A METHOD TO DETERMINE THE CAUCHY INDEX OF A REAL RATIONAL FUNCTION

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ABSTRACT

This work on a method to determine the "Cauchy Index" of a real rational function R(x) is in the tradition of Cauchy's original work entitled "CALCUL DES INDICES DES FONCTIONS". As a result, it is related to the theory of separation of roots of algebraic equations, and is concerned mainly with the combination of some existing mathematical results into a rational procedure involving extensive application of the determinants called the Hankel determinants for finding the Cauchy index of R(x) between the limits a and b ($a < b$; a, b are real numbers or $\pm \infty$).

One of the procedures suggested by Aitken for obtaining the Hankel determinants from a recurrence formula that permits one to avoid numerical computation of all the determinants creates the special problem of finding the first two diagonals whereas, the other is likely to be unstable. To overcome these difficulties, a method (a modified form of Aitken's method) is developed; it permits to generate the determinants $H_k^{(n)}$ recursively and entirely bypasses the computation of all the Hankel determinants.

Jacobi's method is used to transform a given quadratic form to the one that involves only the square

terms. For when, taking as basis the Law of Inertia of quadratic forms, this method is applied to a real Hankel form of finite rank; it produces the relationships for the signature, the rank of the form, and the Hankel determinants.

Consequently, in this supplementary work, a method is established that is exclusively based on the result of Jacobi, relating the signature of a quadratic form, the determinants $H_k^{(n)}$, and its application to the problem of finding the Cauchy index of a real rational function.

It is then applied in the problem of the separation of roots of algebraic equations.

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CHAPTER I

DECOMPOSITION OF MATRICES INTO TRIANGULAR FACTORS

This Chapter is concerned mainly with the decomposition of a square matrix into the product of two triangular matrices. The method that is formulated is analogously equivalent to Gauss' Elimination Algorithm [8, p.128] (carried out for a fixed order of the leading element). Collected in the first section are the basic concepts from linear algebra which are met with most frequently in the sequel.

To avoid constant repetition in our explanations, we state at the outset that throughout this work, all numbers, scalars, elements of matrices, and all the coefficients of the forms referred to in the text belong to a given field F.

1.1 BASIC CONCEPTS

MATRICES:

An aggregate of numbers α_{ij} ($i=0,1,2,\ldots,m-1$; $j=0,1,2,\ldots,n-1$), real or complex, which is arranged in the form of a rectangle table containing m rows and n columns,

$$\begin{bmatrix} \alpha_{00} & \alpha_{01} & \alpha_{02} & \cdots & \alpha_{0n-1} \\ \alpha_{10} & \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n-1} \\ \alpha_{20} & \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n-1} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \alpha_{m-10} & \alpha_{m-11} & \alpha_{m-12} & \cdots & \alpha_{m-1n-1} \end{bmatrix},$$

is called a matrix. In general, the matrix is rectangular (of dimension mxn). But, when $m = n$, the matrix is square and the number $m = n$ is said to be its order.

As a convenient notation, such a matrix is abbreviated to

(1.1-1) $$B = [\alpha_{ij}] \quad (i=0, 1, 2, \ldots, m-1; \; j=0, 1, 2, \ldots, n-1).$$

Two matrices are called equal if their corresponding elements are equal.

When B is a square matrix of order m, we write

(1.1-2) $$B = [\alpha_{ij}]_0^{m-1},$$

and, in this case, the determinant of the matrix B is denoted by

(1.1-3) $$|B| = |\alpha_{ij}|_0^{m-1}.$$

As a convenient and concise notation for the determinants formed from the elements of a given matrix, we take

(1.1-4) $$B\begin{pmatrix} i_0 & i_1 & i_2 \cdots & i_{r-1} \\ j_0 & j_1 & j_2 \cdots & j_{r-1} \end{pmatrix} = \begin{vmatrix} \alpha_{i_0 j_0} & \alpha_{i_0 j_1} & \alpha_{i_0 j_2} & \cdots & \alpha_{i_0 j_{r-1}} \\ \alpha_{i_1 j_0} & \alpha_{i_1 j_1} & \alpha_{i_1 j_2} & \cdots & \alpha_{i_1 j_{r-1}} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \alpha_{i_{r-1} j_0} & \alpha_{i_{r-1} j_1} & \alpha_{i_{r-1} j_2} & \cdots & \alpha_{i_{r-1} j_{r-1}} \end{vmatrix},$$

which is called a minor of B of order r, provided that

$$\begin{matrix} 0 \le i_0 < i_1 < i_2 < \dots < i_{r-1} < m \\ 0 \le j_0 < j_1 < j_2 < \dots < j_{r-1} < n \end{matrix} \quad (r \le m, n).$$

Accordingly, the determinant of a square matrix,

$$B = [\alpha_{ij}]_0^{m-1} ,$$

can be written as

$$(1.1\text{-}5) \qquad |B| = |\alpha_{ij}|_0^{m-1}$$

$$= B\begin{pmatrix} 0 & 1 & 2 & \dots & m-1 \\ 0 & 1 & 2 & \dots & m-1 \end{pmatrix}.$$

<u>Definitions</u>:

The square matrix of order m having unity in each place on its leading diagonal and zero elsewhere is called the <u>Unit Matrix</u> of order m, and it is denoted by I.

A matrix of order m whose only non-zero elements occur in its leading diagonal is called a <u>Diagonal Matrix</u>; it is usually denoted by

$$(1.1\text{-}6) \qquad D = \text{diag}\,[d_0, d_1, d_2, \dots, d_{m-1}].$$

A square matrix

$$B = [\alpha_{ij}]_0^{m-1}$$

is said to be <u>singular</u> or <u>non-singular</u> according as $|B| = 0$ or $|B| \neq 0$.

From a given matrix B we may construct a new matrix the rows of which are the columns of B (and, hence, the columns of which are the rows of B). The resulting matrix is called the <u>Transpose</u> of the matrix B; it is denoted by B^T.

Thus, when $B = [\alpha_{ij}]$, $B^T = [\alpha_{ji}]$.

When $B = B^T$, that is $\alpha_{ij} = \alpha_{ji}$, the matrix B is called <u>Symmetric.</u>

THE INVERSE OF A MATRIX

When $B = [\alpha_{ij}]_0^{m-1}$, the determinant obtained by suppressing the ith row and jth column of $|B|$ and giving the sign $(-1)^{i+j}$, is denoted by B_{ij}; it is called the <u>Co-factor</u> of α_{ij} in $|B|$. The determinant B_{ij} without giving the sign $(-1)^{i+j}$ is called a minor of B.

It is a well-known result [21, p.32] in the theory of determinants that

$$(1.1\text{-}7) \qquad \Sigma \alpha_{ij} B_{kj} = \begin{cases} 0 & \text{when } i \neq k \\ |B| & \text{when } i = k \text{ (i fixed).} \end{cases}$$

Hence, when B is non-singular,

$$[\alpha_{ik}] \times [B_{ki}/|B|] = [\Sigma \alpha_{ij} B_{kj}/|B|]$$

$$= I$$

and, similarly, working with columns instead of rows as lines in (1.1-7),

$$[B_{ki}/|B|] \times [\alpha_{ik}] = I.$$

Accordingly, the matrix

$$[B_{ki}/|B|]$$

is called the <u>Inverse</u> of B and is denoted by

(1.1-8) $$B^{-1} = [B_{ki}/|B|] .$$

<u>RANK OF A MATRIX</u>

<u>Definition</u>

If all minors of order r+1 contained in B (not necessarily square) are zero while at least one minor of order r is non-zero, B is said to be of rank r (≥ 1).

Thus, a square matrix $B = [\alpha_{ij}]_0^{m-1}$ of rank r (≥ 1) is singular or not according as $r < m$ or $r = m$.

<u>RANK AND LINEAR DEPENDENCE</u>

<u>Definition</u>

Consider a set of vectors $X_i (i=0,1,2,\ldots,m-1)$ each having n components, real or complex numbers.

Then, the vectors X_i are said to be linearly dependent with respect to a given field F if there exist numbers $\alpha_i (i=0, 1, 2, \ldots, m-1)$ not all zeros, such that

(1.1-9) $$\Sigma \alpha_i X_i = 0 \quad (i=0, 1, 2, \ldots, m-1).$$

If such a linear dependence does not hold, the vectors X_i are said to be linearly independent with respect to F.

Let us assume that the vectors $X_i (i=0,1,2,\ldots,m-1)$ are linearly dependent, i.e.,

$$\Sigma \alpha_i X_i = 0 \quad (i=0,\ 1,\ 2,\ \ldots\ ,\ m-1), \tag{1.1-10}$$

where at least one of the numbers α_i is non-zero.

Obviously, the vector equation (1.1-10) is equivalent to the following set of scalar equations:

$$\begin{aligned} \alpha_0 x_{00} + \alpha_1 x_{01} + \cdots + \alpha_{m-1} x_{0m-1} &= 0 \\ \alpha_0 x_{10} + \alpha_1 x_{11} + \cdots + \alpha_{m-1} x_{1m-1} &= 0 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad & \cdot \\ \alpha_0 x_{n-10} + \alpha_1 x_{n-11} + \cdots + \alpha_{m-1} x_{n-1m-1} &= 0 \end{aligned} \tag{1.1-11}$$

wherein X_i is a vector with $x_{0i}, x_{1i}, \ldots, x_{n-1i}$ as its components.

As is well-known [9 , p. 100-101], the set (1.1-11) of homogeneous linear equations for α_i has a non-zero solution if and only if the rank of the coefficient matrix is less than the number of unknowns.

This precisely implies that a necessary and sufficient condition for the vectors X_i to be linearly independent is, therefore, that this rank should be m.

Thus:

<u>THEOREM (1.1-1)</u>

In order that the vectors $X_i (i=0,1,2,\ldots,m-1)$

be linearly independent it is necessary and sufficient that the rank of the matrix formed from the components of these vectors

(1.1-12) $A = [x_{ij}]$ $(i=0,1,2,\ldots,n-1;\ j=0,1,2,\ldots,m-1)$

be equal to the number of vectors.

THEOREM (1.1-2)

In an arbitrary rectangular matrix the number of linearly independent columns is always equal to the number of linearly independent rows and equal to the rank of the matrix.

Proof:

By Theorem (1.1-1), the linear independence of the vectors X_i means the columns of the matrix (1.1-12) are linearly independent, since the $j+1^{th}$ column consists of the components x_{ij} $(i=0, 1, 2, \ldots, n-1)$.

Hence, it follows that the rank of a matrix coincides with the number of linearly independent columns in the matrix.

Since under transposition the rows of a matrix become its columns and the rank remains unchanged, the number of linearly independent rows of a matrix is also equal to the rank of the matrix.

Thus, in a matrix the number of linearly independent rows coincides with the number of linearly independent columns and this proves the theorem.

1.2 DECOMPOSITION OF MATRICES INTO TRIANGULAR FACTORS

Let

(1.2-1) $$A = [\alpha_{ij}]_0^{m-1}$$

be given matrix of rank r. In accordance with the definition (1.1-5), we denote the successive principal minors of the matrix A by

(1.2-2) $$D_k^{(0)} = A\begin{pmatrix} 0 & 1 & 2 & \ldots & k-1 \\ 0 & 1 & 2 & \ldots & k-1 \end{pmatrix} \quad (k=1,2,3,\ldots, m).$$

Then:

THEOREM (1.2-1)

The matrix $A = [\alpha_{ij}]_0^{m-1}$ may be represented as the product of a lower and an upper triangular matrix provided that the inequality

(1.2-3) $$D_k^{(0)} \neq 0 \quad (k=1, 2, 3, \ldots , m),$$

holds for the matrix A.

For the proof consider a system of m linear equations in m unknowns

(1.2-4) $$\Sigma\, \alpha_{ij}\, x_j = b_i \quad (i,j=0, 1, 2, \ldots, m-1).$$

We write the system of equations in (1.2-4) in the matrix form:

$$Ax = B,$$

where $A = [\alpha_{ij}]_0^{m-1}$ is a non-singular matrix, x and B are, respectively, the column matrices composed of the unknowns x_i and the constants b_i .

The Gauss' method [12 p.23], carried out for a fixed order of the leading elements, consists in replacing a given system by an equivalent triangular system using a linear combination of the equations; this reduces to combining linearly the rows of A and B. To carry out the method (using the single-division scheme), besides dividing by the leading elements one adds to row elements numbers proportional to the elements of the preceding rows, that is, to effect upon matrix A elementary transformations of the type (a) and (b') [8 , p.14].

The result of several transformations of this form is equivalent to pre-multiplying the matrix A by the lower triangular matrix [8 p.16]:

$$L = \begin{bmatrix} a_{00} & 0 & 0 & \cdots & 0 \\ a_{10} & a_{11} & 0 & \cdots & 0 \\ a_{20} & a_{21} & a_{22} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m-10} & a_{m-11} & a_{m-12} & \cdots & a_{m-1m-1} \end{bmatrix}$$

As a result of these transformations one arrives at a system with the upper triangular matrix

$$U = \begin{bmatrix} b_{00} & b_{01} & b_{02} & \cdots & b_{0m-1} \\ 0 & b_{11} & b_{12} & \cdots & b_{1m-1} \\ 0 & 0 & b_{22} & \cdots & b_{2m-1} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & b_{m-1m-1} \end{bmatrix}$$

wherein $b_{ii} = 1$, such that

(1.2-5) $$LA = U.$$

Since L is non-singular, by (1.2-5):

(1.2-6) $$A = L^{-1}U.$$

Since the diagonal elements of the upper triangular matrix U are equal to one, such a representation is unique, that is, both L and U are uniquely determined by A. We have thus represented A in the form of a product of a lower triangular matrix L^{-1} and an upper triangular matrix U.

1.3 CONGRUENT MATRICES

Definition

When the matrix B is related to the matrix A by a relation

(1.3-1) $$B = E^T AE,$$

the matrix B is said to be congruent to A, where E is a n[illegible]ngular matrix and E^T is the transpose of E.

We note that the relation of congruence is

(i) symmetric, for

$$A = (E^T)^{-1}BE^{-1}$$

$$= (E^{-1})^T BE^{-1} \quad ,$$

and (ii) transitive; for when $B = E^TAE$ and $C = F^TBF$,

then $C = F^TE^TAEF$

$$= (EF)^T A(EF).$$

LEMMA 1

When A is symmetric and B is congruent to A, then B is also symmetric.

By hypothesis,

$$A = A^T,$$

and $B = E^TAE$

then

$$B^T = E^TA^TE$$

$$= E^TAE$$

$$= B \; .$$

Now, the problem of the decomposition of a symmetric matrix A into triangular factors may completely be answered as follows:

THEOREM (1.3-1)

Every symmetric matrix

$$A = [\alpha_{ij}]_0^{m-1} \qquad (\alpha_{ij} = \alpha_{ji})$$

of rank r, for which the inequality

$$D_k^{(0)} \neq 0 \quad (k=1, 2, 3, \ldots, r)$$

holds, can be represented in the form of a product of a lower and an upper triangular matrix, i.e.,

(1.3-2) $LA = U$.

In this case,

$$(1.3\text{-}3) \quad a_{00} = \frac{1}{D_1^{(0)}}, \; a_{11} = \frac{D_1^{(0)}}{D_2^{(0)}}, \; \ldots, \; a_{r-1r-1} = \frac{D_{r-1}^{(0)}}{D_r^{(0)}}.$$

Proof:

The first part of the theorem can easily be inferred from the proof of Theorem (1.2-1), that is,

$$LA = U.$$

Then

(1.3-4) $LAL^T = UL^T$

is both upper triangular, since the product of two triangular matrices of like structure is again a triangular matrix of the same structure; and symmetric (by Lemma 1, p. 11).

The matrix $D = UL^T$ is clearly congruent to A, hence, by Theorem 28 [10 , p. 137] a diagonal matrix.

Now, making use of the formulas for the minors of the product of two matrices we find, from (1.3-2),

$$(1.3\text{-}5)\qquad A\begin{pmatrix}0 & 1 & 2 & \cdots & k\text{-}2 & g\\ 0 & 1 & 2 & \cdots & k\text{-}2 & k\text{-}1\end{pmatrix}$$

$$= \Sigma\, L^{-1}\begin{pmatrix}0 & 1 & 2 \ldots & k\text{-}2 & g\\ \alpha_0 & \alpha_1 & \alpha_2 \cdots & \alpha_{k-2} & \alpha_{k-1}\end{pmatrix} U\begin{pmatrix}\alpha_0 & \alpha_1 & \alpha_2 & \cdots & \alpha_{k-1}\\ 0 & 1 & 2 & \ldots & k\text{-}1\end{pmatrix},$$

$(\alpha_0 < \alpha_1 < \ldots < \alpha_{k-1})$

$k= 1, 2, 3, \ldots, r;\ g=k\text{-}1, k, \ldots, m\text{-}1$ [12, p.9].

Since U is an upper triangular matrix, the first k columns of it contain only one non-vanishing minor of order k, namely $U\begin{pmatrix}0 & 1 & 2 & \ldots & k\text{-}1\\ 0 & 1 & 2 & \ldots & k\text{-}1\end{pmatrix}$.

Thus, equation (1.3-5) can be written as

$$A\begin{pmatrix}0 & 1 & 2 & \ldots & k\text{-}2 & g\\ 0 & 1 & 2 & \ldots & k\text{-}2 & k\text{-}1\end{pmatrix} = L^{-1}\begin{pmatrix}0 & 1 & 2 & \ldots & k\text{-}2 & g\\ 0 & 1 & 2 & \ldots & k\text{-}2 & k\text{-}1\end{pmatrix} U\begin{pmatrix}0 & 1 & 2 \ldots & k\text{-}1\\ 0 & 1 & 2 \ldots & k\text{-}1\end{pmatrix}$$

$$= \frac{1}{a_{00}\, a_{11}\, a_{22} \cdots a_{gk-1}}.$$

We put g=k-1 in this equation, obtaining

$$(1.3\text{-}6)\qquad D_k^{(0)} = \frac{1}{a_{00}\, a_{11}\, a_{22} \cdots a_{k-1\,k-1}},\ (k=1,2,3,\ldots,r),$$

and the relations (1.3-3) follow.

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CHAPTER II

QUADRATIC AND CANONICAL FORMS

Jacobi's Method [12 , p.300] for the transformation of a quadratic form to a form that involves only the squared terms is described in this Chapter. This latter form is referred to as a Canonical Form. The method of the decomposition of a square matrix into triangular factors (described in Chapter I) is used to develop the Jacobi's method. For when this method is applied to a symmetric matrix $B = [\alpha_{ij}]_0^{m-1}$ of rank r, it produces a diagonal matrix D, having $D_k^{(0)}/D_{k+1}^{(0)}$ $(k = 0, 1, 2, \ldots, r-1;\ D_0^{(0)}=1)$ as the elements, that is congruent to B and, the quadratic form B(x,x) changes into one that is associated with the matrix D(the form involving only square terms), wherein no one of the coefficients is zero. Also formulated are the relations for the signature and the rank of quadratic forms.

[Throughout this Chapter, unless contrary is expressly stated, F denotes field of numbers].

2.1 QUADRATIC FORMS

Definition

A linear homogeneous expression, in m variables $x_i (i = 0, 1, 2, \ldots, m-1)$, such as

$$(2.1\text{-}1) \qquad \Sigma\, \alpha_{ij}\, x_j \ (j = 0, 1, 2, \ldots, m-1)$$

is called a Linear Form in the variables x_i; an expression such as

(2.1-2) $\Sigma\, \alpha_{ij}\, x_i\, x_j$ ($\alpha_{ij} = \alpha_{ji}$; $i,j = 0, 1, 2, \ldots, m-1$),

a homogeneous polynomial of the second degree in the variables x_i ($i = 0, 1, 2, \ldots, m-1$), is called a <u>Quadratic Form</u>. (The variables are assumed to commute with one another and with coefficient in a form).

When x is a single-column matrix with elements $x_i (i = 0, 1, 2, \ldots, m-1)$, then Bx is the single-column matrix having

$$\alpha_{i0}\, x_0 + \alpha_{i1}\, x_1 + \ldots + \alpha_{im-1}\, x_{m-1}$$

as the element in its $(i+1)$-th row, where $B = [\alpha_{ij}]_0^{m-1}$ (symmetric matrix) denotes the matrix of the quadratic form (2.1-2), and $x^T Bx$ is a single-element matrix, the single element being

$$\sum_{i=0}^{m-1} x_i(\alpha_{i0}\, x_0 + \alpha_{i1}\, x_1 + \ldots + \alpha_{im-1}\, x_{m-1}) = \sum_{i,j=0}^{m-1} \alpha_{ij}\, x_i\, x_j .$$

Thus the quadratic form is represented by the single-element matrix $x^T Bx$, and is conveniently abbreviated to $B(x,x)$, that is

(2.1-3) $B(x,x) = \Sigma\, \alpha_{ij}\, x_i\, x_j$ $(i,j = 0, 1, 2, \ldots, m-1)$.

$$= x^T Bx,$$

wherein $B = [\alpha_{ij}]_0^{m-1}$.

<u>Definition</u>

The rank of the matrix $B = [\alpha_{ij}]_0^{m-1}$ $(\alpha_{ij} = \alpha_{ji})$

is called the rank of the form $B(x,x)$; the determinant $|B| = |\alpha_{ij}|_0^{m-1}$ is called the Discriminant of the form. The form is singular or non-singular according as its discriminant is zero or not.

A form in which both coefficients and variables are necessarily real numbers is said to be a Real Form.

In what follows we shall exclusively be concerned with real quadratic forms.

LINEAR TRANSFORMATIONS

The set of equations

$$X_i = \Sigma\, \beta_{ij}\, x_j \quad (i,j = 0, 1, 2, \ldots , m-1), \tag{2.1-4}$$

where β_{ij} are given scalars, is called a linear transformation connecting the variables x_i and the variables X_i. It is most conveniently written as

$$X = Ax, \tag{2.1-5}$$

a matrix equation in which X and x denote single-column matrices with elements X_i and x_i $(i = 0, 1, 2, \ldots , m-1)$ respectively, wherein

$$A = [\beta_{ij}]_0^{m-1} .$$

Definition

The determinant, whose elements are the coefficients β_{ij} of the transformation (2.1-4), is called the Modulus of Transformation.

A linear transformation is said to be singular or non-singular according as its modulus is zero or not.

In the sequel we shall make use exclusively of non-singular, real transformations of the variables.

TRANSFORMATIONS OF QUADRATIC FORMS

THEOREM (2.1-1)

If in a quadratic form $B(x,x) = x^T Bx$ of matrix B the x's are subjected to a linear transformation of matrix A ($x = AX$), the result is a quadratic form $C(X,X)$ whose matrix is

$$C = A^T BA \quad . \tag{2.1-6}$$

Proof:

By hypothesis, $x = AX$, and this implies $x^T = X^T A^T$ (See Theorem 23; [9 , p.83]). Hence, by the associated law of multiplication of matrices

$$\begin{aligned} B(x,x) &= x^T Bx \\ &= X^T A^T BAX \\ &= X^T CX \\ &= C(X,X), \end{aligned} \tag{2.1-7}$$

wherein C is the matrix $A^T BA$.

Moreover, to prove that the matrix C of the form $C(X,X)$ is symmetric so that $C(X,X)$ is a quadratic form, one observes that, when $C = A^T BA$,

$$\begin{aligned} C^T &= (A^T BA)^T \\ &= A^T B^T A \\ &= A^T BA \quad (\text{since } B = B^T) \\ &= C, \end{aligned}$$

that is to say, $c_{ij} = c_{ji}$, with $C = [c_{ij}]$. Thus $C(X,X)$ is a quadratic form, having matric $C = A^T BA = [c_{ij}]$ $(c_{ij} = c_{ji})$.

COROLLARY (2.1-2)

When in a quadratic form $B(x,x)$ we subject the x's to a linear transformation $x = AX$, with $|A| = M$, the discriminant of the resulting quadratic form $C(X,X)$ is determined by

(2.1-8) $$|C| = M^2 \cdot |B| .$$

Proof:

By Theorem (2.1-1), the matrix of the form $C(X,X)$ is $C = A^T BA$. Therefore (See [9 ; p.78]),

$$|C| = |A^T| \cdot |B| \cdot |A| .$$

But $|A| = M$, and since the value of a determinant is unchanged when its rows and columns are interchanged, $|A^T|$ is also equal to M. Hence

$$|C| = M^2 |B| .$$

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Assume that A is a matrix of m columns (rows), so that the product AB (BA) can be formed. By Theorem 33 [9 , p. 110] the rank of the product cannot exceed the rank of either factor. Also, when A is a non-singular square matrix of the same order as the square matrix B, the matrices B, AB, BA all have the same rank [12 ; p.17].

Hence, from (2.1-6),

(2.1-9) $$\text{rank}_C \leq \text{rank}_B \ (|A| \mp 0).$$

When A is non-singular, it has an inverse A^{-1} and

(2.1-10) $$B = (A^T)^{-1} \ CA^{-1},$$

The argument used above, then, gives

(2.1-11) $\text{rank}_B \leq \text{rank}_C$,

The inequalities in (2.1-9) and (2.1-11)imply

$$\text{rank } B = \text{rank } C .$$

Thus, from the proceeding remarks, it follows that:

THEOREM (2.1-3)

The rank of a quadratic form B(x,x) is invariant under a linear transformation on the variables.

By Theorem 28 [10] , since B is symmetric there exists a non-singular matrix A such that the matrix

$$C = A^T BA,$$

is a diagonal matrix. The matrices B and C are congruent, (See definition, p.10).

Thus by Theorem (2.1-1), a whole class of congruent symmetric matrices is associated with every quadratic form. All these matrices (Theorem 2.1-3) have one and the same rank, the rank of the form.

Another invariant, in case of real quadratic forms, is the signature of the form. We shall now proceed to introduce this concept.

2.2 REDUCTION OF A QUADRATIC FORM TO A CANONICAL FORM

It will now be shown that a quadratic form B(x,x), with $B = [\alpha_{ij}]_0^{m-1}$, of rank r, when to the variables x_i a linear transformation x = AX is applied, changes into a form

(2.2.1) $$B(x,x) = \Sigma\, \gamma_i X_i^2 \quad (i = 0, 1, 2, \ldots, r-1)$$

wherein no one of γ_i (i = 0, 1, 2, ..., r-1) is zero.

The Form (2.2-1) is referred to as a Canonical Form.

First, it is of interest to note that a quadratic form can be transformed into one whose coefficient of the term x_0^2 is not zero.

To establish this fact, consider the transformation

(2.2-2) $$x_0 = X_{m-1}, x_{m-1} = X_0; \ x_k = X_k \ (k \neq 0, m-1).$$

It is certainly a non-singular linear transformation. Moreover, each of its coefficient is either 1 or 0, and so belongs to F.

If in the form $\Sigma \alpha_{ij} x_i x_j$ one of the coefficients $\alpha_{00}, \alpha_{11}, \ldots, \alpha_{m-1\,m-1}$ is not zero, then either $\alpha_{00} \neq 0$ or a suitable linear transformation of the type (2.2-2) will change the given quadratic form into a form $C(X,X)$, with $C = [c_{ij}]$, in which $c_{00} \neq 0$.

When all the coefficients $\alpha_{00}, \alpha_{11}, \alpha_{22}, \ldots, \alpha_{m-1\,m-1}$ are zero, but one of α_{ij} $(i \neq j)$ is not zero, then a suitable linear transformation of the type (2.2-2) will change the form into a new form $C(X,X)$, wherein each of c_{ii} is zero, but one of the numbers $c_{01}, c_{02}, \ldots, c_{0m-1}$ is not zero.

In such a case, one may assume that

(2.2-3) $$X_0 = Y_0 + Y_k, \ X_k = Y_0 - Y_k, \ X_\ell = Y_\ell \ (\ell \neq 0, k),$$

which is certainly a non-singular linear transformation. Each of its coefficient is 1, 0 or -1 and so belongs to F.

Then, the product of two linear transformations, a transformation of type (2.2-2) followed by one of type (2.2-3) changes the form $B(x,x)$ directly into the form in which the cofficient of Y_0^2 is not zero.

The results under the foregoing linear transformations may be summerized as follows:

THEOREM (2.2-1)

A quadratic form $B(x,x) = \Sigma\, \alpha_{ij}\, x_i x_j$, having α_{ij} not zero, can be transformed by a linear transformation to a quadratic form, wherein the coefficient of X_0^2 is not zero.

The statement made in the beginning of the section is now answered in part by the following result.

THEOREM (2.2.2)

A quadratic form may be transformed by a linear transformation to the canonical form.

(2.2-4) $$B(x,x) = \Sigma\, \gamma_i\, X_i^2 \; (i = 0, 1, 2, \ldots, k-1; k \leq m)$$

wherein $\gamma_i \in F$, and $X_i (i = 0, 1, 2, \ldots, m-1)$ are variables connected with the variables x_i for the transformation in question.

Proof:

For m=1, the result is trivial. Assume that for form of m-1 variables the theorem is valid, with this assumption it will be shown that the theorem is also true for form of m variables. Suppose given a quadratic form

$$B(x,x) = \Sigma\, \alpha_{ij}\, x_i\, x_j \; (i,j = 0, 1, 2, \ldots, m-1).$$

If one α_{ij} is not zero (Theorem (2.2-1)), there is a linear transformation that transforms B(x,x) to

$$B(x,x) = \gamma_0 X_0^2 + 2 \sum_{i=1}^{m-1} \alpha_{0i} X_0 X_i + \sum_{i,j=2}^{m-1} \alpha_{ij} X_i X_j .$$

This may be expressed as

$$(2.2\text{-}5) \qquad B(x,x) = \gamma_0 (X_0 + \frac{\alpha_{01}}{\gamma_0} X_1 + \ldots + \frac{\alpha_{0m-1}}{\gamma_0} X_{m-1})^2 + \Sigma \beta_{ij} X_i X_j ,$$

$(i,j = 1, 2, 3, \ldots , m-1)$.

On the basis of the induction assumption

$$\sum_{i,j=1}^{m-1} \beta_{ij} X_i X_j = \Sigma \gamma_\ell Y_\ell^2 \ (\ell = 1, 2, 3, \ldots k-1;\ k \leq m-1)$$

where

$$\begin{bmatrix} Y_1 \\ Y_2 \\ \cdot \\ \cdot \\ \cdot \\ Y_{m-1} \end{bmatrix} = B_1 \begin{bmatrix} X_1 \\ X_2 \\ \cdot \\ \cdot \\ \cdot \\ X_{m-1} \end{bmatrix}$$

and $_1$ is a non-singular matrix.

Assume $Y_0 = (X_0 + \frac{\alpha_{01}}{\gamma_0} X_1 + \ldots + \frac{\alpha_{0m-1}}{\gamma_0} X_{m-1})$, then

$$Y = B_2 X,$$

where

$$B_2 = \begin{bmatrix} 1 & \frac{\alpha_{01}}{\gamma_0} & \frac{\alpha_{02}}{\gamma_0} \cdots \frac{\alpha_{0m-1}}{\gamma_0} \\ 0 & & \\ 0 & & \\ \cdot & & \\ \cdot & & B_1 \\ \cdot & & \\ 0 & & \end{bmatrix}$$

It is clear that B_2 is a non-singular matrix. With the new variables the quadratic form has the form

(2.2-6) $B(x,x) = \Sigma \gamma_i Y_i^2$ $(i = 0, 1, 2, \ldots, k-1; k \leq m)$

and setting, in (2.2-5), $Y_i = X_i$, one gets the desired form.

THEOREM (2.2-3)

The rank r of a quadratic form $B(x,x) = x^T Bx$ is equal to the number of squares in a Canonical form of $B(x,x)$.

Proof:

By Theorem (2.2-2), the quadratic form $B(x,x)$ under a linear transformation may be transformed to the form

(2.2-5') $\gamma_0 X_0^2 + \gamma_1 X_1^2 + \ldots + \gamma_{k-1} X_{k-1}^2$ $(k \leq m)$.

To prove that the number k in (2.2-4) is equal to the rank r of $B(x,x)$, let A denote the matrix of the transformation whereby we pass from $B(x,x)$ to the form (2.2-4). The matrix of the form (2.2-4) is then $A^T BA$ (Theorem (2.1-1)). Since A is non-singular, the matrix $A^T BA$ has the same rank as B (Theorem 34 [9 , p.110]), see also Theorem (2.1-3)), that is, r. But the rank of the quadratic form (2.2-4), that contains only the squared terms, is precisely the number of non-zero coefficients, for the matrix of this form consists of $\gamma_0, \gamma_1, \gamma_2, \ldots, \gamma_{k-1}$ in the first k places of the principal diagonal and zero elsewhere, so that its rank is k. Hence, $k = r$.

Theorems (2.2-1), (2.2-2) and (2.2-3) lead fairly quickly to the following well-known result in the theory of quadratic forms.

THEOREM (2.2-4)

A quadratic form $B(x,x)$ of rank r, with $B = [\alpha_{ij}]_0^{m-1}$, when we subject the x's to a transformation, changes in a Canonical form

$$B(x,x) = \Sigma\, \gamma_i\, X_i^2 \; (i=0,\ 1,\ 2,\ \ldots,\ r-1;\ r \leq m), \tag{2.2-4}$$

wherein $\gamma_i \neq 0$ $(i=0, 1, 2, \ldots, r-1)$.

Thus, it is proved that every quadratic form of rank r can be represented, under a non-singular linear transformation, in the form

$$\Sigma\, \gamma_i\, X_i^2 \; (i=0,\ 1,\ 2,\ \ldots,\ r-1)$$

where no one of $\gamma_i (i=0, 1, 2, \ldots, r-1)$ is zero. It may be noted there are, in general, an infinite number of ways in carrying out such a representation, even at the first step one has a wide choice as to which non-zero element α_{ij} one selects to become the non-zero element c_{00} or d_{00}, as the case may be.

However, so long as we confine ourselves to real quadratic forms under non-singular, real transformations, the number of positive squares (and, hence, the number of negative squares) in the Canonical form is unique. This result is known as the Law of Inertia for quadratic forms.

THEOREM (2.2-5) (THE LAW OF INERTIA).

If a quadratic form $B(x,x) = x^T Bx$, of rank r, is reduced by two linear transformations to two distinct Canonical forms, the number of positive (and hence, the number of negative) coefficients in one is equal to the number of positive (negative) coefficients in the other.

For a complete proof of this theorem see Browne [5, p. 114-115]; also see Gantmacher [12, p. 297].

Definition

The number of positive squares in the Canonical form is usually called the Index of inertia, or merely the Index, of the form.

Since the rank of a quadratic form is invariant under a linear transformation (Theorem(2.1-3)) and is equal to the number of squares in the Canonical representation (Theorem (2.2-3)), it follows from Theorem(2.2-5) that not only is the total number of squares invariant in the various Canonical forms of a quadratic form under linear transformations, but also are the index of the form and the number of negative squares.

It is, thus, proved that, associated with every quadratic form are two numbers μ and υ, the index of the form and the number of negative squares in any Canonical form. The sum of the numbers μ and υ is the rank of the form (Theorem(2.2-3)), i.e. $r = \mu + \upsilon$. Also, it is shown that r, μ and υ constitute a set of invariants of the form under non-singular, real linear transformations.

Definition

The difference $\sigma = \sigma\,\{B(x,x)\}$ between μ and υ is called the Signature of the form $B(x,x)$. [In Symbols: $\sigma = \mu - \upsilon$].

Since

$$\mu + \upsilon = r \text{ and } \sigma = \mu - \upsilon,$$

it is clear that r and σ uniquely determine and are determined by μ and υ or by μ and r. Hence, the Signature σ is also an invariant of the form under non-singular, real linear transformations.

From the discussion above of real quadratic forms and in particular from the Law of Inertia of forms, we assert that, in order to determine the rank and signature of a form, it is sufficient to transform it into a Canonical form.

In the next section, therefore, we propose to discuss a method for the reduction of real quadratic forms to the Canonical form under non-singular, real linear transformation. It then, furnishes a rule for determining the Signature from the coefficients of the quadratic form.

This method is referred to as Jacobi's Method:

2.3 JACOBI'S METHOD

Suppose given a quadratic form

$$B(x,x) = \Sigma \alpha_{ij} x_i x_j \ (i,j=0, 1, 2, \ldots, m-1),$$
$$\equiv x^T Bx,$$

of rank r ($\leq m$); where $B = [\alpha_{ij}]_0^{m-1}$ is a symmetric matrix.

On condition that the leading sub-matrices of B are non-singular, i.e., that

$$\overset{(0)}{D}_k = B\begin{pmatrix}0 & 1 & 2 & \ldots k-1\\ 0 & 1 & 2 & \ldots k-1\end{pmatrix} \neq 0 \ (k=1,\ 2,\ 3,\ \ldots\ ,\ r),$$

B may be represented as the product of two triangular matrices (Theorem:(1,2-1)). Thus, there is a lower triangular matrix L and an upper triangular matrix U such that LB = U.

Then

$$(2.3\text{-}1) \qquad LBL^T = UL^T = D$$

is both upper triangular and symmetric matrix, hence diagonal having (Theorem (1.3-1)),

$$\overset{(0)}{D}_k \Big/ \overset{(0)}{D}_{k+1} \quad (k=0,\ 1,\ 2,\ \ldots,\ r-1;\ \overset{(0)}{D}_0 = 1),$$

as elements.

It follows, from the equation (2.3-1), that

$$(2.3\text{-}2) \qquad B = L^{-1}D(L^{-1})^T \quad (|L| \neq 0)$$

$$= E^T DE \qquad (E = (L^{-1})^T).$$

Since every symmetric matrix may be associated with a certain quadratic form, from (2.3-2) it follows that the quadratic form,

$$D(y,y) = \Sigma y_k^2\ \overset{(0)}{D}_k \Big/ \overset{(0)}{D}_{k+1} \quad (k=0,\ 1,\ 2,\ldots r-1; \overset{(0)}{D}_0 = 1),$$

wherein y is a single-column matrix, having y_0, y_1, $y_2 \ldots, y_{r-1}$ as elements, goes over into the form B(x,x) under the linear transformation

$$y = Ex;$$

and this implies

$$(2.3\text{-}3) \qquad B(x,x) = D(y,y) = \Sigma y_k^2\ \overset{(0)}{D}_k \Big/ \overset{(0)}{D}_{k+1} \quad (k=0,1,2,\ldots,r-1; \overset{(0)}{D}_0 = 1)$$

The set of linearly independent forms

$$X_k = D_k y_k \ (k=0,\ 1,\ 2,\ \ldots,\ r-1;\ \overset{(0)}{D}_0 = 1)$$

changes (2.3-3) into the form

$$(2.3\text{-}4)\quad B(x,x) = \Sigma\, X_k^2 \Big/ \overset{(0)}{D_k}\ \overset{(0)}{D_{k+1}} \quad (k=0,\ 1,\ 2,\ \ldots,\ r\text{-}1;\ \overset{(0)}{D_0} = 1).$$

This formula (2.3-4) gives us a representation of B(x,x) in the form involving only square terms and is usually referred to as the Jacobi's Formula.

Example;

Suppose given a quadratic form

$$B(x,x) = x_0^2 + 3x_1^2 - 3x_3^2 - 4x_0\,x_1 + 2x_0\,x_2 - 2x_0\,x_3 - 6x_1\,x_2 + 8x_1\,x_3 + 2x_2\,x_3$$

The matrix

$$B = \begin{vmatrix} 1 & -2 & 1 & -1 \\ -2 & 3 & -3 & 4 \\ 1 & -3 & 0 & 1 \\ -1 & 4 & 1 & -3 \end{vmatrix}$$

is of rank r=2. Also $\overset{(0)}{D_1} = 1$, $\overset{(0)}{D_2} = -1$ etc.

Thus, Jacobi's formula (2.3-4) yields

$$B(x,x) = (x_0 - 2x_1 + x_2 - x_3)^2 - (-x_1 - x_2 + 2x_3)^2 ,$$

wherein

$$X_0 = x_0 - 2x_1 + x_2 - x_3 \text{ and } X_1 = -x_1 - x_2 + 2x_3 .$$

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The determinants $\overset{(0)}{D_k}$ (the determinants of the leading sub-matrices of a symmetric matrix), the number of positive and negative squares in a Canonical form and the Signature of a quadratic form may be related by combining the Law of Inertia for quadratic forms of finite rank with the Jacobi's Formula.

Wherefrom we arrive at the following result, obtained by Jacobi:

THEOREM (2.3-1) (Jacobi)

If for a quadratic form

$$B(x,x) = x^T Bx$$

of rank r the inequality

$$\overset{(0)}{D_k} = B\begin{pmatrix} 0 & 1 & 2 & \cdots & k-1 \\ 0 & 1 & 2 & \cdots & k-1 \end{pmatrix} \neq 0 \quad (k=1,\ 2,\ 3 \ldots r),$$

holds, then the index and the number of negative squares of $B(x,x)$ coincide, respectively, with the number P of permanences of sign and the number V of variations of sign in the sequence

$$\overset{(0)}{D_0}, \overset{(0)}{D_1}, \overset{(0)}{D_2}, \ldots, \overset{(0)}{D_r},$$

i.e.,

$$(2.3\text{-}5) \quad \mu = P\,(\overset{(0)}{D_0}, \overset{(0)}{D_1}, \overset{(0)}{D_2}, \ldots, \overset{(0)}{D_r});$$

$$\upsilon = V\,(\overset{(0)}{D_0}, \overset{(0)}{D_1}, \overset{(0)}{D_2}, \ldots, \overset{(0)}{D_r}),$$

and the Signature

$$(2.3\text{-}6) \quad \sigma = r - 2V\,(\overset{(0)}{D_0}, \overset{(0)}{D_1}, \overset{(0)}{D_2}, \ldots, \overset{(0)}{D_r});\ (\overset{(0)}{D_0} = 1).$$

REMARK 1

If in the set $\{\overset{(0)}{D_k} : k=0, 1, 2, \ldots, r;$ $\overset{(0)}{D_0} = 1$ and $\overset{(0)}{D_r} \neq 0\}$ there are zeros, but not three in succession, then the Signature of the form can be determined by the formula:

$$\sigma = r - 2V\,(\overset{(0)}{D}_0, \overset{(0)}{D}_1, \overset{(0)}{D}_2, \ldots, \overset{(0)}{D}_r)$$

omitting the zero $\overset{(0)}{D}_k$ provided $\overset{(0)}{D}_{k-1}\,\overset{(0)}{D}_{k+1} \neq 0$; and setting

$$V(\overset{(0)}{D}_{k-1}, \overset{(0)}{D}_k, \overset{(0)}{D}_{k+1}, \overset{(0)}{D}_{k+2}) = \begin{cases} 1, & \text{when } \dfrac{\overset{(0)}{D}_{k+2}}{\overset{(0)}{D}_{k-1}} < 0 \\[2ex] 2, & \text{when } \dfrac{\overset{(0)}{D}_{k+2}}{\overset{(0)}{D}_{k-1}} > 0 \end{cases} \tag{2.3-7}$$

when $\overset{(0)}{D}_k = \overset{(0)}{D}_{k+1} = 0$.

This rule was established in the case of a single zero $\overset{(0)}{D}_k$ by Gundelfinger [5, p. 127], and the case of two consecutive zeros $\overset{(0)}{D}_k$ by FROBENIUS [11, p. 241-56,; 407-31].

REMARK 2

If in the set $\{\overset{(0)}{D}_k : k=0, 1, 2, \ldots, r;\ \overset{(0)}{D}_0 = 1;\ \overset{(0)}{D}_r \neq 0\}$ there are three consecutive zeros $\overset{(0)}{D}_k$, the signature cannot be immediately determined by (2.3-6). However, as Frobenius has established, for a particular type of forms, there is a rule that enables one to apply the Jacobi's Theorem in general[11, p.407-31; also 12, p.343-44].

REMARK 3

When $\overset{(0)}{D}_r = 0$, but $\overset{(0)}{D}_k \neq 0$ $(k=0, 1, 2, \ldots, r-1)$, the signature of the quadratic form cannot be determined from the signs of $\overset{(0)}{D}_1, \overset{(0)}{D}_2, \overset{(0)}{D}_3, \ldots, \overset{(0)}{D}_{r-1}$. [12, p. 304].

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CHAPTER III

INFINITE HANKEL MATRICES OF FINITE RANK AND HANKEL DETERMINANTS

3.1 INFINITE HANKEL MATRICES OF FINITE RANK

A symmetric matrix constructed from a sequence of numbers c_i $(i = 0, 1, 2, \ldots)$:

(3.1-1) $$C = [c_{i+j}]_0^\infty ,$$

which may be of finite or of infinite rank, is said to be an infinite Hankel Matrix. Clearly, each leading sub-matrix of the Hankel matrix C is also a Hankel matrix. of finite order.

The quadratic form associated with an infinite Hankel matrix of finite rank is called a Hankel Form. It is usually written as

(3.1-2) $$C(x,x) = \Sigma c_{i+j} x_i x_j \ (i,j = 0,1,2,\ldots)$$

where $C = [c_{i+j}]_0^\infty$ is a symmetric matrix of finite rank.

If x is a single-column matrix having $x_0, x_1, x_2, \ldots,$ as the elements, the form $C(x,x)$ is conveniently written as

(3.1-2') $$C(x,x) = \Sigma c_{i+j} x_i x_j \ (i,j = 0,1,2,\ldots)$$

$$= x^T Cx,$$

wherein $C = [c_{i+j}]_0^\infty$ is a symmetric matrix of finite rank.

In the present Chapter, we are concerned with some fundamental properties of the infinite Hankel matrices of finite rank. Among them is the connection between the Hankel matrices and real rational functions; and the general recurrence formulas for generating the minors of these matrices.

For a given sequence of numbers to generate an infinite Hankel matrix of finite rank, a necessary and sufficient condition is given as follows:

THEOREM (3.1-1)

The infinite matrix $C = [c_{i+j}]_0^\infty$ is of finite rank r if and only if there exists an integer r and numbers $\gamma_1, \gamma_2, \ldots, \gamma_r$, such that

(3.1-3) $$c_q = \sum_{g=1}^{r} \gamma_g c_{q-g} \quad (q=r, r+1, r+2, \ldots),$$

and r is the least number having this property (see [13, p.205]).

COROLLARY (3.1-2)

If the infinite Hankel matrix

$$C = [c_{i+j}]_0^\infty$$

is of finite rank r, then

(3.1-4) $$C\begin{pmatrix} 0 & 1 & 2 & \ldots & r-1 \\ 0 & 1 & 2 & \ldots & r-1 \end{pmatrix} = |c_{i+j}|_0^{r-1} \neq 0.$$

For it follows from the relations (3.1-3) that every row (Column) of C is a linear combination of the first r rows (columns). Every minor of C of order r, may be, therefore, represented in the form

$$\alpha C \begin{pmatrix} 0 & 1 & 2 & \cdots & r-1 \\ 0 & 1 & 2 & \cdots & r-1 \end{pmatrix},$$ wherein α is a constant. Hence it follows that

$$C \begin{pmatrix} 0 & 1 & 2 & \cdots & r-1 \\ 0 & 1 & 2 & \cdots & r-1 \end{pmatrix} \neq 0.$$

The determinant in (3.1-4) is usually denoted by $H_r^{(0)}$ and is most conveniently written as

$$(3.1\text{-}5) \quad H_r^{(0)} = C \begin{pmatrix} 0 & 1 & 2 & \cdots & r-1 \\ 0 & 1 & 2 & \cdots & r-1 \end{pmatrix},$$

the determinant of the leading sub-matrix of order r in C.

REMARK:

The inequality $H_r^{(0)} \neq 0$ may not necessarily hold for a Hankel matrix of finite order m and of rank $r < m$.

As a corroborating example, take the Hankel matrix

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \\ 4 & 8 & 6 \end{bmatrix}$$

It is of rank 2, for $\begin{vmatrix} 4 & 8 \\ 8 & 6 \end{vmatrix} \neq 0$, whereas $H_2^{(0)} = \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 0.$

However, if in a Hankel matrix of finite order m, the first r rows are linearly independent, but the first $r + 1$ rows linearly dependent, then

$$(3.1\text{-}6) \qquad c_q = \sum_{g=1}^{r} \gamma_g \, c_{q-g} \qquad (q=r,\ r+1,\ r+2,\ \ldots,\ r+m-1).$$

The matrix constructed from the first r linearly independent rows of the Hankel matrix of finite order m is

$$(3.1\text{-}7) \qquad \begin{bmatrix} c_0 & c_1 & c_2 & \cdot & \cdot & \cdot & c_{m-1} \\ c_1 & c_2 & c_3 & \cdot & \cdot & \cdot & c_m \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ c_{r-1} & c_r & c_{r+1} & \cdot & \cdot & \cdot & c_{r+m-2} \end{bmatrix}$$

which, by hypothesis, is of rank r ($< m$).

On the other hand, by (3.1-6), every column of the matrix can be expressed linearly in terms of the proceeding r columns and, hence, in the terms of the first r columns. But, since the rank of the matrix is r, these first r columns in (3.1-7) must then be linearly independent. Thus, there is a determinant $H_r^{(0)}$, in the Hankel matrix of order m, with first r linearly independent rows, as well as columns; and this implies

$$H_r^{(0)} \neq 0.$$

That is;

THEOREM (3.1-3)

If the first r rows of a Hankel matrix of finite order are linearly independent, but the first $r + 1$ rows are linearly dependent, then the first r columns are linearly independent, and

$$H_r^{(0)} \neq 0.$$

REAL HANKEL FORMS

Let us consider a real Hankel form

$$C(x,x) = \Sigma\, c_{i+j}\, x_i\, x_j \ (i,j = 0,\ 1,\ 2,\ \ldots),$$

$$= x^T C x$$

of finite rank r, wherein $C = [c_{i+j}]_0^\infty$ is a real matrix.

Theorems analogous to theorems for quadratic forms(described in Chapter II) hold for the real Hankel forms. Namely:

THEOREM 1.

Every real Hankel form $C(x,x)$ of finite rank may be changed into a canonical form under a non-singular, real linear transformations of variables.

THEOREM 2. (Law of Inertia)

The number of negative and positive coefficients in a canonical expression of a real Hankel form $C(x,x)$ are independent of the choice of the representation.

THEOREM 3. (Jacobi's Theorem).

Since, μ, υ, and σ have the usual meanings with respect to the real Hankel forms $C(x,x)$, so that

$$\mu + \upsilon = r \quad \text{and} \quad \sigma = \mu - \upsilon,$$

then, from Theorem (2.3-1) it follows that the signs of the successive principal minors $H_k^{(0)}$ (k=0, 1, 2,..., r-1; $H_0^{(0)} = 1$) of the Hankel matrix C will determine the values of μ, υ and σ by the formulas:

$$\begin{aligned} \mu &= P(H_0^{(0)}, H_1^{(0)}, H_2^{(0)}, \ldots, H_r^{(0)}), \\ \upsilon &= V(H_0^{(0)}, H_1^{(0)}, H_2^{(0)}, \ldots, H_r^{(0)}), \\ \sigma &= r-2V(H_0^{(0)}, H_1^{(0)}, H_2^{(0)}, \ldots, H_r^{(0)}). \end{aligned} \tag{3.1-8}$$

Proofs of these theorems coincide almost exactly with proofs of the analogous theorems for quadratic forms.

These formulas in (3.1-8) will fail to determine the value of μ, υ, σ, when any three consecutive $H_k^{(0)}$ are zero. However, as Frobenius has established, for the forms in question there is a rule that guaranties the application of the formulas (3.1-8), in general [11 p. 407-31].

3.2 CONNECTION OF REAL RATIONAL FUNCTIONS WITH REAL INFINITE HANKEL MATRICES

Suppose given a proper real rational function

$$R(x) = \frac{g(x)}{f(x)},$$

where $g(x) = b_0 x^{m-1} + b_1 x^{m-2} + \ldots + b_{m-1}$ and

$f(x) = a_0 x^m + a_1 x^{m-1} + \ldots + a_m$ ($a_0 \neq 0$; $a_m \neq 0$).

We write the expansion of $R(x)$ in a series of negative powers of x:

(3.2-1) $R(x) = \Sigma\, c_n/x^{n+1}$ ($n=0, 1, 2, \ldots$).

If all the poles of $R(x)$ are in the interval $-a \leq x \leq a$, then the series on the right hand side of the expansion (3.2-1) converges for all $x \notin [-a, a]$.

We, therefore, obtain by multiplying on both sides by the denominator

(3.2-2) $(\Sigma\, c_n/x^{n+1})\,(a_0 x^m + a_1 x^{m-1} + \ldots + a_m) =$

$$b_0 x^{m-1} + b_1 x^{m-2} + \ldots + b_{m-1}.$$

When the coefficients of equal powers of x are equated, the following equations are obtained:

(3.2-3)
$$\begin{aligned} b_0 &= a_0 c_0 \\ b_1 &= a_0 c_1 + a_1 c_0 \\ b_2 &= a_0 c_2 + a_1 c_1 + a_2 c_0 \\ &\ldots\ldots\ldots\ldots \\ b_{m-1} &= a_0 c_{m-1} + a_1 c_{m-2} + \ldots\ldots + a_{m-1} c_0, \end{aligned}$$

and

(3.2-3') $0 = a_0 c_q + a_1 c_{q-1} + \ldots + a_m c_{q-m}$

($q = m, m+1, m+2 \ldots$).

Taking

$$\gamma_g = -a_g/a_0 \quad (g=1, 2, 3, \ldots m),$$

the relations (3.2-3') may be written in the form of (3.1-3), i.e.,

$$c_q = \sum_{g=1}^{m} \gamma_g c_{q-g} \quad (q=m, m+1, \ldots).$$

Therefore, by Theorem (3.1-1), the Hankel matrix

$$C = [c_{i+j}]_0^{\infty}$$

constructed from the coefficients c_i (i=0, 1, 2, ...), is of finite rank.

Conversely, if the matrix formed from the element c_i(i=0, 1, 2, ...) is of finite rank r, by Theorem (3.1-1) there is a set of real numbers

$\gamma_1, \gamma_2, \gamma_3, \ldots \gamma_r$; no one of γ_i(i=1, 2, 3,...;r) is zero, such that

$$(3.1\text{-}3) \qquad c_q = \sum_{g=1}^{r} \gamma_g c_{q-g} \quad (q=r, r+1, \ldots).$$

Setting

$$\gamma_g = -a_g/a_0 \quad (g=1, 2, 3, \ldots, r) \text{ and } r = m,$$

the relations (3.1-3) may be written in the form (3.2-3'), i.e.,

$$0 = a_0 c_q + a_1 c_{q-1} + \ldots + a_m c_{q-m} (q=m, m+1, \ldots).$$

Thus, when the real numbers $b_0, b_1, b_2, \ldots, b_{m-1}$ (r = m) are defined by the relations (3.2-3), the expansion

$$R(x) = \frac{b_0 x^{m-1} + b_1 x^{m-2} + \ldots + b_{m-1}}{a_0 x^m + a_1 x^{m-1} + \ldots + a_m} = \Sigma\, c_n / x^{n+1} \quad (a_0 \neq 0;\ a_m \neq 0)$$

is obtained.

The least degree m of the denominator $f(x)$ for which the above expansion holds, by Theorem (3.1-1), is the rank of the Hankel matrix

$$C = [c_{i+j}]_0^\infty \ .$$

From this, one asserts that:

<u>THEOREM (3.2-1)</u>

If $R(x) = \Sigma\, c_n / x^{n+1}$ is real rational,

$$(3.2\text{-}4) \qquad R(x) = \frac{b_0 x^{m-1} + b_1 x^{m-2} + \ldots + b_{m-1}}{a_0 x^m + a_1 x^{m-1} + \ldots + a_m} ,$$

$(a_0 \neq 0;\ a_m \neq 0)$. Then, the real matrix

$$C = [c_{i+j}]_0^\infty$$

is of finite rank.

Conversely, if the matrix C is of finite rank, then the function $R(x)$ is a real rational function of the form (3.2-4), with $a_0 \neq 0;\ a_m \neq 0$. In this case the rank of C is the same as the number of poles of $R(x)$, counting each pole with its proper multiplicity.

COROLLARY (3.2-2)

Let $R(x) = \Sigma c_n/x^{n+1}$ be rational,

(3.2-5) $$R(x) = b_0x^p+b_1x^{p-1}+\dots+b_p/a_0x^q+a_1x^{q-1}+\dots+a_q ,$$

where $a_0 \neq 0$, $a_q \neq 0$. Then

(3.2-6) $$C\begin{pmatrix} n & n+1 & n+2 & \dots & n+q \\ 0 & 1 & 2 & \dots & q \end{pmatrix} = 0, \text{ wherein } c_{ij}=c_{i+j},$$

$n \geq \max(0, p+1-q)$.

Conversely, if there is an integer p such that

$$C\begin{pmatrix} n & n+1 & n+2 & \dots & n+q-1 \\ 0 & 1 & 2 & \dots & q-1 \end{pmatrix} \neq 0;$$

the determinants in (3.2-6) is zero for $n \geq p+1-q$, then $R(x)$ is a real rational function of the form (3.2-5), with $a_0 \neq 0$; $a_q \neq 0$.

CONSTRUCTION OF HANKEL MATRICES CORRESPONDING TO R(x)

Since an arbitrary real rational function $R(x)$ can always be expressed in the form

(3.2-7) $$R(x)=R_1(x) + \sum_{i=0}^{p-1} \left\{\frac{A_1^{(i)}}{x-\alpha_i} + \frac{A_2^{(i)}}{(x-\alpha_i)^2} + \dots + \frac{A_{\eta_i}^{(i)}}{(x-\alpha_i)^{\eta_i}}\right\} ,$$

where all α and A are real numbers $(A_j^{(i)} \neq 0$; $i=0, 1, 2, \dots, p-1)$ and $R_1(x)$ has no real poles. We shall show how to construct the Hankel matrix C corresponding to $R(x)$ from the numbers α and A.

We write its expansion in a series of descending powers of x:

$$R(x) = \Sigma\, c_n/x^{n+1} \quad . \tag{3.2-8}$$

The sequence of the coefficients $c_i(i=0, 1, 2, \ldots)$ determines the Hankel matrix $C = [c_{i+j}]_0^\infty$, and this establishes a correspondence between a real rational $R(x)$ and an infinite Hankel matrix C; it is denoted by

$$R(x) \sim C \quad .$$

Obviously, two rational functions whose difference is a rational function without real poles, correspond to one and the same Hankel matrix C. However, it is not necessary that every infinite matrix $C = [c_{i+j}]_0^\infty$ may correspond to some real rational function. In Theorem (3.2-1), it is shown that an infinite Hankel matrix C corresponds to $R(x)$ if and only if it is of finite rank. Thus, by means of the expansion $R(x) = \Sigma\, c_n/x^{n+1}$, there is a one to one correspondence between real, proper rational function $R(x) = g(x)/f(x) = \Sigma\, c_n/x^{n+1}$ and the matrix $C = [\, c_{i+j}]_0^\infty$ of finite rank.

We note two immediate properties of the correspondence:

1. If $R_1(x) \sim C_1$ and $R_2(x) \sim C_2$, then for arbitrary real numbers k_1, k_2

$$k_1 R_1(x) + k_2 R_2(x) \sim k_1 C_1 + k_2 C_2 \quad . \tag{3.2-9}$$

Since, sometimes we deal with the case wherein the coefficients of the functions $g(x)$ and $f(x)$ are integral rational functions of a parameter α, and so the numbers $c_0, c_1, c_2, \ldots$ depend rationally on α, we therefore differentiate $R(x) = \Sigma\, c_n/x^{n+1}$ term by term with respect to α, and obtain that:

2. If $R(x,\alpha) \sim C(\alpha)$, then

$$\frac{\partial R}{\partial \alpha} \sim \frac{\partial C}{\partial \alpha} \quad . \tag{3.2-10}$$

To carry out the method for the construction of C, we consider first the simple rational function

$$R(x) = \frac{1}{x-\alpha}$$

$$= \Sigma\, \alpha^n/x^{n+1} \quad (n=0, 1, 2, \ldots).$$

It corresponds to the Hankel matrix

$$C = [\alpha^{i+j}]_0^\infty \quad .$$

The form associated with C is, therefore,

$$C(x,x) = \Sigma\, \alpha^{i+j} x_i x_j \; (i,j=0,1,2,\ldots).$$

If

$$\frac{g(x)}{f(x)} = R(x) = R_1(x) + \Sigma(A_k/x-\alpha_k) \quad (k=0, 1, 2, \ldots, p-1),$$

then by (3.2-9) the corresponding matrix C is determined by the formula

$$C = [\Sigma A_k \alpha_k^{i+j}]_0^\infty, \; k=0, 1, 2, \ldots, p-1.$$

Since, in this case R(x) has p simple poles, we can find the A_k as follows:

$$R(x) = \Sigma \frac{g(\alpha_k)}{f'(\alpha_k)} \cdot \frac{1}{x-\alpha_k} \quad (k=0, 1, 2, \ldots, p-1)$$

$$= \Sigma\Sigma \frac{g(\alpha_k)}{f'(\alpha_k)} \cdot \frac{\alpha_k^n}{x^{n+1}},$$

$n=0, 1, 2, \ldots; k=0, 1, 2, \ldots, p-1.$

Thus, the Hankel matrix C corresponding to the function

$$R(x) = \Sigma A_k/x-\alpha_k \quad (k=0, 1, 2, \ldots, p-1),$$

may be determined by the formula

$$(3.2.11) \qquad C = [\Sigma \frac{g(\alpha_k)}{f'(\alpha_k)} \cdot \alpha_k^{i+j}]_0^\infty .$$

Since we have the correspondence

$$\frac{1}{x-\alpha} \sim [\alpha^{i+j}]_0^\infty = C_\alpha ,$$

and if it is differentiated h-1 times term by term, by the properties (3.2-9) and (3.2-10) the following correspondence is obtained:

$$\frac{1}{(x-\alpha)^{h}} \sim \frac{1}{(h-1)!} \cdot \frac{\partial^{h-1} C_{\alpha}}{\partial \alpha^{h-1}} ; \frac{\partial C}{\partial \alpha} = \left[\frac{\partial c_{i+j}}{\partial \alpha} \right]_{0}^{\infty}$$

Thus, in general case, we obtain

$$R(x) \sim C$$

$$= \sum_{k=0}^{p-1} \left(A_1^{(k)} + A_2^{(k)} \frac{\partial}{\partial \alpha_k} + \ldots + \frac{1}{(\eta_k - 1)!} A_{\eta_k}^{(k)} \frac{\partial^{\eta_k - 1}}{\partial \alpha_k^{\eta_k - 1}} \right) C_{\alpha_k}$$

By carrying out the differentiation, we obtain the Hankel form corresponding to the matrix C:

(3.2-12) $C(x,x) = \Sigma\, c_{i+j}\, x_i\, x_j \; (i,j=0, 1, 2, \ldots),$

with

(3.2-13) $$C = \left[\Sigma \left(A_1^{(k)} \alpha_k^{i+j} + A_2^{(k)} \frac{\partial \alpha_k^{i+j}}{\partial \alpha_k} + \ldots + \frac{1}{(\eta_k - 1)!} A_{\eta_k}^{(k)} \frac{\partial^{\eta_k - 1} \alpha_k^{i+j}}{\partial \alpha_k^{\eta_k - 1}} \right) \right]_0^{\infty}$$

$$= [c_{i+j}]_0^{\infty} \; .$$

3.3 HANKEL DETERMINANTS

As it is described in Section (3.2), if a real rational function R(x) is expressed in a series of descending powers of x:

$$R(x) = R_1(x) + \Sigma c_n / x^{n+1} \; (n=0, 1, 2, \ldots),$$

wherein $R_1(x)$ has no real poles, the coefficients $c_n (n=0, 1, 2, \ldots)$ determine the corresponding Hankel matrix

$$C = [c_{i+j}]_0^{\infty} \; ,$$

and the matrix C is of finite rank m; the number of poles of R(x) (multiplicities taken into account).

To determine the Cauchy index (See Sections (5.2; 5.3)) of R(x) we shall require the determinants $H_k^{(0)}$:

$$(3.3\text{-}1) \qquad H_k^{(0)} = C\begin{pmatrix} 0 & 1 & 2 & \dots & k-1 \\ 0 & 1 & 2 & \dots & k-1 \end{pmatrix}$$

$$= |c_{i+j}|_0^{k-1} \quad (k=1,\ 2,\ 3,\ \dots,\ m;\ H_0^{(0)} = 1):$$

the determinants of the leading sub-matrices of the Hankel matrix C.

In this section, we shall formulate a recurrence relation that may be used to obtain the determinants $H_k^{(n)}$ recursively.

For this purpose, we define in general, the Hankel determinants as follows:

For n=0, 1, 2, ... ; we set $H_0^{(n)} = 1$, and

$$(3.3\text{-}2) \qquad H_k^{(n)} = \begin{vmatrix} c_n & c_{n+1} & c_{n+2} & \dots & c_{n+k-1} \\ c_{n+1} & c_{n+2} & c_{n+3} & \dots & c_{n+k} \\ \cdot & \cdot & \cdot & \dots & \cdot \\ c_{n+k-1} & c_{n+k} & c_{n+k+1} & & c_{n+2k-2} \end{vmatrix} \quad (k=1,2,3,\dots).$$

SOME PROPERTIES OF HANKEL DETERMINANTS

Lemma 1.

For n=0, 1, 2, ...; and k=1, 2, 3, ..., the following relation holds:

$$(3.3\text{-}3) \qquad H_{k+1}^{(n-1)} H_{k-1}^{(n+1)} = H_k^{(n+1)} H_k^{(n-1)} - (H_k^{(n)})^2 .$$

Proof:

This relation has been known for a long time [4, 14, 20]. A verification for k=2 is sketched in [17 p.116]. In here, we carry out the verification for the validity of this formula modelled after the one given in [14, p. 120].

Let us consider the Hankel determinant:

$$(3.3\text{-}4) \quad H_{k+1}^{(n-1)} = \begin{vmatrix} c_{n-1} & c_n & c_{n+1} & \cdots & c_{n+k-1} \\ c_n & c_{n+1} & c_{n+2} & \cdots & c_{n+k} \\ c_{n+1} & c_{n+2} & c_{n+2} & \cdots & c_{n+k+1} \\ . & . & . & \cdots & . \\ c_{n+k-1} & c_{n+k} & c_{n+k+1} & \cdots & c_{n+2k-1} \end{vmatrix} .$$

Since $H_k^{(n-1)}$, $H_k^{(n+1)}$, and $H_k^{(n)}$ are the minors of the elements c_{n+2k-1}, c_{n-1} and c_{n+k-1} respectively in (3.3-4); $H_{k-1}^{(n+1)}$ is the (k-1) - rowed minor obtained from (3.3.4) by deleting the first and last rows and the first and last columns, we have by Corollary (17.4) [5, p.45],

$$(3.3\text{-}5) \qquad H_{k+1}^{(n-1)} \cdot H_{k-1}^{(n+1)} = \begin{vmatrix} H_k^{(n+1)} & H_k^{(n)} \\ H_k^{(n)} & H_k^{(n-1)} \end{vmatrix}$$

The equality (3.3-5) precisely establishes the recurrence formula in (3.3-3).

~ ~

If we assume that the coefficient c_n with $n < 0$ are zero, we may define the Hankel determinants $H_k^{(n)}$ of the coefficients of a power series $\Sigma\, c_n/x^{n+1}$ of a rational function R(x) also for negative n without destroying the validity of formula (3.3-3). Thus, if we set

$$H_1^{(n)} = \begin{cases} c_n & \text{for all } n \geq 0 \\ 0 & \text{for all } n < 0 \end{cases} ,$$

the following result is valid.

COROLLARY (3.1)

For all n and non-negative k,

$$H_{k+2}^{(n-1)} H_k^{(n+1)} = H_{k+1}^{(n-1)} H_{k+1}^{(n+1)} - (H_{k+1}^{(n)})^2 , \tag{3.3-6}$$

wherein $H_k^{(-n)} = 0$ for all $n \geq k$.

In the proceeding sections we have seen that an infinite Hankel matrix may be constructed corresponding to a real rational function R(x) from the real numbers α and A, and the determinants $H_k^{(n)}$ are connected by the recurrence formula (3.3-3). In the next Chapter we shall show how the formula (3.3-3) can be used in different ways to obtain the determinants $H_k^{(n)}$ recursively, and how one can obtain the determinants $H_k^{(n)}$ directly from the coefficients of the numerator and denominator of $R(x)$.

~ ~ ~

CHAPTER IV

THE PROGRESSIVE FORM OF AITKEN'S METHOD

Aitken recommended ([1, p.293]; [2, p.81]) two different procedures, making use of the formula (3.3-3), to obtain the Hankel determinants $H_k^{(n)}$ (for which he used a different notation). There is, however, a difficulty in obtaining the starting values for one procedure whereas other is likely to be unstable. In Section (4.2), we shall show how a modified form of Aitken's method (referred to as the progressive form of Aitken's method) is developed that overcomes these difficulties and entirely bypasses the numerical computation of all the determinants. Later, we shall establish some properties of the polynomials $H_k^{(n)}(\lambda)$ (that can be built up from certain determinants of Hankel's type), which are analogous to those of the Hankel determinants discussed in Chapter III.

4.1 AITKEN'S SCHEME

Aitken remarked [1, p.293] that the formula (3.3-3) may be used to generate the determinants $H_k^{(n)}$ recursively. This may easily be seen if we arrange the determinants in a double-entry table [15, p.29]:

$$
\begin{array}{lccccc}
 & 1 & & & & \\
 & 1 & H_1^{(0)} & & & \\
 & 1 & H_1^{(1)} & H_2^{(0)} & & \\
(4.1\text{-}1) & 1 & H_1^{(2)} & H_2^{(1)} & H_3^{(0)} & \\
 & 1 & H_1^{(3)} & H_2^{(2)} & H_3^{(1)} & H_4^{(0)} \\
 & \vdots & \vdots & \vdots & \vdots & \vdots \quad \ddots
\end{array}
$$

The procedure used in arranging the determinants $H_k^{(n)}$ in double-entry table (4.1-1) is referred to as Aitken's method. To carry out the scheme in (4.1-1), he suggested that one may use the formula (3.3-3) in two different ways:

(1) One may start with the first two columns (noting that $H_1^{(n)} = c_n$) and work from the left to the right, adding one new column at a time [1, p. 293].

(2) Secondly, if the first two diagonals of the scheme are known, one may solve for the "lowest" term in (3.3-3) and add a new diagonal, thus working from the top to the bottom. There is, however, a difficulty of obtaining the first two diagonals in this "progressive" form of his scheme. Aitken [1] has indicated how to obtain these starting values for the determination of the roots of a polynomial.

4.2 PROGRESSIVE FORM OF AITKEN'S METHOD

The procedure for the new scheme is developed as follows.

Suppose given a real rational function

$$R(x) = \frac{g(x)}{f(x)} = \Sigma\, c_n/x^{n+1} \quad (n=0,\ 1,\ 2,\ \ldots)$$

$$(4.2\text{-}1) \qquad = \frac{b_0\, x^{m-1} + b_1\, x^{m-2} + \ldots + b_{m-1}}{a_0\, x^m + a_1\, x^{m-1} + \ldots + a_m} \quad (a_0 \neq 0;\ a_m \neq 0).$$

By Theorem (3.2-1), the Hankel matrix

$$C = [c_{i+j}]_0^{\infty}$$

is of finite rank and in this case, the least degree m of the denominator $f(x)$ is the rank of C; and this implies $H_m^{(0)} \neq 0$.

Moreover, by (3.2-3; 3.2-3') we have the relations:

$$(4.2\text{-}2) \quad \begin{aligned} b_0 &= a_0 c_0 \\ b_1 &= a_0 c_1 + a_1 c_0 \\ b_2 &= a_0 c_2 + a_1 c_1 + a_2 c_0 \\ &\cdots\cdots\cdots \\ b_{m-1} &= a_0 c_{m-1} + \cdots + a_{m-1} c_0 , \end{aligned}$$

and

$$(4.2\text{-}2') \quad 0 = a_0 c_q + a_1 c_{q-1} + a_m c_{q-m} \quad (q = m, m+1, \ldots).$$

Since $a_0 \neq 0$, the coefficients c_n are determined by the formula (See [14, p. 136]):

$$(4.2\text{-}3) \quad c_n = \frac{1}{(-1)^n a_0^{n+1}} \begin{vmatrix} b_0 & a_0 & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ b_1 & a_1 & a_0 & 0 & \cdot & \cdot & \cdot & 0 \\ b_2 & a_2 & a_1 & a_0 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & a_0 \\ b_n & a_n & a_{n-1} & a_{n-2} & \cdot & \cdot & \cdot & a_1 \end{vmatrix},$$

$n = 0, 1, 2, \ldots$; $a_{k+1} = b_k = 0$ for all $k \geq m$.

On the other hand, if we write the expansion of $F(x) = \frac{1}{R(x)}$ in a series of descending power of x:

$$(4.2\text{-}4) \qquad F(x) = \frac{f(x)}{g(x)} = d_{-2}x + d_{-1} + \frac{d_0}{x} + \frac{d_1}{x_2} + \ldots,$$

the coefficients d_n may be obtained by the formula that is analogous to one in (4.2-3): i.e.,

$$(4.2\text{-}5) \qquad d_{k-2} = \frac{1}{(-1)^k b_0^{k+1}} \begin{vmatrix} a_0 & b_0 & 0 & 0 & \ldots & 0 \\ a_1 & b_1 & b_0 & 0 & \ldots & 0 \\ a_2 & b_2 & b_1 & b_0 & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot & \ldots & b_0 \\ a_k & b_k & b_{k-1} & b_{k-2} & \ldots & b_1 \end{vmatrix} \quad (k=0, 1, 2, \ldots),$$

where $b_0 \neq 0$; $b_k = 0$ for all $k > m-1$ (the least degree of the denominator $g(x)$ in (4.2-4)).

But, by (4.2-1); (4.2-4),

$$(4.2\text{-}7) \qquad F(x) = \frac{1}{\Sigma c_n/x^{n+1}} \quad (n=0, 1, 2, \ldots)$$

$$= d_{-2}x + d_{-1} + \frac{d_0}{x} + \frac{d_1}{x^2} + \ldots;$$

computing the coefficients d_n, we find that

$$(4.2\text{-}8) \qquad d_{k-2} = \frac{1}{(-1)^k c_0^{k+1}} \begin{vmatrix} c_1 & c_0 & 0 & \ldots & 0 \\ c_2 & c_1 & c_0 & \ldots & 0 \\ c_3 & c_2 & c_1 & \ldots & \cdot \\ \cdot & \cdot & \cdot & \ldots & \cdot \\ \cdot & \cdot & \cdot & \ldots & \cdot \\ c_{k-1} & c_{k-2} & c_{k-3} & \ldots & c_0 \\ c_k & c_{k-1} & c_{k-2} & \ldots & c_1 \end{vmatrix},$$

where $k = 1, 2, \ldots$; $c_0 \neq 0$, and $d_{-2} = c_0^{-1}$.

In the notation of Hankel determinants, the formula (4.2-8) turns out to be

$$(4.2\text{-}8') \qquad d_{k-2} = \frac{(-1)^{\frac{1}{2}k(k-1)}}{(-1)^k c_0^{k+1}} \cdot H_k^{(-k+2)} \quad (k=0, 1, 2, \ldots),$$

wherein it is assumed that coefficients c_n are zero for negative n.

From the equalities (4.2-5) and (4.2-8') we obtain that

$$(4.2\text{-}9) \qquad H_k^{(-k+2)} = \frac{c_0^{k+1}}{(-1)^{\frac{1}{2}k(k-1)} \cdot b_0^{k+1}} \cdot \Delta_k ,$$

where

$$(4.2\text{-}10) \quad \Delta_k = \begin{vmatrix} a_0 & b_0 & 0 & 0 & \cdots & \\ a_1 & b_1 & b_0 & 0 & \cdots & \\ a_2 & b_2 & b_1 & b_0 & \cdots & \\ \cdot & \cdot & \cdot & \cdot & \cdots & \\ \cdot & \cdot & \cdot & \cdot & \cdots & \\ \cdot & \cdot & \cdot & \cdot & \cdots & b_0 \\ a_k & b_k & b_{k-1} & \cdot & \cdot & b_1 \end{vmatrix} ; \; b_k=0 \text{ for all } k > m-1$$

Noting that $b_0 = a_0 c_0$, we find that (4.2-9) is as follows:

$$(4.2\text{-}11) \qquad H_k^{(-k+2)} = \frac{\Delta_k}{(-1)^{\frac{1}{2}k(k-1)} a_0^{k+1}} \quad (k=0, 1, 2, \ldots).$$

Clearly, since coefficients c_n are zero for negative n,

$$(4.2\text{-}12)\qquad \begin{aligned} H_k^{(-k+1)} &= (-1)^{\frac{1}{2}k(k-1)} \cdot c_0^k \; ; \\ H_k^{(-n)} &= 0 \qquad (n \geq k). \end{aligned}$$

1. From (4.2-11) and (4.2-12)(noting $H_1^{(n)} = c_n$; $H_0^{(n)} = 1$) one can compute the first three rows for the new scheme directly from the coefficients of the numerator and denominator of the function R(x). Thus, working from the top to the bottom, one can add a new row at a time.

This method (a modification of Aitken's methods corresponding to (1) and (2) above) has the advantage of yielding all rows of the scheme simultaneously. It is of interest to note for numerical applications that the modified form of Aitken's method appears to be stable. This assertion of stability is, however, based only on numerical rather than analytical evidence.

If the Hankel determinants are arranged in a double-entry table analogous to Aitken scheme (4.1-1), the scheme corresponding to the progressive form of Aitken's method can be visualized as a "progressive form of Aitken's scheme" as follows:

$$
(4.2\text{-}13) \qquad
\begin{array}{cccccccc}
1 & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\
1 & H_1^{(0)} & H_2^{(-1)} & H_3^{(-2)} & H_4^{(-3)} & \cdot & \cdot & \cdot \\
1 & H_1^{(1)} & H_2^{(0)} & H_3^{(-1)} & H_4^{(-2)} & \cdot & \cdot & \cdot \\
1 & H_1^{(2)} & H_2^{(1)} & H_3^{(0)} & H_4^{(-1)} & \cdot & \cdot & \cdot \\
1 & H_1^{(3)} & H_2^{(2)} & H_3^{(1)} & H_4^{(0)} & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}
$$

4.3 THE POLYNOMIAL $H_k^{(n)}(\lambda)$

The set of polynomials $\{H_k^{(0)}(\lambda)$: $k=0, 1, 2, \ldots; H_0^{(0)}(\lambda) = 1\}$ will advantageously be used for the determination of the "Cauchy index" in a finite interval (see Section (5.3)). Some properties of the polynomials $H_k^{(n)}(\lambda)$ $(n=0, 1, 2, \ldots; k=1, 2, 3, \ldots)$, λ is a real parameter, which will be required for this purpose are discussed in this Section. For $n=0, 1, 2, \ldots$, we define the polynomials in question by

$$
(4.3\text{-}1) \qquad H_0^{(n)}(\lambda) = 1 \quad ,
$$

$$(4.3\text{-}2)\quad H_k^{(n)}(\lambda) = \begin{vmatrix} c_n & c_{n+1} & c_{n+2} & \cdots & c_{n+k} \\ c_{n+1} & c_{n+2} & c_{n+3} & \cdots & c_{n+k+1} \\ . & . & . & \cdots & . \\ . & . & . & \cdots & . \\ . & . & . & \cdots & . \\ c_{n+k-1} & c_{n+k} & c_{n+k+1} & \cdots & c_{n+2k-1} \\ 1 & \lambda & \lambda^2 & \cdots & \lambda^k \end{vmatrix},$$

$k = 1, 2, 3, \ldots.$

SOME PROPERTIES OF THE POLYNOMIALS $H_k^{(n)}(\lambda)$

If we agree to consider $c_n = 0$ for negative n, we note two immediate consequences of (4.3-2):

1. When $\lambda = 0$

$$(4.3\text{-}3)\qquad H_k^{(n)}(\lambda) = (-1)^k H_k^{(n+1)} \quad \text{for all } n; k=1,2,3,\ldots.$$

2. When $\lambda \neq 0$,

$$(4.3\text{-}4)\qquad H_k^{(-n)}(\lambda) = \begin{cases} (-1)^{k+1} c_0^k & \text{for } n = k \\ 0 & \text{for } n > k, \end{cases}$$

$n=0, 1, 2, \ldots; k=1, 2, 3, \ldots$.

LEMMA 1

For all n and $k=1, 2, 3, \ldots$, the following relation holds:

$$(4.3\text{-}5)\qquad H_{k-1}^{(n+1)} H_k^{(n-1)}(\lambda) - \lambda H_k^{(n-1)} H_{k-1}^{(n+1)}(\lambda) + H_k^{(n)} H_{k-1}^{(n)}(\lambda) = 0.$$

This property of the polynomials $H_k^{(n)}(\lambda)$ is analogous to that of the Hankel determinants established in Section (3.3).

Proof:

The proof of this Lemma is by the same method as the proof of Lemma 1 in Section (3.3).

Let us consider the determinant

$$H_k^{(n-1)}(\lambda) = \begin{vmatrix} c_{n-1} & c_n & c_{n+1} & \cdots & c_{n+k-1} \\ c_n & c_{n+1} & c_{n+2} & \cdots & c_{n+k} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ c_{n+k-2} & c_{n+k-1} & c_{n+k} & \cdots & c_{n+2k-2} \\ 1 & \lambda & \lambda^2 & \cdots & \lambda^k \end{vmatrix} .$$

In analogy to (3.3-5) we then can write

$$H_{k-1}^{(n+1)} H_k^{(n-1)}(\lambda) = \begin{vmatrix} \lambda H_{k-1}^{(n+1)}(\lambda) & H_{k-1}^{(n)}(\lambda) \\ H_k^{(n)} & H_k^{(n-1)} \end{vmatrix} ;$$

and this implies the recurrence formula (4.3-5).

~ ~ ~

We shall now explain certain connections between infinite Hankel matrices (having the polynomials $H_k^{(n)}(\lambda)$ as the minors), and rational functions.

To this end we define a rational function $F(x)$ by

(4.3-6) $$F(x) = (\lambda - x)\, R(x)$$

where (see Section 3.2))

$$R(x) = \frac{g(x)}{f(x)} = \frac{b_0x^{m-1}+b_1x^{m-2}+\ldots+b_{m-1}}{a_0x^m+a_1x^{m-1}+\ldots+a_m}$$

$$= \Sigma\, c_n/x^{n+1}. \quad (n=0,1,2,\ldots).$$

Then

(4.3-7) $$F(x) = -c_0 + \Sigma(c_n\lambda - c_{n+1})/x^{n+1} \ (n=0,1,2,\ldots).$$

The sequence of numbers $\{c_{i+j}\lambda - c_{i+j+1}: i,j=0,1,2,\ldots\}$ determines an infinite Hankel matrix:

(4.3-8) $$C(\lambda) = [c_{i+j}\lambda - c_{i+j+1}]_0^\infty ,$$

and we, therefore, have the relation (see Section (3.2))

$$F(x) \sim C(\lambda) .$$

The matrix $C(\lambda)$ is of rank m, since m, being the degree of $f(x)$, is equal to the poles of $F(x)$ (multiplicities taken in account).

Thus, we have the following properties of the matrices $C(\lambda)$ that parallel the properties of the Hankel matrices discussed in Theorem (3.1-1), Corollary (3.1-2), and Theorem (3.2-1) respectively.

LEMMA 2

The infinite Hankel matrix

$$C(\lambda) = [c_{i+j}\lambda - c_{i+j+1}]_0^\infty$$

is of finite rank if there exists an integer r and number $\gamma_1, \gamma_2, \ldots, \gamma_r$, such that

$$c_q\lambda - c_{q+1} = \sum_{g=1}^{r} \gamma_g (c_{q-g}\lambda - c_{q-g+1}) \quad (q=r, r+1, \ldots),$$

and r is the least number having this property.

COROLLARY (4.3-1)

If the infinite Hankel matrix $C(\lambda)$ is of finite rank r, then

$$H_r^{(0)}(\lambda) \neq 0.$$

LEMMA 3

The matrix

$$C(\lambda) = [c_{i+j}\lambda - c_{i+j+1}]_0^\infty$$

is of finite rank if and only if the sum of the series

$$F(x) = (c_{-1}\lambda - c_0) + \frac{c_0\lambda - c_1}{x} + \frac{c_1\lambda - c_2}{x^2} + \ldots \quad (c_{-1}=0)$$

is a rational function of x. In this case the rank of $C(\lambda)$ is equal to the number of poles of $F(x)$, counting each pole with its proper multiplicity.

REAL HANKEL FORMS ASSOCIATED WITH MATRICES $C(\lambda)$

In accordance with (3.1-2), a real Hankel form associated with a real matrix $C(\lambda)$ is

$$(4.3\text{-}9)\quad C(x,\lambda,x) = \Sigma(c_{i+j}\lambda - c_{i+j+1})\, x_i x_j \;(i,j=0,\,1,\,2,\,\ldots),$$

$$= x^T C(\lambda)x,$$

wherein x denotes the single-column matrix with elements x_i (i=0, 1, 2, . . .).

We note a few theorems for the forms $C(x,\lambda,x)$:

THEOREM (4.3-2)

Every form $C(x,\lambda,x)$ of finite rank r may be changed by a linear transformation of variables into a canonical form (Jacobi's method).

THEOREM (4.3-3) (Law of Inertia)

In a canonical representation of a form $C(x,\lambda,x)$ of finite rank, the number of positive and the number of negative squares are independent of the choice of the representation.

THEOREM (4.3-4) (Jacobi's Theorem)

If for the form $C(x,\lambda,x)$ of finite rank r the inequality

$$H_k^{(0)}(\lambda) \neq 0 \;(k=1,\,2,\,3,\,.\,.\,.\,,\,r)$$

holds, then μ, υ, and σ are determined by the formulas:

$$\mu = P(H_0^{(0)}(\lambda),\, H_1^{(0)}(\lambda),\, .\,.\,.\,,H_r^{(0)}(\lambda)),$$

$$(4.3\text{-}10)\quad \upsilon = V(H_0^{(0)}(\lambda),\, H_1^{(0)}(\lambda),\, .\,.\,.\,,H_r^{(0)}(\lambda)),$$

and

$$\sigma = r-2V(H_0^{(0)}(\lambda),\, H_1^{(0)}(\lambda),\, .\,.\,.\,,\, H_r^{(0)}(\lambda)).$$

Proofs of these theorems are by the same method as the proofs of the analogous theorems for quadratic forms.

THE PROGRESSIVE FORM OF AITKEN'S METHOD FOR THE POLYNOMIALS $H_k^{(n)}(\lambda)$

For the similar reasons to those that led us to establish the Progressive form of Aitken's method (Section (4.2)), we do not deal again directly with Aitken's method (corresponding to (1) and (2) Section 4.1) to compute the polynomials $H_k^{(n)}(\lambda)$, but with another modified form of his method.

The recurrence formula (4.3-5) is practically significant for the computation of the modified form of Aitken's scheme of the polynomials in question for a rational function of the form (4.3-6).

To motivate the "progressive form" of the method, assume given a rational function (4.3-6):

$$F(x) = (\lambda - x)\, R(x).$$

By the procedure that is analogous to the one explained in Section (4.2) we then can write:

$$(4.3\text{-}11) \qquad H_k^{(-k+1)}(\lambda) = \frac{\Delta_k(\lambda)}{(-1)^{\frac{1}{2}k(k-1)}\, a_0^{k+1}} \quad (k=0,\ 1,\ 2,\ \ldots)$$

where

$$\Delta_k(\lambda) = \begin{vmatrix} a_0 & b_{-1}\lambda-b_0 & 0 & 0 \cdots \\ a_1 & b_0\lambda-b_1 & b_{-1}\lambda-b_0 & 0 \cdots \\ a_2 & b_1\lambda-b_2 & b_0\lambda-b_1 & \cdots \\ \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdots 0 \\ \cdot & \cdot & \cdot & \cdots b_{-1}\lambda-b_0 \\ a_k & b_{k-1}\lambda-b_k & b_{k-2}\lambda-b_{k-1} & \cdots b_0\lambda-b_1 \end{vmatrix};$$

$b_{-1} = 0$ and $b_k = 0$ for all $k \geq m$, and $a_{k+1} = 0$ for all $k \geq m$.

Thus:

2. If use is made (4.3-3), (4.3-4), and (4.3-11), it is obvious that by the recurrence relation (4.3-5) the complete progressive form of Aitken scheme for the polynomials $H_k^{(n)}(\lambda)$ can be built up working from the top to the bottom, adding a new row at a time. As a result the polynomials $H_k^{(n)}(\lambda)$ are generated progressively, i.e. starting from first two rows and working in the downward direction.

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CHAPTER V

DETERMINATION OF THE CAUCHY INDEX OF A REAL RATIONAL FUNCTION

We now proceed to formulate a method to determine the Cauchy index of a real rational function $R(x)$ between the limits a and b (Notation : $I_a^b R(x)$; a and b are real numbers or $\pm\infty$; $a < b$). Its application to the problem of the isolation of roots of algebraic equations will be discussed in section (5.4).

5.1 CAUCHY INDICES

A real rational function $R(x)$, satisfying the equation

(5.1-1) $$\frac{1}{R(x)} = 0 \text{ for a value } \alpha_0 \text{ of } x \; (a < x < b),$$

takes an infinite value at α_0 and there it either

(i) changes its sign from negative to positive,

or (ii) changes its sign from positive to negative,

or (iii) does not change its sign.

The quantity +1 in the first case, -1 in the second, and 0 in the third case, is called the index of $R(x)$ for α_0.

Definition:

We then define the Cauchy Index of $R(x)$ between the limits a and b: [6, p. 176]

(5.1-2) $$I_a^b R(x) = \text{sum of all the indices of } R(x)$$

corresponding to the different roots α_k $(a < \alpha_k < b)$ of the equation (5.1-1).

Since an arbitrary real rational function R(x) can always be expressed as in (3.2-7), in accordance with the definition (5.1-2),

$$(5.1\text{-}3) \qquad I_{-\infty}^{+\infty} R(x) = \sum_{(\eta_k \text{ odd})} \text{sign } A_{\eta_k}^{(k)} ,$$

and, in general,

$$(5.1\text{-}4) \qquad I_a^b R(x) = \sum_{\substack{(a<\alpha_k<b) \\ \eta_k \text{ odd}}} \text{sign } A_{\eta_k}^{(k)} .$$

(Here, sign A_{η_k} means +1, -1, or zero according as $A_{\eta_k} > 0$, $A_{\eta_k} < 0$, or $A_{\eta_k} = 0$).

In particular, let us assume that f(x) is a real polynomial:

$$f(x) = a_0(x-\alpha_0)^{\eta_0} (x-\alpha_1)^{\eta_1} \ldots (x-\alpha_{m-1})^{\eta_{m-1}} ,$$

$\alpha_i \neq \alpha_j$ for $i \neq j$; $i,j = 0, 1, 2, \ldots, m-1$.

If among its roots only the first r are real, then

$$\frac{f'(x)}{f(x)} = \sum \frac{\eta_k}{x-\alpha_k} + R_1(x) \qquad (k=0,1, 2, \ldots,r-1),$$

where $R_1(x)$ is real rational without real poles.

Therefore, according to (5.1-4):

$$I_a^b \frac{f'(x)}{f(x)} = \Sigma \text{ sign } \eta_k \quad (k=0,\ 1,\ 2,\ \ldots,\ r-1) \quad ,$$

= the number of distinct real zeros of the polynomial $f(x)$ in the interval (a,b).

BASIC PROPERTIES OF CAUCHY INDICES

We note a few properties of the Cauchy index $I_a^b R(x)$ that are needed in the sequel.

1. $I_a^b R(x) = - I_b^a R(x)$.

2. If $R(x) = R_1(x)\ R_2(x)$, where $R_1(x)$ and $R_2(x)$ are real rational functions of x, then

$$I_a^b R(x) = \text{sign } R_1(x) \quad I_a^b \ R_2(x) \ ,$$

provided $R_1(x) \neq 0, \infty$ within the interval (a,b).

3. If $a < c < b$, then

$$I_a^b R(x) = I_a^c R(x) + I_c^b R(x) + \lambda_c \ ,$$

where $\lambda_c = 0$ if $R(c)$ is finite and $\lambda_c = \pm 1$ if $R(x)$ becomes infinite at c; in this case $\lambda_c = + 1$ corresponds to a jump from $-\infty$ to $+\infty$ at c and $\lambda_c = -1$ to a jump from $+\infty$ to $-\infty$.

4. If $R(-x) = -R(x)$, then $I_{-a}^{o} R(x) = I_o^a \ R(x)$; and if $R(-x) = R(x)$, then $I_{-a}^{o} R(x) = - I_o^a R(x)$.

These properties follow immediately from the definition of the Cauchy index (see definition: (5.1-2)).

5.2 DETERMINATION OF THE INDEX $I_{-\infty}^{+\infty} R(x)$

In Chapter III, (Section (3.2)) we established a one-to-one correspondence between proper real rational functions $R(x)$ and the Hankel matrices $C = [c_{i+j}]_0^\infty$ of finite rank; moreover, Jacobi's Theorem enabled us to formulate a rule to determine the signature of real Hankel forms $C(x,x)$ (Theorem 3: (3.1-8)). We are thus in a position to establish the fundamental theorem: [16,18,19]:

THEOREM (5.2-1)

If

$$R(x) \sim C$$

and r is the rank of C, the Cauchy index $I_{-\infty}^{+\infty} R(x)$ is equal to the signature of the form $C(x,x)$.

Proof:

Assume that the expansion (3.2-7) holds. Then, by (3.2-9) the corresponding matrix C is determined by the formula (3.2-13).

The Hankel form $C(x,x)$ associated with the matrix C is

$$C(x,x) = \Sigma\, c_{i+j}\, x_i\, x_j \quad (i,j=0,\ 1,\ 2,\ \ldots), \tag{5.2-1}$$

where c_{i+j} is the element in the ith row and jth column of the matrix C in (3.2-13).

We write the form (5.2-1):

$$C(x,x) = \sum_{k=0}^{p-1} T_{\alpha_k}\ (x,x), \tag{5.2-2}$$

where

$$T_{\alpha_k} = \left(A_1^{(k)} + A_2^{(k)} \frac{\partial}{\partial \alpha_k} + \dots + \frac{1}{(\eta_k - 1)!} A_{\eta_k}^{(k)} \frac{\partial^{\eta_k - 1}}{\partial \alpha_k^{\eta_k - 1}} \right) C_{\alpha_k} ,$$

$(C_{\alpha_k} = [\alpha_k^{i+j}]_0^{\infty}$; $k = 0, 1, 2, \dots, p-1)$.

By Theorem (3.2-1) the rank of the form $T_{\alpha_k}(x,x)$ is η_k $(k=0, 1, 2, \dots, p-1)$, and the rank of $C(x,x)$ is then

$$(5.2\text{-}3) \qquad r = \Sigma\, \eta_k \;(k=0, 1, 2, \dots, p-1).$$

But if the rank is equal to the sum of the ranks of the constituents forms, then the same relation holds for the signatures:

$$(5.2\text{-}4) \qquad \sigma\,\{C(x,x)\} = \sum_{\alpha_k \text{ real}} \sigma\, T_{\alpha_k}(x,x) \quad .$$

Since α_k $(k=0, 1, 2, \dots, p-1)$ are real, we obtain [13, p.211]

$$(5.2\text{-}5) \qquad \sigma\{T_{\alpha_k}(x,x)\} = \begin{cases} 0 & \text{for } \eta_k \text{ even} \\ \text{sign } A_{\eta_k}^{(k)} & \text{for } \eta_k \text{ odd} \end{cases}$$

From the qualities (5.2-4) and (5.2-5) it follows that

$$(5.2\text{-}6) \qquad \sigma\{C(x,x)\} = \sum_{\begin{pmatrix}\alpha_j & \text{real} \\ \eta_k & \text{odd}\end{pmatrix}} \text{Sign } A^{(j)}_{\eta_k} .$$

By (5.1-3) the sum on the right-hand of (5.2-7) is

$I^{+\infty}_{-\infty} R(x)$.

Hence:

$$I^{+\infty}_{-\infty} R(x) = \sigma\{C(x,x)\}$$

$$(5.2\text{-}7) \qquad = r - 2V(H_0^{(0)}, H_1^{(0)}, H_2^{(0)}, \ldots, H_r^{(0)}),$$

where $H_0^{(0)} = 1$.

5.3 DETERMINATION OF THE CAUCHY INDEX IN A FINITE INTERVAL

Let the signature of the real Hankel form $C(x,\lambda,x)$, the number of positive and the number of negative squares in its canonical form be denoted by σ_λ, μ_λ, and υ_λ respectively.

Then

$$(5.3\text{-}1) \qquad \sigma_\lambda = \mu_\lambda - \upsilon_\lambda$$ (See definition p.25).

Since $F(x) = (\lambda - x)\, R(x) \sim C(\lambda)$, in analogy to (5.2-8) we then can write

(5.3-2) $$\sigma_\lambda = I_{-\infty}^{+\infty} \; F(x)$$

$$= I_{-\infty}^{+\infty} \; (\lambda - x) \; R(x)$$

$$= I_{-\infty}^{\lambda} \; (\lambda - x) \; R(x) + I_{\lambda}^{+\infty} (\lambda - x) R(x) \quad (-\infty < \lambda < +\infty).$$

Thus, for $a < b$,

$$\sigma_a = I_{-\infty}^{a} (a-x) R(x) + I_{a}^{+\infty} (a-x) \; R(x)$$

$$= I_{-\infty}^{a} R(x) - I_{a}^{b} R(x) - I_{b}^{+\infty} R(x) \; ;$$

and

$$\sigma_b = I_{-\infty}^{b} (b-x) R(x) + I_{b}^{+\infty} (b-x) \; R(x)$$

$$= I_{-\infty}^{a} R(x) + I_{a}^{b} R(x) - I_{b}^{\infty} R(x) \; ,$$

wherefrom

(5.3-3) $$\tfrac{1}{2} (\sigma_b - \sigma_a) = I_{a}^{b} R(x) \; ;$$

and this implies (See Jacobi's Theorem).

(5.3-4) $$I_{a}^{b} R(x) = V(a) - V(b)$$

where

$$V(\lambda) = V(H_0^{(0)}(\lambda), H_1^{(0)}(\lambda), \ldots H_r^{(0)}(\lambda)) \; ,$$

and r is the least degree of the denominator of $R(x)$.

Thus we are led to the following fundamental result:

THEOREM (5.3-1)

If

$$F(x) = (\lambda - x)\, R(x) \sim C(\lambda)$$

and $C(\lambda)$ is of finite rank r, then the Cauchy index of R(x) in a finite interval (a,b) is determined by the formula

$$I_a^b\, R(x) = V(a) - V(b)$$

where $V(\lambda) = V(H_0^{(0)}(\lambda), H_1^{(0)}(\lambda), \ldots, H_r^{(0)}(\lambda)$; λ is a real parameter.

5.4 ISOLATION OF ROOTS OF ALGEBRAIC EQUATIONS

To isolate the roots of an algebraic equation

$$f(x) = a_0x^m + a_1x^{m-1} + \ldots + a_m = 0 ,$$

it is usual to find first the bounds of regions in which all roots are contained. There are various ways to this end [3 p. 75]. This region is then divided into smaller regions so that repeating the process one can find a partial region which does not contain more than one root. It will now be shown that the theory of Cauchy indices discussed in the preceeding Chapter can advantageously be used to carry out this process.

ISOLATION OF THE REAL ROOTS

Since the number of distinct real roots of $f(x) = 0$ in the interval (a,b), as we have seen, is precisely $I_a^b f'(x)/f(x)$; this number may then be obtained by the formula (5.3-4).

Thus, if N is this number,

$$N = I_a^b \frac{f'(x)}{f(x)} = V(a) - V(b) \quad (5.4\text{-}1)$$

By continuing the process of division for the interval (a,b) into smaller intervals, and if use is made of (5.4-1), one can arrive at the interval that contained only one root.

ISOLATION OF THE COMPLEX ROOTS

The concept of Cauchy index can also be used for the isolation of complex roots of a real polynomial,

$$f(x) = a_0x^m + a_1x^{m-1} + \dots + a_m \ .$$

The usual procedure is to consider a complex function f(z) and investigate the number of its complex roots which are contained in various regions of the z-plane.

To achieve this end, first we seek the connection of the Cauchy index with the "Argument Principle".

Let us therefore consider the equation

$$\begin{aligned} f(z) &= a_0z^m + a_1z^{m-1} + \dots + a_m \\ &= a_0(z-z_0)(z-z_1)(z-z_2) \dots (z-z_{m-1}) \\ &= 0 \end{aligned} \quad (5.4\text{-}2)$$

If z be taken positively round a closed circuit in the z-plane which encloses k of the points $z_0, z_1, \dots, z_{m-1}$, the argument of f(z) will be increased by $2k\pi$. Consequently, the number of roots of $f(z) = 0$ which lie inside a given circuit can be ascertained by determining the change in the argument of $f(z)$ when z passes round the circuit.

Let z describe a contour consisting of

$$(5.4\text{-}3) \qquad \begin{aligned} x &= \varphi(t), \\ y &= \psi(t). \end{aligned} \quad (-\infty < t < +\infty)$$

Then

$$(5.4\text{-}4) \qquad \begin{aligned} f(t) &= P(t) + iQ(t) \\ &= R(t)\, e^{i\theta(t)} \end{aligned}$$

where $P(t)$ and $Q(t)$ are real function of t,

$R(t) = \sqrt{P^2(t) + Q^2(t)}$, and $\tan\,\theta(t) = \frac{Q(t)}{P(t)}$.

$R(t)$ and $\theta(t)$ are continuous functions of t as t changes from $-\infty$ to $+\infty$, whilst $\tan\,\theta(t) = \frac{Q(t)}{P(t)}$ has infinite discontinuities at points t_i which are the real roots of the polynomial $P(t)$. The discontinuities of $\frac{Q(t)}{P(t)}$ can have one of the following forms:

$$(5.4\text{-}5) \qquad \begin{aligned} &(1)\ \lim_{t \to t_i \mp 0} \frac{Q(t)}{P(t)} = +\infty \quad (2)\ \lim_{t \to t_i \mp 0} \frac{Q(t)}{P(t)} = -\infty \\ &(3)\ \lim_{t \to t_i \mp 0} \frac{Q(t)}{P(t)} = \pm\infty \quad (4)\ \lim_{t \to t_i \mp 0} \frac{Q(t)}{P(t)} = \mp\infty . \end{aligned}$$

To find the increase in $\theta(t)$ as t changes from $-\infty$ to $+\infty$, we consider all the real roots of the polynomial $P(t)$:

$$-\infty < t_0 < t_1 < \dots < t_{k-1} < +\infty$$

and consider the sequence of intervals:

$$(-\infty, t_0), (t_0, t_1), \dots, (t_{k-2}, t_{k-1}), (t_{k-1}, +\infty).$$

On each interval (t_{i-1}, t_i) $\arctan \frac{Q(t)}{P(t)} = \theta(t)$ is a continuous function. On passing through t_i the value of $\theta(t)$, (a), either remains in the same interval (in the event of a type (1) or type (2) discontinuity), or else leaves it. Here, (b) in the case of a type (3) discontinuity $\theta(t)$ increases on leaving, and (c) in the case of a type (4) discontinuity, $\theta(t)$ decreases.

Thus, the total variation on transition through t_i is $2\pi\eta_i$ where

$$\eta_i = \begin{cases} +1 & \text{for a type (3) discontinuity} \\ -1 & \text{for a type (4) discontinuity} \\ 0 & \text{for a type (1) or (2) discontinuity.} \end{cases}$$

But, in accordance with the definition p. 63 η_i exactly coincides with the index of $\frac{Q(t)}{P(t)}$ for t_i.

So, by ascribing to $\theta(t)$ the principal values in each interval and keeping the track of the change in $\theta(t)$ on passing through the points $t_i (i=0, 1, 2, \dots, k-1)$, we then can find the increase in $\theta(t)$ when t varies from $-\infty$ to $+\infty$, that is,

$$(5.4\text{-}6) \qquad \Delta_c\,(\arg f(z)) = 2\pi\Sigma\eta_i \quad (i=0,\ 1,\ 2,\ \ldots,\ k-1),$$

wherein c is defined by (5.4-3).

In analogy to (5.1-2), we thus can write

$$(5.4\text{-}7) \qquad \frac{1}{2\pi}\Delta_c\,(\arg f(z)) = \Sigma\eta_i$$

= the sum of all the indices of $\frac{Q(t)}{P(t)}$ corresponding to t_i $(i=0,\ 1,\ 2,\ \ldots,\ k-1)$

$$= I_{-\infty}^{+\infty} \frac{Q(t)}{P(t)}.$$

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