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A Study of Errors Involved in Seismic Analysis of Secondary Systems

by

Shrikrishna M. Kulkarni

A THESIS
SUBMITTED UNDER THE SUPERVISION OF
PROF. S. F. NG
IN PARTIAL FULFILMENT OF THE
REQUIREMENTS FOR THE DEGREE OF
MASTER OF APPLIED SCIENCE
IN
CIVIL ENGINEERING

Department of Civil Engineering
University of Ottawa, Ottawa, Canada. K1N 6N5

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ISBN 0-315-60563-4

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To my parents

Acknowledgements

The author wishes to express his deepest gratitude to Professor S.F. Ng for his supervision, continuous encouragement and generous support throughout the course of this study.

Special appreciation is expressed to Mr. Ketan R. Shah and Mr. Chandrashaker, both graduate students at Carleton University, Ottawa for numerous helpful and enlightening discussions throughout this study. Discussions with Mr. Shah helped consolidate the concept of the Error-Parameter. Mr. Chandrashaker's constructive criticism of the work on chapters 4 and 5 is appreciated. The author is indebted to both of them for their help.

Financial assistance provided by the National Science and Engineering Research Council of Canada under grant no. PEN-1PO-C99 is gratefully acknowledged. Various teaching assistantships provided by the Department of Civil Engineering are also gratefully acknowledged.

Abstract

The present study deals with the sources of error in seismic modal analysis of primary-secondary coupled systems using the 'combination of modal properties' approach. Four sources of error are studied: non-classical damping, evaluation of modal properties of the combined system, combination of modal responses and response contribution from the high frequency modes (which is normally neglected in a modal superposition analysis).

An approximate procedure for analysis of non-classically damped systems is to perform normal modal analysis after ignoring the off-diagonal terms in the transformed damping matrix. A study of error involved in this procedure of diagonalization is made with reference to a two-degree-of-freedom shear building system. It is shown that harmonic response of a non-classically damped system provides a valuable insight into its response to a general load. A quantity called the 'error-parameter' is defined as the area under a curve obtained by plotting the error in diagonalization versus frequency of harmonic excitation. The error parameter is shown to have a good correlation with the error in diagonalization for a system when it is subjected to a random load. For a non-harmonic load, the error involved in diagonalization depends on its frequency content. Results of this study indicate that the dependence of the error on the applied load increases with the severity of non-classical nature. The error-parameter is used to study the pattern of variation of the error in diagonalization when the dynamic properties of a system are varied. The error-parameter is shown to be an effective tool for comparison of performance of modal damping estimation procedures. A comparison of three such methods is reported.

It is shown that use of approximate mode shapes obtained by perturbation is an attempt to transform the equations of motion using a transformation matrix that is different from the exact-mode-shape matrix. The error involved in an attempted modal superposition analysis for such a case mainly stems from the neglect of off-diagonal terms in the transformed mass, stiffness and damping matrices. It is suggested that Clough and Mojtahedi's [14] method can be effectively used in the case of primary-secondary coupled systems to eliminate the error involved in using approximate mode-shapes, at least for the modes considered in the analysis.

The phenomenon of closely spaced modes in primary-secondary coupled systems and consequent correlation between modal responses is discussed. Correlation between the responses of high-frequency modes is also discussed. Need for a combination rule that considers the types of correlation is emphasized. It is demonstrated that the response from high frequency modes becomes important in systems in which the variation of stiffness and mass properties is large. (Primary-secondary coupled systems form a typical case.)

An improvement is suggested in Clough and Mojtahedi's [14]method for analysis of non-classically damped systems. The improved method takes into account the response from high-frequency modes. Numerical examples indicate that errors in the response of a (classically or non-classically damped) system obtained using the proposed technique are within acceptable limits even if the response from the higher modes is important. It is suggested that the proposed technique be used as a cost-effective alternative to direct time integration.

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Glossary

A = amplitude of harmonic ground excitation;

c = damping coefficient of a 1-DOF system;

c_j = a damping coefficient for a shear building;

$[C]$ = damping matrix of a system;

$[C^*]$ = transformed damping matrix;

$[D]$ = diagonal matrix obtained by ignoring off-diagonal terms in the transformed damping matrix;

D_j = diagonal element of $[D]$;

D_{jl} = off-diagonal element of $[D]$ (which is equated to zero);

$[D_p]$ = parent matrix obtained from $[D]$;

$\{F\}$ = dynamic force matrix;

F_j = effective dynamic force acting on a system due to ground excitation;

$\{F^*\}$ = transformed force matrix;

i = the imaginary number defined by $\sqrt{-1}$;

k = stiffness of a 1-DOF system;

$[K]$ = stiffness matrix of a system;

k_j = an inter-storey stiffness in a shear building;

$[K']$ = weighted stiffness matrix in the stiffness weighted damping method;

$[K^*]$ = transformed stiffness matrix;

m = mass of a 1-DOF system;

m_j = mass of a floor in a shear building;

$[M]$ = mass matrix of a system;

$[M']$ = weighted mass matrix in the mass weighted damping method;

$[M^*]$ = transformed mass matrix;

N = total number of degrees of freedom;

r = ratio of excitation frequency to natural frequency of a 1-DOF system;

R = a response (such as displacement, elastic force) at a particular DOF of a system;

R_j = modal maximum response in the mode j;

S_{Dj} = spectral displacement corresponding to ω_j and ξ_j;

t = time;

$\ddot{u}_g$ = horizontal ground acceleration;

$\{U_b\}$ = base excitation vector;

$\{U_{br}\}$ = component of $\{U_b\}$ formed by contributions from the higher modes;

y = horizontal displacement, relative to ground, of a 1-DOF system;

y_j = horizontal displacement, relative to ground, of a DOF;

$\{y_j\}$ = displacement response in the j th mode of a system;

$\dot{y}_j$ = first time derivative of y_j;

$\ddot{y}_j$ = second time derivative of y_j;

$\bar{\omega}$ = circular frequency of harmonic ground excitation considered;

Y_j' = amplitude of harmonic response of the DOF j;

Y_j = amplitude of harmonic response of the DOF j per unit amplitude of harmonic ground motion;

$\{z\}$ = vector of modal coordinates;

z_j = element of $\{z\}$;

$\dot{z}_j$ = first time derivative of z_j;

$\ddot{z}_j$ = second time derivative of z_j;

β_j = damping ratio in an individual (fixed base) 1-DOF system defined as $c_j/2\sqrt{k_j m_j}$;

γ_j = j th modal participation factor;

$\{\Gamma\}$ = a vector listing modal participation factors;

θ = angle of phase for a 1-DOF system under harmonic excitation;

λ_j = an eigenvalue;

ξ_j = a modal damping ratio of a system;

ϕ_{jl} = modal displacement of j th DOF in l th mode;

$\{\phi_l\}$ = mode shape vector for mode l;

$[\Phi]$ = mode shape matrix;

ω_j = circular natural frequency of vibration;

$[\Omega]$ = a diagonal matrix that lists ω_j^2;

Subscripts

e	denotes exact quantities
a	denotes approximate quantities
max	denotes maximum quantities

Superscripts

$'$	quantities relating the lower modes (frequency $\leq$ 33 Hz)
$''$	quantities relating the higher modes (frequency $>$ 33 Hz)
$*$	matrices obtained by transformation

Acronyms

CSM	closely spaced modes
CQC	complete quadric combination
DOF	degree of freedom
DSC	double sum combination
MDOF	multi-degree-of-freedom
PSDF	power spectral density function
SDOF	single-degree-of-freedom
SRSS	square root of sum of squares
ZPA	zero period acceleration
1-DOF	1 degree of freedom
2-DOF	2 degree of freedom

Chapter 1

Introduction

Seismic analysis of secondary systems has been of great interest in recent years. A secondary system is any important dynamic system that is attached to the main structure. Two most common examples of secondary systems are light equipment in a building and piping in a nuclear reactor. In these examples, the building or the nuclear reactor is called the "Primary system." The combined dynamic system containing the primary and the secondary system is called "Primary-Secondary (P-S) Coupled System." Abbreviations "P-system" and "S-system" are often used in place of the terms "Primary system" and "Secondary system."

1.1 Necessity of Seismic Analysis of Secondary Systems

Damage caused by an earthquake to a structural system is usually manifold. Structural damage constitutes only a part of the total damage. In fact, low intensity earthquakes may not have any significant structural damage associated with them. However, there may be a significant monetary loss and a risk of loss of life due to the damage caused to sensitive equipment such as computers in building structures or delicate equipment in hospital buildings. Piping in nuclear reactors is a good example of a secondary system that can pose a threat to human life (due to the possibility of radiation leak), should the system undergo structural failure during an earthquake. A typical structural system has several secondary systems attached to it. However, consideration of secondary systems becomes important only if their failure during an earthquake is a potential cause of significant monetary loss or loss of life or both. In residential and commercial buildings, the existing secondary systems are usually not of vital importance. However, industrial buildings and other 'common' buildings such as hospitals or laboratories may typically have one or more secondary systems that deserve attention in seismic design. In certain special structures such as nuclear reactors, piping and vital equipment need very careful consideration. The need for analysis of secondary systems was felt most strongly by engineers and consultants dealing with seismic design of nuclear reactors. However, now it is being recognized on an ongoing basis that the available techniques for seismic analysis of secondary systems should be used to analyse all important secondary systems. With the current knowledge of secondary systems analysis, it is possible to analyse them independent of the primary systems. Also, the additional cost of analysis is not very high and is certainly justified by the expected protection against loss of human life and monetary loss.

Damage caused by an earthquake to equipment in ordinary structures (residential buildings, laboratories, hospitals) is often accepted without further thought by common people and even structural engineers. The structural engineering community is not very aware of the need for secondary system seismic analysis. In the geographic zones which are described as "mild earthquake zones," seismic analysis of the primary system itself is considered expensive. Secondary systems analysis is usually not done either because of ignorance of the necessity of such an analysis or because it implies a more costly analysis. In strong seismic zones, secondary systems analysis is usually done only for very important S-systems. It should however be emphasized that rational design of important secondary systems is necessary and certainly cost-justified. The primary duty of an earthquake engineer is to arrive at a design that would minimize the damage caused by an earthquake. It should be understood by the structural engineering community that performing seismic design of secondary systems is a responsibility, since it is one way of reducing the damage caused by an earthquake. With the available methods of analysis, the extra computational effort required is not very large. Moreover, current computer software packages can easily be modified to incorporate analysis of secondary systems. The methods available today treat P-S coupled systems as ordinary dynamic systems with certain special properties. Once the special properties are accounted for, the analysis follows the same procedure as standard seismic analysis of ordinary structures. In summary, seismic analysis of secondary systems is necessary, cost-effective, relatively simple and must be carried out for all important secondary systems.

1.2 Difficulties in the Analysis of the Combined System

A P-S coupled system can certainly be considered as a single dynamic system. Specifically, one can easily construct important matrices such as mass, stiffness and damping matrices for such a system. Currently available techniques for seismic analysis of ordinary structures can certainly be applied to this system. The results of such an analysis, barring numerical inaccuracies, are the "exact" results and will frequently be referred to in the thesis. Using these results, both the primary and the secondary system can now be designed for the maximum seismic forces. This approach, however, is rarely followed because of the following reasons.

1. The combined dynamic model can become very large. The cost of step-by-step integration or eigensolution can become prohibitive for such a model. This is especially true for secondary systems with large number of degrees of freedom such as piping systems.

2. In general, the two systems are made up of different materials and have stiffness and inertial properties of different orders of magnitude. Usually, the secondary system is much lighter than the primary. Secondary system stiffness can be either much smaller than or comparable to the primary system

stiffness. Hence, the stiffness as well as the mass matrix of the combined system consists of elements of vastly different magnitudes. This may lead to numerical inaccuracies in the analysis.

3. Primary and secondary systems are generally designed by different contractors and primary system design usually precedes secondary system design. The engineer dealing with primary system design may not have access to secondary system properties which are necessary to construct a combined model to find out the response of the combined system.

4. The designer of the secondary system may wish to consider several alternative configurations of the secondary system before arriving at the optimum configuration. It is impractical and prohibitively expensive to analyse the combined models for each different design configuration of the secondary system.

In view of these facts, a solution using the combined dynamic model is rarely attempted. However, such a solution can effectively be used, for the purpose of research, as the "exact solution." Obviously, the usual assumptions made in structural dynamics such as the presence of viscous damping and other assumptions involved in the basic "spring-mass-damper" model are present in this analysis. The analysis is exact only to the extent that no further approximations (due to the fact that the system being considered is a P-S coupled system) are introduced.

1.3 A Solution Technique

Throughout the development of analysis techniques for secondary systems, attempt was made to separate ("decouple") the secondary system analysis from that of the primary system. Several approaches will be discussed later in the thesis. A popular approach available today treats the P-S coupled system as a single dynamic system. However, no attempt is made either to perform a step-by-step time integration of the combined system or to solve an eigenproblem for combined system. Instead, modal properties of the combined system are obtained from those of the individual systems by applying certain special techniques such as perturbation techniques. The basic idea is to now consider the combined system as any other dynamic system and apply either modal superposition or response spectrum techniques to it. This approach will be referred to as "Combination of Modal Properties Approach"[1] in this thesis. Steps involved in this approach can be summarized as follows.

1. Stiffness and mass matrices of the individual systems are constructed and a standard eigensolution is performed to obtain modal properties of the primary system and the secondary system separately.
2. Modal properties of the combined system are obtained from the individual modal properties using perturbation techniques.
3. Damping properties of the combined system are obtained from those of the individual systems.
4. Once modal frequencies, damping ratios and mode shapes of the combined system are known, modal superposition or response spectrum procedure may then be followed in order to evaluate the maximum seismic responses.

This approach is certainly not free of error. In fact, there is some error introduced at every stage in the analysis. The possible sources of error are described below.

1. With the availability of digital computers and efficient and accurate eigensolvers, it is relatively easy to obtain modal properties of individual P and S systems without any significant error. The first step will therefore be considered to be free of error for the purpose of all further discussion.
2. Perturbation techniques are approximate methods. Modal properties of the combined system obtained using perturbation techniques (or other approximate techniques) and the response obtained from these modal properties

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[1] Note that this is not a popular term

(using either the modal superposition or the response spectrum method) is bound to be erroneous. This error will subsequently be referred to as "Error in Combined System Modal Properties".

3. Damping is one the most significant sources of error. In the above discussion, it was implicitly assumed that the combined system is classically damped. This is often not valid since primary and secondary systems generally have substantially different damping properties (different energy absorption rates). However, it is not possible to use the standard modal superposition (or response spectrum) techniques without making the assumption of presence of classical damping. The error involved in such an assumption will subsequently be referred to as "Error in Classical Damping Assumption."

4. Both in the response spectrum and the modal superposition method, it is usually assumed that a significant part of the total response is contained in the first few modes. However, in P-S coupled systems, it is observed that higher modes contribute significantly to the total response of the structure. This is called "Higher Mode Effect." The error associated with the use of a truncated set of modes (first few modes) will be called "Error due to Higher Mode Effect".

5. Response spectrum method being extremely popular, its use for the case of P-S coupled systems must be studied in detail. The most widely known method of combining maximum modal responses is the "Square Root of Sum of Squares (SRSS)" method. Though this method can certainly be used justifiably for common structures such as tall buildings, an attempt to apply it to P-S coupled systems often leads to a significant error. This is due to a phenomenon known as "Closely Spaced Modes (CSMs)" which is commonly observed in P-S coupled systems.

1.4 Objectives of the Thesis

The primary objective of this thesis is to investigate the major sources of error involved in seismic analysis of secondary systems using the "Combination of Modal Properties" approach. Four major sources of error (modal properties, damping, modal combination and higher mode effect) are studied in sufficient detail with the aim of obtaining a clearer understanding of the nature of these errors. Possibility of minimizing these errors is studied and recommendations are made to that effect. Based on the study of nature of the errors involved, new analysis techniques are proposed in some cases, to minimize some of the errors.

1.5 Scope and Limitations

The study of the errors involved is limited to a general discussion about the source of each error, some explanatory notes and the author's contribution to the current knowledge. The discussion about the errors is not always of analytical nature and is by no means complete. The attempt is to bring out the important sources of errors, discuss them in brief and recommend (existing or proposed) solution techniques to minimize the errors. The study presented in this thesis being of a basic nature, several simplifying assumptions have been made to reduce the numerical complexity of the example problems taken and to make the arguments involved easier to understand. However, generality of the problem is believed to have been maintained. The assumptions and simplifications[2] are as below.

1. Ground motion consists of a horizontal excitation in only one direction.

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[2] An implicit assumption in all secondary systems analysis is that the secondary system is elastically attached to the primary system. In other words, the forces of interaction between the two systems are well known stress-resultants such as shear forces and bending moments. If the secondary system is not attached to the primary in this way (e.g. if the secondary system simply 'rests' on the primary by virtue of frictional forces), then the standard techniques of seismic analysis of secondary systems cannot be applied to it. The discussion of such a system is beyond the scope of this thesis.

2. Excitation is uniform for all supports.

3. Soil-Structure interaction is absent.

4. Damping is of a viscous nature.

5. For numerical examples, shear building models are used. In other words, horizontal displacement is considered as the only degree of freedom at each nodal point. (Figs. 5.4 and 5.5).

In this thesis, elastic analysis is used. Though structures cannot be expected to remain elastic during a major earthquake, elastic analysis is still popular today because of its simplicity and the prospect of reducing computation using techniques such as modal superposition. A simple but rational method of seismic design is to evaluate the maximum forces on a structure using elastic analysis and then design individual members using the ductility factor approach [15]. According to the ductility factor approach, design forces on a member are reduced, based on its ductility, and the member is 'allowed' to undergo inelastic deformation. A non-linear seismic analysis of a P-S coupled system is prohibitively expensive.

1. The number of degrees of freedom involved is very large.

2. If different secondary systems are present, different material models (hysteretic models) need to be constructed.

3. Many secondary systems are so delicate and important that one might not wish to allow any inelastic deformation in them. In this case, an expensive inelastic solution can be considered to be under-utilized

1.6 Organization of the Thesis

A comprehensive review of literature on different aspects of seismic analysis of secondary systems is presented in Chapter 2.

Chapter 3 deals with the phenomenon of non-classical damping in primary-secondary coupled systems. A two-degree-of-freedom shear building system is studied in detail in order to gain insight into the nature of approximation involved in neglecting the presence of non-classical damping.

Chapter 4 deals with the error involved in using approximate mode-shapes for a mode-superposition analysis of primary-secondary coupled systems. A new method that can minimize this error is suggested.

Importance of a suitable rule of modal combination is discussed in Chapter 5. Importance of higher modes in primary-secondary coupled systems is also discussed.

An improvement over an existing method for analysis of non-classically damped systems is presented in Chapter 6. The new method is shown to yield acceptable results.

Conclusions of the present work are summarized in Chapter 7. Also, suggestions for further development of the ideas presented in this thesis are given.

Listings of computer programs developed for the purpose of this thesis, as well as sample data and output files are supplied in the appendix.

Chapter 2

Literature Review

As mentioned previously, it is impractical to perform seismic analysis using the combined dynamic model of the primary and the secondary. For this reason, attempts were made by investigators in this area to separate the analysis of the secondary system from that of the primary system. This 'decoupling' implies that the primary system can be analyzed as if the secondary was non-existent.

A pertinent question is that of validity of the assumption of decoupling. Decoupling would be valid if interaction between the two systems is small. It may be intuitively obvious that interaction between the two systems would be significant if a secondary system frequency matches one the frequencies of the primary. This is in fact so. The phenomenon in which the two frequencies match is known as tuning. In terms of modal properties, interaction can be observed in the form of change in modal properties (say, frequencies) of the primary because of the addition of the secondary. In general, for light secondary systems, the change in modal properties can be expected to be small. However, in tuned cases, this is not so. Thus, criteria for decoupling can be arrived at on the basis of the expected changes in the frequencies of the primary. Some earliest efforts in this direction were made by Pickel [58, 1972] and Lin and Liu [49, 1975]. These methods were approximate and lacked sufficient justification. This subject has since then received considerable attention and many contributions have been made. Important recent contributions are: Hadjian [34, 1977], Chen [13, 1980], Gupta [33, 1984] and Hadjian [36, 1986]. The current trend is to limit the change in response in addition to the change in the frequencies of the primary system.

2.1 The Floor Response Spectrum Approach

Once the assumption of decoupling is made, the primary system can be analyzed as if the secondary system was non-existent. A popular approach is to use the response of the primary as the input motion for the secondary and analyze the secondary as if it rests on a ground that has the same motion as that of the supporting floor. This model of the secondary system is known as the 'cascaded model'. (Other models such as the 'lumped mass' model are available but will not be discussed here). In a cascaded secondary system, the floor motion is usually obtained in the form of response spectrum of the floor; such a spectrum is popularly known as 'floor response spectrum', or in short, as 'FRS'.

A direct but not elegent method of obtaining the FRS is to perform a time-history analysis of the primary from which motion of the connecting floor(s) is obtained as a function of time. This, in turn, is used to construct a response spectrum. Quite often, input excitation of the primary is supplied in the form of a response spectrum instead of a time-history. In this case, a time-history that is consistent with the response spectrum of the ground is obtained and is used to analyse the primary. The major problem in this method is that one can find several time-histories that are consistent with a given response spectrum, each of which may evoke different response in the primary. Thus, an ensemble of time-histories consistent with the ground spectrum must be used. Since a single time-history analysis itself is costly, this approach could be prohibitive in terms of cost.

For this reason, several different alternatives were explored. Some of the earliest were those by Biggs [8, 1971] and Kapur and Shao [45, 1973]. Biggs' method, which consisted in using some magnification curves, was an important discovery;

it opened up an entirely different approach to the problem, which was followed later on by other investigators. Biggs first demonstrated that dynamic properties of the structure could be used to obtain the FRS directly from the ground response spectrum, without going through an intermediate time-history analysis. Several contributions have been made since then; many of them based on random vibration approach. Some useful references are: Singh [68, 1975], Peters et al. [57, 1977], Der Kiureghian et al. [19, 1983] and [18, 1983], Gupta and Jaw [31, 1986], Suarez and Singh [75, 1987] and [76, 1987]. In [76] Suarez and Singh employed a modal synthesis approach.

Demerits of the FRS approach are as follows:

1. It considers only a one-way interaction between the two systems. This is certainly conservative for the design of secondary systems. However, it might lead to overconservatism in certain cases.

2. For multiply supported secondary systems, the treatment becomes too complex; sometimes much simplification is necessary to avoid tediousness in computation.

3. For some tuned systems, decoupling may not be permitted; depending on the mass and the frequency ratios between the two systems. This means that the FRS approach is not applicable.

Certain merits of the FRS approach are:

1. Once the FRS is obtained, it permits usage of the most popular response spectrum method for the secondary system.

2. In some cases, the secondary system data is not known at the time of the design of the primary. In such cases, a method such as the combination of modal properties method is at a disadvantage. However, FRS method does not encounter any problem since using this method, secondary system design can be done at any suitable time after completion of the design of the primary system.

Despite its disadvantages, the FRS approach is still very popular in the industry even today, perhaps due to historical reasons: This method has been in existence for a relatively long time now and has received considerable attention from researchers in past and thus is well known to the consulting engineers. This is unlike the combination of modal properties approach, which received significant attention only in the latter part of the last decade.

A review of the contributions made to the area of combination of modal properties appears below.

2.2 The "Combination of Modal Properties" Approach

One of the earliest studies on the design of equipment mounted on structures was made by Newmark [52, 1966]. His aim was to arrive at shock response spectra for equipment. He made the assumption of smallness of the mass ratio and 'cascading' nature of the equipment. His study was of a general nature and addressed to several problems such as multiple-support excitation for multiply supported secondary systems and the determination (estimation) of the dynamic properties of the equipment. He considered several possible variations of the structure and the equipment mass and stiffness and suggested design procedures in each case. For the case of flexible structure (SDOF) and small mass ratio, he considered the problem of tuning. Newmark suggested some empirical rules to determine equipment shock spectra, taking into account possible tuning of the equipment. His methods were conservative and were intended for design.

Later, in 1972, Newmark [53] studied the SDOF-SDOF primary-secondary system further. He also studied the problem of tuning in an MDOF-SDOF system. He illustrated the frequency shifts and changes in mode shapes of the primary due to the addition of an SDOF equipment. This concept was later used by other researchers to provide a formal perturbation approach. Newmark assumed that the mode shapes of the primary remain practically unchanged by the addition of the equipment except for the addition of modal displacements corresponding to the equipment. However, he did not follow a rigorous perturbation procedure. For an MDOF primary system, Newmark used the concept of effective mass ratio.

Amin et al. [2, 1971] solved the general problem of multiply supported secondary systems, though in an approximate manner. They proposed a response spectrum procedure which made use of the individual system modal properties. However, this method involves construction of a response spectrum for the secondary system in an approximate manner. Using their approach, it is possible to obtain maximum modal response of the secondary system to individual modal excitations of the primary. Amin et al. ignored the dynamic interaction effects between the primary and the secondary. They studied these effects and concluded that for mass ratio less than 0.01, interaction was not significant.

The turning point in the study of modal properties of the combined system was the method by Peters et al. [57, 1977] for determination of floor response spectra based on 'coupled' system modal properties. They considered the equipment as a massless SDOF oscillator, which means that the modal properties of the structure remain unchanged, except that appropriate modal displacement terms corresponding to the equipment must be included. Also, it is necessary to include a new modal vector for the new mode generated. This method ignores interaction and is not reliable for the tuned case. Nonetheless, the procedure to obtain modal properties was later used by other researchers.

Interesting study about interaction between a tuned equipment and the structure was given by Kelly and Sackman [46, 1978]. They supplied a very simple but accurate amplification factor for evaluating the peak response of a tuned equipment. For non-tuned systems, they recommended using a conventional approach like the FRS approach. Sackman and Kelly made very significant contributions to the area of secondary systems response. They, for the first time, rationally tackled the problem of tuned and slightly detuned equipment [64, 1979].

Sackman et al. [63, 1983] presented a perturbation approach to evaluate modal properties of a combined equipment-structure system, with an SDOF equipment. They recommended using ground floor spectra to evaluate maximum responses of the combined system. In their approach, non-classical damping nature of the combined system was neglected. Der Kiureghian et al. later [19, 1983] used these properties to evaluate response of the combined system to a stochastic input.

Der Kiureghian et al. [18, 1983] presented a new method for analyzing multiply supported secondary systems. They introduced the concept of cross-cross floor spectra to account for correlation between support motions. They employed perturbation methods to evaluate mode shapes.

Hernried and Sackman [39, 1984] solved a more general problem of MDOF multiply supported secondary systems (with light damping) by employing a perturbation approach.

Gupta [23, 1984] developed a perturbation technique to evaluate modal properties of a combined system containing an MDOF primary and a multiply supported MDOF secondary system. He accounted for the non-classical nature and the static constraints introduced by the secondary system.

Igusa and Der Kiureghian [42, 1985] conducted a thorough investigation of dynamic behaviour of a 2-DOF equipment-structure system. They developed mathematical expressions and criteria to measure three important characteristics: tuning, interaction and non-classical damping. These criteria provide valuable insight into the behaviour of this system. In an accompanying paper [41, 1985] they developed a perturbation approach for eigenproperties of a coupled system consisting of an MDOF primary and an MDOF multiply supported secondary system. They accounted for all important properties: tuning, interaction, non-classical damping and spatial coupling (the fact that the secondary system receives multiple-support excitation)

As an improvement over their earlier method, Gupta and Jaw [29, 1986] presented a perturbation scheme to evaluate complex modal properties of equipment-structure systems. Later [30, 1986], they summarised their earlier contributions to the topic of response spectrum analysis of multiply supported MDOF secondary sytems. They formalized their procedure for such an analysis and presented a computational algorithm for the purpose of a computer code.

Applicability of component mode synthesis to the problem of evaluation of combined system modal properties was shown by Villaverde [83, 1986] [84, 1986].

Singh and Suarez [74, 1986] gave a systematic and formal perturbation scheme for eigenproperties of an MDOF-SDOF primary-secondary system. They used higher order terms and increased the applicability range for the perturbation procedure. (e.g. their method is applicable for greater mass ratios).

2.3 Combination of Modal Properties: Associated Difficulties

Attempt to analyze the combined system using modal properties of individual systems leads to many difficulties. One of the most important sources of error is the fact that the combined system is usually non-classically damped. The P-system and the S-system generally have different damping properties. This leads to a damping matrix that is not diagonalizable by the normal mode shapes of the combined system.

The combined system frequently has what are known as closely spaced modes. This is observed when some of the frequencies of the secondary are close to some of the frequencies of the primary. If a response spectrum approach is attempted, the conventional SRSS rule cannot be used; instead, an appropriate rule that takes into account the closeness in frequencies must be used.

A typical coupled system, due to the large variation in stiffness throughout the system, is likely to receive a non-negligible contribution from high frequency modes. Hence, in any mode-superposition technique (either modal time history or response spectrum), one must include the response contribution from higher mode in a rational way.

In view of these facts, the current knowhow about non-classical damping, combination rules for modal maxima as well as response contribution from higher modes must be used in order to do an accurate analysis of a P-S coupled system. In what follows, a review of available literature in these areas will be attempted.

2.4 Non-classical Damping

The question of existence of classical modes in damped structures was studied by Lord Rayleigh [59, 1945]. He showed that there exists a class of damped systems which possess classical normal modes. His original concept of obtaining a classical damping matrix as a linear combination of stiffness and mass matrices is known today as "Rayleigh damping". Caughey and O'Kelly [11, 1965] derived the necessary and sufficient conditions for a damped system to possess classical normal modes. They supplied a general expression for obtaining a classical damping matrix, which yields Rayleigh's expression as a specific case.

Once classical damping assumption is made, the equations of motion get uncoupled. Each modal equation is then similar to the equation of motion of an SDOF system with a certain percentage critical damping which is usually called "modal damping". In practice, modal damping ratios can be estimated from past experience. For the purpose of time integration, one must obtain the complete damping matrix from these modal damping ratios. Wilson and Penzien [88, 1972] provided two procedures for numerical evaluation of such a matrix.

Systems that do not possess an orthogonal damping matrix are called non-classically damped systems. Classical normal modes (or undamped modes)do not exist in these systems. Foss [21, 1958] developed an ingenious method by which the equations of motion of such a system may be uncoupled. He also demonstrated that orthogonality exists in a certain sense between the mode shapes of a non-classically damped system.

Foss's method involves computation in the complex domain and an eigenvalue problem of a size that is double the size of the (real valued) eigenvalue problem in case of a classically damped system. This is not very attractive to the practicing engineer. Hence other possible techniques (perhaps approximate) were explored. The problem of non-classical damping is very severe in two important types of dynamic systems: soil-structure interaction systems and primary-secondary coupled systems. However, much of the early work aimed at finding a suitable alternative was done by investigators working in the area of soil-structure interaction.

A well known approximate procedure is to use normal-mode analysis by ignoring the off-diagonal terms in the transformed damping matrix. Diagonal terms on the transformed damping matrix provide a set of modal damping ratios. Analysis is then carried out exactly as in case of classically damped systems.

Efforts have also been made to estimate modal damping ratios of the system from those of its individual components, in an approximate manner [9] [60]. Once the modal damping ratios of the combined system are estimated this way, classical normal modes are used to effect a modal superposition solution. Obviously, this procedure ignores any off-diagonal terms that would have appeared by a transformation of the damping matrix by eigenvectors. In addition to this error, there is error involved in the estimation of modal damping.

Biggs and Roesset et al. [60] proposed rules to estimate the modal damping ratios of a non-classically damped system from those of the individual components. There is no rational basis for these rules, which are now known to yield diversified results.

Some other methods for obtaining a diagonal damping matrix (more involved) were reported and compared by Thomson et al. [77, 1974]. They studied frequency response of certain systems with diagonal damping matrix of each system being computed by three different procedures: 1. Ignoring off diagonal terms in the transformed damping matrix. 2. Using an optimization algorithm which minimizes the mean square error of the frequency response. 3. Matching the peak amplitudes of coupled and uncoupled systems. Thomson et al. rank the three methods as 2, 3, 1 on the basis of their performance.

Tsai [79, 1974] proposed a method for estimating the modal damping coefficients in a foundation-structure interaction system. His approach was to match the exact and classical mode solutions of the transfer function at a certain (damping-sensitive) location at all frequencies (in the range of interest).

Assuming that a nonclassical damping matrix is almost diagonalized by the mode shape matrix (i.e. that the off-diagonal terms are small), Cronin [17, 1976] suggested an improvement over the method of ignoring off-diagonal terms for the case of harmonic excitation. According to Cronin, the method of ignoring off-diagonal terms performs very poorly near resonance, especially if the frequencies are closely spaced. His approximate method was shown by him to yield acceptable results in these cases.

Interesting facts about modal coupling in lightly damped structures were described by Hasselman [38, 1976]. He concluded that the degree of coupling between two classical normal modes depends not only on the ratio of the off-diagonal terms to diagonal terms of the transformed damping matrix, but also on the percentage critical damping in the two modes and the difference in the frequencies of the modes. If percentage damping is high then the frequency separation must also be high to justify decoupling. This is in agreement with Cronin's [17, 1976] conclusion. Hasselman also concluded that if adequate frequency separation exists, then the equations of motion will be decoupled for all practical purposes. He proposed certain criteria which when satisfied would ensure that uncoupling does not lead to significant errors.

Warburton and Soni [86, 1977] studied the errors involved in ignoring off-diagonal terms. They proposed certain criteria, which ensure the errors due to diagonalization to be within acceptable limits.

Yan [89, 1979] studied the problem of composite modal damping in detail. He suggested a simple equation for modal damping in case of lightly damped modes.

This expression yields the values of modal damping ratios identical to those obtained from the transformed damping matrix, where transformation is done using undamped mode shapes.

Balendra et al. [5, 1982] proposed a procedure to estimate modal dampings for torsionally coupled buildings on elastic foundation. Their approach was similar to that of Tsai [79, 1974].

Studies reported above and numerous other studies indicate that the method of using classical normal modes by ignoring off-diagonal terms in the transformed damping matrix can be quite erroneous and hence is not reliable.

An entirely different approach was taken by Clough and Mojtahedi [14, 1976] who proposed a simple but interesting and useful technique for seismic analysis of non-classically damped systems. They suggested direct integration of the transformed coupled equations of motion where the transformation matrix consists of only a truncated set of modal shape vectors. This procedure fully accounts for modal interaction between the modes considered, but ignores any damping coupling between this set of modes and the higher modes. The time required for integration is certainly more than that in the case of uncoupled modes, but is much less than that for a complete step-by-step integration of the original equations of motion.

Duncan and Taylor [20, 1979] investigated the effect of damping coupling in a non-classically damped system. They concluded that there may be a significant error associated with truncation of modes in such a system because of the damping coupling between the modes considered in the analysis and the modes that are not considered (i.e., the higher modes). Thus, a method such as Clough and Mojtahedi's [14] must be used with caution; Duncan and Taylor recommended inclusion of a much larger number of modes in the analysis than normally considered necessary for classically damped systems.

Following is a brief review of analysis procedures suggested by different investigators using damped modes shapes. These procedures are certainly accurate but are rather expensive and more involved than the procedures mentioned earlier. The former type of procedures involves solution of an eigenproblem to obtain damped (complex) frequencies and mode shapes. A modal superposition procedure is then formulated using these modal quantities.

Itoh [44, 1973] suggested the use of Fourier transform analysis for solving the modal equations of motion. He advised using the Fast Fourier Transform (FFT) procedure in numerical analysis. Singh [70, 1980] used a random vibration approach to formulate a response spectrum procedure for nonclassically damped systems. He modelled earthquake motion as a stationary random process.

A comparison of four different procedures for modal time history analysis of nonclassically damped systems was made by Traill-Nash [78, 1981]. Of the four methods considered: 1. Mode displacement using normal modes 2. Force summation using normal modes 3. Mode displacement using damped modes. 4. Force summation using damped modes, he recommends the fourth method, i.e. force summation using damped modes; this conclusion may be considered to be an expected one.

Igusa et al. [43, 1984] developed a modal decomposition method for nonclassically damped systems subjected to a stationary excitation. Their expressions are more simple and direct compared to those of Singh [70, 1980]. Also, their method gives a more complete description of the response than Singh's method. They provided closed form solutions for the special case of white noise input.

A different modal superposition procedure was suggested by Gupta and Jaw [28, 1986]. Their method replaces a complex modes shape into two real modal vectors and modal equations are integrated in a way similar to those for classically damped structures. In a companion paper, they extended this procedure into a response spectrum procedure. [26, 1986]. Their approach was deterministic.

At about the same time, Singh and Ghafory-Ashtiany [72, 1986] proposed a new modal superposition procedure using time integration for individual modes. They used complex eigenproperties but the final step-by-step algorithm was given in terms of real quantities. They presented algorithms for both: mode displacement and mode acceleration approaches and recommended mode acceleration approach as the better alternative of the two.

An overview of the modal analysis procedure for non-classically damped systems along with certain physical interpretations and useful discussion was given by Veletsos and Ventura [81, 1986]

Borino and Muscolino [10, 1986] developed a new method for modal analysis of both classically and non-classically damped structures. They named this method as "dynamic correction method". This method was shown by them to yield better results than both the mode displacement and the mode acceleration approach.

Villaverde [85, 1988] extended Rosenblueth's [61] original rule of modal combination to systems with non-classical damping, within the frame-work of complex modal superposition.

2.5 Combination of Modal Responses

Several attempts at finding a suitable rule to combine the maximum modal responses in buildings have been made in the history of earthquake engineering. Most of the suggested rules are based on random vibration theory. A necessary assumption in all such methods is that earthquakes can be represented by a stationary random process. Though earthquake excitation is essentially non-stationary, this assumption is known to give acceptable results.

The most widely known 'square root of sum of squares' (or SRSS) method for combination of maximum modal responses was originally proposed by Emilio Rosenblueth in his Ph.D. thesis at the University of Illinois at Urbana-Champaign and was later published by Goodman et al. [22] in 1955. The details of derivation and a list of assumptions can be found from this reference. Goodman et al. observe that this rule would normally provide sufficiently accurate estimate of the maximum response except when the ratio of the fundamental period of a structure to the earthquake duration is large. This method is still extremely popular and is widely used in the industry even today.

However, this method has a number of limitations. It cannot take into account, for example, the correlation between two modes having close frequencies. Penzien [56, 1969] studied buildings with a setback. He concluded that SRSS method overestimated the response in comparison with a time-history solution when the setback was in resonance with the building.

One of the earliest attempts to formulate a suitable rule for modal combination for structures with closely spaced frequencies was made by Rosenblueth and Elorduy [61]. They treated earthquakes as stationary gaussian white noise. Assuming classical modes and small damping, they arrived at a combination rule which is well known today as the double sum method. This method accounts for correlation between modal responses by introducing modal correlation coefficients. For modes with widely spaced frequencies, the correlation coefficients are very small and can be neglected. In this case, the double sum method degenerates into the SRSS method. The principle disadvantage of this method was that it required a quantity called the duration of white noise which is generally not known if the earthquake input is specified in the form of a response spectrum. Ronsenblueth and Elorduy observed that the SRSS method overestimated the torques in single storey buildings with torsional responses (due to the formation of closely spaced modes.)

Ruiz [62, 1970] proposed a similar method, again based on random vibration concepts. He supplied charts for obtaining modal correlation coefficients. This was perhaps an unattractive aspect, since simple equations are always preferable to the design engineer.

Amin and Ang [1, 1971] developed another modal combination rule based on random vibration approach. However, their approach required the knowledge of the power spectral density function of the input motion.

Singh et al. [67, 1973] compared the performance of the SRSS and the double sum method against time history solution. They concluded that SRSS method consistently underestimated the responses of buildings with CSMs. This was in contrast with the findings of Penzien [56, 1969] and Rosenblueth and Elorduy [61].

A useful rule for modal combination was given by Singh and Chu [71, 1976]. Unlike some earlier approaches, Singh's method required only the assumption of stationarity: the assumption of white noise input was not necessary. According to Singh and Mehta [73, 1983] this method provides exact mathematical results (of at least the mean square response) if all modes are included in the analysis. This method requires only the knowledge of the response spectrum and dynamic properties of the structure. No further information about the earthquake is necessary. (except for the assumption of stationarity).

A discussion on existing modal combination procedures was given by Chen [12, 1977]. He pointed out that SRSS method can give either an overestimate or an underestimate of the response depending on the signs of cross-coupling terms. If the contribution of cross-coupling terms is negative, SRSS would yield conservative results. Otherwise, it would underestimate the response. He discussed the question of validity of the SRSS method for the case of CSMs. As a matter of surprise, he concluded that the SRSS method may be suitable for such a case. He observed: the built-in conservatism in using a smooth design spectrum more or less balances any underestimation done by SRSS. He suggested a statistical procedure to verify this hypothesis. However no formal proof of this hypothesis was given by him.

Kelly and Sackman [47, 1980] studied the behaviour of closely spaced modes and supplied a very interesting understanding of modal interaction in case of CSMs. They pointed out that significant energy transfer exists between different parts of a P-S coupled system in case of tuning. They concluded that SRSS method underestimated the response for undamped or very lightly damped systems, but it could yield extremely conservative results for heavily damped systems An important new rule for combination of modal maxima was given by Wilson et al. [87, 1981]. The method is known as Complete Quadric Combination (or CQC) method. Wilson et al. assumed that the duration of the earthquake is long compared to the periods of the structure and that the earthquake spectrum is smooth over a wide range of frequencies (both of which are usually valid). Based on their observation of performance of SRSS and CQC methods, they suggested that use of SRSS be immediately discontinued in favour of a better method like CQC. Similar to Singh and Chu's method [71, 1976] CQC method does not require any knowledge of the properties of the earthquake excitation. (such as duration of white noise or the PSDF)

At about the same time, Hadjian [35, 1981] studied the problem of modal combination for high frequency modes. Modes with frequencies beyond a certain cut-off frequency have an amplification factor of almost unity. Thus all modes above the cutoff frequency essentially move in phase with the ground and are completely correlated with each other. Modes in the intermediate range also have a certain degree of correlation between them due to their rigidity. Hadjian studied modal interaction in some simple systems in detail and proposed a method which separates what are called rigid and flexible components of response and adds them separately. Hadjian's method faces numerical problems in case of very flexible structures. He suggests the use of conventional rules of modal combination in these cases. For simplicity, Hadjian considered only well spaced modes. Using this method, response from higher modes (above cut-off frequency) can be obtained without actually computing them.

Gupta and Cordero [24, 1981] proposed a simple but interesting method of combination. The method is heuristic and the coefficients suggested are empirically based. This method accounts for correlation between modes due to closeness of frequency as well as due to rigidity. One of the most important features of this method is that it clearly demonstrates the fact that correlation due to rigidity exists even below the cutoff frequency.

Singh and Mehta [73, 1983] presented a new method of modal combination which accounted for modal correlation due to closeness of frequency as well as rigidity. To avoid calculation of higher modes, they use what is known as mode acceleration method.

A comparison of modal combination methods, especially from the point of view of higher modes was made by Gupta and Chen [25, 1984]. They compared SRSS, Hadjian's [35, 1981]method, Gupta's [24, 1981] and another method which they refer to as Kennedy's method. According to Gupta and Chen, Kennedy's method is a simple rule in which all modal responses in modes with frequencies below the cutoff frequency are combined by SRSS (if CSMs do not exist) and those in modes with frequencies beyond the cutoff frequency are combined algebraically. They rank these methods in order of merit as 1. Gupta's method 2. Hadjian's method 3. Kennedy's method and 4. SRSS method.

In 1986, Gupta and Jaw [27] applied Gupta's method of modal combination to a complex 3-D piping network and showed that this method estimates maximum responses very accurately for this system.

Maison et al. [51, 1983] compared four methods of modal combination in order to arrive at some recommendations on their relative performance. They compared SRSS, CQC, double sum and absolute sum methods. Absolute sum method assumes that all modes reach their maxima simultaneously and thus this method provides an upperbound for the maximum response obtained by the response spectrum method. Maison et al concluded that both the double sum method and the CQC method provided good response estimates even for cases with CSMs. SRSS method provided good response prediction for cases with evenly spaced modes. They pointed out that if the duration of white noise in the double sum method is set to infinity, this method becomes virtually identical to CQC. They also gave suggestions regarding the period range for which the SRSS, DSC and CQC rules are most appropriate. Based on their observations, they recommended both the double sum and the CQC method.

2.6 Consideration of Higher Modes

Maddox [50, 1975], suggested a method for taking account of response in higher modes. The 'dynamic truncation method' which he suggested considers the response from first few modes for which inertia loads are significant but uses an approximate method to estimate response from high-frequency modes. (He used the idea of quasi-static response) This method needed computation of all higher modes, which is not an attractive feature.

Hansteen and Bell [37, 1979] pointed out that the number of modes which should be included in the analysis depends on both the frequency content and the distribution of loading. They emphasized that the effect of higher modes may be approximated by a quasi-static analysis only if the predominant frequencies of the load are much lower than the (higher) frequencies of the structure. They describe a technique for computing the static contribution from higher modes. This method does not require computation of modal properties of higher modes, unlike that of Maddox.

A systematic approach and a general procedure for inclusion of higher modes in modal superposition as well as response spectrum analysis was given by Vashi [80] in 1981. Using this method, only the modes below the cut-off frequency (the frequency at which acceleration amplification becomes almost unity, usually specified as 33 Hz) need be calculated. The higher modes are included by a static analysis.

Salmonte [65, 1982] studied the modal contributions of response for structures with local high-frequency modes. He recommended a method which he called 'residual rigid mode' which accounts for high frequency mode contribution to the total response in such structures. According to him, setting up of a cutoff frequency a-priori must be checked carefully and enough number of modes must be included even if some higher modes may have frequencies greater than the cut-off frequency. He cautioned that his residual mode method was less reliable for displacements than for forces. Gupta [27, 1986] applied the residual response concept to a complex piping system. He concluded that consideration of both correlation between modal responses and higher mode response is essential to get a good estimate of the response of piping systems.

The concept of including the effect of higher modes by a static analysis is also known as the mode acceleration method. Thus, all the above methods are some variants of the mode acceleration method. The traditional method of modal superposition analysis, which considers only a truncated set of modes, but does not include the effect of higher modes is known as mode displacement method.

Cornwell et al. [16, 1983] studied the accuracy of mode displacement and mode acceleration methods in terms of the effect of damping levels and excitation frequencies. They concluded that mode-acceleration method performs better than mode-displacement method in almost all cases. Their study shows that mode-acceleration method requires fewer modes compared to the mode-displacement method, for the same accuracy level and that mode acceleration method has better features as far as response of damped structures is considered.

Leger and Wilson [48, 1988] studied two computational variants of the classical mode-acceleration method. They recommend that the pseudo static component be obtained from an expansion of the flexibility matrix in terms of the set of eigenvectors considered in the analysis. They also gave numerous recommendations regarding computer application of the mode-acceleration method.

Chapter 3

Non-classical Damping Considerations

Modal superposition has been a very popular method for seismic analysis of structures. This is due to the simplicity of the method and due to the fact that by using this method, response of a structure can usually be obtained using only the first few modes of the structure, which implies a great economy in solution. This method is well understood by the practicing engineer since it provides a physical understanding of the response in terms of modal responses.

Though applicability of normal modal superposition procedure to 'regular' structures such as tall buildings or chimneys is well known, its applicability to certain other important systems such as soil-structure interaction systems or primary-secondary coupled systems is greatly limited due to a phenomenon which is known as 'non-proportional damping' or more correctly as 'non-classical damping', which prevents use of normal modes for a standard modal superposition solution for such structures.

In view of the popularity of modal superposition, many attempts were made to investigate the possibility of using normal mode shapes in the case of non-classically damped structures, perhaps in an approximate way. A study of seismic analysis of secondary systems must consider the phenomenon of non-classical damping since it is one of the most important sources of error. This chapter is devoted to the problem of non-classical damping. A brief discussion of several techniques used to tackle the problem is provided. A popular solution technique known as "ignoring off-diagonal terms" is discussed in detail. Contribution made by the author to this area is reported.

3.1 Introduction

3.1.1 Normal Modal Superposition

The standard normal modal superposition method basically consists of an orthogonal transformation of the equations of motion using normal modal shapes of a dynamic system. Normal modal shapes are obtained by an eigensolution for the system with damping neglected and hence are also referred to as "undamped" modal shapes.

Equations of motion of a general MDOF dynamic system can be represented in the form of a matrix equation as:

$$[M]\{\ddot{y}\} + [C]\{\dot{y}\} + [K]\{y\} = \{F\} \tag{3.1}$$

where, $[M]$, $[C]$ and $[K]$ are the mass, damping and stiffness matrices respectively of the system,$\{F\}$ is a vector specifying the applied dynamic loads and $\{y\}$ is a vector of generalized displacements. A dot over a quantity denotes differentiation with respect to time.

In standard modal superposition, we use the transformation

$$\{y\} = [\Phi]\{z\} \tag{3.2}$$

Substituting into Eq. 3.1 and premultiplying by $[\Phi]^T$, we get

$$[\Phi]^T[M][\Phi]\{\ddot{z}\} + [\Phi]^T[C][\Phi]\{\dot{z}\} + [\Phi]^T[K][\Phi]\{z\} = [\Phi]^T\{F\} \tag{3.3}$$

Denoting the three triple matrix products as $[M^*]$, $[C^*]$ and $[K^*]$ respectively, and the matrix $[\Phi]^T\{F\}$ as $\{F^*\}$, we get

$$[M^*]\{\ddot{z}\} + [C^*]\{\dot{z}\} + [K^*]\{z\} = \{F^*\} \tag{3.4}$$

Assuming that each modal vector in $[\Phi]$ is normalized so as to yield a unit modal mass for the corresponding mode, we get,

$$[M^*] = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix} \tag{3.5}$$

$$[K^*] = \begin{bmatrix} \omega_1^2 & & \\ & \ddots & \\ & & \omega_N^2 \end{bmatrix} \tag{3.6}$$

Here, both the matrices are of size N, where N=total number of degrees of freedom in the structure.

Undamped mode shapes are orthogonal to the mass and the stiffness matrices, but not necessarily to the damping matrix. In other words, matrices $[M^*]$ and $[K^*]$ are diagonal but matrix $[C^*]$ is not necessarily diagonal. The common practice is to *assume* a damping matrix in such a way that $[C^*]$ is rendered diagonal. For lightly damped structures, damping forces are generally small and hence their distribution throughout the system is considered to be unimportant. Damping is considered merely as a mechanism for energy dissipation. Thus, an equivalent damping matrix that represents the same overall rate of energy dissipation is considered to be acceptable. This form of damping is known as classical damping.

Based on this concept, equivalent diagonalizable damping matrices have been used in the past. If the damping matrix is obtained as a linear combination of mass and stiffness matrices, then it is automatically diagonalized by normal mode shapes. This form of damping is known as Rayleigh damping [59]. More general expressions for obtaining the damping matrix such that it may be diagonalized are available [11]. When classical damping assumption is used, damping is not treated as an independent structural property. The damping matrix obtained merely consists of terms that would give a good approximation to overall energy dissipation.

Once proportional damping is assumed, $[C^*]$ becomes diagonal which implies that the equations of motion get uncoupled. Each 'modal' equation is then solved separately and the resulting modal responses are combined to yield the response of the structure.

3.1.2 Non-classically Damped Systems

In some important systems such as soil-structure interaction systems or primary-secondary coupled systems, classical damping assumption is not valid. Such systems are described as "non-classically damped" systems or as "non-proportionally damped" systems. In a soil-structure interaction system, the soil has a great amount of damping due to what is known as "radiation damping", whearas the structure is relatively much less damped. In a P-S coupled system, the secondary system tends to have much less damping than the primary system. Because of drastic variation of dynamic properties, especially damping properties, classical damping matrices which account for energy dissipation only in an overall sense cannot represent the energy dissipation in these systems. In non-classically damped systems, damping must be treated as an independent structural property, which is not necessarily proportional to the elastic or the inertial properties.

If undamped mode shapes are still used for non-classically damped systems, they result in a system of equations (Eq. 3.4) that are coupled due to the presence of off-diagonal terms in the matrix $[C^*]$.

3.1.3 Techniques for Analysis of Non-classically Damped Systems

Foss [21] developed a modal superposition solution technique of these systems, using complex eigenproperties. Foss showed that complex modal vectors do possess orthogonality in a certain sense. Using complex modal shapes, equations of motion can be uncoupled and then final response may be obtained by superposition. This approach though undoubtedly accurate, requires solution of a complex eigenvalue problem of size $2N$, where N is the number of degrees of freedom. The approach has not gained much popularity partly because it results in a larger size eigenproblem and partly because it involves the necessity to deal with complex numbers.

Clough and Mojtahedi [14] demonstrated that undamped mode shapes can be used to perform a modal superposition time history analysis of non-classically damped structures. They proposed direct integration of the coupled equations of motion such as Eq. 3.4, where the transformation matrix $[\Phi]$ consists of only a truncated set of undamped modes. This takes full account of the modal coupling due to the presence of off diagonal terms in the transformed damping matrix, at least for the modes considered in the analysis.

A popular method of solution is to simply ignore the off-diagonal terms in the transformed damping matrix obtained using undamped modes. With this assumption, normal modal superposition can be used and all the advantages of that method are available.

An approximate method is to completely ignore non-classical nature and assume that modal superposition is valid. The modal damping ratios required are obtained from modal damping ratios of individual components of the system. This method is more approximate than the "ignoring off-diagonal terms" method, since it introduces an approximation in the diagonal terms in addition to that in the off-diagonal terms. In some cases, the errors may cancel to provide good results, but in most cases, this is not so.

3.2 The "Ignoring Off-diagonal Terms" Approach

The nature of approximation involved in ignoring off-diagonal terms has been studied by Hasselman [38], Warburton and Soni [86], Thomson et al [77], Duncan and

Taylor [20]. Hasselman[38] studied modal coupling in lightly damped structures. He concluded that as long as frequency of separation among the modes is adequate, modal equations of motion may be considered uncoupled even if the transformed damping matrix is not diagonal. He suggested criteria to determine the possibility of ignoring modal coupling without significant loss of accuracy. Warburton and Soni[86] also proposed some criteria which could be used to ensure that the error involved in ignoring the off-diagonal terms is reasonably small. They compared their criteria with those of Hasselman [38] and concluded that both were similar in nature. Duncan and Taylor [20] examined the damping coupling for a 2-DOF system. They considered the response of the system over a range of frequencies of applied harmonic load.

In [86] and [20], conclusions were based on a study of harmonic response of certain structures. Using harmonic loads to study non-classical behaviour is an attractive idea because of the simplicity and elegance in the analysis and since any general load can be considered to be a combination of harmonic loads using Fourier series. Thus, if response of a structure is considered over a wide range of frequencies of harmonic excitation, it could represent response to any arbitrary load.

The present study is a systematic investigation of the error involved in ignoring off-diagonal terms in the matrix $[C^*]$. A simple 2-DOF shear building system (Fig. 3.1) is studied in detail with the intention of providing insight into non-classical behaviour of more complex systems. In a strict sense, conclusions made are applicable only to this system. However, some of the conclusions are believed to be of a general nature. Moreover, the present study can be easily extended to consider more complex systems. Such an extended study is believed to provide interesting insight into non-classical behaviour of these systems.

A classically damped system possesses a damping matrix that can be diagonalized by orthogonal transformation and obviously there is no error involved in attempted diagonalization. For a non-classically damped system, however, significant error is likely to be associated with diagonalization, depending on the severity of the non-classical nature. For this reason, the terms "error involved in diagonalization" and "severity of non-classical nature" will henceforth be used interchangeably. Also, it should be noted that the term "diagonalization" will henceforth be used to imply the "ignoring off-diagonal terms" method.

3.3 Study of a Two-DOF System for Error Involved in Diagonalization

Fig. 3.1 depicts a two-DOF shear building that can represent the simplest primary-secondary coupled system. This system was studied in detail in order to examine its non-classical behaviour. It was thought that if a single quantity could be obtained to measure (i.e., represent) the severity of non-classical nature of the system, then that quantity could be used to understand the behaviour of the system by means of parametric studies. In what follows, such a quantity called "error-parameter" is introduced and tested. Attempt is made to explain the behaviour of the parameter, its significance and its utility.

The absolute maximum acceleration of a mass is taken as the response quantity of interest. This choice is partly justified by the fact that final seismic forces depend on the absolute maximum acceleration of the masses. No attempt has been made to fix any of the two masses as the "reference mass". Hence the "error in diagonalization" or "the error parameter" depends upon the mass under consideration. However, conclusions made are of a general nature and before arriving at them, both masses were taken into consideration.

Response of the system in Fig. 3.1, was studied for an earthquake excitation in the form a sine wave (Fig. 3.2).. The response quantity selected was the peak acceleration of the top (or bottom) mass divided by the amplitude of the ground acceleration. Division by the amplitude of the ground acceleration ensures non-dimensionality. Also, any influence of the amplitude of the ground acceleration on the response-quantity of interest is eliminated. Response of the system was found for various frequencies of the applied load using both the exact and the approximate (diagonalized) damping matrices.

The error involved in diagonalization was obtained from these two responses. A curve was obtained by plotting the error in diagonalization as function of the frequency of the applied load. A typical curve obtained is shown in Fig. 3.3. As may be expected, error is maximum near the two resonant frequencies, where the response is more sensitive[55] to the damping matrix used. Away from those frequencies, diagonalization seems to have only a small effect on the response. In view of the variation of the error with the frequency of the applied harmonic load, the error in diagonalization in the case of an arbitrary load can be expected to depend on the frequency content of the load.

The area under the error-frequency curve was chosen as the "error-parameter". The area under the error-frequency curve can be expected to have a very good correlation with the error involved in diagonalization in case of an arbitrary load, which consists of a certain combination of harmonics.

3.3.1 Exact Solution

Consider the building system depicted in Fig. 3.1. The equation of motion for the system is

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \end{Bmatrix} + \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix} \begin{Bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} \tag{3.7}$$

where, y_1 and y_2 are horizontal displacements (relative to the ground) of the two masses. Let A be the amplitude and $\bar{\omega}$ be the frequency of the excitation. Then $F_1 = -m_1 A\ sin\bar{\omega}t$ and $F_2 = -m_2 A \sin \bar{\omega}t$. We shall henceforth ignore the negative sign in the expressions for F_1 and F_2 since it is of little significance. We represent

$$A \sin \bar{\omega}t = \text{Im}\left[Ae^{i\bar{\omega}t}\right]$$

where $i = \sqrt{-1}$ and Im[ ] represents the imaginary part of a complex number. We write the response as $y_1 = Y_1' e^{i\bar{\omega}t}$ and $y_2 = Y_2' e^{i\bar{\omega}t}$ and we retain only the imaginary part. Substituting in the original equation, cancelling $e^{i\bar{\omega}t}$ and dividing throughout by A, we get

$$\left[[K] - \bar{\omega}^2 [M] + i\bar{\omega} [C]\right] \left\{ \begin{array}{c} Y_1 \\ Y_2 \end{array} \right\} = \left\{ \begin{array}{c} m_1 \\ m_2 \end{array} \right\} \tag{3.8}$$

where $[M]$, $[C]$ and $[K]$ are the mass, damping and stiffness matrices respectively and $Y_1 = Y_1'/A$ and $Y_2 = Y_2'/A$. Thus Y_1 and Y_2 are the amplitudes of mass 1 and 2 per unit amplitude of the ground acceleration. Acceleration amplification is $\bar{\omega}^2 Y_1$ for the bottom mass and $\bar{\omega}^2 Y_2$ for the top mass. We are interested in these quantities as the "response quantities".

We write the damping matrix as

$$[C] = \begin{bmatrix} c_{11} & c_{12} \\ c_{12} & c_{22} \end{bmatrix} \tag{3.9}$$

so that, $c_{11} = c_1 + c_2$, $c_{12} = -c_2$ and $c_{22} = c_2$. Using Eqs. 3.7, 3.8 and 3.9 we get

$$\begin{bmatrix} k_1 + k_2 - \bar{\omega}^2 m_1 + i\bar{\omega} c_{11} & -k_2 + i\bar{\omega} c_{12} \\ -k_2 + i\bar{\omega} c_{12} & (k_2 - \bar{\omega}^2 m_2) + i\bar{\omega} c_{22} \end{bmatrix} \left\{ \begin{array}{c} Y_1 \\ Y_2 \end{array} \right\} = \left\{ \begin{array}{c} m_1 \\ m_2 \end{array} \right\} \tag{3.10}$$

Solution of Eq. 3.10 yields

$$Y_1 = [(k_2 - \bar{\omega}^2 m_2) m_1 + k_2 m_2 + i\bar{\omega}(c_{22} m_1 - c_{12} m_2)]/\Delta \tag{3.11}$$

$$Y_2 = [k_2 m_1 + (k_1 + k_2 - \bar{\omega}^2 m_1) m_2 + i\bar{\omega}(-c_{12} m_1 + c_{11} m_2)]/\Delta \tag{3.12}$$

where

$$\Delta = \Delta_1 + i\Delta_2$$

$$\Delta_1 = (k_1 + k_2 - m_1\bar{\omega}^2)(k_2 - m_2\bar{\omega}^2) - \bar{\omega}^2(c_{22}c_{11} - c_{12}^2) - k_2^2 \tag{3.13}$$

$$\Delta_2 = \bar{\omega}[c_{11}(k_2 - m_2\bar{\omega}^2) + c_{22}(k_1 + k_2 - m_1\bar{\omega}^2) + 2k_2 c_{12}] \tag{3.14}$$

Exact solution is the one in which exact damping matrix of Eq. 3.7 is used.

Diagonalized Damping Matrix

Consider the diagonalization approximation. Let ω_1 and ω_2 be the natural frequencies of this system. Let $[\Phi]$ represent the normal mode shape matrix where each modal vector is scaled so as to yield a unit modal mass for that mode.

$$[\Phi] = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \tag{3.15}$$

Using standard modal superposition procedure, let $\{y\} = [\Phi]\{z\}$ where $\{z\}$ is a vector of normal modal co-ordinates. We denote

$$[\Phi]^T[C][\Phi] = \begin{bmatrix} D_1 & D_{12} \\ D_{12} & D_2 \end{bmatrix} \tag{3.16}$$

where, a superscripted T indicates transpose of a matrix. We have,

$$D_1 = c_{11}\phi_{11}^2 + 2c_{12}\phi_{11}\phi_{21} + c_{22}\phi_{21}^2 \tag{3.17}$$

$$D_{12} = c_{11}\phi_{11}\phi_{12} + c_{12}(\phi_{12}\phi_{21} + \phi_{11}\phi_{22}) + c_{22}\phi_{21}\phi_{22} \tag{3.18}$$

$$D_2 = c_{11}\phi_{12}^2 + 2c_{12}\phi_{12}\phi_{22} + c_{22}\phi_{22}^2 \tag{3.19}$$

In the method of ignoring off diagonal terms, D_{12} is simply put as zero. Thus the matrix that is used for modal superposition procedure is

$$[D] = \begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix} \tag{3.20}$$

Modal superposition method can now be used as :

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{z}_1 \\ \ddot{z}_2 \end{Bmatrix} + \begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix} \begin{Bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{Bmatrix} + \begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix} \begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix} = [\Phi]^T \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} \tag{3.21}$$

For the purpose of future discussion, this method (the method of ignoring off diagonal terms) will be called method I.

A Clarification

It is of interest to see under what conditions a 2-DOF system will be classically damped (i.e. $D_{12} = 0$). From Eq. 1.18 we must have for this purpose,

$$c_{11}\phi_{11}\phi_{12} + c_{12}(\phi_{22}\phi_{11} + \phi_{21}\phi_{12}) + c_{22}\phi_{21}\phi_{22} = 0$$

Since $c_{11} = c_1 + c_2$, $c_{12} = -c_2$ and $c_{22} = c_2$, we get after simplification,

$$c_2 = \frac{\alpha_1\alpha_2}{\alpha_1 + \alpha_2 - \alpha_1\alpha_2 - 1}c_1 \tag{3.22}$$

where $\alpha_1 = \phi_{11}/\phi_{21}$ and $\alpha_2 = \phi_{12}/\phi_{22}$.

Thus for a given mass and stiffness matrix (which define the mode shapes), the system is classically damped if and only if one of the damping coefficients is a multiple of the other as specified by Eq. 3.22. For all other combinations of the damping coefficients, the system is non-classically damped. It is important to note that if c_2 is the required multiple of c_1 then the system is classically damped even if both of them are of very large magnitudes.

3.3.2 Evaluation of the Error-Parameter

An indirect approach was used for finding out the approximate response. From the matrix $[D]$, a parent matrix $[D_p]$ is obtained in such a way that

$$[\Phi]^T[D_p][\Phi] = [D] \tag{3.23}$$

$[D_p]$ can be found to be

$$[D_p] = \frac{1}{(\phi_{11}\phi_{22} - \phi_{12}\phi_{21})^2}\begin{bmatrix} D_1\phi_{22}^2 + D_2\phi_{21}^2 & -(D_1\phi_{22}\phi_{12} + D_2\phi_{21}\phi_{11}) \\ -(D_1\phi_{22}\phi_{12} + D_2\phi_{21}\phi_{11}) & D_1\phi_{12}^2 + D_2\phi_{11}^2 \end{bmatrix} \tag{3.24}$$

Using the matrix $[D_p]$ in place of [C], if the procedure used in the exact solution is followed, the response obtained is the approximate response by the diagonalization method. Accurate response is obtained by using the matrix $[C]$ and Eqs. 3.11 and 3.12. Error is calculated as

$$\frac{\text{approximate response} - \text{accurate response}}{\text{accurate response}}$$

A computer program was written for numerical evaluation of the error parameter for a given system. Simpson's rule was used to calculate the area under the error-frequency curve. Accuracy of numerical integration was verified by increasing the points of integration and observing the difference. Frequency range considered was $0.1\omega_1$ to $4.0\omega_2$; which was verified to be sufficiently large by increasing it and observing the difference.

3.3.3 Eigenvalues and Eigenvectors

Let us denote $\lambda_j = \omega_j^2$. We get the following expression for the eigenvalues λ_j.

$$\lambda_j = \frac{1}{2}(\frac{k_1+k_2}{m_1} + \frac{k_2}{m_2}) \mp \frac{1}{2}\sqrt{(\frac{k_1+k_2}{m_1} + \frac{k_2}{m_2})^2 - \frac{4k_1k_2}{m_1m_2}} \qquad (3.25)$$

Normalized eigenvectors can be obtained, in terms of λ_j as

$$\{\phi_j\} = \left\{ \begin{array}{c} \frac{1}{\sqrt{m_1+m_2(\frac{k_1+k_2-m_1\lambda_j}{k_2})^2}} \\ \frac{\frac{k_1+k_2-m_1\lambda_j}{k_2}}{\sqrt{m_1+m_2(\frac{k_1+k_2-m_1\lambda_j}{k_2})^2}} \end{array} \right\}$$

3.4 Merits of The Error-Parameter

3.4.1 Numerical Accuracy

The procedure that was used in calculating the numerical value of the error parameter has a high degree of numerical accuracy in view of the following reasons:

1. Analytical expressions were derived for eigenvalues, eigenvectors, the diagonalized damping matrix [D] and the parent matrix $[D_p]$. These analytical expressions were then coded for the computer.

2. Numerical procedures (such as Simpson's rule) were used only when absolutely necessary and when they were used, it was made sure that the accuracy obtained was to expected standards.

3. Numerical procedures happened to be used after obtaining the analytical results of responses. Thus there was no propagation of error.

4. The entire program was written in double precision.

3.4.2 Other Merits

The proposed parameter is believed to be a good measure of the error involved in diagonalization (or of the severity of the non-classical nature of a given system) in view of the following considerations :

1. Since earthquake type of excitation was used for obtaining the numerical value of the error parameter, conclusions made using the error-parameter can be confidently used for earthquake loads, blast loads or any other loads that are distributed more or less similar to seismic loads on a structure.

2. All modes were obviously considered in the analysis. Hence influence of the number of modes considered was eliminated. The reader is referred to [20].

3. Both the analytical and the numerical components of the procedure for obtaining the error-parameter ensure that it depends only on the dynamic properties of the system.

3.5 A Test of Reliability of the Error-Parameter

A test was conducted to examine whether the "error-parameter" does truly represent the error involved in diagonalization when a system is analyzed for actual earthquake excitations. (This error will henceforth be called "physical error" since this is the actual error in a practical analysis as opposed to the error-parameter, which is a quantity that has been defined so as to be indicative of the expected error.)

For this purpose, the following procedure was conducted. For an example system,

1. The error parameter was found using the procedure laid out earlier in this chapter.
2. The system was subjected to actual earthquake excitation and
    (a) exact response was found using the exact damping matrix of Eq. 3.7.
    (b) approximate response was found using the approximate damping matrix $[D]$.
    (c) percentage error in response was obtained from the accurate and the approximate responses.

Here, "response" indicates absolute maximum acceleration of the mass under consideration. Steps 1 and 2 provide one point on a curve such as Fig. 3.10. By varying dynamic properties of example building systems 1, 2 and 3 (Figs. 3.4 to 3.6), and following steps 1 and 2 above, Figs. 3.10 through 3.17 were obtained. With reference to Fig. 3.1, let us denote $\beta_1 = c_1/2\sqrt{k_1 m_1}$ and $\beta_2 = c_2/2\sqrt{k_2 m_2}$. The two-DOF system shown in Fig. 3.1 can be considered as a combination of two single-DOF systems. Ratios β_1 and β_2 are the damping ratios for the individual systems (and *not* the modal damping ratios in the combined system, which will be denoted as ξ_1 and ξ_2). Three different excitations were considered for the purpose of this testing.

1. The 1934 El Centro earthquake (N-S component)
2. The 1952 Taft earthquake (N-E component)
3. A blast load in the form of a half sine wave $\ddot{u}_g = sin25t$, where t is time in seconds.

The parameter can be said to have passed this test if it is found that the physical error increases(in magnitude) as the error parameter increases and vice versa. Such a monotonic relationship was indeed observed and can be seen from Figs. 3.10 through 3.17. One may note the order of magnitude of the error parameters and the physical errors in Figs. 3.11 and 3.12. In Fig. 3.11 the physical error values are very small compared to those in Fig. 3.12. It may be noted that the error parameter shows a similar variation in its magnitude.

As discussed earlier, the error in diagonalization depends on the frequency content of the applied load. Also, it is apparent from these curves that the dependence of the error on the applied load is more for systems with more severe non-classical damping. (i.e. when the error parameter values are large.) While it is difficult to quantify the dependence of the error on the applied load, it appears certain that the error is likely to be quite different for two different earthquakes with significantly different frequency contents, especially if the system under consideration has severe non-classical damping.

Warburton and Soni[86] have demonstrated that the error involved in diagonalization of the damping matrix is a function of frequency separation between modes, the magnitudes of off diagonal terms in relation to those of the diagonal terms as well as the overall amount of damping.

Warburton and Soni[86] used harmonic loads applied only at a certain DOF of the structure. Their final expression (criteria for ignoring off-diagonal terms) does not contain any load term. They asserted that the errors associated with aperiodic or random loads are usually of a smaller magnitude than those associated with a (resonant) harmonic load. Tables presented by them indeed show such a behaviour. Based on this observation, Warburton and Soni used harmonic loads to arrive at their criteria, hoping that the assumption made is on the conservative side.

The error curve in Fig. 3.3 indicates that the error in diagonalization has sharp peaks near resonant frequencies and that away from resonance, the error is quite small. It is due to this phenomenon that the error associated with a resonant harmonic load is usually larger than that associated with a random load which has a wide Fourier spectrum. Exceptions are the cases in which dominant frequencies of the applied random load are near the resonant frequencies.

Warburton and Soni's[86] criteria can still be considered to be applicable for systems with a small amount of non-classical damping (especially systems with 'light damping' in an overall sense.) This is because for such systems, the error in diagonalization, as indicated previously, does not depend appreciably on the applied load (Figs. 3.10 through 3.17). If diagonalization assumption is reasonably valid for a given system according to Warburton and Soni's criteria, the system may be considered to be having a small (and hence negligible) amount of non-classical damping. Thus, their neglect of the dependence of the error on frequency content of the applied load may be justified. However, it should be understood that in general, study of non-classically damped systems must take into account the dependence of the error on the applied load.

The fact that error in diagonalization depends on the applied load further emphasizes the utility of the error parameter. The proposed error parameter is not based on any specific load. It must be recognized that the error in diagonalization depends on numerous factors and hence in order to study its behaviour, we should look for a quantity that is as "basic" to the system as possible. As mentioned earlier, the error parameter has been defined to provide such a "basic" quantity and thus its use in studying the behaviour of the system is believed to be very reliable.

3.6 Use of the Error-Parameter

After establishing credibility of the error parameter in representing non-classical nature, it may now be used to study the variation of error in diagonalization as the dynamic properties of a system are varied. Numerical value of the error parameter will henceforth be taken to sufficiently represent the severity of the non-classical damping and that numerical value will simply be termed as "the error" for brevity.

Another important use of the proposed error-parameter can be found as an effective tool for comparing performance of modal damping estimation procedures. It was mentioned earlier that modal damping estimation is an approximate method for analysis of non-classically damped systems. In this method, Eq. 3.21 is used except that the quantities D_1 and D_2 are not obtained from the Eqs. 3.17 and 3.19, but are estimated. Therefore modal damping estimation procedures are similar to the "ignoring off-diagonal terms" method (Method I) in that all of them result in an equation similar to Eq. 3.21. If the procedure mentioned earlier for computing the error-parameter is followed using the $[D]$ matrix obtained by modal-damping estimation procedures, the value obtained will be different from the error parameter obtained using method I for getting the $[D]$ matrix. The difference in error parameter due to different methods of obtaining the $[D]$ matrix indicates a way of comparing performance of these methods.

3.7 Parametric Study

Numerical study was conducted using the error parameter to investigate behaviour of some non-classically damped systems. Figs. 3.18 through 3.28 show the outcome of this study. The numerical examples chosen were such as to represent important non-classically damped systems such as soil-structure interaction systems or primary-secondary coupled systems. Fig. 3.18 emphasizes the fact that for a given system such as in Fig. 3.1, there exists a unique ratio c_1/c_2 for which the system is classically damped. In this plot, c_1 was fixed by fixing $\beta_1 = 0.075$. Thus, we get a unique $\beta_2 = 0.075$ for which the error is zero. (i.e., system is classically damped.) A detailed discussion of this system is given in [86].

Fig. 3.19 shows the effect of the "level of damping" on non-classical nature. Since β_2 is fixed as being equal to $0.5\beta_1$, we might say that the "distribution" of damping within the system is maintained constant. We see that if both β_1 and β_2 are small then the error involved is small; whereas high values of β_1 and β_2 yield a high error. The increase in error is almost linear.

Error parameter variation for a P-S coupled system with a very flexible primary system and a relatively stiff secondary system is shown in Fig 3.24. Maintaining $\beta_1 = \beta_2$, both are increased in order to see the effect of the level of damping on this system. Error increases almost linearly as in Fig. 3.19.

Fig. 3.20 symbolizes a soil-structure interaction system for high values of β_1. The ratio β_2 was fixed to zero in order to represent the fact that structural damping is very small compared to soil damping [89]. Error seems to increase almost linearly with β_1. Note that near $\beta_1 = 0.2$ the error is about ten times as much as that near $\beta_1 = 0.02$. This indicates the effect of extremely uneven distribution of damping on non-classical nature.

A dynamic vibration absorber is symbolized[89] in Fig. 3.21. Error seems to increase linearly with β_2. It may be noted that near $\beta_2 = 0.2$, the error becomes ten times as much as that near $\beta_2 = 0.02$.

Fig. 3.27 represents a secondary system tuned to the primary system. Error is plotted with respect to the mass ratio m_2/m_1. The error increases slowly with the mass ratio. The damping ratios are almost equal and are maintained constant. Thus the "distribution" and the "level" of damping may be said to be maintained constant. This increase in the error with respect to the mass ratio may be an expected result. The dynamic interaction between the primary and the secondary is known to increase with the mass ratio (for a fixed frequency ratio). This means that the damping interaction between the primary and the secondary is greater if the mass ratio is greater. In other words, the error involved in using an approximate damping matrix such as $[D_p]$ in place of the exact matrix $[C]$ increases with the mass ratio.

Fig. 3.28 shows a case similar to that in Fig. 3.27 except that β_2 is now very small. The error involved in diagonalization increases with the mass-ratio in a non-linear fashion. The errors obtained are much higher compared to those in Fig. 3.27. This is because the distribution of damping (and to some extent, the level of damping) is now significantly different. This reiterates the influence of the distribution of damping on non-classical nature of a system.

Figs. 3.25 and 3.26 show the variation of error parameter with respect to the mass ratio for a different (tuned) P-S coupled system. This system was chosen to be a very flexible system. Error-parameter variation in Figs. 3.25 and 3.26 is similar to that in Figs. 3.27 and 3.28 respectively.

Error-parameter variation for a P-S coupled system in which tuning is absent is plotted in Figs. 3.22 and 3.23 as a function of the mass ratio m_2/m_1. The error is smaller in Fig. 3.23 for which the damping ratios β_1 and β_2 are apart compared to the error in Fig. 3.22 for which these ratios are closer to each other. This is due to the fact that a given system is classically damped (i.e. the error in diagonalization is small) at or near a particular ratio c_1/c_2 as given by Eq. 3.22, which in turn gives a ratio β_1/β_2. The ratio β_1/β_2 is governed by the stiffness and inertial properties of the two individual systems, and need not necessarily be in the vicinity of unity in order for the error to be small.

It should be noted that it has been possible to make certain observations and arrive at certain conclusions without referring to the absolute numerical values of the error-parameter. Its *variation* gives us important information regarding the behaviour of the system.

3.8 Comparison of Modal Damping Estimation Procedures

Some procedures for estimation of modal damping were mentioned in an earlier chapter. Here, we deal with some methods described by Yan [89]. Three well known procedures for estimation of modal damping ratios are

1. Mass weighted damping :Modal damping ratios are obtained as

$$\xi_j = \frac{\{\phi_j\}^T[M']\{\phi_j\}}{\{\phi_j\}^T[M]\{\phi_j\}}$$

where $[M']$ is a weighted mass matrix in which its submatrices are multiplied by the damping ratios of their corresponding elements before assembly is done. This procedure will henceforth be called method II.

2. Stiffness weighted damping :Modal damping ratios are obtained as

$$\xi_j = \frac{\{\phi_j\}^T[K']\{\phi_j\}}{\{\phi_j\}^T[K]\{\phi_j\}}$$

where $[K']$ is a weighted stiffness matrix in which the submatrices are multiplied by the damping ratios of the corresponding elements before assembly is done. This procedure will henceforth be called method III.

3. Yan's [89]method : This method gives accurate values for modal damping ratios. The results obtained using this method during the course of this study were found to be basically as accurate as those obtained using method I. For this reason, no distinction is made henceforth between this method and method I. This method will henceforth be called method IV.

In order to compare performance of these methods, error parameter values were obtained for a given system, with its dynamic properties varied in a suitable way to obtain curves such as in Fig. 3.29. The method that consistently yields a lower value of error parameter is superior to a method that yields a higher value of the parameter. In order to arrive at such a conclusion, one must consider a large number of cases. Only a few typical cases are presented here. However, the conclusions are based on a more comprehensive study.

Figs. 3.29 to 3.36 clearly indicate that the stiffness weighted damping method (method III) is inferior to the other two methods. Mass weighted damping method (method II) is sometimes better than method I, but in most cases, it yields a higher error parameter compared to method I. Hence, these methods may be ranked as I, II and III on the basis of their merit. Also, method IV yields results identical to those obtained from method I. Hence, it is given the same rank as method I. Validity of this conclusion for other complex structures is not claimed. However, as mentioned earlier, extension of this study to consider larger systems is fairly obvious. Such an extended study will provide sufficient information regarding the validity of this conclusion for larger and more complex systems.

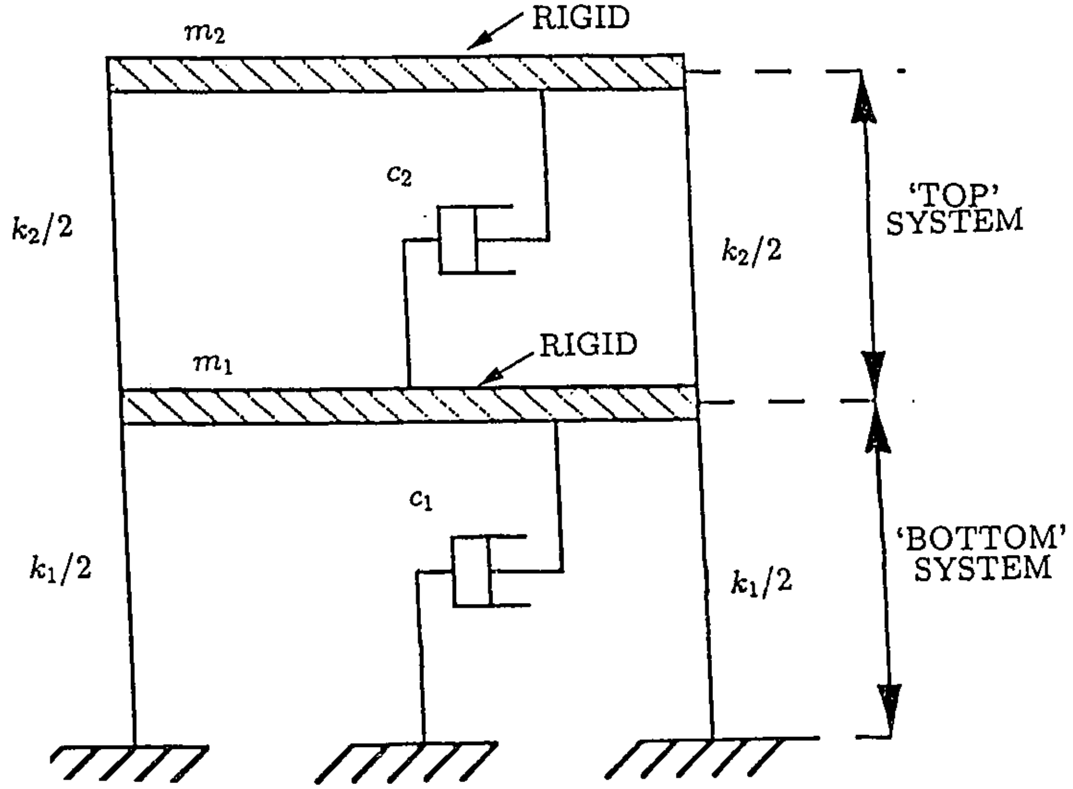


Figure 3.1: The Two-degree-of-freedom Shear Building System Under Study

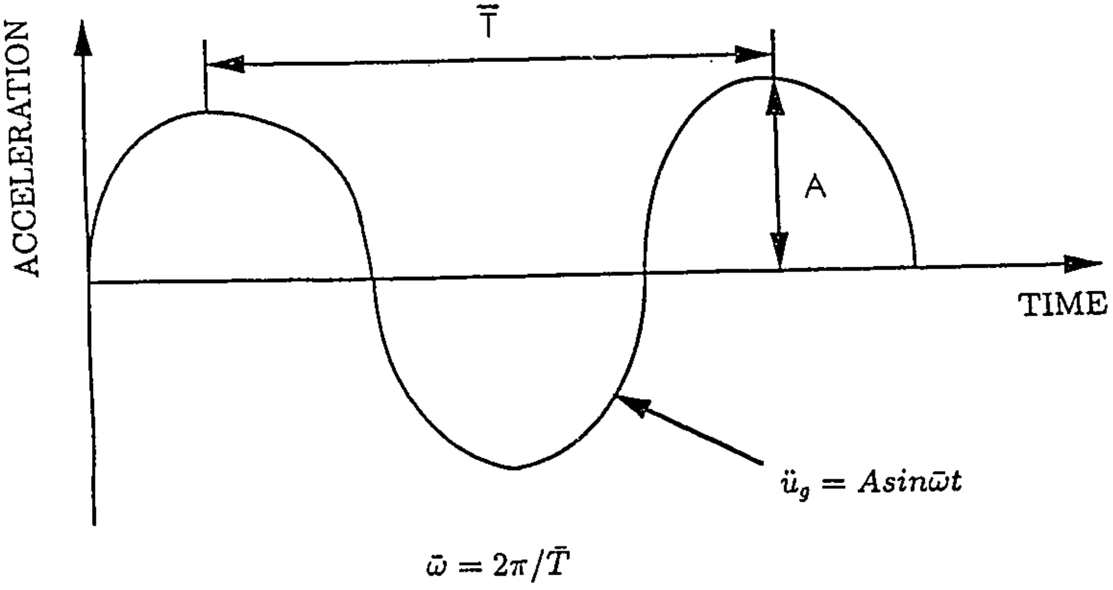


Figure 3.2: Harmonic Ground Excitation

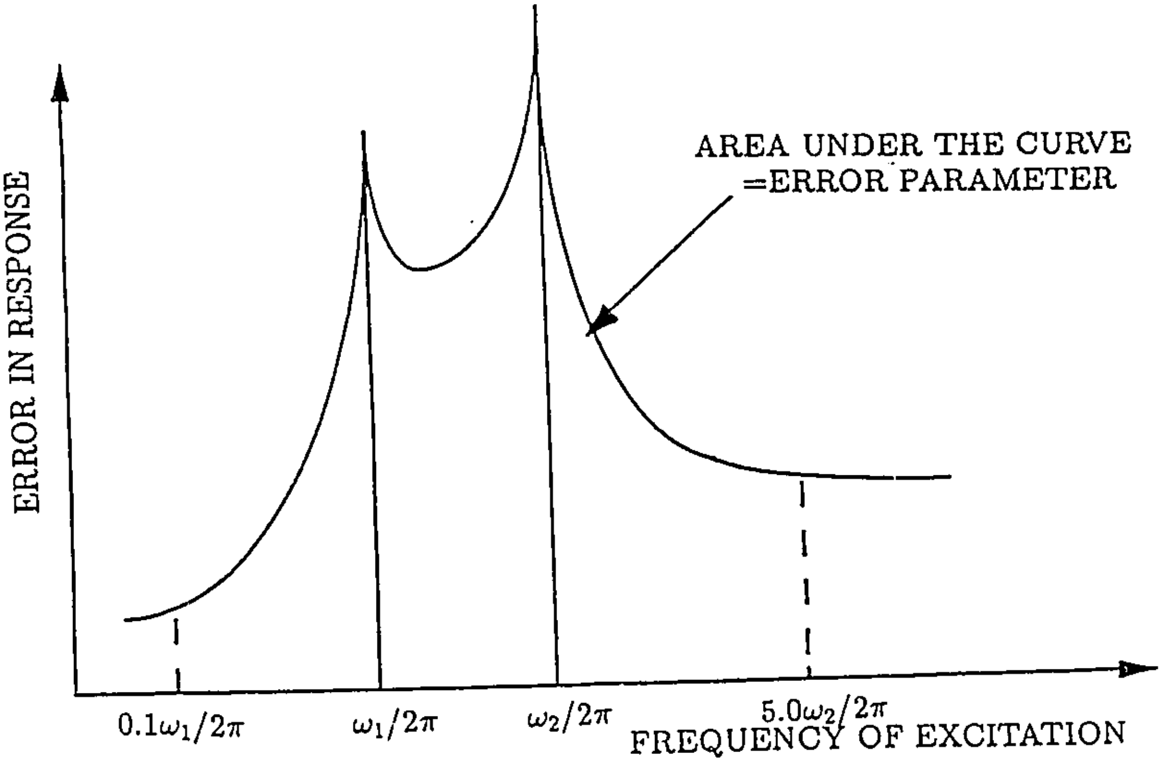
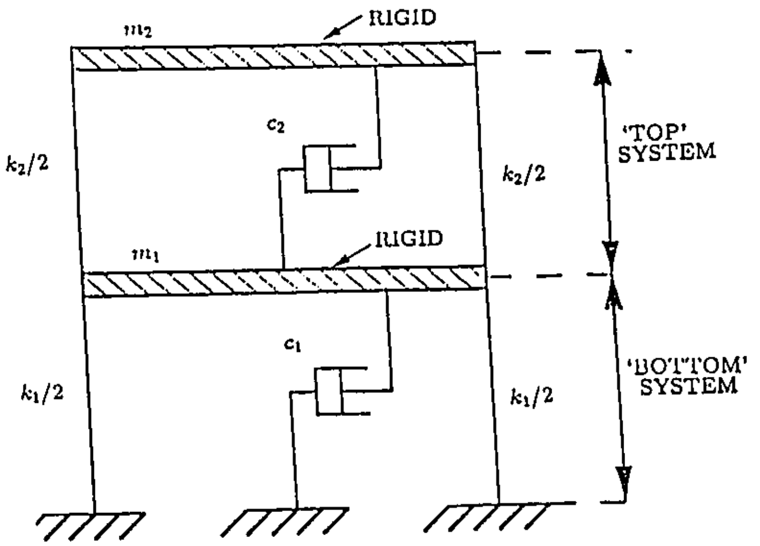


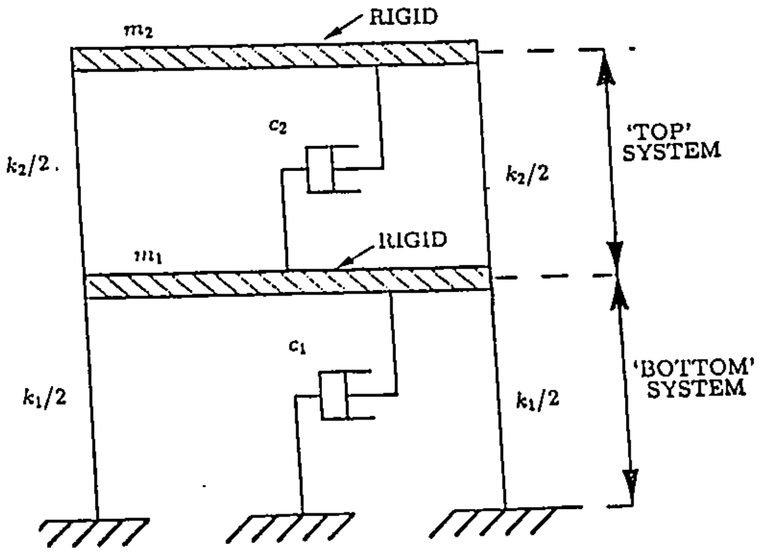
Figure 3.3: A Typical Error-frequency Curve



$k_1 = k_2 = 10^6\ N/m$

$m_1 = m_2 = 10^4\ kg$

Figure 3.4: Building System No. 1

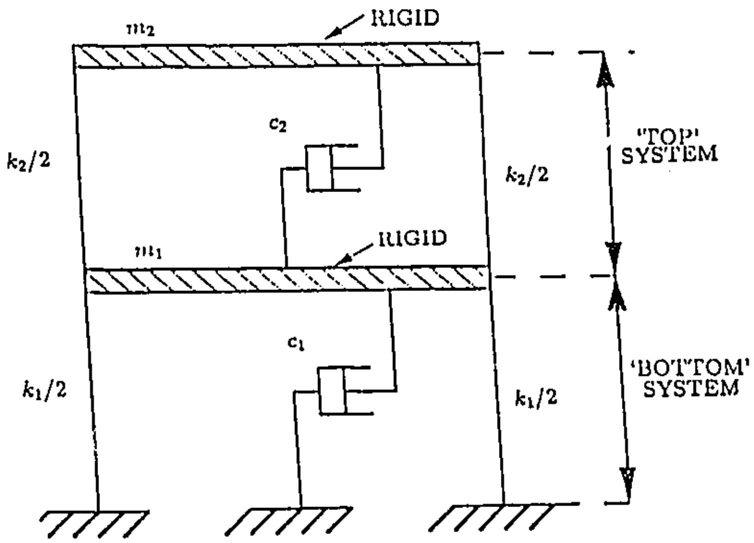


$k_1 = 10^7\ N/m$

$m_1 = 10^4\ kg$

$k_2 = 10^3 m_2$

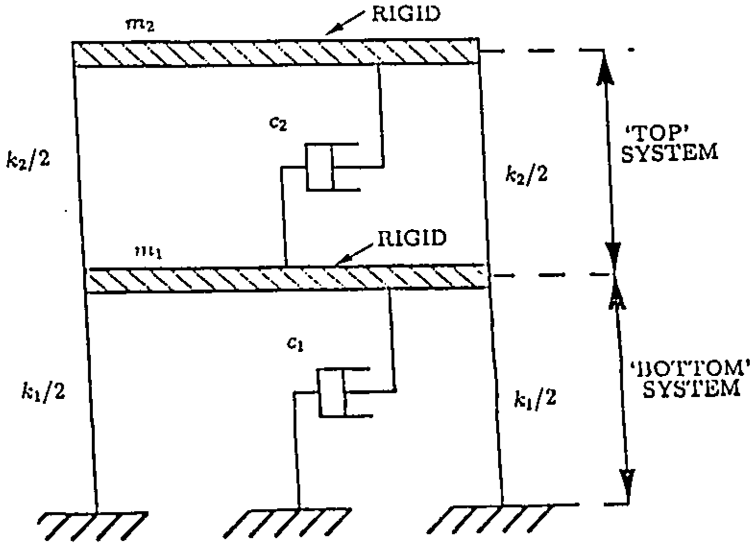
Figure 3.5: Building System No. 2



$k_1 = 104880.0\ N/m$ $\qquad k_2 = 238.2\ N/m$

$m_1 = 1000\ kg$ $\qquad m_2 = 2.271\ kg$

Figure 3.6: Building System No. 3

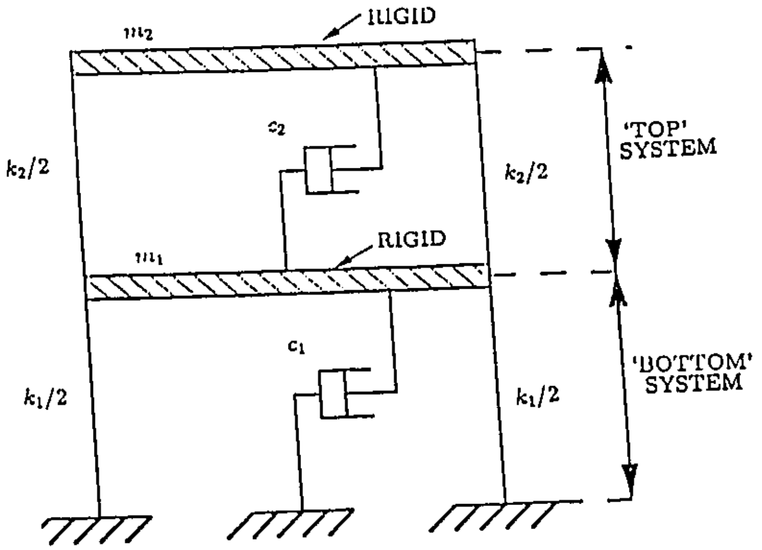


$k_1 = 10^6\ N/m$

$m_1 = 10^6\ kg$

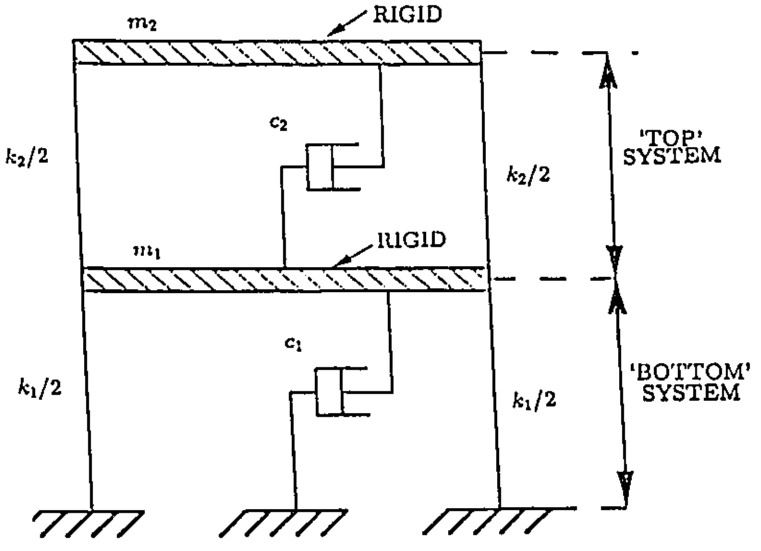
$k_2 = m_2$

Figure 3.7: Building System No. 4



$k_1 = 10^6\ N/m$
$m_1 = 5000\ kg$
$k_2 = 20 m_2$

Figure 3.8: Building System No. 5



$k_1 = 10^6\ N/m$ $k_2 = 10^5\ N/m$
$m_1 = 10^6\ kg$ $m_2 = 10^3\ kg$

Figure 3.9: Building System No. 6

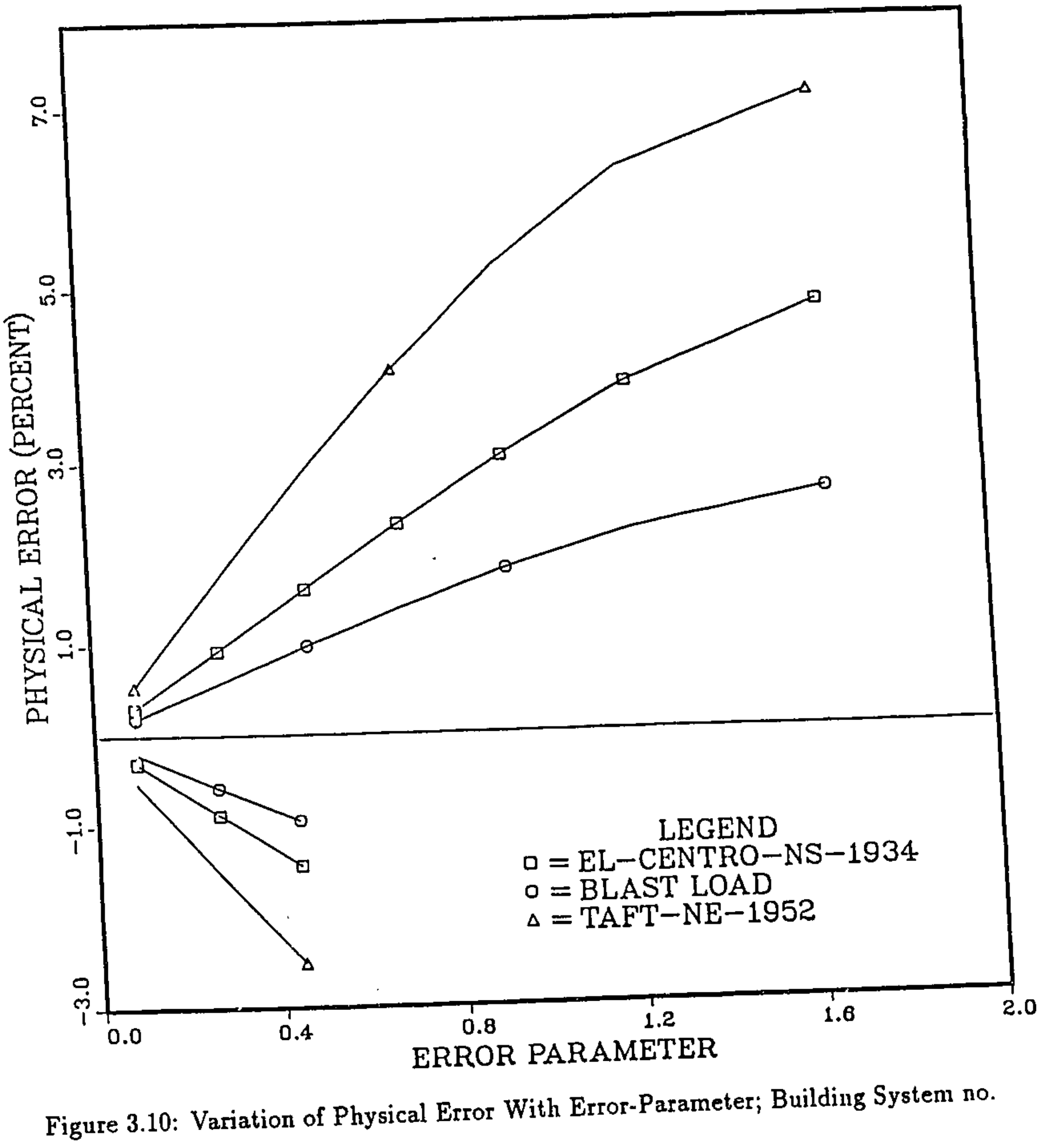


Figure 3.10: Variation of Physical Error With Error-Parameter; Building System no. 1,; Mass no. 1, $\beta_1 = 0.075$, β_2 is varied

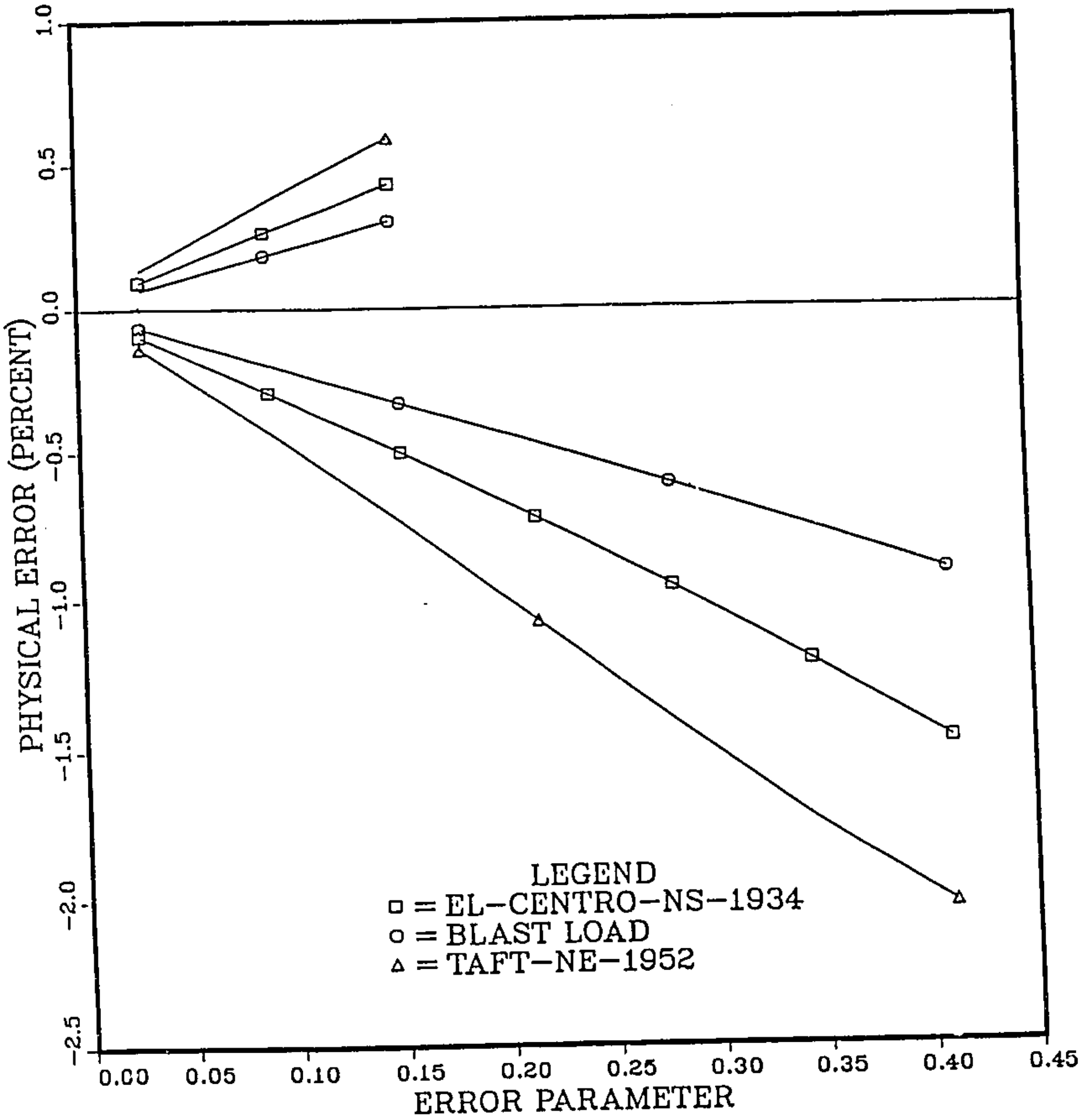


Figure 3.11: Variation of Physical Error With Error-Parameter; Building System no. 1; Mass no. 2, $\beta_1 = 0.075$, β_2 is varied

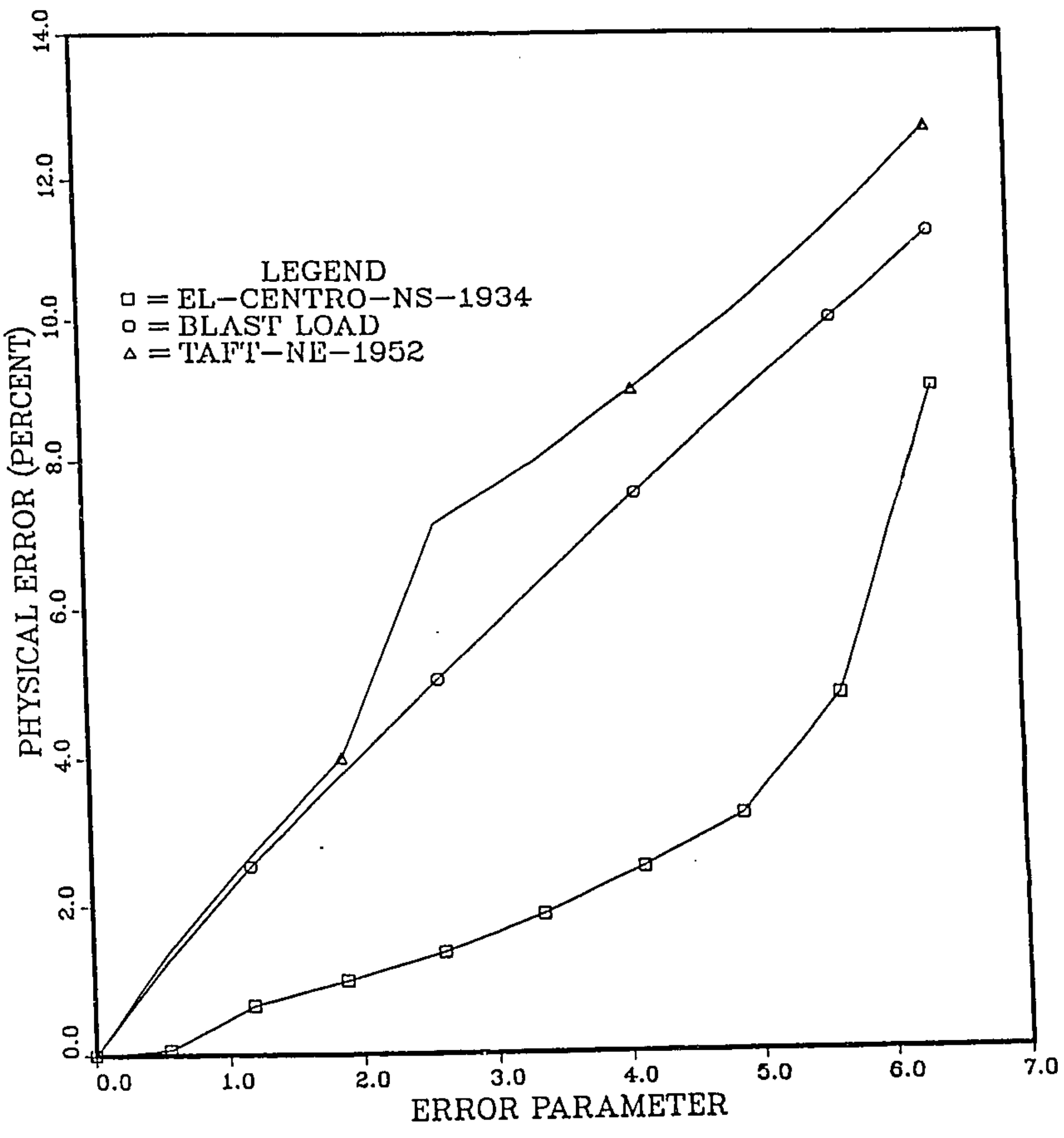


Figure 3.12: Variation of Physical Error With Error-Parameter; Building System no. 2; Mass no. 1, Mass Ratio=0.05 $\beta_2 = 0.01$, β_1 is varied

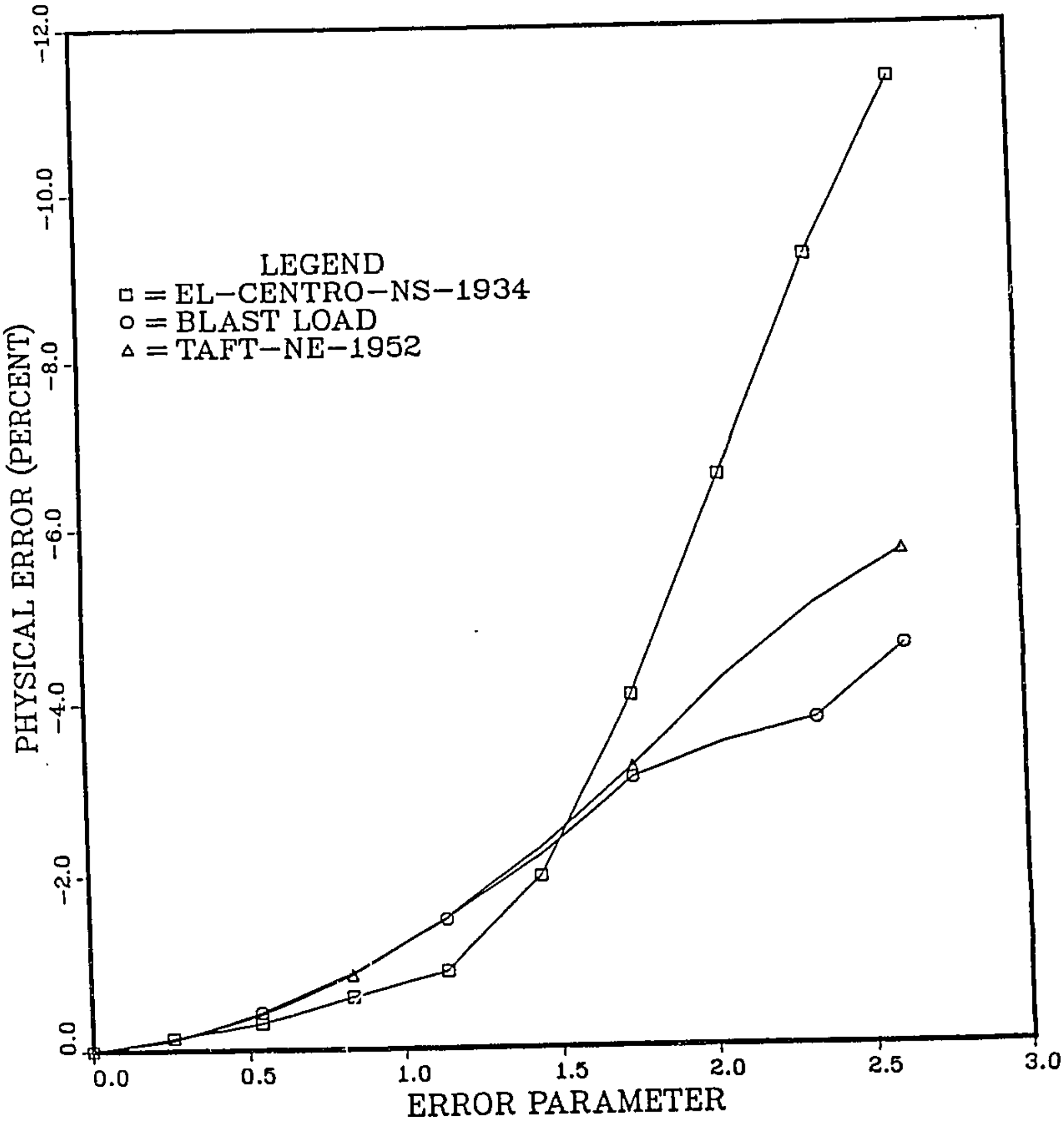


Figure 3.13: Variation of Physical Error With Error-Parameter; Building System no. 2; Mass no. 2, Mass Ratio=0.05 $\beta_2 = 0.01$, β_1 is varied

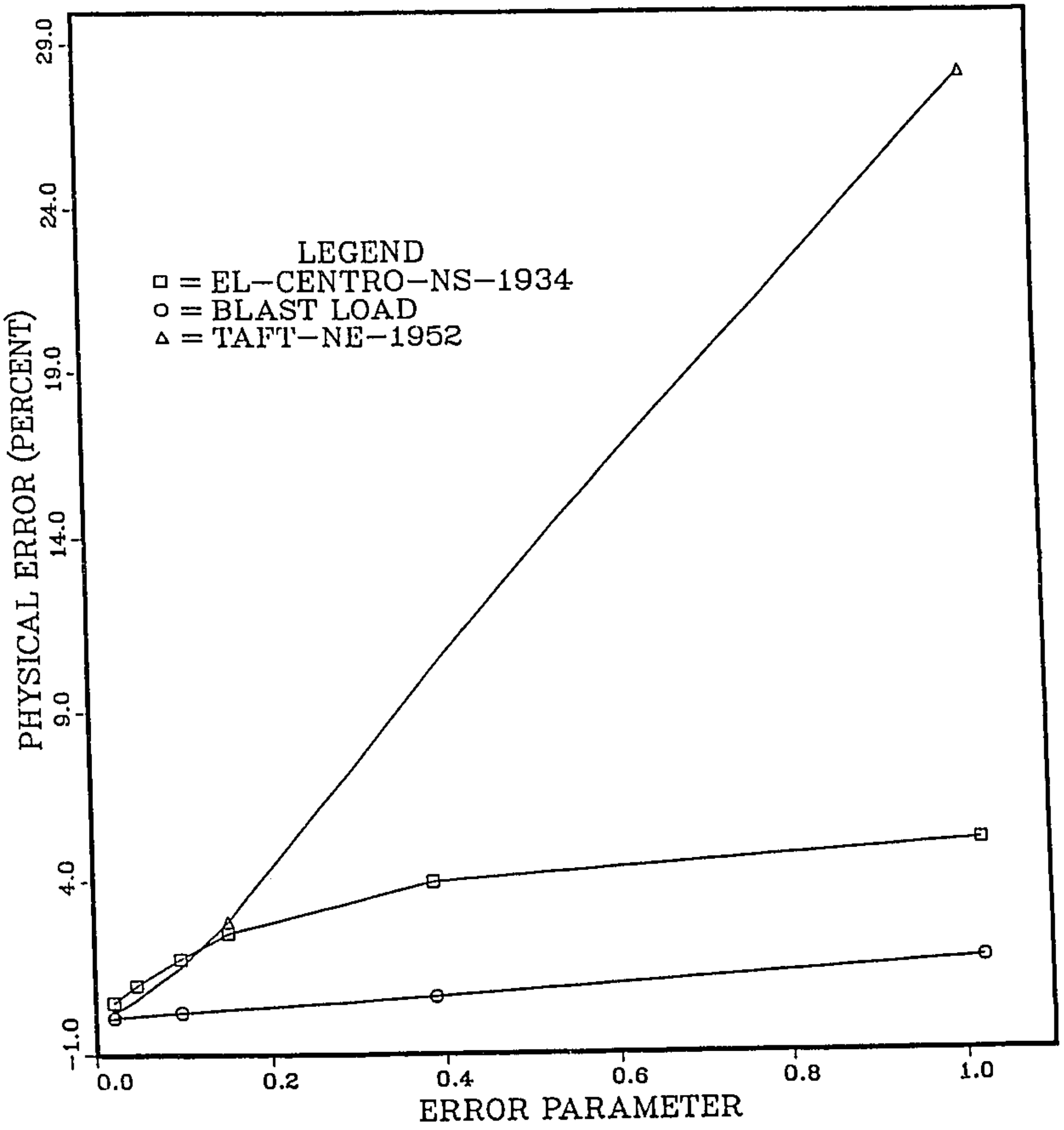


Figure 3.14: Variation of Physical Error With Error-Parameter; Building System no. 3; Mass no. 1, $\beta_2 = 0.0021498$, β_1 is varied

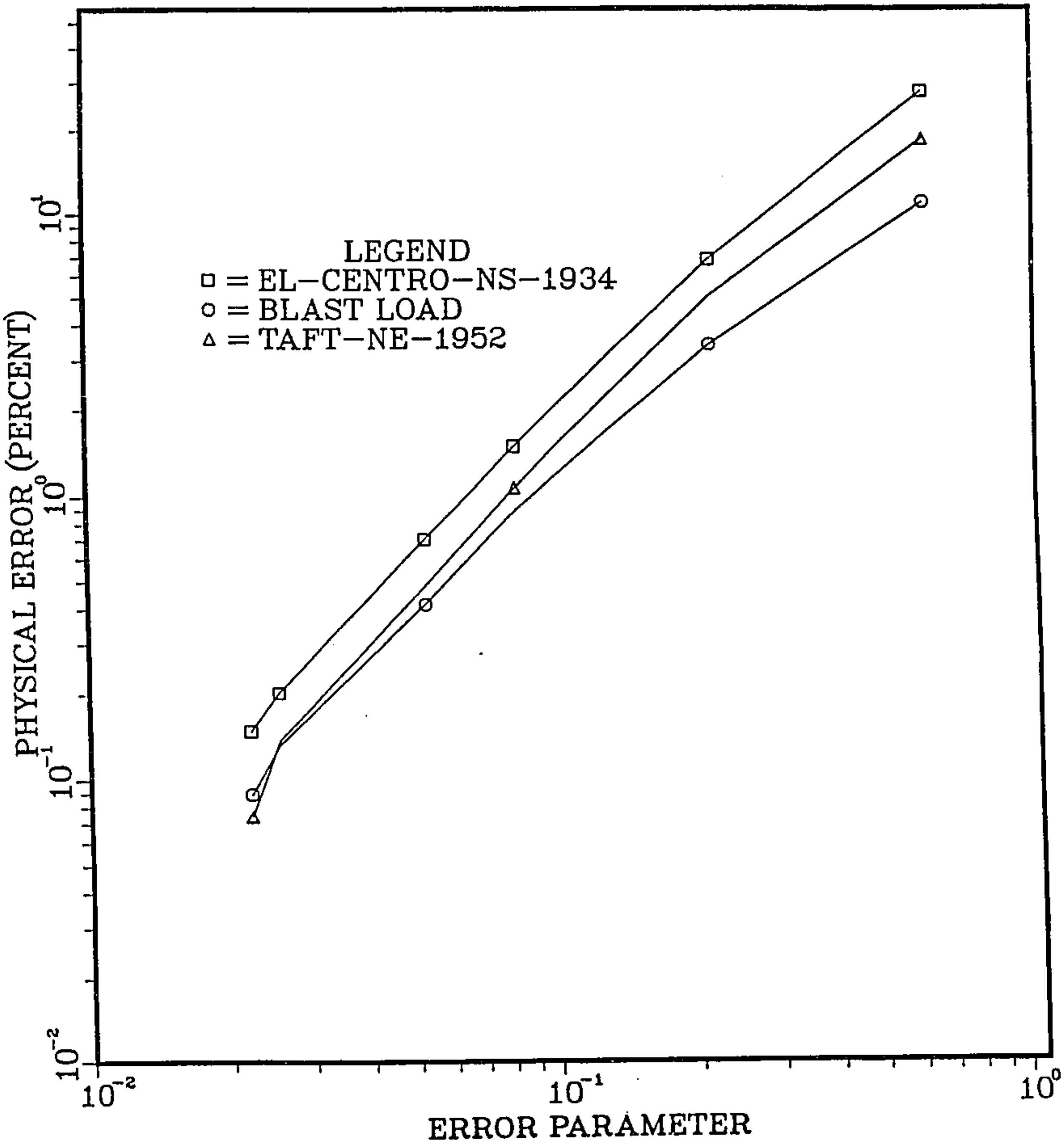


Figure 3.15: Variation of Physical Error With Error-Parameter; Building System no. 3; Mass no. 2, $\beta_2 = 0.0021498$, β_1 is varied

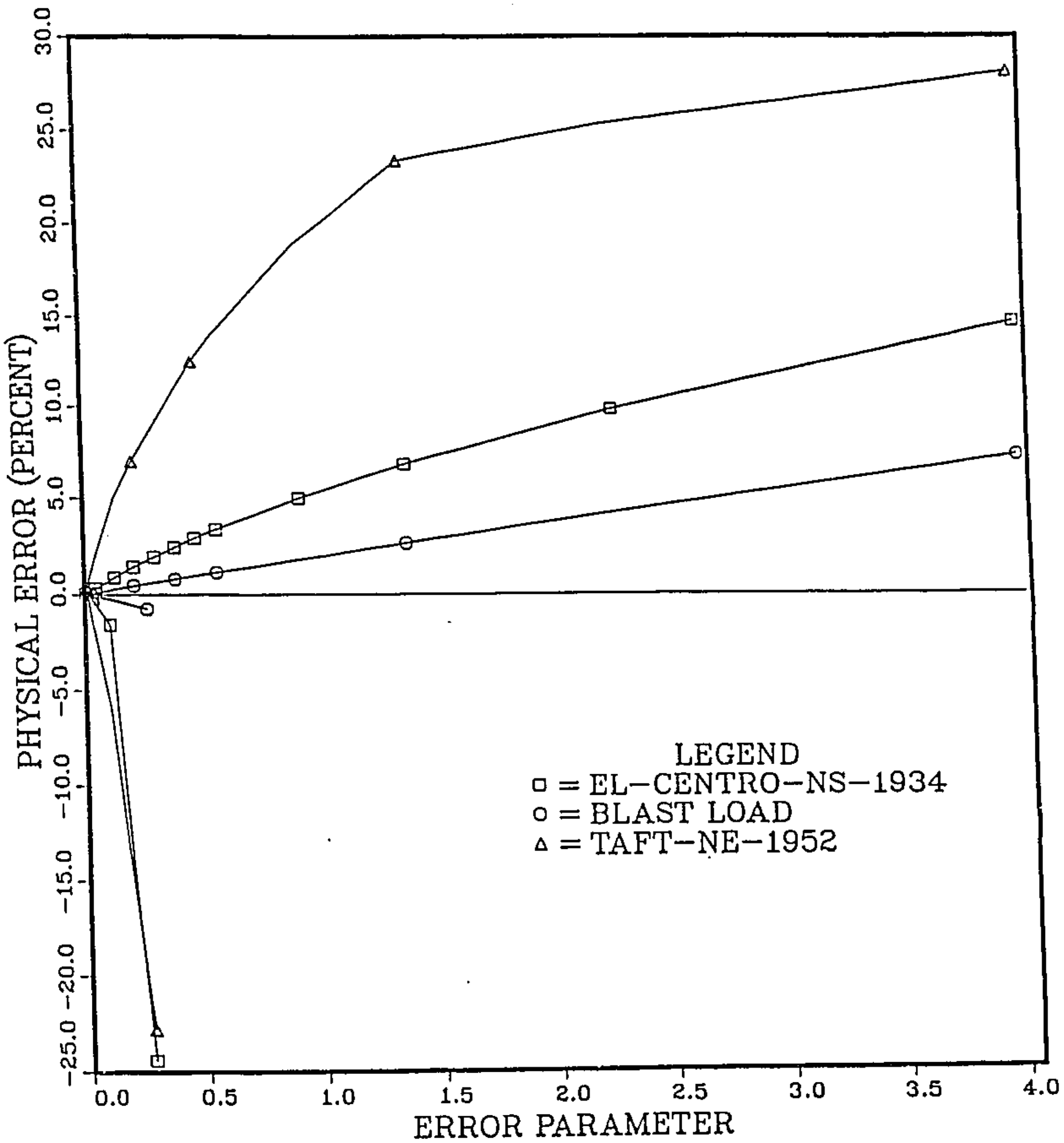


Figure 3.16: Variation of Physical Error With Error-Parameter; Building System no. 3; Mass no. 1, $\beta_2 = 0.021498$, β_1 is varied

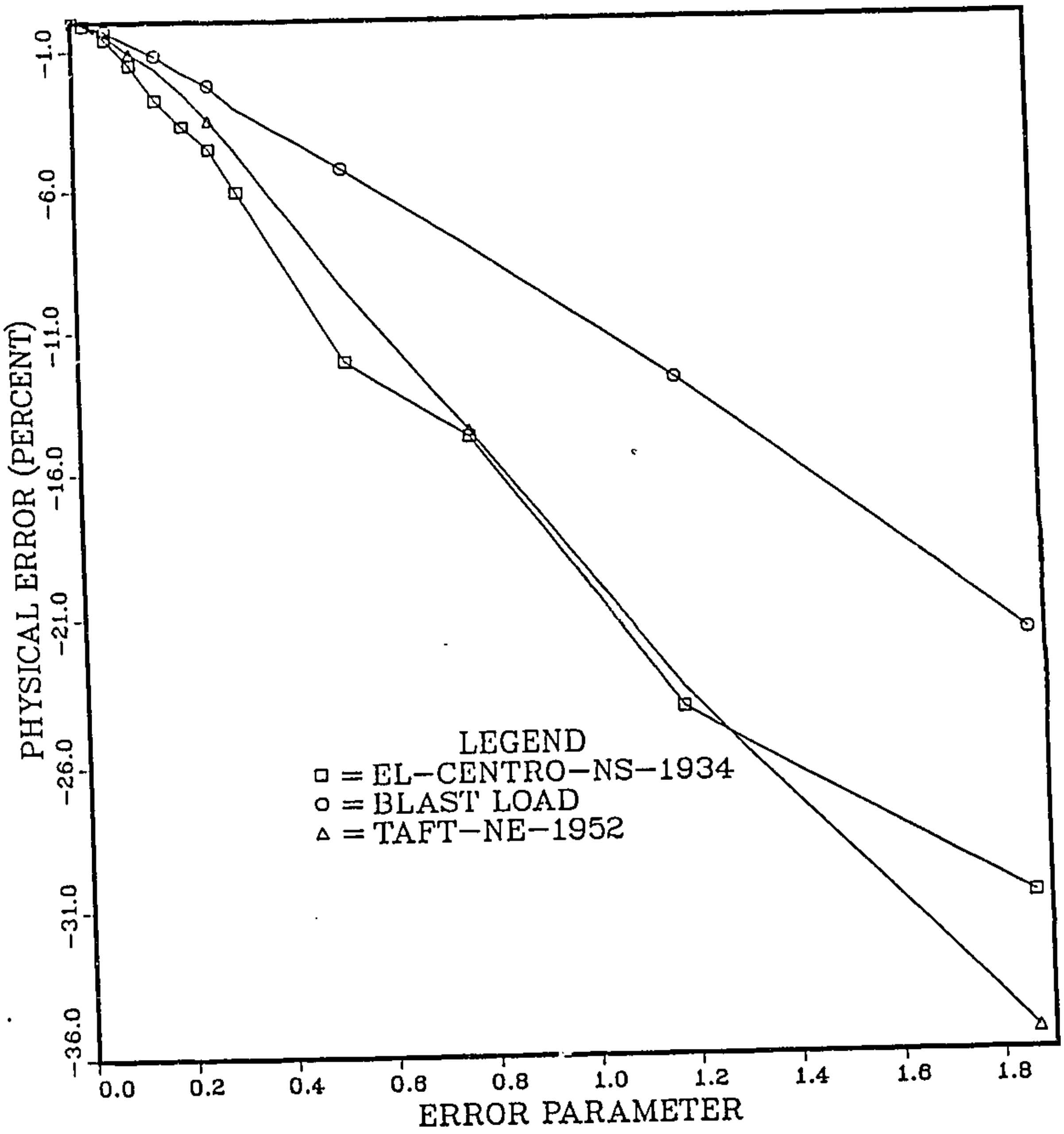


Figure 3.17: Variation of Physical Error With Error-Parameter; Building System no. 3; Mass no. 2, $\beta_2 = 0.021498$, β_1 is varied

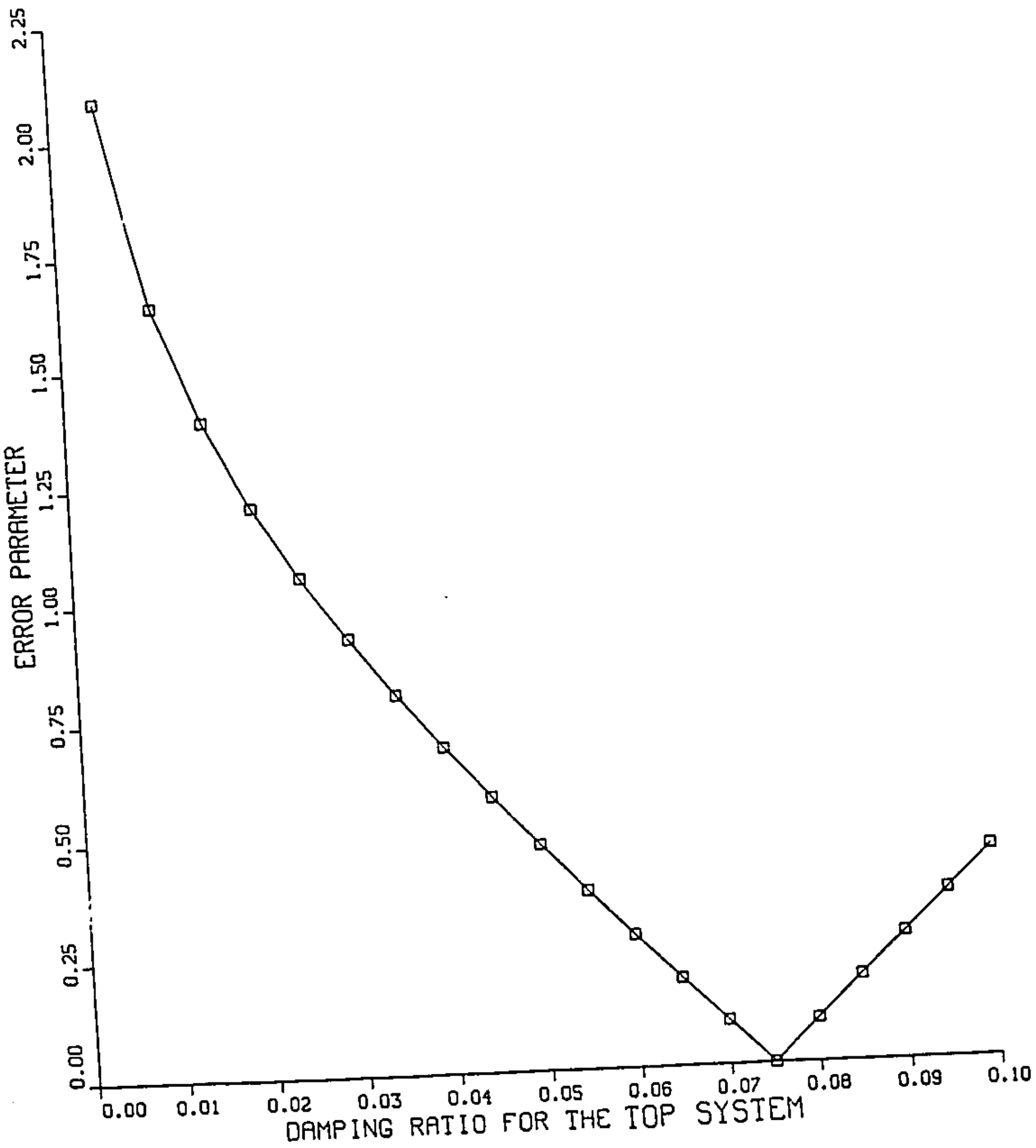


Figure 3.18: Variation of Error-Parameter; Building System no. 1; Mass no. 1, $\beta_1 = 0.075$, β_2 is varied

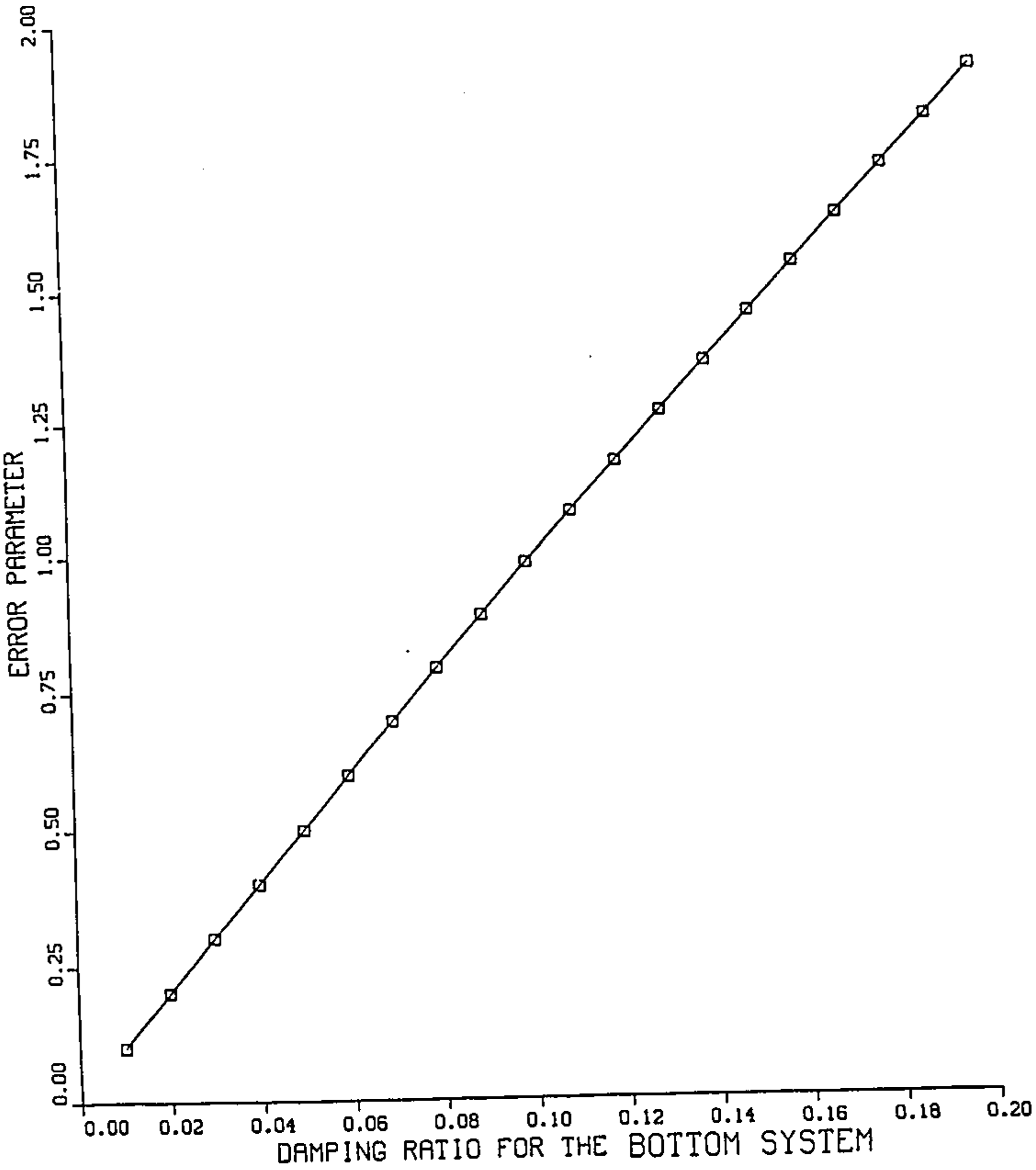


Figure 3.19: Variation of Error-Parameter; Building System no. 1; Mass no. 1, $\beta_2 = 0.5\beta_1$, β_1 is varied

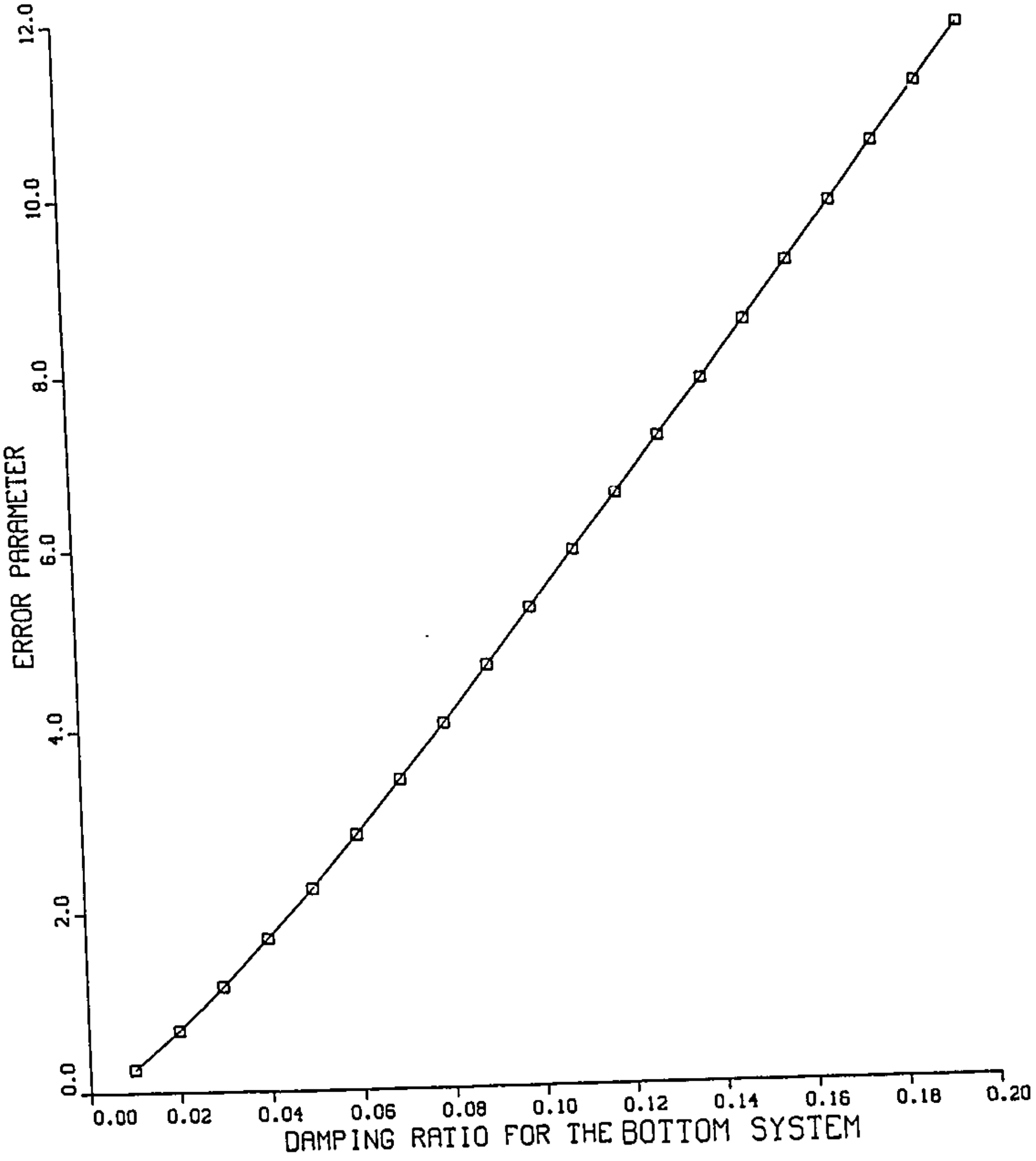


Figure 3.20: Variation of Error-Parameter; Building System no. 1; Mass no. 1, $\beta_2 = 0.0$, β_1 is varied

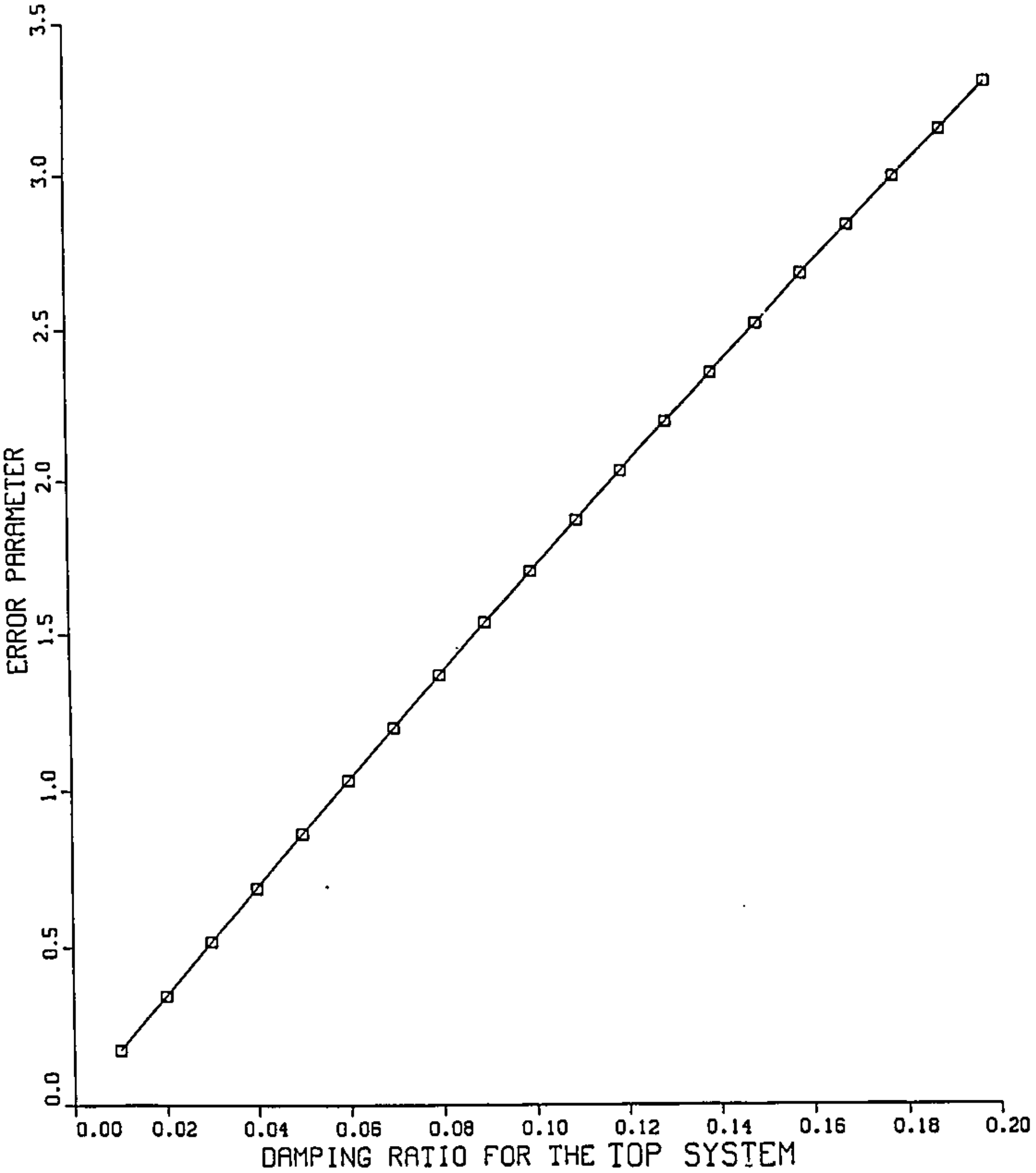


Figure 3.21: Variation of Error-Parameter; Building System no. 1; Mass no. 1, $\beta_1 = 0.0$, β_2 is varied

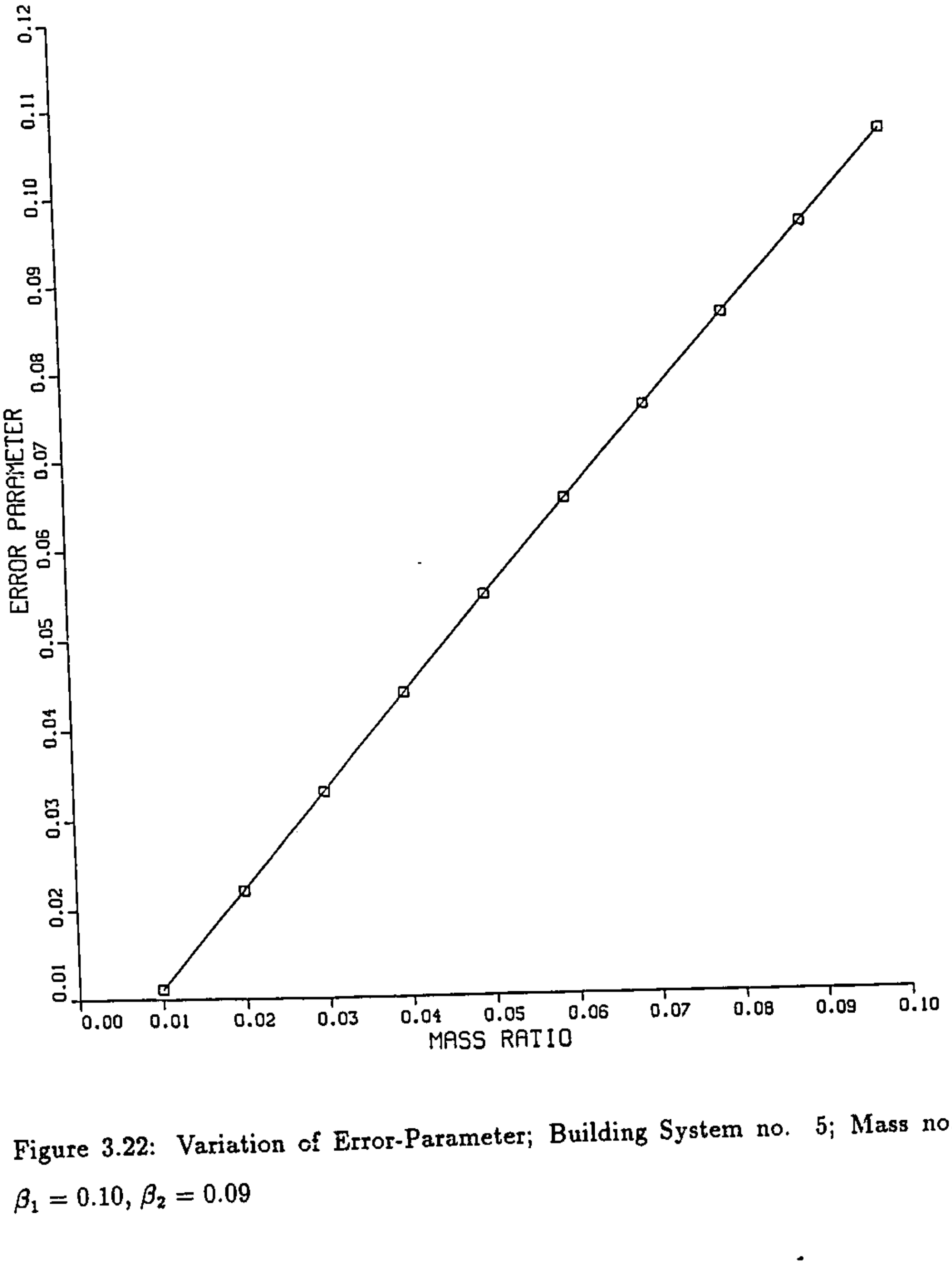


Figure 3.22: Variation of Error-Parameter; Building System no. 5; Mass no. 1, $\beta_1 = 0.10$, $\beta_2 = 0.09$

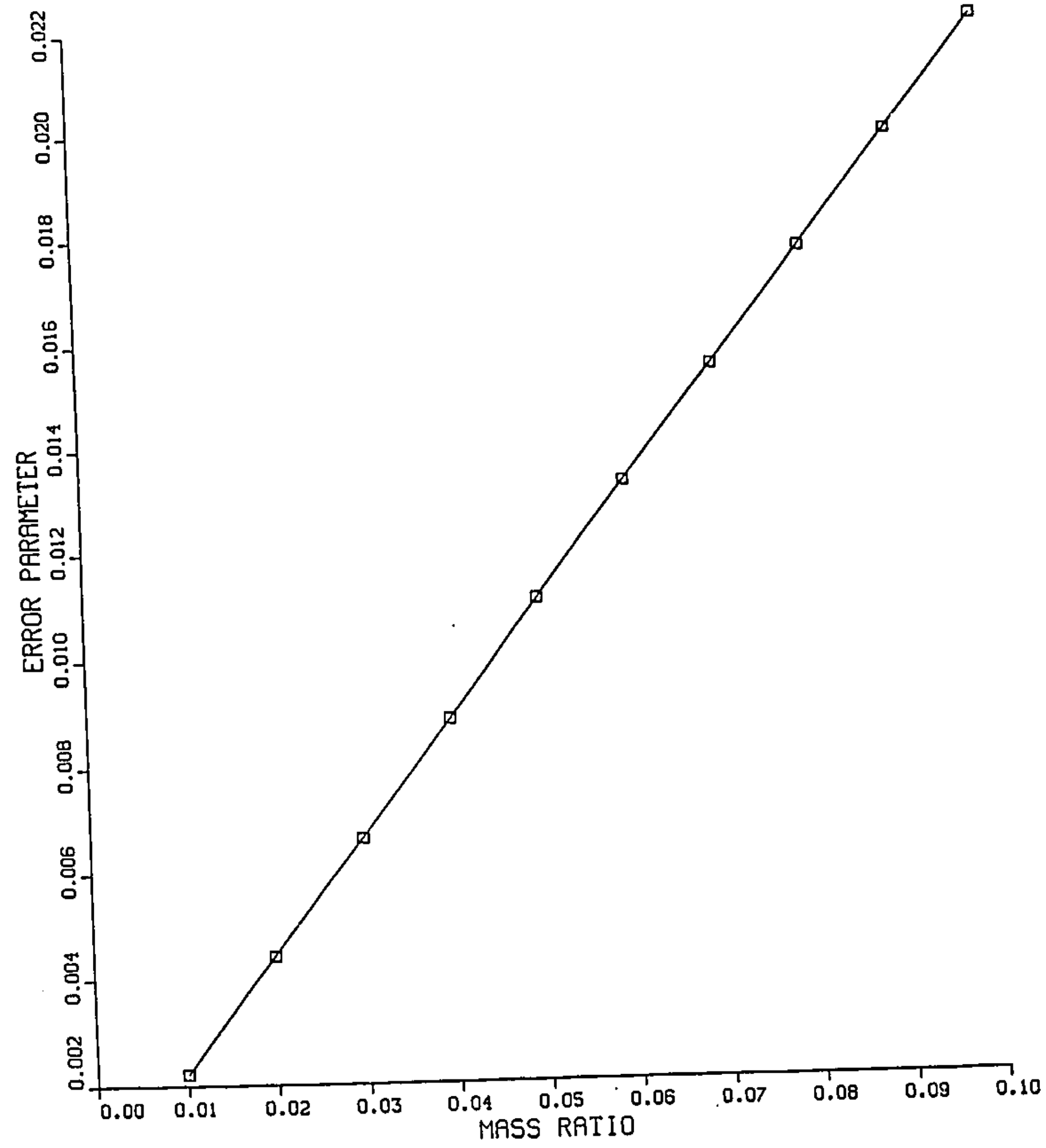


Figure 3.23: Variation of Error-Parameter; Building System no. 5; Mass no. 1, $\beta_1 = 0.10$, $\beta_2 = 0.02$

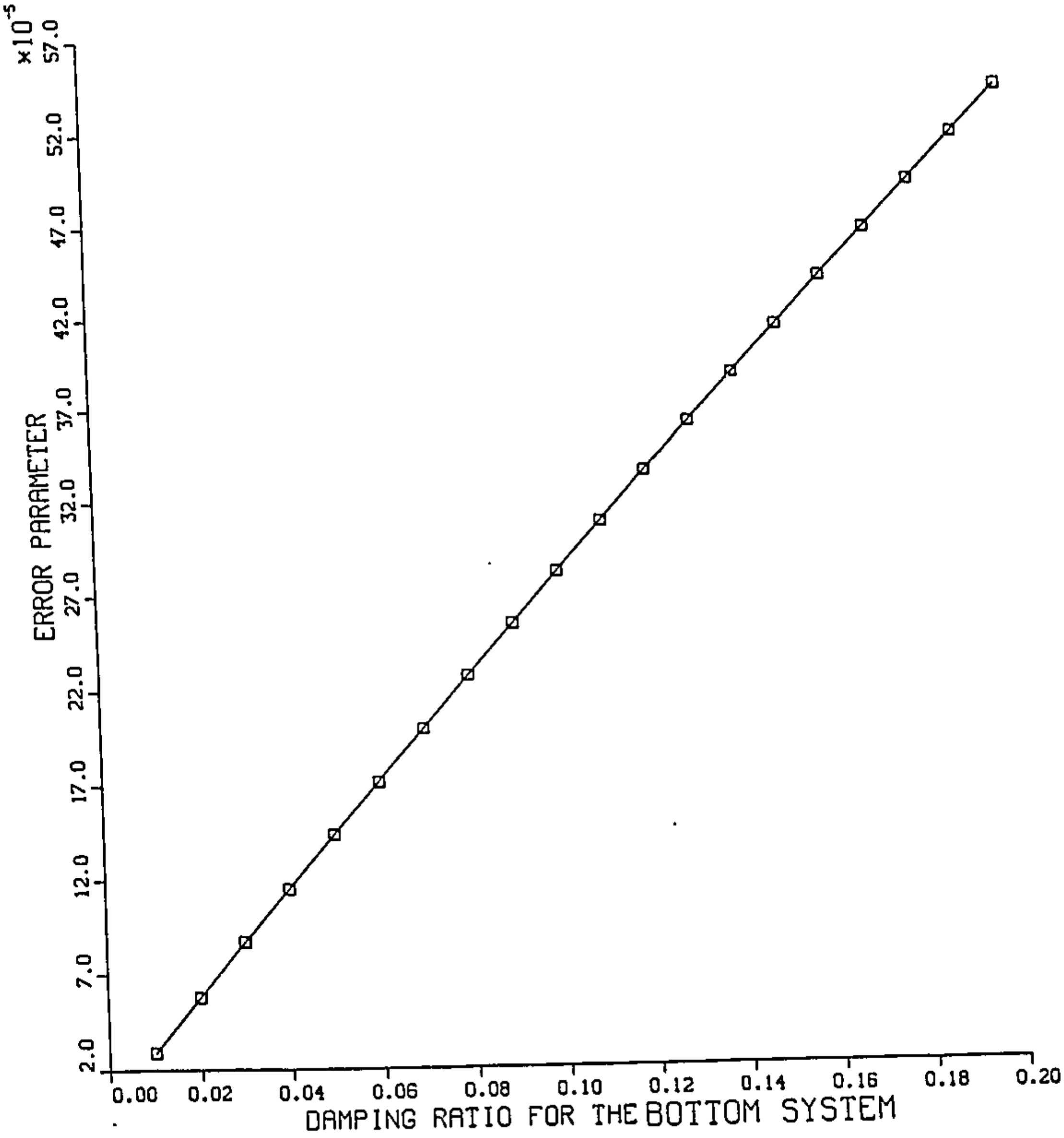


Figure 3.24: Variation of Error-Parameter; Building System no. 6; Mass no. 1, $\beta_1 = \beta_2$, β_2 is varied

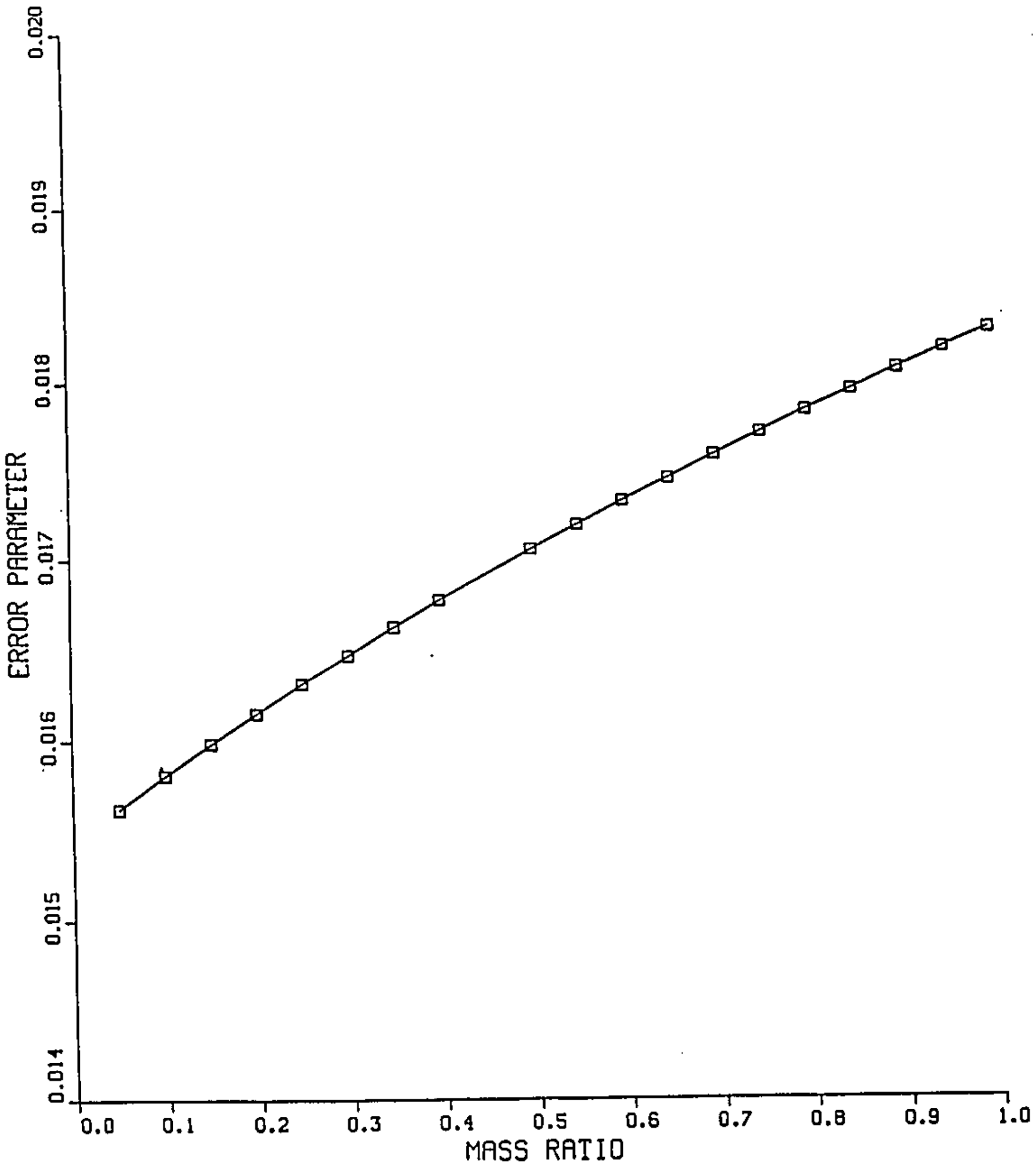


Figure 3.25: Variation of Error-Parameter; Building System no. 4; Mass no. 1, $\beta_1 = 0.10$, $\beta_2 = 0.09$

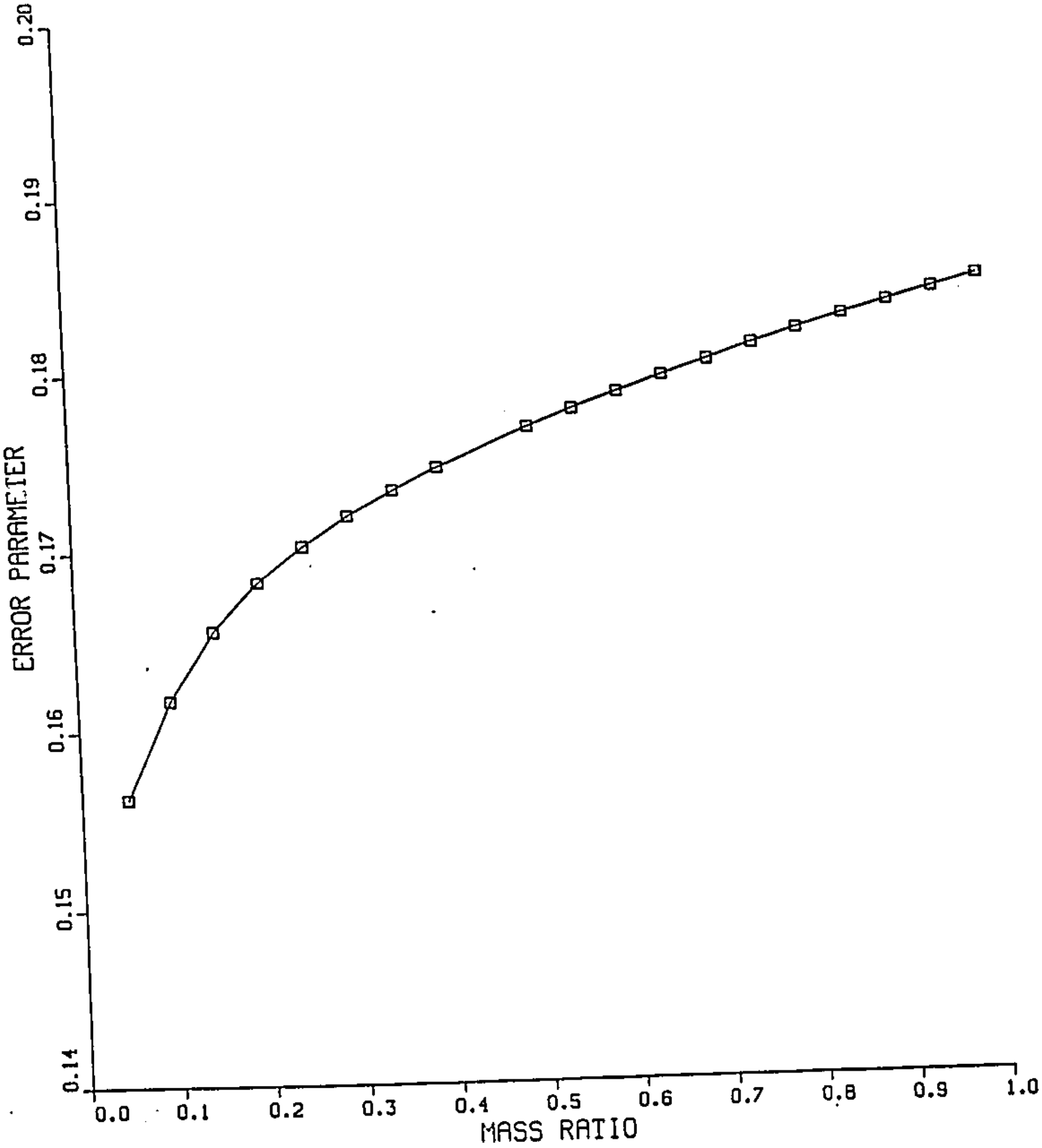


Figure 3.26: Variation of Error-Parameter; Building System no. 4; Mass no. 1, $\beta_1 = 0.10$, $\beta_2 = 0.02$

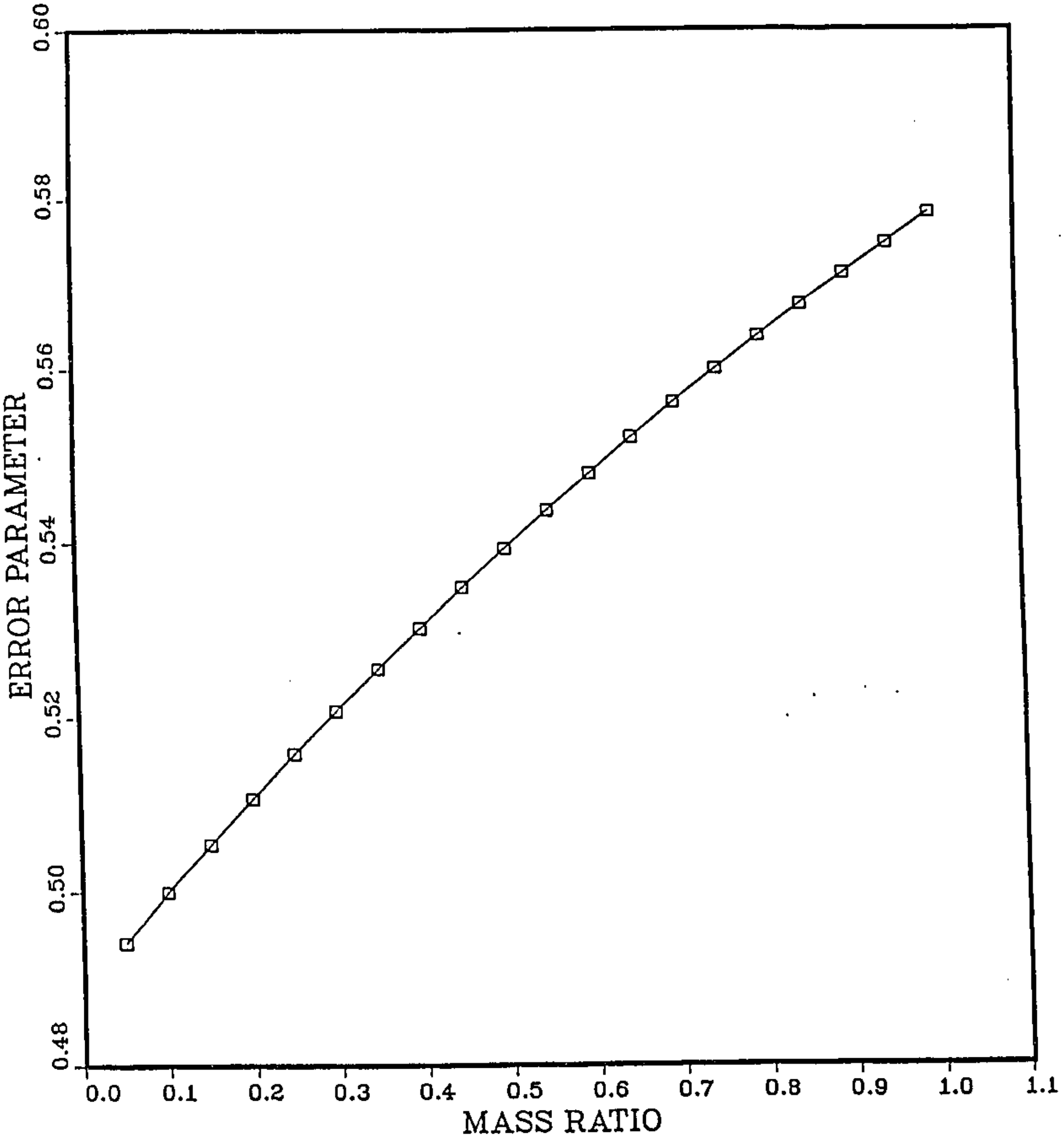


Figure 3.27: Variation of Error-Parameter; Building System no. 2; Mass no. 1, $\beta_1 = 0.10$, $\beta_2 = 0.09$

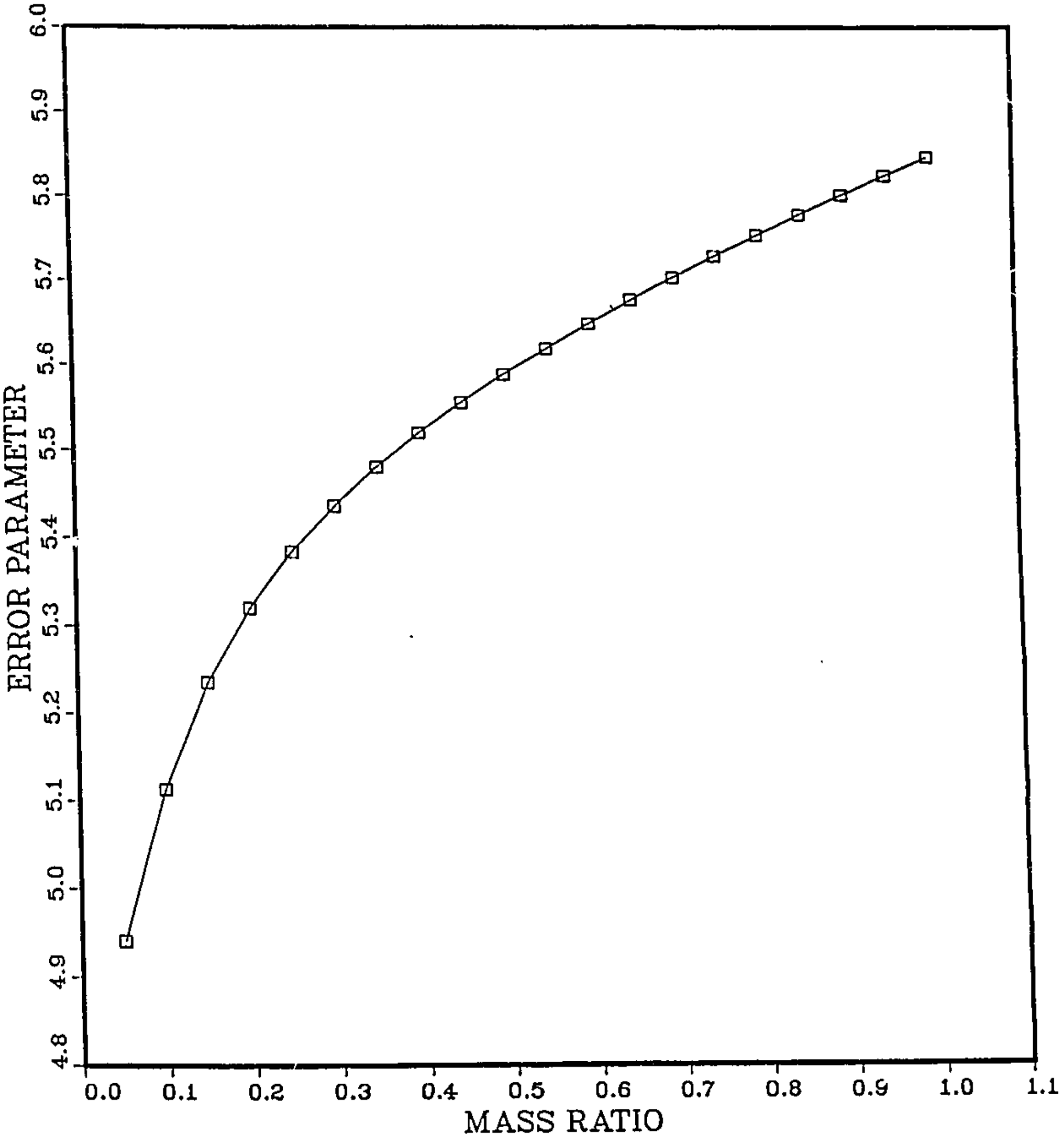


Figure 3.28: Variation of Error-Parameter; Building System no. 2; Mass no. 1, $\beta_1 = 0.10$, $\beta_2 = 0.02$

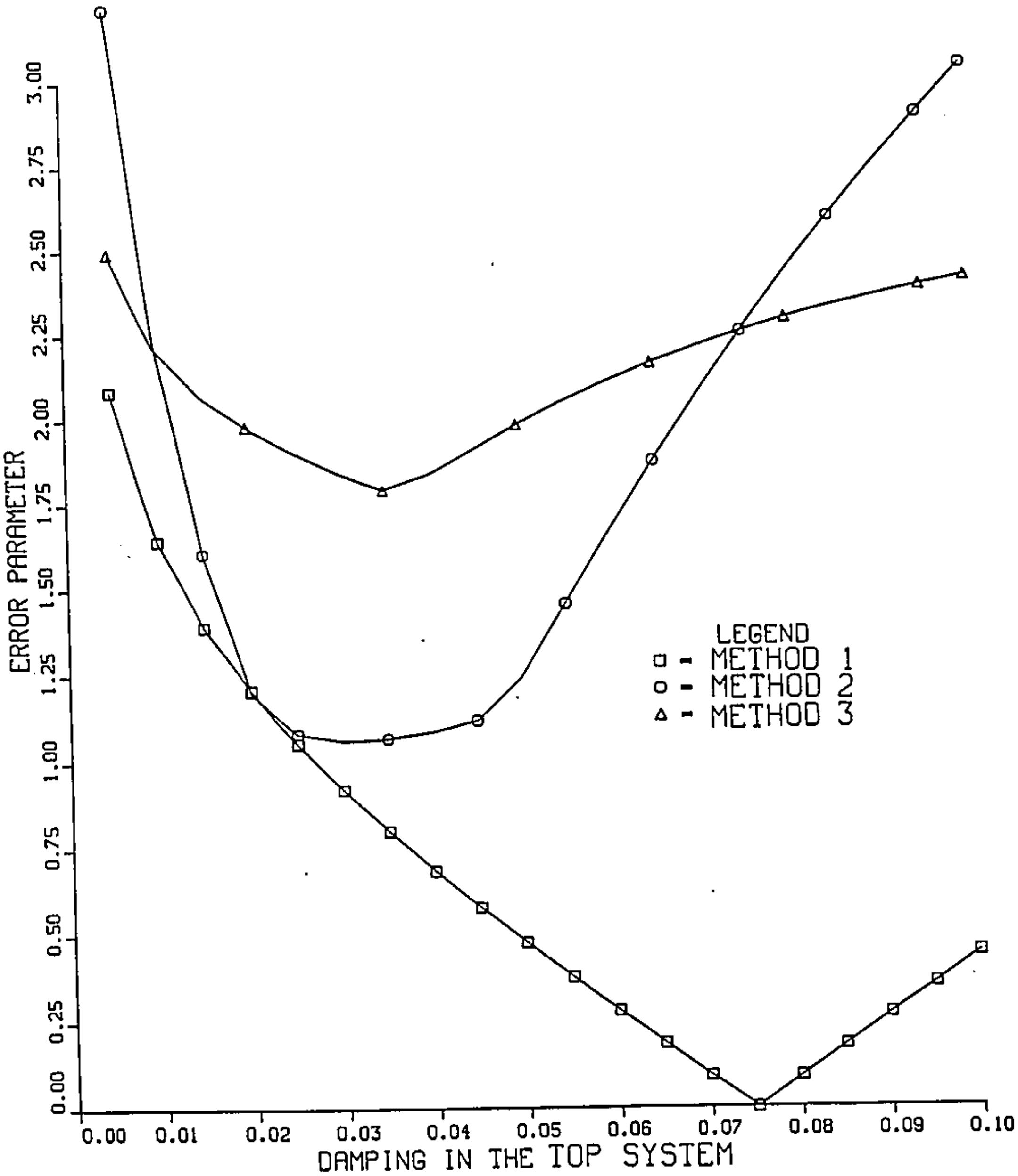


Figure 3.29: Performance of Modal Damping Estimation Procedures; Building System no. 1; Mass no. 1, $\beta_1 = 0.075$, β_2 is varied.

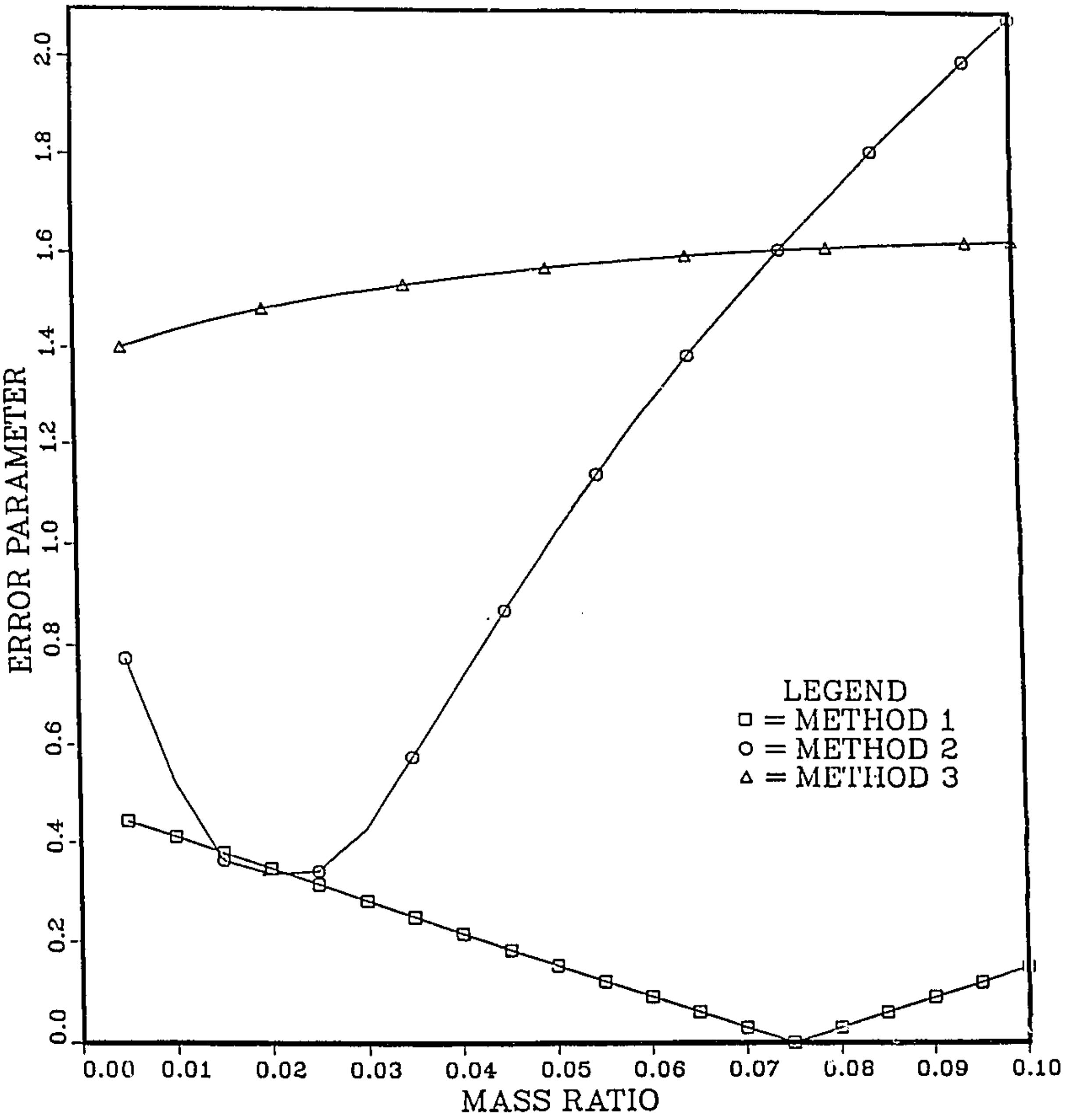


Figure 3.30: Performance of Modal Damping Estimation Procedures; Building System no. 1; Mass no. 2, $\beta_1 = 0.075$, β_2 is varied.

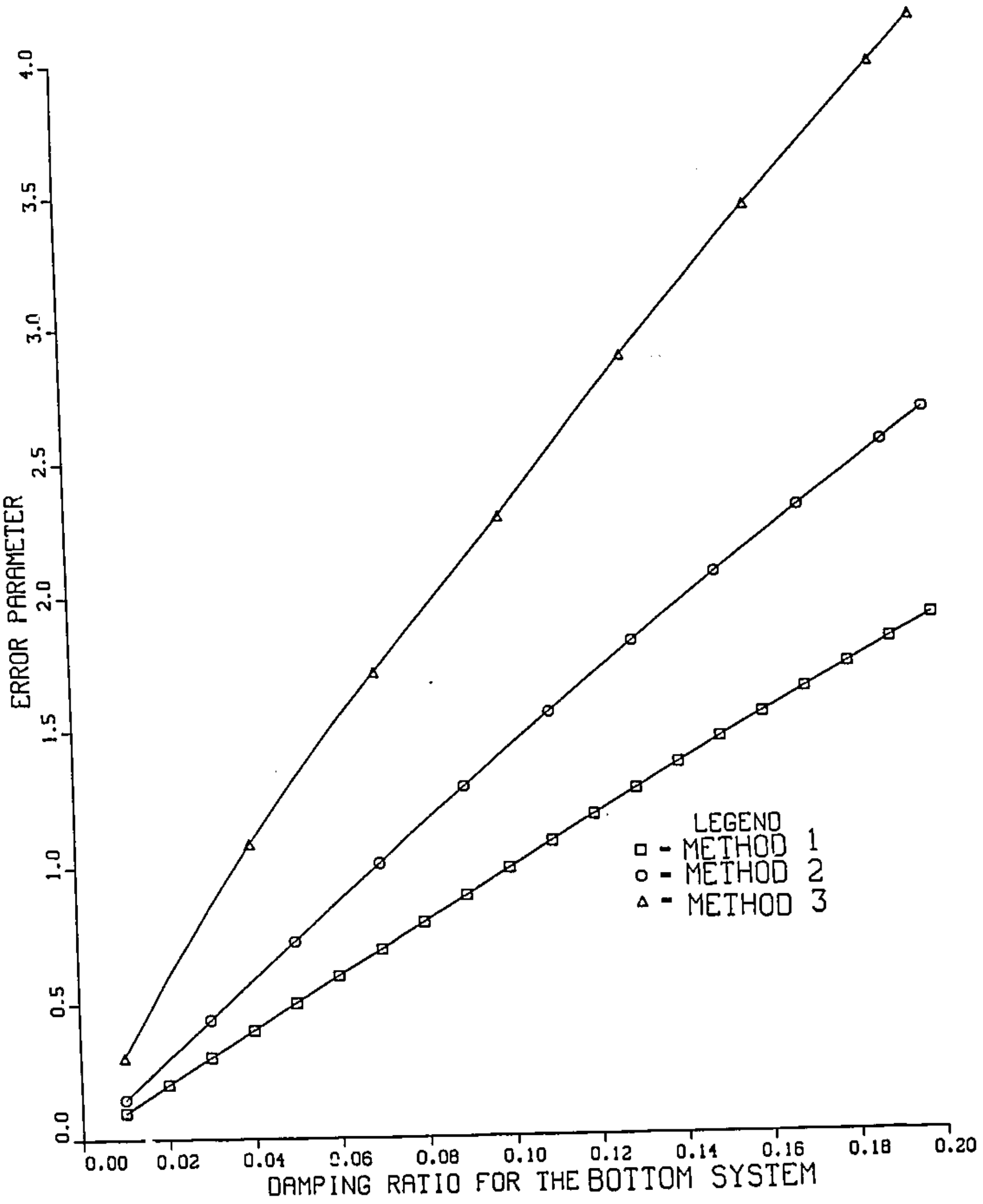


Figure 3.31: Performance of Modal Damping Estimation Procedures; Building System no. 1; Mass no. 1, $\beta_2 = 0.5\beta_1$, β_1 is varied.

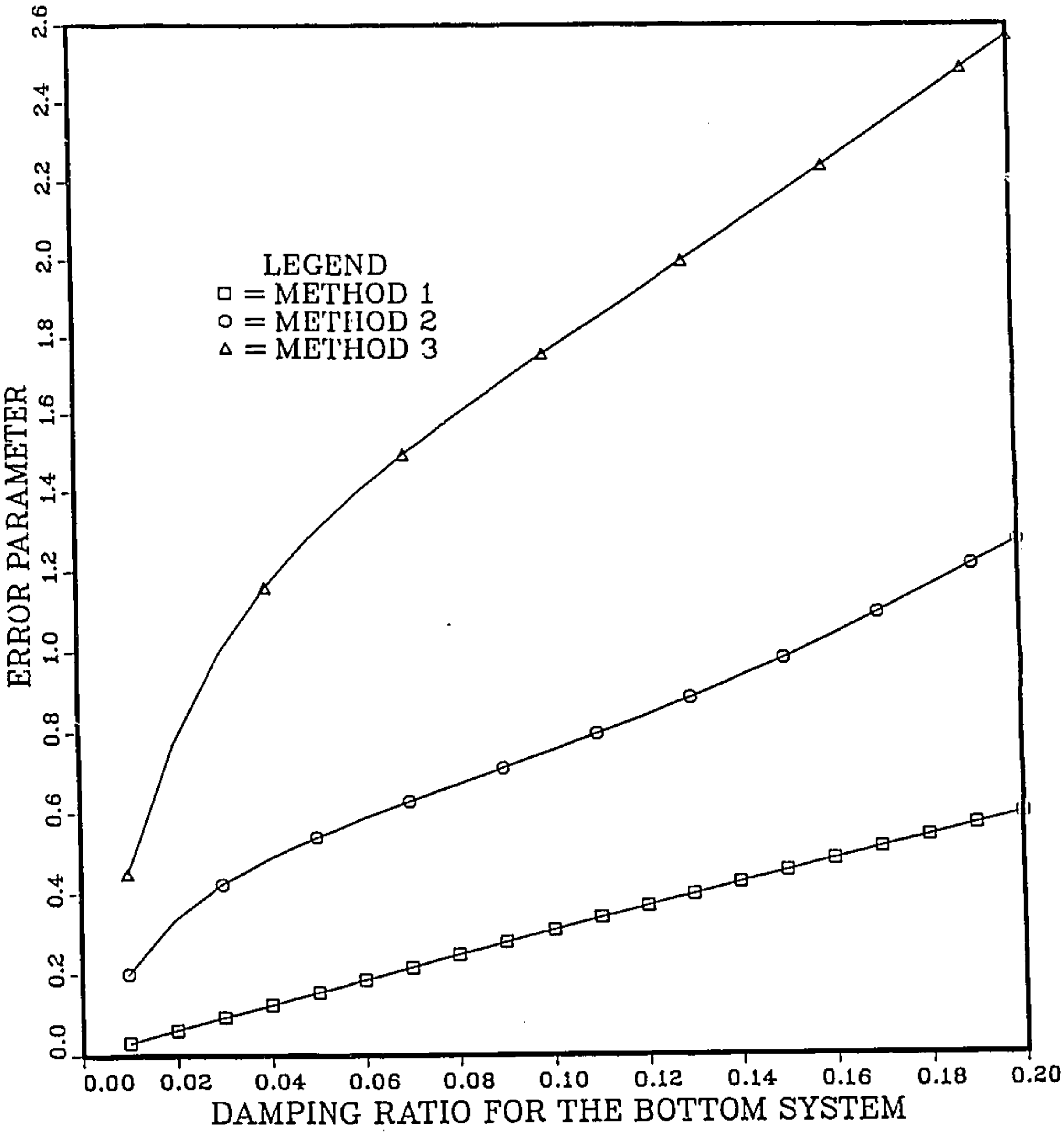


Figure 3.32: Performance of Modal Damping Estimation Procedures; Building System no. 1; Mass no. 2, $\beta_2 = 0.5\beta_1$, β_1 is varied.

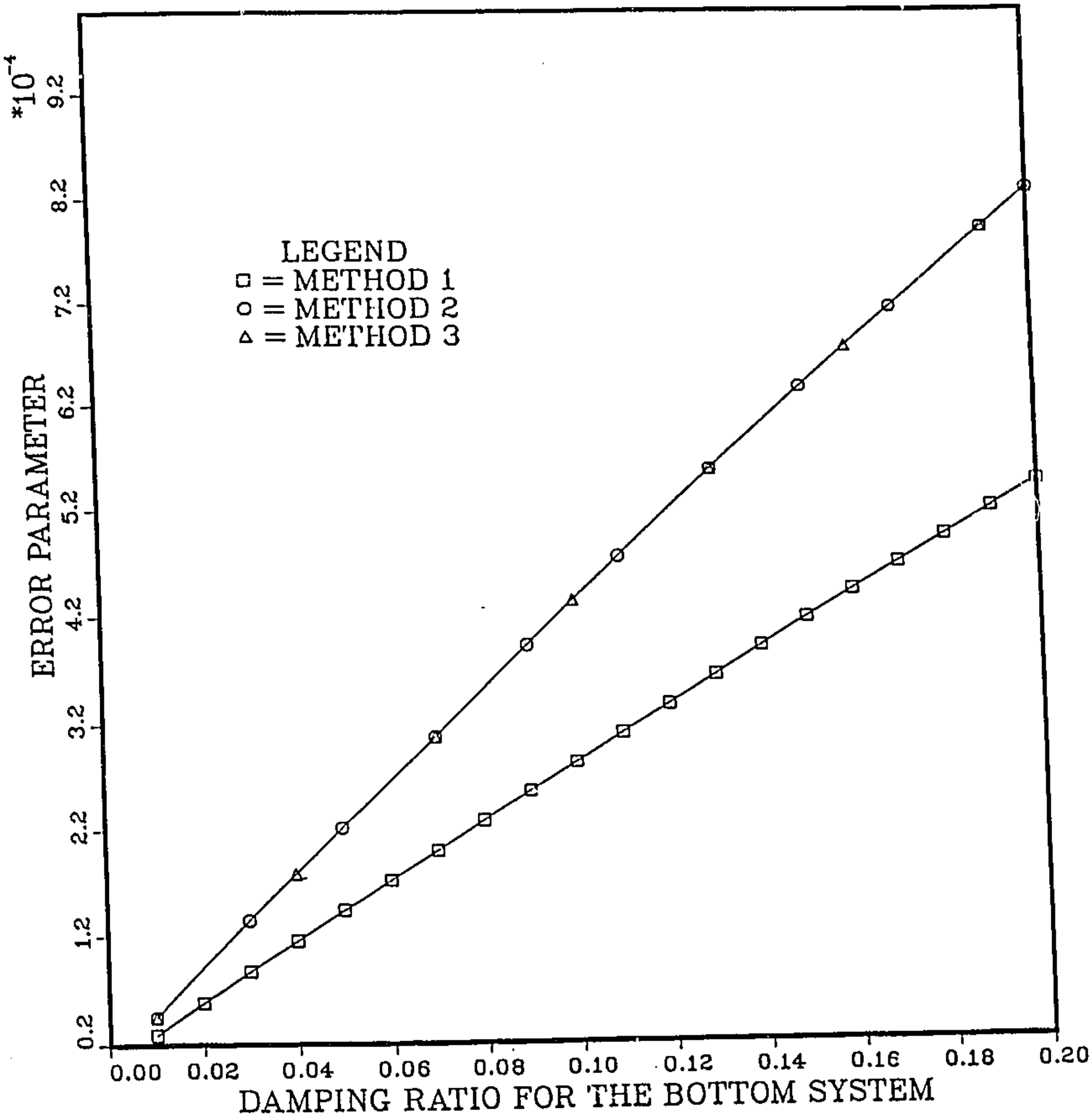


Figure 3.33: Performance of Modal Damping Estimation Procedures; Building System no. 6; Mass no. 1, $\beta_1 = \beta_2$, β_2 is varied.

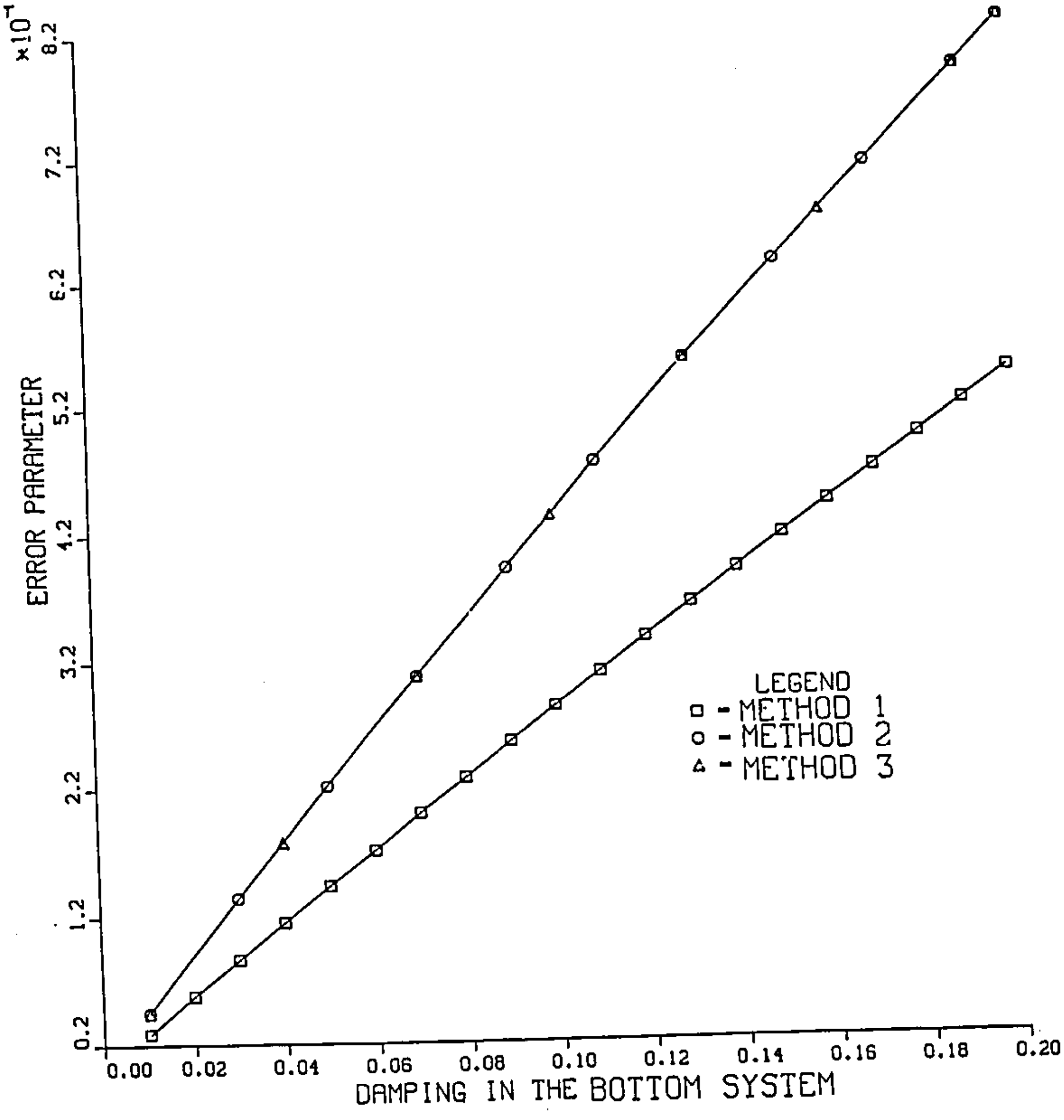


Figure 3.34: Performance of Modal Damping Estimation Procedures; Building System no. 6; Mass no. 2, $\beta_1 = \beta_2$, β_2 is varied.

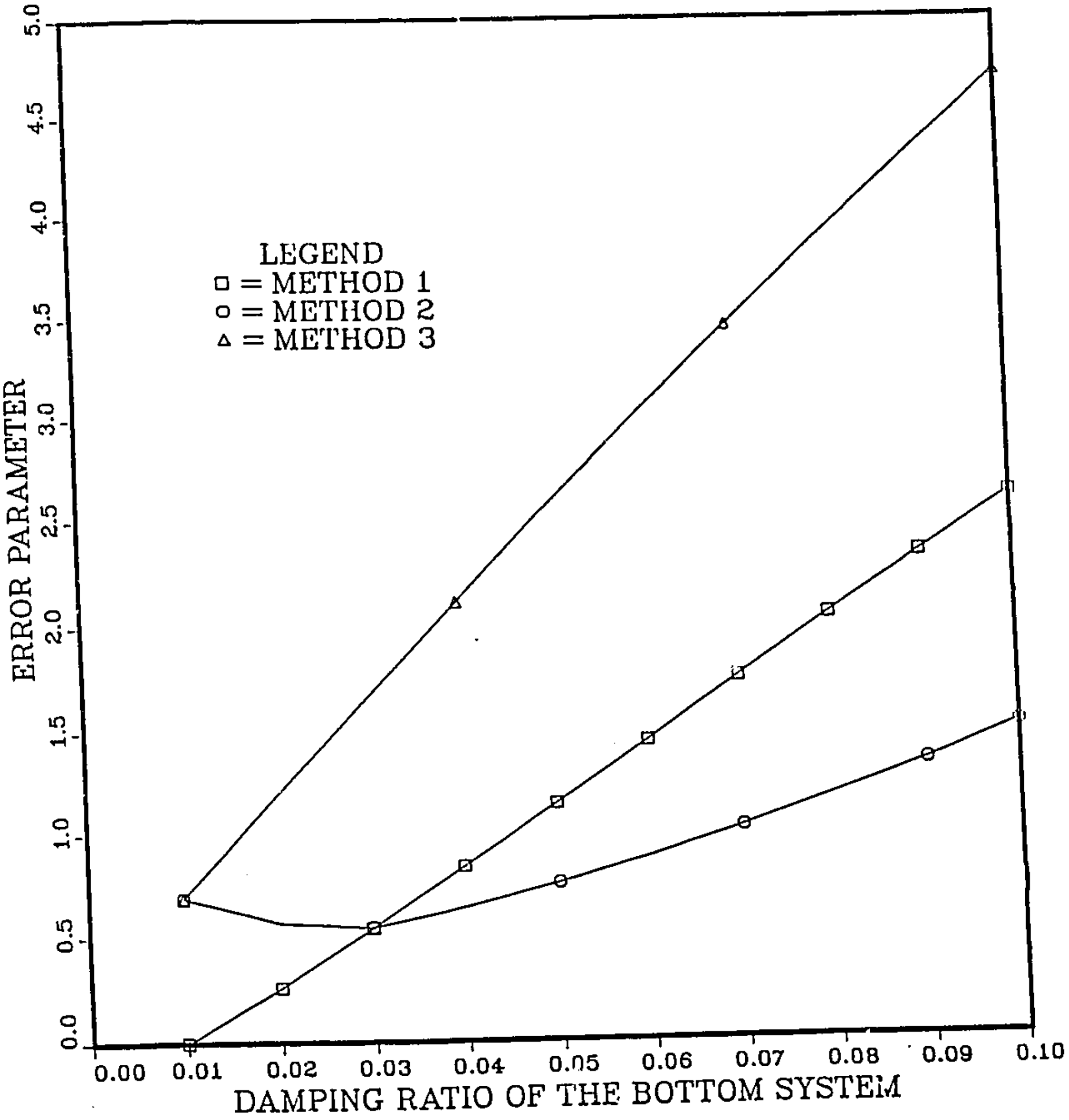


Figure 3.35: Performance of Modal Damping Estimation Procedures; Building System no. 2; Mass no. 1, Mass Ratio=0.05, $\beta_2 = 0.01$, β_1 is varied.

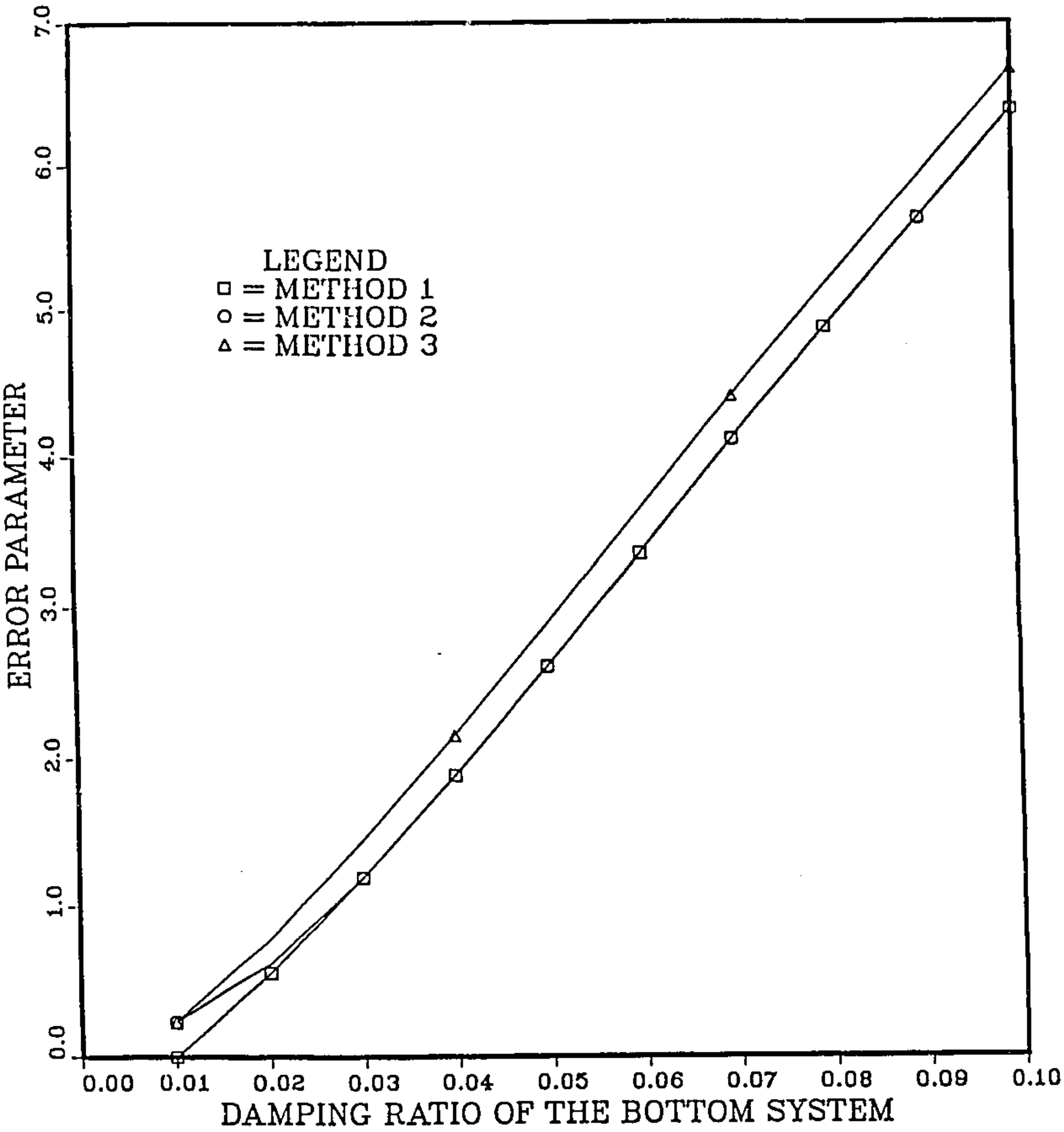


Figure 3.36: Performance of Modal Damping Estimation Procedures; Building System no. 2; Mass no. 2, Mass Ratio=0.05, $\beta_2 = 0.01$, β_1 is varied.

Chapter 4

Modal Properties of the Combined System

The "Combination of Modal Properties" approach involves obtaining modal properties of the combined system using those of the individual systems using a suitable combination technique. Perturbation and component mode synthesis are the two main techniques that have been explored by various investigators. Modal properties of the combined system obtained using any one of these methods are used to perform a modal superposition or a response spectrum analysis of the combined system. Perturbation techniques are more popular and are widely used. In the discussion that follows, it is assumed that eigenproperties of the combined system are obtained using perturbation. However, the discussion is equally applicable to the combined system modal properties obtained using component mode synthesis.

Modal properties obtained using perturbation can only be an approximation to the "exact" modal properties of the combined system. This chapter deals with the error in response (as obtained using either the modal superposition or the response spectrum method) due to the approximation inherent in using approximate modal properties obtained from perturbation techniques. No literature on this topic is known to the author. Content of this chapter is the author's contribution to this area.

Strictly speaking, the modal superposition procedure is not applicable to non-classically damped system. It is possible to use modal superposition for such systems in an approximate sense. However, for the purpose of the discussion in this chapter, only classically damped P-S coupled systems are considered.

4.1 Classically Damped P-S Coupled Systems

Consider a P-S coupled system in which classical damping assumption is valid. Equation of motion of the combined dynamic system subjected to a horizontal earthquake excitation (acting only in one direction) may be written as:

$$[M]\{\ddot{y}\} + [C]\{\dot{y}\} + [K]\{y\} = -[M]\{U_b\}\ddot{u}_g \tag{4.1}$$

where, $[M]$, $[C]$ and $[K]$ are the mass, damping and stiffness matrices respectively of the system, $\{U_b\}$ is a vector that lists static displacement of each DOF due to a unit static displacement of the support. $\ddot{u}_g$ is the ground acceleration. We denote $\{F\} = -[M]\{U_b\}\ddot{u}_g$. Using standard modal superposition, let

$$\{y\} = [\Phi]\{z\} \tag{4.2}$$

So that,

$$[\Phi]^T[M][\Phi]\{\ddot{z}\} + [\Phi]^T[C][\Phi]\{\dot{z}\} + [\Phi]^T[K][\Phi]\{z\} = [\Phi]^T\{F\} \tag{4.3}$$

Or

$$[M^*]\{\ddot{z}\} + [C^*]\{\dot{z}\} + [K^*]\{z\} = \{F^*\} \tag{4.4}$$

Subscript 'e' will henceforth be used to denote 'exact' quantities and subscript 'a' to denote approximate quantities. Consider the exact mode-shape matrix $[\Phi_e]$ of the combined system. If $[\Phi_e]$ is used for modal superposition, then Eq. 4.3 becomes:

$$\begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}\{\ddot{z}\} + \begin{bmatrix} 2\xi_1\omega_{e1} & & \\ & \ddots & \\ & & 2\xi_N\omega_{eN} \end{bmatrix}\{\dot{z}\} + \begin{bmatrix} \omega_{e1}^2 & & \\ & \ddots & \\ & & \omega_{eN}^2 \end{bmatrix}\{z\} = [\Phi]^T\{F\} \tag{4.5}$$

where $\omega_{e1} \ldots \omega_{eN}$ are the exact natural frequencies and $\xi_1 \ldots \xi_N$ are the modal damping ratios.

The equations of motion are uncoupled. An individual 'modal' equation of motion may be written as:

$$\ddot{z}_j + 2\xi_j \omega_{ej} \dot{z}_j + \omega_{ej}^2 z_j = -\gamma_{ej} \ddot{u}_g \tag{4.6}$$

where, γ_{ej} is the jth modal participation factor defined by:

$$\gamma_{ej} = \{\phi_{ej}\}^T [M] \{U_b\} \tag{4.7}$$

If all modes are used in the analysis and if no numerical errors are present, then the solution of the Eq. 4.5 and subsequent substitution in Eq. 4.2 would yield the exact response.

Engineers dealing with P-S coupled systems generally do not have access to the exact modal properties of the combined system. Instead, approximate modal properties obtained using a suitable perturbation technique are employed. Response obtained using approximate modal properties is expected to be approximate.

If an approximate mode-shape matrix $[\Phi_a]$ is used, then we seek the approximate response $\{y_a\}$ as:

$$\{y_a\} = [\Phi_a]\{z_a\} \tag{4.8}$$

If the above procedure of modal superposition is followed using $[\Phi_a]$, we get the following equations:

$$[\Phi_a]^T[M][\Phi_a]\{\ddot{z}_a\} + [\Phi_a]^T[C][\Phi_a]\{\dot{z}_a\} + [\Phi_a]^T[K][\Phi_a]\{z_a\} = -\{\Gamma_a\}\ddot{u}_g \tag{4.9}$$

Or

$$[M_a^*]\{\ddot{z}_a\} + [C_a^*]\{\dot{z}_a\} + [K_a^*]\{z_a\} = -\{\Gamma_a\}\ddot{u}_g \tag{4.10}$$

where, $\{\Gamma_a\}$ lists all the participation factors γ_{aj}.

Though matrices $[M^*]$, $[C^*]$ and $[K^*]$ above are diagonal, matrices $[M_a^*]$, $[C_a^*]$ and $[K_a^*]$ cannot, in general, be expected to be diagonal. This renders the equations of motion Eq. 4.10 coupled due to the presence of off-diagonal terms in the three matrices.

4.1.1 Sources of Error

If modal superposition solution is desired using approximate mode shapes, the off-diagonal terms in Eq. 4.10 must be ignored. In other words, modal coupling introduced by approximate mode shapes must be ignored. The decoupled equations of motion are then solved and substituted back into Eq. 4.8.

Error involved in using approximate mode shapes mainly stems from any approximation involved in the solution of Eq. 4.10. One of the most important sources of error is the neglect of 'modal coupling' explained above.

Modal superposition solution using approximate mode shapes implies solution of 'modal' equations of motion of the form:

$$\ddot{z}_{aj} + 2\xi_j\omega_{aj}\dot{z}_{aj} + \omega_{aj}^2 z_{aj} = -\gamma_{aj}\ddot{u}_g \tag{4.11}$$

In other words, $[C_a^*]$ is replaced by

$$[C_{aa}^*] = \begin{bmatrix} 2\xi_1\omega_{a1} & & \\ & \ddots & \\ & & 2\xi_N\omega_{aN} \end{bmatrix} \tag{4.12}$$

and $[K_a^*]$ is replaced by

$$[K_{aa}^*] = \begin{bmatrix} \omega_{a1}^2 & & \\ & \ddots & \\ & & \omega_{aN}^2 \end{bmatrix} \tag{4.13}$$

It may be noted that there is additional approximation (i.e. in addition to that involved in off-diagonal terms) associated with the fact that the diagonal terms in the matrices in Eq. 4.10 are being replaced by quantities such as $2\xi_j\omega_{aj}$ or ω_{aj}^2. Such replacement will henceforth be referred to as 'modification of the diagonal terms'. It is emphasized that the j th diagonal term on $[K_a^*]$ is *not* equal to ω_{aj}^2 and the j th diagonal term on $[C_a^*]$ is *not* equal to $2\xi_j\omega_{aj}$. This is due to the fact that the approximate eigenproperties cannot be expected to satisfy similar properties that are satisfied by 'exact' eigenproperties. In a perturbation scheme, eigenvalues and eigenvectors are obtained separately in an approximate manner. This is unlike an exact eigensolution, where eigenvalues and eigenvectors are connected through stiffness and mass matrices by mathematical identities; if eigenvalues are known, eigenvectors can be computed exactly and vice versa.

It may be noted that there is no approximation involved in assuming that modal masses are equal to unity, since the approximate mode shapes can always be scaled so as to yield unit modal masses.

Also, there is no error associated with the fact that original modal participation factors have now been replaced with approximate participation factors. Discussion on this statement appears later in this chapter.

Experience shows that the error involved in ignoring the off-diagonal terms is most severe. Error involved in modifying diagonal terms is generally very small and may often be neglected.

4.2 A Different Perspective

It was demonstrated earlier that the error involved in using approximate mode shapes is mainly due to approximations made in order to solve Eq. 4.10; most severe error is associated with the neglect of off-diagonal terms. (In future discussion, no reference will be made to the error involved in 'modifying' off-diagonal terms as in Eqs. 4.12 and 4.13. This does not imply that this error is being neglected. The understanding is that this error exists in addition to the error involved in neglecting off-diagonal terms.)

A pertinent question is whether this is the only source of error. In other words, if there is a way of eliminating the error involved in ignoring off-diagonal terms, is it possible to obtain exact response using approximate mode shapes? The answer to this question is yes.

Modal superposition method is basically a transformation method. In this method, the original equations of motion in physical co-ordinates are transformed into a set of modal equations; the transformation matrix being the modal matrix

$[\Phi]$. However, it is possible to use other transformation matrices in place of $[\Phi]$. Normal modes shapes furnish an extremely convenient transformation matrix since they render the stiffness and mass matrices diagonal. For classically damped systems, they also render the damping matrix diagonal. This makes it possible to deal with 'modal' equations of motion, which are very simple to solve. However, the fact that equations of motion can be decoupled using normal mode shapes is only an additional advantage. The only basic requirement for a matrix to be suitable for use as a transformation matrix is that it must be *non-singular*. Thus, any non-singular matrix can be used to transform the equations of motion; however, the transformed equations of motion will, in general, be coupled. If the coupled equations of motion are solved in an exact manner, then the exact response in physical co-ordinates can be obtained from that in the 'transformed' co-ordinates.

For numerical verification of the above principles, Clough and Mojtahedi's [14] method provides an excellent tool. This method was originally proposed by Clough and Mojtahedi as a tool for analysis of non-classically damped systems. However, the concept is more general and provides a tool to verify the above statements.

4.2.1 Clough and Mojtahedi's Method

Clough and Mojtahedi proposed this method as a way of accounting for modal interaction caused by off-diagonal terms in the transformed damping matrix of a non-classically damped system. They suggested direct integration of the transformed equations of motion such as Eq. 4.3. At each time step, the response obtained in terms of the transformed co-ordinates is transformed back to get exact physical response. This fully accounts for any interaction due to the presence of off-diagonal terms in the transformed matrices $[M^*]$, $[C^*]$ and $[K^*]$. In their paper, Clough and Mojtahedi concerned themselves only with the modal interaction caused by off-diagonal terms in the damping matrix. If exact eigenvectors are used, this is the only form of interaction present. However, if approximate eigenvectors are used, off-diagonal terms are present in all the three matrices. Clough and Mojtahedi's method, though not originally intended for off-diagonal terms in the mass and stiffness matrices, can certainly be used to account for 'interaction' caused by their presence.

During the course of this study, several numerical examples were solved using non-conventional transformation matrices. A program was developed to analyze an MDOF system using Clough and Mojtahedi's method. The response results were compared with those obtained by direct time integration of original equations of motion. In all the cases studied, both the responses were identical. (All modes were considered in the analysis.)

4.3 Mode Shape Matrix Obtained by Perturbation

In view of these facts, the mode shape matrix obtained by perturbation may be considered as a valid transformation matrix. Mode shapes obtained using a reliable perturbation technique are usually very close to the exact mode shapes. As mentioned earlier, it is possible to use an arbitrary transformation matrix, however it must always be ensured that the matrix is non-singular. It is assumed here that the mode shapes obtained using perturbation resemble the exact mode shapes closely enough so that the chance of the mode shape matrix $[\Phi_a]$ being singular may be neglected.

If the transformed equations of motion are solved using Clough and Mojtahedi's method, then exact results may be expected. This is in fact so. A number of numerical examples solved during the course of this study showed that response obtained using perturbation mode shapes and Clough and Mojtahedi's method is identical to that obtained using direct time integration of original equations of motion of the combined dynamic system, provided all modes are considered in the analysis. Thus, the error involved in using approximate mode shapes is *not* associated with a transformation as in Eq. 4.8; instead it is mainly due to the neglect of off-diagonal terms in the transformed matrices.

It was mentioned earlier that additional error is involved if diagonal terms on the transformed matrices are modified. This error is independent of the error involved in forcing the off-diagonal terms to zero. In a certain sense, modification of diagonal terms is unwarranted. It is more correct to use the diagonal terms generated by the transformation. However, from the point of view of a practising engineer, actual triple-matrix multiplication as in Eq 4.3 is never conducted in a mode superposition analysis. The engineer deals directly with modal equations such as Eq 4.6 or 4.11. In this equation, it is convenient to think of the coefficients involved in terms of their physical meaning. This is the reason for the said modification. For the purpose of research, the modification may be easily avoided by performing the actual triple-matrix multiplications. For this reason, discussion of the error associated with modification of diagonal terms has been kept to the minimum.

As mentioned earlier, no error is associated with using approximate modal participation factors γ_{aj}. This is due to the fact that evaluation of γ_{aj} is entirely consistent with the process of transformation of the equations of motion Eq. 4.1 using $[\Phi_a]$ is a valid transformation matrix. In other words, γ_{aj} were obtained by a valid transformation of the equations of motion; no approximations were made in their evaluation. The fact that γ_{aj} are different from γ_{ej} is attributed entirely to the fact that they were obtained by two different transformation procedures, both transformations being equally capable of giving exact physical response.

4.4 A Suggested Technique

The foregoing discussion suggests that it is possible to obtain exact response using approximate mode shapes if a method such as the Clough and Mojtahedi's method is used. Specifically, this method accounts for all possible modal interaction including the substantial damping interaction in the case of non-classically damped systems.

Being a time-history method, the question of error in modal combination is not relevant to this method; no error is involved in modal combination.

This method is computationally more expensive than the conventional modal superposition method but is much less expensive than direct time integration. Application of this method is extremely simple and direct. A program for this method may be obtained by making minor modifications to a standard direct time integration program. (A program for Clough and Mojtahedi's method appears in the appendix.)

In the light of above discussion, Clough and Mojtahedi's method can be considered as a potential alternative to the costly direct time integration method and the conventional response spectrum method. Though the response spectrum method is extremely popular because of its simplicity and cost-effectiveness, its accuracy is questionable especially for non-classically damped systems. For certain important P-S coupled systems such as those in nuclear reactors, it may be necessary to obtain more reliable results. In such cases Clough and Mojtahedi's method provides an attractive alternative to direct time integration.

It is suggested here that Clough and Mojtahedi's method with approximate eigenproperties be used as an alternative solution technique for P-S coupled systems. Briefly, the steps involved in the proposed method are:

1. Evaluation of modal properties of the combined system using perturbation.
2. Transformation of the equations of motion using a truncated set of 'approximate' mode shapes. (Eq. 4.8)
3. Solution of the coupled equations of motion (Eq. 4.10).
4. Obtaining response in physical coordinates using the response in modal coordinates (using Eq 4.8).

Possible sources of error in the suggested method are:

1. Use of a truncated set of eigenvectors implies that interaction between the set of modes considered and the set of higher modes cannot be taken into

account. A partial remedy is to consider a larger set of modes in the analysis.

2. Clough and Mojtahedi's method, in its present form, cannot take into account response contribution from higher modes.

Further research is necessary in order to examine the suitability of this method for general use. At present, the data necessary for a comprehensive numerical verification of the suggested technique is not available with the author. It is recommended here that a thorough numerical investigation of suitability of the suggested method for general use be undertaken in future.

Chapter 5

Modal Combination

Dynamic properties of P-S coupled systems exhibit certain characteristic features due to the peculiar composition of these systems. An important feature, namely non-classical damping was discussed earlier. In this chapter, two other important features namely, 'closely spaced modes' and 'higher mode effect' will be discussed. A study of these features is important from the point of view of modal superposition analysis of these systems. Discussion in this chapter is of a general nature and is mainly in the form of comments on the sources of error and possible remedies.

5.1 Response Spectrum Method

The Response Spectrum Method is basically a modal superposition method in which superposition is done in an approximate way. In the modal superposition-time history method, superposition is done in the time domain; it leads to the exact response if all modes are considered. A modal superposition-response spectrum method cannot be expected to yield exact results even if all modes are considered in the analysis.

The Response Spectrum Method consists in solving an equation such as

$$\ddot{z}_j + 2\omega_j\xi_j\dot{z}_j + \omega_j^2 z_j = -\gamma_j\ddot{u}_g \tag{5.1}$$

to get the maximum response $z_{j,max}$ as:

$$z_{j,max} = \gamma_j S_{Dj} \tag{5.2}$$

where, S_{Dj} is the spectral displacement corresponding to ω_j and ξ_j. For the purpose of design, smoothed response spectra that envelope the spectra of many known strong motion earthquakes are used.

With reference to Eq. 4.2, let us denote:

$$\{y\} = \sum_{j=1}^{N} \{y_j\} \tag{5.3}$$

where, N is the total number of degrees of freedom, $\{y_j\}$ is the response in the j th mode given by

$$\{y_j\} = \{\phi_j\} z_j \tag{5.4}$$

In the response spectrum method, only the maxima of y_j are obtained.

$$\{y_{j,max}\} = \{\phi_j\} z_{j,max} \tag{5.5}$$

The maxima are combined using a rational combination rule such as the 'Square Root of Sum of Squares' rule.

If force response is desired instead of displacement response, the same procedure (obtaining modal maxima and combining them using a suitable combination rule) is followed. We will denote a modal maximum response as R_j and the total response as R. The quantity R could represent any general response, for example, the elastic force on the top floor of a building.

Using the SRSS rule,

$$R^2 = \sum_j R_j^2 \tag{5.6}$$

5.2 Closely Spaced Modes

Maximum modal responses do not occur at the same time. However, if this assumption is made, we get an upperbound to the total response (as obtained from the response spectrum method). This concept is known as the absolute sum method.

$$R = \sum_j |R_j| \tag{5.7}$$

The absolute sum method is a very approximate method and should be used only to find an upperbound to the response.

Random vibration concepts have been effectively used for the purpose of deriving several modal combination rules. Most of the modal combination rules proposed so far have their basis in the random vibration theory.

The most popular SRSS rule originally proposed by Goodman et al. [22] also has its basis in random vibration theory. One of the most important assumptions made in order to arrive at this rule is the assumption that each modal response is statistically independent of the other. In other words, correlation between modal responses is zero. This assumption is known to be valid for conventional building structures or other structures that are more or less 'uniform'.

One of the most important sources of correlation between modal responses is the closeness in frequencies between two modes. Two modes having very close frequencies can be expected to behave essentially in phase with each other; there is a high degree of correlation between these modal responses. As modal frequencies get closer, modal correlation tends to unity.

Closely spaced modes are known to occur, for instance, in buildings with torsional response and in P-S coupled systems. The former situation has been discussed in [61] and will not be discussed here. In P-S coupled systems, if some frequencies of the fixed-base secondary system match those of the primary system, then the combined system exhibits closely spaced modes (CSMs).

Use of a rule such as the SRSS rule for modal combination in the case of systems having closely spaced modes may lead to large errors owing to the fact that the significant modal correlation in such systems is neglected by the SRSS rule. In order to arrive at a reliable estimate of the maximum response of such systems, a combination procedure that takes modal correlation into account must be used. Several efforts were made by investigators working in this area to arrive at a reliable modal combination rule. These have been reported in an earlier chapter. Such procedures essentially introduce correlation coefficients to account for the modal correlation.

Response obtained using the Response Spectrum Method can, at best, be a good approximation to the exact response obtained by direct integration. Some well known combination rules such as the double sum rule, the 'Complete Quadric Combination' (CQC) rule or Gupta's rule are known to give excellent results for systems with closely spaced modes. Application of these methods is extremely simple and direct; the additional computational expense involved is negligible. Considering the improvement in accuracy they provide, there is little justification in continuing to use a potentially erroneous rule such as the SRSS rule. This comment is intended for regular structures too. Use of a rule such as the CQC rule, in the worst case can yield response as accurate as that obtained by the SRSS rule. In many instances, significant improvement over the SRSS rule may be expected. It is strongly recommended by researchers in this area [87] [27] that use of SRSS rule should be immediately discontinued in favour of a rule such as the CQC rule.

Incorporation of these improved rules into a computer program is extremely simple. A subroutine for modal combination using these rules may be obtained with virtually no effort.

5.3 Modal Correlation due to Rigidity

Another important source of correlation between modal responses is rigidity of modes. High frequency modes of a structures respond essentially in phase with the ground. This means that they are almost completely correlated with each other. This correlation between modes is independent of the correlation due to closeness in frequency; it exists even if the higher modes are widely spaced.

Response of higher modes can be studied with reference to harmonic response of a single-degree-of-freedom structure. Consider an SDOF shear building structure

Fig. 5.1 under the action of harmonic excitation

$$\ddot{u}_g = A \sin \bar{\omega} t \tag{5.8}$$

where, A is the amplitude and $\bar{\omega}$ is the frequency of the harmonic excitation. Equation of motion of the system is

$$m\ddot{y} + c\dot{y} + ky = mA \sin \bar{\omega} t \tag{5.9}$$

where, m and k are the mass and stiffness respectively of the structure and y is the horizontal displacement relative to the ground. The negative sign on the right hand side is neglected. Steady state response [55] of the system is:

$$y(t) = \frac{mA/k}{\sqrt{(1-r^2)^2 + (2r\xi)^2}} \sin(\bar{\omega} t - \theta) \tag{5.10}$$

where, $r = \bar{\omega}/\omega$, $\omega = \sqrt{k/m}$ and $\xi = c/(2\sqrt{km})$. The phase angle θ is given as

$$\tan \theta = \frac{2r\xi}{1 - r^2} \tag{5.11}$$

We assume that the excitation is a low frequency excitation such as earthquake excitation. In that case, for a very stiff structure, $\bar{\omega}/\omega$ is a small fraction; its square may be neglected in comparison with unity. For moderately damped structures, $2r\xi$ is a second order term and its square may be neglected in comparison with unity. Hence, for very stiff structures, $\tan \theta \approx 0$ or $\theta \approx 0$. The response can be expressed as:

$$y(t) = \frac{mA}{k} \sin \bar{\omega} t \tag{5.12}$$

We note:

1. The response is essentially in phase with the ground motion.

2. The response can be obtained in a quasi-static manner: by dividing the effective inertial force $mA \sin \bar{\omega} t$ by the stiffness, as if the inertial force was applied statically.

3. Behaviour of damped structures is almost the same as that of undamped structures. In the case of damped structures there exists a phase difference θ, but its numerical value is close to zero.
4. The quantity $\omega^2 y(t)_{max}$ is equal to A. That is, the spectral acceleration is equal to the maximum ground acceleration; in other words, amplification factor for the spectral acceleration is unity.

As frequency of a structure increases, the spectral acceleration approaches the maximum ground acceleration. This is true in general and is reflected in a typical response spectrum. The frequency at which the amplification factor for the spectral acceleration becomes almost exactly equal to unity is referred to as the "cut-off frequency". This frequency is usually specified as 33 Hz. Beyond the cut-off frequency, all modes can be considered to be 'rigid'. Spectral acceleration at or beyond the rigid frequency is called "Zero Period Acceleration (ZPA)".

Correlation due to rigidity does not cease to exist suddenly below the cut-off frequency. It is demonstrated by Gupta and Cordero [24] that modal correlation due to rigidity slowly drops down to zero at a frequency well below the cut-off frequency. Modes in the intermediate range (10 Hz to 33 Hz) exhibit considerable correlation. Singh and Mehta [73] and Gupta and Cordero have proposed modal combination methods that take into account modal correlation due to rigidity. These methods are quite simple in application and consistently yield better results as compared to the SRSS rule. Hence it is strongly recommended by researchers in this area that such rules be used instead of the SRSS rule.

5.4 Higher Mode Effect

A basic premise in the conventional modal superposition method is that a significant part of the response of a dynamic system is contained in the first few modes. This assumption is known to give acceptable results for 'regular' structures such as tall buildings or towers, especially for the displacement response. Frequency content of earthquakes is such that they evoke little excitation in modes with very high frequencies. This is the observation which forms the basis of the above assumption.

However, it is well known that stress resultants obtained from the same number of modes are usually much less accurate [37] [50]. One solution to this problem is to consider a larger number of modes in the analysis than is usually thought necessary for displacement response. This is associated with the following difficulties.

1. Extra computation is needed to obtain a larger number of modes.
2. Accuracy of the higher frequencies and mode shapes of a discretized model diminishes quite rapidly.
3. Extra computation is needed to solve a larger set of modal equations of motion.
4. In a response spectrum approach, combination of response from higher modes must be done with care because of the high amount of correlation among the responses of higher modes. This difficulty may be eliminated by using a proper method of modal combination.
5. In a mode-superposition time history method, very small time steps must be taken for higher modes to ensure accuracy.

Owing to these difficulties, direct inclusion of the response from higher modes is generally discouraged.

5.5 Inclusion of Response from Higher Modes

Higher modes make an essentially static contribution to the total response. This fact may be exploited in order to arrive at a simple procedure to include the response from higher modes. A number of such attempts have been made [50] [37] [80] [35] [27]. Details of these methods can be found from these references. A subsequent chapter deals with one of these methods. In all of these methods, response from the lower modes may be obtained as usual, but the response from the higher modes is included by a quasi-static analysis.

5.6 Special Case of P-S Coupled Systems

Salmonte [65] demonstrated that the higher modes contribute significantly to the total response in the case of structures having local stiff parts. In this paper, he has solved a dynamic model that can represent a building structure in which foundation interaction is present. Foundation stiffnesses are generally very high compared to structural stiffnesses. In Salmonte's example, the horizontal reaction at the translational spring receives a contribution from the ninth mode (whose frequency is well beyond the cut-off frequency of 33 Hz.) that is comparable to the contribution from the fundamental mode.

Systems in which stiffness and mass properties vary substantially from one part to another exhibit behaviour similar to that described by Salmonte. In the case of 'regular' structures such as tall buildings, response contribution decreases with the frequency of successive modes. Thus response from higher modes does not play a significant role, with the exception, perhaps, of stress resultants. It necessary to realize that the assumption of considering only the first few modes is not valid uniformly for all structures and its validity as far as stress resultants are concerned is questionable for any structure.

A P-S coupled system consists of two or more systems with essentially different dynamic properties. Secondary system stiffness may be extremely small or extremely large compared to the primary system stiffness. The general case of multiply supported secondary system (or many such secondary systems) consists of components that have entirely different stiffness and mass properties, these components being connected together in a complex manner. From this point of view, it is apparent that the system possesses characteristics similar to those of the system studied by Salmonte [65]. Specifically, higher modes contribute significantly to the total response.

A reference to this problem was made by Gupta and Jaw [27] in connection with response spectrum analysis of a complex 3-D piping network. No other specific reference is known to the author.

Solution of the abovementioned problem is rather simple. One should simply include all the higher modes by a pseudo-static analysis, using a suitable technique mentioned earlier. However, recognition of the existance of the problem is important. In the absence of such recognition, no further development in this area can be expected. It is the belief of the author that this problem has not received due attention of the investigators in this area.

5.7 A Demonstration of Higher Mode Effect

An attempt was made during the course of this study to effectively demonstrate the importance of higher modes. For this purpose, the following approach was taken.

Displacement response contribution of the j th mode is:

$$\{y_j\} = \{\phi_j\}\gamma_j S_{Dj} \tag{5.13}$$

If the variation of $\{y_j\}$ is studied for successive modes of some typical P-S coupled systems, it can effectively demonstrate the effect of higher modes.

Systems in Figs. 5.2 to 5.5 were analysed for this purpose using the response spectrum of the 1934 El-Centro earthquake (N-S component). (Fig. 5.6). Modal damping ratios for all modes of all the systems were taken to be equal to 0.05. Exact eigenvalues and eigenvectors of all the systems were obtained. The eigensolution algorithm used was Jacobian iteration which is known to provide great accuracy. In this study, accuracy of the eigensolver was tested by comparing its results with analytical results obtained for a large number of systems using a symbolic algebra program MAPLE at the University of Ottawa.

Figs. 5.7 to 5.10 show the response contributions from successive modes for the systems depicted in Figs. 5.2 to 5.5 respectively. (The plots are drawn for the DOF 1).

From Fig. 5.7 it may be seen that modal contribution to the total response diminishes rapidly for successive modes in the case of a uniform shear building system such as in Fig. 5.2.

Fig. 5.3 depicts a typical (practical) multi-storey building. In practice, floor masses and column stiffnesses decrease gradually for successive floors. This is simulated in Fig. 5.3 . The corresponding plot Fig. 5.8 shows that the system behaves almost similar to the uniform system. This may explain why the assumption behind using the first few modes is usually valid for typical multi-storey building systems.

Figs. 5.9 and 5.10 indicate that for P-S coupled systems contribution from the higher modes is significant. The curves obtained are neither smooth nor monotonically decreasing indicating that the premise of using only the first few modes is questionable for such systems.

Tables 5.1 to 5.4 provide some numerical data for the above-mentioned systems. Data for the degree of freedom 10 appears in these tables. It may be observed that this DOF is not as much affected by higher mode effect. As a general observation, all degrees of freedom are not equally affected by the higher mode effect. However, for the degrees of freedom affected by it, significant error may be associated with the neglect of the higher modes.

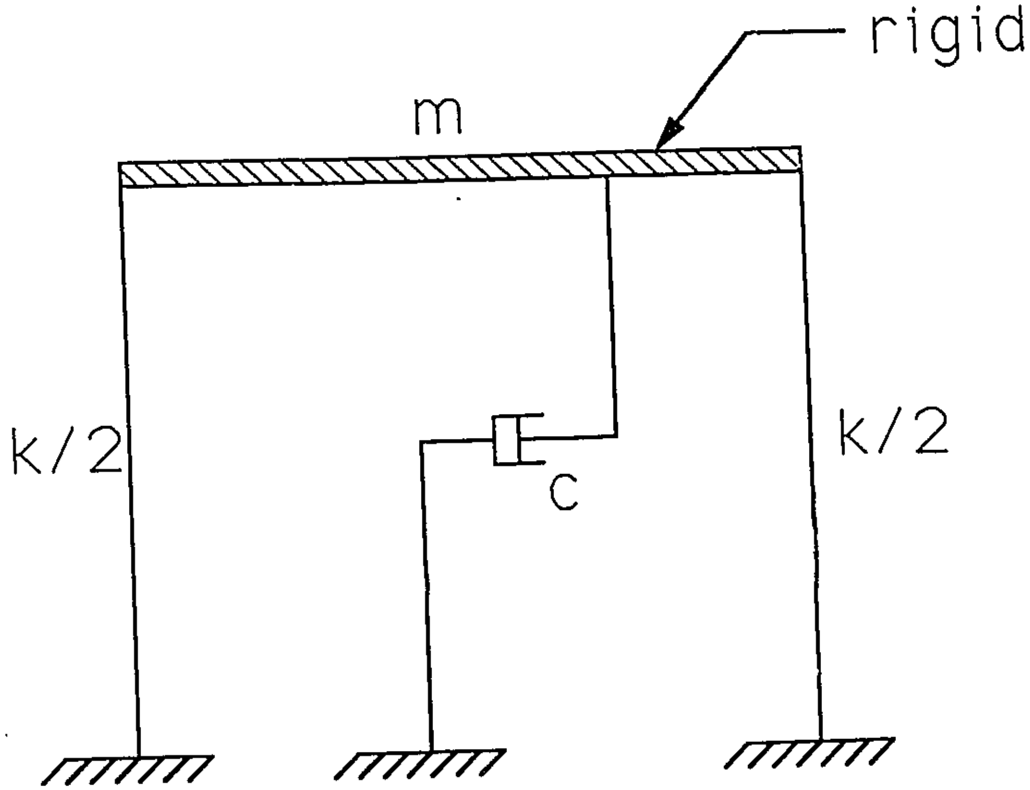
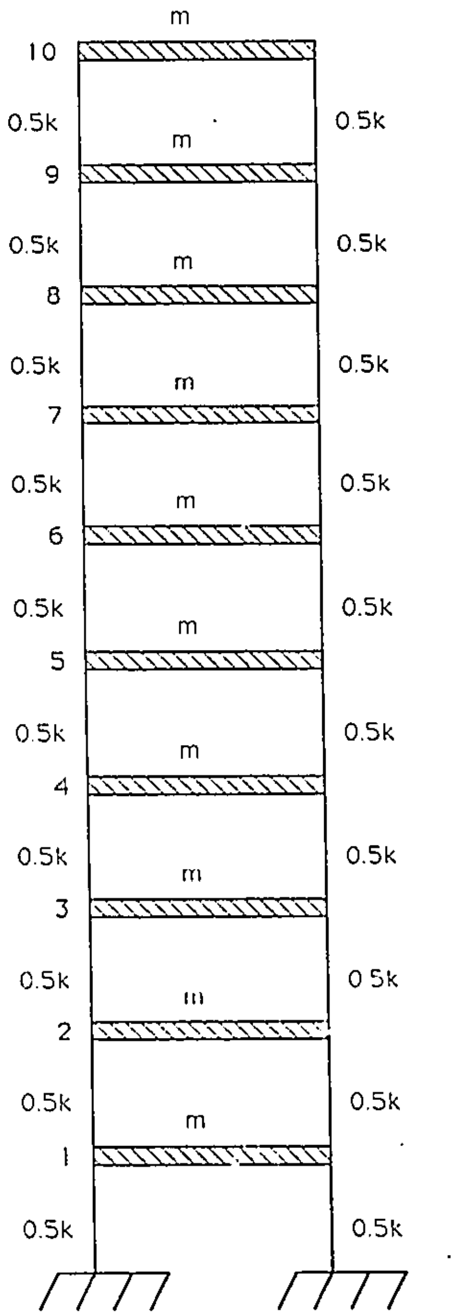


Figure 5.1: A Single Degree of Freedom Shear Building

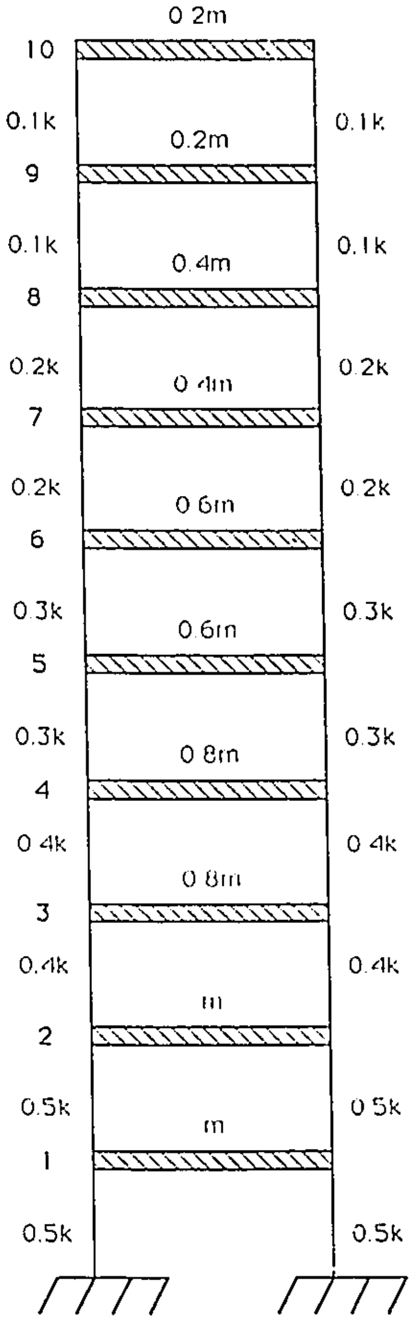


k=1.0E7 N/m
m=1.0E4 kg

All floors rigid

Bold faced number near each mass indicates the corresponding degree of freedom

Figure 5.2: A Uniform Shear Building System



k=1.0E7 N/m
m=1.0E4 kg

All floors
rigid

Bold faced number
near each mass
indicates the
corresponding
degree of freedom

Figure 5.3: Shear Building with Gradual Variation in Mass and Stiffness

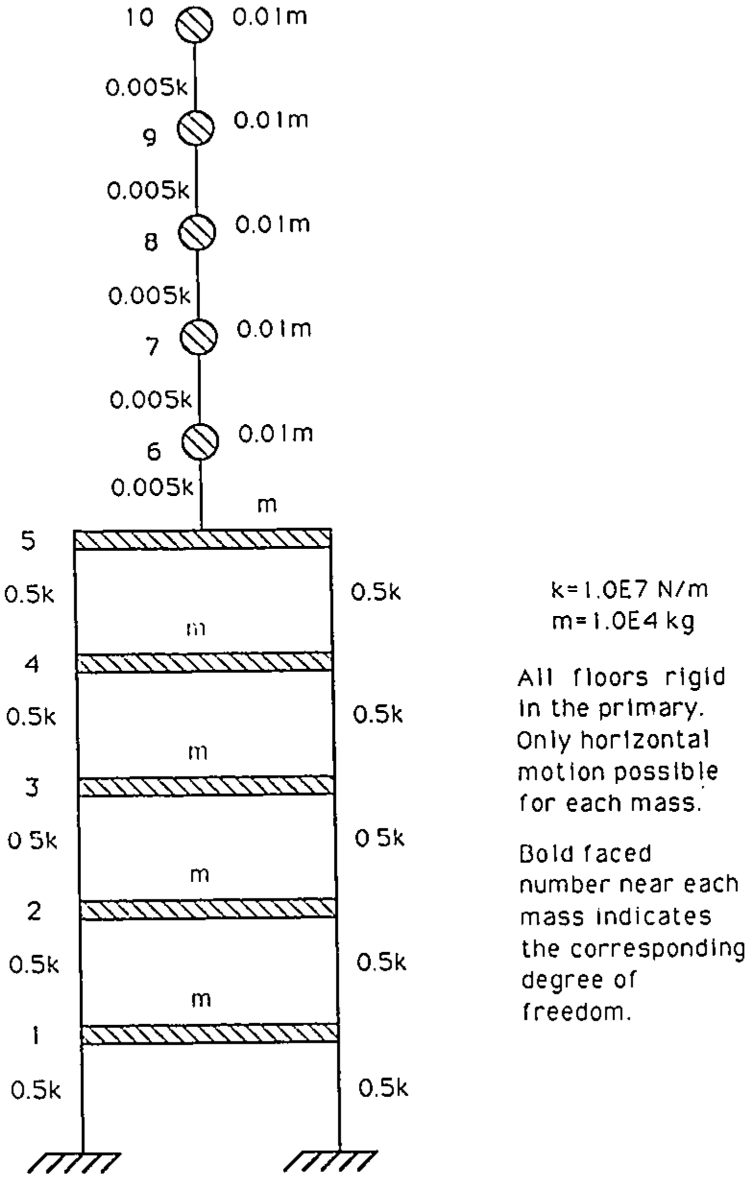
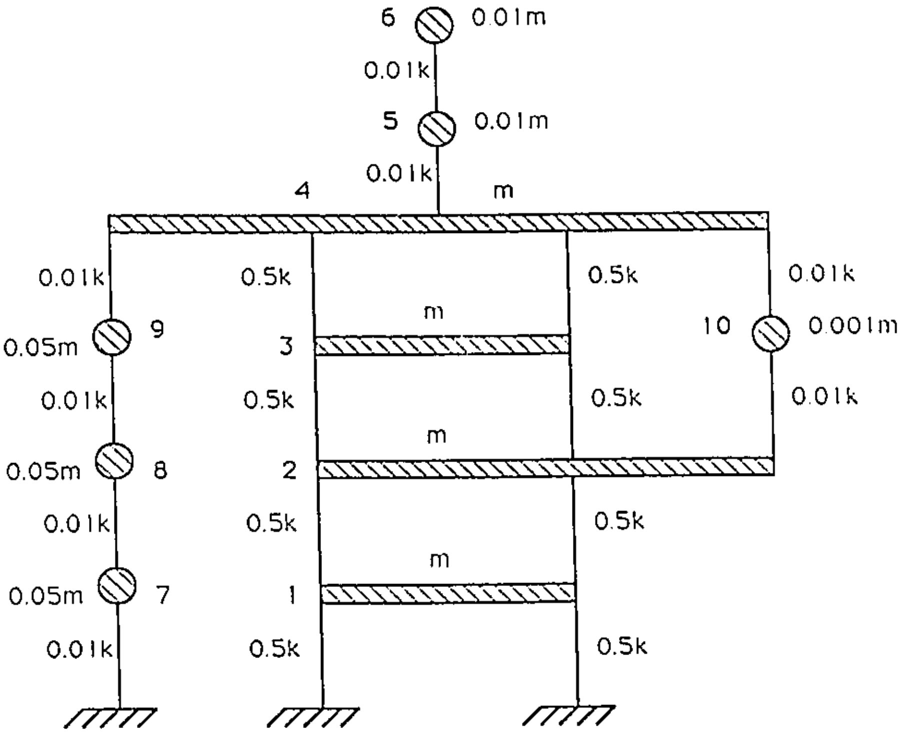


Figure 5.4: A Singly Supported Secondary System and the Primary System



k=1.0E7 N/m
m=1.0E4 kg

All primary floors rigid.
Only horizontal motion
possible for each mass.

Bold faced number near
each mass indicates the
corresponding degree of
freedom.

Figure 5.5: Multiply Suppported Secondary Systems and the Primary System

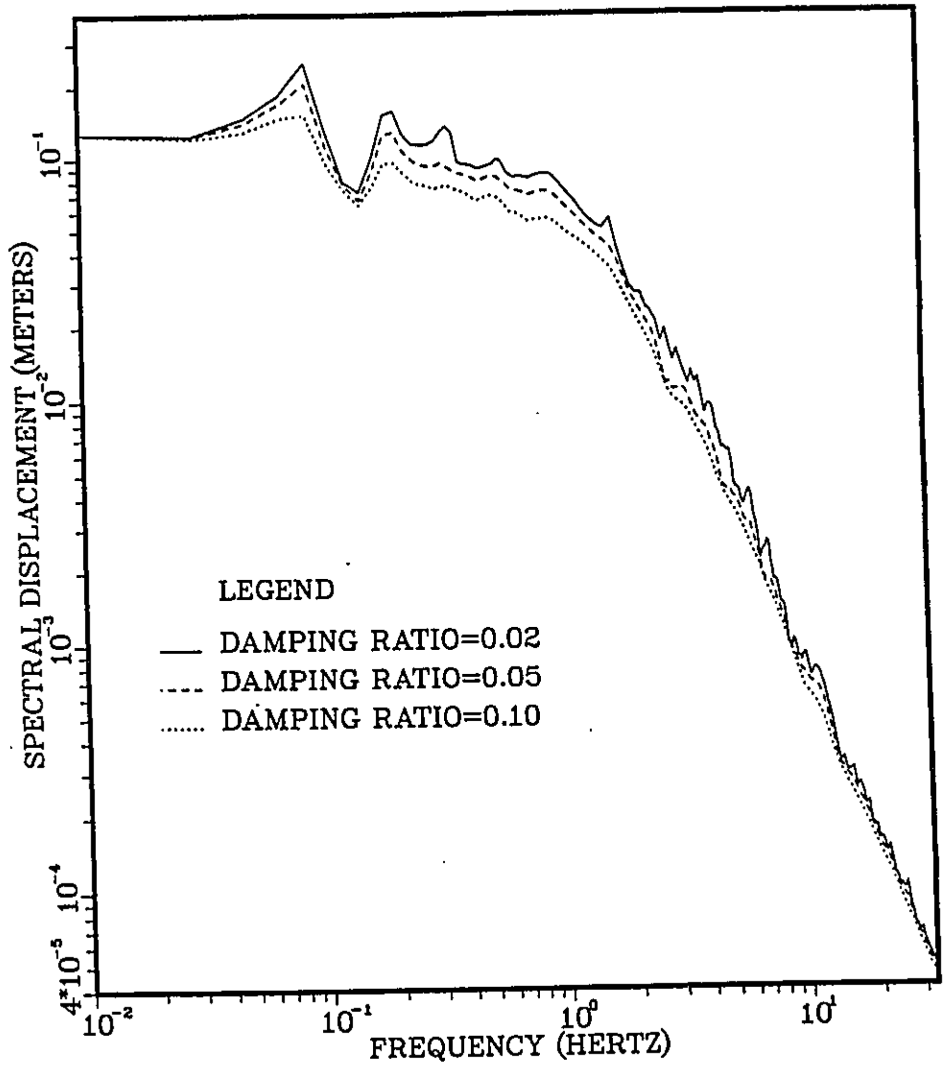


Figure 5.6: Displacement Response Spectrum for the 1934 El-Centro Earthquake, N-S component

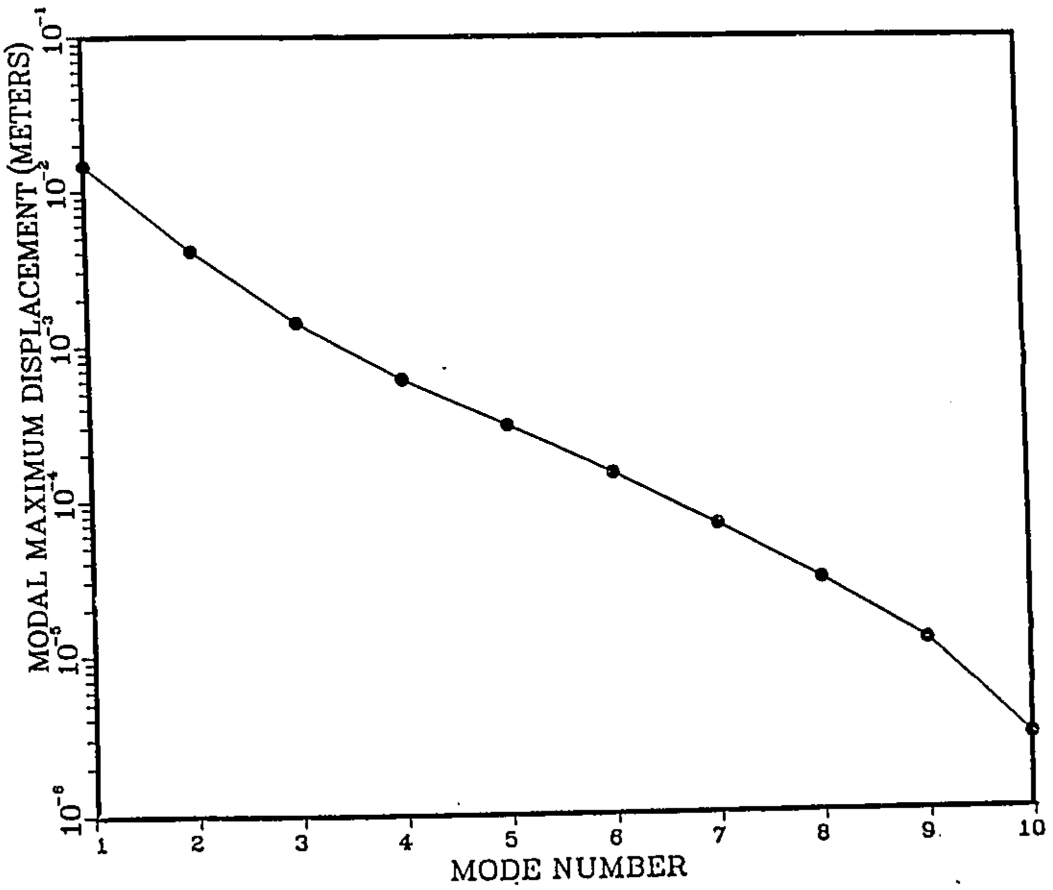


Figure 5.7: Displacement Contribution from Various Modes for the System in Fig. 5.2, DOF 1

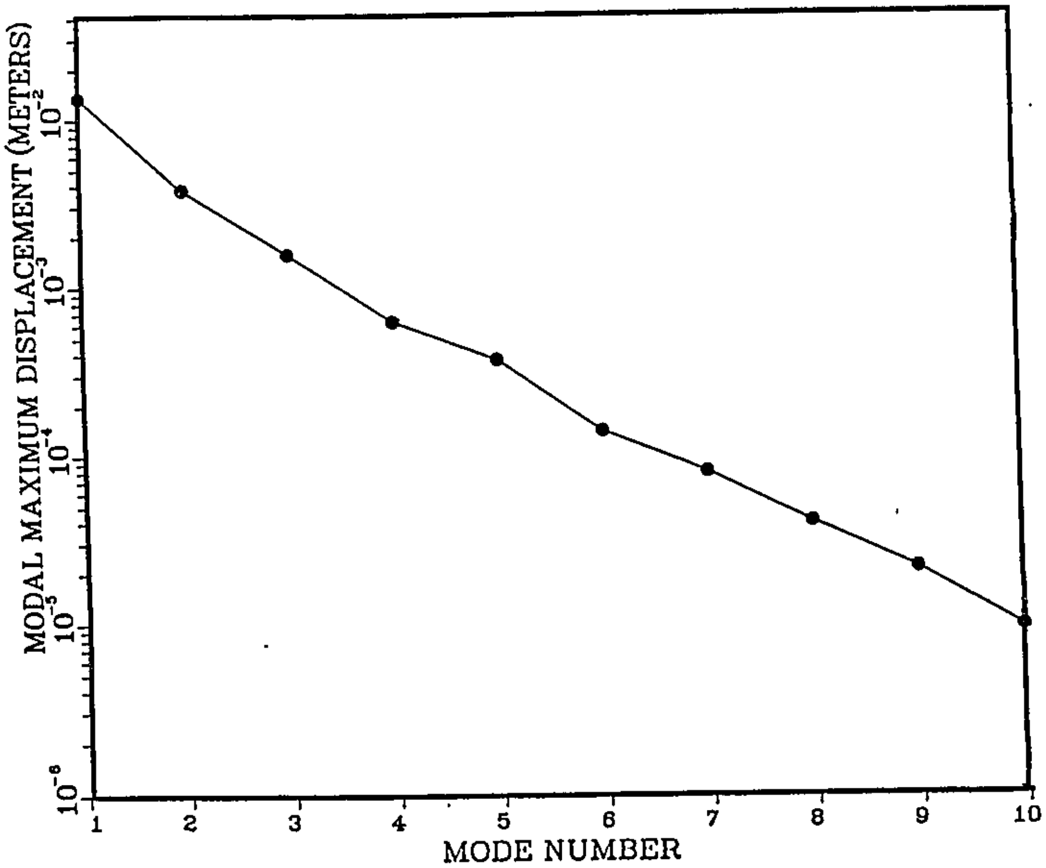


Figure 5.8: Displacement Contribution from Various Modes for the System in Fig. 5.3, DOF 1

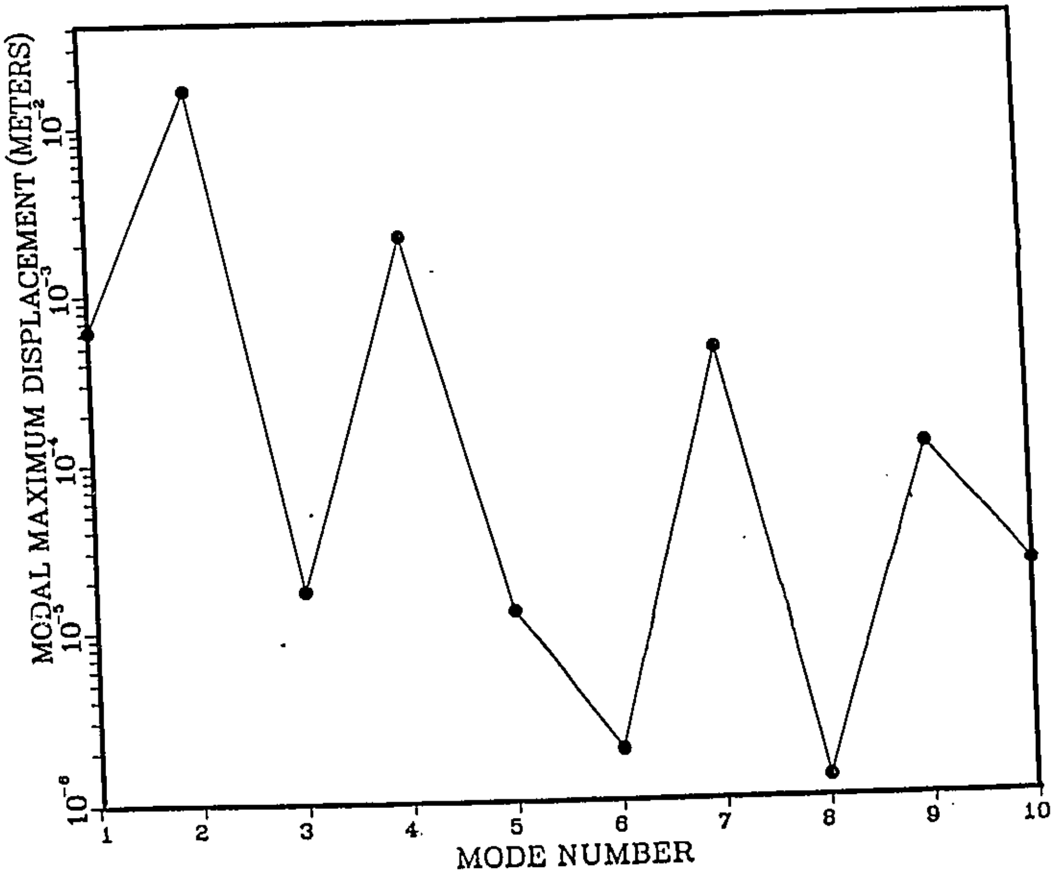


Figure 5.9: Displacement Contribution from Various Modes for the System in Fig. 5.4, DOF 1

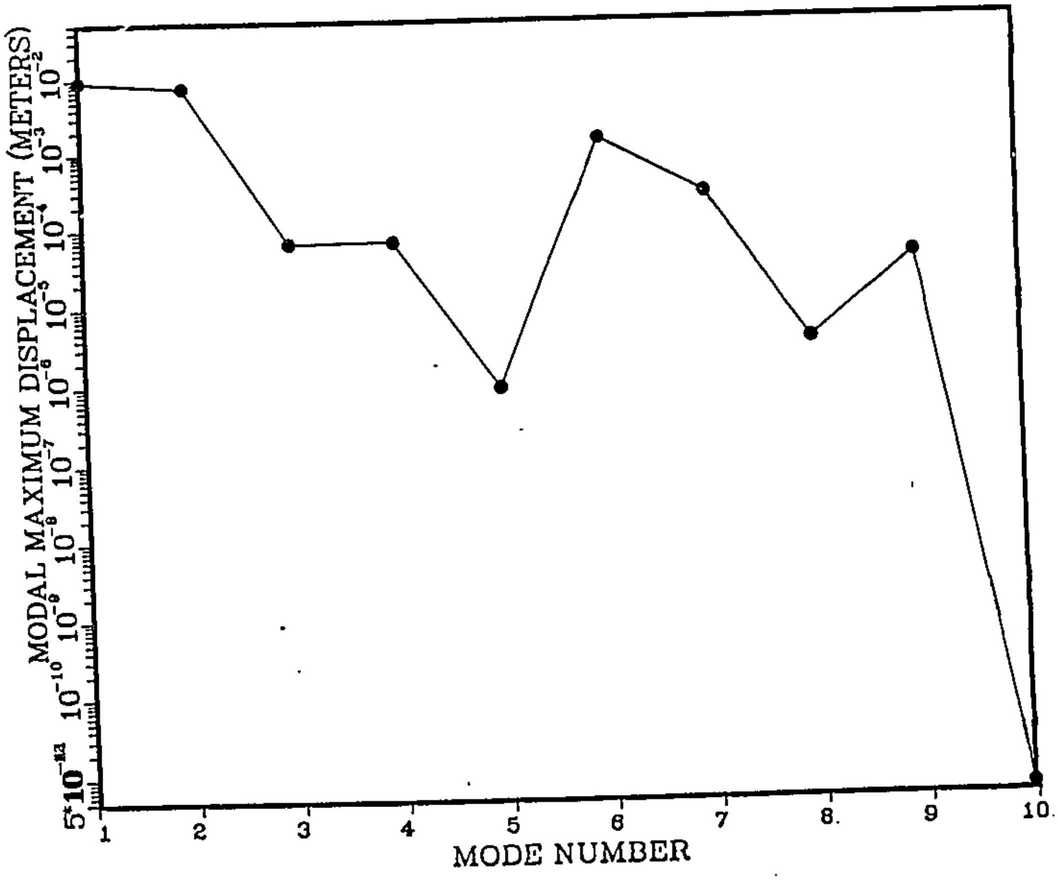


Figure 5.10: Displacement Contribution from Various Modes for the System in Fig. 5.5, DOF 1

Mode No.	Eigenvalue (ω^2), $(rad/sec)^2$	Participation Factor	Displ. Contrib. in meters DOF 1	Displ. Contrib. in meters DOF 10
1	0.22338E+02	0.29119E+03	0.14682E-01	0.98236E-01
2	0.19806E+03	-0.95608E+02	0.40860E-02	-0.91813E-02
3	0.53390E+03	0.55601E+02	0.14342E-02	0.19628E-02
4	0.10000E+04	-0.37796E+02	0.61791E-03	-0.61791E-03
5	0.15550E+04	-0.27364E+02	0.31274E-03	0.25080E-03
6	0.21495E+04	0.20248E+02	0.15103E-03	-0.10301E-03
7	0.27307E+04	0.14878E+02	0.70825E-04	0.42860E-04
8	0.32470E+04	-0.10509E+02	0.31294E-04	-0.17367E-04
9	0.36525E+04	0.67311E+01	0.12468E-04	0.65241E-05
10	0.39111E+04	0.32891E+01	0.31202E-05	-0.15777E-05

Table 5.1: Modal maximum displacements for the system in Fig. 5.2

Mode No.	Eigenvalue (ω^2), $(rad/sec)^2$	Participation Factor	Displ. Contrib. in meters DOF 1	Displ. Contrib. in meters DOF 10
1	0.40891E+02	0.21205E+03	0.13487E-01	0.11617E+00
2	0.20594E+03	-0.91722E+02	0.37753E-02	-0.21002E-01
3	0.48767E+03	0.57078E+02	0.15613E-02	0.59820E-02
4	0.87160E+03	-0.40162E+02	0.62834E-03	-0.15351E-02
5	0.13467E+04	-0.29729E+02	0.37182E-03	0.47758E-03
6	0.21956E+04	0.19616E+02	0.14042E-03	-0.17348E-03
7	0.25978E+04	0.15390E+02	0.78920E-04	0.14372E-03
8	0.29951E+04	0.11754E+02	0.39723E-04	-0.62798E-04
9	0.33447E+04	-0.86086E+01	0.20892E-04	0.20071E-04
10	0.36306E+04	0.57971E+01	0.92038E-05	-0.31706E-05

Table 5.2: Modal maximum displacements for the system in Fig. 5.3

Mode No.	Eigenvalue (ω^2), $(rad/sec)^2$	Participation Factor	Displ. Contrib. in meters DOF 1	Displ. Contrib. in meters DOF 10
1	0.39846E+02	0.45750E+02	0.62117E-03	0.20112E+00
2	0.82045E+02	0.20582E+03	0.16433E-01	-0.90294E-01
3	0.34528E+03	0.63611E+01	0.17082E-04	0.42576E-02
4	0.69029E+03	-0.65647E+02	0.21203E-02	-0.25452E-02
5	0.85942E+03	-0.55562E+01	0.12442E-04	0.11806E-02
6	0.14153E+04	-0.20765E+01	0.19022E-05	-0.21078E-03
7	0.17168E+04	-0.34661E+02	0.44431E-03	0.14351E-03
8	0.18418E+04	0.19239E+01	0.12664E-05	-0.60456E-04
9	0.28315E+04	0.19364E+02	0.11433E-03	0.11047E-06
10	0.36827E+04	0.88492E+01	0.21698E-04	-0.24722E-08

Table 5.3: Modal maximum displacements for the system in Fig. 5.4

Mode No.	Eigenvalue (ω^2), $(rad/sec)^2$	Participation Factor	Displ. Contrib. in meters DOF 1	Displ. Contrib. in meters DOF 10
1	0.10464E+03	0.15089E+03	0.92265E-02	0.22737E-01
2	0.13397E+03	0.11982E+03	0.72982E-02	0.16984E-01
3	0.38456E+03	-0.11967E+02	0.64215E-04	0.85277E-04
4	0.40101E+03	-0.16683E+02	0.65931E-04	0.83914E-04
5	0.68265E+03	-0.48861E+00	0.90799E-06	0.46039E-06
6	0.10146E+04	-0.56956E+02	0.13911E-02	-0.13693E-04
7	0.23486E+04	-0.27615E+02	0.26445E-03	0.46155E-04
8	0.26254E+04	-0.32192E+01	0.34319E-05	0.76262E-06
9	0.35337E+04	-0.12106E+02	0.40657E-04	-0.51102E-04
10	0.20011E+05	0.49175E-02	0.61248E-11	0.19677E-06

Table 5.4: Modal maximum displacements for the system in Fig. 5.5

Chapter 6

A Suggested Technique for Analysis of Non-classically Damped Systems

6.1 General

Study of non-classical damping and analysis procedures for non-classically damped systems is of great importance in the study of primary-secondary coupled systems. This is due to the fact that presence of non-classical damping is one of the major sources of error in the (conventional) analysis of these systems. Error involved in neglecting the presence of off-diagonal terms was discussed in Chapter 3. It is well known today that neglecting non-classical nature of damping can cause unacceptable errors. An efficient analysis technique for non-classically damped systems is today's need.

Classically damped structures have been studied thoroughly in the past and techniques for their analysis are fairly well established. Several textbooks [55] [15] refer to these techniques as standard analysis procedures for classically damped structures. Analysis of non-classically damped structures has not received as much attention. However, several alternative techniques do exist today and were discussed in earlier chapters. In this chapter, an improvement is suggested over an existing techniqe for analysis of non-classically damped structures.

A discussion on merits and demerits of the available procedures for analysis of non-classically damped systems appears in Chapter 3. It was mentioned that direct time integration, despite being the most accurate method, is not always a feasible alternative, owing to cost considerations. Complex modal superposition is certainly one of the better methods, its limitations being the necessity of use of complex numbers and solution of an eigenproblem of size $2N$. The method of ignoring off-diagonal terms in the transformed damping matrix is extremely simple to use, but is unreliable in many cases.

Clough and Mojtahedi's [14] method for analysis of non-classically damped systems has been discussed in Chapters 3, 4 and 5. They proposed direct integration of the transformed equations of motion, where the transformation matrix consists of a truncated set of modal vectors. This method exploits the fact that response of a dynamic system is generally sufficiently represented by the response from the first few modes. Clough and Mojtahedi attempted to solve the first few modal equations of motion in an 'exact' way (i.e., by direct integration). It was, however, shown by Duncan and Taylor [20] that damping coupling with the higher set of modes may be significant; implications of using a truncated set of modes for a non-classically damped system are different from those for a classically damped system. They suggested using a larger number of modes in the analysis of non-classically damped systems. Higher modes play an especially significant role in the case of special systems such as P-S coupled systems and soil-structure interaction systems. For these reasons, response from higher modes must be included if an accurate estimate of the exact response is desired for a general system.

The only assumption involved in Clough and Mojtahedi's method is the neglect of response from the higher modes. If this response is added to the response obtained by Clough and Mojtahedi's method, the resulting response may be expected to be very close to the exact response. Based on this idea, an improved technique is suggested here for analysis of a general elastic dynamic system. Using this technique, systems with non-classical damping as well as those having a predominant 'higher mode effect' may be analysed with good accuracy.

6.2 Formulation

Consider a general multi-degree-of-freedom dynamic system subjected to a horizontal earthquake excitation (acting only in one direction). The equations of motion of the system can be expressed in a matrix form as:

$$[M]\{\ddot{y}\} + [C]\{\dot{y}\} + [K]\{y\} = -[M]\{U_b\}\ddot{u}_g \tag{6.1}$$

The undamped mode shapes and frequencies of the system can be obtained by solving the eigenproblem

$$[K][\Phi] = [M][\Phi][\Omega] \tag{6.2}$$

where, $[\Omega]$ is a diagonal matrix, the diagonal terms being the eigenvalues ω_j^2 of the system. We assume, without loss of generality, that the mode shapes are scaled so as to yield unit modal masses for all modes.

As demonstrated by Clough and Mojtahedi [14], these undamped mode-shapes can certainly be used to transform the equations of motion, though the transformed equations are, in general, expected to be coupled. We write

$$\{y\} = \sum_{j=1}^{N}\{y_j\} \tag{6.3}$$

where,

$$\{y_j\} = \{\phi_j\}z_j \tag{6.4}$$

The static displacement vector $\{U_b\}$ can be expressed [15] in terms of the undamped mode shapes as

$$\{U_b\} = \sum_{j=1}^{N}\{\phi_j\}\gamma_j \tag{6.5}$$

where, γ_j is the j th modal participation factor. Proof of Eq. 6.5 can be found in [27]. Substituting the above expression for $\{U_b\}$ in the original equations of motion and using Eq. 6.3, we get

$$[M]\sum_j\{\ddot{y}_j\}+[C]\sum_j\{\dot{y}_j\}+[K]\sum_j\{y_j\}=-\sum_j[M]\{\phi_j\}\gamma_j\ddot{u}_g \tag{6.6}$$

In the case of classically damped systems, modal equations are uncoupled and it is possible to write

$$[M]\{\ddot{y}_j\}+[C]\{\dot{y}_j\}+[K]\{y_j\}=-[M]\{\phi_j\}\gamma_j\ddot{u}_g \tag{6.7}$$

This equation represents the response of a structure in its j th mode of vibration. In the case of non-classically damped structures, such decoupling is not permitted.

6.2.1 Response from the Lower Modes

Consider the set of first N' modes ($N' < N$). In the context of the suggested technique, this set should contain *all* the modes of a structure having frequency below the 'rigid' frequency mentioned in Chapter 5. (Usually, this frequency is specified as 33Hz). The reader is referred to [27] for a detailed explanation. More modes can be included if desired, but are not necessary.

The suggested improvement consists in assuming that the total response $\{y\}$ can be obtained by superposition of two responses $\{y'\}$ and $\{y''\}$, where the superposition is done in the time domain.

$$\{y\}=\{y'\}+\{y''\} \tag{6.8}$$

Here, $\{y'\}$ is the response in the lower modes obtained assuming the absense of coupling with the higher modes. This response can be obtained by using Clough and Mojtahedi's method. The component $\{y''\}$ is the response from the higher modes obtained independently of the lower modes.

The above assumption amounts to assuming that Eq. 6.6 can be replaced by the following two independent equations.

$$[M]\{\ddot{y}'\}+[C]\{\dot{y}'\}+[K]\{y'\}=-\sum_{j=1}^{N'}[M]\{\phi_j\}\gamma_j\ddot{u}_g \tag{6.9}$$

and

$$[M]\{\ddot{y}''\} + [C]\{\dot{y}''\} + [K]\{y''\} = -\sum_{j=N'+1}^{N} [M]\{\phi_j\}\gamma_j\ddot{u}_g \tag{6.10}$$

where,

$$\{y'\} = \sum_{j=1}^{N'}\{\phi_j\}z_j \tag{6.11}$$

and

$$\{y''\} = \sum_{j=N'+1}^{N} \{\phi_j\}z_j \tag{6.12}$$

We write

$$\{y'\} = [\Phi']\{z'\} \tag{6.13}$$

where, $[\Phi']$ lists the first N' modes. In order to solve Eq. 6.9, we use Eq. 6.13 and premultiply Eq. 6.9 by $[\Phi']^T$. This yields,

$$[\Phi']^T[M][\Phi']\{\ddot{z}'\} + [\Phi']^T[C][\Phi']\{\dot{z}'\} + [\Phi']^T[K][\Phi']\{z'\} = -[\Phi']^T\sum_{j=1}^{N'}[M]\{\phi_j\}\gamma_j\ddot{u}_g \tag{6.14}$$

Direct integration of the coupled modal equations Eq. 6.14 and subsequent substitution in Eq. 6.11 yields $\{y'\}$. This is, in essense, Clough and Mojtahedi's approach.

6.2.2 Response from the Higher Modes

In order to solve Eq. 6.10 it is assumed that the response from the higher modes may be obtained in a quasi-static manner. For high frequency modes, inertial and damping forces can be neglected in comparison with elastic forces. This was demonstrated in Chapter 5. Using this assumption, we can neglect the first two terms on the left hand side of Eq. 6.10. This gives,

$$[K]\{y''\} = -[M]\sum_{j=N'+1}^{N} \{\phi_j\}\gamma_j\ddot{u}_g \tag{6.15}$$

Let us denote

$$\{U_{br}\} = \sum_{j=N'+1}^{N} \{\phi_j\}\gamma_j \tag{6.16}$$

From Eq. 6.5 we get

$$\{U_{br}\} = \{U_b\} - \sum_{j=1}^{N'} \{\phi_j\}\gamma_j \tag{6.17}$$

Using Eqs. 6.15 and 6.16 we get,

$$\{y''\} = -[K]^{-1}[M]\{U_{br}\}\ddot{u}_g \tag{6.18}$$

A large part of the above derivation is after Gupta and Jaw [27]. Their derivation was intended for classically damped systems. Solution of Eq. 6.18 is simple. The inversion of the stiffness matrix needs to be done only once. The only time-dependent quantity on the right hand side of this equation is $\ddot{u}_g$ which is a scalar quantity. Therefore, once the response for the first time point is obtained, the response for the subsequent time points can be obtained by a simple scaling using the ground acceleration at those time-points. At each time step, the responses $\{y'\}$ and $\{y''\}$ are added algebraically to get the response $\{y\}$.

6.3 Evaluation of the Suggested Technique

6.3.1 Assumptions

Assumptions involved in the suggested technique are listed below for easy reference.

1. It is assumed that Eq. 6.6 can be decoupled into Eqs. 6.9 and 6.10.
2. It is assumed that the response from the higher modes can be obtained in a quasi-static manner. Validity of this assumption is well known [27] [80].

6.3.2 Merits of the Suggested Technique

1. It permits use of undamped mode shapes unlike the method of complex modal superposition.
2. Only the modes with frequency below the 'rigid' (or 'cut-off') frequency need be computed.

3. Response from all the higher modes is obtained.

4. Modal coupling due to non-classical damping is considered, though in an approximate manner.

5. This technique is extremely easy to apply. A computer program for this method may be obtained by minor modifications to an existing program for step-by-step time integration.

6. As will be shown later, This technique is more accurate compared to Clough and Mojtahedi's original method, which itself is shown by them to be quite accurate.

7. The suggested technique is much less expensive than direct time integration.

6.3.3 Demerits

1. This technique involves direct time integration of the transformed equations of motion, though only a few such equations are generally involved. However, it was shown in Ref. [14] that integration of a limited number of coupled equations is not much more expensive than integration of the corresponding uncoupled equations, and it is much less expensive than direct integration of the equations of motion or solution of an eigenproblem of size $2N$. (This remark is valid only for typical earthquakes and for systems with a large number of degrees of freedom.)

2. The technique is more expensive compared to Clough and Mojtahedi's method since it involves inversion of the stiffness matrix.

Demerits of the proposed improved technique mainly stem from the extra computational expense involved. However, this increase in expense can certainly considered to be justified by the accuracy achieved. In order to obtain a reliable response for the most general system, some sacrifice in terms of cost of solution must be made. However, a solution technique must exploit all known properties of dynamic systems in order to reduce the computational expense as much as possible. The proposed technique uses the following properties to advantage

1. Response of a system is generally contained in the first few modes
2. Higher modes respond essentially in a quasi-static manner.
3. Eq. 6.5 is used in order to avoid computation of high frequency modes.

6.4 Numerical Verification

Performance of the suggested improved technique was tested on several dynamic systems and was found to be satisfactory. Results of two numerical examples will be presented here.

6.4.1 Systems Analysed

Systems in Figs. 6.1 and 6.2 were subjected to the 1940 El-Centro earthquake. The record for this earthquake was obtained from Paz [54]. Time step used in the analysis was 0.003 sec. This was found to be sufficiently small by decreasing it and observing the difference in the response obtained. Linear interpolation between the available data points was done in order to arrive at the required time step of 0.003 sec.

The systems chosen are primary-secondary coupled systems. Horizontal displacement was taken as the only degree of freedom at each nodal point. In other words, 'shear building' model was used. Masses of the columns were considered to be lumped at floor levels. The stiffness and mass properties appear in Figs. 6.1 and 6.2. We will refer to these systems as example-systems 1 and 2 respectively.

The damping matrices chosen were such as to provide a small amount of non-classical damping for the example-system 1 and a large amount for the example-system 2.

For the example-system 1

$$[C] = \text{diag}\{1.0E4, 1.0E4, 1.0E4, 1.0E4, 1.0E4, 1.0E2, 1.0E2, 2.0E2, 3.0E2, 2.0E2\}$$

For the example-system 2

$$[C] = \text{diag}\{6.0E5, 1.0E6, 9.0E4, 1.0E4, 1.0E4, 0.1E4, 0.1E4, 0.1E4, 0.1E4, 2.0E1\}$$

where, diag{} denotes a diagonal matrix listing on its diagonal the elements inside the braces.

6.4.2 Analyses Performed

For each system, four analyses were performed.

A Direct integration of the equations of motion.

B Analysis using the proposed technique. All the modes having frequency below the cut-off frequency were considered *explicitly.* All the higher modes were included by a quasi-static analysis.

C Analysis using Clough and Mojtahedi's [14] method. All the modes having frequency below the rigid frequency were considered in the analysis.

D Modal analysis by ignoring the off-diagonal terms in the transformed damping matrix. All the modes having frequency below the rigid frequency were considered in the analysis.

Newmark's constant average acceleration scheme was used for integration of the equations, both coupled and uncoupled. Rigid frequency was taken to be 33 Hz. For the example-system 1, the first 6 modes had frequency below 33Hz. For the example-system 2 the number was 7.

Eigensolution was done using the Jacobi iteration algorithm. This makes it difficult to compare to the performance of the abovementioned methods in terms of the cost of solution since the Jacobi method evaluates all the mode shapes. In practical situations, algorithms such as sub-space iteration are used, which compute only a specified number of modes. This makes it possible to distinguish between the analysis methods A, B,C and D on the basis of the number of modes required to be computed. If Jacobian iteration is used, this facility cannot be availed. For this reason, no comparison of the computer time required for solution was made. However, such a comparison appears in [14]. Suffice it to add that the additional cost (involved in using the proposed technique) over Clough and Mojtahedi's method is the cost of inversion of the stiffness matrix of the system.

All the four methods A, B, C and D are time-domain approaches. In the actual analyses, time histories of displacements as well as elastic forces at all nodal points were obtained. However, only the maxima of the displacements and the elastic forces were used for comparison. The assumption behind this decision is questionable if the time step of integration is large. However, the time step used in this study can be considered to be very small. Thus comparison of the maxima may be considered to sufficiently represent the comparison of overall responses.

6.4.3 Results of the Analyses and Discussion

Displacements and elastic forces were obtained at each degree of freedom. The results obtained are shown in tables 6.1 to 6.4. The exact response obtained by direct integration (method A) appears in the second column of each table. Percentage errors in the response obtained using methods B, C and D are listed in the next three columns of each table.

We observe:

1. In all the three methods B, C and D, the displacement response is more accurate than the corresponding elastic force response. This phenomenon is well known.

2. For the example-system 1, the responses obtained using methods C and D are in close aggrement. This is expected since the system has a small amount

of non-classical damping. Ignoring it would not make a significant difference. This is unlike the example-system 2 in which the response of certain degrees of freedom (5, 6) is entirely different if non-classical damping is neglected. The results obtained for the example-system 2 (using methods C and D) demonstrate the error involved in neglecting the presence of non-classical damping.

3. In almost all the cases, the response obtained using method B (the proposed technique) is more accurate than that obtained using method C or method D. In individual cases where this is not so, the difference in the errors is small.

4. Substantial errors may occur if the higher mode response is neglected, depending on the dynamic properties of the system considered. This can be seen from the response of the degrees of freedom 1 and 2 in both the systems.

5. All the degrees of freedom are not equally affected by either non-classical damping nature or the higher mode effect. For instance, the degrees of freedom 1 and 2 (in the example-systems considered) are more sensitive to the number of modes considered, than other degrees of freedom. No general statements can be made in this regard. However, it is a noteworthy observation that parts of a structure that are stiff and are associated with heavy masses (for example, the primary system in the examples considered) are more sensitive to the higher mode effect.

6. Errors in the response obtained using method B (the proposed technique) are within acceptable limits for all the cases considered. In many instances, the errors are very close to zero. Performance of the proposed technique as far as elastic forces are concerned is quite impressive.

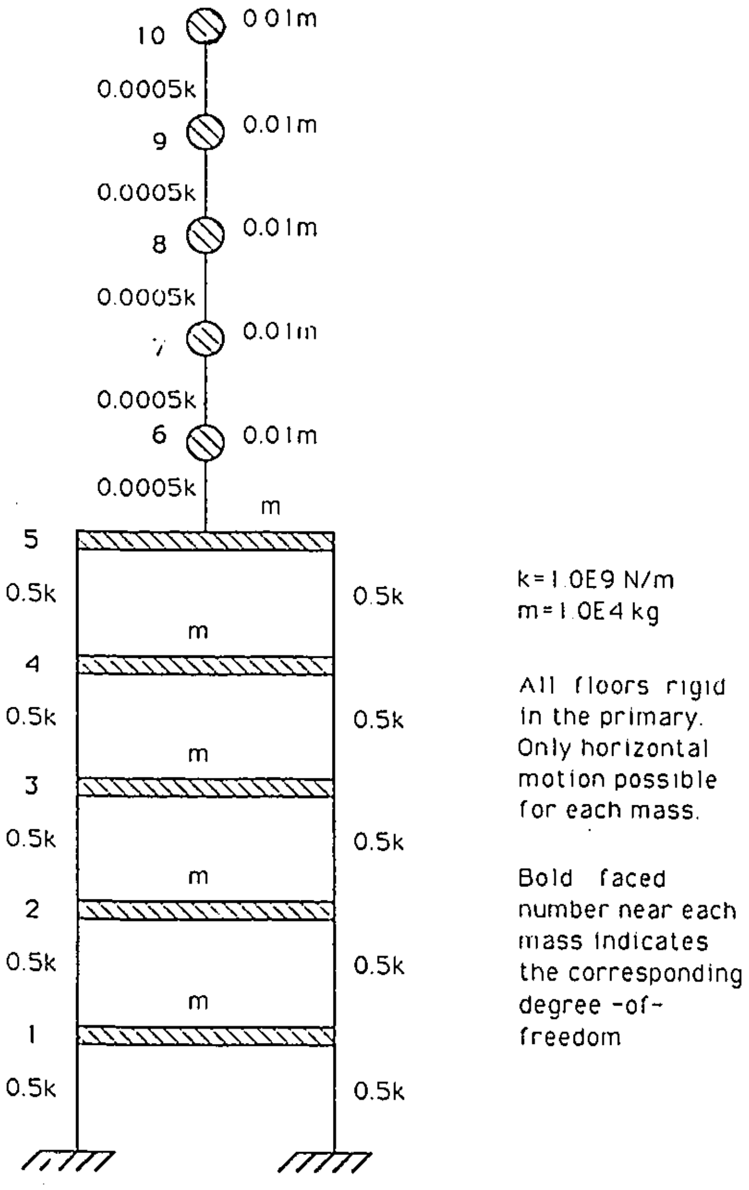
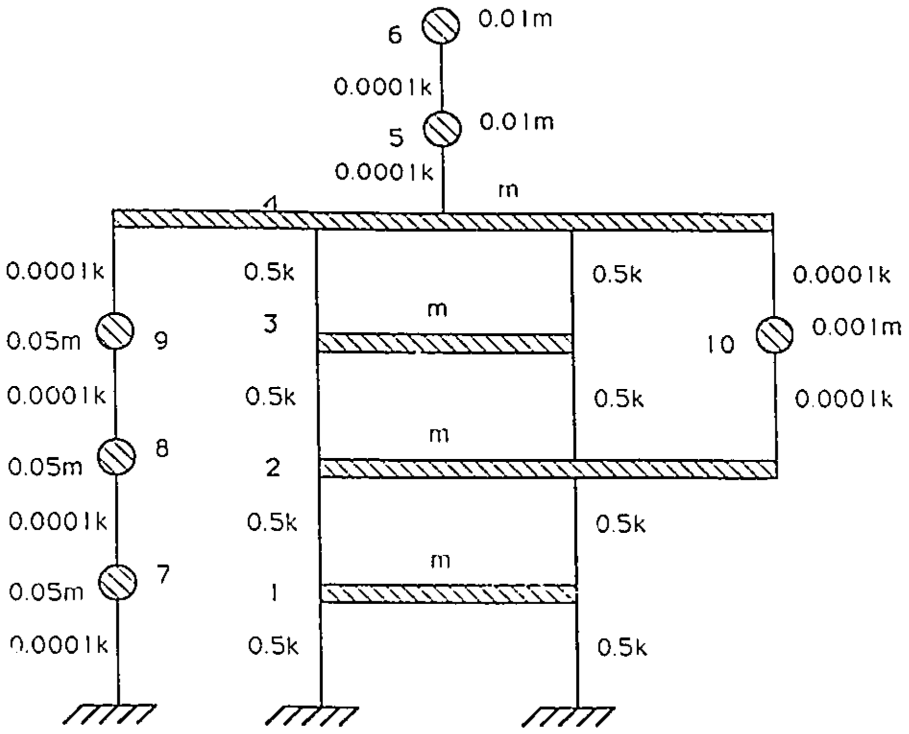


Figure 6.1: Example System no. 1



k= 1.0E9 N/m
m= 1.0E4 kg

All primary floors rigid.
Only horizontal motion
possible for each mass.

Bold faced number near
each mass indicates the
degree of freeodm.

Figure 6.2: Example System no. 2

DOF	Exact Response (meters)	% Error in B	% Error in C	% Error in D
1	-0.26193E-03	1.72	-0.46	-1.04
2	-0.49962E-03	1.18	0.12	-0.47
3	-0.69813E-03	0.40	0.12	-0.46
4	-0.84088E-03	-0.23	0.00	-0.59
5	-0.91483E-03	-0.54	-0.07	-0.65
6	0.75111E-02	0.00	0.00	-0.53
7	0.14369E-01	0.00	0.00	-0.12
8	0.20522E-01	0.00	0.00	0.22
9	0.24966E-01	0.00	0.00	0.00
10	0.27503E-01	0.00	0.00	-0.12

Table 6.1: Comparison of displacement response obtained by different methods for the system in Fig. 6.1

DOF	Exact Response (Newtons)	% Error in B	% Error in C	% Error in D
1	-0.36126E+05	-1.75	-41.24	-41.60
2	-0.45673E+05	-4.28	-10.82	-11.36
3	-0.57776E+05	-0.73	-1.46	-2.06
4	-0.68796E+05	-2.47	-0.40	-1.00
5	-0.78653E+05	-8.24	-5.22	-5.78
6	0.23041E+04	0.03	0.05	-0.39
7	0.83030E+03	0.01	0.07	3.06
8	-0.23104E+04	0.00	0.00	-0.68
9	0.14354E+04	0.00	0.00	-0.43
10	-0.21539E+04	0.00	0.00	-0.72

Table 6.2: Comparison of elastic force response obtained by different methods for the system in Fig. 6.1

DOF	Exact Response (meters)	% Error in B	% Error in C	%Error in D
1	-0.13601E-03	0.26	-8.08	-8.18
2	-0.24125E-03	0.27	-2.77	-2.87
3	-0.31163E-03	0.70	1.11	0.99
4	-0.34568E-03	0.95	3.12	3.00
5	-0.41248E-02	-0.10	-0.10	11.17
6	-0.73050E-02	-0.10	-0.10	-7.13
7	-0.50179E-01	-0.10	-0.10	-0.12
8	-0.73786E-01	-0.10	-0.10	-0.10
9	-0.50118E-01	-0.08	-0.08	-0.08
10	-0.58315E-03	-0.08	-0.08	1.07

Table 6.3: Comparison of displacement response obtained by different methods for the system in Fig. 6.2

DOF	Exact Response (Newtons)	% Error in B	% Error in C	% Error in D
1	-0.32499E+05	-0.75	-52.46	-52.51
2	-0.34861E+05	-2.65	-16.71	-16.79
3	-0.36334E+05	1.16	7.66	7.54
4	-0.38079E+05	0.45	16.78	16.67
5	0.36953E+03	-0.15	-0.10	-162.32
6	-0.41051E+03	-0.10	-0.10	-45.20
7	-0.27713E+04	-0.10	-0.10	-0.10
8	-0.47989E+04	-0.10	-0.10	-0.09
9	-0.27876E+04	-0.11	-0.11	-0.15
10	0.65774E+02	-1.89	-1.89	2.63

Table 6.4: Comparison of elastic force response obtained by different methods for the system in Fig. 6.2

Chapter 7

Conclusions and Recommendations for Further Research

In earlier chapters, several aspects of seismic analysis of secondary systems were discussed and contributions made by the present study to the existing knowhow were reported. The present study covers several aspects of seismic analysis of secondary systems. For this reason, conclusions of the different segments of this study are reported in different sections in this chapter.

7.1 Error Involved in Diagonalization of the Damping Matrix

Conclusions of Chapter 3 which deals with the error involved in diagonalization of the transformed damping matrix are listed below.

1. Response of a non-classically damped system to harmonic excitation provides valuable insight into its response to a general excitation.

2. For a given system, the error involved in diagonalization depends on the frequency of the harmonic excitation applied. Harmonic response of a system is most sensitive to the damping properties near the resonant frequency (frequencies). Hence, the error involved in using an approximate damping matrix (such as the diagonal matrix $[D]$) is maximum near the resonant frequencies.
3. The error involved in diagonalization depends on the frequency content of the applied load, if the applied load is non-harmonic.
4. The spectrum of the error involved in diagonalization obtained by subjecting a system to a harmonic excitation, the frequency of the excitation being varied over a large range (Fig. 3.3), is an important dynamic characteristic of a system. The area under such a curve has been defined as the error-parameter in this study.
5. In view of the dependence of the error in diagonalization on the frequency content of the applied load, consideration of a specific type of load in a study of the error should be discouraged. The error-parameter defined in this study does not consider any specific loads; hence its use in examining the pattern of variation of the error can be considered to be very reliable.
6. The error-parameter bears a significant correlation with the error involved in diagonalization when a system is subjected to a random load such as earthquake load. As the error-parameter of a system increases, the magnitude of the 'physical' error (in an earthquake analysis) also increases and vice versa.
7. It was found that dependence of the error in diagonalization on the applied load increases with the severity of the non-classical damping. For systems with severe non-classical damping, different excitations cause entirely different errors in diagonalization.
8. The error-parameter is an effective tool for the study of the variation of the error involved in diagonalization as dynamic properties of a system are varied.

9. If distribution of the damping properties of a system is maintained constant, the error involved in diagonalization increases with the overall level of damping in the system. (Validity of this conclusion is assured only for a 2-DOF shear building system.)
10. If the damping ratios in the primary and the secondary as well as the frequency ratio between the primary and the secondary (in the 2-DOF shear building system considered) are maintained constant, the error in diagonalization increases with the mass ratio (m_2/m_1).
11. The error parameter is an effective tool for comparison of performance of modal damping estimation techniques. Results of this study indicate that among the methods considered [89], Yan's method (which yields the same results as the method of ignoring off-diagonal terms) is the best method. Mass weighted damping is the next best and stiffness weighted damping is the least accurate method. Validity of this conclusion is claimed only for the two-DOF shear building system considered.

7.2 Modal Properties of the Combined System

Conclusions reached at in Chapter 4 are:

1. Use of approximate mode shapes obtained by perturbation techniques (or component mode synthesis) can be considered as an attempt to solve the equations of motion by transforming them into a set of 'modal' co-ordinates, where the transformation is done using the mode shape matrix consisting of the approximate mode shapes.
2. The transformed equations obtained as above are generally coupled owing to the fact that the approximate mode shapes, in general, do not possess the property of orthogonality with respect to the mass, stiffness and damping matrices.

3. If the coupled 'modal' equations obtained as above are solved exactly (i.e., by direct integration), then exact response is obtained after transforming back to the generalized co-ordinates.

4. When a modal superposition method such as the modal time history method or the response spectrum method is used, the transformed equations of motion must be uncoupled. This means that the off-diagonal terms in the matrices $[M_a^*]$, $[C_a^*]$ and $[K_a^*]$ must be neglected. This is the principal source of error involved in using approximate mode shapes for a modal superposition analysis.

5. A minor source of error is the fact that the diagonal terms, in addition to the off-diagonal terms, are 'modified' in a mode-superposition analysis. If no modification of the diagonal terms is made, the response obtained is closer to the exact response.

6. Clough and Mojtahedi's method [14], with approximate mode shapes used for transformation, is suggested as a useful method for a modal time-history analysis of primary-secondary coupled systems. In the absence of higher mode effect, this method is expected to yield very accurate results. The method accounts for all possible modal interaction, at least in the modes considered in the analysis.

7.3 Modal Combination

In Chapter 5, it was concluded that systems in which stiffness and mass properties exhibit substantial (spatial) variation have a significant part of the total response in the higher modes. In an ideally uniform system (Fig. 5.2) or in a system that is nearly so (Fig. 5.3), modal contribution to the total response decreases monotonically as one considers successive modes. In the case of non-uniform systems, an erratic variation in the contribution of successive modes to the total response is observed.

7.4 The Proposed Technique For Analysis of Non-classically Damped Systems

An improved technique for analysis of non-classically damped systems was presented in Chapter 6. It was concluded that this technique effectively accounts for non-classical damping as well as the higher mode effect, while it benefits from the advantages of a normal modal superposition approach. The suggested technique is very general and is a reliable alternative to direct integration.

The technique may be considered as an effective way of reducing computation by means of transformation using normal mode shapes and by exploiting all the well known properties of dynamic response of structures.

7.5 Final Remarks

In the present study, reference was made to several possible sources of error in using the "combination of modal properties" approach to seismic analysis of secondary systems. Some of the sources of error (modal properties of the combined system, higher mode effect) are not widely referred to in the current literature. It is emphasized here that consideration should be given to *all* the sources of error discussed in the thesis in order to obtain a reliable estimate of the exact seismic response of P-S coupled systems.

7.6 Recommendations For Further Research

Following improvements in the present work are suggested:

1. The concept of the error-parameter defined in Chapter 3 may easily be extended to multi-degree-of-freedom systems. Study of the pattern of variation of the error parameter for these systems is expected to provide a better understanding of their non-classical behaviour. Comparison of the error-parameter

with the 'physical' error in multi-degree-of-freedom systems may give rise to conclusions more general than than those arrived at in this study. As suggested in Chapter 3, comparison of the performance of modal damping estimation procedures for multi-degree-of-freedom systems should enable one to arrive at more general conclusions about superiority or inferiority of a method in comparison with another.

2. The only method of diagonalization of the damping matrix considered in the present work is the 'ignoring off-diagonal terms' approach. There are other methods of diagonalization available [77], which are known to provide better results than this method. The concept of the error-parameter can easily be extended to study the error involved in these methods of diagonalization.

3. The concept of examining the error in the response with the aid of the proposed error-parameter is in fact extremely general. The error-parameter has been defined in this work as the area under an error-curve. The 'error' is taken as the relative error in the response obtained using the 'approximate' damping matrix. The approximate damping matrix could be *any* approximation to the exact damping matrix; it may not necessarily be obtained by using the diagonalization approximation.

4. More research needs to be done on the relation between the magnitude of the error-parameter and the magnitude of the 'physical' error in an earthquake analysis. Specifically, an attempt should be made to answer the following question: 'Is it possible to specify an approximate upper limit to the error-parameter below which the damping coupling in the system may be considered to be negligible?' A subsequent question that may be investigated is: 'If such a criterion can be specified, is it consistent with Warburton and Soni's [86] or Hasselman's [38] criterion?'

5. Implications of the use of approximate (normal) mode shapes in non-classically damped systems need to be studied in more detail.

6. It was suggested in Chapter 4 that Clough and Mojtahedi's [14] method, with approximate modes shapes, be used to analyse P-S coupled systems. However, only the lower modes were expected to be considered in the analysis. Attempt to include all the higher modes by a quasi-static analysis is a worthwhile effort.

7. Extensive numerical study may be undertaken to examine the implications of the assumptions in the extension suggested to Clough and Mojtahedi's [14] method in Chapter 6. Such numerical studies should examine the errors in the response obtained using the present extension against the exact response obtained by direct integration. The trend of numerical results presented in this work indicates that the suggested extended method should perform well in such studies.

Appendix A: Computer Programs

Listing of a Program
For Evaluation of Error-parameter

```
      PROGRAM MAIN

      IMPLICIT DOUBLE PRECISION(A-H,O-Z)
      DIMENSION XK(2,2),XM(2,2),C(2,2),DC(2,2),PHI(2,2),
     %EIGEN(2),ERR1D(1000),ERR2D(1000),ERR1Y(1000),ERR2Y(1000),
     %CYAN(2,2),XI(2),FINAL1(50),FINAL2(50),COEF(50,2)

      WRITE(6,1001)
C
C***  READ THE BASIC INPUT DATA
C     INPUT VARIABLES:
C     XK1, XK2 =STIFFNESS OF THE BOTTOM AND THE TOP FLOOR OF THE
C               SHEAR BULIDING CONSIDERED.
C     XM1,XM2  =MASS OF THE BOTTOM AND THE TOP FLOOR RESPECTIVELY.
C     NXI      =THE NUMBER OF SETS OF DAMPING RATIOS FOR WHICH ANALYSIS
C                NEEDS TO BE CARRIED OUT. SUPPLY NXI SETS OF DAMPING
C                RATIOS XI1 AND XI2
C     XI1,XI2  =DAMPING RATIO OF THE BOTTOM FLOOR AND THE TOP FLOOR RESP
C
      READ(5,*)XK1,XK2
      READ(5,*)XM1,XM2
C
C **  FORM THE STIFFNESS AND MASS MATRICES
C
      XK(1,1)=XK1+XK2
      XK(1,2)=-XK2
      XK(2,1)=-XK2
      XK(2,2)=XK2

      XM(1,1)=XM1
      XM(2,2)=XM2
      XM(1,2)=0.0D00
      XM(2,1)=0.0D00
C
C      FIND EIGENVALUES AND EIGENVECTORS
C
      CALL EIGENV(XK1,XK2,XM1,XM2,EIGEN,PHI)
C
C***   FIND THE RANGE OF FREQUENCIES  OVER WHICH
C      COMPUTATIONS NEED TO BE DONE
C
      CALL WRANGE(EIGEN,W1L,W1R,W2L,W2R)

      SQK1M1=DSQRT(XK1*XM1)
      SQK2M2=DSQRT(XK2*XM2)
C
C****  WRITE THE INPUT DATA AND SOME INTERMEDIATE RESULTS
C
      WRITE(6,3)
 3    FORMAT(/,5X,'STIFFNESS MATRIX')
      WRITE(6,1)((XK(I,J),J=1,2),I=1,2)
      WRITE(6,4)
 4    FORMAT(/,5X,'MASS MATRIX')
      WRITE(6,1)((XM(I,J),J=1,2),I=1,2)
      WRITE(6,5)
 5    FORMAT(/,5X,' EIGENVALUES')
      WRITE(6,2)EIGEN(1),EIGEN(2)
      WRITE(6,27)
 27   FORMAT(/,5X,'EIGENVECTORS')
      WRITE(6,2)((PHI(I,J),J=1,2),I=1,2)
      WRITE(6,6)
 6    FORMAT(/,5X,' W1L,   W1R,   W2L,  W2R')
      WRITE(6,*)W1L,W1R,W2L,W2R
```

```
      WRITE(6,28)
 28   FORMAT(/,5X,'ERROR PARAMETER= AREA UNDER ERROR CURVE.'/
     *,5X,'QUANTITY MEASURED= ABS VALUE OF ACCELERATION AMPLIFICATION'/
     *,5X,'NO OF POINTS FOR INTEGRATION= 901'/)
C
C**** READ THE NO. OF SETS OF DAMPING RATIOS.
C
      READ(5,*)NXI

      DO 510 IM2=1,NXI
      READ(5,*)XI1,XI2
      COEF(IM2,1)=XI1
      COEF(IM2,2)=XI2
C
C**** FORM THE DAMPING MATRIX
C
      C1=XI1*(2.0D00)*SQK1M1
      C2=XI2*(2.0D00)*SQK2M2
      C(1,1)=C1+C2
      C(1,2)=-C2
      C(2,1)=-C2
      C(2,2)=C2


C
C***  FORM THE PARENT MATRIX D_P
C
      CALL DIAG(C1,C2,PHI,DC)

      WRITE(6,25)XI1,XI2
  25  FORMAT(/,5X,'XI1=',E15.6,5X,'XI2=',E15.6)


      D1=C1*PHI(1,1)*PHI(1,1)+C2*((PHI(1,1)-PHI(2,1))**2)
      D2=C1*PHI(1,2)*PHI(1,2)+C2*((PHI(1,2)-PHI(2,2))**2)
      D12=C1*PHI(1,1)*PHI(1,2)+C2*(PHI(1,1)-PHI(2,1))*(PHI(1,2)
     %   -PHI(2,2))
C
C***  EVALUATE THE PARENT DAMPING MATRIX USING YAN'S METHOD AND STORE
C      IT SEPERATELY
C
      CALL YAN(XK,XM,C,EIGEN,PHI,XI,CYAN)
      WRITE(6,*)'     DAMPING MATRIX BY YAN METHOD'
      WRITE(6,1)((CYAN(I,J),J=1,2),I=1,2)

      EPAR1D=0.0D00
      EPAR2D=0.0D00
      EPAR1Y=0.0D00
      EPAR2Y=0.0D00
C
C****  START A LOOP FOR COMPUTATION OF THE ERROR-PARAMETER
C
      DO 530 IPEAK=1,2
      DW=(W1R-W1L)/9.00D2
      W=W1L

      IF(IPEAK.EQ.2)THEN
      DW=(W2R-W2L)/9.00D2
      W=W2L
      ENDIF

      DO 540 IW=1,901
      CALL HARMO2(W,XK,XM,C,Y1R,Y1I,Y2R,Y2I)
      RESP1A=DSQRT(Y1R*Y1R+Y1I*Y1I)
      RESP2A=DSQRT(Y2R*Y2R+Y2I*Y2I)

      CALL HARMO2(W,XK,XM,DC,Y1R,Y1I,Y2R,Y2I)
      RESP1D=DSQRT(Y1R*Y1R+Y1I*Y1I)
      RESP2D=DSQRT(Y2R*Y2R+Y2I*Y2I)
```

```
      ERR1D(IW)=DABS((RESP1D-RESP1A)/RESP1A)
      ERR2D(IW)=DABS((RESP2D-RESP2A)/RESP2A)
      CALL HARMO2(W,XK,XM,CYAN,Y1R,Y1I,Y2R,Y2I)
      RESP1Y=DSQRT(Y1R*Y1R+Y1I*Y1I)
      RESP2Y=DSQRT(Y2R*Y2R+Y2I*Y2I)
      ERR1Y(IW)=DABS((RESP1Y-RESP1A)/RESP1A)
      ERR2Y(IW)=DABS((RESP2Y-RESP2A)/RESP2A)

      W=W+DW
  640 CONTINUE

      IF(IPEAK.EQ.1)THEN
      CALL SIMP(W1L,W1R,900,ERR1D,AREA1)
      CALL SIMP(W1L,W1R,900,ERR2D,AREA2)
      CALL SIMP(W1L,W1R,900,ERR1Y,AREA3)
      CALL SIMP(W1L,W1R,900,ERR2Y,AREA4)
      ELSE
      CALL SIMP(W2L,W2R,900,ERR1D,AREA1)
      CALL SIMP(W2L,W2R,900,ERR2D,AREA2)
      CALL SIMP(W2L,W2R,900,ERR1Y,AREA3)
      CALL SIMP(W2L,W2R,900,ERR2Y,AREA4)
      ENDIF

      EPAR1D=EPAR1D+AREA1
      EPAR2D=EPAR2D+AREA2
      EPAR1Y=EPAR1Y+AREA3
      EPAR2Y=EPAR2Y+AREA4

  530 CONTINUE

      WRITE(6,1005)
      WRITE(6,1004)
      WRITE(6,1003)EPAR1D,EPAR1Y
      WRITE(6,1003)EPAR2D,EPAR2Y

      FINAL1(IM2)=EPAR1D
      FINAL2(IM2)=EPAR2D

  510 CONTINUE


C     *********** FORMAT STATEMENTS******

 1001 FORMAT(//,5X,'****HARMONIC RESPONSE OF A TWO DEGREE OF FREEDOM SYS
     STEM****')
 1002 FORMAT(//,5X,'MIN DIAG ELEMENT',5X,'OFF.DIAG ELEMENT',5X,'AREA'//
     X)
 1003 FORMAT(7X,E18.10,10X,E18.10,10X,E18.10)
 1004 FORMAT(/,5X,'NEGLECTING OFF-DIAGONAL TERMS',5X,'METHOD BY YAN'/)
 1005 FORMAT(/,5X,'***VALUES OF ERROR PARAMETER***')
  1   FORMAT(/,2(5X,E22.14))
  2   FORMAT(/,2(5X,E22.14))

      WRITE(7,1011)XK1,XK2
      WRITE(7,1012)XM1,XM2
 1011 FORMAT(/,5X,'XK1=',D25.16,5X,'XK2=',D25.16)
 1012 FORMAT(/,5X,'XM1=',D25.16,5X,'XM2=',D25.16)
      WRITE(7,1007)
 1007 FORMAT(//,5X,'MASS NO. 1')

      WRITE(7,1008)
 1008 FORMAT(//,8X,'XI1',8X,'XI2',8X,'E-PAR BY DIAGONALIZATION'/,1X)
      WRITE(7,1009)(COEF(I,1),COEF(I,2),FINAL1(I),I=1,NXI)
 1009 FORMAT(4X,E12.6,4X,E12.6,9X,E14.8)
      WRITE(7,1010)
 1010 FORMAT(//,5X,'MASS NO. 2')
      WRITE(7,1008)
      WRITE(7,1009)(COEF(I,1),COEF(I,2),FINAL2(I),I=1,NXI)
```

```
      STOP
      END


      SUBROUTINE EIGENV(XK1,XK2,XM1,XM2,EIGEN,PHI)
C
C***  COMPUTES EIGENPROPERTIES OF THE TWO-DOF SYSTEM CONSIDERED.
C      THIS SUBROUTINE IS NOT SUITABLE FOR OTHER SYSTEMS.
C     THIS SUBROUTINE IS NOTHING BUT A DIRECT CODING OF
C     THE FORMULAE FOR EIGENPROPERTIES OF A 2-DOF SYSTEM
C     DELOPED IN CHAPTER 3 OF THE THESIS.
C
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DIMENSION EIGEN(2),PHI(2,2)

      EXPR1=(XK2/XM2)+(XK1+XK2)/XM1
      EXPR2=DSQRT(EXPR1**2-(4.0D00)*XK1*XK2/(XM1*XM2))

      EIGEN(1)=(EXPR1-EXPR2)/2.0D00
      EIGEN(2)=(EXPR1+EXPR2)/2.0D00

      PHI(1,1)=1.0D00
      PHI(2,1)=(XK1+XK2-EIGEN(1)*XM1)/XK2
      PHI(1,2)=1.0D00
      PHI(2,2)=(XK1+XK2-EIGEN(2)*XM1)/XK2

      SQMM1=DSQRT(XM1*PHI(1,1)*PHI(1,1)+XM2*PHI(2,1)*PHI(2,1))
      SQMM2=DSQRT(XM1*PHI(1,2)*PHI(1,2)+XM2*PHI(2,2)*PHI(2,2))

      PHI(1,1)=PHI(1,1)/SQMM1
      PHI(2,1)=PHI(2,1)/SQMM1
      PHI(1,2)=PHI(1,2)/SQMM2
      PHI(2,2)=PHI(2,2)/SQMM2

      RETURN
      END


      SUBROUTINE HARMO2(W,XK,XM,C,Y1R,Y1I,Y2R,Y2I)
C
C***  EVALUATES EXACT HARMONIC RESPONSE OF THE TWO-DOF SYSTEM.
C     NOT SUITABLE FOR OTHER SYSTEMS.
C     THIS SUBROUTINE DIRECTLY CODES THE FORMULAE DEVELOPED
C     IN CHAPTER 3 OF THE THESIS FOR HARMONIC RESPONSE OF A
C     2-DOF SYSTEM.
C
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DIMENSION XK(2,2),XM(2,2),C(2,2)
      XK2=XK(2,2)
      XK1=XK(1,1)-XK(2,2)
      XM1=XM(1,1)
      XM2=XM(2,2)
      C1=C(1,1)
      C12=C(1,2)
      C2=C(2,2)

      W2=W*W

      DETR=(XK1+XK2-XM1*W2)*(XK2-XM2*W2)-W2*(C2*C1-C12*C12)-XK2*XK2
      DETI=W*(C1*(XK2-XM2*W2)+C2*(XK1+XK2-XM1*W2)+(2.0D00)*XK2*C12)

      RN1=(XK2-W2*XM2)*XM1+XK2*XM2
      CN1=W*(C2*XM1-C12*XM2)
      RN2=XK2*XM1+(XK1+XK2-W2*XM1)*XM2
      CN2=W*(-C12*XM1+C1*XM2)
```

```
      DEN=DETR*DETR+DETI*DETI
      Y1R=(RN1*DETR+CN1*DETI)/DEN
      Y1I=(CN1*DETR-RN1*DETI)/DEN
      Y2R=(RN2*DETR+CN2*DETI)/DEN
      Y2I=(CN2*DETR-RN2*DETI)/DEN
      RETURN
      END


      SUBROUTINE SIMP(X1,XN,NDIV,Y,AREA)
C
C     CALCULATES AREA UNDER A CURVE GIVEN THE Y CO-ORDINATES AT REGULAR
C     INTERVALS OF LENGTH DX.
C     X1=STARTING X CO-ORDINATE. XN=X CO-ORDINATE OF THE END OF THE INTE
C     RVAL. NDIV=NO. OF DIVISIONS OF THE X AXIS.  AREA=AREA UNDER CURVE
C     Y= AN ARRAY OF Y CO-ORDINATES AT SPECIFIED INTERVALS

      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DIMENSION Y(1000)
      NPOIN=NDIV+1
      DX=(XN-X1)/DFLOAT(NDIV)
      AREA=0.0D00
      IF(NPOIN.LT.3)THEN
      WRITE(6,1001)
      STOP
      ENDIF

      IF(NPOIN.EQ.3)THEN
      AREA=(DX/(3.0D00))*(Y(1)+(4.0D00)*Y(2)+Y(3))
      RETURN
      ENDIF

      NN=(NPOIN/2)*2
      IF(NN.EQ.NPOIN)THEN
      AREA=(Y(NPOIN)+Y(NPOIN-1))*DX/2.0D00
      NPOIN=NPOIN-1
      NDIV=NDIV-1
      ENDIF

      SUM1=Y(1)+Y(NPOIN)
      SUM2=0.0D00
      DO 10 I=2,NDIV,2
      SUM2=SUM2+Y(I)
 10   CONTINUE
      SUM3=0.0D00

      NDM1=NDIV-1
      DO 20 I=3,NDM1,2
      SUM3=SUM3+Y(I)
 20   CONTINUE
      AREA=(DX/(3.0D00))*(SUM1+(4.0D00)*SUM2+(2.0D00)*SUM3)+AREA
 1001 FORMAT(//,5X,'ERROR.... NPOIN LESS THAN 3')
      RETURN
      END

      SUBROUTINE DIAG(C1,C2,PHI,DC)
C
C***  EVALUATES THE PARENT DAMPING MATRIX
C     THIS SUBROUTINE IS A DIRECT CODING OF THE FORMULAE
C     DEVELOPED IN CHAPTER 3 OF THE THESIS FOR ELEMENTS
C     OF THE PARENT DAMPING MATRIX OF THE 2-DOF SHEAR BUILDING SYSTEM
C     UNDER STUDY.
C
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DIMENSION PHI(2,2),DC(2,2)
      D1=C1*PHI(1,1)*PHI(1,1)+C2*((PHI(1,1)-PHI(2,1))**2)
      D2=C1*PHI(1,2)*PHI(1,2)+C2*((PHI(1,2)-PHI(2,2))**2)
```

```
      DENO=(PHI(1,1)*PHI(2,2)-PHI(1,2)*PHI(2,1))**2
      IF(DENO.LT.1.0D-25)THEN
      WRITE(6,*)'ERROR... DIVISION BY AN EXTREMELY SMALL QUANTITY'
      STOP
      ENDIF
      DC(1,1)=(D1*PHI(2,2)*PHI(2,2)+D2*PHI(2,1)*PHI(2,1))/DENO
      DC(2,1)=(-D1*PHI(2,2)*PHI(1,2)-D2*PHI(2,1)*PHI(1,1))/DENO
      DC(2,2)=(D1*PHI(1,2)*PHI(1,2)+D2*PHI(1,1)*PHI(1,1))/DENO
      DC(1,2)=DC(2,1)
      RETURN
      END


      SUBROUTINE YAN(XK,XM,C,EIGEN,PHI,XI,CYAN)
C
C***  FORMS THE PARENT DAMPING MATRIX CORRESPONDING TO THE DIAGONAL
C     DAMPING MATRIX OBTAINED BY YAN'S METHOD.
C     THE BASIC PROCEDURE IS SIMILAR TO THE SUBROUTINE 'DIAG'.
C     REFER TO : 'M.J. YAN: COMPOSITE MODAL DAMPING IN STRUCTURES.'
C                PRESENTED AT PVP CONFERENCE, SAN FRANCISCO, CA.
C                 JUNE 1979.
C
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DIMENSION XK(2,2),XM(2,2),C(2,2),EIGEN(2),PHI(2,2),CYAN(2,2),XI(2)

      C1=C(1,1)
      C2=C(2,2)
      C12=C(1,2)
      XK1=XK(1,1)
      XK2=XK(2,2)
      XK12=XK(1,2)

      DO 10 IMODE=1,2
      CMODE=C1*PHI(1,IMODE)*PHI(1,IMODE)+C12*PHI(1,IMODE)*PHI(2,IMODE)
     #      *(2.0D00)+C2*PHI(2,IMODE)*PHI(2,IMODE)
      XKMODE=XK1*PHI(1,IMODE)*PHI(1,IMODE)+XK12*PHI(1,IMODE)
     #       *PHI(2,IMODE)*(2.0D00)+XK2*PHI(2,IMODE)*PHI(2,IMODE)
      XMMODE=XM(1,1)*PHI(1,IMODE)*PHI(1,IMODE)+XM(2,2)*PHI(2,IMODE)
     #      *PHI(2,IMODE)
      XI(IMODE)=(0.5D00)*(CMODE)/DSQRT(XKMODE*XMMODE)
   10 CONTINUE

      D1=(2.0D00)*XI(1)*DSQRT(EIGEN(1))
      D2=(2.0D00)*XI(2)*DSQRT(EIGEN(2))
      DENO=(PHI(1,1)*PHI(2,2)-PHI(1,2)*PHI(2,1))**2
      IF(DENO.LT.1.0D-25)THEN
      WRITE(6,*)'ERROR... DIVISION BY AN EXTREMELY SMALL QUANTITY'
      STOP
      ENDIF

      CYAN(1,1)=(D1*PHI(2,2)*PHI(2,2)+D2*PHI(2,1)*PHI(2,1))/DENO
      CYAN(2,1)=(-D1*PHI(2,2)*PHI(1,2)-D2*PHI(2,1)*PHI(1,1))/DENO
      CYAN(2,2)=(D1*PHI(1,2)*PHI(1,2)+D2*PHI(1,1)*PHI(1,1))/DENO
      CYAN(1,2)=CYAN(2,1)
      RETURN
      END


      SUBROUTINE WRANGE(EIGEN,W1L,W1R,W2L,W2R)
C
C***  EVALUATES THE FREQUENCY RANGE OVER WHICH  COMPUTATIONS NEED
C     TO BE DONE.  THE RANGE IS SPECIFIED AS BETWEEN W1L AND W2R.
C
      IMPLICIT DOUBLE PRECISION(A-H,O-Z)
      DIMENSION EIGEN(2)

      OMGA1=DSQRT(EIGEN(1))
```

```
OMGA2=DSQRT(EIGEN(2))
RATIO=OMGA2/OMGA1
W1L=(0.1D00)*OMGA1
W2R=(4.0D00)*OMGA2

IF(RATIO.LT.(4.0D00))THEN
W1R=OMGA1*(0.5D00)*((1.0D00)+RATIO)
ELSE IF ((RATIO.GE.(4.0D00)).AND.(RATIO.LT.(6.0D00)))THEN
W1R=OMGA1*((0.7D00)+(0.3D00)*RATIO)
ELSE IF ((RATIO.GE.(6.0D00)).AND.(RATIO.LT.(9.0D00)))THEN
W1R=OMGA1*((0.8D00)+(0.2D00)*RATIO)
ELSE IF ((RATIO.GE.(9.0D0)).AND.(RATIO.LT.(11.0D0)))THEN
W1R=OMGA1*((0.85D0)+(0.15D0)*RATIO)
ENDIF
W2L=W1R

IF(RATIO.GE.(11.0D0))THEN
W1R=OMGA1*(3.5D00)
W2L=OMGA2*(0.25D00)
ENDIF

RETURN
END
```

Sample Data File Applicable to the Building System no. 1 in Chapter 3.

```
1.0d6,1.0d6
1.0d4,1.0d4
2
0.05,0.02
0.10,0.02
```

Output File

```
****HARMONIC RESPONSE OF A TWO DEGREE OF FREEDOM SYSTEM****

STIFFNESS MATRIX

  0.20000000000000E+07      -0.10000000000000E+07
 -0.10000000000000E+07       0.10000000000000E+07

MASS MATRIX

  0.10000000000000E+05       0.00000000000000E+00
  0.00000000000000E+00       0.10000000000000E+05

 EIGENVALUES

  0.38196601125011E+02       0.26180339887499E+03

EIGENVECTORS

  0.52573111211913E-02       0.85065080835204E-02
  0.85065080835204E-02      -0.52573111211913E-02

  W1L,   W1R,   W2L,   W2R
0.618033988749894861        11.1803398874989479
11.1803398874989479        64.7213595499957961

 ERROR PARAMETER= AREA UNDER ERROR CURVE.
```

```
QUANTITY MEASURED= ABS VALUE OF ACCELERATION AMPLIFICATION
NO OF POINTS FOR INTEGRATION= 901


XI1=   0.500000E-01     XI2=   0.200000E-01
 DAMPING MATRIX BY YAN METHOD

  0.11600000000000E+05       -0.52000000000000E+04
 -0.52000000000000E+04        0.64000000000000E+04

***VALUES OF ERROR PARAMETER***

NEGLECTING OFF-DIAGONAL TERMS     METHOD BY YAN

    0.6186387596E+00             0.6186387596E+00
    0.1872715642E+00             0.1872715642E+00

XI1=   0.100000E+00     XI2=   0.200000E-01
 DAMPING MATRIX BY YAN METHOD

  0.17600000000000E+05       -0.72000000000000E+04
 -0.72000000000000E+04        0.10400000000000E+05

***VALUES OF ERROR PARAMETER***

NEGLECTING OFF-DIAGONAL TERMS     METHOD BY YAN

    0.1849006620E+01             0.1849006620E+01
    0.5055400637E+00             0.5055400637E+00
```

Listing of a Program for Clough and Mojtahedi's Method With the Suggested Extension

```
C       PROGRAM NUMETHOD
C
C      AN IMPROVEMENT WAS SUGGESTED IN THE PRESENT WORK ON CLOUGH
C      AND MOJTAHEDI'S METHOD, THIS PROGRAM  INCORPORATES THE
C      SUGGESTED IMPROVEMENT. COMMENTS THROUGHOUT THE PROGRAM
C      INDICATE THE CHANGES REQUIRED IN ORDER TO INCORPORATE THE
C      SUGGESTED IMPROVEMENT. IF THE CORRESPONDING STATEMENTS ARE
C      DROPPED, THIS PROGRAM CAN BE USED AS A PROGRAM FOR CLOUGH
C      AND MOJTAHEDI'S ORIGINAL METHOD.
C      THIS PROGRAM CALLS A SUBROUTINE CLOUGH WHICH DOES STEP-BY-
C      STEP TIME INTEGRATION OF AN MDOF SYSTEM FOR A SUPPLIED
C      EXCITATION USING CLOUGH'S METHOD. FOR REFERENCE, SEE

C        "EARTHQUAKE RESPONSE ANALYSIS CONSIDERING NON-
C         PROPORTIONAL DAMPING"

C        BY  RAY W. CLOUGH AND SOHEIL MOJTAHEDI
C

C       IN THIS METHOD, FIRST FEW MODES ARE USED BUT MODAL
C       INTERACTION DUE TO NON-CLASSICAL DAMPING IS CONSIDERED.
C       THE COUPLED EQUATIONS OF MOTION FOR THE FIRST FEW MODES
C       ARE INTEGRATED USING, SAY, NEWMARK'S STEP-BY-STEP METHOD.
C       THE NO. OF MODES TO BE CONSIDERED IS AN INPUT PARAMETER.
C
C       THIS METHOD PROVIDES A WAY TO ACCOUNT FOR  NON-CLASSICAL
C       DAMPING, THOUGH APPROXIMATELY, WHILE CONSIDERING ONLY THE
C       SIGNIFICANT MODES WHILE IT IS A BIT MORE EXPENSIVE THAN
C       MODAL SUPERPOSITION, IT HAS THE ADVANTAGE OF CONSIDERING
C       NON-CLASSICAL DAMPING.
C
C       THE SUGGESTED IMPROVEMENT CONSISTING IN ADDING THE
C       RESPONSE OF THE HIGHER MODES BY A QUASI-STATIC ANALYSIS.
C
C       THIS PROGRAM IS NOT A SOPHISTICATED PROGRAM IN TERMS OF
C       COST OF SOLUTION. E.G., ALL MATRICES ARE STORED IN FULL.
C       EFFORT IS MADE TO ILLUSTRATE THE SMALL AMOUNT OF CHANGES
C       NEEDED TO INCORPORATE THE SUGGESTIONS OF THE PRESENT
C       WORK. NOTE THAT THE A SEGMENT OF THE PROGRAM BETWEEN TWO
C       LINES +++++++++++++++ADDITION++++++++++++++++++++++
C       AND ++++++++++++++ADDITION ENDS+++++++++++++++++ IS
C       AN EXTRA SEGMENT NEEDED IN ORDER TO INCORPORATE THE
C       SUGGESTIONS. IF ALL SUCH SEGMENTS ARE DROPPED, THEN THIS
C       PROGRAM CAN BE USED FOR CLOUGH'S ORIGINAL METHOD.
C


      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      REAL*8 MASS,MASS1
      DIMENSION STIFF(20,20),MASS(20,20),DAMP(20,20),EIGEN(20),
     *PHI(20,20),STIFF1(20,20),MASS1(20,20),IDISP(20),
     *NEF(20),ISPF(20),HEIGHT(20),EIGEN1(20),PHI1(20,20)
      DATA MPDOF/20/
C
C***  THE EARTHQUAKE DATA IS READ FROM A SEPERATE FILE ON UNIT 7.
C
      OPEN(UNIT=5,FILE='NUMETHOD.DAT',STATUS='OLD')
      OPEN(UNIT=6,FILE='NUMETHOD.OUT',STATUS='NEW')
      OPEN(UNIT=7,FILE='RECORD.MASTER',STATUS='OLD')
      READ(5,*)NDOF,NM,GR
C
```

```
C     NM:    NO. OF MODES TO BE CONSIDERED IN CLOUGH'S ANALYSIS
C     MDOF:  TOTAL NO. OF DEGREES OF FREEDOM
C     GR:    GRAVITATIONAL ACCELERATION (SAY, 9.8M/SEC)
C     STIFF: STIFFNESS MATRIX
C     MASS:  MASS MATRIX
C     DAMP:  THE DAMPING MATRIX
C     MWDISP:MAXIMUM NO. OF WANTED MAXIMUM DISPLACEMENTS
C     MWEF:  MAXIMUM NO. OF WANTED MAXIMUM ELASTIC FORCES
C     INDSPF:INDEX FOR WHETHER MAXIMUM SPRING FORCES ARE DESIRED
C             =0 IF MAXIMUM SPRING FORCES ARE NOT DESIRED.
C             =1 IF MAXIMUM SPRING FORCES ARE DESIRED.
C     MWSPF: MAXIMUM NO. OF SPRING FORCES DESIRED
C     IBS:   INDEX FOR WHETHER MAXIMUM BASE SHEAR IS DESIRED
C              =0 IF NOT DESIRED
C              =1 IF DESIRED.
C     IOM:    INDEX FOR WHETHER MAXIMUM OVERTURNING MOMENT IS DESIRED
C             =0 IF NOT DESIRED
C             =1 IF DESIRED
C     HEIGHT:AN ARRAY STORING THE HEGHTS OF EACH FLOOR IN CASE OF
C            A SHEAR BUILDING SYSTEM. IN CASE OF VERY UNUSUAL SYSTEMS
C            THIS DOESN'T HAVE MUCH SIGNIFICANCE. THIS VECTOR IS MAINLY
C            USED TO COMPUTE THE OVERTURNING MOMENT.
C            THIS IS PROVIDED ONLY IF IOM=1.
C     IDISP: AN ARRAY WHICH STORES THE DOFS FOR WHICH MAX.
C            DISPLACEMENTS ARE DESIRED.
C     IEF:   AN ARRAY WHICH STORES THE DOFS FOR WHICH MAX. ELASTIC
C            FORCES ARE DESIRED.
C     ISPF:  AN ARRAY WHICH STORES THE SPRING ELEMENT
C            NOS. FOR WHICH MAX. SPRING FORCES ARE DESIRED.
C
C***  READ BASIC QUANTITIES LIKE STIFFNESS, MASS AND
C     DAMPING PROPERTIES. READ THE MATRICES IN FULL.

C
      READ(5,*)((STIFF(I,J),J=1,MDOF),I=1,MDOF)
      READ(5,*)((MASS(I,J),J=1,MDOF),I=1,MDOF)
      READ(5,*)((DAMP(I,J),J=1,MDOF),I=1,MDOF)
C
C***  COPY STIFFNESS AND MASS MATRICES INTO DUMMY MATRICES
C     IN ORDER TO PREPARE TO USE JACOB. (JACOB DESTROYS STIFFNESS
C     AND MASS MATRICES
C
      DO 10 I=1,MDOF
      DO 20 J=1,MDOF
      MASS1(I,J)=MASS(I,J)
      STIFF1(I,J)=STIFF(I,J)
   20 CONTINUE
   10 CONTINUE
C
C***  GET EIGENPROPERTIES USING THE JACOBI ALGORITHM.
C
      CALL JACOB(STIFF1,MASS1,EIGEN,PHI,MDOF,MPDOF)

C
C***  SORT THE EIGENVALUES IN THE ASCENDING ORDER
C
      CALL SORT(EIGEN,PHI,MDOF,MPDOF)
C
C***  READ DATA WHICH TELLS THE PROGRAM THE REQUIRED RESPONSE
C     QUANTITIES.
C
      READ(5,*)MWDISP,MWEF,INDSPF,IBS,IOM
      IF(MWDISP.GE.1)READ(5,*)(IDISP(I),I=1,MWDISP)
      IF(MWEF.GE.1)READ(5,*)(IEF(I),I=1,MWEF)
      IF (INDSPF.EQ.1)THEN
      READ(5,*)MWSPF
      READ(5,*)(ISPF(I),I=1,MWSPF)
      ENDIF
```

```
      IF(IOM.EQ.1)READ(5,*)(HEIGHT(I),I=1,MDOF)
C
C***  WRITE THE DATA INPUT
C

      WRITE(6,*)'  STIFFNESS MATRIX'
      WRITE(6,1)((STIFF(I,J),J=1,MDOF),I=1,MDOF)
      WRITE(6,*)'  MASS MATRIX'
      WRITE(6,1)((MASS(I,J),J=1,MDOF),I=1,MDOF)
      WRITE(6,*)'  DAMPING MATRIX'
      WRITE(6,1)((DAMP(I,J),J=1,MDOF),I=1,MDOF)
      WRITE(6,*)'MWDISP,MWEF,INDSPF,MWSPF,IBS,IOM'
      WRITE(6,*)MWDISP,MWEF,INDSPF,MWSPF,IBS,IOM
      WRITE(6,*)'DISPLACEMENTS NEEDED AT THE FOLLOWING DOFS'
      WRITE(6,*)(IDISP(I),I=1,MWDISP)
      WRITE(6,*)'ELASTIC FORCES NEEDED AT THE FOLLOWING DOFS'
      WRITE(6,*)(IEF(I),I=1,MWEF)
      IF(INDSPF.EQ.1)THEN
      WRITE(6,*)'SPRING FORCES NEEDED AT THE FOLLOWING DOFS'
      WRITE(6,*)(ISPF(I),I=1,MWSPF)
      ENDIF
      IF(IOM.EQ.1)THEN
      WRITE(6,*)'HEIGHTS OF THE STOREYS'
      WRITE(6,*)(HEIGHT(I),I=1,MDOF)
      ENDIF

  1   FORMAT(4(2X,D16.8))
C
C***  CALL SUBROUTINE CLOUGH TO GET THE STEP-BY-STEP RESPONSE.
C
       CALL CLOUGH(STIFF,STIFF1,MASS,DAMP,EIGEN,PHI,MDOF,MPDOF,NM
     *,GR,MWDISP,MWEF,INDSPF,MWSPF,IBS,IOM,IDISP,IEF,ISPF,HEIGHT)
      STOP
      END


      SUBROUTINE CLOUGH(STIFF,STIFF1,MASS,DAMP,EIGEN,PHI,SIZE,
     *MPDOF,NM,
     *GR,MWDISP,MWEF,INDSPF,MWSPF,IBS,IOM,IDISP,IEF,ISPF,HEIGHT)
C
C***  DOES DIRECT TIME INTEGRATION OF MDOF SHEAR BUILDING USING
C     CLOUGH' METHOD.
C     PRECISION:   DOUBLE
C     REFERENCE:   EARTHQUAKE ANALYSIS CONSIDERING NON-
C                  PROPORTIONAL DAMPING
C                  RAY W. CLOUGH AND SOHEIL MOJTAHEDI
C                  EARTHQUAKE ENGINEERING AND STRUCTURAL
C                  DYNAMICS  VOL.4 489-496 (1976)


C
C     NOTES:    1. STIFF, MASS,DAMP ARE STIFFNESS,MASS AND
C                  DAMPING MATRICES IN THE USUAL SENSE OF THEIR
C                  MEANING. XK, XM, C ARE ACTUALLY OBTAINED BY
C                  DOING THE TRIPLE
C                              T
C                  PRODUCT [PHI] [MATRIX][PHI]. THESE ARE THE
C                  ACTUL MATRICES THAT  ARE USED IN THE ANALYSIS
C                  BY NEWMARK'S METHOD.
C               2. DISP,VEL,ACN,DISPN,VELN,ACNN ARE THE VECTORS
C                  USED TO DENOTE (OLD,NEW RESPECTIVELY)
C                  DISPLACEMENT, VELOCIT AND ACCELERATION
C                  VECTORS THAT ARE USED IN NEWMARK'S METHOD
C                  ACTUAL PHYSICAL DISPLACEMENT, VELOCITY AND
C                  ACCELERATION ARE DENOTED {DP},{VP} AND {AP}
C                  RESPECTIVELY.
C
C                  BASE EXCITATION VECTOR IS ASSUMED TO BE {1}
C                  ONLY FOR THE PURPOSE OF COMPUTING THE
```

```
C                  INERTIAL LOAD ON THE STRUCTURE. CHANGE THAT
C                  STATEMENT TO MAKE THIS PROGRAM SUITABLE FOR
C                  ANY OTHER TYPE OF STRUCTURE
C

      IMPLICIT REAL*8(A-H,O-Z)
      INTEGER SIZE
      REAL*8 MED,MEF,MSPF,MASS,LP
      DIMENSION DISP(20),VEL(20),ACN(20),DISPN(20),VELN(20),ACNN(20),
     $XK(20,20),XM(20,20),DAMP(20,20),XLDN(20),XLDN1(20),
     $VEC1(20),VEC2(20),
     $VEC3(20),VEC4(20),XK1(20,20),GACC(4000),TIME(4000),XMDUM(20,20),
     $IDISP(20),IEF(20),ISPF(20),HEIGHT(20),MED(20),MEF(20),MSPF(20),
     $EFORC(20),SPF(20),EIGEN(20),PHI(20,20),STIFF(20,20),MASS(20,20)
     $,LP(20),DP(20),VP(20),AP(20),C(20,20),DUMMY(20),RDISP(20),
     $UB(20,1),PARTI(20,1), UBO(20,1),STIFF1(20,20),DPR(20)


C+++++++++++++++++ADDITION++++++++++++++++++++++++++++++++++
C
C**** INVERT STIFFNESS MATRIX
C     FIRST COPY [STIFF] INTO [STIFF1].
C
      DO 600 I=1,SIZE
      DO 610 J=1,SIZE
        STIFF1(I,J)=STIFF(I,J)
  610 CONTINUE
  600 CONTINUE

      CALL MATINV(STIFF1,SIZE,MPDOF)
C
C***  PUT THE PRODUCT OF {DUMMY} AND inv[K] IN THE ARRAY
C     [RDISP]. THIS WILL SUBSEQUENTLY BE SCALED TO GET THE
C     RESIDUAL RIGID RESPONSE.
C
C
C***  FILL THE VECTOR {UB} WITH 1'S.
C

      DO 660 I=1,SIZE
  660 UB(I,1)=1.0D0
C
C***  CALL A SUBROUTINE THAT EVALUATES THE PARTICIPATION
C     FACTORS.
C
      CALL PFACT(MASS,PHI,UB,PARTI,SIZE,NM,MPDOF)
C
C***   GETTING THE VECTOR {UBO}
C      FORM THE SUMMATION OF (PARTICIPATION FACTORS  X  THE
C      MODAL VECTORS) AND STORE IT TEMPORARILY IN {UBO}
C
       DO 670 I=1,SIZE
       DO 680 J=1,NM
       UBO(I,1)=UBO(I,1)+PARTI(J,1)*PHI(I,J)
 680   CONTINUE
  670  CONTINUE
C
C**    SUBTRACT THE ABOVE SUM FROM {UB} TO GET THE REAL{UBO}.
C

       DO 690 I=1,SIZE
  690  UBO(I,1)=UB(I,1)-UBO(I,1)

      DO 620 I=1,SIZE
      DUMMY(I)=0.0D0
      DO 630 J=1,SIZE
      DUMMY(I)=DUMMY(I)+MASS(I,J)*UBO(J,1)
 630  CONTINUE
```

```
 620  CONTINUE

      DO 640 I=1,SIZE
      RDISP(I)=0.0
      DO 650 J=1,SIZE
      RDISP(I)=RDISP(I)+STIFF1(I,J)*DUMMY(J)
 650  CONTINUE
  640 CONTINUE

C+++++++++++++++++++++ADDITION ENDS++++++++++++++++++++++++++
C
C***  PREPARE THE MATRICES XK,XM,C ACCORDING TO CLOUGH'S METHOD
C

C***  FORM XM MATRIX BY USING SUB. TRIPLE ON [MASS]
C
      CALL TRIPLE(MASS,PHI,XM,SIZE,MPDOF,NM)
C
C
C***  FORM XK MATRIX BY DOING TRIPLE PRODUCT ON [STIFF]
C
      CALL TRIPLE(STIFF,PHI,XK,SIZE,MPDOF,NM)
C
C***  FORM C MATRIX BY EMPLOYING THE SUBROUTINE TRIPLE ON THE
C     MATRIX DAMP
C
      CALL TRIPLE(DAMP,PHI,C,SIZE,MPDOF,NM)
C
C
C**** NEWMARK'S CONSTANT AVERAGE ACCELERATION METHOD WILL BE
C     USED TO INTEGRATE THE COUPLED EQUATIONS OF MOTION. FOR
C     REFERENCE, SEE
C
C     "NUMERICAL METHODS IN FINITE ELEMENT ANALYSIS"
C            BY K.J. BATHE AND E.L. WILSON.
C
C***  INITIALIZE THE DISPLACEMENT AND VELOCITY VECTORS TO ZERO.
C
      DO 10 I=1,NM
      DISP(I)=0.0D00
      VEL(I)=0.0D00
 10   CONTINUE
C
C***  CONSTANTS
C
      DELTA=0.5D00
      ALPHA=0.25D00


C
C***  READ GROUND ACCELERATION AND MULTIPLY IT BY THE
C     GRAVITATION CONSTANT.
C     NLP= NO. OF TIME POINTS IN THE EXCITATION.
C     TIME= AN ARRAY CONTAINING THE TIME POINTS.
C     GACC= AN ARRAY CONTAINING THE CORRESPONDING GROUND
C     ACCELERATIONS.
C
      READ(7,*)NLP
      READ(7,*)(TIME(I),GACC(I),I=1,NLP)
      DO 250 I=1,NLP
      GACC(I)=GACC(I)*GR
  250 CONTINUE
      NSTEP=NLP-1
C
C
C***  INTIAL QUANTITIES
C
      GACCI=GACC(1)
      T=TIME(1)
```

```
C
C***  COMPUTE THE LOAD VECTOR FOR T=0.  ASSUME THE BASE
C     EXCITATION VECTOR TO BE  {1}. CHANGE THIS SEGMENT IF
C     YOU WANT TO CONSIDER A GENERAL LASE EXCITATION VECTOR.
C
      DO 200 I=1,SIZE
      SUM=0.0D0
      DO 210 J=1,SIZE
      SUM=SUM+MASS(I,J)
  210 CONTINUE
      LP(I)=SUM*GACCI
  200 CONTINUE
C
C***  MULTIPLY THE PHYSICAL FORCE VECTOR BY THE TRANSPOSE OF
C     [PHI] TO GET THE [XLDN]
C
      DO 420 I=1,NM
      XLDN(I)=0.0D0
      DO 430 J=1,SIZE
      XLDN(I)=XLDN(I)+PHI(J,I)*LP(J)
 430  CONTINUE
 420  CONTINUE

C
C***  ACCELRATION VECTOR AT THE TIME T=0
C
C     FIRST, COPY [XM] MATRIX INTO A DUMMY MATRIX FOR THE
C     PURPOSE OF INVERSION
C
      DO 220 I=1,NM
      DO 230 J=1,NM
      XMDUM(I,J)=XM(I,J)
  230 CONTINUE
  220 CONTINUE

      CALL MATINV(XMDUM,NM,MPDOF)
C
C
C***  MULTIPLY INVERSE OF THE DUMMY XM MATRIX BY [XLDN]TO
C     GET THE ACCLERATION VECTOR AT THE TIME T=0
C
      DO 260 I=1,NM
      SUM=0.0D0
      DO 270 J=1,NM
      SUM=SUM+XMDUM(I,J)*XLDN(J)
  270 CONTINUE
      ACN(I)=SUM
  260 CONTINUE

C
C***   BEFORE STARTING THE LOOP OVER TIME INTIALIZE THE ARRAYS
C      THAT STORE THE MAXIMUM VALUE OF CERTAIN QUANTITIES AS A
C      FUNCTION OF TIME
C
      DO 480 I=1,SIZE
      MED(I)=0.0D0
      MEF(I)=0.0D0
      MSPF(I)=0.0D0
  480 CONTINUE
      BSM=0.0D0
      OTMM=0.0D0

C
C**** LOOP OVER TIME
C
      DO 40 ISTEP=1,NSTEP
```

```
      GACCN=GACC(ISTEP+1)
      TN=TIME(ISTEP+1)
      DT=TN-T
C
C*** EVALUATE CONSTANTS
C
      A0=(1.0D00)/(ALPHA*DT*DT)
      A1=DELTA/(ALPHA*DT)
      A2=(1.0D00)/(ALPHA*DT)
      A3=(0.5D00)/ALPHA-1.0D00
      A4=(DELTA/ALPHA)-1.0D00
      A5=(0.5D00*DT)*(DELTA/ALPHA-2.0D00)
      A6=DT*(1.0D00-DELTA)
      A7=DELTA*DT

C
C***  FORM THE EFFECTIVE XK MATRIX I.E. [XK1]
C

      DO 20 I=1,NM
      DO 30 J=1,NM
      XK1(I,J)=XK(I,J)+A0*XM(I,J)+A1*C(I,J)
   30 CONTINUE
   20 CONTINUE
C
C
C***  INVERT [XK1] FOR LATER USE
C
      CALL MATINV(XK1,NM,MPDOF)
C
C***  FORM THE EFFECTIVE LOAD VECTOR. (PHYSICAL) FOR THE NEW
C     VALUE OF GROUND ACCELERATION
C
      DO 150 I=1,SIZE
      ADD=0.0D00
      DO 160 J=1,SIZE
      ADD=ADD+MASS(I,J)
  160 CONTINUE
      LP(I)=ADD*GACCN
  150 CONTINUE
C                              T
C***  FORM [XLDN] VECTOR AS [PHI] [LP]
C
      DO 440 I=1,NM
      XLDN(I)=0.0D0
      DO 450 J=1,SIZE
      XLDN(I)=XLDN(I)+PHI(J,I)*LP(J)
  450 CONTINUE
  440 CONTINUE
C
C***  FORMATION OF THE 'EFFECTIVE FORCE VECTOR'
C     AS IN NEWMARK'S METHODS

C
C     WRITE(6,*)' LOAD VECTOR'
C     WRITE(6,*)(XLDN(I),I=1,2)

      DO 50 I=1,NM
      VEC1(I)=A0*DISP(I)+A2*VEL(I)+A3*ACN(I)
      VEC2(I)=A1*DISP(I)+A4*VEL(I)+A5*ACN(I)
   50 CONTINUE
C
C***  PREMULTIPLY [VEC1] BY [XM] AND [VEC2] BY [C]
C
      DO 60 I=1,NM
      SUM1=0.0D00
      SUM2=0.0D00
      DO 70 J=1,NM
```

```
      SUM1=SUM1+XM(I,J)*VEC1(J)
      SUM2=SUM2+C(I,J)*VEC2(J)
   70 CONTINUE
      VEC3(I)=SUM1
      VEC4(I)=SUM2
   60 CONTINUE

C
C     WRITE(6,*)'   VEC3 AND VEC4'
C     WRITE(6,*)(VEC3(I),I=1,2)
C     WRITE(6,*)(VEC4(I),I=1,2)
C
C***  'EFFECTIVE FORCE VECTOR' FOR THIS STEP AS IN NEWMARK'S
C      METHOD
C
      DO 80 I=1,NM
      XLDN1(I)=XLDN(I)+VEC3(I)+VEC4(I)
   80 CONTINUE
C
C***  FIND THE NEW DISPLACEMENT VECTOR AS [XK1][XLDN1] WHERE
C     [XK1] ACTUALLY CONTAINS THE INVERSE OF THE ORIGINAL
C     MATRIX [XK1]
C
      DO 140 I=1,NM
      SUM=0.0D00
      DO 170 J=1,NM
      SUM=SUM+XK1(I,J)*XLDN1(J)
  170 CONTINUE
      DISPN(I)=SUM
  140 CONTINUE
C
C***  FIND THE NEW ACCELERATION AND VELOCITY VECTORS
C
      DO 180 I=1,NM
      ACNN(I)=A0*(DISPN(I)-DISP(I))-A2*VEL(I)-A3*ACN(I)
      VELN(I)=VEL(I)+A6*ACN(I)+A7*ACNN(I)
  180 CONTINUE
  100 FORMAT(5X,'TIME=',2X,F10.6)
 110  FORMAT(5X,3(5X,D20.13))
C
C***   GET THE PHYSICAL DISPLACEMENT,VELOCITY AND
C      ACCELERATION FROM THE 'FICTITIOUS' QUANTITIES
C
       DO 460 I=1,SIZE
       DP(I)=0.0D0
       VP(I)=0.0D0
       AP(I)=0.0D0
       DO 470 J=1,NM
       DP(I)=DP(I)+PHI(I,J)*DISPN(J)
       VP(I)=VP(I)+PHI(I,J)*VELN(J)
       AP(I)=AP(I)+PHI(I,J)*ACNN(J)
   470 CONTINUE
   460 CONTINUE
C
C+++++++++++++++++++ADDITION+++++++++++++++++++++++++++++++
C
C**   ADD THE CONTRIBUTION OF THE RIGID MODES TO THIS
C     DISPLACEMENT
C
       DO 700 I=1,SIZE
       DPR(I)=RDISP(I)*GACC(ISTEP+1)
   700 CONTINUE
       DO 710 I=1,SIZE
       DP(I)=DP(I)+DPR(I)
   710 CONTINUE

C
C+++++++++++++++++ADDITION ENDS+++++++++++++++++++++++++++++
```

```
C
C***  FIND ELASTIC FORCES FROM DISPLACEMENTS
C
      DO 290 I=1,SIZE
      EFORC(I)=0.0D0
      DO 390 J=1,SIZE
      EFORC(I)=EFORC(I)+STIFF(I,J)*DP(J)
  390 CONTINUE
  290 CONTINUE
C
C***  EXTRACTING MAXIMUM ELASTIC DISPLACEMENTS
C
      DO 300 I=1,MWDISP
      ID=IDISP(I)
      IF(DABS(DP(ID)).GT.DABS(MED(ID)))MED(ID)=DP(ID)
 300  CONTINUE

C
C***  EXTRACTING MAXIMUM ELASTIC FORCES
C
      DO 310 I=1,MWEF
      JJ=IEF(I)
      IF(DABS(EFORC(JJ)).GT.DABS(MEF(JJ)))MEF(JJ)=EFORC(JJ)
  310 CONTINUE
C
C***  USING THE ELASTIC FORCE VECTOR FOR THIS TIME-STEP TO
C     FIND OUT THE BASE SHEAR AND OVERTURNING MOMENT,IF
C     DESIRED ALSO FIND THEIR MAXIMA (IF DESIRED).
C
      IF(IBS.EQ.1)THEN
      BS=0.0D0
      DO 320 I=1,SIZE
      BS=BS+EFORC(I)
  320 CONTINUE
      IF(DABS(BS).GT.DABS(BSM))BSM=BS
      ENDIF
C
C***  FIND OVERTURNING MOMENT, AND ITS MAX. VALUE BY
C     REPLACEMENT (IF DESIRED)
C
      IF(IOM.EQ.1)THEN
      OTM=0.0D0
      DO 330 I=1,SIZE
      OTM=OTM+EFORC(I)*HEIGHT(I)
  330 CONTINUE
      IF(DABS(OTM).GT.DABS(OTMM))OTMM=OTM
      ENDIF
C
C
C***  FIND SPRING FORCES, IF REQUESTED
C
      IF(INDSPF.EQ.1)THEN
      DO 340 I=1,MWSPF
      II=ISPF(I)
      SPF(II)=0.0D0
      DO 350 J=II,SIZE
      SPF(II)=SPF(II)+EFORC(J)
  350 CONTINUE
      IF(DABS(SPF(II)).GT.DABS(MSPF(II)))MSPF(II)=SPF(II)
  340 CONTINUE
      ENDIF

C
C***  COMING BACK TO THE 'FICTITIOUS' QUANTITIES, FILL THE
C     ARRAY FOR THE 'OLD' QUANTITIES SUCH AS [DISP], [VEL]
C     WITH THE RECENTLY FOUND 'NEW' QUANTITIES SUCH AS
C     [DISPN],[VELN] IN ORDER TO BEGIN A NEW STEP
C
```

```
      DO 120 I=1,NM
      DISP(I)=DISPN(I)
      VEL(I)=VELN(I)
      ACN(I)=ACNN(I)
  120 CONTINUE
      T=TN

  40  CONTINUE
C
C***  WRITE THE MAXIMUM QUANTITIES THUS OBTAINED
C
      WRITE(6,1004)
      DO 360 I=1,MWDISP
      ID=IDISP(I)
  360 WRITE(6,1001)ID,MED(ID)

      WRITE(6,1005)
      DO 370 I=1,MWEF
      JJ=IEF(I)
  370 WRITE(6,1001)JJ,MEF(JJ)

      IF(INDSPF.EQ.1)THEN
      WRITE(6,1006)
      DO 380 I=1,MWSPF
      II=ISPF(I)
  380 WRITE(6,1001)II,MSPF(II)
      ENDIF

      IF(IBS.EQ.1)THEN
      WRITE(6,1002)BSM
      ENDIF
      IF(IOM.EQ.1)THEN
      WRITE(6,1003)OTMM
      ENDIF


C
C***  FORMAT STATEMENTS
C
 1001 FORMAT(3X,I5,5X,D25.16)
 1002 FORMAT(3X,'MAXIMUM BASE SHEAR=',2X,D25.16)
 1003 FORMAT(3X,'MAXIMUM OVERTURNING MOMENT=',2X,D25.16)
 1004 FORMAT(3X,'MAXIMUM ELASTIC DISPLACEMENTS'/,4X,'DOF',10X,
     *'DISPLACEMENT')
 1005 FORMAT(3X,'MAXIMUM ELASTIC FORCES'/,4X,'DOF',10X,
     *'ELASTIC FORCES')
 1006 FORMAT(3X,'MAXIMUM SPRING FORCE'/,4X,'DOF',10X,
     *'SPRING FORCE')

      RETURN
      END

      SUBROUTINE MATINV(A,N,NMAX)
C
C *** MATRIX A IS INVERTED IN PLACE.
C     WHEN CONTROL IS TRANSFERRED TO THE CALLING PROGRAM,
C     MATRIX A CONTAINS THE INVERSE OF THE ORINGINAL MATRIX A.
C     N= SIZE OF THE MATRIX.
C     NMAX= THE MAXIMUM POSSIBLE VALUE OF N. GIVE THE ‘LEADING’
C           DIMENSION OF A.
C
C     PRECISION:    DOUBLE
C
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION A(NMAX,NMAX)
C
C***  CHANGE THE FOLLOWING DIMENSION STATEMENT IF CHANGE IN
C     SIZE OF THE PROBLEM IS DESIRED
```

```
C
      DIMENSION INDEX(20,2)

      DO 108 I=1,N
      INDEX(I,1)=0
  108 CONTINUE
      II=0
  109 AMAX=-1.0D00
      DO 110 I=1,N
      IF(INDEX(I,1))110,111,110
  111 DO 112 J=1,N
      IF(INDEX(J,1))112,113,112
  113 TEMP=DABS(A(I,J))
      IF(TEMP-AMAX)112,112,114
  114 IROW=I
      ICOL=J
      AMAX=TEMP
  112 CONTINUE
  110 CONTINUE
      IF(AMAX)225,115,116

  116 INDEX(ICOL,1)=IROW
      IF(IROW-ICOL)119,118,119
  119 DO 120 J=1,N
      TEMP=A(IROW,J)
      A(IROW,J)=A(ICOL,J)
      A(ICOL,J)=TEMP
  120 CONTINUE
      II=II+1
      INDEX(II,2)=ICOL
  118 PIVOT=A(ICOL,ICOL)
      A(ICOL,ICOL)=1.0D00
      PIVOT=1.0D00/PIVOT
      DO 121 J=1,N
      A(ICOL,J)=A(ICOL,J)*PIVOT
  121 CONTINUE
      DO 122 I=1,N
      IF(I-ICOL)123,122,123
  123 TEMP=A(I,ICOL)
      A(I,ICOL)=0.0D00
      DO 124 J=1,N
  124 A(I,J)=A(I,J)-A(ICOL,J)*TEMP
  122 CONTINUE
      GO TO 109

  125 ICOL=INDEX(II,2)
      IROW=INDEX(ICOL,1)
      DO 126 I=1,N
      TEMP=A(I,IROW)
      A(I,IROW)=A(I,ICOL)
  126 A(I,ICOL)=TEMP
      II=II-1
  225 IF(II)125,127,125
  115 WRITE(6,1001)
 1001 FORMAT(//,2X,'ZERO PIVOT')
  127 CONTINUE
      RETURN
      END

C
      SUBROUTINE JACOB(A,B,EIGV,X,N,NMAX)

C     SUBROUTINE JACOB CALCULATES THE FREQUENCIES AND MODE SHAPES
C     OF AN MDOF DYNAMIC SYSTEM. ALL OF THEM ARE CALCULATED.
C     THIS PROCEDURE IS NUMERICALLY STABLE AND EFFICIENT FOR
C     SMALL SYSTEMS. NUMERICAL ERRORS ARE MINIMUM.
C
```

```
C     REFERENCE: BATHE AND WILSON "NUMERICAL METHODS IN FINITE
C     ELEMENT ANALYSIS".
C
C     VARIABLES:
C     A(I,J):STIFFNESS MATRIX
C     B(I,J):MASS MATRIX
C     EIGEN(I):MATRIX CONTAINING EIGENVALUES
C     X(I,J):MODE SHAPE MATRIX, CONTAINS COLUMNS OF
C     EIGENVECTORS.
C     N:     THE SIZE OF [A] OR [B].
C     NMAX:  LEADING DIMENSION OF [A].
C     NSMAX: MAX. NO. OF SWEEPS ALLOWED
C
C     NOTES:
C               1. RTOL IS THE TOLERANCE LEVEL. CHANGE IT TO
C                  SUIT YOUR ACCURACY NEEDS

C               2. EPS IS ANOTHER TOLERANCE LEVEL THAT
C                  DETERMINES THE SMALLNESS IF THE DIAGONAL
C                  ELEMENTS IN AN ITERATION. CHANGE IT TO
C                  SUIT YOUR NEEDS.
C               3. CHANGE NSMAX TO SUIT YOUR NEEDS.
C
C

      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION A(NMAX,NMAX),B(NMAX,NMAX),X(NMAX,NMAX),
     *EIGV(NMAX)
C
C***  CHANGE THE FOLLOWING DIMENSION STATEMENT IF INCREASE IN
C     THE SIZE OF THE PROBLEM IS DESIRED.
C
      DIMENSION D(20)
C
C     INITIALIZE EIGENVALUE AND EIGENVECTOR MATRICES.
C

 2    FORMAT(4X,3F15.6)
      RTOL=1.OD-16
      NSMAX=20
      DO 10 I=1,N
      IF((A(I,I).GT.0.0d0) .AND. (B(I,I).GT.0.0d0)) GO TO 4
      WRITE(6,2020)
      STOP
  4   D(I)=A(I,I)/B(I,I)
 10   EIGV(I)=D(I)
      DO 30 I=1,N
      DO 20 J=1,N
      X(I,J)=0.00d00
 20   CONTINUE
      X(I,I)=1.00d00
 30   CONTINUE
      IF(N.EQ.1) RETURN
C
C     INITIALIZE SWEEP COUNTER AND BEGIN ITERATION
C

      NSWEEP=0
      NR=N-1
 40   NSWEEP=NSWEEP+1
C
C     CHECK IF PRESENT OFF-DIAGONAL ELEMENT IS LARGE ENOUGH
C     TO REQUIRE  ZEROING
C

      EPS=(1D-16*NSWEEP)**2
      DO 210 J=1,NR
      JJ=J+1
```

```
      DO 205 K=JJ,N
      EPTOLA=(A(J,K)*A(J,K))/(A(J,J)*A(K,K))
      EPTOLB=(B(J,K)*B(J,K))/(B(J,J)*B(K,K))
C
      IF((EPTOLA.LT.EPS).AND.(EPTOLB.LT.EPS)) GO TO 205
C
C     IF ZEROING IS REQUIRED THEN CALCULATE THE ROTATION
C     MATRIX ELEMENT CA AND CG.


      AKK=A(K,K)*B(J,K)-B(K,K)*A(J,K)
      AJJ=A(J,J)*B(J,K)-B(J,J)*A(J,K)
      AB=A(J,J)*B(K,K)-A(K,K)*B(J,J)
      CHECK=(AB*AB+4.0d00*AKK*AJJ)/4.0d0
      IF(CHECK)50,60,60
   50 WRITE(6,2020)
      STOP

   60 SQCH=DSQRT(CHECK)
      D1=AB/2.0+SQCH
      D2=AB/2.0-SQCH
      DEN=D1
      IF(DABS(D2).GT.DABS(D1))DEN=D2
      IF(DEN)80,70,80
   70 CA=0.0
      CG=-A(J,K)/A(K,K)
      GO TO 90
   80 CA=AKK/DEN
      CG=-AJJ/DEN
C
C     PERFORM THE GENERALIZED ROTATION TO ZERO THE PRESENT
C     OFF-DIAGONAL ELEMENT
C
   90 IF(N-2)100,190,100
  100 JP1=J+1
      JM1=J-1
      KP1=K+1
      KM1=K-1
      IF(JM1-1)130,110,110
  110 DO 120 I=1,JM1
      AJ=A(I,J)
      BJ=B(I,J)
      AK=A(I,K)
      BK=B(I,K)
      A(I,J)=AJ+CG*AK
      B(I,J)=BJ+CG*BK
      A(I,K)=AK+CA*AJ

  120 B(I,K)=BK+CA*BJ
  130 IF(KP1-N)140,140,160
  140 DO 150 I=KP1,N
      AJ=A(J,I)
      BJ=B(J,I)
      AK=A(K,I)
      BK=B(K,I)
      A(J,I)=AJ+CG*AK
      B(J,I)=BJ+CG*BK
      A(K,I)=AK+CA*AJ
  150 B(K,I)=BK+CA*BJ
C
  160 IF(JP1-KM1)170,170,190
  170 DO 180 I=JP1,KM1
      AJ=A(J,I)
      BJ=B(J,I)
      AK=A(I,K)
      BK=B(I,K)
      A(J,I)=AJ+CG*AK
      B(J,I)=BJ+CG*BK
```

```
      A(I,K)=AK+CA*AJ
  180 B(I,K)=BK+CA*BJ
C

  190 AK=A(K,K)
      BK=B(K,K)
      A(K,K)=AK+2.0*CA*A(J,K)+CA*CA*A(J,J)
      B(K,K)=BK+2.0*CA*B(J,K)+CA*CA*B(J,J)
      A(J,J)=A(J,J)+2.0*CG*A(J,K)+CG*CG*AK
      B(J,J)=B(J,J)+2.0*CG*B(J,K)+CG*CG*BK
      A(J,K)=0.0d0
      B(J,K)=0.0d0
C
C     UPDATE THE EIGENVECTOR MATRIX AFTER EACH ROTATION
C

      DO 200 I=1,N
      XJ=X(I,J)
      XK=X(I,K)
      X(I,J)=XJ+CG*XK
      X(I,K)=XK+CA*XJ
  200 CONTINUE
  205 CONTINUE
  210 CONTINUE
C     WRITE(6,*)X(2,2)
C
C     UPDATE THE EIGENVALUES AFTER EACH SWEEP
C
      DO 220 I=1,N
      IF((A(I,I).GT.0.0).AND.(B(I,I).GT.0.0)) GO TO 220
      WRITE(6,2020)
      STOP
 220  EIGV(I)=A(I,I)/B(I,I)
C
C      CHECK FOR CONVERGENCE
C

  230 DO 240 I=1,N
      TOL=RTOL*D(I)
      DIF=ABS(EIGV(I)-D(I))
      IF(DIF.GT.TOL)GO TO 280
  240 CONTINUE
C
C     FILL OUT THE SECTION TRIANGLE OF RESULTANT MATRICES
C     AND SCALE EIGENVECTORS.
C
  255 DO 260 I=1,N
      DO 260 J=1,N
      A(J,I)=A(I,J)
  260 B(J,I)=B(I,J)
C     WRITE(6,*)((X(I,J),J=1,2),I=1,2)
      DO 270 J=1,N
      BB=DSQRT(B(J,J))
      DO 270 K=1,N
  270 X(K,J)=X(K,J)/BB
C 270 WRITE(6,*)X(K,J)
      WRITE(6,*)'      NO OF SWEEPS MADE'
      WRITE(6,*)NSWEEP
      RETURN
C
C     UPDATE D MATRIX AND START NEW SWEEP IF ALLOWED
C

  280 DO 290 I=1,N
  290 D(I)=EIGV(I)
      IF(NSWEEP.LT.NSMAX) GO TO 40
      GO TO 255
```

```
 2000 FORMAT(5X,'SWEEP NUMBER IN *JACOBI* =',I4)
 2010 FORMAT(2X,6E20.12)
 2020 FORMAT(4X,'ERROR IN SOLUTION '/9X,'MATRICES NOT POSITIVE DEFINITE
     @')
 2030 FORMAT(5X,'CURRENT EIGENVALUES IN JACOBI ARE'/1X)
      END


      SUBROUTINE SORT(A,B,N,NMAX)
C
C     SUBROUTINE JACOBI DOESN'T CALCULATE THE FREQUENCIES AND
C     MODE SHAPE IN ANY PARTICULAR ORDER. HENCE SUBROUTINE SORT
C     IS USED TO DO A SORTING OF THEM IN ASCENDING ORDERS OF
C     FREQUENCIES. IT RETURNS BACK THE EIGENPROPERTIES IN THE
C     PROPER ORDER.
C
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION A(NMAX),B(NMAX,NMAX)

C
C     CHANGE THE FOLLOWING DIMENSION STATEMENT IF INCREASE IN
C     THE SIZE OF THE PROBLEM IS DESIRED.
C
      DIMENSION TEMP(20)

      NM1=N-1
      DO 100 J=1,NM1

      XMIN=A(J)
      IPOS=J
      JP1=J+1

      DO 200 K=JP1,N
      IF(A(K).LT.XMIN)THEN
      XMIN=A(K)
      IPOS=K
      ENDIF
  200 CONTINUE
      T=A(IPOS)
      A(IPOS)=A(J)
      A(J)=T

      DO 300 I=1,N
      TEMP(I)=B(I,J)
      B(I,J)=B(I,IPOS)
      B(I,IPOS)=TEMP(I)
  300 CONTINUE

  100 CONTINUE
      RETURN
      END


      SUBROUTINE TRIPLE(MAT,PHI,PROD,ND,MPDOF,NM)

C
C***  THIS SUBROUTINE PERFORMS THE TRIPLE MATRIX PRODUCT
C          T
C     [PHI] [MAT][PHI] AND PUTS THE RESULTS IN THE MATRIX [PROD]
C
C     PRECISION:  DOUBLE
C
C     NOTES:    1. MAT MUST BE DIMENSIONED ND X ND. BUT PHI IS
C                  ALLOWED TO HAVE SMALLER DIMENSIONS.(ND X NM)
C
C
```

```
      IMPLICIT REAL*8(A-H,O-Z)
      REAL*8 MAT
      DIMENSION MAT(MPDOF,MPDOF),PHI(MPDOF,MPDOF),PROD(MPDOF,MPDOF)
C
C***  CHANGE THIS DIMENSION STATEMENT IF YOU WANT TO CHANGE THE
C     SIZE OF THE PROBLEM
C
      DIMENSION P1(20,20)
C
C***  MULTIPLY [MAT]BY [PHI] TO GET [P1]
C
      DO 10 I=1,ND
      DO 20 J=1,NM
      SUM=0.0D0
      DO 30 K=1,ND
      SUM=SUM+MAT(I,K)*PHI(K,J)
   30 CONTINUE
      P1(I,J)=SUM
   20 CONTINUE
   10 CONTINUE
C                   T
C***  MULTIPLY [PHI]  BY [P1] TO GET THE REQUIRED TRIPLE PRODUCT
C
      DO 40 I=1,NM
      DO 50 J=1,NM
      SUM=0.0D0
      DO 60 K=1,ND
      SUM=SUM+PHI(K,I)*P1(K,J)
   60 CONTINUE
      PROD(I,J)=SUM
   50 CONTINUE
   40 CONTINUE

      RETURN
      END


      SUBROUTINE PFACT(MASS,PHI,UB,PARTI,MDOF,NMODE,MPDOF)
      IMPLICIT REAL*8(A-H,O-Z)
      REAL*8 MASS
      DIMENSION MASS(MPDOF,MPDOF),PHI(MPDOF,MPDOF),UB(MPDOF,1)
     *,PARTI(MPDOF,1)
C
C**** THIS SUBROUTINE EVALUATES THE PARTICIPATION FACTORS FROM
C     THE MASS MATRIX, MODE SHAPE MATRIX AND BASE EXCITATION
C     MATRIX SUPPLIED.
C
C     PUT THE DIMENSIONS OF PTM AND PHITRA SAME AS MDOF
C
      DIMENSION PTM(20,20),PHITRA(20,20)

      CALL MATTRA(PHI,PHITRA,MDOF,NMODE,MPDOF,MPDOF)
      CALL MATMUL(PHITRA,MASS,PTM,NMODE,MDOF,MDOF,MPDOF,MPDOF,MPDOF)
      CALL MATMUL(PTM,UB,PARTI,NMODE,MDOF,1,MPDOF,MPDOF,1)
      RETURN
      END


      SUBROUTINE MATTRA(MAT,TMAT,IROW,ICOL,MNROW,MNCOL)
      IMPLICIT REAL*8 (A-H,O-Z)
      REAL*8 MAT
C
C***  EVALUATES THE TRANSPOSE OF A MATRIX
C
      DIMENSION MAT(MNROW,MNCOL),TMAT(MNCOL,MNROW)
      DO 10 I=1,IROW
      DO 20 J=1,ICOL
```

```
          TMAT(J,I)=MAT(I,J)
   20 CONTINUE
   10 CONTINUE
      RETURN
      END

      SUBROUTINE MATMUL(MAT1,MAT2,MAT12,L,M,N,LMAX,MMAX,NMAX)
      IMPLICIT REAL*8(A-H,O-Z)
      REAL*8 MAT1,MAT2,MAT12
      DIMENSION MAT1(LMAX,MMAX),MAT2(MMAX,NMAX),MAT12(LMAX,NMAX)


C
C**** PERFORMS MATRIX MULTIPLICATION
C
      DO 10 II=1,L
      DO 20 JJ=1,N

         SUM=0.0D0
         DO 30 KK=1,M
         SUM=SUM+MAT1(II,KK)*MAT2(KK,JJ)
 30      CONTINUE
      MAT12(II,JJ)=SUM
 20   CONTINUE
 10   CONTINUE

      RETURN
      END
```

A Sample Data File Applicable to the Example System 1 in Chapter 6

```
10,6,9.8
 2.0d9 -1.0d9  0.0d0  0.0d0   0.0d0   0.0d0   0.0d0   0.0d0   0.0d0   0.0d0
-1.0d9  2.0d9 -1.0d9  0.0d0   0.0d0   0.0d0   0.0d0   0.0d0   0.0d0   0.0d0
 0.0d0 -1.0d9  2.0d9 -1.0d9   0.0d0   0.0d0   0.0d0   0.0d0   0.0d0   0.0d0
 0.0d0  0.0d0 -1.0d9  2.0d9  -1.0d9   0.0d0   0.0d0   0.0d0   0.0d0   0.0d0
.0d0 .0d0 .0d0 -1.0d9 1.0005d9 -0.005d8  0.0d0   0.0d0   0.0d0   0.0d0
.0d0 .0d0 .0d0 0.0d0 -0.005d8  0.010d8 -.005d8   0.0d0   0.0d0   0.0d0
.0d0 .0d0 .0d0 0.0d0 0.0d0 -0.005d8 0.010d8 -0.005d8 0.0d0 0.0d0
.0d0 .0d0 .0d0 0.0d0 0.0d0 0.0d0 -0.005d8 0.010d8 -0.005d8 0.0d0
.0d0 .0d0 .0d0 0.0d0 0.0d0 0.0d0 0.0d0  -0.005d8 0.010d8 -0.005d8
.0d0 .0d0 .0d0 0.0d0 0.0d0 0.0d0 0.0d0  0.0d0  -0.005d8  0.005d8

1.0d4 0.0d0 0.0d0 0.0d0 0.0d0 0.0d0 0.0d0 0.0d0 0.0d0 0.0d0
0.0d0 1.0d4 0.0d0 0.0d0 0.0d0 0.0d0 0.0d0 0.0d0 0.0d0 0.0d0
0.0d0 0.0d0 1.0d4 0.0d0 0.0d0 0.0d0 0.0d0 0.0d0 0.0d0 0.0d0
0.0d0 0.0d0 0.0d0 1.0d4 0.0d0 0.0d0 0.0d0 0.0d0 0.0d0 0.0d0
0.0d0 0.0d0 0.0d0 0.0d0 1.0d4 0.0d0 0.0d0 0.0d0 0.0d0 0.0d0
0.0d0 0.0d0 0.0d0 0.0d0 0.0d0 1.0d2 0.0d0 0.0d0 0.0d0 0.0d0
0.0d0 0.0d0 0.0d0 0.0d0 0.0d0 0.0d0 1.0d2 0.0d0 0.0d0 0.0d0
0.0d0 0.0d0 0.0d0 0.0d0 0.0d0 0.0d0 0.0d0 1.0d2 0.0d0 0.0d0
0.0d0 0.0d0 0.0d0 0.0d0 0.0d0 0.0d0 0.0d0 0.0d0 1.0d2 0.0d0
0.0d0 0.0d0 0.0d0 0.0d0 0.0d0 0.0d0 0.0d0 0.0d0 0.0d0 1.0d2

1.0d4 0.0d0 0.0d0 0.0d0 0.0d0 0.0d0 0.0d0 0.0d0 0.0d0 0.0d0
0.0d0 1.0d4 0.0d0 0.0d0 0.0d0 0.0d0 0.0d0 0.0d0 0.0d0 0.0d0
0.0d0 0.0d0 1.0d4 0.0d0 0.0d0 0.0d0 0.0d0 0.0d0 0.0d0 0.0d0
0.0d0 0.0d0 0.0d0 1.0d4 0.0d0 0.0d0 0.0d0 0.0d0 0.0d0 0.0d0
0.0d0 0.0d0 0.0d0 0.0d0 1.0d4 0.0d0 0.0d0 0.0d0 0.0d0 0.0d0
0.0d0 0.0d0 0.0d0 0.0d0 0.0d0 1.0d2 0.0d0 0.0d0 0.0d0 0.0d0
0.0d0 0.0d0 0.0d0 0.0d0 0.0d0 0.0d0 1.0d2 0.0d0 0.0d0 0.0d0
0.0d0 0.0d0 0.0d0 0.0d0 0.0d0 0.0d0 0.0d0 2.0d2 0.0d0 0.0d0
```

```
0.0d0 0.0d0 0.0d0 0.0d0 0.0d0 0.0d0 0.0d0 0.0d0 3.0d2 0.0d0
0.0d0 0.0d0 0.0d0 0.0d0 0.0d0 0.0d0 0.0d0 0.0d0 0.0d0 2.0d2
10,10,0,0,0
1,2,3,4,5,6,7,8,9,10
1,2,3,4,5,6,7,8,9,10
```

Output File

```
    NO OF SWEEPS MADE
      6
 STIFFNESS MATRIX
  0.20000000D+10 -0.10000000D+10  0.00000000D+00  0.00000000D+00
  0.00000000D+00  0.00000000D+00  0.00000000D+00  0.00000000D+00
  0.00000000D+00  0.00000000D+00 -0.10000000D+10  0.20000000D+10
 -0.10000000D+10  0.00000000D+00  0.00000000D+00  0.00000000D+00
  0.00000000D+00  0.00000000D+00  0.00000000D+00  0.00000000D+00
  0.00000000D+00 -0.10000000D+10  0.20000000D+10 -0.10000000D+10
  0.00000000D+00  0.00000000D+00  0.00000000D+00  0.00000000D+00
  0.00000000D+00  0.00000000D+00  0.00000000D+00  0.00000000D+00
 -0.10000000D+10  0.20000000D+10 -0.10000000D+10  0.00000000D+00
  0.00000000D+00  0.00000000D+00  0.00000000D+00  0.00000000D+00
  0.00000000D+00  0.00000000D+00  0.00000000D+00 -0.10000000D+10
  0.10005000D+10 -0.50000000D+06  0.00000000D+00  0.00000000D+00
  0.00000000D+00  0.00000000D+00  0.00000000D+00  0.00000000D+00
  0.00000000D+00  0.00000000D+00 -0.50000000D+06  0.10000000D+07
 -0.50000000D+06  0.00000000D+00  0.00000000D+00  0.00000000D+00
  0.00000000D+00  0.00000000D+00  0.00000000D+00  0.00000000D+00
  0.00000000D+00 -0.50000000D+06  0.10000000D+07 -0.50000000D+06
  0.00000000D+00  0.00000000D+00  0.00000000D+00  0.00000000D+00
  0.00000000D+00  0.00000000D+00  0.00000000D+00  0.00000000D+00
 -0.50000000D+06  0.10000000D+07 -0.50000000D+06  0.00000000D+00
  0.00000000D+00  0.00000000D+00  0.00000000D+00  0.00000000D+00
  0.00000000D+00  0.00000000D+00  0.00000000D+00 -0.50000000D+06
  0.10000000D+07 -0.50000000D+06  0.00000000D+00  0.00000000D+00
  0.00000000D+00  0.00000000D+00  0.00000000D+00  0.00000000D+00
  0.00000000D+00  0.00000000D+00 -0.50000000D+06  0.50000000D+06
 MASS MATRIX
  0.10000000D+05  0.00000000D+00  0.00000000D+00  0.00000000D+00
  0.00000000D+00  0.00000000D+00  0.00000000D+00  0.00000000D+00
  0.00000000D+00  0.00000000D+00  0.00000000D+00  0.10000000D+05
  0.00000000D+00  0.00000000D+00  0.00000000D+00  0.00000000D+00
  0.00000000D+00  0.00000000D+00  0.00000000D+00  0.00000000D+00
  0.00000000D+00  0.00000000D+00  0.10000000D+05  0.00000000D+00
  0.00000000D+00  0.00000000D+00  0.00000000D+00  0.00000000D+00
  0.00000000D+00  0.00000000D+00  0.00000000D+00  0.00000000D+00
  0.00000000D+00  0.10000000D+05  0.00000000D+00  0.00000000D+00
  0.00000000D+00  0.00000000D+00  0.00000000D+00  0.00000000D+00
  0.00000000D+00  0.00000000D+00  0.00000000D+00  0.00000000D+00
  0.10000000D+05  0.00000000D+00  0.00000000D+00  0.00000000D+00
  0.00000000D+00  0.00000000D+00  0.00000000D+00  0.00000000D+00
  0.00000000D+00  0.00000000D+00  0.00000000D+00  0.10000000D+03
  0.00000000D+00  0.00000000D+00  0.00000000D+00  0.00000000D+00
  0.00000000D+00  0.00000000D+00  0.00000000D+00  0.00000000D+00
  0.00000000D+00  0.00000000D+00  0.10000000D+03  0.00000000D+00
  0.00000000D+00  0.00000000D+00  0.00000000D+00  0.00000000D+00
  0.00000000D+00  0.00000000D+00  0.00000000D+00  0.00000000D+00
  0.00000000D+00  0.10000000D+03  0.00000000D+00  0.00000000D+00
  0.00000000D+00  0.00000000D+00  0.00000000D+00  0.00000000D+00
  0.00000000D+00  0.00000000D+00  0.00000000D+00  0.00000000D+00
  0.10000000D+03  0.00000000D+00  0.00000000D+00  0.00000000D+00
  0.00000000D+00  0.00000000D+00  0.00000000D+00  0.00000000D+00
  0.00000000D+00  0.00000000D+00  0.00000000D+00  0.10000000D+03
 DAMPING MATRIX
  0.10000000D+05  0.00000000D+00  0.00000000D+00  0.00000000D+00
  0.00000000D+00  0.00000000D+00  0.00000000D+00  0.00000000D+00
  0.00000000D+00  0.00000000D+00  0.00000000D+00  0.10000000D+05
```

```
  0.00000000D+00   0.00000000D+00   0.00000000D+00   0.00000000D+00
  0.00000000D+00   0.00000000D+00   0.00000000D+00   0.00000000D+00
  0.00000000D+00   0.00000000D+00   0.10000000D+05   0.00000000D+00
  0.00000000D+00   0.00000000D+00   0.00000000D+00   0.00000000D+00
  0.00000000D+00   0.00000000D+00   0.00000000D+00   0.00000000D+00
  0.00000000D+00   0.10000000D+05   0.00000000D+00   0.00000000D+00
  0.00000000D+00   0.00000000D+00   0.00000000D+00   0.00000000D+00
  0.00000000D+00   0.00000000D+00   0.00000000D+00   0.00000000D+00
  0.10000000D+05   0.00000000D+00   0.00000000D+00   0.00000000D+00
  0.00000000D+00   0.00000000D+00   0.00000000D+00   0.00000000D+00
  0.00000000D+00   0.00000000D+00   0.00000000D+00   0.10000000D+03
  0.00000000D+00   0.00000000D+00   0.00000000D+00   0.00000000D+00
  0.00000000D+00   0.00000000D+00   0.00000000D+00   0.00000000D+00
  0.00000000D+00   0.00000000D+00   0.10000000D+03   0.00000000D+00
  0.00000000D+00   0.00000000D+00   0.00000000D+00   0.00000000D+00
  0.00000000D+00   0.00000000D+00   0.00000000D+00   0.00000000D+00
  0.00000000D+00   0.20000000D+03   0.00000000D+00   0.00000000D+00
  0.00000000D+00   0.00000000D+00   0.00000000D+00   0.00000000D+00
  0.00000000D+00   0.00000000D+00   0.00000000D+00   0.00000000D+00
  0.30000000D+03   0.00000000D+00   0.00000000D+00   0.00000000D+00
  0.00000000D+00   0.00000000D+00   0.00000000D+00   0.00000000D+00
  0.00000000D+00   0.00000000D+00   0.00000000D+00   0.20000000D+03
NWDISP,NWEF,INDSPF,NWSPF,IBS,ION
         10          10           0           0           0           0
DISPLACEMENTS NEEDED AT THE FOLLOWING DOFS
          1           2           3           4           5           6
          7           8           9          10
ELASTIC FORCES NEEDED AT THE FOLLOWING DOFS
          1           2           3           4           5           6
          7           8           9          10
  MAXIMUM ELASTIC DISPLACEMENTS
   DOF          DISPLACEMENT
     1      -0.2664271071238833D-03
     2      -0.5055415026830719D-03
     3      -0.7009350084186856D-03
     4      -0.8389771406795898D-03
     5      -0.9099200467910252D-03
     6       0.7511273298913202D-02
     7       0.1436888299010062D-01
     8       0.2052237717922342D-01
     9       0.2496604847747470D-01
    10       0.2750290381049333D-01
  MAXIMUM ELASTIC FORCES
   DOF          ELASTIC FORCES
     1      -0.3549237626099643D+05
     2      -0.4372088982357489D+05
     3      -0.5735137347470956D+05
     4      -0.6709922614946878D+05
     5      -0.7217431103060889D+05
     6       0.2304859072301348D+04
     7       0.8303345434649507D+03
     8      -0.2310431487127278D+04
     9       0.1435443675715597D+04
    10      -0.2153890519687161D+04
```

The File RECORD.MASTER

A sample of the file RECORD.MASTER is given here. This file is not complete, but only serves as an example.

```
3480
   0.000000     0.01080000   0.003000     0.01017143   0.006000     0.00954286
   0.009000     0.00891429   0.012000     0.00828572   0.015000     0.00765714
```

```
 0.018000    0.00702857  0.021000    0.00640000  0.024000    0.00577143
 0.027000    0.00514286  0.030000    0.00451429  0.033000    0.00388572
 0.036000    0.00325714  0.039000    0.00262857

        .         .        .      .      .     .
        .         .        .      .      .     .
        .         .        .      .      .     .
        .         .        .      .      .     .
        .         .        .      .      .     .
        .         .        .      .      .     .
        .         .        .      .      .     .
        .         .        .      .      .     .
        .         .        .      .      .     .

10.150000    0.00240000 10.152857    0.00587143 10.155714    0.00934286
10.158571    0.01281429 10.161428    0.01628571 10.164286    0.01975714
10.167143    0.02322857 10.170000    0.02670000 10.172857    0.03017143
10.175714    0.03364285 10.178572    0.03711428 10.181429    0.04058571
10.184286    0.04405713 10.187143    0.04752856
```

Appendix B: Example Systems in Chapter 3

Physical Significance of Example-Systems

Various example systems were used in Chapter 3 for the purpose of parametric study. Physical significance of these systems is explained below.

Building System No. 1

This a uniform system. It may represent a 2-storeyed structure. However, this system may also be used to represent a soil-structure interaction system (Fig. 3.20) or a dynamic vibration absorber arrangement (Fig. 3.21). This representation may be questionable as far as mass and stiffness properties are concerned. However, this representation has been used in past [89].

Building System No. 2

This represents a tuned P-S coupled system. Both the primary and the secondary are rather rigid. Mass ratio has been varied from 0.05 to 1.00 in order to study variation of the error with respect to mass ratio.

Building System No. 3

This represents a tuned P-S coupled system. The two modes are very closely spaced. This system is taken from Warburton and Soni [86]. This system has been used only for "testing" the error parameter.

Building System No. 4

This represents a tuned P-S coupled system. Both the primary and the secondary are very flexible. Mass ratio is varied from 0.05 to 1.00 in order to study the variation of the error with respect to mass ratio.

Building System No. 5

This represents an untuned P-S coupled system. Mass ratio is varied from 0.01 to 1.00 in order to study the variation of the error with respect to mass ratio.

Building System No. 6

This represents an untuned P-S coupled system. Mass ratio is varied from 0.01 to 1.00 in order to study the variation of the error with respect to mass ratio. The primary system is taken to be a very flexible system.

Bibliography

[1] Amin, M. and Ang, A.H.S. (1971), "Statistical Treatment of Seismic Loading on Reactor Building and Equipment", Proceedings, *1st International Conference on Structural Mechanics in Reactor Technology*, Berlin, West Germany, Paper no. K3/1.

[2] Amin, M., Hall, W.J., Newmark, N.M. and Kassawara, R.P. (May, 1971), "Earthquake Response of Multiply Connected Light Secondary Systems by Spectrum Methods," Proceedings, *ASME First National Congress on Pressure Vessels and Piping*, San Francisco, California.

[3] Aziz, T.S. and Duff, C.G. (June, 1978), "Mass Coupling Effects in the Dynamic Analysis of Nuclear Power Plant Systems," Presented at the *Pressure Vessels and Piping Conference*, Montreal, Canada, Published by ASME, New York, N.Y.

[4] Aziz, T.S. and Duff, C.G. (June, 1978), "Decoupling Criteria for Seismic Analysis of Nuclear Power Plant Systems," Presented at the *Pressure Vessels and Piping Conference*, Montreal, Canada, Published by ASME, New York, N.Y.

[5] Balendra, T., Tat, C.W. and Lee, S.L. (1982), "Modal Damping for Torsionally Coupled Buildings on Elastic Foundation," *Earthquake Engineering and Structural Dynamics*, Vol. 10, pp. 735-756.

[6] Bathe K.J. and Wilson, E.L. (1987), Numerical Methods in Finite Element Analysis, Prentice-Hall of India, New Delhi, India.

[7] Beredugo, Y.O. (1976), "Modal Analysis of Coupled Motion of Horizontally Excited Embedded Footings," *Earthquake Engineering and Structural Dynamics*, Vol. 4, pp. 403-410.

[8] Biggs, J.M. (Sept., 1971), "Seismic Response Spectra for Equipment Design in Nuclear Power Plants", Proceedings, *1st International Conference on Structural Mechanics in Reactor Technology*, Berlin, West Germany, Paper no. K4/7.

[9] Biggs, J.M. and Whitman, R.V. (1970), "Soil-Structure Interaction in Nuclear Power Plants," *3rd Japanese Symposium on Earthquake Engineering*, Tokyo, Japan.

[10] Borino, G. and Muscolino, G. (1986), "Mode-superposition Methods in Dynamic Analysis of Classically and Non-classically Damped Linear Systems", *Earthquake Engineering and Structural Dynamics*, Vol. 14, pp. 705-717.

[11] Caughey, T.K. and O'Kelly, M.E.J. (September, 1965), "Classical Normal Modes in Damped Linear Dynamic Systems", *Journal of Applied Mechanics*, ASME, Vol. 27, pp. 583-588.

[12] Chen, C. (September, 1977), "The SRSS and the Static Coefficient Methods for Seismic Resistant Design of Equipment and Structures," Presented at the *Energy Technology Conference and Exhibit*, Houston, Texas, Published by ASME, New York, N.Y.

[13] Chen, C. (1980), "The Uncoupling Criteria for Subsystem Seismic Analysis," *Nuclear Engineering and Design*, Vol. 57, pp. 245-252.

[14] Clough, R.W. and Mojtahedi, S. (1976), "Earthquake Response Analysis Considering Non-proportional Damping," *Earthquake Engineering and Structural Dynamics*, Vol. 4, pp. 489-496.

[15] Clough R.W. and Penzien J. (1986), Dynamics of Structures, International Edition, McGraw Hill.

[16] Cornwell, R.E., Craig-Jr, R.R. and Johnson, C.P. (1983), "On the Application of the Mode-Acceleration Method to Structural Engineering Problems", *Earthquake Engineering and Structural Dynamics*, Vol. 11, pp. 679-688.

[17] Cronin, D.L. (February, 1976), "Approximation for Determining Harmonically Excited Response of Nonclassically Damped Systems," *Journal of Engineering for Industry*, ASME, No. 75-DE-P., pp. 43-47.

[18] Der Kiureghian, A., Asfura, A., Sackman, J.L., and Kelly, J.M. (August, 1983), "Seismic Response of Multiply Supported Piping System", Proceedings, *7th International Conference on Structural Mechanics in Reactor Technology*, Paper no. K7/7, Chicago, pp. 495-502.

[19] Der Kiureghian, A., Sackman, J.L. and Nour-Omid, B. (February, 1983), "Dynamic Analysis of Light Equipment in Structures: Response to Stochastic Input," *Journal of Engineering Mechanics*, ACSE, Vol. 109, No. 1, pp. 90-110.

[20] Duncan, P.E. and Taylor, R.E. (1979), "A Note on the Dynamic Analysis of Non-proportionally Damped Systems," *Earthquake Engineering and Structural Dynamics*, Vol. 7, pp. 99-105.

[21] Foss, K.A. (September, 1958), "Co-ordinates Which Uncouple the Equations of Motion of Damped Linear Dynamic Systems", *Journal of Applied Mechanics*, ASME, Vol. 25, pp. 361-364.

[22] Goodman L.E., Rosenblueth, E. and Newmark, N.M. (1955), "Aseismic Design of Firmly Founded Elastic Structures", *Transactions of the ASCE*, Paper no. 2762, pp. 782-802.

[23] Gupta, A.K. (1984), "Seismic Response of Multiply Connected MDOF Primary and MDOF Secondary Systems," *Nuclear Engineering and Design*, Vol. 81, pp. 385-394.

[24] Gupta, A.K. and Cordero, K. (1981), "Combination of Modal Responses", Proceedings, *6th International Conference on Structural Mechanics in Reactor Technology*, Paris, France, Paper no. K7/5.

[25] Gupta, A.K. and Chen, D.C. (1984), "Comparison of Modal Combination Methods," *Nuclear Engineering and Design*, Vol. 78, pp. 53-68.

[26] Gupta, A.K. and Jaw, J-W (1986) "Response Spectrum Method for Nonclassically Damped Systems," *Nuclear Engineering and Design* ,Vol. 91, pp. 161-169.

[27] Gupta, A.K. and Jaw, J.W. (1986), "Modal Combination in Response Spectrum Analysis of Piping Systems," *Journal of Pressure Vessel Technology*, ASME, Vol. 108, pp. 73-77.

[28] Gupta, A.K. and Jaw, J-W (1986), "Seismic Response of Nonclassically Damped Systems," *Nuclear Engineering and Design*, Vol. 91, pp. 153-159.

[29] Gupta, A.K. and Jaw, J.W. (1986), "Complex Modal Properties of Coupled Moderately Light Equipment Systems," *Nuclear Engineering and Design*, Vol. 91, pp. 171-178.

[30] Gupta, A.K. and Jaw, J.W. (1986), "Coupled Response Spectrum Analysis of Secondary Systems Using Uncoupled Modal Properties," *Nuclear Engineering and Design*, Vol. 92, pp. 61-68.

[31] Gupta, A.K. and Jaw, J.W. (1986), "A New Instructure Response Spectrum (IRS) Method For Multiply Connected Secondary Systems with Coupling Effects," *Nuclear Engineering and Design*, Vol 96, pp. 63-80.

[32] Gupta, A.K. and Tembulkar, J.M. (1984), "Dynamic Decoupling of Secondary Systems," *Nuclear Engineering and Design*, Vol. 81, pp. 359-373.

[33] Gupta, A.K. and Tembulkar, J.M. (1984), " Dynamic Decoupling of Multiply Connected MDOF Secondary Systems," *Nuclear Engineering and Design*, Vol. 81, pp.375-383.

[34] Hadjian, A.H. (1977), "On the Decoupling of Secondary Systems for Seismic Analysis", Proceedings, *Sixth World Conference on Earthquake Engineering*, New Delhi, India.

[35] Hadjian, A.H. (1981), "Seismic Response of Structures by the Response Spectrum Method," *Nuclear Engineering and Design*, Vol. 66, pp. 179-201.

[36] Hadjian, A.H. and Ellison, B. (February, 1986), "Decoupling of Secondary Systems for Seismic Analysis," *Journal of Pressure Vessel Technology*, ASME, Vol. 108, pp. 78-85.

[37] Hansteen, O.E. and Bell, K. (1979), "On the Accuracy of Mode Superposition Analysis in Structural Dynamics," *Earthquake Engineering and Structural Dynamics*, Vol. 7, pp. 405-411.

[38] Hasselman, T.K. (November, 1976), "Modal Coupling in Lightly Damped Structures", *AIAA Journal*, Vol. 14, No. 11, pp. 1627-1628.

[39] Hernried, A.G. and Sackman, J.L. (1984), "Response of Secondary Systems in Structures Subjected to Transient Excitation", *Earthquake Engineering and Structural Dynamics*, Vol. 12, pp. 737-748.

[40] Hurty, W.C. and Rubinstein, M.F. (1964), Dynamics of Structures, Prentice-Hall, Inc., New Jersey.

[41] Igusa, T. and Der Kiureghian, A. (January, 1985), "Dynamic Response of Multiply Supported Secondary Systems," *Journal of Engineering Mechanics*, ASCE, Vol. 111, No. 1, pp. 20-41.

[42] Igusa, T. and Der Kiureghian, A. (January, 1985), "Dynamic Characterization of Two-Degree-of-Freedom Equipment-Structure Systems," *Journal of Engineering Mechanics*, ASCE, Vol. 111, No. 1, pp. 1-19.

[43] Igusa, T., Der Kiureghian, A. and Sackman, J.L. (1984), "Modal Decomposition Method For Stationary Response of Non-classically Damped Systems," *Earthquake Engineering and Structural Dynamics*, Vol. 12, pp. 121-136.

[44] Itoh, T. (1973), "Damped Vibration Mode Superposition Method for Dynamic Response Analysis," *Earthquake Engineering and Structural Dynamics*, Vol. 2, pp. 47-57.

[45] Kapur, K.K. and Shao, L.C. (December 1973), "Generation of Seismic Floor Response Spectra for Equipment Design", Proceedings, *Specialty Conference on Structural Design of Nuclear Power Plant Facilities*, Chicago, Ill.

[46] Kelly, J.M. and Sackman, J.L. (1978), "Response Spectra Design Methods For Tuned Equipment-Structure Systems", *Journal of Sound and Vibration*, Vol. 59, No. 2, pp. 171-179.

[47] Kelly, J.M. and Sackman, J.L.(1980), "Conservatism in Summation Rules for Closely Spaced Modes," *Earthquake Engineering and Structural Dynamics*, Vol. 8, pp. 63-74.

[48] Leger, P. and Wilson, E.L. (1988), "Modal Summation Methods for Structural Dynamic Computations", *Earthquake Engineering and Structural Dynamics*, Vol. 16, pp. 23-27.

[49] Lin, C.W. and Liu, T.H. (Sept., 1975), "A Discussion of Coupling and Resonant Effects for Integrated Systems Consisting of Subsystems", Proceedings, *Extreme Load Conditions and Limit Analysis Procedures for Structural Reactor Safeguards and Containment Structures*, Paper no. U3-3, Berlin, West Germany.

[50] Maddox, N.R. (June, 1975), "On the Number of Modes Necessary for Accurate Response and Resulting Forces in Dynamic Analyses," *Journal of Applied Mechanics Division*, ASME, Vol. 42, pp. 516-517.

[51] Maison, B.F., Neuss, C.F. and Kasai, K. (1983), "The Comparative Performance of Seismic Response Spectrum Combination Rules in Building Analysis", *Earthquake Engineering and Structural Dynamics*, Vol. 11, pp. 623-647.

[52] Newmark, N.M. (1966), "Notes on Shock Isolation Concepts", in *Vibrations in Civil Engineering*, Ed. B.O. Skipp, Butterworths, London, U.K.

[53] Newmark, N.M.(1972), "Earthquake Response Analysis of Reactor Structures," *Nuclear Engineering and Design*, Vol. 20, pp. 303-322.

[54] Paz, M. (1986), Micro-computer Aided Engineering: Structural Dynamics, Van Nostrand Reinhold, New York, N.Y.

[55] Paz, M. (1987), Structural Dynamics: Theory and Computation, 2nd Ed., CBS Publishers and Distributors, New Delhi, India.

[56] Penzien, J. (1969), "Earthquake Response of Irregularly Shaped Buildings", Proceedings, *4th World Conference on Earthquake Engineering*, Santiago, Chile, Vol. 2, A-3, pp. 75-89.

[57] Peters, K.A., Schmitz, D. and Wagner, U. (1977), "Determination of Floor Response Spectra on the Basis of the Response Spectrum Method," *Nuclear Engineering and Design*, Vol. 44, pp. 255-262.

[58] Pickel, T.W., Jr. (1972),"Evaluation of Nuclear System Requirements for Accomodating Seismic Effects", *Nuclear Engineering and Design*, Vol. 20, pp. 323-337.

[59] Rayleigh, Lord (1945), Theory of Sound, Vol.1 , Dover Publications, New York, N.Y.

[60] Roesset, J.M., Whitman, R.V. and Dobry, R. (March, 1973), "Modal Analysis for Structures With Foundation Interaction," *Journal of the Structural Division*, ASCE, Vol. 99, No. ST3, pp. 399-416.

[61] Rosenblueth, E. and Elorduy, J. (1969), "Responses of Linear Systems to Certain Transient Disturbances," *4th World Conference on Earthquake Engineering*, Vol. 1, Chile.

[62] Ruiz, P. (1970), "On the Maximum Response of Structures Subjected to Earthquake Excitations," *4th Symposium of Earthquake Engineering*, University of Roorkee, Roorkee, India.

[63] Sackman, J.L., Der Kiureghian, A. and Nour-Omid, B. (February, 1983), "Dynamic Analysis of Light Equipment in Structures: Modal Properties of the Combined System," *Journal of Engineering Mechanics*, ASCE, Vol. 109, No. 1, pp. 73-89.

[64] Sackman, J.L. and Kelly, J.M. (1979), "Seismic Analysis of Internal Equipment and Components in Structures", *Engineering Structures*, Vol. 1, pp. 179-190.

[65] Salmonte, A.J. (1982), "Considerations on the Residual Contribution in Modal Analysis", *Earthquake Engineering and Structural Dynamics*, Vol. 10, pp. 295-304.

[66] Scavuzzo, R.J. and Lam, P. (November, 1977), "Decoupling of Undamped Vibration Systems," *Journal of Pressure Vessel Technology*, ASME, pp. 553-562.

[67] Singh, A.K., Chu, S.L. and Singh, S. (1973), "Influence of Closely Spaced Modes in Response Spectrum Method of Analysis", Proceedings, *1st ASCE Specialty Conference on Structural Design of Nuclear Power Plant Facilities*, Chicago, Illinois.

[68] Singh, M.P. (October, 1975), "Generation of Seismic Floor Spectra," *Journal of the Engineering Mechanics Division*, ASCE, Vol. 101, No. EM5, pp. 593-607.

[69] Singh, M.P. (1980), "Seismic Design Input For Secondary Systems", *Journal of the Structural Division*, ASCE, Vol. 106, No. ST2, pp. 505-517.

[70] Singh, M.P. (1980), "Seismic Response by SRSS for Nonproportional Damping," *Journal of the Engineering Mechanics Division*, ASCE, Vol. 106, No. EM6, pp. 1405-1419.

[71] Singh, M.P. and Chu, S.L. (1976), "Stochastic Considerations in Seismic Analysis of Structures", *Earthquake Engineering and Structural Dynamics*, Vol. 4, pp. 295-307.

[72] Singh, M.P. and Ghafory-Ashtiany, M. (1986), "Modal Time History Analysis of Non-classically Damped Structures for Seismic Motions," *Earthquake Engineering and Structural Dynamics*, Vol. 14, pp. 133-146.

[73] Singh, M.P. and Mehta, K.B. (1983), "Seismic Design Response by an Alternative SRSS Rule", *Earthquake Engineering and Structural Dynamics*, Vol. 11, pp. 771-783.

[74] Singh, M.P. and Suarez, L.E. (1986), "A Perturbation Analysis of the Eigenproperties of Equipment-Structure Systems," *Nuclear Engineering and Design*, Vol. 97, pp. 167-185.

[75] Suarez, L.E. and Singh, M.P. (1987), "Eigenproperties of Non-classically Damped Primary Structure and Equipment Systems by a Perturbation Approach", *Earthquake Engineering and Structural Dynamics*, Vol. 15, pp. 565-583.

[76] Suarez, L.E. and Singh, M.P. (1987), "Floor Response Spectra With Structure-Equipment Interaction Effects by a Mode Synthesis Approach", *Earthquake Enginering and Structural Dynamics*, Vol. 15, pp. 141-158.

[77] Thomson, W.T., Calkins T. and Caravani, P. (1974), "A Numerical Study of Damping," *Earthquake Engineering and Structural Dynamics*, Vol. 3, pp. 97-103.

[78] Traill-Nash, R.W. (1981), "Modal Methods in the Dynamics of Systems with Non-classical Damping," *Earthquake Engineering and Structural Dynamics*, Vol. 9, pp. 153-169.

[79] Tsai, N.C. (April, 1974), "Modal Damping for Soil-Structure Interaction," *Journal of the Engineering Mechanics Division*, ASCE, Vol. 100, No. EM2, pp. 323-341.

[80] Vashi, K.M. (February, 1981), "Computation of Seismic Response from Higher Frequency Modes," *Journal of Pressure Vessel Technology*, ASME, Vol. 103, pp. 16-19.

[81] Veletsos, A.S. and Ventura, C.E. (1986), "Modal Analysis of Non-classically Damped Linear Systems", *Earthquake Engineering and Structural Dynamics*, Vol. 14, pp. 217-243.

[82] Ventura, C.E. and Veletsos, A.S. (1986), "Steady-State and Transient Responses of Non-classically Damped Linear Systems," *Earthquake Engineering and Structural Dynamics*, Vol. 14, pp. 595-608.

[83] Villaverde, R. (1986), "Simplified Seismic Analysis of Secondary Systems," *Journal of Structural Engineering*, ASCE, Vol. 112, No. 3, pp. 588-604.

[84] Villaverde, R. (1986), "Simplified Seismic Analysis of Piping or Equipment Mounted on Two Points of a Multistory Structure," *Nuclear Engineering and Design*, Vol. 92, pp. 37-50.

[85] Villaverde, R. (1988), "Rosenblueth's Modal Combination Rule for Systems With Non-classical Damping", *Earthquake Engineering and Structural Dynamics*, Vol. 16, pp. 315-328.

[86] Warburton, G.B. and Soni, S.R. (1977), "Errors in Response Calculations For Non-Classically Damped Structures", *Earthquake Engineering and Structural Dynamics*, Vol. 5, pp. 365-376

[87] Wilson, E.L., Der Kiureghian, A. and Bayo, E.P. (1981), "A Replacement for the SRSS Method in Seismic Analysis", *Earthquake Engineering and Structural Dynamics*, Vol. 9, pp. 187-194.

[88] Wilson, E.L. and Penzien, J. (1972), "Evaluation of Orthogonal Damping Matrices", *International Journal for Numerical Methods in Engineering*, Vol. 4, pp. 5-10.

[89] Yan, M.J. (June, 1979), "Composite Modal Damping in Structures," Presented at the *Pressure Vessels and Piping Conference*, San Francisco, California, Published by ASME, New York, N.Y.