Categorical linear logic and linear distributivity

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Abstract

Linear logic was introduced by Girard in 1987 as a sub-structural logic that retains the constructivism of intuitionistic logic and the symmetry of classical logic. The categorical semantics of linear logic subsequently became an active area of research. In 1992, Cockett and Seely introduced linearly distributive categories (LDC) as semantics for the multiplicative fragment of linear logic. These categories take multiplicative conjunction (tensor) and disjunction (par), along with their interaction via linear distributivities, as the primitive categorical concepts. This highlighted the importance of linear distributivity which provide precisely necessary structure to model the logical cut rule. The theory of LDCs has continued to develop and remains an active area of research within categorical linear logic.

This thesis explores two key areas of categorical linear logic and linear distributivity that have received less attention in recent years, but remain important and worthy of further investigation. The first concerns the relationship between linear distributivity and cartesian structures. This was central topic in the early development of LDCs, notably with the introduction of cartesian linearly distributive categories (CLDC), but their study was quickly abandoned. Moreover, Cockett, Koslowski and Seely introduced linear bicategories in 2000, the bicategorical analogue of LDCs, as natural categorical semantics for non-commutative linear logic. Despite their promising theoretical foundation, further development in this direction was limited. However, recent efforts in categorical logic, particular in classical logic and relational semantics, have renewed interest and highlighted the need for continued research in these directions.

The study of linear distributivity and cartesian structures presented includes the development of a linearly distributive analogue to the Fox theorem, a central result in monoidal category theory characterizing cartesian categories, and a re-examination of CLDCs, which investigates their unique properties and the possible classes of examples. The study of linear bicategories was reinvigorated through the construction of previously unrecognized examples, by leveraging the theory of quantales and quantaloids within the framework of categorical linear logic.

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Contents

Chapter 1

Introduction

This thesis is concerned with the further development of categorical linear logic, with a particular focus on linearly distributive categories, introduced by Cockett and Seely in 1992 [27], and on linear bicategories, developed in 2000 by Cockett, Koslowski, and Seely [22]. Both structures model multiplicative linear logic by treating multiplicative conjunction and disjunction, along with their interaction via linear distributivity, as the primitive categorical notions of interest.

The thesis consists of three articles: *Linearly Distributive Fox Theorem*, *Cartesian Linearly Distributive Categories: Revisited*, and *Constructing Linear Bicategories*. Each contributes to the advancement of categorical linear logic, from perspectives that have received relatively little attention in recent years. The first two articles revisit the relationship between linear distributivity and cartesian structure, through the development of a linearly distributive analogue to the traditional Fox theorem and the re-investigation of the notion of cartesian linearly distributive categories. The third article reinvigorates the study of linear bicategories by exploring examples through the lens of quantale and quantaloid theory.

To provide appropriate context for this work, the following sections offer an overview of the development of the relevant fields.

1.1 Linear Logic

Categorical logic is the intersection of two fields studying mathematical foundations: logic and category theory. It is the branch of mathematics in which concepts and perspectives from category theory are applied to the study of logical structures and vice versa.

Many logical systems can best be modeled by going beyond set-theoretical model theory. Categorical logic is concerned with identifying the categorical concepts which provide the best interpretation for a given logical system, determining the additional structures or properties an arbitrary category must satisfy to model the logic [64]. These categories are known as the *categorical semantics* of the logic in question. The objects of such categories are in essence the formulas, while the morphisms are equivalence classes of proofs, or derivations of sequents.

It was in considering a particular categorical model for intuitionistic logic that Girard developed a new logic in 1987 [34]. Indeed, Girard introduced and studied the category **Cohs** of coherent spaces and stable maps.

Definition 1.1.1. [34, Def 3.1, 3.2, IV.I]

- A *coherent space* X is a collection of subsets of some set $|X|$ such that
  1. $\varnothing \in X$,
  2. If $a \in X$ and $b \subseteq a$, then $b \in X$, and
  3. If $A \subset X$ formed of pairwise compatible elements, then $\bigcup A \in X$, where $a, b \in X$ are said to be *compatible* with respect to X when $a \cup b \in X$.

  Equivalently, a coherent space is a pair $X = (|X|, \frown_X)$ where $|X|$ is a set, called the web of X, and $\frown_X$ is a binary reflexive and symmetric relation on X. Note that a *clique* of X is a subset $u \subseteq |X|$ such that $\forall x, x' \in u, x \frown_X x'$.

- A *stable map* between coherent spaces $F : X \to Y$ is a function $F : |X| \to |Y|$ satisfying
  1. If $a \subseteq b \in X$, then $F(a) \subseteq F(b)$,
  2. If $a \cup b \in X$, then $F(a \cap b) = F(a) \cap F(b)$, and
  3. F commutes with directed unions.

When considering the function space $X \Rightarrow Y$ of stable maps between two coherent spaces X and Y, Girard noticed that it could be described using a space of repetitions $!X$ and the function space $X \multimap Y$ of a full subcategory of *linear* maps **Cohl**:

$$X \Rightarrow Y = !X \multimap Y$$

Definition 1.1.2. [34, Def IV.3, IV.5, IV.6]

- A stable map between coherent spaces $F : X \to Y$ is *linear* if it additionally satisfies
  1. If $a \cup b \in X$, then $F(a \cup b) = F(a) \cup F(b)$, and
  2. $F(\varnothing) = \varnothing$.

- Consider coherent spaces $X = (|X|, \frown_X)$ and $Y = (|Y|, \frown_Y)$. Define $X \multimap Y$ by

$$|X \multimap Y| = |X| \times |Y|$$

$$(x, y) \frown_{X \multimap Y} (x', y') \quad \text{iff} \quad x \frown_X x' \Rightarrow (y \frown_Y y' \wedge y = y' \Rightarrow x = x')$$

- Consider coherent spaces $X = (|X|, \frown_X)$, define $!X$ by

$$|!X| = X_{fin}, \text{the set of finite cliques of } X$$

$$x \frown_{!X} x' \quad \text{iff} \quad x \cup x' \text{ is a clique of } X$$

Viewing **Cohs** as a model for intuitionistic logic, the function space $X \Rightarrow Y$ represents standard implication. This suggested to Girard that intuitionistic implication could be recovered from the subcategory **Cohl** of linear maps, representing a more restricted form of "linear" implication, and a unary operator !.

Moreover, when considering $\multimap$ applied to the duals $X^\perp$ of a coherent spaces X, defined as follows, Girard noticed that $Y^\perp \multimap X^\perp \cong X \multimap Y$, which suggested that linear implication $\multimap$ was symmetric in a classical sense, i.e. symmetric between inputs and outputs.

Definition 1.1.3. [34, Def 3.5] Consider a coherent space $X = (|X|, \frown_X)$. Define $X^\perp$ by

$$|X^\perp| = |X|$$
$$x \frown_{X^\perp} x' \quad \text{iff} \quad x \frown_X x' \Rightarrow x = x'$$

Altogether, this prompted Girard to develop a logic which retained both the symmetry inherent to classical logic and the constructivist nature of intuitionistic logic. To do so, Girard omitted the traditional structure rules of contraction and weakening, which are necessary for non-constructivist proofs. In reference to his newly created logical connectives, linear implication $\multimap$ abd linear negation $(-)^\perp$, Girard named it *linear logic.*

While Girard first introduced a Gentzen style sequent calculus for linear logic which uniquely used right-sided sequents, the version introduced here will be with two-sided sequents as this presentation more readily relates to the categorical semantics to follow.

Definition 1.1.4. [73, Def 1.1] Propositional linear logic sequent calculus consists of formulas and sequents. The formulas are generated by the binary connectives and by the unary operations from variables and a set of constants given below.

Connectives			
	Multiplicative	Additive	Exponential
Conjunction	$\otimes$	$\&$	!
Disjunction	$⅋$	$\oplus$	?
Linear Implication	$\multimap$		
Linear Negation	$(-)^\perp$		

Constants		
	Multiplicative	Additive
Truth	$\mathbf{1}$	$\top$
Falsity	$\perp$	$\mathbf{0}$

Sequents are ordered pairs of finite sequences of formulas, generated by the following rules from the axioms.

Axioms.

$$A \vdash A\ (id) \qquad A \vdash A^{\perp\perp}\ (D) \qquad A^{\perp\perp} \vdash A\ (D^{-1})$$
$$\vdash \mathbf{1}\ (\mathbf{1}R) \qquad \perp \vdash\ (\perp L) \qquad \Gamma \vdash \top, \Delta\ (\top R) \qquad \Gamma, \mathbf{0} \vdash \Delta\ (\mathbf{0}L)$$

Structural Rules.

$$\frac{\Gamma \vdash \Delta}{\sigma\Gamma \vdash \tau\Delta}\ (\text{PERM}) \qquad \frac{\Gamma \vdash A, \Delta \qquad A, \Theta \vdash \Psi}{\Gamma, \Theta \vdash \Psi, \Delta}\ (\text{CUT})$$

Multiplicative Rules.

$$\frac{\Gamma \vdash \Delta}{\Gamma, \mathbf{1} \vdash \Delta}\ (\mathbf{1}\text{L}) \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \bot, \Delta}\ (\bot\text{R})$$

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \otimes B \vdash \Delta}\ (\otimes\text{L}) \qquad \frac{\Gamma \vdash A, \Delta \qquad \Theta \vdash B, \Psi}{\Gamma, \Theta \vdash A \otimes B, \Delta, \Psi}\ (\otimes\text{R})$$

$$\frac{\Gamma \vdash A, \Delta \qquad \Theta, B \vdash \Psi}{\Gamma, \Theta, A \parr B \vdash \Delta, \Psi}\ (\parr\text{L}) \qquad \frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \parr B, \Delta}\ (\parr\text{R})$$

Additive Rules.

$$\frac{\Gamma, A \vdash \Delta}{\Gamma, A \,\&\, B \vdash \Delta} \qquad \frac{\Gamma, B \vdash \Delta}{\Gamma, A \,\&\, B \vdash \Delta}\ (\&\text{L}) \qquad \frac{\Gamma \vdash A, \Delta \qquad \Gamma \vdash B, \Delta}{\Gamma \vdash A \,\&\, B, \Delta}\ (\&\text{R})$$

$$\frac{\Gamma, A \vdash \Delta \qquad \Gamma, B \vdash \Delta}{\Gamma, A \oplus B \vdash \Delta}\ (\oplus\text{L}) \qquad \frac{\Gamma \vdash A, \Delta}{\Gamma \vdash A \oplus B, \Delta} \qquad \frac{\Gamma \vdash B, \Delta}{\Gamma \vdash A \oplus B, \Delta}\ (\oplus\text{R})$$

Negation Rule.

$$\frac{\Gamma, A \vdash B, \Delta}{\Gamma, B^{\bot} \vdash A^{\bot}, \Delta}\ ({}^{\bot}\text{VAR})$$

Implication Rules.

$$\frac{\Gamma \vdash A, \Delta \qquad \Theta, B \vdash \Psi}{\Gamma, \Theta, A \multimap B \vdash \Psi, \Delta}\ (\multimap\text{L}) \qquad \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \multimap B, \Delta}\ (\multimap\text{R})$$

Exponential Rules

$$\frac{\Gamma, A \vdash \Delta}{\Gamma, !A \vdash \Delta}\ (!\text{D}) \qquad \frac{\Gamma \vdash \Delta}{\Gamma, !A \vdash \Delta}\ (!\text{W})$$

$$\frac{\Gamma, !A, !A \vdash \Delta}{\Gamma, !A \vdash \Delta}\ (!\text{C}) \qquad \frac{!\Gamma \vdash A, ?\Delta}{!\Gamma \vdash !A, ?\Delta}\ (!\text{R})$$

$$\frac{\Gamma \vdash A, \Delta}{\Gamma \vdash ?A, \Delta}\ (?\text{D}) \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash ?A, \Delta}\ (?\text{W})$$

$$\frac{\Gamma \vdash ?A, ?A, \Delta}{\Gamma \vdash ?A, \Delta}\ (?\text{C}) \qquad \frac{!\Gamma, A \vdash ?\Delta}{!\Gamma, ?A \vdash \Delta}\ (?\text{L})$$

Girard additionally introduced quantifiers to linear logic, although they will not be discussed in this thesis.

Linear logic is known as a sub-structural logic since some of the traditional structural rules are omitted, in this case the contraction and the weakening rules. It is in fact their absence which causes conjunction and disjunction to "split" into context-free, known as the multiplicatives, and, context-dependent connectives, known as the additives, i.e. $\otimes$ (read as tensor) and $⅋$ (read as par) versus $\&$ (read as with) and $\oplus$ (read as plus) [80].

The absence of weakening and contraction for the multiplicative connectives has a significant consequence: linear logic no longer consists of stable truths. It is a logic of resources. The premise of linear implication can only be used once. It cannot be duplicated (in other words reused to prove another statement) [80].

However, the exponential connectives ! (read as "of course" or "bang") and ? (read as "why not" or "whimper") reintroduces the notions of stable truth by allowing weakening and contraction, but only for specific formulas [80].

This allows propositional linear logic to recover traditional propositional intuitionistic logic, consisting of the symbols $\wedge, \vee, \Rightarrow$ and $\neg$, through the following identifications.

$$A \wedge B = A \,\&\, B \qquad A \vee B = (!A) \oplus (!B)$$
$$A \Rightarrow B = (!A) \multimap B \qquad \neg A = (!A) \multimap \mathbf{0}$$

There is some redundancy in the structure introduced above. Negation can alternatively be defined by the following *de Morgan dualities*.

$$\begin{aligned}
\mathbf{1}^\perp &:= \perp & \perp^\perp &:= \mathbf{1}\\
\top^\perp &:= \mathbf{0} & \mathbf{0}^\perp &:= \top\\
(A \otimes B)^\perp &:= A^\perp ⅋ B^\perp & (A ⅋ B)^\perp &:= A^\perp \otimes B^\perp\\
(A \,\&\, B)^\perp &:= A^\perp \oplus B^\perp & (A \oplus B)^\perp &:= A^\perp \,\&\, B^\perp\\
(!A)^\perp &:= ?A^\perp & (?A)^\perp &:= !A^\perp\\
(A)^\perp &= A^\perp & (A^\perp)^\perp &= A
\end{aligned}$$

Furthermore, linear implication can be defined by

$$A \multimap B := A^\perp ⅋ B$$

The influence of linear logic since its advent in mathematical logic, category theory and theoretical computer science cannot be overstated.

Specific fragments of linear logic became fields of study in and of themselves. This thesis will be concerned with multiplicative linear logic (MLL) which is the fragment restricted to the multiplicative connectives $\otimes$ and $⅋$, and constants $\mathbf{1}$ and $\perp$, and multiplicative linear logic with negation (MLL+Neg) which additionally includes linear negation $(-)^\perp$ and linear implication $\multimap$. Other important fragments are multiplicative linear logic with exponentials (MELL) and multiplicative linear logic with additives (MALL).

Furthermore, multiple variants of linear logic have been developed. Multiplicative linear logic with the mix rule (MLL+Mix) is an extension of the fragment MLL which adds the mix rule defined as follows [26].

$$\frac{\Gamma \vdash \Delta \qquad \Theta \vdash \Psi}{\Gamma, \Theta \vdash \Delta, \Psi} \text{(MIX)}$$

Linear logic's more sophisticated structural rules allow one to consider non-commutative logic, i.e. where "A and B" no longer necessarily implies "B and A". In traditional logics, this cannot be done because the exchange rule is derivable from the rest of the system. This gives logic a temporal quality. "A and B" can be thought of as "A and then B".

As such, important variants come from considering non-commutative connectives, such as non-commutative multiplicative linear logic and non-commutative multiplicative linear logic with negation, the latter known as bilinear logic (BILL), developed by Lambek for applications in linguistics [49]. On another hand, Retoré showed one can extend linear logic by adding an additional non-commutative connective $\triangleleft$ [65]. This logic, known as pomset logic, was also developed by considering the categorical model of linear logic given by coherent spaces.

When studying pomset logic, Guglielmi recognized that it could not be expressed in sequent calculus. This led Guglielmi to develop the calculus of structures, based on the principle of deep inference, as an alternative logic formalism to sequent calculus, along with system BV to mirror pomset logic in the calculus of structures [38]. The introduction of deep inference spurned many more developments including a formulation of multiple variants of linear logic in the calculus of structures by Straßburger [75].

Moreover, Hyland and de Paiva developed full intuitionistic linear logic (FILL) in order to understand the fragment of linear logic consisting of the multiplicative connectives and linear implication, without negation [43].

Another variant of note is Girard, Scedrov and Scott's bounded linear logic (BLL). Motivated by programming considerations, it replaces the exponentials, which allow an unlimited reuse of formulas, by bounded exponentials, which allow a limited number of re-uses [37].

A variant of linear logic that has received a tremendous amount of attention is Erhard and Regnier's differential linear logic (DiLL), which extends linear logic by inference rules allowing the differentiation of proofs [31]. It was in fact developed when it was noticed that a form of differentiation was present in some categorical models of linear logic.

Linear logic and the branch of mathematics it gave birth to is perhaps one of the greatest examples of the power of studying categorical semantics and of how categorical semantic considerations can impact mathematical logic.

For an overview of linear logic and an exploration of some of the advances discussed above, see [36, 80].

1.2 $*$-Autonomous Categories

In his original paper on linear logic, Girard introduced two specific categorical semantics: coherent spaces and linear maps (discussed above) and phase spaces, the latter presented slightly later in the text [34]. However, the broader question of the general categorical semantics of linear logic remained open, namely, what structures and properties must a category possess to serve as a model of linear logic?

Broadly, viewing objects of a category as formulas and morphisms as proofs, logical constants are modeled by distinguished objects in the category, while connectives behave as binary operations acting on both objects and morphisms. This concept is captured by the monoidal category, as first introduced by Bénabou [6], although subsequently correctly axiomatized by Mac Lane [55], and later refined by Kelly [47].

Definition 1.2.1. [56, Sec 7.1] A **monoidal category** $(\mathcal{X}, \oslash, I, \alpha, \rho, \lambda)$ consists of:

- a category $(\mathcal{X}, ;, 1_X)$,
- the **product** and the **unit** functors

$$\oslash : \mathcal{X} \times \mathcal{X} \to \mathcal{X} \qquad\qquad I : \mathbf{1} \to \mathcal{X}$$

- the **associator**, the **right unitor** and the **left unitor** natural isomorphisms

$$\alpha_{X,Y,Z} : (X \oslash Y) \oslash Z \to X \oslash (Y \oslash Z)$$
$$\rho_X : X \to X \oslash I$$
$$\lambda_X : X \to I \oslash X$$

such that the associativity pentagon and unit triangle identities hold.

As such, monoidal categories provide the foundation for categorical semantics of any logic. Prototypical examples of monoidal categories include the category of sets and relations Rel with the cartesian product, and the category of vector spaces over a field k and linear maps Vect_k with the standard tensor product of vector spaces. Moreover, any category with finite products is monoidal with its monoidal product given by the binary categorical product and its unit given by the terminal object. Such categories are called *cartesian*.

The question of general categorical semantics had already been answered for intuitionistic logic. Lambek showed that there is an equivalence between the positive intuitionistic propositional calculus and cartesian closed categories with a weak natural numbers object [51]. Combined with the Curry-Howard proofs-as-programs correspondence, this led to what is now known as the Curry-Howard-Lambek correspondence, marking a foundational convergence of logic, computation, and category theory [52].

In 1989, Seely endeavored to solve this question for linear logic and demonstrated that $*$-autonomous categories provided the categorical semantics for linear logic [73]. Now, $*$-autonomous categories had been introduced before the advent of linear logic in 1979 by Barr [3]. At the time, Barr was interested in capturing the structure of categories such as certain categories of vector spaces and uniform spaces.

Definition 1.2.2. [3, 4.3] A *$*$-autonomous category* is a symmetric monoidal category $(\mathcal{X}, \otimes, \top)$ equipped with a full and faithful functor $(-)^* : \mathcal{X}^{op} \to \mathcal{X}$ such that there are natural isomorphisms $\mathcal{X}(A \otimes B, C^*) \cong \mathcal{X}(A, (B \otimes C)^*)$.

To be precise, Seely demonstrated that $*$-autonomous categories provide the categorical semantics for MLL+Neg, where $\otimes$ models multiplicative conjunction, $\top$ models multiplicative truth and $(-)^*$ models linear negation. Multiplicative disjunction and falsity are then subsequently defined by the de Morgan dualities, while linear implication is defined by the formula given in the previous section. In order to additionally model the additive fragment, one must consider $*$-autonomous categories with finite products, i.e. with binary products $A \times B$ and terminal object $\mathbf{1}$. $\times$ then models additive conjunction and $\mathbf{1}$ models additive truth, and additive disjunction and falsity are defined via the de Morgan dualities. Finally, the categorical semantics of full propositional linear logic, including the exponentials, are given by what Seely called Girard categories.

Definition 1.2.3. [73, Def 2.2] A *Girard category* is a $*$-autonomous category $\mathcal{X}$ with finite products and a comonad $! : \mathcal{X} \to \mathcal{X}$ satisfying the following:

1. for each $A \in \mathcal{X}$, $!A$ is a $\otimes$-comonoid,
2. there are natural isomorphisms $!A \otimes !B \to !(A \times B)$ and $\top \to !\mathbf{1}$, known as the *Seely isomorphisms*, which preserve the comonoid structures.

Seely further showed how one can recover a cartesian closed category from a Girard category, in the same way one recovers intuitionistic logic from linear logic: by considering sequents of the form $!A \vdash B$ since $A \Rightarrow B = !A \multimap B$.

Proposition 1.2.4. [73, Def 2.2] Let $\mathcal{X}$ be a Girard category, then the Kleisli category $\mathcal{X}_!$ of the comonad ! is cartesian closed.

Then, the category of coherent spaces and linear maps **Cohl** is a Girard category and its Kleisli category is the category of coherent spaces and stable maps **Cohs**, the model of intuitionistic logic which began everything.

The theory of $*$-autonomous categories continued to be developed with respect to linear logic, extended to non-symmetric models of linear logic [4], but also found purchase in the fields of categorical topology [5], categorical algebraic geometry and quantum theory [14, 30], and recursion theory [39]. Of note is the Chu construction, which yields a $*$-autonomous category from any closed symmetric monoidal category [18], which continues to be studied.

A detail-oriented reader may have noticed at this point that the symbols used by Seely do not correspond exactly with those introduced by Girard. Unfortunately, the notation is not consistent across logic and category theory. This thesis will stick to one notational convention throughout all the articles, which will be introduced shortly.

1.3 Linearly Distributive Categories

The categorical semantics of MLL+Neg provided by $\ast$-autonomous categories are categories equipped with two monoidal structures: one modeling multiplicative conjunction (tensor) and truth, and the other modeling multiplicative disjunction (par) and falsity. The occurrence of two monoidal products within a single category is not unusual. In many areas of category theory, mathematical logic, theoretical computer science, and theoretical physics, it is common to encounter multiple binary operations of interest. In the context of monoidal category theory, there is an active research program focused on studying categories with two monoidal structures and the ways in which they interact via *distributivities*.

Perhaps the most well-know definition in this branch is that of the distributive category:

Definition 1.3.1. [20, Sec 3] A *distributive category* is a category $\mathbb{D}$ with finite products $(\times, \mathbf{1})$ and with finite coproducts $(+, \mathbf{0})$ such that the product distributes over the coproduct: the canonical natural transformation

$$d^R_{A,B,C} = [\iota^0_{A,B} \times 1_C, \iota^1_{A,B} \times 1_C] : (A \times C) + (B \times C) \to (A + B) \times C$$

is an isomorphism.

However, the interaction via distributivities between tensor and par is not absent from the definition of $\ast$-autonomous categories. In 1992, Cockett and Seely pursued alternative categorical semantics for MLL, which promoted the role of tensor and par, and their interaction via distributivity sequents

$$A \otimes (B \parr C) \dashv (A \otimes B) \parr C \qquad (A \parr B) \otimes C \vdash A \parr (B \otimes C)$$

which can be derived as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{B \vdash B}\,(\mathrm{ID}) \quad \dfrac{}{C \vdash C}\,(\mathrm{ID})}{B \parr C \vdash B, C}\,(\parr\mathrm{L}) \qquad \dfrac{\dfrac{}{A \vdash A}\,(\mathrm{ID}) \quad \dfrac{}{B \vdash B}\,(\mathrm{ID})}{A, B \vdash A \otimes B}\,(\otimes\mathrm{R})}{A, B \parr C \dashv A \otimes B, C}\,(\mathrm{CUT})}{A \otimes (B \parr C) \dashv A \otimes B, C}\,(\otimes\mathrm{L})}{A \otimes (B \parr C) \dashv (A \otimes B) \parr C}\,(\parr\mathrm{R})$$

With tensor, par and these distributivities as the primitive notions of their categorical semantics for MLL, Cockett and Seely introduced the *linearly distributive category* (LDC) [27].

Definition 1.3.2. [27, Sec 2.1] A *linearly distributive category* $(\mathbb{X}, \otimes, \top, \oplus, \bot)$ consists of:

- a category $(\mathbb{X}, ;, 1_A)$,
- a *tensor* monoidal structure $(\mathbb{X}, \otimes, \top)$
- a *par* monoidal structure $(\mathbb{X}, \oplus, \bot)$

- left and right *linear distributivity* natural transformations

$$\delta^R_{A,B,C} : (A \oplus B) \otimes C \to A \oplus (B \otimes C) \qquad \delta^L_{A,B,C} : A \otimes (B \oplus C) \to (A \otimes B) \oplus C$$

satisfying coherence conditions.

Note here that Cockett and Seely introduced new notation, in contrast to Girard's original notation.

	Tensor	Par	With	Plus
Cockett & Seely	$\otimes . \top$	$\oplus, \bot$	$\times, 1$	$+, 0$
Girard	$\otimes, \mathbf{1}$	$\parr, \bot$	$\&, \top$	$\oplus, \mathbf{0}$

There has been much discussion in the community about which notational convention is best. As the work in this thesis is based on LDCs, Cockett and Seely's conventions shall be used throughout.

The distributivity between tensor and par had not been emphasized prior to the work of Cockett and Seely, potentially due to the use of right-sided sequents within mathematical logic. Now, MLL without negation is essentially the logical system with two binary operations whose sequent calculus includes only right and left introduction rules, and the cut rule. As such, LDCs provide the categorical semantics for a minimal logic with two connectives, making them foundational in the development of categorical semantics. Linear distributivities are precisely the maps needed to model the cut rule.

The LDC was originally named a weakly distributive category as they initially believed linear distributivity to be a "weakening" of the traditional distributivity of distributive categories. It was thought that certain LDCs would coincide with distributive categories, in particular the *cartesian linear distributive categories* (CLDC) [25].

Definition 1.3.3. [27, Sec 2] A *cartesian linearly distributive category* $\mathbb{X}$ is a symmetric linearly distributive category whose tensor structure is the categorical product $\times$ in $\mathbb{X}$, with the terminal object $\mathbf{1}$ in $\mathbb{X}$, and par structure is the categorical coproduct $+$ in $\mathbb{X}$, with the initial object $\mathbf{0}$ in $\mathbb{X}$.

It became clear that this was in fact untrue and, in an amended version of their paper, Cockett and Seely demonstrated that a CLDC is a distributive category if and only if it is a poset [27]. As such, they changed the name to distance from distributive categories and to promote their relevance to linear logic.

Cockett and Seely additionally demonstrated in their original paper how to recover the categorical semantics of MLL+Neg, in other words $*$-autonomous categories, from LDCs.

Definition 1.3.4. [27, Def 4.1] A symmetric linearly distributive category *has negation* if it is equipped with an object function $(-)^\bot$ with parametrized families of maps

$$A^\bot \otimes A \to \bot \qquad A \otimes A^\bot \to \bot \qquad \top \to A^\bot \oplus A \qquad \top \to A \oplus A^\bot$$

satisfying coherence conditions.

The notions of symmetric LDCs with negation and $*$-autonomous categories coincide. In particular, this implies that LDCs provide a more general framework than $*$-autonomous categories. A key motivation for the use of LDCs is that their axioms are often more straightforward to verify than those of $*$-autonomous categories. This idea was further extended to the non-symmetric setting in the amended paper by Cockett and Seely in order to model non-commutative MLL+Neg [27].

Although the notion of CLDCs was ultimately abandoned due to their incompatibility with distributive categories, the theory of LDCs continued to be extensively developed by Cockett, Seely, and collaborators.

Blute, Cockett, Seely, and Trimble introduced a diagrammatic calculus for LDCs, representing morphisms via two-sided proof nets, otherwise known as circuit diagrams. This led to a coherence theorem for both LDCs and $*$-autonomous categories [10]. While this calculus will not be employed in this thesis, it has become a central tool in categorical linear logic.

Furthermore, Blute, Cockett, and Seely explored modeling the exponentials ! and ? within LDCs to extend the categorical semantics of MLL to include exponentials [8]. This work culminated in the development of *linear functors* and *linear transformations*, the appropriate notions of functors and natural transformations between LDCs, leading to the definition of the 2-category **LDC** [28]. Cockett and Seely demonstrated that the appropriate categorical semantics for MLL with exponentials are LDCs with storage, i.e. a comonad $(!, ?)$ in **LDC** carrying a compatible cocommutative comonoid structure. They further demonstrate how to include the additive connectives to the semantics by considering LDCs with *linear products*, which are the appropriate linear functor analogue of categorical products.

Finally, Cockett and Seely demonstrated which features needed to be added to LDCs to model important variants of linear logic, in particular MLL+Mix, Lambek's BILL, de Paiva and Hyland's FILL [26].

Following this series of papers, LDCs have maintained a strong presence in categorical logic. Notable developments include the introduction of traces and Girard's geometry of interaction in the context of LDCs by Blute, Cockett, and Seely [9], the study of categorical semantics for Guglielmi's linear logic extension BV by Blute, Panangaden and Slanov [11], the formulation of linear actegories as categorical semantics for concurrency in theoretical computer science by Cockett and Pastro [24], the study of comonads by Pastro [63] and Hopf monads on LDCs by Hasegawa and Lemay [40], the introduction of dagger LDCs by Cockett, Comfort, and Srinivasan, offering a categorical framework for infinite-dimensional quantum mechanics [21].

Even within papers which build upon $*$-autonomous categories, linear distributivity is at the forefront of the discussions thanks to the efforts of Cockett and Seely.

1.4 Linear Bicategories

Commutative logical connectives are modeled categorically by symmetric monoidal products. While non-commutative connectives can be modeled by non-symmetric monoidal products, one could also consider composition in bicategories. Indeed, bicategories are essentially monoidal categories whose objects are given typed domains and codomains. Therefore, the monoidal product between two objects is only defined if the types line up. Composition is a fundamentally non-symmetric operation, making it the most natural model for non-commutative connectives.

It is from this point of view that Cockett, Kowslowski and Seely introduced the *linear bicategory* in 2000, as the bicategorical analogue to LDCs and the natural categorical semantics for non-commutative MLL [22].

Definition 1.4.1. [22, Sec 2.1] A *linear bicategory* $(\mathcal{B}, \otimes, \top_X, \oplus, \bot_X)$ consists of:

- a class of 0-cells $\mathcal{B}_0$
- a category of 1-cells and 2-cells $\mathcal{B}_1$, with functors $\mathrm{dom}, \mathrm{cod} : \mathcal{B}_1 \to \mathcal{B}_0$, where the following full subcategories are denoted by

$$\mathcal{B}(X, Y) = \{f \in \mathcal{B}_1 \mid \mathrm{dom}(f) = X, \mathrm{cod}(f) = Y\}$$

- a *tensor* bicategorical structure $(\mathcal{B}, \otimes, \top_X)$ with composition functor and identity 1-cells

$$\otimes : \mathcal{B}(X, Y) \times \mathcal{B}(Y, Z) \to \mathcal{B}(X, Z) \qquad \top_X : X \to X$$

- a *par* bicategorical structure $(\mathcal{B}, \oplus, \bot_X)$ with composition functor and identity 1-cells

$$\oplus : \mathcal{B}(X, Y) \times \mathcal{B}(Y, Z) \to \mathcal{B}(X, Z) \qquad \bot_X : X \to X$$

- left and right *linear distributivity* transformations

$$\delta^R_{A,B,C} : (A \oplus B) \otimes C \to A \oplus (B \otimes C) \qquad \delta^L_{A,B,C} : A \otimes (B \oplus C) \to (A \otimes B) \oplus C$$

satisfying coherence conditions.

In their paper, they developed the appropriate analogue of many notions from the theory of LDCS, in particular linear functors and negation. They demonstrated that the relationship between a formula and its negation is best represented by *linear adjoints*, the linear bicategorical analogue of adjoints in a bicategory. Given the foundational nature of adjoints in bicategory theory, this provided a very clear understanding of negation in non-commutative linear logic.

Cockett, Koslowski and Seely wrote a follow-up paper which introduced the definitions of morphisms between linear functors, the linear transformations and the linear modules, although the theory was mostly developed in terms of poly-bicategories [23].

The theory linear bicategories was mostly left undeveloped afterwards, perhaps due to a lack of examples and the complexity of the structures. Recently, however there has ben some renewed attention paid to linear bicategories by graduate students of Blute. Vandeven developed the theory of lax extensions of Set-based functors to linear bicategories [81]. Naeimabadi introduced cartesian linear bicategories [58], the notion combining linear bicategories and cartesian bicategories, in the sense of Carboni and Walters [17].

Moreover, there has been recent work re-visiting the *calculus of relations*, as introduced by De Morgan and Peirce, which provides a complete axiomatisation of the calculus based on the concepts of cartesian and linear bicategories [12]. In each cases, the motivating structure is the bicategory Rel of sets and relations, which is the prototypical example of a linear bicategory and a key model of non-commutative MLL.

1.5 Categorical Classical Logic

A research program which made great use of the developments of $*$-autonomous categories and LDCs is that of categorical classical logic. Unlike intuitionistic logic, the categorical semantics for classical logic remain an open problem.

The reason there has yet to be a universally recognized solution to this question is that if one were to take the categorical semantics of intuitionistic logic, cartesian closed categories, and extend them by adding classical negation, one would get trivial semantics. More precisely, a cartesian closed category with an involutive functor is a necessarily a Boolean algebra. This means in other words that the most straightforward categorical semantics are posetal, all morphisms, or proofs, between objects, or formulas, are identified. As such, these would be a model for provability in classical logic, not of proofs as desired. This theorem is more widely known as *Joyal's paradox* [52].

Multiple attempts have been made to remedy the situation. One such development was the extension of λ-calculus to the $\lambda\mu$-calculus by Parigot to provide well-behaved proof formalism for classical logic [62]. The categorical semantics of this system were studied by Hofmann and Streicher [42], and finally given by Selinger's *control categories* [74]. This point of view in fact related to Girard's formalism **LC**, an improvement of Gentzen's classical logic system LK, which is based on the concept of polarity of formulas [35]. Laurent proved that Girard's LC and Parigot's $\lambda\mu$-calculus are equivalent accounts by introducing the notion of polarity to linear logic proof nets [53]. However, as these models remain based on cartesian closed categories, the complete symmetry of classical logic had to be abandoned.

An alternative approach was to keep the symmetry, but instead "loosen" the cartesian nature of the monoidal products, implied by the weakening and contraction structural rules present in intuitionistic and classical logic.

It is well-known that the monoidal product $\oslash$ of a symmetric monoidal category $(\mathcal{X}, \oslash, I)$ is given by the categorical product if and only if each object is equipped with a canonical cocommutative comonoid structure, i.e. is equipped with coherent diagonal and counit maps

$$\Delta_A : A \to A \oslash A \qquad e_A : A \to I,$$

and the morphisms preserve the comonoid structures. This follows from a more general result known as Fox's theorem, published in 1976 and named after its author, which proves that taking the category of cocommutative comonoids and comonoid morphisms of a symmetric monoidal category provides a right adjoint to the inclusion functor of cartesian categories into the category of symmetric monoidal categories [32].

As such, multiple researchers have developed potential semantics for classical logic by considering categories whose objects have comonoid structures modeling conjunction and dually monoid structures modeling disjunction, while ultimately not being precisely cartesian or co-cartesian. Moreover, they build upon either the foundation of a LDC or a $*$-autonomous category, as they provide the minimal framework necessary to discuss two connectives with left and right introductions rules, and CUT as the only structure rule.

In particular, Führmann and Pym defined *Dummet categories* to be a symmetric LDC $(\mathbb{X}, \otimes, \top, \oplus, \bot)$ where each object is endowed with a $\otimes$-comonoid and $\oplus$-monoid structure, which is further poset enriched, satisfying additional conditions, such that the maps interact coherently with the comonoid and monoid structures, while not preserving it precisely [33]. Then, they defined a *classical category* to be a Dummet category with negation.

Similarly, Straßburger defined B_5*-categories* to be $*$-autonomous categories where each object has a canonical comonoid and monoid structure, but whose maps do not preserve these structures precisely [77], while Lamarche defined $*$*-autonomous categories with finitary medial and the absorption laws* and then considered its category of *germane and weakly idempotent bimonoids* [50]. In the latter case, germane and weakly idempotent bimonoids are particular "intrinsic" coherent comonoid and monoid structures on an object, such that all maps between the objects canonically preserve some of the structure. In each attempt, linear distributivity, sometimes called switch, plays an essential role and its interaction with the "cartesian" structures is investigated.

Another key aspect of the efforts of Straßburger and Lamarche is the presence of the *medial rule*, alongside linear distributivity, which is modeled by the following maps

$$(A \otimes B) \oplus (C \otimes D) \to (A \oplus C) \otimes (B \oplus D)$$

The medial rule was first introduced within the paradigm of deep inference, specifically in the local system for classical logic SKS developed by Brünnler and Tiu [15]. Locality refers to the concept that rules do not need a global view of formulae to be applied and is achieved in SKS by reducing non-local rules, such as contraction and weakening, to their atomic forms. Their general form should still however be admissible and this is guaranteed by the medial rule. The medial rule was also present in Straßburger's local system of linear logic SLLS and Tiu's local system for intuitionistic logic SISgq [79]. The medial rule has subsequently been studied as a term-rewriting rule, sometimes studied alongside linear distributivity [16, 29, 76].

1.6 Quantales and Quantaloids

The theory of quantales began with their introduction by Mulvey in 1986 [57], as a general setting to discuss constructive foundations for quantum mechanics and the non-commutative analogue of the maximal spectrum of a C^*-algebra.

Definition 1.6.1. [57, Sec 3] A *quantale* is a complete lattice Q with an associative binary operation $\otimes$ which is distributive on both sides over arbitrary joins $\bigvee$, i.e. $\forall a, \{b_\alpha\} \in Q$,

$$a \otimes (\bigvee b_\alpha) = \bigvee (a \otimes b_\alpha) \qquad\qquad (\bigvee b_\alpha) \otimes a = \bigvee (b_\alpha \otimes a)$$

Examples of quantales are plentiful. A key example are locales (otherwise known as frames) are complete lattices which satisfy the infinitary distributive law, i.e. $a \wedge \bigvee b_i = \bigvee a \wedge b_i$. In other words, they are quantales with the operation $\otimes$ given by the meet $\wedge$.

Locales were of particular interest to many category theorists at the time as they provided the foundations for algebraic topology and topos theory, as the lattice of open sets of a given topology is a locale [44, 46]. Quantales began being studied in this context by Borceux and Van den Bossche, although they restricted their attention to right-sided and idempotent quantales [13]. The categorical theory of quantales began in earnest by Niefield and Rosenthal, who were concerned with constructions on lattices of ideals [59, 60, 61].

Quantales entered the realm of categorical linear logic thanks to the work of Yetter, who noticed that Girard's phase space semantics for linear logic were quantales [83].

Definition 1.6.2. [34, Sec 1]

- A *phase space* consists of a commutative monoid P, whose elements are called *phases*, and a specified subset $\bot_P \subset P$, whose elements are called *antiphases*.
- Let $G \subseteq P$, then its dual $G^\perp$ is defined by

$$G^\perp = \{p \in P \,|\, pG \subseteq \bot_P\}$$

- A *fact* is a subset G of P such that $G^{\perp\perp} = G$. Its elements are called the *phases* of G.

In particular, the poset of facts of a given phase space is a particular type of quantale, introduced by Yetter:

Definition 1.6.3. [83, Def 1.6] A *Girard* quantale is a quantale Q, together a cyclic dualizing element $\bot$, i.e. there is a $\bot \in Q$ such that $\forall a \in Q$,

$$\bot \circ\!\!- a = a -\!\!\circ \bot \qquad \text{and} \qquad (a -\!\!\circ \bot) -\!\!\circ \bot = a\,.$$

It was in fact this article which standardized the terminology of cyclic linear logic, a particular non-commutative variant of linear logic with one notion of negation and a cyclic permutation rule. Rosenthal continued the study of Girard quantales and demonstrated that Girard's phase quantales are precisely the most general form of a Girard quantale [66]. The relationship of quantales to linear logic was stated clearly by Abramsky and Vickers [1] when they wrote:

Quantales are to linear logic as frames are to intuitionistic logic.

The bicategorical analogue of the quantale was introduced shortly after in 1993 by Abramsky and Vickers, when studying quantales and process semantics, in order to capture the idea of process with types.

Definition 1.6.4. [1, Def 8.1] A *quantaloid* is a small-suplattice enriched category $\mathcal{Q}$, in other words, it is a small category such that

- each hom-set is a sup-lattice, and
- composition of morphisms is distributive on both sides over arbitrary joins.

The theory of quantaloids was then further developed by Rosenthal [68, 70], in particular with respect to the semantics of non-commutative linear logic with the introduction of *Girard quantaloids* [69].

For a detailed survey on the theory of quantales and of quantaloids, see [67] and [71].

More recently, the notions of quantales and quantaloids has found renewed prominence within the field of monoidal topology, a unifying framework for ordered, metric and topological structures provided by the methods of monoidal category theory [41]. Quantales made their way into the theory as the appropriate generalization of the real half-line with opposite ordering $[0,\infty]^{op}$, considered by Lawvere to categorify metric spaces as enriched categories [48, 54]. Indeed, Clementino, Hofmann and Seal considered this construction for arbitrary quantales [19, 72]. In particular, they defined the 2-category of quantale-valued relations Q-Rel as the foundational framework for monoidal topology, although they did not use this terminology at the time.

Definition 1.6.5. [41, Sec III (1.1)] Consider a quantale Q with unit $\top$, then define Q-Rel to be the quantaloid whose objects are sets and arrows $R\colon X \nrightarrow Y$ are functions $R\colon X \times Y \to Q$, called *Q-relations.* Given $R\colon X \nrightarrow Y$ and $S\colon Y \nrightarrow Z$, the composition $R \otimes S\colon X \nrightarrow Z$ is defined by

$$R \otimes S(x,z) = \bigvee_{y \in Y} R(x,y) \otimes S(y,z)$$

Note that the use of $\otimes$ on the left refers to composition in Q-Rel and on the right refers to multiplication in Q. Identities are given by

$$\top_X(x,x') = \begin{cases} \top & \text{if } x = x' \\ \mathbf{0} & \text{if } x \neq x' \end{cases}$$

where $\mathbf{0}$ denoted the bottom element of Q.

For a comprehensive overview of monoidal topology, see [41].

1.7 Motivation and Contributions

With the background on the relevant fields established, the motivation for and main contributions of each article included in this thesis can now be elaborated upon.

The theory presented in the first article, "Linearly Distributive Fox Theorem", was initiated through discussions with Dr. Naeimabadi and Dr. Blute concerning the interaction between cartesian structures and linear distributivity, within the context of the development of cartesian linear bicategories. With further research, the critical relationship between cartesian categories and LDCs in the context of categorical classical logic was recognized. However, CLDCs had still seen no development since their introduction by Cockett and Seely. This observation motivated the pursuit of an analogue of the Fox Theorem for LDCs, which would characterize CLDCs as LDCs satisfying specific properties, thereby deepening the understanding of the interaction between cartesian products and linear distributivity.

In order to prove such a theorem, *medial linearly distributive categories* (MLDC) were introduced. These are LDCs that additionally satisfy the logical medial rule, and therefore can be seen as the categorical semantics for an extension of MLL, or equivalently as the appropriate categorical structure which is both linearly distributive and duoidal in nature. The article presents the theory of MLDCs: discussing how to coherently add symmetry and negation, develops examples and proves unique properties. Furthermore, *medial linear functors* and *medial linear transformations*, along with Frobenius versions of them, are introduced to provide the appropriate notion of 1-cell and 2-cell in a 2-category of symmetric MLDCs (SMLDC). The concept of *medial bimonoids* are then presented, as the appropriate analogue of the comonoids in the Fox theorem. It is demonstrated that the category of bicommutative medial bimonoids $\mathrm{B}[\mathbb{X}]$ of a SMLDC $\mathbb{X}$ is a CLDC and that this construction is in fact 2-functorial. Altogether, this culminates with the desired result:

Linearly Distributive Fox Theorem. (Ch. 2, Thm 7.16) The inclusion 2-functor from CLDCs into SMLDCs has a right 2-adjoint, $\mathrm{inc} \dashv \mathrm{B}[-] : \mathbf{CLDC} \to \mathbf{SMLDC}$.

As a corollary to the above theorem, we get a characterization of CLDCs as SMLDCs whose objects have canonical bicommutative medial bimonoid structures and whose morphisms preserve that structure.

While revisiting examples of CLDCs to construct MLDCs for the first article, an error in Cockett and Seely's revised paper introducing LDCs [27] was discovered, in collaboration with Dr. Pacaud Lemay. The paper claimed that while distributive categories are only CLDCs if they are posets, the Kleisli category of the exception monad of a distributive category is a CLDC. This was found to be incorrect: although the Kleisli category of the exception monad is indeed a LDC, its tensor product does not coincide with the categorical product. This observation spurred a deeper investigation into CLDCs directly.

This led to the development of a second article, co-authored with Dr. Pacaud Lemay, which explores the properties and potential examples of CLDCs in greater detail. The first major property proved is that the initial object is subterminal, while the terminal object is preinitial. This in turn implies all CLDCs are mix LDCs (a known result whose proof is streamlined by the previous observation). It also allows us to demonstrate that an object in a CLDC is preinitial if and only if it is subterminal. The article then focuses on the two main classes of examples of CLDCs: bounded distributive lattices and semi-additive categories. There are two well-known collapses of CLDCs to posets, Joyal's paradox when adding negation and the orthogonality of linear distributivity with traditional distributivity (both discussed previously). Mirroring these results, we prove two new collapses to the semi-additive case, when the CLDC has invertible linear distributivities and when the CLDC is isomix. Together, these four results illustrate why these two types of examples are central to the theory of CLDCs. We provide a construction which maps a CLDC to a bounded distributive lattice, obtained by considering its *semizero* objects, and additionally maps a CLDC to a semi-additive category, obtained by taking the slice over the initial object, or equivalently the coslice under the terminal object. In the final section, new examples of CLDCs are introduced based on the Grothendieck construction and that CLDCs are closed under products.

Finally, the third article, "Constructing Linear Bicategories", arose from the stalled progress in the development of linear bicategories as natural categorical semantics for non-commutative MLL, in part due to a scarcity of known examples. Motivated by the well-established connections between non-commutative linear logic and quantales and quantaloids, and the renewed interest in these structures due to their role in monoidal topology, this work, co-authored with Dr. Blute and Dr. Niefield, constructs new examples of linear bicategories, by leveraging the theory of quantales and quantaloids within the framework of categorical linear logic.

In particular, the LD-quantale and the linear quantaloid are introduced. These are the appropriate generalization of the Girard quantale and the Girard quantaloid to the linearly distributive context respectively. Equivalently, these are the appropriate notions at the intersection of the quantale and LDC, and the quantaloid and linear bicategory. Several different examples of LD-quantales are developed, of particular note are the complete bi-Heyting algebras, i.e. locales which satisfy the opposite infinitary law. We then consider the category of sets and Q-relations Q-Rel, which is shown to be a linear quantaloid if and only if Q is a LD-quantale itself. The previous examples of LD-quantales are then utilized to present examples of linear quantaloids. Furthermore, the concept of enrichment in a Girard quantaloid is generalized to enrichment in a linear quantaloid $\mathcal{Q}$ by introducing, in particular, the category $\mathcal{Q}$-Mod of linear $\mathcal{Q}$-categories and the linear $\mathcal{Q}$-modules. It is further proved that $\mathcal{Q}$-Mod is a linear quantaloid if and only if $\mathcal{Q}$ is a linear quantaloid. The previous examples of Q-Rel are then used to construct new more sophisticated linear quantaloids via the $\mathcal{Q}$-Mod construction. Finally, in an effort to provide non-locally posetal examples of linear bicategories, it is shown that quantales (respectively small quantaloids), modules, and module homomorphisms form a cyclic $*$-autonomous bicategory, in other words a linear bicategory with negation.

Chapter 2

Linearly Distributive Fox Theorem

This chapter consists of a paper authored by Rose Kudzman-Blais, prepared for submission to *Theory and Applications of Categories*, a journal dedicated to contributions to category theory and to categorical methods applied to the mathematical sciences. The article develops the theory of medial linearly distributive categories and presents a linearly distributive analogue of the Fox theorem.

LINEARLY DISTRIBUTIVE FOX THEOREM

ROSE KUDZMAN-BLAIS

Abstract. Linearly distributive categories (LDC), introduced by Cockett and Seely to model multiplicative linear logic, are categories equipped with two monoidal structures that interact via linear distributivities. A seminal result in monoidal category theory is the Fox theorem, which characterizes cartesian categories as symmetric monoidal categories whose objects are equipped with canonical comonoid structures. In order to extend the Fox theorem to LDCs and characterize the subclass of cartesian LDCs (CLDC), we introduce the concepts of medial linearly distributive categories (MLDC), medial linear functors, and medial linear transformations. The former are LDCs which respect the logical medial rule, appearing frequently in deep inference, or alternatively are the appropriate structure at the intersection of LDCs and duoidal categories. We then prove the linearly distributive Fox theorem, which states that the inclusion 2-functor of CLDCs into MLDCs has a right adjoint given by constructing the category of bicommutative medial bimonoids.

1. Introduction

Within monoidal category theory, *cartesian categories* refer to categories whose monoidal product is given by the categorical product and whose monoidal unit is the terminal object. Perhaps the most well-known result concerning cartesian categories is the Fox theorem. In his seminal 1976 paper "Coalgebras and cartesian categories", Fox demonstrated that there is a right adjoint to the inclusion functor from cartesian categories into the category of symmetric monoidal categories [21]. This right adjoint functor consists of taking the cocommutative comonoids of a symmetric monoidal category $(\mathcal{X}, \oslash, I)$, and their maps, in other words considering objects A equipped with coherent diagonal and counit maps:

$$\Delta_A : A \to A \oslash A \qquad\qquad e_A : A \to I$$

As a corollary of this adjunction, we see that cartesian categories are precisely the symmetric monoidal categories whose objects have a canonical comonoid structure and whose maps preserve these structures.

The author would like to thank Richard Blute for suggesting this topic as part of their doctoral thesis and for his support during the development of the theory, and for many instructive discussions on the topic, along with fellow student Shayesteh Naimabadi. The author also wishes to thank Jean-Simon Pacaud Lemay, whose support and help were instrumental in developing the Examples and Properties Section of the paper, as well as the entire category theory group at Macquarie University for the opportunity to visit and present on the topic. The author also acknowledges the support of the Natural Sciences and Engineering Research Council of Canada (NSERC), under the grant awarded to Richard Blute.

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Cartesian categories have played a key role in the field of categorical logic. Indeed, one of the greatest successes of categorical logic was Lambek's equivalence between intuitionistic propositional calculus and cartesian closed categories [32]. If we extend this paradigm and consider cartesian closed categories with a notion of involutive negation to capture the semantics of classical logic, these categories collapse to Boolean algebras due to Joyal's paradox. Thus, there has not yet been a definitive framework for the categorical semantics of classical logic.

The field of categorical logic continued to thrive with the development of linear logic by Girard [23]. The categorical semantics for linear logic were subsequently explored by Seely, who identified that the appropriate definition was given by $*$-autonomous categories [40], previously introduced by Barr [3]. *Linearly distributive categories* (LDC) were later introduced by Cockett and Seely as alternative categorical semantics for linear logic, which take the notions of multiplicative conjunction and disjunction, and their interaction via linear distributivity, as primitive [15]. In particular, LDCs are categories $\mathbb{X}$ with two monoidal structures $(\mathbb{X}, \otimes, \top)$ and $(\mathbb{X}, \oplus, \bot)$, the former known as the tensor structure and the latter as the par structure, equipped with linear distributivity natural transformations:

$$\delta^L_{A,B,C} : A \otimes (B \oplus C) \to (A \otimes B) \oplus C \qquad \delta^R_{A,B,C} : (A \oplus B) \otimes C \to A \oplus (B \otimes C).$$

The theory of categorical linear logic proved useful in developing categorical semantics for classical logic. LDCs provide the minimal framework necessary to discuss two connectives with left and right introduction rules, and cut as the only structural rule. Since any reasonable model of classical logic should at least include such structure, categorical models for classical logic can be built upon the foundations of LDCs (or $*$-autonomous categories, if negation is instead taken as primitive).

As the rules of contraction and weakening are central structural rules of classical logic, which are traditionally modeled by cartesian monoidal structures, a recent paradigm of categorical classical logic has been to add "cartesian" structure to LDCs (or $*$-autonomous categories). In this vein, Führmann and Pym developed the notions of classical categories and Dummett categories, poset-enriched $*$-autonomous categories and LDCs respectively whose objects have coherent notions of $\otimes$-diagonals and counits, and $\oplus$-codiagonals and units [22]. While, Alternatively, Straßburger considered $*$-autonomous categories whose objects also have coherent $\otimes$-diagonals and counits, and $\oplus$-codiagonals and units, but whose maps do not necessarily preserve these structures [44]. While, Lamarche developed kinds of $*$-autonomous categories where an "intrinsic" notion of comonoid/monoid structure can be defined on objects [31].

More recently, Naeimabadi developed the theory of cartesian linear bicategories [35] and recent work has investigated De Morgan and Peirce's calculus of relations, providing an axiomatisation based on the concepts of cartesian and linear bicategories [7]. In fact, the central examples of linear bicategories, the bicategorical analogues of LDCs introduced by Cockett, Koslowski, and Seely [17], are also cartesian bicategories and the Fox theorem is central to the theory of cartesian bicategories, as introduced by Carboni and Walters [12].

It was within this context that the idea arose to investigate *cartesian linearly distributive categories* (CLDC) and to develop a linearly distributive version of the Fox theorem. CLDCs refer to LDCs whose tensor structure is cartesian and par structure is cocartesian. They were first introduced alongside LDCs by Cockett and Seely, though the notion received little attention beyond the first paper [15].

In developing a linearly distributive analogue of the Fox theorem, it quickly became apparent that no such construction could be adapted to all LDCs. For the same reason that the Fox theorem is an adjunction between cartesian categories and symmetric monoidal categories, we needed to restrict to LDCs equipped with additional structure maps:

$$\Delta_\bot : \bot \to \bot \otimes \bot \qquad \nabla_\top : \top \oplus \top \to \top \qquad m : \bot \to \top$$

$$\mu_{A,B,C,D} : (A \otimes B) \oplus (C \otimes D) \to (A \oplus C) \otimes (B \oplus D)$$

Viewed as a logical entailment, the latter map is known as the *medial rule*, first introduced by Brünnler and Tiu for a system of classical logic within the deep inference framework [10]. Moreover, medial maps were central to both Straßburger's and Lamarche's categorical models for classical logic.

Within monoidal category theory, these medial maps are better known as instances of the interchange maps of *duoidal categories*. Duoidal categories are categories $\mathcal{X}$ with two monoidal structures $(\mathcal{X}, \diamond, I)$ and $(\mathcal{X}, \star, J)$, equipped with precisely the structure maps introduced above [1]. The theory of duoidal categories first appeared in the work of Joyal and Street while studying braided monoidal categories [28].

With this perspective in mind, we introduce the notion of *medial linearly distributive categories* (MLDC), inspired both by the work of Straßburger and Lamarche in categorical classical logic and by the theory of duoidal categories. MLDCs can be regarded both as the categorical semantics for multiplicative linear logic with the medial rule, and as a appropriate intersection point between LDCs and duoidal categories.

Leveraging the well-developed theory of LDCs and duoidal categories, we define the appropriate functors and transformations between MLDCs. Within this framework, we consider an analogue to comonoids: *medial bimonoids*, objects A equipped with coherent $\otimes$-diagonal, $\otimes$-counit, $\oplus$-multiplication, and $\oplus$-unit maps,

$$\Delta_A : A \to A \otimes A \qquad e_A : A \to \top \qquad \nabla_A : A \oplus A \to A \qquad u_A : \bot \to A$$

Considering the bicommutative medial bimonoids and their maps in a symmetric MLDC gives the construction needed to define a right adjoint 2-functor to the inclusion 2-functor from CLDCs to the 2-category of symmetric MLDCs. This yields the desired linearly distributive Fox theorem and, as a corollary, a characterization of CLDCs as symmetric MLDCs whose objects have coherent medial bimonoid structure.

In the process of developing this paper, it became evident how poorly understood CLDCs are. This prompted the author to collaborate with Dr. Pacaud Lemay to explore the nature of CLDCs beyond their characterization via the linearly distributive Fox theorem. This resulted in a separate paper, "Cartesian Linearly Distributive Categories: Revisited" [29]. While both papers are written to be independently comprehensible, we recommend reading them in tandem to gain a complete perspective on cartesian and cocartesian structure in the context of LDCs.

Outline. Given that the results developed in this paper are based on the theories of the Fox theorem, LDCs, and duoidal categories, the paper begins with three substantial preliminary sections. Section 2 presents the basics of monoidal categories, cartesian categories, and Fox's theorem. Section 3 reviews the literature on LDCs, including linear functors and linear transformations. Section 4 introduces the necessary background on duoidal categories. Then, Section 5 begins by motivating the introduction of the medial rule and subsequently defines MLDCs. The theory of MLDCs is developed: symmetric MLDCs are defined, adding negation is discussed, examples are introduced, and properties are presented. Section 6 defines the appropriate linear functors and linear transformations between symmetric MLDCs. Section 7 introduces medial bimonoids and shows that considering such structures and their maps does, in fact, yield a CLDC, and finally proves the main result of this paper: the linearly distributive Fox theorem.

Conventions. There are a few notational choices made in this paper and we take a moment to detail the most important.

- Composition of morphisms/arrows in a category is denoted by the symbol ; and is written in diagrammatic order. Objects in a category are denoted by capital letters, while the morphisms are denoted by lowercase letters.
- Monoidal products and monoidal units will be denoted with various different symbols to help distinguish the particular type of category in question. In particular:
  - $\oslash$ and I are used for standard monoidal categories $\mathcal{X}$.
  - $\otimes$ and $\top$ are used for the tensor monoidal structure, and $\oplus$ and $\bot$ for the par monoidal structure in a linearly distributive category $\mathbb{X}$, in agreement with the notation introduced by Cockett and Seely in [15], as opposed to Girard's notation [23].

– $\diamond$ and I are used for the first monoidal structure and $\star$ and J for the second in a duoidal category $\mathcal{X}$, in agreement with the notation introduced by Aguiar and Mahajan [1].

– $\times$ and $\mathbf{1}$ denote the binary categorical product and the terminal object respectively, while $+$ and $\mathbf{0}$ denote the binary categorical coproduct and the initial object in a category.

2. Fox Theorem

We begin by giving a fairly complete description of the Fox theorem for cartesian categories, without proof, as the logic behind the traditional version of the result will be a guiding thread to generalizing it to the context of linearly distributive categories. We invite any reader who is well versed in the Fox theorem to skip this subsection.

2.1. Monoidal Categories.

We briefly outline the foundational definitions of monoidal category theory, primarily to establish the notation used throughout this paper and to provide a reference for the axioms, which will frequently appear in the proofs that follow.

2.2. Definition. *[34, Sec 7.1] A* **monoidal category** $(\mathcal{X}, \oslash, I, \alpha, \rho, \lambda)$ *consists of:*

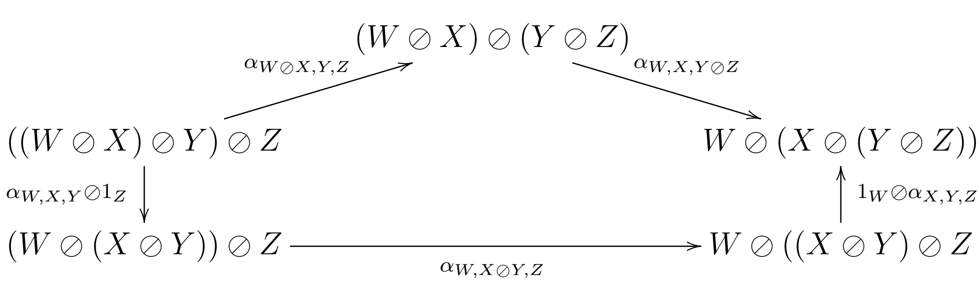
- *a category* $(\mathcal{X}, ;, 1_X)$,
- *the* **product** *and the* **unit** *functors*

$$\oslash : \mathcal{X} \times \mathcal{X} \to \mathcal{X} \qquad\qquad I : \mathbf{1} \to \mathcal{X}$$

- *the* **associator**, *the* **right unitor** *and the* **left unitor** *natural isomorphisms*

$$\begin{aligned} &\alpha : (\oslash \times 1_{\mathcal{X}}); \oslash \Rightarrow (1_{\mathcal{X}} \times \oslash); \oslash && \alpha_{X,Y,Z} : (X \oslash Y) \oslash Z \to X \oslash (Y \oslash Z) \\ &\rho : 1_{\mathcal{X}} \Rightarrow (1_{\mathcal{X}} \times I); \oslash && \rho_X : X \to X \oslash I \\ &\lambda : 1_{\mathcal{X}} \Rightarrow (I \times 1_{\mathcal{X}}); \oslash && \lambda_X : X \to I \oslash X \end{aligned}$$

such that the associativity pentagon (1) *and unit triangle identities* (2) *hold.*



(1)

$$\begin{array}{ccc} & & (X \oslash I) \oslash Y \\ & \overset{\rho_X \oslash 1_Y}{\nearrow} & \\ X \oslash Y & & \Big\downarrow \alpha_{X,I,Y} \\ & \underset{1_X \oslash \lambda_Y}{\searrow} & \\ & & X \oslash (I \oslash Y) \end{array} \tag{2}$$

2.3. Definition. *[34, Sec 9.1] A monoidal category is* **braided** *if it has a* braiding *natural isomorphism*

$$\sigma : \oslash \Rightarrow switch_{\mathcal{X},\mathcal{X}} ; \oslash \qquad\qquad \sigma_{X,Y} : X \oslash Y \to Y \oslash X$$

which behaves coherently with associators (3) *and with unitors* (4).

$$\begin{CD} (X\oslash Y)\oslash Z @>{\alpha_{X,Y,Z}}>> X\oslash(Y\oslash Z) @>{\sigma_{X,Y\oslash Z}}>> (Y\oslash Z)\oslash X \\ @V{\sigma_{X,Y}\oslash 1_Z}VV @. @VV{\alpha_{Y,Z,X}}V \\ (Y\oslash X)\oslash Z @>>{\alpha_{Y,X,Z}}> Y\oslash(X\oslash Z) @>>{1_Y\oslash\sigma_{X,Z}}> Y\oslash(Z\oslash X) \\ \\ X\oslash(Y\oslash Z) @>{\alpha^{-1}_{X,Y,Z}}>> (X\oslash Y)\oslash Z @>{\sigma_{X\oslash Y,Z}}>> Z\oslash(X\oslash Y) \\ @V{1_X\oslash\sigma_{Y,Z}}VV @. @VV{\alpha^{-1}_{Z,X,Y}}V \\ X\oslash(Z\oslash Y) @>>{\alpha^{-1}_{X,Z,Y}}> (X\oslash Z)\oslash Y @>>{\sigma_{X,Z}\oslash 1_Y}> (Z\oslash X)\oslash Y \end{CD} \tag{3}$$

$$\begin{array}{ccc} X & \xrightarrow{\rho_X} & X \oslash I \\ & \underset{\lambda_X}{\searrow} & \Big\downarrow \sigma_{X,I} \\ & & I \oslash X \end{array} \tag{4}$$

A monoidal category is **symmetric** *if it is braided and the braiding is self-inverse* (5).

$$\begin{array}{ccc} X \oslash Y & \xrightarrow{\sigma_{X,Y}} & Y \oslash X \\ & \searrow\!\!\searrow & \Big\downarrow \sigma_{Y,X} \\ & & X \oslash Y \end{array} \tag{5}$$

Given a braided monoidal category, we can define a composite of associators and braidings, sometimes known as the "canonical flip". This natural isomorphism is key to the Fox theorem, therefore, to improve readability, we define it here.

2.4. Definition. *Let $\mathcal{X}$ be a braided monoidal category. Define the natural isomorphism*

$$\tau_{W,X,Y,Z} : (W \oslash X) \oslash (Y \oslash Z) \Rightarrow (W \oslash Y) \oslash (X \oslash Z)$$

to be the following equivalent composites

$$\begin{array}{ccccc}
(W \oslash X) \oslash (Y \oslash Z) & & & & \\
\downarrow{\scriptstyle \alpha_{W,X,Y\oslash Z}} & & & & \\
W \oslash (X \oslash (Y \oslash Z)) & \xrightarrow{1_W \oslash \alpha^{-1}_{X,Y,Z}} & W \oslash ((X \oslash Y) \oslash Z) & \xrightarrow{1_W \oslash (\sigma_{X,Y} \oslash 1_Z)} & W \oslash ((Y \oslash X) \oslash Z) \\
\downarrow{\scriptstyle 1_W \oslash \sigma_{X,Y\oslash Z}} & & & & \downarrow{\scriptstyle 1_W \oslash \alpha_{Y,X,Z}} \\
W \oslash ((Y \oslash Z) \oslash X) & \xrightarrow[1_W \oslash \alpha_{Y,Z,X}]{} & W \oslash (Y \oslash (Z \oslash X)) & \xrightarrow[1_W \oslash (1_Y \oslash \sigma_{X,Z})]{} & W \oslash (Y \oslash (X \oslash Z)) \\
& & & & \downarrow{\scriptstyle \alpha^{-1}_{W,Y,X\oslash Z}} \\
& & & & (W \oslash Y) \oslash (X \oslash Z)
\end{array}$$

2.5. Monoidal Functors and Transformations.

The Fox theorem is an adjunction between the 2-categories of symmetric monoidal categories and cartesian categories. As such, we need to the appropriate notion of functors and natural transformations between monoidal categories.

2.6. Definition. *[34, Sec 9.2] Consider monoidal categories $(\mathcal{X}, \oslash, I)$ and $(\mathcal{Y}, \oslash, I)$. A* **lax monoidal functor** *$(F, m_I, m_\oslash) : (\mathcal{X}, \oslash, I) \to (\mathcal{Y}, \oslash, I)$ is a functor $F : \mathcal{X} \to \mathcal{Y}$ equipped with structure maps:*

- *a morphism $m_I : I \to F(I)$, and*
- *a natural transformation*

$$m_\oslash : (F \times F); \oslash \Rightarrow \oslash; F \qquad\qquad m_{\oslash A,B} : F(A) \oslash F(B) \to F(A \oslash B)$$

satisfying associativity (6) and unitality (7) coherence conditions.

$$\begin{array}{ccccc}
(F(X) \oslash F(Y)) \oslash F(Z) & \xrightarrow{\alpha_{F(X),F(Y),F(Z)}} & F(X) \oslash (F(Y) \oslash F(Z)) & \xrightarrow{1_{F(X)} \oslash m_{\oslash Y,Z}} & F(X) \oslash F(Y \oslash Z) \\
\downarrow{\scriptstyle m_{\oslash X,Y} \oslash 1_{F(Z)}} & & & & \downarrow{\scriptstyle m_{\oslash X,Y\oslash Z}} \\
F(X \oslash Y) \oslash F(Z) & \xrightarrow[m_{\oslash X\oslash Y,Z}]{} & F((X \oslash Y) \oslash Z) & \xrightarrow[F(\alpha_{X,Y,Z})]{} & F(X \oslash (Y \oslash Z))
\end{array} \tag{6}$$

$$\begin{array}{ccc}
F(X) & \xrightarrow{\rho_{F(X)}} & F(X) \oslash I \\
\downarrow{\scriptstyle F(\rho_X)} & & \downarrow{\scriptstyle 1_{F(X)} \oslash m_I} \\
F(X \oslash I) & \xleftarrow[m_{\oslash X,I}]{} & F(X) \oslash F(I)
\end{array}
\qquad
\begin{array}{ccc}
F(X) & \xrightarrow{\lambda_{F(X)}} & I \oslash F(X) \\
\downarrow{\scriptstyle F(\lambda_X)} & & \downarrow{\scriptstyle m_I \oslash 1_{F(X)}} \\
F(I \oslash X) & \xleftarrow[m_{\oslash I,X}]{} & F(I) \oslash F(X)
\end{array} \tag{7}$$

A **colax monoidal functor** $F = (F, n_I, n_\oslash) : (\mathcal{X}, \oslash, I) \to (\mathcal{Y}, \oslash, I)$ *is a lax monoidal functor* $(\mathcal{X}^{op}, \oslash, I) \to (\mathcal{Y}^{op}, \oslash, I)$.

If $\mathcal{X}$ *and* $\mathcal{Y}$ *are symmetric, then a lax monoidal functor* $(F, m_I, m_\oslash) : (\mathcal{X}, \oslash, I) \to (\mathcal{Y}, \oslash, I)$ *is a* **symmetric** *if it additionally interacts coherently with the braidings* (8).

$$\begin{array}{ccc} F(X) \oslash F(Y) & \xrightarrow{\sigma_{F(X),F(Y)}} & F(Y) \oslash F(X) \\ {\scriptstyle m_{\oslash X,Y}} \downarrow & & \downarrow {\scriptstyle m_{\oslash Y,X}} \\ F(X \oslash Y) & \xrightarrow[F(\sigma_{X,Y})]{} & F(Y \oslash X) \end{array} \tag{8}$$

A **monoidal functor** *is a lax (or equivalently colax) monoidal functor whose structure maps* m_I *and* $m_\oslash$ *(or* n_I *and* $n_\oslash$*) are isomorphisms.*

2.7. Remark. Given a symmetric lax monoidal functor $(F, m_I, m_\oslash) : (\mathcal{X}, \oslash, I) \to (\mathcal{Y}, \oslash, I)$, we note that the following commuting diagram involving the canonical flip for use in a future proof:

$$\begin{array}{ccccc} (F(W) \oslash F(X)) \oslash (F(Y) \oslash F(Z)) & \xrightarrow{m_{\oslash W,X} \oslash m_{\oslash Y,Z}} & F(W \oslash X) \oslash F(Y \oslash Z) & \xrightarrow{m_{\oslash W \oslash X, Y \oslash Z}} & F((W \oslash X) \oslash (Y \oslash Z)) \\ {\scriptstyle \tau_{F(W),F(X),F(Y),F(Z)}} \downarrow & & & & \downarrow {\scriptstyle F(\tau_{W,X,Y,Z})} \\ (F(W) \oslash F(Y)) \oslash (F(X) \oslash F(Z)) & \xrightarrow[m_{\oslash W,Y} \oslash m_{\oslash X,Z}]{} & F(W \oslash Y) \oslash F(X \oslash Z) & \xrightarrow[m_{\oslash W \oslash Y, X \oslash Z}]{} & F((W \oslash Y) \oslash (X \oslash Z)) \end{array} \tag{9}$$

2.8. Definition. *[34, Sec 9.2] Let* $(F, m_I^F, m_\oslash^F), (G, m_I^G, m_\oslash^G) : \mathcal{X} \to \mathcal{Y}$ *be lax monoidal functors. A* **monoidal transformation** $\alpha : (F, m_I^F, m_\oslash^F) \Rightarrow (G, m_I^G, m_\oslash^G)$ *is a natural transformation* $\alpha : F \Rightarrow G$ *whose component morphisms behave coherently with the monoidal products and the units* (10).

$$\begin{array}{ccc} F(X) \oslash F(Y) & \xrightarrow{\alpha_X \oslash \alpha_Y} & G(X) \oslash G(Y) \\ {\scriptstyle m^F_{\oslash X,Y}} \downarrow & & \downarrow {\scriptstyle m^G_{\oslash X,Y}} \\ F(X \oslash Y) & \xrightarrow[\alpha_{X \oslash Y}]{} & G(X \oslash Y) \end{array} \qquad \begin{array}{ccc} I & \xrightarrow{m_I^F} & F(I) \\ & {\scriptstyle m_I^G} \searrow & \downarrow {\scriptstyle \alpha_I} \\ & & G(I) \end{array} \tag{10}$$

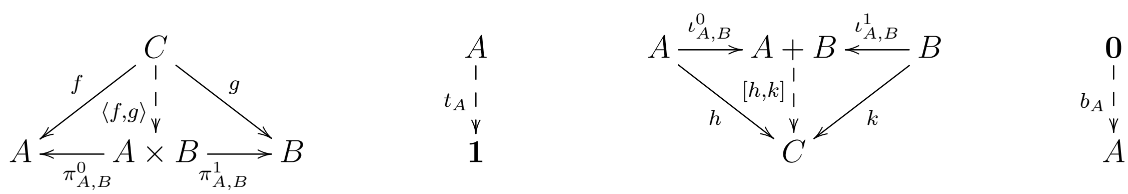
Let $(F, n_I^F, n_\oslash^F), (G, n_I^G, n_\oslash^G) : \mathcal{X} \to \mathcal{Y}$ *be colax monoidal functors. A* **monoidal transformation** $\alpha : (F, n_I^F, n_\oslash^F) \Rightarrow (G, n_I^G, n_\oslash^G)$ *is a monoidal transformation between the lax monoidal functors* $(F, n_I^F, n_\oslash^F), (G, n_I^G, n_\oslash^G) : \mathcal{X}^{op} \to \mathcal{Y}^{op}$.

2.9. Proposition. *There is a 2-category* **SMC** *of symmetric monoidal categories, symmetric monoidal functors and monoidal transformations.*

2.10. CARTESIAN CATEGORIES.

The Fox theorem is concerned with characterizing the symmetric monoidal categories which are cartesian, therefore we quickly introduce the appropriate sub-2-category **CART**.

2.11. DEFINITION. *A* **cartesian category** *is a category with finite categorical products and a* **cocartesian category** *is a category with finite categorical coproducts, where binary products, terminal objects, binary coproducts and initial objects, with the corresponding unique maps, are denoted by*



Cartesian categories have a canonical monoidal structure given by the binary categorical product and the terminal object, and similarly for the cocartesian categories:

2.12. PROPOSITION. *[34, Sec 3.5] Consider a cartesian category $\mathcal{X}$, then it is a symmetric monoidal category $(\mathcal{X}, \times, \mathbf{1})$ whose monoidal product is the categorical product of $\mathcal{X}$ and monoidal unit is the terminal object of $\mathcal{X}$:*

$$(X, Y) \mapsto X \times Y$$
$$(f : X \to X', g : Y \to Y') \mapsto f \times g = \langle \pi^0_{X,Y}; f, \pi^1_{X,Y}; g\rangle : X \times Y \to X' \times Y'$$

with isomorphisms

$$\alpha_{X,Y,Z} = \langle \pi^0_{X\times Y,Z}; \pi^0_{X,Y}, \pi^1_{X,Y} \times 1_Z\rangle : (X \times Y) \times Z \to X \times (Y \times Z)$$
$$\rho_X = \langle 1_X, t_X\rangle : X \to X \times \mathbf{1}$$
$$\lambda_X = \langle t_X, 1_X\rangle : X \to \mathbf{1} \times X$$
$$\sigma_{X,Y} = \langle \pi^1_{X,Y}, \pi^0_{X,Y}\rangle : X \times Y \to Y \times X$$

Consider a cocartesian category $\mathcal{X}$, then it is a symmetric monoidal category $(\mathcal{X}, +, \mathbf{0})$ whose monoidal product is the categorical coproduct of $\mathcal{X}$ and monoidal unit is the initial object:

$$(X, Y) \mapsto X + Y$$
$$(f : X \to X', g : Y \to Y') \mapsto f + g = [f; \iota^0_{X',Y'}, g; \iota^1_{X',Y'}] : X + Y \to X' + Y'$$

with isomorphisms

$$\alpha_{X,Y,Z} = [\iota^0_{X,Y}; \iota^0_{X+Y,Z}, \iota^1_{X,Y} + 1_Z] : X + (Y + Z) \to (X + Y) + Z$$
$$\rho_X = [1_X, b_X] : X + \mathbf{0} \to X$$
$$\lambda_X = [b_X, 1_X] : \mathbf{0} + X \to X$$
$$\sigma_{X,Y} = [\iota^1_{Y,X}, \iota^0_{Y,X}] : X + Y \to Y + X$$

2.13. Proposition. *There is a 2-category* **CART** *of cartesian categories, symmetric monoidal functors and monoidal transformations, known as* **cartesian functors** *and* **cartesian transformations** *in this context. It is a full sub-2-category of* **SMC**.

2.14. Comonoids.

With the 2-categories now appropriately defined, we can introduce the construction which provides the adjoint to the inclusion.

2.15. Definition. *[21] Let $\mathcal{X}$ denote a monoidal category.*

- *A* **comonoid** *in $\mathcal{X}$ is a triple $\langle X, \Delta_X, e_X \rangle$ of an object A in $\mathcal{X}$ equipped with two morphisms, the* **diagonal** *and the* **counit**

$$\Delta_X : X \to X \oslash X \qquad\qquad e_X : X \to I$$

satisfying

$$\begin{array}{ccccc} X & \xrightarrow{\Delta_X} & X \oslash X & \xrightarrow{1_X \oslash \Delta_X} & X \oslash (X \oslash X) \\ {\scriptstyle \Delta_X}\downarrow & & & & \uparrow{\scriptstyle \alpha_{X,X,X}} \\ X \oslash X & & \xrightarrow{\Delta_X \oslash 1_X} & & (X \oslash X) \oslash X \end{array} \tag{11}$$

$$\begin{array}{ccccc} X & \xrightarrow{\Delta_X} & X \oslash X & \xrightarrow{1_X \oslash e_X} & X \oslash I \\ {\scriptstyle \Delta_X}\downarrow & & \searrow\!\!\!\!=\!= & & \downarrow{\scriptstyle \rho_X^{-1}} \\ X \oslash X & \xrightarrow[e_X \oslash 1_X]{} & I \oslash X & \xrightarrow[\lambda_X^{-1}]{} & X \end{array} \tag{12}$$

- *A* **comonoid morphism** $f : \langle X, \Delta_X, e_X \rangle \to \langle Y, \Delta_Y, e_Y \rangle$ *is a morphism* $f : X \to Y$ *in $\mathcal{X}$ such that*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle \Delta_X}\downarrow & & \downarrow{\scriptstyle \Delta_Y} \\ X \oslash X & \xrightarrow[f \oslash f]{} & Y \oslash Y \end{array} \qquad\qquad \begin{array}{ccc} X & \xrightarrow{f} & Y \\ & {\scriptstyle e_X}\searrow & \downarrow{\scriptstyle e_Y} \\ & & I \end{array} \tag{13}$$

- *If $\mathcal{X}$ is a symmetric monoidal category, a comonoid $\langle X, \Delta_X, e_X \rangle$ is* **cocommutative** *if*

$$\begin{array}{ccc} X & \xrightarrow{\Delta_X} & X \oslash X \\ & {\scriptstyle \Delta_X}\searrow & \downarrow{\scriptstyle \sigma_{X,X}} \\ & & X \oslash X \end{array} \tag{14}$$

We denote the category of cocommutative comonoids and comonoid morphisms in $\mathcal{X}$ by $\mathrm{C}[\mathcal{X}]$.

2.16. Lemma. *Given a cartesian category* $(\mathcal{X}, \times, \mathbf{1})$*, every object* A *in* $\mathcal{X}$ *has a canonical unique cocommutative* $\times$*-comonoid structure*

$$\langle A, \langle 1_A, 1_A \rangle : A \to A \times A, t_A : A \to \mathbf{1} \rangle$$

and every arrow $f : A \to B$ *in* $\mathcal{X}$ *is a comonoid morphism with respect to these structures.*

The category of cocommutative comonoids in $\mathcal{X}$ inherits a symmetric monoidal structure from $\mathcal{X}$, which is in fact cartesian.

2.17. Lemma. *[21] Given a pair of cocommutative comonoids* $\langle X, \Delta_X, e_X \rangle$ *and* $\langle X', \Delta_{X'}, e_{X'} \rangle$ *in* $\mathcal{X}$*, then the triple* $\langle X \oslash X', \Delta_{X \oslash X'}, e_{X \oslash X'} \rangle$ *defined by*

$$\Delta_{X \oslash X'} = X \oslash X' \xrightarrow{\Delta_X \oslash \Delta_{X'}} (X \oslash X) \oslash (X' \oslash X') \xrightarrow{\tau_{X,X,X',X'}} (X \oslash X') \oslash (X \oslash X')$$

$$e_{X \oslash X'} = X \oslash X' \xrightarrow{e_X \oslash e_{X'}} I \oslash I \xrightarrow{\rho_I^{-1}} I$$

is a cocommutative comonoid. Consequently, $\mathrm{C}[\mathcal{X}]$ *is a symmetric monoidal category and the monoidal structure is given by the categorical product. Thus,* $\mathrm{C}[\mathcal{X}]$ *is a cartesian category.*

Further, this extends canonically to strong symmetric monoidal functors and monoidal transformations. While the original Fox theorem is only an adjunction between categories, it is well-known that it easily extends to include transformations.

2.18. Lemma. *[21]*

1. *Given a symmetric monoidal functor* $F : \mathcal{X} \to \mathcal{Y}$ *and a cocommutative comonoid* $\langle X, \Delta_X, e_X \rangle$ *in* $\mathcal{X}$*, the triple* $\langle F(X), \Delta_{F(X)}, e_{F(X)} \rangle$ *defined by*

$$\Delta_{F(X)} = F(X) \xrightarrow{F(\Delta_X)} F(X \oslash X) \xrightarrow{m_{\oslash}^{-1}{}_{X,X}} F(X) \oslash F(X)$$

$$e_{F(X)} = F(X) \xrightarrow{F(t_X)} F(I) \xrightarrow{m_I^{-1}} I$$

   is a cocommutative comonoid. As such, F *canonically extends to a cartesian functor between the categories of cocommutative comonoids* $\mathrm{C}[F] : \mathrm{C}[\mathcal{X}] \to \mathrm{C}[\mathcal{Y}]$*.*

2. *Given a monoidal transformation between symmetric monoidal functors* $\alpha : F \Rightarrow G : \mathcal{X} \to \mathcal{Y}$ *and a cocommutative comonoid* $\langle X, \Delta_X, e_X \rangle$ *in* $\mathcal{X}$*, the morphism* $\alpha_X : F(X) \to G(X)$ *is a comonoid morphism. As such,* α *canonically extends to cartesian transformation* $\mathrm{C}[\alpha] : \mathrm{C}[F] \Rightarrow \mathrm{C}[G]$*.*

We are finally ready to state the Fox theorem:

2.19. Theorem. *[21] The functor* $\mathrm{C}[-] : \mathbf{SMC} \to \mathbf{CART}$*, mapping a symmetric monoidal category to its category of cocommutative comonoids and comonoid morphisms, is right adjoint to the inclusion functor* $\mathrm{inc} : \mathbf{CART} \to \mathbf{SMC}$*.*

An incredibly useful corollary is the following characterization of cartesian categories:

2.20. Corollary. *[21] A symmetric monoidal category $\mathcal{X}$ is cartesian if and only if it is isomorphic to its category of cocommutative comonoids* $\mathrm{C}[\mathcal{X}]$.

This previous result can be unwrapped as a statement about the existence of certain natural transformations, as was stated in [26].

2.21. Corollary. *[26, Thm 4.28] A symmetric monoidal category $\mathcal{X}$ is cartesian if and only if there are natural transformations*

$$\begin{array}{ll} e : 1_{\mathcal{X}} \Rightarrow I : \mathcal{X} \to \mathcal{X} & \qquad e_A : A \to I \\ \Delta : 1_{\mathcal{X}} \Rightarrow copy_{\mathcal{X}} ; \oslash : \mathcal{X} \to \mathcal{X} & \qquad \Delta_A : A \to A \oslash A \end{array}$$

such that, $\forall A \in \mathcal{X}, \langle A, \Delta_A, e_A \rangle$ is a cocommutative comonoid and, $\forall A, B \in \mathcal{X}$,

$$\begin{array}{ll} e_{A \oslash B} = (e_A \oslash e_B); \rho_I^{-1} & \qquad e_I = 1_I \\ \Delta_{A \oslash B} = (\Delta_A \oslash \Delta_B); s_{A,A,B,B} & \qquad \Delta_I = \rho_I \end{array}$$

Of course, there is a dual statement for cocartesian categories, by taking the category of commutative monoids and monoid morphisms in a symmetric monoidal category. We quickly outline these definitions as they will be used throughout the work.

2.22. Definition. *Let $\mathcal{X}$ denote a monoidal category.*

- *A* **monoid** *in $\mathcal{X}$ is a comonoid in $\mathcal{X}^{op}$, i.e. a triple $\langle X, \nabla_X, u_X \rangle$ of an object A in $\mathcal{X}$ equipped with two morphisms, the* **multiplication** *and the* **unit**

  $$\nabla_X : X \oslash X \to X \qquad u_X : I \to X$$

  satisfying

  $$\begin{array}{ccccc} (X \oslash X) \oslash X & \xrightarrow{\alpha_{X,X,X}} & X \oslash (X \oslash X) & \xrightarrow{1_X \oslash \nabla_X} & X \oslash X \\ {\scriptstyle \nabla_X \oslash 1_X}\downarrow & & & & \downarrow{\scriptstyle \nabla_X} \\ X \oslash X & & \xrightarrow{\nabla_X} & & X \end{array} \tag{15}$$

  $$\begin{array}{ccccc} X & \xrightarrow{\rho_X} & X \oslash I & \xrightarrow{1_X \oslash u_X} & X \oslash X \\ {\scriptstyle \lambda_X}\downarrow & & {\searrow}\!\!= & & \downarrow{\scriptstyle \nabla_X} \\ I \oslash X & \xrightarrow[u_X \oslash 1_X]{} & X \oslash X & \xrightarrow[\nabla_X]{} & X \end{array} \tag{16}$$

- *A* **monoid morphism** *$f : \langle X, \nabla_X, e_X \rangle \to \langle Y, \nabla_Y, u_Y \rangle$ is a morphism $f : X \to Y$ in $\mathcal{X}$ such that*

  $$\begin{array}{ccc} X \oslash X & \xrightarrow{f \oslash f} & Y \oslash Y \\ {\scriptstyle \nabla_X}\downarrow & & \downarrow{\scriptstyle \nabla_Y} \\ X & \xrightarrow[f]{} & Y \end{array} \qquad \begin{array}{ccc} & & I \\ & {\scriptstyle u_X}\nearrow & \uparrow{\scriptstyle u_Y} \\ X & \xrightarrow[f]{} & Y \end{array} \tag{17}$$

- *If $\mathcal{X}$ is a symmetric monoidal category, a monoid $\langle X, \Delta_X, e_X \rangle$ is* **commutative** *if*

$$\begin{array}{ccc} X \oslash X & \xrightarrow{\nabla_X} & X \\ {\scriptstyle \sigma_{X,X}} \downarrow & \nearrow_{\nabla_X} & \\ X \oslash X & & \end{array} \tag{18}$$

3. Linearly Distributive Categories

Girard introduced linear logic in 1987 to capture the constructive nature of intuitionistic logic and the inherent dualities of classical logic [23]. The categorical semantics for linear logic were subsequently explored by Seely, who identified that the appropriate framework were $*$-autonomous categories [40], previously introduced by Barr [3]. In this context, the monoidal product is the multiplicative conjunction, while the involutive functor $*$ is negation.

Linearly distributive categories (LDC) were introduced by Cockett and Seely as an alternative framework for the semantics of linear logic to $*$-autonomous categories. [15]. Unlike $*$-autonomous categories, LDCs take multiplicative conjunction and disjunction as primitive. Therefore, they consist of two monoidal structures, which interact via linear distributivities. Monoidal products connected through negation and de Morgan duality inherently possess such a linear distribution. Thus, $*$-autonomous categories are LDCs. However, there are significant examples of LDCs without negation, making them more general.

The current work generalizes the Fox theorem to the context of LDCs in order to characterize the notable subclass of cartesian LDCs (CLDC). In this section, we introduce all the necessary background on LDCs needed, as developed by co-authors Cockett and Seely in the series of papers [15], [14] and [16].

3.1. Definition, Negation, Mix and Examples.

3.2. Definition. *[15, Sec 2.1] A* **linearly distributive category***, or LDC,* $(\mathbb{X}, \otimes, \top, \oplus, \bot)$ *consists of:*

- *a category* $(\mathbb{X}, ;, 1_A)$*,*
- *a* **tensor** *monoidal structure* $(\mathbb{X}, \otimes, \top)$
  - *the* **tensor** *product and* **top** *unit functor*

$$\otimes : \mathbb{X} \times \mathbb{X} \to \mathbb{X} \qquad\qquad \top : \mathbf{1} \to \mathbb{X}$$

– *the* $\otimes$*-associator, right* $\otimes$*-unitor, and left* $\otimes$*-unitor natural isomorphisms*

$$\begin{array}{ll} \alpha_\otimes : (\otimes \times 1_\mathbb{X});\otimes \Rightarrow (1_\mathbb{X} \times \otimes);\otimes & \alpha_{\otimes A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C) \\ u^R_\otimes : 1_\mathbb{X} \Rightarrow 1_\mathbb{X} \otimes \top & u^R_{\otimes A} : A \to A \otimes \top \\ u^L_\otimes : 1_\mathbb{X} \Rightarrow \top \otimes 1_\mathbb{X} & u^L_{\otimes A} : A \to \top \otimes A \end{array}$$

such that $(\mathbb{X}, \otimes, \top, \alpha_\otimes, u^R_\otimes, u^L_\otimes)$ *is a monoidal category,*

- *a* **par** *monoidal structure* $(\mathbb{X}, \oplus, \bot)$

– *the* **par** *product and* **bottom** *unit functor*

$$\oplus : \mathbb{X} \times \mathbb{X} \to \mathbb{X} \qquad\qquad \bot : \mathbf{1} \to \mathbb{X}$$

– *the* $\oplus$*-associator, right* $\oplus$*-unitor, and left* $\oplus$*-unitor natural isomorphisms*

$$\begin{array}{ll} \alpha_\oplus : (\otimes \times 1_\mathbb{X});\otimes \Rightarrow (1_\mathbb{X} \times \otimes);\otimes & \alpha_{\oplus A,B,C} : A \oplus (B \oplus C) \to (A \oplus B) \oplus C \\ u^R_\oplus : 1_\mathbb{X} \oplus \bot \Rightarrow 1_\mathbb{X} & u^R_{\oplus A} : A \oplus \bot \to A \\ u^L_\oplus : \bot \oplus 1_\mathbb{X} \Rightarrow 1_\mathbb{X} & u^L_{\oplus A} : \bot \oplus A \to A \end{array}$$

such that $(\mathbb{X}, \oplus, \bot, \alpha_\oplus{}^{-1}, u^{R\,-1}_\oplus, u^{L\,-1}_\oplus)$ *is a monoidal category,*

- *left and right* linear distributivity *natural transformations*

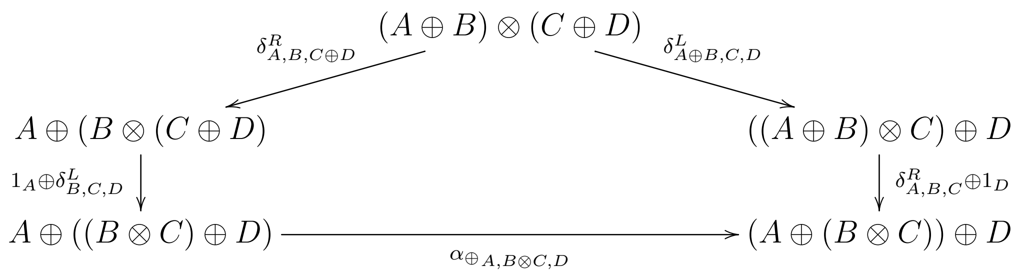
$$\begin{array}{ll} \delta^R : (\oplus \times 1_\mathbb{X});\otimes \Rightarrow (1_\mathbb{X} \times \otimes);\oplus & \delta^R_{A,B,C} : (A \oplus B) \otimes C \to A \oplus (B \otimes C), \\ \delta^L : (1_\mathbb{X} \times \oplus);\otimes \Rightarrow (\otimes \times 1_\mathbb{X});\oplus & \delta^L_{A,B,C} : A \otimes (B \oplus C) \to (A \otimes B) \oplus C, \end{array}$$

satisfying coherence conditions between units and linear distributivities (19)*, associativities and linear distributivities* (20)*, and left and right linear distributivities* (21)*.*

$$\begin{array}{ccc} A \oplus B & \xrightarrow{u^L_{\otimes A\oplus B}} & \top \otimes (A \oplus B) \\ & \searrow^{u^L_{\otimes A} \oplus 1_B} & \downarrow \delta^L_{\top,A,B} \\ & & (\top \otimes A) \oplus B \end{array} \qquad \begin{array}{l} u^L_{\otimes A\oplus B}; \delta^L_{\top,A,B} = u^L_{\otimes A} \oplus 1_B \\ u^R_{\otimes A\oplus B}; \delta^R_{A,B,\top} = 1_A \oplus u^R_{\otimes B} \\ \delta^R_{\bot,A,B}; u^L_{\oplus A\otimes B} = u^L_{\oplus A} \otimes 1_B \\ \delta^L_{A,B,\bot}; u^R_{\oplus A\otimes B} = 1_A \otimes u^R_{\oplus B} \end{array} \tag{19}$$

$$\begin{array}{ccc} (A \otimes B) \otimes (C \oplus D) & \xrightarrow{\alpha_{\otimes A,B,C\oplus D}} & A \otimes (B \otimes (C \oplus D)) \\ & & \downarrow 1_A \otimes \delta^L_{B,C,D} \\ \delta^L_{A\otimes B,C,D} \downarrow & & A \otimes ((B \otimes C) \oplus D) \\ & & \downarrow \delta^L_{A,B\otimes C,D} \\ ((A \otimes B) \otimes C) \oplus D & \xrightarrow[\alpha_{\otimes A,B,C} \oplus 1_D]{} & (A \otimes (B \otimes C)) \oplus D \end{array} \tag{20}$$

$$\delta^L_{A\otimes B,C,D}; (\alpha_{\otimes A,B,C} \oplus 1_D) = \alpha_{\otimes A,B,C\oplus D}; (1_A \otimes \delta^L_{B,C,D}); \delta^L_{A,B\otimes C,D}$$
$$\alpha_{\otimes A\oplus B,C,D}; \delta^R_{A,B,C\otimes D} = (\delta^R_{A,B,C} \otimes 1_D); \delta^R_{A,B\otimes C,D}; (1_A \oplus \alpha_{\otimes B,C,D})$$
$$\delta^L_{A,B,C\oplus D}; \alpha_{\oplus A\otimes B,C,D} = (1_A \otimes \alpha_{\oplus B,C,D}); \delta^L_{A,B\oplus C,D}; (\delta^L_{A,B,C} \oplus 1_D)$$
$$(\alpha_{\oplus A,B,C} \otimes 1_D); \delta^R_{A\oplus B,C,D} = \delta^R_{A,B\oplus C,D}; (1_A \oplus \delta^R_{B,C,D}); \alpha_{\oplus A,B,C\otimes D}$$



(21)

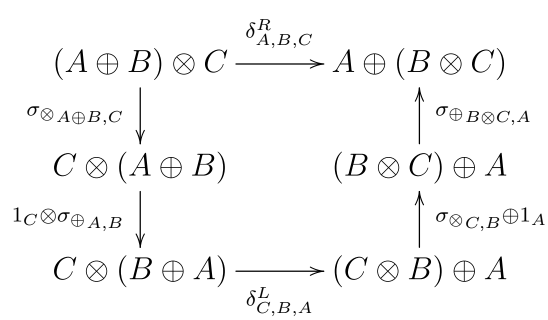
$$\delta^L_{A\oplus B,C,D}; (\delta^R_{A,B,C} \oplus 1_D) = \delta^R_{A,B,C\oplus D}; (1_A \oplus \delta^L_{B,C,D}); \alpha_{\oplus A,B\otimes C,D}$$
$$(\delta^L_{A,B,C} \otimes 1_D); \delta^R_{A\otimes B,C,D} = \alpha_{\otimes A,B\oplus C,D}; (1_A \otimes \delta^R_{B,C,D}); \delta^L_{A,B,C\otimes D}$$

3.3. Remark. The reason the structure isomorphisms of the $\oplus$-monoidal structure having the domain and codomain in a different direction that for the $\otimes$-monoidal structure is due to how these inferences are proved in multiplicative linear logic and the graphical language associated with LDCs [5]. The $\oplus$-unitors and $\oplus$-associator arise more "naturally" in the direction chosen and are in agreement with the notation introduced in Cockett and Seely's later work with Koslowski on linear bicategories [17].

3.4. Definition. *[15, Sec 3] A linearly distributive category* $(\mathbb{X}, \otimes, \top, \oplus, \bot)$ *is* **symmetric**, *or a SLDC, if*

- $(\mathbb{X}, \otimes, \top)$ *is a symmetric monoidal category with* $\otimes$*-braiding*
$$\sigma_\otimes : \otimes \Rightarrow switch_{\mathbb{X},\mathbb{X}}; \otimes \qquad \sigma_{\otimes A,B} : A \otimes B \to B \otimes A$$
- $(\mathbb{X}, \oplus, \bot)$ *is a symmetric monoidal category with* $\oplus$*-braiding*
$$\sigma_\oplus : \oplus \Rightarrow switch_{\mathbb{X},\mathbb{X}}; \oplus \qquad \sigma_{\oplus A,B} : A \oplus B \to B \oplus A$$

such that the linear distributivities interact coherently with the braidings (22).



(22)

Many models of multiplicative linear logic, in particular the ones we are interested in, also satisfy an additional rule: the binary MIX rule $A \otimes B \vdash A \oplus B$. With the cut rule present, satisfying the binary MIX rule is equivalent to satisfying the nullary MIX rule $\bot \vdash \top$. In terms of LDCs, this amounts to the existence of a map $m : \bot \to \top$ such that the two possible induced maps $A \otimes B \to A \oplus B$ are equal. This map will be essential to the following discussions.

3.5. Definition. *[14, Def 6.2] A linearly distributive category* $\mathbb{X}$ *is* **mix** *if there is a morphism* $m : \bot \to \top$ *such that*

$$\begin{array}{ccccc}
A \otimes B & \xrightarrow{1_A \otimes {u^L_{\oplus B}}^{-1}} & A \otimes (\bot \oplus B) & \xrightarrow{1_A \otimes (m \oplus 1_B)} & A \otimes (\top \oplus B) \\
\downarrow {\scriptstyle {u^R_{\oplus A}}^{-1} \otimes 1_B} & & & & \downarrow {\scriptstyle \delta^L_{A,\top,B}} \\
(A \oplus \bot) \otimes B & & & & (A \otimes \top) \oplus B \\
\downarrow {\scriptstyle (1_A \oplus m) \otimes 1_B} & & & & \downarrow {\scriptstyle {u^R_{\otimes A}}^{-1} \oplus 1_B} \\
(A \oplus \top) \otimes B & \xrightarrow[\delta^R_{A,\top,B}]{} & A \oplus (\top \otimes B) & \xrightarrow[1_A \oplus {u^L_{\otimes B}}^{-1}]{} & A \oplus B
\end{array} \tag{23}$$

3.6. Lemma. *Consider a mix linearly distributive category* $\mathbb{X}$*, then the following diagrams commute:*

$$\begin{array}{ccc}
A \otimes (B \otimes C) & \xrightarrow{\alpha_{\otimes}{}^{-1}_{A,B,C}} & (A \otimes B) \otimes C \\
\downarrow {\scriptstyle 1_A \otimes \mathrm{mix}_{B,C}} & & \downarrow {\scriptstyle \mathrm{mix}_{A \otimes B, C}} \\
A \otimes (B \oplus C) & \xrightarrow{\delta^L_{A,B,C}} & (A \otimes B) \oplus C \\
\downarrow {\scriptstyle \mathrm{mix}_{A, B \oplus C}} & & \downarrow {\scriptstyle \mathrm{mix}_{A,B} \oplus 1_C} \\
A \oplus (B \oplus C) & \xrightarrow[\alpha_{\oplus A,B,C}]{} & (A \oplus B) \oplus C
\end{array}
\qquad
\begin{array}{ccc}
(A \otimes B) \otimes C & \xrightarrow{\alpha_{\otimes A,B,C}} & A \otimes (B \otimes C) \\
\downarrow {\scriptstyle \mathrm{mix}_{A,B} \otimes 1_C} & & \downarrow {\scriptstyle \mathrm{mix}_{A, B \otimes C}} \\
(A \oplus B) \otimes C & \xrightarrow{\delta^R_{A,B,C}} & A \oplus (B \otimes C) \\
\downarrow {\scriptstyle \mathrm{mix}_{A \oplus B, C}} & & \downarrow {\scriptstyle 1_A \oplus \mathrm{mix}_{B,C}} \\
(A \oplus B) \oplus C & \xrightarrow[\alpha_{\oplus}{}^{-1}_{A,B,C}]{} & A \oplus (B \oplus C)
\end{array} \tag{24}$$

Proof. This follows by the coherence conditions (20) and the definition of the mix maps in (23). ∎

In the process of developing categorical semantics for classical logic based on LDCs, Führmann and Pym gave a rather useful way to determine whether or not a LDC is mix.

3.7. Lemma. *[22, Lem 3.10] A linearly distributive category* $\mathbb{X}$ *with a map* $m : \bot \to \top$ *is mix if and only if*

$$(1_\bot \otimes m); {u^R_{\otimes \bot}}^{-1} = (m \otimes 1_\bot); {u^L_{\otimes \bot}}^{-1} : \bot \otimes \bot \to \bot \tag{25}$$

If the LDC is symmetric, the above lemma was then utilized to determine that, if $\bot$ has a $\otimes$-comonoid structure or if $\top$ has a $\oplus$-monoid structure, the LDC is mix.

3.8. THEOREM. *[22, Thm 3.11] Every symmetric linearly distributive category with a $\otimes$-comonoid*

$$\Delta_\bot : \bot \to \bot \otimes \bot \qquad e_\bot : \bot \to \top$$

is mix with $m = e_\bot$. Dually, every symmetric linearly distributive category with a $\oplus$-monoid

$$\nabla_\top : \top \oplus \top \to \top \qquad u_\top : \bot \to \top$$

is mix with $m = u_\top$.

There is a stronger version of the nullary MIX rule, which often holds in categorical models: $\bot \dashv\vdash \top$. In this case, $\bot \cong \top$ and we call such LDCs isomix.

3.9. DEFINITION. *[14, Def 6.5] An linearly distributive category is* **isomix** *if it is mix and $m : \bot \to \top$ is an isomorphism.*

In fact, we do not need to check (23) if the two monoidal units are isomorphic:

3.10. LEMMA. *[14, Lem 6.6] A linearly distributive category where $\top \cong \bot$ is isomix.*

We can also consider LDCs where the tensor monoidal product and the par monoidal product are isomorphic to one another.

3.11. DEFINITION. *[18, Sec 2.3] A linearly distributive category is* **compact** *if it is isomix and the mix maps are isomorphisms.*

Note that by Lemma 3.6, the linear distributivities of a compact LDC are the associators (modulo the mix maps).

While LDCs make multiplicative conjunction and disjunction primitive, we also want to be able to consider negation in this categorical framework and have it recover $*$-autonomous categories. We shall present negation in terms of complemented objects.

3.12. DEFINITION. *[16, Def A.5] Consider a linearly distributive category $\mathbb{X}$. The objects A, A^c in $\mathbb{X}$ form a complementation pair if there are morphisms*

$$\gamma : A \otimes A^c \to \bot \qquad \tau : \top \to A^c \oplus A$$

such that

$$\begin{array}{ccc} A & \xrightarrow{1_A} & A \\ {\scriptstyle u^R_{\otimes A}}\downarrow & & \uparrow{\scriptstyle u^L_{\oplus A}} \\ A \otimes \top & & \bot \otimes A \\ {\scriptstyle 1_A \otimes \tau}\downarrow & & \uparrow{\scriptstyle \gamma \oplus 1_A} \\ A \otimes (A^c \oplus A) & \xrightarrow[\delta^L_{A,A^c,A}]{} & (A \otimes A^c) \oplus A \end{array} \qquad \begin{array}{ccc} A^c & \xrightarrow{1_{A^c}} & A^c \\ {\scriptstyle u^L_{\otimes A^c}}\downarrow & & \uparrow{\scriptstyle u^R_{\oplus A^c}} \\ \top \otimes A^c & & A^c \oplus \bot \\ {\scriptstyle \tau \otimes 1_{A^c}}\downarrow & & \uparrow{\scriptstyle 1_{A^c} \oplus \gamma} \\ (A^c \oplus A) \otimes A^c & \xrightarrow[\delta^R_{A^c,A,A^c}]{} & A^c \oplus (A \otimes A^c) \end{array} \tag{26}$$

The pair is denoted by (A, A^c, γ, τ). A is said to be the left complement and A^c is the right complement.

3.13. Definition. *[15, Def 4.1, 4.3] A symmetric linearly distributive category has* **negation** *if there is an object function* $(-)^\perp$*, together with the following parametrized family of maps*

$$\gamma^R_A : A \otimes A^\perp \to \bot \qquad\qquad \tau^R_A : \top \to A \oplus {}^\perp A$$

which additionally induce the following families,

$$\gamma^L_A = A^\perp \otimes A \xrightarrow{\sigma_{\otimes A^\perp, A}} A \otimes A^\perp \xrightarrow{\gamma^R_A} \bot \qquad\qquad \tau^L_A = \top \xrightarrow{\tau^R_A} A \oplus A^\perp \xrightarrow{\sigma_{\oplus A, A^\perp}} A^\perp \oplus A$$

such $(A, A^\perp, \gamma^R_A, \tau^L_A)$ *and* $(A^\perp, A, \gamma^L_A, \tau^R_A)$ *form a complementation pairs.*

3.14. Remark. There is a some redundancy in the above definition. In fact, one need only prove the first diagrams, or equivalently the second diagrams, of (26) for each complementation pairs $(A, A^\perp, \gamma^R_A, \tau^L_A)$ and $(A^\perp, A, \gamma^L_A, \tau^R_A)$.

Notice that the above definition does not require $(-)^\perp$ to act on morphisms, nor does it ask for the many natural isomorphisms which connect the multiplicative connectives and negation together. These are in fact a consequence of the above structure, and excluding them from the definition allows it to be much simpler to verify in examples. We outline in the following lemma how $(-)^\perp$ defines a functor and some of the isomorphisms we will need in Section 5.11.

3.15. Lemma. *[15, Lem 4.4] In a symmetric linearly distributive category with negation* $(\mathbb{X}, \otimes, \top, \oplus, \bot)$*, the object function* $(-)^\perp$ *is a contravariant functor, which acts on morphisms as follows. Given* $f : A \to B$ *in* $\mathbb{X}$*,* $f^\perp : B^\perp \to A^\perp$ *is defined by*

$$\begin{array}{ccccc}
B^\perp & \xrightarrow{u^R_{\otimes B^\perp}} & B^\perp \otimes \top & \xrightarrow{1_{B^\perp} \otimes \tau^R_A} & B^\perp \otimes (A \oplus A^\perp) \\
 & & & & \downarrow {\scriptstyle 1_{B^\perp} \otimes (f \oplus 1_{A^\perp})} \\
\downarrow {\scriptstyle f^\perp} & & & & B^\perp \otimes (B \oplus A^\perp) \\
 & & & & \downarrow {\scriptstyle \delta^L_{B^\perp, B, A^\perp}} \\
A^\perp & \xleftarrow[u^L_{\oplus A^\perp}]{} & \bot \oplus A^\perp & \xleftarrow[\gamma^L_B \oplus 1_{A^\perp}]{} & (B^\perp \otimes B) \oplus A^\perp
\end{array}$$

and we have the following adjunctions

$$A \otimes - \dashv A^\perp \oplus - \qquad\qquad A^\perp \otimes - \dashv A \oplus -$$
$$- \otimes B \dashv - \oplus B^\perp \qquad\qquad - \otimes B^\perp \dashv - \oplus B$$

corresponding to the following bijections

$$\frac{A \otimes B \to B}{B \to A^\perp \oplus C} \qquad\qquad \frac{A^\perp \otimes B \to C}{B \to A \oplus C}$$
$$\frac{A \otimes B \to C}{A \to C \oplus B^\perp} \qquad\qquad \frac{A \otimes B^\perp \to C}{A \to C \oplus B}$$

Moreover, there are following invertible morphisms and natural isomorphisms

$$\beta_\top : \bot^\bot \to \top \qquad \epsilon_{A,B} : (A \oplus B)^\bot \to B^\bot \otimes A^\bot$$
$$\chi_\bot : \bot \to \top^\bot \qquad \phi_{A,B} : A^\bot \oplus B^\bot \to (B \otimes A)^\bot$$

3.16. Definition. *A* $*$-autonomous category *is a symmetric monoidal category* $(\mathbb{X}, \otimes, \top)$ *equipped with a full and faithful functor*

$$(-)^\bot : \mathbb{X}^{op} \to \mathbb{X}$$

such that there is a natural isomorphism $\mathbb{X}(A \otimes B, C^\bot) \cong \mathbb{X}(A, (B \otimes C)^\bot)$.

3.17. Theorem. *[15, Thm 4.5] The notions of symmetric linearly distributive categories with negation and* $*$*-autonomous categories coincide.*

3.18. Example. Let us list three important classes of examples of LDCs.

1. Every monoidal category $(\mathcal{X}, \oslash, I)$ can be viewed as a linearly distributive category, when taking the tensor and par structures to be equal to the original monoidal structure, i.e. $\otimes = \oplus = \oslash$ and $\top = \bot = I$. In this case, the linear distributivities are just $\oslash$-associators. We call such LDCs $(\mathcal{X}, \oslash, I, \oslash, I)$ **degenerate**.

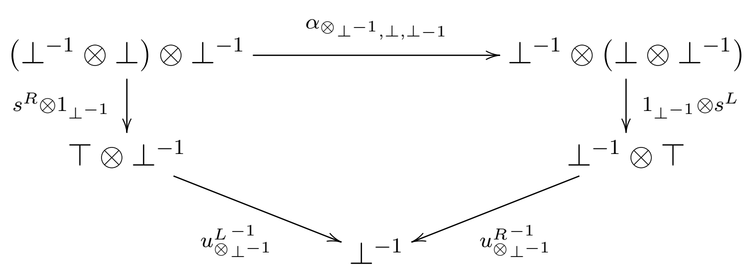
2. By Theorem 3.17, every $*$-autonomous category $(\mathcal{X}, \oslash, I, \star)$ is a linearly distributive category, with the tensor structure given by the original monoidal structure and the par structure given by the de Morgan dual, i.e. $A \oplus B = (B^\star \oslash A^\star)^\star$ and $\bot = I^\star$. Some notable examples of $*$-autonomous categories include the category of finite-dimensional vector spaces, the category of sup-lattices, the category of coherence spaces and the category of finiteness spaces.

3. Cockett and Seely introduced a class of examples known as **shifted tensor** LDCs in [13]. These are the categorical analogue of shift monoids and are defined as follows.

   3.19. Definition. *[16, Sec 5.2] Consider a monoidal category* $(\mathbb{X}, \otimes, \top)$. *An object* $\bot \in \mathbb{X}$ *is said to have a* **tensor inverse** *if there is an object* $\bot^{-1}$ *equipped with two isomorphisms*

   $$s^L : \bot \otimes \bot^{-1} \to \top \qquad s^R : \bot^{-1} \otimes \bot \to \top$$

   such that



   Suppose there is an object $\bot \in \mathbb{X}$ *with a tensor inverse, then define a monoidal product by* $A \oplus B = A \otimes (\bot^{-1} \otimes B)$, *known as the* $\bot$**-shifted tensor**.

3.20. Proposition. *[13, Prop 5.3, 5.4] Consider a monoidal category* $(\mathbb{X}, \otimes, \top)$ *with an object* $\bot \in \mathbb{X}$ *with a tensor inverse, then* $(\mathbb{X}, \otimes, \top, \oplus, \bot)$*, where* $\oplus$ *is the* $\bot$*-shifted tensor, is a linearly distributive category with invertible linear distributivities. Moreover, every linearly distributive categories with invertible linear distributivities has a tensor inverse for* $\bot$ *whose shift tensor product is naturally equivalent to the par product.*

3.21. Linear Functors and Transformations.

In the same spirit as the Fox theorem, we will need the appropriate notions of functors and transformations between linearly distributive categories.

3.22. Definition. *[16, Def 1] Let* $\mathbb{X}$ *and* $\mathbb{Y}$ *be linearly distributive categories. A* **bilax linear functor** $F = (F_\otimes, F_\oplus) : \mathbb{X} \to \mathbb{Y}$ *consists of:*

- *a lax monoidal functor* $(F_\otimes, m_\top, m_\otimes) : (\mathbb{X}, \otimes, \top) \to (\mathbb{Y}, \otimes, \top)$*, equipped with*
  - *a morphism* $m_\top : \top \to F_\otimes(\top)$*,*
  - *a natural transformation*

$$m_\otimes : (F_\otimes \times F_\otimes); \otimes \Rightarrow \otimes; F_\otimes \qquad m_{\otimes A,B} : F_\otimes(A) \otimes F_\otimes(B) \to F_\otimes(A \otimes B)$$

- *a colax monoidal functor* $(F_\oplus, n_\bot, n_\oplus) : (\mathbb{X}, \oplus, \bot) \to (\mathbb{Y}, \oplus, \bot)$*, equipped with*
  - *a morphism* $n_\bot : F_\oplus(\bot) \to \bot$*,*
  - *a natural transformation*

$$n_\oplus : \oplus; F_\oplus \Rightarrow (F_\oplus \times F_\oplus); \oplus \qquad n_{\oplus A,B} : F_\oplus(A \oplus B) \to F_\oplus(A) \oplus F_\oplus(B)$$

- *four natural transformations, known as* **linear strengths***,*

$$\begin{array}{ll}
\nu^R_\otimes : \oplus; F_\otimes \Rightarrow (F_\oplus \times F_\otimes); \oplus & \nu^R_{\otimes A,B} : F_\otimes(A \oplus B) \to F_\oplus(A) \oplus F_\otimes(B) \\
\nu^L_\otimes : \oplus; F_\otimes \Rightarrow (F_\otimes \times F_\oplus); \oplus & \nu^L_{\otimes A,B} : F_\otimes(A \oplus B) \to F_\otimes(A) \oplus F_\oplus(B) \\
\nu^R_\oplus : (F_\otimes \times F_\oplus); \otimes \Rightarrow \otimes; F_\oplus & \nu^R_{\oplus A,B} : F_\otimes(A) \otimes F_\oplus(B) \to F_\oplus(A \otimes B) \\
\nu^L_\oplus : (F_\oplus \times F_\otimes); \otimes \Rightarrow \otimes; F_\oplus & \nu^L_{\oplus A,B} : F_\oplus(A) \otimes F_\otimes(B) \to F_\oplus(A \otimes B)
\end{array}$$

subject to various coherence conditions determining the interaction of the linear strengths with the units (27), with the associativities (28), with the linear distributivities (29), and with other linear strengths via associators (30) and via linear distributivities (31).

$$\begin{array}{ccc}
F_\otimes(\bot \oplus A) & \xrightarrow{F_\otimes(u^L_{\oplus A})} & F_\otimes(A) \\
\downarrow{\scriptstyle \nu^R_{\otimes \bot, A}} & & \uparrow{\scriptstyle u^L_{\oplus F_\otimes(A)}} \\
F_\oplus(\bot) \oplus F_\otimes(A) & \xrightarrow[n_\bot \oplus 1_{F_\otimes(A)}]{} & \bot \oplus F_\otimes(A)
\end{array} \tag{27}$$

$$\nu^R_{\otimes \bot,A}; (n_\bot \oplus 1_{F_\otimes(A)}); u^L_{\oplus F_\otimes(A)} = F_\otimes(u^L_{\oplus A})$$
$$\nu^L_{\otimes A,\bot}; (1_{F_\otimes(A)} \oplus n_\bot); u^R_{\oplus F_\otimes(A)} = F_\otimes(u^R_{\oplus A})$$
$$u^L_{\otimes F_\oplus(A)}; (m_\top \otimes 1_{F_\oplus(A)}); \nu^R_{\oplus \top,A} = F_\oplus(u^L_{\otimes A})$$
$$u^R_{\otimes F_\oplus(A)}; (1_{F_\oplus(A)} \otimes m_\top); \nu^L_{\oplus \top,A} = F_\oplus(u^R_{\otimes A})$$

$$\begin{array}{ccc}
F_\otimes(A \oplus (B \oplus C)) & \xrightarrow{F_\otimes(\alpha_{\oplus A,B,C})} & F_\otimes((A \oplus B) \oplus C) \\
\downarrow{\scriptstyle \nu^R_{\otimes A,B\oplus C}} & & \downarrow{\scriptstyle \nu^R_{\otimes A\oplus B,C}} \\
F_\oplus(A) \oplus F_\otimes(B \oplus C) & & F_\oplus(A \oplus B) \oplus F_\otimes(C) \\
\downarrow{\scriptstyle 1_{F_\oplus(A)} \oplus \nu^R_{\otimes B,C}} & & \downarrow{\scriptstyle n_{\oplus A,B} \oplus 1_{F_\otimes(C)}} \\
F_\oplus(A) \oplus (F_\oplus(B) \oplus F_\otimes(C)) & \xrightarrow[\alpha_{\oplus F_\oplus(A),F_\oplus(B),F_\otimes(C)}]{} & (F_\oplus(A) \oplus F_\oplus(B)) \oplus F_\otimes(C)
\end{array} \tag{28}$$

$$\nu^R_{\otimes A,B\oplus C}; (1_{F_\oplus(A)} \oplus \nu^R_{\otimes B,C}); \alpha_{\oplus F_\oplus(A),F_\oplus(B),F_\otimes(C)} = F_\otimes(\alpha_{\oplus A,B,C}); \nu^R_{\otimes A\oplus B,C}; (n_{\oplus A,B} \oplus 1_{F_\otimes(C)})$$
$$\nu^L_{\otimes A,B\oplus C}; (1_{F_\otimes(A)} \oplus n_{\oplus B,C}); \alpha_{\oplus F_\otimes(A),F_\oplus(B),F_\oplus(C)} = F_\otimes(\alpha_{\oplus A,B,C}); \nu^L_{\otimes A\oplus B,C}; (\nu^L_{\oplus A,B} \oplus 1_{F_\oplus(C)})$$
$$\alpha_{\otimes F_\otimes(A),F_\otimes(B),F_\oplus(C)}; (1_{F_\otimes(A)} \otimes \nu^R_{\oplus B,C}); \nu^R_{\oplus A,B\otimes C} = (m_{\otimes A,B} \otimes 1_{F_\oplus(C)}); \nu^R_{\oplus A\otimes B,C}; F_\oplus(\alpha_{\otimes A,B,C})$$
$$\alpha_{\otimes F_\oplus(A),F_\otimes(B),F_\otimes(C)}; (1_{F_\oplus(A)} \otimes m_{\otimes B,C}); \nu^L_{\oplus A,B\otimes C} = (\nu^L_{\oplus A,B} \otimes 1_{F_\otimes(C)}); \nu^R_{\oplus A\otimes B,C}; F_\oplus(\alpha_{\otimes A,B,C})$$

$$\begin{array}{ccc}
F_\otimes(A \oplus B) \otimes F_\otimes(C) & \xrightarrow{m_{\otimes A\oplus B,C}} & F_\otimes((A \oplus B) \otimes C) \\
\downarrow{\scriptstyle \nu^R_{\otimes A,B} \otimes 1_{F_\otimes(C)}} & & \downarrow{\scriptstyle F_\otimes(\delta^R_{A,B,C})} \\
(F_\oplus(A) \oplus F_\otimes(B)) \otimes F_\otimes(C) & & F_\otimes(A \oplus (B \otimes C)) \\
\downarrow{\scriptstyle \delta^R_{F_\oplus(A),F_\otimes(B),F_\otimes(C)}} & & \downarrow{\scriptstyle \nu^R_{\otimes A,B\otimes C}} \\
F_\oplus(A) \oplus (F_\otimes(B) \otimes F_\otimes(C)) & \xrightarrow[1_{F_\oplus(A)} \oplus m_{\otimes B,C}]{} & F_\oplus(A) \oplus F_\otimes(B \otimes C)
\end{array} \tag{29}$$

$$m_{\otimes A\oplus B,C}; F_\otimes(\delta^R_{A,B,C}); \nu^R_{\otimes A,B\otimes C} = (\nu^R_{\otimes A,B} \otimes 1_{F_\otimes(C)}); \delta^R_{F_\oplus(A),F_\otimes(B),F_\otimes(C)}; (1_{F_\oplus(A)} \oplus m_{\otimes B,C})$$
$$m_{\otimes A,B\oplus C}; F_\otimes(\delta^L_{A,B,C}); \nu^L_{\otimes A\otimes B,C} = (1_{F_\otimes(A)} \otimes \nu^L_{\otimes B,C}); \delta^R_{F_\otimes(A),F_\otimes(B),F_\oplus(C)}; (m_{\otimes A,B} \oplus 1_{F_\oplus(C)})$$
$$\nu^R_{\oplus A,B\oplus C}; F_\oplus(\delta^L_{A,B,C}); n_{\oplus A\otimes B,C} = (1_{F_\otimes(A)} \otimes n_{\oplus B,C}); \delta^L_{F_\otimes(A),F_\oplus(B),F_\oplus(C)}; (\nu^R_{\oplus A,B} \oplus 1_{F_\oplus(C)})$$
$$\nu^L_{\oplus A\oplus B,C}; F_\oplus(\delta^R_{A,B,C}); n_{\oplus A,B\otimes C} = (n_{\oplus A,B} \oplus 1_{F_\otimes(C)}); \delta^R_{F_\oplus(A),F_\oplus(B),F_\otimes(C)}; (1_{F_\oplus(A)} \oplus \nu^L_{\oplus B,C})$$

$$
\begin{array}{ccc}
F_\otimes(A \oplus (B \oplus C)) & \xrightarrow{F_\otimes(\alpha_{\oplus A,B,C})} & F_\otimes((A \oplus B) \oplus C) \\
\downarrow {\scriptstyle \nu^R_{\otimes A,B\oplus C}} & & \downarrow {\scriptstyle \nu^L_{\otimes A\oplus B,C}} \\
F_\oplus(A) \oplus F_\otimes(B \oplus C) & & F_\otimes(A \oplus B) \oplus F_\oplus(C) \\
\downarrow {\scriptstyle 1_{F_\oplus(A)} \oplus \nu^L_{\otimes B,C}} & & \downarrow {\scriptstyle \nu^R_{\otimes A,B} \oplus 1_{F_\oplus(C)}} \\
F_\oplus(A) \oplus (F_\otimes(B) \oplus F_\oplus(C)) & \xrightarrow[\alpha_{\oplus F_\oplus(A),F_\otimes(B),F_\oplus(C)}]{} & (F_\oplus(A) \oplus F_\otimes(B)) \oplus F_\oplus(C)
\end{array}
\tag{30}
$$

$$\nu^R_{\otimes A,B\oplus C}; (1_{F_\oplus(A)} \oplus \nu^L_{\otimes B,C}); \alpha_{\oplus F_\oplus(A),F_\otimes(B),F_\oplus(C)} = F_\otimes(\alpha_{\oplus A,B,C}); \nu^L_{\otimes A\oplus B,C}; (\nu^R_{\otimes A,B} \oplus 1_{F_\oplus(C)})$$
$$\alpha_{\otimes F_\otimes(A),F_\oplus(B),F_\otimes(C)}; (1_{F_\otimes(A)} \otimes \nu^L_{\oplus B,C}); \nu^R_{\oplus A,B\otimes C} = (\nu^R_{\oplus A,B} \otimes 1_{F_\otimes(C)}); \nu^L_{\oplus A\otimes B,C}; F_\oplus(\alpha_{\otimes A,B,C})$$

$$
\begin{array}{ccc}
F_\otimes(A) \otimes F_\otimes(B \oplus C) & \xrightarrow{1_{F_\otimes(A)} \otimes \nu^R_{\otimes B,C}} & F_\otimes(A) \otimes (F_\oplus(B) \oplus F_\otimes(C)) \\
\downarrow {\scriptstyle m_{\otimes A,B\oplus C}} & & \downarrow {\scriptstyle \delta^L_{F_\otimes(A),F_\oplus(B),F_\otimes(C)}} \\
F_\otimes(A \otimes (B \oplus C)) & & (F_\otimes(A) \otimes F_\oplus(B)) \oplus F_\otimes(C) \\
\downarrow {\scriptstyle F_\otimes(\delta^L_{A,B,C})} & & \downarrow {\scriptstyle \nu^R_{\oplus A,B} \oplus 1_{F_\otimes(C)}} \\
F_\otimes((A \otimes B) \oplus C) & \xrightarrow[\nu^R_{\otimes A\otimes B,C}]{} & F_\oplus(A \otimes B) \oplus F_\otimes(C)
\end{array}
\tag{31}
$$

$$m_{\otimes A,B\oplus C}; F_\otimes(\delta^L_{A,B,C}); \nu^R_{\otimes A\otimes B,C} = (1_{F_\otimes(A)} \otimes \nu^R_{\otimes B,C}); \delta^L_{F_\otimes(A),F_\oplus(B),F_\otimes(C)}; (\nu^R_{\oplus A,B} \oplus 1_{F_\otimes(C)})$$
$$m_{\otimes A\oplus B,C}; F_\otimes(\delta^R_{A,B,C}); \nu^L_{\otimes A,B\otimes C} = (\nu^R_{\otimes A,B} \otimes 1_{F_\otimes(C)}); \delta^R_{F_\oplus(A),F_\otimes(B),F_\otimes(C)}; (\nu^L_{\oplus A,B} \oplus 1_{F_\otimes(C)})$$
$$\nu^R_{\oplus A\oplus B,C}; F_\oplus(\delta^R_{A,B,C}); n_{\oplus A,B\otimes C} = (\nu^R_{\otimes A,B} \otimes 1_{F_\oplus(C)}); \delta^R_{F_\oplus(A),F_\otimes(B),F_\oplus(C)}; (1_{F_\oplus(A)} \oplus \nu^R_{\oplus B,C})$$
$$\nu^L_{\oplus A,B\oplus C}; F_\oplus(\delta^L_{A,B,C}); n_{\oplus A\otimes B,C} = (1_{F_\oplus(A)} \otimes \nu^L_{\otimes B,C}); \delta^L_{F_\oplus(A),F_\otimes(B),F_\oplus(C)}; (\nu^L_{\oplus A,B} \oplus 1_{F_\oplus(C)})$$

3.23. DEFINITION. *[16, Def 1] If $\mathbb{X}$ and $\mathbb{Y}$ are symmetric linearly distributive categories, then a bilax linear functor $F = (F_\otimes, F_\oplus)$ is* **symmetric** *if*

- *$F_\otimes : (\mathbb{X}, \otimes, \top) \to (\mathbb{Y}, \otimes, \top)$ is a symmetric lax monoidal functor,*
- *$F_\oplus : (\mathbb{X}, \oplus, \bot) \to (\mathbb{Y}, \oplus, \bot)$ is a symmetric colax monoidal functor, and*
- *the linear strengths interact coherently with the braidings (32)*

$$\begin{array}{ccc} F_\otimes(A\oplus B) & \xrightarrow{v^R_{\otimes A,B}} & F_\oplus(A)\oplus F_\otimes(B) \\ {\scriptstyle F_\otimes(\sigma_{\oplus A,B})}\downarrow & & \downarrow{\scriptstyle \sigma_{\oplus F_\oplus(A),F_\otimes(B)}} \\ F_\otimes(B\oplus A) & \xrightarrow[v^L_{\otimes B,A}]{} & F_\otimes(B)\oplus F_\oplus(A) \\ & & \\ F_\otimes(A)\otimes F_\oplus(B) & \xrightarrow{v^R_{\oplus A,B}} & F_\oplus(A\otimes B) \\ {\scriptstyle \sigma_{\otimes F_\otimes(A),F_\oplus(B)}}\downarrow & & \downarrow{\scriptstyle F_\oplus(\sigma_{\otimes A,B})} \\ F_\oplus(B)\otimes F_\otimes(A) & \xrightarrow[v^L_{\oplus B,A}]{} & F_\oplus(B\otimes A) \end{array} \tag{32}$$

3.24. Remark. In the symmetric case, we can alternatively drop the requirement that the left (or right) linear strengths exist instead of adding (32), as we can always define the missing pair of linear strengths via the braiding. Therefore, we often only give one pair of linear strengths.

3.25. Definition. *[16, Def 3] Let $F, G : \mathbb{X} \to \mathbb{Y}$ be bilax linear functors between linearly distributive categories. A* **linear transformation** *$\alpha : F \Rightarrow G$ consists of:*

- *a monoidal transformation $\alpha_\otimes : (F_\otimes, m^F_\top, m^F_\otimes) \Rightarrow (G_\otimes, m^G_\top, m^G_\otimes)$ and*
- *a monoidal transformation $\alpha_\oplus : (G_\oplus, n^G_\perp, n^G_\oplus) \Rightarrow (F_\oplus, n^F_\perp, n^F_\oplus)$,*

which commute coherently with the linear strengths of F and G (33).

$$\begin{array}{ccc} F_\otimes(A\oplus B) & \xrightarrow{\alpha_{\otimes A\oplus B}} & G_\otimes(A\oplus B) \\ & & \downarrow{\scriptstyle v^{R^G}_{\otimes A,B}} \\ {\scriptstyle v^{R^F}_{\otimes A,B}}\downarrow & & G_\oplus(A)\oplus G_\otimes(B) \\ & & \downarrow{\scriptstyle \alpha_{\oplus A}\oplus 1_{G_\otimes(B)}} \\ F_\oplus(A)\oplus F_\otimes(B) & \xrightarrow[1_{F_\oplus(A)}\oplus\alpha_{\otimes B}]{} & F_\oplus(A)\oplus G_\otimes(B) \end{array} \tag{33}$$

$$\begin{aligned} \alpha_{\otimes A\oplus B}; v^{R^G}_{\otimes A,B}; (\alpha_{\oplus A} \oplus 1_{G_\otimes(B)}) &= v^{R^F}_{\otimes A,B}; (1_{F_\oplus(A)} \oplus \alpha_{\otimes B}) \\ \alpha_{\otimes A\oplus B}; v^{L^G}_{\otimes A,B}; (1_{G_\otimes(A)} \oplus \alpha_{\oplus B}) &= v^{L^F}_{\otimes A,B}; (\alpha_{\otimes A} \oplus 1_{F_\oplus(B)}) \\ (1_{G_\oplus(A)} \otimes \alpha_{\otimes B}); v^{L^G}_{\oplus A,B}; \alpha_{\oplus A\otimes B} &= (\alpha_{\oplus A} \otimes 1_{F_\otimes(B)}); v^{L^F}_{\oplus A,B} \\ (\alpha_{\otimes A} \otimes 1_{G_\oplus(B)}); v^{R^G}_{\oplus A,B}; \alpha_{\oplus A\otimes B} &= (1_{F_\otimes(A)} \otimes \alpha_{\oplus B}); v^{R^F}_{\oplus A,B} \end{aligned}$$

3.26. Proposition. *[16, Prop 4] Linearly distributive categories, bilax linear functors, and linear transformations form a 2-category, which is denoted by* **LDC**. *Restricting to symmetric linearly distributive categories and symmetric bilax linear functors equally gives a 2-category, denoted* **SLDC**. *Both these 2-categories are closed under products.*

3.27. Definition. *A* **linear functor** $F = (F_\otimes, F_\oplus) : \mathbb{X} \to \mathbb{Y}$ *is a bilax linear functor such that*

- $F_\otimes : (\mathbb{X}, \otimes, \top) \to (\mathbb{Y}, \otimes, \top)$ *is a monoidal functor, and*
- $F_\oplus : (\mathbb{X}, \oplus, \bot) \to (\mathbb{Y}, \oplus, \bot)$ *is a comonoidal functor.*

Note that Cockett and Seely do not use the terminology bilax [16]. Their linear functor refers to our bilax linear functor. However, in order to stay consistent with the terminology of monoidal categories introduced and of duoidal categories to come, we will make use of the term bilax.

We will also consider linear functors whose component functors are equal. These were first defined by [6] and then further explored by [18].

3.28. Definition. *[18, Def 3.1] Consider linearly distributive categories* $\mathbb{X}$ *and* $\mathbb{Y}$*, a bilax linear functor* $F = (F_\otimes, F_\oplus) : \mathbb{X} \to \mathbb{Y}$ *is* **Frobenius** *if*

1. $F_\otimes = F_\oplus$,
2. $\nu^R_{\otimes A,B} = \nu^L_{\otimes A,B} = n_{\oplus A,B}$*, and*
3. $\nu^R_{\oplus A,B} = \nu^L_{\oplus A,B} = m_{\otimes A,B}$.

Given the degeneracy, we can give an alternative characterization of such linear functors.

3.29. Proposition. *[18, Lem 3.2] Consider linearly distributive categories* $\mathbb{X}$ *and* $\mathbb{Y}$*, then the following notions coincide:*

- *bilax Frobenius linear functors* $F = (F_\otimes, F_\oplus) : \mathbb{X} \to \mathbb{Y}$*, and*
- $\otimes$*-lax and* $\oplus$*-colax monoidal functors* $(F, m_\otimes, m_\top, n_\oplus, n_\bot) : \mathbb{X} \to \mathbb{Y}$ *satisfying* (34).

$$\begin{array}{ccccc}
F(A)\otimes F(B\oplus C) & \xrightarrow{m_{\otimes A,B\oplus C}} & F(A\otimes(B\oplus C)) & \xrightarrow{F(\delta^L_{A,B,C})} & F((A\otimes B)\oplus C)\\
{\scriptstyle 1_{F(A)}\otimes n_{\oplus B,C}}\downarrow & & & & \downarrow{\scriptstyle n_{\oplus A\otimes B,C}}\\
F(A)\otimes(F(B)\oplus F(C)) & \xrightarrow[\delta^L_{F(A),F(B),F(C)}]{} & (F(A)\otimes F(B))\oplus F(C) & \xrightarrow[m_{\otimes A,B}\oplus 1_{F(C)}]{} & F(A\otimes B)\oplus F(C)\\
\\
F(A\oplus B)\otimes F(C) & \xrightarrow{m_{\otimes A\oplus B,C}} & F((A\oplus B)\otimes C) & \xrightarrow{F(\delta^R_{A,B,C})} & F(A\oplus(B\otimes C))\\
{\scriptstyle n_{\oplus A,B}\otimes 1_{F(C)}}\downarrow & & & & \downarrow{\scriptstyle n_{\oplus A,B\otimes C}}\\
(F(A)\oplus F(B))\otimes F(C) & \xrightarrow[\delta^R_{F(A),F(B),F(C)}]{} & F(A)\oplus(F(B)\otimes F(C)) & \xrightarrow[1_{F(A)}\oplus m_{\otimes B,C}]{} & F(A)\oplus F(B\otimes C)
\end{array} \tag{34}$$

If the LDCs are mix, the definition of Frobenius linear functors can be slightly extended to guarantee that they preserve the mix maps.

3.30. Definition. *[18, Def 3.4] Consider mix linearly distributive categories* $\mathbb{X}$ *and* $\mathbb{Y}$*, then a bilax Frobenius linear functor* $F = (F, F) : \mathbb{X} \to \mathbb{Y}$ *is* mix *if the following diagram commutes.*

$$\begin{array}{ccc} F(\bot) & \xrightarrow{n_\bot} & \bot \\ {\scriptstyle F(m)}\downarrow & & \downarrow{\scriptstyle m} \\ F(\top) & \xleftarrow[m_\top]{} & \top \end{array} \tag{35}$$

With the above definition, we can easily see that every compact LDC is equivalent to a degenerate one.

3.31. Lemma. *Every compact linearly distributive category* $(\mathbb{X}, \otimes, \top, \oplus, \bot)$ *is isomorphic to degenerate linearly distributive categories* $(\mathbb{X}, \otimes, \top, \otimes, \top)$ *and* $(\mathbb{X}, \oplus, \bot, \oplus, \bot)$*, via mix Frobenius linear functors.*

If we then consider linear transformations between Frobenius linear functors, they often correspond a natural isomorphism paired with its inverse.

3.32. Lemma. *[18, Lem 3.6] Suppose* $F, G : \mathbb{X} \to \mathbb{Y}$ *are bilax Frobenius linear functors and* $\alpha = (\alpha_\otimes, \alpha_\oplus) : F \Rightarrow G$ *is a linear transformation, then the following are equivalent:*

1. *One of the diagrams in (36) and one of the diagrams in (37) hold.*
2. $\alpha_\oplus = \alpha_\oplus^{-1}$

$$\begin{array}{ccc} \top & \xrightarrow{m_\top^G} & G(\top) \\ & {\scriptstyle m_\top^F}\searrow & \downarrow{\scriptstyle \alpha_{\oplus\top}} \\ & & F(\top) \end{array} \qquad \begin{array}{ccc} F(\bot) & \xrightarrow{n_\bot^F} & \bot \\ {\scriptstyle \alpha_{\otimes\top}}\downarrow & \swarrow{\scriptstyle n_\bot^G} & \\ G(\bot) & & \end{array} \tag{36}$$

$$\begin{array}{ccc} G(A) \otimes F(B) & \xrightarrow{1_{G(A)} \otimes \alpha_{\otimes B}} & G(A) \otimes G(B) \\ {\scriptstyle \alpha_{\oplus A} \otimes 1_{F(B)}}\downarrow & & \\ F(A) \otimes F(B) & & \downarrow{\scriptstyle m^G_{\otimes A,B}} \\ {\scriptstyle m^F_{\otimes A,B}}\downarrow & & \\ F(A \otimes B) & \xrightarrow[\alpha_{\otimes A \otimes B}]{} & G(A \otimes B) \end{array} \qquad \begin{array}{ccc} F(A) \otimes G(B) & \xrightarrow{\alpha_{\otimes A} \otimes 1_{G(B)}} & G(A) \otimes G(B) \\ {\scriptstyle 1_{F(A)} \otimes \alpha_{\oplus B}}\downarrow & & \\ F(A) \otimes F(B) & & \downarrow{\scriptstyle m^G_{\otimes A,B}} \\ {\scriptstyle m^F_{\otimes A,B}}\downarrow & & \\ F(A \otimes B) & \xrightarrow[\alpha_{\otimes A \otimes B}]{} & G(A \otimes B) \end{array}$$

$$\begin{array}{ccc} G(A \oplus B) & \xrightarrow{n^G_{\oplus A,B}} & G(A) \oplus G(B) \\ {\scriptstyle \alpha_{\oplus A \oplus B}}\downarrow & & \\ F(A \oplus B) & & \downarrow{\scriptstyle 1_{G(A)} \oplus \alpha_{\oplus B}} \\ {\scriptstyle n^F_{\oplus A,B}}\downarrow & & \\ F(A) \oplus F(B) & \xrightarrow[\alpha_{\otimes A} \oplus 1_{F(B)}]{} & G(A) \oplus F(B) \end{array} \qquad \begin{array}{ccc} G(A \oplus B) & \xrightarrow{n^G_{\oplus A,B}} & G(A) \oplus G(B) \\ {\scriptstyle \alpha_{\oplus A \oplus B}}\downarrow & & \\ F(A \oplus B) & & \downarrow{\scriptstyle \alpha_{\oplus A} \oplus 1_{G(B)}} \\ {\scriptstyle n^F_{\oplus A,B}}\downarrow & & \\ F(A) \oplus F(B) & \xrightarrow[1_{F(A)} \oplus \alpha_{\otimes B}]{} & F(A) \oplus G(B) \end{array} \tag{37}$$

3.33. Lemma. *[18, Lem 3.10] Suppose $F, G : \mathbb{X} \to \mathbb{Y}$ are bilax Frobenius linear functors and $\alpha : F \Rightarrow G$ is a $\otimes$-monoidal and $\oplus$-monoidal natural isomorphism, then $(\alpha, \alpha^{-1}) : F \Rightarrow G$ is a linear transformation.*

As Frobenius linear functors compose and identity linear functors are Frobenius, restricting to such 1-cells determines a sub-2-category.

3.34. Proposition. *Linearly distributive categories, bilax Frobenius linear functors and linear transformations form a sub-2-category of* **LDC***, denoted by* **FLDC***.*

3.35. Cartesian Linearly Distributive Categories.

We now introduce cartesian LDCs, the subclass we are interested in characterizing via the linearly distributive Fox theorem.

3.36. Definition. *[15, Sec 2] A* **cartesian linearly distributive category***, or a CLDC, $\mathbb{X}$ is a symmetric linearly distributive category whose tensor $\otimes$ is the categorical product $\times$ in $\mathbb{X}$, with top $\top$ the terminal object $\mathbf{1}$ in $\mathbb{X}$, and par $\oplus$ is the categorical coproduct $+$ in $\mathbb{X}$, with $\perp$ the initial object $\mathbf{0}$ in $\mathbb{X}$.*

The development of the linearly distributive Fox theorem renewed interest in cartesian LDCs, which were further explored in their own right by the author and co-author Jean-Simon Pacaud Lemay in the article "Cartesian Linearly Distributive Categories: Revisited" [29]. This work examines the distinctive structures and properties of cartesian LDCs, discusses key classes of examples, and reassesses previously proposed instances of cartesian LDCs. We recommend reading both articles, as they complement each other, though each is designed to be accessible and valuable independently.

Cartesian LDCs themselves form a 2-category with the appropriate notion of linear functor as 1-cells, mirroring the monoidal functors in the 2-category of cartesian categories.

3.37. Proposition. *There is a sub 2-category of* **LDC** *consisting of cartesian linearly distributive categories, symmetric linear functors, known as* **cartesian linear functors***, and linear transformations, denoted by* **CLDC***, closed under products. Moreover, there is a sub 2-category of* **LDC** *consisting of cartesian linearly distributive categories, symmetric Frobenius linear functors and linear transformations, denoted by* **FCLDC***.*

4. Duoidal Categories

Linearly distributive categories are of course not the only categories with two monoidal structures. Duoidal categories represent another variant, characterized by two monoidal products that interact via additional structure maps, most notably the interchange law. Unlike LDCs, the duoidal structure arises canonically when considering finite products and coproducts. The theory of duoidal categories will play a crucial role in the definitions and results presented in this paper, so we present all the necessary background in this section.

The earliest appearance of a form of the interchange law between monoidal structures in the literature is found in Joyal and Street's work on braided monoidal categories [28]. They demonstrate that a duoidal category with structure maps that are isomorphisms corresponds to a braided monoidal category via a categorical Eckmann-Hilton argument. Subsequently, Balteanu, Fiedorowicz, Schwänzl, and Vogt defined 2-fold monoidal categories as monoids in the monoidal category of monoidal categories [2]. In their definition, the interchange law is not required to be an isomorphism, but the unit objects of both monoidal structures are assumed to coincide, and the monoidal structures themselves are strict. Building on this foundation, Forcey, Siehler, and Sowers generalized the definition by allowing distinct unit objects and non-strict associators, while retaining equalities for unitors and certain structure maps [20].

The currently accepted definition of duoidal category was first introduced by Aguiar and Mahajan under the name 2-monoidal categories [1][1]. The background on duoidal categories presented in this section is almost entirely based on Chapter 6 of their monograph, and readers seeking additional details are encouraged to consult it. The now-standard name for duoidal categories was suggested by Street and first appeared in the literature in [4] and [8].

4.1. Definition, Symmetry and Examples.

4.2. Definition. *[1, Def 6.1] A* **duoidal category** $(\mathcal{X}, \diamond, I, \star, J)$ *is category* $\mathcal{X}$ *with two monoidal structures* $(\mathcal{X}, \diamond, I, \alpha_\diamond, \rho_\diamond, \lambda_\diamond)$ *and* $(\mathcal{X}, \star, J, \alpha_\star, \rho_\star, \lambda_\star)$ *equipped with morphisms*

$$\Delta_I : I \to I \star I \qquad \mu_J : J \diamond J \to J \qquad \iota : I \to J$$

and an **interchange** *natural transformation*

$$\zeta : (\star \times \star); \diamond \Rightarrow s_{\mathcal{X},\mathcal{X},\mathcal{X},\mathcal{X}}; (\diamond \times \diamond); \star \quad \zeta_{A,B,C,D} : (A \star B) \diamond (C \star D) \to (A \diamond C) \star (B \diamond D)$$

such that

- (J, ∇_J, ι) *is a* $\diamond$*-monoid,*
- (I, Δ_I, ι) *is a* $\star$*-comonoid,*
- *interchange maps interact coherently with associativities* (38) *and with unitors* (39).

$$\begin{array}{ccccc} ((A \star B) \diamond (C \star D)) \diamond (E \star F) & \xrightarrow{\alpha_{\diamond A\star B, C\star D, E\star F}} & (A \star B) \diamond ((C \star D) \diamond (E \star F)) & \xrightarrow{1_{A\star B} \diamond \zeta_{C,D,E,F}} & (A \star B) \diamond ((C \diamond E) \star (D \diamond F)) \\ \downarrow{\scriptstyle \zeta_{A,B,C,D} \diamond 1_{E\star F}} & & & & \downarrow{\scriptstyle \zeta_{A,B,C\diamond E,D\diamond F}} \\ ((A \diamond C) \star (B \diamond D)) \diamond (E \star F) & \xrightarrow[\zeta_{A\diamond C,B\diamond D,E,F}]{} & ((A \diamond C) \diamond E) \star ((B \diamond D) \diamond F) & \xrightarrow[\alpha_{\diamond A,C,E} \star \alpha_{\diamond B,D,F}]{} & (A \diamond (C \diamond E)) \star (B \diamond (D \diamond F)) \end{array} \tag{38}$$

[1] We will refrain from using the 2-monoidal category terminology introduced by Aguiar and Mahajan to avoid confusion with monoidal 2-categories, which are 2-categories with a coherent monoidal structure.

$$\alpha_{\diamond A\star B,C\star D,E\star F};(1_{A\star B}\diamond\zeta_{C,D,E,F});\zeta_{A,B,C\diamond E,D\diamond F} = (\zeta_{A,B,C,D}\diamond 1_{E\star F});\zeta_{A\diamond C,B\diamond D,E,F};(\alpha_{\diamond A,C,E}\star\alpha_{\diamond B,D,F})$$
$$\zeta_{A\star B,CD\star E,F};(\zeta_{A,B,D,E}\star 1_{C\diamond F});\alpha_{\star A\diamond D,B\diamond E,C\diamond F} = (\alpha_{\star A,B,C}\diamond\alpha_{\star D,E,F});\zeta_{A,B\star C,D,E\star F};(1_{A\diamond D}\star\zeta_{B,C,E,F})$$

$$\begin{array}{ccc} A\star B & \xrightarrow{\lambda_{\diamond A\star B}} & I\diamond(A\star B) \\ {\scriptstyle \lambda_{\diamond A}\star\lambda_{\diamond B}}\downarrow & & \downarrow{\scriptstyle \Delta_I\diamond 1_{A\star B}} \\ (I\diamond A)\star(I\diamond B) & \xleftarrow[\zeta_{I,I,A,B}]{} & (I\star I)\diamond(A\star B) \end{array} \tag{39}$$

$$\lambda_{\diamond A\star B};(\Delta_I\diamond 1_{A\star B});\zeta_{I,I,A,B} = \lambda_{\diamond A}\star\lambda_{\diamond B} \qquad \rho_{\diamond A\star B};(1_{A\star B}\diamond\Delta_I);\zeta_{A,B,I,I} = \rho_{\diamond A}\star\rho_{\diamond B}$$
$$(\lambda_{\star A}\diamond\lambda_{\star B});\zeta_{J,A,J,B};(\mu_J\star 1_{A\diamond B}) = \lambda_{\star A\diamond B} \qquad (\rho_{\star A}\diamond\rho_{\star B});\zeta_{A,J,B,J};(1_{A\diamond B}\star\mu_J) = \rho_{\star A\diamond B}$$

4.3. Definition. *A duoidal category* $(\mathcal{X},\diamond,I,\star,J)$ *is* **normal** *if* $\iota : I\to J$ *is invertible. Further, a duoidal category is* **strong** *if all its structure maps are invertible.*

4.4. Definition. *[1, Def 6.5] A duoidal category* $(\mathcal{X},\diamond,I,\star,J)$ *is* **braided** *if*

- $(\mathcal{X},\diamond,I)$ *is a braided monoidal category with* $\diamond$*-braiding*

$$\sigma_\diamond : \diamond \Rightarrow switch_{\mathcal{X},\mathcal{X}};\diamond \qquad \sigma_{\diamond A,B} : A\diamond B\to B\diamond A$$

- $(\mathcal{X},\star,J)$ *is a braided monoidal category with* $\oplus$*-braiding*

$$\sigma_\star : \star \Rightarrow switch_{\mathcal{X},\mathcal{X}};\star \qquad \sigma_{\star A,B} : A\star B\to B\star A$$

such that

- (J,μ_J,ι) *is a braided* $\diamond$*-monoid and* (I,Δ_I,ι) *is a cobraided* $\star$*-comonoid, i.e. (40),*
- *the interchange maps interact coherently with the braidings (41).*

$$\begin{array}{ccc} J\diamond J & \xrightarrow{\mu_J} & J \\ {\scriptstyle \sigma_{\diamond J,J}}\downarrow & \nearrow{\scriptstyle \mu_J} & \\ J\diamond J & & \end{array} \qquad \begin{array}{ccc} I & \xrightarrow{\Delta_I} & I\star I \\ & {\scriptstyle \Delta_I}\searrow & \downarrow{\scriptstyle \sigma_{\star I,I}} \\ & & I\star I \end{array} \tag{40}$$

$$\begin{array}{ccc} (A\star B)\diamond(C\star D) & \xrightarrow{\zeta_{A,B,C,D}} & (A\diamond C)\star(B\diamond D) \\ {\scriptstyle \sigma_{\star A,B}\diamond\sigma_{\star C,D}}\downarrow & & \downarrow{\scriptstyle \sigma_{\star A\diamond C,B\diamond D}} \\ (B\star A)\diamond(D\star C) & \xrightarrow[\zeta_{B,A,D,C}]{} & (B\diamond D)\star(A\diamond C) \end{array} \tag{41}$$

$$\zeta_{A,B,C,D};\sigma_{\star A\diamond C,B\diamond D} = (\sigma_{\star A,B}\diamond\sigma_{\star C,D});\zeta_{B,A,D,C}$$
$$\zeta_{A,B,C,D};(\sigma_{\diamond A,C}\star\sigma_{\diamond B,D}) = \sigma_{\diamond A\star B,C\star D};\zeta_{C,D,A,B}$$

The notion of symmetric duoidal categories has not yet appeared in the literature as such, since the motivation for duoidal structures is often braided monoidal categories, although it is a trivial extension of braided duoidal categories. Symmetry is essential to the Fox theorem and therefore will be a key component of this work.

4.5. DEFINITION. *A braided duoidal category* $(\mathcal{X}, \diamond, I, \star, J)$ *is* **symmetric** *if*

- $(\mathcal{X}, \diamond, I)$ *is a symmetric monoidal category, and*
- $(\mathcal{X}, \star, J)$ *is a symmetric monoidal category.*

Given a symmetric duoidal category, there are two canonical flips, one for $\diamond$ and for $\star$. We note here how these canonical flips interact with the duoidal structure maps.

4.6. PROPOSITION. *The following diagrams commute in any symmetric duoidal category:*

$$\begin{array}{ccc} I & \xrightarrow{\Delta_I} & I \star I \\ & & \downarrow{\scriptstyle \Delta_I \star \Delta_I} \\ \downarrow{\scriptstyle \Delta_I} & & (I \star I) \star (I \star I) \\ & & \downarrow{\scriptstyle \tau^{\star}_{I,I,I,I}} \\ I \star I & \xrightarrow[\Delta_I \star \Delta_I]{} & (I \star I) \star (I \star I) \end{array} \qquad \begin{array}{ccc} (J \diamond J) \diamond (J \diamond J) & \xrightarrow{\mu_J \diamond \mu_J} & J \diamond J \\ \downarrow{\scriptstyle \tau^{\diamond}_{J,J,J,J}} & & \\ (J \diamond J) \diamond (J \diamond J) & & \downarrow{\scriptstyle \mu_J} \\ \downarrow{\scriptstyle \mu_J} & & \\ J \diamond J & \xrightarrow[\mu_J]{} & J \end{array} \tag{42}$$

$$\begin{array}{ccccc} \begin{array}{c}((A \star A') \star (B \star B')) \diamond \\ ((C \star C') \star (D \star D'))\end{array} & \xrightarrow{\zeta_{A\star A', B\star B', C\star C', D\star C'}} & \begin{array}{c}((A \star A') \diamond (C \star C')) \star \\ ((B \star B') \diamond (D \star D'))\end{array} & \xrightarrow{\zeta_{A,A',C,C'} \star \zeta_{B,B',D,D'}} & \begin{array}{c}((A \diamond C) \star (A' \diamond C')) \star \\ ((B \diamond D) \star (B' \diamond D'))\end{array} \\ \downarrow{\scriptstyle \tau^{\diamond}_{A\star A', B\star B', C\star C', D\star D'}} & & & & \downarrow{\scriptstyle \tau^{\star}_{A\diamond C, A'\diamond C', B\diamond D, B'\diamond D'}} \\ \begin{array}{c}((A \star B) \star (A' \star B')) \diamond \\ ((C \star D) \star (C' \star D'))\end{array} & \xrightarrow[\zeta_{A\star B, A'\star B', C\star D, C'\star D'}]{} & \begin{array}{c}((A \star B) \star (C \star D)) \star \\ ((A' \star B') \star (C' \star D'))\end{array} & \xrightarrow[\zeta_{A,B,C,D} \star \zeta_{A',B',C',D'}]{} & \begin{array}{c}((A \diamond C) \star (B \diamond D)) \star \\ ((A' \diamond C') \star (B' \diamond D'))\end{array} \end{array} \tag{43}$$

$$\begin{aligned} &\zeta_{A\star A', B\star B', C\star C', D\star C'}; (\zeta_{A,A',C,C'} \star \zeta_{B,B',D,D'}); \tau^{\star}_{A\diamond C, A'\diamond C', B\diamond D, B'\diamond D'} \\ &\qquad = (\tau^{\star}_{A,A',B,B'} \diamond \tau^{\star}_{C,C',D,D'}); \zeta_{A\star B, A'\star B', C\star D, C'\star D'}; (\zeta_{A,B,C,D} \star \zeta_{A',B',C',D'}) \\ &\tau^{\diamond}_{A\star A', B\star B', C\star C', D\star D'}; (\zeta_{A,A',C,C'} \diamond \zeta_{B,B',D,D'}); \zeta_{A\diamond C, A'\diamond C', B\diamond D, B'\diamond D'} = \\ &\qquad (\zeta_{A,A',B,B'} \diamond \zeta_{C,C',D,D'}); \zeta_{A\diamond B, A'\diamond B', C\diamond D, C'\diamond D'}; (\tau^{\diamond}_{A,B,C,D} \star \tau^{\diamond}_{A',B',C',D'}) \end{aligned}$$

4.7. EXAMPLE. We list here some examples of duoidal categories.

1. [1, Prop 6.10] Every monoidal category $(\mathcal{X}, \oslash, I)$ with a braiding σ gives a duoidal category $(\mathcal{X}, \oslash, I, \oslash, I)$ with structure maps given by unitors, an identity map, and the canonical flip:

$$\Delta_I = u_{\otimes I} : I \to I \oslash I \qquad \mu_I = u_{\otimes}^{-1}{}_I : I \oslash I \to I \qquad \iota = 1_I : I \to I$$
$$\zeta_{A,B,C,B} = \tau_{A,B,C,D} : (A \oslash B) \oslash (C \oslash D) \to (A \oslash C) \oslash (B \oslash D)$$

Here we state Joyal and Street's result about braided monoidal categories [28] in the terminology of duoidal categories:

4.8. Proposition. *[1, Prop 6.11] Given a strong duoidal category $(\mathcal{X}, \diamond, I, \star, J)$, $(\mathcal{X}, \diamond, I)$ and $(\mathcal{X}, \star, J)$ are isomorphic braided monoidal categories and the interchange natural transformation is the canonical flip.*

4.9. Duoidal Functors and Transformations.

Aguiar and Mahajan define two different types of functors between duoidal categories: bilax duoidal functors and double lax duoidal functors. We shall only need the former.

4.10. Definition. *[1, Def 6.50] Let $(\mathcal{X}, \diamond, I, \star, J)$ and $(\mathcal{Y}, \diamond, I, \star, J)$ denote duoidal categories.*
A **bilax duoidal functor** $(F, p_I, p_\diamond, q_J, q_\star) : (\mathcal{X}, \diamond, I, \star, J) \to (\mathcal{Y}, \diamond, I, \star, J)$ *is a functor* $F : \mathcal{X} \to \mathcal{Y}$ *such that*

- $(F, p_I, p_\diamond) : (\mathcal{X}, \diamond, I) \to (\mathcal{Y}, \diamond, I)$ *is a monoidal functor with*
  - *a morphism* $p_I : I \to F(I)$,
  - *a natural transformation*

$$p_\diamond : (F \times F); \diamond \Rightarrow \diamond; F \qquad\qquad p_{\diamond A,B} : F(A) \diamond F(B) \to F(A \diamond B)$$

- $(F, q_J, q_\star) : (\mathcal{X}, \star, J) \to (\mathcal{Y}, \star, J)$ *is a comonoidal functor with*
  - *a morphism* $q_J : F(J) \to J$,
  - *a natural transformation*

$$q_\star : \star; F \Rightarrow (F \times F); \star \qquad\qquad q_{\star A,B} : F(A \star B) \to F(A) \star F(B)$$

such that (44) *and* (45).

$$\begin{array}{ccccc}
F(A \star B) \diamond F(C \star D) & \xrightarrow{q_{\star A,B} \diamond q_{\star C,D}} & (F(A) \star F(B)) \diamond (F(C) \star F(D)) & \xrightarrow{\zeta_{F(A),F(B),F(C),F(D)}} & (F(A) \diamond F(C)) \star (F(B) \diamond F(D)) \\
{\scriptstyle p_{\diamond A\star B, C\star D}} \downarrow & & & & \downarrow {\scriptstyle p_{\diamond A,C} \star p_{\diamond B,D}} \\
F((A \star B) \diamond (C \star D)) & \xrightarrow[F(\zeta_{A,B,C,D})]{} & F((A \diamond C) \star (B \diamond D)) & \xrightarrow[q_{\star A\diamond C, B\diamond D}]{} & F(A \diamond C) \star F(B \diamond D)
\end{array} \tag{44}$$

$$\begin{array}{ccccc}
I & \xrightarrow{p_I} & F(I) & \xrightarrow{F(\Delta_I)} & F(I \star I) \\
{\scriptstyle \Delta_I} \downarrow & & & & \downarrow {\scriptstyle q_{\star I,I}} \\
I \star I & & \xrightarrow[p_I \star p_I]{} & & F(I) \star F(I)
\end{array}
\qquad
\begin{array}{ccc}
I & \xrightarrow{p_I} & F(I) \\
{\scriptstyle \iota} \downarrow & & \downarrow {\scriptstyle F(\iota)} \\
J & \xleftarrow[q_J]{} & F(J)
\end{array}$$

$$\begin{array}{ccccc}
F(J) \diamond F(J) & \xrightarrow{p_{\diamond J,J}} & F(J \diamond J) & \xrightarrow{F(\mu_J)} & F(J) \\
{\scriptstyle q_J \diamond q_J} \downarrow & & & & \downarrow {\scriptstyle q_J} \\
J \diamond J & & \xrightarrow[q_J]{} & & J
\end{array} \tag{45}$$

4.11. Definition. *[1, Def 6.51] A* **duoidal transformation** $\alpha : (F, p^F_I, p^F_\diamond, q^F_J, q^F_\star) \Rightarrow (G, p^G_I, p^G_\diamond, q^G_J, q^G_\star)$ *is a natural transformation* $\alpha : F \to G$ *such that*

- $\alpha : (F, p^F_I, p^F_\diamond) \Rightarrow (G, p^G_I, p^G_\diamond)$ *is a monoidal transformation and*
- $\alpha : (F, q^F_J, q^F_\star) \Rightarrow (G, q^G_J, q^G_\star)$ *is a monoidal transformation.*

4.12. Proposition. *[1, Prop 6.52] There is a 2-category of duoidal categories, bilax duoidal functors and duoidal transformations, denoted* **DUO***.*

Just as a monoid in a monoidal category $\mathcal{X}$ can be equivalently defined as a monoidal functor from the terminal category to $\mathcal{X}$, Aguiar and Mahajan define two types of "monoid-like" structures in a duoidal category based upon their two type of functors: duoidal bimonoids and duoidal double monoids. Once more, we only introduce the former.

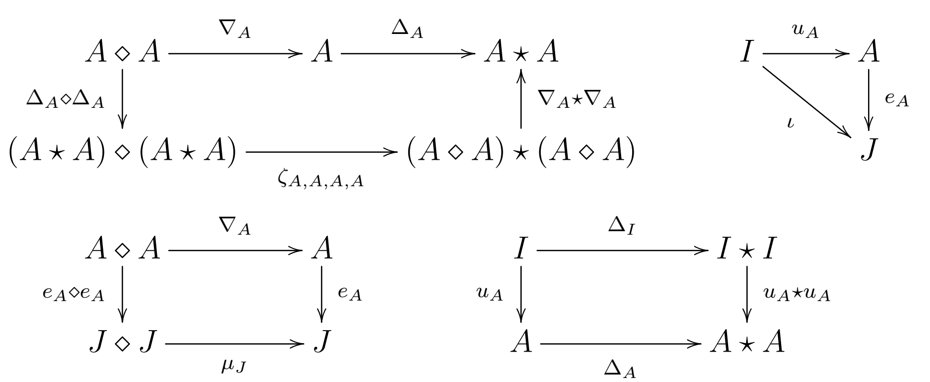
4.13. Definition. *[1, Def 6.25]*

- *A* **duoidal bimonoid** *in duoidal category* $\mathcal{X}$ *is a quintuple* $\langle A, \nabla_A, u_A, \Delta_A, e_A\rangle$ *of an object* A *in* $\mathcal{X}$ *equipped with four morphisms*

$$\nabla_A : A \diamond A \to A \qquad u_A : I \to A \qquad \Delta_A : A \to A \star A \qquad e_A : A \to J$$

  - $\langle A, \nabla_A, u_A\rangle$ *is a* $\diamond$*-monoid,*
  - $\langle A, \Delta_A, e_A\rangle$ *is a* $\star$*-comonoid*

  such that the two structures are compatible (46).



(46)

- *A* morphism of duoidal bimonoids *is a morphism of the underlying* $\diamond$*-monoids and* $\star$*-comonoid.*

4.14. Proposition. *[1, Cor 6.53] A bilax duoidal functor preserves duoidal bimonoids and morphisms of duoidal bimonoids.*

4.15. Cartesian and Cocartesian Duoidal Categories.

As previously stated, duoidal structure arises canonically whenever a monoidal category has finite products or finite coproducts. We detail the construction here.

4.16. Proposition. *[1, Ex 6.19]*

1. *Consider a monoidal category* $(\mathcal{X}, \oslash, I)$ *with finite products, then* $(\mathcal{X}, \oslash, I, \times, \mathbf{1})$ *is a duoidal category with structure maps*
$$\Delta_I = \langle 1_I, 1_I \rangle : I \to I \times I \qquad \mu_{\mathbf{1}} = t_{\mathbf{1}\oslash\mathbf{1}} : \mathbf{1} \oslash \mathbf{1} \to \mathbf{1} \qquad \iota = t_I : I \to \mathbf{1}$$
and interchange natural transformation
$$\zeta_{A,B,C,D} = \langle \pi^0_{A,B} \oslash \pi^0_{C,D}, \pi^1_{A,B} \oslash \pi^1_{C,D} \rangle : (A \times B) \oslash (C \times D) \to (A \oslash C) \times (B \oslash D)$$
2. *Consider a monoidal category* $(\mathcal{X}, \oslash, I)$ *with finite coproducts, then* $(\mathcal{X}, +, \mathbf{0}, \oslash, I)$ *is a duoidal category with structure maps*
$$\Delta_{\mathbf{0}} = b_{\mathbf{0}\oslash\mathbf{0}} : \mathbf{0} \to \mathbf{0} \oslash \mathbf{0} \qquad \mu_I = [1_I, 1_I] : I + I \to I \qquad \iota = b_I : \mathbf{0} \to I$$
and interchange natural transformation
$$\zeta_{A,B,C,D} = [\iota^0_{A,C} \oslash \iota^0_{B,D}, \iota^1_{A,C} \oslash \iota^1_{B,D}] : (A \oslash B) + (C \oslash D) \to (A + C) \oslash (B + D)$$
3. *Consider now a category* $\mathcal{X}$ *with finite products and coproducts, then* $(\mathcal{X}, +, \mathbf{0}, \times, \mathbf{1})$ *is of course a symmetric duoidal category with structure maps*
$$\Delta_{\mathbf{0}} = b_{\mathbf{0}\times\mathbf{0}} = \langle 1_{\mathbf{0}}, 1_{\mathbf{0}} \rangle : \mathbf{0} \to \mathbf{0} \times \mathbf{0} \qquad \mu_{\mathbf{1}} = t_{\mathbf{1}+\mathbf{1}} = [1_{\mathbf{1}}, 1_{\mathbf{1}}] : \mathbf{1} + \mathbf{1} \to \mathbf{1}$$
$$\iota = t_{\mathbf{0}} = b_{\mathbf{1}} : \mathbf{0} \to \mathbf{1}$$
and interchange natural transformation
$$\zeta_{A,B,C,D} = \langle \pi^0_{A,B} + \pi^0_{C,D}, \pi^1_{A,B} + \pi^1_{C,D} \rangle =$$
$$[\iota^0_{A,C} \times \iota^0_{B,D}, \iota^1_{A,C} \times \iota^1_{B,D}] : (A \times B) + (C \times D) \to (A + C) \times (B + D)$$

This idea extends to monoidal functors between monoidal categories with finite products or finite coproducts as follows.

4.17. Lemma. *Given cartesian categories* $\mathcal{X}$ *and* $\mathcal{Y}$*, any functor* $F : \mathcal{X} \to \mathcal{Y}$ *becomes a symmetric colax monoidal functor with structure maps*
$$q_{\mathbf{1}} = t_{F(\mathbf{1})} : F(\mathbf{1}) \to \mathbf{1} \qquad q_{\times A,B} = \langle F(\pi^0_{A,B}), F(\pi^1_{A,B}) \rangle : F(A \times B) \to F(A) \times F(B)$$
Further, any natural transformation $\alpha : F \Rightarrow G$ *becomes a monoidal transformation between the induced symmetric colax monoidal functors.*

Dually, given cocoartesian categories $\mathcal{X}$ *and* $\mathcal{Y}$*, any functor* $G : \mathcal{X} \to \mathcal{Y}$ *becomes a symmetric lax monoidal functor with structure maps*
$$p_{\mathbf{0}} = b_{F(\mathbf{0})} : \mathbf{0} \to F(\mathbf{0}) \qquad p_{+A,B} = [F(\iota^0_{A,B}), F(\iota^1_{A,B})] : F(A) + F(B) \to F(A + B)$$
Any natural transformation $\alpha : F \Rightarrow G$ *becomes a monoidal transformation between the induced symmetric lax monoidal functors.*

4.18. PROPOSITION. *[1, Ex 6.68]*

1. *Consider duoidal categories* $(\mathcal{X}, \oslash, I, \times, \mathbf{1})$ *and* $(\mathcal{Y}, \oslash, I, \times, \mathbf{1})$ *whose second monoidal structures are cartesian as described above. Then, consider a lax monoidal functor* $(F, p_I, p_\oslash) : (\mathcal{X}, \oslash, I) \to (\mathcal{Y}, \oslash, I)$*. By the above Lemma,* $(F, q_\mathbf{1}, q_\times) : (\mathcal{X}, \times, \mathbf{1}) \to (\mathcal{Y}, \times, \mathbf{1})$ *is a colax monoidal functor. This further determines a bilax duoidal functor*
$$(F, p_I, p_\oslash, q_\mathbf{1}, q_\times) : (\mathcal{X}, \oslash, I, \times, \mathbf{1}) \to (\mathcal{Y}, \oslash, I, \times, \mathbf{1})$$
2. *Consider duoidal categories* $(\mathcal{X}, +, \mathbf{0}, \oslash, I)$ *and* $(\mathcal{Y}, +, \mathbf{0}, \oslash, I)$ *whose first monoidal structures are cocartesian. Then, any colax monoidal functor* $(F, q_I, q_\oslash) : (\mathcal{X}, \oslash, I) \to (\mathcal{Y}, \oslash, I)$ *is equally a lax monoidal functor* $(F, p_\mathbf{0}, p_+) : (\mathcal{X}, +, \mathbf{0}) \to (\mathcal{Y}, +, \mathbf{0})$ *by the above Lemma. This determines a bilax duoidal functor*
$$(F, p_\mathbf{0}, p_+, q_I, q_\oslash) : (\mathcal{X}, +, \mathbf{0}, \oslash, I) \to (\mathcal{Y}, +, \mathbf{0}, \oslash, I)$$
3. *If we now consider a categories* $\mathcal{X}$ *and* $\mathcal{Y}$ *with finite products and coproducts, then every functor* $F : \mathcal{X} \to \mathcal{Y}$ *is canonically a bilax duoidal functor.*

4.19. REMARK. Given colax monoidal functors $(F, q^F_\mathbf{1}, q^F_\times) : (\mathcal{X}, \times, \mathbf{1}) \to (\mathcal{Y}, \times, \mathbf{1})$ induced as described above, the standard composition of colax monoidal functors is equal to the induced colax monoidal structure of the composed functor $F; G$. Similarly, for the lax monoidal functors described above.

5. Medial Linearly Distributive Categories

We can immediately provide a characterization of cartesian LDCs $(\mathbb{X}, \times, \mathbf{1}, +, \mathbf{0})$ by recognizing that $(\mathbb{X}, \times, \mathbf{1})$ is a cartesian monoidal structure and $(\mathbb{X}, +, \mathbf{0})$ is a cocartesian monoidal structure, therefore Corollary 2.21 applies to the former and its dual applies to the latter.

5.1. PROPOSITION. *A symmetric linearly distributive category* $\mathbb{X}$ *is cartesian if and only if there are natural transformations*
$$\Delta_A : A \to A \otimes A \qquad e_A : A \to \top \qquad \nabla_A : A \oplus A \to A \qquad u_A : \bot \to A$$
such that, $\forall A, B \in \mathbb{X}$,

- $\langle A, \Delta_A, e_A \rangle$ *determines a cocommutative* $\otimes$*-comonoid,*
- $\langle A, \nabla_A, u_A \rangle$ *determines a commutative* $\oplus$*-monoid, and*

$$\Delta_{A\otimes B} = (\Delta_A \otimes \Delta_B); \tau^{\otimes}_{A,A,B,B} \qquad e_{A\otimes B} = (e_A \otimes e_B); u^{R^{-1}}_{\otimes\top}$$
$$\nabla_{A\oplus B} = \tau^{\oplus}_{A,B,A,B}; (\nabla_A \oplus \nabla_B) \qquad u_{A\oplus B} = u^{R^{-1}}_{\oplus\bot}; (u_A \oplus u_B)$$

$$\Delta_\top = u^R_{\otimes\top} \qquad e_\top = 1_\top \qquad \nabla_\bot = u^R_{\oplus\bot} \qquad u_\bot = 1_\bot$$

The natural transformations of course have component morphisms for all objects in the symmetric LDC. Therefore, there will be maps

$$e_\bot, u_\top : \bot \to \top \qquad u_{\bot\otimes\bot}, \Delta_\bot : \bot \to \bot \otimes \bot \qquad e_{\top\oplus\top}, \nabla_\top : \top \oplus \top \to \top$$

and, given $A, B \in \mathbb{X}$, there will be a maps

$$\Delta_{A\oplus B} : A \oplus B \to (A \oplus B) \otimes (A \oplus B) \qquad \nabla_{A\otimes B} : (A \otimes B) \oplus (A \otimes B) \to A \otimes B$$

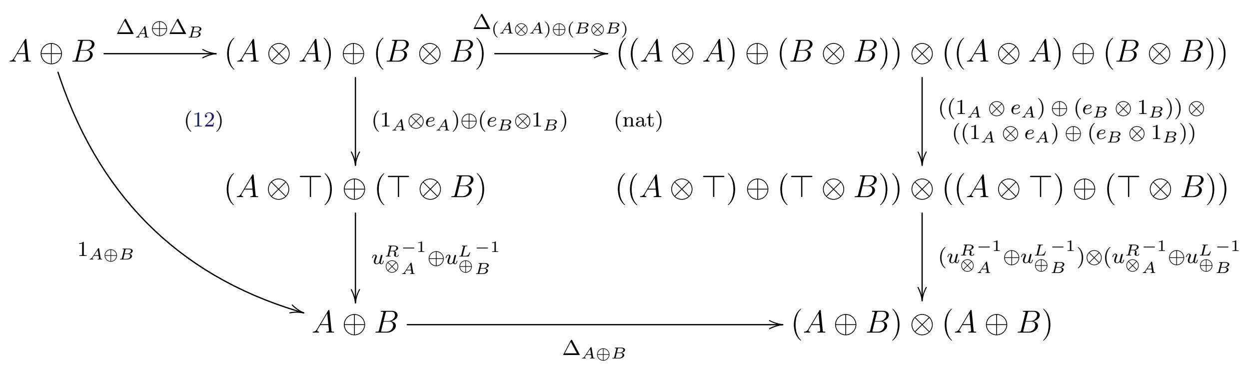
The conditions of Proposition 5.1 give us additional information about these maps. Indeed, by naturality of e and equation $e_\top = 1_\top$, we see that

$$e_\bot = u_\top \qquad e_{\top\oplus\top} = \nabla_\top \qquad e_{A\oplus B} = (e_A \oplus e_B); \nabla_\top$$

Similarly, by naturality of u and equation $u_\bot = 1_\bot$,

$$u_\top = e_\bot \qquad u_{\bot\otimes\bot} = \Delta_\bot \qquad u_{A\otimes B} = \Delta_\bot; (u_A \otimes u_B)$$

Furthermore, we can determine a formula for diagonal morphism $\Delta_{A\otimes B}$:



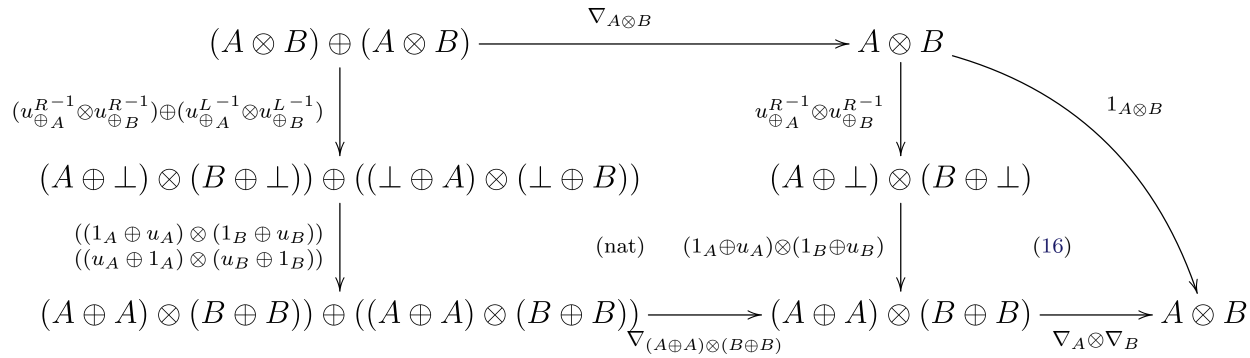
Therefore,

$$\Delta_{A\oplus B} = A \oplus B \xrightarrow{\Delta_A \oplus \Delta_B} (A \otimes A) \oplus (B \otimes B) \xrightarrow{\mu^0_{A,A,B,B}} (A \oplus B) \otimes (A \oplus B)$$

for some natural transformation

$$\mu^0_{A,B,C,D} : (A \otimes B) \oplus (C \otimes D) \to (A \oplus C) \otimes (B \oplus D)$$
$$= \Delta_{(A\otimes B)\oplus(C\otimes D)}; (((1_A \otimes e_B); u^{R^{-1}}_{\otimes A} \oplus (1_C \otimes e_D); u^{R^{-1}}_{\otimes C}) \otimes ((e_A \otimes 1_B); u^{L^{-1}}_{\otimes B} \oplus (e_C \otimes 1_D); u^{L^{-1}}_{\otimes D}))$$

Similarly, by naturality of ∇ and monoid coherence conditions (16),



This determines that

$$\nabla_{A\otimes B}=(A\otimes B)\oplus(A\otimes B)\xrightarrow{\mu^1_{A,B,A,B}}(A\oplus A)\otimes(B\oplus B)\xrightarrow{\nabla_A\otimes\nabla_B}A\otimes B$$

where $\mu^1_{A,B,A,B}$ is the natural transformation

$$\mu^1_{A,B,C,D}:(A\otimes B)\oplus(C\otimes D)\to(A\oplus C)\otimes(B\oplus D)$$
$$=((u_{\oplus A}^{R-1};(1_A\otimes e_B)\oplus u_{\oplus C}^{R-1};(1_C\otimes e_D))\otimes(u_{\oplus B}^{L-1};(e_A\otimes 1_B)\oplus u_{\oplus D}^{L-1};(e_C\otimes 1_D));\nabla_{(A\oplus C)\otimes(B\oplus D)}$$

We can then demonstrate that $\mu^0_{A,B,C,D}=\mu^1_{A,B,C,D}$, though the detailed computations are omitted here due to their length.

5.2. Corollary. *Consider a symmetric linearly distributive category $\mathbb{X}$ satisfying the conditions of Proposition 5.1, then there are maps*

$$m:\bot\to\top\qquad\Delta_\bot:\bot\to\bot\otimes\bot\qquad\nabla_\top:\top\oplus\top\to\top$$

and a natural transformation

$$\mu:(\otimes\times\otimes);\oplus\Rightarrow s_{\mathbb{X},\mathbb{X},\mathbb{X},\mathbb{X}};(\oplus\times\oplus);\otimes\qquad\mu_{A,B,C,D}:(A\otimes B)\oplus(C\otimes D)\to(A\oplus C)\otimes(B\oplus D)$$

such that

$$\Delta_{A\oplus B}=(\Delta_A\oplus\Delta_B);\mu_{A,A,B,B}\qquad e_{A\oplus B}=(e_A\oplus e_B);\Delta_\bot$$
$$\nabla_{A\otimes B}=\mu_{A,B,A,B};(\nabla_A\otimes\nabla_B)\qquad u_{A\otimes B}=\nabla_\top;(u_A\otimes u_B)$$
$$e_\bot=u_\top=m$$

The maps indicated above will be central to our main definition and therefore we take the time to discuss their meaning in the following section.

5.3. Medial Rule and Main Definition.

Considering that linearly distributive categories provide the categorical semantics for multiplicative linear logic, it is important to examine the logical significance of the map $(A \otimes B) \oplus (C \otimes D) \to (A \oplus C) \otimes (B \oplus D)$. If we consider, for example, Boolean logic and take $\otimes = \wedge$ and $\oplus = \vee$, then $(A \wedge B) \vee (C \wedge D) \vdash (A \vee C) \wedge (B \vee D)$ is a valid implication.

This implication appears prominently as an inference rule in multiple deductive systems based on deep inference. Deep inference, more specifically the calculus of structures, is a proof formalism first developed by Guglielmi to generalize Gentzen's sequent calculus, inspired by key concepts from linear logic [25].

This rule was first introduced as an inference rule by Brünnler and Tiu [10] for a local system of classical logic, SKS, and was named the **medial rule**. In this context, locality refers to the idea that rules do not require a global view of formulae of potentially unbounded size, such as contraction within Gentzen's system for classical logic [9]. Locality in SKS is achieved by ensuring that non-local rules, including contraction, are presented in atomic form. However, the general form of contraction must still be admissible in SKS, which necessitates the medial rule. The medial rule also appears in Straßburger's local system of linear logic, SLLS, to ensure that contraction and co-contraction remain atomic [43], and similarly in Tiu's local system for intuitionistic logic, SISgq [46]. Following its introduction in deep inference, the medial rule has also been studied as a term-rewriting rule, both in isolation [45] and alongside the linear distributivity and mix rules [19, 11].

In categorical logic, the medial rule has been examined in the study of categorical semantics for classical logic. Unlike intuitionistic logic, which is modeled by cartesian closed categories, and linear logic, which is modeled by $*$-autonomous or linearly distributive categories, the categorical semantics for classical logic remain unresolved. A naive definition of cartesian $*$-autonomous categories leads only to Boolean algebras as models due to Joyal's paradox.

There has been successful work extending the axioms of cartesian closed categories to capture classical logic. Proof-theoretically, Parigot developed $\lambda\mu$-calculus to extend the "proofs-as-programs" paradigm to classical proofs [37] and Girard's sequent calculus LC, an improvement of Gentzen's classical logic system LK, which is based on the concept of polarity of formulas [24]. BY developing polarized proof-nets, Laurent demonstrated that both these accounts are in fact equivalent [33]. Categorical semantics were developed by Hofmann and Streicher's continuation models [27], and then further generalized by Selinger's control categories [41]. However, by keeping to cartesian closed categories as the framework, the full symmetry of classical logic must be abandoned. Consequently, the categorical semantics for classical logic remains an active area of research.

The medial rule plays a key role in Straßburger's Boolean categories [44] and in

Lamarche's $*$-autonomous categories with finitary medial and the absorption law [31]. In both cases, the authors begin with $*$-autonomous categories and introduce medial maps alongside the $\bot$ contraction map $\bot \to \bot \otimes \bot$ and the $\top$ co-contraction map $\top \oplus \top \to \top$ to construct categorical models of classical logic. Furthermore, in both constructions, the categories include the nullary mix map $\bot \to \top$. In particular, Lamarche's categories, which will be shown to be related to our main definition in Section 5.11, were a major inspiration for the present work.

From a categorical perspective, the medial map can be recognized as an instance of the well-known interchange law in duoidal categories, taking $\diamond = \oplus$, $J = \bot$, $\star = \oplus$ and $I = \bot$. When combined with $\bot$ contraction, $\top$ co-contraction and the nullary mix map $\bot \to \top$, they provide all the necessary structures maps for a duoidal category, beyond the standard maps for monoidal structures.

To establish a linearly distributive Fox theorem, we cannot begin with the 2-category of symmetric LDCs alone. Instead, we must consider symmetric LDCs that include additional medial maps, including nullary mix, $\bot$ contraction and $\top$ co-contraction. This necessity aligns with the role of the medial rule in SLLS, where it enables the atomic contraction and co-contraction rules to extend to all formulae. Therefore, we must first define what it means for a linearly distributive category to have coherent medial maps.

5.4. Definition. *A* **medial linearly distributive category***, or a MLDC,* $(\mathbb{X}, \otimes, \top, \oplus, \bot)$ *consists of:*

- *a category* $(\mathbb{X}, ;, 1_A)$,
- *a* **tensor** *monoidal structure* $(\mathbb{X}, \otimes, \top)$.
- *a* **par** *monoidal structure* $(\mathbb{X}, \oplus, \bot)$,
- $\bot$**-contraction***,* $\top$**-cocontraction** *and* **nullary mix** *morphisms,*

$$\Delta_\bot : \bot \to \bot \otimes \bot \qquad \nabla_\top : \top \oplus \top \to \top \qquad m : \bot \to \top$$

- *a* **medial** *natural transformation, and*

$$\mu : (\otimes\times\otimes); \oplus \Rightarrow s_{\mathbb{X},\mathbb{X},\mathbb{X},\mathbb{X}}; (\oplus\times\oplus); \otimes \qquad \mu_{A,B,C,D} : (A\otimes B)\oplus(C\otimes D) \to (A\oplus C)\otimes(B\oplus D)$$

- *left and right* **linear distributivity** *natural transformations*

$$\delta^R : (\oplus \times 1_{\mathbb{X}}); \otimes \Rightarrow a_{\mathbb{X},\mathbb{X},\mathbb{X}}; (1_{\mathbb{X}} \times \otimes); \oplus \quad \delta^R_{A,B,C} : (A \oplus B) \otimes C \to A \oplus (B \otimes C),$$
$$\delta^L : (1_{\mathbb{X}} \times \oplus); \otimes \Rightarrow a_{\mathbb{X},\mathbb{X},\mathbb{X}}; (\otimes \times 1_{\mathbb{X}}); \oplus \quad \delta^L_{A,B,C} : A \otimes (B \oplus C) \to (A \otimes B) \oplus C,$$

such that

- $(\mathbb{X}, \otimes, \top, \oplus, \bot)$ *is a mix linearly distributive category,*
- $(\mathbb{X}, \oplus, \bot, \otimes, \top)$ *is a duoidal category, and*
- *the* **medial maps** *interact coherently with the linear distributivities* (47).

$$\begin{array}{ccc}
((A \otimes B) \oplus (C \otimes D)) \otimes X & \xrightarrow{\mu_{A,B,C,D} \otimes 1_X} & ((A \oplus C) \otimes (B \oplus D)) \otimes X \\
\downarrow{\scriptstyle \delta^R_{A \otimes B, C \otimes D, X}} & & \downarrow{\scriptstyle \alpha_{\otimes A \oplus C, B \oplus D, X}} \\
(A \otimes B) \oplus ((C \otimes D) \otimes X) & & (A \oplus C) \otimes ((B \oplus D) \otimes X) \\
\downarrow{\scriptstyle 1_{A \otimes B} \oplus \alpha_{\otimes C,D,X}} & & \downarrow{\scriptstyle 1_{A \oplus C} \otimes \delta^R_{B,D,X}} \\
(A \otimes B) \oplus (C \otimes (D \otimes X)) & \xrightarrow[\mu_{A,B,C,D \otimes X}]{} & (A \oplus C) \otimes (B \oplus (D \otimes X))
\end{array} \tag{47}$$

$$(\mu_{A,B,C,D} \otimes 1_X);\alpha_{\otimes A \oplus C, B \oplus D, X}; (1_{A \oplus C} \otimes \delta^R_{B,D,X}) = \delta^R_{A \otimes B, C \otimes D, X}; (1_{A \otimes B} \oplus \alpha_{\otimes C,D,X}); \mu_{A,B,C,D \otimes X}$$
$$(1_X \otimes \mu_{A,B,C,D});\alpha_{\otimes}{}^{-1}_{X, A \oplus C, B \oplus D}; (\delta^L_{X,A,C} \otimes 1_{B \oplus D}) = \delta^L_{X, A \otimes B, C \otimes D}; (\alpha_{\otimes}{}^{-1}_{X,A,B} \oplus 1_{C \otimes D}); \mu_{X \otimes A, B, C, D}$$
$$(\delta^R_{X,A,B} \oplus 1_{C \otimes D});\alpha_{\oplus}{}^{-1}_{X, A \otimes B, C \otimes D}; (1_X \oplus \mu_{A,B,C,D}) = \mu_{X \oplus A, B, C, D}; (\alpha_{\oplus}{}^{-1}_{X,A,C} \otimes 1_{B \oplus D}); \delta^R_{X, A \oplus C, B \oplus D}$$
$$(1_{A \otimes B} \oplus \delta^L_{C,D,X});\alpha_{\oplus A \otimes B, C \otimes D, X}; (\mu_{A,B,C,D} \oplus 1_X) = \mu_{A,B,C,D \oplus X}; (1_{A \oplus C} \otimes \alpha_{\oplus B,D,X}); \delta^L_{A \oplus C, B \oplus D, X}$$

5.5. Remark. In the above definition, $(\mathbb{X}, \oplus, \bot, \otimes, \top)$ being a duoidal category corresponds to

- $(\bot, \Delta_\bot, m)$ is a $\otimes$-comonoid and $(\top, \nabla_\top, m)$ is a $\oplus$-monoid (48),
- the medial maps interact coherently with the associators (49) and with the unitors (50).

$$\begin{array}{ll}
\Delta_\bot; (1_\bot \otimes \Delta_\bot) = \Delta_\bot; (\Delta_\bot \otimes 1_\bot); \alpha_{\otimes \bot,\bot,\bot} & \\
(1_\top \oplus \nabla_\top); \nabla_\top = \alpha_{\oplus \top,\top,\top}; (\nabla_\top \oplus 1_\top); \nabla_\top & \\
& \\
\Delta_\bot; (1_\bot \otimes m) = u^R_{\otimes \bot} & (1_\top \oplus m); \nabla_\top = u^R_{\oplus \top} \\
\Delta_\bot; (m \otimes 1_\bot) = u^L_{\otimes \bot} & (m \oplus 1_\top); \nabla_\top = u^L_{\oplus \top}
\end{array} \tag{48}$$

$$\begin{array}{l}
\mu_{A \otimes B, C, D \otimes E, F}; (\mu_{A,B,D,E} \otimes 1_{C \oplus F}); \alpha_{\otimes A \oplus D, B \oplus E, C \oplus F} \\
\quad = (\alpha_{\otimes A,B,C} \oplus \alpha_{\otimes D,E,F}); \mu_{A, B \otimes C, D, E \otimes F}; (1_{A \oplus D} \otimes \mu_{B,C,E,F}) \\
\\
\alpha_{\oplus A \otimes B, C \otimes D, E \otimes F}; (\mu_{A,B,C,D} \oplus 1_{E \otimes F}); \mu_{A \oplus C, B \oplus D, E, F} \\
\quad = (1_{A \otimes B} \oplus \mu_{C,D,E,F}); \mu_{A,B,C \oplus E, D \oplus F}; (\alpha_{\oplus A,C,E} \otimes \alpha_{\oplus B,D,F})
\end{array} \tag{49}$$

$$
\begin{aligned}
&(\Delta_\bot \oplus 1_{A\otimes B}); \mu_{\bot,\bot,A,B}; (u^L_{\oplus A} \otimes u^L_{\oplus B}) = u^L_{\oplus A\otimes B}\\
&(1_{A\otimes B} \oplus \Delta_\bot); \mu_{A,B,\bot,\bot}; (u^R_{\oplus A} \otimes u^R_{\oplus B}) = u^R_{\oplus A\otimes B}\\
&(u^L_{\otimes A} \oplus u^L_{\otimes B}); \mu_{\top,A,\top,B}; (\nabla_\top \otimes 1_{A\oplus B}) = u^L_{\otimes A\oplus B}\\
&(u^R_{\otimes A} \oplus u^R_{\otimes B}); \mu_{A,\top,B,\top}; (1_{A\oplus B} \otimes \nabla_\top) = u^R_{\otimes A\oplus B}
\end{aligned} \tag{50}
$$

We will refrain from giving examples at this point and first continue developing the theory of MLDCs. Examples will be the focus of Section 5.15.

As discussed in the introduction, MLDCs provide a suitable framework for categories that are both independently linearly distributive and duoidal in a coherent manner. We use the term medial to emphasize that, while certain duoidal categories can inherit a linearly distributive structure from their interchange maps (this construction will be reviewed in Section 5.22), this is not what we intend to axiomatize. We are concerned with categories whose the duoidal and linearly distributive structures do not imply one another, yet nonetheless interact coherently. This perspective will be particularly useful, as it allows us to leverage the well-developed theory of duoidal categories, which was be introduced in Section 4.

Alternatively, MLDCs can be viewed as providing categorical semantics for the logical system MLL+Medial, an extension of multiplicative linear logic with MIX, enriched by the addition of the medial rule. This variant may be incompatible with standard sequent calculus, as the medial rule is rather unconventional: its premise is a disjunction of formulae, while its conclusion is a conjunction. However, MLL+Medial is likely expressible within the calculus of structures, where linear distributivity and medial maps play a central role in several systems within the deep inference paradigm. Overall, the logical study of MLL+Medial remains an open area for future research.

5.6. Remark. This is perhaps a similar situation to that of pomset logic and BV. Pomset logic is a non-commutative variant of MLL which includes a non-commutative connective $\lhd$ introduced by Retoré, motivated by Girard's categorical model for linear logic, coherence spaces, and the study of proof nets [39]. While pomset logic was expressible with proof nets, it did not seem to be properly expressible in a sequent calculus. In an attempt to better understand the situation, Guglielmi determined it could not be done and that lead to the development to of the calculus of structures, as an alternative to sequent calculus based on the idea of deep inference, along with the logic BV [25]. It was conjectured in said paper that BV and pomset logic were one and the same. It was determined that, like pomset logic, BV was inexpressible in the sequent calculus [47]. Recent work has conclusively showed that these two logics are in fact different, while detailing exactly why there are difficulties expressing these logics in sequent calculus [36].

5.7. Symmetry.

If we revisit Proposition 5.1, which provides a preliminary characterization of CLDCs, we observe that for any objects A and B in a CLDC, the diagonal $\Delta_{A\otimes B}$ and the multiplication $\nabla_{A\oplus B}$ are constructed using the canonical flip maps associated with the symmetric $\otimes$ and $\oplus$ monoidal structures. By the same reasoning that led us to define MLDCs to ensure the presence of the necessary medial maps, we now define symmetric MLDCs to guarantee the existence of these canonical flip maps.

5.8. Definition. *A medial linear distributive category is* **symmetric***, or a SMLDC, if* $(\mathbb{X}, \otimes, \top)$ *and* $(\mathbb{X}, \oplus, \bot)$ *are symmetric monoidal categories such that* (22)*,* (40) *and* (41) *hold, meaning* $(\mathbb{X}, \otimes, \top, \oplus, \bot)$ *is a symmetric linearly distributive category and* $(\mathbb{X}, \oplus, \bot, \otimes, \top)$ *is a symmetric duoidal category.*

There is however an equivalent definition for SMLDCs which does not introduce the nullary mix map $m : \bot \to \top$ nor asks for it to be mix. Indeed, in the presence of the braidings, we can define it in terms of the other structure maps and get the mix structure.

It is well-known that given a duoidal category $(\mathbb{X}, \oplus, \bot, \otimes, \top)$, $m : \bot \to \top$ is defined by the rest of the structure.

5.9. Proposition. *[1, Prop 6.9] The structure map* $m : \bot \to \top$ *is equal to*

$$\begin{array}{c}
\bot \xrightarrow{u_{\oplus\bot}^{-1}} \bot\oplus\bot \xrightarrow{u^R_{\otimes\bot}\oplus u^L_{\otimes\bot}} (\bot\otimes\top)\oplus(\top\otimes\bot) \xrightarrow{\mu_{\bot,\top,\top,\bot}} (\bot\oplus\top)\otimes(\top\oplus\bot) \xrightarrow{u^L_{\oplus\top}\otimes u^R_{\oplus\top}} \top\otimes\top \xrightarrow{u_{\otimes\top}^{-1}} \top \\
\bot \xrightarrow{u_{\oplus\bot}^{-1}} \bot\oplus\bot \xrightarrow{u^L_{\otimes\bot}\oplus u^R_{\otimes\bot}} (\top\otimes\bot)\oplus(\bot\otimes\top) \xrightarrow{\mu_{\top,\bot,\bot,\top}} (\top\oplus\bot)\otimes(\bot\oplus\top) \xrightarrow{u^R_{\oplus\top}\otimes u^L_{\oplus\top}} \top\otimes\top \xrightarrow{u_{\otimes\top}^{-1}} \top
\end{array} \tag{51}$$

The proof that both these composites are equivalent, and themselves equal to m follows from the conditions that $(\top, \nabla_\top, m)$ is a $\oplus$-monoid and $(\bot, \Delta_\bot, m)$ is a $\otimes$-comonoid. Therefore, the definition of an arbitrary duoidal category must include the nullary mix map.

However, in the presence of braidings, it can be excluded. It was first noticed by Lamarche in Proposition 2.11 of [31] that, given a category with two symmetric monoidal structures $(\mathbb{X}, \otimes, \top)$ and $(\mathbb{X}, \oplus, \bot)$, and a coherent medial natural transformation $\mu_{A,B,C,D}$, i.e. (41) holds, the above composites are equal.

Further, in the presence of $\bot$-contraction $\Delta_\bot : \bot \to \bot \otimes \bot$ such that $\bot$ is a cocommutative $\otimes$-cosemigroup and $\top$-cocontraction map $\nabla_\top : \top \oplus \top \to \top$ such that $\top$ with commutative $\oplus$-semigroup, which interact coherently with the medial maps, i.e. (39) holds, then the equivalent composites $\bot \to \top$ further endow $\bot$ and $\top$ with $\otimes$-comonoid and $\oplus$-monoid structures respectively, as shown in Proposition 2.19 [31].

Then, as $\bot$ is endowed with a $\otimes$-comonoid structure and $\top$ is endowed with a $\oplus$-monoid structure in a symmetric LDC, Fuhrmann and Pym's theorem in [22] implies their counit and unit, which are equal in this instances, induce a mix structure.

Altogether, these observations allows us to provide an alternative definition of SMLDCs which excludes the nullary mix map and defines it entirely based on the rest of the structure:

5.10. Proposition. *A symmetric medial linearly distributive category* $(\mathbb{X}, \otimes, \top, \oplus, \bot)$ *is equivalently defined as a symmetric linearly distributive category, equipped with*

- $\bot$*-contraction and* $\top$*-cocontraction morphisms,*

$$\Delta_\bot : \bot \to \bot \otimes \bot \qquad \nabla_\top : \top \oplus \top \to \top$$

- *a medial natural transformation, and*

$$\mu : (\otimes \times \otimes); \oplus \Rightarrow s_{\mathbb{X},\mathbb{X},\mathbb{X},\mathbb{X}}; (\oplus \times \oplus); \otimes \qquad \mu_{A,B,C,D} : (A \otimes B) \oplus (C \otimes D) \to (A \oplus C) \otimes (B \oplus D)$$

such that

- $(\bot, \Delta_\bot)$ *is a cocommutative* $\otimes$*-cosemigroup and* $(\top, \nabla_\top)$ *is commutative* $\oplus$*-semigroup* (52)*,*
- *the medial maps interact coherently with the braidings* (53)*, the associators* (54) *and with the unitors* (56)*,*
- *the medial maps interact coherently with the linear distributivities* (56)*, and*
- *the medial maps, the linear distributivities and the braidings interact coherently together* (57)*.*

$$\begin{array}{ll} \Delta_\bot = \Delta_\bot; \sigma_{\otimes \bot,\bot} & \Delta_\bot; (1_\bot \otimes \Delta_\bot) = \Delta_\bot; (\Delta_\bot \otimes 1_\bot); \alpha_{\otimes \bot,\bot,\bot} \\ \nabla_\top = \sigma_{\oplus \top,\top}; \nabla_\top & (1_\top \oplus \nabla_\top); \nabla_\top = \alpha_{\oplus \top,\top,\top}; (\nabla_\top \oplus 1_\top); \nabla_\top \end{array} \tag{52}$$

$$\begin{array}{l} \mu_{A,B,C,D}; \sigma_{\otimes A \oplus C, B \oplus D} = (\sigma_{\otimes A,B} \oplus \sigma_{\otimes C,D}); \mu_{B,A,D,C} \\ \mu_{A,B,C,D}; (\sigma_{\oplus A,C} \otimes \sigma_{\oplus B,D}) = \sigma_{\oplus A \otimes B, C \otimes D}; \mu_{C,D,A,B} \end{array} \tag{53}$$

$$\mu_{A\otimes B,C,D\otimes E,F};(\mu_{A,B,D,E}\otimes 1_{C\oplus F});\alpha_{\otimes A\oplus D,B\oplus E,C\oplus F} = (\alpha_{\otimes A,B,C}\oplus\alpha_{\otimes D,E,F});\mu_{A,B\otimes C,D,E\otimes F};(1_{A\oplus D}\otimes\mu_{B,C,E,F})$$

$$\alpha_{\oplus A\otimes B,C\otimes D,E\otimes F};(\mu_{A,B,C,D}\oplus 1_{E\otimes F});\mu_{A\oplus C,B\oplus D,E,F} = (1_{A\otimes B}\oplus\mu_{C,D,E,F});\mu_{A,B,C\oplus E,D\oplus F};(\alpha_{\oplus A,C,E}\otimes\alpha_{\oplus B,D,F}) \tag{54}$$

$$\begin{aligned}(1_{A\otimes B}\oplus\Delta_\perp);\mu_{A,B,\perp,\perp};(u^R_{\oplus A}\otimes u^R_{\oplus B}) &= u^R_{\oplus A\otimes B}\\ (u^R_{\otimes A}\oplus u^R_{\otimes B});\mu_{A,\top,B,\top};(1_{A\oplus B}\otimes\nabla_\top) &= u^R_{\otimes A\oplus B}\end{aligned} \tag{55}$$

$$(1_X\otimes\mu_{A,B,C,D});\alpha_{\otimes}^{-1}{}_{X,A\oplus C,B\oplus D};(\delta^L_{X,A,C}\otimes 1_{B\oplus D}) = \delta^L_{X,A\otimes B,C\otimes D};(\alpha_{\otimes}^{-1}{}_{X,A,B}\oplus 1_{C\otimes D});\mu_{X\otimes A,B,C,D}$$

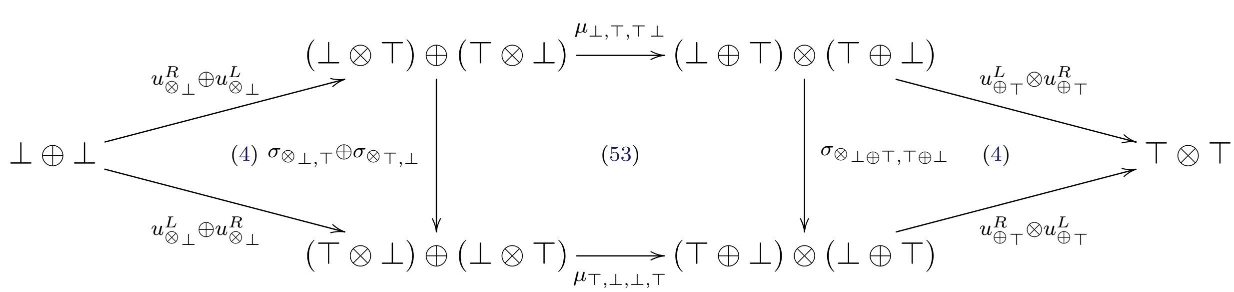
$$(1_{A\otimes B}\oplus\delta^L_{C,D,X});\alpha_{\oplus A\otimes B,C\otimes D,X};(\mu_{A,B,C,D}\oplus 1_X) = \mu_{A,B,C,D\oplus X};(1_{A\oplus C}\otimes\alpha_{\oplus B,D,X});\delta^L_{A\oplus C,B\oplus D,X} \tag{56}$$

$$(1_X\otimes\mu_{A,B,C,D});\alpha_{\otimes}^{-1}{}_{X,A\oplus C,B\oplus D};(\sigma_{\otimes X,A\oplus C}\otimes 1_{B\oplus D});\alpha_{\otimes A\oplus C,X,B\oplus D};(1_{A\oplus C}\otimes\delta^L_{X,B,D}) = \delta^L_{X,A\otimes B,C\otimes D};(\alpha_{\otimes}^{-1}{}_{X,A,B}\oplus 1_{C\otimes D});((\sigma_{\otimes X,A}\otimes 1_B)\oplus 1_{C\otimes D});(\alpha_{\otimes A,X,B}\oplus 1_{C\otimes D});\mu_{X\otimes A,B,C,D}$$

$$(\delta^L_{A,B,X}\oplus 1_{C\otimes D});\alpha_{\oplus}^{-1}{}_{A\otimes B,X,C\otimes D};(1_{A\otimes B}\oplus\sigma_{\oplus X,C\otimes D});\alpha_{\oplus A\otimes B,C\otimes D,X};(\mu_{A,B,C,D}\oplus 1_X) = \mu_{A,B\oplus X,C,D};(1_{A\oplus C}\otimes\alpha_{\oplus}^{-1}{}_{B,X,D});(1_{A\oplus C}\otimes(1_B\oplus\sigma_{\oplus X,D}));(1_{A\oplus C}\otimes\alpha_{\oplus B,D,X});\delta^L_{A\oplus C,B\oplus D,X} \tag{57}$$

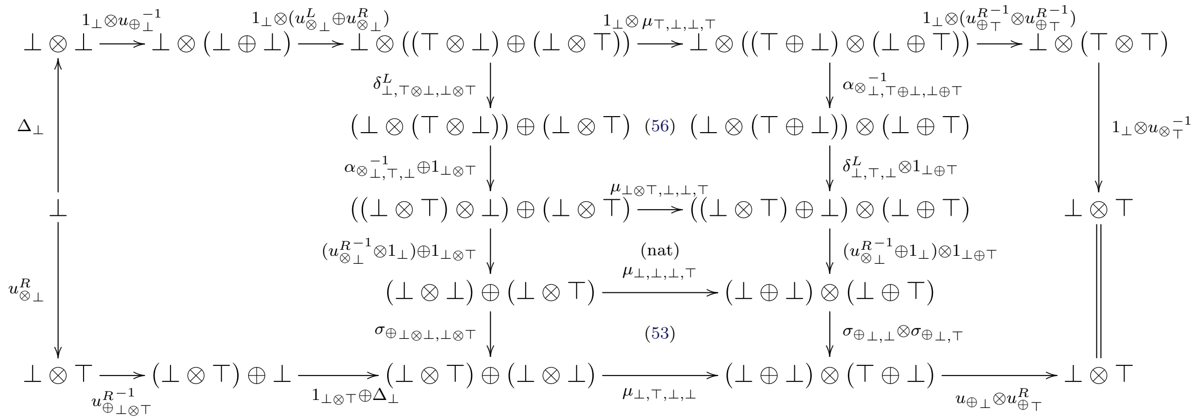
Proof. It is immediate that Definition 5.8 implies the definition presented in the Proposition, since the former is a version of the latter with more structures and conditions. Coherence condition (57) is obtained by replacing the right linear distribution in (47) by its definition as a composite of braidings and the linear distributions, and simplifying using (3) and (53).

Now, consider a SLDC $(\mathbb{X},\otimes,\top,\oplus,\perp)$ as outlined in the Proposition. Then, the following diagram commutes:



As such, we can define $m : \bot \to \top$ to be the equivalent composites in Proposition 5.9.

$(\bot, \Delta_\bot, m)$ is a cocommutative $\otimes$-comonoid as $\Delta_\bot; (1_\bot \otimes m) = u^R_{\otimes \bot}$ holds as follows:



where the left-hand and right-hand diagrams pentagons commute by the interaction between δ^L and the unitors (19), by the naturality of δ^L and by various coherence conditions for the structure maps of the symmetric monoidal categories $(\mathbb{X}, \otimes, \top)$ and $(\mathbb{X}, \oplus, \bot)$.

The upper composite in the diagram is equal to $\Delta_\bot; (1_\bot \otimes m)$, while the lower composite

$$u^R_{\otimes\bot}; u^{R-1}_{\oplus\bot\otimes\top}; (1_{\bot\otimes\top} \oplus \Delta_\bot); \mu_{\bot,\top,\bot,\bot}; (u^R_{\oplus\bot} \otimes u^R_{\oplus\top}) = u^R_{\otimes\bot}; u^{R-1}_{\oplus\bot\otimes\top}; u^R_{\oplus\bot\otimes\top} = u^R_{\otimes\bot}$$

by (55). Thus, $\Delta_\bot; (1_\bot \otimes m) = u^R_{\otimes\bot}$ as promised.

Similarly, we can show that $m : \bot \to \top$ endows $\top$ with a commutative $\oplus$-monoid structure. Then, by Theorem 3.8, the symmetric LDC is mix.

Finally, it straightforward to prove the symmetric versions of (54), (55) using the braidings. The only technical computation is showing that the coherence conditions (53), (56) and (57) imply the right linear distribution, defined as the composite of the left linear distribution and braidings, interacts coherently with the medial maps, in (47), completing the proof. ■

This alternative definition is of particular use because it involves fewer coherence conditions to check when looking at examples and it will be helpful in showing that SMLDCs with negation are equivalent to Lamarche's definition detailed in [31] in the following section.

5.11. Adding Negation.

As linear negation is central to linear logic and a component of many models of multiplicative linear logic, it was essential when developing LDCs that Cockett and Seely demonstrate how to include negation in their categorical framework to recover $*$-autonomous categories. Indeed, many of the key examples of LDCs are $*$-autonomous.

The same holds true in our extension to MLDCs. It is important we describe how the duoidal structure interacts with linear negation and recover Lamarche's definition of $*$-autonomous categories with finitary (binary and nullary) medial, and with the absorption law [31].

5.12. Definition. *A symmetric medial linearly distributive category* $(\mathbb{X}, \otimes, \top, \oplus, \bot)$ **has negation** *if*

- *it has negation as a symmetric linearly distributive category (Definition 3.13),*
- $\bot$*-contraction* $\Delta_\bot : \bot \to \bot \otimes \bot$ *and* $\top$*-cocontraction* $\nabla_\top : \top \oplus \top \to \top$ *are dual to one another* (58)*, and*
- *the medial maps are self-dual* (59)*.*

$$
\begin{array}{ccccc}
(\bot \otimes \bot)^\bot & \xleftarrow{\phi_{\bot,\bot}} & \bot^\bot \oplus \bot^\bot & \xrightarrow{\beta_\top \oplus \beta_\top} & \top \oplus \top \\
{\scriptstyle (\Delta_\bot)^\bot}\downarrow & & & & \downarrow{\scriptstyle \nabla_\top} \\
\bot^\bot & & \xrightarrow{\quad\beta_\top\quad} & & \top
\end{array}
\tag{58}
$$

$$
\begin{array}{ccc}
(A^\bot \otimes B^\bot) \oplus (C^\bot \oplus D^\bot) & \xrightarrow{\epsilon^{-1}_{B,A} \oplus \epsilon^{-1}_{D,C}} & (B \oplus A)^\bot \oplus (D \oplus C)^\bot \\
{\scriptstyle \mu_{A^\bot, B^\bot, C^\bot, D^\bot}}\downarrow & & \downarrow{\scriptstyle \phi_{B \oplus A, D \oplus C}} \\
(A^\bot \oplus C^\bot) \otimes (B^\bot \oplus D^\bot) & & ((D \oplus C) \otimes (B \oplus A))^\bot \\
{\scriptstyle \phi_{A,C} \otimes \phi_{B,D}}\downarrow & & \downarrow{\scriptstyle \mu^\bot_{D,B,C,A}} \\
(C \otimes A)^\bot \otimes (D \otimes B)^\bot & \xrightarrow[\epsilon^{-1}_{D \otimes B, C \otimes A}]{} & ((D \otimes B) \oplus (C \otimes A))^\bot
\end{array}
\tag{59}
$$

5.13. Lemma. *A symmetric medial linearly distributive category with negation* $(\mathbb{X}, \otimes, \top, \oplus, \bot)$ *has a self-dual nullary mix map* $m : \bot \to \top$*, i.e.*

$$
\begin{array}{ccc}
\bot & \xrightarrow{\chi_\top} & \top^\bot \\
{\scriptstyle m}\downarrow & & \downarrow{\scriptstyle (m)^\bot} \\
\top & \xrightarrow[\beta_\top^{-1}]{} & \bot^\bot
\end{array}
\tag{60}
$$

Proof. This follows from the definition of $m : \bot \to \top$ as a composite of unitors and the medial map, i.e. Proposition 5.9, and the fact that the medial maps are self-dual themselves. ∎

By Theorem 3.17, which states the correspondence between SLDCs with negation and $*$-autonomous categories, and by comparing Lamarche's definition to the alternative definition of a SMLDC in Proposition 5.10, it is immediate that:

5.14. PROPOSITION. *The notions of symmetric medial linearly distributive categories with negation and of with that $*$-autonomous categories with finitary medial, and with the absorption law, as defined in [31], coincide.*

We will however also be particularly invested in SMLDCs which do not have a notion of negation. This is due to the ultimate goal of providing a 2-functor which maps SMLDCs to CLDCs as part of the linearly distributive Fox theorem. If this construction preserves the negation of SMLDCs, then by Joyal's paradox, the resulting CLDCs would necessarily be posets. This will be discussed in greater detail in Section 7.1.

5.15. EXAMPLES.

The most important class example of MLDCs is of course the CLDCs, as the former were defined precisely to be an appropriate generalization of the latter.

The canonical duoidal structure of a CLDC was discussed by this author and co-author Pacaud Lemay. It is precisely the one for cartesian and co-cartesian categories as described in Proposition 4.16.

5.16. PROPOSITION. *[29, Prop 3.19] A cartesian linearly distributive category* $(\mathbb{X}, \times, \top, +, \bot)$ *is a duoidal category, with structure maps*

$$\Delta_\bot = b_{\bot\times\bot} = \langle 1_\bot, 1_\bot\rangle : \bot \to \bot \times \bot \qquad \nabla_\top = t_{\top+\top} = [1_\top, 1_\top] : \top + \top \to \top$$
$$m = t_\bot = b_\top : \bot \to \top$$

and interchange natural transformation

$$\begin{aligned}\mu_{A,B,C,D} &: (A \times B) + (C \times D) \to (A + C) \times (B + D)\\ &= \langle \pi^0_{A,B} + \pi^0_{C,D}, \pi^1_{A,B} + \pi^1_{C,D}\rangle = [\iota^0_{A,C} \times \iota^0_{B,D}, \iota^1_{A,C} \times \iota^1_{B,D}]\end{aligned}$$

Moreover, Proposition 3.20 in [29] details the how this duoidal structure interacts with the linear distributivities. The resulting commuting diagrams imply that:

5.17. THEOREM. *Every cartesian linearly distributive category* $(\mathcal{X}, \times, \mathbf{1}, +, \mathbf{0})$ *is canonically a symmetric medial linearly distributive category.*

Another immediate class of examples are the braided monoidal categories, as every monoidal category is a degenerate LDC and every braided category is a strong duoidal category.

5.18. Proposition. *Consider a braided monoidal category $(\mathcal{X}, \oslash, I)$, then it is a medial linearly distributive category $(\mathcal{X}, \oslash, I, \oslash, I)$ with*

$$\nabla_I = u_{\oslash}^{-1}{}_I : I \oslash I \to I \qquad \Delta_I = u_{\oslash I} : I \to I \oslash I \qquad m = 1_I : I \to I$$

$$\mu_{A,B,C,D} = \tau^{\oslash}_{A,B,C,D} : (A \oslash B) \oslash (C \oslash D) \to (A \oslash C) \oslash (B \oslash D)$$

$$\delta^L_{A,B,C} = \alpha_{\oslash}^{-1}{}_{A,B,C} : A \oslash (B \oslash C) \to (A \oslash B) \oslash C$$

$$\delta^R_{A,B,C} = \alpha_{\oslash A,B,C} : (A \oslash B) \oslash C \to A \oslash (B \oslash C)$$

Proof. By Example 3.18, every monoidal category is a degenerate LDC with associators for linear distributivities. By Example 4.7 and Proposition 4.8, we know ever braided monoidal category is a strong duoidal category with the canonical flip as its interchange map. Therefore, it only remains to verify (47), which follow immediately by the coherence theorem for braided monoidal categories. ∎

Our final source of examples stems from generalizing Q-coherences to the linearly distributive context, as first introduced by Lamarche [30] to capture Girard's coherence spaces and Ehrhard's hypercoherences, and later investigated in the context of $*$-autonomous categories with finitary (binary and nullary) medial, and with the absorption law [31].

Firstly, let $(\mathbb{P}, \otimes, \top, \oplus, \bot)$ be a posetal SMLDC. In other words, $\mathbb{P}$ is a poset with two commutative, associative and unital operations $\otimes$ and $\oplus$, equipped with units $\top$ and $\bot$ respectively, such that the following inequalities hold $\forall a, b, c, d \in \mathbb{P}$.

$$\top \oplus \top \leq \top \qquad \bot \leq \bot \otimes \bot \qquad \bot \leq \top$$

$$(a \otimes b) \oplus (c \otimes d) \leq (a \oplus c) \otimes (b \oplus d)$$

$$(a \oplus b) \otimes c \leq a \oplus (b \otimes c) \qquad a \otimes (b \oplus c) \leq (a \otimes b) \oplus c$$

While this may seem at first glance to be an involved definition, one can quickly see that every bounded distributive lattice is such a poset with $\otimes = \wedge$ and $\oplus = \vee$. Indeed, as proved in [29], a CLDC is a poset if and only if it is a bounded distributive lattice, and a posetal CLDC is of course a posetal SMLDC as previously discussed.

5.19. Definition. *[31, Def 4.5]*

- *A* $\mathbb{P}$-coherence $A = (|A|, \rho_A)$ *consists of a pair of a poset* $(|A|, \sqsubseteq)$ *and a symmetric monotone function* $\rho_A : |A| \times |A| \to \mathbb{P}$.
- *A* $\mathbb{P}$-coherence map $f : A \to B$ *is a relation* $f : |A| \nrightarrow |B|$ *which is*
  1. *down-closed in the source:* $(a, b) \in f \wedge a' \sqsubseteq a \implies (a', b) \in f$,
  2. *up-closed in the target:* $(a, b) \in f \wedge b \sqsubseteq b' \implies (a, b') \in f$, *and*
  3. *compatible with the monotone functions:* $(a, b) \in f \wedge (a', b') \in f \implies \rho_A(a, a') \leq \rho_B(b, b')$

5.20. DEFINITION. *[31, Def 4.5] Let* $\mathbb{P}$-Coh *be denote the following structure:*

- *Objects:* $\mathbb{P}$*-coherences* $A = (|A|, \rho_A)$
- *Morphisms:* $\mathbb{P}$*-coherence maps* $f : A \to B$
- *Composition: standard composition of relations, i.e. given* $f : A \to B$ *and* $g : B \to C$*,* $f;g : A \to C$ *is the relation* $f;g : |A| \nrightarrow |C|$ *defined as*

$$(a,c) \in f;g \iff \exists b \in B, (a,b) \in f \wedge (b,c) \in g$$

- *Identities: given* $\mathbb{P}$*-coherence* $A = (|A|, \rho_A)$*,* $1_A : A \to A$ *is defined by*

$$(a,a') \in 1_A \iff a \sqsubseteq a' \in |A|$$

- *Tensor product:* $\otimes : \mathbb{P}\text{-}\mathsf{Coh} \times \mathbb{P}\text{-}\mathsf{Coh} \to \mathbb{P}\text{-}\mathsf{Coh}$ *is defined by*

$$(A = (|A|, \rho_A), B = (|B|, \rho_B)) \mapsto A \otimes B = (|A| \times |B|, \rho_{A\otimes B})$$

$$(f : A \to A', g : B \to B') \mapsto (f \otimes g : A \otimes B \to A' \otimes B')$$

where the monotone function $\rho_{A\otimes B} : (|A| \times |B|) \times ((|A| \times |B|) \to \mathbb{P}$ *is given by*

$$\rho_{A\otimes B}((a,b),(a',b')) = \rho_A(a,a') \otimes \rho_B(b,b')$$

and where $f \otimes g$ *is given by the relation* $f \times g : |A| \times |B| \nrightarrow |A'| \times |B'|$*, i.e.*

$$((a,b),(a',b)) \in f \times g \iff (a,a') \in f \wedge (b,b') \in g$$

- *Tensor unit:* $\mathbb{P}$*-coherence* $\top = (\{*\}, \rho_\top)$ *where* $\rho_\top(*,*) = \top \in \mathbb{P}$
- *Par product:* $\oplus : \mathbb{P}\text{-}\mathsf{Coh} \times \mathbb{P}\text{-}\mathsf{Coh} \to \mathbb{P}\text{-}\mathsf{Coh}$ *is defined by*

$$(A = (|A|, \rho_A), B = (|B|, \rho_B)) \mapsto A \oplus B = (|A| \times |B|, \rho_{A\oplus B})$$

$$(f : A \to A', g : B \to B') \mapsto (f \oplus g : A \oplus B \to A' \oplus B')$$

where the monotone function $\rho_{A\oplus B} : (|A| \times |B|) \times ((|A| \times |B|) \to \mathbb{P}$ *is given by*

$$\rho_{A\oplus B}((a,b),(a',b')) = \rho_A(a,a') \oplus \rho_B(b,b')$$

and where $f \oplus g$ *is given by the relation* $f \times g : |A| \times |B| \nrightarrow |A'| \times |B'|$*, i.e.*

$$((a,b),(a',b)) \in f \times g \iff (a,a') \in f \wedge (b,b') \in g$$

- *Par unit:* $\mathbb{P}$*-coherence* $\top = (\{*\}, \rho_\bot)$ *where* $\rho_\bot(*,*) = \bot \in \mathbb{P}$

- *Bottom-contraction map:* $\Delta_\bot : \bot \to \bot \otimes \bot$ *is relation* $|\bot| \nrightarrow |\bot| \times |\bot|$ *given by*

$$(*, (*, *)) \in \Delta_\bot$$

- *Top-cocontraction map:* $\nabla_\top : \top \oplus \top \to \top$ $|\top| \times |\top| \nrightarrow |\top|$ *given by*

$$((*, *), *) \in \nabla_\top$$

- *Nullary mix map:* $m : \bot \to \top$ *is the relation* $|\bot| \nrightarrow |\top|$ *given by*

$$(*, *) \in m$$

- *Medial maps:* $\mu_{A,B,C,D} : (A \otimes B) \oplus (C \otimes D) \to (A \oplus D) \otimes (C \oplus D)$ *is relation* $(|A| \times |B|) \times (|C| \times |D|) \nrightarrow (|A| \times |C|) \times (|B| \times |D|)$ *given by*

$$(((a, b), (c, d)), ((a', c'), (b', d'))) \in \mu_{A,B,C,D} \iff a \sqsubseteq a' \wedge b \sqsubseteq b' \wedge c \sqsubseteq c' \wedge d \sqsubseteq d'$$

- *Left linear distributivity:* $\delta^L_{A,B,C} : A \otimes (B \oplus C) \to (A \otimes B) \oplus C$ *is relation* $|A| \times (|B| \times |C|) \nrightarrow (|A| \times |B|) \times |C|$ *given by*

$$(((a, (b, c)), ((a', b'), c')) \in \delta^L_{A,B,C} \iff a \sqsubseteq a' \wedge b \sqsubseteq b' \wedge c \sqsubseteq c'$$

5.21. Proposition. $\mathbb{P}$-Coh *is a symmetric medial linearly distributive category.*

As the above construction is simply a small generalization of Lamarche's work, we shall not include the proof in this paper.

5.22. Properties.

Notice that in a SMLDC, we have the canonical flip maps for the symmetric $\otimes$-monoidal product and symmetric $\oplus$-monoidal product. The interaction between the medial maps, the linear distributivities and the canonical flip maps are given below.

5.23. Proposition. *The following diagrams commute in any symmetric medial linearly distributive category:*

$$\begin{array}{ccc}
(A \otimes B) \otimes ((C \otimes D) \oplus (E \otimes F)) & \xrightarrow{\delta^L_{A\otimes B, C\otimes D, E\otimes F}} & ((A \otimes B) \otimes (C \otimes D)) \oplus (E \otimes F) \\
\downarrow{\scriptstyle 1_{A\otimes B} \otimes \mu_{C,D,E,F}} & & \downarrow{\scriptstyle \tau^\otimes_{A,B,C,D} \oplus 1_{E\otimes F}} \\
(A \otimes B) \otimes ((C \oplus E) \otimes (D \oplus F)) & & ((A \otimes C) \otimes (B \otimes D)) \oplus (E \otimes F) \\
\downarrow{\scriptstyle \tau^\otimes_{A,B,C\oplus E,D\oplus F}} & & \downarrow{\scriptstyle \mu_{A\otimes C, B\otimes D, E, F}} \\
(A \otimes (C \oplus E)) \otimes (B \otimes (D \oplus F)) & \xrightarrow[\delta^L_{A,C,E} \otimes \delta^L_{B,D,F}]{} & ((A \otimes C) \oplus E) \otimes ((B \otimes D) \oplus F)
\end{array} \tag{61}$$

$$\delta^L_{A\otimes B,C\otimes D,E\otimes F};(\tau^{\otimes}_{A,B,C,D}\oplus 1_{E\otimes F});\mu_{A\otimes C,B\otimes D,E,F}$$
$$=(1_{A\otimes B}\otimes\mu_{C,D,E,F});\tau^{\otimes}_{A,B,C\oplus E,D\oplus F};(\delta^L_{A,C,E}\otimes\delta^L_{B,D,F})$$

$$\mu_{A,B\oplus C,D,E\oplus F};(1_{A\oplus D}\otimes\tau^{\oplus}_{B,C,E,F});\delta^L_{A\oplus D,B\oplus E,C\oplus F}=$$
$$(\delta^L_{A,B,C}\oplus\delta^L_{D,E,F});\tau^{\oplus}_{A\otimes B,C,D\otimes E,F};(\mu_{A,B,D,E}\oplus 1_{C\oplus F})$$

Proof. The proof of the first equality is given by the commuting diagram in Figure 1, in the Appendix A due to its size, and the other equality follows similarly. ■

5.24. Proposition. *Given a symmetric medial linearly distributive category, the following diagram commutes*

$$\begin{array}{ccc}
(A\otimes B)\oplus(C\otimes D) & \xrightarrow{\mu_{A,B,C,D}} & (A\oplus C)\otimes(B\oplus D)\\
{\scriptstyle \mathrm{mix}_{A,B}\oplus\mathrm{mix}_{C,D}}\downarrow & & \downarrow{\scriptstyle \mathrm{mix}_{A\oplus C,B\oplus D}}\\
(A\oplus B)\oplus(C\oplus D) & \xrightarrow[\tau^{\oplus}_{A,B,C,D}]{} & (A\oplus C)\oplus(B\oplus D)
\end{array}$$

$$\begin{array}{ccc}
(A\otimes B)\otimes(C\otimes D) & \xrightarrow{\tau^{\otimes}_{A,B,C,D}} & (A\otimes C)\otimes(B\otimes D)\\
{\scriptstyle \mathrm{mix}_{A\otimes B,C\otimes D}}\downarrow & & \downarrow{\scriptstyle \mathrm{mix}_{A,C}\otimes\mathrm{mix}_{B\otimes D}}\\
(A\otimes B)\oplus(C\otimes D) & \xrightarrow[\mu_{A,B,C,D}]{} & (A\oplus C)\otimes(B\oplus D)
\end{array} \qquad (62)$$

Proof. The proof of the first equality is given by the commuting diagram in Figure 2 in the Appendix A due to its size, and the other equality follows similarly. ■

If we consider a MLDC with invertible linear distributivities, we know it must know the $\oplus$-monoidal product must be equivalent to the $\bot$-shifted tensor, but in the presence of medial structure, we can say even more:

5.25. Lemma. *Consider a medial linearly distributive category* $(\mathbb{X},\otimes,\top,\oplus,\bot)$ *with invertible linear distributivities, then its linearly distributive structure is compact.*

Proof. Consider a MLDC $(\mathbb{X},\otimes,\top,\oplus,\bot)$ whose linear distributivities $\delta^L_{A,B,C}$ and $\delta^R_{A,B,C}$ have inverses. By Proposition 3.20 and the details of the proof in [15], $\top\oplus\top$ is the tensor inverse of $\bot$ with isomorphisms

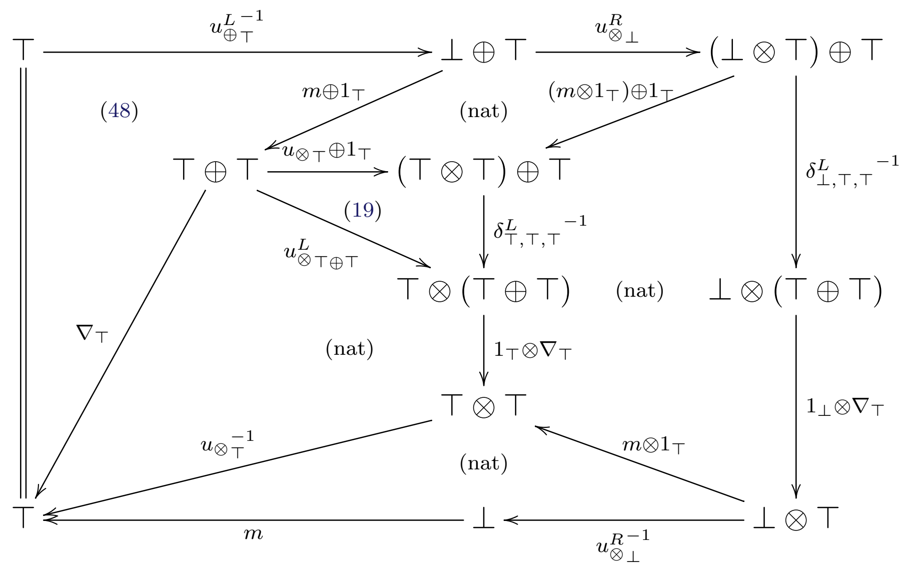
$$s^L=\bot\otimes(\top\oplus\top)\xrightarrow{\delta^L_{\bot,\top,\top}}(\bot\otimes\top)\oplus\top\xrightarrow{u^{R\,-1}_{\otimes\bot}\oplus 1_\top}\bot\oplus\top\xrightarrow{u^L_{\oplus\top}}\top$$

$$s^R=(\top\oplus\top)\otimes\bot\xrightarrow{\delta^R_{\top,\top,\bot}}\top\oplus(\top\otimes\bot)\xrightarrow{1_\top\oplus u^{L\,-1}_{\otimes\bot}}\top\oplus\bot\xrightarrow{u^R_{\oplus\top}}\top$$

This implies that the MLDC is isomix as $m:\bot\to\top$ has inverse defined by

$$m^{-1}=\top\xrightarrow{s^{L\,-1}}\bot\otimes(\top\oplus\top)\xrightarrow{1_\bot\otimes\nabla_\top}\bot\otimes\top\xrightarrow{u^{R\,-1}_{\otimes\bot}}\bot$$

$m^{-1}; m = 1_\top$ holds by the following commuting diagram



Similarly, for $m; m^{-1} = 1_\perp$.

As the linear distributivities are isomorphisms and $m : \perp \to \top$ is invertible, $\mathrm{mix}_{A,B} : A \otimes B \to A \oplus B$ is an isomorphism. ■

Given a normal duoidal category, there is a construction providing a linearly distributive structure. Indeed, let $(\mathcal{X}, \diamond, I, \star, J)$ be a duoidal category where $\iota : I \to J$ is an isomorphism, we can define the following linear distributivities:

$$\partial^L_{A,B,C} = A \diamond (B \star C) \cong (A \star J) \diamond (B \star C) \xrightarrow{\zeta_{A,J,B,C}} (A \diamond B) \star (J \diamond C) \cong (A \diamond B) \star C$$
$$\partial^R_{A,B,C} = (A \star B) \diamond C \cong (A \star B) \diamond (J \star C) \xrightarrow{\zeta_{A,B,J,C}} (A \diamond J) \star (B \diamond C) \cong A \star (B \diamond C)$$

Then, $(\mathcal{X}, \diamond, I, \star, J)$ is an isomix LDC [42].

Consider now an isomix MLDC $(\mathbb{X}, \otimes, \top, \oplus, \perp)$ with linear distributivities

$$\delta^L_{A,B,C} : A \otimes (B \oplus C) \to (A \otimes B) \oplus C \qquad \delta^R_{A,B,C} : (A \oplus B) \otimes C \to A \oplus (B \oplus C)$$

Then, it is in particular a normal duoidal category $(\mathbb{X}, \oplus, \perp, \otimes, \top)$. By above, we know that $(\mathbb{X}, \oplus, \perp, \otimes, \top)$ is an isomix LDC with linear distributivities

$$\partial^L_{A,B,C} : A \oplus (B \otimes C) \to (A \oplus B) \otimes C \qquad \partial^R_{A,B,C} : (A \otimes B) \oplus C \to A \otimes (B \oplus C)$$

Notice the direction of the latter linear distributivities is opposite to the former. It turns out they are inverses of each other and, therefore, when isomix, the linearly distributive structure is compact.

5.26. Lemma. *Consider an isomix medial linearly distributive category* $(\mathbb{X}, \otimes, \top, \oplus, \bot)$, *then its linearly distributive structure is compact.*

Proof. Consider an isomix MLDC $(\mathbb{X}, \otimes, \top, \oplus, \bot)$ and define the natural transformations ∂^L and ∂^R as follows.

$$\begin{aligned}
\partial^L_{A,B,C} =& A \oplus (B \otimes C) \xrightarrow{u^R_{\otimes A} \oplus 1_{B \otimes C}} (A \otimes \top) \otimes (B \otimes C) \xrightarrow{\mu_{A,\top,B,C}} (A \oplus B) \otimes (\top \oplus C) \\
& \xrightarrow{1_{A \oplus B} \otimes (m^{-1} \oplus 1_C)} (A \oplus B) \otimes (\bot \oplus C) \xrightarrow{1_{A \oplus B} \otimes u^L_{\oplus C}} (A \oplus B) \otimes C \\
\partial^R_{A,B,C} =& (A \otimes B) \oplus C \xrightarrow{1_{A \otimes B} \oplus u^R_{\otimes C}} (A \otimes B) \oplus (\top \otimes C) \xrightarrow{\mu_{A,B,\top,C}} (A \oplus \top) \otimes (B \oplus C) \\
& \xrightarrow{(1_A \oplus m^{-1}) \otimes 1_{B \oplus C}} (A \oplus \bot) \otimes (B \oplus C) \xrightarrow{u^R_{\oplus A} \otimes 1_{B \oplus C}} A \otimes (B \oplus C)
\end{aligned}$$

Then, $\partial^R_{A,B,C}$ is the inverse of $\delta^L_{A,B,C}$ and $\partial^L_{A,B,C}$ is the inverse of $\delta^R_{A,B,C}$. The proof that $\delta^L_{A,B,C}; \partial^R_{A,B,C} = 1_{A \otimes (B \oplus C)}$ follows from the commuting diagram in Figure 3 in the Appendix A due to its size. The other equalities follow similarly.

As the linear distributivities are isomorphisms and $m : \bot \to \top$ is invertible, $\text{mix}_{A,B} : A \otimes B \to A \oplus B$ is an isomorphism. ■

All together, recognizing that a compact LDC is necessarily is isomix by definition and has invertible linear distributivities by Lemma 3.6, we get the following result.

5.27. Proposition. *Consider a medial linearly distributive category* $(\mathbb{X}, \otimes, \top, \oplus, \bot)$, *then the following are equivalent:*

1. *it is an isomix linearly distributive category,*
2. *the linear distributivities are isomorphisms, and*
3. *the linearly distributive structure is compact.*

In the presence of symmetry, the above proposition can be extended to consider the duoidal structure as well.

5.28. Proposition. *Consider a symmetric medial linearly distributive category* $(\mathbb{X}, \otimes, \top, \oplus, \bot)$, *then the following are equivalent:*

1. *it is an isomix linearly distributive category,*
2. *the linear distributivities are isomorphisms,*
3. *the linearly distributive structure is compact, and*
4. *the duoidal structure is strong.*

Proof. It only remains to show that if the linearly distributive structure is compact, then the duoidal structure is strong, and that in turn, the duoidal structure being strong implies it is isomix.

The former follows by Proposition 5.24: as the mix maps are isomorphisms, then so are the medial maps. Moreover, as the nullary mix map $m : \bot \to \top$ is invertible, $\Delta_\bot$ and $\nabla_\top$ are given by

$$\Delta_\bot = u^R_{\otimes\bot}; (1_\bot \otimes m^{-1}) \qquad\qquad \nabla_\top = (1_\top \oplus m^{-1}); u^R_{\oplus\top}$$

Therefore, they are invertible.

The latter follows immediately as a strong duoidal structure implies $m : \bot \to \top$ is invertible, which implies it is isomix by Lemma 3.10. ■

5.29. Remark. A SMLDC $(\mathbb{X}, \otimes, \top, \oplus, \bot)$ being compact and strong means

- $m : \bot \cong \top$,
- $\text{mix}_{A,B} : A \otimes B \cong A \oplus B$,
- the linear distributivities are associators (modulo the invertible mix maps),
- the $\bot$-contraction/$\top$-cocontraction are unitors (modulo the invertible nullary mix map), and
- the medial maps are the canonical flip (modulo the invertible mix maps).

In other words, the SMLDC is isomorphic to the medial linearly distributive structure induced by the symmetric monoidal structure $(\mathbb{X}, \otimes, \top)$, as described in Proposition 5.18, and alternatively by the symmetric monoidal structure $(\mathbb{X}, \oplus, \bot)$.

6. Medial Linear Functors and Transformations

With the definition of MLDCs well established, we now turn to the appropriate functors between these categories and the transformations between such functors. Similar to the 2-category of cartesian LDCs **CLDC**, the corresponding 2-category would form a sub-2-category of **LDC**.

As such, these "medial linear functors" are linear functors that also interact coherently with the duoidal structure. Given that there are two established definitions of duoidal functors, there are naturally at least two distinct types of "medial linear functors". However, we shall define only the specific class required for the linearly distributive Fox theorem. It is important to emphasize that these should not be regarded as the only valid choices for 1-cells between MLDCs. Rather, they are the appropriate ones within the specific context of our work.

6.1. Definition. *Let $\mathbb{X}$ and $\mathbb{Y}$ be symmetric linearly distributive categories. A* **symmetric medial linear functor** $F = (F_\otimes, F_\oplus) : \mathbb{X} \to \mathbb{Y}$ *consists of:*

- *a functor $F_\otimes : \mathbb{X} \to \mathbb{Y}$, equipped with*
  - *an invertible morphism $m_\top : \top \xrightarrow{\sim} F_\otimes(\top)$,*
  - *a natural isomorphism*

$$m_\otimes : (F_\otimes \times F_\otimes); \otimes \Rightarrow \otimes; F_\otimes \qquad m_{\otimes A,B} : F_\otimes(A) \otimes F_\otimes(B) \xrightarrow{\sim} F_\otimes(A \otimes B)$$

  - *a morphism $m_\bot : \bot \to F_\otimes(\bot)$,*
  - *a natural transformation*

$$m_\oplus : (F_\otimes \times F_\otimes); \oplus \Rightarrow \otimes; F_\oplus \qquad m_{\oplus A,B} : F_\otimes(A) \oplus F_\otimes(B) \to F_\otimes(A \oplus B)$$

  such that
  - $(F_\otimes, m_\top, m_\otimes) : (\mathbb{X}, \otimes, \top) \to (\mathbb{Y}, \otimes, \top)$ *is a symmetric monoidal functor,*
  - $(F_\otimes, m_\bot, m_\oplus) : (\mathbb{X}, \oplus, \bot) \to (\mathbb{Y}, \oplus, \bot)$ *is a symmetric lax monoidal functor,*
- *a functor $F_\oplus : \mathbb{X} \to \mathbb{Y}$, equipped with*
  - *an invertible morphism $n_\bot : F_\oplus(\bot) \xrightarrow{\sim} \bot$,*
  - *a natural isomorphism*

$$n_\oplus : \oplus; F_\oplus \Rightarrow (F_\oplus \times F_\oplus); \oplus \qquad n_{\oplus A,B} : F_\oplus(A \oplus B) \xrightarrow{\sim} F_\oplus(A) \oplus F_\oplus(B)$$

  - *a morphism $n_\top : F_\oplus(\top) \to \top$,*
  - *a natural transformation*

$$n_\oplus : \otimes; F_\oplus \Rightarrow (F_\oplus \times F_\oplus); \otimes \qquad n_{\otimes A,B} : F_\oplus(A \otimes B) \to F_\oplus(A) \otimes F_\oplus(B)$$

  such that
  - $(F_\oplus, n_\bot, n_\oplus) : (\mathbb{X}, \oplus, \bot) \to (\mathbb{Y}, \oplus, \bot)$ *is a symmetric monoidal functor,*
  - $(F_\oplus, n_\top, n_\otimes) : (\mathbb{X}, \oplus, \bot) \to (\mathbb{Y}, \oplus, \bot)$ *is a symmetric colax monoidal functor,*
- *linear strength natural transformations*

$$v^R_\otimes : \oplus; F_\otimes \Rightarrow (F_\oplus \times F_\otimes); \oplus \qquad v^R_{\otimes A,B} : F_\otimes(A \oplus B) \to F_\oplus(A) \oplus F_\otimes(B)$$
$$v^R_\otimes : (F_\otimes \times F_\oplus); \otimes \Rightarrow \otimes; F_\oplus \qquad v^R_{\oplus A,B} : F_\otimes(A) \otimes F_\oplus(B) \to F_\oplus(A \otimes B)$$

  satisfying the coherence conditions ensuring that

– $(F_\otimes, m_\perp, m_\oplus, m_\top^{-1}, m_\otimes^{-1}) : (\mathbb{X}, \oplus, \perp, \otimes, \top) \to (\mathbb{Y}, \oplus, \perp, \otimes, \top)$ *is a bilax duoidal functor, i.e.* (63) *holds,*

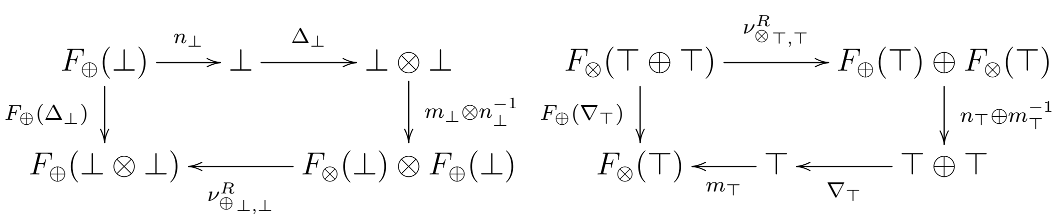
– $(F_\oplus, n_\perp^{-1}, n_\oplus^{-1}, n_\top, n_\otimes) : (\mathbb{X}, \oplus, \perp, \otimes, \top) \to (\mathbb{Y}, \oplus, \perp, \otimes, \top)$ *is a bilax duoidal functor, i.e.* (64) *holds,*

– $F = (F_\otimes, F_\oplus) : \mathbb{X} \to \mathbb{Y}$ *is a symmetric linear functor, and*

– *the linear strengths interact coherently with the* $\perp$*-contraction and* $\top$*-cocontraction maps* (65)*, with the medial transformation* (66)*, and* $m_\oplus$ *and* $n_\otimes$ (67)*.*

$$\begin{aligned}
&\Delta_\perp; (m_\perp \otimes m_\perp); m_{\otimes\perp,\perp} = m_\perp; F_\otimes(\Delta_\perp)\\
&(m_\top \oplus m_\top); m_{\oplus\top,\top}; F_\otimes(\nabla_\top) = \nabla_\top; m_\top\\
&m; m_\top = m_\perp; F_\otimes(m)\\
&(m_{\otimes A,B} \oplus m_{\otimes C,D}); m_{\oplus A\otimes B, C\otimes D}; F_\otimes(\mu_{A,B,C,D})\\
&= \mu_{F_\otimes(A),F_\otimes(B),F_\otimes(C),F_\otimes(D)}; (m_{\oplus A,C} \otimes m_{\oplus B,D}); m_{\otimes A\oplus C, B\oplus D}
\end{aligned} \tag{63}$$

$$\begin{aligned}
&F_\oplus(\Delta_\perp); n_{\otimes\perp,\perp}; (n_\perp \otimes n_\perp) = n_\perp; \Delta_\perp\\
&n_{\oplus\top,\top}; (n_\top \oplus n_\top); \nabla_\top = F_\oplus(\nabla_\top); n_\top\\
&n_\perp; m = F_\oplus(m); n_\top\\
&F_\oplus(\mu_{A,B,C,D}); n_{\otimes A\oplus C, B\oplus D}; (n_{\oplus A,C} \otimes n_{\oplus B,D})\\
&= n_{\oplus A\otimes B, C\otimes D}; (n_{\otimes A,B} \oplus n_{\otimes C,D}); \mu_{F_\oplus(A),F_\oplus(B),F_\oplus(C),F_\oplus(D)}
\end{aligned} \tag{64}$$



(65)

$$\begin{array}{ccc}
F_\otimes((A \otimes B) \oplus (C \otimes D)) & \xrightarrow{F_\otimes(\mu_{A,B,C,D})} & F_\otimes((A \oplus C) \otimes (B \oplus D)) \\
\downarrow {\scriptstyle \nu^R_{\otimes A\otimes B, C\otimes D}} & & \downarrow {\scriptstyle m_\otimes^{-1}{}_{A\oplus C, B\oplus D}} \\
F_\oplus(A \otimes B) \oplus F_\otimes(C \otimes D) & & F_\otimes(A \oplus C) \otimes F_\otimes(B \oplus D) \\
\downarrow {\scriptstyle n_{\otimes A,B} \oplus m_\otimes^{-1}{}_{C,D}} & & \downarrow {\scriptstyle \nu^R_{\otimes A,C} \otimes \nu^R_{\otimes B,D}} \\
\begin{array}{c}(F_\oplus(A) \otimes F_\oplus(B)) \oplus \\ (F_\otimes(C) \otimes F_\otimes(D))\end{array} & \xrightarrow[\mu_{F_\oplus(A),F_\oplus(B),F_\otimes(C),F_\otimes(D)}]{} & \begin{array}{c}(F_\oplus(A) \oplus F_\otimes(C)) \oplus \\ (F_\oplus(B) \oplus F_\otimes(D))\end{array}
\end{array}$$

$$\begin{array}{ccc}
\begin{array}{c}(F_\otimes(A) \otimes F_\oplus(B)) \oplus \\ (F_\otimes(C) \otimes F_\oplus(D))\end{array} & \xrightarrow{\nu^R_{\oplus A,B} \oplus \nu^R_{\oplus C,D}} & F_\oplus(A \otimes B) \oplus F_\oplus(C \otimes D) \\
\downarrow {\scriptstyle \mu_{F_\otimes(A),F_\oplus(B),F_\otimes(C),F_\oplus(D)}} & & \downarrow {\scriptstyle n_\oplus^{-1}{}_{A\otimes B, C\otimes D}} \\
\begin{array}{c}(F_\otimes(A) \oplus F_\otimes(C)) \oplus \\ (F_\oplus(B) \oplus F_\oplus(D))\end{array} & & F_\oplus((A \otimes B) \oplus (C \otimes D)) \\
\downarrow {\scriptstyle m_{\oplus A,C} \otimes n_\oplus^{-1}{}_{B,D}} & & \downarrow {\scriptstyle F_\oplus(\mu_{A,B,C,D})} \\
F_\otimes(A \oplus C) \otimes F_\oplus(B \oplus D) & \xrightarrow[\nu^R_{\oplus A\oplus C, B\oplus D}]{} & F_\oplus((A \oplus C) \otimes (B \oplus D))
\end{array} \tag{66}$$

$$\begin{array}{ccc}
F_\otimes(A \oplus B) \oplus F_\otimes(C) & \xrightarrow{m_{\oplus A\oplus B, C}} & F_\otimes((A \oplus B) \oplus C) \\
\downarrow {\scriptstyle \nu^R_{\otimes A,B} \oplus 1_{F_\otimes(C)}} & & \downarrow {\scriptstyle F_\otimes(\alpha_\oplus^{-1}{}_{A,B,C})} \\
(F_\oplus(A) \oplus F_\otimes(B)) \oplus F_\otimes(C) & & F_\otimes(A \oplus (B \oplus C)) \\
\downarrow {\scriptstyle \alpha_\oplus^{-1}{}_{F_\oplus(A),F_\otimes(B),F_\otimes(C)}} & & \downarrow {\scriptstyle \nu^R_{\otimes A, B\oplus C}} \\
F_\oplus(A) \oplus (F_\otimes(B) \oplus F_\otimes(C)) & \xrightarrow[1_{F_\oplus(A)} \oplus m_{\oplus B,C}]{} & F_\oplus(A) \oplus F_\otimes(B \oplus C)
\end{array} \tag{67}$$

$$\begin{array}{ccc}
F_\otimes(A) \otimes F_\oplus(B \otimes C) & \xrightarrow{\nu^R_{\oplus A, B\otimes C}} & F_\otimes(A \otimes (B \otimes C)) \\
\downarrow {\scriptstyle 1_{F_\otimes(A)} \otimes n_{\otimes B,C}} & & \downarrow {\scriptstyle F_\oplus(\alpha_\otimes{}^{-1}_{A,B,C})} \\
F_\otimes(A) \otimes (F_\oplus(B) \otimes F_\otimes(C)) & & F_\otimes((A \otimes B) \otimes C) \\
\downarrow {\scriptstyle \alpha_\otimes^{-1}{}_{F_\otimes(A),F_\oplus(B),F_\oplus(C)}} & & \downarrow {\scriptstyle n_{\otimes A\otimes B, C}} \\
F_\otimes(A) \otimes (F_\oplus(B) \otimes F_\oplus(C)) & \xrightarrow[\nu^R_{\oplus A,B} \otimes 1_{F_\oplus(C)}]{} & F_\otimes(A \otimes B) \otimes F_\otimes(C)
\end{array}$$

We record here some commuting diagrams that will of use in the next Section 7.11, involving the linear strengths, the duoidal functor structure maps and the canonical flip.

6.2. Proposition. *Given a symmetric medial linear functor $F = (F_\otimes, F_\oplus) : \mathbb{X} \to \mathbb{Y}$ between symmetric medial linearly distributive categories, the following diagrams always commute:*

$$
\begin{array}{ccc}
F_\otimes(A \oplus B) \oplus F_\otimes(C \oplus D) & \xrightarrow{\nu^R_{\otimes A,B} \oplus \nu^R_{\otimes C,D}} & (F_\oplus(A) \oplus F_\otimes(B)) \oplus (F_\oplus(C) \oplus F_\otimes(D)) \\
{\scriptstyle m_{\oplus A \oplus B, C \oplus D}} \downarrow & & \downarrow {\scriptstyle \tau^\oplus_{F_\oplus(A), F_\otimes(B), F_\oplus(C), F_\otimes(D)}} \\
F_\otimes(A \oplus B) \oplus (C \oplus D)) & & (F_\oplus(A) \oplus F_\oplus(C)) \oplus (F_\otimes(B) \oplus F_\otimes(D)) \\
{\scriptstyle F_\otimes(\tau^\oplus_{A,B,C,D})} \downarrow & & \downarrow {\scriptstyle n_\oplus^{-1}{}_{A,C} \oplus m_{\oplus B,D}} \\
F_\oplus((A \oplus C) \oplus (B \oplus D)) & \xrightarrow[\nu^R_{\otimes A \oplus C, B \oplus D}]{} & F_\oplus(A \oplus C) \oplus F_\otimes(B \oplus D)
\end{array}
\tag{68}
$$

$$
\begin{aligned}
&(\nu^R_{\otimes A,B} \oplus \nu^R_{\otimes C,D}); \tau^\oplus_{F_\oplus(A), F_\otimes(B), F_\oplus(C), F_\otimes(D)}; (n_\oplus^{-1}{}_{A,C} \oplus m_{\oplus B,D}) \\
&= m_{\oplus A \oplus B, C \oplus D}; F_\otimes(\tau^\oplus_{A,B,C,D}); \nu^R_{\otimes A \oplus C, B \oplus D}
\end{aligned}
$$

$$
\begin{aligned}
&\nu^R_{\oplus A \otimes B, C \otimes D}; F_\oplus(\tau^\otimes_{A,B,C,D}); n_{\otimes A \otimes C, B \otimes D} \\
&= (m_\otimes^{-1}{}_{A,B} \otimes n_{\otimes C,D}); \tau^\otimes_{F_\otimes(A), F_\otimes(B), F_\oplus(C), F_\oplus(D)}; (\nu^R_{\oplus A,C} \otimes \nu^R_{\oplus B,D})
\end{aligned}
$$

Proof. The proof for the first equality is given by the commuting diagram in Figure 4, in the Appendix A due to its size, and the second equality is proved similarly. ■

Now, we turn to the discussion of transformations. As there are several potential notions of medial linear functors, there are multiple definitions of "medial linear transformations". However, we will introduce only the specific definition relevant to our treatment of MLDCs.

6.3. Definition. *Let $F, G : \mathbb{X} \to \mathbb{Y}$ be symmetric medial linear functors. A* **medial linear transformation** *$\alpha = (\alpha_\otimes, \alpha_\oplus) : F \Rightarrow G$ consists of:*

- *a natural transformation*

$$\alpha_\otimes : F_\otimes \Rightarrow G_\otimes \qquad \alpha_{\otimes A} : F_\otimes(A) \to G_\otimes(A)$$

  such that

  - $\alpha_\otimes : (F_\otimes, m^F_\top, m^F_\otimes) \Rightarrow (G_\otimes, m^G_\top, m^G_\otimes)$ *is a monoidal transformation,*
  - $\alpha_\otimes : (F_\otimes, m^F_\perp, m^F_\oplus) \Rightarrow (G_\otimes, m^G_\perp, m^G_\oplus)$ *is a monoidal transformation,*

- *a natural transformation*

$$\alpha_\oplus : G_\oplus \Rightarrow F_\oplus \qquad \alpha_{\oplus A} : G_\oplus(A) \to F_\oplus(A)$$

  such that

  - $\alpha_\oplus : (G_\oplus, n^G_\perp, n^G_\oplus) \Rightarrow (F_\oplus, n^F_\perp, n^F_\oplus)$ *is a monoidal transformation,*
  - $\alpha_\oplus : (G_\oplus, n_\top{}^G, n_\otimes{}^G) \Rightarrow (F_\oplus, n^F_\top, n_\otimes{}^F)$ *is a monoidal transformation,*

satisfying the coherence conditions ensuring that $\alpha = (\alpha_\otimes, \alpha_\oplus)$ is a linear transformation, i.e. (33) holds.

6.4. Remark. Notice that the conditions that $\alpha_\otimes : F_\otimes \Rightarrow G_\otimes$ satisfies in the above definition are equivalent to asking that

$$\alpha_\otimes : (F_\otimes, m_\perp^F, {m_\oplus}^F, {m_\top^{-1}}^F, {m_\otimes^{-1}}^F) \Rightarrow (G_\otimes, m_\perp^G, {m_\oplus}^G, {m_\top^{-1}}^G, {m_\otimes^{-1}}^G)$$

is a duoidal transformation and, similarly, the conditions that $\alpha_\oplus : G_\oplus \Rightarrow F_\oplus$ satisfies is equivalent to

$$\alpha_\oplus : (G_\oplus, {n_\perp^{-1}}^G, {n_\oplus^{-1}}^G, {n_\oplus}^G, {n_\top}^G) \Rightarrow (F_\oplus, {n_\perp^{-1}}^F, {n_\oplus^{-1}}^F, {n_\oplus}^F, {n_\top}^F)$$

being a duoidal transformation.

6.5. Frobenius Medial Linear Functors and Transformations.

As in the case of Frobenius linear functors, we can consider symmetric medial linear functors whose pair of functors $F_\otimes$ and $F_\oplus$ are equal.

6.6. Definition. *A symmetric medial linear functor $F = (F_\otimes, F_\oplus) : \mathbb{X} \to \mathbb{Y}$ is* **Frobenius** *if*

1. $F_\otimes = F_\oplus$,
2. $m_\perp = n_\perp^{-1}$, $m_{\oplus A,B} = {n_\oplus^{-1}}_{A,B}$, $n_\top = m_\top^{-1}$, *and* $n_{\otimes A,B} = {m_\otimes^{-1}}_{A,B}$,
3. $\nu^R_{\otimes A,B} = n_{\oplus A,B}$, *and* $\nu^R_{\oplus A,B} = m_{\otimes A,B}$.

Given the degeneracy, we can once more give an alternative characterization of such linear functors.

6.7. Proposition. *Consider $\mathbb{X}$ and $\mathbb{Y}$ be symmetric linearly distributive categories, then the following notions coincide:*

- *symmetric Frobenius medial linear functors $F = (F_\otimes, F_\oplus) : \mathbb{X} \to \mathbb{Y}$, and*
- *symmetric $\otimes$-monoidal and symmetric $\oplus$-monoidal functors $(F, m_\top, m_\otimes, n_\perp, n_\oplus) : \mathbb{X} \to \mathbb{Y}$ satisfying (34), (35) and (69).*

$$\begin{array}{ccccc} F(\perp) & \xrightarrow{n_\perp} & \perp & \xrightarrow{\Delta_\perp} & \perp \otimes \perp \\ {\scriptstyle F(\Delta_\perp)}\downarrow & & & & \downarrow{\scriptstyle n_\perp^{-1} \otimes n_\perp^{-1}} \\ F(\perp \otimes \perp) & & \xleftarrow[m_{\otimes \perp,\perp}]{} & & F(\perp) \otimes F(\perp) \end{array} \qquad \begin{array}{ccccc} F(\top \oplus \top) & & \xrightarrow{n_{\oplus \top,\top}} & & F(\top) \oplus F(\top) \\ {\scriptstyle F(\nabla_\top)}\downarrow & & & & \downarrow{\scriptstyle m_\top^{-1} \oplus m_\top^{-1}} \\ F(\top) & \xleftarrow[m_\top]{} & \top & \xleftarrow[\nabla_\top]{} & \top \oplus \top \end{array}$$

$$\begin{array}{ccc} F((A \otimes B) \oplus (C \otimes D)) & \xrightarrow{F(\mu_{A,B,C,D})} & F((A \oplus C) \otimes (B \oplus D)) \\ {\scriptstyle n_{\oplus A \otimes B, C \otimes D}}\downarrow & & \downarrow{\scriptstyle {m_\otimes^{-1}}_{A \oplus C, B \oplus D}} \\ F(A \otimes B) \oplus F(C \otimes D) & & F(A \oplus C) \otimes F(B \oplus D) \\ {\scriptstyle {m_\otimes^{-1}}_{A,B} \oplus {m_\otimes^{-1}}_{C,D}}\downarrow & & \downarrow{\scriptstyle n_{\oplus A,C} \otimes n_{\oplus B,D}} \\ (F(A) \otimes F(B)) \oplus (F(C) \otimes F(D)) & \xrightarrow[\mu_{F(A),F(B),F(C),F(D)}]{} & (F(A) \oplus F(C)) \otimes (F(B) \oplus F(D)) \end{array} \qquad (69)$$

6.8. Remark. Recall that (34) is equivalent to asking that $(F, m_\otimes, m_\top, n_\oplus, n_\perp) : \mathbb{X} \to \mathbb{Y}$ is a bilax Frobenius linear functor. Moreover, (35) and (69) are equivalent to asking that $(F, n_\perp^{-1}, n_\oplus^{-1}, m_\top^{-1}, m_\otimes^{-1}) : (\mathbb{X}, \oplus, \perp, \otimes, \top) \to (\mathbb{Y}, \oplus, \perp, \otimes, \top)$ is a bilax duoidal functor. As bilax Frobenius linear functors and bilax duoidal functors compose, we know symmetric Frobenius medial linear functors compose as well.

If we then consider medial linear transformations between such linear functors, we can see that the paired transformations must in fact be a section-retraction pair.

6.9. Proposition. *Consider symmetric Frobenius medial linear functors $F, G : \mathbb{X} \to \mathbb{Y}$, then*

- *$\alpha_\otimes : F \Rightarrow G$ and $\alpha_\oplus : G \Rightarrow F$ are a $\otimes$-monoidal and $\oplus$-monoidal transformations, and*
- *for each object $A \in \mathbb{X}$, $\alpha_{\otimes A} : F(A) \to G(A)$ is a section of $\alpha_{\oplus A} : G(A) \to F(A)$, i.e. $\alpha_\otimes ; \alpha_\oplus = 1_F$.*

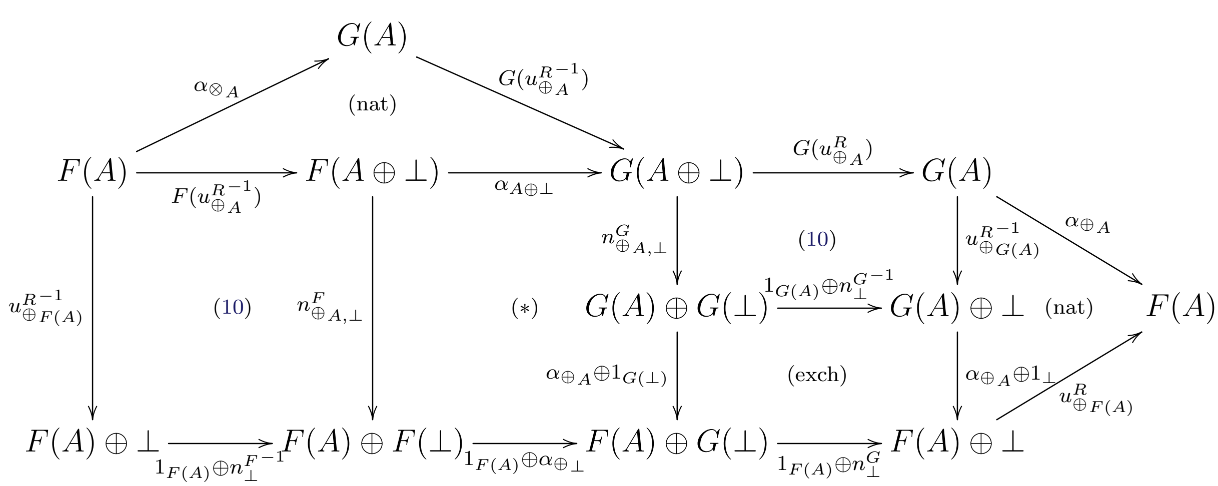
Proof. Let $\alpha = (\alpha_\otimes, \alpha_\oplus) : F \Rightarrow G$ be a medial linear transformation between symmetric Frobenius medial linear functors. This means in particular that:

- $\alpha_\otimes : (F, m_\top^F, m_\otimes^F) \Rightarrow (G, m_\top^F, m_\otimes^G)$ is a $\otimes$-monoidal transformation
- $\alpha_\otimes : (F, n_\perp^{F^{-1}}, n_\oplus^{F^{-1}}) \Rightarrow (G, n_\perp^{G^{-1}}, n_\oplus^{G^{-1}})$ is a $\oplus$-monoidal transformation, meaning $\alpha_\otimes : (F, n_\perp^F, n_\oplus^F) \Rightarrow (G, n_\perp^G, n_\oplus^G)$

Similarly, for $\alpha_\oplus$. Further, as $\alpha = (\alpha_\otimes, \alpha_\oplus) : F \Rightarrow G$ is a linear transformation, the diagrams in (33) holds, the first of which is as follows in the context of Frobenius linear functors:

$$\alpha_{\otimes A\oplus B} ; n^G_{\oplus A,B} ; (\alpha_{\oplus A} \oplus 1_{G(B)}) = n^F_{\oplus A,B} ; (1_{F(A)} \oplus \alpha_{\otimes B}) \qquad (*)$$

Letting $B = \perp$ implies that $\alpha_\otimes ; \alpha_\oplus = 1_{F(A)}$ by the following commuting diagram.



The top composite is $\alpha_\otimes ; \alpha_\oplus$, while the bottom composite is $1_{F(A)}$ since $\alpha_\oplus$ is $\oplus$-monoidal.

■

An immediate situation where one gets such a linear transformation is when considering an invertible $\otimes$-monoidal and $\oplus$-monoidal natural transformation.

6.10. Proposition. *Given an invertible $\otimes$-monoidal and $\oplus$-monoidal natural transformations $\alpha : F \Rightarrow G$ between symmetric Frobenius medial linear functors, the pair $(\alpha, \alpha^{-1}) : F \Rightarrow G$ is a medial linear transformation.*

Proof. Consider an invertible $\otimes$-monoidal and $\oplus$-monoidal natural transformations $\alpha : F \Rightarrow G$ between symmetric Frobenius medial linear functors, then by Lemma 3.33, $(\alpha, \alpha^{-1}) : F \Rightarrow G$ is a linear transformation. To be a medial linear transformation, it remains to show $\alpha : (F_\otimes, m_\perp^F, m_\oplus^F) \Rightarrow (G_\otimes, m_\perp^G, m_\oplus^G)$ is a $\oplus$-monoidal transformation and $\alpha^{-1} : (G_\oplus, n_\top^G, n_\otimes^G) \Rightarrow (F_\oplus, n_\top^F, n_\otimes^F)$ is a $\otimes$-monoidal transformation.

Since F and G are Frobenius, this is in fact requiring that $\alpha : (F, n_\perp^{F^{-1}}, n_\oplus^{F^{-1}}) \Rightarrow (G, n_\perp^{G^{-1}}, n_\oplus^{G^{-1}})$ is a $\oplus$-monoidal transformation and $\alpha^{-1} : (G, m_\top^{G^{-1}}, m_\otimes^{G^{-1}}) \Rightarrow (F, m_\top^{F^{-1}}, m_\otimes^{F^{-1}})$ is a $\otimes$-monoidal transformation, which is true as $\alpha : (F, n_\perp^F, n_\oplus^F) \Rightarrow (G, n_\perp^G, n_\oplus^G)$ is a $\oplus$-monoidal transformation and $\alpha : (F, m_\top^F, m_\otimes^F) \Rightarrow (G, m_\top^G, m_\otimes^G)$ is a $\otimes$-monoidal transformation respectively. ■

6.11. 2-Category **SMLDC**.

We now prove that these functors and transformations as 1-cells and 2-cells, with SMLDCs as 0-cells, form a 2-category.

6.12. Theorem. *Symmetric medial linearly distributive categories, strong symmetric medial linear functors, and strong medial linear transformations form a 2-category, which is denoted by* **SMLDC**.

Proof. Consider SMLDCs $\mathbb{X}$, $\mathbb{Y}$ and $\mathbb{Z}$. Let $F = (F_\otimes, F_\oplus) : \mathbb{X} \to \mathbb{Y}$ and $G = (G_\otimes, G_\oplus) : \mathbb{Y} \to \mathbb{Z}$ be symmetric medial linear functor. Then, their horizontal composite $F; G = (F_\otimes; G_\otimes, F_\oplus; G_\oplus) : \mathbb{X} \to \mathbb{Y}$ is given by:

- functor $F_\otimes; G_\otimes : \mathbb{X} \to \mathbb{Z}$, equipped with

$$m_\top^G; G_\otimes(m_\top^F) : \top \to G_\otimes(F_\otimes(\top))$$
$$m_{\otimes_{F_\otimes(A), F_\otimes(B)}}^G; G_\otimes(m_{\otimes_{A,B}}^F) : G_\otimes(F_\otimes(A)) \otimes G_\otimes(F_\otimes(B)) \to G_\otimes(F_\otimes(A \otimes B))$$

$$m_\perp^G; G_\otimes(m_\perp^F) : \perp \to G_\otimes(F_\otimes(\perp))$$
$$m_{\oplus_{F_\otimes(A), F_\otimes(B)}}^G; G_\otimes(m_{\oplus_{A,B}}^F) : G_\otimes(F_\otimes(A)) \oplus G_\otimes(F_\otimes(B)) \to G_\otimes(F_\otimes(A \oplus B))$$

- a functor $F_\oplus ; G_\oplus : \mathbb{X} \to \mathbb{Z}$, equipped with

$$G_\oplus(n^F_\bot); n^G_\bot : G_\oplus(F_\oplus(\bot)) \to \bot$$
$$G_\otimes(n^F_{\oplus A,B}); n^G_{\oplus F_\oplus(A), F_\oplus(B)} : G_\oplus(F_\oplus(A \oplus B)) \to G_\oplus(F_\oplus(A)) \oplus G_\oplus(F_\oplus(B))$$

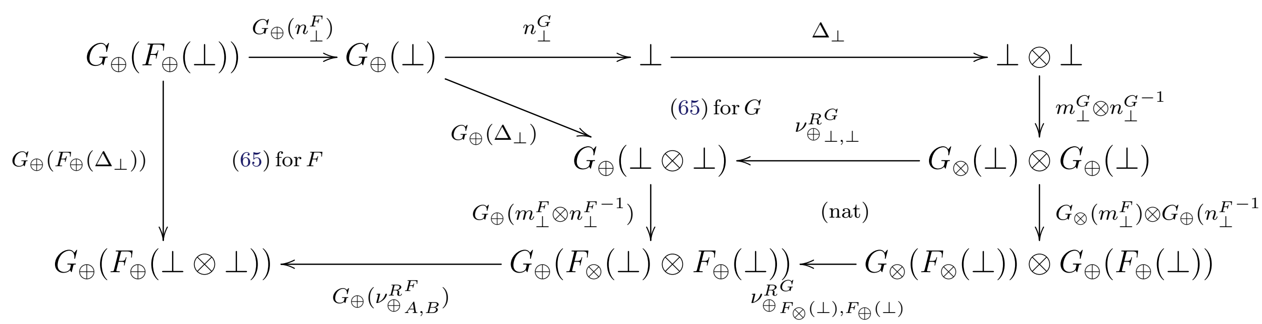
$$G_\oplus(n^F_\top); n^G_\top : G_\oplus(F_\oplus(\top)) \to \top$$
$$G_\otimes(n^F_{\otimes A,B}); n^G_{\otimes F_\oplus(A), F_\oplus(B)} : G_\oplus(F_\oplus(A \otimes B)) \to G_\oplus(F_\oplus(A)) \otimes G_\oplus(F_\oplus(B))$$

- linear strength natural transformations

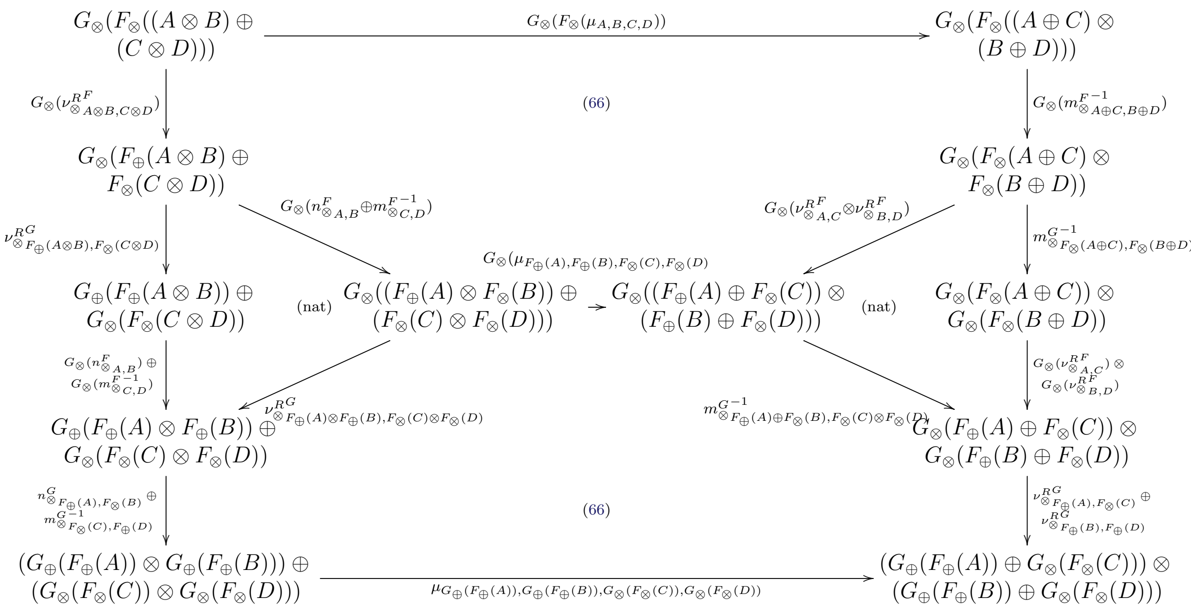
$$G_\otimes(v^{R\,F}_{\otimes A,B}); \nu^{R\,G}_{\otimes F_\oplus(A), F_\otimes(B)} : G_\otimes(F_\otimes(A \oplus B)) \to G_\oplus(F_\oplus(A)) \oplus G_\otimes(F_\otimes(B))$$
$$\nu^{R\,G}_{\oplus F_\otimes(A), F_\oplus(B)}; G_\oplus(\nu^{R\,F}_{\oplus A,B}) : G_\otimes(F_\otimes(A)) \otimes G_\oplus(F_\oplus(B)) \to G_\oplus(F_\oplus(A \otimes B))$$

It is immediate that $F; G : \mathbb{X} \to \mathbb{Z}$ is a symmetric linear functor, and that $F_\otimes ; G_\otimes : \mathbb{X} \to \mathbb{Z}$ and $F_\oplus ; G_\oplus : \mathbb{X} \to \mathbb{Z}$ are bilax duoidal functors, as the composition is inherited from **LDC** and **DUO** in the appropriate ways. It remains only to show the additional coherence conditions also hold for $F; G : \mathbb{X} \to \mathbb{Z}$.

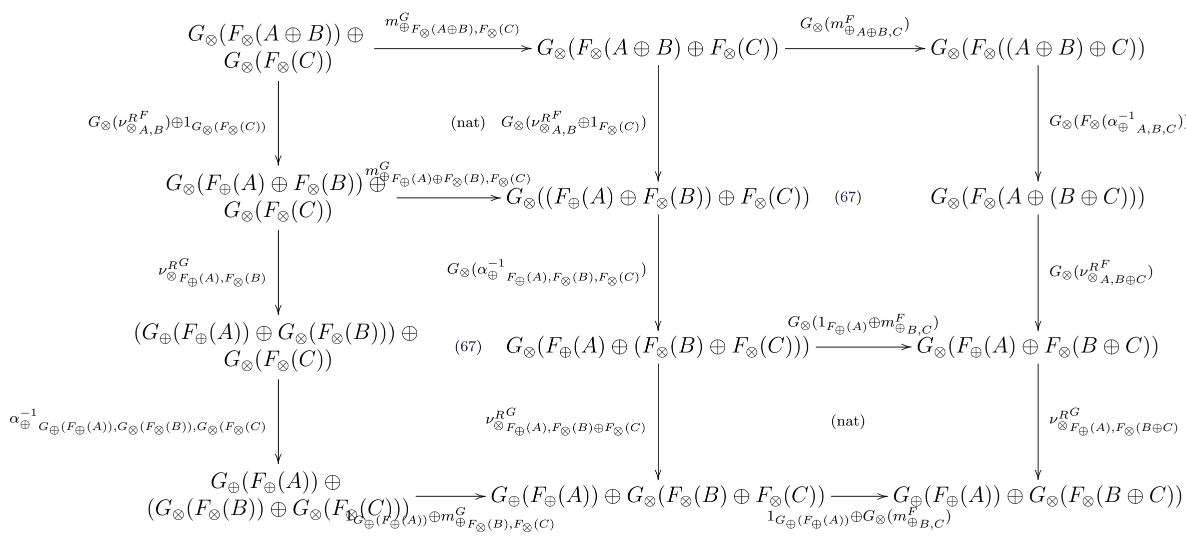
The first diagram of (65) holds by the following commuting diagram and the second diagram holds similarly.



The first diagram (66) holds by the following commuting diagram,



The first diagram of (67) holds by the following commuting diagram,



The second diagram of (66) and of (67) for $F;G:\mathbb{X}\to\mathbb{Z}$ hold similarly.

Finally, it is immediate to show that the identity linear functors and the identity linear transformations are inherited by **SMLDC** and that medial linear transformations are closed under the standard vertical and horizontal composition of linear transformations. ∎

6.13. PROPOSITION. *The inclusion map* inc : **CLDC** $\to$ **SMLDC** *determines a 2-functor.*

PROOF. By Theorem 5.17, every CLDC has a canonical SMLDC structure.

Consider a cartesian linear functor

$$F = (F_\times, F_+) : (\mathbb{X}, \times, \mathbf{1}, +, \mathbf{0}) \to (\mathbb{Y}, \times, \mathbf{1}, +, \mathbf{0})$$

Then, in particular, $(F_\times, m_{\mathbf{1}}^{-1}, m_\times^{-1}) : (\mathbb{X}, \times, \mathbf{1}) \to (\mathbb{Y}, \times, \mathbf{1})$ and $(F_+, n_{\mathbf{0}}^{-1}, n_+^{-1}) : (\mathbb{X}, +, \mathbf{0}) \to (\mathbb{Y}, +, \mathbf{0})$ are respectively colax monoidal and lax monoidal functors. Then, by Proposition 4.18, the following are bilax duoidal functors,

$$(F_\times, p_{\mathbf{0}}, p_+, m_{\mathbf{1}}^{-1}, m_\times^{-1}) : (\mathbb{X}, +, \mathbf{0}, \times, \mathbf{1}) \to (\mathbb{Y}, +, \mathbf{0}, \times, \mathbf{1})$$

$$(F_+, n_{\mathbf{0}}^{-1}, n_+^{-1}, q_{\mathbf{1}}, q_\times) : (\mathbb{X}, +, \mathbf{0}, \times, \mathbf{1}) \to (\mathbb{Y}, +, \mathbf{0}, \times, \mathbf{1})$$

where $p_{\mathbf{0}}, p_\times, q_{\mathbf{1}}$ and $q_\times$ are defined in Lemma . In order to prove $F = (F_\times, F_+)$ is a symmetric medial linear functor, it remains to show the additional coherence conditions.

(65) holds by the universal property of initial and terminal objects:

$$\mathbf{0} \xrightarrow{b_{F_+(\mathbf{0}\times\mathbf{0})}} F_+(\mathbf{0} \times \mathbf{0}) = \mathbf{0} \xrightarrow{\Delta_{\mathbf{0}}} \mathbf{0} \times \mathbf{0} \xrightarrow{p_{\mathbf{0}} \times n_{\mathbf{0}}^{-1}} F_\times(\mathbf{0}) \times F_\times(\mathbf{0}) \xrightarrow{\nu^R_{+\mathbf{0},\mathbf{0}}} F_+(\mathbf{0} \times \mathbf{0})$$
$$= \mathbf{0} \xrightarrow{n_{\mathbf{0}}^{-1}} F_+(\mathbf{0}) \xrightarrow{F_+(\Delta_{\mathbf{0}})} F_+(\mathbf{0} \times \mathbf{0})$$

$$F_\times(\mathbf{1} + \mathbf{1}) \xrightarrow{t_{F_\times(\mathbf{1}+\mathbf{1})}} \mathbf{1} = F_\times(\mathbf{1} + \mathbf{1}) \xrightarrow{\nu^R_{\times\mathbf{1},\mathbf{1}}} F_+(\mathbf{1}) + F_\times(\mathbf{1}) \xrightarrow{q_{\mathbf{1}} + m_{\mathbf{1}}^{-1}} \mathbf{1} + \mathbf{1} \xrightarrow{\nabla_{\mathbf{1}}} \mathbf{1}$$
$$= F_\times(\mathbf{1} + \mathbf{1}) \xrightarrow{F_\times(\nabla_{\mathbf{1}})} F_\times(\mathbf{1}) \xrightarrow{m_{\mathbf{1}}^{-1}} \mathbf{1}$$

The first coherence condition of (66) is

$$\begin{aligned}&F_\times(\mu_{A,B,C,D}); {m_\times^{-1}}_{A+C,B+D}; ({\nu^R_\times}_{A,C} \times {\nu^R_\times}_{B,D})\\ &= {\nu^R_\times}_{A\times B,C\times D}; ({q_\times}_{A,B} + {m_\times^{-1}}_{C,D}); \mu_{F_+(A),F_+(B),F_\times(C),F_\times(D)}\end{aligned}$$

and it holds as follows. The left-hand side and right-hand sides composed with the 0^{th} projection are given by

$$\begin{aligned}&F_\times(\mu_{A,B,C,D}); {m_\times^{-1}}_{A+C,B+D}; ({\nu^R_\times}_{A,C} \times {\nu^R_\times}_{B,D}); \pi^0_{F_+(A)+F_\times(C),F_+(B)+F_\times(D)}\\ &= F_\times(\mu_{A,B,C,D}); {m_\times^{-1}}_{A+C,B+D}; \pi^0_{F_\times(A+C),F_\times(B+D)}; {\nu^R_\times}_{A,C}\\ &= F_\times(\mu_{A,B,C,D}); F_\times(\pi^0_{A+C,B+D}); {\nu^R_\times}_{A,C} = F_\times(\pi^0_{A,B} + \pi^0_{C,D}); {\nu^R_\times}_{A,C}\end{aligned}$$

$$\begin{aligned}&{\nu^R_\times}_{A\times B,C\times D}; ({q_\times}_{A,B} + {m_\times^{-1}}_{C,D}); \mu_{F_+(A),F_+(B),F_\times(C),F_\times(D)}; \pi^0_{F_+(A)+F_\times(C),F_+(B)+F_\times(D)}\\ &= {\nu^R_\times}_{A\times B,C\times D}; ({q_\times}_{A,B} + {m_\times^{-1}}_{C,D}); (\pi^0_{F_+(A),F+(B)} + \pi^0_{F_\times(C),F_\times(D)})\\ &= {\nu^R_\times}_{A\times B,C\times D}; (F_+(\pi^0_{A,B}) + F_\times(\pi^0_{C,D})) = F_\times(\pi^0_{A,B} + \pi^0_{C,D}); {\nu^R_\times}_{A,C}\end{aligned}$$

Similarly, when composed with the 1^{st} projection,

$$F_\times(\mu_{A,B,C,D}); m_{\times\ A+C,B+D}^{-1}; (\nu^R_{\times\ A,C} \times \nu^R_{\times\ B,D}); \pi^0_{F_+(A)+F_\times(C),F_+(B)+F_\times(D)}$$
$$= F_\times(\pi^1_{A,B} + \pi^1_{C,D}); \nu^R_{\times\ B,D}$$

$$\nu^R_{\times\ A\times B,C\times D}; (q_{\times\ A,B} + m_{\times\ C,D}^{-1}); \mu_{F_+(A),F_+(B),F_\times(C),F_\times(D)}; \pi^1_{F_+(A)+F_\times(C),F_+(B)+F_\times(D)}$$
$$= F_\times(\pi^1_{A,B} + \pi^1_{C,D}); \nu^R_{\times\ B,D}$$

Then, by the universal properties of products, the first coherence condition of (66) holds. The second condition similarly by the universal properties of coproducts.

The first condition of (67) is

$$\nu^R_{+\ A,B\times C}; F_+(\alpha_{\times\ A,B,C}^{-1}); q_{\times\ A\times B,C}$$
$$= (1_{F_\times(A)} \times q_{\times\ B,C}); \alpha_{\times\ F_\times(A),F_+(B),F_+(C)}^{-1}; (\nu^R_{+\ A,B} \times 1_{F_+(C)})$$

The left-hand side and right-hand sides composed with the projections are given by

$$\nu^R_{+\ A,B\times C}; F_+(\alpha_{\times\ A,B,C}^{-1}); q_{\times\ A\times B,C}; \pi^0_{F_+(A\times B),F_+(C)} = \nu^R_{+\ A,B\times C}; F_+(\alpha_{\times\ A,B,C}^{-1}); F_+(\pi^0_{A\times B,C})$$
$$= \nu^R_{+\ A,B\times C}; F_+(1_A \times \pi^0_{B,C}) = (1_{F_\times(A)} \times F_+(\pi^0_{B,C})); \nu^R_{+\ A,B}$$

$$(1_{F_\times(A)} \times q_{\times\ B,C}); \alpha_{\times\ F_\times(A),F_+(B),F_+(C)}^{-1}; (\nu^R_{+\ A,B} \times 1_{F_+(C)}); \pi^0_{F_+(A\times B),F_+(C)}$$
$$= (1_{F_\times(A)} \times q_{\times\ B,C}); \alpha_{\times\ F_\times(A),F_+(B),F_+(C)}^{-1}; \pi^0_{F_\times(A)\times F_+(B),F_+(C)}; \nu^R_{+\ A,B}$$
$$= (1_{F_\times(A)} \times q_{\times\ B,C}); (1_{F_\times(A)} \times \pi^0_{F_+(B),F_+(C)}); \nu^R_{+\ A,B} = (1_{F_\times(A)} \times F_+(\pi^0_{B,C})); \nu^R_{+\ A,B}$$

$$\nu^R_{+\ A,B\times C}; F_+(\alpha_{\times\ A,B,C}^{-1}); q_{\times\ A\times B,C}; \pi^1_{F_+(A\times B),F_+(C)} = \nu^R_{+\ A,B\times C}; F_+(\alpha_{\times\ A,B,C}^{-1}); F_+(\pi^1_{A\times B,C})$$
$$= \nu^R_{+\ A,B\times C}; F_+(\pi^1_{A,B\times C}); F_+(\pi^1_{B,C}) = \nu^R_{+\ A,B\times C}; F_+(!_A \times 1_{B\times C}); F_+(u^{L\ -1}_{\times B\times C}); F_+(\pi^1_{B,C})$$
$$= (F_\times(!_A) \times F_+(1_{B\times C})); \nu^R_{+\mathbf{1},B\times C}; F_+(u^{L\ -1}_{\times B\times C}); F_+(\pi^1_{B,C})$$
$$= (F_\times(!_A) \times F_+(1_{B\times C})); (m_{\mathbf{1}}^{-1} \times 1_{F_+(B\times C)}); u^{L\ -1}_{\times F_+(B\times C)}; F_+(\pi^1_{B,C}) = \pi^1_{F_\times(A),F_+(B\times C)}; F_+(\pi^1_{B,C})$$

$$(1_{F_\times(A)} \times q_{\times\ B,C}); \alpha_{\times\ F_\times(A),F_+(B),F_+(C)}^{-1}; (\nu^R_{+\ A,B} \times 1_{F_+(C)}); \pi^1_{F_+(A\times B),F_+(C)}$$
$$= (1_{F_\times(A)} \times q_{\times\ B,C}); \alpha_{\times\ F_\times(A),F_+(B),F_+(C)}^{-1}; \pi^1_{F_\times(A)\times F_+(B),F_+(C)}$$
$$= (1_{F_\times(A)} \times q_{\times\ B,C}); \pi^1_{F_\times(A),F_+(B)\times F_+(C)}; \pi^1_{F_+(B),F_+(C)}$$
$$= \pi^1_{F_\times(A),F_+(B\times C)}; q_{\times\ B,C}; \pi^1_{F_+(B),F_+(C)} = \pi^1_{F_\times(A),F_+(B\times C)}; F_+(\pi^1_{B,C})$$

Then, by the universal property of products, the first condition of (67) holds. The second condition of (67) follows by the universal properties of coproducts for the same reasons as its counterpart.

Consider a linear transformation $\alpha = (\alpha_\times, \alpha_+) : F \Rightarrow G : \mathbb{X} \to \mathbb{Y}$ between cartesian linear functors. In particular, it is a pair consisting of a monoidal transformation $\alpha_\times : (F_\times, m^F_{\mathbf{1}}, m^F_\times) \Rightarrow (G_\times, m^G_{\mathbf{1}}, m^G_\times)$ and $\alpha_+ : (F_+, n^F_{\mathbf{0}}, n^F_+) \Rightarrow (G_+, n^G_{\mathbf{0}}, n^G_+)$. By Proposition 6.11, $\alpha_\times : (F, p^F_{\mathbf{0}}, p^F_+) \Rightarrow (G, p^G_{\mathbf{0}}, p^G_+)$ is a monoidal transformation and $\alpha_+ : (F_+, q^F_{\mathbf{1}}, q^F_\times) \Rightarrow (G_+, q^G_{\mathbf{1}}, q^G_\times)$ is a monoidal transformation. As such, α is a medial linear transformation.

Therefore, the inclusion map $\mathrm{inc} : \mathbf{CLDC} \to \mathbf{SMLDC}$ is well-defined. The proof that it is moreover a 2-functor follows by examining that canonical medial linear functor structure of a cartesian linear functor described above respects composition and identities. ■

It is fairly immediate to see that the above results can restricted to Frobenius linear functors.

6.14. Corollary. *Symmetric medial linearly distributive categories, symmetric Frobenius medial linear functors and medial linear transformations form a sub 2-category of* **SMLDC***, denoted by* **SFMLDC**. *Further, the inclusion map* $\mathrm{inc} : \mathbf{CLDC} \to \mathbf{SMLDC}$ *restricts to an inclusion 2-functor* $\mathrm{inc} : \mathbf{FCLDC} \to \mathbf{SFMLDC}$

Proof. The only technicality to verify is that given a Frobenius cartesian linear functor $F = (F_\times, F_+) : \mathbb{X} \to \mathbb{Y}$, the induced symmetric medial linear functor is Frobenius, in other words that the canonical $+$-lax monoidal structure on $(F_\times, p_{\mathbf{0}}, p_+) : (\mathbb{X}, \mathbf{0}, +) \to (\mathbb{Y}, \mathbf{0}, +)$ corresponds to $(F_+, n_{\mathbf{0}}^{-1}, n_+^{-1}) : (\mathbb{X}, \mathbf{0}, +) \to (\mathbb{Y}, \mathbf{0}, +)$ and similarly for the canonical $\times$-colax monoidal structure on F_+. This is immediate however as $F_+ = F_\times$. While the structure maps $n_{\mathbf{0}}$ and $n_{+A,B}$ are not guaranteed to have inverses, if they do, they must be the canonical ones $p_{\mathbf{0}}$ and $p_{+A,B}$. Similarly, $q_{\mathbf{1}} = {m_{\mathbf{1}}}^{-1}$ and $q_{\times A,B} = {m_\times^{-1}}_{A,B}$. ■

7. Linearly Distributive Fox Theorem

It is now time to discuss the construction on a SMLDCs which will yield a CLDCs, mirroring the category of cocommutative comonoids of a symmetric monoidal category to create a cartesian category in the standard Fox theorem.

If we review our earlier characterization of CLDCs in Proposition 5.1, which followed as a direct consequence of the Fox theorem and its dual, we see that a CLDC $\mathbb{X}$ is in particular a SLDC, where each object $A \in \mathbb{X}$ is equipped with four morphisms

$$\Delta_A : A \to A \otimes A \qquad e_A : A \to \top \qquad \nabla_A : A \oplus A \to A \qquad u_A : \bot \to A$$

such that $\langle A, \Delta_A, e_A \rangle$ is a cocommutative $\otimes$-comonoid and $\langle A, \nabla_A, u_A \rangle$ is a commutative $\oplus$-monoid.

To someone well-versed in duoidal category theory, this structure will be familiar. Indeed, these are the same maps and structures of a duoidal bimonoid in a duoidal category. This is the key to the theorem and why the duoidal structure of the MLDCs is essential.

7.1. Medial Bimonoids.

7.2. Definition. *Let $\mathbb{X}$ be a symmetric medial linearly distributive category.*

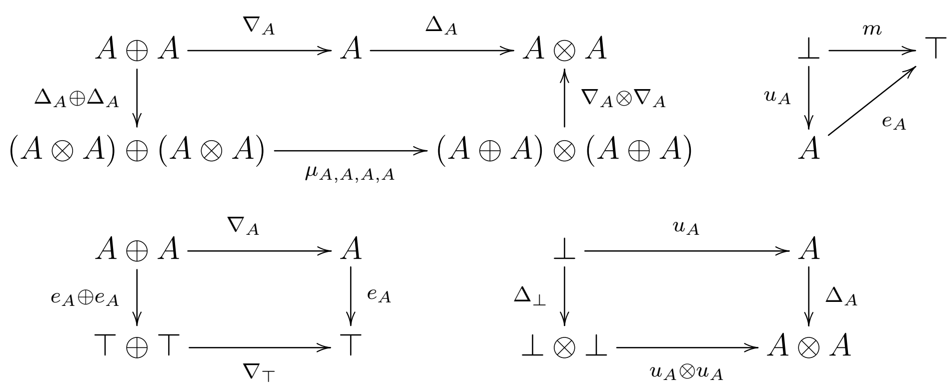
- *A* **medial bimonoid** *in $\mathbb{X}$ is a quintuple $\langle A, \Delta_A, u_A, \nabla_A, e_A \rangle$ consisting of an object A and four morphisms*

  $$\Delta_A : A \to A \otimes A \qquad e_A : A \to \top \qquad \nabla_A : A \oplus A \to A \qquad u_A : \bot \to A$$

  in $\mathbb{X}$ such that

  - $\langle A, \Delta_A, e_A \rangle$ *is a $\otimes$-comonoid,*
  - $\langle A, \nabla_A, u_A \rangle$ *is a $\oplus$-monoid,*

  satisfying coherence conditions between $\otimes$-comonoid and $\oplus$-monoid structure maps (70).



(70)

  Alternatively, a medial bimonoid in $\mathbb{X}$ is a duoidal bimonoid in the duoidal category $(\mathbb{X}, \oplus, \bot, \otimes, \top)$.

- *A medial bimonoid $\langle A, \Delta_A, e_A, \nabla_A, u_A \rangle$ is* **bicommutative** *if $\langle A, \Delta_A, e_A \rangle$ is cocommutative $\otimes$-comonoid and $\langle A, \nabla_A, u_A \rangle$ is a commutative $\oplus$-monoid.*

- *A* **bimonoid morphism** *$f : \langle A, \Delta_A, u_A, \nabla_A, e_A \rangle \to \langle B, \Delta_B, u_B, \nabla_B, e_B \rangle$ is a morphism $f : A \to B$ in $\mathbb{X}$ such that*

  - *$f : \langle A, \Delta_A, e_A \rangle \to \langle B, \Delta_B, e_B \rangle$ is a $\otimes$-comonoid morphism, and*

– *$f : \langle A, \nabla_A, u_A \rangle \to \langle B, \nabla_B, u_B \rangle$ is a $\oplus$-monoid morphism*

It is immediate that the $\otimes$ and $\oplus$ units in a SMLDC have canonical medial bimonoid structures given by the duoidal structure maps, as the $\otimes$-comonoid structure of $\bot$ and $\oplus$-monoid structure for $\top$ are part of the definition of a MLDC, and the other coherence conditions follow easily.

7.3. Proposition. *Given a symmetric medial linearly distributive category $\mathbb{X}$, the following*

$$\langle \top, u_{\otimes\top} : \top \to \top \otimes \top, 1_\top : \top \to \top, \nabla_\top : \top \oplus \top \to \top, m : \bot \to \top \rangle$$
$$\langle \bot, \Delta_\bot : \bot \to \bot \otimes \bot, m : \bot \to \top, u_{\oplus\bot} : \bot \oplus \bot \to \bot, 1_\bot : \bot \to \bot \rangle$$

are bicommutative bimonoids.

Furthermore, medial bimonoids combine together via the tensor and par monoidal products to produce new medial bimonoids as follows.

7.4. Proposition. *Consider a pair of bicommutative medial bimonoids $\langle A, \Delta_A, e_A, \nabla_A, u_A \rangle$ and $\langle B, \Delta_B, e_B, \nabla_B, u_B \rangle$ in a symmetric medial linearly distributive category $\mathbb{X}$. Then, the object $A \otimes B$ equipped with morphisms*

$$\Delta_{A\otimes B} = A \otimes B \xrightarrow{\Delta_A \otimes \Delta_B} (A \otimes A) \otimes (B \otimes B) \xrightarrow{\tau^{\otimes}_{A,A,B,B}} (A \otimes B) \otimes (A \otimes B)$$
$$t_{A\otimes B} = A \otimes B \xrightarrow{e_A \otimes e_B} \top \otimes \top \xrightarrow{u_{\otimes\top}^{-1}} \top$$
$$\nabla_{A\otimes B} = (A \otimes B) \oplus (A \otimes B) \xrightarrow{\mu_{A,B,A,B}} (A \oplus A) \otimes (B \oplus B) \xrightarrow{\nabla_A \otimes \nabla_B} A \otimes B$$
$$u_{A\otimes B} = \bot \xrightarrow{\Delta_\bot} \bot \otimes \bot \xrightarrow{u_A \otimes u_B} A \otimes B$$

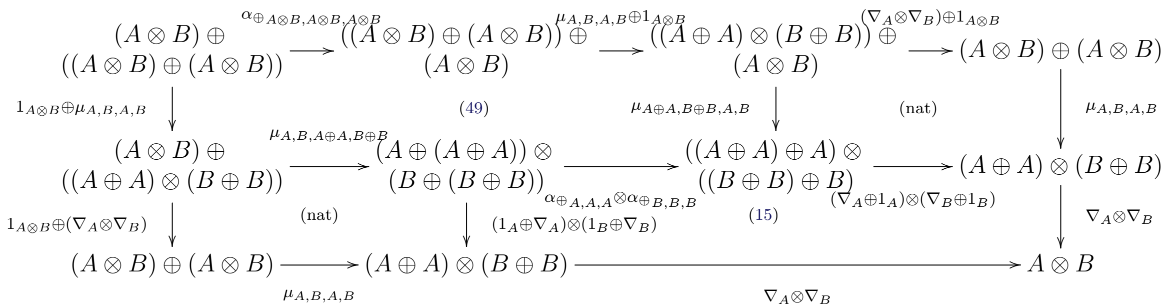
and the object $A \oplus B$ equipped with morphisms

$$\Delta_{A\oplus B} = A \oplus B \xrightarrow{\Delta_A \oplus \Delta_B} (A \otimes A) \oplus (B \otimes B) \xrightarrow{\mu_{A,A,B,B}} (A \oplus B) \otimes (A \oplus B)$$
$$t_{A\oplus B} = A \oplus B \xrightarrow{e_A \oplus e_B} \top \oplus \top \xrightarrow{\nabla_\top} \top$$
$$\nabla_{A\oplus B} = (A \oplus B) \oplus (A \oplus B) \xrightarrow{\tau^{\oplus}_{A,B,A,B}} (A \oplus A) \oplus (B \oplus B) \xrightarrow{\nabla_A \oplus \nabla_B} A \oplus B$$
$$u_{A\oplus B} = \bot \xrightarrow{u_{\oplus\bot}^{-1}} \bot \oplus \bot \xrightarrow{u_A \oplus u_B} A \oplus B$$

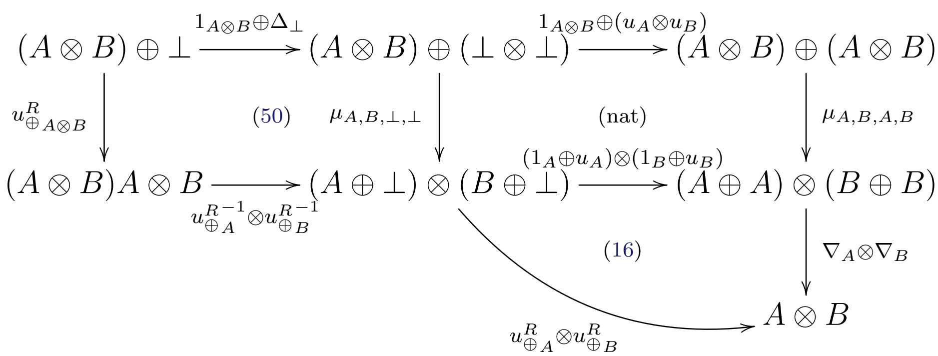
are bicommutative medial bimonoids.

Proof. It follows immediately, by Lemma 2.17 that $\langle A \otimes B, \Delta_{A\otimes B}, t_{A\otimes B} \rangle$ is a cocommutative $\otimes$-comonoid. $\langle A \otimes B, \nabla_{A\otimes B}, e_{A\otimes B} \rangle$ is a commutative $\oplus$-monoid as follows:

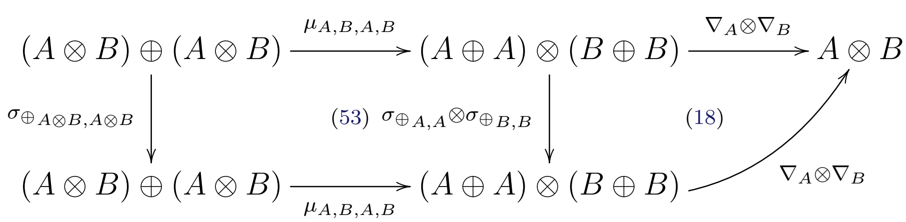
$\nabla_{A\otimes B}$ satisfies the associativity coherence condition in (15) by the commuting diagram below.



$\nabla_{A\otimes B}$ and $e_{A\otimes B}$ satisfy the unitality coherence condition in (16) by the commuting diagram below.

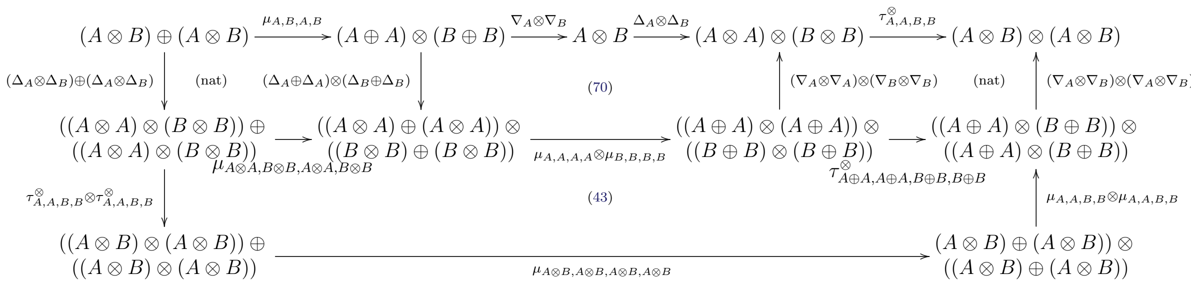


Finally, $\nabla_{A\otimes B}$ satisfies the commutativity condition (18) by the commuting diagram below.

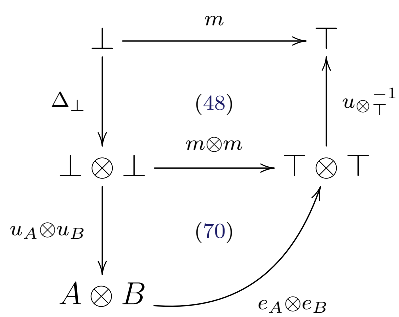


It remains to show the four coherence conditions in (70) hold.

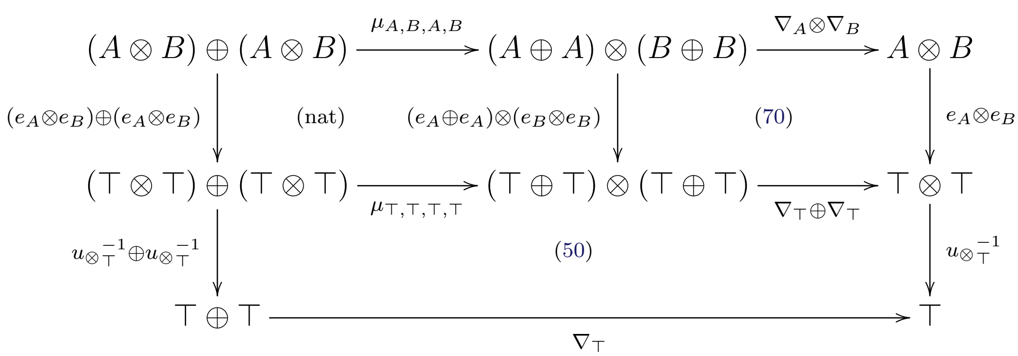
$\Delta_{A\otimes B}$ and $\nabla_{A\otimes B}$ satisfy the first diagram in (70) by the following commuting diagram:



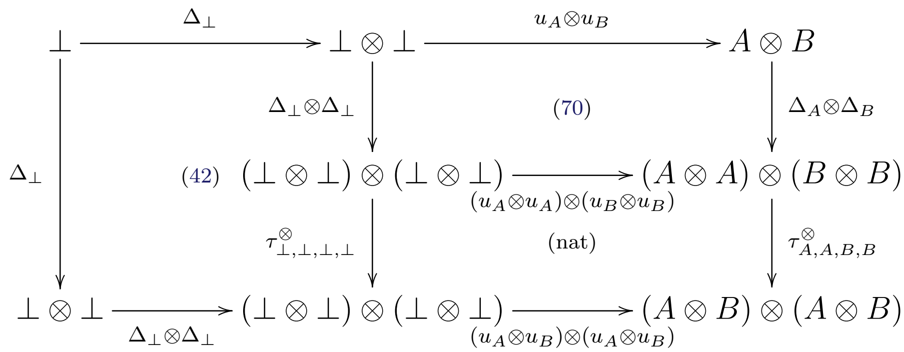
$u_{A\otimes B}$ and $e_{A\otimes B}$ satisfy the second diagram in (70) by the following commuting diagram:



$\nabla_{A\otimes B}$ and $e_{A\otimes B}$ satisfy the third diagram in (70) by the following commuting diagram:



Finally, $\Delta_{A\otimes B}$ and $u_{A\otimes B}$ satisfy the fourth diagram in (70) by the following commuting diagram:



The proof that $A \oplus B$ is a bicommutative medial bimonoid follows similarly. ■

Let us for a moment consider our most important class of SMLDCs, the CLDCs, and discuss the medial bimonoids present in these cases.

7.5. Proposition. *Every object A in a cartesian linearly distributive category $\mathbb{X}$ has a canonical unique bicommutative medial bimonoid structure*

$$\langle A, \langle 1_A, 1_A \rangle : A \to A \times A, t_A : A \to \mathbf{1}, [1_A, 1_A] : A + A \to A, b_A : \mathbf{0} \to A \rangle$$

and every arrow $f : A \to B$ in $\mathbb{X}$ is a medial bimonoid morphism with respect to these canonical structures.

Proof. By Lemma 2.16, it is clear $\langle A, \langle 1_A, 1_A \rangle, t_A \rangle$ is the unique cocommutative $\times$-comonoid structure on A and, dually, $\langle A, [1_A, 1_A], b_A \rangle$ is the unique commutative $+$-monoid structure on A. The proof the structure maps equally satisfy the additional medial bimonoid coherence conditions is a simple exercises using the canonical medial structure of a cartesian LDC and universal properties. Once more, by Lemma 2.16 and its dual, every arrow is a medial bimonoid morphism with respect to these structures. ■

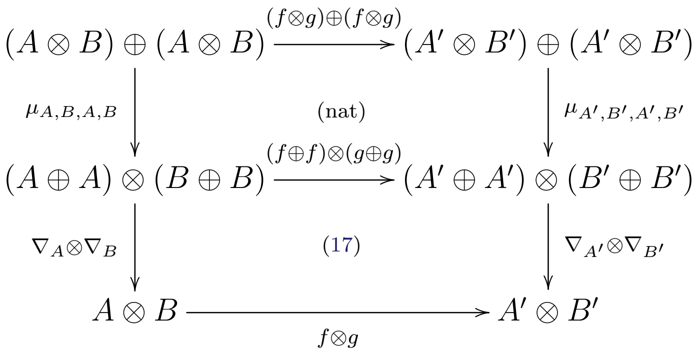
7.6. Definition. *Let $\mathbb{X}$ be a symmetric medial linearly distributive category. Define* $\mathrm{B}[\mathbb{X}]$ *to be the category of bicommutative medial bimonoids and medial bimonoid morphisms in $\mathbb{X}$.*

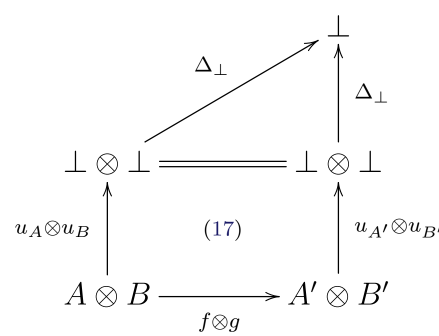
The fact that this is in fact a category is an easy exercise, as the category of commutative $\oplus$-monoids and of cocommutative $\otimes$-comonoids in $\mathbb{X}$ would themselves be categories. The above propositions will be the key to showing that $\mathrm{B}[\mathbb{X}]$ is more than a simple category, but is in fact a SLDC.

7.7. Lemma. *Let $\mathbb{X}$ be a symmetric medial linearly distributive category, then* $\mathrm{B}[\mathbb{X}]$ *is a symmetric linearly distributive category, with $\otimes$ and $\oplus$ monoidal structures as defined in Propositions 7.3 and 7.4.*

Proof. It is sufficient to show that given medial bimonoid morphisms $f : A \to B$ and $g : A' \to B'$, $f \otimes g : A \otimes B \to A' \otimes B'$ and $f \oplus g : A \oplus B \to A' \oplus B'$ are medial bimonoid morphisms, and that all the LDC structure maps are bimonoid morphisms as well to prove that B[$\mathbb{X}$] is a LDC, as composition is inherited directly from $\mathbb{X}$.

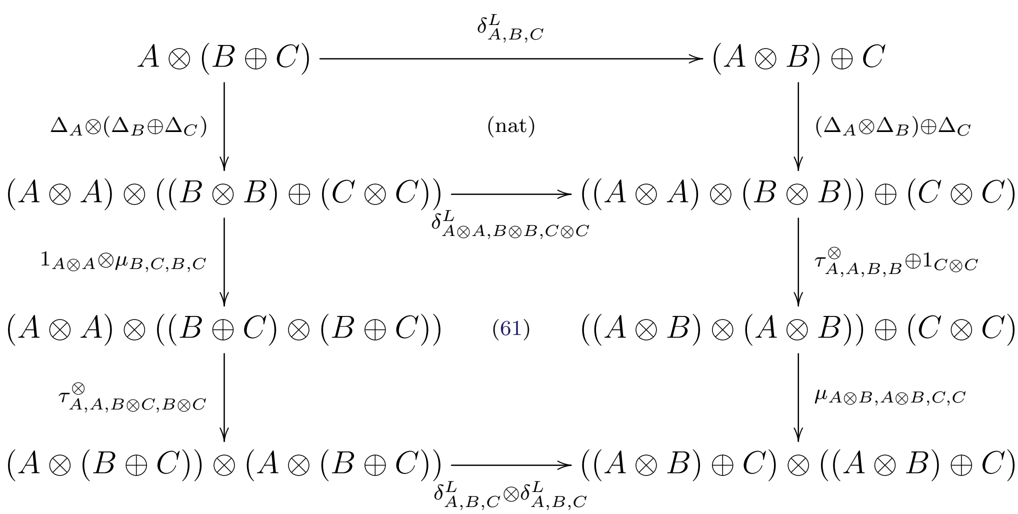
Firstly, it follows from Lemma 2.17 that $f \otimes g$ is $\otimes$-comonoid morphism, thus we need only show it is also a $\oplus$-monoid morphism, i.e. satisfies (17), which is proved by the following commuting diagrams:

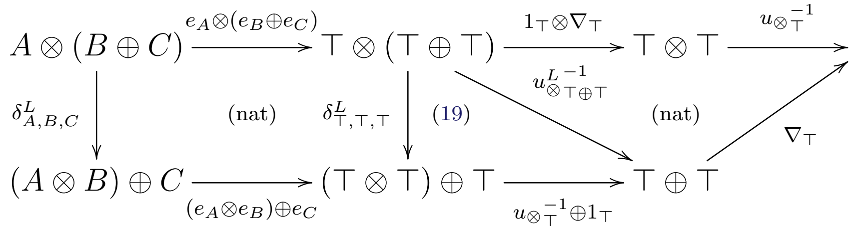


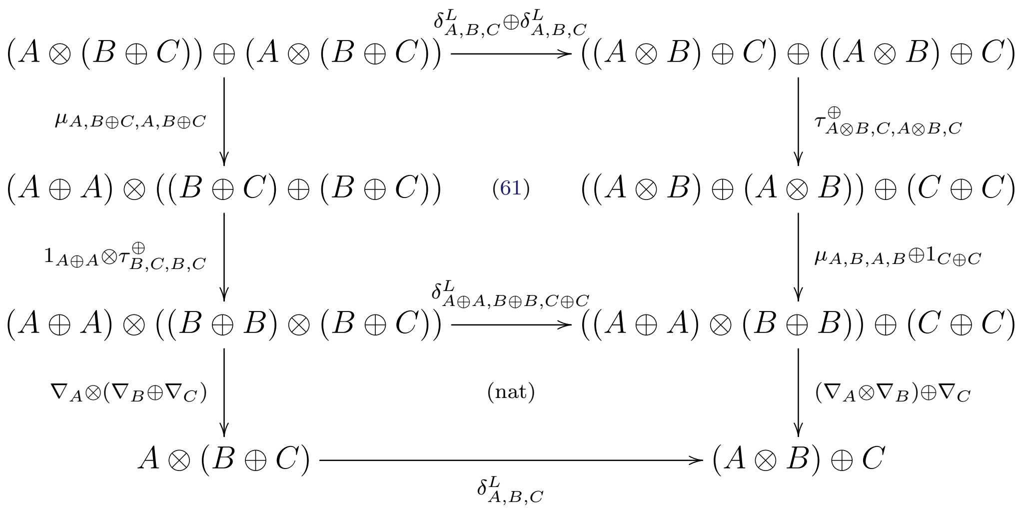


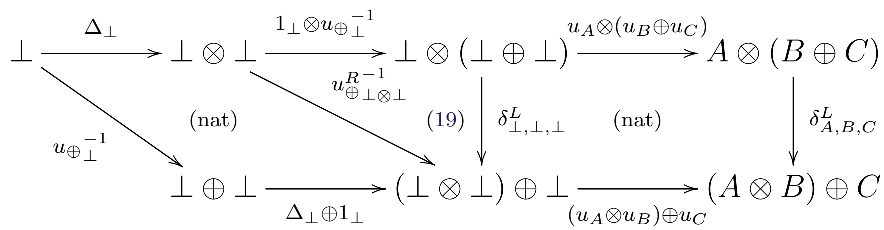
The proof that $f \oplus g$ is a medial bimonoid morphism follows similarly.

To show that the left linear distributivity is a bimonoid morphism, we need to show it is a $\otimes$-comonoid morphism and a $\oplus$-monoid morphism, which follows respectively from the commuting diagrams below:









The proofs that the other structure maps, i.e. the associators, unitors and braidings for the tensor and par monoidal products, follow similarly, involving only coherence conditions from the duoidal structure of $\mathbb{X}$. They are left up to the reader. ■

7.8. THEOREM. *Let $\mathbb{X}$ be a symmetric medial linearly distributive category, then* $\mathrm{B}[\mathbb{X}]$ *is a cartesian linearly distributive category.*

PROOF. We will prove this by showing $\mathrm{B}[\mathbb{X}]$ satisfies the characterization in Proposition 5.1.

Given a bicommutative medial bimonoid $\langle A, \Delta_A, e_A, \nabla_A, u_A \rangle$, then the coherence conditions in (70) ensure that the four bimonoid maps Δ_A, e_A, ∇_A and u_A are themselves medial bimonoid morphisms. These give exactly the component maps of the four necessary natural transformations. These in fact determine natural transformations as the naturality squares correspond exactly to the conditions for every map to be a bimonoid morphism.

The conditions that the component maps form comonoid and monoid structures is immediate by the definition of a bicommutative medial bimonoid and the remaining coherence conditions are satisfied as they correspond exactly to the construction of the tensor and par monoidal structures in B[$\mathbb{X}$] as outlined in Propositions 7.3 and 7.4. ∎

And with this theorem, the first part of the linearly distributive Fox theorem has been proven. It remains only to show that the B[−] construction extends canonically to the symmetric medial linear functors and the medial linear transformation, giving a right adjoint to the inclusion functor outlined in Proposition 6.13.

7.9. Example. Consider a symmetric monoidal category $(\mathcal{X}, \oslash, I)$ viewed as a symmetric medial linearly distributive category $(\mathcal{X}, \oslash, I, \oslash, I)$, then B[$\mathcal{X}$] is the category of bicommutative bimonoids of $\mathcal{X}$, in the sense of [38]. For example, if we consider the category of vector spaces Vect with its standard tensor product, then B[Vect] is the category of bialgebras, in the traditional sense.

It is also important to note that negation is preserved by the B[−] construction.

7.10. Proposition. *Let* $\mathbb{X}$ *be a symmetric medial linearly distributive category with negation, then* B[$\mathbb{X}$] *is a cartesian linearly distributive category with negation, and therefore a bounded distributive lattice.*

Proof. As $\mathbb{X}$ is a symmetric medial LDC with negation, there is a full and faithful functor $(-)^\perp : \mathbb{X}^{op} \to \mathbb{X}$ such that $\mathbb{X}(A \otimes B, C^\perp) \cong \mathbb{X}(A, (B \oplus C)^\perp)$ defined in , in other words $\mathbb{X}$ is a $*$-autonomous category. We shall show that $(-)^\perp$ extends to a functor on B[$\mathbb{X}$], meaning B[$\mathbb{X}$] is a $*$-autonomous category whose monoidal product is cartesian. Then, by Joyal's paradox, it is a bounded distributive lattice.

Consider a bicommutative medial bimonoid $\langle A, \Delta_A, e_A, \nabla_A, u_A \rangle$, then define $\langle A^\perp, \Delta_{A^\perp}, e_{A^\perp}, \nabla_{A^\perp}, u_{A^\perp} \rangle$ by

$$\Delta_{A^\perp} = A^\perp \xrightarrow{(\nabla_A)^\perp} (A \oplus A)^\perp \xrightarrow{\epsilon_{A,A}} A^\perp \otimes A^\perp \qquad e_{A^\perp} = A^\perp \xrightarrow{(u_A)^\perp} \bot^\perp \xrightarrow{\beta_\top} \top$$

$$\nabla_{A^\perp} = A^\perp \oplus A^\perp \xrightarrow{\phi_{A,A}} (A \otimes A)^\perp \xrightarrow{(\Delta_A)^\perp} A^\perp \qquad u_{A^\perp} = \bot \xrightarrow{\chi_\bot} \top^\perp \xrightarrow{(e_A)^\perp} A^\perp$$

This is a bicommutative medial bimonoid as follows.

$\langle A^\perp, \Delta_{A^\perp}, e_{A^\perp}\rangle$ is a cocommutative $\otimes$-comonoid as:

$$\begin{aligned}
\Delta_{A^\perp}; (1_{A^\perp} \otimes e_{A^\perp}) &= (\nabla_A)^\perp; \epsilon_{A,A}; (1_{A^\perp} \otimes (u_A)^\perp); (1_{A^\perp} \otimes \beta_\top) && \text{by definition}\\
&= (\nabla_A)^\perp; (u_A \oplus 1_A)^\perp; \epsilon_{\perp,A}; (1_{A^\perp} \otimes \beta_\top) && \text{by naturality of } \epsilon\\
&= ((u_A \oplus 1_A); \nabla_A)^\perp; \epsilon_{\perp,A}; (1_{A^\perp} \otimes \beta_\top) && \text{by functoriality of } (-)^\perp\\
&= (u^A_{\otimes\perp})^\perp; \epsilon_{\perp,A}; (1_{A^\perp} \otimes \beta_\top) && \text{as } A \text{ is a medial bimonoid}\\
&= u^R_{\otimes A^\perp} && \text{by } (*)
\end{aligned}$$

$$\begin{aligned}
&\Delta_{A^\perp}; (\Delta_{A^\perp} \otimes 1_{A^\perp}); \alpha_{\otimes A^\perp, A^\perp, A^\perp}\\
&= (\nabla_A)^\perp; \epsilon_{A,A}; ((\nabla_A)^\perp \otimes 1_{A^\perp}); (\epsilon_{A,A} \otimes 1_{A^\perp}); \alpha_{\otimes A^\perp, A^\perp, A^\perp} && \text{by definition}\\
&= (\nabla_A)^\perp; (1_A \oplus \nabla_A)^\perp; \epsilon_{A,A\oplus A}; (\epsilon_{A,A} \otimes 1_{A^\perp}); \alpha_{\otimes A^\perp, A^\perp, A^\perp} && \text{by naturality of } \epsilon\\
&= ((1_A \oplus \nabla_A); \nabla_A)^\perp; \epsilon_{A,A\oplus A}; (\epsilon_{A,A} \otimes 1_{A^\perp}); \alpha_{\otimes A^\perp, A^\perp, A^\perp} && \text{by functoriality of } (-)^\perp\\
&= (\alpha_{\oplus A,A,A}; (\nabla_A \oplus 1_A); \nabla_A)^\perp; \epsilon_{A,A\oplus A};\\
&\qquad (\epsilon_{A,A} \otimes 1_{A^\perp}); \alpha_{\otimes A^\perp, A^\perp, A^\perp} && \text{as } A \text{ is a medial bimonoid}\\
&= (\nabla_A)^\perp; (\nabla_A \oplus 1_A)^\perp; (\alpha_{\oplus A,A,A})^\perp;\\
&\qquad \epsilon_{A,A\oplus A}; (\epsilon_{A,A} \otimes 1_{A^\perp}); \alpha_{\otimes A^\perp, A^\perp, A^\perp} && \text{by functoriality of } (-)^\perp\\
&= (\nabla_A)^\perp; (\nabla_A \oplus 1_A)^\perp; \epsilon_{A\oplus A,A}; (1_{A^\perp} \otimes \epsilon_{A,A}) && \text{by } (*)\\
&= (\nabla_A)^\perp; \epsilon_{A,A}; (1_{A^\perp} \otimes (\nabla_A)^\perp); (1_{A^\perp} \otimes \epsilon_{A,A}) && \text{by naturality of } \epsilon\\
&= \Delta_{A^\perp}; (1_{A^\perp} \otimes \Delta_{A^\perp}) && \text{by definition}
\end{aligned}$$

$$\begin{aligned}
\Delta_{A^\perp}; \sigma_{\otimes A^\perp, A^\perp} &= (\nabla_A)^\perp; \epsilon_{A,A}; \sigma_{\otimes A^\perp, A^\perp} && \text{by definition}\\
&= (\nabla_A)^\perp; (\sigma_{\oplus A,A})^\perp; \epsilon_{A,A} && \text{by } (*)\\
&= (\sigma_{\oplus A,A}; \nabla_A)^\perp; \epsilon_{A,A} && \text{by functoriality of } (-)^\perp\\
&= (\nabla_A)^\perp; \epsilon_{A,A} && \text{as } A \text{ is a medial bimonoid}\\
&= \Delta_{A^\perp} && \text{by definition}
\end{aligned}$$

The equalities marked with $(*)$ are incredibly long to show directly, but fairly straightforward if one uses the circuit diagram (or proof net) calculus introduced by Blute, Cockett, Seely and Trimble [5].

Similarly, $\langle A, \nabla_{A^\perp}, u_{A^\perp}\rangle$ is a commutative $\oplus$-monoid.

The first coherence condition of (70) holds as:

$$\begin{aligned}
\nabla_{A^\perp}; \Delta_{A^\perp} &= \phi_{A,A}; (\Delta_A)^\perp; (\nabla_A)^\perp; \epsilon_{A,A} && \text{by definition}\\
&= \phi_{A,A}; (\nabla_A; \Delta_A)^\perp; \epsilon_{A,A} && \text{by functoriality of } (-)^\perp\\
&= \phi_{A,A}; ((\Delta_A \oplus \Delta_A); \mu_{A,A,A,A}; (\nabla_A \oplus \nabla_A))^\perp; \epsilon_{A,A} && \text{as } A \text{ is a medial bimonoid}\\
&= \phi_{A,A}; (\nabla_A \oplus \nabla_A)^\perp (\mu_{A,A,A,A})^\perp; (\Delta_A \oplus \Delta_A)^\perp; \epsilon_{A,A} && \text{by functoriality of } (-)^\perp\\
&= ((\nabla_A)^\perp \otimes (\nabla_A)^\perp); \phi_{A\oplus A, A\oplus A};\\
&\quad (\mu_{A,A,A,A})^\perp; \epsilon_{A\otimes A, A\otimes A}; ((\Delta_A)^\perp \otimes (\Delta_A)^\perp) && \text{by naturality of } \phi \text{ and } \epsilon\\
&= ((\nabla_A)^\perp \otimes (\nabla_A)^\perp); (\epsilon_{A,A} \otimes \epsilon_{A,A});\\
&\quad \mu_{A^\perp,A^\perp,A^\perp,A^\perp}; (\phi_{A,A} \oplus \phi_{A,A}); ((\Delta_A)^\perp \otimes (\Delta_A)^\perp) && \text{by (59)}\\
&= (\Delta_{A^\perp} \oplus \Delta_{A^\perp}); \mu_{A^\perp,A^\perp,A^\perp,A^\perp}; (\nabla_{A^\perp} \otimes \nabla_{A^\perp}) && \text{by definition}
\end{aligned}$$

The second coherence condition of (70) holds as:

$$\begin{aligned}
u_{A^\perp}; e_{A^\perp} &= \chi_\perp; (e_A)^\perp; (u_A)^\perp; \beta_\top && \text{by definition}\\
&= \chi_\perp; (u_A; e_A)^\perp; \beta_\top && \text{by functoriality of } (-)^\perp\\
&= \chi_\perp; (m)^\perp; \beta_\top && \text{as } A \text{ is a medial bimonoid}\\
&= m && \text{by (60)}
\end{aligned}$$

The third coherence condition of (70) holds as:

$$\begin{aligned}
\nabla_{A^\perp}; e_{A^\perp} &= \phi_{A,A}; (\Delta_A)^\perp; (u_A)^\perp; \beta_\top && \text{by definition}\\
&= \phi_{A,A}; (u_A; \Delta_A)^\perp; \beta_\top && \text{by functoriality of } (-)^\perp\\
&= \phi_{A,A}; (\Delta_\perp; (u_A \otimes u_A))^\perp; \beta_\top && \text{as } A \text{ is a medial bimonoid}\\
&= \phi_{A,A}; (u_A \otimes u_A)^\perp; (\Delta_\perp)^\perp; \beta_\top && \text{by functoriality of } (-)^\perp\\
&= (u_A)^\perp \oplus (u_A)^\perp; \phi_{\perp,\perp}; (\Delta_\perp)^\perp; \beta_\top && \text{by naturality of } \phi\\
&= (u_A)^\perp \oplus (u_A)^\perp; (\beta_\top \oplus \beta_\top); \nabla_\top && \text{by (58)}\\
&= (e_{A^\perp} \oplus e_{A^\perp}); \nabla_\top && \text{by definition}
\end{aligned}$$

The final coherence condition of (70) holds as:

$$
\begin{aligned}
u_{A^\perp}; \Delta_{A^\perp} &= \chi_\perp; (e_A)^\perp; (\nabla_A)^\perp; \epsilon_{A,A} && \text{by definition} \\
&= \chi_\perp; (\nabla_A; e_A)^\perp; \epsilon_{A,A} && \text{by functoriality of } (-)^\perp \\
&= \chi_\perp; ((e_A \oplus e_A); \nabla_\top)^\perp; \epsilon_{A,A} && \text{as } A \text{ is a medial bimonoid} \\
&= \chi_\perp; (\nabla_\top)^\perp; (e_A \oplus e_A)^\perp; \epsilon_{A,A} && \text{by functoriality of } (-)^\perp \\
&= \chi_\perp; (\nabla_\top)^\perp; \epsilon_{\top,\top}; ((e_A)^\perp \otimes (e_A)^\perp) && \text{by naturality of } \epsilon \\
&= \Delta_\perp; (\chi_\perp \otimes \chi_\perp); ((e_A)^\perp \otimes (e_A)^\perp) && \text{by (58)} \\
&= \Delta_\perp; (u_{A^\perp} \oplus u_{A^\perp}) && \text{by definition}
\end{aligned}
$$

Now, let $f : \langle A, \Delta_A, e_A, \nabla_A, u_A \rangle \to \langle B, \Delta_B, e_B, \nabla_B, u_B \rangle$ be a medial bimonoid morphism, then $f^\perp : B^\perp \to A^\perp$ is a $\otimes$-comonoid morphism as follows.

$$
\begin{aligned}
\Delta_{B^\perp}; (f^\perp \otimes f^\perp) &= (\nabla_B)^\perp; \epsilon_{B,B}; (f^\perp \otimes f^\perp) && \text{by definition} \\
&= (\nabla_B)^\perp; (f \oplus f)^\perp; \epsilon_{A,A} && \text{by naturality of } \epsilon \\
&= ((f \oplus f); \nabla_B)^\perp; \epsilon_{A,A} && \text{by functoriality of } (-)^\perp \\
&= (\nabla_A; f)^\perp; \epsilon_{A,A} && \text{as } f \text{ is a medial bimonoid morphism} \\
&= f^\perp; (\nabla_A)^\perp; \epsilon_{A,A} && \text{by functoriality of } (-)^\perp \\
&= f^\perp; \Delta_{A^\perp}
\end{aligned}
$$

$$
\begin{aligned}
f^\perp; e_{A^\perp} &= f^\perp; (u_A)^\perp; \beta_\top && \text{by definition} \\
&= (u_A; f)^\perp; \beta_\top && \text{by functoriality of } (-)^\perp \\
&= (u_B)^\perp; \beta_\top && \text{as } f \text{ is a medial bimonoid morphism} \\
&= u_{B^\perp} && \text{by definition}
\end{aligned}
$$

Similarly, $f^\perp : B^\perp \to A^\perp$ is a $\oplus$-monoid morphism and therefore a medial bimonoid morphism.

■

7.11. The Right Adjoint 2-Functor B [-].

7.12. Lemma. *Consider a symmetric medial linear functor* $F = (F_\otimes, F_\oplus) : \mathbb{X} \to \mathbb{Y}$ *between symmetric medial linearly distributive categories, then it canonically extends to a cartesian linear functor* $\mathrm{B}[F] = (\mathrm{B}[F]_\otimes, \mathrm{B}[F]_\oplus) : \mathrm{B}[\mathbb{X}] \to \mathrm{B}[\mathbb{Y}]$.

Proof. Consider a symmetric medial linear functor $F = (F_\otimes, F_\oplus) : \mathbb{X} \to \mathbb{Y}$. Recall that

$$(F_\otimes, m_\perp, m_\oplus, m_\top^{-1}, m_\otimes^{-1}),\ (F_\oplus, n_\perp^{-1}, n_\oplus^{-1}, n_\top, n_\oplus) : (\mathbb{X}, \oplus, \perp, \otimes, \top) \to (\mathbb{Y}, \oplus, \perp, \otimes, \top)$$

are bilax duoidal functors and therefore preserve medial bimonoids and medial bimonoid morphisms by Proposition 4.14.

Thus, given a bicommutative medial bimonoid $\langle A, \Delta_A, e_A, \nabla_A, u_A\rangle$ in $\mathbb{X}$, the object $F_\otimes(A)$ equipped with the four maps

$$\begin{aligned}
\Delta_{F_\otimes(A)} &= F_\otimes(A) \xrightarrow{F_\otimes(\Delta_A)} F_\otimes(A\otimes A) \xrightarrow{m_{\otimes}{}_{A,A}^{-1}} F_\otimes(A)\otimes F_\otimes(A)\\
e_{F_\otimes(A)} &= F_\otimes(A) \xrightarrow{F_\otimes(e_A)} F_\otimes(\top) \xrightarrow{m_\top^{-1}} \top\\
\nabla_{F_\otimes(A)} &= F_\otimes(A)\oplus F_\otimes(A) \xrightarrow{m_{\oplus A,A}} F_\otimes(A\oplus A) \xrightarrow{F_\otimes(\nabla_A)} F_\otimes(A)\\
u_{F_\otimes(A)} &= \bot \xrightarrow{m_\bot} F_\otimes(\bot) \xrightarrow{F_\otimes(u_A)} F_\otimes(A)
\end{aligned}$$

and the object $F_\oplus(A)$ equipped with the four maps

$$\begin{aligned}
\Delta_{F_\oplus(A)} &= F_\oplus(A) \xrightarrow{F_\oplus(\Delta_A)} F_\oplus(A\otimes A) \xrightarrow{n_{\otimes A,A}} F_\oplus(A)\otimes F_\oplus(A)\\
e_{F_\oplus(A)} &= F_\oplus(A) \xrightarrow{F_\oplus(e_A)} F_\oplus(\top) \xrightarrow{n_\top} \top\\
\nabla_{F_\oplus(A)} &= F_\oplus(A)\oplus F_\oplus(A) \xrightarrow{n_{\oplus}{}_{A,A}^{-1}} F_\oplus(A\oplus A) \xrightarrow{F_\oplus(\nabla_A)} F_\oplus(A)\\
u_{F_\oplus(A)} &= \bot \xrightarrow{n_\bot^{-1}} F_\oplus(\bot) \xrightarrow{F_\oplus(u_A)} F_\oplus(A)
\end{aligned}$$

are bicommutative medial bimonoid in $\mathbb{Y}$. Moreover, given a medial bimonoid morphism

$$f : \langle A, \Delta_A, e_A, \nabla_A, u_A\rangle \to \langle B, \Delta_B, e_B, \nabla_B, u_B\rangle$$

in $\mathbb{X}$, then

$$F_\otimes(f) : \langle F_\otimes(A), \Delta_{F_\otimes(A)}, e_{F_\otimes(A)}, \nabla_{F_\otimes(A)}, u_{F_\otimes(A)}\rangle \to \langle F_\otimes(B), \Delta_{F_\otimes(B)}, e_{F_\otimes(B)}, \nabla_{F_\otimes(B)}, u_{F_\otimes(B)}\rangle,$$

$$F_\oplus(f) : \langle F_\oplus(A), \Delta_{F_\oplus(A)}, e_{F_\oplus(A)}, \nabla_{F_\oplus(A)}, u_{F_\oplus(A)}\rangle \to \langle F_\oplus(B), \Delta_{F_\oplus(B)}, e_{F_\oplus(B)}, \nabla_{F_\oplus(B)}, u_{F_\oplus(B)}\rangle$$

are medial bimonoid morphisms in $\mathbb{Y}$.

As such, let $\mathrm{B}[F]_\otimes : \mathrm{B}[\mathbb{X}] \to \mathrm{B}[\mathbb{Y}]$ denote the functor mapping bicommutative medial bimonoid on A to the bicommutative medial bimonoid on $F_\otimes(A)$ and mapping medial bimonoid morphism f to $F_\otimes(f)$. Similarly, let $\mathrm{B}[F]_\oplus : \mathrm{B}[\mathbb{X}] \to \mathrm{B}[\mathbb{Y}]$ denote the functor mapping bicommutative medial bimonoid on A to the bicommutative medial bimonoid on $F_\oplus(A)$ and mapping medial bimonoid morphism f to $F_\oplus(f)$.

Further, $\mathrm{B}[F]_\otimes$ is a symmetric monoidal functor $(\mathrm{B}[\mathbb{X}], \otimes, \langle\top\rangle) \to (\mathrm{B}[\mathbb{Y}], \otimes, \langle\top\rangle)$ when equipped with

- morphism $m_{\langle\top\rangle} : \langle\top\rangle \to \mathrm{B}[F]_\otimes(\langle\top\rangle)$, defined to be $m_\top : \top \to F_\otimes(\top)$, and

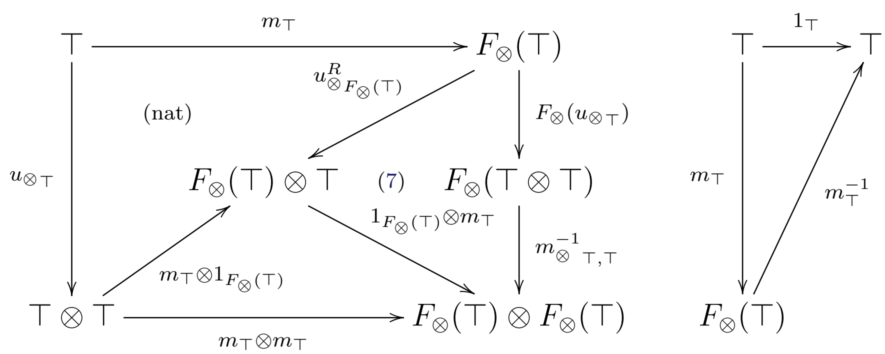
- natural transformation

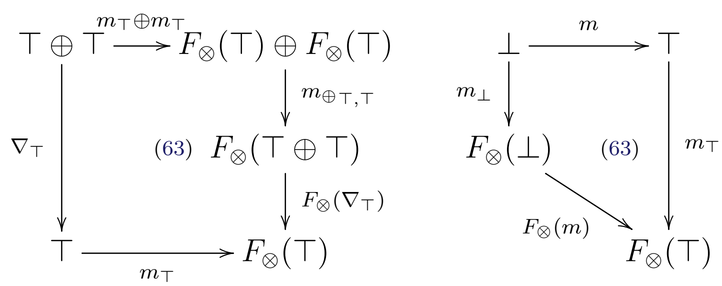
$$m_{\otimes\langle A\rangle,\langle B\rangle} : \mathrm{B}[F]_\otimes(\langle A\rangle) \otimes \mathrm{B}[F]_\otimes(\langle B\rangle) \to \mathrm{B}[F]_\otimes(\langle A\rangle \otimes \langle B\rangle)$$

  defined to be $m_{\otimes A,B} : F_\otimes(A) \otimes F_\otimes(B) \to F_\otimes(A \otimes B)$

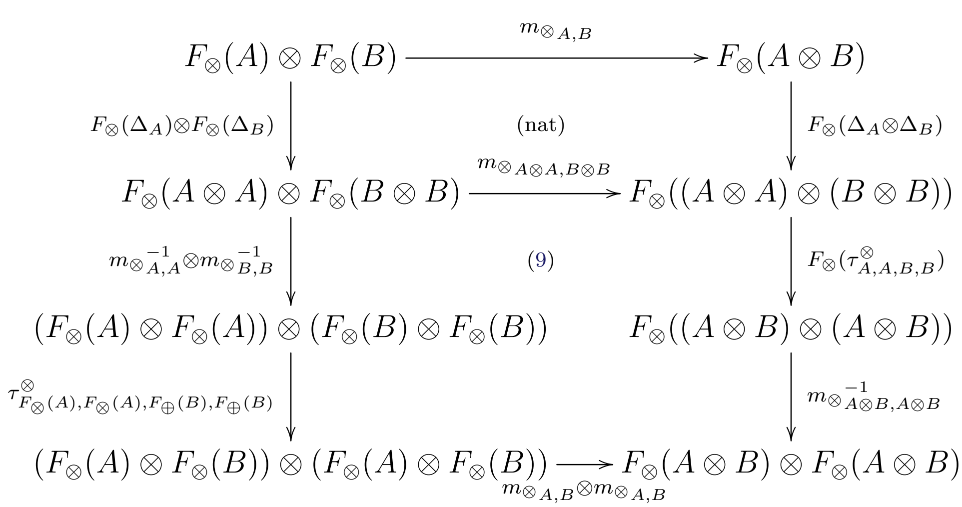
as $(F_\otimes, m_\top, m_\otimes) : (\mathbb{X}, \otimes, \top) \to (\mathbb{Y}, \otimes, \top)$ is a symmetric monoidal functor by definition of symmetric medial linear functor, and its structure morphisms are bimonoid morphisms as follows.

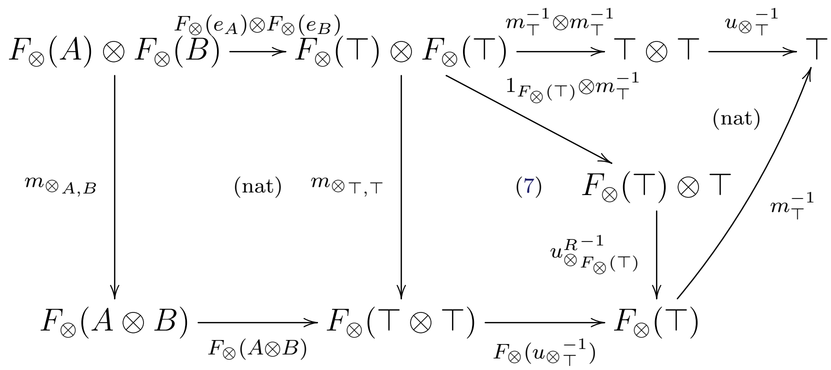
$m_\top : \top \to F_\otimes(\top)$ is a medial bimonoid morphism:

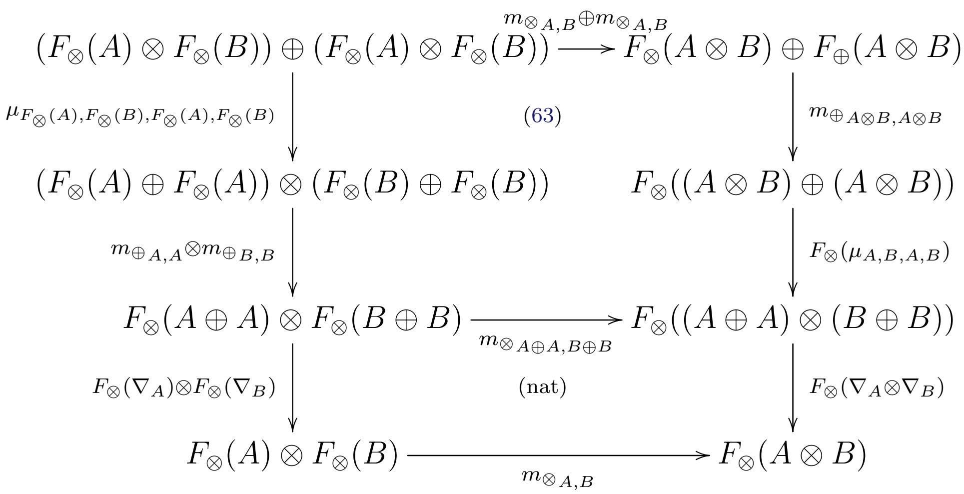


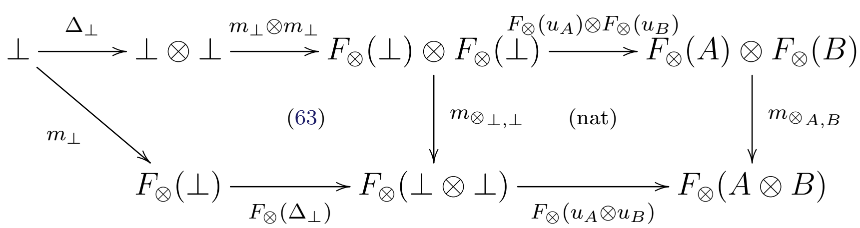


and $m_{\otimes A,B} : F_\otimes(A) \otimes F_\otimes(B) \to F_\otimes(A \otimes B)$ is a medial bimonoid morphism:









We show similarly that $\mathrm{B}[F]_\oplus$ is a symmetric monoidal functor $(\mathrm{B}[\mathbb{X}], \oplus, \langle\bot\rangle) \to (\mathrm{B}[\mathbb{Y}], \oplus, \langle\bot\rangle)$ when equipped with

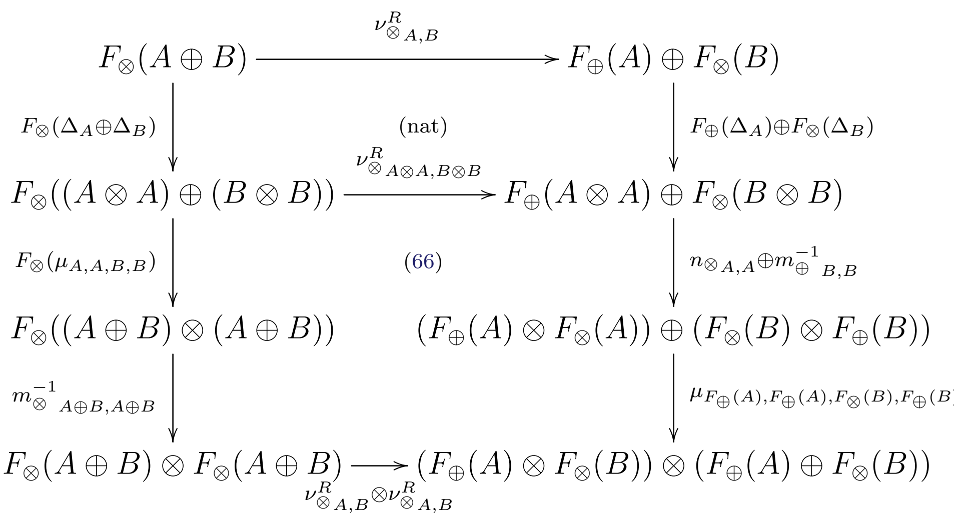
- morphism $n_{\langle\bot\rangle} : \mathrm{B}[F]_\oplus(\langle\bot\rangle)\rangle\langle\bot\rangle$, defined to be $n_\bot : F_\oplus(\bot) \to \bot$, and
- natural transformation
$$n_{\oplus\langle A\rangle,\langle B\rangle} : \mathrm{B}[F]_\oplus(\langle A\rangle \oplus \langle B\rangle) \to \mathrm{B}[F]_\oplus(\langle A\rangle) \oplus \mathrm{B}[F]_\oplus(\langle B\rangle)\,,$$
defined to be $n_{\oplus A,B} : F_\oplus(A \oplus B) \to F_\oplus(A) \oplus F_\oplus(B)$.

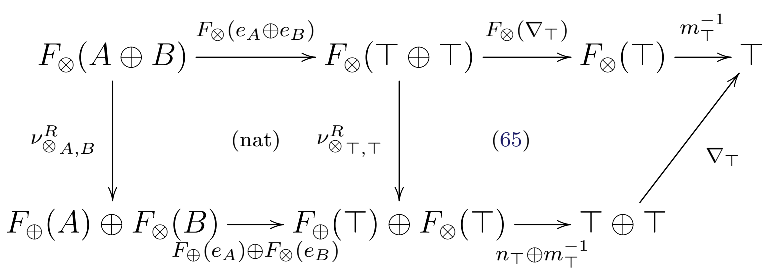
Then, $\mathrm{B}[F] = (\mathrm{B}[F]_\otimes, \mathrm{B}[F]_\oplus) \colon \mathrm{B}[\mathbb{X}] \to \mathrm{B}[\mathbb{Y}]$ is a symmetric linear functor, with linear strengths:

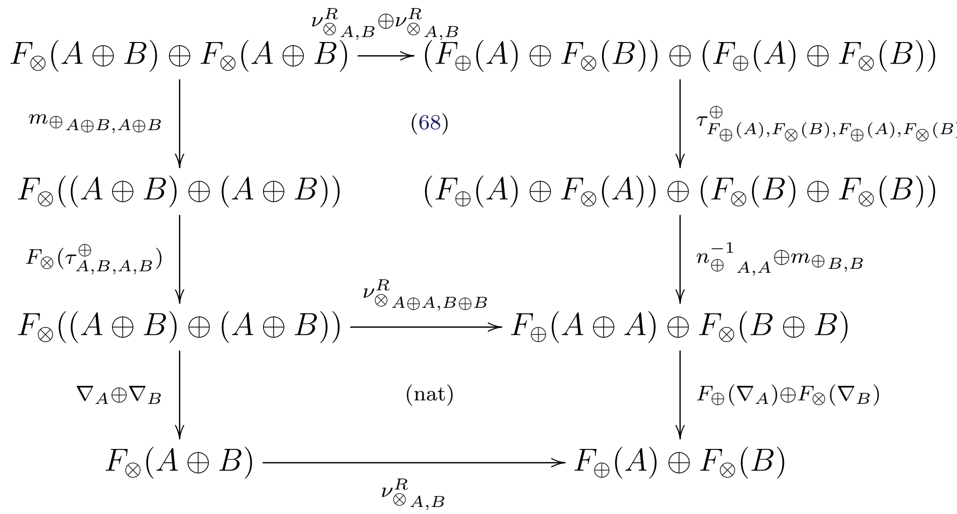
$$\upsilon^R_{\otimes\langle A\rangle,\langle B\rangle} : \mathrm{B}[F]_\otimes(\langle A\rangle \oplus \langle B\rangle) \to \mathrm{B}[F]_\oplus(\langle A\rangle) \oplus \mathrm{B}[F]_\otimes(\langle B\rangle)$$
$$\upsilon^R_{\oplus\langle A\rangle,\langle B\rangle} : \mathrm{B}[F]_\otimes(\langle A\rangle) \otimes \mathrm{B}[F]_\oplus(\langle B\rangle) \to \mathrm{B}[F]_\oplus(\langle A\rangle \otimes \langle B\rangle)$$

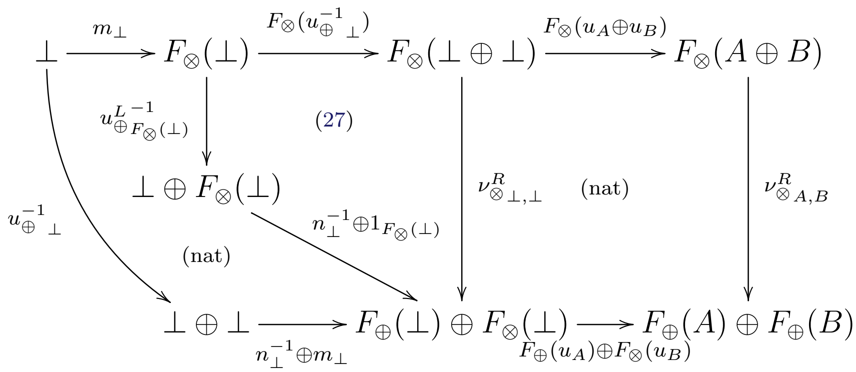
defined to be $\nu^R_{\otimes A,B} : F_\otimes(A \oplus B) \to F_\oplus(A) \oplus F_\otimes(B)$ and $\nu^R_{\oplus A,B} : F_\otimes(A) \otimes F_\oplus(B) \to F_\oplus(A \otimes B)$ respectively in $\mathbb{Y}$, as

$\nu^R_{\otimes A,B}$ is a medial bimonoid morphism by the following commuting diagrams:









and $v^R_{\oplus A,B}$ is a medial bimonoid morphism similarly. ■

7.13. Lemma. *Given a medial linear transformations $\alpha = (\alpha_\otimes, \alpha_\oplus) : F \Rightarrow G$ between symmetric medial linear functors, then it canonically extends to a linear transformation $\mathrm{B}[\alpha] = (\mathrm{B}[\alpha]_\otimes, \mathrm{B}[\alpha]_\oplus) : \mathrm{B}[F] \Rightarrow \mathrm{B}[G]$ between cartesian linear functors.*

Proof. Consider a medial linear transformations $\alpha = (\alpha_\otimes, \alpha_\oplus) : F \Rightarrow G$ between symmetric medial linear functors. Define $\mathrm{B}[\alpha] = (\mathrm{B}[\alpha]_\otimes, \mathrm{B}[\alpha]_\oplus) : \mathrm{B}[F] \Rightarrow \mathrm{B}[G]$, where

- $\mathrm{B}[\alpha]_\otimes : \mathrm{B}[F]_\otimes \to \mathrm{B}[G]_\otimes$ has component morphisms $\alpha_{\otimes A} : F_\otimes(A) \to G_\otimes(A)$ in $\mathbb{Y}$,
- $\mathrm{B}[\alpha]_\oplus : \mathrm{B}[G]_\oplus \to \mathrm{B}[F]_\oplus$ has component morphisms $\alpha_{\oplus A} : G_\oplus(A) \to F_\oplus(A)$ in $\mathbb{Y}$.

This is well-defined as $\alpha_{\otimes A} : F_\otimes(A) \to G_\otimes(A)$ is a medial bimonoid morphism since $\alpha_\otimes : (F_\otimes, m^F_\top, m^F_\otimes) \Rightarrow (G_\otimes, m^G_\top, m^G_\otimes)$ is a monoidal transformation and therefore is a $\otimes$-monoid morphism and $\alpha_\otimes : (F_\otimes, m^F_\perp, m^F_\oplus) \Rightarrow (G_\otimes, m^G_\perp, m^G_\oplus)$ is a monoidal transformation and therefore is a $\oplus$-comonoid morphism.

Similarly, $\alpha_{\oplus A} : G_\oplus(A) \to F_\oplus(A)$ is a medial bimonoid morphism. ■

With the previous lemmas, we can state that the medial bimonoid constructions is a 2-functor between the 2-category of SMLDCs and the 2-category of CLDCs as desired.

7.14. PROPOSITION. $\mathrm{B}[-] : \mathbf{SMLDC} \to \mathbf{CLDC}$ *is a 2-functor.*

PROOF. By Theorem 7.8, Lemma 7.12 and Lemma 7.13, $\mathrm{B}[-]$ determines well-defined maps from the 0-cells, 1-cells and 2-cells respectively of **SMLDC** to **CLDC**. The proof that this is 2-functorial follows immediately as composition in the 2-categories are defined similarly, the composite of two medial bimonoid morphisms is a bimonoid morphism, and 1-cells and 2-cells in **SMLDC** preserve medial bimonoid morphisms. ■

It is fairly immediate that given a symmetric Frobenius medial linear functor $F = (F_\otimes, F_\oplus) : \mathbb{X} \to \mathbb{Y}$, the induced cartesian linear functor $\mathrm{B}[F] : \mathrm{B}[\mathbb{X}] \to \mathrm{B}[\mathbb{Y}]$ is itself Frobenius as $F_\otimes(A) = F_\oplus(A)$ and the medial bimonoid structure maps of $F_\otimes(A)$ are equal to the ones of $F_\oplus(A)$ in the Frobenius contexts. Therefore:

7.15. COROLLARY. $\mathrm{B}[-] : \mathbf{SMLDC} \to \mathbf{CLDC}$ *restricts to a 2-functor* $\mathrm{B}[-] : \mathbf{SFMLDC} \to \mathbf{FCLDC}$.

We have finally arrived at the main result of our paper which motivated all the work:

7.16. THEOREM. [The Linearly Distributive Fox Theorem] $\mathrm{B}[-]$ *is the right adjoint to the inclusion 2-functor,* $\mathrm{inc} \dashv \mathrm{B}[-] \colon \mathbf{CLDC} \to \mathbf{SMLDC}$.

PROOF. To prove this 2-adjunction, we will provide the unit and counit 2-transformations.

Firstly, let us consider the unit $\boldsymbol{\eta} : 1_{\mathbf{CLDC}} \Rightarrow \mathrm{inc}; \mathrm{B}[-]$. Let $(\mathbb{X}, \times, \mathbf{1}, +, \mathbf{0})$ be a CLDC. Recall that every object $A \in \mathbb{X}$ has a canonical bicommutative medial bimonoid structure

$$\langle A, \langle 1_A, 1_A \rangle : A \to A \times A, t_A : A \to \mathbf{1}, [1_A, 1_A] : A + A \to A, b_A : \mathbf{0} \to A \rangle$$

and every arrow $f : A \to B \in \mathbb{X}$ is a medial bimonoid morphism with respect to these canonical medial bimonoid structures, as discussed in Proposition 7.5. Then, let $\eta_\mathbb{X} : \mathbb{X} \to \mathrm{B}[\mathbb{X}]$ be the functor that sends the objects in $\mathbb{X}$ to their canonical medial bimonoid structures and that is identity on the arrows in $\mathbb{X}$.

$\eta_\mathbb{X}$ is further a symmetric monoidal functor between $(\mathbb{X}, \times, \mathbf{1}) \to (\mathrm{B}[\mathbb{X}], \times, \langle \mathbf{1} \rangle)$ when equipped with

- morphism $m_\mathbf{1} : \langle \mathbf{1} \rangle \to \eta_X(\mathbf{1})$ to be $1_\mathbf{1} : \mathbf{1} \to \mathbf{1}$
- natural transformation with component morphisms $m_{\times A,B} : \eta_X(A) \times \eta_X(B) \to \eta_X(A \times B)$ given by $1_{A \times B} : A \times B \to A \times B$

as the canonical medial bimonoid structure on $\mathbf{1}$ described above is the same as as the medial bimonoid structure on $\mathbf{1}$ when $\mathbb{X}$ is viewed as MLDC:

$$\langle \mathbf{1}, u_{\times \mathbf{1}}, 1_\mathbf{1}, \nabla_\mathbf{1}, m \rangle = \langle \mathbf{1}, \langle 1_\mathbf{1}, 1_\mathbf{1} \rangle, t_\mathbf{1}, [1_\mathbf{1}, 1_\mathbf{1}], b_\mathbf{1} \rangle$$

and the canonical medial bimonoid structure on $A\times B$ is the same as the medial bimonoid structure of the tensor product $\times$ in B[$\mathbb{X}$] between the canonical structures on A and B, i.e.

$$\langle A\times B,([1_A,1_A]\times[1_B,1_B]);\tau^{\times}_{A,A,B,B},(t_A\times t_B);u_{\times}{}_{\mathbf{1}}^{-1},$$
$$\mu_{A,B,A,B};([1_A,1_A]\times[1_B,1_B]),\Delta_{\perp};(b_A\times b_B)\rangle$$

$$=\langle A\times B,\langle 1_{A\times B},1_{A\times B}\rangle,t_{A\times B},[1_{A\times B},1_{A\times B}],b_{A\times B}\rangle$$

Similarly, η_X is a symmetric monoidal functor between $(\mathbb{X},+,\mathbf{0})\to(\mathrm{B}[\mathbb{X}],+,\langle\mathbf{0},\rangle)$ when equipped with

- morphism $n_{\mathbf{0}}:\eta_X(\mathbf{0})\to\langle\mathbf{0}\rangle$ to be $1_{\mathbf{0}}:\mathbf{0}\to\mathbf{0}$
- natural transformation with component morphisms $n_{+A,B}:\eta_X(A+B)\to\eta_X(A)+\eta_X(B)$ given by $1_{A+B}:A+B\to A+B$

Then, $\boldsymbol{\eta}_{\mathbb{X}}=(\eta_{\mathbb{X}},\eta_{\mathbb{X}}):\mathbb{X}\to\mathrm{B}[\mathbb{X}]$ is a cartesian linear functor, when further equipped with identity linear strengths.

Let $\boldsymbol{\eta}:1_{\mathbf{CLDC}}\Rightarrow\mathrm{inc};\mathrm{B}[-]$ be defined to be the family of maps $\boldsymbol{\eta}_{\mathbb{X}}=(\eta_{\mathbb{X}},\eta_{\mathbb{X}}):\mathbb{X}\to\mathrm{B}[\mathbb{X}]$. This is a 2-transformation because

- $\boldsymbol{\eta}$ satisfies 1-cell naturality:
  Let $\mathbb{X}$ and $\mathbb{X}$' be CLDCS, and $F=(F_\times,F_+):\mathbb{X}\to\mathbb{X}'$ be a cartesian linear functor between them. 1-cell naturality in this case means

  $$\eta_{\mathbb{X}};\mathrm{B}[\mathrm{inc}(F)]_\times=F_\times;\eta_{\mathbb{X}}\qquad\eta_{\mathbb{X}};\mathrm{B}[\mathrm{inc}(F)]_+=F_+;\eta_{\mathbb{X}}$$

  Now, these functors are defined as follows

  $\eta_{\mathbb{X}}$;B[inc$(F)]_\times$:
  $A\mapsto\langle F_\times(A),F_\times(\langle 1_A,1_A\rangle);m_\times^{-1}{}_{A,A},F_\times(t_A);m_{\mathbf{1}}^{-1},p_{+A,A};F_\times([1_A,1_A]),p_{\mathbf{0}};F_\times(b_A)\rangle$
  $f\mapsto F_\times(f)$

  $F_\times$;$\eta_{\mathbb{X}}$:
  $A\mapsto\langle F_\times(A),\langle 1_{F_\times(A)},1_{F_\times(A)}\rangle,t_{F_\times(A)},[1_{F_\times(A)},1_{F_\times(A)}],b_{F_\times(A)}\rangle$
  $f\mapsto F_\times(f)$

  which are equal as the medial bimonoid structures on $F_\times(A)$ is unique since $\mathbb{X}$' is a CLDC. One shows similarly that the other equality holds as both sides map to the unique medial bimonoid structure on $F_+(A)$.

- $\boldsymbol{\eta}$ satisfies 2-cell naturality: Let $\mathbb{X}$ and $\mathbb{X}$' be CLDCS, $F = (F_\times, F_+), G = (G_\times, G_+) : \mathbb{X} \to \mathbb{X}'$ be a cartesian linear functor between them, and $\alpha = (\alpha_\times, \alpha_+) : F \Rightarrow G$ be a linear transformation between them. 2-cell naturality in this context means

$$1_{\eta_\mathbb{X}}; \mathrm{B}[\mathrm{inc}(\alpha)]_\times = \alpha_\times; 1_{\eta_\mathbb{X}} \qquad 1_{\eta_\mathbb{X}}; \mathrm{B}[\mathrm{inc}(\alpha)]_+ = \alpha_+; 1_{\eta_\mathbb{X}}$$

Now, these natural transformations as component morphisms for each object $A \in \mathbb{X}$ the medial bimonoid morphisms given by

$$(1_{\eta_\mathbb{X}}; \mathrm{B}[\mathrm{inc}(\alpha)]_\times)_A = \alpha_{\times A} = (\alpha_\times; 1_{\eta_\mathbb{X}})_A$$

and therefore are equal. Similarly, for the second equality.

Secondly, let us consider the counit $\boldsymbol{\epsilon} : \mathrm{B}[-]; \mathrm{inc} \Rightarrow 1_{\mathbf{SMLDC}}$. Let $(\mathbb{X}, \otimes, \top, \oplus, \bot)$ be a SMLDC, then define $\epsilon_\mathbb{X} : \mathrm{B}[\mathbb{X}] \to \mathbb{X}$ to be the forgetful functor

$$\langle A, \Delta_A, e_A, \nabla_A, s_A \rangle \mapsto A \qquad f \mapsto f$$

Now, $(\epsilon_\mathbb{X}, 1_\top, 1_{-\otimes-}) : (\mathrm{B}[\mathbb{X}], \otimes, \langle \top \rangle) \to (\mathbb{X}, \otimes, \top)$ is a symmetric monoidal functor when equipped with identity structure maps, and similarly $(\epsilon_\mathbb{X}, 1_\bot, 1_{-\oplus-}) : (\mathrm{B}[\mathbb{X}], \oplus, \langle \bot \rangle) \to (\mathbb{X}, \oplus, \bot)$ is a symmetric monoidal functor. Further, $(\epsilon_\mathbb{X}, 1_\bot, 1_{-\oplus-}, 1_\top, 1_{-\otimes-}) : (\mathrm{B}[\mathbb{X}], \oplus, \langle \bot \rangle, \otimes, \langle \top \rangle) \to (\mathbb{X}, \oplus, \bot, \otimes, \top)$ is a bilax duoidal functor.

Therefore, $\boldsymbol{\epsilon}_\mathbb{X} = (\epsilon_\mathbb{X}, \epsilon_\mathbb{X}) : \mathrm{B}[\mathbb{X}] \to \mathbb{X}$ is a symmetric medial linear functor when equipped with identity linear strengths.

Let $\boldsymbol{\epsilon} : \mathrm{B}[-]; \mathrm{inc} \Rightarrow 1_{\mathbf{SMLDC}}$ be defined to be the family of maps $\boldsymbol{\epsilon}_\mathbb{X} = (\epsilon_\mathbb{X}, \epsilon_\mathbb{X}) : \mathrm{B}[\mathbb{X}] \to \mathbb{X}$. It satisfies 1-cell and 2-cell naturality fairly trivially and therefore it is a 2-natural transformation.

To prove the desired 2-adjunction, it remains solely to show the triangle inequalities hold.

$$\mathrm{inc}(\boldsymbol{\eta}); \boldsymbol{\epsilon}_{\mathrm{inc}} = 1_{\mathrm{inc}} \qquad \boldsymbol{\eta}_{\mathrm{B}[-]}; \mathrm{B}[\boldsymbol{\epsilon}] = 1_{\mathrm{B}[-]}$$

Consider a CLDC $\mathbb{X}$, then

$$(\mathrm{inc}(\boldsymbol{\eta}_\mathbb{X}); \boldsymbol{\epsilon}_{\mathrm{inc}(\mathbb{X})})_\times = (1_{\mathrm{inc}(\mathbb{X})})_\times \qquad (\mathrm{inc}(\boldsymbol{\eta}_\mathbb{X}); \boldsymbol{\epsilon}_{\mathrm{inc}(\mathbb{X})})_+ = (1_{\mathrm{inc}(\mathbb{X})})_+$$

as the left-hand side functors behave as identities:

$$(\mathrm{inc}(\boldsymbol{\eta}_\mathbb{X}); \boldsymbol{\epsilon}_{\mathrm{inc}(\mathbb{X})})_\times = (\mathrm{inc}(\boldsymbol{\eta}_\mathbb{X}); \boldsymbol{\epsilon}_{\mathrm{inc}(\mathbb{X})})_+ : \qquad \begin{array}{l} A \mapsto \langle A, \langle 1_A, 1_A \rangle, t_A, [1_A, 1_A], b_A \rangle \mapsto A \\ f \mapsto f \mapsto f \end{array}$$

Consider a SMLDC $\mathbb{X}$, then

$$(\boldsymbol{\eta}_{\mathrm{B}[\mathbb{X}]}; \mathrm{B}[\boldsymbol{\epsilon}])_\otimes = (1_{\mathrm{B}[\mathbb{X}]})_\otimes \qquad (\boldsymbol{\eta}_{\mathrm{B}[\mathbb{X}]}; \mathrm{B}[\boldsymbol{\epsilon}])_\oplus = (1_{\mathrm{B}[\mathbb{X}]})_\oplus$$

as the left-hand side functors behave as identities:

$$\begin{aligned}(\boldsymbol{\eta}_{\mathrm{B}[\mathbb{X}]};\mathrm{B}[\boldsymbol{\epsilon}])_{\otimes} =& (\boldsymbol{\eta}_{\mathrm{B}[\mathbb{X}]};\mathrm{B}[\boldsymbol{\epsilon}])_{\oplus} : \\ & \langle A, \Delta_A, e_A, \nabla_A, u_A\rangle \\ & \mapsto \langle\langle A\rangle, \langle 1_{\langle A\rangle}, 1_{\langle A\rangle}\rangle, t_{\langle A\rangle}, [1_{\langle A\rangle}, 1_{\langle A\rangle}], b_{\langle A\rangle}\rangle \\ & \mapsto \langle \epsilon_{\mathbb{X}}(\langle A\rangle), \epsilon_{\mathbb{X}}(\langle 1_{\langle A\rangle}, 1_{\langle A\rangle}\rangle); 1_{A\otimes A}, \epsilon_{\mathbb{X}}(t_{\langle A\rangle}); 1_{\top}, 1_{A\oplus A}; \epsilon_{\mathbb{X}}([1_{\langle A\rangle}, 1_{\langle A\rangle}]), 1_{\bot}; \epsilon_{\mathbb{X}}(b_{\langle A\rangle})\rangle \\ & = \langle A, \Delta_A, e_A, \nabla_A, u_A\rangle\end{aligned}$$

since, by Theorem 7.8, the unique maps induced by the universal properties of products and coproducts in $\mathrm{B}[\mathbb{X}]$ are given by the structure maps of the medial bimonoids themselves, i.e.

$$\langle 1_{\langle A\rangle}, 1_{\langle A\rangle}\rangle, = \Delta_A \qquad t_{\langle A\rangle} = e_A \qquad [1_{\langle A\rangle}, 1_{\langle A\rangle}] = \nabla_A \qquad b_{\langle A\rangle} = u_A$$

Therefore, $\boldsymbol{\eta}$ and $\boldsymbol{\epsilon}$ are the unit and counit of the 2-adjunction $\mathrm{inc} \vdash \mathrm{B}[-]\colon \mathbf{CLDC} \to \mathbf{SMLDC}$. ■

Perhaps the most commonly used result of the traditional of the Fox theorem is that a symmetric monoidal category is cartesian if and only if it is isomorphic to its category of comonoids, as stated in Corollary 2.20. We can of course state an analogous Corollary to our 2-adjunction for SMLDCs:

7.17. Corollary. *A symmetric medial linearly distributive category is cartesian if and only if it is isomorphic to its category of bicommutative medial bimonoids and medial bimonoid morphisms.*

Proof. Let $\mathbb{X}$ be a SMLDC. The backwards direction is immediate by Theorem 7.8. As for the forward direction, suppose it is a CLDC, then $\boldsymbol{\eta}_{\mathbb{X}} = (\eta_{\mathbb{X}}, \eta_{\mathbb{X}}) : \mathbb{X} \to \mathrm{B}[\mathbb{X}]$ is a well-defined cartesian linear functor. It is a well-known fact that the unit $\boldsymbol{\eta}_{\mathbb{X}}$ will be an isomorphism as inc is fully faithful and the left adjoint in the adjunction in question. Therefore, $\mathbb{X}$ and $\mathrm{B}[\mathbb{X}]$ are isomorphic as CLDCs.

■

The above corollary can be unwrapped to give an explicit description in terms of medial bimonoid maps, as in the case of traditional cartesian categories.

7.18. Corollary. *A symmetric medial linearly distributive category is cartesian if and only if there are natural transformations*

$$\Delta_A : A \to A \otimes A \qquad e_A : A \to \top \qquad \nabla_A : A \oplus A \to A \qquad u_A : \bot \to A$$

such that, $\forall A, B \in \mathbb{X}, \langle A, \Delta_A, e_A, \nabla_A, u_A\rangle$ determines a bicommutative medial bimonoid, and

$$\Delta_{A\otimes B} = (\Delta_A \otimes \Delta_B); \tau^{\otimes}_{A,A,B,B} \qquad e_{A\otimes B} = (e_A \otimes e_B); u^{R^{-1}}_{\otimes\top}$$
$$\nabla_{A\otimes B} = \mu_{A,B,A,B}; (\nabla_A \otimes \nabla_B) \qquad u_{A\otimes B} = \nabla_{\top}; (u_A \otimes u_B)$$

$$\nabla_{A\oplus B} = \tau^{\oplus}_{A,B,A,B}; (\nabla_A \oplus \nabla_B) \qquad u_{A\oplus B} = u^{R^{-1}}_{\oplus\bot}; (u_A \oplus u_B)$$
$$\Delta_{A\oplus B} = (\Delta_A \oplus \Delta_B); \mu_{A,A,B,B} \qquad e_{A\oplus B} = (e_A \oplus e_B); \Delta_\bot$$

$$\Delta_\top = u^R_{\otimes\top} \qquad e_\top = 1_\top \qquad \nabla_\bot = u^R_{\oplus\bot} \qquad u_\bot = 1_\bot$$
$$e_\bot = u_\top = m$$

This is in essence an amalgamations of Proposition 5.1 and Corollary 5.2, developed at the start to get a better understanding of the underlying structure of CLDCs.

Appendix

A. Large Commuting Diagrams.

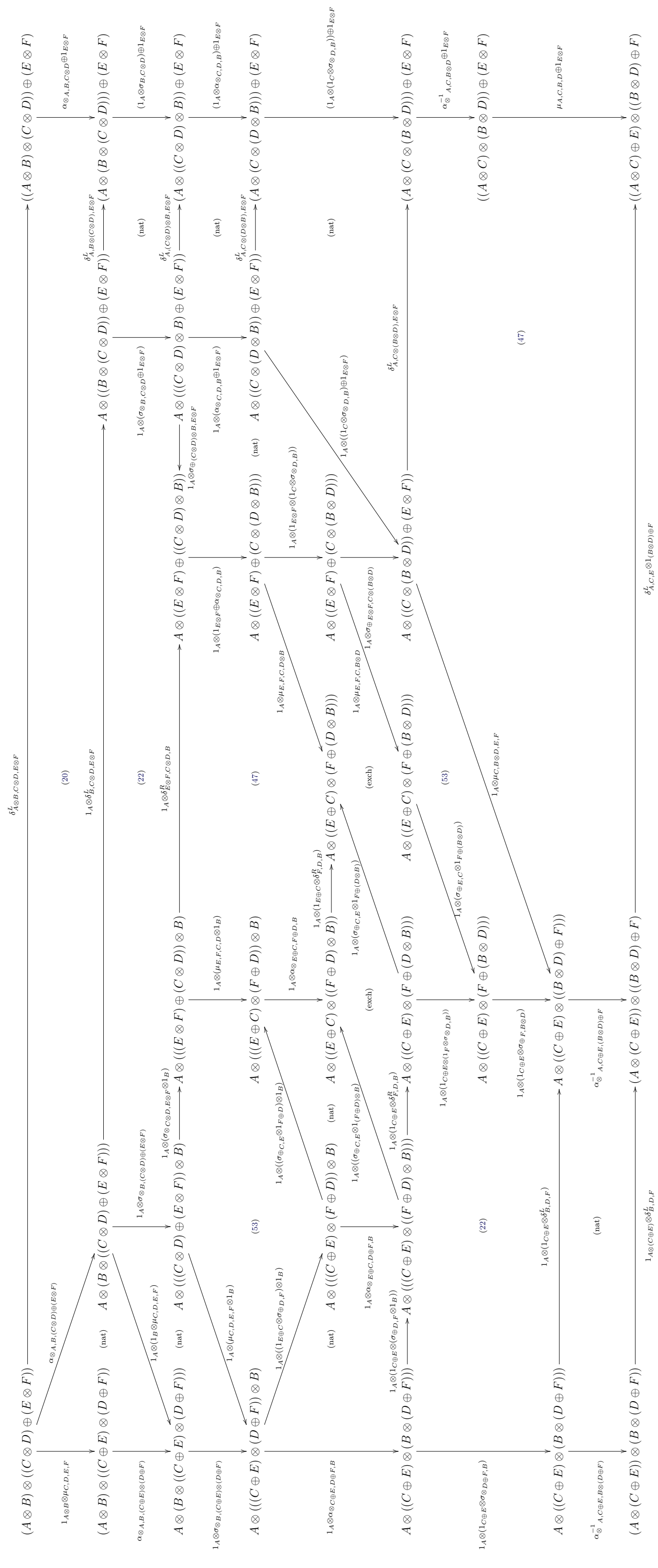


Figure 1: Proof of the first equality in Proposition 5.23

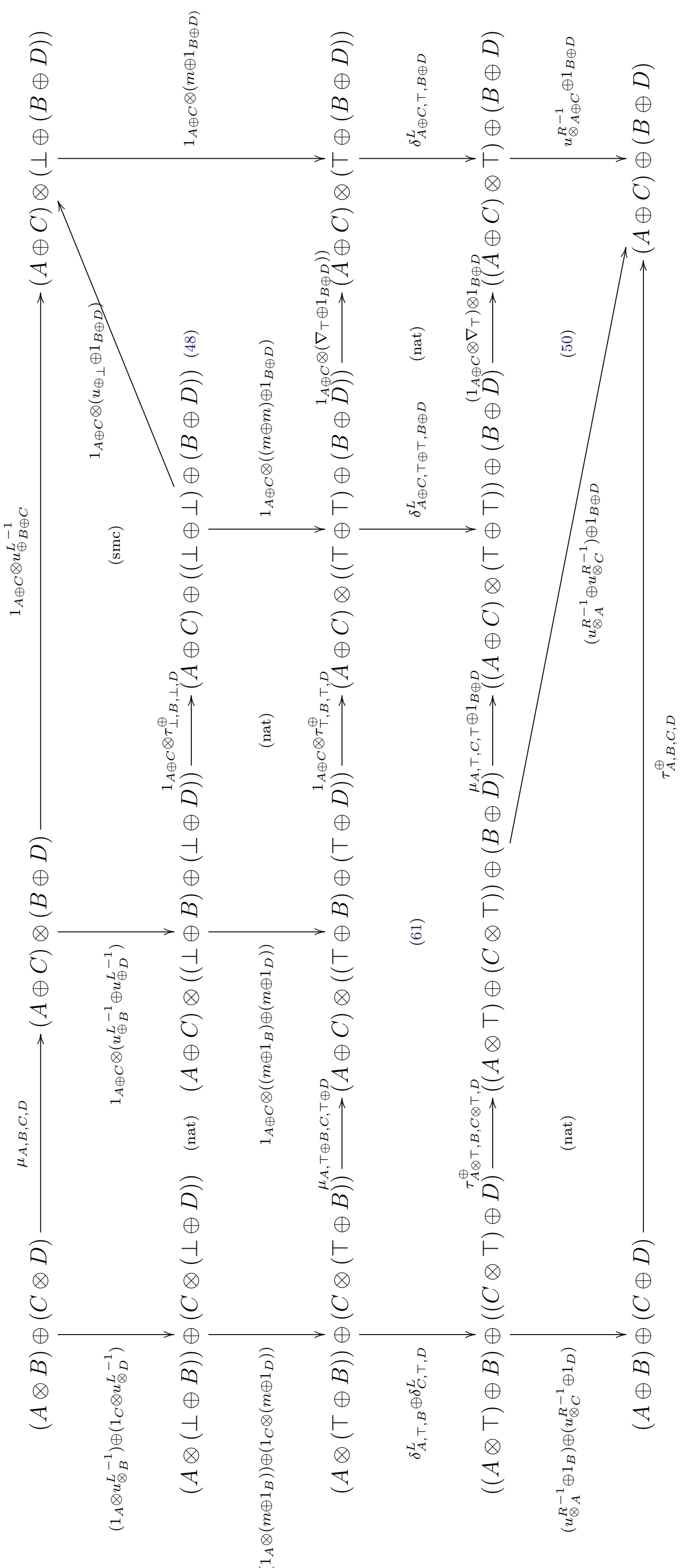


Figure 2: Proof of the first diagram in Proposition 5.24

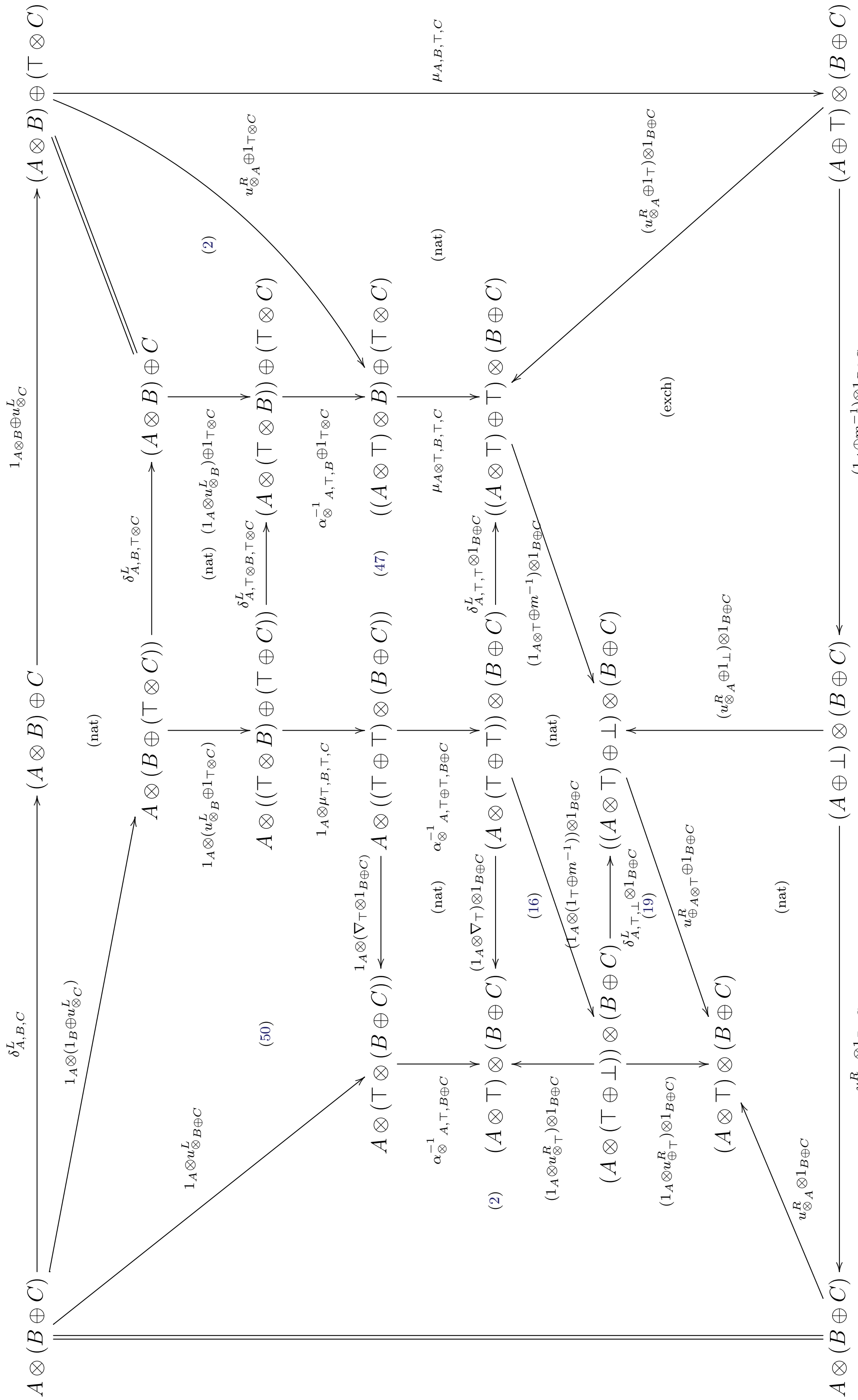


Figure 3: Proof that ∂^R is the inverse of δ^L in Proposition 5.26

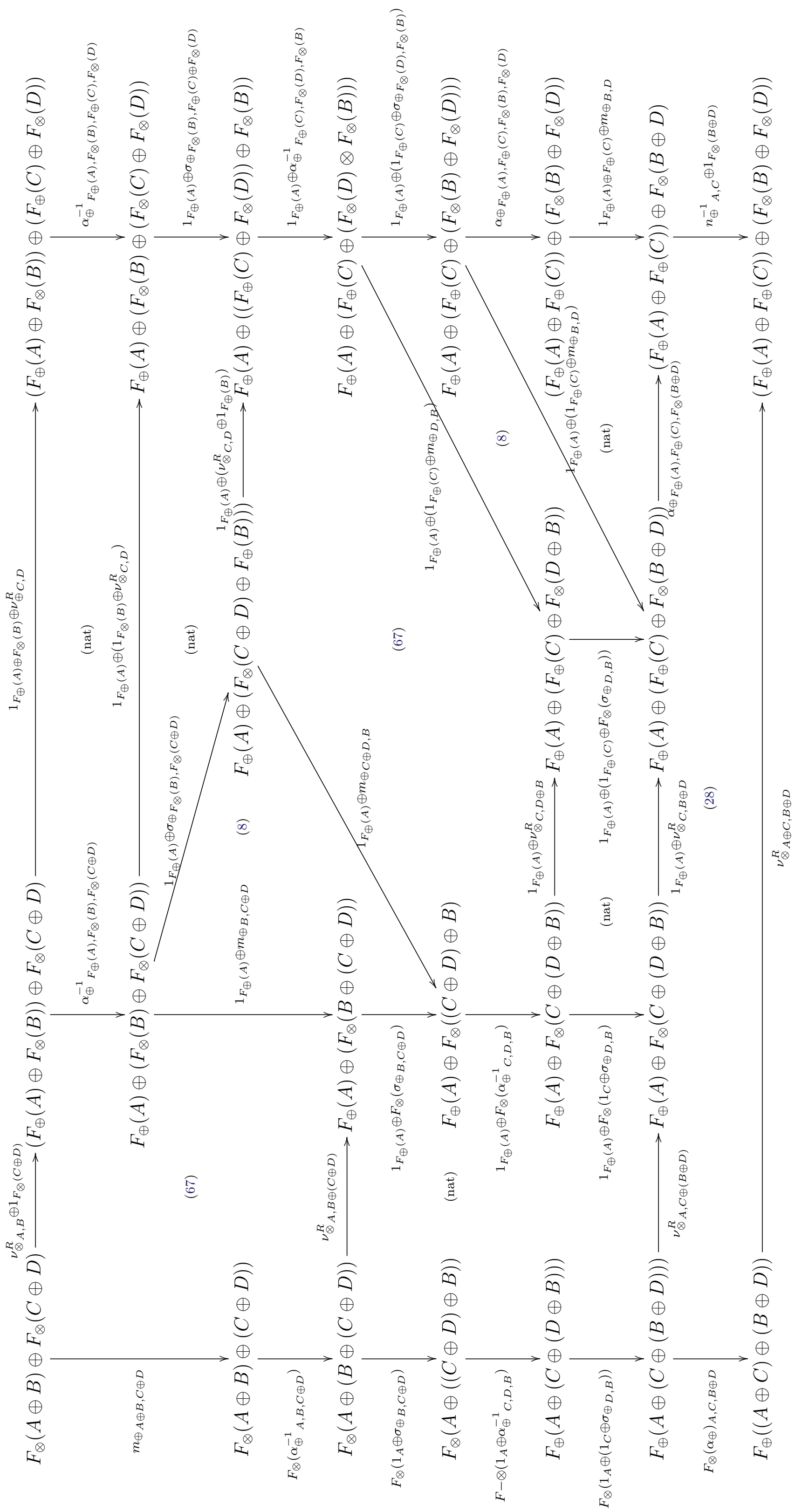


Figure 4: Proof of the first equality in Proposition 6.2

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Chapter 3

Cartesian Linearly Distributive Categories: Revisited

This chapter contains an article co-authored by Rose Kudzman-Blais and Jean-Simon Pacaud Lemay, prepared for submission to *Applied Categorical Structures*, a journal dedicated to the application of categorical methods in algebra, analysis, computer science, logic, order, and topology. The mathematical content was developed collaboratively by both authors, while the manuscript itself was written solely by the first author. The paper revisits the notion of cartesian linearly distributive categories, presenting new results on the topic.

Cartesian Linearly Distributive Categories: Revisited

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Abstract

Linearly distributive categories (LDC) were introduced by Cockett and Seely as alternative categorical semantics for multiplicative linear logic, as developed by Girard. In contrast to Barr's ∗-autonomous categories, LDCs take multiplicative conjunction and disjunction, along with their interaction via linear distributivities, as primitive notions. Given that a LDC has two monoidal products, it is natural to ask when these coincide with categorical products and coproducts. Such LDCs, known as cartesian linearly distributive categories (CLDC), were introduced alongside LDCs. Initially, it was believed that CLDCs and distributive categories would coincide, but this was later found not to be the case. Consequently, the study of CLDCs was largely abandoned. This work revisits CLDCs, demonstrating strong structural properties they all satisfy and investigating two key classes of examples: bounded distributive lattices and semi-additive categories. Additionally, it re-examines a previously assumed class of CLDCs, the Kleisli categories of exception monads of distributive categories, and shows that they do not, in fact, form CLDCs.

Keywords: Categorical linear logic, linearly distributive categories, cartesian, lattices, semi-additive

MSC Classification: 18M45

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1 Introduction

Most categories of interest, whether from logic, theoretical computer science, algebra, or category theory more broadly, exhibit multiple monoidal products worth investigating. Within monoidal category theory, a significant body of research explores the distributivity of monoidal structures, that is, how two monoidal products interact. In general, this interaction is mediated by a natural transformation, known as a distributivity map, which may or may not be an isomorphism. Frequently, one or both of the monoidal structures is cartesian or cocartesian, arising canonically from finite products or coproducts, respectively.

Perhaps the most well-known definition in this area is that of a distributive category [1], a cartesian and cocartesian category whose products $\times$ distribute over its coproducts $+$ by an isomorphism:

$$(A \times C) + (B \times C) \to (A + B) \times C$$

This concept was generalized to distributive monoidal categories and extended further with the introduction of rig categories, also known as bimonoidal categories: categories equipped with two arbitrary monoidal structures whose distributivity is mediated by monomorphisms [2]. They are named as such because they categorify rings without negatives, modeling the distributivity of products over sums [3].

Another important notion of a category with two monoidal structures is that of a duoidal category, which arises canonically when a category is either cartesian or cocartesian [4]. In this setting, distributivity is a natural transformation referred to as interchange:

$$(A \star B) \diamond (C \star D) \to (A \diamond C) \star (B \diamond D)$$

The theory of duoidal categories emerged in the study of braided monoidal categories, where the interchange law generalizes the canonical flip induced by braidings [5].

The development of linearly distributive categories (LDC) fits naturally within this research program. LDCs were first introduced by Cockett and Seely in [6] as categorical semantics for multiplicative linear logic (MLL), as developed by Girard [7]. In this framework, the multiplicative conjunction $\otimes$ and disjunction $\parr$ are taken as primitive notions, along with their interactions via linear distributivities:

$$A \otimes (B \parr C) \to (A \otimes B) \parr C$$

This contrasts with $*$-autonomous categories, introduced by Barr [8] and later shown by Seely [9] to provide categorical semantics for linear logic. $*$-autonomous categories take multiplicative conjunction and negation as primitive, defining disjunction via de Morgan duality. Consequently, linear distributivity is implicit, following from the duality between conjunction and disjunction. However, linear distributivity is a fundamental concept, as it precisely captures the interaction between two binary operations necessary to model Gentzen's cut rule. The significance of explicitly considering linear distributivities is evident in the extensive and continuously growing literature on LDCs since their introduction.

Initially, linearly distributive categories were referred to as weakly distributive categories, as linear distributivity was thought to be a weakening of the classical distributivity of products over coproducts found in distributive categories. It was believed that specializing to a subclass of LDCs known as cartesian linearly distributive categories (CLDC), where the tensor structure is cartesian and the par structure is cocartesian, would recover the notion of a distributive category. However, this turned out to be false: linear distributivity and classical distributivity are, in some sense, orthogonal notions [10]. Following this realization, the study of CLDCs was largely abandoned, while research on LDCs has remained central to categorical linear logic.

The current work revisits the notion of CLDCs and develops their theory. Just as distributive categories relate to rig categories, CLDCs stand in relation to LDCs and remain an interesting structure worth exploring.

Our interest in CLDCs originated from the first author's efforts to better understand the interaction between cartesian structures and linear distributivities by developing a linearly distributive Fox theorem [11]. After presenting this result at a conference and discussing it further with the second author, a potential connection was observed to the second author's recent work on classical distributive restriction categories [12]. Specifically, every classical distributive restriction category is the Kleisli category of the exception monad of a distributive category and these were thought to be CLDCs according to [10], it seemed natural to explore this link. During the first author's visit to Macquarie University, we began investigating this connection. However, it soon became apparent that something was amiss: Kleisli categories of exception monads for distributive categories are not, in fact, examples of CLDCs.

This realization led us to investigate CLDCs more closely. In doing so, we uncovered structural properties that had not been previously explored, revealing distinctive and nuanced behaviour within these categories. In particular, the terminal object in a CLDC is always preinitial and, dually, the initial object is always subterminal. This implies that every CLDC is a mix LDC and further the mix maps can be used to characterize when an object is preinitial or subterminal. Remarkably, these conditions are equivalent in a CLDC.

As we attempted to develop examples of CLDCs, two key classes emerged: bounded distributive lattices and semi-additive categories. Indeed, in many cases, a CLDC must fall into one of these two categories. There are two well-known collapses to posets: the first being Joyal's paradox, and the second stemming from the orthogonality between linear and traditional distributivity. In addition to these, we discovered two new collapse theorems: a CLDC must be semi-additive if it either has invertible linear distributivities or if it is isomix. These four collapse results severely constrain the landscape of possible CLDCs. Nevertheless, we introduce a Grothendieck construction that utilizes semi-additive categories and bounded distributive lattices to generate new examples of CLDCs.

Outline. Section 2 reviews the necessary theory on LDCs and the morphisms between them. Section 3 introduces the notion of CLDCs, develops the appropriate notions of morphisms between CLDCs, and explores fundamental properties shared by all CLDCs. In Section 4, the first significant class of CLDC examples is investigated: bounded distributive lattices. Section 5 examines the second key class of CLDCs: semi-additive categories. Two new collapse theorems are presented, along with a construction that maps a CLDC to a semi-additive category. Section 6 revisits the Kleisli category of the exception monad of a distributive category and explores why it does not have a CLDC structure, despite still being a well-defined LDC. Finally, Section 7 discusses two simple constructions that provide further examples of CLDCs, products and fibrations.

Conventions. We outline here some of the notation used throughout the paper.

- Composition of morphisms in a category is denoted by the symbol ; and is written in diagrammatic order. Objects in a category are denoted by capital letters, while the morphisms are denoted by lowercase letters.
- Binary products are given by $A \times B$, while binary coproducts are $A + B$. The terminal object is $\mathbf{1}$ and the initial object is $\mathbf{0}$, pronounced top and bottom respectively. Binary biproducts are represented by $A \sqcup B$, while a zero object is $\emptyset$, chosen not conflict with the following notation.
- Our linear logic notation aligns with the notation used by Cockett and Seely in [6], in contrast to the symbols introduced by Girard [7]: multiplicative conjunction is represented by $\otimes$, known as tensor, with unit $\top$, and multiplicative disjunction is $\oplus$, known as par, with unit $\bot$.

2 Preliminaries

Before introducing the main structure of interest, we need to provide some important background on LDCs. As the necessary background is spread out among these papers, we collect in the following section all the definitions and results on LDCs required for this work.

2.1 Linearly Distributive Categories

Beyond their introduction in [10], the theory of LDCs was further developed in a subsequent series articles by Cockett and Seely, sometimes alongside co-authors Blute and Trimble [13–15].

Definition 2.1 *[10, Sec 2.1] A* **linearly distributive category***, or a LDC,* $(\mathbb{X}, \otimes, \top, \oplus, \bot)$ *consists of:*

- *a category* $(\mathbb{X}, ;, 1_A)$
- *a* **tensor** *monoidal structure* $(\mathbb{X}, \otimes, \top)$
  - *the* **tensor** *product and* **top** *unit functor*

$$\otimes : \mathbb{X} \times \mathbb{X} \to \mathbb{X} \qquad\qquad \top : 1 \to \mathbb{X}$$

  - *the $\otimes$-associator, right $\otimes$-unitor, and left $\otimes$-unitor natural isomorphisms*

$$\alpha_\otimes : (\otimes \times 1_\mathbb{X}); \otimes \Rightarrow (1_\mathbb{X} \times \otimes); \otimes \qquad \alpha_{\otimes A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$$
$$u^R_\otimes : 1_\mathbb{X} \Rightarrow 1_\mathbb{X} \otimes \top \qquad u^R_{\otimes A} : A \to A \otimes \top$$
$$u^L_\otimes : 1_\mathbb{X} \Rightarrow \top \otimes 1_\mathbb{X} \qquad u^L_{\otimes A} : A \to \top \otimes A$$

  such that $(\mathbb{X}, \otimes, \top, \alpha_\otimes, u^R_\otimes, u^L_\otimes)$ *is a monoidal category,*
- *a* **par** *monoidal structure* $(\mathbb{X}, \oplus, \bot)$
  - *the* **par** *product and* **bottom** *unit functor*

$$\oplus : \mathbb{X} \times \mathbb{X} \to \mathbb{X} \qquad\qquad \bot : 1 \to \mathbb{X}$$

  - *the $\oplus$-associator, right $\oplus$-unitor, and left $\oplus$-unitor natural isomorphisms*

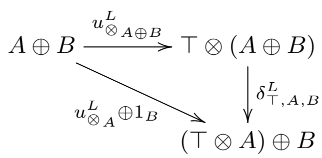
$$\alpha_\oplus : (\otimes \times 1_\mathbb{X}); \otimes \Rightarrow (1_\mathbb{X} \times \otimes); \otimes \qquad \alpha_{\oplus A,B,C} : A \oplus (B \oplus C) \to (A \oplus B) \oplus C$$
$$u^R_\oplus : 1_\mathbb{X} \oplus \bot \Rightarrow 1_\mathbb{X} \qquad u^R_{\oplus A} : A \oplus \bot \to A$$
$$u^L_\oplus : \bot \oplus 1_\mathbb{X} \Rightarrow 1_\mathbb{X} \qquad u^L_{\oplus A} : \bot \oplus A \to A$$

  such that $(\mathbb{X}, \oplus, \bot, {\alpha_\oplus}^{-1}, {u^R_\oplus}^{-1}, {u^L_\oplus}^{-1})$ *is a monoidal category,*

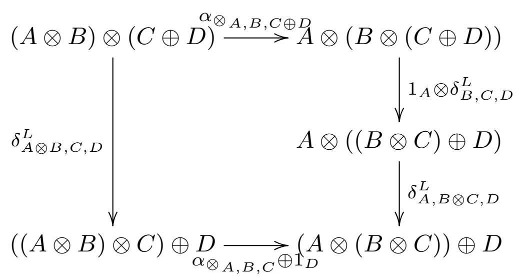
- *left and right* linear distributivity *natural transformations*

$$\delta^R : (\oplus \times 1_{\mathbb{X}}); \otimes \Rightarrow (1_{\mathbb{X}} \times \otimes); \oplus \qquad \delta^R_{A,B,C} : (A \oplus B) \otimes C \to A \oplus (B \otimes C),$$
$$\delta^L : (1_{\mathbb{X}} \times \oplus); \otimes \Rightarrow (\otimes \times 1_{\mathbb{X}}); \oplus \qquad \delta^L_{A,B,C} : A \otimes (B \oplus C) \to (A \otimes B) \oplus C,$$

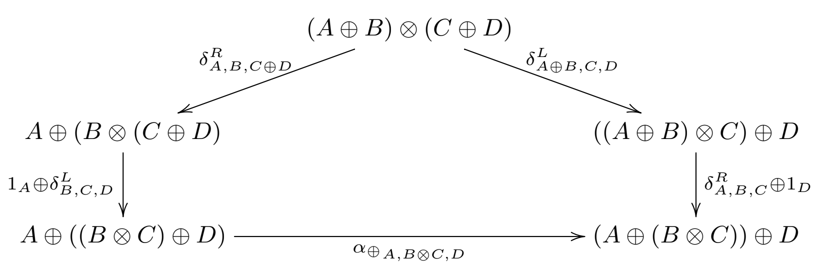
satisfying coherence conditions between units and linear distributivities (1)*, associativities and linear distributivities* (2)*, and left and right linear distributivities* (3)*.*



$$\begin{aligned} u^L_{\otimes A \oplus B}; \delta^L_{\top,A,B} &= u^L_{\otimes A} \oplus 1_B \\ u^R_{\otimes A \oplus B}; \delta^R_{A,B,\top} &= 1_A \oplus u^R_{\otimes B} \\ \delta^L_{A,B,\bot}; u^R_{\oplus A \otimes B} &= 1_A \otimes u^R_{\oplus B} \\ \delta^R_{\bot,A,B}; u^L_{\oplus A \otimes B} &= u^L_{\oplus A} \otimes 1_B \end{aligned} \tag{1}$$



$$\begin{aligned} \delta^L_{A \otimes B,C,D}; (\alpha_{\otimes A,B,C} \oplus 1_D) &= \alpha_{\otimes A,B,C \oplus D}; (1_A \otimes \delta^L_{B,C,D}); \delta^L_{A,B \otimes C,D} \\ (\alpha_{\oplus A,B,C} \otimes 1_D); \delta^R_{A \oplus B,C,D} &= \delta^R_{A,B \oplus C,D}; (1_A \oplus \delta^R_{B,C,D}); \alpha_{\oplus A,B,C \otimes D} \\ \alpha_{\otimes A \oplus B,C,D}; \delta^R_{A,B,C \otimes D} &= (\delta^R_{A,B,C} \otimes 1_D); \delta^R_{A,B \otimes C,D}; (1_A \oplus \alpha_{\otimes B,C,D}) \\ \delta^L_{A,B,C \oplus D}; \alpha_{\oplus A \otimes B,C,D} &= (1_A \otimes \alpha_{\oplus B,C,D}); \delta^L_{A,B \oplus C,D}; (\delta^L_{A,B,C} \oplus 1_D) \end{aligned} \tag{2}$$



$$\begin{aligned} \delta^L_{A \oplus B,C,D}; (\delta^R_{A,B,C} \oplus 1_D) &= \delta^R_{A,B,C \oplus D}; (1_A \oplus \delta^L_{B,C,D}); \alpha_{\oplus A,B \otimes C,D} \\ (\delta^L_{A,B,C} \otimes 1_D); \delta^R_{A \otimes B,C,D} &= \alpha_{\otimes A,B \oplus C,D}; (1_A \otimes \delta^R_{B,C,D}); \delta^L_{A,B,C \otimes D} \end{aligned} \tag{3}$$

We are often interested in specifically modeling commutative multiplicative linear logic. This is accomplished by considering symmetric LDCs.

Definition 2.2 *[10, Sec 3] A linearly distributive category* $(\mathbb{X}, \otimes, \top, \oplus, \bot)$ *is* **symmetric***, or a SLDC, if*

- $(\mathbb{X}, \otimes, \top)$ *is a symmetric monoidal category with* $\otimes$*-braiding*

$$\sigma_\otimes : \otimes \Rightarrow \mathit{switch}_{\mathbb{X},\mathbb{X}}; \otimes \qquad \sigma_{\otimes A,B} : A \otimes B \to B \otimes A$$

- $(\mathbb{X}, \oplus, \bot)$ *is a symmetric monoidal category with* $\oplus$*-braiding*

$$\sigma_\oplus : \oplus \Rightarrow \mathit{switch}_{\mathbb{X},\mathbb{X}}; \oplus \qquad \sigma_{\oplus A,B} : A \oplus B \to B \oplus A$$

such that the linear distributivities interact coherently with the braidings (4).

$$\begin{array}{ccc}
(A \oplus B) \otimes C & \xrightarrow{\delta^R_{A,B,C}} & A \oplus (B \otimes C) \\
\downarrow{\scriptstyle \sigma_{\otimes A\oplus B,C}} & & \uparrow{\scriptstyle \sigma_{\oplus B\otimes C,A}} \\
C \otimes (A \oplus B) & & (B \otimes C) \oplus A \\
\downarrow{\scriptstyle 1_C \otimes \sigma_{\oplus A,B}} & & \uparrow{\scriptstyle \sigma_{\otimes C,B} \oplus 1_A} \\
C \otimes (B \oplus A) & \xrightarrow[\delta^L_{C,B,A}]{} & (C \otimes B) \oplus A
\end{array} \tag{4}$$

Considering the above definition, we can suppress one of the linear distributivities and simply define the other via (4).

There are also variants of multiplicative linear logic that need to be modelled categorically, many of which are developed in [14], one of the most important of which is MLL with the MIX rule $A \otimes B \dashv A \oplus B$, which is equivalent to the nullary MIX rule $\bot \dashv \top$ in the presence of the cut rule.

Definition 2.3 *[14, Def 6.2] A linearly distributive category* $\mathbb{X}$ *is* **mix** *if there is a morphism* $m : \bot \to \top$ *such that*

$$\begin{array}{ccccc}
A \otimes B & \xrightarrow{1_A \otimes u^{L\,-1}_{\oplus B}} & A \otimes (\bot \oplus B) & \xrightarrow{1_A \otimes (m \oplus 1_B)} & A \otimes (\top \oplus B) \\
\downarrow{\scriptstyle u^{R\,-1}_{\oplus A} \otimes 1_B} & & & & \downarrow{\scriptstyle \delta^L_{A,\top,B}} \\
(A \oplus \bot) \otimes B & & & & (A \otimes \top) \oplus B \\
\downarrow{\scriptstyle (1_A \oplus m) \otimes 1_B} & & & & \downarrow{\scriptstyle u^{R\,-1}_{\otimes A} \oplus 1_B} \\
(A \oplus \top) \otimes B & \xrightarrow[\delta^R_{A,\top,B}]{} & A \oplus (\top \otimes B) & \xrightarrow[1_A \oplus u^{L\,-1}_{\otimes B}]{} & A \oplus B
\end{array} \tag{5}$$

in which case there is a natural transformation $\mathrm{mix}_{A,B} : A \otimes B \to A \oplus B$*, known as* **mix maps***, defined by the equivalent composites above.*

Lemma 2.4 *[14, Lem 6.2] A linearly distributive category is mix if and only if* (5) *for any one of the following cases:* $A = B = \top$*,* $A = B = \bot$*,* $A = \bot$ *and* $B = \top$*, or* $A = \top$ *and* $B = \bot$*.*

There is a stronger version of the nullary MIX rule, which often holds in categorical models: $\bot \dashv\vdash \top$. In this case, $\bot \cong \top$ and we call such LDCs isomix.

Definition 2.5 *[14, Def 6.5] A linearly distributive category is* **isomix** *if it is mix and $m : \bot \to \top$ is an isomorphism.*

In fact, we do not need to check (5) if the two monoidal units are isomorphic.

Lemma 2.6 *[14, Lem 6.6] A linearly distributive category where $\top \cong \bot$ is isomix.*

We can further consider the case where $\otimes$ and $\oplus$ monoidal structures are isomorphic.

Definition 2.7 *[16, Sec 2.3] A linearly distributive category is* **compact** *if it is isomix and the mix maps $\mathrm{mix}_{A,B} : A \otimes B \to A \oplus B$ are isomorphisms.*

The first categorical semantics for linear logic were demonstrated by Seely [9] to be $*$-autonomous categories, previously introduced by Barr [8]. Unlike LDCs, $*$-autonomous categories make multiplicative conjunctions and negation primitive, defining multiplicative disjunction by de Morgan dual. LDCs offer multiple advantages over $*$-autonomous categories, as will be discussed in Section 3, however it is still essential to be able to consider models with negation and recover the notion of $*$-autonomous categories.

Before introducing negation, we take a slight detour and talk about complemented objects.

Definition 2.8 *[15, Def A.5] Consider a linearly distributive category $\mathbb{X}$. The objects A, A^c in $\mathbb{X}$ form a complementation pair if there are morphisms*

$$\gamma : A \otimes A^c \to \bot \qquad \tau : \top \to A^c \oplus A$$

such that

$$\begin{array}{ccc} A & \xrightarrow{1_A} & A \\ {\scriptstyle u^R_{\otimes A}}\downarrow & & \uparrow{\scriptstyle u^L_{\oplus A}} \\ A \otimes \top & & \bot \otimes A \\ {\scriptstyle 1_A \otimes \tau}\downarrow & & \uparrow{\scriptstyle \gamma \oplus 1_A} \\ A \otimes (A^c \oplus A) & \xrightarrow[\delta^L_{A,A^c,A}]{} & (A \otimes A^c) \oplus A \end{array} \qquad \begin{array}{ccc} A^c & \xrightarrow{1_{A^c}} & A^c \\ {\scriptstyle u^{L\,c}_{\otimes A}}\downarrow & & \uparrow{\scriptstyle u^R_{\oplus A^c}} \\ \top \otimes A^c & & A^c \oplus \bot \\ {\scriptstyle \tau \otimes 1_{A^c}}\downarrow & & \uparrow{\scriptstyle 1_{A^c} \oplus \gamma} \\ (A^c \oplus A) \otimes A^c & \xrightarrow[\delta^R_{A^c,A,A^c}]{} & A^c \oplus (A \otimes A^c) \end{array} \qquad (6)$$

The pair is denoted by (A, A^c, γ, τ). A is said to be the left complement and A^c is the right complement.

Lemma 2.9 *[10, Lem 4.4] Let (A, A^c, γ, τ) be a complementation pair in a linearly distributive category $\mathbb{X}$, then there an adjunction $A \otimes - \dashv A^c \oplus -$, in other words there is a bijection between the hom-sets $\mathbb{X}(A \otimes B, C) \cong \mathbb{X}(B, A^c \oplus C)$.*

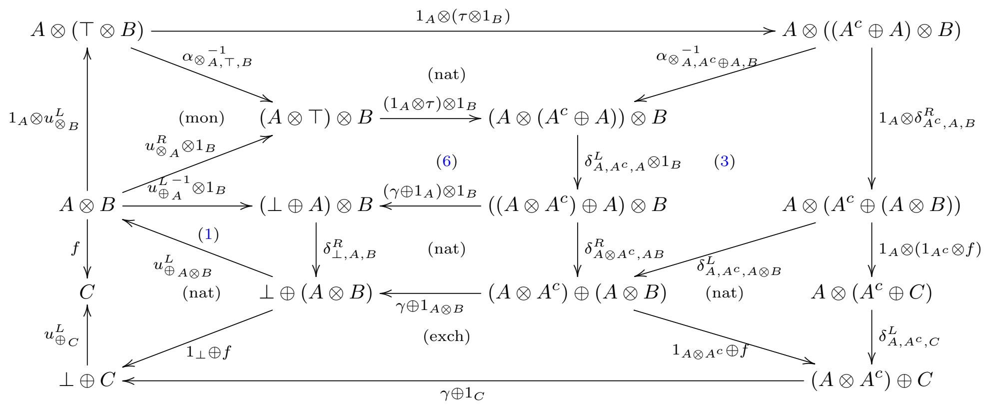
Proof Consider a morphism $f : A \otimes B \to C$ in $\mathbb{X}$, then define a morphism $B \to A^c \oplus C$ by

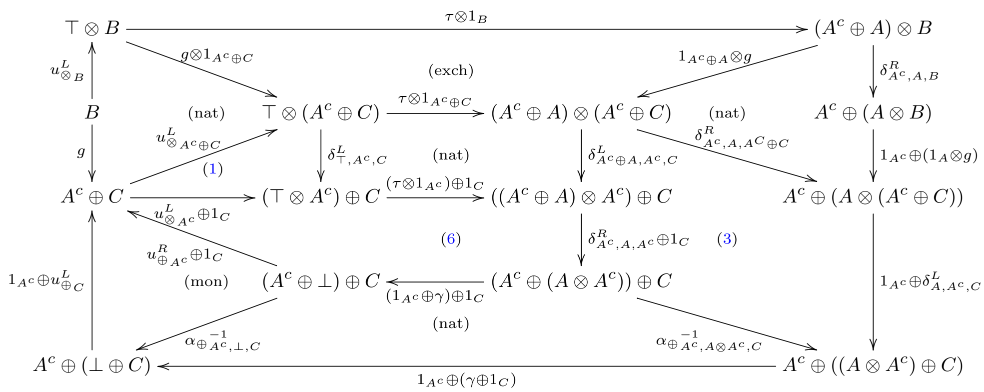
$$B \xrightarrow{u^R_{\otimes B}} \top \otimes B \xrightarrow{\tau \otimes 1_B} (A^c \oplus A) \otimes B \xrightarrow{\delta^R_{A^c,A,B}} A^c \oplus (A \otimes B) \xrightarrow{1_{A^c} \oplus f} A^c \oplus C$$

Consider a morphism $g : B \to A^c \oplus C$, then define a morphism $A \otimes B \to C$ by

$$A \otimes B \xrightarrow{1_A \otimes g} A \otimes (A^c \oplus C) \xrightarrow{\delta^L_{A,A^c,C}} (A \otimes A^c) \oplus C \xrightarrow{\gamma \oplus 1_C} \bot \oplus C \xrightarrow{u^R_{\oplus C}} C$$

These constructions are inverses of one another, giving the desired bijection, by the following commuting diagrams.

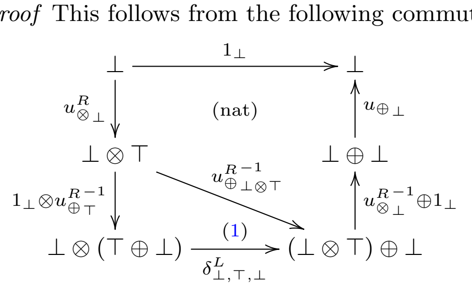


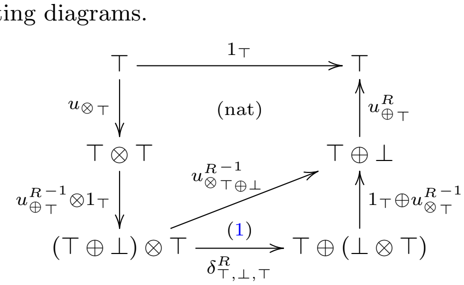


$\square$

Proposition 2.10 *[15, Prop A.7] In any linearly distributive category,* $(\bot, \top, u_{\oplus \top}^{R-1} : \top \to \top \oplus \bot, u_{\otimes \bot}^{R-1} : \bot \otimes \top \to \bot)$ *forms a complementation pair.*

Proof This follows from the following commuting diagrams.





$\square$

The left complement of an object is essentially the "left negation" of said object and similarly for the right complement. Therefore, a LDC having negation means all objects have negations a in coherent manner.

Definition 2.11 *[10, Def 4.1, 4.3] A symmetric linearly distributive category has* **negation** *if there is an object function $(-)^{\perp}$, together with the following parametrized family of maps*

$$\gamma_A^R : A \otimes A^{\perp} \to \perp \qquad\qquad \tau_A^R : \top \to A \oplus {}^{\perp}A$$

which additionally induce the following families,

$$\gamma_A^L = A^{\perp} \otimes A \xrightarrow{\sigma_{\otimes A^{\perp},A}} A \otimes A^{\perp} \xrightarrow{\gamma_A^R} \perp \qquad\qquad \tau_A^L = \top \xrightarrow{\tau_A^R} A \oplus A^{\perp} \xrightarrow{\sigma_{\oplus A,A^{\perp}}} A^{\perp} \oplus A$$

such $(A, A^{\perp}, \gamma_A^R, \tau_A^L)$ and $(A^{\perp}, A, \gamma_A^L, \tau_A^R)$ form a complementation pairs.

Example 2.12 *Let us list three important class of examples of LDCs, as presented in [10].*

1. *Every monoidal category $(\mathcal{X}, \oslash, I)$ can be viewed as a LDC, when taking the tensor and par structures to be equal to the original monoidal structure, i.e. $\otimes = \oplus = \oslash$ and $\top = \perp = I$. In this case, the linear distributivities are just $\oslash$-associators. These LDCs $(\mathcal{X}, \oslash, I, \oslash, I)$ are known as* **degenerate***.*

2. *Every $*$-autonomous category $(\mathcal{X}, \oslash, I, \star)$ is a LDC, with the tensor structure given by the original monoidal structure and the par structure given by the de Morgan dual, i.e. $A \oplus B = (B^{\star} \oslash A^{\star})^{\star}$ and $\perp = I^{\star}$. Some notable examples of $*$-autonomous categories include the category of finite-dimensional vector spaces, the category of sup-lattices, the category of coherence spaces and the category of finiteness spaces. These symmetric LDCs are precisely the ones with negation.*

   Theorem 2.13 *[10, Thm 4.5] The notions of symmetric linearly distributive categories with negation and $*$-autonomous categories coincide.*

3. *Cockett and Seely introduced a class of examples known as* **shifted tensor** *LDCs. These are the categorical analogue of shift monoids and are defined as follows.*

   Definition 2.14 *[10, Sec 5.2] Consider a monoidal category $(\mathbb{X}, \otimes, \top)$. An object $\perp \in \mathbb{X}$ is said to have a* **tensor inverse** *if there is an object $\perp^{-1}$ equipped with two isomorphisms*

   $$s^L : \perp \otimes \perp^{-1} \to \top \qquad s^R : \perp^{-1} \otimes \perp \to \top$$

   satisfying

   $$\begin{array}{ccc} (\perp^{-1} \otimes \perp) \otimes \perp^{-1} & \xrightarrow{\alpha_{\otimes \perp^{-1}, \perp, \perp^{-1}}} & \perp^{-1} \otimes (\perp \otimes \perp^{-1}) \\ {\scriptstyle s^R \otimes 1_{\perp^{-1}}} \downarrow & & \downarrow {\scriptstyle 1_{\perp^{-1}} \otimes s^L} \\ \top \otimes \perp^{-1} & & \perp^{-1} \otimes \top \\ & \searrow {\scriptstyle u^{L\,-1}_{\otimes \perp^{-1}}} \quad \perp^{-1} \quad {\scriptstyle u^{R\,-1}_{\otimes \perp^{-1}}} \swarrow & \end{array} \tag{7}$$

   Suppose there is an object $\perp \in \mathbb{X}$ with a tensor inverse, then define a monoidal product by $A \oplus B = A \otimes (\perp^{-1} \otimes B)$, known as the **$\perp$-shifted tensor***.*

Proposition 2.15 *[6, Prop 5.3, 5.4] Consider a monoidal category* $(\mathbb{X}, \otimes, \top)$ *with an object* $\bot \in \mathbb{X}$ *with a tensor inverse, then* $(\mathbb{X}, \otimes, \top, \oplus, \bot)$*, where* $\oplus$ *is the* $\bot$*-shifted tensor, is a linearly distributive category with invertible linear distributivities. Moreover, every linearly distributive categories with invertible linear distributivities has a tensor inverse for* $\bot$ *whose shift tensor product is naturally equivalent to the par product.*

2.2 Linear Functors

Cockett and Seely further developed the appropriate notion of morphisms between LDCs. The definition was determined in order to capture the notion of a monoidal functor between $*$-autonomous categories, the pair of exponentials, bang ! and why not ?, and the additive connectives of linear logic.

Definition 2.16 *[15, Def 1] Let* $\mathbb{X}$ *and* $\mathbb{Y}$ *be linearly distributive categories.*

- *A* **bilax linear functor** $F = (F_\otimes, F_\oplus) : \mathbb{X} \to \mathbb{Y}$ *consists of:*
  - *a lax monoidal functor* $(F_\otimes, m_\top, m_\otimes) : (\mathbb{X}, \otimes, \top) \to (\mathbb{Y}, \otimes, \top)$*, equipped with*
    - *a morphism* $m_\top : \top \to F_\otimes(\top)$*,*
    - *a natural transformation*

      $$m_\otimes : (F_\otimes \times F_\otimes); \otimes \Rightarrow \otimes; F_\otimes \qquad m_{\otimes A,B} : F_\otimes(A) \otimes F_\otimes(B) \to F_\otimes(A \otimes B)$$
  - *a colax monoidal functor* $(F_\oplus, n_\bot, n_\oplus) : (\mathbb{X}, \oplus, \bot) \to (\mathbb{Y}, \oplus, \bot)$*, equipped with*
    - *a morphism* $n_\bot : F_\oplus(\bot) \to \bot$*,*
    - *a natural transformation*

      $$n_\oplus : \oplus; F_\oplus \Rightarrow (F_\oplus \times F_\oplus); \oplus \qquad n_{\oplus A,B} : F_\oplus(A \oplus B) \to F_\oplus(A) \oplus F_\oplus(B)$$
  - *four natural transformations, known as* **linear strengths***,*

    $$v^R_\otimes : \oplus; F_\otimes \Rightarrow (F_\oplus \times F_\otimes); \oplus \qquad v^R_{\otimes A,B} : F_\otimes(A \oplus B) \to F_\oplus(A) \oplus F_\otimes(B)$$
    $$v^L_\otimes : \oplus; F_\otimes \Rightarrow (F_\otimes \times F_\oplus); \oplus \qquad v^L_{\otimes A,B} : F_\otimes(A \oplus B) \to F_\otimes(A) \oplus F_\oplus(B)$$
    $$v^R_\oplus : (F_\otimes \times F_\oplus); \otimes \Rightarrow \otimes; F_\oplus \qquad v^R_{\oplus A,B} : F_\otimes(A) \otimes F_\oplus(B) \to F_\oplus(A \otimes B)$$
    $$v^L_\oplus : (F_\oplus \times F_\otimes); \otimes \Rightarrow \otimes; F_\oplus \qquad v^L_{\oplus A,B} : F_\oplus(A) \otimes F_\otimes(B) \to F_\oplus(A \otimes B)$$

  subject to various coherence conditions detailed in [15].
- *If* $\mathbb{X}$ *and* $\mathbb{Y}$ *are symmetric linearly distributive categories, then a bilax linear functor* $F = (F_\otimes, F_\oplus)$ *is* **symmetric** *if*
  - $F_\otimes : (\mathbb{X}, \otimes, \top) \to (\mathbb{Y}, \otimes, \top)$ *is a symmetric lax monoidal functor,*
  - $F_\oplus : (\mathbb{X}, \oplus, \bot) \to (\mathbb{Y}, \oplus, \bot)$ *is a symmetric colax monoidal functor, and*
  - *the linear strengths interact coherently with the braidings* (8)

$$
\begin{array}{ccc}
F_\otimes(A \oplus B) & \xrightarrow{v^R_{\otimes A,B}} & F_\oplus(A) \oplus F_\otimes(B) \\
{\scriptstyle F_\otimes(\gamma_{\oplus A,B})}\downarrow & & \downarrow{\scriptstyle \gamma_{\oplus F_\oplus(A), F_\otimes(B)}} \\
F_\otimes(B \oplus B) & \xrightarrow[v^L_{\otimes B,A}]{} & F_\otimes(B) \oplus F_\oplus(A) \\
& & \\
F_\otimes(A) \otimes F_\oplus(B) & \xrightarrow{v^R_{\oplus A,B}} & F_\oplus(A \otimes B) \\
{\scriptstyle \gamma_{\otimes F_\otimes(A), F_\oplus(B)}}\downarrow & & \downarrow{\scriptstyle F_\oplus(\gamma_{\otimes A,B})} \\
F_\oplus(B) \otimes F_\otimes(A) & \xrightarrow[v^L_{\oplus B,A}]{} & F_\oplus(B \otimes A)
\end{array}
\tag{8}
$$

Definition 2.17 *A* **linear functor** $F = (F_\otimes, F_\oplus) : \mathbb{X} \to \mathbb{Y}$ *is a bilax linear functor such that*

- $F_\otimes : (\mathbb{X}, \otimes, \top) \to (\mathbb{Y}, \otimes, \top)$ *is a monoidal functor, and*
- $F_\oplus : (\mathbb{X}, \oplus, \bot) \to (\mathbb{Y}, \oplus, \bot)$ *is a monoidal functor.*

We shall also be interested in restricting our attention to linear functors whose component functors are in fact equal, as in many cases, the morphisms in a standard category consist of one function or functor, and not a pair of interacting functions or functors. These were first defined by [17] and called degenerate linear functors. They were renamed *Frobenius* linear functors by [16] as they generalize Frobenius monoidal functors. We shall use the latter terminology.

Definition 2.18 *[16, Def 3.1] Consider linearly distributive categories* $(\mathbb{X}, \otimes, \top, \oplus, \bot)$ *and* $(\mathbb{Y}, \otimes, \top, \oplus, \bot)$*, a bilax linear functor* $F = (F_\otimes, F_\oplus) : \mathbb{X} \to \mathbb{Y}$ *is* **Frobenius** *if*

1. $F_\otimes = F_\oplus$,
2. $\nu^R_{\otimes A,B} = \nu^L_{\otimes A,B} = n_{\oplus A,B}$, *and*
3. $\nu^R_{\oplus A,B} = \nu^L_{\oplus A,B} = m_{\otimes A,B}$.

Given the degeneracy, we can give an alternative characterization of such linear functors.

Proposition 2.19 *[16, Lem 3.2] Consider linearly distributive categories* $(\mathbb{X}, \otimes, \top, \oplus, \bot)$ *and* $(\mathbb{Y}, \otimes, \top, \oplus, \bot)$*, then the following notions coincide:*

- *bilax Frobenius linear functors* $F = (F_\otimes, F_\oplus) : \mathbb{X} \to \mathbb{Y}$*, and*
- $\otimes$*-lax and* $\oplus$*-colax functors* $(F, m_\otimes, m_\top, n_\oplus, n_\bot) : \mathbb{X} \to \mathbb{Y}$ *satisfying* (9)

$$
\begin{array}{ccccc}
F(A)\otimes F(B\oplus C) & \xrightarrow{m_{\otimes A,B\oplus C}} & F(A\otimes(B\oplus C)) & \xrightarrow{F(\delta^L_{A,B,C})} & F((A\otimes B)\oplus C) \\
\downarrow{\scriptstyle 1_{F(A)}\otimes n_{\oplus B,C}} & & & & \downarrow{\scriptstyle n_{\oplus A\otimes B,C}} \\
F(A)\otimes(F(B)\oplus F(C)) & \xrightarrow[\delta^L_{F(A),F(B),F(C)}]{} & (F(A)\otimes F(B))\oplus F(C) & \xrightarrow[m_{\otimes A,B}\oplus 1_{F(C)}]{} & F(A\otimes B)\oplus F(C) \\
\\
F(A\oplus B)\otimes F(C) & \xrightarrow{m_{\otimes A\oplus B,C}} & F((A\oplus B)\otimes C) & \xrightarrow{F(\delta^R_{A,B,C})} & F(A\oplus(B\otimes C)) \\
\downarrow{\scriptstyle n_{\oplus A,B}\otimes 1_{F(C)}} & & & & \downarrow{\scriptstyle n_{\oplus A,B\otimes C}} \\
(F(A)\oplus F(B))\otimes F(C) & \xrightarrow[\delta^R_{F(A),F(B),F(C)}]{} & F(A)\oplus(F(B)\otimes F(C)) & \xrightarrow[1_{F(A)}\oplus m_{\otimes B,C}]{} & F(A)\oplus F(B\otimes C)
\end{array}
\tag{9}
$$

If the LDCs are mix, the definition of Frobenius linear functors can be slightly extended to guarantee that they preserve the mix maps.

Definition 2.20 *[16, Def 3.4] Consider mix linearly distributive categories* $(\mathbb{X},\otimes,\top,\oplus,\bot)$ *and* $(\mathbb{Y},\otimes,\top,\oplus,\bot)$*, then a bilax Frobenius linear functor* $F=(F,F):\mathbb{X}\to\mathbb{Y}$ *is* **mix** *if the following diagram commutes.*

$$
\begin{array}{ccc}
F(\bot) & \xrightarrow{n_\bot} & \bot \\
\downarrow{\scriptstyle F(m)} & & \downarrow{\scriptstyle m} \\
F(\top) & \xleftarrow[m_\top]{} & \top
\end{array}
\tag{10}
$$

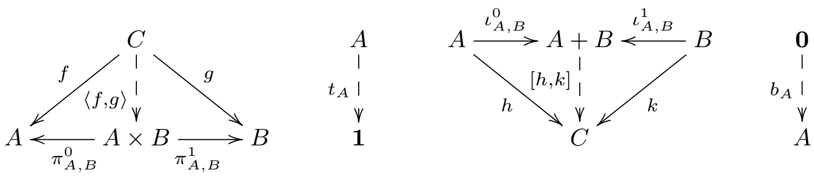
Proposition 2.21 *[16, Lem 3.5] A mix bilax Frobenius linear functor* $F=(F,F):\mathbb{X}\to\mathbb{Y}$ *preserves the mix maps of the mix linearly distributive categories* $(\mathbb{X},\otimes,\top,\oplus,\bot)$ *and* $(\mathbb{Y},\otimes,\top,\oplus,\bot)$*, i.e. the following diagram commutes.*

$$
\begin{array}{ccc}
F(A)\otimes F(B) & \xrightarrow{\mathrm{mix}_{F(A),F(B)}} & F(A)\oplus F(B) \\
\downarrow{\scriptstyle m_{\otimes A,B}} & & \uparrow{\scriptstyle n_{\oplus A,B}} \\
F(A\otimes B) & \xrightarrow[F(\mathrm{mix}_{A,B})]{} & F(A\oplus B)
\end{array}
$$

3 Cartesian Linearly Distributive Categories

We start by briefly outlining the standard structure of cartesian and cocartesian monoidal categories to establish the terminology used throughout this paper.

A **cartesian category** is a category with finite categorical products and a **cocartesian category** is a category with finite categorical coproducts, where binary products, terminal objects, binary coproducts and initial objects, with the corresponding unique maps, are denoted by



A cartesian category $\mathcal{X}$ canonically becomes a symmetric monoidal category $(\mathcal{X}, \times, \mathbf{1})$ whose monoidal product is the categorical product of $\mathcal{X}$ and monoidal unit is the terminal object of $\mathcal{X}$, i.e.

$$\times : (X, Y) \mapsto X \times Y, \qquad (f : X \to X', g : Y \to Y') \mapsto f \times g = \langle \pi^0_{X,Y}; f, \pi^1_{X,Y}; g\rangle$$

with isomorphisms

$$\begin{aligned}
&\alpha_{X,Y,Z} = \langle \pi^0_{X\times Y,Z}; \pi^0_{X,Y}, \pi^1_{X,Y} \times 1_Z\rangle : (X \times Y) \times Z \to X \times (Y \times Z)\\
&\rho_X = \langle 1_X, t_X\rangle : X \to X \times \mathbf{1}\\
&\lambda_X = \langle t_X, 1_X\rangle : X \to \mathbf{1} \times X\\
&\sigma_{X,Y} = \langle \pi^1_{X,Y}, \pi^0_{X,Y}\rangle : X \times Y \to Y \times X
\end{aligned}$$

Similarly, a cocartesian category $\mathcal{X}$ canonically becomes a symmetric monoidal category $(\mathcal{X}, +, \mathbf{0})$ whose monoidal product is the categorical coproduct of $\mathcal{X}$ and monoidal unit is the initial object, i.e.

$$+ : (X, Y) \mapsto X + Y, \qquad (f : X \to X', g : Y \to Y') \mapsto f + g = [f; \iota^0_{X',Y'}, g; \iota^1_{X',Y'}]$$

with isomorphisms

$$\begin{aligned}
&\alpha_{X,Y,Z} = [\iota^0_{X,Y}; \iota^0_{X+Y,Z}, \iota^1_{X,Y} + 1_Z] : X + (Y + Z) \to (X + Y) + Z\\
&\rho_X = [1_X, b_X] : X + \mathbf{0} \to X\\
&\lambda_X = [b_X, 1_X] : \mathbf{0} + X \to X\\
&\sigma_{X,Y} = [\iota^1_{Y,X}, \iota^0_{Y,X}] : X + Y \to Y + X
\end{aligned}$$

Remark 3.1 *For improved readability, we introduce the following notation for the "canonical flip" induced by the symmetry of cartesian and cocartesian structures:*

$$\begin{aligned}
\tau^\times_{W,X,Y,Z} &= \langle \pi^0_{W,X} \times \pi^0_{Y,Z}, \pi^1_{W,X} \times \pi^1_{Y,Z}\rangle : (W \times X) \times (Y \times Z) \to (W \times Y) \times (X \times Z)\\
\tau^+_{W,X,Y,Z} &= [\iota^0_{W,Y} + \iota^0_{X,Z}, \iota^1_{W,Y} + \iota^1_{X,Z}] : (W + X) + (Y + Z) \to (W + Y) + (X + Z)
\end{aligned}$$

We can now introduce the main definition of interest:

Definition 3.2 *[10, Sec 2] A* **cartesian linearly distributive category***, or a CLDC, $\mathbb{X}$ is a symmetric linearly distributive category whose tensor structure is the categorical product $\times$ in $\mathbb{X}$, with the terminal object $\mathbf{1}$ in $\mathbb{X}$, and the par structure is the categorical coproduct $+$ in $\mathbb{X}$, with the initial object $\mathbf{0}$ in $\mathbb{X}$.*

It is a straightforward concept, but we shall see that, in the presence of finite products and coproducts, linear distributivity gives rise to a highly peculiar form of distributivity, leading to surprisingly strong properties and results. We refrain from giving examples at this stage as that will be a key topic discussed in Sections 4, 5 and 7.

We begin by presenting a useful result concerning the interaction between the left distributivity, and certain projections and injections.

Lemma 3.3 *In any cartesian linearly distributive category, the following equations hold.*

$$\begin{array}{ll} \delta^L_{A,B,C};(\pi^1_{A,B}+1_C)=\pi^1_{A,B+C} & (1_A\times\iota^0_{B,C});\delta^L_{A,B,C}=\iota^0_{A\times B,C} \\ \delta^R_{A,B,C};(1_A+\pi^0_{B,C})=\pi^0_{A+B,C} & (\iota^1_{A,B}\times 1_C);\delta^R_{A,B,C}=\iota^1_{A,B\times C} \end{array} \tag{11}$$

Proof The first equation follows from the tensor cartesian structure as

$$\begin{aligned} \delta^L_{A,B,C};(\pi^1_{A,B}+1_C) &= \delta^L_{A,B,C};((t_A\times 1_B)+1_C);({u^L_{\times}}^{-1}_B+1_C) && \text{by definition of } \pi^1 \\ &= (t_A\times 1_{B+C});\delta^L_{\mathbf{1},B,C};({u^L_{\times}}^{-1}_B+1_C) && \text{by naturality of } \delta^L \\ &= (t_A\times 1_{B+C});{u^L_{\times}}^{-1}_{B+C} && \text{by (1)} \\ &= \pi^1_{A.B+C} && \text{by definition of } \pi^1 \end{aligned}$$

The second equation follows similarly by the par cocartesian structure. □

The appropriate definition of cartesian linear functors mirrors the definition of cartesian functors. We shall also be particularly interested in restricting our attention to Frobenius cartesian linear functors.

Proposition 3.4 *There is a category of cartesian linearly distributive categories and linear functors, known as cartesian linear functors, denoted by* **CLDC**. *There is a subcategory of cartesian linearly distributive categories and Frobenius cartesian linear functors, denoted by* **FCLDC**.

3.1 Subterminal Initial and Preinitial Terminal

Objects may not be strictly terminal or initial, but may still exhibit a key characteristic of such objects. These are known as subterminal and preinitial objects, defined as follows.

Definition 3.5 *[1, Sec 3] A* **preinitial object** *is an object with at most one map to each given object.*

Lemma 3.6 *[1, Lem 3.2] In a category with binary coproducts, the following are equivalent:*

(i) *A is preinitial,*

(ii) $\nabla_A=[1_A,1_A]:A+A\to A$ *is an isomorphism,*

(iii) $\iota^0_{A,A}=\iota^1_{A,A}:A\to A+A$.

Dually:

Definition 3.7 *A* **subterminal object** *is an object with at most one map from each given object.*

Lemma 3.8 *In a category with binary products, the following are equivalent:*

(i) *A is subterminal,*

(ii) $\Delta_A = \langle 1_A, 1_A \rangle : A \to A \times A$ *is an isomorphism,*

(iii) $\pi^0_{A,A} = \pi^1_{A,A} : A \times A \to A$.

Proof $(i) \Rightarrow (iii)$: by uniqueness of maps $A \times A \to A$, $\pi^0_{A,A} = \pi^1_{A,A}$.

$(iii) \Rightarrow (i)$: Suppose there are two maps $f, g : B \to A$, then

$$f = \Delta_A; \pi^0_{A,A}; f = \Delta_A; (f \times g); \pi^0_{A,A} = \Delta_A; (f \times g); \pi^1_{A,A} = \Delta_A; \pi^0_{A,A}; g = g.$$

$(ii) \Rightarrow (iii)$: $\Delta_A; \pi^0_{A,A} = 1_A$ and $\Delta_A; \pi^1_{A,A} = 1_A$, thus $\pi^0_{A,A} = \Delta_A^{-1} = \pi^1_{A,A}$.

$(iii) \Rightarrow (ii)$: $\Delta_A; \pi^0_{A,A} = 1_A$ and $\pi^0_{A,A}; \Delta_A = 1_{A \times A}$ as

$$\pi^0_{A,A}; \Delta_A; \pi^0_{A,A} = \pi^0_{A,A}; 1_A = \pi^0_{A,A} \qquad \pi^0_{A,A}; \Delta_A; \pi^1_{A,A} = \pi^0_{A,A}; 1_A = \pi^1_{A,A}.$$

□

The first key result that exemplifies the uniqueness of CLDCs is that the initial object is always subterminal and the terminal object is always preinitial. Of course, this is not the case for most categories with finite products and coproducts, giving us the first indication that examples of CLDCs may be few and far between.

Proposition 3.9 *The initial object* $\mathbf{0}$ *is subterminal and the terminal object* $\mathbf{1}$ *is preinitial in any cartesian linearly distributive category* $\mathbb{X}$.

Proof By Proposition 2.10, $\mathbf{0}$ and $\mathbf{1}$ form a complementation pair.

By Lemma 2.9, there is therefore a bijection between the hom-sets $\mathbb{X}(\mathbf{0} \times \mathbf{0}, \mathbf{0}) \cong \mathbb{X}(\mathbf{0}, \mathbf{1} + \mathbf{0})$. As $\mathbf{0}$ is the initial object of $\mathbb{X}$, the latter hom-sets must be singletons. Consider the projections $\pi^0_{\mathbf{0},\mathbf{0}}$ and $\pi^1_{\mathbf{0},\mathbf{0}}$. They must be equal as they both belong to $\mathbb{X}(\mathbf{0} \times \mathbf{0}, \mathbf{0})$. By Lemma 3.8, $\mathbf{0}$ is subterminal.

Similarly, by 2.9, $\mathbb{X}(\mathbf{1}, \mathbf{1} \oplus \mathbf{1}) \cong \mathbb{X}(\mathbf{0} \otimes \mathbf{1}, \mathbf{1})$ and they are singleton sets as $\mathbf{1}$ is terminal. Therefore, $\iota^0_{\mathbf{1},\mathbf{1}} = \iota^1_{\mathbf{1},\mathbf{1}}$ as they both below to the first hom-set. By Lemma 3.6, $\mathbf{1}$ is preinitial.
□

Remark 3.10 *The above argument clearly applies to more than just CLDCs. Given any LDC, if* $\mathbf{0}$ *is initial, it is also subterminal and dually, if* $\mathbf{1}$ *is terminal, it is preinitial.*

As $\mathbf{0}$ is subterminal and $\mathbf{1}$ is preinitial, we can give a description of objects of the form $A \times \mathbf{0}$ and $A + \mathbf{1}$:

Lemma 3.11 *Consider an object A in a cartesian linearly distributive category, then*

- $A \cong A \times \mathbf{0}$ *if and only if there exists a map* $A \to \mathbf{0}$*, and*
- $A \cong A + \mathbf{1}$ *if and only if there exists a map* $\mathbf{1} \to A$*.*

Proof Suppose $A \cong A \times \mathbf{0}$, then $A \cong A \times \mathbf{0} \xrightarrow{\pi^1_{A,\mathbf{0}}} \mathbf{0}$. Suppose there exists a map $a : A \to \mathbf{0}$, then $\pi^0_{A,\mathbf{0}} : A \times \mathbf{0} \to A$ and $\langle 1_A, a \rangle : A \to A \times \mathbf{0}$ are inverses as

$$\langle 1_A, a \rangle; \pi^0_{A,\mathbf{0}} = 1_A$$

$$\pi^0_{A,\mathbf{0}}; \langle 1_A, a \rangle; \pi^0_{A,\mathbf{0}} = \pi^0_{A,\mathbf{0}}; 1_A = \pi^0_{A,\mathbf{0}}$$

$$\pi^0_{A,\mathbf{0}}; \langle 1_A, a \rangle; \pi^1_{A,\mathbf{0}} = \pi^0_{A,\mathbf{0}}; a = \pi^1_{A,\mathbf{0}}$$

as there is a unique map $A \times \mathbf{0} \to \mathbf{0}$ since $\mathbf{0}$ is subterminal.

Similarly, if $A + \mathbf{1} \cong A$ or if there is a map $\mathbf{1} \to A$. □

3.2 Mix Structure

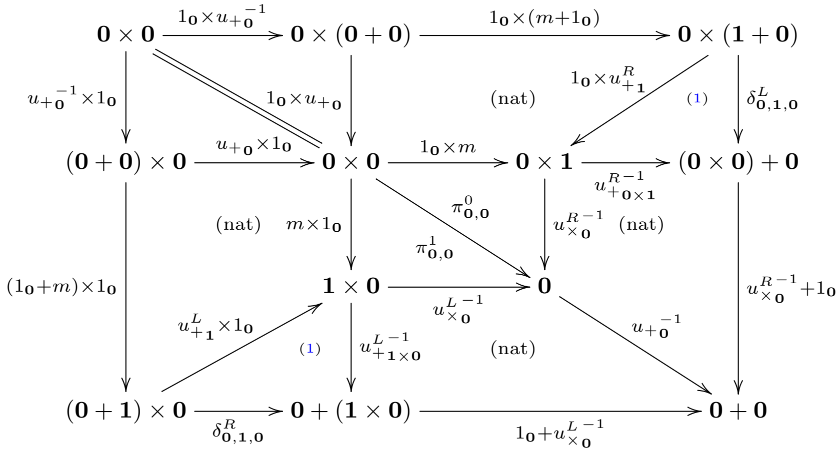
It was first proved in [14] that a CLDC is mix. The proof proceeds by considering the subcategory of all complemented objects, known as the nucleus, of a symmetric LDC and proving that is a $*$-autonomous category, or more precisely a symmetric LDC with negation. Moreover, it was shown that a LDC is mix if and only if its nucleus is mix.

Then, considering a SLDC whose tensor is cartesian results in a cartesian $*$-autonomous category, and therefore a Boolean algebra by Joyal's paradox. A Boolean algebra is a mix LDC as all diagrams commute. Thus, a symmetric LDC whose tensor is cartesian is mix.

We give here an alternative streamlined version of their proof, which utilizes the core of the idea put forth by Cockett and Seely.

Proposition 3.12 *[14, Rem 5.4] A cartesian linearly distributive category is mix with* $m = b_{\mathbf{1}} = t_{\mathbf{0}} : \mathbf{0} \to \mathbf{1}$.

Proof By Lemma 2.4, it suffices to prove (5) for $\mathbf{0} \times \mathbf{0} \to \mathbf{0} + \mathbf{0}$, which holds by the following commuting diagram.



□

Therefore, for all pairs of objects A, B in a CLDC, there exists a mix map from their product to their coproduct:

$$\mathrm{mix}_{A,B} : A \times B \to A + B$$

Note that cartesian Frobenius linear functors preserver the mix maps as:

Lemma 3.13 *A cartesian Frobenius linear functor is a mix Frobenius linear functor.*

Proof Note that $n_{\mathbf{0}}$ and $m_{\mathbf{1}}$ are invertible as the cartesian linear functor is strong. Then, as $\mathbf{0}$ is initial (or $\mathbf{1}$ is terminal), we get $m = n_{\mathbf{0}}^{-1}; F(m); m_{\mathbf{1}}^{-1}$, meaning (10) holds. □

With the existence of mix maps, we can give an alternative characterization of preinitial and subterminal objects in a CLDC.

Lemma 3.14 *Consider an object A in a cartesian linearly distributive category, then*

- *A is preinitial if and only if the following equations hold.*

$$\Delta_A; \mathrm{mix}_{A,A}; \nabla_A = 1_A \qquad \nabla_A; \Delta_A; \mathrm{mix}_{A,A} = 1_{A+A}$$

- *A is subterminal if and only if the following equations hold.*

$$\Delta_A; \mathrm{mix}_{A,A}; \nabla_A = 1_A \qquad \mathrm{mix}_{A,A}; \nabla_A; \Delta_A = 1_{A\times A}$$

Proof Suppose that A is preinitial, then $\Delta_A; m_{A,A}; \nabla_A = 1_A$ by uniqueness of arrows from A. Now, by Lemma 3.6, ∇_A^{-1} exists, so

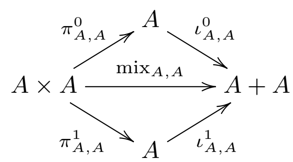
$$\nabla_A; \Delta_A; \mathrm{mix}_{A,A} = \nabla_A; \Delta_A; \mathrm{mix}_{A,A}; \nabla_A; \nabla_A^{-1} = \nabla_A; 1_A; \nabla_A^{-1} = 1_{A+A}$$

Suppose $\Delta_A; \mathrm{mix}_{A,A}; \nabla_A = 1_A$ and $\nabla_A; \Delta_A; \mathrm{mix}_{A,A} = 1_{A+A}$, then ∇_A is an isomorphism with inverse $\Delta_A; \mathrm{mix}_{A,A} : A \to A + A$. By Lemma 3.6, A is preinitial.

Similarly if A is instead subterminal. □

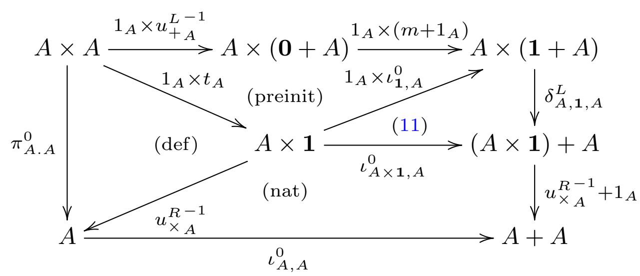
We can give an even further characterization about such objects in relation to the mix maps.

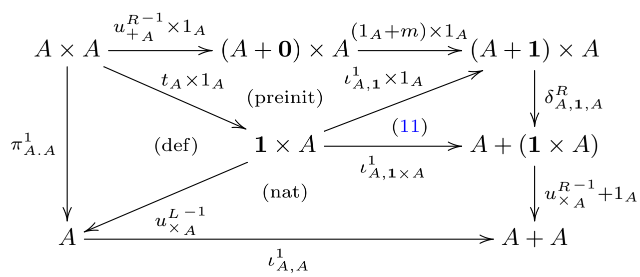
Lemma 3.15 *Consider an object A in a cartesian linearly distributive category, then A is preinitial (or subterminal) if and only if the following diagram commutes.*



(12)

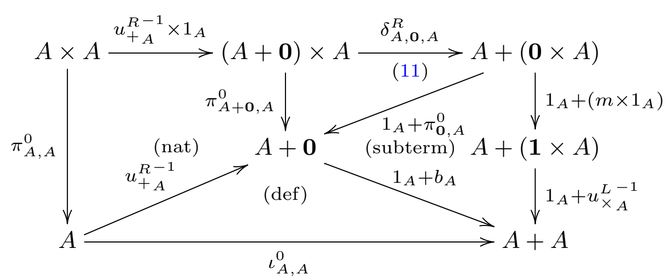
Proof Suppose A is preinitial, then the following diagrams commute

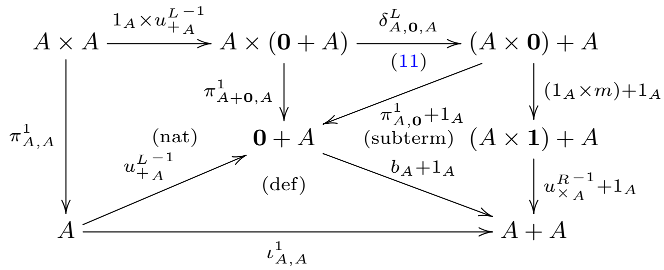




and the upper composites are two equivalent definitions of $\text{mix}_{A,A}$.

Similarly, suppose A is subterminal, then the following diagrams commute





and the upper composites are two other equivalent definitions of $m_{A,A}$ (by naturality of the linear distributivities).

Suppose (12) holds. Then,

$$\Delta_A; \text{mix}_{A,A}; \nabla_A = \Delta_A; \pi^j_{A,A}; \iota^j_{A,A}; \nabla_A = 1_A; 1_A = 1_A$$

and $\nabla_A; \Delta_A; \text{mix}_{A,A} = 1_{A+A}$ as

$$\iota^0_{A,A}; \nabla_A; \Delta_A; \text{mix}_{A,A} = \iota^0_{A,A}; \nabla_A; \Delta_A; \pi^0_{A,A}; \iota^0_{A,A} = 1_A; 1_A; \iota^0_{A,A} = \iota^0_{A,A}$$

$$\iota^1_{A,A}; \nabla_A; \Delta_A; \text{mix}_{A,A} = \iota^1_{A,A}; \nabla_A; \Delta_A; \pi^1_{A,A}; \iota^1_{A,A}$$

Thus, A is preinitial. Similarly, $\text{mix}_{A,A}; \nabla_A; \Delta_A = 1_{A\times A}$ as

$$\text{mix}_{A,A}; \nabla_A; \Delta_A; \pi^0_{A,A} = \pi^0_{A,A}; \iota^0_{A,A}; \nabla_A; \Delta_A; \pi^0_{A,A} = \pi^0_{A,A}; 1_A; 1_A = \pi^0_{A,A}$$

$$\text{mix}_{A,A}; \nabla_A; \Delta_A; \pi^1_{A,A} = \pi^1_{A,A}; \iota^1_{A,A}; \nabla_A; \Delta_A; \pi^1_{A,A} = \pi^1_{A,A}; 1_A; 1_A = \pi^1_{A,A}$$

□

Of course, since being preinitial and being subterminal are equivalent to the same condition in the above lemma, we can conclude that:

Proposition 3.16 *An object in a cartesian linearly distributive category is preinitial if and only if it is subterminal.*

3.3 Duoidal Structure

As discussed at the start of Section 3, duoidal categories are another important type of category with two monoidal structures:

Definition 3.17 *[4, Def 6.1] A* **duoidal category** $(\mathcal{X}, \diamond, I, \star, J)$ *is category* $\mathcal{X}$ *with two monoidal structures*
$(\mathcal{X}, \diamond, I, \alpha_\diamond, \rho_\diamond, \lambda_\diamond)$ *and* $(\mathcal{X}, \star, J, \alpha_\star, \rho_\star, \lambda_\star)$ *equipped with morphisms*

$$\Delta_I : I \to I \star I \qquad \mu_J : J \diamond J \to J \qquad \iota : I \to J$$

and an **interchange** *natural transformation*

$$\zeta : (\star \times \star); \diamond \Rightarrow s_{\mathcal{X},\mathcal{X},\mathcal{X},\mathcal{X}}; (\diamond \times \diamond); \star \quad \zeta_{A,B,C,D} : (A \star B) \diamond (C \star D) \to (A \diamond C) \star (B \diamond D)$$

such that

- (J, ∇_J, ι) *is a* $\diamond$*-monoid,*
- (I, Δ_I, ι) *is a* $\star$*-comonoid,*

- *interchange maps interact coherently with associativities* (13) *and with unitors* (14).

$$
\begin{aligned}
&\alpha_{\diamond A\star B,C\star D,E\star F};(1_{A\star B}\diamond\zeta_{C,D,E,F});\zeta_{A,B,C\diamond E,D\diamond F}\\
&=(\zeta_{A,B,C,D}\diamond 1_{E\star F});\zeta_{A\diamond C,B\diamond D,E,F};(\alpha_{\diamond A,C,E}\star\alpha_{\diamond B,D,F})\\
\\
&\zeta_{A\star B,CD\star E,F};(\zeta_{A,B,D,E}\star 1_{C\diamond F});\alpha_{\star A\diamond D,B\diamond E,C\diamond F}\\
&=(\alpha_{\star A,B,C}\diamond\alpha_{\star D,E,F});\zeta_{A,B\star C,D,E\star F};(1_{A\diamond D}\star\zeta_{B,C,E,F})
\end{aligned}
\tag{13}
$$

$$
\begin{aligned}
&\lambda_{\diamond A\star B};(\Delta_I\diamond 1_{A\star B});\zeta_{I,I,A,B}=\lambda_{\diamond A}\star\lambda_{\diamond B}\\
&(\lambda_{\star A}\diamond\lambda_{\star B});\zeta_{J,A,J,B};(\mu_J\star 1_{A\diamond B})=\lambda_{\star A\diamond B}\\
&\rho_{\diamond A\star B};(1_{A\star B}\diamond\Delta_I);\zeta_{A,B,I,I}=\rho_{\diamond A}\star\rho_{\diamond B}\\
&(\rho_{\star A}\diamond\rho_{\star B});\zeta_{A,J,B,J};(1_{A\diamond B}\star\mu_J)=\rho_{\star A\diamond B}
\end{aligned}
\tag{14}
$$

Now, duoidal structures arise canonically whenever monoidal categories have finite products or finite coproducts. Therefore, every category which is both cartesian and cocartesian is a duoidal category, as detailed in Example 6.19 of [4]. So, in particular:

Proposition 3.18 *A cartesian linearly distributive category* $(\mathbb{X},\times,\mathbf{1},+,\mathbf{0})$ *is a duoidal category, with structure maps*

$$\Delta_{\mathbf{0}}=b_{\mathbf{0}\times\mathbf{0}}=\langle 1_{\mathbf{0}},1_{\mathbf{0}}\rangle:\mathbf{0}\to\mathbf{0}\times\mathbf{0}\qquad \nabla_{\mathbf{1}}=t_{\mathbf{1}+\mathbf{1}}=[1_{\mathbf{1}},1_{\mathbf{1}}]:\mathbf{1}+\mathbf{1}\to\mathbf{1}$$
$$m=t_{\mathbf{0}}=b_{\mathbf{1}}:\mathbf{0}\to\mathbf{1}$$

and interchange natural transformation

$$
\begin{aligned}
\mu_{A,B,C,D}&=\langle\pi^0_{A,B}+\pi^0_{C,D},\pi^1_{A,B}+\pi^1_{C,D}\rangle=[\iota^0_{A,C}\times\iota^0_{B,D},\iota^1_{A,C}\times\iota^1_{B,D}]\\
&:(A\times B)+(C\times D)\to(A+C)\times(B+D)
\end{aligned}
$$

These interchange maps interact coherently with the symmetry of $+$ and $\times$, and with the linear distributivities central to the definition of a CLDC.

Proposition 3.19 *The following diagrams hold in any cartesian linearly distributive category:*

$$
\begin{array}{ccc}
(A\times B)+(C\times D) & \xrightarrow{\mu_{A,B,C,D}} & (A+C)\times(B+D)\\
{\scriptstyle \sigma_{\times A,B}+\sigma_{\times C,D}}\downarrow & & \downarrow{\scriptstyle \sigma_{\times A+C,B+D}}\\
(B\times A)+(C\times D) & \xrightarrow[\mu_{B,A,D,C}]{} & (B+D)\times(A+C)\\
\\
(A\times B)+(C\times D) & \xrightarrow{\mu_{A,B,C,D}} & (A+C)\times(B+D)\\
{\scriptstyle \sigma_{+A\times B,C\times D}}\downarrow & & \downarrow{\scriptstyle \sigma_{+A,C}\times\sigma_{+B,D}}\\
(C\times D)+(A\times B) & \xrightarrow[\mu_{C,D,A,B}]{} & (C+A)\times(D+B)
\end{array}
\tag{15}
$$

$$
\begin{array}{ccc}
X \times ((A \times B) + (C \times D)) & \xrightarrow{\delta^L_{X,A\times B,C\times D}} & (X \times (A \times B)) + (C \times D) \\
\downarrow{\scriptstyle 1_X \times \mu_{A,B,C,D}} & & \downarrow{\scriptstyle \alpha_{\times}{}^{-1}_{X,A,B} + 1_{C\times D}} \\
X \times ((A + C) \times (B + D)) & & ((X \times A) \times B) + (C \times D) \\
\downarrow{\scriptstyle \alpha_{\times}{}^{-1}_{X,A+C,B+D}} & & \downarrow{\scriptstyle \mu_{X\times A,B,C,D}} \\
(X \times (A + C)) \times (B + D) & \xrightarrow[\delta^L_{X,A,C} \times 1_{B+D}]{} & ((X \times A) + C) \times (B + D)
\end{array}
$$

$$
\begin{aligned}
&(1_X \times \mu_{A,B,C,D}); \alpha_{\times}{}^{-1}_{X,A+C,B+D}; (\delta^L_{X,A,C} \times 1_{B+D}) \\
&= \delta^L_{X,A\times B,C\times D}; (\alpha_{\times}{}^{-1}_{X,A,B} + 1_{C\times D}); \mu_{X\times A,B,C,D}
\end{aligned}
$$

$$
\begin{aligned}
&(1_{A\times B} + \delta^L_{C,D,X}); \alpha_{+A\times B,C\times D,X}; (\mu_{A,B,C,D} + 1_X) \\
&= \mu_{A,B,C,D+X}; (1_{A+C} \times \alpha_{+B,D,X}); \delta^L_{A+C,B+D,X} \\
\\
&(\mu_{A,B,C,D} \times 1_X); \alpha_{\times A+C,B+D,X}; (1_{A+C} \times \delta^R_{B,D,X}) \\
&= \delta^R_{A\times B,C\times D,X}; (1_{A\times B} + \alpha_{\times C,D,X}); \mu_{A,B,C,D\times X} \\
&(\delta^R_{X,A,B} + 1_{C\times D}); \alpha_{+}{}^{-1}_{X,A\times B,C\times D}; (1_X + \mu_{A,B,C,D}) \\
&= \mu_{X+A,B,C,D}; (\alpha_{+}{}^{-1}_{X,A,C} \times 1_{B+D}); \delta^R_{X,A+C,B+D}
\end{aligned}
\tag{16}
$$

Proof The diagrams in (15) holds by the universal properties of products and the "cartesian" definition of $\mu_{A,B,C,D}$ in Proposition 3.18:

$$
\begin{aligned}
&(\sigma_{\times A,B} + \sigma_{\times C,D}); \mu_{B,A,D,C} \\
&= (\sigma_{\times A,B} + \sigma_{\times C,D}); \langle \pi^0_{A,B} + \pi^0_{C,D}, \pi^1_{A,B} + \pi^1_{C,D} \rangle \\
&= \langle \sigma_{\times A,B}; \pi^0_{A,B} + \sigma_{\times C,D}; \pi^0_{C,D}, \sigma_{\times A,B}; \pi^1_{A,B} + \sigma_{\times C,D}; \pi^1_{C,D} \rangle \\
&= \langle \pi^1_{A,B} + \pi^1_{C,D}, \pi^0_{A,B} + \pi^0_{C,D} \rangle \\
&= \langle \mu_{A,B,C,D}; \pi^1_{A+C,B+D}, \mu_{A,B,C,D}; \pi^0_{A+C,B+D} \rangle \\
&= \mu_{A,B,C,D}; \langle \pi^1_{A+C,B+D}, \pi^0_{A+C,B+D} \rangle \\
&= \mu_{A,B,C,D}; \sigma_{\times A+C,B+D}
\end{aligned}
$$

$$
\begin{aligned}
&\mu_{A,B,C,D}; (\sigma_{+A,C} \times \sigma_{+B,D}) \\
&= \mu_{A,B,C,D}; \langle \pi^0_{A+C,B+D}; \sigma_{+A,C}, \pi^1_{A+C,B+D}; \sigma_{+B,D} \rangle \\
&= \langle \mu_{A,B,C,D}; \pi^0_{A+C,B+D}; \sigma_{+A,C}, \mu_{A,B,C,D}; \pi^1_{A+C,B+D}; \sigma_{+B,D} \rangle \\
&= \langle (\pi^0_{A,B} + \pi^0_{C,D}); \sigma_{+A,C}, (\pi^1_{A,B} + \pi^1_{C,D}); \sigma_{+B,D} \rangle \\
&= \langle \sigma_{+A\times B,C\times D}; (\pi^0_{C,D} + \pi^0_{A,B}), \sigma_{+A\times B,C\times D}; (\pi^1_{C,D} + \pi^1_{A,B}) \rangle
\end{aligned}
$$

$$= \sigma_{+A\times B,C\times D}; \langle \pi^0_{C,D} + \pi^0_{A,B}, \pi^1_{C,D} + \pi^1_{A,B} \rangle$$
$$= \sigma_{+A\times B,C\times D}; \mu_{C,D,A,B}$$

Alternatively, the above diagrams can be proven using the universal properties of coproducts and the "cocartesian" definition of $\mu_{A,B,C,D}$ in Proposition 3.18.

The first diagram in (16) holds by tensor cartesian structure:

$$\delta^L_{X,A\times B,C\times D}; (\alpha_{\times}^{-1}{}_{X,A,B} + 1_{C\times D}); \mu_{X\times A,B,C,D}$$
$$= \langle \delta^L_{X,A\times B,C\times D}; (\alpha_{\times}^{-1}{}_{X,A,B} + 1_{C\times D}); (\pi^0_{A,B} + \pi^0_{C,D}), \delta^L_{X,A\times B,C\times D};$$
$$(\alpha_{\times}^{-1}{}_{X,A,B} + 1_{C\times D}); (\pi^1_{A,B} + \pi^1_{C,D}) \rangle$$
$$= \langle \delta^L_{X,A\times B,C\times D}; ((1_X \times \pi^0_{A,B}) + \pi^0_{C,D}), \delta^L_{X,A\times B,C\times D}; (\pi^1_{X,A\times B} + 1_{C\times D}); (\pi^1_{A,B} + \pi^1_{C,D}) \rangle$$
$$= \langle (1_X \times (\pi^0_{A,B} + \pi^0_{C,D})); \delta^L_{X,A,C}, \pi^1_{X,(A\times B)+(C\times D)}; (\pi^1_{A,B} + \pi^1_{C,D}) \rangle$$
$$= \langle 1_X \times \mu_{A,B,C,D}; \pi^0_{A+C,B+D}); \delta^L_{X,A,C}, \pi^1_{X,(A\times B)+(C\times D)}; \mu_{A,B,C,D}; \pi^1_{A+C,B+D} \rangle$$
$$= \langle 1_X \times \mu_{A,B,C,D}; \pi^0_{A+C,B+D}); \delta^L_{X,A,C}, (1_X \times \mu_{A,B,C,D}); \pi^1_{X,(A+C)\times(B+D)}; \pi^1_{A+C,B+D} \rangle$$
$$= (1_X \times \mu_{A,B,C,D}); \langle (1_X \times \pi^0_{A+C,B+D}); \delta^L_{X,A,C}, \pi^1_{X,(A+C)\times(B+D)}; \pi^1_{A+C,B+D} \rangle$$
$$= (1_X \times \mu_{A,B,C,D}); \langle \alpha_{\times}^{-1}{}_{X,A+C,B+D}; \pi^0_{X\times(A+C),B+D}; \delta^L_{X,A,C},$$
$$\alpha_{\times}^{-1}{}_{X,A+C,B+D}; \pi^1_{X\times(A+C),B+D} \rangle$$
$$= (1_X \times \mu_{A,B,C,D}); \alpha_{\times}^{-1}{}_{X,A+C,B+D}; \langle \pi^0_{X\times(A+C),B+D}; \delta^L_{X,A,C}, \pi^1_{X\times(A+C),B+D} \rangle$$
$$= (1_X \times \mu_{A,B,C,D}); \alpha_{\times}^{-1}{}_{X,A+C,B+D}; (\delta^L_{X,A,C} + 1_{B+D})`$$

The second diagram in (16) holds by par cocartesian structure:

$$\mu_{A,B,C,D+X}; (1_{A+C} \times \alpha_{+B,D,X}); \delta^L_{A+C,B+D,X}$$
$$= [(\iota^0_{A,C} \times \iota^0_{B,D+X}); (1_{A+C} \times \alpha_{+B,D,X}); \delta^L_{A+C,B+D,X}, (\iota^1_{A,C} \times \iota^1_{B,D+X});$$
$$(1_{A+C} \times \alpha_{+B,D,X}); \delta^L_{A+C,B+D,X})]$$
$$= [(\iota^0_{A,C} \times \iota^0_{B,D}); (1_{A+C} \times \iota^0_{B+D,X}); \delta^L_{A+C,B+D,X}, (\iota^1_{A,C} \times (\iota^1_{B,D} + 1_X)); \delta^L_{A+C,B+D,X})]$$
$$= [(\iota^0_{A,C} \times \iota^0_{B,D}); \iota^0_{(A+C)\times(B+D),X}, \delta^L_{C,D,X}; ((\iota^1_{A,C} \times \iota^1_{B,D}) + 1_X)]$$
$$= [\iota^0_{A\times B,C\times D}; \mu_{A,B.C,D}; \iota^0_{(A+C)\times(B+D),X}, \delta^L_{C,D,X}; (\iota^1_{A\times B,C\times D}; \mu_{A,B,C,D} + 1_X)]$$
$$= [\iota^0_{A\times B,C\times D}; \iota^0_{(A\times B)+(C\times D),X}, \delta^L_{C,D,X}; (\iota^1_{A\times B,C\times D} + 1_X)]; (\mu_{A,B,C,D} + 1_X)$$
$$= [\iota^0_{A\times C,(C\times D)+X}, \delta^L_{C,D,X}; \iota^1_{A\times B,(C\times D)+X}]; \alpha_{+A\times B,C\times D,X}; (\mu_{A,B,C,D} + 1_X)$$
$$= (1_{A\times B} + \delta^L_{C,D,X}); \alpha_{+A\times B,C\times D,X}; (\mu_{A,B,C,D} + 1_X)$$

The last two coherence conditions involving the right distributivities can be obtained similarly. □

The interchange maps should feel vaguely familiar as they resemble the canonical flips induced by the $\times$ and $+$ braidings, introduced in Remark 3.1. In fact, the notion of a duoidal category and the interchange law appeared for the first time when studying braided categories and the canonical flip [5]. In the context of CLDCs, we have mix maps from $A \times B$ to $A + B$, allowing us to see that the interchange maps is essentially the canonical flip with mix maps.

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Proposition 3.20 *Given a cartesian linearly distributive category, the following diagram commutes*

$$
\begin{array}{ccc}
(A \times B) + (C \times D) & \xrightarrow{\mu_{A,B,C,D}} & (A + C) \times (B + D) \\
{\scriptstyle \mathrm{mix}_{A,B} + \mathrm{mix}_{C,D}} \downarrow & & \downarrow {\scriptstyle \mathrm{mix}_{A+C,B+D}} \\
(A + B) + (C + D) & \xrightarrow[\tau^{+}_{A,B,C,D}]{} & (A + C) + (B + D) \\
 & & \\
A \times B) \times (C \times D) & \xrightarrow{\tau^{\times}_{A,B,C,D}} & (A \times C) \times (B \times D) \\
{\scriptstyle \mathrm{mix}_{A \times B, C \times D}} \downarrow & & \downarrow {\scriptstyle \mathrm{mix}_{A,C} \times \mathrm{mix}_{B \times D}} \\
(A \times B) + (C \times D) & \xrightarrow[\mu_{A,B,C,D}]{} & (A + C) \times (B + D)
\end{array}
\tag{17}
$$

Proof The proof that the first diagram commutes follows fairly easily by the naturality of the mix maps, and the definitions of the interchange maps and the canonical flips:

$$
\begin{aligned}
&\mu_{A,B,C,D}; \mathrm{mix}_{A+C,B+D} \\
&= [(\iota^0_{A,C} \times \iota^0_{B,D}); \mathrm{mix}_{A+C,B+D}, (\iota^1_{A,C} \times \iota^1_{B,D}); \mathrm{mix}_{A+C,B+D}] \\
&= [\mathrm{mix}_{A,B}; (\iota^0_{A,C} + \iota^0_{B,D}), \mathrm{mix}_{C,D}; (\iota^0_{A,C} + \iota^0_{B,D})] \\
&= [\mathrm{mix}_{A,B}; \iota^0_{A+B,C+D}; \tau^{+}_{A,B,C,D}, \mathrm{mix}_{C,D}; \iota^1_{A+B,C+D}; \tau^{+}_{A,B,C,D}] \\
&= [\mathrm{mix}_{A,B}; \iota^0_{A+B,C+D}, \mathrm{mix}_{C,D}; \iota^1_{A+B,C+D}]; \tau^{+}_{A,B,C,D} \\
&= (\mathrm{mix}_{A,B} + \mathrm{mix}_{C,D}); \tau^{+}_{A,B,C,D}
\end{aligned}
$$

The second diagram commutes similarly. □

Within the field of logic, the interchange law is known as the medial rule and appears prominently in the study of local logical systems within deep inference, as developed by Guglielmi. The medial rule has also appeared in categorical logic, notably in the work of the first author. Propositions 3.18 and 3.19 in fact imply that every CLDC is a symmetric medial linearly distributive category (SMLDC). For a detailed look at the interaction between medial maps and linear distributivities, and a linearly distributive version of the Fox theorem, characterizing CLDCs as particular SMLDCs, see [11].

Remark 3.21 *Given that duoidal structure arises canonically when finite products exist, any LDC with a cartesian tensor structure is a duoidal category. Moreover, both* (15) *and the first diagram in* (16) *hold in such an LDC, as their proofs rely solely on the universal properties of products. Similarly, any LDC whose par structure is cocartesian is also a duoidal category, with* (15) *and the second diagram in* (16) *holding, since they can be established purely through the universal properties of coproducts.*

4 Bounded Distributive Lattices

We now present the first key class of examples of cartesian LDCS: bounded distributive lattices.

By which we mean, the lattice $\mathcal{B}$ has a top element $\mathbf{1}$ and a bottom element $\mathbf{0}$, and the join $\vee$ and the meet $\wedge$ distribute over one another:

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \qquad \forall a, b, c \in \mathcal{B}$$

Definition 4.1 *[18, Sec I.1.] Consider bounded distributive lattices* $(\mathcal{B}, \wedge, \mathbf{1}, \vee, \mathbf{0})$ *and* $(\mathcal{L}, \wedge, \mathbf{1}, \vee, \mathbf{0})$*. A lattice homomorphism* $f : (\mathcal{B}, \wedge, \mathbf{1}, \vee, \mathbf{0}) \to (\mathcal{L}, \wedge, \mathbf{1}, \vee, \mathbf{0})$ *is a function* $f : \mathcal{B} \to \mathcal{L}$ *such that*

$$f(a \wedge b) = f(a) \wedge f(b) \qquad f(\mathbf{1}) = \mathbf{1}$$
$$f(a \vee b) = f(a) \vee f(b) \qquad f(\mathbf{0}) = \mathbf{0}$$

Let **BDL** *denote the category of bounded distributive lattices and lattice homomorphisms.*

Such lattices are examples of CLDCs:

Example 4.2 *Every bounded distributive lattice* $\mathcal{L}$ *with bottom element* $\mathbf{0}$*, top element* $\mathbf{1}$*, binary meets* $\wedge$*, and binary joins* $\vee$ *is a posetal CLDC* $(\mathcal{L}, \wedge, \mathbf{1}, \vee, \mathbf{0})$ *with linear distributivities given by*

$$\delta^R_{A,B,C} : (A \vee B) \wedge C = (A \wedge C) \vee (B \wedge C) \leq A \vee (B \wedge C)$$
$$\delta^L_{A,B,C} : A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C) \leq (A \wedge B) \vee C$$

In this case, the linear distributivity coherence conditions hold trivially as there is at most one morphism between two objects.

4.1 Collapse to Posets

There are two well-known theorems describing how certain CLDCs collapse to posets: Joyal's paradox and "orthogonality of linear distributivity and standard distributivity". We will take the time to detail these results here, even though they are not new, as they are integral to our understanding of CLDCs.

Before, we give a quick lemmas about posetal (or thin) CLDCs.

Lemma 4.3 *A (small) cartesian linearly distributive category is posetal if and only if it is a bounded distributive lattice.*

Proof A bounded distributive lattice $(\mathcal{L}, \wedge, \mathbf{1}, \vee, \mathbf{0})$ is a CLDC by 5.8 and of course it is posetal. Now given, a posetal CLDC, the only axiom of bounded distributive lattices which is not immediate is distributivity

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \qquad a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$$

The former distributivity law follows as $(a \wedge b) \vee (a \wedge c) \leq a \wedge (b \vee c)$ holds for any lattice and, using linear distributivities, we get the other inequality:

$$\begin{aligned} a \wedge (b \vee c) &= (a \wedge a) \wedge (b \vee c) = a \wedge (a \wedge (b \vee c)) \leq a \wedge ((a \wedge b) \vee c) \\ &= ((a \wedge b) \vee c) \wedge a \leq (a \wedge b) \vee (c \wedge a) = (a \wedge b) \vee (a \wedge c) \end{aligned}$$

The second distributivity law follows similarly, as $a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$ holds for any lattice and the other inequality follows from the linear distributivities:

$$\begin{aligned} (a \vee b) \wedge (a \vee c) &= (a \vee b) \wedge (c \vee a) \leq ((a \vee b) \wedge c) \vee \leq (a \vee (b \wedge c)) \vee a \\ &= a \vee (a \vee (b \wedge c)) = (a \vee a) \vee (b \wedge c) = a \vee (b \wedge c) \end{aligned}$$

□

Now, we consider Joyal's Paradox, one of the most famous results in categorical logic states that a cartesian closed category with involution is a Boolean algebra, in other words a poset. This means that the naive definition for categorical semantics of classical logic, generalizing the categorical semantics of intuitionistic logic to the classical case, just provide semantics of provability and not proofs. For an in-depth discussion of Joyal's Paradox in its standard form, see Appendix B in [19]. We shall state the result here, in the appropriate language of CLDCs.

Theorem 4.4 (Joyal's Paradox) *A (small) cartesian linearly distributive category has negation if and only if it is a Boolean algebra.*

Proof Consider a Boolean algebra $\mathcal{L}$, then it is a bounded distributive lattice and therefore is a CLDC 4.3. It is immediate that it has negation as a Boolean algebra are complemented lattices.

Consider a CLDC $\mathbb{X}$ with negation and an object A. Then, there is a complementation pair $(A, A^{\perp}, \gamma_A^R, \tau_A^L)$. Consider then the hom-set $\mathbb{X}(A, B)$ for some object B:

$$\mathbb{X}(A, B) \cong \mathbb{X}(A \times \mathbf{1}, B) \cong \mathbb{X}(\mathbf{1}, A^{\perp} + B)$$

By Proposition 3.9, $\mathbf{1}$ is preinitial, meaning $\mathbb{X}(\mathbf{1}, A^{\perp} + B)$ and by consequence $\mathbb{X}(A, B)$ is either the empty set or a singleton set. Thus, A is preintial. As this is true for all objects A in $\mathbb{X}$, there is at most one morphism between two objects, thus $\mathbb{X}$ is a poset and therefore a complemented bounded distributive lattice, by Lemma 4.3, in other words, a Boolean algebra. □

Next, we give the well-known result by Cockett and Seely, introduced in the corrected version of "Weakly Distributive Categories" [10]. As previously mentioned, it was initially thought that all distributive categories were CLDCs, where a distributive category is defined as follows.

Definition 4.5 *[1, Sec 3] A* **distributive category** *is a category $\mathbb{D}$ with finite products and with finite coproducts such that the product distributes over the coproduct: the canonical natural transformation*

$$d^R_{A,B,C} = [\iota^0_{A,B} \times 1_C, \iota^1_{A,B} \times 1_C] : (A \times C) + (B \times C) \to (A + B) \times C$$

is an isomorphism. It follows that the canonical natural transformation

$$d^L_{A,B,C} = [1_A \times \iota^0_{B,C}, 1_A \times \iota^1_{B,C}] : (A \times B) + (A \times C) \to A \times (B + C)$$

is also an isomorphism.

It then became clear that this is only true if the category is a poset. This means binary products and coproducts can only satisfy linear distributivities and the standard distributivities in the above definition if all morphisms between certain objects are identified.

We utilize a characterization of preinitial objects in a distributive category from [1] and our result that the terminal object is preinitial in a CLDC to give an alternative, rather simple, proof of the collapse in question.

Proposition 4.6 *[1, Prop 3.3] In any distributive category, the following are equivalent:*

- *A is preinitial,*
- *there is an object B such that $\iota^0_{A,B} : A \to A + B$ is an isomorphism,*
- *$t_A; \iota^0_{\mathbf{1},\mathbf{1}} = t_A; \iota^1_{\mathbf{1},\mathbf{1}} : A \to \mathbf{1} + \mathbf{1}$*

Theorem 4.7 *[10, Prop 3.1] A (small) cartesian linearly distributive category is a distributive category if and only if it is a bounded distributive lattice.*

Proof It is immediate that a bounded distributive lattice is both an example of a CLDC and a distributive category.

Consider now a CLDC $\mathbb{X}$ which is a distributive category, then for all objects $A \in \mathbb{X}$, $t_A; \iota^0_{\mathbf{1},\mathbf{1}} = t_A; \iota^1_{\mathbf{1},\mathbf{1}}$ as $\iota^0_{\mathbf{1},\mathbf{1}} = \iota^1_{\mathbf{1},\mathbf{1}}$ by Proposition 3.9 and Lemma 3.6. Therefore, by Proposition 4.6, every object A is preintial, implying $\mathbb{X}$ is a poset. Therefore, $\mathbb{X}$ is a bounded distributive lattice by Lemma 4.3. □

4.2 Semizero Objects

In a CLDC, preintial and subterminal objects play a unique role, in particular, if an object is preinitial, it is subterminal and vice versa. We will call such objects, which are both preinitial and subterminal, **semizero**.

Definition 4.8 *Let $\mathbb{X}$ be a cartesian linearly distributive category, then define* SZ[$\mathbb{X}$] *to be the full sub-category of semizero objects in $\mathbb{X}$.*

Proposition 4.9 *Given a (small) cartesian linearly distributive category $\mathbb{X}$,* SZ[$\mathbb{X}$] *is a posetal cartesian linearly distributive category, in particular it is a bounded distributive lattice.*

Proof It suffices to note that $\mathbf{1}$ and $\mathbf{0}$ are semizero objects by Proposition 3.9 and that semizero objects in $\mathbb{X}$ are closed under binary products and coproducts as follows.

Consider subzero objects A and B in $\mathbb{X}$, then A and B are preinitial and, by Lemma 3.6, ∇_A and ∇_B are isomorphisms. Now, $\nabla_{A+B} = \tau^+_{A,B,A,B}; (\nabla_A + \nabla_B)$ is also an isomorphism as a composite of invertible arrows. Thus, $A + B$ is preinitial and, by Proposition 3.16, it is subzero. Dually for $A \times B$.

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Then, as every object is subzero, $\mathbb{X}$ is posetal and, by Lemma 4.3, $\mathbb{X}$ is a bounded distributive lattice. $\square$

We shall show that the construction of restricting to subzero objects is in fact a right adjoint to the inclusion functor of bounded distributive lattices into CLDCs.

Now, a cartesian linear functor $F = (F_\times, F_\parr) : \mathbb{X} \to \mathbb{Y}$ consists in particular to a cartesian functors $F_\times : \mathbb{X} \to \mathbb{Y}$ and $F_+ : \mathbb{X} \to \mathbb{Y}$. Given a semizero object A, $F_\times(A)$ is subterminal as cartesian functors preserve subtemrinal objects and $F_\parr(A)$ is preinitial as cartesian functors preserve preintial objects. Therefore, $F_\times(A)$ and $F_\parr(A)$ are both semizero objects in CLDC $\mathbb{Y}$.

Lemma 4.10 *Cartesian linear functors preserve semizero objects in cartesian linearly distributive categories.*

And as such:

Lemma 4.11 *A Frobenius cartesian linear functor* $F : (\mathbb{X}, \times, \mathbf{1}, +, \mathbf{0}) \to (\mathbb{Y}, \times, \mathbf{1}, +, \mathbf{0})$ *between (small) cartesian linearly distributive categories canonically extends to a lattice homomorphism* $F : (\mathrm{SZ}[\mathbb{X}], \wedge, \mathbf{1}, \vee, \mathbf{0}) \to (\mathrm{SZ}[\mathbb{Y}], \wedge, \mathbf{1}, \vee, \mathbf{0})$ *between bounded distributive lattices.*

Now, we can conclude that this semizero objects construction is the right adjoint to the "forgetful" functor from bounded distributive lattices and cartesian linearly distributive categories.

Theorem 4.12 *The functor* $\mathrm{SZ}[-] : \mathbf{FCLDC} \to \mathbf{BDL}$ *mapping a (small) cartesian linearly distributive category to the bounded distributive lattice of its semizero objects is right adjoint to the inclusion functor* $U : \mathbf{BDL} \to \mathbf{FCLDC}$.

Proof Consider a bounded distributive lattice $\mathcal{L}$ and a CLDC $\mathbb{X}$, there is an obvious natural bijection between hom-sets $\mathbf{BDL}(\mathcal{L}, \mathrm{SZ}[\mathbb{X}]) \cong \mathbf{FCLDC}(U(\mathcal{L}), \mathbb{X})$.

Given a lattice homomorphism $f : \mathcal{L} \to \mathrm{SZ}[\mathbb{X}]$, it is immediate that $\mathcal{L} \xrightarrow{f} \mathrm{SZ}[\mathbb{X}] \hookrightarrow \mathbb{X}$ defines a Frobenius cartesian linear functor. Given a Frobenius linear functor $F : \mathcal{L} \to \mathbb{X}$, it canonically becomes a lattice homomorphism $F : \mathrm{SZ}[\mathcal{L}] \to \mathrm{SZ}[\mathbb{X}]$ and, as every object in a bounded distributive lattice is a semizero object when the lattice is viewed as a CLDC, this is a lattice homomorphism $F : \mathcal{L} \to \mathrm{SZ}[\mathbb{X}]$. $\square$

5 Semi-Additive Categories

We now turn to the second key class of examples of CLDCS.

Definition 5.1 *[20, Sec I.5] A* **zero object** *(or null object) in a category $\mathcal{X}$ is an object $\emptyset$, which is both a terminal and initial object.*

Lemma 5.2 *A category has a zero object precisely if it has an initial object $\mathbf{0}$ and a terminal object $\mathbf{1}$, and the unique morphism $b_{\mathbf{1}} = t_{\mathbf{0}} : \mathbf{0} \to \mathbf{1}$ is an isomorphism.*

If a category has a zero object, then there is a unique morphism between every pair of objects X, Y which factors through the zero object: $0_{X,Y} : X \xrightarrow{t_X} \emptyset \xrightarrow{b_Y} Y$.

Definition 5.3 *[21] Consider a category with a zero object $\emptyset$, with finite products and finite coproducts, then there is a canonical natural transformation*

$$\psi_{X,Y} = [\langle 1_X, 0_{X,Y}\rangle, \langle 0_{Y,X}, 1_Y\rangle] = \langle [1_X, 0_{Y,X}], [0_{X,Y}, 1_Y]\rangle : X + Y \to X \times Y$$

The category is said to be **semi-additive** *if the maps $\psi_{X,Y}$ are invertible. Then, the identified isomorphic objects $X + Y$ and $X \times Y$ are called the biproduct of X and Y.*

Interestingly, any isomorphisms between binary coproducts and binary products is sufficient for a category to be semi-additive.

Proposition 5.4 *[21, Thm 5] If a category has all finite products, all finite coproducts and a natural family of isomorphisms $A + B \to A \times B$, then it is semi-additive.*

Definition 5.5 *A functor $F : \mathcal{X} \to \mathcal{Y}$ between semi-additive categories is* **semi-additive** *if*

- *$(F, m_\emptyset, m_\times) : (\mathcal{X}, \times, \emptyset) \to (\mathcal{Y}, \times, \emptyset)$ is a strong monoidal functor,*
- *$(F, n_\emptyset, n_+) : (\mathcal{X}, +, \emptyset) \to (\mathcal{Y}, +, \emptyset)$ is a strong comonoidal functor*

satisfying the following commuting diagrams

$$\begin{array}{ccc} F(A+B) & \xrightarrow{n_{+\,A,B}} & F(A)+F(B) \\ {\scriptstyle F(\psi_{A,B})}\downarrow & & \downarrow{\scriptstyle \psi_{F(A),F(B)}} \\ F(A\times B) & \xleftarrow[m_{\times\,A,B}]{} & F(A)\times F(B) \end{array} \qquad \begin{array}{ccc} F(A)\times F(B) & \xrightarrow{m_{\times\,A,B}} & F(A\times B) \\ {\scriptstyle \psi^{-1}_{F(A),F(B)}}\downarrow & & \downarrow{\scriptstyle F(\psi^{-1}_{A,B})} \\ F(A)+F(B) & \xleftarrow[n_{+\,A,B}]{} & F(A+B) \end{array} \tag{18}$$

Let **SAdd** *denote the category of semi-additive categories and semi-additive functors.*

We can also define binary biproducts between two object as this identified object, equipped with relevant morphisms, making it both a binary product and coproduct.

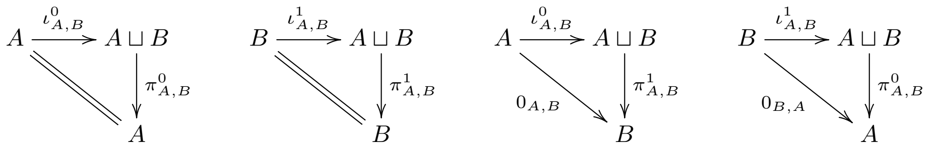
Definition 5.6 *[22] A* **biproduct** *of a pair of objects A and B in a category $\mathcal{X}$ is a an object $A \sqcup B$ equipped with four morphisms*

$$\pi^0_{A,B} : A \sqcup B \to A \qquad \pi^1_{A,B} : A \sqcup B \to B \qquad \iota^0_{A,B} : A \to A \sqcup B \qquad \iota^1_{A,B} : B \to A \sqcup B$$

such that

- *$A \sqcup B$ equipped with $\pi^0_{A,B}$ and $\pi^1_{A,B}$ is a product digram,*

- *$A \sqcup B$ equipped with $\iota^0_{A,B}$ and $\iota^1_{A,B}$ is a coproduct digram, and*
- *the following commuting diagrams hold:*



Lemma 5.7 *Consider a monoidal category $(\mathcal{X}, \oslash, I)$ such that this monoidal structure is both cartesian and cocartesian, then I is a zero object $\emptyset$ and $A \oslash B$ is the binary biproduct $A \sqcup B$.*

We may now introduce the second key class of CLDC examples: the semi-additive categories.

Example 5.8 *(i) Every category $\mathbb{B}$ with all finite biproducts, i.e. a zero object $\emptyset$ and binary biproducts $A \sqcup B$ in the sense of Definition 5.6, induces a Frobenius cartesian LDC $(\mathbb{B}, \sqcup, \emptyset, \sqcup, \emptyset)$ with linear distributivities given by*

$$\delta^R_{A,B,C} = \alpha_{\sqcup A,B,C} = \langle \pi^0_{X\times Y,Z}; \pi^0_{X,Y}, \pi^1_{X,Y} \sqcup 1_Z \rangle = [1_A \sqcup \iota^0_{B,C}, \iota^1_{B,C}; \iota^1_{A,B\sqcup C}]$$
$$: (A \sqcup B) \sqcup C \to A \sqcup (B \sqcup C)$$
$$\delta^L_{A,B,C} = \alpha_{\sqcup}{}^{-1}_{A,B,C} = \langle \pi^0_{A,B} \sqcup 1_C, \pi^1_{B,C}; \pi^1_{A,B\sqcup C} \rangle = [\iota^0_{X,Y}; \iota^0_{X+Y,Z}, \iota^1_{X,Y} \sqcup 1_Z]$$
$$: A \sqcup (B \sqcup C) \to (A \sqcup B) \sqcup C$$

By Lemma 5.7, all degenerate cartesian LDC are of this form.

(ii) Every semi-additive category $\mathbb{B}$, with a zero object $\emptyset$ and isomorphic binary products and coproducts $\psi_{A,B} : A + B \cong A \times B$ can be viewed as a cartesian LDC $(\mathbb{B}, \times, \emptyset, +, \emptyset)$ with linear distributivities given by

$$\delta^R_{A,B,C} = (A+B)\times C \xrightarrow{\psi_{A,B}\times 1_C} (A\times B)\times C \xrightarrow{\alpha_{\times A,B,C}} A\times(B\times C) \xrightarrow{\psi^{-1}_{A,B\times C}} A\times(B+C)$$

$$= (A+B)\times C \xrightarrow{\psi^{-1}_{A+B,C}} (A+B)+C \xrightarrow{\alpha_{+}{}^{-1}_{A,B,C}} A+(B+C) \xrightarrow{1_A+\psi_{B,C}} A+(B\times C)$$

$$\delta^L_{A,B,C} = A\times(B+C) \xrightarrow{1_A\times\psi_{B,C}} A\times(B\times C) \xrightarrow{\alpha_{\times}{}^{-1}_{A,B,C}} (A\times B)\times C \xrightarrow{\psi^{-1}_{A\times B,C}} (A\times B)+C$$

$$= A\times(B+C) \xrightarrow{\psi^{-1}_{A,B+C}} A+(B+C) \xrightarrow{\alpha_{+A,B,C}} (A+B)+C \xrightarrow{\psi_{A,B}+1_C} (A\times B)+C$$

It is clearly isomix as $\emptyset = \mathbf{1} = \mathbf{0}$ and, further, it is compact as the mix maps in this context are the inverses of the canonical maps $\psi_{A,B}$:

$$m_{A,B} = A\times B \xrightarrow{1_A\times\iota^1_{\emptyset,B}} A\times(\emptyset+B) \xrightarrow{\delta^L_{A,\emptyset,B}} (A\times\emptyset)+B \xrightarrow{\pi^0_{A,\emptyset}+1_B} A+B = A\times B \xrightarrow{\psi^{-1}_{A,B}} A+B$$

5.1 Collapse to Biproducts

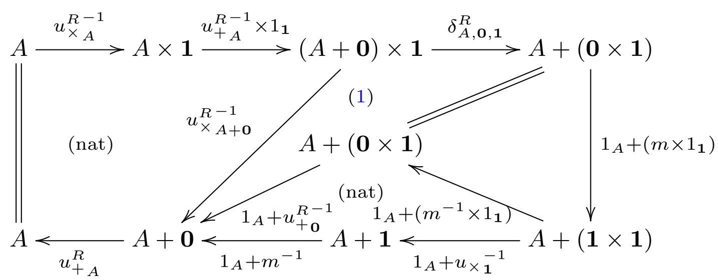
We now turn to the new collapses resulting in semi-additivity.

Lemma 5.9 *A cartesian linearly distributive category is compact if and only if it is a semi-additive category.*

Proof A semi-additive category $\mathbb{B}$ is a compact CLDC $(\mathbb{B}, \times, \emptyset, +, \emptyset)$ as detailed in Example 5.8. Consider a compact CLDC $(\mathbb{X}, \times, \mathbf{1}, +, \mathbf{0})$, then, by Definition 2.7, it is isomix, i.e. $\mathbf{0} \cong \mathbf{1}$ and the natural family of mix maps $\text{mix}_{A,B} : A \times B \to A + B$ are isomorphisms. By Proposition 5.4, $\mathbb{X}$ is semi-additive. In particular, the canonical maps are the inverses of the mix maps. This follows from the proof of Proposition 5.4, which demonstrates that

$$\psi_{A,B} = (u^R_{\times A} + u^L_{\times B}); (\text{mix}_{A,\mathbf{1}} + \text{mix}_{\mathbf{1},B}); ((1_A + m^{-1}) + (m^{-1} + 1_B)); (u^R_{+A} + u^L_{+B}); \text{mix}^{-1}_{A,B}$$

and $u^R_{\times A}; \text{mix}_{A,\mathbf{1}}; (1_A + m^{-1}); u^R_{+A} = 1_A$ by the commuting diagram



Similarly, $u^L_{\times A}; \text{mix}_{\mathbf{1},B}; (m^{-1} + 1_B); u^R_{+B} = 1_B$ and therefore $\psi_{A,B} = \text{mix}^{-1}_{A,B}$. □

The above result is not too surprising, however it turns out that this collapse to a zero object and isomorphic tensor and par structures holds in greater contexts, the first of which is when the linear distributivity is an isomorphism.

Theorem 5.10 *A cartesian linearly distributive category has invertible linear distributivities if and only if it is a semi-additive category.*

Proof It is immediate that a semi-additive category has invertible linear distributivities as they are composites of isomorphisms, as detailed in Example 5.8.

Conversely, consider a CLDC $(\mathbb{X}, \times, \mathbf{1}, +, \mathbf{0})$ such that $\delta^L_{A,B,C}$ is invertible. By Proposition 2.15 and the details of the proof in [10], $\mathbf{1} + \mathbf{1}$ is the tensor inverse of $\mathbf{0}$ with

$$s^L = \mathbf{0} \times (\mathbf{1} + \mathbf{1}) \xrightarrow{\delta^L_{\mathbf{0},\mathbf{1},\mathbf{1}}} (\mathbf{0} \times \mathbf{1}) + \mathbf{1} \xrightarrow{u^{R\,-1}_{\times \mathbf{0}} + 1_{\mathbf{1}}} \mathbf{0} + \mathbf{1} \xrightarrow{u^L_{+\mathbf{1}}} \mathbf{1}$$

$$s^R = (\mathbf{1} + \mathbf{1}) \times \mathbf{0} \xrightarrow{\delta^R_{\mathbf{1},\mathbf{1},\mathbf{0}}} \mathbf{1} + (\mathbf{1} \times \mathbf{0}) \xrightarrow{1_{\mathbf{1}} + u^{L\,-1}_{\times \mathbf{0}}} \mathbf{1} + \mathbf{0} \xrightarrow{u^R_{+\mathbf{1}}} \mathbf{1}$$

and the associated shift tensor product naturally equivalent to the par product via following the natural family of isomorphisms

$$A \times ((\mathbf{1}+\mathbf{1}) \times B) \xrightarrow{1_A \times \delta^R_{\mathbf{1},\mathbf{1},B}} A \times (\mathbf{1} + (\mathbf{1} \times B)) \xrightarrow{\delta^L_{A,\mathbf{1},\mathbf{1}\times B}} (A \times \mathbf{1}) + (\mathbf{1} \times B) \xrightarrow{u^{R^{-1}}_{\times A} + u^{L^{-1}}_{\times B}} A + B$$

Now, $\mathbf{0} \times (\mathbf{1}+\mathbf{1}) \cong \mathbf{1}$ implies that $\mathbf{0} \cong \mathbf{1}$ via $t_{\mathbf{0}} : \mathbf{0} \to \mathbf{1}$ and $s^{L^{-1}}; \pi^0_{\mathbf{0},\mathbf{1}+\mathbf{1}} : \mathbf{1} \to \mathbf{0}$ as follows:

$$s^{L^{-1}}; \pi^0_{\mathbf{0},\mathbf{1}+\mathbf{1}}; t_{\mathbf{0}} = 1_{\mathbf{1}}$$

by the universal property of terminal objects, and

$$\begin{aligned} t_{\mathbf{0}}; s^{L^{-1}}; \pi^0_{\mathbf{0},\mathbf{1}+\mathbf{1}} &=< t_{\mathbf{0}}, 1_{\mathbf{0}} >; \pi^0_{\mathbf{1},\mathbf{0}}; s^{L^{-1}}; \pi^0_{\mathbf{0},\mathbf{1}+\mathbf{1}} \\ &=< t_{\mathbf{0}}, 1_{\mathbf{0}} >; (s^{L^{-1}} \times 1_{\mathbf{0}}); \pi^0_{\mathbf{0}\times(\mathbf{1}+\mathbf{1}),\mathbf{0}}; \pi^0_{\mathbf{0},\mathbf{1}+\mathbf{1}} \\ &=< t_{\mathbf{0}}, 1_{\mathbf{0}} >; (s^{L^{-1}} \times 1_{\mathbf{0}}); \alpha_{\times \mathbf{0},\mathbf{1}+\mathbf{1},\mathbf{0}}; \pi^0_{\mathbf{0},\mathbf{1}} \\ &= 1_{\mathbf{0}} \end{aligned}$$

by (7). Therefore, $\mathbb{X}$ is isomix with $m : \mathbf{0} \cong \mathbf{1}$.

The mix maps $\text{mix}_{A,B} : A \times B \to A + B$ are isomorphisms in this context as they are defined as composites of unitors, linear distributivities and the nullary mix map m, which are all isomorphisms. Therefore, the mix maps are isomorphisms and $\mathbb{X}$ is a compact CLDC. By Lemma 5.9, $\mathbb{X}$ is a semi-additive category. □

Finally, we see that a CLDC whose terminal and initial objects are in fact isomorphic, in other words a zero object, is also a category with finite biproducts.

Theorem 5.11 *A cartesian linearly distributive category is isomix if and only if it is a semi-additive category.*

Proof It is immediate that a semi-additive category is an isomix CLDC.

Conversely, consider an isomix CLDC, i.e. $m : \mathbf{0} \to \mathbf{1}$ is invertible. We shall show that the linear distributivity δ^L is invertible, and consequently, via the symmetries, δ^R is invertible.

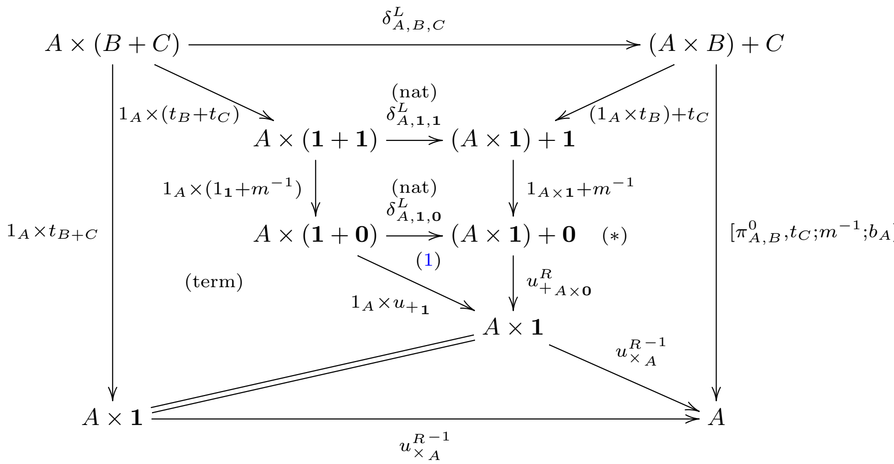
Define the following natural transformation

$$\begin{aligned} \partial^R_{A,B,C} &= \langle [\pi^0_{A,B}, t_C; m^{-1}; b_A], \pi^1_{A,B} + 1_C \rangle = [1_A \times \iota^0_{B,C}, \langle t_C; m^{-1}; b_A, \iota^1_{B,C} \rangle] \\ &: (A \times B) + C \to A \times (B + C) \end{aligned}$$

Then, $\partial^R_{A,B,C}$ is the inverse of $\delta^L_{A,B,C}$:

$$\begin{aligned} &\delta^L_{A,B,C}; \partial^R_{A,B,C} = \delta^L_{A,B,C}; \langle [\pi^0_{A,B}, t_C; m^{-1}; b_A], \pi^1_{A,B} + 1_C \rangle \\ &= \langle \delta^L_{A,B,C}; [\pi^0_{A,B}, t_C; m^{-1}; b_A], \delta^L_{A,B,C}; (\pi^1_{A,B} + 1_C) \rangle \\ &= \langle \pi^0_{A,B+C}, \pi^1_{A,B+C} \rangle = 1_{A \times (B+C)} \end{aligned}$$

with the equality in the first entry by the following commuting diagram and the equality in the second entry by Lemma 3.3.



where $(*)$ commutes by

$$
\begin{aligned}
&((1_A \times t_B) + t_C); (1_{A\times\mathbf{1}} + m^{-1}); u^R_{+A\times\mathbf{1}}; u^{R^{-1}}_{\times A} \\
&= [(1_A \times t_B); \iota^0_{A\times\mathbf{1},\mathbf{0}}, t_C; m^{-1}; \iota^1_{A\times\mathbf{1},\mathbf{0}}]; [1_{A\times\mathbf{1}}, b_{A\times\mathbf{1}}]; u^{R^{-1}}_{\times A} \\
&= [(1_A \times t_B); u^{R^{-1}}_{\times A}, t_C; m^{-1}; b_{A\times\mathbf{1}}; u^{R^{-1}}_{\times A}] = [\pi^0_{A,B}, t_C; m^{-1}; b_A]
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\partial^R_{A,B,C}; \delta^L_{A,B,C} = [1_A \times \iota^0_{B,C}, \langle t_C; m^{-1}; b_A, \iota^1_{B,C}\rangle]; \delta^L_{A,B,C} \\
&= [(1_A \times \iota^0_{B,C}); \delta^L_{A,B,C}, \langle t_C; m^{-1}; b_A, \iota^1_{B,C}\rangle; \delta^L_{A,B,C}] \\
&= [\iota^0_{A\times B,C}, \iota^1_{A\times B,C}] = 1_{A\times(B+C)}
\end{aligned}
$$

Then, by Theorem 5.10, it is a semi-additive category.

Alternatively, we can directly determine the inverse of the mix maps $m_{A,B}$, providing the desired natural family of isomorphisms $A \times B \cong A + B$:

$$\mathrm{mix}^{-1}_{A,B} = A + B \xrightarrow{u^R_{\times A} + 1_B} (A \times \mathbf{1}) + B \xrightarrow{\partial^R_{A,B,C}} A \times (\mathbf{1} + B) \xrightarrow{1_A \times (m^{-1} + 1_B)} A \times (\mathbf{0} + B) \xrightarrow{1_A \times u^L_{+B}} A \times B$$

which can be simplified to

$$\mathrm{mix}^{-1}_{A,B} = \langle [1_A, t_B; m^{-1}; b_A], [t_A; m^{-1}; b_B, 1_B] \rangle = [\langle 1_A, t_A; m^{-1}; b_B \rangle, \langle t_B; m^{-1}; b_A, 1_B \rangle]$$

Note these are precisely the canonical isomorphisms of a semi-additive category, as detailed in Definition 5.3. □

Remark 5.12 *There is a construction from normal duoidal categories, i.e. duoidal categories whose units are isomorphic, to isomix LDCs [23]. Indeed, given a duoidal category $(\mathcal{X}, \diamond, I, \star, J)$ where $\iota : I \to J$ is an isomorphism, we can define the following linear distributivities:*

$$\partial^L_{A,B,C} = A \diamond (B \star C) \cong (A \star J) \diamond (B \star C) \xrightarrow{\zeta_{A,J,B,C}} (A \diamond B) \star (J \diamond C) \cong (A \diamond B) \star (I \diamond C) \cong (A \diamond B) \star C$$

$$\partial^R_{A,B,C} = (A \star B) \diamond C \cong (A \star B) \diamond (J \star C) \xrightarrow{\zeta_{A,B,J,C}} (A \diamond J) \star (B \diamond C) \cong (A \diamond I) \star (B \diamond C) \cong A \star (B \diamond C)$$

Then, $(\mathcal{X}, \diamond, I, \star, J)$ is an isomix LDC.

An interesting observation is that, given any isomix CLDC $(\mathbb{X}, \times, \mathbf{1}, +, \mathbf{0})$, there is a normal duoidal structure $(\mathbb{X}, +, \mathbf{0}, \times, \mathbf{1})$ and therefore $(\mathbb{X}, +, \mathbf{0}, \times, \mathbf{1})$ is an isomix LDC with linear distributivities ∂^L and ∂^R (not a CLDC as the tensor product is the binary coproduct and the par product is the binary product in this case). These linear distributivities ∂^L and ∂^R are exactly the inverses of δ^R and δ^L respectively, and corresponding the mix maps are $m^{-1}_{A,B}$, utilized in the proof of 5.11.

5.2 Slice over Initial and Coslice under Terminal

A construction which will be key to our discussion is that of the slice category and its dual the coslice category, and how they inherit cocartesian and cartesian structures respectively.

Definition 5.13 *[20, Chap II.6] Consider a category $\mathcal{X}$ and an object $X \in \mathcal{X}$. The* **slice category** *of $\mathcal{X}$ over X, denoted $\mathcal{X}/X$ (or $\mathcal{X} \downarrow X$), consists of*

- *Objects: morphisms $f : A \to X$ in $\mathcal{X}$ for some object $A \in \mathcal{X}$,*
- *Morphisms: $g : (f : A \to X) \to (f' : A' \to X)$ is a morphism $g : A \to A'$ in $\mathcal{X}$ such that $g; f' = f$.*

The **coslice category** *of $\mathcal{X}$ under X, denoted by $X \downarrow \mathcal{X}$ consists of*

- *Objects: morphisms $f : X \to A$ in $\mathcal{X}$ for some object $A \in \mathcal{X}$,*
- *Morphisms: $g : (f : X \to A) \to (f' : X \to A')$ is a morphism $g : A \to A'$ in $\mathcal{X}$ such that $f; g = f'$.*

Lemma 5.14 *Consider a category $\mathcal{X}$ and an object $X \in \mathcal{X}$.*

- *The slice category $\mathcal{X}/_X$ has $1_X : X \to X$ as a terminal object.*
- *If $\mathcal{X}$ has an initial object $\mathbf{0}$, then the slice category $\mathcal{X}/_X$ has $b_X : \mathbf{0} \to X$ as its initial object.*
- *If $\mathcal{X}$ has binary coproducts $A + B$, then the slice category $\mathcal{X}/_X$ has binary coproducts defined by: given objects $f : A \to X$ and $f' : A' \to X$, then their coproduct is $(f + f'); \nabla_X : A + A' \to X + X \to X$.*
- *The coslice category $X \downarrow \mathcal{X}$ has $1_X : X \to X$ as an initial object.*
- *If $\mathcal{X}$ has a terminal object $\mathbf{1}$, then the colsice category $X \downarrow \mathcal{X}$ has $t_X : X \to \mathbf{1}$ as its terminal object.*
- *If $\mathcal{X}$ has binary products $A \times B$, then the coslice category $X \downarrow \mathcal{X}$ has binary products defined by: given objects $f : X \to A$ and $f' : X \to A'$, their product is $\Delta_X; (f \times f') : X \to X \times X \to A \times A'$.*

Given a CLDC $\mathbb{X}$, we shall first consider its slice category over the initial object $\mathbf{0}$.

Proposition 5.15 *Let $\mathbb{X}$ be a cartesian linearly distributive category, then $\mathbb{X}/\mathbf{0}$, the slice category of $\mathbb{X}$ over the initial object $\mathbf{0}$, is an isomix cartesian linearly distributive category, in particular it is a semi-additive category.*

Proof Let $a : A \to \mathbf{0}, b : B \to \mathbf{0}, c : C \to \mathbf{0}$ be objects in $\mathbb{X}/\mathbf{0}$ and $f : (c : C \to \mathbf{0}) \to (a : A \to \mathbf{0})$ and $g : (c : C \to \mathbf{0}) \to (b : B \to \mathbf{0})$ be morphisms in $\mathbb{X}/\mathbf{0}$.

By Lemma 5.14, $\mathbb{X}/\mathbf{0}$ has terminal object $1_{\mathbf{0}} : \mathbf{0} \to \mathbf{0}$, with unique induced arrows $t_a : (a : A \to \mathbf{0}) \to (1_{\mathbf{0}} : \mathbf{0} \to \mathbf{0})$ defined by $a : A \to \mathbf{0}$. $\mathbb{X}/\mathbf{0}$ has initial object $1_{\mathbf{0}} : \mathbf{0} \to \mathbf{0}$, with unique induced arrows $b_a : (1_{\mathbf{0}} : \mathbf{0} \to \mathbf{0}) \to (a : A \to \mathbf{0})$ defined by $b_A : \mathbf{0} \to A$.

Finally, $\mathbb{X}/\mathbf{0}$ has binary coproducts $(a +_{/\mathbf{0}} b)$ given by $(a + b); \nabla_{\mathbf{0}} : A + B \to \mathbf{0}$, with injections $\iota^0_{a,b}$ and $\iota^1_{a,b}$ defined to be $\iota^0_{A,B}$ and $\iota^1_{A,B}$, and with unique induced arrows $(a +_{/\mathbf{0}} b : A + B \to \mathbf{0}) \to (c : C \to \mathbf{0})$ defined by $[f, g]$.

Moreover, as $\mathbf{0}$ is subterminal, meaning $\pi^0_{\mathbf{0},\mathbf{0}} = \pi^1_{\mathbf{0},\mathbf{0}} = \pi_{\mathbf{0},\mathbf{0}}$, $\mathbb{X}/\mathbf{0}$ also inherits the binary products as follows. Given $a : A \to \mathbf{0}, b : B \to \mathbf{0} \in \mathbb{X}/\mathbf{0}$, their binary product $(a \times_{/\mathbf{0}} b)$ is $(a \times b); \pi_{\mathbf{0},\mathbf{0}} : A \times B \to \mathbf{0} \times \mathbf{0} \to \mathbf{0}$ with projections $\pi^0_{a,B}$ and $\pi^1_{a,B}$ defined to be $\pi^0_{A,B}$ and $\pi^1_{A,B}$ respectively since

$$\pi^0_{A,B}; a = (a \times b); \pi^0_{\mathbf{0},\mathbf{0}} = (a \times b); \pi_{\mathbf{0},\mathbf{0}} \qquad \pi^1_{A,B}; b = (a \times b); \pi^1_{\mathbf{0},\mathbf{0}} = (a \times b); \pi_{\mathbf{0},\mathbf{0}}$$

Further, the unique induced morphism $(c : C \to \mathbf{0}) \to ((a \times b); \pi_{\mathbf{0},\mathbf{0}}) : A \times B \to \mathbf{0})$ is $\langle f, g \rangle$ since

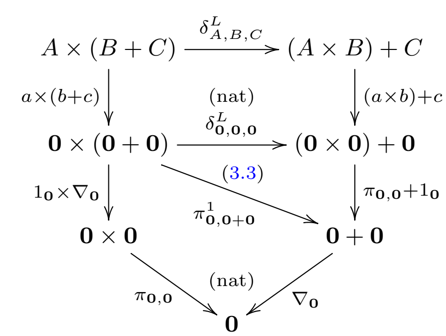
$$\langle f, g \rangle; (a \times b); \pi_{\mathbf{0},\mathbf{0}} = \langle f, g \rangle; \pi^0_{A,B}; a = f; a = c$$

or

$$\langle f, g \rangle; (a \times b); \pi_{\mathbf{0},\mathbf{0}} = \langle f, g \rangle; \pi^1_{A,B}; b = g; b = c$$

Therefore, $\mathbb{X}/\mathbf{0}$ is a cartesian and cocartesian category.

It remains only to show it also inherits the linear distributivities to conclude that $\mathbb{X}/\mathbf{0}$ is a CLDC. The left linear distributivity $\delta^L_{a,b,c}$ is given by $\delta^L_{A,B,C}$ as



Alternatively, recall that $\mathbf{0}$ is subterminal and therefore there is only unique map $A \times (B + C) \to \mathbf{0}$.

Now, $\mathbb{X}/\mathbf{0}$ is an isomix CLDC as $1_{\mathbf{0}} : \mathbf{0} \to \mathbf{0}$ is a zero object. By Theorem 5.11, $\mathbb{X}/\mathbf{0}$ is a semi-additive category. It is useful to note now the canonical isomorphisms mediating the binary biproducts $a +_{/\mathbf{0}} b \cong a \times_{/\mathbf{0}} b$:

$$\psi_{a,b} = \mathrm{mix}^{-1}_{a,b} : a +_{/\mathbf{0}} b \to a \times_{/\mathbf{0}} b \quad \text{is defined to be} \quad \langle [1_A, b; b_A], [a; b_B, 1_B] \rangle : A + B \to A \times B$$
$$\psi^{-1}_{a,b} = \mathrm{mix}_{a,b} : a \times_{/\mathbf{0}} b \to a +_{/\mathbf{0}} b \quad \text{is defined to be} \quad \mathrm{mix}_{A,B} : A \times B \to A + B$$

□

Corollary 5.16 *Consider a semi-additive category $\mathbb{B}$, then it is isomorphic to its slice category over the zero object $\mathbb{B}/\mathbf{0}$, when viewed wither as CLDCs or semi-additive categories, by*

$$F_{\mathbb{B}} : \mathbb{B} \to \mathbb{B}/\mathbf{0}, \qquad A \mapsto (b_A : A \to \mathbf{0}) \qquad (f : A \to B) \mapsto (f : b_A \to b_B)$$

Essentially, the construction is taking the full sub-category of all objects in $\mathbb{X}$, which have a map to the initial object. This map is necessarily unique, in other words $a : A \to \mathbf{0}$ is entirely determined by A, as $\mathbf{0}$ is subterminal. By restricting to such objects, $\mathbf{0}$ is terminal, as the existence of maps to $\mathbf{0}$ is guaranteed. This means the sub-category is an isomix CLDC, which by Theorem 5.11, means it is a semi-additive category.

By Lemma 3.11, we know the full-subcategory of objects with a map to $\mathbf{0}$ can be equivalently described as the objects of the form $A \times \mathbf{0}$. Therefore, we can give an equivalent description the slice over the initial object construction using the full-subcategory of such objects.

Proposition 5.17 *Consider a cartesian linearly distributive category $\mathbb{X}$, then there is an equivalence of semi-additive categories $\mathbb{X}/\mathbf{0} \simeq \mathbb{X} \times \mathbf{0}$, where $\mathbb{X} \times \mathbf{0}$ denotes the full subcategory of objects of the form $A \times \mathbf{0}$ for some A in $\mathbb{X}$.*

Proof Consider the semi-additive functor $F : \mathbb{X}/\mathbf{0} \to \mathbb{X} \times \mathbf{0}$ defined by

$$(a : A \to \mathbf{0}) \mapsto \mathrm{dom}(a) \times \mathbf{0} \qquad (f : a \to b) \mapsto (f \times 1_{\mathbf{0}} : \mathrm{dom}(a) \times \mathbf{0} \to \mathrm{dom}(b) \times \mathbf{0})$$

and the semi-additive functor $G : \mathbb{X} \times \mathbf{0} \to \mathbb{X}/\mathbf{0}$ defined by

$$A \times \mathbf{0} \mapsto (\pi^1_{A,\mathbf{0}} : A \times \mathbf{0} \to \mathbf{0}) \qquad (f : A \times \mathbf{0} \to B \times \mathbf{0}) \mapsto (f \times 1_{\mathbf{0}} : \pi^1_{A\times\mathbf{0}} \to \pi^1_{B\times\mathbf{0}})$$

Then, F and G provide the equivalence, when equipped with the natural isomorphisms $\alpha : 1_{\mathbb{X}/\mathbf{0}} \to F; G$ defined by

$$\alpha_a = \langle 1_A, a \rangle : (a : A \to \mathbf{0}) \to (\pi^1_{A,\mathbf{0}} : A \times \mathbf{0} \to A)$$

and $\beta : G; F \to 1_{\mathbb{X}\times\mathbf{0}}$ defined by

$$\beta_A = \langle 1_{A\times\mathbf{0}}, \pi^0_{A,\mathbf{0}} \rangle : A \times \mathbf{0} \to (A \times \mathbf{0}) \times \mathbf{0}$$

□

Proposition 5.18 *A Frobenius cartesian linear functor $F : (\mathbb{X}, \times, \mathbf{1}, +, \mathbf{0}) \to (\mathbb{Y}, \times, \mathbf{1}, +, \mathbf{0})$ induces a semi-additive functor $F/\mathbf{0} : \mathbb{X}/\mathbf{0} \to \mathbb{Y}/\mathbf{0}$ defined by*

$$(a : A \to \mathbf{0}) \mapsto (F(a); n_{\mathbf{0}} : F(A) \to F(\mathbf{0}) \to \mathbf{0})$$
$$(f : (a : A \to \mathbf{0}) \to (b : B \to \mathbf{0})) \mapsto (F(f) : (F(a); n_{\mathbf{0}} : F(A) \to \mathbf{0}) \to (F(b); n_{\mathbf{0}} : F(B) \to \mathbf{0}))$$

Proof By definition of a Frobenius cartesian linear functor $F : (\mathbb{X}, \times, \mathbf{1}, +, \mathbf{0}) \to (\mathbb{Y}, \times, \mathbf{1}, +, \mathbf{0})$, it is a $\times$-monoidal functor $(F, m_{\mathbf{1}}, m_\times) : (\mathbb{X}, \times, \mathbf{1}) \to (\mathbb{Y}, \times, \mathbf{1})$ and a $+$-monoidal functor $(F, n_{\mathbf{0}}, n_+) : (\mathbb{X}, +, \mathbf{0}) \to (\mathbb{Y}, +, \mathbf{0})$ satisfying (9). Now, $F/\mathbf{0} : \mathbb{X}/\mathbf{0} \to \mathbb{Y}/\mathbf{0}$ as defined above is a semi-additive functor as follows.

$(F/\mathbf{0}, m_{1_{\mathbf{0}}}, m_{\times/\mathbf{0}}) : (\mathbb{X}/\mathbf{0}, \times/\mathbf{0}, 1_{\mathbf{0}}) \to (\mathbb{Y}/\mathbf{0}, \times/\mathbf{0}, 1_{\mathbf{0}})$ is a $\times/\mathbf{0}$-monoidal functor with

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- $m_{1_{\mathbf{0}}} : 1_{\mathbf{0}} \to F/\mathbf{0}(1_{\mathbf{0}})$ defined to be $b_{F(\mathbf{0})} : \mathbf{0} \to F(\mathbf{0})$ with inverse $n_{\mathbf{0}}$,
- $m_{\times/\mathbf{0}_{a,b}} : F/\mathbf{0}(a) \times_{/\mathbf{0}} F/\mathbf{0}(b) \to F/\mathbf{0}(a \times_{/\mathbf{0}} b)$ defined to be $m_{\times A,B} : F(A) \times F(B) \to F(A \times B)$ with inverse $m_{\times}^{-1}{}_{A,B}$.

This is a monoidal functor as

$$\begin{array}{ccc} F/\mathbf{0}(a) & \xrightarrow{u^R_{\times/\mathbf{0}\,F/\mathbf{0}(a)}} & F/\mathbf{0}(a) \times_{/\mathbf{0}} 1_{\mathbf{0}} \\ {\scriptstyle F/\mathbf{0}(u^R_{\times/\mathbf{0}_a})} \downarrow & & \downarrow {\scriptstyle 1_{F/\mathbf{0}(a)} \times m_{1_{\mathbf{0}}}} \\ F/\mathbf{0}(a \times_{/\mathbf{0}} 1_{\mathbf{0}}) & \xrightarrow[m_{\times/\mathbf{0}_{a,1_{\mathbf{0}}}}]{} & F/\mathbf{0}(a) \times_{/\mathbf{0}} F/\mathbf{0}(1_{\mathbf{0}}) \end{array}$$

translates to $\langle 1_{F(A)}, F(a); n_{\mathbf{0}}\rangle; (1_{F(A)} \times b_{F(\mathbf{0})}) = F(\langle 1_A, a\rangle); m_{\times}{}^{-1}_{A,\mathbf{0}}$, which holds by the universal properties of products and $(F, m_{\mathbf{1}}, m_{\times})$ being a monoidal functor. The dual diagram follows similarly and the associativity hexagon condition follows immediately as $\alpha_{\times/\mathbf{0}_{a,b,c}} = \alpha_{\times A,B,C}$.

$(F/\mathbf{0}, n_{1_{\mathbf{0}}}, n_{+/\mathbf{0}}) : (\mathbb{X}/\mathbf{0}, +/\mathbf{0}, 1_{\mathbf{0}}) \to (\mathbb{Y}/\mathbf{0}, +/\mathbf{0}, 1_{\mathbf{0}})$ is a $+/\mathbf{0}$-monoidal functor with

- $n_{1_{\mathbf{0}}} : F/\mathbf{0}(1_{\mathbf{0}}) \to \mathbf{0}$ defined to be $n_{\mathbf{0}} : F(\mathbf{0}) \to \mathbf{0}$ with inverse $b_{F_{\mathbf{0}}}$,
- $n_{+/\mathbf{0}_{a,b}} : F/\mathbf{0}(a +_{/\mathbf{0}} b) \to F/\mathbf{0}(a) +_{/\mathbf{0}} F/\mathbf{0}(b)$ defined to be $n_{+A,B} : F(A + B) \to F(A) + F(B)$ with inverse $n_{+}^{-1}{}_{A,B}$.

It is immediate that it is a monoidal functor as $(F, n_{\mathbf{0}}, n_{+})$ is one.

It remains now to show that (18) holds, which are equivalent to the equations

$$\begin{aligned} &n_{+A,B}; \langle [1_{F(A)}, F(b); n_{\mathbf{0}}; b_{F(A)}], [F(a); n_{\mathbf{0}}; b_{F(B)}, 1_{F(B)}]\rangle; m_{\times A,B} \\ &= F(\langle [1_A, a; b_A], [f; b_B, 1_B]\rangle) \end{aligned}$$

$$m_{\times A,B}; F(\mathrm{mix}_{A,B}); n_{+A,B} = \mathrm{mix}_{F(A),F(B)}$$

The former is equivalent to

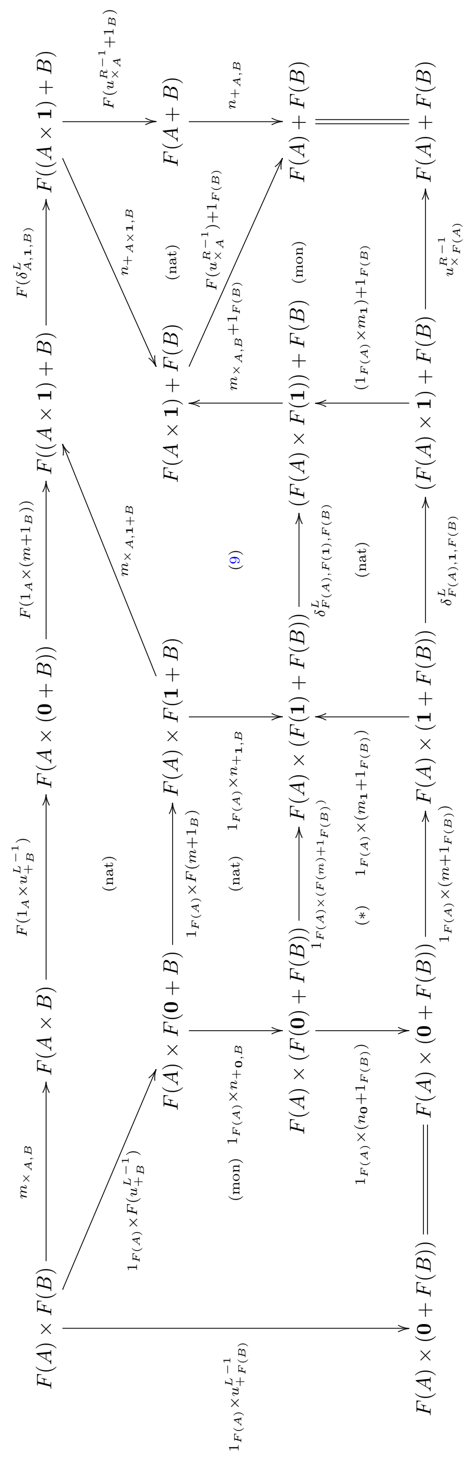
$$\begin{aligned} &\langle [1_{F(A)}, F(b); n_{\mathbf{0}}; b_{F(A)}], [F(a); n_{\mathbf{0}}; b_{F(B)}, 1_{F(B)}]\rangle \\ &= n_{+}^{-1}{}_{A,B}; F(\langle [1_A, b; b_A], [a; b_B, 1_B]\rangle); m_{\times}{}^{-1}_{A,B} \end{aligned}$$

which holds as

$$\begin{aligned} &\iota^i_{F(A),F(B)}; n_{+}^{-1}{}_{B}; F(\langle [1_A, b; b_A], [a; b_B, 1_B]\rangle); m_{\times}{}^{-1}_{A,B}; \pi^j_{F(A),F(B)} \\ &= F(\iota^i_{A,B}; \langle [1_A, b; b_A], [a; b_B, 1_B]\rangle; \pi^j_{A,B}) \end{aligned}$$

and $F(c); b_B = F(c); n_{\mathbf{0}}; b_{F(B)}$ for any $c : C \to \mathbf{0}$.

The latter holds by the following commuting diagram.



F(A) × F(B)
F(A × B)
F(A × (0 + B))
F((A × 1) + B)
F(A) × F(0 + B)
F(A) × F(1 + B)
F(A × 1) + F(B)
F(A + B)
F(A) × (F(0) + F(B))
F(A) × (F(1) + F(B))
(F(A) × F(1)) + F(B)
F(A) + F(B)
F(A) × (0 + F(B))
F(A) × (1 + F(B))
(F(A) × 1) + F(B)
(nat)
(mon)
(9)
(*)

where the $(*)$ square commutes as $\mathbf{1}$ is terminal (or $\mathbf{0}$ is initial) and therefore $n_{\mathbf{0}}^{-1}; F(m); m_{\mathbf{1}}^{-1} = m : \mathbf{0} \to \mathbf{1}$. $\square$

This further entails that $[-]/\mathbf{0} : \mathbf{FCLDC} \to \mathbf{SAdd}$ is a functor.

If we envision $\mathbb{X}/\mathbf{0}$ as the full subcategory of objects with a map to $\mathbf{0}$, or equivalently as the full subcategory of objects $A \cong A \times \mathbf{0}$, the above proposition is saying that a Frobenius cartesian linear functor restricts to semi-additive functors between these full subcategories and.

Now, $[-]/\mathbf{0} : \mathbf{FCLDC} \to \mathbf{SAdd}$ does not however provide a right or left adjoint to the forgetful functor $U : \mathbf{SAdd} \to \mathbf{FCLDC}$. The problem is that neither of the following functors are Frobenius cartesian linear functors

$$F : \mathbb{X} \to \mathbb{X}/\mathbf{0}, \qquad A \mapsto \pi^1_{A,\mathbf{0}} : A \times \mathbf{0} \to \mathbf{0} \qquad (f : A \to B) \mapsto (f \times 1_{\mathbf{0}} : \pi^1_{A,\mathbf{0}} \to \pi^1_{B,\mathbf{0}})$$

$$G : \mathbb{X}/\mathbf{0} \to \mathbb{X}, \qquad (a : A \to \mathbf{0}) \mapsto A \qquad (f : a \to b) \mapsto (f : A \to B)$$

In the first case, $F : \mathbb{X} \to \mathbb{X}/\mathbf{0}$ is not $+$-monoidal as that would be mean there is a natural transformation with component maps $n_{+a,b} : \pi^1_{A+B,\mathbf{0}} \to \pi^1_{A,\mathbf{0}} +_{/\mathbf{0}} \pi^1_{B,\mathbf{0}}(A+B)$, in other words a map $(A+B) \times \mathbf{0} \to (A \times \mathbf{0}) + (B \times \mathbf{0})$ in $\mathbb{X}$, which does exist in an arbitrary CLDC. In the second case, $G : \mathbb{X}/\mathbf{0} \to \mathbb{X}$ is not $\times$-monoidal as that would entail a map $m_{\mathbf{1}} : \mathbf{1} \to \mathbf{0}$, which does not exist in an arbitrary CLDC.

If we were to restrict our attention to their $\times$-monoidal functors or $+$-monoidal functors between CLDCs, then the slice over bottom construction would provide a left or right adjoint respectively.

This is in some sense to be expected. No construction $\mathbf{FCLDC} \to \mathbf{SAdd}$ which is equivalent to providing a subcategory can provide an adjunction. Suppose we consider the lattice with two elements $\{\mathbf{0}, \mathbf{1}\}$, then we would need to either consider the sublattice $\{\mathbf{0}\}$ or the sublattice $\{\mathbf{1}\}$, in order to construct a semi-additive category. In the first case, the construction preserves only the cocartesian structure, while in the second case only the cartesian structure is preserved.

There is likely still a construction which provides an adjunction to the inclusion functor $U : \mathbf{SAdd} \to \mathbf{FCLDC}$, although it is not equivalent to a sub-category. A possibility would be to consider the localization a CLDC with respect to isomorphisms and its mix maps, although it would be difficult to describe and, therefore, we leave it as a future project.

Of course, we can repeat this thought process, but instead by taking the coslice category under the terminal object. Indeed, in this case we are restricting to the full subcategory of objects with a (necessarily unique) map from the terminal object to them. As such, the terminal object becomes initial.

Proposition 5.19 *Let $\mathbb{X}$ be a cartesian linearly distributive category, then $\mathbf{1}/\mathbb{X}$, the coslice category of $\mathbb{X}$ under the terminal object $\mathbf{1}$, is an isomix cartesian linearly distributive category, in particular it is a semi-additive category.*

Furthermore, Frobenius cartesian linear functor $F : \mathbb{X} \to \mathbb{Y}$ induces a semi-additive functor $\mathbf{1}/F : \mathbf{1}/\mathbb{X} \to \mathbf{1}/\mathbb{Y}$ defined by

$$(a : \mathbf{1} \to A) \mapsto (m_{\mathbf{1}}; F(a) : \mathbf{1} \to F(\mathbf{1}) \to F(A))$$

$$(f : (a : \mathbf{1} \to A) \to (b : \mathbf{1} \to B)) \mapsto (F(f) : (m_{\mathbf{1}}; F(a) : \mathbf{1} \to F(A)) \to (m_{\mathbf{0}}; F(b) : \mathbf{1} \to F(B)))$$

Corollary 5.20 *Consider a semi-additive category $\mathbb{B}$, then it is isomorphic to its coslice category over the zero object $\mathbf{0}/\mathbb{B}$, when viewed wither as CLDCs or semi-additive categories.*

Proposition 5.21 *Consider a cartesian linearly distributive category $\mathbb{X}$, then there is an equivalence of semi-additive categories $\mathbf{1}/\mathbb{X} \simeq \mathbb{X} + \mathbf{1}$, where $\mathbb{X} + \mathbf{1}$ denotes the full subcategory of objects of the form $A + \mathbf{1}$ for some A in $\mathbb{X}$.*

6 Revisiting the Kleisli Category of the Exception Monad

As previously stated, it was initially thought that the notions of CLDCs and distributive categories would coincide. It was then demonstrated not to be the case. However, in an effort to relate distributive categories to CLDCs and provide a rich source of examples, Cockett and Seely discussed the well-known exception monad.

Let $\mathbb{D}$ be a distributive category, in the sense of Def 4.5.

Proposition 6.1 *[24, Ex 5.4] The triple $(\cdot + \mathbf{1}, \eta, \mu)$ defined by*

$$\begin{array}{lll} \cdot + \mathbf{1} : \mathbb{D} \to \mathbb{D} & A \mapsto A + \mathbf{1} & f \mapsto f + \mathbf{1} = [f; \iota^0_{B,\mathbf{1}}, \iota^1_{B,\mathbf{1}}] \\ \eta : 1_{\mathbb{D}} \Rightarrow E & & \eta_A = \iota^0_{A,\mathbf{1}} : A \to A + \mathbf{1} \\ \mu : E; E \Rightarrow E & & \mu_A = [1_{A+\mathbf{1}}, \iota^1_{A,\mathbf{1}}] : (A + \mathbf{1}) + \mathbf{1} \to A + \mathbf{1} \end{array}$$

is a monad on $\mathbb{D}$, known as the **exception monad***.*

The exception monad, sometimes known as the maybe monad, is named as such, because from a computer science perspective, the monad models a programming exception, in other words a computational failure which is however handled in a controlled way. Given computations from one data type A to another B represented by morphisms $A \to B$, the effect of possibly throwing an exception is handled by applying the exception monad to B and considering $B + \mathbf{1}$, where the terminal object $\mathbf{1}$ represents the controlled failure. As such, we are interested in the Kleisli category of the exception monad in particular.

For clarity, when discussing the Kleisli category of the exception monad $\mathbb{D}_{\cdot+\mathbf{1}}$, we shall use the interpretation brackets of [12]. A map in $\mathbb{D}_{\cdot+\mathbf{1}}$ will be denoted by $[\![f]\!] : A \to B + \mathbf{1}$, e.g. $[\![1_A]\!] = \eta_A$.

Lemma 6.2 *[12, Prop 7.2] The Kleisli category $\mathbb{D}_{\cdot+\mathbf{1}}$ has finite coproducts, with*

- *initial object $\mathbf{0}$ - the initial object in $\mathbb{D}$,*
- *the unique arrow from the initial object to some object A in $\mathbb{D}_{\cdot+\mathbf{1}}$ is given by*

$$[\![b_A]\!] = b_{A+\mathbf{1}} : \mathbf{0} \to A + \mathbf{1}$$

- *the binary coproduct of objects A and B in $\mathbb{D}_{\cdot+\mathbf{1}}$ is $A + B$ - same as in $\mathbb{D}$,*

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- *the coproduct injections are defined by*

$$[\![\iota^0_{A,B}]\!] = \iota^0_{A,B};\iota^0_{A+B,\mathbf{1}} : A \to (A+B)+\mathbf{1} \qquad [\![\iota^1_{A,B}]\!] = \iota^1_{A,B};\iota^0_{A+B} : B \to (A+B)+\mathbf{1}$$

- *given arrows* $f : A \to C+\mathbf{1}$ *and* $g : B \to C+\mathbf{1}$ *in* $\mathbb{D}._{+\mathbf{1}}$*, the unique induced arrow is*

$$[\![[f,g]]\!] = [f,g] : A+B \to C+\mathbf{1}$$

The Kleisli category $\mathbb{D}._{+\mathbf{1}}$ also has finite products, although they are rather unique. A great deal of attention is paid to their peculiarity in Cockett and Lemay's work on restriction categories in [12].

Lemma 6.3 *[12, Lem 7.3] The Kleisli category* $\mathbb{D}._{+\mathbf{1}}$ *has finite products, with*

- *terminal object* $\mathbf{0}$ *- the initial object in* $\mathbb{D}$*,*
- *the unique arrow from the terminal object to some object* A *in* $\mathbb{D}._{+\mathbf{1}}$ *is given by*

$$[\![t_A]\!] = t_A;\iota^1_{\mathbf{0},\mathbf{1}} : A \to \mathbf{0}+\mathbf{1}$$

- *the binary product of objects* A *and* B *in* $\mathbb{D}._{+\mathbf{1}}$ *is*

$$A \,\&\, B = (A+B)+(A\times B)$$

- *the product projections are defined by*

$$[\![\pi^0_{A,B}]\!] = [[\iota^0_{A,\mathbf{1}}, t_B;\iota^1_{A,\mathbf{1}}], \pi^0_{A,B};\iota^0_{A,\mathbf{1}}] : (A+B)+(A\times B) \to A+\mathbf{1}$$
$$[\![\pi^1_{A,B}]\!] = [[t_A;\iota^1_{B,\mathbf{1}}, \iota^0_{B,\mathbf{1}}], \pi^1_{A,B};\iota^0_{B,\mathbf{1}}] : (A+B)+(A\times B) \to B+\mathbf{1}$$

- *given arrows* $f : C \to A+\mathbf{1}$ *and* $g : C \to V+\mathbf{1}$ *in* $\mathbb{D}._{+\mathbf{1}}$*, the unique induced arrow is*

$$[\![\langle f,g\rangle]\!] = C \xrightarrow{\langle f,g\rangle} (A+\mathbf{1})\times(B+\mathbf{1}) \cong ((A+B)+(A\times B))+\mathbf{1}$$

Example 6.4 *[12, Ex 2.6, Ex 3.9] Potentially the most well-known example of this construction is the Kleisli category of the exception monad on* Set*. It isomorphic to* Par*, the category of sets and partial functions.*

- *The zero object* $\emptyset$ *is the empty set.*
- *Binary coproducts* $X \sqcup Y$ *are given by the disjoint union of sets, with injections* $\iota^0_{X,Y} : X \to X \sqcup Y$ *and* $\iota^1_{X,Y} : Y \to X \sqcup Y$ *defined by*

$$\iota^0_{X,Y}(x) = x \quad \forall x \in X \qquad \iota^1_{X,Y}(y) = y \quad \forall y \in Y$$

- *Binary products are given by the classical product* $X\&Y = X \sqcup Y \sqcup (X\times Y)$*, where* $X \times Y$ *is the cartesian product of sets, with projections* $\pi^0_{X,Y} : X\&Y \to X$ *and* $\pi^1_{X,Y} : X\&Y \to Y$ *defined by*

$$\begin{array}{lll} \pi^0_{X,Y}(x) = x & \pi^1_{X,Y}(x) = \uparrow & \forall x \in X \\ \pi^0_{X,Y}(y) = \uparrow & \pi^1_{X,Y}(y) = y & \forall y \in Y \\ \pi^0_{X,Y}(x,y) = x & \pi^0_{X,Y}(x,y) = y & \forall (x,y) \in X \times Y \end{array}$$

where $\uparrow$ *means undefined.*

The peculiarity of this product comes from the fact that the "naive definition" of & on morphisms is not the correct one:

$$[\![f \,\&\, g]\!] \neq [\![(f+g)+(f\times g)]\!]$$

For a detailed discussion of this fact, see Example 3.11 in [12].

Lemma 6.5 *[12, Prop 7.2]*

- *The Kleisli category of its exception monad is a cocartesian monoidal category* $(\mathbb{D}_{\cdot+\mathbf{1}}, +, \emptyset)$*, where given morphisms* $[\![f]\!] : A \to A' + \mathbf{1}$*,* $[\![g]\!] : B \to B' + \mathbf{1}$*,*

  $$[\![f+g]\!] = [[\![f]\!];(\iota^0_{A',B'} + 1_{\mathbf{1}}), [\![g]\!];(\iota^1_{A',B'} + 1_{\mathbf{1}}))]$$

  $$= \; A+B \xrightarrow{f+g} (A'+\mathbf{1})+(B'+\mathbf{1}) \xrightarrow{\tau^{+}_{A',\mathbf{1},B',\mathbf{1}}} (A'+B')+(\mathbf{1}+\mathbf{1}) \xrightarrow{1_{A'+B'}+t_{\mathbf{1}+\mathbf{1}}} (A'+B')+\mathbf{1}$$

- *The Kleisli category of its exception monad is a symmetric monoidal category* $(\mathbb{D}_{\cdot+\mathbf{1}}, \times, \mathbf{1})$*, where given morphisms* $[\![f]\!] : A \to A' + \mathbf{1}$*,* $[\![g]\!] : B \to B' + \mathbf{1}$*, then*

  $$\begin{aligned}[\![f\times g]\!] =& ([\![f]\!]\times[\![g]\!]); {d^{R}}^{-1}_{A',\mathbf{1},B'+\mathbf{1}}; ({d^{L}}^{-1}_{A',B',\mathbf{1}} + 1_{\mathbf{1}\times(B'+\mathbf{1})});\\ & \alpha^{-1}_{+\;A'\times B',A'\times\mathbf{1},\mathbf{1}\times(B'+\mathbf{1})}; (1_{A'\times B'} + t_{(A'\times\mathbf{1})+(\mathbf{1}\times(B'+\mathbf{1}))})\end{aligned}$$

Now, the Kleisli category $\mathbb{D}_{\cdot+\mathbf{1}}$ is a LDC, as stated without proof by Cockett and Seely in [10].

Proposition 6.6 *Consider a distributive category* $\mathbb{D}$*, the Kleisli category of its exception monad is a symmetric isomix linearly distributive category* $(\mathbb{D}_{\cdot+\mathbf{1}}, \curlyvee, \mathbf{0}, +, \mathbf{0})$*, where*

$$\curlyvee : \mathbb{D}_{\cdot+\mathbf{1}} \times \mathbb{D}_{\cdot+\mathbf{1}} \to \mathbb{D}_{\cdot+\mathbf{1}} \qquad \begin{aligned}(A,B) &\mapsto (A+B)+(A\times B)\\ ([\![f]\!],[\![g]\!]) &\mapsto [\![(f+g)+(f\times g)]\!]\end{aligned}$$

and left linear distributivity

$$[\![\delta^L_{A,B,C}]\!] : \quad \begin{aligned}& A\curlyvee(B+C) \to ((A\curlyvee B)+C)+\mathbf{1}\\ & (A+(B+C))+(A\times(B+C)) \to (((A+B)+(A\times B))+C)+\mathbf{1}\end{aligned}$$

is defined as the unique morphism induced by the following maps:

$$A \xrightarrow{\iota^0_{A,B}} A+B \xrightarrow{\iota^0_{A+B,A\times B}} A\curlyvee B \xrightarrow{\iota^0_{A\curlyvee B,C}} (A\curlyvee B)+C \xrightarrow{\iota^0_{(A\curlyvee B)+C,\mathbf{1}}} ((A\curlyvee B)+C)+\mathbf{1}$$

$$B \xrightarrow{\iota^1_{A,B}} A+B \xrightarrow{\iota^0_{A+B,A\times B}} A\curlyvee B \xrightarrow{\iota^0_{A\curlyvee B,C}} (A\curlyvee B)+C \xrightarrow{\iota^0_{(A\curlyvee B)+C,\mathbf{1}}} ((A\curlyvee B)+C)+\mathbf{1}$$

$$C \xrightarrow{\iota^1_{A\curlyvee B,C}} (A\curlyvee B)+C \xrightarrow{\iota^0_{(A\curlyvee B)+C,\mathbf{1}}} ((A\curlyvee B)+C)+\mathbf{1}$$

$$A\times(B+C) \xrightarrow{{d^L}^{-1}_{A,B,C}} (A\times B)+(A\times C) \xrightarrow{1_{A\times B}+t_{A\times C}} (A\times B)+\mathbf{1} \xrightarrow{\iota^1_{A+B,A\times B}+1_{\mathbf{1}}} (A\curlyvee B)+\mathbf{1} \xrightarrow{\iota^0_{A\curlyvee B,C}+1_{\mathbf{1}}} ((A\curlyvee B)+C)+\mathbf{1}$$

The above theorem follows from a larger result that every symmetric distributive monoidal category with a zero object induces such a symmetric isomix LDC, the details of which can be found in Appendix A.

Therefore, given any distributive category, we can build a LDC. However, it is stated in [10] not only that it is a LDC, but that it is a CLDC.

This is, it turns out, not true. Indeed, the tensor product of the LDC $(\mathbb{D}_{\cdot+\mathbf{1}}, \curlyvee, \mathbf{0}, +, \mathbf{0})$ is not the cartesian product. At this point, one could think that $(\mathbb{D}_{\cdot+\mathbf{1}}, \&, \mathbf{0}, +, \mathbf{0})$ remains an LDC, with everything the same, except the tensor product definition on morphisms. However, the linear distributivities defined in Theorem 6.6 are no longer natural when considering the cartesian product & on $\mathbb{D}_{\cdot+\mathbf{1}}$.

Indeed, let us consider $(\mathrm{Par}, \curlyvee, \emptyset, \sqcup, \emptyset)$ a symmetric isomix LDC with linear distributivity

$$\begin{aligned}\delta^L_{X,Y,Z} : X \curlyvee (Y \sqcup Z) &\to (X \curlyvee Y) \sqcup Z \\ X \sqcup Y \sqcup Z \sqcup (X \times Y) \sqcup (X \times Z) &\to X \sqcup Y \sqcup Z \sqcup (X \times Y)\end{aligned}$$

defined by

$$\delta^L_{X,Y,Z}(u) = \begin{cases} u & \text{if } u \in X, Y, Z, X \times Y \\ \uparrow & \text{otherwise} \end{cases}$$

We will show here that this does not however define a natural transformation $X \,\&\, (Y \sqcup Z) \to (X \,\&\, Y) \sqcup Z$. Thanks to Example 3.11 in [12], we know how & is defined on morphisms in Par. Given partial functions $f : X \to X', g : Y \to Y', h : Z \to Z'$, then

$$(f \,\&\, (g \sqcup h))(u) = \begin{cases} f(u) & \text{if } u \in X \text{ and } f(u)\downarrow \\ g(u) & \text{if } u \in Y \text{ and } g(u)\downarrow \\ h(u) & \text{if } u \in Z \text{ and } h(u)\downarrow \\ (f(u_0), g(u_1)) & \text{if } u = (u_0, u_1) \in X \times Y \text{ and } f(u_0)\downarrow \text{ and } g(u_1)\downarrow \\ (f(u_0), h(u_1)) & \text{if } u = (u_0, u_1) \in X \times Z \text{ and } f(u_0)\downarrow \text{ and } h(u_1)\downarrow \\ f(u_0) & \text{if } u = (u_0, u_1) \in X \times Y \text{ and } f(u_0)\downarrow \text{ and } g(u_1)\uparrow \\ f(u_0) & \text{if } u = (u_0, u_1) \in X \times Z \text{ and } f(u_0)\downarrow \text{ and } h(u_1)\uparrow \\ g(u_1) & \text{if } u = (u_0, u_1) \in X \times Y \text{ and } f(u_0)\uparrow \text{ and } g(u_1)\downarrow \\ h(u_1) & \text{if } u = (u_0, u_1) \in X \times Z \text{ and } f(u_0)\uparrow \text{ and } h(u_1)\downarrow \\ \uparrow & \text{otherwise} \end{cases}$$

while

$$((f \,\&\, g) \sqcup h)(u) = \begin{cases} f(u) & \text{if } u \in X \text{ and } f(u)\downarrow \\ g(u) & \text{if } u \in Y \text{ and } g(u)\downarrow \\ h(u) & \text{if } u \in Z \text{ and } h(u)\downarrow \\ (f(u_0), g(u_1)) & \text{if } u = (u_0, u_1) \in X \times Y \text{ and } f(u_0)\downarrow \text{ and } g(u_1)\downarrow \\ f(u_0) & \text{if } u = (u_0, u_1) \in X \times Y \text{ and } f(u_0)\downarrow \text{ and } g(u_1)\uparrow \\ g(u_1) & \text{if } u = (u_0, u_1) \in X \times Y \text{ and } f(u_0)\uparrow \text{ and } g(u_1)\downarrow \\ \uparrow & \text{otherwise} \end{cases}$$

Now, consider $u = (u_0, u_1) \in X \times Z$, $(\delta^L_{X,Y,Z}; ((f \,\&\, g) \sqcup h))(u) = \uparrow$, while

$$((f \,\&\, (g \sqcup h)); \delta^L_{X',Y',Z'})(u) = \begin{cases} f(u_0) & \text{if } u = (u_0, u_1) \in X \times Z \text{ and } f(u_0) \downarrow \text{ and } h(u_1) \uparrow \\ h(u_1) & \text{if } u = (u_0, u_1) \in X \times Z \text{ and } f(u_0) \uparrow \text{ and } h(u_1) \downarrow \\ \uparrow & \text{otherwise} \end{cases}$$

Therefore, $\delta^L_{X,Y,Z}; ((f \,\&\, g) \sqcup h) \neq (f \,\&\, (g \sqcup h)); \delta^L_{X',Y',Z'}$

This is still not a complete proof that the Kleisli category of the exception monad cannot be a a CLDC. Perhaps the only problem was the definition of the linear distributivities given in [10] (and in Theorem 6.6).

However, as a Corollary to Theorem 5.11, we see that there are no possible linear distributivities $A+(B \,\&\, C) \to (A+B) \,\&\, C$ such that $(\mathbb{D}_{\cdot+\mathbf{1}}, \&, \mathbf{0}, +, \mathbf{0})$ is a CLDC. If $(\mathbb{D}_{\cdot+\mathbf{1}}, \&, \mathbf{0}, +, \mathbf{0})$ was a CLDC, it would be isomix and therefore, by above, it would be semi-additive. This is not the case, therefore:

Corollary 6.7 *The Kleisli category of the exception monad of a distributive category is a cartesian linearly distributive category if and only if it is trivial.*

7 Further Examples

Given all the results of the previous sections, it is reasonable to question whether the only possible CLDCs are bounded distributive lattices or semi-additive categories. We can quickly see that is not the case as CLDCs are closed under products:

Proposition 7.1 *[10, Sec 3] Given cartesian linearly distributive categories $\mathbb{X}$ and $\mathbb{Y}$, their product category $\mathbb{X} \times \mathbb{Y}$ is a cartesian linearly distributive category.*

Therefore, by taking the product of a bounded distributive lattice $(\mathcal{L}, \wedge, \mathbf{1}, \vee, \mathbf{0})$ with a semi-additive category $(\mathbb{B}, \times, \mathbf{0}, +, \mathbf{0})$, we get a CLDC which is neither posetal nor compact.

We will expand upon this idea, using lattices and categories with biproducts as building blocks, to construct more examples of CLDCs. In particular, we will consider the Grothendieck construction for a functor from a semi-additive category to the category of bounded distributive lattices **BDL** [1].

Consider a semi-additive category $\mathbb{B}$ and a functor $F : \mathbb{B}^{op} \to \mathbf{BDL}$.

Definition 7.2 *The Grothendieck construction for F is the category $\int F$ consisting of:*

- *Objects: pairs (A, a), where $A \in \mathbb{B}$ and $a \in F(A)$*
- *Morphisms: $f : (A, a) \to (B, b)$ is a morphism $f : A \to B$ in $\mathbb{B}$ such that $a \leq F(f)(b)$*

[1]This example is due to valuable discussions between the authors and Richard Garner whose idea it was to look at fibrations for additional examples.

Note that composition is inherited directly from $\mathbb{B}$ and is well-defined as, given $f : (A, a) \to (B, b)$ and $g : (B, b) \to (C, c)$ in $\int F$, then $f; g : (A, a) \to (C, c)$:

$$a \leq F(f)(b) \leq F(f)(F(g)(c)) = F(f; g)(c)$$

The inequality above follows as $F(f) : F(B) \to F(A)$ is a bounded distributive lattice homomorphism.

Lemma 7.3 *$\int F$ is a cartesian and cocartesian category with:*

- *Terminal object: $(\mathbf{0}, \mathbf{1}_{F(\mathbf{0})})$, where $\mathbf{1}_{F(\mathbf{0})}$ is the top element of lattice $F(\mathbf{0})$*
- *Binary products: $(A, a) \times (B, b)$ is given by*

$$(A \times B, F(\pi^0_{A,B})(a) \wedge F(\pi^1_{A,B})(b))$$

- *Initial object: $(\mathbf{0}, \mathbf{0}_{F(\mathbf{0})})$, where $\mathbf{1}_{F(\mathbf{0})}$ is the bottom element of lattice $F(\mathbf{0})$*
- *Binary coproducts: $(A, a) + (B, b)$ is given by*

$$(A + B, F(\psi_{A,B}; \pi^0_{A,B})(a) \vee F(\psi_{A,B}; \pi^1_{A,B})(b))$$

Proof Consider an object $(A, a) \in \int F$, then there is a unique morphism $t_{(A,a)} : (A, a) \to (\mathbf{0}, \mathbf{1}_{F(\mathbf{0})})$ given by $t_A : A \to \mathbf{0}$ in $\mathbb{B}$, as $a \leq \mathbf{1}_{F(A)} = F(t_A)(\mathbf{1}_{F(\mathbf{0})})$ (since $F(t_A)$ is a lattice homomorphism). Similarly, there is a unique morphism $b_{(A,a)} : (\mathbf{0}, \mathbf{0}_{F(\mathbf{0})}) \to (A, a)$ given by $b_A : \mathbf{0} \to A$ in $\mathbb{B}$, as $F(b_A)(\mathbf{0}_{F(A)}) = \mathbf{0}_{F(\mathbf{0})} \leq a$.

Consider $(A, a), (B, b) \in \int F$. Then, $(A, a) \times (B, b)$ is equipped with projections $\pi^0_{(A,a),(B,b)} : (A, a) \times (B, b) \to (A, a)$ and $\pi^1_{(A,a),(B,b)} : (A, a) \times (B, b) \to (B, b)$ given by $\pi^0_{A,B} : A \times B \to A$ and $\pi^1_{A,B} : A \times B \to B$ as

$$F(\pi^0_{A,B})(a) \wedge F(\pi^1_{A,B})(b) \leq F(\pi^0_{A,B})(a), \;\; F(\pi^1_{A,B})(b)$$

This is a product diagram since, given morphisms $f : (C, c) \to (A, a)$ and $g : (C, c) \to (B, b)$, there is a unique induced morphism $(C, c) \to (A, a) \times (B, b)$ given by $\langle f, g\rangle : C \to A \times B$ since

$$\begin{aligned} c &\leq F(f)(a) \wedge F(g)(b) = F(\langle f, g\rangle; \pi^0_{A,B})(a) \wedge F(\langle f, g\rangle; \pi^1_{A,B})(b) \\ &= F(\langle f, g\rangle)(F(\pi^0_{A,B})(a) \wedge F(\pi^1_{A,B})(b)) \end{aligned}$$

Notice that $F(\psi_{A,B}; \pi^j_{A,B}) \dashv F(\iota^j_{A,B})$, in other words, given $a \in F(A), b \in F(B), c \in F(A + B)$:

$$F(\psi_{A,B}; \pi^0_{A,B})(a) \leq c \iff a \leq F(\iota^0_{A,B})(x) \qquad F(\psi_{A,B}; \pi^1_{A,B})(a) \leq c \iff a \leq F(\iota^1_{A,B})(c)$$

Finally, $(A, a) + (B, b)$ is equipped with injections $\iota^0_{(A,a),(B,b)} : (A, a) \to (A, a) + (B, b)$ and $\iota^1_{(A,a),(B,b)} : (B, b) \to (A, a) + (B, b)$ given by $\iota^0_{A,a} : A \to A + B$ and $\iota^1_{A,B} : B \to A + B$ respectively as

$$\begin{aligned} &a \leq F(1_A)(a) \vee F(0_{A,B})(b) = F(\iota^0_{A,B}; \psi_{A,B}; \pi^0_{A,B})(a) \vee F(\iota^0_{A,B}; \psi_{A,B}; \pi^1_{A,B})(a) \\ &= F(\iota^0_{A,B})(F(\psi_{A,B}; \pi^0_{A,B})(a) \vee F(\psi_{A,B}; \pi^1_{A,B})(b) \\ &b \leq F(0_{A,B})(a) \vee F(1_B)(b) = F(\iota^1_{A,B}; \psi_{A,B}; \pi^0_{A,B})(a) \vee F(\iota^1_{A,B}; \psi_{A,B}; \pi^1_{A,B})(a) \\ &= F(\iota^1_{A,B})(F(\psi_{A,B}; \pi^0_{A,B})(a) \vee F(\psi_{A,B}; \pi^1_{A,B})(b)) \end{aligned}$$

This is a coproduct diagram since, given morphisms $h : (A, a) \to (C, c)$ and $k : (B, b) \to (C, c)$, there is a unique induced morphism $(A, a) + (B, b) \to (C, c)$ given by $[h, k] : (A, a) + (B, b) \to (C, c)$ since

$$a \le F(h)(c) = F(\iota^0_{A,B})F([h, k])(c) \implies F(\psi_{A,B}; \pi^0_{A,B})(a) \le F([h, k])(c)$$

$$b \le F(k)(c) = F(\iota^1_{A,B})F([h, k])(c) \implies F(\psi_{A,B}; \pi^1_{A,B})(a) \le F([h, k])(c)$$

Therefore,

$$F(\psi_{A,B}; \pi^0_{A,B})(a) \vee F(\psi_{A,B}; \pi^1_{A,B})(a) \le F([h, k])(c)$$

□

Proposition 7.4 *$\int F$ is a cartesian linearly distributive category.*

Proof By the above Lemma, it remains only to define the linear distributivities and prove the coherence conditions hold. Consider $(A, a), (B, b)$ and (C, c) in $\int F$, then

$$\delta^L_{(A,a),(B,b),(C,c)} : (A, a) \times ((B, b) + (C, c)) \to ((A, a) \times (B, b)) + (C, c)$$

is $\delta^L_{A,B,C} : A \times (B + C) \to (A \times B) + C$ since it is a morphism in $\int F$ as it satisfies:

$$\begin{aligned}&F(\pi^0_{A,B+C})(a) \wedge (F(\pi^1_{A,B+C}; \psi_{B,C}; \pi^0_{B,C})(b) \vee F(\pi^1_{A,B+C}; \psi_{B,C}; \pi^1_{B,C})(c))\\ &\le F(\delta^L_{A,B,C})((F(\psi_{A\times B,C}; \pi^0_{A\times B,C}; \pi^0_{A,B})(a) \wedge F(\psi_{A\times B,C}; \pi^0_{A\times B,C}; \pi^1_{A,B})(b))\\ &\quad \vee F(\psi_{A\times B,C}; \pi^1_{A\times B,C})(c))\end{aligned}$$

follows as

$$\begin{aligned}&F(\delta^L_{A,B,C}; \psi_{A\times B,C}; \pi^0_{A\times B,C}; \pi^0_{A,B})(a)\\ &= F((1_A \times \psi_{B,C}); \alpha^{-1}_{A,B,C}; \pi^0_{A\times B,C}; \pi^0_{A,B})(a) = F(\pi^0_{A,B+C})(a)\end{aligned}$$

$$\begin{aligned}&F(\delta^L_{A,B,C}; \psi_{A\times B,C}; \pi^0_{A\times B,C}; \pi^1_{A,B})(b)\\ &= F((1_A \times \psi_{B,C}); \alpha^{-1}_{A,B,C}; \pi^0_{A\times B,C}; \pi^1_{A,B})(b) = F(\pi^1_{A,B+C}; \psi_{B,C}; \pi^0_{B,C})(b)\end{aligned}$$

$$\begin{aligned}&F(\delta^L_{A,B,C}; \psi_{A\times B,C}; \pi^1_{A\times B,C})(c)\\ &= F((1_A \times \psi_{B,C}); \alpha^{-1}_{A,B,C}; \pi^1_{A\times B,C})(a) = F(\pi^1_{A,B+C}; \psi_{B,C}; \pi^1_{B,C})(c)\end{aligned}$$

and for any d, e, f in the bounded distributive lattice $F(A+(B\times C))$, $d\wedge(e\vee f) \le (d\wedge e)\vee f$. □

Therefore, given any contravariant functor from a semi-additive category to the category of bounded distributive lattices, we can consider its Grothendieck construction as an example of a CLDC.

Example 7.5 1. *Consider* Rel*, the category of sets and relations. It has finite biproducts, with zero object the empty set $\emptyset$ and binary biproducts the disjoint union $\sqcup$. Let*

$\mathcal{P} : \mathrm{Rel}^{op} \to \mathbf{BDL}$ be the contravariant powerset functor, mapping sets X to their powerset $\mathcal{P}(X)$ and mapping relations $R : X \to Y$ to the "preimages" $\mathcal{P}(R) : \mathcal{P}(Y) \to \mathcal{P}(X)$ defined, for $V \subseteq Y$, to be

$$\mathcal{P}(R)(V) = \{x \in X \mid \exists y \in V, xRy\}$$

Then, $\int \mathcal{P}$ has objects which are pairs of sets (X, U) with $U \subseteq X$ and morphisms $R : (X, U) \to (Y, V)$ which are relations $R : X \to Y$ such that $U \subseteq \{x \in X \mid \exists y \in V, xRy\}$, in other words the relation R can restrict to a well-defined relation $R|_U : U \to V$.

It is an isomix CLDC, and consequently a semi-additive category, as its terminal object is also its initial object : $(\emptyset, \emptyset)$.

2. *In order to get a CLDC which is not compact or posetal, consider now the double powerset functor $\mathcal{P}^2 : \mathrm{Rel}^{op} \to \mathbf{BDL}$, mapping sets X to the powerset of their powerset $\mathcal{P}(\mathcal{P}(X))$ and mapping relations $R : X \to Y$ to the "preimages of their images" $\mathcal{P}^2(R) : \mathcal{P}(\mathcal{P}(Y)) \to \mathcal{P}(\mathcal{P}(X))$ defined for $\mathbb{V} \subseteq \mathcal{P}(Y)$, to be*

$$\mathcal{P}^2(R)(\mathbb{V}) = \{U \in \mathcal{P}(X) \mid R(U) \in \mathbb{V}\}$$

where $R(U) = \{y \in Y, \mid \exists x \in U, xRy\}$.

Then, $\int \mathcal{P}^2$ has objects which are pairs of sets $(X, \mathbb{U})$ with $\mathbb{U} \subseteq \mathcal{P}(X)$ and morphisms $R : (X, \mathbb{U}) \to (Y, \mathbb{Y})$ which are relations $R : X \to Y$ such that $\forall U \in \mathbb{U}, R(U) \in \mathbb{V}$.

Its CLDC structure is neither compact nor posetal, and is as follows:

- *Terminal object: $(\emptyset, \{\emptyset\})$*
- *Binary products: $(X, \mathbb{U}) \times (Y, \mathbb{V})$ is given by*

$$(X \sqcup Y, \mathcal{P}^2(\pi^0_{X,Y})(\mathbb{U}) \cap \mathcal{P}^2(\pi^1_{X,Y})(\mathbb{V})) = (X \sqcup Y, \mathbb{U} \times \mathbb{V})$$

- *Initial object: $(\emptyset, \emptyset)$,*
- *Binary coproducts: $(X, \mathbb{U}) + (Y, \mathbb{V})$ is given by*

$$(X \sqcup Y, \mathcal{P}^2(\pi^0_{X,Y})(\mathbb{U}) \cup \mathcal{P}^2(\pi^1_{X,Y})(\mathbb{V})) = (X \sqcup Y, (\mathbb{U} \times \mathcal{P}(Y)) \cup (\mathcal{P}(X) \times \mathbb{V}))$$

Appendix A Symmetric distributive monoidal category with a zero object

As previously stated, the fact that the Kleisli category of the exception monad for a distributive category $\mathbb{D}$ is an isomix symmetric LDC $(\mathbb{D}_{\cdot+\mathbf{1}}, \curlyvee, \mathbf{0}, +, \mathbf{0})$ follows from a similar result for symmetric distributive categories with zero objects.

Definition A.1 *[3, Def 2.3.1] A symmetric monoidal category $(\mathcal{X}, \oslash, I)$ is* **distributive** *if*

- *$\mathcal{X}$ has finite coproducts, and*
- *the following canonical transformations*

$$d^L_{A,B,C} = [1_A \oslash \iota^0_{B,C}, 1_A \oslash \iota^1_{B,C}] : (A \oslash B) + (A \oslash C) \to A \oslash (B + C)$$
$$d^R_{A,B,C} = [\iota^0_{A,B} \oslash 1_C, \iota^1_{A,B} \oslash 1_C] : (A \oslash C) + (B \oslash C) \to (A + B) \oslash C$$
$$\lambda_A = b_{\mathbf{0}\oslash A} : \mathbf{0} \to \mathbf{0} \oslash A \qquad \rho_A = t_{A\oslash\mathbf{0}} : \mathbf{0} \to A \oslash \mathbf{0}$$

are isomorphisms.

Symmetric distributive monoidal categories are a generalization of distributive categories to a non-cartesian monoidal product, which describe the behaviour of the Kleisli category of the exception monad for a distributive category, since $\times$ is a well-defined monoidal product on $\mathbb{D}_{\cdot+\mathbf{1}}$, but not the binary product.

There is a well-known folklore result that given such categories, we can build another symmetric monoidal product, which behaves as an "either-or-both" product.

Lemma A.2 *Given a distributive symmetric monoidal category $(\mathcal{X}, \oslash, I)$, the functors*

$$\curlyvee = (\cdot + \cdot) + (\cdot \oslash \cdot) : \mathcal{X} \times \mathcal{X} \to \mathcal{X} \qquad\qquad \mathbf{0} : 1 \to \mathcal{X}$$

along with the following natural isomorphisms defines a symmetric monoidal category $(\mathcal{X}, \curlyvee, \mathbf{0})$.

$$u^R_{\curlyvee A} = u^{R^{-1}}_{+A}; u^{R^{-1}}_{+A+\mathbf{0}}; (1_{A+\mathbf{0}} + \rho_A) = \iota^0_{A,\mathbf{0}}; \iota^0_{A+\mathbf{0},A\oslash\mathbf{0}} : A \to A \curlyvee \mathbf{0}$$
$$u^L_{\curlyvee A} = u^{L^{-1}}_{+A}; u^{R^{-1}}_{+\mathbf{0}+A}; (1_{\mathbf{0}+A} + \lambda_A) = \iota^1_{\mathbf{0},A}; \iota^0_{\mathbf{0}+A,\mathbf{0}\oslash A} : A \to \mathbf{0} \curlyvee A$$
$$\sigma_{\curlyvee A,B} = \sigma_{+A,B} + \sigma_{\oslash A,B} : A \curlyvee B \to B \curlyvee A$$
$$\alpha_{\curlyvee A,B,C} : (A \curlyvee B) \curlyvee C \to A \curlyvee (B \curlyvee C)$$

is defined as the unique morphisms induced by the following maps

$$A \xrightarrow{\iota^0_{A,B\curlyvee C}} A + (B \curlyvee C) \xrightarrow{\iota^0_{A+(B\curlyvee C),A\oslash(B\curlyvee C)}} A \curlyvee (B \curlyvee C)$$
$$B \xrightarrow{\iota^0_{B,C}} B + C \xrightarrow{\iota^0_{B+C,B\oslash C}} B \curlyvee C \xrightarrow{\iota^1_{A,B\curlyvee C}} A + (B \curlyvee C) \xrightarrow{\iota^0_{A+(B\curlyvee C),A\oslash(B\curlyvee C)}} A \curlyvee (B \curlyvee C)$$
$$A \oslash B \xrightarrow{1_A \oslash \iota^0_{B,C}} A \oslash (B + C) \xrightarrow{1_A \oslash \iota^0_{B+C,B\oslash C}} A \oslash (B \curlyvee C) \xrightarrow{\iota^1_{A+(B\curlyvee C),A\oslash(B\curlyvee C)}} A \curlyvee (B \curlyvee C)$$
$$C \xrightarrow{\iota^1_{B,C}} B + C \xrightarrow{\iota^0_{B+C,B\oslash C}} B \curlyvee C \xrightarrow{\iota^1_{A,B\curlyvee C}} A + (B \curlyvee C) \xrightarrow{\iota^0_{A+(B\curlyvee C),A\oslash(B\curlyvee C)}} A \curlyvee (B \curlyvee C)$$
$$((A + B) + (A \oslash B)) \oslash C \xrightarrow{{d^R_{A+B,A\oslash B,C}}^{-1}} ((A + B) \oslash C) + ((A \oslash B) \oslash C)$$
$$\xrightarrow{{d^R_{A,B,C}}^{-1} + 1_{(A\oslash B)\oslash C}} ((A \oslash C) + (B \oslash C)) + ((A \oslash B) \oslash C) \xrightarrow{\chi_{A,B,C}} A \curlyvee (B \curlyvee C)$$

where $\chi_{A,B,C}$ is given as the unique morphisms induced by the following maps

$$A \oslash C \xrightarrow{1_A \oslash \iota^1_{B,C}} A \oslash (B + C) \xrightarrow{1_A \oslash \iota^0_{B+C,B\oslash C}} A \oslash (B \curlyvee C) \xrightarrow{\iota^1_{A+(B\curlyvee C),A\oslash(B\curlyvee C)}} A \curlyvee (B \curlyvee C)$$
$$B \oslash C \xrightarrow{\iota^1_{B+C,B\oslash C}} B \curlyvee C \xrightarrow{\iota^1_{A,B\curlyvee C}} A + (B \curlyvee C) \xrightarrow{\iota^0_{A+(B\curlyvee C),A\oslash(B\curlyvee C)}} A \curlyvee (B \curlyvee C)$$
$$(A \oslash B) \oslash C \xrightarrow{\alpha_{\oslash A,B,C}} A \oslash (B \oslash C) \xrightarrow{1_A \oslash \iota^1_{B+C,B\oslash C}} A \oslash (B \curlyvee C)$$
$$\xrightarrow{\iota^1_{A+(B\curlyvee C),A\oslash(B\curlyvee C)}} A \curlyvee (B \curlyvee C)$$

Lemma A.3 *The Kleisli category of the exception monad for a distributive category $(\mathbb{D}_{\cdot+\mathbf{1}}, \curlyvee, \mathbf{0})$ is a symmetric distributive monoidal category with a zero object $\mathbf{0}$.*

Theorem A.4 *Consider a symmetric distributive monoidal category* $(\mathcal{X}, \oslash, I)$ *with a zero object* $\emptyset$*, then* $(\mathcal{X}, \curlyvee, \emptyset, +, \emptyset)$ *is a symmetric isomix linearly distributive category with left linear distributivity*

$$\delta^L_{A,B,C} : A \curlyvee (B + C) \to (A \curlyvee B) + C$$

defined as the unique morphism induced by the following maps

$$A \xrightarrow{\iota^0_{A,B}} A + B \xrightarrow{\iota^0_{A+B,A\oslash B}} A \curlyvee B \xrightarrow{\iota^0_{A\curlyvee B,C}} (A \curlyvee B) + C$$

$$B \xrightarrow{\iota^1_{A,B}} A + B \xrightarrow{\iota^0_{A+B,A\oslash B}} A \curlyvee B \xrightarrow{\iota^0_{A\curlyvee B,C}} (A \curlyvee B) + C$$

$$C \xrightarrow{\iota^1_{A\curlyvee B,C}} (A \curlyvee B) + C$$

$$A \oslash (B+C) \xrightarrow{1_A \oslash (1_B + t_C)} A \oslash (B+\emptyset) \xrightarrow{1_A \oslash u^R_{+B}} A \oslash B \xrightarrow{\iota^1_{A+B,A\oslash B}} A \curlyvee B \xrightarrow{\iota^0_{A\curlyvee B,C}} (A \curlyvee B)+C$$

and right linear distributivity

$$\delta^R_{A,B,C} : (A + B) \curlyvee C \to A + (B \curlyvee C)$$

defined as the unique morphism induced by the following maps

$$A \xrightarrow{\iota^0_{A,B\curlyvee C}} A + (B \curlyvee C)$$

$$B \xrightarrow{\iota^0_{B,C}} B + C \xrightarrow{\iota^0_{B+C,B\oslash C}} B \curlyvee C \xrightarrow{\iota^1_{A,B\curlyvee C}} A + (B \curlyvee C)$$

$$C \xrightarrow{\iota^1_{B,C}} B + C \xrightarrow{\iota^0_{B+C,B\oslash C}} B \curlyvee C \xrightarrow{\iota^1_{A,B\curlyvee C}} A + (B \curlyvee C)$$

$$(A+B) \oslash C \xrightarrow{(t_A + 1_B) \oslash 1_C} (\emptyset + B) \oslash C \xrightarrow{u^L_{+B} \oslash 1_C} B \oslash C \xrightarrow{\iota^1_{B+C,B\oslash C}} B \curlyvee C \xrightarrow{\iota^1_{A,B\curlyvee C}} A+(B \curlyvee C)$$

Proof First, we show the left linear distributivity is a natural transformation. Consider $f : A \to A', g : B \to B'$ and $h : C \to C'$ in $\mathcal{X}$, then

$$\delta^L_{A,B,C}; ((f \curlyvee g) + h) = (f \curlyvee (g + h)); \delta^L_{A',B',C'}$$

holds by considering the left-hand side and right hand-side pre-composed with the relevant injections as follows.

$$\begin{aligned}
&\iota^0_{A,B+C}; \iota^0_{A+(B+C),A\oslash(B+C)}; \delta^L_{A,B,C}; ((f \curlyvee g) + h) \\
&= \iota^0_{A,B}; \iota^0_{A+B,A\oslash B}; \iota^0_{A\curlyvee B,C}; ((f \curlyvee g) + h) \\
&= f; \iota^0_{A',B'}; \iota^0_{A'+B',A'\oslash B'}; \iota^0_{A'\curlyvee B',C'}
\end{aligned}$$

$$\begin{aligned}
&\iota^0_{B,C}; \iota^1_{A,B+C}; \iota^0_{A+(B+C),A\oslash(B+C)}; \delta^L_{A,B,C}; ((f \curlyvee g) + h) \\
&= \iota^1_{A,B}; \iota^0_{A+B,A\oslash B}; \iota^0_{A\curlyvee B,C}; ((f \curlyvee g) + h) \\
&= g; \iota^1_{A',B'}; \iota^0_{A'+B',A'\oslash B'}; \iota^0_{A'\curlyvee B',C'}
\end{aligned}$$

$$\begin{aligned}
&\iota^1_{B,C}; \iota^1_{A,B+C}; \iota^0_{A+(B+C),A\oslash(B+C)}; \delta^L_{A,B,C}; ((f \curlyvee g) + h) \\
&= \iota^1_{A\curlyvee B,C}; ((f \curlyvee g) + h) \\
&= h; \iota^1_{A'\curlyvee B',C'}
\end{aligned}$$

$$\begin{aligned}
&\iota^1_{A+(B+C),A\oslash(B+C)};\delta^L_{A,B,C};((f\curlyvee g)+h)\\
&=(1_A\oslash(1_B+t_C);u^R_{+B});\iota^1_{A+B,A\oslash B};\iota^0_{A\curlyvee B,C};((f\curlyvee g)+h)\\
&=(1_A\oslash(1_B+t_C);u^R_{+B});(f\oslash g);\iota^1_{A'+B',A'\oslash B'};\iota^0_{A'\curlyvee B',C'}\\
&=(f\oslash(g+t_C);u^R_{+B'});\iota^1_{A'+B',A'\oslash B'};\iota^0_{A'\curlyvee B',C'}
\end{aligned}$$

$$\begin{aligned}
&\iota^0_{A,B+C};\iota^0_{A+(B+C),A\oslash(B+C)};(f\curlyvee(g+h));\delta^L_{A',B',C'}\\
&=f;\iota^0_{A',B'+C'};\iota^0_{A'+(B'+C'),A'\oslash(B'+C')};\delta^L_{A',B',C'}\\
&=f;\iota^0_{A',B'};\iota^0_{A'+B',A'\oslash B'};\iota^0_{A'\curlyvee B',C'}
\end{aligned}$$

$$\begin{aligned}
&\iota^0_{B,C};\iota^1_{A,B+C};\iota^0_{A+(B+C),A\oslash(B+C)};(f\curlyvee(g+h));\delta^L_{A',B',C'}\\
&=g;\iota^0_{B',C'};\iota^1_{A',B'+C'};\iota^0_{A'+(B'+C'),A'\oslash(B'+C')};\delta^L_{A',B',C'}\\
&=g;\iota^1_{A',B'};\iota^0_{A'+B',A'\oslash B'};\iota^0_{A'\curlyvee B',C'}
\end{aligned}$$

$$\begin{aligned}
&\iota^1_{B,C};\iota^1_{A,B+C};\iota^0_{A+(B+C),A\oslash(B+C)};(f\curlyvee(g+h));\delta^L_{A',B',C'}\\
&=h;\iota^1_{B',C'};\iota^1_{A',B'+C'};\iota^0_{A'+(B'+C'),A'\oslash(B'+C')};\delta^L_{A',B',C'}\\
&=h;\iota^1_{A'\curlyvee B',C'}
\end{aligned}$$

$$\begin{aligned}
&\iota^1_{A+(B+C),A\oslash(B+C)};(f\curlyvee(g+h));\delta^L_{A',B',C'}\\
&=(f\oslash(g+h));\iota^1_{A'+(B'+C'),A'\oslash(B'+C')};\delta^L_{A',B',C'}\\
&=(f\oslash(g+h));(1_{A'}\oslash(1_{B'}+t_{C'});u^R_{+B'});\iota^1_{A'+B',A'\oslash B'};\iota^0_{A'\curlyvee B',C'}\\
&=(f\oslash(g+t_C);u^R_{+B'});\iota^1_{A'+B',A'\oslash B'};\iota^0_{A'\curlyvee B',C'}
\end{aligned}$$

Then, by the universal properties of coproducts, the left-hand side composite and right-hand side composite are equal, meaning δ^L is natural.

Second, we show that the right linear distributivity is natural and that this construction will be a symmetric LDC at the same time, by proving (4) holds, i.e.

$$\delta^R_{A,B,C}=\sigma_{\curlyvee A+B,C};(1_C\curlyvee\sigma_{+A,B});\delta^L_{C,B,A};(\sigma_{\curlyvee C,B}+1_A);\sigma_{+B\curlyvee C,A}$$

We consider the right-hand side pre-composed with the appropriate injections.

$$\begin{aligned}
&\iota^0_{A,B};\iota^0_{A+B,C};\iota^0_{(A+B)+C,(A+B)\oslash C};\sigma_{\curlyvee A+B,C};(1_C\curlyvee\sigma_{+A,B});\delta^L_{C,B,A};(\sigma_{\curlyvee C,B}+1_A);\sigma_{+B\curlyvee C,A}\\
&=\iota^0_{A,B};\iota^1_{C,A+B};\iota^0_{C+(A+B),C\oslash(A+B)};(1_C\curlyvee\sigma_{+A,B});\delta^L_{C,B,A};(\sigma_{\curlyvee C,B}+1_A);\sigma_{+B\curlyvee C,A}\\
&=\iota^0_{A,B};\sigma_{+A,B};\iota^1_{C,B+A};\iota^0_{C+(B+A),C\oslash(B+A)};\delta^L_{C,B,A};(\sigma_{\curlyvee C,B}+1_A);\sigma_{+B\curlyvee C,A}\\
&=\iota^1_{B,A};\iota^1_{C,B+A};\iota^0_{C+(B+A),C\oslash(B+A)};\delta^L_{C,B,A};(\sigma_{\curlyvee C,B}+1_A);\sigma_{+B\curlyvee C,A}\\
&=\iota^1_{C\curlyvee B,A};(\sigma_{\curlyvee C,B}+1_A);\sigma_{+B\curlyvee C,A}\\
&=\iota^1_{B\curlyvee C,A};\sigma_{+B\curlyvee C,A}
\end{aligned}$$

$= \iota^0_{A,B\curlyvee C}$

$\iota^1_{A,B};\iota^0_{A+B,C};\iota^0_{(A+B)+C,(A+B)\oslash C};\sigma_{\curlyvee A+B,C};(1_C \curlyvee \sigma_{+A,B});\delta^L_{C,B,A};(\sigma_{\curlyvee C,B}+1_A);\sigma_{+B\curlyvee C,A}$
$= \iota^1_{A,B};\iota^1_{C,A+B};\iota^0_{C+(A+B),C\oslash(A+B)};(1_C \curlyvee \sigma_{+A,B});\delta^L_{C,B,A};(\sigma_{\curlyvee C,B}+1_A);\sigma_{+B\curlyvee C,A}$
$= \iota^1_{A,B};\sigma_{+A,B};\iota^1_{C,B+A};\iota^0_{C+(B+A),C\oslash(B+A)};\delta^L_{C,B,A};(\sigma_{\curlyvee C,B}+1_A);\sigma_{+B\curlyvee C,A}$
$= \iota^1_{C,B};\iota^1_{C,B+A};\iota^0_{C+(B+A),C\oslash(B+A)};\delta^L_{C,B,A};(\sigma_{\curlyvee C,B}+1_A);\sigma_{+B\curlyvee C,A}$
$= \iota^1_{C,B};\iota^0_{C+B,C\oslash B};\iota^0_{C\curlyvee B,A};(\sigma_{\curlyvee C,B}+1_A);\sigma_{+B\curlyvee C,A}$
$= \iota^1_{C,B};\iota^0_{C+B,C\oslash B};\sigma_{\curlyvee C,B};\iota^0_{B\curlyvee C,A};\sigma_{+B\curlyvee C,A}$
$= \iota^0_{B,C};\iota^0_{B+C,B\oslash C};\iota^1_{A,B\curlyvee C}$

$\iota^1_{A+B,C};\iota^0_{(A+B)+C,(A+B)\oslash C};\sigma_{\curlyvee A+B,C};(1_C \curlyvee \sigma_{+A,B});\delta^L_{C,B,A};(\sigma_{\curlyvee C,B}+1_A);\sigma_{+B\curlyvee C,A}$
$= \iota^0_{C,A+B};\iota^0_{C+(A+B),C\oslash(A+B)};(1_C \curlyvee \sigma_{+A,B});\delta^L_{C,B,A};(\sigma_{\curlyvee C,B}+1_A);\sigma_{+B\curlyvee C,A}$
$= \iota^0_{C,B};\iota^0_{C+B,C\oslash B};\iota^0_{C\curlyvee B,A};(\sigma_{\curlyvee C,B}+1_A);\sigma_{+B\curlyvee C,A}$
$= \iota^0_{C,B};\iota^0_{C+B,C\oslash B};\sigma_{\curlyvee C,B};\iota^0_{B\curlyvee C,A};\sigma_{+B\curlyvee C,A}$
$= \iota^1_{B,C};\iota^0_{B+C,B\oslash C};\iota^1_{A,B\curlyvee C}$

$\iota^1_{(A+B)+C,(A+B)\oslash C};\sigma_{\curlyvee A+B,C};(1_C \curlyvee \sigma_{+A,B});\delta^L_{C,B,A};(\sigma_{\curlyvee C,B}+1_A);\sigma_{+B\curlyvee C,A}$
$= \sigma_{\oslash A+B,C};\iota^1_{C+(A+B),C\oslash(A+B)};(1_C \curlyvee \sigma_{+A,B});\delta^L_{C,B,A};(\sigma_{\curlyvee C,B}+1_A);\sigma_{+B\curlyvee C,A}$
$= \sigma_{\oslash A+B,C};(1_C \oslash \sigma_{+A,B});\iota^1_{C+(B+A),C\oslash(B+A)};\delta^L_{C,B,A};(\sigma_{\curlyvee C,B}+1_A);\sigma_{+B\curlyvee C,A}$
$= \sigma_{\oslash A+B,C};(1_C \oslash \sigma_{+A,B});(1_C \oslash (1_B+t_A);u^R_{+B});\iota^1_{C+B,C\oslash B};\iota^0_{C\curlyvee B,A};(\sigma_{\curlyvee C,B}+1_A);\sigma_{+B\curlyvee C,A}$
$= \sigma_{\oslash A+B,C};(1_C \oslash (t_A+1_B);\sigma_{+\mathbf{1},B};u^R_{+B});\iota^1_{C+B,C\oslash B};\sigma_{\curlyvee C,B};\iota^0_{B\curlyvee C,A};\sigma_{+B\curlyvee C,A}$
$= \sigma_{\oslash A+B,C};(1_C \oslash (t_A+1_B);u^L_{+B});\sigma_{\oslash C,B};\iota^1_{B+C,B\oslash C};\iota^1_{A,B\curlyvee C}$
$= ((t_A+1_B);u^L_{+B}\oslash 1_C);\sigma_{\oslash B,C};\sigma_{\oslash C,B};\iota^1_{B+C,B\oslash C};\iota^1_{A,B\curlyvee C}$
$= ((t_A+1_B);u^L_{+B}\oslash 1_C);\iota^1_{B+C,B\oslash C};\iota^1_{A,B\curlyvee C}$

It remains now to show the coherence conditions between units and linear distributivities (1), associativities and linear distributivities (2), and left and right linear distributivities (3).

The first condition from (1) is $u^L_{\curlyvee A+B};\delta^L_{\emptyset,A,B} = u^L_{\curlyvee A}+1_B$ and holds as follows.

$$\begin{aligned}
&u^L_{\curlyvee A+B};\delta^L_{\emptyset,A,B}\\
&= \iota^1_{\emptyset,A+B};\iota^0_{\emptyset+(A+B),\emptyset\oslash(A+B)};\delta^L_{\emptyset,A,B}\\
&= [\iota^1_{\emptyset,A};\iota^0_{\emptyset+A,\emptyset\oslash A};\iota^0_{\emptyset\curlyvee A,B},\iota^1_{\emptyset\curlyvee,B}]\\
&= \iota^1_{\emptyset,A};\iota^0_{\emptyset+A,\emptyset\oslash A}+1_B\\
&= u^L_{\curlyvee A}+1_B
\end{aligned}$$

The second condition from (1) is $\delta^L_{A,B,\emptyset}; u^R_{+A\curlyvee B} = 1_A \curlyvee u^R_{+A}$. It holds by the universal properties of coproducts and pre-composing with the relevant injections.

$$\begin{aligned}
&\iota^0_{A,B+\emptyset}; \iota^0_{A+(B+\emptyset);A\curlyvee(B+\emptyset)}; \delta^L_{A,B,\emptyset}; u^R_{+A\curlyvee B}\\
&= \iota^0_{A,B}; \iota^0_{A+B,A\oslash B}; \iota^0_{A\curlyvee B,\emptyset}; u^R_{+A\curlyvee B}\\
&= \iota^0_{A,B}; \iota^0_{A+B,A\oslash B}
\end{aligned}$$

$$\begin{aligned}
&\iota^0_{B,\emptyset}; \iota^1_{A,B+\emptyset}; \iota^0_{A+(B+\emptyset);A\curlyvee(B+\emptyset)}; \delta^L_{A,B,\emptyset}; u^R_{+A\curlyvee B}\\
&= \iota^1_{A,B}; \iota^0_{A+B,A\oslash B}; \iota^0_{A\curlyvee B,\emptyset}; u^R_{+A\curlyvee B}\\
&= \iota^1_{A,B}; \iota^0_{A+B,A\oslash B}
\end{aligned}$$

$$\begin{aligned}
&\iota^1_{B,\emptyset}; \iota^1_{A,B+\emptyset}; \iota^0_{A+(B+\emptyset);A\curlyvee(B+\emptyset)}; \delta^L_{A,B,\emptyset}; u^R_{+A\curlyvee B}\\
&= \iota^1_{A\curlyvee B,\emptyset}; u^R_{+A\curlyvee B}\\
&= b_{A\curlyvee C}
\end{aligned}$$

$$\begin{aligned}
&\iota^1_{A+(B+\emptyset);A\curlyvee(B+\emptyset)}; \delta^L_{A,B,\emptyset}; u^R_{+A\curlyvee B}\\
&= (1_A \oslash (1_B + t_\emptyset)); (1_A \oslash u^R_{+B}); \iota^1_{A+B,A\oslash B}; \iota^0_{A\curlyvee B,\emptyset}; u^R_{+A\curlyvee B}\\
&= (1_A \oslash u^R_{+B}); \iota^1_{A+B,A\oslash B}
\end{aligned}$$

$$\begin{aligned}
&1_A \curlyvee u^R_{+B}\\
&= (1_A + u^R_{+B}) + (1_A \oslash u^R_{+B})\\
&= [[\iota^0_{A,B}, u^R_{+B}; \iota^1_{A,B}]; \iota^0_{A+B,A\oslash B}, (1_A \oslash u^R_{+B}); \iota^1_{A+B,A\oslash B}]\\
&= [[\iota^0_{A,B}, [1_B, b_B]; \iota^1_{A,B}]; \iota^0_{A+B,A\oslash B}, (1_A \oslash u^R_{+B}); \iota^1_{A+B,A\oslash B}]\\
&= [[\iota^0_{A,B}; \iota^0_{A+B,A\oslash B}, [\iota^1_{A,B}; \iota^0_{A+B,A\oslash B}, b_{A\curlyvee C}]]; , (1_A \oslash u^R_{+B}); \iota^1_{A+B,A\oslash B}]\\
&= \delta^L_{A,B,\emptyset}; u^R_{+A\curlyvee B}
\end{aligned}$$

The third and fourth conditions from (1) then follow by the definition of δ^R as a composite of braidings and δ^L.

The first coherence condition from (2) is

$$\delta^L_{A\curlyvee B,C,D}; (\alpha_{\curlyvee A,B,C} + 1_D) = \alpha_{\curlyvee A,B,C+D}; (1_A \curlyvee \delta^L_{B,C,D}); \delta^L_{A,B\curlyvee C,D}$$

Once more, it holds by the universal properties of coproducts and pre-composing with the relevant injections.

Firstly, the left-hand side:

$$\begin{aligned}
&\iota^0_{A,B}; \iota^0_{A+B,A\oslash B}; \iota^0_{A\curlyvee B,C+D}; \iota^0_{(A\curlyvee B)+(C+D),(A\curlyvee B)\oslash(C+D)}; \delta^L_{A\curlyvee B,C,D}; (\alpha_{\curlyvee A,B,C} + 1_D)\\
&= \iota^0_{A,B}; \iota^0_{A+B,A\oslash B}; \iota^0_{A\curlyvee B,C}; \iota^0_{(A\curlyvee B)+C,(A\curlyvee B)\oslash C}; \iota^0_{(A\curlyvee B)\curlyvee C,D}; (\alpha_{\curlyvee A,B,C} + 1_D)\\
&= \iota^0_{A,B}; \iota^0_{A+B,A\oslash B}; \iota^0_{A\curlyvee B,C}; \iota^0_{(A\curlyvee B)+C,(A\curlyvee B)\oslash C}; \alpha_{\curlyvee A,B,C}; \iota^0_{A\curlyvee(B\curlyvee C),D}\\
&= \iota^0_{A,B\curlyvee C}; \iota^0_{A+(B\curlyvee C),A\oslash(B\curlyvee C)}; \iota^0_{A\curlyvee(B\curlyvee C),D}
\end{aligned}$$

$$
\begin{aligned}
&\iota^1_{A,B};\iota^0_{A+B,A\oslash B};\iota^0_{A\curlyvee B,C+D};\iota^0_{(A\curlyvee B)+(C+D),(A\curlyvee B)\oslash(C+D)};\delta^L_{A\curlyvee B,C,D};(\alpha_{\curlyvee A,B,C}+1_D)\\
&=\iota^1_{A,B};\iota^0_{A+B,A\oslash B};\iota^0_{A\curlyvee B,C};\iota^0_{(A\curlyvee B)+C,(A\curlyvee B)\oslash C};\alpha_{\curlyvee A,B,C};\iota^0_{A\curlyvee(B\curlyvee C),D}\\
&=\iota^0_{B,C};\iota^0_{B+C,B\oslash C};\iota^1_{A,B\curlyvee C};\iota^0_{A+(B\curlyvee C),A\oslash(B\curlyvee C)};\iota^0_{A\curlyvee(B\curlyvee C),D}
\end{aligned}
$$

$$
\begin{aligned}
&\iota^1_{A+B,A\oslash B};\iota^0_{A\curlyvee B,C+D};\iota^0_{(A\curlyvee B)+(C+D),(A\curlyvee B)\oslash(C+D)};\delta^L_{A\curlyvee B,C,D};(\alpha_{\curlyvee A,B,C}+1_D)\\
&=\iota^1_{A+B,A\oslash B};\iota^0_{A\curlyvee B,C};\iota^0_{(A\curlyvee B)+C,(A\curlyvee B)\oslash C};\iota^0_{(A\curlyvee B)\curlyvee C,D};(\alpha_{\curlyvee A,B,C}+1_D)\\
&=\iota^1_{A+B,A\oslash B};\iota^0_{A\curlyvee B,C};\iota^0_{(A\curlyvee B)+C,(A\curlyvee B)\oslash C};\alpha_{\curlyvee A,B,C};\iota^0_{A\curlyvee(B\curlyvee C),D}\\
&=(1_A\oslash\iota^0_{B,C});(1_A\oslash\iota^0_{B+C,B\oslash C});\iota^1_{A+(B\curlyvee C),A\oslash(B\curlyvee C)};\iota^0_{A\curlyvee(B\curlyvee C),D}
\end{aligned}
$$

$$
\begin{aligned}
&\iota^0_{C,D};\iota^1_{A\curlyvee B,C+D};\iota^0_{(A\curlyvee B)+(C+D),(A\curlyvee B)\oslash(C+D)};\delta^L_{A\curlyvee B,C,D};(\alpha_{\curlyvee A,B,C}+1_D)\\
&=\iota^1_{A\curlyvee B,C};\iota^0_{(A\curlyvee B)+C,(A\curlyvee B)\oslash C};\iota^0_{(A\curlyvee B)\curlyvee C,D};(\alpha_{\curlyvee A,B,C}+1_D)\\
&=\iota^1_{A\curlyvee B,C};\iota^0_{(A\curlyvee B)+C,(A\curlyvee B)\oslash C};\alpha_{\curlyvee A,B,C};\iota^0_{A\curlyvee(B\curlyvee C),D}\\
&=\iota^1_{B,C};\iota^0_{B+C,B\oslash C};\iota^1_{A,B\curlyvee C};\iota^0_{A+(B\curlyvee C),A\oslash(B\curlyvee C)};\iota^0_{A\curlyvee(B\curlyvee C),D}
\end{aligned}
$$

$$
\begin{aligned}
&\iota^1_{C,D};\iota^1_{A\curlyvee B,C+D};\iota^0_{(A\curlyvee B)+(C+D),(A\curlyvee B)\oslash(C+D)};\delta^L_{A\curlyvee B,C,D};(\alpha_{\curlyvee A,B,C}+1_D)\\
&=\iota^1_{(A\curlyvee B)\curlyvee C,D};(\alpha_{\curlyvee A,B,C}+1_D)\\
&=\iota^1_{A\curlyvee(B\curlyvee C),D}
\end{aligned}
$$

$$
\begin{aligned}
&\iota^1_{(A\curlyvee B)+C,(A\curlyvee B)\oslash C};\delta^L_{A\curlyvee B,C,D};(\alpha_{\curlyvee A,B,C}+1_D)\\
&=(1_{A\curlyvee B}\oslash(1_C+t_D));(1_{A\curlyvee B}\oslash u^R_{+C});\iota^1_{(A\curlyvee B)+C,(A\curlyvee B)\oslash C};\iota^0_{(A\curlyvee B)\curlyvee C,D};(\alpha_{\curlyvee A,B,C}+1_D)\\
&=(1_{A\curlyvee B}\oslash(1_C+t_D));(1_{A\curlyvee B}\oslash u^R_{+C});\iota^1_{(A\curlyvee B)+C,(A\curlyvee B)\oslash C};\alpha_{\curlyvee A,B,C};\iota^0_{A\curlyvee(B\curlyvee C),D}\\
&=(1_{A\curlyvee B}\oslash(1_C+t_D));(1_{A\curlyvee B}\oslash u^R_{+C});d^{R^{-1}}_{A+B,A\oslash B,C};(d^{R^{-1}}_{A,B,D}+1_{(A\oslash B)\oslash C});\chi_{A,B,C};\iota^0_{A\curlyvee(B\curlyvee C),D}\\
&=d^{R^{-1}}_{A+B,A\oslash B,C+D};(d^{R^{-1}}_{A,B,C+D}+1_{(A\oslash B)\oslash(C+D)});\\
&\quad(((1_A\oslash(1_C+t_D);u^R_{+C})+(1_B\oslash(1_C+t_D);u^R_{+C}))+(1_{A\oslash B}\oslash(1_C+t_D);u^R_{+C}));\chi_{A,B,C};\iota^0_{A\curlyvee(B\curlyvee C),D}
\end{aligned}
$$

Secondly, the right-hand side:

$$
\begin{aligned}
&\iota^0_{A,B};\iota^0_{A+B,A\oslash B};\iota^0_{A\curlyvee B,C+D};\iota^0_{(A\curlyvee B)+(C+D),(A\curlyvee B)\oslash(C+D)};\alpha_{\curlyvee A,B,C+D};(1_A\curlyvee\delta^L_{B,C,D});\delta^L_{A,B\curlyvee C,D}\\
&=\iota^0_{A,B\curlyvee(C+D)};\iota^0_{A+(B\curlyvee(C+D)),A\oslash(B\curlyvee(C+D))};(1_A\curlyvee\delta^L_{B,C,D});\delta^L_{A,B\curlyvee C,D}\\
&=\iota^0_{A,(B\curlyvee C)+D};\iota^0_{A+((B\curlyvee C)+D),A\oslash((B\curlyvee C)+D)};\delta^L_{A,B\curlyvee C,D}\\
&=\iota^0_{A,B\curlyvee C};\iota^0_{A+(B\curlyvee C),A\oslash(B\curlyvee C)};\iota^0_{A\curlyvee(B\curlyvee C),D}
\end{aligned}
$$

$$
\iota^1_{A,B};\iota^0_{A+B,A\oslash B};\iota^0_{A\curlyvee B,C+D};\iota^0_{(A\curlyvee B)+(C+D),(A\curlyvee B)\oslash(C+D)};\alpha_{\curlyvee A,B,C+D};(1_A\curlyvee\delta^L_{B,C,D});\delta^L_{A,B\curlyvee C,D}
$$

$= \iota^0_{B,C+D}; \iota^0_{B+(C+D),B\oslash(C+D)}; \iota^1_{A,B\curlyvee(C+D)}; \iota^0_{A+(B\curlyvee(C+D)),A\oslash(B\curlyvee(C+D))}; (1_A \curlyvee \delta^L_{B,C,D}); \delta^L_{A,B\curlyvee C,D}$
$= \iota^0_{B,C+D}; \iota^0_{B+(C+D),B\oslash(C+D)}; \delta^L_{B,C,D}; \iota^1_{A,(B\curlyvee C)+D}; \iota^0_{A+((B\curlyvee C)+D),A\oslash((B\curlyvee C)+D)}; \delta^L_{A,B\curlyvee C,D}$
$= \iota^0_{B,C}; \iota^0_{B+C,B\oslash C}; \iota^0_{B\curlyvee C,D}; \iota^1_{A,(B\curlyvee C)+D}; \iota^0_{A+((B\curlyvee C)+D),A\oslash((B\curlyvee C)+D)}; \delta^L_{A,B\curlyvee C,D}$
$= \iota^0_{B,C}; \iota^0_{B+C,B\oslash C}; \iota^1_{A,B\curlyvee C}; \iota^1_{A,B\curlyvee C}; \iota^0_{A+(B\curlyvee C),A\oslash(B\curlyvee C)}; \iota^0_{A\curlyvee(B\curlyvee C),D}$

$\iota^1_{A+B,A\oslash B}; \iota^0_{A\curlyvee B,C+D}; \iota^0_{(A\curlyvee B)+(C+D),(A\curlyvee B)\oslash(C+D)}; \alpha_{\curlyvee A,B,C+D}; (1_A \curlyvee \delta^L_{B,C,D}); \delta^L_{A,B\curlyvee C,D}$
$= (1_A \oslash \iota^0_{B,C+D}); (1_A \oslash \iota^0_{B+(C+D),B\oslash(C+D)}); \iota^1_{A+(B\curlyvee(C+D)),A\oslash(B\curlyvee(C+D))}; (1_A \curlyvee \delta^L_{B,C,D}); \delta^L_{A,B\curlyvee C,D}$
$= (1_A \oslash \iota^0_{B,C+D}; \iota^0_{B+(C+D),B\oslash(C+D)}; \delta^L_{B,CD}); \iota^1_{A+((B\curlyvee C)+D),A\oslash((B\curlyvee C)+D)}; \delta^L_{A,B\curlyvee C,D}$
$= (1_A \oslash \iota^0_{B,C}; \iota^0_{B+C,B\oslash C}; \iota^0_{B\curlyvee C,D}); (1_A \oslash (1_{B\curlyvee C} + t_D); u^R_{+B\curlyvee C}); \iota^1_{A+(B\curlyvee C),A\oslash(B\curlyvee C)}; \iota^0_{A\curlyvee(B\curlyvee C),D}$
$= (1_A \oslash \iota^0_{B,C}; \iota^0_{B+C,B\oslash C}); (1_A \oslash \iota^0_{B\curlyvee C,\emptyset}); (1_A \oslash u^R_{+B\curlyvee C}); \iota^1_{A+(B\curlyvee C),A\oslash(B\curlyvee C)}; \iota^0_{A\curlyvee(B\curlyvee C),D}$
$= (1_A \oslash \iota^0_{B,C}); (1_A \oslash \iota^0_{B+C,B\oslash C}); \iota^1_{A+(B\curlyvee C),A\oslash(B\curlyvee C)}; \iota^0_{A\curlyvee(B\curlyvee C),D}$

$\iota^0_{C,D}; \iota^1_{A\curlyvee B,C+D}; \iota^0_{(A\curlyvee B)+(C+D),(A\curlyvee B)\oslash(C+D)}; \alpha_{\curlyvee A,B,C+D}; (1_A \curlyvee \delta^L_{B,C,D}); \delta^L_{A,B\curlyvee C,D}$
$= \iota^0_{C,D}; \iota^1_{B,C+D}; \iota^0_{B+(C+D),B\oslash(C+D)}; \iota^1_{A,B\curlyvee(C+D)}; \iota^0_{A+(B\curlyvee(C+D)),A\oslash(B\curlyvee(C+D))}; (1_A \curlyvee \delta^L_{B,C,D}); \delta^L_{A,B\curlyvee C,D}$
$= \iota^0_{C,D}; \iota^1_{B,C+D}; \iota^0_{B+(C+D),B\oslash(C+D)}; \delta^L_{B,C,D}; \iota^1_{A,(B\curlyvee C)+D}; \iota^0_{A+((B\curlyvee C)+D),A\oslash((B\curlyvee C)+D)}; \delta^L_{A,B\curlyvee C,D}$
$= \iota^1_{B,C}; \iota^0_{B+C,B\oslash C}; \iota^0_{B\curlyvee C,D}; \iota^1_{A,(B\curlyvee C)+D}; \iota^0_{A+((B\curlyvee C)+D),A\oslash((B\curlyvee C)+D)}; \delta^L_{A,B\curlyvee C,D}$
$= \iota^1_{B,C}; \iota^0_{B+C,B\oslash C}; \iota^1_{A,B\curlyvee C}; \iota^0_{A+(B\curlyvee C),A\oslash(B\curlyvee C)}; \iota^0_{A\curlyvee(B\curlyvee C),D}$

$\iota^1_{C,D}; \iota^1_{A\curlyvee B,C+D}; \iota^0_{(A\curlyvee B)+(C+D),(A\curlyvee B)\oslash(C+D)}; \alpha_{\curlyvee A,B,C+D}; (1_A \curlyvee \delta^L_{B,C,D}); \delta^L_{A,B\curlyvee C,D}$
$= \iota^1_{C,D}; \iota^1_{B,C+D}; \iota^0_{B+(C+D),B\oslash(C+D)}; \iota^1_{A,B\curlyvee(C+D)}; \iota^0_{A+(B\curlyvee(C+D)),A\oslash(B\curlyvee(C+D))}; (1_A \curlyvee \delta^L_{B,C,D}); \delta^L_{A,B\curlyvee C,D}$
$= \iota^1_{C,D}; \iota^1_{B,C+D}; \iota^0_{B+(C+D),B\oslash(C+D)}; \delta^L_{B,C,D}; \iota^1_{A,(B\curlyvee C)+D}; \iota^0_{A+((B\curlyvee C)+D),A\oslash((B\curlyvee C)+D)}; \delta^L_{A,B\curlyvee C,D}$
$= \iota^1_{B\curlyvee C,D}; \iota^1_{A,(B\curlyvee C)+D}; \iota^0_{A+((B\curlyvee C)+D),A\oslash((B\curlyvee C)+D)}; \delta^L_{A,B\curlyvee C,D}$
$= \iota^1_{A\curlyvee(B\curlyvee C),D}$

$\iota^1_{(A\curlyvee B)+(C+D),(A\curlyvee B)\oslash(C+D)}; \alpha_{\curlyvee A,B,C+D}; (1_A \curlyvee \delta^L_{B,C,D}); \delta^L_{A,B\curlyvee C,D}$
$= {d^R}^{-1}_{A+B,A\oslash B,C+D}; ({d^R}^{-1}_{A,B,C+D} + 1_{(A\oslash B)\oslash(C+D)}); \chi_{A,B,C+D}; (1_A \curlyvee \delta^L_{B,C,D}); \delta^L_{A,B\curlyvee C,D}$

It remains to show that
$(((1_A \oslash (1_C + t_D); u^R_{+C}) + (1_B \oslash (1_C + t_D); u^R_{+C})) + (1_{A\oslash B} \oslash (1_C + t_D); u^R_{+C})); \chi_{A,B,C}; \iota^0_{A\curlyvee(B\curlyvee C),D}$
$= \chi_{A,B,C+D}; (1_A \curlyvee \delta^L_{B,C,D}); \delta^L_{A,B\curlyvee C,D}$
which we show by pre-composing by injections once more.

$\iota^0_{A,\oslash(C+D),B\oslash(C+D)}; \iota^0_{(A\oslash(C+D))+(B\oslash(C+D)),(A\oslash B)+(C+D)}; LHS$

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$$= (1_A \oslash (1_C + t_D); u^R_{+C}); \iota^0_{A\oslash C,B\oslash C}; \iota^0_{(A\oslash C)+(B\oslash C),(A\oslash B)+C}; \chi_{A,B,C}; \iota^0_{A\curlyvee(B\curlyvee C),D}$$
$$= (1_A \oslash (1_C + t_D); u^R_{+C}); (1_A \oslash \iota^1_{B,C}); (1_A \oslash \iota^0_{B+C,B\oslash C}); \iota^1_{A+(B\curlyvee C),A\oslash(B\curlyvee C)}; \iota^0_{A\curlyvee(B\curlyvee C),D}$$
$$= (1_A \oslash [\iota^0_{C,\emptyset}, t_D; \iota^1_{C,\emptyset}]; u^R_{+C}); (1_A \oslash \iota^1_{B,C}); (1_A \oslash \iota^0_{B+C,B\oslash C}); \iota^1_{A+(B\curlyvee C),A\oslash(B\curlyvee C)}; \iota^0_{A\curlyvee(B\curlyvee C),D}$$
$$= (1_A \oslash [\iota^1_{B,C}; \iota^0_{B+C,B\oslash C}, t_D]); \iota^1_{A+(B\curlyvee C),A\oslash(B\curlyvee C)}; \iota^0_{A\curlyvee(B\curlyvee C),D}$$

$$\iota^1_{A,\oslash(C+D),B\oslash(C+D)}; \iota^0_{(A\oslash(C+D))+(B\oslash(C+D)),(A\oslash B)+(C+D)}; LHS$$
$$= (1_B \oslash (1_C + t_D); u^R_{+C}); \iota^1_{A\oslash C,B\oslash C}; \iota^0_{(A\oslash C)+(B\oslash C),(A\oslash B)+C}; \chi_{A,B,C}; \iota^0_{A\curlyvee(B\curlyvee C),D}$$
$$= (1_B \oslash (1_C + t_D); u^R_{+C}); \iota^1_{B+C,B\oslash C}; \iota^1_{A,B\curlyvee C}; \iota^0_{A+(B\curlyvee C),A\oslash(B\curlyvee C)}; \iota^0_{A\curlyvee(B\curlyvee C),D}$$

$$\iota^1_{(A\oslash(C+D))+(B\oslash(C+D)),(A\oslash B)+(C+D)}; LHS$$
$$= (1_{A\oslash B} \oslash (1_C + t_D); u^R_{+C}); \iota^1_{(A\oslash C)+(B\oslash C),(A\oslash B)+C}; \chi_{A,B,C}; \iota^0_{A\curlyvee(B\curlyvee C),D}$$
$$= (1_{A\oslash B} \oslash (1_C + t_D); u^R_{+C}); \alpha_{\oslash A,B,C}; (1_A \oslash \iota^1_{B+C,B\oslash C}); \iota^1_{A+(B\curlyvee C),A\oslash(B\curlyvee C)}; \iota^0_{A\curlyvee(B\curlyvee C),D}$$

$$\iota^0_{A,\oslash(C+D),B\oslash(C+D)}; \iota^0_{(A\oslash(C+D))+(B\oslash(C+D)),(A\oslash B)+(C+D)}; RHS$$
$$= (1_A \oslash \iota^1_{B,C+D}; \iota^0_{B+(C+D),B\oslash(C+D)}); \iota^1_{A+(B\curlyvee(C+D)),A\oslash(B\curlyvee(C+D))}; (1_A \curlyvee \delta^L_{B,C,D}); \delta^L_{A,B\curlyvee C,D}$$
$$= (1_A \oslash \iota^1_{B,C+D}; \iota^0_{B+(C+D),B\oslash(C+D)}; \delta^L_{B,C,D}); \iota^1_{A+((B\curlyvee C)+D),A\oslash((B\curlyvee C)+D)}; \delta^L_{A,B\curlyvee C,D}$$
$$= (1_A \oslash [\iota^1_{B,C}; \iota^0_{B+C,B\oslash C}; \iota^0_{B\curlyvee C,D}, \iota^1_{B\curlyvee C,D}]); (1_A \oslash (1_{B\curlyvee C} + t_D); u^R_{+B\curlyvee C}); \iota^1_{A+(B\curlyvee C),A\oslash(B\curlyvee C)}; \iota^0_{A\curlyvee(B\curlyvee C),D}$$
$$= (1_A \oslash [\iota^1_{B,C}; \iota^0_{B+C,B\oslash C}, t_D]); \iota^1_{A+(B\curlyvee C),A\oslash(B\curlyvee C)}; \iota^0_{A\curlyvee(B\curlyvee C),D}$$

$$\iota^1_{A,\oslash(C+D),B\oslash(C+D)}; \iota^0_{(A\oslash(C+D))+(B\oslash(C+D)),(A\oslash B)+(C+D)}; RHS$$
$$= \iota^1_{B+(C+D),B\oslash(C+D)}; \iota^1_{A,B\curlyvee(C+D)}; \iota^0_{A+(B\curlyvee(C+D)),A\oslash(B\curlyvee(C+D))}; (1_A \curlyvee \delta^L_{B,C,D}); \delta^L_{A,B\curlyvee C,D}$$
$$= \iota^1_{B+(C+D),B\oslash(C+D)}; \delta^L_{B,C,D}; \iota^1_{A,(B\curlyvee C)+D}; \iota^0_{A+((B\curlyvee C)+D),A\oslash((B\curlyvee C)+D)}; \delta^L_{A,B\curlyvee C,D}$$
$$= (1_B \oslash (1_C + t_D); u^R_{+C}); \iota^1_{B+C,B\oslash C}; \iota^0_{B\curlyvee C,D}; \iota^1_{A,(B\curlyvee C)+D}; \iota^0_{A+((B\curlyvee C)+D),A\oslash((B\curlyvee C)+D)}; \delta^L_{A,B\curlyvee C,D}$$
$$= (1_B \oslash (1_C + t_D); u^R_{+C}); \iota^1_{B+C,B\oslash C}; \iota^1_{A,B\curlyvee C}; \iota^0_{A+(B\curlyvee C),A\oslash(B\curlyvee C)}; \iota^0_{A\curlyvee(B\curlyvee C),D}$$

$$\iota^1_{(A\oslash(C+D))+(B\oslash(C+D)),(A\oslash B)+(C+D)}; RHS$$
$$= \alpha_{\oslash A,B,C+D}; (1_A \oslash \iota^1_{B+(C+D),B\oslash(C+D)}); \iota^1_{A+(B\curlyvee(C+D)),A\oslash(B\curlyvee(C+D))}; (1_A \curlyvee \delta^L_{B,C,D}); \delta^L_{A,B\curlyvee C,D}$$
$$= \alpha_{\oslash A,B,C+D}; (1_A \oslash \iota^1_{B+(C+D),B\oslash(C+D)}; \delta^L_{B,C,D}); \iota^1_{A+((B\curlyvee C)+D),A\oslash((B\curlyvee C)+D)}; \delta^L_{A,B\curlyvee C,D}$$
$$= \alpha_{\oslash A,B,C+D}; (1_A \oslash (1_B \oslash (1_C + t_D); u^R_{+C})); (1_A \oslash \iota^1_{B+C,B\oslash C}; \iota^0_{B\curlyvee C,D});$$
$$(1_A \oslash (1_{B\curlyvee C} + t_D); u^R_{+B\curlyvee C}); \iota^1_{A+(B\curlyvee C),A\oslash(B\curlyvee C)}; \iota^0_{A\curlyvee(B\curlyvee C),D}$$
$$= (1_{A\oslash B} \oslash (1_C + t_D); u^R_{+C}); \alpha_{\oslash A,B,C}; (1_A \oslash \iota^1_{B+C,B\oslash C}; \iota^0_{B\curlyvee C,D});$$
$$(1_A \oslash (1_{B\curlyvee C} + t_D); u^R_{+B\curlyvee C}); \iota^1_{A+(B\curlyvee C),A\oslash(B\curlyvee C)}; \iota^0_{A\curlyvee(B\curlyvee C),D}$$

$= (1_{A \oslash B} \oslash (1_C + t_D)); u^R_{+C}); \alpha_{\oslash A,B,C}; (1_A \oslash \iota^1_{B+C,B \oslash C}); \iota^1_{A+(B \curlyvee C),A \oslash (B \curlyvee C)}; \iota^0_{A \curlyvee (B \curlyvee C),D}$
This completes the proof of the first coherence condition from (2).

The remaining coherence conditions of (2) and (3) hold by similar calculations. They are not included simply because of their length and repetitive nature. □

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Chapter 4

Constructing Linear Bicategories

The content of this chapter consists of a paper co-authored by Richard Blute, Rose Kudzman-Blais and Susan Niefield, submitted to a special edition of the journal *Mathematical Structures in Computer Science* in honor of Philip Scott. The journal publishes work in mathematical fields, including category theory, which advances the fields of theoretical and applied computer science. The co-authors each contribute equally to the mathematical content of the article. The main contributions of the second author are the notion of the linear quantaloid, the examples of LD-quantales, except the shifted monoids, the examples of categories of quantale-valued relations for LD-quantales, and Section 6, the theory of enriching in a linear quantaloid. All authors contributed equally to the redaction of the article, although the editing process was mostly undertaken by the second author.

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ARTICLE

Constructing linear bicategories

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Abstract

Linearly distributive categories were introduced to model the tensor/par fragment of linear logic, without resorting to the use of negation. *Linear bicategories* are the bicategorical version of linearly distributive categories. Essentially, a linear bicategory has two forms of composition, each determining the structure of a bicategory, and the two compositions are related by a linear distribution. After the initial paper on the subject, there was little further work as there seemed to be a lack of examples. The main goal of this paper is to demonstrate that there are in fact a great many examples, which are obtained by considering quantales and quantaloids, and by extending familiar constructions from the (ordinary) bicategorical setting. It is standard in the field of *monoidal topology* that the category of quantale-valued relations is a bicategory. Here we begin by showing that a quantale is *Girard* if and only if the corresponding bicategory is a Girard quantaloid, which is an example of linear bicategory. The *tropical* and *arctic semiring* structures fit together into a Girard quantale, so this construction is likely to have multiple applications. More generally, we define LD-quantales, which are sup-lattices with two quantale structures related by a linear distribution, and their bicategorical analogue, linear quantaloids. We show that Q-Rel is a linear quantaloid if and only if Q is an LD-quantale. We then consider several standard constructions from enriched bicategory theory, and show that these lift to the linear quantaloid setting and produce new examples of linear bicategories. In particular, we consider linear $\mathcal{Q}$-categories, matrices in $\mathcal{Q}$ and linear monads in $\mathcal{Q}$, where $\mathcal{Q}$ is a linear quantaloid. We develop non-locally posetal examples as well, $\mathcal{Q}$uant, the bicategory of quantales, modules and module homomorphisms, and $\mathcal{Q}$tld, the bicategory of quantaloids, modules and module homomorphisms. These turn out to be cyclic $*$-autonomous bicategories, which are in essence a closed version of linear bicategories.

Keywords: Categorical linear logic, linear bicategories, $*$-autonomous, quantales, quantaloids

Dedicated to the memory of our friend Phil Scott

1. Introduction

The idea behind the theory of linearly distributive categories (LDC) as introduced by Cockett and Seely (1997) is that (the multiplicative fragment of) linear logic is best modeled by taking the two multiplicative connectives, $\otimes$ (tensor) and $\oplus$ (par), as primitive. One then obtains a category with two monoidal structures, related by *linear distributions*. A linear distribution is a natural transformation, not an isomorphism typically, of the following form or a symmetric equivalent:

$$A \otimes (B \oplus C) \to (A \otimes B) \oplus C$$

This approach to the model theory of linear logic differs from the original approach, using the $*$-autonomous categories of Barr (1979), of taking tensor and negation as primitive and then defining the par by de Morgan duality.

Just as monoidal categories can be viewed as one-object bicategories, one can ask for the bicategorical version of linearly distributive categories. These are the *linear bicategories* of Cockett et al. (2000), which provide a natural semantics for non-commutative linear logic. The primary goal of this paper is to give new classes of linear bicategories arising from several different sources.

Before diving into the formal definitions, we describe the example that led to the more general discussion below. The structures we consider are two ordered semiring structures on the extended integers $\mathbb{Z}_\infty = \mathbb{Z} \cup \{+\infty, -\infty\}$, as examined by Golan (2003). (One could just as well consider the extended reals.) This set in fact has two semiring structures and these are typically called the *tropical* and *arctic* semirings. They are of great use in the theory of synchronization as considered in Baccelli et al. (1992). See Droste and Kuich (2009); Droste et al. (2009) for how extensively these structures arise.

In both, multiplication is given by the usual addition of integers. But we must be careful in defining $\infty + -\infty$. In the first structure, we define $-\infty +_1 \infty = -\infty = \infty +_1 -\infty$. The addition for this structure is given by max. This gives $\mathbb{Z}_\infty$ the structure of an ordered semiring with $\mathbb{Z}_\infty$ equipped with its usual order.

For the second structure, we again have that the multiplication is given by addition. But analogously we now must have $-\infty +_2 \infty = \infty = \infty +_2 -\infty$. The addition for this structure is given by min. This gives $\mathbb{Z}_\infty$ the structure of an ordered semiring with $\mathbb{Z}_\infty$ equipped with the opposite of its usual order.

If X and Y are sets, we will define a $\mathbb{Z}_\infty$-*relation* from X to Y to be a function $R\colon X \times Y \to \mathbb{Z}_\infty$. As in the category of relations, we consider this as a morphism $R\colon X \nrightarrow Y$. Then the two semiring constructions above allow us to define two distinct relational compositions.

Explicitly, given $X \overset{A}{\nrightarrow} Y \overset{B}{\nrightarrow} Z$, we define

$$A \otimes B(x,z) = \bigvee_{y \in Y} (A(x,y) +_1 B(y,z)) \qquad\qquad A \oplus B(x,z) = \bigwedge_{y \in Y} (A(x,y) +_2 B(y,z))$$

We are of course using the fact that both $\mathbb{Z}_\infty$ and $\mathbb{Z}_\infty^{op}$ are not just semirings with the above structures, but are in fact *quantales*. We are thus following the program of *monoidal topology* as described in Hofmann et al. (2014).

The current project began with the observation that the two compositions described above are related by a *linear distribution* and in fact determine a locally posetal linear bicategory. On the other hand, we have the observation that $\mathbb{Z}_\infty$ with the above operations is a Girard quantale, as investigated by Yetter (1990); Rosenthal (1990) and the two structures are related by the Girard duality.

We first introduce the quantale analogue of linearly distributive categories, and the quantaloidal analogue of linear bicategories, in Section 4, which we call LD-*quantales* and *linear quantaloids* respectively. These definitions will underlie most of our main results. All Girard quantales are LD-quantales and all Girard quantaloids are linear quantaloids.

Our first result in Section 5 is that the category Q-Rel is a Girard quantaloid, defined by Rosenthal (1992), and therefore a locally posetal linear bicategory, if and only if Q is a Girard quantale. This extends naturally to the case where Q is an arbitrary LD-quantale. We then provide concrete examples of Q-Rel as a linear quantaloid.

In Section 6, following the general theory of enriching in a bicategory and the work of Rosenthal (1992), we introduce the quantaloid $\mathcal{Q}$-Mod whose 0-cells are $\mathcal{Q}$-categories, 1-cells are $\mathcal{Q}$-modules and 2-cells are point-wise inequalities, where $\mathcal{Q}$ is a Girard quantaloid. $\mathcal{Q}$-Mod is then itself a Girard quantaloid. This leads to considering enrichment in a linear quantaloid $\mathcal{Q}$ and the introduction of the bicategory $\mathcal{Q}$-Mod of linear $\mathcal{Q}$-categories and linear $\mathcal{Q}$-modules. It is shown to be a linear quantaloid if and only if $\mathcal{Q}$ is itself linear. This result proceeds by first proving the corresponding theorems for linear monads in $\mathcal{Q}$ and matrices in $\mathcal{Q}$. Given these new constructions, we provide more examples of locally posetal linear bicategories, using the linear quantaloids presented in the previous section.

We finally develop non-locally posetal examples as well. This is done in Section 7. We begin by considering $\mathcal{L}$oc, the bicategory whose objects are locales, 1-cells are bimodules and two-cells are bimodule homomorphisms, which we use to illustrate a more general notion. This turns out to be whatCockett et al. (2000) refer to as cyclic $*$-autonomous bicategories, which are linear bicategories. We show that a number of classic examples of bicategories fit into this framework. In particular, the bicategories of quantales and of quantaloids (with their respective modules) are linear bicategories.

Remark 1.1. There are unfortunate notational conflicts between linear logic notation and the usual notation for ordered structures, as well as within the linear logic community. We now give our choice for notation for the remainder of the paper, chosen to be in line with the notation of Cockett and Seely (1997) and Cockett et al. (2000).

- For partially ordered sets, we will denote the top element by **1** and the bottom element by **0**, if they exist.
- We will denote quantales by Q and quantaloids by $\mathcal{Q}$.
- Composition of arrows will be written with diagrammatic ordering.
- In a monoidal category, we will use $\otimes$ to denote the tensor product and $\top$ to denote the unit, including in the case of quantales, as opposed to & and 1 used in Rosenthal (1990). Moreover, we will use $\otimes$ to denote composition in a bicategory and $\top_X$ or $\mathbb{T}_X$ to denote the identity 1-cell on X, in particular for quantaloids, as opposed to $\circ$ and i_X used in Rosenthal (1996).
- In a $*$-autonomous or linearly distributive category, there is a second monoidal structure, which we will denote by $\oplus$, rather than the $⅋$ of Girard (1987). This includes in the case of Girard quantales, as opposed to $\sqcup$ and d in Rosenthal (1990) and LD-quantales. The unit of this second monoidal product will be denoted by $\bot$. $\oplus$ will also denote the second composition in the context of linear bicategories, in particular in Girard or linear quantaloids, and $\bot_X$ or $\perp\!\!\!\perp_X$ will be the identity 1-cell on X. Note that $\bot$, $\bot_X$ and $\perp\!\!\!\perp_X$ will also be used to denoted cyclic dualizing elements and families in Girard quantales and quantaloids.

2. Preliminaries

2.1 Quantales and quantaloids

See Niefield and Rosenthal (1988); Rosenthal (1990, 1996) for a detailed discussion about quantales and quantaloids.

Definition 2.1. • A *quantale* (coined by Mulvey (1986)) is a partially ordered set Q with all suprema and an associative multiplication $\otimes\colon Q\times Q\to Q$ such that for all subsets $P\subseteq Q$ and all elements $a\in Q$, we have

$$\big(\bigvee P\big)\otimes a=\bigvee_{p\in P} p\otimes a \qquad \text{and} \qquad a\otimes\big(\bigvee P\big)=\bigvee_{p\in P} a\otimes p$$

Note that Q necessarily satisfies $a\otimes\mathbf{0}=\mathbf{0}=\mathbf{0}\otimes a$.

- Since the operations $(-)\otimes a$ and $a\otimes(-)$ preserve all sups, they have right adjoints for all $a\in Q$, known as the left and right *residuations*. We denote them by $(-)\circ\!\!-a$ and $a\!-\!\!\circ(-)$ respectively and they are defined by:

$$c\circ\!\!-a=\bigvee\{b\in Q\mid b\otimes a\le c\} \qquad \text{and} \qquad a\multimap c=\bigvee\{b\in Q\mid a\otimes b\le c\}$$

- An element $\top\in Q$ is a *unit* if for all $a\in Q$,

$$\top\otimes a=a\otimes\top=a$$

in which case Q is called *unital*.

The following definitions and result are due to Yetter (1990):

- An element $\bot\in Q$ is a *cyclic dualizing element* if for all $a\in Q$, we have

$$\bot\circ\!\!-a=a\multimap\bot \qquad \text{and} \qquad (a\multimap\bot)\multimap\bot=a$$

A *Girard quantale* is a pair $(Q,\bot)$ where Q is a quantale and $\bot$ is a chosen cyclic dualizing element. We denote $a\multimap\bot$ as $a^{\bot}$.

- If Q is a Girard quantale, it has a second multiplication defined by the linear logic version of de Morgan duality:

Lemma 2.2. Let $(Q,\bot)$ be a Girard quantale, then it is unital with $\top=\bot^{\bot}$ and the operation $(-)^{\bot}$ is a contravariant isomorphism. Q^{op} is thus a unital quantale with multiplication

$$a\oplus b=(b^{\bot}\otimes a^{\bot})^{\bot}$$

and unit $\bot$. Evidently this operation satisfies:

$$(\bigwedge P)\oplus a=\bigwedge_{p\in P} p\oplus a \qquad \text{and} \qquad a\oplus(\bigwedge P)=\bigwedge_{p\in P} a\oplus p$$

Example 2.3. (1) Every locale (or frame) L, i.e., complete lattice satisfying the infinite distributive law, $a\wedge(\bigvee b_i)=\bigvee(a\wedge b_i)\ \forall a,b_i\in L$, is a unital quantale with $\otimes=\wedge$ and $\top=1$. As discussed by Niefield and Rosenthal (1988), a quantale is a locale if and only if it is commutative, right-sided and idempotent. Following is a list of important locales we will consider in this paper:

a. Truth value two-chain $\Omega=\{\mathbf{0},\mathbf{1}\}$

b. Totally ordered 3-chain $3=\{\mathbf{0},1/2,\mathbf{1}\}$, with residuations given by

$$c\circ\!\!-a=\begin{cases}\mathbf{1} & \text{if } a\le c\\ c & \text{if } a>c\end{cases}$$

c. Extended real half-line with opposite ordering $\mathsf{P}_{\max}=([0,\infty]^{op},\max,0)$, with residuations defined by Lawvere (1973) to be

$$c\circ\!\!-a=\begin{cases}c & \text{if } a<c\\ 0 & \text{if } a\ge c\end{cases}$$

d. Lattice of open sets $\mathscr{O}(X)$ in a topological space X

(2) The set of relations $\mathsf{Rel}(X)$ on a set X is a unital quantale with standard relational composition as its operation and with the diagonal relation Δ_X as its unit, i.e., given relations $R, S\colon X \nrightarrow X$,

$$(x, x'') \in R \otimes S \quad \text{if and only if} \quad \exists x' \quad (x, x') \in R \ \text{ and } \ (x', x'') \in S$$

(3) The extended real half-line with opposite ordering can be equipped with other quantale structures. In particular, consider $\mathsf{P}_+ = ([0, \infty]^{op}, +, 0)$, with its operation standard addition extended by $a + \infty = \infty + a = \infty$. It is often called Lawvere's quantale of positive real numbers, as it was first introduced by Lawvere (1973). Residuation is given by "truncated subtraction":

$$c \circ\!\!- a = \begin{cases} c - a & \text{if } a \leq c < \infty \\ 0 & \text{if } a \geq c \\ \infty & \text{if } a < c = \infty \end{cases}$$

(4) The unit interval with multiplication $([0,1], \cdot, 1)$ is isomorphic to $\mathsf{P}_+ = ([0, \infty]^{op}, +, 0)$ under the map $x \mapsto -\ln(x)$, a unital homomorphism of quantales (function preserving arbitrary sups, quantale operation $\otimes$ and the unit $\top$) with inverse $y \mapsto \exp(-y)$. Residuations are given "truncated division", as defined by Hofmann and Reis (2013):

$$c \circ\!\!- a = \begin{cases} c/a & \text{if } 0 \neq a > c \\ 1 & \text{otherwise} \end{cases}$$

Definition 2.4.

- A *quantaloid*, coined by Abramsky and Vickers (1993), is a category $\mathscr{Q}$ enriched over the category of complete lattices and sup-preserving maps.
- As in the case of quantales, for all arrows $f\colon a \to b \in \mathscr{Q}$, the functors $(-) \otimes f\colon \mathscr{Q}(a', a) \to \mathscr{Q}(a', b)$ and $f \otimes (-)\colon \mathscr{Q}(b, b') \to \mathscr{Q}(a, b')$ preserve all sups and thus have right adjoints, also known as residuations, denoted by $(-) \circ\!\!- f\colon \mathscr{Q}(a', b) \to \mathscr{Q}(a', a)$ and $f \multimap (-)\colon \mathscr{Q}(a, b') \to \mathscr{Q}(b, b')$ respectively.

  The following definitions and result are due to Rosenthal (1992):
- A family of 1-cells $\mathscr{D} = \{\bot_a\colon a \to a \mid a \in \mathscr{Q}\}$ is a *cyclic family* if $f \multimap \bot_a = \bot_b \circ\!\!- f$, for all $f\colon a \to b$, and let $f^\perp$ denote their common value. Then $\mathscr{D}$is called a *cyclic dualizing family* if $f^{\perp\perp} = f$, for all f.
  A *Girard quantaloid* is a quantaloid $\mathscr{Q}$ together with a *cyclic dualizing family* $\mathscr{D}$.
- Mirroring Lemma 2.2, if $\mathscr{Q}$ is Girard, it has a second composition $\oplus$ given by the linear logic version of the Morgan duality, such that $\mathscr{Q}^{co}$ is also a quantaloid (where $-^{co}$ denotes the reversal of 2-cells).

Remark 2.5. Quickly, we introduce a bit of notation that will be used throughout this paper. Suppose $(\mathscr{V}, \otimes, \top)$ is a monoidal category then let $\mathscr{B}(\mathscr{V})$ denote its suspension, i.e., the bicategory with one object $\star$ whose 1-cells and 2-cells are the objects and morphisms of $\mathscr{V}$, respectively, with composition given by the tensor product $\otimes$ and $\top_\star$ given by the unit $\top$ of $\mathscr{V}$.

The next preliminary subsections will outline constructions that give rise to various examples of quantaloids, but we outline here three key examples.

Example 2.6. (Rosenthal, 1996)

(1) $\mathscr{B}(Q)$, the suspension of a unital quantale, is a quantaloid with one object. Note that a Girard quantaloid with one object is a Girard quantale.

(2) Rel, the locally posetal bicategory of sets and relations, is a quantaloid with hom-sets ordered under inclusion and standard relational composition: if we have $R\colon X \nrightarrow Y$ and $S\colon Y \nrightarrow Z$ then define

$$(x,z) \in R \otimes S \quad \text{if and only if} \quad \exists y \quad (x,y) \in R \;\text{ and }\; (y,z) \in S$$

(3) Ord, the locally posetal bicategory of preordered sets and order ideals, is a quantaloid with the standard relational composition: recall that a preordered set is a set X endowed with a reflexive and transitive relation $\leq_X$ and order ideals are relations $R\colon X \nrightarrow Y$ such that

$$x \leq_X x', \quad x'Ry \quad \Longrightarrow \quad xRy \quad \text{and} \quad y \leq_X y', \quad xRy \quad \Longrightarrow \quad xRy'$$

Definition 2.7. (Rosenthal, 1991, Def 2.2) If $\mathscr{Q}$ and $\mathscr{Q}'$ are quantaloids, then a *quantaloid homomorphism* is a functor $F\colon \mathscr{Q} \to \mathscr{Q}'$ such that on hom-sets it induces a sup-lattice morphism $\mathscr{Q}(a,b) \to \mathscr{Q}'(F(a),F(b))$ for all $a,b \in \mathscr{Q}$.

2.1.1 The category Q-Rel

A relation $R: X \nrightarrow Y$ assigns a truth value to each pair in $X \times Y$, as such it can be understood as a function from $X \times Y$ to the two-chain quantale. Rel can thus be generalized to arbitrary quantales by considering quantale-valued relations as follows, giving rise to a multitude of quantaloid examples.

Definition 2.8. If Q is a quantale, we can form the category Q-Rel whose objects are sets and arrows $R\colon X \nrightarrow Y$ are functions $R\colon X \times Y \to Q$, called Q-relations. Given $R\colon X \nrightarrow Y$ and $S\colon Y \nrightarrow Z$, the composition $R \otimes S : X \nrightarrow Z$ is defined by

$$R \otimes S(x,z) = \bigvee_{y \in Y} R(x,y) \otimes S(y,z)$$

Note that the use of $\otimes$ on the left refers to composition in Q-Rel and on the right refers to multiplication in Q. Identities are given by

$$\top_X(x,x') = \begin{cases} \top & \text{if } x = x' \\ \mathbf{0} & \text{if } x \neq x' \end{cases}$$

Lemma 2.9. (Hofmann et al., 2014, Section 3, 1.1) If Q is a unital quantale, then Q-Rel is a quantaloid under point-wise ordering, so in particular it is a locally posetal bicategory.

As Q-Rel is a quantaloid, given a Q-relation $R\colon X \nrightarrow Y$, there exists residuation functors $(-)\circ\!\!-R$ and $R\!-\!\!\circ(-)$, defined for $T : X \nrightarrow Z$ and $U : W \nrightarrow Y$ by

$$R \multimap T(y,z) = \bigwedge_{x \in X} R(x,y) \multimap T(x,z) \qquad\qquad U \circ\!\!- R(w,z) = \bigwedge_{y \in Y} U(w,y) \circ\!\!- R(x,y)$$

Example 2.10. (1) Rel $\cong$ 2-Rel, the quantaloid of sets and relations.

(2) P_+-Rel, a quantaloid of sets and extended distance relations $d\colon X \times Y \to [0,\infty]$, with composition $\otimes$ defined, for $D_1 : X \times Y \to [0,\infty]$ and $D_2 : Y \times Z \to [0,\infty]$, by

$$(D_1 \otimes D_2)(x,z) = \bigwedge_{y \in Y} D_1(x,y) + D_2(y,z)$$

and identities $\top_X$ defined by

$$\top_X(x,x') = \begin{cases} 0 & \text{if } x = x' \\ \infty & \text{if } x \neq x' \end{cases}$$

Alternatively, one can consider $[0,1]$-Rel, isomorphic to P_+-Relas the map $[0,1] \cong \mathsf{P}_+$ extends to an isomorphism of quantaloids, with composition $\otimes$ defined, for $D_1 \colon X \times Y \to [0,1]$ and $D_2 \colon Y \times Z \to [0,1]$, by

$$(D_1 \otimes D_2)(x,z) = \bigvee_{y \in Y} D_1(x,y) \cdot D_2(y,z)$$

and identities $\top_X$ defined by

$$\top_X(x,x') = \begin{cases} 1 & \text{if } x = x' \\ 0 & \text{if } x \neq x' \end{cases}$$

(3) $\mathsf{P}_{\mathsf{max}}$-$\mathsf{Rel}$, another quantaloid of sets and extended distance relations $d \colon X \times Y \to [0,\infty]$, but with a different composition $\otimes$ defined, for $D_1 \colon X \times Y \to [0,\infty]$ and $D_2 \colon Y \times Z \to [0,\infty]$, by

$$(D_1 \otimes D_2)(x,z) = \bigwedge_{y \in Y} \mathsf{max}(D_1(x,y), D_2(y,z))$$

The last two examples will connect to Lawvere metric spaces. This will be discussed in Example 2.31.

Note that there is a quantaloid embedding (quantaloid homomorphism which is injective on objects and a faithful functor) from the suspension of the quantale $\mathscr{B}(Q)$ into Q-Rel:

$$\mathscr{B}(Q) \hookrightarrow Q\text{-}\mathsf{Rel} : \qquad \star \mapsto 1 = \{*\} \qquad a \mapsto R_a \colon 1 \nrightarrow 1 \quad \text{where} \quad R_a(*,*) = a$$

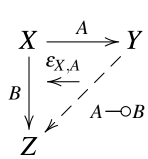
2.2 Modules, matrices and monads

We assume the reader is familiar with the theory of bicategories, introduced by Benabou. Appropriate references are Betti et al. (1983); Leinster (1998). In this section, we introduce three standard constructions in bicategory theory, but first we must review the notion of a biclosed bicategory.

2.2.1 Biclosed bicategories

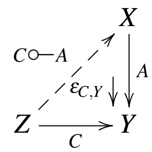
Recall the definitions of right extensions and right liftings, in the sense of Street (2014).

Definition 2.11. Let $\mathscr{B}$ be a bicategory. A *right extension* of $B \colon X \to Z$ along $A \colon X \to Y$ in $\mathscr{B}$ is a 1-cell which we denote $A \multimap B$, also denoted by $X\mathrm{Mod}(A,B)$, together with a 2-cell



inducing a bijection (called the *transpose*) between 2-cells $C \to A \multimap B$ and $A \otimes C \to B$, for all $C \colon Y \to Z$.

A *right lifting* of $C\colon Z \to Y$ through $A\colon X \to Y$ is a right extension in $\mathscr{B}^{op}$, i.e., a 1-cell which we denote $C \circ\!\!- A$, also denoted by Mod$Y(A,C)$, together with a 2-cell



inducing a bijection (called the *transpose*) between 2-cells $B \to C \circ\!\!- A$ and $B \otimes A \to C$, for all $B\colon Z \to X$.

A bicategory $\mathscr{B}$ is called *biclosed* if it admits all right extensions and right liftings.

Remark 2.12. We have chosen to use the more traditional notation and terminology from bicategory theory. Cockett et al. (2000) chose their terminology to be in agreement with Lambek's non-commutative linear logic rather than the theory of bicategories. We note that our notion of right extension is the same as their notion of *right hom* and our notion of right lifting is their *left hom*.

Example 2.13. Suppose $(\mathscr{V}, \otimes, \top)$ is a symmetric monoidal closed category and consider its suspension $\mathscr{B}(\mathscr{V})$. Taking $A \multimap B = \mathrm{Hom}(A,B)$ and $C \circ\!\!- A = \mathrm{Hom}(A,C)$, we get: $\mathscr{B}(\mathscr{V})$ is a biclosed bicategory.

Proposition 2.14. If $\mathscr{B}$ is a biclosed bicategory, then the induced 2-cells

$$(A \otimes C) \multimap B \to C \multimap (A \multimap B) \qquad \text{and} \qquad D \circ\!\!- (A \otimes C) \to (D \circ\!\!- C) \circ\!\!- A$$

are invertible and natural in $A\colon X \to Y$, $B\colon X \to W$, $C\colon Y \to Z$ and $D\colon W \to Z$.

Proof. This follows from the universal properties of the extensions and liftings. □

2.2.2 $\mathscr{B}$-categories and modules

The following definition was first introduced for locally posetal bicategories by Walters (1981) and then defined for more general bicategories by Street (2005).

Definition 2.15. Let $\mathscr{B}$ be a biclosed bicategory.

- A *$\mathscr{B}$-category M* consists of the following data:
  - for each 0-cell $A \in \mathscr{B}$, a set M_A "over A",
  - for each pair of elements x, x' over 0-cells A, B respectively, a 1-cell $M(x,x')\colon A \to B$ in $\mathscr{B}$,
  - for each triple of elements x, x', x'' over 0-cells A, B, C respectively, 2-cells $\eta\colon 1_A \to M(x,x)$ and $\mu\colon M(x,x') \otimes M(x',x'') \to M(x,x'')$ in $\mathscr{B}$

  satisfying the axioms of left and right identities, and associativity.
  Alternatively, a $\mathscr{B}$-category is lax functor from the union of sets M_A viewed as an indiscrete bicategory to $\mathscr{B}$.
- A *$\mathscr{B}$-module* $\Theta\colon M \nrightarrow N$ assigns:
  - to each pair $x \in M_A$ and $y \in N_B$ over 0-cells A and B, a 1-cell $\Theta(x,y)\colon A \to B$ in $\mathscr{B}$,

  - to each triple $x,x' \in M_A$ and $y \in N_B$ over 0-cells A and B, a left action 2-cell $\rho\colon M(x,x') \otimes \Theta(x',y) \to \Theta(x,y)$ in $\mathscr{B}$,
  - to each triple $x \in M_A$ and $y,y' \in N_B$ over 0-cells A and B, a right action 2-cell $\lambda\colon \Theta(x,y) \otimes N(y,y') \to \Theta(x,y')$ in $\mathscr{B}$,

  satisfying five compatibility axioms.

- Given $\mathscr{B}$-modules $\Theta, \Phi\colon M \nrightarrow N$, a morphism of $\mathscr{B}$-modules $\alpha\colon \Theta \to \Phi$ is a family of 2-cells $\alpha\colon \Theta(x,y) \to \Phi(x,y)$ in $\mathscr{B}$ compatible with the left and right actions λ, ϕ.
- Given $\mathscr{B}$-modules $\Theta\colon M \nrightarrow N$ and $\Pi : N \nrightarrow P$, the composite $\Theta \otimes \Pi\colon M \nrightarrow N$ is defined, for $x \in M_A$ and $z \in P_C$ over A and C, by $\Theta \otimes \Pi(x,z)$ as a colimit in $\mathscr{B}(A,C)$.
- By restricting to certain bicategories $\mathscr{B}$, such that the colimit involved in the definition exists, we can define the bicategory $\mathscr{B}$-Mod of $\mathscr{B}$-categories and $\mathscr{B}$-modules. For more details, see Street (2005).

Suppose $(\mathscr{V}, \otimes, \top)$ is a complete and cocomplete symmetric monoidal closed category. Then we can consider $\mathscr{B}(\mathscr{V})$-Mod $\cong \mathscr{V}$-$\mathscr{P}$rof, the bicategory of $\mathscr{V}$-categories, $\mathscr{V}$-profunctors (sometimes called $\mathscr{V}$-distributors), and $\mathscr{V}$-transformations.

Note that composition $A \otimes_R B$ of $\mathscr{V}$-profunctors $A\colon Q \nrightarrow R$ and $B\colon R \nrightarrow S$ can be described by the coend

$$(A \otimes_R B)(q,s) = \int^r A(q,r) \otimes B(r,s)$$

and $Q(-,-)\colon Q \nrightarrow Q$ is the identity 1-cell.

2.2.3 $\mathscr{B}$-matrices

The work of categories enriched in bicategories was further developed by Betti et al. (1983). They introduced the bicategory of $\mathscr{B}$-matrices as a stepping stone to the study of $\mathscr{B}$-Mod.

Definition 2.16. Let $\mathscr{B}$ be a locally small-cocomplete bicategory with a small set of objects $\mathscr{B}_0$.

- Given a family $X = (X_A)_{A \in \mathscr{B}_0}$ of small sets indexed by $\mathscr{B}_0$, an element $x \in X_A$ is said to be an *element of X over A*.
- Given a pair of families $X = (X_A)_{A \in \mathscr{B}_0}$ and $Y = (Y_A)_{A \in \mathscr{B}_0}$, a *$\mathscr{B}$-matrix* $S\colon X \nrightarrow Y$ assigns to each pair x,y of elements over $A, B \in \mathscr{B}_0$ a 1-cell $S(x,y)\colon A \to B$ in $\mathscr{B}$. Composition of $\mathscr{B}$-matrices $S\colon X \nrightarrow Y$ and $T\colon Y \nrightarrow Z$ is by matrix multiplication, i.e., $(S \otimes T)(x,z) = \coprod_{y \in Y} S(x,y) \otimes T(y,z)$.
- A *morphism of $\mathscr{B}$-matrices* $\alpha\colon S \to S'$ is a family of 2-cells $\alpha_{x,y}\colon S(x,y) \to S'(x,y)$ in $\mathscr{B}$.
- Define Matr$\mathscr{B}$ to be the bicategory with families of small sets indexed by $\mathscr{B}_0$ as 0-cells, $\mathscr{B}$-matrices as 1-cells and $\mathscr{B}$-matrix morphisms as 2-cells.

Suppose $(\mathscr{V}, \otimes, \top)$ is a symmetric monoidal closed category with set-indexed products and coproducts. Then, we can consider Matr$\mathscr{B}(\mathscr{V}) \cong \mathscr{V}$-Matr the bicategory of sets, $\mathscr{V}$-matrices, and $\mathscr{V}$-matrix morphisms.

Recall from Carboni et al. (1987) that a $\mathscr{V}$-matrix $A\colon X \nrightarrow Y$ is a function $A\colon X \times Y \to \mathrm{ob}\mathscr{V}$. A morphism $f\colon A \to B$ of $\mathscr{V}$-matrices $X \nrightarrow Y$ is a family

$$\{f(x,y)\colon A(x,y) \to B(x,y) \mid (x,y) \in X \times Y\}$$

of morphisms of $\mathscr{V}$. Composition of $A\colon X \nrightarrow Y$ and $B\colon Y \nrightarrow Z$ is given by matrix multiplication

$$(A \otimes_Y B)(x,z) = \textstyle\coprod_{y\in Y} A(x,y) \otimes B(y,z)$$

and the identity $X \nrightarrow X$ is

$$\top_X(x,x') = \begin{cases} \top & x = x' \\ \mathbf{0} & \text{otherwise} \end{cases}$$

where $\top$ is the unit for $\otimes$ and $\mathbf{0}$ is initial in $\mathscr{V}$.

Given $\mathscr{V}$-matrices $A\colon X \nrightarrow Y$, $B\colon X \nrightarrow Z$, and $C\colon Z \nrightarrow Y$, taking

$$A \multimap B(y,z) = \prod_{x\in X} \mathscr{V}(A(x,y), B(x,z)) \qquad C \circ\!\!- A(z,x) = \prod_{y\in Y} \mathscr{V}(A(x,y), C(z,y))$$

we get the well-known result within folklore, see Blute et al. (1996) and Lack (2010):

Lemma 2.17. If $\mathscr{V}$ is a symmetric monoidal closed category with set-indexed products and coproducts, then $\mathscr{V}$-Matr is a biclosed bicategory.

2.2.4 Monads in $\mathscr{B}$ and modules

Carboni et al. (1987) demonstrated that $\mathscr{B}$-Mod can be broken into the constructions of Matr$\mathscr{B}$ and of Mon$\mathscr{B}$, the bicategory of monads and modules in $\mathscr{B}$ as defined below.

Definition 2.18. Let $\mathscr{B}$ be a biclosed bicategory with local equalizers and coequalizers stable under composition.

- A *monad* in $\mathscr{B}$ is a 1-cell $Q\colon X \to X$ together with two 2-cells $e\colon 1_X \to Q$ and $m\colon Q \otimes Q \to Q$ satisfying the usual associativity and identity axioms. Note that a monad (X,Q) (denoted by just Q when X is understood) is a monoid in the category $\mathscr{B}(X,X)$.
- A (Q,R)-*module* $A\colon (X,Q) \nrightarrow (Y,R)$ is a 1-cell $A\colon X \to Y$ together with action 2-cells $\lambda\colon Q \otimes A \to A$ and $\rho\colon A \otimes R \to A$ satisfying (one-sided) associative and unit laws, and a diagram expressing the commutativity of the two actions.
- A (Q,R)-*module morphism* is a 2-cell $f\colon A \to B$ satisfying compatibility conditions with respect to the actions.
- Let Mon$\mathscr{B}$ denote the bicategory of monads, modules, and module morphisms; with the composition $A \otimes_R B$ of $A\colon (X,Q) \nrightarrow (Y,R)$ and $B\colon (Y,R) \nrightarrow (Z,S)$ given by the local coequalizer

$$A \otimes R \otimes B \underset{A\otimes\lambda_B}{\overset{\rho_A\otimes B}{\rightrightarrows}} A \otimes B \longrightarrow A \otimes_R B$$

  in $\mathscr{B}(X,Z)$, and identity 1-cell $\top_{(X,Q)} = Q\colon (X,Q) \nrightarrow (X,Q)$.

Note that identity 1-cell $\top_X\colon X \to X$ in $\mathscr{B}$ is a trivial monad, every 1-cell $A\colon X \to Y$ is an $(\top_X, \top_Y)$-module, and every 2-cell $f\colon A \to B$ is a $(\top_X, \top_Y)$-module homomorphism. Moreover, $X \mapsto \top_X$ defines a pseudo functor which is a left pseudo adjoint to the forgetful (lax) functor Mon$\mathscr{B} \to \mathscr{B}$.

Given $A\colon (X,Q) \nrightarrow (Y,R)$, $B\colon (X,Q) \nrightarrow (Z,S)$, and $C\colon (Z,S) \nrightarrow (Y,R)$, the right extension $B \multimap_Q A$ in Mon$\mathscr{B}$ of B along A is obtained by taking the local equalizer.

$$A \multimap_Q B \rightarrow A \multimap B \underset{\hat{\lambda}_B}{\overset{\lambda_A^*}{\rightrightarrows}} (Q \otimes A) \multimap B$$

where $\lambda_A^* = \lambda_A \multimap B$ and $\hat{\lambda}_B$ is the transpose of

$$(Q \otimes A) \otimes (A \multimap B) \xrightarrow{\sim} Q \otimes (A \otimes (A \multimap B)) \xrightarrow{Q \otimes \varepsilon_{X,A}} Q \otimes B \xrightarrow{\lambda_B} B$$

Similarly the right lifting $C \circ\!\!-_R A$ in Mon$\mathscr{B}$ of C through A is

$$C \circ\!\!-_R A \rightarrow C \circ\!\!- A \underset{\hat{\rho}_B}{\overset{\rho_A^*}{\rightrightarrows}} C \circ\!\!- (A \otimes R)$$

Note that given morphisms $A\colon X \to Y$, $B\colon X \to Z$ and $C\colon Z \to Y$ in $\mathscr{B}$ the 2-cells

$$A \multimap_{\top_X} B \to A \multimap B \qquad \text{and} \qquad C \circ\!\!-_{\top_Y} A \to C \circ\!\!- A$$

defining the extensions and liftings in Mon$\mathscr{B}$, are invertible, and without loss of generality, we can take them to be equalities.

Moreover, given the above definitions, the following is a well-known result, see Betti et al. (1983) and Lack (2010):

Lemma 2.19. If $\mathscr{B}$ is a biclosed bicategory with local equalizers and coequalizers stable under composition, then Mon$\mathscr{B}$ is a biclosed bicategory.

By examining the definitions of the three constructions, it is immediate that:

Proposition 2.20. (Carboni et al., 1987) If $\mathscr{B}$ is a distributive bicategory (locally cocomplete bicategory with colimits preserved by composition on both sides), $\mathscr{B}$-Mod is biequivalent to MonMatr$\mathscr{B}$.

Furthermore, the constructions are idempotent.

Proposition 2.21. (Carboni et al., 1987) If $\mathscr{B}$ is a distributive bicategory, then ($\mathscr{B}$-Mod)-Mod is biequivalent to $\mathscr{B}$-Mod. If $\mathscr{B}$ is a bicategory with local coequalizers stable under composition, then Mon(Mon$\mathscr{B}$) is biequivalent to Mon$\mathscr{B}$. If $\mathscr{B}$ is a bicategory with small local coproducts stable under composition, then Matr(Matr$\mathscr{B}$) is biequivalent to Matr$\mathscr{B}$.

Thus, by Lemmas 2.17 and 2.19, and calculating the liftings and extensions by the ends

$$(A \multimap B)(r,s) = \int_q \mathrm{Hom}(A(q,r), B(q,s)) \qquad (C \circ\!\!- A)(s,q) = \int_r \mathrm{Hom}(A(q,r), B(s,r))$$

for $\mathscr{V}$-profunctors $A\colon Q \nrightarrow R$, $B\colon Q \nrightarrow S$ and $C\colon S \nrightarrow R$, we get:

Lemma 2.22. Suppose $\mathscr{V}$ is a complete and cocomplete symmetric monoidal closed category, then $\mathscr{V}$-$\mathscr{P}$rof is a biclosed bicategory.

2.2.5 Enrichment in a quantaloid $\mathcal{Q}$

The previous bicategorical constructions can of course be considered in the special case of a quantaloid $\mathcal{Q}$, as done by Rosenthal (1992). We take the time to describe them in detail as a main result of this paper is their generalization to the linear context.

Definition 2.23. Let $\mathcal{Q}$ be a quantaloid.

- A *$\mathcal{Q}$-category* is a pair $M = (X, \rho)$ where:
  - X is a set,
  - ρ is a function $\rho: X \to \mathsf{ob}\mathcal{Q}$, and
  - there is a function, called the *enrichment* assigning to each pair $(x,x') \in X \times X$ a morphism $M(x,x'): \rho(x) \to \rho(x')$ such that $\forall x, x', x'' \in X$
  $$\top_{\rho(x)} \leq M(x,x) \qquad M(x,x') \otimes M(x',x'') \leq M(x,x'')$$

  Alternatively, a $\mathcal{Q}$-category is a lax functor from set X to $\mathcal{Q}$, when X is viewed as an indiscrete bicategory.
- Consider $\mathcal{Q}$-categories $M = (X, \rho_M)$ and $N = (Y, \rho_N)$. A *$\mathcal{Q}$-module* $\Theta: M \nrightarrow N$ consists of an assignment of a morphism $\Theta(x,y): \rho_M(x) \to \rho_N(y)$ to every pair $(x,y) \in X \times Y$ such that $\forall x, x' \in X, y, y' \in Y$
$$\Theta(x,y) \otimes N(y,y') \leq \Theta(x,y') \qquad M(x,x') \otimes \Theta(x',y) \leq \Theta(x,y)$$
- Define the category $\mathcal{Q}$-Mod whose objects are $\mathcal{Q}$-categories and arrows are $\mathcal{Q}$-modules. Given $M \overset{\Theta}{\nrightarrow} N \overset{\Pi}{\nrightarrow} P$, composition $\Theta \otimes \Pi: M \nrightarrow P$ is defined by
$$(\Theta \otimes \Pi)(x,z) = \bigvee_{y \in Y} \Theta(x,y) \otimes \Pi(y,z)$$

  Identity 1-cells $\top\!\!\!\top_M: M \nrightarrow M$ are defined by $\top\!\!\!\top_M(x,x') = M(x,x')$. Note that the use of $\otimes$ on the left refers to composition in $\mathcal{Q}$-Mod and on the right refers to composition in $\mathcal{Q}$.

Remark 2.24. We have chosen to denote the bicategory of $\mathcal{Q}$-categories and $\mathcal{Q}$-modules as $\mathcal{Q}$-Mod in agreement with previously introduced notation, while it is called $\mathsf{Bim}(\mathcal{Q})$ by Rosenthal (1992).

Definition 2.25. Let $\mathcal{Q}$ be a quantaloid. Define the category $\mathsf{Matr}\mathcal{Q}$ whose objects are small families of objects in $\mathcal{Q}$, i.e., pairs (X,γ) of a set X and a function $\gamma: X \to ob\mathcal{Q}$, and arrows are $\mathcal{Q}$-matrices $r: (X,\gamma) \nrightarrow (Y,\phi)$, i.e., families of morphisms $(r_{x,y}: \gamma(x) \to \phi(y))_{(x,y) \in X \times Y}$ in $\mathcal{Q}$. Given $\mathcal{Q}$-matrices $(X,\gamma) \overset{r}{\nrightarrow} (Y,\phi) \overset{s}{\nrightarrow} (Z,\chi)$, composition $r \otimes s: (X,\gamma) \nrightarrow (Z,\chi)$ is defined by
$$(r \otimes s)_{x,z} = \bigvee_{y \in Y} r_{x,y} \otimes s_{y,z}$$

Identity 1-cells $\top\!\!\!\top_{(X,\gamma)}: (X,\gamma) \nrightarrow (X,\gamma)$ are defined by $\top\!\!\!\top_{(X,\gamma)_{x,x'}} = \begin{cases} \top_{\gamma(x)} & \text{if } x = x' \\ \mathbf{0}_{\gamma(x),\phi(x')} & \text{if } x \neq x' \end{cases}$.

Example 2.26. (1) It is a standard result that the category of $\mathcal{B}(\Omega)$-matrices, $\mathsf{Matr}(\mathcal{B}(\Omega))$, is isomorphic to Rel.

(2) Consider $\mathsf{Matr}(\mathcal{B}(\Omega))$. The objects are pairs (X,γ) of a set X and a trivial function mapping the set to $\star$. The $\mathcal{B}(Q)$-matrices are families $(r_{x,y}: \star \to \star)_{(x,y) \in X \times Y}$, a collections of

elements in Q, i.e., a function $X \times Y \to Q$, and composition of $\mathscr{B}(Q)$-matrices is exactly the one of Q-relations. As such, MatrMatr$(\mathscr{B}(Q)) \cong Q$-Rel, generalizing the first example.

Definition 2.27. Let $\mathscr{Q}$ be a quantaloid.

- A *monad* (a,m) in $\mathscr{Q}$ is an object a equipped with a morphism $m\colon a \to a$ such that

$$\top_a \leq m \qquad m \otimes m \leq m$$

  Alternatively, a monad in $\mathscr{Q}$ is a lax functor from the terminal bicategory 1 to $\mathscr{Q}$.
- A (m,n)-*module* $f\colon (a,m) \nrightarrow (b,n)$ is a morphism $f\colon a \to b$ in $\mathscr{Q}$ such that

$$m \otimes f \leq f \qquad f \otimes n \leq f$$

- Mon$\mathscr{Q}$ is the category of monads and monad modules. in $\mathscr{Q}$. Composition is directly inherited from $\mathscr{Q}$ and the identity 1-cell $\top\!\!\top_{a,m}\colon (a,m) \nrightarrow (a,m)$ is $m : a \to a$.

Example 2.28. (1) It is immediate that monads in Rel are preordered sets and monad modules are order ideals, thus Mon(Rel) $\cong$ Ord.

(2) The above example can be generalized to Q-Rel. Consider Mon(Q-Rel): monads in Q-Rel are Q-categories, pairs (X,m), set X equipped with a relation $m\colon X \nrightarrow X$ which is Q-reflexive and Q-transitive, meaning

$$\top \leq m(x,x) \qquad \text{and} \qquad m(x,x') \otimes m(x',x'') \leq m(x,x'') \qquad \forall x,x',x'' \in X$$

while monad modules are Q-profunctors, relations $S\colon (X,m) \nrightarrow (Y,n)$

$$m(x,x') \otimes S(x',y) \leq S(x,y) \quad \text{and} \quad S(x,y') \otimes n(y',y) \leq S(x,y) \quad \forall x,x' \in X, y,y' \in Y$$

These constructions are more than just categories:

Lemma 2.29. (Rosenthal, 1992) $\mathscr{Q}$-Mod, Matr$\mathscr{Q}$ and Mon$\mathscr{Q}$ are quantaloids, so in particular they are locally posetal bicategories.

As $\mathscr{Q}$-Mod, Matr$\mathscr{Q}$ and Mon$\mathscr{Q}$ are quantaloids, there exist residuation functors. In the case of $\mathscr{Q}$-Mod and Matr$\mathscr{Q}$, the formulas are similar to Q-Rel, the inf of the point-wise residuations, while Mon$\mathscr{Q}$ inherits residuations directly from $\mathscr{Q}$.

Propositions 2.20 and 2.21 apply to the case of quantaloids:

Proposition 2.30. (Rosenthal, 1992) $\mathscr{Q}$-Mod is biequivalent to MonMatr$\mathscr{Q}$ and the constructions are idempotent.

Thus, we see that Ord $\cong \mathscr{B}(\Omega)$-Mod $\cong$ Rel-Mod and Q-$\mathscr{P}$rof $\cong \mathscr{B}(Q)$-Mod $\cong$ (Q-Rel)-Mod.

Example 2.31. (1) Consider $Q = \mathsf{P}_+$, then P_+-$\mathscr{P}$rof is the quantaloid of Lawvere metric spaces and P_+-bimodules (in the sense of Lawvere (1973)). Objects are sets X equipped with extended distance functions $m\colon X \times X \to [0,\infty]$ which satisfy the axiom of point inequality and the triangle inequality:

$$m(x,x) \leq 0 \qquad \text{and} \qquad m(x,x'') \leq m(x,x') + m(x',x'') \qquad \forall x,x',x'' \in X$$

In other words the objects are extended quasi-pseudo-metric spaces, or simply Lawvere metric spaces. The arrows $(X,m) \nrightarrow (Y,n)$ are real-valued functions $F\colon X \times Y \to [0,\infty]$

satisfying

$$F(x,y) \le m(x,x') + F(x',y) \quad \text{and} \quad F(x,y) \le F(x,y') + n(y',y) \quad \forall x,x' \in X, y,y' \in Y$$

The standard category Met of Lawvere metric spaces and non-expansive maps is equivalent to the category of P_+-categories and P_+-functors. Every P_+-functor gives rise to a pair of adjoint P_+-bimodules, therefore P_+-$\mathscr{P}$rof is a quantaloid of Lawvere metric spaces with a more general notion of morphisms.

Alternatively, take $Q = [0,1]$, then $[0,1]$-$\mathscr{P}$rof $\cong \mathsf{P}_+$-$\mathscr{P}$rof. Objects are sets X equipped with a functions $m \colon X \times X \to [0,1]$ satisfying

$$1 \le m(x,x) \qquad \text{and} \qquad m(x,x') \cdot m(x',x'') \le m(x,x'') \qquad \forall x,x',x'' \in X$$

while arrows $(X,m) \nrightarrow (Y,n)$ are functions $F \colon X \times Y \to [0,1]$ satisfying

$$m(x,x') \cdot F(x',y) \le F(x,y) \quad \text{and} \quad F(x,y') \cdot n(y',y) \le f(x,y) \quad \forall x,x' \in X, y,y' \in Y$$

(2) If instead, we consider $Q = \mathsf{P}_{\mathsf{max}}$, then $\mathsf{P}_{\mathsf{max}}$-$\mathscr{P}$rof is the quantaloid of Lawvere ultrametric spaces and $\mathsf{P}_{\mathsf{max}}$-bimodules. The metric spaces (X,m) satisfy the strengthened triangle inequality:

$$m(x,x'') \le \mathsf{max}(m(x,x'), m(x',x'')) \qquad \forall x,x',x'' \in X$$

and the arrows $(X,m) \nrightarrow (Y,n)$ are real-valued functions $F \colon X \times Y \to [0,\infty]$ satisfying

$$F(x,y) \le \mathsf{max}(m(x,x'), F(x',y)) \quad \text{and} \quad F(x,y) \le \mathsf{max}(F(x,y'), n(y',y)) \quad \forall x,x' \in X, y,y' \in Y$$

For more details, see Lawvere (1973).

Note that there is a quantaloid embedding of $\mathscr{Q}$ into $\mathscr{Q}$-Mod,

$$\mathscr{Q} \hookrightarrow \mathscr{Q}\text{-}\mathsf{Mod}: \qquad a \mapsto M_a = (1, \rho_a) \quad \text{where} \quad 1 = *, \quad \rho_a(*) = a \quad \text{and} \quad M_a(*,*) = \top_a$$

$$f \colon a \to b \mapsto \Theta_f \colon M_a \nrightarrow M_b \quad \text{where} \quad \Theta_f(*,*) = f$$

of $\mathscr{Q}$ into $\mathsf{Matr}\mathscr{Q}$,

$$\mathscr{Q} \hookrightarrow \mathsf{Matr}\mathscr{Q}: \qquad a \mapsto (1, \gamma_a) \quad \text{where} \quad 1 = *, \quad \gamma_a(*) = a$$

$$f \colon a \to b \mapsto f \colon (1,\gamma_a) \nrightarrow (1,\gamma_b) \quad \text{where} \quad f_{*,*} = f$$

and of $\mathscr{Q}$ into $\mathsf{Mon}\mathscr{Q}$,

$$\mathscr{Q} \hookrightarrow \mathsf{Mon}\mathscr{Q}: \qquad a \mapsto (a, \top_a) \ \text{ the trivial monad}$$

$$f \colon a \to b \mapsto f \colon (a,\top_a) \nrightarrow (b,\top_b)$$

3. Linear bicategories

We introduce the theory of *linear bicategories* as defined by Cockett et al. (2000). The material in this subsection is entirely from that paper.

Linear bicategories are an extension of the theory of *linearly distributive categories*, due to Cockett and Seely Cockett and Seely (1997). Linearly distributive categories axiomatize the multiplicative fragment of linear logic in a way that is closer to the syntax. So the two binary connectives, $\otimes$ and $\oplus$, are taken as primitives, and negation can be added if one wishes.

As usual with bicategories, one begins with a class of *0-cells* which we will denote $\mathscr{B}_0 = \{X,Y,Z,\ldots\}$. Then for every pair of 0-cells, one has a category $\mathscr{B}(X,Y)$. The objects of $\mathscr{B}(X,Y)$ are called *1-cells* and the arrows are called *2-cells*. But now we have two composition functors:

$$\otimes, \oplus \colon \mathscr{B}(X,Y) \times \mathscr{B}(Y,Z) \longrightarrow \mathscr{B}(X,Z)$$

Each of these compositions gives a bicategory structure. Thus for each composition we have all of the morphisms and coherences that this entails. In particular we must have identity 1-cells for each of the two compositions:

$$\top_X : X \to X \qquad \text{and} \qquad \bot_X : X \to X$$

These two bicategory structures are related by a linear distribution as follows. Given 0-cells X,Y,Z,W we have two functors:

$$-\otimes(-\oplus-), (-\otimes-)\oplus- \colon \mathscr{B}(X,Y)\times\mathscr{B}(Y,Z)\times\mathscr{B}(Z,W) \longrightarrow \mathscr{B}(X,W)$$

and we require a natural transformation between them, which is not necessarily an isomorphism:

$$\delta^L : -\otimes(-\oplus-) \Longrightarrow (-\otimes-)\oplus-$$

Symmetrically, we also need a natural transformation

$$\delta^R : (-\oplus-)\otimes- \longrightarrow -\oplus(-\otimes-).$$

All of this structure must satisfy coherence requirements detailed in Cockett et al. (2000).

One of the main goals of this paper will be to provide new examples of linear bicategories, but we mention here the quintessential example Rel, as it will be the example on which ours are based.

Rel is a locally posetal bicategory with its first composition being the standard one. But we have a second composition: for $R\colon X \nrightarrow Y$ and $S\colon Y \nrightarrow Z$, define

$$(x,z) \in R \oplus S \quad \text{if and only if} \quad \forall y \quad (x,y)\in R \quad \text{or} \quad (y,z) \in S$$

We quickly mention here that the appropriate notion of functor between linear bicategories was developed, although they will only make a brief appearance in this article.

Definition 3.1. (Cockett et al., 2000, Def 2.4) Let $\mathscr{B}$ and $\mathscr{B}$’ be linearly distributive categories. A *bilax linear functor* $F = (F_\otimes, F_\oplus)\colon \mathscr{B} \to \mathscr{B}'$ consists of:

- a lax monoidal functor $(F_\otimes, m_\top, m_\otimes)\colon (\mathscr{B},\otimes,\top) \to (\mathscr{B}',\otimes,\top)$, equipped with
- a colax monoidal functor $(F_\oplus, n_\bot, n_\oplus)\colon (\mathscr{B},\oplus,\bot) \to (\mathscr{B}',\oplus,\bot)$, equipped with
- four natural transformations, known as *linear strengths*,

$$\begin{array}{ll}
v^R_\otimes \colon \oplus; F_\otimes \to (F_\oplus \times F_\otimes);\oplus & v^R_{\otimes A,B}\colon F_\otimes(A\oplus B) \to F_\oplus(A)\oplus F_\otimes(B)\\
v^L_\otimes \colon \oplus; F_\otimes \to (F_\otimes \times F_\oplus);\oplus & v^L_{\otimes A,B}\colon F_\otimes(A\oplus B) \to F_\otimes(A)\oplus F_\oplus(B)\\
v^R_\oplus \colon (F_\otimes \times F_\oplus);\otimes \to \otimes; F_\oplus & v^R_{\oplus A,B}\colon F_\otimes(A)\otimes F_\oplus(B) \to F_\oplus(A\otimes B)\\
v^L_\oplus \colon (F_\oplus \times F_\otimes);\otimes \to \otimes; F_\oplus & v^L_{\oplus A,B}\colon F_\oplus(A)\otimes F_\otimes(B) \to F_\oplus(A\otimes B)
\end{array}$$

subject to various coherence conditions.

3.0.1 Cyclic $*$-autonomous bicategories

The notion of a $*$-autonomous category was originally introduced independently of linear logic by Barr (1979), who was trying to capture some of the dualities present in various categories of topological vector spaces. It was only later seen to be appropriate to model the multiplicative fragment of linear logic by Seely (1989). Barr's definition of $*$-autonomous category was a symmetric monoidal closed category with a dualizing object. But linear logic suggested an equivalent definition as a linearly distributive category with negation, see Cockett and Seely (1997). Similarly, as linearly distributive categories have been generalized to linear bicategories, it is natural to generalize the notion of cyclic $*$-autonomous categories as well.

Definition 3.2. (Cockett et al., 2000, [Def 3.14) A bicategory $\mathscr{B}$ is a *cyclic $*$-autonomous bicategory* if

- for any pair of 0-cells X,Y, there is an adjoint equivalence $(-)^* \dashv ((-)^*)^{op} : \mathscr{B}(X,Y) \to \mathscr{B}(Y,X)^{op}$, and
- for any 1-cell $A : X \to Y$, the 1-cell A^* is the right extension of $\top_X^*$ along A, such that these extensions are natural in A.

As discussed by Cockett et al. (2000), we can equivalently define cyclic $*$-autonomous bicategory by introducing the concept of dualizing 1-cells.

Definition 3.3. Suppose $\mathscr{D} = \{\bot_X \colon X \to X \mid X \in \mathscr{B}\}$ is a family of 1-cells in a biclosed bicategory $\mathscr{B}$. Given $A\colon X \to Y$, applying Definition 2.11, we get 2-cells

$$A \xrightarrow{\delta_{A,Y}} (\bot_Y \circ\!\!- A) -\!\!\circ \bot_Y \qquad \text{and} \qquad A \xrightarrow{\delta_{X,A}} \bot_X \circ\!\!- (A -\!\!\circ \bot_X)$$

- A family $\mathscr{D}$ is called *dualizing* if the 2-cells $\delta_{X,A}$ and $\delta_{A,Y}$ are invertible, for all $A\colon X \to Y$.
- A dualizing family $\mathscr{D}$ is called *cyclic* if there are invertible 2-cells $\theta_A \colon \bot_Y \circ\!\!- A \xrightarrow{\sim} A -\!\!\circ \bot_X$, natural in $A\colon X \to Y$, such that the following diagram commutes

$$\begin{array}{ccc} & & \bot_X \circ\!\!- (A -\!\!\circ \bot_X) \\ & \overset{\delta_{X,A}}{\nearrow} & \downarrow {\scriptstyle \bot_X \circ\!\!- \theta_A} \\ A & & \bot_X \circ\!\!- (\bot_Y \circ\!\!- A) \\ & \underset{\delta_{A,Y}}{\searrow} & \downarrow {\scriptstyle \theta_{\bot_Y \circ\!\!- A}} \\ & & (\bot_Y \circ\!\!- A) -\!\!\circ \bot_Y \end{array}$$

In this case, we let $A^{\bot} = A -\!\!\circ \bot_X$.

Proposition 3.4. $\mathscr{B}$ is a cyclic $*$-autonomous bicategory if and only if it admits a cyclic dualizing family.

Example 3.5. Suppose $\mathscr{V}$ is a $*$-autonomous category with cyclic dualizing object $\bot$, one can show that $\mathscr{D} = \{\bot\colon \star \longrightarrow \star\}$ is a cyclic dualizing family for its suspension $\mathscr{B}(V)$, since $A -\!\!\circ \bot \cong A^{\bot} \cong \bot \circ\!\!- A$ and $A \cong (A^{\bot})^{\bot}$, for all objects $A \in \mathscr{V}$. So $\mathscr{B}(V)$ is a cyclic $*$-autonomous bicategory.

In particular, consider $\mathscr{V} = \mathrm{Sup}$, the category of suplattices and suplattice homomorphisms (functions that preserve all joins). Sup is a $*$-autonomous category with cyclic dualizing object Ω^{op}, with $A \circ\!\!- \Omega^{op} \cong \Omega^{op} -\!\!\circ A \cong A^{\circ}$, where A° denotes the opposite poset of A, and $A \cong (A^{\circ})^{\circ}$, for all A. Therefore, $\mathscr{B}(\mathrm{Sup})$ is a cyclic $*$-autonomous bicategory with cyclic dualizing family $\mathscr{D} = \{\Omega^{op} : \star \to \star\}$.

It will come as no surprise then that every cyclic $*$-autonomous bicategory is a linear bicategory by the de Morgan equations. It is remarked in Cockett et al. (2000) without a detailed proof as it would largely follow the same computation described in Cockett and Seely (1997), in the case of LDCs and $*$-autonomous categories. We describe below some of the key details.

Lemma 3.6. If $\mathscr{B}$ is a cyclic $*$-autonomous bicategory, then there is an invertible 2-cell

$$A^\perp \multimap B \xrightarrow{\sim} (B^\perp \otimes A^\perp)^\perp \qquad \text{and} \qquad B \multimapinv C^\perp \xrightarrow{\sim} (C^\perp \otimes B^\perp)^\perp$$

for all $A\colon X \to Y$, $B\colon Y \to Z$ and $C\colon Z \to W$.

Proof. Suppose $\mathscr{D} = \{\perp_X : X \to X\}$ is a cyclic dualizing family for $\mathscr{B}$. Then composition with $\delta_{B,Z}$ induces the transpose invertible 2-cell

$$A^\perp \multimap B \xrightarrow{\sim} A^\perp \multimap ((\perp_Z \multimapinv B) \multimap \perp_Z)$$

Since $\perp_Z \multimapinv B \cong B^\perp$ and by Proposition 2.14, it follows that

$$A^\perp \multimap ((\perp_Z \multimapinv B) \multimap \perp_Z) \xrightarrow{\sim} (B^\perp \otimes A^\perp) \multimap \perp_Z = (B^\perp \otimes A^\perp)^\perp$$

and the desired 2-cell follows. Similarly for the second result. □

While the following result follows from the discussion by Cockett et al. (2000) in Section 3.4, we outline the broad strokes of the proof for readers less initiated in categorical linear logic.

Proposition 3.7. (Cockett et al., 2000, Prop 3.13) Every cyclic $*$-autonomous bicategory is a linear bicategory.

Proof. Suppose $\mathscr{D} = \{\perp_X : X \to X\}$ is a cyclic dualizing family for $\mathscr{B}$. Given $A\colon X \to Y$ and $B\colon Y \to Z$, define $A \oplus B = (B^\perp \otimes A^\perp)^\perp$.

Then $\oplus$ is associative and has identity 1-cells $\perp_X = \top_X^\perp : X \to X$:

$$\begin{aligned} A \oplus (B \oplus C) &= A \oplus (C^\perp \otimes B^\perp)^\perp \cong ((C^\perp \otimes B^\perp) \otimes A^\perp)^\perp \\ &\cong (C^\perp \otimes (B^\perp \otimes A^\perp))^\perp \cong (C^\perp \otimes (A \oplus B)^\perp)^\perp \\ &= (A \oplus B) \oplus C \end{aligned}$$

and $\top_X^\perp \oplus A \cong A \cong A \oplus \top_Y^\perp$.

To see that $\mathscr{B}$ is a linear bicategory, we will define the left linear distribution

$$A \otimes (B \oplus C) \to (A \otimes B) \oplus C$$

Since $A \otimes (B \oplus C) \cong A \otimes (B^\perp \multimap C)$ and $(A \otimes B) \oplus C \cong (A \otimes B)^\perp \multimap C \cong (B^\perp \multimapinv A) \multimap C$, by Lemma 3.6; it suffices to define a 2-cell

$$A \otimes (B^\perp \multimap C) \to (B^\perp \multimapinv A) \multimap C$$

or equivalently, its transpose

$$(B^\perp \multimapinv A) \otimes (A \otimes (B^\perp \multimap C)) \to C$$

For this, we can use the associator and the evaluation maps

$$(B^\perp \multimapinv A) \otimes (A \otimes (B^\perp \multimap C)) \to ((B^\perp \multimapinv A) \otimes A) \otimes (B^\perp \multimap C) \to B^\perp \otimes (B^\perp \multimap C) \to C$$

Similarly, we get a 2-cell $(A \oplus B) \otimes C \to A \oplus (B \otimes C)$, as desired. It remains to show the coherence conditions, in particular the ones involving the linear distributions. □

The last section of this article will be a discussion of several examples of non-posetal cyclic $*$-autonomous bicategories, including $\mathscr{Q}$uant and $\mathscr{Q}$tld.

Cyclic $*$-autonomous bicategories are unique in that every 1-cell has a canonical cyclic linear adjoint, in the following sense:

Definition 3.8. (Cockett et al., 2000, Def 3.1) A *linear adjunction* $A \dashv\!\!| B \colon X \to Y$ consists of a pair of 1-cells $A\colon X \to Y$ and $B\colon Y \to X$, equipped with 2-cells $\tau\colon \top_X \to A \oplus B$, known as the unit, an $\gamma\colon B \otimes A \to \bot_Y$, known as the conunit, satisfying the snake equations.

Definition 3.9. (Cockett et al., 2000, Def 4.1) A *cyclic linear adjunction* $A \| \!\!\mapsto\!\! \| B$ is a pair of linear adjoints $A \dashv\!\!| B$ and $B \dashv\!\!| A$.

Proposition 3.10. (Cockett et al., 2000, Cor 3.4) Any two right (respectively left) linear adjoints to a 1-cell are isomorphic, in the sense that there is a unique 2-cell mediating the isomorphism.

Then,

Proposition 3.11. (Cockett et al., 2000, Rem 4.10) Consider a cyclic $*$-autonomous bicategory, then every 1-cell $A\colon X \to Y$ has a unique (up to isomorphism) cyclic linear adjoint given by $A^{\perp}\colon Y \to X$.

4. LD-quantales and linear quantaloids

It is immediate that Girard quantaloids are locally posetal cyclic $*$-autonomous bicategories. In particular, the second composition is defined as the dual of the first. An alternate approach to the model theory of linear logic is that of linear bicategories where the tensor and par are taken as primitive. We define what the analogous structure would be for quantaloids below.

Definition 4.1. A *linear quantaloid* is a locally small category $\mathscr{Q}$ whose hom-sets are complete lattices with binary operations $\otimes$ and $\oplus$, and families of distinguished morphisms $\{\top_a \mid a \in \mathrm{ob}\mathscr{Q}\}$ and $\{\bot_a \mid a \in \mathrm{ob}\mathscr{Q}\}$ such that

- $(\mathscr{Q}, \otimes, \top_a)$ and $(\mathscr{Q}^{co}, \oplus, \bot_a)$ are quantaloids,
- for all $f\colon a \to b$, $g\colon b \to c$, $h\colon c \to d \in \mathscr{Q}$,

$$f \otimes (g \oplus h) \leq (f \otimes g) \oplus h \quad \text{and} \quad (f \oplus g) \otimes h \leq f \oplus (g \otimes h)$$

Every Girard quantaloid is a linear quantaloid and we have the following obvious observation.

Lemma 4.2. A linear quantaloid is a linear bicategory.

Definition 4.3. If $\mathscr{Q}$ and $\mathscr{Q}$' are linear quantaloids, a *linear quantaloid homomorphism* $F = (F_{\otimes}, F_{\oplus})\colon \mathscr{Q} \to \mathscr{Q}'$ is a bilax linear functor such that $F_{\otimes}$ and $F_{\oplus}$ are quantaloid homomorphisms.

A main result of the present work will be new examples of linear bicategories which are linear quantaloids. To construct these, we need the definition of the analogous linear structure for quantales. These are the LD-quantales as defined below.

Definition 4.4. An LD-*quantale* $(Q, \otimes, \top, \oplus, \bot)$ is a complete lattice Q with operations $\otimes$ and $\oplus$ and elements $\top$ and $\bot$ such that

- $(Q, \otimes, \top)$ and $(Q^{op}, \oplus, \bot)$ are quantales.
- for all $a, b, c \in Q$,

$$a \otimes (b \oplus c) \leq (a \otimes b) \oplus c \quad \text{and} \quad (a \oplus b) \otimes c \leq a \oplus (b \otimes c)$$

Clearly a Girard quantale is a LD-quantale.

The notion of an LD-quantale is not truly a new one. It has previously appeared in some form in the literature, following the introduction of LDCs by Cockett and Seely (1997). In particular, it has been considered within the field of algebraic logic, when discussing ordered algebras, wherein linear distributivity was called *hemi-distributivity* by Dunn and Hardegree (2001).

Indeed, within this context, a LD-quantale refers to a lattice-ordered bimonoid $\langle \mathbb{A}, \wedge, \vee, \cdot, 1, +, 0 \rangle$, where all joins and meets are admissible in the multiplicative pomonoid $\mathbb{A}_{\cdot}$ and in the additive pomonoid $\mathbb{A}_{+}$ respectively, as defined by Galatos and Přenosil (2023).

Now, every locale is a quantale and Rosenthal (1992) remarks that a locale is a Girard quantale if and only if it is a Boolean algebra with $\mathbf{0}$ its dualizing element. We can extend this remark to LD-quantales, but fist we must introduce a slightly less well-known type of lattice.

Definition 4.5. (Reyes and Zolfaghari, 1996, Def 1.1)

- A *Heyting* algebra is a bounded distributive lattice $\mathscr{L}$ with an "implication" operator $\to : \mathscr{L} \times \mathscr{L} \to \mathscr{L}$ with the following property $\forall a, b, c \in \mathscr{L}$:

$$a \leq b \to c \iff a \wedge b \leq c$$

- A *co-Heyting* algebra is a bounded distributive lattice $\mathscr{L}$ with an "subtraction" operator $\backslash : \mathscr{L} \times \mathscr{L} \to \mathscr{L}$ with the following property $\forall a, b, c \in \mathscr{L}$:

$$a \backslash b \leq c \iff a \leq b \vee c$$

- A *bi-Heyting* algebra is a bounded distributive lattice that is both a Heyting and a co-Heyting algebra.

Then:

Proposition 4.6. A locale is a LD-quantale $(\mathscr{L}, \wedge, \mathbf{1}, \vee, \mathbf{0})$ if and only if it is a complete bi-Heyting algebra.

Proof. Note that a complete Heyting algebra is the same notion as a locale. Moreover, a locale is a LD-quantale if and only if the opposite infinitary law holds,

$$(\bigwedge_{i \in I} a_i) \vee b = \bigwedge_{i \in I} (a_i \vee b)$$

This is equivalent to requiring a right adjoint to $(-) \vee a \colon \mathscr{L}^{op} \times \mathscr{L}^{op} \to \mathscr{L}^{op}$, in other words $\mathscr{L}$ has a subtraction operation and is a co-Heyting algebra. □

Example 4.7. (1) $\mathsf{Rel}(X)$, the poset of relations on a set X, is a Girard quantale with cyclic dualizing element Δ_X^c, see Proposition 1.2 in Rosenthal (1992), and therefore is a LD-quantale.

(2) Consider the following locales which are LD-quantales.
- Two-chain Ω is a Girard quantale, and therefore a LD-quantale.
- Three-chain 3 is the smallest locale, which is not Boolean as the law of excluded middle doesn't hold:

$$1/2 \vee (0 \multimapinv 1/2) = 1/2 \vee 0 = 1/2 \neq 1$$

  As such it is not a Girard quantale, but it is nonetheless a LD-quantale as it is a bi-Heyting algebra.
- $\mathsf{P}_{\mathsf{max}}$ is a LD-quantale, as it is a bi-Heyting algebra, with $\oplus = \min$ and $\bot = \infty$, but it is not a Girard quantale as ∞ is not a dualizing element:

$$\text{if} \quad a \neq \infty, \quad a \multimap \infty = \infty \implies (a \multimap \infty) \multimap \infty = 0$$

- Reyes and Zolfaghari (1996) demonstrate that given an oriented irreflexive multigraph, the lattice of its subgraphs is a bi-Heyting algebra, and therefore an LD-quantale.
- Borceux et al. (2006) defined the notion of a bi-Heyting topos to be a topos such that the lattice of its subobjects is precisely a bi-Heyting algebra. An important example is the topos of presheafs $[\mathscr{C}^{op}, Set]$ for any small category $\mathscr{C}$. As such, its lattice of subobjects is a LD-quantale.

(3) Consider the unit interval with multiplication $([0,1], \cdot, 1)$. It becomes a LD-quantale $([0,1], \cdot, \oplus)$ when considering truncated addition $a \oplus b = \mathsf{min}(a+b, 1)$ for its par structure with unit $\bot = 0$. $([0,1]^{op}, \oplus, 0)$ is a quantale since

$$\bigwedge_\alpha \mathsf{min}(a + b_\alpha, 1) = \mathsf{min}(a + \bigwedge_\alpha b_\alpha, 1)$$

Linear distributivities hold as

$$a \cdot (b + c) = (a \cdot b) + (a \cdot c) \leq (a \cdot b) + c \qquad \forall a \in [0,1]$$

It is not however a Girard quantale as 0 is not a dualizing element:

$$\text{if} \quad a \neq 0, \quad a \multimap 0 = 0 \implies (a \multimap 0) \multimap 0 = 1$$

Since $([0,1], \cdot, 1) \cong \mathsf{P}_+$, we can use the isomorphims to define a par structure on P_+: $a \oplus b = \mathsf{max}(-\mathsf{ln}(\mathsf{exp}(-a) + \mathsf{exp}(-b)), 0)$ and $\bot = \infty$. Then, $\mathsf{P}_+ = ([0,\infty]^{op}, +, \oplus)$ is a LD-quantale isomorphic to $([0,1], \cdot, \oplus)$.

Remark 4.8. Given a locale L, its booleanization $\mathsf{Bool}(L) = \{a \in L \mid (a \multimap \mathbf{0}) \multimap \mathbf{0} = a\}$ is a complete Boolean algebra by Banaschewski and Pultr (1996). Therefore, viewing any locale as a LD-quantale, there is always a sub-locale $\mathsf{Bool}(L)$ which is a Girard quantale.

The following is another example of an LD-quantale which requires a little more discussion.

4.1 Shift monoids

Cockett and Seely (1997) introduce the notion of a *shift monoid*. These provide examples of discrete linearly distributive categories which are not $*$-autonomous.

Definition 4.9.
- A *shift monoid* consists of a 4-tuple $\mathscr{M} = (M, +, \top, a)$ where $(M, +, \top)$ is a commutative monoid and a is an invertible element of M.
- If $\mathscr{M}$ is a shift monoid, define a second multiplication by

$$x \cdot y = x + y - a$$

Then this is a second monoid structure on M with unit given by a.

It is trivial to see that $x \cdot (y + z) = (x \cdot y) + z$ and so every shift monoid is a discrete linearly distributive category $(M, +, \cdot)$. We modify the notion of shift monoid as follows in order to construct LD-quantales. Recall that a commutative monoid M is *cancellative* if for all $a, b, c \in M$, one has

$$a + b = a + c \Rightarrow b = c$$

Proposition 4.10.
- Let $(M, +, \top)$ be a cancellative commutative monoid. We view M as a discrete poset and then add top and bottom elements which we denote $\mathbf{1}$ and $\mathbf{0}$. Denote this set as M^+, We then extend the addition on M:

$$\mathbf{1} + b = \mathbf{1} \text{if } b \in M \text{ or } b = \mathbf{1} \qquad \mathbf{0} + b = \mathbf{0} \text{ for all } b \in M^+$$

Then $(M^+, +, \top)$ is a commutative quantale.
- Let $\mathscr{M} = (M, +, \top, a)$ be furthermore a cancellative commutative shift monoid, then extend the second operation in the dual way:

$$\mathbf{0} \cdot b = \mathbf{0} \text{ if } b \in M \text{ or } b = \mathbf{0} \qquad \mathbf{1} \cdot b = \mathbf{1} \text{ for all } b \in M^+$$

Then $(M^+, +, \cdot)$ is a LD-quantale.

Remark 4.11. The cancellative property is needed to ensure that $+$ preserves all suprema in M^+.

5. Q-Rel as a linear bicategory

We now demonstrate that if Q is a Girard quantale, or more generally an LD-quantale, Q-Rel determines a linear quantaloid, providing new examples of linear bicategories.

Proposition 5.1. Q-Rel is a Girard quantaloid with cyclic dualizing family $\mathscr{D} = \{\bot_X : X \nrightarrow X\}$ if and only if Q is a Girard quantale with cyclic dualizing element $\bot$, where $\bot = \bot_1(*, *)$, $\bot_1 : 1 \nrightarrow 1$ being the constant map between singleton sets $1 = \{*\}$, and

$$\bot_X(x, x') = \begin{cases} \bot & x = x' \\ \mathbf{1} & x \neq x' \end{cases} \qquad (*)$$

Proof. Suppose Q-Rel is a Girard quantaloid. Given $a \in Q$, let $R_a : 1 \nrightarrow 1$ denote the a-valued constant relation. Since $R_a \multimap \bot_1 = \bot_1 \circ\!\!- R_a$ and $R_a^{\bot\bot} = R_a$, it follows that $\bot = \bot_1(*, *)$ is a cyclic dualizing element for Q, and so Q is a Girard quantale.

Consider a set X. Let $x' \in X$ and define Q-relation $R\colon 1 \nrightarrow X$ by

$$R(*,x) = \begin{cases} \top & x = x' \\ \mathbf{1} & x \neq x' \end{cases}$$

Now $(\bot_X \circ\!- R)(x,*) = \bigwedge_{\bar{x}} \bot_X(x,\bar{x}) \circ\!- R(*,\bar{x}) = \bot_X(x,x')$ and

$$(R -\!\circ \bot_1)(x,*) = R(*,x) -\!\circ \bot = \begin{cases} \bot & x = x' \\ \mathbf{1} & x \neq x' \end{cases}$$

since $\top -\!\circ \bot = \bot$. As $\mathscr{D}$is cyclic, $\bot_X \circ\!- r = r -\!\circ \bot_1$ and consequently, (*) holds.

Suppose Q is a Girard quantale with cyclic dualizing element $\bot$. Define a family of Q-relations $\mathscr{D}$by $(*)$. Consider $R\colon X \nrightarrow Y$, then

$$(R -\!\circ \bot_X)(y,x) = \bigwedge_{\bar{x}} R(\bar{x},y) -\!\circ \bot_X(\bar{x},x) = R(x,y) -\!\circ \bot = R(x,y)^{\bot}$$

$$(\bot_Y \circ\!- r)(y,x) = \bigwedge_{\bar{y}} d_Y(y,\bar{y}) \circ\!- r(x,\bar{y}) = \bot \circ\!- R(x,y) = R(x,y)^{\bot}$$

and so $\mathscr{D}$is a dualizing family for Q-Rel as $R -\!\circ \bot_X = \bot_Y \circ\!- R$. $\mathscr{D}$being cyclic follows immediately and as such Q-Rel is a Girard quantaloid. □

Consequently, if Q is a Girard quantale, there is a second categorical structure on Q-Rel determining a linear bicategory. Indeed its second composition is obtained as the de Morgan dual of its standard composition

$$R \oplus S(x,z) = (\bigvee_{y \in Y} S(y,z)^{\bot} \otimes R(x,y)^{\bot})^{\bot} = \bigwedge_{y \in Y} (S(y,z)^{\bot} \otimes R(x,y)^{\bot})^{\bot} = \bigwedge_{y \in Y} R(x,y) \oplus S(y,z)$$

with identities $\bot_X$.

This generalizes further. Suppose $(Q,\otimes,\top)$ and $(Q^{op},\oplus,\bot)$ are quantales. Two notions of composition

$$\otimes,\oplus\colon \mathscr{B}(X,Y) \times \mathscr{B}(Y,Z) \longrightarrow \mathscr{B}(X,Z)$$

are defined as follows: given $R\colon X \nrightarrow Y$ and $S\colon Y \nrightarrow Z$,

$$R \otimes S(x,z) = \bigvee_{y \in Y} (R(x,y) \otimes S(y,z)) \qquad R \oplus S(x,z) = \bigwedge_{y \in Y} (R(x,y) \oplus S(y,z))$$

The identity 1-cells are given by:

$$\top_X(x,x') = \begin{cases} \top & \text{if } x = x' \\ \mathbf{0} & \text{if } x \neq x' \end{cases} \qquad \bot_X(x,x') = \begin{cases} \bot & \text{if } x = x' \\ \mathbf{1} & \text{if } x \neq x' \end{cases}$$

and we get:

Theorem 5.2. Q is an LD-quantale if and only if Q-Rel is a linear quantaloid.

Proof. By Lemma 2.9, $(Q\text{-Rel},\otimes,\top_X)$ and $(Q^{op}\text{-Rel},\oplus,\bot_X) \cong (Q\text{-Rel}^{co},\oplus,\bot_X)$ are quantaloids as $(Q,\otimes,\top)$ and $(Q^{op},\oplus,\bot)$ are quantales, inheriting their structure from Q point-wise.

Suppose $(\mathsf{Q}, \otimes, \oplus)$ is an LD-quantale. Given

$$W \overset{R}{\nrightarrow} X \overset{S}{\nrightarrow} Y \overset{T}{\nrightarrow} Z$$

we have $R \otimes (S \oplus T) \leq (R \otimes S) \oplus T$ if and only if, for all w, z,

$$\bigvee_x [R(w,x) \otimes (S \oplus T)(x,z)] \leq \bigwedge_y [(R \otimes S)(w,y) \oplus T(y,z)]$$

if and only if $R(w,x) \otimes (S \oplus T)(x,z) \leq (R \otimes S)(w,y) \oplus T(y,z)$, for all w,x,y,z. But,

$$\begin{aligned} R(w,x) \otimes (S \oplus T)(x,z) &= R(w,x) \otimes \bigwedge_y [S(x,y) \oplus T(y,z)] \\ &\leq R(w,x) \otimes [S(x,y) \oplus T(y,z)] \\ &\leq [R(w,x) \otimes S(x,y)] \oplus T(y,z) \\ &\leq \bigvee_x [R(w,x) \otimes S(x,y)] \oplus T(y,z) \\ &\leq (R \otimes S)(w,y) \oplus T(y,z) \end{aligned}$$

The other inequality follows similarly and we conclude that Q-Rel is a linear quantaloid.

Conversely, suppose Q-Rel is a linear quantaloid. Then a, b, c in Q induces $1 \overset{R_a}{\nrightarrow} 1 \overset{R_b}{\nrightarrow} 1 \overset{R_c}{\nrightarrow} 1$ in Q-Rel, and $R_a \otimes (R_b \oplus R_c) \leq (R_a \otimes R_b) \oplus R_c$ implies $a \otimes (b \oplus c) \leq (a \otimes b) \oplus c$. Thus Q is an LD-quantale. □

Q-Rel is a linear quantaloid for all locales Q as they are LD-quantales.

Example 5.3. (1) 2-$\mathsf{Rel} \cong \mathsf{Rel}$ is a Girard quantaloid, with its par composition given by de Morgan duality, the same as previously introduced in the preliminaries section.

(2) 3-Rel is a linear quantaloid, which is not a Girard quantaloid, as 3 is a complete bi-Heyting algebra, but not a Boolean algebra.

(3) $\mathsf{P}_{\mathsf{max}}$-$\mathsf{Rel}$, the quantaloid of sets and "extended" distance relations, is a linear quantaloid, which is not Girard, with its par composition defined, for $D_1 \colon X \nrightarrow Y$ and $D_2 \colon Y \nrightarrow X$ by

$$(D_1 \oplus D_2)(x,z) = \bigvee_{y \in Y} \mathsf{min}(D_1(x,y), D_2(y,z))$$

and par identities given by

$$\perp_X(x,x') = \begin{cases} \infty & \text{if } x = x' \\ 0 & \text{if } x \neq x' \end{cases}$$

(4) $[0,1]$-Rel is another linear quantaloid, which is not Girard, of sets and relations, with its par composition defined, for $D_1 \colon X \nrightarrow Y$ and $D_2 \colon Y \nrightarrow X$ by

$$(D_1 \oplus D_2)(x,z) = \bigwedge_{y \in Y} \mathsf{min}(D_1(x,y) + D_2(y,z), 1)$$

and par identities given by

$$\perp_X(x,x') = \begin{cases} 0 & \text{if } x = x' \\ 1 & \text{if } x \neq x' \end{cases}$$

Alternatively, P_+-Rel with its par composition defined, for $D_1 \colon X \nrightarrow Y$ and $D_2 \colon Y \nrightarrow X$ by

$$(D_1 \oplus D_2)(x,z) = \bigvee_{y \in Y} \mathsf{max}(-\mathsf{ln}(\mathsf{exp}(-D_1(x,y)) + \mathsf{exp}(-D_2(y,z))), 0)$$

and par identities given by

$$\perp_X(x,x') = \begin{cases} \infty & \text{if } x = x' \\ 0 & \text{if } x \neq x' \end{cases}$$

6. Enriched in a linear quantaloid

Rosenthal (1992) demonstrated that the definition of a Girard quantaloid was closed under various constructions, in particular if $\mathcal{Q}$ is a Girard quantaloid, then $\mathcal{Q}$-Mod is one as well. This can be easily extended into a necessary and sufficient condition.

Proposition 6.1. $\mathcal{Q}$-Mod is a Girard quantaloid with cyclic dualizing family $\{\perp\!\!\perp_M : M \nrightarrow M\}$ if and only if $\mathcal{Q}$ is a Girard quantaloid with cyclic dualizing family $\{\perp_a : a \to a\}$, where $\perp_a = \perp\!\!\perp_{M_a}(*,*)$, $M_a = (1,\rho_a)$ being the $\mathcal{Q}$-category defined by $\rho_a(*) = a$ and $M_a(*,*) = \top_a$, and

$$\perp\!\!\perp_M(x,x') = M(x',x) \multimap \perp_{\rho(x')}$$

Proof. Suppose $\mathcal{Q}$-Mod is a Girard quantaloid with cyclic dualizing family $\{\perp\!\!\perp_M : M \nrightarrow M\}$. Given an object a in $\mathcal{Q}$, define the $\mathcal{Q}$-category $M_a = (1,\rho_a)$ as indicated above. Define a family of morphisms $\mathcal{D} = \{\perp_a : a \to a\}$ in $\mathcal{Q}$ by $\perp_a = \perp\!\!\perp_{M_a}(*,*)$.

Given a morphism $f : a \to b$ in $\mathcal{Q}$, consider its image $\Theta_f : M_a \nrightarrow M_b$ under the embedding of $\mathcal{Q}$ into $\mathcal{Q}$-Mod. Then,

$$f \multimap \perp_a = \Theta_f(*,*) \multimap \perp\!\!\perp_{M_a}(*,*) = \bigwedge_{x\in\{*\}} \Theta_f(x,*) \multimap \perp\!\!\perp_{M_a}(x,*) = (\Theta_f \multimap \perp\!\!\perp_{M_a})(*,*) = \Theta_f^{\perp}(*,*)$$

$$\perp_b \multimapinv f = \perp\!\!\perp_{M_b}(*,*) \multimapinv \Theta_f(*,*) = \bigwedge_{x\in\{*\}} \perp\!\!\perp_{M_b}(*,x) \multimapinv \Theta_f(*,x) = (\perp\!\!\perp_{M_b} \multimapinv \Theta_f)(*,*) = \Theta_f^{\perp}(*,*)$$

implying $\mathcal{D}$is a cyclic family of morphisms in $\mathcal{Q}$ and $\mathcal{D}$being dualizing follows similarly.

The sufficiency proof is shown in Theorem 3.1 in Rosenthal (1992). □

Notice that if $\mathcal{Q}$ is a Girard quantaloid, then for each $\mathcal{Q}$-category $M = (X,\rho)$ and $(x,x') \in X$, there is a morphism $M(x',x)^{\perp} : \rho(x) \to \rho(x')$ satisfying $\forall x,x',x'' \in X$

$$M(x,x)^{\perp} \leq \perp_{\rho(x)} \qquad M(x,x'')^{\perp} \leq M(x',x'')^{\perp} \oplus M(x,x')^{\perp}$$

Therefore the map $M^{\perp} : (x,x') \mapsto M(x',x)^{\perp}$, coupled with (X,ρ) is a $\mathcal{Q}^{co}$-category, where $\mathcal{Q}^{co}$ is the quantaloid with the de Morgan dual composition $\oplus$ and identities $\perp_X$.

$M^{\perp}$ is moreover a $\mathcal{Q}$-module $M \nrightarrow M$: $\forall x,x',x'' \in X$

$$M(x',x)^{\perp} \otimes M(x',x'') \leq M(x'',x)^{\perp} \qquad M(x,x') \otimes M(x',x'')^{\perp} \leq M(x'',x)^{\perp}$$

While M becomes a $\mathcal{Q}^{co}$-module $M^{\perp} \nrightarrow M^{\perp}$: $\forall x,x',x'' \in X$

$$M(x,x'') \leq M(x',x)^{\perp} \oplus M(x',x'') \qquad M(x,x'') \leq M(x,x') \oplus M(x'',x')^{\perp}$$

All together, this entails that $(M,M^{\perp}) : X \to \mathcal{Q}$ is a bilax linear functor, when X is viewed as a degenerate indiscrete linear bicategory.

Each $\mathcal{Q}$-module $\Theta\colon M \nrightarrow N$ interacts coherently with this structure and it determines a $\mathcal{Q}^{co}$-module $M^{\perp} \nrightarrow N^{\perp}$: $\forall x, x' \in X, y, y' \in Y$

$$\Theta(x, y') \leq M(x, x')^{\perp} \oplus \Theta(x, y) \qquad \Theta(x, y) \leq \Theta(x, y') \oplus N(y, y')^{\perp}$$

As with every Girard quantaloid, we can define a second composition on $\mathcal{Q}$-Mod. Given $M \overset{\Theta}{\nrightarrow} N \overset{\Pi}{\nrightarrow} P$, $\Theta \otimes \Pi\colon M \nrightarrow P$ and $\Theta \oplus \Pi\colon M \nrightarrow P$ are defined by

$$(\Theta \otimes \Pi)(x, z) = \bigvee_{y \in Y} \Theta(x, y) \otimes \Pi(y, z) \quad \text{and} \quad (\Theta \oplus \Pi)(x, z) = \bigwedge_{y \in Y} \Theta(x, y) \oplus \Pi(y, z)$$

The identities 1-cells for $\otimes$ and $\oplus$ are given by $\top\!\!\top_M$ and $\perp\!\!\perp_M^{\perp}$ respectively, where

$$\top\!\!\top_M(x, x') = M(x, x') \quad \text{and} \quad \perp\!\!\perp_M(x, x') = M(x', x)^{\perp}$$

The above discussion motivates the following new definitions which generalize $\mathcal{Q}$-Mod to the case where $\mathcal{Q}$ is a linear quantaloid as follows.

Let $\mathcal{Q}$ be a linear quantaloid.

Definition 6.2. A *linear $\mathcal{Q}$-category* is a pair $M = (X, \rho)$ where:

- X is a set.
- ρ is a function $X \to ob\mathcal{Q}$,
- there is a function, called the $\otimes$-*enrichment*, assigning a morphism $M_{\otimes}(x, x')\colon \rho(x) \to \rho(x')$ in $\mathcal{Q}$ to each pair $(x, x') \in X \times X$ such that $\forall x, x', x'' \in X$
$$\top_{\rho(x)} \leq M_{\otimes}(x, x) \qquad M_{\otimes}(x, x') \otimes M_{\otimes}(x', x'') \leq M_{\otimes}(x, x'')$$
- there is a function, called the $\oplus$-*enrichment*, assigning a morphism $M_{\oplus}(x, x')\colon \rho(x) \to \rho(x')$ in $\mathcal{Q}$ to each pair $(x, x') \in X \times X$ such that $\forall x, x', x'' \in X$
$$M_{\oplus}(x, x) \leq \perp_{\rho(x)} \qquad M_{\oplus}(x, x'') \leq M_{\oplus}(x, x') \oplus M_{\oplus}(x', x'')$$
satisfying the following inequalities $\forall x, x', x'' \in X$,
$$M_{\otimes}(x, x'') \leq M_{\oplus}(x, x') \oplus M_{\otimes}(x', x'') \qquad M_{\otimes}(x, x'') \leq M_{\otimes}(x, x') \oplus M_{\oplus}(x', x'')$$
$$M_{\otimes}(x, x') \otimes M_{\oplus}(x', x'') \leq M_{\oplus}(x, x'') \qquad M_{\oplus}(x, x') \otimes M_{\otimes}(x', x'') \leq M_{\oplus}(x, x'')$$

More succinctly, if $\mathcal{Q}$ is a linear quantaloid, a linear $\mathcal{Q}$-category is a bilax linear functor from X to $\mathcal{Q}$, where X is viewed as a degenerate indiscrete linear bicategory.

Note that given a linear $\mathcal{Q}$-category $M = (X, \rho)$, $M_{\otimes} = (X, \rho)$ is a $\mathcal{Q}$-category and $M_{\oplus} = (X, \rho)$ is a $\mathcal{Q}^{co}$-category. Moreover, the $\otimes$-enrichment and $\oplus$-enrichment together assign cyclic linear adjoints in $\mathcal{Q}$ as follows.

Proposition 6.3. Consider a linear $\mathcal{Q}$-category $M = (X, \rho)$ and a pair $(x, x') \in X \times X$, then $\otimes$-enrichment and $\oplus$-enrichments provide a cyclic linear adjunctions $M_{\otimes}(x, x') \Vdash\!\!\!\!\to\!\!\!\Vdash M_{\oplus}(x', x)\colon \rho(x) \to \rho(x')$.

Proof. To show $M_{\otimes}(x, x') \dashv\!\!\!\Vdash M_{\oplus}(x', x)$, we need only provide the unit and co-unit 2-cells, which are inequalities in this context.

$$\top_{\rho(x)} \leq M_{\otimes}(x, x) \leq M_{\otimes}(x, x') \oplus M_{\oplus}(x', x)$$

$$M_{\oplus}(x',x) \otimes M_{\otimes}(x,x') \leq M_{\oplus}(x',x') \leq \bot_{\rho(x')}$$

Similarly, $M_{\oplus}(x',x) \dashv\!\Vert M_{\otimes}(x,x')$. □

Definition 6.4. Consider linear $\mathscr{Q}$-categories $M = (X, \rho_M)$ and $N = (Y, \rho_N)$. A *linear $\mathscr{Q}$-module* $\Theta\colon M \nrightarrow N$ consists of an assignment of a morphism $\Theta(x,y)\colon \rho_M(x) \to \rho_N(y)$ to every pair $(x,y) \in X \times Y$ such that $\forall x, x' \in X, y, y' \in Y$,

$$\Theta(x,y) \otimes N_{\otimes}(y,y') \leq \Theta(x,y') \qquad M_{\otimes}(x,x') \otimes \Theta(x',y) \leq \Theta(x,y)$$

$$\Theta(x,y) \leq M_{\oplus}(x,x') \oplus \Theta(x',y) \qquad \Theta(x,y) \leq \Theta(x,y') \oplus N_{\oplus}(y',y)$$

The above definition of a linear $\mathscr{Q}$-module can be simplified, as the four inequalities are not independent: the top two imply the bottom two, and vice versa.

Proposition 6.5. Consider linear $\mathscr{Q}$-categories $M = (X, \rho_M)$ and $N = (Y, \rho_N)$, then $\Theta\colon M \nrightarrow N$ is a *linear $\mathscr{Q}$-module* if one of the following conditions holds:

(1) Θ is a $\mathscr{Q}$-module $M_{\otimes} \nrightarrow N_{\otimes}$, and
(2) Θ is a $\mathscr{Q}^{co}$-module $M_{\oplus} \nrightarrow N_{\oplus}$.

Proof. Suppose Θ is a $\mathscr{Q}$-module $M_{\otimes} \nrightarrow N_{\otimes}$, in other words $\Theta(x,y) \otimes N_{\otimes}(y,y') \leq \Theta(x,y')$ and $M_{\otimes}(x,x') \otimes \Theta(x',y) \leq \Theta(x,y)$ holds $\forall x, x' \in X, y, y' \in Y$. Then,

$$\begin{aligned}\Theta(x,y) = \Theta(x,y) \otimes \top_{\rho_N(y)} &\leq \Theta(x,y) \otimes N_{\otimes}(y,y) \leq \Theta(x,y) \otimes (N_{\otimes}(y,y') \oplus N_{\oplus}(y',y)) \\ &\leq (\Theta(x,y) \otimes N_{\otimes}(y,y')) \oplus N_{\oplus}(y',y) \leq \Theta(x,y') \oplus N_{\oplus}(y',y)\end{aligned}$$

$$\begin{aligned}\Theta(x,y) = \top_{\rho_X(x)} \otimes \Theta(x,y) &\leq M_{\otimes}(x,x) \otimes \Theta(x,y) \leq (M_{\oplus}(x,x') \oplus M_{\otimes}(x',x)) \otimes \Theta(x,y) \\ &\leq M_{\oplus}(x,x') \oplus (M_{\otimes}(x',x) \otimes \Theta(x,y)) \leq M_{\oplus}(x,x') \oplus \Theta(x',y)\end{aligned}$$

The other direction follows similarly. □

Remark 6.6. It was shown by Cockett, Koslowski and Seely that representable poly-bicategories are linear bicategories and poly-functors between them are bilax linear functors. When viewing linear $\mathscr{Q}$-categories as a poly-functors between representable poly-bicategories, a linear $\mathscr{Q}$-module is a *poly-module* between the poly-functors, as defined in Def 4.1 Cockett et al. (2003).

We can now define bicategory structures attached to the above data.

Definition 6.7. Define $\mathscr{Q}$-Mod to consist of the following data:

- 0-cells are linear $\mathscr{Q}$-categories
- given a pair of $\mathscr{Q}$-categories M and N, there is a category with 1-cells the linear $\mathscr{Q}$-modules $M \nrightarrow N$ and 2-cells the point-wise inequalities, i.e.,

$$\Theta \leq \Pi \Leftrightarrow \forall (x,y) \in X \times Y, \quad \Theta(x,y) \leq \Pi(x,y)$$

- given $\mathscr{Q}$-categories M, N and P, there are two composition functors $\otimes, \oplus$ defined for $\Theta\colon M \nrightarrow N$ and $\Pi\colon N \nrightarrow P$ by

$$(\Theta \otimes \Pi)(x,z) = \bigvee_{y \in Y} \Theta(x,y) \otimes \Pi(y,z) \qquad \text{and} \qquad (\Theta \oplus \Pi)(x,z) = \bigwedge_{y \in Y} \Theta(x,y) \oplus \Pi(y,z)$$

- given every $\mathcal{Q}$-category M, there are identity 1-cells $\top\!\!\top_M, \perp\!\!\perp_M : M \nrightarrow M$ defined by

$$\top\!\!\top_M(x,x') = M_{\otimes}(x,x') \qquad \text{and} \qquad \perp\!\!\perp_M(x,x') = M_{\oplus}(x,x')$$

Note that the assumption that $\mathcal{Q}$ was a linear quantaloid was not entirely necessary to define $\mathcal{Q}$-ModẆe did not need to in fact assume the linear distributivity inequalities held, which allows us to state the following theorem. An analogous remark can be made for the other constructions we will consider.

Theorem 6.8. $(\mathcal{Q}\text{-}\mathsf{Mod}, \otimes, \oplus)$ is a linear quantaloid if and only if $(\mathcal{Q}, \otimes, \oplus)$ is a linear quantaloid.

This will be an immediate consequences of similar results for linear $\mathcal{Q}$-matrices and linear monads in $\mathcal{Q}$, so we will delay proving the above theorem.

Another construction that was shown to be Girard by Rosenthal (1992) is the quantaloid of $\mathcal{Q}$-matrices:

Proposition 6.9. $\mathsf{Matr}\mathcal{Q}$ is a Girard quantaloid with cyclic dualizing family $\{\perp\!\!\perp_{(X,\gamma)} : (X,\gamma) \nrightarrow (X,\gamma)\}$ if and only if $\mathcal{Q}$ is a Girard quantaloid with cyclic dualizing family $\{\perp_a : a \to a\}$, where $\perp_a = \perp\!\!\perp_{(1,\gamma_a)_{*,*}}$, $(1,\gamma_a)$ consisting of the singleton set 1 and the function $\gamma_a : * \mapsto a$, and

$$\perp\!\!\perp_{(X,\gamma)_{x,x'}} = \begin{cases} \perp_{\gamma(x)} & x = x' \\ \mathbf{1}_{\gamma(x),\gamma(x')} & x \neq x' \end{cases}$$

Proof. The forward direction follows similarly to the proof in the case of $\mathcal{Q}$-Mod, while the backwards direction proved by Theorem 3.2 in Rosenthal (1992). □

If $\mathcal{Q}$ is a linear quantaloid, then $\mathsf{Matr}\mathcal{Q}$ inherits a second bicategorical structure. The two notions of composition are defined as follows: given $\mathcal{Q}$-matrices $(X,\gamma) \overset{r}{\nrightarrow} (Y,\phi) \overset{s}{\nrightarrow} (Z,\chi)$, $r \otimes s, r \oplus s : (X,\gamma) \nrightarrow (Z,\chi)$ are defined by

$$(r \otimes s)_{x,z} = \bigvee_{y \in Y} r_{x,y} \otimes s_{y,z} \qquad \text{and} \qquad (r \oplus s)_{x,z} = \bigwedge_{y \in Y} r_{x,y} \oplus s_{y,z}$$

Identity 1-cells $\top\!\!\top_{(X,\gamma)}, \perp\!\!\perp_{(X,\gamma)} : (X,\gamma) \nrightarrow (X,\gamma)$ are defined by

$$\top\!\!\top_{(X,\gamma)_{x,x'}} = \begin{cases} \top_{\gamma(x)} & \text{if } x = x' \\ \mathbf{0}_{\gamma(x),\phi(x')} & \text{if } x \neq x' \end{cases} \qquad \text{and} \qquad \perp\!\!\perp_{(X,\gamma)_{x,x'}} = \begin{cases} \perp_{\gamma(x)} & \text{if } x = x' \\ \mathbf{1}_{\gamma(x),\phi(x')} & \text{if } x \neq x' \end{cases}$$

Theorem 6.10. $\mathsf{Matr}\mathcal{Q}$ is a linear quantaloid if and only if $\mathcal{Q}$ is a linear quantaloid.

Proof. The proof is identical to that of Theorem 5.2, replacing objects in a quantale Q by morphisms in $\mathcal{Q}$. □

Remark 6.11. As in the case of ordinary quantales, it is immediate that for an LD-quantale Q, the linear quantaloid $\mathsf{Matr}\mathcal{B}(Q)$ is isomorphic to Q-Rel.

Finally, as one might expect, taking monads and modules in a Girard quantaloid remains Girard.

Proposition 6.12. $\mathsf{Mon}\mathcal{Q}$ is a Girard quantaloid with cyclic dualizing family $\{\perp\!\!\!\perp_{(a,m)}\colon (a,m) \twoheadrightarrow (a,m)\}$ if and only if $\mathcal{Q}$ is a Girard quantaloid with cyclic dualizing family $\{\perp_a\colon a \to a\}$, where $\perp_a = \perp\!\!\!\perp_{(a,\top_a)}$, $(a,\top_a)$ being the trivial monad on a, and

$$\perp\!\!\!\perp_{(a,m)} = m \multimap \perp_a$$

Proof. The forward direction is immediate from the trivial monad embedding of $\mathcal{Q}$ into $\mathsf{Mon}\mathcal{Q}$, while the backwards direction is proven by Theorem 3.3 in Rosenthal (1992). □

As in the case of $\mathcal{Q}$-Mod, this can be generalized to the linear setting.

Let $\mathcal{Q}$ be a linear quantaloid.

Definition 6.13. (Cockett et al., 2000, Def 4.13) A *linear monad* $(a,m) = (a,m_\otimes,m_\oplus)$ in $\mathcal{Q}$ is a pair of compatible $\otimes$-monad $(a,m_\otimes)$ and $\oplus$-comonad $(a,m_\oplus)$, i.e., it consists of

- an object a
- a morphism $m_\otimes : a \to a$ such that

$$\top_a \leq m_\otimes \qquad \text{and} \qquad m_\otimes \otimes m_\otimes \leq m_\otimes$$

- a morphism $m_\oplus : a \to a$ such that

$$m_\oplus \leq \perp_a \qquad \text{and} \qquad m_\oplus \leq m_\oplus \oplus m_\oplus$$

  satisfying the following inequalities:

$$m_\otimes \leq m_\oplus \oplus m_\otimes \qquad m_\otimes \leq m_\otimes \oplus m_\oplus$$

$$m_\otimes \otimes m_\oplus \leq m_\oplus \qquad m_\oplus \otimes m_\otimes \leq m_\oplus$$

Definition 6.14. Let (a,m) and (b,n) be linear monads in $\mathcal{Q}$. A *linear (m,n)-module* (or monad module) $f\colon (a,m) \twoheadrightarrow (b,n)$ is a morphism $f\colon a \to b$ in $\mathcal{Q}$ which is $\otimes$-monad module $f\colon (a,m_\otimes) \twoheadrightarrow (b,n_\otimes)$ and a $\oplus$-comonad module $f\colon (a,m_\oplus) \twoheadrightarrow (b,n_\oplus)$, i.e., satisfies the following inequalities:

$$f \otimes n_\otimes \leq f \qquad m_\otimes \otimes f \leq f$$

$$f \leq m_\oplus \oplus f \qquad f \leq f \oplus n_\oplus$$

For the same reasons as in the case of linear $\mathcal{Q}$-modules, the $\otimes$ and $\oplus$ monad maps are cyclic linear adjoints and the above definition can be simplified. A morphism $f\colon a \to b$ in $\mathcal{Q}$ is $\otimes$-monad module $f\colon (a,m_\otimes) \twoheadrightarrow (b,n_\otimes)$ if and only if it is a $\oplus$-comonad module $f\colon (a,m_\oplus) \twoheadrightarrow (b,n_\oplus)$.

Definition 6.15. Define $\mathsf{Mon}\mathcal{Q}$ to consist of the following data:

- 0-cells are linear monads in $\mathcal{Q}$
- given a pair of linear monads (a,m) and (b,n), there is a category with 1-cells the linear (m,n)-modules and 2-cells are inherited from $\mathcal{Q}$
- given linear monads (a,m), (b,n) and (c,p), there are two composition functors $\otimes,\oplus$ which are inherited from $\mathcal{Q}$

- given every linear monad (a,m), there are identity 1-cells $\top\!\!\top_{(a,m)}, \perp\!\!\perp_{(a,m)}\colon (a,m) \nrightarrow (a,m)$ defined by

$$\top\!\!\top_{(a,m)} = m_{\otimes} \qquad \text{and} \qquad \perp\!\!\perp_{(a,m)} = m_{\oplus}$$

Lemma 6.16. If $\mathscr{Q}$ is a linear quantaloid, then $(\mathsf{Mon}\mathscr{Q}, \otimes, \top\!\!\top_{(a,m)})$ and $(\mathsf{Mon}\mathscr{Q}^{co}, \oplus, \perp\!\!\perp_{(a,m)})$ are quantaloids.

Proof. $(\mathsf{Mon}\mathscr{Q}, \otimes, \top\!\!\top_{(a,m)})$ is a category with a well-defined composition $\otimes$: given linear monad modules $f\colon (a,m) \nrightarrow (b,n)$ and $g\colon (b,n) \nrightarrow (c,p)$, it is immediate that $f \otimes g\colon a \to c$ is a $\otimes$-monad module since f, g are $\otimes$-monad modules and, by linear distributivity, $f \otimes g\colon a \to c$ is a $\oplus$-comonad module as follows.

$$f \otimes g \leq (m_{\oplus} \oplus f) \otimes g \leq m_{\oplus} \oplus (f \otimes g) \qquad \text{and} \qquad f \otimes g \leq f \otimes (g \oplus p_{\oplus}) \leq (f \otimes g) \oplus p_{\oplus}$$

as f, g are $\oplus$-comonad modules. Identities $\top\!\!\top_{(a,m)}\colon (a,m) \nrightarrow (a,m)$ are also well-defined as $m_{\otimes}$ is a linear monad module:

$$m_{\otimes} \otimes m_{\otimes} \leq m_{\otimes} \quad , \qquad m_{\otimes} \leq m_{\oplus} \oplus m_{\otimes} \qquad \text{and} \qquad m_{\otimes} \leq m_{\otimes} \oplus m_{\oplus}$$

by the definition of linear $\mathscr{Q}$-categories. The quantaloid structure is directly inherited from $(\mathscr{Q}, \otimes, \top)$. Similarly, $(\mathsf{Mon}\mathscr{Q}^{co}, \oplus, \perp\!\!\perp_{(a,m)})$ is a quantaloid. □

Then we get this result:

Theorem 6.17. $\mathsf{Mon}\mathscr{Q}$ is a linear quantaloid if and only if $\mathscr{Q}$ is a linear quantaloid.

Proof. Suppose $\mathscr{Q}$ is a linear quantaloid, then it is immediate that $\mathsf{Mon}\mathscr{Q}$ is a linear quantaloid as compositions $\otimes$ and $\oplus$ are inherited directly.

Suppose $\mathsf{Mon}\mathscr{Q}$ is a linear quantaloid, consider the mapping of each object $a \in \mathscr{Q}$ to the trivial linear monad $(a, \top_a, \perp_a)$ and each morphism $f\colon a \to b$ to itself viewed as the trivial linear monad module $f\colon (a, \top_a, \perp_a) \nrightarrow (b, \top_b, \perp_b)$. Then, $\forall f\colon a \to b, g\colon b \to c, h\colon c \to d$, we know $f \otimes (g \oplus h) \leq (f \otimes g) \oplus h$ and $(f \oplus g) \otimes h \leq f \oplus (g \otimes h)$, since the inequalities hold when they are viewed as 1-cells in $\mathsf{Mon}\mathscr{Q}$. □

We now return to the proof of Theorem 6.8:

Proof. Suppose $\mathscr{Q}$ is a linear quantaloid, then $\mathsf{Matr}\mathscr{Q}$ is a linear quantaloid and therefore we can perform the linear monad construction and see that $\mathsf{MonMatr}\mathscr{Q}$ is a linear quantaloid. By looking at the definitions, it is immediate that $(\mathsf{MonMatr}\mathscr{Q}, \otimes, \top\!\!\top_{(X,\gamma),m)})$ is biequivalent to $(\mathscr{Q}\text{-}\mathsf{Mod}, \otimes, \top\!\!\top_M)$ and $(\mathsf{MonMatr}\mathscr{Q}^{co}, \oplus, \perp\!\!\perp_{(X,\gamma),m)})$ is biequivalent to $(\mathscr{Q}\text{-}\mathsf{Mod}^{co}, \oplus, \perp\!\!\perp_M)$. Thus $\mathscr{Q}\text{-}\mathsf{Mod}$ and $\mathscr{Q}\text{-}\mathsf{Mod}^{co}$ are quantaloids.

Moreover, given linear $\mathscr{Q}$-modules $\Theta\colon M \nrightarrow N$, $\Pi\colon N \nrightarrow P$ and $\Sigma\colon P \nrightarrow R$,

$$\Theta \otimes (\Pi \oplus \Sigma) \leq (\Theta \otimes \Pi) \oplus \Sigma \quad \text{and} \quad (\Theta \oplus \Pi) \otimes \Sigma \leq \Theta \oplus (\Pi \otimes \Sigma)$$

since the inequalities hold when they are viewed as linear (M,N), (N,P) and (P,R)-monad modules in $\mathsf{MonMatr}\mathscr{Q}$.

Suppose $\mathscr{Q}\text{-}\mathsf{Mod}$ is a linear quantaloid, then consider the mapping of each object $a \in \mathscr{Q}$ to the trivial linear $\mathscr{Q}$-category $M_a = (1, \rho_a)$ where $1 = \{*\}$ is the singleton set, $\rho_a(*) = a$, $M_{a\otimes} = \top_a$ and $M_{a\oplus} = \perp_a$, each morphism to $f\colon a \to b$ to the trivial linear $\mathscr{Q}$-module $\Theta_f\colon M_a \nrightarrow M_b$, where $\Theta_f(*,*) = f$. This mapping ensures the linear distributivities hold in $\mathscr{Q}$ as they hold $\mathscr{Q}\text{-}\mathsf{Mod}$. □

Remark 6.18. As expected, the constructions are related: let $\mathcal{Q}$ be a linear quantaloid, then $\mathcal{Q}\text{-}\mathsf{Mod} \cong \mathsf{MonMatr}\mathcal{Q}$. Therefore, given any LD-quantale Q, we can then consider linear quantaloid $\mathsf{Mon}(Q\text{-}\mathsf{Rel})$ which is isomorphic to $\mathsf{MonMatr}\mathscr{B}(Q)$ and $\mathscr{B}(Q)\text{-}\mathsf{Mod}$.

Example 6.19. Let Rel be the Girard quantaloid of sets and relations. The linear monads in Rel are preordered sets $(X,\leq)$ additionally endowed with the relation $\leq^{\perp}$ defined by

$$x \leq^{\perp} y \iff x \nleq y$$

The linear monad modules are order ideals $R\colon (X,\leq_X) \to (Y,\leq_Y)$. In other words, the linear quantaloid of linear monads in Rel is isomorphic to the quantaloid of ordered sets and ideals $\mathsf{Mon}(\mathsf{Rel}) \cong \mathsf{Ord}$.

The above example follows from a more general result about Girard quantaloids (and even further cyclic $*$-autonomous categories). Given a linear monad $(a, m_{\otimes}, m_{\oplus})$ in a quantaloid $\mathcal{Q}$, the monad maps $m_{\otimes}\colon a \to a$ and $m_{\oplus}\colon a \to a$ are cyclic linear adjoints. Recall that these are unique (up to isomorphism, which is equality in the posetal context) and, as a cyclic $*$-autonomous bicategory, cyclic linear adjoints are canonically given by the $(-)^{\perp}$. Thus, $m_{\oplus} = m_{\otimes}^{\perp}$. Furthermore, monad modules $f\colon (a,m) \to (b,n)$ automatically become linear monad modules $f\colon (a,m,m^{\perp}) \to (b,n,n^{\perp})$.

Therefore, the quantaloid of linear monads and linear monad modules of a Girard quantaloid is isomorphic to the quantaloid of monad and monad modules. In other words, considering Girard quantaloids does not give us any truly new quantaloids.

Example 6.20. Let $\mathsf{P}_{\mathsf{max}}$-Rel be the linear quantaloid of sets and "extended" distance relations. Then consider $\mathsf{Mon}(\mathsf{P}_{\mathsf{max}}\text{-Rel})$, a linear quantaloid of sets X endowed with a relation $m_{\otimes}\colon X \times X \to [0,\infty]$ satisfying

$$m_{\otimes}(x,x) \leq 0 \qquad \text{and} \qquad m_{\otimes}(x,x'') \leq \mathsf{max}(m_{\otimes}(x,x'), m_{\otimes}(x',x''))$$

and relation $m_{\oplus}\colon X \times X \to [0,\infty]$ satisfying

$$\infty \leq m_{\oplus}(x,x) \qquad \text{and} \qquad \mathsf{min}(m_{\oplus}(x,x'), m_{\oplus}(x',x'')) \leq m_{\oplus}(x,x'')$$

such that

$$\mathsf{min}(m_{\oplus}(x,x'), m_{\otimes}(x',x'')) \leq m_{\otimes}(x,x'') \qquad , \qquad \mathsf{min}(m_{\otimes}(x,x'), m_{\oplus}(x',x'')) \leq m_{\otimes}(x',x'')$$

$$m_{\oplus}(x,x'') \leq \mathsf{max}(m_{\otimes}(x,x'), m_{\oplus}(x',x'')) \qquad \text{and} \qquad m_{\oplus}(x,x'') \leq \mathsf{max}(m_{\oplus}(x,x'), m_{\otimes}(x',x''))$$

These are Lawvere ultrametric spaces $(X, m_{\otimes})$ with an additional relation $m_{\oplus}$ which interact coherently with $m_{\otimes}$, in particular they are cyclic linear adjoints. The arrows $(X,m) \nrightarrow (Y,n)$ are real-valued functions $f\colon X \times Y \to [0,\infty]$ such that

$$f(x,y') \leq \mathsf{max}(f(x,y), n_{\otimes}(y,y') \qquad \text{and} \qquad f(x,y) \leq \mathsf{max}(m_{\otimes}(x,x'), f(x',y))$$

$$\mathsf{min}(f(x,y'), n_{\oplus}(y',y)) \leq f(x,y) \qquad \text{and} \qquad \mathsf{min}(m_{\otimes}(x,x'), f(x',y)) \leq f(x,y)$$

Note that the top two inequalities imply the bottom two and vice versa. We can of course do the same construction with P_{+}-Rel and [0, 1]-Rel and get similar examples of "linearized" metric spaces.

7. Non-locally posetal examples, $\mathscr{L}$oc, $\mathscr{Q}$uant and $\mathscr{Q}$tld

In this section, we present examples of linear bicategories which are not locally partially ordered.

While the bicategory of locales being a linear bicategory will be a consequence of Theorem 7.8, we start with case of locales as it is the setting in which these greater results were first investigated.

Definition 7.1. (Joyal and Tierney, 1984) Let W, X and Y be locales.

- An (X,Y)-*module* $A\colon X \nrightarrow Y$ is a sup lattice A which is a left X-module and a right Y-module satisfying $x(ay) = (xa)y$.
- If B is a (W,Y)-module, then the sup lattice of right Y-module homomorphisms, denoted by $B \circ\!\!- A$, becomes a (W,X)-module via $(wf)(a) = wf(a)$ and $(fx)(a) = f(xa)$. Dually, if C is a (X,Z)-module, $A \multimap C$ is a (Y,Z)-module.
- A function $f\colon A \to A'$ is a *module homomorphism* if it is a left X-module and right Y-module homomorphism.
- Suppose A is an (X,Y)-module. Then the opposite lattice A° becomes a (Y,X)-module as follows. Given $x \in X$, the function $x \cdot -\colon A \to A$ is sup-preserving map, and hence, has a right adjoint $-/x$. Thus, A° becomes a right X-module via $(a,x) \mapsto a/x$ and a left Y-module via $(y,a) \mapsto r\backslash a$, where $y\backslash -$ is right adjoint to $- \cdot y$, and $(A^\circ)^\circ \cong A$.

While not explicitly stated, Joyal and Tierney (1984) develop the required results to say that:

Theorem 7.2. There is a bicategory whose objects are locales, 1-cells are modules, and 2-cells are module homomorphisms. Composition of $A\colon X \nrightarrow Y$ and $B\colon Y \nrightarrow Z$ is given by

$$A \otimes B = A \otimes_Y B \cong (B^\circ \circ\!\!- A)^\circ \cong (B \multimap A^\circ)^\circ$$

with identity 1-cells $X\colon X \nrightarrow X$. We denote this bicategory by $\mathscr{L}$oc.

Now, we note that $\mathscr{L}$oc admits another composition of 1-cells

$$A \oplus B = (B^\circ \otimes A^\circ)^\circ = (B^\circ \otimes_Y A^\circ)^\circ \cong A^\circ \multimap B$$

with identity 1-cells $X^\circ\colon X \nrightarrow X$, since $X^\circ \oplus A \cong A \cong A \oplus Y^\circ$ and which is associative,

$$A \oplus (B \oplus C) = ((C^\circ \otimes B^\circ) \otimes A^\circ)^\circ \cong (C^\circ \otimes (B^\circ \otimes A^\circ))^\circ = (A \oplus B) \oplus C$$

for all $C\colon Z \nrightarrow W$. To see that $\mathscr{L}$oc is a linear bicategory, we will define the linear distributivity

$$A \otimes (B \oplus C) \to (A \otimes B) \oplus C$$

Since $A \otimes (B \oplus C) \cong A \otimes_Y (B^\circ \multimap C)$ and $(A \otimes B) \oplus C \cong (B^\circ \circ\!\!- A) \multimap C$, it suffices to define a 2-cell $A \otimes_Y (B^\circ \multimap C) \to (B^\circ \circ\!\!- A) \multimap C$ or equivalently,

$$(B^\circ \circ\!\!- A) \otimes_X (A \otimes_Y (B^\circ \multimap C)) \to C$$

For this, we can use the evaluation maps

$$(B^\circ \circ\!\!- A) \otimes_X (A \otimes_Y (B^\circ \multimap C)) \cong ((B^\circ \circ\!\!- A) \otimes_X A) \otimes_Y (B^\circ \multimap C) \xrightarrow{\varepsilon_{A,Y}} B^\circ \otimes_Y (B^\circ \multimap C) \xrightarrow{\varepsilon_{Z,B^\circ}} C$$

Similarly, we get a 2-cell $(A \oplus B) \otimes C \to A \oplus (B \otimes C)$, as desired.

Theorem 7.3. Under the above operations, $\mathscr{L}$oc is a linear bicategory.

7.1 $\mathscr{V}$-Mat, Mon$\mathscr{B}$ *and* $\mathscr{V}$-$\mathscr{P}$rof

Recall $\mathscr{V}$-$\mathscr{M}$at, the bicategory of sets, $\mathscr{V}$-matrices and $\mathscr{V}$-matrix morphisms, which is biclosed when $\mathscr{V}$ is a symmetric monoidal closed category with set-indexed products and coproduct by

Lemma 2.17. If $\mathscr{V}$ is further $*$-autonomous, $\mathscr{V}$-$\mathscr{M}$at becomes a linear bicategory, as stated by Cockett et al. (2000):

Proposition 7.4. Consider $\mathscr{V}$-$\mathscr{M}$at, where $\mathscr{V}$ is a $*$-autonomous category with set-indexed products and coproducts. Given a set X, define $\perp\!\!\!\perp_X : X \nrightarrow X$ by

$$\perp\!\!\!\perp_X(x,x') = \begin{cases} \perp & x = x' \\ \top & \text{otherwise} \end{cases}$$

For $\mathscr{V}$-matrix $A\colon X \nrightarrow Y$, defining $A^\perp\colon Y \nrightarrow X$ by $A^\perp(y,x) = A(x,y)^\perp = A(x,y) \multimap \perp$, one can show that $(A \multimap \perp\!\!\!\perp_Y) \cong A^\perp \cong (\perp\!\!\!\perp_X \circ\!\!- A)$ and $A \cong (A^\perp)^\perp$, and so $\mathscr{V}$-$\mathscr{M}$at is a cyclic $*$-autonomous bicategory with cyclic dualizing family $\mathscr{D} = \{\perp\!\!\!\perp_X : X \nrightarrow X\}$.

Now consider Mon$\mathscr{B}$, the bicategory of monads in $\mathscr{B}$, modules and module morphisms, which is biclosed if $\mathscr{B}$ is biclosed with local equalizers and coequalizers stable under composition by Lemma 2.19. Then Mon$\mathscr{B}$ can be a linear bicategory:

Proposition 7.5. Suppose $\mathscr{B}$ is a cyclic $*$-autonomous bicategory with local equalizers and coequalizers stable under composition, then Mon$\mathscr{B}$ is a cyclic $*$-autonomous bicategory.

Proof. Suppose $\mathscr{D} = \{\perp_X : X \to X \mid X \in \mathscr{B}\}$ is a cyclic dualizing family for $\mathscr{B}$. Define the family of modules

$$\mathscr{D}^\perp = \{Q^\perp : (X,Q) \nrightarrow (X,Q) \mid (X,Q) \in \mathrm{Mon}\mathscr{B}\}$$

This family is well-defined since the 1-cell $Q^\perp = Q \multimap \perp_X : X \to X$ is a module $(X,Q) \nrightarrow (X,Q)$ with action $\lambda : Q \otimes Q^\perp \to Q^\perp$ defined by the transpose of

$$Q \otimes (Q \otimes Q^\perp) \xrightarrow{\sim} (Q \otimes Q) \otimes Q^\perp \xrightarrow{m \otimes Q^\perp} Q \otimes Q^\perp \xrightarrow{\varepsilon_{X,Q}} \perp_X$$

and action ρ defined as $Q^\perp \otimes Q \xrightarrow{\sim} (\perp_X \circ\!\!- Q) \otimes Q \xrightarrow{\rho'} \perp_X \circ\!\!- Q \xrightarrow{\sim} Q^\perp$, where ρ' is the transpose of

$$((\perp_X \circ\!\!- Q) \otimes Q) \otimes Q \xrightarrow{\sim} (\perp_X \circ\!\!- Q) \otimes (Q \otimes Q) \xrightarrow{(\perp_X \circ\!\!- Q) \otimes m} (\perp_X \circ\!\!- Q) \otimes Q \xrightarrow{\varepsilon_{Q,X}} \perp_X$$

Given a monad module $A\colon (X,Q) \nrightarrow (Y,R)$, applying Proposition 2.14, we see that

$$A \multimap_Q Q^\perp = A \multimap_Q (Q \multimap \perp_X) = A \multimap_Q (Q \multimap_{\top_X} \perp_X) \cong A \multimap_{\top_X} \perp_X = A \multimap \perp_X = A^\perp$$

and

$$R^\perp \circ\!\!-_R A \cong (\perp_Y \circ\!\!- R) \circ\!\!-_R A = (\perp_Y \circ\!\!-_{\top_Y} R) \circ\!\!-_R A \cong \perp_Y \circ\!\!-_{\top_Y} A = \perp_Y \circ\!\!- A \cong A^\perp$$

Since all these 2-cells are natural in A, $\mathscr{D}^\perp$ is a cyclic dualizing family for Mon$\mathscr{B}$. □

Finally, recalling that $\mathscr{V}$-$\mathscr{P}$rof can be constructed as Mon($\mathscr{V}$-$\mathscr{M}$at) by Proposition 2.20, and by the above two results:

Proposition 7.6. If $\mathscr{V}$ is a complete and cocomplete $*$-autonomous category, then $\mathscr{V}$-$\mathscr{P}$rof is a cyclic $*$-autonomous bicategory.

7.2 Linear bicategories $\mathscr{Q}$uant *and* $\mathscr{Q}$tld

Joyal and Tierney (1984) observed that for a fixed commutative unital quantale Q, the category of right Q-modules and module homomorphisms is a $*$-autonomous category. Rosenthal (1994) expanded this result to show that the category of (Q,Q)-modules is equally $*$-autonomous and that, given a small quantaloid $\mathcal{Q}$, the category of $(\mathcal{Q},\mathcal{Q})$-modules is $*$-autonomous.

Now that we have access to the notion of linear bicategories, Q and $\mathcal{Q}$ no longer need to be fixed and we can claim the following examples of cyclic $*$-autonomous bicategories.

Definition 7.7.

- Given unital quantales Q and R, a (Q,R)-module $Q \nrightarrow R$ is a suplattice A which is a left Q-module and a right R-module, i.e., there are suplattice homomorphisms $\star\colon Q\times A\to A$ and $\cdot\colon A\times R\to A$ such that

$$(q\otimes q')\star a = q\star(q'\star a) \quad , \quad a\cdot(r\otimes r') = (a\cdot r)\cdot r'$$

$$\top\star a = a \quad , \quad a\cdot\top = a \quad \text{and} \quad (q\star a)\cdot r = q\star(a\cdot r)$$

- Given (Q,R)-modules A and B, a module homomorphism is a suplattice homomorphism $f\colon A\to B$ such that

$$f(q\star a) = q\star f(a) \qquad \text{and} \qquad f(a\cdot r) = f(a)\cdot r$$

- Let $\mathcal{Q}$uant denote the bicategory of unital quantales, modules and module morphisms. Then, $\mathcal{Q}$uant is $\mathsf{Mon}\mathscr{B}(\mathrm{Sup})$, the bicategory of monoids, monoid modules and module morphisms in Sup.

If $\mathscr{V}$ is $*$-autonomous category with equalizers and coequalizers, its suspension $\mathscr{B}(V)$ is a cyclic $*$-autonomous bicategory with local equalizers and co-equalizers stable under composition. Then, by Proposition 7.5, the bicategory $\mathsf{Mon}(\mathscr{B}(V))$ is a cyclic $*$-autonomous bicategory. Taking $\mathscr{V} = \mathrm{Sup}$, we get:

Theorem 7.8. The bicategory $\mathcal{Q}$uant is a cyclic $*$-autonomous bicategory.

Definition 7.9.

- Given small quantaloids $\mathcal{Q}$ and $\mathscr{R}$, a $(\mathcal{Q},\mathscr{R})$-module $A\colon \mathcal{Q}\nrightarrow\mathscr{R}$ consists of, for each $q,q'\in \mathrm{ob}\mathcal{Q}$, $r,r'\in\mathrm{ob}\mathscr{R}$:
  - a suplattice $A(q,r)$,
  - a left action suplattice homomorphism $\star\colon \mathcal{Q}(q,q')\times A(q',r)\to A(q,r)$ such that given $a\in A(q,r)$

    $$\top_q\star a = a \quad \text{for} \quad q=q' \qquad (f\otimes g)\star a = f\star(g\star a) \quad \text{for} \quad f\colon q\to q', g\colon q'\to q''$$

  - a right action suplattice homomorphism $\cdot\colon A(q,r)\times\mathscr{R}(r,r')\to A(q,r')$ such that given $a\in A(q,r)$

    $$a\cdot\top_r = a \quad \text{for} \quad r=r' \qquad a\cdot(h\otimes k) = (a\cdot h)\cdot k \quad \text{for} \quad h\colon r\to r', k\colon r'\to r''$$

  satisfying $\forall a\in A(q,r), f\colon q\to q'\in\mathcal{Q}, h\colon r\to r'\in\mathscr{R}$,

  $$(f\star a)\cdot h = f\star(a\cdot h)$$

- Given $(\mathcal{Q},\mathscr{R})$-modules A and B, a module homomorphism $f\colon A\to B$ is a family of suplattice homomorphisms $f_{q,r}\colon A(q,r)\to B(q,r)$ satisfying, for $f\colon q\to q'\in\mathcal{Q}, a'\in A(q',r), a\in A(q,r), h\colon r\to r'\in\mathscr{R}$,

  $$f_{q,r}(f\star a') = q\star f_{q',r}(a') \qquad \text{and} \qquad f_{q,r'}(a\cdot h) = f_{q,r}(a)\cdot h$$

- Let $\mathcal{Q}$tld denote the bicategory of small quantaloids, modules and module homomorphisms. Then, $\mathcal{Q}$tld is Sup-$\mathscr{P}$rof, the bicategory of Sup-categories, Sup-profunctors and Sup-transformations.

By Proposition 7.6 and taking $\mathscr{V} = \mathrm{Sup}$, we get:

Theorem 7.10. The bicategory $\mathscr{Q}$tld is a cyclic $*$-autonomous bicategory.

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Chapter 5

Conclusion

Linear logic is a sub-structural logic introduced by Girard, which omits the rules of contraction and weakening, in an effort to capture the constructivism of intuitionistic logic and the symmetry of classical logic [34]. The study of its categorical semantics became an active and on-going area of research. Seely demonstrated that $*$-autonomous categories, previously introduced by Barr [3], provide general categorical semantics for the multiplicative fragment of linear logic with negation [73].

Cockett and Seely subsequently introduced linearly distributive categories (LDC) as alternative semantics for multiplicative linear logic (MLL), which consider multiplicative conjunction and multiplicative disjunction, otherwise known as tensor and par respectively, as primitive, along with their interaction via linear distributivities [27]. Precisely, LDCs are categories $\mathbb{X}$ with a tensor monoidal structure $(\mathbb{X}, \otimes, \top)$ and a par monoidal structure $(\mathbb{X}, \oplus, \bot)$, linked by linear distributivity natural transformations

$$\delta^L_{A,B,C} : A \otimes (B \oplus C) \to (A \otimes B) \oplus C \qquad \delta^R_{A,B,C} : (A \oplus B) \otimes C \to A \oplus (B \otimes C)$$

This thesis presents advancements in the field of categorical linear logic with respect to linear distributivity and its categorical implications.

5.1 Linearly Distributivity and Cartesian Structures

The first part of the thesis is concerned with exploring cartesian structures in the presence of linear distributivity and, in particular, developing the study of cartesian linearly distributive categories (CLDC), as first introduced by Cockett and Seely [27]. In monoidal category theory, cartesian categories refer to monoidal categories whose product is given by the categorical binary product and whose unit is given by the terminal object. CLDCs therefore refer to LDCs whose tensor monoidal structure is cartesian and whose par monoidal structure is cocartesian.

A key result about cartesian categories is Fox's theorem. In his seminal paper, Fox proved there is an adjunction between the category of symmetric monoidal categories and the category of cartesian categories [32]. This adjunction is given by the inclusion functor and its right adjoint, the functor mapping a symmetric monoidal category to its category of cocommutative comonoids and comonoid morphisms. As a corollary, cartesian categories are characterized as the symmetric monoidal categories $(\mathcal{X}, \oslash, I)$ whose objects A are endowed with a canonical cocommutative comonoid structure

$$\Delta_A\colon A \to A \oslash A \qquad e_A\colon A \to I$$

and whose maps preserve the structures.

In an effort to characterize CLDCs, the first article in this thesis, "Linearly Distributive Fox Theorem", proves an analogue to the Fox theorem for CLDCs. Comparably to traditional Fox result, the linearly distributive version can only be formulated if we consider a subclass of LDCs, in particular symmetric LDCs which additionally satisfy the medial rule:

$$\mu_{A,B,C,D}\colon (A \otimes B) \oplus (C \otimes D) \to (A \oplus C) \otimes (B \oplus D)$$

The medial rule has been included in multiple logical systems within the paradigm of deep inference, as introduced by Guglielmi [38], and been crucial in recent efforts to provide categorical semantics for classical logic [50, 77]. Alternatively, we can view the medial rule as an instance of the interchange law of duoidal categories [2], developed as a generalization of braided monoidal categories [45].

As such, the concept of *medial linearly distributive categories* (MLDC) was introduced, capturing the structure at the intersection of LDCs and duoidal categories, and providing the categorical semantics for a variant of MLL with the medial rule. The theory of MLDCs was then developed, including the definition of their symmetric variants with negation, the creation of examples, and the presentation of their unique properties. Importantly, the appropriate 1-cells and 2-cells for a 2-category of symmetric MLDCs were defined: *symmetric strong medial linear functors* and *strong medial linear transformations*.

Mirroring the cocommutative comonoids of the Fox theorem, a symmetric MLDC is shown to induce a CLDC when taking the category of bicommutative medial bimonoids, i.e. objects A equipped with the following coherent maps

$$\Delta_A\colon A \to A \otimes A \qquad e_A\colon A \to \top \qquad \nabla_A\colon A \oplus A \to A \qquad u_A\colon \bot \to A,$$

This construction is demonstrated to be a right adjoint to the inclusion 2-functor from symmetric MLDCs to the 2-category of CLDCs, yielding the *linearly distributive Fox theorem*. As a corollary to this theorem, we get the desired characterization of CLDCs as symmetric MLDCs whose objects are canonically bicommutative medial bimonoids and whose maps preserve these structures.

The second article, "Cartesian Linearly Distributive Categories: Revisited", co-authored with Dr. Pacaud Lemay, investigates CLDCs directly, beyond their above characterization. Given a CLDC $(\mathbb{X}, \times, \mathbf{1}, +, \mathbf{0})$, the first key property proved is that the terminal object $\mathbf{1}$ is always preinitial and the initial object $\mathbf{0}$ is always subterminal. This implies a CLDC is a mix LDC, meaning the map $m : \mathbf{0} \to \mathbf{1}$ induces a unique mix natural transformation

$$\mathrm{mix}_{A,B}\colon A \times B \to A + B$$

This further entails that every object in a CLDC is preinitial if and only if it is subterminal.

The two key classes of examples of CLDCs were then investigated: bounded distributive lattices and semi-additive categories, the latter being cartesian and cocartesian categories in which products and coproducts are isomorphic. It was established that a CLDC is a distributive category if and only if it is a bounded distributive lattice by Cockett and Seely. It was also known that Joyal's paradox could be adapted to the context of LDCs to show that a CLDC has negation if and only if it is a bounded distributive lattice. However, there were no analogous results for semi-additive categories.

It was proved that a CLDC has invertible linear distributivities if and only if it is semi-additive, and perhaps even more significantly, that a CLDC is isomix, i.e. has a zero object, if and only if it is semi-additive. Taken together, these four theorems highlight how tightly constrained the class of possible CLDCs must be. From a logical perspective, if a CLDC is understood to model a variant of MLL where the multiplicative connectives permit weakening and contraction, then the theorems suggest that in many contexts, one must either identify all proofs between certain formulas or identify the connectives.

It was then shown how to construct a bounded distributive lattice from any CLDC by taking its category of *semizero objects* and how to construct a semi-additive category by taking the slice of the CLDC over the initial object. The first construction provides an adjunction, while the latter does not.

Moreover, it was previously stated that the Kleisli category of the exception monad on a distributive category would be an example of CLDC [27]. It is shown in the article that the Kleisli category is indeed a LDC, although not a CLDC as its tensor monoidal structure is not cartesian.

Although the previous work developed in the article seems to suggest that lattices and semi-additive categories might be the only possible examples of CLDCs, that is not the case. Indeed, the product of a bounded distributive lattice and a semi-additive category is itself a CLDC. This idea is leveraged to construct new examples of CLDCs using the Grothendieck construction applied to a contravariant functor from a semi-additive category into the category of bounded distributive lattices.

5.2 Linear Bicategories, Quantales and Quantaloids

The second part of this thesis endeavors to re-invigorate the study of linear bicategories, introduced by Cockett, Koslowski and Seely, as the bicategorical analogue of LDCs [22]. Specifically, a linear bicategory $\mathcal{B}$ is a collection of 0-cells, 1-cells and 2-cells, endowed with a tensor bicategorical structure $(\mathcal{B}, \otimes, \top_X)$ and a par bicategorical structure $(\mathcal{B}, \oplus, \bot_X)$, whose interaction is mediated by linear distributivity transformations. As a non-commutative logical connective is most naturally modeled by bicategorical composition, linear bicategories provide categorical semantics for non-commutative MLL. The prototypical example of a linear bicategory is the category of sets and relations Rel, however few other examples had been developed.

The third article "Constructing Linear Bicategories", co-authored with Dr. Blute and Dr. Niefield, attempts to fill this gap by creating new examples of linear bicategories from the perspective of quantale and quantaloid theory, as first introduced by Mulvey [57], and Abramsky and Vickers [1] respectively. A quantale is a complete lattice Q with an associative binary operation $\otimes$, which distributes on both sides over arbitrary joins $\bigvee$, while a quantaloid $\mathcal{Q}$ is the bicategorical analogue of a quantale.

The theory of quantales and quantaloids had been developed to provide categorical semantics for MLL with negation, with the introduction of Girard quantales by Yetter [83] and Girard quantaloids by Rosenthal [69].

These notions were generalized to the linearly distributive context and named *LD-quantale* and *linear quantaloid* respectively. Precisely, a LD-quantale is a complete lattice with two unital associative binary operations $\otimes$ and $\oplus$ such that $\otimes$ distributes on both sides over arbitrary joins $\bigvee$ and $\oplus$ distributes on both sides over arbitrary meets $\bigwedge$, and linear distributivity inequalities hold

$$a \otimes (b \oplus c) \leq (a \otimes b) \oplus c \qquad (a \oplus b) \otimes c \leq a \oplus (b \otimes c)$$

A LD-quantale is the appropriate definition for a structure which is both a quantale and LDC, while a linear quantaloid is the appropriate definition a quantaloid which is also a linear bicategory. Examples of LD-quantales are developed, of particular interest are the complete bi-Heyting algebras, in other words locales satisfying the opposite infinitary distributive law. The construction of linear quantaloids is then the central idea for most of the remainder of the paper.

The first construction arises from the category of quantale-valued relations, denoted Q-Rel, which has become central to the field of monoidal topology [41], for a LD-quantale Q. This category is shown to in fact be a linear quantaloid, which subsequently has been featured in recent work on linear bicategories [58, 81].

We then explore multiple constructions from the theory of enrichment over a bicategory, shown to be closed under Girard quantaloids by Rosenthal, and adapt them to the setting of linear quantaloids. In particular, we demonstrate that given a linear quantaloid $\mathcal{Q}$, the quantaloid of *linear* $\mathcal{Q}$*-categories* and *linear* $\mathcal{Q}$*-modules*, the quantaloid of $\mathcal{Q}$*-matrices*, and the quantaloid of *linear monads* and *linear modules* in $\mathcal{Q}$ are linear quantaloids themselves. Considering the linear quantaloid to be Q-Rel for certain LD-quantales Q provides new interesting examples of linear bicategories.

Finally, it is shown that the bicategory of quantales, bimodules and bimodules morphisms, and the bicategory of quantaloids, bimodules and bimodule morphisms are themselves linear bicategories, which are cyclic $*$-autonomous in nature.

5.3 Further Work

The introduction of MLDCs opens new avenues for investigating MLL+Medial as a variant of linear logic. The first question to address is whether this system is expressible within the traditional sequent calculus, or whether, like pomset logic and BV, it necessitates an alternative logical formalism, such as the calculus of structures. A subsequent question would then be whether MLL+Medial satisfies desirable logical properties like cut elimination. It would also be valuable to extend the proof net calculus of LDCs, developed by Blute, Cockett, Seely and Trimble [10], to MLDCs and investigate the possibility of coherence results in this setting.

The re-examination of CLDCs clarified the known examples of CLDCs and explained why it remains difficult to identify any others beyond bounded distributive lattices and semi-additive categories. Notably, this difficulty is equally present in the theory of MLDCs. Many of the properties which constrain possible classes of CLDCs also apply to MLDCs.

The research did not however suggest no other examples exist. Therefore, the primary direction for on-going research on CLDCs is undoubtedly the discovery of new examples. This is closely tied with identifying new examples of MLDCs. If new flavors of symmetric MLDCs are found, the linearly distributive Fox theorem would then yield new CLDCs, provided no additional collapses occur.

It also became apparent that a large number of LDCs are either $\otimes$-cartesian or $\oplus$-cocartesian. Indeed, while not a CLDC, the construction on symmetric distributive monoidal categories detailed in "Cartesian Linearly Distributive Categories: Revisited", which includes the Kleisli category of the exception monad on a distributive category, is a $\oplus$-cocartesian LDC. Furthermore, the category of sets and functions is a $\otimes$-cartesian LDC with its tensor product the cartesian product of sets, $\otimes = \times$, and its par product $\oplus$ defined as follows in [24].

$$A \oplus B = \begin{cases} B & A = \varnothing \\ A & B = \varnothing \\ \mathbf{1} & \text{otherwise} \end{cases}$$

There has as of yet been no focused work investigating $\otimes$-cartesian or $\oplus$-cocartesian LDCs, but it is likely such efforts would improve our understanding of linearly distributivity in relation to cartesian structures.

As for linear bicategories, there are numerous possible directions for future research. In fact, much of the theory developed for LDCs could be extended to the bicategorical setting. In terms of the particular work developed in this thesis, a natural continuation of this project would be extending the constructions of enrichment over a linear quantaloid to enrichment over a linear bicategory. Indeed, the theory of enrichment over a bicategory $\mathcal{B}$, in particular the concepts of $\mathcal{B}$-categories, $\mathcal{B}$-modules, $\mathcal{B}$-matrices and monads in $\mathcal{B}$, is already well-established [7, 17, 78, 82]. Their generalization to linear bicategories $\mathcal{B}$ would be not necessarily be straightforward however.

The definition of linear monads in a linear bicategory $\mathcal{B}$ has been established [22] and the notion of modules between linear functors $F : \mathcal{B} \to \mathcal{B}'$ was the main motivation for Cockett, Koslowski and Seely's follow-up paper "Morphisms and modules for poly-bicategories" [23]. While they do settle the appropriate definition for poly-bicategories and poly-functors, the translation of these ideas to the representable context, in other words to linear bicategories, remains incomplete and would be the first step in this project. The difficulty defining such modules did potentially contribute to the stagnation of the development of linear bicategories. Regardless, with the renewed interest, the theory of enrichment in a linear bicategory is undoubtedly an important and promising research path, which would yield new linear bicategories.

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