Type theoretic methods for higher structures

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Abstract

The present work is concerned with the study of higher structures through the lens of dependent type theory and categorical logic. We distinguish two main themes, but type theory can be seen as the common ground.

The first part presents some aspects of the theory of $(\infty, 1)$-categories in the framework of simplicial homotopy type theory of Riehl and Shulman. We present the theory of limits and colimits in this setting and prove some results, such as universal properties and preservation under adjunctions. We also include the computation of the limit of a family of spaces. In this setting, we also study and characterize exponentiable functors, where our main tools are Segal and Rezk completion of types, which we develop throughout. Importantly, we emphasize the consistency of our definitions by providing explicit comparison results with the bisimplicial sets model of simplicial homotopy type theory.

In the second part, given any weak model category, we associate to it the notion of an infinitary first-order logic. The idea, originally due to Simon Henry, is motivated by Makkai's FOLDS. To build the logic, we use the syntactic specification of dependent type theory of Cartmell's generalized algebraic theories. In the language we build, the cofibrant objects of the weak model category play the role of contexts from which the formulas are constructed. The fibrant objects are the models which validate the formulas. Thus, the formulas refer to properties of the fibrant objects of the weak model category. We prove what we call *invariance theorems* to show that the language we define is homotopically well-behaved: homotopic variables satisfy the same formulas, equivalent models satisfy the same formulas, equivalent contexts produce equivalent sets of formulas, and Quillen equivalent weak model categories have equivalent languages. The language avoids the so-called "evil properties", and can be seen as a homotopic version of the language for categories introduced by Blanc and Freyd.

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"...vámonos poco a poco, pues ya en los nidos de antaño no hay pájaros de hogaño."

M.C. Saavedra.

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Contents

Preface

This is a thesis by articles. The document contains the results obtained over the past 5 years, and that were prepared for publication. The content is faithful to those texts.

Chapter 2 consists of the paper [5], as submitted to Mathematical Structures in Computer Science. In the paper, we develop and present the theory of limits and colimits in synthetic ∞-categories within the type theory of Riehl and Shulman. In the same setting, we study exponentiable functors between synthetic ∞-categories—also known as Conduché fibrations. This is the content of chapter 3, which is the paper [6], as submitted to Mathematical Structures in Computer Science.

Finally, we establish a new connection between higher structures and categorical logic: it builds a language from any weak model category. The content encompasses the article [7], which is written jointly with Simon Henry. The paper provides a proof of the main conjecture on the language of a weak model category. Section 2 and appendix C were written by Henry. The rest of the article contains the contributions of the author of this thesis.

Chapter 1

Introduction

The main topic of this thesis is *∞-categories* and related *higher structures*. Firstly, consider the transition from categories to 2-categories; in a category, we have objects and arrows between objects, but in a 2-category we have 0-cells (objects), 1-cells between 0-cells, and 2-cells between 1-cells. We can define similar structures for any $n \in \mathbb{N}$, we get *strict n-categories* or *weak n-categories* depending on the axioms we might impose. In strict n-categories, the composition of k-cells is strictly associative and unital, and the interchange law also holds strictly. This just means that the axioms hold up to equality. In contrast, in a weak n-category these axioms hold up to a higher cell, which is subject to additional coherence axioms. Adding cells at all levels should result in what we would call an ∞-category. We often want to consider (∞, n)-categories, where the n in the notation informs us about the k-cells which ought to be invertible in our (∞, n)-category. For instance, when $n = 0$ we require that all k-cells for $k > 0$ are invertible, these $(\infty, 0)$-categories are what we should call the *∞-groupoids*. Understanding what an (∞, n)-category is the motive of active research.

It would be unsatisfactory to justify the existence of ∞-categories simply as higher versions of categories, even though such structures appear naturally. One can find more compelling reasons for their relevance in many key areas in mathematics, theoretical computer science and physics. On the mathematical side, for example, ∞-categories seem to be a highly convenient tool to do homotopy theory in many contexts [50].

The usual way to present ∞-categories is through *model categories*, a fundamental notion due to Quillen [54]:

Definition 1.0.1. A *model category* is a category $\mathcal{C}$ that has three distinguished classes of maps: *cofibrations* $\mathrm{Cof}(\mathcal{C})$, *fibrations* $\mathrm{Fib}(\mathcal{C})$ and *weak equivalences* $\mathcal{W}_{\mathcal{C}}$ satisfying the following axioms :

1. $\mathcal{C}$ is finitely complete and finitely cocomplete.
2. Given any diagram

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d, "i"'] & \bullet \arrow[d, "p"] \\
\bullet \arrow[r] \arrow[ur, dashed] & \bullet
\end{tikzcd}$$

where $i \in \textsc{Cof}(\mathcal{C}) \cap \mathcal{W}_{\mathcal{C}}$ and $p \in \textsc{Fib}(\mathcal{C})$, $i \in \textsc{Cof}(\mathcal{C})$ and $p \in \textsc{Fib}(\mathcal{C}) \cap \mathcal{W}_{\mathcal{C}}$, then the diagonal map exists and makes both triangles commutative.

3. All arrows $f \in \mathcal{C}$ admit factorizations $f = pi$ where $i \in \textsc{Cof}(\mathcal{C}) \cap \mathcal{W}_{\mathcal{C}}$ and $p\textsc{Fib}(\mathcal{C})$, or, $i \in \textsc{Cof}(\mathcal{C})$ and $p \in \textsc{Fib}(\mathcal{C}) \cap \mathcal{W}_{\mathcal{C}}$.
4. $\textsc{Cof}(\mathcal{C})$, $\textsc{Fib}(\mathcal{C})$ and $\mathcal{W}_{\mathcal{C}}$ are closed under retracts.
5. $\mathcal{W}_{\mathcal{C}}$ satisfies the 2-out-of-3 axiom: if any two maps of fg, f and g are in $\mathcal{W}_{\mathcal{C}}$, so is the third.

Remark 1.0.2. Quillen calls the previous definition "closed model category." In modern literature, this is what we call "model category" or "Quillen model category." Moreover, one can require the category to have all small limits and small colimits, not only finite ones. We will consider a more general notion called *weak model categories* discovered by Simon Henry [29].

An object in a model category is called *cofibrant* if the unique map from the initial object is a cofibration. An object is called *fibrant* if the unique map to the terminal object is a fibration. We often single out ∞-categories as fibrant objects of a model structure; we now look at some examples.

Let us denote by **sSet** the category of simplicial sets. A *Kan complex* is a simplicial set such that for all $n \in \mathbb{N}$ and $0 \leqslant i \leqslant n$, any lifting problem against the horn inclusion

$$\begin{tikzcd}
\Lambda^n_i \arrow[r] \arrow[d, hook] & X \\
\Delta^n \arrow[ur, dashed] &
\end{tikzcd}$$

has a diagonal filler as depicted above. The category **sSet** has a Quillen model structure, called the Kan-Quillen model structure. The fibrant objects of this model are exactly the Kan complexes. The Kan-Quillen model structure is a presentation of ∞-groupoids. One way to see this is through the *nerve* functor $N : \mathbf{Cat} \to \mathbf{sSet}$: for a category $\mathcal{C}$, the simplicial set $N(\mathcal{C})$ is a Kan complex if and only if $\mathcal{C}$ is a groupoid. Although we are not intrinsically interested in this fact, it is important to mention that there are other proposed models for ∞-groupoids. In the meantime, we move to discuss $(\infty, 1)$-categories.

One of the first attempts to define what an $(\infty, 1)$-category is implicit in the work of Boardman and Vogt [14, Definition 4.8]: a simplicial set satisfies the *restricted Kan condition* if for all $0 < i < n$, any lifting problem

$$\begin{tikzcd}
\Lambda^n_i \arrow[r] \arrow[d, hook] & X \\
\Delta^n \arrow[ur, dashed] &
\end{tikzcd}$$

has a solution.

Joyal [35], [36], [38] coined the term *quasi-category* for simplicial sets satisfying the restricted Kan condition. Subsequent work by Lurie [49], building on Joyal's work, extended the theory of quasi-categories greatly. This is probably one of the most developed models of $(\infty, 1)$-categories. Quasi-categories are the fibrant objects of the Joyal model structure. Again, a compelling reason why we think of quasi-categories is through the nerve functor, similar to the case for Kan complexes.

Another popular model for $(\infty, 1)$-categories is that of *complete Segal spaces* due to Rezk [59]. The theory of (complete) Segal spaces has seen some development in [57]. There are many other definitions of $(\infty, 1)$-categories, all of which have been shown to be *Quillen equivalent* (the right notion of equivalence between model categories.) In general, presentations of (∞, n)-categories are given as fibrant objects in some suitable Quillen model category. There is a general consensus of what a model for (∞, n)-categories should be, due to work by Barwick–Schommer-Pries [10].

Remark 1.0.3. There is another relation between model categories and quasi-categories: to any model category, $\mathcal{C}$ one can associate a quasi-category $\mathcal{C}^{\infty}$, for example, by applying the machinery of [21]. Moreover, a Quillen equivalence between model categories $\mathcal{A}$ and $\mathcal{B}$, gives rise to an equivalence between the associated quasi-categories $\mathcal{A}^{\infty}$ and $\mathcal{B}^{\infty}$ [31]. In practical terms, a model category can also be thought of as an ∞-category. But also it can be used to organize and study ∞-categories, since ∞-categories can be seen as fibrant objects of a model category. We do not use the former point of view, at most we are interested in the last one.

This is where we point out an issue: whenever a new model arises, we usually have to show that it is Quillen equivalent to another previously "validated" model. This begs the question: Can we identify properties of Quillen model categories that are preserved under Quillen equivalences? An answer to this question would tell us which parts of the theory of higher categories are also valid in a Quillen equivalent model. We venture to give an answer to this question. A further practical issue is that all the proposed definitions employ complex tools, which make it difficult to use $(\infty, 1)$-categories and even limit their applications. An ideal scenario would be to have a model-independent approach to deal with (∞, n)-categories.

One approach to model independence is the work of Riehl and Verity [63], which introduces the concept of ∞-*cosmos*. An ∞-cosmos is a 2-category $\mathcal{C}$ whose objects are called ∞-*categories*, and for every two objects $A, B \in \mathcal{C}$ the *morphisms* $f : A \to B$ are objects of a simplicial set $\mathsf{Fun}(A, B)$, which is furthermore a quasi-category. The category $\mathcal{C}$ contains a distinguished collection of morphisms called *isofibrations*, which satisfy *completeness* and *isofibration* axioms. The ∞-cosmoi formalism includes many models for $(\infty, 1)$-categories, for example, quasi-categories and complete Segal spaces, and even (∞, n)-categories. In ∞-cosmoi, we can develop basic $(\infty, 1)$-category theory: limits and colimits, the comma construction, and representables; one can develop the theory of fibrations of different flavours. By design, all the notions and the theory are internal to an ambient 2-category, hence it is dubbed the *formal theory of* ∞-*categories*. This is analogous to the work of Street and Walters [68] on Yoneda structures in a 2-category and the work of Wood [76, 77] on proarrow equipments.

In [60] Emily Riehl points out that "The fundamental obstacle to giving a uniform definition of a ∞-category is that our traditional set-based mathematics are not really suitable for reasoning about ∞-categories." The essence of the argument is that all constructions pertaining to ∞-categories often involve accounting for an infinite amount of coherence data, and such constructions will be well-defined only up to equivalence. The ∞-cosmoi technology overcomes the set-theoretic limitations, but still employs 2-categorical tools usually known only by experts. With the widespread use of ∞-categories, we might attempt to look for alternative approaches that are more accessible.

Riehl [60] argues in favour of a *synthetic theory of* ∞*-categories* that can be "integrated into the undergraduate curriculum." There is a framework proposed by Rielh and Shulman [61] that makes use of Martin-Löf dependent type theory. More precisely, to reason about ∞-categories, using type theory, they construct an extension of *homotopy type theory* (HoTT) [70]. Before continuing, we make a quick interlude to give an overview of homotopy type theory.

HoTT is a Martin-Löf dependent type theory enhanced with Voevodsky's *Univalence Axiom.* Firstly, the basic type theory consists of primitives

$$\Gamma\ \mathsf{Ctxt} \qquad \Gamma \vdash A\ \mathsf{Type} \qquad \Gamma \vdash a : A \qquad \Gamma \vdash A \equiv B \qquad \Gamma \vdash a \equiv a' : A,$$

with rules that ensure the consistency of the system. It is important to remark that $\equiv$ is a *judgmental equality*, or *definitional equality*, just as the other rules, it is in the metatheory. For consistency with the literature, we will often use the word *identification* instead of equality. This identification is given, as the name suggests, by definition. On the other hand, for any type A there is a *type constructor* id_A called *identity type.* We can think of the identity type as an internal identification. In HoTT, if two elements a, b are identical by definition, then they are also identical internally, in the sense that we can construct a proof $p : \mathsf{id}_A(a, b)$. The implication in the other direction is not postulated, which gives to HoTT its *intentional* character. Also, we can construct dependent types

$$x : A \vdash B(x)\ \mathsf{Type}.$$

Given a dependent type, there are *dependent products* and *dependent sums*

$$\prod_{x:A} B(x) \qquad \sum_{x:A} B(x).$$

For two types A, B, we can also build the *sum* $A + B$. The *empty* type and the *single point* type are also postulated. We can also construct types that contain types, called *universes*, and denoted $\mathcal{U}$, which can be assumed to be hierarchical. Each type constructor can come with introduction, elimination, computation, and uniqueness rules. We also stress another aspect of the theory; *proof relevance.* For example, the judgment $p : \mathsf{id}_A(a, b)$ should also be thought of as providing a proof that a is identical to b. In general, this is just to say that to provide a proof of a statement, one builds an element of a certain type. Two functions $f, g : A \to B$ between types are said to be *homotopic* if the type

$$f \sim g :\equiv \prod_{x:A} \mathsf{id}_A(fx, gx)$$

has a proof. A function $f : A \to B$ is an *equivalence* if the type

$$\left(\sum_{g:B\to A} f \circ g \sim Id_B\right) \times \left(\sum_{h:B\to A} h \circ f \sim Id_A\right)$$

is inhabited. One can then form the type of equivalences between two types A and B, which is denoted

$$A \simeq B$$

and consists of all the functions that are equivalences.

The *Univalence Axiom* relates the identity in the universe $\mathcal{U}$ and the equivalences. Specifically, it states that the induced map

$$\mathsf{id}_{\mathcal{U}}(A, B) \to (A \simeq B)$$

is an equivalence. In HoTT, the presence of univalence enforces that all constructions are invariant under equivalence.

Remark 1.0.4. The Univalence Axiom is a realization of the statement: *Equivalent objects/structures can be identified.* Objects that are identical enjoy the same properties. Therefore, equivalent objects ought to satisfy the same properties. Note that it is a common thing to do in mathematical practice. For example, if two groups are isomorphic, we can use them interchangeably. Notice that what we mean by equivalent structures varies depending on the structures, or objects, we are interested in. Ahrens–North–Shulman–Tsementzis [2] establish a *univalence principle* for many relevant structures, and give a precise formulation of the notion of *identification* between structures.

Simplicial homotopy type theory (sHoTT), introduced in [61], is an extension of homotopy type theory. In this extension it is possible to reason about $(\infty, 1)$-categories. With the rise of *univalent foundations* (UF) it is more natural to expect that type theory is a reasonable prerequisite for studying ∞-categories. Moreover, the fact this theory is grounded in type theory allows computer formalizations [43].

To sum up, we aim to use and develop formal methods to study ∞-categories and model categories; these methods are linked by dependent type theory. In Chapter 2 and Chapter 3, we contribute to the development of the theory of $(\infty, 1)$-categories within the framework of sHoTT. In Chapter 4, we associate to any (weak) model category a first-order language, which is built from a dependent type theory with no constructors. Using the notion of generalized algebraic theory [17], we obtain a first-order language that is compatible with the homotopy theory of the category. In addition, we also prove that this language is itself respected by *equivalences* (Quillen equivalences) between model categories. We see the results contained in chapter 4 as an effort pursuing model independence of ∞-categories.

1.1 Synthetic ∞-categories

Simplicial homotopy type theory (sHoTT), as developed by Riehl and Shulman in [61], proposes a framework to define synthetic $(\infty, 1)$-categories. This theory has been further studied by Bulchholtz and Weinberger in [16]. We must clarify that we can consider sHoTT, as formulated in [61], as "incomplete" as we are unable to perform important categorical constructions such as the opposite ∞-category, the core ∞-groupoid construction or the twisted arrow ∞-category. Implementing these constructions is a highly non-trivial task. Recent work of Bulchholtz–Gratzer–Weinberger [25] implements some of these constructions by enhancing sHoTT with modalities; the resulting type theory is called *triangular type theory* $\mathsf{TT}_{\boxslash}$. In an even more recent development, [26] gives the twisted arrow category construction, the Yoneda embedding. Future work should also provide us with universes classifying (co)cartesian fibrations and Conduché fibrations.

In [5], which is Chapter 2 in this document, we presented the theory of limits and colimits in this setting of higher categories. This work, and [6] (Chapter 3) on exponentiable functors, started as an experimentation of how much category theory of $(\infty, 1)$-categories can be done in the original formulation of sHoTT. In particular, we have not explored further implications of this work in $\mathsf{TT}_{\boxslash}$, but we expect no change in the validity of the results.

Simplicial HoTT is powerful enough to express the expected properties from a theory of limits, but it faces some difficulties for certain computations. In the bare formulation of sHoTT, it is not possible to construct the correct type of spaces (discrete types) within the theory. To be more precise: the type consisting of discrete types (∞-groupoids) is not a Rezk type ($(\infty, 1)$-category), and does not coincide with the $(\infty, 1)$-category of ∞-groupoids when interpreted in the bisimplicial sets model. However, we have shown:

Theorem 1.1.1. Under the mild assumption of the existence of the universe of spaces, it is possible to compute the limit of spaces as a dependent product.

The formal statement and technicalities can be found in [5]. This is an expected result that attests to the adequacy of the theory of limits. The construction of the universe of spaces in [25] implies that the computation above can simply take place in said universe.

While remaining in sHoTT, in [6] we have the following characterization of (synthetic) exponentiable functors.

Theorem 1.1.2. Let $f : E \to B$ a map between synthetic ∞-categories. The following are equivalent:

1. f is exponentiable,
2. f satisfies the Conduché condition.

This is a type-theoretic version of a result by Ayala-Francis-Rozenblyum [4] that characterizes the same class of functors between quasi-categories. However, the conclusions of [6] are partial due to the lack of some of the fundamental constructions in sHoTT mentioned above. This represents a major obstruction than some we encountered for theorem 1.1.1. However, we expect an enhancement of theorem 1.1.2 within $\mathsf{TT}_{\boxslash}$ or in further extensions of the type theory.

Remark 1.1.3. A formalization project by Kudazov–Riehl–Weinberger [43] for sHoTT came to light. In particular, it supports the univalent foundations [70], and evidently sHoTT. The overall goal is to, within the present framework, obtain computer-verified proofs of results about ∞-categories. Using the new proof assistant `Rzk`, the project aims to formalize the existing work on sHoTT, which includes [61], [16], [5] and [6]. Further development on proof assistants will be required in order to capture new developments like $\mathsf{TT}_{\boxslash}$, and further enhancements that include other desirable features such as higher inductive types.

1.2 The language of a model category

We recall that Blanc [13] and Freyd [23] characterized the properties (or formulas) of categories that are invariant under equivalence of categories. Such formulas are those that do not involve equality between objects. In this particular case, these formulas are written in the FOLDS (First Order Logic with Dependent Sorts) of categories [52]. This is the origin of some of the ideas by Simon Henry [28] to obtain similar results for model categories. Along the way, this also puts some results of [52] under the lens of homotopy theory. Succinctly, we can understand this work as identifying formulas in a model category that are preserved when compared to an equivalent model. It is important to mention that [28] sketches a proof for one of the main theorems (*Fourth invariance theorem*) that uses fundamental results yet to appear. However, our work [7] (Chapter 4) altogether is the realization of all these ideas, and more importantly, it includes full proofs via "elementary" results.

As we saw at the beginning, model categories are sometimes crucial in defining the different types of higher categories that we aim to study. Furthermore, whenever we introduce new models for ∞-categories, we aim to confirm their validity by verifying they are equivalent to a previously known and validated model. When it comes to $(\infty, 1)$-categories, the existing definitions have been demonstrated to be Quillen equivalent, for example the case of quasi-categories and complete Segal spaces was addressed in [40]. Many other comparisons are now available in the literature, which include models for (∞, n)-categories. The need for new models is primarily motivated by the desire to study aspects of higher categories that remain hidden in past models. Then we might need to transfer results from a model into another one. For instance: quasi-categories have a well-known notion of limit, and a corresponding definition exists for Segal spaces. Once we prove theorems about limits of quasi-categories, instead of crafting a new argument for Segal spaces to verify that the analogous (and expected) result also holds, we could hope for a theorem that makes this automatic. In general, given a Quillen equivalence between model categories, we anticipate that theorems should be transferable between them. This is basically what in theory we achieve.

For any model category, we associate the notion of a (logical) theory, and we define a language from it. More precisely, we start by extending the Generalized Algebraic Theories of Cartmell [17]. The main reason we require this generalization is so that we can capture the transfiniteness aspect of model categories. This gives us the appropriate notion of context over which the constructed formulas can take variables from. Now, if we start with a (cofibrantly generated) model category, cofibrant objects will turn out to be the contexts, while fibrant objects play the role of models. In this case, variables for a formula are simply maps from a cofibrant object to a fibrant object. Therefore, a formula written in this language refers to properties of fibrant objects. The main results we prove in [7] can be informally stated in the following:

Theorem 1.2.1.

- **First invariance:** Homotopic maps satisfy the same formulas.
- **Second invariance:** Homotopically equivalent fibrant objects validate the same formulas.
- **Third invariance:** Homotopically equivalent cofibrant objects (contexts) give equivalent sets of formulas.
- **Fourth invariance:** Two Quillen equivalent model categories have equivalent languages.

The first, second and third invariances can also be seen as verifying that the language is sensible to the homotopy theory. We can also read the second invariance theorem as establishing a *structure identity principle (SIP)* for fibrant objects of a model category in the sense that equivalent fibrant objects satisfy the same properties. As a sanity check, when we apply our construction to the canonical model structure on the category of small categories, the language we obtain coincides with that of Blanc [13] and Freyd [23].

The fourth invariance theorem also gives us the promised result; that between Quillen equivalent model categories we can transfer properties or theorems. We also point out that all the constructions and results work for a more general notion of homotopy theory, namely weak model categories [29].

For the proofs of the first three invariance theorems, the classical notion of model category is sufficient. However, for the proof of the last invariance theorem, it seems necessary to use weak model categories. This is because in the proof we need to perform a *right Bousfield localization* of a certain category of diagrams. A right Bousfield localization of a model category $\mathcal{M}$ with respect to a class of maps produces a new model category $R\mathcal{M}$ and a right Quillen functor, $\mathcal{M} \to R\mathcal{M}$ which satisfies a universal property. It is known that if $\mathcal{M}$ is a model category that is cellular and right proper, then the right Bousfield localization exists [32, Theorem 5.1.1]. One can weaken the cellularity condition but not right properness, and still we are left with set-theoretic conditions on $\mathcal{M}$. Clark Barwick [9] proves that one can drop the right properness condition, at the cost of considering *right semi-model categories*. Even in this situation, there are still conditions, such as the right semi-model category $\mathcal{M}$ being tractable, that guarantee the existence of a right semi-model category $R\mathcal{M}$ with the expected universal property; for the precise statement, we refer to [9, Theorem 5.22]. In contrast, the Fourth Invariance Theorem is valid with no special assumptions on the (weak) model category. Therefore, we are only able to prove that the resulting categories that we need in the proof have a weak model structure.

Chapter 2

Limits and colimits in synthetic ∞-categories

The content of this chapter consists of the paper [5] as submitted to Mathematical Structures in Computer Science. In the paper, we developed the theory of limits and colimits in synthetic ∞-categories within the type theory of Riehl and Shulman.

2.1 Introduction

Defining the notion of ∞-category is often a difficult task. There have been many different proposed definitions, to mention a few, by Boardman and Vogt [14] with the idea of quasi-categories, later exploited by Joyal [36] and Lurie [49], or by Rezk in [59] in the form of *complete Segal spaces*. In the case of $(\infty, 1)$-categories, most of the known definitions have been shown to be equivalent in an appropriate homotopy theoretic sense; see, for example, the work by Toën [69], Bergner [11] and Joyal-Tierney [40].

Recent work by Riehl and Shulman [61] develops a *synthetic theory of* $(\infty, 1)$*-categories*. They achieve this by implementing an extension of homotopy type theory. Moreover, Riehl explores in [60] the novelty and advantages that support the philosophy behind this approach. We highlight some of these points here. Firstly, all constructions are by design invariant under equivalence. Another aspect of this synthetic theory is that it is by construction compatible with the Univalence Axiom. Moreover, the standard model of the theory, the category of bisimplicial sets, has a well-known homotopy theory. Furthermore, it is possible to carry this synthetic theory internally to any Grothendieck ∞-topos (see Theorem 2.6.5). Finally, another pleasant consequence of this synthetic account is that it somewhat simplifies some definitions and constructions, streamlining the development of the theory of $(\infty, 1)$-categories. To begin with, the synthetic definition of an $(\infty, 1)$-category is fairly succinct in comparison with any other given before.

Previously, Voevodsky's simplicial model for HoTT [42] interpreted types as ∞-groupoids. However, we do not have a similar interpretation of types as ∞-categories, where by ∞-category we mean $(\infty, 1)$-category. To overcome this difficulty, [61] introduced a new type theory by adding to HoTT a strict interval object and the novel idea, due to Shulman and Lumsdaine (unpublished), of a new type former called *extension type*. We call this *simplicial homotopy type theory*, or sHoTT for short. This type theory can be seen as an instance of a cubical type theory; however, it is different from the one presented in [19].

In this setting, it is possible to define *Rezk types* as certain special types that play the role of ∞-categories. In this synthetic treatment they retrieve some expected results from higher categories, including a theory for adjunctions, the analogue of left (right) fibrations, which are called *(contravariant) covariant families*, and the Yoneda lemma for these fibrations. Additional work by Buchholtz and Weinberger [16] studies *synthetic cocartesian fibrations*, a generalization of covariant families. Further treatment can be found in Weinberger's PhD thesis [72]. The cited work also includes *two-sided fibrations*.

Exploiting results from [65], it is shown in [61] that sHoTT has a model in bisimplicial sets where Rezk types correspond to complete Segal spaces (also called Rezk spaces).

However, we are not in a position to claim that sHoTT is indeed able to capture all category theory of $(\infty, 1)$-categories. For example, the construction of the opposite of an $(\infty, 1)$-category is not implemented yet, and we lack a Yoneda embedding in which an $(\infty, 1)$-category is embedded into an $(\infty, 1)$-category of presheaves of spaces. Another important problem is that the type of *discrete types* is a *Segal type* that does not coincide with the ∞-category of ∞-groupoids in the bisimplicial sets model.

Despite the current inherent limitations of the type theory, we have tried to explore which other categorical properties can be obtained with the theory as is. Throughout this paper we intend to present a reasonable theory of limits and colimits of diagrams of $(\infty, 1)$-categories. In this setting, we can prove most of the expected properties that are familiar from category theory. Under the mild assumption of the existence of a type-theoretic universe of spaces, we can compute the limit of a space-valued diagram as a dependent product (see Theorem 2.5.14).

2.1.1 Limits and colimits

Limits and colimits have been studied extensively for quasi-categories in [49], and for Segal spaces. A short presentation appears in [57]. We introduce the definitions of limits and colimits within this synthetic theory and verify that they are consistent with the current definition for Rezk spaces in the bisimplicial model of sHoTT. We also prove some expected results, such as Theorem 2.3.7, which can be phrased as the universal property of colimits. We also present the interaction between limits and colimits with *adjoints*, Theorem 2.3.9 showing that right adjoints preserve limits.

We prove in Theorem 2.4.6 that in a Rezk type, the *type of all limits* of a given diagram is a proposition. This can be understood as a uniqueness property. Lastly, in Section 2.5 we show that in an appropriate sense any limit of spaces can be computed simply as a dependent product. The goal is to replicate the fact that for any diagram of spaces $G : I \to \infty\text{-}\mathbf{Gpd}$ where I is a set, $\lim_I G = \prod_{i \in I} G_i$. The difficulty carrying out this computation in sHoTT is that we are unable to construct the correct type of spaces (discrete types) within the theory. We implement this using a directed formulation of the Univalence Axiom, due to Cavallo, Riehl and Sattler, which allows us to assume the existence of a type with the desired properties.

The reader will appreciate that the study of limits and colimits in the synthetic setting is relatively simple. Our only prerequisite is some knowledge of homotopy type theory. This is in line with the goal of having a synthetic theory of ∞-categories where these are somewhat basic objects out of which we can effortlessly develop a robust synthetic theory.

2.1.2 Outline

In Section 2.2 we begin with an introduction to the work of Riehl-Shulman. The material we present here is not exhaustive, but for the understanding and development of this work it will be enough. All results contained in this section are due to Riehl-Shulman and can be found in [61]. The reader interested in the details of simplicial homotopy type theory is invited to read the mentioned reference; the experienced reader may skip this whole section.

Having the basic theory at hand, we define in Section 2.3 limits and colimits. We prove Theorem 2.3.7, which can be understood as the universal property of limits and colimits. We also prove the analogous result from category theory that right adjoints preserve limits and left adjoints preserve limits.

In Section 2.4 we study limits in Rezk types. This special case yields Theorem 2.4.6, which is the uniqueness of limits up to equality. Up to this point all the results and proofs can be dualized. Finally, in Section 2.5 we carry out the computation of the limit of a diagram of "spaces" as a dependent product. Here we use *univalent covariant families*, due to Cavallo, Riehl and Sattler (unpublished), to make sense of the ∞-category of spaces. Because simply taking the type of all Rezk types does not yield the correct object.

Finally, in Section 3.4 we verify that our definitions are consistent with the semantics. The general procedure we follow is to first interpret our type-theoretic definition in the intended semantics of bisimplicial sets and then prove that the resulting statement is equivalent to the existing definition. For limits and colimits, we do this in Section 2.6.1.

Computer-verified proofs. While preparing this paper, a formalization project for sHoTT [43] came to light. The overall goal is to, within the present framework, obtain computer-verified proofs of results about ∞-categories. Using the new proof assistant `Rzk`, the project aims to formalize the content of [61], [16], and the content of the present work.

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2.2 Extension types, Segal types and covariant families

In this section, we introduce the necessary material of the type theory that we use in the paper. We refrain from providing detailed proofs. The best reference for this is the original paper [61] where simplicial homotopy type theory was first introduced. We indicate precisely where each result can be found in *Ibidem.*

2.2.1 Extension types

Essentially, simplicial homotopy type theory is ordinary homotopy type theory augmented with some axioms postulating the existence of a strict interval object $\mathbb{2}$. That is, a type $\mathbb{2}$ equipped with a total order, a smallest element 0 and a distinct largest element 1. In a general type X, an arrow in X can be defined as a map $f : \mathbb{2} \to X$, with the source and target of f being the image of the endpoints $0, 1 : \mathbb{2}$. However, we would like to be able to talk about the family of hom types from x to y, for $x, y : X$. This would not be achievable in ordinary type theory; it can be done thanks to a new type former called an *extension type*. However, the inner workings of extension types force us to single out the interval $\mathbb{2}$ by putting it in a separate layer of a layered type theory.

More concretely, simplicial type theory is built as a three-layer type theory: the layer of **cubes**, the layer of **topes**, and the layer of **types**. All rules for each layer can be found in greater detail in our main reference [61], here we just give a small account together with some of the fundamental results we will need.

The first layer is a type theory with finite products of types. Some rules of this type theory include the following:

$$\frac{}{\mathbf{1}\ \text{cube}} \qquad \frac{}{\Gamma \vdash \star : \mathbf{1}}$$

$$\frac{I\,\text{cube} \quad J\,\text{cube}}{I \times J\ \text{cube}.}$$

The second layer is an intuitionistic logic over the layer of cubes. Its types are called **topes**. Topes can be regarded as polytopes embedded in a cube. They admit operations of finite conjunction and disjunction, but negation, implication, or quantifiers are not part of this theory. The complete rules for topes can be found in [61, Figure 2]. There is a "tope equality" that appears in the third layer as a "strict equality." For example, this tope equality is used to define some *shapes* below, and it is denoted with the symbol "$\equiv$."

In simplicial type theory, the **cubes** layer is given as follows: It starts by postulating the cube $\mathbb{2}$, which has two elements $0 : \mathbb{2}$ and $1 : \mathbb{2}$. This cube also comes with a tope inequality

$$\frac{x : \mathbb{2} \quad y : \mathbb{2}.}{(x \leqslant y)\,\mathsf{tope}}$$

In addition, there are axioms that turn the inequality tope into a total order relation on $\mathbb{2}$ with distinct endpoints 0 and 1. Recall that these axioms are:

$$\frac{x:\mathbb{2}}{\vdash x\leqslant x}\qquad \frac{x:\mathbb{2},y:\mathbb{2},z:\mathbb{2}}{(x\leqslant y),(y\leqslant z)\vdash(x\leqslant z)}\qquad \frac{x:\mathbb{2},y:\mathbb{2}}{(x\leqslant y),(y\leqslant x)\vdash(x\equiv y)}$$

$$\frac{x:\mathbb{2},y:\mathbb{2}}{\vdash(x\leqslant y)\vee(y\leqslant x)}\qquad \frac{x:\mathbb{2}}{\vdash(0\leqslant x)}\qquad \frac{x:\mathbb{2}}{\vdash(x\leqslant 1)}\qquad \frac{(0\equiv 1)}{\vdash\bot}\ .$$

The rest of the cubes are generated by finite products of the cube $\mathbb{2}$. Using cubes and topes, we can introduce **shapes** as follows:

$$\frac{I\quad \mathsf{cube}\quad t:I\vdash\phi\quad \mathsf{tope}}{\{t:I\mid\phi\}\quad \mathsf{shape.}}$$

Some relevant shapes are:

$$\begin{aligned}
&\Delta^0 :\equiv \{t:\mathbf{1}|\top\},\\
&\Delta^1 :\equiv \{t:\mathbb{2}|\top\},\\
&\Delta^2 :\equiv \{\langle t_1,t_2\rangle:\mathbb{2}\times\mathbb{2}|t_2\leqslant t_1\},\\
&\partial\Delta^1 :\equiv \{t:\mathbb{2}|(t\equiv 0)\vee(t\equiv 1)\},\\
&\partial\Delta^2 :\equiv \{\langle t_1,t_2\rangle:\Delta^2|(0\equiv t_2\leqslant t_1)\vee(t_1\equiv t_2)\vee(t_2\leqslant t_1\equiv 1)\},\\
&\kappa^2_1 :\equiv \{\langle t_1,t_2\rangle:\Delta^2|(t_1\equiv 1)\vee(t_2\equiv 0)\}.
\end{aligned}$$

More can be said about this strict interval: the category of simplicial sets is the classifying topos for such strict intervals [51]. Furthermore, as explained in [61], by embedding simplicial sets into the category of bisimplicial sets, it is possible to show that the latter presents the "classifying $(\infty,1)$-topos" of strict intervals.

Finally, there is a third layer of types that has all the ordinary type formers of homotopy type theory and one additional type former, the *extension type*, that involves the previous layers. Its formation rule is:

$$\frac{\begin{array}{ccc}\{t:I\mid\phi\}\,\mathsf{shape} & \{t:I\mid\psi\}\,\mathsf{shape} & t:I\mid\phi\vdash\psi\\ \Xi\mid\Phi\vdash\Gamma\,\mathsf{Ctxt} & \Xi,t:I\mid\Phi,\psi\mid\Gamma\vdash A\,\mathsf{Type} & \Xi,t:I\mid\Phi,\phi\mid\Gamma\vdash a:A\end{array}}{\Xi\mid\Phi\mid\Gamma\vdash\left\langle\prod_{t:I\,\mid\,\psi}A(t)\Big|^{\phi}_{a}\right\rangle\,\mathsf{Type.}}$$

We refer to [61] for the precise formulation of its rules. The name extension types are suggestive of how to construe them. We can think of $\{t:I\mid\phi\}$ as a "sub-shape" of $\{t:I\mid\psi\}$ and read the judgment $\Xi,t:I\mid\Phi,\phi\mid\Gamma\vdash a:A$ as a function $\phi\to A$. We could represent a point in an extension type with a dashed arrow in the commutative diagram:

$$\begin{tikzcd}
\phi \arrow[r] \arrow[d, hook] & A \\
\psi. \arrow[ur, dashed] &
\end{tikzcd}$$

This dashed arrow does not have to be unique in any sense. The benefit of this diagrammatic representation of extension types will be obvious later on when we introduce *Segal types*. Note that the Hom type in Theorem 2.2.6 is a special case of an extension type. As noted in [16], it is possible to obtain a similar theory by considering HoTT plus a strict interval theory and replacing extension types using identity types (see [61, Pag. 119] and [16, Section 2.4]), but this makes the theory harder to use. In this paper we will follow the original Riehl-Shulman formulation of sHoTT using extension types.

Remark 2.2.1. There is a connection of sHoTT with cubical type theory [19]. We refer to [61, Remark 3.2] where this is explained further. We just mention that the cubical path type $\mathsf{Path}_A(x, y)$ can be expressed as the extension type $\langle \prod_{t:\mathbb{I}} A|^{t\equiv 0 \vee t\equiv 1}_{\mathrm{rec}_\vee(x,y)}\rangle$. This is analogous to the Hom types we introduce below in Theorem 2.2.6.

Extension types behave very much like dependent product types. In particular, one can think of extension types as $\prod$-types with a judgmental equality added to them. Furthermore, many results that hold for $\prod$-types are also valid for extension types. For example:

Theorem 2.2.2. If $t : I \mid \phi \vdash \psi$, $X : \{t : I \mid \psi\} \to \mathcal{U}$ and $Y : \prod\limits_{t:I \mid \psi} (X \to U)$, while $a : \prod_{t:I \mid \psi} X(t)$ and $b : \prod_{t:I \mid \phi} Y(t, x(t))$, then

$$\left\langle \textstyle\prod_{t:I \mid \psi} (\sum_{x:X(t)} Y(t,x)) \Big|^{\phi}_{\lambda t.(a(t),b(t))} \right\rangle \simeq \left(\textstyle\sum_{f:\langle \prod_{t:I \mid \psi} X(t)|^{\phi}_{a}\rangle} \left\langle \prod_{t:I \mid \psi} Y(t, f(t)) \Big|^{\psi}_{f} \right\rangle \right).$$

Proof. This is [61, Theorem 4.3]. ∎

This is similar to [70, Theorem 2.15.7]. The functions needed for the proof come from introduction and computation rules for extension types; these rules are quite similar to those of dependent products. We will need this idea of the proof later in Theorem 2.4.1. Also, we have

Theorem 2.2.3. ([61, Theorem 4.4]) Suppose that we have $t : I \mid \phi \vdash \psi$, $t : I \mid \psi \vdash \chi$, $X : \{t : I \mid \chi\} \to \mathcal{U}$ and $a : \prod\limits_{t:I \mid \phi} X(t)$. Then

$$\left\langle \prod_{t:I \mid \chi} X \Bigg|^{\phi}_{a} \right\rangle \simeq \sum_{f:\langle \prod_{t:I \mid \psi} X|^{\phi}_{a}\rangle} \left\langle \prod_{t:I \mid \chi} X \Bigg|^{\psi}_{f} \right\rangle.$$

2.2.1.1 Relative function extensionality axiom

In homotopy type theory we use the function extensionality axiom for dependent functions. We recall this formulation.

Axiom 2.2.4. Let $B : A \to \mathcal{U}$ be a dependent type family over a type A. For $f, g : \prod_{a:A} B(a)$, the canonical map $f = g \to \prod_{a:A}(fa = ga)$ is an equivalence.

Note that the equality on the left is in the dependent product. An element of $\prod_{a:A}(fa = ga)$ is usually referred to as a homotopy, and often this type is written as $f \sim g$. There is a version of function extensionality for extension types.

Axiom 2.2.5. Suppose $t : I \mid \phi \vdash \psi$ and that $A : \{t : I \mid \psi\} \to \mathcal{U}$ is such that each $A(t)$ is contractible, and moreover, there is an element $a : \prod_{t:I \mid \phi} A(t)$, then $\left\langle \prod_{t:I \mid \psi} A(t)|^{\phi}_{a} \right\rangle$ is contractible.

Further discussion and different formulations on this axiom can be found in [61, 4.4]. Moreover, taking ϕ to be $\bot$ in Theorem 2.2.5 gives us Theorem 2.2.4. This is because for $\phi \equiv \bot$ extension types behave like ordinary dependent products (see [61, Section 2.2]). In the paper, we assume Theorem 2.2.5, hence we also have function extensionality, and we use the notation $f \sim g$ from above.

2.2.2 Segal types

With extension types and simplices at hand, it is feasible to introduce "morphisms" of a type as "points" in a "hom space." The coherences that witness "composition" between these morphisms are elements in another suitable type. We can use the above to define *Segal types.*

Let us recall first that from the rules presented in [61, Figure 3] it is possible to prove the following: Given a type A, in order to construct an element $a : A$ in context $\partial\Delta^1$, we can just give two elements $x, y : A$, and vice versa, any two elements $x, y : A$ determine another element $a : A$ in context $\partial\Delta^1$. Similarly, the construction of an element $a : A$ in context $\partial\Delta^2$ is equivalent to giving three terms $a_0, a_1, a_2 : A$ in context $t : \mathbb{2}$ such that $a_0[0/t] \equiv a_1[0/t]$, $a_0[1/t] \equiv a_2[0/t]$ and $a_1[1/t] \equiv a_2[1/t]$. For more details on this, we refer to [61, Section 3.2]. The last construction can be seen as the "boundary of a triangle in A", see Theorem 2.2.7 below.

Definition 2.2.6. Given $x, y : A$, determining a term $[x, y] : A$ in context $\partial\Delta^1$, define

$$\mathsf{Hom}_A(x, y) :\equiv \left\langle \Delta^1 \to A\Big|^{\partial\Delta^1}_{[x,y]} \right\rangle.$$

A term in this type is called an **arrow** in A.

Note that this is just an extension type where the type family has constant value A. An element $f : \mathsf{Hom}_A(x, y)$ has the property that $f(0) \equiv x$ and $f(1) \equiv y$. Thus, such an element captures the idea of what an arrow in A from x to y should be.

Definition 2.2.7. For $x, y, z : A$ and $f : \mathsf{Hom}_A(x, y), g : \mathsf{Hom}_A(y, z)$ and $h : \mathsf{Hom}_A(x, z)$ there is a term $[x, y, z, f, g, h] : A$ in context $\partial\Delta^2$, define

$$\mathsf{Hom}^2_A\left(\begin{tikzcd} & y \arrow[dr, dash, "g"] & \\ x \arrow[ur, dash, "f"] \arrow[rr, dash, "h"'] & & z \end{tikzcd}\right) :\equiv \left\langle \Delta^2 \to A\Big|^{\partial\Delta^2}_{[x,y,z,f,g,h]} \right\rangle.$$

For an interpretation of the types above, we use the bisimplicial sets model of sHoTT. We give more details in Section 2.6.1.

Definition 2.2.8. A **Segal type** is a type A such that for all $x, y, z : A$ and $f : \mathsf{Hom}_A(x, y), g : \mathsf{Hom}_A(y, z)$ the type

$$\sum_{h:\mathsf{Hom}_A(x,z)} \mathsf{Hom}^2_A \left(\begin{tikzcd} & y \arrow[dr, dash, "g"] & \\ x \arrow[ur, dash, "f"] \arrow[rr, dash, "h"'] & & z \end{tikzcd} \right)$$

is contractible.

Informally, A is a Segal type if any pair of composable arrows have an essentially unique composite.

The first component of the center of contraction of this type is the **composition** of f and g, we adopt the usual notation $g \circ f$ for such a term. This composition of arrows is associative whenever it is defined. For any type A we can define $\mathsf{id}_x : \mathsf{Hom}_A(x, x)$ as $\mathsf{id}_x(t) \equiv x$ for all $t : \Delta^1$. Furthermore, under the assumption that A is a Segal type, this element is the **identity** for the composition (see [61, Section 5.2 and 5.3]). We can also get a useful characterization of Segal types:

Theorem 2.2.9. ([61, Theorem 5.5]) A type A is a Segal type if and only if the restriction map

$$(\Delta^2 \to A) \to (\kappa^2_1 \to A)$$

is an equivalence.

Moreover, this allows us to show that if $A : X \to \mathcal{U}$ is a family over a type or shape X such that for all $x : X$ the type $A(x)$ is a Segal type then the dependent product of the family A over X is again a Segal type [61, Corollary 5.6]. This means that the dependent product of Segal types is again a Segal type.

Let A, B be types and $\phi : A \to B$ a function. For any $x, y : A$ there is an induced function

$$\phi_{\#} : \hom_A(x, y) \to \hom_B(\phi(x), \phi(y))$$

defined by precomposition. We use ϕ for this induced function instead of $\phi_{\#}$ when there is no risk of confusion. We have:

Proposition 2.2.10. Any function $\phi : A \to B$ between Segal types preserves identities and composition. This is for all $a, b, c, d : A$ and $f : \mathsf{Hom}_A(a, b)$, $g : \mathsf{Hom}_A(b, c)$, then we have $\phi(g \circ f) = \phi(g) \circ \phi(f)$ and $\phi(\mathsf{id}_d) = \mathsf{id}_{\phi(d)}$.

Proof. See [61, Proposition 6.1]. ∎

This proposition makes the functorial behaviour of functions between Segal types apparent. From Theorem 2.2.9 the type of functions $A \to B$ is a Segal type when B is itself a Segal type and A is a shape or type. Given two functions, $f, g : A \to B$ we can select a term $\alpha : \mathsf{Hom}_{A\to B}(f, g)$, spelling out the definition of such term, we get for all $s : \mathbb{2}$ a function $\alpha(s) : A \to B$ such that $\alpha(0) \equiv f$ and $\alpha(1) \equiv g$. Additionally, for all $a : A$ we have

$$\lambda(s : \mathbb{2}).\alpha(s)(a) : \mathsf{Hom}_B(fa, ga).$$

We denote this function as α_a and we refer to it as the component of α at $a : A$. An element $\alpha : \mathsf{Hom}_{A\to B}(f, g)$ is called a **natural transformation** between f and g. Such terminology is supported by the following remarks:

Remark 2.2.11. Let A be a type or shape, B a Segal type and $f, g : A \to B$. Then the induced function

$$\mathsf{Hom}_{A\to B}(f, g) \to \prod_{a:A} \mathsf{Hom}_B(fa, ga)$$

is an equivalence [61, Proposition 6.3]. And also [61, Proposition 6.5], if $\alpha : \mathsf{Hom}_{A\to B}(f, g), \beta : \mathsf{Hom}_{A\to B}(g, h)$ and $x : A$ then

$$(\beta \circ \alpha)_x = \beta_x \circ \alpha_x \text{ and } (\mathsf{id}_f)_x = \mathsf{id}_{f(x)}.$$

This is to say that natural transformations are component-wise defined and their composition is performed point-wise.

Remark 2.2.12. Under the same assumptions of the previous remark, if $\alpha : \mathsf{Hom}_{A\to B}(f, g)$ and $k : \mathsf{Hom}_A(x, y)$ then $\alpha_y \circ fk = gk \circ \alpha_x$.

Hence we can say that a natural transformation is truly natural in the categorical sense.

2.2.2.1 Discrete types

Definition 2.2.13. [61, Definition 7.1] For any type A, there is a map

$$\mathsf{idtoarr} : \prod_{x,y:A} (x =_A y) \to \mathsf{Hom}_A(x, y)$$

given by path induction $\mathsf{idtoarr}_{x,x}(\mathsf{refl}_x) :\equiv \mathsf{id}_x$. We say that the type A is discrete if $\mathsf{idtoarr}_{x,y}$ an equivalence for all $x, y : A$.

The first aspect to note about discrete types is that they are also Segal types [61, Proposition 7.3].

2.2.2.2 Rezk types

In the category of bisimplicial sets, Segal spaces are, intuitively, bisimplicial sets containing categorical and homotopical information. These data are not necessarily compatible. Complete Segal spaces fix this problem. More precisely, adopting the terminology and notation of [59], we have X_{hoequiv} as the *space of homotopy equivalences* of X. This space X_{hoequiv} is contained in X_1 and consists of *homotopy equivalences*. Then there is an obvious map $u : X_0 \to X_{\text{hoequiv}}$, if u is an equivalence of spaces we say that the Segal space is **complete**. The map u is not always a weak equivalence; an example of such a Segal space can be found in [56, Example 2.57]. A similar phenomenon occurs for Segal types. These have an associative, unital composition but are not "univalent", just as Segal spaces are not necessarily complete. To amend this problem and obtain "better behaved" types, we need to introduce *Rezk types*.

Let A be a Segal type and $f : \mathsf{Hom}_A(x, y)$, we define the type

$$\mathsf{isiso}(f) :\equiv \left(\sum_{g:\mathsf{Hom}_A(y,x)} g \circ f = \mathsf{id}_x \right) \times \left(\sum_{h:\mathsf{Hom}_A(y,x)} f \circ h = \mathsf{id}_y \right)$$

and say that f is an **isomorphism** if this type is inhabited. The reader might note that we do not ask for the left and right inverses of an isomorphism to be equal. The reason behind this can be found in [61, Section 10.1] There is an immediate characterization of isomorphisms:

Proposition 2.2.14. Let A be a Segal type and $f : \mathsf{Hom}_A(x, y)$. Then, f is an isomorphism if and only if we have $g : \mathsf{Hom}_A(y, x)$ with $g \circ f = \mathsf{id}_x$ and $f \circ g = \mathsf{id}_y$.

Proof. This is [61, Proposition 10.1]. ∎

The type $\mathsf{isiso}(f)$ is a proposition [61, Proposition 10.2] hence, we can define the type of isomorphisms between any two terms x, y in a Segal type A as

$$x \cong y :\equiv \sum_{f:\mathsf{Hom}_A(x,y)} \mathsf{isiso}(f).$$

Given a Segal type A and $x : A$ is clear that id_x is an isomorphism. We get a family of functions

$$\mathsf{idtoiso}_A : \prod_{x,y:A} (x = y \to x \cong y)$$

constructed by path induction via $\mathsf{idtoiso}_A(\mathsf{refl}_x) :\equiv \mathsf{id}_x$. Although the function $\mathsf{idtoiso}_A$ is dependent on the type A, we will drop A from the notation since it is always clear over which type the function is defined.

Definition 2.2.15. A Segal type A is a **Rezk type** if $\mathsf{idtoiso}$ is an equivalence. We denote the inverse by rezk.

We will need the following fact about Rezk types from [61, Proposition 10.9]

Proposition 2.2.16. If B is a Rezk type then for any type or shape X, the type B^X is a Rezk type.

2.2.3 Covariant families

Left fibrations were introduced by Joyal in [37], and further studied in [49]. These are the quasi-categorical versions of functors cofibered in groupoids. In turn, the category of bisimplicial sets also has a well-known theory of left fibrations (see [57]). In sHoTT we study them as *covariant families*, and they play a central role in subsequent sections.

Let $C : A \to \mathcal{U}$ be a type family over a type A. Two elements $x, y : A$, $f : \mathsf{Hom}_a(x, y)$ and $u : C(x), v : C(y)$ define the type

$$\mathsf{Hom}_{C(f)}(u, v) :\equiv \left\langle \prod_{t:\mathbb{2}} C(f(t)) \middle|^{\partial\Delta^1}_{[u,v]} \right\rangle .$$

This type represents the type of arrows from u to v over f in the total type of C.

Definition 2.2.17. A type family $C : A \to \mathcal{U}$ over a Segal type A is **covariant** if for any $f : \mathsf{Hom}_A(x, y)$ and $u : C(x)$ the type

$$\sum_{v:C(y)} \mathsf{Hom}_{C(f)}(u, v)$$

is contractible. Similarly, if for any $f : \mathsf{Hom}_A(x, y)$ and $v : C(y)$ the type

$$\sum_{u:C(x)} \mathsf{Hom}_{C(f)}(u, v)$$

is contractible, we say that C is **contravariant**.

In the situation of the definition above, we denote the center of contraction of each type as $(f_*u, \mathsf{trans}_{f,u})$ and $(f^*v, \mathsf{trans}_{f,v})$, respectively.

From [61, Remark 8.3] we have that if $g : B \to A$ is a function and $C : A \to \mathcal{U}$ is a covariant type family, then $\lambda b.C(g(b)) : B \to \mathcal{U}$ is also covariant. This just means that a covariant family is stable under substitution.

The first important example of a covariant family is given by the Hom type (see [61, Proposition 8.13]):

Proposition 2.2.18. Let A be a type and fix $a : A$. Then the type family $\lambda x.\mathsf{Hom}_A(a, x) : A \to \mathcal{U}$ is covariant if and only if A is a Segal type.

We can show that if $C : A \to \mathcal{U}$ is a covariant family over a Segal type A, then the total type

$$\sum_{x:A} C(x)$$

is a Segal type [61, Theorem 8.8]. Also, as previously advertised, if the covariant family is defined over a Segal type then it is functorial in the following sense:

Proposition 2.2.19. Suppose A is a Segal type and $C : A \to \mathcal{U}$ is covariant. Then, given $f : \mathsf{Hom}_A(x, y)$, $g : \mathsf{Hom}_A(y, z)$, and $u : C(x)$, we have $g_*(f_*u) = (g \circ f)_*u$ and $(\mathsf{id}_x)_*u = u$.

Proof. This is [61, Proposition 8.16]. ∎

Furthermore, naturality holds:

Proposition 2.2.20. Let $C, D : A \to \mathcal{U}$ be two covariant families and a fiber-wise map $\phi : \prod_{x:A} C(x) \to D(x)$. Then for any $f : \mathsf{Hom}_A(x, y)$ and $u : C(x)$,

$$\phi_y(f_* u) = f_*(\phi_x(u)).$$

Proof. See [61, Proposition 8.17]. ∎

If we have a covariant family $C : A \to \mathcal{U}$ over a Segal type, a path $p : x = y$ and $u : C(x)$ then the element $(\mathsf{idtoiso}(p))_* u : C(y)$ can be understood as transporting u along the arrow $\mathsf{idtoiso}(p)$. The covariant transport and the usual type-theoretic transport along paths coincide; by this we mean that they are propositionally equal:

Lemma 2.2.21. If A is Segal and $C : A \to \mathcal{U}$ is covariant and $e : x =_A y$, then for any $u : C(x)$ we have

$$\mathsf{idtoiso}(e)_* u = \mathsf{transport}^C(e, u).$$

Proof. [61, Lemma 10.7]. ∎

2.2.3.1 Yoneda lemma

The Yoneda lemma is a fundamental result of category theory. In the context of ∞-categories it has been proved for different models, such as quasi-categories in [49], [36], for Segal spaces in [57] and ∞-cosmoi in [62]. In simplicial type theory, as noted in [61], the result can be viewed as a directed version of path transport. Also, there is a dependent version of the Yoneda lemma, which can be seen as the analogous to path induction for covariant families. Although we will need only one of its consequences, namely *representability*, we quickly review it in full generality.

Let $C : A \to \mathcal{U}$ a covariant family and a fix term $a : A$. Then there are functions

$$\mathsf{evid}_a^C :\equiv \lambda\phi.\phi(a, \mathsf{id}_a) : \left(\prod_{x:A} \mathsf{Hom}_A(a, x) \to C(x)\right) \to C(a),$$

$$\mathsf{yon}_a^C :\equiv \lambda u.\lambda x.\lambda f.f_* u : C(a) \to \left(\prod_{x:A} \mathsf{Hom}_A(a, x) \to C(x)\right).$$

Theorem 2.2.22. (Yoneda lemma, [61, Theorem 9.1]) If A is a Segal type, then for any covariant family $C : A \to \mathcal{U}$ and $a : A$ the functions yon_a^C and evid_a^C are mutual inverses.

We have the dependent version of Yoneda:

Theorem 2.2.23. ([61, Theorem 9.5]) If A is a Segal type, $a : A$ and $C : \prod_{x:A}(\mathsf{Hom}_a(a,x) \to \mathcal{U})$ is covariant, then the function

$$\mathsf{evid}_a^C :\equiv \lambda\phi.\phi(a, \mathsf{id}_a) : \left(\prod_{x:A}\ \prod_{f:\mathsf{Hom}_A(a,x)} C(x,f)\right) \to C(a, \mathsf{id}_a)$$

is an equivalence.

One particular instance of this equivalence is when $C :\equiv \mathsf{Hom}_A(a', -)$, where A is a Segal type and $a' : A$. This gives the so-called **Yoneda embedding**. The concept of representability is closely related to this embedding.

Definition 2.2.24. A covariant family $C : A \to \mathcal{U}$ over a Segal type is **representable** if there exists $a : A$ and a family of equivalences

$$\prod_{x:A}(\mathsf{Hom}_A(a,x) \simeq C(x)).$$

We conclude this section with a result that is of particular interest to us:

Definition 2.2.25. A term $b : B$ is **initial** if for any $x : B$ the type $\mathsf{Hom}_B(b,x)$ is contractible. That is, if the type

$$\mathsf{isinitial}(b) :\equiv \prod_{x:B} \mathsf{iscontr}(\mathsf{Hom}_B(b,x))$$

is inhabited. Also, a term $b : B$ is **terminal** if for any $x : B$ the type $\mathsf{Hom}_B(x,b)$ is contractible. That is, if the type

$$\mathsf{isterminal}(b) :\equiv \prod_{x:B} \mathsf{iscontr}(\mathsf{Hom}_B(x,b))$$

is inhabited.

In a Segal type, initial and terminal elements are unique up to isomorphism.

Proposition 2.2.26. Let A be a Segal type. The following is true:

1. If $a : A$ is an initial element, then it is unique up to isomorphism.
2. If $a : A$ is a terminal element, then it is unique up to isomorphism.

Proof. Assume that A is a Segal type and let $a, b : A$ be initial elements. The types $\mathsf{Hom}_A(a,b), \mathsf{Hom}_A(b,a)$ are contractible, in particular, inhabited, say, by f, g, respectively. We get the terms $f \circ g : \mathsf{Hom}_A(b,b), g \circ f : \mathsf{Hom}_A(a,a)$ and $\mathsf{id}_b : \mathsf{Hom}_A(b,b), \mathsf{id}_a : \mathsf{Hom}_A(a,a)$. Again, these hom types are contractible so $f \circ g = \mathsf{id}_b$ and $g \circ f = \mathsf{id}_a$. By Theorem 2.2.14 f is an isomorphism between a and b. Therefore, initial terms in Segal types are unique up to isomorphism. Similarly, a terminal element is unique up to isomorphism. ∎

Proposition 2.2.27. A covariant type family $C : A \to \mathcal{U}$ over a Segal type A is representable if and only if there exists an initial element (a, u) in the type $\sum\limits_{x:A} C(x)$, in which case

$$\mathsf{yon}_a^C(u) : \prod_{x:A} (\mathsf{Hom}_A(a, x) \to C(x))$$

defines an equivalence.

Proof. This is [61, Proposition 9.10]. ∎

The majority of the results we have cited are stated in terms of covariant families. However, for contravariant families we have similar results. The proofs for these statements are completely analogous.

2.3 Limits and colimits

There is a well-known path to define limits and colimits in ∞-categories, as in [49] for quasi-categories or in Segal spaces [57]. The strategy is first to establish an adequate notion of cones and cocones. The second step is to define initial and terminal objects. Then a *limit* is a "terminal cone," and a *colimit* is an "initial cocone." We follow the same approach to define them in sHoTT using the machinery of covariant families.

For types A, B and $b : B$, define the constant function

$$\triangle b :\equiv \lambda a.b : A \to B.$$

Remark 2.3.1. If B is a Segal type then by Theorem 2.2.9, $B^A :\equiv A \to B$ is also a Segal type. Let $f : A \to B$, from Theorem 2.2.18 and the fact that covariance and contravariance are stable under substitution (precomposition), then the type families

$$\lambda b.\mathsf{Hom}_{B^A}(f, \triangle b) : B \longrightarrow \mathcal{U} \quad \text{and} \quad \lambda b.\mathsf{Hom}_{B^A}(\triangle b, f) : B \longrightarrow \mathcal{U}$$

are covariant and contravariant, respectively.

Definition 2.3.2. Let $f : A \to B$ be a function. Define the type of **cocones** of f as the type

$$\mathsf{cocone}(f) :\equiv \sum_{b:B} \mathsf{Hom}_{B^A}(f, \triangle b),$$

and the type of **cones** of f as the type

$$\mathsf{cone}(f) :\equiv \sum_{b:B} \mathsf{Hom}_{B^A}(\triangle b, f).$$

Observation 2.3.3. From Theorem 2.2.11 it follows that

$$\mathsf{cocone}(f) \simeq \sum_{b:B} \prod_{a:A} \mathsf{Hom}_B(fa, b)$$

and

$$\mathsf{cone}(f) \simeq \sum_{b:B} \prod_{a:A} \mathsf{Hom}_B(b, fa).$$

We can recognize the type of cones and cocones of a function as direct analogues to cones and cocones in category theory. In Section 2.6.1 we will verify how the definition above and the following are consistent with the semantics.

Definition 2.3.4. A **colimit** for $f : A \to B$ is an initial term of the type:

$$\mathsf{cocone}(f) \equiv \sum_{x:B} \mathsf{Hom}_{B^A}(f, \triangle x).$$

A **limit** for $f : A \to B$ is a terminal term of the type:

$$\mathsf{cone}(f) \equiv \sum_{x:B} \mathsf{Hom}_{B^A}(\triangle x, f)$$

Remark 2.3.5. Since the total space of a covariant family over a Segal type is a Segal type, Theorem 2.3.1 implies that the type of cocones of a function $f : A \to B$ is a Segal type. Similarly, we also get that the type of cones is a Segal type. We will make use of these two facts where needed without explicit mention.

We can show the following:

Corollary 2.3.6. Let B be a Segal type and $f : A \to B$ a function. Limits and colimits are unique up to a unique isomorphism if they exist.

Proof. It follows immediately from Theorem 2.3.5 and Theorem 2.2.26 above. ∎

We will obtain in Theorem 2.4.6 uniqueness property up to equality under the additional assumption that B is a Rezk type. The following characterization is more flexible in terms of computations. We can think of it as the universal property of limits and colimits.

Proposition 2.3.7. Let B a Segal type and $f : A \to B$ be a function. There exists a colimit (b, β) for f if and only if

$$\prod_{x:B} (\mathsf{Hom}_B(b, x) \simeq \mathsf{Hom}_{B^A}(f, \triangle x)).$$

Similarly, there exists a limit (a, α) for f if and only if

$$\prod_{x:B} (\mathsf{Hom}_B(x, a) \simeq \mathsf{Hom}_{B^A}(\triangle x, f)).$$

Proof. The existence of a colimit (b, β) for the function f *i.e.,* (b, β) is an initial term of the type $\sum_{x:B} \mathsf{Hom}_{B^A}(f, \triangle x)$, by Theorem 2.2.27 is equivalent to say that the covariant family

$$\mathsf{Hom}_{B^A}(f, \triangle -) : B \to \mathcal{U}$$

is representable. The proof for the second part is similar. ∎

Remark 2.3.8. We can get an explicit description of the cone and cocones in theorem 2.3.7. Under the same assumptions, given a family of equivalences

$$\phi : \prod_{x:B} (\mathsf{Hom}_B(x, a) \simeq \mathsf{Hom}_{B^A}(\triangle x, f)),$$

we can take $\phi_a : \mathsf{Hom}_B(a, a) \simeq \mathsf{Hom}_{B^A}(\triangle a, f)$. This give us the cone $\phi_a(\mathsf{id}_a) : \mathsf{Hom}_{B^A}(\triangle a, f)$. Following the usual categorical argument, we can show that $(a, \phi_a(\mathsf{id}_a))$ is the limit for f. Similarly we can get the cocone for the apex b.

In the situation of Theorem 2.3.7, it is common practice to say that "b" is the colimit and that "a" is the limit of $f : A \to B$, respectively. This leaves implicit the data of the cocone and the cone, respectively.

2.3.1 Preservation of limits and colimits

The main goal of this section is to reproduce the classical result that limits are preserved under right adjoints. The same argument will show that left adjoints preserve colimits. Different presentations of adjunction are studied in [61], namely, **transposing** and **diagrammatic** adjunctions, and it is shown that they are equivalent. The first kind of adjunction corresponds to the standard definition of adjunction in category theory in terms of hom sets, whereas the second resembles the triangle identities. For us, it will be enough to know that a **quasi-transposing adjunction** between types A, B consists of functions $f : A \to B$ and $u : B \to A$ and a family of maps

$$\phi : \prod_{a:A, b:B} \mathsf{Hom}_B(fa, b) \to \mathsf{Hom}_A(a, ub)$$

with quasi-inverses given by the maps

$$\psi : \prod_{a:A, b:B} \mathsf{Hom}_A(a, ub) \to \mathsf{Hom}_B(fa, b)$$

and homotopies

$$\xi : \prod_{a,b,k} \phi_{a,b}(\psi_{a,b}(k)) = k, \quad \zeta : \prod_{a,b,l} \psi_{a,b}(\phi_{a,b}(l)) = l.$$

In this situation, u together with the data above is a **quasi-transposing right adjoint** for f. Using this notation we have:

Theorem 2.3.9. Let A, B be Segal types and functions $g : J \to B$, $f : A \to B$, $u : B \to A$. If g has a limit (b, β), and u is a quasi-transposing right adjoint of f, then $(u(b), u\beta)$ is a limit for $ug : J \to A$.

Proof. In view of Theorem 2.3.7, we have

$$\gamma : \prod_{x:B} (\mathsf{Hom}_B(x, b) \simeq \mathsf{Hom}_{B^J}(\triangle x, g)),$$

and we only need to show that

$$\prod_{y:A} (\mathsf{Hom}_A(y, u(b)) \simeq \mathsf{Hom}_{A^J}(\triangle y, ug)).$$

Recall that $\mathsf{Hom}_{B^J}(\triangle x, g) \simeq \prod_{j:J} \mathsf{Hom}_B(x, g(j))$. Then, for any $y : A$ we get

$$\gamma_{f(y)} : \mathsf{Hom}_B(f(y), b) \simeq \prod_{j:J} \mathsf{Hom}_B(f(y), g(j)).$$

Using the adjunction we get a chain of equivalences:

$$\mathsf{Hom}_A(y, u(b)) \simeq \mathsf{Hom}_B(f(y), b) \simeq \prod_{j:J} \mathsf{Hom}_B(f(y), g(j)) \simeq \prod_{j:J} \mathsf{Hom}_A(y, ug(j)) \simeq \mathsf{Hom}_{A^J}(\triangle y, ug).$$

This give us the family of equivalences

$$\Gamma : \prod_{y:A} (\mathsf{Hom}_A(y, u(b)) \simeq \mathsf{Hom}_{A^J}(\triangle y, ug)).$$

To complete the proof, note that the cone for $u(b)$ can be obtained from $\beta : \mathsf{Hom}_{B^J}(\triangle b, g)$, since $u\beta : \mathsf{Hom}_{A^J}(\triangle u(b), ug)$. Following Theorem 2.3.8, we must have that $\Gamma_{u(b)}(\mathsf{id}_{u(b)}) = u\beta$. ∎

We also have:

Corollary 2.3.10. If A, B are Segal types and $f : A \to B$ has a left quasi-transposing adjunct, then f preserves colimits of functions $g : J \to A$.

Proof. Similar to the previous theorem. ∎

2.4 (Co)Limits in Rezk types

In this section, we restrict our attention to colimits in a Rezk type, which allows us to improve some of our results.

In what follows, for $C : A \to \mathcal{U}$ a covariant (or contravariant) family, we denote by

$$\tilde{C} :\equiv \sum_{x:A} C(x)$$

the dependent sum over A.

Lemma 2.4.1. Assume that $C : A \to \mathcal{U}$ is a covariant family over a Segal type A, let $(a,u),(b,v) : \sum\limits_{x:A} C(x)$. Then

$$((a,u) \cong (b,v)) \simeq \sum_{f:a\cong b} f_*u = v.$$

Proof. First, we have the equivalence

$$\mathsf{Hom}_{\tilde{C}}((a,u),(b,v)) \simeq \sum_{f:\mathsf{Hom}_A(a,b)} \mathsf{Hom}_{C(f)}(u,v). \tag{2.4.1}$$

Recall that this follows from Theorem 2.2.2

$$\left\langle \prod_{t:I\mid\psi} \Big(\sum_{x:X(t)} Y(t,x)\Big) \middle|^{\phi}_{\lambda t.(a(t),b(t))} \right\rangle \simeq \sum_{f:\left\langle \prod_{t:I\mid\psi} X(t) \middle|^{\phi}_{a}\right\rangle} \left\langle \prod_{t:I\mid\psi} X \middle|^{\psi}_{f} \right\rangle$$

where the proof is established by following the composition of the maps given from left to right by

$$h \mapsto (\lambda t.\pi_1(h(t)), \lambda t.\pi_2(h(t))),$$

and from right to left by

$$(f,g) \mapsto \lambda t.(f(t),g(t)).$$

This applied to (2.4.1) yields in the first coordinate the projection

$$\pi_1 : \sum_{x:A} C(x) \to A.$$

Since $C : A \to \mathcal{U}$ is a covariant family over the Segal type A, $\tilde{C}$ is a Segal type [61, Theorem 8.8]. By Theorem 2.2.10 the projection respects identities and composition. In particular, if $\bar{f} : (a,u) \cong (b,v)$ then $\pi_1(\bar{f}) : a \cong b$. Furthermore, covariance of C implies that $(v, \pi_2(\bar{f})) = (\pi_1(\bar{f})_*u, \mathsf{trans}_{\pi_1(\bar{f}),u})$ hence $\pi_1(\bar{f})_*u = v$. Therefore, we define

$$\phi : ((a,u) \cong (b,v)) \to \sum_{f:a\cong b} f_*u = v$$

to be $\phi :\equiv \lambda\bar{f}.(\pi_1(\bar{f}), p)$ where $p : \pi_1(\bar{f})_*u = v$.
In order to construct the function in the other direction, we observe the following: if $f : a \cong b$ then there exist $g : \mathsf{Hom}_A(b,a)$ with $f \circ g = \mathsf{id}_b$ and $g \circ f = \mathsf{id}_a$. The covariance of C implies that the types

$$\sum_{w:C(b)} \mathsf{Hom}_{C(f)}(u,w) \text{ and } \sum_{w:C(a)} \mathsf{Hom}_{C(g)}(v,w)$$

are contractible with center of contraction $(f_*u, \mathsf{trans}_{f,u})$ and $(g_*v, \mathsf{trans}_{g,v})$, respectively. Note also that

$$u = (\mathsf{id}_a)_*u = (g \circ f)_*u = g_*(f_*u) = g_*v.$$

We get elements

$$(f, \mathsf{trans}_{f,u}) : \sum_{h:\mathsf{Hom}_A(a,b)} \mathsf{Hom}_{C(h)}(u, v)$$

and

$$(g, \mathsf{trans}_{g,v}) : \sum_{h:\mathsf{Hom}_B(b,a)} \mathsf{Hom}_{C(h)}(v, u).$$

Using the equivalence

$$\left(\sum_{k:\mathsf{Hom}_A(x,y)} \mathsf{Hom}_{C(k)}(w, z) \right) \simeq \mathsf{Hom}_{\tilde{C}}((x, w), (y, z)) \tag{2.4.2}$$

we get $\bar{f} : \mathsf{Hom}_{\tilde{C}}((a, u), (b, w))$ and $\bar{g} : \mathsf{Hom}_{\tilde{C}}((b, v), (a, u))$. Since $\tilde{C}$ is a Segal type then $\bar{g} \circ \bar{f} : \mathsf{Hom}_{\tilde{C}}((a, u), (a, u))$. We have the composition in A and the dependent composition, so we obtain the 2-simplex

$$\mathsf{comp}_{g,f} : \mathsf{Hom}^2_A \left(\begin{tikzcd} & b \arrow[dr, dash, "g"] & \\ a \arrow[ur, dash, "f"] \arrow[rr, dash, "\mathsf{id}_a"'] & & a \end{tikzcd} \right)$$

and

$$\mathsf{comp}_{\mathsf{trans}_{g,v},\mathsf{trans}_{f,u}} : \mathsf{Hom}^2_{C(t)} \left(\begin{tikzcd} & v \arrow[dr, dash] & \\ u \arrow[ur, dash] \arrow[rr, dash, "\mathsf{id}_u"'] & & u \end{tikzcd} \right).$$

We get a term $\left(\overline{g \circ f}, (\mathsf{comp}_{g,f}, \mathsf{comp}_{\mathsf{trans}_{g,v},\mathsf{trans}_{f,u}})\right)$ in

$$\sum_{h:\mathsf{Hom}_{\tilde{C}}((a,u),(a,u))} \mathsf{Hom}^2_{\tilde{C}} \left(\begin{tikzcd} & (b, v) \arrow[dr, dash, "\bar{g}"] & \\ (a, u) \arrow[ur, dash, "\bar{f}"] \arrow[rr, dash, "h"'] & & (a, u) \end{tikzcd} \right).$$

Since this type is contractible, in particular we have $\bar{g} \circ \bar{f} = \overline{g \circ f}$. Under the equivalence (2.4.2), id_a is mapped to $\mathsf{id}_{(a,u)}$, so

$$\bar{g} \circ \bar{f} = \overline{g \circ f} = \overline{\mathsf{id}}_a = \mathsf{id}_{(a,u)}.$$

Similarly, $\bar{f} \circ \bar{g} = \mathsf{id}_{(b,v)}$. Therefore we define

$$\psi : \sum_{f:a \cong b} f_* u = v \to ((a, u) \cong (b, v))$$

as $\psi :\equiv \lambda(f, p).\bar{f}$. The functions ϕ and ψ give us the required equivalence.

∎

A related result is [70, Theorem 2.7.2], instead of identity types, we now have isomorphism types. Thus, an isomorphism in Σ-types is determined by an isomorphism in the base together with a dependent isomorphism in the total type lying over it. We formulate the result but give no proof corresponding to contravariant families.

Lemma 2.4.2. Assume that $C : A \to \mathcal{U}$ is a contravariant family over a Segal type A, let $(a, u), (b, v) : \sum_{x:A} C(x)$. Then

$$((a, u) \cong (b, v)) \simeq \sum_{f:a \cong b} u = f^* v.$$

Theorem 2.4.3. If A is a Rezk type and $C : A \to \mathcal{U}$ is a covariant (contravariant) family over A, then $\sum_{x:A} C(x)$ is a Rezk type.

Proof. Let $(a, u), (b, v) : \sum_{x:A} C(x)$. We have

$$\begin{aligned}\big((a, u) = (b, v)\big) &\simeq \sum_{f:a=b} \mathsf{transport}^C(f, u) = v \\ &\simeq \sum_{\mathsf{idtoiso}(f):a\cong b} \mathsf{idtoiso}(f)_* u = v \\ &\simeq (a, u) \cong (b, v),\end{aligned}$$

the middle equivalence follows from A being Rezk type and from Theorem 2.2.21 where we have the equality $\mathsf{idtoiso}(e)_* u = \mathsf{transport}^C(e, u)$. The function

$$\mathsf{idtoiso} : (a, u) = (b, v) \to (a, u) \cong (b, v)$$

is exactly the equivalence above. Indeed, we can use path induction to see this. If $(a, u) = (a, u)$ then $\mathsf{refl}_{(a,u)}$ is mapped to $(\mathsf{refl}_a, \mathsf{refl}_u)$. This goes to $(\mathsf{id}_a, \mathsf{refl}_u)$ which finally corresponds to $\mathsf{id}_{(a,u)} : (a, u) \cong (a, u)$. Therefore, $\sum_{x:A} C(x)$ is Rezk. ∎

Remark 2.4.4. Theorem 2.4.3 can also be deduced from Proposition 4.2.6 and Corollary 6.1.4 of [16]. Their proof makes use of their theory of *(co)cartesian fibrations* in sHoTT that they develop throughout their work. Ours is more elementary and simply uses the characterization of the type of isomorphisms between elements in a σ-type we provided in Theorem 2.4.1.

Definition 2.4.5. Let B is a Segal type, $f : A \to B$ a function. We define the type:

$$\mathsf{hasColimit}(f) :\equiv \sum_{w:\mathsf{cocone}(f)} \mathsf{isinitial}(w)$$

and also

$$\mathsf{hasLimit}(f) :\equiv \sum_{w:\mathsf{cone}(f)} \mathsf{isterminal}(w).$$

In Theorem 2.3.6 we showed that limits and colimits are unique up to isomorphism. Now, we can show further:

Corollary 2.4.6. If B is a Rezk type and we have a function $f : A \to B$, then the types $\mathsf{hasColimit}(f)$ and $\mathsf{hasLimit}(f)$ are propositions.

Proof. Let $(a, \alpha), (b, \beta)$ be limits for f. By Theorem 2.3.6 we have that $(a, \alpha) \cong (b, \beta)$. Theorem 2.4.3 applied to the covariant family

$$\mathsf{Hom}_{B^A}(f, \triangle -) : B \to \mathcal{U}$$

implies that $\mathsf{cone}(f)$ is also a Rezk type. Therefore:

$$\big((a, \alpha) = (b, \beta)\big) \simeq \big((a, \alpha) \cong (b, \beta)\big)$$

hence $\mathsf{hasLimit}(f)$ is a proposition. A similar proof shows that $\mathsf{hasColimit}(f)$ is also a proposition. ∎

2.5 Limit of spaces as dependent product

Let $\{G_i\}_{i \in I}$ be a family of ∞-groupoids indexed by a set I. Denote by G the obvious diagram $I \to \infty$-**Gpd**, then we have that

$$\lim_I G_i = \prod_{i \in I} G_i.$$

We can think for simplicity that the calculation above takes place in bisimplicial sets, *i.e.,* this is a family of complete Segal spaces that are homotopically constant. This could also be happening in simplicial sets.

Given the previous result, it seems natural to expect that a similar result is true in sHoTT. However, such computation is not straightforward: The obstruction is that in our type theory we do not have an ∞-category of ∞-groupoids in which we can take this limit. The way address this issue is by using the notion of *univalent covariant families* originally due to Cavallo-Riehl-Sattler [18] to sufficiently axiomatize the properties of the ∞-category of spaces.

Let B a Segal type. For any covariant family $E : B \to \mathcal{U}$ and $a, b : B$, using the notation of Theorem 2.2.17 we obtain a function

$$\mathsf{arrtofun} : \mathsf{Hom}_B(a, b) \to \big(E(a) \to E(b)\big)$$

where for any $f : \mathsf{Hom}_B(a, b)$ we set

$$\mathsf{arrtofun}(f) :\equiv f_* \equiv \lambda u. f_* u.$$

Definition 2.5.1. Given $E : B \to \mathcal{U}$ a covariant family over a Segal type B. We say that E is **univalent** if for all $a, b : B$ the map $\mathsf{arrtofun}$ is an equivalence. Denote its inverse by

$$\mathsf{dua} : \big(E(a) \to E(b)\big) \to \mathsf{Hom}_B(a, b).$$

We can understand the type B as satisfying a "directed univalent axiom." The prime example of a *univalent fibration* in the semantics should be the ∞-category of ∞-groupoids. In [61, Remark A.27] it is shown that covariant families correspond to left fibrations. Cavallo, Riehl, and Sattler announced in [18] (unpublished) the existence of a fibrant universe (bisimplicial set) of left fibrations. This implies the consistency of Theorem 2.5.1, see also Theorem 2.5.14.

The result in [61, Proposition 8.18] shows that; if $E : B \to \mathcal{U}$ is a covariant type family over a Segal type B, then for any $b : B$ the type $E(b)$ is discrete. Furthermore, if E is univalent, we could think of B as a Segal type "that has terms discrete types." We will make this idea precise in Theorem 2.5.8. In this part of the paper, we use the extensionality axiom and the notation introduced in Section 2.2.1.1. In particular, we use $f \sim g$ introduced there.

Lemma 2.5.2. Let B a Segal type and $E : B \to \mathcal{U}$ a univalent covariant family and $a, b : B$. If $\delta : E(a) \to E(b)$ is an equivalence, then $\mathsf{dua}(\delta)$ is an isomorphism.

Proof. By Theorem 2.2.19 for any $x : B$ and $u : E(x)$ we have $(\mathsf{id}_x)_*(u) = u$. Therefore, $\mathsf{arrtofun}(\mathsf{id}_x) = \mathsf{id}_{E(x)}$. This also implies that

$$\mathsf{dua}(\mathsf{id}_{E(x)}) = \mathsf{dua}(\mathsf{arrtofun}(\mathsf{id}_x)) = \mathsf{id}_x.$$

Treat $\delta : E(a) \to E(b)$ as a bi-invertible function, so there is $\gamma : E(b) \to E(a)$ such that $\delta\gamma \sim \mathsf{id}_{E(b)}$ and $\gamma\delta \sim \mathsf{id}_{E(a)}$. From the extensionality axiom we can further assume $\delta\gamma = \mathsf{id}_{E(b)}$ and $\gamma\delta = \mathsf{id}_{E(a)}$. Hence, $\mathsf{dua}(\delta\gamma) = \mathsf{id}_b$ and $\mathsf{dua}(\gamma\delta) = \mathsf{id}_a$. Observe that again from Theorem 2.2.19, for any $f : \mathsf{Hom}_B(x, y)$, $g : \mathsf{Hom}_B(y, z)$, and $u : E(x)$, we have $g_*(f_* u) = (g \circ f)_* u$, and as a consequence

$$\mathsf{arrtofun}(\mathsf{dua}(\delta) \circ \mathsf{dua}(\gamma)) = \mathsf{arrtofun}(\mathsf{dua}(\delta))\mathsf{arrtofun}(\mathsf{dua}(\gamma))) = \delta\gamma = \mathsf{id}_{E(b)}$$

it follows that $\mathsf{dua}(\delta) \circ \mathsf{dua}(\gamma) = \mathsf{dua}(\delta\gamma) = \mathsf{id}_b$. Similarly,

$$\mathsf{dua}(\gamma) \circ \mathsf{dua}(\delta) = \mathsf{dua}(\gamma\delta) = \mathsf{id}_a.$$

This concludes the proof of the statement. ∎

Definition 2.5.3. Given a type A and a family $E : B \to \mathcal{U}$ over a type B, we define the type

$$\mathsf{issmall}_B^E(A) :\equiv \sum_{b:B} \big(E(b) \simeq A\big).$$

We say that A is B-small if $\mathsf{issmall}_B^E(A)$ is inhabited.

A related notion appears in [64, Definition 17.1.3] where they define what it means for a type A to be small with respect to a univalent universe $\mathcal{U}$; a type A is said to be **$\mathcal{U}$-small** if there is type $X : \mathcal{U}$ and an equivalence between A and X. The same definition appears in [12, Section 2.19] under the name *essentially $\mathcal{U}$-small.*

Remark 2.5.4. In the rest of the paper, we use Theorem 2.5.3 under the assumption that the family $E : B \to \mathcal{U}$ is covariant. One could also think of assuming that E is contravariant, but we have no use for it.

Remark 2.5.5. Since we will require a covariant family, we can express whether a discrete type $A : \mathcal{U}$ is B-small. Firstly, [61, Proposition 8.18] proves that for a covariant type family over a Segal type $E : B \to \mathcal{U}$ each $E(b)$ is discrete. Therefore, if we have a proof of $\mathsf{issmall}^E_B(A)$, then A is also discrete since it is equivalent to a discrete type and of course it is B-small.

A more general and concrete example of this can be seen in the semantics. Given a Grothendieck universe U, one could build a sub-universe U' (an ∞-category) that consists of κ-small ∞-groupoids, where κ is a regular cardinal. By construction, elements of U' are exactly the U'-small elements of U. Of course, this would be one way to justify the consistency of Theorem 2.5.1.

The idea of Theorem 2.5.3 is that B has an element representing the type A. We can think of A as "a type in B", hence the name B-small. To make sense of this, we need to note some simple facts. The first lemma simply computes $\mathsf{transport}^D$ for the type family $D : B \to \mathcal{U}$ as defined below.

Lemma 2.5.6. Let $E : B \to \mathcal{U}$ be a univalent covariant family over a Rezk type B. For a type A, define $D : B \to \mathcal{U}$ as $D(b) :\equiv E(b) \simeq A$. Then for any $b, b' : B$, $p : b = b'$ and $\sigma : E(b) \simeq A$

$$\mathsf{transport}^D(p, \sigma) = \sigma(\mathsf{arrtofun}(\mathsf{idtoiso}(p^{-1}))).$$

Proof.

By path induction we can assume that $p \equiv \mathsf{refl}_b : b = b$. In this case,

$$\mathsf{transport}^D(\mathsf{refl}_b, \sigma) \equiv \mathsf{id}_{D(b)}(\sigma) = \sigma.$$

On the right-hand side we have

$$\begin{aligned} \sigma(\mathsf{arrtofun}(\mathsf{idtoiso}(p^{-1}))) &\equiv \sigma(\mathsf{arrtofun}(\mathsf{idtoiso}(\mathsf{refl}_b))) \\ &\equiv \sigma(\mathsf{arrtofun}(\mathsf{id}_b)) \\ &\equiv \sigma \mathsf{id}_{E(b)} \\ &= \sigma \end{aligned}$$

hence, we can simply use $\mathsf{refl}_\sigma : \sigma = \sigma$. ∎

Remark 2.5.7. Under the same assumptions of the previous lemma, we also have by path induction that for any $p : b = b'$,

$$\mathsf{arrtofun}(\mathsf{idtoiso}(p^{-1})) = \mathsf{arrtofun}(\mathsf{idtoiso}(p))^{-1}$$

since if $p \equiv \mathsf{refl}_b$ then both sides of the equality are $\mathsf{id}_{E(b)}$. Here, $\mathsf{arrtofun}(\mathsf{idtoiso}(p))^{-1}$ abusively denotes an inverse for $\mathsf{arrtofun}(\mathsf{idtoiso}(p))$.

The two results above have the following consequence:

Corollary 2.5.8. Let $E : B \to \mathcal{U}$ be a univalent covariant family over a Rezk type B. Then, for any type A, the type $\mathsf{issmall}^E_B(A)$ is a proposition.

Proof. Let $(b,\sigma),(b',\sigma') : \sum_{b:B}\big(E(b) \simeq A\big)$. We have

$$\big((b,\sigma) = (b',\sigma')\big) \simeq \sum_{p:b=b'} \sigma =^p \sigma'$$

therefore is enough to give a term of the right-hand side. We treat σ and σ' as bi-invertible functions. Let $\delta : A \to E(b)$, $\delta' : A \to E(b')$ be their inverses, respectively. We get an equivalence $\delta'\sigma : E(b) \simeq E(b')$, by Theorem 2.5.2 $\mathsf{dua}(\delta'\sigma) : b \cong b'$. Since B is a Rezk type, $\mathsf{rezk}(\mathsf{dua}(\delta'\sigma)) : b = b'$. From the Theorem 2.5.6 above and the fact that $((\mathsf{arrtofun})(\mathsf{idtoiso}))((\mathsf{rezk})(\mathsf{dua})) = \mathsf{id}_{E(b')^{E(b)}}$ we obtain:

$$\begin{aligned}
\mathsf{transport}^D(\mathsf{rezk}(\mathsf{dua}(\delta'\sigma)),\sigma) &= \sigma(\mathsf{arrtofun}(\mathsf{idtoiso}(\mathsf{rezk}(\mathsf{dua}(\delta'\sigma))^{-1})))\\
&= \sigma(\mathsf{arrtofun}(\mathsf{idtoiso}(\mathsf{rezk}(\mathsf{dua}(\delta'\sigma))))^{-1})\\
&= \sigma(\delta'\sigma)^{-1}\\
&= \sigma(\delta\sigma')\\
&= \mathsf{id}_A\sigma'\\
&= \sigma'
\end{aligned}$$

where again $(\delta'\sigma)^{-1}$ denotes an inverse for $\delta'\sigma$. Since $\delta\sigma'$ is also an inverse, the two are equal. ∎

Remark 2.5.9. In the situation of Theorem 2.5.8, if a type A is B-small then $\mathsf{issmall}^E_B(A)$ is contractible. This entails the existence of a center of contraction (b_0,σ_0), where b_0 is a term of type B and σ_0 is an equivalence between $E(b_0)$ and A. Hence, A is uniquely related to a term in B.

Therefore, given a univalent covariant family of types $E : B \to \mathcal{U}$, one can think of B as being the ∞-category of “small spaces”, or at least a full subcategory of it. However, note that we do not assume that B is closed under limits. The next proposition shows that as long as a dependent product of small types is itself small, it corresponds to a limit in B.

Proposition 2.5.10. Let B a Rezk type, a function $f : D \to B$ and assume that $E : B \to \mathcal{U}$ is a univalent covariant family such that $\prod_{d:D} E(f(d))$ is B-small. If (b_0,σ_0) is the center of contraction of $\mathsf{issmall}^E_B(\prod_{d:D} E(f(d)))$, then b_0 is the limit for the function f.

Proof. Indeed, using Theorem 2.3.7 it is enough to show that for all $b : B$

$$\mathsf{Hom}_B(b,b_0) \simeq \mathsf{Hom}_{B^D}(\triangle b, f).$$

For the right-hand side, we have

$$\mathsf{Hom}_{B^D}(\triangle b, f) \simeq \left(\prod_{d:D}\mathsf{Hom}_B(b,f(d))\right) \simeq \left(\prod_{d:D} E(b) \to E(f(d))\right)$$

The last equivalence follows because E is univalent. For the same reason together with the assumption $\sigma_0 : E(b_0) \simeq \prod_{d:D} E(f(d))$ the left-hand side is equivalent to

$$\big(E(b) \to E(b_0)\big) \simeq \left(E(b) \to \prod_{d:D} E(f(d))\right).$$

Moreover, we have the equivalence

$$\left(E(b) \to \prod_{d:D} E(f(d))\right) \simeq \left(\prod_{d:D} E(b) \to E(f(d))\right).$$

∎

For the rest of the section we want to show that the condition over the dependent product of the fiber being B-small in Theorem 2.5.10 is necessary for the existence of the limit.

Observation 2.5.11. Suppose that $D : B \to \mathcal{U}$ is a constant covariant family at $D : \mathcal{U}$ over a Segal type B. Take any $f : \mathsf{Hom}_B(x, y)$ and $w : D$, then the type

$$\sum_{v:D} \mathsf{Hom}_{D(f)}(w, v)$$

has center of contraction $(f_* w, \mathsf{trans}_{f,w})$. The point $(w, \lambda(t : \mathbb{2}).w)$ belongs to the aforementioned type, which therefore must be equal to the center of contraction. In particular, this implies that $f_* w = w$.

This observation has the following consequence:

Lemma 2.5.12. Let $B : \mathcal{U}$ a Segal type and $E : B \to \mathcal{U}$ a covariant univalent family over B. If there is $u : E(b_0)$ for some $b_0 : B$, then for all $b_1 : B$

$$E(b_1) \simeq \prod_{b:B} \mathsf{Hom}_B(b, b_1).$$

Proof. The covariant family $E : B \to \mathcal{U}$ is univalent, so instead we may prove that for all $b_1 : B$

$$E(b_1) \simeq \left(\prod_{b:B} E(b) \to E(b_1)\right).$$

Let $b_1 : B$ and define

$$F : E(b_1) \to \left(\prod_{b:B} E(b) \to E(b_1)\right),$$

where $F(v) :\equiv F_v$ is the dependent function $\lambda(b : B).\lambda(w : E(b)).v$. The function in the other direction

$$G : \left(\prod_{b:B} E(b) \to E(b_1)\right) \to E(b_1)$$

for each $\phi : \left(\prod_{b:B} E(b) \to E(b_1)\right)$ is defined by $G(\phi) :\equiv \phi_{b_0}(u)$. It is immediate that

$$G(F(v)) \equiv G(F_v) \equiv (F_v)_{b_0}(u) \equiv v,$$

so the composition is the identity. For any $\phi : \left(\prod_{b:B} E(b) \to E(b_1)\right)$ we have: $F(G(\phi)) \equiv F(\phi_{b_0}(u)) \equiv F_{\phi_{b_0}(u)}$. We need to show that this dependent function is equal to ϕ. Consider the family

$$D : B \to \mathcal{U}$$

$D(b) :\equiv E(b_1)$; this defines a covariant family. For any $b : B$ define $f : E(b) \to E(b_0)$ as $f(w) :\equiv u$. Since the family E is univalent, $\mathsf{dua}(f) : \mathsf{Hom}_B(b, b_0)$ and

$$\mathsf{dua}(f)_* \equiv \mathsf{arrtofun}(\mathsf{dua}(f)) = f \tag{2.5.1}$$

where the definitional equality follows by definition of $\mathsf{arrtofun}$.

By naturality, as in Theorem 2.2.20, applied to the function $\phi : \prod_{b:B} E(b) \to D(b)$ and the arrow $\mathsf{dua}(f) : \mathsf{Hom}_B(b, b_0)$, we have the following commutative square

$$\begin{tikzcd}
E(b) \arrow[r, "\mathsf{dua}(f)_*"] \arrow[d, "\phi_b"'] & E(b_0) \arrow[d, "\phi_{b_o}"] \\
D(b) \arrow[r, "\mathsf{dua}(f)_*"'] & D(b_0)
\end{tikzcd}$$

where the horizontal functions are given by the specified type families. We now evaluate at any $w : E(b)$. Since D is constant over B, from Theorem 2.5.11 we get the first equality below, the second follows by naturality, and the third equality comes from (2.5.1),

$$\begin{aligned}
\phi_b(w) &= \mathsf{dua}(f)_*(\phi_b(w)) \\
&= \phi_{b_0}(\mathsf{dua}(f)_*(w)) \\
&= \phi_{b_0}(f(w)) \\
&= \phi_{b_0}(u) \\
&\equiv (F_{\phi_{b_0}(u)})_b(w),
\end{aligned}$$

while the second-to-last and last equality follow by definition of f and F, respectively. This establishes the equality $\phi = F_{\phi_{b_0}(u)}$. Therefore, FG is the identity on $\prod_{b:B}(E(b) \to E(b_1))$, hence the equivalence. ∎

Finally, we have:

Proposition 2.5.13. Let B a Rezk type and a function $f : D \to B$ with limit (b_1, α). Assume that $E : B \to \mathcal{U}$ is a univalent covariant family such that for some $b_0 : B$ there is $u : E(b_0)$. Then $E(b_1) \simeq \prod_{d:D} E(f(d))$.

This just means that as long as the family $E : B \to \mathcal{U}$ has at least one fiber that is inhabited, then the sufficient condition of Theorem 2.5.10 is also necessary for the existence of a limit.

Proof. From Theorem 2.5.12 we have

$$E(b_1) \simeq \prod_{b:B} \mathsf{Hom}_B(b, b_1),$$

since b_1 is the limit:

$$\mathsf{Hom}_B(b, b_1) \simeq \prod_{d:D} \mathsf{Hom}_B(b, f(d)).$$

Combining this equivalence, and using Theorem 2.5.12 again gives us:

$$\begin{aligned} E(b_1) &\simeq \prod_{b:B} \prod_{d:D} \mathsf{Hom}_B(b, f(d)) \\ &\simeq \prod_{d:D} \prod_{b:B} \mathsf{Hom}_B(b, f(d)) \\ &\simeq \prod_{d:D} E(f(d)). \end{aligned}$$

∎

Remark 2.5.14. In a recent paper [25], the authors construct the $(\infty, 1)$-category of ∞-groupoids. They achieve this by enhancing sHoTT with modalities. The type $\mathcal{S}$ they construct satisfies exactly Theorem 2.5.1. Our results in this section simply say that a diagram with values in $\mathcal{S}$ has a limit in $\mathcal{S}$. This limit is given by the dependent product. We could rewrite our results by replacing our type B by $\mathcal{S}$, and each type $E(b)$ for $b : B$ by an element of $\mathcal{S}$. However, we keep our presentation as it is since we think it showcases a "resizing" technique, by means of Theorem 2.5.3, that could be useful in other contexts.

2.6 The bisimplicial sets semantics of sHoTT

In this section, we verify that our synthetic definitions of limit a colimit are semantically correct. We use the fact the semantics of sHoTT is the category of bisimplicial sets **ssSet**.

In [61, Appendix A] it is shown that sHoTT can be interpreted in the category **ssSet**. Let us first recall the following result from [65]:

Theorem 2.6.1. For any elegant Reedy category $\mathcal{C}$, the Reedy model structure on $\mathbf{sSet}^{\mathcal{C}^{op}}$ supports a model of intentional type theory with dependent sums and products, identity types, and as many univalent universes as there are inaccessible cardinals greater than $|\mathcal{C}|$, where $|\mathcal{C}|$ is the cardinal of the set of objects of the category $\mathcal{C}$.

Let us first review how to interpret some usual types. The one point type is the easiest; it agrees with the terminal object in our model. According to [65] we can take a category $\mathcal{M}$ that is locally cartesian closed and has a model structure that is simplicial, right proper, and the cofibrations are the monomorphisms; this is the situation of Theorem 2.6.1. If we have a map $f : A \to B$ then the pullback functor $f^* : \mathcal{M}/B \to \mathcal{M}/A$ has both a right and a left adjoint, which are denoted by $\prod_f$ and $\sum_f$, respectively. Since fibrations are closed under compositions, the functor $\sum_f$ preserves fibrations if f is itself a fibration. The functor f^* preserves cofibrations since these are exactly the monomorphisms, thus $\prod_f$ preserving acyclic fibrations. Furthermore, one can also see that in combination with right properness $\prod_f$ preserves fibrations, see [3, Theorem 26]. This is how dependent sums and products are interpreted.

The Reedy model structure on bisimplicial sets gives rise to a **comprehension category with shapes** in which the judgment $\Gamma \vdash A\,\mathsf{Type}$ is directly interpreted as a fibration $\Gamma \to A$. Here we are overlooking the fact that our types also may depend on topes and shapes. To see more details and the correct definitions, we refer the reader to our cited main reference [61]. The technical result is:

Theorem 2.6.2. ([61, Theorem A.18]) The comprehension category constructed from any model category with **$\mathfrak{T}$-shapes** has **pseudo-stable coherent tope logic** with type eliminations for tope disjunction and also **pseudo-stable extension types** satisfying relative function extensionality.

[61] uses the result on $\mathfrak{T}$ the coherent theory of the strict interval of sHoTT presented in Section 2.2.1 or [61, Section 3.1]. Furthermore, Weinberger shows in [73] that the semantical interpretation extension types is strictly stable under pullbacks and not only pseudo-stable as in the theorem above. Moreover, [61, A.3] proves:

Theorem 2.6.3. Segal types correspond to Segal spaces in **ssSet** and Rezk types to Rezk spaces (a.k.a. complete Segal spaces).

Remark 2.6.4. In Theorem 2.2.13 we defined discrete types. However, the terminology might be confusing. According to [61, Proposition 10.10] a type A is discrete if and only if A is Rezk and all its arrows are isomorphisms. On the other hand, [59, Corollary 6.6] gives a similar characterization for complete Segal spaces, such simplicial spaces are *constant*. A simplicial space X is **constant** if for any map $[n] \to [m]$, the induced map $X_m \to X_n$ is an equivalence of spaces. In particular, for all n we have $X_n = X_0$. Hence, discrete types correspond to constant simplicial spaces. As a consequence, we interpret discrete types into the bisimplicial sets model as constant simplicial spaces.

Remark 2.6.5. The interpretation of sHoTT can be done in greater generality. This is essentially a consequence of the main result in [66], where it is shown that any Grothendieck $(\infty, 1)$-topos gives a model for homotopy type theory. If $\mathcal{C}$ is a *type-theoretic model topos* then one can consider simplicial objects internal to $\mathcal{E}$. The resulting category of internal presheaves gives us again a model for sHoTT. For more details we refer to [73, Section 2].

Now we have the semantical overview of sHoTT. This gives us the opportunity to say a few words on the necessity of contrasting the definitions we make in the type theory with the existing ones in the bisimplicial sets model.

Remark 2.6.6. As we mentioned above, sHoTT can be interpreted in the category of bisimplicial sets. However, note that a type theoretic statement translates internally to the bisimplicial sets model. A concrete example of this phenomenon is the characterization of Segal types. In general, each type is interpreted as a Reedy fibrant bisimplicial set. On the one hand, the equivalence $A^{\Delta^2} \to A^{\kappa_1^2}$ from Theorem 2.2.9 is interpreted as a trivial fibration in the Reedy model structure on **ssSet**. On the other hand, Segal spaces are defined by the Segal condition: a bisimplicial set A is a Segal space if, for all $n \geqslant 2$, the map $A_n \to A_1 \times_{A_0} \cdots \times_{A_0} A_1$ is a trivial fibration in the Kan-Quillen model structure on **sSet**. [61, A.3] shows that both conditions are equivalent.

Informally speaking, an internal statement in the model could be stronger under interpretation in the intended semantics. The results we present in the following sections show that this is not the case, *i.e.,* the internal and external notions coincide.

2.6.1 Limits and colimits

We verify the consistency of our definitions of synthetic limits and colimits introduced in Theorem 2.3.4. The notion of limits and colimits in Segal spaces we are considering are the ones presented in [57]. The goal of this section is prove the next result:

Theorem 2.6.7. The definitions of limits and colimits from Theorem 2.3.4, under interpretation in the category of bisimplicial sets, coincide with the definitions of limits and colimits from [57].

Firstly, let us recall some standard notation and terminology we will use:

- Δ will denote the category whose objects are the non-empty linearly ordered finite sets, $[n] :\equiv \{0 \leqslant 1 \leqslant \cdots \leqslant n\}$ with $n \in \mathbb{N}$, and morphisms the order-preserving maps between linearly ordered sets.
- The representable presheaf $\Delta^{op} \to \mathbf{Set}$ given by $[n] \in \Delta$ is noted by $\Delta[n]$.
- For each $n \in \mathbb{N}$, $F(n) : \Delta^{op} \times \Delta^{op} \to \mathbf{Set}$ is defined as

  $$F(n)_{k,l} :\equiv \Delta([k],[n]).$$

  This just means we are looking at the constant bisimplicial set $\Delta([k],[n])$. Two particular examples are $F(1)$ and $F(0)$, the last of which is the terminal object.
- The category of simplicial sets is cartesian closed: for $X, Y \in \mathbf{sSet}$, the **mapping simplicial set** is defined as

  $$Map(X,Y)_n :\equiv \mathbf{sSet}(X \times \Delta[n], Y).$$
- The category of bisimplicial sets is cartesian closed: for $X, Y \in \mathbf{ssSet}$, the **mapping simplicial set** is defined as

  $$(Y^X)_{kl} :\equiv \mathbf{ssSet}(F(k) \times \Delta[l] \times X, Y).$$

- Decorated arrows " $\twoheadrightarrow$" indicate fibrations in the model structure. We will use this notation for the different models involved, but there is no room for confusion as the type of fibration is clear from the context.

Using Theorem 2.6.2, it is possible to conclude that simplicial type theory can be interpreted in bisimplicial sets. In particular, Hom types do coincide with *hom* spaces. From [59]; if T is a Segal space, then the mapping space between two objects $a, b \in T_0$ is constructed via the pullback square of spaces:

$$\begin{tikzcd}
map_T(a,b) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & T_1 \arrow[d, two heads] \\
\Delta[0] \arrow[r, "(a,b)"'] & T_0 \times T_0.
\end{tikzcd} \tag{2.6.1}$$

On the other hand, from Theorem 2.2.3 we get

$$\Delta^1 \to A \simeq \sum_{k:\partial\Delta^1 \to A} \langle \Delta^1 \to A|^{\partial\Delta^1}_{k} \rangle,$$

and it is immediate that the type of functions $\partial\Delta^1 \to A$ is equivalent to $A \times A$. Therefore, if A is a Segal type, then A^{Δ^1} is the total space of a family over $A \times A$ whose fibers are Hom types. This family is exactly $\mathsf{Hom}(-,-) : A \times A \to \mathcal{U}$. Under the standard interpretation, Δ^1 is $F(1)$ and $\mathbf{1}$ is $F(0)$. Given $a, b : A$ we obtain $\mathsf{Hom}_A(a,b)$ as the substitution along $(a,b) : \mathbf{1} \to A \times A$ into $\mathsf{Hom}_A(-,-)$. When we interpret our type theory into simplicial spaces, this gives us the pullback square of bisimplicial sets

$$\begin{tikzcd}
hom_A(a,b) \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & A^{F(1)} \arrow[d, two heads] \\
F(0) \arrow[r, "(a,b)"'] & A \times A.
\end{tikzcd}$$

We obtain the following comparison result:

Proposition 2.6.8. Let A be a Segal type and $a, b : A$. The interpretation of $\mathsf{Hom}_A(a,b)$ into bisimplicial sets is a constant simplicial space with value the Kan complex $map_A(a,b)$.

Proof. Note that $hom_A(a,b)_0 = map_A(a,b)$. Furthermore, from [61, Proposition 8.29] $\mathsf{Hom}_A(-,-) : A \to A \to \mathcal{U}$ is a two-sided discrete fibration, so it is $A^{F(1)} \twoheadrightarrow A \times A$. Hence, since each fiber $\mathsf{Hom}_A(a,b)$ is discrete, it follows from Theorem 2.6.4 that $hom_A(a,b)$ is a constant simplicial space at $map_A(a,b)$. ∎

The definition of a colimit from [57] is formulated as follows: let $f : K \to A$ be a map of bisimplicial sets where A is a Segal space. The **space of cocones under** K is the Segal space

$$A_{f/} :\equiv F(0) \times_{A^K} (A^K)^{F(1)} \times_{A^K} A.$$

If $a \in A$ then we can construct the space of objects under a by taking the map $a : F(0) \to A$ and applying the previous definition.

The map $f : K \to A$ has a colimit if the Segal space $A_{f/}$ has an initial object. This is to say that there exists an object $\sigma \in A_{f/}$ such that the map $(A_{f/})_{\sigma/} \to A_{f/}$ is a trivial Reedy fibration. Let $a \in A$ be an initial object. Therefore, for any object $x \in A$ we have the pullback:

$$\begin{tikzcd}
hom_A(a,x) \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & A_{a/} \arrow[d, two heads] \\
F(0) \arrow[r, "x"] & A.
\end{tikzcd}$$

The assumption that the map on the right is also an equivalence implies that $hom_A(a,x)$ is equivalent to $F(0)$. This is to say that $hom_A(a,x)$ is contractible, and it means that up to homotopy there is a unique map from a to x. Let us see first why this prospect of initiality matches with Theorem 2.2.25. In Theorem 2.6.8 we showed that $hom_A(a,b)$ is a bisimplicial set constant at $map_A(a,b)$. It remains to show that equivalences are translated correctly. This follows immediately from [65, Lemma 4.3]:

Lemma 2.6.9. For a map f between fibrations $p_1 : E_1 \to B$ and $p_2 : E_2 \to B$ the following are equivalent:

1. f is a weak equivalence.
2. $\mathsf{IsEquiv}_B(f) \to B$ has a section.
3. There is a map $\mathbf{1} \to \prod_B \mathsf{IsEquiv}_B(f)$.
4. $\mathsf{IsEquiv}_B(f) \to B$ is an acyclic fibration.

Theorem 4.4.3 and Theorem 4.2.6 of [70] imply that a function being an equivalence is equivalent to having contractible fibers. The object $\mathsf{IsEquiv}_B(f)$ encodes this last fact, as is shown throughout [65, Section 4]. Therefore, if a type is contractible, then under the standard interpretation it is also contractible in the model. We can conclude that our notion Theorem 2.2.25 of initiality coincides with the one in the semantics. All there is left to do is to compare the cocones with Theorem 2.3.2. It is useful to see that the space $A_{f/}$ is constructed with the successive pullbacks

$$\begin{tikzcd}
F(0) \times_{A^K} (A^K)^{F(1)} \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & (A^K)^{F(1)} \arrow[d, two heads] \\
F(0) \arrow[r, "f"'] & A^K
\end{tikzcd}
\qquad
\begin{tikzcd}
A_{f/} \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & F(0) \times_{A^K} (A^K)^{F(1)} \arrow[d, two heads] \\
A \arrow[r, "\triangle"'] & A^K.
\end{tikzcd}$$

Similar to the case with Hom types: $(A^K)^{\Delta^1}$ is the total space of the type family $\mathsf{Hom}_{A^K}(-,-)$ over $A^K \times A^K$. Then if we take $f : K \to A$ and any $x : A$ the type $\mathsf{Hom}_{A^K}(f, \triangle x)$ is the substitution of $\mathsf{Hom}_{A^K}(-,-)$ along $(f, \triangle) : A \to A^K \times A^K$. Thus, this is the pullback in the semantics:

$$\begin{tikzcd}
A_{f/} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & (A^X)^{F(1)} \arrow[d] \\
A \times F(0) \arrow[r, "\triangle \times f"] & A^X \times A^X.
\end{tikzcd}$$

Chapter 3

Exponentiable functors between synthetic ∞-categories

Once more, within the framework of simplicial homotopy type theory, we study exponentiable functors, a.k.a. Conduché fibrations between synthetic ∞-categories. The content of this chapter consists of the paper [6] as submitted to Mathematical Structures in Computer Science.

3.1 Introduction

3.1.1 Synthetic ∞-category theory

A proposal for a synthetic theory $(\infty, 1)$-categories using homotopy type theory appears in the seminal work of Riehl and Shulman [61], called simplicial homotopy type theory, or sHoTT for short. They define *Segal* and *Rezk types*, which play the role of $(\infty, 1)$-precategories and $(\infty, 1)$-categories. The paper develops categorical properties of said types and also studies discrete fibrations and adjunctions. Further work [16] and [5] present (co)cartesian fibration and (co)limits, respectively.

The standard semantics of sHoTT is the category of bisimplicial sets **ssSet** with the Reedy model structure. [61] shows that Segal types correspond to Segal spaces and Rezk types to complete Segal spaces. Furthermore, the main result in [66] implies that if $\mathcal{E}$ is a Grothendieck $(\infty, 1)$-topos, which is in particular a model of homotopy type theory, then we can produce a model of sHoTT in the (internal) category of simplicial presheaves $\mathcal{E}^{\Delta^{\mathrm{op}}}$ [73].

This is the general framework of synthetic category theory in which our work takes place. We study an important class of functors, the *exponentiable* ones, which we introduce shortly after. This is also a continuation of [5], which started as an exploration on how far we can go in the development of synthetic $(\infty, 1)$-category theory without enhancing sHoTT.

3.1.2 Exponentiable functors

An exponential object, or more generally, exponentiable map, can be defined in multiple ways. If $\mathcal{C}$ is a category with binary products, we say a map $f : E \to B$ in $\mathcal{C}$ is exponentiable if the pullbacks along f exist and the functor $f^* : \mathcal{C}/B \to \mathcal{C}/E$ has a right adjoint $\prod_f$, so it induces an adjoint triple

$$\begin{tikzcd}
\mathcal{C}/B \arrow[r, "f^*" description] & \mathcal{C}/E \arrow[l, bend right=50, "\Sigma_f" description] \arrow[l, bend left=50, "\Pi_f" description] \arrow[l, phantom, bend right=25, "\perp"] \arrow[l, phantom, bend left=25, "\perp"]
\end{tikzcd}$$

where $\sum_f$ is given by composition with f. More generally, a **locally cartesian closed category** is a category in which every map is exponentiable. Exponentiable maps in the category of small categories **Cat** are also known as Conduché fibrations. The literature on the topic is extensive, and they appear, for example, in [20]. In the context of ∞-categories exponentiable functors have been studied in [4, Lemma 5.16] and [50, Appendix B.3]. This result is in the same spirit as Theorem 3.1.1 below.

Let us recall the case for categories, for which we first introduce some notation. Given $f : \mathcal{E} \to \mathcal{B}$ a functor and $a \in \mathcal{B}$, we denote its fiber as $\mathcal{E}_a$. The category $\mathcal{E}_a$ has objects $e \in \mathcal{E}$ and morphisms $k : e_1 \to e_2 \in \mathcal{E}$, such that $f(e) = a$ and $f(k) = Id_a$. If $u : a \to b$ in $\mathcal{B}$ and $x \in \mathcal{E}_a$, $y \in \mathcal{E}_b$, then the set of arrows in $\mathcal{E}$ over u with source x and target y is denoted as $hom^u_{\mathcal{E}}(x, y)$; if $j \in hom^u_{\mathcal{E}}(x, y)$, then $f(j) = u$. Therefore, we can define a profunctor $hom_{\mathcal{E}} : \mathcal{E}_a \times \mathcal{E}_b^{op} \to \mathbf{Set}$ in the obvious way. The following statement characterizes exponentiable functors between categories and it is due to Conduché [20] and Giraud [24].

Theorem 3.1.1. For a functor $f : \mathcal{E} \to \mathcal{B}$, the following conditions are equivalent:

1. The functor $f : \mathcal{E} \to \mathcal{B}$ is exponentiable.
2. For all a, b, $c \in \mathcal{B}$, $u \in hom_{\mathcal{B}}(a, b)$, $v \in hom_{\mathcal{B}}(b, c)$, $x \in \mathcal{E}_a$, $z \in \mathcal{E}_c$, the induced map
$$\left(\int^{y \in \mathcal{E}_b} hom^u_{\mathcal{E}}(x, y) \times hom^v_{\mathcal{E}}(y, z) \right) \to hom^{v \circ u}_{\mathcal{E}}(x, z)$$
is an isomorphism.

The result we prove in Theorem 3.3.2 can be seen as analogous to the previous statement while partially recovering the result in [4] which is the ∞-categorical statement of Theorem 3.1.1. Partial results about exponentiable functors also appear in [50, Appendix B.3]. Condition 2 in Theorem 3.1.1 states that the composition of the *hom*-profunctors is given by the coend formula where the map is induced by composition. This is also what [4] proves in their result, which they do using the language of *correspondences*. In Section 3.4.2.1 we explain how this is reflected in our Theorem 3.3.2.

3.1.3 Outline

To define *exponentiable functors*, throughout Section 3.2, we study *Segal type completions*. This notion is essential to correctly formulate Condition 5 in Theorem 3.3.2. This is exactly what we should think of as the composition of profunctors. In Section 3.3 we present Theorem 3.3.2, which is the characterization of *exponentiable functors* between Segal types. We then specialize this result to Rezk types in Theorem 3.3.6.

Finally, in Section 3.4 we verify that our definitions are consistent with the semantics: we first interpret our type-theoretic definition in the standard semantics, bisimplicial sets, and then verify that the resulting statement is equivalent to the existing definition. We do this for the Segal type completion in Section 3.4.1. Finally, in Section 3.4.2 we verify that our definition of exponentiable functor is semantically sound.

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3.2 The Segal type completion

The notion we study in this section is essential to correctly formulate Condition 5 in Theorem 3.3.2. We start by establishing some basic definitions and mentioning some properties.

Definition 3.2.1. A **Segal type completion** for a type A consists of a Segal type S and a map $\iota : A \to S$ such that for any Segal type X the map

$$\iota^* :\equiv \_ \circ \iota : (S \to X) \to (A \to X) \tag{3.2.1}$$

is an equivalence *i.e.,* if

$$\mathsf{isCompletion}^A(S, \iota) :\equiv \mathsf{isSegal}(S) \times \left(\prod_{X:\mathcal{U}} \mathsf{isSegal}(X) \to \mathsf{IsEquiv}(\iota^*) \right).$$

It is immediate to see that whenever (S, ι) exists, it is unique up to equivalence.

Definition 3.2.2. Let A a type. We define

$$\mathsf{Completion}(A) :\equiv \sum_{S:\mathcal{U}} \sum_{\iota:A\to S} \mathsf{isCompletion}^A(S, \iota).$$

The following result is a direct consequence of the Univalence Axiom:

$$\mathsf{isProp}(\mathsf{Completion}(A)).$$

We call the Segal type in this proposition the **Segal type completion** of the type A. In the Segal space model structure on **ssSet** this corresponds to the fibrant replacement of a Reedy fibrant bisimplicial set by a Segal space, see Theorem 3.4.5. We can define in the same way the **Rezk type completion** of a type. The results that we get for Segal type completions are also true for Rezk type completions. We formally state the relative version of Rezk completions in Theorem 3.2.4.

It is more convenient to work with Segal type completions at an informal level, a trade-off between clarity and formality. We favour this approach as it is more intuitive and it reflects the nature of the semantical counterpart of these types. The equivalence (3.2.1) tells us that the fibers are contractible; for any $\psi : A \to X$ we have

$$\mathsf{isContr}\left(\sum_{\varphi : S \to X} \varphi \circ \iota = \psi\right).$$

Unfolding the above means that for any $\psi : A \to X$, where X is a Segal type, there exists a unique $\varphi : S \to X$ such that $\varphi \circ \iota = \psi$. We can put this pictorially by saying that any $\psi : A \to X$ uniquely factors through ι as in the diagram below:

$$\begin{tikzcd}
A \arrow[r, "\iota"] \arrow[dr, "\psi"'] & S \arrow[d, dashed, "\varphi"] \\
& X.
\end{tikzcd}$$

We will often refer only to the Segal space S and assume that the map $\iota : A \to S$ is given and available for use. By uniqueness in the above, we simply mean that the type

$$\mathsf{isContr}\left(\sum_{\varphi : S \to X} \psi \sim \varphi \circ \iota\right)$$

is inhabited. The homotopy in the centre of contraction is omitted most of the time, and we only make reference to the function. We carry this convention forward below.

We can also consider a relative version of this universal property. For the time being, fix a Segal type B. We use the following notation:

$$\mathcal{U}/B :\equiv \sum_{S:\mathcal{U}} S \to B.$$

Furthermore, we will refer to an element (S, ϕ) of this type by leaving implicit the type S and mentioning that we have a map of type $S \to B$.

Recall from [16] that the **relative function type** between functions $\pi : A \to B$ and $\xi : E \to B$ is given by the pullback diagram:

$$\begin{tikzcd}
\mathsf{Fun}_{/B}(A, E) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E^A \arrow[d, "\xi^A"] \\
\mathbf{1} \arrow[r, "\pi"'] & B^A.
\end{tikzcd}$$

Note that if we assume further that E is a Segal type and A is a type or shape, then $\mathsf{Fun}_{/B}(A, E)$ is a Segal type. An element $\iota : \mathsf{Fun}_{/B}(A, E)$ is a function making the following diagram commute:

$$\begin{tikzcd}
A \arrow[r, "\iota"] \arrow[dr, "\pi"'] & E \arrow[d, "\xi"] \\
& B.
\end{tikzcd}$$

Thus, we call the elements of $\mathsf{Fun}_{/B}(A, E)$ **functions over** B.

Definition 3.2.3. Let $A \to B$ a type over B. A **relative Segal type completion** for $A \to B$ consists of a Segal type over B, $S \to B$, and a map $\iota : \mathsf{Fun}_{/B}(A, S)$ such that for any Segal type X over B the map

$$\iota^*_{/B} : \mathsf{Fun}_{/B}(S, X) \to \mathsf{Fun}_{/B}(A, X) \tag{3.2.2}$$

is an equivalence, *i.e.,* if

$$\mathsf{isCompletion}^A_{/B}(S, \iota) :\equiv \mathsf{isSegal}(S) \times \left(\prod_{X:\mathcal{U}/B} \mathsf{isSegal}(X) \to \mathsf{IsEquiv}(\iota^*_{/B}) \right).$$

It is immediate to see that whenever $S \to B$ exists then it is unique up to equivalence. Following Theorem 3.2.3 we state:

Definition 3.2.4. Let $A \to B$ a type over B. A **relative Rezk type completion** for $A \to B$ consist of a Rezk type over B, $R \to B$ and a map $\iota : \mathsf{Fun}_{/B}(A, R)$ such that for any Rezk type X over B the map

$$\iota^*_{/B} : \mathsf{Fun}_{/B}(R, X) \to \mathsf{Fun}_{/B}(A, X) \tag{3.2.3}$$

is an equivalence *i.e.,* if

$$\mathsf{isRCompletion}^A_{/B}(S, \iota) :\equiv \mathsf{isRezk}(R) \times \left(\prod_{X:\mathcal{U}/B} \mathsf{isRezk}(X) \to \mathsf{IsEquiv}(\iota^*_{/B}) \right).$$

It is again immediate to see that whenever $R \to B$ exists, it is unique up to equivalence.

Definition 3.2.5. Let $A \to B$ a type over B. We define

$$\mathsf{Completion}_{/B}(A) :\equiv \sum_{S:\mathcal{U}/B} \sum_{\iota:\mathsf{Fun}_{/B}(A,S)} \mathsf{isCompletion}^A_{/B}(S, \iota).$$

We will say often that S is a Segal type completion relative to the type B leaving implicit the map $\xi : S \to B$, and that $\iota : \mathsf{Fun}_{/B}(A, S)$ exists. The equivalence (3.2.3) tells us that the fibers are contractible: for any $\psi : \mathsf{Fun}_{/B}(A, X)$ we have

$$\mathsf{isContr}\left(\sum_{\varphi : \mathsf{Fun}_{/B}(S,X)} \varphi \circ \iota = \psi \right).$$

Just as we did before, unfolding the above means that for any $\psi : \mathsf{Fun}_{/B}(A, X)$, where $\delta : X \to B$ is a map between Segal types, there exists a unique $\varphi : \mathsf{Fun}_{/B}(S, X)$ such that $\varphi \circ \iota = \psi$. The picture for this situation is the commutative diagram below:

$$\begin{tikzcd}
A \arrow[rr, "\psi"] \arrow[dr, "\iota"] \arrow[ddr, "\pi"'] & & X \arrow[ddl, "\delta"] \\
& S \arrow[ur, dashed, "\varphi"'] \arrow[d, dashed, "\xi"] & \\
& B. &
\end{tikzcd}$$

For the next section it will be useful to know that for a type A its associated Segal type completion S is also universal relative to any Segal type B, and vice versa. Informally, the categorical interpretation we give to this is that having the Segal type completion over the single point Segal space is equivalent to having it over any slice (by a Segal space). This is the content of:

Proposition 3.2.6. Let A be any type and B a Segal type. Assume further we have a commutative diagram

$$\begin{tikzcd}
A \arrow[rr, "\iota"] \arrow[dr, "\pi"'] & & S \arrow[dl, "\xi"] \\
& B. &
\end{tikzcd}$$

where S is a Segal type. Then:

$$\mathsf{isCompletion}(S, \iota) \simeq \mathsf{isCompletion}^{A}_{/B}(S, \iota).$$

Proof. Since S is a Segal type it is enough to show that

$$\left(\prod_{X : \mathcal{U}} \mathsf{isSegal}(X) \to \mathsf{IsEquiv}(\iota^*) \right) \simeq \left(\prod_{X : \mathcal{U}/B} \mathsf{isSegal}(X) \to \mathsf{IsEquiv}(\iota^*_{/B}) \right).$$

It is possible to construct a function

$$\varphi : \left(\prod_{X : \mathcal{U}} \mathsf{isSegal}(X) \to \mathsf{IsEquiv}(\iota^*) \right) \to \left(\prod_{X : \mathcal{U}/B} \mathsf{isSegal}(X) \to \mathsf{IsEquiv}(\iota^*_{/B}) \right)$$

after some preliminary observations we now make.

Let $\delta : X \to B$, where X a Segal type, and $j : \mathsf{Fun}_{/B}(A, X)$. We wish to show that

$$\mathsf{isContr}\left(\mathsf{fib}_{\iota^*_{/B}}(j)\right).$$

By assumption we have the equivalence $\iota^* : X^S \to X^A$, this gives us unique functions $h : S \to B$ and $g : S \to X$ which makes the following diagrams commutative:

$$\begin{tikzcd}
A \arrow[r, "\iota"] \arrow[dr, "j"'] & S \arrow[d, dashed, "g"] & & A \arrow[r, "\iota"] \arrow[dr, "\pi"'] & S \arrow[d, dashed, "h"] \\
& X & & & B.
\end{tikzcd}$$

Since $j : \mathsf{Fun}_{/B}(A, X)$ then, by uniqueness of $h : S \to B$, we must have $h = \delta \circ g$. It is now clear that g must be unique with this property. Thus, we have the following diagram:

$$\begin{tikzcd}
A \arrow[rr, "\iota"] \arrow[dr, "j"] \arrow[ddr, "\pi"'] & & S \arrow[dl, dashed, "g"'] \arrow[ddl, "h"] \\
& X \arrow[d, "\delta" description] & \\
& B. &
\end{tikzcd}$$

If we were to unfold $\mathsf{IsEquiv}(\iota^*_B)$ and $\mathsf{IsEquiv}(\iota^*_{/B})$, we would be able to give an explicit formula for φ, then the above shows that φ is well-defined. We will not do this since it does not give any more clarity on the result.

Likewise, it is possible to construct

$$\psi : \left(\prod_{X:\mathcal{U}/B} \mathsf{isSegal}(X) \to \mathsf{IsEquiv}(\iota^*_{/B})\right) \to \left(\prod_{X:\mathcal{U}} \mathsf{isSegal}(X) \to \mathsf{IsEquiv}(\iota^*)\right).$$

Let $j : A \to X$ be a function where X is a Segal type. Now we want to observe that

$$\mathsf{isContr}\left(\mathsf{fib}_{i*}(j)\right).$$

Consider $j \times \mathrm{Id}_B : A \times B \to X \times B$. We construct the commutative diagram:

$$\begin{tikzcd}
A \arrow[r, "(\mathrm{Id}_A, \pi)"] \arrow[dr, "\pi"'] & A \times B \arrow[r, "j \times \mathrm{Id}_B"] \arrow[d, "p_2" description] & X \times B \arrow[dl, "p_2"] \\
& B. &
\end{tikzcd}$$

This implies that $(j, \pi) : \mathsf{Fun}_{/B}(A, X \times B)$. Since S is the Segal type completion of A relative to B there exists a unique function $f : S \to X \times B$ making the following diagram commutative:

$$\begin{tikzcd}
A \arrow[rr, "{(j,\pi)}"] \arrow[dr, "\iota"] \arrow[ddr, "\pi"'] & & X \times B \arrow[ddl, "p_2"] \\
& S \arrow[ur, dashed, "f"] \arrow[d, dashed, "\xi"] & \\
& B. &
\end{tikzcd}$$

From this we obtain:

$$\begin{tikzcd}
A \arrow[r, "\iota"] \arrow[dr, "j"'] & S \arrow[d, dashed, "p_1 \circ f"] \\
& X.
\end{tikzcd}$$

To show uniqueness, if we had a map $g : S \to X$ fitting in the triangle above, then we certainly get:

$$\begin{tikzcd}
A \arrow[rr, "{(j,\pi)}"] \arrow[dr, "\iota"] \arrow[ddr, "\pi"'] & & X \times B \arrow[ddl, "p_2"] \\
& S \arrow[ur, "{(g,\xi)}"] \arrow[d, dashed, "\xi"] & \\
& B. &
\end{tikzcd}$$

Uniqueness implies that $f = (g, \xi)$, from which we finally conclude $p_1 \circ f = g$.
After obtaining φ and ψ we can then show that $\varphi \circ \psi$ and $\psi \circ \varphi$ are homotopic. This follows using the universality of each completion. ∎

3.3 Conduché's theorem

The next theorem characterizes, and at the same time, allows defining exponentiable functors between Segal types. As we mentioned in the introduction, this result is analogous to the one given by Ayala, Francis and Rozenblyum in [4, Lemma 5.16] for quasi-categories. Its proof is the main focus of this section.

Recall from [16] that a type family $Q : B \to \mathcal{U}$ is an **inner family** if

$$\mathsf{isInner}(Q) :\equiv \prod_{\alpha:\Delta^2 \to B} \prod_{\delta:\prod_{t:\kappa_1^2} Q(\alpha(i(t)))} \mathsf{isContr}\left(\left\langle \prod_{t:\Delta^2} Q(\alpha(t)) \middle|_{\delta}^{\kappa_1^2} \right\rangle\right).$$

The significance of inner families can be understood via the following result, which can be found in [16, Proposition 4.1.5 and 4.1.6]:

Proposition 3.3.1. Let $Q : B \to \mathcal{U}$ be a type family over a Segal type B. Then $\mathsf{isSegal}\left(\sum_{b:B} Q(b)\right)$ if and only if $\mathsf{isInner}(Q)$.

Let us explain the proposition above. Firstly, given a type A, we can think of it as a type family over $\mathbf{1}$, *i.e.,* $\lambda * .A : \mathbf{1} \to \mathcal{U}$. This gives us a function $\pi : A \to \mathbf{1}$. Then the proposition above says that A is a Segal type if and only if the diagram

$$\begin{tikzcd}
\Lambda_1^2 \arrow[r] \arrow[d] & A \arrow[d, "\pi"] \\
\Delta^2 \arrow[r] \arrow[ur, dashed] & \mathbf{1}
\end{tikzcd}$$

has a diagonal filler that is unique up to homotopy. Theorem 3.3.1 establishes a relative version of this condition. If $Q : B \to \mathcal{U}$ is a type family over the Segal type B and $\pi : \tilde{Q} \to B$ is the canonical projection from its total type, then Theorem 3.3.1 can be rephrased by saying that the diagram

$$\begin{tikzcd}
\Lambda_1^2 \arrow[r, "\delta"] \arrow[d] & \tilde{Q} \arrow[d, "\pi"] \\
\Delta^2 \arrow[r, "\alpha"'] \arrow[ur, dashed] & B
\end{tikzcd}$$

has a diagonal filler that is unique up to homotopy, see [16, Observation 2.4.1]. We proceed to prove the main result of this section.

Theorem 3.3.2. Let be $E : B \to \mathcal{U}$ an inner family over a Segal type B, the following are equivalent:

1. For any inner family $P : B \to \mathcal{U}$,

$$\mathsf{isSegal}\left(\sum_{b:B} (E(b) \to P(b))\right).$$

2. For any inner family $P : B \to \mathcal{U}$, the type family $Q :\equiv \lambda b.(E(b) \to P(b)) : B \to \mathcal{U}$ is inner.
3. For any inner family $P : E \to \mathcal{U}$, the type family $Q :\equiv \lambda b. \prod_{e:E(b)} P(e) : B \to \mathcal{U}$ is inner.
4. For any Segal type X, the type family $Q :\equiv \lambda b.E(b) \to X : B \to \mathcal{U}$ is inner.
5. For any map $\alpha : \Delta^2 \to B$, together with the inclusion $i : \kappa_1^2 \to \Delta^2$. Let $F_1 :\equiv \sum_{t:\Delta^2} E(\alpha(t))$ and $F_2 :\equiv \sum_{t:\kappa_1^2} E(\alpha(i(t)))$ then

$$\mathsf{isCompletion}_{/B}^{F_2}(F_1, \iota),$$

   where $\iota :\equiv \lambda(t,e).(i(t),e) \ : F_2 \to F_1$.

Before proceeding with the proof, we give some motivations for the conditions of the theorem. If a map $f : E \to B \in \mathcal{C}$ is exponentiable then we have a triple adjunction

$$\begin{tikzcd}
\mathcal{C}/B \arrow[rr, "f^*" description] & & \mathcal{C}/E. \arrow[ll, bend right=50, "\Sigma_f" description] \arrow[ll, bend left=50, "\prod_f" description] \arrow[ll, phantom, shift right=4ex, "\perp"] \arrow[ll, phantom, shift left=4ex, "\perp"]
\end{tikzcd}$$

In this situation, the internal hom in the slice $\mathcal{C}/B$ is given by

$$[f : E \to B, -]_{/B} :\equiv \prod_f \circ f^*.$$

In general, this formula is the semantic interpretation of the type family in (2) of Theorem 3.3.2, $\lambda b.E(b) \to P(b)$. In our framework of sHoTT, the extra condition of being an inner family is a natural one since maps between Segal types are equivalent to inner families (Theorem 3.3.1). Conditions (2)-(3) above simply express the fact that exponentiation happens over the point, over B or over a type dependent on B. An explanation behind (5) can be found in Section 3.4.2.1 below.

Proof. (1)⇔(2). This is immediate from [16, Proposition 4.1.5], where it is proved that the total space of a type family $P : B \to \mathcal{U}$ is a Segal type if and only if P is an inner family.
(4) ⇔ (3) ⇔ (2) is a classical result that can be found, for example, in [53].
(1)⇒(5). Consider the projections $p_2 : F_2 \to \kappa_1^2$, $p_1 : F_1 \to \Delta^2$. Denote by $q : F_1 \to \sum_{b:B} E(b)$ the map $\lambda(t, e).(\alpha(t), e)$. Let $f : (\sum_{b:B} E(b)) \to B$ the projection, thus $\alpha \circ p_1 = f \circ q$.

We prove that $f \circ q : F_1 \to B$ is the Segal type completion relative to B of $f \circ q \circ \iota : F_2 \to B$. We will constantly make use of the commutative diagram

$$\begin{tikzcd}
F_2 \arrow[r, "\iota"] \arrow[d, "p_2"'] & F_1 \arrow[r, "q"] \arrow[d, two heads, "p_1"'] & E \arrow[d, two heads, "f"] \\
\Lambda_1^2 \arrow[r, "i"'] & \Delta^2 \arrow[r, "\alpha"'] & B.
\end{tikzcd}$$

In other words, we use that $\alpha \circ p_1 = f \circ q$ and $\alpha \circ i \circ p_2 = f \circ q \circ \iota$.

If there is a map $k : X \to B$ with X a Segal type and $\psi : F_2 \to X$ over B, then we can assume we have the canonical projection $\pi : (\sum_{b:B} P(b)) \to B$, where $P : B \to \mathcal{U}$ is an inner family given by $\lambda b.\mathsf{fib}_k(b)$, *i.e.*, by taking the fiber of k over each $b : B$. This gives us the map $\psi : F_2 \to \sum_{b:B} P(b)$ such that $\alpha \circ i \circ p_2 = \pi \circ \psi$. Our first goal is to construct:

$$\nu : \prod_{t:\kappa_1^2} \left(E(\alpha(i(t))) \to P(\alpha(i(t)))\right).$$

We can assume that $\psi(t, e) \equiv (b_t, e_t) : \sum_{b:B} P(b)$. For each $(t, e) : F_2$ there is a path $p_t : \alpha(i(t)) = b_t$ given by $\alpha \circ i \circ p_2 = \pi \circ \psi$. Then we can consider the transport map

$$p_t^* : P(\alpha(i(t))) \to P(b_t)$$

together with its inverse

$$(p_t^{-1})^* : P(b_t) \to P(\alpha(i(t))).$$

Define $\nu :\equiv \lambda t.\lambda e.(p_t^{-1})^*(e_t)$. This gives us the lifting problem:

$$\begin{tikzcd}
\kappa_1^2 \arrow[r, "\nu"] \arrow[d, "i"'] & \sum_{b:B}(E(b) \to P(b)) \arrow[d, "\pi'"] \\
\Delta^2 \arrow[r, "\alpha"'] & B.
\end{tikzcd}$$

By assumption, there exists a unique

$$\xi : \left\langle \prod_{t:\Delta^2}(E(\alpha(t)) \to P(\alpha(t)))\middle|^{\kappa_1^2}_{\nu} \right\rangle.$$

The construction of

$$\varphi : \left(\sum_{t:\Delta^2} E(\alpha(t))\right) \to \left(\sum_{b:B} P(b)\right)$$

is simply given by the formula: $\lambda(t,e).(\alpha(t), \xi_t(e))$. It remains to show $\phi \circ \iota \sim \psi$. Consider $(t,e) : \sum_{t:\Delta^2} E(\alpha(t))$. Then by definition we have

$$\phi \circ \iota(t,e) \equiv (\alpha(i(t)), \xi_{i(t)}(e)) \text{ and } \psi(t,e) \equiv (b_t, e_t).$$

Using the characterization of paths in the total space we obtain $p_t : \alpha(i(t)) = b_t$, and also there is an equality

$$p_t^*(\xi_{i(t)}(e)) \equiv p_t^*(\nu_t(e)) \equiv p_t^*((p_t^{-1})^*(e_t)) = e_t.$$

Therefore, $\phi \circ \iota(t,e) = \psi(t,e)$, it gives us the required homotopy. To prove uniqueness, let

$$\varphi' : \left(\sum_{t:\Delta^2} E(\alpha(t))\right) \to \left(\sum_{b:B} P(b)\right)$$

over B and a homotopy $\bar{r} : \varphi' \circ \iota \sim \psi$. We can assume that $\varphi'(t,e) \equiv (b'_t, e'_t)$. There is a homotopy $q : \alpha \circ p_1 \sim g \circ \varphi'$. For any $(t,e) : \sum_{t:\Delta^2} E(\alpha(t))$ we get a path $q_t : \alpha(t) = b'_t$, which gives rise to the transport map $q_t^* : P(\alpha(t)) \to P(b'_t)$.
Similarly, if $(t,e) : \sum_{t:\kappa_1^2} E(\alpha(i(t)))$ then

$$\bar{r}_t : \varphi'(\iota(t,e)) \equiv (b'_{i(t)}, e'_t) = (b_t, e_t) \equiv \psi(t,e).$$

This is a path in a total space, so it is given by $r_t : b'_{i(t)} = b_t$ and $d_t : r_t^*(e'_t) = e_t$, where $r_t^* : P(b'_{i(t)}) \to P(b_t)$ is the transport map.

We first make the following observation:

$$\prod_{(t,e):F_2} \prod_{p_t:\alpha(i(t))=b_t} \prod_{q_{i(t)}:\alpha(i(t))=b'_{i(t)}} \prod_{r_t:b'_{i(t)}=b_t} (q_{i(t)}^{-1})^*((r_t^{-1})^*(e_t)) = (p_t^{-1})^*(e_t). \tag{3.3.1}$$

By induction on the paths, we assume that $p_t \equiv \mathsf{refl} : \alpha(i(t)) = \alpha(i(t))$, $q_{i(t)} \equiv \mathsf{refl} : \alpha(i(t)) = \alpha(i(t))$, and $r_t \equiv \mathsf{refl} : \alpha(i(t)) = \alpha(i(t))$. In this case the transport maps are identities, so we can use $\mathsf{refl} : e_t = e_t$. Using the map φ' we construct

$$\xi' : \left\langle \prod_{t:\Delta^2} (E(\alpha(t)) \to P(\alpha(t)))\middle|^{\kappa_1^2}_{\nu} \right\rangle$$

by $\xi' \equiv \lambda t.\lambda e.(q_t^{-1})^*(e'_t)$. To see this indeed gives the correct type to ξ' we evaluate on $t : \kappa_1^2$. Our aim is to construct for each $e : E(\alpha(i(t)))$ a path $\xi'_{i(t)}(e) = \nu_t(e)$. This is

$$(q_{i(t)}^{-1})^*(e'_{i(t)}) = (p_t^{-1})^*(e_t).$$

Note that this follows from (3.3.1) in combination with the path $d_t : r_t^*(e'_t) = e_t$. The uniqueness of ξ gives us the equality $\xi' = \xi$.

We now show that for all $(t, e) : \sum_{t:\Delta^2} E(\alpha(t))$, $\varphi(t, e) = \varphi'(t, e)$, so we need $\varphi(t, e) \equiv (\alpha(t), \xi_t(e)) = (b'_t, e'_t) \equiv \varphi'(t, e)$. First, there is a path $q_t : \alpha(t) = b'_t$. Observe that

$$q_t^*(\xi_t(e)) = q_t^*(\xi'_t(e)) = q_t^*((q_t^{-1})^*(e'_t)) = e'_t.$$

The above proves that F_1 is a Segal type completion for F_2.

(5)$\Rightarrow$ (1). Let $P : B \to \mathcal{U}$ an inner type family. Using the equivalence (1)$\Leftrightarrow$(2), it is enough to show that the type family

$$P :\equiv \lambda b.E(b) \to P(b) : B \to \mathcal{U}$$

is an inner family. This amounts to showing that the projection map

$$\pi : \left(\sum_{b:B} (E(b) \to P(b))\right) \to B$$

is right orthogonal to the horn inclusion $i : \kappa_1^2 \to \Delta^2$. Consider a lifting problem

$$\begin{tikzcd}
\kappa_1^2 \arrow[r] \arrow[d, "i"'] & \sum_{b:B} \big(E(b) \to P(b)\big) \arrow[d, "\pi"] \\
\Delta^2 \arrow[r, "\alpha"'] & B.
\end{tikzcd}$$

This means we have a partial section $\delta : \prod_{t:\kappa_1^2} \big(E(\alpha(i(t))) \to P(\alpha(i(t)))\big)$. We define the function

$$\psi : \left(\sum_{t:\kappa_1^2} E(\alpha(i(t)))\right) \to \left(\sum_{b:B} P(b)\right)$$

as $\psi :\equiv \lambda t.\lambda e.(\alpha(i(t)), \delta_t(e))$. We illustrate this in a commutative diagram:

$$\begin{tikzcd}
 & \sum_{b:B} P(b) \arrow[ddr, "g"] & \\
F_2 \arrow[ur, "\psi"] \arrow[r, "\iota"'] \arrow[d, "p_2"'] & F_1 \arrow[r] \arrow[d, two heads, "p_1"'] & \sum_{b:B} E(b) \arrow[d, two heads, "f"] \\
\Lambda^2_1 \arrow[r, "i"'] & \Delta^2 \arrow[r, "\alpha"'] & B.
\end{tikzcd}$$

Since P is an inner family, the universal property of F_1 implies we can complete this diagram to a unique map $\varphi : F_1 \to \sum_{b:B} P(b)$ over B. In what follows we can assume that $\varphi(t, e) \equiv (b_t, e_t)$. We have paths

$$p_t : \alpha(t) = b_t \text{ and } \bar{q}_t : \psi(t, e) \equiv (\alpha(i(t)), \delta_t(e)) = (b_{i(t)}, e_{i(t)}) \equiv \varphi(\iota(t, e)).$$

The second path amounts to,

$$q_t : \alpha(i(t)) = b_{i(t)} \text{ and } d_t : q_t^*(\delta_t(e)) = e_{i(t)}$$

where again $q_t^* : P(\alpha(i(t))) \to P(b_{i(t)})$ is the transport map. The element

$$\xi : \left\langle \prod_{t:\Delta^2} (E(\alpha(t)) \to P(\alpha(t))) \middle|^{\kappa_1^2}_{\delta} \right\rangle$$

is given by the formula $\xi_t(e) :\equiv (p_t^{-1})^*(e_t)$. We have a function in

$$\prod_{(t,e):F_2} \prod_{p_{i(t)}:\alpha(i(t))=b_{i(t)}} \prod_{q_t:\alpha(i(t))=b_{i(t)}} p^*_{i(t)} = q^*_t. \tag{3.3.2}$$

Indeed, by path induction we can assume that $p_{i(t)} \equiv \mathsf{refl} : \alpha(i(t)) = \alpha(i(t))$ and $q_t \equiv \mathsf{refl} : \alpha(i(t)) = \alpha(i(t))$. Moreover, in this case the transport maps are identities; therefore, the claimed equality holds. From (3.3.2) we get for all $(t, e) : F_2$,

$$\xi_{i(t)}(e) \equiv (p_{i(t)}^{-1})^*(e_{i(t)}) = (q_t^{-1})^*(e_{i(t)}) = \delta_t(e).$$

Assume an element

$$\xi' : \left\langle \prod_{t:\Delta^2} (E(\alpha(t)) \to P(\alpha(t))) \middle|^{\kappa_1^2}_{\delta} \right\rangle.$$

Next, the function

$$\varphi' : \left(\sum_{t:\Delta^2} E(\alpha(t)) \right) \to \left(\sum_{b:B} P(b) \right)$$

is defined as $\lambda(t, e).(\alpha(t), \xi'_t(e))$. We observe that if $(t, e) : \sum_{t:\kappa_1^2} E(\alpha(i(t)))$ then

$$\varphi'(\iota(t, e)) \equiv \varphi'(i(t), e) \equiv (\alpha(i(t)), \xi'_{i(t)}(e)) \equiv (\alpha(i(t)), \delta_t(e)) \equiv \psi(t, e).$$

where the middle definitional equality holds since $\xi'_{i(t)}(e) \equiv \delta_t(e)$. By uniqueness we have $\varphi = \varphi'$. This means that

$$\varphi(t, e) \equiv (b_t, e_t) = (\alpha(t), \xi'_t(e)) \equiv \varphi'(t, e)$$

This equality implies that for $p_t^{-1} : b_t = \alpha(t)$ we also get an equality $(p_t^{-1})^*(e_t) = \xi'_t(e)$. Therefore $\xi_t(e) = \xi'_t(e)$, proving the uniqueness of the extension ξ. This shows that the type $\sum_{b:B}(E(b) \to P(b))$ is a Segal type. ∎

Definition 3.3.3. An inner family $E : B \to \mathcal{U}$ over a Segal type B is said to be **Segal exponentiable** if it satisfies any of the equivalent conditions of Theorem 3.3.2. Moreover, a function $f : E \to B$ between Segal types is **Segal exponentiable** if the family $\lambda b.\mathsf{fib}_f(b) : B \to \mathcal{U}$ is Segal exponentiable, where $\mathsf{fib}_f(b)$ denotes the fiber of f over $b : B$.

Remark 3.3.4. Observe that in the semantics, Segal spaces correspond to $(\infty, 1)$-precategories. Therefore, the above Theorem 3.3.2 refers to exponentiability of functors between $(\infty, 1)$-precategories, hence, the name we suggest. Nima Rasekh pointed out to the author that, as the theorem shows, completeness does not play any role in exponentiability. This characteristic is new and does not appear in quasicategories. Certainly this aspect is not apparent in categories either. We move swiftly to specialize our theorem to Rezk types.

From [16] recall that a type family $P : B \to \mathcal{U}$ over a Segal type B is called **isoinner family** if the following proposition is true:

$$\mathsf{isIsoinner}(P) :\equiv \mathsf{isInner}(P) \times \prod_{b:B} \mathsf{isRezk}(P(b)).$$

For our interests we can always assume the types involved are Rezk types. From [61, Proposition 10.9] that if B is a Rezk type and X is any type or shape, then B^X is also a Rezk type.

Definition 3.3.5. Let $A \to B$ a type over B. We define

$$\mathsf{RCompletion}_{/B}(A) :\equiv \sum_{R:\mathcal{U}/B} \sum_{\iota:\mathsf{Fun}_{/B}(A,R)} \mathsf{isRCompletion}^{A}_{/B}(R, \iota).$$

Corollary 3.3.6. Let $E : B \to \mathcal{U}$ an isoinner family over a Rezk type B, the following are equivalent:

1. For any isoinner type family $P : B \to \mathcal{U}$,

$$\mathsf{isRezk}\left(\sum_{b:B}(E(b) \to P(b))\right).$$

2. For any isoinner family $P : B \to \mathcal{U}$, the type family $Q :\equiv \lambda b.E(b) \to P(b) : B \to \mathcal{U}$ is isoinner.

3. For any isoinner family $P : E \to \mathcal{U}$, the type family $Q :\equiv \lambda b. \prod_{e:E(b)} P(e) : B \to \mathcal{U}$ is isoinner.

4. For any Rezk type X, the type family $Q :\equiv \lambda b.E(b) \to X : B \to \mathcal{U}$ is isoinner.

5. For any map $\alpha : \Delta^2 \to B$, together with the inclusion $i : \kappa_1^2 \to \Delta^2$. Let $F_1 :\equiv \sum_{t:\Delta^2} E(\alpha(t))$ and $F_2 :\equiv \sum_{t:\kappa_1^2} E(\alpha(i(t)))$, then
$$\mathsf{isRCompletion}^{F_2}_{/B}(F_1, \iota),$$
where $\iota :\equiv \lambda(t, e).(i(t), e) \ : F_2 \to F_1$.

Proof. (2)⇒(1). Follows from [16, Proposition 4.2.6] which proves that the total space of an inner family over a Rezk type, is a Rezk type.
(1)⇒(2). Theorem 3.3.2 shows that $\mathsf{isInner}(Q)$. For each $b : B$ we have $\mathsf{isRezk}(E(b) \to P(b))$ since such fiber can be obtained as a pullback from the Rezk type $\sum_{b:B} (E(b) \to P(b))$.
(2) ⇔ (3) ⇔ (4) is a classical result.
(5)⇒(2). From Theorem 3.3.2 the family is inner. Since for each b, $\mathsf{isRezk}(P(b))$ then each fiber $E(b) \to P(b)$ is also a Rezk type.
(1)⇒(5). The proof that F_1 is the completion is the same as in Theorem 3.3.2. We just need to show that it is Rezk. This again follows from [16, Proposition 4.2.6] and the pullback stability of Rezk types. ∎

Definition 3.3.7. An isoinner family $E : B \to \mathcal{U}$ over a Rezk type B is said to be **exponentiable** if it satisfies any of the equivalent conditions of Theorem 3.3.6. Moreover, a function $f : E \to B$ between Rezk types is an **exponentiable functor** if the family $\lambda b.\mathsf{fib}_f(b) : B \to \mathcal{U}$ is exponentiable, where $\mathsf{fib}_f(b)$ denotes the fiber of f over $b : B$.

The terminology "functor" in the definition above is justified by the functorial behaviour of functions between Segal types, see [61, Proposition 6.1]. We have reserved the name exponentiable functor till this point in view of Theorem 3.3.4. From Theorem 3.3.2 and Theorem 3.3.6 it would seem that we have two notions of exponential functors, one for Segal types and the other for Rezk types. However, both coincide when we restrict to Rezk types.

Corollary 3.3.8. Let $E : B \to \mathcal{U}$ an isoinner type family over a Rezk type B. Then E is Segal exponentiable if and only if E is exponentiable.

Proof. To prove this, we observe that Condition 2, respectively in Theorem 3.3.2 and Theorem 3.3.6, are equivalent. The forward direction is obvious since, by definition, any isoinner family is in particular an inner family. Conversely, we just need to show that any fiber $E(b) \to P(b)$ is Rezk. But this follows from [61, Proposition 10.9] since each $P(b)$ is Rezk. ∎

3.4 The bisimplicial sets semantics of sHoTT

In this section we check that our synthetic definitions of Segal type completion and exponentiable functors are semantically correct. In Section 3.4.1 verify that the Segal type completion is consistent with the usual semantics. We finalize with exponentiable functors in Section 3.4.2. We use the fact the semantics of sHoTT is the category of bisimplicial sets **ssSet**. The details of the semantics are found in [61]. We also recommend [5, Section 6] for a discussion.

3.4.1 Segal type completion and Segal space completion

Here we check the soundness of a Segal type completion by comparing it with the Segal space completion defined in bisimplicial sets. Furthermore, since in general we want to consider dependent types, we need a relative version. Given a Segal space B we consider the induced model structure on the slice $\mathbf{ssSet}/B$. The existence of such a Segal completion is given by the fibrant replacement in the Segal model structure on **ssSet**, and in the relative case, the fibrant replacement in the slice $\mathbf{ssSet}/B$ (see Theorem 3.4.5.)

Recall that for objects $\pi : A \to B$ and $\xi : S \to B$ in $\mathbf{ssSet}/B$, the relative mapping space is denoted as $Map_{/B}(A,S)$. This space is obtained by the pullback square:

$$\begin{tikzcd}
Map_{/B}(A,S) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Map(A,S) \arrow[d, "\xi^A"] \\
\Delta[0] \arrow[r, "\pi"'] & Map(A,B).
\end{tikzcd}$$

On the other hand, the Segal space of functions between π and ξ is given by the following pullback square:

$$\begin{tikzcd}
Fun_{/B}(A,S) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & S^A \arrow[d, "\xi^A"] \\
F(0) \arrow[r, "\pi"'] & B^A.
\end{tikzcd}$$

Observe that $Fun_{/B}(A,S)_0 = Map_{/B}(A,S)$.

Definition 3.4.1. Let $\pi : A \to B$ and $\xi : S \to B$ in $\mathbf{ssSet}/B$. Assume further that S is a Segal space and there is a map $\iota : A \to S \in \mathbf{ssSet}/B$. We say that S is a **Segal space completion relative** to B for A if for any Segal space over B, $\delta : X \to B$, the induced map

$$Map_{/B}(S,X) \to Map_{/B}(A,X)$$

is an equivalence of spaces.

This definition is the generalization of the completion of a Segal space into a complete one as defined by Rezk in [59]. A related notion of completion of a precategory into a category in the context of homotopy type theory due to Ahrens, Kapulkin and Shulman appears in [1], where the authors use the suggestive name “Rezk completion.”

Observation 3.4.2. Note that the interpretation of the map (3.2.3) from Theorem 3.2.3 into bisimplicial sets gives us an equivalence between Segal spaces,

$$Fun_{/B}(S,X) \to Fun_{/B}(A,X),$$

which is just to say that we have a level-wise equivalence of spaces. Thus, for any function $\pi : A \to B$, with B a Segal type, the relative Segal type completion $\xi : S \to B$ for the type A gives us a Segal space completion relative to B for the Reedy fibrant bisimplicial set A.

Furthermore, these two notions coincide. Firstly, we introduce notation from [62, Proposition 1.2.22]. The slice **ssSet**$/B$ is cotensored over simplicial sets *i.e.,* we have $Map_{/B}(F(n) \times S, X) \simeq Map_{/B}(S, F(n) \pitchfork_B X)$ (here we think of $F(n)$ as a space). Note that since X is a Segal space over B so it is $F(n) \pitchfork_B X$ as is constructed via the pullback

$$\begin{tikzcd}
F(n) \pitchfork_B X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X^{F(n)} \arrow[d, "\delta^{F(n)}"] \\
B \arrow[r, "cst"'] & B^{F(n)}.
\end{tikzcd}$$

Lemma 3.4.3. Let $\pi : A \to B$ and $\xi : S \to B$ in **ssSet**$/B$, where B is a Segal space. Assume that there is a map $\iota : A \to S \in$ **ssSet**$/B$ showing S as the Segal space completion relative to B for A. Then for any Segal space X together with $\delta : X \to B$, the induced map

$$Fun_{/B}(S, X) \to Fun_{/B}(A, X)$$

is an equivalence of Segal spaces.

Proof. Firstly, for any $n \geqslant 0$ we have the following:

$$\begin{aligned}
Fun_{/B}(S, X)_n &= Map(F(n), Fun_{/B}(S, X)) \\
&= Map(F(n), X^S \times_{B^S} F(0)) \\
&= Map(F(n), X^S) \times_{Map(F(n), B^S)} F(0) \\
&= Map(F(n) \times S, X) \times_{Map(F(n) \times S, B)} F(0) \\
&= Map_{/B}(F(n) \times S, X) \\
&= Map_{/B}(S, F(n) \pitchfork_B X),
\end{aligned}$$

We are relying on the fact that $F(n) \pitchfork_B X$ is a Segal space (see the previous paragraph). Similarly, we get that $Fun_{/B}(A, X)_n = Map_{/B}(A, F(n) \pitchfork_B X)$ for all $n \geqslant 0$. By assumption, $S \to B$ is the Segal space completion relative to B for A. Hence, for $F(n) \pitchfork_B X$ we have an equivalence of spaces

$$Map_{/B}(S, F(n) \pitchfork_B X) \to Map_{/B}(A, F(n) \pitchfork_B X)$$

for all $n \geqslant 0$. This gives us the equivalence between Segal spaces:

$$Fun_{/B}(S, X) \to Fun_{/B}(A, X).$$

∎

Corollary 3.4.4. Given a Segal type B and any type A over B. The notion of a relative Segal type completion for A is consistent with the semantics.

Proof. This is immediate from Theorem 3.4.2 and Theorem 3.4.3. ∎

Of course, we also have the non-relative version of this soundness result, and it is enough to take B to be the terminal object. To finalize this section, we observe that the fibrant replacement in $\mathbf{ssSet}/B$ coincides with Segal completion relative to B. Since the model structure on $\mathbf{ssSet}/B$ is induced by the one from $\mathbf{ssSet}$, it will be enough to verify this fact for $B = 1$.

Proposition 3.4.5. Let A be a Reedy fibrant bisimplicial set. If the Segal space completion of A exists, then it coincides with its fibrant replacement in the Segal space model structure $\mathbf{ssSet}_{SS}$.

Proof. Recall that a map $i : A \to S$ is a weak equivalence in $\mathbf{ssSet}_{SS}$ if is a local map, *i.e.,* a map such that

$$i^* : Map(S, X) \to Map(A, X)$$

is an equivalence of spaces for any Segal space X. Then, it is clear that if S is a fibrant replacement, it must be a Segal space completion.
Conversely, if S is the Segal space completion, then it induces equivalences like the above. Therefore, i is indeed a weak equivalence. S is a Segal space by assumption, so it must be a fibrant replacement in $\mathbf{ssSet}_{SS}$. ∎

3.4.2 Exponentiable functors

Here we verify that our notion of exponentiable functor is semantically correct. Ayala, Francis and Rozenblyum prove in [4] the result below that characterizes exponentiable functors between ∞-categories. This is our reference point.

Theorem 3.4.6. The following conditions on a functor $\pi : \mathcal{E} \to \mathcal{B}$ between ∞-categories are equivalent.

1. The functor π is an exponentiable fibration.
2. For each functor $[2] \to \mathcal{B}$, the diagram of pullbacks

$$\begin{tikzcd}
\mathcal{E} \times_{\mathcal{B}} \{1\} \arrow[r] \arrow[d] & \mathcal{E} \times_{\mathcal{B}} \{1 < 2\} \arrow[d] \\
\mathcal{E} \times_{\mathcal{B}} \{0 < 1\} \arrow[r] & \mathcal{E} \times_{\mathcal{B}} [2]
\end{tikzcd}$$

   is a pushout of ∞-categories.

For a full explanation of the theorem we recommend the original reference [4]. We will focus on Condition 2 to see this is exactly Condition 5 of Theorem 3.3.6. This last condition involves objects that are defined by the following pullback square

$$\begin{tikzcd}
E_{F(1)\sqcup_{F(0)}F(1)} \arrow[r, "\iota"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E_{F(2)} \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, two heads, "f"] \\
F(1) \sqcup_{F(0)} F(1) \arrow[r, "i"'] & F(2) \arrow[r, "\alpha"'] & B.
\end{tikzcd}$$

We remark that the arrow on the far left is not a fibration because $F(1) \sqcup_{F(0)} F(1)$ is not a Segal space. Nevertheless, the diagrams express the fact that $E_{F(1)\sqcup_{F(0)}F(1)}$ and $E_{F(2)}$ are the fibers of f over $F(1) \sqcup_{F(0)} F(1)$ and $F(2)$, respectively.

The map $E_{F(1)\sqcup_{F(0)}F(1)} \to E_{F(2)}$ shows $E_{F(2)}$ as the Segal type completion of $E_{F(1)\sqcup_{F(0)}F(1)}$. Therefore, when we interpret the square in Condition 5 from Theorem 3.3.6 into $\mathbf{ssSet}_{Segal}$. This shows that $E_{F(2)}$ is the fibrant replacement of $E_{F(1)} \coprod_{F(0)} E_{F(1)}$ in $\mathbf{ssSet}_{Segal}$. This is just to say that the diagram

$$\begin{tikzcd}
E_{F(0)} \arrow[r] \arrow[d] & E_{F(1)} \arrow[d] \\
E_{F(1)} \arrow[r] & E_{F(2)}
\end{tikzcd}$$

is a pushout square of Segal spaces. When E and B are Rezk spaces this is exactly Condition 2 of Theorem 3.4.6. On the other hand, the category of simplicial sets can be embedded into bisimplicial sets via $p_1^* : \mathbf{sSet} \to \mathbf{ssSet}$ as defined in [40]. Furthermore, this is shown to provide a Quillen adjunction between the Joyal model structure on **sSet**, and the complete Segal space model on **ssSet**. This inclusion preserves exponentials.

3.4.2.1 On profunctors and correspondences

Due to the limitations of sHoTT we can not yet incorporate all conditions of Theorem 3.4.6 into our Theorem 3.3.2. The composition of profunctors appears naturally in Conduché's theorem. The condition 5 in Theorem 3.3.6 carries similar information in the synthetic framework. Given its relevance, in this last section we explain why this is not yet a theorem.

The result in Theorem 3.4.6 is expressed and proved using correspondences between ∞-categories. If we have categories $\mathcal{C}$ and $\mathcal{D}$, a **correspondence** from $\mathcal{C}$ to $\mathcal{D}$ is category $\mathcal{M}$ which contains $\mathcal{C}$ and $\mathcal{D}$ as full subcategories, it is equipped with a functor $\pi : \mathcal{M} \to \{0 < 1\}$ such that $\mathcal{C} = \pi^{-1}(0)$ and $\mathcal{D} = \pi^{-1}(1)$. While a **profunctor** from $\mathcal{C}$ to $\mathcal{D}$ is a functor $P : \mathcal{C} \times \mathcal{D}^{op} \to \mathbf{Set}$. There is a bicategorical equivalence between profunctors from $\mathcal{C}$ to $\mathcal{D}$ and correspondences from $\mathcal{C}$ to $\mathcal{D}$. Switching to the realm of ∞-categories:

Definition 3.4.7. A **correspondence** between ∞-categories $\mathcal{C}$ and $\mathcal{D}$ is a pair of pullbacks:

$$\begin{tikzcd}
\mathcal{C} \arrow[r] \arrow[d] & \mathcal{M} \arrow[d] & \mathcal{D} \arrow[l] \arrow[d] \\
\{0\} \arrow[r] & \{0 < 1\} & \{1\} \arrow[l]
\end{tikzcd}$$

This is simply a functor between ∞-categories $\mathcal{M} \to \{0 < 1\}$ with fibers $\mathcal{C}$ over 0 and $\mathcal{D}$ over 1.

It is a well-known fact that a profunctor $P : \mathcal{C} \times \mathcal{D}^{op} \to \mathbf{Set}$ can also be defined as a two-sided discrete fibration over $\mathcal{C} \times \mathcal{D}$. This is a functor $\mathcal{E} \to \mathcal{C} \times \mathcal{D}$ that is a discrete Grothendieck fibration over $\mathcal{D}$ and a discrete Grothendieck opfibration over $\mathcal{C}$.

Taking into account the limitations of sHoTT, for us it would make sense to momentarily think of profunctors as two-sided discrete fibrations. Let $P : A \to B \to \mathcal{U}$ be a two-variable type family over Segal types A and B. From [61], we say that P is a **two-sided discrete fibration** if for all $a : A$ and $b : B$ the type families

$$\lambda x.P(x, b) : A \to \mathcal{U} \text{ and } \lambda y.P(a, y) : B \to \mathcal{U}$$

are contravariant and covariant, respectively. The most famous two-sided discrete fibration over a Segal type B is the "Hom" type family

$$\lambda x.\lambda y.\mathsf{Hom}_B(x, y) : B \to B \to \mathcal{U}.$$

More generally, let $f : E \to B$ be a function between Segal types, a, $b : B$, and $u : \mathsf{Hom}_B(a, b)$ then

$$\lambda x.\lambda y.\mathsf{Hom}^u_E(x, y) : E_a \to E_b \to \mathcal{U}$$

is a two-sided discrete fibration. The type $\mathsf{Hom}^u_E(x, y)$ denotes the type of arrows in E that start at $x : E_a$ and end at $y : E_b$.

Weinberger provides in [74] the following characterization of two-sided discrete families:

Proposition 3.4.8. Given $P : A \to B \to \mathcal{U}$ a two-side type family over Rezk types, the following are equivalent:

1. The family P is a two-sided discrete fibration.
2. The family P is cartesian over A and cocartesian over B, and for all $a : A$, $b : B$ the bifibers $P(a, b)$ are discrete types.

We have not introduced cartesian and cocartesian type families in sHoTT, this is the main topic of [16]. However, the meaning of such concepts is in practice the same one as for ∞-categories. Therefore, the first two conditions of the second point in Theorem 3.4.8 simply mean that the type families

$$\lambda x.P(x, b) : A \to \mathcal{U} \text{ and } \lambda y.P(a, y) : B \to \mathcal{U}$$

are cartesian and cocartesian, respectively, for all $a : A$ and $b : B$ together with some compatibility condition. This is what [74] defines as two-sided cartesian family. Given $P : A \to B \to \mathcal{U}$ and $Q : B \to C \to \mathcal{U}$ two two-sided type families, there is a natural composition to obtain another two-sided type family:

$$Q \odot P \equiv \lambda a.\lambda c. \sum_{b:B} P(a, b) \times Q(b, c) : A \to C \to \mathcal{U}.$$

The result in [74, Proposition 5.5] shows that if the families P and Q are two-sided cartesian, then $Q \odot P$ is again a two-sided cartesian family. Unfortunately, even if both P and Q are two-sided discrete fibrations it does not follow that $Q \odot P$ is a two-sided discrete fibration. Instead, to make sense of the composition in this case, we consider the discrete type completion of $Q \odot P$.

If we have a function $f : E \to B$ between Segal types, a condition we would like to add to Theorem 3.3.2 is the following: For any a, b, $c : B$, $u : \mathsf{Hom}_B(a, b)$, $v : \mathsf{Hom}_B(b, c)$ and $x : E_a$, $z : E_c$ the canonical map induced by the composition

$$\left(\sum_{y : E_b} \mathsf{Hom}^u_E(x, y) \times \mathsf{Hom}^v_E(y, z) \right) \to \mathsf{Hom}^{v \circ u}_E(x, z)$$

exhibits $\mathsf{Hom}^{v \circ u}_E(x, z)$ as the discrete type completion of

$$\sum_{y : E_b} \mathsf{Hom}^u_E(x, y) \times \mathsf{Hom}^v_E(y, z).$$

The problem arises because Condition 5 of Theorem 3.3.6 encodes the composition of correspondences in sHoTT. These are Segal (Rezk) types over Δ^1. In [67] it is shown that correspondences from $\mathcal{C}$ to $\mathcal{D}$ are the same as $\mathcal{C} \times \mathcal{D}^{op} \to \mathcal{S}$, where $\mathcal{S}$ denotes the ∞-category of spaces, and furthermore, are the same as a *bifibration.* This is done by endowing the category of correspondences $Corr(\mathcal{C}, \mathcal{D})$ and the category $\mathbf{sSet}/(\mathcal{C} \times \mathcal{D})$, with model structures, respectively, such that they are Quillen equivalent and where the fibrant objects of $\mathbf{sSet}/(\mathcal{C} \times \mathcal{D})$ are the bifibrations. Both of these models are Quillen equivalent to $\mathbf{sSet}/(\mathcal{C} \times \mathcal{D}^{op})$ endowed with the covariant model structure, *i.e.,* this encodes profunctors.

We venture to say that until sHoTT is further enhanced, to be more expressive, the analogous result from [67] is out of reach. By this we just mean we cannot yet establish a full and precise relation between correspondences and two-sided discrete fibrations and profunctors

Chapter 4

Homotopy languages

This chapter builds on original ideas due to Simon Henry. The content consists of a paper [7], which was prepared for submission, was written jointly with Simon Henry, where section 4.2 and section 4.7 were written by him. The rest of the paper contains the contributions of this thesis author.

4.1 Introduction

It is a well-known result in category theory (see for example [23], [13]) that any property of a category, or of objects and morphisms of this category, that does not use equality between objects is automatically invariant both under equivalence of categories, and under substitution of all the objects and morphisms involved by isomorphic ones consistently.

For example, because the notion of limit in a category is naturally formulated without using equality between objects we automatically know that equivalences of categories preserves limits, or that if two diagrams are naturally isomorphic then a limit for one is also a limit for the other.

To be a little more precise, the above-mentioned results are about first-order formula in which we can have quantifiers over all objects of the category, or over all morphisms in a given hom-set "$\mathrm{hom}(X,Y)$". We can use equality between two terms taken from the same $\mathrm{hom}(X,Y)$, but not between two terms of type "objects", or two terms that are in different hom-set.

For example, the property of an object X to be a terminal object, which can be written as

$$\texttt{isTerminal}(X) := \forall y \in \mathrm{Ob}, (\exists v \in \mathrm{Hom}(y,X) \text{ and } \forall u,w \in \mathrm{Hom}(y,X), u = w)$$

is an instance of such formulas, but the following formula

$$\forall X,Y \in \mathrm{Ob}, \forall f \in \mathrm{Hom}(X,Y), \forall g \in \mathrm{Hom}(Y,X), \qquad (f \circ g = \mathsf{id}_Y \text{ and } g \circ f = \mathsf{id}_X \Rightarrow X = Y)$$

which say that the category we are working with is skeletal, or the formula

$$\forall X, Y \in \mathrm{Ob}, \forall f \in \mathrm{Hom}(X,Y), \forall g \in \mathrm{Hom}(Y,X), \\ \qquad (f \circ g = \mathsf{id}_Y \text{ and } g \circ f = \mathsf{id}_X \Rightarrow f = \mathsf{id}_X)$$

which express that identities are the only isomorphisms, are not of this form: the first one involves equality $X = Y$, and the second one involve an equality $f = \mathsf{id}_X$ that is not correctly typed as $f \in \mathrm{Hom}(X,Y)$. And these two formulas are indeed not invariant under equivalence of categories[1].

Note that in order for this to make sense, it is key to use a notion of "dependent type". Indeed, we need to be able to formulate the idea that a morphism f is in $\mathrm{Hom}(X,Y)$, without being able to say that $s(f) = X$ and $t(f) = Y$ as this would involve using equality between objects. So, given two objects X and Y, we need to be able to consider the type of arrows from X to Y as a primitive notion.

Now, it is natural to expect that similar results can be generalized to higher categories. For example, we expect (and it can be shown) that a property of 2-categories or bicategories that does not use equality between objects or between 1-arrows will also be invariant under biequivalences. One can also expect it can be generalized to other sort of higher structures, for example a result about multicategories not using equality between objects should also have similar invariance properties.

The main goal of this paper is, informally, to establish a version of this result for essentially any kind of higher structure independently of the type of structure or the "categoricity level". The only requirement is that the sort of higher structure we are considering must be organized as the fibrant objects of a model category (or semi-model category, or weak model category).

That is, we will attach to every (semi/weak) model category a "first-order language", whose formulas are statements about objects of the category (possibly with parameters) such that

- Replacing the value of the parameters by homotopically equivalent parameters does not change the validity of a formula.
- Two weakly equivalent fibrant objects satisfy the same formulas.

We call these two results respectively the first and second invariance theorem, and their precise statement is given as theorem 4.2.44. We will now go into a little more detail of how this language is defined, and explain the role of the different section of the paper.

As mentioned above, we need to use dependent types. Our starting point is a "Generalised algebraic theory" T in the sense of Cartmell ([17]) as our basis–if we compare to

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[1] As they are formula with no free parameters, invariance under substitution by isomorphic objects does not really make sense.

traditional model theory–T plays a role similar to a signature. However, it is crucial that the theory T can be any generalized algebraic theory, in particular the theory T can include equality axioms.

Starting from this T, we build in section 4.2.1 the first-order language $\mathcal{L}^T$, as well as its quotient, $\mathbb{L}^T$ by a fairly weak "provably equivalent" relation.

The idea is that for each formula, the (free) variables are taken from a context of the theory T, and there can be no equality at all. In particular, the theory T itself can have axioms that are not part of this first order language $\mathcal{L}^T$. We will see through example how in some cases, some form of equality, for example the case of equality between morphism in the same $\mathrm{Hom}(X, Y)$ in the example of categories we started from, can be recovered indirectly using certain equality axioms in the theory T itself.

Since we want to be able to do infinitary logic, we use everywhere an infinitary generalization of the notion of Generalized algebraic theory that is introduced in section 4.5. However, a reader familiar with generalized algebraic theories can probably guess how it works. The logic $\mathcal{L}^T$ we introduce can include arbitrary disjunction and conjunction, as well as quantifiers ranging on infinitely many variables. We will denote by $\mathcal{L}^T_{\lambda,\kappa}$ the language where the formulas only include disjunction and conjunction on less than λ subformulas and where quantifier quantifies on less than κ many variables at the same time. The κ is very often omitted from the notation for technical reasons, but see theorem 4.2.13.

In section 4.2.2 we review quickly some important properties of the category of models of a generalized algebraic theory, or equivalently of the category of models of a "clan" (in the sense of Joyal), most notably their canonical weak factorization system. In section 4.2.3 we explain how the language defined in section 4.2.1 can be given an alternative categorical definition that can be applied to any clan. Note that every clan can be shown to be the syntactic category of a generalized algebraic theory (and we prove more generally that in our infinitary setting any "κ-clan" is the syntactic category of a generalized κ-algebraic theory, this is in section 4.7,) and the category theoretic definition of the language of the clan is equivalent to the syntactic definition of the language of any such generalized algebraic theory.

This reinterpretation is the key to associate a language to any model category: Given a (weak) model category $\mathcal{M}$ we take the category $\mathcal{M}^{\mathrm{Cof}}$ of cofibrant objects and cofibration between them. This category constitutes a co-clan (the opposite of a clan) and we can take the language associated to it. This is what we call the language of the model category $\mathcal{M}$. We review briefly the general theory of weak model category in section 4.7.1 and in section 4.2.4 we explain in details how this language of $\mathcal{M}$ actually talks about the objects of $\mathcal{M}$ and prove the first two invariance theorems mentioned above.

To give a general picture of how this language works, if $\mathcal{M}$ is our model category, each formula in the language has a "context" C, which informally can be thought of as the list of free variables that can appear in the formula as well as their types. This "context" C is concretely just a cofibrant object of $\mathcal{M}$. An interpretation of the context C into an object $X \in \mathcal{M}$ is just a map $v : C \to X$. And given ϕ a formula in context C and $v : C \to X$ a map, $\phi(v)$ can be either true or false. We write

$$M \vdash \phi(v)$$

if it is true.

Section 4.2 ends with our first two invariance theorems, stated as theorem 4.2.44: The first invariance theorem asserts that if X is fibrant and $v : C \to X$ is homotopic to $v' : C \to X$ then $M \vdash \phi(v) \Leftrightarrow M \vdash \phi(v')$. The second invariance theorem states that if $F : X \to Y$ is a weak equivalence between fibrant objects, then $X \vdash \phi(v) \Leftrightarrow Y \vdash \phi(f(v))$.

To give a more concrete example of all this, when $\mathcal{M}$ is the canonical or folk model structure on categories, our construction recovers the language of categories as in [23] or [13]. Now, the formula

$$\forall Z \in \mathrm{Ob}, \forall g, h \in \mathrm{Hom}(Y, Z), g \circ f = h \circ f \Rightarrow g = h$$

is a formula in context $X, Y \in \mathrm{Ob}, f \in \mathrm{Hom}(X, Y)$ which corresponds to the (cofibrant) category $\mathcal{C}$ which has two objects X and Y and a unique non-identity arrow $f : X \to Y$. A map from $\mathcal{C}$ to another category $\mathcal{D}$ is the choice of an arrow f in $\mathcal{D}$ and $\phi(f)$ is true if and only if f is an epimorphism. The second invariance theorem says (in this special case) that equivalence of categories preserves epimorphisms, and the first invariance theorem that if f is isomorphic to another arrow then one is an epimorphism if and only if the other is.

In section 4.3 we show how these notions specialize to many classical model structures, and we also discuss briefly some general tools to construct this language explicitly for any model structure.

Finally, in section 4.4 we prove two more invariance theorems (theorem 4.4.2), that are this time about the expressive power of the language:

1. The 3^{rd} invariance theorem, informally, says that if A and B are two cofibrant objects of $\mathcal{M}$, then each formula in context A can be translated into a formula in context B that is "equivalent" in the sense that its interpretation in any fibrant object is the same.
2. The 4^{th} invariance theorem, informally, says that if $\mathcal{M}$ and $\mathcal{N}$ are two Quillen equivalent weak model categories, then any formula in the language of $\mathcal{M}$ can be similarly translated into an equivalent formula in the language of $\mathcal{N}$.

More details on this will be given in the introduction to section 4.4.

The paper has three appendices that serve to review or introduce basic material. They can either be read first, or skipped entirely: Section 4.5 review Cartmell's notion of generalized algebraic theory, and generalize it to the infinitary case. The goal of section 4.6 is to establish the link between Generalized κ-algebraic theory and a notion of κ-clan, with a notion of κ-contextual category as intermediate. This result is absolutely crucial for the paper, but is a very expected generalization of what happens in the finitary case. Finally, section 4.7 reviews some material on weak model categories and introduces a notion of Reedy model categories in that context, which is only used in section 4.4.

We finish by mentioning that this work is closely related to Makkai "First-order logic with dependent sorts" or FOLDS from [52]. In a sense, Makkai's FOLDS corresponds to the special case where T is the theory of presheaves on a direct category I, encoded using dependent type axioms only, with an additional equality predicate for the types corresponding to maximal objects of I. Because Makkai does not make assumption about the existence of a model structure he only establishes an invariance theorem for what he call "very surjective maps" (our "anodyne fibrations"), that is the analogous to our theorem 4.2.38, more general notions of equivalence and homotopy are not clearly available in his settings.

In conclusion, the present work is at the same time considering a more general algebraic setting (by allowing terms and type in T), but also is restricting the setting by assuming the presence of a model structure that gives a good homotopy theory to be invariant under, and allows obtaining much more interesting results. This seems to make our approach considerably more usable in practice, given the richness of examples it potentially covers.

It should be noted however that there are some results in [52] that we have not yet been able to generalize to this new setting: Makkai established several results that essentially say that any formula that has the desired invariance properties is equivalent to one in the language introduced. Similar results are also given in [23] and [13], and this paper contains no analogous to these results.

4.2 The homotopy invariant language

4.2.1 Syntactic approach: The first-order language of a generalized algebraic theory

In this section, we give a very classic syntactical approach to the language we consider in this paper. We start from a generalized algebraic theory, and we build its first-order language on top of it.

Since we aim to do infinitary logic, we enhance Cartmell's notion of generalized algebraic theory to what we call *generalized κ-algebraic theory* for κ a regular cardinal, which we develop in detail in section 4.5. Nevertheless, this generalization is straightforward and a reader familiar with Cartmell's formalism should be able to guess how it works and read this section directly.

We fix κ, λ two regular cardinals and T a generalized κ-algebraic theory. We will define the first-order language of T with λ-small conjunction and disjunction, denoted $\mathcal{L}^T_\lambda$ or $\mathcal{L}^T_{\lambda,\kappa}$.

More precisely, for each context Γ of T, we will define a set $\mathcal{L}^T_\lambda(\Gamma)$ of "T-formulas in context Γ". Essentially, these are first-order formulae with λ-small conjunctions and disjunctions whose free variables are the variables of the context Γ, in particular, they have less than κ-variables.

Definition 4.2.1. The sets $\mathcal{L}^T_\lambda(\Gamma)$ of T-formulas in context Γ are defined inductively using the following rules:

1. For each context Γ, the true formula $\top$ and false formula $\bot$ are in $\mathcal{L}^T_\lambda(\Gamma)$.
2. If $\Phi \in \mathcal{L}^T_\lambda(\Gamma)$ then $\neg\Phi \in \mathcal{L}^T_\lambda(\Gamma)$.
3. For each collection of formulas $\Phi_i \in \mathcal{L}^T_\lambda(\Gamma)$, indexed by a λ-small set I, the conjunction and disjunction
$$\bigvee_{i\in I} \Phi_i \qquad \bigwedge_{i\in I} \Phi_i$$
are in $\mathcal{L}^T_\lambda(\Gamma)$.
4. Given two ordinals $\gamma < \alpha < \kappa$: If $\Gamma' \equiv \{x_\beta : \Gamma_\beta\}_{\beta<\alpha}$ is a context of length α, and $\Gamma \equiv \{x_\beta : \Gamma_\beta\}_{\beta<\gamma}$ is the subcontext of length γ, then for any formula $\Phi \in \mathcal{L}^T_\lambda(\Gamma')$ we have formulas
$$\exists\{x_\beta : \Gamma_\beta\}_{\gamma\leqslant\beta<\alpha}\Phi \qquad \forall\{x_\beta : \Gamma_\beta\}_{\gamma\leqslant\beta<\alpha}\Phi$$
in $\mathcal{L}^T_\lambda(\Gamma)$.

The collection of all formulas $\{\mathcal{L}^T_\lambda(\Gamma)\}_{\Gamma\in T}$ is what we call *the language of* T. Often, we will simply refer to it by $\mathcal{L}^T_\lambda$.

Remark 4.2.2. The key point in theorem 4.2.1 is that we are not including atomic formulas other than $\top$ and $\bot$. In particular, the language *does not include any equality.* At this point it might be unclear how we get non-trivial formulae in this language as it seems that applying quantifiers, conjunction or disjunction to formulas that are either $\bot$ or $\top$ will never produce any formulas that are not immediately interpreted as $\bot$ or $\top$. Or even, on how we might obtain formulas with free variables. The central idea is that free variables appear thanks to the fact we quantify over dependent types, that is, types in which free variables can appear. The following examples will demonstrate these phenomena.

Example 4.2.3. Let *Cat* be the generalized ω-algebraic theory of categories as introduced in theorem 4.5.7. Then, in the context $(x : \mathsf{Ob})$ we can write the formula
$$\phi(x) := (\forall y : \mathsf{Ob}, \exists f : \mathsf{Hom}(x, y), \top)$$
which expresses that for any object y there is an arrow from x to y. This simply means that x is a weakly initial object. Indeed, $\top$ is a formula in context $(x : \mathsf{Ob},\ y : \mathsf{Ob}, f : \mathsf{Hom}(x, y))$, so that $\exists f : \mathsf{Hom}(x, y), \top$ is a formula in context $(x : \mathsf{Ob},\ y : \mathsf{Ob})$, and $\forall y : \mathsf{Ob}, \exists f : \mathsf{Hom}(x, y), \top$ is a formula in context $(x : \mathsf{Ob})$.

The logic is still not strong enough to express many of the interesting category theoretic notions. For example, without any kind of equality predicate on morphisms there is no way to write down a formula for an initial object, or a limit. In the next example, we show how modifying the theory *Cat* allows recovering equality on morphisms:

Example 4.2.4. We consider the theory $Cat_=$ obtained by adding to the theory Cat the following:

$$x, y : \mathsf{Ob}, f, g : \mathsf{Hom}(x, y) \vdash \mathsf{Eq}(f, g)\mathsf{Type}$$

$$x, y : \mathsf{Ob}, f : \mathsf{Hom}(x, y) \vdash r_f : \mathsf{Eq}(f, f)$$

$$x, y : \mathsf{Ob}, f, g : \mathsf{Hom}(x, y), a : \mathsf{Eq}(f, g) \vdash f \equiv g$$

$$x, y : \mathsf{Ob}, f, g : \mathsf{Hom}(x, y), a : \mathsf{Eq}(f, g) \vdash a \equiv r_f$$

One easily see that a model of $Cat_=$ is just a category, with the type $\mathsf{Eq}(f, g)$ being empty if $f \neq g$ and $\{r_f\}$ if $f = g$. In this new theory, we can now form a formula "$f = g$" in context $(x, y : \mathsf{Ob}, f, g : \mathsf{Hom}(x, y))$ which is defined as

$$(f = g) := (\exists v : \mathsf{Eq}(f, g), \top).$$

Therefore, in the language $\mathcal{L}_\omega^{Cat_=}$ we can form formulas involving equality between parallel morphisms. Then, we recover the "language of categories" as studied in [13] and [23]. For example, we can form the formula "x is initial" in context $(x : \mathsf{Ob})$ as

$$\mathsf{isInitial}(x) := \forall y : \mathsf{Ob}, (\exists f : \mathsf{Hom}(x, y)) \wedge (\forall f, g : \mathsf{Hom}(x, y), f = g).$$

Construction 4.2.5. If $f : \Delta \to \Gamma$ is a context morphism and $\phi \in \mathcal{L}_\lambda^T(\Gamma)$, then we can define its pullback $f^*\phi$. This pullback is obtained by substituting the free variables of ϕ by the components of f. Formally, this is defined inductively as:

1. $f^*\top := \top$ and $f^*\bot := \bot$.
2. $f^*(\neg\Phi) := \neg f^*\Phi$.
3. $f^*\left(\bigvee_{i \in I} \Phi_i\right) := \bigvee_{i \in I} f^*\Phi_i$ and $f^*\left(\bigwedge_{i \in I} \Phi_i\right) := \bigwedge_{i \in I} f^*\Phi_i$.
4. If $\Gamma' \equiv (\Gamma, x_1 \in X_1, \ldots, x_\alpha \in X_\alpha)$ then

$$f^*\left(\exists(x_1 \in X_1, \ldots, x_\alpha \in X_\alpha)\Phi\right) := \exists(x_1 \in f^*X_1, \ldots, x_\alpha \in f^*X_\alpha)f^*\Phi,$$

$$f^*\left(\forall(x_1 \in X_1, \ldots, x_\alpha \in X_\alpha)\Phi\right) := \forall(x_1 \in f^*X_1, \ldots, x_\alpha \in f^*X_\alpha)f^*\Phi,$$

where f^*X_i denotes the pullback of types, obtained by substitution, that is, the types appearing in the canonical pullback of the generalized display map:

$$\begin{tikzcd}
(\Delta, f^*X_1, \ldots, f^*X_\alpha) \arrow[r] \arrow[d, two heads] & (\Gamma, X_1, \ldots, X_\alpha) \arrow[d, two heads] \\
\Delta \arrow[r] & \Gamma.
\end{tikzcd}$$

Definition 4.2.6. For each context Γ in T we define the relation $\vdash_\Gamma$ on $\mathcal{L}_\lambda^T(\Gamma)$ as the smallest family of relations such that:

1. $\vdash_\Gamma$ is a transitive and reflexive relation on $\mathcal{L}_\lambda^T(\Gamma)$.

2. $\forall \Phi \in \mathcal{L}^T_\lambda(\Gamma)$, $\Phi \vdash_\Gamma \top$ and $\bot \vdash_\Gamma \Phi$.

3. $\forall \Phi \in \mathcal{L}^T_\lambda(\Gamma), \Phi \wedge \neg\Phi \vdash \bot$ and $\top \vdash \Phi \vee \neg\Phi$.

4. For any λ-small family $(\Phi_i)_{i \in I} \in \mathcal{L}^T_\lambda(\Gamma)$ we have
$$\bigvee_{i \in I} \Phi_i \vdash_\Gamma \Psi \Leftrightarrow \forall i, (\Phi_i \vdash_\Gamma \Psi)$$
$$\Psi \vdash \bigwedge_{i \in I} \Phi_i \Leftrightarrow \forall i, (\Psi \vdash_\Gamma \Phi_i)$$

5. For $\Gamma' \equiv \left(\Gamma, \left\{x_\beta : \Gamma'_\beta\right\}_{\gamma \leqslant \beta < \alpha}\right)$ a context extension, with $p : \Gamma' \twoheadrightarrow \Gamma$ the corresponding generalized display map, $\Psi \in \mathcal{L}^T_\lambda(\Gamma')$ and $\Phi \in \mathcal{L}^T_\lambda(\Gamma)$ we have
$$\exists\{x_\beta : \Gamma_\beta\}_{\gamma \leqslant \beta < \alpha} \Psi \vdash_\Gamma \Phi \Leftrightarrow \Psi \vdash_{\Gamma'} p^*\Phi,$$
$$\Phi \vdash_\Gamma \forall\{x_\beta : \Gamma_\beta\}_{\gamma \leqslant \beta < \alpha} \Psi \Leftrightarrow p^*\Phi \vdash_{\Gamma'} \Psi.$$

While we have not included the following in the definition, we can show that:

Proposition 4.2.7. If $f : \Delta \to \Gamma$ is a context morphism in T, and $\Phi \vdash_\Gamma \Psi$ then $f^*\Phi \vdash_\Delta f^*\Psi$.

Proof. We can show that if we define the relations $\Phi \vdash'_\Gamma \Delta$ to be "For all $f : \Delta \to \Gamma$, we have $f^*\Phi \vdash_\Delta f^*\Psi$" then it satisfies all the conditions from theorem 4.2.6. Which shows that $\vdash \Rightarrow \vdash'$ and hence concludes the proof. ∎

Definition 4.2.8. A *model* of a generalized κ-algebraic theory T is simply a contextual functor $X : \mathbb{C}_T \to \mathbf{Set}$. We will usually write $X : T \to \mathbf{Set}$.

Construction 4.2.9. Given a model X of our theory T, Γ a context, $x \in X(\Gamma)$ and $\Phi \in \mathcal{L}^T_\lambda(\Gamma)$, we can interpret $\Phi(x)$ as a proposition *i.e.,* true or false in the obvious way by substituting the components of x into ϕ and interpreting all the logic symbols in the usual way. Formally we have:

1. If $\Phi = \top$, then $\Phi(x)$ is true and if $\Phi = \bot$ then $\Phi(x)$ is false,

2. If $\Phi = \neg\Psi$, then $\Phi(x)$ is true if and only if $\Psi(x)$ is false,

3. If $\Phi = \bigvee \Phi_i$, then $\Phi(x)$ is true if and only if $\Phi_i(x)$ is true for some i,

4. If $\Phi = \bigwedge \Phi_i$, then $\Phi(x)$ is true if and only $\Phi_i(x)$ is true for all i,

5. $\Phi = \exists\{x_\beta : \Gamma_\beta\}_{\gamma \leqslant \beta < \alpha} \Psi$ for $\Gamma' = \left(\Gamma, \left\{x_\beta : \Gamma'_\beta\right\}_{\gamma \leqslant \beta < \alpha}\right)$ a context extension, with $p : \Gamma' \twoheadrightarrow \Gamma$ the corresponding generalized display map, then $\Phi(x)$ is true if there exists a $y \in X(\Gamma')$ such that $p(y) = x$ and $\Psi(y)$,

6. If $\Phi = \forall \{x_\beta : \Gamma_\beta\}_{\gamma \leqslant \beta < \alpha} \Psi$ in the same situation as above, then $\Phi(x)$ is true if for any $y \in X(\Gamma')$ such that $p(y) = x$ we have $\Psi(y)$.

The following lemma is immediate by induction, the proof is left to the reader.

Lemma 4.2.10. Let X be a model of a generalized κ-algebraic theory T.

1. For $\Phi, \Psi \in \mathcal{L}^T_\lambda(\Gamma)$ and $x \in X(\Gamma)$, then if $\Psi \vdash_\Gamma \Phi$ and $\Psi(x)$ then $\Phi(x)$.
2. If $f : \Gamma \to \Delta$ is any context morphism and $\Phi = f^*\Psi$ and $x \in X(\Gamma)$ then $\Phi(x) \Leftrightarrow \Psi(f(x))$.

Definition 4.2.11. We write $\Psi \dashv\vdash_\Gamma \Phi$ to mean both $\Psi \vdash_\Gamma \Phi$ and $\Phi \vdash_\Gamma \Psi$. We denote by

$$\mathbb{L}^T_\lambda(\Gamma) := \mathcal{L}^T_\lambda(\Gamma)/(\dashv\vdash_\Gamma)$$

the quotient.

Note that $(\dashv\vdash_\Gamma)$ is indeed an equivalence relation, as $\vdash_\Gamma$ is transitive and reflexive.

Remark 4.2.12. It follows from theorem 4.2.7 that for a context morphism $f : \Delta \to \Gamma$ the f^* operation from $\mathcal{L}^T_\lambda(\Gamma) \to \mathcal{L}^T_\lambda(\Delta)$ is compatible with the relation $\dashv\vdash$, and hence it descends to an operation

$$f^* : \mathbb{L}^T_\lambda(\Gamma) \to \mathbb{L}^T_\lambda(\Delta).$$

It is also easy to see from theorem 4.2.6 that the relation $\vdash$ is compatible with all the logical operations on $\mathcal{L}^T_\lambda$, that is $\neg, \bigvee, \bigwedge, \exists, \forall$ in the sense that for example, if $\Phi_i \vdash \Psi_i$ for all $i \in I$ then $\bigvee_{i \in I} \Phi_i \vdash \bigvee_{i \in I} \Psi_i$ and hence they all descend into operations on L^T_λ.

Construction 4.2.13. At the beginning of the section, we have briefly called the language $\mathcal{L}^T_{\lambda,\kappa}$ before dropping the κ from the notation, as it can be read from the fact that T is a generalized κ-algebraic theory. However, we can consider $\mathcal{L}^T_{\lambda,\kappa'}$ for any $\kappa' \geqslant \kappa$. Indeed, given T a generalized κ-algebraic theory, we can define a generalized κ'-algebraic theory $T_{\kappa'}$ by taking a set of axioms for T and seeing them as axioms for a generalized κ'-algebraic theory. A model of $T_{\kappa'}$ is the same as a model of T. We then define

$$\mathcal{L}^T_{\lambda,\kappa'} := \mathcal{L}^{T_{\kappa'}}_{\lambda,\kappa'} = \mathcal{L}^{T_{\kappa'}}_{\lambda},$$

as well as its quotient

$$\mathbb{L}^T_{\lambda,\kappa'} := \mathbb{L}^{T_{\kappa'}}_{\lambda,\kappa'} = \mathbb{L}^{T_{\kappa'}}_{\lambda}.$$

Example 4.2.14. Let Σ be a signature in the sense of traditional model theory, that is a set of formal symbols for types, functions and relations. Then we can consider the generalized algebraic theory $T_{\Sigma,=}$, which has one type in empty context of each sort symbol X in the signature. Each of these types have an equality predicate as the one constructed in theorem 4.2.4, a term for each function symbol, and for each relation symbol $R \subset X_1, \dots, X_n$ a type axiom

$$x_1 : X_1, \dots, x_n : X_n \vdash R(x_1, \dots, x_n)\mathsf{Type}$$

with the additional axiom

$$x_1 : X_1, \dots, x_n : X_n, t_1, t_2 : R(x_1, \dots, x_n) \vdash t_1 = t_2.$$

Models of this theory are exactly Σ-structures, and elements of $\mathbb{L}^{T_{\Sigma,=}}_{\omega,\omega}$ are essentially the same as usual first-order formula in this signature. Elements of $\mathbb{L}^{T_{\Sigma,=}}_{\lambda,\kappa}$ corresponds to infinitary first-order formulas using λ-small conjunction and disjunction and where $\exists$ and $\forall$ quantifier can quantify over κ-small set of variables.

4.2.2 Models of Clans and their weak factorization system

We recall that:

Definition 4.2.15. A *clan*, or *ω-clan*, is a category $\mathcal{C}$ endowed with a class of maps called *fibrations* such that:

1. $\mathcal{C}$ has a terminal object 1, and for every $X \in \mathcal{C}$ the unique map $X \twoheadrightarrow 1$ is a fibration,
2. Isomorphisms are fibrations, the composite of two fibrations is a fibrations,
3. Pullback of fibrations exists and are fibrations.

For κ a regular cardinal, a κ-clan is a clan which further satisfies:

4 For any ordinal $\lambda < \kappa$, if $A_\bullet : \lambda^{\mathrm{op}} \to \mathcal{C}$ is a diagram in which all the transition maps $A_\beta \twoheadrightarrow A_\alpha$ for $\alpha < \beta$ are fibrations, then the limits

$$\mathtt{Lim}_{\alpha<\lambda} A_\alpha$$

exists, and all the projection maps $\pi_\beta : \mathtt{Lim}_{\alpha<\lambda} A_\alpha \twoheadrightarrow A_\beta$ are fibrations. We refer to these as *limits of κ-small chains of fibrations.*

A *morphism of clans* is a functor that send fibrations to fibrations, preserve the terminal object and pullback of fibrations. A *morphism of κ-clans* is in addition required to preserves the limits of κ-small chains of fibrations.

Fibrations will be denoted with a double-headed arrow $\twoheadrightarrow$.

Remark 4.2.16. We define *coclans* and *κ-coclans* dually, as the category $\mathcal{C}$ endowed with a class of *cofibrations* whose opposite category is a clan or a κ-clan, respectively.

Definition 4.2.17. If $\mathcal{C}$ is a κ-clan, a *model* X of $\mathcal{C}$ is a functor $X : \mathcal{C} \to \mathbf{Set}$ that preserves the terminal object, pullback of fibrations and limits of κ-small chains of fibrations. The category $\mathrm{Mod}(\mathcal{C})$ of models of $\mathcal{C}$ is defined as a full subcategory of the category $\mathrm{Fun}(\mathcal{C}, \mathbf{Set})$ of all functors.

Remark 4.2.18. A key observation is of course that if T is a generalized κ-algebraic theory and $\mathcal{C}_T$ is its contextual category, then $\mathcal{C}_T$ can be seen as a κ-clan where fibrations are the maps that are isomorphic to generalized display maps. Moreover, the models of T are exactly the models of this clan $\mathrm{Mod}(T) = \mathrm{Mod}(\mathcal{C}_T)$, so that models of generalized algebraic theories are special cases of models of clans. Also note that:

- By theorem 4.6.55 every κ-clan $\mathcal{C}$ is equivalent to a κ-contextual category,
- By theorem 4.6.46 every κ-contextual category is isomorphic to the contextual category $\mathcal{C}_T$ of a generalized κ-algebraic theory.

Combining these two results, every κ-clan is equivalent to one of the form $\mathcal{C}_T$ for T a generalized κ-algebraic theory. Hence, there is no fundamental difference between the models of a clan and the models of a generalized κ-algebraic theory.

Construction 4.2.19. Let $\mathcal{C}$ be a κ-clan and $\text{よ}_\bullet : \mathcal{C}^{\mathrm{op}} \to Fun(\mathcal{C}, \mathbf{Set})$ be the contravariant Yoneda embedding. Note that for every $A \in \mathcal{C}^{\mathrm{op}}$ the functor $\text{よ}_A : \mathcal{C} \to \mathbf{Set}$ preserves all limits, so in particular it is a model. Therefore, we have an embedding $\text{よ}_\bullet : \mathcal{C}^{\mathrm{op}} \to \mathrm{Mod}(\mathcal{C})$. Note that by the Yoneda lemma we have a natural isomorphism

$$\mathrm{Hom}(\text{よ}_A, X) \simeq X(A)$$

for $X \in \mathrm{Mod}(\mathcal{C})$ and $A \in \mathcal{C}$.

Remark 4.2.20. The category of models of a κ-clan $\mathcal{C}$ is characterized by preservation of certain κ-small limits. This implies, by general category theoretic results that, for a small κ-clan $\mathcal{C}$:

- The category $\mathrm{Mod}(\mathcal{C})$ is locally κ-presentable,
- The representable models よ_A for $A \in \mathcal{C}$ are κ-presentable objects.

Indeed, the category $\mathrm{Mod}(\mathcal{C}) \subset \mathrm{Fun}(\mathcal{C}, \mathbf{Set})$ is closed under κ-filtered colimits because κ-filtered colimits commute with κ-small limits, which because of the isomorphism $\mathrm{Hom}(\text{よ}_A, X) \simeq X(A)$, implies that the object よ_A is κ-presentable in $\mathrm{Mod}(\mathcal{C})$. Moreover, since every $X \in \mathrm{Mod}(\mathcal{C})$ can be written as $X = \mathtt{Colim}_{\text{よ}_A \to X} \text{よ}_A$. This implies that the category $\mathrm{Mod}(\mathcal{C})$ is locally κ-accessible, and hence locally κ-presentable, as it is also closed under small limits.

Remark 4.2.21. More generally, any κ-presentable category $\mathcal{C}$ is equivalent to the category of functors $\mathcal{C}_\kappa^{\mathrm{op}} \to \mathbf{Set}$ that preserves κ-small limits, where $\mathcal{C}_\kappa$ is the (essentially small) category of κ-presentable objects of $\mathcal{C}$. In particular, every κ-presentable category is the category of models of a κ-clan: One can take the category $\mathcal{C}_\kappa^{\mathrm{op}}$, with all maps being fibrations. However, the category $\mathrm{Mod}(\mathcal{C})$ of models of a κ-clans comes with an additional structure that is more specific:

Definition 4.2.22. Given a κ-clan $\mathcal{C}$, we consider the weak factorization on the category $\mathrm{Mod}(\mathcal{C})$ which is cofibrantly generated by the maps

$$\text{よ}_A \hookrightarrow \text{よ}_B$$

where $B \twoheadrightarrow A$ is a fibration in $\mathcal{C}$. The element of the left class will be called *cofibrations* and the element of right class *anodyne fibrations.*

Remark 4.2.23. In most of the paper, we will work with a model category instead of clan (or at least weak model category.) In this case, the anodyne fibrations will be called trivial cofibrations as usual. However, we want to reserve the use of "trivial fibration" to the case where there is indeed a (weak) model category involved.

Remark 4.2.24. In the special case $\kappa = \omega$, this weak factorization was defined in [27, Definition 2.4.2] and extensively studied in [22]. In particular, Jonas Frey gave in [22] a complete characterization of which pairs of a category and a weak factorization can be obtained in this way from an ω-clan. The methods used by Frey can be extended to the κ-case to obtain a similar characterization. Frey also shows that (in the $\kappa = \omega$ case) the ω-presentable cofibrant object in $\mathrm{Mod}(\mathcal{C})$ are exactly the retracts of representable models. The same proof generalizes to the κ-case to show that if $\mathcal{C}$ is a κ-clan, then κ-presentable cofibrant objects are exactly the retracts of representables. We only mention these result for context, we will not directly use them.

Lemma 4.2.25. Given $\mathcal{C}$ a clan, a morphism $f : M \to N$ of $\mathcal{C}$-models is an anodyne fibration if and only if for every fibration $p : X \twoheadrightarrow Y$ in $\mathcal{C}$, the naturality square:

$$\begin{tikzcd}
M(X) \arrow[r] \arrow[d] & M(Y) \arrow[d] \\
N(X) \arrow[r] & N(Y)
\end{tikzcd}$$

is a *weak pullback square*, that is, if the induced map $M(X) \to N(X) \times_{N(Y)} M(Y)$ is a surjection.

Proof. By the Yoneda lemma, there is a one-to-one correspondence between elements of $M(X)$ and morphisms of models $\text{よ}_X \to M$. The map $M(X) \to M(Y)$ is obtained as the composite $\text{よ}_Y \to \text{よ}_X \to M$ and the map $M(X) \to N(X)$ as the composite $\text{よ}_X \to M \to N$. An element of $N(X) \times_{N(Y)} M(Y)$ is hence the data of maps $\text{よ}_X \to N$ and $\text{よ}_Y \to M$ such that the composite $\text{よ}_Y \to M \to N$ and $\text{よ}_Y \to \text{よ}_X \to N$ coincide. This is exactly a commutative square:

$$\begin{tikzcd}
\text{よ}_Y \arrow[r] \arrow[d, "\text{よ}_p"'] & M \arrow[d, "f"] \\
\text{よ}_X \arrow[r] & N.
\end{tikzcd}$$

An element of $M(X)$ whose image in $N(X) \times_{N(Y)} M(Y)$ is the square above is then exactly a dotted diagonal filling:

$$\begin{tikzcd}
よ_Y \arrow[r] \arrow[d, "よ_p"'] & M \arrow[d, "f"] \\
よ_X \arrow[r] \arrow[ur, dotted] & N.
\end{tikzcd}$$

Hence the surjectivity of this map is equivalent to the fact that f has the right lifting property against $よ_Y \to よ_X$ for all fibrations $X \twoheadrightarrow Y$, which concludes the proof. ∎

4.2.3 The Category theoretic approach: The first-order language of a κ-clans

In this section we present another equivalent approach to the definition of the language, which is more categorical in spirit, and strongly inspired from Lawvere's theory of Hyperdoctrines ([46], [47]). This approach, while much more abstract, has several advantages over the syntactic one. Mainly, it allows working directly with the more general notion of a clan (see section 4.7), instead of a generalized κ-algebraic theory. This will be useful later on to define the language of a model category without having to build explicitly a syntax for it.

As before, we fix λ a regular cardinal. A λ-boolean algebra is a boolean algebra which admits joins (and hence intersections) of λ-small families. We denote by $\mathbf{Bool}_\lambda$ the category whose objects are λ-boolean algebras and whose morphisms are boolean algebra morphisms preserving λ-small joins (and hence intersections).

We introduce the notion of λ-boolean algebra over a clan $\mathcal{C}$, which we can think as an axiomatization of the structure that the $\mathbb{L}_\lambda^T$ from section 4.2.1 have over the contextual category of T.

Definition 4.2.26. Given $\mathcal{C}$ a clan and λ a regular cardinal, a λ-*boolean algebra* over $\mathcal{C}$ is a functor

$$\mathcal{B} : \mathcal{C}^{op} \to \mathbf{Bool}_\lambda$$

such that:

1. For each fibration $\pi : Z \twoheadrightarrow X$ in $\mathcal{C}$, $\pi^* : \mathcal{B}(X) \to \mathcal{B}(Z)$ has a left adjoint:

$$\exists_\pi : \mathcal{B}(Z) \leftrightarrows \mathcal{B}(X) : \pi^*.$$

2. The Beck-Chevalley condition holds for each pullback square along a fibration. That is, given any pullback square:

$$\begin{tikzcd}
Z' \arrow[r, "f'"] \arrow[d, two heads, "\pi'"'] \arrow[dr, phantom, "\lrcorner", very near start] & Z \arrow[d, two heads, "\pi"] \\
X' \arrow[r, "f"'] & X
\end{tikzcd}$$

with π a fibration, we have $f^*\exists_\pi = \exists_{\pi'} f'^*$.

Morphisms of λ-boolean algebras over $\mathcal{C}$ are natural transformations that commute with the $\exists_\pi$. We call weak morphisms the natural transformations with no additional conditions.

Remark 4.2.27. If $\mathcal{B}$ is a λ-boolean algebra over $\mathcal{C}$, then for each $X \in \mathcal{C}$, the negation $\neg : \mathcal{B}(X) \to \mathcal{B}(X)^{op}$ is a contravariant equivalence. Therefore, if $\pi : Z \to X$ is a fibration, then the map $\pi^* : \mathcal{B}(X) \to \mathcal{B}(Z)$ also has a right adjoint defined by:

$$\forall_\pi(\phi) := \neg(\exists_\pi \neg\phi).$$

From this definition, we immediately have the other Beck-Chevalley condition $f^*(\forall_\pi) = \forall_\pi f^*$ and the fact that morphisms of boolean algebras over $\mathcal{C}$ are also compatible with $\forall_\pi$, simply because f^* is compatible with both $\exists_\pi$ and the negation.

Remark 4.2.28. Theorem 4.2.26 will in practice be applied to $\mathcal{C}$ a κ-clan (and not just a clan), the only reason it is stated like that is because the definition actually does not explicitly involve κ. This is related to the fact that the dependencies in κ of the language defined in the previous subsection is only through the choice of which context can our variables (including bound variables) be taken from: taking a larger κ means we can quantify over more variables at the same time. Similarly, the dependency on κ is hidden in the dependency on $\mathcal{C}$, as $\mathcal{C}$ is playing the role of the category of κ-contexts.

Let us start with our main example of such boolean algebra over a clan, which is the motivating example for the notion:

Theorem 4.2.29. Let T be a generalized κ-algebraic theory and $\mathcal{C}_T$ the corresponding κ-contextual category, seen as a clan. Then the construction $X \mapsto \mathbb{L}^T_\lambda(X)$ from theorem 4.2.11 (see also theorem 4.2.1 and 4.2.6) is a λ-boolean algebra over $\mathcal{C}_T$. In fact, it is an initial object in the category of λ-boolean algebras over $\mathcal{C}_T$.

Proof. We first check that $\mathcal{L}^T_\lambda$ is a λ-boolean algebra over $\mathcal{C}_T$. We have mentioned in theorem 4.2.12 that all the logical operations $\vee, \wedge, \neg, \exists$ and so on are compatible with the equivalence relation $\dashv\vdash$. Therefore, they all induce operations on the quotient $\mathbb{L}^T_\lambda$. The first four points of theorem 4.2.6 immediately shows that each $\mathbb{L}^T_\lambda(X)$ is a boolean algebra whose order relation is given by $\vdash$, and with λ-small unions. By theorem 4.2.5, the map $f^* : \mathcal{L}^T_\lambda(X) \to \mathcal{L}^T_\lambda(Y)$ is compatible with all the logical operations, so it gives rise to a morphism of boolean algebras $\mathbb{L}^T_\lambda(X) \to \mathbb{L}^T_\lambda(Y)$. We get a functor $\mathcal{C}_T \to \mathbf{Bool}_\lambda$, the conditions $(g \circ f)^*(\phi) = f^* g^*(\phi)$ and $id^*(\phi) = \phi$ follow immediately by induction. Next, the last two conditions of theorem 4.2.6 show that $\exists$ and $\forall$ defines left and right adjoint to π^*. Finally, the Beck-Chevalley condition follows from how f^* is defined on formulas starting with a $\exists$ quantifier:

$$f^*\left(\exists\{x_\beta : \Gamma_\beta\}_{\gamma \leqslant \beta < \alpha}\Phi\right) = \exists\{x_\beta : f^*\Gamma_\beta\}_{\gamma \leqslant \beta < \alpha} f^*\Phi,$$

which (after passing to the quotient $\mathcal{L} \to \mathbb{L}$) exactly says that $f^*\exists_\pi = \exists\pi f^*$ where π is the generalized display map corresponding to forgetting the variables $\{x_\beta\}_{\gamma \leqslant \beta < \alpha} \in X_\alpha$.

We now check that it is an initial object in the category of λ-boolean algebras over $\mathcal{C}_T$. Let $\mathcal{B}$ be any λ-boolean algebra over $\mathcal{C}$. Any morphism $v : \mathbb{L}^T_\lambda \to \mathcal{B}$ has to satisfy:

1. $v(\bot) = \bot_{\mathcal{B}}$ and $v(\top) = \top_{\mathcal{B}}$.
2. $v(\neg\Phi) = \neg v(\Phi)$.
3. $v(\bigvee_{i\in I} \Phi_i) = \bigvee_{i\in I} v(\Phi_i)$ and $v(\bigwedge_{i\in I} \Phi_i) = \bigwedge_{i\in I} v(\Phi_i)$.
4. $$v(\exists\{x_\beta : \Gamma_\beta\}_{\gamma\leqslant\beta<\alpha}\Phi) = \exists\{x_\beta : \Gamma_\beta\}_{\gamma\leqslant\beta<\alpha} v(\Phi)$$
   and
   $$v(\forall\{x_\beta : \Gamma_\beta\}_{\gamma\leqslant\beta<\alpha}\Phi) = \forall\{x_\beta : \Gamma_\beta\}_{\gamma\leqslant\beta<\alpha} v(\Phi).$$

These form an inductive definition for a function $\mathcal{L}^T_\lambda \to \mathcal{B}$. So there is a unique such function $v : \mathcal{L}^T_\lambda \to \mathcal{B}$. To conclude, we only need to check that this function v descent to a function $\mathbb{L}^T_\lambda \to \mathcal{B}$ and is a morphism of λ-boolean algebras over $\mathcal{C}$. But this is rather immediate: We first observe, by induction over theorem 4.2.6, that if $\Phi \vdash \Psi$ then $v(\Phi) \leqslant v(\Psi)$. This implies that if $\Phi \dashv\vdash \Psi$ then $v(\Phi) = v(\Psi)$, so v does define a function $\mathbb{L}^T_\lambda \to \mathcal{B}$. The naturality condition

$$v(f^*(\Phi)) = f^*(v(\Phi))$$

can be proved by induction on the formula Φ, and the compatibility of v with all the boolean algebra operations and the quantifiers follows immediately from the definition of v. ∎

Proposition 4.2.30. Given any (small) clan $\mathcal{C}$ and λ a regular cardinal, there is an initial λ-boolean algebra over $\mathcal{C}$, which we denote by $\mathbb{L}^{\mathcal{C}}_\lambda$.

Note that by theorem 4.2.29, if T is a generalized κ-algebraic theory, with $\mathcal{C}_T$ its κ-contextual category them

$$\mathbb{L}^{\mathcal{C}_T}_\lambda = \mathbb{L}^T_\lambda.$$

This provides a way to define (or at least to characterize) the first-order language of any clan without having to explicitly give a syntactic description of the clan.

Proof. We can either remark that the λ-boolean algebras over $\mathcal{C}$ are (by their definition) the models of a multisorted λ-algebraic theory (with one sort for each object $c \in \mathcal{C}$) and hence there is an initial object by usual results on algebraic theories. Alternatively, we can use (see section 4.7) that every clan is equivalent to the contextual category of a generalized algebraic theory and use theorem 4.2.29 to conclude. ∎

Next, we mention a few more examples:

Example 4.2.31.

1. Let **Set** be the category of sets, considered as a clan where every arrow is a fibration. The contravariant power-set functor $\mathcal{P} : \mathbf{Set}^{op} \to \mathbf{Bool}_\lambda$ is a λ-Boolean algebra over **Set**. The Beck-Chevalley condition follows from theorem 4.2.32 below.
2. Given $F : \mathcal{C} \to \mathcal{D}$ a morphism of clans, if $\mathcal{B}$ is a λ-boolean algebra over $\mathcal{D}$, then $F^*\mathcal{B}$ defined by $F^*\mathcal{B}(\Gamma) = \mathcal{B}(F(\Gamma))$ is a λ-boolean algebra over $\mathcal{C}$.

3. Combining the two observations above, given any model M of a clan $\mathcal{C}$, that is a morphism of clans $M : \mathcal{C} \to \mathbf{Set}$, one has a boolean algebra $\mathcal{P}(M)$ over $\mathcal{C}$ given by pulling back example 1 along the morphism $M : \mathcal{C} \to \mathbf{Set}$. More explicitly:

$$\begin{array}{rccc} \mathcal{P}(M): & \mathcal{C}^{op} & \to & \mathbf{Set} \\ & \Gamma & \mapsto & \mathcal{P}(M(\Gamma)). \end{array}$$

Lemma 4.2.32. Given a square of sets,

$$\begin{tikzcd} W \arrow[r, "f"] \arrow[d, "g"] & X \arrow[d, "h"] \\ Y \arrow[r, "k"] & Z, \end{tikzcd}$$

then the power set functor satisfies the Beck-Chevalley condition on this square, *i.e.,* $k^*\exists_h = \exists_g f^*$ as maps $\mathcal{P}(X) \to \mathcal{P}(Y)$ if and only if the square is a weak pullback square *i.e.,* if and only if the cartesian gap map $W \to Y \times_Z X$ is surjective.

Proof. Given a subset $P \subset X$ one has:

$$k^* h_! P = \{y \in Y | k(y) = h(p) \text{ for some } p \in P\},$$

$$g_! f^* P = \{g(w) | f(w) \in P\}.$$

Surjectivity of the map $W \to Y \times_Z X$ gives a canonical way to make any element of $k^* h_! P$ into an element of $g_! f^* P$, and conversely, applying the equality to $P = \{p\}$ produces the surjectivity of $W \to Y \times_Z X$. ∎

In this new setting with just a clan $\mathcal{C}$, one can still define the set of formulas $\mathbb{L}^{\mathcal{C}}_{\lambda}$ as the initial λ-boolean algebra over $\mathcal{C}$. We now explain what it means for formulas defined this way to be "true" or "false" given a model and an interpretation of its variables in the model.

Construction 4.2.33. Given a clan $\mathcal{C}$ and a model of $M : \mathcal{C} \to \mathbf{Set}$ we have, as explained in theorem 4.2.31, a λ-boolean algebra over $\mathcal{C}$ defined by $c \mapsto \mathcal{P}(M(c))$. By initiality of the κ-boolean algebra $\mathbb{L}^{\mathcal{C}}_{\lambda}$, there exists a unique morphism of λ-boolean algebras over $\mathcal{C}$:

$$| - |_M : \mathbb{L}^{\mathcal{C}}_{\lambda} \to \mathcal{P}(M).$$

This morphism associates each formula ϕ in context Γ to a subset $|\phi|_M \subseteq M(\Gamma)$. An element $x \in M(\Gamma)$ is said to *satisfy* ϕ if $x \in |\phi|_M$, with some abuse of notation, we say that "$\phi(x)$ is true" in this case. We also write

$$M \vdash \phi(x)$$

when we want to insist on which model we are talking about. When Γ is the terminal object of $\mathcal{C}$ *i.e.,* ϕ is a closed formula, then $M(\Gamma) = \{*\}$. Therefore, $\mathcal{P}(M(\Gamma)) = \{\perp, \top\}$ so that $|\phi|_M$ is simply a proposition. One then says that M satisfies ϕ, and we write $M \vdash \phi$.

Lemma 4.2.34. When $\mathcal{C} = \mathcal{C}_T$ is the κ-contextual category of a κ-generalized algebraic theory, then through the identification $\mathbb{L}^T_\lambda = \mathbb{L}^{\mathcal{C}}_\lambda$, the two definitions of validity of a formula on elements of a model given by theorem 4.2.9 and theorem 4.2.33 are equivalent.

Proof. Defining the validity of formulas as in theorem 4.2.33 it is immediate to verify all the explicit conditions of the inductive definition given in theorem 4.2.9 simply because the map $\mathbb{L}^{\mathcal{C}}_\lambda \to \mathcal{P}(M)$ is a morphism of λ-boolean algebras. Hence, it immediately follows by induction on formulas that the two definitions are equivalent. ∎

Construction 4.2.35. Let $F : \mathcal{C} \to \mathcal{D}$ be a morphism of clans. And let $\mathbb{L}^{\mathcal{C}}_\lambda$ and $\mathbb{L}^{\mathcal{D}}_\lambda$ be their respective initial λ-boolean algebras. From the fact that $\mathbb{L}^{\mathcal{C}}_\lambda$ is initial, there is a morphism of λ-boolean algebras

$$\alpha^F : \mathbb{L}^{\mathcal{C}}_\lambda \to F^*\left(\mathbb{L}^{\mathcal{D}}_\lambda\right).$$

For any $\Gamma \in \mathcal{C}$ and any formula $\Phi \in \mathbb{L}^{\mathcal{C}}_\lambda(\Gamma)$ we denote $F(\Phi) := \alpha^F_\Gamma(\Phi)$ which is a formula in context $F(\Gamma)$ *i.e.*, an element of $\mathbb{L}^{\mathcal{D}}_\lambda(F(\Gamma))$. The following is immediate from the definition above:

Proposition 4.2.36. Let $M : \mathcal{D} \to \mathbf{Set}$ a model of the clan $\mathcal{D}$, $\Phi \in \mathbb{L}^{\mathcal{C}}_\lambda(\Gamma)$ a formula in context Γ and $x \in M(F(\Gamma))$. Then, $M \vdash \alpha_F(\Phi)(x)$ if and only if $F^*M \vdash \Phi(x)$.

Finally, we finish this section by showing the key property of invariance of formulas along anodyne fibrations. An invariance property will be established in the next section assuming we are working with a model category, but this first invariance property is purely algebraic. This is also the key observation in Makkai FOLDS [52] and it is directly inspired from it.

We start with the following observation: Let $\mathcal{C}$ be a clan and $f : M \to N$ a morphism of two $\mathcal{C}$-models, then we have an obvious map $f^* : P(N) \to P(M)$ which sends a subset $A \subset N(c)$ for $c \in \mathcal{C}$ to

$$f_c^{-1}(A) \subset M(c)$$

this map is easily seen to be a *weak* morphism of boolean algebras over $\mathcal{C}$. It is compatible with the boolean algebra operations and the ordinary contravariant functoriality, but it does not have to be compatible with the covariant functoriality $\exists_\pi$ along fibrations. However, one has:

Lemma 4.2.37. Let $\mathcal{C}$ be a clan and let $f : M \to N$ be a morphism between two $\mathcal{C}$-models. Then f is an anodyne fibration if and only if $f^* : P(N) \to P(M)$ is a morphism of λ-boolean algebras.

Proof. We only need to show that for every fibration $p : X \to Y$ the following square

$$\begin{tikzcd}
P(N(X)) \arrow[r, "f^*_X"] \arrow[d, "\exists"] & P(M(X)) \arrow[d, "\exists"] \\
P(N(Y)) \arrow[r, "f^*_Y"] & P(M(Y)).
\end{tikzcd}$$

commutes. From theorem 4.2.32 this is equivalent to say that the dotted map in

$$\begin{tikzcd}
M(X) \arrow[dr, dashed] \arrow[drr, bend left, "f_X"] \arrow[ddr, bend right, "\pi_*"'] & & \\
& P \arrow[r] \arrow[d] & N(X) \arrow[d, "\pi_*"] \\
& M(Y) \arrow[r, "f_Y"] & N(Y)
\end{tikzcd}$$

is surjective. But this is exactly the characterization of anodyne fibrations given in theorem 4.2.25. ∎

This allows us to deduce the key result of invariance of formulae along anodyne fibrations of models. Basically, the validity of formulae is preserved by anodyne fibrations of models:

Corollary 4.2.38. Let $\mathcal{C}$ be a clan and let $f : M \twoheadrightarrow N$ be an anodyne fibration between two $\mathcal{C}$-models. For $c \in \mathcal{C}$, let $x \in M(c)$ and $\phi \in \mathbb{L}^{\mathcal{C}}_{\lambda}$ be any formula. Then

$$M \vdash \phi(x) \Leftrightarrow N \vdash \phi(f(x))$$

Proof. As $f : M \to N$ is an anodyne fibration, it follows from theorem 4.2.37 that the map $f^* : \mathcal{P}(N) \to \mathcal{P}(M)$ is a morphism of boolean algebra over $\mathcal{C}$. Hence, by initiality of $\mathbb{L}^{\mathcal{C}}_{\lambda}$, the unique morphism $|\_|_M : \mathbb{L}^{\mathcal{C}}_{\lambda} \to \mathcal{P}(M)$ is obtained as a composite

$$\mathbb{L}^{\mathcal{C}}_{\lambda} \xrightarrow{|\_|_M} \mathcal{P}(M) \xrightarrow{f^*} \mathcal{P}(N).$$

By definition, $M \vdash \phi(x)$ means that $x \in |\phi|_M$ while $N \vdash \phi(f(x))$ means that $x \in f^*|\phi|_N$, hence the result immediately follows. ∎

4.2.4 The language of a weak model category and two invariance theorems

Construction 4.2.39. Given $\mathcal{M}$ a weak model category, the category $\mathcal{M}^{\textsc{Cof}}$ of cofibrant objects with cofibrations between them forms a coclan. We define the language of $\mathcal{M}$ to be the language of the coclan $\mathcal{M}^{\textsc{Cof}}$. For any regular cardinal λ, we denote by $\mathbb{L}^{\mathcal{M}}_{\lambda}$ the λ-boolean algebra $\mathbb{L}^{\mathcal{M}^{\textsc{Cof}}}_{\lambda}$ over $\mathcal{M}^{\textsc{Cof}}$.

Note that for each *cofibrant* object $X \in \mathcal{M}$, we have a set (or possibly a class if $\mathcal{M}$ is large) of formulas $\mathbb{L}^{\mathcal{M}}_{\lambda}(X)$.

Remark 4.2.40. There is a size issue to be mentioned here. In most practical examples, $\mathcal{M}^{\textsc{Cof}}$ is a large category while the construction of $\mathbb{L}^{\mathcal{M}^{\textsc{Cof}}}_{\lambda}$ developed in section 4.2.3 assumes it is a small category. We can deal with this by invoking a larger Grothendieck universe, but this has a practical consequence: The set of formulas $\mathbb{L}^{\mathcal{M}}_{\lambda}(X)$ might not be a small set. Indeed, it lives in the same Grothendieck universe as the one in which $\mathcal{M}^{\textsc{Cof}}$ is small.

Construction 4.2.41. If $X \in \mathcal{M}$ then we can define a model of the coclan $\mathcal{M}^{\mathrm{Cof}}$ using the restricted Yoneda embedding:

$$よ_X : \begin{array}{ccc} (\mathcal{M}^{\mathrm{Cof}})^{\mathrm{op}} & \to & \mathbf{Set} \\ c & \mapsto & \mathrm{Hom}(c,X), \end{array}$$

which defines a functor $よ : \mathcal{M} \to \mathrm{Mod}(\mathcal{M}^{\mathrm{Cof}})$.

Definition 4.2.42. Let $\mathcal{M}$ be a weak model category. For $c \in \mathcal{M}$ a cofibrant, and $X \in \mathcal{M}$ any object, $v : c \to X$ and $\phi \in \mathbb{L}^{\mathcal{M}}_{\lambda}(c)$ we write

$$X \vdash \phi(v)$$

to mean

$$よ_X \vdash \phi(v)$$

where v is seen as an element of $よ_X(c) = \mathrm{Hom}(c,X)$.

Remark 4.2.43. In the special case where $\mathcal{M} = \mathrm{Mod}(T)$ is the category of models of a generalized κ-algebraic theory (or more generally of a κ-coclan), then $\mathbb{L}^{\mathcal{M}}_{\lambda}$ is the initial λ-boolean algebra over the coclan of all cofibrant objects of $\mathcal{M}$, while the syntactic category of T is equivalent to a full sub-κ-coclan of that. In particular, there is a morphism of λ-boolean algebra over the syntactic category $\mathcal{C}_T$

$$\mathbb{L}^{T}_{\lambda}(X) \to \mathbb{L}^{\mathcal{M}}_{\lambda}(X) \qquad (\text{For } X \in \mathcal{C}_T).$$

If we denote this map by i then for X any model of T we can easily check that

$$X \vdash \phi(v) \Leftrightarrow X \vdash i(\phi)(v)$$

for any $c \in \mathcal{C}_T$ and $\phi \in \mathbb{L}^{T}_{\lambda}(c)$, where the left-hand side is interpreted in the sense of theorem 4.2.1 while the right-hand side is in terms of theorem 4.2.42.

Note that we do expect these to be the same. Informally, $\mathbb{L}^{T}_{\lambda}$ corresponds to an $\mathcal{L}_{\kappa,\lambda}$ logic, in the sense that quantifier can only be applied to formulas in κ-small context—applied to less than κ-many variables at the same time—while $\mathbb{L}^{\mathcal{M}}_{\lambda}$ corresponds to an $\mathcal{L}_{\infty,\lambda}$ logic, where quantifiers can be applied to arbitrarily many formulas at the same time.

Theorem 4.2.44. Let $\mathcal{M}$ be a weak model category, $c \in \mathcal{M}$ a cofibrant object and $\phi \in \mathbb{L}^{\mathcal{M}}_{\lambda}(c)$.

- 1^{st} **invariance theorem:** Let $v_1, v_2 : c \to X$ be two homotopically equivalent maps with X fibrant. Then
$$X \vdash \phi(v_1) \quad \Leftrightarrow \quad X \vdash \phi(v_2).$$
- 2^{nd} **invariance theorem:** Let $f : X \to Y$ be a weak equivalence between two fibrant objects and $v : c \to X$ any map. Then
$$X \vdash \phi(v) \quad \Leftrightarrow \quad Y \vdash \phi(fv).$$

Proof. We start by first observing that the second invariance theorem in the special case where f is a trivial fibration immediately follows from theorem 4.2.38 as a trivial fibration f has the right lifting property against all core cofibrations and hence is sent to an anodyne fibration in $\mathrm{Mod}(\mathcal{M}^{\textsc{Cof}})$ by the functor from theorem 4.2.41.

We use this to prove the 1^{st} invariance theorem: If $v_1, v_2 : c \to X$ are homotopic then there exists a map h:

$$\begin{tikzcd}
& X \\
c \arrow[ru, "v_2"] \arrow[r, dotted, "h"] \arrow[rd, "v_1"'] & PX \arrow[u, "p_2"'] \arrow[d, "p_1"] \\
& X.
\end{tikzcd}$$

The two maps $p_1, p_2 : PX \to X$ are trivial fibrations (they are both fibrations and weak equivalences), $v_1 = p_1 \circ h$ and $v_2 = p_2 \circ h$. By the observation above, we have:

$$\begin{array}{cccc}
 & X & \vdash & \phi(v_1) \\
\Leftrightarrow & X & \vdash & \phi(p_1 h) \\
\Leftrightarrow & PX & \vdash & \phi(h) \\
\Leftrightarrow & X & \vdash & \phi(p_2 h) \\
\Leftrightarrow & X & \vdash & \phi(v_2)
\end{array}$$

This concludes the proof of the 1^{st} invariance theorem.

Next, we observe it is enough to prove the second invariance theorem when X and Y are both bifibrant. Indeed, starting from $f : X \to Y$ a weak equivalence between fibrant objects, $v : c \to X$ and $\phi \in \mathbb{L}^{\mathcal{M}}_X(c)$ as in the theorem. We can replace both X and Y by bifibrant objects

$$\begin{tikzcd}
X^{\textsc{Cof}} \arrow[r, "\sim", "f'"'] \arrow[d, two heads, "\sim"] & Y^{\textsc{Cof}} \arrow[d, two heads, "\sim"] \\
X \arrow[r, "\sim", "f"'] & Y.
\end{tikzcd}$$

First replacing X by a cofibrant object $X^{\textsc{Cof}}$ and then factoring the map $X^{\textsc{Cof}} \to Y$, which is a weak equivalence, as a trivial cofibration followed by a trivial fibration. The map $v : c \to X$, can be lifted to map $v' : c \to X^{\textsc{Cof}}$. As we can already apply the 2^{nd} invariance theorem to trivial fibrations, we have that:

$$X \vdash \phi(v) \Leftrightarrow X^{\textsc{Cof}} \vdash \phi(v')$$

$$Y \vdash \phi(fv) \Leftrightarrow Y^{\textsc{Cof}} \vdash \phi(f'v').$$

Therefore, it is enough to show the 2^{nd} invariance theorem for bifibrant objects.

This last step is achieved essentially using a "Brown factorization": any weak equivalence between bifibrant objects can be factored as a section of a trivial fibration followed by a trivial fibration. Indeed, if $f : X \to Y$ is a map between bifibrant objects we can form the pullbacks:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "e'"] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "e"] \\
X \times_Y PY \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & PY \arrow[d, two heads] \\
X \times Y \arrow[r] \arrow[d, two heads, "\pi_1"] \arrow[dr, phantom, "\lrcorner", very near start] & Y \times Y \arrow[d, two heads, "\pi_1"] \\
X \arrow[r, "f"] & Y.
\end{tikzcd}$$

Note that because the fibrations $PY \to Y$ are trivial fibration, the map $X \times_Y PY \to X$ in the diagram above is also a trivial fibration. The total vertical maps are both the identity. Which gives us a diagram:

$$\begin{tikzcd}
 & X \arrow[dl, "id_X"'] \arrow[d, "e'"] \arrow[dr, "f"] & \\
X & X \times_Y PY \arrow[l, two heads, "\sim", "q"'] \arrow[r, two heads, "p"'] & Y
\end{tikzcd}$$

Where p is the map $X \times_Y PY \twoheadrightarrow X \times Y \overset{\pi_2}{\twoheadrightarrow} Y$. Note that all maps in this diagram are weak equivalences due to the 2-out-of-3 condition. We can now prove the theorem, we have

$$X \vdash \phi(v) \Leftrightarrow X \times_Y PY \vdash \phi(e'v)$$

because $v = qe'v$ and q is a trivial fibration, and

$$X \times_Y PY \vdash \phi(e'v) \Leftrightarrow Y \vdash \phi(fv)$$

because p is a trivial fibration and $fv = pe'v$. Hence, combining the two

$$X \vdash \phi(v) \Leftrightarrow Y \vdash \phi(fv)$$

∎

Finally, we explain how Quillen adjunctions act on formulas. A *Quillen adjunction* between two weak model categories is an adjunction

$$L : \mathcal{C} \leftrightarrows \mathcal{D} : R$$

where the left adjunction L sends cofibrations to cofibrations and the right adjoint R sends fibrations to fibrations.

Remark 4.2.45. There is also a more general notion called "weak Quillen functors" introduced in [29] which is sometimes more convenient. The functor L is only defined on cofibrant objects and R on fibrant objects, and they are only required to preserve core (co)fibrations – all results in this section below, as well as the 4^{th} invariance theorem from section 4.4 apply to weak Quillen adjunctions too. We restrict ourselves to Quillen adjunctions in the paper, unless otherwise stated, for simplicity, and because this already cover most of the applications.

Construction 4.2.46. Given a Quillen adjunction[2] $L : \mathcal{C} \leftrightarrows \mathcal{D} : R$. Then, L restricts to a coclan morphism $L : \mathcal{C}^{\mathrm{Cof}} \to \mathcal{D}^{\mathrm{Cof}}$, which following theorem 4.2.35 we have a (unique) comparison map

$$\alpha_L : \mathbb{L}^{\mathcal{C}}_{\lambda} \to L^*\mathbb{L}^{\mathcal{D}}_{\lambda}$$

obtained from the fact that $\mathbb{L}^{\mathcal{C}}_{\lambda}$ is an initial boolean algebra over $\mathcal{C}$. As before, if $\phi \in \mathbb{L}^{\mathcal{C}}_{\lambda}(C)$, we often write $L(\phi)$ instead of $\alpha_L(\Phi)$. Note that $L(\phi) \in \mathbb{L}^{\mathcal{D}}_{\lambda}(L(C))$.

Finally, exactly as in theorem 4.2.35 we have:

Proposition 4.2.47. For a Quillen adjunction $L : \mathcal{C} \leftrightarrows \mathcal{D} : R$, any[3] object $X \in \mathcal{D}$, and cofibrant object $C \in \mathcal{C}$, any map $v : C \to R(X)$ corresponding to $\tilde{v} : LC \to X$, and $\phi \in \mathbb{L}^{\mathcal{C}}_{\lambda}$ we have

$$R(X) \vdash \phi(v) \Leftrightarrow X \vdash L(\phi)(\tilde{v}).$$

Proof. See theorem 4.2.35. ∎

The 4^{th} invariance theorem theorem that we will establish in section 4.4 as theorem 4.4.2 show that for a Quillen equivalence, this construction gives an equivalence between the language of $\mathcal{C}$ and of $\mathcal{D}$ in an appropriate sense.

4.3 Examples of languages of model categories

In this section, we examine some examples of the language associated to a model category by applying the construction as described in section 4.2. We include examples we believe to be of interest. Furthermore, we start with some general considerations which allows us to construct the language of a model category.

When applying the theory introduced in section 4.2 to a model category $\mathcal{M}$, we have two possible approaches: we can manipulate formulas as element of the free Boolean algebra over $\mathcal{M}^{\mathrm{Cof}}$, following the approach from section 4.2.3, or we can try to build a generalized algebraic theory whose first language is the same as the language of $\mathcal{M}$. For example, we could try to realize $\mathcal{M}$ as the category of models of some generalized κ-algebraic theory, or if that is not possible try to realize the category of κ-presentable cofibrant objects of $\mathcal{M}$ as the opposite of the syntactic category some generalized κ-algebraic theory.

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[2] Or more generally a weak Quillen adjunction in the sense of [29].

[3] If L and R are only a weak Quillen adjunction, then X needs to be fibrant.

We believe that, once we are familiar with how this language works the first approach is simpler. But in order to build familiarity with the languages, in all the examples we will cover below we will try to use the second approach and build a more or less explicit generalized algebraic theory associated to each example, in order to show the reader what can be done in the logic of each case.

It is shown in section 4.6 that any κ-clan is equivalent to the syntactic category of a generalized κ-algebraic theory. So in general, given $\mathcal{M}$ a combinatorial (weak) model category, we can always find a regular cardinal κ and a generalized κ-algebraic theory so that the language associated to $\mathcal{M}$ is the language of this generalized algebraic theory. Unfortunately, the construction of this theory following section 4.6 is extremely unexplicit.

What we would like to do here is to give some tools to help "guess" a simpler generalized algebraic theory that works on concrete examples. Given that our goal is only to guess the correct theory for a few examples, we will not try to make this completely formal and rigorous – though it might be possible.

To that end, let us recall some facts about a generalized κ-algebraic theory T, and of the κ-contextual category $\mathbb{C}_T$ associated to it. Theorem 4.5.3 states inductively what it means for a judgment $\Gamma \vdash \Delta\,\mathsf{Type}$ in a κ-pretheory to be well-formed in T; this is the case whenever Γ is a context, which itself entails that any constituent of Γ is obtained from a derived rule of the κ-pretheory T. In turn, each derived rule is deduced from the list of theorem 4.5.4, or using a rule previously derived. In a generalized κ algebraic theory, each type introduction axiom (derived judgment) is well-formed by theorem 4.5.12. Concretely, this means that in order to build new types in context Γ' we must know that all the variables used in Γ' must be previously be constructed in some context Γ. In a sense, each type must be constructed from more primitive types.

We can use the above in the following:

Remark 4.3.1. Let T be a generalized κ-algebraic theory and $\mathbb{C}_T$ the syntactic κ-contextual category of T with the natural κ-clan structure *i.e.,* in which the fibrations are the generalized display maps. Each type axiom $\Gamma \vdash A\,\mathsf{Type}$ of T corresponds to a display map $(\Gamma.A \twoheadrightarrow \Gamma)$. Now, the type of axioms of T admit a well-founded transitive relation $<$ such that for each type axiom $\Gamma \vdash A\,\mathsf{Type}$ we can show that Γ is a context using only type axioms "smaller" than $\Gamma \vdash A\,\mathsf{Type}$. In particular, it means that only types "smaller than A" can appear in the context Γ. Formulated categorically, this means that the map $\Gamma \to 1$ can be constructed as κ-small composite of pullback of display map $\Gamma'.B \to \Gamma'$, for $\Gamma' \vdash B\,\mathsf{Type}$ type axioms that are smaller than $\Gamma \vdash A\,\mathsf{Type}$. Recall from theorem 4.2.22 that $\mathrm{Mod}(T)$ has a weak factorization system which is cofibrantly generated by the set

$$I = \{\, \text{よ}_A \hookrightarrow \text{よ}_B \in \mathrm{Mod}(T) | B \twoheadrightarrow A \in \mathbb{C}_T\}.$$

Given that every display map is a κ-small composite of pullback of the display map corresponding to type axioms. We can restrict the set of generators to the display maps corresponding to type axioms, which then comes with this additional well-founded relation.

The previous example motivates:

Definition 4.3.2. Let $\mathcal{C}$ be model category and $\textsc{Cof}(\mathcal{C})$ the class of cofibrations. Assume that the cofibrations are generated by a set I. We say that the set of generating cofibrations is *well-founded* if there exists a well-founded relation $<$ on I such that for all $i \in I$, the map $\varnothing \to Dom(i)$ can be written as a κ-composite of pushouts of maps $j \in I$ with $j < i$.

Example 4.3.3. As explained in theorem 4.3.1, if T is a generalized κ-algebraic theory, then the weak factorization from theorem 4.2.22 on $\mathrm{Mod}(T)$ has a well-founded set of generators corresponding to the type of axioms of T.

The general idea is; if we start from a combinatorial weak factorization system, and we want to see it as coming from an explicitly given generalized algebraic theory, we start by finding a well-founded set of generators, and then we build a theory whose type axioms correspond to these generators.

Note that in particular, we need the factorization system to be generated by map with "cofibrant" domain, that is we need the model category to be "tractable". Most model structures we work with in practice, in fact all the examples we will encounter here are tractable. But in general this is not an obstruction, this can be achieved using lemma 4.7 of [30]:

Proposition 4.3.4. [30, 4.7 Lemma]. Fix κ an uncountable regular cardinal. Let (L_1, R_1) and (L_2, R_2) two κ-accessible weak factorization systems on a locally κ-presentable category $\mathcal{C}$ such that $L_1 \subset L_2$ or $R_2 \subset R_1$. There is a κ-accessible weak factorization system (L_3, R_3) on $\mathcal{C}$ such that R_3 is the class of maps tha have the right lifting property against all L_1-maps whose domain is L_2-cofibrant. If (L_1, R_1) is κ-combinatorial, then (L_3, R_3) is also κ-combinatorial.

Observation 4.3.5. If $\mathcal{M}$ is a combinatorial weak model category, then there exists a tractable combinatorial weak model category structure on $\mathcal{M}$ with the same core cofibrations and core acyclic cofibrations. In order to see this, we apply theorem 4.3.4 taking (L_1, R_1)=(acyclic cofibrations, fibrations) and (L_2, R_2)=(cofibrations, acyclic fibrations). This produces a weak factorization system (L_3, R_3) where the class R_3 of fibrations is generated by acyclic cofibrations with cofibrant domain. We apply the the result again, but on (cofibrations, acyclic fibrations)=(L_2, R_2)=(L_1, R_1) to get another weak factorization system (L_3', R_3') where the class R_3' is generated by cofibrations with cofibrant domain. Note this process does not change the core (acyclic) cofibrations or core (acyclic) fibrations.

Once we have generating cofibrations with cofibrant domain, there is always an easy way to get a well-founded set of generators:

Example 4.3.6. If L is a set of generating cofibrations with cofibrant domain of a combinatorial weak model category, then we can get a well-founded class of cofibrations by setting $L' := \{\varnothing \to Dom(l) | l \in L\} \coprod L$. In this case, we can set $(\varnothing \to Dom(l)) < f$ for $f \in L$ and $l \in L$.

Theorem 4.3.3 shows that starting with a κ-clan, one can get a cofibrantly generated weak factorization system on the category of models $\mathrm{Mod}(\mathcal{C})$ such that the generating set of cofibrations is well-founded. We can reverse this process in the sense that if we are given a weak factorization system with a well-founded set of generating cofibrations, then we can produce a generalized κ-algebraic theory from it, and therefore the κ-clan associated to it.

The next example is similar to theorem 4.3.1.

Construction 4.3.7. Let $\mathcal{C}$ be a κ-clan. Assume that $\mathcal{C}$ has a weak factorization system that is cofibrantly generated by a set I with a well-founded relation. Recall that this means that for a cofibration $i : A \hookrightarrow B$ the map $\varnothing \to A$ is a κ-composite of pushouts of maps $j \in I$ with $j < i$. Therefore, we can introduce a type axiom:

$$\overline{A} \vdash \overline{B}\,\mathsf{Type}$$

for $i : A \hookrightarrow B \in I$. The notation $\overline{A}$ denotes the context in which the new type $\overline{B}$ is built, and the context $\overline{A}$ is obtained using types strictly smaller than $\overline{B}$, which reflects the decomposition of the map $\varnothing \hookrightarrow A$ as κ-composite of pushouts of maps $j \in I$ smaller than i.

We can think of this construction as similar to the functor $U : \kappa\text{-CON} \to \kappa\text{-GAT}$ from section 4.6.3.2 which produces a generalized κ-algebraic theory $U(\mathcal{C})$ from a κ-contextual category $\mathcal{C}$. In particular, for a display map $B_{\lambda+1} \twoheadrightarrow B_{\lambda} \in \mathcal{C}$ it gives a type axiom $\overline{B_{\lambda}} \vdash \overline{B_{\lambda+1}}\,\mathsf{Type}$.

Remark 4.3.8. For each of the examples below, we start with a Quillen model category $\mathcal{M}$ and apply theorem 4.3.7 to obtain a theory $T_{\mathcal{M}}$. In general, this is the guiding principle that will allow us to identify the statements, and the language, to which the invariance theorems apply.

Furthermore, using the theory $T_{\mathcal{M}}$ we can consider the category $\mathrm{Mod}(T_{\mathcal{M}})$ and use theorem 4.2.22 to obtain a weak factorization system. Through this process, the cofibrations and trivial fibrations we obtain coincide with the ones from the Quillen model category we start with. However, in general we do not have an equivalence of categories $\mathrm{Mod}(T_{\mathcal{M}}) \cong \mathcal{M}$.

4.3.1 Categories

Let us illustrate our construction on this prime example we have been referring to throughout the paper. Recall that $\mathbf{0}$ is the empty category, $\mathbf{1} := \{0\}$ is the category with a single object, $\mathbf{2} := \{0 \to 1\}$ the arrow category and $P := \{0 \rightrightarrows 1\}$ the category with two parallel arrows. Finally, $\mathcal{J} := \{0 \leftrightarrows 1\}$ denotes the walking isomorphism category. The following result appears in [58].

Theorem 4.3.9. There is Quillen model structure on the category **Cat** such that:

1. Weak equivalences are the equivalences of categories,
2. Cofibrations are the functors injective on objects,

3. Fibrations are the isofibrations.

Furthermore, this models structure is cofibrantly generated. The sets

$$I := \{\mathbf{0} \xrightarrow{u} \mathbf{1},\ \{0\} \sqcup \{1\} \xrightarrow{v} \mathbf{2},\ P \xrightarrow{w} \mathbf{2}\} \text{ and } J := \{\mathbf{1} \to \mathcal{J}\}$$

are the generating cofibration and trivial cofibrations respectively.

In this model structure all objects are cofibrant. We can immediately associate for each generator in I a sort in the following way:

$$\begin{aligned} \mathbf{0} \to \mathbf{1} &\longmapsto \vdash \mathsf{Ob}\,\mathsf{Type} \\ \{0\} \sqcup \{1\} \to \mathbf{2} &\longmapsto x, y : \mathsf{Ob} \vdash \mathsf{Hom}(x, y)\ \mathsf{Type} \\ P &\longmapsto x, y : \mathsf{Ob}, f, g : \mathsf{Hom}(x, y) \vdash \mathsf{Eq}(f, g)\ \mathsf{Type} \end{aligned}$$

Note that while the type Ob has no dependencies, the type $\mathsf{Hom}(x, y)$ depends on two elements of type Ob, which is encoded in the cofibration $\{0\} \sqcup \{1\} \to \mathbf{2}$. The same situation applies with the type Eq which furthermore has dependencies on the types Ob and Hom, now the cofibration $P \hookrightarrow \mathbf{2}$ expresses this.

The resulting theory is what we introduced earlier $Cat_{=}$ which by convenience we recall here. This is defined as:

1. Type of objects: $\vdash \mathsf{Ob}\,\mathsf{Type}$.
2. Type of morphisms: $x : \mathsf{Ob},\ y : \mathsf{Ob} \vdash \mathsf{Hom}(x, y)\ \mathsf{Type}$.
3. Equality type: $x, y : \mathsf{Ob}, f, g : \mathsf{Hom}(x, y) \vdash \mathsf{Eq}(f, g)\mathsf{Type}$
4. Composition operation: $x : \mathsf{Ob},\ y : \mathsf{Ob},\ z : \mathsf{Ob},\ f : \mathsf{Hom}(x, y),\ g : \mathsf{Hom}(y, z) \vdash g \circ f : \mathsf{Hom}(x, z)$.
5. Identity operator: $x : \mathsf{Ob} \vdash \mathsf{id}_x : \mathsf{Hom}(x, x)$.

Subject to the following axioms:

- $x : \mathsf{Ob},\ y : \mathsf{Ob},\ f : \mathsf{Hom}(x, y) \vdash \mathsf{id}_y \circ f \equiv f$.
- $x : \mathsf{Ob},\ y : \mathsf{Ob}, f : \mathsf{Hom}(x, y) \vdash f \circ \mathsf{id}_x \equiv f$.
- $x : \mathsf{Ob},\ y : \mathsf{Ob},\ z : \mathsf{Ob},\ w : \mathsf{Ob},\ f : \mathsf{Hom}(x, y), g : \mathsf{Hom}(y, z), h : \mathsf{Hom}(z, w) \vdash (h \circ g) \circ f \equiv h \circ (g \circ f)$.
- $x, y : \mathsf{Ob}, f : \mathsf{Hom}(x, y) \vdash r_f : \mathsf{Eq}(f, f)$.
- $x, y : \mathsf{Ob}, f, g : \mathsf{Hom}(x, y), a : \mathsf{Eq}(f, g) \vdash f \equiv g$.

- $x, y : \mathsf{Ob}, f, g : \mathsf{Hom}(x, y), a : \mathsf{Eq}(f, g) \vdash a \equiv r_f$.

As remarked in theorem 4.2.4 the language we obtain is the same as the one given by [13] and [23]. In the introduction we presented the formula for an object x to be terminal:

$$\forall y \in \mathrm{Ob}, (\exists v \in \mathrm{Hom}(y, x) \wedge \forall u, w \in \mathrm{Hom}(y, x), \mathsf{Eq}(u, w)) .$$

Such formula is written in the language of categories.

Observation 4.3.10. We verify the above differently to showcase the fact that we do not need to explicitly know the language (type theory) associated to a model category, we only need to know that can be constructed out of cofibrations. The formula above is constructed by first quantifying universally over the cofibration $\mathbf{0} \to \mathbf{1}$ to give $\forall y \in \mathsf{Ob}$. Note that applying the existential quantifier to $\{0\} \sqcup \{1\} \to \mathbf{2}$ give us $\exists v \in \mathrm{Hom}(y, x)$ and the universal quantifier on $\mathbf{1} \to \mathcal{J}$. In the end, the formula can be seen as a composition pushouts "in context x." Building the context of a formula is not an easy task, however, it might be easier to describe a pushout.

Remark 4.3.11. We mentioned at the beginning of the section that the association we do from cofibrations to types is not extremely formal. Again, the reason is that the equivalence between κ-clans and generalized κ-algebraic theories, section 4.6, is not explicit. The association we make, for categories and the other examples below, is the obvious one and ad-hoc to the expected theory. From the start, we know what our intended models are, so once we have the types we define the operations and impose the equations that our intended models satisfy. We stress that this is informal and not very precise.

Remark 4.3.12. In general, a cofibration in a model category could be decomposed as a pushouts of cofibrations in more than one way. Depending on our choices, it might happen that we end up with different, but equivalent, theories.

One of the worst case scenarios is when we do not have a straightforward well-ordering, see the case for unbounded chain complexes below section 4.3.4.

Although these remarks deserve a proof, we choose not give one as this would divert us from the objective of the paper.

4.3.2 2-categories and Bicategories

In this section we examine the language associated to the canonical model structures on the categories **2-Cat** and $\mathbf{Bicat_s}$, respectively. The model structure for these two categories was defined in [44] and [45].

Given a category C its suspension $\sum C$, is defined as the 2-category with two objects X, Y, the hom categories are $\sum C(X, X) = \sigma C(Y, Y) = \sum C(Y, X) = \varnothing$ and $\sum C(X, Y) = C$. Furthermore, each bicategory $\mathscr{B} \in \mathbf{Bicat_s}$ has an underlying **Cat-graph**, in the sense of [75]. This induces a functor $U : \mathbf{Bicat_s} \to \mathbf{Cat\text{-}graph}$ which has left adjoint F, this gives us the free bicategory generated by a **Cat**-graph. The suspension of a category C can be seen as a **Cat**-graph associated to C. The free bicategory generated by the suspension of a category is denoted by $\sum\mathscr{C}$. Moreover, this construction is functorial.

[45, Theorem 3] constructs a model structure for the category of bicategories. This model structure is cofibrantly generated with generating cofibrations given by the suspension of the generating cofibrations of the canonical model structure on **Cat** and an additional functor we specify below. Finally, $\mathscr{E}$ is the "free-living adjoint equivalence " is the bicategory with objects x, y, freely generated by 1-cells $f : x \to y$ and $g : y \to x$, and two invertible 2-cells $\eta : 1_x \Rightarrow gf$, $\varepsilon : fg \Rightarrow 1_y$ satisfying the familiar triangle identities.

Theorem 4.3.13. There is a model structure on the category $\mathbf{Bicat_s}$ of bicategories and strict bifunctors such that:

1. Weak equivalences are the biequivalences,
2. Fibrations are the strict bifunctors with the equivalence lifting property.

Furthermore, the model structure is cofibrantly generated by the sets

$$I := \{\nvdash \to \mathbb{1}, \Sigma u, \Sigma v, \Sigma w\} \text{ and } J := \{\mathbb{1} \to \mathscr{E}\}$$

where $\nvdash$ is the empty bicategory, $\mathbb{1}$ is the bicategory with a single object and no non-identity 2-cells, the functors u, v, w come from theorem 4.3.9, and the bifunctor in J picks the object x.

When we analyze the set of generating cofibrations I we rediscover the generalized algebraic theory of bicategories $Bicat_{=}$:

- $\nvdash \to \nVdash \longmapsto\ \vdash \mathsf{Ob}\,\mathsf{Type}$
- $\{x\} \sqcup \{y\} \overset{\sum u}{\to} \{x \to y\} \mapsto x, y : \mathsf{Ob} \vdash \mathsf{Hom}(x, y)$
- $$\begin{tikzcd} x \arrow[r, bend left, "0"] \arrow[r, bend right, "1"'] & y \arrow[r, "\sum v"] & x \arrow[r, bend left, "0", ""{name=U, below}] \arrow[r, bend right, "1"', ""{name=D}] \arrow[Rightarrow, from=U, to=D] & y \arrow[r, mapsto] & x, y : \mathsf{Ob},\ f, g : \mathsf{Hom}(x, y) \vdash \mathsf{Hom}(f, g)\,\mathsf{Type} \end{tikzcd}$$
- $$\begin{tikzcd} x \arrow[r, bend left, "0", ""{name=U1, below}] \arrow[r, bend right, "1"', ""{name=D1}] \arrow[Rightarrow, from=U1, to=D1, shift right] \arrow[Rightarrow, from=U1, to=D1, shift left] & y \arrow[r, "\sum w"] & x \arrow[r, bend left, "0", ""{name=U, below}] \arrow[r, bend right, "1"', ""{name=D}] \arrow[Rightarrow, from=U, to=D] & y \arrow[r, mapsto] & \begin{cases} x, y : \mathsf{Ob},\ f, g : \mathsf{Hom}(x, y), \\ \alpha, \beta : \mathsf{Hom}(f, g) \vdash \mathsf{Eq}(\alpha, \beta)\,\mathsf{Type} \end{cases} \end{tikzcd}$$

Moreover, we can also introduce the composition and identity operations for arrows and cells:

- Composition operation for arrows: $x : \mathsf{Ob}$, $y : \mathsf{Ob}$, $z : \mathsf{Ob}$, $f : \mathsf{Hom}(x, y)$, $g : \mathsf{Hom}(y, z) \vdash g \circ f : \mathsf{Hom}(x, z)$.
- Identity operator for arrows: $x : \mathsf{Ob} \vdash \mathsf{id}_x : \mathsf{Hom}(x, x)$.

- Vertical composition of cells: $x, y : \mathsf{Ob}, f, g, h : \mathsf{Hom}(x, y), \alpha : \mathsf{Hom}(f, g), \beta : \mathsf{Hom}(g, h) \vdash \beta \circ \alpha : \mathsf{Hom}(f, h)$.
- Horizontal composition of cells: $x, y, z : \mathsf{Ob}, f, g : \mathsf{Hom}(x, y), h, k : \mathsf{Hom}(y, z), \alpha : \mathsf{Hom}(f, g), \beta : \mathsf{Hom}(h, k) \vdash \alpha * \beta : \mathsf{Hom}(h \circ f, k \circ g)$.
- Identity operator for cells: $x, y : \mathsf{Ob}, f : \mathsf{Hom}(x, y) \vdash \mathsf{id}_f : \mathsf{Hom}(f, f)$.

One can also attempt to list all the axioms that the above theory ought to satisfy, with the risk of running out of space. We simply exemplify this with the associator:

$$\begin{aligned} & w, x, y, z : \mathsf{Ob}, f : \mathsf{Hom}(w, x), g : \mathsf{Hom}(x, y), h : \mathsf{Hom}(y, z), \\ & \qquad \alpha : \mathsf{Hom}((h \circ g) \circ f, h \circ (g \circ f)), \beta : \mathsf{Hom}((h \circ (g \circ f), h \circ g) \circ f) \\ & \qquad\qquad \vdash r : \mathsf{Eq}(\alpha \circ \beta, \mathsf{id}_{(h \circ (g \circ f))}) \wedge s : \mathsf{Eq}(\beta \circ \alpha, \mathsf{id}_{(h(\circ g) \circ f}). \end{aligned}$$

We also include the axioms for Eq, the same ones as for categories, that give us the expected behaviour.

Remark 4.3.14. If we now try to obtain the associated theory $2Cat_=$ using the generating cofibration of [45], we see that the resulting theory has similar types and operations as the theory $Bicat_=$ of bicategories. The notable differences is that we do not need associators or unitors, but we need to include equations for the associativity and unitality of the composition of arrows and cells, and also the interchange law relating horizontal and vertical composition of cells. All these axioms are the appropriate ones to obtain 2-categories as the models of the theory $2Cat_=$.

Definition 4.3.15. Let $\mathcal{C}$ be a 2-category. An object $x \in \mathcal{C}$ is *bi-terminal* if for all $y \in \mathcal{C}$ there is an equivalence of categories $\mathcal{C}(y, x) \cong \mathbb{1}$.

Note that $f : a \to b$ being an equivalence can be written as

$$\exists h : \mathrm{Hom}(b, a), \exists \eta : \mathrm{Hom}(\mathsf{id}_a, h \circ f), \exists \varepsilon : \mathrm{Hom}(h \circ f, \mathsf{id}_b), \mathsf{isIso}(\eta) \wedge \mathsf{isIso}(\epsilon), \top.$$

Observe that the statement $\mathsf{isIso}(\eta)$, which says that $\eta : f \Rightarrow g$ is a natural isomorphism, only involves equality of natural transformations:

$$\mathsf{isIso}(\eta)) := \exists \epsilon : \mathrm{Hom}(g, f), s : \mathsf{Eq}(\epsilon \circ \eta, \mathsf{id}_f) \wedge r : \mathsf{Eq}(\eta \circ \epsilon, \mathsf{id}_g), \top.$$

We can then conclude that the notion of bi-terminal object is invariant.

Remark 4.3.16. Other natural, but somewhat different, higher categories to consider in this progression are double categories. Fortunately, this question has been described in Paula Verdugo's PhD thesis [71]. In particular, she builds a model structure on double categories where the fibrant objects are the *equipments*. The language for this model structure will produce formulas that express properties of equipments. Therefore, we can use our invariance theorems for this language of equipments. For the consequences of this, we refer to *Ibidem*.

4.3.3 Bounded below chain complexes

In this section, examine the language of the projective model structure on bounded below chain complexes $Ch(R)$ over a commutative ring R. We start by recalling some facts about this model structure. The detailed proofs can be found elsewhere, e.g. [33].

Given an R-module M for each $n \in \mathbb{Z}$ define $S^n(M) \in Ch(R)$ by

$$S^n(M)_k := \begin{cases} M, \; k = n \\ 0, \; k \neq n. \end{cases}$$

Similarly, $D^n(M) \in Ch(R)$ is defined as

$$D^n(M)_k := \begin{cases} M, \; k = n-1, \, n \\ 0, \quad \text{otherwise.} \end{cases}$$

where the only non-trivial differential $d_n : M \to M$ is the identity. Obviously, we get an inclusion $S^{n-1}(M) \to D^n(M)$.

These constructions induce functors $S^n : R\text{-}Mod \to Ch(R)$ and $D^n : R\text{-}Mod \to Ch(R)$ for each $n \in \mathbb{Z}$. Both functors have right adjoint $Z_n : Ch(R) \to R\text{-}Mod$ and $Ev_n : Ch(R) \to R\text{-}Mod$, respectively, where $Z_nX := Ker(d_n)$ and $Ev_nX := X_n$.

In particular, when $M = R$ the chains above are denoted by S^n and D^n, respectively. We can define the sets

$$I := \{S^{n-1} \to D^n | n \in \mathbb{Z}\} \text{ and } J := \{0 \to D^n | n \in \mathbb{Z}\}.$$

All constructions above work on unbounded chain complexes too. In the next result we restrict to bounded below chains, *i.e.,* $n \geqslant 0$, By definition $(D^0)_{-1} = 0$, so that $S^0 = D^0$. With this information, what we need to know about the projective model structure is summarized in the following:

Theorem 4.3.17. The category of chain complexes $Ch(R)$ admits a model structure where:

1. Weak equivalences are the quasi-isomorphisms
2. Fibrations are the degree-wise epimorphisms.
3. Cofibrations are the degree-wise monomorphims with projective cokernel.

Furthermore, this model structure is proper, cofibrantly generated and combinatorial. Cofibrations and trivial cofibrations are generated by I and J, respectively.

The cofibrant objects in the mode structure from theorem 4.3.17 are complexes such that each R-module is projective. However, this is not the case for unbounded chain complexes, where not every chain complex with projective modules is cofibrant. Nevertheless, in both cases, all objects are fibrant.

Remark 4.3.18. Using the adjunction $S^n \dashv Z_n$, for any chain complex X, a map $S^n \to X$ is simply a map $R \to Z_n X$ of R-modules. And from $D^n \dashv Ev_n$, a map $D^n \to X$ corresponds to $y \in X_n$. Therefore, a commutative square

$$\begin{tikzcd}
S^{n-1} \arrow[r, "x"] \arrow[d, "i_n"'] & X \arrow[d, "f"] \\
D^n \arrow[r, "Y"'] & Y
\end{tikzcd}$$

means that $x \in Z_{n-1}X \subseteq X_{n-1}$ *i.e.,* $d_{n-1}x = 0$ and that $fx = y \in Y_n$. Therefore taking a pushout simply means we freely add $(n-1)$-cycles to X_{n-1} with a specified boundary.

The first element *i.e.,* $n = 0$, of the set I is the cofibration

$$\begin{tikzcd}
0 \arrow[d, "i_0"'] & 0 \arrow[d] & 0 \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] & \cdots \arrow[l] \\
D^0 & 0 & R \arrow[l] & 0 \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

For any $n \geqslant 1$ we have cofibrations i_n

$$\begin{tikzcd}
S^{n-1} \arrow[d, "i_n"'] & 0 \arrow[d] & \cdots \arrow[l] & R \arrow[l] \arrow[d, "1_R"] & 0 \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] & \cdots \arrow[l] \\
D^n & 0 & \cdots \arrow[l] & R \arrow[l] & R \arrow[l, "1_R"'] & 0 \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

We then see immediately that I has a natural, well-founded, order, where we can set i_0 to be the minimal element of the set.

From theorem 4.3.18, we get cycles $y \in X_n$ and for each $x \in X_{n-1}$ such that $dx = 0$ and $\mathsf{C}_n(x) := \{y \in X_n | dy = x\}$, this is for each generating cofibration $i_n : S^{n-1} \to D^n$. This tells us that the ω-generalized algebraic theory has types $\mathsf{C}_n(x)$ for $n \geqslant 1$. We sum up the discussion in the following table:

$$\begin{array}{ccc}
i_0 : 0 \to D^0 & \mapsto & \vdash \mathsf{C}_0 \,\mathsf{Type} \\
i_n : S^{n-1} \to D^n & \mapsto & x : \mathsf{C}_{n-1}(0) \vdash \mathsf{C}_n(x) \,\mathsf{Type}
\end{array}$$

for $n \geqslant 1$. Note that differential is already included in the information that define the types $\mathsf{C}_n(x)$. We should also add, not included in the table, "+" operations on each type $\mathsf{C}_n(x)$, and axioms, that ensure is an abelian group:

$$a : \mathsf{C}_n(x),\, b : \mathsf{C}_n(y) \vdash a + b : \mathsf{C}_n(x + y).$$

Observation 4.3.19. It is important to note that in the theory we do not have equality between chains. The only possibility is to consider $\mathsf{C}_n(x)$ for $x : \mathsf{C}_{n-1}(0)$. However, this is enough to speak about chains satisfying a boundary condition $x - y = d_n z$ which is written in our language as

$$\exists z : \mathsf{C}_n(x - y), \top.$$

4.3.4 Unbounded chain complexes

When we work with unbounded chain complexes, with the obvious modifications, theorem 4.3.17 becomes:

Theorem 4.3.20. The category of chain complexes $Ch(R)$ admits a model structure where:

1. Weak equivalences are the quasi-isomorphisms
2. Fibrations are the degree-wise epimorphisms.
3. Cofibrations are the retracts of monomorphims with projective cokernel.

Furthermore, this model structure is proper, cofibrantly generated and combinatorial. Cofibrations and trivial cofibrations are generated by I and J, respectively.

Unlike the case for bounded chains, the cofibrations, or I, is not well-founded. However, we can obtain a new generating set of cofibrations following theorem 4.3.6. We consider the new set $I' := I \cup \{0 \to S^n | n \in \mathbb{Z}\}$. Note that since $0 \to S^n$ is a cofibration, we are not altering the model structure. The resulting theory is similar to the bounded case, we now must have the following association:

$$\begin{aligned} 0 \to S^n &\quad\mapsto\quad \vdash \mathsf{Z}_n\ \mathsf{Type} \\ i_n : S^{n-1} \to D^n &\quad\mapsto\quad x : \mathsf{Z}_{n-1} \vdash \mathsf{C}_n(x)\ \mathsf{Type} \end{aligned}$$

for $n \in \mathbb{Z}$.

Again, we need to add some non-type axioms. For example, we need each Z_n to contain an element 0, and $C_n(0) = Z_n$, then each C_n has an abelian group structure as in the case of bounded complexes.

4.3.5 Topological spaces

Here we recall the Quillen model structure on the category of topological spaces **Top** [55]. Recall that a map $f : X \to Y \in$ **Top** is a *weak homotopy equivalence* if for all $x \in X$ and $n \geqslant 1$ the induced map $f_* : \pi_n(X, x) \to \pi_n(Y, f(x))$ is an isomorphism of groups and for $n = 0$ is a bijection. Additionally, the map f is a *Serre fibration* if for any CW-complex W the following square has a diagonal filler:

$$\begin{tikzcd} A \times \{0\} \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ A \times [0,1] \arrow[r] \arrow[ur, dashed] & Y. \end{tikzcd}$$

Theorem 4.3.21. The category **Top** has a model category structure such that:

1. Weak equivalences are the weak homotopy equivalences.
2. fibrations are the Serre fibrations.
3. Cofibrations are the maps with the left lifting property against trivial fibrations.

Moreover, this model structure is cofibrantly generated. The generating cofibrations is the set of boundary inclusions $\{S^{n-1} \to D^n | n \in \mathbb{N}\}$. The set $\{D^n \to D^n \times [0,1] | n \in \mathbb{N}\}$ generates trivial cofibrations.

We can immediately write some of the relevant type axiom of the resulting theory:

- $\vdash \text{0-CW}\,\mathsf{Type}$.
- $x, y : \text{0-CW} \vdash \text{1-CW}(x, y)\,\mathsf{Type}$.
- $x : \text{0-CW}, \gamma : \text{1-CW}(x, x) \vdash \text{2-CW}(x, \gamma)\,\mathsf{Type}$.
- $\vdots$

Note that the language associated to the model structure allow us to express properties of topological spaces without relying on a specific set of axioms. However, this presents a limitation coming from the fact that we do not have an equality type. It is a classic result that there is no finitary presentation of a topological space. But in our setting, when X is a CW-complex *i.e.,* it is obtained as an iterated pushout of cells, then a continuous map $D^n \to X$ can be written in the language above.

Example 4.3.22. We can not write the formula

$$\exists x : \text{0-CW}\,\forall y : \text{0-CW},\ x = y.$$

The only possibility is to write

$$\forall x, y : \text{0-CW}\,\exists \alpha : \text{1-CW}(x, y), \top$$

which simply says that a space is path-connected. Moreover, we can not say that two paths $\alpha, \beta : \text{1-CW}(x, x)$ are homotopic in the usual sense, only that there exists $\sigma : \text{2-CW}(x, \alpha, \beta)$.

4.3.6 Kan complexes and quasi-categories

In this section, we analyze two very well-known model structures on the category of simplicial sets **sSet**; the Kan–Quillen and the Joyal model structures. One interesting feature is that we obtain the same theory for both models, but under the light of theorem 4.2.44 meaningful statements are delimited by the fibrant objects. In the first model we are interested in Kan complexes, while in the second model in the quasi-categories. The first model appears in [55] and the second in [37]. These are the first references one can find, but the literature is ample for both models.

Recall that a map $f : X \to Y$ between simplicial sets is a *Kan fibration* if it has the right lifting property for all horn inclusions, *i.e.,* the solid diagram below a diagonal filler

$$\begin{tikzcd}
\Lambda^k[n] \arrow[r] \arrow[d, hook] & X \arrow[d, "f"] \\
\Delta[n] \arrow[r] \arrow[ur, dashed] & Y
\end{tikzcd}$$

for all $0 \leqslant k \leqslant n \in \mathbb{N}$. The simplicial set X is a *Kan complex* if the unique map to the terminal presheaf is a Kan fibration. This is the result from [55]:

Theorem 4.3.23. The category of simplicial sets **sSet** carries a model structure in which:

1. Weak equivalences are maps $f : X \to Y$ whose geometric realization $|f| : |X| \to |Y|$ is a weak homotopy equivalence in the category of topological spaces **Top**. These are called Kan equivalences.
2. Fibrations are the Kan fibrations.
3. Cofibrations are the monomorphisms.

The class of cofibrations is generated by $I := \{\partial^n \hookrightarrow \Delta[n] | n \in \mathbb{N}\}$ and trivial cofibrations are generated by $J := \{\Lambda^k[n] \to \Delta[n] | n \in \mathbb{N} \text{ and } 0 \leqslant k \leqslant n\}$.

Similarly, a map $f : X \to Y$ between simplicial sets is an *inner Kan fibration* if it has the right lifting property for all inner horn inclusions, *i.e.,* the solid diagram below a diagonal filler

$$\begin{tikzcd}
\Lambda^k[n] \arrow[r] \arrow[d, hook] & X \arrow[d, "f"] \\
\Delta[n] \arrow[r] \arrow[ur, dashed] & Y
\end{tikzcd}$$

for all $0 < k < n \in \mathbb{N}$. The simplicial set X is a *quasi-category* if the unique map to the terminal presheaf is an inner Kan fibration. This is the result from [37]:

Theorem 4.3.24. The category of simplicial sets **sSet** carries a model structure in which:

1. Weak equivalences are the weak categorical equivalences.
2. Fibrations are the inner Kan fibrations.
3. Cofibrations are the monomorphisms.

The class of cofibrations is generated by $I := \{\partial\Delta[n] \hookrightarrow \Delta[n] | n \in \mathbb{N}\}$, the set of boundary inclusions.

Notice that both model structures have the same class of generating cofibrations. Hence, we expect that they have the same theories. We get a type for each cofibration in I. The first elements in this list of types are:

- $\vdash$ 0-simplex Type.
- σ_0, σ_1 : 0-simplex $\vdash$ 1-simplex(σ_0, σ_1) Type.
- $\sigma_0, \sigma_1, \sigma_2$: 0-simplex, σ_{01} : 1-simplex(σ_0, σ_1), σ_{12} : 1-simplex(σ_1, σ_2), σ_{02} : 1-simplex$(\sigma_0, \sigma_2) \vdash$ 2-simplex$(\sigma_0, \sigma_1, \sigma_2, \sigma_{01}, \sigma_{12}, \sigma_{02})$ Type.
- $\vdots$

The picture we should have in mind on the dependency of types is the usual one about simplices. A 1-simplex depend on two 0-simplicies, a 2-simplex consist of three 0-simplicies and three 1-simplicies and so forth.

One can see that the faces of an n-simplex are obtained via the dependencies, or context in which is defined. However, we can still adopt the usual notation for faces. Specifically, for each $n \in \mathbb{N}$ one has the faces $d_i(\sigma_{0123\ldots(i-1)i(i+1)\ldots n}) := \sigma_{0123\ldots(i-1)(i+1)\ldots n}$ is the $(n-1)$-simplex "opposite" to the i-th vertex of $\sigma_{012\ldots n}$. This simplex is already defined, and it used in the construction of $\sigma_{012\ldots n}$. We emphasize that this is not part of the theory, but just a convenient and familiar shortcut.

The *degeneracy* operator is part of the theory and needs to be introduced:

$$\sigma_{0123\ldots(i-1)i(i+1)\ldots n} : n\text{-simplex} \vdash s_i(\sigma_{0123\ldots(i-1)i(i+1)\ldots n}) : (n+1)\text{-simplex}$$

where $s_i(\sigma_{0123\ldots(i-1)i(i+1)\ldots n}) := \sigma_{0123\ldots(i-1)\hat{i}(i+1)\ldots n}$ is the $(n+1)$-simplex that contains $\sigma_{0123\ldots(i-1)i(i+1)\ldots n}$ as its i-th and $(i+1)$-faces. We have one of such operations for $0 \leqslant i \leqslant n$. The way we have introduced this operation is not completely correct as we are missing the dependencies for n-simplex and $(n+1)$-simplex and the context, nevertheless we can infer them. For example:

$$x, y : 0\text{-simplex},\ f : 1\text{-simplex}(x, y) \vdash s_1(f) : 2\text{-simplex}(x, y, y, f, s_0(y), f)$$

where $s_0(y)$ is the degeneracy of y or the "identity of y" and is constructed previously.

We also expect the simplicial identities to be satisfied. However, we do not need to postulate all of them as axioms of the theory since some of them are given via dependencies or by operation typing. The only equation we postulate is $s_i s_j = s_{j+1} s_i$ for $i \leqslant j$. On the one hand, the usual equation $d_i d_j = d_{j-1} d_i$ for $i < j$ only involves faces, therefore everything is encoded in the dependency. On the other hand, the equation

$$d_i s_j = \begin{cases} s_{j-1} d_i, & i < j \\ Id, & i = j, j+1 \\ s_j d_{i-1}, & i > j+1 \end{cases}$$

is valid from the definition of degeneracies and dependency of the faces. As we anticipated, the only way to tell apart which formulas are meaningful is through the fibrant objects, quasi-categories and Kan complexes, respectively.

Example 4.3.25. A Kan complex X is contractible if is weakly homotopy equivalent to $\mathbf{1}$. This is just to say that for any $n \geqslant 0$ we can find a lift

$$\begin{tikzcd}
\partial\Delta^n \arrow[r] \arrow[d] & X \arrow[d] \\
\Delta^n \arrow[r] \arrow[ur, dashed] & \mathbf{1}
\end{tikzcd}$$

which expresses the fact that the unique map $X \to \mathbf{1}$ is a weak homotopy equivalence. Note that X must satisfy an infinite number of conditions:

- For $n = 0$ says: $\exists \sigma_0 : \mathsf{0\text{-}simplex}$,
- For $n = 1$ says: $\forall \sigma_0, \sigma_1 : \mathsf{0\text{-}simplex}$, $\exists \sigma_{01} : \mathsf{1\text{-}simplex}(\sigma_0, \sigma_1)$,
- For $n = 2$ says:

$$\forall \sigma_0, \sigma_1 : \mathsf{0\text{-}simplex}\, \sigma_{01} : \mathsf{1\text{-}simplex}(\sigma_0, \sigma_1), \sigma_{12} : \mathsf{1\text{-}simplex}(\sigma_1, \sigma_2),$$
$$\sigma_{02} : \mathsf{1\text{-}simplex}(\sigma_0, \sigma_2), \exists \sigma_{012} : \mathsf{2\text{-}simplex}(\sigma_0, \sigma_1, \sigma_2, \sigma_{01}, \sigma_{12}, \sigma_{02}).$$

One continues unpacking the conditions and take the infinite conjunction of the formulas.

Alternatively, we can note that the domain of a trivial cofibration $i_n : \partial\Delta^n \hookrightarrow \Delta^n$ give us the context, or hypotheses, of the statement. In this case, the codomain gives us the type where the conclusion holds. If we accept this, let us write, $t \in \mathbb{L}^{\mathbf{sSet}}(\partial\Delta^n)$ for a term (formula) which expresses a property in the context $\partial\Delta^n$, similarly $t' \in \mathbb{L}^{\mathbf{sSet}}(\Delta^n)$ for a formula in the context Δ^n. With this convention, we do not have to use the theory explicitly. When we apply the quantifiers, universal or existential, we move these formulas to $\mathbb{L}^{\mathbf{sSet}}(\varnothing)$ and ask whether a fibrant object satisfies the resulting formula. For $\top \in \mathbb{L}^{\mathbf{sSet}}(\Delta^n)$ then for $i_n : \partial\Delta^n \hookrightarrow \Delta^n$ and $j_n : \varnothing \to \partial\Delta^n$ we get maps

$$\exists_{i_n} : \mathbb{L}^{\mathbf{sSet}}(\Delta^n) \to \mathbb{L}^{\mathbf{sSet}}(\partial\Delta^n) \text{ and } \forall_{j_n} : \mathbb{L}^{\mathbf{sSet}}(\partial\Delta^n) \to \mathbb{L}^{\mathbf{sSet}}(\varnothing),$$

and thus the formula $\forall_{j_n}\exists_{i_n}\top : \mathbb{L}^{\mathbf{sSet}}(\varnothing)$ would say that a Kan complex satisfies the corresponding lifting problem. For a Kan complex to be contractible, it needs to satisfy formulas for all $n \in \mathbb{N}$. Therefore,

$$\mathsf{isContr}(X) := (X \vdash \bigwedge_{n\in\mathbb{N}} \forall_{j_n}\exists_{i_n}\top).$$

We are now convinced that contractibility can be written in the language we just described. Theorem 4.3.25 indicates that we might not need to get an explicit syntax from the generating set of cofibrations. Instead, we might just quantify over the required cofibrations. The main reason this is preferable over the syntax is that in general such syntax is complicated to write, see for example section 4.3.8. The previous example shows that we might prefer to choose simplifications that make our sentences easier to read. This is specially true for contexts like the ones covered in the following section.

4.3.7 Reedy languages

The purpose of this subsection is to describe the language for the category $\mathcal{M}^{K^{\mathrm{op}}}$ where K is a Reedy category and $\mathcal{M}$ is a model category whose language we know. This encompasses some of the previous examples and opens the door to further applications.

Recall that if $\mathcal{M}$ is a cofibrantly generated model category whose cofibrations are generated by a well-founded set of cofibrations I then for each cofibration $A \hookrightarrow B \in I$ we can associate a type introduction axiom $\bar{A} \vdash \bar{B}\ \mathsf{Type}$, where $\bar{A}$ is a well-formed context previously constructed.

Let K be a Reedy category with degree function $\deg : K \to \omega$. This restriction is artificial since we could consider more general Reedy categories, however, for the examples this construction is aimed at, this is enough. The objects of K have a well-founded order relation induced by the degree function.

Construction 4.3.26. Let $\partial よ_k$ be the latching object of the representable functor $よ_k$ and $d_k : \partial よ_k \to よ_k$ the induced map. There is a bifunctor

$$\otimes : \mathbf{Set}^{K^{\mathrm{op}}} \times \mathcal{M} \to \mathcal{M}^{K^{\mathrm{op}}}$$

defined by $(A \otimes X)_k := \coprod_{A_k} X$. Let I be as above, given $i : X \to Y \in I$ and $k \in K$ we apply the usual Leibniz construction and obtain the dashed arrow below

$$\begin{tikzcd}
\partial よ_k \otimes X \arrow[r] \arrow[d] & よ_k \otimes X \arrow[d] \arrow[ddr, bend left] & \\
\partial よ_k \otimes Y \arrow[r] \arrow[ddrr, bend right, phantom] \arrow[drr, bend right] & \partial よ_k \otimes Y \coprod_{\partial よ_k \otimes X} よ_k \otimes X \arrow[dr, dashed, "d_k \hat{\otimes} i"] & \\
& & よ_k \otimes Y.
\end{tikzcd}$$

We now consider the set of maps $K \hat{\otimes} I := \{d_k \hat{\otimes} i | k \in K,\ i \in I\}$. By identifying each map $d_k \hat{\otimes} i \in K \hat{\otimes} I$ with a pair (k, i), we see that $K \hat{\otimes} I$ is also a well-founded relation, which we denote by $\leqslant_{\otimes}$. Here the relation is defined entry by entry *i.e.*, $(k', i') \leqslant_{\otimes} (k, i)$ if and only if $\deg(k') \leqslant \deg(k)$ and $i' \leqslant_I i$, where $\leqslant_I$ is the well-founded relation on I.

The previous construction is further justified by [8, Proposition 2.3.22] for premodel categories, but a similar description is abundant in the literature for Quillen model categories.

Proposition 4.3.27. The Reedy weak factorization system on $\mathcal{M}^{K^{\mathrm{op}}}$ is generated by $K \hat{\otimes} I$, and therefore the Reedy model category structure on $\mathcal{M}^{K^{\mathrm{op}}}$ is combinatorial whenever $\mathcal{M}$ is combinatorial.

A useful result we can have in mind is the following:

Lemma 4.3.28. Given any $i : A \to B \in \mathcal{M}$, a morphism $f : X \to Y \in \mathcal{M}^{K^{\mathrm{op}}}$ has the lifting property with respect to $d_k \hat{\otimes} i$ if and only if $\hat{f}^k : X_k \to Y_k \times_{M_k Y} M_k X$ has the right lifting property with respect to i.

Proof. As written, this is [8, Lemma 2.3.21], but it is also a classical result found in [33]. ∎

Remark 4.3.29. The matching objects in theorem 4.3.28 are computed with respect to the Reedy structure of K^{op}. This means that the relevant diagram in $M_k X$ in given by maps in $(K^{\mathrm{op}})_- = K_+$.

Observation 4.3.30. Many models for higher categories are built starting with presheaves over a Reedy category. Then to obtain the desired model one takes a left Bousfield localization for an appropriate class of maps. Importantly, this localization does not change the generating cofibrations. This is just to say that the language of $\mathcal{M}^{K^{\mathrm{op}}}$ remains unchanged after localization.

The cofibrations for the Reedy model structure are usually rather complicated, we can sometimes proceed as in theorem 4.3.25. This is, if $\Gamma' \hookrightarrow \Gamma$ is a generating cofibration then we might simply consider a formula $\phi' \in \mathbb{L}^{\mathcal{M}^{K^{\mathrm{op}}}}(\Gamma')$ or $\phi \in \mathbb{L}^{\mathcal{M}^{K^{\mathrm{op}}}}(\Gamma)$ with no explicit description of the type associated to the cofibration.

As an interesting case, in the following section we examine the Reedy language for Segal spaces. However, the construction applies to any other model category constructed similarly.

4.3.8 Segal spaces

We denote **ssSet** $:= [\Delta^{\mathrm{op}}, \mathbf{sSet}] = [\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}, \mathbf{Set}]$ the category of simplicial spaces, or bisimplicial sets. This category has two model structures that are obtained as left Bousfield localizations of the Reedy model structure. For both of these localizations, we use the Kan–Quillen model structure from the previous section. Recall that this model structure is cofibrantly generated. The set of generating cofibrations are the boundary inclusions. We will use the following facts and notation.

- There is an adjunction of two variables $\square : \mathbf{sSet} \times \mathbf{sSet} \to \mathbf{ssSet}$ defined as $(X \square Y)_{mn} := X_m \times Y_n$ for each $m, n \in \mathbb{N}$. This is called the box product.
- **sSet** can be seen as vertically embedded into **ssSet**. If $X \in \mathbf{sSet}$, then it can be seen as a simplicial space $X \square \Delta[0]$. There is also a horizontal embedding by setting $\Delta[0] \square X$.
- For $[m] \in \Delta$ we write $F(n) := \Delta[n] \square \Delta[0]$ and $\partial F(n) := \partial\Delta[n] \square \Delta[0]$.
- The simplicial spaces $F(n)$ represent the n-th mapping space functors, respectively $Map(F(n), X) = X_n$.

There is map $\iota : F(1) \coprod_{F(0)} \cdots \coprod_{F(0)} F(1) \to F(n)$, where the colimit on left has n factors. The following two model category structures were constructed by Rezk [59].

Theorem 4.3.31. The category admits a unique simplicial model category structure such that:

1. The cofibrations are the monomorphisms.

2. Fibrant objects are simplicial spaces X such that the map
$$X_n \to X_1 \times_{X_0} \cdots \times_{X_0} X_1$$
induced by ι is a Kan equivalence. These objects are called Segal spaces.

3. The weak equivalences are the maps $f : X \to Y \in \mathbf{ssSet}$ such that
$$Map(f, W) : Map(Y, W) \to Map(X, W)$$
is a Kan equivalence for every Segal space W.

4. A map $f : X \to Y$ between Segal spaces is a fibration (weak equivalence) if and only if is a Reedy fibration (Reedy weak equivalence).

Recall that $\mathcal{J}$ denotes the category with two objects and two arrows that are mutually inverses. It is usual to denote by $E(1)$ to the Segal space which is obtained by considering the nerve $N\mathcal{J}$ as a discrete simplicial space. This produces a map $F(1) \to E(1)$.

Theorem 4.3.32. The category admits a unique simplicial model category structure such that:

1. The cofibrations are the monomorphisms.

2. Fibrant objects are Segal spaces X such that the map
$$Map(E(1), X) \to Map(F(0), X)$$
is a Kan equivalence. These objects are called complete Segal spaces.

3. The weak equivalences are the maps $f : X \to Y \in \mathbf{ssSet}$ such that
$$Map(f, W) : Map(Y, W) \to Map(X, W)$$
is a Kan equivalence for every complete Segal space W.

4. A map $f : X \to Y$ between complete Segal spaces is a fibration (weak equivalence) if and only if is a Reedy fibration (Reedy weak equivalence).

These models are cofibrantly generated. The set of generating cofibrations can be described using the box product [40, Proposition 2.2]. This set is given by $\hat{I} := \{d_m \hat{\Box} d_n | m, n \in \mathbb{N}\}$. Explicitly a map in $\hat{I}$ is of the form
$$d_m \hat{\Box} d_n : \partial\Delta[m] \Box \Delta[n] \coprod_{\partial\Delta[m] \Box \partial\Delta[n]} \Delta[m] \Box \partial\Delta[n] \to \Delta[m] \Box \Delta[n]$$

We can obtain the generalized algebraic theory for (complete) Segal space. The domains of these maps provide the context in which a new type is formed. To get a sense of the theory, consider the following picture of a bisimplicial set X:

$$\begin{tikzcd}
X_{00} \arrow[r, shift left=2, leftarrow] \arrow[r] \arrow[r, shift right=2, leftarrow] \arrow[d, shift right=2, leftarrow] \arrow[d] \arrow[d, shift left=2, leftarrow] & X_{01} \arrow[r, shift left=4, leftarrow] \arrow[r, shift left=2] \arrow[r, leftarrow] \arrow[r, shift right=2] \arrow[r, shift right=4, leftarrow] \arrow[d, shift right=2, leftarrow] \arrow[d] \arrow[d, shift left=2, leftarrow] & \cdots \\
X_{10} \arrow[r, shift left=2, leftarrow] \arrow[r] \arrow[r, shift right=2, leftarrow] \arrow[d, shift right=4, leftarrow] \arrow[d, shift right=2] \arrow[d, leftarrow] \arrow[d, shift left=2] \arrow[d, shift left=4, leftarrow] & X_{11} & \\
\vdots & & \ddots
\end{tikzcd}$$

The arrows indicate the degeneracy and face maps. Now we go back to consider the maps $d_m \square d_n$. When $m = n = 0$ then we simply get a map $\varnothing \to \Delta[0] \square \Delta[0]$, and allow us to introduce the type

$$\vdash \textsf{-space}_{00}\ \textsf{Type}.$$

When $n = 0$ the resulting subset of maps is of the form

$$d_m \hat{\square} \Delta[0] : \partial\Delta[m] \square \Delta[0] \to \Delta[m] \square \Delta[0].$$

In this setting, since for $m = 0$ we obtain the previous cofibration $\varnothing \to \mathbf{1}$, for each $m \geqslant 1$ we can write the following types:

- $x, y : \textsf{-space}_{00} \vdash \textsf{-space}_{10}(x, y)\ \textsf{Type}.$
- $x, y, z : \textsf{-space}_{00}, f : \textsf{-space}_{10}(x, y), g : \textsf{-space}_{10}(y, z), h : \textsf{-space}_{10}(x, z) \vdash \textsf{-space}_{20}(x, y, z, f, g, h).$
- $\vdots$

When $m = 0$ we obtain the theory of the categorical direction. Now suppose that $m = 1 = n$, then resulting generating cofibration is the map

$$d_1 \hat{\square} d_1 : \partial\Delta[1] \square \Delta[1] \coprod_{\partial\Delta[1] \square \partial\Delta[1]} \Delta[1] \square \partial\Delta[1] \to \Delta[1] \square \Delta[1]$$

From here we see that the type associated to this map has the following form:

$$\begin{aligned} x_0, x_1, x_2, x_3 : \textsf{-space}_{00}, f_{01} : \textsf{-space}_{01}(x_0, x_1), f_{23} : \textsf{-space}_{01}(x_2, x_3), f_{02} : \textsf{-space}_{10}(x_0, x_2), \\ f_{13} : \textsf{-space}_{10}(x_1, x_3) \vdash \textsf{-space}_{11}(x_0, x_1, x_2, x_3, f_{01}, f_{23}, f_{02}, f_{13}). \end{aligned}$$

We think of this new type as the type of squares where the solid boundary is the given context

$$\begin{tikzcd}
x_0 \arrow[r, "f_{01}"] \arrow[d, "f_{02}"'] & x_1 \arrow[d, "f_{13}"] \\
x_2 \arrow[r, "f_{23}"'] & x_3
\end{tikzcd}$$

For different m, n the context are simply more involved, but the dependencies can be inferred. Note we still need to add the degeneracy operators satisfying the usual axioms. We can see that as we build more complex contexts, it will be computationally difficult to obtain an explicit description of the types. We might instead proceed as in theorem 4.3.25.

Example 4.3.33. Two elements $x, y : \text{-space}_{00}$ are said to be *homotopic* if there exists $\alpha : \text{-space}_{10}(x, y)$. Such sentence only involves types in the language of Segal spaces. In contrast to topological spaces, we can express the fact that two maps are homotopic.

Remark 4.3.34. Note in particular that the language of spaces or Kan complexes is available for us to use. This in combination with our construction in section 4.3.7 allow us to realize many properties of (complete) Segal spaces, for example the ones found in [57], are written in this language.

4.3.9 Functors and Isofibrations

We denote $[1] := \{0 \to 1\}$ the category with two objects and single non-identity arrow. This category can be viewed as a Reedy category in two ways. The first one respects the direction of the arrow, so we take $[1]_+$ to be the non-identity map. While for the second we take the same map to be in $[1]_-$. Recall that if K is a Reedy category then K^{op} is also a Reedy category where $(K^{\mathsf{op}})_+ = K_-$ and $(K^{\mathsf{op}})_- = K_+$. In order to match the computations of theorem 4.3.26, we use the same notation as there. By which we mean that for a model category $\mathcal{C}$ we use $\mathcal{C}^{([1]_+)^{\mathsf{op}}}$ and $\mathcal{C}^{([1]_-)^{\mathsf{op}}}$ with the corresponding Reedy model structures, ignoring the fact that $\mathcal{C}^{([1]_+)^{\mathsf{op}}} = \mathcal{C}^{[1]_-}$ and $\mathcal{C}^{([1]_-)^{\mathsf{op}}} = \mathcal{C}^{[1]_+}$.

Proposition 4.3.35. The Reedy model structure on $\mathcal{C}^{([1]_-)^{\mathsf{op}}}_{Reedy}$ coincides with the projective model structure. In particular, weak equivalences and fibrations are the level-wise weak equivalences and fibrations in $\mathcal{C}$.

Proof. This is a classical and well-known a result. ∎

We are interested in the particular case of $\mathcal{C} = \mathbf{Cat}$. It is immediate to see that all objects are fibrant. The language we obtain should be the language for functors. Since $\mathbf{Cat}$ is cofibrantly generated by $I = \{\mathbf{0} \xrightarrow{u} \mathbf{1}, \{0\} \sqcup \{1\} \xrightarrow{v} \mathbf{2}, P \xrightarrow{w} \mathbf{2}\}$ we have that $[1] \hat{\otimes} I$ generates $\mathcal{C}^{([1]_-)^{\mathsf{op}}}_{Reedy}$, by theorem 4.3.26. This gives us the set of maps

$$\{d_0 \hat{\otimes} u, d_0 \hat{\otimes} v, d_0 \hat{\otimes} w, d_1 \hat{\otimes} u, d_1 \hat{\otimes} v, d_1 \hat{\otimes} w\}.$$

To explain what it means for a map $f : X \to Y$ to have the lifting property against these cofibration we can use theorem 4.3.28, for which we need the matching objects. We observe from theorem 4.3.29 that $M_0 X = 1 = M_1 X$ since $([1]_-)_+$ has no non-identity maps, and the same applies to Y. Therefore, for $i \in I$ and $k = 0, 1$ we have $(d_k \hat{\otimes} i) \pitchfork f$ in $\mathbf{Cat}^{[1]^{\mathsf{op}}_-}$ if and only if $i \pitchfork \hat{f}^k$, but $\hat{f}^k$ is either $X_0 \to Y_0$ or $X_1 \to Y_1$. Diagrammatically we have:

$$\begin{tikzcd}
\partial \text{よ}_k \otimes b \coprod_{\partial \text{よ}_k \otimes a} \text{よ}_k \otimes a \arrow[r] \arrow[d, "d_k \hat{\otimes} i"'] & X \arrow[d, "f"] & \Longleftrightarrow & a \arrow[r] \arrow[d, "i"'] & X_k \arrow[d, "\hat{f}^k"] \\
\text{よ}_k \otimes b \arrow[r] \arrow[ur, dashed] & Y & & b \arrow[r] \arrow[ur, dotted] & Y_k
\end{tikzcd}$$

Specializing to $Y = \mathbf{1}$, it gives us an idea of how types are introduced:

$$\begin{tikzcd}
\mathbf{0} \arrow[r] \arrow[d, "u"'] & X_k & \{0\} \sqcup \{1\} \arrow[r] \arrow[d, "v"'] & X_k & P \arrow[r] \arrow[d, "w"'] & X_k \\
\mathbf{1} & & \mathbf{2} & & \mathbf{2} &
\end{tikzcd}$$

for $k = 0, 1$. This means that we introduce objects, arrows between two objects and equality between arrows to X_0 or X_1. This indicates that corresponding generating cofibration produce the following type axioms:

$$\vdash X_0 \,\mathsf{Type} \qquad a, b : X_0 \vdash X_0(a, b) \,\mathsf{Type} \qquad a, b : X_0, f, g : X_0(a, b) \vdash f =_{X_0} g \,\mathsf{Type}$$

$$\vdash X_1 \,\mathsf{Type} \qquad a, b : X_k \vdash X_k(a, b) \,\mathsf{Type} \qquad a, b : X_1, f, g : X_k(a, b) \vdash f =_{X_k} g \,\mathsf{Type}$$

and we introduce the operation symbol for the functor as an operation

$$a : X_0 \vdash Fa : X_1 \qquad f : X_0(a, b) \vdash Ff : X_1(Fa, Fb)$$

On top of it, we add the usual axioms that ensure we have the expected behaviour with respect to the identity and composition operations. Let us call denote this language by $\mathbb{L}^{Fun}$.

Now we examine the language for the other model structure.

Proposition 4.3.36. The Reedy model structure on $\mathcal{C}^{([1]_+)^{\mathrm{op}}}_{Reedy}$ coincides with the injective model structure. In particular, weak equivalences and cofibrations are the level-wise weak equivalences and cofibrations in $\mathcal{C}$.

Proof. The result is folklore. ∎

We find that fibrant objects are those such that $X_0 \to X_1$ is an isofibration. Therefore, the language in this case refers to isofibrations. Again, this model structure has generating cofibrations

$$\{d_0 \hat{\otimes} u, d_0 \hat{\otimes} v, d_0 \hat{\otimes} w, d_1 \hat{\otimes} u, d_1 \hat{\otimes} v, d_1 \hat{\otimes} w\}.$$

Next, observe that $\partial \text{よ}_0 = 0$ and $\partial \text{よ}_1 = \text{よ}_0$. We have the maps $d_0 : 0 \to \text{よ}_0$ and $d_1 : \text{よ}_0 \to \text{よ}_1$. Therefore, if $i : a \to b \in I$, then this give us the following cofibrations

- $\text{よ}_0 \otimes a \to \text{よ}_0 \otimes b$,

- $\text{よ}_1 \otimes a \coprod_{\text{よ}_0 \otimes a} \text{よ}_0 \otimes b \to \text{よ}_1 \otimes b$.

The map $\text{よ}_0 \otimes a \to \text{よ}_0 \otimes b$ for $i \in I$ corresponds to the following type introduction:

$$\vdash X_0 \,\mathsf{Type} \qquad x, y : X_0 \vdash X_0(x, y) \,\mathsf{Type} \qquad x, y : X_0, f, g : X_0(x, y) \vdash f =_{X_0} g \,\mathsf{Type}$$

which we can think of as a category. The analysis of the second map is more intricate. Let us denote the evaluation of the representables by よ_{k0} and よ_{k1} for $k = 0, 1$, and for simplicity we keep the '$\otimes$' symbol. Evaluating the cofibration $\text{よ}_1 \otimes a \coprod_{\text{よ}_0 \otimes a} \text{よ}_0 \otimes b \to \text{よ}_1 \otimes b$ at $[1]^{\mathsf{op}}_+$ give us the square,

$$\begin{tikzcd}
\text{よ}_{11} \otimes a \coprod_{\text{よ}_{10} \otimes a} \text{よ}_{10} \otimes b \arrow[r] \arrow[d] & \text{よ}_{01} \otimes a \coprod_{\text{よ}_{00} \otimes a} \text{よ}_{00} \otimes b \arrow[d] \\
\text{よ}_{11} \otimes b \arrow[r] & \text{よ}_{01} \otimes b,
\end{tikzcd}$$

where the horizontal arrows are induced by the diagram $[1]^{\mathsf{op}}_+$. This simplifies to

$$\begin{tikzcd}
a \arrow[r] \arrow[d] & a \coprod_a b \arrow[d] \\
b \arrow[r] & b,
\end{tikzcd}$$

which we now compute for $i \in I$, so the pictures take the following form:

$$\begin{tikzcd}
\mathbf{0} \arrow[r] \arrow[d] & \mathbf{1} \arrow[d] \\
\mathbf{1} \arrow[r] & \mathbf{1}
\end{tikzcd}
\qquad
\begin{tikzcd}
\{0\} \sqcup \{1\} \arrow[r] \arrow[d] & \mathbf{2} \arrow[d] \\
\mathbf{2} \arrow[r] & \mathbf{2}
\end{tikzcd}
\qquad
\begin{tikzcd}
P \arrow[r] \arrow[d] & \mathbf{2} \arrow[d] \\
\mathbf{2} \arrow[r] & \mathbf{2}.
\end{tikzcd}$$

From the above we deduce that the type axioms introduced by these cofibrations take, respectively, the following form:

$$x : X_0 \vdash X_1(x) \,\mathsf{Type},$$

$$x, y : X_0, f : X_0(x, y), a : X_1(x), b : X_1(y) \vdash X_1(a, b, f) \,\mathsf{Type},$$

$$x, y : X_0, f : X_0(x, y), a : X_1(x), b : X_1(y), j, k : X_1(a, b, f) \vdash j =_{X_1(a,b,f)} k \,\mathsf{Type}.$$

Unlike the language for functors $\mathbb{L}^{Fun}$, here we do not need a symbol for $F : X_0 \to X_1$. We denote this language for isofibrations as $\mathbb{L}^{Iso}$.

For the observation below, it will be useful to remember that given a functor $F : X \to Y$, an arrow $f : x \to y \in X$ is *cartesian* if for any $h : x' \to y$ and $w : F(x') \to F(x)$ with

$F(f) \circ w = F(h)$, there exists a unique $u : x' \to x$ such that $f \circ u = h$. The following diagram illustrates this definition:

$$\begin{tikzcd}
X \arrow[ddd, "F"'] & x' \arrow[rrd, bend left, "\forall h"] \arrow[rd, dashed, "\exists ! u"'] & & \\
& & x \arrow[r, "f"'] \arrow[dd, mapsto] & y \\
& Fx' \arrow[rrd, bend left, "Fh"] \arrow[rd, "\forall w"'] & & \\
Y & & Fx \arrow[r, "Ff"'] & Fy
\end{tikzcd}$$

A *Grothendieck fibration* is a functor $F : X \to Y$ such that for any $y \in Y$ and $f : a \to F(y)$, there exists a cartesian arrow $\phi_f : f^*y \to y$ such that $F(\phi_f) = f$. The functor $F : X \to Y$ is a *Street fibration* if for any $y \in Y$ and $f : a \to F(y)$, there exists a cartesian arrow $\hat{f} : e \to y$ and an isomorphism $F(e) \cong a$ that makes the resulting triangle commutative.

Remark 4.3.37. It is a classical result that Grothendieck fibration is the same as a Street fibration which is also an isofibration. On the one hand, note that a Grothendieck fibration can be written in the language $\mathbb{L}^{Iso}$ of isofibrations, but not in $\mathbb{L}^{Fun}$ of functors since it contains an equality between objects, such equality of is salvaged in $\mathbb{L}^{Iso}$ thanks to the dependencies. On the other hand, a Street fibration is a formula in $\mathbb{L}^{Fun}$. We also know that the two Reedy model structures on the category $\mathbf{Cat}^{[1]}$ are Quillen equivalent. The above result can also be automatically obtained as an elementary application of 4^{th} invariance theorem, whose proof is the heart of the next section.

4.4 Language invariance under Quillen equivalences

4.4.1 The third and fourth invariance theorem

The main goal of this section is to show two more invariance property of the first order language from section 4.2.4 that we can phrase informally[4] as:

1. 3^{rd} invariance theorem: If two cofibrant objects X and Y are equivalent then any formula in context X can be translated into a formula in context Y.
2. 4^{th} invariance theorem: If two (weak) model categories $\mathcal{M}$ and $\mathcal{N}$ are Quillen equivalents, then any formula in the language of $\mathcal{M}$ can be translated into a formula in the language of $\mathcal{N}$.

These "translations" are equivalent to the original formula in the sense that they interpreted in the same way in any fibrant model, but they might not be equivalent in the more syntactic sense introduced in theorem 4.2.11. More precisely, we introduce the following equivalence relation on formulas:

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[4] The precise statement is just below as theorem 4.4.2.

Definition 4.4.1. Let A be a cofibrant object of $\mathcal{M}$. Two formulas $\phi, \psi \in \mathbb{L}_\lambda^{\mathcal{M}}(A)$ are said to be *semantically equivalent* if for all fibrant objects $X \in \mathcal{M}$ we have $|\phi|_X = |\psi|_X$. In this situation we write $\phi \approx \psi$.

We define $h\mathbb{L}_\lambda^{\mathcal{M}}(A)$ to be the quotient of $\mathbb{L}_\lambda^{\mathcal{M}}(A)$ by the relation $\approx$. We easily check that this is still a Boolean algebra.

By definition of $\approx$ we have that for $\phi, \psi \in \mathbb{L}_\lambda^{\mathcal{M}}(\Gamma)$, then $\phi \approx \psi$, if and only if all maps $v : \Gamma \to X$ with X fibrant.

$$\Gamma \vdash \phi(v) \Leftrightarrow \Gamma \vdash \psi(v)$$

We can now state our theorems.

Theorem 4.4.2.

- 3^{rd} **invariance theorem:** Let $A, B \in \mathcal{M}$ two cofibrant objects of a weak Quillen model category $\mathcal{M}$ and $f : A \to B$ a weak equivalence between them. Then the map $f^* : \mathbb{L}_\lambda(B) \to \mathbb{L}_\lambda(A)$ induces a bijection
$$h\mathbb{L}_\lambda(B) \simeq h\mathbb{L}_\lambda(A).$$
- 4^{th} **invariance theorem:** If $F : \mathcal{M} \to \mathcal{N}$ is a left Quillen equivalence between two weak model categories, then for any cofibrant object $A \in \mathcal{M}$ the induced map
$$h\mathbb{L}F_A : h\mathbb{L}_\lambda^{\mathcal{M}}(A) \to h\mathbb{L}_\lambda^{\mathcal{N}}(FA)$$
from theorem 4.4.5 is an isomorphism.

Remark 4.4.3. Note that if $F : \mathcal{M} \rightleftarrows \mathcal{N} : G$ a Quillen equivalence between weak model categories and B is a cofibrant object of $\mathcal{N}$ which his not of the form $F(A)$ for $A \in \mathcal{M}$ then one can still use the 4^{th} invariance theorem to transfers formula in $h\mathbb{L}(B)$ to a formula in $\mathcal{M}$ by first finding an object of the form $F(A)$ which is homotopically equivalent to B, which is always possible as F is a Quillen equivalence, and first transferring our formula $\phi \in h\mathbb{L}(B)$ to a formula in $h\mathbb{L}(F(A))$ using the 3^{rd} invariance theorem.

Observation 4.4.4. For any cofibrant object $\Gamma \in \mathcal{M}$, $\phi, \psi \in \mathbb{L}_\lambda^{\mathcal{M}}(\Gamma)$ we defined $\phi \approx \psi$ if and only if $|\phi|_X = |\psi|_X$ for all fibrant objects. However, note that if we take a cofibrant replacement X^{Cof} of X, then by theorem 4.2.44 (2^{nd} invariance theorem) we have, $X \vdash \phi(fv)$ if and only if $X^{\mathrm{Cof}} \vdash \phi(v)$ where $f : X^{\mathrm{Cof}} \overset{\sim}{\twoheadrightarrow} X$ and $v : \Gamma \to X^{\mathrm{Cof}}$.

Therefore, when testing the relation $\approx$, it is enough to use bifibrant objects. More precisely, define $\phi \approx_b \psi$ if $|\phi|_X = |\psi|_X$ for any bifibrant object X. Then

$$\phi \approx \psi \text{ if and only if } \phi \approx_b \psi.$$

We now explain the construction of the map $h\mathbb{L}F_A : h\mathbb{L}_\lambda^{\mathcal{M}}(A) \to h\mathbb{L}_\lambda^{\mathcal{N}}(FA)$ mentioned in the 4^{th} invariance theorem.

Construction 4.4.5. The map $h\mathbb{L}F_A$ in the 4^{th} invariance theorem is the map coming from $\mathbb{L}F_A : \mathbb{L}_\lambda^{\mathcal{M}}(A) \to \mathbb{L}_\lambda^{\mathcal{N}}(FA)$ constructed in theorem 4.2.46. It just comes from the fact that $\mathbb{L}_\lambda^{\mathcal{M}}(A)$ is an initial object. Recall that it satisfies the formula:

$$G(X) \vdash \phi(v) \Leftrightarrow X \vdash F(\phi)(\tilde{v}).$$

for any object $X \in \mathcal{D}$, and cofibrant object $C \in \mathcal{C}$, any map $v : C \to G(X)$ corresponding to $\tilde{v} : F(C) \to X$, and $\phi \in \mathbb{L}_\lambda^{\mathcal{C}}$.

This immediately imply the following proposition that shows that the map $h\mathbb{L}_A$ mentioned in the 4^{th} invariance theorem is well-defined.

Proposition 4.4.6. For any left Quillen adjunction $F : \mathcal{M} \leftrightarrows \mathcal{N} : G$ and $A \in \mathcal{M}$ a cofibrant object, the map $F : \mathbb{L}_\lambda(A) \to \mathbb{L}_\lambda(FA)$ is compatible to the relation $\approx$ and induce a morphism of λ-boolean algebra

$$F : h\mathbb{L}_\lambda(A) \to h\mathbb{L}_\lambda(FA).$$

Proof. If ϕ and ψ are semantically equivalent formulas in $\mathbb{L}_\lambda(A)$, then for any fibrant object $X \in \mathcal{N}$, and maps $\tilde{v} : FA \to X$ corresponding to $v : A \to GX$ we have

$$X \vdash F(\phi)(\tilde{v}) \Leftrightarrow G(X) \vdash \phi(v) \Leftrightarrow G(X) \vdash \psi(v) \Leftrightarrow X \vdash F(\psi)(\tilde{v})$$

which shows that $F(\phi) \approx F(\psi)$ and concludes the proof. ∎

We are now ready prove the 3^{rd} invariance theorem. We start with a special case:

Lemma 4.4.7. Let $\Gamma, \Gamma' \in \mathcal{M}^{\mathrm{Cof}}$ and $\pi : \Gamma \overset{\sim}{\hookrightarrow} \Gamma'$ be a core trivial cofibration, then the induced map $h\mathbb{L}_\lambda^{\mathcal{M}}(\Gamma) \to h\mathbb{L}_\lambda^{\mathcal{M}}(\Gamma')$ is an isomorphism of λ-boolean algebras.

Proof. Assume that $\pi : \Gamma \overset{\sim}{\hookrightarrow} \Gamma'$ is a core trivial cofibration. Since to define the language of $\mathcal{M}$ we take the κ-clan $(\mathcal{M}^{\mathrm{Cof}})^{\mathrm{op}}$, when constructing the language we get a covariant functor $\mathcal{M}^{\mathrm{Cof}} \to \mathbf{Bool}_\lambda$. Therefore, we obtain a map $\pi^* : \mathbb{L}_\lambda^{\mathcal{M}}(\Gamma) \to \mathbb{L}_\lambda^{\mathcal{M}}(\Gamma')$ and its left adjoint $\exists_\pi : \mathbb{L}_\lambda^{\mathcal{M}}(\Gamma') \to \mathbb{L}_\lambda^{\mathcal{M}}(\Gamma)$, that furthermore descends to the adjoint pair $h\exists_\pi : h\mathbb{L}_\lambda^{\mathcal{M}}(\Gamma') \rightleftarrows h\mathbb{L}_\lambda^{\mathcal{M}}(\Gamma) : h\pi^*$ between the λ-boolean algebras.

We claim that $h\exists_\pi$ is the inverse for $h\pi^*$. It is enough to show that for any $\phi : \mathbb{L}_\lambda^{\mathcal{M}}(\Gamma)$ and $\psi \in \mathbb{L}_\lambda^{\mathcal{M}}(\Gamma')$ we have $\exists_\pi \pi^*(\phi) \approx \phi$ and $\pi^* \exists_\pi(\phi) \approx \phi$.

Firstly, let $X \in \mathcal{M}^{\mathrm{Fib}}$ be a fibrant object and $x : \Gamma \to X$. Note that $x \in |\exists_\pi \psi|_X \subseteq \hom_{\mathcal{M}}(\Gamma, X)$ if an only there exists $x' : \Gamma' \to X$ such that $x' \in |\psi|_X \subseteq \hom_{\mathcal{M}}(\Gamma', X)$ and that makes the following triangle commutative:

$$\begin{tikzcd}
\Gamma \arrow[r, "x"] \arrow[d, hook, "\sim", "\pi"'] & X \\
\Gamma'. \arrow[ur, "x'"'] &
\end{tikzcd}$$

Since X is fibrant, the map x' always exists. Such x' is not necessarily unique, however, in a situation in which we have two arrows

$$\begin{tikzcd}
\Gamma \arrow[r, "x"] \arrow[d, hook, "\pi"', "\sim"] & X \\
\Gamma' \arrow[ur, shift left, "y"] \arrow[ur, shift right, "z"'] &
\end{tikzcd}$$

that make the triangle commutative, then using that π is a trivial cofibration we see that y and z are homotopic. By the first invariant theorem (theorem 4.2.44) we have $y \in |\psi|_X$ if and only if $z \in |\psi|_X$. Therefore, the existence of $x' \in |\psi|_X$ is independent of choices.

From here, the result is immediate: $x \in |\exists_\pi \pi^* \phi|_X$ if and only if there exists $x' : \Gamma' \to X$ such that $x'\pi = x$ such that $X \vdash \phi(\pi^* x')$ *i.e.,* if and only $x \in |\phi|_X$. This shows that $|\exists_\pi \pi^* \phi|_X = |\phi|_X$ for any fibrant object. Conversely, for $y : \Gamma' \to X$ we have $y \in |\pi^* \exists_\pi \psi|$ if and only if there exists $z : \Gamma' \to X$ such that $z\pi = y\pi$ and $X \vdash \psi(z)$, which is equivalent to $y \in |\psi|_X$, showing that $|\exists_\pi \pi^* \psi|_X = |\psi|_X$. This concludes the proof that $h\exists_\pi$ is the inverse for $h\pi^*$. ∎

We can now ready to prove the 3^{rd} invariance theorem:

Proof of the 3^{rd} invariance theorem: The idea is to use theorem 4.4.7 together with Brown's factorization lemma from [15], or rather an adaptation of it to the setting of weak model structures that we present now. If $f : X \to Y$ is a weak equivalence between cofibrant objects in a weak model category, In general we cannot form a cylinder object for X, but instead a "weak cylinder" for X, that is a diagram:

$$\begin{tikzcd}
X \coprod X \arrow[r, "\nabla"] \arrow[d, hook] & X \arrow[d, hook, "\sim"] \\
IX \arrow[r, "\sim"] & DX,
\end{tikzcd}$$

we then take the pushout of this whole diagram by the map $X \to Y$, using either of the two canonical maps $X \to X \coprod X$:

$$\begin{tikzcd}
X \coprod Y \arrow[r, "(id, f)"] \arrow[d, hook] & Y \arrow[d, hook, "\sim"] \\
IX \coprod_X Y \arrow[r, "\sim"] & DX \coprod_X Y
\end{tikzcd} \tag{4.4.1}$$

and by precomposing with the coproduct inclusion $X \to X \coprod Y$, we obtain a diagram:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, hook] & Y \arrow[d, hook, "\sim"] \\
IX \coprod_X Y \arrow[r, "\sim"] & DX \coprod_X Y
\end{tikzcd}$$

three of the four maps here are weak equivalence, so it follows by 2-out-of-3 that the left vertical map is also a weak equivalence, hence a trivial cofibration. Applying $h\mathbb{L}$ we obtain a diagram:

$$\begin{tikzcd}
h\mathbb{L}(X) & h\mathbb{L}(Y) \arrow[l, "f^*"'] \\
h\mathbb{L}\left(IX \coprod_X Y\right) \arrow[u] & h\mathbb{L}\left(DX \coprod_X Y\right) \arrow[l] \arrow[u]
\end{tikzcd}$$

The two vertical arrows are bijections because of theorem 4.4.7, so in order to show that f^* is a bijection, it is enough to show that the bottom map is a bijection.

This bottom horizontal map fit into a commutative diagram:

$$\begin{tikzcd}
 & Y \arrow[dl, hook, "\sim"'] \arrow[d, hook, "\sim"] \\
IX \coprod_X Y \arrow[r, "\sim"] & DX \coprod_X Y,
\end{tikzcd}$$

where the arrow $Y \to IX \coprod_X Y$ is obtained as the pushout:

$$\begin{tikzcd}
X \arrow[r] \arrow[d, hook, "\sim"] \arrow[dr, phantom, "\ulcorner"] & Y \arrow[d, hook, "\sim"] \\
IX \arrow[r] & IX \coprod_X Y
\end{tikzcd}$$

Applying the $h\mathbb{L}$ functor, we get a triangle:

$$\begin{tikzcd}
 & h\mathbb{L}\left(Y\right) \\
h\mathbb{L}\left(IX \coprod_X Y\right) \arrow[ur] & h\mathbb{L}\left(DX \coprod_X Y\right) \arrow[l] \arrow[u]
\end{tikzcd}$$

the two vertical and diagonal arrows are bijections because of theorem 4.4.7, and so the third, horizontal, arrows also is, which concludes the proof.

∎

We can also, show that the injectivity part of the 4^{th} invariance theorem.

Lemma 4.4.8. Let $F : \mathcal{M} \rightleftarrows \mathcal{N} : G$ a Quillen equivalence. Then, for any cofibrant object $\Gamma \in \mathcal{M}$, the induced map $h\mathbb{L}F_\Gamma : h\mathbb{L}^{\mathcal{M}}_\lambda(\Gamma) \to h\mathbb{L}^{\mathcal{N}}_\lambda(F\Gamma)$ is injective.

Proof. Let ϕ and ψ be formulas in $\mathbb{L}^{\mathcal{M}}_\lambda(\Gamma)$ such that $F(\phi) \approx F(\psi)$ *i.e.,* $F(\phi)$ and $F(\psi)$ are equal in $h\mathbb{L}^{\mathcal{N}}_\lambda(F\Gamma)$. We must show that $\psi \approx \phi$. Alternatively, by theorem 4.4.4 we can show that $\psi \approx_b \phi$. The Quillen equivalence induces an equivalence between homotopy categories $Ho(G) : Ho(\mathcal{N}^{\mathrm{Bif}}) \to Ho(\mathcal{M}^{\mathrm{Bif}})$. Hence, there is a bifibrant object $Y \in \mathcal{N}$ such that GY is isomorphic to X in $Ho(\mathcal{M}^{\text{Bif}})$. Given any $x : \Gamma \to X$, denote by $y : \Gamma \to GY$ any map such that the following triangle

$$\begin{tikzcd} A \arrow[r, "x"] \arrow[dr, "y"'] & X \arrow[d, "\cong"] \\ & GY \end{tikzcd}$$

commutes in $\mathrm{Ho}(\mathcal{M}^{\text{Bif}})$. Lastly, let $y' : F\Gamma \to Y$ the transpose of y via the Quillen adjunction. It follows from the first invariance theorem theorem 4.2.44 that $X \vdash \phi(x)$ if and only if $GY \vdash \phi(y)$. From theorem 4.4.6, this is equivalent to $Y \vdash F(\psi)(y')$. By assumption $F(\phi) \approx F(\psi)$, so $Y \vdash F(\psi)(y')$. Again, this is $GY \vdash \psi(y)$ and $X \vdash \psi(x)$. This establishes the equality $|\phi|_X = |\psi|_X$ for all $X \in \mathcal{M}$ bifibrant, which proves $\psi \approx_b \phi$, and hence $\psi \approx \phi$. This concludes the proof of the statement. ∎

We now explain our strategy to prove the rest of theorem 4.4.2, that is the surjectivity part of the 4^{th} invariance theorem.

In [8], Reid Barton constructs a model 2-category structure on the 2-category of simplicial model categories. The trivial fibrations satisfy a property, that Barton called “extensible” (see theorem 4.4.9). In this section, we introduce a version of these in the non-enriched case, and we call those functors *Barton trivial fibrations*. In section 4.4.2 we show that the result holds for Barton trivial fibrations. After that, the idea is to use the same strategy as for the proof of the 3^{rd} invariance theorem based on this modified Brown factorization lemma to conclude the result holds for general Quillen equivalences. We could do this immediately for combinatorial simplicial model categories using Brown lemma in Barton's model structure, but for the general case we give a direct proof of the existence of the appropriate diagram which is inspired by how it would be done in Barton's model structure, but without relying on it directly. This is done in theorem 4.4.50 using section 4.4.3.

4.4.2 Invariance along Barton trivial fibrations

In this section we introduce a class of left Quillen functor that we call *Barton trivial fibrations* as they are essentially a non-simplicial version of the trivial fibrations of the model structure constructed by Barton in [8], and we establish that theorem 4.4.2 holds for these particular functors.

Definition 4.4.9. Let $F : \mathcal{C} \to \mathcal{D}$ a morphism between κ-coclans. We say that F is *extensible* if for every object in $X \in \mathcal{C}$ and for any cofibration $g : FX \hookrightarrow Y \in \mathcal{D}$ there exists $f : X \to Z$ and an isomorphism $F(Z) \cong Y$ making the obvious triangle commutative.

Dually, $F : \mathcal{C} \to \mathcal{D}$ a morphism between κ-clans is *extensible* if the induced map of κ-coclans $F^{\text{op}} : \mathcal{C}^{\text{op}} \to \mathcal{D}^{\text{op}}$.

In our setting, a functor $F : \mathcal{M} \to \mathcal{N}$ between weak model categories will be called extensible if the morphism of coclans $F : \mathcal{M}^{\text{Cof}} \to \mathcal{N}^{\text{Cof}}$ is extensible.

The terminology *extensible* in the definition above for both clans and coclans, instead of "extensible" and "co-extensible", is simply because it is always clear whether refers to fibrations or cofibrations. This is because, for example, when considering a morphism between clans the relevant structure that ought to be preserved is that related to fibrations. The name extensible from theorem 4.4.9 is adapted from Reid Barton's PhD thesis [8, Definition 8.3.1].

Extensible functors always induce a surjection between the languages of clans.

Lemma 4.4.10. Let $F : \mathcal{M} \to \mathcal{N}$ be an extensible morphism between κ-clans and $\Gamma \in \mathcal{M}$. Then, any formula $\Phi \in \mathbb{L}^{\mathcal{N}}_{\lambda}(F\Gamma)$ is the image by F of a formula $\Phi_0 \in \mathbb{L}^{\mathcal{M}}_{\lambda}(\Gamma)$.

Proof. Since every κ-clan is of the form $\mathbb{C}_T$ for some T generalized κ-algebraic theory it is enough to show the result is valid for the syntactic definition of language as in theorem 4.2.1. We prove by induction on formulas $\Phi \in \mathbb{L}^{\mathcal{N}}_{\lambda}(\Delta)$ that, given any context Γ and $f : \Delta \cong F(\Gamma)$, there is a formula $\Phi_0 \in \mathbb{L}^{\mathcal{M}}_{\lambda}(\Gamma)$ such that $f^*(F\Phi_0) = \Phi$.

1. When $\Phi = \top$ or $\Phi = \bot$, then this can clearly be lifted to $\top$ and $\bot$.

2. If $\Phi = \neg\Psi$ or $\Phi = \bigvee_{i\in I} \Psi_i$ or $\Phi = \bigwedge_{i\in I} \Psi_i$ then it is also clear that Φ can be lifted. Indeed, we can simply use the inductive hypothesis to lift each Ψ_i and then use the boolean algebra structure to conclude.

3. Suppose that Φ is of the form $\exists_\pi\Psi$ or $\forall_\pi\Psi$ for some fibration $\pi : \Gamma' \twoheadrightarrow F(\Gamma)$. The formula $\Psi \in \mathbb{L}^{\mathcal{N}}_{\lambda}(\Gamma')$, so $\Phi \in \mathbb{L}^{\mathcal{N}}_{\lambda}(F\Gamma)$. Furthermore, we assume that Ψ can be lifted. Since F is a trivial fibration, there is a lift $\bar{\pi} : \bar{\Gamma}' \to \Gamma \in \mathcal{M}$ of $\pi : \Gamma' \twoheadrightarrow F(\Gamma)$, which comes with an isomorphism $g : \Gamma' \cong F(\bar{\Gamma}')$ such that the following triangle commutes

$$\begin{tikzcd}
\Gamma' \arrow[r, two heads, "\pi"] \arrow[d, "g", "\cong"'] & F(\Gamma) \\
F(\bar{\Gamma}'). \arrow[ur, two heads, "F(\bar{\pi})"'] &
\end{tikzcd}$$

Therefore, we get a commutative square as in the left, and at the level of languages as on the right

$$\begin{tikzcd}
\Gamma' \arrow[r, two heads, "\pi'"] \arrow[d, "g", "\cong"'] & \Delta \arrow[d, "f"', "\cong"] \\
F(\bar{\Gamma}') \arrow[r, two heads, "F(\bar{\pi})"'] & F(\Gamma)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{L}^{\mathcal{N}}_{\lambda}(F(\bar{\Gamma}')) \arrow[r, "\exists_{\pi'}"] \arrow[d, "g^*"'] & \mathbb{L}^{\mathcal{N}}_{\lambda}(F(\Gamma)) \arrow[d, "f^*"] \\
\mathbb{L}^{\mathcal{N}}_{\lambda}(\Gamma') \arrow[r, "\exists_{F(\bar{\pi})}"'] & \mathbb{L}^{\mathcal{N}}_{\lambda}(\Delta).
\end{tikzcd}$$

By assumption $\psi \in \mathbb{L}^{\mathcal{N}}_{\lambda}(\Gamma')$ can be lifted. Hence, there is a formula $\Psi_0 \in \mathbb{L}^{\mathcal{M}}_{\lambda}(\bar{\Gamma}')$ such that $g^*(F\Psi_0) = \Psi$. Using the right hand square above, one can see that $\exists_{\bar{\pi}}\Psi_0$ is a lift for Φ.

This shows that the map is surjective. ∎

As an immediate consequence of theorem 4.4.10, we can establish the 4^{th} invariance theorem in the special case where $F : \mathcal{M} \to \mathcal{N}$ is a *Barton trivial fibration* as defined below. Before proving theorem 4.4.15, we give sufficient conditions to obtain a left Quillen equivalence. We will use this result to be able to establish 4^{th} invariance theorem for the general case later on.

Definition 4.4.11. A left Quillen functor $F : \mathcal{M} \to \mathcal{N}$ between weak model categories is called *weakly conservative* if for any core cofibration $x \hookrightarrow y \in \mathcal{M}^{\mathrm{Cof}}$ such that $h : Fx \overset{\sim}{\hookrightarrow} Fy$ is a trivial cofibration, the map $x \hookrightarrow y$ is a trivial cofibration.

The 'weakly' part in the previous definition does not come from weak model categories, but rather from the fact that core trivial cofibrations are weak equivalences.

Lemma 4.4.12. Let be $F : \mathcal{M} \to \mathcal{N}$ a left Quillen functor which is extensible and weakly conservative. Suppose there are diagrams

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, hook, "i"'] & C \\
B &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
FA \arrow[r, "Ff"] \arrow[d, hook, "Fi"'] & FC \arrow[d, hook, "v"', "\sim"] \\
FB \arrow[r, "u"'] & Z
\end{tikzcd}$$

in $\mathcal{M}$ and $\mathcal{N}$, respectively, where $C \in \mathcal{M}^{\mathrm{Bif}}$ and $Z \in \mathcal{N}^{\mathrm{Bif}}$ are bifibrant and the right square is commutative. Then, there exists $g : B \to C$ that makes the triangle commutative and such that in the diagram

$$\begin{tikzcd}
FA \arrow[r, "Ff"] \arrow[d, hook, "Fi"'] & FC \arrow[d, hook, "v"', "\sim"] \\
FB \arrow[r, "u"'] \arrow[ur, dashed, "Fg"] & Z
\end{tikzcd}$$

the lower triangle commutes up to homotopy relative to FA.

Proof. Since F is left Quillen then we have $F(B\coprod_A C) \cong FB\coprod_{FA} FC$ and is cofibrant. Up to this isomorphism, we factor the map $F(B\coprod_A C) \to Z$ as $F(B\coprod_A C) \hookrightarrow Y \overset{\sim}{\twoheadrightarrow} Z$. Since F is extensible we can lift this cofibration to a cofibration $B\coprod_A C \hookrightarrow D$ together with the isomorphism $FD \cong Y$ making the resulting triangle commutative, which also implies that FD is bifibrant since Y is. Furthermore, this produces a commutative diagram as on the left,

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, hook, "i"'] & C \arrow[d, hook] \arrow[ddr, "h"] & & & FC \arrow[r, hook, "\sim"] \arrow[d, hook, "Fh"'] \arrow[dr, hook] & Z \\
B \arrow[r] \arrow[drr, "k"] & B\coprod_A C \arrow[dr, hook, "l"] & {} \arrow[r, mapsto, "F"] & {} & FD \arrow[r, dashed, "\cong"] & Y \arrow[u, two heads, "\sim"'] \\
& & D & & &
\end{tikzcd}$$

while the diagram on the right is the result of applying F, we introduce the name $\rho : FD \overset{\sim}{\twoheadrightarrow} Z$ for the evident resulting trivial fibration. We can use the 2-out-of-3 property of weak equivalences between cofibrant-fibrant objects to conclude that $FC \hookrightarrow Y$ is a weak equivalence, and hence a trivial cofibration. Since F is weakly conservative, the map $C \hookrightarrow D$ must be a weak equivalence too. Using that C is bifibrant we can obtain a dashed arrow which is a homotopy inverse of h

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, hook, "i"'] & C \arrow[r, "Id"] \arrow[d, hook, "h"', "\sim"] & C \\
B \arrow[r, "k"'] & D, \arrow[ur, dashed, "r"'] &
\end{tikzcd}$$

we can take $g := rk$ to be a diagonal filler of the square. Observe that when we apply F to the resulting diagram, it gives us the square and the diagonal in the diagram

$$\begin{tikzcd}
FA \arrow[r, "Ff"] \arrow[d, "Fi"'] & FC \arrow[d, hook, "Fh"', "\sim"] \arrow[dr, "\sim", "v"'] & \\
FB \arrow[ur, "Fg"] \arrow[r, "Fk"] \arrow[rr, bend right, "u"'] & FD \arrow[r, two heads, "\sim", "\rho"'] & Z
\end{tikzcd}$$

where a priori the outer triangle involving u is not commutative. However, we can realize this diagram in the homotopy category $\mathrm{Ho}(FA/\mathcal{N})$. So working in the homotopy category we have $hr = Id$ and $FhFr = Id$. By construction, we also get $Fg = FrFk$, therefore $FhFg = FhFrFk = Fk$ in the homotopy category, and $\rho : FD \overset{\sim}{\twoheadrightarrow} Z$ becoming an isomorphism implies $vFg = u$ up to homotopy relative to FA. ∎

Corollary 4.4.13. Let $F : \mathcal{M} \to \mathcal{N}$ a left Quillen functor between weak model categories. Assume that $F : \mathcal{M}^{\mathrm{Cof}} \to \mathcal{N}^{\mathrm{Cof}}$ is extensible and weakly conservative, then F is a left Quillen equivalence.

Proof. We show directly that F induces an equivalence of categories between the homotopy categories.

Assume that $X \in \mathcal{N}^{\mathrm{Cof}}$ is cofibrant. Then we can use that F is extensible for the cofibration $0 \hookrightarrow X$ to obtain a cofibrant object $A \in \mathcal{M}^{\mathrm{Cof}}$ and an isomorphism $FA \cong X \in \mathcal{N}$. This shows that the induced functor is essentially surjective.

We now show that for $\mathrm{Ho}(\mathcal{M}) \to \mathrm{Ho}(\mathcal{N})$ is full. Let $B, C \in \mathcal{M}^{\mathrm{Cof}}$ cofibrant objects. We could take a fibrant replacement C^{Fib} and use this instead, so we can freely assume that C is bifibrant. A map $FB \to FC \in \mathrm{Ho}(\mathcal{N})$ can be represented by a cospan

$$FB \to (FC)^{\mathrm{Fib}} \overset{\sim}{\hookleftarrow} FC \in \mathcal{N}.$$

Therefore, we can use theorem 4.4.12 to find a map $B \to C$ in $\mathrm{Ho}(\mathcal{M})$ which is in the preimage.

Lastly, we see that the induced functor is faithful. Let $A, C \in \mathcal{M}^{\mathrm{Cof}}$ cofibrant and two maps $f, g : A \to C \in \mathcal{M}$ which become equal in $\mathrm{Ho}(\mathcal{N})$ under the induced functor by F. This is just saying that the maps $F\bar{f}, F\bar{g} : FA \to F(C^{\mathrm{Fib}})$ are homotopic where $\bar{f}, \bar{g} : A \to C^{\mathrm{Fib}}$ are maps in $\mathcal{M}$. It will be enough to show that $\bar{f}$ and $\bar{g}$ are homotopic *i.e.,* there is a diagonal filler for the diagram

$$\begin{tikzcd}
A \coprod A \arrow[r, "(\bar{f},\bar{g})"] \arrow[d, hook] & C^{\mathrm{Fib}} \\
IA &
\end{tikzcd}$$

where IA is a weak cylinder object for A. Since F is a left Quillen functor, we can assume that cylinders are preserved. Furthermore, homotopies are independent of the choice of cylinders. We can express the homotopy between of $F\bar{f}$ and $F\bar{g}$ in $\mathcal{N}$ as the commutative square

$$\begin{tikzcd}
F(A \coprod A) \arrow[r, "(F\bar{f},F\bar{g})"] \arrow[d, hook] & F(B^{\mathrm{Fib}}) \arrow[d, hook, "\sim"] \\
F(IA) \arrow[r, "h"'] & F(B^{\mathrm{Fib}})^{\mathrm{Fib}},
\end{tikzcd}$$

where h is the homotopy, and the fibrant replacement $F(C^{\mathrm{Fib}})^{\mathrm{Fib}}$ is necessary since $F(C^{\mathrm{Fib}})$ is not fibrant as F is only left Quillen. The assumptions of theorem 4.4.12 are now satisfied, so this produces a diagonal as on the left whose image fits on the right square up to homotopy:

$$\begin{tikzcd}
A \coprod A \arrow[r, "(\bar{f},\bar{g})"] \arrow[d, hook] & C^{\mathrm{Fib}} \\
IA \arrow[ur, "H"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
F(A \coprod A) \arrow[r, "(F\bar{f},F\bar{g})"] \arrow[d, hook] & F(C^{\mathrm{Fib}}) \arrow[d, hook, "\sim"] \\
F(IA) \arrow[ur, "FH"] \arrow[r, "h"'] & F(C^{\mathrm{Fib}})^{\mathrm{Fib}}
\end{tikzcd}$$

The above shows that $\mathrm{Ho}(\mathcal{M}) \to \mathrm{Ho}(\mathcal{N})$ is faithful, concluding the proof that F is a left Quillen equivalence. ∎

Definition 4.4.14. Let $F : \mathcal{M} \to \mathcal{N}$ a left Quillen functor between weak model categories. We say that F is a *Barton trivial fibration* if it is extensible as a morphism between of the coclans $\mathcal{M}^{\mathrm{Cof}}$ and $\mathcal{N}^{\mathrm{Cof}}$ and weakly conservative.

Barton trivial fibrations which are also simplicial Quillen functor between combinatorial simplicial model categories are exactly the trivial fibrations in [8] in the model 2-category of pre-model categories. As the reader might anticipate, the notion of fibration between (simplicial) model categories exists as well, but we will make no use of it.

We now return to show the 4^{th} invariance theorem for the case in which the functor is a Barton trivial fibration.

Theorem 4.4.15. Let $F : \mathcal{M} \to \mathcal{N}$ be a Barton trivial fibration between weak model categories. Then for any cofibrant $\Gamma \in \mathcal{M}$ the induced map $h\mathbb{L}F_A : h\mathbb{L}^{\mathcal{M}}_{\lambda}(\Gamma) \to h\mathbb{L}^{\mathcal{N}}_{\lambda}(F\Gamma)$ is an isomorphism.

Proof. By the previous theorem 4.4.8 we know that $h\mathbb{L}F_\Gamma : h\mathbb{L}^{\mathcal{M}}_{\lambda}(\Gamma) \to h\mathbb{L}^{\mathcal{N}}_{\lambda}(F\Gamma)$ is injective. Next we can use theorem 4.4.10 by observing that this surjectivity also descends at the level of $h\mathbb{L}F_\Gamma : h\mathbb{L}^{\mathcal{M}}_{\lambda}(\Gamma) \to h\mathbb{L}^{\mathcal{N}}_{\lambda}(F\Gamma)$. ∎

Since our goal is to prove the third invariance theorem, with theorem 4.4.15 at hand, we simply need to reduce our problem to the case in which we have Barton trivial fibrations. The constructions to come are essentially the necessary steps for this reduction process.

4.4.3 Path objects for weak model categories

The next step is to build some sort of "path object" for (weak) model category so that we can emulate Brown Factorization lemma to factor a general Quillen equivalence into a retract of a Barton trivial fibration followed by a Barton fibration. Ideally, we would want for a model category $\mathcal{M}$, we would like to build a diagram of left Quillen functors

$$\mathcal{M} \to P\mathcal{M} \to \mathcal{M} \times \mathcal{M}$$

where the maps $P\mathcal{M} \to \mathcal{M}$ are Barton trivial fibrations, and then try to use it to follow the proof of Brown's factorization. Unfortunately, that is not going to be quite possible: we will not be able to construct a map $\mathcal{M} \to P\mathcal{M}$. Instead, we will construct, a diagram of the form

$$\begin{tikzcd}
R\mathcal{M} \arrow[r] \arrow[d, "p"] & P\mathcal{M} \arrow[d] \\
\mathcal{M} \arrow[r] & \mathcal{M} \times \mathcal{M}
\end{tikzcd}$$

where the arrow p is a Barton Trivial fibration. This will turn out to be sufficient to build our desired Brown style factorization. These weak model categories will be constructed $R\mathcal{M}$ and $P\mathcal{M}$ will be constructed as certain category of functor $\mathcal{M}^J$ and $\mathcal{M}^I$, equipped with certain localization of Reedy model structure. So we get a diagram

$$\begin{tikzcd}
\mathcal{M}^J \arrow[r] \arrow[d, two heads, "\sim"'] & \mathcal{M}^I \arrow[d] \\
\mathcal{M} \arrow[r] & \mathcal{M} \times \mathcal{M}
\end{tikzcd}$$

were the arrow on the left and the two maps $\mathcal{M}^I \to \mathcal{M}$ induced by the projections are Barton trivial fibrations. More precisely, the construction we do takes as input a left Quillen equivalence $F : \mathcal{M} \to \mathcal{N}$ between weak model categories and produces a diagram

$$\begin{tikzcd}
\mathcal{M}^J \arrow[r] \arrow[d, two heads, "\sim"'] & \mathcal{N}^I_F \arrow[d] \\
\mathcal{M} \arrow[r] & \mathcal{N} \times \mathcal{M}
\end{tikzcd}$$

were again the arrow on the left and the two maps $\mathcal{N}^I_F \to \mathcal{N}$ induced by the projections are Barton trivial fibrations. Hence, the first diagram is a particular case when $F = Id_{\mathcal{M}}$. This can be seen as the analogue (or rather a dual) of the diagram (4.4.1) that appear in the proof of the 3^{rd} invariance theorem, and it will play the exact same role.

The bulk of the work lies in endowing the categories $\mathcal{M}^I$ and $\mathcal{M}^J$ with the correct weak model structure. This can be summarized as follows: We start with the Reedy weak model structure on the category $\mathcal{M}^J$, or $\mathcal{N}^I$, and perform a "right Bousfield localization" to obtain our desired models.

Remark 4.4.16. The weak model structure on $\mathcal{N}^I$ encodes a pair of objects A, B in $\mathcal{N}$ with a "correspondence" between them; that is, a homotopy equivalence encoded by a cofibration $A \coprod B \to C$ where both maps $A \to C$ and $B \to C$ are trivial cofibrations. The weak model structure we obtain on $\mathcal{M}^J$ encodes objects X in $\mathcal{M}$ equipped with a (weak) cylinder object, so that we can send such an object X with a cylinder IX to the correspondence $X \coprod X \to IX$.

4.4.3.1 Weak model for objects with weak cylinders

We start by fixing a weak model category $\mathcal{M}$ and let J be the category

$$\begin{tikzcd} a \arrow[r, shift left, "i"] \arrow[r, shift right, "j"'] & b \arrow[r, "k"] & c \end{tikzcd}$$

such that $ki = kj$. Consider the degree function making J into a direct category, $\deg(a) = 0$, $\deg(b) = 1$, $\deg(c) = 2$. Our first goal is to prove:

Theorem 4.4.17. The category of diagrams $\mathcal{M}^J$ has a weak model structure where:

1. A map between diagrams $X \to Y$ is a cofibration if
   (a) It is a Reedy cofibration,
   (b) $Y_a \sqcup_{X_a} X_c \overset{\sim}{\hookrightarrow} Y_c$ and $Y_b \sqcup_{X_b} X_c \overset{\sim}{\hookrightarrow} Y_c$ are trivial cofibrations in $\mathcal{M}$.
2. Fibrations are level-wise fibrations.

Remark 4.4.18. The theorem above make reference to Reedy cofibrations, therefore we must justify first that $\mathcal{M}^J$ carries the Reedy weak model structure. Fortunately, this has been addressed in theorem 4.7.11.

Notation 4.4.19. For the sake of clarity, we denote by $\mathcal{M}^J_{Reedy}$ when referring to the Reedy weak model structure and $\mathcal{M}^J_{Loc}$ for the weak model structure of theorem 4.4.17. Of course, *a priori,* we have yet to prove that the last is indeed a weak model structure. Therefore, whenever we say, for example, that a map $f : X \to Y$ is a cofibration we just mean that f satisfies the corresponding condition of theorem 4.4.17.

We will justify that the following construction, which is simply the conditions of the theorem, is the correct one.

Observation 4.4.20. One can verify that in this new model structure, the core fibrations and core trivial cofibrations coincide with the ones in the Reedy weak model structure (see theorem 4.4.23).

The reader might suspect that this is not a fortuitous coincidence, these suspicions are well justified. As we mentioned, what we have done is a right Bousfield localization of a

Reedy weak model structure on $\mathcal{M}^J$. Such localizations are studied in [30] in the case when $\mathcal{M}$ is a combinatorial (accessible) weak model category. Due to the lack of a general theorem that justifies the existence of these localizations indeed produce a weak model category, we verify all required conditions by hand.

We examine the class of cofibrations. For a diagram $X \in \mathcal{M}^J$, the latching objects are $L_a X = \varnothing$, $L_b X = X_a \sqcup X_a$ and $L_c X = X_b \sqcup_{X_a} X_b$. These are cofibrant in $\mathcal{M}$. Then a map $f : X \to Y$ being a cofibration means that $X_a \hookrightarrow Y_a$,

$$X_b \sqcup_{X_a \sqcup X_a} (Y_a \sqcup Y_a) \hookrightarrow Y_b \text{ and } X_c \sqcup_{(X_b \sqcup_{X_a} X_b)} (Y_b \sqcup_{Y_a} Y_b) \hookrightarrow Y_c$$

are cofibrations in $\mathcal{M}$, and additionally $Y_a \sqcup_{X_a} X_c \overset{\sim}{\hookrightarrow} Y_c$ and $Y_b \sqcup_{X_b} X_c \overset{\sim}{\hookrightarrow} Y_c$ are trivial cofibrations in $\mathcal{M}$.

Therefore, a diagram $Y \in \mathcal{M}^J$ is *cofibrant* if Y_a is a cofibrant object in $\mathcal{M}$,

$$Y_a \sqcup Y_a \hookrightarrow Y_b \text{ and } Y_b \sqcup_{Y_a} Y_b \hookrightarrow Y_c$$

are cofibrations, and additionally $Y_a \overset{\sim}{\hookrightarrow} Y_c$ and $Y_b \overset{\sim}{\hookrightarrow} Y_c$ are trivial cofibrations. Spelling out the second Reedy condition give us the following commutative diagram:

$$\begin{tikzcd}
\varnothing \arrow[r, hook] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner", very near end] & Y_a \arrow[d, hook] \arrow[ddr, bend left] & \\
Y_a \arrow[r, hook] \arrow[drr, bend right] & Y_a \sqcup Y_a \arrow[dr, hook] & \\
& & Y_b
\end{tikzcd}$$

This says that both maps $Y_a \overset{Yi}{\underset{Yj}{\rightrightarrows}} Y_b$ are cofibrations. We can use this on the following diagram

$$\begin{tikzcd}
Y_a \arrow[r, hook] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner", very near end] & Y_b \arrow[d, hook] \arrow[ddr, bend left] & \\
Y_b \arrow[r, hook] \arrow[drr, bend right] & Y_b \sqcup_{Y_a} Y_b \arrow[dr, hook] & \\
& & Y_c
\end{tikzcd}$$

to conclude that $Y_b \hookrightarrow Y_c$ is a cofibration. Of course this is in principle not necessary since we also have $Y_b \overset{\sim}{\hookrightarrow} Y_c$ is a trivial cofibration, the novel aspect is that this follows only from Reedy cofibrancy. We also have a trivial cofibration $Y_a \overset{\sim}{\hookrightarrow} Y_c$, by the two-out-of-three property the maps $Y_a \overset{Yi}{\underset{Yj}{\rightrightarrows}} Y_b$ are trivial cofibrations. We collect the above in the following:

Remark 4.4.21. If Y is cofibrant then we obtain the following diagram:

$$\begin{tikzcd}
Y_a \sqcup Y_a \arrow[r, "\triangledown"] \arrow[d, hook] & Y_a \arrow[d, hook, "\sim"] \\
Y_b \arrow[r, hook, "\sim"] & Y_c.
\end{tikzcd}$$

This is just to say that cofibrant diagrams of $\mathcal{M}^J_{Loc}$ encode objects of $\mathcal{M}$ for which a weak cylinder exists in the sense of theorem 4.7.6.

We reiterate that our goal is to show that the category of diagrams $\mathcal{M}^J_{Loc}$ has a weak model structure on it, where the cofibrations are the ones as specified in theorem 4.4.17. We begin by showing the following lemmas which are expected results in the theory of right Bousfield localizations.

Lemma 4.4.22. Let $X, Y \in \mathcal{M}^J_{Loc}$ cofibrant. Then, a map $X \to Y$ is a cofibration in $\mathcal{M}^J_{Loc}$ if and only if it is a cofibration in $\mathcal{M}^J_{Reedy}$.

Proof. We only prove the interesting direction; assume that X, Y are cofibrant in $\mathcal{M}^J_{Loc}$ and that $X \to Y \in \mathcal{M}^J_{Reedy}$ is a Reedy cofibration. Remains to show that

$$X_c \sqcup_{X_a} Y_a \to Y_c \text{ and } X_c \sqcup_{X_b} Y_b \to Y_c$$

are trivial cofibrations. The fact that the maps are weak equivalences follow by applying the 2-out-of-3 property of to the diagrams:

$$\begin{tikzcd}
X_a \arrow[r, hook] \arrow[d, hook, "\sim"] & Y_a \arrow[d, hook, "\sim"] \arrow[ddr, hook, bend left, "\sim"] & \\
X_c \arrow[r, hook] \arrow[drr, hook] & X_c \sqcup_{X_a} Y_c \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dashed] & \\
 & & Y_c
\end{tikzcd}
\qquad
\begin{tikzcd}
X_b \arrow[r, hook] \arrow[d, hook, "\sim"] & Y_b \arrow[d, hook, "\sim"] \arrow[ddr, hook, bend left, "\sim"] & \\
X_c \arrow[r, hook] \arrow[drr, hook] & X_c \sqcup_{X_b} Y_b \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dashed] & \\
 & & Y_c
\end{tikzcd}$$

The vertical maps $X_a \xhookrightarrow{\sim} X_c$, $X_b \xhookrightarrow{\sim} X_c$, $Y_a \xhookrightarrow{\sim} Y_c$ and $Y_b \xhookrightarrow{\sim} Y_c$, are trivial cofibrations since X and Y are cofibrant in $\mathcal{M}^J_{Loc}$. Remains to see that they are cofibrations. From the Reedy condition we have that the map $X_c \sqcup_{L_c X} L_c Y \hookrightarrow Y_c$ is a cofibration, and observe that the domains of the maps $X_c \sqcup_{X_a} Y_a \to Y_c$ and $X_c \sqcup_{X_b} Y_b \to Y_c$ contained in the colimit $X_c \sqcup_{L_c X} L_c Y$. Therefore, the maps factor as composition of cofibrations

$$X_c \sqcup_{X_a} Y_a \hookrightarrow X_c \sqcup_{L_c X} L_c Y \hookrightarrow Y_c \text{ and } X_c \sqcup_{X_b} Y_b \hookrightarrow X_c \sqcup_{L_c X} L_c Y \hookrightarrow Y_c,$$

which concludes the proof. ∎

Lemma 4.4.23. Let $X \in \mathcal{M}^J_{Loc}$ cofibrant and $X \to Z \in \mathcal{M}^J_{Reedy}$ a Reedy trivial cofibration. Then Z is cofibrant in $\mathcal{M}^J_{Loc}$. Furthermore, $X \to Z$ is a trivial cofibration in $\mathcal{M}^J_{Loc}$.

Proof. Since $X \overset{\sim}{\hookrightarrow} Z$ is a Reedy trivial cofibration then $X_a \overset{\sim}{\hookrightarrow} Z_a$, $X_b \sqcup_{X_a \sqcup X_a} (Z_a \sqcup Z_a) \overset{\sim}{\hookrightarrow} Z_b$ and $X_c \sqcup_{(X_b \sqcup_{X_a} X_b)} (Z_b \sqcup_{Z_a} Z_b) \overset{\sim}{\hookrightarrow} Z_c$ are trivial cofibrations. We then obtain the following diagram:

$$\begin{tikzcd}
X_a \sqcup X_a \arrow[r] \arrow[d, hook, "\sim"] & X_b \arrow[d, hook, "\sim"] \arrow[ddr, bend left] & \\
Z_a \sqcup Z_a \arrow[r] & \bullet \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, hook, "\sim"] & \\
& & Z_b
\end{tikzcd}$$

This shows that $X_b \overset{\sim}{\hookrightarrow} Z_b$ is a trivial cofibration. Since X is cofibrant then all the maps in the diagram

$$\begin{tikzcd}
X_a \arrow[r, shift left] \arrow[r, shift right] & X_b \arrow[r] & X_c
\end{tikzcd}$$

are trivial cofibrations. Consider the commutative diagram where the back and front faces are pushouts

$$\begin{tikzcd}
X_a \arrow[rr, hook, "\sim"] \arrow[dd, hook, "\sim"] \arrow[dr, hook, "\sim"] & & X_b \arrow[dd, hook, "\sim"] \arrow[dr, hook, "\sim"] & \\
& Z_a \arrow[rr, hook, "\sim"] \arrow[dd, hook, "\sim"] & & Z_b \arrow[dd, hook, "\sim"] \\
X_b \arrow[rr, hook, "\sim"] \arrow[dr, hook, "\sim"] & & X_b \sqcup_{X_a} X_b \arrow[dr] \arrow[uull, phantom, "\ulcorner", very near start] & \\
& Z_b \arrow[rr, hook, "\sim"] & & Z_b \sqcup_{Z_a} Z_b, \arrow[uull, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

which, by the two-out-of-three, shows that $X_b \sqcup_{X_a} X_b \overset{\sim}{\hookrightarrow} Z_b \sqcup_{Z_a} Z_b$ is a trivial cofibration. Remains to prove that $Z_b \overset{\sim}{\hookrightarrow} Z_c$ is a trivial cofibration. The pushout

$$\begin{tikzcd}
X_b \sqcup_{X_a} X_b \arrow[r] \arrow[d, hook, "\sim"] & X_c \arrow[d, hook, "\sim"] \arrow[ddr, bend left] & \\
Z_b \sqcup_{Z_a} Z_b \arrow[r] & \bullet \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, hook, "\sim"] & \\
& & Z_c
\end{tikzcd}$$

shows that $X_c \overset{\sim}{\hookrightarrow} Z_c$ is a trivial cofibration. Note that Z is Reedy cofibrant, hence $Z_b \hookrightarrow Z_c$ is a cofibration. By the two-out-of-three property, we can conclude that $Z_b \overset{\sim}{\hookrightarrow} Z_c$ is indeed a trivial cofibration. The above says that Z is cofibrant.

The second part is also true, since $X \to Z$ is a level-wise weak equivalence. ∎

Corollary 4.4.24. Any map between diagrams $f : X \to Y$, where X is a cofibrant diagram X and Y is a fibrant diagram in $\mathcal{M}^J_{Loc}$, can be factored as a trivial cofibration followed by a fibration.

Proof. We factor $f : X \to Y$ in $\mathcal{M}^J_{Reedy}$ to obtain $X \overset{\sim}{\hookrightarrow} Z \twoheadrightarrow Y$. $Z \twoheadrightarrow Y$ is also a fibration in $\mathcal{M}^J_{Loc}$ as is it is level-wise. Finally, $X \overset{\sim}{\hookrightarrow} Z \in \mathcal{M}^J_{Loc}$ by the previous theorem 4.4.23. ∎

For the factorization of a diagram map $f : X \to Y$ in $\mathcal{M}^J$, with X cofibrant and Y fibrant, into a cofibration followed by a trivial fibration we will need an auxiliary class of diagrams.

Construction 4.4.25. Denote by K the category J with the opposite Reedy structure given above (the degree function reversed). We endow $\mathcal{M}^K$ with the Reedy model structure. Then a diagram $Y \in \mathcal{M}^K_{Reedy}$ is fibrant if $Y_c \twoheadrightarrow 1$, $Y_b \twoheadrightarrow Y_c$ and $Y_a \twoheadrightarrow Y_b \times_{Y_c} Y_b$ are fibrations in $\mathcal{M}$. In this situation Y_b is also fibrant.

The limit of a diagram $Y \in \mathcal{M}^K$ is simply the equalizer $Eq(Y_i, Y_j)$. Note that the following pullback also computes the limit of Y:

$$\begin{tikzcd}
P \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & Y_a \arrow[d, two heads] \\
Y_b \arrow[r] & Y_b \times_{Y_c} Y_b.
\end{tikzcd}$$

From this we conclude that $\mathtt{Lim}\,Y$ is a fibrant object of $\mathcal{M}$ if $Y \in \mathcal{M}^K_{Reedy}$ is fibrant, and letting Z to denote the constant diagram at $\mathtt{Lim}\,Y$ then this comes with a diagram map $Z \to Y$ of the following form

$$\begin{tikzcd}
\mathtt{Lim}\,Y \arrow[r, shift left] \arrow[r, shift right] \arrow[d, two heads] & \mathtt{Lim}\,Y \arrow[r] \arrow[d, two heads] & \mathtt{Lim}\,Y \arrow[d, two heads] \\
Y_a \arrow[r, shift left] \arrow[r, shift right] & Y_b \arrow[r] & Y_c
\end{tikzcd}$$

where all top arrows are identities. Finally, note that Y being fibrant in $\mathcal{M}^K_{Reedy}$ implies that both maps $Y_a \rightrightarrows Y_b$ are fibrations. This can be deduced from the following diagram:

$$\begin{tikzcd}
Y_a \arrow[dr, two heads] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& Y_b \times_{Y_c} Y_b \arrow[r, two heads] \arrow[d, two heads] & Y_b \arrow[d, two heads] \\
& Y_b \arrow[r, two heads] & Y_c
\end{tikzcd}$$

Observation 4.4.26. Recall that the fibrations in $\mathcal{M}^J_{Loc}$ are the level-wise fibrations. Since $Z \in \mathcal{M}^K$ is point-wise fibrant then it is Reedy fibrant in $\mathcal{M}^J_{Loc}$. Similarly, Y is Reedy fibrant in $\mathcal{M}^K_{Reedy}$, in particular, implies that is object-wise fibrant, so it is fibrant in $\mathcal{M}^J_{Loc}$. We will use this diagram Z throughout this section.

Lemma 4.4.27. The map $Z \to Y$ from above is a trivial fibration in $\mathcal{M}^J_{Loc}$.

Proof. We show that the map has the right lifting property against any cofibration $A \hookrightarrow B \in \mathcal{M}^J_{Loc}$. First, assume that $A = \varnothing$, B is a cofibrant object in $\mathcal{M}^J_{Loc}$ and Y a fibrant diagram in $\mathcal{M}^K_{Reedy}$. We consider the lifting problem in $\mathcal{M}^J_{Loc}$:

$$\begin{tikzcd}
\varnothing \arrow[r] \arrow[d, hook] & Z \arrow[d] \\
B \arrow[r] & Y
\end{tikzcd}$$

From the discussion above we obtain the following commutative diagram:

$$\begin{tikzcd}
B_a \arrow[r, hook, shift left, "\sim"] \arrow[r, hook, shift right, "\sim"'] \arrow[d] & B_b \arrow[r, hook, "\sim"] \arrow[d] & B_c \arrow[d] \\
Y_a \arrow[r, two heads, shift left] \arrow[r, two heads, shift right] & Y_b \arrow[r, two heads] & Y_c
\end{tikzcd}$$

Thus, we obtain the following lifts:

$$\begin{tikzcd}
B_a \arrow[r] \arrow[d, hook, "Bi"', "\sim"] & Y_a \arrow[d, two heads, "Yi"] \\
B_b \arrow[r] \arrow[ur, dashed, "l_i"'] & Y_b
\end{tikzcd}
\qquad
\begin{tikzcd}
B_a \arrow[r] \arrow[d, hook, "Bj"', "\sim"] & Y_a \arrow[d, two heads, "Yj"] \\
B_b \arrow[r] \arrow[ur, dashed, "l_j"'] & Y_b
\end{tikzcd}
\qquad
\begin{tikzcd}
B_b \arrow[r] \arrow[d, hook, "Bk"', "\sim"] & Y_b \arrow[d, two heads, "Yk"] \\
B_c \arrow[r] \arrow[ur, dashed, "l_k"'] & Y_c
\end{tikzcd}$$

Using this we can construct the following commutative diagram:

$$\begin{tikzcd}
B_a \arrow[r, hook, "\sim"] \arrow[d, hook, "\sim"] \arrow[dr, phantom, "\ulcorner"] & B_b \arrow[d, hook, "\sim"] \arrow[dr, "l_j"] & & \\
B_b \arrow[r, hook, "\sim"] \arrow[dr, "B_k"'] & B_b \sqcup_{B_a} B_b \arrow[r] \arrow[d, hook, "\sim"] & Y_a \arrow[d, two heads] \arrow[dr, "Y_j"] & \\
& B_c \arrow[r] \arrow[dr, "l_k"'] & Y_b \sqcup_{Y_c} Y_b \arrow[r, two heads] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner"] & Y_b \arrow[d, two heads] \\
& & Y_b \arrow[r, two heads] & Y_c
\end{tikzcd}$$

where the middle trivial cofibration and fibration come from B being cofibrant in $\mathcal{M}^J_{Loc}$ and Y being fibrant in $\mathcal{M}^K_{Reedy}$ respectively. Then there exist a map $B_c \xrightarrow{r} Y_a$ that fits in the diagram. Furthermore, we readily see from the diagram that $Y_j r = l_k = Y_i r$. Therefore, there is a unique arrow $B_c \xrightarrow{t} Eq(Y_i, Y_j) = \mathtt{Lim}\, Y$ making the obvious triangle commutative. By taking the appropriate compositions with the map t we can construct a diagram map $B \to Z$ such that is a solution to the lifting problem.

For the general case

$$\begin{tikzcd}
A \arrow[r] \arrow[d, hook] & Z \arrow[d] \\
B \arrow[r] & Y
\end{tikzcd}$$

one can play the same game, the only change is that the diagram is a bit more involved. ∎

The diagram Z from theorem 4.4.25 is not necessarily Reedy cofibrant, but it is almost cofibrant in $\mathcal{M}^J_{Loc}$ as the maps in it are trivial cofibrations. The only missing part is that $\lim Y$ is not cofibrant in $\mathcal{M}$. In order to obtain cofibrant diagram in $\mathcal{M}^J_{Loc}$, we include the following result.

Lemma 4.4.28. If $Y \in \mathcal{M}^K_{Reedy}$ is fibrant then there exists a trivial fibration $W \twoheadrightarrow Y \in \mathcal{M}^J_{Loc}$ with $W \in \mathcal{M}^J_{Loc}$ cofibrant.

Proof. Since Y is fibrant in $\mathcal{M}^K_{Reedy}$, then it is fibrant in $\mathcal{M}^J_{Loc}$ as these are point-wise fibrant. Similarly, Z from theorem 4.4.25 is fibrant in $\mathcal{M}^J_{Loc}$, which also comes with a trivial fibration $Z \overset{\sim}{\twoheadrightarrow} Y$ by theorem 4.4.27. We can take a Reedy cofibrant replacement $W \overset{\sim}{\twoheadrightarrow} Z$. Since this last map is in particular a level-wise weak equivalence, it implies that the maps in W are weak equivalences. By 2-out-of-3 property, the maps in W are trivial cofibrations. This makes W a cofibrant replacement in $\mathcal{M}^J$ of Y by composing the trivial fibrations $W \overset{\sim}{\twoheadrightarrow} Z \overset{\sim}{\twoheadrightarrow} Y$. ∎

Before giving the factorization, we need a technical result that follows from the next lemma.

Remark 4.4.29. From [29, 2.1.11 Proposition], if $A \in \mathcal{M}$ is cofibrant then the coslice category $A/\mathcal{M}$ inherits a weak model structure from $\mathcal{M}$ where a map in $A/\mathcal{M}$ is cofibration, fibration and weak equivalences if is one in $\mathcal{M}$. Dually, one induces a weak model structure on the slice $\mathcal{M}/Y$ if Y is fibrant.

Construction 4.4.30. Consider a map $f : A \to Y$ in $\mathcal{M}$ where A is cofibrant and Y is fibrant. Consider $A/\mathcal{M}$ with the weak model described in the previous theorem 4.4.29.

The map $f : A \to Y$ allows us to see Y as an of object in $A/\mathcal{M}$, which is fibrant as Y is fibrant in $\mathcal{M}$. So, we can take the slice $(A/\mathcal{M})/Y$. Objects of $(A/\mathcal{M})/Y$ are factorizations of the form

$$\begin{tikzcd}
A \arrow[d] \arrow[dr, "f"] & \\
W \arrow[r] & Y.
\end{tikzcd}$$

Let two objects in this category

$$\begin{tikzcd}
A \arrow[d] \arrow[dr, "f"] & & & A \arrow[r] \arrow[dr, "f"'] & X \arrow[d] \\
B \arrow[r] & Y & \text{and} & & Y
\end{tikzcd}$$

which we refer to as B and X. A map from B to X is a diagonal filler of the resulting commutative square:

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & X \arrow[d] \\
B \arrow[r] \arrow[ur, dashed] & Y
\end{tikzcd}$$

A cofibrant object in $(A/\mathcal{M})/Y$ is one in which the first map is a cofibration in $\mathcal{M}$, and a fibrant object when the last map is a fibration *i.e.,*

$$\begin{tikzcd} A \arrow[d, hook] \arrow[dr, "f"] & & & & A \arrow[r] \arrow[dr, "f"'] & X \arrow[d, two heads] \\ B \arrow[r] & Y & \text{and} & & & Y \end{tikzcd}$$

respectively. Also note that the category $(A/\mathcal{M})/Y$ coincides with $A/(\mathcal{M}/Y)$, both as categories and as model categories.

Observation 4.4.31. [29, 2.4.3 Proposition] observed that the Quillen adjunction descends to the homotopy categories: If $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ is a Quillen pair, then we obtain a natural isomorphism

$$\mathrm{Ho}(\mathcal{C}^{\mathrm{B{\scriptsize IF}}})(W, G(Z)) \cong \mathrm{Ho}(\mathcal{D}^{\mathrm{B{\scriptsize IF}}})(F(W), Z)$$

of the homotopy categories.

The category $\mathrm{Ho}(\mathcal{C}^{\mathrm{B{\scriptsize IF}}})$ is the localization of the subcategory of bifibrant objects at trivial (cofibrations) fibrations. This is the content of [29, 2.2.6 Theorem], which also proves that there are equivalences

$$\mathrm{Ho}(\mathcal{C}^{\mathrm{C{\scriptsize OF}}}) \cong \mathrm{Ho}(\mathcal{C}^{\mathrm{B{\scriptsize IF}}}) \cong \mathrm{Ho}(\mathcal{C}^{\mathrm{F{\scriptsize IB}}})$$

where the first category is the localization of $\mathcal{C}^{\mathrm{C{\scriptsize OF}}}$ at trivial cofibrations, and the second is the localization of $\mathcal{C}^{\mathrm{F{\scriptsize IB}}}$ at trivial fibrations. Therefore, up to these equivalences of categories, we say that $\mathrm{Ho}(F) : \mathrm{Ho}(\mathcal{C}^{\mathrm{C{\scriptsize OF}}}) \to \mathrm{Ho}(\mathcal{D}^{\mathrm{C{\scriptsize OF}}})$ and $\mathrm{Ho}(G) : \mathrm{Ho}(\mathcal{D}^{\mathrm{F{\scriptsize IB}}}) \to \mathrm{Ho}(\mathcal{C}^{\mathrm{F{\scriptsize IB}}})$ are "adjoint".

Lemma 4.4.32. For all $i : A \hookrightarrow B$ and $i' : A' \hookrightarrow B'$ cofibrations between cofibrant objects, for all $p : X \twoheadrightarrow Y$ fibration between fibrant objects, if there is a commutative diagram:

$$\begin{tikzcd} A \arrow[r, "\sim", "k"'] \arrow[d, hook, "i"'] & A' \arrow[d, hook, "i'"] \\ B \arrow[r, "\sim", "l"'] & B' \end{tikzcd}$$

then $i \pitchfork p$ if and only if $i' \pitchfork p$. The dual statement also holds: For all $i : A \hookrightarrow B$ core cofibrations, for all $p : X \twoheadrightarrow Y$ and $p' : X' \twoheadrightarrow Y'$ fibrations between fibrant objects, if there is a commutative diagram:

$$\begin{tikzcd} X \arrow[r, "\sim", "m"'] \arrow[d, two heads, "p"'] & X' \arrow[d, two heads, "p'"] \\ Y \arrow[r, "\sim", "n"'] & Y' \end{tikzcd}$$

then $i \pitchfork p$ if and only if $i \pitchfork p'$.

Proof. We prove the first part of the lemma, the second part is dual. We have the following commutative squares

$$\begin{tikzcd} A \arrow[r, "k", "\sim"'] \arrow[d, hook, "i"'] & A' \arrow[d, hook, "i'"] & A \arrow[r, "f"] \arrow[d, hook, "i"'] & X \arrow[d, two heads, "p"] & A' \arrow[r, "f'"] \arrow[d, hook, "i'"'] & X \arrow[d, two heads, "p"] \\ B \arrow[r, "\sim", "l"'] & B' & B \arrow[r, "g"'] & Y & B' \arrow[r, "g'"'] & Y \end{tikzcd}$$

The proof relies heavily on theorem 4.4.30: The middle square above corresponds to a pair of objects B, X in a double slice category $A/\mathcal{M}/Y$, and a diagonal filler witnessing that $i \pitchfork p$ is a map in this double slice category.

We start with the induced weak model structure on the slice $\mathcal{M}/Y$. Note that from [29, 2.4.2 Example] the weak equivalence $k : A \to A'$ induces a weak Quillen equivalence $P_k : A/(\mathcal{M}/Y) \leftrightarrows A'/(\mathcal{M}/Y) : U_k$. Observe that B, B' are cofibrant and Y is fibrant. In what follows we leave Y implicit as we work in the slice $(A/\mathcal{M})/Y$, here we use that $(A/\mathcal{M})/Y = A/(\mathcal{M}/Y)$ from theorem 4.4.30.

The functor P_k takes a cofibration $A \hookrightarrow C$ along $k : A \to A'$, while U_k precomposes with k. Using the following diagram, since P_kB is cofibrant, by the two-out-of-three property

$$\begin{tikzcd}
A \arrow[r, "k"] \arrow[d, hook, "i"'] \arrow[dr, phantom, "\lrcorner", very near end] & A' \arrow[d, hook] \arrow[ddr, hook] & \\
B \arrow[r, "\sim"'] \arrow[ddrr, phantom] \arrow[drr, bend right, "\sim"'] & P_kB \arrow[dr, dashed] & \\
 & & B'
\end{tikzcd}$$

we see that there is a weak equivalence $P_kB \xrightarrow{\sim} B'$, this implies they are isomorphic in $\mathrm{Ho}(A'/(\mathcal{M}/Y))$. We have:

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{Ho}(A'/(\mathcal{M}/Y))}(B', X) &\cong \mathrm{Hom}_{\mathrm{Ho}(A'/(\mathcal{M}/Y))}(P_k(B), X) \\
&\cong \mathrm{Hom}_{\mathrm{Ho}(A/(\mathcal{M}/Y))}(B, U_k(X)) \\
&\cong \mathrm{Hom}_{\mathrm{Ho}(A/(\mathcal{M}/Y))}(B, X).
\end{aligned}$$

The first isomorphism follows from $B' \cong P_k(B)$ in $\mathrm{Ho}(A'/(\mathcal{M}/Y))$, the second is the weak Quillen adjunction $P_k \dashv U_k$ applied to the cofibrant object $B \in (A/\mathcal{M})/Y$ and the fibrant object $X \in (A'/\mathcal{M})/Y$. We crucially use theorem 4.4.31, so the second isomorphism is really up some equivalence of categories.

Now we use $\mathrm{Hom}_{\mathrm{Ho}(A'/(\mathcal{M}/Y))}(B', X) \cong \mathrm{Hom}_{\mathrm{Ho}(A/(\mathcal{M}/Y))}(B, X)$ to conclude. First, recall that a diagonal filler of

$$\begin{tikzcd}
A \arrow[r] \arrow[d, hook] & X \arrow[d, two heads] \\
B \arrow[r] & Y
\end{tikzcd}$$

is the same as a map $B \to X$ in $A/\mathcal{M}/Y$, and similarly for B' and X. Assume that $i \pitchfork p$, this give us a map $B \to X$ in $\mathrm{Ho}(A/\mathcal{M}/Y)$. Using the isomorphism, we have a map $B' \to X$ in $\mathrm{Ho}(A'/\mathcal{M}/Y)$, from which we can select a representative of the homotopy class, which implies that $i' \pitchfork p$. Similarly, we get that $i' \pitchfork p$ implies $i \pitchfork p$. ∎

Lemma 4.4.33. Let $X \to Y$ be a map in $\mathcal{M}^J$ with X cofibrant and Y fibrant. Then such a map can be factored as a cofibration followed by a trivial fibration.

Proof. Observe first that Y can be assumed to be Reedy cofibrant in $\mathcal{M}^J$. Indeed, we can simply take a Reedy cofibrant replacement $Y' \overset{\sim}{\twoheadrightarrow} Y$, and instead use the dashed arrow

$$\begin{tikzcd}
0 \arrow[r, hook] \arrow[d, hook] & Y' \arrow[d, two heads, "\sim"] \\
X \arrow[r] \arrow[ur, dashed] & Y.
\end{tikzcd}$$

Under this assumption, Y is point-wise cofibrant, whence Reedy cofibrant in $\mathcal{M}^K$. Therefore, we can take a fibrant replacement in $\mathcal{M}^K$, $Y \overset{\sim}{\hookrightarrow} Y'$. Using [29, Corollary 2.4.4] equivalences are preserved under pullbacks along fibrations, so we get the pullback square

$$\begin{tikzcd}
LY \arrow[r, "\sim"] \arrow[d, two heads] & W \arrow[d, two heads] \\
Y \arrow[r, hook, "\sim"] & Y'.
\end{tikzcd}$$

Furthermore, we know from theorem 4.4.28 that $W \twoheadrightarrow Y'$ is a trivial fibration in $\mathcal{M}^J$. Therefore, it has the right lifting property against any cofibration between cofibrant objects in $\mathcal{M}^J$. We can use theorem 4.4.32 to conclude that $LY \twoheadrightarrow Y$ satisfies the same property, *i.e.,* it is a trivial fibration in $\mathcal{M}^J$. Since X is cofibrant, we obtain a lift

$$\begin{tikzcd}
0 \arrow[r, hook] \arrow[d, hook] & LY \arrow[d, two heads, "\sim"] \\
X \arrow[r] \arrow[ur, dashed] & Y.
\end{tikzcd}$$

The map $X \to LY$ can be factored in the Reedy model structure $\mathcal{M}^J$ as $X \hookrightarrow X' \overset{\sim}{\twoheadrightarrow} LY$. The diagram X' is cofibrant in $\mathcal{M}^J$ since is equivalent to the cofibrant diagram LY, and X is cofibrant by assumption. Therefore, it follows from theorem 4.4.23 that the Reedy cofibration $X \hookrightarrow X'$ is a cofibration in the model $\mathcal{M}^J$. This gives us the desired factorization in $\mathcal{M}^J$, $X \hookrightarrow X' \overset{\sim}{\twoheadrightarrow} Y$. ∎

All the previous work can be summarized in the following proof of theorem 4.4.17. This proves that the category of diagrams $\mathcal{M}^J$ has a weak model structure with the specified cofibrations and fibrations, which, as explained above, encodes objects with a weak cylinder object. We remark that our proof will show that the conditions of [29, 2.1.10 Definition] are satisfied instead of theorem 4.7.1. The reason is for this is that in theorem 4.4.17 we do not have an explicit class of weak equivalences. More precisely, we will use [29, 2.3.3 Proposition] which gives some alternative criteria to obtain a weak model structure in this sense.

Proof. (theorem 4.4.17) Note first that we have the Reedy weak model structure on $\mathcal{M}^J$ by virtue of theorem 4.7.11. Also, the existence of initial and terminal diagrams is clear. We must justify that the class of (co)fibrations form a class of (co)fibrations in $\mathcal{M}^J$. For fibrations, since these are level-wise, it is immediate that: the terminal diagram is fibrant, any isomorphism with fibrant codomain is a fibration, it is closed under compositions, and stable under pullbacks along maps between fibrant objects.

The dual conditions must be verified for the class of cofibrations. That the initial diagram is cofibrant it is immediate to verify. To see other stability conditions, we observe these are true for $\mathcal{M}^J_{Reedy}$. In addition, for stability under isomorphisms we use repeatedly that maps in $\mathcal{M}$ isomorphic to trivial cofibration are also trivial cofibrations. This simply because the new condition we added involves the requirement that certain maps trivial cofibrations. Stability under pushouts follows from the stability in $\mathcal{M}^J_{Reedy}$ and the fact that trivial cofibrations in the weak model $\mathcal{M}$ are pushout stable.

The factorization of a map $f : X \to Y$, where X is cofibrant and Y is fibrant, into a cofibration followed by a trivial fibration is the content of theorem 4.4.33.

The factorization of a map $f : X \to Y$, where X is cofibrant and Y is fibrant, into a trivial cofibration followed by a fibration is the content of theorem 4.4.24.

In order to conclude, we use [29, 2.3.3 Proposition]. For which we need to verify that a cofibration $X \to Y \in \mathcal{M}^J_{Loc}$ with X cofibrant and Y fibrant admit a relative strong cylinder object. Firstly, we know that the map admits a relative cylinder object in $\mathcal{M}^J_{Reedy}$:

$$\begin{tikzcd}
Y \coprod_X Y \arrow[r] \arrow[d, hook] & Y \\
I_X Y \arrow[ur] &
\end{tikzcd}$$

with $Y \hookrightarrow Y \coprod_X Y \hookrightarrow I_X Y$ a Reedy trivial cofibration. Since Y is cofibrant in $\mathcal{M}^J_{Loc}$ we can use theorem 4.4.23 to conclude that $I_X Y$ is also cofibrant in $\mathcal{M}^J_{Loc}$, and that the map $Y \to I_X Y$ is a trivial cofibration in $\mathcal{M}^J_{Loc}$. Now we have cofibrant objects $Y \coprod_X Y$, $I_X Y$ in $\mathcal{M}^J_{Loc}$ and a Reedy cofibration between them, so we use theorem 4.4.22 to conclude it is actually a cofibration in $\mathcal{M}^J_{Loc}$. This gives us the relative cylinder objects.

Finally, the 2-out-of-3 property for trivial cofibrations between bifibrant objects follow using that $\mathcal{M}^J_{Reedy}$ is a weak model category, so the property is true in this Reedy weak model structure. By which we mean that the property is true for the underlying Reedy trivial cofibrations between bifibrant objects of $\mathcal{M}^J_{Loc}$. Theorem 4.4.23 allows us to conclude that such Reedy trivial cofibrations are indeed trivial cofibrations in $\mathcal{M}^J_{Loc}$. Now [29, 2.3.3 Proposition] allows us to conclude that $\mathcal{M}^J_{Loc}$, with the specified classes of maps, is a weak model category. ∎

4.4.3.2 Weak model on correspondences

Next, we consider another diagram category I:

$$0 \to 2 \leftarrow 1$$

Where $\deg(0) = \deg(1) = 0$ and $\deg(2) = 1$. Similarly to the previous section, we construct a "right Bousfield localization" of the Reedy weak model structure on $\mathcal{N}^I$.

Theorem 4.4.34. There is a weak model structure $\mathcal{N}^I_{Loc}$ on the category of diagrams $\mathcal{N}^I$ obtained from the Reedy weak model structure $\mathcal{N}^I_{Reedy}$, where:

1. A map between diagrams $X \to Y$ is a cofibration if
   (a) It is a Reedy cofibration,
   (b) $X_2 \sqcup_{X_1} Y_1 \xhookrightarrow{\sim} Y_2$ and $X_2 \sqcup_{X_0} Y_0 \xhookrightarrow{\sim} Y_2$ are trivial cofibrations in $\mathcal{M}$.
2. Fibrations are level-wise fibrations.

It will be useful to have in mind that for an object $X \in \mathcal{N}^I$ we have $L_0X = 0$ and $L_1X = X_0 \sqcup X_1$. So a map $X \to Y$ is a Reedy cofibration if the maps $X_0 \hookrightarrow Y_0$, $X_1 \hookrightarrow Y_1$ and $(Y_0 \sqcup Y_1) \sqcup_{(X_0 \sqcup X_1)} X_2 \hookrightarrow Y_2$ are cofibrations.

Observation 4.4.35. Unwinding the definitions, a diagram $X \in \mathcal{N}^I_{Loc}$ is cofibrant if both maps $X_0 \xhookrightarrow{\sim} X_2$ and $X_1 \xhookrightarrow{\sim} X_2$ are trivial cofibrations.

The proof of the theorem is completely analogous to theorem 4.4.17. We state the lemmas necessary for this and only comment on the proofs when adequate.

Lemma 4.4.36. Let $X, Y \in \mathcal{N}^I_{Loc}$ cofibrant. Then, a map $X \to Y$ is a cofibration in $\mathcal{N}^I_{Loc}$ if and only if it is a cofibration in $\mathcal{N}^I_{Reedy}$.

Proof. Just as in theorem 4.4.22 we only prove the interesting direction; assume that X, Y are cofibrant in $\mathcal{N}^J_{Loc}$ and that $X \to Y \in \mathcal{N}^I_{Reedy}$ is a Reedy cofibration. Remains to show that

$$X_2 \sqcup_{X_0} Y_0 \to Y_2 \text{ and } X_2 \sqcup_{X_1} Y_1 \to Y_2$$

are trivial cofibrations. Again, the fact that the maps are weak equivalences follow from X, Y being cofibrant and the 2-out-of-3 property. To see that they are cofibrations we can use the Reedy condition just as in theorem 4.4.22. ∎

Lemma 4.4.37. Let $X \in \mathcal{N}^I_{Loc}$ cofibrant and $X \to Z \in \mathcal{N}^I_{Reedy}$ a Reedy trivial cofibration. Then Z is cofibrant in $\mathcal{N}^I_{Loc}$. Furthermore, $X \to Z$ is a trivial cofibration in $\mathcal{N}^I_{Loc}$.

Proof. The difficult part is to show that. Since $X \to Z$ is a Reedy trivial cofibration, then by theorem 4.7.16 we have it is a levelwise trivial cofibration. Then Z is cofibrant by the 2-out-of-3 property. ∎

Corollary 4.4.38. Any map between diagrams $f : X \to Y$, where X is a cofibrant diagram X and Y is a fibrant diagram in $\mathcal{N}^I_{Loc}$, can be factored as a trivial cofibration followed by a fibration.

Proof. Now that we have theorem 4.4.37, we can proceed as in theorem 4.4.24 by first taking the factorization in $\mathcal{N}^I_{Reedy}$. ∎

Construction 4.4.39. Denote by K' the category I with the opposite Reedy structure given above (the degree function reversed). We endow $\mathcal{N}^{K'}$ with the Reedy model structure. Then a diagram $Y \in \mathcal{N}^{K'}_{Reedy}$ is fibrant if $Y_2 \twoheadrightarrow 1$, $Y_0 \twoheadrightarrow Y_2$ and $Y_1 \twoheadrightarrow Y_2$ are fibrations in $\mathcal{N}$.

In this situation we can see that $\lim Y = Y_0 \times_{Y_2} Y_1$ and is fibrant in $\mathcal{N}$. We can again take a $Z \in \mathcal{N}^I$ to be the correspondence with constant value $\lim Y$. So it comes with a map $Z \to Y$.

Lemma 4.4.40. The map $Z \to Y$ from above is a trivial fibration in $\mathcal{N}^I_{Loc}$.

Proof. The same idea as in theorem 4.4.27 carries over here. The diagrams are even simpler. ∎

Lemma 4.4.41. If $Y \in \mathcal{N}^{K'}_{Reedy}$ is fibrant then there exists a trivial fibration $W \twoheadrightarrow Y \in \mathcal{N}^I_{Loc}$ with $W \in \mathcal{N}^I_{Loc}$ cofibrant.

Proof. The argument of theorem 4.4.28 applies here too. ∎

Lemma 4.4.42. Let $X \to Y$ be a map in $\mathcal{N}^I$ with X cofibrant and Y fibrant. Then such a map can be factored as a cofibration followed by a trivial fibration.

Proof. We have all ingredients to proceed as in theorem 4.4.33. Firstly, we can assume that Y is Reedy cofibrant in $\mathcal{N}^I$ and we can take a fibrant replacement in $\mathcal{N}^{K'}$. So we can construct the following pullback square:

$$\begin{tikzcd}
LY \arrow[r, "\sim"] \arrow[d, two heads, "\sim"'] \arrow[dr, phantom, "\lrcorner", very near start] & W \arrow[d, two heads, "\sim"'] \\
Y \arrow[r, hook, "\sim"] & Y'.
\end{tikzcd}$$

Then we can obtain a map $X \to LY$. Factoring this map as $X \hookrightarrow X' \overset{\sim}{\twoheadrightarrow} LY$, the first map is moreover a cofibration in $\mathcal{N}^I_{Loc}$ in view of theorem 4.4.37. This produces the factorization $X \hookrightarrow X' \overset{\sim}{\twoheadrightarrow} Y$. ∎

The proof of theorem 4.4.34 is a carbon copy from the one of theorem 4.4.17, the lemmas of this section provide us with all the required steps.

4.4.3.3 Projections are Barton trivial fibrations

Lemma 4.4.43. The functor $\mathcal{N}^I \to \mathcal{N}$ such that $A \to B \leftarrow C \in \mathcal{N}^I \mapsto A \in \mathcal{N}$, is extensible. Also, the functor $\mathcal{N}^I \to \mathcal{N}$ such that $A \to B \leftarrow C \in \mathcal{N}^I \mapsto C \in \mathcal{N}$ is extensible.

Proof. Let $A := a \overset{\sim}{\hookrightarrow} b \overset{\sim}{\hookleftarrow} c \in \mathcal{N}^I_{Loc}$ be a cofibrant diagram and $x \in \mathcal{N}^{\mathrm{Cof}}$ a cofibrant object and a cofibration $a \hookrightarrow x$. We take the fibrant replacement of x and consider the pushout as indicated below, and we obtain a solution to the lifting problem on the right:

$$\begin{tikzcd}
& a \arrow[r, hook] \arrow[d, hook, "\sim"'] & x \arrow[r, hook, "\sim"] \arrow[d, hook, "\sim"] & x^{fib} \\
c \arrow[r, hook, "\sim"] & b \arrow[r] & b \sqcup_a x \arrow[ul, phantom, "\ulcorner", very near start] \arrow[ur, dashed] &
\end{tikzcd}$$

The resulting map $c \to x^{fib}$ can be factored as $c \hookrightarrow z \overset{\sim}{\twoheadrightarrow} x^{fib}$. We can take further pushouts

$$\begin{tikzcd}
 & a \arrow[r, hook] \arrow[d, hook, "\sim"'] & x \arrow[r, hook, "\sim"] \arrow[d, hook, "\sim"] & x^{fib} \\
c \arrow[r, hook, "\sim"] \arrow[d, hook] & b \arrow[r, hook] \arrow[d, hook] & b \sqcup_a x \arrow[ur] \arrow[d, hook] \arrow[ul, phantom, very near start, "\lrcorner"] & \\
z \arrow[r, hook, "\sim"'] \arrow[urrr, two heads, bend right=60, "\sim"'] & z \sqcup_c b \arrow[r, hook] \arrow[ul, phantom, very near start, "\lrcorner"] & P. \arrow[ul, phantom, very near start, "\lrcorner"] &
\end{tikzcd}$$

There is a map $P \to x^{fib}$ which we can factor as $P \hookrightarrow y \overset{\sim}{\twoheadrightarrow} x^{fib}$, and the resulting diagram we get

$$\begin{tikzcd}
 & a \arrow[r, hook] \arrow[d, hook, "\sim"'] & x \arrow[r, hook, "\sim"] \arrow[d, hook, "\sim"] & x^{fib} \\
c \arrow[r, hook, "\sim"] \arrow[d, hook] & b \arrow[r, hook] \arrow[d, hook] & b \sqcup_a x \arrow[ur] \arrow[d, hook] \arrow[ul, phantom, very near start, "\lrcorner"] & y \arrow[u, two heads, "\sim"'] \\
z \arrow[r, hook, "\sim"'] & z \sqcup_c b \arrow[r, hook] \arrow[ul, phantom, very near start, "\lrcorner"] & P \arrow[ur, hook] \arrow[ul, phantom, very near start, "\lrcorner"] &
\end{tikzcd}$$

Furthermore, there is a map $b \sqcup_a x \to y$ which is a cofibration as it is the composite of the two cofibrations. Using the 2-out-of-3 property repeatedly, one concludes that the map $z \sqcup_c b \to y$ is a trivial cofibration. Thus, we have constructed the cofibrant object $X := z \overset{\sim}{\hookrightarrow} y \overset{\sim}{\hookleftarrow} x \in \mathcal{N}^I_{Loc}$. The induced map $A \to X$ is a level-wise cofibration. The maps $b \sqcup_a x \to y$ and $b \sqcup_a z \to y$ are trivial cofibrations.

Remains to show that $A \to X$ is a Reedy cofibration. We already have that $a \to x$ and $c \to z$ are cofibrations. We now need to show that the induced map

$$\begin{tikzcd}
a \sqcup c \arrow[r] \arrow[d, hook] & b \arrow[d, hook] \arrow[ddr, bend left] & \\
x \sqcup z \arrow[r] \arrow[drr, bend right] & (x \sqcup z) \sqcup_{a \sqcup c} b \arrow[dr, dashed] & \\
 & & y
\end{tikzcd}$$

is a cofibration. By diagram chasing, one can show that the diagram

$$\begin{tikzcd}
a \sqcup c \arrow[r] \arrow[d] & b \arrow[d] \\
x \sqcup z \arrow[r] & (z \sqcup_c b) \sqcup_b (b \sqcup_a x)
\end{tikzcd}$$

commutes. One shows that the bottom right corner computes the pushout of the span. Using that the map $P \hookrightarrow y$ is a cofibration one concludes that $(x \sqcup) \sqcup_{a \sqcup c} b \to y$ is also a cofibration.

This concludes the proof that $A \to X$ is a Reedy core cofibration in $\mathcal{N}^I$. Therefore, it must a cofibration. We summarize our construction with the following diagram:

$$\begin{tikzcd}
c \arrow[rr, hook] \arrow[d, hook, "\sim"] & & z \arrow[d, hook, "\sim"] \\
b \arrow[rr, hook] & & y \\
a \arrow[rr, hook] \arrow[u, hook, "\sim"] & {} \arrow[d, mapsto] & x \arrow[u, hook, "\sim"] \\
a \arrow[rr, hook] & {} & x
\end{tikzcd}$$

This cofibration is a (strict) lift of $a \hookrightarrow x$, showing that the functor $\mathcal{N}^I \to N$ is an extensible functor. The second part of the lemma is analogous. ∎

Observation 4.4.44. Note that in the previous theorem 4.4.43, using 2-out-of-3 property, if we start with a trivial cofibration $a \overset{\sim}{\hookrightarrow} x$ then we obtain a level-wise equivalence between cofibrant objects in $\mathcal{N}^I_{Loc}$. We conclude that the projections are weakly conservative.

Corollary 4.4.45. The functor $\mathcal{N}^I \to \mathcal{N}$ such that $A \to B \leftarrow C \in \mathcal{N}^I \mapsto A \in \mathcal{N}$, is as Barton trivial fibration. Also, the functor $\mathcal{N}^I \to \mathcal{N}$ such that $A \to B \leftarrow C \in \mathcal{N}^I \mapsto C \in \mathcal{N}$, is a Barton trivial fibration.

Proof. We saw in theorem 4.4.43 that the projections are extensible and from theorem 4.4.44 that is weakly conservative. It is also straightforward to see that it preserve cofibrations and trivial cofibrations. ∎

We now want to see that any left Quillen functor $F : \mathcal{M} \to \mathcal{N}$ part of a Quillen equivalence between weak model categories admits a Brown-like factorization. To this end, consider the following:

Construction 4.4.46. We define the category of diagrams

$$\mathcal{N}^I_F := \{Fa \to b \leftarrow c | a \in \mathcal{M}^{\mathrm{Cof}},\, b, c \in \mathcal{N}\}.$$

The weak model structure on this category is similar to that of $\mathcal{N}^I$, the only difference is that $X \to Y$ is a cofibration if $X_b \sqcup_{FX_a} FY_a \to Y_b$ is a trivial cofibration.

When F is the identity functor we recover $\mathcal{N}^I$ from theorem 4.4.34. A cofibrant object in $\mathcal{N}^I_F$ is a diagram of the form

$$Fa \overset{\sim}{\hookrightarrow} b \overset{\sim}{\hookleftarrow} c.$$

Observation 4.4.47. With the set up above, it follows from theorem 4.4.45 that the projection $\pi_1 : \mathcal{N}^I_F \to \mathcal{M}$, sending each diagram $Fa \to b \leftarrow c$ to a, is a Barton trivial fibration.

To show that the projection from $\pi_2 : \mathcal{N}^I_F \to \mathcal{N}$ sending each diagram $Fa \to b \leftarrow c$ to $c \in \mathcal{N}$ is a trivial fibration we make use of the following:

Lemma 4.4.48. Let $F : \mathcal{M} \to \mathcal{N}$ be a left Quillen equivalence between weak model categories. For any objects $x \in \mathcal{M}^{\text{Cof}}$, $y \in \mathcal{N}^{\text{Fib}}$ and a map $f : Fx \to y$ there exists an object $z \in \mathcal{M}^{\text{Cof}}$ such that f factors as

$$\begin{tikzcd}
Fx \arrow[rr, "f"] \arrow[dr, hook] & & y \\
& Fz. \arrow[ur, "\sim"'] &
\end{tikzcd}$$

Proof. We know that there is an isomorphism

$$\varphi : \mathrm{Hom}_{\mathcal{N}}(Fx, y) \simeq \mathrm{Hom}_{\mathcal{M}}(x, Gy) : \varphi^{-1}$$

given by the Quillen adjunction, natural in $x \in \mathcal{M}^{\text{Cof}}$ and $y \in \mathcal{N}^{\text{Fib}}$. Recall from [29, 2.4.3 Proposition] that $F : \mathcal{M}^{\text{Cof}} \to \mathcal{N}^{\text{Cof}}$ and $G : \mathcal{N}^{\text{Fib}} \to \mathcal{M}^{\text{Fib}}$ preserve equivalences. Take φf the adjoint transpose of f. We can take a factorization

$$\begin{tikzcd}
& z \arrow[dr, two heads, "s", "\sim"'] & \\
x \arrow[ur, hook, "r"] \arrow[rr, "\varphi f"'] & & Gy
\end{tikzcd}$$

By naturality, one checks that $f = \varphi^{-1} sFr$ where Fr is a cofibration. Since the Quillen pair is an equivalence, we deduce from [29, 2.4.5 Proposition (i)] that $\varphi^{-1}s$ is an equivalence. ∎

Corollary 4.4.49. Let $F : \mathcal{M} \rightleftarrows \mathcal{N} : G$ be a Quillen equivalence. Then the projection $\pi_2 : \mathcal{N}^I_F \to \mathcal{N}$sending each diagram $Fa \to b \leftarrow c$ to $c \in \mathcal{N}$ is a Barton trivial fibration.

Proof. We show that in a situation as in the diagram

$$\begin{tikzcd}
Fa \arrow[d, hook, "\sim"] & \\
b & \\
c \arrow[u, hook, "\sim"'] \arrow[d, mapsto] & \\
c \arrow[r, hook] & z,
\end{tikzcd}$$

there is a cofibrant object over z that projects onto $c \hookrightarrow z$. By taking a fibrant replacement, we can assume that the diagram is point-wise fibrant. From [29, 2.2.3 Proposition] there exists a homotopy inverse of $c \overset{\sim}{\hookrightarrow} b$, this give us a map $Fa \to c$. Using theorem 4.4.48 this last map can be factored as $Fa \hookrightarrow Fx \overset{\sim}{\to} c$. The rest of the proof continues as in theorem 4.4.45. ∎

Theorem 4.4.50. Given $F : \mathcal{M} \rightleftarrows \mathcal{N}$ be a left Quillen equivalence between weak model categories. Then, we have a diagram of week model categories

$$\begin{tikzcd}
\mathcal{M}^J \arrow[r, "H"] \arrow[d, "B"'] & \mathcal{N}_F^I \arrow[d, "(\pi_1,\pi_2)"] \\
\mathcal{M} \arrow[r, "(Id_{\mathcal{M}},F)"'] & \mathcal{M} \times \mathcal{N},
\end{tikzcd}$$

where π_1 and π_2 are Barton trivial fibrations.

Proof. The work we have done produces a diagram as on the left below, and the action of the functors on objects is spelled out on the right:

$$\begin{tikzcd}
\mathcal{M}^J \arrow[r, "H"] \arrow[d, "B"'] & \mathcal{N}_F^I \arrow[d, "(\pi_1,\pi_2)"] \\
\mathcal{M} \arrow[r, "(Id_{\mathcal{M}},F)"'] & \mathcal{M} \times \mathcal{N}
\end{tikzcd}
\qquad
\begin{tikzcd}
X_a \rightrightarrows X_b \to X_c \arrow[r, mapsto, "H"] \arrow[d, mapsto, "B"'] & FX_a \rightrightarrows FX_b \arrow[d, mapsto] \\
X_a \arrow[r, mapsto] & (X_a, FX_a)
\end{tikzcd}$$

We have shown in theorem 4.4.45 and theorem 4.4.49 that both projections are Barton trivial fibrations. ∎

It will be essential to highlight that there is a diagonal functor which is a Barton trivial fibration, making the lower triangle commutative.

Corollary 4.4.51. Let $F : \mathcal{M} \to \mathcal{M}$ be a left Quillen equivalence. There exists a Barton trivial fibration $P : \mathcal{N}_F^I \to \mathcal{M}$.

Proof. Theorem 4.4.50 can be further specialized to a diagram

$$\begin{tikzcd}
\mathcal{M}^J \arrow[r] \arrow[d] & \mathcal{N}_F^I \arrow[d, "\pi_1"] \\
\mathcal{M} \arrow[r, "Id_{\mathcal{M}}"'] & \mathcal{M}
\end{tikzcd}$$

from which we see that there is a functor $P : \mathcal{N}_F^I \to \mathcal{M}$. This is an immediate consequence of theorem 4.4.50. ∎

4.4.4 Proof of main theorem

Theorem 4.4.52. Let $F : \mathcal{M} \rightleftarrows \mathcal{N} : G$ a Quillen equivalence. Then, for any cofibrant object $A \in \mathcal{M}$. The induced map $h\mathbb{L}F_A : h\mathbb{L}_\lambda^{\mathcal{M}}(A) \to h\mathbb{L}_\lambda^{\mathcal{N}}(FA)$ is an isomorphism.

Proof. Recall from theorem 4.4.8 that for any cofibrant object A the induced map $h\mathbb{L}F_A$ is injective. Remains to show that it is surjective. Using theorem 4.4.51, we obtain a diagram

$$\begin{tikzcd}
\mathcal{M}^J \arrow[r] \arrow[d] & \mathcal{N}_F^I \arrow[d, "\pi_2"] \arrow[dl, "P"] \\
\mathcal{M} \arrow[r, "F"'] & \mathcal{N}
\end{tikzcd}$$

where P is a Barton trivial fibration. $P : \mathcal{N}_F^I \to \mathcal{M}$ induces, for any cofibrant object $X \in \mathcal{N}_F^I$, an isomorphism $(h\mathbb{L}\pi_1)_X : h\mathbb{L}_\lambda^{\mathcal{N}_F^I}(X) \to h\mathbb{L}_\lambda^{\mathcal{M}}(\pi_1 X)$. Indeed, this follows from theorem 4.4.15. Similarly, the map $(h\mathbb{L}\pi_2)_X : h\mathbb{L}_\lambda^{\mathcal{N}_F^I}(X) \to h\mathbb{L}_\lambda^{\mathcal{N}}(\pi_2 X)$ is an isomorphism of λ-boolean algebras. For $A \in \mathcal{M}^{\mathrm{Cof}}$ cofibrant we can get a correspondence in $C_{FA} \in \mathcal{N}_F^I$ with all objects FA and maps the identities. We can conclude that $h\mathbb{L}F_A$ is surjective by chasing through the maps $(h\mathbb{L}\pi_2)_{C_A}$ and $(h\mathbb{L}P)_{C_A}$ which we already know are isomorphisms. ∎

It is an immediate that:

Corollary 4.4.53. For any Quillen equivalence $F : \mathcal{M} \rightleftarrows \mathcal{N} : G$. The functors $Ho(F) \circ h\mathbb{L}_\lambda^{\mathcal{M}}$ and $h\mathbb{L}_\lambda^{\mathcal{N}} : Ho(\mathcal{N}) \to \mathbf{Bool}_\lambda$ are naturally isomorphic via $h\mathbb{L}F$.

4.5 Infinitary Cartmell theories

We introduce a generalization of *Cartmell theories*, also known as *generalized algebraic theories*, Cartmell [17]. This is straightforward and most of the proofs will be omitted since they are similar to those in [17], in very few cases we will need to provide new proofs. We claim no originality other than the generalization itself. We begin by recalling some definitions given in *Ibidem.* We assume to have a set of variables V whose size is $\aleph_0$ and an alphabet A. Informally, a *Cartmell generalized algebraic theory* consists of:

i) A set S, called the set of *sort symbols*,

ii) A set O, called the set of *operation symbols*,

iii) An introductory rule for each sort symbol,

iv) An introductory rule for each operation symbol,

v) A set of axioms.

To understand our generalization let us examine the previous definition in more detail, for this we need some preliminary notions. An *expression* is a finite sequence of $A \cup V \cup \{(\} \cup \{)\} \cup \{,\}$, inductively:

i) Elements of V and A are expressions,

ii) If $f \in A$ and $e_1, e_2, ..., e_n$ are expressions, then $f(e_1, e_2, ..., e_n)$ is an expression.

The set of expression is denoted by E. This is simply to say that an expression is a finite string taken from the set $A \cup V \cup \{(\} \cup \{)\} \cup \{,\}$. A *premise* is a finite (possibly empty) sequence of $V \times E$. A *conclusion* will be an n-tuple of expressions, i.e. any element of E^n for some $n \in \mathbb{N}$. Finally, a *rule* is given by a premise P and a conclusion C. Rules are written as: $P \vdash C$. This intends to convey the idea that under the premise P, the conclusion C is a valid expression. Whenever P is a premise we will write $x_1 : \Delta_1, x_2 : \Delta_2, ..., x_n : \Delta_n$. For a conclusion, this is slightly more involved since we differentiate depending on the size of the tuple. For example, if we have a 1-tuple Δ, then we write $\Delta\mathsf{Type}$. We favour the notation ":" from type theory instead of the set theoretic one "ϵ" used by Cartmell. Furthermore, we will take advantage of conventions and notation from type theory.

The most important definition we will need to change is that of a *context*. In a Cartmell theory, a *context* is the premise such that a rule

$$x_1 : \Delta_1,\, x_2 : \Delta_2(x_1), ..., x_n : \Delta_n(x_1, x_2, \cdots, x_{n-1}) \vdash \Delta(x_1, x_2, \cdots, x_n)\ \mathsf{Type}$$

is a *derived rule.*

The only difference between Cartmell theories and infinitary Cartmell theories is that in the contexts we allow infinitely many variables. Just as any Cartmell theory gives rise to a contextual category, the same is true for the infinitary case with the appropriate generalized version of a contextual category.

4.5.1 Generalized algebraic theories

In this section, we give the formal definition of an infinitary Cartmell theory. We follow Cartmell [17] to develop the theory, there will be some instances where a change has to be made. We could say that by changing in the definition every instance of "finite" by "size strictly less than κ" we get the correct notion, this is indeed the case. We carve out the definition with a fair amount of details, since the applications we have in mind benefit from having an explicit syntax. The technicalities and motivations for introducing a generalized algebraic in the following way are presented in Cartmell [17].

From now on we fix a regular cardinal κ, unless otherwise stated, all other ordinals mentioned will be strictly smaller than κ.
Let V be a set such that $|V| = \kappa$, this set will be called the set of *variables*. We make an additional assumption on this set: Its elements have *canonical names*, this is $V = \{x_\alpha\}_{\alpha<\kappa}$, This also known as an *enumeration*. This is a minor assumption that allows to change variables. Otherwise, we would need to prove a result similar [17, Corollary, pp 1.32][5]. Let A be any set, which as before is called *alphabet*. Following [17] we define inductively

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[5] This result states that under the substitution property the derived rules are stable under substitution of variables by another variables

the collection of *expressions* A^* over the alphabet A. An expression any λ-sequence of $A \cup V \cup \{(\} \cup \{)\} \cup \{,\}$ subject to:

i) If $x_\alpha \in V$ then $x_\alpha \in A^*$,

ii) If $F \in A$ then $F \in A^*$,

iii) If $F \in A$ and $\{e_\alpha\}_{\alpha<\lambda} \subseteq A^*$ then $F(e_\alpha)_{\alpha<\lambda} \in A^*$.

A *premise* is any λ-sequence of $V \times A^*$. We will usually write premises as $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}$, where x_α are variables and Δ_α are expressions for $\alpha < \lambda$. Suppose we have a premise Γ, or later a *context*, and we need an extra premise (or *context*), according to our variable numeration, formally, we must write Γ, $\{x_\alpha : \Delta_\alpha\}_{\lambda\leqslant\alpha<\mu}$, where λ represents the number of variables in Γ. This is clearly a problem when the expression complexity increases. In order to avoid overloading the notation, we choose to reset the variable counting to only essential variables in use. Under this convention, we will write Γ, $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}$ instead. We will freely assume that Γ is a premise unless otherwise specified.

Definition 4.5.1. A *judgment* is an expression over the alphabet A that has one of the following forms:

1. Type judgment: $\Gamma \vdash \Delta\ \mathsf{Type}$.
2. Element judgment: $\Gamma \vdash t : \Delta$.
3. Type equality judgment: $\Gamma \vdash \Delta \equiv \Delta'$.
4. Term equality judgment: $\Gamma \vdash t \equiv_\Delta t'$.

where Γ is a premise.

Given a premise Γ, $\{e_\alpha\}_{\alpha<\lambda}$ expression and $\{x_\alpha\}_{\alpha<\lambda}$ variables then the new expression

$$\Gamma[e_\alpha|x_\alpha]_{\alpha<\lambda}$$

it is obtained by simultaneously changing the variables in Γ by the expressions. This process, unsurprisingly, is called *substitution* of variables. Along with the infinitary substitutions, we will also allow operations to have possibly infinite arity. This is made explicit:

Definition 4.5.2. A *κ-pretheory* T consist of the following data:

i) A set S, called the set of *sort symbols*,

ii) A set O, called the set of *operation symbols*,

iii) For each sort symbol B, a judgment of the form:

$$\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash B(x_\alpha)_{\alpha<\lambda}\ \mathsf{Type}$$

where λ is some ordinal strictly smaller than κ,

iv) For each operator symbol F, a judgment:

$$\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash F(x_\alpha)_{\alpha<\lambda} : \Delta$$

where λ is some ordinal strictly smaller than κ,

v) A set of judgments, each of which is either a type equality judgment or term equality judgment, listed in theorem 4.5.1. This is the set of *axioms* of the κ-pretheory.

The following definitions are of inductive nature:

Definition 4.5.3. 1. A premise $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}$ is a *context* if the judgment

$$\{x_\beta : \Delta_\beta\}_{\beta<\alpha} \vdash \Delta_\alpha \,\mathsf{Type}$$

is a *derived judgment* of T for every $\alpha < \lambda$. Whenever we want to specify that a premise Γ is a context we will write $\vdash \Gamma \,\mathsf{Ctxt}$.

2. The judgment

$$\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash \Delta \,\mathsf{Type}$$

is a *well-formed judgment* of T if and only if $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}$ is a context.

3. The judgment

$$\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash t : \Delta$$

is *well-formed* if and only if

$$\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash \Delta \,\mathsf{Type}$$

is a *derived judgment* of T.

Definition 4.5.4. Let T be a κ-pretheory. The set of *derived judgments* of T are the ones that can be derived from the following list:

1.

$$\frac{\Gamma \vdash A \,\mathsf{Type}}{\Gamma \vdash A \equiv A}$$

2.

$$\frac{\Gamma \vdash t : A}{\Gamma \vdash t \equiv_A t}$$

3.

$$\frac{\Gamma \vdash A_1 \equiv A_2}{\Gamma \vdash A_2 \equiv A_1}$$

4.

$$\frac{\Gamma \vdash t_1 \equiv_A t_2}{\Gamma \vdash t_2 \equiv_A t_1}$$

5.

$$\frac{\Gamma \vdash A_1 \equiv A_2 \qquad \Gamma \vdash A_2 \equiv A_3}{\Gamma \vdash A_1 \equiv A_3}$$

6.

$$\frac{\Gamma \vdash t_1 \equiv_A t_2 \qquad \Gamma \vdash t_2 \equiv_A t_3}{\Gamma \vdash t_1 \equiv_A t_3}$$

7.

$$\frac{\Gamma \vdash A_1 \equiv A_2 \qquad \Gamma \vdash t_1 \equiv_{A_1} t_2}{\Gamma \vdash t_2 \equiv_{A_2} t_1}$$

8.

$$\frac{\Gamma \vdash A_1 \equiv A_2 \qquad \Gamma \vdash t : A_1}{\Gamma \vdash t : A_2}$$

9.

$$\frac{\Gamma,\, \{x_\delta : A_\delta\}_{\delta<\beta<\lambda} \vdash A_\beta \,\mathsf{Type}}{\Gamma,\, \{x_\alpha : A_\alpha\}_{\alpha<\lambda} \vdash x_\alpha : A_\alpha}$$

10.

$$\frac{\{x_\alpha : A_\alpha\}_{\alpha<\lambda} \vdash B(x_\lambda)\,\mathsf{Type}, \quad \vdash \Gamma\,\mathsf{Ctxt}, \quad \Gamma \vdash t_\alpha : B[t_\alpha | x_\alpha]}{\Gamma \vdash B(t_\lambda)\,\mathsf{Type}}$$

This is true for any B sort symbol with a well-formed introduction type judgment.

11.

$$\frac{\Gamma,\, \{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash F(x_\lambda) : \Delta, \quad \Gamma \vdash t_\alpha : \Delta_\alpha[t_\alpha | x_\alpha]}{\Gamma,\, \{t_\alpha : \Delta_\alpha[t_\alpha | x_\alpha]\}_{\alpha<\lambda} \vdash F(t_\lambda) : \Delta[t_\lambda \mid x_\lambda]}$$

This is true for any F operator symbol with a well-formed introduction type element judgment.

12.

$$\frac{\begin{array}{c}\vdash \Gamma\,\mathsf{Ctxt} \qquad \{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash \Delta \equiv \Delta' \\ \Gamma,\, t_\alpha : \Delta_\alpha[t_\beta \mid x_\beta]_{\beta<\alpha},\, t'_\alpha : \Delta'_\alpha[t'_\beta \mid x_\beta]_{\beta<\alpha} \vdash t_\alpha \equiv_{\Delta_\alpha[t_\beta | x_\beta]_{\beta<\alpha}} t'_\alpha\end{array}}{\begin{array}{c}\Gamma,\, \{t_\alpha : \Delta_\alpha[t_\beta \mid x_\beta]_{\beta<\alpha}\}_{\alpha<\lambda},\, \{t'_\alpha : \Delta'_\alpha[t'_\beta \mid x_\beta]_{\beta<\alpha}\}_{\alpha<\lambda} \\ \vdash \Delta[t_\alpha | x_\alpha]_{\alpha<\lambda} \equiv \Delta'[t'_\alpha | x_\alpha]_{\alpha<\lambda}\end{array}}$$

13.
$$\frac{\vdash \Gamma\,\mathsf{Ctxt} \qquad \{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash t \equiv_\Delta t' \\ \Gamma,\ s_\alpha : \Delta_\alpha[s_\beta \mid x_\beta]_{\beta<\alpha},\ s'_\alpha : \Delta_\alpha[s'_\beta \mid x_\beta]_{\beta<\alpha} \vdash s_\alpha \equiv_{\Delta_\alpha[s'_\beta \mid x_\beta]_{\beta<\alpha}} s'_\alpha}{\Gamma,\ \{s_\alpha : \Delta_\alpha[s_\beta \mid x_\beta]_{\beta<\alpha}\}_{\alpha<\lambda},\ \{s'_\alpha : \Delta_\alpha[s'_\beta \mid x_\beta]_{\beta<\alpha}\}_{\alpha<\lambda} \\ \vdash t[s_\alpha \mid x_\alpha]_{\alpha<\lambda} \equiv_{\Delta[s_\alpha \mid x_\alpha]_{\alpha<\lambda}} t'[s'_\alpha \mid x_\alpha]_{\alpha<\lambda}}$$

14. If $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash \Delta \equiv \Delta'$ is an axiom then

$$\frac{\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash \Delta\,\mathsf{Type} \qquad \{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash \Delta'\,\mathsf{Type},}{\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash \Delta \equiv \Delta'}$$

15. If $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash t \equiv_\Delta t'$ is an axiom then

$$\frac{\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash t : \Delta \qquad \{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash t' : \Delta}{\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash t \equiv_\Delta t'}$$

We are now ready for the following:

Definition 4.5.5. A κ-pretheory T is *well-formed* if all its rules are well-formed. A *generalized κ-algebraic theory* or *κ-Cartmell theory* is a well-formed κ-pretheory.

Remark 4.5.6. Observe that a generalized algebraic theory as defined by Cartmell [17] is the same as an ω-generalized algebraic theory in our sense.

We introduce an important example of κ-algebraic theories.

Example 4.5.7. Let *Cat* denote the ω-algebraic theory defined in the following way:

1. Type of objects: $\vdash\ \mathsf{Ob}\,\mathsf{Type}$.
2. Type of morphisms: $x : \mathsf{Ob},\ y : \mathsf{Ob} \vdash\ \mathsf{Hom}(x,y)\ \mathsf{Type}$.
3. Composition operation: $x : \mathsf{Ob},\ y : \mathsf{Ob},\ z : \mathsf{Ob},\ f : \mathsf{Hom}(x,y),\ g : \mathsf{Hom}(y,z) \vdash g \circ f : \mathsf{Hom}(x,z)$.
4. Identity operator: $x : \mathsf{Ob} \vdash\ \mathsf{id}_x : \mathsf{Hom}(x,x)$.

Subject to the following axioms:

$$\frac{x : \mathsf{Ob},\ y : \mathsf{Ob},\ f : \mathsf{Hom}(x,y)}{\mathsf{id}_y \circ f \equiv f} \qquad \frac{x : \mathsf{Ob},\ y : \mathsf{Ob}, f : \mathsf{Hom}(x,y)}{f \circ \mathsf{id}_x \equiv f}$$

$$\frac{x : \mathsf{Ob},\ y :: \mathsf{Ob},\ z : \mathsf{Ob},\ w : \mathsf{Ob},\ f : \mathsf{Hom}(x,y), g : \mathsf{Hom}(y,z), h : \mathsf{Hom}(z,w)}{(h \circ g) \circ f \equiv h \circ (g \circ f)}$$

4.5.2 Substitution property

Let T be a κ-Cartmell theory. Recall that given Δ, $\{t_\alpha\}_{\alpha<\lambda}$ expressions and $\{x_\alpha\}_{\alpha<\lambda}$ variables, then the new expression $\Delta[e_\alpha|x_\alpha]_{\alpha<\lambda}$ denotes the substitution of variables by the expressions.

Definition 4.5.8. Let $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash \Delta$ be a derived judgment of T. We say that this judgment has the *substitution property* if for every $\vdash \Gamma\,\mathsf{Ctxt}$ and expressions $\{t_\alpha\}_{\alpha<\lambda}$, such that for all $\alpha < \lambda$

$$\Gamma,\ \{t_\beta : \Delta_\beta[t_\gamma|x_\gamma]_{\gamma<\beta}\}_{\beta<\alpha} \vdash t_\alpha : \Delta_\alpha[t_\beta|x_\beta]_{\beta<\alpha}$$

are derived rules, then

$$\Gamma \vdash \Delta[t_\alpha|x_\alpha]_{\alpha<\lambda}$$

is a derived rule of T.

In [17] it is proven that all derived judgment of a generalized algebraic theory satisfy the substitution property. This is done through a series of results that can be generalized to our setting. The proofs are omitted since they are the same as in the original reference.

Lemma 4.5.9. If $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash \Delta$ is a derived judgment of T then the variables that appear in Δ is a subset of $\{x_\alpha\}_{\alpha<\lambda}$

Proof. See [17, Lemma 1, Section 1.7]. ∎

Lemma 4.5.10. 1. The premise of a derived judgment is a context.

2. If $\vdash \{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}\mathsf{Ctxt}$ then for $\alpha < \lambda$, we have

$$\{x_\beta : \Delta_\beta\}_{\beta<\alpha} \vdash \Delta_\alpha\,\mathsf{Type}$$

Proof. See [17, Lemma 2, Section 1.7]. ∎

Theorem 4.5.11. Every derived judgment of a κ-Cartmell theory has the substitution property.

Proof. The same as proof as in [17, 1.7] applies. This goes by proving that each judgment has the substitution property. For the last two judgments in theorem 4.5.1 this is part of theorem 4.5.4. While for the first two it is done by induction on the derivations. It is shown that each derivation rule of theorem 4.5.4 preserve the substitution property. ∎

This result has similar consequences of those in [17]. The proofs are analogous or the same. For us, it is only relevant to know that our κ-Cartmell theories are well-defined. Meaning:

Proposition 4.5.12. The derived judgments of a κ-Cartmell theory are well-formed.

Proof. Again, by induction on the derivations [17, pp. 1.33]. ∎

Both the statement and proof of the next lemma are the same as The Derivation Lemma [17, pp. 1.34]. The proof does not rely on the context size.

Lemma 4.5.13. 1. Every derived type judgment of T is of the form

$$\{x_\beta : \Omega_\beta\}_{\beta<\mu} \vdash A(t_\alpha)_{\alpha<\lambda}$$

for some type symbol A with introductory rule

$$\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash A(x_\alpha)_{\alpha<\lambda} \,\mathsf{Type}$$

and $\{t_\alpha\}_{\alpha<\lambda}$ are expressions such that for all $\alpha < \lambda$ the rule

$$\{x_\beta : \Omega_\beta\}_{\beta<\mu} \vdash t_\alpha : \Delta_\alpha[t_\delta \mid x_\delta]_{\delta<\alpha}.$$

2. Every type element judgment of T is of the form

$$\{x_\beta : \Omega_\beta\}_{\beta<\mu} \vdash x_\beta : \Omega$$

for some x_β and such that $\{x_\beta : \Omega_\beta\}_{\beta<\mu} \vdash \Omega_\beta \equiv \Omega$, or is of the form

$$\{x_\beta : \Omega_\beta\}_{\beta<\mu} \vdash f(t_\alpha)_{\alpha<\lambda} : \Omega$$

for some operator symbol f of T with introductory judgment of the form

$$\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash f(x_\alpha)_{\alpha<\lambda} : \Delta$$

such that for each $\alpha < \lambda$ the rules

$$\{x_\beta : \Omega_\beta\}_{\beta<\mu} \vdash t_\alpha : \Delta_\alpha[t_\delta \mid x_\delta]_{\delta<\alpha}$$

and

$$\{x_\beta : \Omega_\beta\}_{\beta<\mu} \vdash \Delta[t_\alpha \mid x_\alpha]_{\alpha<\lambda} \equiv \Omega$$

are derived rules of T.

Proof. This is follows from theorem 4.5.4 (10) and (11). ∎

4.5.3 Equivalence relation on judgments

Trough out this section we work in an κ-Cartmell theory. We first introduce a relation that allows us to identify context which express the same meaning, but differ on the variables that are used in it [17, 1.13].

There is a relation defined on the judgments of the κ-Cartmell theory T.

Definition 4.5.14. Let $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash \Delta_\lambda \,\mathsf{Type}$, $\{x_\beta : \Omega_\beta\}_{\beta<\mu} \vdash \Omega_\mu \,\mathsf{Type}$ be two type judgments of T. We say that

$$\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash \Delta_\lambda \,\mathsf{Type} \approx \{x_\beta : \Omega_\beta\}_{\beta<\mu} \vdash \Omega_\mu \,\mathsf{Type}$$

if either:

1. Both ordinals are successors such that $\lambda = \mu = \nu + 1$ and for all $\alpha \leqslant \nu$ we have

   $$\{x_\delta : \Delta_\delta\}_{\delta<\alpha} \vdash \Delta_\alpha \equiv \Omega_\alpha$$

   is a derived rule of T.

2. Both ordinals are limits with $\lambda = \mu$ and for any successor ordinal $\nu + 1 < \lambda$ we have

   $$\{x_\alpha : \Delta_\alpha\}_{\alpha<\nu} \vdash \Delta_\nu \,\mathsf{Type} \approx \{x_\beta : \Omega_\beta\}_{\beta<\nu} \vdash \Omega_\nu \,\mathsf{Type}.$$

Lemma 4.5.15. The relation $\approx$ is an equivalence relation on type judgments of the theory T.

Proof. This is an immediate result, since we have assumed canonical names for variables. Otherwise, we could repeat the argument as in [17, 1.13].

∎

Definition 4.5.16. Let $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}$ and $\{x_\beta : \Omega_\beta\}_{\beta<\mu}$ be two contexts. We say that

$$\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \approx \{x_\beta : \Omega_\beta\}_{\beta<\mu}$$

if and only if $\lambda = \mu$ and for all $\alpha < \lambda$

$$\{x_\delta : \Delta_\delta\}_{\delta<\alpha} \vdash \Delta_\alpha \,\mathsf{Type} \approx \{x_\gamma : \Omega_\gamma\}_{\gamma<\alpha} \vdash \Omega_\alpha \,\mathsf{Type}$$

It follows that this induces an equivalence relation on contexts.

Definition 4.5.17. We say that

$$\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash t : \Delta \approx \{x_\beta : \Omega_\beta\}_{\beta<\mu} \vdash s : \Omega$$

if and only if $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash \Delta \,\mathsf{Type} \approx \{x_\beta : \Omega_\beta\}_{\beta<\mu} \vdash \Omega \,\mathsf{Type}$ and $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash t \equiv s$.

Remark 4.5.18. Let $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}$ and $\{x_\beta : \Omega_\beta\}_{\beta<\mu}$ be two contexts. Assume further that

$$\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \approx \{x_\beta : \Omega_\beta\}_{\beta<\mu}.$$

Then for all derived rules

$$\{x_\beta : \Omega_\beta\}_{\beta<\mu} \vdash \Omega,$$

the rule

$$\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash \Omega$$

is also a derived rule.

Regardless of its simplicity, this remark is useful in the next:

Corollary 4.5.19. The relation $\approx$ is an equivalence relation on judgments of the form $\{x_\beta : \Delta_\beta\}_{\beta<\mu} \vdash t : \Delta$.

Proof. Reflexivity is a consequence of 2 from theorem 4.5.4. Assume that $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash t : \Delta \approx \{x_\alpha : \Omega_\alpha\}_{\alpha<\lambda} \vdash s : \Omega$. Hence, the contexts satisfy $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \approx \{x_\alpha : \Omega_\alpha\}_{\alpha<\lambda}$. Applying the symmetry of the relation $\approx$ to contexts, and using theorem 4.5.18, we see that $\{x_\alpha : \Omega_\alpha\}_{\alpha<\lambda} \vdash t \equiv s$. Then we must have $\{x_\alpha : \Omega_\alpha\}_{\alpha<\lambda} \vdash s : \Delta$ and $\{x_\alpha : \Omega_\alpha\}_{\alpha<\lambda} \vdash \Omega \equiv \Delta$. We can apply 4 from theorem 4.5.4 to conclude that $\{x_\alpha : \Omega_\alpha\}_{\alpha<\lambda} \vdash s \equiv t$, thus proving symmetry. Transitivity is a straightforward application of theorem 4.5.18. ∎

Definition 4.5.20. A *morphism* between contexts

$$\langle t_\beta \rangle_{\beta<\mu} : \{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \to \{x_\beta : \Omega_\beta\}_{\beta<\mu}$$

is μ-sequence of terms $\{t_\beta\}_{\beta<\mu}$ such that for all $\beta < \mu$ we have

$$\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash t_\beta : \Omega_\beta[t_\gamma | x_\gamma]_{\gamma<\beta}.$$

Just as in the finite case, with the substitution as composition and the obvious identity, it can be shown that contexts form a category with morphism as defined above. This is called the *category of realizations* of the theory T. The composition of

$$\langle t_\beta \rangle_{\beta<\mu} : \{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \to \{x_\beta : \Omega_\beta\}_{\beta<\mu}$$

and

$$\langle s_\delta \rangle_{\delta<\nu} : \{x_\beta : \Omega_\beta\}_{\beta<\mu} \to \{x_\delta : \Omega'_\delta\}_{\delta<\nu}$$

is the map

$$\langle s_\delta \rangle_{\delta<\nu} \circ \langle t_\beta \rangle_{\beta<\mu} : \{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \to \{x_\delta : \Omega'_\delta\}_{\delta<\nu}$$

defined as the sequence $\langle s_\delta[\langle t_\beta | x_\beta \rangle_{\beta<\mu}] \rangle_{\delta<\nu}$.

Using the previous relation $\approx$ on contexts and rules we induce one on morphisms between contexts. If we have morphisms

$$\langle t_\beta \rangle_{\beta<\mu} : \{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \to \{x_\beta : \Omega_\beta\}_{\beta<\mu} \text{ and } \langle t'_\beta \rangle_{\beta<\mu} : \{x_\alpha : \Delta'_\alpha\}_{\alpha<\lambda} \to \{x_\beta : \Omega'_\beta\}_{\beta<\mu}$$

Then

$$\langle t_\beta \rangle_{\beta<\mu} \approx \langle t'_\beta \rangle_{\beta<\mu}$$

if and only if

$$\{x_\beta : \Omega_\beta\}_{\beta<\mu} \approx \{x'_\beta : \Omega'_\beta\}_{\beta<\mu}$$

and for all $\gamma < \mu$

$$\{x_\beta : \Delta_\beta\}_{\beta<\mu} \vdash t_\gamma : \Omega_\gamma[t_{\gamma'} | x_{\gamma'}]_{\gamma'<\gamma} \approx \{x_\beta : \Delta'_\beta\}_{\beta<\mu} \vdash t'_\gamma : \Omega'_\gamma[t'_{\gamma'} | x_{\gamma'}]_{\gamma'<\gamma}.$$

Unfolding the definition this means that

$$\{x_\beta : \Delta_\beta\}_{\beta<\mu} \vdash \Omega_\gamma[t_{\gamma'} | x_{\gamma'}]_{\gamma'<\gamma}\ \mathsf{Type} \approx \{x_\beta : \Delta'_\beta\}_{\beta<\mu} \vdash \Omega'_\gamma[t'_{\gamma'} | x_{\gamma'}]_{\gamma'<\gamma}\ \mathsf{Type}$$

and that $\{x_\beta : \Delta_\beta\}_{\beta<\mu} \vdash t_\gamma \equiv t'_\gamma$ for all $\gamma < \mu$.

The following remarks are results from [17] whose proofs are completely similar. However, it is important to make them explicit, since they imply that we can define a composition operation of equivalence classes of morphisms between contexts.

Remark 4.5.21. Let $\langle t_\beta\rangle_{\beta<\mu} : \{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \to \{x_\beta : \Omega_\beta\}_{\beta<\mu}$ and $\langle t'_\beta\rangle_{\beta<\mu} : \{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \to \{x_\beta : \Omega'_\beta\}_{\beta<\mu}$ two morphisms between contexts with $\langle t_\beta\rangle_{\beta<\mu} \approx \langle t'_\beta\rangle_{\beta<\mu}$.

1. If $\{x_\beta : \Omega_\beta\}_{\beta<\mu} \vdash \Omega\,\mathsf{Type}$ and $\{x_\beta : \Omega'_\beta\}_{\beta<\mu} \vdash \Omega'\,\mathsf{Type}$ are derived judgment of the theory such that
$$\{x_\beta : \Omega_\beta,\, x_\mu : \Omega\}_{\beta<\mu} \approx \{x_\beta : \Omega'_\beta,\, x_\mu : \Omega'\}_{\beta<\mu}$$
then
$$\{x_\alpha : \Delta_\alpha,\, x_\mu : \Omega[t_\beta|x_\beta]_{\beta<\mu}\}_{\alpha<\lambda} \approx \{x_\alpha : \Delta'_\alpha,\, x_\mu : \Omega'[t'_\beta|x'_\beta]_{\beta<\mu}\}_{\alpha<\lambda}$$
This follows by unwinding the relation $\approx$ and applying the principle 12 from theorem 4.5.4. This simply means that we can extend contexts by a fresh variable. Moreover, there is a more general result:

   For all $\varepsilon > 0$, if $\{x_\beta : \Omega_\beta\}_{\beta<\mu+\varepsilon}$ and $\{x_\beta : \Omega'_\beta\}_{\beta<\mu+\varepsilon}$ are contexts then
$$\{x_\alpha : \Delta_\alpha,\, x_\beta : \Omega_\beta[t_\gamma|x_\gamma]_{\gamma<\beta}\}_{\substack{\alpha<\lambda,\\ \mu\leqslant\beta<\mu+\varepsilon}} \approx \{x_\alpha : \Delta'_\alpha,\, x_\beta : \Omega'_\beta[t'_\gamma|x_\gamma]_{\gamma<\beta}\}_{\substack{\alpha<\lambda,\\ \mu\leqslant\beta<\mu+\varepsilon}}$$

2. If $\{x_\beta : \Omega_\beta\}_{\beta<\mu} \vdash s : \Omega$ and $\{x_\beta : \Omega'_\beta\}_{\beta<\mu} \vdash s' : \Omega'$ are derived judgment such that
$$\{x_\beta : \Omega_\beta\}_{\beta<\mu} \vdash s \equiv_\Omega s'.$$
Then
$$\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash s[t_\beta|x_\beta]_{\beta<\mu} \equiv_{\Omega[t_\beta|x_\beta]_{\beta<\mu}} s'[t'_\beta|x_\beta]_{\beta<\mu}.$$
Observe that the principle 13 from theorem 4.5.4 implies this result.

Remark 4.5.22.

1. Let $\langle t_\beta\rangle_{\beta<\mu} : \{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \to \{x_\beta : \Omega_\beta\}_{\beta<\mu}$ be a morphism between two contexts. If
$$\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \approx \{x'_\alpha : \Delta'_\alpha\}_{\alpha<\lambda} \text{ and } \{x_\beta : \Omega_\beta\}_{\beta<\mu} \approx \{x'_\beta : \Omega'_\beta\}_{\beta<\mu}$$
then $\langle t_\beta\rangle_{\beta<\mu} : \{x'_\alpha : \Delta'_\alpha\}_{\alpha<\lambda} \to \{x'_\beta : \Omega'_\beta\}_{\beta<\mu}$ is also a morphism between these contexts.

2. If we have a context $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda+1}$ and $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \approx \{x'_\alpha : \Delta'_\alpha\}_{\alpha<\lambda}$ then we can extend the context $\{x'_\alpha : \Delta'_\alpha\}_{\alpha<\lambda}$ to $\{x'_\alpha : \Delta'_\alpha\}_{\alpha<\lambda+1}$ such that $x'_\alpha : \Delta'_\alpha$ is $x_\lambda : \Delta_\lambda$.

Remark 4.5.23. Let $\langle t_\beta\rangle_{\beta<\mu+1} : \{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \to \{x_\beta : \Omega_\beta\}_{\beta<\mu+1}$ and $\langle s_\beta\rangle_{\beta<\mu} : \{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \to \{x_\beta : \Omega_\beta\}_{\beta<\mu}$ be morphisms between contexts. Then we have a morphism
$$\langle s_\beta\rangle_{\beta<\mu+1} : \{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \to \{x_\beta : \Omega_\beta\}_{\beta<\mu+1}$$
where $s_\mu \equiv t_\mu$, and such that $\{s_\beta\}_{\beta<\mu+1} \approx \{t_\beta\}_{\beta<\mu+1}$.

4.5.4 The category of κ-Cartmell theories

We construct a category where the objects are κ-Cartmell theories with maps *interpretations*. This is analogous to the category that Cartmell constructs in [17, 1.11], all the results can be copied from there to our setting. Since we work with different theories, the alphabets, expressions and rules are marked accordingly. If T is a theory then these sets are denoted $Alp(T)$, $Exp(T)$, $Rul(T)$ respectively.

Let T and T' two κ-Cartmell theories. Let any function $I : Alp(T) \to Exp(T')$. Using this function, we can define a *preinterpretation* $\widetilde{I} : Exp(T) \to Exp(T')$ by induction on the construction of expressions:

1. If $x \in V$
$$\widetilde{I}(x) := x,$$
2. If $F \in Alp(T)$
$$\widetilde{I}(F) := I(F),$$
3. If $L \in Alp(T)$ alphabet symbol and $\{t_\alpha\}_{\alpha<\lambda}$ are expressions
$$\widetilde{I}(L(t_\alpha)_{\alpha<\lambda}) := I(L)(\widetilde{I}(t_\alpha))_{\alpha<\lambda}.$$

Definition 4.5.24. Given a preinterpretation $\widetilde{I}$ we define a new function $\widehat{I} : Rul(T) \to Rul(T')$.

1. $\widehat{I}(\Gamma \vdash \Delta\ \mathsf{Type}) := \widetilde{I}(\Gamma) \vdash \widetilde{I}(\Delta)\ \mathsf{Type}$
2. $\widehat{I}(\Delta \vdash t : \Delta) := \widetilde{I}(\Delta) \vdash \widetilde{I}(t) : \widetilde{I}(\Delta)$
3. $\widehat{I}(\Delta,\, \Delta' \vdash \Delta \equiv \Delta') := \widetilde{I}(\Delta),\, \widetilde{I}(\Delta') \vdash \widetilde{I}(\Delta) \equiv \widetilde{I}(\Delta')$.
4. $\widehat{I}(\Delta,\, t, t' : \Delta \vdash t \equiv_\Delta t') := \widetilde{I}(\Delta),\, \widetilde{I}(t), \widetilde{I}(t') : \widetilde{I}(\Delta) \vdash \widetilde{I}(t) \equiv_{\widetilde{I}(\Delta)} \widetilde{I}(t')$.

This function is an *interpretation* from T into T' if all introductory judgment and axioms of T are sent to introductory judgment and axioms of T', we will simply denote this as $I : T \to T'$.

Just as in [17] it is possible to prove that:

Lemma 4.5.25. If I is an interpretation from T to T', then it preserves the derived judgments of the theory T.

Proof. From Lemma 2 [17, pp 1.52]. To illustrate how this is done, we show that the derived judgment theorem 4.5.4 (13) it is preserved by I. Consider the derived judgment

$$\frac{\begin{array}{c}\vdash \Gamma\ \mathsf{Ctxt} \qquad \{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash t \equiv_\Delta t' \\ \Gamma,\, s_\alpha : \Delta_\alpha[s_\beta \mid x_\beta]_{\beta<\alpha},\, s'_\alpha : \Delta_\alpha[s'_\beta \mid x_\beta]_{\beta<\alpha} \vdash s_\alpha \equiv_{\Delta_\alpha[s'_\beta|x_\beta]_{\beta<\alpha}} s'_\alpha\end{array}}{\begin{array}{c}\Gamma,\, \{s_\alpha : \Delta_\alpha[s_\beta \mid x_\beta]_{\beta<\alpha}\}_{\alpha<\lambda},\, \{s'_\alpha : \Delta_\alpha[s'_\beta \mid x_\beta]_{\beta<\alpha}\}_{\alpha<\lambda} \\ \vdash t[s_\alpha \mid x_\alpha]_{\alpha<\lambda} \equiv_{\Delta[s_\alpha|x_\alpha]_{\alpha<\lambda}} t'[s'_\alpha \mid x_\alpha]_{\alpha<\lambda}\end{array}}$$

in the theory T. We may assume that the context Γ is of the form $\{x_\beta : \Omega_\beta\}_{\beta<\mu}$, so we get

$$\frac{\begin{array}{c}\vdash \{x_\beta : \Omega_\beta\}_{\beta<\mu}\,\mathsf{Ctxt} \qquad \{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash t \equiv_\Delta t' \\ \{x_\beta : \Omega_\beta\}_{\beta<\mu},\ s_\alpha : \Delta_\alpha[s_\beta \mid x_\beta]_{\beta<\alpha},\ s'_\alpha : \Delta_\alpha[s'_\beta \mid x_\beta]_{\beta<\alpha} \vdash s_\alpha \equiv_{\Delta_\alpha[s'_\beta|x_\beta]_{\beta<\alpha}} s'_\alpha\end{array}}{\begin{array}{c}\{x_\beta : \Omega_\beta\}_{\beta<\mu},\ \{s_\alpha : \Delta_\alpha[s_\beta \mid x_\beta]_{\beta<\alpha}\}_{\alpha<\lambda},\ \{s'_\alpha : \Delta_\alpha[s'_\beta \mid x_\beta]_{\beta<\alpha}\}_{\alpha<\lambda} \\ \vdash t[s_\alpha \mid x_\alpha]_{\alpha<\lambda} \equiv_{\Delta[s_\alpha|x_\alpha]_{\alpha<\lambda}} t'[s'_\alpha \mid x_\alpha]_{\alpha<\lambda}\end{array}}$$

Applying the I to the hypothesis and by theorem 4.5.26 we obtain the following derivations in T'.

- $\vdash \{x_\beta : \widetilde{I}(\Omega_\beta)\}_{\beta<\mu}\,\mathsf{Ctxt}$,
- $\{x_\alpha : \widetilde{I}(\Delta_\alpha)\}_{\alpha<\lambda} \vdash \widetilde{I}(t) \equiv_\Delta \widetilde{I}(t')$,
- $\{x_\beta : \widetilde{I}(\Omega_\beta)\}_{\beta<\mu},\ s_\alpha : \widetilde{I}(\Delta_\alpha)[\widetilde{I}(s_\beta) \mid x_\beta]_{\beta<\alpha},\ \widetilde{I}(s'_\alpha) : \widetilde{I}(\Delta_\alpha)[\widetilde{I}(s'_\beta) \mid x_\beta]_{\beta<\alpha} \vdash \widetilde{I}(s_\alpha) \equiv_{\widetilde{I}(\Delta_\alpha)[\widetilde{I}(s'_\beta)|x_\beta]_{\beta<\alpha}}$ $\widetilde{I}(s'_\alpha)$.

We have all the requirements to use theorem 4.5.4 (13) for the theory T'. Thus,

$$\frac{\begin{array}{c}\vdash \{x_\beta : \widetilde{I}(\Omega_\beta)\}_{\beta<\mu}\,\mathsf{Ctxt} \qquad \{x_\alpha : \widetilde{I}(\Delta_\alpha)\}_{\alpha<\lambda} \vdash \widetilde{I}(t) \equiv_\Delta \widetilde{I}(t') \\ \{x_\beta : \widetilde{I}(\Omega_\beta)\}_{\beta<\mu},\ s_\alpha : \widetilde{I}(\Delta_\alpha)[\widetilde{I}(s_\beta) \mid x_\beta]_{\beta<\alpha},\ \widetilde{I}(s'_\alpha) : \widetilde{I}(\Delta_\alpha)[\widetilde{I}(s'_\beta) \mid x_\beta]_{\beta<\alpha} \\ \vdash \widetilde{I}(s_\alpha) \equiv_{\widetilde{I}(\Delta_\alpha)[\widetilde{I}(s'_\beta)|x_\beta]_{\beta<\alpha}} \widetilde{I}(s'_\alpha)\end{array}}{\begin{array}{c}\{x_\beta : \widetilde{I}(\Omega_\beta)\}_{\beta<\mu},\ \{\widetilde{I}(s_\alpha) : \widetilde{I}(\Delta_\alpha)[\widetilde{I}(s_\beta) \mid x_\beta]_{\beta<\alpha}\}_{\alpha<\lambda},\ \{\widetilde{I}(s'_\alpha) : \widetilde{I}(\Delta_\alpha)[\widetilde{I}(s'_\beta) \mid x_\beta]_{\beta<\alpha}\}_{\alpha<\lambda} \\ \vdash \widetilde{I}(t)[\widetilde{I}(s_\alpha) \mid x_\alpha]_{\alpha<\lambda} \equiv_{\widetilde{I}(\Delta)[\widetilde{I}(s_\alpha)|x_\alpha]_{\alpha<\lambda}} \widetilde{I}(t')[\widetilde{I}(s'_\alpha) \mid x_\alpha]_{\alpha<\lambda}\end{array}}$$

is a derived rule of T'. Therefore, the rule is preserved by the interpretation I.

∎

The following lemma fills the gap:

Lemma 4.5.26. If I is an interpretation of T into T' and we have expressions f and $\{t_\alpha\}_{\alpha<\lambda}$ on the alphabet A_T, then

$$\widetilde{I}(f[t_\alpha \mid x_\alpha]_{\alpha<\lambda}) = \widetilde{I}(f)[\widetilde{I}(t_\alpha) \mid x_\alpha]_{\alpha<\lambda}.$$

Proof. This is done by induction on the length of f in [17, Lemma 1, pp. 1.52]. The interesting case is when $f = F(e_\beta)_{\beta<\mu}$ for some F in the alphabet and expressions $\{e_\beta\}_{\beta<\mu}$. We assume inductively the result true for the expressions $\{e_\beta\}_{\beta<\mu}$. Then we have:

$$\begin{aligned}\widetilde{I}(f[t_\alpha \mid x_\alpha]_{\alpha<\lambda}) &= \widetilde{I}(F(e_\beta[t_\alpha \mid x_\alpha]_{\alpha<\lambda})_{\beta<\mu}) \\ &= I(F)(\widetilde{I}(e_\beta[t_\alpha \mid x_\alpha]_{\alpha<\lambda}))_{\beta<\mu} \\ &= I(F)(\widetilde{I}(e_\beta)[\widetilde{I}(t_\alpha) \mid x_\alpha]_{\alpha<\lambda})_{\beta<\mu}, \text{ by induction hypothesis} \\ &= I(F)(\widetilde{I}(e_\beta))_{\beta<\mu}[\widetilde{I}(t_\alpha) \mid x_\alpha]_{\alpha<\lambda}\end{aligned}$$

$$
\begin{aligned}
&= \widetilde{I}(F(e_\beta)_{\beta<\mu})[\widetilde{I}(t_\alpha) \mid x_\alpha]_{\alpha<\lambda} \\
&= \widetilde{I}(f)[\widetilde{I}(t_\alpha) \mid x_\alpha]_{\alpha<\lambda}
\end{aligned}
$$

∎

There is also a notion of composition of interpretations: If $I : S \to T$ and $J : T \to U$ are interpretations, then there is an interpretation $J \circ I : S \to U$ that is defined in the obvious way. It is also easy to infer what is the identity for this composition. A crucial result to define these compositions is:

Lemma 4.5.27. If $I : S \to T$ and $J : T \to U$ are interpretations then $\widetilde{J \circ I}(e) = \widetilde{J}(\widetilde{I}(e))$

Proof. This is by induction of the expression e see [17, Lemma 3, pp. 1.55]. ∎

We can define the category κ-GAT of κ-generalized algebraic theories. There is an equivalence relation on interpretations between two theories T and T'. If I, $J : T \to T'$ are two interpretations, then $I \approx J$ if an only if for every rule $r \in R_U$ we have $I(r) \approx J(r)$ in the theory T'.

Lemma 4.5.28. If I and J are interpretations from T to T' such that $I \approx J$ then for all type and element judgment $\mathcal{J}$ of U, $\widehat{I}(\mathcal{J}) \approx \widehat{J}(\mathcal{J})$ in T'.

Proof. See [17, Lemma 1, Section 1.14]. ∎

Then theorem 4.5.28 implies that the compositions as given is well-defined. Finally, in order to get the correct morphisms, we need to know that the equivalence relation on interpretations is compatible with the composition. Another advantageous consequence is that this it gives us criteria to establish whether two interpretations are equivalent.

Corollary 4.5.29. If I and J are interpretations from T to T' then $I \approx J$ if and only if for any type element judgment r, $\widehat{I}(r) \approx \widehat{J}(r)$.

Proof. This follows from theorem 4.5.28 and (3) of theorem 4.5.3. ∎

Corollary 4.5.30. If I and J are interpretations from T to T' and I' and J' are interpretations from T' to T'' then from $I \approx J$ and $I' \approx J'$ we conclude that $I' \circ I \approx J' \circ J$.

Proof. [17, pp. 1.72]. ∎

The category κ-GAT has morphisms equivalence classes of interpretations [17, pp. 1.72].

4.5.5 Construction and properties of the category $\mathbb{C}_T$

Let be T an κ-Cartmell theory. The category $\mathbb{C}_T$ has the following data:

- Objects: Equivalence classes of contexts under the relation $\approx$. If $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}$ is a context then the object in $\mathbb{C}_T$ is denoted $[\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}]$.
- Morphisms: A morphism between $[\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}]$ and $[\{x_\beta : \Omega_\mu\}_{\beta<\mu}]$ it is the equivalence class of a map
$$\langle t_\beta\rangle_{\beta<\mu} : \{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \to \{x_\beta : \Omega_\beta\}_{\beta<\mu}$$
induced by the relation $\approx$. We denote this set by
$$\hom_{\mathbb{C}_T}([\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}], [\{x_\beta : \Omega_\mu\}_{\beta<\mu}]).$$
- Composition: This is induced by the composition of maps between contexts. This is again well-defined in view of 2 of theorem 4.5.21.
- Identity: For a context $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}$ its identity is the equivalence class $[\{x_\alpha\}_{\alpha<\lambda}]$.

Remark 4.5.31. The category $\mathbb{C}_T$ has a unique object $1 := [\varnothing]$, the equivalence class of the empty context. Note that this is a terminal object.

Remark 4.5.32. Let $[\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}]$ an object of $\mathbb{C}_T$. Then for any $\mu < \lambda$ we get a morphism $[\langle x_\beta\rangle_{\beta<\mu}] : [\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}] \to [\{x_\beta : \Delta_\beta\}_{\beta<\mu}]$. Indeed, since $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}$ is a context then for any $\beta < \lambda$ we have $\{x_\delta : \Delta_\delta\}_{\delta<\delta} \vdash \Delta_\beta\ \mathsf{Type}$. Therefore, it follows from (theorem 4.5.4, 9) that $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash x_\alpha : \Delta_\alpha$ for all $\alpha < \lambda$. In particular this is true for all $\beta < \mu$, this gives the morphism above.

Following the same argument, if $\nu < \mu$, then we also we have map $[\langle x_\gamma\rangle_{\gamma<\nu}] : [\{x_\beta : \Delta_\beta\}_{\beta<\mu}] \to [\{x_\gamma : \Delta_\gamma\}_{\gamma<\nu}]$. Furthermore, we get a commutative diagram:

$$\begin{tikzcd}
{[\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}]} \arrow[r, "{[\langle x_\beta\rangle_{\beta<\mu}]}"] \arrow[rd, "{[\langle x_\gamma\rangle_{\gamma<\nu}]}"'] & {[\{x_\beta : \Delta_\beta\}_{\beta<\mu}]} \arrow[d, "{[\langle x_\gamma\rangle_{\gamma<\nu}]}"] \\
 & {[\{x_\gamma : \Delta_\gamma\}_{\gamma<\nu}]}
\end{tikzcd}$$

Remark 4.5.33. Since these morphisms are somewhat canonical we will use the notation " $\twoheadrightarrow$ ", and whenever we use this arrow for a morphism it must be assumed that such map is of this form. These morphisms are called display, which is Cartmell's terminology. In contrast, our 'display' maps can be of arbitrary length, which we will often refer to as *generalized display* maps.

Suppose there is a context $[\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda+\varepsilon}]$ with $\varepsilon \geqslant 0$. Then we can consider an ε-indexed sequence of display morphisms:

$$\begin{tikzcd}
\cdots & {[\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda+2}]} \arrow[r, two heads] & {[\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda+1}]} \arrow[r, two heads] & {[\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}]}
\end{tikzcd}$$

Also, there is a display map $[\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda+\varepsilon}] \twoheadrightarrow [\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}]$. This display morphism will be by definition the composition for the sequence. If $\varepsilon = 0$, then this maps is simply the identity. We also get a factorization of the map $[\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}] \twoheadrightarrow 1$ via display maps for any $\lambda \geqslant 0$.

Observation 4.5.34. From the previous theorem 4.5.32 we can observe that if λ is a limit ordinal then $[\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}]$ is the limit of the sequence

$$\begin{tikzcd} \cdots & {[\{x_1 : \Delta_1, x_2 : \Delta_2\}]} \arrow[r, two heads] & {[\{x_1 : \Delta_1\}]} \arrow[r, two heads] & 1. \end{tikzcd}$$

If there is another context $[\{x_\delta : \Gamma_\delta\}_{\delta<\gamma}]$ and maps

$$[\langle t_\beta\rangle_{\beta<\alpha}] : [\{x_\delta : \Gamma_\delta\}_{\delta<\gamma}] \to [\{x_\beta : \Delta_\beta\}_{\beta<\alpha}]$$

for all $\alpha < \lambda$ then we can simply take the map

$$[\langle t_\alpha\rangle_{\alpha<\lambda}] : [\{x_\delta : \Gamma_\delta\}_{\delta<\gamma}] \to [\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}].$$

This can be shown the cone map (which is unique). This verifies our claim.

Using theorem 4.5.32 we can define a function:

$$\nu : Ob(\mathbb{C}_T) \longrightarrow \kappa$$

as $\nu([\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}]) := \lambda$. We call this the *length function.* We can use ν to construct a filtration on the objects of $\mathbb{C}_T$: we define

$$Ob_\lambda(\mathbb{C}_T) := \nu^{-1}(\lambda)$$

then $Ob(\mathbb{C}_T) = \coprod_{\lambda<\kappa} Ob_\lambda(\mathbb{C}_T)$, and so if $\alpha \leqslant \beta$ then $Ob_\alpha(\mathbb{C}_T) \subseteq Ob_\beta(\mathbb{C}_T)$. Furthermore, if $p : A \twoheadrightarrow B$ is a display morphism, then $\nu(B) \leqslant \nu(A)$.

For $\alpha < \beta$ there are functions

$$\pi_\beta : Ob_\beta(\mathbb{C}_T) \to Ob_\alpha(\mathbb{C}_T)$$

that are defined in the obvious way. Additionally, $1 \in Ob_0(\mathbb{C}_T)$ is unique.

The proof of the following lemma is the same as in [17].

Lemma 4.5.35. The pullback of a display map along arbitrary morphisms in $\mathbb{C}_T$ exists, and it is also display.

Proof. We use induction over the context length. Assume we have the following diagram in $\mathbb{C}_T$:

$$\begin{tikzcd} & {[\{x_\beta : \Omega_\beta\}_{\beta<\mu+1}]} \arrow[d, two heads, "{[\langle x_\beta\rangle_{\beta<\mu}]}"] \\ {[\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}]} \arrow[r, "{[\langle t_\beta\rangle_{\beta<\mu}]}"'] & {[\{x_\beta : \Omega_\beta\}_{\beta<\mu}]} \end{tikzcd}$$

Then the pullback is given using theorem 4.5.21, the context is

$$[\{x_\alpha : \Delta_\alpha,\, x_\mu : \Omega_\mu[t_\beta \mid x_\beta]_{\beta<\mu}\}_{\alpha<\lambda}].$$

Therefore we have a commutative square

$$\begin{tikzcd}
{[\{x_\alpha : \Delta_\alpha,\, x_\mu : \Omega_\mu[t_\beta \mid x_\beta]_{\beta<\mu}\}_{\alpha<\lambda}]} \arrow[r, "{[\langle t_\beta, x_\mu\rangle_{\beta<\mu}]}"] \arrow[d, two heads, swap, "{[\langle x_\alpha\rangle_{\alpha<\lambda}]}"] & {[\{x_\beta : \Omega_\beta\}_{\beta<\mu+1}]} \arrow[d, two heads, "{[\langle x_\beta\rangle_{\beta<\mu}]}"] \\
{[\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}]} \arrow[r, swap, "{[\langle t_\beta\rangle_{\beta<\mu}]}"] & {[\{x_\beta : \Omega_\beta\}_{\beta<\mu}]}
\end{tikzcd} \tag{4.5.1}$$

Note that by definition the left vertical morphism is also display. If there is another commutative square

$$\begin{tikzcd}
{[\{x_\zeta : \Gamma_\zeta\}_{\zeta<\xi}]} \arrow[r, "{[\langle g_\beta\rangle_{\beta<\mu+1}]}"] \arrow[d, swap, "{[\langle f_\alpha\rangle_{\alpha<\lambda}]}"] & {[\{x_\beta : \Omega_\beta\}_{\beta<\mu+1}]} \arrow[d, two heads, "{[\langle x_\beta\rangle_{\beta<\mu}]}"] \\
{[\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}]} \arrow[r, swap, "{[\langle t_\beta\rangle_{\beta<\mu}]}"] & {[\{x_\beta : \Omega_\beta\}_{\beta<\mu}],}
\end{tikzcd}$$

the map

$$[\langle f_\alpha, g_\mu\rangle_{\alpha<\lambda}] : [\{x_\zeta : \Gamma_\zeta\}_{\zeta<\xi}] \to [\{x_\alpha : \Delta_\alpha,\, x_\mu : \Omega_\mu[t_\beta \mid x_\beta]_{\beta<\mu}\}_{\alpha<\lambda}]$$

shows that the square (4.5.1) is the pullback.
Next, assume that we have a diagram

$$\begin{tikzcd}
 & {[\{x_\beta : \Omega_\beta\}_{\beta<\mu}]} \arrow[d, two heads, "{[\langle x_\beta\rangle_{\beta<\mu}]}"] \\
{[\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}]} \arrow[r, swap, "{[\langle t_\beta\rangle_{\beta<\nu}]}"] & {[\{x_\beta : \Omega_\beta\}_{\beta<\nu}]}
\end{tikzcd}$$

where μ is a limit ordinal strictly larger than ν. We simplify the notation as follows:

$$\begin{tikzcd}
 & B_\mu \arrow[d, two heads] \\
A_\lambda \arrow[r, swap, "\langle t_\beta\rangle_{\beta<\nu}"] & B_\nu
\end{tikzcd}$$

Assume that the factorization of the map $B_\mu \twoheadrightarrow B_\nu$ is of the form

$$\ldots \twoheadrightarrow B_{\nu+2} \twoheadrightarrow B_{\nu+1} \twoheadrightarrow B_\nu$$

and therefore B_μ is the limit (obtained similarly as in theorem 4.5.34 and theorem 4.5.32).

Then we can take the successive pullback

$$\begin{tikzcd}
f^*B_\mu \arrow[r, "q(f,B_\mu)"] \arrow[dr, phantom, "\lrcorner", very near start] & B_\mu \\
\vdots & \vdots \\
q(f,B_{\nu+1})^*B_{\nu+2} \arrow[r, "q(q(f,B_{\nu+1}),B_{\nu+2})"] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & B_{\nu+2} \arrow[d, two heads] \\
f^*B_{\nu+1} \arrow[r, "q(f,B_{\nu+1})"] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & B_{\nu+1} \arrow[d, two heads] \\
A_\lambda \arrow[r, "f"'] & B_\nu
\end{tikzcd} \tag{4.5.2}$$

where at each successor stage it is given as before, $f := \langle t_\beta \rangle_{\beta<\nu}$, the context

$$f^*B_\mu := [\{x_\alpha : \Delta_\alpha,\, x_\beta : \Omega_\beta[t_\delta \mid x_\delta]_{\delta<\beta}\}_{\substack{\alpha<\lambda \\ \nu<\beta<\mu}}]$$

is the limits of the sequence on the left-hand side, with the obvious display maps to each object in the sequence, and

$$q(f, B_\mu) := [\langle t_\beta,\, x_\gamma \rangle_{\beta<\nu<\gamma<\mu}].$$

This makes the outer rectangle in (4.5.2) commutative. Moreover, the map $q(f, B_\mu)$ is the unique cone map induced by the family of maps

$$\{[\langle t_\beta,\, x_\gamma \rangle_{\beta<\nu<\gamma<\delta}] : f^*B_\mu \to B_\delta\}_{\nu<\delta<\mu}.$$

∎

Using the same notation as in the lemma above, we have:

Remark 4.5.36. 1. If $f = Id_{B_\nu}$ then $(Id_{B_\nu})^*B_\mu = B_\mu$ and $q(Id_{B_\nu}, B_\mu) = Id_{B_\mu}$.

2. For a diagram

$$\begin{tikzcd}
 & & A \arrow[d, two heads, "p"] \\
D \arrow[r, "g"] & C \arrow[r, "f"] & B,
\end{tikzcd}$$

we have that $g^*(f^*(A)) = (fg)^*(A)$ and $q(fg, A) = q(f, A)(g, f^*A)$.

We will refer to the category $\mathbb{C}_T$ as the *syntactic category* associated to the κ-Cartmell theory T.

Observation 4.5.37. We note that theorem 4.5.35 give us an explicit construction of pullbacks in $\mathbb{C}_T$, as well the pullback of the maps and an explicit description of $q(f, B_\mu)$.

We finish this section by characterizing the display maps in the category $\mathbb{C}_T$. This result says that display maps are somehow generic. We start with a preparatory result.

Lemma 4.5.38. Let T a κ-Cartmell theory and $\mathbb{C}_T$ its syntactic κ-contextual category. Assume that there is a $f : \Delta \to \Gamma$, then any display map $B \twoheadrightarrow \Delta$ of length 1 can be obtained as a pullback of the form

$$\begin{tikzcd}
B \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & \Gamma' \arrow[d, two heads] \\
\Delta \arrow[r, "f"'] & \Gamma
\end{tikzcd}$$

where $\Gamma' \twoheadrightarrow \Gamma$ is of length 1.

Proof. This simply a reformulation of theorem 4.5.13. Assume that

$$f = [\langle t_\beta \rangle_{\beta<\mu}] : [\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}] \to [\{x_\beta : \Gamma_\beta\}_{\beta<\mu}].$$

Therefore, when the display map is of the form

$$[\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda+1}] \twoheadrightarrow [\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}].$$

We can construct the square

$$\begin{tikzcd}
{[\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda+1}]} \arrow[r, "\langle t_\beta, x_\lambda \rangle_{\beta<\mu}"] \arrow[d, two heads] & {[\{x : \Gamma_\beta, x_\lambda : \Delta_\lambda\}_{\beta<\mu}]} \arrow[d, two heads] \\
{[\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}]} \arrow[r, "\langle t_\beta \rangle_{\beta<\mu}"'] & {[\{x : \Gamma_\beta\}_{\beta<\mu}].}
\end{tikzcd}$$

Since for all $\beta < \mu$, x_β does not occur in Δ_λ we have that $\Delta_\lambda[t_\beta | x_\beta]_{\beta<\mu} \equiv \Delta_\lambda$. Hence, it follows from the construction of pullbacks in $\mathbb{C}_T$ (theorem 4.5.35) that the square above is indeed a pullback diagram. ∎

We are ready to give the full description of display maps.

Proposition 4.5.39. Every Display map $B \twoheadrightarrow \Delta$ in $\mathbb{C}_T$ is a limit of a κ-small tower $V : \lambda \to \mathbb{C}_T$ where for each limit ordinal $\beta < \lambda$

$$V(\beta) = \mathtt{Lim}_{\alpha<\beta} V(\alpha)$$

and the map $V(\alpha + 1) \to V(\alpha)$ is a pullback of a length one display map of the form $(\Gamma, A) \twoheadrightarrow \Gamma$ where $\Gamma \vdash A\, \mathsf{Type}$ is a type axiom of the theory T.

Proof. Each display map in $\mathbb{C}_T$ has a length λ. Just as in theorem 4.5.32 it admits a decomposition into display maps. It will be enough to prove the second claim, but this follows by an inductive argument in conjunction with the previous theorem 4.5.38. The inductive step provides us with the required map $f : V(\alpha) \to \Gamma$ in theorem 4.5.38. ∎

4.6 Contextual categories and Cartmell theories

This section is the most relevant part. We will show that from the syntax of a κ-Cartmell theory we can construct a category, called κ-*Contextual category*, which we now introduce.

4.6.1 κ-contextual categories

The discussion in section 4.5.5 on the properties of the syntactic category $\mathbb{C}_T$ can be summarized with the next definition, which is the natural generalization of Cartmell's [17] or [41]. We present our definition in the same way as in the latter. Recall that κ is a regular cardinal.

Definition 4.6.1. A category $\mathcal{C}$ is said to be a *κ-contextual category* if:

1. The objects of $\mathcal{C}$ have grading $Ob(\mathcal{C}) = \coprod_{\lambda<\kappa} Ob_\lambda(\mathcal{C})$. This grading determines the *height* of any object $B \in \mathcal{C}$, which we write as $ht(B)$.
2. There is a terminal object $1 \in \mathcal{C}$ and it is unique up to equality with height 0.
3. There is a wide subcategory $Dis(\mathcal{C})$ with distinguished maps " $\twoheadrightarrow$ " called *display morphisms*,
4. The subcategory $Dis(\mathcal{C})$ is closed under transfinite compositions: if we have

$$\begin{tikzcd}\cdots \arrow[r, two heads] & B_3 \arrow[r, two heads] & B_2 \arrow[r, two heads] & B_1 \arrow[r, two heads] & B_0\end{tikzcd}$$

   a λ-sequence of display maps, then there is a unique object B in $Dis(\mathcal{C})$ with height λ and for each $\mu \leqslant \lambda$ a display map $B \twoheadrightarrow B_\mu$ such that for any $\alpha < \lambda$ we have a factorization

$$\begin{tikzcd}B \arrow[rr, two heads] \arrow[dr, two heads] & & B_0 \\ & B_\alpha \arrow[ur, two heads] & \end{tikzcd}$$

5. The inclusion functor preserve $i : Dis(\mathcal{C}) \hookrightarrow \mathcal{C}$ transfinite compositions.
6. If $A \twoheadrightarrow B$ is an arrow in $Dis(\mathcal{C})$ then $B \in Ob_\mu(\mathcal{C})$ and $A \in Ob_\lambda(\mathcal{C})$ for some ordinals λ, μ with $\mu \leqslant \lambda$.
7. For any object $A \in Ob_\lambda(\mathcal{C})$ and any $\mu \leqslant \lambda$ there exists a unique object $B \in Ob_\mu(\mathcal{C})$ and a unique display map $A \twoheadrightarrow B$. The *length* of this display map is the unique ordinal α such that $\lambda = \mu + \alpha$, is such situation, we write $lt(p)$.
8. For any $A \in Ob_\lambda(\mathcal{C})$, a map $A \twoheadrightarrow B$ and any map $f : C \to B$ there is a pullback square

$$\begin{tikzcd}f^*A \arrow[r, "q(f,A)"] \arrow[d, two heads, "f^*p"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, two heads, "p"] \\ C \arrow[r, "f"'] & B\end{tikzcd}$$

   called *canonical pullback* of A along f, and we require $lt(f^*p) = lt(p)$.
9. Canonical pullbacks are strictly functorial: for ordinals with $\mu \leqslant \lambda$, $A \in Ob_\lambda(\mathcal{C})$

   (a) If $f = id_B$ then $id_B^* A = A$ and $q(id_B, A) = id_A$.
   (b) For a diagram

$$\begin{tikzcd}
 & & A \arrow[d, two heads, "p"] \\
D \arrow[r, "g"] & C \arrow[r, "f"] & B,
\end{tikzcd}$$

   we have that $g^*(f^*(A)) = (fg)^*(A)$ and $q(fg, A) = q(f, A)(g, f^*A)$.

10. Given display maps $p : A \twoheadrightarrow B$ and $q : B \to C$ and any $f : X \to C$, in the diagram

$$\begin{tikzcd}
q(f,B)^*A \arrow[r, "q(q(f,B),A)"] \arrow[d, two heads, "q(f,B)^*p"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, two heads, "p"] \\
f^*B \arrow[r, "q(f,B)"'] \arrow[d, two heads, "f^*r"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, two heads, "r"] \\
X \arrow[r, "f"'] & C,
\end{tikzcd}$$

   we have that $f^*r \circ (q(f, B)^*p) = f^*(r \circ p)$ and $q(q(f, B), A) = q(f, A)$.

Remark 4.6.2. We use the term "display map" in a rather different way to Cartmell. For us, a display map can have any height, and it is only bounded by the regular cardinal κ.

We have already seen one example of such a category.

Corollary 4.6.3. For any κ-Cartmell theory T the syntactic category $\mathbb{C}_T$ is a κ-contextual category.

Proof. This is done throughout section 4.5.5. ∎

Remark 4.6.4. It follows from theorem 4.6.1 that for any object $B \in \mathcal{C}$ the map $B \twoheadrightarrow 1$ can be decomposed as a transfinite composition of display maps

$$B_\lambda \twoheadrightarrow \ldots \twoheadrightarrow B_1 \twoheadrightarrow 1.$$

The length of decomposition above is given by the degree of B. This is what [17] calls the tree structure of the category. Whenever we refer to objects in a κ-contextual category as above, we will emphasize its height by writing B_λ. Likewise, we will denote the display maps as $p_\alpha : B_\lambda \twoheadrightarrow B_\alpha$ for each $\alpha < \lambda$.

The following lemma is a consequence of theorem 4.6.1 and theorem 4.6.4.

Lemma 4.6.5. Let $B \in Ob_\lambda(\mathcal{C})$ such that λ is a limit ordinal. Then B itself is a limit object in $\mathcal{C}$.

Proof. From theorem 4.5.32 we obtain a sequence

$$\cdots \twoheadrightarrow B_3 \twoheadrightarrow B_2 \twoheadrightarrow B_1 \twoheadrightarrow 1.$$

It follows from Axiom 4 of theorem 4.6.1 that B must be the limit of the sequence. Finally, we use that the inclusion $Dis(\mathcal{C}) \to \mathcal{C}$ preserve limits. ∎

Definition 4.6.6. Let $\mathcal{C}$, $\mathcal{D}$ contextual categories. A functor $F : \mathcal{C} \to \mathcal{D}$ it is called *contextual functor* if it satisfies the following conditions:

1. $F(Ob_\lambda(\mathcal{C})) \subseteq Ob_\lambda(\mathcal{D})$ for all $\lambda < \kappa$,
2. F restricts to a functor $Dis(\mathcal{C}) \to Dis(\mathcal{D})$,
3. F preserve canonical pullbacks up to equality, meaning that for any square in $\mathcal{C}$

$$\begin{tikzcd}
f^*A \arrow[r, "q(f,A)"] \arrow[d, two heads, "f^*p"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, two heads, "p"] \\
C \arrow[r, "f"'] & B
\end{tikzcd}$$

we have $F(f^*A) = (Ff)^*(FA)$ and $F(q(f,A)) = q(Ff, FA)$.

Since the degree of each object is preserved by a κ-contextual functor, it makes sense to denote $F(A_\lambda) := F(A)_\lambda$ for $A_\lambda \in \mathcal{C}$. Another piece of notation we can introduce is from the functor $F : Dis(\mathcal{C}) \to Dis(\mathcal{D})$; since any display map $p_\alpha : A_\lambda \twoheadrightarrow A_\alpha$ is sent to a display map $F(p_\alpha) : F(A)_\lambda \twoheadrightarrow F(A)_\alpha$, and the degrees are preserved, we agree to omit F on these maps. Contextual functors are the morphisms of the category of κ-contextual categories, we will denote it as κ-CON.

4.6.2 Interlude: categorical facts

We collect and recall some categorical facts about general κ-contextual categories.

Proposition 4.6.7 (The slice κ-contextual category)**.** Let $\mathcal{C}$ be a κ-contextual category. For any object $B \in Ob_\mu(\mathcal{C})$ there is a κ-contextual category which is a full subcategory of the slice $\mathcal{C}_{/B}$ which has objects display maps $A \twoheadrightarrow B$ where $A \in Ob_\lambda(\mathcal{C})$ with $\lambda \geqslant \mu$.

Since we will rarely use categories other than κ-contextual categories, we will employ the slice notation $\mathcal{C}_{/B}$ for the category from the previous proposition.

Proof. The proof is completely formal. The important fact to remember is that the pullback of a display map is also display. ∎

It is a well known fact that the pasting of two pullbacks give us a pullback, in our case consider the following diagram:

$$\begin{tikzcd}
f^*B_\mu \arrow[r, "q(f,B_\mu)"] \arrow[dr, phantom, "\lrcorner", very near start] & B_\mu \\
\vdots & \vdots \\
q(f,B_{\nu+1})^*B_{\nu+2} \arrow[r, "q(q(f,B_{\nu+1}),B_{\nu+2})"] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & B_{\nu+2} \arrow[d, two heads] \\
f^*B_{\nu+1} \arrow[r, "q(f,B_{\nu+1})"] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & B_{\nu+1} \arrow[d, two heads] \\
A_\lambda \arrow[r, "f"'] & B_\nu
\end{tikzcd}$$

Then if μ is a limit ordinal, the object B_μ is the limit of the sequence on the right-hand side. Thus, f^*B_μ is the limit of the sequence on the left-hand side. Note that pairwise we have $q(f,B_{\nu+1})^*B_{\nu+2} = f^*B_{\nu+2}$ and $q(f,B_{\mu+2}) = q(q(f,B_{\mu+1}),B_{\mu+2})$.
If $f : A_\lambda \to B_\nu$ and $p_\nu : B_\mu \twoheadrightarrow B_\nu$ is a display map with $\mu = \nu + 1$, using the universal property of the pullback we can construct the following diagram:

$$\begin{tikzcd}
A_\lambda \arrow[drr, bend left, "f"] \arrow[dr, dashed, "\delta_f^\nu"] \arrow[ddr, bend right, "Id_{A_\lambda}"'] & & \\
& (p_\nu f)^*B_\mu \arrow[r] \arrow[d, two heads, "(p_\nu f)^*p_\nu"'] \arrow[dr, phantom, "\lrcorner", very near start] & B_\mu \arrow[d, two heads, "p_\nu"] \\
& A_\lambda \arrow[r, "p_\nu f"'] & B_\nu.
\end{tikzcd}$$

The map δ_f^ν makes both triangles commutative. We will focus on the fact that $((f_\nu)^*p_\nu)\delta_f^\nu = Id_{A_\lambda}$, where $f_\nu = p_\nu f$. Assume that we have a map $p : B_\mu \twoheadrightarrow B_\nu$ with μ a limit ordinal, in particular the length of p is a limit ordinal. Then a map $f : A_\lambda \to B_\mu$ is determinate by a family of maps $\{f_\gamma : A_\lambda \to B_\gamma\}$. Then we obtain:

$$\begin{tikzcd}
A_\lambda \arrow[rrd, bend left, "f"] \arrow[rd, dashed, "\delta_f^\nu"] \arrow[rddddd, bend right, "Id"'] & & \\
& f^*B_\mu \arrow[r, "q(f,B_\mu)"] \arrow[rd, phantom, "\lrcorner", very near start] & B_\mu \arrow[dddd, bend left, two heads, "p"] \\
& \vdots & \vdots \\
& q(f,B_{\nu+1})^*B_{\nu+2} \arrow[r, "q(q(f,B_{\nu+1}),B_{\nu+2})"] \arrow[d, two heads] \arrow[rd, phantom, "\lrcorner", very near start] & B_{\nu+2} \arrow[d, two heads] \\
& f^*B_{\nu+1} \arrow[r, "q(f,B_{\nu+1})"] \arrow[d, two heads] \arrow[rd, phantom, "\lrcorner", very near start] & B_{\nu+1} \arrow[d, two heads] \\
& A_\lambda \arrow[r, "f_\nu"'] & B_\nu,
\end{tikzcd}$$

where the map δ_f^ν is given as the family of maps $(\delta_f^\nu)_\gamma$, each given by an intermediate pullback square in the diagram above.

Notation 4.6.8. If the situation above, for $f : A_\lambda \to B_\mu$ we denote

$$\Gamma(B_\nu^\mu) := \{h : A_\lambda \to (p_\nu f)^* B_\mu \mid ((p_\nu f)^* p_\nu) h = Id_{A_\lambda}\}.$$

We can consider a more general case, if $A_\lambda \in Ob_\lambda(\mathcal{C})$ and $B_\mu \in Ob_\mu(\mathcal{C})$ with $\lambda < \mu$, then there is a unique display map $p : B_\mu \twoheadrightarrow A_\lambda$. We set

$$\Gamma(B_\lambda^\mu) := \{s : A_\lambda \to B_\mu \mid ps = Id_{A_\lambda}\}$$

for this situation as well, since the object A_λ will be inferred from the context.

If the contextual category is $\mathbb{C}_T$, then recalling theorem 4.5.35, we can give an explicit description of the map δ_f^ν.

Lemma 4.6.9. Assume that $f := [\langle t_\beta \rangle_{\beta<\nu}] : [\{x_\alpha : A_\alpha\}_{\alpha<\lambda}] \to [\{x_\beta : B_\beta\}_{\beta<\nu}]$ and there is a display map $p : [\{x_\beta : B_\beta\}_{\beta<\mu}] \twoheadrightarrow [\{x_\beta : B_\beta\}_{\beta<\nu}]$ then $\delta_f^\nu = [\langle x_\alpha, t_\beta \rangle_{\substack{\alpha<\lambda \\ \nu<\beta<\mu}}]$.

Proof. This follows by induction on μ and the explicit construction of pullbacks from theorem 4.5.35. ∎

In certain situations, the property above characterizes the map δ_f^ν.

Lemma 4.6.10. If $[\{x_\beta : B_\beta\}_{\beta<\mu}]$ is an object of $\mathbb{C}_T$ and $\nu < \mu$ then $f \in \Gamma(B_\nu^\mu)$ if and only if $f = [\langle x_\beta, t_\gamma \rangle_{\beta<\nu<\gamma<\mu}]$, where for all $\nu < \gamma < \mu$, the rule $\{x_\beta : B_\beta\}_{\beta<\nu}, \{t_{\gamma'} : B_{\gamma'}\}_{\gamma'<\gamma} \vdash t_\gamma : B_\gamma$ is a derived rule.

The next result follows from the previous lemma, and it is used in theorem 4.6.41.

Lemma 4.6.11. Let A_λ, B_μ objects of $\mathcal{C}$ and for each $\beta < \mu$ we have maps $r_{\beta+1} \in \Gamma(r_\beta^* \cdots r_1^* p^* B_{\beta+1})$ then there exists a unique sequence of maps $\{g_\beta : A_\lambda \to B_\beta\}_{\beta<\mu}$ such that for all $\beta < \mu$ we have $p_\beta g_{\beta+1} = g_\beta$ such that $\delta_{g_\beta} = r_\beta$.

Some words about the previous lemma are in order. The expression $r_\beta^* \cdots r_1^* p^* B_{\beta+1}$ can be illustrated by the first two steps:

$$\begin{tikzcd}
p^*B_2 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B_2 \arrow[d, two heads] & & \\
p^*B_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B_1 \arrow[d, two heads] & r_1^*p^*B_2 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & p^*B_2 \arrow[d] \\
A_\lambda \arrow[r, "p"'] \arrow[u, bend left, "r_1"] & 1 & A_\lambda \arrow[r, "r_1"'] \arrow[u, bend left, "r_2"] & p^*B_1
\end{tikzcd}$$

4.6.3 The equivalence between κ-GAT and κ-CON

4.6.3.1 The functor $\mathbb{C} : \kappa$-GAT $\to \kappa$-CON

To establish this equivalence of categories, we first define a functor $\mathbb{C} : \kappa\text{-GAT} \to \kappa\text{-CON}$ using the construction of section 4.5.5. The proof again comes from ([17], section 2.4.1). We register all preliminary results needed to define this functor, however again we omit the proofs since they are similar to the original ones given by Cartmell.

On objects $\mathbb{C} : \kappa\text{-GAT} \to \kappa\text{-CON}$ is defined as $\mathbb{C}_T$ for T a κ-Cartmell theory. For a map $[I] : T \to T'$ between theories, we need functor $\mathbb{C}(I) : \mathbb{C}_T \to \mathbb{C}_{T'}$:

1. On objects; $\mathbb{C}(I)([\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}]) := [\{x_\alpha : \widetilde{I}(\Delta_\alpha)\}_{\alpha<\lambda}]$,
2. On morphisms: If $[\langle t_\beta \rangle_{\beta<\mu}] : [\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}] \to [\{x_\beta : \Delta_\beta\}_{\beta<\mu}]$ then $\mathbb{C}(I)([\langle t_\beta \rangle_{\beta<\mu}]) := [\langle \widetilde{I}(\langle t_\beta \rangle_{\beta<\mu})]$.

If there is an interpretation J in the equivalence class $[I]$, then by theorem 4.5.28 any rule r of T we get $\widehat{I}(r) \approx \widehat{J}(r)$. Therefore, the definition of $\mathbb{C}(I)$ does not depend on the representative of $[I]$.

Remains to verify that $\mathbb{C}(I)$ is indeed a contextual functor. Firstly, it is primordial to verify it is well-defined.

Lemma 4.6.12. Let $[I] : T \to T'$ be a map in κ-GAT then the following hold:

1. The interpretation I preserves contexts: If $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}$ is a context in the theory T then $\{x_\alpha : \overline{I}(\Delta_\alpha)\}_{\alpha<\lambda}$ is a context in the theory T'.
2. The interpretation I preserves the equivalence relation $\approx$ between contexts: If $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}$ and $\{x_\alpha : \Omega_\alpha\}_{\alpha<\lambda}$ are contexts in the theory U with $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \approx \{x_\alpha : \Omega_\alpha\}_{\alpha<\lambda}$ then $\{x_\alpha : \overline{I}(\Delta_\alpha)\}_{\alpha<\lambda} \approx \{x_\alpha : \overline{I}(\Omega_\alpha)\}_{\alpha<\lambda}$.

3. The interpretation I preserves morphisms between contexts: If $\langle t_\beta \rangle_{\beta<\mu}\} : \{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \to \{x_\beta : \Omega_\beta\}_{\beta<\mu}$ is a morphism between contexts in the theory T then $\langle \overline{I}(t_\beta)\rangle_{\beta<\mu}\} : \{x_\alpha : \overline{I}(\Delta_\alpha)\}_{\alpha<\lambda} \to \{x_\beta : \overline{I}(\Omega_\beta)\}_{\beta<\mu}$ is a morphism between contexts in the theory T'.
4. The interpretation I preserves the equivalence relation $\approx$ between morphisms of contexts: If $\langle s_\beta \rangle_{\beta<\mu}, \langle t_\beta \rangle_{\beta<\mu} : \{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \to \{x_\beta : \Omega_\beta\}_{\beta<\mu}$ are morphisms between contexts in the theory T with $\langle s_\beta \rangle_{\beta<\mu} \approx \langle t_\beta \rangle_{\beta<\mu}$ then $\langle \overline{I}(s_\beta)\rangle_{\beta<\mu} \approx \langle \overline{I}(t_\beta)\rangle_{\beta<\mu}$.

Proof. The proof of each statement is consequence of theorem 4.5.26 or theorem 4.5.25. Our enumeration of variables give us a notation simplification of the proof given by [17]. For example, to prove 4; we have by assumption that $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash t_\gamma \equiv_{\Omega_\gamma[t_\beta|x_\beta]_{\beta<\gamma}} s_\gamma$ for all $0 < \gamma \leqslant \mu$. Therefore, since the interpretation preserves this rule oT we get that $\{x_\alpha : \overline{I}(\Delta_\alpha)\}_{\alpha<\lambda} \vdash \overline{I}(t_\gamma) \equiv_{\overline{I}(\Omega_\gamma)[\overline{I}(t_\beta)|x_\beta]_{\beta<\gamma}} \overline{I}(s_\gamma)$ for all $0 < \gamma \leqslant \mu$. This exactly establishes $\langle \overline{I}(s_\beta)\rangle_{\beta<\mu} \approx \langle \overline{I}(t_\beta)\rangle_{\beta<\mu}$. ∎

We have seen that the definition of $\mathbb{C}(I)$ give us the correct objects and morphisms. Now we show that it is indeed a contextual functor.

Lemma 4.6.13. Let $I : T \to T'$ be a morphism in κ-GAT. Then the map $\mathbb{C}(I) : \mathbb{C}_T \to \mathbb{C}_{T'}$ is a contextual functor.

Proof. The map is a functor trivially. That it preserves the grading and restricts to a functor between the display subcategories $Dis(\mathbb{C}_T)$ and $Dis(\mathbb{C}_{T'})$, it is also immediate. To prove it preserves canonical pullbacks, consider the following pullback square in the category $\mathbb{C}_T$:

$$\begin{tikzcd}
\left[\{x_\alpha : \Delta_\alpha,\, x_\gamma : \Omega_\gamma[t_\beta \mid x_\beta]_{\beta<\mu}\}_{\substack{\alpha<\kappa,\\ \mu\leqslant\gamma<\mu+\varepsilon}}\right] \arrow[r, "{[\langle t_\beta, x_\gamma\rangle_{\substack{\beta<\mu,\\ \mu\leqslant\gamma<\mu+\varepsilon}}]}"] \arrow[d, "{[\langle x_\alpha\rangle_{\alpha<\kappa}]}"'] & \left[\{x_\beta : \Omega_\beta\}_{\beta<\mu+\varepsilon}\right] \arrow[d, two heads, "{[\langle x_\beta\rangle_{\beta<\mu}]}"] \\
\left[\{x_\alpha : \Delta_\alpha\}_{\alpha<\kappa}\right] \arrow[r, "{[\langle t_\beta\rangle_{\beta<\mu}]}"'] & \left[\{x_\beta : \Omega_\beta\}_{\beta<\mu}\right]
\end{tikzcd}$$

Then a straightforward computation, using the definition of $\mathbb{C}(I)$, shows that this is sent to a pullback square in the category $\mathbb{C}_{T'}$. ∎

Corollary 4.6.14. There is a functor $\mathbb{C} : \kappa\text{-GAT} \to \kappa\text{-CON}$.

4.6.3.2 The functor $U : \kappa$-CON $\to \kappa$-GAT

We now turn to construct a functor that to each κ-contextual category $\mathcal{C}$ associates a κ-generalized algebraic theory $U(\mathcal{C})$, this is part of [17, Section 2.4]. We will use the notation introduced in theorem 4.6.4. This means we identify each object by its height, say B_λ, and write display maps as $p_\alpha : B_\lambda \twoheadrightarrow B_\alpha$ if $\lambda > 0$ and $\alpha < \lambda$. If $\alpha = 0$ then $B_0 = 1$ the terminal object. A morphism $f : A_\lambda \to B_\mu$ is trivial when B_μ is trivial, i.e $\mu = 0$.

Definition 4.6.15. We define $U(\mathcal{C}) \in \kappa$-GAT as:

1. For each non-trivial object B_μ with $\mu = \lambda + 1$ a type symbol $\overline{B_\mu}$ with introductory rule: $\{x_\beta : \overline{B_\beta}\}_{\beta<\mu} \vdash \overline{B_\mu}(x_\beta)_{\beta<\mu}\ \mathsf{Type}$. The notation emphasizes the fact that $\overline{B_\mu}$ depends on the indicated variables.

2. If $f : A_\lambda \to B_\mu$ is morphism of $\mathcal{C}$ with $\mu = \nu + 1$ we get an operator symbol $\overline{f}$. It has introductory rule;

   - If $f : A_\lambda \to B_{\mu+1}$, denote by $\rho_\mu : B_{\mu+1} \twoheadrightarrow B_\mu$. Then the operator symbol has introductory rule

   $$\{x_\alpha : \overline{A}_\alpha\}_{\alpha<\lambda} \vdash \overline{f}(x_\alpha)_{\alpha<\lambda} : \overline{(\rho_\mu f)^* B_{\mu+1}}(x_\alpha)_{\alpha<\lambda}.$$

   This does not clash with the notation from the previous point since it always refer to an object of $\mathcal{C}$ and in this case refers to map.

Subject to the following axioms in $U(\mathcal{C})$:

1. Let A_λ, B_μ, $C_{\nu+1}$ be objects of $\mathcal{C}$ and maps $f : A_\lambda \to B_\mu$, $g : B_\mu \to C_{\nu+1}$:

   $$\{x_\alpha : \overline{A}_\alpha\}_{\alpha<\lambda} \vdash \overline{gf}(x_\alpha)_{\alpha<\lambda} \equiv_{\overline{(p_\nu gf)^* C_{\nu+1}}(x_\alpha)_{\alpha<\lambda}} \overline{g}\big(\overline{p_\beta f}(x_\alpha)_{\alpha<\lambda}\big)_{\beta<\mu}.$$

2. Let B_μ be a non-trivial object of $\mathcal{C}$. For each $\delta < \mu$ we have

   $$\{x_\beta : \overline{B}_\beta\}_{\beta<\mu} \vdash \overline{p_\delta}(x_\beta)_{\beta<\mu} \equiv_{\overline{B}_\delta(x_\beta)_{\beta<\delta}} x_\delta.$$

3. Let A_λ, $B_{\mu+1}$ objects of $\mathcal{C}$ and a map $f : A_\lambda \to B_\mu$ then

   $$\{x_\alpha : \overline{A}_\alpha\}_{\alpha<\lambda} \vdash \overline{f^* B_{\mu+1}}(x_\alpha)_{\alpha<\lambda} \equiv \overline{B_{\mu+1}}\big(\overline{p_\beta f}(x_\alpha)_{\alpha<\lambda}\big)_{\beta<\mu}$$

   and

   $$\{x_\alpha : \overline{A}_\alpha,\ x_\delta : \overline{f^* B_{\mu+1}}(x_\alpha)_{\alpha<\lambda}\}_{\alpha<\lambda} \vdash \overline{q(f, B_{\mu+1})}(x_\alpha, x_\delta)_{\alpha<\lambda} \equiv_{\overline{f^* B_\mu}(x_\alpha)_{\alpha<\lambda}} x_\delta.$$

Observation 4.6.16. It is immediate to observe that $U(\mathcal{C})$ as defined is a κ-pretheory. We have sort symbol and operator symbols introduced by type judgment and type element judgments respectively. Note that the list of axioms we provided are well-formed rules. This is because the premise of each axiom is by definition a context.

Remark 4.6.17. If $f : A_\lambda \to B_\mu$ is a map in $\mathcal{C}$, where μ is a limit ordinal, *i.e.,* B_μ is a limit object, then we get a family of maps $\{f_\nu : A_\lambda \to B_\nu\}_{\nu<\mu}$. Therefore, the associated operator $\overline{f}$ is uniquely determined by the operator $\overline{f_\nu}$, for which in this case we can assume that ν is a successor ordinal.

If $F : \mathcal{C} \to \mathcal{D}$ is a functor between κ-contextual categories, then we need an interpretation $U(F) : U(\mathcal{C}) \to U(\mathcal{D})$;

1. For an object A_λ, the interpretation is defined as

$$U(F)(\overline{A_\lambda}) := \overline{FA_\lambda}(x_\alpha)_{\alpha<\lambda}.$$

2. For a morphism $f : A_\lambda \to B_{\mu+1}$, the operator $\overline{f}$ is interpreted as

$$U(F)(\overline{f}) := \overline{F(f)}(x_\alpha)_{\alpha<\lambda}.$$

The next step is to prove that this is indeed an map between the κ-Cartmell theories, this is done in [17, pp 2.29]. For this, it is enough to show that rules and axioms of $U(\mathcal{C})$ are sent to rules of $U(\mathcal{D})$. The functoriality of $U : \kappa\text{-CON} \to \kappa\text{-GAT}$ is also immediate from its definition. This is tested on each type and operator symbol. It is then enough to take the equivalence class $[U(F)]$.

4.6.3.3 The natural isomorphism $U \circ \mathbb{C} \cong Id_{\kappa\text{-}GAT}$

For each $T \in \kappa$-GAT we want to define an interpretation $[\varphi_T] : T \to U(\mathbb{C}_T)$, we do this by defining a preinterpretation $\varphi_T : Exp(T) \to Exp(U(\mathbb{C}_T))$:

1. If Δ is a type symbol of T with introduction rule

$$\{x_\alpha : \Delta_\beta\}_{\beta<\mu} \vdash \Delta(x_\beta)_{\beta<\mu}\ \mathsf{Type}$$

then

$$\varphi_T(\Delta) := \overline{[\{x_\beta : \Delta_\beta,\, x_\delta : \Delta(x_\beta)_{\beta<\mu}\}_{\beta<\mu}]}(x_\beta)_{\beta<\mu}$$

2. If f is an operator symbol with introductory rule

$$\{x_\alpha : \Delta_\beta\}_{\beta<\mu} \vdash f(x_\beta)_{\beta<\mu} : \Delta,$$

then

$$\varphi_T(f) := \overline{[\langle x_\beta, f(x_\beta)_{\beta<\mu}\rangle_{\beta<\mu}]}(x_\beta)_{\beta<\mu},$$

where $\langle x_\beta, f(x_\beta)_{\beta<\mu}\rangle_{\beta<\mu}$ is the morphism $\{x_\alpha : \Delta_\beta\}_{\beta<\mu} \to \{x_\alpha : \Delta_\beta, x_\delta : \Delta\}_{\beta<\mu}$.

We proceed to verify that as defined $\varphi_T : T \to U(\mathbb{C}_T)$ is an interpretation. This is a crucial point in the proof, so we spell out some details in theorem 4.6.26. The results below are the technical steps towards it.

Lemma 4.6.18. If $\mathcal{C}$ is a contextual category, objects A_λ, B_μ and $f : A_\lambda \to B_\mu$ is map with $\mu = \nu + 1$ (in particular it is non-trivial) then the rule

$$\{x_\alpha : \overline{A}_\alpha(x_\gamma)_{\gamma<\alpha}\}_{\alpha<\lambda} \vdash \overline{f}(x_\alpha)_{\alpha<\lambda} : \overline{B_\mu}\big(\overline{p_\beta \circ f}(x_\alpha)_{\alpha<\lambda}\big)_{\beta<\mu}$$

is a derived rule of $U(\mathcal{C})$.

Proof. We have the axiom

$$\{x_\alpha : \overline{A}_\alpha\}_{\alpha<\lambda} \vdash \overline{f^*B_\mu}(x_\alpha)_{\alpha<\lambda} \equiv \overline{B_\mu}\big(\overline{p_\beta \circ f}(x_\alpha)_{\alpha<\lambda}\big)_{\beta<\mu}$$

for $U(\mathcal{C})$ and the derivation rule for κ-GAT

$$\frac{\Gamma \vdash A_1 \equiv A_2 \qquad t : A_1}{\Gamma \vdash t : A_2}.$$

These put together give us the result. ∎

Lemma 4.6.19. Let $\mathcal{C}$ a κ-contextual category, objects $\{A_\alpha\}_{\alpha<\lambda}$, $\{B_\beta\}_{\beta<\mu+1}$, $\{C_\gamma\}_{\gamma<\varepsilon}$ and a commutative diagram

$$\begin{tikzcd}
C_\varepsilon \arrow[r, "l"] \arrow[d, "k"'] & B_{\mu+1} \arrow[d, two heads, "p"] \\
A_\lambda \arrow[r, "f"'] & B_\mu.
\end{tikzcd}$$

If $h : C_\varepsilon \to f^*B_{\mu+1}$ is the unique map given by the pullback, then the rule

$$\{x_\gamma : \overline{C_\gamma}(x_\delta)_{\delta<\gamma}\}_{\gamma<\varepsilon} \vdash \overline{h}(x_\gamma)_{\gamma<\varepsilon} \equiv_{\overline{(fk)^*B_{\mu+1}}(x_\gamma)_{\gamma<\varepsilon}} \overline{l}(x_\gamma)_{\gamma<\varepsilon}$$

is a derived rule of $U(\mathcal{C})$.

Proof. The proof is the same as [17, Lemma 2 pp. 2.32] using theorem 4.6.18. ∎

Lemma 4.6.20. Let $\mathcal{C}$ a κ-contextual category, objects $\{A_\alpha\}_{\alpha<\lambda}$, $\{B_\beta\}_{\beta<\mu}$, $\{C_\gamma\}_{\gamma<\varepsilon}$ and for $0 < \nu < \mu$ a commutative diagram

$$\begin{tikzcd}
C_\varepsilon \arrow[r, "l_\nu"] \arrow[d, "k_\nu"'] & B_\mu \arrow[d, two heads, "p_\nu"] \\
A_\lambda \arrow[r, "f"'] & B_\nu.
\end{tikzcd}$$

If $h_\nu : C_\varepsilon \to f^*B_\mu$ is the unique map given by the pullback, then the rule

$$\{x_\gamma : \overline{C_\gamma}(x_\delta)_{\delta<\gamma}\}_{\gamma<\varepsilon} \vdash \overline{h_\nu}(x_\gamma)_{\gamma<\varepsilon} \equiv_{\overline{(fk_\nu)^*B_\mu}(x_\gamma)_{\gamma<\varepsilon}} \overline{l_\nu}(x_\gamma)_{\gamma<\varepsilon}$$

is a derived rule of $U(\mathcal{C})$.

Proof. This by induction on the height of p_ν. When it is a successor ordinal, this is the previous theorem 4.6.20. When it is a limit ordinal B_μ is a limit object, therefore the result reduces to the inductive hypothesis, which is the successor case again. ∎

Recall from section 4.6.2 we defined the set of maps $\Gamma(B)$. It follows from the previous result that

Corollary 4.6.21. If $\mathcal{C}$ is a κ-contextual category and $f : A_\lambda \to B_\mu$ is a map in $\mathcal{C}$, then for all $\nu < \mu$

$$\{x_\alpha : A_\alpha(x_\delta)_{\delta<\alpha}\}_{\alpha<\lambda} \vdash \overline{\delta^\nu_f}(x_\alpha)_{\alpha<\lambda} \equiv \overline{f}(x_\alpha)_{\alpha<\lambda}.$$

is a derived rule of $U(\mathcal{C})$.

If we specialize theorem 4.6.21 to the syntactic κ-contextual category of a κ-Cartmell theory T, then

Corollary 4.6.22. Assume that $\{x_\beta : B_\beta\}_{\beta<\mu}$ is a context, $\nu < \mu$ and

$$f_\nu := [\langle t_\beta \rangle_{\beta<\nu}] : [\{x_\alpha : A_\alpha\}_{\alpha<\lambda}] \to [\{x_\beta : B_\beta\}_{\beta<\nu}]$$

a map in $\mathbb{C}_T$ then

$$\{x_\alpha : \overline{A_\alpha}(x_\gamma)_{\gamma<\alpha}\}_{\alpha<\lambda} \vdash \overline{[\langle x_\alpha, t_\varepsilon \rangle_{\substack{\alpha<\lambda \\ \nu\leqslant\varepsilon<\mu}}]} \equiv \overline{[\langle t_\beta, t_\varepsilon \rangle_{\beta<\nu\leqslant\varepsilon<\mu}]}.$$

is a derived rule of $U(\mathbb{C}_T)$.

Proof. This follows from theorem 4.6.21 and the explicit description of $\delta^\nu_{f_\nu}$ given in theorem 4.6.9. ∎

Lemma 4.6.23. If A_λ, B_μ are objects and $f_\nu : A_\lambda \to B_\nu$, with $\nu < \mu$, is a map in a κ-contextual category $\mathcal{C}$, then:

1. The rule
$$\{x_\alpha : \overline{A_\alpha}(x_\delta)_{\delta<\alpha}\}_{\alpha<\lambda} \vdash \overline{f^*_\nu B_\mu}(x_\alpha)_{\alpha<\lambda} \equiv \overline{B}(\delta^\gamma_{(p_\gamma f)}(x_\alpha)_{\alpha<\lambda})_{\gamma<\nu}$$
is a derived rule of $U(\mathcal{C})$.
2. If $g : \Gamma(B^\mu_\nu)$ then the rule
$$\{x_\alpha : \overline{A_\alpha}(x_\delta)_{\delta<\alpha}\}_{\alpha<\lambda} \vdash \overline{\delta^\nu_{gf}}(x_\alpha)_{\alpha<\lambda} \equiv \overline{\delta^\nu_g}(\overline{\delta^\gamma_{p_\gamma f}}(x_\alpha)_{\alpha<\lambda})_{\gamma<\nu}$$
is a derived rule of $U(\mathcal{C})$.

Corollary 4.6.24. If T is a κ-Cartmell theory, $\{x_\beta : B_\beta\}_{\beta<\mu}$ is a context, $\nu < \mu$ and

$$f_\nu := [\langle t_\beta \rangle_{\beta<\nu}] : [\{x_\alpha : A_\alpha\}_{\alpha<\lambda}] \to [\{x_\beta : B_\beta\}_{\beta<\nu}]$$

is a map in $\mathbb{C}_T$ then;

1. $$\frac{\{x_\alpha : \overline{A_\alpha}(x_\delta)_{\delta<\alpha}\}_{\alpha<\lambda}}{\overline{[\{x_\alpha, x_\gamma : B_\gamma[t_\delta|x_\delta]_{\delta<\gamma}\}_{\substack{\alpha<\lambda \\ \nu\leqslant\gamma<\mu}}]}(x_\alpha)_{\alpha<\lambda} \equiv \overline{[\{x_\beta : B_\beta\}_{\beta<\nu}]}(\overline{g_\beta}(x_\alpha)_{\alpha<\lambda})_{\beta<\nu}}$$
   where for each $\beta < \nu$ the map $g_\beta := [\langle x_\alpha, t_\beta \rangle_{\alpha<\lambda}]$.

2. If for all γ, with $\nu < \gamma < \mu$, the rule

$$\{x_\beta : B_\beta\}_{\beta<\nu}, \{t_{\gamma'} : B_{\gamma'}\}_{\gamma'<\gamma} \vdash t_\gamma : B_\gamma$$

   is a derived rule then

$$\{x_\alpha : \overline{A_\alpha}(x_\delta)_{\delta<\alpha}\}_{\alpha<\lambda} \vdash \overline{[\langle x_\alpha, t_\gamma[t_{\gamma'} \mid x_{\gamma'}]_{\gamma'<\gamma}\rangle_{\substack{\alpha<\lambda \\ \nu<\gamma<\mu}}]} \equiv \overline{h}(\overline{g_\beta}(x_\alpha)_{\alpha<\lambda})_{\beta<\nu}$$

   where g_β is defined as in the previous point and $h := [\langle x_\beta, t_\gamma\rangle_{\substack{\beta<\nu \\ \nu<\gamma<\mu}}]$.

Proof. This is a direct application of theorem 4.6.23. We remark that the assumption of point (2) simply give us an element of $\Gamma(B^\mu_\nu)$ and the map on the left depend on variables that according to our convention we leave implicit. ∎

The following lemma is key to prove that we have an interpretation $\varphi_T : T \to U(\mathbb{C}_T)$, the results above are used to prove:

Lemma 4.6.25. If T is a κ-Cartmell theory then:

1. If $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash \Delta\,\mathsf{Type}$ is a type judgment of T, then the rule

$$\{x_\alpha : \overline{A_\alpha}(x_\delta)_{\delta<\alpha}\}_{\alpha<\lambda} \vdash \overline{A}(x_\alpha)_{\alpha<\lambda+1} \equiv \widetilde{\varphi_T}(\Delta)$$

   is a derived rule of $U(\mathbb{C}_T)$ where $A := \{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda+1}$ and $A_\alpha := \{x_\delta : \Delta_\delta\}_{\delta\leqslant\alpha}$.

2. If $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash t : \Delta$ is a type element judgment of T, then the rule

$$\{x_\alpha : \overline{A_\alpha}(x_\delta)_{\delta<\alpha}\}_{\alpha<\lambda} \vdash \overline{\langle x_\alpha, t\rangle_{\alpha<\lambda}}(x_\alpha)_{\alpha<\lambda+1} \equiv_{\overline{A}(x_\alpha)_{\alpha<\lambda}} \widetilde{\varphi_T}(t)$$

   is a derived rule of $U(\mathbb{C}_T)$.

Proof. The proof is by induction on the derivations, by showing that rule derivation preserves the properties above. ∎

The important result of this section is the following.

Corollary 4.6.26. For every κ-Cartmell theory T, the map $\varphi_T : U \to U(\mathbb{C}_T)$ is an interpretation.

Proof. We see that the function $\widehat{\varphi_T} : \mathit{Rul}(T) \to \mathit{Rul}(U(\mathbb{C}_T))$ is well-defined. We start with a rule $\mathcal{J}$ of T and show that $\widehat{\varphi_T}(\mathcal{J})$ is a rule of $U(\mathbb{C}_T)$

1. Type judgment: Assume that $\mathcal{J} := \{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash \Delta\,\mathsf{Type}$ is a rule of T, from theorem 4.5.24 it follows that

$$\widehat{\varphi_T}(\mathcal{J}) = \{x_\alpha : \widetilde{\varphi}(\Delta_\alpha)\}_{\alpha<\lambda} \vdash \widetilde{\varphi_T}(\Delta)\,\mathsf{Type}.$$

From theorem 4.6.25 we have for any $\gamma < \lambda + 1$ the rule

$$\{x_\alpha : \overline{\Delta_\alpha}(x_\delta)_{\delta<\alpha}\}_{\alpha<\lambda} \vdash \overline{A_{\gamma+1}}(x_\alpha)_{\alpha<\gamma+1} \equiv \widetilde{\varphi_T}(\Delta_\gamma)$$

is a derived rule of $U(\mathbb{C}_T)$. Thus, so it is

$$\{x_\alpha : \widetilde{\varphi_T}(\Delta_\alpha)(x_\delta)_{\delta<\alpha}\}_{\alpha<\lambda} \vdash \overline{A_{\gamma+1}}(x_\alpha)_{\alpha<\lambda+1} \equiv \widetilde{\varphi_T}(\Delta).$$

Then it must be the case that $\{x_\alpha : \widetilde{\varphi}(\Delta_\alpha)\}_{\alpha<\lambda} \vdash \widetilde{\varphi_T}(\Delta)\ \mathsf{Type}$ is a rule of $U(\mathbb{C}_T)$.

2. Element judgment: $\Gamma \vdash t : \Delta$. This very similar the previous rule.
3. Type equality judgment: $\Gamma \vdash \Delta \equiv \Delta'$. Also follows from theorem 4.6.25.
4. Term equality judgment: $\Gamma \vdash t \equiv_\Delta t'$. The same argument works.

∎

Corollary 4.6.27. For every κ-Cartmell theory T, the map $[\varphi_T] : U \to U(\mathbb{C}_T)$ is morphism in the category κ-GAT.

Next, we will show that $[\varphi_-] : Id_{\kappa\text{-GAT}} \Rightarrow U \circ \mathbb{C}$ is a natural transformation.

Lemma 4.6.28. Let T, T' two κ-Cartmell theories and $I : T \to T'$ an interpretation between them. Then, we have a commutative diagram

$$\begin{tikzcd}
T \arrow[r, "{[\varphi_T]}"] \arrow[d, "{[I]}"'] & U(\mathbb{C}_T) \arrow[d, "U(\mathbb{C}(I))"] \\
T' \arrow[r, "{[\varphi_{T'}]}"'] & U(\mathbb{C}_{T'}).
\end{tikzcd}$$

Proof. We use theorem 4.5.29. Therefore, it will be enough to test the commutativity of the diagram on type element judgments. Let $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash t : \Delta_\lambda$ a type element judgment of T. For any $\alpha \leqslant \lambda$ we denote $A_\alpha := [\{x_\delta : \Delta_\delta\}_{\delta \leqslant \alpha}]$. It follows from theorem 4.6.25 that

$$\widehat{\varphi_T}\left(\frac{\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}}{t : \Delta_\lambda}\right) \approx \frac{\{x_\alpha : \overline{A_\alpha}\}_{\alpha<\lambda}}{\overline{[\langle x_\alpha, t\rangle_{\alpha<\lambda}]} : \overline{A_\lambda}(x_\alpha)_{\alpha<\lambda}}.$$

We conclude that

$$U(\mathbb{C}(I))\left(\widehat{\varphi_T}\left(\frac{\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}}{t : \Delta_\lambda}\right)\right) \approx \frac{\{x_\alpha : \overline{\mathbb{C}(I)(A_\alpha)}\}_{\alpha<\lambda}}{\overline{\mathbb{C}(I)([\langle x_\alpha, t\rangle_{\alpha<\lambda}])} : \overline{\mathbb{C}(I)(A_\lambda)}(x_\alpha)_{\alpha<\lambda}}.$$

Looking at the other composition: we get

$$\widehat{I}\left(\frac{\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}}{t : \Delta_\lambda}\right) = \frac{\{x_\alpha : \widetilde{I}(\Delta_\alpha)\}_{\alpha<\lambda}}{\widetilde{I}(t) : \widetilde{I}(\Delta_\lambda)}.$$

A second use of theorem 4.6.25 give us that

$$\widehat{\varphi_{T'}}\left(\widehat{I}\left(\frac{\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}}{t : \Delta_\lambda}\right)\right) \approx \frac{\{x_\alpha : \overline{B_\alpha}\}_{\alpha<\lambda}}{\overline{[\langle x_\alpha, \widetilde{I}(t)\rangle_{\alpha<\lambda}]} : \overline{B_\lambda}(x_\alpha)_{\alpha<\lambda}},$$

where for $\alpha \leqslant \lambda$, $B_\alpha := [\{x_\delta : \widetilde{I}(\Delta_\delta)\}_{\delta\leqslant\alpha}]$. However, by definition we have $\mathbb{C}(I)(A_\alpha) = B_\alpha$ for $\alpha \leqslant \lambda$. This completes our verification. ∎

Remains to show that $[\varphi_T]$ is an isomorphism, and natural in T. We proceed to give an inverse $\psi_T : U(\mathbb{C}_T) \to T$. Recall that a type symbol of $U(\mathbb{C}_T)$ is of the form $\overline{A_\lambda} = \overline{[\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}]}$. If $\lambda = \nu + 1$ then by choosing a representative of this equivalence class of the context we can define $\psi_T(\overline{A_\lambda}) := \Delta_\nu$.
If λ is a limit ordinal once we chose a representative $\Delta_\lambda = \{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}$. Then we know that $[\Delta_\lambda] = \lim_{\alpha<\lambda}[\Delta_\alpha]$ in $\mathbb{C}_T$, and this limit is unique. In this case, the value of ψ_T is determined by non-limit ordinals $\alpha < \lambda$, which are $\psi_T(\overline{\Delta_\alpha}) = \Delta_\alpha$. Therefore, we define $\psi_T(\overline{[\Delta_\lambda]}) := \Delta_\lambda$ for some choice of a representative of the equivalence class. However, note that the successor case determinate the limit case.
Operator symbols of $U(\mathbb{C}_T)$ come from morphisms of $\mathbb{C}_T$. Therefore, for a morphism $\overline{f} := [\langle t_\beta\rangle_{\beta<\mu}] : [\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}] \to [\{x_\beta : \Omega_\beta\}_{\beta<\mu}]$ in order to define ψ_T on the associated operator, it is enough to assume that μ is a successor ordinal. Firstly, we need to make choices for the contexts and morphism. However, the definition does not depend on these choices because of (1) from theorem 4.5.22. This allows to define ψ_T as

$$\psi_T(\overline{f}) := t_\mu$$

where $t_\mu : \Omega_\mu[t_\beta|x_\beta]_{\beta<\mu}$.

Lemma 4.6.29. The function ψ_T is an interpretation from $U(\mathbb{C}_T) \to T$.

Proof. We need to check that rules and axioms are preserved by ψ_T. It will be enough to deal with the case where $\lambda = \nu + 1$. Suppose that $\overline{A_\lambda}$ has

$$\frac{\{x_\alpha : \overline{A_\alpha}(x_\delta)_{\delta<\alpha}\}_{\alpha<\nu}}{\overline{A_\nu}(x_\alpha)_{\alpha<\nu}\ \mathsf{Type}}$$

Furthermore, we assume that $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}$ is such that $A_\lambda = [\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}]$. By definition,

$$\widehat{\psi_T}\left(\frac{\{x_\alpha : \overline{A_\alpha}(x_\delta)_{\delta<\alpha}\}_{\alpha<\nu}}{\overline{A_\lambda}(x_\alpha)_{\alpha<\lambda}\ \mathsf{Type}}\right) = \frac{\{x_\alpha : \Delta_\alpha\}_{\alpha<\nu}}{\Delta_\nu\ \mathsf{Type}}.$$

This is obviously a derived rule of T. Preservation of the rule for operator symbols is straightforward. ∎

Lemma 4.6.30. For any κ-Cartmell theory T we have $\psi_T \circ \varphi_T \approx Id_T$.

Proof. From theorem 4.5.29 it is enough to verify the statement on type element judgments. Let $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash t : \Delta_\lambda$ a type element judgment. For any $\alpha \leqslant \lambda$ we denote $A_\alpha := [\{x_\delta : \Delta_\delta\}_{\delta\leqslant\alpha}]$. It follows from theorem 4.6.25 that

$$\widehat{\varphi_T}\left(\frac{\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}}{t : \Delta_\lambda}\right) \approx \frac{\{x_\alpha : \overline{A_\alpha}\}_{\alpha<\lambda}}{\overline{[\langle x_\alpha, t\rangle_{\alpha<\lambda}]} : \overline{A_\lambda}(x_\alpha)_{\alpha<\lambda}}.$$

Hence

$$\widehat{\psi_T}\left(\widehat{\varphi_T}\left(\frac{\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}}{t : \Delta_\lambda}\right)\right) \approx \widehat{\psi_T}\left(\frac{\{x_\alpha : \overline{A_\alpha}\}_{\alpha<\lambda}}{\overline{[\langle x_\alpha, t\rangle_{\alpha<\lambda}]} : \overline{A_\lambda}(x_\alpha)_{\alpha<\lambda}}\right) = \frac{\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}}{t : \Delta_\lambda}.$$

∎

Lemma 4.6.31. For any κ-Cartmell theory T we have $\psi_T \circ \varphi_T \approx Id_T$.

Proof. The proof is similar to the previous lemma. All the definitions and technical results have been established, specially theorem 4.6.25. ∎

Corollary 4.6.32. There is a natural isomorphism $Id_{\kappa\text{-GAT}} \Rightarrow U \circ \mathbb{C}$.

Proof. We have constructed $[\varphi_{-}] : Id_{\kappa\text{-GAT}} \Rightarrow U \circ \mathbb{C}$. ∎

4.6.3.4 The natural isomorphism $\mathbb{C} \circ U \cong Id_{\kappa\text{-}CON}$

In this section, we aim to construct a natural isomorphism $\eta : Id_{\kappa\text{-CON}} \Rightarrow \mathbb{C} \circ U$. Let $\mathcal{C}$ be a κ-contextual category. For this, we first construct a κ-contextual functor $\eta_{\mathcal{C}} : \mathcal{C} \to \mathbb{C}_{U(\mathcal{C})}$. Recall that if A_λ is an object in $\mathcal{C}$ then for any $\alpha \leqslant \lambda$ we denoted $p_\alpha : A_\lambda \twoheadrightarrow A_\alpha$ to the canonical display map that exists. Then we can make the following definition:

1. For $\eta_{\mathcal{C}}(1) := 1$.
2. If A_μ is an object with $\mu = \lambda + 1$ then
$$\eta_{\mathcal{C}}(A_\mu) := [\{x_\alpha : \overline{A_\alpha}(x_\delta)_{\delta<\alpha}\}_{\alpha\leqslant\mu}].$$
3. For an object A_λ, we define $\eta_{\mathcal{C}}(p_0) := \eta_{\mathcal{C}}(p)_0$ where $\eta_{\mathcal{C}}(p)_0 : \eta_{\mathcal{C}}(A) \twoheadrightarrow 1$.
4. If A_λ, B_μ are non-trivial objects, with μ a successor ordinal, and $f : A_\lambda \to B_\mu$ is a morphism in $\mathcal{C}$ then
$$\eta_{\mathcal{C}}(f) := [\langle \overline{p_\beta f}(x_\alpha)_{\alpha<\lambda}\rangle_{\beta\leqslant\mu}].$$

We observe that if μ is a limit ordinal, then any map $f : A_\lambda \to B_\mu$ is determined by a family of maps $\{f_\nu : A_\lambda \to B_\nu\}_{\nu<\mu}$. Thus, in order to define η on such map it is enough to do it on ordinals $\nu < \mu$ which we can assume to be successor ordinals. The map $\eta(f)$ is the map induced by the family of maps $\{\eta(f_\nu) : \eta(A_\lambda) \to \eta(B_\nu)\}_{\nu<\mu}$. In conclusion, we simply need to prove properties of η for successor ordinals. The property for limit ordinals follows using the universal property of the limit object.

Lemma 4.6.33. For any $\mathcal{C}$, $\eta_{\mathcal{C}} : \mathcal{C} \to \mathbb{C}_{U(\mathcal{C})}$ is a κ-contextual functor.

Proof. First we verify that it is a functor. Since for any $\alpha < \lambda$ we have $\overline{p_\alpha}(x_\alpha)_{\alpha<\lambda} = x_\alpha$, then it is immediate to see that $\eta_{\mathcal{C}}$ preserves the identities.
Assume we have non-trivial morphisms $f : A_\lambda \to B_\mu$ and $g : B_\mu \to C_\nu$ then

$$\eta_{\mathcal{C}}(gf) = [\langle \overline{p_\gamma g f}(x_\alpha)_{\alpha<\lambda} \rangle_{\beta \leqslant \nu}]$$

From the first axiom in theorem 4.6.15 $U(\mathcal{C})$ it follows that the above must be $\eta_{\mathcal{C}}(g)\eta_{\mathcal{C}}(f)$ whenever μ and ν are successor ordinals. When we have limits
Now we must verify that it preserves display maps and canonical pullbacks. Both statements are direct consequences of the definitions. Furthermore, the proof from [17] works without mayor changes.
For the preservation of pullbacks: We let $f : A_\lambda \to B_{\mu+1}$ then

$$\begin{aligned}
\eta_{\mathcal{C}}(f^*B) &= [\langle x_\alpha : \overline{A_\delta}(x_\gamma)_{\gamma<\alpha},\, x_\epsilon : \overline{f^*B_{\mu+1}}(x_\alpha)_{\alpha<\lambda} \rangle_{\alpha<\lambda}] \\
&= [\langle x_\alpha : \overline{A_\delta}(x_\gamma)_{\gamma<\alpha},\, x_\epsilon : \overline{B_{\mu+1}}\left(\overline{p_\beta f}(x_\alpha)_{\alpha<\lambda}\right)_{\beta<\mu} \rangle_{\alpha<\lambda}] \\
&= [\langle \overline{p_\beta f}(x_\alpha)_{\alpha<\lambda} \rangle_{\beta \leqslant \mu}]^*[\langle x_\beta : \overline{B_\beta}(x_\gamma)_{\gamma<\beta} \rangle_{\beta \leqslant \mu}] \\
&= \eta_{\mathcal{C}}(f)^*\eta_{\mathcal{C}}(B).
\end{aligned}$$

For a display map of $p_\nu : B_\mu \twoheadrightarrow B_\nu$ with height a successor ordinal, the same argument shows that the pullback along $f_\nu : A_\lambda \to B_\nu$ is preserved. When the height is a limit ordinal, we combine the previous case and the fact that in any κ-contextual category canonical pullbacks are unique. ∎

Lemma 4.6.34. Let $\mathcal{C}$, $\mathcal{C}'$ be κ-contextual categories and a contextual functor $F : \mathcal{C} \to \mathcal{C}'$. Then the following diagram is commutative:

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "\eta_{\mathcal{C}}"] \arrow[d, "F"'] & \mathbb{C}_{U(\mathcal{C})} \arrow[d, "\mathbb{C}(U(F))"] \\
\mathcal{C}' \arrow[r, "\eta_{\mathcal{C}'}"'] & \mathbb{C}_{U(\mathcal{C}')}.
\end{tikzcd}$$

Proof. If $f : A_\lambda \to B_\mu$ is a map in $\mathcal{C}$ then

$$\begin{aligned}
\mathbb{C}(U(F))(\eta_{\mathcal{C}}(f)) &= \mathbb{C}(U(F))([\langle \overline{p_\beta f}(x_\alpha)_{\alpha<\lambda} \rangle_{\beta \leqslant \mu}]) \\
&= [\langle \overline{F(p_\beta f)}(x_\alpha)_{\alpha<\lambda} \rangle_{\beta \leqslant \mu}] \\
&= [\langle \overline{p_\beta F(f)}(x_\alpha)_{\alpha<\lambda} \rangle_{\beta \leqslant \mu}] \\
&= \eta_{\mathcal{C}'}(f).
\end{aligned}$$

∎

Corollary 4.6.35. There is a natural transformation $Id_{\kappa\text{-CON}} \Rightarrow \mathbb{C} \circ U$.

Remains to show that this natural transformation is an isomorphism. For each κ-contextual category $\mathcal{C}$ we construct a κ-contextual functor

$$\xi_{\mathcal{C}} : \mathbb{C}_{U(\mathcal{C})} \to \mathcal{C}$$

which is a two-sided inverse to $\eta_{\mathcal{C}}$. From theorem 4.5.13 we see that:

1. Every derived type judgment of $U(\mathcal{C})$ is of the form

   $$\{x_\beta : \Omega_\beta\}_{\beta<\mu} \vdash \overline{A_\lambda}(t_\alpha)_{\alpha<\lambda} \ \mathsf{Type}$$

   for some object A_λ of $\mathcal{C}$ where for $\alpha \leqslant \lambda$ the rule

   $$\{x_\beta : \Omega_\beta\}_{\beta<\mu} \vdash t_\alpha : \overline{A_\alpha}[t_\delta \mid x_\delta]_{\delta<\alpha}$$

   is a derived rule of $U(\mathcal{C})$.

2. Every type element judgment of T is of the form

   $$\{x_\beta : \Omega_\beta\}_{\beta<\mu} \vdash x_\beta : \Omega_\beta$$

   for some $\beta < \mu$, or is of the form

   $$\{x_\beta : \Omega_\beta\}_{\beta<\mu} \vdash \overline{f}(t_\alpha)_{\alpha<\lambda} : \Omega$$

   for some map $f : A_\lambda \to B_\mu$ of $\mathcal{C}$ such that for each $\alpha < \lambda$ the rules

   $$\{x_\beta : \Omega_\beta\}_{\beta<\mu} \vdash t_\alpha : \overline{A_\alpha}[t_\delta \mid x_\delta]_{\delta<\alpha}$$

   and

   $$\{x_\beta : \Omega_\beta\}_{\beta<\mu} \vdash \overline{B_\mu}(t_\beta)_{\beta<\mu} \equiv \Omega$$

   are derived rules of $U(\mathcal{C})$.

We may assume that $\mu = \nu + 1$, the limit case will follow induction. Let $\mathcal{R}_{\mathcal{C}}$ be the set of type and element type judgments of $U(\mathcal{C})$. Next, we define $\mathcal{J} : \mathcal{R}_{\mathcal{C}} \to \mathcal{C}$ inductively. First we get maps:

1. A rule $r_{\Omega_\mu} := \{x_\beta : \Omega_\beta\}_{\beta<\mu} \vdash \Omega_\mu$ is sent an object $\mathcal{J}(r_{\Omega_\mu}) \in \mathcal{C}$.

2. For any $\alpha < \lambda$ the judgment $r_{t_\alpha} := \{x_\beta : \Omega_\beta\}_{\beta<\mu} \vdash t_\alpha : \overline{A_\alpha}[t_\delta|x_\delta]_{\delta<\alpha}$ is sent to a map $\mathcal{J}(r_{t_\alpha})$.

The we can make the following definitions:

1. $\mathcal{J}(r_{A_\mu}) := (\mathcal{J}(t_\alpha)_{\alpha<\lambda})^* A_\mu$,
   where $\mathcal{J}(t_\alpha)_{\alpha<\lambda}$ denotes the pullbacks as in theorem 4.6.11.

2. $\mathcal{J}(\{x_\beta : \Omega_\beta\}_{\beta<\mu} \vdash \overline{f}(t_\alpha)_{\alpha<\lambda} : \Omega) := (\mathcal{J}(t_\alpha)_{\alpha<\lambda})^* \delta_f^\nu$.

3. $\mathcal{J}(\{x_\beta : \Omega_\beta\}_{\beta<\mu} \vdash x_\beta : \Omega) := \delta^{\beta}_{p_\beta}$ where $p_\beta : \mathcal{J}(r_{\Omega_\mu}) \to \mathcal{J}(r_{\Omega_\beta})$.

The burden of the proof falls into showing that the function $\mathcal{J}$ is well-defined. The proof is by induction on the derived rules of $U(\mathcal{C})$. We will focus on writing down the inductive hypothesis H as in [17] for this induction.

- For rules r_{Ω_μ} of the form $\{x_\beta : \Omega_\beta\}_{\beta<\mu} \vdash \Omega_\mu\ \mathsf{Type}$ then $H(r_{\Omega_\mu})$ is either:
    1. If the premise of r_{Ω_μ} is a non-empty context then $H(r_{\Omega_\beta})$ for all $\beta < \mu$.
    2. If r_{Ω_μ} is the rule $\vdash \Delta\ \mathsf{Type}$ then $ht(\mathcal{J}(r_{\Omega_\mu})) = 1$. Otherwise for all $\beta < \mu$ we have $ht(\mathcal{J}(r_{\Omega_\beta})) < ht(\mathcal{J}(r_{\Omega_\mu}))$.
    3. For a map $\langle t_\beta \rangle_{\beta<\mu} : \{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \to \{x_\beta : \Omega_\beta\}_{\beta<\mu}$. If for each $\beta + 1 < \mu$ we have $\mathcal{J}(r_{t_\beta+1}) \in \Gamma(\mathcal{J}(r_{\Omega_{\beta+1}[t_\gamma|x_\gamma]_{\gamma\leqslant\beta}}))$ where $r_{\Omega_{\beta+1}[t_\gamma|x_\gamma]_{\gamma\leqslant\beta}}$ is the rule $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash \Omega_{\beta+1}[t_\gamma|x_\gamma]_{\gamma\leqslant\beta}\ \mathsf{Type}$ then
$$\mathcal{J}(r_{\Omega_\mu[t_\beta|x_\beta]_{\beta<\mu}}) = (\mathcal{J}(t_\beta)_{\beta<\mu})^*\mathcal{J}(r_{\Omega_\mu})$$
- For rules r_{t_μ} of the form $\{x_\beta : \Omega_\beta\}_{\beta<\mu} \vdash t_\mu : \Omega_\mu$ then $H(r_{t_\mu})$ is either:
    1. $H(r_{\Omega_\mu})$.
    2. $\mathcal{J}(r_{t_\mu}) \in \Gamma(\mathcal{J}(r_{\Omega_\mu}))$.
    3. For a map $\langle t_\beta \rangle_{\beta<\mu} : \{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \to \{x_\beta : \Omega_\beta\}_{\beta<\mu}$. If for each $\beta + 1 < \mu$ we have $\mathcal{J}(r_{t_\beta+1}) \in \Gamma(\mathcal{J}(r_{\Omega_{\beta+1}[t_\gamma|x_\gamma]_{\gamma\leqslant\beta}}))$ then
$$\mathcal{J}(r_{t_\mu[t_\beta|x_\beta]_{\beta<\mu}}) = (\mathcal{J}(t_\beta)_{\beta<\mu})^*\mathcal{J}(r_{t_\mu})$$
    where $r_{t_\mu[t_\beta|x_\beta]_{\beta<\mu}}$ is the rule $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash t_\mu[t_\beta|x_\beta]_{\beta<\mu} : \Omega_\mu[t_\beta|x_\beta]_{\beta<\mu}$.
- For rules $r_\equiv$ or of the form $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash \Delta \equiv \Delta'$, the hypothesis $H(r_\equiv)$ is either:
    1. $H(r_{\Delta'})$ and $\mathcal{J}(r_\Delta) = \mathcal{J}(r_{\Delta'})$.
    2. $H(r_\Delta)$ and $\mathcal{J}(r_\Delta) = \mathcal{J}(r_{\Delta'})$.
- For rules r_ϵ or of the form $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash t \equiv_\Delta t'$, the hypothesis $H(r_\epsilon)$ is either:
    1. $H(r_t)$ and $\mathcal{J}(r_t) = \mathcal{J}(r_{t'})$.
    2. $H(r_{t'})$ and $\mathcal{J}(r_t) = \mathcal{J}(r_{t'})$.

Lemma 4.6.36. Let $\{x_\beta : \Omega_\beta\}_{\beta<\mu} \vdash \Omega$ a rule such that H is satisfied. If $\langle t_\beta \rangle_{\beta<\mu} : \{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \to \{x_\beta : \Omega_\beta\}_{\beta<\mu}$ is a map such that $H(r_{t_\beta})$ for all $\beta < \mu$ then $H(\{x_\beta : \Omega_\beta\}_{\beta<\mu} \vdash \Omega[t_\beta|x_\beta]_{\beta<\mu})$

Proof. By induction on μ and treating all different cases for H. The proof in [17, Lemma 11 pp.2.56] works here too. ∎

Lemma 4.6.37. 1. For any object $A_\lambda \in \mathcal{C}$, we have:

   (a) $A\lambda = \mathcal{J}(\{x_\alpha : \overline{A_\alpha}(x_\gamma)_{\gamma<\alpha}\}_{\alpha<\lambda} \vdash \overline{A_\lambda}(x_\alpha)_{\alpha<\lambda}\,\mathsf{Type})$.
   (b) For all $\alpha < \lambda$, $\delta_{p^\lambda_\alpha} = \mathcal{J}(\{x_\alpha : \overline{A_\alpha}(x_\gamma)_{\gamma<\alpha}\}_{\alpha<\lambda} \vdash x_\alpha : \overline{A_\alpha}(x_\gamma)_{\gamma<\alpha})$ where $p^\lambda_\alpha : A_\lambda \twoheadrightarrow A_\alpha$.

2. For any non-trivial object A_λ and $f : A_\lambda \to B_{\mu+1}$, $\delta_f = \mathcal{J}(\{x_\alpha : \overline{A_\alpha}(x_\gamma)_{\gamma<\alpha}\}_{\alpha<\lambda} \vdash \overline{f}(x_\alpha)_{\alpha<\lambda}\overline{(p_\mu f)^*B}(x_\alpha)_{\alpha<\lambda})$ where $p_\mu : B_{\mu+1} \twoheadrightarrow B_\mu$.

Proof. This is [17, Lemma 12 pp.263]. ∎

Lemma 4.6.38. Every derived rule of $U(\mathcal{C})$ satisfies the hypothesis H.

Proof. This is by induction on derived rules of $U(\mathcal{C})$. Indeed, [17, Lemma pp.2.65] shows that every derivation from theorem 4.5.4 preserves H. ∎

Corollary 4.6.39. 1. For any type symbol $\overline{A_\lambda}$ of the theory $U(\mathcal{C})$ we have $H(\{x_\alpha : \overline{A_\alpha}(x_\gamma)_{\gamma<\alpha}\}_{\alpha<\lambda} \vdash \overline{A_\lambda}(x_\alpha)_{\alpha<\lambda}\,\mathsf{Type})$.

2. For every operator symbol $\overline{f}$ in $U(\mathcal{C})$ where $f : A_\lambda \to B_{\mu+1}$ we have $H(\{x_\alpha : \overline{A_\alpha}(x_\gamma)_{\gamma<\alpha}\}_{\alpha<\lambda} \vdash \overline{f}(x_\alpha)_{\alpha<\lambda}\overline{(p_\mu f)^*B}(x_\alpha)_{\alpha<\lambda})$.

The foremost important result which summarizes the above is:

Corollary 4.6.40. 1. If $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}$ is a context of the theory then for any $\alpha < \delta < \lambda$ we have $ht(r_{\Delta_\alpha}) < ht(r_{\Delta_\beta})$.

2. If there is a map $\langle t_\beta \rangle_{\beta<\mu} : \{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \to \{x_\beta : \Omega_\beta\}_{\beta<\mu}$ then for each $\beta < \mu$ we have $\mathcal{J}(r_{t_\beta}) \in \Gamma(\mathcal{J}(r_{\Omega_\beta[t_\gamma|x_\gamma]_{\gamma<\beta}}))$ where $r_{\Omega_\beta[t_\gamma|x_\gamma]_{\gamma<\beta}}$ is the rule $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash \Omega_\beta[t_\gamma|x_\gamma]_{\gamma<\beta}\,\mathsf{Type}$.

3. If $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \equiv \{x_\alpha : \Delta'_\alpha\}_{\alpha<\lambda}$ then $\mathcal{J}(r_{\Delta_\lambda}) = \mathcal{J}(r_{\Delta'_\lambda})$.

4. If $\langle t_\alpha \rangle_{\alpha<\lambda} \equiv \langle t'_\alpha \rangle_{\alpha<\lambda}$ then for each $\beta < \mu$, $\mathcal{J}(r_{t_\beta}) = \mathcal{J}(r_{t'_\beta})$.

We are almost ready to define a contextual functor $\xi_\mathcal{C} : \mathcal{C}_{U(\mathcal{C})} \to \mathcal{C}$. We only need the next:

Observation 4.6.41. Let a map $\langle t_\beta \rangle_{\beta<\mu} : \{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \to \{x_\beta : \Omega_\beta\}_{\beta<\mu}$ then there are maps $\{g_\beta : \mathcal{J}(r_{\Delta_\lambda}) \to \mathcal{J}(r_{\Omega_\beta})\}_{\beta<\mu}$ with $\delta_{g_\beta} = \mathcal{J}(r_{t_beta})$ and $pg_{\beta+1} = g_\beta$. This is a consequence of theorem 4.6.40 and theorem 4.6.11. Therefore, there exists a unique $g : \mathcal{J}(r_{\Delta_\lambda}) \to \mathcal{J}(r_{\Omega_\mu})$ such that for all $\beta < \mu$ we have $\delta_{pg} = \mathcal{J}(r_{t_\beta})$ where $p : \mathcal{J}(r_{\Delta_\lambda}) \to \mathcal{J}(r_{\Omega_\beta})$.

Definition 4.6.42. We define a function

$$\xi_\mathcal{C} : \mathcal{C}_{U(\mathcal{C})} \to \mathcal{C}$$

by:

1. For an object $[\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}] \in \mathcal{C}_{U(\mathcal{C})}$,

$$\xi([\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}]) := \mathcal{J}(r_{\Delta_\lambda}).$$

2. For a morphism $[\langle t_\beta\rangle_{\beta<\mu}] : [\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}] \to [\{x_\beta : \Omega_\beta\}_{\beta<\mu}]$

$$\xi([\langle t_\beta\rangle_{\beta<\mu}]) := g,$$

where $g : \mathcal{J}(r_{\Delta_\lambda}) \to \mathcal{J}(r_{\Omega_\mu})$ is the unique map from theorem 4.6.41.

Lemma 4.6.43. 1. If $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash \Delta_\lambda \,\mathsf{Type}$ is a derived rule of $U(\mathcal{C})$ then for all $\alpha \leqslant \lambda$, $\{x_\gamma : \Delta_\gamma\}_{\gamma<\lambda} \vdash \Delta_\alpha \equiv \mathcal{J}(r_{\Delta_\alpha})(x_\gamma)_{\gamma<\alpha}$ is a derived rule of $U(\mathcal{C})$.

2. If $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash t_\lambda : \Delta_\lambda$ is a derived rule of $U(\mathcal{C})$ then $\{x_\gamma : \Delta_\gamma\}_{\gamma<\lambda} \vdash t \equiv \mathcal{J}(r_{t_\lambda})(x_\alpha)_{\alpha<\lambda}$ is a derived rule of $U(\mathcal{C})$.

Proof. See [17, Lemma 15 pp. 2.74]. ∎

Corollary 4.6.44. As functions, we have that $\eta_\mathcal{C}\xi_\mathcal{C} = id_{\mathcal{C}_{U(\mathcal{C})}}$ and $\xi_\mathcal{C}\eta_\mathcal{C} = Id_\mathcal{C}$

The results needed for this have been introduced throughout the section. Using that we have a bijection and that $\eta_\mathcal{C}$ is already a functor it follows:

Corollary 4.6.45. The function $\xi_\mathcal{C} : \mathcal{C}_{U(\mathcal{C})} \to \mathcal{C}$ is a contextual functor.

The main result that is of our interest is:

Theorem 4.6.46. There is a natural isomorphism $\mathcal{C}_{\_} \circ U \cong Id_{\kappa\text{-CON}}$.

Finally,

Corollary 4.6.47. The categories κ-CON of κ-contextual categories and κ-GAT of κ-algebraic theories are equivalent.

4.6.4 Coclans and contextual categories

In this section, we prove that every κ-contextual category can be obtained by strictification of a κ-clan. Clans were introduced in [39], a related definition appears in [29] under the name category with fibrations.

Definition 4.6.48. We say that a category $\mathcal{C}$ is a κ-*coclan* if it has a collection of maps $\textsc{Cof}(\mathcal{C})$ satisfying the following conditions:

1. $\mathcal{C}$ has initial object 0.

2. For any $X \in \mathcal{C}$, the map $0 \to X \in \textsc{Cof}(\mathcal{C})$.

3. Any isomorphism is an element of $\textsc{Cof}(\mathcal{C})$.

4. $\textsc{Cof}(\mathcal{C})$ is closed under compositions.

5. $\textsc{Cof}(\mathcal{C})$ is closed under pushouts: If $f : A \to C$ is a morphism in $\mathcal{C}$ and $A \to B \in \textsc{Cof}(\mathcal{C})$ then the map $C \to C \coprod_A B \in \textsc{Cof}(\mathcal{C})$.

6. $\textsc{Cof}(\mathcal{C})$ is closed under transfinite compositions: for any $\lambda < \kappa$ and any λ-diagram of maps in $\textsc{Cof}(\mathcal{C})$

$$A_0 \longrightarrow A_1 \longrightarrow A_2 \longrightarrow \cdots$$

   $\mathtt{Colim}_\lambda A_\alpha$ exists and the map $A_0 \to \mathtt{Colim}_\lambda A_\alpha$ belongs to $\textsc{Cof}(\mathcal{C})$.

As is usual, maps in $\textsc{Cof}(\mathcal{C})$ are called *cofibrations* and they are indicated by arrows " $\rightarrowtail$ ".

Dually, a category $\mathcal{C}$ is *κ-clan* if $\mathcal{C}^{op}$ is a κ-coclan. The distinguished maps are called *fibrations* and they are denoted by $\textsc{Fib}(\mathcal{C})$. The fibrations are indicated by arrows " $\twoheadrightarrow$ ". When working with κ-clans we keep the terminology "transfinite compositions" from κ-coclans as there is no risk of confusion.

Observation 4.6.49. The κ-contextual category $\mathbb{C}_T$ associated to a κ-generalized algebraic theory T has a natural κ-clan structure. Indeed, we can take $\textsc{Fib}(\mathbb{C}_T)$ as the display maps. All the axioms are easily verified. Moreover, this is true for any κ-contextual category not only for $\mathbb{C}_T$.

Recall that a *comprehension category* consists of a category $\mathcal{C}$, a fibration $p : \mathcal{E} \to \mathcal{C}$ and a functor $F : \mathcal{E} \to \mathcal{C}^{\to}$ such that:

1. $\partial_0 F = p$.
2. If f is a cartesian arrow in $\mathcal{E}$ then Ff is a pullback in $\mathcal{C}$, equivalently Ff is a cartesian arrow with respect to the codomain functor $\partial_0 : \mathcal{C}^{\to} \to \mathcal{C}$.

The fibration p is *cloven* if it comes with a choice of cartesian lifts. The comprehension category is said to be *split* is p is a split fibration. We also say that is *full* if F is fully faithful, we use the notation $(\mathcal{C}, \mathcal{E}, p, F)$ for a comprehension category.

The following example appears in [34, Example 4.5], we rewrite it in our setting of κ-clans. Let us fix a κ-clan $\mathcal{C}$, then the inclusion functor $\iota : \textsc{Fib}(\mathcal{C}) \hookrightarrow \mathcal{C}^{\to}$ and $P = \partial_0 \iota$ form a full comprehension category. More precisely: $\textsc{Fib}(\mathcal{C})$ has objects fibrations in $\mathcal{C}$ and arrows between two fibrations $\alpha : f \to g$ are commutative squares of the form

$$\begin{tikzcd}
A \arrow[r, "k"] \arrow[d, two heads, "f"'] & B \arrow[d, two heads, "g"] \\
\Delta \arrow[r, "l"'] & \Gamma.
\end{tikzcd}$$

Hence, an object in $\textsc{Fib}(\mathcal{C})_\Gamma$ over $\Gamma \in \mathcal{C}$ is a fibration $A \twoheadrightarrow \Gamma$. Observe that an arrow $\alpha : f \to g$ as above is cartesian if and only if it is a pullback square in $\mathcal{C}$. In conclusion, for an arrow $l : \Delta \to \Gamma$ and $B \twoheadrightarrow \Gamma \in \textsc{Fib}(\mathcal{C})_\Gamma$, a cartesian lift in $\textsc{Fib}(\mathcal{C})$ is a pullback square

$$\begin{tikzcd}
A \arrow[r, "k"] \arrow[d, two heads, "f"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, two heads, "g"] \\
\Delta \arrow[r, "l"'] & \Gamma.
\end{tikzcd}$$

This comprehension category is not necessarily split, reflecting the fact that taking pullbacks is not strictly functorial. Nevertheless, we can replace it by a split one via the functor

$$(-)_! : \mathbf{CompCat}(\mathcal{C}) \to \mathbf{SplCompCat}(\mathcal{C})$$

from the category of comprehension categories over $\mathcal{C}$ to the category of split comprehension categories over $\mathcal{C}$, the description of this functor appears in [48, 3.1] which we now recall. This produces a split comprehension category $(\mathcal{C}_!, \textsc{Fib}(\mathcal{C})_!, p_!, F_!)$ which is equivalent to the one we started with. Unfolding the result, we take the $\mathcal{C}_!$ to be simply $\mathcal{C}$.

The category $\textsc{Fib}(\mathcal{C})_!$ has:

- Objects: for each $\Gamma \in \mathcal{C}$ is a tuple $A := (V_A, E_A, f_A)$ where $V_A \in \mathcal{C}$, $E_A \twoheadrightarrow V_A \in \textsc{Fib}(\mathcal{C})_{V_A}$ and $f_A : \Gamma \to V_A \in \mathcal{C}$. We also employ the notation $[A] := f_A^* E_A$ given by taking the pullback of $E_A \twoheadrightarrow V_A$ along f_A, so we get a fibration $[A] \twoheadrightarrow \Gamma$. In addition, we write $(E_A)_{f_A}$ for the arrow $[A] \to E_A$. Thus, an object over Γ is a diagram in $\mathcal{C}$ of the form

$$\begin{tikzcd} & E_A \arrow[d, two heads] \\ \Gamma \arrow[r, "f_A"'] & V_A. \end{tikzcd}$$

- Morphisms: A map between $(V_B, E_B, f_B) \to (V_A, E_A, f_A)$ over $\sigma : \Delta \to \Gamma$ is a map in $\mathcal{E}$ between $[B] \twoheadrightarrow \Delta$ and $[A] \twoheadrightarrow \Gamma$ i.e. a commutative square

$$\begin{tikzcd} {[B]} \arrow[r] \arrow[d, two heads] & {[A]} \arrow[d, two heads] \\ \Delta \arrow[r, "\sigma"'] & \Gamma. \end{tikzcd}$$

- Composition is induced by the composition in $\mathcal{E}$, consequently, given by pasting commutative squares.

- The identity for (V_A, E_A, f_A) is the identity of $[A] \twoheadrightarrow \Gamma$ as an object in $\mathcal{C}^{\to}$.

We now unpack the cartesian lifts for the induced functor $p_! : \textsc{Fib}(\mathcal{C})_! \to \mathcal{C}_!$. Let $\sigma : \Delta \to \Gamma$ and $(V_A, E_A, f_A) \in \textsc{Fib}(\mathcal{C})_!$ over Γ. Set $A[\sigma] := (V_A, E_A, f_A\sigma)$, pulling back along $f_A\sigma$, we obtain the commutative outer rectangle below

$$\begin{tikzcd} {[A[\sigma]]} \arrow[r, dashed] \arrow[d, two heads] \arrow[rr, bend left] & {[A]} \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & E_A \arrow[d, two heads] \\ \Delta \arrow[r, "\sigma"'] & \Gamma \arrow[r, "f_A"'] & V_A. \end{tikzcd}$$

The universal property of the pullback on the right give us the unique map $A_\sigma : [A[\sigma]] \to [A]$. Therefore, a lift for σ is given by the evident map $A_\sigma : (V_A, E_A, f_A\sigma) \to (V_A, E_A, f_A)$. From the definition of A_σ the square

$$\begin{tikzcd}
{[A[\sigma]]} \arrow[r, "A_\sigma"] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & {[A]} \arrow[d, two heads] \\
\Delta \arrow[r, "\sigma"'] & \Gamma
\end{tikzcd}$$

is a pullback, this implies that the square as a map in $\text{FIB}(\mathcal{C})_!$ is a cartesian lift of σ for $p_!$. Most importantly, this lift is uniquely determined by the composition $f_A\sigma$. Note that the transfinite composition of fibrations play no role in the construction. We summarize the discussion above in the following:

Theorem 4.6.50. For any κ-clan $\mathcal{C}$ there exist a full split comprehension category $(\mathcal{C}', \mathcal{E}, p_!, \iota_!)$ equivalent to $(\mathcal{C}, \text{FIB}(\mathcal{C}), p, \iota)$.

Proof. We apply the previous construction, this give us $(\mathcal{C}_!, \text{FIB}(\mathcal{C})_!, p_!)$. Since the putative cartesian map is uniquely determined by the composition $f_A\sigma$, we can use a slight abuse of notation and write $A_\sigma := f_A\sigma$. Thus, if $\chi : \Xi \to \Delta$ is another map then $f(\sigma\chi) = (f\sigma)\chi$. This shows that the fibration $p_! : \text{FIB}(\mathcal{C})_! \to \mathcal{C}_!$ is split. The functor $\iota_! : \text{FIB}(\mathcal{C})_! \to \mathcal{C}^\to$ is defined as $\iota_!(V_A, E_A, f_A) := \iota([A] \twoheadrightarrow \Gamma) = [A] \twoheadrightarrow \Gamma$, similarly for arrows. The comprehension category $(\mathcal{C}_!, \text{FIB}(\mathcal{C})_!, p_!, \iota_!)$ is full, since $(\mathcal{C}, \text{FIB}(\mathcal{C}), p, \iota)$ is full. ∎

A *category with attributes* is a comprehension category $(\mathcal{C}, \mathcal{E}, p, F)$ such that p is a discrete fibration. Equivalently, a category with attributes can be defined as:

1. A category $\mathcal{C}$ with a terminal object 1,
2. A presheaf $\mathsf{Ty} : \mathcal{C}^{op} \to \mathbf{Set}$,
3. A function that assigns to each object $A \in \mathsf{Ty}(\Gamma)$, an object $\Gamma.A \in \mathcal{C}$ together with a map $\Gamma.A \to \Gamma$,
4. For each $A \in \mathsf{Ty}(\Gamma)$ and $\sigma : \Delta \to \Gamma$, a pullback square

$$\begin{tikzcd}
\sigma^*\Gamma.A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Gamma.A \arrow[d] \\
\Delta \arrow[r, "\sigma"'] & \Gamma
\end{tikzcd}$$

Corollary 4.6.51. For any κ-clan $\mathcal{C}$ there exist a category with attributes equivalent to $\mathcal{C}$.

Proof. Theorem 4.6.50 give us a full split comprehension category $(\mathcal{C}_!, \text{FIB}(\mathcal{C})_!, p_!, \iota_!)$. We take the category to be $\mathcal{C}_! = \mathcal{C}$. The additional data is given in the obvious way. Defining $\mathsf{Ty}(\Gamma) := (\text{FIB}(\mathcal{C})_!)_\Gamma$, for each $A \in \mathsf{Ty}(\Gamma)$, we get $[A] \twoheadrightarrow \Gamma$ as described above. The required pullbacks are given by the cartesian lifts of $p_!$. Furthermore, these pullbacks are computed strictly along compositions, since $p_!$ is a split fibration. ∎

Our next goal is from the category with attributes given by theorem 4.6.51 define a κ-contextual equivalent to $\mathcal{C}$. In particular, for each object $\Gamma \in \mathcal{C}$ we get a κ-contextual category $\mathcal{C}(\Gamma)$. We start with the following observation:

Definition 4.6.52. The category structure is given by the following data:

- **Objects**: For each ordinal $\mu < \kappa$, we define the set $Ob_\mu(\mathcal{C}(\Gamma))$ inductively over μ;
  - If $\mu = \lambda + 1$ then we define $Ob_\mu(\mathcal{C}(\Gamma)) := \mathsf{Ty}([A_\lambda])$. More explicitly, an object $A_\mu \in Ob_\mu(\mathcal{C}(\Gamma))$ can be represented as the sequence
    $$A_\mu \twoheadrightarrow A_\lambda \twoheadrightarrow \cdots \twoheadrightarrow \Gamma$$
    and comes with a fibration $A_\mu \twoheadrightarrow \Gamma$.
  - If μ is a limit ordinal then $Ob_\mu(\mathcal{C}(\Gamma))$ is the collection of objects of the form $A_\mu := \mathtt{Lim}_{\lambda<\mu} A_\lambda$ obtained as the transfinite composition of a sequence
    $$\cdots \twoheadrightarrow A_\lambda \twoheadrightarrow \cdots \twoheadrightarrow \Gamma.$$
    Each object comes with a fibration $A_\mu \twoheadrightarrow \Gamma$. This is given by the transfinite composition axiom of $\mathcal{C}$.
- **Morphisms**: For ordinals $\mu \leqslant \lambda < \kappa$ and objects $B_\lambda \in Ob_\lambda(\mathcal{C}(\Gamma)), A_\mu \in Ob_\mu(\mathcal{C}(\Gamma))$ we set
  $$Hom_{\mathcal{C}(\Gamma)}(B_\lambda, A_\mu) := Hom_{\mathcal{C}/\Gamma}(B_\lambda, A_\mu).$$
- The rest of the structure of $\mathcal{C}(\Gamma)$ is induced by $\mathcal{C}/\Gamma$, in particular, the transfinite composition is that of $\mathcal{C}/\Gamma$.

Before proving that this gives us a κ-contextual category, let us explain the objects of this category. Recall that for $A \in \mathsf{Ty}(\Gamma)$ means we have a diagram of the form

$$\begin{tikzcd} & E_A \arrow[d, two heads] \\ \Gamma \arrow[r, "f_A"'] & V_A. \end{tikzcd}$$

When we identify this object with $[A]$, then $\mathsf{Ty}([A])$ is the set of objects of the form

$$\begin{tikzcd} & E_B \arrow[d, two heads] \\ {[A]} \arrow[r, "(E_A)_{f_A}"'] & E_A. \end{tikzcd}$$

Each of such objects give $(V_A, f_A, E_B) \in \mathsf{Ty}(\Gamma)$ where $E_B \twoheadrightarrow V_A$ is the composition $E_B \twoheadrightarrow E_A \twoheadrightarrow V_A$. Equivalently, this is the composition $[B] \twoheadrightarrow [A] \twoheadrightarrow \Gamma$. Furthermore, if we write $\Gamma.A := [A]$, then we can rewrite this in a more familiar fashion $\Gamma.A.B \twoheadrightarrow \Gamma.A \twoheadrightarrow \Gamma$. This illustrates the general procedure for successor ordinals. A related construction appears in [41, Definition 4.3].

Lemma 4.6.53. For any κ-clan $\mathcal{C}$ and any $\Gamma \in \mathcal{C}$, the category $\mathcal{C}(\Gamma)$ is a κ-contextual category.

Each axiom can be verified more or less immediately. We start with the category with attributes we obtained in theorem 4.6.51 and the construction from theorem 4.6.52.

Proof.

1. The objects of $\mathcal{C}(\Gamma)$ have grading $Ob(\mathcal{C}(\Gamma)) = \coprod_{\mu<\kappa} Ob_\mu(\mathcal{C}(\Gamma))$ as in theorem 4.6.52. This grading determines the height of each object.
2. The terminal object is Γ.
3. Given ordinals $\mu \leqslant \lambda < \kappa$ and objects $A_\lambda, A_\mu \in \mathcal{C}(\Gamma)$, the display maps between them are the maps in $Hom_{\mathcal{C}(\Gamma)}(A_\lambda, A_\mu)$ which are also fibrations of $\mathcal{C}$. We group these maps and objects in $Dis(\mathcal{C}(\Gamma))$, which is easily seen to be a subcategory.
4. $Dis(\mathcal{C}(\Gamma))$ is closed under transfinite compositions, since $\mathcal{C}$ is itself closed under such compositions.
5. The inclusion functor $i : Dis(\mathcal{C}(\Gamma)) \hookrightarrow \mathcal{C}(\Gamma)$ preserve transfinite compositions.
6. If $A \twoheadrightarrow B$ is an arrow in $Dis(\mathcal{C}(\Gamma))$ then $B \in Ob_\mu(\mathcal{C}(\Gamma))$ and $A \in Ob_\lambda(\mathcal{C}(\Gamma))$ for some ordinals λ, μ with $\mu \leqslant \lambda$: This follows directly by definition of the objects of $\mathcal{C}(\Gamma)$
7. For any object $A \in Ob_\lambda(\mathcal{C}(\Gamma))$ and any $\mu \leqslant \lambda$ there exists a unique object $B \in Ob_\mu(\mathcal{C}(\Gamma))$ and a unique display map $A \twoheadrightarrow B$: We can easily obtain this by induction on λ and verify that the map has the correct length
8. Canonical pullbacks: This is given by the category with attributes structure on $\mathcal{C}$, as explained in theorem 4.6.51.
9. Canonical pullbacks are strictly functorial: This is exactly what theorem 4.6.51 achieves.
10. It follows from the description of objects given above.

∎

Before we can state our main result, we first need to state the appropriate notion of equivalence between κ-clans. We borrow the definitions from [39] adapted to our setting. Let $\mathcal{C}$ and $\mathcal{E}$ be two κ-coclans. We say that a functor $F : \mathcal{C} \to \mathcal{E}$ is a *morphism of κ-coclans* if

1. sends initial objects to initial objects,
2. preserves cofibrations,
3. preserves pushouts of cofibrations along any map

4. preserves transfinite compositions.

Furthermore, a morphism between κ-coclans $F : \mathcal{C} \to \mathcal{E}$ is an *equivalence of κ-coclans* if there exists another morphism of κ-coclans $G : \mathcal{E} \to \mathcal{C}$ and natural isomorphisms $GF \cong Id_{\mathcal{C}}$ and $FG \cong Id_{\mathcal{E}}$.

Similarly, $F : \mathcal{C} \to \mathcal{E}$ is a *morphism of κ-clans* simply if $F^{op} : \mathcal{C}^{op} \to \mathcal{E}^{op}$ morphism of κ-coclans, and an *equivalence of κ-clans* if $F^{op} : \mathcal{C}^{op} \to \mathcal{E}^{op}$ is an equivalence κ-coclans.

Proposition 4.6.54. A morphism of clans $F : \mathcal{C} \to \mathcal{E}$ is equivalence of clans if and only if F reflects fibrations and transfinite compositions in $Dis(\mathcal{E})$, this is; if $F(\mathtt{Lim}_\lambda A_\alpha) \twoheadrightarrow F(A_0)$ is the transfinite composition of the sequence

$$F(\mathtt{Lim}_\lambda A_\alpha) \cdots \twoheadrightarrow FA_2 \twoheadrightarrow FA_1 \twoheadrightarrow FA_0$$

then $\mathtt{Lim}_\lambda A_\alpha \twoheadrightarrow A_0$ is the transfinite composition of the sequence

$$\cdots \twoheadrightarrow A_2 \twoheadrightarrow A_1 \twoheadrightarrow A_0.$$

The equivalence of theorem 4.6.50 give us an equivalence between clans.

Corollary 4.6.55. For any κ-coclan $\mathcal{C}$ there exists a κ-contextual category equivalent to it.

Proof. Let us take the κ-clan given by $\mathcal{D} := \mathcal{C}^{op}$. We can then observe that $\mathcal{D} \cong \mathcal{D}(1)$ where $\mathcal{D}(1)$ is the κ-contextual category obtained from theorem 4.6.53. We can take the opposites again to get $\mathcal{C}$. ∎

4.7 Weak model categories

The most general setting in which we will show good homotopy theoretic properties of the language introduced in section 4.2 is for the weak model categories introduced in [29], which we will briefly recall here. In practice this extra-generality compared to Quillen model structure is not extremely useful - all the examples we will consider in section 4.3 are Quillen model structures, so it would not be unreasonable to skip the present subsection. There are two reasons we need weak model categories:

- A key construction towards the proof of the third invariance theorem in section 4.4 is in general only a weak model structure, and we need to use its language as an intermediate tool.
- Future applications to left and right semi-model structure: actual weak model structure that are not left or right semi-model structures are fairly uncommon, but the weak model categories which include both left and right semi-model structure at the same time, are considerably more common.

4.7.1 Review

Definition 4.7.1. A *weak model category* is a category $\mathcal{M}$ with three classes of maps, *cofibrations*, *fibrations* and *weak equivalences* satisfying the following conditions:

1. $\mathcal{M}$ has an initial object 0 and a terminal object 1, the identity of 0 is a cofibration, the identity of 1 is a fibration.
2. A composite of cofibrations with cofibrant domain is a cofibration. A composite of fibrations with fibrant codomain is a fibration.
3. Given two composable arrows $X \xrightarrow{f} Y \xrightarrow{g} Z$ where each one of X,Y and Z are fibrant or cofibrant, if two of f, g, $g \circ f$ are weak equivalences, then the third also is.
4. Every isomorphism between objects that are either fibrant or cofibrant is a weak equivalence.
5. Given a solid diagram:

   $$\begin{tikzcd} A \arrow[r] \arrow[d, tail, "i"] & B \arrow[d, tail, dotted, "j"] \\ C \arrow[r, dotted] & D \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

   Where i is a cofibration and A and B are cofibrant, then the pushout j exists and is a cofibration.
6. The dual of condition 5 holds for fibrations between fibrant objects.
7. Every arrow isomorphic to a fibration, cofibration, or weak equivalence is also one.
8. Every arrow from a cofibrant to a fibrant object can be factored as a cofibration followed by a trivial fibration.
9. Every arrow from a cofibrant to a fibrant object can be factored as a trivial cofibration followed by a fibration.
10. Given a solid square:

    $$\begin{tikzcd} A \arrow[r] \arrow[d, tail, "i"] & X \arrow[d, two heads, "p"] \\ B \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

    Where A and B are cofibrant, i is a cofibration, X and Y are fibrant, p is a fibration and either p or i is a weak equivalence, then there exists a dotted map that makes the diagram to commute.

Remark 4.7.2. In theorem 4.7.1 we use the usual conventions: a *cofibrant object* is an object such that the unique map $0 \to X$ is a cofibration, and a *fibrant object* is an object such that the unique map $X \to 1$ is a fibration. A trivial (co)fibration is a map which is both an equivalence and a (co)fibration. We will also use the term *core cofibrations* to mean "cofibration between cofibrant objects" and *core fibrations* to mean "fibration between fibrant objects".

Remark 4.7.3. It is crucial to observe that theorem 4.7.1 only involve the core cofibrations, core fibrations and weak equivalences between objects that are either fibrant or cofibrant. By that we mean that if given $\mathcal{M}$ a category with these three class of maps, then ($\mathcal{M}$, cofibrations, fibrations, weak equivalences) is a weak model structure if and only if ($\mathcal{M}$, core cofibrations, core fibrations, weak equivalences between objects that are either fibrant or cofibrant) is a model structure.

For this reason, we generally consider that only core cofibrations, core fibrations and weak equivalence between objects that are either fibrant or cofibrant are to be treated as relevant notions. Nothing we will do here depends on the three class of maps outside these restrictions. In [29] it was even considered that the words cofibrations, fibrations and weak equivalences to mean "core cofibrations", "core fibrations" and "weak equivalences between fibrant or cofibrant objects".

Remark 4.7.4. The definition of weak model structure in [29] is different from theorem 4.7.1, but it is equivalent. It is stated without reference to the class of weak equivalence and using the notion of (weak relative) path object and cylinder object. It is easy to show that a weak model structure in the sense of theorem 4.7.1 is a weak model structure in the sense of [29] by constructing the cylinder and path objects as factorization of the codiagonal and diagonal maps (see 4.7.5 below). Conversely, it is shown in [29] that given a weak model structure, it admits a (unique[6]) class of weak equivalences such that all conditions of theorem 4.7.1 are satisfied.

It is shown in [29] that most of the basic theory of Quillen model categories carries over to weak model categories, with only some additional care taken - mostly replacing objects by fibrant and cofibrant replacement of objects before applying the usual construction. The main significant difference is that the homotopy category (defined in terms of homotopy class of maps between bifibrant objects as we will recall below) is no longer equivalent to $\mathcal{M}[W^{-1}]$ - the localization of $\mathcal{M}$ at weak equivalence, but only to $\mathcal{M}^{\text{cof}\vee\text{fib}}[W^{-1}]$ the localization the full subcategory of objects that are either fibrant or cofibrant at the weak equivalences. The problem is that the axioms of a weak model category allows us to take a fibrant replacement of a cofibrant object C as a (trivial cofibration/fibration) factorization of $C \to 1$, and similarly we can take a cofibrant replacement of a fibrant objects, but there is no way to do similar replacement with an object which is neither fibrant nor cofibrant.

We now quickly go over some aspects of the construction of the homotopy category of a weak model category, the result mentioned below are all proved in section 2.1 and 2.2 of [29].

Construction 4.7.5. If X is a bifibrant object (i.e. fibrant and cofibrant), we can form a *cylinder objects* IX for X as a (cofibration, trivial fibration) factorization:

$$X \coprod X \hookrightarrow IX \overset{\sim}{\twoheadrightarrow} X$$

---

[6] Keeping in mind theorem 4.7.3. Only the class of weak equivalence between fibrant or cofibrant objects is uniquely defined, outside of this, there are no restriction whatsoever on weak equivalence from theorem 4.7.1.

and a path objects for X as a (trivial cofibration, fibration) factorization

$$X \xhookrightarrow{\sim} PX \twoheadrightarrow X \times X.$$

Given a pair of maps $f, g : X \rightrightarrows Y$ between bifibrant objects, we say they are homotopic if there is a dotted map h making the diagram below commutative:

$$\begin{tikzcd}
X \arrow[d] \arrow[dr, "f"] & \\
IX \arrow[r, dotted, "h"] & Y \\
X \arrow[u] \arrow[ur, "g"'] &
\end{tikzcd}$$

or equivalently a map h

$$\begin{tikzcd}
& Y \\
X \arrow[ur, "g"] \arrow[r, dotted, "h"] \arrow[dr, "f"'] & PY \arrow[u] \arrow[d] \\
& Y.
\end{tikzcd}$$

This is an equivalence relation, and the homotopy category $\mathrm{Ho}(\mathcal{M})$ of $\mathcal{M}$ can be defined as the category of bifibrant objects with homotopy class of maps between them. Moreover, this category is equivalent to the formal localization $\mathcal{M}^{\mathrm{cof} \vee \mathrm{fib}}[W^{-1}]$.

Construction 4.7.6. Note that if an object $C \in \mathcal{M}$ is only cofibrant and not fibrant we cannot define a cylinder object in the same was as above, as the factorization axiom does not allow us to factor the maps $X \coprod X \to X$ if X is not fibrant. In place of this, we can consider a fibrant replacement $X \xhookrightarrow{\sim} X^{\mathrm{Fib}} \twoheadrightarrow 1$, and then form a factorization:

$$\begin{tikzcd}
X \coprod X \arrow[r, hook] \arrow[d, "\nabla"] & IX \arrow[d, two heads, "\sim"] \\
X \arrow[r, hook, "\sim"] & X^{\mathrm{Fib}}.
\end{tikzcd}$$

This object IX, and more generally any object fitting into a diagram:

$$\begin{tikzcd}
X \coprod X \arrow[r, hook] \arrow[d, "\nabla"] & IX \arrow[d, "\sim"] \\
X \arrow[r, hook, "\sim"] & DX
\end{tikzcd}$$

is called a weak cylinder object. Dually, if Y is fibrant we define a weak path object of Y as any object PY that fits into a diagram:

$$\begin{tikzcd}
TX \arrow[r, "\sim"] \arrow[d, two heads, "\sim"] & PX \arrow[d, two heads] \\
X \arrow[r, "\Delta"] & X \times X
\end{tikzcd}$$

We can then show that for a pair of maps $X \rightrightarrows Y$ from a cofibrant object X to a fibrant object Y the following are equivalent:

- f is homotopic to g in terms of a weak cylinder object for X.
- f is homotopic to g in terms of a weak path objects for Y.
- f and g are equal in the localization $\mathcal{M}^{\mathrm{cof} \vee \mathrm{fib}}[W^{-1}]$.

Moreover, any arrow $X \to Y$ in the localization $\mathcal{M}^{\mathrm{cof} \vee \mathrm{fib}}[W^{-1}]$ comes from an arrow $X \to Y$ in $\mathcal{M}$.

4.7.2 Weak Reedy model structure

Before doing all the constructions, we need to set up the formalism needed for such. In this section, we study Reedy weak model categories. These are, as the name suggests, the counterpart of Reedy model categories. Most of the proofs are straightforward adaptation of the classical ones, so they are omitted.

Definition 4.7.7. A *Reedy category* is a category R together with two wide subcategories R_+ and R_- and a functor $deg : R \to \alpha$, where α is an ordinal, such that:

1. For every $a \to b \in R_+$ a non-identity arrow, $\deg(a) < \deg(b)$.
2. For every $a \to b \in R_-$ a non-identity arrow, $\deg(b) < \deg(a)$.
3. Every arrow in R factors uniquely as an arrow in R_- followed by an arrow in R_+.

When the subcategory R_- consists of identity arrows only, then R is called *direct category.* Similarly, when the subcategory R_+ consists of identity arrows only, then R is called *inverse category*

Let R be a Reedy category and $\mathcal{M}$ be a weak model category. Consider $\mathcal{M}^R$ the category of R-shaped diagram in $\mathcal{M}$. Given $X : R \to \mathcal{M}$ such a diagram and $r \in R$ any object. The *latching object* at r is the colimit (if it exists)

$$L_r X := \mathtt{Colim}_{s \in (R_+/r) - \{Id_r\}} X_s.$$

Dually, the *matching object* at r is the limit (if it exists)

$$M_r X := \mathtt{Lim}_{s \in (r/R_-) - \{Id_r\}} X_s.$$

Definition 4.7.8. A map $f : X \to Y$ in $\mathcal{M}^R$ is said to be a *(trivial) Reedy cofibration* at $r \in R$ if the colimit $L_rY \sqcup_{L_rX} X_r$ exists and the induced dotted map in the diagram below

$$\begin{tikzcd}
L_rX \arrow[r] \arrow[d] & X_r \arrow[d] \arrow[ddr, bend left] & \\
L_rY \arrow[r] \arrow[drr, bend right] & L_rY \sqcup_{L_rX} X_r \arrow[dr, dashed] \arrow[ul, phantom, "\ulcorner", very near start] & \\
 & & Y_r
\end{tikzcd}$$

is a (trivial) cofibration in $\mathcal{M}$.

Dually, $f : X \to Y$ in $\mathcal{M}^R$ is said to be a *(trivial) Reedy fibration* at $r \in R$ if the limit $M_rX \times_{M_rY} Y_r$ exists and the induced dotted map in the diagram below

$$\begin{tikzcd}
X_r \arrow[dr, dashed] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
 & M_rX \times_{M_rY} Y_r \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y_r \arrow[d] \\
 & M_rX \arrow[r] & M_rY
\end{tikzcd}$$

exists and is a (trivial) fibration in $\mathcal{M}$.

A map is said to be a (trivial) Reedy (co)fibration if it is one at each $r \in R$.

Remark 4.7.9. We want to clarify that in theorem 4.7.8 the colimit $L_rY \sqcup_{L_rX} X_r$ is considered as a single colimit not as a pushout using the object L_rX and L_rY. It is possible that $L_rY \sqcup_{L_rX} X_r$ exists without the colimit L_rY or L_rX existing. Explicitly, it is the colimits of all the X_i for $i \in R^+/r$ and of the Y_i for $i \in R^+/r - \{id_r\}$. with all the maps coming from the functoriality in i and the natural map $X_i \to Y_i$. We apply the same logic to the limit $M_rX \times_{M_rY} Y_r$.

Definition 4.7.10. A Reedy category is said to be *locally finite* if for any object $X \in R$ the categories (R_+/X) and (R_-/X) are finite.

It is a classical result that for any Quillen model category $\mathcal{M}$ and a Reedy category R that the category of functors $\mathcal{M}^R$ carries a model structure in which the weak equivalences are the level-wise weak equivalences, the (trivial) (co)fibrations are precisely the Reedy (trivial) (co)fibrations. The same result can be obtained if we simply assume that the base category carries a weak model structure.

Theorem 4.7.11. Assume that $\mathcal{M}$ is a weak model category and that R is a locally finite Reedy category. Then there is a weak model structure on $\mathcal{M}^R$ such that a map $f : X \to Y$ it is:

1. A *weak equivalence* if and only if $f_r : X_r \to Y_r$ is a weak equivalence for all $r \in R$.
2. An *(trivial) cofibration* if it is a (trivial) Reedy cofibration.
3. An *(trivial) fibration* if it is a (trivial) Reedy fibration.

Remark 4.7.12. When the Reedy category is directed, this model structure coincides with the projective weak model structure. It is straightforward to define this last weak model category. In this weak model, the weak equivalences and the fibrations are the level-wise weak equivalences and fibrations respectively. Similarly, when the Reedy category is an inverse category, then the Reedy weak model structure is Quillen equivalent to the injective model structure. In this other case, weak equivalences and cofibrations are given level-wise.

We now prove the theorem:

Lemma 4.7.13. Let I be a direct category and $X : I \to \mathcal{M}$ be a diagram. Let $U \subset V \subset I$ be two sieves[7] of I, such that $V - U$ has a finite number of objects. Assume that the colimit

$$X(U) \coloneqq \mathtt{Colim}_{u \in U} X(u)$$

exists and is cofibrant, and that for each $v \in V - U$. The latching object $L_v X$ exists and is cofibrant, and the map $L_v X \to X(v)$ is a cofibration. Then $X(V)$ exists and the comparison map $X(U) \to X(V)$ is a cofibration. If $L_v X \to X(v)$ is actually a trivial cofibration for every $v \in V - U$, then $X(U) \to X(V)$ is a trivial cofibration.

Proof. This is immediate by induction on the number of objects of $V - U$. If it only has one object, then $X(U) \to X(V)$ can be seen to be a pushout of the core cofibration $L_v X \to X_v$ to the cofibrant object $X(U)$. If $V - U$ has several objects, we iterate this process once for each object of $V - U$. ∎

Corollary 4.7.14. Let R be a locally finite Reedy category, $X : R \to \mathcal{M}$ be a diagram and let $k \in R$ an object. Assume that X is Reedy cofibrant at every r such that $\deg(r) < \deg(k)$. Then the latching object $L_k(X)$ exists and is cofibrant.

Proof. Using a proof by induction on $\deg(x)$, we can freely assume that all the latching object $L_r(X)$ are cofibrant for all r such that $\deg(r) < \deg(x)$. We can then just apply the theorem 4.7.13 to the finite direct category $I = R^+/x$ and $U = \varnothing$, $V = I$. ∎

Corollary 4.7.15. Let I be a finite direct category, and let $X : I \to \mathcal{M}$ be a Reedy cofibrant diagram and $U \subset I$ be a sieve. Then $\mathtt{Colim}_I X$ and $\mathtt{Colim}_U X$ exists, are cofibrant and the obvious comparison map $\mathtt{Colim}_U X \to \mathtt{Colim}_I X$ is a cofibration.

If furthermore the latching map $L_r X \to X(r)$ is a trivial cofibration for each $r \in I - U$, then the map $\mathtt{Colim}_U X \to \mathtt{Colim}_I X$ is a trivial cofibrations.

Proof. By theorem 4.7.14 all the latching objects of X are cofibrant, so we can simply apply theorem 4.7.13 and conclude. ∎

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[7] That is subcategories with the properties that if there is an arrow $x \to x'$ and $x' \in V$ then $x \in V$.

Corollary 4.7.16. Let R be a locally finite Reedy category.

- Any core (trivial) Reedy cofibration $X \to Y$ in $\mathcal{M}^R$ is in particular a levelwise (trivial) cofibration. That is, the map $X(r) \to Y(r)$ are (trivial) cofibrations for any $r \in R$.
- A map $X \to Y$ in $\mathcal{M}^R$ which is both a core Reedy cofibration and a level-wise weak equivalence is a trivial Reedy cofibration.

Dually, the same is true for fibrations and trivial fibrations.

Proof. As both statement only depends on the restriction to the subcategory R^+, we can freely assume that R is a (locally finite) direct category. In both cases, we consider the natural transformation $X \to Y$ as a diagram $T : R \times \{0 < 1\} \to \mathcal{M}$. We then observe that the latching map of T at an object $(r, 0)$ is just $L_r X \to X$, and the latching map of T at $(r, 1)$ is

$$L_r Y \sqcup_{L_r X} X(r) \to Y(r)$$

Hence the assumption that $X \to Y$ is a core Reedy cofibration translate into the fact that T is Reedy cofibrant. For any object $r \in R$, the composite $R \times \{0 < 1\}/(r, 1) \to R \times \{0 < 1\} \to \mathcal{M}$ is immediately seen to be Reedy cofibrant as well, and we can then apply theorem 4.7.15 to the Sieve $U = R/r \times \{0\}$ to conclude that $X(r) \to Y(r)$ is a cofibration.

If $X \to Y$ is further assumed to be trivial, then the latching map of T at all objects of the form $(r, 1)$ are trivial, and hence using the “trivial” case of theorem 4.7.15 we conclude that $X(r) \to Y(r)$ is trivial.

If instead we assume that $X(r) \to Y(r)$ is a weak equivalence for all r, then we proceed by strong induction on $\deg(r)$. Assume that we already know that at all k such that $\deg(k) < \deg(r)$.

If $\deg(r) = 0$, then the latching map is just $X(r) \to Y(r)$ itself, so it is a trivial cofibration as it is a cofibration and a weak equivalence. Assume now that we already know that all the latching maps

$$L_r Y \sqcup_{L_r X} X(r) \to Y(r)$$

are trivial cofibrations for any r such that $\deg(r) < \deg(k)$. We can then deduce by the same argument as above that the map $L_k(X) \to L_k(Y)$ is a core trivial cofibration, which shows that the map $X(r) \to L_r Y \sqcup_{L_r X} X(r)$ is a trivial cofibration, hence an equivalence, and hence by 2-out-of-3 for equivalences, the map $L_r Y \sqcup_{L_r X} X(r) \to Y(r)$, is both an equivalence and a core cofibration, so it is a weak equivalence. ∎

Note that we have also proved that:

Lemma 4.7.17. Let R be a locally finite Reedy category, and $i : X \to Y$ be a core Reedy cofibration in $\mathcal{M}^R$. Then the domain of the latching map $L_r Y \sqcup_{L_r X} X(r)$ is cofibrant.

Proof. At the beginning of the proof of theorem 4.7.16 we observed that it could be written as a latching object $L_{(r,1)}T$ of a cofibrant Reedy diagram T. Hence, the result follows from theorem 4.7.14. ∎

Proposition 4.7.18. For any locally finite Reedy category R, in $\mathcal{M}^R$, the composite of two Reedy core cofibrations is a Reedy core cofibrations.

Proof. We use a strategy very similar to the proof of theorem 4.7.16. Here again, the result only depends on the restriction to R^+ so we can freely assume that R is a direct category. Let $X \to Y \to Z$ be two composable Reedy core cofibrations in $\mathcal{M}^R$. We consider this as a diagram $T : R \times \{0 < 1 < 2\} \to \mathcal{M}$. As in the proof of theorem 4.7.16. We observe that the latching map at an element of the form $(r, 0)$ is the latching map $L_r X \to X$ of X hence is a cofibration as X is Reedy cofibrant. The latching map at an element $(r, 1)$ is the map

$$L_r Y \sqcup_{L_r X} X(r) \to Y(r)$$

which is a cofibration as $X \to Y$ is assumed to be a Reedy cofibration. And finally, the latching map at $(r, 2)$ is the map

$$L_r Z \sqcup_{L_r Y} Y(r) \to Z(r)$$

which is also a cofibration. So this diagram $R \times \{0 < 1 < 2\} \to \mathcal{M}$ is Reedy cofibrant. It immediately follows that, for any $r \in R$ the composite $R \times \{0 < 1 < 2\}/(r, 2) \to R^- \times \{0 < 1 < 2\} \to \mathcal{M}$ is a Reedy cofibrant diagram. Hence, applying theorem 4.7.15, we can deduce that the map

$$\mathtt{Colim}_U T \to Z(r)$$

is a cofibration, where $U \subset R \times \{0 < 1 < 2\}/(r, 2)$ is the sieve containing all the objects except $(r, 1)$ and $(r, 2)$. But this map can be seen to be exactly

$$L_r Z \sqcup_{L_r X} X(r) \to Z(r)$$

by theorem 4.7.12. This conclude the proof, as this can be applied to any object $r \in R$. ∎

Proposition 4.7.19. Consider a cospan $Y \leftarrow X \to Z$ of diagram $R \to \mathcal{M}$, such that X, Y, Z are all Reedy cofibrant and the arrow $X \to Y$ is a Reedy cofibration. Then the (level-wise) pushout $Y \sqcup_X Z$ exists in $\mathcal{M}^R$ and the natural transformation $Z \to Y \sqcup_X Z$ is a Reedy cofibration.

Proof. It follows from theorem 4.7.16 that for each $r \in R$ the three objects in the diagram $Y(r) \leftarrow X(r) \to Z(r)$ are cofibrant and the map $X(r) \to Y(r)$ is a cofibration, so the levelwise pushout $Y(r) \sqcup_{X(r)} Z(r)$ exists and by general category theoretic results is functorial in r and is a pushout in the category of diagrams $\mathcal{M}^R$. We only need to check that the map $Z(r) \to Y(r) \sqcup_{X(r)} Z(r)$ is a Reedy cofibration. For this observe that as colimits commutes with colimits we have:

$$L_r(Y \sqcup_X Z) = \mathtt{Colim}_{r' \to r \in R^+} Y(r') \sqcup_{X(r')} Z(r') = L_r Y \sqcup_{L_r X} L_r Z$$

So that in the latching map

$$L_r(Y \sqcup_X Z) \sqcup_{L_r Z} Z \to Y \sqcup_X Z$$

the domain can be identified with

$$(L_rY \sqcup_{L_rX} L_rZ) \sqcup_{L_rZ} Z = L_rY \sqcup_{L_rX} Z = (L_rY \sqcup_{L_rX} X) \sqcup_X Z$$

so the latching map is

$$(L_rY \sqcup_{L_rX} X) \sqcup_X Z \to Y \sqcup_X Z$$

which is a pushout of the latching map $L_rY \sqcup_{L_rX} X \to Y$, which is itself a core cofibration as $X \to Y$ is a core Reedy cofibration. Hence, this concludes the proof. $\blacksquare$

We are now ready to prove theorem 4.7.11:

Proof. We go over all the conditions of theorem 4.7.1. The validity of conditions 1, 3, 7 and 4 is trivial. Condition 2 is theorem 4.7.18 together with its dual. Condition 5 is theorem 4.7.19, and condition 6 is the dual statement.

The proof of conditions 10 is essentially the same as the proof for ordinary model categories, as for example in Chapter 15 of [32] or in Chapter 5.2 of [33]. The key step in the proof is that in order to construct a diagonal lift in a square:

$$\begin{tikzcd}
A \arrow[r] \arrow[d, tail, "i"] & X \arrow[d, two heads, "p"] \\
B \arrow[r] \arrow[ur, dotted] & Y
\end{tikzcd}$$

where say i is a core cofibration and p is a core fibration, one of them being a (level-wise) weak equivalence. Then we proceed by induction as in the usual proof, at each step we need to produce a diagonal lift in a square of the form

$$\begin{tikzcd}
A(r) \sqcup_{L_rA} L_r(B) \arrow[r] \arrow[d, tail] & X(r) \arrow[d, two heads] \\
B(r) \arrow[r] \arrow[ur, dotted] & Y(r) \times_{M_rY} M_rX
\end{tikzcd}$$

Now by theorem 4.7.17 (and its dual) the object $A(r) \sqcup_{L_rA} L_r(B)$ is cofibrant and $Y(r) \times_{M_rY} M_rX$ is fibrant, by definition of Reedy cofibration and fibration, the left vertical map is a cofibration and the right vertical is a fibration, and if one of i or p (say i) is a weak equivalence, then the second point of theorem 4.7.16 show that the left vertical map is a trivial cofibration, hence the square admit a diagonal lift, which concludes the proof.

The proof of condition 8 and (dually of condition 9), also follows very closely the classical proof, as in Chapter 15 of [32] or in Chapter 5.2 of [33]. Given $A \to X$ a map from a Reedy cofibrant diagram to a Reedy fibrant diagram that we want to factor as a core trivial Reedy cofibration followed and a core Reedy fibration, $A \to B \to X$. We proceed by induction to construct the diagram, the object $B(r)$, and the maps $A(r) \to B(r) \to X(r)$ gradually by induction on the degree of r. Following the classical proof, at each stage, we need to construct a factorization of a map in $\mathcal{M}$:

$$A(r) \sqcup_{L_rA} L_rB \to X(r) \times_{M_rX} M_rB$$

as a trivial cofibration followed by a fibration. But as observed above, the domain is cofibrant and the target is fibrant, so this is indeed possible in $\mathcal{M}$. The case of condition 9 is done in the exact same way, but factoring the map above as a cofibration followed by a trivial fibration.

∎

Conclusions

In this thesis, we have compiled research that uses type theory to study a wide class of higher structures, *i.e.,* $(\infty, 1)$-categories and weak model structures.

The important aspect to remark is the special treatment of the equality, different to that of classical logic. On the one hand, we have the intentional equality from Martin-Löf type theory (MLTT.) In this setting, our objects can be "equal" in different ways; via definition or via the equality of the type theory. MLTT enhanced with Voevodsky's Univalence Axiom give us homotopy type theory (HoTT), in which we have an additional way of thinking about the equality via the type of equivalences. Moreover, the equality type in MLTT has the structure of a weak ω-category. All the extra features of this type theoretic equality are not shared with the set-theoretic foundations equality.

Indeed, chapter 2 ([5]) and chapter 3 ([6]) use univalent foundations to study some aspects of $(\infty, 1)$-categories. The type theory we use, due to Riehl and Shulman, is known as simplicial homotopy type theory (sHoTT). In chapter 2, we defined limits and colimits of a functor between $(\infty, 1)$-categories. Then, we proved basic fundamental results about these constructions. Furthermore, we also compute the limit of a diagram of ∞-groupoids in a (proposed) $(\infty, 1)$-category of ∞-groupoids.

Throughout chapter 3, we studied a class of functors called Conduché fibrations or exponentiable functors. Our main result is the characterization of such functors. Overall, when we use sHoTT to reason about $(\infty, 1)$-categories, we see massive simplifications in the arguments compared to the existing literature. The simplifications are not only at a technical level, but also in the space for these arguments. It is evident that sHoTT either simplifies or avoids completely the combinatorics of any particular model presenting $(\infty, 1)$-categories. It is important to remark that we have used the original formulation of sHoTT, and this is enough for our initial goals. However, we have also seen that in order to further develop the theory of $(\infty, 1)$-categories and its applications, we will certainly need to use, at the very least, an extension of sHoTT.

In chapter 4 ([7]), we presented the findings on a new relation between type theory and model categories. Furthermore, the constructions allow us to include even more general structures: weak model categories. From each weak model category we can construct an infinitary generalized algebraic theory. We construct a logic whose formulas have variables in the contexts of the generalized algebraic theory. In this logic, we also take a non-traditional approach to the equality between objects. However, unlike the equality of HoTT or ZFC, this new equality is not postulated, but rather emergent and only dictated by the weak model category in question. This is encoded in the cofibrations of the weak model category.

We proved that the logic is compatible with the (weak) model structure; this is what we have called “invariance theorems.” In particular, the 4^{th} invariance theorem establishes that the language we construct is itself invariant under Quillen equivalences. We see this work as a step towards model independence. In addition, it opens the door to new relations between logic and homotopy theory.

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