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OPTION PRICING FOR INFINITE VARIANCE DATA

By

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Abstract

Infinite variance distributions are among the competing models used to explain the non-normality of stock price changes (Mandelbrot, 1963; Fama, 1965; Mandelbrot and Taylor, 1967; Rachev and Samorodnitsky, 1993). We investigate the asymptotic option price formula in infinite variance setting for both independent and correlated data using point processes. As we shall see the application of point process models can also lead us to investigate a more general option price formula. We also apply a recursion technique to quantify various characteristics of the resulting formulas. It shows that such formulas, and even their approximations, may be difficult to apply in practice. A nonparametric bootstrap method is proposed as one alternative approach and its asymptotic consistency is established under a resampling scheme of $m = o(n)$. Some empirical evidence is provided showing the method works in principle, although large sample sizes appear to be needed for accuracy. This method is also illustrated using publicly available financial data.

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Dedication

This thesis is dedicated to the memory of my mother Simindokt Aliabadi and my brother Behzad Jahandideh.

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Chapter 1

Introduction

1.1 Related Research Works

Modelling stock returns has been a very interesting topic for a long time. One reason is that some important models in financial theory critically rely on the distribution form for the returns of underlying stocks, such as mean-variance portfolio theory, capital asset pricing models, and prices of derivative securities. In the search for satisfactory descriptive models of stock returns, many distributions have been tried and some new distributions have been created over the past several decades. Prior to the work of Mandelbrot (1963) the usual assumption was the distribution of price changes in a speculative series is approximately normal. In the best-known theoretical exposition of the normal hypothesis, Osborne (1959) used arguments based on the Central Limit Theorem to support the assumption of normality. The Osborne model begins by assuming that price changes from transaction to transaction in an individual security are independent, identically distributed random variables. It further assumes that transactions are fairly uniformly spread across time, and that the distribution of price changes from transaction to transaction has finite variance. If the number of transactions per day, week, or month is very large, then price changes across these differencing intervals will be sums of many independent variables. Under these conditions the central-limit theorem leads us to expect that the daily, weekly, and monthly price changes will have normal or Gaussian distribution. Moreover, the variances of the distributions will be proportional to the respective time intervals.

Empirical evidence in support of the Gaussian hypothesis has been offered by Kendall (1953) and Moore (1962). Kendall found that weekly price changes for Chicago wheat and British common stocks were approximately normally distributed, and Moore reported similar results for the weekly change in log price of a sample of stocks from the New York Stock Exchange.

The Gaussian hypothesis was not seriously questioned until recently when the work of Mandelbrot (1963) first began to appear. In his work, Mandelbrot contends that those past researches have overemphasized agreements between empirical distribution of price changes and the normal distribution and have neglected certain departures from normality which are consistently observed. In particular, in most empirical work, Kendall's and Moore's included, it has been found that the extreme tails of empirical distributions are higher (i.e., contain more of the probability) than those of the normal distribution. Mandelbrot feels that these departures from normality are sufficient to warrant a radically new approach to the theory of random walks in speculative prices. This new approach, which shall be called the stable Paretian hypothesis, makes two basic assumptions:

- the variances of the empirical distribution behave as if they were infinite, and
- the empirical distribution conform best to the non-Gaussian members of a family of limiting distributions which Mandelbrot has called stable Paretian.

From the path-breaking studies of Mandelbrot (1963) and Fama (1965) in the 60's, it is now clear that the empirical distribution of many observed economic and financial data deviates from the ideal Gaussian Law, since they often exhibit skewness (asymmetry) and fat tails. Of the many procedures introduced for dealing with non-normality, the α-stable subordinated process discussed in Mandelbrot and Taylor (1967) has received special attention. The Mandelbrot-Taylor model is based on the assumption that the log price differences of a stock is a subordinated process $W(\tau(t))$, where $W(t)$ is a Gaussian process with mean zero, and $\tau(t)$, the directing process, is a positive $(\alpha/2)$-stable process. The value of $W(t)$ is interpreted as the stock price difference on a time scale measured in volume of transactions, and $\tau(t)$ is the cumulative volume, or number of transactions, up to time $t > 0$. The Mandelbrot-Taylor model ensures that $W(\tau(t))$ is an $0 < \alpha < 2$ stable process. This is attractive because it is compatible with empirical evidence of the heavy tailed nature of stock price differences over fixed time intervals, while simultaneously, it embraces the theory that sequences of price changes should exhibit a random walk behavior. See Mandelbrot and Taylor (1967) for further discussion on this point.

Rachev and Samorodnitsky (1993) considered the Mandelbrot-Taylor model in more details, describing a discrete time approximation to this process involving innovations which are assumed to have a symmetric Pareto distribution. Using a "random up-down" variation of the Cox-Ross-Rubenstein approximation method (Cox,Ross and Rubenstein, (1979)), Rachev and Samorodnitsky derived a binomial option pricing formula for stock price differences given by

$$C_n = E\left\{\frac{[S_0 \exp(\sigma(X_{1,n} + \cdots + X_{n,n})) - K]_+}{2^{-n}\prod_{i=1}^{n}(\exp(\sigma X_{i,n}) + \exp(-\sigma X_{i,n}))}\right\} \tag{1.1}$$

where n is the number of time periods, S_0 is the current price of the stock, C_n is the value of the call after n time periods, K is the striking price, σ is a positive real number, and $X_{i,n}$, $i = 1, \ldots, n$, are i.i.d. symmetric Pareto random variables defined by

$$P\{|X_{i,n}| > x\} = n^{-1}x^{-1/\alpha}, \qquad x \geq n^{-1/\alpha}, \; 1 < \alpha < 2.$$

Let $\Gamma_i = \sum_{j=1}^{i} E_j$ where $\{E_j\}$ is a sequence of independent exponential random variables with mean equal to one. A simple application of Proposition 13.15 from Breiman (1992) leads to the following equivalent expression

$$C_n = E\left\{\frac{\left[S_0 \exp\left(\sigma\left(\frac{\Gamma_{n+1}}{n}\right)^{1/\alpha}\sum_{i=1}^{n}\delta_i\Gamma_i^{-1/\alpha}\right) - K\right]_+}{\prod_{i=1}^{n}\cosh\left(\sigma\left(\frac{\Gamma_{n+1}}{n}\right)^{1/\alpha}\Gamma_i^{-1/\alpha}\right)}\right\}, \tag{1.2}$$

where $\{\delta_i\}$ is an i.i.d. sequence of Rademacher random variables, independent of $\{\Gamma_i\}$, i.e.

$$P\{\delta_i = 1\} = \frac{1}{2} = 1 - P\{\delta_i = -1\}.$$

Rachev and Samorodnitsky's Theorem 1 shows that $C_n \to C$ as $n \to \infty$, where

$$C = E\left\{ \frac{\left[S_0 \exp\left(\sigma \sum_{i=1}^{\infty} \delta_i \Gamma_i^{-1/\alpha}\right) - K\right]_+}{\prod_{i=1}^{\infty} \cosh\left(\sigma\, \Gamma_i^{-1/\alpha}\right)} \right\}. \tag{1.3}$$

Equation (1.3) is the limiting option pricing formula when log stock price differences follows a symmetric Pareto distribution. This result was later generalized by LePage, Podgórski and Ryznar (1997) who showed that (1.3) remains valid for properly normalized innovations X_i, that are i.i.d. and are in the domain of attraction of a symmetric α-stable law with index $\alpha \in (0, 2)$. This result greatly increases the scope of applicability of the formula.

1.2 Organization of the Thesis

In this thesis we apply point process theory to investigate asymptotic option pricing formulae for Equation (1.3). Before we present our approach we first review some background theory about α-stable distributions, point processes and option pricing models in Chapters 2, 3 and 4.

In Chapter 5, we show that the value C, defined by Equation (1.3), can be derived by the use of point process methods. This method is quite versatile, and in fact, allows us not only to give a streamlined proof of the LePage, Podórski and Ryznar result, but will also be used to verify the consistency of a nonparametric bootstrap procedure for estimating (1.3).

In Chapter 6, we presents several simulations for option pricing models and we discuss some practical difficulties surrounding computation of those models.

Using a recursion method discussed in Banjevic, Ishwaran and Zarepour (2002) we quantify the behavior of the values of C_n in Chapter 7. For example, we show that $\prod_{i=1}^{\infty} \cosh(\sigma\Gamma_i^{-1/\alpha})$ possesses all infinite positive moments, thus helping to explain the variability that can be seen in $Z_N = \sum_{i=1}^{N} \ln(\cosh(\sigma\Gamma_i^{-1/\alpha}))$.

In Chapter 8, we introduce a nonparametric bootstrap procedure which is an alternative method for calculating C and is of some practical importance, as C appears next to impossible to compute in closed form. Then, we apply our bootstrap method to show it will possess none of the problems discussed in Chapter 7. This is because the bootstrap works directly with the data and the bootstrap method works without requiring a closed form expression for G^{-1} defined in Breiman (1992) Proposition 13.15.

In Chapter 9, we present examples of asset returns to show the significance of their autocorrelation functions. This suggests the investigation of a more general option pricing formula. We show that this can be done by applying a convergence result (cf. Davis and Resnick (1985) Theorem 2.4.) for point processes based on moving averages.

In Chapter 10, we briefly discuss some problems which might be considered in the future works.

Chapter 2

Preliminaries

In this chapter, we recall some concepts from probability theory, probability measures and financial derivatives. We assume familiarity with a working knowledge of measure and probability theory at the level of Halmos (1950), and of real analysis at the level of Royden (1968).

2.1 Probability space

Let Ω be a nonempty set. Let $\mathcal{A}$ be a σ-field of subsets of Ω, that is, a nonempty class of subsets of Ω which contains Ω and is closed under countable union and taking complement.

Let $\mathbf{P}$ be a measure defined on $\mathcal{A}$ satisfying $\mathbf{P}(\Omega) = 1$. Then the triple $(\Omega, \mathcal{A}, \mathbf{P})$ is called a *probability space*, and $\mathbf{P}$, a *probability measure.* The set Ω is the *sure event*, and elements of $\mathcal{A}$ are called *events.* Singleton sets $\{\omega\}$ are called *elementary events.* The symbol $\emptyset$ denotes the empty set and is known as the *null* or *impossible* event

2.2 Convergence concepts

Let $(\Omega, \mathcal{A}, \mathbf{P})$ be a probability space, and let $\{X_n, n \geq 1\}$ be a sequence of random variables defined on it.

Almost sure convergence

The sequence of random variables $\{X_n\}$ is said to converge almost surely to the random variable X if and only if there exists a set $E \in \mathcal{A}$ with $\mathbf{P}(E) = 0$, such that, for every $\omega \in E^c$,

$$|X_n(\omega) - X(\omega)| \to 0 \quad \text{as} \quad n \to \infty .$$

In that case we write $X_n \overset{a.s.}{\to} X$.

Convergence in probability

A somewhat weaker concept of convergence is that of convergence in probability. The sequence of random variables $\{X_n\}$ is said to converge in probabilty to the random variable X if for every $\epsilon > 0$

$$\mathbf{P}\{\omega \in \Omega ; |X_n(\omega) - X(\omega)| \geq \epsilon\} \to 0 \quad \text{as} \quad n \to \infty .$$

In this case we write $X_n \overset{P}{\to} X$. It is known that

$$X_n \overset{a.s.}{\to} X \quad \text{implies} \quad X_n \overset{P}{\to} X ,$$

but the converse is not true in general (e.g. see, Laha and Rohatgi (1979) page 47).

Weak convergence of distribution functions

Let $\{F_n\}$ be a sequence of uniformly bounded, non-decreasing, right-continuous functions defined on $\Re$. We say that F_n converges weakly to a bounded, non-decreasing, right-continuous function F on $\Re$ if

$$F_n(x) \to F(x) \quad \text{as} \quad n \to \infty$$

at all continuity points x of F. In this case we write $F_n \overset{w}{\to} F$.

Weak convergence of probability measures

Although the concept of weak convergence of distribution functions is tied to the real line (or to Euclidean space, at any rate), the concept of weak convergence of probability measures can be formulated for the general metric space. Suppose Ω is a metric space and suppose $\mathcal{A}$ is the class of all Borel sets in Ω. Let $\{P_n\}$ be a sequence of probability measures defined on $\mathcal{A}$ and let P be another probability measure defined on $\mathcal{A}$. The sequence $\{P_n\}$ is said to converge weakly to P if and only if

$$\int_\Omega f dP_n \to \int_\Omega f dP \quad \forall f \in C(\Omega),$$

where $C(\Omega)$ denotes the class of all bounded, continuous real valued functions defined on Ω. In this case we write $P_n \Rightarrow P$. Note that

$$X_n \xrightarrow{d} X$$

if the distributions P_n of the X_n converge weakly to the distribution P of X. Recall that by the distribution Q of a random variable Y we mean the probability measure Q defined by

$$Q(A) = \mathbf{P}(Y^{-1}A) = \mathbf{P}\{\omega : Y(\omega) \in A\} = \mathbf{P}\{Y \in A\}, \quad A \in \mathcal{A}.$$

Continuous Mapping Theorem

Billingsley (1968) *Let Ω_1 and Ω_2 be two metric spaces and suppose X_n, $n \geq 0$ are random elements of Ω_1 such that*

$$X_n \xrightarrow{d} X.$$

If a function $h : \Omega_1 \to \Omega_2$ is measurable and satisfies

$$\mathbf{P}(D_h) = 0,$$

where D_h is the set of discontinuities of h, then

$$h(X_n) \xrightarrow{d} h(X).$$

2.3 Weak convergence in $C[0,1]$ and $D[0,1]$

Assume $C[0,1]$ is the space of all continuous real-valued functions on $[0,1]$. On this space define the uniform metric as follows:

$$d(x,y) = ||x-y|| = \sup\{|x(t)-y(t)| : t \in [0,1]\}.$$

On $C[0,1]$, the uniform metric is the most common metric in statistical context. Unfortunately, most of the empirical stochastic processes do not live in $C[0,1]$; instead they belong to $D[0,1]$, the space of right continuous functions with finite left limits. In some cases, weak convergence of some stochastic processes will not hold with respect to uniform metric. For these cases Skorohod introduced several other metrics of which the following is the most common:

$$d(x,y) = \inf\{||x - y \circ \lambda|| \vee ||y - I|| \, : \, \lambda \in \Lambda[0,1]\},$$

where $I(x) = x$ and $\Lambda[0,1]$ is the space of all strictly increasing mappings of $[0,1]$ onto itself, (thus $\lambda(0) = 0$ and $\lambda(1) = 1$). It can be shown that $(D[0,1], d)$ is a separable metric space, but it is not complete. To resolve this problem, Skorohod introduced an equivalent metric (i.e. topologies are the same) which makes this space complete and separable. Since the topologies are the same, if $X_n \xrightarrow{w} X$ with respect to one metric then this convergence will hold with respect to the other metric as well. Thus for the Skorohod topology we can use either of these metrics. For detailed account of this subject see Bilingsley (1968).

2.4 Bootstrap Methods

In 1979, Efron revolutionized the field of statistics with his invention of the bootstrap, which he introduced to the world with his paper in the Annals of Statistics. The bootstrap broadly refers to a continually growing collection of methodology in which data are resampled in order to incorporate, into statistical inference, the information contained in the data regarding its probability distribution. Conceptually simple yet computationally intense, the bootstrap owes much of its rise in popularity over the last twenty years to the advent of the personal computer over the same period of time. As computers become faster and more powerful, the bootstrap becomes a more practical and indispensable tool for the data analyst.

The bootstrap idea

The original sample represents the population from which it was drawn. So resamples from this sample represent what we would get if we took many samples from the population. The bootstrap distribution of a statistic, based on many resamples, represents the sampling distribution of the statistic, based on many samples.

Procedure for bootstraping

The bootstrap is first of all a way of finding the sampling distribution, at least approximately, from just one sample. Here is the procedure:

- **Resample.** Create hundreds of new samples, called *bootstrap samples* or *resamples,* by sampling with replacement from the original random sample. Each resample is the same size as the original random sample.
- **Calculate the bootstrap distribution.** Calculate the statistic for each resample.
- **Use the bootstrap distribution.** The bootstrap distribution gives information about the shape, center, and spread of the sampling distribution of the statistic.

To be more specific, let $X_1, X_2, \ldots, X_n$ be independent identically distributed random variables with a common distribution F. Let $\hat{F}_n$ be the empirical distribution based on the sample $X_1, X_2, \ldots, X_n$, that is

$$\hat{F}_n(x) = \frac{1}{n} \sum_{i=1}^{n} I(X_i \leq x) .$$

To bootstrap a statistic (for example, the sample mean), we draw an i.i.d sample, say $X_1^*, X_2^*, \ldots, X_n^*$, from $\hat{F}_n$ and calculate the statistic for that sample. We repeat this resampling a number of times and save the resample statistic into a vector. We then inspect the bootstrap distribution of the resampled statistic.

Note that it might seem that the bootstrap creates data out of nothing. This seems suspicious. But we are not using the resampled observations as if they were real data. We emphasis that the bootstrap is not a substitute for gathering more data to improve accuracy. Instead, the bootstrap idea is to use the resample means to estimate how the sample mean of a sample of some size from this population varies because of random sampling.

Chapter 3

Stochastic Processes Applied to Finance

The use of probability in the financial world is not a surprise to anybody nowadays. What is probably more important to realize is that the use of continuous-time models (e.g., stochastic differential equations) necessitated from the globalization of the economy, i.e., from the fact that markets become larger in scope and size.

Brownian motion is a fundamental stochastic process, and is, in a sense, the identity in the spaces of processes. Being so fundamental it pops up everywhere, much as a normal random variable is used often when the only information about a distribution is merely the mean and the variance.

In this chapter, we start by explaining Brownian motion. However, it has been recognized that Brownian motion is often inadequate as a model, largely because it fails to approximate real phenomena, phenomena observed in practice. Although Brownian motion can capture large fluctuations, it can not capture large jumps because it is continuous. Fortunately, not all processes sharing some similarities with Brownian motion are continuous. In fact, if we maintain the fundamental property of independence and stationary of increments we obtain the so-called stable (Lévy) process, named after Paul Lévy who understood a lot about these processes even before modern tools were available. So, we will introduce a brief study of stable processes. In fact, several of stochastic calculus tools carry over to processes and financial problems can be studied using stable processes and related models. Finally, since in this thesis we use a point process approach to derive the asymptotic formula for speculative prices, we will also introduce a brief review of point processes.

3.1 Wiener Processes

Definition 3.1. Wiener-process *A Wiener process (often called Brownian motion) on the interval* $[0,T]$ *is a random variable* $W(t)$ *that depends continuously on* $t \in [0,T]$ *and satisfies the following:*

- $W(0) = 0$.
- *For* $0 \leq s < t \leq T$
$$W(t) - W(s) \overset{d}{=} \sqrt{t-s}N(0,1)\,.$$
where $N(0,1)$ *is a normal distribution with zero mean and unit variance. Because the normal distribution is used, the process is often referred to as Gaussian.*
- *For* $0 \leq s < t < u < v \leq T$*, the random variables* $W(t) - W(s)$ *and* $W(v) - W(u)$ *are independent.*

A Wiener process is, in fact, a particular type of Markov process with a mean change of zero and a variance rate per unit time.

Now consider the process

$$X(t) = \mu t + \sigma W(t)\,, \quad t \geq 0\,,$$

for constants $\sigma > 0$ and a real number μ. It is clear that $X(t)$ is a Gaussian process with expectation and covariance functions

$$\mu_X(t) = \mu t \quad \text{and} \quad c_X(t,s) = \sigma^2 \min(t,s), \quad s,t \geq 0\,.$$

The expectation function $\mu_X(t) = \mu t$ essentially determines the characteristic shape of the sample paths. Therefore, X is called a *Brownian motion with linear drift* or a generalized Wiener process.

3.2 Stable Processes

A random variable X is said to be *stable* if for X_1 and X_2 independent copies of X and any positive constants a and b,

$$aX_1 + bX_2 \stackrel{d}{=} cX + d\,, \tag{1}$$

for some positive real number c and some real number d. The random variable X is said to be *strictly stable* if (1) holds with $d = 0$ for all choices of a and b. A random variable X is *symmetric stable* if it is stable and symmetrically distributed around 0, i.e. $X \stackrel{d}{=} -X$.

There are three cases where one can write down a closed form expressions for the density and verify directly that they are stable.

- Normal distributions. $X \sim N(\mu, \sigma^2)$ if it has a density

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)\,, \quad -\infty < x < +\infty\,.$$

- Cauchy distributions. $X \sim Cauchy(\gamma, \delta)$ if it has density

$$f(x) = \frac{1}{\pi} \frac{\gamma}{\gamma^2 + (x-\delta)^2}\,, \quad -\infty < x < +\infty\,.$$

- Lévy distributions. $X \sim \text{Lévy}(\gamma, \delta)$ if it has density

$$f(x) = \sqrt{\frac{\gamma}{2\pi}} \frac{1}{(x-\delta)^{3/2}} \exp\left(-\frac{\gamma}{2(x-\delta)}\right)\,, \delta < x < +\infty\,.$$

Other than the normal distribution, the Cauchy distribution, the Lévy distribution, and the reflection of the Lévy distribution, there are no known closed form expressions for general stable densities and it is unlikely that any other stable distributions have closed forms for their densities. However there are computer programs to compute quantities of interest for stable distributions which means it is possible to use them in practical problems. The following is another equivalent definition of stable random variables whose proof can be found in Samorodnitsky and Taqqu (1992).

A random variable X is stable (α-stable) if and only if there exist a random variable Z with characteristic function

$$E\exp(iuZ) = \begin{cases} \exp(-|u|^\alpha[1 - i\beta \tan \frac{\pi\alpha}{2}(sign\, u)]) & \alpha \neq 1 \\ \exp(-|u|[1 + i\beta \frac{2}{\pi}(sign\, u \ln |u|)]) & \alpha = 1 \end{cases},$$

where $0 < \alpha \leq 2$ and $-1 \leq \beta \leq 1$, and two real numbers a and b, with $a > 0$, such that

$$X \stackrel{d}{=} aZ + b\,.$$

These distributions are symmetric around zero when $\beta = 0$ and $b = 0$, in which case the characteristic function of aZ has the simpler form

$$\phi(u) = e^{-a^\alpha |u|^\alpha}\,.$$

It is known that a $N(\mu, \sigma^2)$ distribution is stable with ($\alpha = 2, \beta = 0, a = \sigma^2/2, b = \mu$), a Cauchy($\gamma, \delta$) distribution is stable with ($\alpha = 1, \beta = 0, a = \gamma, b = \delta$) and a Lévy($\gamma, \delta$) distribution is stable with ($\alpha = 1/2, \beta = 1, a = \gamma, b = \delta$).

The above Definition shows that a general stable distribution requires four parameters to describe:

- An *index of stability* or *characteristic exponent* $\alpha \in (0, 2]$,
- A *skewness parameter* $\beta \in [-1, 1]$,
- A *scale parameter* $\sigma > 0$,
- A *shift parameter* $\mu \in \Re$.

In most of the recent literature,the notation $S_\alpha(\sigma, \beta, \mu)$, or simply S_α, is used for the class of stable laws. We say that the distribution function F of a random variable X is *stable Paretian* if X is α-stable.

In analogy to the one-dimensional case, an $\Re^d$-valued random vector X is said to follow a *multivariate stable distribution* if for any positive real numbers a and b there exist a positive real number c and a vector $D \in \Re^d$ such that

$$aX_1 + bX_2 \stackrel{d}{=} cX + D$$

where X_1 and X_2 are independent copies of X.

Definition 3.2. *Let $X(t)$ be a random variable dependent on time t. Then the stochastic process $X(t)$, for $0 < t < \infty$, is a* stable process *if the finite-dimensional distribution of $X(t)$ is stable. Recall that the finite-dimensional distribution of a stochastic process X are the distributions of the finite-dimensional vectors*

$$(X(t_1), \ldots, X(t_n)),$$

where $t_1 < \ldots < t_n < \infty$.

Domain of attraction of stable distributions

The classic Central Limit Theorem states that if $X_1, \cdots, X_n$ are i.i.d. random variables with $E(X_1) = 0$ and $E(X_1^2) = 1$, then

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i \xrightarrow{d} Z,$$

where Z has a standard normal distribution and "$\xrightarrow{d}$" means "convergence in distribution".

It turns out that a similar phenomenon occurs for certain classes of random variables with $E(X^2) = \infty$. In particular, under certain conditions, there exists constants $a_n > 0$ and b_n such that

$$a_n^{-1} \sum_{i=1}^{n} (X_i - b_n) \xrightarrow{d} Y_\alpha,$$

where Y_α is a stable random variable with index $\alpha \in (0, 2]$.

Definition 3.3. *The distribution function F of a random variable X is in the domain of attraction of the α-stable distribution S_α, if for any sequence $X_1, X_2, \ldots$ of i.i.d. random variables with common distribution function F there are sequences of constants $a_n > 0$ and $b_n \in \Re$ such that*

$$Z_n := a_n^{-1} \sum_{i=1}^{n} (X_i - b_n) \xrightarrow{d} Y_\alpha,$$

where Y_α is an α- stable random variable with $\alpha \in (0, 2]$.

It follows from the definition of a stable random variable that S_α belongs to its own domain of attraction. Before giving conditions for a stable random variable to be in a domain of attraction of a stable law, we present the concept of a slowly varying function.

Definition 3.4. *A positive function $L(x)$ is said to be slowly varying at infinity if for all $t > 0$,*

$$\lim_{x \to \infty} \frac{L(tx)}{L(x)} = 1 .$$

In the sequel we assume the term "slowly varying" means "slowly varying at infinity". A slowly varying function has the basic property that it grows slower than any power of x and decays slower than any power of x as x gets large; more precisely, for any $\epsilon > 0$,

$$\lim_{x \to \infty} x^{-\epsilon} L(x) \; = \; 0.,$$

and

$$\lim_{x \to \infty} x^{\epsilon} L(x) \; = \; \infty.$$

A constant c, $\ln x$, and $(\ln x)^a$, $a > 0$, are all examples of a slowly varying function. We can now state necessary and sufficient conditions for a random variable to be in a domain of attraction for $\alpha \in (0, 2)$.

Proposition 3.1. *The random variable X is in the domain of attraction of a stable law with index $\alpha \in (0, 2)$ if and only if*

(a) $P[|X| > x] = x^{-\alpha} L(x)$ *for some slowly varying function L and,*

(b) $\lim_{x \to \infty} \frac{P[X > x]}{P[|X| > x]} \; = \; p$

where p is a real number with $0 \leq p \leq 1$.

Condition (a) says that the tails of the distribution function of X are going to zero smoothly while condition (b) insures that the left and right tails are balanced in some way.

3.3 Point Processes

A point process is a random collection of points, where each point represents the time and location of an event. Examples of events include incidents of disease, sightings or births of a species, or the occurrences of fires, earthquakes, lightning strikes or volcanic eruptions. Typically the spatial locations are recorded in d-spatial coordinates, though sometimes only one or two spatial coordinates are available or of interest. A point process ξ is mathematically defined as follows:

Let E be some subset of $\Re^d$, and let $\mathcal{E}$ be the σ-algebra generated by open subsets of E. Suppose that $A \in \mathcal{E}$ and $\{X_n, n \geq 0\}$ are random elements of E which represent random points in the state space E. If we define the discrete measure ε_{X_n} by

$$\varepsilon_{X_n}(A) = \begin{cases} 1 & \text{if} \quad X_n \in A, \\ 0 & \text{if} \quad X_n \notin A, \end{cases}$$

then, by summing over n, we get the total number of random points X_n which fall in A. A *point measure* m is defined to be a measure of the form $m = \sum_n \varepsilon_{X_n}$ which is non-negative integer-valued and finite on relatively compact subsets of E. The class of such point measures is denoted by $M_P(E)$ and $\mathcal{M}_P(E)$ is the smallest σ-algebra making the evaluation maps $m \to m(A)$ measurable where $m \in M_P(E)$ and $A \in \mathcal{E}$. A *point process* on E is a measurable map ξ from a probability space $(\Omega, \mathcal{A}, P)$ to $(M_P(E), \mathcal{M}_P(E))$.

Vague Convergence

Let $C_K^+(E)$ be the set of all continuous functions $E \to \Re_+$ with compact support. A useful topology for $M_P(E)$ is the vague topology which renders $M_p(E)$ a complete separable metric space (cf. Resnick (1987)). If $\mu_n \in M_p(E)$ is a sequence of point measures, then μ_n converges vaguely to μ_0, (written $\mu_n \xrightarrow{v} \mu_0$) if

$$\mu_n(f) \longrightarrow \mu_0(f) \qquad \text{for all } f \in C_K^+(E),$$

where by $\mu(f)$ we mean

$$\mu(f) = \int f \, d\mu \, .$$

If Q is a probability measure on $(M_p(E), \mathcal{M}_p(E))$ the Laplace functional of Q is

$$\Psi(f) = \int_{M_p(E)} \exp\left\{-\int_E f(x)\mu(dx)\right\} Q(d\mu)$$

where $f : E \longrightarrow R_+$ is bounded. Alternatively if ξ is a point process the Laplace transform of ξ is

$$\Psi_E(f) = E\exp\{-\xi(f)\} = E\exp\left\{-\int_E f(x)\xi(dx, w)\right\} .$$

An important result for us is that given a sequence of point processes ξ_n, $n \geq 0$, and a point process ξ_0,

$$\xi_n \Rightarrow \xi_0 \quad \text{iff} \quad \Psi_{\xi_n}(f) \rightarrow \Psi_{\xi_0}(f), \;\; \forall \; f \in C_K^+(E) \, ,$$

where by "$\Rightarrow$" we mean "converges weakly" [Resnick (1987)].
The following Proposition is one of the basic convergence results that will be used later [Resnick (1987)].

Proposition 3.2. *For each n suppose $(X_{n,j})_{j\geq 0}$ are i.i.d. random elements of $(E, \mathcal{E})$ and μ is a Radon measure on $(E, \mathcal{E})$. Define*

$$\xi_n = \sum_{j=1}^{\infty} \varepsilon_{(jn^{-1}, X_{n,j})}$$

and suppose ξ is a Poisson random measure on $(0, \infty) \times E$ with mean measure $dt \times d\mu$. Then $\xi_n \Rightarrow \xi$ in $M_p((0, \infty) \times E)$ if and only if

$$nP[X_{n,1} \in \cdot] \xrightarrow{v} \mu \, .$$

Under regularly varying tail for distributions, several functionals from $M_P(E)$ into $D[0, \infty)$ are studied in Resnick (1986) Section 3 to investigate problems related to extremal processes. One of these functionals , which is defined in the next proposition, has a crucial rule in deriving our option pricing formulas.

Proposition 3.3 *Suppose $X_{k,n} = a_n^{-1} X_k$, where X_k's are in the domain of attraction of a stable law with index $\alpha \in (0, 2)$. Define $T_\delta : M_P(\Re^0) \longrightarrow \Re$ by*

$$T_\delta(\xi_n) = \sum_k X_k I\{|X_k| > \delta\},$$

where $\xi_n = \sum_k \varepsilon_{X_k}$. Then T_δ is almost surely continuous for each fixed $\delta > 0$.

The most tractable and commonly used point process model is the Poisson process and we discuss this in the next part.

The Poisson Process

A Poisson process on $(E, \mathcal{E})$ with mean measure μ is a point process ξ satisfying:

(1) For $A \in \mathcal{E}$

$$P[\xi(A) = k] = \begin{cases} \frac{\exp(-\mu(A))(\mu(A))^k}{k!} & \text{if } \mu(A) < \infty \\ 0 & \text{if } \mu(A) = \infty \end{cases}$$

(2) If $A_1, \ldots, A_n$ are mutually disjoint, then $\xi(A_1), \ldots, \xi(A_n)$ are independent random variables.

When the mean measure μ is a multiple of Lebesgue measure, we call the process homogeneous. Thus, in the homogeneous case, there is a parameter $\alpha > 0$ such that for any A we have $\xi(A)$ Poisson distributed with mean $E\xi(A) = \alpha|A|$, where $|A|$ is the Lebesgue measure of A. When $E = [0, \infty)$ the parameter α is called the *rate* of the (homogeneous) Poisson process.

The Poisson process ξ can be identified by the characteristic form of its Laplace functional:

$$\Psi_\xi(f) = E\exp\{-\xi(f)\} = \exp\left\{-\int(1 - e^{-f})d\mu\right\}.$$

We may use the abbreviation PRM for a Poisson random measure ξ with mean measure μ on $(E, \mathcal{E})$.

Transforming Poisson Processes

Some very useful results arise by considering what happens to a Poisson process under various types of transformations. Suppose T is some transformation with domains E and range E', where E' is another subset of Euclidian space:

$$T : E \longrightarrow E'.$$

Given the measures ξ and μ defined on subsets of E, we may use T to define induced measures ξ' and μ' on subsets of E' as follows: For $A' \subset E'$ define

$$\xi'(A') = \xi(T^{-1}(A')),$$

$$\mu'(A') = \mu(T^{-1}(A')).$$

If ξ has points $\{X_n\}$, then ξ' has points $\{X_n'\} = \{T(X_n)\}$, since for $A' \subset E'$

$$\xi'(A') = \xi(T^{-1}(A')) = \sum_n \varepsilon_{X_n}(T^{-1}(A'))$$

$$= \sum_n I[X_n \in T^{-1}(A')] = \sum_n I[T(X_n) \in A'] = \sum_n \varepsilon_{T(X_n)}(A').$$

The above fact together with the following proposition say that if you shift the points of a Poisson process around you still have a Poisson process.

Proposition 3.4. *Suppose*

$$T : E \longrightarrow E'$$

is a mapping of one Euclidean space, E, into another, E' with the property that if $B' \subset E'$ is bounded in E', then $T^{-1}(B')$ is bounded in E. If ξ is a Poisson random measure with mean measure μ on E and with points $\{X_n\}$, then $\xi' := \xi \circ T^{-1}$ is a Poisson random measure with $\mu' := \mu \circ T^{-1}$ on E' and with points $\{T(X_n)\}$.

Example 3.1. Let $\xi = \sum_{n=1}^{\infty} \varepsilon_{\Gamma_n}$, where $\Gamma_n = \sum_{i=1}^{n} E_i$ and $\{E_i\}_{i=1}^{n}$ is a finite sequence of i.i.d whose distribution F is exponential with unit rate. It is known that ξ is a homogeneous Poisson process on $[0, \infty)$ of unit rate. The mean measure μ is

Lebesgue measure and, in particular, $\mu([0,t]) = t$. If $T(x) = x^{-1/\alpha}$, $0 < \alpha < 2$, then $\sum_n \varepsilon_{\Gamma_n^{-1/\alpha}}$ ia a PRM, and the mean measure μ' is given by

$$\mu'((x,\infty]) = \mu\{s : x < Ts \leq \infty\} = \mu[0, x^{-\alpha}) = x^{-\alpha}.$$

Note that μ' ia absolutely continuous with respect to the Lebesgue measure and its Radon-Nykodym derivative is

$$f(x) = \alpha\, x^{-\alpha-1}.$$

Example 3.2. Let ξ be the same random measure as the one defined in Example 1 and let

$$\xi' = \sum_n \varepsilon_{\delta_n \Gamma_n^{-1/\alpha}},$$

where, for each n, $\delta_n = 1$ with probability p or $\delta_n = -1$ with probability $q = 1 - p$. Thus

$$\mu'((x,\infty]) = px^{-\alpha} I(X > 0) + q(-x)^{-\alpha} I(x < 0),$$

and hence μ' has a Radon-Nykodym derivative of the form

$$f(x) = \alpha \left(p\, x^{-\alpha-1} I(x > 0) + q(-x)^{-\alpha-1} I(x < 0)\right).$$

Example 3.3. For each n let $\{X_{j,n} : j \geq 1\}$ be a sequence of i.i.d. random variables with the p.d.f.

$$f_n(x;\alpha) = \frac{x^{\alpha/n-1}\exp(-x)}{\Gamma(\alpha/n)} I\{x > 0\}.$$

Since

$$n\, P(X_{j,n} \in \cdot\,) = n \frac{x^{\alpha/n-1}\exp(-x)}{\Gamma(\alpha/n)}\, dx \xrightarrow{v} x^{-1}\exp(-x)dx,$$

from Proposition 3.2 we have

$$\xi_n := \sum_{j=1}^{\infty} \varepsilon_{(j\,n^{-1}, X_{j,n})}(\cdot) \xrightarrow{d} \xi := \sum_{j=1}^{\infty} \varepsilon_{(U_j, N^{-1}(\Gamma_j))}(\cdot),$$

where

$$N(x,\infty) = \alpha \int_x^\infty \frac{\exp(-u)}{u}\, du\,.$$

The following proposition (see Resnick (1987) Proposition 3.18) considers a special case of transforming point processes. It plays an important role in deriving our main results in the sequel.

Proposition 3.5. *Suppose that E, E' are two spaces which are locally compact with countable bases. Suppose $T : E \longrightarrow E'$ is continuous and satisfies $T^{-1}(K')$ is compact in E for every compact K' in E'.*

Then $\hat{T} : M_+(E) \longrightarrow M_+(E')$ defined by

$$\hat{T}(\mu) = \mu \circ T^{-1}$$

is continuous. Note $\hat{T}$ restricted to $M_p(E)$ is of the form

$$\hat{T}\left(\sum \varepsilon_{X_i}\right) = \sum \varepsilon_{TX_i}\,.$$

3.4 Lévy Processes

A *Lévy process* (process with stationary, independent increments) is an independent sum of a Wiener process and a centered sum of Poisson jumps. The Lévy process $\{X(t) : t \geq 0\}$ in $\Re$ has the characteristic function

$$E\exp\{isX(t)\} = \exp\left\{t\left[ias - \frac{cs^2}{2} + \int_{|x|>1} (\exp(isx) - 1)\nu(dx) + \int_{0<|x|\leq 1} (\exp(isx) - 1 - isx)\nu(dx)\right]\right\}$$

where $s \in \Re$, $a \in \Re$, $c > 0$ and ν (called the Lévy measure) is a σ-finite measure on $\Re^0 := (-\infty, o) \cup (0,\infty)$ with the property that

$$\int (x^2 \wedge 1)\nu(dx) < \infty$$

(cf. Gnedenko and Kolomogorov (1968), p. 84). The structure of $X(\cdot)$ is as follows:

Let $\xi = \sum \varepsilon_{(t_k, j_k)}$ be a PRM on $\Re_+ \times \Re^0$ with mean measure $dt \times dv$. Suppose (as before) that $W(\cdot)$ is standard Brownian motion independent of ξ. The Itô representation of $X(\cdot)$ (cf. Itô (1969), p. 1.7.7) is

$$X(t) = at + W(ct) + \sum_{t_k \le t} j_k 1_{[|j_k|>1]} + \lim_{\delta \downarrow 0} \left[\sum_{t_k \le t} j_k 1_{[|j_k| \in (\delta, 1]]} - t \int_{|s| \in (\delta, 1]} s\nu(ds) \right]$$

where for almost all ω, the convergence is uniform on compact t-sets.

As an special case, consider $j_k = \delta_k \Gamma_k^{-1/\alpha}$ in which $\{\delta_i\}$ is an i.i.d sequence of Rademacher random variables, independent of $\{\Gamma_i\}$, with

$$P(\delta_i = 1) = p = 1 - P(\delta_i = -1)$$

and let $c = 0$. Then we can see that this, subclass of the Lévy processes, is in fact the class of stable processes for which the Lévy measure is of the form

$$\nu(x, \infty) = Kpx^{-\alpha} \quad \text{and} \quad \nu(-\infty, -x) = Kqx^{-\alpha},$$

where $x > 0$, $K > 0$, $0 \le p, q \le 1$, $p + q = 1$, and $0 < \alpha < 2$.

Consider the simple but important case where $X(\cdot)$ is a non-decreasing Lévy process in R_+ without drift and $X(0) = 0$. A standard fact is that a Lévy process can be non-decreasing iff the Wiener component in its Itô representation is absent and the Lévy measure ν satisfies

$$\int_0^\infty (x \wedge 1)\nu(dx) < \infty.$$

In this case the centering in its Itô representation converge, so we may remove them, and thus it is convenient to give the Itô representation as

$$X(t) = \sum_{t_k \le t} j_k.$$

This simple case of Lévy process is called a Lévy motion which can also be described as follows:

Definition 3.5. Lévy motion *The counterpart of Brownian motion in the stable setting can be defined as a process $X(t)$ such that*

- $X(0) = 0$ *almost surely;*
- *The increment of* $X(t)$ *are independent;*
- *The increments* $X(t) - X(s)$ *are distributed like* $S_\alpha(|t-s|^{1/\alpha}, \beta, 0)$.

3.5 Subordinated Processes

The subordination approach goes back to Bochner (1955), see also Feller (1966). The method of stochastic subordination, or random time indexing, has been recently applied to Wiener process price processes to model financial returns. As an example we briefly describe Mandelbrot and Taylor (1967) model in this section.

The crucial feature of the Mandelbrot-Taylor model is that the return process $(W(t))_{t\geq 0}$, is measured in relation to the transaction volume and not physical or calendar time. Let $\{W(t) : T \geq 0\}$ be a Brownian motion with zero drift and variance v^2, which is viewed as the process of stock log prices on the time scaled measured in volume of transactions. The cumulative volume $(T(t))_{t\geq 0}$, i.e., the number of transactions up to calendar time t is assumed to follow a positive $\frac{\alpha}{2}$-stable process with characteristic function

$$E\exp(i\theta T(t)) = \exp\left\{-\nu t|\theta|^{\frac{\alpha}{2}}(1 - i(\theta/|\theta|)\tan(\pi\alpha/4))\right\} \qquad 0 < \alpha < 2\,, \quad \nu > 0\,.$$

The subordinated process $Z(t) = W(T(t))$ representing the return process with respect to calendar time is then an α-stable Lévy motion with characteristic function

$$E\exp(i\theta Z(t)) = \exp(-t|\sigma\theta|^{\alpha})\,,$$

where

$$\sigma^{\alpha} = \nu(v^2/2)^{\alpha/2}/\cos(\frac{\pi\alpha}{4})$$

(cf. Clark (1973), Samuelson (1955) and Osborne (1959)). As returns are defined as the consecutive differences of the logarithms of the prices, $S(t) = \exp\{Z(t)\}$ represents the price process in the Mandelbrot-Taylor model.

A discrete version of the Mandelbrot-Taylor model is considered in Rachev and Samorodnitsky (1993). There, they derived a model for option pricing with heavy-tailed distributed returns. In chapter 5, we discuss Rachev and Samorodnitsky model in more details by using a point process approach and derive an asymptotic formula for speculative prices. There are other alternative option pricing formulae based on the Mandelbrot-Taylor price process such as, *generalized binomial option pricing model*, *generalized Mandelbrot-Taylor option pricing model*, and *subordinated asset pricing model for option valuation*. For a detailed description of these models see Rachev and Mittnik (2000).

Chapter 4

Option Pricing Under Alternative Stable Models

4.1 The option pricing problem

During the last 25 years options have become an important financial instrument. For example, the Chicago Board of Option Exchange (CBOE) was opened on April 26,1973; and 911 option contracts were made on that day. One year later about 20,000 contracts were made per day; and in 1987 the average number of contracts on the 550 stocks and various indexes reached 770,000 per day (cf Hull (1997), Willmott, Dewynne and Howisson (1993) and the review of Shiryaev (1994) and the references therein).

An Option represents the right (but not the obligation) to buy or sell a security or other asset under specified terms. An Option to buy is known as a "*Call*" , and an Option to sell is called a "*Put*". You can purchase Options (the right to buy or sell the security in question) or sell (write) Options. As a seller, you would become obligated to sell a security to, or buy a security from, the party that purchased the Option. Both types of options will have an *exercise price* and an *exercise time*.

There are two standard conditions under which options operate: *European* options can be utilized only at the exercise time, whereas *American* options can be utilized at any time up to the exercise time.

Options trade on many different exchanges throughout the world. The underlying assets include stocks, foreign currencies, stock indices, and many different futures contracts.

Factors affecting option prices

There are six factors affecting the price of a stock option:

- The current stock price.
- The strike price.
- The time of expiration.
- The volatility of the stock price.
- The risk-free interest rate.
- The dividends expected during the life of the option.

Stock price and strike price

If it is exercised at some time in the future, the payoff from a call option is the amount by which the stock price exceeds the strike price. Call options, therefore, become more valuable as the stock price increase and less valuable as the strike price increase. For a put option, the payoff on exercise is the amount by which the strike price exceeds the stock price. Put options, therefore, behave in the opposite way to call option. They become less valuable as the stock price increases and more valuable as the strike price increases.

Time to expiration

Consider next the effect of the expiration date. Both put and call American options become more valuable as the time to expiration increases. For example, consider two options that differ only with respect to expiration date. The owner of the long-life option has all the exercise opportunities open to the owner of the short-life option and more. The long-life option must, therefore, always be worth at least as much as the short-life option.

European put and call options do not necessarily become more valuable as the time to expiration increases. This is because the owner of a long-life European option do not have all the exercise opportunities open to the owner of a short-life European option. The owner of the long-life European option can exercise only at the maturity of that option.

Volatility

The volatility of a stock price, σ, is defined so that $\sigma\sqrt{\Delta t}$ is the standard deviation of the return on the stock in a short length of time, Δt. It is a measure of how much uncertain we are about future stock price movements. As volatility increases, the chance that the stock will do very well or very poorly increases.

Risk-free interest rate

The risk-free interest rate affects the price of an option in a less clear-cut way. As interest rates in the economy increase, the expected growth rate of the stock price tends to increase. However, the present value of any future cash flows received by the holder of the option decreases. These two effects tend to decrease the value of a put option. Hence, put option prices decline as the risk-free interest rate increases. In the case of calls, the first effect tends to increase the price and the second effect tends to decrease it. It can be shown that the first effect always dominates the second effect: that is, the price of a call always increases as the risk-free interest rate increases.

Assumptions

We now move on to derive some relationship between option prices that do not require any assumption about volatility and the probabilistic behavior of stock prices. We assume that there are some market participants, such as large investment banks, for which

- There are no transactions costs.

- All trading profits are subject to the same tax rate.
- Borrowing and lending at the risk-free interest rate is possible.

We assume that these market participants are prepared to take advantage of arbitrage opportunities as they arise. This means that any available arbitrage opportunities disappear very quickly. For the purpose of our analysis, it is, therefore, reasonable to assume that there are no arbitrage opportunities.

Upper bounds for option prices

An American or European call option gives the holder the right to buy one share of a stock for a certain price. No matter what happens, the option can never be worth more than the stock. Hence, the stock price is an upper bound to the option price:

$$C \leq S_0 \,\text{and}\, c \leq S_0 ,$$

where S_0 is the current stock price, c is the value of American call option to buy one share, and C is the value of European call option to buy one share. If these relationships do not hold, an arbitrageur can easily make a riskless profit buy buying the stock and selling the call option.

4.2 Discrete-time option pricing: Binomial option pricing formulae

A discrete-time approach to option valuation is a lattice based models of the type developed by Cox, Ross and Rubinstein (1979). The Cox-Ross-Rubinstein (CRR) Binomial Model is a discrete model. The price of the underlying stock is assumed to take one of two states, up or down (possibly at different rates) at the end of each period in a single-period binomial model.

The CRR binomial model was developed in 1976, three years after Black-Scholes-Merton model. It's strengths were its relative simplicity, its flexibility, and the fact that it permitted the pricing of American as well as European options (Black-Scholes-Merton wasn't able to provide fair values for options that could be exercised before expiry).

Under the Cox-Ross-Rubinstein n period Binomial Model, the price of a European call with j periods remaining to expiration is given as;

$$C = \frac{\sum_{j=0}^{n} \frac{n!}{j!(n-j)!} p^j (1-p)^{n-j} \max[0, S_0(1+u)^j (1+d)^{n-j} - k]}{(1+r)^n}$$

where

$n =$ total number of periods,
$u = e^{\sigma\sqrt{(T-t)/n}} - 1 =$ up parameter,
$d = \left(\frac{1}{1+u}\right) - 1 =$ down parameter,

To operate in the binomial framework, the continuously compounded risk-free rate must be converted to a discrete rate r, thus;

$(1-r)^{(T-t)/n} - 1 =$ Risk-free rate.

Other variables are as in the Black-Sholes model given above. The n period binomial model lends itself well to computer solution.

4.3 Continuous-time option pricing: Black-Scholes model

If S is the stock price at time t, the expected drift rate in S should be assumed to be μS for some constant parameter, μ. This means that in a short interval of time, Δt, the expected increase in S is $\mu S \Delta t$. The parameter, μ, is the expected rate of return on the stock.

If the volatility of the stock price is zero, this model implies that

$$\Delta S = \mu S \Delta t\,.$$

In the limit as $\Delta t \longrightarrow 0$ we have $dS = \mu S dt$ or $\frac{dS}{S} = \mu dt$, so that

$$S_T = S_0\, e^{\mu T}\,.$$

where S_0 and S_T are the stock prices at time zero and time T, respectively. This shows that when the variance rate is zero, the stock price grows at a continuously compounded rate of μ per unit of time.

In practice, of course, a stock price does exhibit volatility. A reasonable assumption is that the variability of the percentage return in a short period of time, Δt, is the same regardless of the stock price. In other words, an investor is just as uncertain of the percentage return when the stock price is \$ 50 as when it is \$ 10. This suggest that the standard deviation of the change in a short period of time Δt should be proportional to the stock price and leads to the model

$$dS = \mu S dt + \sigma S dW$$

or

$$\frac{dS}{S} = \mu dt + \sigma dW \,, \tag{4.1}$$

where W is the standard Brownian motion. Equation 4.1 is the most widely used model of stock price behavior. The variable σ is the volatility of the stock price. The variable μ is its expected rate of return.

Itô's Lemma

The price of a stock option is a function of the underlying stock's price and time. More generally, we can say that the price of any derivative is a function of the stochastic variables underlying the derivative and time. We must therefore, acquire some understanding of the behavior of functions of stochastic variables. An important result in this area was discovered by a mathematician, K. Itô, in 1951 [Itô (1951)].

Definitions We say that a variable X follows an *Itô process* if

$$dX(t) = \alpha(t, X(t))dt + \sigma(t, X(t))dW(t) \,,$$

where W is a Wiener process and α and σ are functions of X and t. In such a case we say that X has a drift rate of α and a variance rate of σ^2. Suppose α and σ are two continuous functions. If there is a continuous stochastic process $X(\cdot)$ such that

$$X(t) = X(0) + \int_0^t a(s, X(s))ds + \int_0^t b(s, X(s))dW(s)$$

for any $t \in [0, T]$, we say that $X(\cdot)$ is a solution of the stochastic differential equation

$$dX(t) = \alpha(t, X(t))dt + \sigma(t, X(t))dW(t),$$

with initial condition $X(0)$. The solution $X(\cdot)$ is called an *Itô process.*

Lemma 1 (Itô's Lemma) *Suppose that the value of a variable X follows an Itô process:*

$$dX(t) = \alpha(t, X(t))dt + \sigma(t, X(t))dW(t),$$

where W is a Wiener process and α and σ are functions of X and t. Suppose also that G is a smooth function of X and t. Then G follows the process

$$dG = \left(\frac{\partial G}{\partial X}\alpha + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial X^2}\sigma^2\right) dt + \frac{\partial G}{\partial X}\sigma dW,$$

where W is the same Wiener process as the one that appears in the equation for dX.

An application of Itô's lemma

Earlier we argued that the model:

$$dS = \mu S dt + \sigma S dW,$$

with μ and σ constant, is a reasonable model of stock price movements. From Itô's lemma, it follows that if G is a smooth function of S and t, then the process followed by G is

$$dG = \left(\frac{\partial G}{\partial S}\mu S + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2}{\partial S^2}\sigma^2 S^2\right) dt + \frac{\partial G}{\partial S}\sigma S dW,$$

Now let us define

$$G = \ln S.$$

Because

$$\frac{\partial G}{\partial S} = \frac{1}{S} \qquad \frac{\partial^2 G}{\partial S^2} = -\frac{1}{S^2} \qquad \frac{\partial G}{\partial t} = 0,$$

it follows that the process is

$$dG = \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma dW .$$

Because μ and σ are constant, this equation indicates that G follows a generalized Wiener process. It has constant drift rate $\mu - \sigma^2/2$ and constant variance rate σ^2. The change in G between time zero and some future time, T, is, therefore, normally distributed with mean $(\mu - \sigma^2/2)T$ and variance $\sigma^2 T$.

Now observe that

$$\ln S(t) = \ln S(0) + \int_0^t (\mu - \sigma^2/2) ds + \int_0^t \sigma dW(s)$$

or equivalently

$$\ln S(t) = \ln S(0) + (\mu - \sigma^2/2)t + \sigma W(t) .$$

This means that the natural logarithm of the process $S(t)$ defined by

$$S(t) = S(0) \exp\left(\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W(t)\right) ,$$

is a solution of the stochastic differential equation

$$dG = \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma dW .$$

The process $S(t)$ defined above is in fact a model for stock prices that is widely used in daily markets. We will make use of this model in presenting an elementary proof of the Black-Scholes option pricing model.

The Black-Scholes Option Pricing Model

One of the best-known and most widely used formulas in finance is the Black-Scholes option pricing model. It was originally developed in 1973 by two scientists, Fisher Black and Myron Scholes [see Black and Scholes (1973)]. They designed the model to calculate the price of a European-style call option on non-dividend-paying stocks. (Recall that a European option is one that can be exercised only on the expiration date, not before, as opposed to an American option, which can be executed anytime before the expiration date.) Options traders and others who make their living in the market quickly learned to use the Black-Scholes model to determine the correct price for options and to help them adjust their complicated stock and options combinations. The model can be modified to apply to American options, puts, and options on stocks that pay dividends, as well as to options on other underlying securities such as futures and indexes.

Understanding the Black-Scholes model is fundamental to understanding both the theory of option pricing and the strategies of profitable trading. The mathematical concepts that underlie the model are advanced and complex, but the application of the model is relatively simple.

The foundation of the model rests on the construction of a hypothetical risk-free portfolio, consisting of long call options and short positions in the underlying stock. With proper selection of the number of call options held and the number of stocks sold, the investor can lock in a certain amount of profit. Since this profit is certain, or risk free, the investor earns the risk-free rate on the portfolio. The model uses four directly observable variables (the market price of the stock, the exercise price on the call, the time remaining until expiration on the call, and the risk-free interest rate) and one variable that is fairly easy to estimate (the standard deviation of the stocks returns). With the five variables, the basic Black-Scholes option pricing model calculates the price of a call option as follows:

$$C = S_0\Phi(d_1) - Ke^{(-rT)}\Phi(d_2)\,,$$

where

$\Phi(t) = \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}} \exp(\frac{-z^2}{2})\, dz,$
$C =$ Theoretical call term,
$S_0 =$ Current stock price,
$T =$ Time until stock expiration,
$K =$ Option striking price,
$r =$ Risk-free interest rate,

$d_1 = \frac{\log(S_0/K)+(r+\sigma^2/2)T}{\sigma\sqrt{T}}$,

$d_2 = d_1 - \sigma\sqrt{T}$,

σ = standard deviation of stock's returns,

An elementary proof of Black-Scholes model

The following elementary proof of the Black-Scholes model is given in Hillebrand (2001).

We know that at time zero the price C of the European call with the strike price K and expiration time T is defined to be

$$e^{-rT}E(S(T)-K)_+\,,$$

where by S(T) we mean the stock price at time T and by $(S(T)-K)_+$ we mean $\max\{(S(T)-K),0\}$. Recall that by an application of Itô's Lemma the stock price at any time t can be modelled by

$$S(t) \;=\; S_0\,\exp[(r-\frac{\sigma^2}{2})t+\sigma W(t)]\,,$$

where $W(t)$ is a Brownian motion. Put

$$X \;=\; (r-\frac{\sigma^2}{2})T+\sigma W_T\,,$$

then X is normally distributed with mean $\alpha = (r-\frac{\sigma^2}{2})T$ and variance $\gamma^2 = \sigma^2 T$. Now

$$\begin{aligned} E[S_0\exp(X)-K]_+ \;&=\; \int_A [S_0\exp(x)-K]\,\frac{1}{\sqrt{2\pi}}\,\exp[-\frac{(x-\alpha)^2}{2\gamma^2}]\,dx \\ &= S_0\int_A \frac{1}{\sqrt{2\pi}}\,\exp[x-\frac{(x-\alpha)^2}{2\gamma^2}]\,dx - K\int_A \frac{1}{\sqrt{2\pi}}\,\exp\left[-\frac{(x-\alpha)^2}{2\gamma^2}\right]\,dx\,, \end{aligned}$$

where $A = \{x\in\Re : S_0\exp(x) > K\}$. We now use the properties of the standard normal distribution function Φ to write:

$$\int_A \frac{1}{\sqrt{2\pi}}\,\exp\left[-\frac{(x-\alpha)^2}{2\gamma^2}\right]\,dx \;=\; P\left(X>\log\frac{K}{S_0}\right) \;=\; P\left(\frac{X-\alpha}{\gamma} > \frac{\log\frac{K}{S_0}-\alpha}{\gamma}\right)$$

$$= 1 - \Phi\left(\frac{\log\frac{K}{S_0} - \alpha}{\gamma}\right) = \Phi\left(\frac{\log\frac{S_0}{K} + \alpha}{\gamma}\right)$$

and

$$\int_A \frac{1}{\sqrt{2\pi}} \exp\left[x - \frac{(x-\alpha)^2}{2\gamma^2}\right] dx = \int_A \frac{1}{\sqrt{2\pi}} \exp\left[\alpha + \frac{\gamma^2}{2} - \frac{(x-\alpha-\gamma^2)^2}{2\gamma^2}\right] dx$$

$$= \exp\left(\alpha + \frac{\gamma^2}{2}\right) \int_{\log\frac{K}{S_0}}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left[\frac{(x-\alpha-\gamma^2)^2}{2\gamma^2}\right] dx$$

$$= \int_{\log\frac{K}{S_0}}^{\infty} \Phi(\alpha+\gamma^2, \gamma^2)\, dx = P(X + \gamma^2 > \log\frac{K}{S_0})$$

$$= P\left(\frac{X-\alpha}{\gamma} > \frac{\log\frac{K}{S_0} - \gamma^2 - \alpha}{\gamma}\right) = 1 - \Phi\left(\frac{\log\frac{K}{S_0} - \gamma^2 - \alpha}{\gamma}\right)$$

$$= \Phi\left(\frac{\log\frac{S_0}{K} + \alpha + \gamma^2}{\gamma}\right).$$

This shows that

$$E[S_0\exp(X) - K]_+ = S_0 \exp\left(\alpha + \frac{\gamma^2}{2}\right) \Phi\left(\frac{\log\frac{S_0}{K} + \alpha + \gamma^2}{\gamma}\right) - K\Phi\left(\frac{\log\frac{S_0}{K} + \alpha}{\gamma}\right).$$

Therefore, if we put $X = (r - \frac{\sigma^2}{2})T + \sigma W_T$, $\alpha = (r - \frac{\sigma^2}{2})T$ and $\gamma^2 = \sigma^2 T$ in the above formula after some simple calculations, we get:

$$E\left\{S_0 \exp\left[\left(r - \frac{\sigma^2}{2}\right)T + \sigma W_T\right] - K\right\}^+ = S_0\exp(rT)\Phi(d_1) - K\Phi(d_2),$$

where, $d_1 = \frac{\log(S_0/K) + (r+\sigma^2/2)T}{\sigma\sqrt{T}}$ and $d_2 = d_1 - \sigma\sqrt{T}$. Finally

$$C = \exp(-rt)\, E(S(T) - K)_+ = S_0\Phi(d_1) - Ke^{(-rT)}\Phi(d_2),$$

as desired.

In order to understand the model itself, we divide it into two parts. The first part, $S_0\Phi(d_1)$, derives the expected benefit from acquiring a stock outright. This is found by multiplying stock price S_0 by the change in the call premium with respect to a change in the underlying stock price $\Phi(d_1)$. The second part of the model,

$Ke^{(-rt)}\Phi(d_2)$, gives the present value of paying the exercise price on the expiration day. The fair market value of the call option is then calculated by taking the difference between these two parts.

The two most significant limiting assumptions in this model are that:

1) No dividends or cash flows are generated by the underlying asset.
2) Only one interest rate is input which implies a flat yield curve.

4.4 Stable Model for asset returns and option pricing

Empirical validations of the Black-Scholes formula have produced mixed results. One possible explanation is that, asset-price processes do not satisfy the condition of Black-Scholes model (cf. Platen and Schweizer (1994)). The term of the distribution of asset-price changes represents a controversial issue in modelling functionals of prices. A basic assumption in viewing the Black-Scholes formula as the limiting case of binomial option pricing formula was the price changes were in domain of attraction of the normal law. As has been discussed earlier in the introduction of this thesis, empirical studies typically reject normality for a wide range of financial assets. This holds especially for short-term returns, which are practically relevant for the limiting case (See for example Rachev and Mittnik (2000) pages 509–548).

Although there are many different examples that show considerable deviation from normality for daily asset return data, we present the following two examples as they are among the earliest examples indicating that deviation.

Example 4.1. This example was first introduced in Mandelbrot (1963). It had been suggested [Kendall (1953)] that wheat price relatives follow a Gaussian distribution. Indeed, a casual visual inspection of the histograms of these relatives, as plotted on natural coordinates shows them to be nicely bell shaped. The importance of the tails is, however, notoriously underestimated by plotting the data on natural coordinates. It is, on the contrary, stressed by using probability paper. As seen in Mandelbrot

(1963), probability-paper plots of wheat price relatives are definitely **S**-Shaped and in fact wheat is stable Paretian with an α near 2 but NOT equal 2.

Example 4.2. This example was introduced in Famma (1965). The data that used in that paper consisted of daily prices for each of the thirty stocks of the Dow-Jones Industrial Average. The time periods varied from stock to stock but ran from about the end of 1957 to September 26, 1962. The actual tests were performed on the first differences of the natural logarithms of daily prices. The variable of interest was

$$Z_{i+1} = \ln X_{i+1} - \ln X_i ,$$

where X_{i+1} is the price of the security at the end of day $i+1$ and X_i is the price at the end of day i. By comparing the empirical distribution of each stock and the unit normal distribution it was realized that in each case the empirical distribution is more peaked in the center and have longer tails than the normal distribution.

We now introduce other examples to give additional empirical evidence in favor of stable Paretian model. The following examples are based on the data that we gathered from Yahoo Finance Web-Page.

Examples 4.3. As an illustration of stable modelling of financial returns, we report on the following three time series:

- The daily DOW Jones Industrial Average from March 3, 1997 to December 11, 2003, with sample size $n = 1706$.
- The daily NASDAQ Stock Composite from January 5, 1998 to December 9 2003, with sample size $n = 1493$.
- The daily S & P TSX Composite Index from January 2, 1998 to December 9, 2003, with sample size $n = 1487$.

The levels and returns of each series are shown in Figure 4.1. Note that we use the standard convention and define the return r_t in period t by $r_t = (\ln P_t - \ln P_{t-1}) \times 100$, where P_t is the price of the asset at time t.

Figure 4.2 shows the empirical densities obtained via kernel density estimation along with fitted normal distributions, which is shown in dashed lines. Note that

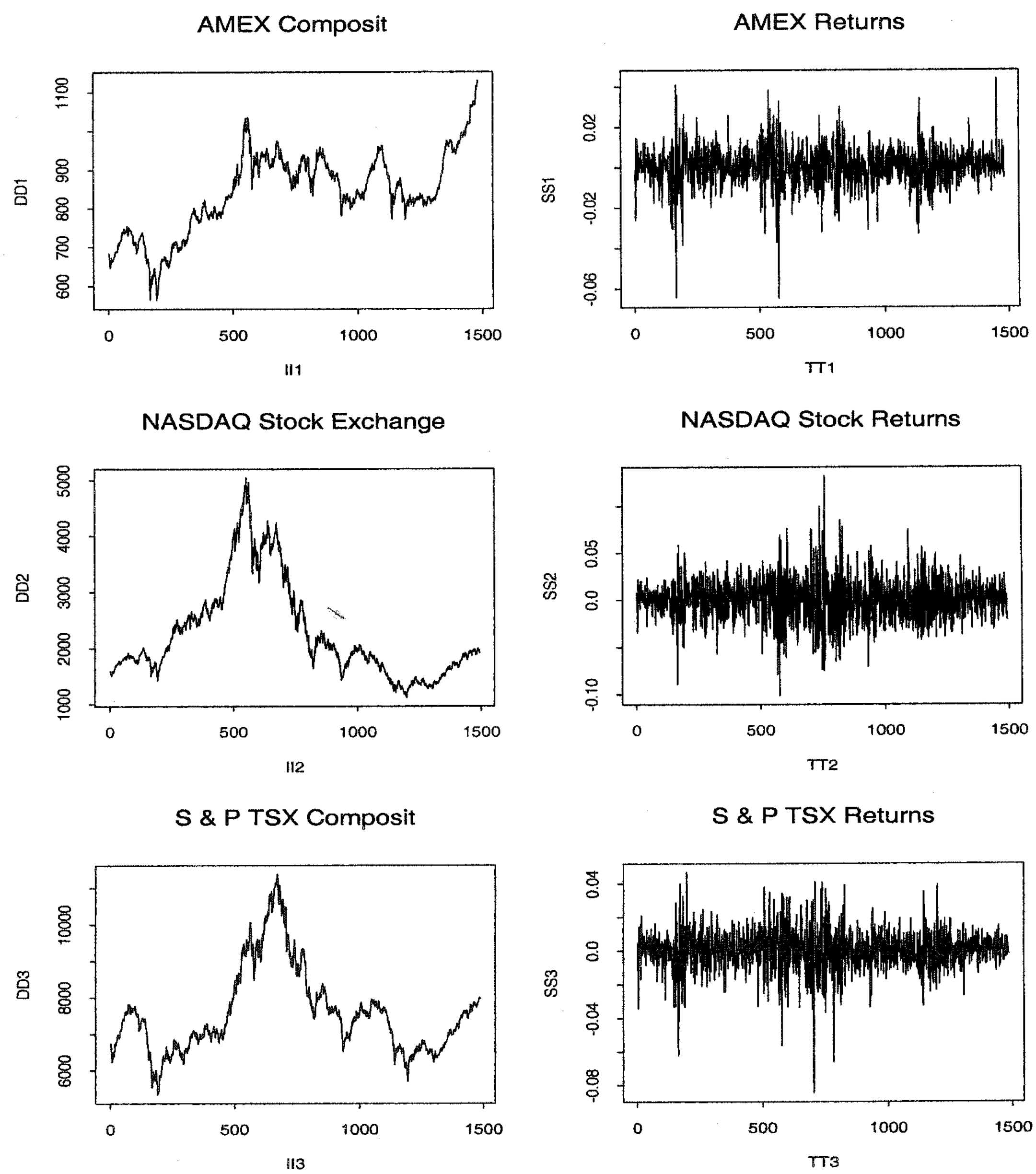

Figure 4.1 Levels and Returns of Time Series.

we used Maximum Likelihood Estimator to estimate μ and σ^2. For all three cases we observe that the normal model does not provide a closer approximation to the empirical densities.

Figure 4.3 shows shows the empirical densities obtained via kernel density estimation along with fitted stable distributions, which is shown in dashed lines. Note that we used Nolan's approximation procedure (see Nolan 2001) to estimate stable parameters. We explain this procedure in Chapter 6.

In Chapter 5 and we will focus on the non-Gaussian stable Paretian case and use an approach based on a binomial option pricing formula with random up's and down's. As we shall see the result will be a formula for continuous pricing subject to subordinated processes of price change.

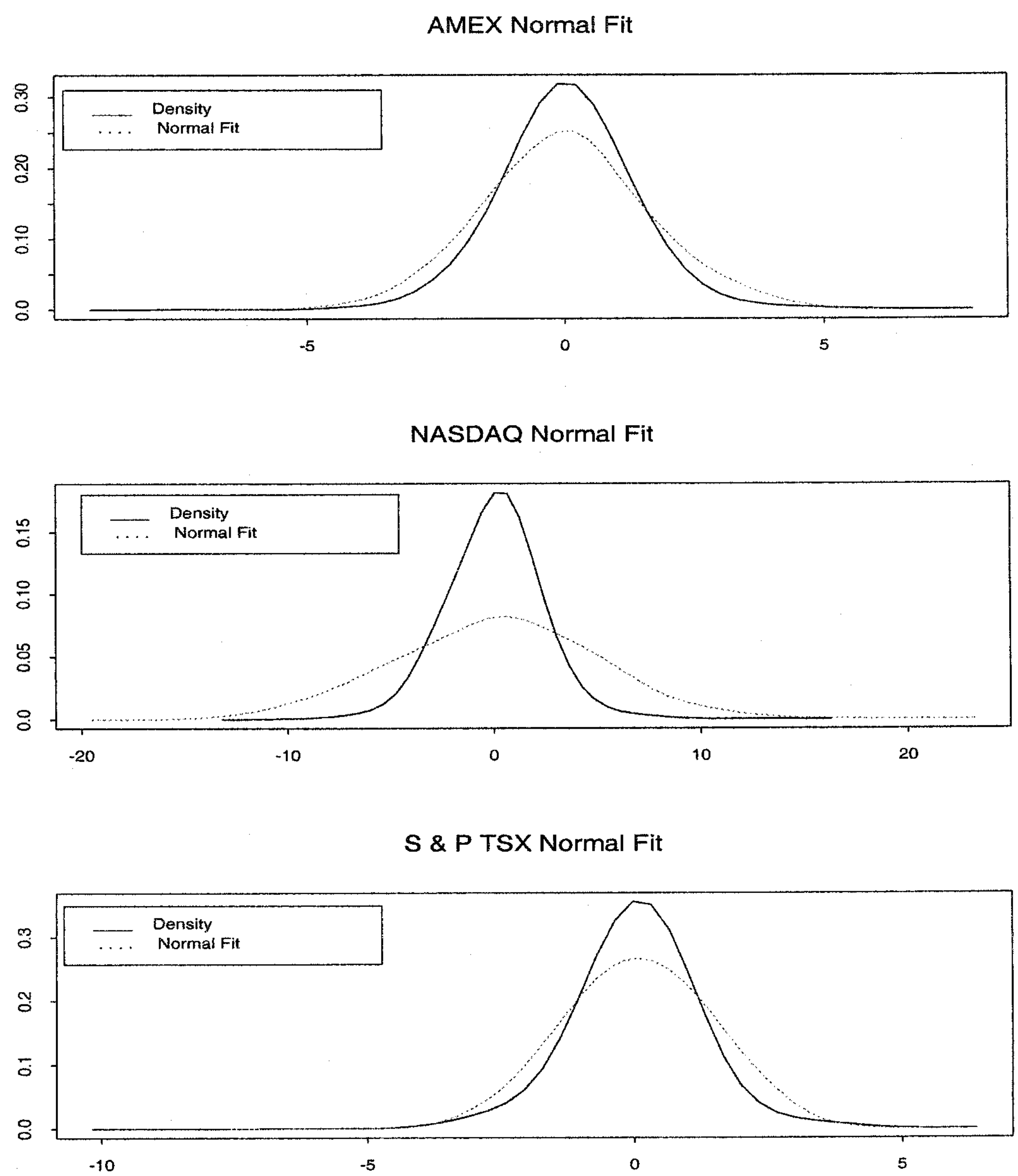

Figure 4.2 Fitted Unconditional Normal Densities .

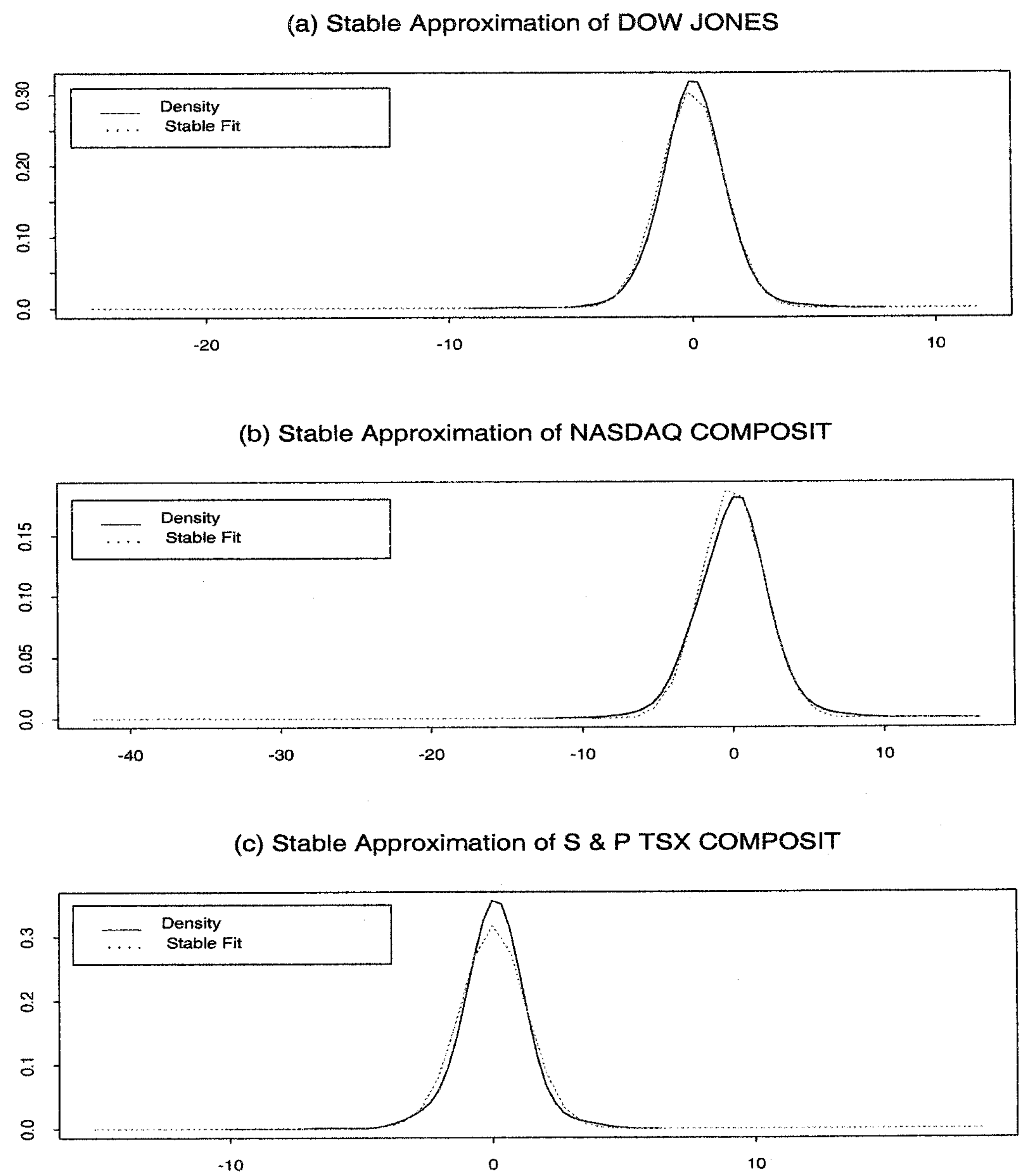

Figure 4.3 (a) - For DOW JONES Industrial Average $\alpha = 1.5$ and $\sigma = 0.2$.
(b) - For NASDAQ Stock Composit $\alpha = 1.47$ and $\sigma = 0.23$.
(c) - For S & P TSX Composite Index $\alpha = 1.4$ and $\sigma = 0.25$.

Chapter 5

Option pricing formula for infinite variance distributions

Infinite variance distributions are among the competing models used to explain the non-normality of the price changes. In this chapter we investigate the option price formula from such models asymptotically by the use of point process methods.

5.1 Option pricing formula for asset returns

Let us recall the model studies in Rachev and Samorodnitsky (1993). Without loss of any generality we can assume that the time of expiration of the call option equals to 1. Next, let us assume that there are n movements of the stock price until the expiration time. The consecutive price movements at moment k/n are determined by

$$S_k \stackrel{d}{=} S_0 \prod_{i=1}^{k} u_i^{\delta_i} d_i^{(1-\delta_i)},$$

or

$$\log(S_k/S_0) \stackrel{d}{=} \sum_{i=1}^{k} (U_i\delta_i + D_i(1-\delta_i)),$$

where $U_i := \log u_i$, $D_i := \log d_i$, and δ_i's are independent identically distributed Bernolli $(\frac{1}{2})$ independent of u_i's and d_i's. We assume that the log-increment of our stock price process are symmetrically distributed,

$$U_i = \sigma|X_{i,n}|, D_i = -U_i,$$

where n represents the number of movements until the terminal time T of a call and $\{X_{i,n} : i = 1, \ldots, n\}$ are independent identically distributed symmetric Pareto random variables with

$$P(|X_{i,n}| > x) = n^{-1}x^{-\alpha}, \; x \geq n^{-1/\alpha}, \; 1 < \alpha < 2.$$

Note: As shown in LePage, Podgórski and Ryznar (1997), Pareto assumption can be generalized to the class of observations in the domain of attraction of a stable law with index $0 < \alpha < 2$. In practice, however, index α should be chosen in $(1, 2)$.

With this assumption we can write,

$$\log(S_k/S_0) \stackrel{d}{=} \sigma \sum_{i=1}^{k} X_{i,n},$$

and thus the process

$$\xi_n(t) = \log(S_k/S_0), \; \frac{k-1}{n} < t \leq \frac{k}{n}, \; k = 1, \ldots, n, (\xi_n(0) = 0),$$

converges weakly to a symmetric α-stable Levy motion in $D[0,1]$ with characteristic function given by

$$E\exp[i\theta\xi(t)] = \exp(-t|\sigma\theta|^{\alpha})$$

as described in the Mandelbrot and Taylor model (cf. Mandelbrot and Taylor (1967)). We know that the random riskless interest rate at the ith period r_i is the average of the price up and down movement, i.e.

$$r_i = \frac{u_i + d_i}{2}.$$

Now, we can write down the option pricing formula from Rachev and Samorodnitsky (1993)

$$C_n = E\left[\frac{(S_n - K)_+}{R_1 \ldots R_n}\right],$$

where K is a striking price at the expiration time (after n movements). Denoting $|\mathbf{X}_n| = (|X_{i,n}|)_{i=1}^n$ we can write

$$R_1 \dots R_n = \frac{1}{2^n} \prod_{i=1}^{n} [\exp(-\sigma|X_{i,n}|) + \exp(\sigma|X_{i,n}|)] .$$

Thus

$$C_n = E \frac{[S_0 \exp(\sigma \sum_{i=1}^n X_{i,n}) - K]_+}{\prod_{i=1}^n \cosh(\sigma X_{i,n})} .$$

In the next step we derive a limiting pricing formula by taking $n \to \infty$. We show that an asymptotic formula for speculative prices can be derived by applying continuous mapping theorem to convergence of a point process. As we shall see in Chapter 8, this approach enables us to conclude asymptotic validity of the bootstrap for these speculative prices.

5.2 Basic Convergence Lemmas

The following Lemma will play an important role in our asymptotic analysis. The result is a simple consequence of the distributional relationship between the order statistics from a uniform distribution with that of sums involving independent exponential random variables (see Breiman (1986)).

Lemma 5.1 *If $Y_1, \cdots, Y_n$ are nonnegative i.i.d. random variables with a common distribution function $1 - G$, then*

$$(Y_{(1)}, \cdots, Y_{(n)}) \stackrel{d}{=} \Big(G^{-1}\big(\frac{\Gamma_1}{\Gamma_{n+1}}\big), \cdots, G^{-1}\big(\frac{\Gamma_n}{\Gamma_{n+1}}\big)\Big)$$

where $Y_{(i)}$ is the ith largest value, i.e.: $Y_{(1)} \geq Y_{(2)} \geq \cdots \geq Y_{(n)}$ and

$$G^{-1}(u) = \inf\{y : G(y) \leq u\} .$$

Lemma 5.1 is handy because it allows us to restate expressions involving random variables $\{X_i\}$ in terms of expressions that involve $\{\Gamma_i\}$. As these two expressions are equivalent in distribution we can then work out expectations and higher order moments by working with a more convenient formulation involving exponential random variables.

The next lemma, concerning heavy tailed distributions, will be crucial in our development, and was first proved in Zarepour (1999a). Hereafter convergence in distribution will be taken with respect to the vague topology, and will be denoted by "$\xrightarrow{v}$" as before. Recall that for X to have a heavy tailed distribution in R^d, there exist positive constants $a_n \to \infty$ such that

$$nP(a_n^{-1}X \in \cdot) \xrightarrow{v} \nu(\cdot)$$

where $\mu(\cdot)$ is a Radon measure on R^d (see Davis, Mulrow and Resnik (1987) for more background on heavy tailed distributions). Equivalently, X has a heavy tailed distribution if there exists an $\alpha > 0$ such that to say that:

$$nP\left\{\left(a_n^{-1}\,||X||, X/||X||\right) \in (dr, ds)\right\} \to \alpha r^{-\alpha-1} dr S(ds)\,, \tag{5.1}$$

where S is a Borel probability measure on unit sphere $\{s \in R^d \ : \ ||s|| = 1\}$. In this thesis $d = 1$.

Lemma 5.2 *Let $\{X_i\}$ be an i.i.d. sequence of random vectors on R^d satisfying* (5.1). *If $G(x) = P(||X_1|| > x)$, then*

$$\nu_n(\cdot) = \sum_{i=1}^{n} \varepsilon_{S_i a_n^{-1} G^{-1}(\Gamma_i/\Gamma_{n+1})}(\cdot) \ \xrightarrow{a.s.} \ \nu(\cdot) = \sum_{i=1}^{\infty} \varepsilon_{S_i \Gamma_i^{-1/\alpha}}(\cdot)\,,$$

where $S_i \stackrel{d}{=} X_i/||X_i||$ for $i = 1, \ldots, n$.

Since in the sequel the factor $\prod_{n=1}^{\infty} \cosh \sigma \Gamma_n^{-1/\alpha}$ plays an important role in the option pricing formula, it is important to investigate if it has a finite value. The next Lemma shows that this is true under a mild condition.

Lemma 5.3 *For each $0 < \alpha < 2$ the infinite product*

$$\prod_{n=1}^{\infty} \cosh \sigma \Gamma_n^{-1/\alpha}$$

converges.

Proof : With no loss of any generality put $\sigma = 1$. We know that $\Gamma_n \to +\infty$ as $n \to +\infty$. Using the fact that

$$\lim_{x\to 0} \frac{\cosh x - 1}{x^2/2} = 1\,,$$

we conclude that

$$\lim_{n\to+\infty} \frac{\cosh \Gamma_n^{-1/\alpha} - 1}{\Gamma_n^{-2/\alpha}} = 2\,.$$

Hence, The series $\sum_{n=1}^{\infty}(\cosh \Gamma_n^{-1/\alpha} - 1)$ converges if the series $\sum_{n=1}^{\infty} \Gamma_n^{-2/\alpha}$ converges. Since

$$\lim_{n\to+\infty} \frac{\Gamma_n}{n} = 1\,,$$

the series $\sum_{n=1}^{\infty} \Gamma_n^{-p}$ behaves like harmonic series. Hence, it converges whenever $p > 1$. Therefore $\sum_{n=1}^{\infty}(\cosh \Gamma_n^{-1/\alpha} - 1)$ converges whenever $\frac{2}{\alpha} > 1$, or equivalently, whenever $0 < \alpha < 2$. By convergence criteria for infinite products, we conclude that $\prod_{n=1}^{\infty} \cosh \Gamma_n^{-1/\alpha}$ converges if $0 < \alpha < 2$. □

By applying a specialized type of continuous mapping theorem we can prove the following useful corollary to Lemma 5.2 for random variables in the domain of attraction of a symmetric stable law (note that we only apply Lemma 5.3 for the case $d = 1$ but for generality Lemma 5.2 is given for arbitrary d). This will be used to establish the option pricing formula.

Note. Throughout this paper we use the important fact that the function $h = \ln\cosh$ is an even function.

Corollary 5.1 *Let $\{X_i\}$ be a sequence of i.i.d. random variables from the domain of attraction of a symmetric stable law with index $0 < \alpha < 2$. Let $G(x) = P\{|X_1| > x\}$. Then,*

$$\prod_{i=1}^{n} \cosh(a_n^{-1} X_i) \stackrel{d}{=} \prod_{i=1}^{n} \cosh\left(a_n^{-1} G^{-1}(\Gamma_i/\Gamma_{n+1})\right) \stackrel{a.s.}{\to} \prod_{i=1}^{\infty} \cosh\left(\Gamma_i^{-1/\alpha}\right)\,.$$

Proof. For $0 < \epsilon < M < \infty$ let $g_{\epsilon,\,M}$ be the real valued function

$$g_{\epsilon,\,M}(x) = \begin{cases} 0 & x \in A_{0,\epsilon,M} \\ L^1_{\epsilon,M} & x \in A_{1,\epsilon,M} \\ \ln(\cosh x) & x \in A_{2,\epsilon,M} \\ L^2_{\epsilon,M} & x \in A_{1,\epsilon,M} \end{cases} .$$

where

$$A_{0,\,\epsilon,\,M} = (-\infty, -M-\epsilon/2) \bigcup (-\epsilon/2, \epsilon/2) \bigcup (M+\epsilon/2, +\infty),$$

$$A_{1,\,\epsilon,\,M} = [-\epsilon, -\epsilon/2] \bigcup [\epsilon/2, \epsilon]\ , \quad A_{2,\,\epsilon,\,M} = (-M, -\epsilon) \bigcup (\epsilon, M),$$

$$A_{3,\,\epsilon,\,M} = [-M - \epsilon/2, -M] \bigcup [M, M + \epsilon/2],$$

and the positive and continuous functions $L^1_{\epsilon,M}$ and $L^2_{\epsilon,M}$ are chosen in such a way that $g_{\epsilon,\,M}$ is continuous, $0 \leq L^1_{\epsilon,M}(x) \leq \ln\cosh(x)$ if $x \in A_{1,\epsilon,M}$ and $0 \leq L^2_{\epsilon,M}(x) \leq \ln\cosh(x)$ if $x \in A_{2,\epsilon,M}$. Note that because of the concavity of the function $h = \ln\cosh$ it is clear that $L^1_{\epsilon,M}$ and $L^2_{\epsilon,M}$ can be chosen in this fashion. A sketch of graph of $g_{\epsilon,M}$ is shown in Figure 5.1.

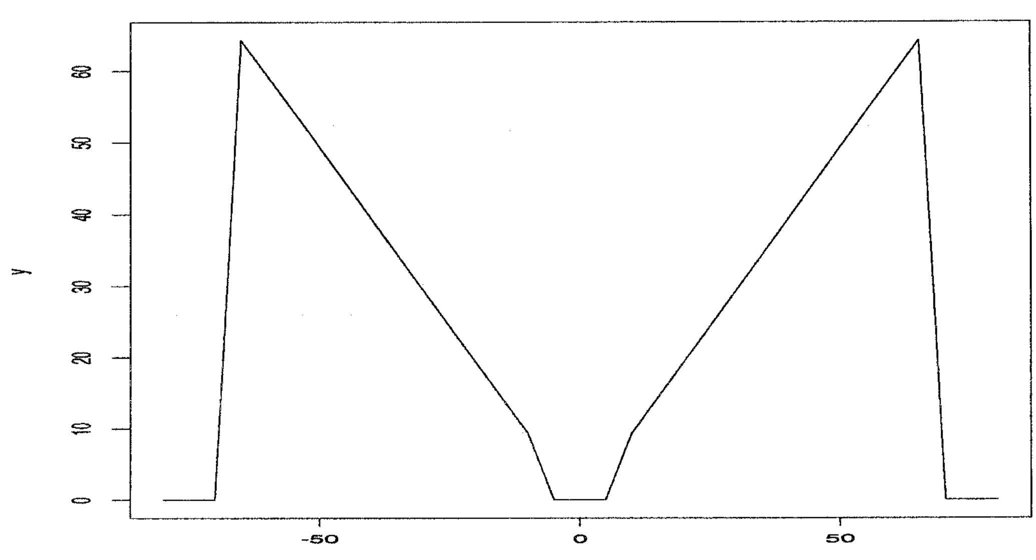

Figure 5.1 A sketch of graph of $g_{\epsilon,\,M}$.

Let $\nu(\cdot)$ and $\nu_n(\cdot)$ be the point processes defined in Lemma 5.2 with $d = 1$ and $S_i = \text{sgn}(X_i)$ (note that in this case $S_i = \delta_i$, independently of $|X_i|$). To prove the corollary we establish that

$$\lim_{M\to+\infty} \lim_{\epsilon\to 0^+} \int g_{\epsilon, M}(x)\nu(dx) \overset{a.s.}{\to} \sum_{i=1}^{\infty} \ln\left(\cosh(\Gamma_i^{-1/\alpha})\right) , \tag{5.2}$$

and

$$\lim_{M\to+\infty} \lim_{\epsilon\to 0^+} \limsup_{n} \left| \int g_{\epsilon, M}(x)\nu_n(dx) - \sum_{i=1}^{n} \ln\cosh[a_n^{-1}G^{-1}(\Gamma_i/\Gamma_{n+1})S_i] \right| \overset{a.s.}{=} 0 . \tag{5.3}$$

As we shall see, to complete the proof, we need to prove (5.2) and (5.3). We begin with (5.2). Observe that

$$\int g_{\epsilon, M}(x)\nu(dx) = \sum_{j=1}^{3} \int_{A_{j,\epsilon,M}} g_{\epsilon, M}(x)\nu(dx) .$$

Because $0 \le g_{\epsilon, M}(x) \le \ln\cosh(x)$ over $A_{j,\epsilon,M}$ whenever $j = 1$ or $j = 3$, we have

$$0 \le \int_{A_{j,\epsilon,M}} g_{\epsilon, M}(x)\nu(dx) \le \sum_{i=1}^{\infty} \ln\cosh\left(\Gamma_i^{-1/\alpha}\right) I\left\{\delta_i\Gamma_i^{-1/\alpha} \in A_{j,\epsilon,M}\right\} .$$

Furthermore, since

$$\sum_{i=1}^{\infty} \ln\cosh\left(\Gamma_i^{-1/\alpha}\right) I\left\{\delta_i\Gamma_i^{-1/\alpha} \in A_{j,\epsilon,M}\right\} \le \sum_{i=1}^{\infty} \ln\cosh\left(\Gamma_i^{-1/\alpha}\right) < \infty ,$$

for $j = 1$ or $j = 3$, an application of the Dominated Convergence Theorem shows that

$$0 \le \lim_{\epsilon\to 0^+} \int_{A_{j,\epsilon,M}} g_{\epsilon, M}(x)\nu(dx) \le \sum_{i=1}^{\infty} \ln\cosh\left(\Gamma_i^{-1/\alpha}\right) \lim_{\epsilon\to 0^+} I\left\{\delta_i\Gamma_i^{-1/\alpha} \in A_{j,\epsilon,M}\right\} = 0$$

whenever $j = 1$ or $j = 3$. Now consider the contribution from $A_{2,\epsilon,M}$. Another application of the Dominated Convergence Theorem shows that

$$\begin{aligned} \lim_{\epsilon\to 0^+} \int_{A_{2,\epsilon,M}} g_{\epsilon, M}(x)\nu(dx) &= \sum_{i=1}^{\infty} \ln\cosh\left(\Gamma_i^{-1/\alpha}\right) \lim_{\epsilon\to 0^+} I\left\{\delta_i\Gamma_i^{-1/\alpha} \in A_2^{\epsilon,M}\right\} \\ &\overset{a.s}{=} \sum_{i=1}^{\infty} \ln\cosh\left(\Gamma_i^{-1/\alpha}\right) I\left\{\delta_i\Gamma_i^{-1/\alpha} \in A_M\right\} , \end{aligned}$$

where $A_M = (-M, M)$ and we have used the fact that $\cosh\left(\delta_i \Gamma_i^{-1/\alpha}\right) = \cosh\left(\Gamma_i^{-1/\alpha}\right)$. Now taking the limit as $M \to \infty$, and once more using the Dominated Convergence Theorem, deduce (5.2).

To prove (5.3), let $Y_{i,n} = a_n^{-1} G^{-1}(\Gamma_i/\Gamma_{n+1})$. By definition of ν_n we have

$$\sum_{i=1}^{n} \ln(\cosh(Y_{i,n})) = \int \ln(\cosh x)\nu_n(dx)\,.$$

Therefore,

$$\left|\int g_{\epsilon,M}(x)\nu_n(dx) - \sum_{i=1}^{n} \ln\cosh(Y_{i,n})\right|$$

$$= \int_{A_{0,\epsilon,M}} \ln\cosh(x)\nu_n(dx) + \left|\int_{A_{1,\epsilon,M}\bigcup A_{3,\epsilon,M}} \left(g_{\epsilon,M}(x) - \ln(\cosh x)\right)\nu_n(dx)\right|$$

$$\leq \int_{A_{0,\epsilon,M}} \ln\cosh(x)\nu_n(dx) + 2\int_{A_{1,\epsilon,M}\bigcup A_{3,\epsilon,M}} \ln(\cosh x)\nu_n(dx)\,.$$

By Lemma 5.2 the second integral on the right-hand side converges almost surely to

$$2\int_{A_{1,\epsilon,M}\bigcup A_{3,\epsilon,M}} \ln(\cosh x)\nu(dx)$$

as $n \to \infty$. Take $\epsilon \to 0^+$ and use the Dominated Convergence Theorem to see that this term converges to zero. Note that the Dominated Convergence Theorem is applicable because

$$\sum_{i=1}^{\infty} \ln\left(\cosh\left(\Gamma_i^{-1/\alpha}\right)\right) < \infty\,.$$

To deal with the first integral, rewrite it as

$$\int_{\{|x|<\epsilon/2\}} \ln(\cosh x)\nu_n(dx) + \int_{\{|x|\geq M+\epsilon/2\}} \ln(\cosh x)\nu_n(dx)\,.$$

For the first term we can write

$$\int_{\{|x|<\epsilon/2\}} \ln(\cosh x)\nu_n(dx) = \sum_{i=1}^{\infty} \ln\cosh\left(Y_{i,n}\right) I\{i \leq n\}\, I\left\{|Y_{i,n}| < \epsilon/2\right\}.$$

Since $|Y_{i,n}| \stackrel{a.s.}{\to} \Gamma_i^{-1/\alpha}$ and $\sum_{i=1}^{\infty} \ln\left(\cosh\left(\Gamma_i^{-1/\alpha}\right)\right) < \infty$, we can use the Dominated Convergence Theorem to show that the first term converges to

$$\sum_{i=1}^{\infty} \ln\cosh\left(\Gamma_i^{-1/\alpha}\right) I\left\{\Gamma_i^{-1/\alpha} < \epsilon/2\right\}.$$

Since

$$\lim_{\epsilon\to 0^+} \sum_{i=1}^{\infty} \ln\cosh\left(\Gamma_i^{-1/\alpha}\right) I\left\{\Gamma_i^{-1/\alpha} < \epsilon/2\right\} = 0$$

we conclude that the first term converges to zero as $n \to \infty$ and $\epsilon \to 0^+$. For the second term, let T_c be the transformation defined by

$$T_c\left(\sum_{i=1}^{\infty} \varepsilon_{x_i}\right) = \sum_{i=1}^{\infty} x_i\, I\{|x_i| \geq c\}.$$

This defines a continuous mapping from $M_p(\Re^+)$ into $\Re$ for each $c > 0$ (see Proposition 3.3). Therefore, since the image of a compact subset of $\Re$ under the function $h = \ln\cosh$ is compact, we get by proposition 3.5 and Lemma 5.2 that

$$\int_{\{|x|\geq M+\epsilon/2\}} \ln(\cosh x)\nu_n(dx) \stackrel{a.s.}{\to} \int_{\{|x|\geq M+\epsilon/2\}} \ln(\cosh x)\nu(dx)$$

$$= \sum_{i=1}^{\infty} \ln\left(\cosh\left(\Gamma_i^{-1/\alpha}\right)\right) I\left\{\left|\delta_i\Gamma_i^{-1/\alpha}\right| \geq M+\epsilon/2\right\}.$$

Take the limit as $\epsilon \to 0^+$ and $M \to \infty$ to deduce by the Dominated Convergence Theorem that

$$\lim_{M\to\infty}\lim_{\epsilon\to 0^+}\lim_{n\to\infty} \int_{A_{0,\epsilon,M}} \ln(\cosh x)\nu_n(dx) = 0,$$

which completes the proof of (5.3).

Now since by Lemma 5.2,

$$\int g_{\epsilon,M}(x)\nu_n(dx) \stackrel{a.s.}{\to} \int g_{\epsilon,M}(x)\nu(dx), \tag{5.4}$$

and since

$$\left|\sum_{i=1}^{n} \ln\left(\cosh\left(a_n^{-1}G^{-1}(\Gamma_i/\Gamma_{n+1})\right)\right) - \sum_{i=1}^{n} \ln\left(\cosh(\Gamma_i^{-1/\alpha})\right)\right|$$

$$\leq \left|\sum_{i=1}^{n} \ln\left(\cosh\left(a_n^{-1}G^{-1}(\Gamma_i/\Gamma_{n+1})\right)\right) - \int g_{\epsilon,M}(x)\nu_n(dx)\right|$$

$$+ \left|\int g_{M,\epsilon}(x)\nu_n(dx) - \int g_{\epsilon,M}(x)\nu(dx)\right|$$

$$+ \left|\int g_{\epsilon,M}(x)\nu(dx) - \sum_{i=1}^{n} \ln\left(\cosh(\Gamma_i^{-1/\alpha})\right)\right|,$$

an application of (5.2),(5.3) and (5.4) to the right-hand side of this inequality shows that

$$\sum_{i=1}^{n} \ln\left(\cosh(a_n^{-1}X_i)\right) \stackrel{d}{=} \sum_{i=1}^{n} \ln\left(\cosh\left(a_n^{-1}G^{-1}(\Gamma_i/\Gamma_{n+1})\right)\right) \stackrel{a.s.}{\to} \sum_{i=1}^{\infty} \ln\left(\cosh(\Gamma_i^{-1/\alpha})\right).$$

Note that the equality in distribution on the left-hand side is due to Lemma 5.1. The result now follows by taking the exponential (a continuous function) of the left and right sides. □

5.3 Main Result

Using Corollary 5.1 we can give a streamlined proof of the option pricing formula from LePage, Podgórski, and Ryznar (1997).

Theorem 5.1. (Option Pricing formula for stock price differences) *Let $\{X_i\}$ be a sequence of i.i.d. random variables from the domain of attraction of a stable law for $0 < \alpha < 2$. Then $C_{(n)} \longrightarrow C$, where*

$$C_{(n)} = E\left\{\frac{[S_0\exp(\sigma a_n^{-1}\sum_{i=1}^{n} X_i) - K]_+}{\prod_{i=1}^{n}\cosh(\sigma a_n^{-1}X_i)}\right\}$$

and

$$C = E\left\{\frac{\left[S_0\,\exp\left(\sigma\sum_{i=1}^{\infty}\delta_i\Gamma_i^{-1/\alpha}\right) - K\right]_+}{\prod_{i=1}^{\infty}\cosh\left(\sigma\,\Gamma_i^{-1/\alpha}\right)}\right\}.$$

Moreover,

$$\frac{[S_0\exp(\sigma a_n^{-1}\sum_{i=1}^{n} X_i) - K]_+}{\prod_{i=1}^{n}\cosh(\sigma a_n^{-1}X_i)} \stackrel{d}{\to} \frac{\left[S_0\,\exp\left(\sigma\sum_{i=1}^{\infty}\delta_i\Gamma_i^{-1/\alpha}\right) - K\right]_+}{\prod_{i=1}^{\infty}\cosh\left(\sigma\,\Gamma_i^{-1/\alpha}\right)}. \tag{5.5}$$

Proof. First note that $X_i = \delta_i|X_i|$ where $\delta_i = X_i/|X_i|$ and $\{\delta_i\}$ and $\{|X_i|\}$ are mutually independent. Therefore by Lemma 5.1

$$\frac{[S_0 \exp(\sigma a_n^{-1} \sum_{i=1}^n X_i) - K]_+}{\prod_{i=1}^n \cosh(\sigma a_n^{-1} X_i)} \stackrel{d}{=} \frac{[S_0 \exp(\sigma a_n^{-1} \sum_{i=1}^n \delta_i G^{-1}(\Gamma_i/\Gamma_{n+1})) - K]_+}{\prod_{i=1}^n \cosh(\sigma a_n^{-1} G^{-1}(\Gamma_i/\Gamma_{n+1}))}. \tag{5.6}$$

where $G(x) = P\{|X_1| > x\}$. To prove (5.5) note that denominator of the right-hand side of (5.6) converges almost surely to the denominator of the right-hand side of (5.5) by Corollary 5.1. It is also straightforward to show that the numerator of the right-hand side of (5.6) converges almost surely to the numerator of the right-hand side of (5.5) [for example see LePage, Woodroofe and Zinn (1981)]. Next to show $C_n \to C$, let

$$R_n = \frac{[S_0 \exp(\sigma a_n^{-1} \sum_{i=1}^n \delta_i G^{-1}(\Gamma_i/\Gamma_{n+1})) - K]_+}{\prod_{i=1}^n \cosh(\sigma a_n^{-1} G^{-1}(\Gamma_i/\Gamma_{n+1}))}$$

and

$$R = \frac{\left[S_0 \exp\left(\sigma \sum_{i=1}^\infty \delta_i \Gamma_i^{-1/\alpha}\right) - K\right]_+}{\prod_{i=1}^\infty \cosh\left(\sigma \Gamma_i^{-1/\alpha}\right)}.$$

Note that $C_n = E\{E(R_n \,|\, \Gamma)\}$ and $C = E\{E(R \,|\, \Gamma)\}$ where $\Gamma = (\Gamma_1, \Gamma_2, \ldots)$ and observe that

$$E(R_n \,|\, \Gamma) \le S_0 .$$

Therefore, by the Dominated Convergence Theorem it suffices to show that

$$E(R_n \,|\, \Gamma) \stackrel{a.s.}{\to} E(R \,|\, \Gamma) \quad \text{for each fixed } \Gamma.$$

Convergence of conditional expectation follows since R_n converges almost surely to R for a fixed Γ (because R_n converges unconditionally) and R_n is conditionally uniformly integrable. This last fact is shown by,

$$E(R_n^2 \,|\, \Gamma) \le S_0^2 \prod_{i=1}^n \cosh\left(2\sigma a_n^{-1} G^{-1}(\Gamma_i/\Gamma_{n+1})\right) + S_0^2 K^2.$$

Note by Corollary 5.1 that the first expression on the right-hand side is convergent almost surely. □.

In practice, some kind of approximation to C is needed because C appears impossible to work out in closed form. In Chapters 7 and 8 we discuss some properties for C in more details.

Chapter 6

Simulation for Option Price Formula

In Chapter 5 we observed that option price formula can be written as the expectation of a random variable which can be simulated approximately. In this chapter we analyze and simulate this formula by giving theoretical and empirical examples.

6.1 Simulation for Truncated Option Price Model

In the case of the Pareto distribution we may use a Monte Carlo estimate of C_n defined by

$$C_n = E\left\{ \frac{\left[S_0 \exp\left(\sigma \left(\frac{\Gamma_{n+1}}{n} \right)^{1/\alpha} \sum_{i=1}^{n} \delta_i \Gamma_i^{-1/\alpha} \right) - K \right]_+}{\prod_{i=1}^{n} \cosh\left(\sigma \left(\frac{\Gamma_{n+1}}{n} \right)^{1/\alpha} \Gamma_i^{-1/\alpha} \right)} \right\}, \tag{6.1}$$

as an estimator for

$$C = E\left\{ \frac{\left[S_0 \exp\left(\sigma \sum_{i=1}^{\infty} \delta_i \Gamma_i^{-1/\alpha} \right) - K \right]_+}{\prod_{i=1}^{\infty} \cosh\left(\sigma\, \Gamma_i^{-1/\alpha} \right)} \right\}. \tag{6.2}$$

A Monte Carlo estimate is easily derived by simulating $\delta_1, \ldots, \delta_n$ and $\Gamma_1, \ldots, \Gamma_{n+1}$. However, for more general data, as in the general stable law considered in LePage, Podgórski and Ryznar (1997), the analogous formula for C_n is difficult to work with because it requires inverting the cumulative distribution function for the normalized innovations (Lepage *et. al.* (1997) in fact comment on the practical difficulties surrounding computation of C, although they do not address the topic in detail: Remark 4 of Lepage *et. al.* (1997)).

To avoid this inversion, one might instead try to approximate C by truncating its product and sum expression to N terms. Call this estimator $\widehat{C_N}$, i.e.,

$$\widehat{C_N} = E\left\{\frac{\left[S_0 \exp\left(\sigma \sum_{i=1}^{N} \delta_i \Gamma_i^{-1/\alpha}\right) - K\right]_+}{\prod_{i=1}^{N} \cosh\left(\sigma \Gamma_i^{-1/\alpha}\right)}\right\}. \tag{6.3}$$

We show, however, that this approach is impractical due to the slow convergence of $\Gamma_i^{-1/\alpha}$ to zero.

Example 6.1 Consider a European call option and suppose that the current share price is \$42 and the exercise price is \$40. We use Program 3 in Appendix A to calculate truncated option prices when $\sigma = 0.2$, $\alpha = 1.4$. Table 6.1, shows that for small values of N truncated option prices can not be reasonably determined. (The value of α is suggested by Rachev and Mittnik (2000) who note often in practice that $1.4 \le \alpha < 2$).

N	2000	4000	6000	8000	10000
$\widehat{C_N}$	7.478249	6.758038	8.131283	6.883622	7.867903

Table 6.1. Truncated option prices for various N's. Observe the variability of $\widehat{C_N}$ in this table.

The variability of $\widehat{C_N}$ can also be seen from Figure 6.1 which is obtained by running Program 3 for the data given in the following two examples.

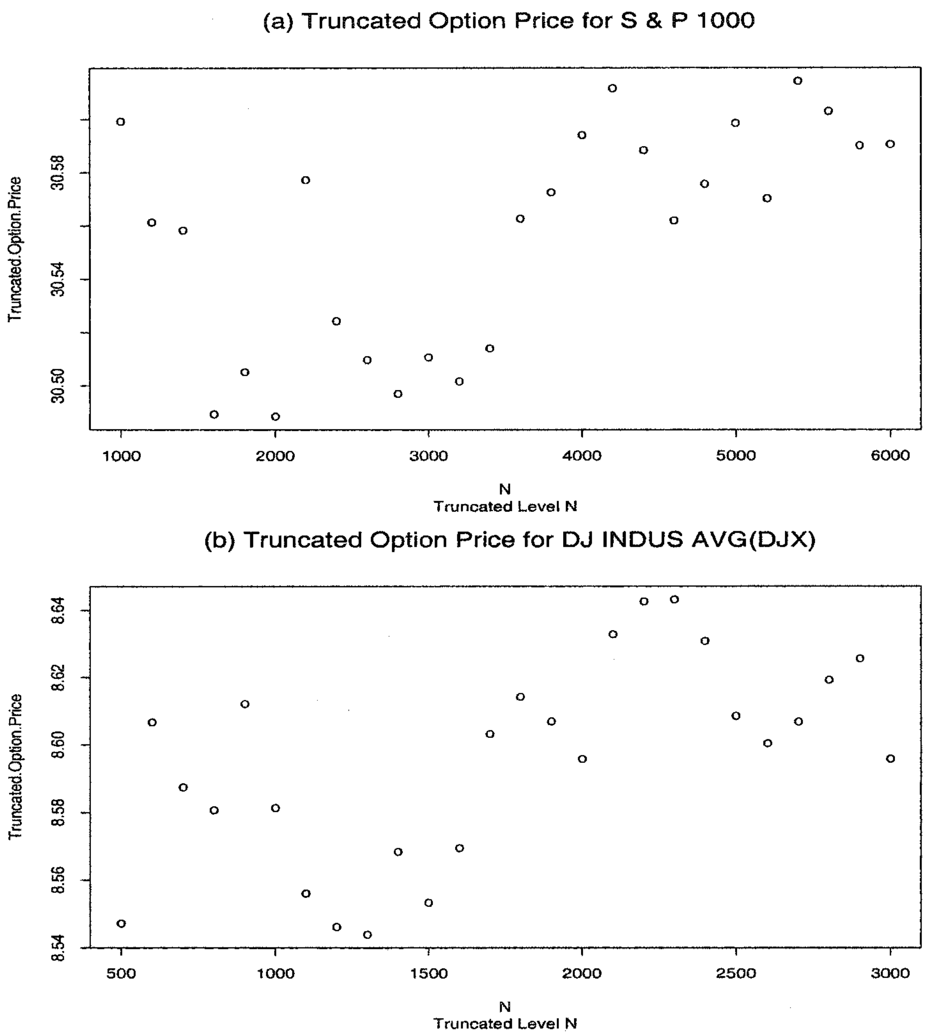

Figure 6.1 Truncated option prices for two European call option estimated by truncated value $\widehat{C_N}$.

Example 6.2 In a table that shows Quotes for Stock Index Options from The Wall Street Journal on August 5, 1998, one October call option on the S & P 100 with a strike price of 520 costs \$33. Suppose the value of the index at the close of trading on August 4, 1998, was 525.64 and assume $\alpha = 1.7$ and $\sigma = 0.035$. We approximate

C by calculating $\widehat{C_N}$ for different values of $N = 1000, 1200, \ldots, 6000$. Figure 6.1-(a) shows the results of this simulation.

Example 6.3 Consider a European call option on the DJ INDUS AVG (DJX) that is two months away from maturity. The current value of the index is 80, the exercise price is 82, and the volatility of the index is 0.08. Here we assume $\alpha = 1.6$ and estimate the call option. The results are summarized in Table 6.2. Once again Figure 6.1-(b) shows the variability of $\widehat{C_N}$ for small values of $N = 500, 600, \ldots, 3000$.

N	1000	1400	1800	2200	2600	3000
$\widehat{C_N}$	8.581414	8.568481	8.614243	8.642552	8.600389	8.595876

Table 6.2. Truncated option prices calculated for Example 6.3.

The variability of $\widehat{C_N}$ seen in Figure 6.1 shows that $\widehat{C_N}$ must be converging slowly to C. In fact both the numerator and denominator of the fraction R_N, i.e.

$$R_N = \frac{\left[S_0 \exp\left(\sigma \sum_{i=1}^{N} \delta_i \Gamma_i^{-1/\alpha}\right) - K\right]_+}{\prod_{i=1}^{N} \cosh\left(\sigma \Gamma_i^{-1/\alpha}\right)},$$

vary significantly with N. Thus, both terms must be contributing to the slow convergence. Moreover, Figure 6.2 and 6.3 which are based on the simulation of a European call option described in Example 6.1, shows that R_N is often Zero. Consequently $\widehat{C_N}$ can only be accurately estimated by using a large number of simulations (another deficiency of the method). In Chapter 7 we use a recursion method to quantify the behavior of these values.

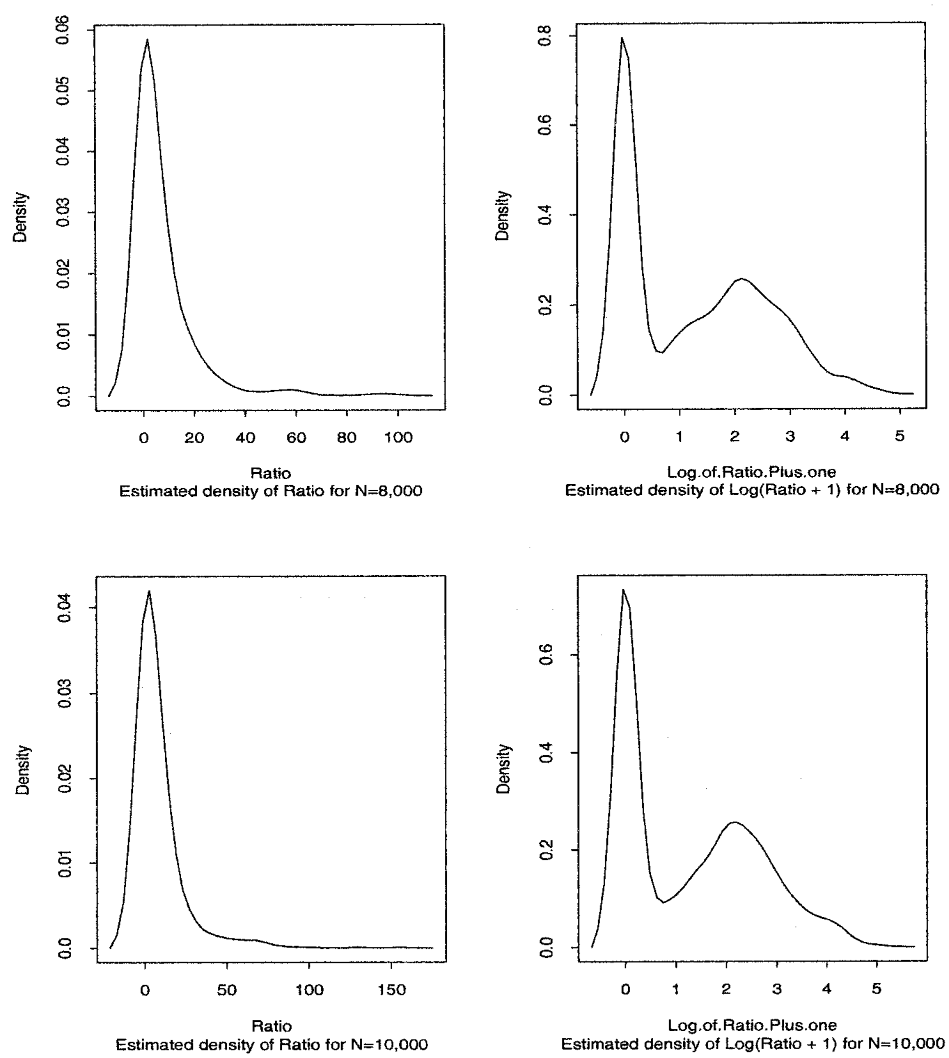

Figure 6.2 Estimated density for R_N and $\ln(1+R_N)$ for $N = 8,000$ and $N = 10,000$. Note the spike at zero which shows R_N equals zero often.

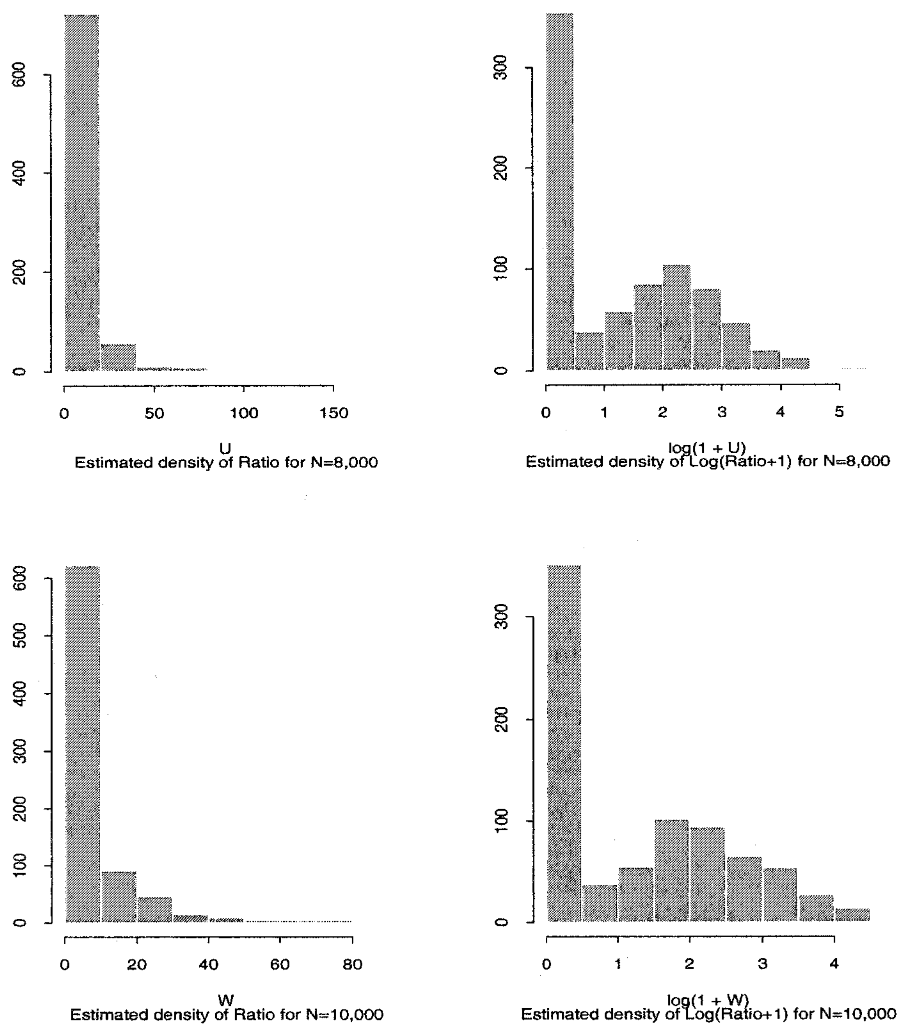

Figure 6.3 Histogram for R_N and $\ln(1+R_N)$ for $N = 8,000$ and $N = 10,000$. This also shows that R_N equals zero often.

6.2 Stopping Value for C

As we already mentioned the estimator $\widehat{C_N}$ might behave poorly due to its slow convergence. We now give a more formal argument for this statement.

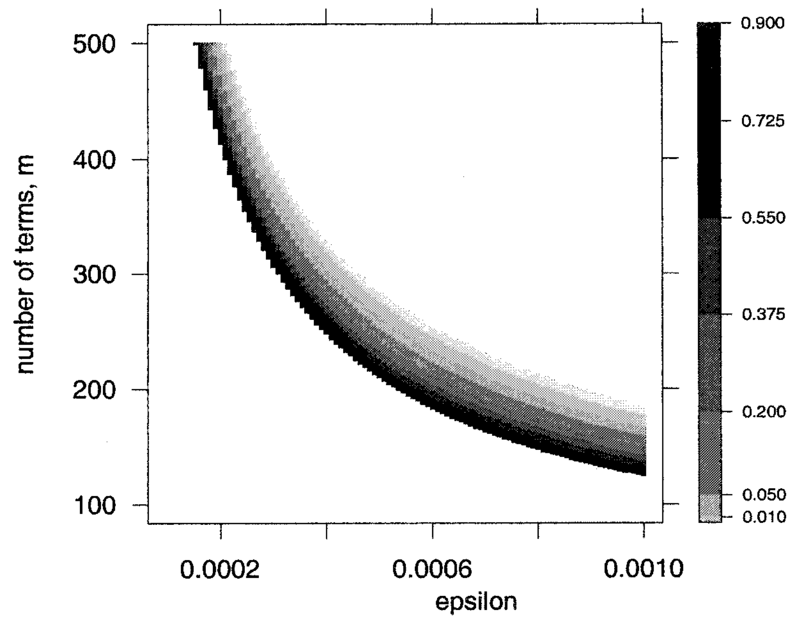

Figure 6.4 Plot p versus number of terms m and target value ϵ such that $P\{N_\epsilon > m\} = p$ (all values computed using $\alpha = 1.4$). Gray scale on right side gives value for p.

It is clear that the accuracy of $\widehat{C_N}$ depends upon how rapidly $\Gamma_i^{-1/\alpha}$ decreases. Therefore, for each $\epsilon > 0$ let

$$N_\epsilon = \inf\{i : \Gamma_i^{-1/\alpha} < \epsilon\}$$

be the truncation value, or stopping value, required to ensure a small enough value for $\Gamma_i^{-1/\alpha}$. Observe that

$$P\{N_\epsilon > m\} = P\{\Gamma_m < \epsilon^{-\alpha}\} = 1 - \sum_{k=0}^{m-1} \frac{\epsilon^{-\alpha} \exp(\epsilon^{-\alpha})}{k!}. \tag{6.4}$$

Also note that

$$E(N_\epsilon) = \epsilon^{-\alpha} + 1,$$

which is very large when ϵ is small, and thus a large truncation value will often be needed for accurate estimation. In fact, (6.4) can be used to determine exactly how many terms m are needed to ensure a small enough tail probability p for N_ϵ. See Figure 6.4. As can be seen, as ϵ becomes small the value needed for m increases rapidly.

In the next section we consider some estimating procedures, proposed in recent literatures, by which one can estimate two main parameters α and σ of the option pricing model. We shall apply them when we work with empirical data in Section 6.4 and other examples in Chapter 8.

6.3 Estimating Stable Parameters

Several methods have been proposed for estimating stable parameters. For the index of stability α, one approach is just to plot the empirical distribution function of the observed data on a log-log scale. Recall from Samorodnitsky and Taqqu (1994), page 16, that the asymptotic tail behavior of stable laws is Pareto when $\alpha < 2$, i.e.

$$\lim_{x\to\infty} x^\alpha P(X > x) = c_\alpha(1+\beta)\sigma^\alpha .$$

That means the tail of the empirical distribution function on a log-log scale should approach a straight line with slope $-\alpha$ if the data is stable. However this method is unreliable in practice, because it has not been known when the Pareto tail behavior actually occurs.

The Hill estimator (see Hill (1975)) have been used recently in a number of studies to measure the tail thickness of financial data. Assuming the right tail of a distribution is asymptotically Pareto, i.e., for large x, $1 - F(x) \approx cx^{-\alpha}$ $(\alpha > 0,\ c > 0)$, the Hill estimator attempts to measure tail thickness α. Given a sample of n observations,

$X_1, X_2, \ldots, X_n$, the Hill estimator is given by

$$\hat{\alpha}_{\text{Hill}} = \frac{1}{(1/k)\sum_{j=1}^{k} \ln(X_{n+1-j:n}) - \ln X_{n-k:n}}$$

with standard error

$$\widehat{STD}\left(\hat{\alpha}_{\text{Hill}}\right) = \frac{k\hat{\alpha}_{\text{Hill}}}{(k-1)(k-2)^{1/2}}, \quad k > 2,$$

where $X_{j:n}$ denotes the jth order statistic of sample $X_1, \ldots, X_n$. Because for $0 < \alpha < 2$ the Pareto distribution is in the domain of attraction of the α-stable Paretian law, the Hill estimator is also commonly used to estimate the stable index α.

There are numerous examples in the literature where the Hill estimator delivers tail-index estimates which are greater than 2, causing authors to reject the α-stable hypothesis (e.g. see DuMouchel (1983) and Lux (1995)). To overcome this difficulty Pickands (1975) proposed an alternative tail index estimator of the form

$$\hat{\alpha}_{\text{Pick}} = \frac{\ln 2}{\ln(X_{n-k+1:n} - X_{n-2k+1:n}) - \ln(X_{n-2k+1:n} - X_{n-4k+1:n})}$$

But, under the stable assumptions $\hat{\alpha}_{\text{Pick}}$ yields very poor estimates of index α (see Drees (1996) and Weissman (1996)). Later Mittnik and Rachev (1996a) introduced a modification of Pickands' estimator. This estimator, denoted by $\hat{\alpha}_{\text{UP}}$, is sometimes referred to as the unconditional Pickands tail estimator. It takes the form

$$\hat{\alpha}_{\text{UP}} = \frac{\ln 2}{\ln X_{n-k+1:n} - \ln X_{n-2k+1:n}}.$$

It is obvious that both $\hat{\alpha}_{\text{Hill}}$ and $\hat{\alpha}_{\text{UP}}$ depends on k. In fact it is suggested in Mittnik and Rachev (2000) that these two estimators recover the true value of α for some value of k. They suggest that it is possible to choose an optimal value of k, in terms of a function in n and α for each estimator.

Another approach in approximating the index α, is Maximum Likelihood Estimation which was first developed by DuMouchel (1971). A program for maximum likelihood estimation of general stable parameters is now described in Nolan (2001).

The log likelihood function ℓ, described in Nolan (2001), for an i.i.d. stable sample $X_1, X_2, \ldots, X_n$, is given by

$$\ell(\theta) = \sum_{i=1}^{n} \log f(X_i \,|\, \vec{\theta}),$$

where $\vec{\theta} = (\alpha, \beta, \sigma, \delta_0)$ and $f(x \,|\, \vec{\theta})$ is the density. As we know the difficulty in evaluating ℓ is that there are no known closed formulas for general stable densities. However, As noticed in Nolan (2001), the program STABLE described in Nolan (1997) gives a reliable computations of stable densities for values of $\alpha > 0.1$ and any β, σ and δ_0. That program was improved to give more accurate density calculations on the tails, which found to be necessary for accurate likelihood calculations. The program now includes a fast, pre-computed spline approximation to stable densities for $\alpha \geq 0.4$, routines for maximum likelihood estimation of stable parameters, and diagnostics for assessing the stability of a data set.

Notice that Groeneboom, Lopuhaá and DeWolf (2003) introduce kernel-type estimators which can be used for estimating the extreme value index over the whole (positive and negative) range.

6.4 Illustrations

In this section we present two examples to show that the α-stable model provides a much closer approximation to the empirical densities. Then we use Formula 6.3 to suggest option prices for the given data. Our first example is copied from Nolan (2001).

"**Example 6.4. Exchange Rate Data** Daily exchange rate data for 15 different currencies were recorded (in U.K. Pounds) over a 16 year period (2 January 1980 to 21 May 1996). The data was transformed by $y_t = \ln(x_{t+1}/x_t)$, giving $n = 4,274$ data values. The transformed data was fit with a stable distribution; results are shown in Table 6.3. The data are likely non-stationary over such a time period and there are questions about the dependence in the values. Nevertheless, a naive fit can be done to illustrate the method."

Country	α	β	σ	δ_0
Australia	1.479 ± 0.047	0.033 ± 0.080	0.00413 ± 0.00013	-0.00015 ± 0.00022
Austria	1.559 ± 0.047	-0.119 ± 0.092	0.00285 ± 0.00009	0.00014 ± 0.00015
Belgium	1.473 ± 0.047	-0.061 ± 0.080	0.00306 ± 0.00010	0.00009 ± 0.00016
Canada	1.574 ± 0.047	-0.051 ± 0.093	0.00379 ± 0.00012	0.00004 ± 0.00020
Denmark	1.545 ± 0.047	-0.119 ± 0.090	0.00272 ± 0.00008	0.00022 ± 0.00014
France	1.438 ± 0.047	-0.146 ± 0.078	0.00245 ± 0.00008	0.00028 ± 0.00013
Germany	1.495 ± 0.047	-0.182 ± 0.085	0.00244 ± 0.00008	0.00019 ± 0.00013
Italy	1.441 ± 0.046	-0.043 ± 0.076	0.00266 ± 0.00009	0.00017 ± 0.00014
Japan	1.511 ± 0.047	-0.148 ± 0.086	0.00368 ± 0.00012	0.00013 ± 0.00019
Netherlands	1.467 ± 0.047	-0.167 ± 0.081	0.00244 ± 0.00008	0.00016 ± 0.00013
Norway	1.533 ± 0.047	-0.070 ± 0.088	0.00253 ± 0.00008	0.00005 ± 0.00013
Spain	1.512 ± 0.047	-0.007 ± 0.083	0.00268 ± 0.00008	0.00012 ± 0.00014
Sweden	1.517 ± 0.047	-0.081 ± 0.085	0.00256 ± 0.00008	0.00006 ± 0.00013
Switzerland	1.599 ± 0.047	-0.179 ± 0.100	0.00295 ± 0.00009	0.00014 ± 0.00016
United States	1.530 ± 0.047	-0.088 ± 0.088	0.00376 ± 0.00012	0.00009 ± 0.00020

Table 6.3. Exchange rate analysis. Parameter estimate and 95% confidence intervals with sample size of $n = 4274$.

As another illustration we report on an example of time series that is given in the following:

Example 6.5. NASDAQ Stock Composite Consider the daily NASDAQ Stock Composite from January 5, 1998 to December 9 2003, with sample size $n = 1493$. In Example 4.3 of we mentioned that the normal model does not provide a closer approximation to the empirical density of the return.

We claim that the α-stable model provides a much closer approximation to the empirical densities. To show this we estimate α by either calculating $\alpha_{\hat{UP}}$ or applying Nolan's STABLE program. The estimated value is $\alpha \approx \hat{\alpha}_{UP} = 1.46$. Nolan's program can also be used to estimate the scale parameter σ which is $\sigma \approx 0.42$, we observe in Figure 6.4 that $S_{1.46}(0.42, 0, 0)$ is fitted with the empirical density of the return.

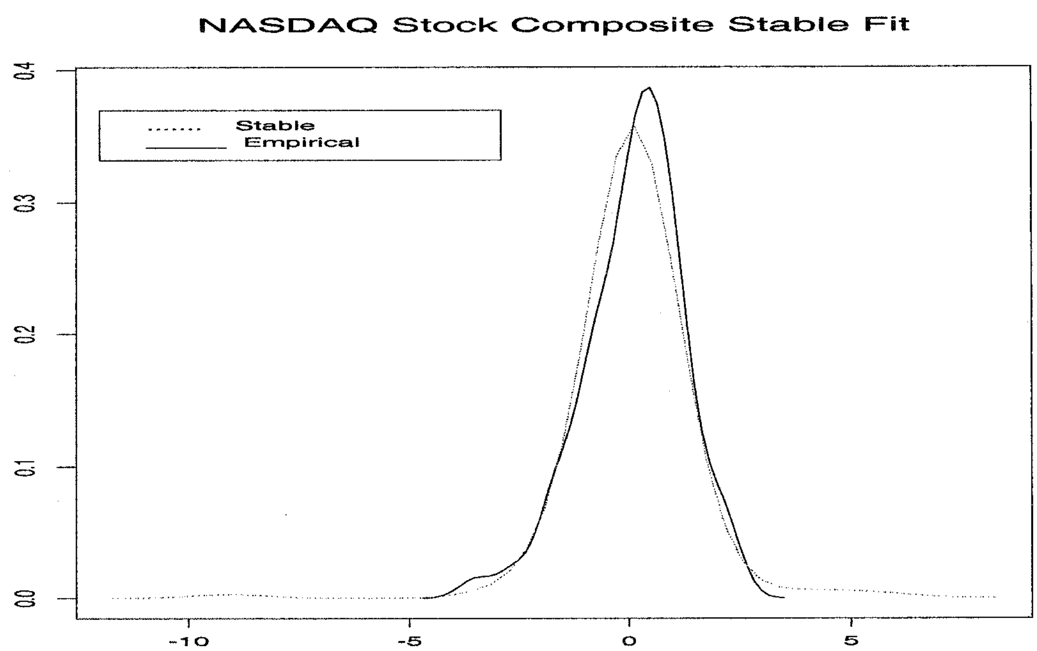

Figure 6.5 Fitted Stable Density.

In practice we may attempt to calculate the call option after finding suitable estimates of α and σ. But, this may not be easy, as we may have only one sample and we can not generate a sample of suitable size for the ratio inside expectation, in the formula for option price, to estimate the call option. In the next chapter we show that a nonparametric bootstap procedure can help us to overcome this difficulty.

Chapter 7

Some Properties for Option Pricing Formula

By studying the properties of the numerator and denominator of the option pricing formula we can gain further insight into why the estimator $\widehat{C_N}$ works so poorly. We use the recursive method of Banjevic, Ishwaran and Zarepour (2002) for this purpose.

7.1 More on Variability of C

Our first result shows that the random variable

$$D = \prod_{i=1}^{\infty} \cosh\left(\sigma \Gamma_i^{-1/\alpha}\right)$$

in the denominator possesses no positive finite moments. This again explains the variability of C seen in Figure 6.1.

Lemma 7.1. *We have*

$$\exp\left[-\int_0^\infty \left(1 - \left(\cosh\frac{\sigma}{v^{1/\alpha}}\right)^u\right) dv\right] = \begin{cases} < +\infty & \text{if } \ u \leq 0 \\ +\infty & \text{if } \ u > 0 \end{cases},$$

where $0 < \alpha < 2$, $\sigma \in R$.

Proof. Let $u > 0$. Since

$$\frac{\sigma}{v^{1/\alpha}} > 2\ln 2 \Leftrightarrow v < \left(\frac{\sigma}{2\ln 2}\right)^{\alpha},$$

if we put $a = \left(\frac{\sigma}{2\ln 2}\right)^{\alpha}$ we have,

$$\int_0^{\infty} \left[\left(\cosh \frac{\sigma}{v^{1/\alpha}}\right)^{u} - 1\right] dv = \int_0^{a} \left[\left(\cosh \frac{\sigma}{v^{1/\alpha}}\right)^{u} - 1\right] dv + B,$$

where

$$B = \int_a^{\infty} \left[\left(\cosh \frac{\sigma}{v^{1/\alpha}}\right)^{u} - 1\right] dv.$$

Since

$$(\cosh w)^u - 1 \geq \left(1 + \frac{w^2}{2}\right)^u - 1$$

and since

$$\left(1 + \frac{w^2}{2}\right)^u - 1 \geq \frac{uw^2}{2} \quad \text{whenever } u > 0$$

we have

$$\int_0^{\infty} \left[\left(\cosh \frac{\sigma}{v^{1/\alpha}}\right)^{u} - 1\right] dv \geq \frac{\sigma^2 u}{2} \int_0^{a} v^{-2/\alpha}\, dv + B.$$

Now observe that

$$\frac{u\,\sigma^2}{2} \int_0^{a} v^{-2/\alpha}\, dv = A + \frac{u\alpha\sigma^2}{2(2-\alpha)} \lim_{v \to 0^+} v^{\frac{\alpha-2}{\alpha}},$$

where $A = \frac{u\alpha\sigma^2}{2(2-\alpha)} \left(\frac{\sigma}{2\ln 2}\right)^{\frac{\alpha-2}{\alpha}}$. Since $0 < \alpha < 2$, we have

$$\frac{u\alpha\sigma^2}{2(2-\alpha)} \lim_{v \to 0^+} v^{\frac{\alpha-2}{\alpha}} = +\infty.$$

Therefore,

$$\int_0^{\infty} \left[\left(\cosh \frac{\sigma}{v^{1/\alpha}}\right)^{u} - 1\right] dv = +\infty,$$

as A is a finite real number and $B > 0$.

Now let $u < 0$. Since $\cosh \frac{\sigma}{v^{1/\alpha}} > 1$, we have $\left(\cosh \frac{\sigma}{v^{1/\alpha}}\right)^u < 1$. Hence,

$$\int_0^{\infty} \left(1 - \left(\cosh \frac{\sigma}{v^{1/\alpha}}\right)^u\right) dv \geq 0.$$

This shows that

$$\exp\left[-\int_0^\infty \left(1-\left(\cosh\frac{\sigma}{v^{1/\alpha}}\right)^u\right)\,dv\right] = M\,,$$

where M is either 0 or a positive real number less than 1. ☐

Theorem 7.1. *The u^{th} moment of the random variable D is of the form*

$$\mu_D(u) = E(D^u) = \exp\left[-\int_0^\infty \left(1-\left(\cosh\frac{\sigma}{v^{1/\alpha}}\right)^u\right)\,dv\right].$$

Furthermore, $\mu_D(u)$ is finite whenever $u \le 0$. It is infinite whenever $u > 0$.

Proof. For each $u \in R$ and $t > 0$ put $K(u,t) = E(D(t)^u)$, where

$$D(t) = \prod_{i=1}^{\infty} \cosh\frac{\sigma}{(\Gamma_i + t)^{1/\alpha}}\,.$$

Then uth moment exists if and only if

$$K(u,t) = E(E(D(t)^u\,|\,\Gamma_1)) = \int_0^\infty E(D(t)^u\,|\,\Gamma_1 = x)\,e^{-x}dx\,.$$

Therefore, this is equivalent to

$$\begin{aligned}
K(u,t) &= \int_0^\infty e^{-x} E\left(\left(\prod_{i=1}^{\infty}\cosh\frac{\sigma}{(\Gamma_i+t)^{1/\alpha}}\right)^u \,\Big|\,\Gamma_1 = x\right)dx \\
&= \int_0^\infty e^{-x}\left(\cosh\frac{\sigma}{(x+t)^{1/\alpha}}\right)^u E\left(\left(\prod_{i=1}^{\infty}\cosh\frac{\sigma}{(\Gamma_i+x+t)^{1/\alpha}}\right)^u\right)dx \\
&= \int_0^\infty e^{-x}\left(\cosh\frac{\sigma}{(x+t)^{1/\alpha}}\right)^u K(u, x+t)\,dx \\
&= \int_t^\infty e^{t-v}\left(\cosh\frac{\sigma}{v^{1/\alpha}}\right)^u K(u,v)dv
\end{aligned}$$

Hence,

$$e^{-t}K(u,t) = \int_t^\infty e^{-v}\left(\cosh\frac{\sigma}{v^{1/\alpha}}\right)^u K(u,v)dv\,.$$

We now differentiate both sides of the above equation with respect to t to obtain

$$e^{-t}\frac{d}{dt}K(u,t) - e^{-t}K(u,t) = -\,e^{-t}\left(\cosh\frac{\sigma}{v^{1/\alpha}}\right)^u K(u,t)\,,$$

or equivalently,

$$\frac{d}{dt}K(u,t) - K(u,t) \;=\; -\left(\cosh\frac{\sigma}{v^{1/\alpha}}\right)^{u} K(u,t)\,.$$

Notice that, by the monotone convergence theorem, K is a continuous function with respect to the parameter t and hence the above differentiation is possible by the Fundamental Theorem of the Calculus.

By solving the above differential equation we derive the following expression for $K(u,t)$:

$$K(u,t) \;=\; A\,\exp\left[-\int_t^{\infty}\left(1-\left(\cosh\frac{\sigma}{v^{1/\alpha}}\right)^{u}\right)dv\right],$$

and by using the fact that $\lim_{t\to\infty} K(u,t) \;=\; 1$ we conclude that $A=1$. Hence,

$$\mu_D(u) \;=\; K(u,0) \;=\; \exp\left[-\int_0^{\infty}\left(1-\left(\cosh\frac{\sigma}{v^{1/\alpha}}\right)^{u}\right)dv\right],$$

by letting $t\downarrow 0$. Therefore,

$$\mu_D(u) \;=\; +\infty \qquad \text{for all} \quad u>0\,,$$

and

$$\mu_D(u) \;<\; +\infty \qquad \text{for all} \quad u\le 0\,,$$

by Lemma 7.1. □

7.2 Characteristic Functions of Expressions in C

In this section we use the recursion technique used in the proof of Theorem 7.1 to derive the joint characteristic functions of major components of C.

Theorem 7.2. *Let* $Y \;=\; \sigma\sum_{i=1}^{\infty}\delta_i\Gamma_i^{-1/\alpha}$ *and* $Z \;=\; \sum_{i=1}^{\infty}\ln\left(\cosh\sigma\Gamma_i^{-1/\alpha}\right)$ *. Then the joint characteristic function* $\phi_{Y,Z}(\theta_1,\theta_2)$ *of* Y *and* Z *is*

$$\phi_{Y,Z}(\theta_1,\theta_2) \;=\; \exp\left\{\int_0^{\infty}\left[\exp\left(i\theta_2\ln\cosh\sigma\,t^{-1/\alpha}\right)\cos\left(\sigma\,\theta_1 t^{-1/\alpha}\right) \;-\; 1\right]dv\right\}.$$

Proof. Let

$$Y(t) = \sigma \sum_{i=1}^{\infty} \delta_i(\Gamma_i + t)^{-1/\alpha} \quad \text{and} \quad Z(t) = \sum_{i=1}^{\infty} \ln(\cosh \sigma(\Gamma_i + t)^{-1/\alpha})$$

for each $t > 0$. Let $\phi_{Y,Z}(\theta_1, \theta_2, t)$ denote the joint characteristic function for the pair $(Y(t), Z(t))$. Conditioning on Γ_1, and using a similar style of argument as in the proof of Theorem 7.1, it follows that

$$\begin{aligned} &\exp(-t)\phi_{Y,Z}(\theta_1, \theta_2, t) \\ &= \frac{1}{2}\int_t^{\infty} \exp(-v) \exp\left[i\sigma\,\theta_1(v+t)^{-1/\alpha} + i\theta_2 \ln\cosh\sigma\,(v+t)^{-1/\alpha}\right] \phi_{Y,Z}(\theta_1, \theta_2, v)\,dv \\ &+ \frac{1}{2}\int_t^{\infty} \exp(-v) \exp\left[-i\sigma\,\theta_1(v+t)^{-1/\alpha} + i\theta_2 \ln\cosh\sigma\,(v+t)^{-1/\alpha}\right] \phi_{Y,Z}(\theta_1, \theta_2, v)dv\,, \end{aligned}$$

where the value of $\frac{1}{2}$ outside the two integrals is due to conditioning on $\delta_1 = \pm 1$. Differentiate both sides of the above equation with respect to t to get

$$\frac{\partial}{\partial t}\phi_{Y,Z}(\theta_1, \theta_2, t) = \phi_{Y,Z}(\theta_1, \theta_2, t)\left\{\exp\left(i\theta_2 \ln\cosh\sigma\, t^{-1/\alpha}\right)\cos\left(\sigma\,\theta_1 t^{-1/\alpha}\right) - 1\right\}.$$

Since $\lim_{t\to\infty}\phi_{Y,Z}(\theta_1, \theta_2, t) = 1$, the unique solution to this differential equation is

$$\phi_{Y,Z}(\theta_1, \theta_2, t) = \exp\left\{\int_t^{\infty}\left[\exp\left(i\theta_2 \ln\cosh\sigma\, t^{-1/\alpha}\right)\cos\left(\sigma\,\theta_1 t^{-1/\alpha}\right) - 1\right]dv\right\}.$$

Set $t = 0$ to obtain the desired result. □

Notice that Theorem 7.2 gives a closed form expression for the characteristic function of $\log D$, where $D = \prod_{i=1}^{\infty}\cosh\left(\sigma\Gamma_i^{-1/\alpha}\right)$. In fact:

Corollary 7.1. *Let Y and Z be those random variables defined in Theorem 7.2. Then*

$$\phi_Z(\theta) = \exp\left\{\int_0^{\infty}\left(\left[\cosh(\sigma v^{-1/\alpha})\right]^{i\theta} - 1\right)dv\right\}$$

and

$$\phi_Y(\theta) = \exp\left\{\int_0^{\infty}\left[\cos\left(\sigma\theta u\right) - 1\right]dN_\alpha(u)\right\},$$

where

$$dN_\alpha(u) = \alpha u^{-(\alpha+1)}du\,.$$

Proof. Since $\phi_X(\theta) = \phi_{Y,Z}(0,\theta)$, from Theorem 7.2 we have

$$\phi_Z(\theta) = \exp\left\{\int_0^\infty \left[\exp\left(i\theta \ln\cosh\sigma\, t^{-1/\alpha}\right) - 1\right] dv\right\}.$$

But

$$\exp\left\{i\theta\ln(\cosh(\sigma v^{-1/\alpha}))\right\} = \left[\cosh(\sigma v^{-1/\alpha})\right]^{i\theta},$$

and that means

$$\phi_Z(\theta) = \exp\left\{\int_0^\infty \left(\left[\cosh(\sigma v^{-1/\alpha})\right]^{i\theta} - 1\right) dv\right\}.$$

Also since $\phi_Y(\theta) = \phi_{Y,Z}(\theta,0)$, another application of Theorem 7.2 yields

$$\phi_Y(\theta) = \exp\left\{\int_0^\infty \left[\cos\left(\sigma\,\theta v^{-1/\alpha}\right) - 1\right] dv\right\}.$$

Using integration by substitution, we can rewrite this as

$$\phi_Y(\theta) = \exp\left\{\int_0^\infty [\cos(\sigma\theta u) - 1]\, dN_\alpha(u)\right\},$$

where

$$dN_\alpha(u) = \alpha u^{-(\alpha+1)} du.$$

□

Remark 7.1. As is well known, the characteristic function of Y shows that Y is a symmetric stable law. It is also clear that

$$Y_N = \sigma\sum_{i=1}^{N}\delta_i\Gamma_i^{-1/\alpha} \xrightarrow{a.s.} Y \quad \text{as } N \to \infty.$$

7.3 A Differential Equation for C

The following theorem gives a differential equation that can be interpreted as another representation of the option price formula.

Theorem 7.3 *for $t > 0$ put*

$$I(K,t) = E\left[\frac{\left(S_0 \exp\left(\sum_{j=1}^{\infty} \delta_j \sigma(\Gamma_j + t)^{-1/\alpha}\right) - K\right)_+}{\Pi_{j=1}^{\infty} \cosh \sigma(\Gamma_j + t)^{-1/\alpha}}\right].$$

Then the function I satisfies the following differential equation:

$$\frac{\partial}{\partial t} I(K,t) = I(K,t) - \frac{1}{2}\left[I(K_1(t),t) + I(K_2(t),t)\right]$$

where $K_1(v) = \frac{K}{\exp(\sigma v^{\frac{-1}{\alpha}})}$, *and* $K_2(v) = \frac{K}{\exp(-\sigma(v^{-\frac{1}{\alpha}}))}$(Note: $I(K,0) = C$).

Proof . Let

$$X(t) = S_0 \exp\left(\sum_{j=1}^{\infty} \delta_j \sigma(\Gamma_j + t)^{-1/\alpha}\right),$$

$$Y(t) = \Pi_{j=1}^{\infty} \cosh \sigma(\Gamma_j + t)^{-1/\alpha},$$

and

$$W(t) = \frac{(X(t) - K)_+}{Y(t)}.$$

Then $I(K,t) = E(W(t))$ and hence,

$$I(K,t) = \int_0^{\infty} P(W(t) > w)\, dw.$$

Therefore, if we condition on Γ_1, and use a similar style of argument as in the proof of Theorem 7.1 once again, we derive

$$I(K,t) = \int_0^{\infty}\int_0^{\infty} P\left[\frac{\left(\left(\exp \delta_1 \sigma(x+t)^{\frac{-1}{\alpha}}\right) X(x+t) - K\right)_+}{\left(\cosh \sigma(x+t)^{\frac{-1}{\alpha}}\right) Y(x+t)} > w\right] e^{-x} dx dw.$$

Thus if we put $x + t = v$ we get

$$I(K,t) =$$

$$\frac{1}{2}\int_0^{\infty}\int_t^{\infty} \left\{P\left[\frac{(X(v) - K_1(v))_+}{Y(v)} > y\right] + P\left[\frac{(X(v) - K_2(v))_+}{Y(v)} > z\right]\right\} e^{t-v} dv dw$$

where $y = \frac{w \cosh\left(\sigma v^{-\frac{1}{\alpha}}\right)}{\exp(\sigma v^{-\frac{1}{\alpha}})}$ and $z = \frac{w \cosh\left(\sigma v^{-\frac{1}{\alpha}}\right)}{\exp(-\sigma(v^{-\frac{1}{\alpha}}))}$. Using simple calculations, we observe that:

$$I(K,t) = \frac{1}{2}\int_t^{\infty} \left[I(K_1(v),v) + I(K_2(v),v)\right] e^{t-v} dv$$

or equivalently

$$I(K,t)\,e^{-t} = \frac{1}{2}\int_t^{\infty} \left[I(K_1(v)\,,v) + I(K_2(v)\,,v)\right] e^{-v} dv$$

We now differentiate both sides of the above equation with respect to t to get

$$\frac{\partial}{\partial t} I(K,t) = I(K,t) - \frac{1}{2}\left[I(K_1(t)\,,t) + I(K_2(t)\,,t)\right] ,$$

as desired. □

A close form solution seems to be impossible. A numerical solution may lead us to better understanding of the behavior of C, but this is a challenging problem and we do not address it here.

Chapter 8

Nonparametric Bootstrap for Option Pricing

As we explained in the previous two chapters the option pricing formulae derived in Rachev and Samarodnitsky(1993) and LePage, Podgórski, and Ryznar (1997), and even their approximations, may be difficult to apply in practice. In this chapter we use the properties of the bootstrap for the mean of heavy-tailed distributions to introduce another formula for option pricing. Our development will once again rely on point process methods using a result analogous to Lemma 5.2. As we shall see, this simple and easy to implement nonparametric bootstrap method is an alternative approach in estimating call options in practice.

8.1 Basic Convergence Lemmas

In this section we recall two lemmas based on which we derive another formula for option pricing. The first lemma, which follows from Zarepour and Knight (1999), is about the weak limit of the conditional point process $\sum_{k=1}^{n} \varepsilon_{a_n^{-1} X_k^*}(\cdot)$ for large value of n, where $\{X_k^*\}$ is a bootstrap sample.

Lemma 8.1. *Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables with a heavy tailed distribution F satisfying Equation* 5.1. *Let $\hat{F}_n$ be the empirical distribution based* *on the sample $X_1, X_2, \ldots, X_n$ and let $X_1^*, X_2^*, \ldots, X_n^*$ be a bootstrap sample of size $m = m(n)$ from $\hat{F}_n$. If $m = n$, then*

$$\sum_{i=1}^{m} \varepsilon_{a_m^{-1} X_k^*}(\cdot) \xrightarrow{d} \sum_{i=1}^{\infty} M_i^* \, \varepsilon_{\delta_i \Gamma_i^{-1/\alpha}}(\cdot) \quad \text{in distribution}$$

where $\{M_i^*\}$ are i.i.d. with a Poisson distribution such that $E(M_i^*) = 1$.

The second lemma has been proven by Feigin and Resnick (1997) and Zarepour and Knight (1999). This is also about the weak limit of a conditional point process in a slightly different way.

Lemma 8.2. *Assume all the conditions stated in Lemma 8.1. If $m \to \infty$ such that $m/n \to 0$ (i.e. $m = o(n)$), then*

$$\sum_{i=1}^{m} \varepsilon_{a_m^{-1} X_k^*}(\cdot) \xrightarrow{d} \sum_{i=1}^{\infty} \varepsilon_{\delta_i \Gamma_i^{-1/\alpha}}(\cdot) \quad \text{in probability.}$$

If $\frac{m \log\log n}{n} \to 0$, then in the above, the convergence in distribution holds for almost all sample paths.

Note. In practice the term $\log\log n$ does not play an important role, as for example $\log\log 10^6 \approx 2.67$. Therefore, from now on we only assume $m = o(n)$.

8.2 Main Result

Using Lemma 8.1 and Lemma 8.2, and applying the same techniques used in the proof of Theorem 5.1, we can establish the following new option pricing formula for the nonparametric bootstrap.

Theorem 8.1. *Let $\{X_i\}$ be a sequence of i.i.d. random variables from the domain of attraction of an α symmetric stable law for $0 < \alpha < 2$. If $m = n$, then $C_n^* \to C^*$ where*

$$C_n^* = E\left\{ \frac{[S_0 \exp(\sigma a_m^{-1} \sum_{i=1}^{m} X_i^*) - K]_+}{\prod_{i=1}^{m} \cosh(\sigma a_m^{-1} X_i^*)} \right\}$$

and

$$C^* = E\left\{\frac{\left[S_0 \exp\left(\sigma \sum_{i=1}^{\infty} M_i^* \delta_i \Gamma_i^{-1/\alpha}\right) - K\right]_+}{\prod_{i=1}^{\infty}\left(\cosh\left(\sigma \Gamma_i^{-1/\alpha}\right)\right)^{M_i^*}}\right\},$$

where M_i^* is defined as in Lemma 8.1. Furthermore if $m = o(n)$, then the previous result holds with M_i^* replaced by 1. That is, $C_n^* \to C$ for C defined as before, i.e.,

$$C = E\left\{\frac{\left[S_0 \exp\left(\sigma \sum_{i=1}^{\infty} \delta_i \Gamma_i^{-1/\alpha}\right) - K\right]_+}{\prod_{i=1}^{\infty} \cosh\left(\sigma \Gamma_i^{-1/\alpha}\right)}\right\}.$$

Notice that $\{M_{i,n}\}$ and $\{X_{i,n}\}$ are independent. Also $\{M_i\}$, $\{\Gamma_i\}$, and $\{\delta_i\}$ are mutually independent.

Before we prove Theorem 8.1 we need to recall the following generalization of Skorohod Representation Theorem (cf. Kallenberg (1997) Theorem 3.30). The importance of this theorem is that it relates convergence in distribution to almost sure convergence.

(Coupling, Skorohod, Dudley) *Let $\zeta, \zeta_1, \zeta_2, \ldots$ be random elements in a separable metric space (S, ρ) such that $\zeta_n \xrightarrow{d} \zeta$. Then, on a suitable probability space, there exist some random elements $\eta \stackrel{d}{=} \zeta$ and $\eta_n \stackrel{d}{=} \zeta_n$, $n \in N$, with*

$$\eta_n \longrightarrow \eta \quad a.s..$$

Proof of Theorem 8.1. To prove that $C_n^* \to C^*$ when $m = n$, first notice that by the definition of the bootstrap, and by Lemma 8.1,

$$\sum_{i=1}^{n} \varepsilon_{a_n^{-1} X_k^*}(\cdot) \stackrel{d}{=} \sum_{i=1}^{n} M_{i,n}^* \varepsilon_{\delta_i a_n^{-1} G^{-1}(\Gamma_i/\Gamma_{n+1})}(\cdot)$$

where $G(x) = P\{|X_1| > x\}$ and $(M_{1,n}^*, \ldots, M_{n,n}^*)$ is a multinomial random vector from n trials in which each multinomial cell has probability $1/n$ of occurring. It is well known (see Knight (1989) for example) that

$$(M_{1,n}^*, \ldots, M_{n,n}^*, 0, 0, \ldots) \xrightarrow{d} (M_1^*, M_2^*, \ldots).$$

Use Lemma 8.1 and the Skorohod Representation Theorem to find a probability space such that

$$(M^*_{1,n}, \ldots, M^*_{n,n}) \stackrel{d}{=} (M_{1,n}, \ldots, M_{n,n}),$$

$$(M^*_1, M^*_2, \ldots) \stackrel{d}{=} (M_1, M_2, \ldots),$$

$$(M_{1,n}, \ldots, M_{n,n}, 0, 0, \ldots) \stackrel{a.s}{\rightarrow} (M_1, M_2, \ldots)$$

and

$$\sum_{i=1}^{n} \varepsilon_{a_n^{-1} X_i^*}(\cdot) \stackrel{d}{=} \sum_{i=1}^{n} M_{i,n}\, \varepsilon_{\delta_i a_n^{-1} G^{-1}(\Gamma_i/\Gamma_{n+1})}(\cdot)$$

$$\stackrel{a.s.}{\longrightarrow} \sum_{i=1}^{\infty} M_i \varepsilon_{\delta_i \Gamma_i^{-1/\alpha}}(\cdot) \stackrel{d}{=} \sum_{i=1}^{\infty} M_i^* \varepsilon_{\delta_i \Gamma_i^{-1/\alpha}}(\cdot)\,.$$

Let $\nu_n(\cdot)$ and $\nu(\cdot)$ denote the second and third terms in the previous expression. Using a similar argument as in the proof of Corollary 5.1, one can show that

$$\prod_{i=1}^{n} \cosh(\sigma a_n^{-1} X_i^*) \stackrel{d}{=} \prod_{i=1}^{n} \left(\cosh\left(\sigma a_n^{-1} G^{-1}(\Gamma_i/\Gamma_{n+1})\right)\right)^{M_{i,n}} \stackrel{a.s.}{\longrightarrow} \prod_{i=1}^{n} \left(\cosh\left(\sigma \Gamma_i^{-1/\alpha}\right)\right)^{M_i}$$

Now argue in a similar fashion as in the proof of Theorem 5.1 to conclude that $C_n^* \rightarrow C^*$.

Finally, to show that $C_n^* \rightarrow C$ when $\frac{m \log\log n}{n} \rightarrow 0$ use Lemma 8.2 and the Skorohod Representation Theorem to find i.i.d. random variables $\{Y_i\}$ such that

$$\sum_{i=1}^{m} \varepsilon_{a_m^{-1} X_i^*}(\cdot) \stackrel{d}{=} \sum_{i=1}^{m} \varepsilon_{a_m^{-1} Y_i}(\cdot) \stackrel{a.s.}{\longrightarrow} \sum_{i=1}^{\infty} \varepsilon_{\delta_i \Gamma_i^{-1/\alpha}}(\cdot)\,.$$

Arguing in the same manner as above, deduce that $C_n^* \rightarrow C$. □

8.3 Illustration

If $X_{i,n}$ is the log stock price difference, then $X_{i,n} = a_n^{-1} X_i$. By Theorem 8.1 to get a correct limit we must use a bootstrap of $m = o(n)$ values of the data, $X^*_{1,n}, \ldots, X^*_{m,n}$. Therefore, since

$$a_m^{-1} X_i^* = a_n a_m^{-1} X_{i,n}^*\,,$$

we have

$$C_n^* = E\left\{\frac{\left[S_0 \exp\left(\sigma a_n\, a_m^{-1} \sum_{i=1}^m X_{i,n}^*\right) - K\right]_+}{\prod_{i=1}^m \cosh\left(\sigma a_n\, a_m^{-1} X_{i,n}^*\right)}\right\} \to C\,.$$

Thus to get a bootstrap approximation for C we compute the fraction on the left-hand side for each bootstrap sample and average these values over several different samples. Note that this method works without requiring a closed form expression for G^{-1}.

Example 8.1. Consider a European call option on the British pound. Suppose that the current exchange rate is 1.6200, the strike is 1.6000. With $\alpha = 1.46$ and a volatility of %10 and applying Program 8 in Appendix A we calculate the approximated option price as follows:

- We generate a random sample of size $n = 400,000$ of the symmetric α-stable law (i.e. of the $S_{1.46}(1, 0, 0)$).
- For various integers $m \in [70, 105]$ we compute the fraction on the left-hand side for each bootstrap sample and average these values over several different samples.
- The results of this simulation, that are summarized in Table 8.1 and Figure 8.1, show that most of the time a bootstrap approximation is in [1.140, 1.146].

m	70	71	72	73	74	75	76	77	78
C_n^*	0.145	0.147	0.142	0.143	0.146	0.151	0.143	0.141	0.136
m	79	80	81	82	83	84	85	86	87
C_n^*	0.143	0.145	0.146	0.144	0.148	0.144	0.146	0.150	0.146
m	88	89	90	91	92	93	94	95	96
C_n^*	0.145	0.147	0.143	0.144	0.141	0.145	0.146	0.139	0.144
m	97	98	99	100	101	102	103	104	105
C_n^*	0.142	0.143	0.141	0.138	0.144	0.145	0.144	0.144	0.142

Table 8.1. Approximated option price on the British pound.

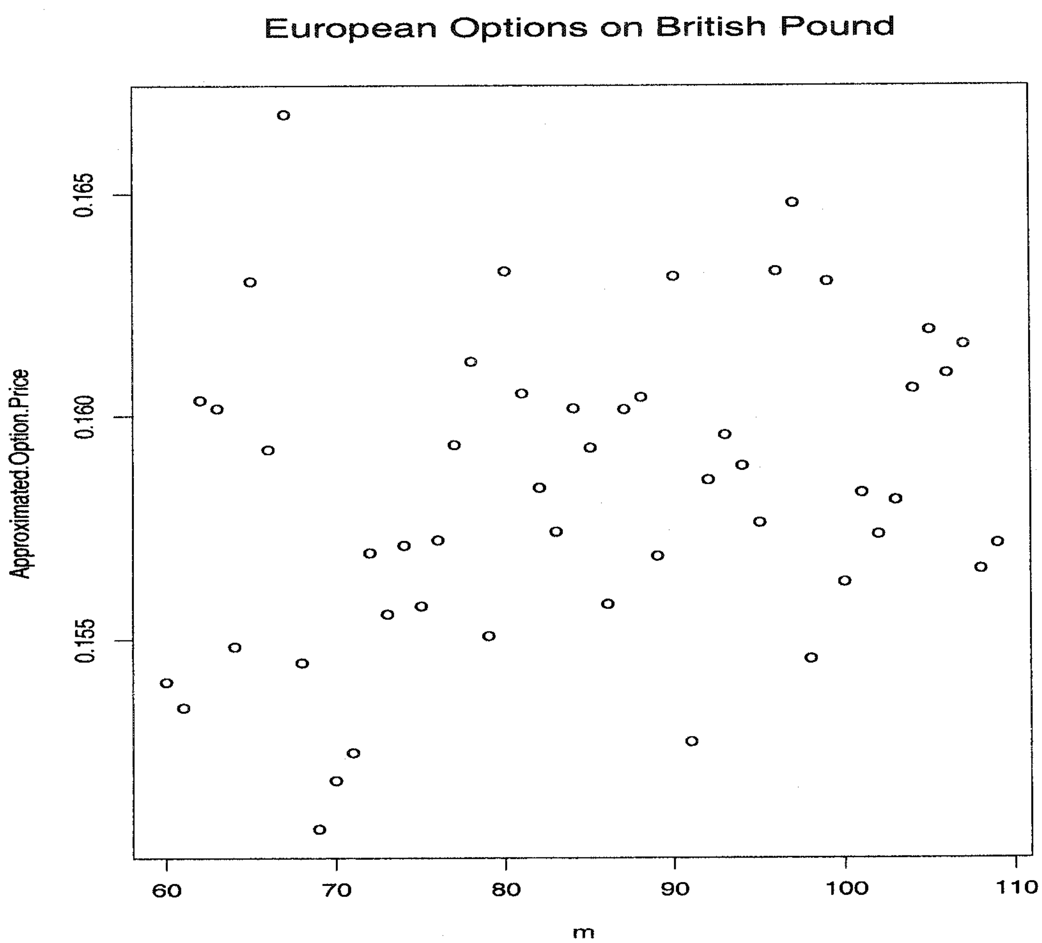

Figure 8.1 A bootstrap approximation for C on British Pound.

Example 8.2. In this example we show how to estimate the option price from financial data using the m-out-of-n-bootstrap. For our example we use the closing price of the Walt Disney Company for trading dates starting from January 3, 1984 to March 15, 2004. We use the standard convention and convert the data to returns by taking the log difference of the closing price on subsequent trading days and multiplying this value by 100. That is, the return at time i is defined as

$$X_i = 100 \times \log\left(\frac{P_i}{P_{i-1}}\right)$$

where P_i is the price of the stock at time i. The total sample size, the number of returns, was $n = 5097$.

We analyze the data using the bootstrap described in Theorem 8.1. We also consider a modified bootstrap technique, defined similarly, but in place of the real data uses what we call *estimate data*; i.e. data that can be used as a reasonable facsimile. The modified procedure is useful because it allows us to choose the sample size at our convenience. As we show later this will help us to gauge the accuracy of the bootstrap. For estimated data we used randomly sampled values from a symmetric α-stable law with $\alpha = 1.45$. The use of a stable law was suggested by a careful analysis of the data, while the value $\alpha = 1.45$ was obtained using the unconditional Pickands' estimator $\hat{\alpha}_{UP}$ (see Section 6.4 and Program 7 in Appendix A). Figure 8.2 shows that this approximation is very reasonable.

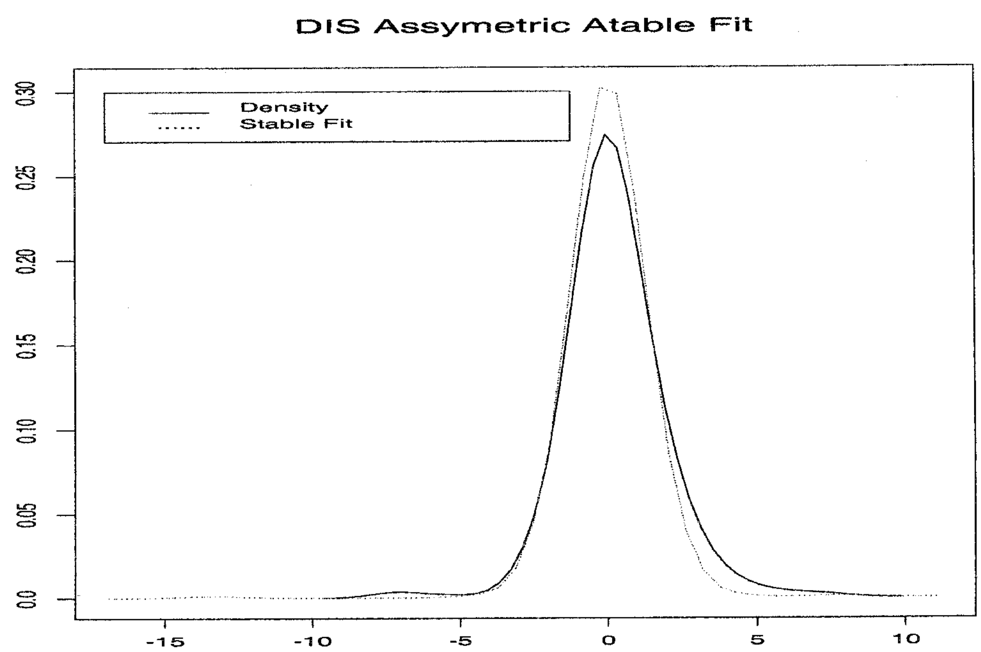

Figure 8.2 Density for Disney stock price data versus symmetric stable law density with $\alpha = 1.45$.

The volatility parameter σ was estimated using stock prices from the last 150 trading days. This follows the common practice of estimating volatility using the last 90 to 180 days of return. While it is preferable to use more data for estimation, it is recognized that volatility can be time dependent and that using stock prices too far in the past can reduce accuracy. Using a software written by Nolan (2000b), described in Section 6.5, we found σ to be 0.34, which was the value we used in the option pricing formula. We also took $S_0 = \$25.33$ and $K = \$20.00$. The value for S_0 represents the closing price of the stock on the last trading day in our data set, March 15, 2004, while K was the strike price offered on a call option the same day.

In implementing the bootstrap we used $a_m = m^{1/\alpha}$ with $\alpha = 1.45$, the normalizing constant for a symmetric $\alpha = 1.45$ stable law. For the estimated data this is the exact normalizing constant, while for the real data this should be reasonably accurate given that the data is well approximated by this law. Figure 8.3 and Figure 8.4 present boxplots for the bootstrap values C_n^* using the two different methods. For each procedure the value $m = m(n)$ was varied from $n^{1/3}$ to $n^{1/2}$. In each case $B = 10,000$ bootstrap samples were used. Figure 8.3 used the real data, while Figure 8.4 is based on the estimated data comprising $n = 50,000$ randomly sampled values from the target stable law. The sample size for the estimated data is roughly 10 times that of the real data. Both plots appear similar in that they show a distinct bimodal distribution, however the bootstrap values from the estimated data exhibit more variability.

Figure 8.5 records the estimated option price as a function of $m(n)$ for both procedures. It is immediately clear that the option price values are much smaller using the real data. In light of the difference in variability seen in Figure 8.3 and Figure 8.4 this is perhaps not too surprising. To be able to assess which of the two estimates are more accurate we estimated the option price using an i.i.d. Monte Carlo approach. This estimate was computed by drawing $X_1, \ldots, X_n$ independently from a symmetric $\alpha = 1.45$ stable law and then averaging

$$R_n = \left\{ \frac{[S_0 \exp(\sigma\, a_n^{-1} \sum_{i=1}^n X_i) - K]_+}{\prod_{i=1}^n \cosh(\sigma a_n^{-1} X_i)} \right\}$$

over B independent such draws. We used two different values for n: (a) the sample size for the real data ($n = 5097$), (b) the sample size for the estimated data ($n = 50,000$).

Estimated values for C using this approach are indicated in Figure 8.5 by arrows, being roughly equal to \$ 9.70 and \$ 10.00 for cases (a) and (b) respectively. Both

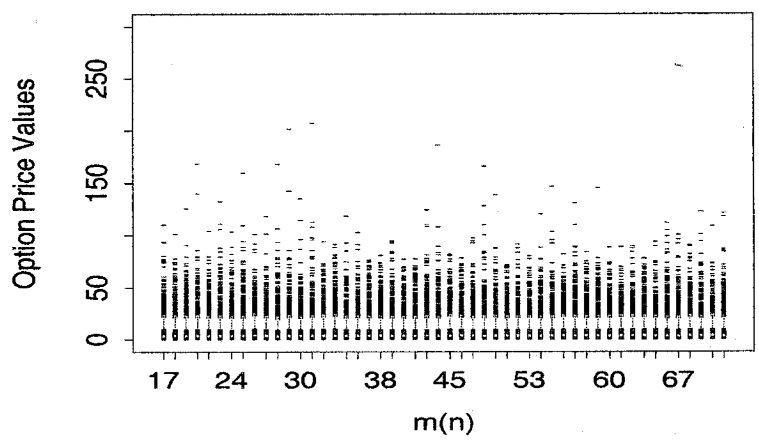

Figure 8.3 Boxplots of sampled bootstrap option pricing values for Disney stock price data using original data. Notice the bimodal nature of the sampled values. Bootstrap sizes $m(n)$ varied from $n^{1/3}$ to $n^{1/2}$. See also Figures 6.2 and 6.3.

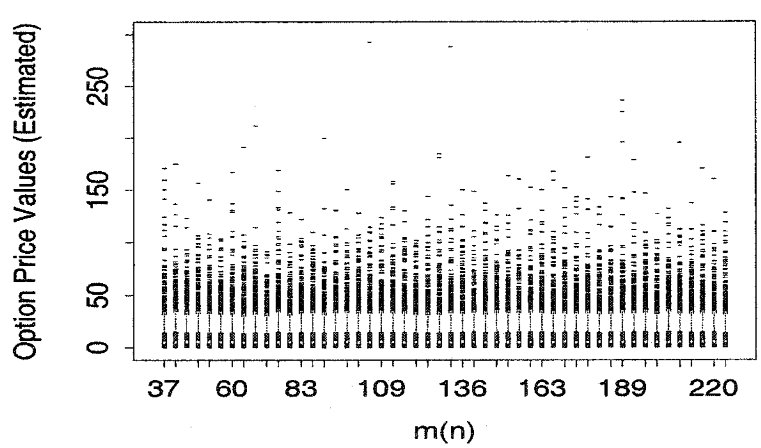

Figure 8.4 Boxplots of sampled bootstrap option pricing values for Disney stock price data using $n = 50,000$ simulated data values from $\alpha = 1.45$ symmetric stable law. Again notice the bimodal nature of the sampled values. Bootstrap sizes $m(n)$ varied from $n^{1/3}$ to $n^{1/2}$.

values agree closely with the modified bootstrap based on the estimated data, showing that the bootstrap is working accurately, and that it is accurate even with sample sizes on the order of a few thousand (as in case (a)).

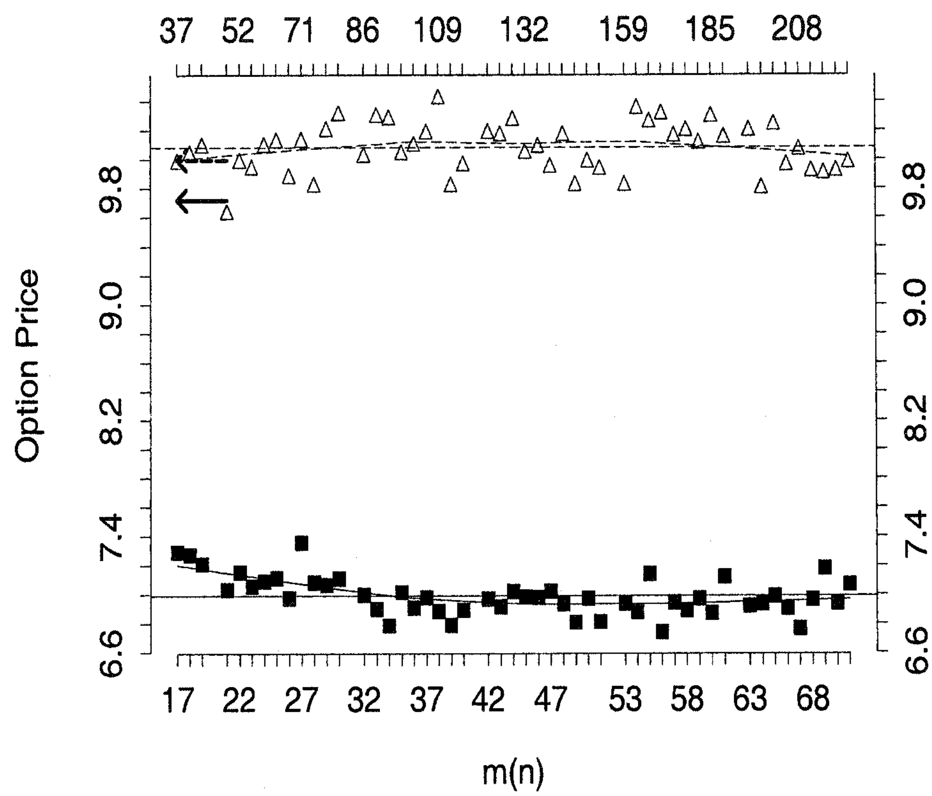

Figure 8.5 Option price for Disney stock price data using $m(n)$ bootstrap. Upper and lower x-axis indicate values for $m(n)$ for original data, respectively. Smoothed lines on graph are lowest estimates, horizontal lines are overall means, arrows indicate i.i.d. Monte Carlo estimates (in all cases thick lines used for original data, dashed lines for estimated data).

The lower values seen for the bootstrap using the real data is interesting. Given that the bootstrap should be accurate for this sample size, we have to suspect that the difference must be due to discrepancies between the observed data and the pseudo-estimated data. In fact it is more than likely that the option price derived using the observed data is more accurate. The average option price from Figure 8.5 is approximately \$ 6.80, which is remarkably close to the trading value of \$ 6.90 seen for a call option that day (using the same strike price of \$ 20.00). This is the real strength of the bootstrap. By working directly with the data we are freed to a great extent from having to make parametric assumption (such as specifying a closed form expression for G) and this can lead to more accurate estimation for the option price in cases where the underlying distribution is misspecified.

Chapter 9

Option Pricing Formula for Linear Moving Averages

In Chapter 5 we discussed a generalized binomial option price model and derived the corresponding approximation for that pricing formula. In this chapter we consider another possible option price model that can be approximated by the binomial models and try to find its corresponding approximations.

9.1 Introduction

Consider a sequence $\{Z_k : -\infty < k < \infty\}$ of real valued independent, identically distributed random variables. We assume

$$P(|Z_k| > x) = x^{-\alpha}L(x) \quad \text{where} \quad L(x) \quad \text{is slowly varying at } \infty \text{ and } \alpha > 0 \quad (9.1)$$

and

$$\frac{P(Z_k > x)}{P(|Z_k| > x)} \to p \quad \text{and} \quad \frac{P(Z_k \le -x)}{P(|Z_k| > x)} \to q \tag{9.2}$$

as $x \to \infty$, $0 \le p \le 1$ and $q = 1 - p$. It is know that under mild conditions on a real sequence $\{c_j : j \ge 0\}$ (cf. Davis and Resnick (1985) and Cline (1983) also see condition (9.5) in Proposition 9.1 below) the series

$$\sum_{j=0}^{\infty} c_j Z_{-j}$$

converges and we may define the stationary sequence of moving averages

$$X_n := \sum_{j=0}^{\infty} c_j Z_{n-j} \tag{9.3}$$

for $n \geq 1$. Further let $\{a_n\}$ be a sequence of positive constants such that

$$nP(|Z_1| > a_n x) \to x^{-\alpha} \quad \text{for all} \quad x > 0\,. \tag{9.4}$$

In fact a_n may be defined as $\inf\{x : P(|Z_k| > x) \leq n^{-1}\}$. On the space $(0,\infty) \times (R\backslash\{0\})$ define the measure

$$\mu(dt, dx) = dt \times \lambda(dx)$$

where

$$\lambda(dx) = \alpha p x^{-\alpha-1} 1_{(0,\infty)}(x)dx + \alpha q(-x)^{\alpha-1} 1_{(-\infty,0)}(x)dx\,,$$

and recall Corollary 3.2 of Resnick (1986) that says

$$\sum_{k=1}^{\infty} \varepsilon_{(k/n,\, a_n^{-1} Z_k)} \Rightarrow \sum_{k=1}^{\infty} \varepsilon_{(t_k,\, \delta_k \Gamma_k^{-1/\alpha})} \quad \text{in} \quad M_p((0,\infty) \times (R\backslash\{0\}))$$

where $\sum_{k=1}^{\infty} \varepsilon_{(t_k, \Gamma_k^{-1/\alpha})}$ is a PRM(μ) on $(0,\infty) \times (R\backslash\{0\})$ and δ_k and Γ_k are defined as in Chapter 4.

In the sequel we apply the following convergence result for point processes based on $\{X_k\}$ (cf. Davis and Resnick (1985) Theorem 2.4.)

Proposition 9.1. *Suppose $\{a_n\}$ satisfies (9.4), $\{c_j\}$ satisfies*

$$\sum_{j=0}^{\infty} |c_j|^{\delta} < \infty \quad \textit{for some} \quad \delta < \alpha\,, \quad \delta \leq 1\,, \tag{9.5}$$

$\{Z_k\}$ satisfies (9.1) and (9.2), and $\{X_n\}$ is given by (9.3). Let $\{(t_k, \delta_k \Gamma_k^{-1/\alpha})\}$ be the points of PRM(μ) on $(0,\infty) \times (R\backslash\{0\})$. Then,

(i) *In* $M_p((0,\infty)\times(R\backslash\{0\})$ *as* $n\to\infty$

$$\sum_{k=1}^{\infty}\varepsilon_{(k/n,\,X_k/a_n)}\xrightarrow{d}\sum_{i=0}^{\infty}\sum_{k=1}^{\infty}\varepsilon_{(t_k,\,\delta_k\Gamma_k^{-1/\alpha}c_i)}\,.$$

(ii) *For any positive integer* ℓ

$$\sum_{k=1}^{\infty}\varepsilon_{(k/n,\,a_n^{-1}(X_k,X_{k-1},\ldots,X_{k-\ell}))}\xrightarrow{d}\sum_{i=0}^{\infty}\sum_{k=1}^{\infty}\varepsilon_{(t_k,\,\delta_k\Gamma_k^{-1/\alpha}(c_i,c_{i-1},\ldots,c_{i-\ell}))}$$

in $M_p\Big((0,\,\infty)\times(R^{\ell+1}\setminus\{(0,0,\ldots,0)\})\Big)$ *where the sum in the limit is taken over those points lying in the state space.*

By applying continuous mapping theorem to Proposition 9.1, a variety of results can be drawn. Some of these results was explored in Davis and Resnick (1985). One of these results is about the asymptotic properties of the sample covariance function and correlation functions, i.e.,

$$\hat{\gamma}(h)=\frac{1}{n}\sum_{t=h+1}^{n}X_tX_{t-h}\,,\qquad\hat{\rho}(h)=\frac{\hat{\gamma}(h)}{\hat{\gamma}(0)}=\frac{\sum_{t=h+1}^{n}X_tX_{t-h}}{\sum_{t=1}^{n}X_t^2}\,,$$

where h is a nonnegative integer. Davis and Resnick (1985) Theorem 4.2 (see Proposition 9.2 below), shows that $na_n^{-2}\hat{\gamma}(h)$ has a stable limit and $\hat{\rho}(h)\to\rho(h)$ in probability.

Proposition 9.2. *Let* $\sum_{k=1}^{\infty}\varepsilon_{\Gamma_k^{-1/\alpha}}$ *be a PRM*(λ) *on* $R\setminus\{0\}$ *with*

$$\lambda(dx)=\alpha px^{-\alpha-1}1_{(0,\infty)}(x)dx+\alpha q(-x)^{\alpha-1}1_{(-\infty,0)}(x)dx\,,\qquad 0<\alpha<2\,.$$

Suppose (1)-(5) *hold with* $0<\alpha<2$. *Then for every nonnegative integer* ℓ, *as* $n\to\infty$*:*

(i) $(n/a)n^2)(\hat{\gamma}(0),\hat{\gamma}(1),\ldots,\hat{\gamma}(\ell))$

$$\Rightarrow\sum_{i=1}^{\infty}j_i^2\cdot\left(\sum_{j=0}^{\infty}c_j^2,\sum_{j=0}^{\infty}c_jc_{j+1},\ldots,\sum_{j=0}^{\infty}c_jc_{j+\ell}\right),$$

and

(ii) $$\hat{\rho(\ell)}\xrightarrow{P}\rho(\ell)=\frac{\sum_{j=0}^{\infty}c_jc_{j+\ell}}{\sum_{j=0}^{\infty}c_j^2}\,.$$

Remark 9.1. Note that $\rho(\ell)$'s are not the autocorrelations, but they play a simmilar role in the infinite variance case. In fact, Proposition 9.2 and the following Corollary (see Davis and Reanick (1985)) suggest that even for random variables with infinite variances we can still define notion of sample covariance and sample correlation functions.

Corollary 9.1. *The same limit laws are attained in Proposition 9.2 if $\hat{\gamma}(h)$ and $\hat{\rho}(h)$ are replaced by their mean corrected versions,*

$$\tilde{\gamma}(h) = (1/n) \sum_{t=h+1}^{n} (X_t - \bar{X})(X_{t-h} - \bar{X}) \quad \text{and} \quad \tilde{\rho}(h) = \frac{\tilde{\gamma}(h)}{\tilde{\gamma}(0)},$$

respectively, where $\bar{X} = (1/n) \sum_{i=1}^{n} X_i$.

A crucial ingredient in deriving our results in the next sections is the following result by Davis and Resnick (1985) Theorem 2.4.

Proposition 9.3. *Suppose that all the condition* (9.1) - (9.5) *hold. Then,*

$$\nu_n(\cdot) = \sum_{k=1}^{n} \varepsilon_{a_n^{-1} X_k}(\cdot) \xrightarrow{d} \sum_{i=0}^{\infty} \sum_{k=1}^{\infty} \varepsilon_{c_i \delta_k \Gamma_k^{-1/\alpha}}(\cdot).$$

Example 9.1. The following four time series justify our attempts in finding an option pricing formula for a sequence of moving averages.

- The daily Applied Material Inc. (AMAT) from January 2, 1998 to December 12, 2003, with sample size $n = 1496$.
- The daily Wall Mart Stores Inc. (WMT) from November 1, 1991 to June 30, 2003, with sample size $n = 2939$.
- The daily S & P TSX Composite Index from January 2, 1998 to December 9, 2003, with sample size $n = 1487$.

- The daily ADC Telecommunication Inc. (ADCT) from October 24, 96 to December 12, 2003, with sample size $n = 1789$.

We inspect sample autocorrelation functions and sample partial autocorrelation functions of the return series as shown in Figure 9.1. The ACF and Partial ACF for all of the above time series exhibit two relatively large spikes at early lags, possibly suggesting either a subset AR(2) or MA(2).

Example 9.1 and many other similar examples suggest that the autocorrelation function of many asset returns are significant and hence an investigation of option pricing formulae for moving averages is necessary.

9.2 The Generalized Cox-Ross-Rubinstien Model

Recall the following consecutive price movements at moment k/n described in Chapter 4.

$$S_k \stackrel{d}{=} S_0 \prod_{i=1}^{k} u_i^{\delta_i} d_i^{(1-\delta_i)},$$

or

$$\log(S_k/S_0) \stackrel{d}{=} \sum_{i=1}^{k} (U_i\delta_i + D_i(1-\delta_i)),$$

where $U_i := \log u_i = \sigma a_n^{-1}|X_i|$, $D_i := \log d_i = -\sigma a_n^{-1}|X_i|$, X_i are independent identically distributed random variables from the domain of a symmetric α-stable law, $\sigma > 0$, the normalizing constants are defined as before, and δ_i's are independent identically distributed Bernolli $(\frac{1}{2})$ independent of u_i's and d_i's. Thus,

$$\log(S_k/S_0) \stackrel{d}{=} \sigma a_n^{-1} \sum_{i=1}^{k} Z_i, \qquad (9.6)$$

where $Z_i = |X_i|\delta_i + |X_i|(\delta_i - 1)$. There are several papers in which the limit distributions of (9.6) are investigated (for example see Rachev and Ruschendorf (1994) and Mittnik and Rachev (1999)). In this section we investigate the option pricing model for a general case of (9.6) by applying Proposition 9.1.

Considering the examples given in Chapter 1 of Rachev and Mittnik (2000) and the examples given in Section 9.1, we may assume that in (9.6) each X_i is a moving

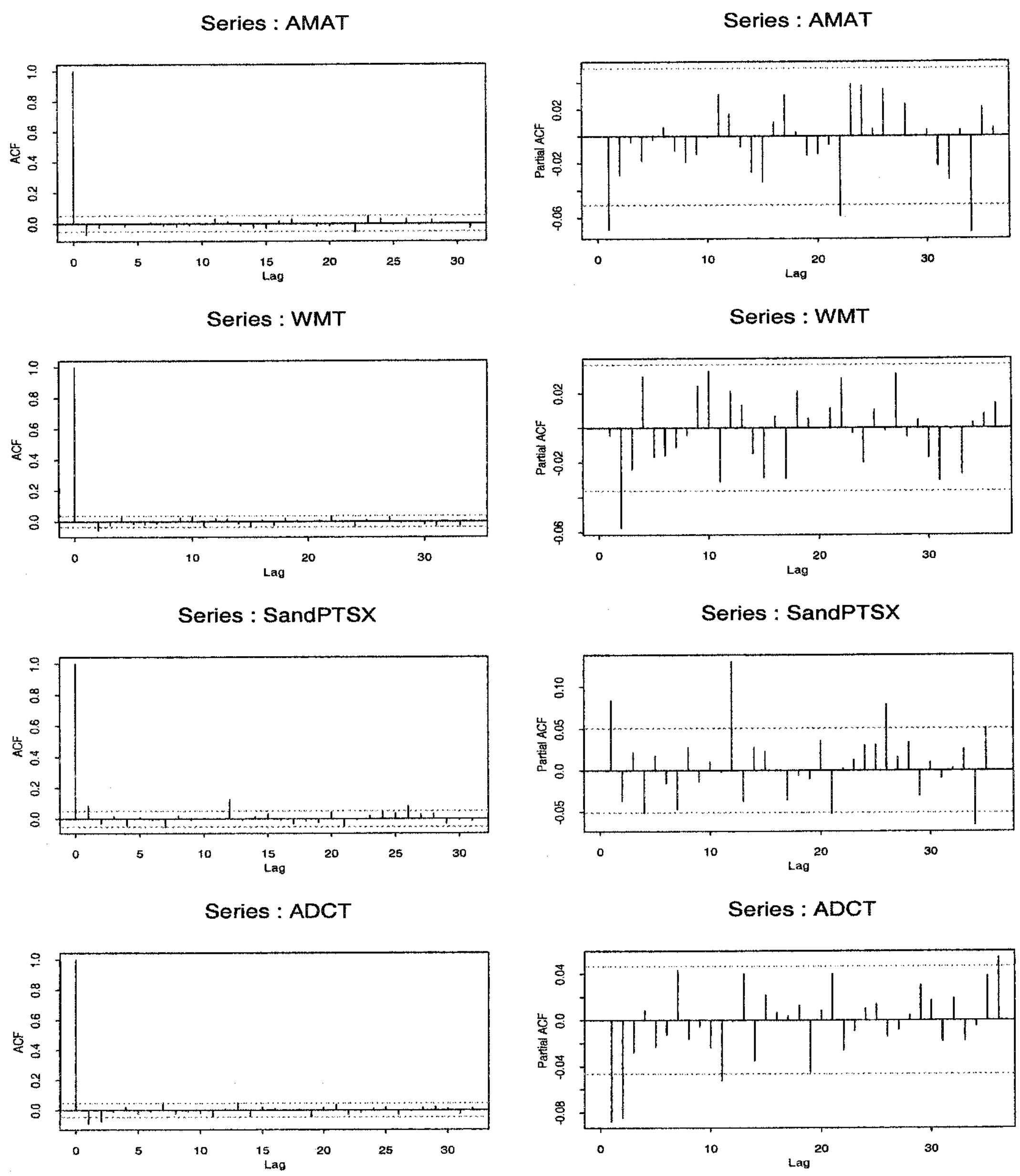

Figure 9.1 Sample Autocorrelation Functions of Returns and Sample of Partial Autocorrelation Functions of Returns.

average of the form $X_i := \sum_{j=0}^{\infty} c_j Z_{i-j}$ in which (9.1) - (9.5) are satisfied. With this in mind we may continue to follow the argument used in Rachev and Samorodnitsky and discuss the following random variable:

$$R_n = \frac{[S_0 \exp(\sigma a_n^{-1} \sum_{i=1}^{n} X_i) - K]_+}{\prod_{i=1}^{n} \cosh(\sigma a_n^{-1} X_i)}$$

In the next section we introduce some convergence theorems which help us to find the limit of C_n as $n \to \infty$.

9.3 Basic Convergences

In this section we use convergence in distribution of random measures to obtain limit theorems for sums of random variables. Before we introduce our first results it is important to prove the following Lemma.

Lemma 9.1. *Let* $\sigma > 0$ *and let the sequence* $\{c_i\}$ *satisfies* (9.5).

$$\text{(a)} \quad \sum_{i=1}^{\infty}\sum_{k=1}^{\infty} \sigma |c_i| \Gamma_k^{-1/\alpha} < \infty \qquad \text{if} \quad \alpha \in (0,1)$$

and

$$\text{(b)} \quad \sum_{i=1}^{\infty}\sum_{k=1}^{\infty} \ln\cosh\left(\sigma c_i \Gamma_k^{-1/\alpha}\right), < \infty \qquad \text{if} \quad \alpha \in (0,2)$$

Proof Without loss of any generality assume $\sigma = 1$. Note that the condition imposed on $\{c_i\}$ is $\sum_{i=1}^{\infty} |\, c_i \,|^{\delta} < \infty$, where
$0 < \delta < 1$. With this condition we certainly have

$$\sum_{i=1}^{\infty} |c_i| < \infty .$$

Since the series $\sum_{k=1}^{\infty} \Gamma_k^{-1/\alpha}$ is convergent for $0 < \alpha < 1$ and since

$$\sum_{i=1}^{\infty}\sum_{k=1}^{\infty} |c_i| \Gamma_k^{-1/\alpha} = \sum_{i=1}^{\infty} |c_i| \sum_{k=1}^{\infty} \Gamma_k^{-1/\alpha} ,$$

we conclude that the double series in part (a) is convergent for all $\alpha \in (0,1)$. To prove part (b) first observe that

$$\lim_{x \to 0} \frac{\ln \cosh x}{\frac{x^2}{2}} = 1,$$

and

$$\Gamma_N^{-1/\alpha} \stackrel{a.s.}{\to} 0.$$

Therefore, for N large enough we have

$$\begin{aligned}\sum_{i=1}^{\infty}\sum_{k=N}^{\infty} \ln\cosh\left(c_i\Gamma_k^{-1/\alpha}\right) &\approx \sum_{i=1}^{\infty}\sum_{k=N}^{\infty} \frac{c_i^2\,\Gamma_k^{-2/\alpha}}{2} \\ &= \left(\sum_{i=1}^{\infty} c_i^2\right)\left(\frac{1}{2}\sum_{k=N}^{\infty}\Gamma_k^{-2/\alpha}\right).\end{aligned}$$

Now since $\sum_{i=1}^{\infty} c_i^2 < \infty$ and since for $0 < \alpha < 2$ the series $\sum_{k=1}^{\infty}\Gamma_k^{-2/\alpha}$ is a convergent series, the double series in part (b) is also convergent. □

The following sets,

$$\begin{aligned}A_{0,\epsilon,M} &= (-\infty, -M-\epsilon/2)\bigcup(-\epsilon/2, \epsilon/2)\bigcup(M+\epsilon/2, +\infty),\\ A_{1,\epsilon,M} &= [-\epsilon, -\epsilon/2]\bigcup[\epsilon/2, \epsilon]\,, \quad A_{2,\epsilon,M} = (-M, -\epsilon)\bigcup(\epsilon, M),\\ A_{3,\epsilon,M} &= [-M-\epsilon/2, -M]\bigcup[M, M+\epsilon/2],\end{aligned}$$

are defined in Chapter 5, but for the sake of convenience we present them here one more time.

Lemma 9.2. (a) For each $\alpha \in (0,1)$ we have

$$\lim_{\epsilon \to 0^+} \sum_{i=0}^{\infty}\sum_{k=1}^{\infty} \sigma c_i \Gamma_k^{-1/\alpha} I\left\{c_i\Gamma_k^{-1/\alpha} \in \left(A_{1,\epsilon,M} \bigcup A_{3,\epsilon,M}\right)\right\} = 0.$$

(b) *For each $\alpha \in (0,2)$ we have*

$$\lim_{\epsilon \to 0^+} \sum_{i=0}^{\infty}\sum_{k=1}^{\infty} \ln\cosh\left(\sigma c_i\Gamma_k^{-1/\alpha}\right) I\left\{c_i\Gamma_k^{-1/\alpha} \in \left(A_{1,\epsilon,M} \bigcup A_{3,\epsilon,M}\right)\right\} = 0.$$

Proof. In both parts use Lemma 9.1 and the Dominated Convergence Theorem for series to deduce that

$$\lim_{\epsilon\to 0^+} \sum_{i=0}^{\infty}\sum_{k=1}^{\infty} \sigma c_i \Gamma_k^{-1/\alpha} I\left\{c_i\Gamma_k^{-1/\alpha} \in \left(A_{1,\epsilon,M} \bigcup A_{3,\epsilon,M}\right)\right\}$$

$$= \sum_{i=0}^{\infty}\sum_{k=1}^{\infty} \sigma c_i \Gamma_k^{-1/\alpha} \lim_{\epsilon\to 0^+} I\left\{c_i\Gamma_k^{-1/\alpha} \in \left(A_{1,\epsilon,M} \bigcup A_{3,\epsilon,M}\right)\right\} = 0\,.$$

and

$$\lim_{\epsilon\to 0^+} \sum_{i=0}^{\infty}\sum_{k=1}^{\infty} \ln\cosh\left(\sigma c_i \Gamma_k^{-1/\alpha}\right) I\left\{c_i\Gamma_k^{-1/\alpha} \in \left(A_{1,\epsilon,M} \bigcup A_{3,\epsilon,M}\right)\right\}$$

$$= \sum_{i=0}^{\infty}\sum_{k=1}^{\infty} \ln\cosh\left(\sigma c_i \Gamma_k^{-1/\alpha}\right) \lim_{\epsilon\to 0^+} I\left\{c_i\Gamma_k^{-1/\alpha} \in \left(A_{1,\epsilon,M} \bigcup A_{3,\epsilon,M}\right)\right\} = 0\,.$$

□

Lemma 9.3. (a) *For each* $\alpha \in (0,1)$ *we have*

$$\lim_{M\to\infty} \lim_{\epsilon\to 0^+} \sum_{i=0}^{\infty}\sum_{k=1}^{\infty} \sigma c_i \Gamma_k^{-1/\alpha} I\left\{c_i\Gamma_k^{-1/\alpha} \in \left(A_{2,\epsilon,M}\right)\right\}$$

$$= \sum_{i=0}^{\infty}\sum_{k=1}^{\infty} \ln\cosh\left(\sigma c_i \Gamma_k^{-1/\alpha}\right)\,.$$

(b) *For each* $\alpha \in (0,2)$ *we have*

$$\lim_{M\to\infty} \lim_{\epsilon\to 0^+} \sum_{i=0}^{\infty}\sum_{k=1}^{\infty} \ln\cosh\left(\sigma c_i \Gamma_k^{-1/\alpha}\right) I\left\{c_i\Gamma_k^{-1/\alpha} \in \left(A_{2,\epsilon,M}\right)\right\}$$

$$= \sum_{i=0}^{\infty}\sum_{k=1}^{\infty} \ln\cosh\left(\sigma c_i \Gamma_k^{-1/\alpha}\right)\,.$$

Proof. In both parts, since

$$\lim_{M\to\infty} \lim_{\epsilon\to 0^+} I\left\{c_i\Gamma_k^{-1/\alpha} \in \left(A_{2,\epsilon,M}\right)\right\} = 1\,,$$

the desired result follows by another application of Lemma 1 and the Dominated Convergence Theorem. □

In the proof of our next Lemma we need to apply the following Proposition (Karamata's Theorem) which is a useful theorem in the theory of slowly varying functions.

Proposition 9.4 (Karamata). *Let F be a distribution function on $[0,\infty)$ with*

$$1 - F(x) = x^{-\alpha} L(x)$$

for some $\alpha > 0$ and some slowly varying function L. Define μ_β by

$$\mu_\beta(x) = \int_0^x y^\beta \, F(dy) \, .$$

If $\mu_\beta(\infty) = \infty$, then

$$\lim_{t\to\infty} \frac{t^\beta(1-F(t))}{\mu_\beta(t)} = \frac{\beta-\alpha}{\alpha} \, .$$

Thus if X is in the domain of attraction of a stable law with index α and F is the distribution of $|X|$, we get for $\beta > \alpha$ and $t = a_N x$

$$\begin{aligned} N a_N^{-\beta} E(|X|^\beta I\{|X| \le a_N x\}) &= \frac{x^\beta E(|X|^\beta I\{|X| \le a_N x\})}{a_N^\beta x^\beta (1 - F(a_N x))} N(1 - F(a_N x)) \\ &\longrightarrow \frac{\alpha}{\beta - \alpha} x^{\beta-\alpha} \quad \text{as } N \to \infty \end{aligned}$$

since $N(1 - F(a_N x)) \to x^{-\alpha}$ from the definition of the norming constants a_N.

Lemma 9.4. *Assume all the conditions* (9.1) - (9.5) *are satisfied and* $0 < \alpha < 1$. *Then*

$$\int_{|x|<\epsilon/2} |x| \, \nu_n(dx) \xrightarrow{P} 0 \, .$$

Proof. Since

$$\int_{|x|<\epsilon/2} |x| \, \nu_n(dx) = \sum_{i=1}^N a_N^{-1} |X_i| I\left\{ a_N^{-1}|X_i| < \frac{\epsilon}{2} \right\},$$

to prove the Lemma it is enough to prove that

$$\sum_{i=1}^{N} E\left(a_N^{-1}|X_i| I\left\{a_N^{-1}|X_i| < \frac{\epsilon}{2}\right\}\right) \to 0 .$$

The latter claim is true because

$$\lim_{N\to\infty} N\, E\left(a_N^{-1}|X_i| I\left\{a_N^{-1}|X_i| < \frac{\epsilon}{2}\right\}\right) = \frac{1-\alpha}{\alpha}\left(\frac{\epsilon}{2}\right)^{1-\alpha},$$

by Proposition 9.4 with $\beta = 1$, and

$$\lim_{\epsilon\to 0^+} \frac{1-\alpha}{\alpha}\left(\frac{\epsilon}{2}\right)^{1-\alpha} = 0,$$

as $\alpha < 1$. □

Lemma 9.5 *Assume* (9.1) - (9.5) *hold for* $\alpha \in (0,1)$, *then*

$$\left(\sum_{k=1}^{n} a_n^{-1} X_n , \sum_{k=1}^{n} \ln\cosh\left(a_n^{-1} X_n\right)\right)$$

$$\xrightarrow{d} \left(\sum_{i=0}^{\infty}\sum_{k=1}^{\infty} c_i \delta_k \Gamma_k^{-1/\alpha} , \sum_{i=0}^{\infty}\sum_{k=1}^{\infty} \ln\cosh\left(c_i \delta_k \Gamma_k^{-1/\alpha}\right)\right).$$

Proof. Let s and t be any two arbitrary real number. Let f be defined by $f(x) = s\,x + t\,\ln\cosh x$. Define a function $g_{\epsilon,M,f}$ by

$$g_{\epsilon,M,f}(x) = \begin{cases} 0 & x \in A_{0,\epsilon,M} \\ L^{(1)}_{\epsilon,M,f}(x) & x \in A_{1,\epsilon,M} \\ f(x) & x \in A_{2,\epsilon,M} \\ L^{(2)}_{\epsilon,M,f}(x) & x \in A_{3,\epsilon,M} \end{cases}.$$

where $0 < \epsilon < M < \infty$, $L^{(1)}_{\epsilon,M,f}$ and $L^{(2)}_{\epsilon,M,f)}$ are two continuous functions chosen so that $g_{\epsilon,M,f}$ is continuous and such that

$$\left|L^{(1)}_{\epsilon,M,f}(x)\right| \le |f(x)| \quad x \in A_{1,\epsilon,M}$$

and

$$\left|L^{(2)}_{\epsilon,M,f}(x)\right| \le |f(x)| \quad x \in A_{2,\epsilon,M}.$$

(see Figure 5.1). Note that by the absolute continuity of f such functions $L^{(1)}_{\epsilon,M,f}$ and $L^{(2)}_{\epsilon,M,f}$ can be constructed.

Let ν_n and ν be defined as in Proposition 9.3. We will show as $n \to \infty$, $\epsilon \to 0^+$ and $M \to \infty$

$$\int g_{\epsilon,M,f}(x)\,\nu_n(dx) \xrightarrow{d} s\sum_{i=0}^{\infty}\sum_{k=1}^{\infty} c_i\delta_k\Gamma_k^{-1/\alpha} + t\sum_{i=0}^{\infty}\sum_{k=1}^{\infty} \ln\cosh\left(c_i\delta_k\Gamma_k^{-1/\alpha}\right) \tag{9.7}$$

and

$$\left|\int g_{\epsilon,M,f}(x)\,\nu_n(dx) - s\sum_{k=1}^{n} a_n^{-1}X_k - t\sum_{k=1}^{n}\ln\cosh\left(a_n^{-1}X_k\right)\right| \xrightarrow{P} 0\,. \tag{9.8}$$

By Theorem 3.2 in Billingsley (1969) the limits (9.7) and (9.8) imply that

$$s\sum_{k=1}^{n} a_n^{-1}X_k + t\sum_{k=1}^{n}\ln\cosh\left(a_n^{-1}X_k\right)$$

$$\xrightarrow{d} s\sum_{i=0}^{\infty}\sum_{k=1}^{\infty} c_i\delta_k\Gamma_k^{-1/\alpha} + t\sum_{i=0}^{\infty}\sum_{k=1}^{\infty} \ln\cosh\left(c_i\delta_k\Gamma_k^{-1/\alpha}\right).$$

An application of the Cramér Wold Device leads to the desired result.

We begin by proving (9.7). Because $g_{\epsilon,M,f}$ is a continuous function with compact support, by Proposition 9.3

$$\int g_{\epsilon,M,f}(x)\,\nu_n(dx) \xrightarrow{d} \int g_{\epsilon,M,f}(x)\,\nu(dx)\,.$$

By the definition of ν this implies that

$$\int g_{\epsilon,M,f}(x)\,\nu_n(dx) \xrightarrow{d} \sum_{j=1}^{3}\sum_{i=0}^{\infty}\sum_{k=1}^{\infty} g_{\epsilon,M,f}\left(c_i\delta_k\Gamma_k^{-1/\alpha}\right) I\left\{c_i\delta_k\Gamma_k^{-1/\alpha} \in (A_{j,\epsilon,M})\right\}$$

By Lemma 9.2, we have, as $\epsilon \to 0^+$

$$\sum_{i=0}^{\infty}\sum_{k=1}^{\infty} g_{\epsilon,M,f}\left(c_i\delta_k\Gamma_k^{-1/\alpha}\right) I\left\{c_i\delta_k\Gamma_k^{-1/\alpha} \in (A_{j,\epsilon,M})\right\} \xrightarrow{d} 0$$

for $j = 1$ and $j = 3$, and By Lemma 9.3 we have, as $M \to \infty$ and $\epsilon \to 0^+$

$$\sum_{i=0}^{\infty}\sum_{k=1}^{\infty} g_{\epsilon,M,f}\left(c_i\delta_k\Gamma_k^{-1/\alpha}\right) I\left\{c_i\delta_k\Gamma_k^{-1/\alpha} \in (A_{2,\epsilon,M})\right\}$$

$$\xrightarrow{d} s\sum_{i=0}^{\infty}\sum_{k=1}^{\infty} c_i\delta_k\Gamma_k^{-1/\alpha} + t\sum_{i=0}^{\infty}\sum_{k=1}^{\infty} \ln\cosh\left(c_i\delta_k\Gamma_k^{-1/\alpha}\right).$$

This shows that (9.7) holds.

To prove (9.8) observe that $\sum_{k=1}^{n} f(a_n^{-1}X_k) = \int f(x)\nu_n(dx)$. Therefore,

$$\left|\int g_{\epsilon,M,f}(x)\nu_n(dx) - \sum_{k=1}^{n} f(a_n^{-1}X_k)\right|$$

$$= \left|\int_{A_{0,\epsilon,M}} f(x)\nu_n(dx) + \int_{\{A_{1,\epsilon,M}\bigcup A_{3,\epsilon,M}\}} (g_{\epsilon,M,f}(x) - f(x))\nu_n(dx)\right|$$

$$\leq \left|\int_{A_{0,\epsilon,M}} f(x)\nu_n(dx)\right| + \int_{\{A_{1,\epsilon,M}\bigcup A_{3,\epsilon,M}\}} (|g_{\epsilon,M,f}(x)| + |f(x)|)\nu_n(dx).$$

Both $g_{\epsilon,M,f}$ and f are bounded over $A_{1,\epsilon,M}\bigcup A_{3,\epsilon,M}$. Call this bound B. Hence by proposition 9.3

$$\int_{A_{1,\epsilon,M}\bigcup A_{3,\epsilon,M}} (|g_{\epsilon,M,f}(x)| + |f(x)|)\nu_n(dx) \to B\nu\left(A_{1,\epsilon,M}\bigcup A_{3,\epsilon,M}\right)$$

Letting $\epsilon \to 0^+$ infer that this term converges to zero. We also have

$$\left|\int_{A_{0,\epsilon,M}} f(x)\nu_n(dx)\right| \leq \int_{|x|<\epsilon/2} |f(x)|\nu_n(dx) + \int_{|x|\geq M+\epsilon/2} |f(x)|\nu_n(dx)$$

$$\leq |s|\int_{|x|<\epsilon/2} |x|\nu_n(dx) + |t|\int_{|x|<\epsilon/2} \ln\cosh(x)\nu_n(dx)$$

$$+ \int_{|x|\geq M+\epsilon/2} |f(x)|\nu_n(dx) \quad .$$

The first term converges, in probability, to zero as $n \to \infty$ and $\epsilon \to 0^+$ by Lemma 9.4. The second term converges, almost surely, to zero as $n \to \infty$ and $\epsilon \to 0^+$ (see the last part of the proof of Corollary 5.1). To prove the third term also converges to zero, use Proposition 3.3, Proposition 3.5 and a similar argument as in the proof of (5.3) in Corollary 5.1 applied separately to functions sh_1 and th_2, where $h_1(x) = \sigma x$ and $h_2(x) = \ln\cosh\sigma x$. □

Remark 9.2. Notice that a similar proof of Davis and Resnick (1985) Theorem 4.1 does not work in here. Therefore our continuous approximation that we use in the proof of Corollary 5.1 and Lemma 9.5 is very useful.

9.4 Main Result

Dependence for stock price data is clearly an empirical fact of life. To address this issue we try to extend the option pricing formula to handle stock price data that are non-independent. Specifically, we consider any data $X_1, \ldots, X_n$ that can be written as a linear moving average process of the form (9.3).

It has been suggested that option pricing should not be affected by dependence in stock price data. The following theorem suggests in fact that the option price formula can be dramatically altered if the data followed a moving average process.

Theorem 9.1. *Suppose that in* (9.6) *each* X_i *is a moving average of the form* $X_i := \sum_{j=0}^{\infty} c_j Z_{i-j}$ *in which* (9.1) - (9.5) *are satisfied for* $0 < \alpha < 1$. *Then* $R_n \xrightarrow{d} R$, *where*

$$R_n = \frac{(S_0 \exp(\sigma a_n^{-1} \sum_{k=1}^{n} X_k) - K)_+}{\prod_{k=1}^{n} \cosh(\sigma a_n^{-1} X_k)}$$

and

$$R = \frac{(S_0 \exp(\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \sigma c_i \delta_j \Gamma_j^{-1/\alpha}) - K)_+}{\prod_{i=1}^{\infty} \prod_{j=1}^{\infty} \cosh(\sigma c_i \Gamma_j^{-1/\alpha})}$$

Proof : By Lemma 9.5 (see also Resnick (1985)) we have

$$\sigma a_n^{-1} \sum_{k=1}^{n} X_k \xrightarrow{d} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \sigma c_i \Gamma_j^{-1/\alpha}.$$

We also have

$$\sum_{k=1}^{n} \ln \cosh(\sigma a_n^{-1} X_k) \xrightarrow{d} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \ln \cosh(\sigma c_i \Gamma_j^{-1/\alpha}).$$

By Lemma 9.5 and the Continuous Mapping Theorem we have

$$\left(S_0 \exp(\sigma a_n^{-1} \sum_{k=1}^{n} X_k) - K \right)_+ \xrightarrow{d} \left(S_0 \exp(\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \sigma c_i \Gamma_j^{-1/\alpha}) - K \right)_+ ,$$

$$\exp\left(\sum_{k=1}^{n} \ln \cosh(\sigma a_n^{-1} X_k) \right) = \prod_{k=1}^{n} \cosh(\sigma a_n^{-1} X_k)$$

$$\xrightarrow{d} \exp\left(\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \ln\cosh(\sigma c_i \Gamma_j^{-1/\alpha})\right) = \prod_{i=1}^{\infty}\prod_{j=1}^{\infty} \cosh(\sigma c_i \Gamma_j^{-1/\alpha}),$$

and

$$\left(\left(S_0 \exp(\sigma a_n^{-1}\sum_{k=1}^{n} X_k) - K\right)_+ , \prod_{k=1}^{n} \cosh(\sigma a_n^{-1} X_k)\right)$$
$$\xrightarrow{d} \left(\left(S_0 \exp(\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \sigma c_i \Gamma_j^{-1/\alpha}) - K\right)_+ , \prod_{i=1}^{\infty}\prod_{j=1}^{\infty} \cosh(\sigma c_i \Gamma_j^{-1/\alpha})\right).$$

Since $\cosh x \geq 1$ for all real numbers x, the sequence $\prod_{k=1}^{n} \cosh(\sigma a_n^{-1} X_k)$ is bounded away from zero. Thus, another application of continuous mapping theorem leads us to conclude that

$$\frac{(S_0 \exp(\sigma a_n^{-1} \sum_{k=1}^{n} X_k) - K)_+}{\prod_{k=1}^{n} \cosh(\sigma a_n^{-1} X_k)} \xrightarrow{d} \frac{\left(S_0 \exp(\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \sigma c_i \Gamma_j^{-1/\alpha}) - K\right)_+}{\prod_{i=1}^{\infty}\prod_{j=1}^{\infty} \cosh(\sigma c_i \Gamma_j^{-1/\alpha})}$$

□

Some comments on this last result are given in Chapter 10.

Chapter 10

Future Works

10.1 Option pricing for price changes in the domain of attraction of the normal law

As an alternative of the Mandelbrot-Taylor log-stable model Clark (1973) proposed the asset price process to be generated via the process Z subordinated to the Wiener process W by a log-normal process T. Clark (1973) modelled the stock process changes by

$$\xi(t) = X(r(t)),$$

that is ξ is subordinated to $X(t)$ with directing process $r(t) \geq 0$ for which

$$E\{[r(t)]^2\} < \infty.$$

Since the choice of a log-normal directed process was not completely justified in Clark (1973), Rachev and Samorodnitsky (1993) assumed only that $r(t)$ has a finite first moment and derived a pricing formula for stock returns governed by such subordinated process (see Rachev and Samorodnitsky (1993) Theorem 3.). These results and the result obtained in Chapter 9, motivate us to think about the validity of the following claim .

Let

$$X_t = \sum_{j=0}^{\infty} c_j Z_{t-j}$$

where Z_i's are in the domain of attraction of normal law. It seems that the option formula can be generalized for this case. One of our future works is to establish the validity of this claim.

10.2 A General Option Pricing Formula for Linear Moving Averages

It seems that Theorem 9.1 is also valid for $1 \le \alpha < 2$. We have not proved this claim yet, but we will investigate its validity in our future works. It is also important to show that $E(R_n) \to E(R)$, where

$$R_n = \frac{(S_0 \exp(\sigma a_n^{-1} \sum_{k=1}^{n} X_k) - K)_+}{\prod_{k=1}^{n} \cosh(\sigma a_n^{-1} X_k)}$$

and

$$R = \frac{(S_0 \exp(\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \sigma c_i \delta_j \Gamma_j^{-1/\alpha}) - K)_+}{\prod_{i=1}^{\infty} \prod_{j=1}^{\infty} \cosh(\sigma c_i \Gamma_j^{-1/\alpha})}$$

Here usual Dominated Convergence Theorem does not work, but it seems that still we can prove $E(R_n) \to E(R)$. Notice that from Lemma 3.11 of Kallenberg (2002), we have

$$E(R) \le \lim_{n\to\infty} \inf E(R_n)\,.$$

This case is still under our investigation. If this result happens to be correct, then the claim in Rachev and Samorodnitsky (1993) that the correlation in the data does not change Option Pricing Formula seems to be incorrect.

10.3 Application of block bootstraping method for the general option pricing formula

Block bootstraping is a variation of bootstrapping that randomly selects with replacement multiple contiguous observations as opposed to individual observations in order to create additional samples. We use block bootstrapping to preserve the serial dependence of the data. Our results in Chapter 8 and Chapter 9 suggest to think about the application of a block bootstraping method for the moving average process of the infinite variance cases. This does not seem to be very practical but the mathematical theory can be developed.

10.4 Numerical Computations Applied to Option Pricing Formula

The recursive method of Banjevic, Ishwaran and Zerepour (2002) can be used to study the properties of the option pricing formula and gain further insight into other properties of option price formula by deriving integral or differential equations. As an example one may consider the differential equation

$$\frac{\partial}{\partial t} I(K,t) = I(K,t) - \frac{1}{2}\left[I(K_1(t)\,,t) + I(K_2(t)\,,t)\right] ,$$

which is derived in Section 7.3. However, sometimes a closed form solution to these equations seems to be impossible. In such cases a numerical solution may lead us to better understanding of the behavior of option prices, but these are challenging problems and we will address them in our future works.

Appendix A

S – Plus Programs

Program 1. Levels and Returns of Time Series

```
Z<-scan("File",what=list(0))
S<-Z[[1]]
n<-length(S)
X<-rep(0,n)
for (i in 0:(n-1)){X[i+1]<-S[n-i]}
U<-rep(0,(n-1))
for (i in 1:(n-1)){U[i]<-log(X[i+1]/X[i])}
x1<-rep(0,n)
for (i in 1:n){x1[i]<-i}
x2<-rep(0,(n-1))
for (i in 1:(n-1)){x2[i]<-i }

plot(x1,X,type="l",main="Levels")
plot (x2,U,type="l",main="Returns")
```

Program 2. Comparison of Empirical and Normal Probability Distributions

```
Z<-scan("File",what=list(0))
S<-Z[[1]]
n<-length(S)
X<-rep(0,n)
for (i in 0:(n-1)){X[i+1]<-S[n-i]}
U<-rep(0,(n-1))
for (i in 1:(n-1)){U[i]<-log(X[i+1]/X[i])}
Y<-rnorm((n-1),mean(U),var(U))
a<-density(X)
b<-density(U)
t1<-a$x
t2<-b$x
y1<-a$y
y2<-b$y
WW<-cbind(t1,t2)
UU<-cbind(y1,y2)

mathplot(WW,UU,type="l",main="Comparison of Empirical and Normal Probabil-
ity Distributions")
```

Program 3. Truncated Option Price Calculation

```
GAMMA<-function(e,n){G<- rep(0,n)
            for (j in 1:n){ G[j]<-sum(e[1:j])}
            return(G)
            }

DELTA<-function(a,n){D<-rep(0,n)
           for (j in 1:n){D[j]<-sample(a,1)}
           return(D)
           }

RATIOG<-function(S0,K,sigma,alpha,x,d){ u<-max(S0*exp(sigma*
            sum(d*(x**(-1/alpha))))-K,0)/prod (cosh(sigma*(x**(-1/alpha))))}
            return(u)
            }

S0<-42
K<-40
alpha<-1.4
sigma<-0.2
a<-c (-1,1)
n<-10000
M<-1000
ratio<-matrix(0,M,L)
MESH<-2000
L<-floor(n/MESH)
CN<-rep(0,L)
N<-rep(0,L)
N0<-0
for (i in 1:M){
             e<-rexp(n)
             x<-GAMMA(e,n)
             d<-DELTA(a,n)
             for (k in 1:L){s<-k*MESH
                         N[k]<-N0+s
                         gamma<-x[1:s]
                         delta<-d[1:s]
                         ratio[i,k]<-RATIOG(S0,K,sigma,alpha,gamma,delta)
                         }
             }

for (k in 1:L){CN[k]<-mean(ratio[,k])}
CN
plot(N,CN,main="Truncated option prices for European call options")
```

Program 4. Comparison of Empirical and Normal Probability Distributions

```
Z1<-scan("c:/AMEX-COMPOSIT-1998-2003.txt",what=list("",0))
S1<-Z1[[2]]
n1<-length(S1)
P1<-rep(0,n1)
for (i in 0:(n1-1)){P1[i+1]<-S1[n1-i]}
SS1<-rep(0,(n1-1))
for (i in 1:(n1-1)){SS1[i]<-100*(log(P1[i+1]/P1[i]))}
Y1<-rnorm((n1-1),mean(SS1),var(SS1))
a1<-density(SS1)
b1<-density(Y1)
x11<-a1$x
x21<-b1$x
y11<-a1$y
y21<-b1$y
WW1<-cbind(x11,x21)
UU1<-cbind(y11,y21)

Z2<-scan("c:/NASDAQ-STOCK-1998-2003.txt",what=list("",0)) S2<-Z2[[2]]
n2<-length(S2)
P2<-rep(0,n2)
for (i in 0:(n2-1)){P2[i+1]<-S2[n1-i]}
SS2<-rep(0,(n2-1))
for (i in 1:(n2-1)){SS2[i]<-100*(log(P2[i+1]/P2[i]))}
Y2<-rnorm((n2-1),mean(SS2),var(SS2))
a2<-density(SS2)
b2<-density(Y2)
x12<-a2$x
x22<-b2$x
y12<-a2$y
y22<-b2$y
WW2<-cbind(x12,x22)
UU2<-cbind(y12,y22)

Z3<-scan("c:/S-and-P-TSX-Composit-index-1998-2003.txt",what=list("",0))
S3<-Z2[[2]]
n3<-length(S3)
P3<-rep(0,n3)
for (i in 0:(n3-1)){P3[i+1]<-S3[n1-i]}
SS3<-rep(0,(n3-1))
for (i in 1:(n3-1)){SS3[i]<-100*(log(P3[i+1]/P3[i]))}
Y3<-rnorm((n3-1),mean(SS3),var(SS3))
a3<-density(SS3)
b3<-density(Y3)
x13<-a3$x
x23<-b3$x
```

```
y13<-a3$y
y23<-b3$y
WW3<-cbind(x13,x23)
UU3<-cbind(y13,y23)

par(mfrow=c(3,1))
matplot(WW1,UU1,type="l",main="AMEX Normal Fit")
matplot(WW2,UU2,type="l",main="NASDAQ Normal Fit")
matplot(WW3,UU3,type="l",main="S & P TSX Normal Fit")
```

Program 5. Estimated Density for Ratio and Log(1+Ratio)

#In this program we call the functions defined in Program 3#

```
S0<-42
K<-40
n<-8000
N<-1000
alpha<-1.4
sigma<-0.1
U<-rep(0,N)
a<-c(-1,1)
for (i in 1:N){e<-rexp(n)
               x<-GAMMA(e,n)
               d<-DELTA(a)
               U[i]<-RATIOG(S0,K,sigma,alpha,x,d)
               }

n<-10000
W<-rep(0,N)
a<-c(-1,1)
for (i in 1:N){e<-rexp(n)
               x<-GAMMA(e,n)
               d<-DELTA(a)
               W[i]<-RATIOG(S0,K,sigma,alpha,x,d)
               }

par(mfrow=c(2,2))

a<-density(U)
Ratio<-a$x
Density<-a$y
plot(Ratio,Density,type="l",sub="Estimated density of Ratio for N=8,000")

CU<-log(1+U)
a<-density(CU)
Log.of.Ratio.Plus.one<-a$x
Density<-a$y
plot(Log.of.Ratio.Plus.one,Density,type="l",sub="Estimated density of
     Log(Ratio + 1) for N=8,000")

a<-density(W)
Ratio<-a$x
Density<-a$y
plot(Ratio,Density,type="l",sub="Estimated density of Ratio for N=10,000")
```

```
CW<-log(1+W)
a<-density(CW)
Log.of.Ratio.Plus.one<-a$x
Density<-a$y
plot(Log.of.Ratio.Plus.one,Density,type="l",sub="Estimated density of
      Log(Ratio + 1) for N=10,000")

par(mfrow=c(2,2))
hist(U,sub="Estimated density of Ratio for N=8,000")
hist(log(1+U),sub="Estimated density of Log(Ratio+1) for N=8,000")
hist(W,sub="Estimated density of Ratio for N=10,000")
hist(log(1+W),sub="Estimated density of Log(Ratio+1) for N=10,000")
```

Program 6. Stable Approximation

```
Z<-scan("c:NASDAQ-STOCK-1998-2003.txt",what=list("",0))
S<-Z[[2]]
n<-length(S)
X<-rep(0,151)
for (i in 0:150){X[i+1]<-S[n-i]}
U<-rep(0,150)
for (i in 1:150){U[i]<-100*(log(X[i+1]/X[i]))}

    # In the following sigma and alpha are approximated by using Nolan's Stable
Program #

Y<-sigma*rstab(150,alpha,0)
a<-density(U)
b<-density(Y)
x1<-a$x
x2<-b$x
y1<-a$y
y2<-b$y
WW<-cbind(x1,x2)
UU<-cbind(y1,y2)
matplot(WW,UU,type="l",main="Stable Approximation of NASDAQ
          Stock Composit")

Z<-0.42*rstab(150,alpha,0)
aa<-density(U)
bb<-density(Z)
t1<-aa$x
t2<-bb$x
r1<4-aa$y
r2<-bb$y
WWW<-cbind(t1,t2)
UUU<-cbind(r1,r2)
matplot(WWW,UUU,type="l",main="NASDAQ Stock Composite Stable Fit")
```

Program 7. Bootstrap Approximation of Option Price

```
BOOT.OPTION<-function(S0,K,x,am,sigma)
                {
                u<-max(S0*exp(sigma*(am** (-1))*sum(x))-K,0)/prod(cosh
                        (sigma*(am**(-1))*x))
                 return(u)
                }

sigma<-0.1
alpha<-1.46
S0<-1.62
K<-1.6
n<-400000
RET.STAB<-rstab(n,alpha,0)

N<-10000
L<-50
Call.Price<-rep(0,L)
I<-rep(0,L)

for (k in 1:L){
            m<-(105-k)
            I[k]<-m
            am<-m**(1/alpha)
            OPT.STAB<-rep(0,N)
            for (i in 1:N){
                        RETURN.STAR<-sample(RET.STAB,m,replace=T)
                        OPT.STAB[i]<-BOOT.OPTION(S0,K,RETURN.STAR,
                         am,sigma)
                        }
            Call.Price[k]<-mean(OPT.STAB)
            }
Call.Price
plot(m,Call.Price,main="European Options on British Pound")
```

Program 8. Graphing Autocorrelation Functions of Returns

```
Z1<-scan("c:/data/AMAT.txt",what=list(0))
S1<-Z1[[1]]
n1<-length(S1)
P1<-rep(0,n1)
for (i in 0:(n1-1)){P1[i+1]<-S1[n1-i]}
AMAT<-rep(0,(n1-1))
for (i in 1:(n1-1)){AMAT[i]<-100*(log(P1[i+1]/P1[i]))}

Z2<-scan("c:/data/WMT.txt",what=list(0))
S2<-Z2[[1]]
n2<-length(S2)
P2<-rep(0,n2)
for (i in 0:(n2-1)){P2[i+1]<-S2[n2-i]}
WMT<-rep(0,(n2-1))
for (i in 1:(n2-1)){WMT[i]<-100*(log(P2[i+1]/P2[i]))}

Z3<-scan("c:/S-and-P-TSX-Composit-index-1998-2003.txt",what=list("",0))
S3<-Z3[[2]]
n3<-length(S3)
P3<-rep(0,n3)
for (i in 0:(n3-1)){P3[i+1]<-S3[n3-i]}
SandPTSX<-rep(0,(n3-1))
for (i in 1:(n3-1)){SandPTSX[i]<-100*(log(P3[i+1]/P3[i]))}

Z4<-scan("c:/data/ADCT.txt",what=list(0))
S4<-Z4[[1]]
n4<-length(S4)
P4<-rep(0,n4)
for (i in 0:(n4-1)){P4[i+1]<-S4[n4-i]}
ADCT<-rep(0,(n2-1))
for (i in 1:(n2-1)){ADCT[i]<-100*(log(P2[i+1]/P2[i]))}

par(mfrow=c(4,2))

acf(AMAT,plot=T)
acf(AMAT,36,type="partial",plot=T)
acf(WMT,plot=T)
acf(WMT,36,type="partial",plot=T)
acf(SandPTSX,plot=T)
acf(SandPTSX,36,type="partial",plot=T)
acf(ADCT,plot=T)
acf(ADCT,36,type="partial",plot=T)
```

Appendix B

Data Sets

In this thesis we use the closing price of the following Companies for trading dates in the given period of time. The data is also available on the YAHOO Finance Web-site, **http://finance.yahoo.com/**.

- The daily AMEX Composite index from March 3, 1997 to December 11, 2003, with sample size $n = 1707$.
- The daily NASDAQ Stock Composite from January 5, 1998 to December 9 2003, with sample size $n = 1493$.
- The daily S & P TSX Composite Index from January 2, 1998 to December 9, 2003, with sample size $n = 1487$.
- The daily Applied Material Inc. (AMAT) from January 2, 1998 to December 12, 2003, with sample size $n = 1496$.
- The daily Wall Mart Stores Inc. (WMT) from November 1, 1991 to June 30, 2003, with sample size $n = 2939$.
- The daily S & P TSX Composite Index from January 2, 1998 to December 9, 2003, with sample size $n = 1487$.
- The daily ADC Telecommunication Inc. (ADCT) from October 24, 96 to December 12, 2003, with sample size $n = 1789$.
- The daily Walt Disney Company (DIS) from January 3, 1984 to March 15, 2004, with sample size $n = 5097$.

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