FACULTÉ DES ÉTUDES SUPÉRIEURES ET POSTOCTORALES

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FACULTY OF GRADUATE AND POSDOCTORAL STUDIES

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GRADE / DEGREE

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Positive Cocycles for Minimal Z^d – Actions on a Cantor Set Resulting from Cut and Project Schemes: The Octogonal Tiling

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POSITIVE COCYCLES FOR MINIMAL $\mathbb{Z}^d$–ACTIONS ON A CANTOR SET RESULTING FROM CUT AND PROJECT SCHEMES: THE OCTOGONAL TILING

By
Christiane Laperrière, B.Sc.
February 2009

A Thesis
submitted to the School of Graduate Studies and Research
in partial fulfillment of the requirements
for the degree of
Masters of Science in Mathematics[1]

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[1] The M.Sc. Program is a joint program with Carleton University, administered by the Ottawa-Carleton Institute of Mathematics and Statistics.

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ISBN: 978-0-494-59903-7
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ISBN: 978-0-494-59903-7

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Abstract

We study the cut and projection method, which is a way to construct tilings. This construction leads to a minimal $\mathbb{Z}^d$-action on the Cantor set. In this thesis, we will focus our attention on two examples that we will describe in full details: the Fibonacci tiling on $\mathbb{R}$ and the octogonal tiling on $\mathbb{R}^2$. For the octogonal tiling, we find small strictly positive cocycles for the minimal action on specific cones of $\mathbb{Z}^2$.

Acknowledgements

J'aimerais tout d'abord remercier mon superviseur, Thierry Giordano, pour sa patience et son aide sans borne. Son enthousiame pour les mathématiques est contagieux. Je tiens également à remercier le CRSNG pour l'aide financière qui a supporté ce projet. Un gros merci à Patrick Boily pour son soutien moral continu, son affection et le partage de ses connaissances et expériences. Finalement, merci à ma famille; Maman, Jay et Boubou, et à mes meilleures amies; Myriam Joyal Boudreau, Termeh Kousha et Kaitlyn Church, pour leur amour et encouragements.

Dedication

À Pépère, parce que cette thèse t'a toujours intéressé.

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Chapter 1

Introduction

This thesis is focused on the minimal $\mathbb{Z}^d$-actions over a Cantor set that arise in the construction of certain tilings in $\mathbb{R}^d$. In the last 35 years, the study of tilings has undergone a major revolution, thanks to an *a priori* unrelated link with crystallography.

1.1 Historical Perspective

The notion of what constitutes a crystal has changed numerous times over the twentieth century. For a while, scientists could not even agree on what a crystal was:

> Four hundred years ago, not only were crystals not distinguished from fossils, they were not always distinguished from living things. The reality of atoms was not fully established until the discovery of the x-ray diffraction in 1912. [27]

Diffraction refers to different phenomena associated with the bending of waves when they interact with obstacles in their path. In a Bragg diffraction pattern, first observed in 1913, each dot is formed from the constructive interference of x-ray waves going through a crystal. The data can be used to determine the symmetry and the

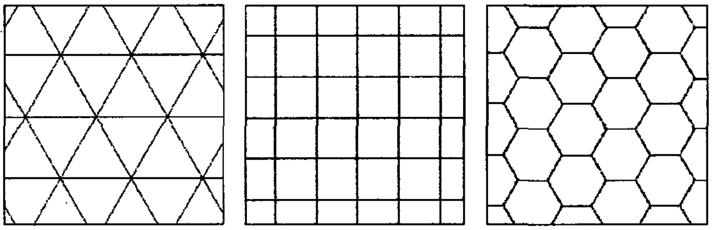

Figure 1.1: Periodic tilings with a single type of regular polygon (modified from [14]).

atomic structure of the crystal. The diffraction pattern is a consequence of the interference of the x-ray waves reflecting from different crystal planes. Informally, then, a **crystal** is a solid with an essentially discrete diffraction diagram.

To understand the link between crystallography and tilings, we need to know what a tiling is:

> A tiling of a space is a collection of tiles which completely covers the space, and such that for each pair of tiles in the tiling the interiors have empty intersection. [23, p.8]

In chapter 4, we will give a formal definition of a tiling. Constructing a tiling is very similar to covering a bathroom floor with ceramic tiles. In the plane, we use polygonal tiles. Squares, equilateral triangles and hexagons are the only regular polygons that tile the entire plane using identical copies of themselves (see Figure 1.1). For example, if one was to attempt to cover the plane with copies of a single regular pentagon, the tilings thus constructed would always have gaps in them (see Figure 1.2).

Many tilings are **periodic**, meaning that a basic unit consisting of one or more tiles is regularly repeated throughout the entire pattern [22].

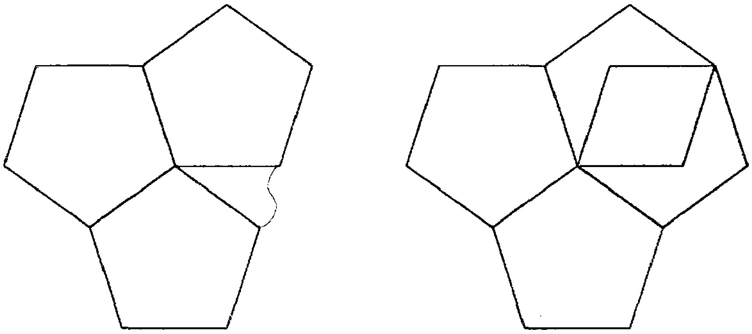

Figure 1.2: A single pentagon cannot tile the plane. A pentagon has interior angles of 108 degrees. If we glue three pentagons around a point, it covers 3 × 108 = 324 < 360 degrees, and four pentagons cover 4 × 108 = 432 > 360 degrees so that the tiles overlap (from [14]).

Non-periodic structures, also known as **aperiodic** structures, are forbidden for crystals because the constituents of the atoms that make up crystals have to fall into an orderly, regular arrangement [21]. The **crystallographic restriction** formalizes this notion by specifying which rotational symmetries are allowed in crystal structures: concretely, crystals can only have twofold, threefold, fourfold, or sixfold axes of rotational symmetry, as periodicity is incompatible with N-fold rotational symmetry for $N = 5$ and $N > 6$ [27, p.6].

In the early 1970's, Roger Penrose was trying to answer what appeared to be an unrelated question: is there a single tile which builds only nonperiodic tilings? We still cannot answer this question for tilings of the plane, however, in 1974, Penrose found examples of several *pairs* of tiles that tile the plane only nonperiodically. Such a pair is the kite and the dart. Figure 1.3 shows the kite and dart tiles, as well as the modified kite and dart tiles which are necessary to build aperiodic tilings:

> One can modify the original quadrilaterals [...], adding bumps and dents to the edges [...]. The two original quadrilaterals can abut to form a

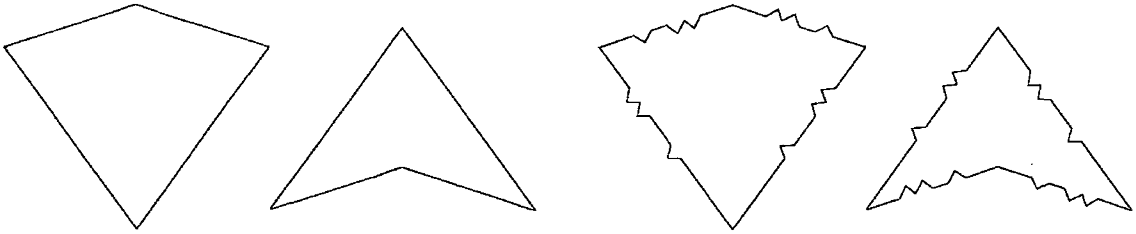

Figure 1.3: The kite and dart and the modified kite and dart tiles (from [23]).

> rhombus from which one can tile the plane in a simple periodic way; this is what adding the bumps manages to avoid. [23, p.36]

Penrose tilings produce a diffraction pattern having a fivefold symmetry (as can be seen in Figure 1.4). Although such tilings are nonperiodic, and so cannot be directly connected to crystals, Penrose started to have doubts about the theory underlying crystal structures. His doubts were justified:

> A few people did take Penrose's ideas seriously, but it took the discovery in 1984 of tiny metallic crystals composed of aluminum and manganese to alert the scientific world to Penrose's tilings. [21]

Indeed, the solid discovered by Dany Schectman *et al* had a diffraction pattern showing icosahedral symmetry, which is the equivalent of fivefold symmetry in three dimensions (see Figure 1.5). According to the crystallographic restriction, the solid in question had a nonperiodic atomic structure and so could not be a crystal. At the time, this was very surprising. As Charles Radin mentions in [23], "Physicists had gotten used to thinking that solids *had* to be crystalline, though they were aware that this state of affairs did not seem to follow from any known physical law."

The newly discovered solid was highly ordered, just like any other crystal, but did not have periodicity.

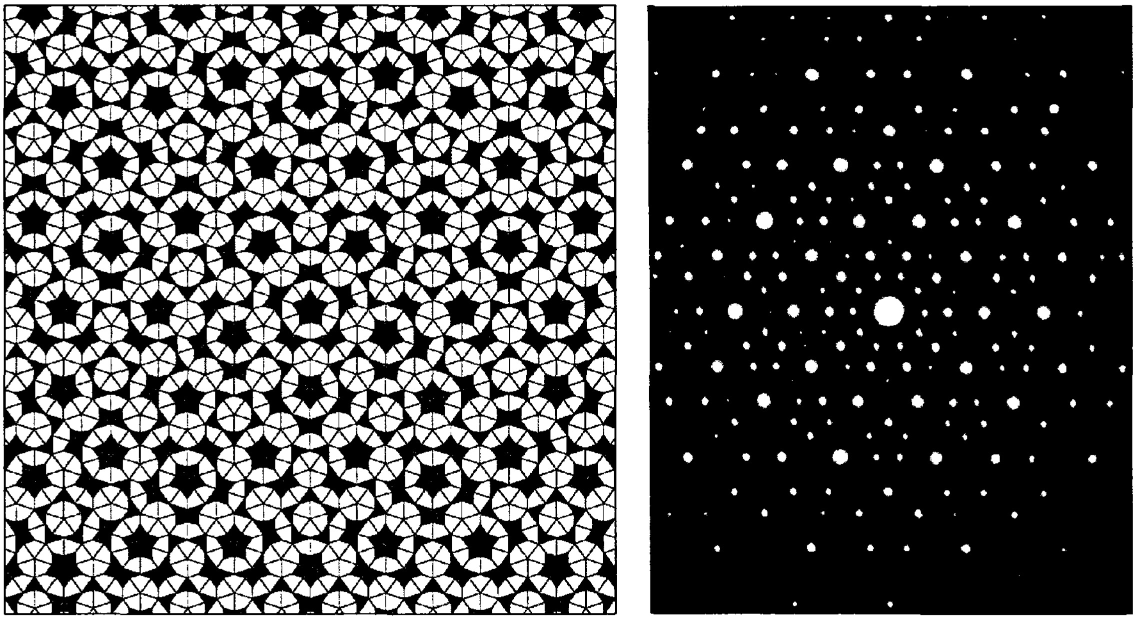

Figure 1.4: A Penrose tiling using the kite and dart tiles and its diffraction pattern. The Penrose tiling is taken from [3] and the diffraction pattern is taken from [2].

After this first discovery, many other such solids were found. They were called quasiperiodic crystals, or simply quasicrystals. Informally, a **quasicrystal** is a solid whose diffraction pattern exhibits symmetry forbidden by the crystallographic restriction.

At that point, people became more interested in Penrose tilings and how they were constructed, as their diffraction patterns were very similar to those of quasicrystals. Later on, other aperiodic tilings of the plane were constructed using two or more tiles. The octogonal tiling studied in this work and the Pinwheel tiling are examples of such tilings (see Figure 1.6 and Figure 1.7). These tilings are constructed using either of two different methods; the **substitution rule** and the **cut and projection method**. We will give a detailed description of the latter method in chapter 4.

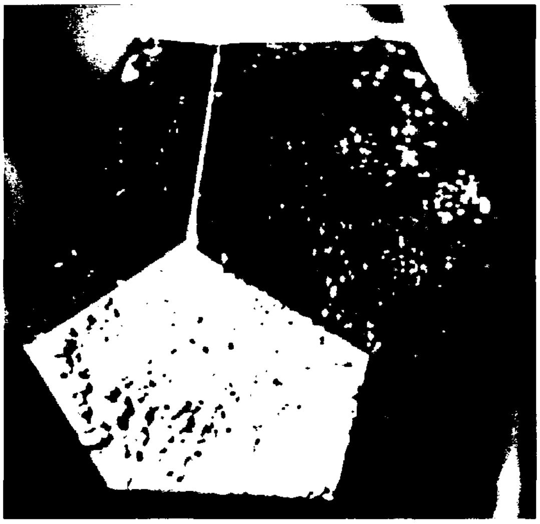

Figure 1.5: The quasicrystal discovered by Schectman (from [3]).

1.2 Thesis Overview

This thesis studies the minimal $\mathbb{Z}^d$-actions over the Cantor set which arises in the construction of the Fibonacci tiling of $\mathbb{R}$ and the octogonal tiling of $\mathbb{R}^2$. We will study these actions and small cocycles associated to these actions. Since the Cantor set will appear often in our work, chapter 2 will start with a reminder of the definition and the theorem of uniqueness of the Cantor set. Next, we present the basic definition and results concerning C^*-algebras that will be of use in subsequent parts of this work: the main objective of this section is to prove the Gelfand-Naimark Theorem which states that every commutative unital C^*-algebra is isomorphic to $C(X)$ for some compact Hausdorff space X. We prove this theorem early on so that we can use it in the last section of chapter 4.

Chapter 3 will give definitions and results about cocycles. In particular, we will give the definition of a small strictly positive cocycle on subsets of $\mathbb{Z}^2$ for free, minimal $\mathbb{Z}^2$-actions on the Cantor set. We will focus our attention on product actions: they

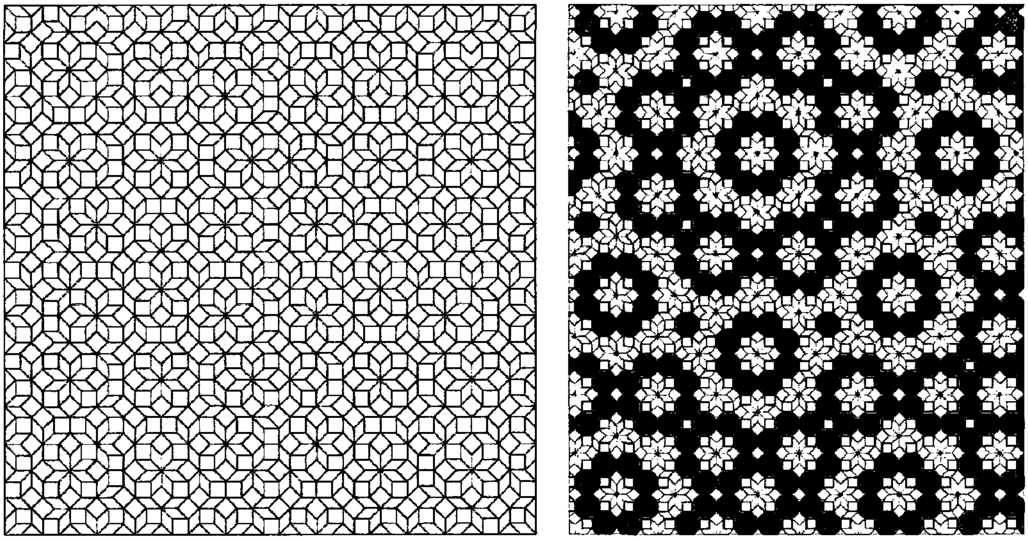

Figure 1.6: Octogonal tilings (from [3]).

will arise from the construction of tilings in chapter 4. We will give three conditions for a minimal free Cantor $\mathbb{Z}^2$-system that will guarantee the existence of a small strictly positive cocycle. We then apply this theorem to product actions on the Cantor set and specific cones. We show that given a free minimal $\mathbb{Z}^2$ product action on the product of two Cantor sets, we can find small strictly positive cocycles, for specific cones in $\mathbb{Z}^2$. We also show that if (X, Φ_1) and (Y, Φ_2) are two minimal Cantor $\mathbb{Z}^2$-dynamical systems and that $F : X \to Y$ is a $\mathbb{Z}^2$–equivariant factor map, then the existence of a small strictly positive cocycle θ on a cone C for (Y, Φ_2) implies the existence of a small strictly positive cocycle on C for (X, Φ_1). This cocycle will be given by $\theta \circ (F \times \mathrm{Id})$.

A cut and projection method is defined by a cut and project scheme which is a triplet of three spaces; the d-dimensional physical space where the tiling will be constructed, the n-dimensional internal space and a lattice. We project specific points of the lattice on the physical space. This gives us a point set from which we construct a tiling, using the Voronoi tesselation method. In the first part of chapter 4, we give

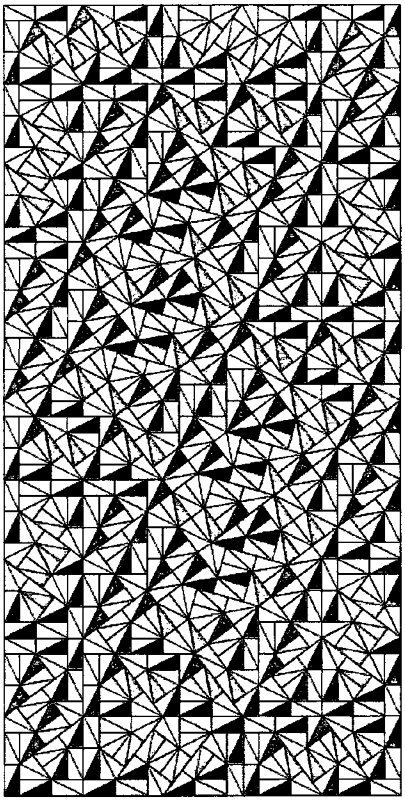

Figure 1.7: A pinwheel tiling (from [3]).

basic notions of tilings, then we discuss the cut and projection method. This theory will be illustrated with the help of the two tilings mentioned above; the Fibonacci tiling and the octogonal tiling.

The second part of chapter 4 explains how, from certain cut and project schemes, we can construct a minimal $\mathbb{Z}^d$-action on a Cantor set. This action will in fact be given by rotations on the product of n Cantor sets.

Chapter 2

Preliminaries

2.1 The Cantor Set

Let A_0 be the closed interval $[0,1]$, $A_1 = A_0 - (\frac{1}{3}, \frac{2}{3}) = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1] = A_{1,1} \cup A_{1,2}$, $A_2 = A_1 - ((\frac{1}{9}, \frac{2}{9}) \cup (\frac{7}{9}, \frac{8}{9})) = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{3}{9}] \cup [\frac{6}{9}, \frac{7}{9}] \cup [\frac{8}{9}, 1] = A_{2,1} \cup A_{2,2} \cup A_{2,3} \cup A_{2,4}$ and in general, define

$$A_n = A_{n-1} - \bigcup_{k=0}^{\infty} \left(\frac{1+3k}{3^n}, \frac{2+3k}{3^n} \right) = \bigcup_{i \in \{1, \ldots, 2^n\}} A_{n,i}.$$

Definition 2.1.1 The **Cantor set**, denoted by C, is the intersection

$$C = \bigcap_{n \in \mathbb{N} \cup \{0\}} A_n,$$

with the induced topology from $\mathbb{R}$. We say that a topological space is a **Cantor space** if it is homeomorphic to the Cantor set.

In fact, we have the following theorem that we could also use as the definition of a Cantor space.

Theorem 2.1.2 A topological space X is a Cantor space if and only if it is a totally disconnected compact metrizable space with no isolated points.

Proof: See p.197 in [13], section 30 in [29], and p.121 in [1]. □

Remark 2.1.3 The totally disconnected condition in the last theorem can be replaced by the existence of a basis of clopen sets. Since every metric compact space is separable (see 3.16.2 in [7]), i.e. has a countable dense subset, the basis of clopen sets can be assumed to be countable.

2.2 C^*-algebras

Most definitions and results in this section are taken from [6, 16]. Let us first recall the definition of an algebra over a field K.

Definition 2.2.1

1. An **algebra** A over a field K is a K-vector space equipped with an associative bilinear operation.
2. An algebra A is **unital** if there is an element $1 \in A$, called a **unit**, such that $1a = a1 = a$ for all $a \in A$.
3. An **involutive** algebra A is an algebra over $\mathbb{C}$ together with a map

$$a \in A \mapsto a^* \in A$$

   (called an involution) satisfying the following properties:

   i. $(a^*)^* = a, \forall\ a \in A$

   ii. $(\lambda a + \mu b)^* = \overline{\lambda} a^* + \overline{\mu} b^*,\ \forall\ a, b \in A,\ \forall\ \lambda, \mu \in \mathbb{C}$

   iii. $(ab)^* = b^* a^*,\ \forall\ a, b \in A$.

Example 2.2.2

1. $M_n(\mathbb{C})$, the set of $n \times n$-matrices with complex coefficients, is a unital involutive algebra, where the involution is given by the conjugate transpose $M^* = (\overline{M})^T$, $M \in M_n(\mathbb{C})$.

2. $C(X)$, the continuous functions from X to $\mathbb{C}$, where X is compact, with the pointwise multiplication of functions, is an involutive unital algebra, where $f(x)^* = \overline{f(x)}$, $\forall\, x \in X$.

3. $C_0(X)$, the continuous functions from X to $\mathbb{C}$ which vanish at infinity, where X is locally compact, is an involutive algebra for the same involution as $C(X)$. Notice that $C_0(X)$ has no unit. See Example 2.2.8 for more details.

Recall that a (complex) **Banach space** is a complete normed vector space over $\mathbb{C}$.

Definition: A complex **Banach algebra** A is an algebra over $\mathbb{C}$ that has a norm $||\cdot||$ relative to which A is a Banach space and such that for all $a, b \in A$, we have $||ab|| \leq ||a||||b||$. Moreover, if A is unital, we require that $||1|| = 1$.

Example 2.2.3

1. $C(X)$ with the sup norm, where X is compact, is a Banach algebra.

2. $(L^1(\mathbb{R}, \lambda), *)$ is a Banach algebra, where λ is the Lebesgue measure and $*$ denotes the multiplication by convolution.

Definition 2.2.4 A C^***-algebra** is a complex Banach space A which is an involutive algebra such that for all $a, b \in A$ one has

1. $||ab|| \leq ||a||||b||$

2. $||a^*a|| = ||a||^2$.

Note that these two conditions imply that $\|a^*\| = \|a\|$. Indeed,

$$\|a\|^2 = \|a^*a\| \leq \|a^*\|\|a\| \Rightarrow \|a\| \leq \|a^*\| \leq \|a^{**}\| = \|a\|.$$

Remark 2.2.5 If A is a C^*-algebra, there is only one norm that will satisfy the second property of the previous definition, that is, norms on C^*-algebras are unique. We prove this in Proposition 2.3.14.

Example 2.2.6 $M_n(\mathbb{C})$ and $C(X)$, where X is compact, are C^*-algebras.

Remark 2.2.7 There exists Banach algebras which are not C^*-algebras for any involution. Such an example is the following Banach algebra:

$$A = \left\{ \begin{pmatrix} \lambda I_n & B \\ 0 & \lambda I_n \end{pmatrix} ; \lambda \in \mathbb{C}, B \in M_n(\mathbb{C}) \right\},$$

with the usual matrix operations and matrix norm. Recall that the Jacobson radical of a ring R is the intersection of the annihilators of all simple right R-modules. We can show that the Jacobson radical of A is not trivial. Since every finite dimensional C^*-algebra must have a trivial Jacobson radical, A cannot be a C^*-algebra for any involution. Notice that there exists an involution on A but it does not satisfy the second property of C^*-algebras.

Example 2.2.8 If X is a locally compact space then we denote by $C_0(X)$ the set of all continuous functions $f : X \longrightarrow \mathbb{C}$ which vanish at infinity, in the sense that for every $\varepsilon > 0$ there exists a compact subset $K \subseteq X$ such that $|f(x)| < \varepsilon$ for every $x \in X \backslash K$. In this example, we show that $C_0(X)$ is a C^*-algebra.

We will use the sup norm and the involution defined by $f^*(x) = \overline{f(x)}$ for all $x \in X$.

If $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $C_0(X)$ then for all $x \in X$, $(f_n(x))_{n\in\mathbb{N}}$ will be a Cauchy sequence in $\mathbb{C}$, since

$$\|f_n - f_m\| = \sup_{x\in X}\{|f_n(x) - f_m(x)|\}, \ \forall\ n, m \in \mathbb{N}.$$

As $\mathbb{C}$ is complete, $(f_n(x))_{n\in\mathbb{N}}$ converges. Define $f(x) = \lim_{n\to\infty} f_n(x)$. Then it is well-known that f_n will converge to f and that f is continuous.

To show that f vanishes at infinity, let $\varepsilon > 0$. Choose n large enough so that $\|f - f_n\| < \frac{\varepsilon}{2}$. For this f_n there exists a compact set K for which $|f_n(x)| < \frac{\varepsilon}{2}$ for all $x \in X\backslash K$. Then

$$\begin{aligned}|f(x)| &= |f(x) - f_n(x) + f_n(x)| \\ &\leq |f(x) - f_n(x)| + |f_n(x)| < \varepsilon,\end{aligned}$$

for all $x \in X\backslash K$. Therefore, $f \in C_0(X)$.

Finally, defining $(fg)(x) = f(x)g(x)$ and $f^*(x) = \overline{f(x)}$ for all $x \in X$ makes $C_0(X)$ into a commutative C^*-algebra. Indeed, the three conditions for the involution are easy to verify and we also have

$$\|fg\| = \sup_{x\in X}\{|f(x)g(x)|\} = \sup_{x\in X}\{|f(x)||g(x)|\} \leq \sup_{x\in X}\{|f(x)|\}\sup_{x\in X}\{|g(x)|\} = \|f\|\|g\|$$

and

$$\begin{aligned}\|ff^*\| &= \sup_{x\in X}\{|f(x)f^*(x)|\} \\ &= \sup_{x\in X}\{|f(x)\overline{f(x)}|\} \\ &= \sup_{x\in X}\{|f(x)|^2\} \\ &= \left(\sup_{x\in X}\{|f(x)|\}\right)^2 \\ &= \|f\|^2,\end{aligned}$$

which conclude this example. □

Example 2.2.9 An **Hilbert space** is a vector space with an inner product that is complete in the associated norm. If H and K are Hilbert spaces then a **bounded operator** from H to K is a linear map $a : H \to K$ for which

$$||a|| := \sup\{||av||; v \in H, (v, v) = 1\} < \infty.$$

We denote $B(H, K)$ the space of all such bounded operators. We have that $B(H, K)$ is a Banach space and when $K = \mathbb{C}$ we write $B(H, \mathbb{C}) = H^*$ (the topological dual of H). For every operator $a \in B(H)$, there is a unique operator $a^* \in B(H)$ such that $(v, a^*w) = (av, w)$ for all $v, w \in H$. We call a^* the adjoint of a and we can check that the application which sends a to a^* is an involution on $B(H)$ such that $||a^*a|| = ||a||^2$. Hence $B(H)$ is a C^*-algebra. □

Theorem 2.2.10 (GNS construction) The involutive subalgebras of $B(H)$ are precisely the Banach algebras with an involution satisfying the second C^*−condition.

Definition 2.2.11 Let A be a unital Banach algebra. The **spectrum** $\sigma(a)$ of $a \in A$ is the set of all $z \in \mathbb{C}$ for which $a - z1$ has no inverse in A.

Theorem 2.2.12 ([16], section 3) The spectrum $\sigma(a)$ of any element $a \in A$ is a non-empty compact subset of $\{z \in \mathbb{C}; |z| \leq ||a||\}$.

2.3 Commutative C^*−algebras and the Gelfand-Naimark Theorem

The following definitions and results are taken from [5].

Definition 2.3.1 A **field** F is a ring in which every nonzero element has a multiplicative inverse.

Definition 2.3.2 A **division ring** is a field in which the multiplication need not be associative.

Theorem 2.3.3 (Gelfand-Mazur) If A is a complex Banach algebra that is also a division ring, then $A = \mathbb{C}$, i.e. $A = \{\lambda 1;\ \lambda \in \mathbb{C}\}$.

Proof: If $a \in A$, then $\sigma(a) \neq \emptyset$. Let $\lambda \in \sigma(a)$ so that $a - \lambda 1$ has no inverse. Since A is a division ring, we have that $a - \lambda 1 = 0$, that is $a = \lambda 1$. ∎

Remark 2.3.4 If A is an abelian complex Banach algebra and $h : A \longrightarrow \mathbb{C}$ is a nonzero homomorphism, then $\ker(h)$ is a maximal ideal. In fact, we have the following proposition.

Proposition 2.3.5 ([5], Proposition 8.2) If A is an abelian Banach algebra and $\mathcal{M}$ is a maximal ideal, then there is a homomorphism $h : A \longrightarrow \mathbb{C}$ such that $\mathcal{M} = \ker(h)$. Conversely, if $h : A \longrightarrow \mathbb{C}$ is a nonzero homomorphism, then $\ker(h)$ is a maximal ideal. Moreover, this correspondence $h \longmapsto \ker(h)$ between homomorphisms and maximal ideals is bijective.

Corollary 2.3.6 If A is an abelian Banach algebra and $h : A \longrightarrow \mathbb{C}$ is a homomorphism, then h is continuous.

Proof: Maximal ideals are closed, hence $\ker(h)$ is closed. ∎

Proposition 2.3.7 If A is an abelian unital Banach algebra and $h : A \longrightarrow \mathbb{C}$ is a nonzero homomorphism, then $\|h\| = 1$.

Proof: Let $a \in A$ and set $\lambda = h(a)$. If $|\lambda| > \|a\|$, then $\|a/\lambda\| < 1$. Hence $1 - a/\lambda$ is invertible. Let $b = (1 - a/\lambda)^{-1}$ so that $1 = b(1 - a/\lambda) = b - ba/\lambda$. Since $h(1) = 1$, we have

$$1 = h(b - ba/\lambda) = h(b) - h(b)h(a)/\lambda = h(b) - h(b) = 0,$$

which is a contradiction. Hence $\|a\| \geq |\lambda| = |h(a)|$. Consequently,

$$\|h\| = \sup\{|h(a)|; a \in A, \|a\| \leq 1\} \leq \sup\{\|a\|; a \in A, \|a\| \leq 1\} = 1,$$

since $1 \in A$. ∎

Definition 2.3.8 If A is an abelian Banach algebra, let $\Delta(A)$ denote the set of all nonzero homomorphisms of $A \rightarrow \mathbb{C}$. Endow $\Delta(A)$ with the relative weak*-topology that it has as a subset of A^*. With this topology, $\Delta(A)$ is called the **maximal ideal space** of A. Basic weak*-neighbourhoods for $h \in \Delta(A)$ are given by sets of the form

$$B^{\varepsilon}_{a_1,\ldots,a_n}(h) = \{j \in \Delta(A); |j(a_i) - h(a_i)| < \varepsilon, \ i = 1, \ldots, n\},$$

where $\varepsilon > 0, n \in \mathbb{N}$, and $a_i \in A$ for $i = 1, \ldots, n$.

Theorem 2.3.9 If A is an abelian Banach algebra, then its maximal ideal space, $\Delta(A)$, is a compact Hausdorff space.

Proof: We first notice that $\Delta(A)$ is contained in the unit ball of A^* (i.e. the bounded linear functionals on A with norm less than or equal to one). By the Alaoglu Theorem (Theorem 9.7.9 in [8]), this ball is compact in the weak*-topology. Hence, to show that $\Delta(A)$ is compact, it suffices to show that $\Delta(A)$ is weak*-closed. Let $\{h_i\}$ be a net in $\Delta(A)$ and suppose that $h_i \rightarrow h$ in the weak*-topology, for some h in the unit ball. For any $a, b \in A$ we have:

$$\begin{tikzcd} h_i(ab) \arrow[d] & = & h_i(a) \arrow[d] & h_i(b) \arrow[d] \\ h(ab) & & h(a) & h(b) \end{tikzcd}$$

and so $h(ab) = h(a)h(b)$. Also, since $h_i(1) = 1$ for all i, then $h(1) = 1$. Hence h is a nonzero homomorphism, i.e. $h \in \Delta(A)$. Therefore, $\Delta(A)$ is compact in the weak* topology. To prove that $\Delta(A)$ is Hausdorff, consider $h \neq k \in \Delta(A)$. By hypothesis, there exists $a \in A$ such that $h(a) \neq k(a)$. Let $\varepsilon = |h(a) - k(a)| > 0$ and consider the two following neighbourhoods of h and k, respectively,

$$B_a^{\frac{\varepsilon}{2}}(h) = \{j \in \Delta(A); |j(a) - h(a)| < \tfrac{\varepsilon}{2}\}$$
$$B_a^{\frac{\varepsilon}{2}}(k) = \{j \in \Delta(A); |j(a) - k(a)| < \tfrac{\varepsilon}{2}\}.$$

These two neighbourhoods are disjoint, for if $j \in B_a^{\frac{\varepsilon}{2}}(h) \cap B_a^{\frac{\varepsilon}{2}}(k)$, then

$$|h(a) - k(a)| \leq |h(a) - j(a)| + |j(a) - k(a)| < \varepsilon,$$

which is a contradiction. This shows that $\Delta(A)$ is Hausdorff. ∎

Theorem 2.3.10 ([5], Theorem 8.7) If X is compact and $\Delta(C(X))$ is the maximal ideal space of $C(X)$, then the map $x \mapsto \delta_x$ is a homeomorphism of X onto $\Delta(C(X))$.

Definition 2.3.11 Let A be an abelian Banach algebra with maximal ideal space $\Delta(A)$. If $a \in A$, then the **Gelfand transform** of a is the function $\hat{a} : \Delta(A) \rightarrow \mathbb{C}$ defined by $\hat{a}(h) = h(a)$.

Definition 2.3.12 The **spectral radius** $r(a)$ of $a \in A$ is defined by

$$r(a) = \sup\{|z| \ ; \ z \in \sigma(a)\}.$$

Proposition 2.3.13 ([16], sections 3 and 4) For each $a \in A$, one has

1. $r(a) = \lim_{n \rightarrow \infty} \|a^n\|^{1/n}$
2. $\sigma(a) = \sigma(\hat{a})$,

where $\sigma(\hat{a}) = \{h(a); \ h \in \Delta(A)\}$.

Proposition 2.3.14 ([20], p.870) If $\|\cdot\|$ is a C^*−norm on a involutive algebra A, then it is given by the expression

$$\|a\| = r(a^*a)^{\frac{1}{2}}, \ \forall\ a \in A.$$

Hence a C^*−norm on a involutive algebra is unique if it exists.

Proof: If $\|\cdot\|$ is a C^*−norm on A, then repeated use of the second C^*−condition gives

$$\|a\|^2 = \|a^*a\| = \|(a^*a)^2\|^{\frac{1}{2}} = \ldots = \|(a^*a)^{2^n}\|^{\frac{1}{2^n}}, \ \forall\ n \in \mathbb{N}.$$

Hence

$$\|a\|^2 = \lim_{n\to\infty} \|(a^*a)^{2^n}\|^{\frac{1}{2^n}} = \lim_{m\to\infty} \|(a^*a)^m\|^{\frac{1}{m}} = r(a^*a),$$

by Proposition 2.3.13. Therefore, $\|a\| = r(a^*a)^{\frac{1}{2}}, \ \forall\ a \in A.$ ∎

Theorem 2.3.15 If A is an abelian Banach algebra with maximal ideal space $\Delta(A)$ and $a \in A$, then the Gelfand transform of a, denoted by $\hat{a}$, belongs to $C(\Delta(A))$. The map $a \mapsto \hat{a}$ of A into $C(\Delta(A))$ is a continuous homomorphism of norm 1.

Proof: If $h_i \to h$ in $\Delta(A)$, then $h_i \to h$ in the weak* topology in A^*. So if $a \in A$,

$$\hat{a}(h_i) = h_i(a) \to h(a) = \hat{a}(h).$$

Thus $\hat{a} \in C(\Delta(A))$.

Define $\gamma : A \to C(\Delta(A))$ by $\gamma(a) = \hat{a}$. If $a, b \in A$, then

$$\gamma(ab)(h) = \widehat{ab}(h) = h(ab) = h(a)h(b) = \hat{a}(h)\hat{b}(h).$$

Therefore, $\gamma(ab) = \gamma(a)\gamma(b)$. It is easy to see that γ is linear so γ is a homomorphism. Also, by Proposition 2.3.7, we know that if $a \in A$, $|\hat{a}(h)| = |h(a)| \le \|a\|$; thus $\|\gamma(a)\|_\infty = \|\hat{a}\|_\infty \le \|a\|$. So γ is continuous and $\|\gamma\| \le 1$. But since $\gamma(1) = 1$, $\|\gamma\| = 1$. ∎

Lemma 2.3.16 If A is a unital C^*-algebra and $a = a^* \in A$ then $h(a) \in \mathbb{R}$ for all $h \in \Delta(A)$.

Proof: Let $h(a) = \alpha + i\beta$, for $\alpha, \beta \in \mathbb{R}$. Let us show that $\beta = 0$. Let $b = a - \alpha 1$. Then for $t \in \mathbb{R}$,

$$\begin{aligned} |h(b+it1)|^2 &\le \|b+it1\|^2 && \text{(by Proposition 2.3.7)} \\ &= \|(b+it1)^*(b+it1)\| && \text{(by the second property of a } C^*\text{–algebra)} \\ &= \|(b-it1)(b+it1)\| && \text{(since } b \text{ is self-adjoint)} \\ &= \|b^2+t^2 1\| \le \|b\|^2 + t^2 \end{aligned}$$

Moreover, we have

$$|h(b+it1)|^2 = |h(b)+it|^2 = |i\beta + it|^2 = |\beta + t|^2 = (\beta+t)^2 = \beta^2 + 2\beta t + t^2.$$

Combining these two inequalities, we get $\beta^2 + 2t\beta \le \|b\|^2$ for all $t \in \mathbb{R}$. This can only hold if $\beta = 0$ which proves the result. ∎

Theorem 2.3.17 If A is an abelian unital C^*-algebra and $\Delta(A)$ is its maximal ideal space, then the Gelfand transform $\gamma : A \longrightarrow C(\Delta(A))$ is an isometric *-isomorphism.

Proof: We first show that $\widehat{a^*} = \overline{\hat{a}}$. That is, for any $h \in \Delta(A)$, $h(a^*)$ must be equal to $\overline{h(a)}$. Let $a \in A$ and $h \in \Delta(A)$. We write $a = \frac{a+a^*}{2} + i\frac{a-a^*}{2i}$ so that $a^* = \frac{a^*+a}{2} + i\frac{a^*-a}{2i} = \frac{a^*+a}{2} - i\frac{a-a^*}{2i}$. Hence

$$h(a^*) = h\left(\frac{a^*+a}{2} - i\frac{a-a^*}{2i}\right) = h\left(\frac{a^*+a}{2}\right) - ih\left(\frac{a-a^*}{2i}\right).$$

Moreover,

$$h(a) = h\left(\frac{a+a^*}{2}\right) + ih\left(\frac{a-a^*}{2i}\right).$$

Since $\frac{a+a^*}{2}$ and $\frac{a-a^*}{2i}$ are self-adjoint, Lemma 2.3.16 tells us that $h\left(\frac{a+a^*}{2}\right)$ and $h\left(\frac{a-a^*}{2i}\right)$ are real, so that

$$\overline{h(a)} = h\left(\frac{a+a^*}{2}\right) - ih\left(\frac{a-a^*}{2i}\right) = h(a^*).$$

Hence γ is a $*$−homomorphism.

We now show that the Gelfand transform is an isometry. First, let $x = x^*$ be a self-adjoint element of A. Then $||x^2|| = ||xx^*|| = ||x||^2$, so that $||x|| = r(x)$ by part 1 of Proposition 2.3.13. But part 2 of the same proposition implies that

$$r(x) := \sup\{|z| \ ; \ z \in \sigma(x)\} = \sup\{|z| \ ; \ z \in \sigma(\hat{x})\} = \sup\{|h(x)| \ ; \ h \in \Delta(A)\} = ||\hat{x}||.$$

Therefore, $||x|| = ||\hat{x}||$ when $x = x^*$. Let $a \in A$. We have

$$\begin{aligned} ||\hat{a}||^2 &= ||\hat{a}^*\hat{a}|| && \text{since } C(\Delta(A)) \text{ is a } C^*\text{−algebra} \\ &= ||\widehat{a^*}\hat{a}|| && \text{since } \gamma \text{ is } *\text{-homomorphism} \\ &= ||\widehat{a^*a}|| && \\ &= ||a^*a|| && \text{since } a^*a \text{ is self-adjoint} \\ &= ||a||^2 && \text{since } A \text{ is a } C^*\text{−algebra} \end{aligned}$$

The Gelfand tranform is thus isometric and hence injective.

Because γ is an isometry, its range is closed. To show that γ is surjective, therefore, it suffices to show that its range is dense. This is accomplished by applying the Stone-Weierstrass theorem. Note that $\hat{1} = 1$, so $\gamma(A)$ is a subalgebra of $C(\Delta(A))$ containing the constant functions. Because γ preserves the involution, $\gamma(A)$ is closed under complex conjugation.

It remains to show that $\gamma(A)$ separates the points of $\Delta(A)$. If h_1 and h_2 are distinct homomorphisms in $\Delta(A)$, then there exists $a \in A$ such that $h_1(a) \neq h_2(a)$, i.e. $\hat{a}(h_1) \neq \hat{a}(h_2)$. ■

Corollary 2.3.18 If A is an abelian C^*–algebra without a unit and $\Delta(A)$ is its maximal ideal space, then the Gelfand transform $\gamma : A \longrightarrow C_0(\Delta(A))$ is an isometric *-isomorphism.

Chapter 3

Cocycles

In this chapter, we define small strictly positive cocycles following [11]. The reason for studying these cocycles is that they can help us determine if an equivalence relation is affable, i.e. orbit equivalent to an approximately finite (AF) equivalence relation. The first section is a reminder of group actions and gives the description of equivalence relations associated to these actions. The second section gives definitions related to cocycles and an existence theorem of such cocycles (Theorem 3.2.4). In the next section, we explain how to use cocycles to get results on equivalence relations associated to minimal, free $\mathbb{Z}^2$-actions on a Cantor set. Finally, in the last section, we give an example of a $\mathbb{Z}^2-$action on a Cantor set for which small strictly positive cocycles exist.

3.1 Group actions and Étale Equivalence Relations

In this section, we recall the notion of a group action and we describe the equivalence relation associated to such actions.

Let G be a countable group with the discrete topology and X be a compact metric space (we will mostly be interested in the case where X is a Cantor set). Let Φ be an action of G on X, i.e. for every $g \in G$, $\Phi(g) \in \text{Homeo}(X)$ and the map $g \mapsto \Phi(g)$ from G to $\text{Homeo}(X)$ is a homomorphism.

Definition 3.1.1 The action Φ is

1. **free** if $\Phi(g)x = x$ for some $x \in X$ if and only if $g = id$;
2. **minimal** if the Φ-orbit $\mathcal{O}_\Phi(x) = \{\Phi(g)x; g \in G\}$ of every point $x \in X$ is dense in X.

Definition 3.1.2 We will call (X, Φ) a minimal, free Cantor $\mathbb{Z}^2-$system, if X is the Cantor set and Φ is a minimal, free $\mathbb{Z}^2-$action on X.

Definition 3.1.3 ([11]) Given two free group actions (X, G, Φ_1) and (Y, H, Φ_2), a topological **orbit equivalence** between them is a homeomorphism $h : X \to Y$ such that, for every $x \in X$, we have

$$h(\mathcal{O}_{\Phi_1}(x)) = \mathcal{O}_{\Phi_2}(h(x)).$$

Before giving the definition of an étale equivalence relation, we consider a motivating example. If Φ is a free continuous action of G on X, let $\mathcal{R}_\Phi$ denote the equivalence relation given by

$$\mathcal{R}_\Phi = \{(x, \Phi(g)x); x \in X, g \in G\} \subset X \times X.$$

If we consider the space $X \times G$ endowed with the product topology, we get a $\sigma-$compact, locally compact space. Since we have a bijection

$$\begin{aligned} X \times G &\to \mathcal{R}_\Phi \\ (x, g) &\mapsto (x, \Phi(g)x) \end{aligned}$$

the equivalence relation $\mathcal{R}_\Phi$ becomes a topological space $(\mathcal{R}_\Phi, \mathcal{T})$, where $\mathcal{T}$ denotes the topology defined by the above bijection. Denote the two canonical projections from $\mathcal{R}_\Phi$ to X by s and r (for source and range), so that

$$s(x, \Phi(g)x) = x, \text{ and } r(x, \Phi(g)x) = \Phi(g)x.$$

Then s and r are local homeomorphisms (that is, for any $(x, \Phi(g)x) \in \mathcal{R}_\Phi$, there exists neighborhoods $U, V \subset \mathcal{R}_\Phi$ of $(x, \Phi(g)x)$ such that $s(U)$ and $r(V)$ are open in X and $s : U \longrightarrow s(U)$ and $r : V \longrightarrow r(V)$ are homeomorphisms). Indeed, let $(x, \Phi(g)x) \in \mathcal{R}_\Phi$ and choose an open neighborhood $U_x \subset X$ of x small enough so that $U_x \cap \Phi(g)(U_x) = \emptyset$ (such a neighborhood exists by freeness of the action). Then $s : U_x \times \{g\} \longrightarrow s(U_x \times \{g\}) = U_x$ and $r : U_x \times \{g\} \longrightarrow r(U_x \times \{g\}) = \Phi(g)U_x$ are homeomorphisms.

As G is countable, $\mathcal{R}_\Phi$ is a **countable equivalence relation**, i.e. each equivalence class

$$[x]_{\mathcal{R}_\Phi} = \{y \in X; (x, y) \in \mathcal{R}_\Phi\}$$

is countable for each $x \in X$.

Definition 3.1.4 Let $\mathcal{R}$ be a countable equivalence relation on a compact metric space X. We say that $(\mathcal{R}, \mathcal{T})$ is **étale** if the maps $s, r : \mathcal{R} \longrightarrow X$ are local homeomorphisms.

3.2 Definition of Cocycles and Results

In this section, we define $\mathbb{Z}$−valued one-cocycles for free $\mathbb{Z}^2$-actions on a compact metric space. We also explain what we mean by a strictly positive cocycle. Theorem 3.2.4 gives us sufficient conditions on minimal, free Cantor $\mathbb{Z}^2$-systems that will guarantee the existence of a stricly positive cocycle. We then apply this existence theorem to give examples where strictly positive cocycles exist.

We consider $G = \mathbb{Z}^2$ endowed with the discrete topology and $X \times \mathbb{Z}^2$ with the product topology, where X is a compact metric space. Recall that a $\mathbb{Z}^2-$action $\Phi : X \times \mathbb{Z}^2 \to X$ is free if for any $x \in X$ and $n \in \mathbb{Z}^2$, we have $\Phi^n(x) = x$ only if $n = 0$. We define a norm on $\mathbb{Z}^2$ by $|(i,j)| = \max\{|i|, |j|\}$. For any $n \in \mathbb{Z}^2$ and positive integer m, we denote the closed ball

$$B(n,m) = \{n' \in \mathbb{Z}^2;\ |n' - n| \leq m\}.$$

Notice that the topology on $\mathbb{Z}^2$, induced by the norm is the discrete topology.

Definition 3.2.1 [11] Let Φ be a free action of $\mathbb{Z}^2$ on the compact metric space X. A $\mathbb{Z}-$valued **one-cocycle** for Φ is a continuous function

$$\theta : X \times \mathbb{Z}^2 \to \mathbb{Z}$$

such that, for all x in X and m, n in $\mathbb{Z}^2$, we have

$$\theta(x, m+n) = \theta(x,m) + \theta(\Phi^m(x), n).$$

Let C be a subset of $\mathbb{Z}^2$. If θ is a cocycle, we say θ is **positive** with respect to C if

$$\theta(X \times C) \geq 0.$$

We say θ is **proper** with respect to C if the map

$$\theta : X \times C \to \mathbb{Z}$$

is proper, that is, the pre-image of any finite set is compact. The usual definition of a proper map is one where the inverse image of compact sets is compact. In this case, a compact set in $\mathbb{Z}$ is a finite set. If θ is proper and positive with respect to C we say that θ is **strictly positive** with respect to C. Let M be a positive integer. We will write $\theta \leq M^{-1}$ if $|\theta(x,n)| \leq 1$ for all $x \in X$ and $n \in B(0,M)$. We say that θ is **small** if $\theta \leq 2^{-1}$.

Definition 3.2.2 A **cone** C of $\mathbb{Z}^2$ is a subset of $\mathbb{Z}^2$ defined by

$$C = \langle a,b \rangle = \{ia + jb;\ i,j \geq 0\},\ \text{for some } a,b \in \mathbb{Z}^2.$$

Lemma 3.2.3 Let $(X,\Phi_1),(Y,\Phi_2)$ be two minimal Cantor $\mathbb{Z}^2$-dynamical systems and let $F : X \to Y$ be a $\mathbb{Z}^2-$equivariant factor map, i.e. F is continuous, surjective and

$$F(\Phi_1(x)) = \Phi_2(F(x)),\ \forall\, x \in X.$$

If θ is a small strictly positive cocycle on a cone C for (Y,Φ_2), then $\theta \circ (F \times \mathrm{Id})$ is a small strictly positive cocycle on C for (X,Φ_1).

Proof: We want to show that $\theta \circ (F \times \mathrm{Id})$ is a cocycle, i.e. we want to verify the following equality:

$$(\theta \circ (F \times \mathrm{Id}))(x, k_1 + k_2) = (\theta \circ (F \times \mathrm{Id}))(x,k_1) + (\theta \circ (F \times \mathrm{Id}))(\Phi_1^{k_1}(x), k_2),$$

for all $x \in X,\ k_1, k_2 \in \mathbb{Z}^2$. We have

$$\begin{aligned}(\theta \circ (F \times \mathrm{Id}))(x, k_1 + k_2) &= \theta(F(x), k_1 + k_2)\\ &= \theta(F(x), k_1) + \theta(\Phi_2^{k_1}(F(x)), k_2) \text{ since } \theta \text{ is a cocycle,}\end{aligned}$$

while

$$\begin{aligned}(\theta \circ (F \times \mathrm{Id}))(x, k_1) + (\theta \circ (F \times \mathrm{Id}))(\Phi_1^{k_1}(x), k_2) &= \theta(F(x), k_1) + \theta(F(\Phi_1^{k_1}(x)), k_2)\\ &= \theta(F(x), k_1) + \theta(\Phi_2^{k_1}(F(x)), k_2),\end{aligned}$$

hence $\theta \circ F$ is a cocycle.

Since θ is positive on C, i.e. $\theta(Y \times C) \geq 0$, we have

$$(\theta \circ (F \times \mathrm{Id}))(X \times C) = \theta(F(X) \times C) = \theta(Y \times C) \geq 0.$$

Therefore, $\theta \circ (F \times \mathrm{Id})$ is also positive on C.

Since θ is small, we have that $|\theta(y,n)| \leq 1$ for all $y \in Y$ and $n \in B(0,2)$. Hence $|\theta(F(x),n)| \leq 1$ for all $x \in X$ and $n \in B(0,2)$, so that $\theta \circ (F \times \mathrm{Id})$ is also small.

Finally, F is proper since it is a continuous function with compact domain X. Indeed, let $K \subseteq Y$ be compact so that K is closed. Hence $F^{-1}(K) \subseteq X$ is closed and hence compact. Moreover, $F \times \mathrm{Id}$ is proper as well. Since the composition of two proper functions is proper, we have that the cocycle $\theta \circ (F \times \mathrm{Id})$ is proper. ∎

Theorem 3.2.4 [11] Let (X, Φ) be a minimal, free Cantor $\mathbb{Z}^2$-system. Let a, b be generators for $\mathbb{Z}^2$. Suppose that for any $N \geq 1$, we may find clopen sets A and B such that

1. A and $\Phi^a(B)$ are disjoint, $\Phi^b(A)$ and B are disjoint,
2. $A \cup \Phi^a(B) = \Phi^b(A) \cup B$,
3. the sets $\Phi^{i(a+b)}(A \cup \Phi^a(B))$, are disjoint for $0 \leq i \leq N$.

Then for any $M \geq 1$, we may find a cocycle θ which is stricly positive on the cone $C = \langle a, b \rangle$ and such that $\theta \leq M^{-1}$.

Proof: We only give a sketch of the proof. For more details, see [11]. Given $M \geq 1$, choose $N \geq 1$ such that the set $\{ia + jb;\ -N \leq i, j \leq N\}$ contains $B(0, M)$ in $\mathbb{Z}^2$. Define $\theta(x, 0) = 0$, for all $x \in X$. Then define $\theta(x, ia + jb)$ inductively on $|i| + |j|$. Let $x \in X$. If $i > 0$, define

$$\theta(x, ia + jb) = \theta(x, (i-1)a + jb) + \chi_{\Phi^{-(i-1)a-jb}(A)}(x).$$

If $i < 0$, define

$$\theta(x, ia + jb) = \theta(x, (i+1)a + jb) - \chi_{\Phi^{-ia-jb}(A)}(x).$$

If $j > 0$, define

$$\theta(x, ia + jb) = \theta(x, ia + (j-1)b) + \chi_{\Phi^{-ia-(j-1)b}(B)}(x).$$

Finally, if $j < 0$, define

$$\theta(x, ia + jb) = \theta(x, ia + (j+1)b) - \chi_{\Phi^{-ia-jb}(B)}(x).$$

We can verify directly that θ is a cocycle and that by definition, it is positive on $\langle a, b \rangle$. The choice of N and the third hypothesis of the theorem will guarantee us that $\theta \leq M^{-1}$. Finally, the two first hypotheses guarantee us that θ will be proper. ∎

Example 3.2.5 [12] Let α, β be two numbers such that $\{1, \alpha, \beta\}$ are linearly independent over the rational numbers. We consider the natural action of $\mathbb{Z}^2$ on the circle, $\mathbb{R}/\mathbb{Z}$, by rotating by α and β. We select a single orbit, say that of 0, and cut the circle at these points, replacing each by two points separated by a gap. The old point will become the right endpoint of the gap and a new point will be the left end of the gap.

Consider the subgroup of $\mathbb{R}$,

$$\mathrm{Cut}_{\alpha,\beta} = \{i + j\alpha + k\beta;\ i, j, k \in \mathbb{Z}\}.$$

We let $\tilde{X} = \mathbb{R} \cup \{a';\ a \in \mathrm{Cut}_{\alpha,\beta}\}$. We give $\tilde{X}$ a linear order by setting $a' < b$, $a < b'$ and $a' < b'$, whenever $a < b$. Finally, we set $a' < a$, for all $a \in \mathbb{R}$. The space $\tilde{X}$ is given the order topology. Notice that for $x < y$ in $\mathrm{Cut}_{\alpha,\beta}$, $[x, y) = [x, y']$ is a clopen set in $\tilde{X}$ and the set of all such sets forms a basis for $\tilde{X}$. The natural action of the group $\mathbb{Z} + \alpha\mathbb{Z} + \beta\mathbb{Z}$ extends in a natural way to $\tilde{X}$. We let $S^1_{\alpha,\beta} = \tilde{X}/\mathbb{Z}$ which is a Cantor set obtained by disconnecting the circle S^1 along the forward orbits of 0: $\{0, \alpha, 2\alpha, 3\alpha, \ldots\}$ and $\{0, \beta, 2\beta, 3\beta, \ldots\}$ (See [17], section 3, for more details). It has an action of $\alpha\mathbb{Z}+\beta\mathbb{Z}$. This is a Cantor minimal $\mathbb{Z}^2$-system, $(S^1_{\alpha,\beta}, \Phi)$, where the action Φ is given by

$$\begin{aligned} &\Phi^{(m,n)}(x) = x - m\alpha - n\beta, \text{ when } x \text{ is a real number, interpreted modulo } \mathbb{Z}, \text{ and} \\ &\Phi^{(m,n)}(x') = (x - m\alpha - n\beta)', \text{ when } x' \in \mathrm{Cut}_{\alpha,\beta}. \end{aligned} \tag{3.2.1}$$

Theorem 3.2.6 ([12], Example 9.4) The action Φ satisfies the hypotheses of Theorem 3.3.6. Therefore, for every generators a, b of $\mathbb{Z}^2$ and $M \geq 1$, there exists a cocycle θ that is strictly positive on $\langle a, b\rangle$, and such that $\theta \leq M^{-1}$.

From now on, we will assume we are working with a minimal, free product $\mathbb{Z}^2$-action, denoted Φ, on the product $X \times X$ of two Cantor sets. The product action, Φ can be written as $\Phi = \varphi \otimes \psi$ where φ and ψ are $\mathbb{Z}-$actions on X. If $(x_1, x_2) \in X \times X$ and $(m, n) \in \mathbb{Z}^2$, Φ is defined in the following way:

$$\Phi^{(m,n)}(x_1, x_2) = (\varphi^m(x_1), \psi^n(x_2)). \tag{3.2.2}$$

Notice that if φ and ψ are minimal, free $\mathbb{Z}$-actions on X then the product action $\varphi \otimes \psi$ will be a minimal, free $\mathbb{Z}^2-$action on $X \times X$. Notice that it is not true that all free, minimal actions on $X \times X$ can be decomposed into a product action. For now, we are only working with actions that can be decomposed in such a way.

Theorem 3.2.7 If $\Phi = \varphi \otimes \psi$ is a minimal, free $\mathbb{Z}^2$-action, like in Equation 3.2.2, then there is a small strictly positive cocycle for Φ on the cone

$$C = \langle a, b\rangle = \{ia + jb; i, j \geq 0\},$$

where $a = (1, 0)$ and $b = (0, 1)$.

Proof: Let $N \geq 1$ and choose a clopen set $A_1 \subset X$ such that $\varphi^i(A_1)$ are disjoint for $0 \leq i \leq N$. Such a clopen set exists since the action is free. Now let $A = A_1 \times X$ and $B = \emptyset$. Let us verify that these subsets of $X \times X$ satisfy the conditions of Theorem 3.2.4. First of all, A and B are obviously clopen since A_1 is clopen. Furthermore,

1. $A \cap \Phi^a(B) = A \cap \emptyset = \emptyset$ and $\Phi^b(A) \cap B = \Phi^b(A) \cap \emptyset = \emptyset$,
2. $A \cup \Phi^a(B) = A \cup \emptyset = A$ and
$$\Phi^b(A) \cup B = \Phi^{(0,1)}(A) \cup \emptyset = \varphi^0(A_1) \times \psi^1(X) = A_1 \times X = A,$$
3. the sets $\Phi^{i(a+b)}(A \cup \Phi^a(B)) = \Phi^{(i,i)}(A_1 \times X) = \varphi^i(A_1) \times \psi^i(X) = \varphi^i(A_1) \times X$ are disjoint for $0 \leq i \leq N$, by the choice of A_1.

Hence, by Theorem 3.2.4, there exists a stricly positive cocycle on the cone C generated by $(1,0)$ and $(0,1)$. ∎

Another way to see this is to actually follow the proof of Theorem 3.2.4, which can be found in [11] and is sketched in Theorem 3.2.4, to construct the cocycle. Given $M \geq 1$, choose $N \geq 1$ such that $\{(i,j); -N \leq i,j \leq N\}$ contains $B(0,M)$ in $\mathbb{Z}^2$. Once again choose a clopen set $A_1 \subset X$ such that $\varphi^i(A_1)$ are disjoint for $0 \leq i \leq N$. Now let $A = A_1 \times X$. We define $\theta : X \times \mathbb{Z}^2 \to \mathbb{Z}$ in the following way:

1. $\theta(x,(i,j)) = \sum_{0 \leq k < i} \chi_{\Phi^{-k}(A)}(x), \ \forall\, x \in X, \ i \geq 0, \forall\, j$
2. $\theta(x,(i,j)) = \sum_{i \leq k < 0} -\chi_{\Phi^{-k}(A)}(x), \ \forall\, x \in X, \ i \leq 0, \forall\, j.$

More generally, we do the same for the cone $C = \langle a, b \rangle$, where $a = (a_1, a_2)$ and $b = (b_1, b_2)$. We assume that a and b satisfy $a_1 \geq b_1 \geq 0$, but we exclude the case $a_1 = b_1 = 0$.

Theorem 3.2.8 Let $C = \langle a, b \rangle$ be the cone where $a = (a_1, a_2)$ and $b = (b_1, b_2)$ are as described at the bottom of the previous page. If $\Phi = \varphi \otimes \psi$ is a minimal, free $\mathbb{Z}^2$-action, like in Equation 3.2.2, then there is a strictly positive cocycle for Φ on the cone C.

Proof: Let $N \geq 1$ and let $Y \subset X$ be a clopen set such that $\varphi^j(Y)$ are disjoint for $j = 0, \ldots, (N+1)(a_1 + b_1) - 1$. Set

$$A_1 = \bigcup_{i \in E} \varphi^i(Y) \quad \text{and} \quad B_1 = \bigcup_{j \in D} \varphi^j(Y),$$

where

$$D = \begin{cases} \{0, \ldots, b_1 - 1\} & \text{if } b_1 \neq 0, \\ \emptyset & \text{otherwise} \end{cases}$$

$$E = \begin{cases} \{0, \ldots, a_1 - 1\} & \text{if } a_1 \neq 0 \\ \emptyset & \text{otherwise.} \end{cases}$$

Let $A = A_1 \times X$ and $B = B_1 \times X$. Since we exclude the case where both a_1 and b_1 are zero, we know that at least one of D or E is nonempty. We verify the properties of A and B. Since Y is clopen, A and B are clopen.

1. We have

$$\begin{aligned} A \cap \Phi^a(B) &= (A_1 \times X) \cap (\varphi^{a_1}(B_1) \times X) = (A_1 \cap \varphi^{a_1}(B_1)) \times X \\ &= \left[\left(\bigcup_{i \in E} \varphi^i(Y) \right) \cap \left(\bigcup_{j \in D + a_1} \varphi^j(Y) \right) \right] \times X = \emptyset \end{aligned}$$

and

$$\begin{aligned} \Phi^b(A) \cap B &= (\varphi^{b_1}(A_1) \times X) \cap (B_1 \times X) = (\varphi^{b_1}(A_1) \cap B_1) \times X \\ &= \left[\left(\bigcup_{i \in E + b_1} \varphi^i(Y) \right) \cap \left(\bigcup_{j \in D} \varphi^j(Y) \right) \right] \times X = \emptyset, \end{aligned}$$

by the choice of Y.

2. We also have

$$\begin{aligned} A \cup \Phi^a(B) &= \left[\left(\bigcup_{i\in E} \varphi^i(Y)\right) \cup \left(\bigcup_{j\in D+a_1} \varphi^j(Y)\right)\right] \times X \\ &= \bigcup_{k\in K} \varphi^k(Y) \times X. \end{aligned}$$

and

$$\begin{aligned} \Phi^b(A) \cup B &= \left[\left(\bigcup_{i\in E+b_1} \varphi^i(Y)\right) \cup \left(\bigcup_{j\in D} \varphi^j(Y)\right)\right] \times X \\ &= \bigcup_{k\in K} \varphi^k(Y) \times X. \end{aligned}$$

where $K = \{0, \ldots, a_1 + b_1 - 1\}$. That is, $A \cup \Phi^a(B) = \Phi^b(A) \cup B$.

3. The sets

$$\begin{aligned} \Phi^{i(a+b)}(A \cup \Phi^a(B)) &= \Phi^{(i(a_1+b_1), i(a_2+b_2))}\left(\bigcup_{k\in K} \varphi^k(Y) \times X\right) \\ &= \varphi^{i(a_1+b_1)}\left(\bigcup_{k\in K} \varphi^k(Y)\right) \times X \\ &= \bigcup_{k\in K} \varphi^{k+i(a_1+b_1)}(Y) \times X \end{aligned}$$

are disjoint for $0 \leq i \leq N$ by the choice of Y.

Hence, by Theorem 3.2.4, there exists a stricly positive cocycle on the cone C generated by a and b. ∎

Remark 3.2.9 Notice that a similar construction works for the cone $C = \langle a, b\rangle$, where $a = (a_1, a_2), b = (b_1, b_2)$, if the following condition holds : $b_1 \leq a_1 \leq 0$ (excluding $a_1 = b_1 = 0$).

The only difference is that we take

$$D = \begin{cases} \{b_1 + 1, \ldots, 0\} & \text{if } b_1 \neq 0, \\ \emptyset & \text{otherwise} \end{cases} \quad \text{and} \quad E = \begin{cases} \{a_1 + 1, \ldots, 0\} & \text{if } a_1 \neq 0 \\ \emptyset & \text{otherwise.} \end{cases}$$

3.3 Cocycles and Affable Equivalence Relations

In this section, we define approximately finite equivalence relations and affable actions. We then state two results that gives us sufficient conditions for a minimal, free $\mathbb{Z}^2$-action on the Cantor set to be affable.

Definition 3.3.1 An étale equivalence relation $\mathcal{R}$ on X is an **AF-relation** (approximately finite equivalence relation) if X is a totally disconnected compact metrizable space (e.g. the Cantor set) and if there are

$$\mathcal{R}_1 \subset \mathcal{R}_2 \subset \cdots$$

such that $\cup_n \mathcal{R}_n = \mathcal{R}$ and $\mathcal{R}_n \subset \mathcal{R}$ is a compact open subequivalence relation, for each $n \geq 1$.

Definition 3.3.2 An action is **affable** if it is orbit equivalent to an action whose associated equivalence relation is an AF-equivalence relation.

The following theorem is the main result from [12].

Theorem 3.3.3 [12] On the Cantor set, any minimal étale equivalence relation which arises from an action of a finitely generated abelian group is affable.

Special cases of this theorem have been obtained in [12] for free, minimal, $\mathbb{Z}^2$-actions on a Cantor set with arbitrary small strictly positive cocycles. In this section, we recall these results. The first theorem states that if a free, minimal $\mathbb{Z}^2$-action on a

Cantor set possesses arbitrarily small cocycles for sufficiently many cones, then its orbit relation is affable. We first start with a definition.

Definition 3.3.4 Let r, r' be positive real numbers. We define

$$C(r,r') = \{(i,j) \in \mathbb{Z}^2; j \leq ri, j \leq r'i\}.$$

The definition can be extended to include the cases $r = +\infty$ and $r = 0$ (using the convention $+\infty \cdot 0 = 0$).

Theorem 3.3.5 (Theorem 4.1 in [12]) Let (X, Φ) be a free, minimal action of $\mathbb{Z}^2$ on a Cantor set. Suppose that there are positive numbers r_∞, s_∞, with $s_\infty^{-1} - r_\infty^{-1} \leq 1$ satisfying the following. For every $\varepsilon > 0$, there are positive $r_\infty + \varepsilon > r > r' > r_\infty$ so that for every $M \geq 1$, there is a cocycle θ on (X, Φ) such that

1. θ is strictly positive on $C(r, r')$, and
2. $\theta \leq M^{-1}$.

Similarly, for every $\varepsilon > 0$, there are positive $s_\infty - \varepsilon < r < r' < s_\infty$ such that for every $M \geq 1$, there is a cocycle θ such that the same conditions hold. Then the equivalence relation $\mathcal{R}_\Phi$ is affable.

The following theorem is similar to the first one but the hypotheses are stronger. Indeed, we ask for the existence of small strictly positive cocycles on any cone of the form $C = \langle a, b \rangle$, where a, b are generators of $\mathbb{Z}^2$, not just on cones of the form $C(r, r')$, where $r, r' \in \mathbb{R}$.

Theorem 3.3.6 (Corollary 4.2 in [12]) Let (X, Φ) be a free, minimal action of $\mathbb{Z}^2$ on a Cantor set. Suppose that for every $a, b \in \mathbb{Z}^2$ which generate $\mathbb{Z}^2$ as a group and $M \geq 1$, there is a cocycle θ such that

1. θ is strictly positive on $\langle a, b\rangle = \{ia + jb; i, j > 0\}$, and
2. $\theta \leq M^{-1}$

Then the equivalence relation $\mathcal{R}_\Phi$ is affable.

Remark 3.3.7 In Theorem 3.2.8, we have shown that if $\Phi = \varphi \otimes \psi$ is a product action on $X \times X$, where X is a Cantor set, and if Φ is minimal and free, then there is a strictly positive cocycle for Φ on the cone $C = \langle a, b\rangle$ where $a = (a_1, a_2)$ and $b = (b_1, b_2)$ satisfy $a_1 \geq b_1 \geq 0$ or $b_1 \leq a_1 \leq 0$, i.e. when $a_1 b_1 \geq 0$ (where we exclude the case $a_1 = b_1 = 0$). We would still need to show it holds for any cone to be able to use Theorem 3.3.6 and conclude the equivalence relation $\mathcal{R}_\Phi$ is affable.

Chapter 4

Tilings and The Cut and Projection Method

4.1 Tilings

In the introduction, we gave a descriptive definition of tilings and showed some pictures of well known tilings. In this section, we give mathematical definitions and results about tilings.

Definition 4.1.1 A **tiling** of $\mathbb{R}^d$ is a countable collection of closed subsets

$$T = \{t_1, t_2, t_3, ...\}$$

of $\mathbb{R}^d$ such that

1. t_i is homeomorphic to the closed unit ball in $\mathbb{R}^d$,
2. $t_i \cap t_j$ has empty interior, $i \neq j$,
3. $\bigcup\limits_{i=1}^{\infty} t_i = \mathbb{R}^d$.

The t_i's are called the **tiles** of the tiling. Let $X \subset \mathbb{R}^d$ and $x' \in \mathbb{R}^d$. Then $X + x' = \{x + x'; x \in X\}$ is called the translate of X by x'. If T is a tiling, its translate by $x \in \mathbb{R}^d$ is defined by $T + x := \cup_{t \in T}(t + x)$. Notice that $T + x$ is also a tiling. We will often consider tilings constructed from a finite set of **prototiles** $\{p_1, p_2, ..., p_n\}$, i.e. if $t \in T$, then $t = p_i + x$ for some $x \in \mathbb{R}^d$.

Definition 4.1.2 [24] A **patch** in a tiling T is a finite subset of tiles

$$\{t_1, t_2, ..., t_m\} \subset T.$$

Remark 4.1.3 Some authors (e.g [25]) add the following condition in the definition of a patch: $\bigcup_{i=1}^{m} t_i$ is connected.

Let $R > 0$ and $x \in \mathbb{R}^d$. We define a special type of patch

$$T \cap B(x, R) := \{t \in T; t \subset B(x, R)\}.$$

A patch has diameter less than R if there is a disk of radius R enclosing the patch.

Remark 4.1.4 If we work with tilings made of a finite set of prototiles which are polygons meeting edge to edge, then the patch $T \cap B(x, R)$ will also be connected, hence will satisfy the stronger definition of a patch.

Definition 4.1.5 We say that a tiling T is **periodic** if its translation group

$$\Gamma_T = \{x \in \mathbb{R}^d; T + x = T\}$$

is a lattice in $\mathbb{R}^d$. A tiling T is **aperiodic** if $\Gamma_T = \{0\}$ (i.e. $T + x \neq T$ for any $x \neq 0 \in \mathbb{R}^d$).

Definition 4.1.6 A tiling T is said to have **finite local complexity** (denoted FLC) if for any $R > 0$, there is only a finite number of patches in T (up to translation) with diameter less than R. In other words, for any $R > 0$, $\{T \cap B(x, R); x \in \mathbb{R}^d\}/\mathbb{R}^d$ (where we mod out by translations) is finite.

Example 4.1.7 A periodic tiling has FLC. $\square$

Let $P = \{T;\ T \text{ is a tiling in } \mathbb{R}^d\}$ be the set of tilings in $\mathbb{R}^d$. We define a metric on P by saying that the distance between two tilings T and T' in P is smaller than ε if the tilings agree on $B(0, 1/\varepsilon)$ up to translations no greater than ε. More precisely, we define $d : P \times P \to \mathbb{R}$ by

$$d(T, T') = \inf\{1, \varepsilon; \exists\ x, x' \text{ with } \|x\|, \|x'\| < \varepsilon, \text{ s.t. } T \cap B(x, 1/\varepsilon) = T' \cap B(x', 1/\varepsilon)\}. \tag{4.1.1}$$

The distance between two tilings is small when the tilings agree on a large ball about the origin, up to a small translation. To prove that d is in fact a metric on P, we first need the following lemma.

Lemma 4.1.8 ([28], p.9) If a and b are positive numbers such that $a + b \leq 1$, then

$$\frac{1}{a+b} \leq \frac{1-ab}{a}.$$

Proof: Since $a, b, a + b \leq 1$, we have

$$\begin{aligned} & 0 \leq 1 - a(a+b) \\ \Rightarrow\ & 0 \leq b - a^2b - ab^2 \\ \Rightarrow\ & a \leq a - a^2b - ab^2 + b = (a+b)(1-ab) \end{aligned}$$

and so $\frac{1}{a+b} \leq \frac{1-ab}{a}$. $\blacksquare$

Theorem 4.1.9 The function $d : P \times P \to \mathbb{R}$ defined in equation 4.1.1 is a metric on P.

Proof: ([28], p.9) By definition, d is symmetric and $d(T,T') \geq 0$, for all T,T' in P. Also, $d(T,T)=0$ for every tiling T since we can find arbitrarily large balls around the origin where T agrees with itself. Conversely, if $d(T,T')=0$, then T and T' must agree, up to small translations, on arbitrarily large balls. This only holds if $T=T'$. What is left to show is the triangle inequality. Let R,S and T be tilings. We need to show that

$$d(T,S) \leq d(T,R)+d(R,S).$$

If $d(T,R)+d(R,S) \geq 1$, then the equality holds since $d(T,S) \leq 1$. Hence we can assume that $d(T,R)+d(R,S)<1$. Choose $\varepsilon>0$ small enough so that

$$d(T,R)+d(R,S)+\varepsilon<1.$$

Take x_{TR} and x'_{TR} in $\mathbb{R}^d$ with

$$\|x_{TR}\|, \|x'_{TR}\| < d(T,R)+\frac{\varepsilon}{2},$$

such that

$$T \cap B\left(x_{TR}, \frac{1}{d(T,R)+\frac{\varepsilon}{2}}\right) = R \cap B\left(x'_{TR}, \frac{1}{d(T,R)+\frac{\varepsilon}{2}}\right).$$

Similarly, take x_{RS} and x'_{RS} in $\mathbb{R}^d$ with

$$\|x_{RS}\|, \|x'_{RS}\| < d(R,S)+\frac{\varepsilon}{2},$$

such that

$$R \cap B\left(x_{RS}, \frac{1}{d(R,S)+\frac{\varepsilon}{2}}\right) = S \cap B\left(x'_{RS}, \frac{1}{d(R,S)+\frac{\varepsilon}{2}}\right).$$

Since $T-x_{TR}$ and $R-x'_{TR}$ agree on $B\left(0, \frac{1}{d(T,R)+\frac{\varepsilon}{2}}\right)$, we have that $T-x_{TR}-x_{RS}$ and $R-x'_{TR}-x_{RS}$ will agree on $B_1 := B\left(-x_{RS}, \frac{1}{d(T,R)+\frac{\varepsilon}{2}}\right)$. Similarly since $R-x_{RS}$ and $S-x'_{RS}$ agree on $B\left(0, \frac{1}{d(R,S)+\frac{\varepsilon}{2}}\right)$, we have that $R-x_{RS}-x'_{TR}$ and $S-x'_{RS}-x'_{TR}$

will agree on $B_2 := B\left(-x'_{TR}, \frac{1}{d(R,S)+\frac{\varepsilon}{2}}\right)$. Combining these two facts, we get that $T - x_{TR} - x_{RS}$ and $S - x'_{RS} - x'_{TR}$ will agree on $B_1 \cap B_2$. The latter intersection contains the origin since $\|x'_{TR}\|, \|x_{RS}\| < 1$ and $\frac{1}{d(S,R)+\frac{\varepsilon}{2}}, \frac{1}{d(T,R)+\frac{\varepsilon}{2}} > 1$. Notice that

$$r_1 := \frac{1}{d(T,R)+\frac{\varepsilon}{2}} - \|x_{RS}\|$$

and

$$r_2 := \frac{1}{d(R,S)+\frac{\varepsilon}{2}} - \|x'_{TR}\|$$

are the largest radii such that $B(0,r_1) \subset B_1$ and $B(0,r_2) \subset B_2$. This means that $r := \min\{r_1, r_2\}$ is such that $B(0,r) \subset B_1 \cap B_2$. We have

$$\begin{aligned} r_1 &= \frac{1}{d(T,R)+\frac{\varepsilon}{2}} - \|x_{RS}\| \\ &\geq \frac{1}{d(T,R)+\frac{\varepsilon}{2}} - (d(R,S)+\frac{\varepsilon}{2}) \\ &= \frac{1-(d(T,R)+\frac{\varepsilon}{2})(d(R,S)+\frac{\varepsilon}{2})}{d(T,R)+\frac{\varepsilon}{2}}. \end{aligned}$$

By Lemma 4.1.8, we conclude that $r_1 \geq \frac{1}{d(T,R)+\frac{\varepsilon}{2}+d(R,S)+\frac{\varepsilon}{2}}$. We can show the same for r_2, and so $r \geq \frac{1}{d(T,R)+d(R,S)+\varepsilon}$. Thus $T - x_{TR} - x_{RS}$ and $S - x'_{RS} - x'_{TR}$ agree on $B\left(0, \frac{1}{d(T,R)+d(R,S)+\varepsilon}\right)$. From the triangle inequality in $\mathbb{R}^d$, we have

$$\|x_{TR} + x_{RS}\| \leq \|x_{TR}\| + \|x_{RS}\| < d(T,R) + d(R,S) + \varepsilon,$$

and

$$\|x'_{RS} + x'_{TR}\| \leq \|x'_{RS}\| + \|x'_{TR}\| < d(T,R) + d(R,S) + \varepsilon,$$

and so we have $d(T,S) \leq d(T,R) + d(R,S)$. ∎

We will use the following lemma to show that the metric space P is complete.

Lemma 4.1.10 If $(S_n)_{n\geq 1}$ is a Cauchy sequence in P then for every $R > 0$, there exists an integer N_R such that

$$B(0,R) \cap S_n = B(0,R) \cap S_{N_R}, \ \forall\ n \geq N_R.$$

Proof: Let $R > 0$ and choose $\varepsilon > 0$ small enough so that for every x with $\|x\| < \varepsilon$, we have

$$B(0, R) \subset B\left(x, \frac{1}{\varepsilon}\right).$$

Let $N_R = N_\varepsilon$ be such that for every $n, m \geq N_\varepsilon = N_R$, we have

$$d(S_n, S_m) < \varepsilon.$$

Such an N_ε exists since $(S_n)_{n\geq 1}$ is Cauchy. By definition of the metric d, there exists x, x', with $\|x\|, \|x'\| < \varepsilon$ such that

$$B\left(x, \frac{1}{\varepsilon}\right) \cap S_n = B\left(x', \frac{1}{\varepsilon}\right) \cap S_{N_R}, \ \forall\, n \geq N_R.$$

By the choice of ε, we then have $B(0, R) \cap S_n = B(0, R) \cap S_{N_R}$, $\forall\, n \geq N_R$. ∎

Theorem 4.1.11 The metric space P is complete.

Proof: Let $(S_n)_{n\geq 1}$ be a Cauchy sequence in P. We need to show that $(S_n)_{n\geq 1}$ converges to a tiling. Let $(R_k)_{k\geq 1}$ be a sequence in $\mathbb{R}$ such that $R_k \to \infty$ when $k \to \infty$ and for each $k \geq 1$ let N_{R_k} be such that

$$B(0, R_k) \cap S_n = B(0, R_k) \cap S_{N_{R_k}}, \ \forall\, n \geq N_{R_k}.$$

We define a tiling S which will be the limit of the Cauchy sequence $(S_n)_{n\geq 1}$ in the following way: For each $k \geq 1$, define $B(0, R_k) \cap S := B(0, R_k) \cap S_{N_{R_k}}$. Let us show that the tiling S is the limit of the sequence $(S_n)_{n\geq 1}$. Let $\varepsilon > 0$ and choose $k \geq 1$ such that $R_k > \frac{1}{\varepsilon}$. Then

$$\begin{aligned} B(0, R_k) \cap S &= B(0, R_k) \cap S_{N_{R_k}} \\ &= B(0, R_k) \cap S_n \text{ for all } n > N_{R_k} \end{aligned}$$

Hence

$$B\left(0, \frac{1}{\varepsilon}\right) \cap S = B\left(0, \frac{1}{\varepsilon}\right) \cap S_n \text{ for all } n > N_{R_k}.$$

Therefore, for all $\varepsilon > 0$ there exists N_{R_k} such that

$$d(S, S_n) < \varepsilon \text{ for all } n > N_{R_k},$$

so that $S_n \to S \in P$. This shows that P is complete. ∎

Definition 4.1.12 Consider $T + \mathbb{R}^d = \{T + x;\ x \in \mathbb{R}^d\} \subset P$ endowed with the tiling metric defined above. Denote the completion of the metric space $(T + \mathbb{R}^d, d)$ by Ω_T, which we will call the **continuous hull** of T.

Theorem 4.1.13 ([24], Lemma 2): Ω_T is compact if T has FLC.

For each $x \in \mathbb{R}^d$ we have a homeomorphism $\varphi_x : \Omega_T \to \Omega_T$ defined by $\varphi_x(T') = T' + x$ for $T' \in \Omega_T$. Since $\varphi_x \circ \varphi_y = \varphi_{x+y}$, $\mathbb{R}^d$ acts on Ω_T. Therefore, given a tiling T of $\mathbb{R}^d$, we have a topological dynamical system $(\Omega_T, \mathbb{R}^d)$. Recall that the dynamical system $(\Omega_T, \mathbb{R}^d)$ is minimal if the orbit of any point (i.e. tiling) in $(\Omega_T, \mathbb{R}^d)$ is dense.

Definition 4.1.14

1. The tiling T is called **repetitive** if for any patch $Q \subset T$, $\exists\, R > 0$ such that for every $x \in \mathbb{R}^d$, $T \cap B(x, R)$ contains a translate of Q.
2. The tiling T will be called minimal if $(\Omega_T, \mathbb{R}^d)$ is minimal.

Definition 4.1.15 A subset $Y \subseteq \mathbb{R}^d$ is **relatively dense** if there exists a compact subset $K' \subset \mathbb{R}^d$ such that $Y + K' = \mathbb{R}^d$.

Example 4.1.16 Let $Y = \mathbb{R}^d \backslash \{0\}$ and K' be the closed ball centered at the origin of radius equal to $r > 0$. Then K' is a compact subset of $\mathbb{R}^d$ such that $Y + K' = \mathbb{R}^d$. Indeed, let $x \in \mathbb{R}^d$. If $x \neq 0$ then $x \in Y$ and $0 \in K'$ so we can write $x = x + 0$. If $x = 0$ then $x = (r, 0, ..., 0) + (-r, 0, ..., 0)$ where $(r, 0, ..., 0) \in Y$ and $(-r, 0, ..., 0) \in K'$. Therefore Y is relatively dense.

Even when T is an aperiodic tiling, it is possible for Ω_T to contain periodic tilings.

Example 4.1.17 Consider tiling the plane with white unit squares, fitting edge to edge. Remove one white tile and replace it with a black tile of the same size. Call this tiling T. This tiling is aperiodic but its hull contains the original tiling by white unit squares. Indeed, let S denote the periodic tiling with white unit squares and let $\varepsilon > 0$. Assume without lost of generality that the black tile of T contains the origin. Let $(T + x_n)_{n \geq 1} \in \Omega_T$ where $x_n = (n, 0)$. Choose $N_\varepsilon > \frac{1}{\varepsilon} + s$, where s is the length of the side of the black tile. Then

$$(T + x_n) \cap B_{\frac{1}{\varepsilon}}(0) = S \cap B_{\frac{1}{\varepsilon}}(0) \; \forall \; n > N_\varepsilon,$$

hence $S = \lim_{n \to \infty} (T + x_n) \in \Omega_T$. □

We say T is **strongly aperiodic** if Ω_T does not contain any periodic tiling.

Proposition 4.1.18 ([15] Proposition 2.4) If the tiling T is aperiodic and minimal, then it is strongly aperiodic.

4.2 Point sets in $\mathbb{R}^d$

In this section, we explain the relation between tilings and point sets. The method used to construct a tiling from a point set is called the Voronoi tesselation and will be described in Definition 4.2.11

Definition 4.2.1 [27] A point set $\Lambda \subset \mathbb{R}^d$ is said to be **uniformly discrete** if there exists a positive real number r such that, for every $x, y \in \Lambda$, $\|x - y\| \geq r$.

Definition 4.2.2 [27] A point set Λ is said to be **relatively dense** in $\mathbb{R}^d$ if there is a positive real number R such that every sphere of radius greater than R contains at least one point of Λ in its interior.

Recall that we gave a definition of relative density for subsets of $\mathbb{R}^d$ in 4.1.15. This definition also holds for point sets and is equivalent to Definition 4.2.2. Let us show this.

Theorem 4.2.3 Let $\Lambda \subset \mathbb{R}^d$ be a point set. The following are equivalent:

1. There is a positive real number R such that every sphere of radius greater than R contains at least one point of Λ.
2. There exists a compact set $K' \subset \mathbb{R}^d$ such that $\Lambda + K' = \mathbb{R}^d$.

Proof: Suppose that (2) holds. In order to get a contradiction, suppose that for every $R > 0$ there exists $x \in \mathbb{R}^d$ such that $B_R(x) \cap \Lambda = \emptyset$. Let $R > 0$ and choose $x \in \mathbb{R}^d$ such that $B_R(x) \cap \Lambda = \emptyset$. Since (2) holds, there exists $y \in \Lambda$ and $k \in K'$ such that $x = y + k$. In particular, $k = x - y$. By the choice of x, we have that $\|x - y\| > R$. Since K' is compact in $\mathbb{R}^d$, it is bounded. Let

$$r = \max\{\|k\|;\ k \in K'\} < \infty.$$

Then $\|x - y\| = \|k\| \leq r$. Therefore, we have $R < \|x - y\| \leq r$, and hence $R < r$. As this holds for any $R > 0$, we get a contradiction. Hence there exists $R > 0$ such that $B_R(x) \cap \Lambda \neq \emptyset$ for any $x \in \mathbb{R}^d$.

Now assume (1) holds. Let $x \in \mathbb{R}^d$. By hypothesis, $B_R(x) \cap \Lambda \neq \emptyset$, so let

$$y \in B_R(x) \cap \Lambda.$$

Consider the compact set $K' := B_R(0)$. Let $k := y - x$ so that

$$\|k\| = \|y - x\| < R,$$

hence $k \in K'$. Therefore, for any $x \in \mathbb{R}^d$, there exists $y \in \Lambda$ and $k \in K'$ such $x = y + k$, i.e. $\Lambda + K' = \mathbb{R}^d$. ∎

Definition 4.2.4 A point set $\Lambda \subset \mathbb{R}^d$ is a **Delone set** if it is uniformly discrete and relatively dense.

As with tilings, we can define a metric on the set of point sets in $\mathbb{R}^d$, where two point sets will be considered close if, after a small translation, they coincide on a large compact region. Moreover, $\mathbb{R}^d$ acts on any point set by translation. Let Λ be a point set in $\mathbb{R}^d$. Denotes by $[\Lambda]_{\mathbb{R}^d}$ the orbit of Λ under the action of $\mathbb{R}^d$. We can complete $[\Lambda]_{\mathbb{R}^d}$ with respect to the metric defined above. Call this set X_Λ. Notice that the action of $\mathbb{R}^d$ extends to X_Λ so that we get a dynamical system $(X_\Lambda, \mathbb{R}^d)$.

The following definitions are taken from [26] and will be used in Theorem 4.2.10.

Definition 4.2.5 A point set $\Lambda \subset \mathbb{R}^d$ is **almost periodic** if for any open neighborhood $U \subset X_\Lambda$ of Λ, the set

$$A_U = \{x \in \mathbb{R}^d; \Lambda + x \in U\}$$

is relatively dense in $\mathbb{R}^d$.

Definition 4.2.6 For a compact subset $K \subset \mathbb{R}^d$ and a point set $\Lambda \subset \mathbb{R}^d$, we define

$$T_K(\Lambda) := \{x \in \mathbb{R}^d; (\Lambda + x) \cap K = \Lambda \cap K\}.$$

Remark 4.2.7 Let K be a compact subset of $\mathbb{R}^d$ and let

$$U = U_K = \{\Lambda' \in X_\Lambda;\ \Lambda' \cap K = \Lambda \cap K\}.$$

Let $A_{U_K} = \{x \in \mathbb{R}^d; \Lambda + x \in U_K\}$. If we have an $x \in A_{U_K}$, i.e. $\Lambda + x \in U_K$, then by definition of U_K, we have that $(\Lambda + x) \cap K = \Lambda \cap K$, i.e. $x \in T_K(\Lambda)$. On the other hand, $x \in T_K(\Lambda)$ implies that $\Lambda + x \in U_K$. Hence $T_K(\Lambda) = A_{U_K}$.

Theorem 4.2.8 A point set $\Lambda \subset \mathbb{R}^d$ is almost periodic if and only if $T_K(\Lambda)$ is relatively dense.

Proof: Suppose that Λ is almost periodic. Let K be a compact subset of $\mathbb{R}^d$ and consider U_K defined in Remark 4.2.7. Since U_K is an open neighborhood of Λ and Λ is almost periodic, the set A_{U_K} is relatively dense. By the same remark, $T_K(\Lambda) = A_{U_K}$ and therefore $T_K(\Lambda)$ is relatively dense.

Now suppose that $T_K(\Lambda)$ is relatively dense for any compact $K \subset \mathbb{R}^d$. Let $U \subset X_\Lambda$ be an open neighborhood of Λ. Let $r > 0$ be small enough so that $B(\Lambda, r) \subset U$ and let K be the closed ball in $\mathbb{R}^d$ centered at the origin of radius $\frac{1}{r}$. Since K is compact, $T_K(\Lambda)$ is relatively dense by hypothesis. If $\Lambda' \in U_K$ then $\Lambda' \cap K = \Lambda \cap K$. By the definition of K, this implies that $d(\Lambda, \Lambda') < r$ and so $\Lambda' \in B(\Lambda, r) \subset U$. This shows that $U_K \subset U$. In general, if $U_1 \subset U_2$ are neighborhoods of Λ, we have $A_{U_1} \subset A_{U_2}$. In our case, we then have $T_K(\Lambda) = A_{U_K} \subset A_U$. Since $T_K(\Lambda)$ is relatively dense, so will be A_U. We have shown that for any open neighborhood U of Λ, A_U is relatively dense. Therefore, Λ is almost periodic. ∎

Definition 4.2.9 [26] A point set Λ is of finite local complexity (FLC) if for every compact $K \subset \mathbb{R}^d$, there is a compact $K' \subset \mathbb{R}^d$ such that, for every $t \in \mathbb{R}^d$, there is some $t' \in K'$ with $(t + \Lambda) \cap K = (t' + \Lambda) \cap K$.

Theorem 4.2.10 ([26], Proposition 3.1) If the point set Λ has FLC then the following conditions are equivalent:

1. $T_K(\Lambda)$ is relatively dense for every compact K,
2. $(X_\Lambda, \mathbb{R}^d)$ is minimal,
3. Λ is repetitive, i.e., for every compact $K \subset \mathbb{R}^d$, there is a compact $K' \subset \mathbb{R}^d$ such that, for every $t_1, t_2 \in \mathbb{R}^d$, there is an $s \in K'$ with

$$(t_1 + \Lambda) \cap K = (s + t_2 + \Lambda) \cap K.$$

Proof: By Theorem 4.2.8, we have that (1) is true if and only if Λ is almost periodic, therefore (1) $\Leftrightarrow$ (2) follows from general theorems on dynamical systems (see [9], Proposition 2.5). Now let us show that (3) $\Rightarrow$ (1). Let $K \subset \mathbb{R}^d$ be compact and assume that (3) holds so that there exists a compact $K' \subset \mathbb{R}^d$ such that for all $x_1, x_2 \in \mathbb{R}^d$, there is an $s \in K'$ with $(x_1 + \Lambda) \cap K = (s + x_2 + \Lambda) \cap K$. In particular, if $x_2 \in \mathbb{R}^d$ and $x_1 = 0$, there exists $-s \in K'$ such that

$$\Lambda \cap K = (-s + x_2 + \Lambda) \cap K.$$

We then have that $-s + x_2 \in T_K(\Lambda)$, that is, there exists $t \in T_K(\Lambda)$ such that $x_2 = s + t$. Hence, for any $x_2 \in \mathbb{R}^d$, $x_2 \in T_K(\Lambda) - K'$. As $-K'$ is compact, condition (1) holds.

Assume now that (1) holds and let $K \subset \mathbb{R}^d$ be compact. Since Λ has FLC we can find a compact K_1 such that for all $t_1 \in \mathbb{R}^d$, there is $t' \in K_1$ with

$$(t_1 + \Lambda) \cap K = (t' + \Lambda) \cap K. \tag{4.2.1}$$

Since (1) holds by hypothesis, we have that $T_{(K-K_1)}(\Lambda)$ is relatively dense. This means that there exists a compact K_2 such that $T_{(K-K_1)}(\Lambda) + K_2 = \mathbb{R}^d$. Then, for any $t_2 \in \mathbb{R}^d$, there exists $t'' \in K_2$ such that $t'' + t_2 \in T_{(K-K_1)}(\Lambda)$, i.e.

$$(t'' + t_2 + \Lambda) \cap (K - K_1) = \Lambda \cap (K - K_1). \tag{4.2.2}$$

Now since $t' \in K_1$, we have that $K - t' \subset K - K_1$. By the equation 4.2.2, we have

$$(t'' + t_2 + \Lambda) \cap (K - t') = \Lambda \cap (K - t'),$$

and so, by equation 4.2.1

$$(t'' + t_2 + t' + \Lambda) \cap K = (t' + \Lambda) \cap K = (t_1 + \Lambda) \cap K. \tag{4.2.3}$$

As $t' + t'' \in K_1 + K_2$, (3) is fulfilled for $K' := K_1 + K_2$ and $s = t' + t''$. ∎

Definition 4.2.11 For $\Lambda \subset \mathbb{R}^d$, a Delone set, and $x \in \Lambda$, the **Voronoi cell** of x is

$$V_x = \{y \in \mathbb{R}^d;\ \|y - x\| \leq \|y - x'\|, \forall\, x' \in \Lambda \backslash \{x\}\}.$$

To construct the Voronoi cell of $x \in \Lambda$, we connect x to every other point in Λ by straight line segments and construct the perpendicular bisector of each of these segments. The Voronoi cell V_x is the largest convex region about x bounded by these hyperplanes. Since Λ is relatively dense, it is infinite. Therefore, we have to connect x to infinitely many points and so, in principle, the Voronoi construction calls for an infinite number of operations. We will see that in fact, a finite number of operations always suffices (see Corollary 4.3.12). Relative density of Λ also implies that the Voronoi cells of every point will be bounded. Discreteness of Λ excludes the possibility for accumulation points in Λ so that V_x is well defined for all $x \in \Lambda$. If we carry out this construction for every point of Λ, we obtain a partition of $\mathbb{R}^d$ into cells called the **Voronoi tesselation** induced by Λ.

4.3 The Cut and Projection Method

In this section, we give the details of a special case of the cut and projection method, a way to construct certain tilings. See [18] for the general cut and projection method.

4.3.1 Lattices in $\mathbb{R}^N$

Let us start with two results about discrete subgroups of $\mathbb{R}^N$.

Definition 4.3.1 A **lattice** in $\mathbb{R}^N$ is a discrete subgroup $\tilde{L}$ of $\mathbb{R}^N$ such that the quotient group $\mathbb{R}^N/\tilde{L}$ has finite volume, i.e. has finite Lebesgue measure. A **uniform lattice** is a lattice such that the quotient $\mathbb{R}^N/\tilde{L}$ is compact.

Theorem 4.3.2 ([19], p.39) Every discrete subgroup $\tilde{L}$ of $\mathbb{R}^N$ is of the form

$$\tilde{L} = \mathbb{Z}v_1 + ... + \mathbb{Z}v_l,$$

where $\{v_i\}$ is some linearly independent set of vectors in $\mathbb{R}^N$.

Let $\tilde{L} = \mathbb{Z}v_1 + \ldots + \mathbb{Z}v_l$ be a discrete subgroup of $\mathbb{R}^N$, where the v_i's are linearly independent. We denote the $\mathbb{R}-$linear span of $\tilde{L}$ in $\mathbb{R}^N$ by

$$\langle\tilde{L}\rangle = \mathrm{Span}_{\mathbb{R}}(\tilde{L}) = \mathrm{Span}_{\mathbb{R}}\{v_1, \ldots, v_l\}.$$

If $\langle\tilde{L}\rangle \neq \mathbb{R}^N$, i.e. $l < N$, then $\mathbb{R}^N/\langle\tilde{L}\rangle \cong \mathbb{R}^{N-l}$. Hence the volume of the quotient space $\mathbb{R}^N/\tilde{L}$, relative to the Lebesgue measure on $\mathbb{R}^N$, is infinite, because already the volume of the quotient space $\mathbb{R}^N/\langle\tilde{L}\rangle$ is infinite. That is, $\tilde{L}$ will be a lattice in $\mathbb{R}^N$ if and only if $l = N$, in which case, it will in fact be a uniform lattice in $\mathbb{R}^N$.

Corollary 4.3.3 [19] Let $\tilde{L}$ be a discrete subgroup of $\mathbb{R}^N$. The following statements are equivalent:

1. $\tilde{L}$ is a lattice in $\mathbb{R}^N$;
2. $\tilde{L}$ is a uniform lattice in $\mathbb{R}^N$;
3. $\langle\tilde{L}\rangle = \mathbb{R}^N$.

4.3.2 The Cut and Projection Method

Definition 4.3.4 [3] A **cut and project scheme** is a triple $(E, E^{\perp}, \mathbb{Z}^N)$ of locally compact abelian groups in $\mathbb{R}^N$ where $E \cong \mathbb{R}^d$ and $E^{\perp} \cong \mathbb{R}^n$ are orthogonal and $\mathbb{Z}^N$ is a lattice in $E \oplus E^{\perp} \cong \mathbb{R}^N$, where $N = d + n$. We also ask that the orthogonal projections π and $\pi^{\perp}$, onto E and $E^{\perp}$ respectively, be such that $\pi|_{\mathbb{Z}^N}$ is injective, and

$\pi^{\perp}(\mathbb{Z}^N)$ is dense in $E^{\perp}$. We get the following diagram:

$$\begin{tikzcd} E & \mathbb{R}^N \arrow[l, "\pi"'] \arrow[r, "\pi^{\perp}"] & E^{\perp} \\ & \mathbb{Z}^N \arrow[u, phantom, "\cup"] & \end{tikzcd} \tag{4.3.1}$$

We call E the **physical space** and $E^{\perp}$ the **internal space**. We will soon explain how we get a tiling from this scheme but for now, let us only mention that the tiling will fill the physical space $E \cong \mathbb{R}^d$.

For example, if we are working in $E \oplus E^{\perp} \cong \mathbb{R}^2$, where $d = n = 1$, we will get a tiling on the real line. To make sure that the conditions on the two projections are satisfied, the physical space E will be a straight line going through the origin with an irrational slope. We will discuss this in more details in Example 4.3.14.

Let M be a subset of $E^{\perp}$ and denote $\Sigma = M + E$. We will sometimes refer to Σ as the **strip** of the tiling. Given M, we construct a point set $\Lambda(M)$ in E in the following way:

$$\Lambda(M) = \{\pi(a);\ a \in \mathbb{Z}^N \cap \Sigma\}. \tag{4.3.2}$$

Equivalently, we have

$$\Lambda(M) = \{\pi(a);\ a \in \mathbb{Z}^N, \pi^{\perp}(a) \in M\}.$$

We say that M is the **acceptance domain** of $\Lambda(M)$.

Remark 4.3.5 We can define a cut and project scheme in a slightly different way than what is defined in Definition 4.3.4. This new scheme is obtained from a cut and project scheme $(E, E^{\perp}, \mathbb{Z}^N)$ to which we apply the linear transformation which sends E onto $\mathrm{Span}_{\mathbb{R}}\{e_1, \ldots, e_d\}$ and $E^{\perp}$ onto $\mathrm{Span}_{\mathbb{R}}\{e_{d+1}, \ldots, e_N\}$, where $\{e_1, \ldots, e_N\}$ is the canonical basis of $\mathbb{R}^N$. The lattice $\mathbb{Z}^N$ will then be sent to another lattice of $\mathbb{R}^N$ denoted by $\tilde{L}$. We denote the orthogonal projections on $\mathbb{R}^d$ and $\mathbb{R}^n$ by π_1 and π_2

respectively. We will have that $\pi_1|_{\tilde{L}}$ is injective and that $\pi_2(\tilde{L})$ is dense in $\mathbb{R}^n$. The triple $(\mathbb{R}^d, \mathbb{R}^n, \tilde{L})$ is then also called and cut and project scheme. This version of a cut and project scheme is described in [18] as a special case of the more general cut and project scheme which looks at a triple of the form $(\mathbb{R}^d, G, \tilde{L})$, where G is a locally compact group in $\mathbb{R}^N$.

The diagram 4.3.1 becomes

$$\begin{tikzcd}
\mathbb{R}^d & \mathbb{R}^d \times \mathbb{R}^n \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & \mathbb{R}^n. \\
& \cup & \\
& \tilde{L} &
\end{tikzcd} \tag{4.3.3}$$

Similarly to the first cut and project scheme, given a subset M of $\mathbb{R}^n$, we define the associated point set

$$\Lambda(M) := \{\pi_1(a);\ a \in \tilde{L}, \pi_2(a) \in M\}.$$

Given a cut and project method $(E, E^{\perp}, \mathbb{Z}^N)$, we can recover a cut and project method of the form $(\mathbb{R}^d, \mathbb{R}^n, \tilde{L})$ and vice versa. Since E and $E^{\perp}$ are $d-$dimensional and $n-$dimensional subspaces of $\mathbb{R}^N$ respectively, we have

$$E = \mathrm{Span}_{\mathbb{R}}\{v_1, ..., v_d\} \quad \text{and} \quad E^{\perp} = \mathrm{Span}_{\mathbb{R}}\{v_{d+1}, ..., v_N\},$$

for some linearly independent v_i's in $\mathbb{R}^N$. Let A be the invertible matrix which sends e_i onto v_i for $i = 1, ..., N$. Then $A \in O(N)$ is an orthogonal matrix. Notice that $E = A(\mathbb{R}^d)$, $E^{\perp} = A(\mathbb{R}^n)$. Let $\tilde{L} := A(\mathbb{Z}^N) = \mathrm{Span}_{\mathbb{Z}}\{v_1, \dots, v_N\}$. By Corollary 4.3.3, $\tilde{L}$ is a lattice in $\mathbb{R}^N = \mathbb{R}^d \oplus \mathbb{R}^n$. We have that $\pi = A \circ \pi_1, \pi^{\perp} = A \circ \pi_2$ while $\pi_1 = A^{-1} \circ \pi$ and $\pi_2 = A^{\perp} \circ \pi^{\perp}$.

Moreover, $\mathbb{R}^N = E \oplus E^{\perp}$, and $\pi|_{\mathbb{Z}^N}$ is injective, while $\pi^{\perp}(\mathbb{Z}^N)$ is dense in $E^{\perp}$. We need to check that $(\mathbb{R}^d, \mathbb{R}^n, \tilde{L})$ satisfies the properties of a cut and project scheme, that is $\pi_1|_{\tilde{L}}$ is injective and $\pi_2(\tilde{L})$ is dense in $\mathbb{R}^n$, where π_1 and π_2 are the projections on $\mathbb{R}^d$ and $\mathbb{R}^n$ respectively. We have the following commutative diagram:

$$\begin{tikzcd}
\tilde{L} \arrow[r, "A^{-1}"] \arrow[d, "\pi_1|_{\tilde{L}}"'] & \mathbb{Z}^N \arrow[d, "\pi|_{\mathbb{Z}^N}"] \\
\mathbb{R}^d \arrow[r, "A"] & E
\end{tikzcd} \tag{4.3.4}$$

We first show that $\pi_1|_{\tilde{L}}$ is injective. Let $x, y \in \tilde{L}$ and suppose that

$$\pi_1(x) = \pi_1(y) = z \in \mathbb{R}^d.$$

Then

$$\pi(A^{-1}(x)) = A(\pi_1(x)) = A(z)$$

and

$$\pi(A^{-1}(y)) = A(\pi_1(y)) = A(z),$$

so that $\pi(A^{-1}(x)) = \pi(A^{-1}(y))$. As $A^{-1}(x), A^{-1}(y)$ are in $\mathbb{Z}^N$ and $\pi|_{\mathbb{Z}^N}$ is injective by hypothesis, it implies that $A^{-1}(x) = A^{-1}(y)$, which in turns implies that $x = y$.

To show that $\pi_2(\tilde{L})$ is dense in $\mathbb{R}^n$, we use a similar commutative diagram:

$$\begin{tikzcd}
\tilde{L} \arrow[r, "A^{-1}"] \arrow[d, "\pi_2"'] & \mathbb{Z}^N \arrow[d, "\pi^{\perp}"] \\
\mathbb{R}^n \arrow[r, "A"] & E^{\perp}
\end{tikzcd} \tag{4.3.5}$$

Let $x \in \mathbb{R}^n$ and consider $z := A(x) \in E^{\perp}$. Since $\pi^{\perp}(\mathbb{Z}^N)$ is dense in $E^{\perp}$, there exists a sequence $(z_n)_{n\in\mathbb{N}} \in \mathbb{Z}^N$ such that $\pi^{\perp}(z_n) \to z$ when $n \to \infty$. But $(A(z_n))_{n\in\mathbb{N}}$ is a sequence in $\tilde{L}$ for which

$$\pi_2(A(z_n)) = A^{-1}(\pi^{\perp}(z_n)) \to A^{-1}(z) = x,$$

when $n \to \infty$. Hence $(A(z_n))_{n\in\mathbb{N}}$ is a sequence in $\tilde{L}$ for which $\pi_2(A(z_n))$ converges to x, which proves the density property.

Conversely, given a cut and project scheme $(\mathbb{R}^d, \mathbb{R}^n, \tilde{L})$, we want to recover a scheme of the form $(E, E^\perp, \mathbb{Z}^N)$. First of all, since $\tilde{L}$ is a lattice of $\mathbb{R}^N$, we have that $\tilde{L}$ is a discrete subgroup of $\mathbb{R}^N$, and hence, by Theorem 4.3.2 and Corollary 4.3.3, there are linearly independent vectors $v_1, ..., v_N$ of $\mathbb{R}^N$ such that

$$\tilde{L} = \mathbb{Z}v_1 + ... + \mathbb{Z}v_N.$$

Let A be the invertible matrix which sends e_i onto v_i for $i = 1, ..., N$. Notice that the v_i's might not be orthogonal under the usual scalar product. We will define a new scalar product on $\mathbb{R}^N$ for which the v_i's will be orthogonal. Let $x, y \in \mathbb{R}^N$ and B be an $N \times N$ matrix. Recall that $(x, y)_B := y^t B x$ defines a scalar product on $\mathbb{R}^N$ if and only if B is symmetric and positive definite. Let

$$B = (AA^t)^{-1}.$$

We have that B is symmetric and positive definite. Moreover,

$$(v_i, v_j)_B = (Ae_i, Ae_j)_B = e_j^t A^t B A e_i = e_j^t e_i = \delta_{i,j},$$

for $1 \leq i, j \leq N$, so that the new scalar product $(\cdot,\cdot)_B$ makes the v_i's orthonormal. We then define

$$E := \mathrm{Span}_{\mathbb{R}}\{v_1, ..., v_d\}$$

and

$$E^\perp := \mathrm{Span}_{\mathbb{R}}\{v_{d+1}, ..., v_N\},$$

which will be two orthogonal subspaces of $\mathbb{R}^N$ with respect to the scalar product $(\cdot,\cdot)_B$. We can prove that $\pi|_{\mathbb{Z}^N}$ is injective and $\pi^\perp(\mathbb{Z}^N)$ is dense in $E^\perp$, by using diagrams very similar to those given in 4.3.4 and 4.3.5. ☐

Throughout this chapter, we will assume that the acceptance domain M is a compact subset of $E^{\perp}$ which is the closure of its interior.

Remark 4.3.6 Since the acceptance domain, M, is assumed to be compact, the boundary of M in $E^{\perp}$ is compact and nowhere dense (i.e. $\mathrm{Int}(\mathrm{Cl}(\partial M)) = \emptyset$). Indeed, we have

$$\begin{aligned} \mathrm{Int}(\mathrm{Cl}(\partial M)) &= \mathrm{Int}(\partial M) \quad \text{since } \partial M \text{ is closed} \\ &= \mathrm{Int}(M \cap (\mathrm{Int}(M))^c) \quad \text{by definition of } \partial M \text{ and since } M \text{ is closed.} \end{aligned}$$

If there exists $y \in \mathrm{Int}(M \cap (\mathrm{Int}(M))^c)$ then there exists $r > 0$ for which the ball $B_r(y)$ is contained in $M \cap (\mathrm{Int}(M))^c$. In particular, $B_r(y) \subset (\mathrm{Int}(M))^c$, which implies that $y \notin \mathrm{Int}(M)$, that is, for all $\varepsilon > 0$, $B_\varepsilon(y) \not\subset M$. But we have that $B_r(y) \subset M$, a contradiction. This shows that $\mathrm{Int}(M \cap (\mathrm{Int}(M))^c$ is indeed empty. □

We will often take $M = \pi^{\perp}([0,1]^N)$. Figure 4.1 illustrates the cut and projection method when $N = 2, d = n = 1$, i.e. when we want to construct a one-dimensional tiling. We use a cut and project scheme of the form $(E, E^{\perp}, \mathbb{Z}^2)$. The tiles of the resulting tiling will be the line segments joining every two consecutive points of $\Lambda(M)$.

Recall that a point set Λ in $\mathbb{R}^d$ is a Delone set if Λ is uniformly discrete and relatively dense.

Theorem 4.3.7 [27] The point set $\Lambda(M)$, defined in equation 4.3.2, is a Delone set in E.

Proof: We must show that $\Lambda(M)$ is uniformly discrete and relatively dense in E. To prove uniform discreteness, it suffices to show that there is a neighborhood of the

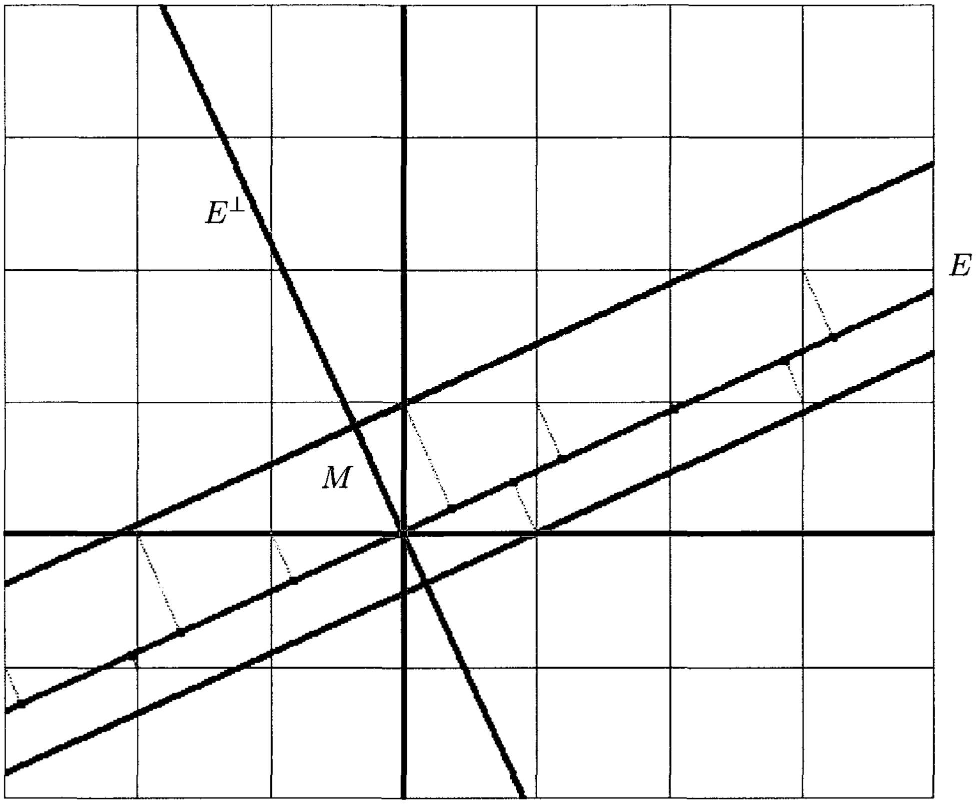

Figure 4.1: The cut and projection method illustrated for the case $d = n = 1$, $N = 2$. In this figure, $M = \pi^{\perp}([0,1]^2)$.

origin in E that contains no other points of $\Lambda(M)$. Let $m > 0$ be any positive real number. Consider $D_m(0)$, the closed ball in E or radius m around the origin. Set $K = \Sigma \cap (D_m(0) + M)$. Then K is compact in $\mathbb{R}^N$ because it is closed and bounded. This and the fact that $\mathbb{Z}^N$ is uniformly discrete implies that $K \cap \mathbb{Z}^N$ is finite and hence $\pi(K \cap \mathbb{Z}^N)$ is also finite. From here, it is easy to choose $r > 0$ small enough so that $B_r(0) \cap \Lambda(M) = \{0\}$ (for example, we can take $r = \frac{1}{2}\min\{\|x\| : x \in \pi(K \cap \mathbb{Z}^N)\}$). Relative density is immediate because $\mathbb{Z}^N$ is relatively dense in $\mathbb{R}^N$ and hence in Σ (therefore $\pi(\mathbb{Z}^N \cap \Sigma)$ is relatively dense since $\pi|_{\mathbb{Z}^N}$ is injective). ∎

Remark 4.3.8 Since, by Theorem 4.3.7, $\Lambda(M)$ is a Delone set in $\mathbb{R}^d$, we can construct the Voronoi cell of every point of $\Lambda(M)$. The Voronoi tesselation will have the following properties ([27], p.44)

1. The cells are convex and fit together along whole faces, no two cells have a common interior point;
2. The points of $\Lambda(M)$ whose Voronoi cells share a vertex v lie on a sphere, centered at v, that has no points of $\Lambda(M)$ in its interior.

The Voronoi tesselation of $\Lambda(M)$ will give us a tiling of $\mathbb{R}^d$.

Definition 4.3.9 [27] A tiling is said to be **normal** if there are positive real numbers r and R such that every tile contains a ball of radius r and every tile is contained in a ball of radius R.

Remark 4.3.10 If a tiling is constructed from a finite set of prototiles, then it will be normal.

Theorem 4.3.11 ([27] p.153) The Voronoi tessellation induced by a Delone set $\Lambda \subset \mathbb{R}^d$ is a normal tiling of $\mathbb{R}^d$.

Proof: Since Λ is a Delone set, it is uniformly discrete and so there exists a real numbers r such that for every $x, y \in \Lambda$, $\|x - y\| \geq 2r$. Therefore, the minimal distance between any two points of Λ is $2r$. Since the edges of V_x lie on the bisector of the lines joining x to its neighbors, the distance from x to the nearest edge of V_x must be greater than or equal to r. Thus a ball of radius r can be inscribed in V_x. Next, since Λ is a Delone set, it is relatively dense, so there exists a positive real number R such that every sphere of radius greater than R contains at least one point of Λ in its interior. Let v be any vertex of V_x. By definition 4.2.11, v is equidistant from x and at least d other points of Λ, those that lie at the centers of the surrounding Voronoi cell. This means that v is the center of a ball of radius $|v - x|$ containing no points of Λ in its interior, so $\|v - x\| \leq R$. Thus V_x is contained in the closed ball $D_R(x)$.■

Corollary 4.3.12 To construct V_x we do not need to join x to all the other points of Λ, but only those that lie inside the ball $D_{2R}(x)$.

The following theorem states that the two conditions on the projections π and $\pi^\perp$ in a cut and project scheme $(E, E^\perp, \mathbb{Z}^N)$ are equivalent to one another and to the fact that $E \cap \mathbb{Z}^N = 0$. We will often use the latter condition to construct cut and project schemes.

Theorem 4.3.13 ([27] p.55) Let $(E, E^\perp, \mathbb{Z}^N)$ be a cut and project scheme. Then the following are equivalent:

1. $\pi(\mathbb{Z}^N)$ is everywhere dense in E;
2. $E \cap \mathbb{Z}^N = 0$;
3. $\pi^\perp|_{\mathbb{Z}^N}$ is injective.

Throughout this section, we will illustrate the theory of the cut and projection method with two examples; the Fibonacci tiling, which tiles the line, and the octogonal tiling, which tiles the plane.

Example 4.3.14 (THE FIBONACCI TILING)
The Fibonacci tiling is a one-dimensional tiling, so $N = 2$ and $d = n = 1$. We use $(E, E^\perp, \mathbb{Z}^2)$ as our cut and project scheme, where E is the line going through the origin with slope equal to $1/\tau$, where $\tau = \frac{1+\sqrt{5}}{2}$ is the golden mean. As $E \cap \mathbb{Z}^2 = 0$, the conditions on the projections π and $\pi^\perp$ will be satisfied, by Theorem 4.3.13.

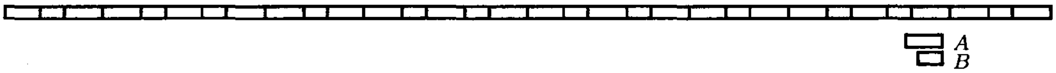

Figure 4.2: Patch of the Fibonacci tiling.

The acceptance domain is $M = \pi^{\perp}([-\frac{1}{2}, \frac{1}{2}]^2)$, i.e. in this case we are not using the usual acceptance domain $\pi^{\perp}([0,1]^2)$. To construct this tiling we first find its point pattern $\Lambda(M)$. That is, we take every point of $\mathbb{Z}^2$ and project it in $E^{\perp}$. If the orthogonal projection falls in M then we project the initial point in E. The tiles will be the line segments joining every two consecutive points of the point pattern $\Lambda(M)$. See Figure 4.1 to get an idea of the construction and see Figure 4.2 for a patch of the Fibonacci tiling. To illustrate the link between the two cut and projection methods defined earlier, let

$$E = \mathrm{Span}_{\mathbb{R}}\{v_1\} = \mathrm{Span}_{\mathbb{R}}\{(\tau, 1)\},$$

$$E^{\perp} = \mathrm{Span}_{\mathbb{R}}\{v_2\} = \mathrm{Span}_{\mathbb{R}}\{(-1, \tau)\}$$

and

$$A = \begin{pmatrix} \tau & -1 \\ 1 & \tau \end{pmatrix}.$$

Define $\tilde{L} = A(\mathbb{Z}^2)$. Then we can also use the cut and project scheme $(\mathbb{R}, \mathbb{R}, \tilde{L})$ to construct the Fibonacci tiling. In that case, the physical space and internal space are the one-dimensional subspaces generated by e_1 and e_2 respectively. $\square$

Example 4.3.15 (THE OCTOGONAL TILING)

The octogonal tiling has $N = 4$ and $d = n = 2$, with cut and projection scheme $(E, E^{\perp}, \mathbb{Z}^4)$, where the physical space E and its orthocomplement $E^{\perp}$ are the vector subspaces of $\mathbb{R}^4$ invariant under the action of the linear map

$$B = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

To give a concrete description of E, we note that

$$\det(B - \lambda I) = \lambda^4 + 1 \text{ and that } \lambda = \frac{\pm\sqrt{2} \pm i\sqrt{2}}{2}$$

are the eigenvalues of B. One choice of eigenvectors in $\mathbb{C}^4$ is

$$u_1 = \left(\frac{1}{2\sqrt{2}}, \frac{1-i}{4}, \frac{-i}{2\sqrt{2}}, \frac{-1-i}{4}\right), \quad u_2 = \left(\frac{1}{2\sqrt{2}}, \frac{1+i}{4}, \frac{i}{2\sqrt{2}}, \frac{-1+i}{4}\right),$$
$$u_3 = \left(\frac{1+i}{4}, \frac{-i}{2\sqrt{2}}, \frac{-1+i}{4}, \frac{1}{2\sqrt{2}}\right), \quad u_4 = \left(\frac{1-i}{4}, \frac{i}{2\sqrt{2}}, \frac{-1-i}{4}, \frac{1}{2\sqrt{2}}\right).$$

As we already mentioned, E and $E^\perp$ will be two-dimensional subspaces of $\mathbb{R}^4$. They are given by

$$E = \operatorname{Span}\{u_1 + u_2, -i(u_1 - u_2)\} = \operatorname{Span}\left\{\left(\frac{1}{\sqrt{2}}, \frac{1}{2}, 0, \frac{-1}{2}\right), \left(0, \frac{-1}{2}, \frac{-1}{\sqrt{2}}, \frac{-1}{2}\right)\right\}$$

and

$$E^\perp = \operatorname{Span}\{u_3 + u_4, -i(u_3 - u_4)\} = \operatorname{Span}\left\{\left(\frac{1}{2}, 0, \frac{-1}{2}, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{2}, \frac{-1}{\sqrt{2}}, \frac{1}{2}, 0\right)\right\},$$

so that $B(E) = E$ and $B(E^\perp) = E^\perp$. Set $M = \pi^\perp([0,1]^4)$. Notice that the acceptance domain, M, in this example, is an octogon (see Figure 4.3). To construct the octogonal tiling, we have to start by computing $\Lambda(M)$, which gives us a point set in $E \cong \mathbb{R}^2$. To get the octogonal tiling from $\Lambda(M)$, we construct its Voronoi tesselation. Recall that for the Fibonacci example, we only had to connect the points of $\Lambda(M)$ to get the tiling. This is equivalent to the Voronoi tesselation in $\mathbb{R}$. Once again, we make

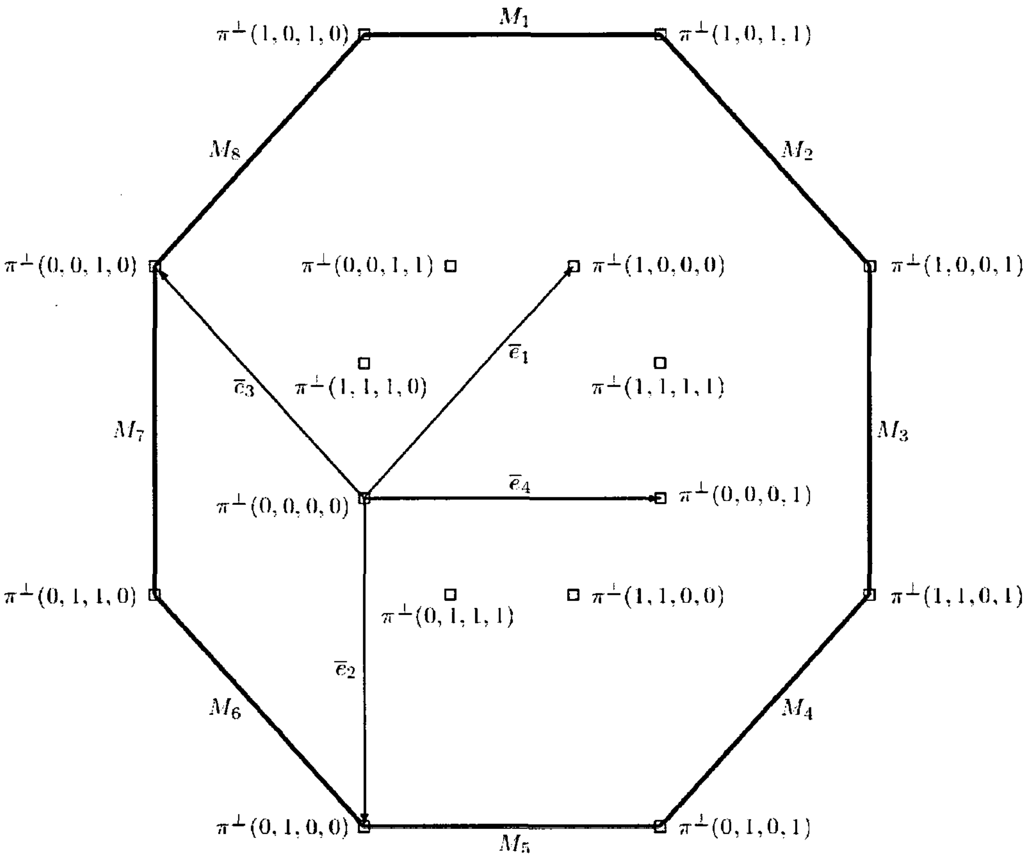

Figure 4.3: Acceptance domain for the octogonal tiling

the connection between the two cut and project schemes. Let

$$A = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \in SO(4).$$

and define $\tilde{L} = A(\mathbb{Z}^4)$. Then we can use the cut and project scheme $(\mathbb{R}^2, \mathbb{R}^2, \tilde{L})$ to construct the octogonal tiling, where the physical and internal spaces are the two-dimensional subspaces generated by e_1, e_2 and e_3, e_4 respectively. □

4.3.3 Non-singular points in $\mathbb{R}^N$

We now introduce the notion of non-singular points of $\mathbb{R}^N$. These points form a dense subset of $\mathbb{R}^N$ that will play a crucial role in section 4.4 when we will define minimal Cantor $\mathbb{Z}^d$–dynamical systems associated to tilings.

Definition 4.3.16 Let $(E, E^\perp, \mathbb{Z}^N)$ be a cut and project scheme. A point $v \in \mathbb{R}^N$ is called **non-singular** if $(v + \mathbb{Z}^N) \cap \partial\Sigma = \emptyset$, where $\partial\Sigma$ is the boundary of the strip $\Sigma = M + E$. We will denote the set of non-singular points by NS.

Remark 4.3.17 The condition of non-singularity can also be expressed as follows : $v \in NS$ if $\pi^\perp(v + \mathbb{Z}^N) \cap \partial M = \emptyset$. Indeed, if there exists $w \in (v + \mathbb{Z}^N) \cap \partial\Sigma$, then

$$w \in \partial\Sigma \Rightarrow \pi^\perp(w) \in \partial M,$$

since $\Sigma = M + E$. We then have $\pi^\perp(w) \in \pi^\perp(v + \mathbb{Z}^N) \cap \partial M$. On the other hand, if there exists $w \in \pi^\perp(v + \mathbb{Z}^N) \cap \partial M$, then

$$w = \pi^\perp(v + m) \in \partial M,$$

for some $m \in \mathbb{Z}^N$, so that $v + m \in \partial\Sigma$. Hence $(v + \mathbb{Z}^N) \cap \partial\Sigma \neq \emptyset$. □

Definition 4.3.18 We can also look at the complement of NS, the **singular** points, by noticing that a point $v' \in \mathbb{R}^N \backslash NS$ if $(v' + \mathbb{Z}^N) \cap \partial\Sigma \neq \emptyset$, i.e. if $\exists\ w \in \mathbb{Z}^N$ such that $v' - w \in \partial\Sigma$, equivalently,

$$v' \in \partial\Sigma + w = \partial M + E + w.$$

Hence, we get the following description of the singular points

$$\mathbb{R}^N \backslash NS = \bigcup_{w \in \mathbb{Z}^N} (\partial M + E + w).$$

We will refer to the singular points either by $\mathbb{R}^N \backslash NS$ or simply by S.

Let $\tilde{P}_v = (v + \mathbb{Z}^N) \cap \Sigma$ be the **strip point pattern** and $P_v = \pi(\tilde{P}_v) \subset E$ be the **projection point pattern**. Recall from 4.3.2 that $\Lambda(M) = \{\pi(a);\ a \in \mathbb{Z}^N \cap \Sigma\}$ so we can see that $\Lambda(M) = P_0$.

Lemma 4.3.19 The set of non-singular points, NS, is a dense G_δ subset of $\mathbb{R}^N$, invariant under translation by E.

proof: It follows from the fact that $\mathbb{R}^N \backslash NS = \bigcup_{w \in \mathbb{Z}^N} (\partial M + E + w)$ and ∂M is nowhere dense. First, NS is an intersection of open sets, hence a G_δ set, since $NS = \bigcap_{w \in \mathbb{Z}^N} (\partial M + E + w)^c$ and $\partial M + E + w$ is closed for all $w \in \mathbb{Z}$.

To show that NS is dense in $\mathbb{R}^N$, we first show that since ∂M is nowhere dense and closed in $E^\perp$, then $(\partial M)^c$ is dense in $E^\perp$. Indeed, since $\operatorname{Int}(\partial M) = \emptyset$ (by Remark 4.3.6), every nonempty open set of $E^\perp$ contains a point of $(\partial M)^c$, which is equivalent to saying that $(\partial M)^c$ is dense in $E^\perp$.

Next,

$$
\begin{aligned}
\mathrm{Cl}_{\mathbb{R}^N}(NS) &= \mathrm{Cl}_{\mathbb{R}^N}\left(\bigcap_{w \in \mathbb{Z}^N} (\partial M + E + w)^c_{\mathbb{R}^N} \right) \\
&= \bigcap_{w \in \mathbb{Z}^N} \mathrm{Cl}_{\mathbb{R}^N}\left((\partial M + E + w)^c_{\mathbb{R}^N} \right) \\
&= \bigcap_{w \in \mathbb{Z}^N} \mathrm{Cl}_{\mathbb{R}^N}\left((\partial M)^c_{E^\perp} + E + w \right) \\
&= \bigcap_{w \in \mathbb{Z}^N} \left(\mathrm{Cl}_{E^\perp}((\partial M)^c_{E^\perp}) + E + w \right) \\
&= \bigcap_{w \in \mathbb{Z}^N} (E^\perp + E + w) \\
&= \mathbb{R}^N,
\end{aligned}
$$

so that the non-singular points are dense in $\mathbb{R}^N$. The invariance under translation by E is clear. ∎

Definition 4.3.20 Let G be a group isomorphic to $\mathbb{Z}^N$ such that G is of finite order in $\mathbb{Z}^N$, i.e. for all $g \in G$, there exists $k \neq 0 \in \mathbb{Z}$ such that $kg \in \mathbb{Z}^N$. Fix a free generating set, $r_1, r_2, \ldots, r_N$ for G and let $F = \mathrm{Span}_{\mathbb{R}}\{r_1, \ldots, r_n\}$, where $n = N - d$. Let

$$G_0 = \mathrm{Span}_{\mathbb{Z}}\{r_1, \ldots, r_n\}$$

and let G_1 be the complementary subgroup generated by the other d generators. Let π' be the skew projection onto F parallel to E and let $M' = \pi'(M)$.

Remark 4.3.21 The set F defined in Definition 4.3.20 will be complementary to E. Indeed, we know that $E \cap \mathbb{Z}^N = 0$ and we will show this implies that $E \cap G = 0$ and therefore $E \cap F = 0$. Let $g \in E \cap G$. Since G is of finite order, there exists $k \neq 0 \in \mathbb{Z}$ such that $kg \in \mathbb{Z}^N$. As E is a vector space, $kg \in E \cap \mathbb{Z}^N = 0$. Therefore $g = 0$, i.e. $E \cap G = 0$. Now let

$$x = \sum_{i=1}^{n} \lambda_i r_i \in E \cap F.$$

We first suppose that $\lambda_i = \frac{p_i}{q_i} \in \mathbb{Q}$, for some $p_i, q_i \in \mathbb{Z}$, $i = 1, \ldots, n$. Hence $(\prod_{i=1}^{n} q_i)x \in E \cap G = 0$. Therefore, $x = 0$. We can then use the fact that the rationals are dense in $\mathbb{R}$ to show that if $x = \sum_{i=1}^{n} \lambda_i r_i \in E \cap F$ with $\lambda_i \in \mathbb{R}$ for $i = 1, \ldots, n$ then $x = 0$. □

Let $g \in G = G_0 \oplus G_1$. Then $G \cong \mathbb{Z}^N$ acts on F in the following way:

$$\begin{aligned} \chi : F \times G &\to F \\ (x, g) &\mapsto x + \pi'(g). \end{aligned} \tag{4.3.6}$$

As the actions induced by the groups G_0 and G_1 commute, we have that $\chi|_{F \times G_1}$ induces a G_1-action on F/G_0 via

$$([x], g_1) \in F/G_0 \times G_1 \mapsto \chi(([x], g_1)) = [x + \pi'(g_1)], \text{ for } g_1 \in G_1. \tag{4.3.7}$$

We can see the quotient F/G_0 as a n-dimensional torus on which G_1 acts.

Example 4.3.22 (THE FIBONACCI TILING)

We now return to our first example, the Fibonacci tiling. Recall that we defined

$$E = \mathrm{Span}_{\mathbb{R}}\{v_1\} = \mathrm{Span}_{\mathbb{R}}\{(\tau, 1)\},$$

and

$$E^{\perp} = \mathrm{Span}_{\mathbb{R}}\{v_2\} = \mathrm{Span}_{\mathbb{R}}\{(-1, \tau)\}.$$

Let $G = \mathbb{Z}^2 = \mathrm{Span}_{\mathbb{Z}}\{e_1, e_2\}$ so that $F = \mathrm{Span}_{\mathbb{R}}\{e_1\}$ is the $x-$axis.

Then $G_0 = \mathrm{Span}_{\mathbb{Z}}\{e_1\}$ and $G_1 = \mathrm{Span}_{\mathbb{Z}}\{e_2\}$. If we denote $xe_1 + ye_2 = (x, y)$, the G_1-action on $F/G_0 = [0, 1)$ is given by

$$\begin{aligned} ([(x,0)],(0,k)) \mapsto\ & [(x,0) + \pi'((0,k))] \\ &= [(x,0) + k\pi'((0,1))] \\ &= [(x,0) + (-k\tau, 0)] \\ &= [(x - k\tau, 0)], \end{aligned} \tag{4.3.8}$$

where $x \in \mathbb{R}$, $k \in \mathbb{Z}$. That is, the action of G_1 on the $F/G_0 = [0, 1)$ is a translation by $\tau\mathbb{Z} \mod [0, 1)$. This is equivalent to a rotation on the circle $[0, 1)$ by angles of the form $2\pi\tau\mathbb{Z}$. Since τ is irrational, the rotation is minimal.

Now we give a detailed description of the singular points for the Fibonacci tiling. As seen in Remark 4.3.17, $v' \in \mathbb{R}^2$ will be singular if $\pi^{\perp}(v' + \mathbb{Z}^2) \cap \partial M \neq \emptyset$, i.e. $v' \in \mathbb{R}^2 \backslash NS$ if there exists $m = (a, b) \in \mathbb{Z}^2$ such that

$$\pi^{\perp}(v' + m) = \pi^{\perp}(v') + a\bar{e}_1 + b\bar{e}_2 \in \partial M,$$

where $\bar{e}_i = \pi^{\perp}(e_i)$, $i = 1, 2$. But when $N = 2$, the boundary of the acceptance domain

$M = \pi^{\perp}([-\frac{1}{2}, \frac{1}{2}]^2)$ is only two points:

$$\begin{aligned}\partial M &= \left\{\pi^{\perp}\left(\frac{1}{2}, -\frac{1}{2}\right), \pi^{\perp}\left(-\frac{1}{2}, \frac{1}{2}\right)\right\} \\ &= \left\{\pi^{\perp}\left(\frac{e_1}{2}\right) + \pi^{\perp}\left(\frac{-e_2}{2}\right), \pi^{\perp}\left(\frac{-e_1}{2}\right) + \pi^{\perp}\left(\frac{e_2}{2}\right)\right\} \\ &= \left\{\frac{\bar{e}_1 - \bar{e}_2}{2}, \frac{\bar{e}_2 - \bar{e}_1}{2}\right\}.\end{aligned}$$

For v' to be singular, we thus need

$$\pi^{\perp}(v') + a\bar{e}_1 + b\bar{e}_2 \in \partial M,$$

that is,

$$\pi^{\perp}(v') + a\bar{e}_1 + b\bar{e}_2 = \frac{\bar{e}_1 - \bar{e}_2}{2} \text{ or } \frac{\bar{e}_2 - \bar{e}_1}{2},$$

for some $a, b \in \mathbb{Z}$. In both cases, we get that $v' \in \mathbb{R}^2 \backslash NS$ if

$$\pi^{\perp}(v') \in \frac{(1+2\mathbb{Z})\bar{e}_1}{2} + \frac{(1+2\mathbb{Z})\bar{e}_2}{2} = \frac{\bar{e}_1}{2}\mathbb{Z}_{\text{odd}} + \frac{\bar{e}_2}{2}\mathbb{Z}_{\text{odd}},$$

where $\mathbb{Z}_{\text{odd}}$ denotes the odd integers.

Next, we would like to compute $(F \cap S)/G_0$, that is, we want to find the singular points in $F/G_0 \cong [0, 1)$. Let $(x, 0) = xe_1 \in F$. For $(x, 0)$ to be singular, we need

$$\pi^{\perp}(x, 0) = x\bar{e}_1 \in \frac{\bar{e}_1}{2}\mathbb{Z}_{\text{odd}} + \frac{\bar{e}_2}{2}\mathbb{Z}_{\text{odd}}.$$

But it is easy to verify that $\bar{e}_2 = -\tau\bar{e}_1$, so that we need

$$x\bar{e}_1 \in \frac{\bar{e}_1}{2}\mathbb{Z}_{\text{odd}} + \frac{\tau\bar{e}_1}{2}\mathbb{Z}_{\text{odd}},$$

i.e. $x = \frac{m+\tau n}{2}$, for some $m, n \in \mathbb{Z}_{\text{odd}}$. Taking the quotient by G_0, we get

$$(F \cap S)/G_0 = \left\{\left(\tfrac{1+\tau n}{2}, 0\right) + G_0;\ n \in \mathbb{Z}_{\text{odd}}\right\}.$$

Recall the G_1−action on F/G_0 given in equation 4.3.8:

$$([(x,0)],(0,k)) \mapsto [(x-k\tau,0)].$$

Note that G_1 also acts on $(F \cap NS)/G_0$. To prove this we only need to show that the action sends elements of NS to elements of NS. Assume that $(x-k\tau,0) \in S$, i.e. we have

$$x-k\tau = \frac{m+\tau n}{2}, \text{ for some } m,n \in \mathbb{Z}_{\text{odd}}.$$

Then $x = \frac{m+\tau(n+2k)}{2}$ for some $m,n \in \mathbb{Z}_{\text{odd}}$. Since $n+2k$ will also be odd, x is in S. □

Example 4.3.23 (THE OCTOGONAL TILING)
Recall from Example 4.3.15 that in the octogonal tiling, we take

$$E = \text{Span}\left\{\left(\frac{1}{\sqrt{2}},\frac{1}{2},0,\frac{-1}{2}\right),\left(0,\frac{-1}{2},\frac{-1}{\sqrt{2}},\frac{-1}{2}\right)\right\}$$

and

$$E^{\perp} = \text{Span}\left\{\left(\frac{1}{2},0,\frac{-1}{2},\frac{1}{\sqrt{2}}\right),\left(\frac{1}{2},\frac{-1}{\sqrt{2}},\frac{1}{2},0\right)\right\},$$

with $M = \pi^{\perp}([0,1]^4)$. Let $\{e_1,...,e_4\}$ denote the canonical basis of $\mathbb{R}^4$ and $\overline{e}_i := \pi^{\perp}(e_i)$, $i = 1,...,4$. It is easy to see that $\{\overline{e}_4,\overline{e}_2\}$ forms a basis for $E^{\perp}$. For this example, $G = \mathbb{Z}^4$, $F = E^{\perp} = \text{Span}_{\mathbb{R}}\{\overline{e}_4,\overline{e}_2\}$ with $G_0 = \text{Span}_{\mathbb{Z}}\{\overline{e}_4,\overline{e}_2\}$ and $G_1 = \text{Span}_{\mathbb{Z}}\{e_1,e_3\}$. Recall that these two groups act on $F = E^{\perp}$ and this action was given in equation 4.3.6.

Recall from the definition of singular points, given in 4.3.18 that $v' \in \mathbb{R}^4$ is singular if there exists $m = (a,b,c,d) \in \mathbb{Z}^4$ such that

$$\pi^{\perp}(v'+m) = \pi^{\perp}(v') + a\overline{e}_1 + b\overline{e}_2 + c\overline{e}_3 + d\overline{e}_4 \in \partial M.$$

In particular, every point on the boundary of M is singular.

The goal of what follows is to describe in more details the singular points in the torus $E^{\perp}/G_0 \cong [0,1) \times [0,1)$.

First consider every point on the edge M_5 (see Figure 4.3). These are all singular since they are on the boundary of M. These points get mapped on the line $t\bar{e}_4$ for $t \in [0,1)$ when we take the quotient by G_0. We can translate these points by $\mathbb{Z}\bar{e}_1$ or $\mathbb{Z}\bar{e}_3$ and still get singular points;

$$\begin{aligned} t\bar{e}_4 + \mathbb{Z}\bar{e}_1 &= \{t\bar{e}_4 + n\bar{e}_1; n \in \mathbb{Z}\} = \left\{(t,0) + n\left(\tfrac{1}{\sqrt{2}}, \tfrac{-1}{\sqrt{2}}\right); n \in \mathbb{Z}\right\} \\ &= \left\{\left(t + \tfrac{n}{\sqrt{2}}, \tfrac{-n}{\sqrt{2}}\right); n \in \mathbb{Z}\right\}. \end{aligned}$$

Similarly,

$$t\bar{e}_4 + \mathbb{Z}\bar{e}_3 = \left\{\left(t + \tfrac{n}{\sqrt{2}}, \tfrac{-n}{\sqrt{2}}\right); n \in \mathbb{Z}\right\}.$$

These are the horizontal lines in the torus with vertical coordinate given by $\frac{1}{\sqrt{2}}\mathbb{Z}$ mod $[0,1)$. It is easy to verify that the edge M_1, when translated by $\mathbb{Z}\bar{e}_1$ or $\mathbb{Z}\bar{e}_3$, gives us the same singular points in the torus.

For the edges M_7 and M_3, we can show in a similar way that when we mod out by G_0, we get the singular points $\left(\frac{n}{\sqrt{2}}, t - \frac{n}{\sqrt{2}}\right) + G_0$. These are the vertical lines with horizontal coordinate equal to $\frac{1}{\sqrt{2}}\mathbb{Z}$ mod $[0,1)$.

Now consider the edge M_6. When we take the quotient by G_0, the edge M_6 and the line $M_6 + \mathbb{Z}\bar{e}_3$ get mapped to the diagonal (t,t) for $t \in [0,1)$. Let us see what singular points we get by translating this diagonal by $\mathbb{Z}\bar{e}_1$;

$$\begin{aligned} (t,t) + \mathbb{Z}\bar{e}_1 &= \{(t,t) + n\bar{e}_1; n \in \mathbb{Z}\} = \left\{(t,t) + n\left(\tfrac{1}{\sqrt{2}}, \tfrac{-1}{\sqrt{2}}\right); n \in \mathbb{Z}\right\} \\ &= \left\{\left(t + \tfrac{n}{\sqrt{2}}, t - \tfrac{n}{\sqrt{2}}\right); n \in \mathbb{Z}\right\} \end{aligned}$$

This is the set of points $(x, y) = \left(t + \frac{n}{\sqrt{2}}, t - \frac{n}{\sqrt{2}}\right)$ for $t \in [0,1)$, $n \in \mathbb{Z}$. Solving for t we get $t = x - \frac{n}{\sqrt{2}} = y + \frac{n}{\sqrt{2}}$, and so $y = x - \frac{2n}{\sqrt{2}} = x - \sqrt{2}n$. Hence all lines with slope equal to 1 and y−intercept equal to $\sqrt{2}n \mod [0,1)$ for some $n \in \mathbb{Z}$ are singular points in the torus. It is easy to see that the edge M_2 (and the line $M_2 + \mathbb{Z}\bar{e}_3$) get mapped to the same singular points than $M_6 + \mathbb{Z}\bar{e}_3$ when translated by $\mathbb{Z}\bar{e}_1$.

We can show that the edges M_4 and M_8 become the singular points given by the lines with slope equal to -1 and y−intercept equal to $\sqrt{2}n \mod [0,1)$ for some $n \in \mathbb{Z}$, when translated by $\mathbb{Z}\bar{e}_3$.

In summary, the singular points in the torus, $(E^{\perp} \cap S)/G_0$, are described as

1. vertical lines with horizontal coordinate given by $\frac{1}{\sqrt{2}}\mathbb{Z} \mod [0,1)$,
2. horizontal lines with vertical coordinate given by $\frac{1}{\sqrt{2}}\mathbb{Z} \mod [0,1)$, (4.3.9)
3. lines with slopes equal to ±1 and y−intercept equal to $\sqrt{2}\mathbb{Z} \mod [0,1)$.

Recall how we defined G_0 and G_1 at the beginning of this example. These two groups act on $E^{\perp}$ by translations. Furthermore, if we take the quotient of $E^{\perp}$ by G_0, the $\mathbb{Z}^2$−action of G_1 on $E^{\perp}$ induces a $\mathbb{Z}^2$−action on the torus $\mathbb{T} = E^{\perp}/G_0$. Let $(m, n) \in \mathbb{Z}^2$ and $(x, y) \in E^{\perp}$. This action is given by

$$\begin{aligned}
\Phi^{(m,n)}((x, y) + G_0) &= (x, y) + m\bar{e}_1 + n\bar{e}_3 + G_0 \\
&= (x, y) + \left(\tfrac{m}{\sqrt{2}}, \tfrac{-m}{\sqrt{2}}\right) + \left(\tfrac{-n}{\sqrt{2}}, \tfrac{-n}{\sqrt{2}}\right) + G_0 \\
&= \left(x + \tfrac{m}{\sqrt{2}} - \tfrac{n}{\sqrt{2}}, y - \tfrac{m}{\sqrt{2}} - \tfrac{n}{\sqrt{2}}\right) + G_0 \\
&= \left(x + \tfrac{m-n}{\sqrt{2}}, y - \tfrac{m+n}{\sqrt{2}}\right) + G_0.
\end{aligned}$$

Now suppose we wished to rewrite elements of $E^{\perp}$, for example $x\bar{e}_4 + y\bar{e}_2$, using another basis $\{w_1, w_2\}$, where $w_1 = \bar{e}_4 + \bar{e}_2$ and $w_2 = -\bar{e}_4 + \bar{e}_2$. Notice that $w_1 = -\sqrt{2}\bar{e}_3$ and $w_2 = -\sqrt{2}\bar{e}_1$. We need to find $\tilde{x}$ and $\tilde{y}$ such that $\tilde{x}w_1 + \tilde{y}w_2 = x\bar{e}_4 + y\bar{e}_2$. It is easy to verify that $\tilde{x} = (x+y)/2$ and $\tilde{y} = (y-x)/2$. Using the basis $\{w_1, w_2\}$ of $E^{\perp}$, we express the $\mathbb{Z}^2-$action on the torus as follows:

$$\begin{aligned}
\Phi^{(m,n)}(\tilde{x}w_1 + \tilde{y}w_2 + G_0) &= \Phi^{(m,n)}\left(\tfrac{x+y}{2}w_1 + \tfrac{y-x}{2}w_2 + G_0\right) \\
&= \Phi^{(m,n)}((x,y) + G_0) \\
&= \left(x + \tfrac{m-n}{\sqrt{2}}, y - \tfrac{m+n}{\sqrt{2}}\right) + G_0 \\
&= \left(\tfrac{x+y}{2} - \tfrac{n}{\sqrt{2}}\right) w_1 + \left(\tfrac{y-x}{2} - \tfrac{m}{\sqrt{2}}\right) w_2 + G_0 \\
&= \left(\tilde{x} - \tfrac{n}{\sqrt{2}}\right) w_1 + \left(\tilde{y} - \tfrac{m}{\sqrt{2}}\right) w_2 + G_0.
\end{aligned}$$

The reason we use the basis $\{w_1, w_2\}$ is that we then get a product action, i.e. $\mathbb{Z}^2$ acts on the two component, $\tilde{x}, \tilde{y}$ independently. The $\mathbb{Z}^2-$action by G_1 on the torus $E^{\perp}/G_0$ can then be defined as two translations by elements of the form $\frac{1}{\sqrt{2}}\mathbb{Z}$. One translation is in the direction of w_1 and the other is in the direction of w_2. If we express the torus using the $\{w_1, w_2\}$ basis of $E^{\perp}$, we get rotations by angles of the form $\frac{2\pi}{\sqrt{2}}\mathbb{Z}$:

$$\begin{aligned}
\varphi \otimes \psi : [0,1) \times [0,1) \times \mathbb{Z}^2 &\to [0,1) \times [0,1) \\
((\tilde{x}, \tilde{y}), (m,n)) &\mapsto \left(\tilde{x} + \frac{m}{\sqrt{2}}, \tilde{y} + \frac{n}{\sqrt{2}}\right)
\end{aligned} \tag{4.3.10}$$

Using the same method as in the Fibonacci tiling example, we can show that G_1 not only acts on the torus $E^{\perp}/G_0$ but also on $(E^{\perp} \cap NS)/G_0$. To prove this, we assume that $\left(x + \frac{m-n}{\sqrt{2}}, y - \frac{m+n}{\sqrt{2}}\right)$ is in S (i.e. it is expressed in one of the three forms given in 4.3.9). It is easy to show that we then have that $(x,y) \in S$ as well. □

4.4 A $\mathbb{Z}^d$–action on the Cantor space

In this section, we will look at a $\mathbb{Z}^d$-action on a Cantor space induced by the action given in equation 4.3.7. Before we start, let us recall some definitions from the last section.

Let $(E, E^{\perp}, \mathbb{Z}^N)$ be a cut and project scheme with acceptance domain M. We let $G \cong \mathbb{Z}^N$ with $G = \mathrm{Span}_{\mathbb{Z}}\{r_1, \ldots, r_N\}$. We consider the two subgroups

$$G_0 = \mathrm{Span}_{\mathbb{Z}}\{r_1, \ldots, r_n\} \text{ and } G_1 = \mathrm{Span}_{\mathbb{Z}}\{r_{n+1}, \ldots, r_N\}.$$

Moreover, we let $F = \mathrm{Span}_{\mathbb{R}}\{r_1, \ldots, r_n\}$. Hence F/G_0 is isomorphic to an n–dimensional torus. We say that $v \in \mathbb{R}^N$ is non-singular (NS) if $(v + \mathbb{Z}^N) \cap \partial\Sigma = \emptyset$, where Σ is the strip $\Sigma = M + E$. We will define a metric on $\mathbb{R}^N$ such that the completion of the quotient space $(F \cap NS)/G_0$ with respect to that metric will be a Cantor space on which G_1 acts.

Let us first define a metric on subsets of Σ. Let $D_r(0)$ be the closed Euclidean ball of radius r around 0 in E. Let $C_r = \pi^{-1}(D_r(0)) \cap \Sigma$ and $dC_r = \pi^{-1}(\partial D_r(0)) \cap \Sigma$. Given a subset A of Σ, define $A[r] = (A \cap C_r) \cup dC_r$. Let d_r' be the Hausdorff metric defined among closed subsets of C_r and define a metric on subsets of Σ by

$$D'(A, A') = \inf\left\{\frac{1}{r+1} \ ; \ d_r'(A[r], A'[r]) < 1/r\right\}.$$

Notice that since the Hausdorff metric is only defined on closed subsets of C_r, the D' metric will only be defined on subsets $A \subset \Sigma$ such that $A[r]$ is closed. We can use the D' metric on subsets of Σ to define a metric $\overline{D'}$ on $\mathbb{R}^N$; let $v, w \in \mathbb{R}^N$ and define

$$\overline{D'}(v, w) = D'(\tilde{P}_v, \tilde{P}_w) + \|v - w\|,$$

where $\tilde{P}_v = (v + \mathbb{Z}^N) \cap \Sigma$. Since $\tilde{P}_v$ is closed for all $v \in \mathbb{R}^N$, there is no problem in using the D' metric.

Let $M\tilde{P}$ be the D'–closure of the space $\{\tilde{P}_v \ ; \ v \in NS\}$ and $\tilde{\Pi}$ be the $\overline{D'}$-completion of NS. We also define

$$M\tilde{P}_u := \overline{\{\tilde{P}_v; \ v \in u + E\}}^{D'} .$$

Let $F^o = F \cap NS$ and $\overline{F}$ be the $\overline{D'}$–closure of F^o in $\tilde{\Pi}$.

Lemma 4.4.1 (3.7 in [10])

1. The canonical injection $\mu : NS \longrightarrow \mathbb{R}^N$ extends to a surjection $\tilde{\mu} : \tilde{\Pi} \longrightarrow \mathbb{R}^N$, which is continuous.
2. The map $v \mapsto \tilde{P}_v$, $v \in NS$ extends to a continuous E-equivariant surjection $\tilde{\eta} : \tilde{\Pi} \longrightarrow M\tilde{P}$, which is an open map.
3. If $a \in M\tilde{P}$ and $b \in \mathbb{R}^N$, then $|\tilde{\eta}^{-1}(a) \cap \tilde{\mu}^{-1}(b)| \leq 1$.

Theorem 4.4.2 (Theorem 9.2 in [10]): With the data above, we have $\overline{F} = \tilde{\mu}^{-1}(F)$ and there is a natural equivalence $\tilde{\Pi} \equiv \overline{F} \times E$ and a surjection $\nu : \overline{F} \longrightarrow F$ which fits into the following commutative square

$$\begin{tikzcd}
\tilde{\Pi} \arrow[r, leftrightarrow] \arrow[d, "\tilde{\mu}"] & \overline{F} \times E \arrow[d, "\nu \times \mathrm{id}"] \\
\mathbb{R}^N \arrow[r, leftrightarrow] & F \times E.
\end{tikzcd}$$

Moreover, the set $\nu^{-1}(v)$ is a singleton whenever $v \in NS \cap F$.

Before we give the next definition, recall that π' denotes the projection on F, parallel to E and that $M' = \pi'(M)$, where M is the acceptance domain.

Definition 4.4.3 Let $\mathcal{A}$ be the algebra of subsets (i.e. closed under finite union, finite intersection and symmetric difference) of $F^o = F \cap NS$ generated by the sets

$$(M' \cap NS) + \pi'(v)$$

as v runs over $\mathbb{Z}^N$. Let $\mathcal{B} = \{\overline{A}\ ;\ A \in \mathcal{A}\}$, where the bar refers to the $\overline{D'}$–closure in $\overline{F}$. Define $\overline{M}$ to be the $\overline{D'}$–closure of $M' \cap NS$.

The following two lemmas will be needed in Propositions 4.4.8 and 4.4.7, respectively.

Lemma 4.4.4 Let X be a topological space with topology τ. Let $\mathcal{B}$ be a basis for τ. If $\mathcal{B}$ is closed under symmetric differences and (X, τ) satisfies the T_1 separation axiom then (X, τ) is Hausdorff.

Proof: Let $x, y \in X$, $x \neq y$. Since (X, τ) satisfies the T_1 separation axioms we can find open sets $U, V \in \mathcal{B}$ such that $x \in U$, $y \in V$ and $x \notin V$, $y \notin U$. As $\mathcal{B}$ is closed under symmetric difference the sets $U' = U \cap (U \bigtriangleup V)$, $V' = V \cap (U \bigtriangleup V) \in \mathcal{B}$. Furthermore, we have that $x \in U', y \in V'$ and $U' \cap V' = \emptyset$. Consequently, (X, τ) is Hausdorff. ∎

Lemma 4.4.5 Let X be a topological space and $A \subset X$. Then

$$\partial A = \mathrm{Cl}(A) \cap (\mathrm{Int}(A))^c = \emptyset$$

if and only if A is clopen in X.

Proof: If A is clopen, then $\mathrm{Cl}(A) = A = \mathrm{Int}(A)$, so that $\partial A = A \cap A^c = \emptyset$. Now assume that $\partial A = \emptyset$. We know that $\mathrm{Int}(A) \subset \mathrm{Cl}(A)$. But if $\mathrm{Cl}(A) \cap (\mathrm{Int}(A))^c = \emptyset$ then we have $\mathrm{Int}(A) = \mathrm{Cl}(A)$ which implies that A is both open and closed. ∎

Lemma 4.4.6 (Lemma 9.6 in [10]) The set $\overline{M} = \overline{M' \cap NS}$ is a compact clopen subset of $\overline{F}$, where the bar refers to the $\overline{D}'$–closure.

Proof: By definition, $\overline{M}$ is closed. To prove compactness, first recall that

$$\tilde{P}_v := (v + \mathbb{Z}^N) \cap \Sigma,$$

$P_v = \pi(\tilde{P}_v)$ and

$$M\tilde{P}_u := \overline{\{\tilde{P}_v;\ v \in u + E\}}^{D'} .$$

By Lemma 3.2 in [10], $M\tilde{P}_u$ is compact in $\overline{F}$. Define $f : M' \cap NS \to M\tilde{P}_u \times M'$ by $f(v) = (\tilde{P}_v, v)$. Recall the definition of the $\overline{D}'$ metric: $\overline{D}'(v,w) = D'(\tilde{P}_v, \tilde{P}_w) + \|v-w\|$. Since $M' \cap NS$ is given the $\overline{D}'$–topology, $M\tilde{P}_u$ is given the D' topology and M' is given the Euclidean topology, the function f is an isometry. Therefore $M' \cap NS$ is embedded $\overline{D}'$-isometrically in $M\tilde{P}_u \times M'$. As the latter is compact, $\overline{M}$ is compact.

To show that $\overline{M}$ is open, let $v \in \overline{M}$. We need to find an $\varepsilon > 0$ such that

$$\overline{D}'(v,w) < \varepsilon \Rightarrow w \in \overline{M}.$$

Suppose such an ε does not exist. Then there exists two sequences $(v_n)_{n\geq 1}$, $(v'_n)_{n\geq 1}$ with $v_n \in M \cap NS$ and $v'_n \in (NS \cap F)\backslash M$ for all $n \geq 1$ such that $(v_n)_{n\geq 1}$ and $(v'_n)_{n\geq 1}$ are $\overline{D}'$-convergent and have the same limit. Let

$$\lim_{n\to\infty} v_n = \lim_{n\to\infty} v'_n = x \in \tilde{\Pi}_u.$$

Then

$$\|v_n - x\| \leq \overline{D}'(v_n, x) \text{ and } \|v'_n - x\| \leq \overline{D}'(v'_n, x),$$

which implies that $\tilde{\mu}(x) \in \partial M$. By construction, $\tilde{P}_{v_n}$ and $\tilde{P}_{v'_n}$ have different D'-limit. This is a contradiction since both limits must be equal to $\tilde{\eta}(x)$. ∎

Proposition 4.4.7 There is an isomorphism of Boolean algebras between $\mathcal{A}$ and $\mathcal{B}$.

Proof: Consider the map $\phi : \mathcal{A} \to \mathcal{B}$, $A \mapsto \overline{A}$, where the bar refers to the $\overline{D}'$ closure in $\overline{F}$. We will show that ϕ is an isomorphism of Boolean algebras, i.e. ϕ is bijective and preserves the operations (finite union, finite intersection and symmetric difference) on $\mathcal{A}$. By definition of $\mathcal{B}$, ϕ is surjective. We always have $\overline{A \cup B} = \overline{A} \cup \overline{B}$, for any metric. In particular, ϕ preserves the union. In general, the same for the intersection does not hold, but we will show it does for the $\overline{D}'$-metric. First, let us show that $\phi(A^c) = \phi(A)^c$, for any $A \in \mathcal{A}$, i.e. $\overline{A^c} = (\overline{A})^c$. For $A \in \mathcal{A}$, we have

$$A \subset \overline{A} \Rightarrow (\overline{A})^c \subset A^c \subset \overline{A^c}.$$

By Lemma 4.4.6, we know that $\overline{A}$ is clopen in $\overline{F}$. Hence, by Lemma 4.4.5, this implies that $\partial A = \mathrm{Cl}(A) \cap (\mathrm{Int}(A))^c = \overline{A} \cap \overline{A^c} = \emptyset$. Therefore, $\overline{A^c} \subset (\overline{A})^c$. This shows that $\overline{A^c} = (\overline{A}^c)$ and therefore, $\phi(A^c) = \phi(A)^c$. From this, we now have that

$$\begin{aligned} \phi(A \cap B) &= \phi((A^c \cup B^c)^c) = \phi(A^c \cup B^c)^c \\ &= (\phi(A^c) \cup \phi(B^c))^c = (\phi(A)^c \cup \phi(B)^c)^c \\ &= \phi(A) \cap \phi(B), \end{aligned}$$

for all $A, B \in \mathcal{A}$. Finally, ϕ is injective. To show this, let $\overline{A} = \overline{B}$. Then we have $A \subset \overline{A} = \overline{B}$, hence $A^c \supset (\overline{B})^c = \overline{B^c} \supset B^c$. Similarly, $B^c \supset A^c$, and so $A = B$. ∎

Proposition 4.4.8 The collection $\mathcal{B}$ is a base of clopen neighborhoods which generates the topology of $\overline{F}$.

Proof: The sets in $\mathcal{B}$ are clopen by Lemma 4.4.6. Therefore, the metric topology on $\overline{F}$ we are considering, let us call it τ, is finer than the topology τ' generated by $\mathcal{B}$ (i.e. $\tau' \subseteq \tau$). Both topologies are invariant under the action of $\mathbb{Z}^N$ so that it suffices to show their equivalence on some closed r-ball X of $\overline{F}$. Let us first show that (X, τ') satisfies the T_1 separation axiom. Then we will combine this with the fact that $\mathcal{B}$ is closed under symmetric differences (because $\mathcal{A}$ and $\mathcal{B}$ are isomorphic as Boolean algebras and $\mathcal{A}$ is closed under symmetric differences) so that (X, τ') is Hausdorff, by Lemma 4.4.4.

If $a, b \in F$, $a \neq b$ then the facts that $\mathrm{Int}(M) \neq \emptyset$, M' is bounded and $\pi'(\mathbb{Z}^N)$ is dense in F, imply that there is some $v \in \mathbb{Z}^N$ such that $a \in \mathrm{Int}(M') + \pi'(v)$ and $b \notin \overline{M'} + \pi'(v)$, where we take the Euclidean closure in F. Hence a and b are separated by the topology induced by $\tilde{\mu}(\mathcal{B})$. In particular, if $x, y \in \overline{F}$ and $y \in \cap\{B \in \mathcal{B}\ ;\ x \in B\}$, then $\tilde{\mu}(x) = \tilde{\mu}(y)$. (Here we assume that y is in every neighborhood of x, i.e. x and y cannot be separated by $\tilde{\mu}(\mathcal{B})$.) But, if $x \neq y$ and $\tilde{\mu}(x) = \tilde{\mu}(y) = v$, then, by part 3 of Lemma 4.4.1, $\tilde{\eta}(x) \neq \tilde{\eta}(y)$ (otherwise $x, y \in \tilde{\mu}^{-1}(x) \cap \tilde{\eta}^{-1}(x)$ would imply $x = y$, which is a contradiction).

Since $\tilde{\mu}(x) = \tilde{\mu}(y)$, then by Theorem 4.4.2, we have that $(\nu \times \mathrm{id})(x) = (\nu \times \mathrm{id})(y)$. Since $\nu^{-1}(u)$ is a singleton for $u \in NS \cap F$ and $x \neq y$, we have that $x, y \notin NS \cap F$, which implies that $v \notin NS \cap F$. By definition, v being a singular point implies that $(\mathbb{Z}^N + v) \cap \partial\Sigma \neq \emptyset$. Let $p \in (\mathbb{Z}^N + v) \cap \partial\Sigma$. As $p \in (\mathbb{Z}^N + v)$, there exists $b \in \mathbb{Z}^N$ such that $p = b + v$. Also, since $\pi'(\mathbb{Z}^N)$ is dense in $F = E^{\perp}$, we can find sequences $v_n, v_n' \in \pi'(\mathbb{Z}^N)$ both converging to v in the Euclidean topology, such that

$$p + (v_n - v) = b + v_n \in \Sigma \quad \text{and} \quad p + (v_n' - v) = b + v_n' \notin \Sigma,$$

for all n large enough. But then, for such a choice of n, $y \notin \overline{A}$ (closure of $A = (NS \cap M') + \pi'(p + v_n)$ in $\overline{D}'$ metric) and $x \in \overline{A}$, a contradiction to the construction of y. This implies that (X, τ') satisfies the T_1 separation axiom.

As mentioned earlier, this in turns implies that (X, τ') is Hausdorff. Consider the identity map $Id : (X, \tau) \to (X, \tau')$ which is continuous. As (X, τ) is compact we get that Id is a homeomorphism (since any bijective continuous function from a compact space to a Hausdorff space is a homeomorphism). This shows the equivalence of the two topologies τ and τ'. ∎

Theorem 4.4.9 $\overline{F}$ is locally a Cantor set.

Proof: We need to show that $\overline{F}$ is locally compact, has a countable basis of clopen sets and has no isolated points. First of all, let $x \in \overline{F}$ so that x can be seen as the $\overline{D}'$–limit of a sequence $(x_n)_{n\geq 1}$, where $x_n \in F \cap NS$ for all $n \geq 1$. We will find a compact neighborhood of x. Since $\mathrm{Int}(M')$ is nonempty and $\pi'(\mathbb{Z}^N)$ is dense in F, there exists $v \in \mathbb{Z}^N$ such that $x \in M' + \pi'(v)$. For N large enough, we have that $x_n \in (M' \cap NS) + \pi'(v)$ for all $n \geq N$. As x is the limit of the later sequence, we have $x \in \overline{M' \cap NS} + \pi'(v) = \overline{M} + \pi'(v)$. From Lemma 4.4.6 , we know that $\overline{M}$ is compact, and hence $\overline{M} + \pi'(v)$, which is a set in $\mathcal{B}$, is a compact neighborhood of x. By Lemma 4.4.6, we also know that the sets in $\mathcal{B}$, which generate the $\tau'-$ topology on $\overline{F}$, form a countable basis of clopen sets in $\overline{F}$. Finally, we need to show that $\overline{F}$ has no isolated points. We can show (see Theorem 9.4 in [10]) that every clopen subset of $\overline{F}$ has $\tilde{\mu}$ image with Euclidean interior in F and so cannot be a single point. ∎

Definition 4.4.10 We will denote $X = \overline{F}/G_0$, which is a space on which G_1 acts continuously.

Recall the action of G_1 on F/G_0, given in equation 4.3.7:

$$([x], g_1) \in F/G_0 \times G_1 \mapsto \chi(([x], g_1)) = [x + \pi'(g_1)], \text{ for } g_1 \in G_1.$$

Theorem 4.4.11 (Theorem 10.3 in [10]) Let $(E, E^{\perp}, \mathbb{Z}^N)$ be a cut and project scheme and let G be a group isomorphic to $\mathbb{Z}^N$. Then X is a Cantor set on which G_1 acts minimally and there is a commutative diagram of G_1 equivariant maps

$$\begin{tikzcd}
\overline{F} \arrow[r, "q"] \arrow[d, "\nu"'] & X \arrow[d, "\nu'"'] \\
F \arrow[r, "q"] & F/G_0.
\end{tikzcd}$$

The space F/G_0 is homeomorphic to an n–dimensional torus. The action of G_1 on this space is by rotation and is minimal.

Proof: We first show that X is compact. Let $q : \overline{F} \longrightarrow X$ be the quotient map. Let

$$Y_0 = \left\{ \sum_{1 \le j \le n} \lambda_j r_j;\ 0 \le \lambda_j < 1 \right\}$$

be a subset of F. Choose $J \subset \mathbb{Z}^N$ finite but large enough that

$$Y_1 := \bigcup_{v \in J} (M' + \pi'(v))$$

contains $\overline{Y}_0$, the Euclidean closure of Y_0. In particular,

$$q\left(\bigcup_{v \in J} (\overline{M} + \pi'(v)) \right) = X,$$

the image of a compact set under a continuous map. So X is also compact. Since G_0 acts isometrically on $\overline{F}$ with uniformly discrete orbits, q is open and locally a homeomorphism and so X inherits the metrizability of $\overline{F}$, a base of clopen sets and the lack of isolated points. Therefore, $X = \overline{F}/G_0$ is a Cantor set. ∎

Remark 4.4.12 Suppose we define $\Delta := E^{\perp} \cap \mathbb{Z}^N$ and $\tilde{\Delta} := U \cap \overline{\pi^{\perp}(\mathbb{Z}^N)}$, where U is the real vector space generated by Δ. In the octogonal tiling, we have $\Delta = 0$ and hence $\tilde{\Delta} = 0$. But this is not the case in general. For example, the Penrose tiling has $\Delta = (e_1 + e_2 + e_3 + e_4 + e_5)\mathbb{Z}$ which is a subgroup of index 5 in $\tilde{\Delta}$ ([10], Example 2.12). In the general case, the previous theorem states that the space F/G_0 is homeomorphic to a finite union of tori each of dimension $N - d - \dim \Delta$. We see this space as a topological group, in which case it is the product of a subgroup of $\tilde{\Delta}/\Delta$ with the $(N - d - \dim \Delta)$–torus.

Theorem 4.4.13 We have a $*$-isomorphism of C^*–algebras $C_0(\overline{F}) \cong C^*(\mathcal{A})$.

Proof: By Proposition 4.4.7 there is an isomorphism of Boolean algebras between $\mathcal{A}$ and $\mathcal{B}$. It follows that $C^*(\mathcal{A}) \cong C^*(\mathcal{B})$, hence we only need to show $C_0(\overline{F}) \cong C^*(\mathcal{B})$. First, if $A \in \mathcal{A}$, we have that the characteristic function $\chi_{\overline{A}} : \overline{F} \longrightarrow \{0, 1\}$ is continuous because $\overline{A}$ is clopen in $\overline{F}$, by Lemma 4.4.6. By the same Lemma, we have that $\overline{A}$ is compact. Consequently, $\chi_{\overline{A}} \in C_0(\overline{F})$ since for $\varepsilon > 0$, we have that

$$|\chi_{\overline{A}}(x)| = 0 < \varepsilon, \ \forall \ x \in \overline{F} \backslash \overline{A}.$$

Let $\mathfrak{B}$ denote the $*$-algebra generated by the characteristic functions χ_B, for $B \in \mathcal{B}$. We have that $\mathfrak{B} \subset C_0(\overline{F})$. In what follows, we will show that $\mathfrak{B}$ is dense in $C_0(\overline{F})$, proving the result. Let $f \in C_0(\overline{F})$ and $\varepsilon > 0$. There exists a compact set $K \subset \overline{F}$ such that

$$|f(x)| < \varepsilon, \ \forall \ x \in \overline{F} \backslash K.$$

For all $x \in K$, define $V_x = \{y \in \overline{F};\ |f(x) - f(y)| < \varepsilon\}$. Since V_x is open in $\overline{F}$ and $\mathcal{B}$ is a basis for the topology in $\overline{F}$, we have that for all $x \in K$, there exists $B_x \in \mathcal{B}$ such that $x \in B_x \subset V_x$. Hence $\{B_x\}_{x \in K}$ is a clopen covering of K. As K is compact, there exists $x_1, \ldots, x_m$ such that

$$K \subset \bigcup_{i=1}^{m} B_{x_i}.$$

By setting $\tilde{B}_{x_1} = B_{x_1}$, $\tilde{B}_{x_2} = B_{x_2} \backslash \tilde{B}_{x_1}$, $\tilde{B}_{x_3} = B_{x_3} \backslash (\tilde{B}_{x_1} \amalg \tilde{B}_{x_2})$, etc., we get a clopen partition $\{\tilde{B}_{x_i}\}_{i=1,\ldots,m}$ of $\bigcup_{i=1}^{m} B_{x_i}$. Finally, we set

$$\tilde{f} = \sum_{i=1}^{m} f(x_i) \chi_{\tilde{B}_{x_i}} \in \mathfrak{B}.$$

We then have that $\|\tilde{f} - f\| < \varepsilon$. This shows that $\mathfrak{B}$ is dense in $C_0(\overline{F})$ and therefore $\overline{\mathfrak{B}} = C^*(\mathcal{B}) \cong C_0(\overline{F})$. ∎

4.5 The M-topology and the $\mathcal{F}-$topology

In this section we will define a new topology on $\mathbb{R}^{d+n} = \mathbb{R}^N$. We will define it in two different ways and show that they are equivalent. The two topologies are called the M-topology and the $\mathcal{F}$-topology and are defined in [4]. The M-topology is related to the cut and projection method. In fact, M stands for the acceptance domain defined in section 4.3. The space $\mathbb{R}^{d+n}$ endowed with he M-topology and the $\mathcal{F}$-topology will be denoted by $\mathbb{R}^{d+n}_M$ and $\mathbb{R}^{d+n}_{\mathcal{F}}$ respectively. We will also define two pseudo tori, $\mathbb{T}^{d+n}_M$ and $\mathbb{T}^{d+n}_{\mathcal{F}}$, on which $\mathbb{R}^{d+n}_M$ and $\mathbb{R}^{d+n}_{\mathcal{F}}$, respectively, act by translation. When M is $\tilde{L}$-compatible, the two topologies are equivalent

4.5.1 The M-topology

We first start with some definitions and results.

Definition 4.5.1 Let $(\mathbb{R}^d, \mathbb{R}^n, \tilde{L})$ be a cut and projection scheme and let M be a bounded subset of $\mathbb{R}^n$ with non-empty interior (M will be the acceptance domain). We say that M is **admissible** if for every ball $B_\varepsilon(x)$ with $\varepsilon > 0$ and $x \in \pi_2(\tilde{L}) \cap M$, there exists a finite family $\{a_1, ..., a_m\} \in \pi_2(\tilde{L})$ such that $M \cap (M + a_1) \cap ... \cap (M + a_m)$ is a subset of $B_\varepsilon(x)$ with non-empty interior.

For instance, any convex polytope (generalization of a convex polygon in higher dimension) is an admissible set (see the following example). Recall that for the octogonal tiling, M is an octogon.

Example 4.5.2 Let us show that any convex polygon is admissible. Let M denote the polygon and M_i its edges for $i = 1, \ldots, n$. Let $x \in M$ and $\varepsilon > 0$ such that

$$\varepsilon < \min_{i=1,\ldots,n} \{\text{length of } M_i\}.$$

For $i = 1, \ldots, n$, let H_i denote the hyperplanes in $\mathbb{R}^N$ passing through and parallel to the edge M_i. Assume that $\langle x, y\rangle \geq 0$ for all $y \in H_i$ and $i = 1, \ldots, n$. Let $H_i^+ := \{y \in \mathbb{R}^2;\ \langle y, z\rangle \geq 0\ \forall\ z \in H_i\}$. We then have $M = \bigcap_{i=1}^n H_i^+$. Let h_i be the projection of x on H_i and let k_i be the point on the line segment joining x and h_i such that $d(x, k_i) = \varepsilon/2$. Choose $a_i \in \pi_2(\tilde{L}) \cap B_{\frac{\varepsilon}{4}}(k_i)$. Let $\tilde{H}_i$ be the translation of H_i by a_i. Then $\displaystyle\bigcap_{i=1}^{n} \tilde{H}_i^+ \subset B_\varepsilon(x)$. ∎

For two functions $f : \mathbb{R}^d \longrightarrow \mathbb{C}$ and $g : \mathbb{R}^n \longrightarrow \mathbb{C}$, we define the function $f \otimes g$ from $\mathbb{R}^N = \mathbb{R}^{d+n}$ by $(f \otimes g)(x, y) = f(x)g(y)$, for $x \in \mathbb{R}^d$, $y \in \mathbb{R}^n$.

We denote by $\mathcal{A}_M$ the C^*-algebra generated by the set of functions

$$\{f \otimes (\mathcal{X}_M \circ T_n^{\pi_2(a)}); a \in \tilde{L}, f \in C_c(\mathbb{R}^d)\},$$

where

- $\mathcal{X}_M$ denotes the characteristic function of M;
- $T_n^{\pi_2(a)}$ denotes the translations in $\mathbb{R}^n$ by $\pi_2(a)$;
- $C_c(\mathbb{R}^d)$ denotes the set of continuous functions $f : \mathbb{R}^d \longrightarrow \mathbb{C}$ with compact support (the support of f is the set $Y \subseteq \mathbb{R}^d$ for which $f(x) = 0\ \forall\ x \in X \backslash Y$).

We will denote by $\mathbb{R}_M^{d+n}$ the locally compact space of all characters of $\mathcal{A}_M$. By Gelfand's theorem (see Section 2.3), we have that $\mathcal{A}_M$ is isomorphic to $C_0(\mathbb{R}_M^{d+n})$.

Lemma 4.5.3 When M is admissible, $C_0(\mathbb{R}^{d+n})$ is a closed subalgebra of $\mathcal{A}_M$.

Proof: Let $\mathcal{B}_M$ be the C^*-algebra generated by the set of functions

$$\{\chi_M \circ T_n^{\pi_2(a)};\ a \in \tilde{L}\}.$$

It is enough to show that $C_0(\mathbb{R}^n) \subset \mathcal{B}_M$. Let $f \in C_0(\mathbb{R}^n)$ and $\varepsilon > 0$. There is a compact $K \subset \mathbb{R}^n$ such that $|f(x)| < \varepsilon$ for $x \in \mathbb{R}^n \backslash K$. For $x \in K$, define

$$V_x := \{y \in \mathbb{R}^n;\ |f(x) - f(y)| < \varepsilon\},$$

which is open in $\mathbb{R}^n$. Assume, without lost of generality, that $x \in M$. Since M is admissible, there exists $a_1^x, a_2^x, \ldots, a_{m_x}^x \in \pi_2(\tilde{L})$ such that

$$x \in \operatorname{Int}(M \cap (M + a_1^x) \cap (M + a_2^x) \cap \ldots \cap (M + a_{m_x}^x)) \subset V_x.$$

Hence

$$K \subset \bigcup_{x \in K} \operatorname{Int}(M \cap (M + a_1^x) \cap (M + a_2^x) \cap \ldots \cap (M + a_{m_x}^x)).$$

As K is compact, we can reduce this to a finite union. Let

$$g := \sum_{\text{finite}} f(x) \chi_{(M + a_1^x)} \chi_{(M + a_2^x)} \cdots \chi_{(M + a_{m_x}^x)}.$$

Then g is in the *-algebra generated by

$$\{\chi_M \circ T_n^{\pi_2(a)};\ a \in \tilde{L}\}.$$

Moreover, $|f(x) - g(x)| < \varepsilon$ for all $x \in \mathbb{R}^n$ so that f can be approximated, as close as we want by functions like g. Therefore, $f \in \mathcal{B}_M$. ∎

Recall the following

Theorem 4.5.4 (Riesz) If X is a locally compact Hausdorff space, then every bounded linear functional Φ on $C_0(X)$ is represented by a unique regular complex Borel measure μ, where $\Phi f = \int_X f \, d\mu$.

Let X be a locally compact set. By Riesz'theorem, we have that

$$C_0(X)_1^* = M_1(X) \supseteq P(X),$$

where $M_1(X)$ is the set of regular probability measures on X and $P(X)$ is the set of Dirac measures on X.

If we know all the Dirac measures on X then we know X. Hence we can recover X from $C_0(X)^*_1$. Now since $C_0(X)^*_1 = M_1(X)$ is the unit ball in $C_0(X)^*$, we have, by the Alaoglu Theorem, that $C_0(X)^*_1$ is a convex $\sigma(C_0(X)^*, C_0(X))$-compact set. Moreover, the Dirac measures are extreme points of $M_1(X)$ (by Theorem 8.4 in [5]).

Lemma 4.5.5 Let X and Y be two topological spaces such that $C_0(Y) \subset C_0(X)$. Then there exist a surjective application from X onto Y.

Proof: Denote the inclusion by $\varphi : C_0(Y) \to C_0(X)$. Then φ is an isometric injection. Define $\varphi^* : C_0(X)^* \to C_0(Y)^*$ by $\varphi^*(\psi) = \psi \circ \varphi$, for $\psi \in C(X)^*$, where

$$\begin{aligned} \psi \circ \varphi : C(Y) &\to \mathbb{C} \\ f &\mapsto \psi(\varphi(f)). \end{aligned}$$

Let us show that φ^* is surjective. Let $\eta \in C_0(Y)^*$. We would like to show that $\eta = \psi \circ \varphi$ for some $\psi \in C_0(X)^*$. Define $\xi : \varphi(C_0(Y)) \to \mathbb{C}$ by $\xi(\varphi(f)) = \eta(f)$, for $f \in C_0(Y)$. By definition, ξ is a linear form and $\xi \circ \varphi = \eta$. Since $\varphi(C_0(Y))$ is a subspace of $C_0(X)$, we can extend ξ to a linear form $\psi : C_0(X) \to \mathbb{C}$, by the Hahn-Banach theorem (see Theorem 3.1.2 p.47 of [8]).

Then, $\psi \in C_0(X)^*$ and $\psi|_{\varphi(C(Y))} = \xi$. In particular,

$$\psi \circ \varphi = \xi \circ \varphi = \eta,$$

and this shows that $\varphi^* : C_0(X)^* \to C_0(Y)^*$ is surjective. Moreover, we have that $\varphi^*(\text{ext}(C_0(X)^*_1)) = \text{ext}(C_0(Y)^*_1)$, where ext denotes the extreme points. This last statements holds since φ^* is a linear application hence it preserve the convex property and extreme points. By restricting φ^* to $P(X)$ we then get a surjection

$$\varphi^* : P(X) \to P(Y),$$

as the extreme points here are the Dirac measures, and so we get a surjection $\varphi^* : X \to Y$. ∎

Hence, from the inclusion $C_0(\mathbb{R}^{d+n}) \subset \mathcal{A}_M \cong C_0(\mathbb{R}^{d+n}_M)$, proven in Lemma 4.5.3, we get a continuous surjection $\mathbb{R}^{d+n}_M \twoheadrightarrow \mathbb{R}^{d+n}$. Therefore, $\mathbb{R}^{d+n}_M$ can be seen as the completion of $\mathbb{R}^{d+n}$ for a finer topology than the usual one, that will be called the **M-topology**.

Lemma 4.5.6 In the M-topology, the sets $\mathbb{R}^d \times (M + a)$, for $a \in \pi_2(\tilde{L})$, are clopen.

Proof: We verify this for $a = 0$, i.e. we verify that $\mathbb{R}^d \times M$ is clopen in $\mathbb{R}^{d+n}_M$. First, $\mathbb{R}^d \times M$ is closed in $\mathbb{R}^{d+n}$ since M is closed in $\mathbb{R}^n$. So $\mathbb{R}^d \times M$ is closed in $\overline{\mathbb{R}^{d+n}}^M = \mathbb{R}^{d+n}_M$ since the M-topology is finer than the usual one.

Now let us show that $\mathbb{R}^d \times M$ is open in $\mathbb{R}^{d+n}_M$. Let $(x, y) \in \mathbb{R}^d \times M$. We would like to find an open set $U \subset \mathbb{R}^{d+n}_M$ such that $(x, y) \in U \subseteq \mathbb{R}^d \times M$. Let $f \in C_c(\mathbb{R}^d)$ such that $f(x) = 1$ and let $0 < \varepsilon < 1$. Then $x \in f^{-1}((1 - \varepsilon, 1 + \varepsilon))$. Consider $g := f \otimes \chi_M \in \mathcal{A}_M$. Since the M-topology makes all the functions in $\mathcal{A}_M$ continuous, g is M-continuous. Moreover, $g((x, y)) = 1$, so that

$$(x, y) \in g^{-1}((1 - \varepsilon, 1 + \varepsilon)) = f^{-1}((1 - \varepsilon, 1 + \varepsilon)) \times M.$$

Since g is M-continuous and $(1 - \varepsilon, 1 + \varepsilon)$ is open in $\mathbb{R}$, we have that

$$f^{-1}((1 - \varepsilon, 1 + \varepsilon)) \times M$$

is open in the M-topology. Hence the set $U := f^{-1}((1-\varepsilon, 1+\varepsilon)) \times M$ does the trick. In conclusion, the set $\mathbb{R}^d \times M$ is clopen in the M-topology. ∎

Because $\mathbb{R}^{d+n}$ is dense in $\mathbb{R}^{d+n}_M$, the continuous function $x \in \mathbb{R}^{d+n} \mapsto x + a \in \mathbb{R}^{d+n}$, for $a \in \tilde{L}$, extends to $\mathbb{R}^{d+n}_M$. Set $\mathbb{T}^{d+n}_M = \mathbb{R}^{d+n}_M / \tilde{L}$. This is called a pseudo-torus. It is a torus in the space $\mathbb{R}^d$ and it is a Cantor set in the space $\mathbb{R}^n$. We can view $\mathbb{T}^{d+n}_M$ as $\mathbb{T} \times X$ where X is a Cantor set in $\mathbb{R}^n$ and $\mathbb{T}$ is a d-dimensional torus.

4.5.2 The $\mathcal{F}$-topology

We assume the acceptance domain, M, is an **$\tilde{L}$-compatible** polytope, i.e. its vertices belong to $\pi_2(\tilde{L})$. Let $F_1, F_2, ..., F_p$ be the hyperplanes of $\mathbb{R}^{d+n}$ parallel to the maximal faces of $\mathbb{R}^d \times M$. For the octogonal tiling, the acceptance domain is an octogon so $p = 8$, where the maximal faces are the edges of the octogon. For each $j \in \{1, ..., p\}$, let $u_j \in \mathbb{R}^{d+n}$ be a unit vector perpendicular to F_j. Define

$$\begin{aligned} F_j^+ &= \{x \in \mathbb{R}^{d+n}; \langle u_j, x\rangle \geq 0\}, \\ F_j^- &= \mathbb{R}^{d+n} \backslash F_j^+ \end{aligned}$$

Let $\mathcal{F}$ be the family of affine hyperplanes $F_j + a$ with $j \in \{1, ..., p\}$ and $a \in \pi_2(\tilde{L})$. Let $\mathbb{R}^d \times \mathbb{R}^n$ be endowed with the coarsest (smallest) topology for which given any $F \in \mathcal{F}$, the closed half-space F^+ is clopen. This is what we call the $\mathcal{F}-$topology. In the same way as before, we can define the pseudo-torus $\mathbb{T}^{d+n}_{\mathcal{F}} = \mathbb{R}^{d+n}_{\mathcal{F}} / \tilde{L}$.

Theorem 4.5.7 If M is $\tilde{L}-$compatible, the $M-$topology and the $\mathcal{F}-$topology are equivalent in $\mathbb{R}^{d+n}$.

Proof: Let us first show that if $x \in \mathbb{R}^{d+n}$ belongs to an $\mathcal{F}-$open set V, we can find an M-open set U such that $x \in U \subseteq V$. It is enough to show this for $V = F^+ = F_j^+$ for some $j \in \{1, ..., p\}$. We will assume that each u_j points away from M for $i \in \{1, ..., p\}$. Since $x \in F_j^+$ then $\langle u_j, x\rangle \geq 0$, i.e. x is on the positive side of F_j. Since M has a nonempty interior and $\pi_2(\tilde{L})$ is dense in $\mathbb{R}^n$, we can find $a \in \pi_2(\tilde{L})$ such that $x \in \pi_2(M + a)$. Moreover, we can assume that $\langle u_j, y\rangle \geq 0$ for all $y \in M + a$. Set $U = \mathbb{R}^d \times (M + a)$. Then U is an $M-$open set such that $x \in U \subseteq V$.

Now we have to show that if $x \in \mathbb{R}^{d+n}$ belongs to an $M-$open set U then we can find an $\mathcal{F}$-open set V such that $x \in V \subseteq U$. It is enough to show this for $U = \mathbb{R}^d \times M$. Let $x \in U$ so that $\pi_2(x) \in M$. Again, if we assume the u_j are pointing away from M, then $x \in \bigcap_{j=1,...,p} F_j^-$. If we set $V = \bigcap_{j=1,...,p} F_j^-$ then V is open in the $\mathcal{F}$-topology and $V \subseteq U$. ∎

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