CONTROLLABILITY, STABILITY AND STABILIZABILITY OF DISTRIBUTED PARAMETER SYSTEMS

By

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ABSTRACT

In this thesis the question of stabilization of perturbed (or uncertain) infinite dimensional linear systems is considered. First, we identify the class of perturbations for which the system remains controllable thereby stabilizable by the same feedback law as for the nominal system. That is, sufficient conditions are presented that guarantee stabilizability of the perturbed system given that the unperturbed system has similar properties. Secondly, we present a methodology for designing feedback controllers such that the feedback system is stable. It is shown that exponential stability can be achieved by choice of suitable additional state feedback controls even in the presence of unbounded and nonlinear perturbations. Both deterministic and stochastic systems are considered. Finally, we apply these stabilization results to a simplified SCOLE model proposed by NASA. Numerical simulations are carried out to illustrate the impact of perturbations on the performance of the space structures and the effectiveness of the stabilizing control.

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CHAPTER 1

INTRODUCTION

1.1 A Brief Literature Review

Dynamic systems which describe processes in science, engineering, physics and economics are usually very complex and the identification of mathematical models is difficult. Consequently, early investigations of systems were confined mainly to the analysis of stability behaviour of very simple models using frequency domain methods. Since the introduction of the state space approach, system theory has enjoyed great success and become very popular in the construction of dynamic models for socio-economic, science and engineering systems. Moreover, many systems theory concepts such as controllability, observability and optimality have been introduced and analysed. However, these concepts are only well understood for simple systems, namely, linear difference equations and linear ordinary differential equations in finite dimensional spaces. In the engineering terminology these systems are called lumped parameter systems and they can be described by linear maps on finite dimensional linear vector spaces. Many scientific and engineering problems, however, are modelled by distributed parameter systems. For example, spacecraft and new generation of satellites which are expected to be deployed in the near future would be very large structures consisting of lumped bodies and distributed flexible appendages, such as a long beam, a solar panel and antenna etc. Like the flexible spacecraft, there are many other systems such as robotic manipulators with elastic arms, power network and fluid flow which can be rigorously described by partial differential equations, delay differential equations and coupled ordinary and partial differential equations. The appropriate state space for the distributed systems is an infinite dimensional function space. In order to treat, in a mathematically unified manner, a variety of physical systems modelled by ordinary or partial differential equations, integro-differential equations and functional differential equations, a good deal of research has focused on control systems defined on infinite dimensional spaces [3, 10, 22, 28, 34, 35, 36, 40, 45, 51, 62, 66, 67, 69, 77, 90, 95, 96, 101, etc.]. The traditional matrices of the finite dimensional theory have therefore been replaced by suitable operators and functional analytic methods have taken over the linear algebraic techniques. Many of these equations can be reformulated as ordinary differential equations on abstract infinite dimensional spaces, for example, Hilbert spaces. This is where semigroup theory plays a central role and provides a unified and powerful tool for the study of a wide class of systems, both finite and infinite dimensional, which include lumped, delay and distributed systems. Semigroups in infinite dimensional spaces are direct extensions of exponential of matrices in finite dimensional Euclidian spaces, where system operators (matrices in finite dimensional space) are called generators of semigroups.

Using the powerful semigroup representation, many authors have worked on generalizing the major system theory concepts of controllability, observability, stability, optimal control, and estimation to linear infinite dimensional systems [3, 10, 17, 22, 28, 30, 35, 37, 40, 45, 48, 49, 52, 58, 68, 69, 72, 77, 89, 90, 92, 94, 97, 101, 102, 104, 106, 110, 112, 116, 118]. We do not consider all the system theory concepts in this thesis, but concentrate on controllability, stability and stabilizability which turn out to be more complicated in infinite dimensions. Although some aspects of these concepts from classical finite dimensional control theory admits a satisfactory generalization to ones in infinite dimensional spaces, aspects such as rank conditions in controllability theory [Triggiani 112,113], and an extension of a well-known theorem of A.M. Lyapunov concerning Hurwitzian matrices ($n \times n$ matrices with eigenvalues in the left half-plane) to strongly continuous semigroups of operators on a complex Hilbert space [Datko 37], there are, however, new phenomena that arise due to the infinite dimensionality of the state and/or control space. For instance, in the case of finite dimensional systems, it is well known that "controllability implies stabilizability". For the infinite dimensional case, this relationship fails and the situation becomes much more complicated. Another example is shown by Pazy [97] that Lyapunov's theorem is not always applicable in infinite dimensional space while it applies to finite dimensional systems in studying the effects of perturbations on asymptotic stability of ordinary differential equations. Another different feature is that there are several types of stability and controllability concepts for infinite dimensional systems which are all equivalent in the finite dimensional case [Benchimol 22, Dolecki 40].

During the last ten years, stability and stabilizability problems for infinite dimensional linear systems have been extensively studied by several authors [Benchimol 21,22, Chen 30, Datko 37,38, Huang 58, Lasiecka 69,70,71, Levan 75,76,77, Littman 89, Littman and Markus 90,91,92, Pazy 97, Pritchard and Zabczyk 101, Rabah and Ionescu 102, Russell 104, Slemrod 106, Triggiani 110,111, Zabczyk 116,117]. Two concepts of stability are mainly considered in infinite dimensional case: one is asymptotic stability (strongly or weakly), another is exponential stability. Concerning exponential stability, Triggiani in [110] gives a stabilizability result based on spectral decomposition of the system mainly applicable to compact semigroups and delay systems which have a finite number of "unstable modes". In [116], Zabczyk used a result of Datko [38] to show that if the system is exactly controllable then, it is exponentially stabilizable. Megan has shown in [94] that for a group (not generally a semigroup), the exponential stability with an arbitrary prefixed decay rate is equivalent to the exact null-controllability of the system. On the other hand, approximate controllability, which will simply be called controllability in the thesis (defined in Chapter 3) plays an important role in asymptotic stabilizability. In [106], Slemrod studied the case in which the system operator generates a strongly C_0-semigroup of contraction operators. He obtained sufficient conditions for strong stabilizability of unitary groups, and for weak stabilizability of contraction semigroups. Slemrod's result were then generalized by Benchimol [21,22] and Levan [75,76,77] who considered weakly and strongly (asymptotically) stabilizable problems for contraction semigroups, respectively. Slemrod used the invariance principle, while Benchimol and Levan's techniques are simpler and more directly semigroup theoretic, relying on a functional decomposition of contraction semigroups. All of these results of asymptotic stability depend on (approximate) controllability of the system. The class of stabilizable problems discussed above is an important one in which the control action used is distributed control. Nevertheless, in infinite dimensional systems, for example, those described by partial differential equations control action is very often applied on the boundary of a given domain. For such boundary control systems some stabilizability results were obtained by Lasiecka, Triggiani and Chen [Chen 30, Lagnese 68, Lasiecka 69,70,71, Lasiecka and Triggiani 72], etc.

While so much effort has been devoted to the study of deterministic infinite dimensional systems, its counterpart–stochastic evolution equations have also attracted much attention due to the fact that in many physical and engineering dynamic systems noise appears distributed both in time and space. The galloping conductor is one such example where the distributed noise could be attributed to the random aerodynamic forces acting on the (transmission line) conductors, arising from the randomness of wind velocity and the irregularity of ice formation on the conductor surface [Riaz, Biswas and Ahmed 103]. Distributed noise arises also in many other practical systems such as flexible satellites and heating furnaces, etc. From the system theory point of view, therefore, stability of stochastic evolution equations is also important. In fact, some results have been obtained in this area [Ahmed 1,3, Biswas and Ahmed 25, Haussmann 53, Ichikawa 59,60,61, Zabczyk 115]. In the linear stochastic equations with noises depending on the state of the system, for example, the stability problem has been studied by Ahmed, Ichikawa [Ahmed 1,3, Ichikawa 60,61]. A necessary and sufficient condition for the exponential stability of the second moment (in the mean square sense) is obtained in terms of a Lyapunov equation in [Ichikawa 60] and the asymptotic stability of sample paths is considered by Haussman [53]. After introducing a proper definition of stochastic integrals with respect to the real Wiener processes such that one can use basic results and arguments in finite dimensions and prove Ito's formula for infinite dimensional stochastic differential equations, Ichikawa [59] has shown that the results for the linear case can be extended to a class of semilinear stochastic evolution equations whose nonlinear terms satisfy a Lipschitz condition. It is known that, using semigroups or evolution operators, infinite dimensional differential equations describe a wide class of parabolic, hyperbolic and functional differential equations, both deterministic and stochastic, and provide a unified treatment of these equations. As a result, more and more attention has been paid to the study of stability of evolution equations.

1.2 Research Objective and Motivation

Mathematical models for physical control systems are developed essentially under many simplifying assumptions giving only a nominal (unperturbed) system

$$\dot{x} = Ax + Bu$$

where A generates a linear or nonlinear semigroup on a Banach space X and B is a linear operator from another Banach space U to X. The true physical system, however, is usually given by the perturbed (uncertain) one

$$\dot{x} = (A + P)x + Bu$$

where the perturbation P of the system operator represents the uncertainties in the modeling process and may be bounded or even unbounded. Usually, uncertainties arise due to variations in parameters (structured uncertainties) that describe the system. Uncertainties also arise due to model truncation at high frequencies (unstructured uncertainties). In stability problems, therefore, an important consideration is to design a controller which guarantees that the system will remain stable even if it is subject to some perturbations. Since any design is based on the nominal system, it is also very interesting to evaluate the sensitivity (robustness) of system stability under the perturbations.

The question of stabilization of uncertain linear systems described by ordinary differential equations in finite dimensions has received considerable attention in current literature [18, 19, 20, 29, 32, 33, 47, 50, 73, 74, 78, 98, 105, 109, etc.]. Although these problems involve controlling a system with uncertain parameters, they differ from stochastic control problems in the following fundamental way: no a priori statistics is assumed for the uncertain parameters; only a bound on the parameter variations is assumed to be given and the objective is to design a deterministic feedback control which guarantees the stability of the uncertain system while the exact characteristics of the perturbing operator is not known. Here the term "guaranteed performance" is used to mean that the resulting closed-loop system will have certain desirable properties (for example, asymptotic stability). This kind of deterministic design procedure for uncertain systems which are nominally time-invariant can be applied to many cases of interest. For instance, see the application to macroeconomics discussed in [Leitmann and Wan 74]. Also, Ryan, Leitmann and Corless [105] developed this theory for application to robotic tracking. The theory, however, is only applicable for systems in the finite dimensional case. As mentioned earlier, many practical systems are described by infinite dimensional differential equations using the semigroup approach. This gives the author initial motivation to extend the theory to infinite dimensional systems with bounded or unbounded perturbations. First, we should identify the class of perturbations for which the system remains controllable and stabilizable by the same feedback controller as for the nominal system. Then we will consider the problem of designing a state feedback linear control law that guarantees exponential stability of the perturbed system, both deterministic and stochastic, uniformly with respect to the perturbation, under the assumption that the nominal system is exponentially stabilizable. In other words, we provide additional controls only dependent on the set of perturbations (not the individual perturbation) preserving exponential stability. Finally we try to apply these results to a simplified Spacecraft Control Laboratory Experiment (SCOLE) model proposed by NASA and present some interesting simulation results.

1.3 Outline of the Thesis and Contributions

The thesis is organized as follows:

Chapter 1 gives a brief review of the literature for stabilizability of infinite dimensional systems and the motivation to the problem of stabilizing perturbed systems in Hilbert spaces. Then the problem treated in this thesis is presented.

In Chapter 2 we will give some important definitions and properties of linear systems in Hilbert space and basic semigroup theory. Also we will introduce different stability concepts induced by infinite dimensionality and discuss the relationships among them.

In Chapter 3, the question of controllability of perturbed linear systems in Hilbert space is considered. Sufficient conditions are presented that guarantee invariance of controllability under bounded or unbounded perturbations.

Based on the results of Chapter 3, we identify (specify) in Chapter 4 the class of perturbations for which the system is robust in the sense that the perturbed system preserves the stability properties of the unperturbed system with the same feedback control as for the nominal system. Numerical results for some distributed systems are given.

In Chapter 5, we present a methodology for designing feedback controllers such that the feedback system is stable. It is shown that exponential stability can be achieved by choice of suitable state feedback controls even in the presence of unbounded and nonlinear perturbations. Our method is based on perturbation theory of semigroups. The results are illustrated by two examples involving heat equation and wave equation. Its counterpart-stabilizability of stochastic evolution equation is presented in Chapter 6. Both linear and semilinear systems are considered. Not only exponential stability in the mean square sense, but also stability of sample paths is obtained for stochastic systems.

In Chapter 7 we apply these stabilization results to a simplified SCOLE system proposed by NASA. After a dynamic model for the spacecraft consisting of a rigid body and a flexible beam is developed, numerical simulations are carried out in order to illustrate the impact of perturbations on the performance of the space structures and the effectiveness of the stabilizing controls suggested in previous chapters.

Finally, concluding remarks and suggestions for further research are presented in Chapter 8.

The original contributions of the thesis include:

(i) Proof that controllability of infinite dimensional systems is invariant under a class of bounded or unbounded perturbations. This generalizes invariant properties of controllability for finite dimensional systems [Ahmed 2, Silverman and Anderson 107]: section 3.2 and 3.3 [Ahmed and Li 5,6,7,8, Li and Ahmed 84].

(ii) Investigation of robustness (sensitivity) of system stability under the perturbations. The class of perturbations for which the system remains stabilizable by the same feedback control as for the nominal system is identified (specified): sections 4.3, 4.4 and parts of 4.2 [Ahmed and Li 5,6,7,8, Li and Ahmed 84,85].

(iii) Development of a methodology for designing feedback controllers such that the feedback systems, both deterministic and stochastic, are stable in the presence of linear or nonlinear, bounded or unbounded perturbations: sections 5.2, 5.3 and 5.4; sections 6.3 and 6.4 [Li and Ahmed 79,80,81,82,83].

(iv) Development of a simplified mathematical model for a hybrid system and its numerical simulations: sections 7.2, 7.3, 7.4 and 7.5 [Li, Ahmed and Lim 85].

CHAPTER 2

LINEAR SYSTEMS ON INFINITE DIMENSIONAL SPACE AND THEIR STABILITY PROBLEMS

2.1 Introduction

In many scientific and engineering problems the system is modelled by partial differential equations, integral equations or coupled ordinary and partial differential equations. It is known that, as mentioned in chapter 1, the semigroup theory gives a unified treatment of these interesting problems known as distributed parameter systems. In fact, they can be described as linear or nonlinear dynamical systems on infinite dimensional Banach spaces using a semigroup approach, or more generally, the evolution operator approach. In this chapter we will give some basic definitions and properties that characterise the semigroups based on their generators of linear systems in Hilbert spaces.

Although some properties of the classical finite dimensional theory have a satisfactory generalization to infinite dimensional space, there are, however many new phenomena and concepts that appear due to the infinite dimensionality of the state space. For example with regards to stability behavior, one of the prominent interests in system theory, some pathologies appear with respect to the corresponding finite dimensional case where a system is known to be asymptotically stable if and only if the spectrum of the system matrix is strictly on the left side of the imaginary axis. Moreover, it implies exponential stability. In infinite dimensional spaces, however, these notions are much more complicated. A system could be asymptotically stable but not exponentially decaying, and so all the spectrum of the system operator can be strictly in the open left-hand side of complex plane, the system may not be exponentially stable [Huang 58, Pazy 96, Pritchard and Zabczyk 101]. On the other hand, Triggiani shows that in [111], if the system operator is bounded on an infinite dimensional Banach space, the system may be asymptotically stable, even though its spectrum contains the imaginary axis (the closed left-hand side of the complex plane). Another purpose of the present chapter, therefore, is intended to introduce different stability concepts induced by infinite dimensionality and discuss the relationships among them.

2.2 Linear Systems and Basic Semigroups Theory

In this entire chapter we denote by X a separable complex Hilbert space, $\mathcal{L}(X)$ ($\mathcal{L}_u(X)$) the space of bounded (not necessarily bounded) linear operators on X and consider one-parameter semigroups of bounded linear operators $T(t)$ on X. By this we understand a family $\{T(t) : t \in R^+\} \subset \mathcal{L}(X)$, usually written as $T(t), t \geq 0$, such that

(i) $T(0) = I$ (I is the identity operator on X);

(ii) $T(t+s) = T(t)T(s)$ for every $t, s \geq 0$ (the semigroup property).

The main purpose of this section is to describe the class of linear systems and related semigroups characterization which we will use later. Another problem we will treat here is how diverse properties of a semigroup can be characterized in terms of its generator. This is a reasonable question from the theoretical point of view (since the generator uniquely determines the semigroup). It is of interest from the practical point of view as well: the generator is the given object, not the semigroup, defined by the abstract differential equation. Therefore, it is useful to dispose of conditions of the generator itself giving information on the solutions (which might not be known explicitely). We will discuss smoothness properties such as analyticity, differentiability, norm continuity and compactness of the semigroup and the generator which characterize these special semigroups. Consider the abstract Cauchy problem

$$\begin{cases} \dot{x}(t) = Ax(t), & t \geq 0 \\ x(0) = x_0 \end{cases} \tag{2.1}$$

where A is a linear operator on X defined by

$$D(A) = \{x \in X : \lim_{t \downarrow 0} \frac{T(t)x - x}{t} \text{ exists}\} \tag{2.2}$$

and

$$Ax = \lim_{t \downarrow 0} \frac{T(t)x - x}{t} \text{ for } x \in D(A). \tag{2.3}$$

We say that A is the infinitesimal generator of the semigroup $T(t)$, $D(A)$ is the domain of A. The operators A which will be considered in our thesis are not necessarily bounded, i.e., $A \in \mathcal{L}_u(X)$. Neverthless, we begin with generators A being bounded linear operators on X.

A semigroup of bounded linear operators $T(t), t \geq 0$, is uniformly continuous if

$$\lim_{t \downarrow 0} \|T(t) - I\| = 0. \tag{2.4}$$

From semigroup properties (i) and (ii), it is not difficult to see that (2.4) implies that $T(t)$ is uniformly continuous for all $t > 0$.

Theorem 2.1[Pazy 96]

A linear operator A is the infinitesimal generator of a uniformly continuous semigroup $T(t)$ if and only if A is a bounded linear operator on X. ■

Theorem 2.1 shows that if A is a bounded linear operator, it generates a uniformly continuous semigroup $T(t), t \geq 0$ which is defined by

$$T(t) = e^{tA} = \sum_{n=0}^{\infty} \frac{(tA)^n}{n!}. \tag{2.5}$$

On the other hand, based on theorem 2.1, if A is unbounded linear operator on X, the semigroup $T(t), t \geq 0$ generated by A is not uniformly continuous. Therefore, we introduce the following definition:

Definition 2.2 (strongly continuous semigroup)

A strongly continuous semigroup is a map $T(t)$ from R^+ to $\mathcal{L}(X)$, which satisfies the semigroup properties (i), (ii) and

$$\|T(t)x - x\|_X \to 0 \quad \text{as } t \downarrow 0, \quad \text{for every } x \in X. \tag{2.6}$$

A strongly continuous semigroup of bounded linear operators on X is called simply a C_0-semigroup.

The relations between C_0-semigroup and its generator are much more difficult to characterize than those of uniformly continuous semigroups. First, recall that if A is a linear, not necessarily bounded, operator in X, the resolvent set $\rho(A)$ of A is the set of all complex numbers λ for which $\lambda I - A$ is invertible, i.e., $(\lambda I - A)^{-1}$ is a bounded linear operator in X. The operator $R(\lambda, A) = (\lambda I - A)^{-1}$ for $\lambda \in \rho(A)$ is a bounded linear operator and is called the resolvent of A. The following theorem is due to Hille and Yosida which gives a complete characterization of those linear operators A which generate C_0 -semigroups.

Theorem 2.3(Hille-Yosida)

A linear operator A is the infinitesimal generator of a C_0 -semigroup $T(t)$ satisfying $\|T(t)\| \leq Me^{wt}, M \geq 1, w \in R$, if and only if

(i) A is closed and $D(A)$ is dense in X;

(ii) The resolvent set $\rho(A)$ of A contains the ray (w, ∞) and

$$\|R(\lambda, A)^n\| \leq \frac{M}{(\lambda - w)^n} \quad \text{for } \lambda > w, n = 1, 2, \cdots. \tag{2.7}$$

Remark 2.4

The condition that every real $\lambda, \lambda > w$ is in the resolvent set of A together with the estimate (2.7) imply that every complex λ satisfying $\mathrm{Re}\lambda > w$ is in the resolvent set of A and

$$\|R(\lambda, A)^n\| \leq \frac{M}{(\mathrm{Re}\lambda - w)^n} \quad \text{for } \mathrm{Re}\lambda > w, n = 1, 2, \cdots. \tag{2.8}$$

From definition 2.2 and theorem 2.3 it follows that there are constants $w \geq 0$ and $M \geq 1$ such that $\|T(t)\| \leq Me^{wt}$. If $w = 0, T(t)$ is called uniformly bounded semigroup, moreover if $M = 1$ it is called a C_0-semigroup of contractions. Hille-Yosida theorem characterizes the generators of general C_0-semigroups. It is, however, not easy to verify the condition like the estimate (2.7). For the generator of a C_0-semigroup of contractions there is a different characterization given by the Lumer-Phillips theorem [Lumer and Phillips 93]. We give this theorem in the following and from there we can derive some general results which are easier to verify.

Definition 2.5

A linear operator A is dissipative if $\mathrm{Re}(Ax, x) \leq 0$ for every $x \in D(A)$. Furthermore, if A is dissipative and the range, $R(I - A)$, of $I - A$ is X $(R(I - A) = X)$, it is called m-dissipative.

Theorem 2.6(Lumer-Phillips)

A linear operator A in X is the infinitesimal generator of a C_0-semigroup of contractions if and only if it is m- dissipative. ■

Remark 2.7

Theorem 2.6 is the version of the Lumer-Phillips theorem in a Hilbert space X. If X is a general Banach space, one has to add the requirement that A is a densely defined operator. This condition in Hilbert space (in fact, reflexive Banach space), however, is guaranteed by m-dissipativity (for details see [Pazy 96]).

Theorem 2.8

A linear operator A is the generator of a C_0-semigroup $T(t)$ satisfying $\|T(t)\| \leq e^{wt}$ if and only if there exists a real number w such that

$$\mathrm{Re}(Ax, x) \leq w\|x\|^2 \quad \text{for } x \in D(A) \tag{2.9}$$

and $R((1+w)I - A) = X$.

Proof

Consider the linear operator $B = A - wI$ and apply theorem 2.6 to the operator B, then the result follows upon noting that B generates the C_0-semigroup $S(t) \equiv e^{-wt}T(t)$ of contractions given that $\{T(t), t \geq 0\}$ is the C_0-semigroup generated by A if and only if B is m-dissipative. ■

The following result is more useful since the conditions are easier to verify.

Theorem 2.9

Let A be a densely defined closed linear operator. Then $T(t), t \geq 0$ generates a C_0-semigroup satisfying $\|T(t)\| \leq e^{wt}$ if and only if there exists a real number w such that

$$\mathrm{Re}(Ax, x) \leq w\|x\|^2 \quad \text{for } x \in D(A) \tag{2.10}$$

and

$$\mathrm{Re}(A^*x, x) \leq w\|x\|^2 \quad \text{for } x \in D(A^*). \tag{2.11}$$

Proof

(If): Letting $A_w \equiv A - wI$, it is obvious that A_w is a densely defined closed linear operator satisfying $\mathrm{Re}(A_w x, x) \leq 0$ for $x \in D(A_w) = D(A)$ and $\mathrm{Re}(A_w^* x, x) = \mathrm{Re}(A^* x, x) - w(x, x) \leq 0$ for $x \in D(A_w^*) = D(A^*)$. Therefore, it follows from corollary 1.4.4 [Pazy 96] that A_w generates a C_0-semigroup of contractions $S(t), t \geq 0$, and hence A generates a C_0-semigroup $T(t) = S(t)e^{wt}, t \geq 0$ in X satisfying $\|T(t)\| \leq e^{wt}$.

(only if): Since A generates a C_0-semigroup $T(t)$ satisfying $\|T(t)\| \leq e^{wt}$, A is a densely defined closed linear operator[Pazy 96] and the semigroup $S(t) \equiv T(t)e^{-wt}, t \geq 0$, corresponding to the generator $A_w = A - wI$, is contractive. Hence, $\mathrm{Re}(A_w x, x) \leq 0$ for $x \in D(A_w)$. On the other hand, since X is a Hilbert space, its dual $S^*(t)$ is generated by A_w^* and it is also contractive as $\|S^*(t)\| = \|S(t)\| \leq 1$. Thus $\mathrm{Re}(A_w^* x, x) \leq 0$ for $x \in D(A_w^*)$. That is, $\mathrm{Re}(Ax, x) \leq -w\|x\|^2$ for $x \in D(A)$ and $\mathrm{Re}(A^* x, x) \leq -w\|x\|^2$ for $x \in D(A^*)$. This completes the proof. ■

Now we give two examples to which theorem 2.9 can be applied.

Example 2.10

Consider the abstract second order differential equation

$$\ddot{y} - B\dot{y} + Ay = 0 \tag{2.12}$$

where A is a self-adjoint, positive operator on a real Hilbert space H, so that the operator $A^{1/2}$ is well defined and $B \in \mathcal{L}(D(A^{1/2}), H)$, with $D(B) \supset D(A^{1/2})$ and

$D(B^*) \supset D(A^{1/2})$. Assume that there exists a real number α such that

$$(y, By) \leq \alpha \|y\|^2, \quad y \in D(B).$$

Then the equation (2.12) can be rewritten as a first order differential equation

$$\dot{z} = \mathcal{A}z \tag{2.13}$$

on Hilbert space $\mathcal{H} = D(A^{1/2}) \times H$ where $\mathcal{A}$ is defined by

$$\mathcal{A}z = \begin{pmatrix} 0 & I \\ -A & B \end{pmatrix} z, \quad z \equiv \begin{pmatrix} y \\ w \end{pmatrix} = \begin{pmatrix} y \\ \dot{y} \end{pmatrix} \tag{2.14}$$

with $D(\mathcal{A}) = D(A) \times D(A^{1/2})$. We shall prove that $\mathcal{A}$ generates a C_0-semigroup $T(t)$ with

$$\begin{aligned} \|T(t)\|_{\mathcal{H}} &\leq e^{\alpha t}, \quad \alpha > 0, \\ \|T(t)\|_{\mathcal{H}} &\leq 1, \quad \alpha \leq 0. \end{aligned} \tag{2.15}$$

First we define the inner product on $\mathcal{H}$ by

$$(z_1, z_2)_{\mathcal{H}} = (A^{1/2}y_1, A^{1/2}y_2)_H + (w_1, w_2)_H,$$

and it is easy to show that

$$\mathcal{A}^*z = \begin{pmatrix} 0 & -I \\ A & B^* \end{pmatrix} z, \quad z \in D(\mathcal{A})$$

with $D(\mathcal{A}^*) = D(\mathcal{A})$ and that $\mathcal{A}$ is closed, densely defined by the conditions on A and B [Pritchard and Zabczyk 101]. Thus

$$(\mathcal{A}z, z)_{\mathcal{H}} = (\mathcal{A}^*z, z)_{\mathcal{H}} = (Bw, w)_H \leq \alpha \|w\|_H^2 \begin{cases} \leq \alpha \|z\|_{\mathcal{H}}^2 & \alpha > 0, \\ \leq 0 & \alpha \leq 0. \end{cases} \tag{2.16}$$

Hence we apply theorem 2.9 to conclude that $\mathcal{A}$ is the generator of a C_0-semigroup satisfying (2.15).

To illustrate this abstract version we have the following one- dimensional wave equation:

$$\begin{cases} \dfrac{\partial^2 y}{\partial t^2} = \dfrac{\partial^2 y}{\partial x^2} + a_1 x \dfrac{\partial^2 y}{\partial t \partial x} + a_2 \dfrac{\partial y}{\partial t}, \quad 0 < x < 1 \\ y(0,t) = y(1,t) = 0 \\ y(x,0) = y_0(x) \\ y_t(x,0) = y_1(x), \end{cases} \tag{2.17}$$

on the real Hilbert space $H = L_2(0,1)$. Then, for fixed $t \geq 0$ we set,

$$A = -\frac{\partial^2}{\partial x^2}, \quad B = a_1 x \frac{\partial}{\partial x} + a_2 I,$$

and $D(A) = H^2(0,1) \cap H_0^1(0,1)$, $D(A^{1/2}) = H_0^1(0,1)$. For $y \in D(B) = D(A^{1/2})$,

$$\begin{aligned}(a_1 x \frac{\partial y}{\partial x}, y) &= \int_0^1 a_1 x y \frac{\partial y}{\partial x} dx \\ &= a_1 x y^2 \mid_0^1 - a_1 \int_0^1 y^2 dx - a_1 \int_0^1 xy \frac{\partial y}{\partial x} dx,\end{aligned}$$

$$\text{hence,} \quad (a_1 x \frac{\partial y}{\partial x}, y) = -\frac{a_1}{2} \int_0^1 y^2 dx = -\frac{a_1}{2} \|y\|^2.$$

This implies

$$(By, y) = (a_1 x \frac{\partial y}{\partial x}, y) + (a_2 y, y) = (-\frac{a_1}{2} + a_2)\|y\|^2.$$

Therefore,

$$(\mathcal{A}z, z)_{\mathcal{H}} = (\mathcal{A}^* z, z)_{\mathcal{H}} \begin{cases} \leq (a_2 - \frac{a_1}{2})\|z\|^2 & \text{if } a_2 > \frac{a_1}{2} \\ \leq 0 & \text{if } a_2 \leq \frac{a_1}{2} \end{cases}$$

where $\mathcal{H} = H_0^1(0,1) \times L_2(0,1)$ and $\mathcal{A}$ generates a C_0-semigroup $T(t)$ with

$$\|T(t)\|_{\mathcal{H}} \leq e^{(a_2 - a_1/2)t} \quad a_2 > \frac{a_1}{2} \tag{2.18}$$

or

$$\|T(t)\|_{\mathcal{H}} \leq 1 \qquad a_2 \leq \frac{a_1}{2}. \tag{2.19}$$

In fact, example 2.10 only uses the sufficient conditions in theorem 2.9 for A to be the generator of a C_0-semigroup satisfying $\|T(t)\| \leq e^{wt}$. However, both necessary and sufficient conditions will be applied to the next example.

Example 2.11

Consider the partial differential equation

$$\begin{cases} \dfrac{\partial z}{\partial t} = \dfrac{\partial z}{\partial x}, & 0 < x < 1 \\ z(0, x) = f_0(x) \end{cases}$$

on the Hilbert space $H = L_2[0, 1]$ with boundary conditions

(i) $z(1, t) = 0$ and $z(0, t)$ free;

or

(ii) $z(0, t) = 0$ and $z(1, t)$ free.

We show that A generates a C_0-semigroup of contractions in H for the case (i). On the other hand, for the case (ii), A is not generator of a C_0-semigroup of contractions in H unless $z(0, t) = z(1, t) = 0$.

Proof

(i) Let $Az = \frac{\partial z}{\partial x}$, $H = L_2[0, 1]$,

$$D(A) = \{z : z \in H^1[0, 1], z(1) = 0\}.$$

Then it is easy to show that $A^* z = -\frac{\partial z}{\partial x}$,

$$D(A^*) = \{z : z \in H^1[0, 1], z(0) = 0\}.$$

Moreover,

$$(Az, z) = -\frac{1}{2}z^2(0) \leq 0$$

$$(A^*z, z) = -\frac{1}{2}z^2(1) \leq 0.$$

Hence, A generates a C_0-semigroup of contractions $T(t)$ on H by applying the sufficient conditions of theorem 2.9.

(ii) As in case (i) we have $Az = \frac{\partial z}{\partial x}$, $H = L_2[0,1]$,

$$D(A) = \{z : z \in H^1[0,1], z(0) = 0\}$$

and $A^*z = -\frac{\partial z}{\partial x}$,

$$D(A^*) = \{z : z \in H^1[0,1], z(1) = 0\}.$$

Furthermore

$$(Az, z) = \frac{1}{2}z^2(1) \geq 0$$

$$(A^*z, z) = \frac{1}{2}z^2(0) \geq 0.$$

Hence, by the necessary conditions of theorem 2.9, A is not the generator of a C_0-semigroup of contractions on H unless $z(0) = z(1) = 0$. ■

So far we have seen that the fundamental Hille-Yosida theorem gives a complete characterization of generators of general C_0-semigroups and the Lumer-Phillips theorem, on the other hand, describes those generators of C_0-semigroups of contractions. In fact, for contraction semigroups, there are some other important properties given by Phillips [99]:

<u>Theorem 2.12</u>

Let $T(t), t \geq 0$, be a semigroup generated by A. Then,

(i) $T(t), t \geq 0$, is an isometric semigroup: $\|T(t)x\| = \|x\|$ for all $t \geq 0$ and all x in X, if and only if A is m-dissipative and $\mathrm{Re}(Ax, x) = 0$ for all $x \in D(A)$.

(ii) $T(t), t \geq 0$, is a unitary semigroup: $\|T(t)x\| = \|x\| = \|T^*(t)x\|$ for all $t \geq 0$ and all $x \in X$, if and only if A is skewadjoint: $A = -A^*$. ■

As mentioned earlier, there are some semigroups with special properties, such as group property, analyticity, differentiability, norm continuity and compactness, etc., which arise naturally in many applications. These properties bacome very important later on in the study of stability problems of the linear system (2.1). Therefore, we wish to consider these special semigroups and characterize them in terms of their generators.

Definition 2.13 (C_0-group of bounded operators)

A one parameter family $T(t), -\infty < t < \infty$, of bounded linear operators on X is a C_0-group of bounded operators if it satisfies

(i) $T(0) = I$.

(ii) $T(t+s) = T(t)T(s)$ for $-\infty < t, s < \infty$.

(iii) $\lim_{t \to 0} T(t)x = x$ for $x \in X$.

For a C_0-group we have the following Hille-Yosida's characterization theorem.

Theorem 2.14[Pazy 96]

A is the infinitesimal generator of a C_0-group of bounded operators $T(t)$ satisfying $\|T(t)\| \leq Me^{w|t|}$ if and only if

(i) A is closed and $\overline{D(A)} = X$.

(ii) Every real $\lambda, |\lambda| > w$, is in the resolvent set $\rho(A)$ of A and for such λ

$$\|R^n(\lambda, A)\| \leq M(|\lambda| - w)^{-n}, n = 1, 2, \cdots. \tag{2.20}$$

We now describe a hierarchy of smoothness conditions on the semigroup, starting with the most restrictive class; namely, analytic semigroups which plays an important role in the study of diffusion, heat transfer and many hydrodynamic problems [Ahmed 3]. These classes of semigroups are also known as holomorphic and parabolic semigroups. The generators of these semigroups can be characterized by a particularly simple condition.

Definition 2.15(Analytic semigroups)

Let $\Delta = \{z : -\delta < arg z < \delta, \delta > 0\}$ and for $z \in \Delta$ let $T(z)$ be a bounded linear operator on a Hilbert space X. The family $T(z), z \in \Delta$ is an analytic semigroup in Δ if

(i) $z \rightarrow T(z)$ is analytic in Δ.

(ii) $T(0) = I$ and $\lim_{z \rightarrow 0; z \in \Delta} T(z)x = x$ for every $x \in X$ (in the strong sense).

(iii) $T(z_1 + z_2) = T(z_1)T(z_2)$ for $z_1, z_2 \in \Delta$.

A semigroup $T(t)$ will be called analytic if it is analytic in some sector Δ containing the nonnegative real axis.

Theorem 2.16[Arendt et al. 12]

Let A be a densely defined operator on X. Then A is the generator of an analytic semigroup of angle $\delta \in (0, \frac{\pi}{2})$ if and only if there exist $M > 0$ and $\gamma \geq 0$ such that whenever $\lambda \in \rho(A)$ we have,

$$\|R(\lambda, A)\| \leq \frac{M}{|\lambda|} \text{ for } \mathrm{Re}\lambda > 0, |\lambda| \geq \gamma. \tag{2.21}$$

Compared with the Hille-Yosida theorem, theorem 2.16 gives a very simple criterion for an operator to be the generator of a (analytic) semigroup. Merely the resolvent and not its powers have to be estimated.

Next we consider semigroups satisfying a less restrictive smoothness condition.

Definition 2.17 (Differentiable semigroups)

Let $T(t)$ be a C_0-semigroup on X. The semigroup $T(t)$ is called differentiable for $t > t_0$ if for every $x \in X, t \to T(t)x$ is differentiable for $t > t_0$. $T(t)$ is called differentiable if it is differentiable for $t > 0$.

Theorem 2.18 [Pazy 96]

A C_0-semigroup $T(t), t \geq 0$, satisfying $\|T(t)\| \leq Me^{\omega t}$ is differentiable if and only if its generator A satisfies the following: for every $b > 0$ there are constants $a_b > 0$ and $M_b > 0$ such that

$$\rho(A) \supset \Sigma_b = \{\lambda : a_b e^{-b\mathrm{Re}\lambda} \leq |\mathrm{Im}\lambda|\} \tag{2.22}$$

and $\|R(\lambda, A)\| \leq M_b|\mathrm{Im}\lambda|$ for $\lambda \in \Sigma_b$, $\mathrm{Re}\lambda \leq \omega$. ■

Definition 2.19 (Norm continuous semigroups)

A C_0-semigroup $T(t)$ is called norm continuous for $t > t_0$ if for $t > t_0, T(t)$ is continuous in the uniform operator norm topology. $T(t)$ is called norm continuous if it is norm continuous for $t > 0$.

A complete characterization of norm continuous semigroups in terms of their generators is not yet known. But we have the following:

Theorem 2.20 [Arendt et al. 12]

Let A be the generator of a norm continuous semigroup. Then for every $b \in R$ the set $\{\lambda \in \sigma(A) : \mathrm{Re}\lambda \geq b\}$ is bounded. ■

Norm continuous semigroups are of particular interest in spectral theory. Moreover their asymptotic behavior is easy to describe (see theorem 2.35).

Definition 2.21 (Compact semigroups)

A C_0-semigroup $T(t)$ on X is called compact for $t > t_0$ if for every $t > t_0, T(t)$ is a compact operator. $T(t)$ is called compact if it is compact for $t > 0$.

Theorem 2.22 [Pazy 96]

A C_0-semigroup $T(t)$ is compact if and only if it is norm continuous and its generator A has compact resolvent. ■

Remark 2.23

Like norm continuous semigroups, theorem 2.22 is not a complete characterization of compact semigroups as it depends on the information of the semigroup itself. Some effort has been attempted by Pavel [95] to give the necessary and sufficient conditions for compact semigroups depending on only the information of their generators, but the result is not quite satisfactory in the sense that some conditions are difficult to verify.

We have considered so far some different classes of semigroups with special properties. Their relations are indicated as follows:

$$\text{analytic} \Longrightarrow \text{differentiable} \Longrightarrow \text{norm continuous} \Longleftarrow \text{compact.} \tag{2.23}$$

2.3 Asymptotic Behavior, Stability and Spectrum of Linear Systems

We start our discussion from the stability of the linear equations (2.1) on the finite dimensional state space $X = R^n$ where A is a $n \times n$ matrix. There are many equivalent conditions for the system to be asymptotically stable (or equivalently exponentially stable). They are based either on properties of the spectrum of the matrix A ($\sigma(A)$) or on the existence of an appropriate Lyapunov function. For example, the following theorem holds:

Theorem 2.24

Let dim $X = n < +\infty$. The origin of the system (2.1) is exponentially stable [Ahmed 2] if and only if all eigenvalues of the matrix A have negative real parts

$$\sup\{\mathrm{Re}\lambda : \lambda \in \sigma(A)\} < 0 \tag{2.24}$$

and if and only if there exists a nonnegative definite matrix $S \geq 0$ such that

$$SA + A^*S = -I. \tag{2.25}$$

■

If the Hilbert space X is infinite dimensional, the above theorem is not always true. For instance, even when A is a bounded linear operator in X, that is, $A \in \mathcal{L}(X)$ and generates a uniformly continuous semigroup (theorem 2.1), Triggiani shows that the system (2.1) may be asymptotically stable, even though $\sup\{\mathrm{Re}\lambda : \lambda \in \sigma(A)\} = 0$. For details we refer to [111] by Triggiani. Our main interest is A being the infinitesimal generator of a C_0-semigroup $T(t), t \geq 0$ on an infinite dimensional Hilbert space X. In this case, in contrast with the classical finite dimensional theory, asymptotic stability is not equivalent to exponential stability. In fact, there are many different notions of stability for infinite dimensional systems as given below:

Definition 2.25(Exponential stability)

Let A generate a C_0-semigroup $T(t), t \geq 0$ in X. Then the system (2.1) (or the generator A) is uniformly exponentially stable if and only if there exist two positive real numbers $M \geq 1$ and $w > 0$ such that, for any t

$$\|T(t)\| \leq Me^{-wt}. \tag{2.26}$$

Definition 2.26(Uniform asymptotic stability)

Let A generate a C_0-semigroup $T(t)$ in X. Then the system (2.1) (or the generator A) is uniformly asymptotically stable if and only if

$$\|T(t)\| \to 0 \quad \text{as } t \to +\infty. \tag{2.27}$$

Definition 2.27 (Strong L_2-stability)

Let A generate a C_0-semigroup $T(t)$ in X. Then the system (2.1) (or the generator A) is strongly L_2-stable if and only if for each $x \in X$,

$$\int_0^\infty \|T(t)x\|^2 dt < \infty. \tag{2.28}$$

Definition 2.28 (Strong asymptotic stability)

Let A generate a C_0-semigroup $T(t)$ in X. Then the system (2.1) (or the generator A) is strongly asymptotically stable if and only if for each $x \in X$

$$\|T(t)x\| \to 0 \quad \text{as} \quad t \to +\infty. \tag{2.29}$$

Definition 2.29 (Weak asymptotic stability)

Let A generate a C_0-semigroup $T(t), t \geq 0$. Then the system (2.1) (or the operator A) is weakly asymptotically stable if and only if for each x and y in X

$$(T(t)x, y) \to 0 \quad \text{as } t \to +\infty. \tag{2.30}$$

Recall that for a finite dimensional linear systems, all the definitions of asymptotic stability given above are equivalent. In infinite dimensional spaces, some of these definitions are equivalent, as will be seen in the following theorem, but others are not.

Theorem 2.30 [Dakto 38]

The following three notions of stability are equivalent:

a) $\|T(t)\| \leq Me^{-wt}$ for some $M \geq 1$ and $w > 0$. (exponential stability)

b) $\|T(t)\| \to 0$ as $t \to \infty$. (uniform stability)

c) $\int_0^\infty \|T(t)x\|^2 dt < \infty$ for each $x \in X$. (strong L_2-stability) ■

Theorem 2.29 holds also for general Banach spaces, and the condition c) can be replaced by the more general condition:

$$\int_0^\infty \alpha(\|T(t)x\|)dt < +\infty \quad \text{for } x \in X \tag{2.31}$$

where $\alpha(\cdot)$ is a strictly increasing, convex function with $\alpha(0) = 0$. This result was given by Zabczyk [117]. Recently, by an explicit constructive proof [Littman 89], Littman has shown that without the convexity condition on α (2.31) is still equivalent to a).

It is trivial to see from definitions 2.26, 2.28 and 2.29, that the uniform or exponential stability implies strong stability which in turn implies weak stability. But the converses are not true.

Example 2.31

Let $X = l^2(R)$. Consider $A : x = (x_n) \to (-x_n/n)$, then it generates the semigroup $T(t)x = (e^{-t/n}x_n), t \geq 0$ where $x = (x_n) = (x_1, x_2, \cdots, x_n, \cdots)$ and $\|x\| \equiv \|x\|_{l^2} = (\sum_{n\geq 1} |x_n|^2)^{1/2}$. It is clear that $\lim_{t\to+\infty} \|T(t)x\| = 0$ (strongly stable) but

$$\|T(t)\| = \sup_{\|x\|=1} \|T(t)x\| \geq \lim_{n\to\infty} e^{-t/n} = 1 \text{ for all } t \geq 0 \text{ (not exponentially stable).}$$

In fact, $\|T(t)\| = 1$ as $\|T(t)x\| \leq \|x\|$ for $x \in X = l^2(R)$.

Example 2.32

Let $X = L_2[0, \infty)$ and

$$T(t)z(x) = \begin{cases} z(x-t) & \text{if } x \geq t \\ 0 & \text{if } x < t. \end{cases}$$

Then for any $z, y \in X$,

$$\begin{aligned} |(T(t)z, y)| &= |\int_t^\infty z(x-t)y(x)dx| \\ &\leq (\int_t^\infty |z(x-t)|^2 dx)^{1/2} (\int_t^\infty |y(x)|^2 dx)^{1/2} \\ &= \|z\| (\int_t^\infty |y(x)|^2 dx)^{1/2} \to 0 \text{ as } t \to \infty, \end{aligned}$$

so it is weakly stable, but it is not strongly stable since

$$\begin{aligned}\|T(t)z\| &= (\int_t^\infty |z(x-t)|^2 dx)^{1/2}\\ &= (\int_0^\infty |z(x)|^2 dx)^{1/2}\\ &= \|z\|.\end{aligned}$$

Hence weak stability does not in general imply strong stability and strong stability does not imply exponential stability. In the light of the above discussion and the conclusion that $\int_0^\infty \|T(t)x\| dt < +\infty$ for all $x \in X$ implies exponential stability of $T(t)$, it is natural to ask whether the weaker condition $\int_0^\infty |(T(t)x, y)| dt < +\infty$ for all $x, y \in X$ implies exponential stability of $T(t)$. This was a open problem posed by Pritchard and Zabczyk [101] and affirmatively answered by Huang in [58] as discussed in the following theorem.

Theorem 2.33

Let $T(t)$ be a C_0-semigroup in a Hilbert space X, then $T(t)$ is exponentially stable if and only if

$$\int_0^\infty |(T(t)x, y)| dt < +\infty \quad \text{for all } x, y \in X. \tag{2.32}$$

■

So far we have discussed different types of stability and relations among them except for Lyapunov type of stability. In fact, the Lyapunov type of stability condition (2.25) in theorem 2.24 has its complete infinite dimensional counterpart. In the case of a Hilbert space X the identity (2.25) is equivalent to the equation

$$(Ax, Sx) + (Sx, Ax) = -(x, x) \quad \text{for } x \in D(A) \tag{2.33}$$

or simply $2\mathrm{Re}(Sx, Ax) = -\|x\|^2$. Therefore, there is yet another equivalent statement in theorem 2.30 [see Datko 38]:

d) there exists a nonnegative operator $0 \leq S \in \mathcal{L}(X)$ such that

$$2\mathrm{Re}(x, SAx) = -\|x\|^2 \quad \text{for } x \in D(A). \tag{2.34}$$

As mentioned earlier, however, the first part of theorem 2.24 is not true for infinite dimensional systems; that is, the condition (2.24) is no longer equivalent to the exponential stability of the infinite dimensional system (2.1).

Example 2.34

We refer to the example 4.4 of Pazy [96] where

$$D(A) = \{u : u \text{ is absolutely continuous, } u' \in X\} \text{ and } Au = u'$$

on $X = E \cap L^p(0,\infty)$ where E is the space of all measurable functions f on $[0,\infty)$ for which $|f|_1 = \int_0^\infty e^s|f(s)|ds < \infty$. X endowed with the norm $\|f\| = |f|_1 + \|f\|_{L^p}$ is easily seen to be a Banach space. Then $T(t)f(x) = f(x+t), t \geq 0$ is a C_0-semigroup generated by A and satisfies $\|T(t)\| = 1$, but $\sup\{\mathrm{Re}\,\lambda : \lambda \in \sigma(A)\} \leq -1$, clearly $\|T(t)\|$ does not decay exponentially.

From this example we conclude that in order to obtain exponential decay of $\|T(t)\|$ from the spectral condition $\sup\{\mathrm{Re}\lambda : \lambda \in \sigma(A)\} < 0$ one has to supplement it with some further conditions on $T(t)$ or A.

Theorem 2.35 [Hille and Phillips 54]

Let A be the infinitesimal generator of a C_0-semigroup $T(t)$ and let $\sigma(A)$ denote its spectrum. Let

$$w_s = \sup\{\mathrm{Re}\lambda : \lambda \in \sigma(A)\}, \tag{2.35}$$

$$w_0 = \inf\{w : \|T(t)\| \leq Me^{wt}, \text{ for some } M \geq 1 \text{ and all } t \geq 0\}. \tag{2.36}$$

Then we have

$$w_0 = \lim_{t\to\infty} \frac{\ln \|T(t)\|}{t} = \inf_{t>0} \frac{\ln \|T(t)\|}{t} < +\infty \tag{2.37}$$

and $w_s \leq w_0$. ■

The above theorem shows that the equality $w_0 = w_s$, which is true in finite dimensional spaces, does not hold in general for infinite dimensions. Consider the example 2.34 again, we know that in this example $w_s = \sup\{\text{Re}\lambda : \lambda \in \sigma(A)\} \leq -1$, but $w_0 = \lim_{t\to\infty} \frac{\ln \|T(t)\|}{t} = lim_{t\to\infty} \frac{\ln 1}{t} = 0$. Hence $w_s < w_0$. See Zabczyk [118] and Huang [58] for more counterexamples. In fact, even if the resolvent $(\lambda - A)^{-1}$ of A is compact, the equality $w_0 = w_s$ does not hold [Huang 58]. It naturally raises the following question: What are the conditions that the generator A must satisfy in order that the equality $w_0 = w_s$ holds? Some positive answers are available as given below:

(i) A generates an analytic semigroup $T(t)$ [Pazy 96].

(ii) A generates a differentiable semigroup $T(t)$[Triggiani 110].

(iii) A generates a compact semigroup $T(t)$ [Zabczyk 118].

Also Russell has proved [104] that for some hyperbolic systems the equality $w_0 = w_s$ holds. Thus in many important cases the knowledge of the spectrum of the infinitesimal generator $A(w_s < 0)$ allows us to predict the asymptotic behavior of the solution of (2.1). In fact, Huang [57] has shown that for norm continuous semigroups the equality $w_0 = w_s$ holds. As shown in (2.23) this class of C_0-semigroups properly contains the analytic, the differentiable and the compact semigroups.

Theorem 2.36

If A generates a norm continuous semigroup $T(t), t \geq 0$ in a Hilbert space X, then $w_0 = w_s$. Hence $T(t)$ is exponentially stable if and only if $w_s < 0$. (In this case $(w_0 = w_s)$ $T(t)$ is said to satisfy the spectrum growth property.) ■

Until now we concentrate on finding the conditions on $T(t)$ or A which satisfy the spectrum growth property. It has been shown that many important classes of semigroups satisfy this property. As the counterexample 2.34 shows, however, it is not true in general for infinite dimensional systems (2.1). Secondly, as a complete characterization of norm continuous or compact semigroups in terms of their generators is still unknown (see the remark 2.23). theorem 2.36 is not a complete answer for exponential stability of C_0-semigroups in Hilbert spaces in terms of the spectral property of its generator. Therefore, one has to search for some supplemental conditions on the generator A, with $w_s < 0$, which may not satisfy the spectrum growth property but still guarantee the exponential stability of the system (2.1). This is precisely stated in the following:

Theorem 2.37 [Huang 57]

A C_0-semigroup $T(t)$ generated by A in a Hilbert space X is exponentially stable if and only if $w_s < 0$ and there exists $w \in (w_s, 0)$ such that $\sup\{\|(\lambda - A)^{-1}\|; \mathrm{Re}\lambda \geq w\} < +\infty$. ■

In fact, based on theorem 2.37 we have the following general result.

Theorem 2.38

Let A generate a C_0-semigroup $T(t)$ on a Hilbert space X. Then the system (2.1) is exponentially stable if and only if $w_s < 0$ and $C = \sup\{\|(\lambda - A)^{-1}\|; \mathrm{Re}\lambda \geq 0\} <$

$+\infty$.

Proof

(only if): Since A generates an exponentially stable C_0 -semigroup $T(t), t \geq 0$, by theorem 2.37, $w_s < 0$ and there exists $w \in (w_s, 0)$ such that $\sup\{\|(\lambda - A)^{-1}\|;\ \mathrm{Re}\lambda \geq w\} < +\infty$. Furthermore, as $\{\|(\lambda I - A)^{-1}\|;\ \mathrm{Re}\lambda \geq 0\} \subset \{\|(\lambda I - A)^{-1}\|;\ \mathrm{Re}\lambda \geq w\}$, we have

$$C = \sup\{\|(\lambda I - A)^{-1}\|; \mathrm{Re}\lambda \geq 0\} \leq \sup\{\|(\lambda I - A)^{-1}\|;\ \mathrm{Re}\lambda \geq w\} < +\infty.$$

(if): Let $w = \max\{w_s/2, -1/2C\}$ and $\lambda = \sigma + i\nu, \sigma \in [w, 0]$. Then

$$\begin{aligned} \lambda I - A &= (\sigma I + i\nu - A) \\ &= [I + \sigma(i\nu - A)^{-1}](i\nu - A). \end{aligned}$$

Since

$$\begin{aligned} \|[I + \sigma(i\nu - A)^{-1}]x\| &\geq \|x\| - |\sigma|\|(i\nu - A)^{-1}x\| \\ &\geq \|x\| - \frac{1}{2C}\cdot C\|x\| = 1/2\|x\|, \end{aligned}$$

we have $\|[I+\sigma(i\nu - A)^{-1}]^{-1}\| \leq 2$. Thus for $\sigma \in [w, 0], -\infty < \nu < +\infty$, $\lambda = \sigma + i\nu \in \rho(A)$ and

$$\begin{aligned} \|(\lambda I - A)^{-1}\| &= \|(i\nu - A)^{-1}[I + \sigma(i\nu - A)^{-1}]^{-1}\| \\ &\leq \|(i\nu - A)^{-1}\|\|[I + \sigma(i\nu - A)^{-1}]^{-1}\| \\ &\leq 2C. \end{aligned}$$

Hence, $\sup\{\|(\lambda I - A)^{-1}\|; \mathrm{Re}\lambda \geq w\} < +\infty$ and by theorem 2.37, $T(t)$ is exponentially stable. ■

2.4 Conclusions

In this chapter, a brief review of the semigroup theory and some related fundamental generation theorems have been presented. In particular, we have considered some semigroups with special properties and characterized them in terms of their generators. Using the Lumer-Phillips theorem, we have derived a couple of general theorems which are easy to apply. Two interesting examples were given to illustrate them. The other issue considered in this chapter is the stability problem of linear systems on Hilbert spaces. Due to the infinite dimensionality of the state space, we have seen that there are many different notions of stability. Some of them are equivalent as shown in the theorems, but others are not. These problems do not exist in the case of finite dimensional linear systems where all of those stability notions are equivalent and stability depends on the spectrum of system operators only. However, in order to obtain the exponential stability of linear systems on infinite dimensional spaces from the spectrum condition, one has to supplement it with some further conditions on the system operator. Theorem 2.38 gave a general result for this problem.

CHAPTER 3

INVARIANCE OF CONTROLLABILITY UNDER BOUNDED OR UNBOUNDED PERTURBATIONS

3.1 Introduction, Definitions and Prelimilaries

In this chapter we consider the controllability problem for autonomous infinite dimensional systems described by the linear controlled system

$$\dot{x} = Ax + Bu, \quad x(0) = x_0, \tag{3.1}$$

and corresponding perturbed system

$$\dot{x} = (A + P)x + Bu, \quad x(0) = x_0 \tag{3.2}$$

where A is a linear operator on a complex Hilbert space X, Bu represents the control action and P is a class of bounded or unbounded perturbations of A.

The concept of controllability, introduced three decades ago by Kalman [63,64,65], plays a central role in modern control theory. They appear as sufficient and sometimes as necessary conditions for the existence of stabilizing controls for finite dimensional systems. For distributed parameter systems similar properties also exist and they will be discussed later. This is one of the resons why we consider the controllability problem in this chapter before studying the stabilizability problem. Essentially, in finite dimensional space, the controlled system is said to be controllable if every desired transition of the plant state can be effected in finite time by some control inputs. It is well known [Silverman and Anderson 107] that the controllability of a linear system may be fully determined from the system solution. However, the solution to such a system is not always available in closed form. Therefore, various criteria to characterize the controllability in terms of the known system operators (coefficient matrices in finite dimensional systems) are developed and controllability theory for linear systems in finite dimensional spaces is well established. One of the important results in this development is that the controllability property is invariant under certain class of perturbations; that is, the perturbed system $(A + P, B)$, where $P = BQ$, is controllable if and only if (A, B) is controllable. For infinite dimensional systems, however, the situation becomes much more complex. First, there are many non-equivalent definitions of controllability for infinite dimensional systems. Among them, the principal controllability notions, which have been extensively used in the literature, are exact controllability, which is the natural extension of the finite dimensional concept of controllability to infinite dimensions but is too strong in general; and approximate controllability, which turns out to be more useful in applications. These concepts are equivalent for finite dimensional systems. In this chapter we will consider the questions of approximate controllability (simply called controllability), which is useful in the later chapters when we deal with stability problems. Secondly, in the infinite dimensional case, the perturbation operator P may be bounded or unbounded. In case it is unbounded, which occurs in many problems, $A + P$ may not even generate a C_0-semigroup.

Before proceeding to our controllability results, we introduce some important notations and definitions.

For any pair of complex Hilbert spaces Y and Z, $\mathcal{L}(Y, Z)(\mathcal{L}_u(Y, Z))$ denotes the space of bounded (not necessarily bounded) linear operators from Y to Z. If Y coincides with Z set $\mathcal{L}(Y, Y) \equiv \mathcal{L}(Y)(\mathcal{L}_u(Y, Y) \equiv \mathcal{L}_u(Y))$. For any $K \in \mathcal{L}_u(Y, Z), K^* \in \mathcal{L}_u(Z, Y)$ denotes the adjoint of the operator K whenever it is defined.

Let X and U be any pair of Hilbert spaces representing respectively the state space and the control space for the systems (3.1) and (3.2). Let $Z(t) = L_2[(0,t);U]$ be the space of strongly measurable U-valued functions $u(\cdot)$ such that $\int_0^t \|u(s)\|^2 ds < \infty$ and define $Z = \cup_{t>0} Z(t)$. Throughout we assume that A is a closed densely defined linear operator in $\mathcal{L}_u(X)$ satisfying the Hille-Yosida conditions so that it is the infinitesimal generator of a C_0-semigroup $T(t), t \geq 0$, of <u>bounded</u> linear operators in X (i.e. $T(t) \in \mathcal{L}(X)$ for $t \geq 0$). With these notations we can state the following basic definitions for the solution of system (3.1):

<u>Definition 3.1</u>

A function $x : [0,t) \to X$ is a <u>classical solution</u> of system (3.1) on $[0,t)$ if x is continuous on $[0,t)$, continuously differentiable on $(0,t), x(s) \in D(A)$ for $0 < s < t$ and (3.1) is satisfied on $[0,t)$ for $x(0) = x_0 \in D(A), Bu \in C^1[(0,t);X]$.

<u>Definition 3.2</u>

A function x which is differentiable almost everywhere on $[0,t]$ such that $\dot{x} \in L_1[(0,t);X]$ is called a <u>strong solution</u> of the problem (3.1) if $x(0) = x_0 \in D(A)$ and $\dot{x}(t) = Ax(t) + Bu$ *a.e.* on $[0,t]$, where $Bu \in L_1[(0,t);X]$ and $\int_0^t T(t-s)Bu(s)ds \in D(A)$.

It is obvious that both classical solutions and strong solutions require rather strong conditions to be satisfied. Thus, we need a weaker one in later applications.

<u>Definition 3.3</u>

Consider the system (3.1), where A generates a C_0 -semigroup $T(t)$ over a Hilbert space X and B is a linear bounded operator from H to X. If $u \in L_2[(0,t);H]$, we say that

$$x(t) = T(t)x_0 + \int_0^t T(t-s)Bu(s)ds \tag{3.3}$$

is a mild solution of (3.1) on $[0, t]$.

With the help of the definition of a mild solution we can introduce the following controllability concepts.

Definition 3.4

Consider the system (3.1) where A generates a C_0- semigroup $T(t)$ over a Hilbert space X and B is a linear bounded operator mapping from U to X. The system (3.1) is exactly controllable if for any x and y in X, there exists a time t and a control $u(\cdot)$ in Z such that

$$y = T(t)x + \int_0^t T(t-s)Bu(s)ds. \tag{3.4}$$

Definition 3.5

Consider the system (3.1) where A generates a C_0-semigroup $T(t)$ over a Hilbert space X and $B \in \mathcal{L}(U, X)$. The system (3.1) is (approximately) controllable if for any $\xi \in X$, and any $\varepsilon > 0$, there exist a $t > 0$ and $u(\cdot)$ in Z such that

$$\|\xi - \int_0^t T(t-s)Bu(s)ds\| < \varepsilon. \tag{3.5}$$

As mentioned before most systems of interest are not exactly controllable. For example, it happens when the semigroup $T(t)$ or the operator B is compact (integral operators, finite rank operators, etc.) [Benchimol 22]. We will, therefore, concentrate on studying approximate controllability(simply called controllability) in our thesis. In the study of controllability, as we will see, the adjoint operator plays a crucial role. Indeed it is known [Ahmed and Teo 10] that the system (3.1) is (approximately) controllable if and only if, for $\xi \in X$, $B^*T^*(t)\xi = 0$ for all $t \geq 0$, implies $\xi = 0$. Thus we may introduce the following definition for controllability.

Definition 3.6

The pair (A, B) or system (3.1) is said to be controllable if for $x \in X$, $B^*T^*(t)x = 0$ for all $t \geq 0$ implies $x = 0$.

Remark 3.7

Definition 3.5 and 3.6 are equivalent, see [Curtain and Pritchard 35] for detail.

To prove our controllability results we present the following perturbation theorem.

Theorem 3.8

Let A be the generator of a C_0-semigroup of contractions in X and suppose $P \in \mathcal{L}_u(X)$, such that $D(P) \subset D(A)$, $A+P$ is dissipative and $\|Px\|_X \leq \alpha\|Ax\|_X + \beta\|x\|_X$ for $0 \leq \alpha < 1, \beta \geq 0$ and all $x \in D(A)$. Then $A + P$ generates a C_0- semigroup of contractions $S(t)$, $t \geq 0$, in X.

Proof

Since A is the generator of a C_0-semigroup of contractions in X, it is m- dissipative. Furthermore, by assumption $A + P$ is dissipative and therefore for every $x \in D(A)$,

$$\mathrm{Re}(t(A+P)x,\ x) \leq 0 \quad \text{for} \quad 0 \leq t \leq 1$$

and

$$(1-t)\mathrm{Re}(Ax,\ x) \leq 0 \quad \text{for} \quad 0 \leq t \leq 1.$$

Hence by adding the two we have, for every $x \in D(A)$, $\mathrm{Re}((A+tP)x,\ x) \leq 0$ for $0 \leq t \leq 1$, that is, $A + tP$ is dissipative for all $t \in [0,\ 1]$. Since A is m-dissipative and

P is A-bounded with A-bound $\alpha \in [0,\ 1)$ it follows from theorem 3.2 [Pazy 96] that $A + tP$ is m-dissipative for all $t \in [0,\ 1]$ and hence $A + P$ is m-dissipative. Then it follows from Lumer-Phillips' theorem that $A + P$ is the infinitesimal generator of a C_0-semigroup of contractions $S(t)$, $t \geq 0$, in X. ∎

Remark 3.9

It is clear that the perturbation P is a bounded operator if $\alpha = 0$ and relatively bounded with respect to A if $\alpha > 0$. In the existing literatures, it is usually assumed that P itself is dissipative (see [Davies 39, Pazy 96]). Theorem 3.8 indicates that the conclusion remains true if we merely assume that $A + P$ is dissipative.

3.2 Invariance of Controllability under Bounded Perturbations

First we consider the controllability problem for the nominal system (3.1). Recall that, since A is the infinitesimal generator of a strongly continuous semigroup $T(t), \sigma(A)$, the spectrum of A, is contained in the halfplane $\mathrm{Re}\lambda < \omega_0$, where

$$\omega_0 = \lim_{t \to \infty} t^{-1} \ln \|T(t)\| = \inf_{t>0}(t^{-1} \ln \|T(t)\|) < +\infty. \tag{3.6}$$

Let $\rho(A)$ (the resolvent set of the operator A) be the complement of $\sigma(A)$ that contains the halfplane $\mathrm{Re}\lambda > \omega_0$ and $R(\lambda, A) \equiv (\lambda I - A)^{-1}$ the resolvent of A corresponding to $\lambda \in \rho(A)$.

Theorem 3.10

Let A be the generator of a C_0-semigroup $T(t)$, $t \geq 0$, in X and $B \in \mathcal{L}(U,\ X)$, then the pair $(A,\ B)$ is controllable if and only if, for some $\lambda_1 \in \rho(A)$, $B^* R^*(\lambda, A)x = 0$ for all $\mathrm{Re}\lambda \geq \lambda_1$ implies $x = 0$.

Proof

In view of definition 3.6 it suffices to prove that for any $x \in X$, $B^*T^*(t)x = 0$ for $t \geq 0$ is equivalent to $B^*R^*(\lambda, A)x = 0$ for all $\mathrm{Re}\lambda \geq \lambda_1$ for some $\lambda_1 \in \rho(A)$. Clearly if $B^*T^*(t)x = 0$ for all $t \geq 0$ then

$$B^*R^*(\lambda, A)x = \int_0^\infty e^{-\lambda t} B^*T^*(t)x dt = 0$$

for all $\lambda \in \rho(A)$. For the converse suppose $B^*R^*(\lambda, A)x = 0$ for all $\mathrm{Re}\lambda \geq \lambda_1$ for some $\lambda_1 \in \rho(A)$. Then for all $\eta \in U$

$$(B^*R^*(\lambda, A)x,\ \eta) = \int_0^\infty e^{-\lambda t}(B^*T^*(t)x,\ \eta)dt = 0$$

for all $\mathrm{Re}\lambda \geq \lambda_1$ or equivalently,

$$(B^*R^*(\mu + \lambda_1, A)x, \eta) = \int_0^\infty e^{-\mu t}(e^{-\lambda_1 t}B^*T^*(t)x,\ \eta)dt = 0$$

for all $\mathrm{Re}\mu \geq 0$. This is possible only if $e^{-\lambda_1 t}(B^*T^*(t)x,\ \eta) = 0$ for all $t \geq 0$ [Dunford and Schwartz 41]. Since $\eta \in U$ is arbitrary this implies that $B^*T^*(t)x = 0$ for all $t \geq 0$. This equivalence proves that the pair $(A,\ B)$ is controllable if and only if $B^*R^*(\lambda, A)x = 0$ for all $\mathrm{Re}\lambda \geq \lambda_1$ for some (or any) $\lambda_1 \in \rho(A)$ implies $x = 0$. ■

Lemma 3.11

Let A be the generator of a C_0-semigroup $T(t)$ and $B \in \mathcal{L}(U, X)$, then the following two statements are equivalent

$$B^*[R^*(\lambda_0, A)]^n x = 0, \quad n = 1, 2, \cdots, \text{ implies } x = 0 \text{ for } \lambda_0 \text{ where } \mathrm{Re}\lambda_0 > \omega_0,$$

if and only if

$$B^*R^*(\lambda, A)x = 0 \text{ implies } x = 0 \text{ for all } \mathrm{Re}\lambda > \omega_0.$$

Proof

(if): Suppose $B^*R^*(\lambda, A)x = 0$ implies $x = 0$ for all λ with $\mathrm{Re}\lambda > \omega_0$. Now assume that $B^*[R^*(\lambda_0, A)]^n x = 0, \quad n = 1, 2, \cdots$ for some $x \neq 0$. Since for every $\lambda \in \rho(A)$, $\lambda \to R(\lambda, A)$ is holomorphic and $\frac{d^n R(\lambda, A)}{d\lambda^n} = (-1)^n n! R(\lambda, A)^{n+1}$ [Pazy 96], we have

$$R^*(\lambda, A)x = \sum_{n=0}^{\infty} (\lambda_0 - \lambda)^n [R^*(\lambda_0, A)]^{n+1} x$$

for suitably small ε such that $|\lambda - \lambda_0| < \varepsilon (\varepsilon > 0)$ and $\mathrm{Re}\lambda > \omega_0$ (Recall that $\mathrm{Re}\lambda_0 > \omega_0$). Therefore

$$\begin{aligned} B^*R^*(\lambda, A)x &= \sum_{n=0}^{\infty} (\lambda_0 - \lambda)^n B^*[R^*(\lambda_0, A)]^{n+1} x \\ &= 0 \quad \text{for } |\lambda - \lambda_0| < \varepsilon, \end{aligned}$$

and then by analytic continuation for all λ with $\mathrm{Re}\lambda > \omega_0$. Thus,

$$B^*R^*(\lambda, A)x = 0,$$

and this implies $x = 0$ which contradicts the assumption.

(only if): suppose $B^*R^*(\lambda, A)x = 0$, for all λ with $\mathrm{Re}\lambda > \omega_0$, then

$$\begin{aligned} B^*R^*(\lambda, A)x &= \sum_{n=0}^{\infty} (\lambda_0 - \lambda)^n B^*[R^*(\lambda_0, A)]^{n+1} x \\ &= 0 \quad \text{for all } \lambda \text{ such that } |\lambda - \lambda_0| < \varepsilon \text{ and } \mathrm{Re}\lambda > \omega_0. \end{aligned}$$

It implies that, for some λ_0 where $\mathrm{Re}\lambda_0 > w_0$, $B^*[R^*(\lambda_0, A)]^n x = 0, n = 1, 2, \cdots$, therefore $x = 0$, which proves the result. ■

Using lemma 3.11, we have another result on the controllability of the system (3.1).

Theorem 3.12

Let A be the generator of a C_0-semigroup $T(t), t \geq 0$, in X and $B \in \mathcal{L}(U, X)$, then the system (3.1) or the pair (A, B) is controllable if and only if, for some λ_0 with $\mathrm{Re}\lambda_0 > \omega_0$, $B^*[R^*(\lambda_0, A)]^n x = 0, n = 1, 2, \cdots$, implies $x = 0$.

Proof

It follows directly from the results of theorem 3.10 and lemma 3.11. ∎

We now apply some of these results to consider the controllability problems of the uncertain (perturbed) system (3.2) under the assumption of bounded perturbations. To prove our results we make use of the following definition on the class of uncertainties or perturbations of the generator A:

Definition 3.13

Let A generate a C_0-semigroup of contractions $T(t), t \geq 0$, in X and $B \in \mathcal{L}(U, X)$, then $\mathcal{P}_b(B)$ denotes the class of bounded operators P satisfying the conditions

(i) $P = BQ$, for $Q \in \mathcal{L}(X, U)$,

(ii) $A + P$ is dissipative.

It should be noted that $\mathcal{P}_b(B)$ is not empty set (see the examples 3.22 and 4.32).

Theorem 3.14

Let A be the generator of a C_0-semigroup of contractions in X, then for each $P \in \mathcal{P}_b(B)$ (where $\mathcal{P}_b(B)$ is as in definition 3.13), the pair $(A + P,\ B)$ is controllable if and only if the pair $(A,\ B)$ is controllable.

Proof

By virtue of theorem 3.10 it suffices to prove that for any real number $\lambda_1 \in \rho(A+P)$, sufficiently large, $B^*R^*(\lambda, A+P)x = 0$ for all $\mathrm{Re}\lambda \geq \lambda_1$ implies $x = 0$. For $\lambda \in \rho(A)$ (therefore, $\bar{\lambda} \in \rho(A)$ [Pazy 96]), we have [Dunford and Schwartz 41],

$$(\bar{\lambda}I - A^* - P^*) = (I - P^*R(\bar{\lambda}, A^*))(\bar{\lambda}I - A^*)$$

$$= (I - P^*R^*(\lambda, A))(\bar{\lambda}I - A^*).$$

Since A is the generator of a C_0-semigroup of contractions $\|R(\lambda, A)\| \leq \frac{1}{\mathrm{Re}\lambda}$, $\mathrm{Re}\lambda > 0$, and by definition 3.13, $\|P\| = \|P^*\| \leq \beta$ [Pazy 96]. Therefore choosing $\lambda_1 \in \rho(A)$ sufficiently large so that $\frac{\beta}{\mathrm{Re}\lambda} < 1$ for all $\mathrm{Re}\lambda \geq \lambda_1$ we have,

$$\|P^*R^*(\lambda, A)\| \leq \|P^*\|\|R^*(\lambda, A)\| = \|P\|\|R(\lambda, A)\| < 1 \quad \text{for all } \mathrm{Re}\lambda \geq \lambda_1.$$

Hence for $\mathrm{Re}\lambda \geq \lambda_1$, $(I - P^*R^*(\lambda, A))$ is invertible and we have,

$$R(\bar{\lambda}, A^* + P^*) = R^*(\lambda, A+P) = R^*(\lambda, A)(I - P^*R^*(\lambda, A))^{-1} \quad \text{for } \mathrm{Re}\lambda \geq \lambda_1.$$

Suppose now that $B^*R^*(\lambda, A+P)x = 0$ for all $\mathrm{Re}\lambda \geq \lambda_1 > 0$. Then

$$B^*R^*(\lambda, A)\eta_\lambda = 0 \quad \text{for all } \mathrm{Re}\lambda \geq \lambda_1 > 0,$$

where $\eta_\lambda = (I - P^*R^*(\lambda, A))^{-1}x$.

On the other hand, we have

$$x = (I - P^*R^*(\lambda, A))\eta_\lambda$$

$$= \eta_\lambda - Q^*B^*R^*(\lambda, A)\eta_\lambda$$

$$= \eta_\lambda, \quad \mathrm{Re}\lambda \geq \lambda_1 > 0. \text{ (Because } B^*R^*(\lambda, A)\eta_\lambda = 0.)$$

Hence, $B^*R^*(\lambda, A)\eta_\lambda = B^*R^*(\lambda, A)x = 0$ for all $\mathrm{Re}\lambda \geq \lambda_1$. Since the pair (A, B) is controllable, this implies that $x = 0$.

For the converse, writing $\tilde{A} \equiv (A + P)$ and then following the same procedure one can prove that $(A, B) \equiv (\tilde{A} - P, B)$ is controllable. This completes the proof. ■

The above result shows that the controllability of the perturbed system (3.2) can be obtained without the knowledge of the solution, provided the system operator A is the generator of a C_0- semigroup of contractions and the perturbations satisfy the so called matching condition [Barmish, Corless and Leitmann 20, Petersen 98] and dissipative property. In fact, the conclusion of controllability invariance under certain class of bounded perturbations also holds for generators of general C_0-semigroups (not necessarily C_0-semigroup of contractions) and the dissipativity in the definition 3.13, therefore, is not essential. We give a more general result, based on definition 3.6 and the variation-of-constants formula (the Volterra integral equation), in the following theorem which extends well known finite dimensional controllability property [Ahmed 2, Silverman and Anderson 107] to infinite dimension.

Theorem 3.15

Let A be the generator of a C_0-semigroup in X and $B \in \mathcal{L}(U, X)$ and $Q \in \mathcal{L}(X, U)$, then

(i) For $\lambda \in R$, $(A + \lambda I, B)$ is controllable if and only if (A, B) is controllable;

(ii) $(A + BQ, B) \equiv (A + P, B)$ is controllable if and only if (A, B) is controllable.

Proof

(i) Suppose A generates a C_0-semigroup $T(t), t \geq 0$, in X, then $A + \lambda I$ generates a C_0-semigroup $T_\lambda(t) = T(t)e^{\lambda t}, t \geq 0$. By definition 3.6, it suffices to show that $B^*T^*(t)x = 0$ is equivalent to $B^*T_\lambda^*(t) = 0$ for $t \geq 0$ and $x \in X$. This is obvious since $e^{\lambda t} \neq 0$.

(ii) Since $Q \in \mathcal{L}(X,U)$ and X is a Hilbert space, it is known that $A + BQ$, A^* and $(A + P)^* = A^* + P^*$ [Dunford and Schwartz 41] are generators of C_0-semigroups $S(t), T^*(t)$ and $S^*(t)$ respectively. By using the variation-of-constants formula and considering the adjoints, we have the Volterra integral equations for $S^*(t)x$ and $T^*(t)x, x \in X$,

$$S^*(t)x = T^*(t)x + \int_0^t T^*(t-\tau)Q^*B^*S^*(\tau)x d\tau, \quad t \geq 0, \tag{3.7}$$

and

$$T^*(t)x = S^*(t)x - \int_0^t S^*(t-\tau)Q^*B^*T^*(\tau)x d\tau, \quad t \geq 0, \tag{3.8}$$

where $P^* = Q^*B^*$.

The result now follows from the identities (3.7), (3.8) and the fact that a linear homogeneous Volterra integral equation has a unique solution which is identically zero (see Ahmed [3], Levan and Rigby [77]). ■

3.3 Invariance of Controllability under Unbounded Perturbations

In the last section we discussed the controllability problem for the nominal system (3.1) and the perturbed system (3.2). It is shown that the controllability property is preserved, as in the finite dimensional systems, under a broad class of bounded perturbations. In this section, we shall consider this invariance property for the system (3.2) with unbounded perturbations. In fact, when $P \equiv BQ$ is an unbounded perturbation, $A+P$ and even its closure may fail to be a generator of a C_0-semigroup. Even if it does, the variation-of-constants formula (the integral equation) may not hold for every $x \in X$ [Davies 39]. However, the result of Theorem 3.15, for bounded perturbations, may be extended to a certain class of unbounded perturbations for which we give the following definition:

Definition 3.16

Let A generate a C_0-semigroup $T(t), t \geq 0$, in X and let $\mathcal{P}_c(B)$ denote the class of closed operators $P \equiv BQ$, for $Q \in \mathcal{L}_u(X, U)$ having well defined adjoints $Q^* \in \mathcal{L}_u(U, X)$, satisfying the conditions

(i) $D(P) \supset D(A)$,

(ii) For each $t > 0$, there exists a finite constant K_t such that $\|PT(t)x\| \leq K_t\|x\|$ for $x \in D(A)$ and $\int_0^1 K_t dt < \infty$.

It should be noted that $\mathcal{P}_c(B) \neq \emptyset$ (see Davies [39], Dunford and Schwartz [41]).

Lemma 3.17 [Dunford and Schwartz 41]

If $P \in \mathcal{P}_c(B)$ then the operator $A + P$ defined on $D(A)$ is closed and is the infinitesimal generator of a C_0-semigroup $S(t)$. Further, $D(P) \supset \bigcup_{t>0} T(t)X$ and

$$S(t) = \sum_{n=0}^{\infty} S_n(t), \quad t \geq 0,$$

where $S_0(t) = T(t)$ and $S_n(t)x = \int_0^t T(t-s)PS_{n-1}(s)x ds$ for $x \in X$ and $n \geq 1$, the series being absolutely convergent, uniformly on finite time intervals. ■

Theorem 3.18

Let $T(t), t \geq 0$, be a C_0-semigroup in X with infinitesimal generator A and $P \in \mathcal{P}_c(B)$, then $(A + P, B)$ is controllable if and only if (A, B) is controllable.

Proof

Let $T(t), t \geq 0$ and $S(t), t \geq 0$, denote the semigroups generated by A and $A + P$ respectively. By Lemma 3.17, for each $x \in X$, we have the integral equation

$$S(t)x = T(t)x + \int_0^t T(t-\tau)PS(\tau)x d\tau, t \geq 0, \tag{3.9}$$

for $S(t)x, t \geq 0$ and, due to the fact that $A = (A + P) - P$, we also have the integral equation

$$T(t)x = S(t)x - \int_0^t S(t-\tau)PT(\tau)xd\tau, t \geq 0, \tag{3.10}$$

for $T(t)x, t \geq 0$. [Davies 39]. Recalling that $P = BQ$, for some $Q \in \mathcal{L}_u(X, U)$, the adjoint equation corresponding to (3.9) is given by

$$S^*(t)x = T^*(t)x + \int_0^t (T(t-\tau)BQS(\tau))^* x d\tau, \quad t \geq 0.$$

Since $T(t)B \in \mathcal{L}(U, X), t \geq 0$, we have

$$S^*(t)x = T^*(t)x + \int_0^t (QS(t-\tau))^* B^* T^*(\tau) x d\tau, \quad t \geq 0. \tag{3.11}$$

Similarly from (3.10), we obtain

$$T^*(t)x = S^*(t)x - \int_0^t (QT(t-\tau))^* B^* S^*(\tau) x d\tau, \quad t \geq 0. \tag{3.12}$$

The assertion of the theorem now follows from the equation (3.11), (3.12) and the same arguments as given in Theorem 3.15. ■

Although the operators in $\mathcal{P}_c(B)$ extend bounded perturbations to a certain class of unbounded perturbations (see [Davies 39, Dunford and Schwartz 41]), its significance, however, is limited by the fact that if A generates a C_0-group, $T(t)^{-1} \in \mathcal{L}(X)$ thus $T(t)X = X$ for all $t > 0$. In this case, by Lemma 3.17, $\mathcal{P}_c(B) \subset \mathcal{L}(X)$ thereby limiting the class of admissible perturbations. Consider now a more general class of unbounded perturbations, say $P \in \mathcal{P}_u(B)$ as given in the following:

<u>Definition 3.19</u>

Let A generate a C_0-semigroup of contractions $T(t), t \geq 0$, in X and let $\mathcal{P}_u(B)$ denote the class of operators $P \equiv BQ$, for $Q \in \mathcal{L}_u(X, U)$ having well defined adjoints $Q^* \in \mathcal{L}_u(U, X)$, satisfying the conditions

(i) $D(P) \supset D(A)$.

(ii) $A + P$ is dissipative and P is relatively bounded with respect to A; that is, there exist $0 \leq \alpha < 1$ and $\beta \geq 0$ such that

$$\|Px\|_X \leq \alpha\|Ax\|_X + \beta\|x\|_X \quad \text{for all } \ x \in D(A),$$

and

(iii) its dual P^* is uniquely defined and also relatively bounded with respect to A^* satisfying

$$\|P^*\xi\| \leq \alpha'\|A^*\xi\| + \beta'\|\xi\| \quad \text{for all } \ \xi \in D(A^*)$$

with relative bounds $0 \leq \alpha' < 1$, and $\beta' \geq 0$.

Remark 3.20

Note that without loss of generality we may take $\alpha' = \alpha, \beta' = \beta$ since, otherwise, we can take $\{a, b\}$ for the relative bounds with $a \equiv \max\{\alpha, \alpha'\}$ and $b \equiv \max\{\beta, \beta'\}$. Hence from now on we assume a common relative bound $\{\alpha, \beta\}$. Also, it should be noted that $\mathcal{P}_u(B) \neq \emptyset$ (see the example 4.31).

In fact, when $P \in \mathcal{P}_c(B)$ it has relative bound zero with respect to A [Davies 39]. However this does not necessarily imply that $\mathcal{P}_c(B) \subset \mathcal{L}(X)$.

As mentioned before, the variation-of-constants formula cannot be applied to general unbounded perturbations. Furthermore, in actual applications, it is the operators A and P which are given and not their semigroups $T(t)$ and $S(t)$. Therefore it is essential to consider controllability problems using the properties of the operators and their resolvents rather than their semigroups.

Theorem 3.21

Suppose A generates a C_0-semigroup of contractions $T(t), t \geq 0$, in X, the class $\mathcal{P}_u(B)$ satisfies the definition 3.19, and there exists a $\lambda_0 \in \rho(A)$ such that

$$\alpha\|A^* R^*(\lambda, A)\| + \beta\|R^*(\lambda, A)\| < 1 \quad \text{for all } \mathrm{Re}\lambda \geq \lambda_0. \tag{3.13}$$

Then, for each $P \in \mathcal{P}_u(B)$, the pair $(A + P, B)$ is controllable if and only if (A, B) is controllable.

Proof

Since each $P \in \mathcal{P}_u(B)$ satisfies the hypotheses of Theorem 3.8, $A + P$ is the infinitesimal generator of a C_0-semigroup of contractions $S(t), t \geq 0$, in X. Assuming that (A, B) is controllable we must prove that $(A + P, B)$ is controllable. Due to theorem 3.10, however, it suffices to prove that

$$B^* R^*(\lambda, A + P)x = 0, \quad \text{for } \mathrm{Re}\lambda > \lambda_0, \quad \text{implies} \quad x = 0.$$

Since the relative boundedness of P^* with respect to A^* with relative bounds $\{\alpha, \beta\}$ satisfies inequality (3.13),

$$\|P^* R^*(\lambda, A)\| \leq \alpha\|A^* R^*(\lambda, A)\| + \beta\|R^*(\lambda, A)\| < 1 \quad \text{for all } \mathrm{Re}\lambda \geq \lambda_0.$$

Hence $I - P^* R^*(\lambda, A)$ is invertible for $\mathrm{Re}\lambda > \lambda_0$ and we have

$$R^*(\lambda, A + P) = R(\bar{\lambda}, (A + P)^*) = R^*(\lambda, A)(I - P^* R^*(\lambda, A))^{-1}, \quad \text{for } \mathrm{Re}\lambda > \lambda_0 > 0.$$

Thus if $B^* R^*(\lambda, A + P)x = 0$, for $\mathrm{Re}\lambda > \lambda_0$, then

$$B^* R^*(\lambda, A)\eta_\lambda = 0, \quad \text{for } \mathrm{Re}\lambda > \lambda_0,$$

where $\eta_\lambda = (I - P^* R^*(\lambda, A))^{-1}x$. Since $P \in \mathcal{P}_u(B)$, there exists a $Q \in \mathcal{L}_u(X, U)$ so that $P = BQ$. Therefore,

$$
\begin{aligned}
x &= (I - P^* R^*(\lambda, A))\eta_\lambda \\
&= \eta_\lambda - Q^* B^* R^*(\lambda, A)\eta_\lambda \\
&= \eta_\lambda, \quad \text{for } \mathrm{Re}\lambda > \lambda_0.
\end{aligned}
$$

Hence, $B^* R^*(\lambda, A)\eta_\lambda = B^* R^*(\lambda, A)x = 0$ for all $\mathrm{Re}\lambda > \lambda_0 > 0$. Since the pair (A, B) is controllable, this implies that $x = 0$.

For the necessary condition, similar to the proof of theorem 3.14, one can prove that (A, B) is controllable whenever $(A + P, B)$ is controllable. This completes the proof. ■

At first sight the inequality (3.13) may appear to be too restrictive. Actually in the case of reflexive Banach spaces (therefore, Hilbert spaces) this inequality is always satisfied if A is m-dissipative (hence A^* is m-dissipative) and $0 \le \alpha < \frac{1}{2}$, since in this situation we can always find a $\gamma \in \rho(A)$ such that

$$
\begin{aligned}
\alpha\|A^* R^*(\lambda, A)\| + \beta\|R^*(\lambda, A)\| &= \alpha\|\lambda R^*(\lambda, A) - I\| + \beta\|R^*(\lambda, A)\| \\
&\le (2\alpha + \frac{\beta}{\lambda}) < 1 \quad \text{for all } \mathrm{Re}\lambda \ge \gamma.
\end{aligned}
\tag{3.14}
$$

Thus the result of theorem 3.21 remains valid without reference to the inequality (3.13) provided $0 \le \alpha < \frac{1}{2}$.

We give the following one dimensional hyperbolic equation to illustrate our results.

<u>Example 3.22</u>

Consider the controlled wave equation

$$\begin{cases} \dfrac{\partial^2 y}{\partial t^2} = \dfrac{\partial^2 y}{\partial x^2} + a_1(x)\dfrac{\partial y}{\partial x} + a_2(x)\dfrac{\partial y}{\partial t} + u, \quad 0 < x < 1 \\ y(0,t) = y(1,t) = 0 \\ y(x,0) = y_0(x) \\ y_t(x,0) = y_1(x) \end{cases} \tag{3.15}$$

on a real Hilbert space $H = L_2(0,1)$. Defining

$$A = -\frac{\partial^2}{\partial x^2}, \; P_1 = a_1(x)\frac{\partial}{\partial x}, \; P_2 = a_2(x)$$

and $D(A) = H^2(0,1) \cap H_0^1(0,1)$, $D(A^{1/2}) = H_0^1(0,1)$ where $a_1(x), a_2(x) \in L_\infty(0,1)$.

Then the equation (3.15) can be rewritten as a controlled first order differential equation

$$\dot{z} = (\mathcal{A} + P)z + Bu \tag{3.16}$$

on the Hilbert space $\mathcal{H} = D(A^{1/2}) \times H$ where $\mathcal{A}$ is defined by

$$\mathcal{A}z = \begin{pmatrix} 0 & I \\ -A & 0 \end{pmatrix} z, \; z \equiv \begin{pmatrix} y \\ w \end{pmatrix} = \begin{pmatrix} y \\ \dot{y} \end{pmatrix}$$

with $D(\mathcal{A}) = D(-A) \times D(A^{1/2})$ and $Pz = \begin{pmatrix} 0 & 0 \\ P_1 & P_2 \end{pmatrix} z$, $B = \begin{pmatrix} 0 \\ I \end{pmatrix}$.

It is known from example 2.10 that $\mathcal{A}$ generates a C_0 -semigroup of contractions on $\mathcal{H}$. Curtain and Pritchard [35] have shown that $(\mathcal{A}, B)$ is controllable (in fact, exactly controllable). We now apply our controllability invariance theorem to verify that the perturbed system $(\mathcal{A} + P, B)$ is controllable instead of going through the long and bitter computation of the solution of (3.16).

Since $D(P_1) = D(A^{1/2})$ and $D(P_2) = H$, P is a bounded operator on $\mathcal{H}$; i.e., $\|Pz\|_{\mathcal{H}} \le \beta \|z\|_{\mathcal{H}}$ for some $\beta > 0$. On the other hand, $P = BQ$ where $Q = (P_1, P_2)$ and it is easy to show that $\mathcal{A} + P$ is dissipative. That is, P satisfies the conditions of theorem 3.14. Therefore, the perturbed system (3.16) is controllable. ■

As mentioned earlier, the adjoint operators are very important in the study of controllability and stabilizability. In fact, one will see that controllability properties of the system (3.1) (or the pair (A, B)) and its adjoint system (A^*, B) play an equally important role in the stabilizability problem on Hilbert space. Hence we should consider the controllability invariance of the adjoint system under bounded or unbounded perturbations. We say that the pair (A^*, B) is controllable, according to definition 3.6, if for $x \in X, B^*T(t)x = 0$ for all $t \geq 0$ implies $x = 0$. Therefore, by theorem 3.10, (A^*, B) is controllable if and only if $B^*R(\lambda, A)x = 0$ for all λ with $\mathrm{Re}\lambda > \lambda_0$ implies $x = 0$, where $\lambda_0 \in \rho(A)$.

Definition 3.23

Let A generate a C_0-semigroup of contractions $T(t), t \geq 0$, in X and let $\mathcal{P}_u(B^*)$ denote the class of operators P such that, $P^* = BQ$ for $Q \in \mathcal{L}_u(X, U)$ having well defined adjoints $Q^* \in \mathcal{L}_u(U, X)$, satisfying the conditions

(i) $D(P) \supset D(A)$;

(ii) $A + P$ is dissipative and P is relatively bounded with respect to A; that is, there exist $0 \leq \alpha < 1$ and $\beta \geq 0$ such that

$$\|Px\|_X \leq \alpha\|Ax\|_X + \beta\|x\|_X \quad \text{for all } x \in D(A).$$

Again, it should be noted that $\mathcal{P}_u(B^*) \neq \emptyset$ (see the example 4.31).

Theorem 3.24

Suppose A generates a C_0-semigroup of contractions $T(t), t \geq 0$, in X and the class $\mathcal{P}_u(B^*)$ satisfies definition 3.23 and there exists a $\lambda_0 \in \rho(A)$ such that

$$\alpha\|AR(\lambda, A)\| + \beta\|R(\lambda, A)\| < 1 \quad \text{for all } \mathrm{Re}\lambda \geq \lambda_0. \tag{3.17}$$

Then, for each $P \in \mathcal{P}_u(B^*)$, the pair $((A+P)^*, B)$ is controllable if and only if (A^*, B) is controllable.

<u>Proof</u>

We only prove the sufficient condition. Similarly one can prove the necessary condition. Since each $P \in \mathcal{P}_u(B^*)$ satisfies the hypotheses of theorem 3.8, $A + P$ generates a C_0-semigroup of contractions $S(t), t \geq 0$, in X. Assuming that (A^*, B) is controllable and due to the statement preceding this theorem, in order to prove that $((A+P)^*, B)$ is controllable, it suffices to show that

$$B^* R(\lambda, A+P)x = 0 \quad \text{for } \mathrm{Re}\lambda > \lambda_0, \quad \text{implies } x = 0.$$

Because of (3.17),

$$\|PR(\lambda, A)\| \leq \alpha \|AR(\lambda, A)\| + \beta \|R(\lambda, A)\| < 1 \quad \text{for all } \mathrm{Re}\lambda \geq \lambda_0.$$

Hence $I - PR(\lambda, A)$ is invertible for $\mathrm{Re}\lambda > \lambda_0$ and we have

$$R(\lambda, A+P) = R(\lambda, A)(I - PR(\lambda, A))^{-1} \quad \text{for } \mathrm{Re}\lambda > \lambda_0 > 0.$$

Thus if $B^* R(\lambda, A+P)x = 0$, for $\mathrm{Re}\lambda > \lambda_0$, then

$$B^* R(\lambda, A)\eta_\lambda = 0, \quad \text{for } \mathrm{Re}\lambda > \lambda_0,$$

where $\eta_\lambda = (I - PR(\lambda, A))^{-1}x$. Since $P \in \mathcal{P}_u(B^*)$, there exists a $Q^* \in \mathcal{L}_u(U, X)$ so that $P = Q^* B^*$. Therefore,

$$\begin{aligned} x &= (I - PR(\lambda, A))\eta_\lambda \\ &= \eta_\lambda - Q^* B^* R(\lambda, A)\eta_\lambda \\ &= \eta_\lambda, \quad \text{for } \mathrm{Re}\lambda > \lambda_0. \end{aligned}$$

Hence, $B^* R(\lambda, A)\eta_\lambda = B^* R(\lambda, A)x = 0$ for all $\mathrm{Re}\lambda > \lambda_0 > 0$. Since the pair (A^*, B) is controllable, this implies that $x = 0$. ■

Remark 3.25

The inequality (3.17) is always satisfied if A generates a C_0 -semigroup of contractions and $0 \leq \alpha < \frac{1}{2}$ (See (3.14)). Therefore, without reference to the inequality (3.17) the result of theorem 3.24 remains valid provided the relatively bound α satisfies $0 \leq \alpha < \frac{1}{2}$.

Remark 3.26

In theorem 3.24, we proved the invariance of controllability property for the adjoint system, which is the counterpart of theorem 3.21, for the class of perturbations $\mathcal{P}_u(B^*)$. In fact, one can similarly obtain all other counterparts of controllability invariance theorems given in section 2 and section 3 for the adjoint system.

3.4 Conclusions

The controllability problems for infinite dimensional systems have been considered in this chapter. First, we obtained some equivalent results which give the necessary and sufficient conditions of the system controllability depending on the system operator and its resolvent instead of its solution (i.e. semigroup).

Then, with these controllability results, it is shown that the property of controllability is preserved under a broad class of bounded or unbounded perturbations which extends the finite dimensional controllability result.

Finally, due to the fact that controllability properties of the system and its adjoint system play an equally important role in the stabilizability problem on Hilbert space, we also considered the controllability of the adjoint system and its invariance properties.

CHAPTER 4

STABILIZABILITY AND ROBUST CONTROL OF PERTURBED INFINITE DIMENSIONAL LINEAR SYSTEMS

4.1 Introduction

It is well known that mathematical models for physical systems are developed essentially under many simplifying assumptions given only a nominal system. Therefore it is essential to study the robustness of the control system designed on the basis of the nominal system dynamics. The question of stability and control of uncertain dynamic systems described by ordinary differential equations has been extensively studied in recent literature [Barmish 18,19, Barmish, Corless and Leitmann 20, Corless and Leitmann 32,33, Leitmann 73, etc.]. With so-called matching conditions for the perturbation (i.e. structural perturbation) feedback control laws are designed based on nominal system that guarantee asymptotic stability of the perturbed system. However, as mentioned in the earlier chapters, a large class of scientific and engineering problems is modelled by partial differential equations, integral equations or coupled ordinary and partial differential equations which can be described as differential equations in infinite dimensional spaces using semigroups. Suppose that the nominal system is governed by the abstract evolution equation

$$\dot{x} = Ax + Bu, \quad x(0) = x_0, \tag{4.1}$$

in a complex Hilbert space X where A is the generator of a C_0-semigroup $T(t)$, $t \geq 0$, in X and B is a bounded linear operator from another Hilbert space U to X and the control u is any strongly measurable U-valued functions defined for $t \geq 0$.

We assume that the true physical system is given by the perturbed equation

$$\dot{x} = (A + P)x + Bu, \quad x(0) = x_0, \tag{4.2}$$

where $P \in \mathcal{P}$ is a family of bounded or unbounded perturbations of A representing the uncertainty in the modelling process.

In this chapter, we consider strong (and weak) stabilizability problems for the perturbed system (4.2) in Hilbert space X and identify (specify) the class of perturbations $\mathcal{P}$ for which the perturbed system remains stabilizable by the same feedback control as for the nominal (unperturbed) system (4.1). In Section 4.2, some definitions and preliminary results for the unperturbed system (4.1) are given. Based on the results of Section 4.2 and Chapter 3, we identify(specify) in Sections 4.3 and 4.4 the class of perturbations $\mathcal{P}$ for which the system is robust in the sense that the perturbed system preserves the stability properties of the unperturbed system with the same feedback control, $u = -B^*x$, as for the nominal system (4.1). Several numerical results are presented in Section 4.5 to illustrate our theory.

4.2 Definitions and Preliminaries

From now on, let X and U be any pair of complex Hilbert spaces representing respectively the state space and the control space for the systems (4.1) and (4.2). Throughout we assume that A is a closed densely defined linear operator in $\mathcal{L}_u(X)$ satisfying the Hille-Yosida conditions so that it is the infinitesimal generator of a C_0-semigroup $T(t)$, $t \geq 0$, of bounded linear operators in X (i.e. $T(t) \in \mathcal{L}(X)$ for $t \geq 0$).

The stabilization problem can be stated simply as following: Given the system (4.1), find conditions on the pair (A, B) which will ensure the existence of a bounded operator K mapping X into U such that the feedback system

$$\dot{x} = (A + BK)x \tag{4.3}$$

is stable in some sense (see the definitions 2.25-2.29). In finite dimension, the answers are

(i) the feedback system (4.3) satisfies the conditions in theorem 2.24; or

(ii) the pair (A, B) is controllable.

In infinite dimensional spaces, the problem has also been extensively studied by many authors [Benchimol 21,22, Datko 37,38, Levan 75,76, Littman 89, Triggiani 110,111, etc.]. There are mainly three types of conditions which guarantee stabilizability of the system (4.1). The first type requires special, structural properties of the generator A. See [Triggiani 110] where he gives a stabilizability result based on spectral decomposition of the system main applicable to compact semigroups which have a finite number of unstable modes. Here we are not going to discuss this case. The second one uses the notions of controllability. Since there are many different notions of controllability and stabilizability in infinite dimensional spaces, the relations between controllability and stabilizability are much more complex. In [117], Zabczyck used a result of Datko [38] to show that if the system is exactly controllable, then it is exponentially stabilizable. Actually, the same condition holds, if we assume only the exact null controllability of the system [Curtain and Pritchard 35]. Furthermore, Megan has shown [94] that for a group generated by the system operator, the exponential stabilizability with any preassigned decay rate is equivalent to the exact controllability of the system. For the weaker notions of stabilizability, first, Slemrod in [106] has shown that if A generates a C_0-contraction semigroup with a compact resolvent, the system (4.1) is weakly stabilizable if it is approximately controllable.

In his proof, he uses the LaSalle invariance principle. This result has been generalized by Benchimal [21,22]. He has shown [21] that these conditions, in fact, guarantee the strongly stabilizability of the system. For weak stabilizability, the approximate controllability of the system is sufficient. His techniques are simpler and more directly semigroup theoretic, relying on a functional decomposition of contraction semigroups, following Sz.-Nagy -Foias. Using similar techniques, Lavan and Rigby has also shown that [77] controllability properties of the system (A, B) and its adjoint system (A^*, B) play an equally important role in the strong stabilizability problem. In this chapter we are mainly interested in this type of conditions which guarantee strong (or weak) stabilizability of the system. The third one is in terms of a Lyapunov method and a corresponding operator Riccati equation which will be considered in the next chapter.

Based on the definitions of stability in Chapter 2, we have the following

Definition 4.1

The pair $(A,\ B)$, or the system (4.1), is said to be strongly stabilizable [weakly stabilizable] if there exists a $K \in \mathcal{L}(X,\ U)$ such that the C_0-semigroup, $S(t)$, $t \geq 0$, on X generated by $(A + BK)$ has the property,

$$s \cdot \lim_{t\to\infty} S(t)\xi = 0 \quad [w \cdot \lim_{t\to\infty} S(t)\xi = 0], \tag{4.4}$$

for every $\xi \in X$.

To prove our stabilizability results we need the following

Lemma 4.2

Let A generate a uniformly bounded C_0-semigroup $T(t)$, $\|T(t)\| \leq M$ for $t \geq 0$, where $M \geq 1$. If the pair (A, B) is controllable and for $x \in X$:

$$\int_0^\infty \|B^*T^*(t)x\|^2 dt < \infty. \tag{4.5}$$

then the semigroup $T(t), t \geq 0$, is w-stable (i.e., $(T(t)x, y) \to 0$ as $t \to \infty, \forall x, y \in X$).

Proof

Let P be the operator defined by

$$(Px, x) = \int_0^\infty \|B^* T^*(t)x\|^2 dt \quad \text{for } x \in X. \tag{4.6}$$

Then it is self-adjoint and positive due to the fact that (A, B) is controllable. Furthermore, as $T^*(t)x \in X$ for $x \in X$,

$$(PT^*(t)x, T^*(t)x) = \int_t^\infty \|B^* T^*(\theta)x\| d\theta. \tag{4.7}$$

Therefore

$$\lim_{t \to \infty} (PT^*(t)x, T^*(t)x) = 0 \quad \text{for } x \in X.$$

Let Q be the positive square root of P, then for $x \in X : \|QT^*(t)x\| \to 0$ as $t \to \infty$. This implies that, for $t \to \infty$,

$$(QT^*(t)x, z) = (T^*(t)x, Qz) \leq \|QT^*(t)x\| \|z\| \to 0.$$

Since the range of P is dense in X by the fact that (A, B) is controllable, so is the range of Q hence,

$$(T^*(t)x, y) \to 0, \; t \to \infty, \quad \text{for } x, y \in X, (i.e., T^*(t) \text{ is w-stable})$$

Finally, since $T^*(t), t \geq 0$, is w-stable then so is $T(t), t \geq 0$. This ends our proof.
■

Remark 4.3

The conditions of lemma 4.2 are sufficient for the uniformly bounded semigroup $T(t), t \geq 0$, to be s-stable if the generator A has compact resolvent, since w-stability implies s-stability in this case.

We now apply these results to stabilizability of the system (4.1) where the system operator A generates a C_0-semigroup of contractions.

Theorem 4.4

Let A be the generator of a C_0-semigroup of contractions on X. If the system (4.1) is controllable then it is weakly stabilizable by the feedback $-PB^*$, where $P \in \mathcal{L}(U)$ is self-adjoint and bounded below, i.e. there exists a $\xi > 0$ such that $(Pu, u) \geq \xi \|u\|^2$ for $u \in U$.

Proof

Since BPB^* is a bounded operator in X and $(BPB^*x, x) = (PB^*x, B^*x) = \|QB^*x\| \geq 0$ where $Q^2 = P$, $A - BPB^*$ is contractive and generates a C_0-semigroup of contractions $S(t), t \geq 0$. So is its adjoint $A^* - BPB^*$. Hence we have

$$\frac{d}{dt}\|S^*(t)x\|^2 = 2\mathrm{Re}((A^* - BPB^*)S^*(t)x, S^*(t)x)$$

$$= 2\mathrm{Re}(A^*S^*(t)x, S^*(t)x) - 2\|QB^*S^*(t)x\|^2 \quad \text{for } x \in D(A^*).$$

Therefore, for $x \in D(A^*)$,

$$\|S^*(t)x\|^2 - \|x\|^2 = \int_0^t 2\mathrm{Re}(A^*S^*(\tau)x, S^*(\tau)x)d\tau - 2\int_0^t \|QB^*S^*(\tau)x\|^2 d\tau, \ t \geq 0.$$

Then, since $S^*(\tau)x \in D(A^*)$ for $x \in D(A^*)$ and A^* is also dissipative, the integrand of the first integral on the right hand side is nonpositive. Thus it follows that

$$2\int_0^t \|QB^*S^*(\tau)x\|^2 dt \leq \|x\|^2 - \|S^*(t)x\|^2 \leq \|x\|^2, \quad \text{for } x \in D(A^*).$$

Further, since $\|Qu\|^2 = (Pu, u) \geq \xi\|u\|^2$, we have

$$\int_0^t \|B^*S^*(\tau)x\|^2 d\tau \leq 1/\xi \int_0^t \|QB^*S^*(\tau)x\|^2 d\tau \leq (1/2\xi)\|x\|^2.$$

Therefore $\int_0^\infty \|B^*S^*(t)x\|^2 dt$ converges for every $x \in X$. On the other hand, $(A - BPB^*, B)$ is controllable as (A, B) is controllable by theorem 3.14. This implies that the semigroup $S(t), t \geq 0$, is w-stable from lemma 4.2, that is, the pair (A, B) is weakly stabilizable by the feedback $-PB^*$. ■

The proof of weak stabilizability of the contractive system (4.1) on a Hilbert space in theorem 4.4 is simpler and more straightforward than those proofs using the LaSalle Invariance Principle or the Nagy-Foias Canonical Decomposition mentioned earlier. In fact, their results (stated in corollary 4.5) are a special case of theorem 4.4:

<u>Corollary 4.5</u>

Suppose A generates a contraction semigroup on X and the pair (A, B) is controllable, then system (4.1) is weakly stabilizable by the feedback control $-B^*$.

<u>Proof</u>

The result directly follows from theorem 4.4 by taking $P = I$. ■

In applications one may wish that the closed loop semigroup $S(t), t \geq 0$, corresponding to the feedback system (4.3) be strongly stable under the condition of controllability. This however is very rare. In fact Triggiani [110] has given a number of counterexamples demonstrating that a system that is (approximately) controllable is not necessarily strongly stabilizable. Levan [76] gave an interesting result that states that, in general, if the resolvent of A is not compact then it is not possible to strongly stabilize the system (4.1) by the feedback control $u = -B^*x$. However, under some additional assumptions on the system operator A, approximate controllability does imply strong stabilizability of the system. Results of this type were first obtained by Slemrod [106] and then were generalized by Benchimol [21]. Levan also investigated the problem by considering controllability of the adjoint system and normality of the semigroup [Levan and Rigby 77].

In the sequel we need the following known perturbation results.

Lemma 4.6 [Benchimol 21]

Let A be the generator of a C_0-semigroup $T(t), t \geq 0$, in X and $D \in \mathcal{L}(X)$. Then $A + D$ generates a C_0-semigroup $S(t), t \geq 0$, in X. Furthermore

(i) if A has compact resolvent, so has $A + D$;

(ii) if $T(t)$ is compact for any $t > 0$, so is $S(t)$;

(iii) if A and D are dissipative, $S(t)$ is a contraction semigroup satisfying $S(t)x = \lim_{n\to\infty}[e^{Dt/n}T(t/n)]^n x$ for $x \in X, t \geq 0$. ■

Theorem 4.7

If the generator A (as in lemma 4.6) has compact resolvent or the semigroup $T(t)$ is compact, then the conditions (i)& (ii) of theorem 4.4 are sufficient for the semigroup $S(t)$ to be strongly or exponentially stabilizable.

Proof

We only have to note that if A has compact resolvent ($T(t)$ is compact), by lemma 4.6, so has $A - BPB^*$ (so is $S(t)$) and then w-stability implies s-stability (w-stability implies e-stability). ■

As mentioned earlier, controllability of (A, B) and (A^*, B) play an equally important role in the problem of strong stabilizability. Before presenting the results of this type, we give the Nagy and Foias' canonical decomposition of contraction semigroups. First, recall the notion of a completely nonunitary (cnu) semigroup:

Definition 4.8

A semigroup $T(t), t \geq 0$ on X is cnu if for each nonzero x in X there is a $t > 0$ such that, either $\|T(t)x\| \neq \|x\|$ or $\|T^*(t)x\| \neq \|x\|$.

Lemma 4.9 [Levan and Rigby 77]

To every contraction semigroup $T(t), t \geq 0$ on X, there are reducing subspaces $H_{cnu}(T)$ and $H_u(T)$ such that

$$X = H_{cnu}(T) \oplus H_u(T) \tag{4.8}$$

uniquely where $H_u(T)$ is the maximal reducing subspace on which the semigroup is unitary, and

$$H_u(T) = \{x \in X : \|T(t)x\| = \|x\| = \|T^*(t)x\|, t > 0\}. \tag{4.9}$$

Hence $T(t), t \geq 0$ admits the decomposition

$$T(t) = T_{cnu}(t) \oplus T_u(t), \quad t \geq 0, \tag{4.10}$$

where $T_{cnu}(t) = T(t)|_{H_{cnu}(T)}$ is cnu and $T_u(t)|_{H_u(T)}$ is unitary.

Using this canonical decomposition of contraction semigroups, Levan has shown the following result

Theorem 4.10 [Levan and Rigby 77]

If for the semigroup $T(t), t \geq 0$ of a contractive system (A, B)

$$\begin{aligned} H_{cnu}(T) &= \{x \in X : \lim_{t \to \infty} \|T(t)x\| = 0\} \\ &= \{x \in X : \lim_{t \to \infty} \|T^*(t)x\| = 0\}, \end{aligned} \tag{4.11}$$

then the system is s-stabilizable by the feedback $-B^*$ if (A, B) and (A^*, B) are controllable. ■

We now consider some special semigroups which are applicable by theorem 4.10. First, noting that when $T(t), t \geq 0$ is normal, $T(t)T^*(t) = T^*(t)T(t)$ for $t \geq 0$. From the spectral theory of normal operators [Levan and Rigby 77], it is known that (4.11) holds for normal semigroups. Hence, we have

Corollary 4.11

A contractive system (A, B) whose semigroup $T(t), t \geq 0$, is normal, and $D(A) = D(A^*)$, is s-stabilizable by the feedback control $-B^*$ if both (A, B) and (A^*, B) are controllable. ■

Corollary 4.11 holds in particular for selfadjoint contraction semigroups. In this case $T(t) = T^*(t)$ for $t \geq 0$: therefore we state

Corollary 4.12

A contractive system (A, B) with $T(t), t \geq 0$ selfadjoint is s-stabilizable by the feedback $-B^*$ if (A, B) is controllable. ■

The class of systems of the form (4.1) is an important one, but their description implies the restriction that the control action used is usually of distributed type. Sometimes for a system described by partial differential equations, control action is applied on the boundary of a given domain Ω. Such systems can be represented in the form (4.1) but with the operator B unbounded (see Curtain and Pritchard [35]). The questions of stabilizability of boundary control systems were extensively studied by Ahmed [3], Lagnese [68], Lasiecka [69,70,71], Lasiecka and Triggiani [72], Pritchard and Zabczyk [101]. Similar results to theorem 4.4 and 4.7 were obtained by Pritchard and Zabczyk [101].

An important class of control systems are those for which the control action is implemented on the boundary through the intermediary of dynamical systems. Delay systems are examples of such a situation [Pritchard and Zabczyk 101]. Another example is the evolution of heat in a region Ω with heat flow controlled on the boundary $\Gamma = \partial\Omega$:

$$\begin{cases} \dfrac{\partial z}{\partial t} = \Delta z, \quad z(0,\cdot) = z_0(\cdot) \in L_2(\Omega) \\ \tau z = y \\ \dfrac{\partial y}{\partial t} = By + Du, \end{cases}$$

where τ is a boundary operator and B is a linear operator. In chapter 7, a practical example of such a system which describes a simplified SCOLE (space-craft control laboratory experiment) is considered and its stability properties are obtained. In fact, it is shown that one can apply the theories developed in this chapter for the system (4.1) and (4.2) to such boundary control systems by properly defining operators and state spaces.

4.3 Robustness of Stabilizing Controller under Bounded Perturbations

In stability problems, an important consideration is to design a controller which guarantees the stability of the system even if it is subject to some perturbations. Since any design is usually based on the nominal system, it is essential to evaluate the sensitivity (robustness) of system stability under the perturbations. In this section, we mainly consider weak (strong) stabilizability of the system (4.2) under bounded perturbations. It is known from last section that the feedback controller $u = -B^*x$ is a weakly (strongly) stabilizing controller for contraction semigroups generated by the system operator A, provided (A, B) is controllable (and the resolvent of A is compact). Based on this fact, we identify the class of bounded perturbations $P \in \mathcal{P}$ for which the system is robust in the sense that the perturbed system (4.2) preserves the stability properties of the unperturbed system with the same feedback control, $u = -B^*x$, as for the nominal system (4.1).

To prove our results we make use of some basic assumptions on the class of bounded perturbations P of the generator A as given below.

Assumption (A1)

For the given operator $B \in \mathcal{L}(U, X)$ and for any finite number β, define

$$\mathcal{P}_b(B) \equiv \{P \in \mathcal{L}(X) : P = BQ \text{ for some } Q \in \mathcal{L}(X, U),$$
$$A + P \text{ is dissipative and } \|P\|_{\mathcal{L}(X)} \leq \beta\}.$$

We say P satisfies assumption (A1) if $P \in \mathcal{P}_b(B)$.

Assumption (A2)

For any finite number β, define

$$\mathcal{P}_b \equiv \{P \in \mathcal{L}(X) : P \text{ is dissipative and } \|P\|_{\mathcal{L}(X)} \leq \beta\}.$$

We say P satisfies assumption (A2) if $P \in \mathcal{P}_b$.

Theorem 4.13

Suppose A is the generator of a contraction semigroup in X, perturbation P satisfies assumption (A1) and the pair (A, B) is controllable. Then the perturbed system (4.2) is weakly stabilizable by the feedback control $u = -B^*x$ uniformly with respect to $P \in \mathcal{P}_b(B)$.

Proof

By theorem 3.8, $A+P$ is the generator of a contraction semigroup and by theorem 3.14, $(A+P, B)$ is controllable for each $P \in \mathcal{P}_b(B)$. Therefore by corollary 4.5 the pair $(A+P, B)$ or equivalently the perturbed system (4.2) is weakly stabilizable by the same feedback control $u = -B^*x$ for all $P \in \mathcal{P}_b(B)$. ■

For strong stabilizability of the perturbed system (4.2) we have the following result.

Theorem 4.14

If the assumptions of theorem 4.13 are satisfied and A has compact resolvent then the perturbed system (4.2) is strongly stabilizable by the control $u = -B^*x$ uniformly with respect to $P \in \mathcal{P}_b(B)$.

Proof

The proof easily follows from theorem 4.13, lemma 4.6 and theorem 4.7. ■

If A is the generator of an analytic semigroup and its resolvent is compact, then the semigroup $T(t), t > 0$, is compact. Often analytic semigroups arise in the study of diffussions and, in general, systems governed by differential equations of parabolic type. In general if $T(t), t > 0$, is compact one may expect stronger stability properties.

Theorem 4.15

If A generates a compact semigroup of contractions, and the conditions of theorem 4.13 hold then the perturbed system (4.2) is stabilizable exponentially uniformly with respect to $P \in \mathcal{P}_b(B)$.

Proof

The proof follows from lemma 4.6 and the fact that for compact semigroups weak stability implies exponential stability. ■

So far we have considered the stabilizability problem for the perturbed system (4.2) under the condition that bounded perturbations satisfy assumption (A1). The matching condition $P = BQ$ in assumption (A1), however, has limited the entrance of perturbations, especially, for the boundary control problems. Therefore, one should try to avoid this condition while considering the robustness of stability for some perturbed systems. In this regard, we have the following result for perturbations satisfying the assumption (A2).

Theorem 4.16

Let A generate a C_0-semigroup of contractions and have compact resolvent. Suppose the perturbation P satisfies assumption (A2) and the pair (A, B) is controllable. Then the perturbed system (4.2) is strongly stabilizable by the feedback control $u = -B^*x$ uniformly with respect to $P \in \mathcal{P}_b$.

Proof

By theorem 4.7, $A - BB^*$ generates a C_0-semigroup of contractions $S(t), t \geq 0$, satisfying $\lim_{t\to\infty} \|S(t)\| = 0$ for $x \in X$. Since P satisfies assumption (A2), by lemma 4.6, $A - BB^* + P$ generates a contraction semigroup $S_p(t), t \geq 0$ satisfying

$$S_p(t)x = \lim_{n\to\infty} [e^{Pt/n} S(t/n)]^n x \text{ for } x \in X, t \geq 0.$$

That is, given any $\varepsilon > 0$, there exists a sufficiently large N such that

$$\|S_p(t)x - (e^{Pt/N} S(t/N))^N x\| \leq \varepsilon/2.$$

Since e^{Pt} and $S(t)$ are contractions, $\|e^{Pt/N}S(t/N)\| \leq 1$. Furthermore, since $A - BB^*$ is strongly stable (which implies $\|S(t/N)x\| \to 0$ as $t \to \infty$), we have

$$\begin{aligned}
\|S_p(t)x\| &= \|S_p(t)x - (e^{Pt/N}S(t/N))^N x + (e^{Pt/N}S(t/N))^N x\| \\
&< \varepsilon/2 + \|(e^{Pt/N}S(t/N))^N x\| \\
&\leq \varepsilon/2 + \|e^{Pt/N}S(t/N)x\| \\
&\leq \varepsilon/2 + \|S(t/N)x\| \to 0 \text{ as } t \to \infty.
\end{aligned}$$

This proves that $A + P - BB^*$ is strongly stable for all $P \in \mathcal{P}_b$. ■

Theorem 4.16 shows that the stabilizing control $u = -B^*x$ remains robust for a class of perturbations without satisfying the matching condition. Therefore, it allows us to apply this result to a large class of boundary value control problems (see chapter 7), which are not applicable by the previous results. On the other hand, in assumption (A2), dissipativity is assumed for P instead of $A + P$. Under this condition, the result of theorem 4.16 is well known for finite dimensional systems. Due to the complexity of infinite dimensional systems and their stability properties as shown in chapter 2, however, theorem 4.16 is not a direct extension of the finite dimensional result. In fact, one will see in the next section that if P is a unbounded perturbation, we require more conditions than assumption (A2) to guarantee the invariance of strong stability of the perturbed system (4.2).

4.4 Robustness of Stabilizing controller under Unbounded Perturbations

In this section we discuss the questions of stabilizability of the (uncertain) system (4.2) under the assumptions that the perturbations P are merely relatively bounded with respect to the generator A of the nominal system. We need the following assumption.

Assumption (A3)

For any finite positive number β and $0 \le \alpha < 1$ define

$\mathcal{P}_u(B) \equiv \{P \in \mathcal{L}_u(X) : D(P) \supset D(A), P = BQ$ for some $Q \in \mathcal{L}_u(X, U)$ and

its adjoint $Q^* \in \mathcal{L}_u(U, X)$ exists, $A + P$ is dissipative and

$\|Px\|_X \le \alpha\|Ax\|_X + \beta\|x\|$ for all $x \in D(A)$. Further,

its adjoint P^* is also relatively bounded with respect to A^* satisfying

$D(A^*) \subset D(P^*)$ and $\|P^*\xi\| \le \alpha'\|A^*\xi\| + \beta'\|\xi\|$ for all $\xi \in D(A^*)$

with relative bounds $0 \le \alpha' < 1$, and $\beta' \ge 0\}$.

We say P satisfies assumption (A3) if $P \in \mathcal{P}_u(B)$.

Remark 4.17

As remark 3.20, without loss of generality, we may take $\alpha' = \alpha, \beta' = \beta$. Hence from now on we assume a common relative bound $\{\alpha, \beta\}$.

Lemma 4.18

Let A be the generator of a C_0-semigroup of contractions $T(t), t \ge 0$, in X, P is relatively bounded with respect to A satisfying

$$\alpha\|AR(\lambda, A)\| + \beta\|R(\lambda, A)\| < 1 \text{ for all } \mathrm{Re}\lambda \ge \lambda_0 > 0 \text{ and } \lambda_0 \in \rho(A) \qquad (4.12)$$

and $A+P$ is dissipative. Then $A+P$ generates a C_0-semigroup of contractions $S(t), t \geq 0$, in X. Furthermore,

(i) if A has compact resolvent, so does $A+P$,

(ii) if $T(t)$ is analytic and $P \in \mathcal{P}_u(B)$ with relative bound zero, so is $S(t)$ and if, in addition, $T(t)$ is compact for $t>0$, then so is $S(t)$.

Proof

(i) By Theorem 3.8, $A+P$ is the generator of a C_0-semigroup of contractions and hence $\rho(A+P) \supset (0,\infty)$. Further, since

$$(\lambda I-(A+P)) = (I-PR(\lambda,A))(\lambda I-A) \quad \text{for} \quad \lambda \in \rho(A) \supset (0,\infty)$$

and the relative boundedness of P with respect to A with relative bounds $\{\alpha,\beta\}$ satisfies inequality (4.12), $\|PR(\lambda,A)\| < 1$ for all $\mathrm{Re}\lambda \geq \lambda_0$. Hence $I-PR(\lambda,A)$ is invertible for $\mathrm{Re}\lambda > \lambda_0$ and the resolvent of $A+P$ can be expressed in the form

$$R(\lambda, A+P) = R(\lambda,A)(I-PR(\lambda,A))^{-1}, \ \mathrm{Re}\lambda > \lambda_0.$$

But $R(\lambda,A)$ is compact for $\mathrm{Re}\lambda > \lambda_0$ and $(I-PR(\lambda,A))^{-1} \in \mathcal{L}(X)$ and therefore $R(\lambda,A+P)$ is also compact for $\mathrm{Re}\lambda > \lambda_0$. Moreover, $R(\lambda,A+P)$ is compact for all $\lambda \in \rho(A+P)$ by the resolvent identity. This proves the first assertion.

(ii) Since $T(t)$ is analytic and every $P \in \mathcal{P}_u(B)$ has relative bound zero with respect to A, the semigroup $S(t)$ generated by $A+P$ is analytic by a result due to Pazy [96, Theorem 2.1, pp.80].

In order to prove that $S(t), t>0$, is compact, we have to show that $S(t), t>0$ is a uniformly continuous semigroup and $R(\lambda, A+P)$ is compact for $\lambda \in \rho(A+P)$ [Pazy 96]. Since $S(t), t>0$, is an analytic semigroup, $S(t)$ is uniformly continuous for $t > 0$. The compactness of the semigroup $T(t), t > 0$, implies that $R(\lambda, A)$ is compact for $\lambda \in \rho(A)$ and hence by (i) $R(\lambda, A+P)$ is compact for $\lambda \in \rho(A+P)$ (see remark 4.21). Therefore we have $S(t), t > 0$, a compact semigroup. ■

For stabilizability of system (4.2), subject to unbounded perturbations, we have the following results.

Theorem 4.19

Suppose A generates a C_0-semigroup of contractions, and P satisfies assumption (A3). If the pair (A, B) is controllable and there exists a $\lambda_1 \in \rho(A)$ such that

$$\alpha\|A^* R^*(\lambda_1, A)\| + \beta\|R^*(\lambda_1, A)\| < 1 \text{ for all } \mathrm{Re}\lambda \geq \lambda_1 > 0. \qquad (4.13)$$

Then the perturbed system (4.2) is weakly stabilizable with the feedback control $u = -B^* x$ uniformly with respect to $P \in \mathcal{P}_u(B)$.

Proof

By Theorem 3.8, $A + P$ generates a C_0-semigroup of contractions in X and by Theorem 3.21 $(A + P, B)$ is controllable. Hence the assertion follows from corollary 4.5. ■

Theorem 4.20

Suppose the conditions of theorem 4.19 are satisfied and the inequality (4.12) holds. If A has compact resolvent, then the perturbed system (4.2) is strongly stabilizable with control law $u = -B^* x$ uniformly with respect to $P \in \mathcal{P}_u(B)$.

Proof

Under the given assumptions the resolvent $R(\lambda, A+P)$ is a compact operator in X

and by theorem 4.19 the pair $(A + P, B)$ is weakly stabilizable. Hence the conclusion follows from theorem 4.7. ■

Remark 4.21

As discussed in chapter 3, in the case of reflexive Banach spaces the inequalities (4.12) and (4.13) are always satisfied if A is m- dissipative and $0 \leq \alpha < 1/2$.

Thus the results of theorem 4.19 and 4.20 remain valid without reference to the inequalities (4.12) and (4.13) provided $0 \leq \alpha < 1/2$. This is stated precisely in the following theorem.

Theorem 4.22

Suppose assumption (A3) holds with $0 \leq \alpha = \alpha' < 1/2$ and the pair (A, B) is controllable. Then the perturbed system (4.2) is weakly stabilizable with feedback control $u = -B^*x$ uniformly with respect to $P \in \mathcal{P}_u(B)$. Further if A has compact resolvent, the system (4.2) is strongly stabilizable with the same control law uniformly with respect to $P \in \mathcal{P}_u(B)$. ■

If $T(t)$, $t > 0$, is compact one may expect stronger stability properties.

Theorem 4.23

Let A generate an analytic semigroup of contractions and let $P \in \mathcal{P}_u(B)$, with relative bound zero, satisfy assumption (A3). If the conditions of Theorem 4.22 are satisfied, the perturbed system (4.2) is stabilizable exponentially uniformly with respect to $P \in \mathcal{P}_u(B)$.

Proof

Since A is the generator of an analytic semigroup and has compact resolvent, the corresponding semigroup is compact. By Lemma 4.18, $S_p(t)$ generated by the operators $(A - BB^* + P)$, is compact and the result follows from Theorem 4.22 and the fact that, for compact semigroups, weak (strong) stability implies exponential stability. ■

It is known [Kato 67] that for a selfadjoint operator A with domain and range in X, and any symmetric operator P that is relatively A-bounded with A-bound α smaller than one, $A + P$ is also selfadjoint. Further if $S(t)$, $t \geq 0$, is a C_0-semigroup of selfadjoint bounded operators in X and if it is weakly stable then it is also strongly stable. This follows from the simple fact that $(S(t)x,\ x) = \|S(\frac{t}{2})x\|^2$, $t \geq 0$, and weak stability: $(S(t)x, x) \to 0$ as $t \to \infty$ for any $x \in X$.

Using these facts we obtain the following result.

Theorem 4.24

Suppose A is selfadjoint, $P = BQ$ is symmetric and relatively bounded with respect to A with relative bound $0 \leq \alpha < 1/2$. If $A + P$ is dissipative and the pair (A, B) is controllable, then the perturbed system (4.2) is strongly stabilizable by the same feedback law $u = -B^*x$.

Proof

Since A is selfadjoint and P is symmetric and relatively A-bounded with A-bound smaller than one, $A + P$ is selfadjoint and moreover for $\xi \in D(A) = D(A^*)$,

$$\|P^*\xi\| \leq \|P\xi\| \leq \alpha\|A\xi\| + \beta\|\xi\| = \alpha\|A^*\xi\| + \beta\|\xi\|$$

for $0 \leq \alpha < \frac{1}{2}$ and $\beta \geq 0$. Hence it follows from Theorems 4.22 that the perturbed system (4.2) is weakly stabilizable. The closed loop semigroups $\{S_p(t),\ t \geq 0\}$, corresponding to the selfadjoint operators $(A + P - BB^*)$, are obviously selfadjoint bounded operators in X and hence weak stability implies strong stability. This ends the proof. ■

Remark 4.25

In fact, by applying the result of corollary 4.12 and theorem 3.21 or theorem 3.24, one may reach the same conclusion.

Furthermore, if A is selfadjoint and has compact resolvent, one may even expect exponential stability which is a stronger conclusion than strong stability.

Theorem 4.26

Let A be selfadjoint and generate a C_0-semigroup of contractions, and let the conditions of theorem 4.24 hold with relative bound zero. If A has a compact resolvent, the perturbed system (4.2) is stabilizable exponentially.

Proof

It is known [Curtain and Pritchard 35] that if A is selfadjoint and has compact resolvent, A generates an analytic semigroup and thus a compact semigroup. Therefore the conclusion follows from Theorems 4.23 and 4.26. ■

Similar to the bounded perturbation case, we now consider a robustness result for some unbounded perturbed system without the matching condition.

Assumption (A4)

Define
$$\mathcal{P}_u \equiv \{P \in \mathcal{L}_u(X) : P \text{ is } m\text{-dissipative and}$$
$$\text{there exist } 0 \le \alpha < 1 \text{ and } \beta \ge 0 \text{ such that}$$
$$\|Px\|_X \le \alpha\|Ax\|_X + \beta\|x\| \text{ for } x \in D(A)\}.$$

We say P satisfies assumption (A4) if $P \in \mathcal{P}_u$. (Note that α and β cannot be zero simultaneously in the assumption.)

Lemma 4.27

Let A, B and Z be generators of C_0-semigroups of contraction $T(t), V(t)$ and $S(t)$ on X, respectively. Suppose that $D(A) \subset D(B)$ and
$$Zx = Ax + Bx$$
for all $x \in D(A)$. Then
$$S(t)x = \lim_{n\to\infty}(V(t/n)T(t/n))^n x$$
for all $x \in X$ and $0 \le t < \infty$. ■

For simplicity we introduce the notation $G(X, M, w)$ to denote the class of infinitesimal generators $\{A\}$ of C_0-semigroups $\{T_A(t), t \ge 0\}$ in the Hilbert space X satisfying $\|T_A(t)\|_{\mathcal{L}(X)} \le Me^{wt}$, for $t \ge 0$ where $M \ge 1$, and $w \in R$ are the stability parameters.

Theorem 4.28

Let $A, P \in G(X, 1, 0)$ and $D(A) \subset D(P)$. Assume that
$$\|Px\| \le \alpha\|Ax\| + \beta\|x\|, \quad x \in D(A) \tag{4.14}$$

for some $0 \leq \alpha < 1$ and $\beta \geq 0$. If A is strongly stable, so is $A + P$.

Proof

Let $T(t)$ and $V(t)$ denote contraction semigroups generated by A and P. By the assumptions $A + P$ generates a C_0-semigroup of contractions $S(t), t \geq 0$, on X. Following lemma 4.27, we have

$$S(t)x = \lim_{n\to\infty}(V(t/n)T(t/n))^n x \quad \text{for } x \in X.$$

That is, given any $\varepsilon > 0$, there exists a sufficiently large N such that

$$\|S(t)x - (V(t/N)T(t/N))^N x\| < \varepsilon/2.$$

Hence

$$\begin{aligned}
\|S(t)x\| &= \|S(t)x - (V(t/N)T(t/N))^N x + (V(t/N)T(t/N))^N x\| \\
&\leq \|S(t)x - (V(t/N)T(t/N))^N x\| + \|(V(t/N)T(t/N))^N x\| \\
&< \varepsilon/2 + \|(V(t/N)T(t/N))^N x\|.
\end{aligned}$$

Since $T(t)$ and $V(t)$ are contractions, $\|V(t/N)T(t/N)\| \leq 1$. Furthermore, due to the fact that A is strongly stable (which implies $\|T(t/N)x\| \to 0$ as $t \to \infty$) we have

$$\begin{aligned}
\|S(t)x\| &< \varepsilon/2 + \|(V(t/N)T(t/N))^N x\| \\
&\leq \varepsilon/2 + \|V(t/N)T(t/N)x\| \\
&\leq \varepsilon/2 + \|T(t/N)x\| \to 0 \quad \text{as } t \to \infty.
\end{aligned}$$

This proves that $A + P$ is strongly stable. ■

Theorem 4.29

Let $A \in G(X,1,0)$ and have compact resolvent. Suppose (A, B) is controllable and $P \in \mathcal{P}_u$ satisfies assumption (A4). Then the perturbed system $(A + P, B)$ is s-stabilizable by the same feedback $-B^*$ as the nominal system (4.1).

Proof

By theorem 4.7, $A - BB^*$ generates a C_0-semigroup of contractions $T(t)$ which is strongly stable. Since $P \in \mathcal{P}_u$, it generates a C_0-semigroup of contractions. Because of theorem 4.28, $A - BB^* + P$ generates a strongly stable semigroup. This completes the proof. ∎

Remark 4.30

We have considered the robustness of the stabilizing controller $u = -B^*x$ for the perturbed system (4.2). In fact , these results are also true for the controller $u = -KB^*x$ [Li, Ahmed and Lim 85] which we will apply to the SCOLE system in chapter 7. Here $K \in \mathcal{L}(U)$ is self-adjoint and strictly positive.

4.5 Examples and Numerical Results

In this section we give some simple and interesting examples to illustrate our results.

Example 4.31

Consider the diffusion equation

$$\begin{cases} \dfrac{\partial z}{\partial t} = \dfrac{\partial^2 z}{\partial x^2} + r_1 \dfrac{\partial z}{\partial x} + r_2 z + u, \qquad x \in (0,1) \\ z(0,t) = z(1,t) = 0 \\ z(x,0) = \sin \pi x. \end{cases}$$

We write it in abstract form as

$$\dot{z} = Az + P(r)z + Bu, \qquad z(0) = \sin \pi x,$$

with the state space $X = L_2(0,1)$. The operator A is given by

$$A = -A_0, \quad A_0 = -\frac{\partial^2}{\partial x^2}, \quad D(A) = \{z \in L_2(0,1); \frac{\partial^2 z}{\partial x^2} \in L_2(0,1), \ z(0) = z(1) = 0\}.$$

It is known that A generates a compact semigroup $T(t), t > 0$ on X where $\|T(t)\| \leq e^{-\pi^2 t} \leq 1$, [Curtain and Pritchard 36], that is, $T(t)$ is a contraction semigroup. The operator $P(r)$ is given by

$$P(r) = P_1(r) + P_2(r) = r_1 \frac{\partial}{\partial x} + r_2$$

where

$$r \in \Omega \equiv \{(r_1, r_2) \in R^2 : |r_1| \leq M_1, |r_2| \leq M_2\}.$$

Since $P_1^*(r) = -P_1(r)$ (skew-adjoint operator), $P_1(r)$ is dissipative and $(x, P_1(r)x) = 0$. It is clear that $D(A) \subset D(P(r))$. Also, from Kato [67], we know that the operator

$D = \frac{\partial}{\partial x}$ is A-bounded with A-bound equal to zero in X. That is,

$$\|\frac{\partial z}{\partial x}\|_X \leq k\|\frac{\partial^2 z}{\partial x^2}\|_X + h(k)\|z\|_X$$

where $k > 0$ is arbitrary, while $h(k) \to \infty$ when $k \to 0$.

Therefore

$$\|P(r)z\|_X \leq kM_1\|Az\|_X + hM_1\|z\|_X + M_2\|z\|_X \quad \text{for all} \quad z \in D(A).$$

Since k can be chosen as small as one desires, we can find $0 \leq k_1 < 1$ and $k_2 \geq 0$ such that

$$\|P(r)z\|_X \leq k_1\|Az\|_X + k_2\|z\|_X \quad \text{for all} \quad z \in D(A).$$

Furthermore, since A is selfadjoint, we also have,

$$\|P^*(r)z\| \leq k_1\|A^*z\| + k_2\|z\|$$

If $M_2 \leq \pi^2$, it follows from the above discussion that $A + P(r)$ is dissipative. Since here B is the identity operator in X, (A, B) is controllable [Curtain and Pritchard 35] and hence the conditions of Theorem 4.22 are satisfied. Simulation results are shown for $P(r) = 0$, $r \in \Omega_1 = \{r \in R^2 : -3 \leq r_1 \leq 0, r_2 \leq 5\}$, $r \in \Omega_2 = \{r \in R^2 : 0 \leq r_1 \leq 4, r_2 \leq 9\}$ and $r \in \Omega_3 = \{r \in R^2 : r_1$ arbitrary, $11 > r_2 > 10\}$. In the last case, assumption (A3) is violated as $A + P(r)$ is not dissipative for $r \in \Omega_3$ and the results (Fig.4.1) show that it is not exponentially stabilizable with $u = -B^*z = -z$.

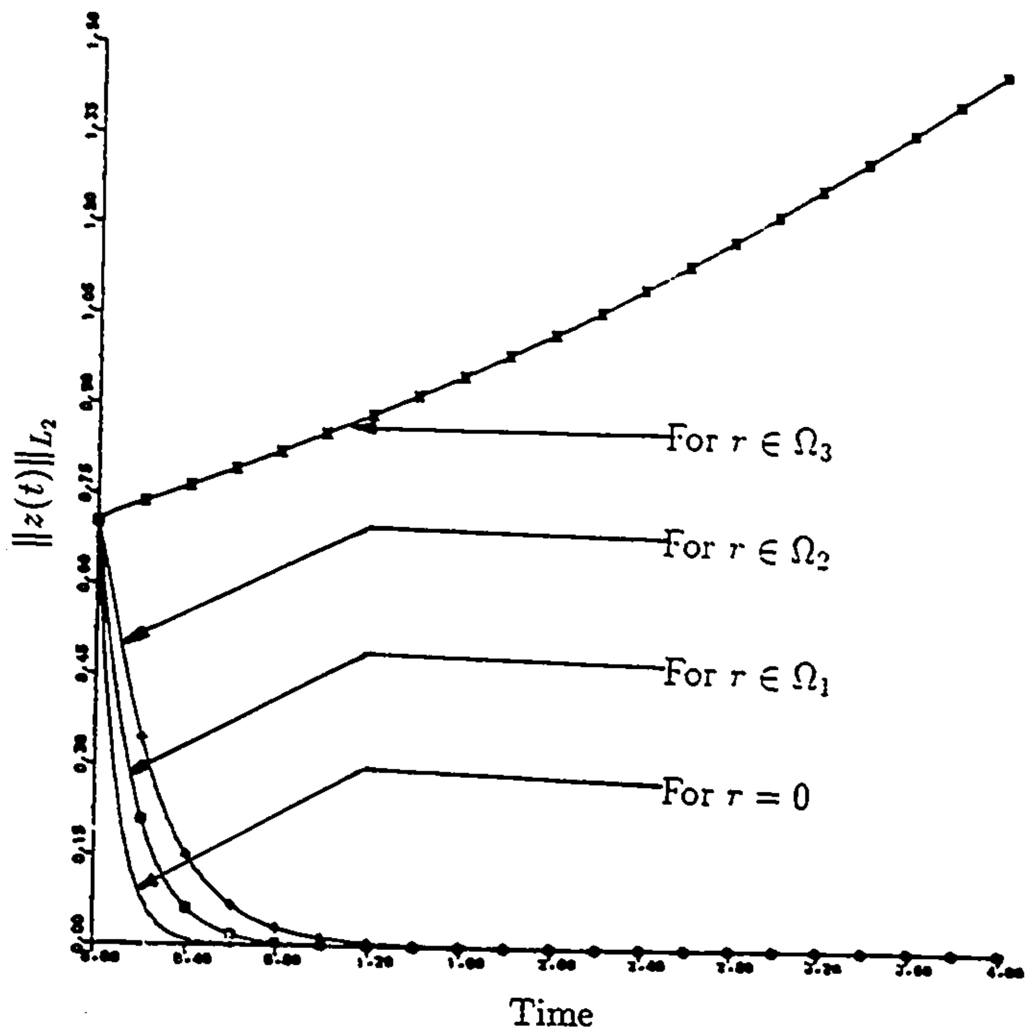

Fig.4.1. Diffusion equation (Example 4.31)

Example 4.32

Consider the controlled scalar wave equation given by

$$\begin{cases} \dfrac{\partial^2 z}{\partial t^2} = \dfrac{\partial^2 z}{\partial x^2} + r\dfrac{\partial z}{\partial t} + u, & x \in (0,\ 1) \\ z(0,t) = z(1,t) = 0, & \end{cases}$$

with initial state

$$z(x,0) = x^2 - x, \quad z_t(x,0) = \sin \pi x, \quad x \in (0,1).$$

We may associate with this equation an abstract evolution equation of the form

$$\dot{y} = Ay + P(r)y + Bu, \quad y(0) = \begin{pmatrix} y_{01} \\ y_{02} \end{pmatrix} = \begin{pmatrix} z(x,0) \\ z_t(x,0) \end{pmatrix},$$

with $X = H^1(0,1) \times L_2(0,1)$, $D(A) = H^2(0,1) \times H_0^1(0,1)$,

$$A = \begin{pmatrix} 0 & 1 \\ -A_0 & 0 \end{pmatrix}, \quad P(r) = \begin{pmatrix} 0 & 0 \\ 0 & r \end{pmatrix}, \quad \text{and} \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

where $r \in \Omega = \{r \in R : -M \leq r \leq 0\}$.

The operator A generates a C_0-semigroup of contractions which has compact resolvent, $P(r) = BQ(r)$ where $Q(r) = (0, r)$ is obviously bounded and dissipative. Furthermore, (A, B) is controllable (exactly controllable [Curtain and Pritchard 35]). Therefore, the conditions of Theorem 4.14 are satisfied. Simulation results are shown for $P(r) = 0$, $r \in \Omega_1 = \{r \in R : -1 \leq r < 0\}$, $r \in \Omega_2 = \{r \in R : -3 \leq r < -1\}$ and $r \in \Omega_3 = \{r \in R : 0 < r \leq 2\}$. Since $\Omega_3 \not\subset \Omega$ and $A + P(r)$ is not dissipative for $r \in \Omega_3$, the results show that it is not stabilizable by $u = -B^* y = -z_t$(Fig.4.2).

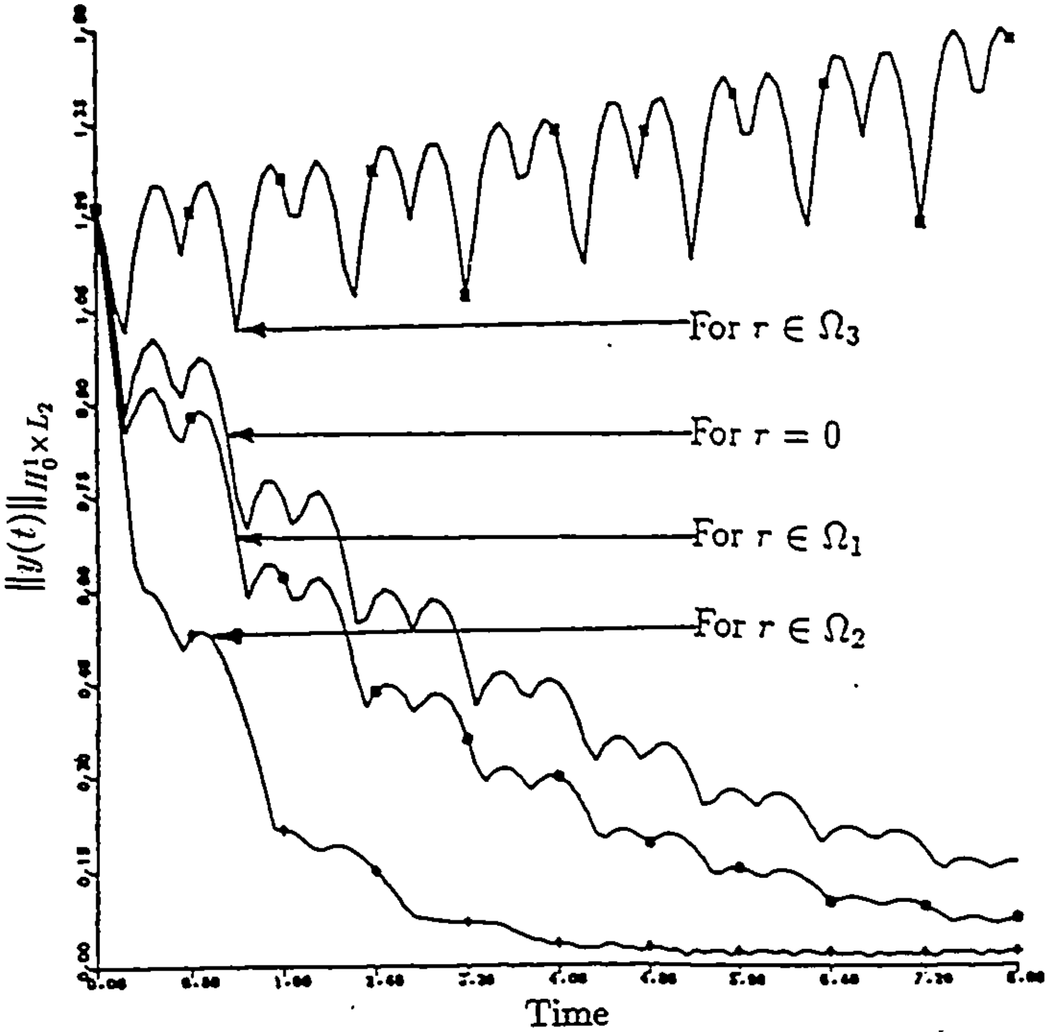

Fig.4.2. Wave equation (Example 4.32)

Example 4.33

Consider the following one-dimensional wave equation

$$\begin{cases} \dfrac{\partial^2 z}{\partial t^2} = \dfrac{\partial^2 z}{\partial x^2} + rx\dfrac{\partial^2 z}{\partial x \partial t} + u, \quad x \in (0,1) \\ z(0,t) = z(1,t) = 0 \\ z(x,0) = x^2 - x \\ z_t(x,0) = \sin \pi x, \end{cases}$$

on the Hilbert space $H = L_2(0,1)$. As in the last example, the equation can be rewritten as a first order differential equation

$$\dot{y} = Ay + P(r)y + Bu, \quad y(0) = \begin{pmatrix} y_{01} \\ y_{02} \end{pmatrix} = \begin{pmatrix} z(\text{[illegible]} \\ z_t\text{[illegible]} \end{pmatrix}$$

on the Hilbert space $X = H^1(0,1) \times L_2(0,1)$, with $D(A) =$ [illegible] $1) \times H_0^1(0,1)$,

$$A = \begin{pmatrix} 0 & 1 \\ -A_0 & 0 \end{pmatrix}, \quad P(r) = \begin{pmatrix} 0 & 0 \\ 0 & r\frac{\partial}{\partial x} \end{pmatrix}, \text{ and } B = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

where $r \in \Omega = \{r \in R : 0 \leq r \leq M\}$.

The operators $A, P(r)$ and $A + P(r)$ are the generators of C_0- semigroups of contractions (see Example 2.10 in Chapter 2). In fact,

$$\begin{aligned} (rx\frac{\partial z_t}{\partial x}, z_t) &= \int_0^1 rxz_t\frac{\partial z_t}{\partial x}dx \\ &= rxz_t^2 \,|_0^1 - r\int_0^1 z_t^2 dx - \int_0^1 rxz_t\frac{\partial z_t}{\partial x}dx. \end{aligned}$$

Hence, $(y, P(r)y) = (z_t, rx\frac{\partial z_t}{\partial x}) = -r/2\|z_t\|^2 \leq 0$, if $r \geq 0$.

Furthermore, (A, B) is controllable and A has a compact resolvent. Therefore the conditions of Theorem 4.29 are satisfied. Simulation results are shown for $P(r) = 0$, $r \in \Omega_1 = \{r \in R : 0 \leq r \leq 1\}$, $r \in \Omega_2 = \{r \in R : 1 \leq r \leq 3\}$ and $r \in \Omega_3 = \{r \in R : -4 \leq r < 0\}$. Since $\Omega_3 \not\subset \Omega$ and $P(r)$ is not dissipative for $r \in \Omega_3$, the results show that it is not stabilizable by $u = -B^*y = -z_t$ (Fig.4.3).

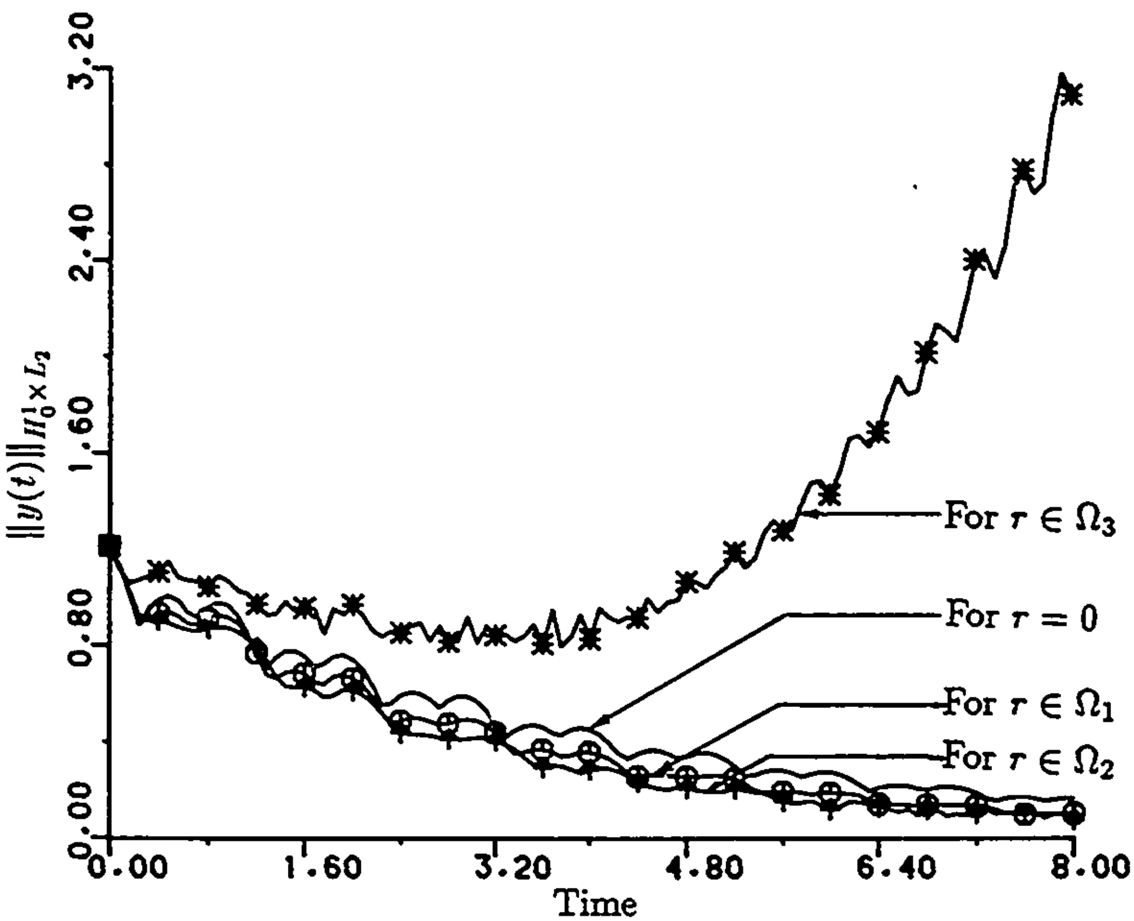

Fig.4.3. Wave equation (Example 4.33)

CHAPTER 5

FEEDBACK STABILIZATION OF DETERMINISTIC EVOLUTION EQUATIONS

5.1 Introduction

In this chapter we consider the question of stabilization of deterministic evolution equations described by the controlled system

$$\dot{x} = Ax + Bu, \quad x(0) = x_0, \tag{5.1}$$

and its perturbed system

$$\dot{x} = Ax + P(x) + Bu, \quad x(0) = x_0 \tag{5.2}$$

where A generates a strongly continuous semigroup $T(t), t \geq 0$, on a real Hilbert space X (i.e. from now on we consider X a real Hilbert space unless stated otherwise (section 5.2)), B is a bounded linear operator from U into X, where U is another real Hilbert space and P is a class of bounded or unbounded perturbations of A, which might be linear or nonlinear. Recall that, in the last chapter, we have considered the question of stabilizability and robust control for perturbed systems where we identify (specify) the class of perturbations $P \in \mathcal{P}$ for which the system remains stabilizable by the same control as for the nominal system. Here we pose the inverse problem. We consider the problem of designing a state feedback linear control law $u = Dx = (D_0 + D_1)x$ for the given perturbation that guarantees stability of the system (5.2) under the assumption that the nominal system (5.1) is stable with $u = D_0x$. In other words, we provide additional controls dependent on the perturbations preserving the stability. First of all, it is natural to present weak and strong stabilization criteria of linear perturbed systems from the results of chapter 4. Then, using perturbation theory of semigroups, we consider the exponential stabilization problem for linear and nonlinear perturbed systems (5.2). Finally, the results are illustrated by some examples involving diffusion and wave equations.

5.2 Weak (Strong) Stabilization of Linear Perturbed Systems

Using the results of chapter 4, we present the following sufficient conditions that guarantee stability of the perturbed system (5.2) on a complex Hilbert space X with a proper choice of control.

Theorem 5.1

Consider the perturbed control system (5.2) where A generates a C_0-semigroup of contractions, $B \in \mathcal{L}(U, X)$ and the linear operator P represents a perturbation of the generator A. Suppose that the following assumptions hold:

(A1) The pair (A, B) is controllable.

(A2) $P = BQ$, for $Q \in \mathcal{L}_u(X, U)$ having a well defined adjoint $Q^* \in \mathcal{L}_u(U, X)$, and $D(A) \subset D(P)$. Both P and its adjoint P^* are relatively bounded with respect to A and A^*, respectively, with relative bound less than one half.

(A3) There exists $K \in \mathcal{L}(U, X)$ such that $((A + P + BK)x, x) \leq 0$.

Then the system (5.2) with the feedback control

$$u = Dx = (-B^* + K)x \tag{5.3}$$

is weakly stable. Furthermore, if A has compact resolvent, it is strongly stable.

Proof

Let $P_1 \equiv P + BK$. Then $D(P_1) \supset D(A)$ as $BK \in \mathcal{L}(X)$ and

$$\begin{aligned}
\|P_1 x\| &\le \|Px\| + \|BKx\| \\
&\le \alpha\|Ax\| + \beta\|x\| + \|BK\|\|x\| \\
&= \alpha\|Ax\| + \beta_1\|x\| \quad \text{for } x \in D(A). \\
\|P_1^* \xi\| &\le \|P^*\xi\| + \|(BK)^*\xi\| \\
&\le \|P^*\xi\| + \|BK\|\|\xi\| \\
&\le \alpha\|A^*\xi\| + \beta_1\|\xi\| \quad \text{for } \xi \in D(A^*)
\end{aligned}$$

where we make use of the fact that $\|S\| = \|S^*\|$ if $S \in \mathcal{L}(X)$, and $\beta_1 = \beta + \|BK\|$. Also, $A + P_1 = A + P + BK$ is dissipative by assumption and $P_1 = P + BK = B(Q + K) = BQ_1$. Therefore, the conditions of theorem 4.22 are satisfied, and so $(A+P_1, B)$ is weakly stabilizable with $u = -B^*x$. That is, the perturbed system (5.2) or $(A+P, B)$ is weakly stabilizable by the control $u = (-B^* + K)x$. Furthermore, if A has compact resolvent, so has $A + P + BK$ (lemma 4.18). Hence $A + P - BB^* + BK$ is strongly stable if it is weakly stable. ■

By applying theorem 4.29 in chapter 4 for which the matching condition $P = BQ$ is not required, we have another stabilization result.

Theorem 5.2

Consider the perturbed system (5.2) where A is the generator of a C_0-semigroup of contractions. Let the following assumptions hold:

(A1) The pair (A, B) is controllable and A has compact resolvent.

(A2) $D(A) \subset D(P)$ and there exist $0 \leq \alpha < 1$ and $\beta \geq 0$ such that

$$\|Px\|_X \leq \alpha\|Ax\|_X + \beta\|x\| \quad \text{for } x \in D(A).$$

(A3) There exists $K \in \mathcal{L}(U, X)$ such that $P + BK$ is m- dissipative.

Then the perturbed system (5.2) is strongly stable with the feedback control $u = (-B^* + K)x$.

Proof

It follows from assumptions (A1)-(A3) and theorem 4.29. ■

5.3 Exponential Stabilization of Linear Perturbed Systems

From now on we discuss the question of exponential stabilizability. In engineering problems this is the most desirable form of stability. As mentioned earlier in chapter 2, an important stability criterion for a finite dimensional system is the one which requires the existence of an appropriate "Lyapunov function". Lyapunov's result can be stated as follows: Let A be a $n \times n$ matrix and A^* its adjoint. Then A has all its characteristic roots lying in the half-plane $\mathrm{Re}(z) < 0$ if and only if the solution S of the matrix equation $A^*S + SA = -I$ (the identity matrix) is a unique positive definite matrix. Datko [37] has extended this well-known theorem of A.M. Lyapunov to strongly continuous semigroups of operators on a Hilbert space. In fact, he has shown that the three statements below are equivalent:

(i) $\|T(t)\| \leq Me^{-wt}, t \geq 0$ for some $M \geq 1$ and $w > 0$ (exponential stability)

(ii) $\int_0^\infty \|T(t)x\|^2 dt < \infty$ for all $x \in X$. (strong L_2-stability)

(iii) There exists a nonnegative and symmetric operator $0 \leq S \in \mathcal{L}(X)$ such that

$$(Ax, Sx) + (Sx, Ax) = -(x, x) \text{ for } x \in D(A). \text{ (Lyapunov stability)}$$

The usefulness of Lyapunov's theorem in ordinary differential equations is that it allows for an explicit representation of a Lyapunov function as a positive definite quadratic form. Using this representation one may then, for example, study the effects of perturbations on an asymptotically stable linear constant coefficient system of ordinary differential equations. Consider say the finite-dimensional perturbed system

$$\dot{x} = Ax + f(x), f(0) = 0.$$

Let A have its eigenvalues in the left half-plane $\mathrm{Re}(z) < 0$. By Lyapunov's theorem, we then have a positive definite matrix S such that

$$\frac{d}{dt}(Sx, x) = -\|x\|^2 + 2(Sx, f(x)).$$

Hence, one may obtain conditions on the linear or nonlinear perturbation $f(x)$ to guarantee that $d(Sx, x)/dt$ is negative definite thus insuring asymptotic stability of the zero equilibrium. Therefore, it is natural to ask if these results still hold for a perturbed <u>infinite</u> dimensional system

$$\begin{cases} \dot{x} = Ax + f(x), \\ x(0) = x_0 \end{cases} \tag{5.4}$$

where $f(0) = 0$. We consider the solutions to (5.4) in the mild sense, namely, as solutions of the associated integral equation

$$x(t) = T(t)x_0 + \int_0^t T(t-s)f(x(s))ds. \tag{5.5}$$

Like the finite dimensional case, the question above is very important from the point of view of Lyapunov's direct method since there are some situations where one may easily find "Lyapunov" functions which are not strictly positive but ensure L_2 stability. To be more specific let $S \geq 0$ satisfy statement (iii). Suppose further that

$$2(Sx, Ax + f(x)) \leq -d\|x\|^2, x \in D(A) \text{ for some } d > 0.$$

Then applying the Lyapunov argument to (5.4) and (Sx, x), it is possible to obtain [Ichikawa 60]

$$(Sx(t; x_0), x(t; x_0))^{1/2} \leq Ne^{-bt}\|x_0\| \text{ for some } N > 0 \text{ and } b > 0,$$

$$\int_0^\infty \|x(t; x_0)\|^2 dt \leq K\|x_0\|^2, x_0 \in X \text{ for some } K > 0,$$

where $x(t; x_0)$ is the mild solution of (5.4). However, we cannot in general conclude that

$$\|x(t; x_0)\| \leq Me^{-wt}\|x_0\| \text{ for some } M \geq 1 \text{ and } w > 0$$

which requires an estimate of the form $(Sx, x) \geq a\|x\|^2$ for some $a > 0$ (strictly positive). In other words, the quadratic form (Sx, x) has to define an equivalent norm on X. If X has finite dimension, this is always the case. It is not, however, in infinite dimension. In fact, Pazy [97] has shown that if A is the generator of an analytic semigroup, (Sx, x) cannot be equivalent to $\|x\|^2$. That is, Lyapunov theory is not always applicable to infinite dimensional systems. We will pose the similar question for stochastic evolution equations in the next chapter and apply Ichikawa's result to the problem. In fact, Ichikawa has shown that, under some assumptions, sufficient conditions for L_p stability assure exponential stability as well. In that case, the function $v(x) = (Sx, x)$ where $0 \leq S \in \mathcal{L}(X)$ is a Lyapunov function even though (Sx, x) may not be equivalent to $\|x\|^2$. In this chapter, however, we present another approach and study the exponential stability problem for linear and nonlinear perturbed deterministic systems (5.2) based on perturbation theory of semigroups.

Theorem 5.3

Consider the linear perturbed control system

$$\dot{x} = Ax + Px + Bu \tag{5.6}$$

where the operator P represents the linear perturbation of the generator A. Suppose the following assumptions hold:

(A1) The pair $\{A, B\}$ is exponentially stabilizable in the sense that there exists an operator $D_0 \in \mathcal{L}(X, U)$ such that $A_0 \equiv A + BD_0 \in G(X, 1, -w)$ for some $w > 0$ so that the corresponding semigroup $T_0(t)$ has the property $\|T_0(t)\| \leq e^{-wt}$.

(A2) P is a closed operator with $D(A) \subset D(P) \subset X$ satisfying

(1) there exist $k \geq 0$ and $\gamma \geq 0$ such that

$$(Px, x) \leq k(Ax, x) + \gamma(x, x) \quad \text{for } x \in D(A); \tag{5.7}$$

(2) there exist $0 \leq \alpha < 1$ and $\beta \geq 0$ such that for $x \in D(A)$

$$\|Px\| \leq \alpha\|Ax\| + \beta\|x\|. \tag{5.8}$$

(A3) The linear operator BB^* is coercive; that is,

$$(x, BB^*x) \geq \lambda\|x\|^2, \quad \text{for all } x \in X, \quad \text{some } \lambda > 0 \tag{5.9}$$

where B^* is the adjoint of the operator B.

Then the system (5.6) with the feedback control

$$u \equiv D_1x \equiv (1 + k)D_0 - (\gamma/\lambda)B^* \tag{5.10}$$

is exponentially stable.

Proof

Using the operator D_1, the feedback system is given by

$$\dot{x} = (A_0 + Q)x \tag{5.11}$$

where $Q \equiv P + kBD_0 - (\gamma/\lambda)BB^*$.

By (A1) and theorem 2.8, $A_0 \in G(X, 1, -w)$ and hence the corresponding semi-group $T_0(t), t \geq 0$, satisfies $\|T_0(t)\| \leq e^{-wt}, t \geq 0$. Furthermore, $A_w \equiv (A_0 + wI)$ is dissipative, and, since A_0 is m-dissipative, A_w is also m-dissipative. Since $D(P) \supset D(A) = D(A_0) = D(A_w)$, and BD_0 and BB^* are bounded operators we have $D(Q) \supset D(A_w)$ and

$$\|Qx\| \leq \alpha\|A_w x\| + \delta\|x\|$$

for $x \in D(A_w)$ where $0 \leq \alpha < 1$, and $\delta \equiv (\alpha + k)\|BD_0\| + (\gamma/\lambda)\|BB^*\| + \alpha w + \beta$. Hence Q is relatively A_w bounded with A_w-bound $\alpha < 1$. We show that $A_w + tQ$ is dissipative for $0 \leq t \leq 1$. Indeed, for $x \in D(A_w) = D(A)$, we have

$$(x, (A_w + tQ)x) = (x, A_w x) + t(x, Qx) \leq t(x, Qx) \quad \text{for } 0 \leq t \leq 1;$$

and it follows from assumptions (A2) and (A3) that

$$\begin{aligned}(x, Qx) &= (x, Px) + k(x, BD_0 x) - (\gamma/\lambda)(BB^* x, x)\\ &\leq k(x, A_0 x) + (x, [\gamma - (\gamma/\lambda)BB^*]x)\\ &\leq -kw\|x\|^2\end{aligned}$$

and hence $(x, (A_w + tQ)x) \leq -tkw\|x\|^2$ for $t \in [0, 1]$. This shows that the operator $A_w + tQ$ is dissipative for all $t \in [0, 1]$. Then, it follows from the well known perturbation theorem [Pazy 96] and m-dissipativity of A_w itself that $A_w + Q$ is m-dissipative.

Hence by theorem 2.6, the operator $A_w + Q \in G(X, 1, 0)$ or equivalently the operator $A_0 + Q \in G(X, 1, -w)$. Therefore the feedback system operator $A_0 + Q$ generates an exponentially decaying contraction semigroup $S_0(t), t \geq 0$, with the decay rate w. This proves the theorem. ■

In the case of bounded perturbations, we have the following:

Corollary 5.4

Consider the system (5.6) and suppose the assumption (A1) and (A3) hold with the perturbing operator P bounded, say, $\|P\| \leq \gamma$ for some $\gamma \geq 0$. Then, the system (5.6) is exponentially stable with respect to the null state for the control $u = (D_0 - (\gamma/\lambda)B^*)x$. ■

Remark 5.5

We have introduced the notions of relative boundedness, which are important in perturbation theory, for quadratic forms (5.7) and operators (5.8) in theorem 5.3. In general, there is not any clear relationship between these two kinds of relative boundedness (see examples in section 5.5). That is why we have to assume relative boundedness for both in the theorem. If we restrict ourselves only to symmetric operators, however, form-relative boundedness is weaker than operator-relative boundedness. More precisely, we have

Lemma 5.6 [Kato 67]

Let A be selfadjoint and nonnegative and T symmetric with $D(T) \supset D(A)$ and

$$\|Tx\| \leq \|Ax\| \quad \text{for } x \in D(A).$$

Then

$$(Tx, x) \leq (Ax, x) \quad \text{for } x \in D(A). \quad ■$$

We replace the assumption (A2) by

(A2)′ $P = P_1 + P_2$, where P_1 is a closed and symmetric operator with $D(A) \subset D(P_1)$ and P_2 is a bounded operator in X satisfying that, there exist $0 \leq k < 1$ and $\gamma \geq 0$ such that for $x \in D(A)$,

$$\|P_1 x\| \leq k\|Ax\|, \tag{5.12}$$

and $\|P_2\| \leq \gamma$.

Under the assumptions (A1), (A2)′ and (A3) it is easy to prove the following result:

Corollary 5.7

Consider the system (5.6) where A is a selfadjoint positive linear operator with feedback control $u = D_1 x$ defined in equation (5.10). If the assumptions (A1), (A2)′ and (A3) are satisfied then $x = 0$ is exponentially stable.

Proof

By lemma 5.6 and assumption (A2)′, it is easy to show that

$$(Px, x) \leq k(Ax, x) + \gamma(x, x) \quad \text{for } x \in D(A).$$

Therefore, the result follows from theorem 5.3. ■

In theorem 5.3, it is assumed that the linear operator BB^* is coercive, which appears too restrictive for applications in some cases. One should relax assumptions (A2) and (A3) to the following

(A2)″ P is a closed operator with $D(A) \subset D(P) \subset X$ satisfying

(1) there exist $k \geq 0$ and $\mu \geq 0$ such that

$$(Px, x) \leq k(Ax, x) + \mu(B^*x, B^*x) \quad \text{for } x \in D(A); \tag{5.13}$$

(2) there exist $0 \leq \alpha < 1$ and $\beta \geq 0$ such that for $x \in D(A)$

$$\|Px\| \leq \alpha\|Ax\| + \beta\|x\|.$$

Assumption (A2)$''$ imposes much weaker conditions on the control operator B than assumptions (A2) and (A3) did. In fact, (5.13) implies that the operator $P - kA + \mu BB^*$ is dissipative. Using this, we have following improved result:

Theorem 5.8

Consider the linear perturbed system (5.6) and let the assumptions (A1) and (A2)$''$ hold. Then the system (5.6) with feedback control

$$u \equiv D_2 x \equiv (1 + k)D_0 - \mu B^* \tag{5.14}$$

is exponentially stable.

Proof

With the control $u = D_2 x$, the feedback system is given by

$$\dot{x} = (A_0 + U)x \tag{5.15}$$

where $U \equiv P + kBD_0 - \mu BB^*$.

Again $A_w \equiv (A_0 + wI)$ is m-dissipative by assumption (A1) and theorem 2.8. Since $D(P) \supset D(A) = D(A_0) = D(A_w)$, and BD_0 and BB^* are bounded operators we have $D(U) \supset D(A_w)$ and

$$\|Ux\| \leq \|Px\| + k\|BD_0x\| + \mu\|BB^*x\|$$

$$\leq \alpha\|Ax\| + [k\|BD_0\| + \mu\|BB^*\| + \beta]\|x\|$$

$$\leq \alpha\|A_wx\| + \epsilon\|x\|$$

for $x \in D(A_w)$ where $0 \leq \alpha < 1$, and $\epsilon \equiv (\alpha + k)\|BD_0\| + \mu\|BB^*\| + \alpha w + \beta$. Hence U is relatively A_w bounded with A_w-bound $\alpha < 1$ and it follows from assumption (A2)$''$ that

$$(x, Ux) = (x, Px) + k(x, BD_0x) - \mu(x, BB^*x)$$

$$\leq k(x, A_0x) + \mu(B^*x, B^*x) - \mu(x, BB^*x)$$

$$\leq -kw\|x\|^2 \leq 0.$$

Hence $A_w + U$ is m-dissipative and $A_0 + U \in G(X, 1, -w)$ by theorem 2.6. This completes our proof. ■

5.4 Exponential Stabilization of Nonlinear Perturbed Systems

We have considered exponential stabilization of linear systems so far. Similar results can be proved for some nonlinear (semilinear) systems. For this we need some results on generation theory of nonlinear semigroups.

Definition 5.9 (Accretive/Dissipative operators)

A single valued nonlinear operator A mapping $D(A)(\neq \emptyset) \subseteq X$ to X is called accretive if

$$(Ax - Ay, x - y) \geq 0 \text{ for every pair } x, y \in D(A). \tag{5.16}$$

It is said to be dissipative if the inequality is reversed or equivalently $-A$ is accretive. It is said to be m-accretive (or m-dissipative) if the range $R(I+\lambda A) = X$ (or $R(I-\lambda A) = X$) for $\lambda > 0$.

Theorem 5.10 [Crandall 34]

Let X be a Hilbert space and A a single valued dissipative operator from $D(A)$ to X such that $R(I-\lambda A) \supset D(A)$ for all small $\lambda > 0$. Then A generates a nonlinear nonexpansive semigroup given by the exponential formula $S(t)x = \lim_{n\to\infty}(I-(t/n)A)^{-n}x$ for $x \in D(A)$, where the convergence is uniform on compact subsets of $R_0 \equiv [0,\infty)$, satisfying

(i) $\|S(t)\xi - S(t)\eta\| \le \|\xi - \eta\|$ for all $\xi, \eta \in D(A)$;

(ii) $\|S(t+h)\xi - S(t)\xi\| \le |h|\|A\xi\|$ for all $\xi \in D(A), t \ge 0, t+h \ge 0$.

If

$$R(I-\lambda A) \supseteq \overline{D(A)} \quad \text{for all small } \lambda > 0, \tag{5.17}$$

then the exponential formula holds on $\overline{D(A)}$ and further if $\overline{D(A)} = X$ then A is m-dissipative and generates a nonexpansive semigroup on X. ■

This is the original Crandall-Liggett generation theorem. The condition $R(I-\lambda A) \supseteq \overline{D(A)}$ is called the "range condition". The following perturbation theory due to Kato [66] is used in the sequel.

Theorem 5.11

Let A be m-dissipative and let f be single valued dissipative with $D(A) \subset D(f)$. Furthermore suppose that f is locally A-bounded and locally A-Lipschitz in the sense that there exist nonnegative constants $\alpha < 1$ and $\beta < \infty$ such that for each $x \in D(A)$ there is a neighbourhood Ψ of x such that

(a) $\|fx\| \le \alpha\|Ax\| + \beta\|x\|$ for $x \in D(A) \cap \Psi$.

(b) $\|fx - fy\| \le \alpha\|Ax - Ay\| + \beta\|x - y\|$ for $x, y \in D(A) \cap \Psi$.

Then $A + f$ is m-dissipative and generates a (nonlinear) nonexpansive semigroup on X. ■

Theorem 5.12

Consider the control system

$$\dot{x} = Ax + P(x) + Bu \tag{5.18}$$

where $A \in G(X), B \in \mathcal{L}(U, X)$ and P is a nonlinear perturbation and not necessarily bounded. Let the following assumptions hold:

(A1) The pair (A, B) is exponentially stabilizable with a feedback operator $D_0 \in \mathcal{L}(X, U)$; that is the semigroup $T_0(t), t \ge 0$, corresponding to $A + BD_0$ has the property $\|T_0(t)\| \le e^{-wt}$ for some $w > 0$.

(A2) P is a closed single valued operator in X satisfying

(1) $P(0) = 0, D(A) \subset D(P)$, and there exist $k \ge 0$ and $\gamma \ge 0$ such that

$$(P(x) - P(y), x - y) \le k(Ax - Ay, x - y) + \gamma(x - y, x - y) \quad \text{for } x, y \in D(A); \tag{5.19}$$

(2) there exist $0 \le \alpha < 1$ and $\beta \ge 0$ such that for $x, y \in D(A)$,

$$\|P(x) - P(y)\| \le \alpha\|Ax - Ay\| + \beta\|x - y\|. \tag{5.20}$$

(A3) The linear operator BB^* is coercive; that is, $(x, BB^*x) \ge \lambda\|x\|^2$, for some $\lambda > 0$ and $x \in X$.

Then the system (5.18) with feedback control $u \equiv D_1 x \equiv [(1+k)D_0 - (\gamma/\lambda)B^*]x$ is exponentially stable with respect to the zero state.

Proof

Define $A_0 \equiv (A + BD_0)$ and

$$f(x) \equiv P(x) + [kBD_0 - (\gamma/\lambda)BB^*]x. \tag{5.21}$$

First we show that (A_0+f) generates a nonlinear (nonexpansive) semigroup $S(t), t \geq 0$, in X. Clearly by (A1), $(A_0 + wI)$ is m-dissipative and hence A_0 is m-dissipative. By theorem 5.11, it suffices to show that f satisfies the conditions (a) and (b) of that theorem and that it is dissipative. For $x, y \in D(A)$ we have

$$\begin{aligned}\|f(x) - f(y)\| &\leq \|P(x) - P(y)\| + [k\|BD_0\| + (\gamma/\lambda)\|BB^*\|]\|x - y\| \\ &\leq \alpha\|Ax - Ay\| + \beta\|x - y\| + \beta_0\|x - y\| \\ &\leq \alpha\|A_0x - A_0y\| + \delta\|x - y\| \quad \text{for } x, y \in D(A),\end{aligned}$$

where $\beta_0 \equiv k\|BD_0\| + (\gamma/\lambda)\|BB^*\|$ and $\delta \equiv [(\alpha + k)\|BD_0\| + \beta + (\gamma/\lambda)\|BB^*\|]$. Since $f(0) = 0$ this also implies that

$$\|f(x)\| \leq \alpha\|A_0x\| + \delta\|x\| \quad \text{for } x \in D(A) = D(A_0).$$

This shows that f is relatively A_0-bounded and A_0-Lipschitz with $\alpha < 1$. We show that f is dissipative. Indeed, by virtue of assumption (A2),

$$\begin{aligned}(x - y, f(x) - f(y)) &\leq (x - y, P(x) - P(y)) + k(x - y, BD_0x - BD_0y) \\ &\quad + (\gamma/\lambda)(x - y, BB^*x - BB^*y) \\ &\leq k(A_0x - A_0y) + [\gamma - (\gamma/\lambda)\|BB^*\|]\|x - y\|^2 \\ &\leq -kw\|x - y\|^2 \leq 0 \quad \text{for } x, y \in D(A) = D(A_0).\end{aligned}$$

Since $D(A) \subset D(f) = D(P) \subset X$ and $\overline{D(A)} = X$, for any $x, y \in D(f)$ one can find sequences $\{x_n\}, \{y_n\} \subset D(A)$ such that $x_n \to x, y_n \to y$. On the other hand, f is closed as P is closed by assumption (A2) and $kBD_0 - (\gamma/\lambda)BB^*$ is bounded. Therefore, we have $f(x_n) \to f(x), f(y_n) \to f(y)$ and

$$(x - y, f(x) - f(y)) = \lim_{n\to\infty}(x_n - y_n, f(x_n) - f(y_n)) \leq 0 \quad \text{for } x, y \in D(f).$$

Thus f is dissipative. In summary, we have A_0 is m-dissipative and f is dissipative. Hence by theorem 5.11, $A_0 + f$ is m-dissipative and so generates a nonexpansive (nonlinear) semigroup $S(t), t \geq 0$, in X. Even more is true; $A_0 + f$ is also strictly dissipative giving

$$([A_0x + f(x)] - [A_0y + f(y)], x - y) \leq -(1 + k)w\|x - y\|^2.$$

Hence $\|S(t)\xi - S(t)\eta\| \leq e^{-(1+k)wt}\|\xi - \eta\|$. But, since $(A_0 + f)(0) = 0$, it follows from this that $\|S(t)\xi\| \leq e^{-(1+k)wt}\|\xi\|$ for all $t \geq 0$. This proves the theorem. ∎

As in the linear case, one should relax assumption (A2) and (A3) to the following:

(A2)$'$ P is a closed single valued operator in X satisfying

(1) $P(0) = 0, D(A) \subset D(P)$, and there exist $k \geq 0$ and $\mu \geq 0$ such that

$$(P(x)-P(y), x-y) \leq k(Ax-Ay, x-y)+\mu(x-y, BB^*(x-y)) \quad \text{for } x, y \in D(A); \tag{5.22}$$

(2) there exist $0 \leq \alpha < 1$ and $\beta \geq 0$ such that for $x, y \in D(A)$

$$\|P(x) - P(y)\| \leq \alpha\|Ax - Ay\| + \beta\|x - y\|. \tag{5.23}$$

Theorem 5.13

Consider the controlled system (5.18) with linear feedback control given by

$$u \equiv D_2 x \quad \text{where} \quad D_2 \equiv (1+k)D_0 - \mu B^*.$$

If assumption (A1) and (A2)$'$ are satisfied, then the mild solution $x(t, x_0)$ of the feedback system

$$\dot{x} = Ax + BD_2 x + P(x), \quad x(0) = x_0, \; t \geq 0 \tag{5.24}$$

is exponentially stable. ■

Remark 5.14

The perturbations considered here are assumed to be globally A-bounded. If only local A-boundedness is assumed in the sense: $D(A) \subset D(P)$, and for each $x \in X$ there is a neighbourhood Ψ of x and constants $0 \leq \alpha < 1, \beta \geq 0$ (depending only on Ψ) such that

$$\|P(x)\| \leq \alpha\|Ax\| + \beta\|x\| \quad \text{for } x \in D(A) \cap \Psi,$$

similar stability results can be obtained.

5.5 Examples and Numerical Results

Example 5.15

Consider a controlled semilinear diffusion equation

$$\begin{cases} z' - c\Delta z + f(z) = u(t,x) & t \geq 0, x \in \Omega \\ z \,|_\Gamma = 0, \quad z(0) = h(x) & \text{for } x \in \Omega \end{cases}$$

on a bounded open subset Ω of R^n, with a smooth boundary $\Gamma = \partial\Omega$, which describes a large class of physical problems such as heat transfer, chemical diffusion, Markov processes, turbulence, etc.. In this numerical example, we present a problem analogous to Burgers' mathematical model of turbulence [Buis 28, Eckhaus 42] in which $n = 1, \Omega = (0,1), c = \frac{1}{R}$ and $f(z) = -\frac{6}{\sqrt{R}}x\frac{\partial z}{\partial x} - (x^2 + \frac{6}{\sqrt{R}})z - R^2(\int_0^1 z^2 dx)^{1/2}$, $h(x) = \sin x$.

One may associate with this equation a semilinear evolution equation of parabolic type

$$\begin{cases} \dfrac{dz}{dt} = Az + P(z) + Bu \\ z(0)(x) = \sin x \end{cases}$$

on real Hilbert space $X = L_2(0,1)$ where $A = \frac{1}{R}\frac{\partial^2}{\partial x^2} + x^2 + \frac{6}{\sqrt{R}}$ with $D(A) = \{z \in X : z \in H_0^2(0,1)\}$ and $B = 1$. If we assume that $R = 2$ and $D_0 = -1$, it is known that $\tilde{A} = A + BD$ generates an exponentially stable contraction semigroup in X; in fact, $(z, \tilde{A}z) \le (-\frac{\pi^2}{2} + x^2 + \frac{6}{\sqrt{2}} - 1)\|z\|^2 \le -\frac{1}{2}\|z\|^2$ for $z \in D(\tilde{A}) = D(A)$. The operator $P(z)$ is given by

$$P(z) = P_1(z) + P_2(z) = \frac{6}{\sqrt{2}}x\frac{\partial z}{\partial x} + 4\|z\|$$

Since $(z, P_1 z) = \frac{6}{\sqrt{2}}\int_0^1 xz\frac{\partial z}{\partial x}dx = \frac{6}{\sqrt{2}}xz^2 \,|_0^1 - \frac{6}{\sqrt{2}}(\int_0^1 xz\frac{\partial z}{\partial x}dx + \int_0^1 z^2 dx)$, therefore

$$(z, P_1 z) = -\frac{3}{\sqrt{2}}\int_0^1 z^2 dx \le 0, \quad z \in D(P_1) \equiv \{z \in X : z \in H_0^1(0,1)\},$$

thus P_1 is dissipative. Furthermore,

$$\begin{aligned} \|P_1 z\|^2 = 18\int_0^1 x^2(\frac{\partial z}{\partial x})^2 dx &\le 18\int_0^1 (\frac{\partial z}{\partial x})^2 dx \\ &\le \delta\int_0^1 (\frac{\partial^2 z}{\partial x^2})^2 dx + b(\delta)\int_0^1 z^2 dx \quad \text{for any } \delta > 0. \end{aligned}$$

where $b(\delta)$ is appropriately chosen.

Hence, one shall verify that for any $\varepsilon > 0$,

$$\|P_1 z\| \leq \varepsilon\|Az\| + \beta(\varepsilon)\|z\| \text{ for } z \in D(A),$$

where $\beta(\varepsilon) \to \infty$ as $\varepsilon \to 0$; that is, P_1 is A-bounded with A-bound zero in X. For $P_2(z)$,

$$\|P_2(v) - P_2(w)\| = 4|\|v\| - \|w\|| \leq 4\|v - w\| \text{ for } v, w \in X.$$

By choosing $\gamma = 4$ and $k = 0$ in equation (5.21), one can verify that all the conditions of theorem 5.12 are satisfied (here $k = 0$, as P_1 is dissipative). Hence the feedback system is exponentially stable. On the other hand, it is unstable with the control $u = D_0 z = -z$ only (Fig.5.1).

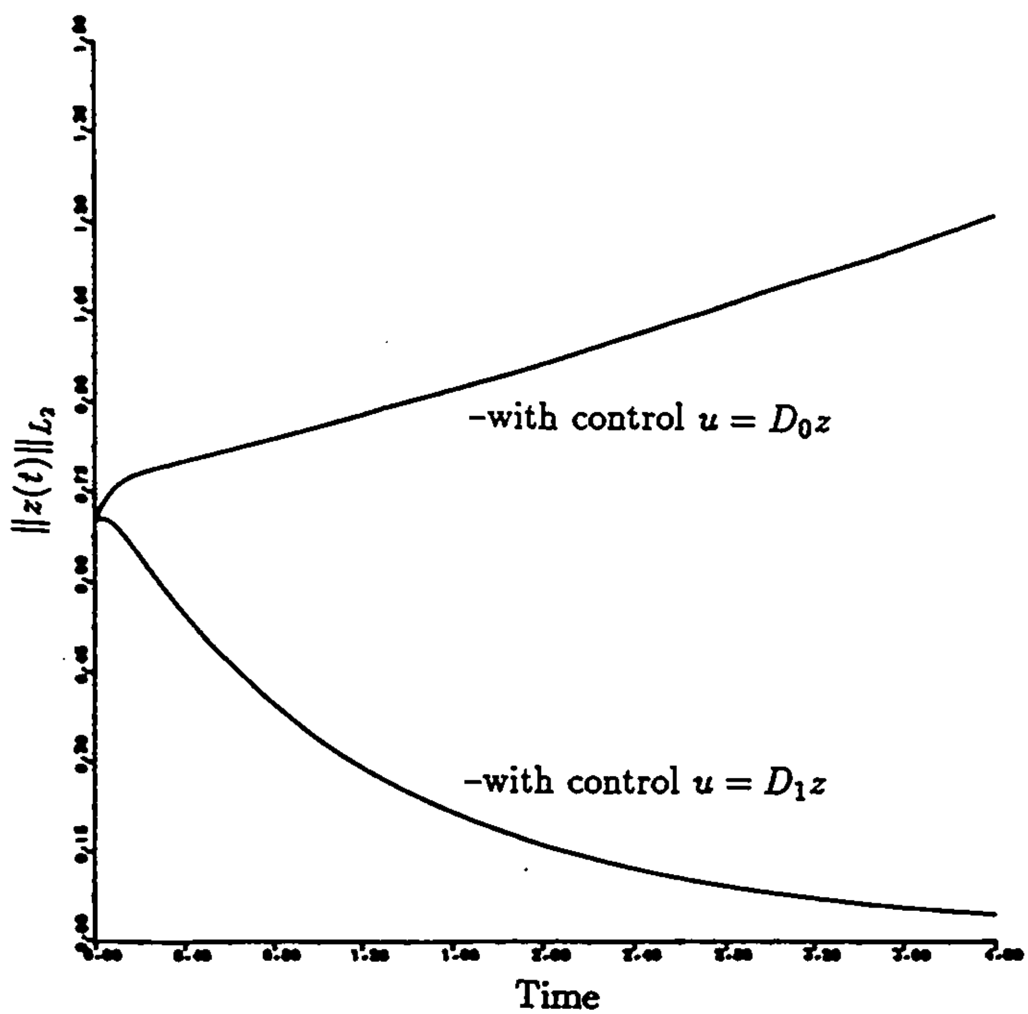

Fig.5.1. Semilinear diffusion equation (Example 5.15)

Example 5.16

Recently many results on semilinear wave equations have been reported [Haraux 52, Lasiecka 70, etc.]. In this example, we consider a model given by Haraux[52]: Let Ω be a bounded open subset of R^n, with a smooth boundary $\Gamma = \partial\Omega$ and consider the semilinear forced wave equation

$$\begin{cases} z'' - \Delta z + f(z) + g(z') = u(t,x), t \geq 0, x \in \Omega, \\ z\,|_\Gamma = 0, \quad t \geq 0, \\ z(0) = h_0(x), \; z_t(0) = h_1(x) \quad x \in \Omega. \end{cases}$$

We may formulate this equation as the following abstract evolution equation:

$$\dot{y} = Ay + P(y) + Bu, \; y(t) = \begin{pmatrix} z \\ z_t \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \; y(0) = \begin{pmatrix} h_0(x) \\ h_1(x) \end{pmatrix}$$

with $X = H_0^1(\Omega) \times L_2(\Omega), D(A) = H^2(\Omega) \times H_0^1(\Omega)$, and

$$A = \begin{pmatrix} 0 & 1 \\ \Delta & 0 \end{pmatrix}, P(y) = \begin{pmatrix} 0 & 0 \\ -f(y_1) & -g(y_2) \end{pmatrix}, B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

As in example 5.15, we consider the case where $n = 1, \Omega = (0,1)$ and let $f(z) = 0$ (if $f \neq 0$, a similar result holds), $g(\xi) = -\frac{4\xi}{1+|\xi|}, h_0(x) = \sin x, h_1(x) = x - x^2$. It is known that if $D = (1,-2), \tilde{A} = A + BD$ generates an exponentially stable group. Since $P(y) = \begin{pmatrix} 0 & 0 \\ 0 & \frac{4y_2}{1+|y_2|} \end{pmatrix}$, we have

$$(v, P(v)) = \frac{4\|v_2\|^2}{(1+|v_2|)} \leq 4\|v_2\|^2$$
$$= 4(v, BB^*v) \quad \text{for} \quad v \in X.$$

Therefore, by taking $\mu = 4$ in Assumption (A2)$'$ all the conditions of theorem 5.13 are satisfied and the feedback system is exponentially stable with the control $u = D_2 y = y_1 - 6y_2$. However, with the nominal control $u = D_0 y = y_1 - 2y_2$, it is unstable (see Fig.5.2).

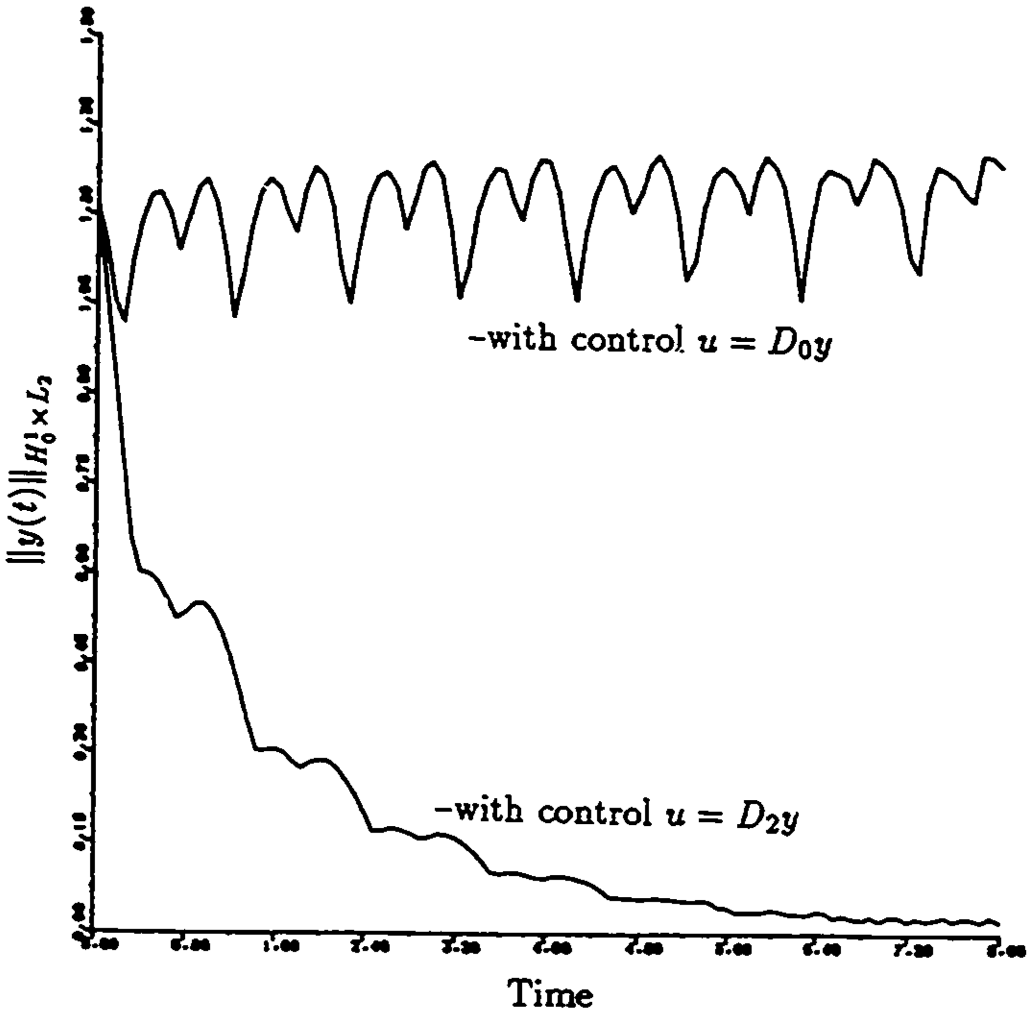

Fig.5.2. Semilinear wave equation (Example 5.16)

CHAPTER 6

FEEDBACK STABILIZATION OF STOCHASTIC EVOLUTION EQUATIONS

6.1 Introduction

We have considered the question of stabilization of deterministic systems in Hilbert space. However, in many physical and engineering problems, noise appears distributed both in time and space. An important example is the galloping conductor where this distributed noise could be attributed to the random aerodynamic forces acting on the (transmission line) conductors, arising because of the randomness of wind velocity and the irregularity of ice formation on the conductor surface [Riaz, Biswas and Ahmed 103]. Distributed noise arises also in many other practical systems such as flexible satellites, heating furnaces, etc. From the system theory point of view, therefore, stability of stochastic evolution equations is also important. In fact, much effort has been made in this area [Ahmed, Ichikawa, $\cdots$, etc]. For example, the question of stability of stochastic evolution equations

$$dx = [Ax + f(x)]dt + G(x)dw(t), \quad x(0) = x_0 \tag{6.1}$$

has been considered by Ichikawa [59] where A generates a strongly continuous semigroup on a real separable Hilbert space, and f, G are linear (or nonlinear) bounded operators. But in many practical situations, f and G might be bounded or unbounded, even uncertain; that is, $f, G \in \mathcal{P}$, where $\mathcal{P}$ is a family of linear (nonlinear) unbounded operators. The problem of stabilization here is to design a state feedback control law that assures exponential stability of the null state uniformly with respect to perturbations $f, G \in \mathcal{P}$. In this chapter, we show that our approach developed for determinstic systems can be extended to stochastic evolution equations based on perturbation theory of semigroups and some results of Ichikawa. Furthermore, in addition to exponential stability in the mean square sense for stochastic systems, stability of sample paths is also obtained.

6.2 Preliminaries and Problem Statements

It is clear that to consider stochastic evolution equations we need some basic abstract probability theory. Let H and Y be real separable Hilbert spaces. We denote by $\langle , \rangle$ the inner products in these Hilbert spaces and by $\| \ \|$ the norms of vectors and operators. We suppose throughout that $(\Omega, \mathcal{F}, \mu)$ is a complete probability space. Let $y : \Omega \to Y$ be a square-integrable random variable, i.e., $y \in L_2(\Omega, \mathcal{F}, \mu; Y)$. The covariance operator of y is

$$C(y) = C(y, y) = E[(y - Ey) \circ (y - Ey)],$$

where E denotes the expectation and $g \circ h \in \mathcal{L}(Y)$ for any $g, h \in Y$ is defined by

$$(g \circ h)k = g\langle h, k\rangle, \qquad k \in Y.$$

Consequently, $C(y)$ is a self-adjoint, nonnegative, trace class (or nuclear) operator [Ichikawa 59] and $trC(y) = E[\|y - Ey\|^2] = E[\|y\|^2] - \|Ey\|^2$, where tr denotes the trace. If $P \in \mathcal{L}(Y)$, then

$$tr[PC(y)] = trC(Py, y) = E\langle P(y - Ey), y - Ey\rangle \tag{6.2}$$

where $C(x, y) = E[(x - Ex) \circ (y - Ey)]$ is the joint covariance of x and y.

A random variable $y \in L_2(\Omega, \mathcal{F}, \mu; Y)$ is Gaussian if $\langle y, e_i \rangle$ is a real Gaussian random variable for all i, where $\{e_i\}, i = 1, 2, \cdots$, is a complete orthonormal basis for Y.

Let $\mathcal{T} = [0, T]$ be a subinterval of $[0, \infty)$. A stochastic process in Y is a family of random variables $y(t), t \in \mathcal{T}$ in Y. A stochastic process $x(t), t \in \mathcal{T}$, is a modification of $y(t)$ if for each $t \in \mathcal{T}, x(t) = y(t)$ with probability one. There are several types of continuity one can define [Ichikawa 59], but all we require is the following:

Definition 6.1

Let $y(t)$ be a Y-valued stochastic process on $[0, T]$. Then

(1) $y(t)$ is continuous in mean square if

$$E[\|y(t+\epsilon) - y(t)\|^2] \to 0 \quad \text{as } \epsilon \to 0,$$

(2) $y(t)$ has continuous sample paths if

$$\mu\{\sup_{0 \le t \le T} \|y(t+\epsilon) - y(t)\| > 0\} \to 0 \quad \text{as } \epsilon \to 0.$$

The process $y(t)$ is measurable if y is measurable relative to $\mathcal{B}(\mathcal{T}) \times \mathcal{F}$, where $\mathcal{B}(\mathcal{T})$ is the Borel field of subsets of $\mathcal{T}$. Let $\mathcal{F}_t, t \in \mathcal{T}$, be a family of increasing sub σ-fields of $\mathcal{F}$. A stochastic process $y(t), t \in \mathcal{T}$, is adapted to $\mathcal{F}_t$ if $y(t)$ is $\mathcal{F}_t$ measurable for all $t \in \mathcal{T}$. It is a martingale with respect to $\mathcal{F}_t$ if it is adapted to $\mathcal{F}_t$ with the properties:

(a) $E\|y(t)\| < \infty$ for all $t \in \mathcal{T}$,

(b) $E[y(t)|\mathcal{F}_s] = y(s)$ for all $s < t, s, t \in \mathcal{T}$, where $E[\cdot|\mathcal{F}_s]$ denotes the conditional expectation with respect to $\mathcal{F}_s$.

A particular example of a martingale is a Wiener process, which is used for modeling white noise disturbances in engineering systems. The following is one of several equivalent definitions.

Definition 6.2 [Ichikawa 59]

A stochastic process $w(t), t \geq 0$, in a real Hilbert space is a Wiener process if:

(i) $w(t) \in \mathcal{L}_2(\Omega, \mathcal{F}, \mu; H)$ and $Ew(t) = 0$ for all $t \geq 0$.

(ii) $C(w(t) - w(s)) = (t-s)Q, Q \in \mathcal{L}(H)$ is a nonnegative nuclear operator.

(iii) $w(t)$ has independent increments.

(iv) $w(t)$ has continuous sample paths.

The operator Q is the incremental covariance operator of the Wiener process $w(t)$.

In fact, one can show that as a consequence of definition 6.2, there exists a complete orthonormal basis $\{e_i\}_{i=1}^{\infty}$ for H such that

$$w(t) = \sum_{i=1}^{\infty} \beta_i(t) e_i, \tag{6.3}$$

where $\beta_i(t)$ are mutually independent real Wiener processes with incremental variance parameter $\lambda_i > 0$ and $trQ = \sum_{i=1}^{\infty} \lambda_i < \infty$. Furthermore, $w(t)$ is Gaussian and

$$E\|w(t) - w(s)\|^2 = \sum_{i=1}^{\infty} \lambda_i (t-s) = trQ(t-s). \tag{6.4}$$

Next we introduce stochastic integrals with respect to $w(t)$, $\mathcal{T} = [0, T], 0 < T < \infty$. Let $\mathcal{F}_t$ be a family of increasing sub σ-fields of $\mathcal{F}$ such that

$$w(t) \text{ is measurable relative to } \mathcal{F}_t \text{ for each } t \in \mathcal{T},$$

$$w(t) - w(s) \text{ is independent of } \mathcal{F}_s \text{ for all } s < t, s, t \in \mathcal{T}.$$

Let $M(H, Y)$ be the space of stochastic processes $G(\cdot,\cdot) : \mathcal{T} \times \Omega \to \mathcal{L}(H, Y)$ which are strongly measurable, i.e., $G(t,\cdot)h$ is a measurable stochastic process for all $h \in H$. Define also

$$M_1(H,Y) = \{G \in M(H,Y) | E[\int_0^t \|G(t)\|^2 dt] < \infty\}. \tag{6.5}$$

Lemma 6.3 [Ichikawa 59]

$$\int_{t_0}^{t_1} G(t)dw(t) = \sum_{i=1}^{\infty} \int_{t_0}^{t_1} G(t)e_i d\beta_i(t), \quad 0 \le t_0 \le t_1 \le T,$$

where $w(t) = \sum_{i=1}^{\infty} \beta_i(t)e_i$, $G \in M_1(H,Y)$, and the limit is in $L_2(\Omega, \mathcal{F}, \mu; Y)$. Furthermore, the integral satisfies

$$E[\int_{t_0}^t G(s)dw(s)] = 0, \tag{6.6}$$

$$\begin{aligned} E[\| \int_{t_0}^t G(s)dw(s)\|^2] &= \int_{t_0}^t E(trG(s)QG^*(s))ds \\ &\le trQ \int_{t_0}^t E\|G(s)\|^2 ds, \end{aligned} \tag{6.7}$$

and $y(t) = \int_0^t G(s)dw(s)$ is a well-defined Y-valued stochastic process (martingale), which has a modification with continuous sample paths. ■

We now consider the stochastic evolution equation

$$dx = (Ax + P(x))dt + G(x)dw(t) + Bu(t)dt, \; x(0) = x_0 \tag{6.8}$$

where, by (6.8), we mean the integral equation

$$x(t) = x_0 + \int_0^t [Ax + P(x) + Bu]ds + \int_0^t G(x)dw(s) \tag{6.9}$$

on a real Hilbert space X, where $A : D(A) \to X$ is the infinitesimal generator of a C_0-semigroup $T(t), t \ge 0$, on X and B is a bounded linear operator from U to X, U

is another real Hilbert space, and $P \in \mathcal{P}$ is a family of linear or nonlinear unbounded operators perturbing the generator A. Let $w(t), t \geq 0$, be a Wiener process taking values in a Hilbert space H(real and separable). Let $\mathcal{F}_t = \sigma(w(s), 0 \leq s \leq t)$ denote the σ-algebra generated by $w(s), 0 \leq s \leq t$ and $G : X \to \mathcal{L}(H, X)$ might be linear or nonlinear. Recall that a stochastic process $x(t), t \geq 0$, is said to be a mild solution of (6.1) if $x(t)$ is $\mathcal{F}_t$-adapted and satisfies the associated integral equation

$$x(t) = T(t)x_0 + \int_0^t T(t-s)f(x(s))ds + \int_0^t T(t-s)G(x(s))dw(s). \tag{6.10}$$

Lemma 6.4

Consider the stochastic evolution equation

$$dx = Axdt + G(x)dw, \quad x(0) = x_0 \in X, t \geq 0, \tag{6.11}$$

and suppose that $A \in G(X), G \in \mathcal{L}(X, \mathcal{L}(H, X))$ and there exists a constant $a > 0$ such that

$$2(x, Ax) + tr\{QG^*(x)G(x)\} \leq -a\|x\|^2 \text{ for } x \in X. \tag{6.12}$$

Then the mild solution of (6.11) (exists [Ahmed 1,3]) is exponentially stable in the sense that there exist $M \geq 1, w > 0$, such that $E\{\|x(t, x_0)\|^2\} \leq Me^{-wt}\|x_0\|^2$.

Proof

Under the given assumptions Ito's lemma applies [Ahmed 3]. Using this one can show that the Ito differential of the functional $V(x) \equiv (1/2)\|x\|^2$ along the solution trajectory $x(t) = x(t, x_0)$ is given by

$$dV(x) = (x, Ax)dt + (x, G(x)dw) + (1/2)tr\{G(x)QG^*(x)\}dt.$$

Hence upon taking the expectation we have

$$dE\|x(t)\|^2 = 2E\{(x(t), Ax(t))\} + E\{tr[G(x)QG^*(x)]\}$$

$$\leq -aE\|x(t)\|^2.$$

Hence the result follows. ■

As mentioned earlier in Chapter 5, Lyapunov theory is not always applicable to infinite dimensional systems. Suppose there exists a Lyapunov operator $S \geq 0$ and a constant $d > 0$ such that

$$2(Sx, Ax + f(x)) + trG^*(x)SG(x)Q \leq -d\|x\|^2, \text{ for } x \in D(A),$$

which ensures L_2 stability; *i.e.* $\int_0^\infty E\|x(t,x_0)\|^2 dt \leq K\|x_0\|^2, x_0 \in X$ for some $K > 0$ where $x(t, x_0)$ is the mild solution of (6.1). However, in general, we cannot conclude from this that $E\|x(t,x_0)\|^2 \leq Me^{-at}\|x_0\|^2$ for some $M \geq 1$ and $a > 0$. But Ichikawa has shown the following

<u>Lemma 6.5</u> [Ichikawa 60]

Suppose there exists a positive continuous function g such that both g and its reciprocal are defined on $[0, \infty)$, and

(C) $E\|x(t,x_0)\|^2 \leq g(t)\|x_0\|^2$, for $x_0 \in X$.

Then the following conditions are equivalent:

(E1) $\int_0^\infty E\|x(t,x_0)\|^2 dt \leq K\|x_0\|^2$ for some $K > 0$.

(E2) $E\|x(t,x_0)\|^2 \leq Me^{-at}\|x_0\|^2$ for some $M \geq 1$ and $a > 0$. ■

Remark 6.6

In [59], Ichikawa assumes that $f(x)$ and $G(x)$ of equation (6.1) satisfy Lipschitz conditions, which guarantee that the inequality (C) holds. In fact, a unique mild solution $x(t, x_0), t \geq 0$, exists, and is continuous in the mean square sense [Ichikawa 60]. Moreover, there exist positive numbers N and b such that

$$E\|x(t, x_0)\|^2 \leq N e^{bt} \|x_0\|^2 \quad \text{for any} \quad x_0 \in X, t \geq 0. \tag{6.13}$$

Therefore, according to lemma 6.5, the Lyapunov method is applicable to system (6.1). In other words, sufficient conditions for L_p stability ensure exponential stability as well in this case.

6.3 Exponential Stabilization of Linear Perturbed Stochastic Systems

Now we are prepared to prove a result on the stability and robustness of the linear perturbed system

$$dx = (A + P)x dt + B u dt + G x dw(t), \quad x(0) = x_0, \ t \geq 0. \tag{6.14}$$

Define the set

$\mathcal{P}_{ub} \equiv \{P \in \mathcal{L}_u(X) : D(A) \subset D(P) \subset X, \ (x, Px) \leq k(x, Ax) + \gamma(x, x)$ for some $k \geq 0$ and $\gamma \geq 0$; and there exist $0 \leq \alpha < 1, \beta \geq 0$ such that $\|Px\| \leq \alpha\|Ax\| + \beta\|x\|$ for all $x \in D(A)\}$.

Theorem 6.7

Consider the perturbed system (6.14) and let the following assumptions hold:

(A1) The generator A is exponentially stabilizable; that is, there exists a $D_0 \in \mathcal{L}(X)$ such that $A_0 \equiv A + BD_0$ generates an exponentially stable contraction semigroup $T_0(t)$ in X satisfying $\|T_0(t)\| \leq e^{-wt}$, for all $t \geq 0$ and some $w > 0$.

(A2) $P \in \mathcal{P}_{ub}$ and there exists a $\nu > 0$ such that $\|G(x)\|_{\mathcal{L}(H,X)} \leq \nu\|x\|_X$ for $x \in X$.

(A3) The linear operator BB^* is coercive; that is, $(x, BB^*x) \geq \lambda\|x\|^2$, for some $\lambda > 0$ and $x \in X$.

Then the system (6.14), with a linear feedback control law given by $u = D_1 x$, where

$$D_1 \equiv (1+k)D_0 - \frac{1}{2\lambda}(2\gamma + \nu^2 trQ)B^* \in \mathcal{L}(X, U), \tag{6.15}$$

is exponentially stable in the mean square sense.

Proof

By virtue of lemma 6.4, it suffices to prove that

(i) $\tilde{A} \equiv A + P + BD_1$ generates a C_0-semigroup in X,

and

(ii) there exists a constant $d > 0$ such that

$$2(x, (A + P + BD_1)x) + trG(x)QG^*(x) \leq -d\|x\|^2$$

for all $x \in D(A)$ and all $P \in \mathcal{P}_{ub}$.

Define

$$P_0 \equiv P + kBD_0 - \frac{1}{2\lambda}(2\gamma + \nu^2 trQ)BB^*. \tag{6.16}$$

By virtue of (A1), A_0 generates a contraction semigroup $T_0(t), t \geq 0$, in X satisfying $\|T_0(t)\| \leq e^{-wt}$. Hence A_0 is (strictly) dissipative. Furthermore, it is easy to verify that $D(A) = D(A_0) \subset D(P_0) = D(P)$ and P_0 is relatively A_0- bounded with the same relative bound α. More precisely, there is a positive constant $\beta_0 \equiv \beta_0(k, \lambda, \gamma, \nu, B, D_0, Q) \geq 0$, depending on the quantities displayed, so that

$$\|P_0\xi\| \leq \alpha\|A_0\xi\| + \beta_0\|\xi\| \quad \text{for all } \xi \in D(A_0).$$

We show that P_0 is dissipative. For $x \in D(A_0)$, it follows from assumptions (A1), (A2) and (A3) that

$$\begin{aligned}(x, P_0x) &\leq (x, (kA + kBD_0)x) + (x, (\gamma - \frac{\gamma}{\lambda}BB^*)x) - 1/2\lambda(x, (\nu^2 trQ)BB^*x) \\ &\leq -kw\|x\|^2 \leq 0.\end{aligned}$$

Thus it follows from the perturbation result [Pazy 96], that $A_0 + P_0$ is the generator of a C_0-semigroup of contractions in the Hilbert space X. In fact, it generates an exponentially stable semigroup. This implies (i). For (ii), we compute

$$\begin{aligned}&2(x, (A + P + BD_1)x) + tr[G(x)QG^*(x)] \\ =&2(x, (A_0 + P_0)x) + tr[G(x)QG^*(x)] \\ \leq& -2(1 + k)w\|x\|^2 - \nu^2 trQ\|x\|^2 + trG(x)QG^*(x) \\ \leq& -d\|x\|^2 \quad \text{for all } P \in \mathcal{P}_{ub} \text{ and } x \in D(A),\end{aligned}$$

where $d = -2(1 + k)w > 0$.

Therefore, the perturbed system (6.14) with the feedback control $u = D_1x$ is exponentially stable in the mean square sense uniformly with respect to the set $\mathcal{P}_{ub}$. ■

The assumption that $G \in \mathcal{L}(X, \mathcal{L}(H, X))$, as given in theorem 6.7, has the limitation that it does not allow unbounded operators while, in application to stochastic partial differential equations, this situation arises if the coefficients of differential operators contain white noise [Ahmed 1.3]. Therefore, it is necessary to extend theorem 6.7 to admit operators G which are relatively A-bounded. Next, we consider a situation where G admits differential operators.

$$dx = (A + P + \frac{1}{2}G^2)xdt + Budt + Gxdw(t), x(0) = x_0, t \geq 0, \qquad (6.17)$$

where G generates a uniformly bounded strongly continuous group $S(t), t \in R = (-\infty, \infty)$ on X satisfying $\|S(t)\| \leq N$ for some $N \geq 1$ and $w(t)$ is a (one dimensional) standard Wiener process. By virtue of theorem 6.7, if the conditions (A1)-(A3) are satisfied, there exists a $D_2 \in \mathcal{L}(X, U)$ such that $\tilde{A} \equiv A + P + BD_2$ generates an exponentially stable contraction semigroup $T_d(t), t \geq 0$, in X. For this stability result, we need the following additional condition:

(A4) $D(\tilde{A}) \subset D(G^2), T_d(t) : D(\tilde{A}) \rightarrow D(\tilde{A})$ and the operators $T_d(t)$, $S(t)$ commute for all $t \geq 0$. Furthermore, $\tilde{A} + \frac{1}{2}G^2$ is dissipative and there exist $1 > \mu \geq 0$ and $\delta \geq 0$ such that $\|G^2x\| \leq 2\mu\|\tilde{A}x\| + \delta\|x\|$, for $x \in D(\tilde{A})$.

Theorem 6.8

Consider the controlled system (6.17) and let the conditions (A1)-(A4) be satisfied. Then there exists a feedback control $u = D_2x, D_2 \in \mathcal{L}(X, U)$, such that the null-state $x = 0$ is exponentially stable in the mean square sense.

Proof

Define $D_2 \equiv (1+k)D_0 - \frac{\gamma}{\lambda}B^* \in \mathcal{L}(X, U)$. Following the proof of theorem 6.7, it is clear that $\tilde{A} = A + P + BD_2$ generates an exponentially stable contraction semigroup $T_d(t), t \geq 0$, in X satisfying $\|T_d(t)\| \leq e^{-dt} \leq 1$ for some $d > 0$ and $t \geq 0$. Since $D(\tilde{A}) \subset D(G^2), \tilde{A} + \frac{1}{2}G^2$ is dissipative and $\|\frac{1}{2}G^2x\| \leq \mu\|\tilde{A}x\| + \frac{1}{2}\delta\|x\|$, it follows by theorem 3.8 that $\tilde{A} + \frac{1}{2}G^2$ generates a C_0-semigroup of contractions $T_G(t), t \geq 0$, in X. Defining

$$\hat{A} \equiv \tilde{A} + \frac{1}{2}G^2, \quad f(x) \equiv Gx,$$

our system equation (6.17) reduces to

$$\begin{cases} dx(t) = \hat{A}xdt + f(x)dw(t), & t \geq 0 \\ x(0) = x_0 \end{cases} \qquad (6.18)$$

where $\hat{A}$ and $f(x)$ satisfy all the conditions of theorem 3 in [Ahmed 1]. Then for each $x_0 \in X$ the evolution equation (6.18) has a unique mild solution given by the solution of the integral equation,

$$x(t) = T_G(t)x_0 + \int_0^t T_G(t-\theta)f(x(\theta))dw(\theta). \qquad (6.19)$$

Furthermore, the solution is continuous with probability one. On the other hand, due to condition (A4), we have $x(t) = T_d(t)S(w(t))x_0$, for $x_0 \in D(A)$ (apply Ito's formula in Hilbert space) [Ichikawa 59]. Hence, there exist $M = M(N) \geq 1$ and $a = a(d) > 0$ such that $E\|x(t)\|^2 \leq Me^{-at}\|x_0\|^2$ for $x_0 \in D(A)$. ■

Again, like deterministic cases, one should obtain the same results without the coercive assumption for the operator BB^*. Define

$\mathcal{P}'_{ub} \equiv \{P \in \mathcal{L}_u(X) : D(A) \subset D(P) \subset X,\ (x, Px) \leq k(x, Ax) + \gamma(x, BB^*x)$ for some $k \geq 0$ and $\gamma \geq 0$; and there exist $0 \leq \alpha < 1, \beta \geq 0$ such that $\|Px\| \leq \alpha\|Ax\| + \beta\|x\|$ for all $x \in D(A)\}$.

We have the following:

<u>Theorem 6.9</u>

Consider the system (6.14) and suppose the following assumptions hold:

(A1) Assumption (A1) of theorem 6.7 holds.

(A2)′ Suppose $P \in \mathcal{P}'_{ub}$ and there exists a $\nu > 0$ such that $\|G(x)\|_{\mathcal{L}(H,X)} \leq \nu\|x\|$ and $tr[G(x)QG^*(x)] - \nu^2 trQ(x, BB^*x) \leq 0$ for all $x \in X$.

Then the feedback system with a control given by $u = D_3x$, where

$$D_3 \equiv (1+k)D_0 - [\gamma + \frac{1}{2}\nu^2 trQ]B^*, \tag{6.20}$$

is exponentially stable in the mean square sense.

<u>Proof</u>

Define

$$P_1 \equiv P + kBD_0 - [\gamma + \frac{1}{2}\nu^2 trQ]BB^*.$$

By assumption (A2)′, P_1 is dissipative and A_0-bounded with the same relative bound α. Furthermore,

$$2(x, (A + P + BD_3)x) + tr[G(x)QG^*(x)] \leq -2(1+k)w\|x\|^2.$$

Then the result follows from arguments similar to those used in theorem 6.7. ■

<u>6.4 Exponential Stabilization of Nonlinear Perturbed Stochastic Systems</u>

In this section, a controlled semilinear stochastic system given by,

$$dx = (Ax + P(x))dt + G(x)dw(t) + Bu(t)dt, \; x(0) = x_0, t \geq 0, \tag{6.21}$$

on a real separable Hilbert space X is considered, where $G : X \to \mathcal{L}(H, X)$ are nonlinear, and $P \in \mathcal{P}$ is a family of nonlinear unbounded operators perturbing the generator A. Our main objective is to find a state feedback control law such that the mild solution of (6.21) is exponentially stable in the mean square sense uniformly with respect to structural perturbations.

Theorem 6.10

Consider the controlled system (6.21) and let the following assumptions hold:

(A1) Assumption (A1) of theorem 6.7 holds.

(A2) $P = P_1 + P_2$, where $P_1 \in \mathcal{P}_{nb}$ and

$P_2 \in \mathcal{P}_{nb} \equiv \{P : P$ maps $X \to X$ and satisfies a Lipschitz condition with $P(0) = 0$; *i.e.*, $\|P(x) - P(y)\| \leq \mu\|x - y\|$ for some $\mu > 0\}$.

Let $\mathcal{P} \equiv \{P : P$ satisfies assumption (A2) $\}$.

(A3) Assumption (A3) of theorem 6.7 holds.

(A4) The operator $G : X \to \mathcal{L}(H, X)$ (the space of all linear bounded operators from H to X) is nonlinear and satisfies a Lipschitz condition together with $G(0) = 0$; that is, $\|G(x) - G(y)\|_{\mathcal{L}(H,X)} \leq \nu\|x - y\|_X$ for all $x, y \in X$, some $\nu > 0$.

Then the mild solution $x(t, x_0)$ of system (6.21) with a linear feedback control law given by $u = D_4 x$, where

$$D_4 \equiv (1 + k)D_0 - \frac{1}{2\lambda}(2\gamma + 2\mu + \nu^2 trQ)B^* \in \mathcal{L}(X, U), \qquad (6.22)$$

is uniformly exponentially stable with respect to perturbations in the mean square sense; that is,

$$E\|x(t, x_0)\|^2 \leq Me^{-at}\|x_0\|^2, \quad x_0 \in X \text{ for some } M \geq 1 \text{ and } a > 0.$$

Proof

By virtue of lemma 6.5, it suffices to prove that

(i) there exists a unique mild solution $x(t, x_0)$ to

$$dx = (Ax + P(x) + BD_4x)dt + G(x)dw(t), \quad x(0) = x_0, t \geq 0, \tag{6.23}$$

which is continuous in the mean square sense, and a positive continuous function $g(t)$ on $[0, \infty)$ such that, for $x(t, x_0)$, the condition (C) in lemma 6.5 is satisfied.

(ii) there exists a constant $d > 0$ such that

$$2(x, (A + BD_4)x + P(x)) + trG(x)QG^*(x) \leq -d\|x\|^2, x \in D(A)$$

holds for all $P \in \mathcal{P}$.

Using the feedback control law (6.22) define

$$A_r \equiv (A + BD_0) + (P_1 + kBD_0 - \gamma BB^*)$$

$$= \tilde{A} + A_p,$$

$$f \equiv P_2 - [(2\mu + \nu^2 trQ)/2\lambda]BB^*.$$

Then the system (6.23) reduces to

$$dx = [A_r x + f(x)]dt + G(x)dw(t), \quad x(0) = x_0 \in X, t \geq 0 \tag{6.24}$$

with $f(0) = 0$ and $G(0) = 0$. Thus for (i), by virtue of Remark 6.6, it suffices to show that A_r generates a C_0-semigroup. By assumption (A1) the semigroup $S(t)$ generated by $\tilde{A}$ satisfies $\|S(t)\| \leq e^{-wt} \leq 1$ for $t \geq 0$. Therefore, if one can show that $A_r \equiv \tilde{A} + A_p$ is dissipative and A_p is relatively bounded with respect to $\tilde{A}$ with relative bound $\alpha < 1$, A_r generates a C_0-semigroup of contractions in X. In fact, by assumption (A2),

$$(x, A_r x) \leq (x, P_1 x) + k(x, BD_0 x) - \gamma(x, BB^*)$$

$$\leq k(x, (A + BD_0)x)$$

$$\leq -kw\|x\|^2 \leq 0 \quad \text{for } x \in D(A) \subset D(A_r).$$

Hence A_r is dissipative. On the other hand,

$$\|A_p x\| \leq \|P_1 x\| + k\|BD_0 x\| + \gamma\|BB^* x\|$$

$$\leq \alpha\|Ax\| + \beta_2\|x\|$$

$$\leq \alpha\|\tilde{A}x\| + \beta_3\|x\| \quad \text{for } x \in D(A) = D(\tilde{A}) \subset D(A_p),$$

where $\beta_2 \equiv \beta + k\|BD_0\| + \gamma\|BB^*\|, \beta_3 \equiv \beta_2 + \alpha\|BD_0\|$. Hence $A_r = \tilde{A} + A_p$ generates a C_0-semigroup of contractions in X.

For (ii), utilizing Assumptions (A1)-(A4), we obtain

$$2(x, (A + BD_3)x + P(x)) + trG(x)QG^*(x)$$

$$\leq 2(x, (A + BD_0)x) + 2(x, (P_1 + kBD_0 - \gamma BB^*)x)$$

$$+2(x, P_2(x) - \frac{\mu}{\lambda}BB^* x) + trQ\|G(x)G^*(x)\| - \frac{\nu^2 trQ}{\lambda}(x, BB^* x)$$

$$\leq -2(1 + k)w\|x\|^2 = -d\|x\|^2, \text{ for } x \in D(A),$$

where $d = 2(1 + k)w > 0$. This completes the proof. ■

Remark 6.11

It is obvious that the class of perturbations $P \in \mathcal{P}$ is a subset of that given in chapter 5; that is, more conditions on the perturbation were imposed in stochastic systems than the corresponding ones in deterministic systems. This is natural on account of the complexity of stochastic systems. However, we believe that further research can be conducted in order to relax the conditions on the perturbation. Furthermore, the assumption on the control operator B might also be relaxed, as was done in theorem 6.9.

<u>Remark 6.12</u>

In this section, it is assumed that G satisfies a Lipschitz condition on X which has the limitation that it does not allow unbounded operators. However, our stability results can be extended to some systems where G might be differential (unbounded) operators as in the linear case.

In practice, one is more concerned with sample path stability than moment stability (i.e., stability in the mean square sense) while mild solution of stochastic systems has continuous sample paths. In fact, under the assumptions of theorem 6.10 and using Ichikawa's result [59], one can show that sample paths of the mild solution of (6.24) are exponentially stable with probability one.

<u>Theorem 6.13</u>

Consider the controlled system (6.21) and suppose the conditions in theorem 6.10 are satisfied. Then the sample paths of the mild solution $x(t, x_0)$ of the feedback system

$$dx = (Ax + BD_4x + P(x))dt + G(x)dw(t),\ x(0) = x_0 \in X \text{ (nonrandom)}, t \geq 0$$

are exponentially stable with probability one; that is, there exists a random time $0 < T(\omega) < \infty$ and constant $M_1 > 0$ such that for all $t \geq T(\omega)$,

$$\|x(t, x_0)\|^2 \leq M_1 e^{-at/4} \|x_0\|^2 \qquad w.p.1$$

where $a \leq 0$ is the same constant as that in theorem 6.10.

Proof

By theorem 6.10, the mild solution $x(t, x_0)$ of feedback system

$$dx = (Ax + P(x) + BD_4 x)dt + G(x)dw(t)$$

$$\equiv (A_r x + f(x))dt + G(x)dw(t), \; x(0) = x_0, t \geq 0,$$

is exponentially stable in the mean square sense such that

$$E\|x(t, x_0)\|^2 \leq Me^{-at}\|x_0\|^2, \; x_0 \in X \text{ for some } M \geq 1 \text{ and } a > 0.$$

Furthermore, $f(x), G(x)$ satisfy Lipschitz conditions with $f(0) = 0$ and $G(0) = 0$. Define $v(x) = \|x\|^2$. If one can show that

(i) $a_1 v(x) \geq \|x\|^2, \; a_1 > 0,$

(ii) $v(x)$ is twice Frechet differentiable and derivatives $v_x(x)$ and $v_{xx}(x)$ are continuous in X and $\mathcal{L}(X)$, respectively,

(iii) $v(x) + \|x\|\|v_x(x)\| + \|x\|^2\|v_{xx}(x)\| \leq a_2\|x\|^2, a_2 > 0,$

(iv) $(v_x(x), A_r x) \leq b_1 v(x)$ for all $x \in D(A_r) = D(A), b_1 \geq 0,$

then by virtue of Ichikawa's result [59, theorem 5.1], there exists a random variable $0 < T(\omega) < \infty$ and a constant $M_1 > 0$ such that for all $t > T(\omega)$,

$$\|x(t, x_0)\|^2 \leq M_1 e^{-at/4}\|x_0\|^2 \qquad w.p.1.$$

However, (i),(ii) and (iii) are obvious. For (iv),

$$(v_x(x), A_r x) = 2(x, A_r x)$$

$$\leq -2(1 + k)wv(x) \leq 0.$$

This completes our proof. ■

6.5 Examples and Numerical Results

Example 6.14

Consider the stochastic version of example 5.15.

$$\begin{cases} dz = Azdt + P(z)dt + Budt + G(z)dw(t) \\ z(0)(x) = \sin x \end{cases}$$

on the real Hilbert space $X = L_2(0,1)$ where $w(t)$ is a (one dimensional) standard Wiener process, $A = (1/2)\frac{\partial^2}{\partial x^2} + x^2 + 6/\sqrt{2}$ with $D(A) = \{z \in X : z \in H_0^2(0,1)\}$ and $B = I$. If we take $D_0 = -1$, it is known that $\tilde{A} = A + BD_0$ generates an exponentially stable semigroup in X. The operator $P(z)$ is given by

$$P(z) = P_1 z + P_2(z) = \frac{6}{\sqrt{2}}(1-x)\frac{\partial z}{\partial x} + 4\|z\|.$$

Since

$$\begin{aligned}(z, P_1 z) &= (6/\sqrt{2})\int_0^1 (1-x)z\frac{\partial z}{\partial x}dx \\ &= (6/\sqrt{2})(1-x)z^2|_0^1 - (6/\sqrt{2})(\int_0^1 (1-x)z\frac{\partial z}{\partial x}dx - \int_0^1 z^2 dx),\end{aligned}$$

therefore $(z, P_1 z) = (3/\sqrt{2})\int_0^1 z^2 dx = (3/\sqrt{2})\|z\|^2$.

Furthermore, it is easy to verify that P_1 is A-bounded with A-bound zero in X. For $P_2(z)$,

$$\|P_2(v) - P_2(w)\| \leq 4\|v - w\| \quad \forall v, w \in X.$$

Assume that $G(\xi) = \sin \xi$, then for $v, w \in X$,

$$\|G(v) - G(w)\| \leq \|v - w\|,$$

Hence, by choosing $\gamma = 3, \mu = 4, k = 0$ and $\nu = 1$, all the conditions of theoerm 6.10 are satisfied. Numerical results are shown in Fig.6.1, where the Gaussian noise process was generated using the IMSL subroutine GGNML and numerical integration was carried out using Runge-Kutta-like algorithm.

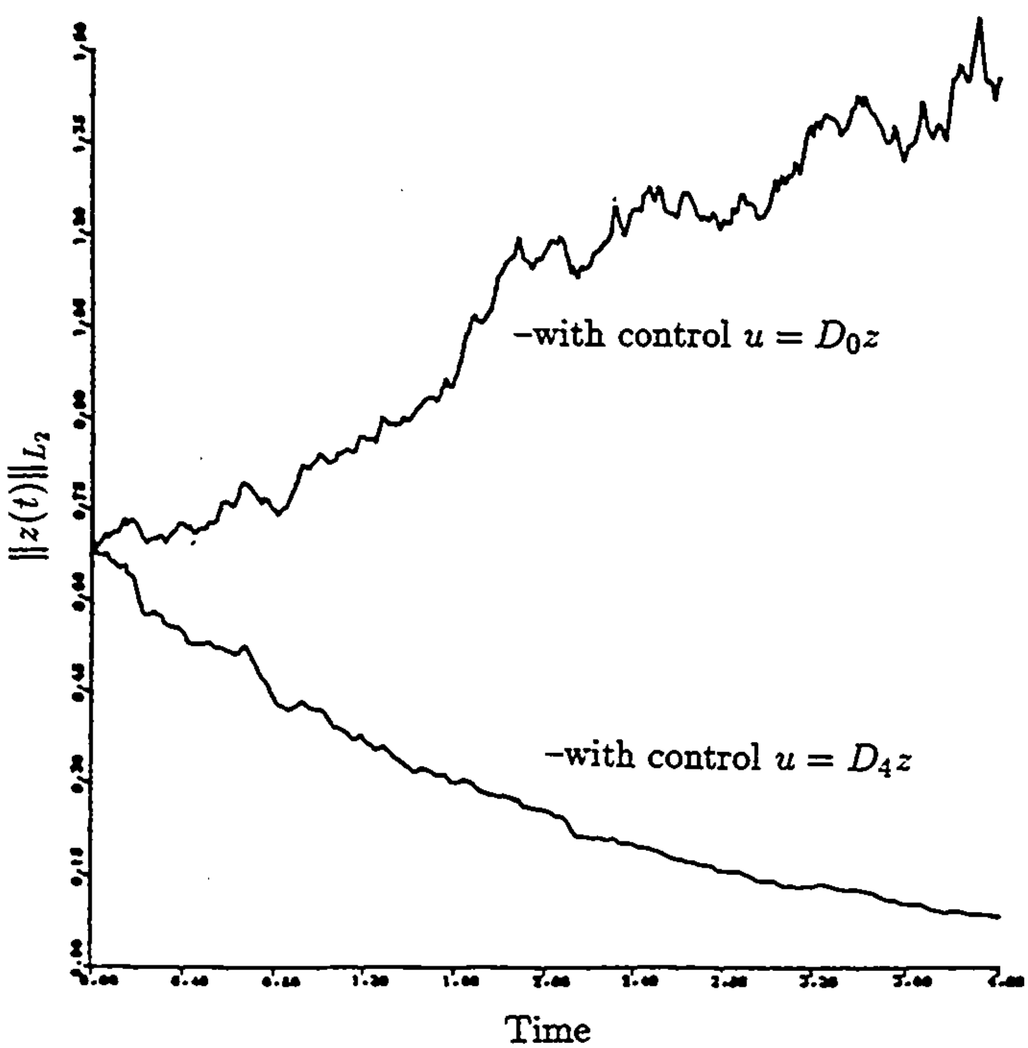

Fig.6.1. Semilinear stochastic diffusion equation (Example 6.14)

Example 6.15

Consider the controlled stochastic parabolic equation

$$\begin{cases} dz = \dfrac{1}{2}\dfrac{\partial^2}{\partial \xi^2} z dt - \dfrac{\partial}{\partial \xi} z dt + (r + \dfrac{1}{2}) z dt + (\dfrac{\partial}{\partial \xi} z - z) dw(t) + u dt, \xi \in (0,1) \\ z(0,t) = z(1,t) = 0 \\ z(\xi, 0) = z_0(\xi), \ \xi \in (0,1). \end{cases}$$

which was considered in [Ichikawa 59, example 1.3] without the perturbation and control terms. It gives an example of theorem 6.8 where $X = L_2(0,1), G = \frac{d}{d\xi} - I$ with $D(G) = \{z \in X : z \in H_0^1(0,1)\}$. Then $z(t) = e^{(r-D_2)t} T(w(t)) z_0$ is the solution for $z_0 \in D(G^2)$, where $T(t) z_0 = z_0(\xi + t) e^{-t}$ is generated by the operator G. Numerical results are shown for $r = 5$ with the control $u = D_2 z = -5.5z$(stable) and without control (unstable)(Fig.6.2).

Example 6.16

Consider the linear stochastic version of example 5.16 given by

$$\begin{cases} dy = [A + P] y dt + B u dt + G y dw, \quad y(t) = \begin{pmatrix} z \\ z_t \end{pmatrix} \\ y(0) = \begin{pmatrix} h_0(x) \\ h_1(x) \end{pmatrix} = \begin{pmatrix} \sin x \\ x - x^2 \end{pmatrix} \end{cases}$$

on the real Hilbert space $X = H_0^1(0,1) \times L_2(0,1)$ with $D(A) = H^2(0,1) \times H_0^1(0,1)$, and

$$A = \begin{pmatrix} 0 & 1 \\ \Delta & 0 \end{pmatrix}, \ P = \begin{pmatrix} 0 & 0 \\ 0 & 2x \end{pmatrix}, \ B = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ and } G = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

It is known that if $D_0 = (1 \quad -2)$, $\tilde{A} = A + BD_0$ generates an exponentially stable group. By choosing $\gamma = 2$ and $\nu = 1$, the conditions of theorem 6.9 are satisfied. Numerical results are shown in Fig 6.3.

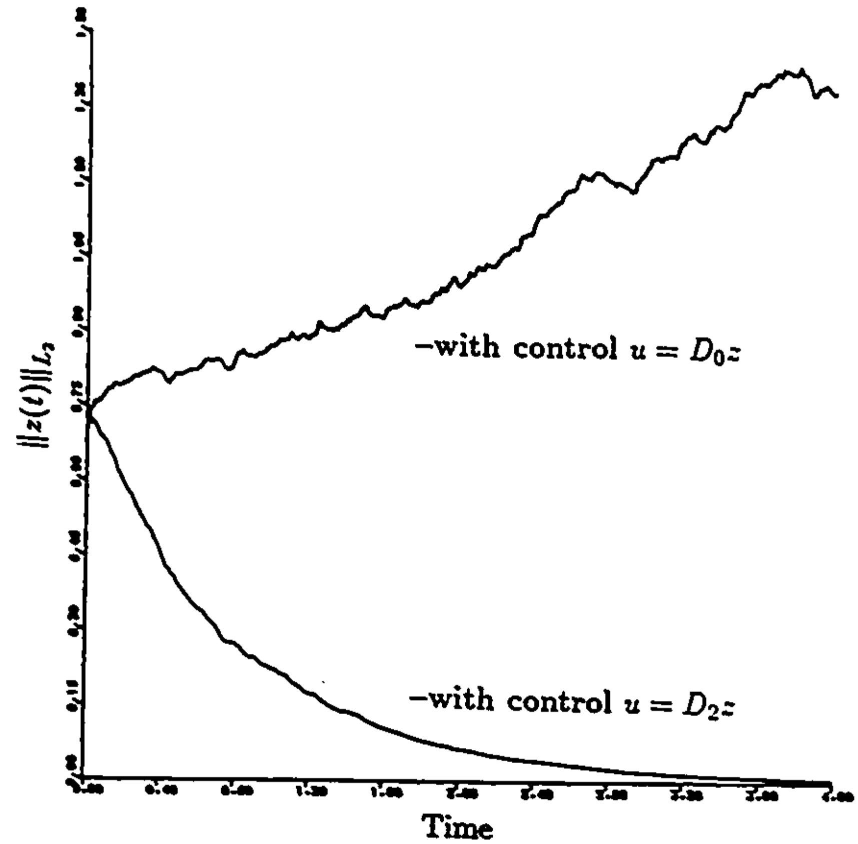

Fig.6.2. Semilinear stochastic parabolic equation (Example 6.15)

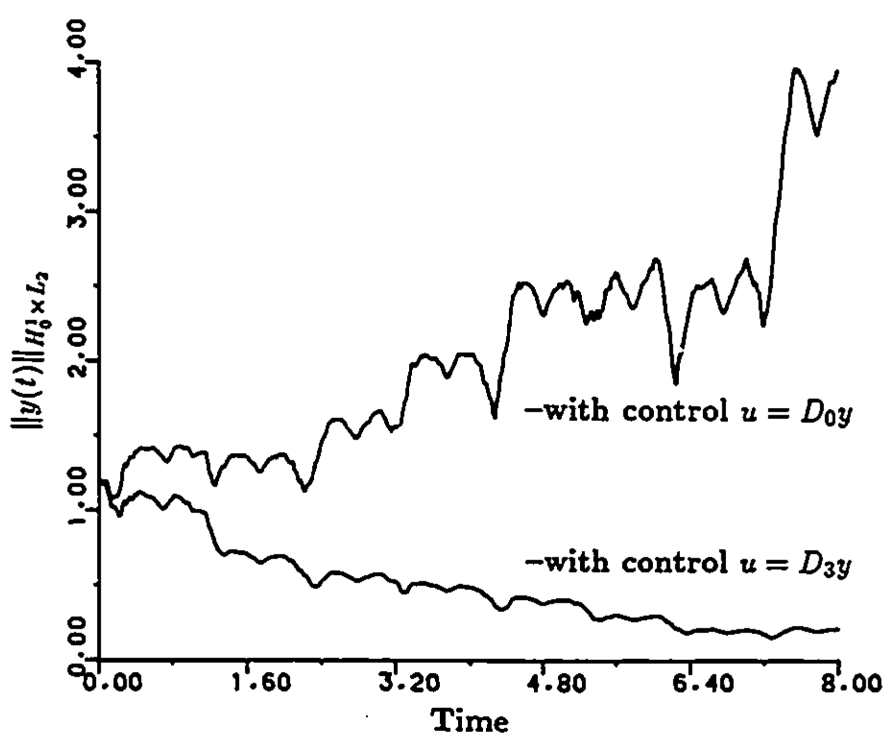

Fig.6.3. Stochastic Wave equation (Example 6.16)

CHAPTER 7

APPLICATION STABILIZATION RESULTS TO A FLEXIBLE STRUCTURAL DYNAMICS–SIMPLIFIED SCOLE MODEL

7.1 Introduction

With the ever-increasing demand for high speed global communication links, very large spacecrafts (so called third generation spacecrafts) are expected to be deployed in the next ten years in order to solve the problem of congestion in the geosynchronous orbit. The large spacecraft involves a high level of mechanical flexibility. Often this is combined with extremely accurate pointing requirements.

Among the variety of large scale flexible space structures, stations and laboratories currently under consideration, a typically model suggested by NASA is the Spacecraft Control Laboratory Experiment (SCOLE) [AIAA 11, Taylor and Balakrishnan 108], consisting of a long flexible mast (M) or an elastic beam which joins two rigid bodies. One rigid body represents the space shuttle orbiter (S), the other represents the antenna reflector (A) (see Figure 7.1).

In recent years considerable attention has been devoted to the problem of modelling and control of these large flexible spacecrafts [Ahmed and Biswas 4,26,27, Ahmed and Lim 9,86,87,88, Balakrishnan 13,14, Balakrishnan and Taylor 15, Littman and Markus 90,91, etc.]. It is obvious that flexibility of various components introduces additional complexities in control and stabilization of the spacecraft. In order to study the effects of structural flexibility on the shuttle motion, it is necessary to develop a model which is mathematically rigorous yet simple enough for theoretical, as well as numerical, investigation. With these objectives in mind a simplified mathematical model for a flexible spacecraft was developed by Littman and Markus [90,91] under certain assumptions. The model consists of a combination of ordinary differential equations-describing the dynamics of the space shuttle orbiter and its antenna reflector, as well as partial differential equations- describing the vibration of the flexible mast, the whole viewed as a unified hybrid dynamical system.

The model, however, does not include various structurally damping effects and the possibility of random disturbances arising from some external sources such as meteorite collisions, radiations in solar pressure, magnetic fields variations, etc.. These perturbations may induce uncertainties and lead to instability of the entire system [Ahmed and Lim 9,86,87].

In this chapter, using Littman and Markus' dynamics, we develop a model that includes those perturbations and apply the stabilization results we obtained in order to prove stability of the system. Some numerical results are presented to illustrate the impact of perturbations on the dynamic behavior of the system and effectiveness of the suggested controls on the suppression of vibration.

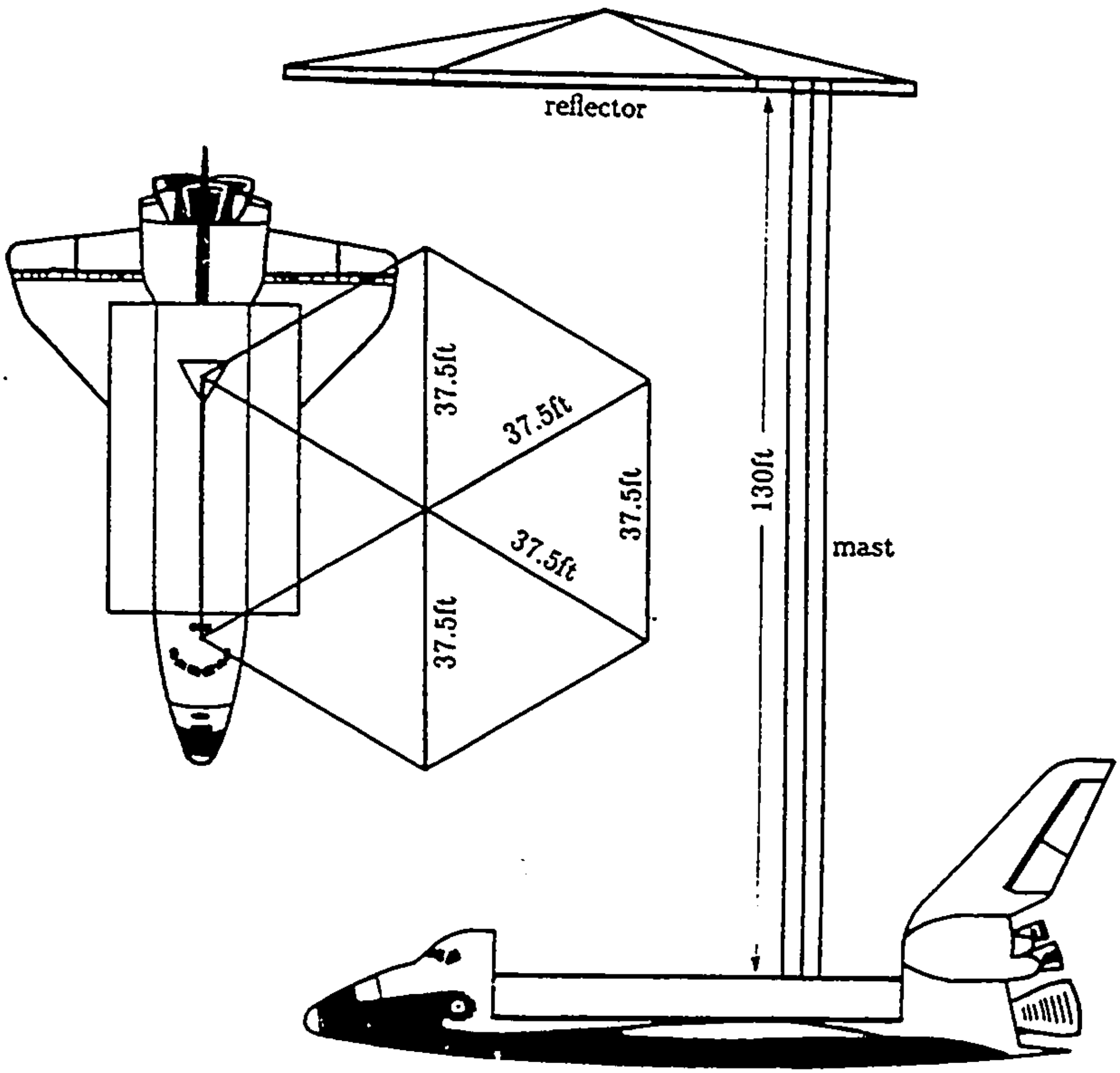

Fig.7.1. Drawing of the Shuttle/ Antenna Configuration (NASA)

7.2 Derivation of the Dynamical Equations of the Hybrid System –Simplified SCOLE Model

We are interested in one specific elastic structure (simplified SCOLE model) that forms a basic type of component in many more complicated and extensive space-environment constructions. In this hybrid-system the elastic beam is a long flexible mast (M), clamped at one end to a massive space-ship (S), and fastened at the other end to a rigid antenna (A) whereupon the control is effected by means of gas-jets (see figure 7.2). The physical problem requires the shuttle (S) to slew rapidly in orientation, while controlling the corresponding motion and structural vibrations of the mast (M), so that the antenna (A) points accurately towards a specified target. The control forces and torques are applied to antenna (A) and/or to the mast (M) by means of cold gas jets, gyros and proof-mass adjustments. These control forces and torques regulate the coupled PDE and ODE as a unified hybrid dynamical system. An important control objective is to minimize the time required (possibly under an energy constraint) to slew the whole physical system (S)+(M)+(A) –and to stabilize the resulting structure vibrations.

As a conceptual scheme for this control dynamic, Littman and Markus [90,91] assume that a definite thrust-torque program has been established for re-orienting (S), about its fixed centroid, to its final specified attitude. Once such a control logic for (S) has been fixed, this is carried forth regardless of the positions, motions, or vibrational modes of (M) and (A) and then the rotational influence on (A) and (M) will be analyzed later as a separate matter. This is reasonable approach as (S) is much more massive then either (A) or (M). The goal is now to regulate the system (M)+(A) towards some specified rest state relative to (S)-both during and after the reorienting of (S)-say, where (M) is represented by a line segment that is orthogonal to the two rigid bodies (S) and (A) at its ends. In a mathematical sense we consider (S) to be stationary in a suitable rotating coordinate frame and the dynamical equations of (M) and (A) are to be analyzed relative to the rotating coordinate frame (x, y) attached rigidly to the shuttle (S).

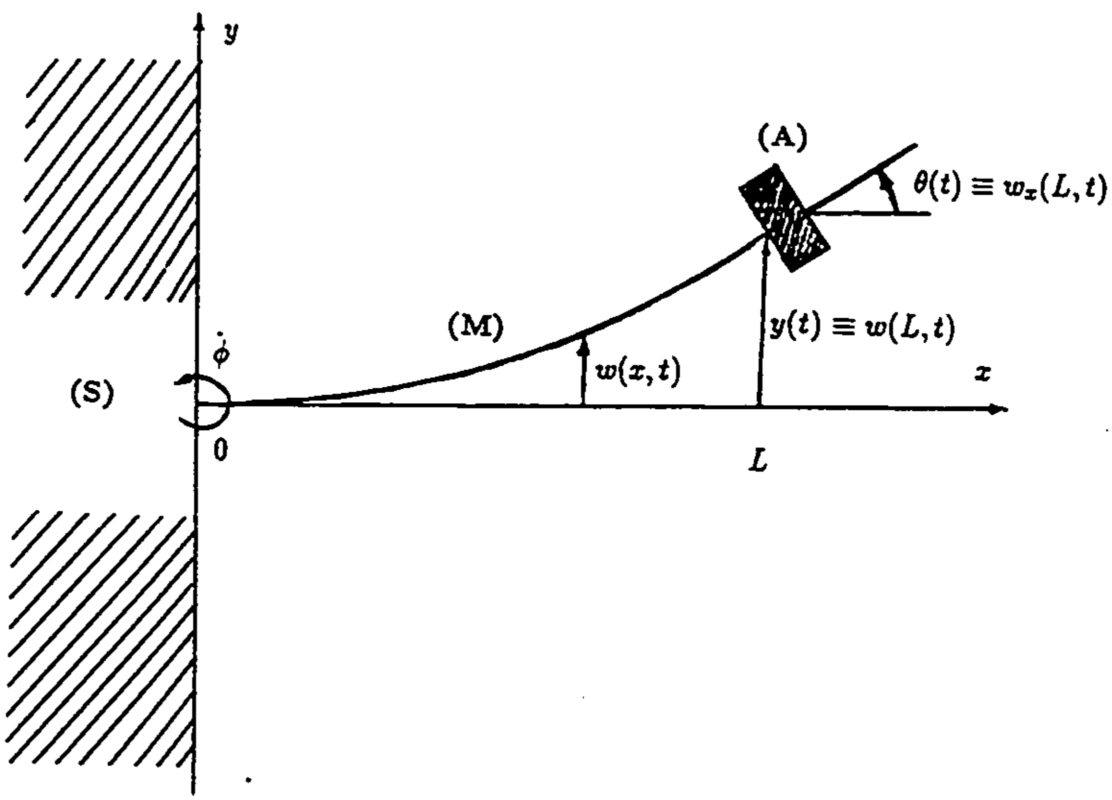

Fig.7.2. Two-dimensional Simplified Version of SCOLE

The system (M)+(A) is regarded as a hybrid control system in that the elastic vibrations of the mast (M) are governed by a partial differential equation (the PDE of Euler-Bernoulli in linear elasticity theory), whereas the oscillations of the antenna (A) are described by ordinary differential equations (the ODE of Newton-Euler in rigid body dynamics).

Assuming that all motions and forces are restricted to a fixed plane, with x-axis normal to the space shuttle (S) which is stationary in the (x, y) coordinates–but which is truly rotating (about the fixed point 0) with an angular velocity $\dot{\phi}(t)$ through absolute space, and with all vibrations along the y-axis orthogonal to the x-axis, then the Euler-Bernoulli theory asserts that the amplitude $w(x, t)$ of a transverse vibration for the elastic beam (M) satisfies the following PDE:

$$\rho w_{tt}(t,x) = -EIw_{xxxx}(t,x) + F(t,x) \quad \text{on } 0 \leq x \leq L, \; t \geq 0.$$

Here the given positive physical constants are:

$\rho =$ linear density of (M)–including contributions referring to the cross-sectional area of the beam

$E =$ Young's elastic modulus of (M)

$I =$ Moments of inertia of a cross-section of (M) (EI is the given elastic bending constant for (M))

and F represents any other transverse force (per mass)– say damping, control, or external forces and perturbations.

In this chapter we develop the control and stabilization of the system only after (S) has come to absolute rest; that is, we assume $\dot{\phi} = 0$ here, and leave the investigations of transient states, while (S) is being rotated, to a later study. (For the analysis of dynamic systems including the rotation of rigid body (S), see Ahmed and Biswas [4,26,27], Ahmed and Lim [9,86,87,88].) We also assume that all (relative) motions are transverse, so $\dot{x} = 0$ and longitudinal forces cancel. Then the PDE for the amplitude $w(x,t)$ of the mast (M), within the frame (x,y), becomes

$$\rho w_{tt}(t,x) = -EIw_{xxxx}(t,x) + p(w_t(t,x)) + f_1(t,x)$$

where $p(\cdot)$ is a linear or nonlinear (differential, integral or multiplication) operator and represents various structurally damping perturbations (see Balakrishnan [14], Bank et al. [16,17], Chen and Russell [31], Huang [55,56,57], Pritchard and Blakeley [100], Wang [114]), $f_1(t,x)$ are control forces acting on (M). The boundary conditions assumed for (M) are:

$$\begin{cases} w(0,t) \equiv 0, \quad w_x(0,t) \equiv 0 & \text{(clamped beam at } x = 0\text{)}, \\ w(L,t) \equiv y(t), \quad w_x(L,t) \equiv \theta(t) & \text{(linked beam at } x = L\text{)}. \end{cases}$$

Here, as indicated in Fig.7.2, $y(t)$ is the transverse displacement of the centroid of the antenna (A) at time $t \geq 0$, and $\theta(t)$ is angular displacement of the antenna (A).

Next the dynamics of the rigid antenna (A) are described by the control ODE based on the Newtonian-Eulerian principles:

$$\begin{cases} m\ddot{y} = EIw_{xxx}(t,L) + f_2(t) \\ J\ddot{\theta} = -EIw_{xx}(t,L) + f_3(t). \end{cases}$$

Here the given positive physical constants are

m = mass of antenna (A),

J = moment of inertia of (A) about its centroid-the point of attachment of the mast (M), and $f_2(t)$ and $f_3(t)$ are the control force and torque (respectively) applied to (A). The terms $EIw_{xxx}(t,L)$ and $-EIw_{xx}(t,L)$ are the force and torque, respectively, that (M) exerts on (A)-according to the classical linear theory of elasticity.

7.3 Simplification of the Mathematical Model

The formulation of the control hybrid system has thus been completed, and next we reduce this problem to a boundary value problem. This is accomplished by incorporating the dynamics of (A) into certain novel boundary conditions for the mast (M), which are introduced by Littman and Markus [90,91].

For this purpose define the following boundary operators

$$(B_1w)(t) = mw_{tt}(t,L) - EIw_{xxx}(t,L)$$

and

$$(B_2w)(t) = Jw_{xtt}(t,L) + EIw_{xx}(t,L).$$

Then set (at $x = L$)

$$B_1w = f_2(t)$$

$$B_2w = f_3(t).$$

We have

$$\begin{cases} \rho w_{tt}(t,x) = -EIw_{xxxx}(t,x) + p(w_t(t,x)) + f_1(t,x) \quad 0 < x < L, t \geq 0, \\ w(0,t) \equiv w_x(0,t) = 0 \\ B_1w = f_2(t), \quad B_2w = f_3(t). \end{cases}$$

Here the controllers f_1, f_2 and f_3 describe the forces and torque applied to the mast (M) and the antenna (A) that bring the full system (M)+(A) to rest.

For simplicity of exposition we shall take the length-scale so $L = 1$. Further adjustments of the time-unit reduces the boundary value problem to the form

$$w_{tt}(t,x) + w_{xxxx}(t,x) + p(w_t(t,x)) = F_1(t,x)$$

with the clamped-end conditions at $x = 0$

$$w(t,0) = 0, \quad w_x(t,0) = 0,$$

and the linked-end conditions at $x = 1$

$$\mu_1 w_{tt}(t,1) - w_{xxx}(t,1) = F_2(t)$$

$$\mu_2 w_{ttx}(t,1) + w_{xx}(t,1) = F_3(t).$$

In order to make our constructions as explicit as possible, we shall set the mass constants $\mu_1 = \mu_2 = 1$. Thus we have our simplified mathematical model to:

$$\begin{cases} w_{tt}(t,x) + w_{xxxx}(t,x) + p(w_t(t,x)) = F_1(t,x) \\ w(t,0) \equiv w_x(t,0) \equiv 0 \\ w_{tt}(t,1) - w_{xxx}(t,1) = F_2(t) \\ w_{ttx}(t,1) + w_{xx}(t,1) = F_3(t). \end{cases} \tag{7.1}$$

Clearly, this system can be described by the following second-order evolution equation (Littman, Markus and You [92]):

$$\frac{d^2y(t)}{dt^2} = -Ay(t) + P(\dot{y}(t)) + Bu(t), t \geq 0 \tag{7.2}$$

where

$$y(t) = \begin{pmatrix} w(\cdot,t) \\ w(1,t) \\ w_x(1,t) \end{pmatrix}, \quad A = \begin{pmatrix} \frac{d^4}{dx^4} & 0 & 0 \\ -\frac{d^3}{dx^3}|_{x=1} & 0 & 0 \\ \frac{d^2}{dx^2}|_{x=1} & 0 & 0 \end{pmatrix},$$

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, P(\cdot) = \begin{pmatrix} -p(\cdot) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and $u(t) = \begin{pmatrix} F_1(t) \\ F_2(t) \\ F_3(t) \end{pmatrix}$, with the state space given by the real Hilbert space $H = L^2(0,1) \times R \times R$ and the domain of A given by

$$D(A) = \{ \begin{pmatrix} w \\ w_1 \\ w_2 \end{pmatrix} \in H^4(0,1) \times R \times R : w(0) = w'(0) = 0, \quad w(1) = w_1 \text{ and } w'(1) = w_2 \}. \tag{7.3}$$

Then, introducing the real Hilbert space $X = V \times H$, where $V \equiv D(A^{1/2})$ is endowed with inner product $(v_1, v_2)_V \equiv (A^{1/2}v_1, A^{1/2}v_2)_H$, we can reformulate the system as an abstract first-order system:

$$\frac{dz(t)}{dt} = \mathcal{A}z(t) + \mathcal{P}(z(t)) + \mathcal{B}u(t), t > 0, \tag{7.4}$$

where

$$z(t) = \begin{pmatrix} y(t) \\ \dot{y}(t) \end{pmatrix}, \quad \mathcal{A} = \begin{pmatrix} 0 & I \\ -A & 0 \end{pmatrix},$$
$$\mathcal{B} = \begin{pmatrix} 0 \\ B \end{pmatrix}, \quad \mathcal{P}(\cdot) = \begin{pmatrix} 0 & 0 \\ 0 & P(\cdot) \end{pmatrix}$$

and $D(\mathcal{A}) = D(A) \times V \subset X$.

In the case of random disturbances arising from some external sources, we have the following stochastic counterpart of (7.4)

$$dz(t) = \mathcal{A}z(t)dt + \mathcal{P}(z(t))dt + \mathcal{B}u(t)dt + \mathcal{G}dw(t) \tag{7.5}$$

where

$$\mathcal{G} = \begin{pmatrix} 0 \\ G \end{pmatrix}, \quad dw(t) = \begin{pmatrix} dw_1(t) \\ dw_2(t) \\ dw_3(t) \end{pmatrix}$$

and $G = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, and $\{w_1(t), t \geq 0\}$ is a generalized Brownian motion taking values from suitable spaces of distributions on the mast and $w_2(t), w_3(t)$ are real Brownian motion acting on the antenna.

7.4 Robust Boundary Control of Hybrid Systems

In this section, we consider that the control forces and torques are only applied to the antenna and thereby to the boundary of the mast, which implies $F_1(t, x) \equiv 0$. These controllers are more realistic than the control applied distributed through the mast which is hard to implement in practice. It is shown that the vibrations in the flexible mast can be completely eliminated by applying appropriate feedback control forces and torques to the antenna; moreover, the boundary feedback control is robust in the sense that the stability properties remain invariant under a broad class of perturbations by the same feedback law. Also we assume that perturbations are linear. With these assumptions, system (7.4) reduces to

$$\frac{dz}{dt} = \mathcal{A}z(t) + \mathcal{P}z(t) + \mathcal{B}_b u(t), t > 0 \tag{7.6}$$

where $\mathcal{B}_b = \begin{pmatrix} 0 \\ B_b \end{pmatrix}$ with $B_b = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$, $u(t) = \begin{pmatrix} F_2(t) \\ F_3(t) \end{pmatrix}$ and $\mathcal{P} \in \mathcal{L}_u(X)$ is a linear operator.

The nominal system corresponding to the perturbed system (7.6) is given by

$$\frac{dz(t)}{dt} = \mathcal{A}z(t) + \mathcal{B}_b u(t), \tag{7.7}$$

This is the dynamic model considered by Littman, Markus and You [92] and they show that the following statements hold for the nominal system:

1) $\mathcal{A}$ generates a strongly continuous unitary group $T(t), t \in R$ on X.

2) $\mathcal{A}$ has compact resolvent.

3) $(\mathcal{A}, \mathcal{B}_b)$ is (approximately) controllable.

Therefore, by theorem 4.7, it is strongly stabilizable by the feedback $-K\mathcal{B}_b^*$, where $K \in \mathcal{L}(U)$ is self-adjoint and strictly positive. Moreover, note that

$$\begin{aligned} u(t) = \begin{pmatrix} F_2(t) \\ F_3(t) \end{pmatrix} &\equiv -K\mathcal{B}_b^* z(t) = -K(0 \ B_b^*) \begin{pmatrix} y(t) \\ \dot{y}(t) \end{pmatrix} \\ &= -KB_b^*\dot{y}(t) = -K \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} w_t(\cdot, t) \\ w_t(1,t) \\ w_{tx}(1,t) \end{pmatrix} \\ &= -K \begin{pmatrix} w_t(1,t) \\ w_{tx}(1,t) \end{pmatrix}, \ t \geq 0. \end{aligned}$$

So if $\mathcal{P}$ satisfies assumption (A4) in chapter 4 (i.e., $\mathcal{P}$ is m-dissipative and $\mathcal{A}$-bounded with relative bound less than one), then by theorem 4.29, the perturbed system (7.6) is strongly stabilizable by the boundary feedback $\begin{pmatrix} F_2(t) \\ F_3(t) \end{pmatrix} = -2 \begin{pmatrix} w_t(1,t) \\ w_{tx}(1,t) \end{pmatrix}$ (here we choose $K = 2I$). Otherwise, it may not be stabilizable by the same feedback $-K\mathcal{B}_b^* z$. This is clearly indicated in the following simulation results.

<u>Case 1</u>

Consider the case where $p(\xi) = 4(1-x)\frac{\partial \xi}{\partial x}$. It is easy to show that for any $\nu \in X$, there exists a z such that $(\lambda I - \mathcal{P})z = \nu$. This proves that $R(\lambda I - \mathcal{P}) = X$ for $\lambda > 0$. Furthermore,

$$(z, \mathcal{P}z) = (y_t, Py_t) = -4 \int_0^1 (1-x) \frac{\partial w_t}{\partial x} w_t dx,$$

but

$$\int_0^1 (1-x)\frac{\partial w_t}{\partial x} w_t dx = (1-x)w_t^2|_0^1 - \int_0^1 (1-x)\frac{\partial w_t}{\partial x} w_t dx + \int_0^1 w_t dx$$
$$= (1/2)\int_0^1 w_t^2 dx.$$

This implies

$$(z, \mathcal{P}z) = -2\int_0^1 w_t^2 dx \leq 0.$$

Hence, $\mathcal{P}$ is m-dissipative. On the other hand,

$$\|\mathcal{P}z\|_X^2 = \|Py_t\|_H^2 \leq 4\int_0^1 (\frac{\partial w_t}{\partial x})^2 dx$$
$$\leq 4\varepsilon \int_0^1 (\frac{\partial^2 w_t}{\partial x^2})^2 dx + b(\varepsilon)\int_0^1 w_t^2 dx + kw_t^2(1,t) + kw_{tx}^2(1,t)$$
$$\leq 4\varepsilon\|\mathcal{A}z\|_X^2 + \beta(\varepsilon)\|z\|_X^2 \quad \text{for } z \in D(\mathcal{A}) = D(A)\times V$$

where $\varepsilon > 0$ is arbitrary, $b(\varepsilon)$ is appropriately chosen, while $\beta(\varepsilon) \to \infty$ when $\varepsilon \to 0$. That is, $\mathcal{P}$ is relatively bounded with respect to $\mathcal{A}$ with relative bound zero. Therefore, $\mathcal{P}$ satisfies all the conditions in assumption (A4). So the perturbed system (7.6) remains strongly stabilizable by the feedback $u(t) = -2\begin{pmatrix} w_t(1,t) \\ w_{tx}(1,t)\end{pmatrix}$ in this case.

<u>Case 2</u>

Let $p(\xi) = -\frac{1}{2}(1-x)^2\frac{\partial^2 \xi}{\partial x^2} + \xi$. It is easy to show that

$$\|\mathcal{P}z\|_X^2 \leq \frac{1}{4}\|\mathcal{A}z\|_X^2 + \beta\|z\|_X^2.$$

That is, $\mathcal{P}$ is relatively bounded with respect to $\mathcal{A}$ with relative bound 1/2 and

$$(z, \mathcal{P}z) = (y_t, Py_t) = \frac{1}{2}\int_0^1 (1-x)^2 \frac{\partial^2 w_t}{\partial x^2} w_t dx - \int_0^1 w_t^2 dx$$

$$= \frac{1}{2}(1-x)^2 \frac{\partial w_t}{\partial x} w_t \mid_0^1 - \frac{1}{2}\int_0^1 (1-x)^2 (\frac{\partial w_t}{\partial x})^2 dx$$

$$+ \int_0^1 (1-x)\frac{\partial w_t}{\partial x} w_t dx - \int_0^1 w_t^2 dx$$

$$= -\frac{1}{2}\int_0^1 (1-x)^2 (\frac{\partial w_t}{\partial x})^2 dx + \frac{1}{2}\int_0^1 w_t^2 dx - \int_0^1 w_t^2 dx \leq 0.$$

Therefore, $\mathcal{P}$ again satisfies all the conditions in assumption (A4) as in case 1 and the perturbed system (7.6) remains strongly stabilizable by the same boundary feedback control.

Case 3

In this case we take $p(\xi) = \frac{1}{2}(1-x)^2 \frac{\partial^2 \xi}{\partial x^2} - (2-x)\frac{\partial \xi}{\partial x}$, then

$$(z, \mathcal{P}z) = -\frac{1}{2}\int_0^1 (1-x)^2 \frac{\partial^2 w_t}{\partial x^2} w_t dx + \int_0^1 (2-x)\frac{\partial w_t}{\partial x} w_t dx$$

$$= \frac{1}{2}\int_0^1 (1-x)^2 (\frac{\partial w_t}{\partial x})^2 dx - \int_0^1 (1-x)\frac{\partial w_t}{\partial x} w_t dx + \int_0^1 (2-x)\frac{\partial w_t}{\partial x} w_t dx$$

$$\geq \frac{1}{2} w_t^2(1,t) \geq 0.$$

Hence, $\mathcal{P}$ is not m-dissipative and the perturbed system (7.6) is not strongly stabilizable by the feedback $u(t) = -2\begin{pmatrix} w_t(1,t) \\ w_{tx}(1,t) \end{pmatrix}$ is this case (see Figures 7.3-7.7).

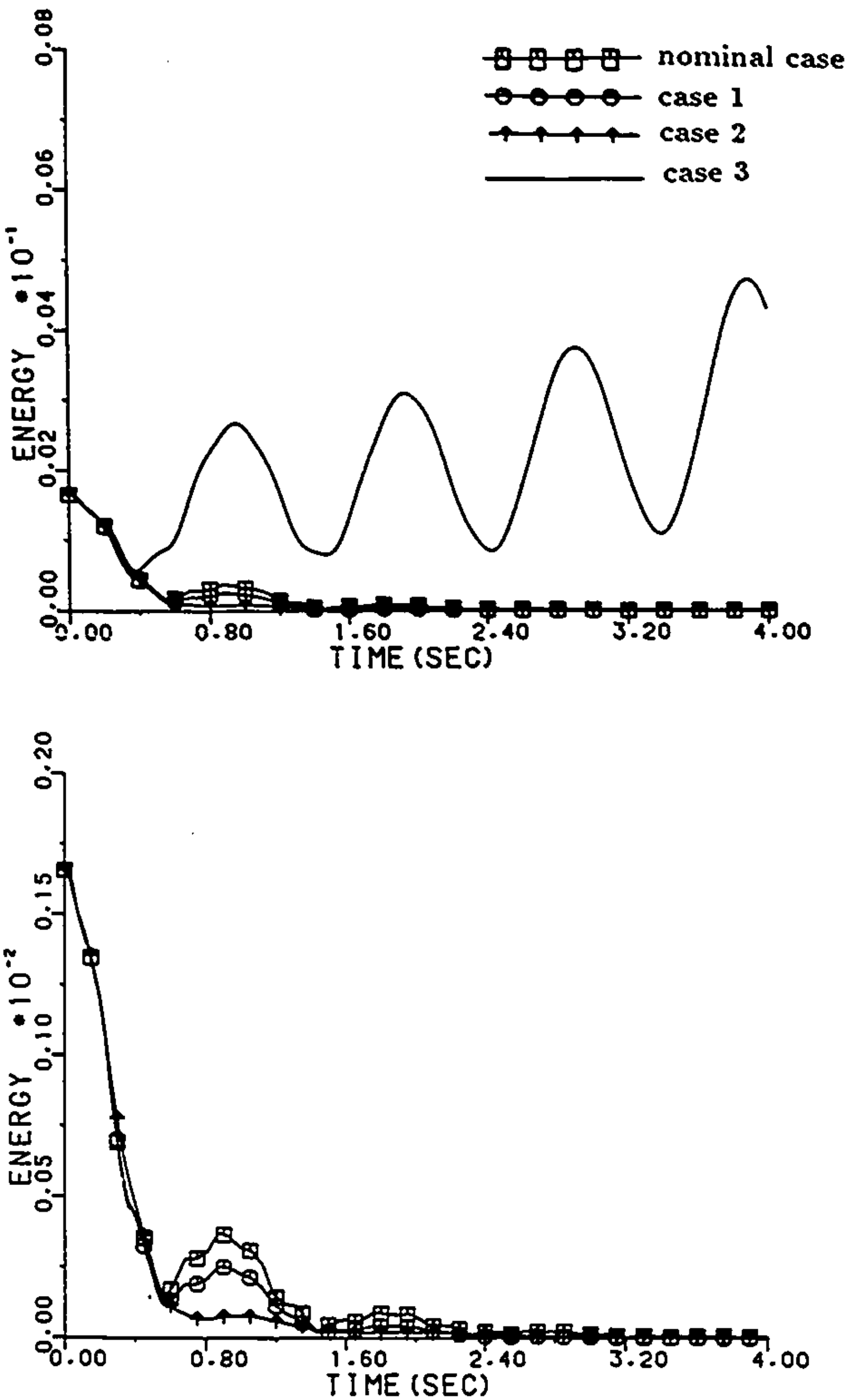

Fig.7.3. Total energy.

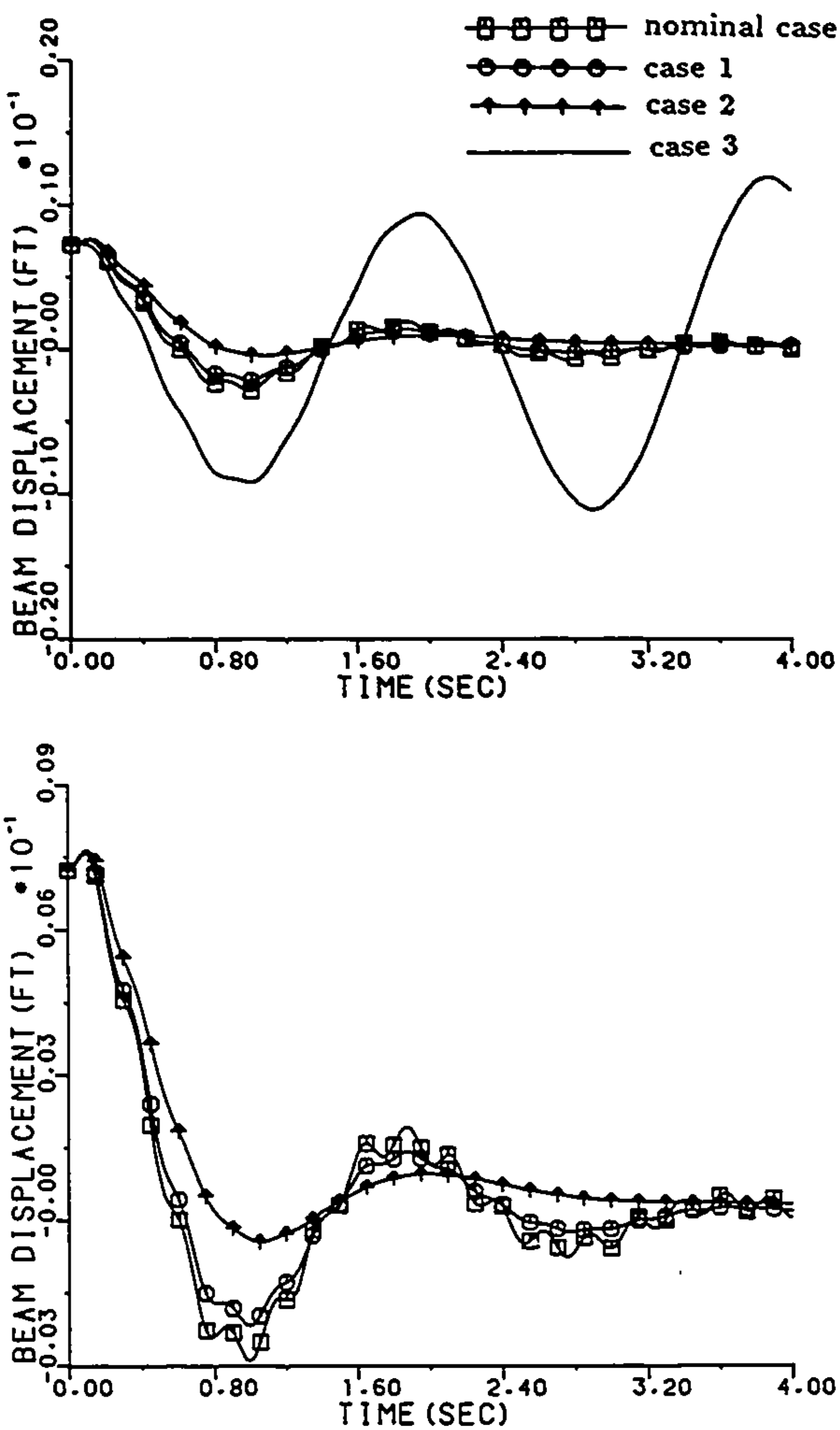

Fig.7.4. Beam displacement $w(1/2, t)$.

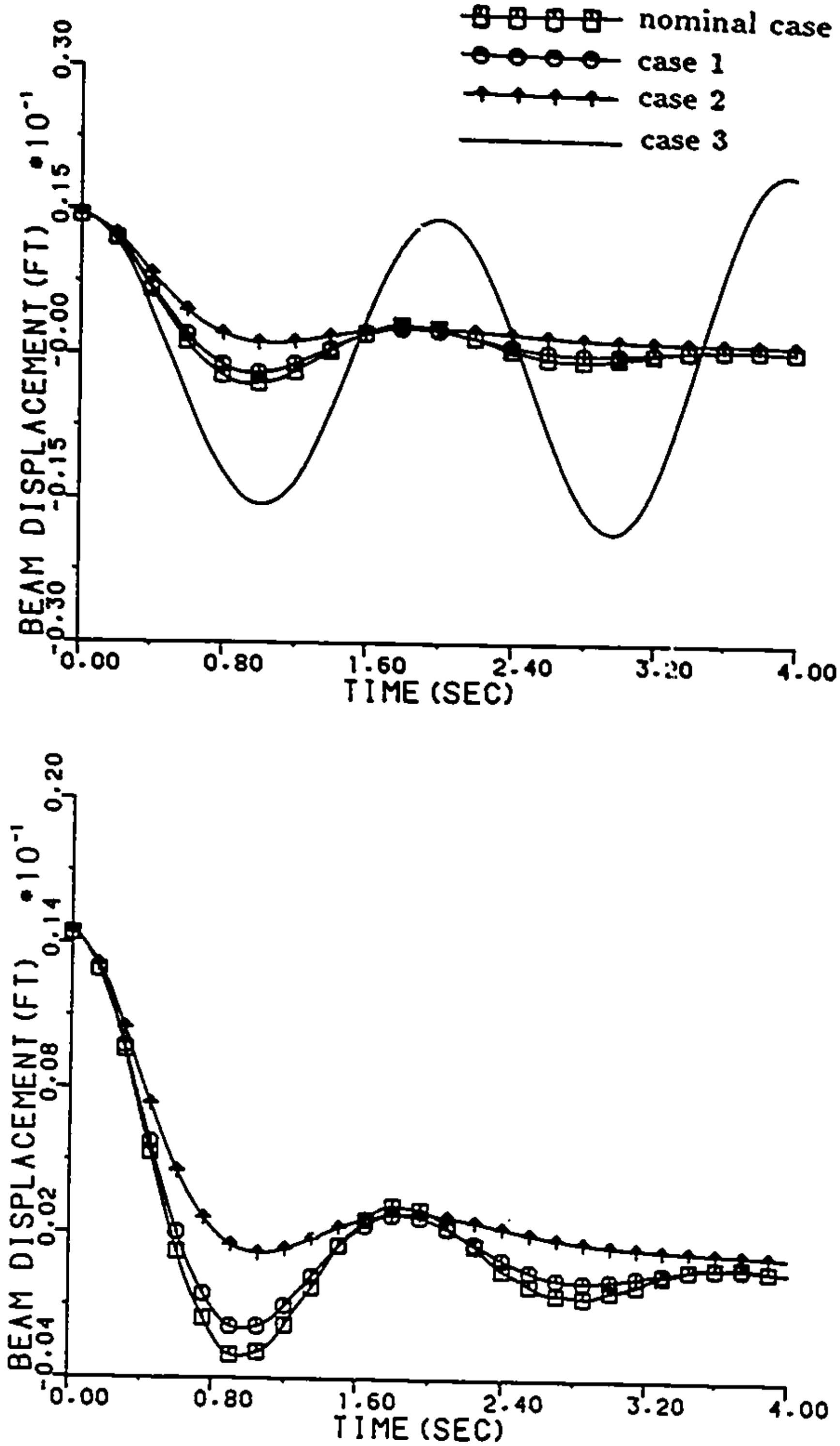

Fig.7.5. Beam displacement $w(1,t)$.

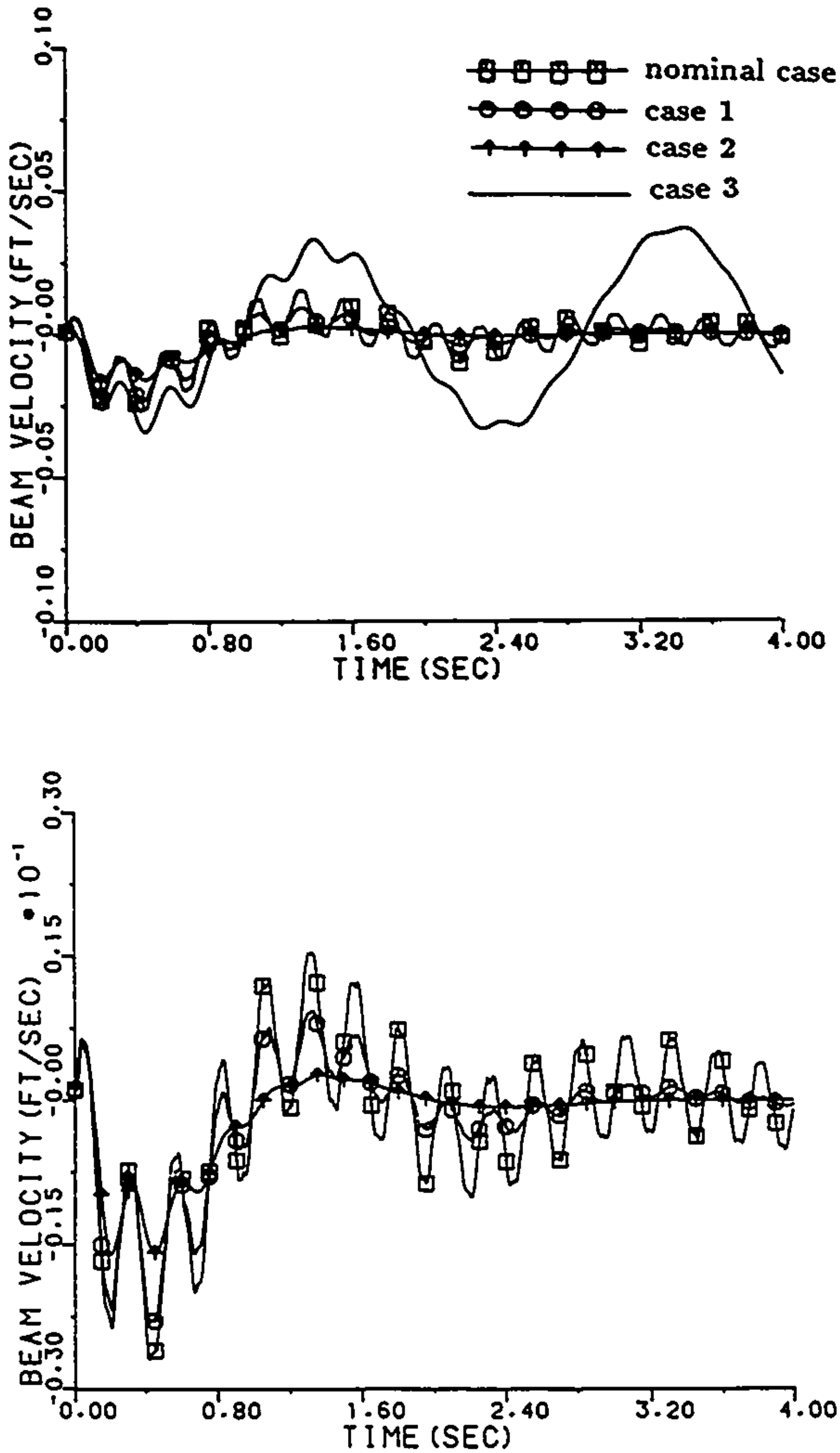

Fig.7.6. Beam velocity $\frac{\partial w}{\partial t}(1/2, t)$.

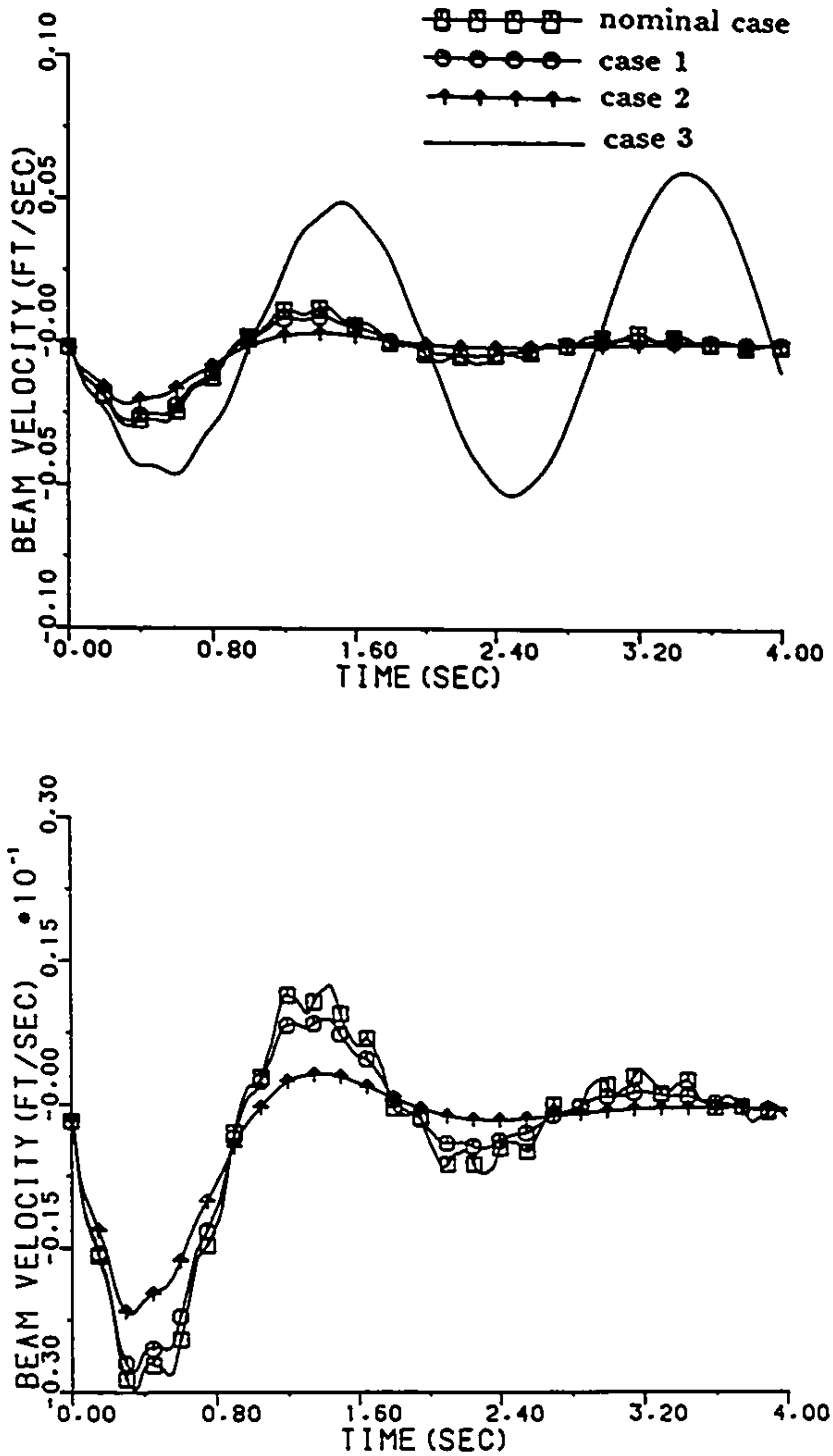

Fig.7.7. Beam velocity $\frac{\partial w}{\partial t}(1,t)$.

7.5 Exponential Stabilization of Perturbed Hybrid Systems

It has been shown that (Littman, Markus and You [92], see also Gibson [48]) the hybrid system given by (7.7) can never be exponentially stabilizable by any bounded linear feedback. In a majority of engineering problems, however, this stronger form of asymptotic stability (i.e. exponentially asymptotic stability) is most desirable. In this section we are mainly concerned with this problem. In other words, we wish to find an operator $D_1 \in \mathcal{L}(X, U)$, such that the perturbed system (7.4) (or (7.5)) with feedback control $u \equiv D_1 x$ is exponentially stable with respect to the zero state (in the mean square sense for (7.5)).

Consider the nominal system corresponding to the perturbed systems (7.4) and (7.5) given by

$$\frac{dz(t)}{dt} = \mathcal{A}z(t) + \mathcal{B}u(t) \tag{7.8}$$

where $\mathcal{A} = \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix}$ generates a strongly continuous unitary group $T(t), t \in R$ on X and $\mathcal{B} = \begin{pmatrix} 0 \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ I \end{pmatrix}$. We know that A is self-adjoint and coercively dissipative (Littman, Markus and You [92]); i.e., there is a constant $\delta > 0$, such that

$$(Ay, y) \leq -\delta \|y\|^2 \quad \text{for } \ y \in D(A).$$

If we choose the feedback operator $D_0 \equiv -\alpha\mathcal{B}^*$ where $\alpha > 0$, then

$$\begin{aligned} u(t) = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} &= -\alpha\mathcal{B}^* z(t) = -\alpha \begin{pmatrix} 0 & B^* \end{pmatrix} \begin{pmatrix} y(t) \\ \dot{y}(t) \end{pmatrix} \\ &= -\alpha B^* \dot{y}(t) = -\alpha \dot{y}(t). \end{aligned}$$

Therefore, we have

$$\mathcal{A} + \mathcal{B}D_0 = \begin{pmatrix} 0 & I \\ A & -\alpha \end{pmatrix}.$$

Based on proposition 3.5 in (Pritchard and Zabczyk [101]), the feedback system $\mathcal{A} + \mathcal{B}D_0$ is exponentially stable (Fig. 7.8-7.12).

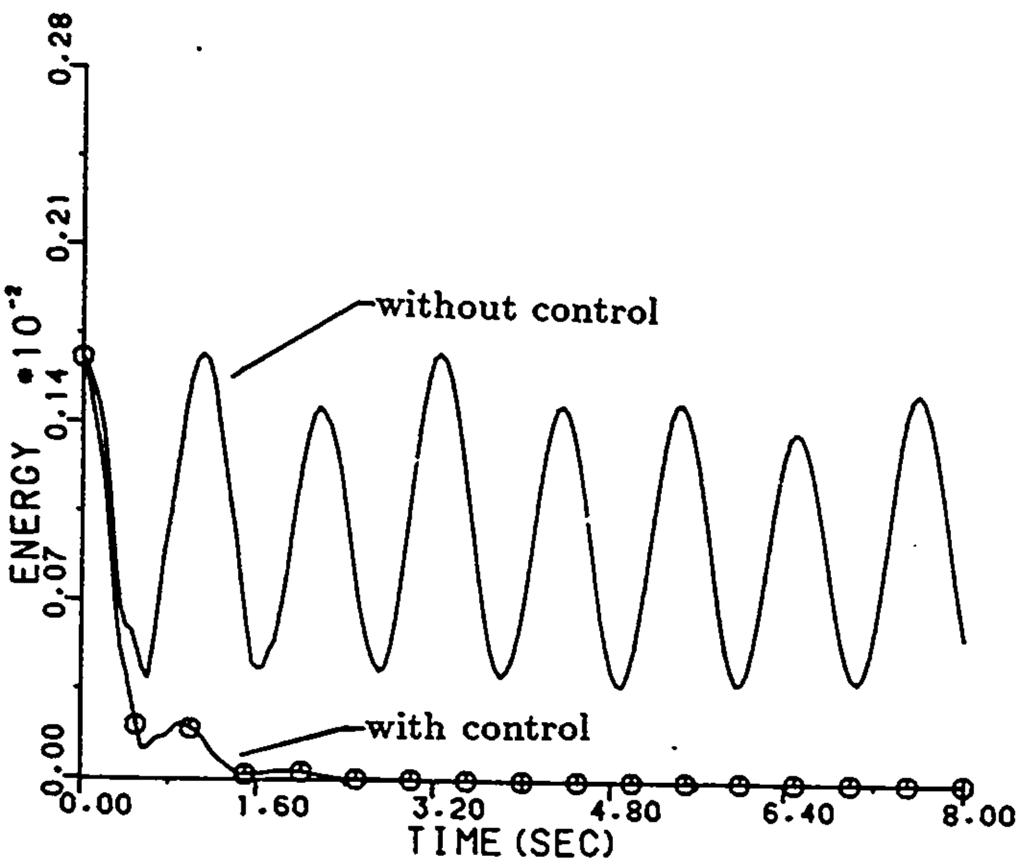

Fig.7.8. Total energy.

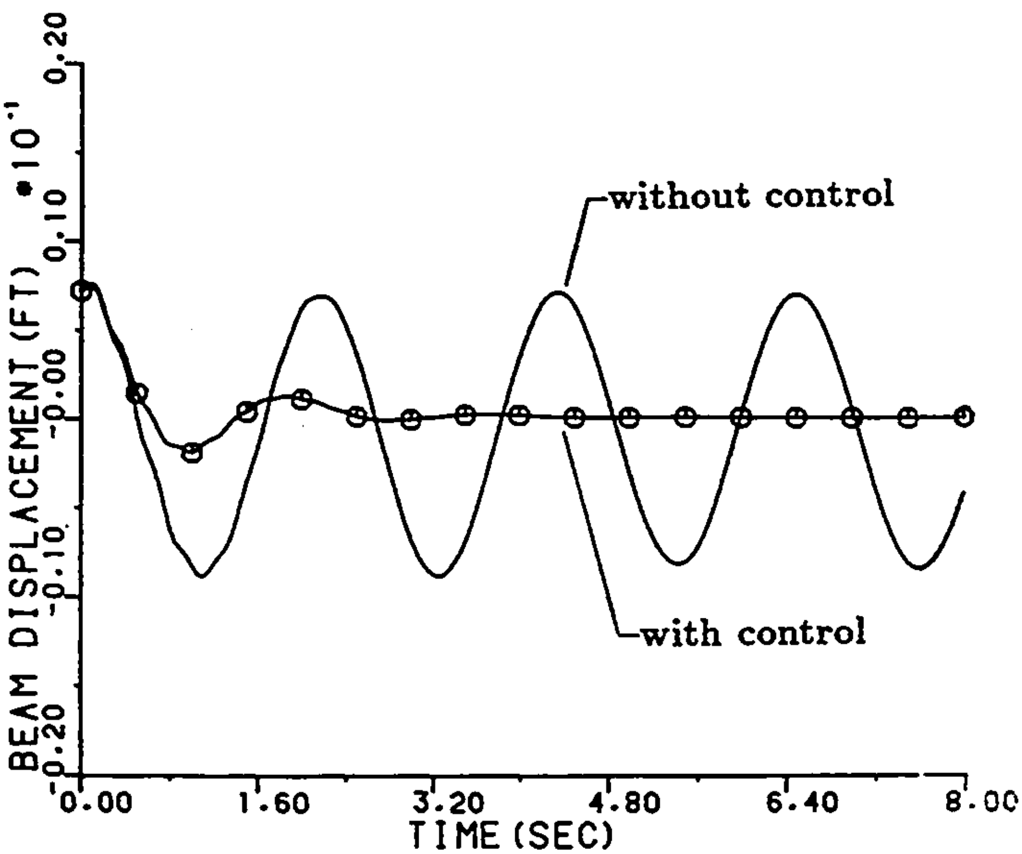

Fig.7.9. Beam displacement $w(1/2, t)$.

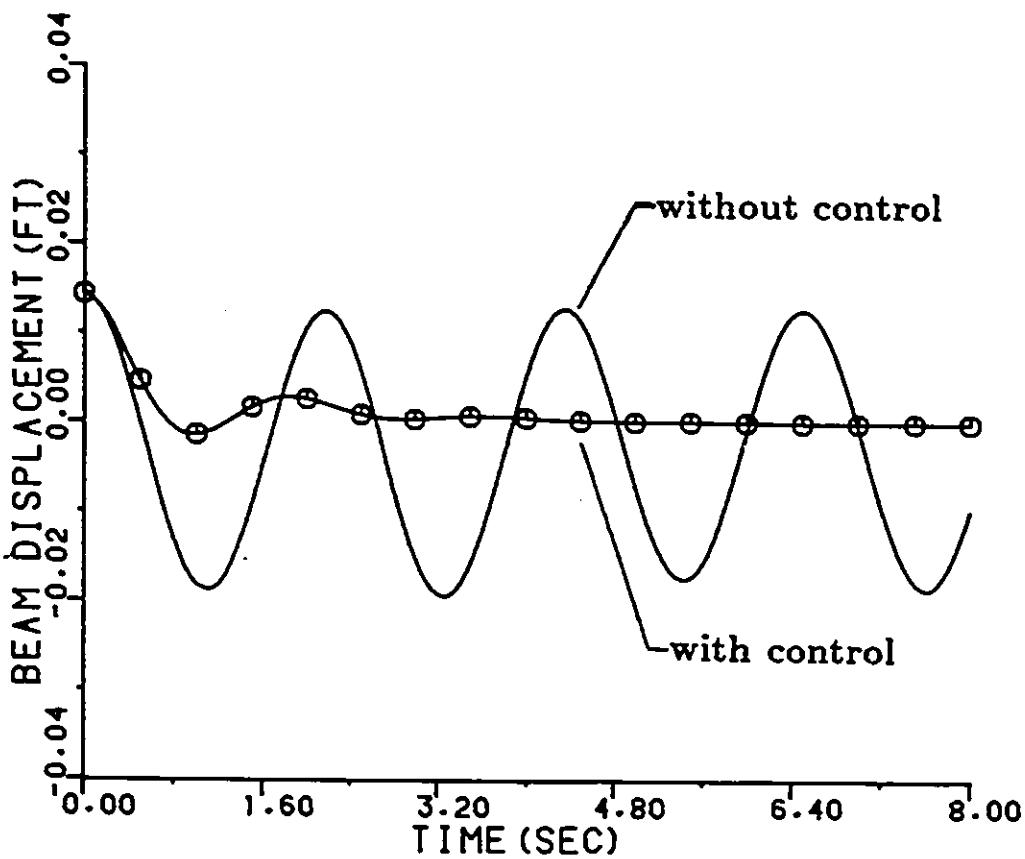

Fig.7.10. Beam displacement $w(1,t)$.

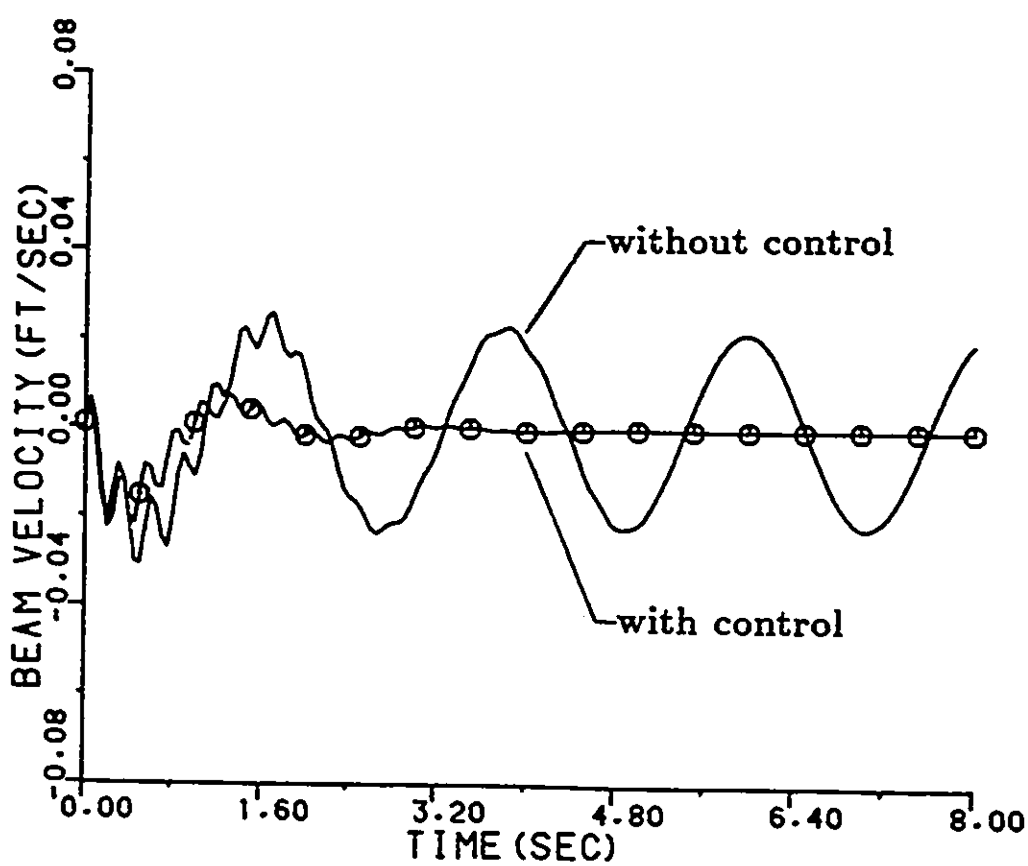

Fig.7.11. Beam velocity $\frac{\partial w}{\partial t}(1/2, t)$.

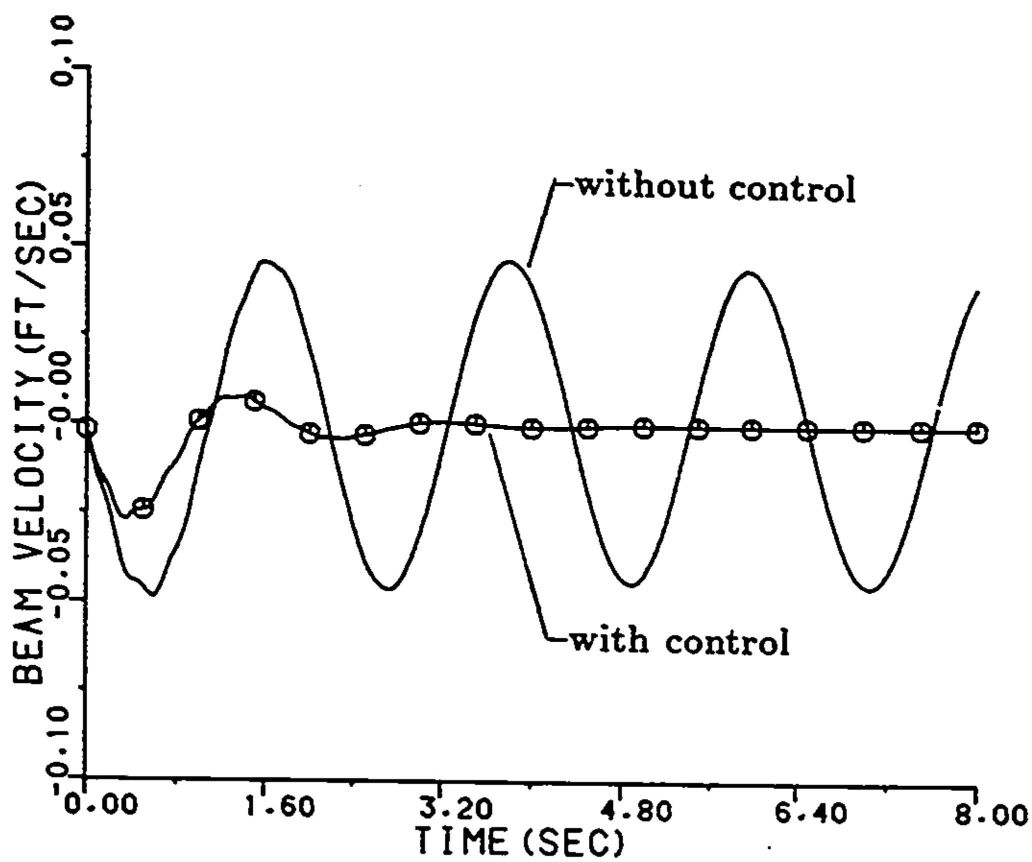

Fig.7.12. Beam velocity $\frac{\partial w}{\partial t}(1,t)$.

Now consider the perturbed system (7.4)

$$\frac{dz}{dt} = \mathcal{A}z(t) + \mathcal{P}z(t) + \mathcal{B}u(t)$$

with damping perturbation $p(\xi) = 1/2(x-1)\frac{\partial^2 \xi}{\partial x^2} - 4(2-x)\frac{\partial \xi}{\partial x}$. As in section 7.4, it can be easily shown that $\mathcal{P}$ is relatively bounded with respect to $\mathcal{A}$ with relative bound less than one. In fact,

$$\begin{aligned}
\|\mathcal{P}z\|_X^2 &\le \frac{1}{4}\|\mathcal{A}z\|_X^2 + 64\int_0^1 (\frac{\partial w_t}{\partial x})^2 dx \\
&\le (\frac{1}{4} + 64\varepsilon)\|\mathcal{A}z\|_X^2 + b(\varepsilon)\int_0^1 w_t^2 dx + kw_t^2(1,t) + kw_{tx}^2(1,t) \\
&\le \alpha(\varepsilon)\|\mathcal{A}z\|_X^2 + \beta(\varepsilon)\|z\|_X^2
\end{aligned}$$

where $\alpha(\varepsilon) = (1/4) + 64\varepsilon$ and $\varepsilon > 0$ is arbitrary. So we can choose ε such that $\alpha(\varepsilon) < 1$. Furthermore,

$$\begin{aligned}
(z, \mathcal{P}z) = (y_t, Py_t) &= (1/2)\int_0^1 (1-x)\frac{\partial^2 w_t}{\partial x^2} w_t dx + 4\int_0^1 (2-x)\frac{\partial w_t}{\partial x} w_t dx \\
&= (1/2)(1-x)\frac{\partial w_t}{\partial x} w_t \,|_0^1 + (1/2)\int_0^1 \frac{\partial w_t}{\partial x} w_t dx \\
&- (1/2)\int_0^1 (1-x)(\frac{\partial w_t}{\partial x})^2 dx + 2(2-x)w_t^2 \,|_0^1 + 2\int_0^1 w_t^2 dx \\
&\le (1/4)w_t^2(1) + 2w_t^2(1) + 2\int_0^1 w_t^2 dx
\end{aligned}$$

Therefore, by choosing $\mu = -4$ in theorem 5.8, we know that the system (7.4) with feedback control $u = D_1 z = (D_0 - \mu\mathcal{B}^*)z$ is exponentially stable (Fig. 7.13-7.17).

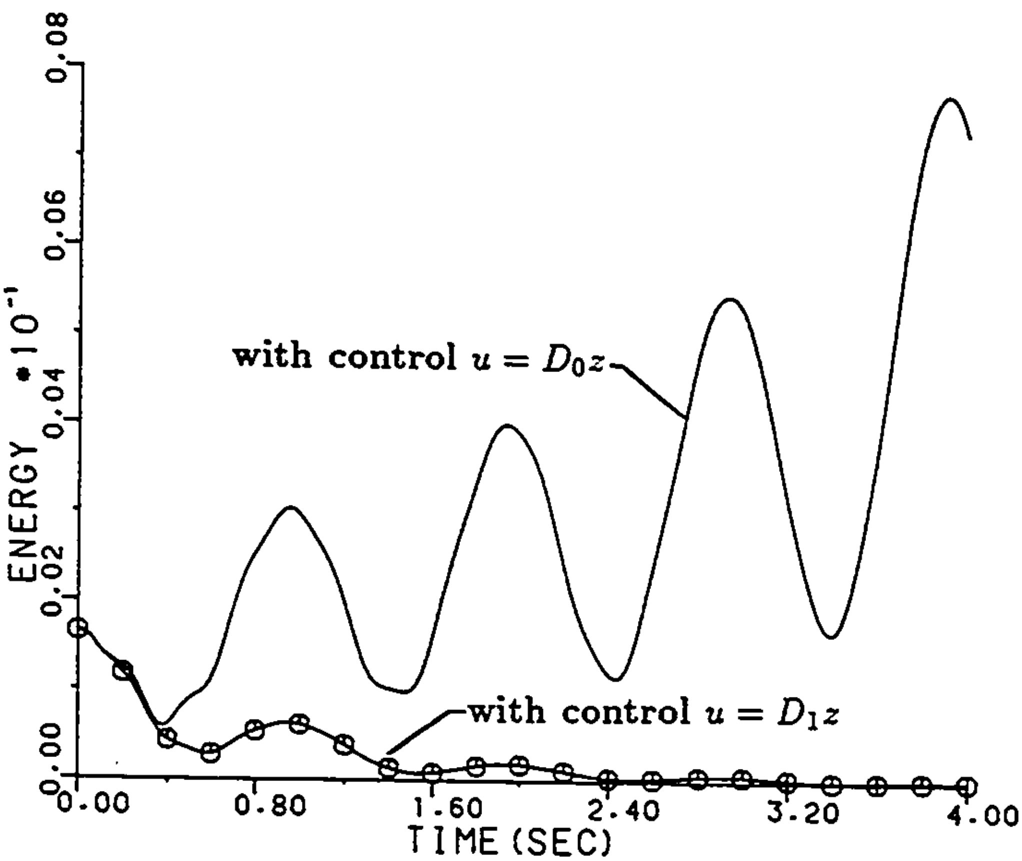

Fig.7.13. Total energy.

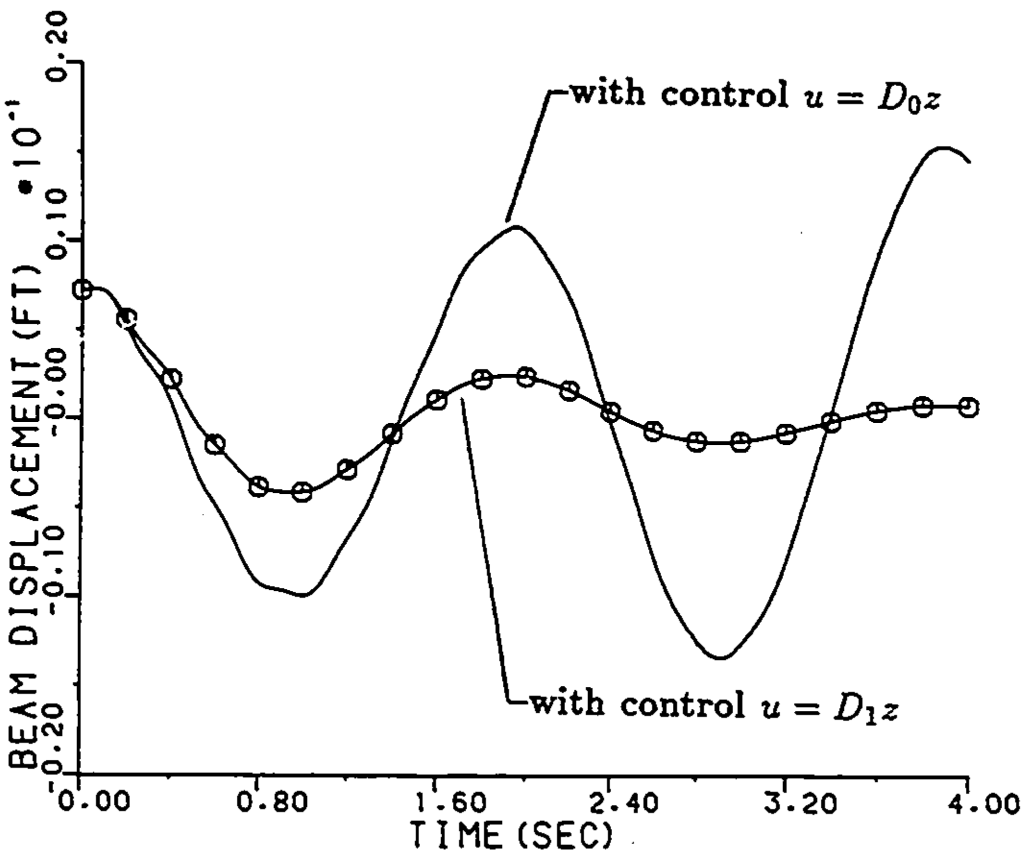

Fig.7.14. Beam displacement $w(1/2, t)$.

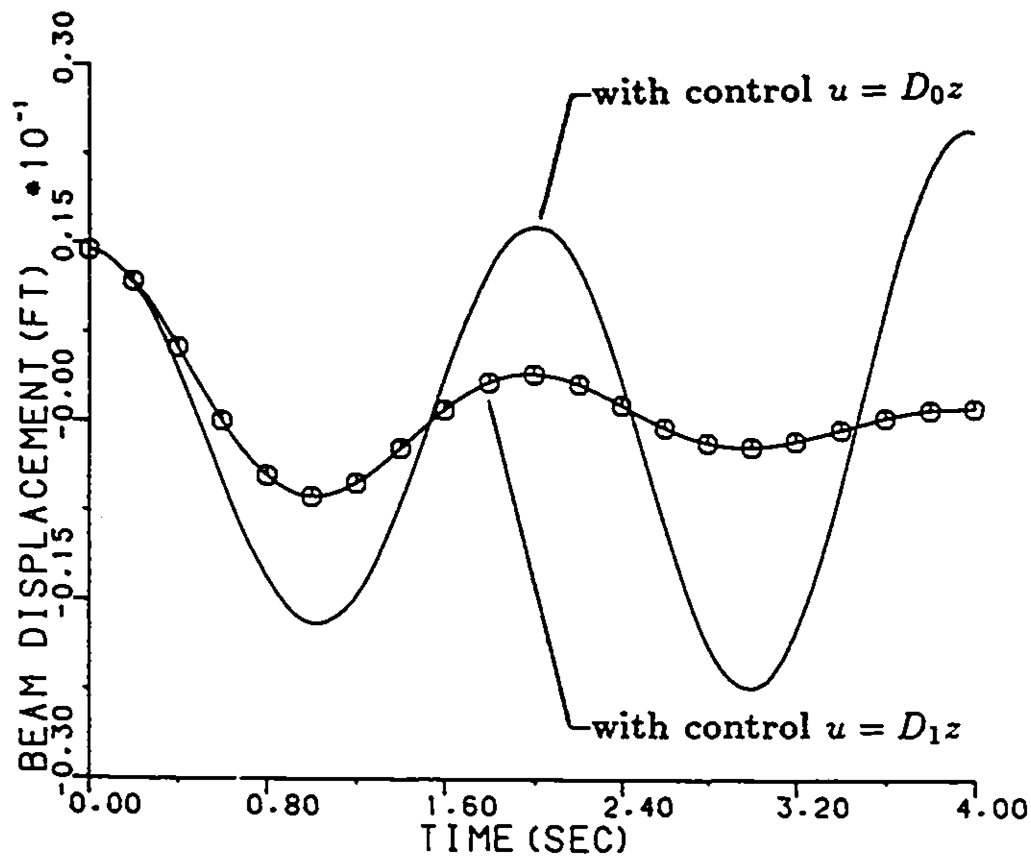

Fig.7.15. Beam displacement $w(1,t)$.

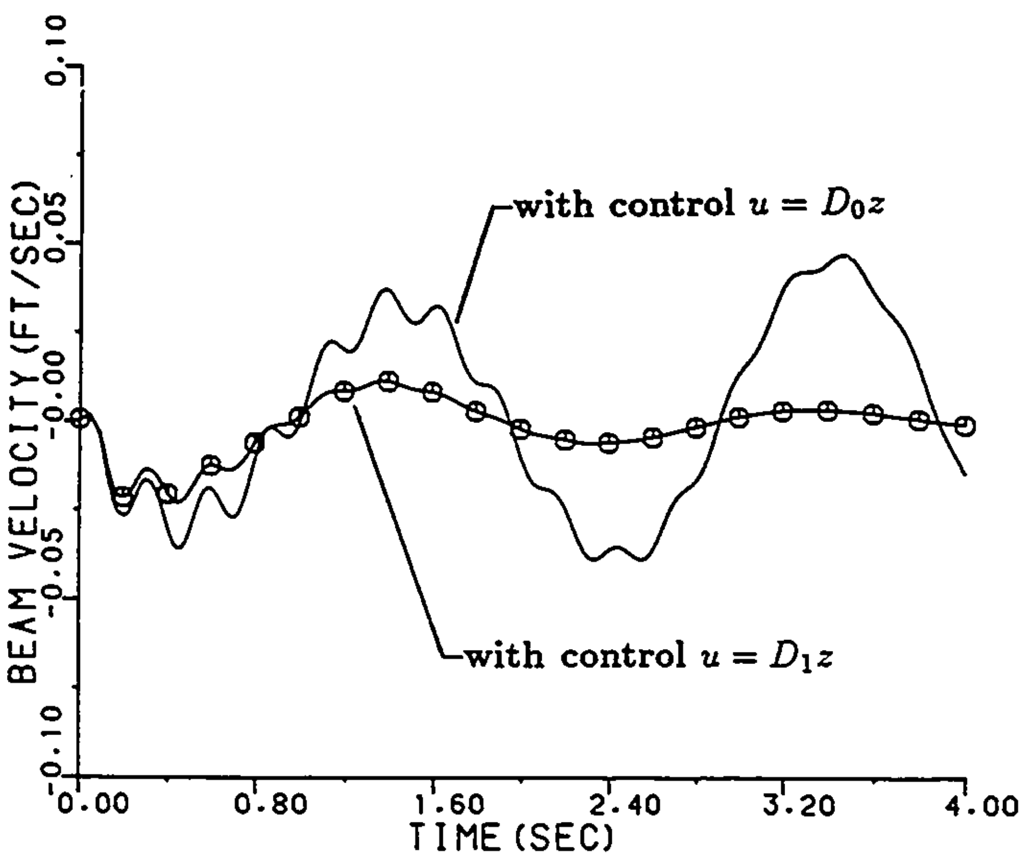

Fig.7.16. **Beam velocity** $\frac{\partial w}{\partial t}(1/2, t)$.

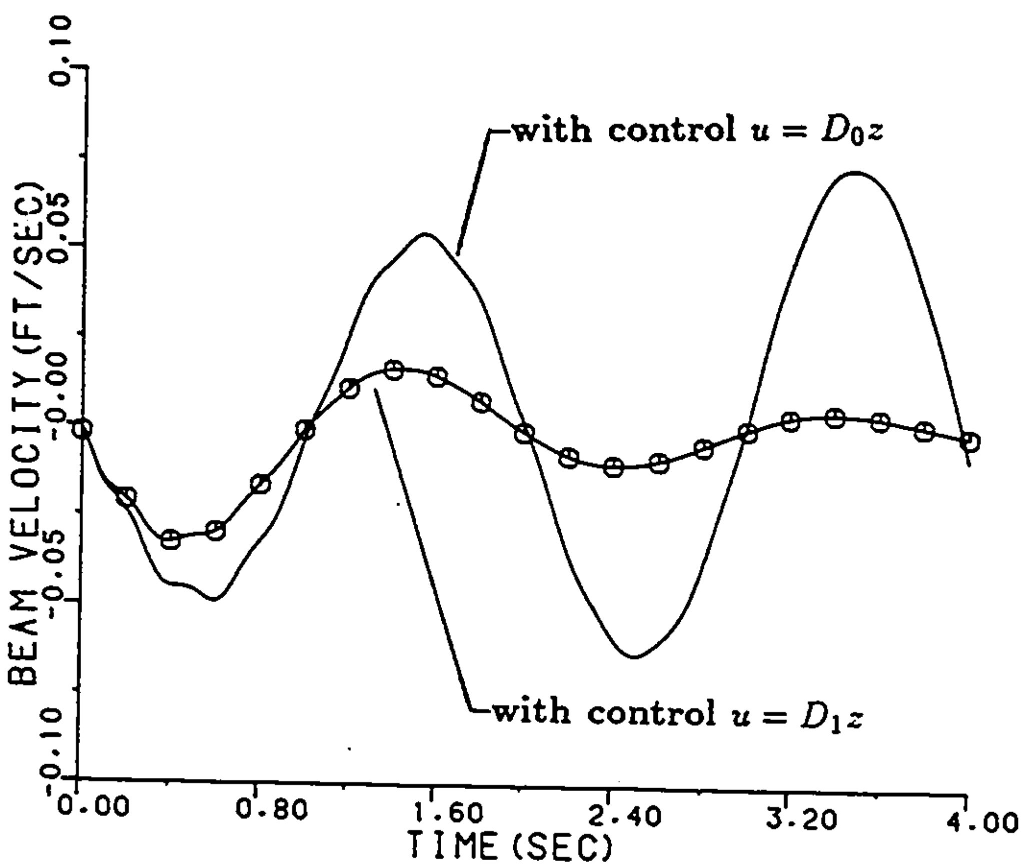

Fig.7.17. Beam velocity $\frac{\partial w}{\partial t}(1, t)$.

As mentioned earlier, flexible structures in space are often subjected to random disturbances arising from various external sources as well as on-board disturbances. This disturbances may induce random distributed forces and random torques on the space structures and hence a system may produce additional additive vibrations of the flexible members. If the disturbances persist, these small motions may lead to instability of the entire systems without appropriate controls. In order to use the spacecraft for any application, for example communication, radiotelescopes, etc., it is necessary that the entire system be brought to a neighborhood of the rest state following random disturbances. In this section, therefore, we also consider the problem of stabilization of the SCOLE model subject to random perturbations. It is shown that, if the system is perturbed by random noises, we can find a proper control based on our stabilization results which will eliminate the noise effect and eventually drive the system back to the rest state.

Let's consider the stochastic version of the perturbed system (7.5)

$$\frac{dz}{dt} = \mathcal{A}z(t) + \mathcal{P}z(t) + \mathcal{B}u(t) + \mathcal{G}dw(t)$$

with the same damping perturbations $D = (1/2)(x-1)\frac{\partial^2}{\partial x^2} - 4(2-x)\frac{\partial}{\partial x}$ and $\mathcal{G} = \begin{pmatrix} 0 \\ G \end{pmatrix}$.

Then, by theorem 6.9 and theorem 6.13, the feedback system (7.5) with a control given by $u = D_2 z = (D_0 - (\mu + (1/2)trQ)\mathcal{B}^*)z$ is exponentially stable in the mean square sense as well as with probability one, where Q is the incremental covariance operator of the generalized Wiener process $w(t)$ (Fig. 7.18-7.22).

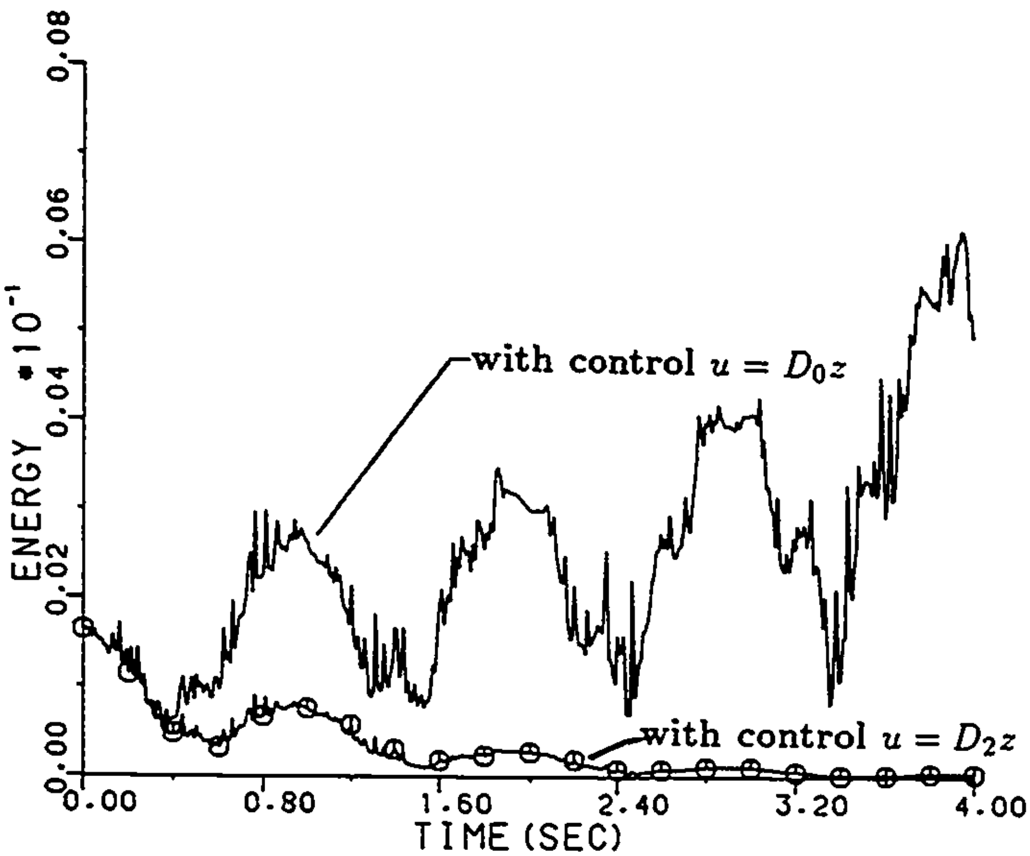

Fig.7.18. Total energy.

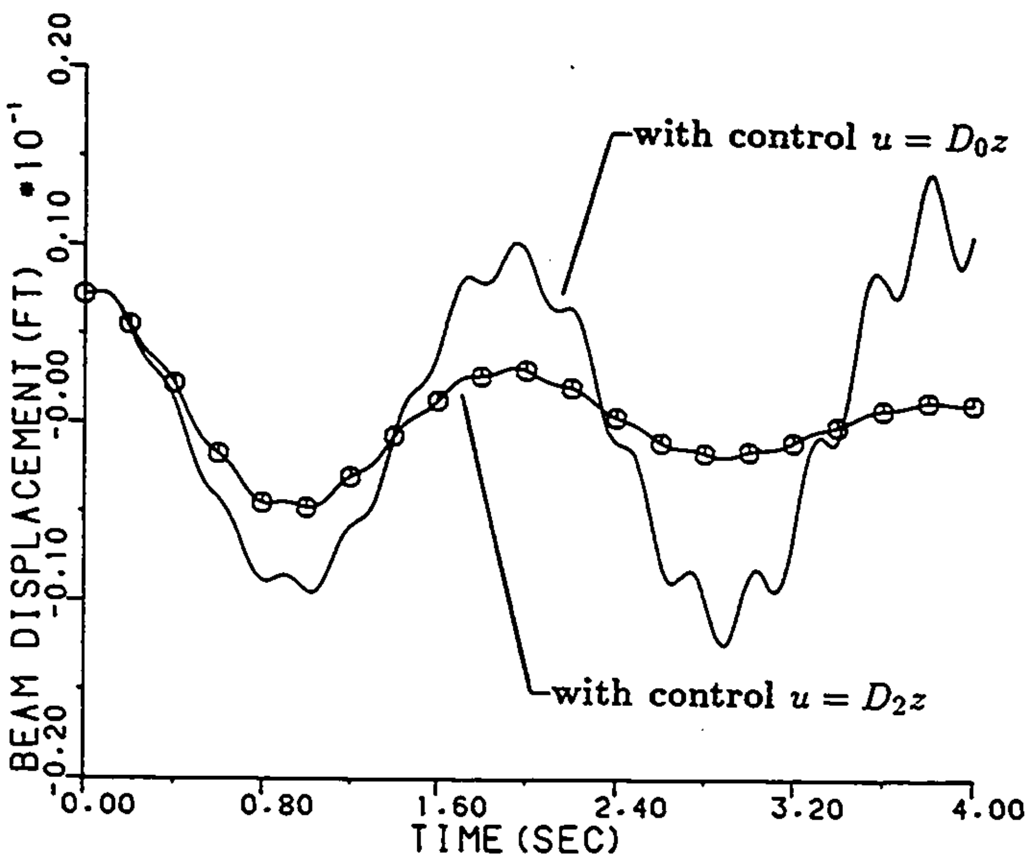

Fig.7.19. Beam displacement $w(1/2, t)$.

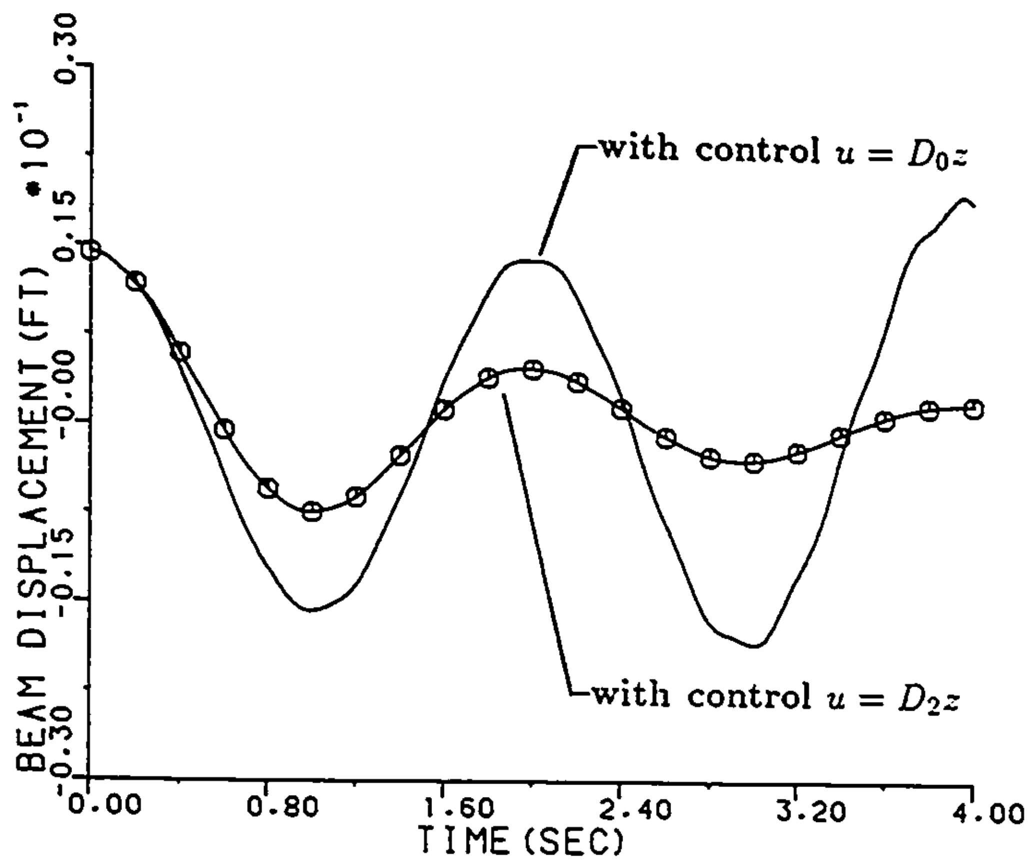

Fig.7.20. Beam displacement $w(1, t)$.

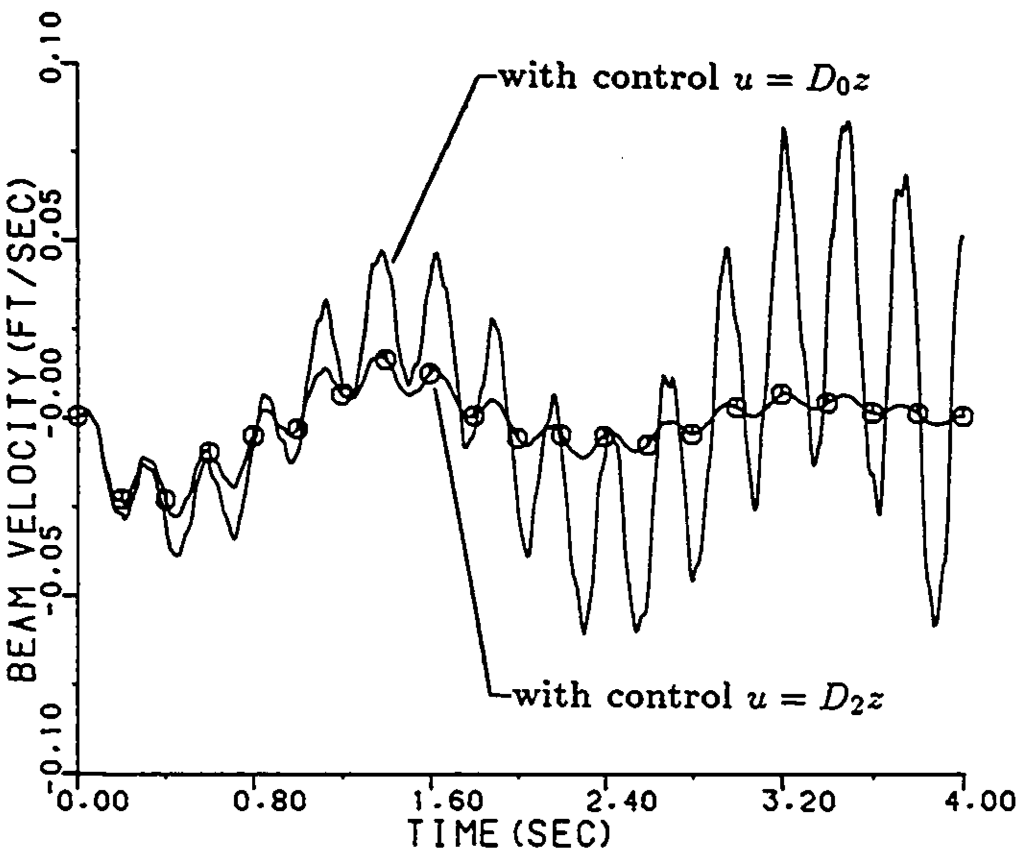

Fig.7.21. Beam velocity $\frac{\partial w}{\partial t}(1/2, t)$.

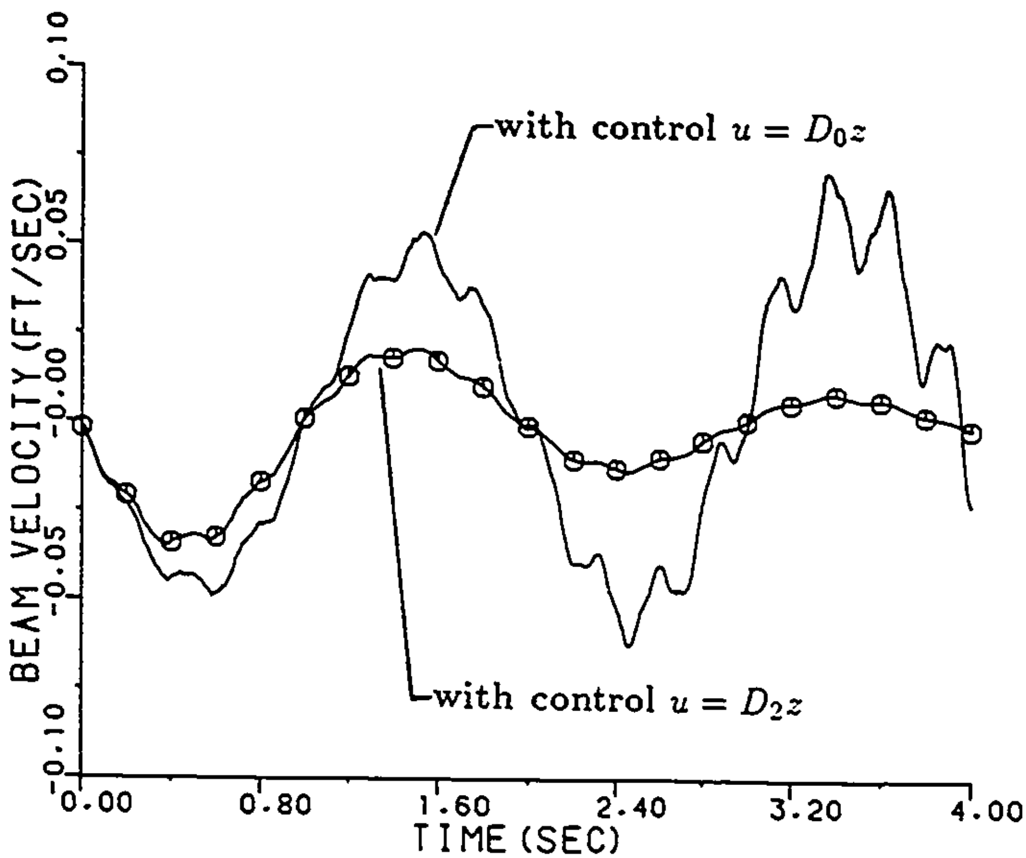

Fig.7.22. Beam velocity $\frac{\partial w}{\partial t}(1,t)$.

CHAPTER 8

CONCLUSIONS AND SUGGESTIONS FOR FUTURE RESEARCH

8.1 Conclusions

Many scientific and engineering problems are extremely complex and hard to model as dynamic systems in finite dimensions (lumped systems). In order to meet this growing challenge, much research effort for infinite dimensional system theory has been made over the past ten years. As part of continuing work in this area, the questions of stability, controllability and stabilizability of infinite dimensional systems (distributed parameter systems) and their perturbed systems have been considered in this thesis. We describe the system dynamics in terms of a strongly continuous semigroup on an appropriate Hilbert space. Using this unifying mathematical approach, we are able to generalize some important finite dimensional system results to infinite dimensions, in both deterministic and stochastic systems. In fact, it is shown that under certain conditions controllability is invariant under bounded or unbounded perturbations. Using these results, we identify the class of perturbations for which the system is robust in the sense that the perturbed system preserves the stability properties of the unperturbed system with the same feedback control as for the nominal system. The theory is illustrated by some numerical results involving both parabolic and hyperbolic equations.

We also consider the inverse problem. That is, instead of identifying the class of perturbations for which the perturbed system remains stabilizable by the same feedback control $u = D_0 x$ as for the nominal system, we present a methodology for designing feedback controllers such that the feedback system is stable. In other words we consider the problem of designing a state feedback linear control law $u = Dx \equiv (D_0 + D_1)x$ that guarantees exponential stability of the perturbed system uniformly with respect to perturbations under the assumption that the nominal system is exponentially stable with $u = D_0x$. It is shown that exponential stability can be achieved by choice of suitable controls even in the presence of unbounded and nonlinear perturbations. The method is mainly based on perturbation theory of semigroups. In this regard, we treat both deterministic and stochastic systems.

Finally, we applied these results to a simplified spacecraft model. Numerical simulations are carried out to illustrate the impact of perturbations on the performance of the space structures and the effectiveness of the stabilizing controls we suggested.

8.2 Suggestions for Future Research

Further research could be conducted in several directions as follows:

(i) As pointed out in Chapter 2, there are two important control problems. One is distributed control, another is boundary control. In practice distributed control may be difficult or costly to implement and very often for systems described by partial differential equations, control action is applied on the boundary of a given domain. Although it is shown our results can be applied to an important class of boundary control systems for which the control action is implemented on the boundary through the intermediary of dynamic systems (section 7.3), development of controllability and stabilizability results for general boundary control systems remains an interesting and challenging problem.

(ii) So far we have discussed system concepts of time-invariant systems; that is, both the system operator and its pertubation operators are independent of time t. But in many practical cases, they are time-varying. Extending the results in the thesis to time-varying systems, therefore, will be a significant problem in applications.

(iii) The controllers developed in the thesis are all based on the assumption that the complete state is available for implementation of the control law. Namely, the feedback control is a function of the complete state variable. In many practical situations, however, only a few output quantities are available. That is, the entire state is not available for direct measurement, only part of the state is given by system output. Therefore, the control law must be based on an estimate of the state, rather that the actual state. In finite dimensional uncertain systems, we have successfully solved this problem by reconstructing the state variable from the available output via a Luenberger observer (Li and Ahmed [78], see also Galimidi and Barmish [47]). Recently, Luenberger observer theory has been extended to distributed parameter systems (El Jai et al. [43,44], Gressang and Lamont [49]). The extension is based on the consideration of sensors. Hence development of robust controllers based on the estimate of the state for perturbed infinite dimensional systems is an outstanding topic for further studies. Meanwhile, the optimal placement of sensors and actuators becomes another interesting problem.

(iv) In the development of the dynamic model for spacecraft, we only consider the vibrational modes of the mast (M) and the antenna (A) regardless of the rotational influence on (A) and (M). In other words, we decouple the whole physical system. In fact, the dynamic equations of motion for the spacecraft are a hybrid system which combines radial equations describing the translational motion of the rigid body (S), attitude equations governing the rotational motion of the system and partial differential equations representing the vibrational dynamics of the flexible members along with very complex boundary conditions (see Ahmed and Biswas

[4,26,27], Ahmed and Lim [9,86,87,88]). These equations are strongly coupled to one another and expressed in implicit form. Transformation of the equations into an explicit form or simplification into abstract equations so that semigroup theory and our results can be applied for system analysis is a totally open and excellent problem from both theoretical and applications points of view.

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