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HOMOGENEOUS LOCALLY NILPOTENT DERIVATIONS AND AFFINE ML-SURFACES

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Abstract

Let $B = \mathbf{k}[X_0, X_1, X_2]$ be the polynomial ring in three variables over an algebraically closed field $\mathbf{k}$ of characteristic zero. We consider the homogeneous case of the problem of describing locally nilpotent derivations of B. Given integers a_0, a_1, a_2 satisfying $\gcd\{a_0, a_1, a_2\} = 1$, we define a $\mathbb{Z}$-grading $\mathfrak{g}$ on B by declaring that X_i is homogeneous of degree a_i (for $i = 0, 1, 2$). In this thesis, we give an explicit description of the $\mathfrak{g}$-homogeneous locally nilpotent derivations of B when the integers a_0, a_1, a_2 are not pairwise relatively prime. In the case where a_0, a_1, a_2 are pairwise relatively prime, we characterize the kernels of $\mathfrak{g}$-homogeneous locally nilpotent derivations of B among all subalgebras of B.

Now assume that $\mathbf{k}$ is an arbitrary field of characteristic zero. In the remainder of this thesis, we study properties of affine $\mathbf{k}$-surfaces which have trivial Makar-Limanov invariant. In particular, we prove that such surfaces have only finitely many singular points. As an application, we also prove that a complete intersection surface with trivial Makar-Limanov invariant is normal; in particular, any hypersurface of the affine space $\mathbb{A}^3_{\mathbf{k}}$ with trivial Makar-Limanov invariant is normal.

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The proofs of several basic known facts given in the background chapters of this thesis follow ideas from Daigle's lecture notes [9] and Freudenburg's book [30]. I thank both of them for preparing excellent material on the theory of locally nilpotent derivations.

I am also grateful to the support staff at the University of Ottawa including Carolynne, Chantal, Michelle, Steven, and Suzanne for helping me out with several things over the past five years.

Without mentioning any names, I would like thank my numerous friends whose constant love and encouragement has made this work possible. Last but not least, I would like to express my deep gratitude to my parents for their constant love and support.

Dedication

To my parents, Aai and Baba:

for your unquestionable love and support with my studies,
for all your patience throughout the course of this work,
for giving me the chance to prove and improve myself
through all my walks of life, and
for being such wonderful parents to me.

Contents

Notation and conventions

The following notations and conventions are used throughout this thesis:

• The natural numbers, integers, rational numbers, real numbers and complex numbers are denoted by symbols $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$, respectively. We include the number zero in $\mathbb{N}$.

• All rings are commutative and associative rings with identity element.

• The group of units of a ring A is denoted A^*.

• Given sets A and B, we write $A \subset B$ to indicate that A is strictly included in B.

• $\mathbf{k}$ always denotes a field, and the algebraic closure of $\mathbf{k}$ is denoted by $\bar{\mathbf{k}}$.

• Given a ring A and an integer $n > 0$, $A^{[n]}$ denotes an A-algebra isomorphic to the polynomial ring in n variables over A.

• By a domain, we mean an integral domain and, UFD means a unique factorization domain. If A is a domain, the field of fractions of A is denoted by $\operatorname{Frac} A$.

• If $A \subseteq B$ are domains, the abbreviation $\operatorname{trdeg}_A B$ means the transcendence degree of $\operatorname{Frac} B$ over $\operatorname{Frac} A$.

• If n is a positive integer, by $\mathbf{k}^{(n)}$, we mean $\operatorname{Frac}(\mathbf{k}^{[n]})$.

• Given rings $A \subseteq B$, we say that B is an affine A-algebra if and only if B is finitely generated as an A-algebra.

• Given a positive integer n, $\mathbb{A}^n_{\mathbf{k}}$ (or $\mathbb{A}^n$) and $\mathbb{P}^n_{\mathbf{k}}$ (or $\mathbb{P}^n$) denote the affine and the projective n-spaces over $\mathbf{k}$, respectively.

• The Krull dimension of a ring B is denoted $\dim B$.

- Given a ring R, $\operatorname{Max} R$ denotes the set of all maximal ideals of the ring R.
- Given a $\mathbf{k}$-algebra B, the group of $\mathbf{k}$-automorphisms of B is denoted $\operatorname{Aut}_{\mathbf{k}}(B)$.
- Given integers $m, n \in \mathbb{Z}$, $\gcd(m, n)$ is the greatest common divisor of m and n, which by convention we take to be a nonnegative integer. By convention, $\gcd(0, 0) = 0$.

Introduction

Fix $\mathbf{k}$ to be an arbitrary field of characteristic zero. As the title suggests, this thesis is broadly divided into two subjects: homogeneous locally nilpotent derivations (of polynomial rings) and affine surfaces with trivial Makar-Limanov invariant. We divide the material of this thesis into 5 chapters.

In Chapter 1, we gather and develop some known facts that we use throughout this thesis. Chapters 2-4 are devoted to the study of homogeneous locally nilpotent derivations of polynomial rings. Chapter 5 deals with the study of affine surfaces with trivial Makar-Limanov invariant and can be read independently of Chapters 2-4. A self-contained appendix on valuation rings is also included at the end of the thesis, in order to provide proofs of certain known facts about polynomial curves stated in Chapter 1.

Given a ring B, a *derivation* D of B is a map $D : B \to B$ satisfying the following conditions for all $x, y \in B$: $D(x+y) = D(x) + D(y)$ and $D(xy) = D(x)y + xD(y)$. A derivation D of B is *locally nilpotent* if, given $b \in B$, $D^n(b) = 0$ for some $n \in \mathbb{N}$. The set of all locally nilpotent derivations of B is denoted by $\text{LND}(B)$. Given $D \in \text{LND}(B)$, $\ker D = \{ b \in B \mid D(b) = 0 \}$ is a subring of B.

As indicated in the introduction of [30], the study of locally nilpotent derivations goes back to the 1960s in the work of Dixmier, Gabriel, Nouazé and Rentschler in Lie algebras and Lie groups. Nowadays, the theory of locally nilpotent derivations provides algebraic machinery to investigate some of the fundamental questions of affine algebraic geometry. In fact, many of these questions can be formulated in terms of (locally nilpotent) derivations. One of them is the famous Jacobian Conjecture. Let $B = \mathbf{k}[x_1, \ldots, x_n] = \mathbf{k}^{[n]}$. We say that a derivation D of B has a *slice* if there exists an element $s \in B$ satisfying $Ds = 1$. Then the Jacobian Conjecture is equivalent to the following problem: Suppose that $f = (f_1, \ldots, f_{n-1})$, where $f_i \in B$. Consider the Jacobian derivation Δ_f of B, which is given by the Jacobian determinant. That is, $\Delta_f(b) = \left|\frac{\partial(f_1,\ldots,f_{n-1},b)}{\partial(x_1,\ldots,x_n)}\right|$ for all $b \in B$. If Δ_f has a slice s, does it follow that $\mathbf{k}[f_1, \ldots, f_{n-1}, s] = B$?

Another important question in this area is the Zariski Cancellation Problem for the affine space: If V is an affine variety and $V \times \mathbb{A}^1 \cong \mathbb{A}^{n+1}$, does it follow that $V \cong \mathbb{A}^n$? This problem is equivalent to the Slice Problem in the theory of locally nilpotent derivations: If $D \in \text{LND}(\mathbf{k}^{[n]})$ has a slice, is $\ker D \cong \mathbf{k}^{[n-1]}$? Not only can locally nilpotent derivations be used to reformulate the statement of the Cancellation Problem, they can also be used to investigate the problem. As one example of this, we mention that Crachiola and Makar-Limanov [5] gave an elementary proof of the case $\dim V = 2$ of the above problem, based on the theory of locally nilpotent derivations.

A special case of Hilbert's Fourteenth Problem is the following question: Given $B = \mathbf{k}^{[n]}$ and $D \in \text{LND}(B)$, is $\ker D$ a finitely generated $\mathbf{k}$-algebra? This question has an affirmative answer when $n \leq 3$ and a negative answer when $n \geq 5$. The case $n = 4$ is still open. Several counterexamples to Hilbert's Fourteenth Problem use locally nilpotent derivations. (See [26, 29, 40, 54, 60, 19].)

Given an affine $\mathbf{k}$-domain B, the Automorphism Problem [2] for B asks for a description of the automorphism group of the $\mathbf{k}$-algebra B, denoted $\text{Aut}_{\mathbf{k}}(B)$. This is equivalent to describing the automorphisms of the algebraic variety $X = \text{Spec}\, B$. One approach for understanding automorphisms of B is the study of algebraic actions of certain algebraic groups on X. Of particular importance is the additive algebraic group $(\mathbf{k}, +)$, also denoted by $\mathbf{G}_a(\mathbf{k})$ or simply $\mathbf{G}_a$. A remarkable fact is the natural

[2]Note that the Automorphism Problem for $B = \mathbf{k}^{[n]}$, $n \geq 3$, is a famous open problem.

bijection between the set of all algebraic $\mathbf{G}_a$-actions on X and the set $\text{LND}(B)$. This is one of the reasons why a classification of locally nilpotent derivations of B is desirable: it would be equivalent to a classification of all $\mathbf{G}_a$-actions on $X = \operatorname{Spec} B$.

The theory of locally nilpotent derivations is also used as a tool for classifying affine algebraic varieties. For instance, there is an ongoing project whose aim is to classify affine surfaces that admit "many" $\mathbf{G}_a$-actions. Chapter 5 of this thesis is devoted to this question.

With these motivations in mind, one wants to describe $\text{LND}(B)$, where B is an affine $\mathbf{k}$-domain. We stress that $\text{LND}(B)$ is only a set, i.e., it does not have any interesting algebraic structure. For instance, it is in general not closed under addition: $X\frac{\partial}{\partial Y}$ and $Y\frac{\partial}{\partial X}$ are two elements of $\text{LND}(\mathbf{k}[X,Y])$ whose sum is not in $\text{LND}(\mathbf{k}[X,Y])$. Also note that $\frac{\partial}{\partial X}$ belongs to $\text{LND}(\mathbf{k}[X,Y])$ but $X\frac{\partial}{\partial X}$ does not. So when we speak of describing $\text{LND}(B)$, it is a set that we want to describe, not an algebraic structure. Also noteworthy is the fact that, in many cases of interest, the problem of describing $\text{LND}(B)$ reduces to that of describing the set

$$\text{KLND}(B) = \left\{ \ker D \mid D \in \text{LND}(B) \text{ and } D \neq 0 \right\};$$

in particular, this is the case when B is a polynomial ring over $\mathbf{k}$ (cf. 1.4.2).

Because of the fundamental importance of polynomial rings (or affine spaces) in algebraic geometry, the description of the sets $\text{LND}(\mathbf{k}^{[n]})$ has been of great interest. The Automorphism Problem for $\mathbf{k}^{[n]}$ is a classical open question of algebraic geometry for $n \geq 3$. The solution to this problem is trivial when $n = 1$; indeed, all automorphisms of $\mathbf{k}[x]$ have the form $ax + b$, where $a \in \mathbf{k}^*$ and $b \in \mathbf{k}$. The case $n = 2$ was settled by Jung and van der Kulk [34, 46]. Their result says that any automorphism of $\mathbf{k}^{[2]}$ is *tame* (cf. 1.3.5). Recently, Shestakov and Umirbaev [57] made some extraordinary progress in $n = 3$ case of the problem. They proved that the famous *Nagata automorphism* of $\mathbf{k}^{[3]}$ (see [51]) is not tame. Despite this development, there are still several unanswered questions related to the structure of the group $\operatorname{Aut}_{\mathbf{k}}(\mathbf{k}^{[3]})$.

The problem of describing $\text{LND}(\mathbf{k}^{[1]})$ is trivial, and $\text{LND}(\mathbf{k}^{[2]})$ was described by Rentschler in 1968 [53].

From now on, let $B = \mathbf{k}^{[3]}$. Although a complete description of the set $\text{LND}(B)$ is out of reach, the homogeneous case of the problem is partially understood. Let G be any torsion-free abelian group. Given a grading $\mathfrak{g} = \{B_i\}_{i \in G}$ of B, a derivation D of B is $\mathfrak{g}$-*homogeneous* if there exists $e \in G$ such that $D(B_i) \subseteq B_{i+e}$ holds for all $i \in G$. In this case, we say that D is homogeneous of degree e. We set

$\text{KLND}(B, \mathfrak{g}) = \{ \ker D \mid D \in \text{LND}(B),\ D \neq 0, \text{ and } D \text{ is } \mathfrak{g}\text{-homogeneous} \}.$

Daigle [10] described the set $\text{KLND}(B, \mathfrak{g})$ when B is graded by an $\mathbb{N}$-grading $\mathfrak{g} = \{B_i\}_{i \in \mathbb{N}}$. In his analysis, he assumed that the grading $\mathfrak{g}$ on B satisfies $B_0 = \mathbf{k}$; his classification splits into two cases depending on the "type" of the grading: $\text{type}(\mathfrak{g}) = 0$ and $\text{type}(\mathfrak{g}) > 0$. Inspired by his work, we consider the problem of describing kernels of homogeneous locally nilpotent derivations of B when B is $\mathbb{Z}$-graded.

In Chapter 2, we study coordinatizable gradings of polynomial rings. Given a torsion-free abelian group G and $B = \mathbf{k}^{[n]}$, where $\mathbf{k}$ is a field of characteristic zero, a grading $\mathfrak{g} = \{B_i\}_{i \in G}$ of B is *coordinatizable* if there exist $\mathfrak{g}$-homogeneous elements $X_1, \dots, X_n \in B$ such that $B = \mathbf{k}[X_1, \dots, X_n]$. For such a coordinatizable grading $\mathfrak{g}$ of B, we then define the notion of $\text{type}(\mathfrak{g})$ (cf. 2.3.9). A result of Chapter 2 is as follows:

Corollary. *Let $\mathfrak{g} = \{R_i\}_{i \in G}$ be a grading on $R = \mathbf{k}[X_1, \dots, X_n] = \mathbf{k}^{[n]}$, where G is a torsion-free abelian group and where we assume that the variables $X_1, \dots, X_n$ are $\mathfrak{g}$-homogeneous. Then the following hold:*

(i) *If $A \in \text{KLND}(R, \mathfrak{g})$, then $|\{X_1, \dots, X_n\} \cap A| \geq \text{type}(\mathfrak{g}) - 1$.*

(ii) *If $\text{type}(\mathfrak{g}) = n$, then $\text{KLND}(R, \mathfrak{g}) = \{A_1, \dots, A_n\}$, where $A_i = \mathbf{k}^{[n-1]}$ is the subalgebra of R generated by $\{X_1, \dots, X_n\} \setminus \{X_i\}$.*

If $\mathbf{k}$ is any algebraically closed field of characteristic zero, it is known (cf. 2.3.6) that for any $\mathbb{Z}$-grading of $B = \mathbf{k}^{[3]}$ there exist homogeneous elements $X, Y, Z \in B$ satisfying

$B = \mathbf{k}[X, Y, Z]$. So the above assertions (i) and (ii) are valid for any $\mathbb{Z}$-grading of B.

In Chapter 3, we give an algebraic characterization of the subalgebras of $B = \mathbf{k}^{[3]}$ which belong to $\text{KLND}(B, \mathfrak{g})$, where $\mathfrak{g}$ is a $\mathbb{Z}$-grading of type 0. The main result of that chapter (with some minor changes in the notation) is as follows:

Theorem 3.1.1 *Let* $\mathbf{k}$ *be an algebraically closed field of characteristic zero. Let* $\mathfrak{g} = \{B_i\}_{i \in \mathbb{Z}}$ *be a* $\mathbb{Z}$*-grading on* $B = \mathbf{k}[X, Y, Z] = \mathbf{k}^{[3]}$ *such that* X, Y, Z *are homogeneous and* $\gcd(\deg X, \deg Y, \deg Z) = 1$. *Suppose that* $f, g \in B$ *are homogeneous, and let* $A = \mathbf{k}[f, g]$. *If* $\gcd(\deg(f), \deg(g)) = 1$ *or* $\text{type}(\mathfrak{g}) = 0$, *then the following conditions are equivalent:*

(1) $A \in \text{KLND}(B, \mathfrak{g})$.

(2) $B_{(fg)} = (A_{(fg)})^{[1]}$ *and* f, g *have integral fibres in* B.

Moreover, if $\text{type}(\mathfrak{g}) = 0$ *then* $\gcd(\deg(f), \deg(g)) = 1$, *whenever these equivalent conditions are satisfied.*

In the above theorem, $B_{(fg)}$ is the degree zero component of the $\mathbb{Z}$-graded ring $S^{-1}B$, where $S = \{1, fg, (fg)^2, \dots\}$.

Note that a special case of the above theorem (where $\mathfrak{g}$ is an $\mathbb{N}$-grading) was proved by Daigle (cf. Theorem 3.5 of [7]).

In Chapter 4 (which can be read independently of Chapter 3), we classify all homogeneous locally nilpotent derivations of $B = \mathbf{k}^{[3]}$ when B is $\mathbb{Z}$-graded by a grading $\mathfrak{g}$ that satisfies $\text{type}(\mathfrak{g}) > 0$. One key result used in our classification is the fact that the kernel of any such derivation contains a variable of B. This nontrivial fact follows from the condition $\text{type}(\mathfrak{g}) > 0$ and is a consequence of a theorem due to Daigle (cf. Theorem 4.1.2). In Theorems 4.2.3 and 4.2.5, we give a complete list of kernels of homogeneous locally nilpotent derivations of B when the type of the grading is positive.

The second part of this thesis is Chapter 5, which is devoted to the study of affine surfaces which have trivial Makar-Limanov invariant. Given an affine $\mathbf{k}$-domain R, the Makar-Limanov invariant of R (denoted $\mathrm{ML}(R)$) is the intersection of the kernels of all locally nilpotent derivations of R. If X is an affine $\mathbf{k}$-variety, we define the Makar-Limanov invariant of X to be the Makar-Limanov invariant of its coordinate algebra $\mathcal{O}_X(X)$. In recent years, this invariant has become a central tool in classifying and distinguishing affine varieties. It provides a method to actually compute the automorphism groups of certain affine $\mathbf{k}$-algebras. Researchers have used this invariant to investigate several important questions including the Linearization Problem for $\mathbb{C}^*$-actions on $\mathbb{C}^3$ (see [36]) and the Cancellation Problem for Varieties (cf. [5], [28]). A $\mathbf{k}$-domain R is said to have trivial Makar-Limanov invariant if $\mathrm{ML}(R) = \mathbf{k}$ and is said to be rigid if $\mathrm{ML}(R) = R$. Varieties of dimension at least two which have trivial Makar-Limanov invariant admit many $\mathbf{G}_a$-actions and have "large" automorphism groups. The affine spaces $\mathbb{A}^n_{\mathbf{k}}$ are the simplest examples of varieties with trivial Makar-Limanov invariant. While it is known that $\mathbb{A}^1_{\mathbf{k}}$ is the only affine curve with trivial Makar-Limanov invariant, the class of affine surfaces with trivial Makar-Limanov invariant contains many surfaces other than $\mathbb{A}^2_{\mathbf{k}}$, some of which are not even normal (cf. Examples 5.3.7.1, 5.3.7.2).

Let $\mathcal{M}(\mathbf{k})$ denote the class of **two-dimensional** affine $\mathbf{k}$-domains with trivial Makar-Limanov invariant. We use the same symbol $\mathcal{M}(\mathbf{k})$ to denote the corresponding class of surfaces. That is, an affine surface $X = \operatorname{Spec} B$ is said to belong to $\mathcal{M}(\mathbf{k})$ if $B \in \mathcal{M}(\mathbf{k})$. A surface in the class $\mathcal{M}(\mathbf{k})$ is also called an ML-surface.

Describing all surfaces in $\mathcal{M}(\mathbf{k})$ is an important problem related to the classification of affine surfaces. (See [3], [11], [15], [22], [32], [47]). In particular, the special cases when the affine surface is smooth or normal have been studied by many researchers. However, it is desirable to understand the algebraic and geometric properties of surfaces in the class $\mathcal{M}(\mathbf{k})$ when we drop the assumption of normality. With this motivation, we explore some properties of surfaces in $\mathcal{M}(\mathbf{k})$ in Chapter 5. A complete list of properties of ML-surfaces proved in this thesis is given in 5.1.3. One important result of Chapter 5 (with minor changes in the notation) is as follows:

Theorem 5.5.6. *Suppose that $R \in \mathcal{M}(\mathbf{k})$ and let I be a height 1 ideal of R. If $A_1, A_2 \in \text{KLND}(R)$ are such that $I \cap A_i \neq 0$ (for $i = 1, 2$), then $A_1 = A_2$.*

The above theorem has some algebraic and geometric consequences:

(i) *A surface in the class $\mathcal{M}(\mathbf{k})$ has finitely many singular points.*

(ii) *Complete intersection surfaces in the class $\mathcal{M}(\mathbf{k})$ are normal. In particular, hypersurfaces of $\mathbb{A}^3_{\mathbf{k}}$ with trivial Makar-Limanov invariant are normal.*

Our proof of Theorem 5.5.6 is given by using the geometry of surfaces in the class $\mathcal{M}(\mathbf{k})$ and studying their fibrations; this allows us to prove the theorem under the additional assumption that $\mathbf{k}$ is algebraically closed (for all geometrical arguments, we assume that the base field is algebraically closed). Then we use some algebraic arguments to deduce that the theorem is valid over any field of characteristic zero. A second proof of Theorem 5.5.6 is given at the end of Chapter 5. This proof is purely algebraic and relies on an unpublished result of Bhatwadekar (Theorem 5.7.3). We thank Professor Bhatwadekar for allowing us to include his result in this thesis.

Finally, we would like to emphasize that the results (i) and (ii) mentioned above are valid over an arbitrary field of characteristic zero. Certain results, for instance Lemma 5.3.6, are known (and are almost trivial) when $\mathbf{k}$ is algebraically closed, but require careful arguments when that assumption is dropped. Also it is interesting to note that surfaces in the class $\mathcal{M}(\mathbf{k})$ do not always behave well when $\mathbf{k}$ is not algebraically closed. For instance, if $X = \operatorname{Spec} R$ belongs to the class $\mathcal{M}(\mathbf{k})$, then X is not necessarily rational and may have very few $\mathbf{k}$-rational points. Moreover, $\bar{\mathbf{k}} \otimes_{\mathbf{k}} R$ is not necessarily an integral domain. (See 5.3.7.3.)

Some results of Chapter 5 will appear in [41] and are used in a joint paper with Daigle [21], where we classify all complete intersection surfaces in the class $\mathcal{M}(\mathbf{k})$.

Chapter 1

Preliminaries

In this chapter, we give some definitions and basic facts about locally nilpotent derivations, commutative algebra and algebraic geometry. This chapter provides basic background required for all other chapters. The material in this chapter is well-known, and our main references are the books and notes of van den Essen [25], Freudenburg [30] and Daigle [9].

1.1 Derivations

Throughout this section, B denotes a ring and A denotes a subring of B.

Definition 1.1.1 A *derivation* of B is a map $D : B \to B$ that satisfies

$$D(x + y) = D(x) + D(y) \quad \text{and} \quad D(xy) = D(x)y + xD(y)$$

for all $x, y \in B$. If D is a derivation of B, we define $\ker D = \{ x \in B \mid D(x) = 0 \}$. In fact, $\ker D$ is a subring of B. A derivation D of B is an A-*derivation* if $D(b) = 0$ for all $b \in A$.

Notation 1.1.2 The set of all derivations of B will be denoted $\mathrm{Der}(B)$ and the set of all A-derivations of B will be denoted $\mathrm{Der}_A(B)$.

We note that $\mathrm{Der}(B)$ is a B-module and $\mathrm{Der}_A(B)$ is a B-submodule of $\mathrm{Der}(B)$.

Definition 1.1.3 A derivation D of B is *locally nilpotent* if for each $b \in B$, there exists a natural number n (depending on b) such that $D^n(b) = 0$.

Notation 1.1.4 The set of all locally nilpotent derivations of B will be denoted by $\mathrm{LND}(B)$. By $\mathrm{LND}_A(B)$, we mean the set of all locally nilpotent derivations of B which are also A-derivations. We also define

$$\mathrm{KLND}(B) = \{ \ker D \mid D \in \mathrm{LND}(B), D \neq 0 \},$$

$$\mathrm{KLND}_A(B) = \{ \ker D \mid D \in \mathrm{LND}_A(B), D \neq 0 \}.$$

Example 1.1.5 Let $B = \mathbf{k}[X, Y, Z] = \mathbf{k}^{[3]}$. Then $D = X^2 \frac{\partial}{\partial Z}$ is a locally nilpotent derivation of B with $\ker D = \mathbf{k}[X, Y]$.

Definition 1.1.6 Let $B = \mathbf{k}[X_1, \dots, X_n] = \mathbf{k}^{[n]}$. Then $D \in \mathrm{Der}(B)$ is *triangular* if $D(X_i) \in \mathbf{k}[X_1, \dots, X_{i-1}]$ for every i. In particular, $D(X_1) \in \mathbf{k}$.

Lemma 1.1.7 *Every triangular derivation is locally nilpotent.*

Proof: See Corollary 1.3.17 of [25], for instance. ∎

Definition 1.1.8 An element $b \in B$ is *algebraic* over A if there exists a nonzero polynomial $f(T) \in A[T]$ such that $f(b) = 0$. If b is not algebraic over A, we say that b is *transcendental* over A. We say that A is *algebraically closed* in B if every element of $B \setminus A$ is transcendental over A.

Definition 1.1.9 Let $A \subseteq B$ be domains. We say that A is *factorially closed* in B if, $x, y \in B$ and $xy \in A \setminus \{0\}$ imply that $x, y \in A$.

1.1.10 Suppose that $A \subseteq B$ are integral domains and A is factorially closed in B. Then the following hold:

(1) *A is algebraically closed in B and $A^* = B^*$. In particular, if* $\mathbf{k}$ *is a field contained in B, then* $\mathbf{k} \subseteq A$.

Proof. Let $b \in B$ be algebraic over A. Write a relation

$$a_n b^n + a_{n-1} b^{n-1} + \cdots + a_1 b + a_0 = 0$$

of least degree n, where $a_i \in A$. If $a_0 = 0$ then $b = 0$ by minimality of n, so $b \in A$. If $a_0 \neq 0$, then

$$b(a_n b^{n-1} + \cdots + a_1) = -a_0 \in A \setminus \{0\}.$$

As A is factorially closed in B, it follows that $b \in A$. Thus, A is algebraically closed in B. Next, $A^* \subseteq B^*$ is clear, so let $b \in B^*$. There exists $b' \in B$ such that $bb' = 1 \in A \setminus \{0\}$. Again A is factorially closed in B implies that $b, b' \in A$, so $b \in A^*$.

(2) *An element of A is irreducible in A if and only if it is irreducible in B.*

Proof. If $a \in A$ is irreducible in B, then it is irreducible in A because $A^* = B^*$. Now let $a \in A$ be irreducible in A and write $a = b_1 b_2$ with $b_1, b_2 \in B$. Then $b_1, b_2 \in A$ because A is factorially closed in B, so one of b_1, b_2 belongs to A^* and hence to B^*. So a is irreducible in B.

(3) *If B is a UFD then so is A.* This follows from the previous property.

(4) $B \cap \operatorname{Frac} A = A$.

Proof. Let $0 \neq b \in B \cap \operatorname{Frac} A$, then $b = a/a'$ for some $a, a' \in A \setminus \{0\}$. So $ba' = a \in A \setminus \{0\}$. As A is factorially closed in B, $b \in A$.

Definition 1.1.11 Let B be a ring and $D \in \text{LND}(B)$. A *slice* of D is an element $s \in B$ satisfying $D(s) = 1$. A *preslice* of D is an element $p \in B$ satisfying $D(p) \neq 0$ and $D^2(p) = 0$.

1.1.12 Given a multiplicatively closed subset $S \subseteq B \setminus \{0\}$, $D \in \text{Der}(B)$ can be extended to a derivation $\mathfrak{D} \in \text{Der}(S^{-1}B)$. If $b \in B$ and $s \in S$, we define

$$\mathfrak{D}\left(\frac{b}{s}\right) = \frac{sD(b) - bD(s)}{s^2}\ .$$

We sometimes use the notation $\mathfrak{D} = S^{-1}D$. If $S \subset \ker D$ then $\mathfrak{D}(\frac{b}{s}) = \frac{D(b)}{s}$, and $D \in \text{LND}(B)$ implies that $\mathfrak{D} \in \text{LND}(S^{-1}B)$.

Definition 1.1.13 A *degree function* on a ring B is a map $\deg : B \to \mathbb{N} \cup \{-\infty\}$ such that for all $x, y \in B$,

(1) $\deg x = -\infty \Leftrightarrow x = 0$ and $\deg(xy) = \deg(x) + \deg(y)$,

(2) $\deg(x + y) \leq \max(\deg x, \deg y)$.

We note that if B admits a degree function then it is a domain. Also, if deg is a degree function on a domain B, then $\{\, x \in B \mid \deg x \leq 0 \,\}$ is a factorially closed subring of B.

It is well-known that exponentiating a derivation gives rise to a ring homomorphism. In the case of locally nilpotent derivations this gives the following fact (for a proof, see Proposition 2.7 of [9], for instance).

Lemma 1.1.14 *Consider rings $B \subseteq C$, where $C \supseteq \mathbb{Q}$. If $D \in \text{LND}(B)$ and $\gamma \in C$, then the map*

$$B \to C, \quad b \mapsto \sum_{n \in \mathbb{N}} \frac{1}{n!} D^n(b)\, \gamma^n$$

is a homomorphism of A-algebras where $A = \ker D$.

1.2 Basic properties of LNDs

Suppose that B is a domain of characteristic zero. Let $D : B \to B$ be a nonzero derivation of B, and let $A = \ker D$. Then the following hold:

(1) *If D is locally nilpotent, then A factorially closed in B. In particular, A is*

algebraically closed in B, and if $\mathbf{k}$ is a field included in B then D is a $\mathbf{k}$-derivation.

Proof. Define a map $\deg_D : B \to \mathbb{N} \cup \{-\infty\}$ by $\deg_D(0) = -\infty$ and $\deg_D(x) = \max\{\, n \in \mathbb{N} \mid D^n(x) \neq 0 \,\}$ for any nonzero $x \in B$. First we claim that $\deg_D$ is a degree function on B. To prove this claim, consider the ring $C = (S^{-1}B)[T]$, where $S = \mathbb{Z} \setminus \{0\}$, and T is an indeterminate. Then $B \subseteq C$ are rings and $C \supseteq \mathbb{Q}$. By Lemma 1.1.14 (with $\gamma = T \in C$), we get a ring homomorphism $\theta : B \to C$, where $\theta(b) = \sum_{n=0}^{\infty} \frac{1}{n!} D^n(b) T^n$. Note that $\theta(b) = 0$ implies that $b = 0$, so θ is injective. Then $\deg_D$ is the composite $B \xrightarrow{\theta} (S^{-1}B)[T] \xrightarrow{\deg_T} \mathbb{N} \cup \{-\infty\}$, which is a degree function on B. Next we observe that $A = \ker D = \{\, x \in B \mid \deg_D x \leq 0 \,\}$. It follows that A is factorially closed in B by the definition of the degree function.

(2) *Let S be a multiplicatively closed subset of $B \setminus \{0\}$, and consider the derivation $\mathfrak{D} = S^{-1}D$ of $S^{-1}B$ defined in* 1.1.12. *Then $\mathfrak{D}$ is locally nilpotent if and only if D is locally nilpotent and $S \subset A$. If D is locally nilpotent and $S \subset A$, then* $\ker \mathfrak{D} = S^{-1}A$ *and $S^{-1}A \cap B = A$.*

Proof. Suppose that $\mathfrak{D}$ is locally nilpotent. As $\ker \mathfrak{D}$ is factorially closed in $S^{-1}B$, we have $S \subset (S^{-1}B)^* \subset \ker \mathfrak{D}$. It follows that $S \subset B \cap \ker \mathfrak{D} = A$. Furthermore, D being a restriction of a locally nilpotent derivation, D is locally nilpotent. To prove the other direction, suppose that D is locally nilpotent and that $S \subset A$. Then $\mathfrak{D}(\frac{b}{s}) = \frac{D(b)}{s}$ for all $b \in B$ and $s \in S$, so $\mathfrak{D}$ is locally nilpotent.

Next assume that $D \in \text{LND}(B)$, $S \subset A$. Given $b \in B$ and $s \in S$ we obtain,

$$\mathfrak{D}\left(\frac{b}{s}\right) = 0 \Leftrightarrow \frac{D(b)}{s} = 0 \Leftrightarrow Db = 0 \Leftrightarrow \frac{b}{s} \in S^{-1}A.$$

It follows that $\ker \mathfrak{D} = S^{-1}A$. Finally, $B \cap S^{-1}A = B \cap \ker \mathfrak{D} = \ker D = A$.

(3) *Assume that D is locally nilpotent and that $\mathbb{Q} \subseteq B$. If s is a slice of D, then $B = A[s] = A^{[1]}$. If p is a preslice of D and $\alpha = Dp$ then $B_\alpha = A_\alpha[p] = A_\alpha^{[1]}$.*

Proof. The first assertion is a well-known result due to D. Wright (cf. Proposition 2.1 of [59]) and the second assertion follows immediately from the first. Indeed, let p be a preslice of D and $\alpha = Dp$. Then $\alpha \in A \setminus \{0\}$. Consider the multiplicatively closed set $S = \{1, \alpha, \alpha^2, \cdots\}$ of B and the derivation $\mathfrak{D} = S^{-1}D \in \text{LND}(S^{-1}B)$. It follows by (2) that $\ker \mathfrak{D} = S^{-1}A$ and $\mathfrak{D}(\frac{p}{\alpha}) = 1$; so the result follows from the first assertion.

(4) *Let $b \in B \setminus \{0\}$. The derivation $bD : B \to B$ is locally nilpotent if and only if D is locally nilpotent and $b \in A$.*

Proof. Assume that D is locally nilpotent, and $b \in A$. Then bD is locally nilpotent because $(bD)^n = b^n D^n$ for all n. In the other direction, suppose that bD is locally nilpotent and nonzero. Then there exists a preslice $p \in B$ of bD. Note that $\ker(bD) = \ker D = A$. As $bD(p) \in A \setminus \{0\}$ and $A = \ker(bD)$ is factorially closed in B, we obtain $b \in A$. It follows that $(bD)^n = b^n D^n$ for all n, and this implies that D is locally nilpotent.

(5) *If D is locally nilpotent then $S^{-1}B = (\text{Frac}\, A)^{[1]}$, where $S = A \setminus \{0\}$; in particular,* $\text{trdeg}_A B = 1$.

Proof. Consider the multiplicative set $S = A \setminus \{0\}$ of B and the extended locally nilpotent derivation $\mathfrak{D} = S^{-1}D$ of $S^{-1}B$. Recall that $\ker \mathfrak{D} = S^{-1}A$ by (2). Choose a preslice p of D and let $\alpha = p/D(p)$. Then α is a slice of $S^{-1}D$ and $S^{-1}B = (S^{-1}A)^{[1]}$ by (3); it follows that $S^{-1}B = (\text{Frac}\, A)^{[1]}$ and $\text{trdeg}_A B = 1$.

(6) *If $A_1, A_2 \in \text{KLND}(B)$ and $A_1 \subseteq A_2$, then $A_1 = A_2$.*

Proof. By (5), $\text{trdeg}_{A_1} B = 1 = \text{trdeg}_{A_2} B$. If $A_1 \subseteq A_2$ then $\text{trdeg}_{A_1} A_2 = 0$, hence A_2 is algebraic over A_1. But A_1 is algebraically closed in B, so $A_1 = A_2$.

1.3 Coordinate systems and automorphisms

Definition 1.3.1 Let $A \subseteq B$ be rings and $B = A^{[n]}$. A *variable* of B over A is an element $f \in B$ such that $B = A[f, f_2, \ldots, f_n]$ for some $f_2, \ldots, f_n \in B$. A *coordinate system* of B over A is an ordered n-tuple $(f_1, \ldots, f_n)$ of elements of B such that $B = A[f_1, \ldots, f_n]$. If B is a polynomial ring over a field $\mathbf{k}$, by a variable of B, we mean a variable of B over $\mathbf{k}$. By a coordinate system of B, we mean a coordinate system of B over $\mathbf{k}$.

Lemma 1.3.2 *Let $A \subseteq B$ be rings, where $B = A^{[n]}$. If $(f_1, \ldots, f_n)$ is a coordinate system of B over A, then $f_1, \ldots, f_n$ are algebraically independent over A.*

Proof: This is well-known, but we recall the proof. The assumption $B = A^{[n]}$ implies that there exists a coordinate system $(x_1, \ldots, x_n)$ of B over A such that the family $(x_1, \ldots, x_n)$ is algebraically independent over A. Then, by the universal property of the polynomial ring, there exists a homomorphism $\theta : B \to B$ of A-algebras such that $\theta(x_i) = f_i$ for all $i = 1, \ldots, n$. Note that θ is surjective. To prove the lemma, it is enough to show that θ is injective. If A is noetherian then so is B, and the result follows from a well-known fact: if R is a noetherian ring, every surjective ring homomorphism $R \to R$ is bijective. In the general case, consider $\beta \in B$ such that $\theta(\beta) = 0$; we show that $\beta = 0$. We can choose a finite subset S of A such that: (i) when β or any f_i is expressed as a polynomial in $x_1, \ldots, x_n$, then all coefficients belong to S; (ii) for each i, one can express x_i as a polynomial expression in $f_1, \ldots, f_n$ with coefficients in S. Let A_0 be the subalgebra $R[S]$ of A, where R is the prime subring of A. We have $A_0[x_1, \ldots, x_n] = A_0[f_1, \ldots, f_n]$, and we denote this subring of B by B_0. Then B_0 is noetherian, and θ restricts to a surjective (hence bijective) ring homomorphism $B_0 \to B_0$. As $\beta \in B_0$ and $\theta(\beta) = 0$, it follows that $\beta = 0$. ∎

Remark 1.3.3 If A and B are domains, then the above lemma can also be proved by considering the transcendence degree of B over A.

Corollary 1.3.4 *Let $A \subseteq B$ be rings, where $B = A^{[n]}$, and let $(f_1, \ldots, f_n)$ and $(g_1, \ldots, g_n)$ be two coordinate systems of B over A. Then there exists a unique $\theta \in \mathrm{Aut}_A(B)$ satisfying $\theta(f_i) = g_i$ for all $i = 1, \ldots, n$.*

1.3.5 Let $B = \mathbf{k}^{[n]}$. The group of algebraic $\mathbf{k}$-automorphisms of B is called the *general affine group* or the *affine Cremona group* in dimension n and is denoted $GA_n(\mathbf{k})$. Choose a coordinate system $(X_1, \dots, X_n)$ of B. An automorphism $F \in GA_n(\mathbf{k})$ can be represented by the n-tuple $(F_1, \dots, F_n)$, where $F_i = F(X_i) \in B$, and the fact that F is an automorphism implies that $(F_1, \dots, F_n)$ is a coordinate system of B. The *affine subgroup* of $GA_n(\mathbf{k})$ is defined by

$$Af_n(\mathbf{k}) = \{\, F \in GA_n(\mathbf{k}) \mid \deg F_i = 1 \text{ for each } i \,\}.$$

The well-known general linear group $GL_n(\mathbf{k})$ can be realized as a subgroup of $Af_n(\mathbf{k})$ consisting of affine automorphisms $(F_1, \dots, F_n)$, where $F_i(0) = 0$ for each i. Elements of $GL_n(\mathbf{k})$ are called *linear automorphisms* of B. The *triangular subgroup* $BA_n(\mathbf{k})$ of $GA_n(\mathbf{k})$ is

$$\{\, F \in GA_n(\mathbf{k}) \mid F_i = a_i X_i + f_i, a_i \in \mathbf{k}^*,\ f_i \in \mathbf{k}[X_1, \dots, X_{i-1}] \text{ for each } i \,\}.$$

Elements of $BA_n(\mathbf{k})$ are called *triangular automorphisms.* The *tame subgroup* $TA_n(\mathbf{k})$ is the subgroup of $GA_n(\mathbf{k})$ generated by $Af_n(\mathbf{k})$ and $BA_n(\mathbf{k})$.

Note that the subgroups $Af_n(\mathbf{k})$, $BA_n(\mathbf{k})$ and $TA_n(\mathbf{k})$ of $GA_n(\mathbf{k})$ depend on the choice of the coordinate system γ. If γ and γ' are coordinate systems of B and F is an automorphism of B which is tame with respect to γ, it is not a priori clear that F is tame with respect to γ'.

1.3.6 Let $\mathbf{k}$ be a field. The variables of $\mathbf{k}^{[n]}$ have the following properties:

(i) If f is a variable of $\mathbf{k}[X_1, \dots, X_n]$ then the ideal of $\mathbf{k}[X_1, \dots, X_n]$ generated by the partial derivatives $\frac{\partial f}{\partial X_j}$ $(1 \le j \le n)$ is equal to $\mathbf{k}[X_1, \dots, X_n]$.

(ii) Let $(f_1, \dots, f_n)$ be a coordinate system of $\mathbf{k}[X_1, \dots, X_n]$ and, for each $i \in \{1, \dots, n\}$, let $L_i = \sum_{j=1}^{n} a_{ij} X_j$ (with $a_{ij} \in \mathbf{k}$) be the linear part of f_i (i.e., the homogeneous component of f_i of degree 1, with respect to the standard grading). Then the $n \times n$ matrix (a_{ij}) has nonzero determinant.

(iii) Let f be a variable of $\mathbf{k}[X,Y]$ such that $f \notin \mathbf{k}[X]$, and write $f = f_0(X) + f_1(X)Y + \cdots + f_n(X)Y^n$, $f_n(X) \neq 0$. Then $f_n(X) \in \mathbf{k}^*$.

1.4 LNDs and their kernels

Definition 1.4.1 Let B be a ring. A derivation $D : B \to B$ is *irreducible* if the only principal ideal of B that contains $D(B)$ is B.

1.4.2 Given a domain B, kernels of locally nilpotent derivations of B are important objects of study, especially, in describing the set $\text{LND}(B)$. Indeed, we have following well-known results. (For the proofs, see 2.19 and 2.20 of [9], for instance.)

Lemma 1. *Let B be a domain of characteristic zero satisfying the ascending chain condition for principal ideals, let $A \in \text{KLND}(B)$ and consider the set*

$$S = \{\, D \in \text{LND}_A(B) \mid D \text{ is an irreducible derivation} \,\}.$$

Then $S \neq \varnothing$ and $\text{LND}_A(B) = \{\, aD \mid a \in A \text{ and } D \in S \,\}$.

Lemma 2. *Let B be a UFD of characteristic zero and let $A \in \text{KLND}(B)$. Then $\text{LND}_A(B)$ contains an irreducible derivation D, unique up to multiplication by a unit. Moreover, for any such D, $\text{LND}_A(B) = \{\, aD \mid a \in A \,\}$.*

By the above lemmas, we observe the following: If B is a noetherian UFD containing $\mathbb{Q}$, then the problem of describing $\text{LND}(B)$ splits into two parts:

(I) Describe the set $\text{KLND}(B)$. That is, answer the question: Which subrings of B are elements of $\text{KLND}(B)$?

(II) For each $A \in \text{KLND}(B)$, give the unique irreducible element of $\text{LND}_A(B)$.

Definition 1.4.3 Let $B = \mathbf{k}^{[n]}$ and $D \in \text{Der}(B)$. The *rank* of D is the least integer $r \in \{0, 1, \ldots, n\}$ for which there exists a coordinate system $(T_1, \ldots, T_{n-r}, X_1, \ldots, X_r)$ of B satisfying $\mathbf{k}[T_1, \ldots, T_{n-r}] \subseteq \ker D$. Given such a coordinate system, we can write

$$D = f_1(T,X)\frac{\partial}{\partial X_1} + \cdots + f_r(T,X)\frac{\partial}{\partial X_r}$$

with $f_i(T, X) \in \mathbf{k}[T_1, \ldots, T_{n-r}, X_1, \ldots, X_r]$ for all i.

Let $B = \mathbf{k}^{[n]}$. The following facts are easy to prove:

- $\operatorname{rank} D = 0$ if and only if $D = 0$; $\operatorname{rank} D = 1$ is equivalent to saying $B = (\ker D)^{[1]}$ and $\ker D = \mathbf{k}^{[n-1]}$.
- $\operatorname{rank} D < n$ if and only if $\ker D$ contains a variable of B.
- Derivations with same kernel have the same rank.

The rank of a locally nilpotent derivation of $\mathbf{k}^{[2]}$ is less than two because of the following theorem [53]:

Theorem 1.4.4 (Rentschler's Theorem). *Let $B = \mathbf{k}^{[2]}$, where $\mathbf{k}$ is a field of characteristic zero. If $0 \neq D \in \text{LND}(B)$, then there exist u and v such that $B = \mathbf{k}[u, v]$ and $\ker D = \mathbf{k}[u]$. Moreover, given such u, v, we have $D = f(u)\frac{\partial}{\partial v}$ for some $f(u) \in \mathbf{k}[u]$.*

We remark that, if $n \neq 2$, there exist locally nilpotent derivations of $\mathbf{k}^{[n]}$ of rank n. (See [31], for instance.)

Theorem 1.4.5 (Miyanishi's Theorem). *Let $\mathbf{k}$ be a field of characteristic zero and $B = \mathbf{k}^{[3]}$. If $A \in \text{KLND}(B)$ then $A = \mathbf{k}^{[2]}$.*

The above result was proved in [50] for the special case when $\mathbf{k} = \mathbb{C}$, and one can reduce the general case to the special case by using [37] (details are given in [20]).

1.4.6 Remark: Kernels of locally nilpotent derivations of $\mathbf{k}^{[n]}$ are not always finitely generated. In fact, a special case of *Hilbert's Fourteenth Problem* can be stated as follows: Let $\mathbf{k}$ be a field of characteristic zero and $B = \mathbf{k}^{[n]}$. If $A \in \text{KLND}(B)$, is A finitely generated as a $\mathbf{k}$-algebra?

The above problem has an affirmative answer when $n \leq 3$ and has a negative answer when $n \geq 5$ (cf. [26, 29, 40, 54, 60, 19]). The case $n = 4$ is still open.

1.5 Some algebraic geometry

For the general background in algebraic geometry, our main reference is [33].

Definition 1.5.1 A *variety* X is an integral separated scheme of finite type over an algebraically closed field $\mathbf{k}$. We say that X is *complete* if it is proper over $\mathbf{k}$.

Remark 1.5.2 It follows from the definition of algebraic varieties that if x is a point of a variety X, then the local ring $\mathcal{O}_{X,x}$ is an integral domain; if $X = \operatorname{Spec} A$ is an affine variety, then A is an integral domain and a finitely generated $\mathbf{k}$-algebra.

Definition 1.5.3 The *dimension* of a topological space X (denoted $\dim X$) is the supremum of all integers n such that there exists a chain $Z_0 \subset \cdots \subset Z_n$ of irreducible closed subsets of X (recall that $\subset$ means strict inclusion).

Definition 1.5.4 The *dimension of a scheme* X, denoted $\dim X$, is its dimension as a topological space. If Z is an irreducible closed subset of X, then the *codimension* of Z in X, denoted $\operatorname{codim}(Z, X)$, is the supremum of integers n such that there exists a chain $Z = Z_0 \subset Z_1 \subset \cdots \subset Z_n$ of distinct closed irreducible subsets of X, beginning with Z. If Y is any closed subset of X, we define

$$\operatorname{codim}(Y, X) = \inf_{Z \subseteq Y} \operatorname{codim}(Z, X),$$

where the infimum is taken over all closed irreducible subsets of Y.

Definition 1.5.5 Given a prime ideal $\mathfrak{p}$ of a ring R, *height of* $\mathfrak{p}$, denoted $\operatorname{ht} \mathfrak{p}$, is the supremum of all integers n such that there exists a chain $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_n = \mathfrak{p}$ of distinct prime ideals of R. If I is a proper ideal of R then the height of I (denoted $\operatorname{ht} I$) is the infimum of the heights of the prime ideals $\mathfrak{p}$ of R which satisfy $\mathfrak{p} \supseteq I$.

1.5.6 Let R be an affine $\mathbf{k}$-domain where $\mathbf{k}$ is any field. Recall that, $\operatorname{trdeg}_{\mathbf{k}} R$ is the transcendence degree of $\operatorname{Frac} R$ over $\mathbf{k}$. The following are well-known facts from commutative algebra:

(a) $\dim R = \mathrm{trdeg}_{\mathbf{k}}(R)$

(b) For every $\mathfrak{p} \in \operatorname{Spec} R$, $\dim(R/\mathfrak{p}) + \operatorname{ht} \mathfrak{p} = \dim R$.

(c) Every maximal ideal of R has height equal to $\dim R$.

If $X = \operatorname{Spec} R$ is an affine variety, $\dim X$ is equal to the Krull dimension of the ring R. Varieties of dimension 1 and 2 are called *curves* and *surfaces*, respectively.

An integral domain is said to be *normal* if it is integrally closed in its field of fractions (see Definition 1.8.1). More generally, a ring R is *normal* if $R_{\mathfrak{p}}$ is a normal domain for every prime ideal $\mathfrak{p}$ of R.

Definition 1.5.7 A variety X is *normal* at a point $x \in X$ if the the local domain $\mathcal{O}_{X,x}$ is normal. We say that X is normal if X is normal at each $x \in X$.

Note that an affine variety $X = \operatorname{Spec} R$ is normal if and only if R is a normal ring.

Definition 1.5.8 Let X be a variety. A point $x \in X$ is *nonsingular* or *regular* or *smooth*, if the local ring $\mathcal{O}_{X,x}$ is a regular local ring (i.e., its maximal ideal can be generated by $\dim \mathcal{O}_{X,x}$ elements). We say that X is nonsingular (or regular or smooth) if it is nonsingular at every point. A point of X which is not nonsingular is said to be *singular*. The set of singular points of X is denoted $\operatorname{Sing} X$.

Example 1.5.9 The curve in $\mathbb{A}^2$ given by the equation $y^2 = x^3$ has only one singular point, namely, $(0, 0)$.

Theorem 1.5.10 *Given a variety X, $\operatorname{Sing} X$ is a proper closed subset of X.*

Proof: See [33, Theorem 5.3], for instance. ∎

Definition 1.5.11 A variety X is *regular in codimension* 1 if $\operatorname{codim}(\operatorname{Sing}(X), X) > 1$. A ring R is *regular in codimension* 1 if $R_{\mathfrak{p}}$ is a regular local ring for every height 1 prime ideal $\mathfrak{p}$ of R.

Note that an affine variety $X = \operatorname{Spec} R$ is regular in codimension 1 if and only if R is regular in codimension 1.

The following theorems are well-known results of algebraic geometry. See Chapter 2, p. 126 and 127 of [56], for instance.

Theorem 1.5.12 *A nonsingular variety is normal.*

Theorem 1.5.13 *A normal variety is regular in codimension* 1.

Consequently, normality and regularity are equivalent conditions for affine curves. If X is a normal surface, then $\operatorname{Sing} X$ is a finite set of points. For an affine surface X,

$$X \text{ is smooth} \Rightarrow X \text{ is normal} \Rightarrow \operatorname{Sing} X \text{ is a finite set.}$$

However, the converses of the above statements do not hold, as can be seen in the following examples:

Example 1.5.14 Consider the affine surface $S \subset \mathbb{A}^3$ defined by the equation $XZ = Y^2$. Then S is normal and has a singular point at the origin.

Example 1.5.15 Consider the 2-dimensional affine $\mathbf{k}$-domain $B = \mathbf{k}[x, xy, y^2, y^3]$. Then the origin is the only singular point of $S = \operatorname{Spec} B$. However, S is not normal.

Definition 1.5.16 Let R be a domain containing a field $\mathbf{k}$. We say that R is a *complete intersection over* $\mathbf{k}$ if it is isomorphic to a quotient

$$\mathbf{k}[X_1, \ldots, X_n]/(f_1, \ldots, f_p)$$

for some $n, p \in \mathbb{N}$, where $(f_1, \ldots, f_p)$ is a prime ideal of $\mathbf{k}[X_1, \ldots, X_n]$ of height p. If R is a complete intersection over $\mathbf{k}$, we call $\operatorname{Spec} R$ a complete intersection over $\mathbf{k}$.

Recall the following criterion for noetherian normal rings due to Serre. (See [48], for instance.)

Theorem 1.5.17 (Serre) *A noetherian ring A is normal if and only if it satisfies*
(R_1) $A_{\mathfrak{p}}$ *is regular for all* $\mathfrak{p} \in \operatorname{Spec} A$ *with* $\operatorname{ht} \mathfrak{p} \leq 1$, *and*
(S_2) $\operatorname{depth} A_{\mathfrak{p}} \geq \min(\operatorname{ht} \mathfrak{p}, 2)$ *for all* $\mathfrak{p} \in \operatorname{Spec} A$.

We obtain the following consequence of Serre's Theorem:

Lemma 1.5.18 *Let B be an affine* $\mathbf{k}$*-domain which is a complete intersection over* $\mathbf{k}$. *If B is regular in codimension* 1, *then B is normal.*

Proof: It is a well-known fact that complete intersections are Cohen-Macaulay (cf. [24, Proposition 18.13]), and so they satisfy (S_2). (See [48, 17.I, p.125].) In view of Serre's theorem, it suffices to prove that B satisfies (R_1). So let $\mathfrak{p} \in \operatorname{Spec} B$. If $\operatorname{ht} \mathfrak{p} = 0$, then clearly $B_{\mathfrak{p}}$ is regular. If $\operatorname{ht} \mathfrak{p} = 1$, $B_{\mathfrak{p}}$ is regular by hypothesis. ∎

1.6 LNDs and $\mathbf{G}_a$-actions

Let $\mathbf{k}$ be a field of characteristic zero and let B be a $\mathbf{k}$-algebra. Then there is a bijection between the set $\text{LND}(B)$ and the set of all $\mathbf{G}_a(\mathbf{k})$-actions on the affine scheme $\operatorname{Spec} B$. This section is devoted to describing this bijection in the case when $\mathbf{k}$ is algebraically closed.
For a proof of the following well-known fact, see Proposition 3.2 and Lemma 3.3 of [9], for instance.

Proposition 1.6.1 *Let B be an algebra over a field* $\mathbf{k}$ *of characteristic zero, $D \in$* $\text{LND}(B)$ *and $A = \ker D$. The map*

$$e^D : B \to B, \quad b \mapsto \sum_{n \in \mathbb{N}} \frac{D^n(b)}{n!}$$

is an automorphism of B *as an* A*-algebra which satisfies* $A = \{\, b \in B \mid e^D(b) = b \,\}$. *The subgroup generated by the set* $\{\, e^D \mid D \in \text{LND}_{\mathbf{k}}(B) \,\}$ *is normal in* $\text{Aut}_{\mathbf{k}}(B)$.

Notation 1.6.2 Let $\mathbf{k}$ be any algebraically closed field of characteristic zero. By $\mathbf{G}_a(\mathbf{k})$ or simply $\mathbf{G}_a$, we mean the additive algebraic group $(\mathbf{k}, +)$. Recall that the coordinate ring of this affine algebraic group is $\mathbf{k}^{[1]}$.

Definition 1.6.3 Let $\mathbf{k}$ be any algebraically closed field of characteristic zero. If X is a $\mathbf{k}$-variety, then an *algebraic* $\mathbf{G}_a$*-action on* X is a morphism $\alpha : \mathbf{G}_a \times X \to X$ of varieties which satisfies for all $x \in X$:

(1) $\alpha(0, x) = x$.

(2) $\alpha(a + b, x) = \alpha(a, \alpha(b, x))$ for all $a, b \in \mathbf{k}$.

1.6.4 Given an algebraically closed field $\mathbf{k}$ of characteristic zero and a $\mathbf{k}$-algebra B, let us define a set map $\text{LND}_{\mathbf{k}}(B) \to \{\mathbf{G}_a(\mathbf{k})\text{-actions on Spec}\, B\}$. Given a locally nilpotent derivation D of B, we have a group homomorphism $f : (\mathbf{k}, +) \to \text{Aut}_{\mathbf{k}}(B)$ given by $f(\lambda) = e^{\lambda D}$. Applying the "functor Spec," we get the group homomorphism

$$(\mathbf{k}, +) \to \text{Aut}_{\mathbf{k}}(\text{Spec}\, B), \quad \lambda \mapsto \text{Spec}\, e^{\lambda D}.$$

To prove that we get a $\mathbf{G}_a$-action, we need to verify that the map

$$\alpha : \mathbf{G}_a \times \text{Spec}\, B \to \text{Spec}\, B, \quad (\lambda, x) \mapsto (\text{Spec}\, e^{\lambda D})(x)$$

is a morphism of varieties. Let $\mathbf{k}[T]$ be the coordinate ring of the algebraic group $\mathbf{G}_a(\mathbf{k})$ where T is an indeterminate. Then

$$\mathbf{G}_a \times \text{Spec}\, B = \text{Spec}(\mathbf{k}[T] \otimes_{\mathbf{k}} B) = \text{Spec}(B[T]).$$

To show that α is a morphism, it suffices to prove that $\alpha = \text{Spec}\, \theta$ for some $\mathbf{k}$-algebra homomorphism $\theta : B \to B[T]$. Define $\theta : B \to B[T]$ by $\theta(b) = \sum_{j \in \mathbb{N}} \frac{D^j(b)}{j!} T^j$. By Lemma 1.1.14, θ is a $\mathbf{k}$-algebra homomorphism and one can verify that $\text{Spec}\, \theta = \alpha$.

So the above discussion defines a set map $\text{LND}(B) \to \{G_a\text{-actions on Spec}\, B\}$, $D \mapsto \alpha$, and one can verify that this map is bijective. Moreover, the ring of invariants B^{G_a} of the action α corresponding to D is equal to $\ker(D)$. Indeed, for $b \in B$ we have

$$b \in B^{G_a} \iff \forall_{\lambda \in \mathbf{k}}\ e^{\lambda D}(b) = b \iff \forall_{\lambda \in \mathbf{k}}\ b \in \ker(\lambda D) \iff b \in \ker D.$$

1.7 Makar-Limanov invariant

Definition 1.7.1 Let $\mathbf{k}$ be a field of characteristic zero. Given a $\mathbf{k}$-domain B, one defines its *Makar-Limanov invariant* by

$$\text{ML}(B) = \bigcap_{D \in \text{LND}(B)} \ker D.$$

Note that $\mathbf{k} \subseteq \text{ML}(B)$ by 1.1.10 (1). If $X = \text{Spec}\, B$ is an affine $\mathbf{k}$-variety, define $\text{ML}(X) = \text{ML}(B)$. We say that B has trivial Makar-Limanov invariant if $\text{ML}(B) = \mathbf{k}$.

The Makar-Limanov invariant plays an important role in classifying and distinguishing affine varieties. Given a domain B over a field $\mathbf{k}$ of characteristic zero, the following properties can be easily obtained:

(i) $\text{ML}(B)$ is a subring of B that is invariant under any automorphism of B.

(ii) $\text{ML}(B)$ is factorially closed in B. Consequently, $B^* \subset \text{ML}(B)$ and $\text{ML}(K) = K$ for any field K.

Example 1.7.2 Let $B = \mathbb{C}[X, Y]$, then $\text{ML}(B) = \mathbb{C}$. If we regard B as a $\mathbb{C}$-algebra, it has trivial ML-invariant. If we regard B as an $\mathbb{R}$-algebra, it does not have trivial ML-invariant.

Example 1.7.3 Affine spaces $\mathbb{A}^n_{\mathbf{k}}$ are varieties with trivial Makar-Limanov invariant. The ring $R = \mathbb{C}[X^2, X^3]$ is a variety satisfying $\text{ML}(R) = R$ (cf. Example 5.1.1 (4)). (Such rings are called rigid rings.)

1.8 Integral closure and valuation rings

Definition 1.8.1 Given rings $R \subseteq S$, we define the *integral closure* of R in S to be the subring of S consisting of all the elements of S that are integral over R. If R is a domain, the integral closure of R in $\operatorname{Frac} R$ is denoted $\tilde{R}$ and is also called the *normalization* of R. We say that a domain R is *normal* if $\tilde{R} = R$.

The following is a well-known fact. See [2, Proposition 5.12], for instance.

Proposition 1.8.2 *Let $A \subseteq B$ be rings, C the integral closure of A in B. If $S \subset A$ is multiplicatively closed, then $S^{-1}C$ is the integral closure of $S^{-1}A$ in $S^{-1}B$.*

Theorem 1.8.3 (Emmy Noether) *Let R be an affine domain over a field $\mathbf{k}$, and let L be a finite extension of $\operatorname{Frac} R$. If T is the integral closure of R in L, then T is a finitely generated R-module; in particular, T is again an affine $\mathbf{k}$-domain.*

Proof: See Corollary 13.13 of [24], for instance. ∎

Definition 1.8.4 A subring R of a field K is a *valuation ring* of K if every nonzero $x \in K$ satisfies $x \in R$ or $x^{-1} \in R$.

1.8.5 A valuation ring R is a local ring, and we denote the maximal ideal of R by M_R. The *residue field* of R is the field R/M_R.

Proposition 1.8.6 [49, Theorem 10.4]. *Let R be a subring of a field K. The integral closure of R in K is the intersection of all valuation rings V of K such that $V \supseteq R$.*

Notation 1.8.7 Given a field extension $F/\mathbf{k}$, the set of valuation rings of F that contain $\mathbf{k}$ will be denoted $\operatorname{Val}(F/\mathbf{k})$. We also define $\mathbb{P}_{F/\mathbf{k}} = \{ R \in \operatorname{Val}(F/\mathbf{k}) \mid R \neq F \}$.

1.8.8 Let A be a domain of trdeg 1 over a field $\mathbf{k}$, and let $F = \operatorname{Frac} A$. Note that the set $E = \{ R \in \mathbb{P}_{F/\mathbf{k}} \mid A \nsubseteq R \}$ is nonempty because, if every element of $\mathbb{P}_{F/\mathbf{k}}$ contains A, then

$$A \subseteq \bigcap_{R \in \operatorname{Val}(F/\mathbf{k})} R = \tilde{\mathbf{k}},$$

where $\tilde{\mathbf{k}}$ is the integral closure of $\mathbf{k}$ in F. This is a contradiction.

Definition 1.8.9 Let $E = \{ R \in \mathbb{P}_{F/\mathbf{k}} \mid A \nsubseteq R \}$ as in 1.8.8. We refer to the elements of E as the "places at infinity of A". If $E = \{R\}$, we say that A has *one place at infinity*, namely R. Furthermore, if the residue field of R is $\mathbf{k}$, we say that A has *one rational place at infinity*.

For the proof of Lemmas 1.8.10 and 1.8.13, refer to Lemmas A.0.11 and A.0.14 of Appendix A, respectively.

Lemma 1.8.10 *For a domain A of* trdeg 1 *over a field* $\mathbf{k}$, *the following are equivalent:*

(1) *The normalization of A is $\mathbf{k}^{[1]}$.*

(2) *A is a subalgebra of $\mathbf{k}^{[1]}$.*

(3) $\operatorname{Frac} A = \mathbf{k}^{(1)}$ *and A has one rational place at infinity.*

Corollary 1.8.11 *Let A be a normal domain of* trdeg 1 *over a field* $\mathbf{k}$. *If $A \subseteq \mathbf{k}^{[1]}$, then $A \cong \mathbf{k}^{[1]}$.*

Definition 1.8.12 If A and $\mathbf{k}$ satisfy the equivalent conditions of Lemma 1.8.10, we say that A is a *polynomial curve* over $\mathbf{k}$.

Lemma 1.8.13 *Let $\mathbf{k}$ be a perfect field and let A be a $\mathbf{k}$-domain. If there exists an algebraic extension $K/\mathbf{k}$ such that $K \otimes_{\mathbf{k}} A$ is a polynomial curve over K, then A is a polynomial curve over $\mathbf{k}$.*

Chapter 2

Homogeneous locally nilpotent derivations of $\mathbf{k}^{[n]}$

In this chapter, we discuss regular gradings and coordinatizable gradings. Then we define the notion of "type" for a coordinatizable grading of a polynomial ring. The notion of type for $\mathbb{N}$-graded polynomial rings was introduced by Daigle in [10]. We generalize this notion for polynomial rings that are graded by an arbitrary torsion-free abelian group G, and then we discuss the special case $G = \mathbb{Z}$ in detail. The material of this chapter serves as background for Chapter 3 and Chapter 4; besides, it also contains some new results related to homogeneous locally nilpotent derivations of the polynomial rings $\mathbf{k}^{[n]}$.

2.1 Gradings and homogeneous derivations

Throughout this section, assume that $(G, +)$ is an abelian group.

Definition 2.1.1 Let R be a ring. A *G-grading* of R is a family $\mathfrak{g} = \{R_i\}_{i \in G}$ of subgroups R_i of $(R, +)$ satisfying $R = \oplus_{i \in G} R_i$ and $R_i R_j \subseteq R_{i+j}$ for all $i, j \in G$. An element of R is *homogeneous* if it belongs to $\bigcup_{i \in G} R_i$. If $x \neq 0$ is a homogeneous element of R, then there is a unique $i \in G$ such that $x \in R_i$; we write $\deg_{\mathfrak{g}}(x) = i$. (Note that $\deg_{\mathfrak{g}}(xy) = \deg_{\mathfrak{g}}(x) + \deg_{\mathfrak{g}}(y)$ for all homogeneous $x, y \in R$ such that $xy \neq 0$.)

Definition 2.1.2 Given a G-graded ring R and $b \in R$, we can write $b = \sum_{i \in G} b_i$ as a decomposition of homogeneous elements of R. The homogeneous elements $b_i \in R_i$ are called the *homogeneous components* of b. A subring S of R is called a *homogeneous subring* if $a \in S$ implies that every homogeneous component of a belongs to S.

Lemma 2.1.3 *If R is a G-graded ring, then 1 is homogeneous of degree zero.*

Proof: Write $1 = \sum_{i \in G} x_i$ with $x_i \in R_i$. If $h \in R_j$ then $h = \sum_{i \in G} x_i h$, where $x_i h \in R_{i+j}$. In particular, we must have $x_0 h = h$. It follows that $x_0 y = y$ for every $y \in R$, so $x_0 = 1$ (and $x_i = 0$ for all $i \in G \setminus \{0\}$). ∎

Definition 2.1.4 Let $\mathfrak{g} = \{B_i\}_{i \in G}$ be a grading of a ring B. A derivation D of B is *homogeneous* (or $\mathfrak{g}$-*homogeneous*) if there exists $j \in G$ such that $D(B_i) \subseteq B_{i+j}$ for all $i \in G$. Such a j is unique if $D \neq 0$ and we say that D is homogeneous of degree j.

Notation 2.1.5 Let $\mathfrak{g} = \{B_i\}_{i \in G}$ be a grading of a ring B. The set of all $\mathfrak{g}$-homogeneous locally nilpotent derivations of B will be denoted by $\text{LND}(B, \mathfrak{g})$. Similarly, $\text{KLND}(B, \mathfrak{g}) = \{\ \ker D \mid D \in \text{LND}(B, \mathfrak{g}) \text{ and } D \neq 0\ \}$.

Lemma 2.1.6 *Let $\mathfrak{g} = \{B_i\}_{i \in G}$ be a grading of a ring B. If D is a homogeneous derivation of B, then $\ker D$ is a homogeneous subring of B.*

Proof: We may assume that $D \neq 0$. Let $d \in G$ be such that D is homogeneous of degree d. Recall that $\ker D$ is a subring of B. Let $0 \neq a \in \ker D$ and write $a = \sum_{j=1}^{n} a_{i_j}$ where $a_{i_j} \in B_{i_j}$ and $i_1, \ldots, i_n$ are distinct elements of G. Then $D(a) = 0$ implies that $D(a_{i_1}) + \cdots + D(a_{i_n}) = 0$ where $D(a_{i_j}) \in B_{i_j + d}$ and $i_1 + d, \ldots, i_n + d$ are distinct elements of G. It follows that $D(a_{i_j}) = 0$ for each $j = 1, \ldots, n$. ∎

Let $B = \mathbf{k}^{[3]}$, where $\mathbf{k}$ is any field of characteristic zero. Recall from Miyanishi's Theorem (cf. 1.4.5) that any nonzero $D \in \text{LND}(B)$ satisfies $\ker D = \mathbf{k}^{[2]}$. We also have:

Lemma 2.1.7 *Let $\mathbf{k}$ be a field, and let $A = \mathbf{k}^{[2]}$ be graded by a $\mathbb{Z}$-grading. Then there exist homogeneous elements $f, g \in A$ such that $A = \mathbf{k}[f, g]$.*

Proof: See [4, 38], for instance. ∎

Applying the above lemma, we obtain the homogeneous version of Miyanishi's Theorem:

Corollary 2.1.8 *Let $\mathbf{k}$ be a field of characteristic zero. Assume that $B = \mathbf{k}[X, Y, Z] = \mathbf{k}^{[3]}$ is $\mathbb{Z}$-graded by a grading $\mathfrak{g}$, and $0 \neq D \in \text{LND}(B, \mathfrak{g})$. Then $\ker D = \mathbf{k}[f, g] = \mathbf{k}^{[2]}$, where f and g are $\mathfrak{g}$-homogeneous elements of B.*

2.2 Regular gradings

Definition 2.2.1 A grading $\mathfrak{g}$ on a ring R is *regular* if $x, y \in R$ are homogeneous whenever xy is a nonzero homogeneous element of R.

Example 2.2.2 Let $\mathbf{k}$ be a field and $R = \mathbf{k}[X]/(X^2)$ (where $\mathbf{k}[X] = \mathbf{k}^{[1]}$). Write $R = \mathbf{k}[x]$, where $x^2 = 0$ and, define a $\mathbb{Z}$-grading on R by setting $R_0 = \mathbf{k}$, $R_1 = \mathbf{k}x$ and $R_i = 0$ for every $i \in \mathbb{Z} \setminus \{0, 1\}$. As $1 + x$ and $1 - x$ are non-homogeneous elements whose product is 1, the grading is not regular.

Example 2.2.3 Consider the set of complex numbers $\mathbb{C}$. Then $\mathbb{C} = \mathbb{R} \oplus i\mathbb{R}$ is a nonregular grading of $\mathbb{C}$ by the group $\mathbb{Z}/2\mathbb{Z}$.

Lemma 2.2.4 *Let* $\mathfrak{g} = \{R_i\}_{i \in G}$ *be a regular grading of a ring* R*. Then,*

(1) *Every unit of* R *is homogeneous.*

(2) *If* $\mathbf{k}$ *is a field included in* R *then* $\mathbf{k} \subseteq R_0$*.*

Proof: If $u \in R^*$ then $uu^{-1} = 1$ is homogeneous, so u is homogeneous by regularity. Let $\mathbf{k}$ be a field included in R and suppose that $\mathbf{k} \nsubseteq R_0$. Pick $\lambda \in \mathbf{k} \setminus R_0$. As λ is homogeneous by the first assertion, $\lambda \in R_i$ for some $i \neq 0$. As $1 \in R_0$, we see that $1 + \lambda$ is not homogeneous (note that $\lambda \neq 0$ since $\lambda \notin R_0$). However $1 + \lambda \in \mathbf{k} \setminus \{0\} \subseteq R^*$, contradicting the first assertion. ∎

Lemma 2.2.5 *An abelian group admits a total order if and only if it is torsion-free.*

Proof: See Proposition 1.1.7 of [1], for instance. ∎

Lemma 2.2.6 *If* G *is a torsion-free abelian group, then every* G*-grading of an integral domain is regular.*

Proof: By Lemma 2.2.5, we may choose a total order on G. Let $\mathfrak{g}$ be a G-grading of an integral domain R. For each $x \in R$ define: $\mathrm{Supp}_{\mathfrak{g}}(x) = \{\, i \in G \mid x_i \neq 0 \,\}$, where $x = \sum_{i \in G} x_i$. Fix an ordering of G and define for each $x \in R \setminus \{0\}$,

$$W^+(x) = \max \mathrm{Supp}_{\mathfrak{g}}(x),\ W^-(x) = \min \mathrm{Supp}_{\mathfrak{g}}(x), \text{ and } W(x) = W^+(x) - W^-(x).$$

Then we have a map $W : R \setminus \{0\} \to G$ with the following properties:

(1) $\forall x \in R \setminus \{0\}$, $W(x) \geq 0$, where equality holds if and only if x is homogeneous.

(2) $W(xy) = W(x) + W(y)$ for all $x, y \in R \setminus \{0\}$.

If $x, y \in R$ are such that $xy \neq 0$ and xy is homogeneous, then $0 = W(xy) = W(x) + W(y)$, which implies that x and y are homogeneous, so we are done. ∎

Notation 2.2.7 Let G be any abelian group. Suppose that $R = \oplus_{i \in G} R_i$ is a G-graded ring, and A is a subset of R. Then $G(A)$ denotes the subgroup of G generated by the set $\{\, i \in G \mid A \cap R_i \neq \{0\} \,\}$. In particular, $G(R)$ is the subgroup of G generated by $\{\, i \in G \mid R_i \neq \{0\} \,\}$, and it is clear that for every subset A of R, $G(A)$ is a subgroup of $G(R)$.

Proposition 2.2.8 *Let G be an abelian group and $\mathfrak{g}$ a G-grading of a domain R. Then $\mathfrak{g}$ is regular if and only if $G(R)$ is torsion-free.*

Proof: Suppose that $G(R)$ contains an element $k \neq 0$ of finite order. We show that $\mathfrak{g}$ is not regular. Let $S = \{\, i \in G \mid R_i \neq \{0\} \,\}$. Note that S is closed under addition because R is an integral domain. As $G(R)$ is the subgroup of G generated by S, there exist distinct elements $i, j \in S$ such that $k = i - j$. Choose $x \in R_i \setminus \{0\}$ and $y \in R_j \setminus \{0\}$. Let $n \geq 2$ be the order of k and define

$$z = \sum_{u=0}^{n-1} x^u y^{n-1-u} \in R. \tag{2.2.1}$$

The element $x - y$ of R is not homogeneous, but $(x - y)z = x^n - y^n$ is homogeneous since $ni = nj$. So to prove that $\mathfrak{g}$ is not regular it suffices to verify that $(x - y)z \neq 0$. Note that each term $x^u y^{n-1-u}$ in (2.2.1) is nonzero and has degree

$$\deg_{\mathfrak{g}}(x^u y^{n-1-u}) = ui + (n-1-u)j = (n-1)j + u(i-j) = (n-1)j + uk;$$

now $u \mapsto (n-1)j + uk$ is an injective map $\{0, 1, \ldots, n-1\} \to G$ (because k has order n), so the terms in the sum (2.2.1) have distinct degrees; in particular $z \neq 0$ and consequently $(x - y)z \neq 0$. So $\mathfrak{g}$ is not regular.

Conversely, if $G(R)$ is torsion-free then $\mathfrak{g}$ is regular by Lemma 2.2.6. ∎

Corollary 2.2.9 *Let R be an integral domain and a finitely generated $\mathbf{k}$-algebra. Suppose that R is endowed with a regular G-grading, where G is an abelian group. Then $G(R)$ is isomorphic to $\mathbb{Z}^r$ for some $r \geq 0$.*

Proof: Write $R = \mathbf{k}[h_1, \ldots, h_n]$, where each h_i is homogeneous. Note that $\mathbf{k} \subseteq R_0$ by Lemma 2.2.4, so $G(R)$ is generated by the finite set $\deg_{\mathfrak{g}}(h_i)$, where $1 \leq i \leq n$. By Proposition 2.2.8, $G(R)$ is torsion-free, so it is a free abelian group of finite rank. ∎

2.3 Coordinatizable gradings and type

Throughout this section, we have $B = \mathbf{k}^{[n]}$ where $\mathbf{k}$ is a field of characteristic zero.

Definition 2.3.1 Let G be an abelian group. A G-grading $\mathfrak{g} = \{B_i\}_{i \in G}$ of B is *coordinatizable* if $\mathbf{k} \subseteq B_0$ and there exists a coordinate system $\gamma = (X_1, \ldots, X_n)$ of B such that each X_i is homogeneous. Such a coordinate system γ is called a *homogeneous coordinate system* of B.

Remark 2.3.2 If G is torsion-free then the condition $\mathbf{k} \subseteq B_0$ is automatically satisfied by Lemmas 2.2.6 and 2.2.4.

Example 2.3.3 Consider the $\mathbb{Z}/2\mathbb{Z}$-grading of $B = \mathbb{C}[X_1, \ldots, X_n]$ given by

$$B = B_0 \oplus B_1, \qquad B_0 = \mathbb{R}[X_1, \ldots, X_n], \quad B_1 = i\,\mathbb{R}[X_1, \ldots, X_n].$$

Then each X_i is homogeneous but $\mathbb{C} \not\subseteq B_0$, so this grading is not coordinatizable.

Remark 2.3.4 Let G be an abelian group, $\mathfrak{g}$ a coordinatizable G-grading of B, $(X_1, \ldots, X_n)$ a homogeneous coordinate system, and h a nonzero homogeneous element of B. Write $h = \sum_i a_i X^i$ where $i = (i_1, \ldots, i_n) \in \mathbb{N}^n$, $X^i = X_1^{i_1} \cdots X_n^{i_n}$ and $a_i \in \mathbf{k}$. Then

$$\deg_{\mathfrak{g}}(X^i) = \deg_{\mathfrak{g}}(h) \text{ for each } i \text{ such that } a_i \neq 0.$$

Remark 2.3.5 To define a coordinatizable G-grading of B, we may proceed as follows. Choose an arbitrary coordinate system $\gamma = (X_1, \ldots, X_n)$ of B and an n-tuple

$(a_1, \dots, a_n) \in G^n$. Then for each $j \in G$ define B_j to be the $\mathbf{k}$-span of the monomials $X_1^{i_1} \cdots X_n^{i_n}$ satisfying $i_1 a_1 + \cdots + i_n a_n = j$. Note that, in particular, we have $\mathbf{k} \subseteq B_0$. Then $\mathfrak{g} = \{B_j\}_{j \in G}$ is a coordinatizable G-grading of B and γ is a homogeneous coordinate system of B. In fact, $\mathfrak{g}$ is the unique G-grading of B which satisfies $\mathbf{k} \subseteq B_0$ and $X_i \in B_{a_i}$ for $i = 1, \dots, n$. Note that every coordinatizable G-grading of B can be obtained by the above procedure.

2.3.6 The following fact is highly nontrivial:

Theorem. *If $\mathbf{k}$ is an algebraically closed field of characteristic zero, then all $\mathbb{Z}$-gradings of $\mathbf{k}^{[3]}$ are coordinatizable.*

The case $\mathbf{k} = \mathbb{C}$ of the above theorem is a consequence of [39, 45, 42, 35, 44, 43, 55, 36]. The generalization to algebraically closed fields of characteristic zero is considered straightforward.

For the remainder of this section, let G be an abelian group and $\mathfrak{g}$ a coordinatizable grading of $B = \mathbf{k}^{[n]}$.

2.3.7 Each homogeneous coordinate system $\gamma = (X_1, \dots, X_n)$ of B determines an element $a(\mathfrak{g}, \gamma) = (a_1, \dots, a_n)$ of G^n, where $a_i = \deg_{\mathfrak{g}}(X_i)$ for each i. The subgroup of G generated by $a_1, \dots, a_n$ is $G(B)$ (in the notation of 2.2.7).

Lemma 2.3.8 *Let γ and γ' be two homogeneous coordinate systems of B. Then $a(\mathfrak{g}, \gamma')$ is a permutation of $a(\mathfrak{g}, \gamma)$.*

Proof: Write $\gamma = (X_1, \dots, X_n)$ and $\gamma' = (X'_1, \dots, X'_n)$. For each $k \in G$, let $I_k = \{ i \mid \deg_{\mathfrak{g}}(X_i) = k \}$ and $I'_k = \{ i \mid \deg_{\mathfrak{g}}(X'_i) = k \}$. It suffices to show that $|I_k| = |I'_k|$ for every $k \in G$. Suppose that $k \in G$ is such that $|I'_k| > |I_k|$. For each $i \in I'_k$, we may express X'_i as a polynomial in $X_1, \dots, X_n$ as follows:

$$X'_i = c_i + L_i + Q_i, \tag{2.3.1}$$

where $c_i \in \mathbf{k}$, L_i is a linear form in $X_1, \dots, X_n$ and Q_i belongs to the ideal $(X_1, \dots, X_n)^2$ of $B = \mathbf{k}[X_1, \dots, X_n]$. As X'_i is homogeneous, L_i belongs to the $\mathbf{k}$-span of $\{ X_j \mid j \in I_k \}$; as $|I'_k| > |I_k|$, it follows that $\{ L_i \mid i \in I'_k \}$ is linearly dependent. Choose a nonzero family $(s_i)_{i \in I'_k}$ of scalars such that $\sum_{i \in I'_k} s_i L_i = 0$ and let $V = \sum_{i \in I'_k} s_i X'_i$. Express V as a polynomial in $X_1, \dots, X_n$, $V = c + L + Q$ as in (2.3.1); then $L = 0$, and this is impossible because V is a variable of B. ∎

Definition 2.3.9 Let $\gamma = (X_1, \dots, X_n)$ be a homogeneous coordinate system of B and let $a(\mathfrak{g}, \gamma) = (a_1, \dots, a_n)$ be as in 2.3.7. For each $i \in \{1, \dots, n\}$, let $G_{\gamma,i}$ be the subgroup of G generated by $\{ a_j \mid j \in \{1, \dots, n\} \setminus \{i\} \}$. So in the notation of 2.2.7, $G_{\gamma,i} = G(\mathbf{k}[X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_n])$. The unordered n-tuple $[G_{\gamma,1}, \dots, G_{\gamma,n}]$ of subgroups of G depends only on $\mathfrak{g}$ by Lemma 2.3.8 (i.e., it is independent of the choice of γ). Consequently, the cardinality of the set $\{ i \mid G_{\gamma,i} \neq G(B) \}$ depends only on $\mathfrak{g}$. We define an integer type$(\mathfrak{g}) \in \{0, \dots, n\}$ by

$$\text{type}(\mathfrak{g}) = |\{ i \mid G_{\gamma,i} \neq G(B) \}|.$$

Remark 2.3.10 Observe that each coordinatizable grading of B has a well-defined type. Also note that the trivial grading on B is of type 0.

2.4 Homogeneous derivations of a UFD

In Section 2.4, we assume that G is a torsion-free abelian group and B is a G-graded UFD containing $\mathbb{Q}$. Let $\mathfrak{g} = \{B_i\}_{i \in G}$ denote the grading of B and observe that $\mathfrak{g}$ is regular by Lemma 2.2.6. Given $A \in \text{KLND}(B, \mathfrak{g})$, recall (from 2.2.7) the notations

$$G(A) = \langle \{ i \in G \mid A \cap B_i \neq \{0\} \} \rangle, \quad G(B) = \langle \{ i \in G \mid B_i \neq \{0\} \} \rangle$$

and define $\mathcal{H}(A)$ to be the set

$$\{ H \mid H \text{ is a homogeneous prime element of } B \text{ and } \deg_{\mathfrak{g}}(H) \notin G(A) \}.$$

Note that $\mathcal{H}(A) = \varnothing$ whenever $G(A) = G(B)$.

The following generalizes Proposition 2.1 of [7].

Proposition 2.4.1 *Let $\mathfrak{g} = \{B_i\}_{i\in G}$ be a G-grading of B. If $A \in \text{KLND}(B, \mathfrak{g})$ satisfies $G(A) \neq G(B)$, then:*

(a) $\mathcal{H}(A) \neq \varnothing$ and any two elements of $\mathcal{H}(A)$ are associates in B.

(b) Let $H \in \mathcal{H}(A)$. Then for any $D \in \text{LND}(B, \mathfrak{g})$ satisfying $\ker D = A$, the set of homogeneous preslices of D is $\{\, aH \mid a \text{ is a nonzero homogeneous element of } A \,\}$.

Proof: As $G(A)$ is a proper subgroup of $G(B)$ and $\{\, g \in G \mid B_g \neq 0 \,\}$ generates $G(B)$, we may choose $g \in G$ such that $B_g \neq 0$ and $g \notin G(A)$. Choose a nonzero $\alpha \in B_g$, and note that $\deg \alpha \notin G(A)$. Since B is a UFD, write $\alpha = H_1 \cdots H_s$ as a product of prime elements of B. As $\alpha = H_1 \cdots H_s$ is homogeneous and the grading is regular, every H_i must be homogeneous. If $\deg H_i \in G(A)$ for all $i \in \{1, \ldots, s\}$, then $\deg \alpha = \sum_{i=1}^{s} \deg H_i \in G(A)$, which is not the case. Hence $\deg H_j \notin G(A)$ for some $j \in \{1, \ldots, s\}$. Then $H_j \in \mathcal{H}(A)$, and $\mathcal{H}(A) \neq \varnothing$. Next, let $D \in \text{LND}(B, \mathfrak{g})$ be such that $\ker D = A$. We claim:

Each element of $\mathcal{H}(A)$ is a homogeneous preslice of D and divides every homogeneous preslice of D. (2.4.1)

Indeed, let $H \in \mathcal{H}(A)$ and let v be a homogeneous preslice of D. Then $\alpha = Dv \in A \setminus \{0\}$ and $B_\alpha = A_\alpha[v] = A_\alpha^{[1]}$ (cf. 1.2 (3)). In particular, there exists an integer $m \geq 0$ such that $\alpha^m H \in A[v]$. As $\alpha^m H$ is homogeneous, we may write $\alpha^m H = \sum_{i\in I} a_i v^i$, where I is a nonempty finite subset of $\mathbb{N}$ and for each $i \in I$, $a_i \in A \setminus \{0\}$ is homogeneous with $\deg(a_i v^i) = \deg(\alpha^m H)$. Hence $\deg a_i + i \deg v = m \deg \alpha + \deg H$ for each $i \in I$. Consequently, $i \deg v \notin G(A)$ because $\deg H \notin G(A)$. In particular, $i > 0$ for all $i \in I$. Therefore we have:

$$\alpha^m H = bv \quad \text{for some } b \in B \setminus \{0\}. \tag{2.4.2}$$

We claim that $H \mid v$ in B. If not, then $H \mid b$ because H is prime in B. Then by (2.4.2), $v \mid \alpha^m$. As $\alpha^m \in A \setminus \{0\}$ and A is factorially closed in B (cf. 1.2 (1)), it follows that $v \in A$. This contradicts the fact that $Dv \neq 0$, hence $H \mid v$ in B. Writing $v = aH$ for some $a \in B$, (2.4.2) gives $ba = \alpha^m \in A \setminus \{0\}$. Again A is factorially closed in B implies that $a \in A$. As $0 \neq v = aH$ is homogeneous and the grading is regular, a is homogeneous. So we have:

$$v = aH \quad \text{for some homogeneous } a \in A \setminus \{0\}. \tag{2.4.3}$$

It follows from (2.4.3) that H is a preslice of D, so claim (2.4.1) is proved. From (2.4.1) we deduce that if H and H' belong to $\in \mathcal{H}(A)$, then $H \mid H'$ and $H' \mid H$, which proves the assertion (a) of the proposition. Next, the assertion (b) follows from (2.4.3). ∎

2.5 Homogeneous derivations of $\mathbf{k}^{[n]}$

Throughout Section 2.5, let $B = \mathbf{k}^{[n]}$, where $\mathbf{k}$ is a field of characteristic zero. We also consider a coordinatizable G-grading $\mathfrak{g} = \{B_i\}_{i \in G}$ of B, where G is a torsion-free abelian group. Note that $\mathfrak{g}$ is regular by Lemma 2.2.6.

Lemma 2.5.1 *Let $\gamma = (X_1, \dots, X_n)$ be a homogeneous coordinate system of B and let $j \in \{1, \dots, n\}$. Then the following hold:*

(1) *If $h \in B$ is a homogeneous element such that $\deg(h) \notin G_{\gamma,j}$, then $X_j \mid h$.*

(2) *If $A \in$ KLND$(B, \mathfrak{g})$ and $G(A) \not\subseteq G_{\gamma,j}$, then $X_j \in A$.*

Proof: If h is a homogeneous element of B such that $X_j \nmid h$, then h contains a monomial of the type $\lambda X_1^{m_1} \dots X_{j-1}^{m_{j-1}} X_{j+1}^{m_{j+1}} \dots X_n^{m_n}$, where $\lambda \in \mathbf{k}^*$ and $m_i \geq 0$ for every $i \neq j$. Since h is homogeneous, $\deg h$ is equal to the degree of the above monomial, which is an element of $G_{\gamma,j}$. This proves (the contrapositive of) the first assertion. To prove (2), let $A \in \text{KLND}(B, \mathfrak{g})$ be such that $G(A) \not\subseteq G_{\gamma,j}$. Then there exists a homogeneous element $h \in A \setminus \{0\}$ such that $\deg(h) \notin G_{\gamma,j}$. It follows that $X_j \mid h$ by the first assertion. As A is factorially closed in B, it follows that $X_j \in A$. ∎

Proposition 2.5.2 *Let $A \in \text{KLND}(B, \mathfrak{g})$ be such that $G(A) \neq G(B)$ and let $D \in \text{LND}(B, \mathfrak{g})$ be such that* $\ker D = A$. *Let $\gamma = (X_1, \dots, X_n)$ be a homogeneous coordinate system of B. Then the following statements are equivalent for any $j \in \{1, \dots, n\}$.*

(1) $\deg_{\mathfrak{g}}(X_j) \notin G(A)$.

(2) X_j is a preslice of D.

(3) $G(A) = G_{\gamma,j}$.

(4) $G(A) \subseteq G_{\gamma,j} \neq G(B)$.

Moreover, there exists a unique $j \in \{1, \dots, n\}$ satisfying the above conditions.

Proof: As each X_j is prime and homogeneous, (1) is equivalent to

$$(2') \ X_j \in \mathcal{H}(A)$$

by definition of the set $\mathcal{H}(A)$, and (2′) is equivalent to (2) by Proposition 2.4.1. Since $G(A)$ is a proper subgroup of $G(B)$, $\deg(X_j) \notin G(A)$ for some $j \in \{1, \dots, n\}$. So at least one j satisfies (1). The elements of $\mathcal{H}(A)$ are associates of each other by Proposition 2.4.1. So at most one j satisfies (2′); thus exactly one $j \in \{1, \dots, n\}$ satisfies the equivalent conditions (1), (2′) and (2).

Now suppose that j satisfies (2′). Then j is the only element of $\{1, \dots, n\}$ which satisfies (1), so each $i \in \{1, \dots, n\} \setminus \{j\}$ satisfies $\deg(X_i) \in G(A)$ and consequently $G_{\gamma,j} \subseteq G(A)$. If $G(A) \not\subseteq G_{\gamma,j}$, then $X_j \in A$ by Lemma 2.5.1. This contradicts the fact that $\deg(X_j) \notin G(A)$, so $G(A) = G_{\gamma,j}$. This proves that (2′) implies (3). It is easy to see that (3) implies (4). Next, if j satisfies (4) then $\deg(X_j) \notin G_{\gamma,j}$ (otherwise $G_{\gamma,j} = G(B)$), so $\deg(X_j) \notin G(A)$ (because $G(A) \subseteq G_{\gamma,j}$). So j satisfies (1). ∎

Corollary 2.5.3 *If* type($\mathfrak{g}$) $= 0$ *then* $G(A) = G(B)$ *for all* $A \in \text{KLND}(B, \mathfrak{g})$.

Proof: Let $\gamma = (X_1, \ldots, X_n)$ be a homogeneous coordinate system of B. Let $A \in \text{KLND}(B, \mathfrak{g})$ be such that $G(A) \neq G(B)$. Then $G(A) = G_{\gamma,j}$ for some j by Proposition 2.5.2. But then $G_{\gamma,j} \neq G(B)$ and hence type($\mathfrak{g}$) > 0. ∎

Corollary 2.5.4 *For any homogeneous coordinate system* $\gamma = (X_1, \ldots, X_n)$ *of* B,

$$\{\, G(A) \mid A \in \text{KLND}(B, \mathfrak{g}) \,\} = \{G_{\gamma,1}, \ldots, G_{\gamma,n}\}.$$

Proof: Let $A_i = \mathbf{k}^{[n-1]}$ be the subalgebra of B generated by $\{X_1, \ldots, X_n\} \setminus \{X_i\}$. Then $\ker\left(\frac{\partial}{\partial X_i}\right) = A_i \in \text{KLND}(B, \mathfrak{g})$ and $G(A_i) = G_{\gamma,i}$, which proves that

$$\{\, G(A) \mid A \in \text{KLND}(B, \mathfrak{g}) \,\} \supseteq \{G_{\gamma,1}, \ldots, G_{\gamma,n}\}.$$

Let $A \in \text{KLND}(B, \mathfrak{g})$. We show that $G(A) = G_{\gamma,j}$ for some $j = 1, \ldots, n$. If $G(A) \neq G(B)$, the result follows by Proposition 2.5.2. So assume that $G(A) = G(B)$, then clearly, $G_{\gamma,j} \subseteq G(A)$ for all $j \in \{1, \ldots, n\}$. As $A \in \text{KLND}(B, \mathfrak{g})$, we have $X_j \notin A$ for some $j \in \{1, \ldots, n\}$. Then (2) of Lemma 2.5.1 implies that $G(A) \subseteq G_{\gamma,j}$. So $G(A) = G_{\gamma,j}$. ∎

Corollary 2.5.5 *Let* $A \in \text{KLND}(B, \mathfrak{g})$. *Then for any homogeneous coordinate system* $\gamma = (X_1, \ldots, X_n)$ *of* B, $|\{X_1, \ldots, X_n\} \cap A| \geq \text{type}(\mathfrak{g}) - 1$. *If we also assume that* $G(A) = G(B)$, *then* $|\{X_1, \ldots, X_n\} \cap A| \geq \text{type}(\mathfrak{g})$.

Proof: By definition, type($\mathfrak{g}$) is the cardinality of the set $S = \{\, i \mid G_{\gamma,i} \neq G(B) \,\}$. If $G(A) = G(B)$, then each $i \in S$ satisfies $G(A) \not\subseteq G_{\gamma,i}$, which implies that $X_i \in A$

by Lemma 2.5.1. So $|\{X_1, \dots, X_n\} \cap A| \geq \text{type}(\mathfrak{g})$ in this case. If $G(A) \neq G(B)$, then by (4) of Proposition 2.5.2, exactly one $j \in S$ satisfies $G(A) \subseteq G_{\gamma,j}$. Then the total number of elements $i \in S$ satisfying $G(A) \not\subseteq G_{\gamma,i}$ is $\text{type}(\mathfrak{g}) - 1$. So $|\{X_1, \dots, X_n\} \cap A| \geq \text{type}(\mathfrak{g}) - 1$. ∎

Corollary 2.5.6 *Every $\mathfrak{g}$-homogeneous locally nilpotent derivation D of $B = \mathbf{k}^{[n]}$ satisfies* $\text{rank}(D) \leq n + 1 - \text{type}(\mathfrak{g})$.

Corollary 2.5.7 *Let $\gamma = (X_1, \dots, X_n)$ be a homogeneous coordinate system of B. If* $\text{type}(\mathfrak{g}) = n$, *then* $\text{KLND}(B, \mathfrak{g}) = \{A_1, \dots, A_n\}$ *where $A_i = \mathbf{k}^{[n-1]}$ is the subalgebra of B generated by $\{X_1, \dots, X_n\} \setminus \{X_i\}$.*

Proof: Clearly, $\text{KLND}(B, \mathfrak{g}) \supseteq \{A_1, \dots, A_n\}$. If $A \in \text{KLND}(B, \mathfrak{g})$ then Corollary 2.5.5 implies that $|\{X_1, \dots, X_n\} \cap A| \geq n - 1$. Hence $A = A_i$ for some i. ∎

2.6 The case $G = \mathbb{Z}$

Some of the definitions and results of Sections 2.3 and 2.5 acquire a simpler form in the special case $G = \mathbb{Z}$. The purpose of this section is to give those simpler statements, with brief explanations.

Throughout this section we let $B = \mathbf{k}^{[n]}$, where $\mathbf{k}$ is a field of characteristic zero, and we consider coordinatizable $\mathbb{Z}$-gradings of B.

As $\mathbb{Z}$ is torsion-free, an arbitrary $\mathbb{Z}$-grading $\mathfrak{g} = \{B_i\}_{i \in \mathbb{Z}}$ of B is regular (cf. Lemma 2.2.6), and in particular satisfies $\mathbf{k} \subseteq B_0$ (cf. Lemma 2.2.4). Thus $\mathfrak{g}$ is coordinatizable if and only if there exists a homogeneous coordinate system of B (i.e., a coordinate system $(X_1, \dots, X_n)$ of B such that each X_i is homogeneous).

If $\mathfrak{g} = \{B_i\}_{i\in\mathbb{Z}}$ is a coordinatizable $\mathbb{Z}$-grading of B then any homogeneous coordinate system $\gamma = (X_1, \dots, X_n)$ of B determines

$$a(\mathfrak{g}, \gamma) = (a_1, \dots, a_n) = (\deg(X_1), \cdots, \deg(X_n)) \in \mathbb{Z}^n$$

which, up to permutation, is independent of the choice of γ (cf. Lemma 2.3.8). The integer $d = \gcd \{ a_i \mid i = 1, \dots, n \}$ is equal to the gcd of the set $\operatorname{Supp}(\mathfrak{g}) = \{ i \in \mathbb{Z} \mid B_i \neq \{0\} \}$, so we will use the notation $d = \gcd \operatorname{Supp}(\mathfrak{g})$ for it. Note that $d = 0$ is equivalent to $\mathfrak{g}$ being the trivial grading. If $\mathfrak{g}$ is not the trivial grading then we may always assume that $d = \gcd \operatorname{Supp}(\mathfrak{g})$ is equal to 1. Indeed, our purpose is to describe $\text{KLND}(B, \mathfrak{g})$, and the grading $\mathfrak{g}' = \{B'_i\}_{i\in\mathbb{Z}}$ of B defined by $B'_i = B_{di}$ satisfies $\gcd \operatorname{Supp}(\mathfrak{g}') = 1$ and $\text{KLND}(B, \mathfrak{g}) = \text{KLND}(B, \mathfrak{g}')$.

Until the end of this section, let $\mathfrak{g} = \{B_i\}_{i\in\mathbb{Z}}$ be a coordinatizable $\mathbb{Z}$-grading of $B = \mathbf{k}^{[n]}$ satisfying $\gcd \operatorname{Supp}(\mathfrak{g}) = 1$.

2.6.1 Let $\gamma = (X_1, \dots, X_n)$ be a homogeneous coordinate system of B. Let $\mathbb{Z}_{\gamma,i}$ be the subgroup of $\mathbb{Z}$ generated by $\{ a_j \mid j \in \{1, \dots, n\} \setminus \{i\} \}$. It follows from Definition 2.3.9 that $\operatorname{type}(\mathfrak{g})$ is the cardinality of the set $\{ i \mid \mathbb{Z}_{\gamma,i} \neq \mathbb{Z} \}$. The notion of the type can be clarified as follows. For $i \in \{1, \dots, n\}$ we define:

$$\alpha_i = \gcd \{ a_j \mid j \in \{1, \dots, n\} \setminus \{i\} \}. \tag{2.6.1}$$

The following facts can be deduced easily:

- The nonnegative integers $\alpha_1, \dots, \alpha_n$ are pairwise relatively prime.
- $\gcd(\alpha_i, a_i) = 1$, and $i \neq j \Rightarrow \alpha_i \mid a_j$.
- The unordered tuple $[\alpha_1, \dots, \alpha_n]$ is independent of the choice of γ, and so is uniquely determined by the grading $\mathfrak{g}$ on B.
- Given $i \in \{1, \dots, n\}$, $\mathbb{Z}_{\gamma,i}$ is the subgroup of $\mathbb{Z}$ generated by α_i. We note the following: $\mathbb{Z}_{\gamma,i} = \mathbb{Z} \iff \alpha_i = 1$. It follows that

$$\operatorname{type}(\mathfrak{g}) \text{ is the cardinality of the set } \{ i \mid \alpha_i \neq 1 \}. \tag{2.6.2}$$

Definition 2.6.2 Given $A \in \text{KLND}(B, \mathfrak{g})$, define $d(A) = \gcd\{\, i \mid i \in \mathbb{Z}(A)\,\}$. Equivalently, $d(A) = \gcd\{\, i \in \mathbb{Z} \mid A_i \neq 0\,\}$.

Note that the subgroup $\mathbb{Z}(A)$ (cf. 2.2.7) of $\mathbb{Z}$ is equal to $d(A)\mathbb{Z}$.

Theorem 2.6.3 *Let $\gamma = (X_1, \ldots, X_n)$ be a $\mathfrak{g}$-homogeneous coordinate system of B. Then we have the following:*

(1) $\{\, d(A) \mid A \in \text{KLND}(B, \mathfrak{g})\,\} = \{\alpha_1, \ldots, \alpha_n\}$. *In particular,* $\text{type}(\mathfrak{g}) = 0$ *implies that* $d(A) = 1$ *for all* $A \in \text{KLND}(B, \mathfrak{g})$.

(2) *Let $i \in \{1, \ldots, n\}$. If $h \in B$ is homogeneous and $\alpha_i \nmid \deg(h)$, then $X_i \mid h$.*

(3) *Let $D \in \text{LND}(B, \mathfrak{g})$ and $A = \ker D$. Let $i \in \{1, \ldots, n\}$ be such that $\alpha_i \neq 1$.*

  (a) *If $d(A) \neq \alpha_i$ then $X_i \in A$.*

  (b) *If $d(A) = \alpha_i$, then the set of all irreducible and homogeneous preslices of D is equal to $\{\, \lambda X_i \mid \lambda \in \mathbf{k}^*\,\}$*

(4) *For each $A \in \text{KLND}(B, \mathfrak{g})$, $|\{X_1, \ldots, X_n\} \cap A| \geq \text{type}(\mathfrak{g}) - 1$.*

Proof: Note that $\mathbb{Z}(A) = \mathbb{Z}_{\gamma,i} \Leftrightarrow d(A) = \alpha_i$. Then (1) follows from Corollary 2.5.4. Moreover, $\text{type}(\mathfrak{g}) = 0$ implies that $\alpha_i = 1$ for all i. Hence $d(A) = 1$.

Next, let $h \in B$ be homogeneous and $\alpha_i \nmid \deg(h)$. Then $\deg(h) \notin \mathbb{Z}_{\gamma,i}$, and $X_i \mid h$ follows from Lemma 2.5.1. So (2) holds.

Let us now prove (3). Let $D \in \text{KLND}(B, \mathfrak{g})$ and $A = \ker D$. Suppose that $i \in \{1, \ldots, n\}$ is such that $\alpha_i \neq 1$. If $d(A) \neq \alpha_i$, then $\alpha_i \nmid d(A)$ (because $d(A) = \alpha_j$ for some $j \neq i$ and $\alpha_i \nmid \alpha_j$ if $i \neq j$) . Then $\mathbb{Z}(A) \not\subseteq \mathbb{Z}_{\gamma,i}$, and $X_i \in A$ holds by Lemma 2.5.1. If $d(A) = \alpha_i$, the desired conclusion follows from Proposition 2.5.2.

Finally, the fact in (4) follows from Corollary 2.5.5.

∎

We remark that Corollaries 2.5.6 and 2.5.7 are also valid here, but since their statements remain the same we will not repeat them. Next, we introduce some notions which will be used in Section 4.2. From now on, fix a homogeneous coordinate system $\gamma = (X_1, \dots, X_n)$ of B.

Definition 2.6.4 Let us define $\nu = \prod_{i=1}^{n} \alpha_i$ and $R = \bigoplus_{i \in \nu\mathbb{Z}} B_i$.

Note that ν is a nonnegative integer, and R is a subring of B. Both ν and R are independent of the choice of γ, and are uniquely determined by the grading $\mathfrak{g}$ on B. The following facts will be used in Theorem 4.2.3.

Lemma 2.6.5 $R = \mathbf{k}[X_1^{\alpha_1}, \dots, X_n^{\alpha_n}]$.

Proof: For any i, $\deg(X_i^{\alpha_i}) = \alpha_i a_i$ is divisible by ν. Hence $\mathbf{k}[X_1^{\alpha_1}, \dots, X_n^{\alpha_n}] \subseteq R$. Conversely, if $X_1^{e_1} \dots X_n^{e_n}$ is a monomial in R, then $\deg(X_1^{e_1} \dots X_n^{e_n}) = e_1 a_1 + \dots + e_n a_n = p\nu$ for some integer p. As $a_2, \dots, a_n, \nu$ are divisible by α_1, so is $a_1 e_1$. But $\gcd(a_1, \alpha_1) = 1$ so $\alpha_1 \mid e_1$. Similarly $\alpha_i \mid e_i$ for all $i = 1, \dots, n$. It follows that $R \subseteq \mathbf{k}[X_1^{\alpha_1}, \dots, X_n^{\alpha_n}]$. ∎

Lemma 2.6.6 *If U denotes the set of homogeneous prime elements of B that do not belong to R then*

$$U = \{\, \lambda X_i \mid \lambda \in \mathbf{k}^* \text{ and } i \in \{1, \dots, n\} \text{ is such that } \alpha_i \neq 1 \,\}. \tag{2.6.3}$$

Proof: The right hand side of (2.6.3) is contained in U by Lemma 2.6.5. Now let $h \in U$. As $h \notin R$, some $\alpha_j \nmid \deg(h)$. It follows that $X_j \mid h$ by Theorem 2.6.3. But h and X_j are prime so $h = \lambda X_j$ for some $\lambda \in \mathbf{k}^*$. Since $h \notin R$, $X_j \notin R$. Also by Lemma 2.6.5, $X_j^{\alpha_j} \in R$, so $\alpha_j \neq 1$. Thus U is contained in the right hand side of (2.6.3). ∎

Chapter 3

An algebraic criterion for kernels in $\mathbf{k}^{[3]}$

3.1 Introduction

Let $\mathbf{k}$ be a field of characteristic zero and $B = \mathbf{k}^{[3]}$. Consider the problem of describing $\text{KLND}(B, \mathfrak{g})$, where $\mathfrak{g}$ is a $\mathbb{Z}$-grading on B. In view of the homogeneous version of Miyanishi's Theorem (see Corollary 2.1.8), each element of $\text{KLND}(B, \mathfrak{g})$ has the form $\mathbf{k}[f, g]$ for some homogeneous elements $f, g \in B$. So the question that has to be answered is the following:

Which $\mathfrak{g}$-homogeneous polynomials $f, g \in B$ satisfy $\mathbf{k}[f, g] \in \text{KLND}(B, \mathfrak{g})$?

A partial answer is given by Daigle in [7], namely:

Theorem. *Let $\mathfrak{g}$ be an $\mathbb{N}$-grading on $B = \mathbf{k}[X, Y, Z]$, where $\mathbf{k}$ is any field of characteristic zero, X, Y, Z are homogeneous, and $\gcd(\deg X, \deg Y, \deg Z) = 1$. Let f, g be homogeneous elements of B satisfying $\gcd(\deg f, \deg g) = 1$ and set $A = \mathbf{k}[f, g]$. Then the following are equivalent:*

(1) *There exists $D \in \text{LND}(B, \mathfrak{g})$ such that $\ker D = A$.*

(2) $B_{(fg)} = (A_{(fg)})^{[1]}$ *and* f, g *are irreducible in* $\bar{\mathbf{k}}[X, Y, Z]$.

The goal of this chapter is to generalize Daigle's theorem to the case of $\mathbb{Z}$-gradings. We will prove the following result:

Theorem 3.1.1 *Let* $\mathbf{k}$ *be an algebraically closed field of characteristic zero. Let* $\mathfrak{g} = \{B_i\}_{i\in\mathbb{Z}}$ *be a* $\mathbb{Z}$*-grading on* $B = \mathbf{k}^{[3]}$ *such that* $\gcd \operatorname{Supp}(\mathfrak{g}) = 1$. *Suppose that* $f, g \in B$ *are homogeneous, and let* $A = \mathbf{k}[f, g]$. *If* $\gcd(\deg(f), \deg(g)) = 1$ *or* $\operatorname{type}(\mathfrak{g}) = 0$, *then the following conditions are equivalent:*

(1) $A \in \text{KLND}(B, \mathfrak{g})$.

(2) $B_{(fg)} = (A_{(fg)})^{[1]}$ *and* f, g *have integral fibres in* B.

Moreover, if $\operatorname{type} \mathfrak{g} = 0$ *then* $\gcd(\deg(f), \deg(g)) = 1$, *whenever these equivalent conditions are satisfied.*

In the above statement, we refer to Definition 3.3.1 for the notion of integral fibres. Theorem 3.1.1 is a consequence of Propositions 3.4.4 and 3.5.4, below.

Remark 3.1.2 (i) In condition (2), $B_{(fg)}$ is the homogeneous localization of B with respect to the multiplicative set $S = \{1, fg, (fg)^2, \cdots\}$ (i.e., $B_{(fg)}$ is the degree zero component of the $\mathbb{Z}$-graded ring $S^{-1}B$).
(ii) Daigle also gives a geometric condition equivalent to the conditions (1) and (2) of his theorem, namely: f and g are irreducible elements of $\bar{B} = \bar{\mathbf{k}}[X, Y, Z]$ and the surface $\operatorname{Proj} \bar{B} \setminus V(fg)$ is isomorphic to $\mathbb{P}^2_{\mathbf{k}}$ minus 2 lines.

3.2 Some known results

The following is Lemma 2.4 of [7] and will be used in the proof of Proposition 3.4.4.

Lemma 3.2.1 *Let $R = \oplus R_n$ be a $\mathbb{Z}$-graded UFD satisfying:*

$$\text{For every } n \in \mathbb{Z}, \text{ if } R_n \neq 0 \text{ then } R_n \cap R^* \neq \varnothing.$$

Then R_0 is a UFD.

The following results will be used later in the proof of Proposition 3.5.4:

Lemma 3.2.2 [7, Lemma 2.5] *Let $R = \bigoplus R_n$ be a $\mathbb{Z}$-graded domain and Q a homogeneous subring of R satisfying*

$$\text{For all } n \in \mathbb{Z}, \text{ if } R_n \neq 0, \text{ then } R_n \cap Q^* \neq \varnothing.$$

Then the following are equivalent:
(1) *There exists a homogeneous element v of R such that $R = Q[v] = Q^{[1]}$;*
(2) $R_0 = (Q_0)^{[1]}$.

Theorem 3.2.3 [8, Theorem 2.1] *Let B be an affine UFD over a field $\mathbf{k}$ of characteristic zero and let $x_1, \dots, x_n$ ($n \geq 2$) be prime elements of B no two of which are associates. Suppose that $B = \mathbf{k}[x_1, \dots, x_n]$ and that $B = \oplus_{i \in \mathbb{Z}} B_i$ is a $\mathbb{Z}$-grading such that $\mathbf{k} \subseteq B_0$, each x_i is homogeneous and*
(i) $\gcd(\deg(x_1), \cdots, \deg(x_{i-1}), \deg(x_{i+1}), \dots, \deg(x_n)) = 1$, *for all $i = 1, \dots, n$.*
Suppose that A is a homogeneous subalgebra of B satisfying $A \not\subseteq B_0$ and the following conditions:
(ii) *$A^* = B^*$, A is a UFD and every homogeneous prime element of A is a prime element of B.*
(iii) *$A = \mathbf{k}[S]$ and $B_{(S)} = A_{(S)}^{[1]}$, for some homogeneous multiplicatively closed subset S of A.*
Then $\gcd\{\, i \mid A_i \neq 0 \,\} = 1$ and A is the kernel of a homogeneous locally nilpotent derivation $D : B \to B$.

The following fact will be used in the proof of Proposition 3.4.4.

Lemma 3.2.4 *Let $\mathcal{B}$ be a UFD containing an algebraically closed field* $\mathbf{k}$ *of characteristic zero. If* $\operatorname{trdeg}_{\mathbf{k}} \mathcal{B} = 2$ *and* $\mathcal{A} \in \text{KLND}(\mathcal{B})$*, then* $\mathcal{B} = \mathcal{A}^{[1]}$.

Proof: Let $\mathcal{A} \in \text{KLND}(\mathcal{B})$. Since $\mathcal{B}$ is a UFD, there is an irreducible $D \in \text{LND}(\mathcal{B})$ such that $\ker D = \mathcal{A}$ (cf. 1.4.2, Lemma 2). In view of 1.2 (3), it suffices to prove that D has a slice. Let E denote the set of all preslices of D and note that $E \neq \varnothing$. Define a set map $\ell : E \to \mathbb{N}$ as follows: given $s \in E$, consider the prime factorization of $D(s)$ in $\mathcal{A}$, $Ds = up_1 \dots p_n$ where $u \in \mathcal{A}^*$ and the p_i are prime elements of $\mathcal{A}$. Set $\ell(s) = n$ (recall that $\mathcal{A}$ is a UFD since it is factorially closed in $\mathcal{B}$, cf. 1.1.10 (3)). Choose $s \in E$ which minimizes the value of $\ell(s)$. We claim that $\ell(s) = 0$. Indeed, suppose the contrary and assume that $Ds = up_1 \dots p_n$, where $n \geq 1$. Write $p = p_n$. As $\mathcal{A}$ is factorially closed in $\mathcal{B}$, p is prime in $\mathcal{B}$ and hence $\bar{\mathcal{B}} = \mathcal{B}/p\mathcal{B}$ is an integral domain. Let $\bar{D} \in \text{LND}(\bar{\mathcal{B}})$ denote D modulo $p\mathcal{B}$; then $\bar{D} \neq 0$, because D is irreducible. As $\operatorname{trdeg}_{\mathbf{k}}(\bar{\mathcal{B}}) = 1$ and $\mathbf{k}$ is algebraically closed, it follows that $\ker(\bar{D}) = \mathbf{k}$. Note that the element $s + p\mathcal{B}$ of $\bar{\mathcal{B}}$ belongs to $\ker \bar{D}$; so there exists $\lambda \in \mathbf{k}$ such that $s + p\mathcal{B} = \lambda + p\mathcal{B}$ in $\bar{\mathcal{B}}$, or equivalently $s - \lambda \in p\mathcal{B}$. Define $s' = (s - \lambda)/p \in \mathcal{B}$, then $D(s') = D(s)/p$. So $s' \in E$ and $\ell(s') = \ell(s) - 1$, contradicting the minimality of $\ell(s)$. Thus $\ell(s) = 0$ and $D(s) \in \mathcal{B}^*$. It follows that D has a slice. Then $\mathcal{B} = \mathcal{A}^{[1]}$ follows by 1.2 (3). ∎

3.3 The notion of integral fibres

Definition 3.3.1 Given a $\mathbf{k}$-algebra B, an element $h \in B$ is said to have *integral fibres* if and only if $h - \lambda$ is a prime element of B for every $\lambda \in \mathbf{k}$. We will sometimes say that h has integral fibres in B when there is a risk of confusion (e.g. if $h \in A \subset B$ where A is a subalgebra of B).

Remark 3.3.2 If B is a UFD containing a field $\mathbf{k}$, $b \in B$ has integral fibres if and only if $b-\lambda$ is irreducible for every $\lambda \in \mathbf{k}$. Note that $h \in B$ has integral fibres implies that h is irreducible, but the converse fails. For example, consider $B = \mathbb{C}[X,Y,Z] = \mathbb{C}^{[3]}$. Then $XY+1 \in B$ is an irreducible element of B which does not have integral fibres.

Lemma 3.3.3 *Let B be a UFD containing an algebraically closed field $\mathbf{k}$ and satisfying $B^* = \mathbf{k}^*$. An element $h \in B$ has integral fibres if and only if $h \notin \mathbf{k}$ and $\mathbf{k}[h]$ is factorially closed in B.*

Proof: Suppose that $h \in B$ has integral fibres. As 0 is not a prime element of B , it is clear that $h \notin \mathbf{k}$. Let us prove that $\mathbf{k}[h]$ is factorially closed in B. Let $bc \in \mathbf{k}[h]\setminus\{0\}$ for some $b, c \in B$. As $\mathbf{k}$ is algebraically closed, we can write $bc = \mu\prod_i(h-\mu_i)$ for some $\mu_i \in \mathbf{k}$ and $\mu \in \mathbf{k}^*$. As h has integral fibres, the elements $h-\mu_i$ are prime elements of B. Using unique factorization in B and $B^* = \mathbf{k}^*$, we conclude that $b, c \in \mathbf{k}[h] \setminus \{0\}$. Thus $\mathbf{k}[h]$ is factorially closed in B. In the other direction, suppose that $h \notin \mathbf{k}$ and that $\mathbf{k}[h]$ is factorially closed in B. Then h is transcendental over $\mathbf{k}$ and for any $\lambda \in \mathbf{k}$, $h-\lambda$ is an irreducible element of $\mathbf{k}[h] = \mathbf{k}^{[1]}$. As $\mathbf{k}[h]$ is factorially closed in B, it follows that $h-\lambda$ is an irreducible element of B (cf. 1.1.10 (2)). So h has integral fibres. ∎

Lemma 3.3.4 *Let B be $\mathbb{Z}$-graded UFD containing a field $\mathbf{k}$ and satisfying $B^* = \mathbf{k}^*$. Let h be a homogeneous element of B. If $\deg(h) \neq 0$ and h is prime in B, then h has integral fibres.*

Proof: Let $\lambda \in \mathbf{k}^*$ and let us show that $h-\lambda$ is irreducible in B. First consider the case where $\deg(h) > 0$. Suppose that $h-\lambda = fg$ where $f, g \in B$. Write $f = \sum_{i=i_0}^{i_1} f_i$ and $g = \sum_{j=j_0}^{j_1} g_j$, where $f_i \in B_i$, $g_j \in B_j$, $i_0 \leq i_1$, $j_0 \leq j_1$, and where f_{i_0}, f_{i_1} , g_{j_0}, and g_{j_1} are all nonzero. Since h and λ are homogeneous of degrees $\deg(h) > 0$ and

$\deg(\lambda) = 0$, respectively, it follows that

$$f_{i_0} g_{j_0} = -\lambda, \quad \text{and} \tag{3.3.1}$$

$$f_{i_1} g_{j_1} = h. \tag{3.3.2}$$

It follows from (3.3.1) and $B^* = \mathbf{k}^*$ that $f_{i_0}, g_{j_0} \in \mathbf{k}^*$, so $i_0 = 0 = j_0$. From (3.3.2) and the fact that h is irreducible we get that one of f_{i_1}, g_{j_1} belongs to $\mathbf{k}^*$. If $f_{i_1} \in \mathbf{k}^*$, then $i_1 = 0 = i_0$. Hence $f = f_{i_0} \in \mathbf{k}^*$. Similarly, if $g_{j_1} \in \mathbf{k}^*$ then $g \in \mathbf{k}^*$. So one of f, g is a unit and $h - \lambda$ is irreducible in B.

In the case where $\deg(h) < 0$, replace the grading of B by its opposite and apply the above paragraph. ∎

3.4 A necessary condition for a kernel in $\mathbf{k}^{[3]}$

In Section 3.4, we assume that $\mathbf{k}$ is an algebraically closed field of characteristic zero. Let $B = \mathbf{k}^{[3]}$ be graded by a $\mathbb{Z}$-grading $\mathfrak{g}$ and let $A \in \text{KLND}(B, \mathfrak{g})$. By Corollary 2.1.8, choose homogeneous elements $f, g \in B$ in B satisfying $A = \mathbf{k}[f, g]$. With these hypotheses, the goal of this section is to prove Proposition 3.4.4.

3.4.1 Recall from Definition 2.6.2 that $d(A) = \gcd\{\, i \mid A_i \neq 0 \,\}$. If $A = \mathbf{k}[f, g]$ where f and g are homogeneous, then one verifies that $d(A) = \gcd(\deg(f), \deg(g))$. So if $\text{type}(\mathfrak{g}) = 0$, then $\gcd(\deg(f), \deg(g)) = d(A) = 1$ by Theorem 2.6.3.

Lemma 3.4.2 *Assume that at least one of* $\deg(f)$, $\deg(g)$ *is nonzero. Let* $p = \frac{\deg(f)}{\gcd(\deg(f), \deg(g))}$, $q = \frac{\deg(g)}{\gcd(\deg(f), \deg(g))}$, *and* $\xi = \frac{f^q}{g^p}$. *Then* ξ *is transcendental over* $\mathbf{k}$ *and* $A_{(fg)} = \mathbf{k}[\xi, \xi^{-1}]$. *In particular,* $A_{(fg)}$ *is a UFD.*

Proof: The integer $d := \gcd(\deg(f), \deg(g)) > 0$ by assumption. So the integers p, q in the statement of the lemma are well defined. Since one of $\deg(f)$ and $\deg(g)$

is nonzero, without loss of generality we assume that $p \neq 0$. As $A = \mathbf{k}[f,g] = \mathbf{k}^{[2]}$, f and g are algebraically independent over $\mathbf{k}$. So ξ is transcendental over $\mathbf{k}$. Clearly, $\xi, \xi^{-1} \in A_{(fg)}$; hence $\mathbf{k}[\xi,\xi^{-1}] \subseteq A_{(fg)}$. Next, let $\alpha = \frac{a}{f^i g^j} \in A_{(fg)}$ for some $a \in A$ and integers i, j, where a is a homogeneous element of degree equal to $\deg(f^i g^j)$. It suffices to prove that $\frac{\lambda f^k g^l}{f^i g^j}$ belongs to $\mathbf{k}[\xi,\xi^{-1}]$ for any monomial $\lambda f^k g^l$ in a (where $\lambda \in \mathbf{k}^*$). Since $\deg(\lambda f^k g^l) = \deg(f^i g^j)$, it follows that $\deg(f)(k-i) = -\deg(g)(l-j)$. Dividing throughout by d, we obtain, $p(k-i) = -q(l-j)$. Since $\gcd(p,q) = 1$, $p \mid l-j$. Let $pl' = l-j$. Then $p(k-i) = -q(l-j) = -qpl'$. As $p \neq 0$, $k-i = -l'q$. It follows that

$$\frac{\lambda f^k g^l}{f^i g^j} = \lambda f^{k-i} g^{l-j} = \lambda f^{-l'q} g^{pl'} = \lambda \xi^{-l'} \in \mathbf{k}[\xi,\xi^{-1}].$$

∎

Corollary 3.4.3 *If* $\gcd(\deg(f),\deg(g)) = 1$, *then* $A_{(fg)} = \mathbf{k}[\xi,\xi^{-1}]$ *where* $\xi = \frac{f^{\deg(g)}}{g^{\deg(f)}}$.

Proposition 3.4.4 *If* $\gcd(\deg(f),\deg(g)) = 1$ *or* $\text{type}(\mathfrak{g}) = 0$, *then* $B_{(fg)} = (A_{(fg)})^{[1]}$ *and* f, g *have integral fibres in* B.

Proof: If $\text{type}(\mathfrak{g}) = 0$, $\gcd(\deg(f),\deg(g)) = 1$ by 3.4.1. So we prove the theorem when $\gcd(\deg(f),\deg(g)) = 1$. Note that B_{fg} is a $\mathbb{Z}$-graded UFD. If the n-th graded component of B_{fg} is nonzero for some $n \in \mathbb{Z}$, then there exist integers i, j such that $\deg(f^i g^j) = n$. Then $f^i g^j \in B_{fg}^* \cap (B_{fg})_n$, and $B_{(fg)}$ is a UFD by Lemma 3.2.1.

Next, let us choose $D \in \text{LND}(B,\mathfrak{g})$ such that $\ker D = A$. We claim that $A_{(fg)}$ is the kernel of some nonzero locally nilpotent derivation of $B_{(fg)}$. As $f, g \in A$, $D_{fg} : B_{fg} \to B_{fg}$ is a nonzero homogeneous locally nilpotent derivation of B_{fg}, and $\ker D_{fg} = A_{fg}$ (cf. 1.2 (2)). Suppose that the derivation D_{fg} has degree n. Since $\gcd(\deg(f),\deg(g)) = 1$, we can choose integers k, l such that $\deg(f^k g^l) = -n$. Then $f^k g^l D_{fg}$ is a locally nilpotent derivation of B_{fg}, which is homogeneous of degree zero. The restriction of $f^k g^l D_{fg}$ to $B_{(fg)}$ is also a locally nilpotent derivation, and has kernel equal to $B_{(fg)} \cap \ker(f^k g^l D_{fg}) = B_{(fg)} \cap A_{fg} = A_{(fg)}$. So the claim is proved. Next observe that

$$\operatorname{trdeg}_{\mathbf{k}} B_{(fg)} = \operatorname{trdeg}_{A_{(fg)}} B_{(fg)} + \operatorname{trdeg}_{\mathbf{k}} A_{(fg)}.$$

By Corollary 3.4.3, $A_{(fg)} = \mathbf{k}[\xi, \xi^{-1}]$ where $\xi = \frac{f^{\deg(g)}}{g^{\deg(f)}}$, and ξ is transcendental over $\mathbf{k}$, so $\operatorname{trdeg}_{\mathbf{k}} A_{(fg)} = \operatorname{trdeg}_{\mathbf{k}} \mathbf{k}[\xi, \xi^{-1}] = 1$. Since $A_{(fg)} \in \text{KLND}(B_{(fg)})$, $\operatorname{trdeg}_{A_{(fg)}} B_{(fg)} = 1$ by 1.2 (5). Thus $\operatorname{trdeg}_{\mathbf{k}} B_{(fg)} = 2$. As $B_{(fg)}$ is a UFD, $B_{(fg)} = (A_{(fg)})^{[1]}$ by Lemma 3.2.4.

Finally, since $A = \mathbf{k}[f, g] = \mathbf{k}^{[2]}$, $f - \lambda$ and $g - \lambda$ are prime in A for all $\lambda \in \mathbf{k}$. As A is factorially closed in B, $f - \lambda$ and $g - \lambda$ are prime in B. It follows that f, g have integral fibres in B.

∎

3.5 A sufficient condition for a kernel in $\mathbf{k}^{[3]}$

In Section 3.5, we assume that $\mathbf{k}$ is algebraically closed and $B = \mathbf{k}^{[3]}$ is $\mathbb{Z}$-graded by a grading $\mathfrak{g}$ satisfying $\gcd \operatorname{Supp}(\mathfrak{g}) = 1$. (cf. 2.6). Let f, g be homogeneous elements of B and let $A = \mathbf{k}[f, g]$. The following assumptions hold throughout Section 3.5:

$$B_{(fg)} = (A_{(fg)})^{[1]} \tag{3.5.1}$$

$$f, g \text{ have integral fibres in } B \tag{3.5.2}$$

$$f, g \text{ satisfy equivalent conditions (1), (2) of Lemma 3.5.1 below.} \tag{3.5.3}$$

Lemma 3.5.1 *Under assumptions* (3.5.1) *and* (3.5.2), *the following conditions are equivalent:*

(1) *At least one of* $\deg(f), \deg(g)$ *is not equal to zero.*

(2) f, g *are algebraically independent over* $\mathbf{k}$.

Proof: Suppose that (1) holds. Since f, g satisfy (3.5.2), $f, g \notin \mathbf{k}$. If f, g are algebraically dependent over $\mathbf{k}$, we can pick a polynomial $\varphi(X, Y) \in \mathbf{k}[X, Y] = \mathbf{k}^{[2]}$ such that $\varphi(f, g) = 0$, and

$$\varphi(f, 0) \neq 0 \text{ or } \varphi(0, g) \neq 0.$$

We may assume that $\varphi(0, g) \neq 0$ and write $\varphi(X, Y)$ as $\varphi(X, Y) = X\varphi_1(X, Y) + \varphi(0, Y)$. Then $\varphi(f, g) = 0$ implies that $f\varphi_1(f, g) = -\varphi(0, g) \in \mathbf{k}[g] \setminus \{0\}$. Since g has integral fibres in B, $\mathbf{k}[g]$ is factorially closed in B by Lemma 3.3.3. So $f \in \mathbf{k}[g] \setminus \{0\}$. As $\mathbf{k}$ is algebraically closed, $f = ag + b$ for some $a \in \mathbf{k}^*$ and $b \in \mathbf{k}$. Since at least one of $\deg(f)$, $\deg(g)$ is nonzero, $b = 0$. Hence f and g are associates of each other. Then $B_{(fg)} = A_{(fg)}^{[1]}$ implies that $B_{(f)} = A_{(f)}^{[1]} = (\mathbf{k}[f]_{(f)})^{[1]} = \mathbf{k}^{[1]}$. This leads to a contradiction because $\operatorname{trdeg}_{\mathbf{k}} B_{(f)} = 2$. Hence f, g are algebraically independent over $\mathbf{k}$ and (2) holds.

Next, suppose that (2) holds. If (1) does not hold, then $\deg(f) = 0 = \deg(g)$. Hence $\deg(fg) = 0$, and $A_{(fg)} = A_{fg}$. Also note that

$$B_{(fg)} = \left\{ b/f^i g^j \mid \deg(b) = \deg(f^i g^j) = 0 \right\} = (B_0)_{fg}.$$

The condition $B_{(fg)} = (A_{(fg)})^{[1]}$ implies that $(B_0)_{fg} = (A_{fg})^{[1]}$. As f, g are algebraically independent over $\mathbf{k}$, $\operatorname{trdeg}_{\mathbf{k}} \left((A_{fg})^{[1]}\right) = 3$. It follows that $\operatorname{trdeg}_{\mathbf{k}}(B_0) = 3$ and $B_0 = B$; this contradicts the fact that $\gcd \operatorname{Supp}(\mathfrak{g}) = 1$.

∎

Lemma 3.5.2 *For any $a \in A$,*

$$f \mid a \text{ in } A \Leftrightarrow f \mid a \text{ in } B, \quad \text{and} \quad g \mid a \text{ in } A \Leftrightarrow g \mid a \text{ in } B.$$

Proof: We prove that $f \mid a$ in $A \Leftrightarrow f \mid a$ in B, and the proof of the other assertion is similar. Clearly if $f \mid a$ in A, then $f \mid a$ in B. Now let us assume that $f \mid a$ in B.

Write $a \in A$ as $a = P(f,g)$ for some polynomial $P(f,g) \in A = \mathbf{k}[f,g]$. There exists a polynomial $\varphi(f,g) \in \mathbf{k}[f,g]$ such that

$$a = f\varphi(f,g) + P(0,g).$$

To prove that $f \mid a$ in A, it suffices to prove that $P(0,g) = 0$. By contradiction, assume that $P(0,g) \in \mathbf{k}[g]\backslash\{0\}$. As $f \mid a$ in B, $f \mid P(0,g)$ in B. Then $fb = P(0,g) \in \mathbf{k}[g]\backslash\{0\}$ for some nonzero $b \in B$. Recall that g has integral fibres by assumption, so $\mathbf{k}[g]$ is factorially closed in B by Lemma 3.3.3. Then $f \in \mathbf{k}[g]$, contradicting the fact that f, g are algebraically independent over $\mathbf{k}$. ∎

Lemma 3.5.3 *If* $\gcd(\deg(f), \deg(g)) = 1$ *or* $\text{type}(\mathfrak{g}) = 0$, *then every homogeneous prime element of* A *is a prime element of* B.

Proof: Let α be a homogeneous prime element of A. We shall prove that α is prime in B. If $f \mid \alpha$ in B then, $f \mid \alpha$ in A by Lemma 3.5.2. As α is a prime element of A, $\alpha = \lambda f$ for some $\lambda \in \mathbf{k}^*$. As f is prime in B, α is prime in B and we are done in this case. Similarly, if $g \mid \alpha$ in B then α is prime in B. From now on we assume that $f \nmid \alpha$ and $g \nmid \alpha$ in B; it follows that

$$\gcd_B(\alpha, fg) = 1. \tag{3.5.4}$$

Note that either $\alpha \in (A_{fg})^*$ or α is prime in A_{fg}. If $\alpha \in (A_{fg})^*$ then $\alpha \mid (fg)^n$ in A (hence in B) for some $n \in \mathbb{N}$, so $\gcd_B(\alpha, (fg)^n) \neq 1$, which contradicts (3.5.4). So α is prime in A_{fg}. Let $d = \gcd(\deg(f), \deg(g))$. As $\deg(\alpha) \in d\mathbb{Z}$, we may choose $i, j \in \mathbb{Z}$ such that $\deg(f^i g^j \alpha) = 0$. It follows that $f^i g^j \alpha$ is a prime in A_{fg}. As $f^i g^j \alpha \in A_{(fg)}$, it follows that $f^i g^j \alpha$ is a prime in $A_{(fg)}$. So $f^i g^j \alpha$ is a prime in $B_{(fg)}$ by (3.5.1). Let us now show that

if $\beta, \gamma \in B$ are homogeneous elements such that $\alpha \mid \beta\gamma$ in B, then $\alpha \mid \beta^d$ or $\alpha \mid \gamma^d$ in B. (3.5.5)

Indeed, there exist $i_1, j_1, i_2, j_2 \in \mathbb{Z}$ such that $f^{i_1}g^{j_1}\beta^d$ and $f^{i_2}g^{j_2}\gamma^d$ belong to $B_{(fg)}$. As $\alpha \mid \beta^d\gamma^d$ in B, it follows that $f^ig^j\alpha$ divides $(f^{i_1}g^{j_1}\beta^d)(f^{i_2}g^{j_2}\gamma^d)$ in B_{fg}, and hence that $f^ig^j\alpha$ divides $(f^{i_1}g^{j_1}\beta^d)(f^{i_2}g^{j_2}\gamma^d)$ in $B_{(fg)}$. As $f^ig^j\alpha$ is a prime in $B_{(fg)}$, we obtain that $f^ig^j\alpha$ divides $f^{i_1}g^{j_1}\beta^d$ or $f^{i_2}g^{j_2}\gamma^d$ in $B_{(fg)}$. It suffices to consider the first case: $f^ig^j\alpha$ divides $f^{i_1}g^{j_1}\beta^d$ in $B_{(fg)}$. Then $\alpha \mid \beta^d$ in B_{fg}, which implies that there exist $m, n \in \mathbb{N}$ such that $\alpha \mid f^mg^n\beta^d$ in B. As $\gcd_B(\alpha, fg) = 1$, we obtain $\alpha \mid \beta^d$ in B and (3.5.5) is proved.

By assumption, $\gcd(\deg(f), \deg(g)) = 1$ (i.e., $d = 1$) or $\text{type}(\mathfrak{g}) = 0$. If $d = 1$ then condition (3.5.5) implies that α is a prime element of B and we are done. From now on assume that $\text{type}(\mathfrak{g}) = 0$. As B is a UFD, condition (3.5.5) implies that $\alpha = P^N$ for some homogeneous prime element P of B and some integer $N > 0$. Choose a homogeneous coordinate system (X, Y, Z) of B; choose distinct elements $u, v \in \{X, Y, Z\}$ such that $u \nmid P$ and $v \nmid P$ in B; then $u \nmid \alpha$ and $v \nmid \alpha$ in B, so $\gcd_B(\alpha, fguv) = 1$. Since $\text{type}(\mathfrak{g}) = 0$, we have $\gcd(\deg(u), \deg(v)) = 1$ and we may therefore choose $r, s \in \mathbb{N}$ such that $\deg(u^rv^sP) \in d\mathbb{Z}$; then choose $i_3, j_3 \in \mathbb{Z}$ such that $f^{i_3}g^{j_3}u^rv^sP \in B_{(fg)}$. As $\alpha = P^N$, $f^ig^j\alpha$ divides $(f^{i_3}g^{j_3}u^rv^sP)^N$ in B_{fg}; thus $f^ig^j\alpha$ divides $(f^{i_3}g^{j_3}u^rv^sP)^N$ in $B_{(fg)}$. Since $f^ig^j\alpha$ is a prime in $B_{(fg)}$, it follows that $f^ig^j\alpha$ divides $f^{i_3}g^{j_3}u^rv^sP$ in $B_{(fg)}$. Then there exist $m, n \in \mathbb{N}$ such that $\alpha \mid f^mg^nu^rv^sP$ in B. As $\gcd_B(\alpha, fguv) = 1$, we obtain $\alpha \mid P$ in B. Thus, α is an associate of P, and hence it is prime in B. ∎

Proposition 3.5.4 *If* $\gcd(\deg(f), \deg(g)) = 1$ *or* $\text{type}(\mathfrak{g}) = 0$, *then* $A \in \text{KLND}(B, \mathfrak{g})$.

Proof: Let $B = \mathbf{k}[X, Y, Z]$ and assume that $\text{type}\,\mathfrak{g} = 0$. Every homogeneous prime element of A is a prime element of B by Lemma 3.5.3. One verifies that all conditions of Theorem 3.2.3 are satisfied by the pair (A, B). Then $A \in \text{KLND}(B, \mathfrak{g})$ follows from Theorem 3.2.3. (Moreover, $\gcd(\deg(f), \deg(g)) = 1$ also holds in this case.)

It now remains to prove the proposition when $\gcd(\deg(f), \deg(g)) = 1$. As $B_{(fg)} = (A_{(fg)})^{[1]}$, $B_{fg} = A_{fg}[v] = A_{fg}^{[1]}$ for some homogeneous element v of B_{fg} (cf. Lemma 3.2.2 with $R = B_{fg}$ and $Q = A_{fg}$). Then $\mathcal{D} := \frac{\partial}{\partial v}$ is a homogeneous locally nilpotent derivation of B_{fg}, and $\ker \mathcal{D} = A_{fg}$. Choose $m, n \in \mathbb{N}$ such that $f^m g^n \mathcal{D}(b) \in B$ for each $b \in \{X, Y, Z\}$. If $\mathcal{D}' := f^m g^n \mathcal{D}$, then $\mathcal{D}'(B) \subseteq B$ and $\mathcal{D}' : B_{fg} \to B_{fg}$ is locally nilpotent and homogeneous. So the restriction D of $\mathcal{D}'$ to B is a homogeneous locally nilpotent derivation of B, and $\ker D = A_{fg} \cap B$. It suffices to prove that $A_{fg} \cap B = A$.

First we claim that $A \cap fB = fA$. To see this, recall that $A = \mathbf{k}[f, g] = \mathbf{k}^{[2]}$ where $\mathbf{k}$ is algebraically closed. As f is prime in B, fB is a prime ideal of B and hence $A \cap fB$ is a prime ideal of A. It is clear that $fA \subseteq A \cap fB$; if this inclusion is strict then $A \cap fB$ must be a maximal ideal of A and hence of the form $(f, g - \lambda)$ for some $\lambda \in \mathbf{k}$. Then $g - \lambda \in fB$. As f and $g - \lambda$ are irreducible (because f, g have integral fibres in B), it follows that they are associates, which contradicts the fact that they are algebraically independent over $\mathbf{k}$. This shows that $A \cap fB = fA$, and similarly $A \cap gB = gA$. Then by induction, one can show that

$$A \cap f^i g^j B = f^i g^j A \quad \forall i, j \in \mathbb{N}.$$

It follows that $A_{fg} \cap B = A$. So $A \in \text{KLND}(B, \mathfrak{g})$. ∎

3.6 Remarks on Theorem 3.1.1

In view of Propositions 3.4.4 and 3.5.4, the proof of Theorem 3.1.1 is now complete. Compare Theorem 3.1.1 with Daigle's Theorem mentioned in the introduction of this chapter. Note that the condition of having integral fibres is necessary in the statement of Theorem 3.1.1. To justify this, we provide the following example:

Example 3.6.1 Let $B = \mathbf{k}[X, Y, Z]$ be graded by a grading $\mathfrak{g}$ of type zero which is defined by declaring $(\deg(X), \deg(Y), \deg(Z)) = (1, -1, 0) \in \mathbb{Z}^3$. Let $f = X$, $g = XY + 1$. Then f, g are algebraically independent over $\mathbf{k}$, and are both homogeneous elements (of degrees one and zero, respectively). Note that g does not have integral fibres. Set $A = \mathbf{k}[f, g]$. Then $A \notin \text{KLND}(B, \mathfrak{g})$. Indeed, if $A = \mathbf{k}[X, XY + 1] \in \text{KLND}(B, \mathfrak{g})$, then $XY \in A$. As A is factorially closed in B, $Y \in A$; this implies that $\mathbf{k}[X, Y] \subseteq \mathbf{k}[X, XY]$, a contradiction.

Next we claim that $B_{(fg)} = (A_{(fg)})^{[1]}$. To prove this, it suffices to prove two things: $A_{(fg)} = \mathbf{k}[g, g^{-1}]$ and $B_{(fg)} = \mathbf{k}[g, g^{-1}, Z]$. The inclusion $\mathbf{k}[g, g^{-1}] \subseteq A_{(fg)}$ is clear. Let $\frac{\alpha}{f^i g^j} \in A_{(fg)}$, where α is homogeneous and $\deg(\alpha) = \deg(f^i g^j) = i$. We can write α as a sum of monomials in $\mathbf{k}[f, g]$, each homogeneous of degree i. It suffices to prove that for each monomial $\lambda f^m g^n$ in α (where $\lambda \in \mathbf{k}^*$), $\frac{\lambda f^m g^n}{f^i g^j} \in \mathbf{k}[g, g^{-1}]$. But this follows immediately because $i = \deg(f^i g^j) = \deg(\lambda f^m g^n) = m$. So $A_{(fg)} = \mathbf{k}[g, g^{-1}]$.

Next, the inclusion $\mathbf{k}[g, g^{-1}, Z] \subseteq B_{(fg)}$ is obvious. In the other direction, assume that $\frac{b}{f^i g^j} \in B_{(fg)}$, for some homogeneous $b \in B$. Let $\lambda X^k Y^l Z^t$ be a monomial in b, where $\lambda \in \mathbf{k}^*$. It suffices to prove that $\frac{\lambda X^k Y^l Z^t}{f^i g^j} \in \mathbf{k}[g, g^{-1}, Z]$. Note that

$$k - l = \deg(\lambda X^k Y^l Z^t) = \deg(f^i g^j) = i,$$

hence $k = l + i$. As $X = f$ and $Y = \frac{g-1}{X} = \frac{g-1}{f}$, we obtain:

$$\frac{\lambda X^k Y^l Z^t}{f^i g^j} = \frac{\lambda f^k (g-1)^l Z^t}{f^{l+i} g^j} = \frac{\lambda (g-1)^l Z^t}{g^j} \in \mathbf{k}[g, g^{-1}, Z].$$

It follows that $B_{(fg)} = \mathbf{k}[g, g^{-1}, Z] = (A_{(fg)})^{[1]}$.

Note also that the condition $\text{type}(\mathfrak{g}) = 0$ or $\gcd(\deg(f), \deg(g)) = 1$ is required in the statement Theorem 3.1.1. For instance, consider the following example:

Example 3.6.2 Let $B = \mathbf{k}[X_0, X_1, X_2] = \mathbf{k}^{[3]}$. Define a grading $\mathfrak{g}$ on B by declaring $\deg(X_0) = 1$, $\deg(X_1) = 0$ and $\deg(X_2) = 0$. Note that $(\alpha_0, \alpha_1, \alpha_2) = (0, 1, 1)$ and consequently, type $\mathfrak{g} = 1$. Let $f = X_1$, $g = X_2$. Note that $\gcd(\deg(f), \deg(g)) = 0$ and

$A = \mathbf{k}[f, g] \in \text{KLND}(B, \mathfrak{g})$. As $B_0 = A$, it follows that $B_{(fg)} = A_{(fg)}$ and Condition (2) of Theorem 3.1.1 is not satisfied.

Chapter 4

Classification of $\mathbb{Z}$-homogeneous derivations of $\mathbf{k}^{[3]}$: Positive type

Let $B = \mathbf{k}^{[3]}$ be the polynomial ring in three variables over an algebraically closed field $\mathbf{k}$ of characteristic zero. In this chapter, we give a classification of kernels of homogeneous locally nilpotent derivations of B when B is graded by a $\mathbb{Z}$-grading $\mathfrak{g}$ that satisfies type$(\mathfrak{g}) > 0$. The main purpose of this chapter is to generalize the following theorem of Daigle [10] when $\mathbf{k}$ is algebraically closed:

Theorem: *Let $\mathbf{k}$ be any field of characteristic zero and $B = \mathbf{k}^{[3]}$. Consider a grading $\mathfrak{g} = \{B_i\}_{i\in\mathbb{N}}$ of B satisfying $B_0 = \mathbf{k}$, $\gcd \operatorname{Supp}(\mathfrak{g}) = 1$, and $\operatorname{type} \mathfrak{g} > 0$. If $A \in \text{KLND}(B, \mathfrak{g})$, then there exists a homogeneous coordinate system (Y_0, Y_1, Y_2) of B such that one of the following holds:*

(i) $A = \mathbf{k}[Y_0, Y_1]$.

(ii) $\gcd(\deg Y_0, \deg Y_1) = 1$ *and* $A = \mathbf{k}[Y_0, Y_0^e Y_2 + \psi(Y_0, Y_1)]$ *for some* $e \in \mathbb{N}$ *and some* $\psi(Y_0, Y_1) \in \mathbf{k}[Y_0, Y_1]$, *such that* $Y_0^e Y_2 + \psi(Y_0, Y_1)$ *is homogeneous and irreducible.*

(ii) $\gcd(\deg Y_0, \deg Y_1) = 1 = \gcd(\deg Y_0, \deg Y_2)$ *and* $A = \mathbf{k}[Y_0, P]$ *for some homogeneous* $P \in B$ *which satisfies* $\gcd_B(P_{Y_1}, P_{Y_2}) = 1$ *and which is a variable of* $\mathbf{k}(Y_0)[Y_1, Y_2]$.

Theorems 4.2.3 and 4.2.5 generalize the above theorem under the additional assumption that $\mathbf{k}$ is algebraically closed. Although the statements of our results are very similar to those which were known in the special case of $\mathbb{N}$-gradings, it does not seem possible to obtain the generalizations simply by repairing the proofs of the special case, as some key steps of the argument break down when applied to the general setting. Finally, we would like to point out that our classification relies on a nontrivial result proved by Daigle (cf. Theorem 4.1.2).

4.1 Preliminary results

Throughout this section, $\mathbf{k}$ denotes an arbitrary field of characteristic zero. We gather some results that we will be using later in the classification of homogeneous locally nilpotent derivations. The following is an immediate consequence of Theorem 2.4 of [18]. This result will be used in the proof of Theorem 4.2.3.

4.1.1 *Let R be a UFD containing $\mathbb{Q}$, let $S = R[X,Y] = R^{[2]}$ and let $K = \operatorname{Frac} R$. For a subring A of S, the following are equivalent:*

(1) $A \in \text{KLND}_R(S)$.

(2) $A = R[P]$ for some $P \in S$ which satisfies $\gcd_S(P_X, P_Y) = 1$ and which is a variable of $K[X,Y]$.

Moreover, if P is an element of S such that $R[P] \in \text{KLND}_R(S)$ then P satisfies (2).

The following result will be used in Lemma 4.2.2.

Theorem 4.1.2 [12, Theorem 5.1] *Let $B = \mathbf{k}^{[3]}$, and let D be an irreducible locally nilpotent derivation of B. If there exists a variable $Y \in B$ such that $DY \neq 0$ and $D^2Y = 0$, then there exist $X, Z \in B$ such that*

$$B = \mathbf{k}[X,Y,Z], \quad DX = 0, \quad DY \in \mathbf{k}[X], \quad DZ \in \mathbf{k}[X,Y].$$

In particular, $\ker D$ contains a variable of B, and D is triangular.

Lemma 4.1.3 *Let G be an abelian group and $R = \oplus_{i\in G} R_i$ be an affine* **k***-domain endowed with a regular G-grading. If A is a graded subring of R such that $R = A^{[1]}$, then there exists a homogeneous element $f \in R$ such that $R = A[f]$.*

Proof: The subgroup $G(R)$ of G generated by $\{\, i \in G \mid R_i \neq 0 \,\}$ is isomorphic to $\mathbb{Z}^r$ for some r by Corollary 2.2.9. So $G(R)$ can be totally ordered. Replacing G by $G(R)$ if necessary, we may assume that G can be ordered. Fix a total ordering of G.

Choose $f \in R$ such that $R = A[f]$, let $f = \sum_{i\in G} f_i$ be the homogeneous decomposition of f and note that the set $\{\, i \in G \mid f_i \notin A \,\}$ is nonempty. Define $e = \max\{\, i \in G \mid f_i \notin A \,\}$. As $R = A[f] = A[f + a]$ for any $a \in A$, we may choose f so that

$$f_e \notin A \text{ and } f_i = 0 \text{ for all } i > e.$$

We shall prove that $R = A[f_e]$. Define a G-grading on the polynomial ring $A[X] = A^{[1]}$ by keeping the original grading on A and declaring that X is homogeneous of degree e, i.e., $A[X] = \oplus_{n\in G} A[X]_n$ where:

$$A[X]_n = \sum_{i+je=n} A_i X^j.$$

If $H(X) \in A[X]_n$ and $H(X) \neq 0$, then $H(f_e) \in R_n \setminus \{0\}$ (note that $R = A^{[1]}$ implies that A is algebraically closed in R, so f_e is transcendental over A and consequently $H(f_e) \neq 0$). As $f_i = 0$ for all $i > e$, we also have:

$$H(f) = H(f_e) + r \quad \text{for some } r \in \oplus_{i<n} R_i.$$

Now let $h \neq 0$ be a homogeneous element of R. Then there exists $P(X) \in A[X] \setminus \{0\}$ such that $P(f) = h$. Let $P(X) = \sum_{n\in G} P_n(X)$ (where $P_n(X) \in A[X]_n$) be the homogeneous decomposition of $P(X)$, and let $N = \max\{\, n \in G \mid P_n(X) \neq 0 \,\}$. Applying the preceding paragraph to each $P_n(X)$ gives

$$P(f) = P_N(f_e) + r \quad \text{for some } r \in \oplus_{i<N} R_i,$$

where $P_N(f_e) \in R_N \setminus \{0\}$. It follows that $P_N(f_e) = h$ (and $r = 0$), so $h \in A[f_e]$, which shows that $R = A[f_e]$. ∎

4.1.4 Let $B = \mathbf{k}[X_1, X_2, X_3] = \mathbf{k}^{[3]}$ and $A \in \text{KLND}(B)$. From Miyanishi's Theorem (cf. 1.4.5), $A = \mathbf{k}[f_1, f_2] = \mathbf{k}^{[2]}$ for some $f_1, f_2 \in B$. One defines the *Jacobian derivation* Δ_A (or $\Delta_{(f_1,f_2)}$) of B by the Jacobian determinant

$$\Delta_{(f_1,f_2)}(b) = \left| \frac{\partial(f_1, f_2, b)}{\partial(X_1, X_2, X_3)} \right|$$

for any $b \in B$. If (X_1', X_2', X_3') is another coordinate system of B and if f_1', f_2' are such that $A = \mathbf{k}[f_1', f_2']$, then $\Delta_{(f_1',f_2')} = \lambda \Delta_{(f_1,f_2)}$ for some $\lambda \in \mathbf{k}^*$.

The following Lemma is Corollary 2.6 of [6].

Lemma. *Let $B = \mathbf{k}^{[3]}$ and $A \in \text{KLND}(B)$. Then the Jacobian derivation $\Delta_A : B \to B$ is locally nilpotent, irreducible and has kernel A. Moreover,*

$$\{ D \in \text{LND}(B) \mid \ker D = A \} = \{ a\Delta_A \mid a \in A \setminus \{0\} \}.$$

Remark. If $\mathfrak{g} = \{B_i\}_{i \in \mathbb{Z}}$ is a $\mathbb{Z}$-grading on $B = \mathbf{k}^{[3]}$ and $A \in \text{KLND}(B, \mathfrak{g})$, then Δ_A is $\mathfrak{g}$-homogeneous. Indeed, if $D \in \text{LND}(B, \mathfrak{g})$ is such that $\ker D = A$, then $D = a\Delta_A$ for some $a \in A$ by above lemma. As D is $\mathfrak{g}$-homogeneous and $\mathfrak{g}$ is regular, it follows that Δ_A is $\mathfrak{g}$-homogeneous.

4.2 Classification

Throughout Section 4.2, we assume that $\mathbf{k}$ is an algebraically closed field of characteristic zero. Let $\mathfrak{g}$ be a $\mathbb{Z}$-grading on $B = \mathbf{k}^{[3]}$ satisfying $\text{type}(\mathfrak{g}) > 0$ and $\gcd \text{Supp}(\mathfrak{g}) = 1$.

4.2.1 Recall from Section 2.6 that a homogeneous coordinate system $\gamma = (X_0, X_1, X_2)$ of B and the grading $\mathfrak{g}$ on B determine the following data:

- the ordered triple $a(\mathfrak{g},\gamma) = (a_0, a_1, a_2) = (\deg_{\mathfrak{g}} X_0, \deg_{\mathfrak{g}} X_1, \deg_{\mathfrak{g}} X_2)$
- $\alpha(\mathfrak{g},\gamma) = (\alpha_0, \alpha_1, \alpha_2) = (\gcd(a_1, a_2), \gcd(a_0, a_2), \gcd(a_0, a_1))$
- $\text{type}(\mathfrak{g}) = |\{\, i \mid \alpha_i \neq 1 \,\}|$ ($\text{type}(\mathfrak{g})$ is independent of the choice of γ)
- the integer $\nu = \alpha_0\alpha_1\alpha_2$ and the ring $R = \oplus_{i\in\nu\mathbb{Z}} B_i$. (Recall also that ν and R are independent of the choice of γ.)

Given $A \in \text{KLND}(B, \mathfrak{g})$, recall also the integer $d(A) = \gcd\{\, i \in \mathbb{Z} \mid A_i \neq 0 \,\}$.

Lemma 4.2.2 *Let the setup be as in the beginning of Section* 4.2. *If $A \in \text{KLND}(B, \mathfrak{g})$, then there exists a homogeneous coordinate system (X, Y, Z) of B such that $X \in A$.*

Proof: Choose a homogeneous coordinate system (Y_0, Y_1, Y_2) of B and let D be a homogeneous locally nilpotent derivation of B such that $A = \ker D$. We may choose D to be irreducible by 1.4.2 (Lemma 2). First we consider the case when $D(s) \in \mathbf{k}^*$ for some $s \in B$. In that case, $B = A^{[1]}$ follows by an easy application of 1.2 (3). As A is a homogeneous subring of B, Lemma 4.1.3 implies that there exists a homogeneous $h \in B$ such that $B = A[h]$. By Corollary 2.1.8, $A = \mathbf{k}[f, g]$ for some homogeneous $f, g \in B$. So (f, g, h) is a homogeneous coordinate system of B and $f \in A$. This shows that the desired claim is true whenever $Ds \in \mathbf{k}^*$ for some $s \in B$.

If $\text{type}(\mathfrak{g}) > 1$, the result follows from Theorem 2.6.3 (4). So suppose that $\text{type}(\mathfrak{g}) = 1$, then there is a unique $i \in \{0, 1, 2\}$ such that $\alpha_i \neq 1$. Without loss of generality, we may assume that $\alpha_0 \neq 1$ and $\alpha_1 = 1 = \alpha_2$. If $d(A) \neq \alpha_0$, then $Y_0 \in A$ by Theorem 2.6.3 (3a) and the result is proved.

If $d(A) = \alpha_0$, Y_0 is a (homogeneous) preslice of D by Theorem 2.6.3 (3b). Then by Theorem 4.1.2, there exists a coordinate system (X, Y_0, Z) of B such that $DX = 0$ and $DY_0 \in \mathbf{k}[X] \setminus \{0\}$. It suffices to prove that X and Z can be chosen to be homogeneous.

We may assume that $DY_0 \notin \mathbf{k}^*$ (otherwise the result follows from the earlier argument). As $\mathbf{k}$ is algebraically closed, we write $DY_0 \in \mathbf{k}[X] \setminus \{0\}$ as

$$DY_0 = \lambda \prod_{i=1}^{m}(X - a_i)$$

for some $\lambda, a_i \in \mathbf{k}$. Since Y_0 is homogeneous, and D is homogeneous, DY_0 is homogeneous. As the grading $\mathfrak{g}$ on B is regular, $X - a_i$ is homogeneous for each i. In particular, $X - a_1$ is homogeneous. Then $DY_0 \in \mathbf{k}[X] = \mathbf{k}[X - a_1]$, where $X - a_1$ is homogeneous. Replacing X by $X - a_1$ if needed (and calling it by X again), we can choose X to be homogeneous. Note that $\mathbf{k}[X, Y_0]$ is a homogeneous subring of B. As $B = \mathbf{k}[X, Y_0, Z] = \mathbf{k}[X, Y_0]^{[1]}$, Z can be chosen to be homogeneous by Lemma 4.1.3. ∎

Theorem 4.2.3 *Let the setup be as in the beginning of Section* 4.2. *Given* $A \in$ KLND$(B, \mathfrak{g})$, *there exists a homogeneous coordinate system* (Y_0, Y_1, Y_2) *of* B *such that one of the following holds:*

(i) $A = \mathbf{k}[Y_0, Y_1]$.

(ii) $\gcd(\deg Y_0, \deg Y_1) = 1 = \gcd(\deg Y_0, \deg Y_2)$ *and* $A = \mathbf{k}[Y_0, P]$, *for some homogeneous* $P \in B$ *which satisfies* $\gcd_B(P_{Y_1}, P_{Y_2}) = 1$ *and which is a variable of* $\mathbf{k}(Y_0)[Y_1, Y_2]$.

(iii) $\gcd(\deg Y_0, \deg Y_1) = 1$, $A = \mathbf{k}[Y_0, \varphi(Y_0)Y_2 + \psi(Y_0, Y_1)]$ *for some* $\varphi \in \mathbf{k}[Y_0] \setminus \{0\}$, *and* $\psi \in \mathbf{k}[Y_0, Y_1]$ *such that* $\gcd(\frac{\partial \psi}{\partial Y_1}, \varphi) = 1$, *and* $\varphi(Y_0)Y_2 + \psi(Y_0, Y_1)$ *is homogeneous and irreducible.*

Proof: If (Y_0, Y_1, Y_2) is a homogeneous coordinate system of B and if A is one of the rings displayed in (i)-(iii) of Theorem 4.2.3, then $A \in$ KLND$(B, \mathfrak{g})$ by 4.1.1.

By Lemma 4.2.2, we choose a homogeneous coordinate system $\gamma = (Y_0, Y_1, Y_2)$ of B so that $Y_0 \in A$. Let $a(\mathfrak{g}, \gamma) = (a_0, a_1, a_2)$ and $\alpha(\mathfrak{g}, \gamma) = (\alpha_0, \alpha_1, \alpha_2)$ be as in 4.2.1. As $B = \mathbf{k}[Y_0]^{[2]}$, 4.1.1 implies that $A = \mathbf{k}[Y_0]^{[1]}$. Since A is graded and $\mathbf{k}[Y_0]$ is a homogeneous subring of A, it follows from Lemma 4.1.3 that $A = \mathbf{k}[Y_0, P]$ for some *homogeneous* element P of B. By 4.1.1, it follows that $\gcd_B(P_{Y_1}, P_{Y_2}) = 1$ and P is a variable of $\mathbf{k}(Y_0)[Y_1, Y_2]$.

Observe that P is irreducible in B (because it is irreducible in $A = \mathbf{k}[Y_0, P] = \mathbf{k}^{[2]}$ and A is factorially closed in B). Consider the case where $\alpha_i \nmid \deg P$ for some $i \in \{0, 1, 2\}$. Then $Y_i \mid P$ follows from Theorem 2.6.3 (2), and this implies that P and Y_i are associates; thus $A = \mathbf{k}[Y_0, Y_i]$ and in particular $i \in \{1, 2\}$. In this case, condition (i) of the Theorem is satisfied.

$$\text{From now on, we assume that } \alpha_i \mid \deg P \text{ for each } i. \tag{4.2.1}$$

It follows that $P \in R = \oplus_{i \in \nu\mathbb{Z}} B_i$, where $\nu = \alpha_0\alpha_1\alpha_2$. We claim that there exists $Q \in B$ homogeneous and irreducible such that

$$\mathbf{k}(Y_0)[P, Q] = \mathbf{k}(Y_0)[Y_1, Y_2]. \tag{4.2.2}$$

Recall that $B = \mathbf{k}[Y_0, Y_1, Y_2]$, $A = \mathbf{k}[Y_0, P]$ and consider the Jacobian derivation (cf. 4.1.4) $\Delta_A : B \longrightarrow B$ which is homogeneous, locally nilpotent, irreducible, and has kernel A. Let $K = \mathbf{k}(Y_0)$. Since P is a variable of $K[Y_1, Y_2]$, there exists $Q' \in K[Y_1, Y_2]$ such that $K[P, Q'] = K[Y_1, Y_2]$. We may assume that $Q' \in B$. As (P, Q') and (Y_1, Y_2) are two coordinate systems of $K^{[2]}$, we obtain:

$$\Delta_A(Q') = \frac{\partial(P, Q')}{\partial(Y_1, Y_2)} \in K^*,$$

and so $\Delta_A(Q') \in B \cap K^* = \mathbf{k}[Y_0] \setminus \{0\}$. Since Δ_A is homogeneous, some homogeneous component Q'_i of Q' satisfies $\Delta_A(Q'_i) \in \mathbf{k}[Y_0] \setminus \{0\}$. Let δ denote the degree function determined by Δ_A. As $\Delta_A(Q'_i) \in \mathbf{k}[Y_0] \setminus \{0\}$, $\delta(Q'_i) = 1$. Write $Q'_i \in B$ as a product of prime factors. A unique prime factor Q of Q'_i satisfies $\delta(Q) = 1$, and the other factors have δ-degree 0 and hence belong to A. Then $Q_i' = aQ$ with $a \in A \setminus \{0\}$, where Q is homogeneous and irreducible; also, $a\Delta_A(Q) = \Delta_A(Q_i') \in \mathbf{k}[Y_0] \setminus \{0\}$ implies that $\Delta_A(Q) \in \mathbf{k}[Y_0] \setminus \{0\}$.

Let $S = \mathbf{k}[Y_0] \setminus \{0\}$ and let $\Delta = S^{-1}\Delta_A \in \text{LND}\big(\mathbf{k}(Y_0)[Y_1, Y_2]\big)$. Then $\Delta(Q)$ is a unit of $\mathbf{k}(Y_0)[Y_1, Y_2]$ and hence $\mathbf{k}(Y_0)[Y_1, Y_2] = (\ker \Delta)[Q] = (\ker \Delta)^{[1]}$ by 1.2 (3). As $\ker \Delta = S^{-1} \ker \Delta_A = \mathbf{k}(Y_0)[P]$, this gives $\mathbf{k}(Y_0)[Y_1, Y_2] = \mathbf{k}(Y_0)[P, Q]$, i.e., Q satisfies (4.2.2). Hence (4.2.2) is proved. Next, recall that the ring $R = \oplus_{i \in \nu\mathbb{Z}} B_i$ equals $\mathbf{k}[Y_0^{\alpha_0}, Y_1^{\alpha_1}, Y_2^{\alpha_2}]$ by Lemma 2.6.5. After choosing Q as in (4.2.2), we make two cases.

• The case $Q \in R$: In this case, $P, Q \in R$. Note that $(\operatorname{Frac} R)(Y_0) \subseteq \mathbf{k}(Y_0, Y_1, Y_2)$. As $\mathbf{k}(Y_0)[P, Q] = \mathbf{k}(Y_0)[Y_1, Y_2]$, and $P, Q \in R$, it follows that $(\operatorname{Frac} R)(Y_0) \supseteq \mathbf{k}(Y_0, Y_1, Y_2)$. Thus we obtain $(\operatorname{Frac} R)(Y_0) = \mathbf{k}(Y_0, Y_1, Y_2)$, and consequently,

$$\mathbf{k}(Y_0, Y_1^{\alpha_1}, Y_2^{\alpha_2}) = \mathbf{k}(Y_0, Y_1, Y_2).$$

It follows that $\alpha_1 = 1 = \alpha_2$ and condition (ii) of the Theorem is satisfied.

• The case $Q \notin R$: In this case, some $\alpha_i \nmid \deg Q$. Then $Y_i \mid Q$ by Theorem 2.6.3 (2); this implies that Q and Y_i are associates. Since (4.2.2) holds, $i \neq 0$, so $i \in \{1, 2\}$. Without loss of generality, we may assume that $i = 1$. Then simplifying (4.2.2), we obtain:

$$\mathbf{k}(Y_0)[Y_1, P] = \mathbf{k}(Y_0)[Y_1, Y_2] = K^{[2]}, \tag{4.2.3}$$

where $K = \mathbf{k}(Y_0)$. In other words, we have $C[P] = C[Y_2]$ where C denotes the ring $\mathbf{k}(Y_0)[Y_1]$; this implies that $P = c_1 Y_2 + c_0$ for some $c_1 \in C^*$ and $c_0 \in C$. Using that $P \in B$, it follows that

$$P = \varphi(Y_0) Y_2 + \psi(Y_0, Y_1) \tag{4.2.4}$$

for some nonzero $\varphi(Y_0) \in \mathbf{k}[Y_0]$, and $\psi(Y_0, Y_1) \in \mathbf{k}[Y_0, Y_1]$. If $\psi(Y_0, Y_1) = 0$, then $\varphi(Y_0) \in \mathbf{k}^*$ because P is irreducible, and A has the form as in (i). Next, assume that $\psi(Y_0, Y_1) \neq 0$, and let us prove that $\alpha_2 = \gcd(a_0, a_1) = 1$. As P is homogeneous

$$\deg P = \deg(\varphi(Y_0) Y_2) = \deg(\varphi(Y_0)) + \deg Y_2. \tag{4.2.5}$$

Note that $\varphi(Y_0)$ is a homogeneous polynomial. As $\alpha_2 \mid \deg P$ (cf. (4.2.1)) and $\alpha_2 \mid a_0$, it follows that $\alpha_2 \mid \deg \varphi(Y_0)$ and $\alpha_2 \mid a_2$ by (4.2.5). Since $\gcd(\alpha_2, a_2) = 1$, it follows that $\alpha_2 = 1$. The condition $\gcd(\frac{\partial \psi}{\partial Y_1}, \varphi) = 1$ follows from (4.2.4) and from the fact that $\gcd_B(P_{Y_1}, P_{Y_2}) = 1$. So A has the form as in (iii). ∎

4.2.4 Let the setup be as in the beginning of Section 4.2. We fix a $\mathfrak{g}$-homogeneous coordinate system $\gamma = (X_0, X_1, X_2)$ of B and describe all the elements of $\text{KLND}(B, \mathfrak{g})$ in terms of γ. Since $\text{type}(\mathfrak{g}) > 0$, we assume that γ has been chosen so that

$$\alpha_0 \neq 1. \tag{4.2.6}$$

Let $W(B, \mathfrak{g}, \gamma)$ be the set of all pairs $(u, v) \in B \times B$ satisfying: (X_0, u, v) is a $\mathfrak{g}$-homogeneous coordinate system of B.

Theorem 4.2.5 *Assume the setup as in the beginning of Section* 4.2 *and the notation as in* 4.2.4. *Then the elements of* $\text{KLND}(B, \mathfrak{g})$ *are:*

(a) $\mathbf{k}[X_0, P]$ *for all homogeneous* $P \in B$ *such that* $\gcd_B(P_{X_1}, P_{X_2}) = 1$ *and such that* P *is a variable of* $\mathbf{k}(X_0)[X_1, X_2]$.

(b) $\mathbf{k}[u, \varphi(u)v + \psi(u, X_0)]$, *for all* $(u, v) \in W(B, \mathfrak{g}, \gamma)$ *such that* $\varphi \in \mathbf{k}[u] \setminus \{0\}$, *and* $\psi \in \mathbf{k}[u, X_0]$, $\gcd(\frac{\partial \psi}{\partial X_0}, \varphi) = 1$, *where* $\varphi(u)v + \psi(u, X_0)$ *is homogeneous and irreducible.*

Proof: If A is one of the rings displayed in part (a) (resp. part (b)) of the statement of the theorem, then we apply 4.1.1 to $B = \mathbf{k}[X_0]^{[2]}$ (resp. to $B = \mathbf{k}[u]^{[2]}$) to conclude that $A \in \text{KLND}(B, \mathfrak{g})$. In the other direction, let $A \in \text{KLND}(B, \mathfrak{g})$. We will prove that A occurs in one of the rings displayed in (a) and (b). If $X_0 \in A$, 4.1.1 implies that $A = \mathbf{k}[X_0]^{[1]}$. As A is graded and $\mathbf{k}[X_0]$ is a homogeneous subring of A, it follows from Lemma 4.1.3 that $A = \mathbf{k}[X_0, P]$ for some homogeneous element P of

B. By 4.1.1, $\gcd_B(P_{X_1}, P_{X_2}) = 1$ and P is a variable of $\mathbf{k}(X_0)[X_1, X_2]$; so A is of the type described in (a). From now on we assume that

$$X_0 \notin A \tag{4.2.7}$$

By Theorem 4.2.3, choose a homogeneous coordinate system $\gamma' = (Y_0, Y_1, Y_2)$ of B so that one of the following holds:

(i) $A = \mathbf{k}[Y_0, Y_1]$.

(ii) $\gcd(\deg Y_0, \deg Y_1) = 1 = \gcd(\deg Y_0, \deg Y_2)$ and $A = \mathbf{k}[Y_0, P]$ for some homogeneous $P \in B$ which satisfies $\gcd_B(P_{Y_1}, P_{Y_2}) = 1$ and which is a variable of $\mathbf{k}(Y_0)[Y_1, Y_2]$.

(iii) $\gcd(\deg Y_0, \deg Y_1) = 1$, $A = \mathbf{k}[Y_0, \varphi(Y_0)Y_2 + \psi(Y_0, Y_1)]$ for some $\varphi \in \mathbf{k}[Y_0] \setminus \{0\}$, and $\psi \in \mathbf{k}[Y_0, Y_1]$ such that $\gcd(\frac{\partial \psi}{\partial Y_1}, \varphi) = 1$, and $\varphi(Y_0)Y_2 + \psi(Y_0, Y_1)$ is homogeneous and irreducible.

Let $(\alpha'_0, \alpha'_1, \alpha'_2) = \alpha(\mathfrak{g}, \gamma')$. Applying Lemma 2.6.6 to each of γ, γ' gives

$$\{\, \lambda X_j \mid \lambda \in \mathbf{k}^* \text{ and } \alpha_j \neq 1 \,\} = \{\, \lambda Y_j \mid \lambda \in \mathbf{k}^* \text{ and } \alpha'_j \neq 1 \,\}. \tag{$*$}$$

As X_0 belongs to the left hand side of $(*)$, there exists $i \in \{0, 1, 2\}$ such that $\alpha'_i \neq 1$ and

$$X_0 = \lambda Y_i \text{ for some } \lambda \in \mathbf{k}^*. \tag{4.2.8}$$

Then $\alpha'_i \neq 1$, where $(\alpha'_0, \alpha'_1, \alpha'_2) = \alpha(\mathfrak{g}, \gamma')$. Note that this i also satisfies the following condition:

$$\text{given distinct } k, j \in \{0, 1, 2\},\ \gcd(\deg Y_j, \deg Y_k) = 1 \Rightarrow i \in \{j, k\}. \tag{4.2.9}$$

Since one of (i)-(iii) holds, $Y_0 \in A$. Then by (4.2.7) and (4.2.8),

$$Y_i \notin A \text{ and } i \in \{1, 2\}. \tag{4.2.10}$$

If (i) holds then $Y_0, Y_1 \in A$ by (4.2.10). So $i = 2$ by (4.2.7) and (4.2.8). Then (X_0, Y_0, Y_1) is a homogeneous coordinate system of B and $(Y_0, Y_1) \in W(B, \mathfrak{g}, \gamma)$. So

$A = \mathbf{k}[u, v]$ for some $(u, v) \in W(B, \mathfrak{g}, \gamma)$, and A appears in (b) with $\varphi(u) = 1$ and $\psi = 0$ (also observe that $\gcd(\frac{\partial\psi}{\partial X_0}, \varphi) = \gcd(0, 1) = 1$).

If (ii) holds then $\gcd(\deg Y_0, \deg Y_1) = 1 = \gcd(\deg Y_0, \deg Y_2)$. Then (4.2.9) implies that $i = 0$, which contradicts (4.2.10). So this case does not arise.

If (iii) holds then $\gcd(\deg Y_0, \deg Y_1) = 1$. Conditions (4.2.9) and (4.2.10) imply that $i = 1$. Then (X_0, Y_0, Y_2) is a homogeneous coordinate system of B and $(Y_0, Y_2) \in W(B, \mathfrak{g}, \gamma)$. In this case,

$$A = \mathbf{k}[Y_0, \varphi(Y_0)Y_2 + \psi(Y_0, Y_1)] = \mathbf{k}[u, \varphi(u)v + \psi(u, \tfrac{1}{\lambda}X_0)]$$

where $(u, v) = (Y_0, Y_2) \in W(B, \mathfrak{g}, \gamma)$, so A appears in (b).

∎

4.3 Examples

Throughout this section, let $B = \mathbf{k}^{[3]}$ where $\mathbf{k}$ is an algebraically closed field of characteristic zero. We consider some examples of $\mathbb{Z}$-gradings $\mathfrak{g}$ of B satisfying $\text{type}(\mathfrak{g}) \geq 1$, and we give explicitly all elements of $\text{KLND}(B, \mathfrak{g})$ for each $\mathfrak{g}$.

In each case, we write $B = \mathbf{k}[X_0, X_1, X_2]$, and we define $\mathfrak{g}$ by declaring that X_0, X_1, X_2 are homogeneous of certain degrees. Note that (X_0, X_1, X_2) is then a homogeneous coordinate system of B.

Example 4.3.1 If $\text{type}(\mathfrak{g}) = 3$, then by Corollary 2.5.7 we obtain:

$$\text{KLND}(B, \mathfrak{g}) = \{\mathbf{k}[X_0, X_1], \mathbf{k}[X_0, X_2], \mathbf{k}[X_1, X_2]\}.$$

Example 4.3.2 Define a grading $\mathfrak{g}$ on B by declaring $\deg(X_0) = -2$, $\deg(X_1) = 3$ and $\deg(X_2) = 0$. Note that $(\alpha_0, \alpha_1, \alpha_2) = (3, 2, 1)$ and consequently, $\text{type}(\mathfrak{g}) = 2$. We show that all the elements of $\text{KLND}(B, \mathfrak{g})$ are:

(i) $\mathbf{k}[X_0, X_1]$

(ii) $\mathbf{k}[X_0, X_2 + f(X_0^3 X_1^2)]$, for some polynomial $f \in \mathbf{k}[X_0^3 X_1^2]$

(iii) $\mathbf{k}[X_1, X_2 + g(X_0^3 X_1^2)]$, for some polynomial $g \in \mathbf{k}[X_0^3 X_1^2]$

Before proving this, observe the consequence: $B = A^{[1]}$ for all $A \in \text{KLND}(B, \mathfrak{g})$. This is a special property of the grading $\mathfrak{g}$, but not all gradings of type 2 have that property.[1]

It is clear that all rings given in (i–iii) belong to $\text{KLND}(B, \mathfrak{g})$. We prove the converse. Let $W(B, \mathfrak{g})$ be the set of all pairs $(u, v) \in B \times B$ such that (X_0, u, v) is a $\mathfrak{g}$-homogeneous coordinate system of B. Given $A \in \text{KLND}(B, \mathfrak{g})$, two cases arise by Theorem 4.2.5:

Case (I) $A = \mathbf{k}[X_0, P]$ for some homogeneous $P \in B$, where $\gcd_B(P_{X_1}, P_{X_2}) = 1$ and such that P is a variable of $\mathbf{k}(X_0)[X_1, X_2]$. Note that P is irreducible in B, because it is irreducible in A and A is factorially closed in B. If $X_1 \mid P$ then X_1, P are associates in B, so $A = \mathbf{k}[X_0, X_1]$ and we are done in this case. From now on, assume that $X_1 \nmid P$. Then P contains a monomial $bX_0^j X_2^k$ for some $b \in \mathbf{k}^*$, $j, k \in \mathbb{N}$. This implies that $\deg(P)$ is even and hence (by homogeneity) $P \in \mathbf{k}[X_0, X_1^2, X_2]$.

This implies that $P_{X_1}(X_0, 0, X_2) = 0$. Then 1.3.6 (i) (and the fact that P is a variable of $\mathbf{k}(X_0)[X_1, X_2]$) implies that $P_{X_2}(X_0, 0, X_2) \in \mathbf{k}(X_0)^*$. If we write

$$P = P_0 + P_1 X_2 + \cdots + P_n X_2^n \quad (P_i \in \mathbf{k}[X_0, X_1],\ P_n \neq 0)$$

then $P_n \in \mathbf{k}[X_0] \setminus \{0\}$ by 1.3.6 (iii), so the fact that $P_{X_2}(X_0, 0, X_2) \in \mathbf{k}(X_0)^*$ implies that $n = 1$ and hence that $P = \mu X_0^e X_2 + \psi(X_0, X_1)$, with $e \in \mathbb{N}$, $\mu \in \mathbf{k}^*$ and $\psi(X_0, X_1) \in \mathbf{k}[X_0, X_1]$. In fact we have $e = 0$. Indeed, if $e > 0$ then $\deg(X_0^e X_2) < 0$, so $\deg(\psi(X_0, X_1)) < 0$, which implies $X_0 \mid \psi(X_0, X_1)$ and hence $X_0 \mid P$; this implies that X_0, P are associates, which contradicts the fact that $A = \mathbf{k}[X_0, P]$. So $e = 0$ and consequently $\deg(P) = \deg(\mu X_2) = 0$. By homogeneity, it follows that $\psi(X_0, X_1) \in \mathbf{k}[X_0^3 X_1^2]$, so A is of the type described in (ii).

[1]Define $\mathfrak{g}$ by $\deg(X_0) = 4$, $\deg(X_1) = 9$ and $\deg(X_2) = 6$, then $\text{type}(\mathfrak{g}) = 2$; let $A = \mathbf{k}[X_0, X_0^3 X_2 + X_1^2]$, then $A \in \text{KLND}(B, \mathfrak{g})$ and $B \neq A^{[1]}$.

Case (II) A has the form $\mathbf{k}[u, \varphi(u)v+\psi(u, X_0)]$, where $(u,v) \in W(B, \mathfrak{g})$, $\varphi(u) \in \mathbf{k}[u] \setminus \{0\}$, $\psi(u, X_0) \in \mathbf{k}[u, X_0]$, $\varphi(u)v + \psi(u, X_0)$ is homogeneous and irreducible and $\gcd(\frac{\partial\psi}{\partial X_0}, \varphi) = 1$. As $(u,v) \in W(B, \mathfrak{g})$, (X_0, u, v) is a $\mathfrak{g}$-homogeneous coordinate system of B. By Lemma 2.3.8, $(\deg(u), \deg(v))$ is either $(3,0)$ or $(0,3)$. We claim:

$$\text{If } (\deg(u), \deg(v)) = (3,0), \text{ then } (u,v) = (aX_1, bX_2 + f(X_0^3X_1^2)) \text{ for some } a, b \in \mathbf{k}^* \text{ and } f \in \mathbf{k}[X_0^3X_1^2]; \tag{4.3.1}$$

and,

$$\text{if } (\deg(u), \deg(v)) = (0,3), \text{ then } (u,v) = (bX_2 + f(X_0^3X_1^2), aX_1) \text{ for some } a, b \in \mathbf{k}^* \text{ and } f \in \mathbf{k}[X_0^3X_1^2]. \tag{4.3.2}$$

It is clear that (4.3.2) can be obtained from (4.3.1) by interchanging u and v, so it is enough to prove (4.3.1). Assume that $(\deg(u), \deg(v)) = (3,0)$. As u is homogeneous and $\deg(u) > 0$, it follows that X_1 divides each monomial of u, so $X_1 \mid u$; as u is irreducible, $u = aX_1$ for some $a \in \mathbf{k}^*$. Write

$$v = F_0 + F_1X_2 + \cdots + F_nX_2^n \quad (F_i \in \mathbf{k}[X_0, X_1],\ F_n \neq 0).$$

As $u = aX_1$, (X_0, X_1, v) is a homogeneous coordinate system of B. In particular, for each choice of $\alpha, \beta \in \mathbf{k}$ the element $\bar{v} = v(\alpha, \beta, X_2) \in \mathbf{k}[X_2]$ satisfies $\mathbf{k}[X_2] = \mathbf{k}[\bar{v}]$, so $\deg_{X_2}(\bar{v}) = 1$. This implies that $n = 1$ and that $F_1 \in \mathbf{k}^*$, so $v = F_0(X_0, X_1) + bX_2$ for some $b \in \mathbf{k}^*$. By homogeneity, $F_0(X_0, X_1) \in \mathbf{k}[X_0^3X_1^2]$, so (4.3.1) is proved. So both (4.3.1) and (4.3.2) are true.

In case (4.3.1), $A = \mathbf{k}[aX_1, Q]$ where $Q = \varphi(aX_1)(bX_2 + f(X_0^3X_1^2)) + \psi(aX_1, X_0)$. As Q is homogeneous, we see that if φ is not a constant then $X_1 \mid Q$, which is not possible; so $\varphi \in \mathbf{k}^*$ and $Q = b'X_2 + g(X_0^3X_1^2)$ (some $b' \in \mathbf{k}^*$), which means that A is as described in (iii).

In case (4.3.2), $A = \mathbf{k}[bX_2 + f(X_0^3X_1^2), Q]$ where

$$Q = \varphi(bX_2 + f(X_0^3X_1^2))(aX_1) + \psi(bX_2 + f(X_0^3X_1^2), X_0).$$

As $\deg(Q) = 3$ and $\deg(\psi(bX_2 + f(X_0^3X_1^2), X_0)) \leq 0$, homogeneity of Q implies that $\psi = 0$. Then irreducibility of Q gives $\varphi \in \mathbf{k}^*$, so Q is a unit times X_1 and A is as in (ii).

Finally, we remark that

$$\{\, d(A) \mid A \in \text{KLND}(B, \mathfrak{g}) \,\} = \{1, 2, 3\} = \{\alpha_0, \alpha_1, \alpha_2\}.$$

Consequently, Theorem 2.6.3 (1) is also verified for this example.

Example 4.3.3 Define a grading $\mathfrak{g}$ on B by declaring $\deg(X_0) = 1$, $\deg(X_1) = 0$ and $\deg(X_2) = 0$. Note that $(\alpha_0, \alpha_1, \alpha_2) = (0, 1, 1)$ and consequently, $\text{type}(\mathfrak{g}) = 1$. Applying Theorem 4.2.5 and by an easy computation, we can show that all the elements of $\text{KLND}(B, \mathfrak{g})$ are:

(i) $\mathbf{k}[X_1, X_2]$.

(ii) $\mathbf{k}[X_0, P]$, where P is a variable of $\mathbf{k}[X_1, X_2]$.

Also note that $\{\, d(A) \mid A \in \text{KLND}(B, \mathfrak{g}) \,\} = \{0, 1\} = \{\alpha_0, \alpha_1, \alpha_2\}$.

Chapter 5

Affine ML-Surfaces

5.1 Introduction

Throughout this chapter, $\mathbf{k}$ denotes an arbitrary field of characteristic zero. Recall from Chapter 1 that the Makar-Limanov invariant of a $\mathbf{k}$-algebra B (denoted $\mathrm{ML}(B)$) is the intersection of the kernels of all locally nilpotent derivations of B. If $X = \operatorname{Spec} B$ is an affine $\mathbf{k}$-variety, one defines $\mathrm{ML}(X) = \mathrm{ML}(B)$. We say that B has trivial Makar-Limanov invariant if $\mathrm{ML}(B) = \mathbf{k}$ and that B is rigid if $\mathrm{ML}(B) = B$. The Makar-Limanov invariant, also known as the AK-invariant, was introduced by Makar-Limanov in 1996 and provides a useful algebraic tool to classify and distinguish varieties. Computing the Makar-Limanov invariant of a given algebra can be very challenging. We discuss examples below, of which some are trivial, but others require significant work.

Example 5.1.1 Let us see the Makar-Limanov invariant of certain domains.

1. $\mathrm{ML}(\mathbf{k}) = \mathbf{k}$ because every locally nilpotent derivation of $\mathbf{k}$ is a $\mathbf{k}$-derivation.

2. Let $B = \mathbf{k}[x_1, \dots, x_n] = \mathbf{k}^{[n]}$. Then $\mathrm{ML}(B) = \mathbf{k}$ because
$$\ker \frac{\partial}{\partial x_1} \cap \dots \cap \ker \frac{\partial}{\partial x_n} = \mathbf{k}.$$

3. Let B be a nonrigid one-dimensional affine $\mathbf{k}$-domain. Then $B \cong \mathbf{k}'^{[1]}$ for some field $\mathbf{k}'$ such that $\mathbf{k} \subseteq \mathbf{k}' \subset B$ and $\mathbf{k}'$ is algebraic over $\mathbf{k}$. Indeed, let $0 \neq D \in \text{LND}(B)$ and let $A = \ker D$. Note that $\text{trdeg}_{\mathbf{k}} A = 0$ because $\text{trdeg}_A B = 1$ by 1.2 (5). Hence $A = \mathbf{k}'$ for some field $\mathbf{k}'$ such that $\mathbf{k}'$ is algebraic over $\mathbf{k}$. Now choose a preslice a of D. Then $0 \neq Da \in A = \mathbf{k}'$. In fact, a is a slice of D. Then $B = \mathbf{k}'^{[1]}$ follows from 1.2 (3).

4. Let $B = \mathbf{k}[x^2, x^3]$. As $\dim B = 1$ and B is not a polynomial ring, B is a one-dimensional rigid $\mathbf{k}$-domain by the previous example.

5. Let $B = \mathbf{k}[x, y, z]/(x^n z - P(y))$, where $\deg P \geq 1$ and $n \in \mathbb{N}$. If $n \leq 1$ or $\deg P(y) = 1$, then $\text{ML}(B) = \mathbf{k}$. If $n \geq 2$ and $\deg P(y) \geq 2$, then $\text{ML}(B) = \mathbf{k}[x]$.

6. The Russell-Koras threefold: Let $R = \mathbb{C}[x, y, z, t]$, where $x + x^2 y + z^2 + t^3 = 0$. Makar-Limanov proved that $\text{ML}(R) = \mathbb{C}[x]$. The proof of this simple looking fact is not easy. This fact played a role in the solution of the important linearization problem for $\mathbb{C}^*$-actions on $\mathbb{C}^3$.

7. If B is a two-dimensional affine factorial $\mathbb{C}$-domain satisfying $\text{ML}(B) = \mathbb{C}$, then $B \cong \mathbb{C}^{[2]}$. So $\mathbb{A}^2_{\mathbb{C}}$ is the only factorial $\mathbb{C}$-surface with trivial Makar-Limanov invariant.

8. Let $B = \mathbb{C}[x, y, z]/(x^2 + y^3 + z^5)$. Then B is a two-dimensional affine $\mathbb{C}$-UFD and $\text{ML}(B) = B$.

9. A result of Bandman and Makar-Limanov [3] states that every smooth hypersurface of $\mathbb{A}^3_{\mathbb{C}}$ with trivial Makar-Limanov invariant is isomorphic (as an algebraic surface) to a surface with equation $xy = p(z)$, where $p(z)$ is a nonconstant polynomial in $\mathbb{C}[z]$ with simple roots.

We note that a variety with trivial Makar-Limanov invariant has a "large" automorphism group (where the word "large" does not have a precise definition). Indeed, if B

is a $\mathbf{k}$-domain of dimension at least two and satisfies $\text{ML}(B) = \mathbf{k}$, then $|\,\text{KLND}(B)| > 1$. It follows that $\text{KLND}(B)$ has infinite cardinality due to the following fact (cf. Proposition 1.10 of [13]):

Proposition. *Let B be a domain of characteristic zero and suppose $\mathbf{k} \subset B$ is a field such that $\text{trdeg}_{\mathbf{k}} B < \infty$. Then $|\,\text{KLND}(B)|$ is either 0, 1 or $|\mathbf{k}|$.*

Consequently, a domain with trivial Makar-Limanov invariant admits many locally nilpotent derivations with distinct kernels. Since a locally nilpotent derivation of B gives rise to an automorphism of B via exponentiation, it is reasonable to consider that the group $\text{Aut}_{\mathbf{k}}(B)$ is large.

As we have already observed, the affine line $\mathbb{A}^1_{\mathbf{k}}$ is the only affine curve with trivial Makar-Limanov invariant. Clearly, this claim fails in higher dimensions. For instance, Example 5.1.1 (5) gives many affine surfaces with trivial Makar-Limanov invariant which are not isomorphic to $\mathbb{A}^2_{\mathbf{k}}$. Although it is desirable to classify all affine varieties of a given dimension which have trivial Makar-Limanov invariant, the problem seems to be hard even in dimension 2.

Notation 5.1.2 Given a field $\mathbf{k}$ of characteristic zero, let $\mathcal{M}(\mathbf{k})$ denote the class of **two-dimensional** affine $\mathbf{k}$-domains which have trivial Makar-Limanov invariant. We say that an affine surface $X = \text{Spec}\, B$ belongs to the class $\mathcal{M}(\mathbf{k})$ if $B \in \mathcal{M}(\mathbf{k})$. Such X is also called an ML-*surface.*

Over the last decade, Bandman, Daigle, Dubouloz, Gurjar, Masuda, Makar-Limanov, Miyanishi and Russell (see [3], [11], [15], [22], [32], [47]) have been actively investigating properties of normal (or smooth) surfaces in the class $\mathcal{M}(\mathbf{k})$. However, it is desirable to understand what happens when we drop the assumption of normality.

In this chapter, we explore some interesting properties of the surfaces in the class $\mathcal{M}(\mathbf{k})$. In 5.1.3, we list some facts and observations obtained in this chapter. Some of these facts are easy to prove (and some special cases are known), while other facts are nontrivial.

5.1.3 Let $B \in \mathcal{M}(\mathbf{k})$, where $\mathbf{k}$ is any field of characteristic zero. Then:

(1) B is not necessarily normal (cf. Examples 5.3.7.1 and 5.3.7.2).

(2) B is not necessarily rational (Remark 5.3.7.3).

(3) If $\mathbf{k}$ is algebraically closed or B is normal, then B is rational (Lemma 5.3.8).

(4) If $A \in \text{KLND}(B)$, then A is not necessarily a polynomial curve over $\mathbf{k}$ (Example 5.3.7.2).

(5) If B is normal and $A \in \text{KLND}(B)$, then $A \cong \mathbf{k}^{[1]}$ (Lemma 5.3.6). If B is rational and $A \in \text{KLND}(B)$, then A is a polynomial curve over $\mathbf{k}$ (Remark 5.3.9).

(6) If $A_1, A_2 \in \text{KLND}(B)$ and I is a height 1 ideal of B satisfying $I \cap A_1 \neq 0$ and $I \cap A_2 \neq 0$, then $A_1 = A_2$ (Theorem 5.5.6). See Proposition 5.5.1.2 for the geometric version of this statement when $\mathbf{k} = \bar{\mathbf{k}}$.

(7) B is regular in codimension 1. Consequently, the affine surface Spec B has finitely many singular points (Corollary 5.5.10).

(8) If B satisfies Serre's condition (S_2), then B is normal (Corollary 5.6.1). Hence, complete intersection surfaces in the class $\mathcal{M}(\mathbf{k})$ are normal (Corollary 5.6.2). In particular, hypersurfaces in $\mathbb{A}^3_{\mathbf{k}}$ with trivial Makar-Limanov invariant are normal.

5.2 Preliminary results

Throughout this section, assume that $\mathbf{k}$ is any field of characteristic zero. The following is a consequence of Lemma 3.1 of [9], but for completeness we include a proof.

Lemma 5.2.1 *Let B be a $\mathbf{k}$-algebra and $f(T) \in B[T]$, where T is an indeterminate.*

(a) *If $f(T)$ has infinitely many roots in $\mathbf{k}$, then $f(T) = 0$.*

(b) *If J is an ideal of B and $f(\lambda) \in J$ for infinitely many $\lambda \in \mathbf{k}$, then $f(T) \in J[T]$.*

Proof: To prove (a), we use induction on $\deg_T(f)$. The result is immediate if $\deg_T(f) \leq 0$. So assume that $\deg_T(f) > 0$. Let $\alpha \in \mathbf{k}$ be such that $f(\alpha) = 0$; since $T - \alpha \in B[T]$ is a monic polynomial, $f(T) = (T - \alpha)g(T)$ for some $g(T) \in B[T]$ such that $\deg_T(g) < \deg_T(f)$. If $\beta \in \mathbf{k} \setminus \{\alpha\}$ is such that $f(\beta) = 0$, then $(\beta - \alpha)g(\beta) = 0$ and $\beta - \alpha \in B^*$, so $g(\beta) = 0$. So $g(\beta) = 0$ holds for infinitely many $\beta \in \mathbf{k}$ and, by the inductive hypothesis, $g(T) = 0$. It follows that $f(T) = 0$.

To prove (b), consider an ideal J of B such that $f(\lambda) \in J$ for infinitely many $\lambda \in \mathbf{k}$. If $\bar{f}(T) \in (B/J)[T]$ denotes the polynomial $f(T)$ modulo J, then $\bar{f}(T)$ is the zero polynomial by (a). It follows that $f(T) \in J[T]$. ∎

Remark 5.2.2 Recall that if I is a proper ideal of a ring R, then $\operatorname{ht} I$ is the infimum of the heights of the prime ideals $\mathfrak{p}$ of R which satisfy $\mathfrak{p} \supseteq I$. Note that $\operatorname{ht} I$ is not defined when $I = R$. Whenever we speak of an ideal I of a certain height, we tacitly assume that I is a proper ideal. For instance, the statement of Lemma 5.2.3 should be interpreted as saying that if I is a *proper* ideal of B of height h, then IB' is a *proper* ideal of B' of height at most h.

Lemma 5.2.3 *Let $B \subseteq B'$ be domains such that B' is integral over B. If I is an ideal of B of height h, then its extension IB' satisfies $\operatorname{ht} IB' \leq h$. In particular, $\operatorname{ht} I = 1$ implies that $\operatorname{ht} IB' = 1$.*

Proof: Given an ideal I of B of height h, choose a prime ideal $\mathfrak{p}$ of B satisfying $I \subseteq \mathfrak{p}$ and $\operatorname{ht} \mathfrak{p} = h$. As B' is integral over B, there exists a prime ideal $\mathfrak{q}$ of B' such that $\mathfrak{q} \cap B = \mathfrak{p}$. Note that $\operatorname{ht} \mathfrak{q} \leq h$ by [49, Ex. 9.8, p. 70]. Also $I \subseteq \mathfrak{p}$ implies that $IB' \subseteq \mathfrak{p}B' \subseteq \mathfrak{q}$ (so in particular, $IB' \neq B'$). Then $\operatorname{ht} IB' \leq \operatorname{ht} \mathfrak{q} \leq h$. If $\operatorname{ht} I = 1$, then $\operatorname{ht} IB' \leq 1$. Since B' is a domain, $\operatorname{ht} IB' \neq 0$ and so $\operatorname{ht} IB' = 1$. ∎

The following lemmas are well-known facts from commutative algebra. See Chapter 5 of [2] (Corollary 5.8 and Exercise 3).

Lemma 5.2.4 *If $A \subseteq B$ is an integral extension of rings, $\mathfrak{p} \in \operatorname{Spec} B$ and $\mathfrak{p}_0 = \mathfrak{p} \cap A \in \operatorname{Spec} A$, then $\mathfrak{p}$ is maximal if and only if $\mathfrak{p}_0$ is maximal.*

Lemma 5.2.5 *Let $f : B \to B'$ be a homomorphism of A-algebras, and let C be an A-algebra. If f is integral, then $f \otimes 1 : B \otimes_A C \to B' \otimes_A C$ is integral.*

Definition 5.2.6 Let R be a ring and $D \in \mathrm{Der}(R)$. An ideal I of R is called an *integral ideal* for D if $D(I) \subseteq I$.

Lemma 5.2.7 *Let R be a noetherian* **k***-algebra, and let $D \in \mathrm{Der}(R)$. If I is an integral ideal for D, then every minimal prime ideal of R that contains I is also an integral ideal for D.*

Proof: This fact is an easy consequence of [12, Lemma 2.10], but we provide a proof for the sake of completeness. Consider the noetherian **k**-algebra $\bar{R} = R/I$. As $D(I) \subseteq I$, D induces a derivation $\bar{D} : \bar{R} \to \bar{R}$, defined by $\bar{D}(r + I) = Dr + I$. Since there is a natural bijection between the set of minimal prime ideals of R containing I and the set of minimal prime ideals of $\bar{R}$, it suffices to prove the case $I = 0$ of the lemma. So it is enough to prove that:

$$D(\mathfrak{p}) \subseteq \mathfrak{p} \text{ for every minimal prime ideal } \mathfrak{p} \text{ of } R. \tag{5.2.1}$$

Let η denote the nilradical of R and recall that $\eta = \cap \mathfrak{p}$, where $\mathfrak{p}$ varies over the set of all minimal prime ideals of R. First we shall prove that $D(\eta) \subseteq \eta$. Consider the ring homomorphism $\varepsilon : R \to R[[t]]$, $\varepsilon(x) = \sum_{i=0}^{\infty} \frac{D^i(x)}{i!} t^i$, where t is an indeterminate. If $x \in \eta$, then x is nilpotent, hence $\varepsilon(x)$ is a nilpotent element of $R[[t]]$. So each coefficient of the power series $\varepsilon(x)$ belongs to η; in particular, $D(x) \in \eta$.

As A is noetherian, it has finitely many minimal prime ideals, say $\mathfrak{p}_1, \dots, \mathfrak{p}_m$. Then we have: $\eta = \bigcap_{i=1}^m \mathfrak{p}_i$. Now let us prove (5.2.1). If $m = 1$, $\eta = \mathfrak{p}_1$, and we are done. So assume that $m > 1$. It suffices to prove that $D(\mathfrak{p}_i) \subseteq \mathfrak{p}_i$ for each $i = 1, \dots, m$. Given $i \in \{1, \dots, m\}$, choose $x \in \mathfrak{p}_i$ and $y \in \bigcap \mathfrak{p}_j$ such that $y \notin \mathfrak{p}_i$. Then $xy \in \eta$, hence $D(xy) \in \eta$. Note that

$$D(x)y^2 + xyD(y) = y(D(x)y + xD(y)) = yD(xy) \in \eta.$$

Since $xyD(y) \in \eta$, we obtain $D(x)y^2 \in \eta \subseteq \mathfrak{p}_i$. Hence $D(x) \in \mathfrak{p}_i$, which proves that $D(\mathfrak{p}_i) \subseteq \mathfrak{p}_i$. ∎

Lemma 5.2.8 *Let R be a $\mathbf{k}$-domain, and let I be a nonzero ideal of R. If $A \in$ $\text{KLND}(R)$, then the following statements are equivalent:*

(1) $I \cap A \neq (0)$.

(2) *There exists $D \in \text{LND}(R)$ such that* $\ker D = A$ *and I is an integral ideal for D.*

Proof: Assume that (1) holds. Let $0 \neq a \in I \cap A$, and let $E \in \text{LND}(R)$ be such that $A = \ker E$. Since $a \in A$, $aE \in \text{LND}(R)$ and aE has kernel A. Moreover, as $a \in I$, $(aE)(b) = a(Eb) \in I$ for all $b \in I$. So $(aE)(I) \subseteq I$, and hence $D := aE$ is the required locally nilpotent derivation of R proving assertion (2).

In the other direction, assume that $D \in \text{LND}(R)$, $\ker D = A$, and $D(I) \subseteq I$. Choose any $b \in I$, $b \neq 0$. Then the set $\{b, Db, D^2b, \dots\}$ is included in I and contains a nonzero element of A. ∎

Lemma 5.2.9 *Let B be a $\mathbf{k}$-algebra, J an ideal of B, and $D \in \text{LND}(B)$. If $e^{tD}(J) \subseteq J$ for some nonzero $t \in \mathbf{k}$, then J is an integral ideal for D.*

Proof: First we observe that if $e^{tD}(J) \subseteq J$ for some nonzero $t \in \mathbf{k}$, then $e^{tD}(J) \subseteq J$ for infinitely many $t \in \mathbf{k}$. Let $f \in J$. We will show that $D(f) \in J$. Let $n = \deg_D(f)$, i.e., n is the maximum nonnegative integer such that $D^n(f) \neq 0$. Define a polynomial $P(T) \in B[T]$ by

$$P(T) = f + D(f)T + \frac{D^2(f)T^2}{2!} + \cdots + \frac{D^n(f)T^n}{n!}.$$

Then for infinitely many $t \in \mathbf{k}$,

$$P(t) = f + D(f)t + \frac{D^2(f)t^2}{2!} + \cdots + \frac{D^n(f)t^n}{n!} = e^{tD}(f) \in J.$$

By Lemma 5.2.1, all the coefficients of $P(T)$ belong to J, so $D(f) \in J$. ∎

Lemma 5.2.10 *Let R be an affine $\mathbf{k}$-domain, and let $D \in \text{LND}(R)$. If $\tilde{R}$ denotes the normalization of R, then there exists $\tilde{D} \in \text{LND}(\tilde{R})$ such that* $\ker \tilde{D} \cap R = \ker D$.

Proof: Let $A = \ker D$ and consider $S = A \setminus \{0\}$. Recall that D extends to a locally nilpotent derivation $\mathfrak{D}$ of $S^{-1}R$ defined by $\mathfrak{D}(\frac{r}{s}) = \frac{Dr}{s}$ for any $r \in R$ and $s \in S$. Consider the integral closure $(S^{-1}R)^{\sim}$ of $S^{-1}R$ in $\text{Frac}\, R$ and note that $(S^{-1}R)^{\sim} = S^{-1}\tilde{R}$. As $S^{-1}R = (\text{Frac}\, A)^{[1]}$ by 1.2 (5), $S^{-1}R$ is a polynomial ring over a field, so it is normal. It follows that $S^{-1}R = (S^{-1}R)^{\sim} = S^{-1}\tilde{R}$. Thus, D extends to a locally nilpotent derivation $\mathfrak{D}$ of $S^{-1}\tilde{R}$.

By Noether's Theorem, $\tilde{R}$ is a finitely generated R-module. Let $x_1, \dots, x_m$ be a set of generators for $\tilde{R}$ as an R-module. For each $1 \le i \le m$, $\mathfrak{D}(x_i) \in S^{-1}\tilde{R}$. So there exist $s_i \in S$ such that $s_i \mathfrak{D}(x_i) \in \tilde{R}$. If we set $s := \prod_{i=1}^{m} s_i$, then $s\,\mathfrak{D}(x) \in \tilde{R}$ for each $x \in \tilde{R}$. As $s \in \ker D \subseteq \ker \mathfrak{D}$, $s\mathfrak{D} \in \text{LND}(S^{-1}\tilde{R})$ by 1.2 (4). Then the restriction $\tilde{D} : \tilde{R} \to \tilde{R}$ of $s\,\mathfrak{D}$ belongs to $\text{LND}(\tilde{R})$ and satisfies $\ker \tilde{D} \cap R = \ker D$. ∎

Lemma 5.2.11 *For a two-dimensional affine $\mathbf{k}$-domain R,*

$$|\,\text{KLND}(R)| > 1 \text{ if and only if } \text{ML}(R) \text{ is algebraic over } \mathbf{k}.$$

Proof: Assume that $\text{ML}(R)$ is algebraic over $\mathbf{k}$. Since $\text{trdeg}_{\mathbf{k}} A = 1$ for any $A \in \text{KLND}(R)$, we obtain $|\,\text{KLND}(R)| > 1$. In the other direction, let A and A' be distinct elements of $\text{KLND}(R)$. As $\text{trdeg}_{\mathbf{k}} A = 1 = \text{trdeg}_{\mathbf{k}} A'$ and $A \cap A'$ is algebraically closed in R, it follows that $A \cap A'$ is algebraic over $\mathbf{k}$. Thus $\text{ML}(R)$ is algebraic over $\mathbf{k}$. ∎

Corollary 5.2.12 *If $R \in \mathcal{M}(\mathbf{k})$, then $\tilde{R} \in \mathcal{M}(\mathbf{k}')$ for some algebraic field extension $\mathbf{k}' \supseteq \mathbf{k}$ such that $\mathbf{k}' \subset \tilde{R}$. In particular, if $\mathbf{k}$ is algebraically closed, then $\mathrm{ML}(\tilde{R}) = \mathbf{k}$.*

Proof: As $R \in \mathcal{M}(\mathbf{k})$, we get $|\,\mathrm{KLND}(R)| > 1$ by Lemma 5.2.11. Let A_1 and A_2 be distinct elements of $\mathrm{KLND}(R)$. There exist $\tilde{A}_1, \tilde{A}_2 \in \mathrm{KLND}(\tilde{R})$ satisfying $\tilde{A}_i \cap R = A_i$ (cf. Lemma 5.2.10), so $|\,\mathrm{KLND}(\tilde{R})| > 1$. Hence $\mathrm{ML}(\tilde{R})$ is algebraic over $\mathbf{k}$ and is a field, say, $\mathrm{ML}(\tilde{R}) = \mathbf{k}'$. Then clearly, $\mathbf{k} \subseteq \mathbf{k}' \subset \tilde{R}$ and $\mathbf{k}'$ is algebraic over $\mathbf{k}$. ∎

5.3 Rationality and normality of ML-surfaces

Throughout this section, unless otherwise stated, $\mathbf{k}$ denotes an arbitrary field of characteristic zero.

Theorem 5.3.1 *Let $f : X \to Y$ be a dominant morphism of integral schemes of finite type over $\mathbf{k}$, and let $e = \dim X - \dim Y$.*

(1) *$f(X)$ is a constructible subset of Y. In particular, $f(X)$ contains a dense open subset of Y.*

(2) *There exists a dense open subset U of Y such that, for each $y \in U$, every irreducible component of $f^{-1}(y)$ has dimension e.*

Proof: Both assertions are due to Chevalley. See [33], Chapter II, Exercises 3.19 and 3.22. ∎

Lemma 5.3.2 *Let B be a $\mathbf{k}$-domain and $A \in \mathrm{KLND}(B)$. Let $\pi : \operatorname{Spec} B \to \operatorname{Spec} A$ be the canonical morphism determined by the inclusion map $A \hookrightarrow B$. Then there exists a nonempty open set $U \subseteq \operatorname{Spec} A$ such that $\pi^{-1}(\mathfrak{p}) \cong \mathbb{A}^1_{\kappa(\mathfrak{p})}$ for every $\mathfrak{p} \in U$, where*

$\kappa(\mathfrak{p})$ is the residue field $A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$. Furthermore, if $\mathbf{k}$ is algebraically closed and A is $\mathbf{k}$-affine, then $\pi^{-1}(\mathfrak{m}) \cong \mathbb{A}^1_{\kappa(\mathfrak{m})} = \mathbb{A}^1_\mathbf{k}$ for every closed point $\mathfrak{m}$ of U.

Proof: Pick a nonzero $D \in \operatorname{LND} B$ such that $A = \ker D$. Then $I := D(B) \cap A$ is a nonzero ideal of A and consequently $U := \operatorname{Spec} A \setminus V(I)$ is a nonempty open subset of $\operatorname{Spec} A$. We claim that $\pi^{-1}(\mathfrak{p}) \cong \mathbb{A}^1_{\kappa(\mathfrak{p})}$ for every $\mathfrak{p} \in U$. Let $\mathfrak{p} \in U$ and consider the commutative diagram

$$\begin{array}{ccc} B & \longrightarrow & \kappa(\mathfrak{p}) \otimes_A B \\ \uparrow & & \uparrow \\ A & \longrightarrow & \kappa(\mathfrak{p}) \end{array} \tag{5.3.1}$$

which defines the fibre of π over $\mathfrak{p}$, i.e., $\pi^{-1}(\mathfrak{p}) \cong \operatorname{Spec}(\kappa(\mathfrak{p}) \otimes_A B)$. As $\mathfrak{p} \notin V(I)$, there exists $a \in I$ such that $a \notin \mathfrak{p}$ (so in particular $a \neq 0$). Then we have

$$B_a = A_a^{[1]}. \tag{5.3.2}$$

Indeed, by the definition of I, there exists a preslice p of D such that $Dp = a$. Then $B_a = A_a[p] = A_a^{[1]}$ by 1.2 (3). As $a \notin \mathfrak{p}$, the horizontal arrows in the Diagram (5.3.1) factor through A_a and B_a respectively:

$$\begin{array}{ccccc} B & \hookrightarrow & B_a & \longrightarrow & \kappa(\mathfrak{p}) \otimes_A B \\ \uparrow & & \uparrow & & \uparrow \\ A & \hookrightarrow & A_a & \longrightarrow & \kappa(\mathfrak{p}) \end{array}$$

Moreover, we claim that

$$\kappa(\mathfrak{p}) \otimes_A B \cong \kappa(\mathfrak{p}) \otimes_{A_a} B_a. \tag{5.3.3}$$

Indeed, we know that $B_a \cong A_a \otimes_A B$, so

$$\kappa(\mathfrak{p}) \otimes_{A_a} B_a \cong \kappa(\mathfrak{p}) \otimes_{A_a} (A_a \otimes_A B) \cong \kappa(\mathfrak{p}) \otimes_A B.$$

Hence (5.3.3) is proved. Putting together (5.3.2) and (5.3.3), we obtain

$$\kappa(\mathfrak{p}) \otimes_A B \cong \kappa(\mathfrak{p}) \otimes_{A_a} B_a \cong \kappa(\mathfrak{p}) \otimes_{A_a} A_a^{[1]} \cong \kappa(\mathfrak{p})^{[1]}.$$

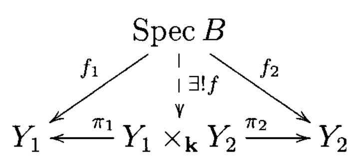
It follows that $\pi^{-1}(\mathfrak{p}) \cong \operatorname{Spec}(\kappa(\mathfrak{p}) \otimes_A B) \cong \operatorname{Spec}(\kappa(\mathfrak{p})^{[1]}) \cong \mathbb{A}^1_{\kappa(\mathfrak{p})}$.

Furthermore, assume that A is a finitely generated $\mathbf{k}$-algebra. If $\mathfrak{m}$ is a closed point of U, then $\kappa(\mathfrak{m})$ is simply $A/\mathfrak{m}$ and hence is a finitely generated $\mathbf{k}$-algebra. By Hilbert's Nullstellensatz, the field extension $\kappa(\mathfrak{m})/\mathbf{k}$ is algebraic. So if $\mathbf{k}$ is algebraically closed, $\kappa(\mathfrak{m}) = \mathbf{k}$ and the result follows by the previous case. ∎

The following fact is used in the proof of Lemma 5.3.6.

Lemma 5.3.3 *Let B be an affine $\mathbf{k}$-domain of dimension 2. Let A_1, A_2 be affine subalgebras of B of dimension 1, and let $\operatorname{Spec} A_1 \xleftarrow{f_1} \operatorname{Spec} B \xrightarrow{f_2} \operatorname{Spec} A_2$ be the morphisms determined by the inclusions $A_1 \hookrightarrow B \hookleftarrow A_2$. If B is algebraic over its subalgebra $\mathbf{k}[A_1 \cup A_2]$, then at most finitely many curves $C \subset \operatorname{Spec} B$ are such that $f_1(C)$ is a point of $\operatorname{Spec} A_1$ and $f_2(C)$ is a point of $\operatorname{Spec} A_2$.*

Proof: Set $Y_i = \operatorname{Spec} A_i$ (for $i = 1, 2$). Consider the product of affine schemes $Y_1 \times_{\mathbf{k}} Y_2 \cong \operatorname{Spec}(A_1 \otimes_{\mathbf{k}} A_2)$. By the universal property of the product, there exists a unique morphism $f : \operatorname{Spec} B \to Y_1 \times_{\mathbf{k}} Y_2$ such that the following diagram commutes:



We leave it to the reader to verify that

no curve in $Y_1 \times_{\mathbf{k}} Y_2$ is shrunk to a point by each of the projection morphisms $Y_1 \xleftarrow{\pi_1} Y_1 \times_{\mathbf{k}} Y_2 \xrightarrow{\pi_2} Y_2$. $(*)$

Now consider a curve $C \subset \operatorname{Spec} B$ which is shrunk to a point by each of f_1, f_2. As C has closed points and the image of a closed point is a closed point, $f_i(C)$ is in fact a closed point of $\operatorname{Spec} A_i$ for each $i = 1, 2$. So, for each $i = 1, 2$, $\pi_i(f(C))$ is a closed point of $\operatorname{Spec} A_i$. It follows that the closure $\overline{f(C)}$ of $f(C)$ in $Y_1 \times_{\mathbf{k}} Y_2$ is shrunk to a closed point by each π_i. By $(*)$, $\overline{f(C)}$ cannot be a curve; as $\overline{f(C)}$ is an irreducible closed subset of $Y_1 \times_{\mathbf{k}} Y_2$ of dimension 0 or 1, it must be a closed point of $Y_1 \times_{\mathbf{k}} Y_2$. We have shown that any curve in Spec B which is shrunk to a point by each of f_1, f_2 is already shrunk to a closed point by f. So it is enough to prove the following claim:

$$\text{At most finitely many curves in Spec } B \text{ are shrunk to a point by } f. \qquad (**)$$

Let $A_1 \otimes_{\mathbf{k}} A_2 \xrightarrow{\varphi} B$ be the homomorphism of $\mathbf{k}$-algebras corresponding to f and let $A \subseteq B$ be the image of φ. For each $i = 1, 2$, the composite $A_i \to A_1 \otimes_{\mathbf{k}} A_2 \xrightarrow{\varphi} B$ is the inclusion map $A_i \hookrightarrow B$, so $\mathbf{k}[A_1 \cup A_2] \subseteq A \subseteq B$ and hence B is algebraic over A. So $\dim A = 2 = \dim B$. Applying the functor Spec to $A_1 \otimes_{\mathbf{k}} A_2 \twoheadrightarrow A \hookrightarrow B$ gives the commutative diagram

$$\begin{array}{ccc} & & Y_1 \times_{\mathbf{k}} Y_2 \\ & \overset{f}{\nearrow} & \uparrow h \\ \operatorname{Spec} B & \xrightarrow[g]{} & \operatorname{Spec} A \end{array}$$

where f is as before, h is a closed immersion, and g is a dominant morphism of surfaces. Now it is well-known that a dominant morphism of surfaces shrinks at most finitely many curves (for instance, this follows from Theorem 5.3.1). As h is in particular injective, $(**)$ is proved and we are done. ∎

Lemma 5.3.4 *Let $K/\mathbf{k}$ be an extension of fields of characteristic zero and let B be a $\mathbf{k}$-algebra. Then each $D \in \text{LND}_{\mathbf{k}}(B)$ extends uniquely to a locally nilpotent K-derivation $\overline{D} : K \otimes_{\mathbf{k}} B \to K \otimes_{\mathbf{k}} B$. Moreover, if $A = \ker D$ then $\ker \overline{D} = K \otimes_{\mathbf{k}} A$.*

Proof: The exact sequence $0 \to A \to B \xrightarrow{D} B$ in the category of $\mathbf{k}$-vector spaces gives rise to the exact sequence $0 \to K \otimes_{\mathbf{k}} A \to K \otimes_{\mathbf{k}} B \xrightarrow{\overline{D}} K \otimes_{\mathbf{k}} B$ in the category of K-vector spaces, where $\overline{D}$ is the K-linear map defined by

$$\overline{D}(\lambda \otimes b) = \lambda \otimes D(b), \quad \text{for all } \lambda \in K \text{ and } b \in B.$$

It is easily verified that $\overline{D} \in \text{LND}_K(K \otimes_{\mathbf{k}} B)$ and that $\overline{D}$ is the only extension of D. ∎

Lemma 5.3.5 [11, Lemma 3.7] *Let R be an integral domain containing a field $\mathbf{k}$ of characteristic zero. If R is normal and $\mathrm{ML}(R) = \mathbf{k}$, then for any field extension K of $\mathbf{k}$ we have:*

(a) *$K \otimes_{\mathbf{k}} R$ is an integral domain.*

(b) $\mathrm{ML}(K \otimes_{\mathbf{k}} R) = K$.

Lemma 5.3.6 *Let $B \in \mathcal{M}(\mathbf{k})$. If B is normal and $A \in \mathrm{KLND}(B)$, then $A \cong \mathbf{k}^{[1]}$.*

Proof: Let $A \in \mathrm{KLND}(B)$. Note that A is a normal domain of transcendence degree 1 over $\mathbf{k}$. In view of Lemma 1.8.11, it suffices to prove that $A \subseteq \mathbf{k}^{[1]}$. First consider the case when $\mathbf{k}$ is algebraically closed. As $B \in \mathcal{M}(\mathbf{k})$, there exists $A' \in \mathrm{KLND}(B)$ such that $A \neq A'$. Let $\mathrm{Spec}\, A \xleftarrow{f} \mathrm{Spec}\, B \xrightarrow{f'} \mathrm{Spec}\, A'$ be the canonical morphisms. Observe that A and A' satisfy the hypotheses of Lemma 5.3.3. Hence by Lemmas 5.3.2 and 5.3.3, we can pick a closed point $y \in \mathrm{Spec}\, A'$ satisfying:

1. $f'^{-1}(y) \cong \mathbb{A}^1_{\mathbf{k}}$.
2. $f'^{-1}(y)$ is not shrunk to a point by f.

Then the composite $f'^{-1}(y) \hookrightarrow \mathrm{Spec}\, B \xrightarrow{f} \mathrm{Spec}\, A$ is a dominant morphism of curves $\mathbb{A}^1_{\mathbf{k}} \to \mathrm{Spec}\, A$. This implies that A is a subalgebra of a polynomial ring $\mathbf{k}^{[1]}$, so we are done in this case.

Next we prove the general case. By Lemma 5.3.5, $\mathcal{B} := \bar{\mathbf{k}} \otimes_{\mathbf{k}} B$ is an integral domain and $\mathrm{ML}(\mathcal{B}) = \bar{\mathbf{k}}$. If $\tilde{\mathcal{B}}$ denotes the normalization of $\mathcal{B}$, then $\mathrm{ML}(\tilde{\mathcal{B}}) = \bar{\mathbf{k}}$ by Corollary 5.2.12. Applying previous case, each element of $\mathrm{KLND}(\tilde{\mathcal{B}})$ is isomorphic to $\bar{\mathbf{k}}^{[1]}$. Given $A \in \mathrm{KLND}(B)$, $\bar{\mathbf{k}} \otimes_{\mathbf{k}} A \in \mathrm{KLND}(\mathcal{B})$ and there exists $D \in \mathrm{LND}(\tilde{\mathcal{B}})$ such that $\ker D \cap \mathcal{B} = \bar{\mathbf{k}} \otimes_{\mathbf{k}} A$ (cf. Lemma 5.2.10). As $\ker D \cong \bar{\mathbf{k}}^{[1]}$, it follows that $\bar{\mathbf{k}} \otimes_{\mathbf{k}} A \subseteq \bar{\mathbf{k}}^{[1]}$. Then $A \subseteq \mathbf{k}^{[1]}$ follows by Lemma 1.8.13. ∎

5.3.7 Let $B \in \mathcal{M}(\mathbf{k})$ and $A \in \text{KLND}(B)$. If B is not normal, A need not be a polynomial ring; in fact, A is not necessarily a polynomial curve over $\mathbf{k}$. To justify these claims, consider the following examples.

Example 5.3.7.1 Consider the 2-dimensional affine $\mathbf{k}$-domain $B = \mathbf{k}[x, xy, y^2, y^3]$. Then $D = x\frac{\partial}{\partial y}$, $E = y^2\frac{\partial}{\partial x}$ are two locally nilpotent derivations of B satisfying $\ker D \cap \ker E = \mathbf{k}$, so $\text{ML}(B) = \mathbf{k}$. Observe that $\ker E = \mathbf{k}[y^2, y^3] \not\cong \mathbf{k}^{[1]}$.

Example 5.3.7.2 Let $B = \mathbb{R}[x, ix, y, iy]$, where $i^2 = -1$. Then B is a two-dimensional affine $\mathbb{R}$-domain. Note that the normalization of B is $\tilde{B} = \mathbb{C}[x, y]$. Also note that $D_1 = x\frac{\partial}{\partial y}$ and $D_2 = y\frac{\partial}{\partial x}$ are two nonzero locally nilpotent derivations of $\mathbb{C}[x, y]$. Furthermore, $D_i(B) \subseteq B$ for every $i = 1, 2$. Let $\delta_i := D_i|_B$. Then $\delta_i \in \text{LND}(B)$ and $A_i := \ker \delta_i = \ker D_i \cap B$ for each $i = 1, 2$. It follows that $A_1 = \mathbb{C}[x] \cap B = \mathbb{R}[x, ix]$, and $A_2 = \mathbb{R}[y, iy]$. So $\text{ML}(B) = \mathbb{R}$. As $\text{Frac}(A_1) = \mathbb{C}(x) = \mathbb{C}^{(1)}$, it follows by Lemma 1.8.10 that A_1 is not a polynomial curve over $\mathbb{R}$.

5.3.7.3 In the previous example, $B \in \mathcal{M}(\mathbb{R})$ and $\text{Frac}\, B = \mathbb{C}(x, y) = \mathbb{C}^{(2)}$. This proves that an affine ML-surface is not necessarily rational. Moreover, one can see that $\text{Spec}\, B$ has only one $\mathbb{R}$-rational point and that $\mathbb{C} \otimes_{\mathbb{R}} B$ is not an integral domain.

The following gives an interesting criterion for the rationality of an affine ML-surface. In particular, the following lemma proves that a normal ML-surface over an arbitrary field of characteristic zero is rational.

Lemma 5.3.8 *Let $\mathbf{k}$ be a field of characteristic zero and $R \in \mathcal{M}(\mathbf{k})$. Then the following are equivalent:*

(i) *R is rational, i.e.,* $\text{Frac}\, R = \mathbf{k}^{(2)}$.

(ii) *$\mathbf{k}$ is algebraically closed in* $\text{Frac}\, R$.

(iii) $\text{ML}(\tilde{R}) = \mathbf{k}$, *where $\tilde{R}$ is the normalization of R.*

Moreover, if R is normal then (i)-(iii) *are satisfied.*

Proof: Assume that (i) holds. As $\mathbf{k}$ is algebraically closed in $\mathbf{k}^{(2)}$, it is clear that $\mathbf{k}$ is algebraically closed in $\operatorname{Frac} R$ and (ii) holds. Now assume that (ii) holds. Recall by Corollary 5.2.12 that $\mathrm{ML}(\tilde{R}) = \mathbf{k}'$ for some field $\mathbf{k}'$ that satisfies $\mathbf{k} \subseteq \mathbf{k}' \subset \tilde{R}$ and $\mathbf{k}'$ is algebraic over $\mathbf{k}$. As $\tilde{R} \subset \operatorname{Frac} R$ and $\mathbf{k}$ is algebraically closed in $\operatorname{Frac} R$, it follows that $\mathbf{k}$ is algebraically closed in $\mathbf{k}'$. Hence $\mathrm{ML}(\tilde{R}) = \mathbf{k}$ and (iii) holds. Finally, assume that (iii) holds and recall that $\mathrm{ML}(R) = \mathbf{k}$. Given $A \in \textsc{klnd}(R)$, there exists $\tilde{A} \in \textsc{klnd}(\tilde{R})$ such that $\tilde{A} \cap R = A$ (cf. Lemma 5.2.10). As $\tilde{A} = \mathbf{k}^{[1]}$ by Lemma 5.3.6, it follows that $\mathbf{k} \subset A \subseteq \mathbf{k}^{[1]}$. So Lüroth's Theorem implies that $\operatorname{Frac} A = \mathbf{k}^{(1)}$. Then $\operatorname{Frac} R = (\operatorname{Frac} A)^{(1)} = \mathbf{k}^{(2)}$ follows by 1.2 (5), proving assertion (i).
The last assertion is clear, as (iii) holds whenever R is normal. ∎

Remark 5.3.9 Let $R \in \mathcal{M}(\mathbf{k})$ and assume that R is rational. Then each element A of $\textsc{klnd}(R)$ is a polynomial curve. Indeed, $\mathrm{ML}(\tilde{R}) = \mathbf{k}$ by Lemma 5.3.8. Also, Lemma 5.2.10 implies that there exists $\tilde{A} \in \textsc{klnd}(\tilde{R})$ such that $\tilde{A} \cap R = A$. Since $\tilde{A} \cong \mathbf{k}^{[1]}$ by Lemma 5.3.6, we have $\mathbf{k} \subset A \subseteq \mathbf{k}^{[1]}$, so A is a polynomial curve.

5.4 Completion of surfaces and fibrations

Throughout this section, we fix $\mathbf{k}$ to be an algebraically closed field of characteristic zero. All varieties are assumed to be $\mathbf{k}$-varieties. In this section, we state some properties of affine normal surfaces, fibrations on such surfaces, and completions of such surfaces. The material of this section is known.

Definition 5.4.1 Let S be a complete normal surface, and let $C_1, \dots, C_n$ be distinct irreducible curves in S (where $n \geq 1$) satisfying:

(i) $C_1 \cup \cdots \cup C_n$ is included in $S \setminus \operatorname{Sing}(S)$;

(ii) each curve C_i is isomorphic to $\mathbb{P}^1$;

(iii) if $i \neq j$ and $C_i \cap C_j \neq \varnothing$ then $C_i \cap C_j$ is one point P and the two curves meet transversally at P (i.e., the local intersection number of C_i and C_j at P is 1);

(iv) if i, j, k are distinct then $C_i \cap C_j \cap C_k = \varnothing$.

Then we use the following terminology:

- The formal sum $D = \sum_{i=1}^{n} C_i$ is called an *SNC-divisor* of S.
- The closed subset $C_1 \cup \cdots \cup C_n$ of S is called the *support* of D and is denoted $\operatorname{Supp}(D)$.
- The curves $C_1, \ldots, C_n$ are called the *irreducible components* of D.

Remark 5.4.2 An SNC-divisor of S is in particular a Weil divisor of S, but for our purposes it is not necessary to define the notion of a Weil divisor.

Definition 5.4.3 1. Let Y be a scheme and $\mathcal{P}$ a property of a point. The phrase "the general point of Y has property $\mathcal{P}$" (or "$\mathcal{P}(y)$ is true for general $y \in Y$") means that there exists a dense open subset U of Y such that every point $y \in U$ has property $\mathcal{P}$.

2. Let $f : X \to Y$ be a morphism of schemes and $\mathcal{P}$ a property of a fibre. The phrase "the general fibre of f has property $\mathcal{P}$" means that there exists a dense open subset U of Y such that, for every point $y \in U$, the fibre $f^{-1}(y)$ has property $\mathcal{P}$. Note that if the general fibre of f has $\mathcal{P}$ and Y is an integral scheme, then the generic fibre of f (i.e., the fibre of f over the generic point of Y) has $\mathcal{P}$.

Definition 5.4.4 An $\mathbb{A}^1$*-fibration* (respectively, a $\mathbb{P}^1$*-fibration*) on a surface S is a surjective morphism $\rho : S \to Z$ on a nonsingular curve Z whose general fibres are isomorphic to $\mathbb{A}^1$ (respectively, to $\mathbb{P}^1$). For our purposes, we will always consider $\mathbb{A}^1$-fibrations whose codomain Z is $\mathbb{A}^1$.

Recall that $\mathbf{k}$ is an algebraically closed field of characteristic zero.

Lemma 5.4.5 *Let $B \in \mathcal{M}(\mathbf{k})$. If B is normal and $A \in \text{KLND}(B)$, then the canonical morphism $\rho : \operatorname{Spec} B \to \operatorname{Spec} A$ induced by the inclusion map $A \hookrightarrow B$ is an $\mathbb{A}^1$-fibration.*

Proof: If $A \in \text{KLND}(B)$, $A = \mathbf{k}[t] = \mathbf{k}^{[1]}$ by Lemma 5.3.6. The inclusion $A \hookrightarrow B$ gives rise to the canonical morphism $\rho : S \to \mathbb{A}^1 = \operatorname{Spec}(\mathbf{k}[t])$. The general fibre of ρ is an affine line by Lemma 5.3.2, so it is enough to show that ρ is surjective. Given a maximal ideal $\mathfrak{p} = (t - \lambda)$ of A (where $\lambda \in \mathbf{k}$), consider its extension $\mathfrak{p}B$ in B. As $B \in \mathcal{M}(\mathbf{k})$, $B^* = \mathbf{k}^*$ and $t - \lambda \notin B^*$. Hence there exists a maximal ideal P of B such that $t - \lambda \in P$. Then $P \cap A = \mathfrak{p}$, so $\mathfrak{p}$ is in the image of ρ. ∎

Definition 5.4.6 Let S be an affine normal surface and $\rho : S \to \mathbb{A}^1$ an $\mathbb{A}^1$-fibration. By a *completion of the pair* (S, ρ), we mean a commutative diagram of morphisms of algebraic varieties

$$\begin{array}{ccc} S & \hookrightarrow & \bar{S} \\ \rho \downarrow & & \downarrow \bar{\rho} \\ \mathbb{A}^1 & \hookrightarrow & \mathbb{P}^1 \end{array} \tag{5.4.1}$$

such that the "$\hookrightarrow$" are open immersions, $\bar{S}$ is a complete normal surface, and $\bar{S} \setminus S$ is the support of an SNC-divisor of $\bar{S}$.

5.4.7 It is well-known that given any affine normal surface S and an $\mathbb{A}^1$-fibration $\rho : S \to \mathbb{A}^1$, there exists a completion of (S, ρ). Indeed, such S can be embedded as an open set into a complete surface S' by Nagata's theorem. By resolving the singularities of S' which are in $S' \setminus S$, we get a complete normal surface S'' which contains S as an open set and such that $S'' \setminus S$ is the support of an SNC-divisor of S'' . An $\mathbb{A}^1$-fibration ρ on S gives rise to a rational map $S'' \dashrightarrow \mathbb{P}^1$. By a suitable succession of blowings-up of points of $S'' \setminus S$, we arrive at a complete normal surface $\bar{S}$ and a morphism $\bar{\rho} : \bar{S} \to \mathbb{P}^1$ such that $(\bar{S}, \bar{\rho})$ is a completion of the pair (S, ρ).

Furthermore, if S is an affine normal surface and ρ_1, ρ_2 are two $\mathbb{A}^1$-fibrations on S, then we can find a complete normal surface $\bar{S}$ such that $(\bar{S}, \bar{\rho}_i)$ is a completion of (S, ρ_i), respectively, for each $i = 1, 2$.

Setup 5.4.8 Throughout Paragraph 5.4.8, we assume:

(i) S is an affine normal surface.

(ii) $\rho : S \longrightarrow \mathbb{A}^1$ is an $\mathbb{A}^1$-fibration.

(iii) $(\bar{S}, \bar{\rho})$ is a completion of (S, ρ), with notation as in Diagram (5.4.1); we let D be the SNC-divisor of $\bar{S}$ whose support is $\bar{S} \setminus S$.

As $\bar{S}$ is complete, $\bar{\rho}$ is a closed map. So given any curve $C \subset \bar{S}$, $\bar{\rho}(C)$ is either a point or all of $\mathbb{P}^1$. Accordingly we have:

Definition 5.4.8.1 A curve $C \subset \bar{S}$ is *$\bar{\rho}$-vertical* if $\bar{\rho}(C)$ is a point. Otherwise, we say that the curve is *$\bar{\rho}$-horizontal*. Thus $C \subset \bar{S}$ is $\bar{\rho}$-horizontal if and only if $\bar{\rho}(C) = \mathbb{P}^1$.

Lemma 5.4.8.2 *Let the setup be as in* 5.4.8. *Then the following hold:*

(a) *The morphism $\bar{\rho}$ is surjective. For a general point $z \in \mathbb{P}^1$, $\bar{\rho}^{-1}(z) \cap S \cong \mathbb{A}^1$ and $\bar{\rho}^{-1}(z) \cong \mathbb{P}^1$. Consequently, $\bar{\rho}$ is a $\mathbb{P}^1$-fibration.*

(b) *A general fibre of $\bar{\rho}$ intersects D in exactly one point.*

(c) *Exactly one irreducible component of D is $\bar{\rho}$-horizontal, and a general fibre of $\bar{\rho}$ intersects this component in exactly one point.*

Proof: As ρ is surjective and $\bar{S}$ is complete, it follows by the commutativity of Diagram (5.4.1) that $\bar{\rho}$ is surjective. Since ρ is an $\mathbb{A}^1$-fibration and Diagram (5.4.1) commutes, a general point $z \in \mathbb{P}^1$ satisfies $\bar{\rho}^{-1}(z) \cap S = \rho^{-1}(z) \cong \mathbb{A}^1$. Note that $\bar{\rho}^{-1}(z)$ is a union of curves for a general point $z \in \mathbb{P}^1$ (cf. Theorem 5.3.1), so write $\bar{\rho}^{-1}(z) = C_1 \cup \cdots \cup C_r$ for some irreducible curves $C_i \subset \bar{S}$. As $\bar{\rho}^{-1}(z) \cap S \cong \mathbb{A}^1$, it follows by irreducibility of $\mathbb{A}^1$ that exactly one $i \in \{1, \ldots, r\}$ satisfies $C_i \cap S \cong \mathbb{A}^1$ and $C_j \cap S = \varnothing$ for $j \neq i$. We say that $z \in \mathbb{P}^1$ is a *bad point* if a component of $\bar{\rho}^{-1}(z)$ is contained in D. As D has only finitely many components and a component of D cannot be mapped to two points of $\mathbb{P}^1$, it follows that the set of bad points of $\mathbb{P}^1$ is finite. Ignoring this finite set of bad points of $\mathbb{P}^1$, we conclude that a general fibre of $\bar{\rho}$ is a complete irreducible curve whose intersection with S is an affine line. This complete curve is nonsingular by the well-known "characteristic zero Bertini Theorem", hence it is isomorphic to $\mathbb{P}^1$. It follows that $\bar{\rho}$ is a $\mathbb{P}^1$-fibration and (a) is proved.

As the general fibre of $\bar{\rho}$ satisfies $\bar{\rho}^{-1}(z) \cong \mathbb{P}^1$ and $\bar{\rho}^{-1}(z) \cap S \cong \mathbb{A}^1$, it follows that $\bar{\rho}^{-1}(z)$ intersects D in *exactly* one point, proving assertion (b).

Finally, we prove (c). If all components of D were $\bar{\rho}$-vertical, $\bar{\rho}(D)$ would be a finite subset of $\mathbb{P}^1$. Then $(\bar{\rho})^{-1}(z) \subseteq S$ for a general point $z \in \mathbb{P}^1$ contradicting (b). Hence there exists an irreducible component of D which is $\bar{\rho}$-horizontal. If H_1 and H_2 are two $\bar{\rho}$-horizontal components, then $\bar{\rho}(H_1) = \mathbb{P}^1 = \bar{\rho}(H_2)$. For a general point $z \in \mathbb{P}^1$, there exist points $P_1 \in H_1$ and $P_2 \in H_2$ such that $\bar{\rho}(P_1) = z = \bar{\rho}(P_2)$. As $(\bar{\rho})^{-1}(z)$ intersects D in only one point, $P_1 = P_2 \in H_1 \cap H_2$. Then $z \in \bar{\rho}(H_1 \cap H_2)$, i.e., a general point of $\mathbb{P}^1$ belongs to $\bar{\rho}(H_1 \cap H_2)$. This is absurd, because $\bar{\rho}(H_1 \cap H_2)$ is a finite set. Hence $\bar{\rho}$ has a unique horizontal component, say H. Since a general fibre of $\bar{\rho}$ meets D in exactly one point and $\bar{\rho}(H) = \mathbb{P}^1$, it follows that a general fibre of $\bar{\rho}$ meets H in exactly one point. ∎

5.5 Geometry of surfaces in the class $\mathcal{M}(\mathbf{k})$

In this section, $\mathbf{k}$ is an arbitrary field of characteristic zero except in 5.5.1 and Corollary 5.5.3, where it is assumed to be algebraically closed.

Setup 5.5.1 The following assumptions and notations are valid throughout Paragraph 5.5.1. Suppose that $\mathbf{k}$ is algebraically closed. Fix $B \in \mathcal{M}(\mathbf{k})$, assume that B is normal, and let $S = \operatorname{Spec} B$. Consider distinct elements $A_1, A_2 \in \text{KLND}(B)$ and recall from Lemma 5.3.6 that $A_i \cong \mathbf{k}^{[1]}$ for $i = 1, 2$. Let $\rho_i : S \to \mathbb{A}^1$ be the morphism determined by the inclusion $A_i \hookrightarrow B$ for $i = 1, 2$. It follows from Lemma 5.4.5 that ρ_1 and ρ_2 are $\mathbb{A}^1$-fibrations, and Lemma 5.3.3 implies that at most finitely many curves in S are contracted to a point by each of ρ_1, ρ_2 (consequently, the general fibre of ρ_1 is not shrunk to a point by ρ_2). Choose a complete normal surface $\bar{S}$ and morphisms $\bar{\rho}_1, \bar{\rho}_2 : \bar{S} \to \mathbb{P}^1$ such that, for each $i = 1, 2$, $(\bar{S}, \bar{\rho}_i)$ is a completion of (S, ρ_i) in the sense of 5.4.6. We also consider the following diagram:

$$\begin{array}{ccc} S & \hookrightarrow & \bar{S} \\ {\scriptstyle \rho_1}\downarrow\downarrow{\scriptstyle \rho_2} & & {\scriptstyle \bar{\rho}_1}\downarrow\downarrow{\scriptstyle \bar{\rho}_2} \\ \mathbb{A}^1 & \hookrightarrow & \mathbb{P}^1 \end{array} \qquad (5.5.1)$$

Let ∞ be such that $\mathbb{P}^1 = \mathbb{A}^1 \cup \{\infty\}$ in Diagram (5.5.1). For $i = 1, 2$, let H_i be the unique irreducible component of $D = \bar{S} \setminus S$ which is $\bar{\rho}_i$-horizontal. (See Lemma 5.4.8.2.)

Lemma 5.5.1.1 *We have: $\bar{\rho}_1(H_2) = \{\infty\}$, and $\bar{\rho}_2(H_1) = \{\infty\}$. In particular, H_1 and H_2 are distinct.*

Proof: Recall that $H_i \subset D$ and $\bar{\rho}_i(H_i) = \mathbb{P}^1$ for each $i = 1, 2$. For a general point $z_1 \in \mathbb{P}^1$, $\bar{\rho}_1^{-1}(z_1) = C_1$ is an irreducible curve in $\bar{S}$ which intersects H_1 in a unique point, say Q, and $C_1 \cap S$ is not shrunk to a point by ρ_2. Then $\bar{\rho}_2(C_1) = \mathbb{P}^1$. Choose $Q_1 \in C_1$ such that $\bar{\rho}_2(Q_1) = \{\infty\}$. Clearly, $Q_1 \in D$. Since C_1 meets D in exactly one point, $C_1 \cap D = \{Q_1\}$. Consequently, $\{Q\} = C_1 \cap H_1 \subseteq C_1 \cap D = \{Q_1\}$. It follows that $\{Q_1\} = C_1 \cap H_1$. Repeating this process for infinitely many points z_i of $\mathbb{P}^1$, we get infinitely many points $Q_i \in H_1$ satisfying $\bar{\rho}_1(Q_i) = z_i$ and $\bar{\rho}_2(Q_i) = \{\infty\}$. So we conclude that $\bar{\rho}_2(H_1) = \{\infty\}$. Similarly, we can prove that $\bar{\rho}_1(H_2) = \{\infty\}$. As

$\bar{\rho}_1(H_1) = \mathbb{P}^1 = \bar{\rho}_2(H_2)$, it follows immediately that H_1 and H_2 are distinct. ∎

Proposition 5.5.1.2 *There does not exist an irreducible curve $C \subset S$ such that $\rho_1(C)$ and $\rho_2(C)$ are points.*

Proof: By contradiction, suppose that there exists an irreducible curve C_0 of S such that $\rho_1(C_0) = a_1$ and $\rho_2(C_0) = a_2$ for some points $a_i \in \mathbb{A}^1$. Consider $C := \bar{C}_0$, the closure of C_0 in $\bar{S}$. Then C is a curve in $\bar{S}$ such that $C \cap D \neq \varnothing$, $\bar{\rho}_1(C) = a_1$, and $\bar{\rho}_2(C) = a_2$ (where $a_1, a_2 \in \mathbb{P}^1 \setminus \{\infty\}$). Since D is connected, there is an integer $k \geq 1$ and a sequence $D_1, \ldots, D_k$ of irreducible components of D satisfying:

- For each $1 \leq i < k$, D_i is $\bar{\rho}_1$-vertical and $\bar{\rho}_2$-vertical, and $D_k \in \{H_1, H_2\}$.
- $C \cap D_1 \neq \varnothing$, and $D_i \cap D_{i+1} \neq \varnothing$ (for $1 \leq i < k$).

Note that $\bar{\rho}_j(D_k) = \infty$ for some $j \in \{1, 2\}$. Since $C \cup D_1 \cup \cdots \cup D_k$ is connected, it follows that $\bar{\rho}_j(C \cup D_1 \cup \cdots \cup D_k)$ is connected and is a finite set of points, i.e., is one point. But $a_j, \infty \in \bar{\rho}_j(C \cup D_1 \cup \cdots \cup D_k)$, so we obtain a contradiction. ∎

For the remainder of this section, assume that $\mathbf{k}$ is an arbitrary field of characteristic zero.

Definition 5.5.2 Let B be an integral domain of characteristic zero. We say that B has property $(*)$ if, for any height 1 ideal I of B (with $I \neq B$) and $A_1, A_2 \in \text{KLND}(B)$, the conditions $I \cap A_1 \neq 0$ and $I \cap A_2 \neq 0$ imply $A_1 = A_2$.

Corollary 5.5.3 *Suppose that $\mathbf{k}$ is algebraically closed and that $B \in \mathcal{M}(\mathbf{k})$ is normal. Then B has property $(*)$.*

Proof: By contradiction, suppose that there exist distinct $A_1, A_2 \in \text{KLND}(B)$ and a height 1 ideal I of B such that $I \cap A_i \neq 0$ for $i = 1, 2$. Pick a height 1 prime ideal $\mathfrak{p}$ of B such that $\mathfrak{p} \supseteq I$, and note that $\mathfrak{p} \cap A_i \neq 0$ for $i = 1, 2$. So the irreducible curve $C = V(\mathfrak{p}) \subset \operatorname{Spec} B$ is mapped to a point by each canonical morphism

$\rho_i : \operatorname{Spec} B \to \operatorname{Spec} A_i$ $(i = 1, 2)$. This contradicts Proposition 5.5.1.2. ∎

Notation 5.5.4 Let $B \subseteq B'$ be integral domains of characteristic zero. We write $B \lhd B'$ to indicate that B' is integral over B and that, for each $A \in \text{KLND}(B)$, there exists $A' \in \text{KLND}(B')$ such that $A' \cap B = A$. Clearly, $\lhd$ is a transitive relation.

Lemma 5.5.5 *Let B, B' be integral domains of characteristic zero such that $B \lhd B'$. If B' has property $(*)$, then so does B.*

Proof: Let $I \neq B$ be a height 1 ideal of B and let $A_1, A_2 \in \text{KLND}(B)$ satisfy $I \cap A_i \neq 0$. As B' is integral over B, $IB' \neq B'$ and $\operatorname{ht} IB' = 1$ by Lemma 5.2.3. Since $B \lhd B'$, there exist $A'_1, A'_2 \in \text{KLND}(B')$ such that $A'_i \cap B = A_i$ for $i = 1, 2$. Moreover, $A'_i \cap IB' \supseteq A_i \cap I \neq 0$. Since B' has property $(*)$, it follows that $A'_1 = A'_2$. Consequently, $A_1 = A_2$. ∎

Theorem 5.5.6 *Each element B of $\mathcal{M}(\mathbf{k})$ has property $(*)$.*

Proof: If $\tilde{B}$ denotes the normalization of B, $B \lhd \tilde{B}$ follows by Lemma 5.2.10. Moreover, Corollary 5.2.12 implies that $\tilde{B} \in \mathcal{M}(\mathbf{k}')$ for some field $\mathbf{k}'$. As $B \lhd \tilde{B}$, it suffices to prove the theorem when B is normal by Lemma 5.5.5.

If B is normal, $\mathcal{B} = \bar{\mathbf{k}} \otimes_{\mathbf{k}} B$ is an integral domain and $\text{ML}(\mathcal{B}) = \bar{\mathbf{k}}$ by Lemma 5.3.5. Then $\tilde{\mathcal{B}} \in \mathcal{M}(\bar{\mathbf{k}})$ by Corollary 5.2.12, so $\tilde{\mathcal{B}}$ has property $(*)$ by Corollary 5.5.3. It suffices to prove that $B \lhd \tilde{\mathcal{B}}$ because then the result follows by Lemma 5.5.5.

As $\bar{\mathbf{k}}$ is integral over $\mathbf{k}$, it follows by Lemma 5.2.5 that $\bar{\mathbf{k}} \otimes_{\mathbf{k}} B$ is integral over $\mathbf{k} \otimes_{\mathbf{k}} B \cong B$. Furthermore, given $A \in \text{KLND}(B)$, $\bar{A} = \bar{\mathbf{k}} \otimes_{\mathbf{k}} A$ belongs to $\text{KLND}(\mathcal{B})$ by Lemma 5.3.4 and satisfies $\bar{A} \cap (\mathbf{k} \otimes_{\mathbf{k}} B) = A$. This proves that $B \lhd \mathcal{B}$. Finally, $\mathcal{B} \lhd \tilde{\mathcal{B}}$ and $\lhd$ is transitive, so it follows that $B \lhd \tilde{\mathcal{B}}$. ∎

Remark 5.5.7 It follows that every two-dimensional affine **k**-domain has property $(*)$. Indeed, let B be such a ring. If $|\,\text{KLND}(B)| \leq 1$, then it is trivial that B has property $(*)$. If $|\,\text{KLND}(B)| > 1$ then $B \in \mathcal{M}(\mathbf{k}')$ for some field $\mathbf{k}'$ (by Corollary 5.2.12), so B has property $(*)$ by the above theorem.

Theorem 5.5.8 [48, Theorem 73, p.246] *Let A an affine domain containing a field. Then $U = \{\, \mathfrak{p} \in \operatorname{Spec} A \mid A_{\mathfrak{p}}$ is a regular local ring$\,\}$ is a nonempty open subset of the affine scheme $X = \operatorname{Spec} A$.*

Proposition 5.5.9 *Let B be an affine **k**-domain. If $\mathfrak{p}$ is a height 1 prime ideal of B such that $B_{\mathfrak{p}}$ is not regular, then $D(\mathfrak{p}) \subseteq \mathfrak{p}$ for every $D \in \text{LND}(B)$.*

Proof: The set $T = \{\, \mathfrak{p} \in \operatorname{Spec} B \mid B_{\mathfrak{p}} \text{ is not regular} \,\}$ is a closed and proper subset of $X := \operatorname{Spec} B$. For every $\mathfrak{p} \in T$ satisfying $\operatorname{ht} \mathfrak{p} = 1$, the closure $\overline{\{\mathfrak{p}\}}$ is an irreducible component of T of codimension 1. Moreover, $\mathfrak{p}$ is the unique generic point of that component. As T has only finitely many irreducible components, it follows that T contains only finitely many prime ideals of height 1. Denote these prime ideals by $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$.

Pick $\mathfrak{p} \in \{\mathfrak{p}_1, \cdots, \mathfrak{p}_n\}$ and $D \in \text{LND}(B)$. We will prove that $D(\mathfrak{p}) \subseteq \mathfrak{p}$. In view of Lemma 5.2.9, it is enough to show that

$$e^{\lambda D}(\mathfrak{p}) \subseteq \mathfrak{p} \text{ for some nonzero } \lambda \in \mathbf{k}. \tag{5.5.2}$$

As the group $\operatorname{Aut}(B)$ acts on the set T, it follows that it acts on $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$. Furthermore, $\mathbf{k} = \bigcup_{i=1}^{n} \{\, \lambda \in \mathbf{k} \mid e^{\lambda D}(\mathfrak{p}) = \mathfrak{p}_i \,\}$. Since $\mathbf{k}$ is infinite, there exists $i \in \{1, \ldots, n\}$ such that $\Omega := \{\, \lambda \in \mathbf{k} \mid e^{\lambda D}(\mathfrak{p}) = \mathfrak{p}_i \,\}$ is infinite. Pick distinct elements λ_1, λ_2 of Ω. Then $e^{(-\lambda_2+\lambda_1)D}(\mathfrak{p}) \subseteq \mathfrak{p}$. So (5.5.2) holds. ∎

Corollary 5.5.10 *If $B \in \mathcal{M}(\mathbf{k})$ and $X = \operatorname{Spec} B$, then the set*

$$\operatorname{Sing}(X) = \{\, \mathfrak{p} \in \operatorname{Spec} B \mid B_{\mathfrak{p}} \text{ is not a regular local ring} \,\}$$

is a finite set of maximal ideals of B. Consequently, B is regular in codimension 1.

Proof: The set $T = \operatorname{Sing} X$ is a proper closed subset of X, so $\dim T \leq 1$. It follows by Proposition 5.5.9 that given a height 1 prime ideal $\mathfrak{p}$ of B belonging to T, $D(\mathfrak{p}) \subseteq \mathfrak{p}$ for every $D \in \text{LND}(B)$. Then Lemma 5.2.8 implies that $\mathfrak{p} \cap \ker D \neq 0$ for every $D \in \text{LND}(B)$. Since B has property $(*)$ by Theorem 5.5.6, we obtain that the set $\text{KLND}(B)$ is a singleton, a contradiction. So T contains no height 1 prime ideal; consequently, B is regular in codimension 1. This also proves that $\dim T = 0$. So T is a finite set of maximal ideals. ∎

5.6 An application to complete intersections

In this section, $\mathbf{k}$ is an arbitrary field of characteristic zero. By Serre's normality criterion and Corollary 5.5.10, we obtain:

Corollary 5.6.1 *If $B \in \mathcal{M}(\mathbf{k})$ and B satisfies Serre's condition (S_2), then B is normal.*

Note also the following consequence of Lemma 1.5.18 and Corollary 5.5.10:

Corollary 5.6.2 *If $B \in \mathcal{M}(\mathbf{k})$ and B is a complete intersection over $\mathbf{k}$ then B is normal.*

Corollary 5.6.3 *Let S be a hypersurface of $\mathbb{A}^3_{\mathbf{k}}$. If S belongs to $\mathcal{M}(\mathbf{k})$, then S is normal.*

5.7 Algebraic approach, Bhatwadekar's theorem

Throughout this section, assume that $\mathbf{k}$ is a field of characteristic zero. We now proceed to give a purely algebraic proof of Theorem 5.5.6 which relies on a result of Bhatwadekar.

Lemma 5.7.1 *Let A be a noetherian normal domain and $f_1, \ldots, f_n \in A$. If no height 1 prime ideal of A contains $f_1, \ldots, f_n$, then $A = A_{f_1} \cap \cdots \cap A_{f_n}$.*

Proof: Let $x \in A_{f_1} \cap \cdots \cap A_{f_n}$. If $\mathfrak{p}$ is a height 1 prime ideal of A, then $f_i \notin \mathfrak{p}$ for some i. Consequently, $A_{f_i} \subseteq A_{\mathfrak{p}}$; hence, $x \in A_{\mathfrak{p}}$. As $A = \bigcap_{\mathrm{ht}\, \mathfrak{p}=1} A_{\mathfrak{p}}$ by [49, Theorem 11.5, p. 81], it follows that $x \in A$. ∎

Lemma 5.7.2 *Let K be a field and A be a normal domain. If $K \subset A \subseteq K[x] = K^{[1]}$ and* $\mathrm{Frac}\, A = K(x)$, *then $A = K[x]$.*

Proof: Since A contains a nonconstant polynomial $f \in K[x]$, it follows that x is integral over A. As $x \in \mathrm{Frac}\, A$ and A is normal, it follows that $x \in A$. ∎

Theorem 5.7.3 (Bhatwadekar) *Let $\mathbf{k}$ be a field of characteristic zero, and let R be a $\mathbf{k}$-affine normal domain of dimension 2. Assume that $\mathbf{k}[x]$ and $\mathbf{k}[y]$ are distinct elements of* $\mathrm{KLND}(R)$. *Then R is integral over $\mathbf{k}[x, y]$.*

Proof: Let $A = \mathbf{k}[x, y]$ and let B be the integral closure of A in $\mathrm{Frac}\, R$. As R is normal and $R \supseteq A$, it follows that $B \subseteq R$. Note that $\mathbf{k}[x]$ is algebraically closed in R (cf. 1.2 (1)), and $y \notin \mathbf{k}[x]$. It follows that $A = \mathbf{k}^{[2]}$, and so $\mathrm{Frac}\, R$ is algebraic over $\mathrm{Frac}\, A$. Since $\mathrm{Frac}\, A \otimes_A B$ is the integral closure of $\mathrm{Frac}\, A$ in $\mathrm{Frac}\, R$, it follows that $\mathrm{Frac}\, A \otimes_A B = \mathrm{Frac}\, R$. Thus, $B \subseteq R$ is a birational inclusion of rings.

Let $S = \mathbf{k}[x] \setminus \{0\}$, then $S^{-1}R = \mathbf{k}(x)^{[1]}$ holds by 1.2 (5). Hence $\mathbf{k}(x) \subseteq S^{-1}B \subseteq S^{-1}R = \mathbf{k}(x)^{[1]}$, where $S^{-1}B$ is normal and birational to $S^{-1}R$. It follows that $S^{-1}B = S^{-1}R$ by Lemma 5.7.2; in particular, $R \subseteq S^{-1}B$. As R is affine, $R \subseteq B_s$ for some $s \in \mathbf{k}[x] \setminus \{0\}$. Similarly, $R \subseteq B_t$ for some $t \in \mathbf{k}[y] \setminus \{0\}$. So $R \subseteq B_s \cap B_t$. To complete the proof of the theorem, it suffices to show that $B_s \cap B_t = B$.

As $s \in \mathbf{k}[x] \setminus \{0\}$, $t \in \mathbf{k}[y] \setminus \{0\}$ and $A = \mathbf{k}[x, y] = \mathbf{k}^{[2]}$, it is clear that no height 1 prime ideal of A contains both s and t. As B is integral over A, it follows that no height 1 prime ideal of B contains s, t. Then Lemma 5.7.1 implies that $B_s \cap B_t = B$, and we are done. ∎

The following gives another proof of Theorem 5.5.6.

Corollary 5.7.4 *Each element B of $\mathcal{M}(\mathbf{k})$ has property $(*)$.*

Proof: As we argued in the first paragraph of the proof of Theorem 5.5.6, it suffices to prove the case where B is normal. So, suppose that B is normal and (by contradiction) suppose that there exist distinct $A_1, A_2 \in \text{KLND}(B)$ and a height 1 ideal I of B such that $I \cap A_i \neq 0$ for $i = 1, 2$. Pick a height 1 prime ideal $\mathfrak{p}$ of B such that $\mathfrak{p} \supseteq I$, and note that $\mathfrak{p} \cap A_i \neq 0$ for $i = 1, 2$. By Lemma 5.3.6 and the fact that B is normal, we have $A_i \cong \mathbf{k}^{[1]}$ for each $i = 1, 2$. Write $A_1 = \mathbf{k}[x]$ and $A_2 = \mathbf{k}[y]$. So the height 1 prime ideal $\mathfrak{p}$ of B satisfies $\mathfrak{p} \cap \mathbf{k}[x] \neq 0$ and $\mathfrak{p} \cap \mathbf{k}[y] \neq 0$. Then the prime ideal $\mathfrak{p}_0 = \mathfrak{p} \cap \mathbf{k}[x, y] = \mathbf{k}^{[2]}$ must be maximal, since it contains a nonzero element of $\mathbf{k}[x]$ and a nonzero element of $\mathbf{k}[y]$. As B is integral over $\mathbf{k}[x, y]$ by Theorem 5.7.3, it follows that $\mathfrak{p}$ is a maximal ideal of B and hence has height 2, a contradiction.

∎

Some concluding remarks

While Chapter 4 describes explicitly the homogeneous locally nilpotent derivations of $B = \mathbf{k}^{[3]}$ in the case of a $\mathbb{Z}$-grading $\mathfrak{g}$ of positive type, it still remains an open question to give such a description in the case type$(\mathfrak{g}) = 0$. The criterion given in Chapter 3 is a useful piece of information, but does not answer the question in a satisfactory way. This becomes clear if we compare Chapter 3 with the extensive theory which exists in the special case of $\mathbb{N}$-gradings of type 0. As that theory (cf. [16] and [17]) is based on the study of the geometry of the weighted projective planes $\text{Proj}\,(B, \mathfrak{g})$, which are defined only when $\mathfrak{g}$ is an $\mathbb{N}$-grading, it is not clear how to carry out a similar investigation when $\mathfrak{g}$ is a $\mathbb{Z}$-grading.

We wish to mention that Theorem 2.6.3 of Chapter 2 is used by Daigle in [14], where he gives a necessary condition that a polynomial $f \in B = \mathbf{k}[X, Y, Z]$ must satisfy if there exists a nonzero locally nilpotent derivation of B satisfying $D^2(f) = 0$.

The main results of Chapter 5 (in particular, Theorem 5.5.6 and its corollaries) should eventually appear in [41], and are used in the joint paper [21] with Daigle. One of the main results of [21] is as follows:

Theorem. *A complete intersection surface over* $\mathbf{k}$ *with trivial Makar-Limanov invariant is isomorphic as an algebraic surface to the hypersurface of* $\mathbb{A}^3_{\mathbf{k}}$ *given by* $XY = P(Z)$*, where* $P(Z) \in \mathbf{k}[Z]$ *is a nonconstant polynomial.*

In this result, $\mathbf{k}$ is any field of characteristic zero and no assumption of smoothness or normality is made on the surface. Corollary 5.5.10 of this thesis is needed in the proof of the above theorem.

We believe that the machinery developed in [21] can be used to investigate the following interesting question :

Classify all nonrigid hypersurfaces of $\mathbb{A}^3_{\mathbf{k}}$.

Next, we say that an affine surface S is *quasihomogeneous* (with respect to $\mathrm{Aut}(S)$) if the natural action of $\mathrm{Aut}(S)$ on S has a Zariski-open orbit with finite complement. It is known that smooth ML-hypersurfaces of $\mathbb{A}^3$ are quasihomogeneous (see [3]). We are confident that similar arguments as in Proposition 5.5.9 and Corollary 5.5.10 of this thesis would prove that an affine ML-surface is quasihomogeneous with respect to its group of automorphisms. Related to this subject, the following question arises naturally:

Which quasihomogeneous surfaces have trivial ML-invariant?

The above question has been answered by Gizatullin, Bertin, Bandman, Makar-Limanov (in the smooth case) and Dubouloz (in the normal case). We hope to investigate this question in the case when the surface under consideration is not necessarily normal.

Appendix A

Valuation rings

The purpose of Appendix A is to prove Lemmas 1.8.10 and 1.8.13 stated in Chapter 1. To prove these results, we gather and develop some basic results in the theory of valuation rings. We refer to Section 1.8 of Chapter 1 for the basic definitions and facts about valuation rings.

Theorem A.0.5 [49, Proposition 10.2] *Let K be a field, $A \subseteq K$ a subring, and $\mathfrak{p}$ a prime ideal of A. Then there exists a valuation ring R of K such that $A \subseteq R$ and $M_R \cap A = \mathfrak{p}$.*

Remark A.0.6 Let R and S be two valuation rings of a field K. If $R \subseteq S$ and $M_R \subseteq M_S$, then $R = S$. Indeed, given $x \in K$,

$$x \notin R \Rightarrow x^{-1} \in M_R \Rightarrow x^{-1} \in M_S \Rightarrow x \notin S.$$

Thus $K \setminus R \subseteq K \setminus S$; it follows that $R = S$.

Given fields $\mathbf{k} \subseteq F$, recall the notations $\mathrm{Val}(F/\mathbf{k})$ and $\mathbb{P}_{F/\mathbf{k}}$ from 1.8.7.

Lemma A.0.7 *Given field extensions $\mathbf{k} \subseteq F \subseteq F'$, we have a surjective set map*

$$\varphi : \mathrm{Val}(F'/\mathbf{k}) \to \mathrm{Val}(F/\mathbf{k}), \quad R' \mapsto R' \cap F.$$

Proof: Consider $R' \in \mathrm{Val}(F'/\mathbf{k})$ and let $R = R' \cap F$. First we prove that $R \in \mathrm{Val}(F/\mathbf{k})$. Clearly $\mathbf{k} \subseteq R$. Given $x \in F^*$, we have to show that $x \in R$ or $x^{-1} \in R$. As $F^* \subseteq F'^*$ and $R' \in \mathrm{Val}(F'/\mathbf{k})$, it follows that $x \in R'$ or $x^{-1} \in R'$. Then $x \in R' \cap F$ or $x^{-1} \in R' \cap F$, which proves that $x \in R$ or $x^{-1} \in R$. So φ is a well defined map. Moreover we have:

$$M_R = R \setminus R^* = (R' \cap F) \setminus (R^* \cap F) = (R' \setminus R'^*) \cap F = M_{R'} \cap F. \tag{A.0.1}$$

To prove that φ is surjective, let $S \in \mathrm{Val}(F/\mathbf{k})$ and note that $\mathbf{k} \subseteq S \subseteq F \subseteq F'$. By Theorem A.0.5, there exists $R' \in \mathrm{Val}(F'/\mathbf{k})$ such that $S \subseteq R'$ and $M_{R'} \cap S = M_S$. Note that S and $R' \cap F$ belong to $\mathrm{Val}(F/\mathbf{k})$. As $S \subseteq R' \cap F$ and $M_S \subseteq M_{R'} \cap F = M_{R' \cap F}$ by Equation (A.0.1), $S = R' \cap F$ follows by Remark A.0.6 and φ is surjective. ∎

Definition A.0.8 Consider fields $\mathbf{k} \subseteq F \subseteq F'$, $R \in \mathrm{Val}(F/\mathbf{k})$, and $R' \in \mathrm{Val}(F'/\mathbf{k})$. We say that R' *lies over* R if $R' \cap F = R$.

The following is a well-known fact; see [58, Theorem I.2.2], for instance.

Lemma A.0.9 *Let* $\mathbf{k}$ *be a field,* $K = \mathbf{k}(t) = \mathbf{k}^{(1)}$*. Then*

$$\mathbb{P}_{K/\mathbf{k}} = \left\{ \mathbf{k}[t]_{\mathfrak{m}} \mid \mathfrak{m} \in \mathrm{Max}\,\mathbf{k}[t] \right\} \cup \{\mathbf{k}[1/t]_{(1/t)}\},$$

where $\mathrm{Max}\,\mathbf{k}[t]$ *denotes the set of all maximal ideals of* $\mathbf{k}[t]$*. In particular,* $\mathbf{k}[t]$ *has only one (rational) place at infinity.*

For a proof of the following fact from field theory, see [52, 1.2].

Theorem A.0.10 (Generalized Lüroth Theorem) *Let* $\mathbf{k} \subseteq K \subseteq \mathbf{k}^{(n)}$ *be field extensions, where* $\mathrm{trdeg}_{\mathbf{k}}(K) = 1$*. Then* $K = \mathbf{k}^{(1)}$*.*

The following results (Lemmas A.0.11, A.0.12, A.0.13 and Theorem A.0.14) are taken from unpublished lecture notes of Daigle.

Lemma A.0.11 *For a domain A of* trdeg 1 *over a field* $\mathbf{k}$*, the following are equivalent:*

(1) *The normalization of A is $\mathbf{k}^{[1]}$.*

(2) *A is a subalgebra of $\mathbf{k}^{[1]}$.*

(3) $\operatorname{Frac} A = \mathbf{k}^{(1)}$ *and A has one rational place at infinity.*

Proof: Set $K = \operatorname{Frac} A$. Clearly, (1) implies (2). To prove that (2) implies (3), choose a ring $\mathbf{k}[t] = \mathbf{k}^{[1]}$ such that $\mathbf{k} \subseteq A \subseteq \mathbf{k}[t]$. Then $K = \mathbf{k}^{(1)}$ by Lüroth's Theorem. As the set $E = \{\, S \in \mathbb{P}_{K/\mathbf{k}} \mid A \nsubseteq S \,\}$ is nonempty, it follows that A has at least one place at infinity, say $A \nsubseteq R$. To prove (3), it remains to show that such R is unique and has residue field $\mathbf{k}$.

Consider the field extensions $\mathbf{k} \subseteq K \subseteq \mathbf{k}(t)$. There exists $R' \in \operatorname{Val}(\mathbf{k}(t)/\mathbf{k})$ such that $R' \cap K = R$ by Lemma A.0.7. If $\mathbf{k}[t] \subseteq R'$ then then $A \subseteq R' \cap K = R$, a contradiction. So $\mathbf{k}[t] \nsubseteq R'$ and R' is a place at infinity of $\mathbf{k}[t]$. As $\mathbf{k}[t]$ has one rational place at infinity, it follows that $R'/M_{R'} \cong \mathbf{k}$.

If R_1 is a place at infinity for A satisfying $R_1 \neq R$, then there exists R_1' lying over R_1, and $R_1 \neq R \Longrightarrow R_1' \neq R'$. As $\mathbf{k}[t]$ has only one place at infinity, it follows that A has only one place at infinity. Furthermore, $\mathbf{k} \subseteq R/M_R \subseteq R'/M_R' \cong \mathbf{k}$; it follows that $R/M_R \cong \mathbf{k}$, and A has a unique rational place at infinity. Thus (2) implies (3).

Finally, we prove that (3) implies (1). Suppose that $K = \mathbf{k}(u) = \mathbf{k}^{(1)}$ and that A has one rational place at infinity, say R. Recall from Lemma A.0.9 that

$$\mathbb{P}_{K/\mathbf{k}} = \{\, \mathbf{k}[u]_{\mathfrak{m}} \mid \mathfrak{m} \in \operatorname{Max} \mathbf{k}[u] \,\} \cup \{\mathbf{k}[1/u]_{(1/u)}\}.$$

If $R = \mathbf{k}[1/u]_{(1/u)}$, then we have:

$$\tilde{A} = \bigcap V = \bigcap_{\mathfrak{m} \in \operatorname{Max} \mathbf{k}[u]} \mathbf{k}[u]_{\mathfrak{m}} = \mathbf{k}[u],$$

where V varies over the set of elements of $\operatorname{Val}(K/\mathbf{k})$ that contain A. So $\tilde{A} = \mathbf{k}^{[1]}$ in this case and we are done. If $R = \mathbf{k}[u]_{\mathfrak{m}}$ for some $\mathfrak{m} \in \operatorname{Max} \mathbf{k}[u]$, then $\mathbf{k}[u]/\mathfrak{m} \cong R/M_R \cong \mathbf{k}$

since R is a rational place at infinity of A. It follows that $\mathfrak{m} = (u - \lambda)$ for for some $\lambda \in \mathbf{k}$. Then $R = \mathbf{k}[u]_{(u-\lambda)} = \mathbf{k}[u - \lambda]_{(u-\lambda)} = \mathbf{k}[1/u']_{(1/u')}$, where $u - \lambda = 1/u'$, and the result follows by the previous case. ∎

Lemma A.0.12 *If A is a polynomial curve over* $\mathbf{k}$*, then A is* $\mathbf{k}$*-affine.*

Proof: Let $\mathbf{k}[t] = \mathbf{k}^{[1]}$ be the normalization of A, and note that t is integral over A. Assume that $t^n + a_1 t^{n-1} + \cdots + a_n = 0$ for some $a_1, \ldots, a_n \in A$. It follows that t is integral over $A_0 = \mathbf{k}[a_1, \ldots, a_n]$. Consequently, $\mathbf{k}[t]$ is a finitely generated module over the noetherian ring A_0. Then the submodule A of $\mathbf{k}[t]$ is finitely generated, so A is $\mathbf{k}$-affine. ∎

Lemma A.0.13 *Suppose that A is an algebra over a field* $\mathbf{k}$ *and that $K/\mathbf{k}$ is a field extension such that $K \otimes_{\mathbf{k}} A$ is K-affine. Then A is* $\mathbf{k}$*-affine.*

Proof: Choose $a_1, \ldots, a_n \in A$ so that $1 \otimes a_1, \ldots, 1 \otimes a_n$ generate $K \otimes_{\mathbf{k}} A$ as a K-algebra. Write $A_0 = \mathbf{k}[a_1, \ldots, a_n]$ and consider the inclusion map $i : A_0 \hookrightarrow A$. As K is a flat $\mathbf{k}$-module, the homomorphism of K-algebras $(1 \otimes i) : K \otimes_{\mathbf{k}} A_0 \hookrightarrow K \otimes_{\mathbf{k}} A$ is injective. Note that $(1 \otimes i)(1 \otimes a_i) = 1 \otimes a_i$ for every $i = 1, \ldots, n$. Since $K \otimes_{\mathbf{k}} A = K[1 \otimes a_1, \ldots, 1 \otimes a_n]$, it follows that $1 \otimes i$ is surjective. This, together with the fact that K is a faithfully flat $\mathbf{k}$-module, implies that i is surjective, so $A = A_0 = \mathbf{k}[a_1, \ldots, a_n]$. Thus A is a finitely generated $\mathbf{k}$-algebra.

∎

Theorem A.0.14 *Let* $\mathbf{k}$ *be a perfect field and let A be a* $\mathbf{k}$*-domain. If there exists an algebraic extension $K/\mathbf{k}$ such that $K \otimes_{\mathbf{k}} A$ is a polynomial curve over K, then A*

is a polynomial curve over $\mathbf{k}$.

Proof: Choose an algebraic extension K of $\mathbf{k}$ such that $K \otimes_{\mathbf{k}} A$ is a polynomial curve over K. Note that $K \otimes_{\mathbf{k}} A$ is K-affine by Lemma A.0.12 and A is $\mathbf{k}$-affine by Lemma A.0.13. Write $A = \mathbf{k}[a_1, \dots, a_n]$. Choose a polynomial ring $K[t] = K^{[1]}$ which contains $K \otimes_{\mathbf{k}} A$ as a subalgebra. For each $i = 1, \dots, n$, write $a_i = f_i(t) \in K[t]$. Let S be the finite subset of K which contains the coefficients of the polynomials $f_1, f_2, \dots, f_n$ and define $\mathbf{k}' = \mathbf{k}[S]$. Then the image of the composite $\mathbf{k}' \otimes_{\mathbf{k}} A \to K \otimes_{\mathbf{k}} A \to K[t]$ is included in $\mathbf{k}'[t]$. So

$$[\mathbf{k}' : \mathbf{k}] < \infty \text{ and } \mathbf{k}' \otimes_{\mathbf{k}} A \text{ is a polynomial curve over } \mathbf{k}'.$$

Define $F = \operatorname{Frac} A$ and $F' = (\mathbf{k}' \otimes_{\mathbf{k}} A) \otimes_A F = \mathbf{k}' \otimes_{\mathbf{k}} F$ and note that all homomorphisms are injective in the following diagram:

$$\begin{array}{ccccc} \mathbf{k}' & \hookrightarrow & \mathbf{k}' \otimes_{\mathbf{k}} A & \hookrightarrow & F' \\ \uparrow & & \uparrow & & \uparrow \\ \mathbf{k} & \hookrightarrow & A & \hookrightarrow & F \end{array} \tag{A.0.2}$$

Since $F' = (\mathbf{k}' \otimes_{\mathbf{k}} A) \otimes_A F$ is a localization of the domain $\mathbf{k}' \otimes_{\mathbf{k}} A$, we have $\mathbf{k}' \otimes_{\mathbf{k}} A \subseteq F' \subseteq \operatorname{Frac}(\mathbf{k}' \otimes_{\mathbf{k}} A)$. As $\mathbf{k}'$ is algebraic over $\mathbf{k}$, $F' = \mathbf{k}' \otimes_{\mathbf{k}} F$ is integral over the field F, and hence is a field. Thus, $F' = \operatorname{Frac}(\mathbf{k}' \otimes_{\mathbf{k}} A)$.

Let $\mathcal{B}$ be a basis of $\mathbf{k}'$ over $\mathbf{k}$ such that $1 \in \mathcal{B}$. Then $\mathcal{B}$ is a basis of F' over F. This implies that

$$[F' : F] = [\mathbf{k}' : \mathbf{k}], \quad F' = \mathbf{k}'F, \quad \mathbf{k}' \cap F = \mathbf{k}.$$

As $\mathbf{k}' \otimes_{\mathbf{k}} A$ is a polynomial curve over $\mathbf{k}'$, it follows that $F' = \mathbf{k}'^{(1)}$. In particular, $\mathbf{k}'$ is algebraically closed in F'. From this and $\mathbf{k}' \cap F = \mathbf{k}$, it follows that $\mathbf{k}$ is algebraically closed in F. In the terminology of [58], $\mathbf{k}$ is the full field of constants of the function field $F/\mathbf{k}$. As $F' = \mathbf{k}'F$, $F'/\mathbf{k}'$ is an algebraic constant field extension of $F/\mathbf{k}$. Now

$\mathbf{k}$ is assumed to be perfect, so all the hypothesis of [58, Theorem III.6.3, p. 103] are satisfied. Hence $F/\mathbf{k}$ has genus zero and F'/F is unramified. It suffices to prove that

$$A \text{ has one rational place at infinity.} \tag{A.0.3}$$

Indeed, if this is true then $F = \mathbf{k}^{(1)}$ (because $F/\mathbf{k}$ has genus 0 and has a rational place), so Lemma A.0.11 implies that A is a polynomial curve over $\mathbf{k}$.

Consider Diagram (A.0.2) and let

$$E = \left\{ R \in \mathbb{P}_{F/\mathbf{k}} \mid A \nsubseteq R \right\} \text{ and } E' = \left\{ R' \in \mathbb{P}_{F'/\mathbf{k}'} \mid \mathbf{k}' \otimes_{\mathbf{k}} A \nsubseteq R' \right\}.$$

If R is any element of E then every $R' \in \mathbb{P}_{F'/\mathbf{k}'}$ lying over R (i.e., satisfying $R' \cap F = R$) must belong to E'. But E' is a singleton, say $E' = \{R'\}$. It follows that E is a singleton, say $E = \{R\}$, and that R' is the only element of $\mathbb{P}_{F'/\mathbf{k}'}$ lying over R. Let κ' and κ be the residue fields of R' and R, respectively. Then $[F' : F] = ef$, where $f = [\kappa' : \kappa]$ and e is the ramification index of R' over R. As F'/F is unramified, we have $e = 1$. Also, as $\mathbf{k}' \otimes_{\mathbf{k}} A$ is a polynomial curve over $\mathbf{k}'$, $\kappa' = \mathbf{k}'$. Hence

$$[\mathbf{k}' : \mathbf{k}] = [F' : F] = ef = [\kappa' : \kappa] = [\mathbf{k}' : \kappa],$$

so $\kappa = \mathbf{k}$ and (A.0.3) is proved.

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