ProQuest Information and Learning
300 North Zeeb Road, Ann Arbor, MI 48106-1346 USA
800-521-0600

UMI®

8C

A DISTORTION THEOREM
FOR ANALYTIC MAPS OF ANNULI

A thesis submitted
by
C.E. Castonguay
to

the Faculty of Pure and Applied Science
of the University of Ottawa

in partial fulfillment of the requirements
for the degree of
Master of Science
in the subject of
Mathematics

1963

TABLE OF CONTENTS

ACKNOWLEDGEMENTS

I wish to express my sincere thanks to Dr. H.G. Helfenstein who, as thesis advisor, suggested this work, and with patience and generous encouragement directed it during its development.

I am indebted to Dr. M. Wyman of Edmonton for help in evaluating some of the integrals of Appendix A.

It is a priviledge to acknowledge the assistance granted me by the International Nickel Company of Canada.

Chapter I

INTRODUCTION

Every abstract Riemann surface $\mathcal{R}$ can be made "concrete" by considering it as a smooth covering surface of the complex plane $\mathcal{W}$ by means of a suitable projection map p. Since this covering map is not unique, it seems natural to single out some such maps by an extremal property. The use of Riemannian metrics compatible with the conformal structure on the given surface $\mathcal{R}$ for the study of $\mathcal{R}$ is well-known; it suggests an investigation of the distortion caused by p between such a metric $ds_{\mathcal{R}}$ and the Euclidean metric of $\mathcal{W}$. A definition of integral (or average) distortion involving the logarithm of the local distortion, already introduced by Nevalinna ([4], p. 249) for a proof of the Picard-Landau theorem, is used. It has the disadvantage of depending on the system of coordinates used on $\mathcal{R}$; however, the corresponding invariant integral involving the area-element of $ds_{\mathcal{R}}$ does not exist in general - not even for the natural metrics of constant curvature.

In this thesis, sharp inequalities are deduced for this distortion for doubly connected surfaces, and the minimizing projection maps are determined. The simply connected case has been treated in [2]. The new result is applied in particular to the natural locally hyperbolic metric and to the pseudosphere.

Applications to the Bergman and Lindelöf metrics as well as to surfaces of higher connectivity will appear elsewhere.

Chapter II

DEFINITIONS

Let $\mathcal{R}$ be a doubly connected Riemann surface - that is, the fundamental group of $\mathcal{R}$ is infinite cyclic. Let $\mathcal{W}$ denote the (finite) complex z-plane, and let

$$\dot{\mathcal{E}} = \{ z \mid 0 < |z| < \infty \} = \text{the punctured Euclidean plane,}$$

$$\dot{\mathcal{H}} = \{ z \mid 0 < |z| < 1 \} = \text{the punctured hyperbolic plane,}$$

$$\mathcal{A}_q = \{ z \mid \sqrt{q} < |z| < \frac{1}{\sqrt{q}} \, , \; 0 < q < 1 \} \, .$$

$\mathcal{R}$ is conformally equivalent to one and only one of the above standard surfaces [5]. Denote this surface by $\mathcal{A}$, and let j be a 1-1 conformal map of $\mathcal{R}$ onto $\mathcal{A}$. For any smooth covering map $p : \mathcal{R} \to \mathcal{W}$, there exists a unique analytic map $f(z)$, $f'(z) \neq 0$ for all z in $\mathcal{A}$, such that the diagram

$$\begin{tikzcd}
\mathcal{R} \arrow[d, "j"'] \arrow[dr, "p"] & \\
\mathcal{A} \arrow[r, "f"'] & \mathcal{W}
\end{tikzcd}$$

is commutative.

Since all the surfaces $\mathcal{A}$ are schlichtartig, $\mathcal{A}$ thus furnishes a global system of coordinates for $\mathcal{R}$, whereby we may define an "integral distortion" for any such map p ,

$$D_z[p] = \iint_{\mathcal{A}} (\ln \delta_f(z))^2 \, dx \, dy \, ,$$

where

$$\delta_f(z) \;=\; \text{local distortion at } z \;=\; |f'(z)| \cdot \gamma(z) \,,$$

$$\gamma(z) \;=\; \frac{|dz|}{ds_{\mathcal{R}}} > 0 \text{ in } \mathcal{A} \,, \text{ and } z \;=\; x + iy \,,$$

with integration in the Lebesgue sense.

In order to make a later statement independent of coordinate system, we define, for an arbitrary real constant C,

$$D_z[\, p , C \,] \;=\; \iint_{\mathcal{A}} (\ln \delta_f - C)^2 \, dx \, dy \;.$$

In the following we restrict ourselves to the cases $\mathcal{A} = \mathcal{A}_q$ or $\dot{\mathcal{H}}$. The preceeding definition yields infinite values for $D_z[p]$ in the case of $\dot{\mathcal{E}}$.

Although we shall not use it here, the following is a suggestion for an invariant definition of distortion. If k is any other 1-1 conformal map of $\mathcal{R}$ onto $\mathcal{A}$, then there exists a unique conformal automorphism h such that the diagram

$$\begin{tikzcd}
\mathcal{A} \arrow[rrd, "g = f \circ h^{-1}"] & & \\
& \mathcal{R} \arrow[lu, "k"'] \arrow[ld, "j"] \arrow[r, "p"] & \mathcal{W} \\
\mathcal{A} \arrow[uu, "h"] \arrow[rru, "f"'] & &
\end{tikzcd}$$

is commutative.

Accordingly, relative to the new system of coordinates $h(z) = \zeta = \xi + i\eta$, we have

$$D_{\zeta}[p] \;=\; \iint_{h(\mathcal{A})} (\ln \delta_g(\zeta))^2 \, d\xi \, d\eta \;.$$

The local distortion being invariant,

$$D_\zeta[p] = \iint_{\mathcal{a}} (\ln \delta_f(z))^2 \left|\frac{d\zeta}{dz}\right|^2 dx\,dy \quad .$$

To make the measure of distortion independent of coordinate system, we may define

$$D[p] = \inf_h D_{h(z)}[p] \quad ,$$ where h ranges over all the conformal automorphisms of $\mathcal{a}$.

Chapter III

DISTORTION THEOREMS

Suppose $\mathcal{R}$ is conformally equivalent to some $\mathcal{A}_q$. Let $\varphi(z) = \ln \gamma(z)$ have in $\mathcal{A}_q$ the Fourier expansion

$$\varphi(z) = \sum_{-\infty}^{\infty} a_k(r)\, e^{ik\theta} \quad ,$$

where

$$a_k(r) = \frac{1}{2\pi} \int_0^{2\pi} \varphi(z)\, e^{-ik\theta}\, d\theta \qquad \text{for all } k ,$$

and $z = re^{i\theta}$ (we will assume later that φ is at least piecewise smooth in θ - that is, $\frac{\partial \varphi}{\partial \theta}$ exists in $\mathcal{A}_q$ and is piecewise continuous there - so that it may be represented by its Fourier series).

Define for all integers k :

$$L_k = \int_{\sqrt{q}}^{\frac{1}{\sqrt{q}}} |a_k(r)|^2\, r dr \quad , \qquad M_k = \int_{\sqrt{q}}^{\frac{1}{\sqrt{q}}} a_k(r)\, r^{k+1}\, dr \quad ,$$

$$N_k = \int_{\sqrt{q}}^{\frac{1}{\sqrt{q}}} a_k(r)\, r^{-k+1}\, dr \quad , \qquad \alpha_k = \int_{\sqrt{q}}^{\frac{1}{\sqrt{q}}} r^{2k+1}\, dr \quad .$$

Clearly, $\alpha_k > 0$ for all k.

Let

$$\beta = \int_{\sqrt{q}}^{\frac{1}{\sqrt{q}}} \ln r \cdot r dr \quad , \qquad \gamma = \int_{\sqrt{q}}^{\frac{1}{\sqrt{q}}} \ln^2 r \cdot r dr \quad ,$$

$$\delta = \int_{\sqrt{q}}^{\frac{1}{\sqrt{q}}} a_0(r) \cdot \ln r \cdot r\,dr \quad , \qquad \Delta_k = \begin{cases} \alpha_k \alpha_{-k} - \alpha_0^2 \ , & k \neq 0 \\ \alpha_0 \gamma - \beta^2 \ , & k = 0 \end{cases} .$$

By the Schwartz inequality, $\Delta_k > 0$ for all k.

Finally, let

$$A_k = \frac{\alpha_0 N_k - \alpha_{-k} M_k}{\Delta k} \quad , \qquad k \neq 0 \quad ,$$

$$A_0 = \text{a nearest integer to } \frac{\beta M_0 - \alpha_0 \delta}{\Delta_0} \quad ,$$

$$B_0 = - \frac{M_0 + \beta A_0}{\alpha_0} \quad ,$$

and
$$C_k = \begin{cases} L_k + \dfrac{1}{\Delta k} \left\{ \alpha_0 (M_k \overline{N}_k + \overline{M}_k N_k) - \alpha_{-k} |M_k|^2 - \alpha_k |N_k|^2 \right\} \ , & k \neq 0 \\ L_0 + \left\{ \dfrac{\left[2\beta\, \delta\, M_0 - \alpha_0 \delta^2 \right] - \gamma\, M_0^2}{\Delta_0} \right\} \ , & k = 0 \end{cases} .$$

<u>Theorem I</u>

a) With notation as above, if φ is piecewise smooth in θ and square-integrable over $\mathcal{A}_q$, then for any smooth covering map $p : \mathcal{R} \to \mathcal{W}$ the following inequality holds:

$$D_z[p] \geq 2\pi \left\{ \frac{\Delta_0}{\alpha_0} \left| A_0 - \frac{\beta M_0 - \alpha_0 \delta}{\Delta_0} \right|^2 + \sum_{-\infty}^{\infty} C_k \right\} \qquad \dots \quad (1)$$

---

b) If in addition $\dfrac{\partial^2 \varphi}{\partial \theta^2}$ exists, is piecewise continuous, and bounded in $\mathcal{A}_q$, then

$$g(z) = \exp\left\{B_0 + i\,\Psi_0\right\} \cdot z^{A_0} \cdot \exp\left\{2 \sum_{-\infty}^{\infty}{}' A_k z^k\right\}$$

is analytic (and one-valued) in $\mathcal{A}_q$, with Ψ_0 an arbitrary real constant and Σ' denoting summation over $k \neq 0$.

c) Under the conditions for b), the inequality (1) is sharp if and only if

$$\oint_{|z|=1} g(z)\,dz = 0 \qquad ,$$

with equality holding then if and only if $p = \int g(z)\,dz \circ j$, Ψ_0 and a constant of integration being arbitrary.

Proof of a)

Given $p = f \circ j : \mathcal{R} \to \mathcal{W}$, set

$$\phi(z) = \ln |f'(z)| = \mathcal{R}\log f'(z) .$$

Under the projection map $j = e \circ j : \mathcal{R} \to \mathcal{W}$, where $e(z) \equiv z$, we have $\phi(z) \equiv 0$ and so $D_z[j] < \infty$, since $\ln \delta_j = \varphi$ is square-integrable over $\mathcal{A}_q$. Thus there exist smooth projection maps with finite distortion. Consequently we will establish inequality (1) among such maps – that is, we will assume $\ln \delta_f = \phi + \varphi$ is square-integrable over $\mathcal{A}_q$; (1) will hold trivially for maps with infinite distortion, since the r.h.s. of (1) must be finite.

$\phi(z)$ being harmonic in $\mathcal{A}_q$, $\phi + \varphi$ is piecewise smooth in θ; let $\sum_{-\infty}^{\infty} b_k(r)\, e^{ik\theta}$ be its Fourier representation in $\mathcal{A}_q$.

$\phi + \varphi$ being square-integrable over $\mathcal{A}_q$, by the Lebesgue-Fubini theorem and the completeness relation we have

$$D_z[p] = \int_{\sqrt{q}}^{\frac{1}{\sqrt{q}}} r\,dr \int_0^{2\pi} (\phi + \varphi)^2 \, d\theta$$

$$= 2\pi \int_{\sqrt{q}}^{\frac{1}{\sqrt{q}}} \sum_{-\infty}^{\infty} |b_k(r)|^2 \, r\,dr \qquad \ldots \quad (2)$$

Let $h_n(r) = \sum_{-n}^{n} |b_k(r)|^2 \cdot r$, and $\lim_{n \to \infty} h_n(r) = h(r)$. From $h_n(r) \leq h(r)$ and (2),

$$\int_{\sqrt{q}}^{\frac{1}{\sqrt{q}}} h_n(r)\,dr \leq \int_{\sqrt{q}}^{\frac{1}{\sqrt{q}}} h(r)\,dr < \infty .$$

$(h_n(r))_{n=0,1,2,\ldots}$ is therefore an increasing sequence of integrable real-valued functions; applying the Monotone Convergence theorem, we may write (2) as

$$D_z[p] = 2\pi \int_{\sqrt{q}}^{\frac{1}{\sqrt{q}}} h(r)\,dr = 2\pi \lim_{n \to \infty} \int_{\sqrt{q}}^{\frac{1}{\sqrt{q}}} h_n(r)\,dr$$

$$= 2\pi \lim_{n \to \infty} \sum_{-n}^{n} \int_{\sqrt{q}}^{\frac{1}{\sqrt{q}}} |b_k(r)|^2 \cdot r\,dr$$

$$= 2\pi \sum_{-\infty}^{\infty} \int_{\sqrt{q}}^{\frac{1}{\sqrt{q}}} |b_k(r)|^2 \cdot r\,dr \qquad \ldots \quad (3)$$

$\phi = \sum_{-\infty}^{\infty} (b_k(r) - a_k(r))\, e^{ik\theta}$ is harmonic in $\mathcal{A}_q$: integrating $\nabla^2 \phi = 0$, we find

$$b_k(r) = \begin{cases} E_k r^k + F_k r^{-k} + a_k(r) & , \quad k \neq 0 \\ E_o \ln r + F_o + a_o(r) & , \quad k = 0 \end{cases} \qquad \dots \quad (4)$$

where E_k, F_k are complex constants depending on $f(z)$.

From (3), (4) we obtain

$$D_z[p] = 2\pi \left\{ \frac{\Delta_o}{\alpha_o} \left| E_o - \frac{\beta H_o - \alpha_o \delta}{\Delta_o} \right|^2 + \frac{1}{\alpha_o} \left| \beta E_o + \alpha_o F_o + H_o \right|^2 + C_o \right.$$

$$+ \sum_{-\infty}^{\infty}{}' \left(\frac{\Delta_k}{\alpha_{-k}} \left| E_k - \frac{\alpha_o H_k - \alpha_{-k} H_k}{\Delta_k} \right|^2 \right.$$

$$\left. \left. + \frac{1}{\alpha_{-k}} \left| \alpha_o E_k + \alpha_{-k} F_k + H_k \right|^2 + C_k \right) \right\} \qquad \dots \quad (5)$$

ϕ and $\phi + \varphi$ are real-valued; therefore $\bar{a}_k = a_{-k}$, and $\bar{b}_k = b_{-k}$. It follows from (4) that

$$\bar{E}_k = F_{-k} , \quad k \neq 0 \quad ; \quad \bar{E}_o = E_o , \quad \bar{F}_o = F_o . \qquad \dots \quad (6)$$

Calculating the harmonic conjugate of

$$\phi(z) = E_o \cdot \ln r + F_o + \sum_{-\infty}^{\infty}{}' (E_k r^k + F_k r^{-k}) \, e^{ik\theta}$$

and using (6), we find that $f'(z)$ is of the form

$$f'(z) = \exp \left\{ F_o + i \Psi_o \right\} \cdot z^{E_o} \cdot \exp \left\{ 2 \sum_{-\infty}^{\infty}{}' E_k z^k \right\} , \qquad \dots \quad (7)$$

Ψ_o some real constant. Since $f(z)$ was one-valued analytic, we conclude that E_o must be an integer.

a) now follows from (5).

Proof of b)

We show $\sum_{-\infty}^{\infty}{}' A_k z^k$ converges uniformly in $\mathcal{Q}_q$.

Writing

$$\sum_{-\infty}^{\infty}{}' A_k z^k = \sum_{k=1}^{\infty} A_k z^k + \sum_{k=1}^{\infty} A_{-k} w^k$$

with $w = \frac{1}{z}$, we show $\sum_{k=1}^{\infty} A_k z^k$ uniformly convergent in $|z| < \frac{1}{\sqrt{q}}$; the uniform convergence of $\sum_{k=1}^{\infty} A_{-k} w^k$ in $|w| < \frac{1}{\sqrt{q}}$ may be established in a similar way.

Let

$$\left|\frac{\partial^2 \varphi}{\partial \theta^2}\right| < D < \infty \quad \text{in } \mathcal{Q}_q ,$$

with D independent of r and θ. Twice repeated integration by parts of $a_k(r) = \frac{1}{2\pi}\int_0^{2\pi} \varphi \, e^{-ik\theta} \, d\theta$ yields

$$|a_k(r)| < \frac{D}{k^2} , \quad k \geq 1 .$$

It is then easy to see that

$$|A_k| , |A_{-k}| < \frac{D}{k^3} \cdot q^{-(k/2 + 1)} \quad \text{for large } k .$$

Also for k large enough, $\Delta_k > \frac{1}{2} \alpha_k \alpha_{-k}$, $\alpha_0 < \alpha_{-k}$, and $\alpha_k > \frac{1}{5k} q^{-(k+1)}$.

Hence for k sufficiently large, say $k \geq k_o(q)$,

$$|A_k| = \frac{1}{\Delta_k} |\alpha_o N_k - \alpha_{-k} M_k|$$

$$< \frac{2}{\alpha_k \cdot \alpha_{-k}} |\alpha_o N_k - \alpha_{-k} M_k|$$

$$< \frac{2}{\alpha_k} (|N_k| + |M_k|)$$

$$< \frac{20D}{k^2} \cdot q^{k/2} \quad .$$

Therefore for $|z| < \frac{1}{\sqrt{q}}$ and $k \geq k_o$,

$$|A_k z^k| < \frac{20D}{k^2} \quad ,$$

and the convergence follows.

Proof of c)

The vanishing of the period $\oint_{|z|=1} g(z)\, dz$ is necessary and sufficient for $\int_1^z g(t)\, dt$ to be one-valued analytic in $\mathcal{A}_q$. The rest is clear from (5), where the variables E_k, F_k appear within squares multiplied by positive coefficients.

Corollary

Suppose $\gamma(z) = \gamma(r)$. Then the inequality (1) holds with $C_k = 0$, $k \neq 0$; (1) is sharp if and only if $A_o \neq -1$, with equality holding then if and only if

$$p = \left(\exp\left\{\frac{B_0 + i\,\psi_0}{A_0 + 1}\right\} \cdot z^{A_0 + 1} + \psi_1\right) \circ j \quad ,$$

with ψ_0 and ψ_1 arbitrary constants, real and complex respectively.

Proof.

Since $a_k(r) \equiv 0 \,, \quad k \neq 0 \,,$

$M_k = N_k = L_k = C_k = 0 \,, \quad k \neq 0 \,.$

Hence

$$g(z) = \exp\left\{B_0 + i\varphi_0\right\} \cdot z^{A_0}$$

and $\int g(z)\, dz$ is one-valued analytic in $\mathcal{A}_q$ if and only if $A_0 \neq -1$.

If $\gamma(z) = \gamma(r)$ and $A_0 = -1$, then the inequality (1) cannot be sharp - that is, in this case there exists no map with minimal distortion. The r.h.s. of (1) may yet be, however, the greatest lower bound for the distortions of the projection maps. Knowing that for any such $p = f \circ j$, f is an analytic map such that (7) holds with E_0 an integer, we see, keeping in mind $\varphi = \varphi(r)$ and (5), that this question is equivalent to the following: with $E_0 = A_0 = -1$, is it possible to choose the complex constants E_k, $k \neq 0$, in such a way that $\sum_{-\infty}^{\infty}{}' \dfrac{\Delta_k}{\alpha_{-k}} |E_k|^2$ is arbitrarily small and $\oint_{|z|=1} f'(z)\, dz = 0$? This problem remains open.

Suppose $\mathcal{R}$ is conformally equivalent to $\dot{\mathcal{H}}$, and that

$\gamma(z) = \gamma(r)$. Defining for $\dot{\mathcal{H}}$ the quantities $\alpha_0, \beta, \gamma_0$, etc., as for $\mathcal{A}_q$ by assigning the appropriate limits to the integrals, we state

<u>Theorem 2</u>

If $\varphi(r)$ is square-integrable over $\dot{\mathcal{H}}$, then for any smooth $p : \mathcal{R} \to \mathcal{W}$,

$$D_z[p] \geq 2\pi \left\{ \frac{\Delta_0}{\alpha_0} \left| A_0 - \frac{\beta\gamma_0 - \alpha_0\delta}{\Delta_0} \right|^2 + C_0 \right\} .$$

This inequality is sharp if and only if $A_0 \neq -1$.

<u>Proof.</u>

We have

$$D_z[p] = 2\pi \int_0^1 \left(|b_0(r)|^2 + \sum_{-\infty}^{\infty}{}' \, |E_k r^k + F_k r^{-k}|^2 \right) r dr$$
$$\geq 2\pi \int_0^1 |b_0(r)|^2 \cdot r dr \; ,$$

and the proof carries on as before.

Under the same assumptions as in the above theorems, one can deduce in the same way as for (1) an inequality

$$D_z[p, C] \geq M$$

for arbitrary real constant C, with M a non-negative finite lower bound independent of C and p . From the mean-value theorem of integration (assuming continuity of the integrand) we deduce an invariant inequality for $\delta_f(z)$: for every real constant C there exists a point

z_o in $\mathcal{A}$ such that

$$\delta_f(z_o) \geq \exp\left\{ \left(\frac{K}{\operatorname{area}(\mathcal{A})}\right)^{\frac{1}{2}} + c \right\},$$

with area($\mathcal{A}$) denoting the Euclidean area of $\mathcal{A}$.

Chapter IV

APPLICATIONS

1. The Locally Hyperbolic Metric

Every Riemann surface $\mathcal{R}$ has a universal covering surface $\mathcal{U}$ which is either the Riemann sphere $\mathcal{S}$, the Euclidean plane $\mathcal{E}$, or the hyperbolic plane $\mathcal{H}$ (Uniformisation Theorem, [5]). Correspondingly, $\mathcal{R}$ is said to be of elliptic, parabolic, or hyperbolic type. Any projection map $p : \mathcal{U} \to \mathcal{R}$ induces on $\mathcal{R}$ a natural Riemannian metric of constant curvature (locally spherical, locally Euclidean, or locally hyperbolic metric).

$\dot{\mathcal{H}}$ is of hyperbolic type. If $\mathcal{H}$ is represented by the upper half ζ-plane with line element $ds_{\mathcal{H}} = \frac{|d\zeta|}{\mathfrak{I}(\zeta)}$, then a covering map of $\mathcal{H}$ onto $\dot{\mathcal{H}}$ is given by $z = e^{i\zeta}$. The induced locally hyperbolic metric on $\dot{\mathcal{H}}$ becomes

$$ds_{\dot{\mathcal{H}}} = \frac{|dz|}{|z| \ln \frac{1}{|z|}} \qquad \ldots \quad (8)$$

Each $\mathcal{A}_q$ is of hyperbolic type. With the same representation for $\mathcal{H}$, $z = \sqrt{q} \cdot \exp\left\{ \frac{\ln \frac{1}{q}}{i\pi} \cdot \ln \zeta \right\}$ is a projection map $\mathcal{H} \to \mathcal{A}_q$.

The induced metric is

$$ds_{\mathcal{A}_q} = \frac{\pi\, |dz|}{\ln \frac{1}{q} \cdot |z| \cdot \cos\left(\frac{\pi \cdot \ln z}{\ln q}\right)}$$

In both these examples , $\gamma(z) = \gamma(r)$; from the calculations which follow it will become evident that φ is square-integrable in both cases, so that we are justified in applying the theorems. We will demonstrate the existence of minimizing conformal representations for the surfaces $\mathcal{H}$ and $\mathcal{A}_q$ with metrics (8) and (9) respectively by estimating Λ_o for each case. We will also give the r.h.s. of (1) for each surface.

---

a. $\mathcal{A} = \mathcal{H}$, $\gamma(z) = r \ln \frac{1}{r}$. Then

$$\varphi(r) = a_o(r) = \ln r + \ln \ln \frac{1}{r}$$

whence

$$M_o = \int_o^1 a_o(r)\, r\, dr$$

$$= \beta + G_1$$

$$\delta = \gamma - G_2$$

$$L_o = \gamma - 2G_2 + G_3 ,$$

where

$$G_1 = \int_o^1 \ln \ln \frac{1}{r} \cdot r\, dr = \int_o^\infty \ln u \cdot e^{-2u}\, du$$

$$= -\frac{(\mathcal{E} + \ln 2)}{2} \qquad \text{(see App. A, ii): } \mathcal{E} = \text{Euler's constant)}$$

$$G_2 = \int_o^\infty \ln u \cdot u \cdot e^{-2u}\, du = \frac{1}{4} + \frac{1}{2}\, G_1$$

$$G_3 = \int_o^\infty \ln^2 u \cdot e^{-2u}\, du = \frac{1}{2}(\mathcal{E} + \ln 2)^2 + \frac{\pi^2}{12} \quad \text{(App. A, ii)} ;$$

also $\alpha_o = \frac{1}{2}$, $\beta = -\frac{1}{4}$, $\gamma = \frac{1}{4}$, $\Delta_o = \frac{1}{16}$.

We find

$$\frac{\beta w_o - \alpha_o \delta}{\Delta_o} = +1 \text{, and so } A_o = 1 ,$$

$$B_o = \mathcal{E} + 1 + \ln 2$$

$$C_o = G_3 - \frac{\gamma G_1^2 + 2\beta G_1 G_2 + \alpha_o G_2^2}{\Delta_o}$$

$$= \frac{\pi^2}{12} - \frac{1}{2} .$$

In view of theorem 2, we may therefore state

<u>Theorem 3</u>

For $\dot{\mathcal{H}}$ with metric given by (8), any smooth covering map $p : \dot{\mathcal{H}} \to \mathcal{W}$ satisfies

$$D_z[p] \geq \pi\left(\frac{\pi^2}{6} - 1\right) ,$$

with equality holding only for

$$p(z) = \exp\left\{\mathcal{E} + 1 + i\psi_o\right\} \cdot z^2 + \psi_1 ,$$

with ψ_o and ψ_1 arbitrary constants, real and complex respectively.

---

b. $$\mathcal{Q} = \mathcal{Q}_q ,\ \gamma(z) = r \cdot \frac{\ln \frac{1}{q}}{\pi} \cdot \cos\left(\frac{\pi \ln r}{\ln q}\right) .$$

Then

$$\varphi(r) = \ln r + \ln\left(\frac{\ln \frac{1}{q}}{\pi}\right) + \ln \cos\left(\frac{\pi \ln r}{\ln q}\right)$$

whence

$$M_o = \beta + s_1 \alpha_o - s\,\theta_1$$

$$\delta = \gamma + s_1 \beta + s^2 \theta_2$$

$$L_o = \gamma + s_1^2 \alpha_o - s\,\theta_3 + 2(s_1 \beta - s_1 s\,\theta_1 + s^2 \theta_2)\,,$$

where

$$s = \frac{\ln \frac{1}{q}}{\pi}\,, \qquad s_1 = \ln s\,,$$

$$\theta_1 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \ln \cos\theta \cdot e^{-2s\,\theta}\, d\theta$$

$$\theta_2 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \ln \cos\theta \cdot \theta \cdot e^{-2s\,\theta}\, d\theta$$

$$\theta_3 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \ln^2 \cos\theta \cdot e^{-2s\,\theta}\, d\theta\ ;$$

from

$$\theta_1 = \frac{d}{dx} \left\{ \frac{\pi \cdot \Gamma(x+1)}{2^x \cdot \Gamma\left(1 + \frac{x + i2s}{2}\right) \cdot \Gamma\left(1 + \frac{x - i2s}{2}\right)} \right\} \Bigg|_{x = 0}$$

(App. A, i)

$$\theta_2 = -\frac{1}{2} \cdot \frac{d\theta_1}{ds}$$

$$\theta_3 = \frac{d^2}{dx^2} \left\{ \frac{\pi \cdot \Gamma(x+1)}{2^x \cdot \Gamma\left(1 + \frac{x + i2s}{2}\right) \cdot \Gamma\left(1 + \frac{x - i2s}{2}\right)} \right\} \Bigg|_{x = 0}$$

(App. A, i)

we obtain upon simplification

$$\theta_1 = -\frac{\sin h\,(\pi s)}{s} \cdot K$$

$$\theta_2 = -\frac{1}{2s^2}\left\{(\sin h\,(\pi s) - \cos h\,(\pi s)\cdot(\pi s))\cdot K + \sin h\,(\pi s)\cdot s\cdot \Im(\Psi'(is))\right\}$$

$$\theta_3 = -\frac{\sin h\,(\pi s)}{s}\cdot\left\{K^2 + \Psi'(1) - \frac{1}{2s^2} - \frac{1}{2}\Re(\Psi'(is))\right\} ,$$

with $K = \mathcal{E} + \ln 2 + \Re(\Psi(is))$ and $\Psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$; also

$$\alpha_o = \sin h\,(\pi s)$$

$$\beta = \frac{1}{2}(\cos h\,(\pi s)\cdot(\pi s) - \sin h\,(\pi s))$$

$$\gamma = \frac{1}{4}(\sin h\,(\pi s)\cdot(\pi s)^2 - 2\cos h\,(\pi s)\cdot(\pi s) + 2\sin h\,(\pi s))$$

$$\Delta_o = \frac{1}{4}(\sin h^2\,(\pi s) - (\pi s)^2) \quad .$$

We find

$$\frac{\beta M_o - \alpha_o\delta}{\Delta_o} = \frac{s(\beta\theta_1 + s\alpha_o\,\theta_2)}{\Delta_o} - 1$$

$$= -\frac{2s\cdot\sin h^2\,(\pi s)}{\sin h^2\,(\pi s) - (\pi s)^2}\cdot\Im(\Psi'(is)) - 1 \quad \ldots (15)$$

We refer to values of q where

$$\frac{\beta M_o - \alpha_o \delta}{\Delta_o} = \frac{2n+1}{2} \quad \text{for some integer } n$$

as critical values of the conformal modulus, where $A_o(q)$ may be either n or $n+1$. We will show that

$$-\frac{1}{2} < \frac{\beta M_o - \alpha_o \delta}{\Delta_o} < \frac{3}{2} \quad \text{for} \quad 0 < s < \infty ,$$

thus proving (Theorem 1, Corollary) the existence of maps with least distortion for each $\mathcal{A}_q$.

Given q, A_o and B_o may then be calculated and the precise form of the minimizing maps obtained. Their integral distortion may be calculated knowing

$$\begin{aligned} C_o &= s\theta_3 - \frac{s^2}{\Delta_o}\left(\gamma\theta_1^2 + 2\beta\theta_1 \cdot s\theta_2 + \alpha_o (s\theta_2)^2\right) \\ &= \sin h\,(\pi s) \cdot \left\{ \Psi'(1) - \frac{1}{2}\left(\mathcal{R}(\Psi'(is)) + \frac{1}{s^2}\right) \right. \\ &\quad \left. - \frac{\sin h^2\,(\pi s)}{\sin h^2\,(\pi s) - (\pi s)^2} \cdot s^2 \cdot \left(\mathfrak{J}(\Psi'(is))\right)^2 \right\} \qquad \ldots (11) \end{aligned}$$

We incorporate these results into

<u>Theorem 4</u>

a) For any annulus $\mathcal{A}_q$ with metric (9), any $p : \mathcal{A}_q \rightarrow \mathcal{W}$ satisfies

$$D_s[p] \geq 2\pi \left\{ \frac{\Delta_o}{\alpha_o} \left| A_o - \frac{\beta M_o - \alpha_o \delta}{\Delta_o} \right|^2 + C_o \right\} \qquad \ldots (12)$$

where $\frac{\beta M_o - \alpha_o \delta}{\Delta_o}$ and C_o are given by (10) and (11) respectively.

This inequality is sharp.

b) There exists only one critical value, $\bar{q} \in (e^{-\frac{.34}{\pi}}, e^{-\frac{.33}{\pi}})$,

and

$$\Lambda_o(q) = \begin{cases} 1 \text{ for } 0 < q < \bar{q} \\ 0 \text{ for } \bar{q} < q < 1 \\ 0 \text{ or } 1 \text{ for } q = \bar{q} \end{cases} .$$

Proof of a)

Using the series representation

$$\Psi(z) = -\mathcal{E} + \sum_{k=0}^{\infty} \left(\frac{1}{k+1} - \frac{1}{z+k}\right)$$

we rewrite (10) in the form

$$c(s) = \frac{\beta M_o - \alpha_o \delta}{\Delta_o} + 1 = 4 \cdot b(s) \cdot \Sigma(s)$$

with

$$b(s) = \frac{\sinh^2(\pi s) \cdot s^2}{\sinh^2(\pi s) - (\pi s)^2}$$

$$\Sigma(s) = \sum_{n=1}^{\infty} \frac{n}{(n^2 + s^2)^2} .$$

To prove the existence of minimizing maps and to give rough bounds for $\Lambda_o(q)$, we show

$$-\frac{1}{2} < \frac{\beta M_o - \alpha_o \delta}{\Delta_o} < \frac{3}{2} ,$$

that is,

$$\frac{1}{2} < c(s) < \frac{5}{2} ,$$

whence A_0 = either 0 or 1.

It is shown in app. B, i) that $b(s)$ is strictly increasing in $(0, \infty)$. We use $\lim_{s \to 0} b(s) = \frac{3}{\pi^2}$ (app. B, ii)) to define $b(0)$ and $c(0)$. We estimate $\Sigma(k)$ for k a positive integer by its Euler-Maclaurin expansion, truncating after the third term, within the following error (app. B, iii)):

$$\Sigma(k) = \frac{1}{2} \cdot \frac{1}{1 + k^2} + \frac{5}{12} \cdot \frac{1}{(1 + k^2)^2} + \frac{1}{3} \cdot \frac{1}{(1 + k^2)^3}$$

$$\pm \frac{5}{9} \cdot \frac{1}{k^6} \qquad \ldots (13)$$

If $I = [s', s'']$ is any finite closed subinterval of $[0, \infty)$, since $b(s)$ is strictly increasing and $\Sigma(s)$ is obviously decreasing there, we have

$$\operatorname*{Min}_{s \in I} c(s) > 4 \left(\operatorname*{Min}_{s \in I} b(s)\right) \cdot \left(\operatorname*{Min}_{s \in I} \Sigma(s)\right) = 4\, b(s')\ \Sigma(s'')$$

$$\operatorname*{Max}_{s \in I} c(s) < 4 \left(\operatorname*{Max}_{s \in I} b(s)\right) \cdot \left(\operatorname*{Max}_{s \in I} \Sigma(s)\right) = 4\, b(s'')\ \Sigma(s') .$$

We will show $c(s) > \frac{1}{2}$ by partitioning $[0, \infty)$ into appropriate subintervals $[s', s'']$ and showing

$$4\, b(s')\, \Sigma(s'') > \frac{1}{2} .$$

for each such interval.

This is easily checked for each of the intervals $[0, \frac{1}{2}]$, $[\frac{1}{2}, 1]$, $[1, \frac{3}{2}]$, and $[\frac{3}{2}, 2]$. Thus for $s \in (0, 2)$, $c(s) > \frac{1}{2}$. $b(0)$ is needed for the first of these calculations.

Suppose now $s \in [n, n+1]$, n an integer ≥ 2. Then

$$c(s) > 4\, b(n)\, \Sigma(n+1) > 4\, n^2 \cdot \Sigma(n+1)$$

$$> \frac{2n^2}{n^2 + 2n + 2} - \frac{20}{9} \cdot \frac{1}{n^4} \quad \text{from (13)}$$

$$> \frac{1}{2} \quad \text{for} \quad n \geq 2 \; .$$

Thus $c(s) > \frac{1}{2}$ for $s \in (0, \infty)$.

The proof of $c(s) < \frac{5}{2}$ is similar to that above. A partition of $[0, 10]$ into intervals of diameter $\frac{1}{10}$ proves sufficiently fine to check that $\text{Max}\; c(s) < \frac{5}{2}$ for $s \in (0, 10)$. If $s \in [n, n+1]$, n an integer ≥ 10, then using $b(n) \approx n^2$ since $\sinh(\pi n) > 10^7$,

$$c(s) < 4\, b(n+1) \cdot \Sigma(n)$$

$$\approx 4\, (n+1)^2 \cdot \Sigma(n)$$

$$< 2 + \frac{12n + 5}{3(1 + n^2)} + \frac{10n + 4}{3(1 + n^2)^2} + \frac{8n}{3(1 + n^2)^3} + \frac{(n+1)^2}{n^6} \cdot \frac{20}{9}$$

from (13)

$$< 2.43 \quad \text{for} \quad n \geq 10 \; .$$

Therefore $c(s) < \frac{5}{2}$ for $s \in (0, \infty)$.

From the above calculations and (11), we may remark that C_o varies approximately as $\sin h\ (\pi s)$; from

$$\frac{\Delta_o}{\alpha_o} = \frac{\sin h^2\ (\pi s) - (\pi s)^2}{4 \sin h\ (\pi s)}$$

we see that therefore the r.h.s. of (12) also varies roughly as $\sin h\ (\pi s)$.

<u>Proof of b)</u>

A partition process, similar to those used above, with mesh diameter $\frac{1}{1000}$, was programmed for an IBM 1650 and yielded

$$c(s) < \frac{3}{2} \quad \text{for} \quad s \in (0, .33]$$

$$c(s) > \frac{3}{2} \quad \text{for} \quad s \in [.35, 3] \quad .$$

In app. E, iv) and v), $c'(s)$ is shown to be positive in the intervals $[.33, .35]$ and $[3, \infty)$ respectively. Also, $c(.34) > \frac{3}{2}$ by direct calculation. Combining these results with those in the proof of a), we conclude that there exists a point $\bar{s} = \frac{\ln \frac{1}{q}}{\pi} \in (.33, .34)$ such that $c(\bar{s}) = \frac{3}{2}$ and

$$\frac{1}{2} < c(s) < \frac{3}{2} \quad \text{for} \quad 0 < s < \bar{s}$$

$$\frac{3}{2} < c(s) < \frac{5}{2} \quad \text{for} \quad \bar{s} < s < \infty \quad .$$

The above inequalities, interpreted in terms of $A_0(q)$ and q, give the sought relation.

Knowing $c'(s) > 0$ in $[.33, .34]$, $\bar{q}$ may be calculated as accurately as desired.

2. The Pseudosphere $\mathcal{P}$.

We use for $\mathcal{P}$ the parametric representation in E^3:

$$x_1 = \frac{\cos v}{\cosh u}, \quad x_2 = \frac{\sin v}{\cosh u}, \quad x_3 = u - \tanh u,$$

$$0 \leq v < 2\pi, \quad -\infty < u < \infty.$$

Because of the singularity at the rim $u = 0$, the Euclidean metric of E^3 does not induce a conformal structure on the total pseudosphere. However, the upper funnel $\mathcal{P}_u$: $u > 0$ is a Riemann surface which is conformally equivalent to $\dot{\mathcal{H}}$, represented by $0 < |z| < e^{-1}$, $z = re^{i\theta}$, through the map $j : \mathcal{P}_u \rightarrow \dot{\mathcal{H}}$ given by

$$r = e^{-\cosh u}, \quad \theta = v.$$

The natural locally hyperbolic metric of $\dot{\mathcal{H}}$ does not coincide with the metric of $\mathcal{P}_u$ induced by the Euclidean metric of E^3 (it is only this latter metric which gives some geometric interest to $\mathcal{P}_u$). Hence $\mathcal{P}_u$ does not fall into one of our classes of doubly connected Riemann surfaces with a complete Riemannian metric in the sense of Hopf and Rinow [3]. However, our methods, applied to $\mathcal{P}_u$ with the abovementioned metric and coordinate system given by j , permit also to determine maps $p : \mathcal{P}_u \to \mathcal{W}$ with minimum distortion.

We find $\gamma(z) = r \ln \frac{1}{r} = \gamma(r)$, and it will become clear from the calculations that follow that $\varphi(r) = \ln(r \ln \frac{1}{r})$ is square-integrable over $\mathcal{H}$.

Easily,

$$M_o = \beta + I_1$$

$$\delta = \gamma - I_2$$

$$L_o = \gamma - 2I_2 + I_3 \quad ,$$

where

$$I_1 = \int_1^\infty \ln u \cdot e^{-2u}\, du = \frac{1}{2} \int_1^\infty \frac{e^{-2u}}{u}\, du$$

$$= -\frac{1}{2} E_i(-2) \quad (E_i(x) \text{ denotes the integral-logarithm})$$

$$I_2 = \int_1^\infty \ln u \cdot u \cdot e^{-2u}\, du = \frac{1}{4\, e^2} + \frac{1}{2} I_1$$

$$I_3 = \int_1^\infty \ln^2 u \cdot e^{-2u}\, du$$

$$= \ln 2 \cdot E_i(-2) - \frac{\ln^2 2}{2\, e^2} + \frac{1}{2} \int_2^\infty \ln^2 v \cdot e^{-v}\, dv$$

$$= \ln 2 \cdot E_i(-2) - \frac{\ln^2 2}{2\, e^2} + \frac{1}{2} \Big\{ \mathcal{E}^2 + \frac{\pi^2}{6}$$

$$+ \sum_{m=1}^{\infty} \frac{(-2)^m}{m!\, m^2} \left[(\ln 2^m - 1)^2 + 1\right] \qquad \text{(app. A, iii))} ;$$

and $\alpha_o = \frac{1}{2\, e^2}$, $\beta = \frac{-3}{4\, e^2}$, $\gamma = \frac{5}{4\, e^2}$, $\Delta_o = \frac{1}{16\, e^4}$.

We obtain

$$\frac{\beta M_o - \alpha_o \delta}{\Delta_o} = \frac{\beta I_1 + \alpha_o I_2}{\Delta_o} - 1$$

$$= 4\, e^2 \cdot E_1(-2) + 1 \approx -0.445 \quad ,$$

whence $A_o = 0$.

Using

$$B_o = \frac{3}{2} + e^2 \cdot E_1(-2)$$

$$C_o = I_3 - \frac{e^2}{2} \left(5\, (E_1(-2))^2 + \frac{4\, E_1(-2)}{e^2} + \frac{1}{e^4}\right) \quad ,$$

we find the minimizing maps and estimate their distortion. With respect to the coordinate system given by j , we have

<u>Theorem 5</u>

For any $p : \mathcal{P}_u \to \mathcal{W}$,

$D_z[p] \geq 0.330$

where the minimum distortion has been estimated accurately to two decimal places. The minimizing maps are $p = h \circ j$,

$$h(z) = \exp\left\{\frac{3}{2} + e^{2} \cdot E_1(-2) + i\,\Psi_o\right\} \cdot z + \Psi_1 ,$$

Ψ_o, Ψ_1 arbitrary as before.

APPENDIX A

i) To evaluate $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\ln \cos t)^n e^{\alpha t} dt$, n a positive integer

and α arbitrary real, consider

$$I(x) = 2\int_0^{\frac{\pi}{2}} (\cos t)^x \cos h\ (\alpha t)\ dt$$

$$= 2\int_0^{\frac{\pi}{2}} (\cos t)^x \cos\ (i\alpha t)\ dt$$

$$= \frac{\pi \cdot \Gamma(x+1)}{2^x \cdot \Gamma(1 + \frac{x+i\alpha}{2}) \cdot \Gamma(1 + \frac{x-i\alpha}{2})} \quad \text{([1], vol. 1, p. 12).}$$

Differentiating n times with respect to x and then setting $x = 0$, one finds

$$\left.\frac{d^n(I(x))}{dx^n}\right|_{x=0} = 2\int_0^{\frac{\pi}{2}} (\ln \cos t)^n \cos h\ (\alpha t)\ dt$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\ln \cos t)^n e^{\alpha t} dt$$

$$= \left(\frac{d}{dx}\right)^n \left.\left[\frac{\pi \cdot \Gamma(x+1)}{2^x \cdot \Gamma(1 + \frac{x+i\alpha}{2}) \cdot \Gamma(1 + \frac{x-i\alpha}{2})}\right]\right|_{x=0}$$

ii) To find $\int_0^{\infty} \ln^2 u \cdot e^{-2u}\ du$, consider

$$K(t) = \int_0^\infty u^t \cdot e^{-2u} \, du = \frac{1}{2} \int_0^\infty \left(\frac{v}{2}\right)^t e^{-v} \, dv$$

$$= \frac{1}{2^{t+1}} \Gamma(t+1) \qquad \text{for } \mathcal{R}(t) > -1 \quad .$$

$$\ln K(t) = \ln \Gamma(t+1) - (t+1) \cdot \ln 2 \quad ,$$

whence

$$\dot{K}(t) = K(t) \cdot (\Psi(t+1) - \ln 2) \quad .$$

We notice that

$$\dot{K}(0) = \int_0^\infty \ln u \cdot e^{-2u} \, du = K(0) \cdot (\Psi(1) - \ln 2)$$

$$= -\left(\frac{\mathcal{E} + \ln 2}{2}\right) \quad .$$

Differentiating further,

$$\ddot{K}(t) = \dot{K}(t) \cdot (\Psi(t+1) - \ln 2) + K(t) \cdot \dot{\Psi}(t+1) \; ,$$

and

$$\ddot{K}(0) = \int_0^\infty \ln^2 u \cdot e^{-2u} \, du$$

$$= K(0) \, (\Psi(1) - \ln 2)^2 + K(0) \cdot \dot{\Psi}(1)$$

$$= \frac{1}{2} (\mathcal{E} + \ln 2)^2 + \frac{\pi^2}{12} \quad .$$

iii) We notice that

$$\int_2^\infty \ln^2 v \cdot e^{-v} \, dv = \left. \frac{\partial^2 \Gamma(a, x)}{\partial a^2} \right|_{(1,\,2)} \quad .$$

From

$$\Gamma(\alpha, x) = \Gamma(\alpha) - \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \cdot \frac{x^{\alpha+n}}{\alpha + n} ,$$

$$\alpha \neq 0, -1, -2, \ldots \qquad ([1], \text{vol. } 2, \text{p. } 135)$$

we obtain

$$\left.\frac{\partial^2 \Gamma(\alpha, x)}{\partial\alpha^2}\right|_{(1,\,2)} = \mathcal{E}^2 + \frac{\pi^2}{6} + \sum_{m=1}^{\infty} \frac{(-2)^m}{m!\, m^2} [(\ln 2^m - 1)^2 + 1] .$$

Another evaluation of $I_3 = \int_1^{\infty} \ln^2 u \cdot e^{-2u}\, du$ is as follows. Integrating by parts and letting $u = \frac{t}{2}$,

$$I_3 = \int_1^{\infty} \frac{\ln u}{u} \cdot e^{-2u}\, du$$

$$= \int_2^{\infty} \frac{\ln t}{t} \cdot e^{-t}\, dt - \ln 2 \int_2^{\infty} \frac{e^{-t}}{t}\, dt .$$

Noticing

$$\left.\frac{\partial \Gamma(\alpha, 2)}{\partial\alpha}\right|_{\alpha=0} = \int_2^{\infty} \frac{\ln t}{t} \cdot e^{-t}\, dt$$

we obtain

$$I_3 = \left[\frac{\partial}{\partial\alpha} \Gamma(\alpha, 2) - \ln 2 \cdot \Gamma(\alpha, 2)\right]\Big|_{\alpha=0}$$

$$= \frac{\partial}{\partial\alpha} [2^{-\alpha} \Gamma(\alpha, 2)]\Big|_{\alpha=0} ,$$

which can be evaluated to any desired degree of accuracy.

APPENDIX B

i) We show $b'(s) > 0$ for $s > 0$.

From

$$b'(s) = 2\pi^3 \cdot (b(s))^2 \cdot \left(\frac{1}{(\pi s)^3} - \frac{\cosh(\pi s)}{\sinh^3(\pi s)}\right),$$

it is sufficient to show

$$h(t) = \frac{\sinh^3 t}{\cosh t} - t^3 > 0 \quad \text{for} \quad t > 0 .$$

This follows from

$$h(0) = \dot{h}(0) = \ddot{h}(0) = 0$$

and

$$\dddot{h}(t) = 2\,\frac{\sinh^4 t}{\cosh^4 t}\,(4\cosh^2 t + 3) > 0 .$$

ii) To evaluate $\lim\limits_{s \to 0} b(s)$, we notice that

$$b(s) = s^2 \cdot \frac{1}{1 - \left(\frac{\pi s}{\sinh(\pi s)}\right)^2} = s^2 \cdot \sum_{n=0}^{\infty} a^n \qquad \ldots \quad (1)$$

where $a = \left(\frac{\pi s}{\sinh(\pi s)}\right)^2 < 1$ for all s.

Applying the Euler-MacLaurin expansion for $s \in (0, 1)$ with $g(x) = a^x$, we have

$$\sum_{n=0}^{\infty} a^n = \sum_{n=0}^{\infty} g(n) = -\frac{1}{\ln a} + \frac{1}{2} - \frac{\ln a}{12}$$

$$+ \frac{\ln^3 a}{720} + \ldots + \frac{B_{2m}}{2m!}(\ln a)^{2m-1} + R_m \qquad \ldots \quad (2)$$

where

$$|E_m| < 2 \left|\frac{B_{2m+2}}{(2m+2)!}(\ln a)^{2m+1}\right|$$

since $g^{(m)}(x)$ is of constant sign in $(0, 1)$ for any m ([6], p. 133).

From

$$B_{2m} = (-1)^m \cdot 2 \cdot \frac{(2m)!}{(2\pi)^{2m}} \cdot \zeta(2m) \qquad ([6], \text{p. } 129)$$

we see

$$|E_m| < \frac{2}{\pi} \cdot \zeta(2m+2) \cdot \left|\frac{\ln a}{2\pi}\right|^{2m+1}$$

and since $\left|\frac{\ln a}{2\pi}\right| < 1$ for $s \in (0, 1)$, we conclude that (2) converges for s near 0. Since $\lim\limits_{s\to 0} (\ln a) = 0$, we have from (1), (2)

$$\lim_{s\to 0} b(s) = \lim_{s\to 0} \frac{-s^2}{\ln a}$$

$$= \frac{1}{2} \lim_{s\to 0} \frac{s^2}{\ln \sinh(\pi s) - \ln(\pi s)}$$

from which, using l'Hopital's rule twice,

$$\lim_{s\to 0} b(s) = \frac{3}{\pi^2}.$$

iii) We estimate $\Sigma(k)$ for k a positive integer. We notice that $f(x) = \frac{x}{(x^2+s^2)^2}$ has as second derivative

$$f''(x) = \frac{12x(x^2 - s^2)}{(x^2+s^2)^4} \begin{cases} \le 0 & \text{for } x \le s \\ \ge 0 & \text{for } x \ge s \end{cases},$$

which permits use to use Euler's formula twice:

$$\sum_{n=1}^{k-1} \frac{n}{(n^2 + k^2)^2} = \sum_{n=1}^{k-1} f(n)$$

$$= \int_1^k f(t)dt - \frac{1}{2} f(t) \Big]_1^k + \frac{1}{12} f'(t) \Big]_1^k + E_1(k) \qquad \ldots \quad (3)$$

$$\sum_{n=k}^{\infty} \frac{n}{(n^2 + k^2)^2} = \int_k^{\infty} f(t)dt + \frac{1}{2} f(k) - \frac{1}{12} f'(k) + E_2(k) \qquad \ldots \quad (4)$$

and estimate

$$|E_1(k)| < 2 \left| \frac{B_4}{4!} (f'''(k) - f'''(1)) \right|$$

$$|E_2(k)| < 2 \left| \frac{B_4}{4!} \cdot f'''(k) \right| \quad .$$

We find

$$f'''(k) = \frac{3}{2} \cdot \frac{1}{k^6}$$

$$|f'''(1)| < \frac{12 \cdot 16}{k^6} \quad ;$$

whence

$$|E_1(k) + E_2(k)| \leq |E_1(k)| + |E_2(k)|$$

$$\leq 2 \left| \frac{B_4}{4!} \right| \cdot (2|f'''(k)| + f'''(1))$$

$$< \frac{5}{9} \cdot \frac{1}{k^6} \quad .$$

Adding (3) and (4) we get

$$\Sigma(k) = \frac{1}{2} \cdot \frac{1}{1+k^2} + \frac{5}{12} \cdot \frac{1}{(1+k^2)^2} + \frac{1}{3} \cdot \frac{1}{(1+k^3)^3} \pm \frac{5}{9} \cdot \frac{1}{k^6} \quad .$$

iv) To show $c'(s) > 0$ in $I_o = [.33, .35]$, we consider the function $p(s)$ defined by

$$c'(s) = \frac{8s \cdot \sin h\,(\pi s)}{(\sin h^2\,(\pi s) - (\pi s)^2)^2} \cdot \quad (\sin h^3\,(\pi s) - \cos h\,(\pi s) \cdot (\pi s)^3) \cdot \Sigma(s)$$

$$- 2s^2 \cdot \sin h\,(\pi s) \cdot (\sin h^2\,(\pi s) - (\pi s)^2) \cdot \sum_{n=1}^{\infty} \frac{n}{(n^2+s^2)^3} \qquad \ldots (5)$$

$$= \frac{8s \cdot \sin h\,(\pi s)}{(\sin h^2\,(\pi s) - (\pi s)^2)^2} \cdot p(s) \quad .$$

We prove $p(s) > 0$ in I_o. Calculations yield

$$p(.35) > p(.33) \geq 0.0135 \quad .$$

We approximate $p(s)$ in I_o with the liner interpolating polynomial

$$P(s) = 0.0135 + (s - 0.0135) \left(\frac{P(.35) - p(.33)}{.35 - .33}\right)$$

with an accuracy

$$|P(s) - p(s)| < \frac{(.35 - .33)^2}{8} \cdot |p''(\xi)| \quad ,$$

where $\xi \in I_o$. One can show $|p''(s)| < 4$ in I_o, whence

$$|P(s) - p(s)| < 2.10^{-4} .$$

Clearly $P(s) \geq 0.0135$ in I_o , hence

$$p(s) > 0.0133 > 0 \text{ in } I_o .$$

v) For $s \geq 3$, $c'(s)$ is very nearly equal to

$$d(s) = \frac{8s \cdot \sin h^2 (\pi s)}{\sin h^2 (\pi s) - (\pi s)^2} \cdot \sum_{n=1}^{\infty} \frac{n(n^2 - s^2)}{(n^2 + s^2)^3} :$$

this can be seen from (5) using $\sin h (\pi s) \approx \cos h (\pi s)$, and $(\pi s)^3 \approx (\pi s)^2$ in comparison to $\sin h^3 (\pi s)$. The Euler expansion of the series term enables us to find a constant r such that

$$d(s) > r > 0 \text{ and } |c'(s) - d(s)| < r \text{ for } s \geq 3 .$$

REFERENCES

1. Erdélyi, A., 1953: "Higher Transcendental Functions", New York.

2. Helfenstein, H.G., 1955: "Conformal Maps with Least Distortion", Can. J. Math., Vol. 7, pp. 306-313.

3. Hopf, H. and Rinow, W., 193 : "Über den Begriff der vollständigen differentialgeometrischen Fläche", Comment. Math. Helv., Vol. , pp.

4. Nevanlinna, R., 1936: "Eindeutige Analytische Funktionen", first edition, Berlin.

5. Nevanlinna, R., 1953: "Uniformisierung", Berlin.

6. Steffensen, T.F., 1927: "Interpolation", New York.

7. Tolstov, G.P., 1962: "Fourier Series", New Jersey.