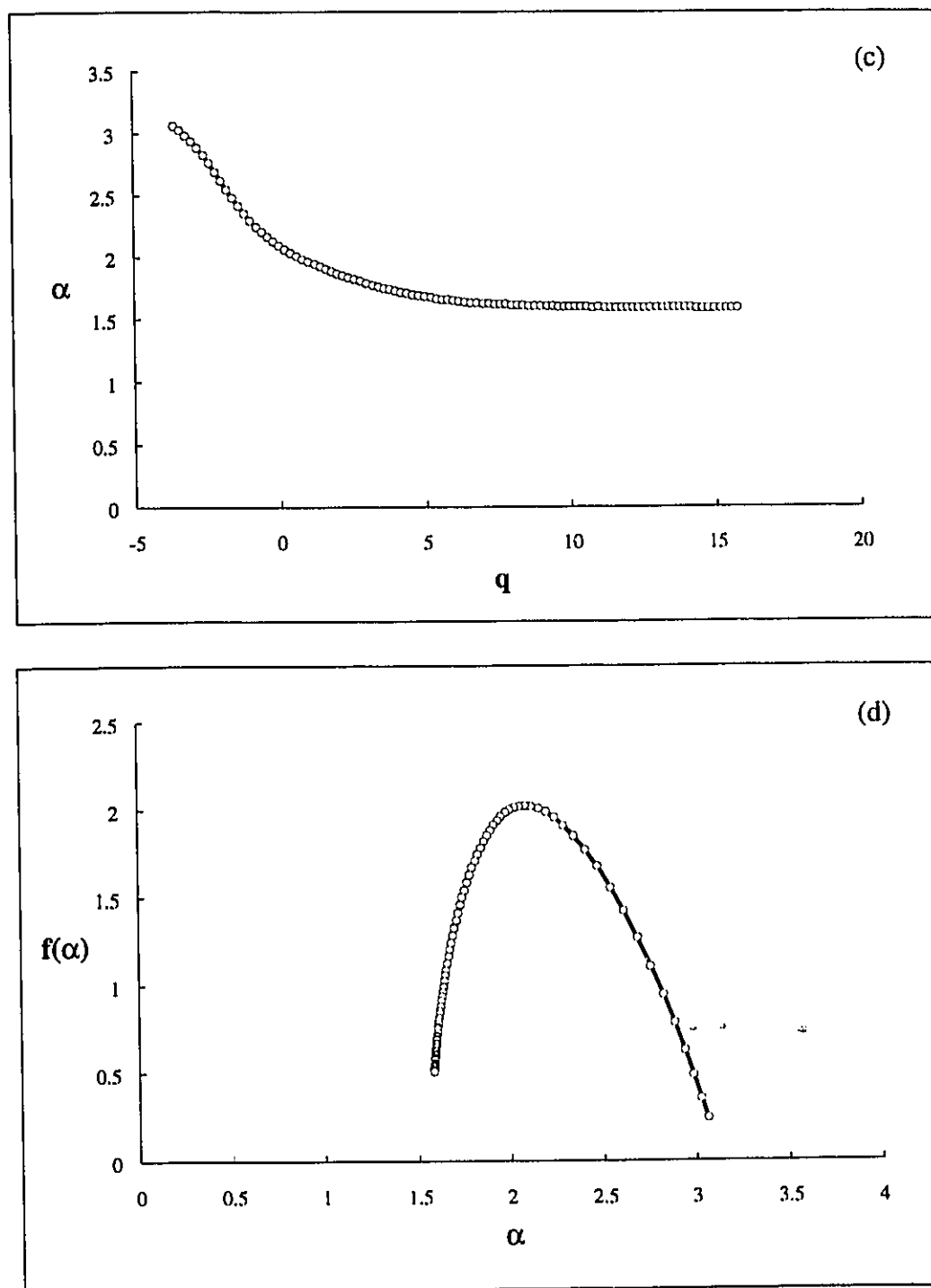


Fig. 7.6.3 Multifractal analysis of Gerrard's example of oaks. (a) Log-log plot of $\chi_q(\epsilon)$ versus ϵ for selected values of q ; (b) $\tau(q)$ versus q ; (c) $\alpha(q)$ versus q ; (d) Multifractal spectrum showing $f(\alpha)$ versus α .

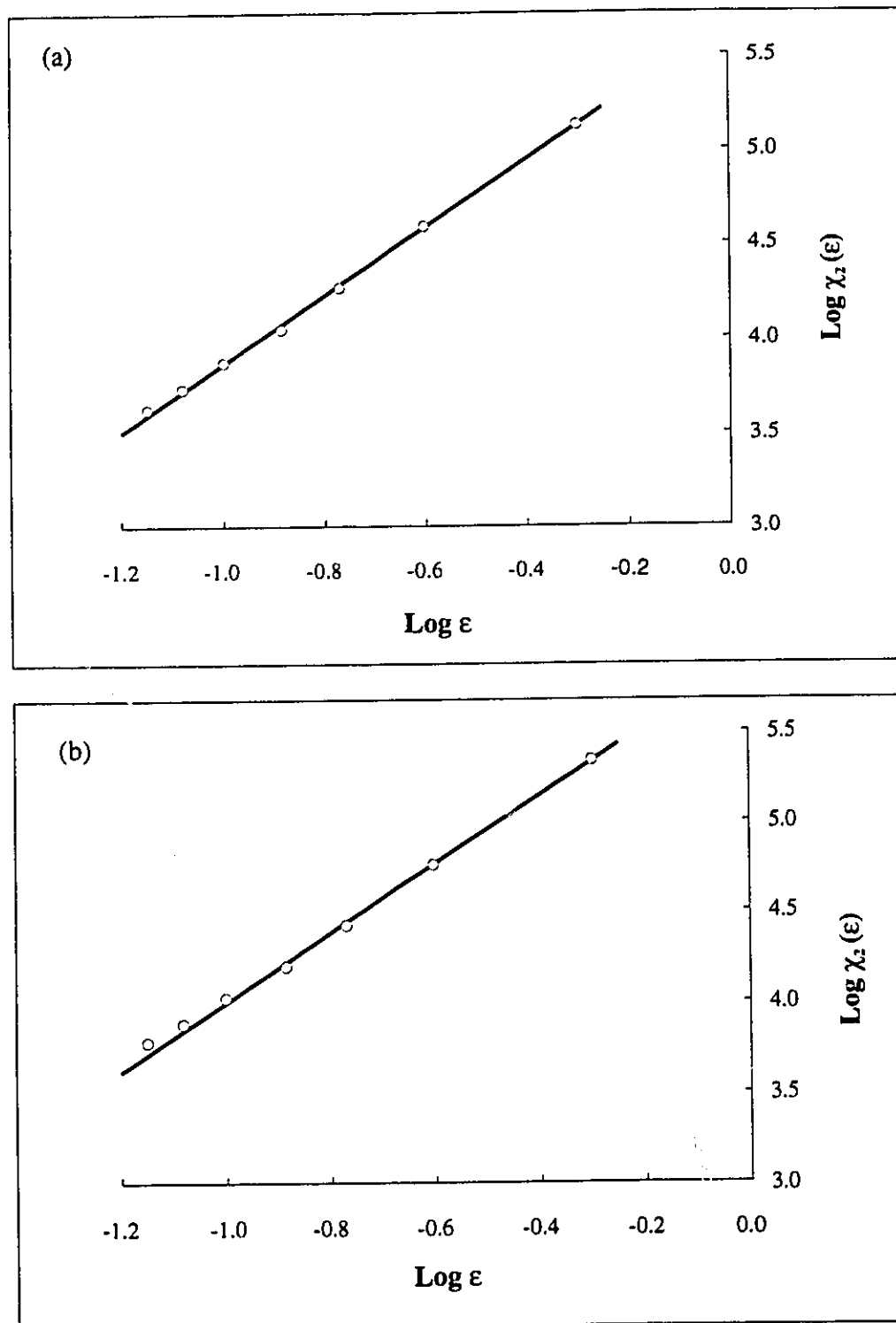
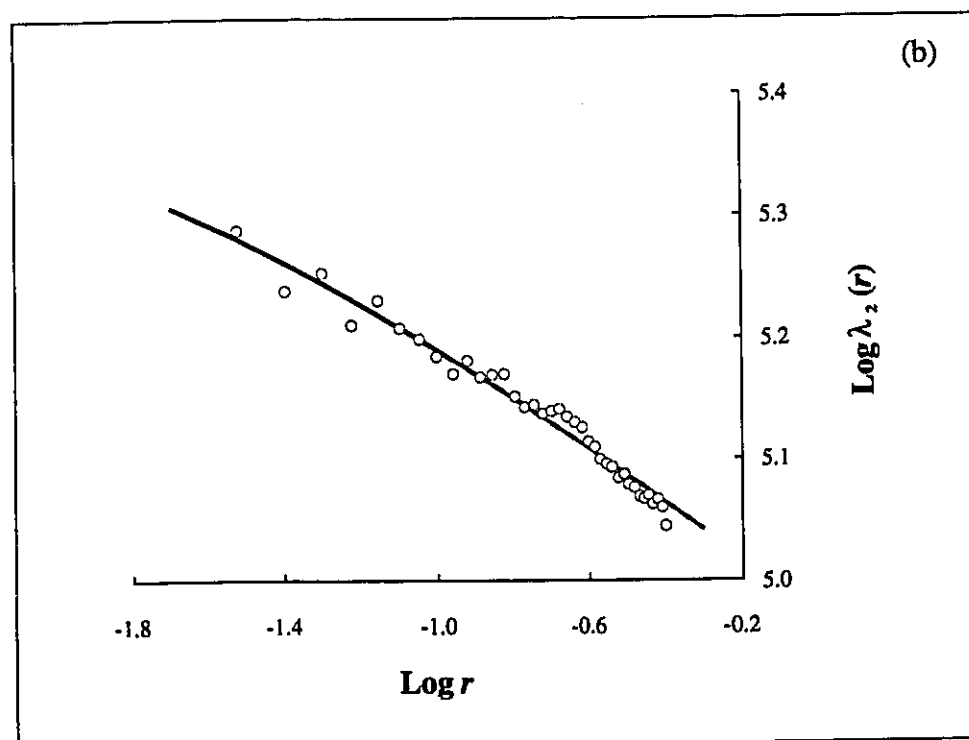
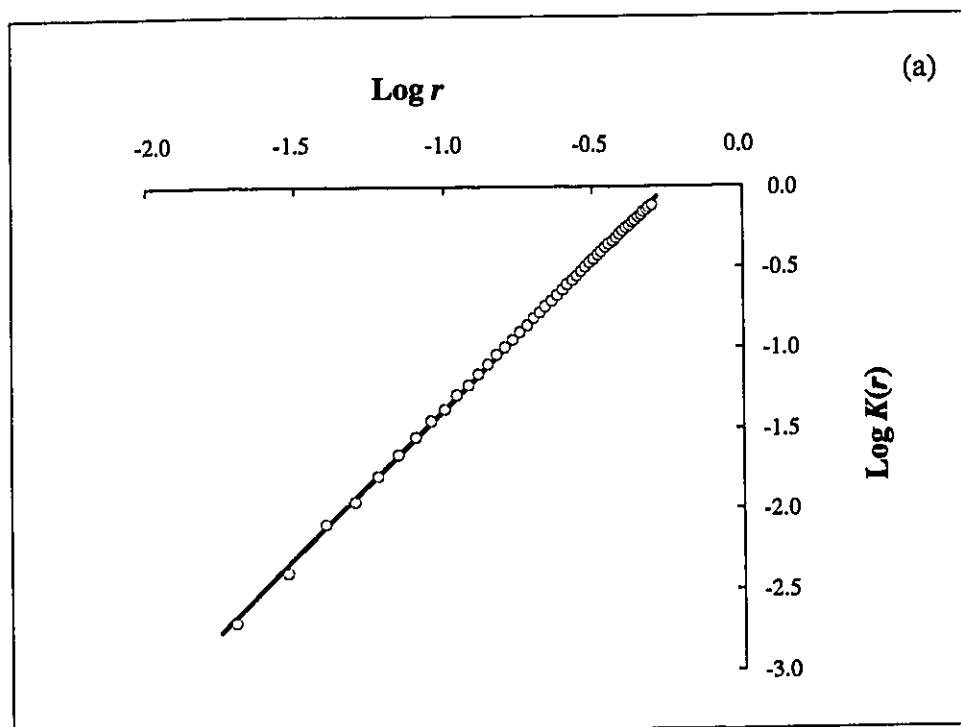


Fig. 7.6.4 Log-log plot of relationship between $\chi_q(\epsilon)$ and ϵ for $q=2$; $\tau(2)$ estimated as slope of least squares (LS) line. (a) hickories; (b) oaks

correction (Ripley, 1988)

$$K(r) = \frac{|A|}{N(T)^2} \sum_{i,j} w_{ij}^{-1} I_r(r_{ij}) \quad (7.6.1)$$

where $I_r(r_{ij})$ is an indicator function assuming the value 1 if $r_{ij} < r$; 0 otherwise. The weight w_{ij} represents the proportion of the circumference of the circle around point i with radius r_{ij} that lies within A . Use was made of the explicit formula for w_{ij} developed for a rectangular study area (Diggle, 1983, p. 72; also see Agterberg, 1994b). In Eq. (7.6.1) it is assumed $\lambda = \text{constant}$. Therefore, the estimated results are consistent with the second relation in Eq. (7.3.16). The resulting estimates of $K(r)$ are shown in Fig. 7.6.5 for hickories and Fig. 7.6.6 for oaks. For $r \leq 0.25$, these estimates are identical to those previously obtained by Diggle (1983, Fig. 62). In Figs. 7.6.5a and 7.6.6a, the values of $K(r)$ were fitted by least squares in accordance with Eq. (7.3.16) yielding second-moment mass exponent $\hat{\tau}(2)=1.835$ for hickories (nearly equal to $\tau(2)=1.821$) and $\hat{\tau}(2)=1.927$ for oaks (close to $\tau(2)=1.913$). Figures 7.6.5b and 7.6.6b show second-order intensities λ_2 for hickories and oaks estimated from successive pairs of values for $K(r)$ by means of the central difference technique. The solid lines in these figures satisfy Eq. (7.3.12) with $\tau(2)=1.835$ and 1.927, respectively. Clearly the straight line model corresponding to Eq.(7.2.12) is less satisfactory for Figs. 7.6.5b and 7.6.6b. Figs. 7.6.5c to 7.6.5d illustrate that the hickories do not satisfy the CSR model. Plots in Figs. 7.6.5c and 7.6.6c are for values of $[K(r)/\pi]^{0.5}$ against r . For oaks the estimated values fall nearly on this line and the CSR model can be accepted in that case. Figs. 7.6.5d and 7.6.6d illustrate the values of $K(r)-\pi r^2$ against r . Similar conclusions can be drawn from these figures.



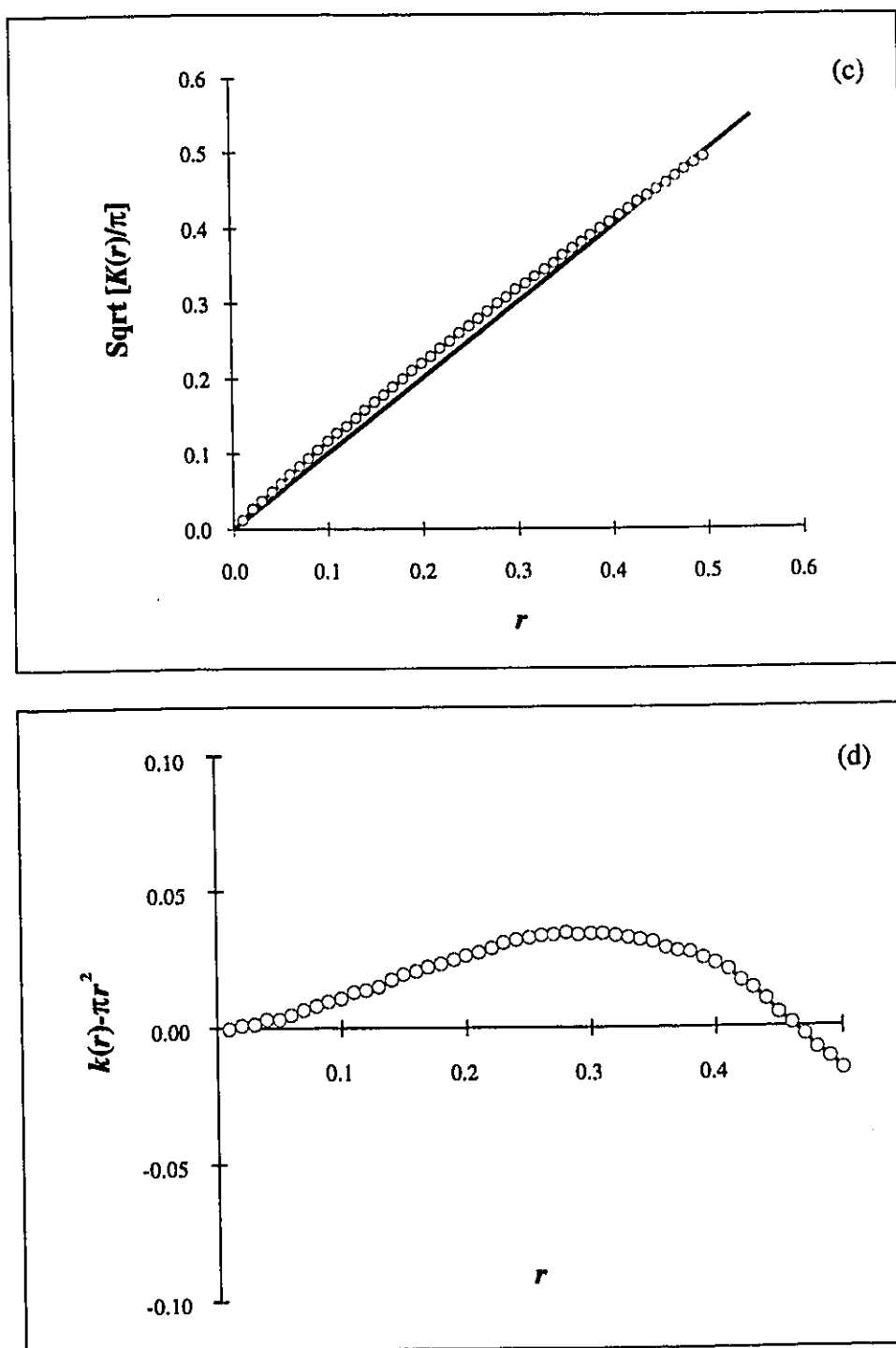
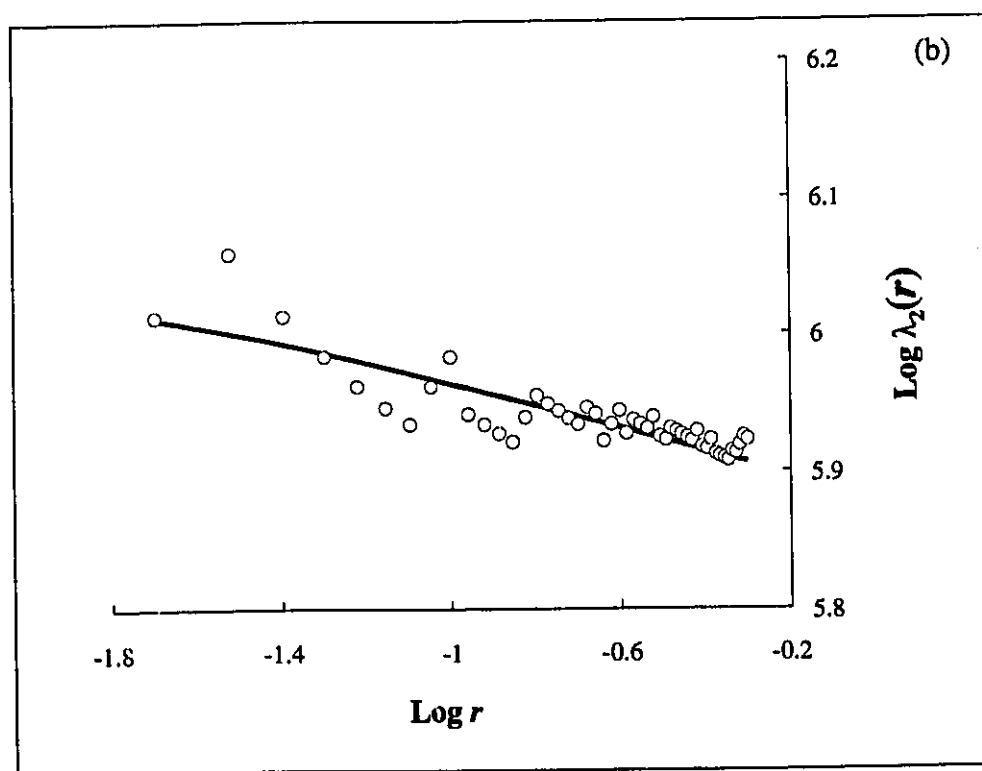
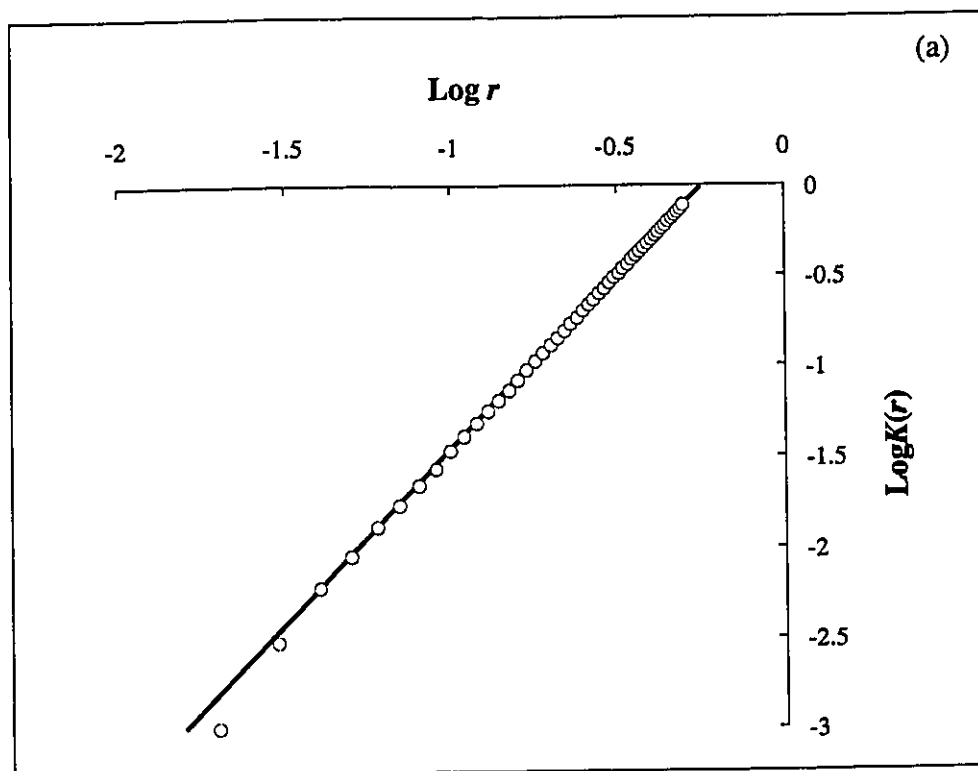


Fig. 7.6.5 Spatial statistics for hickories. (a) Log-log plot of relationship between $K(r)$ and distance r using Ripley's edge effect correction; LS straight line fitted to all points except the first two on the left, indicates approximate power-law relation with exponent 1.835; (b) Log-log plot for relationship between second-order intensity λ_2 and r , solid line satisfies Eq. (7.3.13); (c) Estimated values of $[K(r)/\pi]^{0.5}$ versus r ; (d) Estimated values of $K(r) - \pi r^2$ versus r . (c) and (d) show departure from CSR.



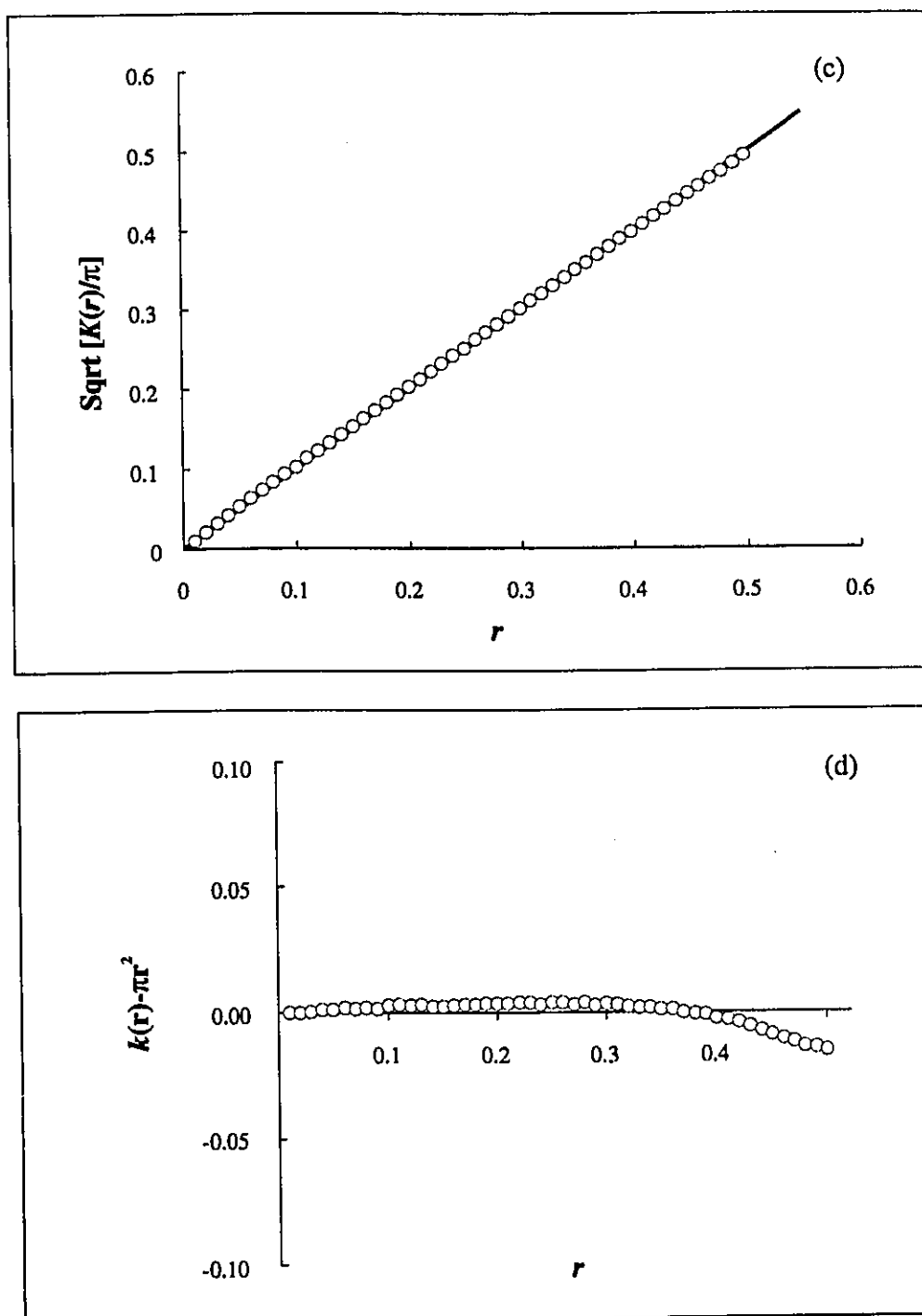


Fig. 7.6.6 Spatial statistics for oaks. (a) Log-log plot of relationship between $K(r)$ and distance r using Ripley's edge effect correction; LS straight line fitted to all points except the first two on the left, indicates approximate power-law relation with exponent 1.927; (b) Log-log plot for relationship between second-order intensity λ_2 and r , solid line satisfies Eq. (7.3.13); (c) Estimated values of $[K(r)/\pi]^{0.5}$ versus r ; (d) Estimated values of $K(r) - \pi r^2$ versus r . (c) and (d) show properties of CSR.

7.6.2 Gold mineral occurrences in the Iskut River map sheet, northwestern British Columbia

There are 183 documented Au mineral occurrences in the Iskut River map sheet (Fig. 7.6.7). The spatial aggregation of these mineral occurrences is clearly shown on the map. In order to use the multifractal model, different cell sizes (ϵ) ranging from 3 to 20 km will be used. The grid maps with cell sizes ranging from 3 to 20 km are illustrated in Fig. 7.6.8, corresponding to different sampling densities using the SPANS point modelling procedure. Due to the Voronoi technique, the cells on each maps may vary in sizes around the mean ϵ_j . Denote α_{ji} as the size of cell i on map j . For correction of the variations of cell size, the following formula can be used (Agterberg, et al., 1994b):

$$\chi_q(\epsilon_j) = \sum_{i=1}^{N(\epsilon_j)} \frac{\alpha_{ji}}{\epsilon_j^2} \left[\frac{\epsilon_j^2}{\alpha_{ji}} \mu_{ji} \right]^q \quad (7.6.2)$$

Better results are obtained by means of the correction in Eq. (7.6.2), especially for relatively large cell sizes. Estimated results are illustrated in Fig. 7.6.9a to 7.6.9d with q ranging from 0 to 4. It shows that the partition function has power-law type relations only for $0 \leq q \leq 4$. It is estimated that $\tau(0)=-1.335 \pm 0.077$, $\tau(1)=0$ and $\tau(2)=1.219 \pm 0.037$. The estimates of $K(r)$ were made using the program CLUST (Agterberg, 1993) for edge effect corrections. The results were plotted in the log-log plots of Fig. 7.6.10a. The values were fitted by the least square method with Eq. (7.3.15) setting $\tau(2)=1.219 \pm 0.037$. It shows that for $r > 1$ km, the values of $K(r)$ approach a straight line on log-log plot corresponding to a power-law type relation. Values of λ_2 were estimated from Fig. 7.6.10a by the central

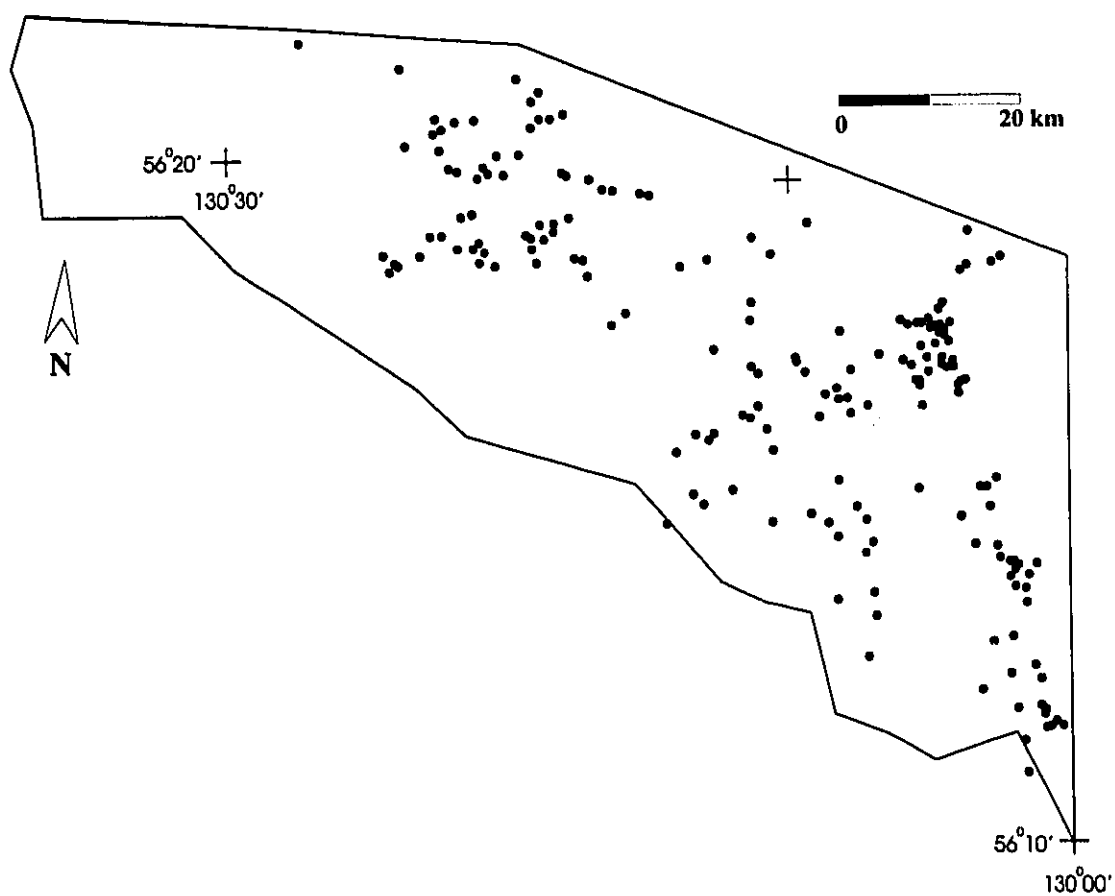
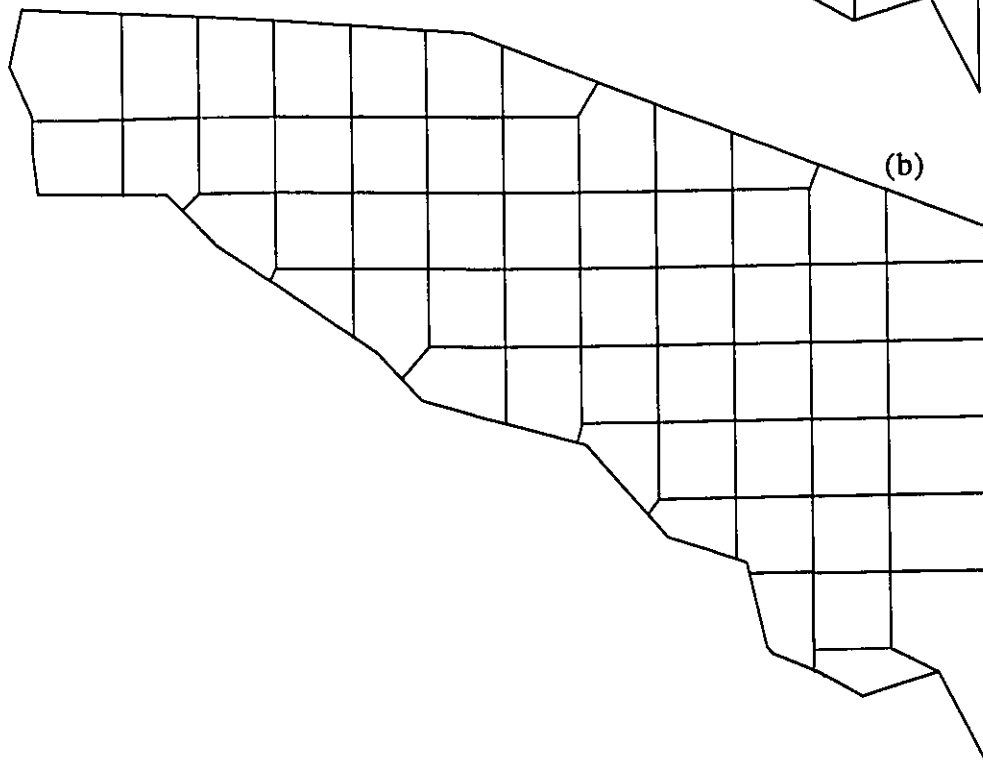
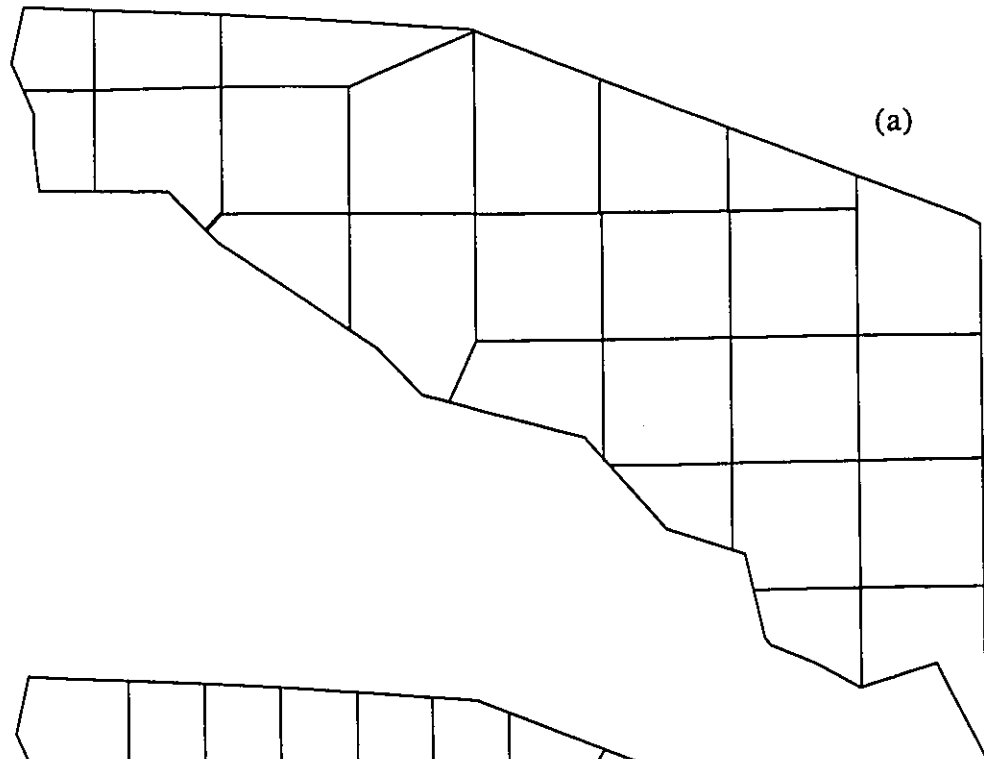


Fig. 7.6.7. Au mineral occurrences in the Iskut River map sheet, northwestern British Columbia (B.C. Minfile Map, 1989)



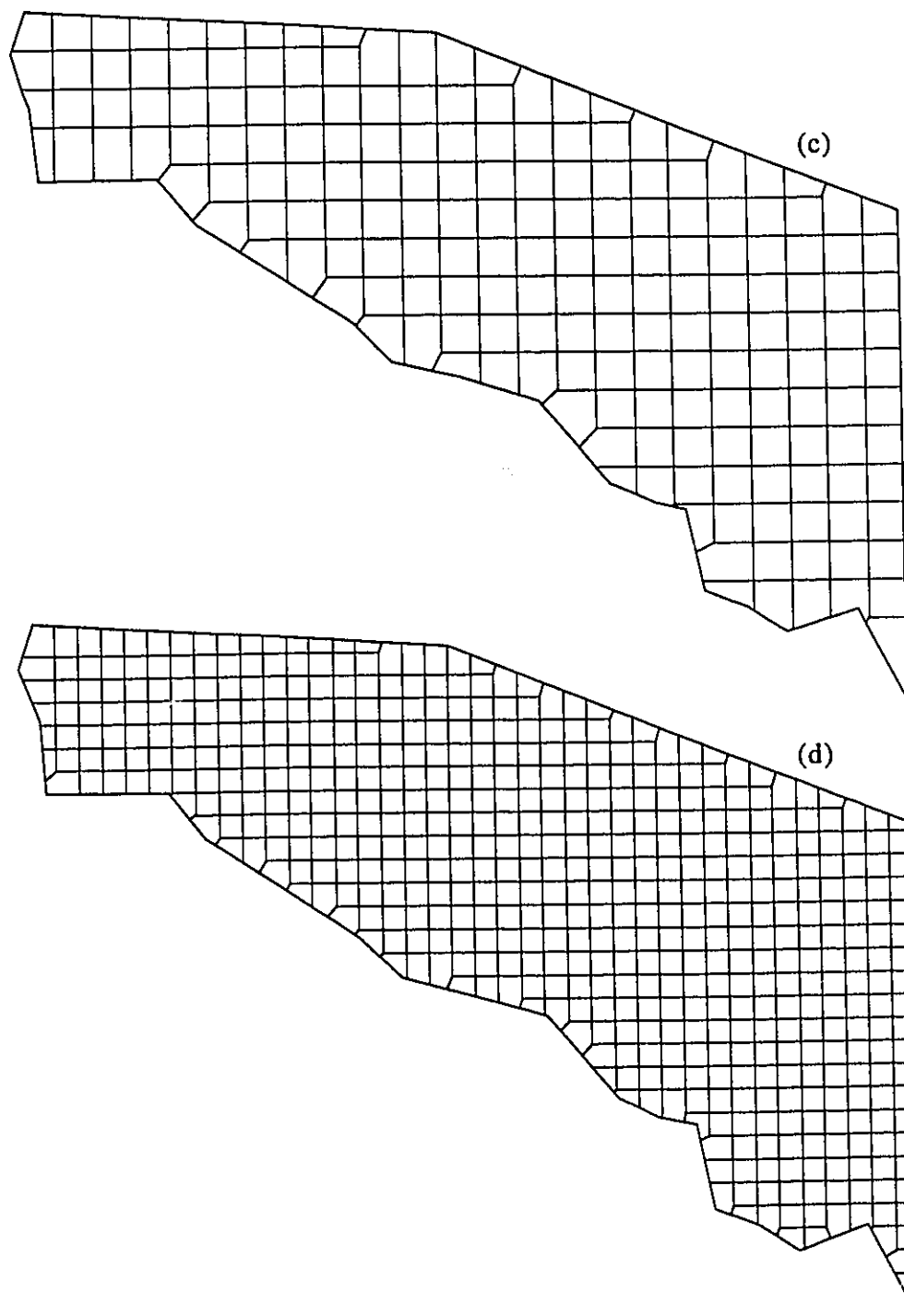
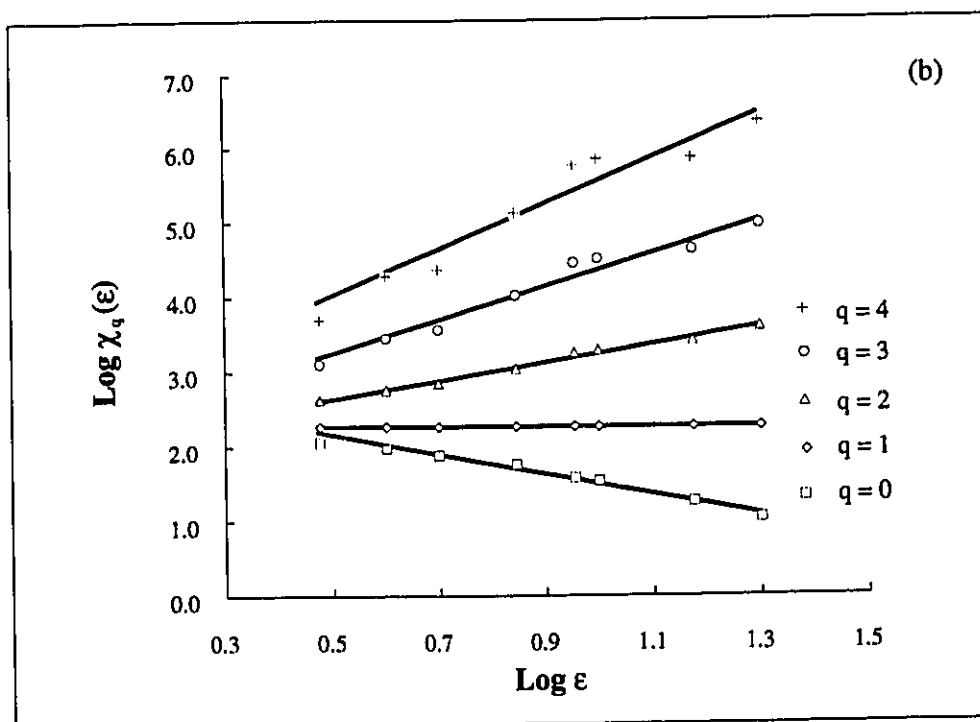
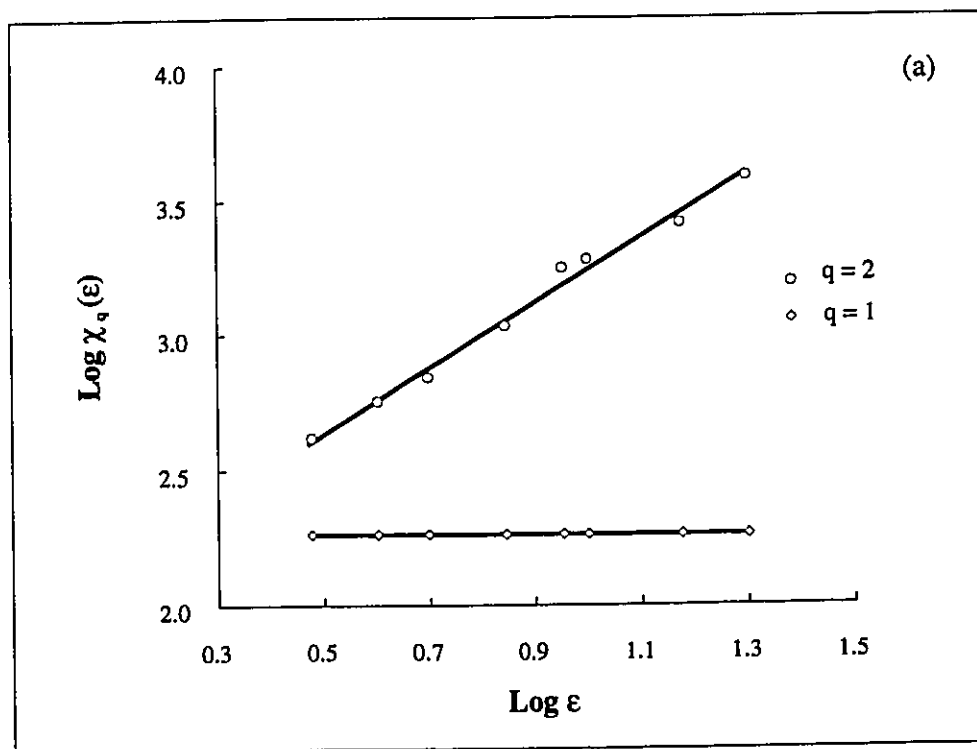


Fig. 7.6.8 Grid maps created by using SPANS point modelling. (a) Sampling density of 225 km² per point; (b) Sampling density of 81 km² per point; (c) Sampling density of 25 km² per point; (d) Sampling density of 9 km² per point



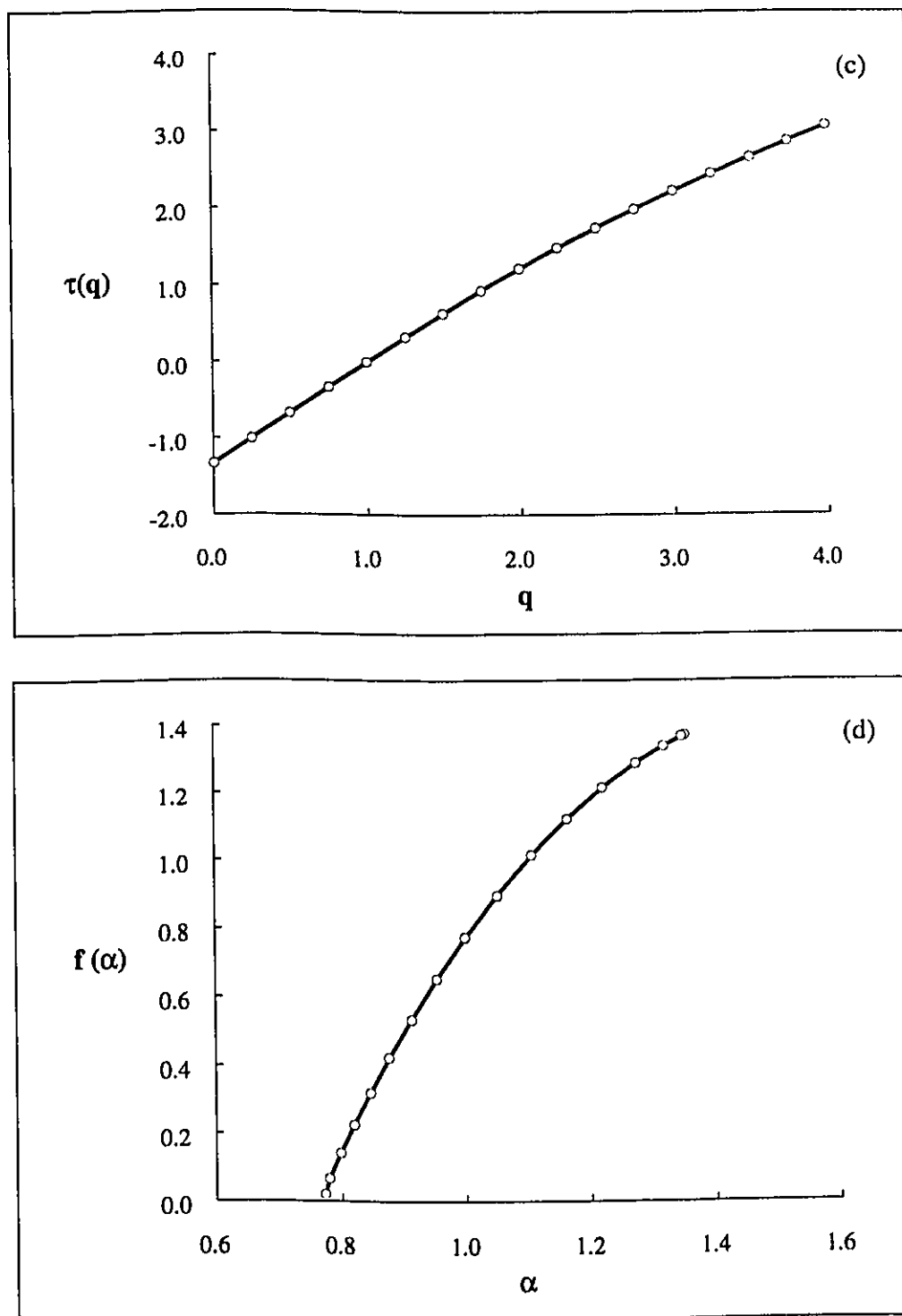
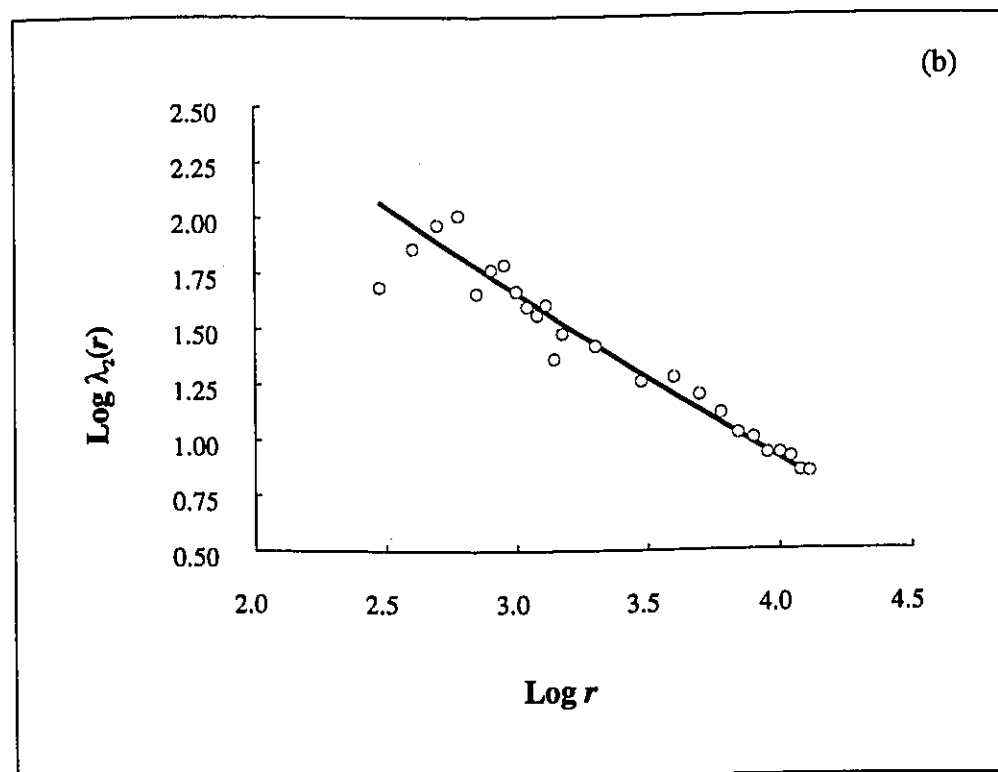
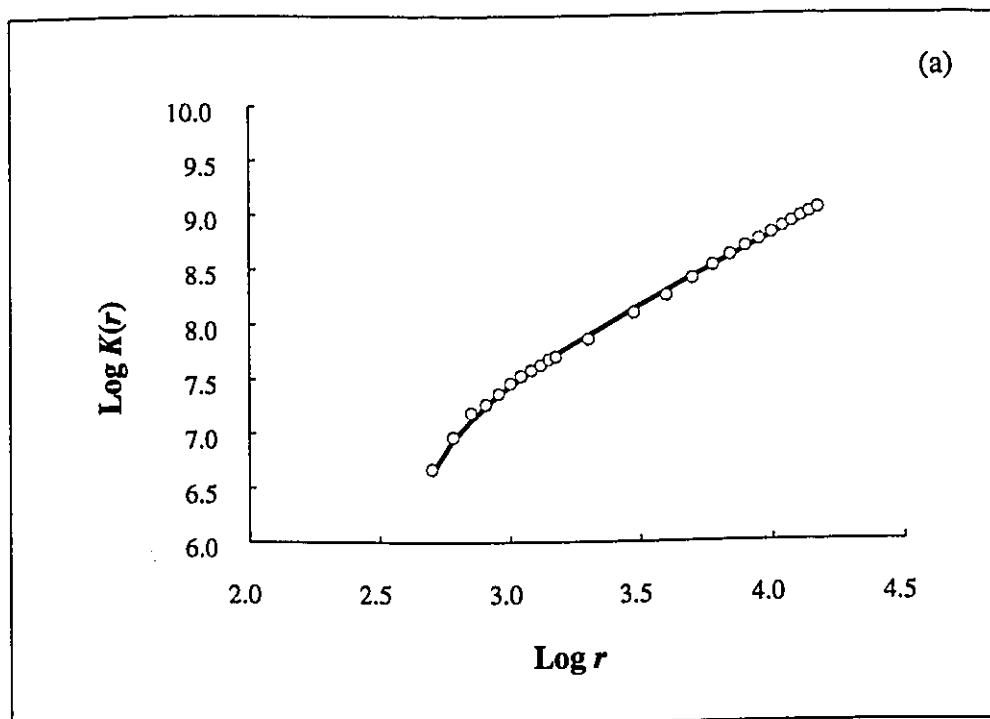


Fig. 7.6.9 Results obtained by multifractal analysis for Au mineral occurrences. (a) $\chi_q(\epsilon)$ versus ϵ for $q = 1$ and 2 on log-log paper, dots indicating estimated values and solid lines indicating LS fitting; (b) $\chi_q(\epsilon)$ versus ϵ ; (c) $\tau(q)$ versus q ; (d) fractal dimension spectrum $f(\alpha)$ versus α .



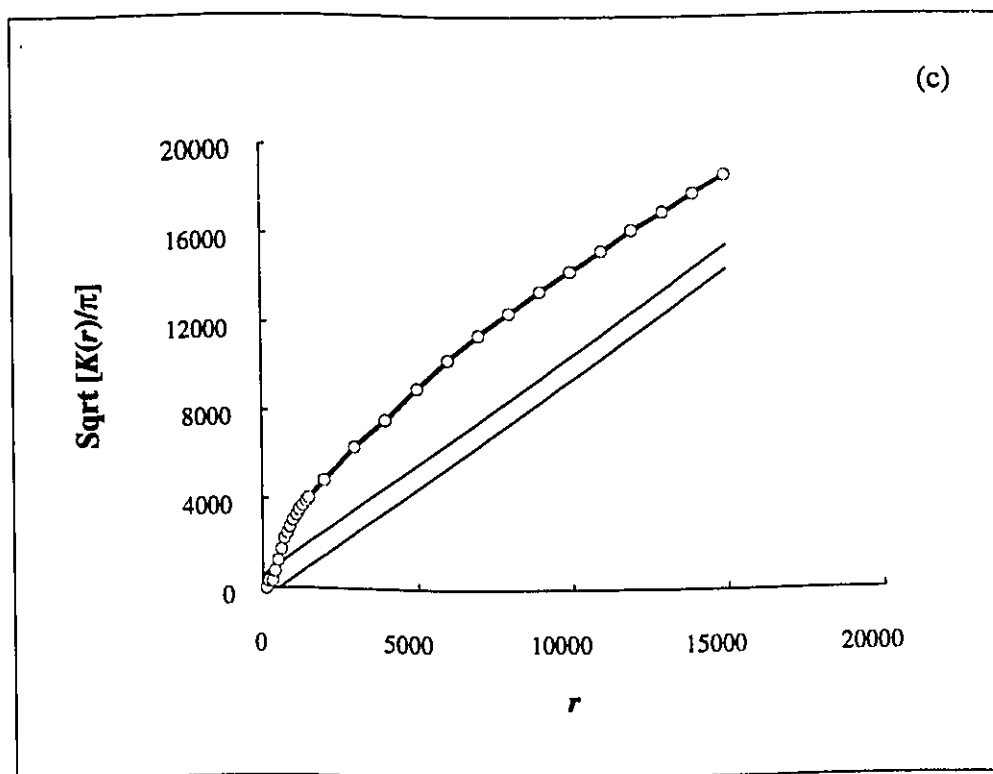


Fig. 7.6.10 Spatial statistics for Au mineral occurrences. (a) Log-log plot of relationship between $K(r)$ and distance r using Ripley's edge effect correction; LS straight line fitted according with Eq. (7.3.15) with $\tau(2)=1.219$; (b) Log-log plot for relationship between second-order intensity λ_2 and r ; solid line satisfies Eq. (7.3.12); (c) Estimated values of $[K(r)/\pi]^{0.5}$ versus r showing departure from CSR.

difference method, as can be seen in Fig. 7.6.10b. The solid line fitted by means of Eq. (7.3.12) gave the exponent $\tau(2)=1.219$. This result also indicates that the mineral occurrences do not have a complete spatial random distribution in the area; instead, they are clustered with a fractal dimension spectrum ($\tau(2) < 2$). This result can also be concluded from Fig. 7.6.10c.

7.7. CONCLUSION

The mass exponent function $\tau(q)$ in the multifractal model is useful for characterizing the underlying spatial structure of the measurements with $-\tau(0)$ corresponding to the box-counting dimension (D_b), and $\tau(2)$ determining the cluster dimension of fractal points (D_c).

The multifractal model results in approximate power-law type functions for first-order intensity, second-order intensity and $K(r)$ in the analysis of spatial point processes. These functions are primarily characterized by the values of $\tau(1)$, $\tau(0)$ and $\tau(2)$ according to Eqs.(7.3.5), (7.3.13) and (7.3.16).

A conclusion from the applications of the methods discussed in this chapter is that Au mineral occurrences in the Iskut River map sheet, northwestern British Columbia, are clustered in space with cluster dimension 1.219 ± 0.037 and box-counting dimension 1.335 ± 0.077 . These results will be used in Chapter 8.

Chapter 8 FRACTAL PATTERN INTEGRATION FOR MINERAL POTENTIAL ESTIMATION

8.1 INTRODUCTION

Geoscience maps of different types are to be integrated for target selection in mineral exploration. The geologist compares these maps and looks for combinations of indicators favourable for occurrence of mineral deposits of different types. The statistical integration of map data usually involves the following stages: superimposing grid cells of the same shape on a geological map; coding relevant geological, geophysical, geochemical and remote sensing features and mineral deposit information; based on these cells statistical models are used to associate mineral deposit information to various geoscience variables coded for each cell. Models constructed in this way are then used in a so-called prospecting region to predict undiscovered mineral deposits of the same type. The results obtained may strongly depend on the definitions of the variables used for the calculations and also on the arrangement, shapes and size of the cells. In order to improve the estimated result, various types of geoscience data are extracted and associated according to a geological model locally characterizing the occurrence of the mineral deposits (Harris, 1984; McCammon et al, 1983; Agterberg et al., 1990; Harris and Pan, 1991; Bonham-Carter et al., 1983, 1989; Cheng, 1985; Cheng and Wang, 1990). Optimization of cell size under a complete spatial randomness (CSR) condition for mineral deposits was studied by Zhao et al. (1983). Other efforts in this field include the one by Cheng and Wang

(1990) and Wang, Cheng and Fan (1990) who used geological objects instead of grid cells for sampling. In the latter study, geological objects at different scales such as ore belts, ore fields and ore deposits, were delineated by a means of group of synthesizing features and mineral resources were estimated for each type of object. These unit objects were called "consistent geological areas" by Harris and Pan (1991).

The representation and calculations required for quantitative analysis of digitized patterns (points, lines and areas) have been greatly aided by the development of geographical information systems (GIS) for the treatment of map data (Bonham-Carter, 1994; Bonham-Carter et al., 1988). In particular the raster data model, available in most of GIS systems, and the ability to convert vector and raster formats are suitable for data integration on the basis of unit cells or pixels. These GIS attributes have led to the development of new methods for statistical and non-statistical pattern integration simulating the practice by exploration geologists of superimposing maps for delineating favourable areas. For instance, the weights of evidence method was proposed and has been intensively used for data integration (Agterberg, 1989, 1990, 1992; Agterberg et al., 1993b; Bonham-Carter, 1994; Bonham-Carter et al., 1988; Cheng et al., 1994c). This method can be implemented by means of GIS on the basis of a polygon map, called the unique condition map, where each polygon represents a unique combination of binary or ternary patterns, although the calculation of the weights are performed by defining a unit cell (see section 5.5). In addition, knowledge-based or subjective techniques have been used, including fuzzy set theory (An et al, 1991, 1992, 1994; Bonham-Carter, 1994; Cheng, 1986) and the

Dempster-Shafer model (Chung et al., 1991, 1992, 1993). Fractal modelling was proposed for dealing with fractal and non-fractal patterns (Cheng, Agterberg and Bonham-Carter, 1994b). Implementation of these methods, especially for data-based or objective models, usually employs measures on the patterns, such as number of points, lengths of lines and areas of polygons. The required prior and conditional probabilities in weights of evidence modelling are estimated by the number of events (mineral deposits) and areas of indicator patterns. A commonly used method is to superimpose a grid on the map which contains patterns of points, lines and polygons for the feature of interest, and then count the number of cells containing these attributes. These numbers, from which the probabilities are estimated, depend on the cell size. In addition, it is usually assumed that the patterns are randomly located with respect to each small unit cell.

Fractal theory has shown that spatial patterns often exhibit non-integer dimension. For fractal patterns, the concepts of number, length and area are not valid. In addition, as discussed in Chapter 7, spatial points may have cluster distribution with fractal dimension. The ordinary data integration methods have to be modified in order to be used for fractal patterns. The purpose of this chapter is to introduce a new procedure proposed by Cheng, Agterberg and Bonham-Carter (1994b) for fractal pattern integration. This method is demonstrated with a case study of Au mineral potential estimation in the Iskut River map sheet, northwestern British Columbia. The method can also be used for evaluating impact areas in environmental studies.

8.2 CHARACTERISTICS OF FRACTAL PATTERNS AND FRACTAL MEASUREMENTS

Suppose a fractal object E is located in $\mathbb{R}^2$. By box-counting, the space can be divided into subareas of same size $\epsilon \times \epsilon$. The number of cells ($N(\epsilon)$) containing the feature of the fractal set E has power-law relation Eq. (3.2.3) where D is the fractal dimension and c is the size of the fractal set measured in D -space. These two measures can be used for estimating the probability and conditional probability that a unit cell randomly chosen from the study area intersects the pattern E , e.g., a unit area containing one or more mineral deposits of a given type.

Suppose a group of points E (mineral occurrences) distributed in the set S , $S \subset \mathbb{R}^2$. The total number of points E may or may not be finite. Two cases can be discussed: 1. the points are CSR in S ; 2. the points have fractal cluster distribution with box-counting fractal dimension D_E . Assume a small cell u (without losing generality u can be taken as rectangular and with size $\epsilon \times \epsilon$) is randomly taken from S . Three random variables can be defined: (1) the number of cells of size $\epsilon \times \epsilon$ covering the set S , denoted as N_S with mean $EN_S = N(S)$; (2) the number of cells containing at least one point, denoted as N_ϵ , a random variable with mean denoted as $EN_\epsilon = N(\epsilon)$; (3) the number of points in a area $A \subset S$, denoted as $N(A)$, a random variable with integral values ($N(A)=0,1,..., n$).

In the case of points with the CSR property, it was shown in Chapter 7 that for any two

sets $A \subset S$ and $B \subset S$ and $A \cap B = \emptyset$, $N(A)$ and $N(B)$ are independent random variables with binomial distributions:

$$P \{N(A) = k\} = \binom{k}{n} \left(\frac{|A|}{S} \right)^k \left(1 - \frac{|A|}{S} \right)^{n-k} \quad (8.2.1)$$

If $n \gg 1$, Eq. (8.2.1) becomes a Poisson distribution

$$P \{N(A) = k\} = \frac{(\lambda |A|)^k}{k!} e^{-\lambda |A|} \quad (8.2.2)$$

where $\lambda = n / S$. Therefore, the probability of a small cell u containing one or more points is

$$P \{N(u) \geq 1\} = 1 - P \{N(u) = 0\} = 1 - \left(1 - \frac{\epsilon^2}{S} \right)^n \quad (8.2.3)$$

or from Eq. (8.2.2)

$$P \{N(u) \geq 1\} = 1 - e^{-\lambda \epsilon^2} \quad (8.2.4)$$

For small ϵ , Eqs. (8.2.3) and (8.2.4) have the approximations

$$P \{N(u) \geq 1\} \approx \frac{n}{S} \epsilon^2 \quad (8.2.5)$$

$$P \{N(u) \geq 1\} \approx \lambda \epsilon^2 \quad (8.2.6)$$

These two relations are identical for large n and small ϵ , and are commonly used as estimates of the probability $P\{N(u) \geq 1\}$. It will be shown that these relations have to be modified in the case of fractal points.

In order to obtain a generalized form of Eqs. (8.2.5) and (8.2.6), the following relation between the variables N_S , N_ϵ and $N(u)$ can be used

$$N_\epsilon = \sum_{i=1}^{N_S} I[N(u_i) - 1] \quad (8.2.7)$$

where $I[x]$ is the characteristic function with $I[x] = 1$, if $x \geq 1$ and otherwise, $I[x] = 0$. u_i represents the i -th cell of size $\epsilon \times \epsilon$. The mean of N_ϵ can be estimated as follows (Cheng, 1989; Cheng and Wang, 1990): Denote the probability of N_S as $P[N_S=k]$. The conditional mean of N_ϵ knowing $N_S = k$ is

$$\begin{aligned} E[N_\epsilon | N_S=k] &= E\left[\sum_{i=1}^k I[N(u_i) - 1]\right] = \\ &= kEI[N(u) - 1] = kP[N(u) \geq 1] \end{aligned} \quad (8.2.8)$$

Therefore,

$$\begin{aligned} N(\epsilon) &= E[E[N_\epsilon | N_S=k]] = \sum_k kP[N(u) \geq 1] P[N_S=k] = \\ &= EN_S P[N(u) \geq 1] = N(S) P[N(u) \geq 1] \end{aligned} \quad (8.2.9)$$

and

$$P\{N(u) \geq 1\} \approx \frac{N(\epsilon)}{N(S)} \quad (8.2.10)$$

which is a general form of Eqs. (8.2.5) and (8.2.6) because for small ϵ , $N(\epsilon) \approx n$, and $\lambda = N(\epsilon)/N(S)$ so that Eq. (8.2.10) becomes identical to Eqs. (8.2.5) and (8.2.6).

In the case of fractal points with box-counting dimension D_E , if $0 < D_E \leq 2$, theoretically,

$$N(\epsilon) \propto \epsilon^{-D_F} \quad (8.2.11)$$

where $N(\epsilon)$ increases infinitely as $\epsilon \rightarrow 0$. In this case, the total number of points (n) in S is infinite. Therefore, relations Eqs. (8.2.5) and (8.2.6) can be replaced by Eq. (8.2.9). In practice, Eq. (8.2.10) holds true only for the limited range $\epsilon \in [\epsilon_{\min}, \epsilon_{\max}]$. Eq. (8.2.10) is affected by irregular edges for $\epsilon > \epsilon_{\max}$ and by limited map scale or computer resolution for $\epsilon < \epsilon_{\min}$. For example, events may be located so close to one another that they cannot be represented separately on a map with given scale. In general, the value of the total number of points may not represent a true property of the spatial distribution of the points. Therefore, relation Eq. (8.2.9) (with $\epsilon > \epsilon_{\min}$ for the case of finite point set E) can be used for estimating the probability $P\{N(u) \geq 1\}$, and can be rewritten as

$$P\{N(u) \geq 1\} \approx \frac{C}{S} \epsilon^{2-D_F} \quad (8.2.12)$$

8.3 FRACTAL PATTERN INTEGRATION

Two patterns A and L are assumed to be associated with events E ; for example, A represents rock type, L represents faults and E represents mineral deposits of a given type. The following notation is used. $P(E)$ represents the unconditional probability that a unit area contains one or more events (E). $P(E|A)$, $P(E|L)$ and $P(E|AL)$ are conditional probabilities for events occurring on the patterns A , L and the overlap of A and L , respectively, which are estimated using Eq. (8.2.10). Suppose A , L , E , and the various

patterns of overlap of A, L and E have dimensions D_A , D_L , D_E , D_{AE} , D_{LE} and D_{ALE} . Then the numbers of cells containing these features can be expressed as:

$$\begin{aligned} N_A(e) &= C_A e^{-D_A} \\ N_L(e) &= C_L e^{-D_L} \\ N_E(e) &= C_E e^{-D_E} \end{aligned} \quad (8.3.1)$$

where c_A , c_L , and c_E are constants. Similarly, the means of numbers of cells containing A and E, L and E, or A, L and E, are

$$\begin{aligned} N_{AE}(e) &= C_{AE} e^{-D_{AE}} \\ N_{LE}(e) &= C_{LE} e^{-D_{LE}} \\ N_{ALE}(e) &= C_{ALE} e^{-D_{ALE}} \end{aligned} \quad (8.3.2)$$

Therefore, using Eq. (8.2.9) the probabilities are estimated as:

$$\begin{aligned} P(E) &= \frac{C_E}{C_S} e^{2-D_E} ; \quad P(\bar{E}) = 1 - \frac{C_E}{C_S} e^{2-D_E} \\ P(A) &= \frac{C_A}{C_S} e^{2-D_A} ; \quad P(\bar{A}) = 1 - \frac{C_A}{C_S} e^{2-D_A} ; \\ P(L) &= \frac{C_L}{C_S} e^{2-D_L} ; \quad P(\bar{L}) = 1 - \frac{C_L}{C_S} e^{2-D_L} \\ P(AE) &= \frac{C_{AE}}{C_S} e^{2-D_{AE}} ; \quad P(LE) = \frac{C_{LE}}{C_S} e^{2-D_{LE}} \end{aligned} \quad (8.3.3)$$

where $c_i = S$, and the conditional probabilities are:

$$\begin{aligned} P(E|A) &= \frac{C_{AE}}{C_S} e^{D_A - D_{AE}} ; \quad P(E|L) = \frac{C_{LE}}{C_S} e^{D_L - D_{LE}} \\ P(E|\bar{A}) &= \frac{C_E e^{2-D_E} - C_{AE} e^{2-D_{AE}}}{C_S - C_A e^{2-D_A}} ; \quad P(E|\bar{L}) = \frac{C_E e^{2-D_E} - C_{LE} e^{2-D_{LE}}}{C_S - C_L e^{2-D_L}} \end{aligned} \quad (8.3.4)$$

These relationships between A, L and E can be expressed by means of a $(2 \times 2 \times 2)$ table of probabilities (Table 8.3.1):

Table 8.3.1 Relationships between A, L and E

	E		$\bar{E}$	
	L	$\bar{L}$	L	$\bar{L}$
A	$P(A LE)$	$P(A \bar{L}E)$	$P(A L\bar{E})$	$P(A \bar{L}\bar{E})$
$\bar{A}$	$P(\bar{A} LE)$	$P(\bar{A} \bar{L}E)$	$P(\bar{A} L\bar{E})$	$P(\bar{A} \bar{L}\bar{E})$

Conditional independence of A and L under E implies:

$$\begin{aligned} P(AL|E) &= P(A|E) P(L|E) ; P(\bar{A}\bar{L}|E) = P(\bar{A}|E) P(\bar{L}|E) ; \\ P(A\bar{L}|E) &= P(A|E) P(\bar{L}|E) ; P(\bar{A}L|E) = P(\bar{A}|E) P(L|E) \end{aligned} \quad (8.3.5)$$

The relations of Eq.(8.3.5) are equivalent to:

$$\begin{aligned} C_{ALE} &= \frac{C_{AE} C_{LE}}{C_E} ; D_{ALE} = D_{AE} + D_{LE} - D_E ; \\ C_{A\bar{L}E} &= \frac{C_{AE} C_{\bar{L}E}}{C_E} ; D_{A\bar{L}E} = D_{AE} + D_{\bar{L}E} - D_E ; \\ C_{\bar{A}LE} &= \frac{C_{\bar{A}E} C_{LE}}{C_E} ; D_{\bar{A}LE} = D_{\bar{A}E} + D_{LE} - D_E ; \\ C_{\bar{A}\bar{L}E} &= \frac{C_{\bar{A}E} C_{\bar{L}E}}{C_E} ; D_{\bar{A}\bar{L}E} = D_{\bar{A}E} + D_{\bar{L}E} - D_E \end{aligned} \quad (8.3.6)$$

If the assumption of conditional independence of A and L holds true, the eight probabilities in the preceding table are mutually interrelated by:

$$\begin{aligned} P(ALE) &= P(A|E) P(L|E) P(E) ; \\ P(A\bar{L}E) &= P(A|\bar{E}) P(L|\bar{E}) P(\bar{E}) \end{aligned} \quad (8.3.7)$$

If odds (O) are used instead of probabilities (P) with $O=P/(1-P)$, then:

$$\begin{aligned}
\log_e O(E|A L) &= W_A^+ + W_L^+ + \log_e O(E) \\
\log_e O(E|\bar{A} L) &= W_A^- + W_L^+ + \log_e O(E) \\
\log_e O(E|A \bar{L}) &= W_A^+ + W_L^- + \log_e O(E) \\
\log_e O(E|\bar{A} \bar{L}) &= W_A^- + W_L^- + \log_e O(E)
\end{aligned} \tag{8.3.8}$$

These are extensions of Bayes' rule which holds only if A and L are conditionally independent. The weights

$$\begin{aligned}
W_A^+ &= \log\left\{\frac{P(A|E)}{P(A|\bar{E})}\right\}; & W_A^- &= \log\left\{\frac{P(\bar{A}|E)}{P(\bar{A}|\bar{E})}\right\} \\
W_L^+ &= \log\left\{\frac{P(L|E)}{P(L|\bar{E})}\right\}; & W_L^- &= \log\left\{\frac{P(\bar{L}|E)}{P(\bar{L}|\bar{E})}\right\}
\end{aligned} \tag{8.3.9}$$

satisfy the following expressions:

$$\begin{aligned}
W_A^+ &= \log\left[\frac{\frac{C_{AE}}{C_E} \epsilon^{D_E - D_{AE}}}{\frac{C_A - C_{AE} \epsilon^{2 - D_{AE}}}{C_S - C_E \epsilon^{2 - D_E}}}\right] \sim \log\left[\frac{C_{AE} C_S \epsilon^{D_E - D_{AE}}}{C_A C_E}\right] \\
W_A^- &= \log\left[\frac{1 - \frac{C_{AE}}{C_E} \epsilon^{D_E - D_{AE}}}{1 - \frac{C_A - C_{AE} \epsilon^{2 - D_{AE}}}{C_S - C_E \epsilon^{2 - D_E}}}\right] \sim \log\left[\frac{C_S - \frac{C_{AE}}{C_E} \epsilon^{D_E - D_{AE}}}{C_S - C_A}\right]
\end{aligned} \tag{8.3.10}$$

It shows that the contrast $C = W^+ - W^-$ (section 5.5; more details can be found in Agterberg, 1990) decreases as ϵ decreases ($C \propto \log \epsilon$), because, in general, $D_{AE} < D_E$. Only if $D_{AE} \approx D_E$ or $D_E = 0$, corresponding to the ordinary point pattern, are the two weights independent of ϵ . In this special case, the values of W_A^+ and W_A^- can be estimated using c_{AE} , c_E , c_A and c_S . For small cell size ϵ and $D_A = 2$, we may have following approximation:

$$\log O(E|A) \sim W_A^+ + \log\left\{\frac{C_E}{C_S} e^{2-D_E}\right\} \quad (8.3.11)$$

or

$$O(E|A) \sim \frac{C_{AE}}{C_A} e^{2-D_{AE}} \quad (8.3.12)$$

where the conditional odds of $O(E|A)$ has power-law relation to cell size ϵ with exponent $2-D_{AE}$.

Standard deviations of weights W_A^+ and W_A^- can be obtained as follows (Bishop et al., 1975; Agterberg et al., 1992):

$$\begin{aligned} S^2(W_A^+) &= \frac{1}{N_{AE}(\epsilon)} + \frac{1}{N_{A\bar{E}}(\epsilon)} ; \\ S^2(W_A^-) &= \frac{1}{N_{AE}(\epsilon)} + \frac{1}{N_{A\bar{E}}(\epsilon)} \end{aligned} \quad (8.3.13)$$

which can be rewritten using Eqs. (8.3.1) and (8.3.2) as:

$$\begin{aligned} S^2(W_A^+) &= \frac{C_A}{(C_A - C_{AE} e^{D_A - D_{AE}}) C_{AE} e^{-D_{AE}}} \\ S^2(W_A^-) &= \frac{C_S - C_A e^{2-D_A}}{(C_E e^{-D_E} - C_{AE} e^{-D_{AE}}) (C_S - C_E e^{2-D_E} - C_A e^{2-D_A} + C_{AE} e^{2-D_{AE}})} \end{aligned} \quad 8.3.14$$

Both $s(W_A^+)$ and $s(W_A^-)$ decrease as cell size ϵ decreases to zero. However, the standardized value of C ($t(C) = C/s(C)$) is approximately $t(C) \propto \epsilon^{-1/2D_E}$ which increases as cell size ϵ decreases. Therefore, the optimal cell size ϵ should be the minimum value yielding both (a) power-law relations which hold true for the fractal patterns, and (b) $t(C)$

is as large as possible.

8.4 GOLD POTENTIAL ESTIMATION IN THE ISKUT RIVER MAP SHEET, NORTHWESTERN BRITISH COLUMBIA

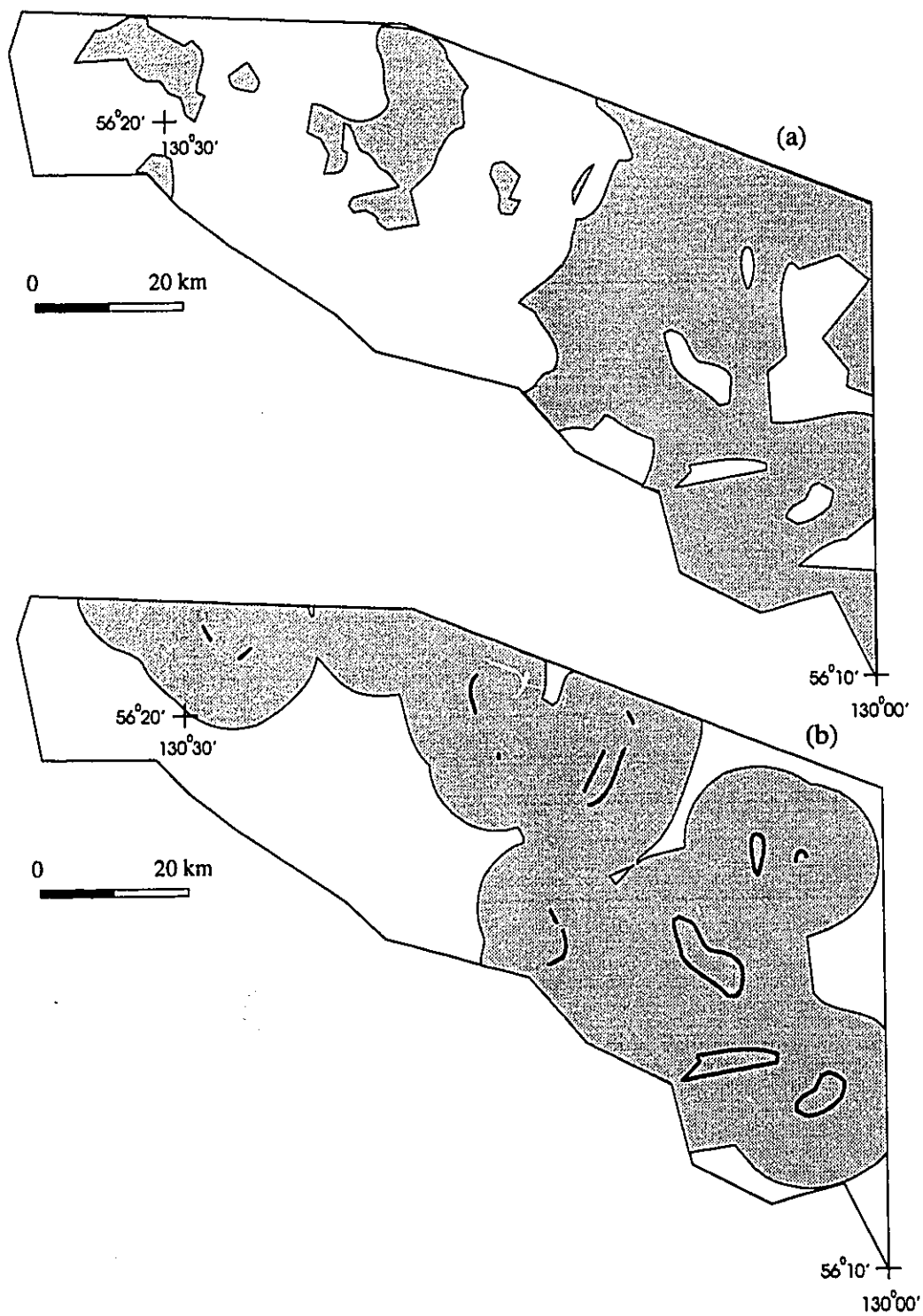
8.4.1 Indicator Patterns and Fractal Characteristics

The area chosen for this study is the Iskut River map sheet (104B), northwestern British Columbia. Geological variables were selected by metallogenic considerations and probabilistic approaches based on fundamental geological principles of hydrothermal deposit formation, combined with regional geochemical, geophysical and mineral inventory data. Datasets used for this study consist of (a) geological map (1:250,000); (b) regional geochemical reconnaissance data (1:250,000); (c) regional aeromagnetic data (250,000); (d) mineral occurrence records (B.C. Minfile, 1989). For the present study, the following indicator patterns will be considered: (1) Paleozoic-Mesozoic sedimentary and volcanic clastic rocks (Fig. 8.4.1a); (2) Buffer zones (<10 km) around the contacts between Paleozoic-Mesozoic sedimentary and volcanic clastic rocks and Mesozoic intrusive rocks (quartz diorite, monzonite, monzodiorite and alkali-feldspar porphyry intrusions)(Fig. 8.4.1b); (3) low magnetic anomalies (<36.9 Nt) (Fig. 8.4.1c); and (4) geochemical anomalies for Au (>30ppb) in stream sediments (Fig. 8.1.d). The binary pattern in Fig. 8.4.1a was compiled using Voronoi mapping based on the geological units coded for 698 stream sediment stations. Buffer zones in Fig. 8.4.1b and binary aero-

magnetic patterns in Fig. 8.4.1c were delineated and optimized in terms of the contrast $C=W^+-W^-$ (cf. Agterberg, 1990), respectively. Au anomalies in Fig. 8.4.1d were delineated by statistical and fractal methods (Chapters 3 and 5, also see Cheng et al., 1994d). To use the method introduced in section 8.3 to integrate these binary patterns (Fig. 8.4.1) with Au mineral occurrences (Fig. 7.6.7), the box-counting method for cell sizes (2 to 20 km) was used for estimating the fractal dimensions and measures. The estimated results for the Au mineral occurrences on the overlaps of the binary patterns are shown in Fig. 8.4.2 and the estimated fractal measurements are summarized in Table 8.4.1. The binary patterns in Fig. 8.4.1 are considered as ordinary non-fractal polygon patterns with $D=2$ and measure c equal to the polygon areas.

8.4.2 Patterns Integration

The mineral occurrences on each binary pattern are fractal sets with box-counting dimensions (D from 0.9023 to 1.2885) and measures (c from 123.83 to 513.94) less than the dimension ($D=1.335\pm0.077$) and measure ($c=732.46\pm137.5$) for all occurrences. From these estimated values of D and c , the weights (W^+ and W^-) and their standard deviations ($s(W^+)$ and $s(W^-)$) can be estimated for each patterns by Eqs. (8.3.10) and (8.3.14). The results for each binary patterns obtained using cell sizes 2, 4 and 6 (km) are shown in Tables 8.4.2 and 8.4.3. Generally, the weights (W^+ and W^-) and their standard deviations ($s(W^+)$ and $s(W^-)$) decrease but t -values increase as the cell size decreases. For comparison, the ordinary weights of evidence method was used to compute the weights



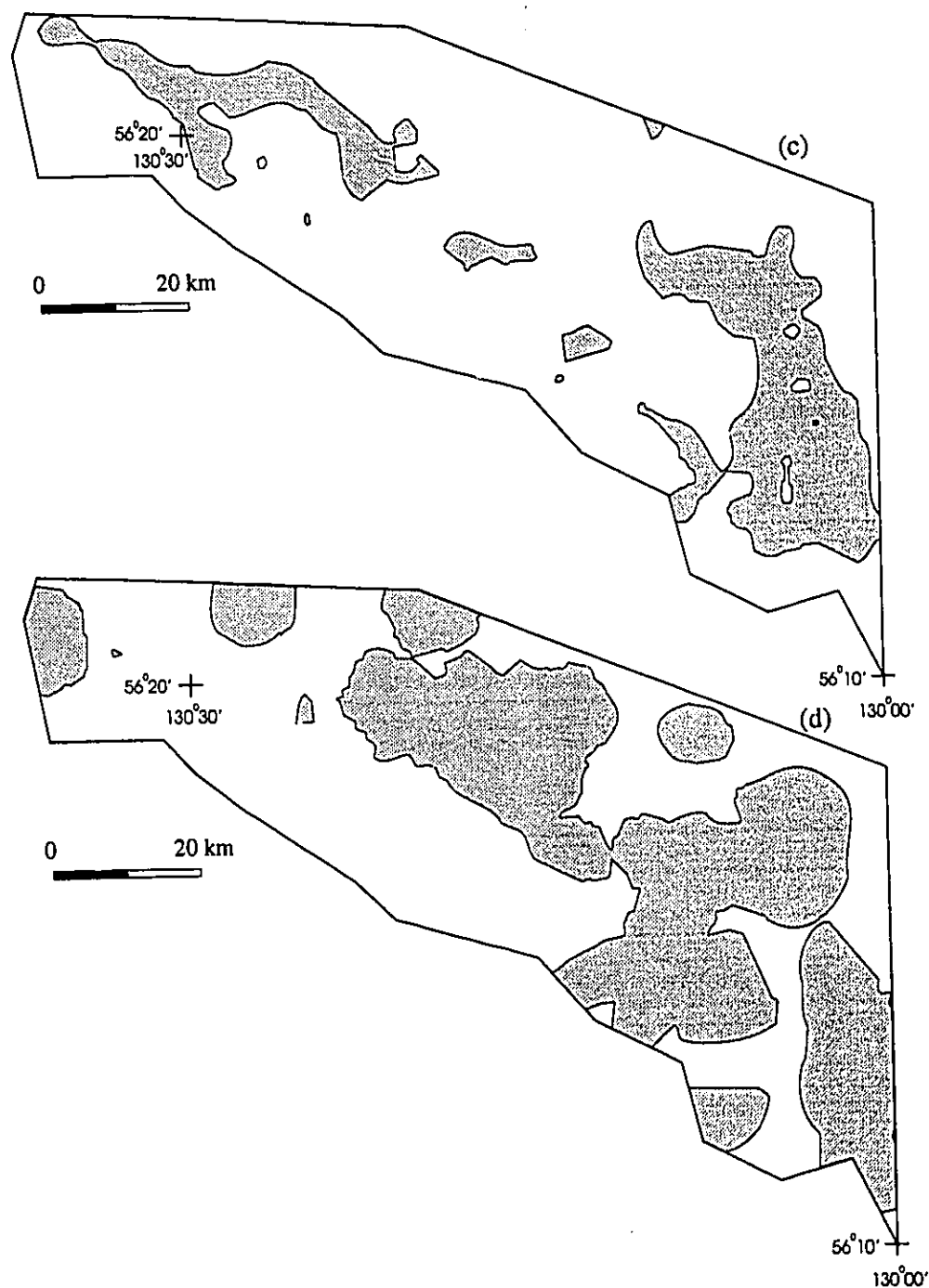


Fig. 8.4.1 Indicator patterns (shaded areas) for Au mineral potential mapping. (a) Paleozoic to Mesozoic sedimentary and volcanic clastic rocks; (b) Buffer zones (≤ 10 km) around the contacts between Paleozoic to Mesozoic rocks (sedimentary and volcanic clastic rocks) and Mesozoic intrusive rocks (Quartz diorite, monzonite, monzodiorite, alkali-feldspar porphyry intrusions); (c) Lower aeromagnetic anomalies (<36.9 Nt); (d) geochemical anomalies for Au (>30 ppb) in stream sediments delineated by fractal methods.

for the binary patterns. The results for binary rock type (Fig. 8.4.1a) with different unit cells are shown in Table 8.4.2. It can be seen that the weights obtained by means of the

Table 8.4.1 Fractal Measurements for the Binary Patterns

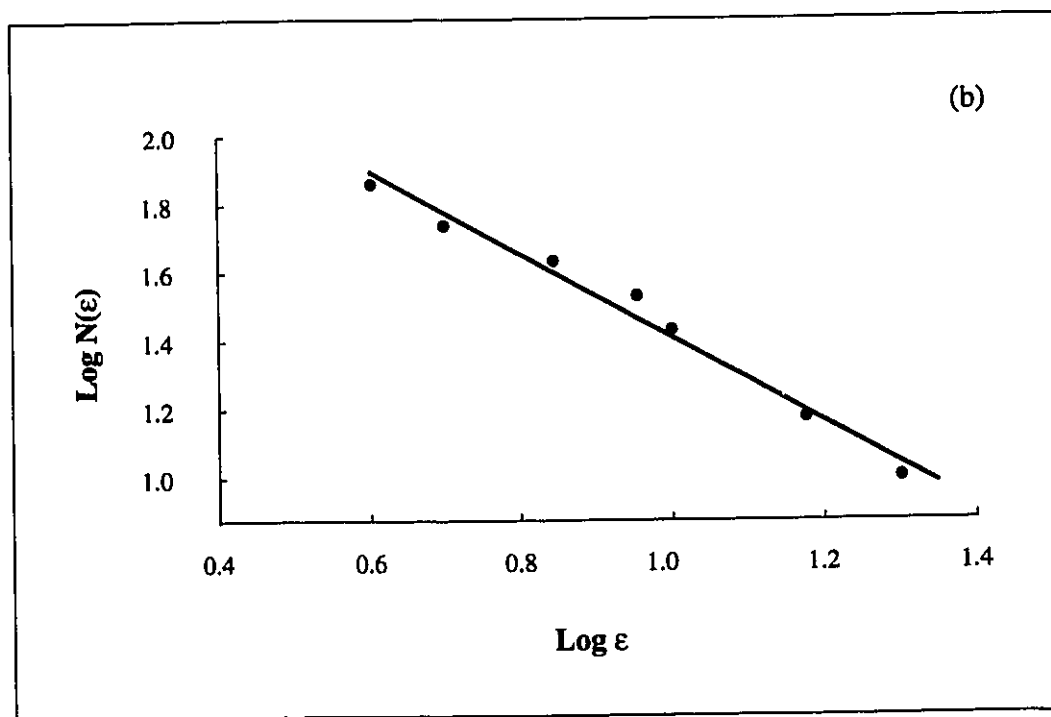
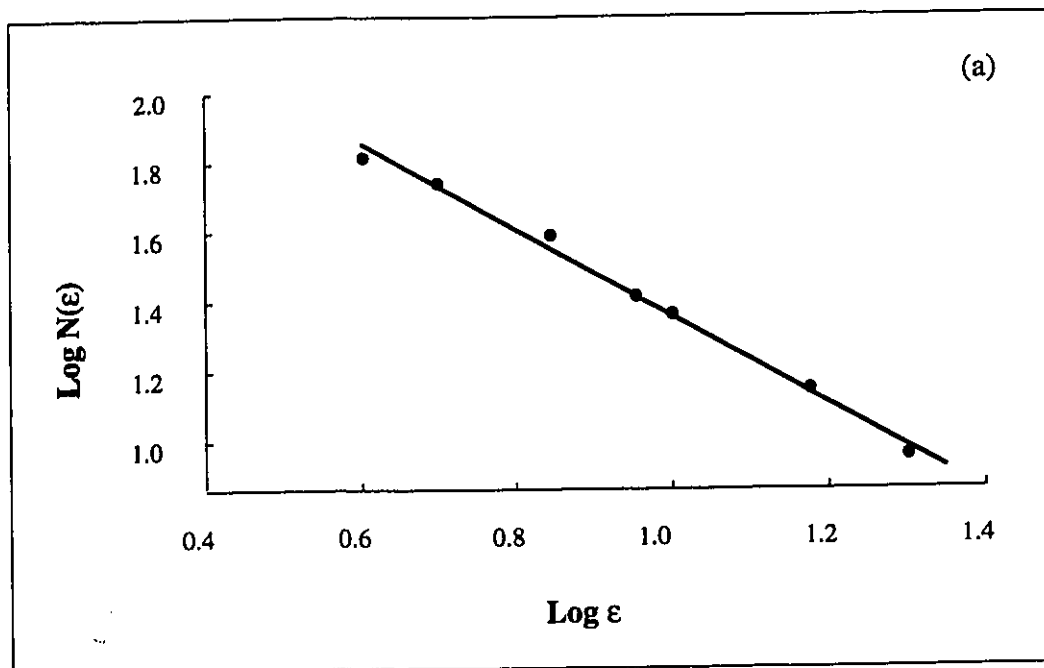
Geological Features	Label	D	c
Au Mineral Occurrences	E	1.335±0.077	732±137.5
P-M Sedimentary Rock	X ₁	2	2387
Buffers Around Contacts	X ₂	2	3144
Magnetic Anomalies	X ₃	2	1128
Au Geochem. Anomalies	X ₄	2	2241
Overlap Conditions	E&X ₁	1.246±0.047	401±44.0
	E&X ₂	1.289±0.069	514±84.8
	E&X ₃	0.902±0.059	124±17.5
	E&X ₄	1.223±0.075	428±78.0
Basemap	S	2	4754

Table 8.4.2 Comparison of Weights for Binary Patterns ($s_1=s(W^+)$ & $s_2=s(W^-)$)

e^2	Ordinary Weights				Fractal Weights			
	W^+	s_1	W^-	s_2	W^+	s_1	W^-	s_2
16	1.4	.29	-1.1	.18	.21	.16	-.25	.19
9	.58	.12	-.78	.16	.16	.13	-.19	.15
6	.47	.11	-.69	.15	.13	.11	-.16	.13
4	.42	.10	-.65	.15	.11	.09	-.13	.11
3	.39	.09	-.62	.15	.09	.08	-.11	.09
2	.37	.09	-.60	.14	.07	.07	-.08	.08
1	.36	.09	-.59	.14	.04	.05	-.05	.06

Table 8.4.3 Weights for Binary Patterns Obtained Using Fractal Method

X	$z(\text{km})$	W^+	$s(W^+)$	W^-	$s(W^-)$	C	$s(C)$	$t(C)$
	8	0.5342	0.4128	-0.4792	0.3643	1.0134	0.5505	1.8408
	6	0.3370	0.2571	-0.3654	0.2787	0.7024	0.3792	1.8520
X_1	4	0.2136	0.1639	-0.2517	0.1931	0.4653	0.2533	1.8375
	2	0.1090	0.0909	-0.1270	0.1056	0.2360	0.1393	1.6940
	1	0.0440	0.0548	-0.0479	0.0592	0.0920	0.0806	1.1404
	8	0.3252	0.3169	-0.6550	0.4895	0.9802	0.5832	1.6809
	6	0.2215	0.2171	-0.5805	0.3669	0.7220	0.4261	1.6942
X_2	4	0.1457	0.1438	-0.3448	0.2486	0.4905	0.2872	1.7078
	2	0.0757	0.0806	-0.1730	0.1325	0.2487	0.1550	1.6041
	1	0.0304	0.0482	-0.0641	0.0728	0.0946	0.0873	1.0833
	8							
	6	0.6343	0.4346	-0.1834	0.2080	0.8178	0.4818	1.6972
X_3	4	0.2584	0.2382	-0.0778	0.1433	0.3362	0.2780	1.2455
	2	0.0001	0.1405	-0.0000	0.0784	0.0000	0.1608	0.0012
	1	-0.1694	0.0952	0.0443	0.0444	-0.2137	0.1051	-2.0330
	8	1.3118	0.8729	-0.7515	0.4195	2.0630	0.9685	2.1305
	6	0.5940	0.3005	-0.5816	0.3106	1.1756	0.4322	2.7202
X_4	4	0.3599	0.1703	-0.4135	0.2091	0.7735	0.2696	2.8691
	2	0.2003	0.0900	-0.2325	0.1110	0.4328	0.1429	3.0277
	1	0.1130	0.0537	-0.1215	0.0612	0.2345	0.0814	2.8806



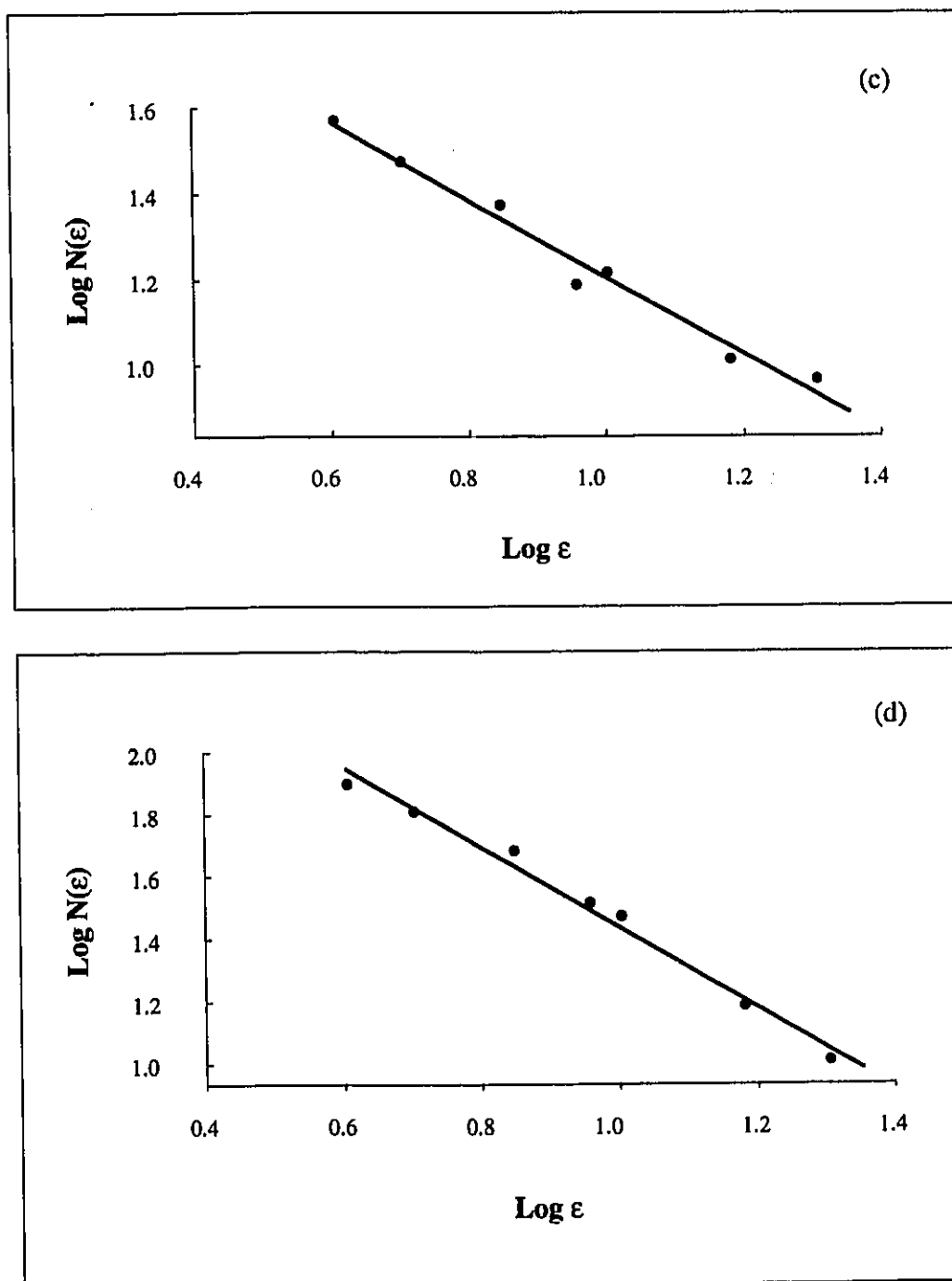


Fig. 8.4.2 Log-log plots showing fractal analysis for mineral occurrences on the binary patterns in Fig. 8.4.1, respectively. Straight lines are the least squares fits. Cell sizes used are 20, 15, 10, 9, 7, 5 and 4 km.

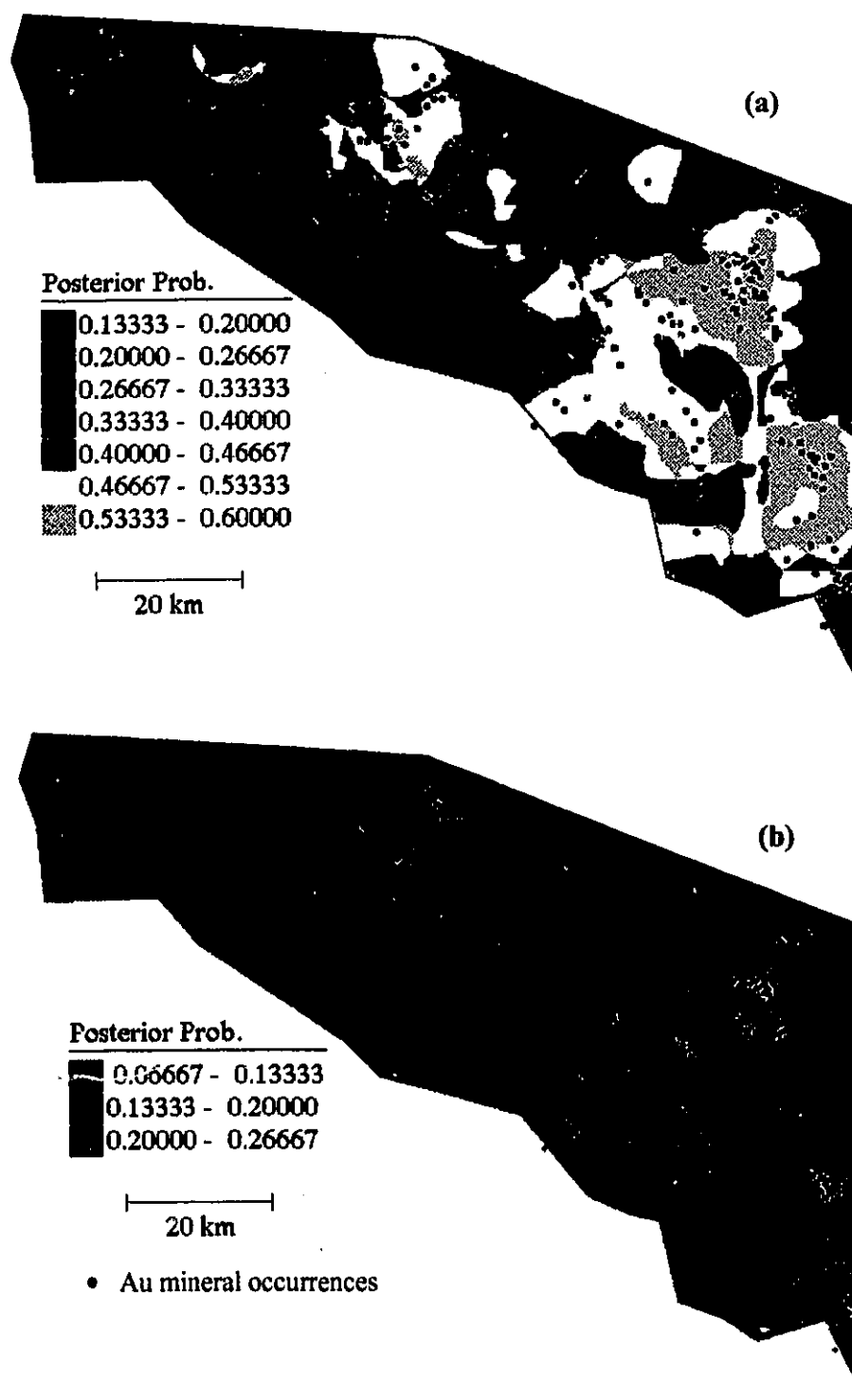


Fig. 8.4.3 Posterior probability map showing mineral potential for Au in the Iskut River map sheet. The results were obtained by using the parameters in Fig. 8.4.2 and Tables 8.4.1 and 8.4.2 for the binary patterns in Fig. 8.4.1. (a) cell size 4 km, (b) cell size 1 km.

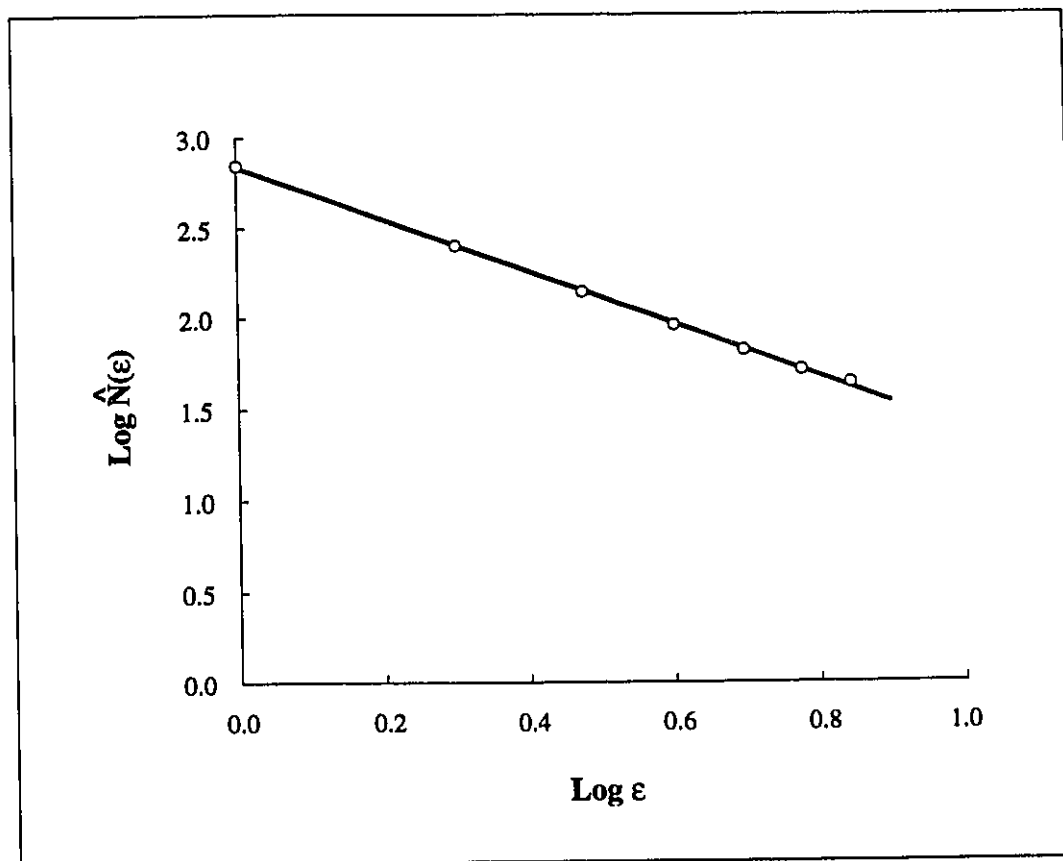


Fig. 8.4.4 Log-log plot showing the relationships between the estimated total number of mineral occurrences ($\hat{N}(\epsilon)$) in the study area computed using the approach introduced in this chapter and the used cell sizes (ϵ).

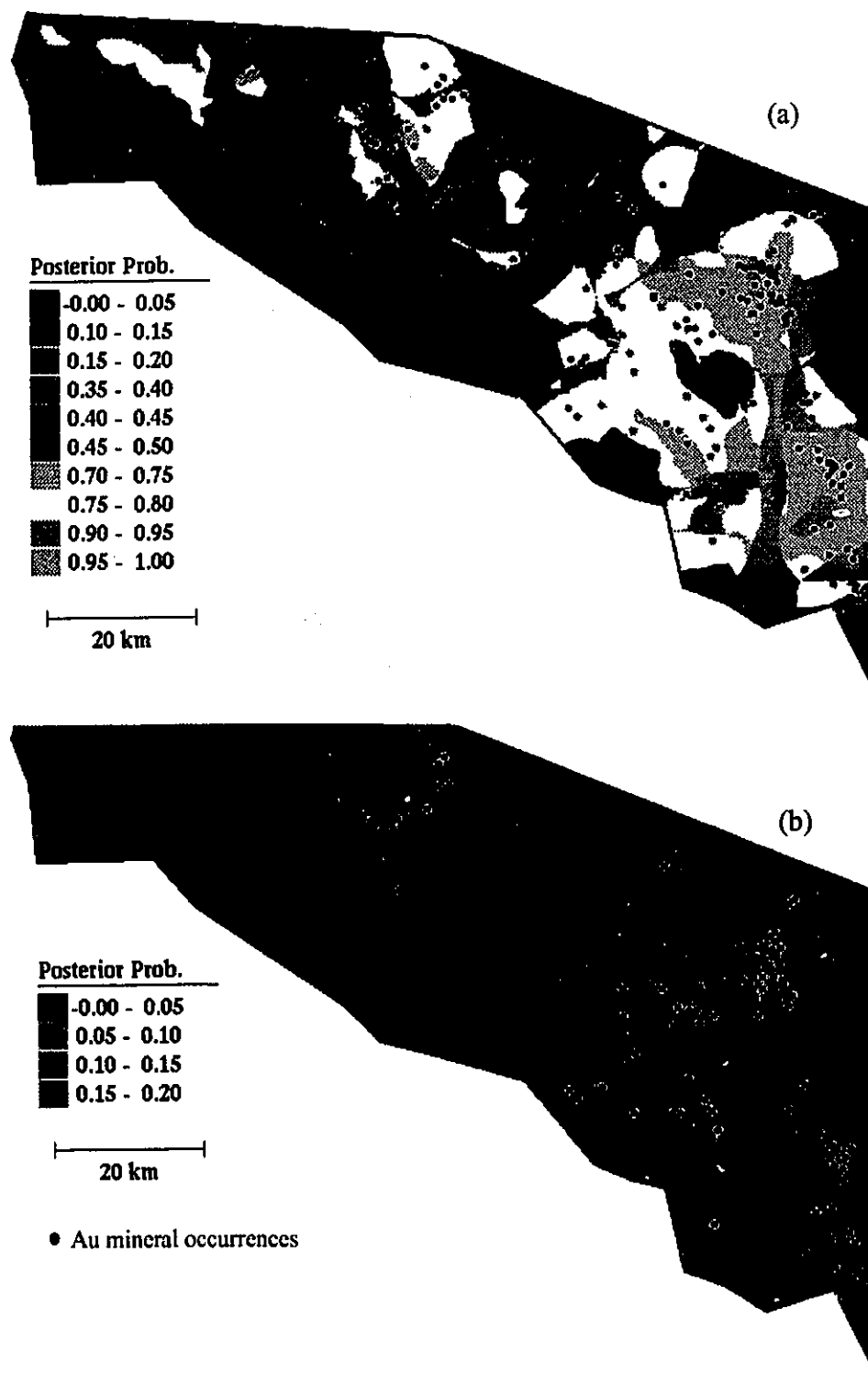


Fig. 8.4.5 Posterior probability map showing Au potential in the Iskut River map sheet. The results were obtained by using the ordinary weights of evidence method. (a) cell sizes 4km; (b) cell size 1km.

ordinary weights of evidence method with various unit cells decrease as cell size ϵ decreases. For small cell size the weights and standard deviations approach constants. Therefore, the optimal cell size should be the minimum value yielding (a) power-law relations for all patterns and (b) maximum t-values for the weights. For the present example, 4 km was chosen as the cell size for computing the posterior probability. The results can be seen in Fig. 8.4.3. The estimated total Au mineral occurrences for various cell sizes are shown in the log-log plot (Fig. 8.4.4) which shows power-law relation with exponent $D=1.3643\pm0.023$ and constant $c=683.899\pm30.4$. These results are slightly less than those obtained from Fig. 7.6.7. The results obtained using the ordinary weights of evidence method with cell size 4 km and 1 km are shown in Fig. 8.4.5.

8.5 CONCLUSION

The method introduced in this chapter can be applied for non-fractal/fractal pattern integration in mineral potential estimation, and is relatively easy to implement by performing box-counting with the aid of GIS.

For mineral potential mapping based on fractal patterns, such as Au mineral occurrences in the study area, the results of the estimated posterior probability and the number of mineral occurrences in the study area are significantly related to the values of the cell sizes.

The optimum cell size was recommended to be the minimum value (4 km for the present example) yielding both (a) power-law relations for fractal patterns and (b) maximum t-value for the contrast C.

Four indicator patterns (Paleozoic to Mesozoic sedimentary and volcanic clastic rocks, contacts between sedimentary/volcanic clastic and Mesozoic intrusive rocks, linear aeromagnetic anomalies and geochemical anomalies for Au in stream sediments) were found to be spatially associated with the spatial distribution of Au mineral occurrences and these patterns were integrated for Au mineral potential estimation.

CONCLUSIONS

Conclusions from Chapter 2 can be summarized as follows:

- ▶ Two different characteristics of the parametric statistics were classified and tested for various statistics. Total misclassification errors for U, χ^2 and F statistics decrease as the size of samples increases infinitely.
- ▶ Spatial U^* -statistics constructed by means of a moving average technique with variable window radius provide good thresholds for Au-associated geochemical anomaly separation in stream sediment samples.

Conclusions from Chapter 3, related to the study of fractal models and their implementation with GIS, are:

- ▶ Various types of fractal models can be used in the application of GIS for analysis of spatial data.
- ▶ Newly derived relationships between perimeters and areas of similar shaped fractal sets show that only if $D_A \approx 2$, corresponding to the case of ordinary areas, can the exponent $D_{AL}=2D_L/D_A$ estimated from the perimeter-area relation be used as unbiased estimate of the fractal dimension D_L . In general, $D_{AL} \geq D_L$ because $2/D_A \geq 1$. The results obtained for

Au, Cu and As concentration values in the Mitchell-Sulphurets mineral district show that Au is more irregularly distributed than Cu and As in this study area.

Several important conclusions resulted from discussions of multifractal models and spatial statistics in Chapters 4, 5, 6, 7 are:

- Newly derived relationships between $f(\alpha)$, $\tau(q)$, α , $C(\gamma)$ and $K(q)$ show that the two principal existing multifractal models (one based on fractal dimension function $f(\alpha)$ and the other on codimension function $C(\gamma)$), are identical except for simple linear relations if $\gamma \geq D - \alpha(0)$ only. For $\gamma < D - \alpha(0)$, however, $f(\alpha)$ is a continuous function, but $C(\gamma) = \text{constant}$. In this sense, the model based on $f(\alpha)$ is more useful than that based on $C(\gamma)$.
- Au concentration values in bed rock samples in the Mitchell-Sulphurets mineral district should be described by multifractal models instead of fractal models.
- Multifractal models can be used not only for characterizing physical and chemical continuous values, but also for spatial objects, such as faults and igneous rocks associated with Au mineralization and Au mineral occurrences in the Iskut River map sheet.
- In general, the multifractal model provides more information about the measurements or fields on spatial objects than fractal models. The multifractal model may lead to several fractal models with different fractal dimensions and various usages; such as, $-\tau(0)$

corresponding to box-counting fractal dimension of the support, $\tau(2)$ associated with the correlation dimension of random variables and cluster dimension for point processes. The Hurst exponent H of a Brownian surface satisfies $H = (1/2) [\tau(2)+1]$.

- The element concentration-area method proposed in Chapter 5, which is based on multifractal theory, can be used for separating geochemical anomalies from background by means of a simple threshold. The method is applicable to elements for which the spatial pattern of concentration values satisfy a multifractal model.

- The concentration-area method has been successfully applied to Au, Cu and other Au associated elements in bed rock samples the Mitchell-Sulphurets mineral district (1:20,000) as well as to Au in stream sediment samples in the Iskut River map sheet (1:250,000). The results were similar to those obtained with the ordinary weights of evidence method.

- The new relations between the multifractal model and spatial statistics derived in Chapters 6 and 7 show that the multifractal model results in (1) approximate power-law type functions for covariance and autocorrelation, and logarithmic type functions for the semivariogram of spatial random variables; (2) approximate power-law functions for the second-order intensity and $K(r)$ of point processes. These functions are primarily characterized by the value of $\tau(2)$.

► Au mineral occurrences in the Iskut River map sheet have multifractal properties with box-counting dimension 1.335 ± 0.077 and cluster dimension 1.219 ± 0.037 . The value $\tau(2) < 2$ can be used for describing the aggregation of the point processes. The Au mineral occurrences were shown to be clustered in this area.

The method introduced in Chapter 8 is useful for fractal and non-fractal pattern integration in mineral potential mapping, and can also be applied for evaluating impact areas in environmental studies.

► It is relatively easy to implement fractal/multifractal models by performing the box-counting method with the aid of GIS.

► The optimum cell size used for calculation of weights can be set at the minimum value yielding both (1) power-law relations for fractal patterns and (2) maximum t-value for the contrast $\bar{C}$.

► Four indicator patterns (Paleozoic to Mesozoic sedimentary and volcanic clastic rocks, contacts between sedimentary/volcanic clastic and Mesozoic intrusive rocks, linear aeromagnetic anomalies and geochemical anomalies for Au in stream sediments) were found to be spatially associated with the spatial distribution of Au mineral occurrences in the Iskut River area and these patterns were integrated for Au mineral potential estimation.

This study has led to new results complementary to the existing theory of fractal and multifractal models. This study also provides a general, systematic methodology for fractal modelling and spatial analysis with GIS. The newly developed methods have been tested using well-known examples from literature and practical examples from the Iskut River map sheet (1:250,000) and the Mitchell-Sulphurets mineral district (1:20,000). It can be anticipated that fractal and multifractal models implemented with GIS will be widely used in the geosciences and other fields.

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