Killing Forms, W-Invariants, and the Tensor Product Map

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Abstract

Associated to a split, semisimple linear algebraic group G is a group of invariant quadratic forms, which we denote $\mathrm{Q}(G)$. Namely, $\mathrm{Q}(G)$ is the group of quadratic forms in characters of a maximal torus which are fixed with respect to the action of the Weyl group of G. We compute $\mathrm{Q}(G)$ for various examples of products of the special linear, special orthogonal, and symplectic groups as well as for quotients of those examples by central subgroups. Homomorphisms between these linear algebraic groups induce homomorphisms between their groups of invariant quadratic forms. Since the linear algebraic groups are semisimple, $\mathrm{Q}(G)$ is isomorphic to $\mathbb{Z}^n$ for some n, and so the induced maps can be described by a set of integers called *Rost multipliers*. We consider various cases of the Kronecker tensor product map between copies of the special linear, special orthogonal, and symplectic groups. We compute the Rost multipliers of the induced map in these examples, ultimately concluding that the Rost multipliers depend only on the dimensions of the underlying vector spaces.

Dedications

To root systems - You made it all worth Weyl.

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Contents

Preface

In this thesis we study certain invariant quadratic forms associated to split semisimple linear algebraic groups. In particular, we study quadratic forms in characters of a split maximal torus which are invariant under the action of the Weyl group. These invariants have been systematically studied by S. Garibaldi, A. Merkurjev, and J.-P. Serre in their celebrated lecture notes on cohomological invariants and Galois cohomology [GMS]. More precisely, they consider a functor Q which takes an appropriate linear algebraic group G (simply connected, smooth) and produces the group of quadratic invariants $\mathrm{Q}(G)$. Then they investigate functorial maps $\mathrm{Q}(\alpha)\colon \mathrm{Q}(G) \to \mathrm{Q}(G')$ where α is a homomorphism between simple, simply connected groups. When the groups are *simple*, their group of invariants are infinite cyclic (isomorphic to $\mathbb{Z}$), and so $\mathrm{Q}(\alpha)$ is determined by a single integer $r(\alpha)$ called the *Rost multiplier* of α. Several results about $r(\alpha)$, and examples of computations for various α between common simple linear algebraic groups can be found in works by A. Merkurjev [Mer], S. Garibaldi and K. Zainoulline [GZ], H. Bermudez and A. Ruozzi [BR], and others. In the case that G is not simple, then the computation of $\mathrm{Q}(G)$ and of $r(\alpha)$ becomes a much more difficult problem as the group $\mathrm{Q}(G)$ turns into a product of several infinite cyclic groups, hence leading to several associated Rost multipliers. Additionally, $\mathrm{Q}(G)$ and Rost multipliers play an important role in the theory of cohomological invariants of linear algebraic groups. They appear in this context in works such as S. Garibaldi's [Gar], S. Baek's [Baek], A. Merkurjev's [Mer], and in his work with A. Neshitov and K. Zainoulline, [MNZ].

In a recent paper [BDZ], S. Baek, R. Devyatov, and K. Zainoulline computed $\mathrm{Q}(G)$ when G is a quotient of some *semisimple but not simple* linear algebraic group by its central subgroup. In this thesis we will be extending such computations to cases dealing with the tensor (Kronecker) product map between various combinations of the special linear, special orthogonal, and symplectic groups. Then, similar to work done by S. Garibaldi and A. Quéguiner-Mathieu in [GQ-M], the computations will also be extended to quotients of these groups by the kernel of the Kronecker product map in addition to other central subgroups.

Our main object of study is the group of invariants $\mathrm{Q}(G)$ associated to a linear algebraic group G. We work with it in the following form, where T denotes a maximal torus in the group G.

$$\mathrm{Q}(G) = \mathrm{S}^2(T^*)^W.$$

$\mathrm{Q}(G)$ consists of quadratic forms on the character group of the maximal torus which are invariant under the action of the Weyl group W. As mentioned above, when the linear algebraic group G is simple, this group of invariants is isomorphic to $\mathbb{Z}$ and hence generated by a single element q, called the normalized Killing form of G. When G is semisimple, the invariants form a free abelian group generated by the normalized Killing forms of the simple components of G. This allows us to describe the maps between groups of invariants which are induced from homomorphisms between groups by a unique collection of integers. We do so in the following cases.

Let V_1, V_2 be vector spaces with finite dimensions d_1, d_2 respectively. Denoting the special linear group with SL, the special orthogonal group with SO, and the symplectic group with Sp, let G_1, G_2, H be linear algebraic groups (with maximal tori T_1, T_2, T_H resp.) in one of the following configurations.

G_1	G_2	H
SL	SL	SL
SO	SO	SO
Sp	SO	Sp
Sp	Sp	SO

Here $G_i = \mathrm{Sp}$ only when V_i is of even dimension. We then have the tensor product map defined by the Kronecker product of matrices

$$\rho\colon G_1(V_1) \times G_2(V_2) \to H(V_1 \otimes V_2)$$
$$(A, B) \mapsto A \otimes B$$

which induces a map between invariant quadratic forms

$$\rho^*\colon \mathrm{S}^2(T_H^*)^{W_H} \to \mathrm{S}^2(T_1^*)^{W_1} \times \mathrm{S}^2(T_2^*)^{W_2}.$$

Since each of G_1, G_2, H are simple groups, their group of invariants are generated by normalized Killing forms q_1, q_2, q_H respectively. In all considered cases we show that the map ρ^* is given by

$$\rho^*(q_H) = (d_2 q_1, d_1 q_2)$$

hence it depends only on the dimensions of the underlying vector spaces. Ultimately we show that this behaviour generalizes to the following result

Theorem. *Let $V_1, V_2, \ldots, V_{n+m}$ be vector spaces with finite dimensions $d_1, d_2, \ldots, d_{n+m}$ respectively. Let $G_1, \ldots, G_{n+m}, H$ be linear algebraic groups in one of the following configurations.*

- $G_1, \ldots, G_{n+m}, H = \mathrm{SL}$, *all groups are the special linear group,*

- $G_1, \ldots, G_{n+m}, H = \mathrm{SO}$, *all groups are the special orthogonal group,*
- $G_1, \ldots, G_n = \mathrm{Sp}$, $G_{n+1}, \ldots, G_{n+m} = \mathrm{SO}$, *and* $H = \mathrm{SO}$ *where* n *is even,*
- $G_1, \ldots, G_n = \mathrm{Sp}$, $G_{n+1}, \ldots, G_{n+m} = \mathrm{SO}$, *and* $H = \mathrm{Sp}$ *where* n *is odd.*

Then when considering the map

$$\begin{aligned}\rho_{n+m} \colon G_1(V_1) \times \ldots \times G_{n+m}(V_{n+m}) &\to H(V_1 \otimes \ldots \otimes V_{n+m}) \\ (A_1, \ldots, A_{n+m}) &\mapsto A_1 \otimes \ldots \otimes A_{n+m}\end{aligned}$$

the induced map between invariant quadratic forms is given by

$$\rho_n^*(q_H) = (d_2 \ldots d_{n+m} q_1, \ldots, d_1 d_2 \ldots \hat{d}_i \ldots d_{n+m} q_i, \ldots, d_1 d_2 \ldots d_{n+m-1} q_{n+m})$$

where $\hat{d}_i$ *indicates ommision.*

Afterwards we provide descriptions of $\mathrm{S}^2(G')^W$ where $G' = G / \mu_k$ when the k$^{\text{th}}$ roots of unity μ_k are a central subgroup of G which is one of our three commonly considered linear algebraic groups, SL, SO, or Sp. We do the same for the invariant quadtraic forms of $G_1 \times G_2 / \mu_k$ when G_1, G_2 are in one of the above listed configurations. In particular this includes the cases $G_1 \times G_2 / \ker(\rho)$.

To reach these conclusions we require several background facts and constructions which we present in Chapter 1. We begin with an exposition on abstract root systems in section 1.1. Following [Hum90] we work from the definition of a root system to the classification theorem of simple root systems. It is here that we introduce the first notion of a Weyl group.

Section 1.2 is dedicated to describing linear algebraic groups. As in [KMRT], they are presented as functors from the category of algebras of an algebraically closed field to the category of groups. Specifically, linear algebraic groups are functors which are represented by finitely generated Hopf algebras, and so section 1.2 begins with definitions of Hopf algebras and their related notions. This section then provides examples of linear algebraic groups among which are the special linear, special orthogonal, and symplectic groups. These examples serve to define these groups.

Next, a connection is made between root systems and linear algberaic groups via discussion of Lie algebras. In section 1.3 we define Lie algebras, describe how a Lie algebra arises from a linear algebraic group, and provide a method of computing that Lie algebra. Here we introduce the Killing form, a bilinear form on a Lie algebra defined in terms of its adjoint representation. The section ends with discussion of how a Lie algbera produces a unique root system, thereby making the connecting from linear algebraic group to root system. However, the root system associated with a linear algebraic group may be obtained in another way, by considering the action of the group on its Lie algebra, and this comprises the majority of section 1.4. This perspective provides another description of the Killing form, one which we use later in this thesis, and so here we summarize the Killiing forms of various linear algebraic groups.

We reach discussion of invariant quadtraic forms in section 1.5. The structure of the group of invariant quadtraic forms is described, that it is an free abelian group of finite rank. The normalized Killing form is introduced in the beginning section 1.6. The majority of section 1.6 is spent developing the notion of $\mathrm{Q}(G)$ from [GMS] to ultimately show that any homomorphism between smooth linear algebraic groups does induce a map between their invariant quadratic forms.

Having covered the preliminaries, Chapter 2 starts by defining the Kronecker product map. This, followed by examples of the kernel and restrictions of the codomain of the Kronecker product map comprise section 2.1. Section 2.2 contains the proof of the above stated Theorem. First the computations for the examples involving the product of two groups are presented, and then these results are used in an inductive argument of the Theorem.

Finally, the descriptions of the group of invariant quadtraic forms of quotient groups are contained in section 2.3. The quotient of single simple groups by roots of unity which form central subgroups are covered first. Such quotients of the product of special linear groups and the product of special orthogonal and/or symplectic groups follow.

Chapter 1

Preliminaries

Our aim in this chapter is to introduce the definitions and notations of the objects we will work without throughout the paper. We also explain various auxiliary results which will be used to prove the main results of Chapter 2. This chapter also provides brief overviews of the theory of root systems, Lie algebras, linear algebraic groups, and invariant quadratic forms. Most of the material presented here is taken from Humphreys' book on root systems and his book on the representation theory of Lie algebras, [Hum90] and [Hum72] respectively, as well as from Knus, Merkurjev, Rost, and Tignol's *The Book of Involutions*, [KMRT].

1.1 Root Systems

We introduce the notion of a root system following the book by Humphreys [Hum90]. We also introduce the notions of simple systems and coxeter graphs of root systems, ultimately concluding with the use of Dynkin diagrams to classify root systems.

Given a real euclidean vector space V, associated to each $\alpha \in V$ there is a reflection s_α defined by

$$s_\alpha(\lambda) = \lambda - 2\frac{(\lambda, \alpha)}{(\alpha, \alpha)}\alpha \quad \forall \lambda \in V$$

where $(-, -)$ denotes a symmetric, positive definite, bilinear form on V. Inspired by geometric realizations of finite reflection groups, we wish to consider finite subsets of V which are stable with respect to each associated reflection. Such collections are called *root systems*.

Definition 1.1.1. *Given a vector space V and a finite set $\Phi \subset V$, we say Φ is a* root system *if*

(R1) $\Phi \cap \text{Span}\{\alpha\} = \{\alpha, -\alpha\} \quad \forall \alpha \in \Phi$,

(R2) $s_\alpha(\Phi) = \Phi \quad \forall \alpha \in \Phi$.

Additionally, we say that the root system is crystallographic *if we also require*

(R3) $\dfrac{2(\alpha, \beta)}{(\beta, \beta)} \in \mathbb{Z} \quad \forall \alpha, \beta \in \Phi$.

Elements of Φ are called roots. Given root systems $\Phi_1, \ldots \Phi_n$ subsets of $V_1, \ldots, V_n$ respectively, $\Phi := \bigcup_{i=1}^{n} \Phi_i$ is a root system in $V := \bigoplus_{i=1}^{n} V_i$. We call $\Phi \subset V$ *irreducible* if it cannot be written as such a sum, and we call it *reduced* if $V = \text{Span}(\Phi)$. Associated to each root system is a finite reflection group generated by the associated reflections

$$W := \langle s_\alpha \mid \alpha \in \Phi \rangle \subseteq \text{End}(V)$$

called the Weyl group of Φ. W acts naturally on V, stabilizing Φ. Two root systems $\Phi \subset V$, $\Phi' \subset V'$ are considered isomorphic if there exists an isomorphism of vector spaces $\varphi \colon V \xrightarrow{\sim} V'$ such that $\varphi(\Phi) = \Phi'$.

All of the information in a root system can be recovered from a small subset of the original roots, called a *simple system*, elements of which will be called *simple roots*. Thus it will be these simple systems which we use to classify root systems.

Definition 1.1.2. *A subset $\Delta \subset \Phi$ is called a* simple system *if*

(i) Δ *is a basis for* $\text{Span}(\Phi) \subseteq V$,

(ii) $\forall \alpha \in \Phi$, $\alpha = \sum_{\gamma \in \Delta} c_\gamma \gamma$ *such that each* $c_\gamma \in \mathbb{R}$, *and either* $c_\gamma \geq 0\ \forall \gamma$ *or* $c_\gamma \leq 0\ \forall \gamma$.

Simple systems exist for all reduced root systems [Hum90, Theorem 1.3], and so in the following discussion we assume all root systems are reduced. Simple systems are not unique, and in fact any two simple systems in Φ are conjugate under the action of W.

Theorem 1.1.3. [Hum90, Theorem 1.4] *If $\Delta, \Delta' \subset \Phi$ are two simple systems, then there exists $w \in W$ such that $w(\Delta) = \Delta'$. (Here $w(\Delta) := \{w(\alpha) \mid \alpha \in \Delta\}$).*

The following theorem and lemma demonstrate how the entire root system Φ can be recovered from any simple system $\Delta \subset \Phi$.

Theorem 1.1.4. [Hum90, Theorem 1.5] *Fix a simple system $\Delta \subset \Phi$. Then W is generated by the set of* simple reflections $\{s_\alpha \mid \alpha \in \Delta\}$. *I.e.* $W =< s_\alpha \mid \alpha \in \Delta >$.

Lemma 1.1.5. [Hum90, Corollary 1.5] *Given* Δ*, for all* $\beta \in \Phi$*, there exists* $w \in W$ *such that* $w(\beta) \in \Delta$*. In particular* $W(\Delta) = \Phi$*.*

Hence given a simple system Δ, we can obtain $W = \langle s_\alpha \mid \alpha \in \Delta \rangle$, and then in turn reconstruct $\Phi = W(\Delta)$. Since Δ is a basis of V, $|\Delta| = \dim(V)$ and is therefore an invariant of the root system called the *rank* of the root system.

Remark 1.1.6. In the case that Φ is crystallographic, W stabilizes the lattice $L := \mathrm{Span}_{\mathbb{Z}}(\Delta)$ and is therefore also crystallographic. ($G \leq \mathrm{GL}(V)$ is called crystallographic if it stabilizes some lattice in V). Furthermore if we consider some $\beta \in \Phi$, by the above lemma there exists $w \in W$ such that $w(\beta) \in \Delta$ and hence in L. This means that $\beta = w^{-1}(w(\beta)) \in L$ and therefore $\Phi \subset L$. In a crystallographic root system, all roots are integral linear combinations of simple roots.

Now, in order to classify root systems we need only classify possible simple systems. To do so we introduce the *Coxeter graph* of a root system.

Definition 1.1.7. *Given* $\Delta \subset \Phi$ *a simple and root system respectively, the* Coxeter graph*,* Γ *(with vertex set* $V(\Gamma)$ *and edge set* $E(\Gamma)$*), is an undirected, edge labeled graph such that*

(i) $V(\Gamma) = \Delta$*,*

(ii) The edge (α, β) *is included in* $E(\Gamma)$ *if* $m(\alpha, \beta) := \mathrm{ord}(s_\alpha s_\beta) \geq 3$*, in which case the edge is labeled with the integer* $m(\alpha, \beta)$*.*

It is important to note here that the above definition does not depend on our choice of simple system Δ. Had we instead chosen the simple system $w(\Delta)$ for some $w \in W$, the map $\Delta \to w(\Delta)$ induces an isomorphism of Coxeter graphs since $m(w\alpha, w\beta) = m(\alpha, \beta)$. (This follows from the fact that for all $\alpha \in \Phi$, $w \in W$, we have $w s_\alpha w^{-1} = s_{w\alpha}$, which implies $(s_{w\alpha} s_{w\beta})^n = w(s_\alpha s_\beta)^n w^{-1}$, and therefore $\mathrm{ord}(s_{w\alpha} s_{w\beta}) = \mathrm{ord}(s_\alpha s_\beta)$.)

Given two root systems Φ, Φ' with isomorphic Coxeter graphs, the graph isomorphism induces an isometry between their vector spaces, and so $\Phi \cong \Phi'$ as root systems [Hum, Proposition 2.1]. Hence our classification of root systems will be presented in terms of possible Coxeter graphs. Since we need only classify irreducible root systems, we note the following.

Proposition 1.1.8. *If* Φ *is an irreducible root system, then its Coxeter graph* Γ *is connected.*

Proof: Let Φ be a root system, let Γ be its Coxeter graph, and assume that Γ is disconnected. Let $\Gamma_1, \Gamma_2 \subset \Gamma$ be two disjoint subgraphs with $\Gamma_1 \cup \Gamma_2 = \Gamma$, and let $\Delta_1, \Delta_2 \subset \Delta$ be the respective vertex sets.

Since there is no edge between any $\alpha \in \Delta_1$ and $\beta \in \Delta_2$, we have that $m(\alpha, \beta) = 2$. Therefore $(\alpha, \beta) = 0$, and so the two sets Δ_1, Δ_2 are mutually orthogonal.

Because Δ is a basis for V, we have a decomposition

$$(V, \Phi) = (\text{Span}(\Delta_1), \Phi_1) \oplus (\text{Span}(\Delta_2), \Phi_2)$$

where $\Phi_i = \Phi \cap \text{Span}(\Delta_i)$. Finally note that since the decomposition is orthogonal, Φ_2 is fixed pointwise by s_α for any $\alpha \in \Delta_1$, and vice versa. Therefore the reflections corresponding to roots in Φ_i permute Φ_i only, and so $\Phi_i \subset \text{Span}(\Delta_i)$ is a root system. The decomposition then means that Φ is reducible, and so by contrapositive, irreducible root systems produce connected Coxeter graphs. ∎

Now we are left with deciding whether a given graph can be obtained as the Coxeter graph of a root system. The decision procedure is to determine if the graph is *positive definite* or not.

Definition 1.1.9. *Given an arbitrary undirected, integer edge labeled graph Γ, let $n := |V(\Gamma)|$, and associate to Γ an $n \times n$ matrix A, where the entry $a_{ij} := -\cos(\frac{\pi}{m(v_i, v_j)})$. Here v_i denote the vertices of Γ, and $m(v_i, v_j)$ equals the edge label between v_i and v_j if they are adjacent, or 2 if they are not. Since Γ is undirected, $a_{ij} = a_{ji}$ and so A defines a symmetric bilinear form, B, on $\mathbb{R}^n$. We call Γ* positive definite (positive semidefinite, negative definite, etc.) *if B is positive definite (positive semidefinite, etc.).*

Proposition 1.1.10. *The matrix A associated to the Coxeter graph of a root system is positive definite.*

Proof: Let Φ be a root system in V with simple system $\Delta = \{\gamma_1, \ldots, \gamma_n\}$ and Coxeter graph Γ. Let A be the matrix associated to Γ, i.e.

$$A = \left[-\cos\left(\frac{\pi}{m(\gamma_i, \gamma_j)}\right)\right]_{i,j=1}^{n}.$$

By [Hum90, Corollary 1.3], if $\alpha \neq \beta \in \Delta$, then $(\alpha, \beta) \leq 0$. Therefore by the cosine formula for an inner product, $(\alpha, \beta) = ||\alpha||\,||\beta|| \cos(\theta) \leq 0$ implies that $\theta \in [\frac{\pi}{2}, \pi)$, where θ is the angle between the roots α and β.

Since the element $s_\alpha s_\beta \in W$ acts as a rotation with order $m(\alpha, \beta)$, it rotates by an angle of $\frac{2\pi}{m(\alpha,\beta)}$, and therefore the angle between the reflecting hyperplanes is $\frac{\pi}{m(\alpha,\beta)}$.

If $\alpha \neq \beta$ then $\theta \geq \frac{\pi}{2}$ meaning that $\theta = \pi - \frac{\pi}{m(\alpha,\beta)}$ and $\cos(\theta) = -\cos(\frac{\pi}{m(\alpha,\beta)})$. If $\alpha = \beta$ then $\theta = 0$ and $m(\alpha, \beta) = 1$ which again means $\cos(\theta) = -\cos(\frac{\pi}{m(\alpha,\beta)})$. Hence

$$(\alpha, \beta) = ||\alpha||\,||\beta|| \left(-\cos\left(\frac{\pi}{m(\alpha, \beta)}\right)\right).$$

Now let $v = (c_1, \dots c_n) \neq 0 \in \mathbb{R}^n$ and let $w = \sum_{i=1}^{n} \frac{c_i}{||\gamma_i||}\gamma_i \in V$. We compute

$$v^T A v = \sum_{i,j=1}^{n} c_i c_j a_{ij} = \sum_{i,j=1}^{n} c_i c_j \left(-\cos\left(\frac{\pi}{m(\gamma_i, \gamma_j)}\right)\right)$$

and since $w \neq 0$

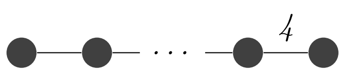
$$\begin{aligned} 0 < (w, w) &= \sum_{i,j=1}^{n} \frac{c_i}{||\gamma_i||}\frac{c_j}{||\gamma_j||}(\gamma_i, \gamma_j) = \sum_{i,j=1}^{n} \frac{c_i}{||\gamma_i||}\frac{c_j}{||\gamma_j||}||\gamma_i||\,||\gamma_j|| \left(-\cos\left(\frac{\pi}{m(\gamma_i, \gamma_j)}\right)\right) \\ &= \sum_{i,j=1}^{n} c_i c_j \left(-\cos\left(\frac{\pi}{m(\gamma_i, \gamma_j)}\right)\right) = v^T A v. \end{aligned}$$

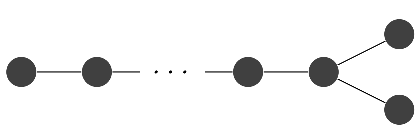
Hence the matrix A associated to a root system is positive definite. ∎

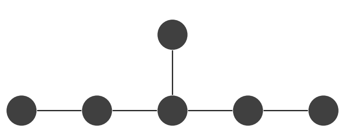
We are now able to classify irreducible root systems.

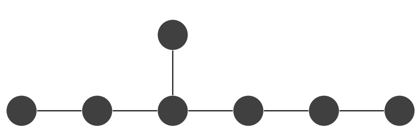
Theorem 1.1.11. [Hum90, Theorem 2.7] *The only connected, positive definite graphs are the following.*

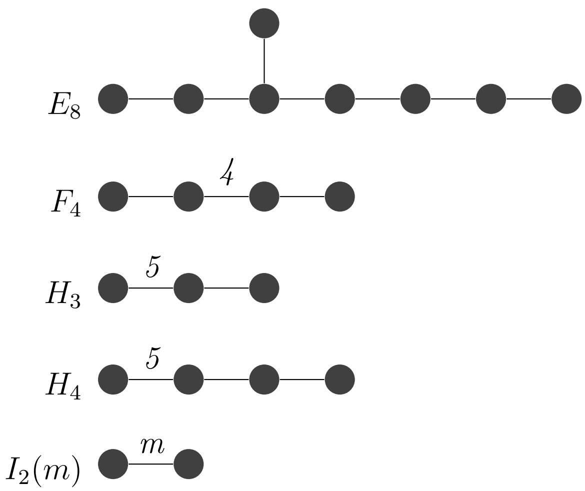
A_n


B_n


D_n


E_6


E_7




Here the letters name the class of the root system/graph, and the subscripted number its rank. As is usually done, we have omitted the edge labels when the edge has $m(\alpha, \beta) = 3$.

The graphs above classify irreducible root systems satisfying (R1) and (R2). There is a similar classification of crystallographic root systems found by refining our current classification. First, some consequences of enforcing (R3).

Proposition 1.1.12. [Hum90, Proposition 2.8] *If Φ, and hence W, is crystallographic each integer $m(\alpha, \beta)$ is in $\{2, 3, 4, 6\}$ when $\alpha \neq \beta \in \Delta$.*

Proposition 1.1.13. *If Φ is irreducible and crystallographic, then each root must have one of at most two lengths. That is, $|\{||\alpha|| \mid \alpha \in \Phi\}| \leq 2$.*

Proof: Let $\alpha, \beta \in \Delta$. Recall that (R3) states $\frac{2(\alpha,\beta)}{(\beta,\beta)} \in \mathbb{Z}$ for all $\alpha, \beta \in \Phi$. Therefore

$$\frac{2(\alpha, \beta)}{(\beta, \beta)} = \frac{2||\alpha|| \, ||\beta||(-\cos(\frac{\pi}{m(\alpha,\beta)}))}{||\beta||^2} = \frac{||\alpha||}{||\beta||} \left(-2\cos\left(\frac{\pi}{m(\alpha, \beta)} \right) \right) \in \mathbb{Z}$$

and symmetrically

$$\frac{||\beta||}{||\alpha||} \left(-2\cos\left(\frac{\pi}{m(\alpha, \beta)} \right) \right) \in \mathbb{Z}.$$

By the previous proposition there are only a small number of possibilities for $\cos(\frac{\pi}{m(\alpha,\beta)})$.

Case 1: $m(\alpha, \beta) = 2 \Rightarrow -2\cos(\frac{\pi}{m(\alpha,\beta)}) = 0$, and no information may be gained.

Case 2: $m(\alpha, \beta) = 3 \Rightarrow -2\cos(\frac{\pi}{m(\alpha,\beta)}) = -1$. Therefore both $\frac{||\alpha||}{||\beta||}$ and $\frac{||\beta||}{||\alpha||}$ are themselves positive integers. Hence they are equal to 1, and $||\alpha|| = ||\beta||$.

Case 3: $m(\alpha,\beta) = 4 \Rightarrow -2\cos(\frac{\pi}{m(\alpha,\beta)}) = -\sqrt{2}$. So there exists $n, m \in \mathbb{Z}$ such that

$$\frac{||\alpha||}{||\beta||} = \frac{n}{\sqrt{2}}, \quad \frac{||\beta||}{||\alpha||} = \frac{m}{\sqrt{2}} \Rightarrow \frac{nm}{2} = 1.$$

Without loss of generality this means that $n = 1, m = 2$ and $||\beta|| = \sqrt{2}||\alpha||$.

Case 4: $m(\alpha,\beta) = 6 \Rightarrow -2\cos(\frac{\pi}{m(\alpha,\beta)}) = -\sqrt{3}$. By a similar argument this produces that $||\beta|| = \sqrt{3}||\alpha||$.

Finally it is enough to notice that in each positive definite graph there are at most two edge labels. Hence each pair of simple roots connected by a path of edges labeled 3 are the same length, and those pairs connected by an edge with a larger label have length related as described above. This ensure that there are at most two root lengths among the simple roots of an irreducible, crystallographic root system. Then by Lemma 1.1.5, and the fact that all $w \in W$ are orthogonal transformations, for all $\alpha \in \Phi$ there exists $\gamma \in \Delta$ such that $||\alpha|| = ||\gamma||$. ∎

Based on these restrictions, Coxeter graphs of type H_3, H_4 and $I_2(m)$ for all $m \notin \{3, 4, 6\}$ do not arise from crystallographic root systems.

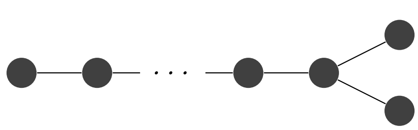
We modify the remaining Coxeter graphs to include the additional information about root lengths. For each edge in the Coxeter graph with label greater than 3, we replace it with a directed edge pointing toward the shorter root. It is also customary to replace the explicit edge labels of 4 and 6 with double and triple edges respectively. The resulting graphs are called *Dynkin diagrams*, and they form our classification of crystallographic root systems.

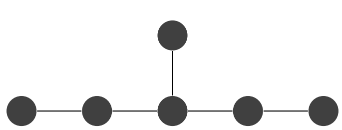
Theorem 1.1.14. *The only graphs arising from irreducible crystallographic root systems are the following.*

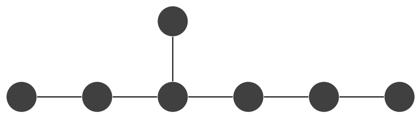
A_n 

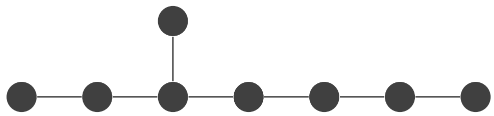
B_n 

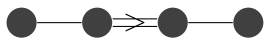
C_n 

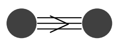
D_n 

E_6 

E_7 

E_8 

F_4 

G_2 

Here single, double, and triple edges represent edge labels of 3, 4, and 6 respectively.

The above theorem states slightly more than we have shown. It indicates that there are in fact root systems giving rise to each of the listed Dynkin diagrams. Producing examples of such root systems is straightforward, simply choose vectors in euclidean space with appropriate dot products and restrict to their span. The following first four examples are found in [KMRT], while the remainder are taken from [Bou].

Let e_i be the i^{th} standard basis vector of $\mathbb{R}^n$.

A_n:

Vector Space	$V = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid \sum_{i=1}^{n+1} x_i = 0\}$
Root System	$\Phi = \{e_i - e_j \mid 1 \leq i \neq j \leq n+1\}$
Simple System	$\Delta = \{e_1 - e_2, e_2 - e_3, \ldots, e_n - e_{n+1}\}$

B_n:

Vector Space	$V = \mathbb{R}^n$
Root System	$\Phi = \{\pm e_i \pm e_j, \pm e_k \mid 1 \leq i < j \leq n, 1 \leq k \leq n\}$
Simple System	$\Delta = \{e_1 - e_2, e_2 - e_3, \ldots, e_{n-1} - e_n, e_n\}$

C_n:

Vector Space	$V = \mathbb{R}^n$
Root System	$\Phi = \{\pm e_i \pm e_j, \pm 2e_k \mid 1 \leq i < j \leq n, 1 \leq k \leq n\}$
Simple System	$\Delta = \{e_1 - e_2, e_2 - e_3, \ldots, e_{n-1} - e_n, 2e_n\}$

D_n:

Vector Space	$V = \mathbb{R}^n$
Root System	$\Phi = \{\pm e_i \pm e_j \mid 1 \leq i < j \leq n\}$
Simple System	$\Delta = \{e_1 - e_2, e_2 - e_3, \ldots, e_{n-1} - e_n, e_{n-1} + e_n\}$

E_6:

Vector Space	$V = \mathbb{R}^8$
Root System	$\{\pm e_i \pm e_j, \pm\frac{1}{2}((-1)^{d_1}, \ldots, (-1)^{d_5}, -1, -1, 1) \mid (i), (ii)\}$
Simple System	$\Delta = \{e_1 + e_2, e_2 - e_1, e_3 - e_2, \ldots, e_5 - e_4, (\frac{1}{2}, -\frac{1}{2}, \ldots, -\frac{1}{2}, \frac{1}{2})\}$

with $(i) : 1 \leq i < j \leq 5$ and $(ii) : \sum_{i=1}^{5} d_i$ is even.

E_7:

Vector Space	$V = \mathbb{R}^8$
Root System	$\{\pm e_i \pm e_j, \pm(e_7 - e_8), \pm\frac{1}{2}((-1)^{d_1}, \ldots, (-1)^{d_6}, 1, -1) \mid (i), (ii)\}$
Simple System	$\Delta = \{e_1 + e_2, e_2 - e_1, e_3 - e_2, \ldots, e_6 - e_5, (\frac{1}{2}, -\frac{1}{2}, \ldots, -\frac{1}{2}, \frac{1}{2})\}$

with $(i) : 1 \leq i < j \leq 6$ and $(ii) : \sum_{i=1}^{6} d_i$ is odd.

E_8:

Vector Space	$V = \mathbb{R}^8$
Root System	$\{\pm e_i \pm e_j, \frac{1}{2}((-1)^{d_1}, \ldots, (-1)^{d_8}) \mid (i), (ii)\}$
Simple System	$\Delta = \{e_1 + e_2, e_2 - e_1, e_3 - e_2, \ldots, e_7 - e_6, (\frac{1}{2}, -\frac{1}{2}, \ldots, -\frac{1}{2}, \frac{1}{2})\}$

with $(i) : 1 \leq i < j \leq 8$ and $(ii) : \sum_{i=1}^{8} d_i$ is even.

F_4:

Vector Space	$V = \mathbb{R}^4$
Root System	$\{\pm e_i \pm e_j, \pm e_k, (\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}) \mid 1 \leq i < j \leq 4, 1 \leq k \leq 4\}$
Simple System	$\Delta = \{(1, -1, 0, 0), (0, 1, -1, 0), (0, 0, 1, 0), (-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})\}$

G_2:

Vector Space	$V = \{(x, y, z) \in \mathbb{R}^3 \quad \lvert x + y + z = 0\}$
Root System	$\{\pm(e_i - e_j), \pm(2, -1, -1), \pm(-1, 2, -1), \pm(-1, -1, 2) \mid 1 \leq i < j \leq 3\}$
Simple System	$\Delta = \{(1, -1, 0), (-1, 2, -1)\}$

1.2 Linear Algebraic Groups and the Hopf Algebras

Our approach to linear algebraic groups will be through the use of group schemes as covered in [KMRT]. In order to introduce group schemes we must first introduce *Hopf algebras*. Throughout this section $\mathbb{F}$ will be a field.

Definition 1.2.1. *By a* Hopf algebra *over* $\mathbb{F}$ *we mean an* $\mathbb{F}$*-bialgebra,* A*, together with an antipode* $i\colon A \to A$*. Specifically, if*

$$m\colon A \otimes_{\mathbb{F}} A \to A$$

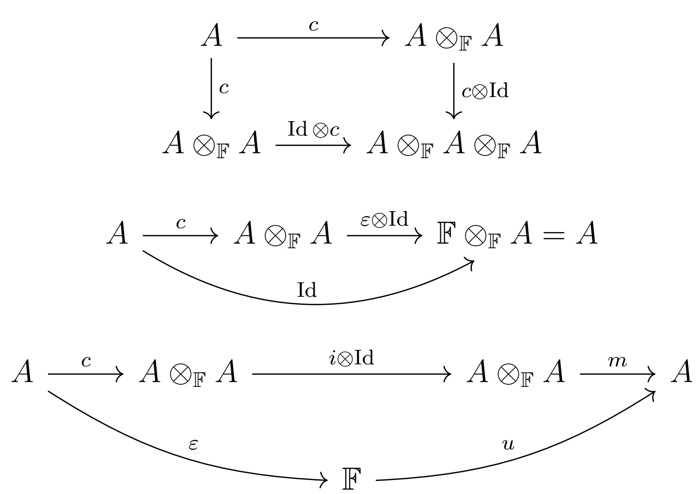
$$u\colon \mathbb{F} \to A$$

are the multiplication and unit maps of A *as an* $\mathbb{F}$*-algebra, then the comultiplication and counit maps*

$$c\colon A \to A \otimes_{\mathbb{F}} A$$

$$\varepsilon\colon A \to \mathbb{F}$$

are $\mathbb{F}$*-algebra homomorphisms such that the following diagrams commute.*



The Hopf algebra is called commutative *if* $m(a \otimes b) = m(b \otimes a)$ *for all* $a, b \in A$*, and it is called* cocommutative *if* $c(a) = \sum x \otimes y = \sum y \otimes x$ *for all* $a \in A$*.*

Definition 1.2.2. *If A and B are Hopf algebras over $\mathbb{F}$, then $f\colon A \to B$ is a* Hopf algebra homomorphism *if it respects the Hopf algebra structures on A and B*

$$f \circ m_A = m_B \circ (f \otimes f),$$

$$f \circ u_A = u_B,$$
$$(f \otimes f) \circ c_A = c_B \circ f,$$
$$\varepsilon_A = \varepsilon_B \circ f,$$
$$f \circ i_A = i_B \circ f.$$

That is, f is an algebra homomorphism, coalgebra homomorphism, and preserves the antipode.

Definition 1.2.3. *If A is an $\mathbb{F}$-Hopf algebra, $J \subset A$ is called a* Hopf ideal *if*

$$m(A \otimes_{\mathbb{F}} J + J \otimes_{\mathbb{F}} A) \subseteq J,$$

$$c(J) \subseteq A \otimes_{\mathbb{F}} J + J \otimes_{\mathbb{F}} A,$$
$$\varepsilon(J) = 0,$$
$$i(J) \subseteq J.$$

J is an algebra ideal, a coalgebra ideal and closed under the antipode.

For every Hopf ideal $J \subset A$ there is a natural Hopf algebra structure on A/J, and the quotient map $q\colon A \to A/J$ is a surjective Hopf algebra homomorphism. Also, as to be expected, if $f\colon A \to B$ is a Hopf algebra homomorphism, then $\ker(f) \subseteq A$ is a Hopf ideal and $A/\ker(f) \cong \mathrm{Img}(f) \subseteq B$.

Hopf algebras are present in the theory of group schemes because their structure allows us to define a group operation on the the set of morphisms from the Hopf algebra to an $\mathbb{F}$-algebra.

Let A be an $\mathbb{F}$-Hopf algebra, R be an $\mathbb{F}$-algebra, and let $\mathrm{Hom}_{\mathbb{F}}(A, R)$ denote the set of $\mathbb{F}$-algebra homomorphism from A to R. For $f, g \in \mathrm{Hom}_{\mathbb{F}}(A, R)$ we define their product to be

$$f \cdot g := m_R \circ (f \otimes g) \circ c_A.$$

With this multiplication $\mathrm{Hom}_{\mathbb{F}}(A, R)$ is a group. Its identity element is $e := u_R \circ \varepsilon_A$, and inverses are given by $f^{-1} := f \circ i_A$.

Hence $\mathrm{Hom}_{\mathbb{F}}(A, -)\colon \mathrm{Alg}_{\mathbb{F}} \to \mathrm{Grp}$ is a functor from the category of $\mathbb{F}$-algebras to the category of groups. These functors are precisely those which we will call *groups schemes.*

Definition 1.2.4. *An* affine group scheme G *over* $\mathbb{F}$ *is a functor*

$$G\colon \text{Alg}_{\mathbb{F}} \to \text{Grp}$$

which is isomorphic to $\text{Hom}_{\mathbb{F}}(A, -)$ *for some* $\mathbb{F}$*-Hopf algebra* A*.*

By Yoneda's Lemma, if a group scheme G is isomorphic to $\text{Hom}_{\mathbb{F}}(A, -)$ for a Hopf algebra A, then A is the unique such Hopf algebra (up to isomorphism). Denote this Hopf algebra by $\mathbb{F}[G]$. G is said to *represented* by $\mathbb{F}[G]$. If G is a group scheme over $\mathbb{F}$ and $R \in \text{Alg}_{\mathbb{F}}$, then the group $G(R)$ is called the *group of R points* of G. A group scheme G is said to be *commutative* if $G(R)$ is commutative for all $R \in \text{Alg}_{\mathbb{F}}$.

Definition 1.2.5. *Let* G, H *be group schemes over* $\mathbb{F}$*. A* group scheme homomorphism $\rho\colon G \to H$ *is a natural transformation of functors between* G *and* H*.*

Yoneda's Lemma ensures that ρ *is completely determined by a unique Hopf algebra homomorphism* $\rho^*\colon \mathbb{F}[H] \to \mathbb{F}[G]$*, called the* comorphism *of* ρ*.*

If $\rho\colon G \to H$ is a group scheme homomorphism, we will denote by $\rho_R\colon G(R) \to H(R)$ the group homomorphism between the groups of R points.

Since the study of group schemes is equivalent to the study of their representing Hopf algebras, we would like to restrict our attention to those group schemes represented by easy to handle Hopf algebras. This leads to the notion of *algebraic groups.*

Definition 1.2.6. *A group scheme* G *over* $\mathbb{F}$ *is called an* algebraic group *if the representing Hopf algebra* $\mathbb{F}[G]$ *is finitely generated as an* $\mathbb{F}$*-algebra.*

The following are some standard examples of algebraic groups.

Example 1.2.7. The *trivial group* scheme, $\mathbb{1}$, defined as

$$\begin{aligned} \mathbb{1}\colon \text{Alg}_{\mathbb{F}} &\to \text{Grp} \\ R &\mapsto \{1\} \end{aligned}$$

is represented by $\mathbb{F}[\mathbb{1}] = \mathbb{F}$, the trivial Hopf algebra. Since $\mathbb{F} \otimes_{\mathbb{F}} \mathbb{F} = \mathbb{F}$, the Hopf algebra structure is given by taking m, u, c, ε, i to all be the identity on $\mathbb{F}$.

Example 1.2.8. Let V be a finite dimensional $\mathbb{F}$-vector space. The functor

$$\begin{aligned} \mathbb{V}\colon \text{Alg}_{\mathbb{F}} &\to \text{Grp} \\ R &\mapsto V \otimes_{\mathbb{F}} R \end{aligned}$$

is represented by $\mathbb{F}[\mathbb{V}] = \mathrm{S}(V^*)$, the symmetric algebra of V dual. It is finitely generated by $\{f_1, f_2, \ldots, f_n\}$, a basis of V^*. The Hopf algebra structure on $\mathrm{S}(V^*)$ is given by

$$\begin{aligned} m(f_i \otimes f_j) &= f_i \otimes f_j, \\ u(a) &= a \cdot 1, \\ c(f_i) &:= f_i \otimes 1 + 1 \otimes f_i, \\ \varepsilon(f_i) &:= 0, \\ i(f_i) &:= -f_i. \end{aligned}$$

A particular case of this functor is when $V = \mathbb{F}$, in which case we denote the functor $\mathbb{G}_a$, called the *additive group*.

$$\begin{aligned} \mathbb{G}_a \colon \mathrm{Alg}_{\mathbb{F}} &\to \mathrm{Grp} \\ R &\mapsto \mathbb{F} \otimes_{\mathbb{F}} R = R. \end{aligned}$$

The group operation induced by the Hopf structure defined above is simply addition in R. That is, if $(R, \cdot, +)$ is an $\mathbb{F}$-algebra, then $\mathbb{G}_a(R) = (R, +)$ is the additive group of R. The representing Hopf algebra of the additive group, $\mathbb{F}[\mathbb{G}_a] = \mathrm{S}(\mathbb{F}^*) = \mathbb{F}[t]$, is the ring of polynomials over $\mathbb{F}$.

Example 1.2.9. Let A be a unital, associative, $\mathbb{F}$-algebra of dimension n. A has a faithful representation as a subalgebra of $\mathrm{M}_n(\mathbb{F})$ ($n \times n$ matricies with entries in $\mathbb{F}$) and hence has a norm map $N : A \to \mathbb{F}$ induced by the determinant of the representatives.

N can be considered as an element of $\mathrm{S}^n(A^*) \subset \mathrm{S}(A^*)$ and so we can form an algebra $B := \mathrm{S}(A^*)[\frac{1}{N}]$. If we equip B with a comultiplication induced by the map $A^* \to A^* \otimes_{\mathbb{F}} A^*$ dual to the multiplication $m_A \colon A \otimes_{\mathbb{F}} A \to A$, (that is for $f \in A^*$, $c(f) = f \circ m_A$) then $\mathrm{Hom}_{\mathbb{F}}(B, R)$ becomes a group for all $R \in \mathrm{Alg}_{\mathbb{F}}$. This condition then implies that B has a unique counit and antipode making it a Hopf algebra [KMRT, Remark 20.1].

Denoting $\mathbb{GL}_1(A) := \mathrm{Hom}_{\mathbb{F}}(B, -)$, and the set of units of A by $A^\times$, we get a group scheme acting as follows.

$$\begin{aligned} \mathbb{GL}_1(A) \colon \mathrm{Alg}_{\mathbb{F}} &\to \mathrm{Grp} \\ R &\mapsto (A \otimes_{\mathbb{F}} R)^\times. \end{aligned}$$

The *general linear group* is a case of the above functor when $A = \mathrm{End}_{\mathbb{F}}(V)$, the algebra of endomorphisms for a finite dimensional $\mathbb{F}$-vector space V. In this case we denote $\mathbb{GL}_1(\mathrm{End}_{\mathbb{F}}(V)) = \mathbb{GL}(V)$, and then the group of R points are given by

$$\begin{aligned} \mathbb{GL}(V) \colon \mathrm{Alg}_{\mathbb{F}} &\to \mathrm{Grp} \\ R &\mapsto \mathrm{GL}(V \otimes_{\mathbb{F}} R) \end{aligned}$$

where $\mathrm{GL}(V \otimes_{\mathbb{F}} R)$ is the classical group of invertible R-linear endomorphisms of $V \otimes_{\mathbb{F}} R$. Further convention is to denote $\mathbb{GL}(\mathbb{F}^n) = \mathbb{GL}_n(\mathbb{F})$.

The pair to the additive group is the *multiplicative group* $\mathbb{G}_m := \mathbb{GL}_1(\mathbb{F})$

$$\begin{aligned} \mathbb{G}_m \colon \mathrm{Alg}_{\mathbb{F}} &\to \mathrm{Grp} \\ R &\mapsto R^{\times}. \end{aligned}$$

$\mathbb{G}_m$ sends an $\mathbb{F}$-algebra $(R, \cdot, +)$ to the multiplicative group of its units $(R^{\times}, \cdot)$. In this case the representing Hopf algebra is $\mathbb{F}[\mathbb{G}_m] = \mathbb{F}[t, t^{-1}]$, the ring of Laurent polynomials over $\mathbb{F}$. The Hopf algebra structure is then simply the ring structure alongside

$$c(t) = t \otimes t,$$

$$\varepsilon(t) = 1,$$

$$i(t) = t^{-1}.$$

Deviating slightly from the content in [KMRT], we will discuss group scheme analogues to some classical *linear algebraic groups* (subgroups of $\mathrm{GL}_n(\mathbb{F})$ which are also an algebraic variety, that is they are the zero locus of some family of polynomials on $\mathrm{M}_n(\mathbb{F})$). The *special linear group* $\mathrm{SL}_n(\mathbb{F})$, the *orthogonal* and *special orthogonal groups*, $\mathcal{O}_n(\mathbb{F})$ and $\mathrm{SO}_n(\mathbb{F})$ respectively, as well as the *symplectic group* $\mathrm{Sp}_{2n}(\mathbb{F})$ are examples. Each of these groups has a realization as a group scheme which is a subgroup of $\mathbb{GL}_n(\mathbb{F})$. First we use [KMRT] to define what we mean by a subgroup of a group scheme, and following that we give the group schemes corresponding to the listed examples.

Definition 1.2.10. *Let H, G be group schemes over $\mathbb{F}$. H is called a* (closed) subgroup of G *if there exists a Hopf ideal $J \subset \mathbb{F}[G]$ such that $\mathbb{F}[H] = \mathbb{F}[G]/_J$.*

This produces a homomorphism of group schemes $\rho\colon H \to G$ induced by the quotient map $q\colon \mathbb{F}[G] \to \mathbb{F}[H]$. ρ is called a (closed) embedding. *Furthermore, since q is surjective, each $\rho_R\colon H(R) \to G(R)$ is injective, allowing us to identify $H(R)$ as a subgroup of $G(R)$ for all $R \in \mathrm{Alg}_{\mathbb{F}}$.*

Example 1.2.11. The special linear group is typically defined as

$$\mathrm{SL}_n(\mathbb{F}) := \{A \in \mathrm{GL}_n(\mathbb{F}) | \det(A) = 1\}.$$

The representing algebra of $\mathbb{GL}_n(\mathbb{F})$ is $\mathbb{F}[\mathbb{GL}_n(\mathbb{F})] = \mathbb{F}[x_{ij}, \frac{1}{\det}]$ (where x_{ij} form a dual basis of $\mathrm{End}_{\mathbb{F}}(V)^*$), and it contains the determinant polynomial, $\det = \det([x_{ij}]_{i,j=1}^n)$. The ring ideal generated by $\det -1$ is also a Hopf ideal in $\mathbb{F}[x_{ij}, \frac{1}{\det}]$ and so we can consider the group scheme represented by the quotient. It follows that

$$\mathrm{Hom}_{\mathbb{F}}\left(\mathbb{F}[x_{ij}, \tfrac{1}{\det}] \Big/ < (\det -1) > , R\right) = \{A \in \mathrm{GL}_n(R) \mid \det(A) = 1\} = \mathrm{SL}_n(R).$$

We call this group scheme the *special linear group*

$$\begin{aligned}\mathbb{SL}_n(\mathbb{F}) : \mathrm{Alg}_{\mathbb{F}} &\to \mathrm{Grp}\\ R &\mapsto \mathrm{SL}_n(R)\end{aligned}$$

and it is a subgroup of $\mathbb{GL}_n(\mathbb{F})$.

The remainder of the listed examples are built similarly. The defining polynomials of the variety are used to generate a Hopf ideal which then corresponds to the subgroup scheme of $\mathbb{GL}_n(\mathbb{F})$.

Example 1.2.12. The orthogonal and special orthogonal groups are given as follows

$$\mathcal{O}_n(\mathbb{F}) = \{A \in \mathrm{GL}_n(\mathbb{F}) \mid A^T A = I\},$$

$$\mathrm{SO}_n(\mathbb{F}) = \{A \in \mathrm{GL}_n(\mathbb{F}) \mid A^T A = I, \det(A) = 1\}.$$

The Hopf ideal generated by polynomials which ensure that columns of the resulting matricies are orthonormal, that is

$$J := \langle -\delta_{ij} + \sum_{k=1}^{n} x_{ki}x_{kj} \mid 1 \leq i, j \leq n\rangle$$

(δ_{ij} is the Kronecker delta function) and the same Hopf ideal with an added generator of $\det -1$,

$$J' := \langle \det -1, -\delta_{ij} + \sum_{k=1}^{n} x_{ki}x_{kj} \mid 1 \leq i, j \leq n\rangle$$

will yield the representing algebras of $\mathbb{O}_n(\mathbb{F})$, the *orthogonal group* scheme, and $\mathbb{SO}_n(\mathbb{F})$, the *special orthogonal group* scheme, respectively.

$$\mathbb{F}[\mathbb{O}_n(\mathbb{F})] = {}^{\textstyle \mathbb{F}[x_{ij}, \frac{1}{\det}]}\big/ {}_{\textstyle J}\ ,$$

$$\mathbb{F}[\mathbb{SO}_n(\mathbb{F})] = {}^{\textstyle \mathbb{F}[x_{ij}, \frac{1}{\det}]}\big/ {}_{\textstyle J'}\ .$$

Example 1.2.13. When our vector space is even dimensional, i.e. we are working with $\mathbb{F}^{2n}$, we can us the following matrix to define the *symplectic group*.

$$K := \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$$

where I_n is the $n \times n$ identity matrix. Using K, the symplectic group is

$$\mathrm{Sp}_{2n}(\mathbb{F}) := \{A \in \mathrm{GL}_n(\mathbb{F}) \mid A^T K A = K\}.$$

Just as with the orthogonal groups we can generate a Hopf ideal by the relevant relations

$$J := \langle 1 + \sum_{i=1}^{n} (x_{n+k,i}\, x_{k,n+i} - x_{k,i}\, x_{n+k,n+i}) \mid 1 \le i \le n \rangle.$$

Then the *symplectic group* scheme, $\mathbb{Sp}_{2n}(\mathbb{F})$, is the group scheme represented by

$$\mathbb{F}[\mathbb{Sp}_{2n}(\mathbb{F})] = \mathbb{F}[x_{ij}, \tfrac{1}{\det}] \,/\, J\ .$$

Remark 1.2.14. The orthogonal and symplectic groups can be described more generally as groups of matricies which preserve a bilinear form, B, on V. The group

$$G = \{A \in \mathrm{End}(V) \mid B(Ax, Ay) = B(x, y)\ \forall x, y \in V\}$$

is called the orthogonal group when B is symmetric, and the symplectic group when B is antisymmetric.

If Ω is a symmetric matrix, it defines a symmetric bilinear form via $B(x, y) = x^T \Omega y$, and therefore an orthogonal group denoted

$$\mathrm{SO}(V, \Omega) = \{A \in \mathrm{End}(V) \mid A^T \Omega A = \Omega\}.$$

Similarily for any antisymmetric matrix J we have a corresponding symplectic group

$$\mathrm{Sp}(V, J) = \{A \in \mathrm{End}(V) \mid A^T J A = J\}.$$

In the case that the bilinear form in question is non-degenerate, the choice of representing matrix, Ω or J, is simply equivalent to a choice of basis for V. This is because all non-degenerate symmetric (resp. antisymmetric) matricies over an algebraically closed field are conjugate in $\mathrm{M}_{\dim(V)}(\mathbb{F})$.

Lemma 1.2.15. *Let Ω be a symmetric matrix representing a non-degenerate bilinear form B on V. Then there exists a matrix P such that $P^T \Omega P = I$.*

Proof: Let $\dim(V) = n$. First we note that there exists $x_0 \in V$ such that $B(x_0, x_0) \neq 0$ (otherwise $B(x, x) = 0$ for all $x \in V \Leftrightarrow B$ is antisymmetric). Let $B(x_0, x_0) = c$ and choose $\alpha \in \mathbb{F}$ such that $\alpha^2 = c^{-1}$. Set $v_1 := \alpha x_0$. Therefore

$$B(v_1, v_1) = B(\alpha x_0, \alpha x_0) = \alpha^2 B(x_0, x_0) = c^{-1} c = 1.$$

We may then induct on the dimension of V by considering the subspace $\{v_1\}^\perp$ which is of lesser dimension. $B|_{\{v_1\}^\perp}$ remains a symmetric, nondegenerate bilinear form and so produces a basis of $\{v_1\}^\perp$, $\{v_2, \ldots, v_n\}$ such that $B(v_i, v_j) = \delta_{ij}$ (an equivalent condition to being conjugate to the identity, as below). This basis can then be extended to one of V sharing the same property, namely $\{v_1, v_2, \dots, v_n\}$ in which $B(v_i, v_j) = \delta_{ij}$.

Finally, by setting $P = [\ v_1 \quad v_2 \quad \dots \quad v_n\ ]$ (basis vectors inserted as columns) we have that

$$P^T \Omega P = [B(v_i, v_j)]_{i,j=1}^{n} = I.$$

∎

Lemma 1.2.16. *Let J be an antisymmetric matrix representing a non-degenerate bilinear form B on V ($\dim(V) = 2n$). Then there exists a matrix P such that $P^T J P = K$.*

Proof: Fix a vector $v_1 \in V$. Since B is non-degenerate, there exists $x_0 \in V$ such that $B(v_1, x_0) = -1$. Set $v_{n+1} = x_0$.

Next, since B remains non-degenerate on $\text{Span}\{v_1, v_{n+1}\}^{\perp}$, we may use induction to know it has a basis $\{v_2, \dots, v_n, v_{n+2}, \dots, v_{2n}\}$ such that $B(v_i, v_{n+i}) = -1$ for $2 \leq i \leq n$, and $B(v_i, v_j) = 0$ otherwise.

Extending this to the set $\{v_1, \dots, v_{2n}\}$, we have a basis of V where

$$B(v_i, v_j) = \begin{cases} -1 & 1 \leq i \leq n, j = n + i \\ 1 & n + 1 \leq i \leq 2n, j = i - n \\ 0 & \text{else.} \end{cases}$$

Therefore if we once again take P to be the matrix with the vectors $v_1, \dots, v_{2n}$ as columns

$$P^T J P = [B(v_i, v_j)]_{i,j=1}^{2n} = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}.$$

∎

This conjugacy of the representing matricies induces a conjugacy of the groups relative these forms. Let Ω_1, Ω_2 be two non-degenerate symmetric (resp. antisymmetric) matricies such that $P^T \Omega_1 P = \Omega_2$ for some P. If $A \in \text{SO}(V, \Omega_2)$ (resp. $\text{Sp}(V, \Omega_2)$), then

$$\begin{aligned} & A^T \Omega_2 A = \Omega_2 \\ \Rightarrow & A^T (P^T \Omega_1 P) A = P^T \Omega_1 P \\ \Rightarrow & (P^{-1})^T A^T P^T \Omega_1 P A P^{-1} = \Omega_1 \\ \Rightarrow & \left(P A P^{-1}\right)^T \Omega_1 \left(P A P^{-1}\right) = \Omega_1 \end{aligned}$$

and so $PAP^{-1} \in \mathrm{SO}(V, \Omega_1)$ (resp. $\mathrm{Sp}(V, \Omega_1)$). Therefore

$$P(\mathrm{SO}(V, \Omega_2))P^{-1} = \mathrm{SO}(V, \Omega_1)$$

and similarly for the symplectic case. Both groups are the special orthogonal or symplectic group of V, however they are expressed in different bases.

The above four subgroup schemes of $\mathbb{GL}_n(\mathbb{F})$ each corresponded to a quotient algebra of $\mathbb{F}[\mathbb{GL}_n(\mathbb{F})]$. Other classical examples of matrix groups which are not subgroups of $\mathrm{GL}_n(\mathbb{F})$ can be found in quotients of $\mathrm{GL}_n(\mathbb{F})$. Due to duality, these quotient groups will correspond to Hopf subalgebras of $\mathbb{F}[\mathbb{GL}_n(\mathbb{F})]$. The two examples we will discuss are group scheme analogues of $\mathrm{PGL}_n(\mathbb{F})$ and $\mathrm{PSp}_{2n}(\mathbb{F})$, the *projective general linear group* and *projective symplectic group.*

Example 1.2.17. Just as projective space is the result of quotienting a vector space by the action of scalars, so to is the *projective general linear group* acquired by taking the quotient of $\mathrm{GL}_n(\mathbb{F})$ by scalar multiples of the identity.

$$\mathrm{PGL}_n(\mathbb{F}) := \mathrm{GL}_n(\mathbb{F}) \Big/ \{\alpha I | \alpha \in \mathbb{F}^\times\} \,.$$

As outlined in [SR, Example 2.16], consider the subalgebra of $\mathbb{F}[\mathbb{GL}_n(\mathbb{F})]$

$$A := \bigoplus_{r=0}^{\infty} \frac{\mathbb{F}[x_{ij}]_{nr}}{\det^r}$$

where by $\mathbb{F}[x_{ij}]_{nr}$ we denote the set of homogenous polynomials of degree nr in variables x_{ij}. Since $\deg(\det^r) = nr$, these polynomials are invariant with respect to scaling all variables, and so intuitively the group represented by A will also have scaling invariant elements. We can then define the *projective general linear group scheme*

$$\begin{aligned} \mathbb{PGL}_n(\mathbb{F})\colon\ \mathrm{Alg}_{\mathbb{F}} &\to \mathrm{Grp} \\ R &\mapsto \mathrm{PGL}_n(R) \end{aligned}$$

which is represented by $\mathbb{F}[\mathbb{PGL}_n(\mathbb{F})] = A$.

Example 1.2.18. As a final collection of examples of linear algebraic groups, for each subgroup scheme of $\mathbb{GL}_n(\mathbb{F})$ there is a corresponding subgroup scheme of $\mathbb{PGL}_n(\mathbb{F})$.

Let $H \leq \mathbb{GL}_n(\mathbb{F})$ such that $\mathbb{F}[H] = \mathbb{F}[\mathbb{GL}_n(\mathbb{F})] \big/ J$ where $J \subset \mathbb{F}[\mathbb{GL}_n(\mathbb{F})]$ is a Hopf ideal. We can then consider the following commutative diagram of Hopf algebras

$$\begin{array}{ccc} \mathbb{F}[\mathbb{GL}_n(\mathbb{F})] \big/ J & \overset{i^*}{\twoheadleftarrow} & \mathbb{F}[\mathbb{GL}_n(\mathbb{F})] \\ \uparrow & & \uparrow q^* \\ \mathrm{Img}(i^* \circ q^*) & \twoheadleftarrow & \mathbb{F}[\mathbb{PGL}_n(\mathbb{F})] \end{array}$$

where i^* is the comorphism of the inclusion map $i : H \to \mathbb{GL}_n(\mathbb{F})$, and q^* is the comorphism of the quotient map $q : \mathbb{GL}_n(\mathbb{F}) \to \mathbb{PGL}_n(\mathbb{F})$.

Via duality, we also have a commutative diagram of group schemes

$$\begin{array}{ccc} H & \xhookrightarrow{\quad i \quad} & \mathbb{GL}_n(\mathbb{F}) \\ \big\downarrow\!\!\!\downarrow & & \big\downarrow\!\!\!\downarrow q \\ \mathrm{Hom}_{\mathbb{F}}(\mathrm{Img}(i^* \circ q^*), -) & \hookrightarrow & \mathbb{PGL}_n(\mathbb{F}) \end{array}$$

where we call $\mathrm{Hom}_{\mathbb{F}}(\mathrm{Img}(i^* \circ q^*), -) := \mathrm{P}H$ the projective version of H.

For instance, if we choose $H = \mathbb{Sp}_{2n}(\mathbb{F})$, then $\mathrm{P}H = \mathbb{PSp}_{2n}(\mathbb{F})$ and the group of $\mathbb{F}$ points is the classical projective symplectic group

$$\mathbb{PSp}_{2n}(\mathbb{F})(\mathbb{F}) = \mathrm{Sp}_{2n}(\mathbb{F}) \big/ \{\pm I\} .$$

1.3 Lie Algebras and the Killing Forms

Associated to any linear algebraic group is a Lie algebra. The structure of this Lie algebra is the bridge connecting linear algebraic groups to root systems, and therefore allows us to classify linear algebraic groups.

Definition 1.3.1. *A* Lie algebra $\mathfrak{L}$ *is a vector space over a field* $\mathbb{F}$ *with an additional binary operation* $[-,-] : \mathfrak{L} \times \mathfrak{L} \to \mathfrak{L}$ *called the* Lie bracket *such that*

(i) $[-,-] \colon \mathfrak{L} \times \mathfrak{L} \to \mathfrak{L}$ *is bilinear,*

(ii) $[x,x] = 0 \ \forall x \in \mathfrak{L}$,

(iii) $[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0 \ \forall x,y,z \in \mathfrak{L}$ *(called the Jacobi identity).*

A Lie subalgebra is simply a subspace which is closed under the bracket, and a Lie ideal is a subalgebra I such that for all $x \in I, y \in \mathfrak{L}$ we have $[x,y] \in I$.

Definition 1.3.2. *A Lie algebra* $\mathfrak{L}$ *is called* simple *if it contains no nontrivial ideals (if* $I \subseteq \mathfrak{L}$ *is an ideal, then* $I = 0, \mathfrak{L}$*), and a Lie algebra is called* semisimple *if it is a direct sum of simple Lie algebras.*

An algebraic group will be called simple (resp. semisimple) in the case that its corresponding Lie algebra is simple (resp. semisimple).

If G is an algebraic group scheme over $\mathbb{F}$ with representing Hopf algebra A, then the definition of its Lie algebra, $\mathrm{Lie}(G)$, as given in [KMRT] relies on *derivations* of the Hopf algebra A.

Definition 1.3.3. *Let* M *be an* A*-module. A* derivation*,* D*, of* A *into* M *is an* $\mathbb{F}$*-linear map* $D \colon A \to M$ *such that*

$$D(ab) = aD(b) + bD(a)$$

for all $a,b \in A$*. The set of all such derivations is denoted* $\mathrm{Der}(A,M)$.

In particular, $D \in \mathrm{Der}(A,A)$ *is called* left-invariant *if*

$$c \circ D = (\mathrm{Id} \otimes D) \circ c.$$

Definition 1.3.4. *The* Lie algebra associated to G *is given by*

$$\mathrm{Lie}(G) = \{D \in \mathrm{Der}(A,A) \mid D \text{ is left-invariant}\}$$

with bracket given by $[D_1, D_2] = D_1 \circ D_2 - D_2 \circ D_1$*, where the operations on the right are the usual composition and difference of linear maps.*

In practice, it is easiest to compute the Lie algebra of a group scheme not by finding left-invariant derivations of A, but by instead computing the kernel of the group homomorphism between certain groups of points of G. The setup is as follows.

By $\mathbb{F}[\delta]$ we denote the $\mathbb{F}$-algebra of *dual numbers*.

$$\mathbb{F}[\delta] = \{a + b\delta \mid a, b \in \mathbb{F}, \delta^2 = 0\}.$$

There exists a unique $\mathbb{F}$-algebra homomorphism $k\colon \mathbb{F}[\delta] \to \mathbb{F}$ defined by $k(\delta) = 0$. We are interested in the kernel of the group homomorphism $G(k)\colon G(\mathbb{F}[\delta]) \to G(\mathbb{F})$, to which we will add a vector space structure.

Clearly if $f \in \ker(G(k))$, we can regard it as $f = \varepsilon + f_1\delta$ where $\varepsilon\colon A \to \mathbb{F}$ is the counit of A and $f_1\colon A \to \mathbb{F}$ is a derivation in $\mathrm{Der}(A, \mathbb{F})$. Then $\ker(G(k))$ is an $\mathbb{F}$-vector space with operations

$$\begin{array}{ll} f + g = f \cdot g & \forall f, g \in \ker(G(k)) \text{ (multiplication in the group } G(\mathbb{F}[\delta])) \\ \alpha(\varepsilon + f_1\delta) = \varepsilon + \alpha f_1\delta & \forall f \in \ker(G(k)), \alpha \in \mathbb{F}. \end{array}$$

By [KMRT, Proposition 21.1], with these operations there exists a natural $\mathbb{F}$-vector space isomorphism $\mathrm{Lie}(G) \cong \ker(G(k))$. It is given by $D \leftrightarrow \varepsilon + (\varepsilon \circ D)\delta$.

Furthermore, there is a method of computing the Lie bracket on $\mathrm{Lie}(G)$ from the elements of $\ker(G(k))$. To compute $[\varepsilon + f_1\delta, \varepsilon + g_1\delta]$, consider the $\mathbb{F}$-algebra

$$\mathbb{F}[\delta, \delta'] = \{a + b\delta + c\delta' \mid \delta^2 = \delta'^2 = 0\}$$

and the two elements $f' = \varepsilon + f_1\delta$ and $g' = \varepsilon + g_1\delta'$ in the group $G(\mathbb{F}[\delta, \delta'])$. $f'g'f'^{-1}g'^{-1}$ will be an element of the form $\varepsilon + h\delta\delta'$, and setting $[\varepsilon + f_1\delta, \varepsilon + g_1\delta] = \varepsilon + h\delta$ makes the isomorphism above one of Lie algebras.

A convenient result is [KMRT, Corollary 21.2] which states that if G is an algebraic group scheme, then $\dim_{\mathbb{F}}(\mathrm{Lie}(G)) < \infty$. This allows us to limit our further discussion of Lie algebras to finite dimensional ones without losing any relevant cases.

Carrying out the above construction on the examples of linear algebraic groups mentioned previously yields these Lie algebras

G	$\mathrm{Lie}(G)$	$[-,-]$
$\mathbb{1}$	$\{0\}$	$[0,0]=0$
$\mathbb{V}$	V	$[A,B]=0$
$\mathbb{G}_a$	$\mathbb{F}$	$[A,B]=0$
$\mathbb{G}_m$	$\mathbb{F}$	$[A,B]=0$
$\mathbb{GL}(V)$	$\mathfrak{gl}(V)=\mathrm{End}(V)$	$[A,B]=AB-BA$
$\mathbb{PGL}(V)$	$\mathfrak{pgl}=\mathrm{End}(V)\big/\{c\,\mathrm{Id}\mid c\in\mathbb{F}^\times\}$	$[A,B]=AB-BA$
$\mathbb{SL}(V)$	$\mathfrak{sl}(V)=\{A\in\mathrm{End}(V)\mid\mathrm{Tr}(A)=0\}$	$[A,B]=AB-BA$
$\mathbb{O}(V)$	$\mathfrak{o}(V)=\{A\in\mathrm{End}(V)\mid A^T=-A\}$	$[A,B]=AB-BA$
$\mathbb{SO}(V)$	$\mathfrak{so}(V)=\{A\in\mathrm{End}(V)\mid A^T=-A,\mathrm{Tr}(A)=0\}$	$[A,B]=AB-BA$
$\mathbb{Sp}(W)$	$\mathfrak{sp}(W)=\{A\in\mathrm{End}(W)\mid(KA)^T=KA\}$	$[A,B]=AB-BA$

where V is any $\mathbb{F}$-vector space and W is a vector space of even dimension.

Let $\mathfrak{L}$ be a Lie algebra over a field $\mathbb{F}$. Denote the adjoint representation of $\mathfrak{L}$ by $\mathrm{ad}\colon \mathfrak{L}\to\mathrm{GL}(\mathfrak{L})$. We define a symmetric bilinear form on $\mathfrak{L}$ called the *Killing form.*

$$\begin{aligned}\mathcal{K}\colon \mathfrak{L}\times\mathfrak{L}&\to\mathbb{F}\\ (x,y)&\mapsto\mathrm{Tr}(\mathrm{ad}(x)\,\mathrm{ad}(y)).\end{aligned}$$

The Killing form has a number of convenient properties, some of which we will state here.

Proposition 1.3.5. *The Killing form is associative with respect to the Lie bracket. That is*

$$\mathcal{K}([x,y],z)=\mathcal{K}(x,[y,z]).$$

Proof: This follows from a simple calculation. Both sides of the equation become

$$\mathrm{Tr}(\mathrm{ad}(x)\,\mathrm{ad}(y)\,\mathrm{ad}(z))-\mathrm{Tr}(\mathrm{ad}(y)\,\mathrm{ad}(x)\,\mathrm{ad}(z)).$$

∎

In particular this means that the radical of the Killing form

$$\mathrm{Rad}(\mathcal{K}):=\{x\in\mathfrak{L}\mid\mathcal{K}(x,y)=0\;\forall y\in\mathfrak{L}\}$$

is an ideal of $\mathfrak{L}$. The Killing form is called *nondegenerate* when $\mathrm{Rad}(\mathcal{K})\neq 0$ and this is equivalent to the semisimplicity of $\mathfrak{L}$.

Theorem 1.3.6. [Hum72, Theorem 5.1] *Let $\mathfrak{L}$ be a nonzero Lie algebra, then*

$$\mathfrak{L} \text{ is semisimple} \Leftrightarrow \mathcal{K} \text{ is nondegenerate.}$$

This next property will be relevant to our future discussion on invariants under group actions on $\mathfrak{L}$.

Proposition 1.3.7. *The Killing form respects automorphism of $\mathfrak{L}$. For all $\sigma \in \mathrm{Aut}(\mathfrak{L})$ and $x, y \in \mathfrak{L}$*

$$\mathcal{K}(\sigma(x), \sigma(y)) = \mathcal{K}(x, y).$$

Proof: Let $z \in \mathfrak{L}$ and consider $\mathrm{ad}(\sigma(x))\,\mathrm{ad}(\sigma(y))(z)$.

$$\begin{aligned}
&\mathrm{ad}(\sigma(x))\,\mathrm{ad}(\sigma(y))(z)\\
=&[\sigma(x), [\sigma(y), z]]\\
=&[\sigma(x), \sigma([y, \sigma^{-1}(z)])]\\
=&\sigma([x, [y, \sigma^{-1}(z)]])\\
=&(\sigma\,\mathrm{ad}(x)\,\mathrm{ad}(y)\sigma^{-1})(z).
\end{aligned}$$

Therefore, $\mathrm{ad}(\sigma(x))\,\mathrm{ad}(\sigma(y)) = \sigma\,\mathrm{ad}(x)\,\mathrm{ad}(y)\sigma^{-1}$ which means that when we evaluate the Killing form

$$\begin{aligned}
&\mathcal{K}(\sigma(x), \sigma(y))\\
=&\,\mathrm{Tr}(\mathrm{ad}(\sigma(x))\,\mathrm{ad}(\sigma(y)))\\
=&\,\mathrm{Tr}(\sigma\,\mathrm{ad}(x)\,\mathrm{ad}(y)\sigma^{-1})\\
=&\,\mathrm{Tr}(\mathrm{ad}(x)\,\mathrm{ad}(y)\sigma^{-1}\sigma)\\
=&\,\mathrm{Tr}(\mathrm{ad}(x)\,\mathrm{ad}(y))\\
=&\mathcal{K}(x, y).
\end{aligned}$$

∎

Clearly this causes the Killing form to be an element of the ring of invariant polynomials on $\mathfrak{L}$ with respect to any group of automorphisms. In particular the Killing form is an invariant of the action of the Weyl group.

Example 1.3.8. The following are examples of the Killing form of some classical Lie algebras, the general linear algebra, special linear algebra, special orthogonal algebra, and symplectic algebra respectively.

$\mathfrak{L}$	$\mathcal{K}(x, y)$
$\mathfrak{gl}(n, \mathbb{F})$	$2n\,\mathrm{Tr}(xy) - 2\,\mathrm{Tr}(x)\,\mathrm{Tr}(y)$
$\mathfrak{sl}(n, \mathbb{F})$	$2n\,\mathrm{Tr}(xy)$
$\mathfrak{so}(n, \mathbb{F})$	$(n-2)\,\mathrm{Tr}(xy)$
$\mathfrak{sp}(2n, \mathbb{F})$	$(4n+2)\,\mathrm{Tr}(xy)$

The Killing form is of importance in the following discussion since it is essentially the inner product with respect to which the roots of a Lie algebra form a root system as in 1.1. We follow the exposition of this theory as in [Hum72].

To begin, let $\mathfrak{L}$ be a finite dimensional semisimple Lie algebra. To construct the roots we require the notion of *toral subalgebras.*

Definition 1.3.9. *An element $x \in \mathfrak{L}$ is called* semisimple *if its adjoint representation,* $\mathrm{ad}(x) \in \mathrm{GL}(\mathfrak{L})$*, is a diagonalizable matrix.*

Definition 1.3.10. *A subalgebra $T \subset \mathfrak{L}$ is called* toral *if all elements of T are semisimple.*

In particular, non-zero toral subalgebras exist in semisimple Lie algebras.

Lemma 1.3.11. [Hum72, 8.1] *A toral subalgebra $T \subset \mathfrak{L}$ is abelian, i.e. $[T, T] = 0$.*

Now, let H be a maximal toral subalgebra of $\mathfrak{L}$. Since it is abelian, $\mathrm{ad}_{\mathfrak{L}}(H)$ is a commuting family of semisimple endomorphism of $\mathfrak{L}$ and hence they are simultaneously diagonalizable. Therefore, for each α in $H^* = \mathrm{Hom}_{\mathbb{F}}(H, \mathbb{F})$ we have subspaces

$$\mathfrak{L}_\alpha = \{x \in \mathfrak{L} \mid [h, x] = \alpha(h)x \quad \forall h \in H\}.$$

These form a decomposition of $\mathfrak{L}$

$$\mathfrak{L} = \bigoplus_{\alpha \in H^*} \mathfrak{L}_\alpha.$$

The set $\Phi = \{\alpha \in H^* \mid \mathfrak{L}_\alpha \neq 0, \alpha \neq 0\}$ are called the *roots* of $\mathfrak{L}$, and the associated $\mathfrak{L}_\alpha$ are called *root spaces.* These root spaces, alongside those elements of $\mathfrak{L}$ which are zero-eigenvectors for the elements of H (its centralizer $C_{\mathfrak{L}}(H) = \{x \in \mathfrak{L} \mid [x, H] = 0\}$), form a decomposition of $\mathfrak{L}$ called the *root space decomposition* or *Cartan decomposition*

$$\mathfrak{L} = C_{\mathfrak{L}}(H) \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{L}_\alpha.$$

It turns out that by [Hum72, Proposition 8.2] we have $H = C_{\mathfrak{L}}(H)$, and that by [Hum72, Proposition 8.4], for each $\alpha \in \Phi$, $\dim_{\mathbb{F}}(\mathfrak{L}_\alpha) = 1$.

[Hum72, Corollary 8.1] gives us that the restriction of the Killing form to H remains non-degenerate. This allows us to transfer the Killing form to H^* via the isomorphism

$$\begin{aligned} H &\cong H^* \\ h &\mapsto \mathcal{K}(h, -). \end{aligned}$$

We denote the inverse image of $\alpha \in H^*$ by $t_\alpha \in H$, and then define the bilinear form on H^* by $(\alpha, \beta) = \mathcal{K}(t_\alpha, t_\beta)$. This leads us to the following formula for the restriction of the Killing form to H.

Lemma 1.3.12. *Let $\lambda, \mu \in H^*$. Then*

$$(\lambda, \mu) = \mathcal{K}(t_\lambda, t_\mu) = \sum_{\alpha \in \Phi} \alpha(t_\lambda)\alpha(t_\mu).$$

Proof: By the root space decomposition, $\mathfrak{L}$ has an $\mathbb{F}$-basis

$$\{w_1, \ldots, w_m\} \cup \{v_\alpha \mid \alpha \in \Phi\}$$

where $\{w_1, \ldots, w_m\}$ is a basis of H and each $v_\alpha \in \mathfrak{L}_\alpha$.

We then consider the form of the matrix $\operatorname{ad}(t_\lambda)\operatorname{ad}(t_\mu)$ with respect to this basis. For each w_i, since it is in H

$$\operatorname{ad}(t_\lambda)\operatorname{ad}(t_\mu)(w_i) = [t_\lambda, [t_\mu, w_i]] = [t_\lambda, 0] = 0$$

and for each v_α, since it is in $\mathfrak{L}_\alpha$ and $t_\lambda, t_\mu \in H$

$$\operatorname{ad}(t_\lambda)\operatorname{ad}(t_\mu)(v_\alpha) = [t_\lambda, [t_\mu, v_\alpha]] = [t_\lambda, \alpha(t_\mu)v_\alpha] = \alpha(t_\lambda)\alpha(t_\mu)v_\alpha.$$

So the matrix of $\operatorname{ad}(t_\lambda)\operatorname{ad}(t_\mu)$ takes the form

$$\operatorname{ad}(t_\lambda)\operatorname{ad}(t_\mu) = \left[\begin{array}{ccc|cccc} 0 & & & & & & \\ & \ddots & & & & & \\ & & 0 & & & & \\ \hline & & & \alpha(t_\lambda)\alpha(t_\mu) & & & \\ & & & & \beta(t_\lambda)\beta(t_\mu) & & \\ & & & & & \ddots & \\ & & & & & & \gamma(t_\lambda)\gamma(t_\mu) \end{array}\right]$$

where α, β, γ represent the elements of Φ. From this it becomes clear that

$$\mathcal{K}(t_\lambda, t_\mu) = \operatorname{Tr}(\operatorname{ad}(t_\lambda)\operatorname{ad}(t_\mu)) = \sum_{\alpha \in \Phi} \alpha(t_\lambda)\alpha(t_\mu)$$

and the result follows. ∎

1.4 Killing Form and Characters

Instead of considering a maximal toral subalgebra and the adjoint action on its Lie algebra, we may also find the root system of a linear algebraic group G by considering the adjoint action of a *maximal torus* $T \subseteq G$ on the Lie algebra Lie(G). To define what is an algebraic torus we first require two preliminary definitions. We again reference [KMRT] in the following.

Definition 1.4.1. *An algebraic group scheme, G, is called* diagonablizable *if*

$$G = \mathrm{Hom}_{\mathbb{F}}(\mathbb{F}\langle H\rangle, -)$$

where H is a concrete group and $\mathbb{F}\langle H\rangle$ is the group algebra equipped with Hopf algebra structure $c(h) = h \otimes h$, $i(h) = h^{-1}$, and $\varepsilon(h) = 1$.

An algebraic group scheme is of multiplicative type *if its restriction to $\mathbb{F}_{\mathrm{sep}}$-algebras, G_{sep}, is a diagonalizable group scheme.*

Definition 1.4.2. *A* torus*, T, is an algebraic group scheme of multiplicative type such that H is a free abelian group of finite rank.*

A torus is called split *if it is already a diagonalizable group scheme, i.e. $T = \mathrm{Hom}_{\mathbb{F}}(\mathbb{F}\langle H\rangle, -)$. In this case T is isomorphic to the group scheme of diagonal matricies, $\mathbb{G}_m \times \ldots \times \mathbb{G}_m$, where the number of factors is the rank of H.*

Remark 1.4.3. If T is a torus, the character group is defined to be $T^* := \mathrm{Hom}_{\mathbb{F}}(T, \mathbb{G}_m)$. In the case that T is split, $T = \mathbb{G}_m \times \ldots \mathbb{G}_m$ with n factors where n is the rank of H. T is then represented by the Hopf algebra $\mathbb{F}[t_1, t_1^{-1}, t_2, t_2^{-1}, \ldots, t_n, t_n^{-1}]$, with structure $c(t_i) = t_i \otimes t_i$, $\varepsilon(t_i) = 1$, and $i(t_i) = t_i^{-1}$.

The group T^* is then isomorphic to the group of comorphisms

$$\mathrm{Hom}_{\mathbb{F}}\left(\mathbb{F}[t, t^{-1}], \mathbb{F}[t_1, t_1^{-1}, \ldots, t_n, t_n^{-1}]\right).$$

These are all maps φ (which are uniquely determined by $\varphi(t)$) such that

- $\varphi(t)$ is invertible,
- $c(\varphi(t)) = \varphi(t) \otimes \varphi(t)$.

Note that the only elements of $\mathbb{F}[t_1, t_1^{-1}, \ldots, t_n, t_n^{-1}]$ exhibiting the property that $c(x) = x \otimes x$ are the monomials $t_1^{d_1} t_2^{d_2} \ldots t_n^{d_n}$ where $d_i \in \mathbb{Z}$. Then since the Hopf algebra structure on $\mathbb{F}[t, t^{-1}]$ induces a group structure on comorphisms defined by pointwise multiplication, we have that for

$$\varphi_1 \equiv t_1^{d_1} \ldots t_n^{d_n} \text{ and } \varphi_2 \equiv t_1^{e_1} \ldots t_n^{e_n}$$

$$\varphi_1 \cdot \varphi_2 = t_1^{d_1+e_1} \dots t_n^{d_n+e_n}.$$

Therefore there is clearly an isomorphism $T^* \cong \mathbb{Z}^n$ given by

$$\varphi \equiv t_1^{d_1} \dots t_n^{d_n} \mapsto (d_1, \dots, d_n).$$

Definition 1.4.4. *Let G be a linear algebraic group. A* maximal torus, *$T \subseteq G$, is a torus which is a subgroup G that is not contained in any larger toral subgroup. If a maximal torus is split, G is also called split.*

Note that all maximal tori in a group G are conjugate [Hall, Theorem 11.9].

Let G be a split semisimple algebraic group and let $T \subseteq G$ be a split maximal torus. Consider the adjoint representation $\mathrm{ad}\colon G \to \mathbb{GL}(\mathrm{Lie}(G))$ as defined in [KMRT, Example 22.19.2].

Let $R \in \mathrm{Alg}_{\mathbb{F}}$, $g \in G(R)$, and $x \in \mathrm{Lie}(G) \otimes_{\mathbb{F}} R$. Since

$$\mathrm{Lie}(G) \otimes_{\mathbb{F}} R = \ker(G(R[\delta]) \to G(R))$$

this means that $x\colon \mathbb{F}[G] \to R[\delta]$ such that $x(\alpha) = \varepsilon(\alpha) + f(\alpha)\delta$ where ε is the counit of $\mathbb{F}[G]$ and

$$f \in \mathrm{Der}(\mathbb{F}[G], R) = \{f\colon \mathbb{F}[G] \to R \mid f(ab) = \varepsilon(b)f(a) + \varepsilon(a)f(b)\}.$$

Then g acting on x is

$$\begin{aligned}(\mathrm{ad}(R)(g))(x)\colon \mathbb{F}[G] &\to R[\delta]\\ \alpha &\mapsto \varepsilon(\alpha) + (gfg^{-1})(\alpha)\delta.\end{aligned}$$

Here the expression gfg^{-1} is a multiplication of maps from $\mathbb{F}[G]$ using its Hopf algebra structure, and $gfg^{-1} \in \mathrm{Der}(\mathbb{F}[G], R)$.

We can restrict this representation to the maximal torus T, and since T is split, $\mathrm{ad}|_T\colon T \to \mathbb{GL}(\mathrm{Lie}(G))$ is a representation of a diagonalizable group. Therefore by the theory of representations of diagonalizable groups [KMRT, Section 22.20], there is a decomposition

$$\mathrm{Lie}(G) = \bigoplus_{\alpha} V_\alpha$$

where the sum is taken over all *weights* $\alpha \in T^*$, and

$$V_\alpha := \{x \in \mathrm{Lie}(G) \mid \mathrm{ad}(g)(x) = \alpha(g)x \ \forall g \in T\}$$

is called the *weight space*.

Definition 1.4.5. *$\alpha \in T^*$ is a weight of the representation* $\mathrm{ad}|_T$ *if the weight space $V_\alpha \neq 0$.*

The non-zero weights are called roots *of G (with respect to T). The set of roots is denoted $\Phi(G)$.*

Theorem 1.4.6. [KMRT, Theorem 25.1] *The set $\Phi(G)$ is a root system in $T^* \otimes_{\mathbb{Z}} \mathbb{R}$. The root system does not depend (up to isomorphism) on the choice of maximal torus T.*

Once we have the root system for an algebraic group, we can distinguish an element of $S^2(T^*)$ as the Killing form of the root system. Motivated by the previous section, we define the Killing form to be

$$\mathcal{K} = \sum_{\alpha \in \Phi(G)} \alpha^2.$$

In these next examples, we compute the root systems and Killing forms of the common linear algebraic groups. We do so by choosing a basis on V and a maximal torus in G which prove convenient in Chapter 2. Note that we will be working with the traditional definitions of these linear algebraic groups. This suffices since $\mathrm{Lie}(G)$ is an $\mathbb{F}$-vector space and the action of G on $\mathrm{Lie}(G)$ is determined by $\mathrm{ad}(\mathbb{F})\colon G(\mathbb{F}) \to \mathrm{GL}(\mathrm{Lie}(G))$ where $G(\mathbb{F})$ is precisely the traditional group.

Example 1.4.7. Let $\dim(V) = n+1$ and choose any basis of V. Following [KMRT, 25.9], we give the root system of

$$\mathrm{SL}(V) = \{A \in \mathrm{End}(V) \mid \det(A) = 1\}.$$

We will denote diagonal matricies by

$$\mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_{n+1}) = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_{n+1} \end{bmatrix}$$

Choose the maximal torus

$$T = \left\{ \mathrm{diag}(\lambda_1, \ldots, \lambda_{n+1}) \mid \prod_{i=1}^{n+1} \lambda_i = 1 \right\} \subseteq \mathrm{SL}(V)$$

which has character group

$$T^* = \langle \chi_i \mid 1 \leq i \leq n+1,\ \chi_1\chi_2\ldots\chi_{n+1} = 1 \rangle$$

where $\chi_i(\mathrm{diag}(\lambda_1, \ldots, \lambda_{n+1})) := \lambda_i$.

Note that

$$\begin{aligned} &T^* \cong \mathbb{Z}^{n+1} \Big/ \langle e_1 + e_2 + \ldots + e_{n+1} \rangle \\ &\chi_i \leftrightarrow \overline{e_i} \end{aligned}$$

and is therefore generated by $\{\overline{e_i} \mid 1 \leq i \leq n\}$.

The Lie algebra of $\mathrm{SL}(V)$ is

$$\begin{aligned}\mathrm{Lie}(\mathrm{SL}(V)) &= \{A \in \mathrm{End}(V) \mid \mathrm{Tr}(A) = 0\} \\ &= \mathrm{Span}\{E_{ij}, E_{ii} - E_{i+1,i+1} \mid 1 \leq i < j \leq n+1\}.\end{aligned}$$

T acts on $\mathrm{Lie}(\mathrm{SL}(V))$ by conjugation. If $A \in T$ and $B \in \mathrm{Lie}(\mathrm{SL}(V)$ then $A \cdot B = ABA^{-1}$. The weight space decomposition is

Weight space	Weight
Diagonal Matricies	$1 \leftrightarrow 0$
$\mathbb{F}\langle E_{ij}\rangle$ for $1 \leq i \neq j \leq n+1$	$\chi_i\chi_j^{-1} \leftrightarrow \overline{e_i} - \overline{e_j}$

This produces a root system of type A_n

$$\Phi(\mathrm{SL}(V)) = \{\overline{e_i} - \overline{e_j} \mid 1 \leq i \neq j \leq n+1\}.$$

The Killing form is then

$$\mathcal{K} = \sum_{\substack{i,j=1 \\ i\neq j}}^{n+1} (\overline{e_i} - \overline{e_j})^2 = 2(n+1)\sum_{i=1}^{n+1}\overline{e_i}^2 = 4(n+1)\sum_{\substack{i,j=1 \\ i\leq j}}^{n}\overline{e_i}\,\overline{e_j}.$$

Example 1.4.8. Let $\dim(V) = 2n$ and choose a basis such that the symmetric bilinear form is represented by the matrix with ones on the second diagonal only.

$$\Omega = \begin{bmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{bmatrix}$$

The special orthogonal group relative to Ω is

$$\mathrm{SO}(V, \Omega) = \{A \in \mathrm{End}(V) \mid A^T\Omega A = \Omega,\ \det(A) = 1\}.$$

This group has a maximal torus

$$T := \{\mathrm{diag}(t_1, t_2, \ldots, t_n, t_n^{-1}, \ldots, t_2^{-1}, t_1^{-1}) \mid t_i \in \mathbb{F}^\times\}$$

which then has character group $T^* = \langle \chi_i \mid 1 \leq i \leq n\rangle \cong \mathbb{Z}^n$ via $\chi_i \leftrightarrow e_i$ where

$$\chi_i(\mathrm{diag}(t_1, t_2, \ldots, t_n, t_n^{-1}, \ldots, t_2^{-1}, t_1^{-1})) = t_i.$$

Relative to our choise of basis

$$\mathrm{Lie}(\mathrm{SO}(V,\Omega)) = \{A \in \mathrm{End}(V) \mid \Omega A^T \Omega = -A\}.$$

The matrix $\Omega A^T \Omega$ is simply the reflection of A along its second diagonal. Denote this by A^C. If $A = [a_{ij}]_{i,j=1}^n$, then $A^C = [a_{n+1-j,n+1-i}]_{i,j=1}^n$. Therefore

$$\begin{aligned}\mathrm{Lie}(\mathrm{SO}(V,\Omega)) &= \{A \in \mathrm{End}(V) \mid A^C = -A\} \\ &= \mathrm{Span}\{E_{ij} - E_{2n+1-j,2n+1-i} \mid 1 \le i \le 2n, 1 \le j \le 2n+1-i\}.\end{aligned}$$

T again acts on $\mathrm{Lie}(\mathrm{SO}(V,\Omega))$ by conjugation producing the following decomposition.

Weight space	Weight
Diagonal Matricies	$1 \leftrightarrow 0$
$\mathbb{F}\langle E_{ij} - E_{2n+1-j,2n+1-i}\rangle$, $1 \le i \ne j \le n$	$\chi_i\chi_j^{-1} \leftrightarrow e_i - e_j$
$\mathbb{F}\langle E_{i,2n+1-j} - E_{j,2n+1-i}\rangle$, $1 \le i,j \le n$	$\chi_i\chi_j \leftrightarrow e_i + e_j$
$\mathbb{F}\langle E_{2n+1-i,j} - E_{2n+1-j,i}\rangle$, $1 \le i,j \le n$	$\chi_i^{-1}\chi_j^{-1} \leftrightarrow -e_i - e_j$

The root system produced is of type D_n

$$\Phi(\mathrm{SO}(V,\Omega)) = \{\pm e_i \pm e_j \mid 1 \le i < j \le n\}.$$

Its Killing form is

$$\begin{aligned}\mathcal{K} &= \sum_{\substack{i,j=1\\ i<j}}^{n} (e_i+e_j)^2 + (-e_i+e_j)^2 + (e_i-e_j)^2 + (-e_i-e_j)^2 \\ &= 4(n-1)\sum_{i=1}^{n} e_i^2.\end{aligned}$$

Example 1.4.9. Let $\dim(V) = 2n+1$, and choose a basis such that the symmetric form is represented by Ω. Again we consider the special orthogonal group

$$\mathrm{SO}(V,\Omega) = \{A \in \mathrm{End}(V) \mid A^T\Omega A = \Omega,\ \det(A) = 1\}.$$

We choose the maximal torus

$$T := \{\mathrm{diag}(t_1, t_2, \ldots, t_n, 1, t_n^{-1}, \ldots, t_2^{-1}, t_1^{-1}) \mid t_i \in \mathbb{F}^\times\}.$$

The character group of T is again $T^* = \langle \chi_i \mid 1 \le i \le n\rangle \cong \mathbb{Z}^n$, however

$$\chi_i(\mathrm{diag}(t_1, t_2, \ldots, t_n, 1, t_n^{-1}, \ldots, t_2^{-1}, t_1^{-1})) = t_i.$$

The Lie algebgra also appears similar.

$$\begin{aligned}\mathrm{Lie}(\mathrm{SO}(V,\Omega)) &= \{A \in \mathrm{End}(V) \mid A^C = -A\} \\ &= \mathrm{Span}\{E_{ij} - E_{2n+2-j,2n+2-i} \mid 1 \le i \le 2n+1, 1 \le j \le 2n+2-i\}.\end{aligned}$$

The difference in dimension of V produces some additional weight spaces in the decomposition of $\mathrm{Lie}(\mathrm{SO}(V,\Omega))$.

Weight space	Weight
Diagonal Matricies	$1 \leftrightarrow 0$
$\mathbb{F}\langle E_{ij} - E_{2n+2-j,2n+2-i}\rangle$, $1 \le i \ne j \le n$	$\chi_i\chi_j^{-1} \leftrightarrow e_i - e_j$
$\mathbb{F}\langle E_{i,2n+2-j} - E_{j,2n+2-i}\rangle$, $1 \le i < j \le n$	$\chi_i\chi_j \leftrightarrow e_i + e_j$
$\mathbb{F}\langle E_{2n+2-i,j} - E_{2n+2-j,i}\rangle$, $1 \le i < j \le n$	$\chi_i^{-1}\chi_j^{-1} \leftrightarrow -e_i - e_j$
$\mathbb{F}\langle E_{i,n+1} - E_{n+1,2n+2-i}\rangle$, $1 \le i \le n$	$\chi_i \leftrightarrow e_i$
$\mathbb{F}\langle E_{n+1,i} - E_{2n+2-i,n+1}\rangle$, $1 \le i \le n$	$\chi_i^{-1} \leftrightarrow -e_i$

The root system is of type B_n

$$\Phi(\mathrm{SO}(V,\Omega)) = \{\pm e_i \pm e_j, \pm e_k \mid 1 \le i < j \le n, 1 \le k \le n\}$$

and the Killing form is

$$\begin{aligned}\mathcal{K} &= \sum_{\substack{i,j=1\\ i<j}}^{n} \left((e_i+e_j)^2 + (-e_i+e_j)^2 + (e_i-e_j)^2 + (-e_i-e_j)^2\right) + \sum_{i=1}^{n}\left(e_i^2 + (-e_i)^2\right) \\ &= (4n-2)\sum_{i=1}^{n} e_i^2.\end{aligned}$$

Example 1.4.10. Let $\dim(V) = 2n$ and choose a basis such that an antisymmetric bilinear form is represented by

$$\Psi = \left[\begin{array}{cc|cc} & 0 & & \begin{smallmatrix} & & 1\\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{smallmatrix} \\ \hline \begin{smallmatrix} & & -1\\ & \cdot^{\cdot^{\cdot}} & \\ -1 & & \end{smallmatrix} & & 0 & \end{array}\right] = \begin{bmatrix} 0 & \Omega_n \\ -\Omega_n & 0 \end{bmatrix}$$

The symplectic group is then

$$\mathrm{Sp}(V,\Psi) = \{A \in \mathrm{End}(V) \mid A^T\Psi A = \Psi\}$$

in which we choose the same maximal torus as in 1.4.8

$$T := \{\mathrm{diag}(t_1, t_2, \ldots, t_n, t_n^{-1}, \ldots, t_2^{-1}, t_1^{-1}) \mid t_i \in \mathbb{F}^\times\}$$

and so the character group $T^* = \langle \chi_i \mid 1 \leq i \leq n \rangle$ is also the same.

Computing the Lie algebra yields

$$\mathrm{Lie}(\mathrm{Sp}(V, \Psi)) = \{A \in \mathrm{End}(V) \mid \Psi A^T \Psi = A\}.$$

For $A = [a_{ij}]_{i,j=1}^{2n} \in \mathrm{End}(V)$, the equality $\Psi A^T \Psi = A$ is equivalent to the following relations.

For $1 \leq i, j \leq n$:

$$a_{ij} = -a_{2n+1-j,2n+1-i}$$

For $1 \leq i \leq j \leq n$:

$$\begin{aligned} a_{i,2n+1-j} &= a_{j,2n+1-i} \\ a_{2n+1-i,j} &= a_{2n+1-j,i}. \end{aligned}$$

Therefore we can express the Lie algebra as

$$\begin{aligned} \mathrm{Lie}(\mathrm{Sp}(V, \Psi)) = \mathrm{Span}(&\{E_{ij} - E_{2n+1-j,2n+1-i} \mid 1 \leq i, j \leq n\} \\ &\cup \{E_{i,2n+1-j} + E_{j,2n+1-i}, E_{2n+1-i,j} + E_{2n+1-j,i} \mid 1 \leq i < j \leq n\} \\ &\cup \{E_{i,2n+1-i} \mid 1 \leq i \leq 2n\}). \end{aligned}$$

The weight space decomposition is

Weight space	Weight
Diagonal Matricies	$1 \leftrightarrow 0$
$\mathbb{F}\langle E_{ij} - E_{2n+1-j,2n+1-i}\rangle$ for $1 \leq i \neq j \leq n$	$\chi_i \chi_j^{-1} \leftrightarrow e_i - e_j$
$\mathbb{F}\langle E_{i,2n+1-j} + E_{j,2n+1-i}\rangle$ for $1 \leq i < j \leq n$	$\chi_i \chi_j \leftrightarrow e_i + e_j$
$\mathbb{F}\langle E_{2n+1-i,j} + E_{2n+1-j,i}\rangle$ for $1 \leq i < j \leq n$	$\chi_i^{-1} \chi_j^{-1} \leftrightarrow -e_i - e_j$
$\mathbb{F}\langle E_{i,2n+1-i}\rangle$ for $1 \leq i \leq n$	$\chi_i^2 \leftrightarrow 2e_i$
$\mathbb{F}\langle E_{2n+1-i,i}\rangle$ for $1 \leq i \leq n$	$\chi_i^{-2} \leftrightarrow -2e_i$

The resulting root system is of type C_n

$$\Phi(\mathrm{Sp}(V, \Psi)) = \{\pm e_i \pm e_j, \pm 2e_k \mid 1 \leq i < j \leq n, 1 \leq k \leq n\}$$

with Killing form

$$\begin{aligned} \mathcal{K} &= \sum_{\substack{i,j=1 \\ i<j}}^{n} \left((e_i + e_j)^2 + (-e_i + e_j)^2 + (e_i - e_j)^2 + (-e_i - e_j)^2\right) + \sum_{i=1}^{n} \left((2e_i)^2 + (-2e_i)^2\right) \\ &= 4(n+1) \sum_{i=1}^{n} e_i^2. \end{aligned}$$

To summarize, the Killing forms associated to the special linear, special orthogonal, and symplectic groups are below.

Group	Killing Form
$\mathrm{SL}(V)$, $\dim(V) = n+1$	$4(n+1)\sum_{\substack{i,j=1\\ i\le j}}^{n} \overline{e_i}\,\overline{e_j}$
$\mathrm{SO}(V)$, $\dim(V) = 2n$	$4(n-1)\sum_{i=1}^{n} e_i^2$
$\mathrm{SO}(V)$, $\dim(V) = 2n+1$	$4(n-2)\sum_{i=1}^{n} e_i^2$
$\mathrm{Sp}(V)$, $\dim(V) = 2n$	$4(n+1)\sum_{i=1}^{n} e_i^2$

1.5 Invariants Under the Action of the Weyl Group

At the end of section 1.3 we obtained an equation for the Killing form of a Lie algebra in terms of its roots. This allows us to consider the Killing form as an element of the symmetric tensor product of H^*, denoted $\mathrm{S}(H^*)$. That is,

$$\mathcal{K} = \sum_{\alpha \in \Phi} \alpha^2 \in \mathrm{S}^2(H^*).$$

Then since all elements of the Weyl group, W, permute the elements of Φ, the Killing form is an invariant under the induced action on $\mathrm{S}(H^*)$. In order to study all such polynomial invariants, i.e. the ring $\mathrm{S}(H^*)^W$, we introduce certain *divided difference operators*, one for each simple reflection corresponding to a chosen basis of H^*.

Definition 1.5.1. *Let $\{\alpha_1, \ldots, \alpha_n\} \subseteq \Phi$ be a basis of H^* and let $\{s_1, \ldots, s_n\} \subseteq W$ be the associated simple reflections. The k^{th} divided difference operator is*

$$\begin{aligned} \Delta_k : \mathrm{S}(H^*) &\to \mathrm{S}(H^*) \\ q &\mapsto \frac{q - s_k(q)}{\alpha_k}. \end{aligned}$$

Observe that $q - s_k(q)$ is divisible by α_k for all k, hence we have

$$\Delta_k^{m+1} : \mathrm{S}^{m+1}(H^*) \to \mathrm{S}^m(H^*).$$

It is clear from the definition how these operators relate to polynomial invariants. $\Delta_k(q) = 0$ if and only if $s_k(q) = q$, and therefore

$$q \in \mathrm{S}(H^*)^W \Leftrightarrow \Delta_k(q) = 0 \quad \forall k \in \{1, \ldots, n\}.$$

Additionally these operators have a few properties which are somewhat reminiscent of differential operators.

Lemma 1.5.2. *Let $x, y \in \mathrm{S}(H^*)$. Then*

- $(\Delta_k)^2(x) = 0$,
- $\Delta_k(xy) = \Delta_k(x)y + s_k(x)\Delta_k(y)$.

Using these divided difference operators we can easily argue the following fact.

Proposition 1.5.3. *Let $\mathfrak{L}$ be a semisimple Lie algebra with maximal toral subalgebra H. Then there are no non-zero W-fixed linear forms on H^*. That is $\mathrm{S}^1(H^*)^W = 0$.*

Proof: Let $q := \sum_{i=1}^{n} c_i\alpha_i$ be a generic linear form on H^*. Then,

$$\Delta_k(q) = \sum_{i=1}^{n} c_i \frac{2(\alpha_i, \alpha_k)}{(\alpha_k, \alpha_k)} = \frac{2(\sum_{i=1}^{n} c_i\alpha_i, \alpha_k)}{(\alpha_k, \alpha_k)}.$$

If $\Delta_k(q) = 0$ for all $k \in \{1, \ldots, n\}$, then this means $(\sum_{i=1}^{n} c_i\alpha_i, \alpha_k) = 0$, and therefore $(\sum_{i=1}^{n} c_i\alpha_i, \sum_{k=1}^{n} d_k\alpha_k) = 0$ for all choices of d_k. Hence $(q, \lambda) = 0$ for all $\lambda \in H^*$, but since the bilinear form is lifted from the Killing form and is nondegenerate on H^*, this forces $q = 0$. ∎

We are also able to speak about the fixed quadratic elements of $\mathrm{S}(H^*)$. We have seen that the Killing form is such a fixed quadratic form and it turns out that among quadratic forms on H^*, the Killing form is essentially unique in this regard.

Theorem 1.5.4. *Let Φ be an irreducible root system in H^*. The Weyl group, W, of the root system acts on Φ which contains a basis of H^*, and hence the action extends naturally to $\mathrm{S}(H^*)$. Under this action the only fixed quadratic forms are multiples of the Killing form.*

$$\mathrm{S}^2(H^*)^W = \{c\mathcal{K} \mid c \in \mathbb{F}\}.$$

Proof: The proof hinges on the following lemma from [Hum90] which is a result from the representation theory of groups.

Lemma 1.5.5. [Hum90, Lemma 6.4(c)] *Let G be a group, E a finite dimensional $\mathbb{R}$-vector space, and $\rho : G \to \mathrm{GL}(E)$ be a group representation such that the only elements of $\mathrm{GL}(E)$ commuting with all of $\rho(G)$ are the scalars. Then, if β, β' are two symmetric, non-degenerate, G-invariant, bilinear forms on E, there exists some $c \in \mathbb{R}$ such that $\beta = c\beta'$.*

The assumption that E be an $\mathbb{R}$-vector space is used by Humphreys in parts (a) and (b) of lemma 6.4, however part (c) may be generalized to E being a vector space over arbitrary $\mathbb{F}$ and the result still holds. We will employ the lemma in this case.

Since the elements of $\mathrm{S}^2(H^*)$ are quadratic forms on H, we wish to apply the lemma to W acting on H. W is already a subset of $\mathrm{GL}(H^*)$, and since H^* is isomorphic to H via the map $\alpha \leftrightarrow t_\alpha$, the Weyl group also acts on H as a subgroup of $\mathrm{GL}(H)$. (For $s \in W, s(t_\alpha) = t_{s(\alpha)}$.) The representation ρ is then simply the restriction to W of the isomorphism $\mathrm{GL}(H^*) \to \mathrm{GL}(H)$. So we aim to show that the centralizer of the image of the Weyl group consists only of the scalars.

To start, choose a simple system $\{s_1, \ldots, s_n\} \subseteq W$. Then $\{\alpha_1, \ldots, \alpha_n\}$ is a basis of H^*, and letting $t_k := t_{\alpha_k}$, $\{t_1, \ldots, t_n\}$ is a basis of H.

Let $A \in \mathrm{GL}(H)$ such that $Aw = wA$ for all $w \in W$. In particular $As_k = s_kA$ for all $k \in \{1, \ldots, n\}$.

$$\begin{aligned}
&\Rightarrow As_k(t_k) = s_kA(t_k)\\
&\Rightarrow -A(t_k) = s_k(At_k)\\
&\Rightarrow At_k \in \{x \in H \mid s_k(x) = -x\} = \{ct_k \mid c \in \mathbb{F}\}\\
&\Rightarrow \forall k, \exists c_k \in \mathbb{F} \text{ such that } At_k = c_kt_k.
\end{aligned}$$

All simple coroots are eigenvectors for the matrix A. Now let $i \neq j$ and consider $s_iA(t_j) = As_i(t_j)$.

$$\begin{aligned}
&\Rightarrow s_i(c_jt_j) = A(t_j - m_{ij}t_i), \text{ where } m_{ij} = \frac{2(t_i, t_j)}{(t_i, t_i)} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}\\
&\Rightarrow c_jt_i - c_jm_{ij}t_i = c_j - m_{ij}c_it_i\\
&\Rightarrow c_j(m_{ij}t_i) = c_i(m_{ij}t_i)\\
&\Rightarrow c_j = c_i \text{ if } m_{ij} \neq 0.
\end{aligned}$$

However, we also know that $m_{ij} = \frac{2(\alpha_i,\alpha_j)}{(\alpha_i,\alpha_i)} \neq 0$ whenever α_i, α_j are adjacent in the Dynkin diagram of the root system. So $c_i = c_j$ for all adjacent roots, and then since irreducible root systems have connected Dynkin diagrams, we know that there exists $c \in \mathbb{F}$ such that $At_k = ct_k$ for all k. Finally the fact that $\{t_1, \ldots, t_n\}$ form a basis for H implies that $A = cI$. Hence the lemma applies to W acting on H.

Now to briefly avoid overlapping notation, there is the Killing form on H, which we will denote $\mathcal{K}'(-,-)$, and there is the associated quadratic form

$$\mathcal{K} = \sum_{\alpha \in \Phi} \alpha^2 \in \mathrm{S}^2(H^*)$$

For $x \in H$, $\mathcal{K}(x, x) = \mathcal{K}'(x)$.

Let $q \in \mathrm{S}^2(H^*)^W$. Then

$$q'(x, y) := q(x + y) - q(x) - q(y)$$

is a bilinear form on H with $q'(x, x) = 2q(x)$. Since q is fixed by the action of W, so is q', and therefore if q' is nondegenerate the lemma applies. That is there exists $c \in \mathbb{F}$ such that $q' = c\mathcal{K}'$. In particular

$$q'(x, x) = c\mathcal{K}'(x, x) \Rightarrow 2q(x) = c\mathcal{K}(x)$$

and so $q \in \{c\mathcal{K} \mid c \in \mathbb{F}\}$.

If q' is degenerate, then consider the set

$$D := \{x \in H \mid q'(x,y) = 0 \ \forall y \in H\} \neq \{0\}.$$

For $x \in D$, $y \in H$, and $w \in W$, since q' is W-invariant

$$0 = q'(x, w^{-1}(y)) = q'(w(x), y)$$

and so $w(x) \in D$. Then because the elements of w are automorphisms of H, D is a W-stable subspace. However in the case that Φ is irreducible, the only such stable subspaces are all of H or $\{0\}$. (Each s_k fixes the entire space and a hyperplane perpendicular to the element t_k. The intersection of these n pairwise non-parallel hyperplanes in n-dimensional space is the origin). Thus $D \neq \{0\}$ implies that $D = H$ and this forces $q' = 0$, and $q = 0$ in turn. So $q \in \{c\mathcal{K} \mid c \in \mathbb{F}\}$ still. ∎

Remark 1.5.6. This result does not require any of the preceding Lie structure and applies to any irreducible root system Φ in a vector space V^*. When the Killing form is defined as $\mathcal{K} = \sum_{\alpha \in \Phi} \alpha^2$, the associated bilinear form on V is

$$\mathcal{K}'(x,y) = \mathcal{K}(x+y) - \mathcal{K}(x) - \mathcal{K}(y) = 2\sum_{\alpha \in \Phi} \alpha(x)\alpha(y).$$

This will be non-degenerate by parallel to the Killing form of a Lie algebra. That is, there exists a Lie algbera $\mathfrak{L}$ of the same root system type as Φ. It will have a maximal toral subalgebra H, and a root system $\Phi' \subseteq H^*$ isomorphic to $\Phi \subseteq V^*$. The Killing form of $\mathfrak{L}$ is non-degenerate and corresponds via the root system isomorphism to $\frac{1}{2}\mathcal{K}'$. Hence $\mathcal{K}'$ is also non-degenerate.

Corollary 1.5.7. *Let Φ be a root system in a vector space V^* which is not irreducible. If $\Phi_i \subseteq V_i^*$ are the irreducible components ($\Phi = \bigcup_{i=1}^{n} \Phi_i$, and $V^* = \oplus_{i=1}^{n} V_i^*$.) with Killing forms $\mathcal{K}_i$, then*

$$\mathrm{S}^2(V^*)^W = \bigoplus_{i=1}^{n} \mathrm{S}^2(V_i^*)^{W_i} = \bigoplus_{i=1}^{n} \{c\mathcal{K}_i \mid c \in \mathbb{F}\}$$

where W_i is the Weyl group of the root system $\Phi_i \subseteq V_i^*$ and as such the whole Weyl group is $W = W_1 \times \ldots \times W_n$.

Proof: Let $q \in \mathrm{S}^2(V^*)^W$ and set q' to be the associated bilinear form on V. Since q' is fixed by W, it is fixed by all subgroups W_i and hence $q'|_{V_i \times V_i} \in \mathrm{S}^2(V_i^*)^{W_i}$. So $q'|_{V_i \times V_i} = c_i\mathcal{K}_i$ for some $c_i \in \mathbb{F}$.

Let $i \neq j$ and take $v_0 \in \Phi_i$, $w_0 \in \Phi_j$. The reflection $s_{w_0} \in W_j \leq W$ fixes v_0 while negating w_0. Because q' is W-invariant, this means

$$q'(v_0, w_0) = q'(s_{w_0}(v_0), s_{w_0}(w_0)) = q'(v_0, -w_0) = -q'(v_0, w_0)$$

and so $q'(v_0, w_0) = 0$. Since the sub-root systems span their respective subspaces, $q'(v, w) = 0$ whenever $v \in V_i, w \in V_j, i \neq j$.

Next let $x, y \in V$. They have decompositions $x = \sum_{i=1}^{n} v_i$, and $y = \sum_{i=1}^{n} w_i$, where $v_i, w_i \in V_i$. By definition

$$\mathcal{K}_i'(x, y) = 2 \sum_{\alpha \in \Phi_i} \alpha(x)\alpha(y).$$

However, for all $\alpha \in \Phi_i$, $\alpha(V_j) = 0$ if $i \neq j$. This implies that

$$\mathcal{K}_i'(x, y) = 2 \sum_{\alpha \in \Phi_i} \alpha(v_i)\alpha(w_i) = \mathcal{K}_i'(v_i, w_i).$$

Finally, consider

$$\begin{aligned}
q'(x, y) &= q'(\sum_{i=1}^{n} v_i, \sum_{i=1}^{n} w_i) \\
&= \sum_{i,j=1}^{n} q'(v_i, w_j) \\
&= \sum_{i=1}^{n} q'(v_i, w_i) + \sum_{\substack{i,j=1 \\ i \neq j}}^{n} q'(v_i, w_j) \\
&= \sum_{i=1}^{n} c_i \mathcal{K}_i'(v_i, w_i) + 0 \\
&= \sum_{i=1}^{n} c_i \mathcal{K}_i'(x, y).
\end{aligned}$$

Hence $q' = \sum_{i=1}^{n} c_i \mathcal{K}_i'$, meaning $q = \sum_{i=1}^{n} c_i \mathcal{K}_i$. The reverse inclusion holds since W_i fixes $\mathcal{K}_i$ and acts trivially on $\mathcal{K}_j, i \neq j$, fixing them also. ∎

1.6 Invariant Quadratic Forms

We may ask about invariants of the Weyl group which are somewhat removed from the Lie algebra setting, and consider the action of the Weyl group on quadratic forms in characters. Namely, let G be a split simple linear algebraic group with maximal torus T. Consider a root system $\Phi(G) \subset T^*$ with Weyl group W and a subgroup of invariant quadratic forms $\mathrm{S}^2(T^*)^W$. The group of characters T^* is a lattice and it may be extended to a vector space by considering $T^* \otimes_{\mathbb{Z}} \mathbb{Q}$. Then $\Phi(G)$ is a root system in $T^* \otimes_{\mathbb{Z}} \mathbb{Q}$ and $\mathrm{S}^2(T^* \otimes_{\mathbb{Z}} \mathbb{Q})^W$ consists of linear combinations of Killing forms of the irreducible components of $\Phi(G)$. We may identify our original invariants as the subgroup $\mathrm{S}^2(T^*)^W \leq \mathrm{S}^2(T^* \otimes_{\mathbb{Z}} \mathbb{Q})^W$ of elements with strictly integer coefficients. In the case that $\Phi(G)$ is irreducible, $\mathrm{S}^2(T^* \otimes_{\mathbb{Z}} \mathbb{Q})^W$ is 1-dimensional and so $\mathrm{S}^2(T^*)^W$ is an infinite cyclic group generated by an element we denote by q_0, called the *normalized Killing form.* Of course when $\Phi(G)$ is not irreducible, the group $\mathrm{S}^2(T^*)^W$ will be generated by the normalized Killing forms of the irreducible components. To summarize,

Proposition 1.6.1. *Let G be a split semisimple linear algebraic group with maximal torus T. Let $\Phi(G) = \bigcup_{i=1}^{n} \Phi_i$ be the root system decomposed into its irreducible components. Let W denote the Weyl group of $\Phi(G)$. If q_i is the normalized killing form of Φ_i, then*

$$\mathrm{S}^2(T^*)^W = \bigoplus_{i=1}^{n} \mathbb{Z}\langle q_i \rangle.$$

Below we summarize the Killing forms and normalized Killing forms of our standard simple linear algebraic groups.

Group	T^*	Killing Form	Normalized Killing Form
$\mathrm{SL}(V)$, $\dim(V) = n+1$	$\langle \overline{e_i} \mid 1 \leq i \leq n \rangle$	$4(n+1) \sum_{\substack{i,j=1 \\ i \leq j}}^{n} \overline{e_i}\,\overline{e_j}$	$\sum_{\substack{i,j=1 \\ i \leq j}}^{n} \overline{e_i}\,\overline{e_j}$
$\mathrm{SO}(V)$, $\dim(V) = 2n$	$\langle e_i \mid 1 \leq i \leq n \rangle$	$4(n-1)\sum_{i=1}^{n} e_i^2$	$\sum_{i=1}^{n} e_i^2$
$\mathrm{SO}(V)$, $\dim(V) = 2n+1$	$\langle e_i \mid 1 \leq i \leq n \rangle$	$4(n-2)\sum_{i=1}^{n} e_i^2$	$\sum_{i=1}^{n} e_i^2$
$\mathrm{Sp}(V)$, $\dim(V) = 2n$	$\langle e_i \mid 1 \leq i \leq n \rangle$	$4(n+1)\sum_{i=1}^{n} e_i^2$	$\sum_{i=1}^{n} e_i^2$

These groups of quadratic invariants turn out to display functorial properties. A map of linear algebraic groups $\varphi : G \to H$ induces a map $\varphi^* : \mathrm{S}^2(T_H^*) \to \mathrm{S}^2(T_G^*)$. In order to demonstrate this we follow the exposition of [GMS].

Definition 1.6.2. *Given an algebraic group G, a* loop *in G is a group homomorphism $\mathbb{G}_m \to G$, and so the set of all loops in G is $G_* := \mathrm{Hom}_{\mathbb{F}}(\mathbb{G}_m, G)$.*

Some needed properties of loops are

1) If $f_1, f_2 \in G_*$ are two loops with commuting images, then the pointwise product $f_1 f_2$ is again a loop. In particular, if $f_1, \ldots, f_n$ are loops with pairwise commuting images and $k_1, \ldots, k_n \in \mathbb{Z}$, then $f_1^{k_1} f_2^{k_2} \ldots f_n^{k_n}$ is again a loop.

2) If $g \in G(\mathbb{F})$ and $f \in G_*$, then ${}^g f$ is again a loop, where for $\alpha \in R \in \mathrm{Alg}_{\mathbb{F}}$, $({}^g f(R))(\alpha) = g f(R)(\alpha) g^{-1}$.

Our object of interest is the group $\mathrm{Q}(G)$ consisting of particular maps defined on these loops.

Definition 1.6.3. *To a linear algebraic group G we associate the abelian group*

$$\mathrm{Q}(G) := \{q : G_* \to \mathbb{Z} \mid \text{functions such that } (i) \text{ and } (ii) \text{ hold}\}$$

(i) : $q({}^g f) = q(f)$ for all $f \in G_*$ and $g \in G(\mathbb{F})$.

(ii) : For all $f_1, \ldots, f_n \in G_*$ with pairwise commuting images, the map

$$\begin{aligned} \mathbb{Z}^n &\to \mathbb{Z} \\ (k_1, \ldots, k_n) &\mapsto q(f_1^{k_1} f_2^{k_2} \ldots f_n^{k_n}) \end{aligned}$$

is a quadratic form on $\mathbb{Z}^n$.

The group operation in $\mathrm{Q}(G)$ is simply pointwise addition of images in $\mathbb{Z}$.

Lemma 1.6.4. *Let T be a split torus, i.e. $T \cong \mathbb{G}_m \times \ldots \times \mathbb{G}_m$ (with n factors). Then the group $\mathrm{Q}(T)$ is isomorphic to $\mathrm{S}^2(T^*)$, where both are groups of quadratic forms on the lattice T_*.*

Proof: Note that since T is abelian, all loops have commuting images and so the set of loops is simply the cocharacter group.

$$\begin{aligned} T^* = \langle \chi_i \mid 1 \le i \le n \rangle &\cong \mathbb{Z}^n \\ \chi_i &\leftrightarrow e_i \\ T_* = \langle \Delta_i \mid 1 \le i \le n \rangle &\cong \mathbb{Z}^n \\ \Delta_i &\leftrightarrow d_i \end{aligned}$$

where χ_i is the projection onto the i^{th} factor, and Δ_i is injection into the i^{th} factor with the identity in all other factors.

When the relations $\chi_i(\Delta_i(t)) = t$, and for $i \neq j$, $\chi_i(\Delta_j(t)) = 1$, are expressed additively, we have that $e_i(d_j) = \delta_{ij}$, and so T^* and T_* are dual lattices. Then naturally, the elements of $\mathrm{S}^2(T^*)$ are quadratic forms on T_*. We can show that these elements of $\mathrm{S}^2(T^*)$ also belong to $\mathrm{Q}(T)$.

Let $q \in \mathrm{S}^2(T^*)$ so $q = \sum_{\substack{i,j=1 \\ i \leq j}}^{n} c_{ij} e_i e_j$ for some $c_{ij} \in \mathbb{F}$.

Since T is abelian, for all $f \in T_*$ and $g \in T(\mathbb{F})$, if $\alpha \in R \in \mathrm{Alg}_{\mathbb{F}}$ then

$$({}^g f(R))(\alpha) = g f(R)(\alpha) g^{-1} = g g^{-1} f(R)(\alpha) = f(R)(\alpha)$$

implying ${}^g f = f$ and so clearly $q({}^g f) = q(g)$, satisfying *(i)* from the definition of $\mathrm{Q}(T)$.

Next take $f_1, \ldots, f_r \in T_*$. Let $f_i = \sum_{k=1}^{n} b_{ik} d_k$. Consider the map $\mathbb{Z}^r \to \mathbb{Z}$ such that $(l_1, \ldots, l_r) \mapsto q(l_1 f_1 + \ldots + l_r f_r)$, which is the map from *(ii)* expressed additively.

$$q(l_1 f_1 + \ldots + l_r f_r) = \left(\sum_{\substack{i,j=1 \\ i \leq j}}^{n} c_{ij} e_i e_j \right) \left(l_1 \sum_{k=1}^{n} b_{1k} d_k + \ldots + l_r \sum_{k=1}^{n} b_{rk} d_k \right).$$

After some rearranging this becomes

$$(l_1, \ldots, l_r) \mapsto \sum_{k,m=1}^{r} \left(\sum_{\substack{i,j=1 \\ i \leq j}}^{n} c_{ij} b_{ki} b_{mj} \right) l_k l_m$$

which is a quadratic form on $\mathbb{Z}^r$. Hence $q \in \mathrm{Q}(T)$.

For the reverse inclusion consider $z \in \mathrm{Q}(T)$. Choose the elements $\Delta_1, \ldots, \Delta_n \in T_*$ to construct the quadratic map $\mathbb{Z}^n \to \mathbb{Z}$ given by $(l_1, \ldots, l_n) \mapsto z(l_1 d_1 + \ldots + l_n d_n)$. Since this is a quadratic map it takes the form

$$z(l_1 d_1 + \ldots + l_n d_n) = \sum_{\substack{i,j=1 \\ i \leq j}}^{n} c_{ij} l_i l_j$$

for some $c_{ij} \in \mathbb{F}$. However this map agrees with $\sum_{\substack{i,j=1 \\ i \leq j}}^{n} c_{ij} e_i e_j \in \mathrm{S}^2(T^*)$ on all points of $\mathbb{Z}^n$, and therefore

$$z = \sum_{\substack{i,j=1 \\ i \leq j}}^{n} c_{ij} e_i e_j \in \mathrm{S}^2(T^*)$$

so the result follows. ∎

Our goal in introducing the group $\mathrm{Q}(G)$ was to use its functorial properties. If $\varphi : G \to H$ is a homomorphism of linear algebraic groups, there is an associated map between loops

$$\begin{aligned} \varphi_* \colon G_* &\to H_* \\ f &\mapsto \varphi \circ f. \end{aligned}$$

This in turn gives rise to the map

$$\begin{aligned} \mathrm{Q}(\varphi) \colon \mathrm{Q}(H) &\to \mathrm{Q}(G) \\ q &\mapsto q \circ \varphi_* \end{aligned}$$

which is a group homomorphism.

Thus, if we have a split maximal torus, T, in a linear algebraic group G, it comes with an inclusion map $i \colon T \hookrightarrow G$ which induces the homomorphism

$$\mathrm{Q}(i) \colon \mathrm{Q}(G) \to \mathrm{Q}(T) \cong \mathrm{S}^2(T^*).$$

Due to condition *(i)* of the definition of $\mathrm{Q}(G)$, the image of this homomorphism is subset of $\mathrm{S}^2(T^*)^W$. The Weyl group is usually defined as the group generated by the reflections with respect to the root system $\Phi(G) \subseteq T^*$, however there is a notion of the Weyl groups of G with respect to a subtorus T', and this is

$$W_{T'} := \mathrm{N}_{G(\mathbb{F})}(T'(\mathbb{F})) \Big/ \mathrm{C}_{G(\mathbb{F})}(T'(\mathbb{F}))$$

the normalizer of the torus modulo the centralizer of the torus in the group of $\mathbb{F}$ points. This Weyl group acts on the torus, and hence on the cocharacters by conjugation. For $s \in W_T$ and $\Delta \in T_*$, $s(\Delta) := {}^g\Delta$ where $g \in \mathrm{N}_{G(\mathbb{F})}(T(\mathbb{F}))$ is any representative of s. There is then an analogous action on the group of characters T^*. For $\chi \in T^*$, $(s\chi)(\Delta) := \chi({}^g\Delta)$. However, in the case that T is a maximal torus, these groups and their actions are isomorphic [Hall]. So we can argue our claim that $\mathrm{Q}(i)$ maps into $\mathrm{S}^2(T^*)^W$.

Let $q \in \mathrm{Q}(G)$, $s \in W$ with representative $g \in \mathrm{N}_{G(\mathbb{F})}(T(\mathbb{F}))$, and let $\Delta \in T_*$.

$$s(\mathrm{Q}(i)(q))(\Delta) = (\mathrm{Q}(i)(q))(s\Delta) = (\mathrm{Q}(i)(q))({}^g\Delta) = q(i \circ {}^g\Delta).$$

Since i is simply an inclusion map, and ${}^g\Delta$ is conjugation in G, we have that $i \circ {}^g\Delta = {}^g(i \circ \Delta)$, and so

$$q(i \circ {}^g\Delta) = q({}^g(i \circ \Delta)) = q(i \circ \Delta) = (\mathrm{Q}(i)(q))(\Delta)$$

and therefore

$$s(\mathrm{Q}(i)(q)) = \mathrm{Q}(i)(q) \Rightarrow \mathrm{Q}(i)(q) \in \mathrm{S}^2(T^*)^W.$$

We will denote this homomorphism by $\beta\colon \mathrm{Q}(G) \to \mathrm{S}^2(T^*)^W$.

Proposition 1.6.5. [GMS, Part 2, Proposition 7.2]

$$\beta\colon \mathrm{Q}(G) \to \mathrm{S}^2(T^*)^W$$

is an isomorphism of groups.

Proof: We start by showing that β is injective.

If $f \in G_*$ is a loop, then its image is either isomorphic to $\mathbb{G}_m$ or is trivial. In both cases the image is contained in a maximal torus conjugate to T, and so there is an element $g \in G(\mathbb{F})$ such that the image of ${}^g f$ is a subgroup of T, i.e. ${}^g f \in T_*$.

Assume $q_1, q_2 \in \mathrm{Q}(G)$ such that $\beta(q_1) = \beta(q_2)$. This assumption is that for all $\Delta \in T_*$, $q_1(i \circ \Delta) = q_2(i \circ \Delta)$. They agree on loops in T. Therefore for all $f \in G_*$

$$q_1(f) = q_1({}^g f) = q_2({}^g f) = q_2(f)$$

and so $q_1 = q_2$. β is injective.

To show that β is surjective, we construct an extension of a map on loops in T to a map on loops in G. Being an extension, β will then simply restrict it to the original map.

Let $q \in \mathrm{S}^2(T^*)^W$. For $f \in G_*$ and $g \in G(\mathbb{F})$ as above, define $\overline{q} \in \mathrm{Q}(G)$ by $\overline{q}(f) := q({}^g f)$. If $\overline{q}$ is well-defined, then β is surjective.

This definition of $\overline{q}$ depends on g, which is an artifct of our choice of maximal torus containing the image of f. Let S, S' be two maximal tori containing the image of f, each of which are conjugate to T via the elements $g, g' \in G(\mathbb{F})$, (${}^g S = T$ and ${}^{g'} S' = T$). Since S, S' are tori containing the image of f, they are both contained in its centralizer $H := \mathrm{C}_G(\mathrm{Img}(f))$, where they are still maximal and hence conjugate via an element $h \in H$, (${}^h S = S'$). Then we have

$${}^{g'hg^{-1}}T = {}^{g'h}S = {}^{g'}S' = T$$

meaning $g'hg^{-1} = w \in \mathrm{N}_{G(\mathbb{F})}(T(\mathbb{F}))$. In particular, $g'h = wg$.

Finally, $q({}^{g'} f) = q({}^{g'h} f) = q({}^{wg} f)$, and since q is invariant under the action of W caused by w, $q({}^{wg} f) = q({}^g f)$. This means $\overline{q}$ is well-defined, and so β is surjective. ∎

Remark 1.6.6. Let $\varphi\colon G \to H$ be a map between linear algebraic groups with maximal tori T_G, T_H and Weyl groups W_G, W_H.

We have the map

$$\mathrm{Q}(\varphi)\colon \mathrm{S}^2(T_H^*)^{W_H} \to \mathrm{S}^2(T_G^*)^{W_G}$$

where for $q \in \mathrm{S}^2(T_H^*)^{W_H}$, $\mathrm{Q}(\varphi)(q) = q \circ \varphi_*$.

We also have the map $\varphi^* : H^* \to G^*$ which restricts to $\varphi^*|_{T_H^*} : T_H^* \to T_G^*$, which in turn extends to

$$\begin{aligned} \varphi^*|_{T_H^*}\colon \mathrm{S}^2(T_H^*) &\to \mathrm{S}^2(T_G^*) \\ q &\mapsto q \circ \varphi. \end{aligned}$$

However note that in the latter instance, $\mathrm{S}^2(T_H^*)$ is being considered as multiplicative maps $T_H \to \mathbb{G}_m$. These are naturally considered as additive maps $T_{H*} \to \mathbb{Z}$ by taking

$$(q \circ \varphi)(f) = q \circ \varphi \circ f \in \mathbb{G}_{m*} \cong \mathbb{Z} \text{ for } f \in T_{H*}.$$

But then since $q \circ \varphi \circ f = q \circ \varphi_*(f)$, our two maps are identified. $\mathrm{Q}(\varphi) = \varphi^*$.

So for any homomorphism between linear algebraic groups, its dual map preserves quadratic invariants.

Chapter 2

Tensor Product Maps

In this chapter we study the behaviour of invariant quadratic forms with respect to the Kronecker tensor product map. To start with introduce the Kronecker tensor product map itself before discussing the kernels and codomains of the map when it is restricted to combinations of the special linear, special orthogonal, and symplectic groups. We continue to directly compute the behavior of the induced map on invariant quadratic forms in several cases culminating in our main result, Theorem 2.2.1, which states that this behaviour depends only on the dimensions of the underlying vector spaces. Some (but not all) computations and proofs of this chapter can be found in [GMS] and [BDZ] which motivated our investigations.

2.1 The Kronecker Product of Matricies

The Kronecker tensor product of two matricies (each representing an endomorphism of a vector space) produces a matrix which represents an endomorphism of the tensor product of the two vector spaces. Here we define this notion precisely for the invertible endomorphisms of the general linear group.

Let V_1, V_2 be finite dimensional vectors spaces over an algebraically closed field $\mathbb{F}$. Consider the following tensor product map.

$$V_1 \oplus V_2 \to V_1 \otimes_{\mathbb{F}} V_2$$
$$(v_1, v_2) \mapsto v_1 \otimes v_2.$$

It induces a map

$$\rho : \mathrm{GL}(V_1) \times \mathrm{GL}(V_2) \to \mathrm{GL}(V_1 \otimes V_2)$$

by requiring that for $(A_1, A_2) \in \mathrm{GL}(V_1) \times \mathrm{GL}(V_2)$

$$\rho((A_1, A_2))(v_1 \otimes v_2) = A_1 v_1 \otimes A_2 v_2.$$

Hence $\rho((A_1, A_2)) = A_1 \otimes A_2$ where the expression on the right is the *Kronecker product of matricies.*

Definition 2.1.1. *Let V_1, V_2 be n-dimensional and m-dimensional vector spaces respectively. Let $A \in \mathrm{GL}(V_1)$ and $B \in \mathrm{GL}(V_2)$. The* Kronecker product *of A and B is $A \otimes B \in \mathrm{GL}(V_1 \otimes V_2)$, where $(A \otimes B)(v \otimes w) := Av \otimes Bw$.*

If $A = [a_{ij}]_{i,j=1}^n$ with respect to a basis $\{v_1, \dots v_n\}$ of V_1, and $B = [b_{ij}]_{i,j=1}^m$ with respect to a basis $\{w_1, \dots, w_m\}$ of V_2, then

$$A \otimes B := [a_{ij}B]_{i,j=1}^n$$

with respect to the basis

$$\{v_1 \otimes w_1, v_1 \otimes w_2, \dots, v_1 \otimes w_m, v_2 \otimes w_1, \dots, v_2 \otimes w_m, \dots, v_n \otimes w_m\} \subseteq V_1 \otimes V_2.$$

That is, $A \otimes B$ is an $nm \times nm$ block matrix whose blocks are of the form $a_{ij}B$.

Example 2.1.2. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$, then

$$A \otimes B = \left[\begin{array}{c:c} aB & bB \\ \hdashline cB & dB \end{array}\right] = \left[\begin{array}{cc:cc} ae & af & be & bf \\ ag & ah & bg & bh \\ \hdashline ce & cf & de & df \\ cg & ch & dg & dh \end{array}\right]$$

Since our map ρ defined by the Kronecker product is a group homomorphism, naturally we ask about its kernel.

Proposition 2.1.3. *We have*

$$\ker(\rho) = \{(cI, c^{-1}I) \mid c \in \mathbb{F}^\times\}.$$

Proof: The kernel of ρ is all pairs $(A, B) \in \mathrm{GL}(V_1) \times \mathrm{GL}(V_2)$ such that $A \otimes B = I$. Let $A = [a_{ij}]_{i,j=1}^n$ and $B = [b_{ij}]_{i,j=1}^m$ with respect to bases as above. Then, if $A \otimes B = [a_{ij}B]_{i,j=1}^n = I_{nm}$ we know,

i) $a_{ij} = 0$ whenever $i \neq j$,

ii) $a_{ii}b_{jj} = 1$ for all i, j,

iii) $b_{ij} = 0$ whenever $i \neq j$.

Hence A, B are both diagonal matricies, and since $a_{11}b_{jj} = 1$ for all j, this means $B = a_{11}^{-1}I$. Then $a_{ii}b_{11} = 1$ for all i implies $A = a_{11}I$.

Hence $\ker(\rho) \subseteq \{(cI, c^{-1}I) \mid c \in \mathbb{F}^\times\}$, with the other inclusion being clear. ∎

We may also consider restrictions of ρ to distinguished linear algebraic subgroups of $\mathrm{GL}(V)$ such as $\mathrm{SL}(V), \mathrm{SO}(V)$, etc. We do this as follows. Let $G_1 \hookrightarrow \mathrm{GL}(V_1)$ and $G_2 \hookrightarrow \mathrm{GL}(V_2)$ be linear algebraic subgroups. Define the restriction of ρ to $G_1 \times G_2$ to be the composition

$$\rho|_{G_1\times G_2} : G_1 \times G_2 \hookrightarrow \mathrm{GL}(V_1) \times \mathrm{GL}(V_2) \xrightarrow{\rho} \mathrm{GL}(V_1 \otimes V_2).$$

Proposition 2.1.4. *We have*

$$\ker(\rho|_{G_1\times G_2}) = \{(cI, c^{-1}I) \mid cI \in G_1, c^{-1}I \in G_2\}.$$

Proof: This merely follows from the fact that the kernel of a restricted homomorphism is the intersection of the original kernel with the domain. That is,

$$\ker(\rho|_{G_1\times G_2}) = \ker(\rho) \cap (G_1 \times G_2) = \{(cI, c^{-1}I) \mid cI \in G_1, c^{-1}I \in G_2\}.$$

∎

In the following examples, let V be an n-dimensional vector space over $\mathbb{F}$.

Example 2.1.5. Consider the restriction of ρ to

$$\mathrm{SL}(V) = \{A \in \mathrm{GL}(V) \mid \det(A) = 1\}.$$

The kernel is then

$$\ker(\rho|_{\mathrm{SL}(V)\times\mathrm{SL}(V)}) = \{(cI, c^{-1}I) \mid cI, c^{-1}I \in \mathrm{SL}(V)\} = \{(cI, c^{-1}I) \mid c^n = 1\}.$$

Hence $\ker(\rho|_{\mathrm{SL}(V)\times\mathrm{SL}(V)})$ is isomorphic to the group of n^{th} roots of unity, $\mu_n \subseteq \mathbb{F}$.

Example 2.1.6. The special orthogonal group is

$$\mathrm{SO}(V) = \{A \in \mathrm{GL}(V) \mid \det(A) = 1, A^T = A^{-1}\}.$$

Therefore

$$\ker(\rho|_{\mathrm{SO}(V)\times\mathrm{SO}(V)}) = \{(cI, c^{-1}I) \mid c^n = 1, c^2 = 1\} = \begin{cases} \{(I, I)\} & n \text{ odd} \\ \{(I, I), (-I, -I)\} & n \text{ even.} \end{cases}$$

To consider restrictions to the symplectic group, we assume that V is $2n$-dimensional.

Example 2.1.7.

$$\mathrm{Sp}(V) = \{A \in \mathrm{GL}(V) \mid A^T K A = K\} \text{ where } K = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}.$$

Hence

$$\ker(\rho|_{\mathrm{Sp}(V)\times\mathrm{Sp}(V)}) = \{(cI, c^{-1}I) \mid cc^{-1}K = K\} = \{(cI, c^{-1}I) \mid c \in \mathbb{F}^\times\}.$$

In proposition 2.1.4 there was no requirement that the linear algebraic subgroups G_1, G_2 be equal, and so we can also consider examples involving two of the common subgroups.

Example 2.1.8.

$$\begin{aligned}\ker(\rho|_{\mathrm{SL}(\mathbb{F}^n)\times\mathrm{SO}(\mathbb{F}^m)}) &= \{(cI, c^{-1}I) \mid c^n = 1, c^{-m} = 1, c^{-2} = 1\} \\ &= \begin{cases} \{(I, I)\} & \gcd(n,m) \text{ odd} \\ \{(I, I), (-I, -I)\} & \gcd(n,m) \text{ even} \end{cases}\end{aligned}$$

In general, the above tensor product map can be extended to a product of n copies of $\mathrm{GL}(V_i)$. That is we consider

$$\begin{aligned}\rho_n : \mathrm{GL}(V_1) \times \ldots \times \mathrm{GL}(V_n) &\to \mathrm{GL}(V_1 \otimes \ldots \otimes V_n) \\ (A_1, A_2, \ldots, A_n) &\to A_1 \otimes A_2 \otimes \ldots \otimes A_n\end{aligned}$$

and $A_1 \otimes \ldots \otimes A_n$ is as we would expect

$$(A_1 \otimes \ldots \otimes A_n)(v_1 \otimes \ldots \otimes v_n) = A_1 v_1 \otimes \ldots \otimes A_n v_n.$$

Proposition 2.1.9. *We have*

$$\ker(\rho_n) = \{(c_1 I, c_2 I, \ldots, c_n I) \mid c_1 c_2 \ldots c_n = 1\}.$$

Proof: We argue inductively assuming the result holds for all ρ_m with $m < n$.

Let $A_1 \otimes \ldots \otimes A_n \in \ker(\rho_n)$. This means that $(A_1 \otimes \ldots \otimes A_{n-1}, A_n)$ is in the kernel of

$$\rho : \mathrm{GL}(V_1 \otimes \ldots \otimes V_{n-1}) \times \mathrm{GL}(V_n) \to \mathrm{GL}(V_1 \otimes \ldots \otimes V_n).$$

$\Rightarrow (A_1 \otimes \ldots \otimes A_{n-1}, A_n) = (c_n^{-1} I, c_n I)$ for some $c_n \in \mathbb{F}^\times$
$\Rightarrow (c_n A_1) \otimes A_2 \otimes \ldots \otimes A_n = I$
$\Rightarrow (c_n A_1, A_2, \ldots, A_{n-1}) \in \ker(\rho_{n-1})$
$\Rightarrow (c_n A_1, A_2, \ldots, A_{n-1}) = (a_1 I, c_2 I, \ldots, c_{n-1} I)$ such that $a_1 c_2 \ldots c_{n-1} = 1$
$\Rightarrow (A_1, A_2, \ldots, A_{n-1}, A_n) = (c_n^{-1} a_1 I, c_2 I, \ldots, c_{n-1} I, c_n I)$
with $c_n^{-1}(a_1 c_2 \ldots c_{n-1}) c_n = c_n^{-1} c_n = 1$.

Hence setting $c_1 := c_n^{-1}a_1$ we obtain that $(A_1, \ldots, A_n) = (c_1 I, \ldots, c_n I)$ such that $c_1 c_2 \ldots c_n = 1$. This implies that

$$\ker(\rho_n) \subseteq \{(c_1 I, c_2 I, \ldots, c_n I) \mid c_1 c_2 \ldots c_n = 1\}$$

with the other inclusion again being clear. ∎

Finally we consider restrictions of ρ_n to $G_1 \times G_2 \times \ldots \times G_n$, where each $G_i \hookrightarrow \mathrm{GL}(V_i)$ is a linear algebraic subgroup. The restricted map, which we shall denote as

$$\rho_n' : G_1 \times \ldots \times G_n \to \mathrm{GL}(V_1 \otimes \ldots \otimes V_n),$$

has the following kernel.

Proposition 2.1.10. *We have*

$$\ker(\rho_n') = \{(c_1 I, \ldots, c_n I) \mid c_1 c_2 \ldots c_n = 1,\ c_i I \in G_i\}.$$

Proof: Just as in the two subgroup case, the identity stems directly from the fact that

$$\ker(\rho_n') = \ker(\rho_n) \cap (G_1 \times \ldots \times G_n)$$

and the result follows. ∎

Example 2.1.11. Let $\dim(V_i) = d_i$ and let $G_i = \mathrm{SL}(V_i)$ with an obvious embedding into $\mathrm{GL}(V_i)$, then

$$\ker(\rho_n) = \{(c_1 I, \ldots, c_n I) \in \mu_{d_1} \times \ldots \times \mu_{d_n} \mid c_1 c_2 \ldots c_n = 1\}.$$

Having considered the kernels of these maps, we move on to considering their images. We are able to restrict the codomain significantly, demonstrating that ρ is a map between distinguished linear algebraic subgroups. First, a useful fact for doing so.

Lemma 2.1.12. *Let V_1, V_2 be two $\mathbb{F}$-vector spaces with $\dim(V_1) = n$ and $\dim(V_2) = m$. Let $A \in \mathrm{GL}(V_1), B \in \mathrm{GL}(V_2)$ be two matrices, then*

$$\det(A \otimes B) = \det(A)^m \det(B)^n.$$

Proof: First we decompose the matrix as $A \otimes B = (A \otimes I_m)(I_n \otimes B)$. Consider $A \otimes I_m$ which has the form

$$\begin{bmatrix} a_{11} & & & a_{1n} & & \\ & \ddots & \cdots & & \ddots & \\ & & a_{11} & & & a_{1n} \\ & \vdots & & & \vdots & \\ a_{n1} & & & a_{nn} & & \\ & \ddots & \cdots & & \ddots & \\ & & a_{n1} & & & a_{nn} \end{bmatrix}$$

which in $m(m-1)/2$ row transpositions can be rearranged to

$$\begin{bmatrix} a_{11} & & & a_{12} & & & & a_{1n} & & \\ \vdots & & & \vdots & & & \cdots & \vdots & & \\ a_{n1} & & & a_{n2} & & & & a_{nn} & & \\ & a_{11} & & & a_{12} & & & & a_{1n} & \\ & \vdots & & & \vdots & & \cdots & & \vdots & \\ & a_{n1} & & & a_{n2} & & & & a_{nn} & \\ & & \ddots & & & \ddots & & & & \ddots \\ & & a_{11} & & & a_{12} & & & & a_{1n} \\ & & \vdots & & & \vdots & \cdots & & & \vdots \\ & & a_{n1} & & & a_{n2} & & & & a_{nn} \end{bmatrix}$$

and in a further $m(m-1)/2$ column transpositions can take the form

$$\begin{bmatrix} A & & \\ & \ddots & \\ & & A \end{bmatrix}$$

which clearly has a determinant of $\det(A)^m$. Finally since we performed an even number of row/column transpositions in total, $m(m-1)$, the determinant of our original matrix agress with this determinant, and so we have

$$\det(A \otimes I_m) = \det(A)^m.$$

The determinant of the other factor is much more straightfoward. We have

$$\det(I_n \otimes B) = \det\left(\begin{bmatrix} B & & \\ & \ddots & \\ & & B \end{bmatrix}\right) = \det(B)^n.$$

Finally

$$\det(A \otimes B) = \det(A \otimes I_m)\det(I_n \otimes B) = \det(A)^m \det(B)^n$$

and so we obtain the result. ∎

This lemma is all that is required to restrict the codomain of $\rho|_{\mathrm{SL}(V_1)\times\mathrm{SL}(V_2)}$. If $(A, B) \in \mathrm{SL}(V_1)\times\mathrm{SL}(V_2)$, then $\det(A) = \det(B) = 1$ which implies that $\det(A\otimes B) = 1$. Hence

$$\rho|_{\mathrm{SL}(V_1)\times\mathrm{SL}(V_2)} : \mathrm{SL}(V_1) \times \mathrm{SL}(V_2) \to \mathrm{SL}(V_1 \otimes V_2)$$

and the argument easily extends via induction, showing that

$$\rho_n : \mathrm{SL}(V_1) \times \ldots \times \mathrm{SL}(V_n) \to \mathrm{SL}(V_1 \otimes \ldots \otimes V_n).$$

In the case of ρ restricted to special orthogonal groups, by definition If $(A, B) \in \mathrm{SO}(V_1) \times \mathrm{SO}(V_2)$, then $A^T A = I_n$, $B^T B = I_m$, and $\det(A) = \det(B) = 1$. Therefore

$$(A \otimes B)^T (A \otimes B) = (A^T \otimes B^T)(A \otimes B) = (A^T A) \otimes (B^T B) = I_n \otimes I_m = I_{nm}$$

and $\det(A \otimes B) = 1$. From this it's clear that $(A \otimes B) \in \mathrm{SO}(V_1 \otimes V_2)$ and so

$$\rho|_{\mathrm{SO}(V_1)\times\mathrm{SO}(V_2)} : \mathrm{SO}(V_1) \times \mathrm{SO}(V_2) \to \mathrm{SO}(V_1 \otimes V_2)$$

and this extends similarily,

$$\rho_n : \mathrm{SO}(V_1) \times \ldots \times \mathrm{SO}(V_n) \to \mathrm{SO}(V_1 \otimes \ldots \otimes V_n).$$

The last of the common linear algebraic subgroups we have been considering, $\mathrm{Sp}(V)$, does not conform in such a straightforward manner. Recall that

$$\mathrm{Sp}(V) = \{A \in \mathrm{GL}(V) \mid A^T K A = K\} \text{ where } K = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}.$$

It is also the case that if $A \in \mathrm{Sp}(V)$ then $\det(A) = 1$. Therefore if

$$(A_1, \ldots, A_n) \in \mathrm{Sp}(V_1) \times \ldots \times \mathrm{Sp}(V_n)$$

then

$$(A_1 \otimes \ldots \otimes A_n)^T (K \otimes \ldots \otimes K)(A_1 \otimes \ldots \otimes A_n) = A_1^T K A_1 \otimes \ldots \otimes A_n^T K A_n = K \otimes \ldots \otimes K$$

and $\det(A_1 \otimes \ldots \otimes A_n) = 1$. So the image of ρ_n lies in the group of matricies of determinant 1 preserving the bilinear form corresponding to $K \otimes \ldots \otimes K$. This is either $\mathrm{SO}(V \otimes \ldots \otimes V, K \otimes \ldots \otimes K)$ if the form is symmetric, or $\mathrm{Sp}(V \otimes \ldots \otimes V, K \otimes \ldots \otimes K)$ if the form is antisymmetric. This depends on the parity of n. Since

$$(K \otimes \ldots \otimes K)^T = K^T \otimes \ldots \otimes K^T = -K \otimes \ldots \otimes -K = (-1)^n (K \otimes \ldots \otimes K)$$

the image lies in SO when n is even, and in Sp when n is odd.

2.2 The Product of Quadratic Invariants

As we have seen already, a map of linear algebraic groups $\varphi : H \to G$ restricts to a map between maximal tori, $\varphi|_{T_H} : T_H \to T_G$. This induces a dual map between character groups $\varphi^*_{T_H} : T^*_G \to T^*_H$, which in turn extends to a map on quadratic forms $\varphi^*_{T_G} : S^2(T^*_H) \to S^2(T^*_H)$ mapping the Weyl group fixed points into Weyl group fixed points.

$$\begin{aligned}\varphi^*_{T_G} : S^2(T^*_H)^{W_H} &\to S^2(T^*_H)^{W_G}\\ q &\mapsto q \circ \varphi_*.\end{aligned}$$

In the case that H, G are split semisimple groups, we have shown that their groups of quadratic invariants are free groups of finite rank generated by the normalized Killing forms of their simple components. Therefore any such map is described entirely by the images of these generators. We now describe this behavior for the map induced by the Kronecker product map, ρ, in our familiar examples.

2.2.1 The $\mathrm{SL} \times \mathrm{SL}$-case

Let $\dim(V_1) = n+1$ and $\dim(V_2) = m+1$. Consider

$$\rho : \mathrm{SL}(V_1) \times \mathrm{SL}(V_2) \to \mathrm{SL}(V_1 \otimes V_2).$$

Let $T_1 \subseteq \mathrm{SL}(V_1)$, $T_2 \subseteq \mathrm{SL}(V_2)$ be maximal tori as in example 1.4.7. Denote the maximal torus in $\mathrm{SL}(V_1 \otimes V_2)$ by

$$T_{(n+1)(m+1)} = \{\mathrm{diag}(t_1, t_2, \dots, t_{(n+1)(m+1)}) \mid t_1 t_2 \dots t_{(n+1)(m+1)} = 1\}$$

$$\text{and} \quad T^*_{(n+1)(m+1)} \cong \mathbb{Z}^{(n+1)^2} / \langle f_1 + f_2 + \dots + f_{(n+1)(m+1)} \rangle$$

with generating elements $\overline{f}_i$ for $1 \le i \le (n+1)(m+1)-1$. Then since $T_1 \times T_2$ is a maximal torus in $\mathrm{SL}(V_1) \times \mathrm{SL}(V_2)$ we compute

$$\rho|_{T_1 \times T_2} : T_1 \times T_2 \to T_{(n+1)(m+1)}.$$

Let $A = \mathrm{diag}(a_1, \dots, a_{n+1}) \in T_1$, $B = \mathrm{diag}(b_1, \dots, b_{m+1}) \in T_2$. We have

$$\begin{aligned}\rho(A, B) = \mathrm{diag}(&a_1 b_1, a_1 b_2, \dots a_1 b_{m+1},\\ &a_2 b_1, a_2 b_2, \dots, a_2 b_{m+1}, \dots\\ &\dots, a_{n+1} b_1, a_{n+1} b_2, \dots, a_{n+1} b_{m+1}).\end{aligned}$$

To find $\rho^*(\overline{f}_i)$, we first decompose $i = k(m+1) + r$ in such a way that $0 \le k \le n$, $1 \le r \le m+1$. Observe that this decomposition is unique. Then we obtain

$$\rho^*(\overline{f}_i)(A, B) = \overline{f}_i(\rho(A, B)) = a_{k+1} b_r$$

and so

$$\rho^*(\overline{f}_i) = (\overline{e}_{k+1}, \overline{e}_r) \in T_1 \times T_2.$$

To apply these arguments to invariants $\mathrm{S}^2(T^*_{(n+1)(m+1)})^W$, note that by example 1.4.7

$$\Phi(\mathrm{SL}(V_1 \otimes V_2) = \{\overline{f}_i - \overline{f}_j \mid 1 \leq i \neq j \leq (n+1)(m+1)\}.$$

Hence the Killing form and normalized Killing form are

$$\mathcal{K}_{\mathrm{SL}(V_1\otimes V_2)} = 2(n+1)(m+1) \sum_{i=1}^{(n+1)(m+1)} \overline{f}_i^2 = 4(n+1)(m+1) \sum_{\substack{i,j=1\\ i\leq j}}^{(n+1)(m+1)-1} \overline{f_i}\,\overline{f_j},$$

$$q_{\mathrm{SL}(V_1\otimes V_2)} = \sum_{\substack{i,j=1\\ i\leq j}}^{(n+1)(m+1)-1} \overline{f_i}\,\overline{f_j} = \frac{1}{2} \sum_{i=1}^{(n+1)(m+1)} \overline{f}_i^2.$$

Since $\rho^*(\overline{f}_i)$ depends on r and k in the decomposition of i, we express the Killing form as a double sum over these variables.

$$\begin{aligned}
q_{\mathrm{SL}(V\otimes V)} &= \frac{1}{2} \sum_{k=0}^{n} \sum_{r=1}^{m+1} \overline{f}^2_{k(n+1)+r} \\
\Rightarrow \rho^*(q_{\mathrm{SL}(V\otimes V)}) &= \frac{1}{2} \sum_{k=0}^{n} \sum_{r=1}^{m+1} \rho^*(\overline{f}^2_{k(n+1)+r}) \\
&= \frac{1}{2} \sum_{k=0}^{n} \sum_{r=1}^{m+1} (\overline{e}_{k+1}, \overline{e}_r)^2 \\
&= \frac{1}{2} \sum_{k=1}^{n+1} \sum_{r=1}^{m+1} (\overline{e}_k^2, \overline{e}_r^2) \\
&= \frac{1}{2} \sum_{k=1}^{n+1} \left((m+1)\overline{e}_k^2, \sum_{r=1}^{m+1} \overline{e}_r^2 \right) \\
&= \frac{1}{2} \left((m+1) \sum_{k=1}^{n+1} \overline{e}_k^2, (n+1) \sum_{r=1}^{m+1} \overline{e}_r^2 \right) \\
&= \left((m+1)\frac{1}{2} \sum_{k=1}^{n+1} \overline{e}_k^2, (n+1)\frac{1}{2} \sum_{r=1}^{m+1} \overline{e}_r^2 \right) \\
&= \left((m+1) q_{\mathrm{SL}(V_1)}, (n+1) q_{\mathrm{SL}(V_2)} \right).
\end{aligned}$$

The Killing form is simply a multiple of the normalized Killing form, and so we obtain

$$\begin{aligned}&\rho^*(\mathcal{K}_{\mathrm{SL}(V_1\otimes V_2)})\\=&4(n+1)(m+1)\rho^*(q_{\mathrm{SL}(V_1\otimes V_2)})\\=&4(n+1)(m+1)\big((m+1)q_{\mathrm{SL}(V_1)},(n+1)q_{\mathrm{SL}(V_2)}\big)\\=&((m+1)^2 4(n+1)q_{\mathrm{SL}(V_1)},(n+1)^2 4(m+1)q_{\mathrm{SL}(V_2)})\\=&((m+1)^2\mathcal{K}_{\mathrm{SL}(V_1)},(n+1)^2\mathcal{K}_{\mathrm{SL}(V_2)}).\end{aligned}$$

2.2.2 The Even $\mathrm{SO}\times\mathrm{SO}$-case

Let $\dim(V_1)=2n$ and $\dim(V_2)=2m$. We repeat the above process for the map

$$\rho:\mathrm{SO}(V_1,\Omega_{2n})\times\mathrm{SO}(V_2,\Omega_{2m})\to\mathrm{SO}(V_1\otimes V_2,\Omega_{4nm}).$$

Let $T_1\subseteq\mathrm{SO}(V_1,\Omega_{2n})$, $T_2\subseteq\mathrm{SO}(V_2,\Omega_{2m})$ be maximal tori as described in 1.4.8. The larger torus is

$$T_{4nm}=\{\mathrm{diag}(t_1,\dots,t_{2nm},t_{2nm}^{-1},\dots,t_1^{-1})\mid t_i\in\mathbb{F}^\times\}$$

$$\text{and}\quad T_{4nm}^*=\langle f_i\mid 1\le i\le 2nm\rangle\cong\mathbb{Z}^{2nm}.$$

The root system of $\Phi(\mathrm{SO}(V_1\otimes V_2,\Omega_{4nm})$ is of type D_{2nm}

$$\Phi(\mathrm{SO}(V_1\otimes V_2,\Omega_{4nm}))=\{\pm f_i\pm f_j\mid 1\le i<j\le 2nm\}$$

and so has Killing form and normalized Killing form

$$\mathcal{K}_{\mathrm{SO}(V_1\otimes V_2,\Omega_{4nm})}=4(2nm-1)\sum_{i=1}^{2nm}f_i^2,\quad q_{\mathrm{SO}(V_1\otimes V_2,\Omega_{4nm})}=\sum_{i=1}^{2nm}f_i^2.$$

Let $A=\mathrm{diag}(a_1,\dots,a_n,a_n^{-1},\dots,a_1^{-1})\in T_1$, $B=\mathrm{diag}(b_1,\dots,b_m,b_m^{-1},\dots,b_1^{-1})\in T_2$. We then have

$$\begin{aligned}\rho(A,B)=\mathrm{diag}(&a_1b_1,a_1b_2,\dots,a_1b_m,a_1b_m^{-1},\dots,a_1b_2^{-1},a_1b_1^{-1},\\&a_2b_1,a_2b_2,\dots,a_2b_m,a_2b_m^{-1},\dots a_2b_2^{-1},a_2b_1^{-1},\dots\\\dots,&a_nb_1,a_nb_2,\dots a_nb_m,a_nb_m^{-1},\dots,a_nb_2^{-1},a_nb_1^{-1},\\&a_n^{-1}b_1,a_n^{-1}b_2,\dots,a_n^{-1}b_m,a_n^{-1}b_m^{-1},\dots,a_n^{-1}b_2^{-1},a_n^{-1}b_2^{-1},\dots\\\dots,&a_1^{-1}b_1,a_1^{-1}b_2,\dots,a_1^{-1}b_m,a_1^{-1}b_m^{-1},\dots,a_1^{-1}b_2^{-1},a_1^{-1}b_1^{-1}).\end{aligned}$$

If $i=k(2m)+r$ is such that $0\le k\le n-1, 1\le r\le 2m$, then

$$\rho^*(f_i)=\begin{cases}(e_{k+1},e_r) & 1\le r\le m\\(e_{k+1},-e_{2m+1-r}) & m+1\le r\le 2m\end{cases}$$

$$\begin{aligned}
\Rightarrow \rho^*(q_{\mathrm{SO}(V_1\otimes V_2,\Omega_{4nm})}) &= \sum_{i=1}^{2nm} \rho^*(f_i^2) \\
&= \sum_{k=0}^{n-1}\left[\sum_{r=1}^{m} \rho^*(f_{k(2n)+r}^2) + \sum_{r=m+1}^{2m} \rho^*(f_{k(2n)+r}^2)\right] \\
&= \sum_{k=0}^{n-1}\left[\sum_{r=1}^{m} (e_{k+1}, e_r)^2 + \sum_{r=m+1}^{2m} (e_{k+1}, -e_{2n+1-r})^2\right] \\
&= \sum_{k=1}^{n}\left[\sum_{r=1}^{m} (e_k^2, e_r^2) + \sum_{r=1}^{m} (e_k^2, e_r^2)\right] \\
&= 2\sum_{k=1}^{n}\sum_{r=1}^{m} (e_k^2, e_r^2) \\
&= 2\left(m\sum_{k=1}^{n} e_k^2, n\sum_{r=1}^{m} e_r^2\right) \\
&= ((2m)q_{\mathrm{SO}(V_1,\Omega_{2n})}, (2n)q_{\mathrm{SO}(V_2,\Omega_{2m})}).
\end{aligned}$$

Applying the map to the Killing form gives

$$\rho^*(\mathcal{K}_{\mathrm{SO}(V_1\otimes V_2,\Omega_{4nm})}) = \left(\frac{4nm^2-2m}{n-1}\mathcal{K}_{\mathrm{SO}(V_1,\Omega_{2n})}, \frac{4n^2m-2n}{m-1}\mathcal{K}_{\mathrm{SO}(V_2,\Omega_{2n})}\right).$$

2.2.3 The Odd SO × SO-case

We now consider the orthogonal case when both vectors spaces have odd dimension. Let $\dim(V_1) = 2n+1$, $\dim(V_2) = 2m+1$.

$$\rho : \mathrm{SO}(V_1, \Omega_{2n+1}) \times \mathrm{SO}(V_2, \Omega_{2m+1}) \to \mathrm{SO}(V_1 \otimes V_2, \Omega_{(2n+1)(2m+1)}).$$

Again let $T_1 \subseteq \mathrm{SO}(V_1, \Omega_{2n+1})$, $T_2 \subseteq \mathrm{SO}(V_2, \Omega_{2m+1})$ be the maximal tori as per example 1.4.9. The maximal torus in $\mathrm{SO}(V_1 \otimes V_2, \Omega_{(2n+1)(2m+1)})$ is

$$T_{(2n+1)(2m+1)} = \{\mathrm{diag}(t_1, t_2, \ldots, t_{2nm+n+m}, 1, t_{2nm+n+m}^{-1}, \ldots, t_2^{-1}, t_1^{-1}) \mid t_i \in \mathbb{F}^\times\}.$$

Let $A = \mathrm{diag}(a_1, \ldots, a_n, 1, a_n^{-1}, \ldots, a_1^{-1}) \in T_1$, $B = \mathrm{diag}(b_1, \ldots, b_m, 1, b_m^{-1}, \ldots, b_1^{-1}) \in T_2$, and let $\rho(A, B) := \mathrm{diag}(c_1, \ldots, c_{(2n+1)(2m+1)})$.

If $i = k(2m+1) + r$ with $0 \le k \le 2n, 1 \le r \le 2m+1$ then

$$c_i = \begin{cases} & 0 \le k \le n-1 \\ a_{k+1}b_r & \quad 1 \le r \le m \\ a_{k+1} & \quad r = m+1 \\ a_{k+1}b^{-1}_{2m+2-r} & \quad m+2 \le r \le 2m+1 \\ & k = n \\ b_r & \quad 1 \le r \le m \\ 1 & \quad r = m+1 \\ b^{-1}_{2m+2-r} & \quad m+2 \le r \le 2m+1 \\ & n+1 \le k \le 2n \\ a^{-1}_{2n+1-k}b_r & \quad 1 \le r \le m \\ a^{-1}_{2n+1-k} & \quad r = m+1 \\ a^{-1}_{2n+1-k}b^{-2}_{2m+2-r} & \quad m+2 \le r \le 2m+1 \end{cases}$$

and so for $1 \le i \le 2nm + n + m$,

$$\rho^*(f_i) = \begin{cases} & 0 \le k \le n-1 \\ (e_{k+1}, e_r) & \quad 1 \le r \le m \\ (e_{k+1}, 0) & \quad r = m+1 \\ (e_{k+1}, -e_{2m+2-r}) & \quad m+2 \le r \le 2m+1 \\ & k = n \\ (0, e_r) & \quad 1 \le r \le m. \end{cases}$$

From example 1.4.9

$$\begin{aligned} &\Phi(\mathrm{SO}(V_1 \otimes V_2, \Omega_{(2n+1)(2m+1)})) \\ &= \{\pm f_i \pm f_j, \pm f_k \mid 1 \le i < j \le 2nm+n+m, 1 \le k \le 2nm+n+m\} \end{aligned}$$

and

$$\mathcal{K}_{\mathrm{SO}(V_1 \otimes V_2, \Omega_{(2n+1)(2m+1)})} = 4(2nm+n+m-2) \sum_{i=1}^{2nm+n+m} f_i^2,$$

$$q_{\mathrm{SO}(V_1 \otimes V_2, \Omega_{(2n+1)(2m+1)})}) = \sum_{i=1}^{2nm+n+m} f_i^2.$$

This implies

$$
\begin{aligned}
\rho^*(q_{\mathrm{SO}(V_1\otimes V_2,\Omega_{(2n+1)(2m+1)})}) &= \sum_{i=1}^{2nm+n+m} \rho^*(f_i)^2 \\
&= \left(\sum_{k=0}^{n-1}\left[\sum_{r=1}^{m}\rho^*(f^2_{k(2m+1)+r}) + \rho^*(f^2_{k(2m+1)+(m+1)}) + \sum_{r=m+2}^{2m+1}\rho^*(f^2_{k(2m+1)+r})\right] + \sum_{r=1}^{m}\rho^*(f^2_{n(2m+1)+r})\right) \\
&= \left(\sum_{k=0}^{n-1}\left[\sum_{r=1}^{m}(e_{k+1},e_r)^2 + (e_{k+1},0)^2 + \sum_{r=m+2}^{2m+1}(e_{k+1},-e_{2m+2-r})^2\right] + \sum_{r=1}^{m}(0,e_r)^2\right) \\
&= \left(\sum_{k=1}^{n}\left[(e_k^2,0) + 2\sum_{r=1}^{m}(e_k^2,e_r^2)\right] + \sum_{r=1}^{m}(0,e_r^2)\right) \\
&= \left(\sum_{k=1}^{n}\left[(e_k^2,0) + 2\left(me_k^2,\sum_{r=1}^{m}e_r^2\right)\right] + \left(0,\sum_{r=1}^{m}e_r^2\right)\right) \\
&= \left(\left[\sum_{k=1}^{n}\left((2m+1)e_k^2, 2\sum_{r=1}^{m}e_r^2\right)\right] + \left(0,\sum_{r=1}^{m}e_r^2\right)\right) \\
&= \left(\left((2m+1)\sum_{k=1}^{n}e_k^2, 2n\sum_{r=1}^{m}e_r^2\right) + \left(0,\sum_{r=1}^{m}e_r^2\right)\right) \\
&= \left((2m+1)\sum_{k=1}^{n}e_k^2, (2n+1)\sum_{r=1}^{m}e_r^2\right) \\
&= \big((2m+1)q_{\mathrm{SO}(V_1,\Omega_{2n+1})}, (2n+1)q_{\mathrm{SO}(V_2,\Omega_{2m+1})}\big)
\end{aligned}
$$

and for the Killing form

$$
\begin{aligned}
\rho^*(\mathcal{K}_{\mathrm{SO}(V_1\otimes V_2,\Omega_{(2n+1)(2m+1)})}) = \\
&\Big(\frac{4nm^2+4nm+2m^2-3m+n-2}{n-2}\mathcal{K}_{\mathrm{SO}(V_1,\Omega_{2n+1})}, \\
&\frac{4n^2m+4nm+2n^2-3n+m-2}{m-2}\mathcal{K}_{\mathrm{SO}(V_2,\Omega_{2m+1})}\Big).
\end{aligned}
$$

2.2.4 The Mixed $\mathrm{SO}\times\mathrm{SO}$-case

The remaining combination of orthogonal groups is when one vector space has even dimension and the other is has odd dimension.
Let $\dim(V_1) = 2n$ and $\dim(V_2) = 2m+1$. We consider

$$\rho\colon \mathrm{SO}(V_1,\Omega_{2n}) \times \mathrm{SO}(V_2,\Omega_{2m+1}) \to \mathrm{SO}(V_1\otimes V_2,\Omega_{2n(2m+1)}).$$

The maximal tori $T_1 \subseteq \mathrm{SO}(V_1, \Omega_{2n})$, $T_2 \subseteq \mathrm{SO}(V_2, \Omega_{2m+1})$ are as in the previous examples. Since $V_1 \otimes V_2$ is even dimensional, the larger maximal torus is

$$T_{2n(2m+1)} = \{\mathrm{diag}(t_1, \dots, t_{n(2m+1)}, t^{-1}_{n(2m+1)}, \dots, t_1^{-1}) \mid t_i \in \mathbb{F}^\times\}$$

$$\text{and} \quad T^*_{2n(2m+1)} = \langle \chi_i \mid 1 \le i \le n(2m+1) \rangle \cong \mathbb{Z}^{n(2m+1)}$$
$$\chi_i \leftrightarrow f_i.$$

As always, it is sufficient to understand $\rho^*(f_i)$. Let $A = \mathrm{diag}(a_1, \dots, a_n, a_n^{-1}, \dots, a_1) \in T_1$ and $B = \mathrm{diag}(b_1, \dots, b_m, 1, b_m^{-1}, \dots, b_1) \in T_2$. If $\rho(A, B) = A \otimes B = \mathrm{diag}(c_1, \dots, c_{2n(2m+1)})$, then for $i = k(2m+1) + r$ where $0 \le k \le 2n-1$, $1 \le r \le 2m+1$

$$c_i = \begin{cases} & \text{if } 0 \le k \le n-1 \\ a_{k+1} b_r & \quad 1 \le r \le m \\ a_{k+1} & \quad r = m+1 \\ a_{k+1} b^{-1}_{2m+2-r} & \quad m+2 \le r \le 2m+1 \\ & \text{if } n \le k \le 2n-1 \\ a^{-1}_{2n-k} b_r & \quad 1 \le r \le m \\ a^{-1}_{2n-k} & \quad r = m+1 \\ a^{-1}_{2n-k} b^{-1}_{2m+2-r} & \quad m+2 \le r \le 2m+1. \end{cases}$$

Therefore for $1 \le i \le n(2m+1)$

$$\rho^*(f_i) = \begin{cases} & \text{if } 0 \le k \le n-1 \\ (e_{k+1}, e_r) & \quad 1 \le r \le m \\ (e_{k+1}, 0) & \quad r = m+1 \\ (e_{k+1}, -e_{2m+2-r}) & \quad m+2 \le r \le 2m+1. \end{cases}$$

We know by example 1.4.8 that $\mathrm{SO}(V_1 \otimes V_2, \Omega_{2n(2m+1)})$ has Killing form and normalized Killing form

$$\mathcal{K}_{\mathrm{SO}(V_1 \otimes V_2, \Omega_{2n(2m+1)})} = 4(n(2m+1) - 1) \sum_{i=1}^{n(2m+1)} f_i^2,$$

$$q_{\mathrm{SO}(V_1 \otimes V_2, \Omega_{2n(2m+1)})} = \sum_{i=1}^{n(2m+1)} f_i^2.$$

So we compute

$$\begin{aligned}
\rho^*(q_{\mathrm{SO}(V_1\otimes V_2,\Omega_{2n(2m+1)})}) = & \sum_{i=1}^{n(2m+1)} \rho^*(f_i)^2 \\
=&\sum_{k=0}^{n-1}\left(\sum_{r=1}^{m}\rho^*(f_{k(2m+1)+r})^2+\rho^*(f_{k(2m+1)+(m+1)})^2+\sum_{r=m+2}^{2m+1}\rho^*(f_{k(2m+1)+r})^2\right) \\
=&\sum_{k=0}^{n-1}\left(\sum_{r=1}^{m}(e_{k+1},e_r)^2+(e_{k+1},0)^2+\sum_{r=m+2}^{2m+1}(e_{k+1},-e_{2m+2-r})^2\right) \\
=&\sum_{k=1}^{n}\left((e_k^2,0)+2\sum_{r=1}^{m}(e_k^2,e_r^2)\right) \\
=&\sum_{k=1}^{n}\left((2m+1)e_k^2,2\sum_{r=1}^{m}e_r^2\right) \\
=&\left((2m+1)\sum_{k=1}^{n}e_k^2,2n\sum_{r=1}^{m}e_r^2\right) \\
=&\left((2m+1)q_{\mathrm{SO}(V_1,\Omega_{2n})},(2n)q_{\mathrm{SO}(V_2,\Omega_{2m+1})}\right).
\end{aligned}$$

As for the Killing forms, we obtain

$$\begin{aligned}
\rho^*(\mathcal{K}_{\mathrm{SO}(V_1\otimes V_2,\Omega_{2n(2m+1)})}) = & \\
& (\frac{4nm^2+4nm-2m-n-1}{n-1}\mathcal{K}_{\mathrm{SO}(V_1,\Omega_{2n})}, \\
& \frac{4n^2m+2n^2-2n}{m-2}\mathcal{K}_{\mathrm{SO}(V_2,\Omega_{2m+1})}).
\end{aligned}$$

2.2.5 The $\mathrm{Sp}\times\mathrm{Sp}$-case

Let $\dim(V_1)=2n$ and $\dim(V_2)=2m$. Consider the product of symplectic groups

$$\rho:\mathrm{Sp}(V_1,\Psi_{2n})\times\mathrm{Sp}(V_2,\Psi_{2m})\to\mathrm{SO}(V_1\otimes V_2,\Psi_{2n}\otimes\Psi_{2m}).$$

In this case we cannot immediately evoke previous results about the root structure of $\mathrm{SO}(V_1\otimes V_2,\Psi_{2n}\otimes\Psi_{2m})$ since this is an orthogonal group with respect to a different orthogonal form, $\Psi_{2n}\otimes\Psi_{2m}\neq\Omega_{4nm}$. We take a brief aside to derive the Lie algebra and root structure.

$$\mathrm{Lie}(\mathrm{SO}(V_1\otimes V_2,\Psi_{2n}\otimes\Psi_{2m}))=\{A\in\mathrm{End}(V_1\otimes V_2)\mid A^T(\Psi_{2n}\otimes\Psi_{2m})=-(\Psi_{2n}\otimes\Psi_{2m})A\}.$$

To analyze this condition we define the function $h(i)$,

$$h(i) = \begin{cases} (-1)^{\lceil \frac{i}{m} \rceil + 1} & 1 \leq i \leq 2nm \\ (-1)^{\lceil \frac{i}{m} \rceil} & 2nm+1 \leq i \leq 4nm \end{cases}$$

and note that

$$\Psi_{2n} \otimes \Psi_{2m} = \begin{bmatrix} & & & h(4nm) \\ & & \iddots & \\ & h(2) & & \\ h(1) & & & \end{bmatrix}$$

If we set $A = [a_{ij}]_{i,j=1}^{4nm}$, then

$$-(\Psi_{2n} \otimes \Psi_{2m})A^T(\Psi_{2n} \otimes \Psi_{2m}) = [-h(i)h(j)a_{4nm+1-j,4nm+1-i}]_{i,j=1}^{4nm}$$

meaning that $A \in \mathrm{Lie}(\mathrm{SO}(V_1 \otimes V_2, \Psi_{2n} \otimes \Psi_{2m}))$ is equivalent to

$$a_{ij} = -h(i)h(j)a_{4nm+1-j,4nm+1-i} \quad \forall 1 \leq i,j \leq 4nm$$

and so $\mathrm{Lie}(\mathrm{SO}(V_1 \otimes V_2, \Psi_{2n} \otimes \Psi_{2m}))$ is spanned by the set

$$\{E_{ij} - h(i)h(j)E_{4nm+1-j,4nm+1-i} \mid 1 \leq i,j \leq 4nm\}.$$

The maximal torus in $\mathrm{SO}(V_1 \otimes V_2, \Psi_{2n} \otimes \Psi_{2m})$ does not differ from the usual.

$$T_{4nm} = \{\mathrm{diag}(t_1, \ldots, t_{2nm}, t_{2nm}^{-1}, \ldots, t_1^{-1}) \mid t_i \in \mathbb{F}^\times\}$$

$$\text{and} \quad T_{4nm}^* = \langle \chi_i \mid 1 \leq i \leq 2nm \rangle.$$

We then obtain a weight space decomposition almost identical to our previous decomposition with respect to Ω_{4nm}.

Weight space	Weight
Diagonal Matricies	$1 \leftrightarrow 0$
$\mathbb{F}\langle E_{ij} - h(i)h(j)E_{4nm+1-j,4nm+1-i}\rangle$, $1 \leq i \neq j \leq 2nm$	$\chi_i\chi_j^{-1} \leftrightarrow e_i - e_j$
$\mathbb{F}\langle E_{i,4nm+1-j} - h(i)h(j)E_{j,4nm+1-i}\rangle$, $1 \leq i,j \leq 2nm$	$\chi_i\chi_j \leftrightarrow e_i + e_j$
$\mathbb{F}\langle E_{4nm+1-i,j} - h(i)h(j)E_{4nm+1-j,i}\rangle$, $1 \leq i,j \leq 2nm$	$\chi_i^{-1}\chi_j^{-1} \leftrightarrow -e_i - e_j$

This produces a root system of type D_{2nm}

$$\Phi(\mathrm{SO}(V_1 \otimes V_2, \Psi_{2n} \otimes \Psi_{2m})) = \{\pm e_i \pm e_j \mid 1 \leq i < j \leq 2nm\}.$$

It was expected that the root systems of SO relative to the different orthogonal forms Ω_{4nm} and $\Psi_{2n} \otimes \Psi_{2m}$ would be isomorphic, but in fact they are directly equal in T^*. In addition, the maximal tori in $\mathrm{SO}(V_i, \Omega)$ and $\mathrm{Sp}(V_i, \Psi)$ are also equal. Therefore we may use our previous calculations about the Killing form, normalized Killing form, images of the roots under ρ^*, etc. from $\mathrm{SO}(V_1 \otimes V_2, \Omega_{4nm})$.

$$\mathcal{K}_{\mathrm{SO}(V_1\otimes V_2,\Psi_{2n}\otimes\Psi_{2m})} = 4(2nm-1)\sum_{i=1}^{2nm} f_i^2, \quad q_{\mathrm{SO}(V_1\otimes V_2,\Psi_{2n}\otimes\Psi_{2m})} = \sum_{i=1}^{2nm} f_i^2$$

This implies

$$\begin{aligned}\rho^*(q_{\mathrm{SO}(V_1\otimes V_2,\Psi_{2n}\otimes\Psi_{2m})}) &= \left(2m\sum_{k=1}^{n} e_k^2, 2n\sum_{r=1}^{m} e_r^2\right)\\ &= \left(2mq_{\mathrm{Sp}(V_1,\Psi_{2n})}, 2nq_{\mathrm{Sp}(V_2,\Psi_{2m})}\right).\end{aligned}$$

On the level of Killing forms the result does differ slightly from the case involving only orthogonal groups. We have

$$\rho^*(\mathcal{K}_{\mathrm{SO}(V_1\otimes V_2,\Psi_{2n}\otimes\Psi_{2m})}) = \left(\frac{4nm^2+2m}{n+1}\mathcal{K}_{\mathrm{Sp}(V_1,\Psi_{2n})}, \frac{4n^2m+2n}{m+1}\mathcal{K}_{\mathrm{Sp}(V_2,\Psi_{2m})}\right).$$

2.2.6 The Even $\mathrm{Sp} \times \mathrm{SO}$-case

Let $\dim(V_1) = 2n$, $\dim(V_2) = 2m$. Consider

$$\rho : \mathrm{Sp}(V_1, \Psi_{2n}) \times \mathrm{SO}(V_2, \Omega_{2m}) \to \mathrm{Sp}(V_1 \otimes V_2, \Psi_{4nm}).$$

Since both the torus

$$T_{4nm} = \{\mathrm{diag}(t_1, \ldots, t_{2nm}, t_{2nm}^{-1}, \ldots, t_1^{-1}) \mid t_i \in \mathbb{F}^\times\}$$

and the normalized Killing form

$$q_{\mathrm{Sp}(V_1\otimes V_2,\Psi_{4nm})} = \sum_{i=1}^{2nm} f_i^2$$

are the same in $\mathrm{Sp}(V_1 \otimes V_2, \Psi_{4nm})$ as they are for $\mathrm{SO}(V_1 \otimes V_2, \Omega_{4nm})$ (as is also the case for $\mathrm{Sp}(V_1, \Psi_{2n})$ and $\mathrm{SO}(V_1, \Omega_{2n})$) we may again directly apply our previous calculations. Therefore

$$\begin{aligned}\rho^*(q_{\mathrm{Sp}(V_1\otimes V_2,\Psi_{4nm})}) &= \left(2m\sum_{k=1}^{n} e_k^2, 2n\sum_{r=1}^{m} e_r^2\right)\\ &= \left(2mq_{\mathrm{Sp}(V_1,\Psi_{2n})}, 2nq_{\mathrm{SO}(V_2,\Omega_{2m})}\right).\end{aligned}$$

The Killing form is then given by

$$\rho^*(\mathcal{K}_{\mathrm{Sp}(V_1\otimes V_2,\Psi_{4nm})}) = \left(\frac{4nm^2+2m}{n+1}\mathcal{K}_{\mathrm{Sp}(V_1,\Psi_{2n})}, \frac{4n^2m+2n}{m-1}\mathcal{K}_{\mathrm{SO}(V_2,\Omega_{2m})}\right).$$

2.2.7 The Odd $\mathrm{Sp} \times \mathrm{SO}$-case

Let $\dim(V_1) = 2n$, $\dim(V_2) = 2m + 1$. Consider

$$\rho : \mathrm{Sp}(V_1, \Psi_{2n}) \times \mathrm{SO}(V_2, \Omega_{2m+1}) \to \mathrm{Sp}(V_1 \otimes V_2, \Psi_{2n(2m+1)}).$$

The argument is analagous to the one used above. Since the tori and normalized Killing forms of $\mathrm{Sp}(V_1, \Psi_{2n})$ and $\mathrm{Sp}(V_1 \otimes V_2, \Psi_{2n(2m+1)})$ are the same as their orthogonal counterparts, we know that

$$\rho^*(q_{\mathrm{Sp}(V_1 \otimes V_2, \Psi_{2n(2m+1)})}) = \big((2m+1) q_{\mathrm{Sp}(V_1, \Psi_{2n})}, (2n) q_{\mathrm{SO}(V_2, \Omega_{2m+1})}\big)$$

and so

$$\begin{aligned}\rho^*(\mathcal{K}_{\mathrm{Sp}(V_1 \otimes V_2, \Psi_{2n(2m+1)})}) =& \Big(\frac{4nm^2 + 4nm + 2m + n + 1}{n+1} \mathcal{K}_{\mathrm{Sp}(V_1, \Psi_{2n})}, \\ & \frac{4n^2 m + 2n^2 + 2n}{m - 2} \mathcal{K}_{\mathrm{SO}(V_2, \Omega_{2m+1})}\Big).\end{aligned}$$

2.2.8 The General Case

All our computations can be summarized as follows.

Let $\dim(V_1) = d_1$ and $\dim(V_2) = d_2$. Suppose G_1, G_2, and G_3 are the following groups (choosing Sp only when the appropriate d_i is even).

G_1	G_2	G_3
SL	SL	SL
SO	SO	SO
Sp	SO	Sp
Sp	Sp	SO

Then the Kronecker product map $\rho\colon G_1(V_1) \times G_2(V_2) \to G_3(V_1 \otimes V_2)$ induces a map between invariant quadratic forms such that

$$\begin{aligned}\rho^*\colon \mathrm{S}^2(T_3^*)^{W_3} &\to \mathrm{S}^2(T_1^*)^{W_1} \oplus \mathrm{S}^2(T_2^*)^{W_2} \\ q_3 &\mapsto (d_2 q_1, d_1 q_2)\end{aligned}$$

where $T_i \subseteq G_i$ are the maximal tori, and W_i are the respective Weyl groups.

This result can be generalized to an arbitrary product of groups.

Theorem 2.2.1. *Let $V_1, \ldots, V_n$ be vector spaces such that $\dim(V_i) = d_i$. Consider linear algebraic groups $G_1, \ldots, G_n, H$ in one of the following configurations*

- $G_1, \ldots, G_n, H = \mathrm{SL}$
- $G_1, \ldots, G_{2m} = \mathrm{Sp}$
  $G_{2m+1}, \ldots, G_n = \mathrm{SO}$
  $H = \mathrm{SO}$
- $G_1, \ldots, G_{2m+1} = \mathrm{Sp}$
  $G_{2m+2}, \ldots, G_n = \mathrm{SO}$
  $H = \mathrm{Sp}$

where $2m$ or $2m+1 \leq n$, and $G_i = \mathrm{Sp}$ only when d_i is even.

Consider the Kronecker product map

$$\rho|_n \colon G_1(V_1) \times \ldots \times G_n(V_n) \to H(V_1 \otimes \ldots \otimes V_n)$$

and let $q_i \in \mathrm{S}^2(T_i^*)^{W_i}$ and $q_H \in \mathrm{S}^2(T_H^*)^{W_H}$ be the normalized Killing forms.

Then

$$\rho|_n^*(q_H) = \Big((d_2 \ldots d_n)q_1, (d_1 d_3 \ldots d_n)q_2, \ldots, (d_1 \ldots \hat{d}_i \ldots d_n)q_i, \ldots, (d_1 \ldots d_{n-1})q_n\Big)$$

where $\hat{d}_i$ represents ommision.

Proof: We proceed inductively on the number of factors, with the base case of two factors already verified. Note that our map $\rho|_n$ factors as

$$\rho|_n \colon G_1(V_1) \times \ldots \times G_n(V_n) \overset{\rho|_{n-1}\times \mathrm{Id}}{\to} H'(V_1 \otimes \ldots \otimes V_{n-1}) \times G_n(V_n) \overset{\rho'}{\to} H(V_1 \otimes \ldots \otimes V_n).$$

Here H' is the appropriate group from among $\mathrm{SL}, \mathrm{SO}, \mathrm{Sp}$ depending on the groups $G_1, \ldots, G_{n-1}$. The map ρ' is a tensor product map from two groups to one, and is therefore included in our previous examples, meaning

$$\rho'^*(q_H) = (d_n q_{H'}, \dim(V_1 \otimes \ldots \otimes V_{n-1})q_n) = (d_n q_{H'}, (d_1 \ldots d_{n-1})q_n).$$

Then by assumption, since the map

$$\rho|_{n-1} \colon G_1(V_1) \times \ldots \times G_{n-1}(V_{n-1}) \to H'(V_1 \otimes \ldots \otimes V_{n-1})$$

involves one less factor, we have

$$\rho|_{n-1}^*(q_{H'}) = ((d_2 \ldots d_{n-1})q_1, \ldots, (d_1 \ldots \hat{d}_i \ldots d_{n-1})q_i, \ldots, (d_1 \ldots d_{n-2})q_{n-1}).$$

Combining these two equalities we obtain

$$\begin{aligned}
\rho|_n^*(q_H) &= \left((\rho|_{n-1}^* \times \mathrm{Id}) \circ \rho'^*\right)(q_H) \\
&= (\rho|_{n-1}^* \times \mathrm{Id})(d_n q_{H'}, (d_1 \ldots d_{n-1})q_n) \\
&= (d_n \rho|_{n-1}^*(q_{H'}), (d_1 \ldots d_{n-1})q_n) \\
&= (d_n\big((d_2 \ldots d_{n-1})q_1, \ldots, (d_1 \ldots d_{n-2})q_{n-1}\big), (d_1 \ldots d_{n-1})q_n) \\
&= \Big((d_2 \ldots d_n)q_1, \ldots, (d_1 \ldots \hat{d}_i \ldots d_n)q_i, \ldots, (d_1 \ldots d_{n-1})q_n\Big)
\end{aligned}$$

and the result follows. ∎

2.3 Quadratic Invariants of Quotient Groups

Let G be a split semisimple linear algebraic group with maximal torus T. If H is a subgroup of T which is normal in G, then we can conisder the following short exact sequences.

$$1 \longrightarrow H \hookrightarrow G \twoheadrightarrow {}^{G}\!/_{H} \longrightarrow 1$$

$$1 \longrightarrow H \xhookrightarrow{\text{inc}} T \twoheadrightarrow {}^{T}\!/_{H} \longrightarrow 1$$

When we consider the character groups of T and ${}^{T}\!/_{H}$ we again get an exact sequence

$$0 \longrightarrow \left({}^{T}\!/_{H}\right)^* \hookrightarrow T^* \xrightarrow{\text{inc}^*} H^* \longrightarrow 0$$

meaning that the character group $\left({}^{T}\!/_{H}\right)^*$ is simply a subset of T^*. Namely it is the kernel of inc^*.

There are two Weyl groups at work here. Let W be the Weyl group of G with respect to T, acting on T^*, and let W' be the analogous Weyl group for ${}^{G}\!/_{H}$, which acts on $\left({}^{T}\!/_{H}\right)^*$. Since $\left({}^{T}\!/_{H}\right)^* \subseteq T^*$, it is also acted upon by W, however these actions are the same since W is isomorphic to W'.

Proof: We will use the definition of the Weyl group relative a torus which refers to normalizers and centralizers. These are

$$W = {}^{\mathrm{N}_G(T)}\Big/_{\mathrm{C}_G(T)}\,,$$

$$W' = {}^{\mathrm{N}_{G/H}({}^{T}\!/_{H})}\Big/_{\mathrm{C}_{G/H}({}^{T}\!/_{H})}\,.$$

At this point we use that the normalizer (resp. centralizer) in the quotient is the quotient of the normalizer (resp. centralizer).

$$\mathrm{N}_{G/H}({}^{T}\!/_{H}) = {}^{\mathrm{N}_G(T)}\!/_{H}\,,$$

$$\mathrm{C}_{G/H}({}^{T}\!/_{H}) = {}^{\mathrm{C}_G(T)}\!/_{H}\,.$$

Finally this means that

$$W' = {}^{\mathrm{N}_{G/H}({}^{T}\!/_{H})}\Big/_{\mathrm{C}_{G/H}({}^{T}\!/_{H})} = \left({}^{\mathrm{N}_G(T)}\!/_{H}\right)\Big/\left({}^{\mathrm{C}_G(T)}\!/_{H}\right)$$
$$\cong {}^{\mathrm{N}_G(T)}\Big/_{\mathrm{C}_G(T)} = W.$$

∎

Hence the invariants $\mathrm{S}^2\left(\left({}^{T}/_{H}\right)^*\right)^W$ are described by a simple intersection

$$\mathrm{S}^2\left(\left({}^{T}/_{H}\right)^*\right)^W = \mathrm{S}^2\left(\left({}^{T}/_{H}\right)^*\right) \cap \mathrm{S}^2(T^*)^W.$$

In the case that G is a split semisimple group, we fully understand the structure of $\mathrm{S}^2(T^*)^W$. It is generated by the normalized Killing forms of the simple components, and so the subring of invariant quadratic forms $\mathrm{S}^2\left(\left({}^{T}/_{H}\right)^*\right)^W$ will be generated by certain multiples of those generators. We now compute these invariants in examples where we consider quotients of our typical linear algebraic groups by central subgroups.

2.3.1 The SL-case

Let $\dim(V) = n$ and consider the group $\mathrm{SL}(V)$ with maximal torus T as in 1.4.7. This contains a subgroup isomorphic to the n^{th} roots of unity,

$$\begin{aligned} \mu_n &\hookrightarrow \mathrm{SL}(V) \\ \zeta &\mapsto \mathrm{diag}(\zeta, \ldots, \zeta) \end{aligned}$$

and so we have the short exact sequences

$$1 \longrightarrow \mu_n \overset{\varphi}{\hookrightarrow} T \longrightarrow\!\!\!\!\!\rightarrow {}^{T}/_{\mu_n} \longrightarrow 1$$

$$0 \longrightarrow \left({}^{T}/_{\mu_n}\right)^* \hookrightarrow T^* \overset{\varphi^*}{\longrightarrow\!\!\!\!\!\rightarrow} \mu_n^* \longrightarrow 0$$

where $T^* \cong \mathbb{Z}^{n-1}$.

Remark 2.3.1. Since $\mathbb{F}$ is algebraically closed, $\mu_n \subset \mathbb{F}$ for all n. The character group consists of the exponential maps, i.e.

$$\mu_n^* = \left\{ \sigma^d \colon \underset{\zeta \mapsto \zeta^d}{\mu \to \mathbb{F}} \mid 0 \le d \le n-1 \right\} \cong {}^{\mathbb{Z}}/_{n\mathbb{Z}}\,.$$

Now to understand the map φ^*, we describe how it acts on the generators of T^*. Let χ_i be such a generator, and let $\zeta \in \mu_n$.

$$\varphi^*(\chi_i^k)(\zeta) = \chi_i^k(\varphi(\zeta)) = \chi_i^k(\mathrm{diag}(\zeta, \ldots, \zeta) = \zeta^k.$$

Therefore $\varphi^*(\chi_i^k) = \sigma^k$, which means that

$$\varphi^*(\chi_1^{k_1} \dots \chi_{n-1}^{k_{n-1}})(\zeta) = \zeta^{k_1+\dots+k_{n-1}}$$

and so the map φ^* expressed additively is

$$\begin{aligned} \varphi^* \colon \mathbb{Z}^{n-1} &\to \mathbb{Z} / n\mathbb{Z} \\ (x_1, x_2, \dots, x_{n-1}) &\mapsto \sum_{i=1}^{n-1} x_i \pmod{n}. \end{aligned}$$

Then since the character group of the quotient torus is the kernel of φ^*, it consists of the points with coordinateds summing to a multiple of n.

$$\left(T / \mu_n\right)^* = \left\{ (x_1, \dots, x_{n-1}) \in T^* \mid \sum_{i=1}^{n-1} x_i \in n\mathbb{Z} \right\}.$$

Lemma 2.3.2. *Let $d \in \mathbb{Z}$ and consider the subset*

$$P := \left\{ \sum_{i=1}^{n} c_i e_i \in \mathbb{Z}^n \mid \sum_{i=1}^{n} c_i \in d\mathbb{Z} \right\}.$$

Then

$$\mathrm{S}^2(P) = \left\{ \sum_{\substack{i,j=1 \\ i \le j}}^{n} c_{ij} e_i e_j \in \mathrm{S}^2(\mathbb{Z}^n) \mid \text{(i), (ii)} \right\}$$

where

(i) $\sum_{\substack{i,j=1 \\ i \le j}}^{n} c_{ij} \in d^2\mathbb{Z}$,

(ii) $c_{i,n} \equiv -c_{1,i} - \dots - c_{i-1,i} - 2c_{ii} - c_{i,i+1} - \dots - c_{i,n-1} \pmod{d}$ *for* $1 \le i \le n-1$.

Proof: P is generated by the elements $\{e_i - e_n, de_n \mid 1 \le i \le n-1\}$, and so $\mathrm{S}^2(P)$ is generated by the tensor products of pairs of these generators.

$$\mathrm{S}^2(P) = \langle e_ie_j - e_ie_n - e_je_n + e_n^2, de_ie_n - de_n^2, d^2e_n \mid 1 \le i \le j \le n-1 \rangle.$$

Hence it is a subset of $\mathrm{S}^2(\mathbb{Z}^n)$ given by conditions (i) and (ii). ∎

Using the description of $\mathrm{S}^2\left(\left(T / \mu_n\right)^*\right)$ as in 2.3.2, namely

$$\mathrm{S}^2\left(\left(T / \mu_n\right)^*\right) = \left\{ \sum_{\substack{i,j=1 \\ i \le j}}^{n-1} c_{ij} e_i e_j \in \mathrm{S}^2(T^*) \mid \text{(i),(ii)} \right\}$$

(i) $\sum_{\substack{i,j=1 \\ i \le j}}^{n-1} c_{ij} \in n^2\mathbb{Z}$,

(ii) $c_{i,n-1} \equiv -c_{1,i} - \ldots - 2c_{ii} - \ldots - c_{i,n-2} \pmod{n}$ for $1 \le i \le n-2$,

we can determine the ring of invariants $\mathrm{S}^2\left(\left(T/\mu_n\right)^*\right)^W$.

The normalized Killing form for $\mathrm{SL}(V)$ is

$$q = \sum_{\substack{i,j=1 \\ i \le j}}^{n-1} e_i e_j.$$

Currently q has the property that

$$\sum_{\substack{i,j=1 \\ i \le j}}^{n-1} c_{ij} = \sum_{\substack{i,j=1 \\ i \le j}}^{n-1} 1 = \frac{n^2-n}{2}.$$

Now if we compute the lowest common multiple of $\frac{n^2-n}{2}$ and n^2 we get

$$\operatorname{lcm}\left\{\frac{n^2-n}{2}, n^2\right\} = \begin{cases} n^3 - n^2 & n \text{ even} \\ \frac{n^3-n^2}{2} & n \text{ odd} \end{cases}$$

and so the least multiple of q which satisfies (i) is $2nq$ when n is even, and nq when n is odd. In both these cases, all coefficients of the multiple of q are divisible by n and so condition (ii) is satisfied trivially. Therefore

$$\mathrm{S}^2\left(\left(T/\mu_n\right)^*\right)^W = \begin{cases} \mathbb{Z}\langle 2nq\rangle & n \text{ even} \\ \mathbb{Z}\langle nq\rangle & n \text{ odd.} \end{cases}$$

2.3.2 The SO or Sp-case

Let $\dim(V) = 2n$ and let G be either $\mathrm{SO}(V, \Omega)$ or $\mathrm{Sp}(V, \Psi)$. In both these cases we have a short exact sequence

$$1 \longrightarrow \mu_2 \hookrightarrow G \twoheadrightarrow G/\mu_2 \longrightarrow 1$$

where $\mu_2 = \{\pm 1\}$ is identified with the diagonal matricies $\{\pm I\}$. Furthermore, we are able to choose the same maximal torus T for both groups, and both groups have the same normalized Killing form. Therefore we may consider both groups simultaneously with the short exact sequence

$$1 \longrightarrow \mu_2 \hookrightarrow T \twoheadrightarrow T/\mu_2 \longrightarrow 1.$$

This has dual sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \left(T/\mu_2\right)^* & \hookrightarrow & T^* & \xrightarrow{\varphi^*} & \mu_2^* & \longrightarrow & 0 \\ & & & & \wr\Vert & & \wr\Vert & & \\ & & & & \mathbb{Z}^n & & \mathbb{Z}/2\mathbb{Z} & & \end{array}$$

where $\varphi^*(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i \pmod 2$, and so

$$\left(T/\mu_2\right)^* = \left\{ \sum_{i=1}^{n} c_i e_i \mid \sum_{i=1}^{n} c_i \in 2\mathbb{Z} \right\}.$$

Again by 2.3.2, this means that

$$\mathrm{S}^2\left(\left(T/\mu_2\right)^*\right) = \left\{ \sum_{\substack{i,j=1 \\ i \leq j}}^{n} c_{ij} e_i e_j \in \mathrm{S}^2(T^*) \mid \text{(i),(ii)} \right\}$$

(i) $\sum_{\substack{i,j=1 \\ i \leq j}}^{n} c_{ij} \in 4\mathbb{Z}$,

(ii) $c_{i,n} \equiv c_{1,i} + \ldots + 2c_{ii} + \ldots + c_{i,n-1} \pmod 2$ for $1 \leq i \leq n-1$.

The normalized Killing form of G is $q = \sum_{i=1}^{n} e_i^2$, which has $\sum_{\substack{i,j=1 \\ i \leq j}}^{n} c_{ij} = n$. The generator of the quotient's invariant quadratic forms then depends on the class of n (mod 4).

$$\mathrm{S}^2\left(\left(T/\mu_2\right)^*\right)^W = \begin{cases} \mathbb{Z}\langle q \rangle & n \equiv 0 \pmod 4 \\ \mathbb{Z}\langle 4q \rangle & n \equiv 1 \pmod 4 \\ \mathbb{Z}\langle 2q \rangle & n \equiv 2 \pmod 4 \\ \mathbb{Z}\langle 4q \rangle & n \equiv 3 \pmod 4. \end{cases}$$

When $n \equiv 0 \pmod 4$ condition (ii) is satisfied by q since for each i, only $c_{ii} = 1$ is non-zero and so the condition requires that $0 \equiv 2 \pmod 2$. When $n \equiv 1, 2, 3$ (mod 4), the generator has even coefficients and so (ii) is satisfied trivially.

2.3.3 The Product of SLs

Here we consider quotients by central subgroups of multiple copies of SL. Let $\dim(V_1) = n + 1$, $\dim(V_2) = m + 1$. Set $T_i \subseteq \mathrm{SL}(V_i)$ to be the maximal tori and consider the short exact sequence

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mu_k & \hookrightarrow & T_1 \times T_2 & \twoheadrightarrow & T_1 \times T_2 /\mu_k & \longrightarrow & 1 \\ & & \zeta & \longmapsto & (\zeta I_{n+1}, \zeta I_{m+1}) & & & & \end{array}$$

where $k| \gcd(n+1, m+1)$. We have the associated dual short exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \left(T_1 \times T_2 /\mu_k\right)^* & \hookrightarrow & T_1^* \times T_2^* & \twoheadrightarrow & \mu_k^* & \longrightarrow & 0 \\ & & & & \wr\| & & \wr\| & & \\ & & & & \mathbb{Z}^{n+m} & \twoheadrightarrow & \mathbb{Z}/k\mathbb{Z} & & \\ & & & & \sum_{i=1}^{n+m} c_i e_i & \longmapsto & \sum_{i=1}^{n+m} c_i \pmod{k} & & \end{array}$$

and so we have a description of $\left(T_1 \times T_2 /\mu_k\right)^*$,

$$\begin{aligned} \left(T_1 \times T_2 /\mu_k\right)^* &= \left\{ \sum_{i=1}^{n+m} c_i e_i \in \mathbb{Z}^{n+m} \mid \sum_{i=1}^{n+m} c_i \equiv 0 \pmod{k} \right\} \\ &= \langle e_i - e_{n+m}, k e_{n+m} \mid 1 \leq i \leq n+m-1 \rangle \end{aligned}$$

and then by 2.3.2

$$\mathrm{S}^2\left(\left(T_1 \times T_2 /\mu_k\right)^* \right) = \left\{ \sum_{\substack{i,j=1 \\ i \leq j}}^{n+m} c_{ij} e_i e_j \mid \text{(i),(ii)} \right\}$$

(i) $\sum_{\substack{i,j=1 \\ i \leq j}}^{n+m} c_{ij} \in k^2\mathbb{Z}$,

(ii) $c_{i,n+m} \equiv -c_{1,i} - \ldots - 2c_{ii} - \ldots - c_{i,n+m-1} \pmod{k}$ for $1 \leq i \leq n+m-1$.

Recall that the normalized Killing forms of $\mathrm{SL}(V_1)$ and $\mathrm{SL}(V_2)$ as elements of $\mathrm{S}^2(T_1^* \times T_2^*)^W$ are

$$q_1 = \sum_{\substack{i,j=1 \\ i \leq j}}^{n} e_i e_j \text{ and } q_2 = \sum_{\substack{i,j=n+1 \\ i \leq j}}^{n+m} e_i e_j$$

respectively. The invariants of the quotient are then elements $d_1 q_1 + d_2 q_2$ with $d_1, d_2 \in \mathbb{Z}$ such that (i) and (ii) are satisfied. Consider such a linear combination.

$$\sum_{\substack{i,j=1 \\ i \leq j}}^{n+m} c_{ij} = \frac{(n+1)n}{2} d_1 + \frac{(m+1)m}{2} d_2$$

so condition (i) is equivalent to $(n+1)nd_1 + (m+1)md_2 \equiv 0 \pmod{2k^2}$.

For condition (ii), first consider the case when $1 \le i \le n$. Then $c_{i,n+m} = 0$ and

$$-c_{1,i} - \ldots - 2c_{ii} - \ldots - c_{i,n+m-1} = -c_{1,i} - \ldots - 2c_{ii} - \ldots - c_{i,n} - 0 = -(n+1)d_1 \equiv 0 \pmod{k}$$

with the last equivalene holding because $k|\gcd(n+1, m+1)$ and so $k|n+1$.

For $n+1 \le i \le n+m$, $c_{i,n+m} = d_2$ and

$$\begin{aligned} -c_{1,i} - \ldots - 2c_{ii} - \ldots - c_{i,n+m-1} &= 0 - c_{n+1,i} - \ldots - 2c_{ii} - \ldots - c_{i,n+m-1} \\ &= -md_2 \equiv -(-1)d_2 \equiv d_2 \pmod{k} \end{aligned}$$

which shows that condition (ii) is trivially satisfied. Hence

$$\mathrm{S}^2\left(\left(T_1 \times T_2 \big/ \mu_k\right)^*\right)^W = \{d_1q_1 + d_2q_2 \mid (n+1)nd_1 + (m+1)md_2 \equiv 0 \pmod{2k^2}\}.$$

This condition is equivalent to the description of these invariants given in [BDZ, Proposition 6.3].

As a slight variation to example 2.3.3, we can consider the group μ_k embedded into $T_1 \times T_2$ such that it is a subgroup of the kernel of the tensor product map. That is

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mu_k & \hookrightarrow & T_1 \times T_2 & \twoheadrightarrow & T_1 \times T_2 \big/ \mu_k & \longrightarrow & 1 \\ & & \zeta & \longmapsto & (\zeta I_{n+1}, \zeta^{-1} I_{m+1}) & & & & \end{array}$$

with dual sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \left(T_1 \times T_2 \big/ \mu_k\right)^* & \hookrightarrow & T_1^* \times T_2^* & \twoheadrightarrow & \mu_k^* & \longrightarrow & 0 \\ & & & & \wr\Vert & & \wr\Vert & & \\ & & & & \mathbb{Z}^{n+m} & \twoheadrightarrow & \mathbb{Z} \big/ k\mathbb{Z} & & \\ & & & & \sum_{i=1}^{n+m} c_i e_i & \longmapsto & \sum_{i=1}^{n} c_i - \sum_{i=n+1}^{n+m} c_i \pmod{k}. & & \end{array}$$

This changes the generators of the quotient torus

$$\left(T_1 \times T_2 \big/ \mu_k\right)^* = \langle e_i + e_{n+m}, e_j - e_{n+m}, ke_{n+m} \mid 1 \le i \le n, n+1 \le j \le n+m-1 \rangle$$

and changes the conditions on quadratic elements

$$\mathrm{S}^2\left(\left(T_1 \times T_2 \big/ \mu_k\right)^*\right) = \left\{ \sum_{\substack{i,j=1 \\ i \le j}}^{n+m} c_{ij} e_i e_j \mid \text{(i),(ii)} \right\}$$

(i) $\sum_{\substack{i,j=1\\i\leq j}}^{n} c_{ij} - \sum_{i=1}^{n}\sum_{j=n+1}^{n+m} c_{ij} + \sum_{\substack{i,j=n+1\\i\leq j}}^{n+m} c_{ij} \in k^2\mathbb{Z}$,

(ii)

$$c_{i,n+m} \equiv c_{1,i} + \ldots + 2c_{ii} + \ldots + c_{in} - c_{i,n+1} - \ldots - c_{i,n+m-1} \pmod{k}$$

for $1 \leq i \leq n$, and

$$c_{i,n+m} \equiv c_{1,i} + \ldots + c_{in} - c_{i,n+1} - \ldots - 2c_{ii} - \ldots - c_{i,n+m-1} \pmod{k}$$

for $n+1 \leq i \leq n+m-1$.

Now the fixed elements are those $d_1q_1 + d_2q_2$ satisfying (i) and (ii). Considering such an element, we have that

$$\sum_{\substack{i,j=1\\i\leq j}}^{n} c_{ij} - \sum_{i=1}^{n}\sum_{j=n+1}^{n+m} c_{ij} + \sum_{\substack{i,j=n+1\\i\leq j}}^{n+m} c_{ij} = \frac{(n+1)n}{2}d_1 - 0 + \frac{(m+1)m}{2}d_2$$

and so we recover the same condition: $(n+1)nd_1 + (m+1)md_2 \equiv 0 \pmod{2k^2}$.

Furthermore, for $1 \leq i \leq n$, $c_{i,n+m} = 0$ and

$$c_{1,i} + \ldots + 2c_{ii} + \ldots + c_{in} - c_{i,n+1} - \ldots - c_{i,n+m-1} = (n+1)d_1 - 0 \equiv 0 \pmod{k}$$

while for $n+1 \leq i \leq n+m-1$, $c_{i,n+m} = d_2$ and

$$c_{1,i} + \ldots + c_{in} - c_{i,n+1} - \ldots - 2c_{ii} - \ldots - c_{i,n+m-1} = 0 - md_2 \equiv d_2 \pmod{k}.$$

Once again, condition (ii) is trivially satisfied, and so the ring of invariant quadratic forms is the same as before.

$$\mathrm{S}^2\left(\left(T_1 \times T_2 \big/ \mu_k\right)^*\right)^W = \{d_1q_1 + d_2q_2 \mid (n+1)nd_1 + (m+1)md_2 \equiv 0 \pmod{2k^2}\}.$$

2.3.4 The Product of SOs and Sps

Let $\dim(V_1) = 2n$, $\dim(V_2) = 2m$ and take $G_1, G_2 \in \{\mathrm{SO}, \mathrm{Sp}\}$. Let $T_i \subseteq G_i(V_i)$ be the maximal tori and consider

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mu_2 & \hookrightarrow & T_1 \times T_2 & \twoheadrightarrow & T_1 \times T_2 \big/ \mu_2 & \longrightarrow & 1 \\ & & \pm 1 & \longmapsto & (\pm I_{2n}, \pm I_{2m}) & & & & \end{array}$$

and dual

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \left(T_1 \times T_2 \big/ \mu_2\right)^* & \hookrightarrow & T_1^* \times T_2^* & \twoheadrightarrow & \mu_2^* & \longrightarrow & 0 \\ & & & & \wr\| & & \wr\| & & \\ & & & & \mathbb{Z}^{n+m} & \twoheadrightarrow & \mathbb{Z} \big/ 2\mathbb{Z} & & \\ & & & & \sum_{i=1}^{n+m} c_i e_i & \longmapsto & \sum_{i=1}^{n+m} c_i \pmod 2 & & \end{array}$$

meaning that

$$\left(T_1 \times T_2 \big/ \mu_2\right)^* = \langle e_i - e_{n+m}, 2e_{n+m} \mid 1 \leq i \leq n+m-1 \rangle$$

$$\mathrm{S}^2\left(\left(T_1 \times T_2 \big/ \mu_2\right)^*\right) = \left\{ \sum_{\substack{i,j=1 \\ i \leq j}}^{n+m} c_{ij} e_i e_j \mid \text{(i),(ii)} \right\}$$

(i) $\sum_{\substack{i,j=1 \\ i \leq j}}^{n+m} c_{ij} \in 4\mathbb{Z}$,

(ii) $c_{i,n+m} \equiv c_{1,i} + \ldots + 2c_{ii} + \ldots + c_{i,n+m-1} \pmod 2$ for $1 \leq i \leq n+m-1$.

Invariant quadratic forms of $G_1 \times G_2$ are generated by the two normalized Killing forms, in this case

$$q_1 = \sum_{i=1}^{n} e_i^2 \text{ and } q_2 = \sum_{i=n+1}^{n+m} e_i^2.$$

If an element $d_1 q_1 + d_2 q_2$ for $d_1, d_2 \in \mathbb{Z}$ satisfies (i) then

$$\sum_{\substack{i,j=1 \\ i \leq j}}^{n+m} c_{ij} = \sum_{i=1}^{n} c_{ii} + \sum_{i=n+1}^{n+m} c_{ii} = nd_1 + md_2 \equiv 0 \pmod 4.$$

Also for such an element, for all $1 \leq i \leq n+m-1$, $c_{i,n+m} = 0$ and

$$c_{1,i} + \ldots + 2c_{ii} + \ldots + c_{i,n+m-1} = 2c_{ii} \equiv 0 \pmod 2.$$

With condition (ii) again being trivially satisfied, this leaves that

$$\mathrm{S}^2\left(\left(T_1 \times T_2 \big/ \mu_2\right)^*\right)^W = \{ d_1 q_1 + d_2 q_2 \mid nd_1 + md_2 \equiv 0 \pmod 4 \}.$$

Bibliography

[Baek] S. Baek. *Chow Groups of Products of Severi-Brauer Varieties and Invariants of Degree 3*. Trans. Amer. Math. Soc. 369 (2017), no. 3, 1757–1771.

[BDZ] S. Baek, R. Devyatov, and K. Zainoulline. *The K-theory of Versal Flags and Cohomological Invariants of Degree 3*. to appear in Documenta Mathematica (preprint version is available at arXiv:1612.07278).

[Bou] N. Bourbaki. *Éléments de Mathématique, Groupes et Algèbres de Lie, Chapitres 4 à 6*. Masson, Paris, 1981.

[BR] H. Bermudez and A. Ruozzi. *Degree Three Cohomological Invariants of Groups that are Neither Simply-Connected nor Adjoint*. J. Ramanujan Math. Soc. 29 (2014), no. 4, 465–481.

[Gar] S. Garibaldi. *Orthogonal Involutions on Algebras of Degree 16 and the Killing Form of E_8*. Contemporary Mathematics Volume 493, 2009.

[GMS] S. Garibaldi, A. Merkurjev, and J.-P. Serre. *Cohomological Invariants in Galois Cohomology*. University Lecture Series 28, AMS, Providence, RI, 2003.

[GQ-M] S. Garibaldi and A. Quéguiner-Mathieu. *Restricting the Rost Invariant to the Center*. St. Petersburg Math J. Vol 19 (2008), no. 2, 197–213.

[GZ] S. Garibaldi and K Zainoulline. *The Gamma-Filtration and the Rost Invariant*. J. Reine und Angew. Math. 696 (2014), 225–244.

[Hall] B.C. Hall. *Lie Groups, Lie Algebras, and Representations (an elementary introduction, second edition)*. Springer, 2015.

[Hum72] J. Humphreys. *Introductions to Lie Algebras and Representation Theory*. Springer-Verlag New York Inc, 1972.

[Hum90] J. Humphreys. *Reflection Groups and Coxeter Groups*. Cambridge University Press, 1990.

[KMRT] M.-A. Knus, A. Merkurjev, M. Rost, and J.-P. Tignol, *The Book of Involutions*. American Mathematical Society, Providence, RI, 1998, With a preface in French by J. Tits.

[Mer] A. Merkurjev. *Degree Three Cohomological Invariants of Semisimple Groups*. J. Eur. Math. Soc. (JEMS), 18 (2016), no. 2, 657–680.

[MNZ] A. Merkurjez, A. Neshitov, and K. Zainoulline. *Invariants of Degree 3 and Torsion in the Chow Group of a Versal Flag*. Composito Mathematica 151 (2015), 1416–1432.

[SR] W.F. Santos and A. Rittatore. *Actions and Invariants of Algebraic Groups*. CRC Press, 2005.