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WAVE BLOCKING PHENOMENON : A DYNAMICAL SYSTEMS APPROACH

By

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Les hommes ont toujours soupçonné qu'il devrait y avoir un domaine de questions dont les réponses seraient -a priori- symétriquement réunies dans une construction close et régulière.

Wittgenstein
Tractatus logico-philosophicus
5.4541

Abstract

Traveling waves are an intricate part of many physical phenomena such as electrical impulses in nervous tissue and cardiac muscle, many chemical reactions and the action potential in certain types of conductors. The problem of propagation failure of such waves due to various perturbations, also known as wave blocking, remains a relatively elusive concept that is nevertheless observed in experiments. Some model specific research has been conducted on the subject, most of which utilizes numerical tools. The object of this thesis is to adopt a symmetry based approach to obtain results that are much more general. We study the effect of a symmetry breaking perturbation acting on an ODE with translational symmetry, modeling a traveling wave. We show that for a large class of such systems, the wave blocking phenomenon can be explained by the presence of saddle-node bifurcations of solutions on a carefully chosen invariant curve. We achieve this by using symmetry reduction tools as well as properties of center manifolds.

Résumé

Les ondes progressives sont une partie intégrante de plusieurs phénomènes physiques comme des impulsions électriques dans les tissus nerveux et les muscles cardiaques, plusieurs réactions chimiques et le potentiel d'action dans certains types de conducteurs. Le problème de défaillance de propagation de ces ondes dû à certaines perturbations, aussi connu sous le nom de blocage, reste un concept relativement mal connu qui est néanmoins observé expérimentalement. Certaines recherches propres à des modèles précis furent menées sur ce sujet, la plupart utilisant des outils numériques. L'objet de cette thèse est d'adopter une approche basée sur la symétrie de tels phénomènes, pour ainsi obtenir des résultats plus généraux. Nous étudions les effets d'une perturbation brisant la symétrie de translation d'une EDO, représentant une onde progressive. Nous montrons que pour une vaste classe de tels systèmes, le phénomène de blocage d'ondes peut être expliqué par la présence de bifurcations "saddle-node" de certaines solutions sur une courbe invariante bien choisie. Pour ce faire, nous employons des outils de réduction symétrique ainsi que des propriétés de variétés du centre.

Keywords, Mots clés

traveling wave, wave blocking, bifurcation theory, differential equation, dynamical systems

onde progressive, blocage d'onde, théorie des bifurcations, équations différentielles, systèmes dynamiques

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Dedication

I dedicate this work to my parents. They supported me during my university studies in more ways that I could have imagined.

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Chapter 1

Introduction

In an effort to harvest the power of mathematics to understand the nature we live in, scientists often engage in the enterprise of mathematical modeling of physical phenomena. The object of this thesis is to study some attributes and intricacies surrounding the modeling of a phenomenon called a *traveling wave*. More precisely, we will study the pathological settings in which these *traveling waves* fail to propagate. We call this : *wave blocking*.

A *traveling wave* is any type of wave that propagates in time, through some medium, in a precise direction. There are several physical examples of such waves like the electrical impulses in nervous tissue, electrocardiac waves and many chemical reactions [10, 19, 14, 16]. We will visit more of these examples in pages to come. Needless to say, this phenomenon is worthy of interest for obvious reasons.

In this chapter, we will explore previous results and studies concerning *wave blocking*. We will discuss their applications and limitations in a brief review on the subject. In an effort to contribute to this literature, we will motivate and put in context the results of this thesis before engaging into its development.

Enjoy !

1.1 Literature Review

As mentioned above, the phenomenon of wave blocking has many physical occurrences. In essence, when a wave propagates in a given medium, it may encounter inhomogeneities that change the way it travels. It will sometimes be slowed down, and sometimes stall completely. When such a wave depicts a nervous impulse for example, the outcome of a perturbation like torn or ill tissue is of prime importance for the owner of the nervous system. The same thing is true for an electrocardiac wave that meets ill cells in the heart's muscle. It is these types of phenomena that mainly motivated researchers to study wave blocking, or *propagation failure* as it is also referred to.

1.1.1 The Gap Model

Most of the research done on wave blocking has been conducted on a model called the *Gap Model*. This is a relatively simple model used in many fields of research like population genetics [3], combustion theory [18] and multiple biological phenomena like the propagation of electrical waves in nervous or heart tissue [15, 10]. It is composed of a scalar, bistable, reaction-diffusion PDE defined as follows :

$$u_t = [d(x)u_x]_x + h(u, x) \quad (1.1)$$

where

$$d(x) = \begin{cases} D & ; 0 \leq x < L \\ 1 & ; else \end{cases}$$

and

$$h(u, x) = \begin{cases} g(u, x) & ; 0 \leq x < L \\ f(u) & ; else \end{cases}.$$

Here, $L > 0$ describes the width of the *gap* and $D > 0$ the strength of the diffusion term in the *gap*. The reaction terms f and g respectively represent the dynamics outside and inside the *gap*.

The presence of wave blocking (propagation failure) for solutions of this model has been studied by J.T. Lewis and J.P. Keener in [9] as well as Ikeda and Mimura in [8].

In [9], f is simplified as

$$f(u) = u(1 - u)(u - \alpha)$$

for $0 < \alpha < \frac{1}{2}$ but some broader conditions on f relative to α are described in order to generalize their analysis. Also, g is taken to be zero which represents the absence of excitability in the gap, leaving only the diffusion term u_{xx} to act on the dynamics.

Lewis and Keener begin by showing that there exists stable solutions to (1.1) representing traveling waves. They then show that a wave block is equivalent to a steady state solution (equilibrium). The problem is now translated to a bifurcation analysis with parameter L . In essence, they ask : for which value L will there exist steady state solutions ?

This problem is often compared to the propagation of a wave in a straight cable for which the diameter is constant except for a portion of length L where the diameter is different. Intuitively, one can see that a given wave would be slowed down by this change of diameter. In fact, in the occurrence of the wave persisting, it would do so with some delay as it goes through the gap. On the other hand, for a long enough L , the wave would be stopped. To better characterize this wave block, one would have to find the threshold value L^* above which propagation fails.

Lewis and Keener conducted multiple numerical experiments that show that there exists such a L^* for equation (1.1). They then engage in analytically showing the existence of the threshold L^* and also, that the appearance of steady state solutions happens via a *limit point* (or *saddle node*) bifurcation. They conduct this analysis for different values of α that appear in the simplified cubic term f . Since the reaction term is directly linked to f and we have control over $0 < \alpha < \frac{1}{2}$, it is only natural to inquire about the different L^* s for different α . The results are as one would suspect and the threshold value L^* augments as the overall excitability of the medium grows

(when $\alpha \rightarrow 0$).[9]

1.1.2 Other Research

Even if a great part of the research conducted on wave blocking focused on reaction-diffusion models similar to the gap model, there are some results on different types of systems. A good example are models of action potential. Goldstein and Rall [4] studied this type of system and the effect of different core geometries on conductors. Sharp and Joyner investigated the effects of inhomogeneities on the electrical action potential of cardiac cells [16]. Further exploration of this field would yield many articles on various specialized problems and we should note that we do not present an exhaustive review.

1.2 Motivation

Although there is an extensive literature on subjects relative to propagation failure, we should keep something in mind. Most of the research has been conducted with important numerical aspects, and on specific systems with a precise problem in mind. For this reason, the paper from Lewis and Keener [9] stands out with somewhat of a general flavored approach to this phenomenon. Indeed, the idea of conducting a bifurcation analysis to characterize wave blocking expands the spectrum of application of their result.

It is in this philosophy that the study presented in this thesis took inspiration. Furthermore, the setting in which we will present the results obtained will be considerably more general as this section aims at explaining. Indeed, the assumptions made on the equations we will study are as broad as possible, in order to capture a mechanism by which most of traveling waves would be blocked.

1.2.1 Parameters

Like Lewis and Keener, we associate the presence of wave blocking with that of equilibrium point solutions. These fixed points are likely to appear via bifurcation relative to carefully chosen parameters.

On the other hand, it would seem like there are some generalities concerning traveling waves that are left unexploited by the existing literature. Indeed, regardless of the choice of model describing a traveling wave, the following intuition remains. It is only natural to think that if a wave encounters a local perturbation, it will be affected by it, in the extent of failing to propagate. For the sake of clarity, let us consider the following analogy.

Suppose that M. Cantor is running on a straight line. He represents a traveling wave. Now suppose that M. Russell and M. Wittgenstein hold a stretched sheet in his path. They represent a perturbation. Two things can happen when Cantor reaches the sheet : either he breaks through or the sheet holds and Cantor is stopped. The outcome depends on two things : the speed at which Cantor runs and the grip that Russell and Wittgenstein have on the sheet. If Cantor runs fast enough, the other two might not be able to hold on and although he will slow down from the sheet encounter, Cantor will go on. On the other hand, if Russell and Wittgenstein fancy a tight grip (for some unknown hypothetical reason) relative to Cantor's speed, then they will stop the runner in it's path.

Although, this might seem like a simplistic parallel, the idea behind it is quite valid. Indeed, every traveling wave has a velocity and every perturbation has some sort of potency. For this reason, we will base our approach on these two parameters, with a bifurcation analysis in mind.

Note that for Lewis and Keener's Gap Model (1.1), the potency of the perturbation is the gap width L and the velocity of the wave depends directly on the term f which in turn, depends directly on the parameter α .

1.2.2 Dynamical System Approach

We now address our goal of generality for the actual equation describing a traveling wave. In order to keep a broad spectrum of validity, we adopt a dynamical system approach. It is clear that a traveling wave is a dynamical system before being the outcome of any specific model. Our intention is to explain the underlying mechanism behind the blocking of a wave. We do not wish to provide a specific computational method but rather an existence proof of such a mechanism. Nevertheless, the results obtained certainly have interesting properties that can be used to address specific problems. We will explore this, but the reader should not have utilitarian expectations in mind.

The first step in the definition of our approach lies in a result of D. Henry concerning the reduction of certain parabolic PDEs to ODEs [7]. In brief, it can be shown that a large class of PDEs on \mathbb{R}^m , for any m , can be expressed in a classical dynamical system form. Indeed, we can represent such PDEs as an ODE on a function space. We write these systems as

$$\dot{x} = F(x)$$

where $x(t)$ is an element of a space X that is, for our purposes, comparable to $C^k(\mathbb{R}^l, \mathbb{R}^m)$ where l is also arbitrary. Most of the equations generating traveling waves can be represented in that form with only mild assumptions. It turns out that in that setting, these types of equations have a common characteristic : translational symmetry. In other words, the dynamics of the resulting flow on X are invariant with respect to a specific translation in X , representing the propagative nature of such waves. Furthermore, the actual wave corresponds to solutions that lie on an invariant line that we call a *relative equilibrium*. This line in X is defined with the same direction as the above mentioned symmetry. We then represent the presence of inhomogeneities in the path of our wave with a *symmetry breaking* term. We get an equation of the form

$$\dot{x} = F(x) + \varepsilon G(x, \varepsilon)$$

where $\varepsilon > 0$ is our perturbation parameter. $G(\cdot, \varepsilon)$ can be any vector field that does not have the same symmetry as F and has a compact support, representing a local nature.

Our goal is to exploit the symmetry attributes of such a representation, as it spans all types of traveling wave models. Not surprisingly, this proves to be a daunting task as the space X can be quite complex in some cases : lack of an orthogonality notion, incompleteness and of course, infinite dimensionality. For these reasons, we will use an important simplification for the purpose of this thesis. We will replace the space X with the finite dimensional euclidean space \mathbb{R}^n for an arbitrary n . Although this might seem like a oversimplification of the problem, the reader should keep in mind that the result we obtain is a first step towards a generalization for infinite dimensional spaces. Indeed, most of the concepts we use on \mathbb{R}^n have an equivalent for spaces like $C^k(\mathbb{R}^l, \mathbb{R}^m)$. We will briefly address possible ways one could arrive to a generalization in the discussion section of this thesis.

Furthermore, aside from the generality of our approach, there is another hands-on advantage attributable to our system. Recall that the results obtained by Lewis and Keener were related to a scalar reaction-diffusion equation. The idea of only considering a scalar potential is somewhat restrictive. Indeed, for biological needs, a traveling wave might carry the change of several quantitative variables at the same time. For example : the concentration of many different substances, along with electrical charge, temperature and so on. A scalar PDE cannot capture this reality. Since we aim our result at being valid for a state space of arbitrary dimensions, we do not get this limitation.

Now that we have made clear what is the flavor of this thesis, we have to define rigorously the system that we will study as well as the result we wish to prove. Before doing so, we present a short preliminary chapter, in which we explore some techniques used in our demonstrations.

Chapter 2

Preliminaries

In this chapter, we present a brief overview of the techniques we will use in this document.

2.1 Symmetry Groups and Relative Equilibria

In order to harvest the symmetry attributes of a dynamical system, one can make use of a group representation of certain transformations in Euclidean Space. The idea is to represent a particular rigid transformation as the *action* of some group on a linear space V such as \mathbb{R}^n . For example, the collection of all possible rotations about the origin in the plane can be viewed as the action of $SO(2)$ on \mathbb{R}^2 . Every possible rotation is associated with an angle $\theta \in [0, 2\pi] \sim [0, 1]$. Any sequence of consecutive rotations of angles θ_i is equivalent to one rotation of angle $\sum_i \theta_i$. Thus, applying multiple rigid transformations to V is equivalent to applying the group operation to the associated group components and then interpreting its *action* on V .

In a more formal way, a group action can be described as follows :

Definition 1 (Group Action) *We say that a group $(\Gamma, *)$ acts on a set V if there exists a map $\psi : \Gamma \times V \rightarrow V$ that respect the following conditions :*

$$\psi(e, v) = v \quad \forall v \in V \quad \text{where } e \text{ is the identity of } \Gamma$$

$$\psi(\gamma, \psi(\mu, v)) = \psi(\gamma * \mu, v) \quad \forall v \in V \quad \text{where } \gamma, \mu \in \Gamma.$$

ψ is called a group action on V .

To stay concise, we will use the notation γv to describe the action of the group element γ on the space element v (in our case a vector of \mathbb{R}^n). i.e.: $\gamma v := \psi(\gamma, v)$.

In order to use this definition, we must investigate the relation between such a *group action* and vector fields. Suppose then that Γ is a group acting on V and that $F : V \rightarrow V$ is a C^k vector field ($k \geq 1$) from which we can construct a dynamical system from the solutions of the differential equation $\dot{x} = F(x)$.

Definition 2 (Symmetry Group) *If the action of Γ with respect to F is as follows : $F(\gamma x) = \gamma F(x) \forall x \in \mathbb{R}^n$ and $\forall \gamma \in \Gamma$, we say that the vector field F is Γ -equivariant. For a dynamical system constructed from the equation $\dot{x} = F(x)$, we say that Γ is a symmetry group for the system.*

This definition is generally used for rigid transformations in euclidean space like a rotation for example. In the case of affine transformations such as a translation, adding a referential coordinate to the linear space makes it a rigid transformation. Nevertheless, the use of the above definition by replacing the concept of *equivariance* with the concept of *invariance* (i.e. $F(\gamma x) = F(x)$) yields the same idea.

In essence, a system that has a symmetry group admits solutions that are mapped to other solutions by the action of the group. We will see later that a system modeling a traveling wave is usually constructed from a vector field that has some group of translations as a symmetry group (under *invariance*).

We move on to the next important concept of this section from which we will be able to construct a form of stability attribute for solutions of systems admitting symmetry groups.

Definition 3 (Relative Equilibrium) *Let $x_0 \in V$ and consider the group orbit $\Gamma x_0 = \{\gamma x_0 | \forall \gamma \in \Gamma\}$. If Γx_0 is invariant under the flow (or semi flow) induced by $\dot{x} = F(x)$, then we say that Γx_0 is a relative equilibrium for that system.*

In other words, we observe the presence of a relative equilibrium when there is one or more solutions to the system that is contained in a group orbit of the symmetry group. Recall the above example in \mathbb{R}^2 with a rotation symmetry. The symmetry group is $SO(2)$, therefore a relative equilibrium can appear in two manners : either there is a periodic solution tracing a circle centered at the origin or there is such a circle for which each point is an equilibrium point. In turn, we can inquire about the stability of relative equilibria just as we do for regular equilibria. We can also conduct asymptotical and bifurcation analysis of these structures, the details of which will be made clear as we go along.

We are now equipped to create some sort of characterization of dynamical systems on a given space. Indeed, we now know that any system of the above form with the same symmetry group will have one thing in common : the derivatives of their solutions are *invariant* or *equivariant* under the action of their symmetry group. This characterization is quite broad in the sense that two systems with the same symmetry groups may behave very differently. Nevertheless, we can be sure that the differences between such systems will not arise in the directions relative to the group action. In addition, if two such systems each have a relative equilibrium, then they may only differ in the stability of these equilibria.

This intuitive conclusion is the basis for a procedure called *symmetry reduction*. The key aspect of this method is to decompose the space in which the dynamical system lives near a relative equilibrium. In essence, we split the space into a part

that is tangent to the group action along the relative equilibrium and into a part that is transverse to it. Since the dynamics have an invariant structure within the tangent space, we can discard this part of the space and concentrate on the part of the system evolving in the transverse space. By doing so, we reduce the dimension of the problem by exactly the number of dimensions of the symmetry group which can considerably simplify the analysis.

Definition 4 (Tangent and Transverse Vector Fields) *Let $x_0 \in V$ and consider the group orbit Γx_0 . Let $f : V \rightarrow V$ be a vector field on the space V .*

Then f is a tangent vector field to Γx_0 if $f(x)$ is tangent to the group orbit Γx_0 $\forall x \in V$.

Let N_x be the space of normal vectors to the group orbit Γx_0 at x when $x \in \Gamma x_0$. Then f is a normal vector field if $\forall x \in \Gamma x_0$, N_x is invariant under the flow induced by $\dot{x} = f(x)$. This concept can be extended to any transverse space N_x to yield the general concept of a transverse vector field.

Definition 5 (Symmetry Reduction) *Consider a system $\dot{x} = f(x)$ that admits a relative equilibrium Γx_0 with respect to a group Γ . It is possible, using the above definitions, to decompose the vector field f into two fields f_{tg} and f_{tr} that are respectively tangent and transverse to Γx_0 .*

Let us investigate a low dimension example to ease the reader into the use of these tools and give a general flavor of the key aspect of this thesis.

2.2 Rotational Symmetry : A Simple Example

Consider the following system on \mathbb{R}^2 written in polar form :

$$\begin{aligned}\dot{r} &= r(1-r), \quad r \geq 0 \\ \dot{\theta} &= 1, \quad \theta \in [0, 2\pi]_{\sim}\end{aligned}$$

The associated vector field is $f(r, \theta) = (r(1 - r), 1)$. Notice that it is invariant under rotation about the origin. Indeed, $f(r, \theta + \gamma) = (r(1 - r), 1) = f(r, \theta)$ for any $\gamma \in [0, 2\pi]_{\sim}$. Hence, it is easy to see that $\Gamma = SO(2)$ is a symmetry group for this system with its usual action on \mathbb{R}^2 . This system admits two relative equilibria. The first one is the solution $(r(t), \theta(t)) \equiv (0, 0)$ which is an equilibrium point at the origin. The group orbit $\Gamma(0, 0) = \{(0, 0)\}$ is trivial and proves unhelpful for our demonstration task. The second relative equilibrium is more interesting.

It is easy to see that $(r(t), \theta(t)) = (1, t \bmod 2\pi)$ is a periodic solution to the system. Also, this solution commutes with the group action since it is sent onto itself under rotation about the origin. Another way to see this is to notice that the group orbit of any point of that solution traces precisely the entire solution curve.

$$\forall \theta \in [0, 2\pi]_{\sim}, \quad \Gamma(1, \theta) = S^1 = \{(1, t \bmod 2\pi) | t \in \mathbb{R}\}$$

Thus, S^1 is a (non-trivial) relative equilibrium. We can now decompose f into a tangent and transverse parts.

The tangent vector field to this relative equilibrium is $f|_{\Gamma(1, \theta)}$ which we can write $f_{tg}(\theta) = 1$. The transverse vector spaces to this relative equilibrium are $N_{(1, \theta)} = \{(r, \theta) | r \geq 0\}$ and the tangent vector field is $f_{tr}(r) = r(1 - r)$. Notice that $r = 1$ is an equilibrium point for the system $\dot{r} = f_{tr}(r)$ as it should be since it represents points on S^1 . We may now conduct a stability analysis of this equilibrium.

$$Df_{tr}(1) = \frac{d}{dr}(r(1 - r))(1) = 1 - 2r|_{r=1} = -1 \quad (2.1)$$

Equation (2.1) tells us that $r = 1$ is a sink for the system $\dot{r} = f_{tr}(r)$. We may now expand this result to our original system and we can conclude that every point on S^1 is a sink for the transverse system on the respective transverse vector space. Therefore, the periodic solution on S^1 must be attracting for nearby solutions. In other words, the relative equilibrium is asymptotically stable.

The crucial thing to remember about this example is the conclusions we drew from the stability analysis conducted on the flow of the transverse vector field. The

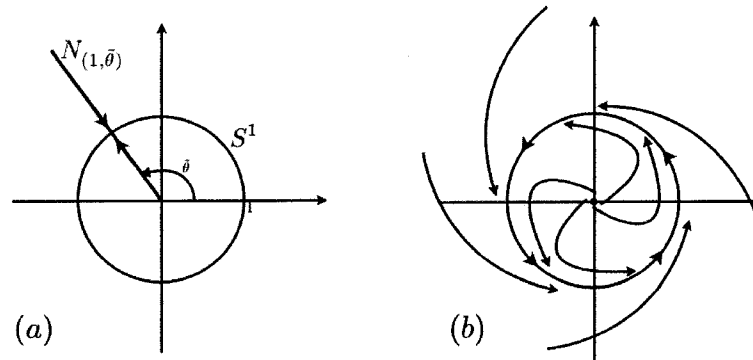


Figure 2.1: (a) Dynamics of the flow of f_{tr} on $N_{(1, \bar{\theta})}$. (b) Dynamics of the flow of f on \mathbb{R}^2 .

rotational symmetry reduction enabled us to study only the radial components of the dynamics in order to come up with a phase portrait of the flow of the original system as Figure 2.1 illustrates.

This concludes the chapter on preliminaries. It is important to note that we will use a similar approach to the above example in our analysis of the dynamics of a traveling wave. The reader should keep in mind that the use of this approach will not be as detailed in the following pages but draws its origin in the above results. For more detailed information on this kind of approach, one should refer to Krupa[11].

Chapter 3

Construction of the Problem

In this chapter, we define the general aspect of the class of systems that we will analyze. We do this by setting a number of conditions that a given system has to respect in order to qualify for the results presented in this document. The goal here is to have conditions as general as possible to give a broader range to these results. We then formulate the results of this research in the form of a concise theorem (Theorem 1). Finally, we design a symmetry reduction of our general system that will enable us to considerably lighten the complexity of the problem.

3.1 Characteristics of Candidate Systems

As discussed earlier, we will only consider an Autonomous Ordinary Differential Equation model for a traveling wave. Nevertheless, a discussion concerning the generalization of these results to Partial Differential Equation and non-autonomous models will follow.

For the rest of this document, we will let the following equation describe the system modeling a traveling wave :

$$\dot{x} = F(x), \quad \text{where } \dot{x} := \frac{dx}{dt} \quad \text{and } x \in \mathbb{R}^n, t \in \mathbb{R} \quad (3.1)$$

Here, $x(t)$ is a time dependent function describing the flow of system (3.1) in n dimensions. F is a vector field on \mathbb{R}^n which is at least C^2 . To incorporate the propagating nature of the wave, we will suppose the presence of a symmetry on the vector field F in the following way :

$$F(x + \alpha v) = F(x), \quad v \in \mathbb{R}^n, \quad \|v\| = 1, \quad \forall x \in \mathbb{R}^n, \quad \alpha \in \mathbb{R}$$

where v is some unit vector in \mathbb{R}^n representing the direction in which the wave propagates.



Figure 3.1: Traveling Wave.

It is important to notice that this feature of F introduces $(\mathbb{R}, +)$ as a symmetry group. The action of this group is described as follows for $\alpha \in \mathbb{R}$ and $x \in \mathbb{R}^n$:

$$\alpha x := x + \alpha v.$$

In turn, this captures the idea that a wave traveling in a given direction v has an invariant dynamical behavior in that direction. In essence, this means that any initial value problem of this system with initial condition points residing on a line with directional vector v , will produce identical unique solutions modulo a translation in direction v .

Indeed, suppose $x_1(t)$ and $x_2(t)$ are two solutions of $\dot{x} = F(x)$ with initial conditions $x_1(t_0) = \tilde{x}_1$, $x_2(t_0) = \tilde{x}_1 + \alpha v$ where $\alpha \in \mathbb{R}$. Then we have :

$$\dot{x}_2(t_0) = F(x_2(t_0)) = F(\tilde{x}_1 + \alpha v) = F(\tilde{x}_1) = \dot{x}_1(t_0).$$

By repeating this process with $x_{1,2}(t_0 + t) \forall t$, we get the desired equivalence.

The next assumption we will make is the presence of a relative equilibrium with respect to the translation symmetry. This is motivated by the physical argument that a traveling wave has a stable propagative wavefront. That is to say, if you find yourself on the wavefront traveling in direction v , you will stay on that course. To include this feature in our system, we suppose that there exists some $x_0 \in \mathbb{R}^n$ such that $F(x_0)$ is parallel to v . This way, we know that there is a solution on the line $D := \{x_0 + \alpha v | \alpha \in \mathbb{R}\}$. We have that D is a relative equilibrium. The traveling wave is the solution on D .

Furthermore, we suppose that this solution attracts others nearby and so, that D is a *asymptotically stable relative equilibrium*. The inclusion of this assumption is preceded by the fact that in many physical systems, the observation of a traveling wave on a relatively long term is intimately linked to the presence of such an attracting invariant structure. Indeed, if the relative equilibrium would not be asymptotically stable, the traveling wavefront would only be observable for a precise choice of initial conditions, which is not the case in most physical occurrences of such a wave.

Figure 3.2 shows a sketch of a possible phase portrait coming from a candidate system. The red curves represent solutions of the system and the arrows represent the direction with respect to time evolution.

Already, we can derive a general characteristic for a system that satisfies the above mentioned assumptions. That is, the speed at which a solution on the relative equilibrium D travels is constant. (Wavefront velocity is constant) To see this, it suffices to notice the following :

$$\forall x \in D, \exists \alpha_x \text{ such that } x = x_0 + \alpha_x v \Rightarrow F(x) = F(x_0 + \alpha_x v) = F(x_0) \forall x \in D.$$

Since $\dot{x} = F(x)$, it follows that the time derivative of the solution on D is the same at every point, which yields the sought constant velocity. As this velocity does not change through space or time, lets assign it a variable : $c = \|F(x_0)\|$. Although

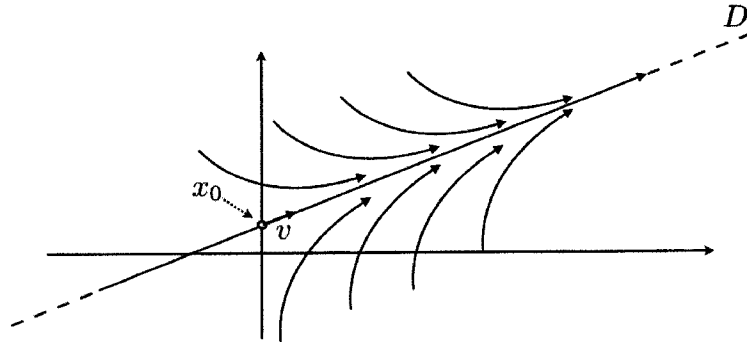


Figure 3.2: Asymptotically Stable Relative Equilibrium.

this velocity depends directly on the nature of F , we will manipulate c as a parameter in future process. It will be made clear as to how we will do this.

It is helpful to note that a reaction-diffusion PDE on a line, which models most traveling waves instances, admits the same type of translational symmetry. Indeed, we can write such a PDE as

$$u_t = \Delta u + f(u)$$

where $u : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^m$ and f is a vector field on \mathbb{R}^m . A translation by α is described by $(T_\alpha u)(z) = u(z - \alpha, t)$ where $z \in \mathbb{R}$. It is clear that this form of equation admits a translational symmetry.

We now have a class of systems with which we can work. Since the goal of this thesis is to try to characterize the wave blocking phenomenon under some symmetry breaking perturbation, we need to define what we will use to model such a perturbation.

In physical instances of such systems, a perturbation will usually be a local change in the structure of the dynamics which can be scaled. For example, consider the electric wave propagating in the heart tissue when the heart beats. In an idealized way, we can see this as a traveling wave on the heart tissue (domain). The direction

of propagation and thus, the relative equilibrium, is characterized by the direction of the fibers in the heart tissue and by its conductivity. Now, suppose that there is some dead or ill tissue somewhere on the heart that changes the conductivity. This dead tissue "spot" will break the symmetry of propagation of the wave and, if big enough, might stop the wave altogether when the wavefront reaches its location.

In order to capture this *symmetry breaking perturbation* idea, we introduce a perturbation term in our equation :

$$\dot{x} = F(x) + \boxed{\varepsilon G(x, \varepsilon)}, \varepsilon \geq 0 \quad (3.2)$$

where $G : \mathbb{R}^n \times \mathbb{R}^+ \rightarrow \mathbb{R}^n$ is at least C^2 , has compact support with respect to x , is bounded and does not admit the same translation symmetry as F . (i.e.: $\forall x \in \mathbb{R}^n, G(x + \alpha v, \varepsilon) \neq G(x, \varepsilon), \alpha \in \mathbb{R}$) Also, ε is a scaling parameter that lets us decide on the potency of the perturbation G .

The compact support assumption for G is motivated by the local nature of the perturbations found in physical systems and is central to the analysis.

We are now equipped to commence our analysis. Table 3.1 gives a overview of the assumptions we made to construct the class of systems we will work with.

$F(x)$	$F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ At least C^2 Translational symmetry : $F(x + \alpha v) = F(x) \forall \alpha \in \mathbb{R}$ where $v \in \mathbb{R}^n, \ v\ = 1$
D	$D = \{x_0 + \alpha v \alpha \in \mathbb{R}\}$ where $F(x_0) \parallel v$ Asymptotically stable relative equilibrium
$G(x, \varepsilon)$	$G : \mathbb{R}^n \times \mathbb{R}^+ \rightarrow \mathbb{R}^n$ At least C^2 Does <u>not</u> admit the same translation symmetry as F Bounded Compact support with respect to x
ε	Perturbation parameter ≥ 0
c	Velocity of the solution on D $ c = \ F(x_0)\ $

Table 3.1: Properties of the perturbed system $\dot{x} = F(x) + \varepsilon G(x, \varepsilon)$.

3.2 Formulation of The Result

The big question now arises : is the relative equilibrium D going to persist as a flow-invariant set under a perturbation and if so, what will be the attributes of the traveling wave solution living on it ? It turns out that we are able to predict the persistence of this invariant curve and that the dynamics of the solutions living on it are characterized by the presence of a specific type of bifurcation. We present this result in the following theorem.

Theorem 1 (Wave Blocking for ODE) *Let F and G be as in Table 3.1 and consider the system $\dot{x} = F(x) + \varepsilon G(x, \varepsilon)$. Then there exists a neighborhood of $(\varepsilon, c) = (0, 0)$ for which the relative equilibrium D will persist in the form of a stable, smooth, flow-invariant curve. There also exists two bifurcation branches separating this neighborhood into sections in which we can respectively observe the presence and absence of equilibrium points for the dynamics of our system on this invariant curve.*

Furthermore, the appearance of such equilibrium points occurs via saddle-node bifurcations.

We prove this theorem in the rest of this document. Furthermore, it will be made clear that the blocking of the traveling wave solution described earlier is directly associated to the presence of equilibrium points on the invariant curve. The next and last section of this chapter constructs the symmetry approach previously mentioned and the general setting in which we will work.

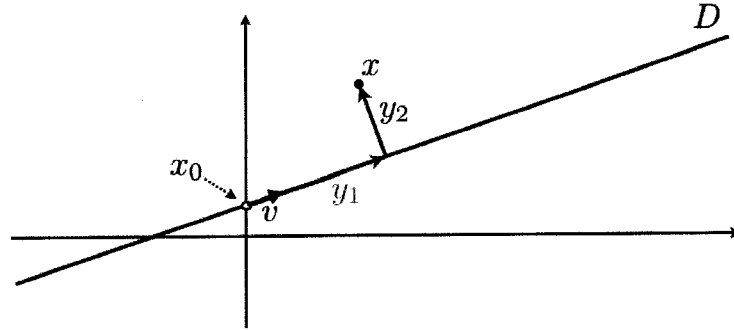
3.3 Symmetry Partitioning Along the Relative Equilibrium

We now have a well defined system and a well defined question to address. We also have a furnished toolbox to do so. In order to be able to analyze the perturbed system under a symmetry reduction, we must begin by studying the unperturbed system $\dot{x} = F(x)$. As we go along, the reader can refer to Table 3.1 for the assumptions we will use.

Since we have a relative equilibrium D that commutes with the group of symmetry of the system, we may define tangent and transverse vector fields. We must decompose \mathbb{R}^n in two spaces in which our new vector fields will live. The first part of this decomposition, the space for our tangent vector field, is quite natural. We will take D . The second part, for the transverse vector field, is subject to a choice. Nevertheless, we will take the simplest approach and choose the orthogonal space to D .

Let $x \in \mathbb{R}^n$ and $\langle \cdot, \cdot \rangle$ denote the usual scalar product on \mathbb{R}^n . We set the two following variables :

$$\begin{aligned} y_1(x) &= \langle x - x_0, v \rangle \\ y_2(x) &= x - x_0 - \langle x - x_0, v \rangle v \end{aligned}$$

Figure 3.3: Decomposition of \mathbb{R}^n .

For legibility purposes, we will drop the argument for y_1 and y_2 . Thus, for any $x \in \mathbb{R}^n$, we can write : $x = x_0 + y_1v + y_2$ as Figure 3.3 illustrates. We have that $y_1 \in \mathbb{R}$ and with a proper rigid change of coordinates, we can view y_2 as an element of \mathbb{R}^{n-1} . Notice that in these new coordinates, the relative equilibrium D is described as $D = \{(y_1, y_2) \in \mathbb{R} \times \mathbb{R}^{n-1} | y_2 = 0\}$. Also, without loss of generality, we shall let $x_0 = 0$ as it is just a matter of a simple coordinate translation. We then have :

$$\begin{aligned} y_1 &= \langle x, v \rangle \\ y_2 &= x - \langle x, v \rangle v \\ x &= y_1v + y_2 \end{aligned}$$

Let us now return to our perturbed system $\dot{x} = F(x) + \varepsilon G(x, \varepsilon)$ and investigate its representation in these new coordinates. We begin with y_1 :

$$\begin{aligned} \dot{y}_1 &= \langle \dot{x}, v \rangle \\ \dot{y}_1 &= \langle F(x) + \varepsilon G(x, \varepsilon), v \rangle \\ \dot{y}_1 &= \langle F(y_1v + y_2) + \varepsilon G(y_1v + y_2, \varepsilon), v \rangle. \end{aligned} \tag{3.3}$$

We now make use of the invariance of F for translations in direction v and properties of the scalar product. Equation (3.3) becomes :

$$\dot{y}_1 = \langle F(y_2), v \rangle + \varepsilon \langle G(y_1 v + y_2, \varepsilon), v \rangle. \quad (3.4)$$

A similar approach for y_2 yields

$$\begin{aligned} \dot{y}_2 &= \dot{x} - \langle \dot{x}, v \rangle v \\ \dot{y}_2 &= F(y_2) + \varepsilon G(y_1 v + y_2, \varepsilon) - \langle F(y_2) + \varepsilon G(y_1 v + y_2, \varepsilon), v \rangle v \\ \dot{y}_2 &= F(y_2) - \langle F(y_2), v \rangle v + \varepsilon [G(y_1 v + y_2, \varepsilon) - \langle G(y_1 v + y_2, \varepsilon), v \rangle v]. \end{aligned} \quad (3.5)$$

From equations (3.4) and (3.5), we define the maps $F_1 : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$, $F_2 : \mathbb{R}^{n-1} \rightarrow \mathbb{R}^{n-1}$, $G_1 : \mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R}^+ \rightarrow \mathbb{R}$ and $G_2 : \mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R}^+ \rightarrow \mathbb{R}^{n-1}$ as follows :

$$\begin{aligned} F_1(y_2) &= \langle F(y_2), v \rangle \\ F_2(y_2) &= F(y_2) - \langle F(y_2), v \rangle v \\ G_1(y_1, y_2, \varepsilon) &= \langle G(y_1 v + y_2, \varepsilon), v \rangle \\ G_2(y_1, y_2, \varepsilon) &= G(y_1 v + y_2, \varepsilon) - \langle G(y_1 v + y_2, \varepsilon), v \rangle v. \end{aligned}$$

Using this new notation, we can write our original equation in the form of a coupled system of equations on $\mathbb{R} \times \mathbb{R}^{n-1}$:

$$\begin{aligned} \dot{y}_1 &= F_1(y_2) + \varepsilon G_1(y_1, y_2, \varepsilon) \\ \dot{y}_2 &= F_2(y_2) + \varepsilon G_2(y_1, y_2, \varepsilon). \end{aligned} \quad (3.6)$$

It is crucial to notice that the system (3.6) has a first part that is tangent to the relative equilibrium $D = \{(y_1, y_2) | y_2 = 0\}$ and a second part that is transverse to it when $\varepsilon = 0$. Nevertheless, the question on the persistence of the relative equilibrium when $\varepsilon \neq 0$ remains. We will investigate this matter in the following chapter as we will define conditions for which the equilibrium will indeed persist.

Chapter 4

Persistence of the Relative Equilibrium

In the last chapter, we have come up with a concise set of properties that precisely characterize the class of systems on which the results of our analysis will be valid. Table 3.1 is a quick reference for these properties. After doing so, we started our analysis by applying symmetry analysis tools to our system in order to get a coupled system of equations on a partitioned version of \mathbb{R}^n that well represents the symmetry attributes of our traveling wave system. We ended up with equation (3.6).

In this chapter, we will carry on with the analysis. First, we will find a way to prove the persistence of the relative equilibrium D under a small perturbation ($\varepsilon > 0$) of the system. Second, we will parametrize this "perturbed" relative equilibrium in an ultimate goal to study the dynamics of our system on it.

The intuitive approach to this problem would be to try and compute the evolution of the solution on D as the parameter ε varies. Indeed, the differentiability of all the maps constituting the system seems to allow for this. Nevertheless, the fact that we deal with abstract vector fields and maps and that we only demand certain attributes as described in table 3.1 makes this approach a very difficult one. In fact, there are

too many unknowns to make this task possible without adding further assumptions. Hence, if we want to keep our generality, we must operate differently. In order to continue, we will need to modify equation (3.6) a little more.

4.1 Parameters

As discussed earlier, the perturbation that we impose on the system is modulated by the parameter ε both in the original equation $\dot{x} = F(x) + \varepsilon G(x, \varepsilon)$ and in the coupled system equations $\dot{y}_1 = F_1(y_2) + \varepsilon G_1(y_1, y_2, \varepsilon)$; $\dot{y}_2 = F_2(y_2) + \varepsilon G_2(y_1, y_2, \varepsilon)$. Needless to say, the behavior of the relative equilibrium D will be closely linked to the value of ε . Nevertheless, returning to a more intuitive approach, we can foresee that the velocity of propagation of a wave might have something to do with its ability to propagate beyond a perturbation in the domain. Also, since our ultimate goal is to characterize blocking of the wave on the relative equilibrium, considering the velocity of the wave in the unperturbed system seems to be appropriate.

In many models, the velocity of a wave is already a parameter. As mentioned earlier, we have assigned a variable to this velocity : $c = \|F(0)\|$. For the rest of this document, we will consider c to be a parameter of our coupled system. We will also allow c to take negative values to represent a propagation in the opposite direction of the parametrization of D . This will prove quite useful in the power series expansion of the maps F_1 and F_2 . This choice of parameter along with ε also provides us with a proper parameter space to investigate the wave blocking phenomenon. Indeed, the situation where $c = 0$ represents a trivial block for any ε since the wave simply does not move. Furthermore, as c grows, it is reasonable to believe that ε will have to grow as well to obtain a block.

In light of this reasoning, we will consider c and ε as the parameters of our system with which we will eventually do a bifurcation analysis to explain the blocking phenomenon of our traveling wave. Of course, this analysis will take place for values

of ε relatively small and in relation with c . As a result, the important part of the parameter space to study will be in a neighborhood of the *organizing center* $(\varepsilon, c) = (0, 0)$. We expect bifurcation branches emanating from that point and separating the parameter space in parts in which we observe wave blocking and parts where we do not. Figure 4.1 sketches a possible bifurcation portrait.

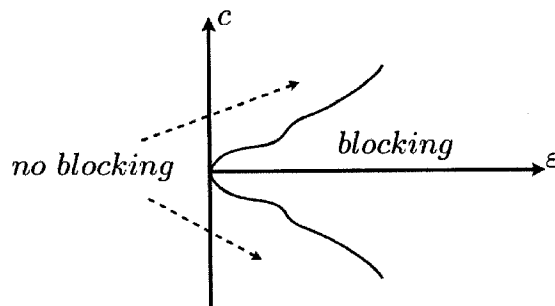


Figure 4.1: Parameter Space with Possible Bifurcation Branches.

Although we had to address the bifurcation analysis to motivate the inclusion of c as a parameter to the system, there is still much work to do before we can think about bifurcations.

4.2 Suspension of the System in Parameter Space

Since parameter space is closely related to the behavior of the system, it can be useful to consider it as a part of the space in which the system lives. We wish to do this without altering the dynamics created by our differential equations and that is why we set the time derivative of ε and c to be null. There are some mathematical reasons for which we do this and they will be quite evident in the following text. Therefore, here is our suspended system living in $\mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R}^+ \times \mathbb{R}$:

$$\begin{aligned}
\dot{y}_1 &= F_1(y_2) + \varepsilon G_1(y_1, y_2, \varepsilon) \\
\dot{y}_2 &= F_2(y_2) + \varepsilon G_2(y_1, y_2, \varepsilon). \\
\dot{\varepsilon} &= 0 \\
\dot{c} &= 0
\end{aligned} \tag{4.1}$$

To make this more usable, we take the Taylor expansion of F_1 and F_2 around $y_2 = 0$. Recall that $\|v\| = 1$, $|c| = \|F(0)\|$ and that $F(0)$ is parallel to v so we can write $F(0) = cv$.

$$\begin{aligned}
F_1(y_2) &= F_1(0) + R_1(y_2) \\
&= \langle F(0), v \rangle + R_1(y_2) \\
&= c \langle v, v \rangle + R_1(y_2) \\
&= c + R_1(y_2)
\end{aligned} \tag{4.2}$$

$$\begin{aligned}
F_2(y_2) &= F_2(0) + R_2(y_2) \\
&= F(0) - \langle F(0), v \rangle v + R_2(y_2) \\
&= cv - cv + R_2(y_2) \\
&= R_2(y_2)
\end{aligned} \tag{4.3}$$

In equations (4.2) and (4.3), R_1 and R_2 denote the rest of the terms in the Taylor expansions, namely, the terms of degree one and above. As a result, it is easy to see that $\boxed{R_1(0) = 0 = R_2(0)}$. This concludes our transformation of the system. Equation (4.1) along with equations (4.2) and (4.3) yield the final version :

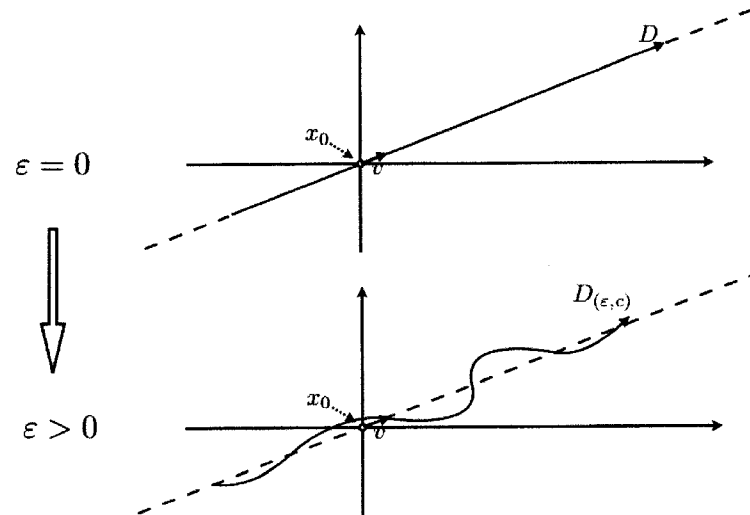
$$\begin{aligned}
\dot{y}_1 &= c + R_1(y_2) + \varepsilon G_1(y_1, y_2, \varepsilon) \\
\dot{y}_2 &= R_2(y_2) + \varepsilon G_2(y_1, y_2, \varepsilon) \\
\dot{\varepsilon} &= 0 \\
\dot{c} &= 0
\end{aligned} \tag{4.4}$$

4.3 Parametrizing the Persisting Relative Equilibrium

Now that we have equation (4.4) to work with, we can inquire about its dynamics under perturbation ($\varepsilon > 0$). Recall that the presence of a traveling wave is closely linked to the existence of the relative equilibrium D in the unperturbed system. In fact, in our abstract model, the traveling wave is represented by the solution on D and by the fact that it is attracting other nearby solutions. Intuitively, both in the physical and in the mathematical sense, progressively adding a perturbation to the system will change the traveling wave until it stops. This is equivalent to the conclusion that the relative equilibrium, as a flow-invariant geometric structure on which a solution lives, persists but changes shape as we augment the potency of the perturbation term in our equation.

In order to show that this is in fact the case, we start by supposing that D has an homologous structure in the system (4.4) for every sufficiently small value of ε . Since the presence of this structure will also depend on the value of c , let's call it $D_{(\varepsilon, c)}$. Our goal is first to show that this object exists for some values of (ε, c) and second, to parametrize it the same way we do with D , that is, with y_1 . Figure 4.2 sketches a possible portrait of $D_{(\varepsilon, c)}$. Before we continue, the reader should notice that $D_{(0, c)} = D = \{y_1 v | y_1 \in \mathbb{R}\} \quad \forall c \in \mathbb{R}$.

Another thing that should be noticed is that for any value of \tilde{y}_1 , the points

Figure 4.2: Evolution of D with respect to ε .

$(y_1, y_2, \varepsilon, c) = (\tilde{y}_1, 0, 0, 0)$ are equilibrium points for the system (4.4). Since all these points constitute our original relative equilibrium D , we are justified to ask if this fact could help us investigate the presence of $D_{(\varepsilon, c)}$ for $\varepsilon, c \neq 0$.

In an attempt to answer this question, we investigate the stability of these points of equilibria for the flow of system (4.4). We do this by computing the linearization of the system around these points. Let \mathcal{F} be the following vector field :

$$\mathcal{F} \begin{pmatrix} y_1 \\ y_2 \\ \varepsilon \\ c \end{pmatrix} = \begin{pmatrix} c + R_1(y_2) + \varepsilon G_1(y_1, y_2, \varepsilon) \\ R_2(y_2) + \varepsilon G_2(y_1, y_2, \varepsilon) \\ 0 \\ 0 \end{pmatrix}.$$

Then our system is written as

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{\varepsilon} \\ \dot{c} \end{pmatrix} = \mathcal{F} \begin{pmatrix} y_1 \\ y_2 \\ \varepsilon \\ c \end{pmatrix} \quad (4.5)$$

and as expected, we get

$$\mathcal{F} \begin{pmatrix} \tilde{y}_1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

In this notation, the linearization of the system around $(\tilde{y}_1, 0, 0, 0)$ is

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{\varepsilon} \\ \dot{c} \end{pmatrix} = A_{\tilde{y}_1} \begin{pmatrix} y_1 - \tilde{y}_1 \\ y_2 \\ \varepsilon \\ c \end{pmatrix} \quad (4.6)$$

where

$$A_{\tilde{y}_1} = D\mathcal{F} \begin{pmatrix} \tilde{y}_1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

The next step is to inspect the spectrum of the $(n+2) \times (n+2)$ matrix $A_{\tilde{y}_1}$ in order to enquire about the stability of the equilibrium point $(\tilde{y}_1, 0, 0, 0)$. Lets take a closer look at that matrix :

$$A_{\tilde{y}_1} = D\mathcal{F} \begin{pmatrix} \tilde{y}_1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & DR_1(0) & G_1(\tilde{y}_1, 0, 0) & 1 \\ 0 & DR_2(0) & G_2(\tilde{y}_1, 0, 0) & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (4.7)$$

The reader should notice that the second row and second column of that upper triangular matrix are $n - 1$ thick. By taking a look at the diagonal, It is easy to see that the spectrum of $A_{\tilde{y}_1}$ unfolds as

$$sp(A_{\tilde{y}_1}) = \{0, sp(DR_2(0)), 0, 0\}.$$

Recall that $R_2(y_2) = 0 + F_2(y_2)$ and that in our assumptions listed in table (3.1), we supposed that the relative equilibrium D was asymptotically stable and hence attracted nearby solutions. This means that in the unperturbed system, at any point $(\tilde{y}_1, 0)$ on D , the transverse dynamics in y_2 is directed at D . Mathematically, this implies that $y_2 = 0$ is a sink for F_2 . Hence, we now know from this assumption that the $n - 1$ eigenvalues of $DR_2(0) = DF_2(0)$ have **strictly negative real parts**. This fact along with the theorem of existence of invariant manifolds (which can be found in Carr [1]) ensures us that there is a $n - 1$ -dimensional invariant Stable Manifold tangent to the cartesian product of the (generalized) eigenspaces of $DR_2(0)$ at $(\tilde{y}_1, 0, 0, 0)$. Any solution of system (4.4) starting on this manifold will stay on it and tend towards the point $(\tilde{y}_1, 0, 0, 0)$. The reader can find some reference of this result in Chicone [2].

In a similar way, the same result guarantees the existence of an invariant 3-dimensional Center Manifold $W_{\tilde{y}_1}$, this time, tangent to the (generalized) eigenspaces associated with the zero eigenvalues of $A_{\tilde{y}_1}$.

At this point, the reader must ask him or herself : what do these manifolds have to do with the curve $D_{(\varepsilon, c)}$? The fact is that the flow-invariant structure of a center manifold, along with the absence of asymptotic behavior of the flow on it, determine the sought curve $D_{(\varepsilon, c)}$. Indeed, since the invariance of the dynamics of system (3.1) on D is the key attribute to model a traveling wave, the persistence of this traveling wave under a perturbation is analogous to the persistence of this invariant curve under perturbation. The above center manifolds, one for every $\tilde{y}_1 \in \mathbb{R}$, exactly depict this invariance for slices of the system (3.2) at each \tilde{y}_1 . To construct the invariant curve $D_{(\varepsilon, c)}$, we must combine these manifolds together with respect to \tilde{y}_1 ,

our parameter for this curve, to get a global portrait.

In essence, for some \tilde{y}_1 , the manifold $W_{\tilde{y}_1}$ is locally represented by a map taking as argument the variables responsible for the zero eigenvalues in $A_{\tilde{y}_1}$, namely y_1 , ε and c . Loosely speaking, say $y_2 = w_{\tilde{y}_1}(y_1, \varepsilon, c)$ is a local representation of $W_{\tilde{y}_1}$ around $(\tilde{y}_1, 0, 0, 0)$. Then, for fixed (ε, c) , the parametrization of $D_{(\varepsilon, c)}$ with parameter \tilde{y}_1 would be

$$D_{(\varepsilon, c)} = \{(\tilde{y}_1, w_{\tilde{y}_1}(\tilde{y}_1, \varepsilon, c), \varepsilon, c) | \tilde{y}_1 \in \mathbb{R}\}.$$

This means that for a given parameter pair (ε, c) , every point on $D_{(\varepsilon, c)}$ will be represented by $w_{\tilde{y}_1}(\tilde{y}_1, \varepsilon, c)$ for some \tilde{y}_1 . Notice that it is not only the argument \tilde{y}_1 that changes as we vary it but also the function $w_{\tilde{y}_1}$ as it depends on \tilde{y}_1 .

Essentially, the idea is to parametrize $D_{(\varepsilon, c)}$ for any (ε, c) in a sufficiently small neighborhood of $(0, 0)$ with the bundle of manifolds $\{W_{\tilde{y}_1} | \tilde{y}_1 \in \mathbb{R}\}$. Each manifold $W_{\tilde{y}_1}$ is a slice at \tilde{y}_1 on which every point $w_{\tilde{y}_1}(\tilde{y}_1, \varepsilon, c)$ coincides with the passing of the curve $D_{(\varepsilon, c)}$ for equal (ε, c) . Figure 4.3 illustrates this idea.

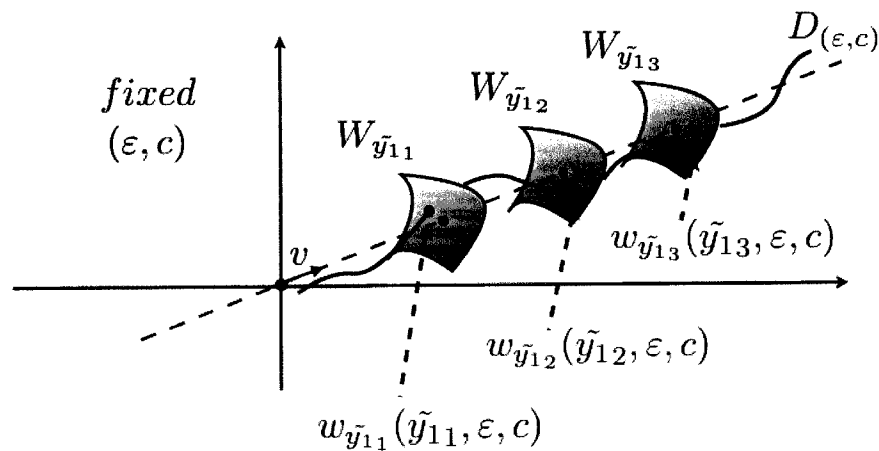


Figure 4.3: Parametrization of $D_{(\varepsilon, c)}$ with $\tilde{y}_{1,2,3} \in \mathbb{R}$.

Now that the way in which we will construct the various $D_{(\varepsilon, c)}$ is made clear,

we must do so formally and rigorously. In the next section, we will demonstrate the existence of such a manifold bundle and discuss on the continuity and differentiability of the "glueing" of the different slices. This becomes crucial since it directly influences the continuity and differentiability of $D_{(\varepsilon,c)}$.

4.4 Slices and Glue

Before we engage in the intricate dance of weaving together changes of coordinates and existence theorems, let us take a step back and look at a strategy.

We look at getting a global portrait of the curve $D_{(\varepsilon,c)}$. Our building blocks to create a smooth map of it are the manifold "slices" $W_{\tilde{y}_1}$. In order to glue these slices together properly, we will need a proper domain. We know that locally, every manifold $W_{\tilde{y}_1}$ is defined for an open neighborhood of $(\tilde{y}_1, 0, 0, 0)$. This is due to the Theorem of Existence of Center Manifolds that can be found in Carr [1] as well as in the Appendix of this document as Theorem 2. Eventually, we will have to bring together these neighborhoods in order to get our global domain.

In addition to defining a domain, we will have to get a well-defined map to characterize $D_{(\varepsilon,c)}$. Again, this task begins by defining proper representations of the manifold slices $W_{\tilde{y}_1}$; functions we labeled $w_{\tilde{y}_1}$. We must then find a way to glue together these different local maps to get a global parametrization with respect to \tilde{y}_1 . We will have to be careful when doing this to ensure the continuity and differentiability of this global map.

4.4.1 Jordan Canonical Form

To get started, let us fix \tilde{y}_1 so we can work on a given slice $W_{\tilde{y}_1}$. In order to get the existence of a center manifold around $(\tilde{y}_1, 0, 0, 0)$ for the system (4.5) :

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{\varepsilon} \\ \dot{c} \end{pmatrix} = \mathcal{F} \begin{pmatrix} y_1 \\ y_2 \\ \varepsilon \\ c \end{pmatrix},$$

we must first express it in a particular set of coordinates in order to apply Theorem 2 (see Appendix A). We begin by looking at its linearization around $(\tilde{y}_1, 0, 0, 0)$: the matrix $A_{\tilde{y}_1}$. Recall from equation (4.7) that the block $DR_2(0)$ on the diagonal is responsible for the eigenvalues with strictly negative real parts. For computational reasons, we must transform this block into Jordan Canonical Form. At the same time, we will deal with the $n - 1$ row vector $DR_1(0)$ and try to simplify it. Recall the matrix we are working with :

$$A_{\tilde{y}_1} = \begin{pmatrix} 0 & DR_1(0) & G_1(\tilde{y}_1, 0, 0) & 1 \\ 0 & DR_2(0) & G_2(\tilde{y}_1, 0, 0) & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Let $\lambda_1, \lambda_2, \dots, \lambda_{n-1}$ be the eigenvalues of $DR_2(0)$. Suppose for now that they are real values. Let u_1, u_2, \dots, u_{n-1} be the associated eigenvectors or generalized eigenvectors. Consider the basis \mathcal{B} of \mathbb{R}^{n+2}

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{\lambda_1} DR_1(0) \circ u_1 \\ u_1 \\ 0 \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} \frac{1}{\lambda_{n-1}} DR_1(0) \circ u_{n-1} \\ u_{n-1} \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \vdots \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \right\}$$

with associated basis change matrix $P_{\mathcal{B}}$ constituted of the above vectors, in order, as columns. Then it is easy to check that

$$P_B^{-1}A_{\tilde{y}_1}P_B = \begin{pmatrix} 0 & 0 & G_1(\tilde{y}_1, 0, 0) & 1 \\ 0 & B & G_2(\tilde{y}_1, 0, 0) & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (4.8)$$

where B is the Jordan Canonical Form of $DR_2(0)$. If the eigenvalues have imaginary parts, the same reasoning holds to get rid of $DR_1(0)$. We can then find another basis of real vectors to express B in Real Jordan Canonical Form with orthogonal blocks on the diagonal for each pair of complex conjugate eigenvalues.

What is crucial to notice is that this first change of coordinates will be valid for all our slices $W_{\tilde{y}_1}$ since it does not depend on the choice of \tilde{y}_1 . For this reason, let us fix this basis as our standard one for the rest of this section and use equation (4.8) to set a new notation given by

$$A_{\tilde{y}_1} = \begin{pmatrix} 0 & 0 & G_1(\tilde{y}_1, 0, 0) & 1 \\ 0 & B & G_2(\tilde{y}_1, 0, 0) & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The next step for us will be to design another change of coordinates, this time dependent on \tilde{y}_1 , to obtain a Jordan Canonical Form of $A_{\tilde{y}_1}$. In order to do this, we must first find the eigenvectors associated with the zero eigenvalues of $A_{\tilde{y}_1}$. In other words, we must find $\ker(A_{\tilde{y}_1})$. Notice that the fact that the function G is bounded assures us that the values of $G_{1,2}(\tilde{y}_1, 0, 0)$ will be finite. Let

$$0 = A_{\tilde{y}_1} \begin{pmatrix} y_1 \\ y_2 \\ \varepsilon \\ c \end{pmatrix} = \begin{pmatrix} \varepsilon G_1(\tilde{y}_1, 0, 0) + c \\ By_2 + \varepsilon G_2(\tilde{y}_1, 0, 0) \\ 0 \\ 0 \end{pmatrix}. \quad (4.9)$$

We know that B is invertible since its eigenvalues are all non zero and so the general solution to equation (4.9) is

$$\ker(A_{\tilde{y}_1}) = \text{span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -B^{-1}G_2(\tilde{y}_1, 0, 0) \\ 1 \\ -G_1(\tilde{y}_1, 0, 0) \end{pmatrix} \right\}.$$

In order to obtain our desired Jordan Form, we are still missing a vector to define a usable generalized eigenbasis. We want a generalized eigenvector associated to the eigenvalue zero. In other words, we want to find

$$\ker(A_{\tilde{y}_1}^2) \setminus \ker(A_{\tilde{y}_1}).$$

Consider the equation

$$0 = A_{\tilde{y}_1}^2 \begin{pmatrix} y_1 \\ y_2 \\ \varepsilon \\ c \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & B^2 & BG_2(\tilde{y}_1, 0, 0) & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \varepsilon \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ B^2 y_2 + BG_2(\tilde{y}_1, 0, 0)\varepsilon \\ 0 \\ 0 \end{pmatrix}. \quad (4.10)$$

This yields

$$B^2 y_2 + BG_2(\tilde{y}_1, 0, 0)\varepsilon = 0 \Rightarrow y_2 = -B^{-1}G_2(\tilde{y}_1, 0, 0)\varepsilon$$

and we get

$$\ker(A_{\tilde{y}_1}^2) = \text{span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -B^{-1}G_2(\tilde{y}_1, 0, 0) \\ 1 \\ -G_1(\tilde{y}_1, 0, 0) \end{pmatrix}, \begin{pmatrix} 0 \\ -B^{-1}G_2(\tilde{y}_1, 0, 0) \\ 1 \\ 0 \end{pmatrix} \right\}.$$

We then get

$$\ker(A_{\tilde{y}_1}^2) \setminus \ker(A_{\tilde{y}_1}) = \text{span} \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -B^{-1}G_2(\tilde{y}_1, 0, 0) \\ 1 \\ 0 \end{pmatrix} \right\}.$$

Since we only need one generalized eigenvector, let's pick the simplest one. Notice that we have a chain of generalized eigenvectors associated to zero :

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

since

$$A_{\tilde{y}_1} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This means that we get the generalized eigenbasis associated to zero

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -B^{-1}G_2(\tilde{y}_1, 0, 0) \\ 1 \\ -G_1(\tilde{y}_1, 0, 0) \end{pmatrix} \right\}.$$

This lets us design a basis of \mathbb{R}^{n+2} in which $A_{\tilde{y}_1}$ will be represented in Jordan Canonical Form. Let

$$\mathcal{B}_{\tilde{y}_1} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -B^{-1}G_2(\tilde{y}_1, 0, 0) \\ 1 \\ -G_1(\tilde{y}_1, 0, 0) \end{pmatrix}, \begin{pmatrix} 0 \\ e_1 \\ 0 \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ e_{n-1} \\ 0 \\ 0 \end{pmatrix} \right\}$$

where e_1, \dots, e_{n-1} are the usual canonical basis vectors of \mathbb{R}^{n-1} . We write the basis change matrix associated to $\mathcal{B}_{\tilde{y}_1}$ along with its inverse as follows :

$$P_{\mathcal{B}_{\tilde{y}_1}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -B^{-1}G_2(\tilde{y}_1, 0, 0) & I_{n-1} \\ 0 & 0 & 1 & 0 \\ 0 & 1 & -G_1(\tilde{y}_1, 0, 0) & 0 \end{pmatrix}, \quad P_{\mathcal{B}_{\tilde{y}_1}}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & G_1(\tilde{y}_1, 0, 0) & 1 \\ 0 & 0 & 1 & 0 \\ 0 & I_{n-1} & B^{-1}G_2(\tilde{y}_1, 0, 0) & 0 \end{pmatrix}.$$

Given the above definition, we get the sought Jordan Form

$$P_{\mathcal{B}_{\tilde{y}_1}}^{-1} A_{\tilde{y}_1} P_{\mathcal{B}_{\tilde{y}_1}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & B \end{pmatrix}. \quad (4.11)$$

4.4.2 Existence of Center Manifolds

Now that we have constructed a basis in which the linearization of our system (4.5) can be expressed into Jordan Canonical Form, we will aim to use this in our quest for the center manifold $W_{\tilde{y}_1}$. We first rewrite system (4.5) (in standard basis) using its linear part around $(\tilde{y}_1, 0, 0, 0)$ and regroup higher order terms of the Taylor expansion of \mathcal{F} :

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{\varepsilon} \\ \dot{c} \end{pmatrix} = \mathcal{F} \begin{pmatrix} y_1 \\ y_2 \\ \varepsilon \\ c \end{pmatrix} = A_{\tilde{y}_1} \begin{pmatrix} y_1 - \tilde{y}_1 \\ y_2 \\ \varepsilon \\ c \end{pmatrix} + \begin{pmatrix} H_1(y_1 - \tilde{y}_1, y_2, \varepsilon, c) \\ H_2(y_1 - \tilde{y}_1, y_2, \varepsilon, c) \\ 0 \\ 0 \end{pmatrix} \quad (4.12)$$

where it is useful to notice that $H_{1,2}(0, 0, 0, 0) = 0$ and $DH_{1,2}(0, 0, 0, 0) = 0$.

The reader should recall that we are still working with an arbitrary fixed \tilde{y}_1 . We now want to apply the change of coordinates that we designed in Subsection 4.4.1. By doing so, we will fulfill the requirements of the Theorem of Existence of Center Manifolds (Theorem 2 in appendix).

Let us inspect the effect of this change of coordinates on the concerned variables. Here is its action :

$$\begin{aligned} & \begin{pmatrix} y_1 - \tilde{y}_1 \\ y_2 \\ \varepsilon \\ c \end{pmatrix} \mapsto P_{\mathcal{B}_{\tilde{y}_1}}^{-1} \begin{pmatrix} y_1 - \tilde{y}_1 \\ y_2 \\ \varepsilon \\ c \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & G_1(\tilde{y}_1, 0, 0) & 1 \\ 0 & 0 & 1 & 0 \\ 0 & I_{n-1} & B^{-1}G_2(\tilde{y}_1, 0, 0) & 0 \end{pmatrix} \begin{pmatrix} y_1 - \tilde{y}_1 \\ y_2 \\ \varepsilon \\ c \end{pmatrix} = \begin{pmatrix} y_1 - \tilde{y}_1 \\ \varepsilon G_1(\tilde{y}_1, 0, 0) + c \\ \varepsilon \\ y_2 + \varepsilon B^{-1}G_2(\tilde{y}_1, 0, 0) \end{pmatrix}. \end{aligned} \quad (4.13)$$

For convenience, we will name the new coordinates in (4.13)

$$\begin{pmatrix} y_1 - \tilde{y}_1 \\ \sigma \\ \varepsilon \\ z \end{pmatrix} = \begin{pmatrix} y_1 - \tilde{y}_1 \\ \varepsilon G_1(\tilde{y}_1, 0, 0) + c \\ \varepsilon \\ y_2 + \varepsilon B^{-1}G_2(\tilde{y}_1, 0, 0) \end{pmatrix}$$

Using this new notation, we rewrite system (4.6) as follows :

$$\begin{pmatrix} \dot{y}_1 \\ \dot{\sigma} \\ \dot{\varepsilon} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} y_1 - \tilde{y}_1 \\ \sigma \\ \varepsilon \end{pmatrix} \quad (4.14)$$

$$\dot{z} = Bz.$$

Equation (4.14) represents the linear part of our original system (4.5) around the point $(\tilde{y}_1, 0, 0, 0)$. We also want to carry on with the nonlinear part of our system in the new coordinates. Recall the representation of system (4.5) written with a linear part in equation (4.12). We must apply the change of coordinates on this representation. Consider

$$P_{\tilde{y}_1}^{-1} \begin{pmatrix} H_1 \\ H_2 \\ 0 \\ 0 \end{pmatrix} \circ P_{\tilde{y}_1} = \begin{pmatrix} H_1 \circ P_{\tilde{y}_1} \\ 0 \\ 0 \\ H_2 \circ P_{\tilde{y}_1} \end{pmatrix}.$$

Then letting

$$\begin{aligned} h_1 &= H_1 \circ P_{\tilde{y}_1} \\ h_2 &= H_2 \circ P_{\tilde{y}_1} \end{aligned}$$

we write our system in the new coordinates as follows :

$$\begin{pmatrix} \dot{y}_1 \\ \dot{\sigma} \\ \dot{\varepsilon} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} y_1 - \tilde{y}_1 \\ \sigma \\ \varepsilon \end{pmatrix} + \begin{pmatrix} h_1(y_1 - \tilde{y}_1, \sigma, \varepsilon, z) \\ 0 \\ 0 \end{pmatrix} \quad (4.15)$$

$$\dot{z} = Bz + h_2(y_1 - \tilde{y}_1, \sigma, \varepsilon, z).$$

Since it is easy to verify that $h_{1,2}(0, 0, 0, 0) = 0$ along with $Dh_{1,2}(0, 0, 0, 0) = 0$, we have now all the necessary criteria to apply Theorem 2 from the Appendix.

Therefore, we know that for some neighborhood of $(\tilde{y}_1, 0, 0, 0)$, say

$$U_{\tilde{y}_1} = \{(y_1, \sigma, \varepsilon, z) \mid \| (y_1 - \tilde{y}_1, \sigma, \varepsilon, z) \| < \beta_{\tilde{y}_1}\}$$

for some $\beta_{\tilde{y}_1} > 0$, there exists a local center manifold $W_{\tilde{y}_1}$ that can be represented by a C^2 map $z = w_{\tilde{y}_1}(y_1, \sigma, \varepsilon)$.

Now that we have shown that there is a center manifold attached to every \tilde{y}_1 , the next step is to find a way to compute them. We will use the standard fixed point method, the details of which can be found in Carr [1].

We start by defining a C^∞ cut off function $\psi : \mathbb{R}^3 \rightarrow [0, 1]$ such that $\psi(x) = 1$ when $\|x\| \leq 1$ and $\psi(x) = 0$ when $\|x\| > 2$. Here, ψ can be any function that respects the above mentioned properties. Carrying on, let $\beta_{\tilde{y}_1} \in \mathbb{R}$ such that $\beta_{\tilde{y}_1} > 0$. We then define

$$\begin{aligned} \tilde{h}_1(y_1 - \tilde{y}_1, \sigma, \varepsilon, z) &= h_1\left(\psi\left(\frac{1}{\beta_{\tilde{y}_1}}(y_1 - \tilde{y}_1, \sigma, \varepsilon)\right)(y_1 - \tilde{y}_1, \sigma, \varepsilon), z\right) \\ \tilde{h}_2(y_1 - \tilde{y}_1, \sigma, \varepsilon, z) &= h_2\left(\psi\left(\frac{1}{\beta_{\tilde{y}_1}}(y_1 - \tilde{y}_1, \sigma, \varepsilon)\right)(y_1 - \tilde{y}_1, \sigma, \varepsilon), z\right). \end{aligned}$$

We set a new system :

$$\begin{aligned} \begin{pmatrix} \dot{y}_1 \\ \dot{\sigma} \\ \dot{\varepsilon} \\ \dot{z} \end{pmatrix} &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} y_1 - \tilde{y}_1 \\ \sigma \\ \varepsilon \end{pmatrix} + \begin{pmatrix} \tilde{h}_1(y_1 - \tilde{y}_1, \sigma, \varepsilon, z) \\ 0 \\ 0 \end{pmatrix} \\ &= Bz + \tilde{h}_2(y_1 - \tilde{y}_1, \sigma, \varepsilon, z). \end{aligned} \quad (4.16)$$

Since we study the presence of center manifolds in small neighborhoods around $(\tilde{y}_1, 0, 0, 0)$, for an appropriate $\beta_{\tilde{y}_1}$, the center manifolds of system (4.15) will coincide with the ones of system (4.16). Therefore, without loss of generality, we will study system (4.16) from now on.

The idea now is to set up an equation by defining $w_{\tilde{y}_1}$ as the fixed point of some function. We start by defining a proper space on which we will work.

Let $p, q \in \mathbb{R}$ such that $p, q > 0$. Define X as the set of all Lipschitz functions $w : \mathbb{R}^3 \rightarrow \mathbb{R}^{n-1}$ with Lipschitz constant p , such that $\|w(x)\| \leq q$ for any $x \in \mathbb{R}^3$ and $w(0) = 0$. Note that X is complete with respect to the metric induced by the supremum norm ($\|\cdot\|_\infty$) [1]. We then let $r_0 \in \mathbb{R}^3$ and $w \in X$. We define the curve $\gamma_{\tilde{y}_1}(t, r_0, w)$ to be the solution of

$$\begin{pmatrix} \dot{y}_1 \\ \dot{\sigma} \\ \dot{\varepsilon} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} y_1 - \tilde{y}_1 \\ \sigma \\ \varepsilon \end{pmatrix} + \begin{pmatrix} \tilde{h}_1(y_1 - \tilde{y}_1, \sigma, \varepsilon, w(y_1 - \tilde{y}_1, \sigma, \varepsilon)) \\ 0 \\ 0 \end{pmatrix}$$

such that $\gamma_{\tilde{y}_1}(0, r_0, w(r_0)) = r_0$.

We now construct a functional expression on X in order to properly pose our fixed point problem. Let $T_{\tilde{y}_1} : X \rightarrow X$ be defined as

$$(T_{\tilde{y}_1} w)(r_0) = \int_{-\infty}^0 e^{-Bs} \tilde{h}_2[\gamma_{\tilde{y}_1}(s, r_0, w), w(\gamma_{\tilde{y}_1}(s, r_0, w))] ds. \quad (4.17)$$

It can be shown, as in the proof of Theorem 2, that if a function $w_{\tilde{y}_1}$ is a fixed point of equation (4.17), then it locally defines a center manifold $W_{\tilde{y}_1}$ around $(\tilde{y}_1, 0, 0, 0)$.

In light of all this, we must investigate the presence of a fixed point for equation (4.17). For a wise choice of p, q and $\beta_{\tilde{y}_1}$ (in fact, for small enough p, q and $\beta_{\tilde{y}_1}$), it can be proved that $T_{\tilde{y}_1}$ is a contraction on X . As mentioned earlier, X is complete. Therefore, using Banach's Fixed Point Theorem (Theorem 3 in appendix), we have the existence of the function $w_{\tilde{y}_1}$. To show that $w_{\tilde{y}_1}$ is C^2 , the same reasoning is applied to the subset of all Lipschitz C^2 functions of X . Banach's Theorem also establishes the uniqueness of $w_{\tilde{y}_1}$ (for our chosen *cut off function*).

The domain of definition of $w_{\tilde{y}_1}$ is shown to be

$$U_{\tilde{y}_1} = \{(y_1, \sigma, \varepsilon) \mid \|\frac{1}{\beta_{\tilde{y}_1}}(y_1 - \tilde{y}_1, \sigma, \varepsilon)\| < 1\},$$

the open ball of radius $\beta_{\tilde{y}_1}$ around $(\tilde{y}_1, 0, 0, 0)$. Notice that $\beta_{\tilde{y}_1}$ and $U_{\tilde{y}_1}$ depend on $T_{\tilde{y}_1}$ which in turn depends on \tilde{y}_1 . Thus, for any $\tilde{y}_1 \in \mathbb{R}$, we have the unique expression for a C^2 , local center manifold $W_{\tilde{y}_1}$ as follows :

$$\begin{aligned} w_{\tilde{y}_1} : U_{\tilde{y}_1} &\rightarrow \mathbb{R}^{n-1} \\ (y_1 - \tilde{y}_1, \sigma, \varepsilon) &\mapsto z = w_{\tilde{y}_1}(y_1 - \tilde{y}_1, \sigma, \varepsilon). \end{aligned}$$

4.4.3 Globalization of Manifolds

Since our goal is to "glue" all these $W_{\tilde{y}_1}$ together, our first step must be to express the different manifolds in our standard set of coordinates $(y_1, y_2, \varepsilon, c)$. After this, we must find a way to combine these manifolds together to design a global one that will parametrize the curve $D_{(\varepsilon, c)}$ for fixed values of (ε, c) . We will also have to construct a global domain in the form of a tubular open set in \mathbb{R}^3 ($\ni (y_1, \varepsilon, c)$) running along all of $y_1 \in \mathbb{R}$.

Before carrying on with these tasks, we should mention an important aspect of the manifold bundle $\{W_{\tilde{y}_1} | \tilde{y}_1 \in \mathbb{R}\}$. An inspection of the construction of these manifolds reveals that the differences between the $W_{\tilde{y}_1}$ s arise from the values of the perturbation terms $G_1(\tilde{y}_1, 0, 0)$ and $G_2(\tilde{y}_1, 0, 0)$ for distinct \tilde{y}_1 . Indeed, the function \tilde{h}_2 which is used to define $T_{\tilde{y}_1}$ directly depends on the value of these terms. Since the original perturbation $G(\cdot, \varepsilon)$ has a compact support (see Table 3.1), the values of \tilde{y}_1 for which we get distinct $w_{\tilde{y}_1}$ also form a compact set. Furthermore, it is easy to see that for \tilde{y}_1 outside the support of $G(\cdot, \varepsilon)$ ($support(G(\cdot, \varepsilon))$), the manifold $W_{\tilde{y}_1}$ becomes trivial since the relative equilibrium D is no longer perturbed. As a result, for $y_1 \notin support(G(\cdot, \varepsilon))$,

$$w_{\tilde{y}_1}(y_1 - \tilde{y}_1, \sigma, \varepsilon) = 0 \quad \forall(\sigma, \varepsilon)$$

and we see the curve $D_{(\varepsilon, c)}$ coincide with D . Figure (4.4) illustrates this idea.

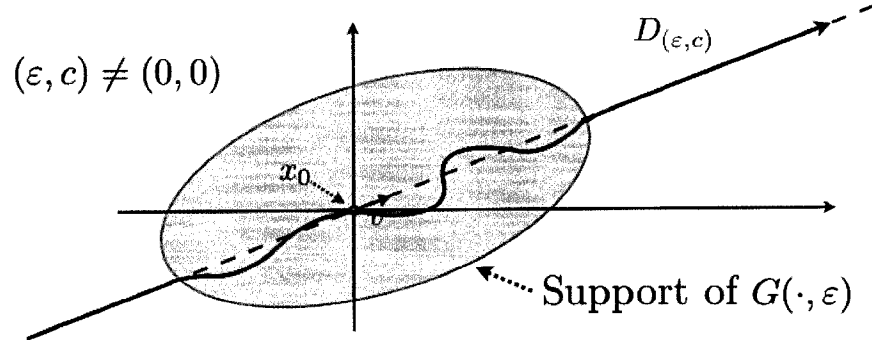


Figure 4.4: Localized action of the perturbation on D due to the compactness of $\text{support}(G(\cdot, \varepsilon))$.

Notice that for values of \tilde{y}_1 outside $\text{support}(G(\cdot, \varepsilon))$, we can assign any radius to the open balls $U_{\tilde{y}_1}$ as long as it does not intersect with $\text{support}(G(\cdot, \varepsilon))$. If it does, the radius becomes subject to the same cut-off argument as before. Notice also that a manifold defined for $\tilde{y}_1 \in \text{support}(G(\cdot, \varepsilon))$ that has a domain spanning outside of it has to vanish for this area of its domain. For these reasons, we must concentrate on defining a global map for $y_1 \in \text{support}(G(\cdot, \varepsilon))$ and the rest will trivially follow.

We begin by resetting our coordinates. Recall

$$\begin{pmatrix} y_1 \\ \sigma \\ \varepsilon \\ z \end{pmatrix} = \begin{pmatrix} y_1 \\ \varepsilon G_1(\tilde{y}_1, 0, 0) + c \\ \varepsilon \\ y_2 + \varepsilon B^{-1} G_2(\tilde{y}_1, 0, 0) \end{pmatrix}$$

and thus, from now on, when we refer to $w_{\tilde{y}_1}$, one should think of

$$w_{\tilde{y}_1}(y_1 - \tilde{y}_1, \varepsilon G_1(\tilde{y}_1, 0, 0) + c, \varepsilon) - \varepsilon B^{-1} G_2(\tilde{y}_1, 0, 0).$$

This new expression gives us a standard representation of the manifold bundle

$\{W_{\tilde{y}_1} | \tilde{y}_1 \in \mathbb{R}\}$ with each $W_{\tilde{y}_1}$ defined by the unique local map

$$\begin{aligned} w_{\tilde{y}_1} : \quad U_{\tilde{y}_1} &\rightarrow \mathbb{R}^{n-1} \\ (y_1, \varepsilon, c) &\mapsto y_2 = w_{\tilde{y}_1}(y_1, \varepsilon, c). \end{aligned}$$

Notice that the change of coordinates does not affect the C^2 derivability of our new expression $w_{\tilde{y}_1}$ since G_1 and G_2 are at least C^2 themselves.

We now want to design a way to combine the different $w_{\tilde{y}_1}$ s into one function that has every $y_1 \in \mathbb{R}$ in its domain. This can be viewed as the gluing of the $W_{\tilde{y}_1}$ s. Unfortunately, there is a problem since the local domains $U_{\tilde{y}_1}$ s overlap. Hence, we have to be careful to which manifold slice we assign a point (y_1, ε, c) if this point is in the domain of several $w_{\tilde{y}_1}$ s. We claim that two $w_{\tilde{y}_1}$ s that have domains which intersect will coincide over this intersection.

In order to verify this claim, we will need the definition of a center manifold that is used by Vanderbauwhede in his proof of the Center Manifold Theorem [17]. This proof is more general than the proof of Carr [1] we have been using so far. We did not need the heavy functional analysis machinery he uses but his approach to the theorem focusses on a simple fact that will prove to be quite useful.

Definition 6 (Center Manifold) *Let A be an $n \times n$ matrix, $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be C^k for $k \geq 1$ with $f(0) = 0$, $Df(0) = 0$ and bounded k -derivatives. Define $x(t, x_0)$ to be the solution of $\dot{x} = Ax + f(x)$ such that $x(0, x_0) = x_0$. Then a Center Manifold for $\dot{x} = Ax + f(x)$ around zero is*

$$W_c = \{x_0 \in \mathbb{R}^n | \sup_{t \in \mathbb{R}} \|\pi_h x(t, x_0)\| < \infty\}$$

where π_h is the standard projection in the hyperbolic subspace of A : the generalized eigenspaces associated with non-zero real part eigenvalues.[17]

In other words, the center manifold must contain any solution that does not have a hyperbolic behavior.

Now, recall system (4.4) :

$$\begin{aligned}\dot{y}_1 &= c + R_1(y_2) + \varepsilon G_1(y_1, y_2, \varepsilon) \\ \dot{y}_2 &= R_2(y_2) + \varepsilon G_2(y_1, y_2, \varepsilon) \\ \dot{\varepsilon} &= 0 \\ \dot{c} &= 0\end{aligned}$$

on which we are working. We verified the existence of local center manifolds for this systems at any $(y_1, 0, 0, 0)$ by showing the existence of global center manifolds for the cut-off versions of the Taylor expansions of (4.4) around each $(y_1, 0, 0, 0)$: system (4.12)

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{\varepsilon} \\ \dot{c} \end{pmatrix} = \mathcal{F} \begin{pmatrix} y_1 \\ y_2 \\ \varepsilon \\ c \end{pmatrix} = A_{\tilde{y}_1} \begin{pmatrix} y_1 - \tilde{y}_1 \\ y_2 \\ \varepsilon \\ c \end{pmatrix} + \begin{pmatrix} H_1(y_1 - \tilde{y}_1, y_2, \varepsilon, c) \\ H_2(y_1 - \tilde{y}_1, y_2, \varepsilon, c) \\ 0 \\ 0 \end{pmatrix}.$$

Note that the manifolds $w_{\tilde{y}_1}$ we computed are unique in the cut-off neighborhoods of $(\tilde{y}_1, 0, 0, 0)$ but lose their unicity if we consider them globally. It is important to see that since we work precisely in these neighborhoods $(U_{\tilde{y}_1})$, we may consider them as unique.

Let us now take two points $\kappa_1, \kappa_2 \in \mathbb{R}$ such that $U_{\kappa_1} \cap U_{\kappa_2} \neq \emptyset$. It follows that the functions w_{κ_1} and w_{κ_2} overlap on this intersection. Chose a point $(y_{10}, \varepsilon_0, c_0) \in U_{\kappa_1} \cap U_{\kappa_2}$ and consider the solution $x_0(t)$ of the system (4.4) such that

$$x_0(0) = (y_{10}, w_{\kappa_1}(y_{10}, \varepsilon_0, c_0), \varepsilon_0, c_0).$$

Let \tilde{x}_0 be the corresponding solution of the cut-off system which is equal to x_0 around $(\kappa_1, 0, 0, 0)$. Since the map w_{κ_1} is a local center manifold for system (4.4) around $(\kappa_1, 0, 0, 0)$, it follows from Definition 6 that the solution $\tilde{x}_0(t)$ cannot have a *hyperbolic behavior* :

$$\sup_{t \in \mathbb{R}} \|\pi_h(\kappa_1) \tilde{x}_0(t)\| < \infty \quad (4.18)$$

where $\pi_h(\kappa_1)$ is the standard projection in the hyperbolic subspace proper to κ_1 .

Let $y_{20}(t)$ and $\varepsilon_0(t)$ be respectively the y_2 and ε components of $\tilde{x}_0(t)$. Note that in our suspended system (4.4), $\dot{\varepsilon} = 0$ and thus, $\varepsilon_0(t) = \varepsilon_0$ for all t . From our coordinate system proper to κ_1 , we can see that

$$\pi_h(\kappa_1)\tilde{x}_0(t) = y_{20}(t) + \varepsilon_0 B^{-1}G_2(\kappa_1, 0, 0). \quad (4.19)$$

We have that (4.18) and (4.19), together with the fact that G_2 is bounded imply that

$$\sup_{t \in \mathbb{R}} \|y_{20}(t)\| < \infty. \quad (4.20)$$

In turn, it follows from (4.20) that

$$\sup_{t \in \mathbb{R}} \|y_{20}(t) + \varepsilon_0 B^{-1}G_2(\kappa_2, 0, 0)\| < \infty$$

which yields

$$\sup_{t \in \mathbb{R}} \|\pi_h(\kappa_2)\tilde{x}_0(t)\| < \infty.$$

According to Definition 6, this means that the solution $\tilde{x}_0(t)$ must also live on w_{κ_2} since $(y_{10}, \varepsilon_0, c_0) \in U_{\kappa_2}$. It follows from the unicity of local center manifolds and the unicity of solutions that

$$x_0(0) = \tilde{x}_0(0) = (y_{10}, w_{\kappa_1}(y_{10}, \varepsilon_0, c_0), \varepsilon_0, c_0) = (y_{10}, w_{\kappa_2}(y_{10}, \varepsilon_0, c_0), \varepsilon_0, c_0).$$

By repeating this argument for all $(y_{10}, \varepsilon_0, c_0) \in U_{\kappa_1} \cap U_{\kappa_2}$, it follows that

$$w_{\kappa_1}|_{U_{\kappa_1} \cap U_{\kappa_2}} = w_{\kappa_2}|_{U_{\kappa_1} \cap U_{\kappa_2}}. \quad (4.21)$$

Figure 4.5 illustrates this idea.

Equation (4.21) now confirms an important intuition one would get by looking at the manifold bundle $\{W_{\tilde{y}_1} | \tilde{y}_1 \in \mathbb{R}\}$. Indeed, it tells us that the $W_{\tilde{y}_1}$ s locally depict a unique manifold W that runs over all of $\tilde{y}_1 \in \mathbb{R}$ and (ε, c) close enough to zero. Again, for fixed (ε, c) close enough to zero, this manifold will parametrize the curve

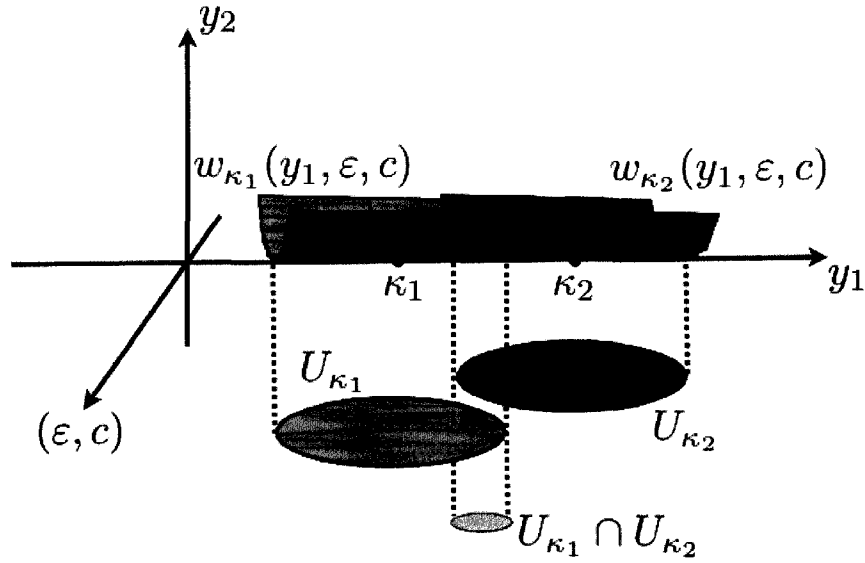


Figure 4.5: Manifolds with overlapping domains.

$D_{(\varepsilon, c)}$. We must now formally express what it means to be close enough to zero and define a map for the manifold W .

Although it is somewhat counterintuitive, we begin by defining an expression for W before its domain. We call this function

$$y_2 = \omega(y_1, \varepsilon, c).$$

It will be defined for all $y_1 \in \mathbb{R}$ and we will find a constant bound (with respect to y_1) for (ε, c) later on.

Again, we know that for any \tilde{y}_1 well outside $\text{support}G(\cdot, \varepsilon)$, $w_{\tilde{y}_1} \equiv 0$. Therefore, we set $\omega(y_1, \varepsilon, c) := 0$ for any (y_1, ε, c) such that $y_1 \notin \text{support}(G(\cdot, \varepsilon))$ and we concentrate on $y_1 \in \text{support}(G(\cdot, \varepsilon))$. Since this set is compact and the collection $\{U_{y_1} | y_1 \in \text{support}(G(\cdot, \varepsilon))\}$ forms an open cover of it, we can find $\{\kappa_1, \kappa_2, \dots, \kappa_m\} \subset \text{support}(G(\cdot, \varepsilon))$ such that $\{U_{\kappa_i}\}_{1 \leq i \leq m}$ is a finite subcover.

Notice that for values of κ_i on the edges of $\text{support}(G(\cdot, \varepsilon))$, $w_{\kappa_i}(y_1, \varepsilon, c) \rightarrow 0$ as y_1 exits $\text{support}(G(\cdot, \varepsilon))$. This is due to the continuity of the perturbation G and the effect it has on the transverse dynamics of System (4.4).

We can now define the map ω :

$$\omega(y_1, \varepsilon, c) = \begin{cases} 0 & \text{if } y_1 \notin \text{support}(G(\cdot, \varepsilon)) \\ w_{\kappa_j}(y_1, \varepsilon, c) & \text{when } y_1 \in U_{\kappa_j}, \text{ for some } \kappa_j, j = 1, 2, \dots, m. \end{cases}$$

It is crucial to notice that if a point lies in more than one U_{κ_j} , it does not matter to which w_{κ_j} we assign it, since the latter must coincide on any intersection. This guarantees well definedness. Furthermore, we know that every w_{κ_i} is C^2 . It follows from construction that ω must also be C^2 .

We move on to define a proper domain V for ω . We wish to find a bound $\xi > 0$ such that if $\|(\varepsilon, c)\| < \xi$, then $\omega(y_1, \varepsilon, c)$ will be defined for any y_1 . Ultimately, we are interested in what happens around $(\varepsilon, c) = (0, 0)$. Therefore, the size of ξ does not matter, as long as it is greater than zero.

Consider every non-empty intersection $U_{\kappa_i} \cap U_{\kappa_j}$, $i, j = 1, 2, \dots, m$. Recall that each U_{κ} is a neighborhood of $(\kappa, 0, 0)$. Since these are intersections of open sets, we may find open balls of radius $\xi_{i,j}$ centered somewhere on the line $(y_1, 0, 0)$ and contained in the respective intersections. Finally, since there are finitely many such intersections, we let $\xi = \min\{\xi_{i,j}\}$. Note that we can always make ξ bigger by adding finitely many U_{κ} s to our finite subcover.

We then define our tubular domain

$$V = \{(y_1, \varepsilon, c) \mid \|(\varepsilon, c)\| < \xi, \forall y_1 \in \mathbb{R}\}$$

which passes through every U_{κ_i} and their intersections. Figure 4.6 sketches the set V .

This domain lets us formally define the map ω as follows

$$\begin{aligned} \omega : \quad V &\rightarrow \mathbb{R}^{n-1} \\ (y_1, \varepsilon, c) &\mapsto y_2 = \omega(y_1, \varepsilon, c) \end{aligned}$$

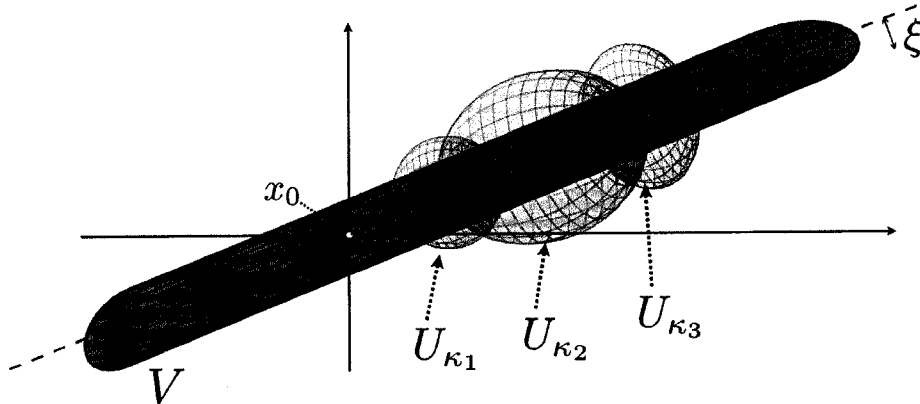


Figure 4.6: Sketch of the domain V and different U_{κ} s for $\kappa_1, \kappa_2, \kappa_3 \in \mathbb{R}$.

and we get $W = \omega(V)$.

We can now represent the sought curves $D_{(\varepsilon, c)}$: for any fixed (ε, c) within V , we get

$$D_{(\varepsilon, c)} = \{\omega(y_1, \varepsilon, c) | y_1 \in \mathbb{R}\}.$$

Recall that we designed $D_{(\varepsilon, c)}$ as the equivalent to the relative equilibrium D for our perturbed system. Since we have constructed this curve $D_{(\varepsilon, c)}$ from center manifolds of a suspended system of (3.2), it follows that the dynamics of that same system will be invariant on the curve. We are interested in the solution(s) that live(s) on $D_{(\varepsilon, c)}$ since it/they represent(s) the evolution of the traveling wave we are studying. The described differentiability of $D_{(\varepsilon, c)}$ allows for these solutions to live on it. As for the stability of the dynamics of system (3.2) on $D_{(\varepsilon, c)}$, it follows from the stability of center manifolds that $D_{(\varepsilon, c)}$ is also stable.

Thus, we can say that for parameters (ε, c) close enough to $(0, 0)$, the relative equilibrium of system (3.1) persists under perturbation (system (3.2)) in the form of a stable flow-invariant C^2 curve in \mathbb{R}^n . In the next chapter, we will investigate the dynamics on $D_{(\varepsilon, c)}$ in an attempt to understand the blocking phenomenon via a bifurcation analysis of this/these solution(s). This concludes Chapter 4.

Chapter 5

Bifurcation Analysis

Before we engage in any form of bifurcation analysis, let us take a look at what we have accomplished so far. We started our study on a system of the form $\dot{x} = F(x) + \varepsilon G(x, \varepsilon)$. In the unperturbed case where $\varepsilon = 0$, we associated the presence of a traveling wave to the existence of a asymptotically stable relative equilibrium D , a flow-invariant line with respect to the dynamics of $\dot{x} = F(x)$. The traveling wave solution of that system is precisely the solution living on D . We then showed that for small enough (ε, c) , this line persists in the form of a C^2 flow-invariant curve in $\mathbb{R}^n : D_{(\varepsilon, c)}$. The traveling wave solution of the perturbed system is represented by the dynamics of our system on this curve.

To address the wave blocking phenomenon of such a system under symmetry breaking perturbation, we must investigate the form of solutions that live on $D_{(\varepsilon, c)}$. In the case where there is a unique solution that spans all of $D_{(\varepsilon, c)}$, we can say that the wave persists although its velocity will not necessarily be constant. On the other hand, if we can observe the appearance of an equilibrium point solution on $D_{(\varepsilon, c)}$, this would mean that the wave comes to a standstill at that given point and hence, be blocked. Due to the general nature of the assumptions we made concerning our system, it is not in our intentions to precisely compute the location

of such equilibrium points but rather to arrive at some conditions as to when they will appear. Recall that the work done in this thesis has for goal to rigorously show and explain the presence of wave blocking for our chosen class of systems.

5.1 Dynamics on the Invariant Curve $D_{(\varepsilon,c)}$

We need to come up with an equation that represents the dynamical behavior of the solution(s) that live(s) on $D_{(\varepsilon,c)}$. Since we have shown that $D_{(\varepsilon,c)}$ is C^2 and we know that it is invariant for the perturbed system, we can conclude that there exists a one-dimensional ODE describing this dynamical system. We write this equation the following way :

$$\dot{y}_1 = Q(y_1, \varepsilon), \quad y_1 \in \mathbb{R}, \quad \varepsilon \text{ fixed.} \quad (5.1)$$

Note that equation (5.1) describes the dynamics on $D_{(\varepsilon,c)}$. Therefore, to get a portrait of a given solution $y_1(t)$ in \mathbb{R}^n , we have to express it via the parametrization of $D_{(\varepsilon,c)}$. The solution then becomes :

$$t \mapsto (y_1(t), \omega_{\varepsilon,c}(y_1(t)))$$

which is a solution of system (3.2) if $(\varepsilon, c) \in V$. Here, $Q : \mathbb{R} \times \mathbb{R}^+ \rightarrow \mathbb{R}$ can be computed by restricting the dynamics of system (3.2) on the curve $D_{(\varepsilon,c)}$. This restriction obviously depends on the nature of F and G in our original system. The assumptions we used up to now are too broad to arrive at a general strategy as to how we can do this precisely. Fortunately, we do not need such a computation in order to carry out our analysis as the existence of such a representation is all we need. One should only keep in mind that Q depends on F and G and is sufficiently smooth.

Let us now take the Taylor expansion of Q around $\varepsilon = 0$:

$$Q(y_1, \varepsilon) = Q(y_1, 0) + \varepsilon P(y_1, \varepsilon)$$

where P is $\mathcal{O}(1)$ in ε . Recall that when $\varepsilon = 0$, $D_{(0,c)} = D$. The restriction of the unperturbed system on the line D yields a solution with constant velocity c . It follows that we must have $Q(y_1, 0) = c$ for any y_1 and we write

$$Q(y_1, \varepsilon) = c + \varepsilon P(y_1, \varepsilon).$$

Therefore, system (5.1) becomes

$$\dot{y}_1 = c + \varepsilon P(y_1, \varepsilon). \quad (5.2)$$

Equation (5.2) represents the dynamics on $D_{(\varepsilon,c)}$ while clearly showing the role of our parameters c and ε . We will use this representation to analyze the possible bifurcations that can occur.

5.2 Bifurcations of the Traveling Wave Solution

As mentioned earlier, the occurrence of a wave block coincides with the presence of a fixed point of the dynamics on $D_{(\varepsilon,c)}$. We must inquire about the values of c and ε for which we can observe such fixed points.

We have that a point $x \in \mathbb{R}$ is left fixed by the dynamics of system (5.2) if

$$\begin{aligned} c + \varepsilon P(x, \varepsilon) &= 0 \\ \Rightarrow \varepsilon P(x, \varepsilon) &= -c. \end{aligned} \quad (5.3)$$

Since we do not know what is P precisely, we can look at its bounds. Notice that for x outside $\text{support}(G(\cdot, \varepsilon))$, $P(x, \varepsilon) = 0$ since there is no perturbation effect on our original system and $|\dot{x}|$ must equal c . Since $\text{support}(G(\cdot, \varepsilon))$ is compact, it follows that $\text{support}(P(\cdot, \varepsilon))$ must also be compact for any $(\varepsilon, c) \in V$. Therefore, we can conclude that for any $(\varepsilon, c) \in V$, $P(x, \varepsilon)$ must reach its minimum and maximum for $x \in \mathbb{R}$. We write

$$\begin{aligned} m(\varepsilon) &= \min_{x \in \mathbb{R}} \{P(x, \varepsilon)\} \\ M(\varepsilon) &= \max_{x \in \mathbb{R}} \{P(x, \varepsilon)\}. \end{aligned} \quad (5.4)$$

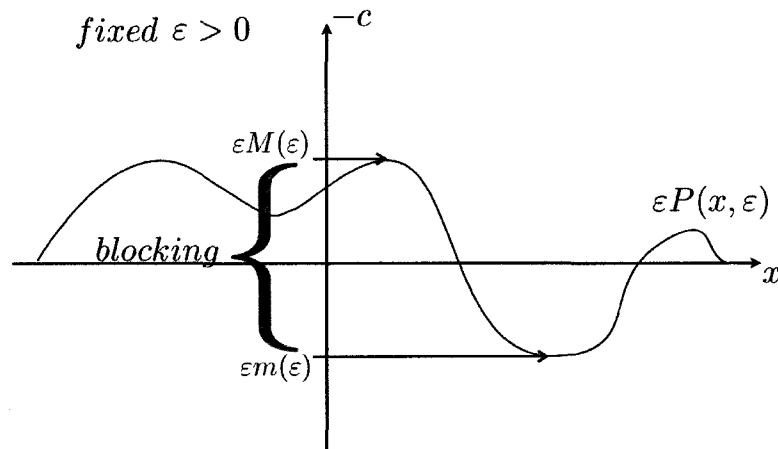


Figure 5.1: For a fixed value of ε , we observe wave blocking for values of $-c$ that can be reached by $\varepsilon P(x, \varepsilon)$.

We can see that if $-c > \varepsilon M(\varepsilon)$ or if $-c < \varepsilon m(\varepsilon)$, then equation (5.3) cannot be satisfied. In addition, since P is at least C^1 , we can use the Implicit Function Theorem to guarantee the presence of a function $x(\varepsilon, c)$ such that

$$\varepsilon P(x(\varepsilon, c), \varepsilon) + c = 0, \quad \forall c \in [-\varepsilon M(\varepsilon), -\varepsilon m(\varepsilon)]. \quad (5.5)$$

Figure 5.1 illustrates this idea for a fixed value of ε . Notice also that $M(\varepsilon) \geq 0$ and $m(\varepsilon) \leq 0$ since $\text{support}(P(\cdot, \varepsilon)) \neq \mathbb{R}$.

We now have all the tools to sketch a bifurcation portrait of our system. We can observe two branches ($-\varepsilon m(\varepsilon)$ and $-\varepsilon M(\varepsilon)$) that separate our parameter space in three regions. One of these regions represents values of (ε, c) for which there will be fixed points for the dynamics of equation (5.2) and hence, wave blocking. Figure 5.2 sketches this idea.

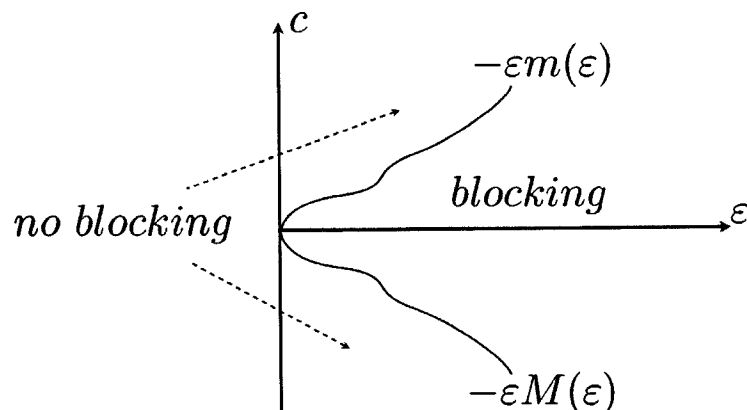


Figure 5.2: Sketch of the region in parameter space where we can observe wave blocking.

It is crucial to notice that the bifurcation diagram presented in figure (5.2) is only valid for $(\varepsilon, c) \in V$, our tubular neighborhood outside of which we are not guaranteed of the existence of the invariant curve $D_{(\varepsilon, c)}$. Recall that we defined V in Chapter 4 as the set $\{(y_1, \varepsilon, c) \mid \|(\varepsilon, c)\| < \xi, \forall y_1 \in \mathbb{R}\}$ with $\xi > 0$ carefully chosen. It follows that our bifurcation analysis is valid in the semi-disk $\{(\varepsilon, c) \mid \|(\varepsilon, c)\| < \xi, \varepsilon > 0\}$ as depicted in Figure 5.3. Notice that the important aspect of this result is to understand what happens around the *organizing center* of bifurcations : $(\varepsilon, c) = (0, 0)$. Therefore, this remark should not be viewed as a limit to our bifurcation analysis.

We now define properly the regions in our half-disk parameter space in which we

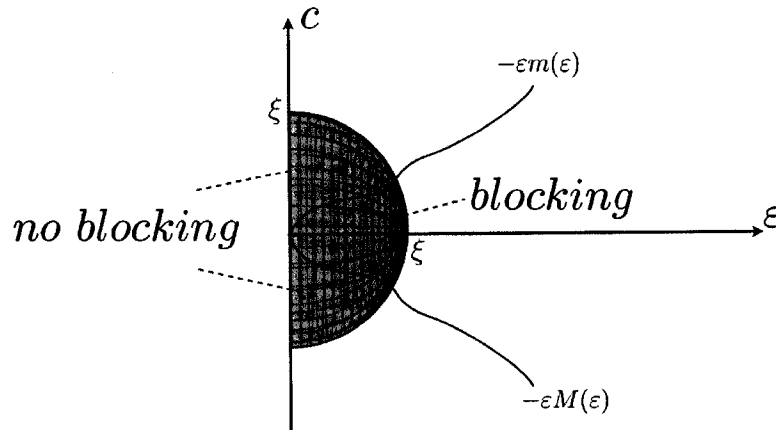


Figure 5.3: Sketch of bifurcation diagram with proper domain.

have topologically distinct solutions on $D_{(\varepsilon, c)}$ as illustrated in Figure 5.3.

1.NO BLOCKING

[unique solution]

$$\{(\varepsilon, c) | c > -\varepsilon m(\varepsilon), \|(\varepsilon, c)\| < \xi\} \cup \{(\varepsilon, c) | c < -\varepsilon M(\varepsilon), \|(\varepsilon, c)\| < \xi\} \quad (5.6)$$

2.BLOCKING

[presence of equilibrium points]

$$\{(\varepsilon, c) | -\varepsilon M(\varepsilon) \leq c \leq -\varepsilon m(\varepsilon), \|(\varepsilon, c)\| < \xi\}$$

As we focus our analysis around the origin, there is another insightful fact worthy of notice. If we take the Taylor expansion of $P(x, \varepsilon)$ around $\varepsilon = 0$ and look at its degree zero part, it can give us a good approximation of the curves $-\varepsilon m(\varepsilon)$ and $-\varepsilon M(\varepsilon)$ for small values of ε since the higher terms become negligible.

$$P(x, \varepsilon) = P(x, 0) + \varepsilon \frac{d}{d\varepsilon} P(x, 0) + \dots$$

We can then write the approximation

$$\begin{aligned}\varepsilon M(\varepsilon) &\approx \max_{x \in \mathbb{R}} \{\varepsilon P(x, 0)\} = \varepsilon M(0) \\ \varepsilon m(\varepsilon) &\approx \min_{x \in \mathbb{R}} \{\varepsilon P(x, 0)\} = \varepsilon m(0).\end{aligned}\tag{5.7}$$

The angle of the wedge between the two lines $-\varepsilon m(0)$ and $-\varepsilon M(0)$ can often prove to be an important characteristic for a given system as it governs the shapes of the regions 1 and 2 of figure (5.3). Figure (5.4) illustrates.

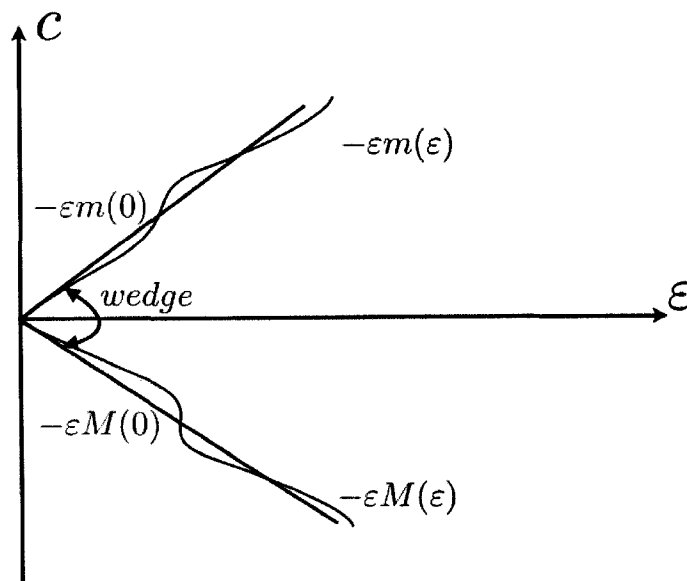


Figure 5.4: Sketch of the approximation of the bifurcation branches of equation (5.7).

In some cases, it might be possible to use this type of approximation by taking truncated Taylor expansions of the original vector field of equation (3.2) in carefully chosen directions.

5.3 Characterizing the Bifurcations

Our final task will be to investigate the kind of bifurcations that the traveling wave solution living on $D_{(\varepsilon,c)}$ will undergo. Recall equation (5.2) : $\dot{x} = c + \varepsilon P(x, \varepsilon)$ where $P(\cdot, \varepsilon)$ is at least C^1 and has compact support for any ε in our domain of interest. We get the presence of equilibrium points on $D_{(\varepsilon,c)}$ if we have values of ε and c such that $c + \varepsilon P(x, \varepsilon) = 0$ has a solution. We may ask ourselves how many of these solutions exist.

To answer this question, let us fix $\varepsilon = \varepsilon_0 > 0$ and investigate what happens if we vary c . Suppose that $P(x, \varepsilon_0)$ reaches its maximum $M(\varepsilon_0)$ at x_M and its minimum $m(\varepsilon_0)$ at x_m . These points might not be unique. We know that $P(\cdot, \varepsilon_0)$ has compact support and thus, must eventually reach zero on both sides of the real line. Let $a \in \mathbb{R}$ and $b \in \mathbb{R}$ be two points outside the support of $P(\cdot, \varepsilon_0)$ and respectively on both sides of it so that the intervals $(-\infty, a]$ and $[b, \infty)$ do not intersect $\text{support}(P(\cdot, \varepsilon_0))$.

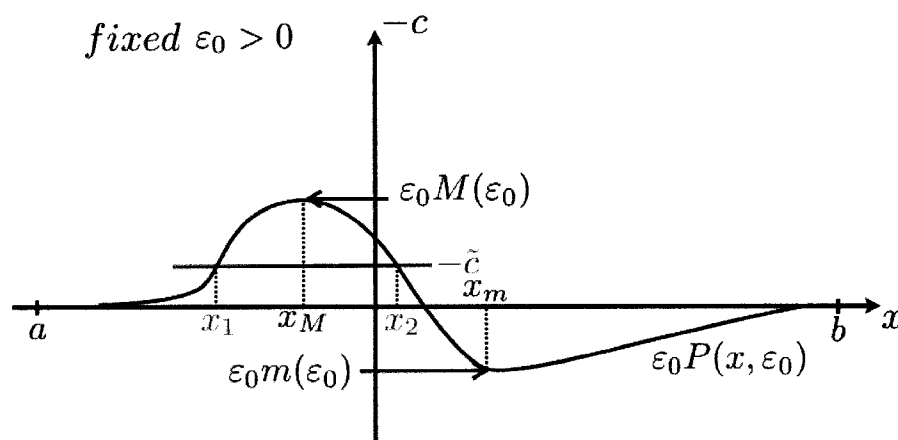


Figure 5.5: Presence of multiple equilibrium points for $\varepsilon_0 m(\varepsilon_0) < -\tilde{c} < \varepsilon_0 M(\varepsilon_0)$.

We use the Intermediate Value Theorem on the intervals $[a, x_M]$ and $[x_M, b]$ and the work done in this chapter to conclude that for any value $-\tilde{c}$ strictly between 0 and $\varepsilon_0 M(\varepsilon_0)$, there must be at least two points x_1 and x_2 such that $-\tilde{c} = \varepsilon_0 P(x_{1,2}, \varepsilon_0)$.

We do the same with x_m for values of $-\tilde{c}$ strictly between $\varepsilon_0 m(\varepsilon_0)$ and 0. Therefore, x_1 and x_2 are distinct fixed points for the dynamics on $D_{(\varepsilon,c)}$. Figure 5.5 sketches this concept.

Suppose for now that the points x_M and x_m are unique. Consider the evolution of $-\tilde{c}$ as it goes from $-\tilde{c} > \varepsilon_0 M(\varepsilon_0)$ to $-\tilde{c} = \varepsilon_0 M(\varepsilon_0)$ and finally to $\varepsilon_0 m(\varepsilon_0) < -\tilde{c} < \varepsilon_0 M(\varepsilon_0)$. For the first stage, there is no fixed point. At the second stage, there is the appearance of a single fixed point that splits into two as $-\tilde{c}$ enters the third stage. We can see that the dynamics on $D_{(\varepsilon,c)}$ undergoes a *saddle-node* bifurcation (see Kuznetsov [12] for details about this type of bifurcation). The stability of these fixed points can easily be deduced from context. Figure (5.6) illustrates this idea.

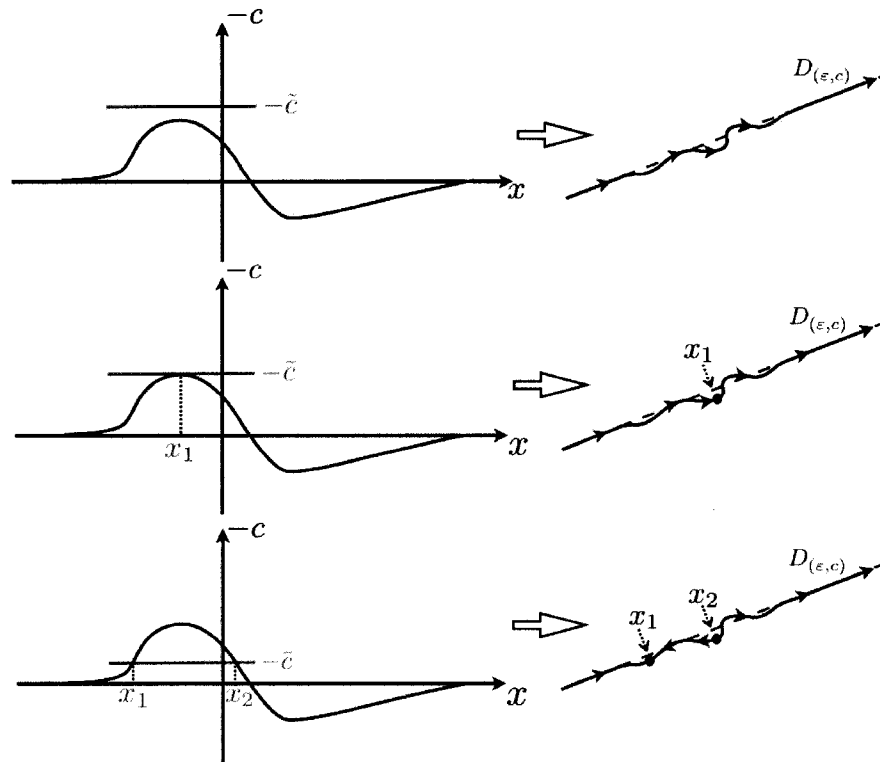


Figure 5.6: As $-\tilde{c}$ reaches $\varepsilon_0 M(\varepsilon_0)$, an equilibrium point appears and then splits into two.

Notice that the same phenomenon occurs when we approach the curve $\varepsilon_0 P(x, \varepsilon_0)$ from underneath. A *saddle-node* bifurcation arises when we reach $\varepsilon_0 m(\varepsilon_0)$ with $-\tilde{c}$.

We must now address the fact that the function $P(x, \varepsilon_0)$ could reach its minimum and/or maximum at more than one point. Following the same reasoning as above, there will be as many *saddle-node* bifurcations as there are such points. Furthermore, the same thing will happen at each local minimum or maximum as Figure 5.7 shows.

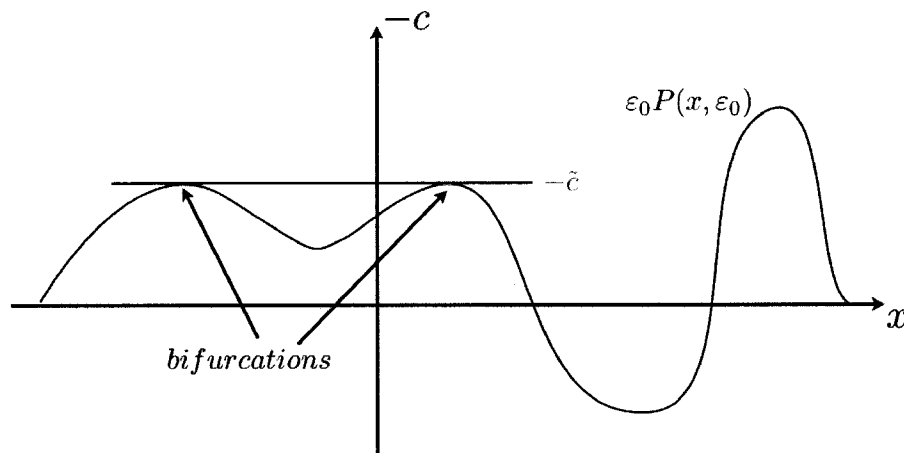


Figure 5.7: Multiple *saddle-node* bifurcations.

The important thing to note is that every change in the dynamics on $D_{(\varepsilon, c)}$ will generically occur via a *saddle-node* bifurcation as we vary c . There might be exceptions for degenerate cases such as when the extremum values $\varepsilon_0 M(\varepsilon_0)$ and $\varepsilon_0 m(\varepsilon_0)$ are reached by step sections of $\varepsilon_0 P(x, \varepsilon_0)$ where we would get fixed point intervals instead of separate pairs of fixed points. Finally, we note that this reasoning is valid for any (ε_0, c) in our domain V .

Although it is worthy of interest to understand which type of bifurcation triggers wave blocking under the effect of a symmetry breaking perturbation, we should keep in mind an important fact. If our only wish is to know if there will be wave blocking

for some parameter values, we only need to ask if there will be at least one equilibrium point on $D_{(\varepsilon,c)}$. Every other "blocking point" is overkill.

On the other hand, if one wishes to build a mechanism to force the traveling wave through one of these "blocking points", it becomes crucial to know that the deeper we venture in the blocking region of parameter space, the greater the chance is that more "blocking points" will appear.

We have now explored the effects of various parameter values on the nature of the solution living on the invariant curve $D_{(\varepsilon,c)}$. We have come up with the general shape of the regions of parameter space that characterize the aspect of the problem of wave blocking.

We can conclude that for systems respecting the assumptions of Table 3.1, the traveling wave solution will be blocked by the symmetry-breaking perturbation for parameter values described in this chapter. Furthermore, we can say that this wave block is due to a *saddle-node* bifurcation of the traveling wave solution which emanates from the organizing center $(\varepsilon, c) = (0, 0)$. This concludes this chapter as well as the proof of Theorem 1.

Chapter 6

Conclusion

6.1 Summary

In this thesis, we wanted to study the phenomenon of wave blocking with generality in mind. Our dynamical system approach harvested important geometric properties of systems where traveling waves are observable. By doing so, we could arrive at a very concise set of conditions a given system has to respect, in order to qualify for our result. These conditions are listed in Table 3.1. We then generated a formulation of our result in the form of Theorem 1.

Essentially, we used the symmetries of our candidate ODE system, describing a traveling wave, in order to decompose the state space into tangent and transverse spaces. We constructed a representation of the dynamics of our system in those spaces. We then used properties of center manifold and a carefully constructed parametrization of an invariant curve to represent the wave under symmetry breaking perturbation of the original ODE. A bifurcation analysis of the possible solutions one could observe on this curve followed. We concluded that wave blocking occurs when there are equilibrium points of the dynamics on the same curve. We also produced the general shape of the regions of parameter space (velocity v.s. perturbation) where these equilibrium points exist. We finished by showing that the appearance of such

immobile solutions must happen via a *saddle-node* bifurcation.

6.2 Discussion

As mentioned earlier, this section has for a goal to foresee possible future research that could be undertaken to extend the results from this thesis. After having gone through the demonstrations of this document, we can now address this in more details.

Recall from Chapter 1 that the general setting for a perturbed PDE system in which we observe a traveling wave is an ODE with the properties listed in Table 3.1. The only difference is that we replace \mathbb{R}^n with a infinite dimensional space [7]. Indeed, the solutions to such a system are elements of a space X , that we can picture as $C^k(\mathbb{R}^l, \mathbb{R}^m)$. In fact, what changes from our approach is the nature of the transverse space to our symmetry (labeled y_2). Instead of being \mathbb{R}^{n-1} and thus have $n-1$ dimensions, we have an infinite dimensional space of functions. The problem in our demonstration, when considering this difference, reveals itself when we use the stability assumption of the relative equilibrium D . Indeed, in finite dimensions, this stability implies that the eigenvalues from the linear operator of the transverse dynamics all have strictly negative real parts. In turn, this fact guarantees the presence of a stable invariant manifold. Although there is an equivalent concept in infinite dimensions, one has to be careful when working with non-finite spectra. In order to achieve a similar result, one would have to derive a precise set of conditions for this spectrum –and ultimately the governing equation– to properly bridge the concept of transverse stability from finite to infinite dimensions. In addition, a special definition of transversality would have to be constructed since the concept of orthogonality cannot always be taken for granted in infinite dimensional spaces. It is not hard to imagine that such a set of conditions exists.

We now consider the inclusion of time-dependent systems to our assumptions. Indeed, some physical phenomena seem to have changing characteristics as time goes by. The equations modeling such phenomena would likely have a dependency on time.

There exists objects called Lagrangian Coherent Structures (LCS) that could come handy. A LCS is a similar structure to an invariant manifold but has the property that it varies in time. They are, for our considerations, evolving invariant manifolds. Good descriptions of such objects and their properties can be found in article by J. Marsden [13] as well as in an article by G. Haller [6].

If we were to consider an ODE with a translational symmetry but that is also time-dependent, we could conceive a parallel between the center manifolds slices $W_{\tilde{y}_1}$ and LCS slices. By following the same procedure as in Chapter 4, it would be conceivable to arrive to a similar representation of the invariant curve $D_{(\varepsilon,c)}$, but with time dependence. Of course, conditions for stability of these LCS and of smoothness of their time-evolution are examples of attributes that have to be attended to. The idea of LCS is a relatively new concept in dynamical system theory. The literature on the subject is emerging and there is still a considerable amount of unknowns preventing results as general as the Center Manifold Theorem. Nevertheless, the inclusion of some special, non-autonomous systems to our assumptions is not so unthinkable. One could distill conditions that guarantee a nice behavior of the LCS slices attached to our relative equilibrium D .

There is another remark worthy of notice. Translational symmetry is a concept linked to the traveling wave phenomenon, but there are many other types of symmetries for which the reasoning we used could hold. For example, a rotational symmetry is linked to rotating and spiral waves. By deriving transverse dynamics to a circular relative equilibrium, one could come up with a family of perturbations that could cause wave blocking. Indeed, instead of having a tubular neighborhood in parameter space to get the existence of the invariant curve $D_{(\varepsilon,c)}$, one could conceive some sort

of toral neighborhood and a closed invariant curve. Parallels with other forms of symmetries are also foreseeable.

As the reader can notice, there are many ways in which the result we presented could be extended. Symmetry approaches to dynamical problems are powerful tools and prove to deliver surprising results. We can only imagine what future research might generate.

Appendix A

Useful Theorems

Theorem 2 (Existence of Center Manifold) *Consider the system*

$$\begin{aligned}\dot{x} &= Ax + f(x, y) \\ \dot{y} &= By + g(x, y)\end{aligned}\tag{A.1}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, A is a $n \times n$ matrix with zero real part eigenvalues, B is a $m \times m$ matrix with negative real part eigenvalues and f, g are C^2 functions such that $f(0, 0) = 0 = g(0, 0)$ and $Df(0, 0) = 0 = Dg(0, 0)$. Then system (A.1) has a local center manifold W^c that can be expressed by $y = w(x)$ for $\|x\| < \delta$ and some $\delta > 0$ and w is a C^2 function.

A proof along with details of this theorem can be found in Carr [1], page 16.

Theorem 3 (Banach's Fixed Point Theorem) *Let F be a contraction on a closed subset U of a complete metric space X . Then there is a unique $\tilde{x} \in U$ such that $F(\tilde{x}) = \tilde{x}$.*

A proof of this theorem can be found in Granas and Dugundji [5].

Appendix B

Table and Figures

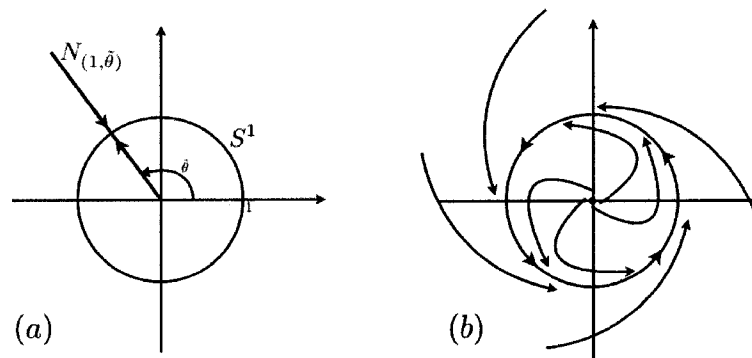


Figure B.1: [Figure (2.1)] (a) Dynamics of the flow of f_{tr} on $N_{(1, \bar{\theta})}$. (b) Dynamics of the flow of f on \mathbb{R}^2 .



Figure B.2: [Figure 3.1] Traveling Wave.

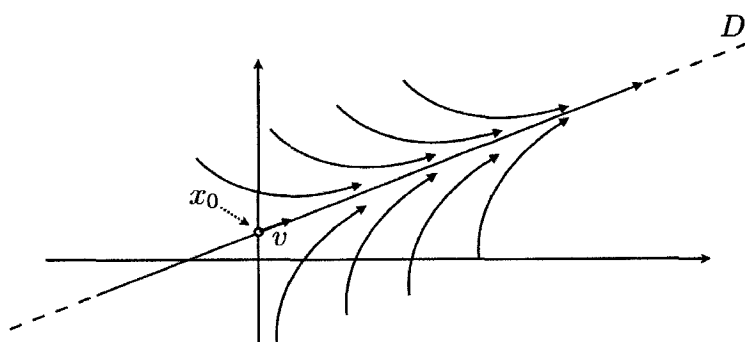


Figure B.3: [Figure 3.2] Asymptotically Stable Relative Equilibrium.

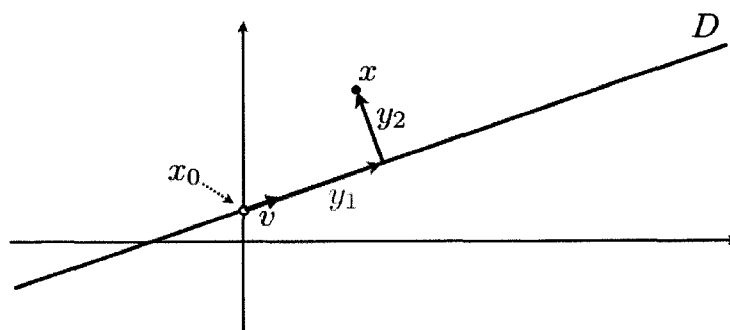


Figure B.4: [Figure 3.3] Decomposition of \mathbb{R}^n .

$F(x)$	$F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ At least C^2 Translational symmetry : $F(x + \alpha v) = F(x) \forall \alpha \in \mathbb{R}$ where $v \in \mathbb{R}^n, \ v\ = 1$
D	$D = \{x_0 + \alpha v \alpha \in \mathbb{R}\}$ where $F(x_0) \parallel v$ Asymptotically stable relative equilibrium
$G(x, \varepsilon)$	$G : \mathbb{R}^n \times \mathbb{R}^+ \rightarrow \mathbb{R}^n$ At least C^2 Does <u>not</u> admit the same translation symmetry as F Bounded Compact support with respect to x
ε	Perturbation parameter ≥ 0
c	Velocity of the solution on D $ c = \ F(x_0)\ $

Table B.1: [Table 3.1] Properties of the perturbed system $\dot{x} = F(x) + \varepsilon G(x, \varepsilon)$.

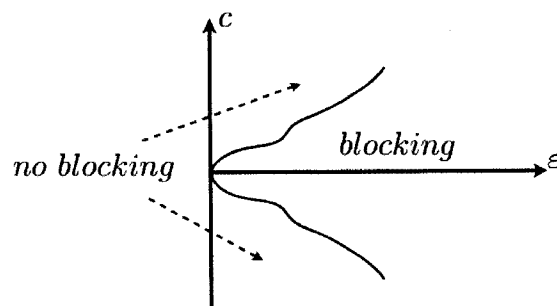


Figure B.5: [Figure 4.1] Parameter Space with Possible Bifurcation Branches.

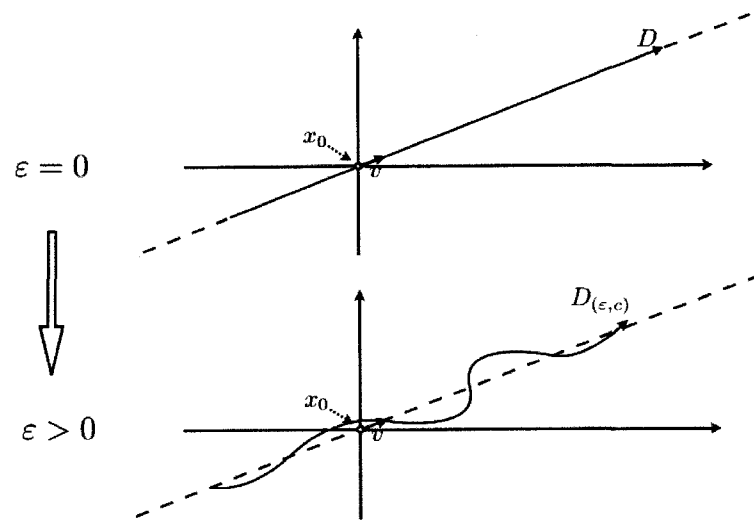


Figure B.6: [Figure 4.2] Evolution of D with respect to ε .

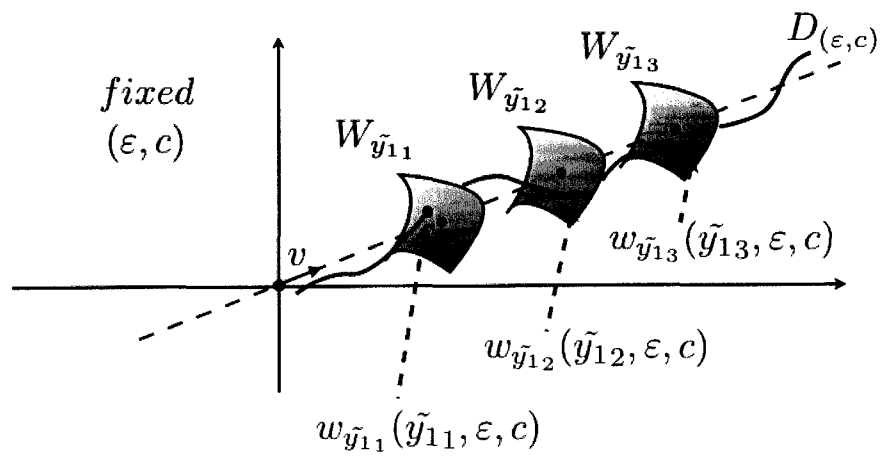


Figure B.7: [Figure 4.3] Parametrization of $D_{(\varepsilon, c)}$ with $\tilde{y}_{1,2,3} \in \mathbb{R}$.

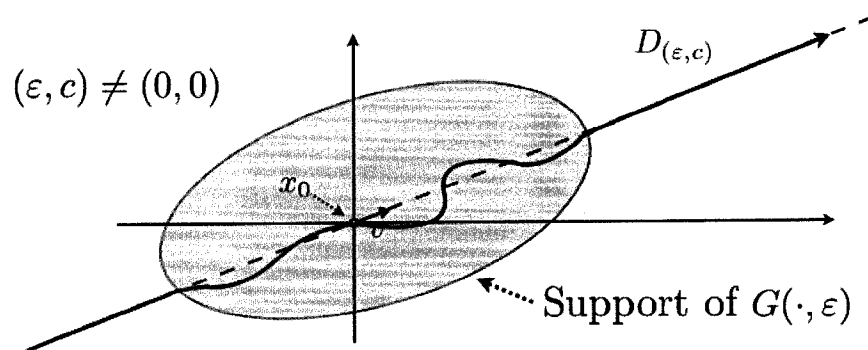


Figure B.8: [Figure 4.4] Localized action of the perturbation on D due to the compactness of $\text{support}(G)$.

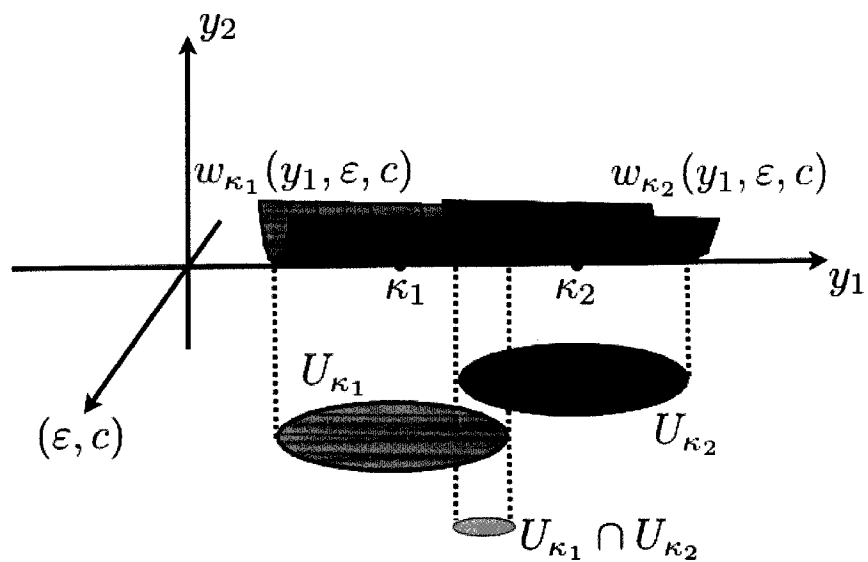


Figure B.9: [Figure 4.5] Manifolds with overlapping domains.

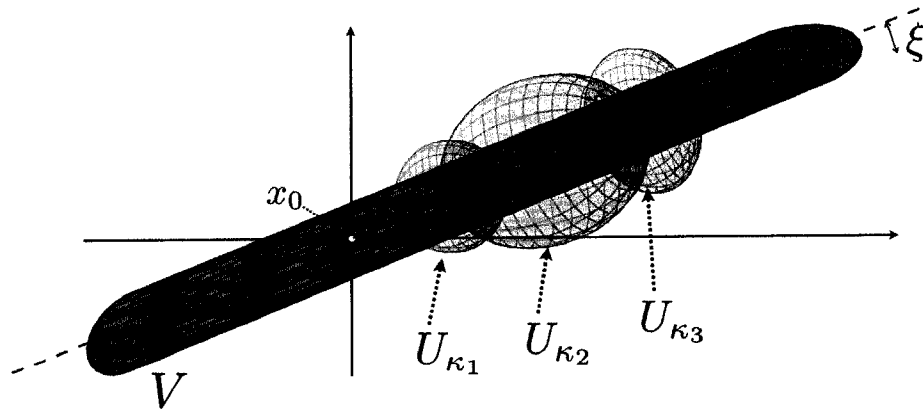


Figure B.10: [Figure 4.6] Sketch of the domain V and different U_{κ} s for $\kappa_1, \kappa_2, \kappa_3 \in \mathbb{R}$.

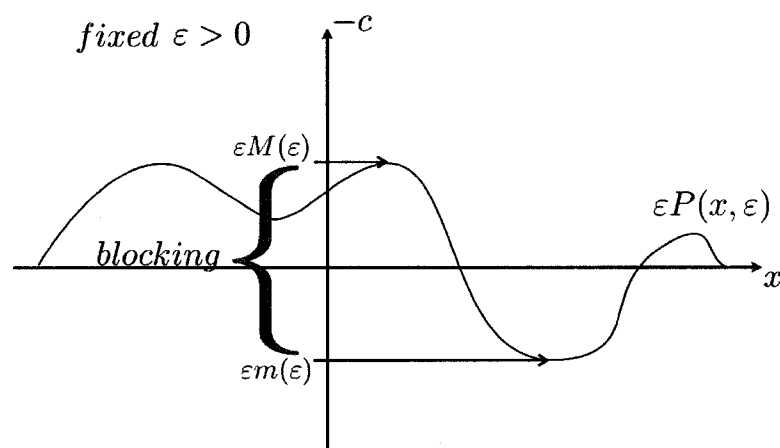


Figure B.11: [Figure 5.1] For a fixed value of ε , we observe wave blocking for values of $-c$ that can be reached by $\varepsilon P(x, \varepsilon)$.

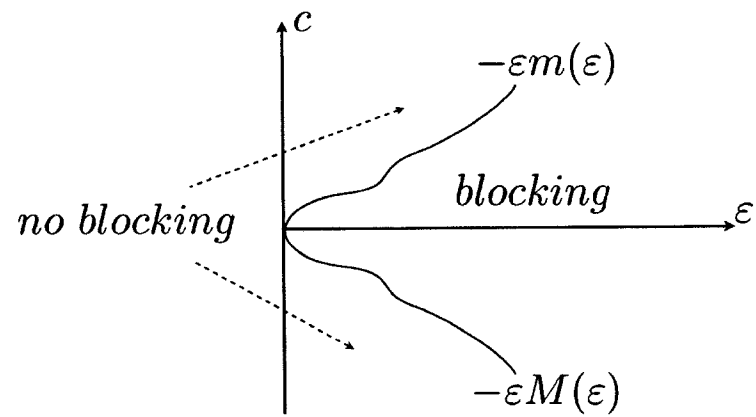


Figure B.12: [Figure 5.2] Sketch of the region in parameter space where we can observe wave blocking.

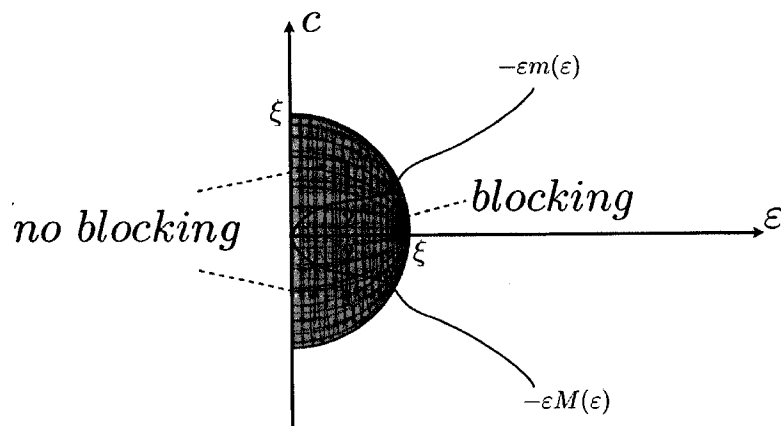


Figure B.13: [Figure 5.3] Sketch of bifurcation diagram with proper domain.

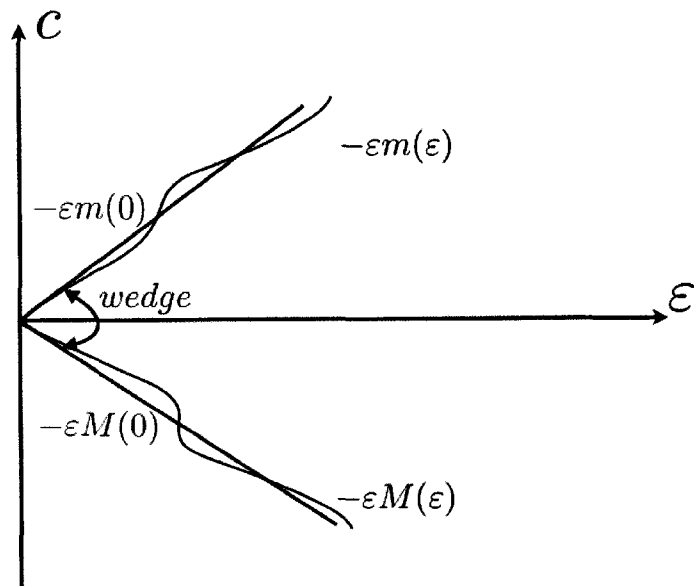


Figure B.14: [Figure 5.4] Sketch of the approximation of the bifurcation branches of equation (5.7).

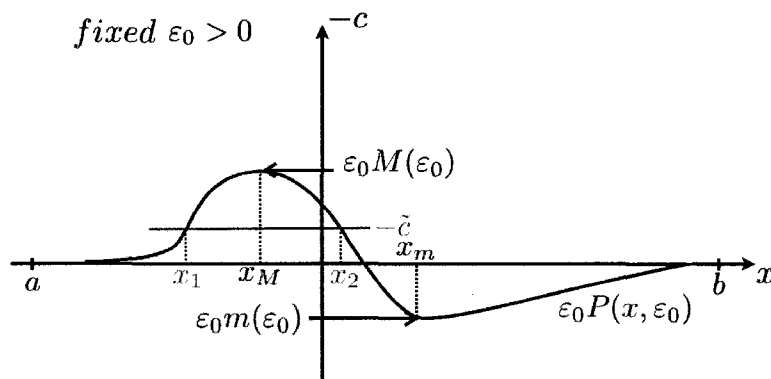


Figure B.15: [Figure 5.5] Presence of multiple equilibrium points for $\varepsilon_0 m(\varepsilon_0) < -\tilde{c} < \varepsilon_0 M(\varepsilon_0)$.

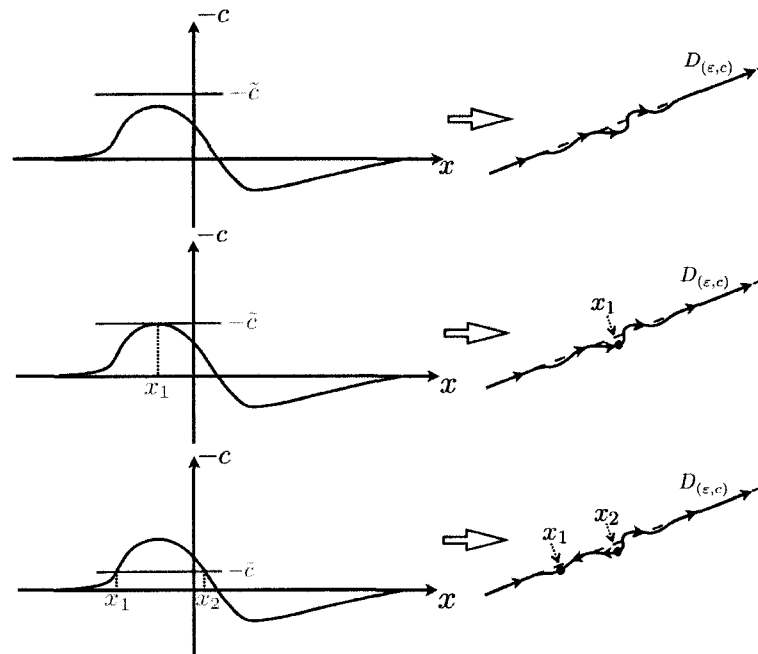


Figure B.16: [Figure 5.6] As $-\tilde{c}$ reaches $\varepsilon_0 M(\varepsilon_0)$, an equilibrium point appears and then splits into two.

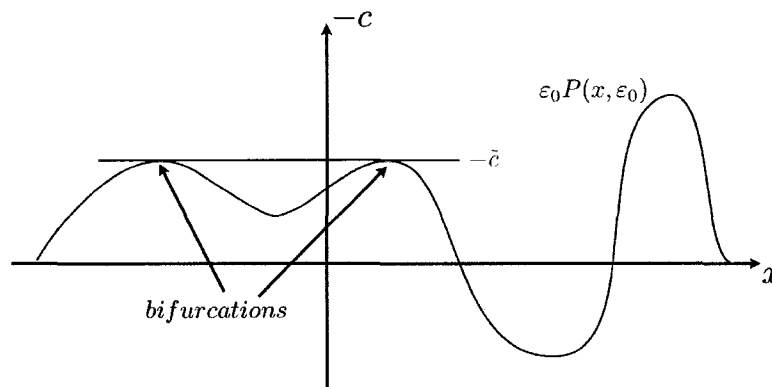


Figure B.17: [Figure 5.7] Multiple *saddle-node* bifurcations.

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